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THE FERMIONIC SIGNATURE OPERATOR AND QUANTUM STATES IN RINDLER SPACE-TIME

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Abstract. The fermionic signature operator is constructed in Rindler space-time. It is shown to be an unbounded self-adjoint operator on the Hilbert space of solutions of the massive Dirac equation. In two-dimensional Rindler space-time, we prove that the resulting fermionic projector state coincides with the Fulling-Rindler vacuum. Moreover, the fermionic signature operator gives a covariant construction of general thermal states, in particular of the Unruh state. The fermionic signature operator is shown to be well-defined in asymptotically Rindler space-times. In four-dimensional Rindler space-time, our construction gives rise to new quantum states.

Contents

1. Introduction

The fermionic signature operator is a symmetric operator on the solution space of the massive Dirac equation in globally hyperbolic space-times. It encodes geometric information [11] and gives a new covariant method for obtaining Hadamard states [12], the so-called fermionic projector (FP) states. The abstract construction in space-times

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of finite and infinite lifetime as given in [13, 14] opens up the research program to explore the fermionic signature operator in various space-times and to verify if the resulting FP states are Hadamard. So far, the fermionic signature operator has been studied in the examples of closed FRW space-times [13], ultrastatic space-times and de Sitter space-time [14], an external potential in Minkowski space [12] and different two-dimensional space-times [11].

As the first example involving a horizon, we here consider two-dimensional Rindler space-time (see [20] or [23, Section 6.4]). This is of physical interest in view of the Unruh effect, which is closely related to the Hawking effect in black hole geometries (see for example [3]). Also, from a mathematical point of view, the example of Rindler space-time is interesting because, although lifetime is infinite, the methods in [14] do not apply. The reason is that the strong mass oscillation property does not hold due to boundary contributions on the horizon (for boundary contributions in a more general setting see [4]). Instead, we adapt the construction for space-times of finite lifetime in [13], making it possible to define the fermionic signature operator as a densely defined unbounded operator. Our construction is covariant in the sense that it does not depend on the choice of specific coordinates. We will show that this operator is indeed a multiple of the Dirac Hamiltonian in Rindler coordinates (for details see Theorem 10.1 below). This means that the construction of the fermionic signature operator "detects" the Killing symmetry of our space-time as described by translations in Rindler time. We thus obtain a covariant construction of the Fulling-Rindler vacuum [15] and of general thermal states like the Unruh state [22] (see Corollaries 11.1 and 11.2; for a general introduction to quantum states in Rindler space-time see for example [24, Chapter 5]).

Extending the above analysis to four-dimensional Rindler space-time, we obtain states which are indeed different from the Fulling-Rindler vacuum and general thermal states (see Section 13). The physical properties of these new states are still under investigation.

It is a main advantage of our construction that it also applies in situations without Killing symmetries. This is made clear by considering asymptotically Rindler space-times (see Theorem 12.1).

2. Preliminaries

In this section, we recall a few basic definitions, mainly using the notation and conventions in [13]. We restrict attention to the two-dimensional situation (for the four-dimensional setting see Section 13). The two-dimensional *Rindler space-time* $(\mathscr{R}, g)$ is isometric to the subset of two-dimensional Minkowski space

$$\mathscr{R} = \big\{ (t,x) \in \mathbb{R}^{1,1} \quad \text{with} \quad |t| < x \big\} \tag{2.1}$$

with the induced line element

$$ds^2 = g_{ij}\, dx^i dx^j = dt^2 - dx^2 \,. \tag{2.2}$$

We let $S\mathscr{R} = \mathscr{R} \times \mathbb{C}^2$ be the trivial spinor bundle. We work in the so-called chiral representation of the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.3}$$

The Dirac matrices are symmetric with respect to the *spin scalar product* defined by

$$\prec\psi|\phi\succ = \langle\psi, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \phi\rangle_{\mathbb{C}^2} \tag{2.4}$$

(where $\langle .,. \rangle_{\mathbb{C}^2}$ is the canonical scalar product on $\mathbb{C}^2$). Introducing the *Dirac operator*

$$\mathcal{D} := i\gamma^j \partial_j \,, \tag{2.5}$$

the *massive Dirac equation* reads

$$(\mathcal{D} - m)\psi = 0 \,, \tag{2.6}$$

where $m > 0$ is the rest mass (we always work in natural units $\hbar = c = 1$).

Rindler space-time is globally hyperbolic (for example, for any $\alpha \in (-1, 1)$, the ray $\{(\alpha x, x)$ with $x > 0\}$ is a Cauchy surface). Taking smooth and compactly supported initial data on a Cauchy surface $\mathscr{N}$ and solving the Cauchy problem, one obtains a Dirac solution in the class $C^\infty_{\text{sc}}(\mathscr{R}, S\mathscr{R})$ of smooth wave functions with spatially compact support. On solutions ψ, ϕ in this class, one defines the scalar product

$$(\psi|\phi) := 2\pi \int_{\mathscr{N}} \prec\psi|\slashed{\nu}\phi\succ|_q \, d\mu_{\mathscr{N}}(q) \,, \tag{2.7}$$

where $\slashed{\nu} = \gamma^j \nu_j$ denotes Clifford multiplication by the future-directed normal ν. Due to current conservation, this scalar product is independent of the choice of $\mathscr{N}$. Forming the completion, we obtain the Hilbert space $(\mathcal{H}, (.|.))$, referred to as the *solution space* of the Dirac equation. We denote the norm on this Hilbert space by $\|\psi\| := \sqrt{(\psi|\psi)}$. Another object which will be important later on is the *space-time inner product*

$$<\psi|\phi> := \int_{\mathscr{R}} \prec\psi|\phi\succ|_q \, d\mu_{\mathscr{R}}(q) \,. \tag{2.8}$$

A priori, this integral may diverge for solutions of the Dirac equation. But the space-time inner product is well-defined for example if one of the wave functions has compact support.

The method for constructing the fermionic signature operator $\mathcal{S}$ is as follows: Let $\mathcal{D}(\mathcal{S})$ be a subspace of $\mathcal{H}$ such that for any $\phi \in \mathcal{D}(\mathcal{S})$, the anti-linear mapping $<.|\phi> : \mathcal{H} \to \mathbb{C}$ given by (2.8) is well-defined and bounded, i.e.

$$\big|<\psi|\phi>\big| \le C(\phi)\, \|\psi\| \qquad \text{for all } \psi \in \mathcal{H}$$

for a suitable constant $C(\phi) < \infty$. Then the Fréchet-Riesz theorem makes it possible to represent this anti-linear mapping by a vector $\mathcal{S}\phi$, i.e.

$$<\psi|\phi> = (\psi \,|\, \mathcal{S}\phi) \qquad \text{for all } \psi \in \mathcal{H} \,. \tag{2.9}$$

Varying $\phi \in \mathcal{D}(\mathcal{S})$, we obtain a linear mapping

$$\mathcal{S} \,:\, \mathcal{D}(\mathcal{S}) \to \mathcal{H} \,,$$

referred to as the *fermionic signature operator*. Obviously, this operator is symmetric on the Hilbert space $\mathcal{H}$. Our goal is to show that in Rindler space-time, the domain $\mathcal{D}(\mathcal{S})$ of this operator can be chosen as a dense subset of $\mathcal{H}$, and that the fermionic signature operator has a self-adjoint extension.

3. Embedding in Minkowski Space

For the subsequent analysis, it is often useful to regard Rindler space-time as a subset of Minkowski space, and to embed the solution space in Rindler space-time into the solution space in Minkowski space. We now explain this construction. Let $(\mathscr{M}, g)$ be the two-dimensional Minkowski space (thus $\mathscr{M} = \mathbb{R}^{1,1}$ with the metric (2.2)). Moreover, we let $S\mathscr{M} = \mathscr{M} \times \mathbb{C}^2$ be the trivial spinor bundle, again with the spin scalar product (2.4). Then the inclusions

$$\mathscr{R} \subset \mathscr{M} \qquad \text{and} \qquad S\mathscr{R} = \mathscr{R} \times \mathbb{C}^2 \subset \mathscr{M} \times \mathbb{C}^2 = S\mathscr{M}$$

are clearly isometries. The Dirac operator and the Dirac equation are again given by (2.5) and (2.6). For clarity, we denote the scalar product (2.7) in Minkowski space with an additional subscript $\mathscr{M}$. For convenience, we always choose $\mathscr{N}$ as the Cauchy surface $\{t = 0\}$, so that

$$(\Psi|\Phi)_{\mathscr{M}} = 2\pi \int_{-\infty}^{\infty} \prec\!\Psi|\gamma^0\Phi\!\succ\!|_{(0,x)}\, dx \tag{3.1}$$

(to avoid confusion, we consistently denote wave functions in Minkowski space by capital Greek letters, whereas wave functions in Rindler space-time are denoted by small Greek letters). The corresponding Hilbert space is denoted by $(\mathcal{H}_{\mathscr{M}}, (.|.)_{\mathscr{M}})$. In order to extend Dirac solutions from Rindler space-time to Minkowski space, let $\psi \in C^\infty_{\text{sc}}(\mathscr{R}, S\mathscr{R})$ be a solution with spatially compact support. Thus restricting it to the ray $\{(0, x) \text{ with } x > 0\}$ gives a smooth function with compact support. We extend this function by zero to the Cauchy surface $\{t = 0\}$, i.e.

$$\Psi_0(x) := \begin{cases} \psi(0,x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \qquad \in C^\infty_0(\mathbb{R}, \mathbb{C}^2)\,.$$

Solving the Cauchy problem in $\mathscr{M}$ with initial data Ψ_0 yields a solution $\Psi(t, x)$ in Minkowski space. We thus obtain an isometric embedding

$$\iota_{\mathscr{M}} \,:\, \mathcal{H} \rightarrow \mathcal{H}_{\mathscr{M}}\,.$$

It is also useful to introduce the operator $\pi_{\mathscr{R}}$ as the restriction to Rindler space-time,

$$\pi_{\mathscr{R}} \,:\, \mathcal{H}_{\mathscr{M}} \rightarrow \mathcal{H}\,, \qquad \pi_{\mathscr{R}}\Psi = \Psi|_{\mathscr{R}}\,.$$

Obviously, the identity

$$\pi_{\mathscr{R}} \circ \iota_{\mathscr{M}} = \mathbb{1}_{\mathcal{H}}$$

holds. Moreover, for every $\Psi \in \mathcal{H}_{\mathscr{M}}$ and $\phi \in \mathcal{H}$,

$$\big(\Psi \,\big|\, \iota_{\mathscr{M}}\phi\big)_{\mathscr{M}} = 2\pi \int_0^\infty \prec\!\Psi|\gamma^0\phi\!\succ\!|_{(0,x)}\, dx = \big(\pi_{\mathscr{R}}\Psi \,\big|\, \phi\big)\,,$$

which can be written as

$$\iota^*_{\mathscr{M}} = \pi_{\mathscr{R}}\,.$$

This relation also shows that the orthogonal complement of the image of $\iota_{\mathscr{M}}$ coincides with the kernel of $\pi_{\mathscr{R}}$, consisting of all Dirac solutions in Minkowski space which vanish on the ray $\{(0, x) \text{ with } x > 0\}$.

In analogy to (2.8), the space-time inner product in Minkowski space is defined by

$$<\!\Psi|\Phi\!>_{\mathscr{M}} := \int_{\mathscr{M}} \prec\!\Psi|\Phi\!\succ\!|_q\, d\mu_{\mathscr{M}}(q)\,.$$

It is not directly related to (2.8) because one integrates over a different space-time region. However, a direct connection can be obtained by inserting the characteristic function of Rindler space-time into the integrand,

$$<\Psi|\Phi>_{\mathscr{R}} := \int_{\mathscr{M}} \chi_{\mathscr{R}}(q) \prec\!\Psi|\Phi\!\succ\!|_q \, d\mu_{\mathscr{M}}(q) \,. \tag{3.2}$$

Then for any $\Psi, \Phi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$,

$$<\Psi|\Phi>_{\mathscr{R}} = <\pi_{\mathscr{R}} \Psi \,|\, \pi_{\mathscr{R}} \Phi> \,.$$

Introducing the *relative fermionic signature operator* $\mathcal{S}_{\mathscr{R}} : \mathcal{D}(\mathcal{S}_{\mathscr{R}}) \subset \mathcal{H}_{\mathscr{M}} \to \mathcal{H}_{\mathscr{M}}$ in analogy to (2.9) by

$$<\Psi|\Phi>_{\mathscr{R}} = (\Psi \,|\, \mathcal{S}_{\mathscr{R}} \Phi)_{\mathscr{M}} \qquad \text{for all } \Psi \in \mathcal{H}_{\mathscr{M}} \,, \tag{3.3}$$

this fermionic signature operator in Rindler space-time is recovered by

$$\mathcal{S} = \pi_{\mathscr{R}} \, \mathcal{S}_{\mathscr{R}} \, \iota_{\mathscr{M}} \qquad \text{with} \qquad \mathcal{D}(\mathcal{S}) = \pi_{\mathscr{R}}\big(\mathcal{D}(\mathcal{S}_{\mathscr{R}})\big) \,. \tag{3.4}$$

With this in mind, in the remainder of the paper we work exclusively in Minkowski space. For notational simplicity, the subscript $\mathscr{M}$ will be omitted in what follows.

4. The Relative Fermionic Signature Operator as an Unbounded Operator

Lemma 4.1. *For every $\Phi \in C^\infty_{sc}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}$, there is a constant $c = c(\Phi)$ such that*

$$\big|<\Psi|\Phi>_{\mathscr{R}}\big| \le c(\Phi)\, \|\Psi\| \qquad \textit{for all } \Psi \in \mathcal{H} \,.$$

Proof. Let $\Phi \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}$. Then its restriction to the Cauchy surface $\{t = 0\}$ is compact, i.e.

$$\operatorname{supp} \Phi(0, .) \subset (-R, R) \,.$$

Due to finite propagation speed, we know that

$$\operatorname{supp} \Phi(t, .) \subset (-R - |t|, R + |t|) \qquad \text{for all } t \in \mathbb{R} \,. \tag{4.1}$$

We now make use of the fact that solutions of the massive Dirac equation for compactly supported initial data decay rapidly in null directions. More precisely, for any $p \in \mathbb{N}$ there is a constant $C = C(\Phi, p)$ such that

$$\big|\Phi(t, x)\big| \le \frac{C}{1 + |t|^p} \qquad \text{for all } t \in \mathbb{R} \text{ and } x \ge |t| \,. \tag{4.2}$$

This inequality can be verified in two ways. One method is to specialize the more general results in asymptotically flat space-times as derived in [21]. Another method is to use that each component of Φ is a solution of the Klein-Gordon equation

$$\big(\partial_t^2 - \partial_x^2 + m^2\big)\Phi(t, x) = 0$$

and to apply the estimates in [16, Theorem 7.2.1], choosing the parameter N in this theorem to be negative and large.

Combining (4.1) and (4.2) with the Schwarz inequality, we obtain the estimate

$$\begin{aligned} \int_{\mathscr{R}} \big|\prec\!\Psi|\Phi\!\succ\big| \, dt\, dx &\le \int_0^\infty dt \int_{|t|}^{|t|+R} dx \, \|\Psi(t,x)\| \, \|\Phi(t,x)\| \\ &\le \int_0^\infty \|\Psi(t,.)\|_{L^2(dx)} \, \frac{C\sqrt{R}}{1+|t|^p} \, dt = C\,\sqrt{R}\, \|\Psi\| \int_0^\infty \frac{dt}{1+|t|^p} \,. \end{aligned}$$

Choosing $p = 2$ gives the desired estimate. ∎

Using this lemma, for any $\Psi \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}$, the Fréchet-Riesz theorem gives a unique vector $\mathcal{S}_{\mathscr{R}} \Phi \in \mathcal{H}$ such that (3.3) holds. This makes it possible to introduce the relative fermionic signature operator as the densely defined operator

$$\mathcal{S}_{\mathscr{R}} \;:\; C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H} \rightarrow \mathcal{H} \,. \tag{4.3}$$

From (3.3) it is obvious that $\mathcal{S}_{\mathscr{R}}$ is symmetric, i.e.

$$(\Psi \,|\, \mathcal{S}_{\mathscr{R}} \Phi) = (\mathcal{S}_{\mathscr{R}} \Psi \,|\, \Phi) \qquad \text{for all } \Psi, \Phi \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H} \,.$$

We point out that the operator $\mathcal{S}_{\mathscr{R}}$ is *unbounded*. This can be understood from the fact that the inequality (4.2) and the subsequent estimate depend essentially on the support of Φ. In particular, if we consider a sequence of wave functions Ψ_n whose support is shifted more and more to the right,

$$\Phi_n(t,x) = \Phi(t, x-n) \,,$$

then the constant $c(\Phi_n)$ in the statement of Lemma 4.1 must be chosen larger and larger if n is increased. This shows that the inequality

$$|{<}\Psi|\Phi{>}_{\mathscr{R}}| \leq c\, \|\Phi\|\, \|\Psi\| \quad \text{for all } \Psi, \Phi \in \mathcal{H} \qquad \text{is violated} \,,$$

no matter how large the constant c is chosen. Using the terminology introduced in [13, Section 3.2], *Rindler space-time is not m-finite*. Nevertheless, the estimate of Lemma 4.1 enables us to introduce the fermionic signature operator as a densely defined, unbounded symmetric operator. This makes it unnecessary to use the mass oscillation methods introduced in [14] for the construction of the fermionic signature operator in space-times of infinite lifetime.

In order to get into the position to employ spectral methods, we must construct a self-adjoint extension of the relative fermionic signature operator. Our method is to compute $\mathcal{S}_{\mathscr{R}}$ in more detail in momentum space. As we shall see, working with plane waves in a suitable parametrization in momentum space, the operator $\mathcal{S}_{\mathscr{R}}$ becomes a multiplication operator, making it possible to construct a self-adjoint extension with standard functional analytic methods.

5. Transformation to Momentum Space

For the following computations, it is most convenient to work in momentum space. We denote the position and momentum variables by $q = (t,x)$ and $p = (\omega, k)$, respectively. Clearly, any smooth and spatially compact Dirac solution $\Psi \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M})$ can be represented as

$$\Psi(q) = \int_{\mathbb{R}^2} \frac{d^2p}{(2\pi)^2}\, \hat{\Psi}(p)\, \delta(p^2 - m^2)\, e^{-ipq} \,, \tag{5.1}$$

where $\hat{\Psi}$ is a smooth function on the mass shell (and $pq = \omega t - kx$ is the Minkowski inner product). In this momentum representation, the Dirac equation (2.6) reduces to the algebraic equation

$$\big(\not{p} - m\big) \hat{\Psi}(p) = 0 \,.$$

The matrix $\not{p} - m$ has eigenvalues 0 and $-2m$. Its kernel is positive definite with respect to the spin scalar product if p is on the upper mass shell, and it is negative definite if p is on the lower mass shell. Thus we can choose a spinor $\mathfrak{f}(p)$ with the properties

$$(\not{p} - m)\, \mathfrak{f}(p) = 0 \qquad \text{and} \qquad \prec \mathfrak{f}(p) | \mathfrak{f}(p) \succ = \epsilon(\omega) \,, \tag{5.2}$$

where ϵ is the sign function $\epsilon(\omega) = 1$ for $\omega \geq 0$ and $\epsilon(\omega) = -1$ otherwise. More specifically, we choose

$$\mathfrak{f}(p) = \frac{1}{\sqrt{2m}} \, \frac{1}{\sqrt{\epsilon(\omega)\,(\omega-k)}} \begin{pmatrix} m \\ \omega - k \end{pmatrix} . \tag{5.3}$$

Lemma 5.1. *The spinor $\mathfrak{f}(p)$ satisfies the relations*

$$\begin{aligned} \prec \!\mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \mathfrak{f}(-\omega,k) \!\succ &= 0 \\ \prec \!\mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \mathfrak{f}(\omega,k) \!\succ &= \frac{|\omega|}{m} \,. \end{aligned}$$

Proof. These relations can be verified by in a straightforward manner using the explicit formulas (5.2) and (2.3). Alternatively, they can also be derived abstractly by applying the anti-commutation relations of the Dirac matrices:

$$\begin{aligned} &\prec \!\mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \mathfrak{f}(-\omega,k) \!\succ = \frac{1}{m} \prec \!\not{p}\, \mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \mathfrak{f}(-\omega,k) \!\succ \\ &\quad = \frac{1}{m} \prec \!\mathfrak{f}(\omega,k) \,|\, \big(\omega \gamma^0 - k \gamma^1\big)\, \gamma^0 \, \mathfrak{f}(-\omega,k) \!\succ \\ &\quad = \frac{1}{m} \prec \!\mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \big(\omega \gamma^0 + k \gamma^1\big)\, \mathfrak{f}(-\omega,k) \!\succ = - \prec \!\mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \mathfrak{f}(-\omega,k) \!\succ \\ &\prec \!\mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \mathfrak{f}(\omega,k) \!\succ = \frac{1}{m} \prec \!\not{p}\, \mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \mathfrak{f}(\omega,k) \!\succ \\ &\quad = \frac{1}{m} \prec \!\mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \big(\omega \gamma^0 + k \gamma^1\big) \mathfrak{f}(\omega,k) \!\succ \\ &\quad = \frac{2\omega}{m} \prec \!\mathfrak{f}(\omega,k) \,|\, \mathfrak{f}(\omega,k) \!\succ - \frac{1}{m} \prec \!\mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \not{p} \mathfrak{f}(\omega,k) \!\succ \\ &\quad = \frac{2\omega}{m} \prec \!\mathfrak{f}(\omega,k) \,|\, \mathfrak{f}(\omega,k) \!\succ - \prec \!\mathfrak{f}(\omega,k) \,|\, \gamma^0 \, \mathfrak{f}(\omega,k) \!\succ . \end{aligned}$$

Using the right relation in (5.2), the result follows. ∎

It is convenient to represent the spinor $\hat{\Psi}(p)$ in (5.1) as a complex multiple of the spinor $\mathfrak{f}(p)$. Thus we write the Fourier integral (5.1) as

$$\Psi(q) = \int_{\mathbb{R}^2} \frac{d^2p}{2\pi} \, \epsilon(\omega)\, \delta(p^2 - m^2)\, g(p)\, \mathfrak{f}(p)\, e^{-ipq} \tag{5.4}$$

with a complex-valued function $g(p)$. In the next two lemmas we specify the regularity of the function $g(p)$ and rewrite the scalar product (3.1) in momentum space.

Lemma 5.2. *For every smooth and spatially compact Dirac solution $\Psi \in C^\infty_{sc}(\mathscr{M}, S\mathscr{M})$, the function g in the representation* (5.4) *is a Schwartz function on the mass shells, i.e.*

$$g_\pm(k) := g\big(\pm \sqrt{k^2+m^2}, k \big) \;\in\; \mathcal{S}(\mathbb{R}, \mathbb{C}) \,.$$

Proof. Evaluating (5.4) at $q^0 = 0$ gives

$$\begin{aligned}
\Psi(0,x) &= \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \epsilon(\omega)\, \delta(p^2-m^2)\, g(p)\, \mathfrak{f}(p)\, e^{ikx} \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{\pm} \frac{\epsilon(\omega)}{\sqrt{k^2+m^2}}\, g(p)\, \mathfrak{f}(p)\, e^{ikx} \Big|_{p=\left(\pm\sqrt{k^2+m^2},k\right)} \\
i\partial_t \Psi(0,x) &= \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \epsilon(\omega)\, p_0\, \delta(p^2-m^2)\, g(p)\, \mathfrak{f}(p)\, e^{ikx} \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{\pm} g(p)\, \mathfrak{f}(p)\, e^{ikx} \Big|_{p=\left(\pm\sqrt{k^2+m^2},k\right)} \,.
\end{aligned}$$

On the other hand, taking the one-dimensional Fourier transformation, we know that

$$\Psi(0,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \widehat{\Phi}_0(k)\, e^{ikx} \qquad \text{and} \qquad \partial_t \Psi(0,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \widehat{\Phi}_1(k)\, e^{ikx}$$

for Schwartz functions $\widehat{\Phi}_0, \widehat{\Phi}_1 \in \mathcal{S}(\mathbb{R}, \mathbb{C}^2)$. Comparing the integrands, we obtain

$$\pm\sqrt{k^2+m^2}\, \widehat{\Phi}_0(k) + \widehat{\Phi}_1(k) = g_\pm(k)\, \mathfrak{f}\big(\pm\sqrt{k^2+m^2}, k\big) \,.$$

Taking the spin scalar product with $\mathfrak{f}$ and using the right equation in (5.2), we get

$$g_\pm(k) = \prec\!\Big(\sqrt{k^2+m^2}\, \widehat{\Phi}_0(k) \pm \widehat{\Phi}_1(k)\Big) \,\big|\, \mathfrak{f}\big(\pm\sqrt{k^2+m^2}, k\big)\!\succ \,.$$

According to (5.3), the spinor $\mathfrak{f}$ is smooth and grows at most linearly for large k (meaning that $\|\mathfrak{f}\|_{\mathbb{C}^2} \leq c(1+|k|)$ for a suitable constant c). This gives the result. □

Lemma 5.3. *In the Fourier representation* (5.4)*, the scalar product* (3.1) *can be written as*

$$(\Psi|\widetilde{\Psi}) = \frac{1}{2m} \int_{\mathbb{R}^2} \overline{g(p)}\, \widetilde{g}(p)\, \delta\big(p^2-m^2\big)\, d^2p \,. \tag{5.5}$$

Proof. We substitute (5.4) into (2.7). In view of the rapid decay of g (see Lemma 5.2), we may commute the integrals using Plancherel's theorem to obtain

$$\begin{aligned}
(\Psi|\widetilde{\Psi}) &= 2\pi \int_{-\infty}^{\infty} dx \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \epsilon(\omega)\, \delta\big(p^2-m^2\big)\, \overline{g(p)} \\
&\quad \times \int_{\mathbb{R}^2} \frac{d^2\widetilde{p}}{2\pi}\, \epsilon(\widetilde{\omega})\, \delta\big(\widetilde{p}^2-m^2\big)\, \widetilde{g}(\widetilde{p}) \prec\!\mathfrak{f}(p) \,|\, \gamma^0\, \mathfrak{f}(\widetilde{p})\!\succ e^{-i(k-\widetilde{k})x} \\
&= 2\pi \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \epsilon(\omega)\, \delta\big(p^2-m^2\big)\, \overline{g(p)} \int_{\mathbb{R}^2} \frac{d^2\widetilde{p}}{2\pi}\, \epsilon(\widetilde{\omega})\, \delta\big(\widetilde{p}^2-m^2\big) \\
&\quad \times 2\pi\delta\big(k-\widetilde{k}\big)\, \widetilde{g}(\widetilde{p}) \prec\!\mathfrak{f}(p) \,|\, \gamma^0\, \mathfrak{f}(\widetilde{p})\!\succ \\
&= \int_{\mathbb{R}^2} d^2p\, \epsilon(\omega)\, \delta\big(p^2-m^2\big)\, \overline{g(p)} \int_{\mathbb{R}^2} d\widetilde{\omega}\, \epsilon(\widetilde{\omega})\, \delta\big(\widetilde{\omega}^2-k^2-m^2\big) \\
&\quad \times \widetilde{g}\big(\widetilde{\omega},k\big) \prec\!\mathfrak{f}(p) \,|\, \gamma^0\, \mathfrak{f}\big(\widetilde{\omega},k\big)\!\succ \\
&= \int_{\mathbb{R}^2} d^2p\, \epsilon(\omega)\, \delta\big(p^2-m^2\big)\, \overline{g(p)}\, \frac{1}{2|\omega|} \\
&\quad \times \sum_{\pm} \epsilon(\pm\omega)\, \widetilde{g}\big(\pm\omega,k\big) \prec\!\mathfrak{f}(p) \,|\, \gamma^0\, \mathfrak{f}\big(\pm\omega,k\big)\!\succ \,.
\end{aligned}$$

Applying Lemma 5.1 gives (5.5). □

We finally choose a convenient parametrization of the mass shells:

Proposition 5.4. *In the parametrization*

$$\begin{pmatrix} \omega \\ k \end{pmatrix} = ms \begin{pmatrix} \cosh\alpha \\ \sinh\alpha \end{pmatrix} \qquad \textit{with} \qquad s \in \{\pm 1\} \textit{ and } \alpha \in \mathbb{R}\,, \tag{5.6}$$

the scalar product (3.1) *takes the form*

$$(\Psi|\tilde{\Psi}) = \frac{1}{4m} \sum_{s=\pm 1} \int_{-\infty}^{\infty} \overline{g(s,\alpha)}\, \tilde{g}(s,\alpha)\, d\alpha\ . \tag{5.7}$$

Proof. We carry out the ω-integration in (5.5),

$$\begin{aligned}
\int_{\mathbb{R}^2} \overline{g(p)}\, \tilde{g}(p)\, \delta\big(p^2-m^2\big)\, d^2p &= \sum_{\pm} \int_{-\infty}^{\infty} \frac{dk}{2\sqrt{k^2+m^2}}\, \big(\overline{g}\, \tilde{g}\big)\big|_{\left(\pm\sqrt{k^2+m^2},k\right)} \\
&= \sum_{s=\pm} \int_{-\infty}^{\infty} m\, \cosh\alpha\, \frac{1}{2m\cosh\alpha}\, \big(\overline{g}\, \tilde{g}\big)\big|_{\left(ms\cosh\alpha, ms\sinh\alpha\right)}\, d\alpha \\
&= \frac{1}{2} \sum_{s=\pm} \int_{-\infty}^{\infty} \overline{g(s,\alpha)}\, \tilde{g}(s,\alpha)\, d\alpha\,.
\end{aligned}$$

This gives the result. ∎

6. The Relative Fermionic Signature Operator in Momentum Space

In this section, we compute the fermionic signature operator more explicitly in momentum space. The first step is to transform the space-time inner product to momentum space.

Proposition 6.1. *For any* $\Psi, \tilde{\Psi} \in C^\infty_{sc}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}$*, the space-time inner product* (3.2) *takes the form*

$$<\!\Psi|\tilde{\Psi}\!>_{\mathscr{R}} = \frac{1}{4m} \sum_{s,\tilde{s}=\pm 1} \int_{-\infty}^{\infty} d\alpha\, \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} d\tilde{\alpha}\; I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\; \overline{g(s,\alpha)}\, \tilde{g}(\tilde{s},\tilde{\alpha})\,, \tag{6.1}$$

where I_ε *is the kernel*

$$I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big) = \frac{1}{4\pi^2\, m} \times \begin{cases} \dfrac{s\cosh\beta}{1-\cosh(2\beta+i\varepsilon s)} & \textit{if } s=\tilde{s} \\[2ex] -\dfrac{s\sinh\beta}{1+\cosh(2\beta)} & \textit{if } s\neq\tilde{s} \end{cases} \tag{6.2}$$

and

$$\beta := \frac{1}{2}\,\big(\alpha-\tilde{\alpha}\big)\,. \tag{6.3}$$

Proof. Using the Fourier representation (5.4) in (3.2), we obtain

$$\begin{aligned}
<\Psi|\tilde{\Psi}>_{\mathscr{R}} &= \int_{\mathscr{M}} dt\, dx\, \chi_{\mathscr{R}}(t,x) \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \epsilon(\omega)\, \delta(p^2-m^2)\, \overline{g(p)} \\
&\qquad \times \int_{\mathbb{R}^2} \frac{d^2\tilde{p}}{2\pi}\, \epsilon(\tilde{\omega})\, \delta(\tilde{p}^2-m^2)\, \tilde{g}(\tilde{p}) \prec\mathfrak{f}(p)\,|\,\mathfrak{f}(\tilde{p})\succ e^{i(p-\tilde{p})q} \\
&= \int_{\mathbb{R}^2} \frac{d^2p}{(2\pi)^2}\, \epsilon(\omega)\, \delta(p^2-m^2)\, \overline{g(p)} \\
&\qquad \times \int_{\mathbb{R}^2} d^2\tilde{p}\, \epsilon(\tilde{\omega})\, \delta(\tilde{p}^2-m^2)\, \tilde{g}(\tilde{p}) \prec\mathfrak{f}(p)\,|\,\mathfrak{f}(\tilde{p})\succ K(p,\tilde{p})\,, \qquad (6.4)
\end{aligned}$$

where the kernel $K(p,\tilde{p})$ is defined by

$$K(p,\tilde{p}) = \int_{\mathscr{M}} \chi_{\mathscr{R}}(t,x)\, e^{i(p-\tilde{p})q}\, dt\, dx\,. \qquad (6.5)$$

Rewriting the integrals in (6.4) in the parametrization (5.6) (exactly as in the proof of Proposition 5.4), we get

$$<\Psi|\tilde{\Psi}>_{\mathscr{R}} = \frac{1}{16\pi^2} \sum_{s,\tilde{s}=\pm 1} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\tilde{\alpha}\; s\,\tilde{s}\; \overline{g(s,\alpha)}\, \tilde{g}(\tilde{s},\tilde{\alpha}) \prec\mathfrak{f}(p)\,|\,\mathfrak{f}(\tilde{p})\succ K(p,\tilde{p})\,. \qquad (6.6)$$

Applying Lemma 6.3 and Lemma 6.4 below, the result follows. □

Comparing (5.7) and (6.1), one can immediately read off the relative fermionic signature operator as defined by (3.3) and (4.3):

Corollary 6.2. *For any* $\tilde{\Psi} \in C^\infty_{sc}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}$,

$$\big(\mathcal{S}_{\mathscr{R}}\, \tilde{\Psi}\big)(s,\alpha) = \sum_{\tilde{s}=\pm 1} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\; \tilde{g}(\tilde{s},\tilde{\alpha})\; d\tilde{\alpha}\,.$$

In the following two lemmas we compute the spin scalar product and the kernel in (6.6).

Lemma 6.3. *In the parametrization* (5.6)*, the spin scalar product of the spinors* (5.3) *is computed by*

$$\prec\mathfrak{f}(s,\alpha)\,|\,\mathfrak{f}(\tilde{s},\tilde{\alpha})\succ = \left\{ \begin{array}{ll} s\cosh\beta & \text{if } s=\tilde{s} \\ s\sinh\beta & \text{if } s\neq\tilde{s}\,. \end{array} \right.$$

Proof. Using (5.3) and (2.4), we have

$$\prec\mathfrak{f}(p)|\mathfrak{f}(\tilde{p})\succ = \frac{1}{2}\, \frac{(\omega-k)+(\tilde{\omega}-\tilde{k})}{\sqrt{\epsilon(\omega)\,(\omega-k)\,\epsilon(\tilde{\omega})\,(\tilde{\omega}-\tilde{k})}}\,.$$

In the parametrization (5.6), we obtain

$$\prec\mathfrak{f}(s,\alpha)|\mathfrak{f}(\tilde{s},\tilde{\alpha})\succ = \frac{1}{2}\, \frac{se^{-\alpha}+\tilde{s}e^{-\tilde{\alpha}}}{e^{-\frac{\alpha}{2}-\frac{\tilde{\alpha}}{2}}} = \frac{1}{2}\left(se^{\beta}+\tilde{s}e^{-\beta}\right).$$

This gives the result. □

Lemma 6.4. *In the parametrization* (5.6)*, the distribution* $K(p,\tilde{p})$ *defined by* (6.5) *has the form*

$$K(s,\alpha;\tilde{s},\tilde{\alpha}) = \frac{1}{m^2} \times \begin{cases} \displaystyle\lim_{\varepsilon\searrow 0} \frac{1}{1-\cosh(\alpha-\tilde{\alpha}-i\varepsilon s)} & \text{if } s=\tilde{s} \\ \displaystyle\frac{1}{1+\cosh(\alpha-\tilde{\alpha})} & \text{if } s\neq\tilde{s}\,. \end{cases}$$

Proof. We first write (6.5) as

$$K(p,\tilde{p}) = \int_{\mathbb{R}^2} dt\,dx\,\chi(x-t)\,\chi(x+t)\,e^{i(p-\tilde{p})q}\,. \tag{6.7}$$

Introducing null coordinates

$$u=\frac{1}{2}\,(t-x) \qquad \text{and} \qquad v=\frac{1}{2}\,(t+x)$$

as well as corresponding momenta

$$p_u = \omega-\tilde{\omega}+k-\tilde{k} \qquad \text{and} \qquad p_v = \omega-\tilde{\omega}-k+\tilde{k}\,,$$

we can compute the integrals in (6.7) to obtain

$$\begin{aligned} K(p_u,p_v) &= 2\int_{\mathbb{R}^2} du\,dv\,\chi(-2u)\,\chi(2v)\,e^{i(p_u u+p_v v)} = 2\int_{-\infty}^0 du\,e^{ip_u u}\int_0^\infty dv\,e^{ip_v v} \\ &= 2\lim_{\varepsilon\searrow 0}\int_{-\infty}^0 du\,e^{ip_u u+\varepsilon u}\,\lim_{\varepsilon'\searrow 0}\int_0^\infty dv\,e^{ip_v v-\varepsilon' v} = 2\lim_{\varepsilon,\varepsilon'\searrow 0}\frac{1}{p_u-i\varepsilon}\,\frac{1}{p_v+i\varepsilon'}\,. \end{aligned}$$

We next express p_u in the parametrization (5.6),

$$\begin{aligned} p_u &= (\omega+k)-(\tilde{\omega}+\tilde{k}) = ms\big(\cosh(\alpha)+\sinh(\alpha)\big) - m\tilde{s}\big(\cosh(\tilde{\alpha})+\sinh(\tilde{\alpha})\big) \\ &= m\big(se^{\alpha}-\tilde{s}e^{\tilde{\alpha}}\big)\,. \end{aligned}$$

This gives

$$\lim_{\varepsilon\searrow 0}\frac{1}{p_u-i\varepsilon} = \lim_{\varepsilon\searrow 0}\frac{1}{m(se^{\alpha}-\tilde{s}e^{\tilde{\alpha}})-i\varepsilon}\,. \tag{6.8}$$

We distinguish the two cases $s\neq\tilde{s}$ and $s=\tilde{s}$. In the case $s\neq\tilde{s}$, the denominator in (6.8) is always non-zero. Therefore, we can take the limit $\varepsilon\searrow 0$ pointwise to obtain

$$\lim_{\varepsilon\searrow 0}\frac{1}{p_u-i\varepsilon} = \frac{1}{ms}\,\frac{1}{e^{\alpha}+e^{\tilde{\alpha}}} = \frac{e^{-\alpha}}{ms}\,\frac{1}{1+e^{-2\beta}} \qquad (s\neq\tilde{s})\,,$$

where β is again given by (6.3). In the remaining case $s=\tilde{s}$, we rewrite (6.8) as

$$\begin{aligned} \lim_{\varepsilon\searrow 0}\frac{1}{p_u-i\varepsilon} &= \frac{1}{ms}\lim_{\varepsilon\searrow 0}\frac{1}{(e^{\alpha}-e^{\tilde{\alpha}})-i\varepsilon s/m} = \frac{e^{-\alpha}}{ms}\lim_{\varepsilon\searrow 0}\frac{1}{1-e^{-2\beta}-i\varepsilon s e^{-\alpha}/m} \\ &= \frac{e^{-\alpha}}{ms}\lim_{\delta\searrow 0}\frac{1}{1-e^{-2\beta}-i\delta s\,e^{-2\beta}} = \frac{e^{-\alpha}}{ms}\lim_{\delta\searrow 0}\frac{1}{1-e^{-2\beta+i\delta s}}\,, \end{aligned}$$

where $\delta=\varepsilon e^{-\alpha+2\beta}/m>0$. We conclude that

$$\lim_{\varepsilon\searrow 0}\frac{1}{p_u-i\varepsilon} = \frac{e^{-\alpha}}{ms}\lim_{\delta\searrow 0}\frac{1}{1-e^{-2\beta+i\delta s}} \qquad (s=\tilde{s})\,.$$

Treating p_v in the same way, we obtain

$$\lim_{\varepsilon\searrow 0} \frac{1}{p_u - i\varepsilon} = \begin{cases} \dfrac{e^{-\alpha}}{ms}\,\dfrac{1}{1+e^{-2\beta}} & \text{if } s \neq \tilde{s} \\ \dfrac{e^{-\alpha}}{ms}\,\lim_{\varepsilon\searrow 0}\dfrac{1}{1-e^{-2\beta+i\varepsilon s}} & \text{if } s = \tilde{s} \end{cases} \tag{6.9}$$

$$\lim_{\varepsilon'\searrow 0} \frac{1}{p_v + i\varepsilon'} = \begin{cases} \dfrac{e^{\alpha}}{ms}\,\dfrac{1}{1+e^{2\beta}} & \text{if } s \neq \tilde{s} \\ \dfrac{e^{\alpha}}{ms}\,\lim_{\varepsilon'\searrow 0}\dfrac{1}{1-e^{2\beta-i\varepsilon' s}} & \text{if } s = \tilde{s}\,. \end{cases} \tag{6.10}$$

When multiplying (6.9) and (6.10), the fact that both limits $\varepsilon, \varepsilon' \searrow 0$ exist in the distributional sense justifies that we can simply set $\varepsilon = \varepsilon'$ and take the limit. Using (6.3), the result follows. $\square$

7. Diagonalizing the Relative Fermionic Signature Operator

In Corollary 6.2, the relative fermionic signature operator $\mathcal{S}_{\mathscr{R}}$ was represented by an integral operator. Since the kernel $I(s,\alpha;\tilde{s},\tilde{\alpha})$ only depends on the difference $\alpha-\tilde{\alpha}$ (see (6.2) and (6.3)), we can diagonalize the fermionic operator with the plane wave ansatz

$$g\big(\tilde{s},\tilde{\alpha}\big) = e^{-i\ell\tilde{\alpha}}\,\hat{g}(\tilde{s},\ell) \tag{7.1}$$

for a real parameter ℓ. Clearly, the plane wave is not a vector in our Hilbert space $\mathcal{H}$. But the corresponding spectral parameter corresponds to a point in the continuous spectrum of $\mathcal{S}_{\mathscr{R}}$. For clarity, we first give the computations. The functional analytic framework will be developed in Section 8 below.

Lemma 7.1. *The integral kernel I_ε, (6.2), satisfies the relation*

$$\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\,e^{-i\ell\tilde{\alpha}}\,d\tilde{\alpha} = e^{-i\ell\alpha}\,\frac{\ell}{2\pi m}\times \begin{cases} \dfrac{2}{1+e^{-2\pi s\ell}} & \text{if } s=\tilde{s} \\ -\dfrac{is}{\cosh(\pi\ell)} & \text{if } s\neq\tilde{s}\,. \end{cases}$$

Using the result of this lemma, one sees that for the plane wave ansatz (7.1), the equation $\mathcal{S}_{\mathscr{R}}\,g = \lambda g$ reduces to the eigenvalue equation for a Hermitian matrix,

$$\hat{\mathcal{S}}_{\mathscr{R}}(\ell)\,\hat{g}(\ell) = \lambda\hat{g}(\ell) \qquad \text{with} \qquad \hat{\mathcal{S}}_{\mathscr{R}}(\ell) = \frac{\ell}{\pi m}\begin{pmatrix} \dfrac{1}{1+e^{-2\pi\ell}} & -\dfrac{i}{2\cosh(\pi\ell)} \\ \dfrac{i}{2\cosh(\pi\ell)} & \dfrac{1}{1+e^{2\pi\ell}} \end{pmatrix}, \tag{7.2}$$

where $\hat{g}(\ell)\in\mathbb{C}^2$ is the vector with components $\hat{g}(1,\ell)$ and $\hat{g}(-1,\ell)$. The matrix $\hat{\mathcal{S}}_{\mathscr{R}}(\ell)$ has the eigenvalues

$$\lambda = 0 \qquad \text{and} \qquad \lambda = \frac{\ell}{\pi m} \tag{7.3}$$

with respective eigenfunctions

$$\hat{g}(\ell) = \begin{pmatrix} ie^{-\pi\ell} \\ 1 \end{pmatrix} \qquad \text{and} \qquad \hat{g}(\ell) = \begin{pmatrix} -ie^{\pi\ell} \\ 1 \end{pmatrix}. \tag{7.4}$$

Proof of Lemma 7.1. In the case $s \neq \tilde{s}$, we have

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} I_\varepsilon\big(s, \alpha; \tilde{s}, \tilde{\alpha}\big)\, e^{-i\ell(\tilde{\alpha}-\alpha)}\, d\tilde{\alpha} \\
&\qquad = 2 \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} I_\varepsilon\big(s, \alpha; \tilde{s}, \tilde{\alpha}\big)\, e^{2i\ell\beta}\, d\beta = -\frac{s}{2\pi^2\, m} \int_{-\infty}^{\infty} \frac{\sinh\beta}{1+\cosh(2\beta)}\, e^{2i\ell\beta}\, d\beta\,.
\end{aligned}$$

The integral can be computed as follows. First, using the transformation

$$\frac{\sinh\beta}{1+\cosh(2\beta)} = \frac{e^\beta - e^{-\beta}}{(e^\beta + e^{-\beta})^2} = -\frac{d}{d\beta}\bigg(\frac{1}{e^\beta + e^{-\beta}}\bigg)\,,$$

we can integrate by parts to obtain

$$\int_{-\infty}^{\infty} \frac{\sinh\beta}{1+\cosh(2\beta)}\, e^{2i\ell\beta}\, d\beta = 2i\ell \int_{-\infty}^{\infty} \frac{e^{2i\ell\beta}}{e^\beta + e^{-\beta}}\, d\beta = i\ell \int_{-\infty}^{\infty} \frac{e^{2i\ell\beta}}{\cosh\beta}\, d\beta\,.$$

Now the integral can be calculated with residues. The variable transformation $\beta \mapsto -\beta$ shows that the last integral is even in ℓ. Therefore, it suffices to consider the case $\ell > 0$. Then we can close the contour in the upper half plane. There the integrand has poles at $\beta_n = i\pi(n+\frac{1}{2})$ with $n \in \mathbb{N}_0$. This gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\sinh\beta}{1+\cosh(2\beta)}\, e^{2i\ell\beta}\, d\beta = -2\pi\ell \sum_{n=0}^{\infty} \text{Res}\Big(\frac{e^{2i\ell\beta}}{\cosh\beta}, \beta_n\Big) \\
&= -2\pi\ell \sum_{n=0}^{\infty} (-i)\,(-1)^n\, e^{-2\pi\ell\,(n+\frac{1}{2})} = 2\pi i\ell\, e^{-\pi\ell} \sum_{n=0}^{\infty} \big(-e^{-2\pi\ell}\big)^n \\
&= 2\pi i\ell\, e^{-\pi\ell}\, \frac{1}{1+e^{-2\pi\ell}} = \frac{i\pi\ell}{\cosh(\pi\ell)}\,. \qquad (7.5)
\end{aligned}$$

In the case $s = \tilde{s}$, we find similarly

$$\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} I_\varepsilon\big(s, \alpha; \tilde{s}, \tilde{\alpha}\big)\, e^{-i\ell(\tilde{\alpha}-\alpha)}\, d\tilde{\alpha} = \frac{s}{2\pi^2\, m} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \frac{\cosh(\beta - \frac{i\varepsilon s}{2})}{1-\cosh(2\beta - i\varepsilon s)}\, e^{2i\ell\beta}\, d\beta\,.$$

Rewriting the integrand as

$$\frac{\cosh(\beta - \frac{i\varepsilon s}{2})}{1-\cosh(2\beta - i\varepsilon s)} = -\frac{e^{\beta - \frac{i\varepsilon s}{2}} + e^{-\beta + \frac{i\varepsilon s}{2}}}{(e^{\beta - \frac{i\varepsilon s}{2}} - e^{-\beta + \frac{i\varepsilon s}{2}})^2} = \frac{d}{d\beta}\bigg(\frac{1}{e^{\beta - \frac{i\varepsilon s}{2}} - e^{-\beta + \frac{i\varepsilon s}{2}}}\bigg)\,,$$

we can again integrate by parts to obtain

$$\int_{-\infty}^{\infty} \frac{\cosh(\beta - \frac{i\varepsilon s}{2})}{1-\cosh(2\beta - i\varepsilon s)}\, e^{2i\ell\beta}\, d\beta = -i\ell \int_{-\infty}^{\infty} \frac{e^{2i\ell\beta}}{\sinh(\beta - \frac{i\varepsilon s}{2})}\, d\beta\,. \qquad (7.6)$$

Now the last integral is odd under the joint transformations

$$\ell \mapsto -\ell \qquad \text{and} \qquad s \mapsto -s\,.$$

Therefore, it again suffices to consider the case $\ell > 0$, where the contour can be closed in the upper half plane. In the case $s = 1$, the contour encloses the poles at the points $\beta_n = i\pi n$ with $n \in \mathbb{N}_0$. This gives

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \frac{\cosh \beta}{1 - \cosh(2\beta - i\varepsilon)}\, e^{2i\ell\beta}\, d\beta = 2\pi\ell \sum_{n=0}^{\infty} \mathrm{Res}\Big(\frac{e^{2i\ell\beta}}{\sinh \beta}, \beta_n \Big) \\
&= 2\pi\ell \sum_{n=0}^{\infty} \big(-e^{-2\pi\ell} \big)^n = \frac{2\pi\ell}{1 + e^{-2\pi\ell}} .
\end{aligned}$$

In the case $s = -1$, the contour does not enclose the pole at $\beta_0 = 0$. We thus obtain

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \frac{\cosh \beta}{1 - \cosh(2\beta - i\varepsilon)}\, e^{2i\ell\beta}\, d\beta = 2\pi\ell \sum_{n=1}^{\infty} \mathrm{Res}\Big(\frac{e^{2i\ell\beta}}{\sinh \beta}, \beta_n \Big) \\
&= 2\pi\ell \sum_{n=1}^{\infty} \big(-e^{-2\pi\ell} \big)^n = -2\pi\ell \frac{e^{-2\pi\ell}}{1 + e^{-2\pi\ell}} = -\frac{2\pi\ell}{1 + e^{2\pi\ell}} .
\end{aligned}$$

This concludes the proof. □

8. A Self-Adjoint Extension of the Relative Fermionic Signature Operator

We let U be the mapping

$$U : \mathcal{H} \to L^2(\mathbb{R}, \mathbb{C}^2) , \qquad g(s, \alpha) \mapsto \hat{g}(s, \ell) = \frac{1}{\sqrt{8\pi m}} \int_{-\infty}^{\infty} g(s, \alpha)\, e^{i\ell\alpha}\, d\alpha .$$

From (5.4) and Plancherel's theorem, one sees immediately that this mapping is unitary. Moreover, its inverse is given by

$$U^{-1} : L^2(\mathbb{R}, \mathbb{C}^2) \to \mathcal{H} , \qquad \hat{g}(s, \ell) \mapsto g(s, \alpha) = \sqrt{\frac{2m}{\pi}} \int_{-\infty}^{\infty} \hat{g}(s, \ell)\, e^{-i\ell\alpha}\, d\ell .$$

Theorem 8.1. *Choosing the domain of definition*

$$\mathcal{D}(\mathcal{S}_{\mathscr{R}}) = U^{-1}\Big(\Big\{ \hat{g} \in L^2(\mathbb{R}, \mathbb{C}^2) \quad \text{with} \quad \hat{\mathcal{S}}_{\mathscr{R}}\, \hat{g} \in L^2(\mathbb{R}, \mathbb{C}^2) \Big\} \Big) \tag{8.1}$$

(where $(\hat{\mathcal{S}}_{\mathscr{R}} \hat{g})(\ell) = \hat{\mathcal{S}}_{\mathscr{R}}(\ell)\, \hat{g}(\ell)$ is the pointwise multiplication by the matrix in (7.2)*), the relative fermionic signature operator is a self-adjoint operator on $\mathcal{H} \equiv \mathcal{H}_{\mathscr{M}}$. Its spectrum consists of a pure point spectrum at zero and an absolutely continuous spectrum,*

$$\sigma_{pp}\big(\mathcal{S}_{\mathscr{R}}\big) = \{0\} , \qquad \sigma_{ac}\big(\mathcal{S}_{\mathscr{R}}\big) = \mathbb{R} .$$

It has the spectral decomposition

$$\mathcal{S}_{\mathscr{R}} = \int_{-\infty}^{\infty} \lambda\, dE_\lambda ,$$

where the spectral measure dE_λ is given by

$$E_U = U^{-1} \Big(\chi_U(0)\, \hat{K} + \chi_U \hat{L} \Big) U .$$

Here χ_U is the characteristic function, and $\widehat{K}$ and $\widehat{L}$ are the multiplication operators

$$\widehat{K}(\ell) = \mathbb{1}_{\mathbb{C}^2} - \widehat{E}(\ell) = \begin{pmatrix} \dfrac{e^{-2\pi\ell}}{1+e^{-2\pi\ell}} & \dfrac{i}{2\cosh(\pi\ell)} \\ -\dfrac{i}{2\cosh(\pi\ell)} & \dfrac{e^{2\pi\ell}}{1+e^{2\pi\ell}} \end{pmatrix} \tag{8.2}$$

$$\widehat{L}(\ell) = \frac{\pi m}{\ell}\, \widehat{\mathcal{S}}_{\mathscr{R}}(\ell) = \begin{pmatrix} \dfrac{1}{1+e^{-2\pi\ell}} & -\dfrac{i}{2\cosh(\pi\ell)} \\ \dfrac{i}{2\cosh(\pi\ell)} & \dfrac{1}{1+e^{2\pi\ell}} \end{pmatrix} . \tag{8.3}$$

Proof. For a Dirac solution $\Psi \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M})$, we know from Lemma 5.2 and Proposition 5.4 that the corresponding function $g(s,\alpha)$ is smooth and that all its derivatives are square integrable. As a consequence, its Fourier transform is pointwise bounded and has rapid decay, i.e.

$$\sup_\ell \big|(1+\ell^2)^p\, \widehat{g}(\ell)\big| < \infty \qquad \text{for all } p\,.$$

Using furthermore that the kernel $I_\varepsilon(s,\alpha,\tilde{s},.)$ given in (6.2) decays exponentially, we may use Fubini to exchange the orders of integration in the following computation,

$$\begin{aligned}
\big(\mathcal{S}_{\mathscr{R}}\,\Psi\big)(s,\alpha) &= \sum_{\tilde{s}=\pm 1} \lim_{\varepsilon\searrow 0} \int_{-\infty}^\infty I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big) \sqrt{\frac{2m}{\pi}} \bigg(\int_{-\infty}^\infty \widehat{g}(s,\ell)\, e^{-i\ell\tilde{\alpha}}\, d\ell \bigg)\, d\tilde{\alpha} \\
&= \sqrt{\frac{2m}{\pi}} \sum_{\tilde{s}=\pm 1} \int_{-\infty}^\infty \widehat{g}(s,\ell) \left(\lim_{\varepsilon\searrow 0} \int_{-\infty}^\infty I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\, e^{-i\ell\tilde{\alpha}}\, d\alpha \right) d\ell \\
&= \sqrt{\frac{2m}{\pi}} \sum_{\tilde{s}=\pm 1} \int_{-\infty}^\infty \big(\widehat{\mathcal{S}}_{\mathscr{R}}(\ell)\, \widehat{g}(\ell)\big)_s\; e^{-i\ell\alpha}\, d\ell = \big(U^{-1}\, \widehat{\mathcal{S}}_{\mathscr{R}}\, U\Psi\big)(s,\alpha)\,,
\end{aligned}$$

where in the last line we applied Lemma 7.1. Therefore, the unitary transformation of $\mathcal{S}_{\mathscr{R}}$ yields a multiplication operator, i.e.

$$\big(U\, \mathcal{S}_{\mathscr{R}}\, U^{-1}\widehat{g}\big)(\ell) = \widehat{\mathcal{S}}_{\mathscr{R}}(\ell)\, \widehat{g}(\ell) \qquad \text{for all } \widehat{g} \in U\big(C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}\big)\,.$$

Obviously, this multiplication operator can be extended to the domain

$$\mathcal{D}\big(\,\widehat{\mathcal{S}}_{\mathscr{R}}\big) := \Big\{ \widehat{g} \in L^2(\mathbb{R}, \mathbb{C}^2) \quad \text{with} \quad \widehat{\mathcal{S}}_{\mathscr{R}}\, \widehat{g} \in L^2(\mathbb{R}, \mathbb{C}^2) \Big\} \tag{8.4}$$

(where again $(\widehat{\mathcal{S}}_{\mathscr{R}}\widehat{g})(l) := \widehat{\mathcal{S}}_{\mathscr{R}}(\ell)\, \widehat{g}(\ell)$). Our task is to prove that with this domain, the multiplication operator $\widehat{\mathcal{S}}_{\mathscr{R}}$ is self-adjoint. Once this has been shown, we obtain the self-adjointness of $\mathcal{S}_{\mathscr{R}}$ with domain (8.1) by unitary transformation. Moreover, the properties of the spectrum and the spectral measure follow immediately by computing the spectral measure of the multiplication operator $\widehat{\mathcal{S}}_{\mathscr{R}}$ and unitarily transforming back to the Hilbert space $\mathcal{H}$.

In order to establish that the multiplication operator $\widehat{\mathcal{S}}_{\mathscr{R}}$ with domain (8.4) is self-adjoint, we need to show that the domain of its adjoint $\widehat{\mathcal{S}}^*_{\mathscr{R}}$ coincides with (8.4). This follows using standard functional methods (see for example [19, 17]), which we here recall for completeness: Let $\Psi \in \mathcal{D}(\widehat{\mathcal{S}}^*_{\mathscr{R}})$. Then there is a vector $v \in \mathcal{H}$ such that

$$\langle \Psi, \widehat{\mathcal{S}}_{\mathscr{R}}\, u \rangle_{L^2(\mathbb{R},\mathbb{C}^2)} = \langle v, \widehat{\mathcal{S}}_{\mathscr{R}}\, u \rangle_{L^2(\mathbb{R},\mathbb{C}^2)} \qquad \text{for all } u \in \mathcal{D}\big(\widehat{\mathcal{S}}_{\mathscr{R}}\big)\,.$$

Therefore, we may use Lebesgue's monotone convergence theorem to obtain

$$\begin{aligned}
\big\|\widehat{\mathcal{S}}^*_{\mathscr{R}}\,\Psi\big\|_{L^2(\mathbb{R},\mathbb{C}^2)} &= \lim_{L\to\infty} \big\|\chi_{[-L,L]}\,\widehat{\mathcal{S}}^*_{\mathscr{R}}\,\Psi\big\|_{L^2(\mathbb{R},\mathbb{C}^2)} = \lim_{L\to\infty} \big\|\chi_{[-L,L]}\,\widehat{\mathcal{S}}^*_{\mathscr{R}}\,\Psi\big\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&= \lim_{L\to\infty} \sup_{\Phi\in\mathcal{H},\,\|\Phi\|=1} \big\langle \Phi,\, \chi_{[-L,L]}\,\widehat{\mathcal{S}}^*_{\mathscr{R}}\,\Psi\big\rangle_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\overset{(*)}{=} \lim_{L\to\infty} \sup_{\Phi\in\mathcal{H},\,\|\Phi\|=1} \big\langle \widehat{\mathcal{S}}_{\mathscr{R}}\,\chi_{[-L,L]}\,\Phi,\, \Psi\big\rangle_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&= \lim_{L\to\infty} \sup_{\Phi\in\mathcal{H},\,\|\Phi\|=1} \int_{-L}^{L} \big\langle \Phi(\ell),\, \widehat{\mathcal{S}}_{\mathscr{R}}(\ell)\,\Psi(\ell)\big\rangle_{\mathbb{C}^2}\, d\ell \\
&= \lim_{L\to\infty} \bigg(\int_{-L}^{L} \big\|\widehat{\mathcal{S}}_{\mathscr{R}}(\ell)\,\Psi(\ell)\big\|^2_{\mathbb{C}^2}\, d\ell \bigg)^{\frac{1}{2}} ,
\end{aligned}$$

where in $(*)$ we used that the function $\chi_{[-L,L]}\,\Phi$ is in the domain of $\widehat{\mathcal{S}}_{\mathscr{R}}$ (see (8.4) and exploited the fact that the matrix $\widehat{\mathcal{S}}_{\mathscr{R}}(\ell)$ in (7.2) is uniformly bounded for $\ell \in [-L, L]$). Again applying Lebesgue's monotone convergence theorem, we conclude that the pointwise product $\widehat{\mathcal{S}}_{\mathscr{R}}(\ell)\,\Psi(\ell)$ is in $L^2(\mathbb{R}, \mathbb{C}^2)$. Using (8.4), it follows that the vector Ψ lies in the domain of $\widehat{\mathcal{S}}_{\mathscr{R}}$. This concludes the proof. □

9. The Fermionic Signature Operator of Rindler Space-Time

Having defined the relative fermionic signature operator $\mathcal{S}_{\mathscr{R}}$ as a self-adjoint operator with dense domain $\mathcal{D}(\mathcal{S}_{\mathscr{R}})$, the fermionic signature operator $\mathcal{S}$ in Rindler space-time is obtained from (3.4). We then have the following result.

Theorem 9.1. *Choosing the domain of definition*

$$\mathcal{D}(\mathcal{S}) = \pi_{\mathscr{R}}\,\mathcal{D}(\mathcal{S}_{\mathscr{R}}) \tag{9.1}$$

(with $\mathcal{D}(\mathcal{S}_{\mathscr{R}})$ according to (8.1)*), the fermionic signature operator $\mathcal{S}$ in Rindler space-time is a self-adjoint operator on $\mathcal{H}_{\mathscr{R}}$. It has an absolutely continuous spectrum with spectral measure dE_λ given by*

$$E_U = \pi_{\mathscr{R}}\, U^{-1}\big(\chi_U \widehat{L}\big) U\, \iota_{\mathscr{M}} ,$$

where $\widehat{L}$ is again the multiplication operator (8.3)*.*

Proof. On the solution space $\mathcal{H}_{\mathscr{M}}$ in Minkowski space, we consider the transformation

$$T_{\text{CPT}} \,:\, \mathcal{H}_{\mathscr{M}} \to \mathcal{H}_{\mathscr{M}} , \qquad \Psi(t,x) \mapsto \gamma^0\gamma^1\,\Psi(-t,-x)$$

(in physics referred to as the CPT transformation [2, Section 5.4]; one verifies directly that this transformation maps again to solutions of the Dirac equation). A direct computation shows that T_{CPT} is unitary and that $T_{\text{CPT}}^2 = -\mathbb{1}$.

The transformation T_{CPT} can be used to describe the Hilbert space $\mathcal{H}_{\mathscr{M}}$ completely in terms of $\mathcal{H}_{\mathscr{R}}$. To see how this comes about, we first note that a solution $\Psi \in \mathcal{H}_{\mathscr{M}}$ is determined uniquely by its Cauchy data at time zero. The restriction to the right half line $\Psi|_{\{t=0,x>0\}}$ gives rise to a unique solution in $\mathcal{H}_{\mathscr{R}}$, and applying $\iota_{\mathscr{M}}$ yields a solution in Minkowski space which vanishes identically on the left half line $\Psi|_{\{t=0,x<0\}}$. Applying T_{CPT} to this solution gives a new solution which vanishes identically on the right half line $\Psi|_{\{t=0,x>0\}}$. In view of (3.1), the solutions which vanish on the right half line are orthogonal to those which vanish on the left half line. We thus obtain the orthogonal direct sum decomposition

$$\mathcal{H}_{\mathscr{M}} = \big(T_{\text{CPT}}\, \iota_{\mathscr{M}}\, \mathcal{H}_{\mathscr{R}}\big) \oplus \big(\iota_{\mathscr{M}}\, \mathcal{H}_{\mathscr{R}}\big) \,.$$

Since the Dirac solutions in $T_{\text{CPT}}\, \iota_{\mathscr{M}}\, \mathcal{H}_{\mathscr{R}}$ vanish identically in the Rindler wedge, it is obvious that

$$\mathcal{S}_{\mathscr{R}}\big|_{T_{\text{CPT}}\, \iota_{\mathscr{M}}\, \mathcal{H}_{\mathscr{R}}} = 0 \qquad \text{and} \qquad \mathcal{S}_{\mathscr{R}}\big(\mathcal{H}_{\mathscr{M}}\big) \subset \iota_{\mathscr{M}} \mathcal{H}_{\mathscr{R}} \,.$$

Moreover, working out T_{CPT} in momentum space, one sees that T_{CPT} leaves the parameter ℓ in (7.1) unchanged and simply maps the trivial and non-trivial eigenspaces of the matrix (7.2) to each other (see (7.3) and (7.4)). This shows that the operator $\iota_{\mathscr{M}}$ in (3.4) maps precisely to the orthogonal complement of the kernel of $\mathcal{S}_{\mathscr{R}}$, and that the image of $\mathcal{S}_{\mathscr{R}}$ is mapped by $\pi_{\mathscr{M}}$ unitarily to $\mathcal{H}_{\mathscr{R}}$. Therefore, the spectral representation of $\mathcal{S}$ is obtained by that of $\mathcal{S}_{\mathscr{R}}$ simply by removing the kernel. This gives the result. ☐

10. Connection to the Hamiltonian in Rindler Coordinates

The fermionic signature operator is closely related to the Dirac Hamiltonian in Rindler coordinates, as we now explain. Recall that the *Rindler coordinates* $\tau \in \mathbb{R}$ and $\rho \in (0, \infty)$ are defined by

$$\begin{pmatrix} t \\ x \end{pmatrix} = \rho \begin{pmatrix} \sinh \tau \\ \cosh \tau \end{pmatrix} \,.$$

In these coordinates, the Rindler line element takes the form

$$ds^2 = \rho^2\, d\tau^2 - d\rho^2 \,.$$

We work intrinsically in Rindler space-time. Translations in the time coordinate τ,

$$\tau \mapsto \tau + \Delta \,, \qquad \rho \mapsto \rho \,, \tag{10.1}$$

describe a Killing symmetry. Therefore, writing the Dirac equation in this time coordinate in the Hamiltonian form

$$i \partial_\tau \psi = H \psi \,, \tag{10.2}$$

the Dirac Hamiltonian is time independent (for details see the proof of Theorem 10.1 below).

Theorem 10.1. *The fermionic signature operator* $\mathcal{S}$ *and the Hamiltonian* H *in Rindler coordinates satisfy the relation*

$$\mathcal{S} = -\frac{H}{\pi m} \,.$$

Proof. One method of deriving the Dirac operator would be to compute the spin connection in this coordinate system. For our purposes, it is more convenient to again take the Dirac operator in the reference frame (t, x) and to express it in the Rindler coordinates (τ, ρ), but without transforming the spinor basis (this Dirac operator coincides with the intrinsic Dirac operator up to a local $\mathrm{U}(1,1)$-gauge transformation; for details in the more general four-dimensional setting see [6]). Using the identities

$$\begin{aligned}
\frac{\partial}{\partial \rho} &= \frac{\partial t}{\partial \rho}\frac{\partial}{\partial t} + \frac{\partial x}{\partial \rho}\frac{\partial}{\partial x} = \sinh \tau\, \frac{\partial}{\partial t} + \cosh \tau\, \frac{\partial}{\partial x} \\
\frac{\partial}{\partial \tau} &= \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} + \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} = \rho \cosh \tau\, \frac{\partial}{\partial t} + \rho \sinh \tau\, \frac{\partial}{\partial x} \,,
\end{aligned}$$

the Dirac operator becomes

$$\begin{aligned}
\mathcal{D} &= \frac{i}{\rho}\Big(\gamma^0 \cosh\tau - \gamma^1 \sinh\tau\Big)\,\partial_\tau + i\Big(-\gamma^0 \sinh\tau + \gamma^1\cosh\tau\Big)\,\partial_\rho \\
&= \frac{i}{\rho}\begin{pmatrix} 0 & e^{-\tau} \\ e^{\tau} & 0 \end{pmatrix}\partial_\tau + i\begin{pmatrix} 0 & e^{-\tau} \\ -e^{\tau} & 0 \end{pmatrix}\partial_\rho \,.
\end{aligned}$$

Consequently, the Dirac Hamiltonian in (10.2) can be written as

$$H = i\rho \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_\rho + m\rho\begin{pmatrix} 0 & e^{-\tau} \\ e^{\tau} & 0 \end{pmatrix} .$$

The time translation in (10.1) must be complemented by the corresponding transformation of the spinors

$$\psi \mapsto \exp\Big(\gamma^0\gamma^1\,\frac{\Delta}{2}\Big)\psi = \begin{pmatrix} e^{-\frac{\Delta}{2}} & 0 \\ 0 & e^{\frac{\Delta}{2}} \end{pmatrix}\psi \,. \tag{10.3}$$

Indeed, by direct computation one verifies that the Dirac operator as well as the Dirac Hamiltonian are invariant under the joint transformations (10.1) and (10.3). If we also change the momentum variables according to

$$\alpha \mapsto \alpha + \Delta \,, \tag{10.4}$$

we know by Lorentz symmetry that the Dirac solutions in our Fourier representation remain unchanged. Therefore, the time evolution in the time coordinate τ is described by the inverse of the transformation (10.4), $\alpha \mapsto \alpha - \Delta$. We conclude that, infinitesimally, the Hamiltonian H is given by $i\partial_\tau = i\partial_\Delta = -i\partial_\alpha$. Using this formula in our plane wave ansatz (7.1), we conclude that

$$H\widehat{g}(s,\ell) = -\ell\,\widehat{g}(s,\ell) \,.$$

Comparing with (7.3), one sees that the eigenvalues of H agree up to a factor $-\pi m$ with that of those of the relative fermionic signature operator. Taking into account that the image of the operator $\iota_{\mathscr{M}}$ in (3.4) coincides with the orthogonal complement of the kernel of $\mathcal{S}_{\mathscr{R}}$ (see Theorem 9.1), we obtain the result. $\square$

11. Applications

11.1. **The FP State and Thermal States.** As explained in [5, 12], the fermionic signature operator can also be used to single out a distinguished fermionic quantum state, sometimes referred to as the fermionic projector state (or FP state). We now recall the construction and show that, in two-dimensional Rindler space-time, this construction gives precisely the Fulling-Rindler vacuum. We again work intrinsically in Rindler space-time. Since the Hamiltonian in the Dirac equation (10.2) is independent of τ, we can separate the τ-dependence with a plane wave ansatz

$$\psi(\tau,\rho) = e^{-i\Omega\tau}\,\chi(\rho) \,. \tag{11.1}$$

The sign of the separation constant Ω gives a splitting of the solution space of the Dirac equation into two subspaces. The Fulling-Rindler vacuum is the unique quantum state corresponding to this "frequency splitting" in the time coordinate τ. Next, the fermionic signature operator as defined by (2.9) is a self-adjoint operator with dense domain $\mathcal{D}(\mathcal{S})$ given by (9.1). Therefore, the functional calculus gives rise to projection operators $\chi_{(-\infty,0)}(\mathcal{S})$ and $\chi_{(0,\infty)}(\mathcal{S})$. Applying Araki's construction in [1] gives the FP state ω, being a pure quasi-free state on the algebra generated by the smeared fields operators (for details see [12, Section 6]). In view of Theorem 10.1, the projection operators $\chi_{(-\infty,0)}(\mathcal{S})$ and $\chi_{(0,\infty)}(\mathcal{S})$ coincide with the above frequency splitting. We thus obtain the following result:

Corollary 11.1. *The pure quasi-free FP state ω obtained from the fermionic signature operator coincides with the Fulling-Rindler vacuum.*

The advantage of working with the fermionic signature operator is that the construction is robust under perturbations of the metric. This connection will be discussed further in Section 12 below.

Applying Theorem 10.1, one can also construct thermal states (see [22] or [24, Chapter 5]; note that in our units the Boltzmann constant $k_B = 1$):

Corollary 11.2. *Applying Araki's construction to the positive operator*

$$W(\beta) = \frac{1}{1+e^{\beta m \pi \mathcal{S}}} \,, \tag{11.2}$$

one obtains a thermal state of temperature $1/\beta$. Choosing $\beta = 2\pi$, we get the Unruh state.

11.2. **The Fermionic Projector and Causal Fermion Systems.** Exactly as explained in [13, Section 3], the *fermionic projector P* is introduced as the operator

$$P = -\chi_{(-\infty,0)}(\mathcal{S})\, k_m \;:\; C^\infty_0(\mathscr{R}, S\mathscr{R}) \rightarrow \mathcal{H} \,,$$

where $\mathcal{S}$ is the fermionic signature operator, and k_m is the *causal fundamental solution* defined as the difference of the advanced and retarded Green's operators,

$$k_m := \frac{1}{2\pi i}\left(s^\vee_m - s^\wedge_m\right) \;:\; C^\infty_0(\mathscr{R}, S\mathscr{R}) \rightarrow C^\infty_{\text{sc}}(\mathscr{R}, S\mathscr{R}) \cap \mathcal{H}_{\mathscr{R}} \,.$$

The fermionic projector P can be represented by a distribution, referred to as the *kernel of the fermionic projector*. Namely, just as in [13, Section 3.5], one shows that there is a unique distribution $\mathcal{P} \in \mathcal{D}'(\mathscr{R} \times \mathscr{R})$ such that

$$<\!\phi | P \psi\!> = \mathcal{P}\big(\overline{\phi} \otimes \psi\big) \qquad \text{for all } \phi, \psi \in C^\infty_0(\mathscr{R}, S\mathscr{R}) \,.$$

Indeed, the bi-distribution $\mathcal{P}$ agrees with the two-point function of the FP state in Corollary 11.1.

After inserting an ultraviolet regularization, the kernel of the fermionic projector gives rise to a *causal fermion system* (see [13, Section 4] or [8, Section 1.2]). The theory of causal fermion systems is an approach to describe fundamental physics (see [7, 8] or the survey papers [9] or [10]). In this context, the kernel of the fermionic projector is used extensively in the analysis of the causal action principle.

12. Asymptotically Rindler Space-Times

The main purpose of this paper was to show that the construction of quantum states with the fermionic signature operator gives agreement with the frequency splitting in Rindler coordinates and the construction of thermal states. We now give an outlook which also explains the advantages of working with the fermionic signature operator.

For the frequency splitting in Rindler coordinates, it is essential that the coordinate τ corresponds to a symmetry of space-time. Therefore, the construction of the Fulling-Rindler vacuum breaks down as soon as space-time no longer has this symmetry. The frequency splitting no longer works even if the Rindler metric is modified by a small

τ-dependent perturbation, simply because the separation ansatz (11.1) can no longer be used. However, using the fermionic signature operator has the major benefit that the constructions in [13, 14] apply to arbitrary space-times, without any symmetry assumptions.

In particular, the above construction applies to curved space-times for which the metric tends to Rindler space-time at infinity with a suitable decay rate. The crucial point for the construction is to establish rapid decay estimates for the Dirac solutions in null directions (4.2). These estimates have been worked out in a more general static setting in the thesis [21], where also sufficient decay properties of the metric perturbations are specified. In particular, Corollary 4.7.5 in this thesis immediately gives the following result.

Theorem 12.1. *Let $\mathscr{R}$ be the Rindler wedge* (2.1) *with the static Lorentzian metric*

$$ds^2 = \big(1 + A(x)\big)^2 \big(dt^2 - dx^2\big) \:,$$

where the metric function A has the following properties:

(i) $\|A\|_{C^k(\mathbb{R}^+)} < \infty$ *for all $k \in \mathbb{N}$.*
(ii) *There are constants $C, \alpha > 0$ such that*

$$\big|A(x)\big|,\; \big|A'(x)\big| \leq \frac{C}{(1+x)^\alpha} \qquad \text{for all } x \in \mathbb{R}^+ \:.$$

Then the relation (2.9) *uniquely defines the fermionic signature operator $\mathcal{S}$ as an operator with dense domain $\mathcal{D}(\mathcal{S}) = C^\infty_{sc}(\mathscr{R}, S\mathscr{R})$.*

We remark that the methods in [21] could be adapted to the non-static situation.

In order to analyze the spectral properties of $\mathcal{S}$, one could adapt the perturbative methods as developed in [12] for an external potential in Minkowski space. We expect that for sufficiently small perturbations, the resulting FP state should again be Hadamard. However, proving this conjecture is more difficult than the construction in [12, Section 5], mainly because the fermionic signature operator in Rindler space-time does not have a spectral gap separating the positive and negative spectrum. Therefore, we leave this problem as a project for future research.

13. Extension to Four-Dimensional Rindler Space-Time

We now explain how our results extend to the case of four-dimensional Rindler space-time. Thus let $\mathscr{M} = \mathbb{R}^{1,3}$ be four-dimensional Minkowski space and $\mathscr{R}$ the subset

$$\mathscr{R} = \big\{ (t,x,y,z) \in \mathbb{R}^{1,3} \text{ with } |t| < x \big\} \:.$$

The Dirac equation in Rindler space-time is formulated as the restriction of the Dirac equation in Minkowski space to $\mathscr{R}$ (we use the same notation and conventions as in [2, 18]). Its solutions are most easily constructed by separating the y-and z-dependence with a plane wave ansatz,

$$\psi(t,x,y,z) = e^{ik_y y + ik_z z}\, \tilde{\psi}(t,x) \:, \tag{13.1}$$

giving the Dirac equation in t and x

$$\big(i \gamma^0 \partial_t + i \gamma^1 \partial_x \big) \tilde{\psi}(t,x) = \big(m + \gamma^2 k_y + \gamma^3 k_z \big) \tilde{\psi}(t,x) \:.$$

Transforming to momentum space, the solutions lie on a mass shell of mass

$$\tilde{m} := \sqrt{m^2 + k_y^2 + k_z^2} \:. \tag{13.2}$$

Thus, similar to (5.1), we can make the ansatz

$$\tilde{\psi}(q) = \int_{\mathbb{R}^2} \frac{d^2p}{(2\pi)^2}\, \hat{\psi}(p)\, \delta(p^2 - \tilde{m}^2)\, e^{-ipq}\,,$$

giving rise to the algebraic equation

$$\big(\omega\gamma^0 - k\gamma^1\big)\hat{\psi} = \big(m + \gamma^2 k_y + \gamma^3 k_z\big)\hat{\psi} \tag{13.3}$$

(where again $p = (\omega, k)$). This equation has a two-dimensional solution space. In analogy to (5.2), we choose a basis of solutions $\mathfrak{f}_1, \mathfrak{f}_2$. In the next lemma it is shown that these spinors can be chosen to have similar properties to those stated in Lemma 5.1 and Lemma 6.3.

Lemma 13.1. *Given k_y and k_z, there are spinors $\mathfrak{f}_a(p)$ with $a = \pm 1$ which solve the Dirac equation* (13.3) *and satisfy the relations*

$$\begin{aligned} \prec\!\mathfrak{f}_a(\omega,k)\,|\,\mathfrak{f}_b(\omega,k)\!\succ &= \epsilon(\omega)\,\delta_{ab} \\ \prec\!\mathfrak{f}_a(\omega,k)\,|\,\gamma^0\,\mathfrak{f}_b(-\omega,k)\!\succ &= 0 \\ \prec\!\mathfrak{f}_a(\omega,k)\,|\,\gamma^0\,\mathfrak{f}_b(\omega,k)\!\succ &= \frac{|\omega|}{m}\,\delta_{ab}\,. \end{aligned}$$

Moreover, in the parametrization (5.6),

$$\prec\!\mathfrak{f}_a(s,\alpha)\,|\,\mathfrak{f}_b(\tilde{s},\tilde{\alpha})\!\succ = s\,\delta_{ab}\,\frac{\tilde{m}}{m}\left\{\begin{array}{ll} \cosh(\beta + i\nu_a) & \text{if } s = \tilde{s} \\ \sinh(\beta + i\nu_a) & \text{if } s \neq \tilde{s}\,, \end{array}\right.$$

where β is again given by (6.3), *and the angle $\nu_a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is defined by*

$$\nu_a = \arctan\left(\frac{a}{m}\,\sqrt{k_y^2 + k_z^2}\right). \tag{13.4}$$

Proof. After rotating our reference frame, we can assume that $k_z = 0$ and $k_y > 0$. Then in the Dirac representation (see for example [2]), the Dirac equation (13.3) takes the form

$$\begin{pmatrix} \omega - m & 0 & 0 & -k + ik_y \\ 0 & \omega - m & -k - ik_y & 0 \\ 0 & k - ik_y & -\omega - m & 0 \\ k + ik_y & 0 & 0 & -\omega - m \end{pmatrix} \hat{\psi} = 0\,. \tag{13.5}$$

Obviously, this matrix has two invariant subspaces: one spanned by the first and fourth spinor components, and the other spanned by the second and third spinor components. Choosing $\mathfrak{f}_1$ in the first and $\mathfrak{f}_{-1}$ in the second of these subspaces, the above inner products all vanish if $a \neq b$. In the remaining case $a = b$, one can restrict attention to two-spinors. In order to get back to the setting in two-dimensional Rindler space-time, we use the identity

$$U \begin{pmatrix} \omega - m & -k \pm ik_y \\ k \pm ik_y & -\omega - m \end{pmatrix} U = \begin{pmatrix} \omega - \tilde{m} & -k \\ k & -\omega - \tilde{m} \end{pmatrix}\,,$$

where U is the matrix

$$U = \begin{pmatrix} \cos(\nu_a/2) & i\sin(\nu_a/2) \\ i\sin(\nu_a/2) & \cos(\nu_a/2) \end{pmatrix}\,.$$

Now the results follow by direct computation. ∎

Using the result of this lemma, we can represent the solution in analogy to (5.4) by

$$\psi(q) = \sum_{a=\pm 1} \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \epsilon(\omega)\, \delta(p^2 - m^2)\, g_a(p)\, \mathfrak{f}_a(p)\, e^{-ipq}$$

with two complex-valued functions $g_{\pm 1}$. The subsequent analysis can be extended in a straightforward way. In particular, the kernel I_ε in Corollary 6.2 is to be replaced by the kernels

$$I^a_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big) = \frac{1}{4\pi^2\, m} \times \begin{cases} \dfrac{s\cosh(\beta + i\nu_a)}{1-\cosh(2\beta + i\varepsilon s)} & \text{if } s = \tilde{s} \\[2ex] -\dfrac{s\sinh(\beta + i\nu_a)}{1+\cosh(2\beta)} & \text{if } s \neq \tilde{s}\,. \end{cases}$$

The residues can be computed as in Lemma 7.1 if one transforms the integrals in the following way,

$$\begin{aligned}
&\int_{-\infty}^\infty \frac{\sinh(\beta + i\nu_a)}{1+\cosh(2\beta)}\, e^{-2i\ell\beta}\, d\beta \\
&= \cos\nu_a \int_{-\infty}^\infty \frac{\sinh(\beta)}{1+\cosh(2\beta)}\, e^{-2i\ell\beta}\, d\beta + i\sin\nu_a \int_{-\infty}^\infty \frac{\cosh(\beta)}{1+\cosh(2\beta)}\, e^{-2i\ell\beta}\, d\beta \\
&= -\cos\nu_a \int_{-\infty}^\infty \frac{d}{d\beta}\left(\frac{1}{e^\beta + e^{-\beta}}\right) e^{-2i\ell\beta}\, d\beta + i\sin\nu_a \int_{-\infty}^\infty \frac{1}{e^\beta + e^{-\beta}}\, e^{-2i\ell\beta}\, d\beta \\
&= \int_{-\infty}^\infty \frac{1}{e^\beta + e^{-\beta}} \left(\cos\nu_a \frac{d}{d\beta} + i\sin\nu_a\right) e^{-2i\ell\beta}\, d\beta \\
&= \int_{-\infty}^\infty \frac{1}{e^\beta + e^{-\beta}} \Big(-2i\ell\cos\nu_a + i\sin\nu_a\Big)\, e^{-2i\ell\beta}\, d\beta\,,
\end{aligned}$$

showing that the integral is obtained from the earlier integral (7.5) if one only replaces the prefactor ℓ by $\tilde{\ell}$ given by

$$\tilde{\ell}_a := \ell\cos\nu_a - \frac{\sin\nu_a}{2}\,. \tag{13.6}$$

The same method also applies to the integral (7.6) and again amounts to the replacement (13.6). We conclude that the matrix in (7.2) is to be replaced by the two matrices

$$\hat{\mathcal{S}}^a_{\mathscr{R}}(\ell) = \frac{\tilde{\ell}_a}{\pi m} \begin{pmatrix} \dfrac{1}{1+e^{-2\pi\ell}} & -\dfrac{i}{2\cosh(\pi\ell)} \\[2ex] \dfrac{i}{2\cosh(\pi\ell)} & \dfrac{1}{1+e^{2\pi\ell}} \end{pmatrix}.$$

These matrices have the eigenvalues

$$\lambda = 0 \qquad \text{and} \qquad \lambda = \frac{\tilde{\ell}_a}{\pi m}\,.$$

As a consequence, the analog of Theorem 10.1 is the following statement:

Theorem 13.2. *After separating the y- and z-dependence by the plane wave ansatz* (13.1)*, the fermionic signature operator $\mathcal{S}$ and the Hamiltonian H in Rindler coordinates satisfy the relations*

$$\mathcal{S} = -\frac{H}{\pi\tilde{m}} - \frac{1}{2\pi\, m\tilde{m}}\, \gamma^0\gamma^1\Big(\gamma^2\partial_y + \gamma^3\partial_z\Big) \tag{13.7}$$

with $\widetilde{m}$ *according to* (13.2).

Proof. Considering again a Lorentz boost, just as in the proof of Theorem 10.1 we find that $H = -\ell$. Therefore, considering as in (13.5) the situation that $k_z = 0$ and $k_y > 0$, we obtain on the first and fourth spinor components that

$$\mathcal{S}^a = -\frac{H}{\pi m}\,\cos\nu_a - \frac{\sin\nu_a}{2\pi m}$$

with $a = 1$. Similarly, on the second and third spinor components, the same formula holds with $a = -1$. Using (13.4), we can simplify these equations to

$$\mathcal{S}^a = -\frac{H}{\pi\widetilde{m}} - \frac{a k_y}{2\pi m\,\widetilde{m}}\,.$$

By direct computation, one verifies that the operator

$$\gamma^0\gamma^1\Big(\gamma^2\partial_y + \gamma^3\partial_z\Big)$$

has an eigenvalue k_y, and the corresponding eigenspace is the subspace spanned by the first and fourth spinor components. Likewise, the subspace spanned by the second and third spinor components is an eigenspace to the eigenvalue $-k_y$. This proves (13.7) for the case $k_z = 0$ and $k_y > 0$. The general case follows immediately because the operator (13.7) is invariant under rotations in the yz-plane. □

We remark that the separation of the y- and z-dependence could be described more mathematically by a Fourier transformation $\psi(t,x,y,z) \mapsto \widetilde{\psi}(t,x,k_y,k_z)$, being a unitary transformation between corresponding Hilbert spaces. Since this procedure is very similar to that at the beginning of Section 8, we leave the details to the reader. Carrying out this procedure, the factors $1/\widetilde{m}$ become multiplication operators in momentum space (see (13.2)). Clearly, in position space, these operators are nonlocal in the variables y and z.

Applying the constructions outlined in Section 11.1, we again get quasi-free quantum states. However, these states are different from the Fulling-Rindler vacuum and the thermal states as obtained in Corollaries 11.1 and 11.2. The physical significance of these new states is presently under investigation.

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