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Abstract We show that the number of numerical semigroups containing two given coprime numbers p and q agrees with a quasipolynomial in q of degree exactly $p-1$ and having constant leading coefficient lying between $\frac{1}{(p-1)!\cdot p!}$ and $\frac{1}{(p-1)\cdot p!}$.

Keywords Numerical semigroup, Ehrhart quasipolynomial, rational polytope, lattice path

1 Introduction

A numerical semigroup is by definition a submonoid (i. e. a subsemigroup containing 0) of $\mathbb{N}$ whose complement in $\mathbb{N}$ is finite.

For coprime numbers $p, q \in \mathbb{N}_{>0}$ there are only finitely many numerical semigroups containing p and q. It is natural to ask for the precise number $n(p, q)$ of such semigroups.

Theorem 1.1. *Let $p \in \mathbb{N}_{\geq 2}$.*

a) $n(p, q)$ agrees (where it is defined, i. e. for coprime p, q) with a quasipolynomial in q of degree exactly $p-1$. The leading coefficient $\lambda(p)$ of this quasipolynomial is constant.

b) The function $n(p, q)$ is increasing in both variables (cf. [4, 4.2]).

c)

$$\frac{1}{(p-1)!\cdot p!} \leq \lambda(p) \leq \frac{1}{(p-1)\cdot p!}.$$

Proof: This follows from 2.3, 2.4, 2.5, 3.2, 3.3 and 4.1. □

Remark 1.2. *a) For a result similar to 1.1 a) see [4, 3.7].*

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b) *For $p = 2, 3$ the upper and lower bound in 1.1 c) coincide. $\lambda(4)$ was determined in [4, 5.5]; it attains the upper bound. For $p = 5, 6, 7$, $\lambda(p)$ lies strictly between the lower and upper bound. For details, see Examples 4.3.*

Recall from [7, section 4.4] that a quasipolynomial of degree d is a function $f : \mathbb{N} \to \mathbb{C}$ of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \ldots + c_0(n)$$

with periodic functions c_i having integer periods, $c_d \neq 0$.

An important example of a quasipolynomial which we will also need is the following one: Given a d-dimensional rational convex polytope $\mathcal{P} \subseteq \mathbb{R}^p$ (in the sense of [7, p. 493], i. e. as the convex hull of finitely many rational points), one might expect that the number of lattice points of $n \cdot \mathcal{P}$ in $\mathbb{Z}^p$ behaves more or less like a polynomial of degree d in n. The precise statement is

Theorem. *(Ehrhart's theorem; [7, Theorem 4.6.8])*
Let $\mathcal{P} \subseteq \mathbb{R}^p$ be a d-dimensional rational convex polytope.

$$i(\mathcal{P}, n) := \text{ number of lattice points of } n \cdot \mathcal{P}$$

is a quasipolynomial of degree d.

Remark 1.3. *We will show that $n(p, q)$ agrees with the Ehrhart quasipolynomial in q of some $(p-1)$-dimensional rational polytope $\mathcal{P} \subseteq \mathbb{R}^{p-1} \times \{1\}$. In other words, for the cone $\mathcal{A}_p$ over $\mathcal{P}$ with vertex in the origin, $n(p, q)$ equals*

$$i(\mathcal{P}, q) = \#\left(\mathcal{A}_p \cap \{x_p = q\} \cap \mathbb{N}^p\right)$$

(lattice points in $\mathcal{A}_p$ whose last coordinate is q). For similar considerations see [2] and [6].

In order to prove theorem 1.1 it will also be useful to identify the set of lattice points in $\mathcal{A}_p$ with a certain class of lattice paths:

2 Small lattice paths

For $p \in \mathbb{N}_{>0}$, we consider the following system of homogenous linear inequalities

$$x_i + x_j \leq \begin{cases} x_{i+j} & \text{if } i + j \leq p \\ x_{i+j-p} + x_p & \text{if } i + j > p \end{cases} \quad (i, j = 1, \ldots, p-1) \tag{1}$$

in the indeterminates $x_1, \ldots, x_p$ and its solution set over $\mathbb{R}_{\geq 0}$

$$\mathcal{A}_p := \left\{ x = (x_1, \ldots, x_p) \in \mathbb{R}_{\geq 0}{}^p \,|\, x \text{ satisfies (1)} \right\},$$

which is a cone.

Proposition 2.1. *1. Every solution $x \in \mathbb{R}^p$ of (1) with $x_1 \geq 0$ is in $\mathcal{A}_p$.*

2. Each $x \in \mathcal{A}_p$ satisfies

$$x_i \leq x_{i+1} \text{ and } px_i \leq ix_p \text{ for } i = 1, \ldots, p-1. \tag{2}$$

3. For $p \geq 3$, the conditions "$x_i + x_j \leq x_p$ if $i + j = p$" are redundant in (1).

The easy proof is left to the reader.

In particular, the set

$$\mathcal{P} := \mathcal{A}_p \cap \{x_p = 1\}$$

is bounded and hence a rational convex polytope. $\mathcal{A}_p$, which is clearly the cone over $\mathcal{P}$, is p-dimensional, since it contains the points $(1, 2, \ldots, p), (0, 1, \ldots, p-1), \ldots, (0, \ldots, 0, 1, 2), (0, \ldots, 0, 1)$. Therefore, $\mathcal{P}$ has dimension $p - 1$. Now, Ehrhart's theorem ([7, Theorem 4.6.8]) says that

$$i(\mathcal{P}, q) = \#\left(\mathcal{A}_p \cap \mathbb{N}^p \cap \{x_p = q\}\right)$$

is a quasipolynomial of degree $p-1$. Its leading coefficient is a nonzero periodic function which we denote by $\lambda(p)$.

By a lattice path we shall mean a path in the lattice $\mathbb{Z}^2$ with unit steps to the right and down. Let $q \in \mathbb{N}$ and denote by $\Lambda(p, q)$ the set of all lattice paths from $(0, p)$ to $(q, 0)$.

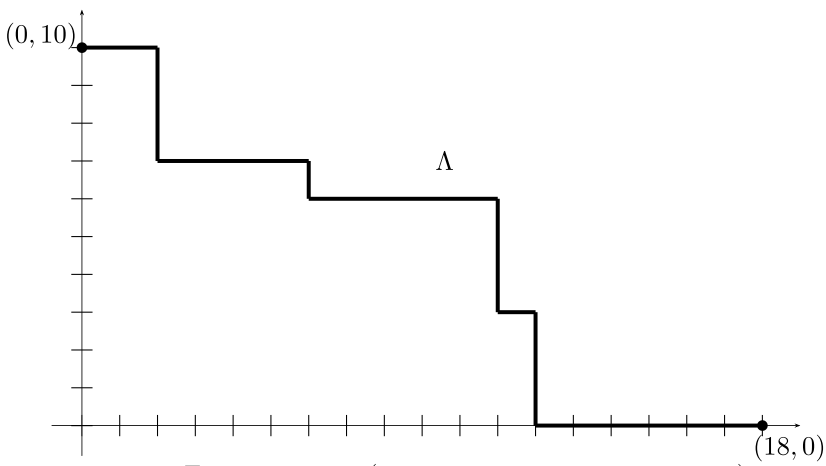

Figure 1: $x_\Lambda = (2, 2, 2, 6, 11, 11, 11, 12, 12, 12, 18)$

Let $\Lambda \in \Lambda(p, q)$. For $i = 0, \ldots, p$, let $P_i = (x_i, p - i)$ be the unique point lying on Λ with height $p - i$ and maximal x-coordinate. Following essentially [5, Ch. 1.6], Λ is uniquely determined by its **vector representation**

$$x_\Lambda := (x_0, \ldots, x_{p-1}, x_p = q).$$

It is easy to see that this defines a bijection

$$x : \bigcup_{q\in\mathbb{N}} \Lambda(p,q) \to \{x \in \mathbb{N}^{p+1} | x_0 \leq \ldots \leq x_p\}.$$

This bijection is even order-preserving, where we take the usual partial order for tuples on the right-hand side and for the paths on the left-hand side we take the partial order induced by inclusion in the following sense:

$$\Lambda_1 \leq \Lambda_2 :\iff \overline{\Lambda_1} \subseteq \overline{\Lambda_2}$$

for two paths Λ_1, Λ_2 starting at $(p, 0)$ and where for a given path $\Lambda \in \Lambda(p,q)$ we denote by $\overline{\Lambda} \subseteq \mathbb{N}^2$ the set of all lattice points in $\mathbb{N}^2$ lying either beneath Λ or on Λ itself:

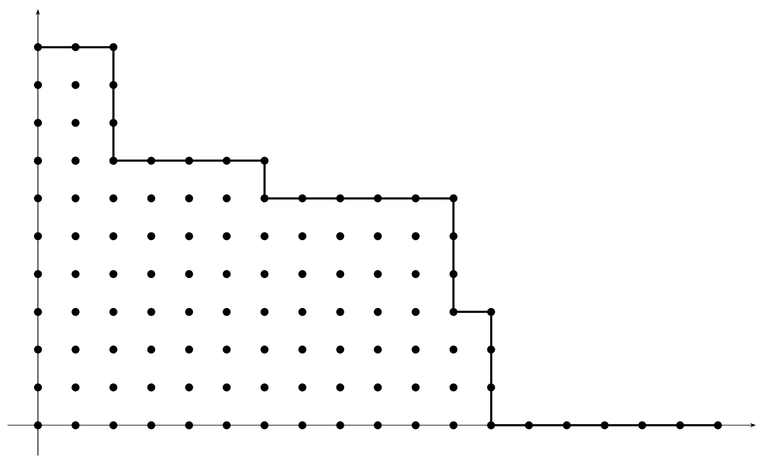

Figure 2: $\overline{\Lambda}$ = set of dots

Note that this order relation $\Lambda_1 \leq \Lambda_2$ corresponds essentially to the notion of domination of the two paths, see [5, 1.6].

The cyclic group $\mathbb{Z}_{p+q}$ of order $p+q$ operates on $\Lambda(p,q)$ by cyclically permuting the steps of a given path (cyclic permutations of paths are also used in [5, p. 8]).

Definition. *We call a given path $\Lambda \in \Lambda(p,q)$* **small** *if Λ is a (necessarily unique) smallest element in its orbit under the above group operation of $\mathbb{Z}_{p+q}$ (notice that not every orbit has a smallest element). By $\Lambda_{small}(p,q)$ we denote the set of all small paths from $(0,p)$ to $(q,0)$. By $\Lambda_{\downarrow}(p,q)$ we denote all paths from $(0,p)$ to $(q,0)$ which start with a step downward.*

It is an easy exercise to see that $\Lambda_{\text{small}}(p,q) \subseteq \Lambda_{\downarrow}(p,q)$. By 'repeating' a given $\Lambda \in \Lambda_{\downarrow}(p,q)$ we get the lattice path $\Lambda_2 := ((0,p)+\Lambda) \cup ((q,0)+\Lambda)$ in $\Lambda_{\downarrow}(2p,2q)$. Note that one trivially has

$$\overline{\Lambda_2} \subseteq \overline{\Lambda} + \overline{\Lambda} := \{v + w | v, w \in \overline{\Lambda}\}.$$

One can check that Λ is small precisely if $\overline{\Lambda_2} = \overline{\Lambda} + \overline{\Lambda}$ holds.

Remark 2.2. *For the smallness of a given $\Lambda \in \Lambda_\downarrow(p, q)$ one only has to show that $P + Q \in \overline{\Lambda_2}$ holds for those lattice points P, Q on Λ which lie behind a step to the right and before a downward step.*

By $\lambda_{\text{small}}(p, q)$ we denote the number of small lattice paths from $(0, p)$ to $(q, 0)$. If we start with a small lattice path from $(0, p)$ to $(q, 0)$ and adhere at its end one additional step to the right it is clear that the resulting lattice path from $(0, p)$ to $(q + 1, 0)$ is again small (e. g. by remark 2.2). In particular:

Remark 2.3. *$\lambda_{small}(p, q)$ is increasing in q.*

Since every small path $\Lambda \in \Lambda_{\text{small}}(p, q)$ starts with a step downward, the first coordinate x_0 of its vector representation is zero and we can omit it: With the notation from above, we set for $\Lambda \in \Lambda_\downarrow(p, q)$

$$x'_\Lambda := (x_1, \ldots, x_{p-1}, x_p = q).$$

By remark 2.2, a given $\Lambda \in \Lambda_\downarrow(p, q)$ is small precisely if $P_i + P_j \in \overline{\Lambda_2}$ holds for $i, j = 1, \ldots, p - 1$. Because of $x'_{\Lambda_2} = (x_1, \ldots, x_p, x_1 + x_p, \ldots, x_p + x_p)$, this means that Λ is small if and only if $x'_\Lambda \in \mathcal{A}_p$. We have thus shown (note that the basic idea of relating paths to lattice points on polyhedra is already mentioned in [5, p. 19]):

Lemma 2.4. *In the above situation, x'_Λ induces a bijection*

$$x' : \text{Set of small lattice paths from } (0, p) \text{ to } (q, 0) \to \mathcal{A}_p \cap \mathbb{N}^p \cap \{x_p = q\}.$$

In particular, $i(\mathcal{P}, q) = \lambda_{small}(p, q)$.

Furthermore, since $\lambda_{\text{small}}(p, q)$ is increasing in q, elementary calculus shows

Lemma 2.5. *The leading coefficient $\lambda(p)$ of the quasipolynomial $i(\mathcal{P}, q)$ is constant.*

Remark 2.6. *We get a first upper bound for $\lambda(p)$ by considering the set $\Lambda_0(p, q)$ of all paths from $(0, p)$ to $(q, 0)$ lying completely in the triangle defined by $(0, p)$, $(q, 0)$ and $(0, 0)$. Clearly,*

$$\Lambda_0(p, q) \subseteq \Lambda_\downarrow(p, q)$$

and a given path $\Lambda \in \Lambda_\downarrow(p, q)$ is in $\Lambda_0(p, q)$ if and only if $x = x'_\Lambda$ satisfies (2). By proposition 2.1,

$$\Lambda_{small}(p, q) \subseteq \Lambda_0(p, q). \tag{3}$$

For coprime p and q each orbit of $\mathbb{Z}_{p+q}$, operating on $\Lambda(p, q)$, consists of precisely $p+q$ elements and contains exactly one element of $\Lambda_0(p, q)$, hence $\#\left(\Lambda_0(p, q)\right) = \frac{1}{p+q}\binom{p+q}{p}$. This can be seen by the method of 'penetrating analysis' explained in [5, Section 1.4] (note that [5, p. 12] also contains recursive formulas for arbitrary p, q). Since $\frac{1}{p+q}\binom{p+q}{p} = \frac{(p+q-1)\cdot\ldots\cdot(q+1)}{p!}$ is a polynomial in q of degree $p - 1$ and leading coefficient $\frac{1}{p!}$, (3) implies that $\lambda(p) \leq \frac{1}{p!}$. However, Proposition 3.2 below gives a sharper upper bound for $\lambda(p)$.

3 Upper and lower bounds for $\lambda(p)$

Lemma 3.1. *Let $f = f(n), g = g(n) : \mathbb{N} \to \mathbb{R}$ be quasipolynomials of the same degree $d \geq 1$, both with constant leading coefficient c_f resp. c_g. Let $\Delta : \mathbb{N} \to \mathbb{R}$ be a function whose absolute value is bounded above by a polynomial of degree $d-1$ and such that, for $n \gg 0$ one has $f(n) + \Delta(n) \leq g(n)$. Then*

$$c_f \leq c_g.$$

Proof: Clear from elementary analysis. □

Note that lemma 3.1 remains valid if f and g are only quasipolynomials for $n \gg 0$; similarly, it remains also valid if the absolute value of their difference is bounded above by a polynomial of degree $d-1$ only for $n \gg 0$.

For the rest of this section, we assume $p \geq 2$.

Proposition 3.2.

$$\lambda(p) \leq \frac{1}{(p-1)\cdot p!}.$$

Proof: We call two functions

$$f = f(n), g = g(n) : \mathbb{N} \to \mathbb{R}$$

'$(p-2)$-equivalent' if the absolute value of their difference is bounded above by a polynomial of degree $p-2$.

We identify the paths Λ in $\Lambda_{\downarrow}(p,q)$ with their truncated vector representations $x'_\Lambda = x = (x_1, \ldots, x_p)$. Obviously the number of small paths x with $x_1 = 0$ is a quasipolynomial in q of degree $p-2$. Therefore,

(A) $\lambda_{\text{small}}(p,q)$ and the cardinality of

$$\Lambda'_{\text{small}}(p,q) = \Lambda_{\text{small}}(p,q) \cap \{x_1 \geq 1\}$$

(as a function of q) are quasipolynomials of the same degree $p-1$ and the same constant leading coefficient $\lambda(p)$.

Furthermore, because of the inequalities (1),

$$\Lambda'_{\text{small}}(p,q) \subseteq \Lambda'(p,q) := \{x \in \mathbb{N}^p | 1 \leq x_1 < \ldots < x_p = q\}$$

where

$$\#\left(\Lambda'(p,q)\right) = \binom{q-1}{p-1}.$$

For $q \geq p$, this is a polynomial in q with leading coefficient $\frac{1}{(p-1)!}$.

We introduce (similarly to [5, Section 1.6]) differences

$$r_i = x_i - x_{i-1} \quad (i = 1, \ldots, p; x_0 := 0)$$

and identify the paths from $\Lambda'(p,q)$ with the p-compositions $(r_1, \ldots, r_p) \in \mathbb{N}^p_{>0}$, $r_1 + \ldots + r_p = q$ of q.

Similarly as above one sees that

(B) The cardinality of

$$\Lambda''_{\text{small}}(p,q) := \Lambda'_{\text{small}}(p,q) \cap \{r_i \neq r_j \text{ for } i \neq j\}$$

is $(p-2)$-equivalent to $\lambda'_{\text{small}}(p,q)$.

Likewise,

(C) The cardinality of

$$\Lambda''(p,q) := \Lambda'(p,q) \cap \{r_i \neq r_j \text{ for } i \neq j\}$$

is $(p-2)$-equivalent to $\lambda'(p,q)$.

The symmetric group $\mathfrak{S}_p$ operates on $\Lambda''(p,q)$ in a natural way, and:

(D) Each orbit consists of $p!$ elements, from which precisely $(p-2)!$ are in

$$\tilde{\Lambda}(p,q) := \{(r_1,\ldots,r_p) \in \Lambda''(p,q) | r_1 \leq r_i \leq r_p \text{ for all } i = 1,\ldots,p-1\}.$$

Furthermore, it is clear from the inequalities (1) that

(E)

$$\Lambda''_{\text{small}}(p,q) \subseteq \tilde{\Lambda}(p,q).$$

Because of (A) – (E), Lemma 3.1 shows that

$$\lambda(p) \leq \frac{1}{(p-1)!} \cdot \frac{(p-2)!}{p!} = \frac{1}{p-1} \cdot \frac{1}{p!}.$$

□

For a suitable $d \in \mathbb{N}$ the polytope $d \cdot \mathcal{P}$ has vertices with integer coefficients. Hence $i(d \cdot \mathcal{P}, q)$ is a polynomial in q of degree $p-1$ with highest coefficient $\text{Vol}_{p-1}(d\mathcal{P})$, see [7, 4.6.13]. It follows that $\lambda(p) = \text{Vol}_{p-1}(\mathcal{P})$ holds as well. Finally, since $\mathcal{P}$ contains the $(p-1)$-simplex Δ with vertices $\frac{1}{p}(1,2,\ldots,p), \frac{1}{p-1}(0,1,\ldots,p-1),\ldots,\frac{1}{2}(0,\ldots,1,2),(0,\ldots,0,1)$, we get

Proposition 3.3.

$$\lambda(p) = \text{Vol}_{p-1}(\mathcal{P}) \geq \text{Vol}_{p-1}(\Delta) = \frac{1}{(p-1)!} \cdot \frac{1}{p!}.$$

Remark 3.4. *For $p \geq 3$, $P_p^* := \{(x_1,\ldots,x_{p-1}) \in \mathbb{R}^{p-1} | x_i + x_j \geq x_{i+j}$ for $i + j \neq p$, with indices reduced modulo $p\}$ is a cone with vertex in the origin and contained in $\mathbb{R}^{p-1}_{\geq 0}$ ([2, Prop. 1.1.a)]). From Proposition 2.1 it follows that $\mathcal{P} = \mathcal{P}_0 \times \{1\}, \mathcal{P}_0 := \left(\frac{1}{p}(1,2,\ldots,p-1) - P_p^*\right) \cap \{x_1 \geq 0\}$, hence*

$$\lambda(p) = \text{Vol}(P_p^* \cap \{x_1 \leq \frac{1}{p}\}).$$

4 Semigroups

For the final step in the proof of theorem 1.1 we relate the numerical semigroups containing p and some q relatively prime to p to the lattice points from $\mathcal{A}_p \cap \mathbb{N}^p \cap \{x_p = q\}$: Given any numerical semigroup H containing p, let

$$\mathcal{AP}(H,p) := \{h \in H | h - p \notin H\}$$

which is called the Apéry set of p in H. We denote by $\mathcal{H}_{pq}$ the set of all numerical semigroups containing p and q..

Let $H \in \mathcal{H}_{pq}$. For $i = 1, \dots p$, let $\widetilde{h_i} = \widetilde{h_i}(H)$ be the smallest number from H with $\widetilde{h_i} \equiv iq \bmod p$. It can be uniquely written in the form

$$\widetilde{h_i} = iq - x_i p \text{ with } x_i \in \mathbb{N}.$$

One has $0 \leq x_1 \leq \ldots \leq x_p = q$, $\widetilde{h_p} = 0$ and $\mathcal{AP}(H,p) = \left\{\widetilde{h_1}, \dots, \widetilde{h_p}\right\}$.

Lemma 4.1. *In the above situation, $H \mapsto \tilde{v}(H) := (x_1, \dots, x_p)$ defines a bijection*

$$\mathcal{H}_{pq} \to \mathcal{A}_p \cap \mathbb{N}^p \cap \{x_p = q\}.$$

Proof: Let $H \in \mathcal{H}_{pq}$. Because of $H = \langle p, \widetilde{h_1}, \dots, \widetilde{h_{p-1}} \rangle_{\mathbb{N}}$, $\tilde{v}$ is injective. By construction of $\tilde{h}(H) := (\widetilde{h_1}, \dots, \widetilde{h_p})$,

$$\widetilde{h_i} + \widetilde{h_j} \geq \begin{cases} \widetilde{h_{i+j}} & \text{if } i + j \leq p \\ \widetilde{h_{i+j-p}} & \text{if } i + j > p \end{cases}. \tag{4}$$

These inequalities can be expressed in terms of $x_1, \dots, x_p$:

- In case $i+j \leq p$: $iq - x_i p + jq - x_j p \geq (i+j)q - x_{i+j}p$, i. e. $x_i + x_j \leq x_{i+j}$.
- In case $i + j > p$: $iq - x_i p + jq - x_j p \geq (i + j - p)q - x_{i+j-p}p$, i. e. $x_i + x_j \leq x_{i+j-p} + q = x_{i+j-p} + x_p$.

This means that $\tilde{v}(H) \in \mathcal{A}_p \cap \mathbb{N}^p \cap \{x_p = q\}$.

Conversely, let $(x_1, \dots, x_p) \in \mathcal{A}_p \cap \mathbb{N}^p \cap \{x_p = q\}$. Set $\widetilde{\widetilde{h_i}} := iq - x_i p$, for $i = 1, \dots, p$. Because of (2), $\widetilde{\widetilde{h_i}} \geq 0$ and, therefore, $H := \langle p, \widetilde{\widetilde{h_1}}, \dots, \widetilde{\widetilde{h_{p-1}}} \rangle \in \mathcal{H}_{pq}$. Similarly to above, (1) for $(x_1, \dots, x_p)$ implies (4) for $(\widetilde{\widetilde{h_1}}, \dots, \widetilde{\widetilde{h_p}})$, i. e. $(\widetilde{\widetilde{h_1}}, \dots, \widetilde{\widetilde{h_p}}) = \tilde{h}(H)$ and, therefore, $(x_1, \dots, x_p) = \tilde{v}(H)$. □

Remark 4.2. *By [3] the semigroups $H \in \mathcal{H}_{pq} \setminus \{\langle p, q \rangle\}$ correspond to certain lattice paths, called "admissible in the (p,q)-system". Moreover the considerations following [3, Lemma 2.2] relate these paths to the small paths from $(0,p)$ to $(q,0)$ not going through $(0,0)$, hence $n(p,q) = \lambda_{small}(p,q)$. Together with Lemma 2.4 this also proves 1.1 a).*

Examples 4.3. *1. In [3, 2.5] it is shown that* $n(2,q) = \frac{q+1}{2}$ *and* $n(3,q) = \lfloor \frac{q^2}{12} + \frac{q}{2} \rfloor + 1$*; hence* $\lambda(2) = \frac{1}{2}$, $\lambda(3) = \frac{1}{12}$.

2. *By [4, 5.5],* $\lambda(4) = \frac{1}{72}$.

3. *Helmut Knebl even computed the quasipolynomials* $n(p,q)$ *for* $p = 4$ *and* $p = 5$*,* p *and* q *coprime (private communication):*

$$n(4,q) = \begin{cases} \frac{1}{72}q^3 + \frac{1}{6}q^2 + \frac{13}{24}q + \frac{5}{18} & \text{if } q \equiv 1 \text{ mod } 6 \\ \frac{1}{72}q^3 + \frac{1}{6}q^2 + \frac{13}{24}q + \frac{1}{2} & \text{if } q \equiv 3 \text{ mod } 6 \\ \frac{1}{72}q^3 + \frac{1}{6}q^2 + \frac{13}{24}q + \frac{7}{18} & \text{if } q \equiv 5 \text{ mod } 6 \end{cases}$$

$$n(5,q) = \begin{cases} \frac{13}{8640}q^4 + \frac{13}{432}q^3 + \frac{31}{144}q^2 + \frac{2}{3}q + \frac{4}{5} & \text{if } q \equiv 0 \text{ mod } 6 \\ \frac{13}{8640}q^4 + \frac{13}{432}q^3 + \frac{59}{288}q^2 + \frac{235}{432}q + \frac{1897}{8640} & \text{if } q \equiv 1 \text{ mod } 6 \\ \frac{13}{8640}q^4 + \frac{13}{432}q^3 + \frac{31}{144}q^2 + \frac{17}{27}q + \frac{83}{135} & \text{if } q \equiv 2 \text{ mod } 6 \\ \frac{13}{8640}q^4 + \frac{13}{432}q^3 + \frac{59}{288}q^2 + \frac{9}{16}q + \frac{171}{320} & \text{if } q \equiv 3 \text{ mod } 6 \\ \frac{13}{8640}q^4 + \frac{13}{432}q^3 + \frac{31}{144}q^2 + \frac{35}{54}q + \frac{88}{135} & \text{if } q \equiv 4 \text{ mod } 6 \\ \frac{13}{8640}q^4 + \frac{13}{432}q^3 + \frac{59}{288}q^2 + \frac{227}{432}q + \frac{3017}{8640} & \text{if } q \equiv 5 \text{ mod } 6 \end{cases}$$

In particular,

$$\frac{1}{4! \cdot 5!} < \lambda(5) = \frac{13}{8640} < \frac{1}{4 \cdot 5!}.$$

4. *Computations using 'polymake' ([1]) show that* $\lambda(6) = \frac{59}{345600} \approx \frac{1}{5858}$ *and* $\lambda(7) = \frac{231349}{15676416000} \approx \frac{1}{67761}$*. Theorem 1.1. c) says that*

$$\frac{1}{86400} \leq \lambda(6) \leq \frac{1}{3600}$$

resp.

$$\frac{1}{3628800} \leq \lambda(7) \leq \frac{1}{30240}.$$

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