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Abstract

In this paper we introduce a new formulation for shape optimization problems in fluids in a diffuse interface setting that can in particular handle topological changes. By adding the Ginzburg–Landau energy as a regularization to the objective functional and relaxing the non-permeability outside the fluid region by introducing a porous medium approach we hence obtain a phase field problem where the existence of a minimizer can be guaranteed. This problem is additionally related to a sharp interface problem, where the permeability of the non-fluid region is zero. In both the sharp and the diffuse interface setting we can derive necessary optimality conditions using only the natural regularity of the minimizers. We also pass to the limit in the first order conditions.

Key words. Shape and topology optimization, phase field method, diffuse interfaces, Stokes flow, fictitious domain.

AMS subject classification. 35R35, 35Q35, 49Q10, 49Q20, 76D07.

1 Introduction

Shape optimization is the problem of minimizing some functional depending on the shape or geometry of certain regions. If the topology is part of the optimization process one refers to this also as shape and topology optimization. Here we work on the specific branch of shape optimization in fluids. This means, that the objective functional depends not only explicitly on certain quantities related to the shape but also implicitly by including physical values describing the motion of some fluid which is located inside the unknown optimal region. Hence the objective functional may depend for instance on the velocity or the pressure of the fluid. In this work we assume that the fluid obeys the Stokes equations. Thus the general problem to be considered here can be written as

$$\min_{(E,\boldsymbol{u})} \int_E f\left(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u},p\right)\,\mathrm{d}x \qquad \text{subject to } -\mu\Delta\boldsymbol{u}+\nabla p=\boldsymbol{f},\ \operatorname{div}\boldsymbol{u}=0, \quad \text{in } E. \tag{1}$$

Here, $\boldsymbol{u}$ denotes the velocity, p the pressure, $\mu>0$ the viscosity of the fluid and $\boldsymbol{f}$ is some general external force.
Due to the broad application fields of shape optimization in fluid mechanics, quite elaborated practical methods have been developed in industry. But advanced numerical methods, like gradient or Newton's method, require gradients of the cost functional. One approach to formulate a gradient in an appropriate Hilbert space setting is the shape

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sensitivity analysis. Several authors derived formulas for the shape derivative in a fluid dynamical setting. But either the calculations are formal, [35], or there are restrictions in terms of geometric or regularity constraints on the reference domain and hence on the minimizing set, compare for instance [3, 32, 33, 37]. In this work we will present a formula for shape derivatives that is verified for very general sets as a reference domain, see Section 5.

Furthermore, it has turned out that most shape optimization problems lack existence of a minimizer and finding well-posed formulations of (1) is not trivial. The right space for the admissible shapes has to be characterized and suitable regularizations or constraints may be necessary. The main contribution here is due to Šverák, [44], who was able to show an existence result in space dimension two. This was then extended to more space dimensions by Bucur and Zolésio, see [11], and applied to a fluid dynamical setting in [5]. But their result needs a restriction on the admissible shapes in terms of not yet characterized geometric constraints. Apart from that, there are also contributions considering compressible fluids, like [34], but again geometric constraints on the admissible shapes are necessary. As indicated in [31, 41], it may not be expected that a minimizer exists for the general problem (1) without any restrictions or regularizations. One idea to overcome this problem was established in the field of finding optimal material configurations by [1]. There, a multiple of the perimeter of the shape is added to the objective functional and the problem is formulated in a setting of Caccioppoli sets. This additional perimeter term gives rise to better compactness properties and prevents oscillations and the occurrence of microstructures, compare also [4]. In addition, by minimizing over all Caccioppoli sets there are almost no restrictions in geometric, regularity or topological terms on the admissible shapes. Anyhow, most problems in shape optimization that have been shown to be well-posed, even by using a perimeter penalization, have special structure, i.e. they can be reformulated to a problem without state equations, see for example [1, 9]. This corresponds in our case to the case of minimizing the total potential power and is already discussed in [24]. For minimizing a general objective functional, the idea of a so called fictitious material approach has been developed in the field of structural optimization, see [1, 9], where the void region is replaced by a very weak material. This idea has been transferred to fluid mechanical setting by [8], where the region outside the fluid is replaced by a porous medium. Anyhow, only applying the porous medium approach gives only a well-posed problem in case of having the above-mentioned special structure, i.e. here minimizing the total potential power in a Stokes flow. As discussed in [21], it is not expected that one can generalize this to general objective functionals or different state equations.

But coupling this porous medium approach to a Ginzburg–Landau penalization, which is the diffuse interface analogue of the perimeter penalization, one can show well-posedness with a general objective functional and also apply different state equations. The resulting problem is then given in a phase field setting. Additionally, we can consider a sharp interface limit and show that under suitable assumptions the obtained minimizers approximate a black-and-white solution of a perimeter penalized sharp interface problem.

The porous medium – phase field formulation of the shape optimization problem (1) with a general objective functional including the velocity of the fluid and its derivative can be roughly outlined as

$$\min_{(\varphi,\boldsymbol{u})} \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2 \,\mathrm{d}x + \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \gamma\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x$$

$$\text{subject to } \int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v},$$

where φ is the phase field function. For details we refer to Section 2.

In this paper we will

- show existence of a minimizer for the resulting phase field problem (see Theorem 1 in Section 2);
- discuss the corresponding perimeter penalized sharp interface problem (see Section 3), which is in a simplified form given as

$$\min_{(\varphi,\boldsymbol{u})} \int_\Omega f\left(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}\right) \mathrm{d}x + \gamma c_0 P_\Omega(\{\varphi = 1\})$$
$$\text{subject to } \int_{\{\varphi=1\}} \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \, \mathrm{d}x = \int_{\{\varphi=1\}} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x \quad \forall \boldsymbol{v};$$

- consider convergence of solutions of the phase field problem to a solution of the sharp interface problem as the interfacial thickness and the permeability of the medium outside the fluid converge to zero (see Section 4);
- derive first order optimality conditions for the phase field and sharp interface shape optimization problems by geometric variations. In the case of the sharp interface problem we can derive the first order conditions under assumptions which are much weaker than conditions which appeared in the literature previously (see Section 5);
- relate the obtained optimality conditions to existing criteria, hence to a variational inequality in the diffuse interface setting, compare Remark 8, and to shape derivatives in the well-known Hadamard form in the sharp interface setting, see Remark 9;
- consider the sharp interface limit in the obtained optimality systems (see Theorem 5 in Section 5);
- discuss the same questions if the objective functional depends additionally on the pressure of the fluid (see Section 6).

A comparable sharp interface limit in the first variation formula has been carried out for instance in [23], where geometric variations of the elastic Ginzburg–Landau energy are considered. We also mention the work [6] where a sharp interface limit in the structural optimization has been carried out by formal asymptotics. But for a setting with state equations, which even depend on the phase field parameter, the rigorous considerations in this paper are new. The generalization to the stationary Navier–Stokes equations will be the subject of a forthcoming paper but is already discussed in [27].

2 Problem formulation

In the following we will minimize a certain objective functional depending on the behaviour of some fluid by varying the shape, geometry and topology of the region wherein the fluid is located. The fluid region is to be chosen inside a fixed container $\Omega \subset \mathbb{R}^d$, which is assumed to fulfill

(A1) $\Omega \subseteq \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded Lipschitz domain with outer unit normal $\boldsymbol{n}$.

The velocity of the fluid has prescribed Dirichlet boundary data on $\partial\Omega$, hence we may impose for instance certain in-or outflow profiles. Additionally we can assume a body force acting on the whole of Ω. And so we fix for the subsequent considerations the following functions:

(A2) Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ denote the applied body force and $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ the given boundary function such that $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}x = 0$.

We remark, that throughout this work $\mathbb{R}^d$-valued functions or function spaces of $\mathbb{R}^d$-valued functions are denoted by boldface letters.

The general functional to be minimized is for the time being given as $\int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}) \, \mathrm{d}x$ and hence depends on the velocity $\boldsymbol{u} \in \boldsymbol{U} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \operatorname{div} \boldsymbol{v} = 0, \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{g}\}$ of the fluid and its derivative. The treatment of the pressure in the objective functional is studied in Section 6. The objective functional is chosen according to the following assumptions:

(A3) We choose $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ as a Carathéodory function, thus fulfilling

- $f(\cdot, v, A) : \Omega \to \mathbb{R}$ is measurable for each $v \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$, and
- $f(x, \cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is continuous for almost every $x \in \Omega$.

Let $p \geq 2$ for $d = 2$ and $2 \leq p \leq 2d/d-2$ for $d = 3$ and assume that there are $a \in L^1(\Omega)$, $b_1, b_2 \in L^\infty(\Omega)$ such that for almost every $x \in \Omega$ it holds

$$|f(x, v, A)| \leq a(x) + b_1(x)|v|^p + b_2(x)|A|^2, \quad \forall v \in \mathbb{R}^d, A \in \mathbb{R}^{d\times d}. \tag{2}$$

Additionally, assume that the functional

$$F : \boldsymbol{H}^1(\Omega) \to \mathbb{R}, \quad F(\boldsymbol{u}) := \int_\Omega f(x, \boldsymbol{u}(x), \mathrm{D}\boldsymbol{u}(x)) \, \mathrm{d}x \tag{3}$$

is weakly lower semicontinuous, $F|_{\boldsymbol{U}}$ is bounded from below, and F is radially unbounded in $\boldsymbol{U}$, which means

$$\lim_{k\to\infty} \|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)} = +\infty \implies \lim_{k\to\infty} F(\boldsymbol{u}_k) = +\infty \tag{4}$$

for any sequence $(\boldsymbol{u}_k)_{k\in\mathbb{N}} \subseteq \boldsymbol{U}$.

Remark 1. *Remark that condition* (2) *implies that* $\boldsymbol{H}^1(\Omega) \ni \boldsymbol{u} \mapsto \int_\Omega f(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}(x)) \, \mathrm{d}x$ *is continuous, see [36].*

The shape to be optimized is here the region filled with fluid and is described by a design function $\varphi \in L^1(\Omega)$. The fluid region then corresponds to $\{x \in \Omega \mid \varphi(x) = 1\}$ and the non-fluid region is described by $\{x \in \Omega \mid \varphi(x) = -1\}$. We will formulate a diffuse interface problem, hence φ is also allowed to take values in $(-1, 1)$, which yields then an interfacial region. The thickness of the interface is dependent on the so-called phase field parameter $\varepsilon > 0$. We impose an additional volume constraint for the fluid region, i.e. $\int_\Omega \varphi \, \mathrm{d}x \leq \beta |\Omega|$, where $\beta \in (-1, 1)$ is an arbitrary but fixed constant. Hence, the design space for the optimization problem is given by

$$\Phi_{ad} := \left\{\varphi \in H^1(\Omega) \mid |\varphi| \leq 1 \text{ a.e. in } \Omega, \int_\Omega \varphi \, \mathrm{d}x \leq \beta |\Omega|\right\}. \tag{5}$$

In order to obtain a well-posed problem, we use the idea of perimeter penalization, see for instance [1]. Thus we add a multiple of the diffuse interface analogue of the perimeter functional, which is the Ginzburg-Landau energy, to the objective functional. To be precise we add

$$\gamma \int_\Omega \frac{\varepsilon}{2} |\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x$$

where $\psi : \mathbb{R} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, given by

$$\psi(\varphi) := \begin{cases} \frac{1}{2}\left(1-\varphi^2\right), & \text{if } |\varphi| \le 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

is the potential and $\gamma > 0$ a fixed weighting parameter for this regularization. As already discussed in the introduction, we will use the porous medium approach introduced by [8] for the optimization problem. Thus the region outside the fluid obeys the equations of flow through porous material with small permeability $(\overline{\alpha}_\varepsilon)^{-1} \ll 1$. Notice that we couple the parameter for the porous medium approach to the phase fiel parameter $\varepsilon > 0$. In the interfacial region we interpolate between the Stokes equations and the porous medium equations by using an interpolation function $\alpha_\varepsilon : [-1,1] \to [0, \overline{\alpha}_\varepsilon]$ fulfilling the following assumptions:

(A4) Let $\alpha_\varepsilon : [-1,1] \to [0,\overline{\alpha}_\varepsilon]$ be decreasing, surjective and continuous for every $\varepsilon > 0$.

It is required that $\overline{\alpha}_\varepsilon > 0$ is chosen such that $\lim_{\varepsilon\searrow 0} \overline{\alpha}_\varepsilon = +\infty$ and α_ε converges pointwise to some function $\alpha_0 : [-1,1] \to [0,+\infty]$. Additionally, we impose $\alpha_\delta(x) \ge \alpha_\varepsilon(x)$ if $\delta \le \varepsilon$ for all $x \in [-1,1]$, $\lim_{\varepsilon\searrow 0} \alpha_\varepsilon(0) < \infty$ and a growth condition of the form $\overline{\alpha}_\varepsilon = o\left(\varepsilon^{-\frac{2}{3}}\right)$.

Remark 2. *For space dimension $d = 2$ we can even choose $\overline{\alpha}_\varepsilon = o(\varepsilon^{-\kappa})$ for any $\kappa \in (0,1)$, compare also Remark 5.*

Thus the overall optimization problem is given as

$$\min_{(\varphi,\boldsymbol{u})} J_\varepsilon(\varphi,\boldsymbol{u}) := \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x + \gamma\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x \tag{6}$$

subject to $(\varphi,\boldsymbol{u}) \in \Phi_{ad} \times \boldsymbol{U}$ and

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{7}$$

where $\boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \operatorname{div}\boldsymbol{v} = 0\}$. The first term which includes the interpolation function α_ε appearing in the objective functional (6) penalizes too large values for $|\boldsymbol{u}|$ outside the fluid region (hence if $\varphi = -1$). This is a result of the choice of $\alpha_\varepsilon(-1) = \overline{\alpha}_\varepsilon \gg 1$. The penalization of too large values for the velocity in the porous medium is in particular important because we want in the limit $\varepsilon \searrow 0$ the velocity $\boldsymbol{u}$ to vanish outside the fluid region, see Section 3. By this we ensure to arrive in the desired black-and-white solutions.

Concerning the state equations (7) we directly find the following solvability result:

Lemma 1. *For every $\varphi \in L^1(\Omega)$ with $|\varphi(x)| \le 1$ a.e. in Ω there exists a unique $\boldsymbol{u} \in \boldsymbol{U}$ such that (7) is fulfilled. This defines a solution operator for the constraints, which will be denoted by $\boldsymbol{S}_\varepsilon : \Phi_{ad} \to \boldsymbol{U}$. Here, we define $\boldsymbol{S}_\varepsilon(\varphi) := \boldsymbol{u}$ if $\boldsymbol{u}$ solves (7).*

Proof. This follows by an application of Lax-Milgram's theorem. For details we refer to [27, Lemma 5.1]. □

Using this existence result for the state equations we can rewrite (6) – (7) into an unconstrained optimization problem by introducing the reduced objective functional $j_\varepsilon : L^1(\Omega) \to \overline{\mathbb{R}}$, which is given as

$$j_\varepsilon(\varphi) := \begin{cases} J_\varepsilon(\varphi, \boldsymbol{S}_\varepsilon(\varphi)) & \text{if } \varphi \in \Phi_{ad}, \\ +\infty & \text{otherwise.} \end{cases} \tag{8}$$

Then (6) – (7) is equivalent to

$$\min_{\varphi \in L^1(\Omega)} j_\varepsilon(\varphi). \tag{9}$$

Due to the regularization by the Ginzburg-Landau energy and the porous medium formulation we obtain, in contrast to most formulations in shape optimization, that the problem (6) – (7) admits a minimizer, even with a general objective functional, as the following theorem shows:

Theorem 1. *There exists at least one minimizer $\varphi_\varepsilon \in \Phi_{ad}$ of j_ε, and hence there exists also a minimizer of* (6) – (7)*.*

Proof. We use the direct method in the calculus of variations. From the boundedness assumption in Assumption **(A3)** we deduce that $J_\varepsilon : \Phi_{ad} \times \boldsymbol{U} \to \mathbb{R}$ is bounded from below by a constant. Thus we can choose an admissible minimizing sequence $(\varphi_k, \boldsymbol{u}_k)_{k\in\mathbb{N}} \subset \Phi_{ad} \times \boldsymbol{U}$, which gives in particular that $\boldsymbol{u}_k = \boldsymbol{S}_\varepsilon(\varphi_k)$ for all $k \in \mathbb{N}$. The coercivity of the objective functional, see (4), yields a uniform bound on $\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)}$.
Moreover, the uniform bound in $(J_\varepsilon(\varphi_k, \boldsymbol{u}_k))_{k\in\mathbb{N}}$ implies that $\sup_{k\in\mathbb{N}} \|\nabla\varphi_k\|_{L^2(\Omega)} < \infty$. Besides, $\varphi_k \in \Phi_{ad}$ for all $k \in \mathbb{N}$, and so $\|\varphi_k\|_{L^\infty(\Omega)} \le 1 \ \forall k \in \mathbb{N}$. Thus we find a subsequence of $(\boldsymbol{u}_k, \varphi_k)_{k\in\mathbb{N}}$, denoted by the same, such that $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}_0$ in $\boldsymbol{H}^1(\Omega)$ and $\varphi_k \rightharpoonup \varphi_0$ in $H^1(\Omega)$ for some element $(\boldsymbol{u}_0, \varphi_0) \in \boldsymbol{U} \times \Phi_{ad}$. Here we used that Φ_{ad} and $\boldsymbol{U}$ are closed and convex and thus weakly closed subspaces of $H^1(\Omega)$ and $\boldsymbol{H}^1(\Omega)$, respectively.
Next we show, that $\boldsymbol{u}_0 = \boldsymbol{S}_\varepsilon(\varphi_0)$. Therefore we use Lebesgue's dominated convergence theorem and the pointwise convergence of the sequences $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ and $(\varphi_k)_{k\in\mathbb{N}}$, which follows after choosing subsequences. From this we find quite easily

$$\lim_{k\to\infty} \int_\Omega \alpha_\varepsilon(\varphi_k)\boldsymbol{u}_k \cdot \boldsymbol{v}\,\mathrm{d}x = \int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Then we can take the limit $k \to \infty$ in the weak formulation of the state equation (7) and see that $\boldsymbol{u}_0$ fulfills (7) with φ replaced by φ_0. In particular, this gives $\boldsymbol{u}_0 = \boldsymbol{S}_\varepsilon(\varphi_0)$ and thus $(\varphi_0, \boldsymbol{u}_0)$ is admissible for (6) – (7).
Similar as above we obtain by using Lebesgue's dominated convergence theorem

$$\lim_{k\to\infty} \int_\Omega \alpha_\varepsilon(\varphi_k) |\boldsymbol{u}_k|^2 \,\mathrm{d}x = \int_\Omega \alpha_\varepsilon(\varphi_0) |\boldsymbol{u}_0|^2 \,\mathrm{d}x.$$

This gives us in view of the lower semicontinuity of the objective functional stated in Assumption **(A3)** the estimate

$$J_\varepsilon(\varphi_0, \boldsymbol{u}_0) \le \liminf_{k\to\infty} J_\varepsilon(\varphi_k, \boldsymbol{u}_k) = \inf_{(\varphi,\boldsymbol{u})\in\Phi_{ad}\times\boldsymbol{U}, \boldsymbol{u}=\boldsymbol{S}_\varepsilon(\varphi)} J_\varepsilon(\varphi, \boldsymbol{u})$$

which implies that $(\varphi_0, \boldsymbol{u}_0)$ minimizes J_ε.

□

Thus we have shown that the phase field model, which is given by (6) – (7), is well-defined in the sense that we have a well-defined solution operator for the constraints and have guaranteed existence of a minimizer for the overall optimization problem.

3 Sharp interface problem

In Section 4 we will consider the limit $\varepsilon \searrow 0$, the so-called sharp interface limit. Hence we want to send both the interface thickness and the permeability of the medium outside the fluid to zero in order to arrive in a sharp interface problem whose solutions can be considered as black-and-white solutions. This means that only pure fluid and pure non-fluid phases exist, and the permeability of the material outside the fluid is zero (thus "real walls", according to [20], can appear). The problem appearing in the limit $\varepsilon \searrow 0$ will be introduced in this section. This turns out to be a sharp interface problem in a setting of Caccioppoli sets with perimeter penalization. In order to formulate this we will briefly introduce some notation. For a detailed introduction into the theory of Caccioppoli sets and functions of bounded variations we refer to [2, 19]. We call a function $\varphi \in L^1(\Omega)$ a function of bounded variation if its distributional derivative is a vector-valued finite Radon measure. The space of functions of bounded variation in Ω is denoted by $BV(\Omega)$, and by $BV(\Omega,\{\pm 1\})$ we denote functions in $BV(\Omega)$ having only the values ± 1 a.e. in Ω. We then call a measurable set $E \subset \Omega$ Caccioppoli set if $\chi_E \in BV(\Omega)$. For any Caccioppoli set E, one can hence define the total variation $|\mathrm{D}\chi_E|(\Omega)$ of $\mathrm{D}\chi_E$, as $\mathrm{D}\chi_E$ is a finite measure. This value is then called the perimeter of E in Ω and is denoted by $P_\Omega(E) := |\mathrm{D}\chi_E|(\Omega)$.

In the sharp interface problem we still define the velocity of the fluid on the whole of Ω, even though there is only a part of it filled with fluid. This is realized by defining the velocity to be zero in the non-fluid region. Hence, the velocity corresponding to some design variable $\varphi \in L^1(\Omega)$ is to be chosen in the space $\boldsymbol{U}^\varphi := \{\boldsymbol{u} \in \boldsymbol{U} \mid \boldsymbol{u}|_{\{\varphi=-1\}} = \boldsymbol{0} \text{ a.e. in } \Omega\}$, where we recall that the fluid regions is given by $\{\varphi = 1\}$ and the non-fluid region by $\{\varphi = -1\}$. Correspondingly we define $\boldsymbol{V}^\varphi := \{\boldsymbol{u} \in \boldsymbol{V} \mid \boldsymbol{u}|_{\{\varphi=-1\}} = \boldsymbol{0} \text{ a.e. in } \Omega\}$. The space $\boldsymbol{U}^\varphi$ may be empty if the conditions $\boldsymbol{u}|_{\{\varphi=-1\}} = \boldsymbol{0}$ and $\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}$ are conflicting. Thus we only allow design variables φ where $\boldsymbol{U}^\varphi \neq \emptyset$. The design space for the sharp interface problem is given as

$$\Phi_{ad}^0 := \left\{ \varphi \in BV(\Omega,\{\pm 1\}) \mid \int_\Omega \varphi \, \mathrm{d}x \le \beta |\Omega|, \boldsymbol{U}^\varphi \neq \emptyset \right\}.$$

We can then write the the sharp interface problem as

$$\min_{(\varphi,\boldsymbol{u})} J_0(\varphi,\boldsymbol{u}) := \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}) \, \mathrm{d}x + \gamma c_0 P_\Omega(\{\varphi = 1\}) \tag{10}$$

subject to $(\varphi,\boldsymbol{u}) \in \Phi_{ad}^0 \times \boldsymbol{U}^\varphi$ and

$$\mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \, \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}^\varphi. \tag{11}$$

Here, $c_0 := \int_{-1}^1 \sqrt{2\psi(s)} \, \mathrm{d}s = \frac{\pi}{2}$ is a constant appearing due to technical reasons in the limit $\varepsilon \searrow 0$, compare Section 4. Recall, that $\gamma > 0$ was an arbitrary weighting parameter for the perimeter penalization. Let us start by considering the state equations.

Lemma 2. *For every $\varphi \in L^1(\Omega)$ such that $\boldsymbol{U}^\varphi \neq \emptyset$ there exists a unique $\boldsymbol{u} \in \boldsymbol{U}^\varphi$ such that* (11) *is fulfilled. This defines a solution operator denoted by $\boldsymbol{S}_0 : \Phi_{ad}^0 \to \boldsymbol{U}$ where we define $\boldsymbol{S}_0(\varphi) := \boldsymbol{u} \in \boldsymbol{U}^\varphi$ if $\boldsymbol{u}$ fulfills* (11).

Proof. This can be shown by an application of Lax-Milgram's theorem, compare [27, Lemma 6.1] for details. □

Using this solution operator we can define the reduced objective functional $j_0 : L^1(\Omega) \to \overline{\mathbb{R}}$ by

$$j_0(\varphi) := \begin{cases} J_0(\varphi, \boldsymbol{S}_0(\varphi)) & \text{if } \varphi \in \Phi_{ad}^0, \\ +\infty & \text{otherwise.} \end{cases} \tag{12}$$

Hence we see that (10) – (11) is equivalent to

$$\min_{\varphi \in L^1(\Omega)} j_0(\varphi). \tag{13}$$

Remark 3. *The existence of a minimizer for this problem may not be guaranteed in general. There are several examples for the Laplace equation, see for instance [12, 15] and included references, indicating this. But we will obtain as a consequence from our sharp interface considerations in Section 4 and the fact that the porous medium – phase field problem introduced in the previous section always admits a minimizer for each $\varepsilon > 0$, that under suitable assumptions also the sharp interface problem* (13) *has a minimizer.*

4 Sharp interface limit

We will show in this section, that the sharp interface problem (13), which was introduced in the previous section, appears in some sense as limit problem of the phase field problems (9) introduced in Section 2 as the phase field parameter ε tends to zero. We directly state the main result of this section:

Theorem 2. *Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be minimizers of $(j_\varepsilon)_{\varepsilon>}$. Then there exists a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$, which is denoted by the same, and an element $\varphi_0 \in L^1(\Omega)$ such that*

$$\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0. \tag{14}$$

If it holds

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x \in \Omega | \varphi_0(x)=1, \varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon) \tag{15}$$

then we obtain moreover

$$\lim_{\varepsilon \searrow 0} j_\varepsilon(\varphi_\varepsilon) = j_0(\varphi_0) \tag{16}$$

and φ_0 is a minimizer of j_0.

Remark 4. *In particular, Theorem 2 implies that if* (15) *is fulfilled, then the sharp interface problem is well-posed in the sense, that there exists a least one minimizer of* (10)-(11)*. This has not been shown so far and is still an open problem for the general shape optimization problem in fluid dynamics, compare also discussion in the introduction and in Remark 3. And so proving a convergence result without any condition as in* (15) *would imply a much stronger result concerning well-posedness of the shape optimization problem that is not expected. In this sense, the result at hand seems currently optimal.*

Before proving this theorem, we start with a preparatory lemma.

Lemma 3. *Let* $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq L^1(\Omega)$ *with* $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ *be such that for* $\varepsilon \searrow 0$

$$\lim_{\varepsilon\searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0, \quad \|\varphi_\varepsilon - \varphi_0\|_{L^1(\{\varphi_0=1,\varphi_\varepsilon<0\})} = \mathcal{O}(\varepsilon) \tag{17}$$

with $\varphi_0 \in BV(\Omega, \{\pm 1\})$, $\boldsymbol{U}^{\varphi_0} \neq \emptyset$ *and* $|\varphi_\varepsilon| \leq 1$ *pointwise almost everywhere in* Ω. *Then there exists a subsequence of* $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ *(denoted by the same) such that*

$$\lim_{\varepsilon\searrow 0} \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{u}_\varepsilon|^2 \, \mathrm{d}x = 0$$

where $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi_0)$.

Proof. We split the proof into several steps:

- *1st step:* First of all we choose a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ that converges pointwise almost everywhere in Ω to φ_0. Then we take some $\delta > 0$, such that $\varepsilon < \delta$ for ε small enough and notice that due to Assumption **(A4)** it holds $\alpha_\delta \leq \alpha_\varepsilon$ pointwise, and therefore we arrive in the pointwise estimate

$$\alpha_\delta(\varphi_0(x)) = \lim_{\varepsilon\searrow 0} \alpha_\delta(\varphi_\varepsilon(x)) \leq \liminf_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)). \tag{18}$$

  This gives, as $\delta \searrow 0$,

$$\begin{aligned}\alpha_0(\varphi_0(x)) &= \lim_{\delta\searrow 0} \alpha_\delta(\varphi_0(x)) = \lim_{\delta\searrow 0}\left(\lim_{\varepsilon\searrow 0} \alpha_\delta(\varphi_\varepsilon(x))\right) \leq \lim_{\delta\searrow 0}\left(\liminf_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x))\right) = \\ &= \liminf_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x))\end{aligned} \tag{19}$$

  for almost every $x \in \Omega$. On the other hand we deduce from $\alpha_\varepsilon \leq \alpha_0$ pointwise almost everywhere

$$\limsup_{\varepsilon\searrow 0} (\alpha_\varepsilon(\varphi_\varepsilon(x))) \leq \limsup_{\varepsilon\searrow 0} (\alpha_0(\varphi_\varepsilon(x))) = \alpha_0(\varphi_0(x)).$$

  We sum up the estimates to obtain

$$\alpha_0(\varphi_0(x)) \leq \liminf_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)) \leq \limsup_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)) \leq \alpha_0(\varphi_0(x))$$

  which holds for almost every $x \in \Omega$ and implies

$$\lim_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)) = \alpha_0(\varphi_0(x)) \qquad \text{for a.e. } x \in \Omega. \tag{20}$$

  This will be used later.

- *2nd step:* Now we show, that for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{v}|_{\{\varphi_0=-1\}} = \boldsymbol{0}$ it holds

$$\lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon) |\boldsymbol{v}|^2 \, \mathrm{d}x = 0.$$

  To this end, we notice first for almost every $x \in \Omega$ that due to (20),

$$\lim_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)) |\boldsymbol{v}(x)|^2 = 0. \tag{21}$$

To apply Lebesgue's convergence theorem and deduce the convergence in $L^1(\Omega)$ we estimate in several steps. Since α_ε is decreasing we find

$$\alpha_\varepsilon(\varphi_\varepsilon(x))|\boldsymbol{v}(x)|^2 \le \alpha_\varepsilon(0)|\boldsymbol{v}(x)|^2 \le \alpha_0(0)|\boldsymbol{v}(x)|^2$$

for almost every $x \in \{\varphi_\varepsilon \ge 0\}$ where we used $\lim_{\varepsilon\searrow 0}\alpha_\varepsilon(0) = \alpha_0(0) < \infty$, see Assumption **(A4)**. From this bound and the pointwise convergence (21) we obtain thanks to Lebesgue's convergence theorem

$$\lim_{\varepsilon\searrow 0}\int_{\{\varphi_\varepsilon\ge 0\}}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2\,\mathrm{d}x = 0. \tag{22}$$

To consider the part of Ω where φ_ε is non-positive, we deduce from $\boldsymbol{v}|_{\{\varphi_0=-1\}} = \mathbf{0}$ that $\{x\in\Omega \mid \boldsymbol{v}(x)\ne\mathbf{0}\} \subseteq \{x\in\Omega \mid \varphi_0(x)=1\}$ and thus we get for almost every $x\in\{\varphi_\varepsilon<0\}$ the estimate

$$\alpha_\varepsilon(\varphi_\varepsilon(x))|\boldsymbol{v}(x)|^2 \le \overline{\alpha}_\varepsilon\underbrace{|\varphi_\varepsilon(x)-\varphi_0(x)|}_{\ge 1}|\boldsymbol{v}(x)|^2\chi_{\{\varphi_0=1\}}(x). \tag{23}$$

Due to the pointwise estimate $|\varphi_\varepsilon|\le 1, |\varphi_0|\le 1$ we have

$$\overline{\alpha}_\varepsilon\int_\Omega \chi_{\{\varphi_0=1\}\cap\{\varphi_\varepsilon<0\}}|\varphi_\varepsilon-\varphi_0||\boldsymbol{v}|^2\,\mathrm{d}x \le C\overline{\alpha}_\varepsilon\|\varphi_0-\varphi_\varepsilon\|^{\frac{2}{3}}_{L^1(\{\varphi_0=1\}\cap\{\varphi_\varepsilon<0\})}\|\boldsymbol{v}\|^2_{\boldsymbol{L}^6(\Omega)}. \tag{24}$$

We combine

$$\|\varphi_\varepsilon-\varphi_0\|_{L^1(\{\varphi_0=1\}\cap\{\varphi_\varepsilon<0\})} = \mathcal{O}(\varepsilon) \tag{25}$$

and $\overline{\alpha}_\varepsilon = \mathrm{o}\left(\varepsilon^{-2/3}\right)$, see Assumption **(A4)**, to get therefrom

$$\lim_{\varepsilon\searrow 0}\int_\Omega \chi_{\{\varphi_0=1\}\cap\{\varphi_\varepsilon<0\}}\overline{\alpha}_\varepsilon|\varphi_\varepsilon-\varphi_0||\boldsymbol{v}|^2\,\mathrm{d}x = 0. \tag{26}$$

And so, in view of (23)

$$\lim_{\varepsilon\searrow 0}\int_{\{\varphi_\varepsilon<0\}}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2\,\mathrm{d}x = 0$$

which gives combined with (22) finally

$$\lim_{\varepsilon\searrow 0}\int_\Omega\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2\,\mathrm{d}x = 0.$$

We notice that for every $\varepsilon>0$ the velocity field $\boldsymbol{u}_\varepsilon\in\boldsymbol{U}$ is the unique solution of

$$\min_{\boldsymbol{v}\in\boldsymbol{U}} F_\varepsilon(\boldsymbol{v}) := \int_\Omega\left(\frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla\boldsymbol{v}|^2 - \boldsymbol{f}\cdot\boldsymbol{v}\right)\mathrm{d}x$$

since the state equation (7) is the first order optimality condition for this optimization problem, which is necessary and sufficient for the convex optimization problem of minimizing the functional F_ε over $\boldsymbol{U}$.

We proceed by defining

$$F_0(\boldsymbol{v}) := \int_\Omega\left(\frac{1}{2}\alpha_0(\varphi_0)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla\boldsymbol{v}|^2 - \boldsymbol{f}\cdot\boldsymbol{v}\right)\mathrm{d}x$$

and notice, that the unique minimizer of F_0 in $\boldsymbol{U}$ is $\boldsymbol{S}_0(\varphi_0)$, since again the state equations are the necessary and sufficient first order optimality conditions for the convex optimization problem $\min_{\boldsymbol{v}\in\boldsymbol{U}} F_0(\boldsymbol{v})$. We use the functionals $(F_\varepsilon)_{\varepsilon>0}$ to show that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded:

- *3rd step:* From $\boldsymbol{U}^{\varphi_0} \neq \emptyset$ we know that can choose some $\boldsymbol{u}_0 \in \boldsymbol{U}^{\varphi_0} \subset \boldsymbol{U}$ and obtain, because $\boldsymbol{u}_\varepsilon$ are minimizers of F_ε, the estimate

$$\begin{aligned}\int_\Omega \left(\frac{\mu}{2}|\nabla \boldsymbol{u}_\varepsilon|^2 + \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_\varepsilon\right)\mathrm{d}x = F_\varepsilon(\boldsymbol{u}_\varepsilon) \le F_\varepsilon(\boldsymbol{u}_0) = \\ = \int_\Omega \left(\frac{\mu}{2}|\nabla \boldsymbol{u}_0|^2 + \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_0|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_0\right)\mathrm{d}x \le \\ \le \int_\Omega \left(\frac{\mu}{2}|\nabla \boldsymbol{u}_0|^2 - \boldsymbol{f}\cdot\boldsymbol{u}_0\right)\mathrm{d}x + \left(\limsup_{\varepsilon\searrow 0}\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_0|^2\,\mathrm{d}x + c\right)\end{aligned} \tag{27}$$

for some constant $c \ge 0$ and $\varepsilon > 0$ small enough.
To see that $\limsup_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_0|^2\,\mathrm{d}x < \infty$ we can use the second step of this proof. And so from (27), the inequalities of Poincaré and Young and the boundary condition on $\boldsymbol{u}_\varepsilon$ we find a constant $C > 0$ independent of ε such that

$$\|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)} < C.$$

The result of the previous step implies in particular the existence of a subsequence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$, which will be denoted by the same, that converges weakly in $\boldsymbol{H}^1(\Omega)$ to some limit element $\boldsymbol{u}_0 \in \boldsymbol{U}$. To see that $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi_0)$, we next claim that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges in $\boldsymbol{U}$ with respect to the weak $\boldsymbol{H}^1(\Omega)$ topology to F_0 as $\varepsilon \searrow 0$.

- *4th step:* We will see, that the constant sequence defines a recovery sequence for $(F_\varepsilon)_{\varepsilon>0}$. Choosing $\boldsymbol{v} \in \boldsymbol{U}$ we can assume that $F_0(\boldsymbol{v}) < \infty$, otherwise it would hold trivially

$$\limsup_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}) \le F_0(\boldsymbol{v}).$$

Therefore, we can assume $\int_\Omega \alpha_0(\varphi_0)|\boldsymbol{v}|^2\,\mathrm{d}x < \infty$ and so $\boldsymbol{v} \in \boldsymbol{U}^{\varphi_0}$. Due to the second step of this proof this yields

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2\,\mathrm{d}x = 0.$$

As the remaining terms of $(F_\varepsilon)_{\varepsilon>0}$ are independent of ε this already implies

$$\limsup_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}) \le F_0(\boldsymbol{v}).$$

- *5th step:* Let $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0} \subseteq \boldsymbol{U}$ be an arbitrary sequence that converges weakly in $\boldsymbol{H}^1(\Omega)$ to some $\boldsymbol{v} \in \boldsymbol{U}$. Due to the compact imbedding of $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{L}^2(\Omega)$ we certainly have a subsequence of $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0}$, which will be denoted by the same, that converges pointwise almost everywhere in Ω to $\boldsymbol{v}$. From this convergence, the pointwise convergence of $\alpha_\varepsilon(\varphi_\varepsilon)$ that was proven in (20) and Fatou's lemma we see

$$\begin{aligned}\int_\Omega \alpha_0(\varphi_0)|\boldsymbol{v}|^2\,\mathrm{d}x = \int_\Omega\left(\liminf_{\varepsilon\searrow 0}\alpha_\varepsilon(\varphi_\varepsilon)\right)\left(\liminf_{\varepsilon\searrow 0}|\boldsymbol{v}_\varepsilon|^2\right)\mathrm{d}x \le \\ \le \int_\Omega \liminf_{\varepsilon\searrow 0}\left(\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}_\varepsilon|^2\right)\mathrm{d}x \le \liminf_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}_\varepsilon|^2\,\mathrm{d}x\end{aligned} \tag{28}$$

which yields

$$F_0(\boldsymbol{v}) \le \liminf_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}_\varepsilon)$$

since the remaining terms are weakly lower semicontinuous in $\boldsymbol{H}^1(\Omega)$.

This proves that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to F_0 as $\varepsilon \searrow 0$ in $\boldsymbol{U}$ with respect to the weak $\boldsymbol{H}^1(\Omega)$ topology. In view of standard results for Γ-convergence, see for instance [14], we see therefrom that the limit point of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is the unique minimizer of F_0, and thus $\boldsymbol{u}_0$ minimizes F_0 in $\boldsymbol{U}$. We find that the first order optimality conditions for the convex optimization problem $\min_{\boldsymbol{v}\in\boldsymbol{U}} F_0(\boldsymbol{u})$ are exactly given by the state equations (11). Thus, the minimizer $\boldsymbol{u}_0 \in \boldsymbol{U}$ of F_0 fulfills (11) and hence $\boldsymbol{u}_0 = \boldsymbol{S}_0(\varphi_0)$.
Due to the Γ-convergence result we have additionally $\lim_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) = F_0(\boldsymbol{u}_0)$ and so

$$\lim_{\varepsilon\searrow 0}\left[\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 + \frac{\mu}{2}|\nabla \boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x\right] = \underbrace{\int_\Omega \frac{1}{2}\alpha_0(\varphi_0)|\boldsymbol{u}_0|^2}_{=0} + \frac{\mu}{2}|\nabla \boldsymbol{u}_0|^2 \,\mathrm{d}x.$$

This gives us in view of (28) and by using Lemma 4 the convergences

$$\lim_{\varepsilon\searrow 0}\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x = 0, \qquad \lim_{\varepsilon\searrow 0}\int_\Omega \frac{\mu}{2}|\nabla \boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x = \int_\Omega \frac{\mu}{2}|\nabla \boldsymbol{u}_0|^2 \,\mathrm{d}x$$

and finally proves the statement of the lemma.

□

In the proof we used the following lemma that can be verified by direct calculations:

Lemma 4. *Let* $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ *be sequences that are bounded from below with* $\lim_{k\to\infty}(a_k + b_k) = (a+b)$ *where* $a, b \in \mathbb{R}$*, such that* $a \le \liminf_{k\to\infty} a_k$ *and* $b \le \liminf_{k\to\infty} b_k$*. Then it holds* $\lim_{k\to\infty} a_k = a$ *and* $\lim_{k\to\infty} b_k = b$.

Remark 5. *If we are in space dimension* $d = 2$ *we can use that* $\boldsymbol{H}^1(\Omega)$ *is imbedded in* $\boldsymbol{L}^{p'}(\Omega)$ *for any* $1 \le p' < \infty$*. Hence we can replace* (24) *for some* $1 < p < \infty$ *by*

$$\overline{\alpha}_\varepsilon \int_\Omega \chi_{\{\varphi_0=1\}\cap\{\varphi_\varepsilon<0\}} |\varphi_\varepsilon - \varphi_0| |\boldsymbol{v}|^2 \,\mathrm{d}x \le C\overline{\alpha}_\varepsilon \|\varphi_0 - \varphi_\varepsilon\|^{1/p}_{L^1(\{\varphi_0=1\}\cap\{\varphi_\varepsilon<0\})} \left\||\boldsymbol{v}|^2\right\|^{1/p'}_{\boldsymbol{L}^{p'}(\Omega)}$$

where $p' = \frac{p}{p-1}$*. Thus to conclude* (26) *from* (25) *it is sufficient to assume* $\overline{\alpha}_\varepsilon = o\left(\varepsilon^{-1/p}\right)$ *for any* $p \in (1, +\infty)$*. And so the condition* $\overline{\alpha}_\varepsilon = o\left(\varepsilon^{-2/3}\right)$ *claimed in Assumption* ***(A4)*** *can be weakened if* $d = 2$*, see also Remark 2.*

Lemma 3 and the Γ-convergence results of [29], where it is shown that a multiple of the perimeter is the $L^1(\Omega)$-Γ-limit of the Ginzburg-Landau energy, give us all essential tools to prove Theorem 2.

Proof of Theorem 2. We split the proof into several steps:

- *1st step:* Assume we have an arbitrary $\varphi \in L^1(\Omega)$ chosen such that $j_0(\varphi) < \infty$. We will show that there exists a sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega)$ converging to φ in $L^1(\Omega)$ as $\varepsilon \searrow 0$ such that
$$\limsup_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) \le j_0(\varphi).$$
We start by approximating $E^\varphi := \{\varphi = 1\}$ by smooth sets. For this purpose we use the result of [29, Lemma 1], which gives a sequence $(E_k)_{k\in\mathbb{N}}$ of open subsets of Ω such that $\partial E_k \cap \Omega \in C^2$, $|E_k| = |E^\varphi|$ for $k \gg 1$, $\lim_{k\to\infty} P_\Omega(E_k) = P_\Omega(E^\varphi)$ and $\lim_{k\to\infty}\|\varphi_k - \varphi\|_{L^1(\Omega)} = 0$ with the convergence rate
$$\|\varphi_k - \varphi\|_{L^1(\Omega)} = \mathcal{O}\left(k^{-1}\right). \tag{29}$$

Here we denoted $\varphi_k := 2\chi_{E_k} - 1$. The convergence rate (29) is not explicitly stated in [29, Lemma 1] but follows easily from the explicit construction in the proof.
We now construct for every k large enough a recovery sequence $\left(\varphi_\varepsilon^k\right)_{\varepsilon>0} \subset L^1(\Omega)$ converging to φ_k in $L^1(\Omega)$ as $\varepsilon \searrow 0$ such that

$$\limsup_{\varepsilon\searrow 0} \int_\Omega \left(\frac{\gamma\varepsilon}{2} \left|\nabla\varphi_\varepsilon^k\right|^2 + \frac{\gamma}{\varepsilon}\psi\left(\varphi_\varepsilon^k\right)\right) \mathrm{d}x \le \gamma c_0 P_\Omega\left(E_k\right) \tag{30}$$

analog as it is done for example in [40, p. 222 ff], [29, Proposition 2] or [7, Proposition 3.11]. To this end we define for $\varepsilon > 0$ small enough the function $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$g_\varepsilon\left(s\right) := \begin{cases} -1, & s < -\varepsilon\frac{\pi}{\sqrt{2}} \\ \sin\left(\frac{s}{\sqrt{2}\varepsilon}\right) & |s| \le \varepsilon\frac{\pi}{\sqrt{2}} \\ 1 & s > \varepsilon\frac{\pi}{\sqrt{2}} \end{cases}.$$

To fulfill the integral constraint, it may be necessary to shift the profile by a constant $\eta_\varepsilon > 0$. Here we choose $\eta_\varepsilon := \varepsilon\frac{\pi}{\sqrt{2}} = \mathcal{O}\left(\varepsilon\right)$ to ensure $\varphi_\varepsilon^k(x) = -1$ if $\varphi(x) < 0$. Thus we define

$$\varphi_\varepsilon^k(x) := g_\varepsilon\left(d_k(x) - \eta_\varepsilon\right).$$

with d_k being the signed distance function to $\Gamma_k := \partial E_k \cap \partial\left(\Omega \setminus E_k\right)$, which means $d_k(x) = d\left(x, \Gamma_k\right)$ for $x \in E_k$ and $d_k(x) = -d(x, \Gamma_k)$ otherwise. Due to our construction, Γ_k defines a C^2-submanifold and thus the signed distance function d_k to Γ_k is a C^2-function. Then we get pointwise $g_\varepsilon(d_k(x) - \eta_\varepsilon) \le \varphi_k(x)$ and so in particular $\int_\Omega \varphi_\varepsilon^k(x)\,\mathrm{d}x \le \int_\Omega \varphi_k(x)\,\mathrm{d}x = \int_\Omega \varphi\,\mathrm{d}x \le \beta\,|\Omega|$ which means, that the integral constraint is fulfilled for φ_ε^k.
Now we use calculations that can be found in more detail in [29, 40, 7] to obtain

$$\lim_{\varepsilon\searrow 0} \left\|\varphi_\varepsilon^k - \varphi_k\right\|_{L^1(\Omega)} = 0, \quad \left\|\varphi_\varepsilon^k - \varphi_k\right\|_{L^1(\Omega)} = \mathcal{O}\left(\varepsilon\right) \tag{31}$$

and that (30) holds.
Then we choose a diagonal sequence $\left(\varphi_{\varepsilon_k}^k\right)_{k\in\mathbb{N}}$ that converges to φ in $L^1(\Omega)$ and fulfills per construction

$$\limsup_{k\to\infty} \int_\Omega \left(\frac{\gamma\varepsilon_k}{2} \left|\nabla\varphi_{\varepsilon_k}^k\right|^2 + \frac{\gamma}{\varepsilon_k}\psi\left(\varphi_{\varepsilon_k}^k\right)\right) \mathrm{d}x \le \gamma c_0 P_\Omega\left(E^\varphi\right)$$

which follows from (30) and $\lim_{k\to\infty} P_\Omega\left(E_k\right) = P_\Omega\left(E^\varphi\right)$. Besides, we conclude from (29) and (31) the following convergence rate $\left\|\varphi_{\varepsilon_k}^k - \varphi\right\|_{L^1(\Omega)} = \mathcal{O}\left(k^{-1}\right)$. We continue with defining $\boldsymbol{u}_k := \boldsymbol{S}_{\varepsilon_k}\left(\varphi_{\varepsilon_k}^k\right)$ and see that $\boldsymbol{U}^\varphi \ne \emptyset$ since $j_0(\varphi) < \infty$. From Lemma 3 we thus get, after possibly choosing a subsequence, that $\left(\boldsymbol{u}_k\right)_{k\in\mathbb{N}}$ converges strongly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{u} := \boldsymbol{S}_0(\varphi)$ and it holds $\lim_{k\to\infty} \int_\Omega \alpha_{\varepsilon_k}\left(\varphi_{\varepsilon_k}^k\right)\left|\boldsymbol{u}_k\right|^2 \mathrm{d}x = 0$. Using the continuity of the objective functional we end up with

$$\limsup_{k\to\infty} j_{\varepsilon_k}(\varphi_{\varepsilon_k}^k) \le j_0(\varphi).$$

- *2nd step:* Next we will show that for any sequence $\left(\varphi_\varepsilon\right)_{\varepsilon>0} \subseteq L^1(\Omega)$ converging to an arbitrary element $\varphi \in L^1(\Omega)$ such that

$$\left\|\varphi_\varepsilon - \varphi\right\|_{L^1(\{x\in\Omega|\varphi(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}\left(\varepsilon\right) \tag{32}$$

it holds

$$j_0(\varphi) \le \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon).$$

Without loss of generality we assume $\liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) < \infty$ and can therefore assume $\varphi \in BV(\Omega, \{\pm 1\})$ and $\int_\Omega \varphi \le \beta|\Omega|$. Moreover we denote $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$.
From Assumption **(A3)** and $\liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) < \infty$ we know that there exists a subsequence, denoted by the same, such that $\left(\|\boldsymbol{u}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)}\right)_{\varepsilon>0}$ is bounded uniformly in $\varepsilon > 0$. So we obtain for a subsequence, which is still indexed by $\varepsilon > 0$, that $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some element $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. Furthermore, we see that

$$\liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) < \infty \implies \liminf_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x < \infty.$$

At the same time we can assume that (after choosing a subsequence) $(\varphi_\varepsilon)_{\varepsilon>0}$ and $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converge pointwise almost everywhere in Ω, and as a consequence we get similar to (28) with Fatou's Lemma

$$\int_\Omega \alpha_0(\varphi)|\boldsymbol{u}|^2 \,\mathrm{d}x \le \liminf_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x < \infty$$

and thus in particular $\boldsymbol{u} = \boldsymbol{0}$ a.e. in $\{\varphi = -1\}$ where we used $\lim_{\varepsilon\searrow 0} \alpha_\varepsilon(\varphi_\varepsilon(x)) = \alpha_0(\varphi(x))$ a.e. in Ω, which follows as in (19)-(20).
We have $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$, which gives us $\boldsymbol{u}_\varepsilon \in \boldsymbol{U}$, and as a consequence $\boldsymbol{u} \in \boldsymbol{U}$. Altogether this implies $\boldsymbol{u} \in \boldsymbol{U}^\varphi$, and thus $\boldsymbol{U}^\varphi \neq \emptyset$ together with $j_0(\varphi) < \infty$.

According to [29, Proposition 1] we have, after rescaling in ε,

$$\gamma c_0 P_\Omega(\{\varphi = 1\}) \le \liminf_{\varepsilon\searrow 0} \int_\Omega \left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right) \mathrm{d}x.$$

After those preparation, we choose a subsequence $(j_{\varepsilon_k}(\varphi_{\varepsilon_k}))_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} j_{\varepsilon_k}(\varphi_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon).$$

We will now apply Lemma 3 to deduce the convergence of a subsequence of $(\boldsymbol{u}_{\varepsilon_k})_{k\in\mathbb{N}}$ in $\boldsymbol{H}^1(\Omega)$. For this purpose, we use in particular the convergence rate of $(\varphi_{\varepsilon_k})_{k\in\mathbb{N}}$ stated in (32). Thus, we obtain the existence of a subsequence $\left(\boldsymbol{u}_{\varepsilon_{k(l)}}\right)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} \left\|\boldsymbol{u}_{\varepsilon_{k(l)}} - \boldsymbol{u}\right\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{l\to\infty} \int_\Omega \alpha_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right)\left|\boldsymbol{u}_{\varepsilon_{k(l)}}\right|^2 \mathrm{d}x = 0$$

where $\boldsymbol{u} = \boldsymbol{S}_0(\varphi)$.
Plugging these results together we end up with

$$j_0(\varphi) \le \liminf_{l\to\infty} j_{\varepsilon_{k(l)}}\left(\varphi_{\varepsilon_{k(l)}}\right) = \lim_{k\to\infty} j_{\varepsilon_k}(\varphi_{\varepsilon_k}) = \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon)$$

and finish the second step.

- *3rd step:* We use the results of the previous steps to finally prove the statement. First of all we see, that the existence of minimizers $(\varphi_\varepsilon)_{\varepsilon>0} \subset \Phi_{ad}$ of $(j_\varepsilon)_{\varepsilon>0}$ with $j_\varepsilon(\varphi_\varepsilon) < \infty$ follows from Theorem 1.
  Let now $\widetilde{\varphi}_\varepsilon \subseteq L^1(\Omega)$ be the sequence constructed in the first step corresponding to some arbitrary $\widetilde{\varphi} \in \Phi^0_{ad}$. Then, as we have shown, there exists a constant $C > 0$ independent of ε such that
  $$j_\varepsilon(\widetilde{\varphi}_\varepsilon) < C.$$
  Since φ_ε is a minimizer of j_ε for every $\varepsilon > 0$ we deduce
  $$j_\varepsilon(\varphi_\varepsilon) \leq j_\varepsilon(\widetilde{\varphi}_\varepsilon) < C$$
  and so we can conclude
  $$\int_\Omega \left(\frac{\gamma\varepsilon}{2} |\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon) \right) \mathrm{d}x < C. \tag{33}$$
  Using the arguments of [29, Proposition 3, case a)], compare also [40, Proposition 3, Remark (1.35)], we get from this uniform estimate that $(\varphi_\varepsilon)_{\varepsilon>0}$ has a subsequence that converges in $L^1(\Omega)$ to an element $\varphi_0 \in L^1(\Omega)$.

  For the next step we assume that the sequence of minimizers $(\varphi_\varepsilon)_{\varepsilon>0}$ fulfills additionally (15). Then we see by the second step of this proof, that
  $$j_0(\varphi_0) \leq \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon). \tag{34}$$
  Taking another arbitrary admissible $\varphi \in L^1(\Omega)$, $j_0(\varphi) < \infty$, we find again by the first step of this proof, that there exists a sequence $(\widehat{\varphi}_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega)$ converging in $L^1(\Omega)$ to φ as $\varepsilon \searrow 0$ such that
  $$\limsup_{\varepsilon\searrow 0} j_\varepsilon(\widehat{\varphi}_\varepsilon) \leq j_0(\varphi).$$
  And thus, by the minimizing property of φ_ε and (34), we end up with
  $$j_0(\varphi_0) \leq \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) \leq \limsup_{\varepsilon\searrow 0} j_\varepsilon(\widehat{\varphi}_\varepsilon) \leq j_0(\varphi) \tag{35}$$
  which implies
  $$j_0(\varphi_0) \leq j_0(\varphi) \quad \forall \varphi \in L^1(\Omega).$$
  And thus φ_0 minimizes j_0. It remains to prove (16). But for this purpose we choose $\varphi \equiv \varphi_0$ in the previous considerations and obtain then from (35) that
  $$j_0(\varphi_0) \leq \liminf_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) \leq \limsup_{\varepsilon\searrow 0} j_\varepsilon(\widehat{\varphi}_\varepsilon) \leq j_0(\varphi_0) \tag{36}$$
  and thus $\lim_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) = j_0(\varphi_0)$. This finally proves the statement of the theorem.

□

5 Optimality conditions

In this section we will derive first order necessary optimality conditions for both the phase field problem (9) and the sharp interface problem (13) by geometric variations of the optimal shape. To be precise, we vary the fluid regions in direction of certain vector fields and calculate the first variation with respect to those geometric transformations. This means that we apply the ideas of shape sensitivity analysis to a setting where the reference domain are only Caccioppoli sets in general. In Theorem 5 we will then show that we can also derive the optimality system for the sharp interface problem as a limit from the corresponding diffuse interface system.

For this purpose, we have to impose additional differentiability assumptions on the data, which have to be assumed throughout this section:

(A5) Assume that $\alpha_\varepsilon \in C^2([-1,1])$ for all $\varepsilon > 0$ and $\boldsymbol{f} \in \boldsymbol{H}^1(\Omega)$.
Assume that $x \mapsto f(x,v,A) \in \mathbb{R}$ is in $W^{1,1}(\Omega)$ for all $(v,A) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$ and the partial derivatives $\mathrm{D}_2 f(x,\cdot,A)$, $\mathrm{D}_3 f(x,v,\cdot)$ exist for all $v \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ and a.e. $x \in \Omega$. Let $p \geq 2$ for $d = 2$ and $2 \leq p \leq {}^{2d}/_{d-2}$ for $d = 3$ and assume that there are $\hat{a} \in L^1(\Omega)$, $\hat{b}_1, \hat{b}_2 \in L^\infty(\Omega)$ such that for almost every $x \in \Omega$ it holds

$$\mathrm{D}_{(2,3)} f(x,v,A) \leq \hat{a}(x) + \hat{b}_1(x)|v|^{p-1} + \hat{b}_2(x)|A| \quad \forall v \in \mathbb{R}^d, A \in \mathbb{R}^{d\times d}. \tag{37}$$

Remark 6. *If the objective functional fulfills Assumption **(A7)**, we find that*

$$F : \boldsymbol{H}^1(\Omega) \ni \boldsymbol{u} \mapsto \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})\,\mathrm{d}x$$

is continuously Fréchet differentiable and that its directional derivative is given in the following form:

$$\mathrm{D}F(\boldsymbol{u})(\boldsymbol{v}) = \int_\Omega \mathrm{D}_{(2,3)} f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u})(\boldsymbol{v},\mathrm{D}\boldsymbol{v})\,\mathrm{d}x \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

For details concerning Nemytskii operators we refer to [36].

As we will derive first order optimality conditions by varying the domain Ω with transformations, we introduce here the admissible transformations and its corresponding velocity fields:

Definition 1 ($\mathcal{V}_{ad}$, $\mathcal{T}_{ad}$)**.** The space $\mathcal{V}_{ad}$ of admissible velocity fields is defined as the set of all $V \in C\left([-\tau,\tau];C\left(\overline{\Omega},\mathbb{R}^d\right)\right)$, where $\tau > 0$ is some fixed, small constant, such that it holds:

- **(V1)** **(V1a)** $V(t,\cdot) \in C^2\left(\overline{\Omega},\mathbb{R}^d\right)$,
  (V1b) $\exists C > 0$: $\|V(\cdot,y) - V(\cdot,x)\|_{C([-\tau,\tau],\mathbb{R}^d)} \leq C|x-y|\ \forall x,y \in \overline{\Omega}$,
- **(V2)** $V(t,x)\cdot\boldsymbol{n}(x) = 0 \quad$ on $\partial\Omega$,
- **(V3)** $V(t,x) = \boldsymbol{0}$ for a.e. $x \in \partial\Omega$ with $\boldsymbol{g}(x) \neq \boldsymbol{0}$.

We will often use the notation $V(t) = V(t,\cdot)$.
Then the space $\mathcal{T}_{ad}$ of admissible transformations for the domain is defined as solutions of the ordinary differential equation

$$\partial_t T_t(x) = V(t,T_t(x)), \qquad T_0(x) = x \tag{38a}$$

for $V \in \mathcal{V}_{ad}$, which gives some $T : (-\tilde{\tau},\tilde{\tau}) \times \overline{\Omega} \to \overline{\Omega}$, with $0 < \tilde{\tau}$ small enough.

Remark 7. *Let $V \in \mathcal{V}_{ad}$ and $T \in \mathcal{V}_{ad}$ be the transformation associated to V by* (38)*. Then T admits the following properties:*

- $T(\cdot,x) \in C^1\left([-\tilde{\tau},\tilde{\tau}],\mathbb{R}^d\right)$ *for all* $x \in \overline{\Omega}$,
- $\exists c > 0, \forall x,y \in \overline{\Omega},\ \|T(\cdot,x) - T(\cdot,y)\|_{C^1([-\tilde{\tau},\tilde{\tau}],\mathbb{R}^d)} \leq c|x-y|$,
- $\forall t \in [-\tilde{\tau},\tilde{\tau}],\ x \mapsto T_t(x) = T(t,x) : \overline{\Omega} \to \overline{\Omega}$ *is bijective,*
- $\forall x \in \overline{\Omega},\ T^{-1}(\cdot,x) \in C\left([-\tilde{\tau},\tilde{\tau}],\mathbb{R}^d\right)$,
- $\exists c > 0, \forall x,y \in \overline{\Omega},\ \left\|T^{-1}(\cdot,x) - T^{-1}(\cdot,y)\right\|_{C([-\tilde{\tau},\tilde{\tau}],\mathbb{R}^d)} \leq c|x-y|$.

This is shown in [16, 17].

We start with stating optimality conditions for the phase field problem (9):

Theorem 3. *For any minimizer $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon) \in \Phi_{ad} \times \boldsymbol{U}$ of* (6) – (7) *there exists a Lagrange multiplier $\lambda_\varepsilon \geq 0$ for the integral constraint such that the following necessary optimality system is fulfilled:*

$$\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0)\,\mathrm{d}x, \quad \lambda_\varepsilon\left(\int_\Omega \varphi_\varepsilon\,\mathrm{d}x - \beta|\Omega|\right) = 0 \tag{39}$$

for all $T \in \mathcal{T}_{ad}$ with velocity $V \in \mathcal{V}_{ad}$. The derivative is given by the following formula:

$$\begin{aligned}\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) &= \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\left(\boldsymbol{u}_\varepsilon \cdot \dot{\boldsymbol{u}}_\varepsilon[V] + \frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 \operatorname{div} V(0)\right)\mathrm{d}x + \\ &+ \int_\Omega [\mathrm{D}f(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon)(V(0),\dot{\boldsymbol{u}}_\varepsilon[V],\mathrm{D}\dot{\boldsymbol{u}}_\varepsilon[V] - \mathrm{D}\boldsymbol{u}_\varepsilon\mathrm{D}V(0)) + \\ &+ f(x,\boldsymbol{u}_\varepsilon,\mathrm{D}\boldsymbol{u}_\varepsilon)\operatorname{div} V(0)]\,\mathrm{d}x + \\ &+ \int_\Omega \left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi_\varepsilon)\right)\operatorname{div} V(0) - \gamma\varepsilon\nabla\varphi_\varepsilon \cdot \nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x\end{aligned} \tag{40}$$

where $\dot{\boldsymbol{u}}_\varepsilon[V] := \partial_t|_{t=0}\left(\boldsymbol{S}_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) \circ T_t\right) \in \boldsymbol{H}_0^1(\Omega)$ is given as the unique solution of

$$\begin{aligned}&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\dot{\boldsymbol{u}}_\varepsilon[V] \cdot \boldsymbol{z} + \mu\nabla\dot{\boldsymbol{u}}_\varepsilon[V] \cdot \nabla\boldsymbol{z}\,\mathrm{d}x = \\ &= \int_\Omega \mu\mathrm{D}V(0)^T\nabla\boldsymbol{u}_\varepsilon : \nabla\boldsymbol{z}\,\mathrm{d}x + \int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon : \mathrm{D}V(0)^T\nabla\boldsymbol{z}\,\mathrm{d}x + \\ &+ \int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon : \nabla\left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z}\right)\mathrm{d}x - \\ &- \int_\Omega \mu\nabla\boldsymbol{u}_\varepsilon : \nabla\boldsymbol{z} \operatorname{div} V(0)\,\mathrm{d}x + \int_\Omega (\nabla\boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z}\,\mathrm{d}x + \int_\Omega \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x - \\ &- \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x\end{aligned} \tag{41}$$

which has to hold for every $\boldsymbol{z} \in \boldsymbol{V}$, together with

$$\operatorname{div}\dot{\boldsymbol{u}}_\varepsilon[V] = \nabla\boldsymbol{u}_\varepsilon : \mathrm{D}V(0). \tag{42}$$

Proof. We start with proving that $\mathbb{R} \supseteq I \ni t \mapsto \boldsymbol{u}_\varepsilon(t) \circ T_t \in \boldsymbol{H}^1(\Omega)$ is differentiable at $t = 0$ if I is a suitably small interval around 0 and $\boldsymbol{u}_\varepsilon(t) := \boldsymbol{S}_\varepsilon(\varphi_\varepsilon \circ T_t^{-1})$. We also obtain that $\dot{\boldsymbol{u}}_\varepsilon[V] := \partial_t|_{t=0}(\boldsymbol{u}_\varepsilon(t) \circ T_t)$ solves the equation stated in the assumption. To this end, we apply the implicit function theorem and start by defining the function

$$F = (F_1, F_2) : I \times \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{g}\} \to \boldsymbol{V}' \times L_0^2(\Omega)$$

by

$$\begin{aligned} F_1\left(t,\boldsymbol{u}\right)\left(\boldsymbol{z}\right) := &\int_\Omega \alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}\cdot\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det\mathrm{D}T_t\,\mathrm{d}x+ \\ &+\int_\Omega \mu \mathrm{D}T_t^{-T}\nabla\boldsymbol{u} : \mathrm{D}T_t^{-T}\nabla\left(\det\mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det\mathrm{D}T_t\,\mathrm{d}x- \\ &-\int_\Omega \boldsymbol{f}\circ T_t\cdot\left(\det\mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\det\mathrm{D}T_t\,\mathrm{d}x \end{aligned} \tag{43}$$

and

$$F_2(t,\boldsymbol{u}) = \left(\mathrm{D}T_t^{-1} : \nabla\boldsymbol{u}\right)\det\mathrm{D}T_t.$$

The function F_2 is motivated by the identity $\left(\mathrm{D}T_t^{-1} : \nabla\boldsymbol{v}\right)\circ T_t^{-1} = \operatorname{div}\left(\boldsymbol{v}\circ T_t^{-1}\right)$. This function is well-defined, since for any $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{v}|_{\partial\Omega} = \boldsymbol{g}$ we have due to Gauß' theorem

$$\begin{aligned} \int_\Omega \left(\mathrm{D}T_t^{-1} : \nabla\boldsymbol{v}\right)\det\mathrm{D}T_t\,\mathrm{d}x &= \int_\Omega \operatorname{div}\left(\boldsymbol{v}\circ T_t^{-1}\right)\circ T_t\det\mathrm{D}T_t\,\mathrm{d}x = \int_\Omega \operatorname{div}\left(\boldsymbol{v}\circ T_t^{-1}\right)\mathrm{d}x = \\ &= \int_{\partial\Omega}\boldsymbol{v}\circ T_t^{-1}\cdot\boldsymbol{n}\,\mathrm{d}x = \int_{\partial\Omega}\boldsymbol{g}\circ T_t^{-1}\cdot\boldsymbol{n}\,\mathrm{d}x = 0 \end{aligned}$$

where we used, that $T_t(x) = x$ if $\boldsymbol{g}(x)\neq\boldsymbol{0}$ and $\int_{\partial\Omega}\boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{d}x = 0$, see Assumption **(A2)**. Fixing some $\boldsymbol{G}\in\boldsymbol{H}^1(\Omega)$ with $\boldsymbol{G}|_{\partial\Omega} = \boldsymbol{g}$ we define

$$(G_1,G_2) = G : I\times\boldsymbol{H}_0^1(\Omega)\to\boldsymbol{V}'\times L_0^2(\Omega),\quad G(t,\boldsymbol{v}) := F\left(t,\boldsymbol{v}+\boldsymbol{G}\right).$$

Direct calculations then show that

$$G(t,\boldsymbol{u}_\varepsilon(t)\circ T_t-\boldsymbol{G}) = F(t,\boldsymbol{u}_\varepsilon(t)) = 0,\quad \forall t\in I.$$

Using additionally

$$\mathrm{D}_uG_1\left(0,\boldsymbol{u}_\varepsilon-\boldsymbol{G}\right)\left(\boldsymbol{v}\right)\left(\boldsymbol{z}\right) = \int_\Omega\alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{v}\cdot\boldsymbol{z}\,\mathrm{d}x+\int_\Omega\mu\nabla\boldsymbol{v}\cdot\nabla\boldsymbol{z}\,\mathrm{d}x,\quad \mathrm{D}_uG_2\left(0,\boldsymbol{u}_\varepsilon-\boldsymbol{G}\right)\boldsymbol{v} = \operatorname{div}\boldsymbol{v}$$

for all $\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega),\boldsymbol{z}\in\boldsymbol{V}$, we find by Lax-Milgram's theorem and [38, Lemma II.2.1.1] that $\mathrm{D}_uG(0,\boldsymbol{u}_\varepsilon+\boldsymbol{G}) : \boldsymbol{H}_0^1(\Omega)\to\boldsymbol{V}'\times L_0^2(\Omega)$ is an isomorphism. Hence, we can apply the implicit function theorem to obtain differentiability of $t\mapsto(\boldsymbol{u}_\varepsilon(t)\circ T_t-\boldsymbol{G})\in\boldsymbol{H}^1(\Omega)$ at $t=0$, and thus of $t\mapsto\boldsymbol{u}_\varepsilon(t)\circ T_t$ at $t=0$, together with

$$\begin{aligned} \partial_t|_{t=0}\left(\boldsymbol{u}_\varepsilon(t)\circ T_t\right) &= \partial_t|_{t=0}\left(\boldsymbol{u}_\varepsilon(t)\circ T_t-\boldsymbol{G}\right) = -\mathrm{D}_uG\left(0,\boldsymbol{u}_\varepsilon-\boldsymbol{G}\right)^{-1}\partial_tG\left(0,\boldsymbol{u}_\varepsilon-\boldsymbol{G}\right) = \\ &= -\mathrm{D}_uG\left(0,\boldsymbol{u}_\varepsilon-\boldsymbol{G}\right)^{-1}\partial_tF\left(0,\boldsymbol{u}_\varepsilon\right). \end{aligned} \tag{44}$$

This means, that $\dot{\boldsymbol{u}}_\varepsilon[V]\in\boldsymbol{H}_0^1(\Omega)$ is the unique solution of (41) – (42).

Hence we can derive the differentiability of $t\mapsto j_\varepsilon(\varphi_\varepsilon\circ T_t^{-1})$ at $t=0$ together with (40) by using standard calculation rules that can be found in books introducing in the field of shape sensitivity analysis, compare for instance [16, 39].

It remains to show the existence of a Lagrange multiplier for the integral constraint such that (39) is fulfilled. Therefore, we distinguish between two cases.
First we assume that $\int_\Omega\varphi_\varepsilon\,\mathrm{d}x<\beta|\Omega|$. Then we find for t small enough and any transformation $T\in\mathcal{T}_{ad}$ that $\int_\Omega\varphi_\varepsilon\circ T_t^{-1}\,\mathrm{d}x<\beta|\Omega|$, and so $\varphi_\varepsilon\circ T_t^{-1}\in\Phi_{ad}$. Thus,

$$j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)\geq j_\varepsilon\left(\varphi_\varepsilon\right)\quad\forall|t|\ll 1$$

and so

$$\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = 0.$$

Hence, (39) is fulfilled for $\lambda_\varepsilon = 0$. Therefore, so we can assume for the following considerations that $\int_\Omega \varphi_\varepsilon \, \mathrm{d}x = \beta\,|\Omega|$.
We follow now a similar idea as in [13, Proof of Proposition 1.17]. Since $\int_\Omega \varphi_\varepsilon \, \mathrm{d}x = \beta\,|\Omega|$, we may find some $W \in \mathcal{V}_{ad}$ with associated transformation $S \in \mathcal{T}_{ad}$ such that

$$-\int_\Omega \varphi_\varepsilon \operatorname{div} W(0) \, \mathrm{d}x = 1.$$

We define $g := [-t_0, t_0] \times [-s_0, s_0] \to \mathbb{R}$ by

$$g(t,s) := -\int_\Omega \varphi_\varepsilon \circ T_t^{-1} \circ S_s^{-1} \, \mathrm{d}x + \beta\,|\Omega|$$

for $t_0, s_0 > 0$ small enough. We want to use the implicit function theorem to find a function $t \mapsto s(t)$ such that $g(t, s(t)) = 0$. To this end, we notice that by assumption it holds $g(0,0) = 0$ and besides

$$\partial_s|_{s=0} g(0,s) = -\partial_s|_{s=0} \int_\Omega \varphi_\varepsilon \det \mathrm{D}S_s \, \mathrm{d}x = -\int_\Omega \varphi_\varepsilon \operatorname{div} W(0) \, \mathrm{d}x = 1 \neq 0. \tag{45}$$

Moreover, since $V, W \in \mathcal{V}_{ad}$ and thus $V(t), W(s) \in C^2\left(\overline{\Omega}, \mathbb{R}^d\right)$ for all $|t| \ll 1$ and $|s| \ll 1$, we see directly that g is continuously differentiable. And so the implicit function theorem yields the existence of some $\tau_0 > 0$ and a continuously differentiable function $s : [-\tau_0, \tau_0] \to \mathbb{R}$ such that

$$g(t, s(t)) = 0, \quad \forall t \in (-\tau_0, \tau_0)\,, \qquad s'(0) = -\partial_s g(0,0)^{-1} \partial_t g(0,0).$$

The last identity can in view of (45) be rewritten as

$$s'(0) = -\partial_t g(0,0). \tag{46}$$

In particular, we obtain that $\varphi_\varepsilon \circ T_t^{-1} \circ S_{s(t)}^{-1} \in \Phi_{ad}$ for all $t \in (-\tau_0, \tau_0)$ and so

$$j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1} \circ S_{s(t)}^{-1}\right) \geq j_\varepsilon\left(\varphi_\varepsilon\right)$$

holds for all t small enough. From this, we see

$$0 = \partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1} \circ S_{s(t)}^{-1}\right) = \partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ \left(S_{s(t)} \circ T_t\right)^{-1}\right). \tag{47}$$

Introducing the notation $\widetilde{T}_t := S_{s(t)} \circ T_t$, we find from $S, T \in \mathcal{T}_{ad}$ that $\widetilde{T} \in \mathcal{T}_{ad}$ with $\partial_t|_{t=0} \widetilde{T}_t = W(0)s'(0) + V(0)$. Now we notice, that by (40) and (41)-(42) the expression $\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right)$ only depends on $\partial_t|_{t=0} T_t$ and that $C^1(\Omega) \ni \partial_t|_{t=0} T_t \mapsto \partial_t|_{t=0} j_\varepsilon(\varphi_\varepsilon \circ T_t^{-1})$ is linear. Thus, (47) reads as

$$\partial_s|_{s=0} j_\varepsilon\left(\varphi_\varepsilon \circ S_s^{-1}\right) s'(0) + \partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = 0.$$

Defining

$$\lambda_\varepsilon := \partial_s|_{s=0} j_\varepsilon\left(\varphi_\varepsilon \circ S_s^{-1}\right) \in \mathbb{R} \tag{48}$$

we thus have

$$\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = -\lambda_\varepsilon s'(0) = \lambda_\varepsilon g'(0) = -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0) \, \mathrm{d}x$$

where we made use of (46). This shows, that (39) is fulfilled for λ_ε, if λ_ε is defined by (48). As $\int_\Omega \varphi_\varepsilon \,\mathrm{d}x = \beta\,|\Omega|$, the complementarity condition of (39) holds trivially. And so it remains to show that $\lambda_\varepsilon \geq 0$. To this end, we recall that $\int_\Omega \varphi_\varepsilon = \beta\,|\Omega|$ and by the particular choice of $W \in \mathcal{V}_{ad}$ we have

$$\partial_s|_{s=0}\left(\int_\Omega \varphi_\varepsilon \circ S_s^{-1}\,\mathrm{d}x\right) = \int_\Omega \varphi_\varepsilon \operatorname{div} W(0) = -1 < 0.$$

Thus, any $s > 0$ small enough fulfills $\int_\Omega \varphi_\varepsilon \circ S_s^{-1}\,\mathrm{d}x \leq \beta\,|\Omega|$, which yields that $\varphi_\varepsilon \circ S_s^{-1} \in \Phi_{ad}$. Hence,

$$j_\varepsilon\left(\varphi_\varepsilon \circ S_s^{-1}\right) \geq j_\varepsilon\left(\varphi_\varepsilon\right) \quad \forall 0 < s \ll 1$$

and thus we obtain

$$\lambda_\varepsilon = \partial_s|_{s=0} j_\varepsilon\left(\varphi_\varepsilon \circ S_s^{-1}\right) \geq 0.$$

So we have shown, that $\lambda_\varepsilon \geq 0$ is a Lagrange multiplier for the integral constraint.
We finally remark that $\lambda_\varepsilon \geq 0$ does not depend on the choice of the transformation $T \in \mathcal{T}_{ad}$ or on its velocity field $V \in \mathcal{V}_{ad}$. This can be seen in the definition of λ_ε, see (48), since the transformation $S \in \mathcal{T}_{ad}$ is chosen independently of T and V.

For some more detailed calculations we refer to [27, Section 7.2]. □

Remark 8. *We want to remark, that one can also consider the phase field problem* (6) – (7) *as an optimal control problem and then derive a variational inequality by parametric variations as in standard optimal control problems, see [43]. This optimality condition is then given by*

$$\mathrm{D}j_\varepsilon\left(\varphi_\varepsilon\right)\left(\varphi - \varphi_\varepsilon\right) + \lambda_\varepsilon \int_\Omega \left(\varphi - \varphi_\varepsilon\right)\mathrm{d}x \geq 0 \quad \forall \varphi \in H^1(\Omega), |\varphi| \leq 1 \text{ a.e. in } \Omega. \tag{49}$$

This criteria can also be rewritten in a more convenient adjoint formulation.
Assuming more regularity on Ω, the boundary data $\boldsymbol{g}$ and the objective functional one can then show, that the optimality conditions derived in Theorem 3 are necessary for the variational inequality. To be precise, if the variational inequality is fulfilled, also (39) *is fulfilled. Roughly speaking, one can insert $\varphi \equiv \varphi_\varepsilon \circ T_{-t}$ into* (49)*, divide by t, and use some rearrangements. For details, we refer to [27, Section 7].*

In the next theorem, we want to state optimality conditions for the sharp interface problem that can be obtained by geometric variations. We point out, that in contrast to existing works [3, 10, 30, 37] no constraints on the reference domain, thus the minimizer, are necessary despite it being only measurable.

As a preparation we prove the following lemmas:

Lemma 5. *Assume $T \in \mathcal{T}_{ad}$. Let $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$ and define*

$$\boldsymbol{v}_t := \left(\det \mathrm{D}T_t^{-1}\right)\left(\mathrm{D}T_t\right)\boldsymbol{v} \circ T_t^{-1}.$$

Then it holds $\operatorname{div} \boldsymbol{v}_t = \left(\det \mathrm{D}T_t^{-1}\right)\left(\operatorname{div} \boldsymbol{v}\right) \circ T_t^{-1} = 0.$

Lemma 6. *Let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $\operatorname{div} \boldsymbol{u} = 0$. Then*

$$\operatorname{div}\left(\operatorname{div} V(0)\boldsymbol{u} + \mathrm{D}\boldsymbol{u}V(0) - \mathrm{D}V(0)\boldsymbol{u}\right) = 0 \qquad \forall V \in \overline{\mathcal{V}}_{ad},$$

where this identity has to be understood in the distributional sense.

Both lemmas can be shown by direct calculations, see [27, Lemma 3.6, Lemma 3.7].

Now we can state necessary optimality conditions for the sharp interface optimization problem (10) – (11):

Theorem 4. *For every minimizer* $(\varphi_0, \boldsymbol{u}_0) \in \Phi^0_{ad} \times \boldsymbol{U}^{\varphi_0}$ *of* (10)–(11) *there exists a Lagrange multiplier* $\lambda_0 \geq 0$ *for the integral constraint such that the following necessary optimality system is fulfilled:*

$$\partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) = -\lambda_0 \int_\Omega \varphi_0 \operatorname{div} V(0)\,\mathrm{d}x, \quad \lambda_0\left(\int_\Omega \varphi_0\,\mathrm{d}x - \beta|\Omega|\right) = 0 \tag{50}$$

for all $T \in \mathcal{T}_{ad}$ *with velocity* $V \in \mathcal{V}_{ad}$*. The derivative is given by the following formula:*

$$\begin{aligned}\partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) = &\int_\Omega [\mathrm{D}f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right)\left(V(0), \dot{\boldsymbol{u}}_0[V], \mathrm{D}\dot{\boldsymbol{u}}_0[V] - \mathrm{D}\boldsymbol{u}_0 \mathrm{D}V(0)\right) + \\ &+ f\left(x, \boldsymbol{u}_0, \mathrm{D}\boldsymbol{u}_0\right) \operatorname{div} V(0)]\,\mathrm{d}x + \\ &+ \gamma c_0 \int_\Omega \left(\operatorname{div} V(0) - \nu \cdot \nabla V(0)\nu\right)\,\mathrm{d}|\mathrm{D}\chi_{E_0}|\end{aligned} \tag{51}$$

with $\nu = \frac{\mathrm{D}\chi_{E_0}}{|\mathrm{D}\chi_{E_0}|}$ *being the generalised unit normal on the Caccioppoli set* $E_0 := \{\varphi_0 = 1\}$*, compare [2].*
Moreover $\dot{\boldsymbol{u}}_0[V] := \partial_t|_{t=0}\left(\boldsymbol{S}_0\left(\varphi_0 \circ T_t^{-1}\right) \circ T_t\right) \in \boldsymbol{H}^1_0(\Omega)$ *with* $\dot{\boldsymbol{u}}_0[V] = \boldsymbol{0}$ *a.e. in* $\Omega \setminus E_0$ *is given as the unique solution of*

$$\begin{aligned}\int_{E_0} \mu \nabla \dot{\boldsymbol{u}}_0[V] : \nabla \boldsymbol{z}\,\mathrm{d}x = &\int_{E_0} \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z}\,\mathrm{d}x + \int_{E_0} \mu \nabla \boldsymbol{u}_0 : \mathrm{D}V(0)^T \nabla \boldsymbol{z}\,\mathrm{d}x + \\ &+ \int_{E_0} \mu \nabla \boldsymbol{u}_0 : \nabla\left(\operatorname{div} V(0)\,\boldsymbol{z} - \mathrm{D}V(0)\,\boldsymbol{z}\right)\,\mathrm{d}x - \int_{E_0} \mu \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z} \operatorname{div} V(0)\,\mathrm{d}x + \\ &+ \int_{E_0} \left(\nabla \boldsymbol{f} \cdot V(0)\right) \cdot \boldsymbol{z}\,\mathrm{d}x + \int_{E_0} \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x\end{aligned} \tag{52}$$

which has to hold for all $\boldsymbol{z} \in \boldsymbol{V}^{E_0}$*, together with*

$$\operatorname{div} \dot{\boldsymbol{u}}_0[V] = \nabla \boldsymbol{u}_0 : \mathrm{D}V(0). \tag{53}$$

Proof. Let us first notice that $\boldsymbol{U}^{\varphi_0(t)} \neq \emptyset$, where $\varphi_0(t) := \varphi_0 \circ T_t^{-1}$, and hence $\boldsymbol{u}_0(t) := \boldsymbol{S}_0(\varphi_0(t))$ is due to Lemma 2 well-defined. Indeed $\left(\det \mathrm{D}T_t^{-1}\right)\left(\mathrm{D}T_t\right)\boldsymbol{u}_0 \circ T_t^{-1}$ is due to Lemma 5 and Definition 1 an element in $\boldsymbol{U}^{\varphi_0(t)}$ since $\boldsymbol{u}_0 \in \boldsymbol{U}^{\varphi_0}$.
Our proof starts with considering the mapping $\mathbb{R} \supseteq I \ni t \mapsto \boldsymbol{u}_0(t) \circ T_t \in \boldsymbol{H}^1(\Omega)$, where I is assumed to by a suitably small interval around 0. The procedure to show differentiability of this mapping at $t = 0$ is to apply some implicit function argument. But the mapping $\left\{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \mid \boldsymbol{v}|_{\Omega \setminus E_0} = \boldsymbol{0}\right\} \ni \boldsymbol{v} \mapsto \operatorname{div} \boldsymbol{v}$ is not surjective onto $L^2_0(\Omega)$ or $L^2_0(E_0)$, as we don't have enough regularity of $E_0 := \{\varphi_0 = 1\}$ (see the counterexample in [22]). Instead, we apply [37, Theorem 6], which is a result for differentiating implicit equation solutions in a linear setting. For this purpose, we define $F : I \times \boldsymbol{V}^{\varphi_0} \to (\boldsymbol{V}^{\varphi_0})' \times L^2(\Omega)$ such that $F(t, \cdot)$ give the weak from of the state equations on $\{\varphi_0(t) = 1\}$ pulled back onto $\{\varphi_0 = 1\} = E_0$ and transformed to a homogeneous problem where. Some additional terms have to be added because we will insert the divergence free pullback $\left(\det \mathrm{D}T_t\right)\left(\mathrm{D}T_t^{-1}\right)\boldsymbol{u}_0(t) \circ T_t$ of $\boldsymbol{u}_0(t)$ onto $\{\varphi_0(t) = 1\}$. To be precise, we define

$$\begin{aligned}F(t,\boldsymbol{v})\boldsymbol{z} := & \int_{E_0} \mu\nabla\boldsymbol{u} : \mathrm{D}T_t^{-T}\nabla\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\mathrm{d}x- \\ & - \int_{E_0}\mu\nabla\left(\det \mathrm{D}T_t \mathrm{D}T_t^{-1}\right)\cdot\boldsymbol{u} : \mathrm{D}T_t^{-T}\nabla\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\cdot\det \mathrm{D}T_t\,\mathrm{d}x = \\ & = \int_{E_0}\mu\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\cdot \mathrm{D}T_t^{-1}\cdot\nabla\boldsymbol{u} : \mathrm{D}T_t^{-T}\nabla\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\cdot\det \mathrm{D}T_t\,\mathrm{d}x- \\ & - \int_{E_0}\mu\nabla\left(\det \mathrm{D}T_t \mathrm{D}T_t^{-1}\right)\cdot\boldsymbol{u} : \mathrm{D}T_t^{-T}\nabla\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\cdot\det \mathrm{D}T_t\,\mathrm{d}x\,\mathrm{d}x \quad \forall\boldsymbol{z}\in\boldsymbol{V}^{\varphi_0}.\end{aligned}$$

Then we observe with Lemma 5 that due to $\boldsymbol{u}_0(t)\in\boldsymbol{U}^{\varphi_0(t)}$ and $T\in\mathcal{T}_{ad}$ it follows $(\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right)\boldsymbol{u}_0(t)\circ T_t\in\boldsymbol{U}^{\varphi_0}$. Moreover, for $\boldsymbol{z}\in\boldsymbol{V}^{\varphi_0}$ arbitrary we get $\boldsymbol{z}_t := \left(\det \mathrm{D}T_t^{-1}\right)(\mathrm{D}T_t)\,\boldsymbol{z}\circ T_t^{-1}\in\boldsymbol{V}^{\varphi_0(t)}$ and thus we find

$$\begin{aligned}&\int_{E_0}\mu\left(\nabla\boldsymbol{u}_0(t)\right)(T_t)\cdot\left(\nabla\boldsymbol{z}_t\right)(T_t)\cdot\det \mathrm{D}T_t\,\mathrm{d}x - \int_{E_0}\boldsymbol{f}\circ T_t\cdot\boldsymbol{z}_t\circ T_t\cdot\det \mathrm{D}T_t\,\mathrm{d}x = \\ &= \int_{T_t(E_0)}\mu\nabla\boldsymbol{u}_0(t)\cdot\nabla\boldsymbol{z}_t\,\mathrm{d}x - \int_{T_t(E_0)}\boldsymbol{f}\cdot\boldsymbol{z}_t\,\mathrm{d}x = 0.\end{aligned}$$

Next we choose some $\boldsymbol{G}\in\boldsymbol{U}^{\varphi_0}$. Then we see by direct calculation that it holds

$$\begin{aligned}&F\left(t,(\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right)\boldsymbol{u}_0(t)\circ T_t-\boldsymbol{G}\right)\boldsymbol{z} = \int_{E_0}\boldsymbol{f}\circ T_t\cdot\boldsymbol{z}_t\circ T_t\cdot\det \mathrm{D}T_t\,\mathrm{d}x - F\left(t,\boldsymbol{G}\right)\boldsymbol{z} = \\ &= \int_{E_0}\boldsymbol{f}\circ T_t\cdot\left(\det \mathrm{D}T_t^{-1}\mathrm{D}T_t\boldsymbol{z}\right)\cdot\det \mathrm{D}T_t\,\mathrm{d}x - F\left(t,\boldsymbol{G}\right)\boldsymbol{z} =: \widetilde{F}(t)\boldsymbol{z}\end{aligned}$$

which defines

$$\widetilde{F}(t)\in\left(\boldsymbol{V}^{\varphi_0}\right)'.$$

Summarizing, we have

$$F\left(t,\cdot\right)\in\mathcal{L}\left(\boldsymbol{V}^{\varphi_0},\left(\boldsymbol{V}^{\varphi_0}\right)'\right)\quad\forall t\in I$$

and

$$F\left(t,(\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right)\boldsymbol{u}_0(t)\circ T_t-\boldsymbol{G}\right) = \widetilde{F}\left(t\right)\quad\forall t\in I.$$

Due to the differentiability assumptions on the transformation $T\in\mathcal{T}_{ad}$ we observe that

$$I\ni t\mapsto F(t,\cdot)\in\mathcal{L}\left(\boldsymbol{V}^{\varphi_0},\left(\boldsymbol{V}^{\varphi_0}\right)'\right)$$

as well as $I\ni t\mapsto\widetilde{F}(t)\in\left(\boldsymbol{V}^{\varphi_0}\right)'$ are differentiable at $t=0$. We see that it holds for all $\boldsymbol{v},\boldsymbol{z}\in\boldsymbol{V}^{\varphi_0}$

$$F(0,\boldsymbol{v})\boldsymbol{z} = \int_{E_0}\mu\nabla\boldsymbol{v}\cdot\nabla\boldsymbol{z}\,\mathrm{d}x. \tag{54}$$

Thus for fixed $\boldsymbol{v}\in\boldsymbol{V}^{\varphi_0}$ we can estimate, using Poincaré's inequality,

$$\left\|F\left(0,\boldsymbol{v}\right)\right\|_{\left(\boldsymbol{V}^{\varphi_0}\right)'} = \sup_{\boldsymbol{z}\in\boldsymbol{V}^{\varphi_0}\setminus\{\boldsymbol{0}\}}\frac{\left|F\left(0,\boldsymbol{v}\right)\boldsymbol{z}\right|}{\left\|\boldsymbol{z}\right\|_{\boldsymbol{H}^1(\Omega)}}\geq\frac{\mu\left\|\nabla\boldsymbol{v}\right\|^2_{\boldsymbol{L}^2(\Omega)}}{\left\|\boldsymbol{v}\right\|_{\boldsymbol{H}^1(\Omega)}}\geq c(\Omega)\left\|\boldsymbol{v}\right\|_{\boldsymbol{H}^1(\Omega)}.$$

And so we can apply [37, Theorem 6] to get differentiability of

$$I\ni t\mapsto\left((\det \mathrm{D}T_t)\left(\mathrm{D}T_t^{-1}\right)\boldsymbol{u}_0(t)\circ T_t-\boldsymbol{G}\right)\in\boldsymbol{H}^1(\Omega)$$

and thus of $t \mapsto \boldsymbol{u}_0(t)\circ T_t \in \boldsymbol{H}^1(\Omega)$ at $t=0$. Besides, we obtain that $\dot{\boldsymbol{u}}_0[V] := \partial_t|_{t=0}(\boldsymbol{u}_0(t)\circ T_t)$ is the unique solution of

$$F\left(0, (\operatorname{div} V(0))\,\boldsymbol{u}_0 - \mathrm{D}V(0)\boldsymbol{u}_0 + \dot{\boldsymbol{u}}_0[V]\right) = \widetilde{F}'(0) - \partial_t|_{t=0} F\left(t, \boldsymbol{u}_0 - \boldsymbol{G}\right)$$

which yields after some calculation (52).

We now proceed by deriving (51). Therefore, we first note that by [26, 10.2] it holds

$$\partial_t|_{t=0} P_\Omega\left(T_t(E_0)\right) = \int_\Omega \left(\operatorname{div} V(0) - \nu\cdot\nabla V(0)\nu\right)\,\mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|.$$

The remaining terms of $\partial_t|_{t=0} j_0\left(\varphi_0\circ T_t^{-1}\right)$ can be calculated directly and hence we arrive in (51).
Finally, the existence of a Lagrange multiplier $\lambda_0 \ge 0$ for the integral constraint can be deduced by the same method as in Theorem 3.

□

Remark 9. *Assume that* $E_0 := \operatorname{int}(\{\varphi_0 = 1\})$ *is a well-defined open subset of* Ω *such that* $\partial E_0\cap\Omega \in C^2$, E_0 *has finitely many connected components,* $\boldsymbol{g}\in \boldsymbol{H}^{\frac{3}{2}}(\partial\Omega)$ *and* $\left(\mathrm{D}_2 f(\cdot,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0) - \operatorname{div}\mathrm{D}_3 f(\cdot,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)\right)\in \boldsymbol{L}^2(E_0)$ *for* $\boldsymbol{u}_0\in\boldsymbol{H}^2(E_0)$. *Then one can also derive the "classical" shape derivatives which can for a large class of possible objective functionals be rewritten in the well-known Hadamard form, compare for instance [16, 39]. In this case, the optimality conditions derived in Theorem 3 can be shown to be equivalent to the following system, which can be obtained by classical calculus:*

$$\begin{aligned}&\int_{E_0} \mathrm{D}\left(f(x,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)\right) V(0)\,\mathrm{d}x + \int_\Omega f(x,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)\,\operatorname{div} V(0)\,\mathrm{d}x + \\ &+ \int_{\partial E_0\cap\Omega} \left(\mu\partial_\nu \boldsymbol{q}_0\cdot\partial_\nu\boldsymbol{u}_0 - (\mathrm{D}_3 f)(x,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)\,\nu\cdot\partial_\nu\boldsymbol{u}_0 + \gamma c_0\kappa + 2\lambda_0\right) V(0)\cdot\nu\,\mathrm{d}x = 0,\end{aligned}$$

which holds for all $V\in\mathcal{V}_{ad}$. *Here,* $\boldsymbol{u}_0\in\boldsymbol{U}^{\varphi_0}$ *solves the state equations* (11) *corresponding to* φ_0 *and* $\boldsymbol{q}_0\in\boldsymbol{H}_0^1(E_0)$ *with* $\boldsymbol{q}_0|_{\partial E_0} = \boldsymbol{0}$ *is the solution of the adjoint equation*

$$\int_{E_0}\mu\nabla\boldsymbol{q}_0\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_{E_0}\mathrm{D}_{(2,3)} f(x,\boldsymbol{u}_0,\mathrm{D}\boldsymbol{u}_0)(\boldsymbol{v},\mathrm{D}\boldsymbol{v})\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{H}_0^1(E_0),\ \operatorname{div}\boldsymbol{v}=0.$$

For details, we refer to [27, Section 8].

So far, we have derived necessary optimality conditions by geometric variations for the phase field problem, see Theorem 3, and also for the sharp interface problem, see Theorem 4. Additionally, we know, that in the diffuse interface setting, where the problem inherits the structure of an optimal control problem, the geometric optimality conditions are fulfilled if the variational inequality, which is obtained by parametric variations, is fulfilled, compare Remark 8. Additionally, we can also show equivalence of the optimality system in the sharp interface to shape derivatives in Hadamard form, compare Remark 9. Thus, the optimality conditions are all consistent with existing approaches towards these problems.
In Section 4 we have connected the phase field problems to the sharp interface problems by showing that as the thickness of the interface tends to zero, also minimizers converge under suitable assumptions. We now complete this picture by showing that also the optimality conditions of the phase field problem can be shown to be an approximation of the derived necessary optimality system in the sharp interface setting. This is the content of the following theorem:

Theorem 5. *Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be the sequence of minimizers of $(j_\varepsilon)_{\varepsilon>0}$ converging to $\varphi_0 \in L^1(\Omega)$ given by Theorem 2. Assume moreover that*

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon)\,. \tag{55}$$

Then the limit element φ_0 is a minimizer of j_0. Moreover it holds

$$\lim_{\varepsilon\searrow 0} \partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) \quad \forall T \in \mathcal{T}_{ad}. \tag{56}$$

If $|\{\varphi_0 = 1\}| > 0$ then we have additionally the following convergence results:

$$\varphi_\varepsilon \overset{\varepsilon\searrow 0}{\longrightarrow} \varphi_0 \qquad \text{in } L^1(\Omega), \tag{57a}$$

$$\boldsymbol{u}_\varepsilon \overset{\varepsilon\searrow 0}{\longrightarrow} \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}_\varepsilon[V] \overset{\varepsilon\searrow 0}{\longrightarrow} \dot{\boldsymbol{u}}_0[V] \qquad \text{in } \boldsymbol{H}^1(\Omega), \tag{57b}$$

$$\lambda_\varepsilon \overset{\varepsilon\searrow 0}{\longrightarrow} \lambda_0, \quad j_\varepsilon(\varphi_\varepsilon) \overset{\varepsilon\searrow 0}{\longrightarrow} j_0(\varphi_0) \qquad \text{in } \mathbb{R}, \tag{57c}$$

where $\boldsymbol{u}_\varepsilon := \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$, $\boldsymbol{u}_0 := \boldsymbol{S}_0(\varphi_0)$, $(\lambda_\varepsilon)_{\varepsilon>0} \subseteq \mathbb{R}^+$ are Lagrange multipliers for the integral constraint defined due to Lemma 3, $\lambda_0 \geq 0$ is a Lagrange multiplier such that it holds (50)*, and thus is a Lagrange multiplier for the integral constraint in the sharp interface according to Theorem 4.*

Remark 10. *We remark that the condition $|\{\varphi_0 = 1\}| > 0$ is only necessary to prove convergence of the Lagrange multipliers $(\lambda_\varepsilon)_{\varepsilon>0}$, whereas the other statements would hold true even if this condition is not fulfilled. But as $|\{\varphi_0 = 1\}| = 0$ means that there is no fluid present at all (up to sets of measure zero) this is not a restrictive assumption. For instance in the case of non-homogeneous boundary data, thus if $\mathcal{H}^{d-1}\left(\{x \in \partial\Omega \mid \boldsymbol{g}(x) \neq \boldsymbol{0}\}\right) > 0$, we find that $|\{x \in \Omega \mid \varphi_0(x) = 1\}| > 0$.*

Proof. We assume for the following considerations that (55) is fulfilled. The existence of a subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ that converges to a minimizer φ_0 of j_0 in $L^1(\Omega)$ follows from Theorem 2. In fact, we even obtain therefrom directly the convergence of the objective functionals, see (57c). Moreover, by using (55) we can apply Lemma 3 to obtain, after possibly choosing a subsequence

$$\lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x = 0, \quad \lim_{\varepsilon\searrow 0} \|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0\|_{\boldsymbol{H}^1(\Omega)} = 0 \tag{58}$$

which shows the first convergence of (57b).
From the second step in the proof of Lemma 3 we even find

$$\lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2 \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \boldsymbol{v}|_{\Omega\setminus E^{\varphi_0}} = \boldsymbol{0}. \tag{59}$$

This result will be used later on in this proof. We proceed by defining the auxiliary functions $\boldsymbol{w}_\varepsilon := (-\operatorname{div} V(0) + \mathrm{D}V(0))\,\boldsymbol{u}_\varepsilon$ for all $\varepsilon > 0$ and obtain from the regularity of V and the already proven convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ directly that $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ converges strongly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{w}_0 := (-\operatorname{div} V(0) + \mathrm{D}V(0))\,\boldsymbol{u}_0$.

We recall, that $\dot{\boldsymbol{u}}_\varepsilon[V] \in \boldsymbol{H}_0^1(\Omega)$ is due Lemma 3 given as the unique solution of (41) – (42). The main idea of the proof is to use the approach of Lemma 3, i.e. we show that $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$ are the unique minimizers of functionals which Γ-converge as $\varepsilon \searrow 0$ in the weak $\boldsymbol{H}^1(\Omega)$-topology. To this end, we define for $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$:

$$F_\varepsilon(\boldsymbol{v}) := \int_\Omega \left(\frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla\boldsymbol{v}|^2\right)\mathrm{d}x - R_\varepsilon(\boldsymbol{v}) + \\ + \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon \cdot \mathrm{D}V(0)\boldsymbol{v}\,\mathrm{d}x - D_\varepsilon(\boldsymbol{w}_\varepsilon)(\boldsymbol{v})$$

where $R_\varepsilon \in \boldsymbol{H}^{-1}(\Omega)$ is given by

$$\begin{aligned} R_\varepsilon(\boldsymbol{z}) &:= \int_\Omega \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \,\mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \mathrm{D}V(0)^T \nabla \boldsymbol{z}\,\mathrm{d}x + \\ &+ \mu \int_\Omega \nabla \boldsymbol{u}_\varepsilon : \nabla\left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z}\right)\mathrm{d}x - \\ &- \int_\Omega \mu \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{z} \operatorname{div} V(0)\,\mathrm{d}x + \int_\Omega (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z} + \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x \end{aligned}$$

and $D_\varepsilon(\boldsymbol{w}_\varepsilon) \in \boldsymbol{H}^{-1}(\Omega)$ is defined by

$$D_\varepsilon(\boldsymbol{w}_\varepsilon)(\boldsymbol{z}) = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{w}_\varepsilon \cdot \boldsymbol{z} + \mu \nabla \boldsymbol{w}_\varepsilon \cdot \nabla \boldsymbol{z}\,\mathrm{d}x.$$

Additionally, we define

$$F_0(\boldsymbol{v}) := \int_\Omega \left(\frac{1}{2}\alpha_0(\varphi_0)|\boldsymbol{v}|^2 + \frac{\mu}{2}|\nabla \boldsymbol{v}|^2\right)\mathrm{d}x - R_0(\boldsymbol{v}) - D_0(\boldsymbol{w}_0)(\boldsymbol{v})$$

where

$$\begin{aligned} R_0(\boldsymbol{z}) &:= \int_\Omega \mu \mathrm{D}V(0)^T \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z}\,\mathrm{d}x + \int_\Omega \mu \nabla \boldsymbol{u}_0 : \mathrm{D}V(0)^T \nabla \boldsymbol{z}\,\mathrm{d}x + \\ &+ \mu \int_\Omega \nabla \boldsymbol{u}_0 : \nabla\left(\operatorname{div} V(0)\boldsymbol{z} - \mathrm{D}V(0)\boldsymbol{z}\right)\mathrm{d}x - \\ &- \int_\Omega \mu \nabla \boldsymbol{u}_0 : \nabla \boldsymbol{z} \operatorname{div} V(0)\,\mathrm{d}x + \int_\Omega (\nabla \boldsymbol{f} \cdot V(0)) \cdot \boldsymbol{z}\,\mathrm{d}x + \int_\Omega \boldsymbol{f} \cdot \mathrm{D}V(0)\boldsymbol{z}\,\mathrm{d}x \end{aligned}$$

and

$$D_0(\boldsymbol{w}_0)(\boldsymbol{z}) = \int_\Omega \alpha_0(\varphi_0)\boldsymbol{w}_0 \cdot \boldsymbol{z} + \mu \nabla \boldsymbol{w}_0 \cdot \nabla \boldsymbol{z}\,\mathrm{d}x.$$

We remark that $(R_\varepsilon)_{\varepsilon>0} \subseteq \boldsymbol{H}^{-1}(\Omega)$ and $R_0 \in \boldsymbol{H}^{-1}(\Omega)$. From the already proven convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ to $\boldsymbol{u}_0$ we find that $(R_\varepsilon)_{\varepsilon>0}$ converges to R_0 (strongly) in $\boldsymbol{H}^{-1}(\Omega)$.

Next we see, that due to Lemma 6 it holds

$$\operatorname{div}\left(\operatorname{div} V(0)\boldsymbol{u}_\varepsilon + \mathrm{D}\boldsymbol{u}_\varepsilon V(0) - \mathrm{D}V(0)\boldsymbol{u}_\varepsilon\right) = 0$$

and so

$$\operatorname{div} \boldsymbol{w}_\varepsilon = \operatorname{div}\left(-\operatorname{div} V(0)\boldsymbol{u}_\varepsilon + \mathrm{D}V(0)\boldsymbol{u}_\varepsilon\right) = \operatorname{div}\left(\mathrm{D}\boldsymbol{u}_\varepsilon V(0)\right) = \mathrm{D}\boldsymbol{u}_\varepsilon : \nabla V(0)$$

where we used for the last step $\operatorname{div} \boldsymbol{u}_\varepsilon = 0$. This implies $\operatorname{div}\left(\dot{\boldsymbol{u}}_\varepsilon[V] - \boldsymbol{w}_\varepsilon\right) = 0$. And so we can conclude from $\dot{\boldsymbol{u}}_\varepsilon[V]|_{\partial\Omega} = \boldsymbol{w}_\varepsilon|_{\partial\Omega} = \boldsymbol{0}$ that $(\dot{\boldsymbol{u}}_\varepsilon[V] - \boldsymbol{w}_\varepsilon) \in \boldsymbol{V}$. In particular, we can insert $(\dot{\boldsymbol{u}}_\varepsilon[V] - \boldsymbol{w}_\varepsilon) \in \boldsymbol{V}$ as a test function into (41) and end up with

$$\begin{aligned} &\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\dot{\boldsymbol{u}}_\varepsilon[V]|^2\,\mathrm{d}x + \int_\Omega \mu|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]|^2\,\mathrm{d}x = R_\varepsilon(\dot{\boldsymbol{u}}_\varepsilon[V] - \boldsymbol{w}_\varepsilon) - \\ &- \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon \cdot \mathrm{D}V(0)(\dot{\boldsymbol{u}}_\varepsilon[V] - \boldsymbol{w}_\varepsilon)\,\mathrm{d}x - D_\varepsilon(\dot{\boldsymbol{u}}_\varepsilon[V])(\boldsymbol{w}_\varepsilon) \le \\ &\le \|R_\varepsilon\|_{\boldsymbol{H}^{-1}(\Omega)}\left(\|\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{w}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)}\right) + \\ &+ C\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\right)^{\frac{1}{2}}\left(\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\dot{\boldsymbol{u}}_\varepsilon[V]|^2\right)^{\frac{1}{2}} + \left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{w}_\varepsilon|^2\right)^{\frac{1}{2}}\right) + \\ &+ C\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{w}_\varepsilon|^2\right)^{\frac{1}{2}}\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\dot{\boldsymbol{u}}_\varepsilon[V]|^2\right)^{\frac{1}{2}} + \mu\|\nabla\boldsymbol{w}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{60}$$

By observing

$$\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{w}_\varepsilon|^2 \,\mathrm{d}x = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\left|(-\operatorname{div} V(0) + \mathrm{D}V(0))\,\boldsymbol{u}_\varepsilon\right|^2 \,\mathrm{d}x \le \\ \le C\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x. \tag{61}$$

we find thanks to Young's inequality from (60)

$$\begin{aligned}&\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\dot{\boldsymbol{u}}_\varepsilon[V]|^2\,\mathrm{d}x + \int_\Omega \frac{\mu}{2}|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]|^2\,\mathrm{d}x \le \\ &\le \underbrace{\|R_\varepsilon\|_{\boldsymbol{H}^{-1}(\Omega)}}_{\le C}\left(\|\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{H}^1(\Omega)} + \underbrace{\|\boldsymbol{w}_\varepsilon\|_{\boldsymbol{H}^1(\Omega)}}_{\le C}\right) + C\underbrace{\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\right)}_{\le C} + \\ &+ \mu\underbrace{\|\nabla\boldsymbol{w}_\varepsilon\|_{\boldsymbol{L}^2(\Omega)}}_{\le C}\|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{62}$$

And so, by using again Young's inequality together with Poincaré's inequality we end up having a uniform bound on $\|\dot{\boldsymbol{u}}_\varepsilon[V]\|_{\boldsymbol{H}^1(\Omega)}$ and

$$\sup_{\varepsilon>0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\dot{\boldsymbol{u}}_\varepsilon[V]|^2\,\mathrm{d}x < \infty. \tag{63}$$

This directly implies the existence of a subsequence of $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$, denoted by the same, that converges weakly in $\boldsymbol{H}^1(\Omega)$ as $\varepsilon \searrow 0$.

After these preparatory steps we notice that $(\dot{\boldsymbol{u}}_\varepsilon[V] - \boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ are the unique minimizers in $\boldsymbol{V}$ of the convex functionals $(F_\varepsilon)_{\varepsilon>0}$, and similarly $(\dot{\boldsymbol{u}}_0[V] - \boldsymbol{w}_0)$ is the unique minimizer of F_0 in $\boldsymbol{V}$. This follows by observing that the linearized state equations (41) – (42) and (52) – (53) are the necessary and sufficient optimality conditions for these convex optimization problems, see also discussion in Lemma 3.

We continue by proving that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to F_0 in $\boldsymbol{V}$ with respect to the weak $\boldsymbol{H}^1(\Omega)$ topology as $\varepsilon \searrow 0$. For this purpose, we will follow closely the arguments of Lemma 3 and only point out the steps which differ from the corresponding parts in the proof of Lemma 3. We conclude in several steps:

Claim: For any $\boldsymbol{v} \in \boldsymbol{V}$ it holds $\limsup_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}) \le F_0(\boldsymbol{v})$.

Proof: Without loss of generality we can assume $F_0(\boldsymbol{v}) < \infty$, which gives $\int_\Omega \alpha_0(\varphi_0)|\boldsymbol{v}|^2 < \infty$. As we know $\alpha_0(\varphi_0) \in \{0,\infty\}$ a.e. in Ω this already implies $\boldsymbol{v} = \boldsymbol{0}$ in $\{\varphi_0 = -1\}$. Using (59) we deduce therefrom

$$\lim_{\varepsilon\searrow 0}\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2\,\mathrm{d}x = \int_\Omega \alpha_0(\varphi_0)|\boldsymbol{v}|^2\,\mathrm{d}x = 0$$

and applying Hölder's inequality we get moreover

$$\left|\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot\mathrm{D}V(0)\boldsymbol{v}\,\mathrm{d}x\right| \le C\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\right)^{\frac{1}{2}}\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{v}|^2\right)^{\frac{1}{2}} \xrightarrow{\varepsilon\searrow 0} 0.$$

Similarly, we get due to (61) that

$$\left|\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\, \boldsymbol{w}_\varepsilon \cdot \boldsymbol{v}\,\mathrm{d}x\right| \le C\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,|\boldsymbol{v}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \xrightarrow{\varepsilon\searrow 0} 0.$$

Combining these results with the convergence of $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ to $\boldsymbol{w}_0$ in $\boldsymbol{H}^1(\Omega)$ we deduce the claim.

Claim: Let $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0} \subset \boldsymbol{V}$ be such that $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{v}$ as $\varepsilon \searrow 0$. Then:

$$F_0(\boldsymbol{v}) \le \liminf_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}_\varepsilon).$$

Proof: We assume $\liminf_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}_\varepsilon) < \infty$, otherwise the claim would be trivial. Following the arguments of Lemma 3, in particular the calculation in (28), we can deduce

$$\int_\Omega \alpha_0(\varphi_0)\,|\boldsymbol{v}|^2\,\mathrm{d}x \le \liminf_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,|\boldsymbol{v}_\varepsilon|^2\,\mathrm{d}x.$$

Next we choose a subsequence such that

$$\lim_{k\to\infty}\int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k})\,|\boldsymbol{v}_{\varepsilon_k}|^2\,\mathrm{d}x = \liminf_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,|\boldsymbol{v}_\varepsilon|^2\,\mathrm{d}x.$$

By Hölder's inequality we find for this subsequence

$$\begin{aligned}&\left|\int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k})\,\boldsymbol{u}_{\varepsilon_k}\cdot \mathrm{D}V(0)\boldsymbol{v}_{\varepsilon_k}\,\mathrm{d}x\right| \le \\ &\le C\left(\int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k})\,|\boldsymbol{u}_{\varepsilon_k}|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\underbrace{\left(\int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k})\,|\boldsymbol{v}_{\varepsilon_k}|^2\,\mathrm{d}x\right)^{\frac{1}{2}}}_{<C}\end{aligned}$$

which gives in view of (58),

$$\lim_{k\to\infty}\left|\int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k})\,\boldsymbol{u}_{\varepsilon_k}\cdot \mathrm{D}V(0)\boldsymbol{v}_{\varepsilon_k}\,\mathrm{d}x\right| = 0.$$

Thus, we obtain

$$\liminf_{\varepsilon\searrow 0}\left|\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{u}_\varepsilon\cdot \mathrm{D}V(0)\boldsymbol{v}_\varepsilon\,\mathrm{d}x\right| \le \lim_{k\to\infty}\left|\int_\Omega \alpha_{\varepsilon_k}(\varphi_{\varepsilon_k})\,\boldsymbol{u}_{\varepsilon_k}\cdot \mathrm{D}V(0)\boldsymbol{v}_{\varepsilon_k}\,\mathrm{d}x\right| = 0$$

and therefrom

$$\liminf_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot \mathrm{D}V(0)\boldsymbol{v}_\varepsilon\,\mathrm{d}x = 0.$$

Similarly, we find by means of (61)

$$0 = \int_\Omega \alpha_0(\varphi_0)\boldsymbol{w}_0\cdot\boldsymbol{v}\,\mathrm{d}x = \liminf_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\,\boldsymbol{w}_\varepsilon\cdot\boldsymbol{v}_\varepsilon\,\mathrm{d}x.$$

Now we can use the strong convergence of $(R_\varepsilon)_{\varepsilon>0}$ to R_0 in $\boldsymbol{H}^{-1}(\Omega)$ and the weakly lower semicontinuity of the remaining terms to deduce the claim.

Combining the previous two claims, we can conclude that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to F_0 in $\boldsymbol{V}$ with respect to the weak $\boldsymbol{H}^1(\Omega)$ topology. And so standard results for Γ-convergence, see for instance [14], imply:

Claim: If $\boldsymbol{v}_\varepsilon \in \boldsymbol{V}$ minimizes F_ε for every $\varepsilon > 0$ and the sequence $(\boldsymbol{v}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{v}_0$, then $\boldsymbol{v}_0$ minimizes F_0 and $\lim_{\varepsilon\searrow 0} F_\varepsilon(\boldsymbol{v}_\varepsilon) = F_0(\boldsymbol{v}_0)$.

We will use this result to show the remaining statements of the theorem. To this end, we recall that $(\dot{\boldsymbol{u}}_\varepsilon[V] - \boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to some element in $\boldsymbol{H}^1(\Omega)$, which has to be a minimizer of F_0 due to the claim above. But since F_0 is a strictly convex function, the minimizer $\dot{\boldsymbol{u}}_0[V] - \boldsymbol{w}_0$ is the only one, and thus $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$ converges weakly in $\boldsymbol{H}^1(\Omega)$ to $\dot{\boldsymbol{u}}_0[V]$ and

$$\lim_{\varepsilon\searrow 0} F_\varepsilon(\dot{\boldsymbol{u}}_\varepsilon[V] - \boldsymbol{w}_\varepsilon) = F_0(\dot{\boldsymbol{u}}_0[V] - \boldsymbol{w}_0). \tag{64}$$

By

$$\left|\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{w}_\varepsilon \cdot \dot{\boldsymbol{u}}_\varepsilon[V]\,\mathrm{d}x\right| \overset{(61)}{\le} C \underbrace{\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x\right)^{\frac12}}_{\xrightarrow{(58)}0} \underbrace{\left(\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\dot{\boldsymbol{u}}_\varepsilon[V]|^2\,\mathrm{d}x\right)^{\frac12}}_{\overset{(63)}{<}C} \tag{65}$$

we also have

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{w}_\varepsilon\cdot\dot{\boldsymbol{u}}_\varepsilon[V]\,\mathrm{d}x = 0.$$

Thanks to the convergence of $(R_\varepsilon)_{\varepsilon>0}$ to R_0 in $\boldsymbol{H}^{-1}(\Omega)$, the strong convergence of $(\boldsymbol{w}_\varepsilon)_{\varepsilon>0}$ in $\boldsymbol{H}^1(\Omega)$ this yields in view of (64)

$$\lim_{\varepsilon\searrow 0}\left[\int_\Omega \frac12\alpha_\varepsilon(\varphi_\varepsilon)|\dot{\boldsymbol{u}}_\varepsilon[V]|^2 + \frac{\mu}{2}|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]|^2\,\mathrm{d}x + \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot \mathrm{D}V(0)\dot{\boldsymbol{u}}_\varepsilon[V]\,\mathrm{d}x\right] =$$
$$= \int_\Omega \frac12\alpha_0(\varphi_0)|\dot{\boldsymbol{u}}_0[V]|^2 + \frac{\mu}{2}|\nabla\dot{\boldsymbol{u}}_0[V]|^2\,\mathrm{d}x.$$

Applying again (63) and (58) we find similar to (65)

$$\lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\mathrm{D}V(0)\dot{\boldsymbol{u}}_\varepsilon[V]\,\mathrm{d}x = 0$$

wherefrom we arrive in

$$\lim_{\varepsilon\searrow 0}\left[\int_\Omega \frac12\alpha_\varepsilon(\varphi_\varepsilon)|\dot{\boldsymbol{u}}_\varepsilon[V]|^2 + \frac{\mu}{2}|\nabla\dot{\boldsymbol{u}}_\varepsilon[V]|^2\right] = \int_\Omega \frac12\alpha_0(\varphi_0)|\dot{\boldsymbol{u}}_0[V]|^2 + \frac{\mu}{2}|\nabla\dot{\boldsymbol{u}}_0[V]|^2\,\mathrm{d}x.$$

Thus, using Lemma 4, we can deduce the strong convergence of $(\dot{\boldsymbol{u}}_\varepsilon[V])_{\varepsilon>0}$ in $\boldsymbol{H}^1(\Omega)$ and

$$\lim_{\varepsilon\searrow 0}\int_\Omega \frac12\alpha_\varepsilon(\varphi_\varepsilon)|\dot{\boldsymbol{u}}_\varepsilon[V]|^2\,\mathrm{d}x = \int_\Omega \frac12\alpha_0(\varphi_0)|\dot{\boldsymbol{u}}_0[V]|^2\,\mathrm{d}x = 0.$$

We continue this proof by considering the terms in the optimality system arising from the Ginzburg-Landau energy. To this end we observe that

$$\lim_{\varepsilon\searrow 0} j_\varepsilon(\varphi_\varepsilon) = j_0(\varphi_0), \quad \lim_{\varepsilon\searrow 0}\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x = 0$$

together with (57b) imply

$$\lim_{\varepsilon\searrow 0}\int_\Omega\left(\frac{\gamma\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon}\psi(\varphi)\right)\mathrm{d}x = \gamma c_0 P_\Omega(\{\varphi_0 = 1\}).$$

Using the same calculations as in [23, Proof of Theorem 4.2] we can deduce therefrom

$$\lim_{\varepsilon\searrow 0}\gamma\int_\Omega\left(\frac{\varepsilon}{2}\left|\nabla\varphi_\varepsilon\right|^2+\frac{1}{\varepsilon}\psi\left(\varphi_\varepsilon\right)\right)\operatorname{div}V(0)\,\mathrm{d}x=\gamma c_0\int_\Omega\operatorname{div}V(0)\,\mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|$$

and

$$\lim_{\varepsilon\searrow 0}\gamma\varepsilon\int_\Omega\nabla\varphi_\varepsilon\cdot\nabla V(0)\nabla\varphi_\varepsilon\,\mathrm{d}x=\gamma c_0\int_\Omega\nu\cdot\nabla V(0)\nu\,\mathrm{d}\left|\mathrm{D}\chi_{E_0}\right|$$

where ν is as usual the generalised unit normal on $E_0 := \{\varphi_0 = 1\}$. The proof in [23] uses ideas of [28] and is based on the Reshetnyak continuity theorem, see [2, Theorem 2.39]. For more details we refer the reader to [23, Proof of Theorem 4.2].

To finish the proof of (56) we deduce from (58) and (63)

$$\left|\int_\Omega\alpha_\varepsilon\left(\varphi_\varepsilon\right)\boldsymbol{u}_\varepsilon\cdot\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\,\mathrm{d}x\right|\leq\left(\int_\Omega\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{u}_\varepsilon\right|^2\right)^{\frac{1}{2}}\left(\int_\Omega\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right|^2\right)^{\frac{1}{2}}\xrightarrow{\varepsilon\searrow 0}0.$$

At the same time, (56) and the regularity of $V\in\mathcal{V}_{ad}$ imply

$$\lim_{\varepsilon\searrow 0}\int_\Omega\frac{1}{2}\alpha_\varepsilon\left(\varphi_\varepsilon\right)\left|\boldsymbol{u}_\varepsilon\right|^2\operatorname{div}V(0)\,\mathrm{d}x=0.$$

Due to the proven convergence results of $\left(\boldsymbol{u}_\varepsilon\right)_{\varepsilon>0}$ and $\left(\dot{\boldsymbol{u}}_\varepsilon\left[V\right]\right)_{\varepsilon>0}$ we thus obtain

$$\lim_{\varepsilon\searrow 0}\partial_t|_{t=0}j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=\partial_t|_{t=0}j_0\left(\varphi_0\circ T_t^{-1}\right). \tag{66}$$

It remains to consider the Lagrange multipliers $\left(\lambda_\varepsilon\right)_{\varepsilon>0}$. In view of (39), we see that the left-hand side of

$$\partial_t|_{t=0}j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=-\lambda_\varepsilon\int_\Omega\varphi_\varepsilon\operatorname{div}V(0)\,\mathrm{d}x$$

converges for every $T\in\mathcal{T}_{ad}$ with velocity field $V\in\mathcal{V}_{ad}$ as $\varepsilon\searrow 0$. We choose a specific velocity field $V\in\mathcal{V}_{ad}$ such that it holds $\int_\Omega\varphi_0\operatorname{div}V(0)\,\mathrm{d}x>0$. This is possible, since $\varphi_0\in\Phi_{ad}$ and thus $\{\varphi_0=1\}\subsetneq\Omega$, and due to the assumption $|\{\varphi_0=1\}|>0$ it holds $\{\varphi_0=-1\}\subsetneq\Omega$. Then we deduce from (66) that

$$\lim_{\varepsilon\searrow 0}-\lambda_\varepsilon\int_\Omega\varphi_\varepsilon\operatorname{div}V(0)\,\mathrm{d}x=\lim_{\varepsilon\searrow 0}\partial_t|_{t=0}j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right)=\partial_t|_{t=0}j_0\left(\varphi_0\circ T_t^{-1}\right).$$

But since

$$\lim_{\varepsilon\searrow 0}\int_\Omega\varphi_\varepsilon\operatorname{div}V(0)\,\mathrm{d}x=\int_\Omega\varphi_0\operatorname{div}V(0)\,\mathrm{d}x>0$$

it follows therefrom that $\left(\lambda_\varepsilon\right)_{\varepsilon>0}$ converges in $\mathbb{R}$, and we call the limit element $\lambda_0\geq 0$. Additionally, we know then that $\lambda_0\geq 0$ fulfills (50). This finally finishes the proof. □

6 Pressure terms in the objective functional

6.1 Phase field problem

As already mentioned in the introduction, we can also include the pressure of the fluid in the objective functional. There are several applications where this is desirable. But in contrast the the velocity of the fluid, we cannot give a meaning to the pressure in the whole of Ω in the sharp interface setting, as we do not know if the pressure vanishes outside the fluid region or how it behaves. And so it only makes sense to consider the pressure in a part where fluid is present. Mathematically, this condition is implemented by including an additional constraint in the admissible regions. To be precise, we prescribe the design variable to have the value one, which corresponds to presence of fluid, at certain given region M_i, $i = 1, \dots, m$. Those regions M_i are given as the parts where the pressure is included in the objective functional.

(A6) Assume to have finitely many fixed disjoint Lipschitz domains $(M_i)_{i=1}^m$, $M_i \subset \Omega$. Let $h_M : \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function, that means here h_M is assumed to be continuous, such that $|h_M(v)| \le C|v|^2$ holds for all $v \in \mathbb{R}^m$ for some constant $C > 0$. Additionally, assume that

$$H : L^2(\Omega) \ni q \mapsto \int_\Omega h_M(q\chi_{M_1}, \dots, q\chi_{M_m})\,\mathrm{d}x \tag{67}$$

is weakly lower semicontinuous and bounded from below. We use the following the notation:

$$\int_\Omega h(p)\,\mathrm{d}x = \int_\Omega h_M(p\chi_{M_1}, \dots, p\chi_{M_m})\,\mathrm{d}x \quad \forall p \in L^2(\Omega).$$

Moreover, we have to assume some compatibility condition such that the admissible set is not empty: $\sum_{i=1}^m |M_i| < \beta|\Omega|$.

The admissible design functions φ for the phase field problem are then chosen in

$$\Phi_p := \{\varphi \in \Phi_{ad} \mid \varphi|_{M_i} = 1, \quad \forall i = 1, \dots, m\}$$

and the pressure is chosen in

$$L^2_M(\Omega) := \left\{q \in L^2(\Omega) \mid \int_{M_i} q\,\mathrm{d}x = 0,\ \forall i = 1, \dots, m,\ q|_{\Omega \setminus \cup_{i=1}^m M_i} = 0\right\}.$$

The choice of the pressure to be zero outside the regions M_i is arbitrary and does not influence the problem, as the objective functional only takes the pressure inside M_i into account. The overall optimization problem in the phase field setting is given as

$$\begin{aligned}\min_{(\varphi,\boldsymbol{u},p)} J_\varepsilon^P(\varphi,\boldsymbol{u},p) := \frac{1}{2}\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}) + h(p)\,\mathrm{d}x + \\ + \gamma\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x\end{aligned} \tag{68}$$

subject to $(\varphi, \boldsymbol{u}, p) \in \Phi_p \times \boldsymbol{U} \times L^2_M(\Omega)$,

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{69}$$

and

$$\int_\Omega \alpha_\varepsilon(\varphi)\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} - p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(M_i), \tag{70}$$

$$i = 1, \dots, m. \tag{71}$$

Remark 11. *Of course, one could also replace the objective functional* $\int_\Omega f(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}) + h(p)\,\mathrm{d}x$ *by* $\int_\Omega \tilde{f}(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u},p)\,\mathrm{d}x$ *for an appropriate chosen function* $\tilde{f}$. *But to simplify the considerations and notation we focus here on the form specified above.*

Remark 12. *By standard results, compare for instance [22, 42], we obtain for an arbitrary bounded Lipschitz domain* $U \subset \mathbb{R}^d$ *the following result: If* $F \in \boldsymbol{H}^{-1}(U)$ *with* $F(\boldsymbol{u}) = 0$ *for all* $\boldsymbol{u} \in \boldsymbol{H}_0^1(U)$ *with* $\operatorname{div} \boldsymbol{u} = 0$*, then there exists some* $p \in L^2(U)$*, which is unique up to a constant, such that* $\nabla p = F$ *in* $\boldsymbol{H}^{-1}(U)$*.*
This result ensures for any $F \in \boldsymbol{H}^{-1}(\Omega)$ *such that* $F(\boldsymbol{u}) = 0$ *for all* $\boldsymbol{u} \in \boldsymbol{H}_0^1(U)$ *with* $\operatorname{div} \boldsymbol{u} = 0$ *the existence and uniqueness of* $p_i \in L^2(M_i)$ *with* $\int_{M_i} p_i \,\mathrm{d}x = 0$ *such that* $\nabla p_i = F|_{\boldsymbol{H}_0^1(M_i)}$ *in* $\boldsymbol{H}^{-1}(M_i)$*. Then we can define* $p := \sum_{i=1}^m p_i \in L^2_M(\Omega)$ *and see that* $\nabla p = F|_{\boldsymbol{H}_0^1(M_i)}$ *in* $\boldsymbol{H}^{-1}(M_i)$ *for all* $i = 1, \ldots, m$*.*

We directly establish the following existence results:

Lemma 7. *For every* $\varphi \in L^1(\Omega)$ *with* $|\varphi(x)| \leq 1$ *a.e. in* Ω *there exists a unique* $\boldsymbol{u} \in \boldsymbol{U}$ *and* $p \in L^2_M(\Omega)$ *such that* (69) – (70) *is fulfilled. This defines a solution operator* $\boldsymbol{S}_\varepsilon^P : \Phi_p \to \boldsymbol{U} \times L^2_M(\Omega)$*,* $\boldsymbol{S}_\varepsilon^P(\varphi) := (\boldsymbol{u}, p)$ *if* $(\boldsymbol{u}, p)$ *solve* (69) – (70)*.*

Proof. By Lemma 1 we obtain for every $\varphi \in L^1(\Omega)$ with $|\varphi| \leq 1$ a.e. in Ω a unique solution $\boldsymbol{u} \in \boldsymbol{U}$ of (69). The pressure $p \in L^2_M(\Omega)$ can then be obtained as outlined in Remark 12. □

Remark 13. *We obtain by standard results, compare for instance [42, Proposition 1.2], in particular that for any* $p \in L^2_M(\Omega)$ *fulfilling* (70) *for some* $\boldsymbol{u} \in \boldsymbol{U}$ *and* $\varphi \in L^1(\Omega)$ *with* $|\varphi| \leq 1$ *a.e. in* Ω *it holds*

$$\|p\|_{L^2(\Omega)} \leq c(\Omega) \left(\sum_{i=1}^m \|\alpha_\varepsilon(\varphi)\boldsymbol{u} - \mu\Delta\boldsymbol{u} - \boldsymbol{f}\|_{\boldsymbol{H}^{-1}(M_i)} \right). \tag{72}$$

This estimate is important for the following considerations.

We can hence define the reduced objective functional $j_\varepsilon^P : L^1(\Omega) \to \overline{\mathbb{R}}$ by

$$j_\varepsilon^P(\varphi) := \begin{cases} J_\varepsilon^P\left(\varphi, \boldsymbol{S}_\varepsilon^P(\varphi)\right) & \text{if } \varphi \in \Phi_p, \\ +\infty & \text{otherwise} \end{cases} \tag{73}$$

and obtain that (68)-(70) is equivalent to

$$\min_{\varphi \in L^1(\Omega)} j_\varepsilon^P(\varphi). \tag{74}$$

Additionally we obtain well-posedness of the optimization problem:

Theorem 6. *There exists at least one minimizer* $\varphi_\varepsilon \in \varphi_p$ *of* j_ε^P*, and hence there exists also a minimizer of* (68)-(70)*.*

Proof. This can be established by the direct method in the calculus of variations by using in particular the pressure estimate (72) and the arguments of Theorem 1, see also [27, Lemma 19.2]. □

6.2 Sharp interface problem

Corresponding to Section 3 we can introduce a corresponding sharp interface problem in a setting of Caccioppoli sets including a perimeter constraint. But before introducing the problem formulation we study the general existence of the pressure in measurable sets. Standard results, compare [42, 22], only ensure the existence of a pressure in a Lipschitz domain. But in our setting we can define some pressure in a measurable set, as the following lemma shows:

Lemma 8. *Let $E \subset \Omega$ be a measurable set and $\boldsymbol{u} \in \boldsymbol{U}$ with $\boldsymbol{u}|_{\Omega\setminus E} = \mathbf{0}$ a.e. such that*

$$\mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}. \tag{75}$$

Then there exists some $p \in L^2(E)$ such that

$$\mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x - \int_E p \operatorname{div} \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\ \boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}. \tag{76}$$

Proof. We denote by $\varphi := 2\chi_E - 1 \in L^1(\Omega, \{\pm 1\})$ the function associated to the measurable set E. For $\varepsilon > 0$ we define $\boldsymbol{u}_\varepsilon \in \boldsymbol{U}$ as a solution to

$$\int_\Omega \alpha_\varepsilon(\varphi) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x + \mu \int_\Omega \nabla \boldsymbol{u}_\varepsilon \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_E \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{77}$$

which exists for example due to Lemma 1 and means that $\boldsymbol{u}_\varepsilon = \boldsymbol{S}_\varepsilon(\varphi)$. Defining $\varphi_\varepsilon := \varphi$ for all $\varepsilon > 0$ we see as in the proof of Lemma 3 that (after possibly choosing a subsequence) $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ converges to $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ as $\varepsilon \searrow 0$ and $\lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi) |\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x = 0$. Now from (77) and using the convergence of $(\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ to $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ we see that $(\alpha_\varepsilon(\varphi)\boldsymbol{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $\boldsymbol{V}'$ and thus there exists some $A \in \boldsymbol{V}'$ such that

$$\lim_{\varepsilon\searrow 0} \int_\Omega \alpha_\varepsilon(\varphi) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x = A(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}$$

and so passing to the limit in (77) gives

$$A(\boldsymbol{v}) + \mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x = \int_E \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \operatorname{div} \boldsymbol{v} = 0.$$

For some $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ with $\boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}$ we obtain

$$\int_\Omega \alpha_\varepsilon(\varphi) \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x = \int_{\{\varphi=-1\}} \alpha_\varepsilon(\varphi) \boldsymbol{u}_\varepsilon \cdot \underbrace{\boldsymbol{v}}_{=\mathbf{0}} \,\mathrm{d}x + \int_{\{\varphi=1\}} \underbrace{\alpha_\varepsilon(\varphi)}_{=0} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x = 0. \tag{78}$$

So we know that we can extend A to a linear, continuous functional on

$$\left(\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}\right).$$

Since $\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}$ is a linear and closed subspace of $\boldsymbol{H}_0^1(\Omega)$ we can extend A to a linear and continuous functional on $\boldsymbol{H}_0^1(\Omega)$ by defining

$$A(\boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \left(\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}\right)^\perp$$

where $\left(\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}\right)^\perp$ denotes the orthogonal complement of $\boldsymbol{V} + \left\{\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{w}|_{\Omega\setminus E} = \mathbf{0}\right\}$ in $\boldsymbol{H}_0^1(\Omega)$.

Using standard results concerning solvability of the gradient equation, compare for instance [22, 42] or Remark 12, we can thus conclude that there exists some $p \in L^2(\Omega)$ such that

$$A(\boldsymbol{v}) + \mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x - \int_\Omega p \operatorname{div} \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{79}$$

Since due to (78) it holds $A(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ such that $\boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}$, this implies in particular

$$\mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x - \int_\Omega p \operatorname{div} \boldsymbol{v} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \boldsymbol{v}|_{\Omega\setminus E} = \mathbf{0}$$

and so $p|_E$ is a pressure associated to $\boldsymbol{u}$ fulfilling (76).

□

One question that arises during these considerations is, if the set $\left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v}|_{\Omega\setminus E} = \boldsymbol{0}\right\}$ can be identified with $\boldsymbol{H}_0^1(\mathrm{int}(E))$, because then Lemma 8 would define a pressure $p \in L^2(\mathrm{int}(E))$ associated to the Stokes equations that are fulfilled in $\mathrm{int}(E)$, whereas $\mathrm{int}(E)$ is not a Lipschitz set as it is necessary for the classical results. In those results the lack of boundary regularity implies that the pressure can only be found in L^2_{loc} of the corresponding subset.
But due to the considerations in [18], see also [27], we find one representative E_c of the equivalence class of E, a so-called "crack-free" representative, such that

$$\left\{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. in } \Omega \setminus E_c\right\} = \boldsymbol{H}_0^1(\mathrm{int}E_c) = \boldsymbol{H}_0^1(\mathrm{int}\overline{E}_c).$$

Now fixing this representative E_c, we can solve the Stokes equations in $\mathrm{int}E_c$ in the sense of (75) and obtain due to Lemma 8 an associated pressure $p \in L^2\left(\mathrm{int}E_c\right)$.

But even though we could define one pressure in the usual way for the sharp interface equation this is not the situation we want to consider because it is not clear which conditions to state to get uniqueness of this pressure, since the Caccioppoli sets in the shape optimization problem may have varying, or even infinitely many, connected components. In particular, we cannot fix the connected components, since topological changes are allowed during the optimization process. Instead, we define the pressure only in the fixed domains M_i, as already done in the previous subsection. Thus the overall optimization problem in the sharp interface formulation is given as

$$\min_{(\varphi,\boldsymbol{u},p)} J_0^P\left(\varphi,\boldsymbol{u},p\right) = \int_\Omega f\left(x,\boldsymbol{u},\mathrm{D}\boldsymbol{u}\right)\,\mathrm{d}x + \int_\Omega h(p)\,\mathrm{d}x + \gamma c_0 P_\Omega\left(\{\varphi = 1\}\right) \tag{80}$$

with

$$(\varphi,\boldsymbol{u},p) \in \Phi_p^0 \times \boldsymbol{U}^\varphi \times L_M^2(\Omega)$$

such that

$$\mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}^\varphi, \tag{81}$$

$$\mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x - \int_\Omega p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(M_i),\, i = 1,\dots,m. \tag{82}$$

The design space Φ_p^0 is given as

$$\Phi_p^0 := \{\varphi \in \Phi_{ad}^0 \mid \varphi|_{M_i} = 1,\ \forall i = 1,\dots,m\}.$$

We directly obtain:

Lemma 9. *For every $\varphi \in L^1(\Omega)$ such that $\boldsymbol{U}^\varphi \neq \emptyset$ there exists a unique $\boldsymbol{u} \in \boldsymbol{U}^\varphi$ and $p \in L_M^2(\Omega)$ such that it holds* (81)-(82). *This defines a solution operator $\boldsymbol{S}_0^P : \overline{\Phi}_p^0 \to \boldsymbol{U}\times L_M^2(\Omega)$, where $\boldsymbol{S}_0^P\left(\varphi\right) := (\boldsymbol{u},p)$ if $(\boldsymbol{u},p)$ fulfill* (81) – (82).

Proof. Existence and uniqueness of $\boldsymbol{u} \in \boldsymbol{U}^\varphi$ follow from Lemma 2, and the existence and uniqueness of p in $L_M^2(\Omega)$ follows then as indicated in Remark 12. □

And so we end up in defining the reduced objective functional for the sharp interface problem by

$$j_0^P : L^1(\Omega) \to \overline{\mathbb{R}}, \quad j_0^P\left(\varphi\right) := \begin{cases} J_0^P\left(\varphi, \boldsymbol{S}_0^P\left(\varphi\right)\right), & \text{if } \varphi \in \Phi_p^0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{83}$$

6.3 Sharp interface limit

We want to show a sharp interface limit result corresponding to Theorem 2 and directly state the main result:

Theorem 7. *Let* $(\varphi_\varepsilon)_{\varepsilon>0}$ *be minimizers of* $\left(j_\varepsilon^P\right)_{\varepsilon>0}$. *Then there exists a subsequence of* $(\varphi_\varepsilon)_{\varepsilon>0}$, *which is denoted by the same, and an element* $\varphi_0 \in L^1(\Omega)$ *such that* $\lim_{\varepsilon\searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0$. *If it holds* $\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon)$ *then we obtain moreover* $\lim_{\varepsilon\searrow 0} j_\varepsilon^P(\varphi_\varepsilon) = j_0^P(\varphi_0)$ *and* φ_0 *is a minimizer of* j_0^P.

We can follow the arguments of Theorem 2 by making in particular use of Lemma 3. The only point that has to be treated more carefully is the construction of the recovery sequence, since we have to ensure that the condition $\varphi_\varepsilon|_{M_i} = 1$ is not violated. And so we will need the following adapted version of [29, Lemma 1]:

Lemma 10. *Let* E *be a measurable subset of* Ω. *If* $(E \setminus \bigcup_{i=1}^m M_i)$ *and* $\Omega \setminus E$ *both contain a non-empty open ball and* $\bigcup_{i=1}^m M_i \subset E$, *then there exists a sequence* $(E_n)_{n\in\mathbb{N}}$ *of open subset of* Ω *such that* $\partial E_n \cap \Omega \in C^2$, $\lim_{n\to\infty} |E_n \Delta E| = 0$, $\lim_{n\to\infty} P_\Omega(E_n) = P_\Omega(E)$, $|E_n| = |E|$ *for* n *large enough, and*

$$\bigcup_{i=1}^m M_i \subseteq E_n, \qquad d(\partial M_i \cap \Omega, \partial E_n \cap \Omega) > 0 \quad \forall n \gg 1, i = 1, \ldots, m.$$

Moreover, we get the convergence rate $|E_n \Delta E| = \mathcal{O}\left(n^{-1}\right)$.

Proof. We adapt the construction of [29, proof of Lemma 1] and roughly sketch the modifications of this proof. We distinguish between two cases:

- *1st case:* Assume that $d(\partial M_i \cap \Omega, \partial E \cap \Omega) > 0$ for all $i = 1, \ldots, m$.
  We define $\varphi := \chi_E$, choose standard mollifiers $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}\phi_\varepsilon \subseteq B_\varepsilon(0)$, $\phi_\varepsilon \geq 0$, $\int_{\mathbb{R}^d} \phi_\varepsilon \,\mathrm{d}x = 1$ and define $\varphi_\varepsilon := \varphi * \phi_\varepsilon$. We then choose the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}} \subset (0,1)$ as in [29, Lemma 1] and define $E_n := \{x \in \Omega \mid \varphi_{\varepsilon_n}(x) > t_n\}$, $F_n := \{x \in \mathbb{R}^d \mid \varphi_{\varepsilon_n}(x) > t_n\}$. Remark, that we may alter E_n by in-or excluding, respectively, balls of certain radii in order to obtain $|E_n| = |E|$ for $n \gg 1$, see [29]. Denoting $M := \bigcup_{i=1}^m M_i$ we obtain that for almost every $x \in \overline{M}$ there exists some $n(x)$ such that $x \in \operatorname{int} F_n$ for all $n \geq n(x)$ and so $\overline{M} \subseteq \bigcup_{x\in\overline{M}} \operatorname{int} F_{n(x)}$. Since $\overline{M}$ is compact, we can choose finitely many $\{F_{n(x_i)} \mid x_i \in \overline{M}, i = 1, \ldots, N\}$ such that $\overline{M} \subseteq \bigcup_{i=1}^N \operatorname{int} F_{n(x_i)}$. Defining $\overline{n} := \max_{i=1,\ldots,N} n(x_i)$ we see that $\overline{M} \subseteq \operatorname{int} F_n$ for all $n \geq \overline{n}$. Then the statement follows from the fact $E_n = F_n \cap \Omega$ and the corresponding parts in the proof of [29, Lemma 1].

- *2nd case:* Now assume we have a general E fulfilling the assumptions of the lemma. Then we choose some $\varepsilon > 0$ such that $B_\varepsilon(E) \cap \Omega \subset \Omega$, $\overline{E} \subsetneq B_\varepsilon(E) \cap \Omega$ and define $F_\varepsilon := B_\varepsilon(E) \cap \Omega$. Using that $d(\partial M_i \cap \Omega, \partial F_\varepsilon \cap \Omega) > 0$ for all $i = 1, \ldots, m$ we find from the first case of the proof that there exists a sequence $(E_n^\varepsilon)_{n\in\mathbb{N}}$ such that the statements of the lemma are fulfilled for E replaced by F_ε. But we do not want the volume of E_n^ε to equal F_ε but merely this of E, which is smaller, and hence we define $\widetilde{E}_n^\varepsilon := E_n^\varepsilon \setminus B_{r_\varepsilon}(x_1)$ with r_ε such that $|B_{r_\varepsilon}(x_1)| = |E_n^\varepsilon| - |E| = \mathcal{O}(\varepsilon)$ and $x_1 \in E \setminus \bigcup_{i=1}^m M_i$ such that $B_{\delta_0}(x_1) \subset E \setminus \bigcup_{i=1}^m M_i$ for some $\delta_0 > 0$. One then obtains by direct calculations and the results of [29] that a diagonal sequence $\left(\widetilde{E}_{n_\varepsilon}^\varepsilon\right)_{\varepsilon>0}$ fulfills the statements of the lemma.

For more details we refer to [27, Lemma 21.1]. □

Proof of Theorem 7. We can follow the arguments of the proof of Theorem 2. We only give some details on the construction of the recovery sequence in the first step of this proof. We approximate for some $\varphi \in L^1(\Omega)$ with $j_0(\varphi) < \infty$ the set $E^\varphi := \{\varphi = 1\}$ by the sets $(E_n)_{n\in\mathbb{N}}$ given by Lemma 10. This ensure in particular that $d_n := d(\bigcup_{i=1}^m \partial M_i \cap \Omega, \partial E_n \cap \Omega) > 0$ for $n \gg 1$. An analogous construction as in Theorem 2 gives for every $n \gg 1$ sequence $(\varphi_\varepsilon^n)_{\varepsilon>0} \subset H^1(\Omega)$ such that

$$\limsup_{\varepsilon\searrow 0} \int_\Omega \left(\frac{\varepsilon}{2} |\nabla \varphi_\varepsilon^n|^2 + \frac{1}{\varepsilon}\psi(\varphi_\varepsilon) \right) \mathrm{d}x \le c_0 P_\Omega(E_n).$$

We observe from this construction in particular that

$$\{\varphi_\varepsilon^n = 1\} \subset E_n, \quad d_\varepsilon^n := d(\{\varphi_\varepsilon n = 1\}, \partial E_n \cap \Omega) = \mathcal{O}(\varepsilon).$$

And so if we choose $\varepsilon_n^0 > 0$ such that $d_{\varepsilon_0^n}^n < d_n$, which implies $d_\varepsilon^n < d_n$ for all $\varepsilon < \varepsilon_n^0$, we find $M_i \subset \{\varphi_\varepsilon^n = 1\}$ for all $\varepsilon < \varepsilon_n^0$ and all $i = 1, \ldots, m$.
We then choose the diagonal sequence $\left(\varphi_{\varepsilon_n}^n\right)_{n\in\mathbb{N}}$ such that $\varepsilon_n < \varepsilon_n^0$. This diagonal sequence is hence admissible for the diffuse interface problem and we can proceed as in the proof of Theorem 2.
In particular we can always deduce the convergence of the pressure in $L^2(\Omega)$ from the convergence of the velocity fields in $\boldsymbol{H}^1(\Omega)$ by using pressure estimates as in Remark 13 and the fact that $\varphi_\varepsilon|_{M_i} = 1$ implies $\alpha_\varepsilon(\varphi_\varepsilon) = 0$ in M_i. □

6.4 Optimality conditions

As in the previous sections, we can derive optimality conditions by geometric variations in the setting including pressure terms in the objective functional, too. For this purpose we have to assume the differentiability assumptions **(A7)** of Section 5 together with

(A7) Assume that $h_M : \mathbb{R}^m \to \mathbb{R}$ is differentiable and that there is some constant $C > 0$ such that $|\mathrm{D}h_M(v)| \le C|v|$ for all $v \in \mathbb{R}^m$.

Remark 14. *If Assumption **(A7)** is fulfilled, we find that* $H : L^2(\Omega) \to \mathbb{R}$, *defined in* (67), *is differentiable with* $\mathrm{D}H(p)(q) = \int_\Omega \mathrm{D}h(p)q\,\mathrm{d}x$ *for all* $p, q \in L^2(\Omega)$, *compare [36].*

For the geometric variations we use transformations $T \in \mathcal{T}_{ad}^p$ which are to be defined by the ordinary differential equation (38) associated to some velocity fields $V \in \mathcal{V}_{ad}^p$. The set $\mathcal{V}_{ad}^p$ is given as

$$\mathcal{V}_{ad}^p := \{V \in \mathcal{V}_{ad} \mid V(t,x) = 0 \text{ for every } x \in M_i, i = 1, \ldots, m\}.$$

Thus we do not vary the domains M_i, which are assumed to be part of the fluid region and hence do not have to be changed. Then we find:

Theorem 8. *For any minimizer* $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon) \in \Phi_p \times \boldsymbol{U} \times L_M^2(\Omega)$ *of* (68)-(70) *the following necessary optimality conditions are fulfilled:*

$$\partial_t|_{t=0} j_\varepsilon^P\left(\varphi_\varepsilon \circ T_t^{-1}\right) = -\lambda_\varepsilon \int_\Omega \varphi_\varepsilon \operatorname{div} V(0)\,\mathrm{d}x, \quad \lambda_\varepsilon \left(\int_\Omega \varphi_\varepsilon\,\mathrm{d}x - \beta|\Omega| \right) = 0 \tag{84}$$

for all $T \in \mathcal{T}_{ad}^p$ *with velocity* $V \in \mathcal{V}_{ad}^p$, *where* $\lambda_\varepsilon \ge 0$ *is a Lagrange multiplier for the integral constraint and the derivative is given by the following formula:*

$$\partial_t|_{t=0} j_\varepsilon^P\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right) + \int_\Omega \mathrm{D}h(p_\varepsilon)\,\dot{p}_\varepsilon[V] + h(p_\varepsilon) \operatorname{div} V(0)\,\mathrm{d}x. \tag{85}$$

Here $\dot{\boldsymbol{u}}_\varepsilon[V] \in \boldsymbol{H}_0^1(\Omega)$ *is given as the solution of* (41)-(42) *and* $\dot{p}_\varepsilon[V] \in L^2(\Omega)$ *with* $\dot{p}_\varepsilon[V] = 0$ *in* $\Omega \setminus \cup_{i=1}^m M_i$ *is the pressure associated to* $\dot{\boldsymbol{u}}_\varepsilon[V]$ *by* (41)−(42) *as described in Remark 12 where the mean value is here chosen according to*

$$\int_{M_i} \dot{p}_\varepsilon[V]\,\mathrm{d}x = -\int_{M_i} p_\varepsilon \operatorname{div} V(0)\,\mathrm{d}x \quad \forall i = 1,\dots,m. \tag{86}$$

Proof. To prove that $\mathbb{R} \supseteq I \ni t \mapsto (p_\varepsilon(t) \circ T_t) \in L^2(\Omega)$ is differentiable at $t = 0$, if $(\boldsymbol{u}_\varepsilon(t), p_\varepsilon(t)) := \boldsymbol{S}_\varepsilon^P\left(\varphi_\varepsilon \circ T_t^{-1}\right)$, we can apply the differentiability result for implicit function equations [37, Theorem 6] to

$$F : I \times \{p \in L^2(\Omega) \mid p|_{\Omega \setminus \cup_{i=1}^m M_i} = 0\} \to \bigtimes_{i=1}^m H^{-1}(M_i) \times \mathbb{R}^m$$

$$F(t,p)(\boldsymbol{z}) = \left(\left(\int_\Omega p\left(\mathrm{D}\boldsymbol{z}_i : \mathrm{D}T_t^{-1}\right)\det \mathrm{D}T_t\,\mathrm{d}x\right)_{i=1}^m, \left(\int_{M_i} p \det \mathrm{D}T_t\,\mathrm{d}x\right)_{i=1}^m\right).$$

We then see that $F(t, p_\varepsilon(t) \circ T_t) = f(t)$ for t small enough and some appropriate chosen function f. The remaining requirements for [37, Theorem 6] can be verified quite easily, compare [27, Theorem 19.2].
For the rest of the proof we can follow the arguments of Theorem 3, where in particular also a formula for $\partial_t|_{t=0} j_\varepsilon\left(\varphi_\varepsilon \circ T_t^{-1}\right)$ is given. □

Correspondingly, we also obtain optimality conditions for the sharp interface problem by geometric variations:

Theorem 9. *For any minimizer* $(\varphi_0, \boldsymbol{u}_0, p_0) \in \Phi_p \times \boldsymbol{U} \times L_M^2(\Omega)$ *of* (80)−(82) *the following necessary optimality conditions are fulfilled:*

$$\partial_t|_{t=0} j_0^P\left(\varphi_0 \circ T_t^{-1}\right) = -\lambda_0 \int_\Omega \varphi_0 \operatorname{div} V(0)\,\mathrm{d}x, \quad \lambda_0\left(\int_\Omega \varphi_0\,\mathrm{d}x - \beta|\Omega|\right) = 0 \tag{87}$$

for all $T \in \mathcal{T}_{ad}^p$ *with velocity* $V \in \mathcal{V}_{ad}^p$*, where* $\lambda_0 \geq 0$ *is a Lagrange multiplier for the integral constraint and the derivative is given by the following formula:*

$$\partial_t|_{t=0} j_0^P\left(\varphi_\varepsilon \circ T_t^{-1}\right) = \partial_t|_{t=0} j_0\left(\varphi_0 \circ T_t^{-1}\right) + \int_\Omega \mathrm{D}h(p_0)\,\dot{p}_0[V] + h(p_0)\operatorname{div} V(0)\,\mathrm{d}x. \tag{88}$$

Moreover $\dot{\boldsymbol{u}}_0[V] \in \boldsymbol{H}_0^1(\Omega)$ *with* $\dot{\boldsymbol{u}}_0[V] = \boldsymbol{0}$ *a.e. in* $\Omega \setminus E_0$ *fulfills* (52)-(53)*and* $\dot{p}_0[V] \in L^2(\Omega)$ *with* $\dot{p}_0[V] = 0$ *in* $\Omega \setminus \cup_{i=1}^m M_i$ *is the pressure associated to* $\dot{\boldsymbol{u}}_0[V]$ *by* (52) − (53) *as described in Remark 12 where the mean value is here chosen according to*

$$\int_{M_i} \dot{p}_0[V]\,\mathrm{d}x = -\int_{M_i} p_0 \operatorname{div} V(0)\,\mathrm{d}x \quad \forall i = 1,\dots,m. \tag{89}$$

Proof. Let's use the notation $(\boldsymbol{u}_0(t), p_0(t)) := \boldsymbol{S}_0^P(\varphi_0(t))$ for t small enough. We know from Theorem 4 that $\mathbb{R} \supset I \ni t \mapsto (\boldsymbol{u}_0(t) \circ T_t) \in \boldsymbol{H}^1(\Omega)$ is differentiable at $t = 0$, if I is a suitable small interval around $t = 0$. Applying the idea of the proof of Theorem 8 to the setting of Theorem 4 we can deduce that $I \ni t \mapsto (p_0(t) \circ T_t) \in L^2(M_i)$ is differentiable at $t = 0$ for all $i = 1,\dots,m$. Then we get by direct calculations and by using the arguments of Theorem 4 the result.

□

Finally, we also obtain the we can pass to the limit $\varepsilon \searrow 0$ in this geometric first variations and obtain a result corresponding to Theorem 5:

Theorem 10. *Let* $(\varphi_\varepsilon)_{\varepsilon>0}$ *be the sequence of minimizers of* $\left(j_\varepsilon^P\right)_{\varepsilon>0}$ *converging to* $\varphi_0 \in L^1(\Omega)$ *given by Theorem 7 and assume that* $\|\varphi_\varepsilon - \varphi_0\|_{L^1(\{x\in\Omega|\varphi_0(x)=1,\varphi_\varepsilon(x)<0\})} = \mathcal{O}(\varepsilon)$. *Then the limit element* φ_0 *is a minimizer of* j_0^P *and it holds* $\lim_{\varepsilon\searrow0}\partial_t|_{t=0}j_\varepsilon\left(\varphi_\varepsilon\circ T_t^{-1}\right) = \partial_t|_{t=0}j_0\left(\varphi_0\circ T_t^{-1}\right)$ *for all* $T\in\mathcal{T}_{ad}^p$. *If* $|\{\varphi_0=1\}|>0$ *then we have additionally the convergence results* (57)*, with* j_ε *and* j_0 *replaced by* j_ε^P *and* j_0^P*, respectively. Additionally, it holds* $\lim_{\varepsilon\searrow0}\|p_\varepsilon - p_0\|_{L^2(\Omega)} = 0$ *and* $\lim_{\varepsilon\searrow0}\|\dot{p}_\varepsilon[V] - \dot{p}_0[V]\|_{L^2(\Omega)} = 0$ *where* $(\boldsymbol{u}_\varepsilon, p_\varepsilon) := \boldsymbol{S}_\varepsilon^P(\varphi_\varepsilon)$ *for all* $\varepsilon\geq0$.

Proof. This can be shown as in Theorem 10, where the pressure terms can all be handled as in the proof of Theorem 7. I.e., we deduce the convergence of $(p_\varepsilon)_{\varepsilon>0}$ and $(\dot{p}_\varepsilon[V])_{\varepsilon>0}$ in $L^2(\Omega)$ from the convergence of the corresponding velocity fields in $\boldsymbol{H}^1(\Omega)$ by using pressure estimates as in Remark 13 and the fact that $\varphi_\varepsilon|_{M_i}=1$ implies $\alpha_\varepsilon(\varphi_\varepsilon)=0$ in M_i. See [27, Theorem 21.2] for more details. □

7 Conclusion and outlook

Summarizing we have found a very general formulation for shape and topology optimization in a Stokes flow. Due to the phase field structure and the porous medium approach this problem can be shown to be well-posed and we arrive in a structure that can be handled with well-known techniques, both mathematically and numerically. In contrast to different formulations we can even use general objective functionals. Additionally, this approach is also applicable to nonlinear state equations like the stationary Navier-Stokes equations, compare [27]. First numerical examples show that this problem is also practicable and the results are comparable to those in literature, see [25]. In addition to the sharp interface limit, we also derived necessary optimality conditions that can be related to classical optimality conditions under suitable regularity assumptions. As also the optimality system can be shown to converge as the phase field parameter tends to zero, we have hence found a consistent approximation of the difficult problem of shape and topology optimization in fluid dynamics which can be used for further investigations in this field.

References

[1] L. Ambrosio and G. Buttazzo. An optimal design problem with perimeter penalization. *Calc. Var. Partial Differential Equations*, 1(1):55–69, 1993.

[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford: Clarendon Press, 2000.

[3] J. Bello, E. Fernández-Cara, J. Lemoine, and J. Simon. The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier–Stokes flow. *SIAM J. Control Optim.*, 35(2):626–640, 1997.

[4] M.P. Bendsøe, R.B. Haber, and C.S. Jog. A new approach to variable-topology shape design using a constraint on perimeter. *Struct. Multidiscip. Optim.*, 11(1-2):1–12, 1996.

[5] L. C. Berselli and P. Guasoni. Some problems of shape optimization arising in stationary fluid motion. *Adv. Math. Sci. Appl.*, 14(1):279–293, 2004.

[6] L. Blank, H. Farshbaf-Shaker, H. Garcke, and V. Styles. Relating phase field and sharp interface approaches to structural topology optimization. *to appear in ESAIM: COCV*, 2014.

[7] J. F. Blowey and C. M. Elliott. The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy Part I: Mathematical analysis. *European J. Appl. Math.*, 2:233–280, 8 1991.

[8] T. Borrvall and J. Petersson. Topology optimization of fluids in Stokes flow. *Internat. J. Numer. Methods Fluids*, 41(1):77–107, 2003.

[9] B. Bourdin and A. Chambolle. Design-dependent loads in topology optimization. *ESAIM Control Optim. Calc. Var.*, 9:19–48, 8 2003.

[10] C. Brandenburg, F. Lindemann, M. Ulbrich, and S. Ulbrich. A Continuous Adjoint Approach to Shape Optimization for Navier Stokes Flow. In K. Kunisch, J. Sprekels, G. Leugering, and F. Tröltzsch, editors, *Optimal Control of Coupled Systems of Partial Differential Equations*, volume 158 of *Internat. Ser. Numer. Math.*, pages 35–56. Birkhäuser, 2009.

[11] D. Bucur and J.P. Zolésio. N-dimensional shape optimization under capacitary constraint. *J. Differential Equations*, 123(2):504 – 522, 1995.

[12] G. Buttazzo and G. Dal Maso. Shape optimization for Dirichlet problems: Relaxed formulation and optimality conditions. *Appl. Math. Optim.*, 23(1):17–49, 1991.

[13] G. Buttazzo, M. Giaquinta, and S. Hildebrandt. *One-dimensional Variational Problems: An Introduction.* Oxford Science Publications, 1998.

[14] G. Dal Maso. *An Introduction to* Γ*-convergence.* Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser, 1993.

[15] G. Dal Maso and U. Mosco. Wiener's criterion and Γ-convergence. *Appl. Math. Optim.*, 15(1):15–63, 1987.

[16] M.C. Delfour and J.P. Zolésio. *Shapes and Geometries: Analysis, Differential Calculus, and Optimization.* Adv. Des. Control. SIAM, 2001.

[17] M.C. Delfour and J.P. Zolésio. Shape derivatives for nonsmooth domains. In K.-H. Hoffmann and W. Krabs, editors, *Optimal Control of Partial Differential Equations*, volume 149 of *Lecture Notes in Control and Inform. Sci.*, pages 38–55. Springer, 1991.

[18] M.C. Delfour and J.P. Zolésio. Uniform fat segment and cusp properties for compactness in shape optimization. *Appl. Math. Optim.*, 55(3):385–419, 2007.

[19] L.C. Evans and R.F. Gariepy. *Measure Theory and Fine Properties of Functions.* Mathematical Chemistry Series. CRC PressINC, 1992.

[20] A. Evgrafov. The Limits of Porous Materials in the Topology Optimization of Stokes Flows. *Appl. Math. Optim.*, 52(3):263–277, 2005.

[21] A. Evgrafov. Topology optimization of slightly compressible fluids. *ZAMM Z. Angew. Math. Mech.*, 86(1):46–62, 2006.

[22] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations.* Springer, 2011.

[23] H. Garcke. The Γ-limit of the Ginzburg-Landau energy in an elastic medium. *AMSA*, 18:345–379, 2008.

[24] H. Garcke and C. Hecht. A phase field approach for shape and topology optimization in Stokes flow. *Preprint-Nr.: 09/2014, Universität Regensburg, Mathematik*, 2014.

[25] H. Garcke, C. Hecht, M. Hinze, and C. Kahle. Numerical approximation of phase field based shape and topology optimization for fluids. *arXiv:1405.3480*, 2014.

[26] E. Giusti. *Minimal surfaces and functions of bounded variation.* Notes on pure mathematics. Dept. of Pure Mathematics, 1977.

[27] C. Hecht. *Shape and topology optimization in fluids using a phase field approach and an application in structural optimization.* Dissertation, University of Regensburg, 2014.

[28] S. Luckhaus and L. Modica. The Gibbs-Thompson relation within the gradient theory of phase transitions. *Arch. Ration. Mech. Anal.*, 107(1):71–83, 1989.

[29] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Ration. Mech. Anal.*, 98(2):123–142, 1987.

[30] B. Mohammadi and O. Pironneau. Shape optimization in fluid mechanics. *Annu. Rev. Fluid Mech.*, 36:255–279, 2004.

[31] F. Murat. Contre-exemples pour divers problèmes où le contrôle intervient dans les coefficients. *Annali di Matematica Pura ed Applicata*, 112(1):49–68, 1977.

[32] O. Pironneau. On optimum profiles in Stokes flow. *J. Fluid Mech.*, 59:117–128, 5 1973.

[33] P. I. Plotnikov and J. Sokolowski. Shape derivative of drag functional. *SIAM J. Control Optim.*, 48(7):4680–4706, 2010.

[34] P.I. Plotnikov and J. Sokolowski. Domain dependence of solutions to compressible navier-stokes equations. *SIAM J. Control Optim.*, 45(4):1165–1197, 2006.

[35] S. Schmidt and V. Schulz. Shape Derivatives for General Objective Functions and the Incompressible Navier–Stokes Equations. *Control Cybernet.*, 39(3):677–713, 2010.

[36] R.E. Showalter. *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations.* Mathematical surveys and monographs, v. 49. American Mathematical Society, 1997.

[37] J. Simon. Domain variation for drag in Stokes flow. In X. Li and J. Yong, editors, *Control Theory of Distributed Parameter Systems and Applications*, volume 159 of *Lecture Notes in Control and Inform. Sci.*, pages 28–42. Springer, 1991.

[38] H. Sohr. *The Navier-Stokes Equations: An Elementary Functional Analytic Approach.* Birkhäuser, 2001.

[39] J. Sokolowski and J.P. Zolésio. *Introduction to Shape Optimization: Shape Sensitivity Analysis.* Springer, 1992.

[40] P. Sternberg. The effect of a singular perturbation on nonconvex variational problems. *Arch. Ration. Mech. Anal.*, 101(3):209–260, 1988.

[41] L. Tartar. Problemes de controle des coefficients dans des equations aux derivees partielles. In A. Bensoussan and J.L. Lions, editors, *Control Theory, Numerical Methods and Computer Systems Modelling*, volume 107 of *Lecture Notes in Economics and Mathematical Systems*, pages 420–426. Springer, 1975.

[42] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis.* Studies in Mathematics and Its Applications. North-Holland, 1977.

[43] F. Tröltzsch. *Optimale Steuerung partieller Differentialgleichungen.* Vieweg, 2009.

[44] V. Šverák. On optimal design. *J. Maths. Pures Appl.*, 72:537–551, 1993.