$L^p$-spectrum of the Dirac operator
on products with hyperbolic spaces

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\textbf{L}^p\text{-SPECTRUM OF THE DIRAC OPERATOR ON PRODUCTS WITH HYPERBOLIC SPACES}

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\textbf{Abstract.} We study the \textit{L}^p-\text{spectrum of the Dirac operator on complete manifolds. One of the main questions in this context is whether this spectrum depends on \( p \). As a first example where \( p \)\-independence fails we compute explicitly the \textit{L}^p-\text{spectrum for the hyperbolic space and its product with compact spaces.}

1. \textbf{Introduction}

The \textit{L}^p\text{-spectrum of the Laplacian and its \( p \)-(in)dependence was and still is studied by many authors, e.g. in \cite{15}, \cite{16}, \cite{19}. On closed manifolds one easily sees that the spectrum is independent of \( p \in [1, \infty) \). For open manifolds, independence only holds under additional geometric conditions. Hempel and Voigt \cite{19}, \cite{20} proved such results for Schrödinger operators in \( \mathbb{R}^n \) with potentials admitting certain singularities. Then Kordyukov \cite{23} generalized this result to uniformly elliptic operators with uniformly bounded smooth coefficients on a manifold of bounded geometry with subexponential volume growth. Independently, Sturm \cite{28} showed the independence of the \textit{L}^p-spectrum for a class of uniformly elliptic operators in divergence form on manifolds with uniformly subexponential volume growth and Ricci curvature bounded from below. Both results include the Laplacian acting on functions. Later the Hodge-Laplacian acting on \( k \)-forms was considered. E.g. under the assumptions of the result by Sturm from above, Charalambous proved the \textit{L}^p\-independence for the Hodge-Laplacian in \cite{12} Proposition 9. The machinery used to obtain these independence results uses estimates for the heat kernel as in \cite{27}.

In contrast, the \textit{L}^p\text{-spectrum of the Laplacian on the hyperbolic space does depend on \( p \) \cite{14} Theorem 5.7.1]. Its \textit{L}^p\text{-spectrum is the convex hull of a parabola in the complex plane, and this spectrum degenerates only for \( p = 2 \) to a ray on the real axis, cf. Remark 10.1.}

In addition to the intrinsic interest of the \( p \)-independence of the \textit{L}^p\text{-spectrum, such results were used to get information on the \textit{L}^2\text{-spectrum by considering the \textit{L}^1\text{-spectrum, as in particular examples the \textit{L}^p\text{-spectrum can be easier to control. The result of Sturm was used for example by Wang \cite{30} Theorem 3]} to prove that the spectrum of the Laplacian acting on functions on complete manifolds with asymptotically non-negative Ricci curvature is \([0, \infty)\). Explicit calculations for the Laplace-Beltrami operator on locally symmetric spaces were carried out recently by Ji and Weber, see e.g. \cite{22, 31}.

About the \textit{L}^p\text{-spectrum of the Dirac operator much less is known. As before, on closed manifold the spectrum is independent on \( p \in [0, \infty] \). Kordyukov’s methods \cite{23} do not apply directly to the Dirac operator \( D \), but following a remark of \cite{23} Page 224 his methods generalize to suitable systems, and thus also to the square \( D^2 \). Unfortunately, the system case is not completely worked out, but it seems to us, that the case of systems is completely analogous to the case of operators on functions. Assuming this, Kordyukov has shown that the spectrum of \( D^2 \) is \( p \)-independent for \( 1 \leq p < \infty \) on manifolds with bounded geometry and subexponential volume growth. For many such manifolds (e.g. for all such manifolds of even dimension or all manifolds of dimension \( 4k + 1 \)), this already implies the
$p$-independence of the $L^p$-spectrum of $D$, see our Lemma 8 together with the following symmetry considerations.

Many of the results and techniques that were constructed up for Laplace operators are not yet developed for Dirac operators. For the Dirac operator such independence results would not only be of interest on their own, e.g., for (classical) Dirac operators certain $L^p$-spaces and $L^p$-spectral gaps naturally occur when considering a spinorial Yamabe-type problem which was our motivation to enter into this subject, see [4].

In this paper we determine explicitly the $L^p$-spectrum for a special class of complete manifolds – products of compact spaces with hyperbolic spaces. More precisely, we study the following manifolds:

Let $(N^n, g_N)$ be a closed Riemannian spin manifold. Let $M = M_c$ be the product manifold $(M_c^{m,k} = H_c^{k+1} \times N^n, g_M = g_M^{c,k+1} + g_N)$ where $H_c^{k+1}$ is the $(k+1)$-dimensional hyperbolic space scaled such that its scalar curvature is $-c^2 k(k+1)$ for $c \neq 0$ and $H_0^{k+1}$ is the $(k+1)$-dimensional Euclidean space. For those manifolds we obtain the following result which is also illustrated in Figure 1:

**Theorem 1.1.** We use the notions from above. Let $p \in [1, \infty]$, and $c \geq 0$. The $L^p$-spectrum of the Dirac operator on $M_c^{m,k}$ is given by the set

$$\sigma_p := \left\{ \mu \in \mathbb{C} \quad \middle| \quad \mu^2 = \lambda_0^2 + \kappa^2, |\text{Im}\kappa| \leq c k \left| \frac{1}{p} - \frac{1}{2} \right| \right\}$$

where $\lambda_0^2$ is the lowest eigenvalue of $(D^N)^2$, $\lambda_0 \geq 0$, and $D^N$ is the Dirac operator on $(N, g_N)$. In particular, the Dirac operator $D : H^1 \rightarrow L^p$ on $M_c^{m,k}$ has a bounded inverse if and only if $\lambda_0 > c k \left| \frac{1}{p} - \frac{1}{2} \right|$.

For an overview of the structure of the proof, see the end of the introduction.

From the Theorem 1.1 one can directly read off the $L^p$-spectrum of $D^2$ and compare it to the known spectrum of the Laplacian acting on functions which is done in Remark 10.1.

![Figure 1](image-url)
In Section 4 the Dirac operator on the model spaces $M^{m,k}_c$ is written in polar coordinates and the action of $\text{Spin}(k+1)$ on $M^{m,k}_c$ is studied. This is used in Section 5 to prove a certain symmetry property of the Green function on $M^{m,k}_c$ and in Section 6 to study its decay. After all these preparations we are ready to prove the main theorem:

**Structure of the proof of Theorem 1.1**

Section 7. We decompose the Green function into a singular part and a smoothing operator. Using the homogeneity of the hyperbolic space we show in Proposition 7.1 that the singular part gives rise to a bounded operator from $L^p$ to itself for all $p \in [1, \infty]$. In Proposition 7.2 we show that under certain assumptions on the decay of the Green function also the smoothing part gives rise to a bounded operator from $L^p$ to $L^p$ for certain $p$.

Section 8. Using the decay estimate obtained in Section 6 we then see that the $L^p$-spectrum of $M^{m,k}_c$ is contained in the set $\sigma_p$ given in Theorem 1.1. Thus, it only remains to show that under certain assumptions on the decay of the Green function also the smoothing operator has spectrum contained in the set $\sigma_p$.

For that we construct test spinors on $H^{k+1}_c$ in Section 9 and finish the proof for product spaces in Section 10.

2. Preliminaries

2.1. **Notations and conventions.** In the article we will use the convention that a spin manifold is a manifold which admits a spin structure together with a fixed choice of spin structure.

Let $(M, g)$ be a spin manifold and $\Sigma_M$ the corresponding spinor bundle, see Section 2.3.

$\Gamma(\Sigma_M)$ denotes the space of spinors, i.e., sections of $\Sigma_M$. The space of smooth compactly supported sections is denoted by $C^\infty_c(M, \Sigma_M)$, or shortly $C^\infty_c(\Sigma_M)$. The hermitian metric on fibers of $\Sigma_M$ is denoted by $\langle ., . \rangle$, the corresponding norm by $|.|$. For $s_1, s_2 \in \Gamma(M, \Sigma_M)$ we define the $L^2$-scalar product

$$(s_1, s_2)_{L^2(g)} := \int_M \langle s_1, s_2 \rangle \, dvol_g.$$  

For $s \in [1, \infty]$ $\|.|\}_{L^s(g)}$ is the $L^s$-norm on $(M^n, g)$. In case the underlying metric is clear from the context we abbreviate shortly by $\|.|$.

$\text{Spec}_{L^2}(D)$ denotes the spectrum of the Dirac operator on $M$ viewed as an operator from $L^2$ to $L^2$, cf. Appendix B.

We denote by $\pi_i : M \times M \rightarrow M$, $i = 1, 2$, the projection to the $i$-th component. Moreover, we set $\Sigma_M \boxtimes \Sigma_M := \pi_1^*(\Sigma_M) \otimes (\pi_2^*(\Sigma_M))$.

$C^i(M)$ denotes the space of $i$-times continuously differentiable functions on $M$.

$B_r(x) \subset M$ is the ball around $x \in M$ of radius $r$ w.r.t. the metric given on $M$.

A Riemannian manifold is of bounded geometry, if its injectivity radius is positive and the curvature tensor and all derivatives are bounded.

The metric on the $k$-dimensional sphere $S^k$ with constant sectional curvature 1 will be denoted by $\sigma^k$. For $S^k$ with metric $r^2\sigma^k$ we write $S^k_r$.

2.2. **Coordinates and notations for $H^{k+1}_c$ and its product spaces.** We introduce coordinates on $H^{k+1}_c$ by equipping $\mathbb{R}^{k+1}$ with the metric $g_{\mathbb{R}^{k+1}} = dr^2 + f(r)^2 \sigma^k$ where $\sigma^k$ is the standard metric on $S^k$ and

$$f(r) := \sinh_c(r) := \begin{cases} \frac{1}{c} \sinh(cr) & \text{if } c \neq 0 \\ r & \text{if } c = 0. \end{cases}$$

In particular, the distance $\text{dist}_{\mathbb{R}^{k+1}}$ of $y$ to 0 w.r.t. $g_{\mathbb{R}^{k+1}}$ coincides with the euclidean one on $\mathbb{R}^{k+1}$. The subset $\{ y \in H^{k+1}_c \mid \text{dist}_{\mathbb{R}^{k+1}}(y, 0) = r \}$ is isometric to $S^k_{f(r)}$ and its (unnormalized) mean curvature is given by
The spinor bundle \( \Sigma \). Dual spinors.

Let \( M \) be a point. Set \( \mathbb{H}^{k+1} \) be a closed Riemannian spin manifold. Note that we include the case where \( N \) is just a point. Set \( \mathcal{M}^{m,k} := \mathbb{H}^{k+1} \times N \), and \( \pi_{\mathcal{M}} \) shall denote the projection of \( \mathcal{M}^{m,k} \) onto its \( \mathbb{H}^{k+1} \)-coordinates.

### 2.3. General preliminaries about spin geometry

The following can e.g. be found in [17]. A spin structure on \( M \) is a pair \((\mathcal{P}_{\text{Spin}}(M), \alpha)\) where \( \mathcal{P}_{\text{Spin}}(M) \) is a principal \( \text{Spin}(m) \)-bundle and where \( \alpha: \mathcal{P}_{\text{Spin}}(M) \to \mathcal{P}_{\text{SO}}(M) \) is a fiber map over the identity of \( M \) that is compatible with the double covering \( \Theta: \text{Spin}(m) \to \text{SO}(m) \) and the corresponding group actions, i.e., the following diagram commutes

\[
\begin{array}{ccc}
\text{Spin}(m) \times \mathcal{P}_{\text{Spin}}(M) & \longrightarrow & \mathcal{P}_{\text{Spin}}(M) \\
\downarrow & & \downarrow \\
\text{SO}(m) \times \mathcal{P}_{\text{SO}}(M) & \longrightarrow & \mathcal{P}_{\text{SO}}(M)
\end{array}
\]

Let \( \Sigma_m \) be an irreducible representation of \( \text{Cl}_m \). In case \( m \) is odd there are two such irreducible representations. Both of them coincide if considered as \( \text{Spin}(m) \)-representations. If \( m \) is even, there is only one irreducible \( \text{Cl}_m \)-representation of \( \Sigma_m \), but it splits into non-equivalent subrepresentations \( \Sigma_m^{(+)} \) and \( \Sigma_m^{(-)} \) as \( \text{Spin}(m) \)-representations.

Let \( \varepsilon \in \{+,-\} \). We use the notation \( \Sigma_m^{(\varepsilon)} \) if \( m \) is odd as well and set in this case \( \Sigma_m^{(+)} = \Sigma_m \).

The spinor bundle \( \Sigma_M \) is defined as \( \Sigma_M = \mathcal{P}_{\text{Spin}}(M) \times_{\rho_m} \Sigma_m \) where \( \rho_m: \text{Spin}(m) \to \text{End}(\Sigma_m) \) is the complex spinor representation. Moreover, the spinor bundle is endowed with a Clifford multiplication, denoted by \( \cdot' \), \( \cdot: TM \to \text{End}(\Sigma_M) \). Then, the Dirac operator acting on the space of smooth sections of \( \Sigma_M \) is defined as the composition of the connection \( \nabla \) on \( \Sigma_M \) (obtained as a lift of the Levi-Civita connection on \( TM \)) and the Clifford multiplication. Thus, in local coordinates this reads as

\[
D = \sum_{i=1}^{m} e_i \cdot \nabla_{e_i}
\]

where \((e_i)_{i=1,\ldots,m}\) is a local orthonormal basis of \( TM \). The Dirac operator is formally self-adjoint as an operator on \( L^2 \), i.e., for \( \psi \in C^{\infty}(M, \Sigma_M) \) and \( \varphi \in C^\infty_c(M, \Sigma_M) \) we have

\[
(\varphi, D\psi) = (D\varphi, \psi).
\]

As \( M \) is complete, the Dirac operator is not only formally self-adjoint, but actually has a self-adjoint extension that is a densely defined operator \( D: L^2 \to L^2 \), see [KK]. From the spectral theorem it then follows that \( D - \mu: L^2 \to L^2 \) is invertible for all \( \mu \notin \mathbb{R} \).

Define \( \omega_M = i^{\frac{m+1}{2}} e_1 \cdot e_2 \cdots e_m \) with \((e_i)_i\) being a positively oriented orthonormal frame on \( M \). If \( m \) is even, \( \omega_M^2 = 1 \) and the corresponding \( \pm 1 \) eigenspaces are the spaces of so-called positive (resp. negative) spinors.

### 2.4. Dual spinors

The hermitian metric induces a natural isomorphism from \( \Sigma_M^* \) to \( \overline{\Sigma}_M \).

In this way we obtain a metric connection and a Clifford multiplication on \( \Sigma_M^* \) and this allows us to define a Dirac operator \( D^*: C^\infty(\Sigma_M^*) \to C^\infty(\overline{\Sigma}_M^*) \). Locally \( D^*f = \sum_i e_i \cdot \nabla_{e_i} f \) where \( f \in C^\infty(\Sigma_M^*) \) and \( e_i \) is a local orthonormal frame on \( M \). Completely analogously
to the proof that the usual Dirac operator is formally self-adjoint, one proves that for $f \in C^\infty(\Sigma^*_M)$, $\varphi \in C^\infty(\Sigma_M)$ such that $\text{supp } f \cap \text{supp } \varphi$ is relatively compact we have

$$\int \mathcal{D}^* f(\varphi) \text{dvol}_g = \int f(\mathcal{D} \varphi) \text{dvol}_g.$$  

2.5. Spinors on product manifolds. In this subsection our notation is close to [7]. Let $(P^{m+n} = M^m \times N^n, g_P = g_M + g_N)$ be a product of Riemannian spin manifolds $(M, g_M)$ and $(N, g_N)$. We have

$$P_{\text{Spin}}(M \times N) = (P_{\text{Spin}}(M) \times P_{\text{Spin}}(N)) \times \xi \Sigma_{m+n}$$

where $\xi : \text{Spin}(m) \times \text{Spin}(n) \to \text{Spin}(m + n)$ is the Lie group homomorphism lifting the standard embedding $\text{SO}(m) \times \text{SO}(n) \to \text{SO}(m + n)$. Note that $\xi$ is not an embedding, its kernel is $(-1, -1)$, where $-1$ denotes the non-trivial element in the kernel of $\text{Spin}(m) \to \text{SO}(m)$ resp. $\text{Spin}(n) \to \text{SO}(n)$.

The spinor bundle can be identified with

$$\Sigma_P = \left\{ \begin{array}{ll} \Sigma_M \otimes (\Sigma_N \oplus \Sigma_N) & \text{if both } m \text{ and } n \text{ are odd} \\ \Sigma_M \otimes \Sigma_N & \text{else,} \end{array} \right.$$  

and the Levi-Civita connection acts as $\nabla^{\Sigma_M \oplus \Sigma_N} = \nabla^{\Sigma_M} \otimes \text{Id}_{\Sigma_N} + \text{Id}_{\Sigma_M} \otimes \nabla^{\Sigma_N}$. This identification can be chosen such that for $X \in TM, Y \in TN, \varphi \in \Gamma(\Sigma_M)$, and $\psi = (\psi_1, \psi_2) \in \Sigma_N \oplus \Sigma_N$ for both $m$ and $n$ odd and $\psi \in \Gamma(\Sigma_N)$ otherwise, we have

$$(X, Y) \cdot_P (\varphi \otimes \psi) = (X \cdot_M \varphi) \otimes (\omega_N \cdot_N \psi_1) + \varphi \otimes (Y \cdot_N \psi)$$

where for both $n$ and $m$ odd we set $\omega_N \cdot_N (\psi_1, \psi_2) := i(\psi_2, -\psi_1)$ and $Y \cdot_N (\psi_1, \psi_2) := (Y \cdot_N \psi_2, Y \cdot_N \psi_1)$.

The Dirac operator is then given by

$$D^P(\varphi \otimes \psi) = (D^M \varphi \otimes \omega_N \cdot_N \psi) + (\varphi \otimes \tilde{D}^N \psi)$$

where $\tilde{D}^N = \text{diag}(D^N, -D^N)$ if both $m$ and $n$ are odd and $\tilde{D}^N = D^N$ otherwise.

Since $\omega_N$ and $\tilde{D}^N$ anticommute, $D^M \otimes \omega_N$ and $\text{id} \otimes \tilde{D}^N$ anticommute as well. Thus

$$(D^P)^2 = (D^M)^2 \otimes \text{id} + \text{id} \otimes (\tilde{D}^N)^2. \quad (1)$$

2.6. A covering lemma.

**Lemma 2.1** (Covering lemma). Let $(M, g)$ be a Riemannian manifold of bounded geometry, and let $R > 0$. Then there are points $(x_i)_{i \in I} \subset M$ where $I$ is a countable index set such that

(i) the balls $B_R(x_i)$ are pairwise disjoint and

(ii) $(B_{2R}(x_i))_{i \in I}$ and $(B_{3R}(x_i))_{i \in I}$ are both uniformly locally finite covers of $M$.

**Proof.** Choose a maximal family of points $(x_i)_{i \in I}$ in $M$ such that the sets $B_R(x_i)$ are pairwise disjoint. Then $\bigcup_{i \in I} B_{2R}(x_i) = M$. For $y \in M$ let $L(y) = \{ i \in I \mid y \in B_{3R}(x_i) \}$. For $i \in L(y)$ we have $B_R(x_i) \subset B_{4R}(y)$ and, thus,

$$\bigcup_{i \in L(y)} B_R(x_i) \subset B_{4R}(y),$$

where $\sqcup$ denotes disjoint union. Comparing the volumes of both sides and using the bounded geometry of $M$ we see that there exists a number $L_R$ such that $|L(y)| \leq L_R$ for all $y \in M$. Thus, the covering by sets $B_{3R}(x_i)$, and hence the one by $B_{2R}(x_i)$, is uniformly locally finite. \qed
2.7. Interpolation theorems.

**Theorem 2.2** (Riesz-Thorin Interpolation Theorem, [32] Theorem II.4.2). Let $T$ be an operator defined on a domain $D$ that is dense in both $L^q$ and $L^p$. Assume that $Tf \in L^q \cap L^p$ for all $f \in D$ and that $T$ is bounded in both norms. Then, for any $r$ between $p$ and $q$ the operator $T$ is a bounded operator from $L^r$ to $L^r$.

**Theorem 2.3** (Stein Interpolation Theorem, [14] Section 1.1.6, [25] Theorem IX.21). Let $p_0, q_0, p_1, q_1 \in [1, \infty]$, $0 < t < 1$, and $S = \{ z \in \mathbb{C} \mid 0 \leq \text{Re} \, z \leq 1 \}$. Let $A_z$ be linear operators from $L^{p_0} \cap L^{p_1}$ to $L^{q_0} + L^{q_1}$ for all $z \in S$ with the following properties:

1. There is $M_0 > 0$ such that $\|A_z f\|_{p_0} \leq M_0 \|f\|_{p_0}$ for all $f \in L^{p_0} \cap L^{p_1}$ and $y \in \mathbb{R}$.
2. There is $M_1 > 0$ such that $\|A_z f\|_{p_1} \leq M_1 \|f\|_{p_1}$ for all $f \in L^{p_0} \cap L^{p_1}$ and $y \in \mathbb{R}$.

Then, for $1/p = t/p_1 + (1 - t)/p_0$ and $1/q = t/q_1 + (1 - t)/q_0$,

$$\|A_z f\|_q \leq M_1^t M_0^{1-t} \|f\|_p$$

for all $f \in L^{p_0} \cap L^{p_1}$. Hence, $A_t$ can be extended to a bounded operator from $L^p$ to $L^q$ with norm at most $M_1^t M_0^{1-t}$.

3. The Green function

In this section, we collect results on existence and properties of the Green function of the Dirac operator $D$ and its shifts $D - \mu$, $\mu \in \mathbb{C}$. They are obvious applications of standard methods, but a suitable reference does not exist yet. Unless otherwise stated we only assume in this section that the Riemannian spin manifold $(M, g)$ is complete.

**Definition 3.1.** [5] Definition 2.1] A smooth section $G_{D - \mu} : M \times M \setminus \Delta \to \Sigma_M \otimes \Sigma_M$ that is locally integrable on $M \times M$ is called a Green function of the shifted Dirac operator $D - \mu$ if

$$\langle (D_x - \mu)(G_{D - \mu}(x, y)) = \delta_y \text{Id}_{\Sigma_M}|_y \rangle$$

in the sense of distributions, i.e., for any $y \in M$, $\psi_0 \in \Sigma_M|_y$, and $\varphi \in C_c(\Sigma_M)$

$$\int_M \langle G_{D - \mu}(x, y) \psi_0, (D - \mu) \varphi(x) \rangle \, dx = \langle \psi_0, \varphi(y) \rangle$$

and $G_{D - \mu}(\cdot, y) \in L^2(M \setminus B_r(y))$ for any $r > 0$.

In case that the operator $D - \mu$ is clear from the context, we shortly write $G = G_{D - \mu}$.

**Proposition 3.2.** If $M$ is a closed Riemannian spin manifold with invertible operator $D : L^2(\Sigma_M) \to L^2(\Sigma_M)$, then a unique Green function exists.

To prove the well-known proposition, one usually starts by showing the existence of a parametrix.

**Lemma 3.3.** [24] III.§4] Let $M$ be a closed Riemannian spin manifold. Then there is a smooth section $P_{D - \mu} : M \times M \setminus \Delta \to \Sigma_M \otimes \Sigma_M^*$, called parametrix, which is $L^1$ on $M \times M$ and which satisfies

$$\langle (D_x - \mu)(P_{D - \mu}(x, y)) \rangle \delta_y \text{Id}_{\Sigma_M}|_y + R(x, y)$$

in the distributional sense for a smooth section $R : M \times M \to \Sigma_M \otimes \Sigma_M^*$. Convoling with $P_{D - \mu}$ thus defines an operator $P_{D - \mu}$ by

$$\langle P_{D - \mu} \psi, \varphi \rangle = \int_M \int_M \langle P_{D - \mu}(x, y) \psi(y), \varphi(x) \rangle \, dx \, dy$$

for all $\psi, \varphi \in C_c(\Sigma_M)$. Then, $P_{D - \mu}$ is a right inverse to $D - \mu$ up to infinitely smoothing operators. We thus call it a right parametrix. The existence of such a right parametrix
Proof of Proposition 3.2. From the last Lemma we have the existence of a parametrix \( P_{\mu}(x, y) \). We will use the notations of that Lemma. Since \( D - \mu \) is assumed to be invertible, there is a section \( P'_{\mu}(x, y) : M \times M \to \Sigma_M \otimes \Sigma^*_M \) with \((D_x - \mu) P'_{\mu}(x, y) = R(x, y)\). By elliptic regularity \( P'_{\mu}(x, y) \) is smooth in \( x \) and \( y \). We set \( G_{\mu}(x, y) = P_{\mu}(x, y) - P'_{\mu}(x, y) \) and obtain \((D_x - \mu)(G_{\mu}(x, y)) = \delta_y \text{Id}_{\Sigma_M|y}\). Moreover, since \( P_{\mu}(x, y) \) is \( L^1 \) on \( M \times M \) and \( P'_{\mu}(x, y) \) is smooth in both entries the Green function \( G_{\mu}(x, y) \) is \( L^1 \) as well. Furthermore, \( P_{\mu}(x, y) \) is smooth on \( M \setminus B_r(y) \) for any \( r > 0 \) and, hence, the same is true for \( G_{\mu}(x, y) \). In particular, \( G_{\mu}(x, y) \in L^2(M \setminus B_r(y)) \). If \( \tilde G_{\mu}(x, y) \) is a possibly different Green function of \( D - \mu \) then \((D_x - \mu)(G_{\mu}(x, y) - \tilde G_{\mu}(x, y)) = 0 \) for all \( y \in M \). As \( D - \mu \) is invertible we have \( G_{\mu}(x, y) \). 

As for \( P_{\mu}(x, y) \), convolution with \( G_{\mu}(x, y) \) defines an operator \( G_{\mu}(x, y) \) by
\[
(G_{\mu}(x, y)\psi, \varphi) = \int_M \int_M \langle G_{\mu}(x, y)\psi(y), \varphi(x) \rangle \, dx \, dy
\]
for all \( \psi, \varphi \in C_c(\Sigma_M) \). By construction \( G_{\mu}(x, y) \) is the right inverse of \( D - \mu \), and is thus even defined on \( L^2 \). Since the inverse of \( D - \mu \) exists by assumption, \( G_{\mu}(x, y) = (D - \mu)^{-1} \), and \( G_{\mu}(x, y) \) is in particular also a left inverse of \( D - \mu \).

Lemma 3.4. Let \( M \) be a closed Riemannian spin manifold, and let \( D - \mu \) be invertible. Then \( G_{\mu}(x, y) \) is the adjoint of \( G_{\bar \mu}(y, x) \), i.e. \( G_{\mu}(x, y) \) is the integral kernel of the adjoint operator of \( G_{\mu}(x, y) \).

**Proof.** Using the definitions and discussions from above and Lemma 3.3(ii) we have \( G_{\mu}(x, y) = ((D - \mu)^{-1})^* = (D - \bar \mu)^{-1} = G_{\bar \mu}(y, x) \). In particular, we get for all \( \psi, \varphi \in L^2(\Sigma_M) \) that
\[
\langle \psi, G_{\mu}(x, y) \varphi \rangle = \langle G_{\mu}(x, y)\psi, \varphi \rangle = \langle (D - \mu)^{-1}\psi, \varphi \rangle = \langle \psi, (D - \mu)^{-1}\varphi \rangle
\]
for all \( \psi, \varphi \in L^2(\Sigma_M) \).

Moreover, we have

Lemma 3.5. In the situation of Lemma 3.4 we have \((D_y - \mu)G_{\mu}(x, y) = \delta_x \text{Id}_{\Sigma_M|x} \), i.e., for \( f_0 \in \Gamma(\Sigma_M|x) \), \( \varphi \in C_c(\Sigma_M) \)
\[
\int \langle (D_y - \mu)G_{\mu}(x, y)f_0, \varphi(y) \rangle \, dy = f_0(\varphi(x)).
\]

**Proof.**
\[
\int \langle (D_y - \mu)G_{\mu}(x, y)f_0, \varphi(y) \rangle \, dy = \int \langle G_{\mu}(x, y)f_0, (D_y - \mu)\varphi(y) \rangle \, dy
\]
\[
= \int f_0(G_{\mu}(x, y)(D_y - \mu)\varphi(y)) \, dy
\]
\[
= f_0(\varphi(x)).
\]
where the last step uses that \( G_{\mu}(x, y) \) is also the left inverse of \( D - \mu \).

Now, \( M \) has no longer to be closed, but we assume bounded geometry.
Proposition 3.6. Let \((M, g)\) be a Riemannian spin manifold of bounded geometry. Let 
\(D - \mu: L^2(\Sigma_M) \to L^2(\Sigma_M)\) be invertible. Then there exists a unique Green function.

Proof. We choose \(R > 0\) such that \(3R\) is smaller than the injectivity radius. Let \((x_i)_{i \in I}\) be 
as in the Covering Lemma 2.1. We define \((M \times M)_\varepsilon := \{(x, y) \in M \times M \mid \text{dist}(x, y) < \varepsilon\}. \)
Because of \(M = \bigcup_{i \in I} B_{3R}(x_i)\) we have 
\[(M \times M)_R \subset \bigcup_{i \in I} B_{3R}(x_i) \times B_{3R}(x_i).\]
We embed each ball \(B_{3R}(x_i)\) isometrically into a closed connected manifold \(M_x\), which is 
diffeomorphic to a sphere and \(D_{M_x} - \mu\) is invertible. This can always be achieved by local 
metric deformation on \(M_x\), see Proposition C.1.

Thus, by Proposition 3.2 the operator \(D_{M_x} - \mu\) possesses a Green function \(G^{x_i}(x, y)\) with 
\((D_{x_i} - \mu)G^{x_i}(x, y) = \delta_y \Id_{\Sigma_y}\). By abuse of notation we will view \(G^{x_i}(x, y)\) for \(x, y \in B_{3R}(x_i)\) also as a partially defined section of \(\Sigma_M \otimes \Sigma_M^* \to M \times M\), which is defined on 
\(B_{3R}(x_i) \times B_{3R}(x_i)\).

Now we choose smooth functions \(a_i\) on \(M \times M\) such that \(\text{supp } a_i \subset B_{3R}(x_i) \times B_{3R}(x_i) \subset (M \times M)_{6R}\) and such that \(\sum_{i \in I} a_i = 1\) on \((M \times M)_R/2\). Now we set 
\[H(x, y) = \sum_{i \in I} a_i(x, y)G^{x_i}(x, y).\]

This implies \(\text{supp } H \subset (M \times M)_{6R}\). Moreover, \(H(., y) \in L^2(M \setminus B_r(y))\) for all \(r > 0\) since 
this is true for each summand.

Our next goal is to prove that \((D_x - \mu)H(x, y) - \delta_y \Id_{\Sigma_y}\) is smooth. Note that \(G^{x_i}(x, y)\) and 
\(G^{x_j}(x, y)\) are both defined for \((x, y) \in (B_{3R}(x_i) \times B_{3R}(x_i)) \cap (B_{3R}(x_j) \times B_{3R}(x_j))\), but 
they will not coincide in general. On the other hand their defining property and the locality of the 
differential operator \(D\) (cp. Lemma 3.3) imply that 
\[(D_x - \mu)(G^{x_i}(x, y) - G^{x_j}(x, y)) = (D^i_y - \mu)(G^{x_i}(x, y) - G^{x_j}(x, y)) = 0.\]

Thus, 
\[\underbrace{(D_x - \mu)^2 + (D^i_y - \mu)^2}_P (G^{x_i}(x, y) - G^{x_j}(x, y)) = 0.\]

Since \(P\) is an elliptic operator, elliptic regularity implies that \(G^{x_i}(x, y) - G^{x_j}(x, y)\) viewed as a 
difference of distributions is a smooth function on \((B_{3R}(x_i) \times B_{3R}(x_i)) \cap (B_{3R}(x_j) \times B_{3R}(x_j))\), 
and thus \(a_i(x, y)(G^{x_i}(x, y) - G^{x_j}(x, y))\) as well. On \(B_{3R}(x_j) \times B_{3R}(x_j)\) we rewrite 
\[H(x, y) = G^{x_j}(x, y) + \sum_{i \in I \setminus \{j\}} a_i(x, y) (G^{x_i}(x, y) - G^{x_j}(x, y)),\]
and we conclude that \((D_x - \mu)H(x, y) = \delta_y \Id_{\Sigma_y} + F(x, y)\) where \(F(x, y)\) is a smooth section of 
\(\Sigma_M \otimes \Sigma_M^*\) with support in \((M \times M)_{6R}\).

There is a unique section \(H'\) of \(\Sigma_M \otimes \Sigma_M^*\) such that \((D_x - \mu)H'(x, y) = F(x, y)\) and such that 
\(H'(., y)\) is \(L^2\) for all \(y\). This follows for each \(y\) from the assumption that \(D - \mu\) is invertible. 
As \(D - \mu\) is a linear operator with continuous inverse and by elliptic regularity \(H'\) is smooth 
in \(x\) and \(y\).

We set \(\hat{G}(x, y) = H(x, y) - H'(x, y)\), and this gives a Green function for \(D - \mu\).
Assume that \(G\) and \(\hat{G}\) are two Green functions for \(D\), then \((D_x - \mu)((G - \hat{G})(., y)) = 0\).
By the invertibility, \(G = \hat{G}\) follows. Smoothness follows by smoothness of all \(G^{x_i}\), and 
smoothness of \(F\) and \(H'\). \(\square\)

Note that due to the last Proposition Lemmata 3.4 and 3.5 also hold true for manifolds \(M\) 
of bounded geometry.

We finish this section by stating another property of the Green function:
Lemma 3.7. Let \((M,g)\) be a Riemannian spin manifold of bounded geometry, and let \(D - \mu\) be invertible. Then the Green function also decays in \(L^2\) in the second entry, i.e., \(G_{D-\mu}(x,\cdot) \in L^2(M \setminus B_r(x))\) for all \(r > 0\).

Proof. The Green function \(G_{D-\mu}(\cdot, x)\) is in \(L^2(M \setminus B_r(x))\) in the first component. Then the claim follows from Lemma 3.4 in the extended version to manifolds \(M\) of bounded geometry. \(\square\)

4. The Dirac operator on hyperbolic space and its products

In this section we examine the Dirac operator on the model spaces \(M_{c,k}^{m,k} = \mathbb{H}^{k+1}_c \times N\). Note that we also allow the case where \(N\) is zero dimensional. First, we introduce polar coordinates on \(\mathbb{H}^{k+1}_c\) and write the Dirac operator in these coordinates. Then, we study the canonical action of \(\text{Spin}(k+1)\) on \(M_{c,k}^{m,k}\) and its spinor bundle.

4.1. The Dirac operator in polar coordinates. Let us introduce some notation, and let us compare it to notation in the existing literature.

In this section we have to work with spinors on various submanifolds of \(\mathbb{H}^{k+1}_c \times N\). So let \((Z_b)_{b \in B}\) a smooth family of pairwise disjoint submanifolds of \(\mathbb{H}^{k+1}_c \times N\). For simplicity of presentation we assume that all \(Z_b\) are isomorphic to \(Z\), in particular we obtain a smooth map \(F: Z \times B \to M\). The tangent space \(TZ_b\) carries an induced connection and similar the normal bundle \(\nu_b \to Z_b\) of \(Z_b\) in \(M\). As vector bundles \(TM|Z_b\) equals \(\nu_b \oplus TZ_b\). The connection on those vector bundles are denoted by \(\nabla^M\) for \(TM|Z_b\) and \(\nabla^\text{int}\) for \(\nu_b \oplus TZ_b\). The difference is essentially the second fundamental form \(\Pi_{Z_b}\) of \(Z_b\) in \(M\).

Putting all these bundles together for various \(b\) we obtain the following bundles over \(Z \times B\):

\[
F^*TM = \bigcup_{b \in B} TM|Z_b, \quad TZ_B := \bigcup_{b \in B} TZ_b, \quad \nu_B := \bigcup_{b \in B} \nu_b.
\]

Again as bundles with scalar products we have \(F^*TM = TZ_B \oplus \nu_B\) but both sides carry different metric connections. The pullback of Levi-Civita connection on \(TM\) to \(F^*TM\) is denoted by \(\nabla^M\) whereas the sum connection on the right hand side is denoted by \(\nabla^\text{int}\) where for \(X \in T_z Z_b, Y \in \mathcal{C}^\infty(TZ_B)\) and \(W \in \mathcal{C}^\infty(\nu_B)\) we have

\[
\nabla^M_X Y - \nabla^\text{int}_X Y = \Pi_{Z_b}(X,Y), \quad (\nabla^M_W \Pi_{Z_b} - \nabla^\text{int}_X W, Y) = -\langle \Pi_{Z_b}(X,Y), W \rangle.
\]

These two connection define connection 1-forms on the pullbacks of the frame bundle of \(M\) and the spin structure of \(M\). So we finally obtain connections, again denoted by \(\nabla^M\) and \(\nabla^\text{int}\), on \(F^*\Sigma M \to Z \times B\).

In particular we have for all \(X \in TZ_B\) and all spinors \(\varphi \in \mathcal{C}^\infty(F^*\Sigma M)\)

\[
\nabla^M_X \varphi = \nabla^\text{int}_X \varphi + \frac{1}{2} \sum_i e_i \cdot \Pi_Z(X,e_i) \cdot \varphi
\]

where \((e_i)_i\) is a local orthonormal frame on \(F^*\Sigma M\), cp. \cite{7} around (9).

Remark 4.1. In \cite{7} a slightly different notation is used, as can be seen in the following dictionary of notations

<table>
<thead>
<tr>
<th>Bär \cite{7}</th>
<th>(Q)</th>
<th>(M)</th>
<th>(\nabla^Q) and (\nabla^\Sigma Q)</th>
<th>(\nabla^M \oplus \nabla^N) and (\nabla^\Sigma M \otimes \text{id} + \text{id} \otimes \nabla^\Sigma N)</th>
<th>(\hat{D})</th>
<th>(\hat{D})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our article</td>
<td>(M = \mathbb{H}^{k+1}_c \times N)</td>
<td>(Z)</td>
<td>(\nabla^M)</td>
<td>(\approx \nabla^\text{int})</td>
<td>(D^Z_\partial)</td>
<td>(D^Z_{\text{int}})</td>
</tr>
</tbody>
</table>

Furthermore, in \cite{7} only the case that \(B\) is a point is formally studied but the calculations in there immediately generalize to our setting.
Also be aware that in [8] a further notation is used which has several advantages if the submanifold is a hypersurface which is not the case in our article. In [8] the Clifford multiplication of the ambient manifold coincides with the Clifford multiplication on the submanifold only up to Clifford multiplication with the normal vector field. In contrast to this in our notation the Clifford multiplication of the ambient space $M$ coincides with the one on the submanifold $Z$.

The partial Dirac operators $D^Z_\partial$ are now defined as $D^Z_\partial = \sum_i e_i \cdot \nabla^M_{e_i}$, and the intrinsic Dirac operators are given by $D^Z_{\text{int}} = \sum_i e_i \cdot \nabla^\text{int}_{e_i}$. As this definition does not depend on the choice of frame, it yields a global definition. The intrinsic Dirac operator is a twisted Dirac operator on the submanifold $N$. In the following applications all normal bundles have a parallel trivialization, hence, in this case the intrinsic Dirac operator coincides on the submanifold with several copies of the Dirac operator on this submanifold. As multiplicities are irrelevant for our discussion we have chosen the name 'intrinsic Dirac operator' for $D_{\text{int}}$, slightly abusing the language.

By [3], the intrinsic Dirac operator $D^Z_{\text{int}}$ is related to the partial Dirac operator $D^Z_\partial$ via

$$D^Z_\partial \varphi = D^Z_{\text{int}} \varphi - \frac{1}{2} \vec{H}_Z \cdot \varphi,$$

where $\vec{H}_Z = \text{tr} II Z$ is the unnormalised mean curvature vector field of $Z$ in $M$, see [7, Lemma 2.1].

We now come to our specific situation $M = \mathbb{M}^m_{c,k}$: We express the hyperbolic metric in polar normal coordinates centered in a fixed point $p_0$ which will be sometimes identified with 0. In these polar coordinates $\mathbb{M}^m_{c,k} \setminus \{(p_0) \times N\}$ is parametrized by $\mathbb{R}^+ \times S^k \times N$. We are especially interested in the submanifolds $Z = \mathbb{M}^m_{c,k}$ that are either $\mathbb{R}^+ \times \{x\} \times \{y\}$ or $\{r\} \times S^k \times \{y\}$ or $\{r\} \times \{x\} \times N^n$, always equipped with the metric induced from $\mathbb{M}^m_{c,k}$. In the following we will address these families of submanifolds shortly by $\mathbb{R}^+$, $S^k$ and $N$. The corresponding spaces $B$ are then $S^k \times N$, $\mathbb{R}^+ \times N$ and $\mathbb{R}^+ \times S^k$, respectively.

On an open set we choose an orthonormal frame $e_1,\ldots,e_m$, $m = n + k + 1 = \dim M$, such that $e_{k+1},\ldots,e_m$ is an orthonormal frame for the submanifolds $N$, and $e_2,\ldots,e_{k+1}$ is an orthonormal frame for $S^k$ and where $e_1 := \partial_r$. The notation should be read such that $\Sigma$ and $\partial_r$ denote essentially the same (radial) vector, but $\partial_r$ is viewed as a vector which acts via Clifford multiplication whereas $\Sigma$ acts as a covariant derivative.

The Dirac operator $D$ on $(r_0,\infty) \times S^k \times N$ is the sum of partial Dirac operators

$$D = \partial_r \cdot \frac{\nabla}{\partial r} + D^S_\partial + D^N_\partial$$

where the partial Dirac operators along $N$ and $S^k$ are locally defined as

$$D^N_\partial \varphi := \sum_{i=1}^n e_i \cdot \nabla^M_{e_i} \varphi, \quad D^S_\partial \varphi := \sum_{i=n+1}^{n+k} e_i \cdot \nabla^M_{e_i} \varphi,$$

for $\varphi \in C^\infty(\Sigma_M)$.

The intrinsic Dirac operators along $N$ and $S^k$ are given by

$$D^N_{\text{int}} \varphi := \sum_{i=1}^n e_i \cdot \nabla^\text{int}_{e_i} \varphi, \quad D^S_{\text{int}} \varphi := \sum_{i=n+1}^{n+k} e_i \cdot \nabla^\text{int}_{e_i} \varphi.$$

We denote the second fundamental form of $S^k$ in $\mathbb{H}^{k+1}_c$ as $\Pi_{S^k}$ and set $\vec{H}_{S^k} := \text{tr} \Pi_{S^k}$. Then $\Pi_{S^k}$ and $\vec{H}_{S^k}$ do not depend on whether they represent the second fundamental form and the mean curvature field of $S^k$ in $\mathbb{H}^{k+1}_c$, or of $S^k$ in $\mathbb{H}^{k+1}_c \times N$ or of $S^k \times N$ in $\mathbb{H}^{k+1}_c \times N$.

Using $\vec{H}_N = 0$ and $f(r) = \sinh_c(r)$, cp. Section 2.2,

$$\vec{H}_{S^k \times N} = \vec{H}_{S^k} = -k \frac{\partial_r f(r)}{f(r)} \partial_r = -k \coth_c(r)$$
we obtain $D^N := D^N_0 = D^N_{\text{int}}$ and $D^{\phi^k}_\partial = D^{\phi^k}_{\text{int}} + \frac{k}{2} \coth_c(r) \partial_r$.

We set $D\phi^k := f(r)D^{\phi^k}_{\text{int}}$ which is on each spherical submanifold up to multiplicity the standard Dirac operator on $S^k$ and obtain
\[
D = \frac{1}{\sinh_c(r)} D\phi^k + \partial_r \cdot \nabla \frac{\partial}{\partial r} + \frac{k}{2} \coth_c(r) \partial_r \cdot + D^N.
\]

**Lemma 4.2.** The following operators anticommute: $D^N$ with $D\phi^k$, $D^N$ with $\partial_r$, $D^N$ with $\partial_r \cdot \nabla$, $D\phi^k$ with $\partial_r$, and $D\phi^k$ with $\partial_r \cdot \nabla$. However, $\partial_r$ commutes with $\partial_r \cdot \nabla$, and $(D\phi^k)^2$ commutes with $D$.

**Proof.** Let $P_{\text{Spin}}(\mathbb{H}^{k+1}) \to P_{SO}(\mathbb{H}^{k+1})$ and $P_{\text{Spin}}(N) \to P_{SO}(N)$ be the fixed spin structures on $\mathbb{H}^{k+1}$ and $N$. Then we write as in Subsection 2.5
\[
\Sigma_{\mathbb{H}^{k+1} \times N} = (P_{\text{Spin}}(\mathbb{H}^{k+1}) \times P_{\text{Spin}}(N)) \times \zeta \Sigma_m
\]
where $\zeta$ is the composition $\text{Spin}(k + 1) \times \text{Spin}(m) \xrightarrow{\zeta} \text{Spin}(m) \xrightarrow{\rho} \text{End}(\Sigma_m)$. The bundle $P$ carries the Levi-Civita connection-1-form $\alpha^L_C$ and another connection-1-form $\alpha^\text{int}$ as explained before.

We obtain a connection preserving bundle homomorphism $I_c$, which is fiberwise an isometric isomorphism, and
\[
\Sigma_{\mathbb{H}^{k+1} \setminus \{p_0\} \times N}, \nabla^\text{int} \xrightarrow{I_c} \Sigma_{\mathbb{R}^+ \times S^k \times N}, \nabla^L_C
\]
commutes. Note that $I_c$ is also compatible with the Clifford multiplication in the sense that for $X \in TZ$ we have
\[
I_c(X \cdot \varphi) = \begin{cases} \frac{f(r)}{r} X \cdot I_c(\varphi) & \text{for } Z = \mathbb{R}^+ \times \{x \times y\} \text{ or } r \times \{x \times y\} \\
X \cdot I_c(\varphi) & \text{for } Z = \{r \times S^k \times y\}. \end{cases}
\]

Then the lemma follows immediately by the corresponding statements for $\Sigma_{\mathbb{R}^+ \times S^k \times N}$. □

We will also use the map $\tilde{I}_c := I_0^{-1} \circ I_c : \Sigma_{\mathbb{H}^{k+1} \setminus \{p_0\} \times N} \to \Sigma_{\mathbb{R}^+ \times S^k \times N}$ which allows to identify $\Sigma_{\mathbb{H}^{k+1} \times N}$ with $\Sigma_{\mathbb{R}^+ \times S^k \times N}(x,y)$ and thus with $\Sigma_{\mathbb{R}^+ \times S^k \times N}(0,y)$, $0 \cong p_0$.

### 4.2. The action of Spin$(k + 1)$ on $\mathbb{H}^{m,k} = \mathbb{H}^{k+1} \times N$.

We identify $T_{p_0} \mathbb{H}^{k+1}$ with $\mathbb{R}^{k+1}$. The left action $a_1$ of the spin group Spin$(k + 1)$ on $\mathbb{R}^{k+1}$ obtained by composing the double covering Spin$(k + 1) \to SO(k + 1)$ with the tautological representation yields a left action on $\mathbb{H}^{k+1}$ via the exponential map $\exp_{p_0} : \mathbb{R}^{k+1} \to \mathbb{H}^{k+1}$ which is a diffeomorphism. As this action is isometric it yields a left action on $P_{\text{Spin}}(\mathbb{H}^{k+1})$ - also called $a_1$. Thus, we obtain a Spin$(k + 1)$-action on $P_{\text{Spin}}(\mathbb{H}^{k+1}) \times P_{\text{Spin}}(N) \times \Sigma_m$ as $a_1 = a_1 \times \text{id} \times \text{id}$. Since $a_1$ and the principal Spin$(k + 1)$-action which acts from the right commute, the $a_1$-action descends to a Spin$(k + 1)$-action from the left - denoted by $a_2$ - on the spinor bundle $\Sigma_{\mathbb{H}^{k+1} \times N} = (P_{\text{Spin}}(\mathbb{H}^{k+1}) \times P_{\text{Spin}}(N)) \times \zeta \Sigma_m$ (for $\zeta$ as in (5)) such that
\[
\Sigma_{\mathbb{H}^{k+1} \times N} \xrightarrow{a_2(\gamma)} \Sigma_{\mathbb{H}^{k+1} \times N}
\]
commutes.

By construction, the action $a_1$ does not depend on $c$. Thus, Diagram (6) commutes with this Spin$(k + 1)$-action.

Moreover, note that $a_1$ preserves the spheres $S^k_c : = \{r \times S^k \times \{y\} \subset \mathbb{H}_c^{k+1} \times N$. Hence, the diagram above can be restricted to this submanifold. In particular, $a_1$ acts transitively.
on $S^k_{r,y}$. Furthermore, $(p_0, y)$ is a fixed point of $a_1 \times \text{id}$ for all $y \in N$. Thus, the $a_2$-action can be restricted to an action that maps $\Sigma_{k+1} \times N(p_0, y)$ to itself.

4.3. Spinors on $\mathbb{R}^k \subset \mathbb{R}^{k+1}$. We will now analyse the special case $N = \{y\}$ and $c = 0$, thus $\mathbb{R}^{k+1} = \mathbb{R}^k \subset \mathbb{R}^{k+1}$. This well-known case is not only important as an example, but will also be used to derive consequences for the general case.

We obtain immediately from (3) and $\Pi_{\mathbb{R}^k} = -\frac{1}{r}g_{\mathbb{R}^k}\partial_r$ where $S^k_r$ is the sphere of radius $r$ canonically embedded in $\mathbb{R}^{k+1}$:

**Lemma 4.3.** Assume that $\varphi$ is a parallel spinor on $\mathbb{R}^{k+1}$. Then for any $X \in T\mathbb{S}^k_r$ we have

$$\nabla^X_r \varphi = -\frac{1}{2r} \partial_r \cdot X \cdot \varphi \text{ and } \nabla^X_r (\partial_r \cdot \varphi) = \frac{1}{2r} \partial_r \cdot X \cdot (\partial_r \cdot \varphi).$$

In particular, we have

$$D^g \varphi = rD^g_{\text{int}} \varphi = -\frac{k}{2} \partial_r \cdot \varphi \text{ and } D^g_{\text{int}} (\partial_r \cdot \varphi) = -\frac{k}{2} \partial_r \cdot (\partial_r \cdot \varphi).$$

Using Lemma 4.2 and $\nabla^X_r \partial_r = 0$ this implies

$$(D^g_{\text{int}})^2 \varphi = \frac{k^2}{4} \varphi \text{ and } (D^g_{\text{int}})^2 (\partial_r \cdot \varphi) = \frac{k^2}{4} (\partial_r \cdot \varphi).$$

5. Modes of Spin$(k+1)$-equivariant maps

We now have a Spin$(k+1)$-action on $\Sigma_{k+1}|_0 \cong \Sigma_{k+1}$, $\{r\} \times S^k$ and $\Sigma_{k+1}|_{\{r\} \times S^k}$, and thus on $C^\infty(S^k, \Sigma_{k+1}|_{\{r\} \times S^k})$ given by $(\gamma \cdot f)(x) = a_2(\gamma)f(a_1(\gamma)^{-1}x)$. To simplify notations we mostly write $S^k$ for $\{r\} \times S^k$.

We now have to classify Spin$(k+1)$-equivariant functions $\Sigma_{k+1}|_0 \to C^\infty(S^k, \Sigma_{k+1}|_{S^k})$.

For $\psi_0 \in \Sigma_{k+1}|_0$ let the parallel spinor on $\mathbb{R}^{k+1}$ with value $\psi_0$ at 0 be denoted by $\Psi_0$. For $k$ even, the positive and negative parts of $\Psi_0$ are denoted by $\Psi_0^{(\pm)}$.

**Lemma 5.1.** Let $F \colon \Sigma_{k+1} \to C^\infty(S^k, \Sigma_{k+1}|_{S^k})$ be a Spin$(k+1)$-equivariant map. Then for $k$ even $F$ has the form

$$\psi_0 \mapsto (a_1 \Psi_0 + a_2 \partial_r \cdot \Psi_0)|_{S^k}$$

and for $k$ odd $F$ has the form

$$\psi_0 \mapsto (a_{11} \Psi_0^{(+)1} + a_{22} \Psi_0^{(+)2} + a_{21} \partial_r \cdot \Psi_0^{(+)1})|_{S^k}$$

for suitable constants $a_i, a_{ij} \in \mathbb{C}$.

**Proof.** First, we note that the maps $F$ described above are actually Spin$(k+1)$-equivariant since $\partial_r$ is a Spin$(k+1)$-equivariant vector field.

Let $A \colon \Sigma_{k+1}^{(\delta)} \to \Sigma_{k+1}^{(\varepsilon)}$ be the inclusion map, $\delta, \varepsilon \in \{+, -\}$. By composition we obtain for fixed $\delta, \varepsilon \in \{+, -\}$ a Spin$(k+1)$-equivariant map

$$\Sigma_{k+1}^{(\delta)} \xrightarrow{A} \Sigma_{k+1}^{(\varepsilon)} \xrightarrow{F} C^\infty(S^k, \Sigma_{k+1}^{(\varepsilon)}|_{\{r\} \times S^k}) \to C^\infty(S^k, \Sigma_{k+1}^{(\varepsilon)}), \quad (7)$$

where in the last step we projected $\Sigma_{k+1}$ to $\Sigma_{k+1}^{(\varepsilon)}$. If we compose this map with evaluation at the north pole of the sphere, then we obtain a Spin$(k)$-equivariant map $\sigma : \Sigma_{k+1}^{(\delta)} \to \Sigma_{k+1}^{(\varepsilon)}$. Because of the Spin$(k+1)$-equivariance of (7) and since Spin$(k+1)$ acts transitively on $S^k$, this map uniquely determines the map $\Sigma_{k+1}^{(\delta)} \to C^\infty(S^k, \Sigma_{k+1}^{(\varepsilon)})$ above.

If $k$ is odd, then $\Sigma_{k+1}^{(\varepsilon)} \cong \Sigma_{k+1}^{(\delta)} \cong \Sigma_k$ as Spin$(k)$-representations, and Schur’s Lemma tells us that there is up to scaling a unique such map $\sigma$. Using the fact that $e_{k+1}^{(\pm)} : \Sigma_{k+1}^{(\pm)} \to \Sigma_{k+1}^{(\pm)}$, $\sigma$ can be written as

$$\tau \in \Sigma_{k+1}^{(\delta)} \mapsto \begin{cases} a_{\delta, \delta} \tau & \text{for } \delta = \varepsilon \\ a_{\delta, \varepsilon} e_{k+1} \cdot \tau & \text{for } \delta \neq \varepsilon \end{cases}$$

where $a_{\delta, \varepsilon} \in \mathbb{C}$.
for suitable $a_{\delta, \varepsilon} \in \mathbb{C}$. As $\partial_r$ is the Spin($k+1$)-equivariant extension of $e_{k+1}$ we obtain the lemma for $k$ odd.

If $k$ is even, then $\Sigma_{k+1} = \Sigma_k = \Sigma_k^{(+)} \oplus \Sigma_k^{(-)}$ as Spin($k$)-representations. In this case $e_{k+1}$ commutes with Spin($k+1$) and preserves the splitting. Using Schur’s Lemma, $e_{k+1}^2 = 1$ and because $e_{k+1}$ is tracefree we know that $e_{k+1}$ acts as $\pm \text{diag}(i^r, -i^r)$. For $\varepsilon = \delta$ we can again apply Schur’s Lemma. As $\Sigma_k^{(+)}$ and $\Sigma_k^{(-)}$ are not equivalent as Spin($k$)-representations the maps $\sigma : \Sigma_k^{(\pm)} \to \Sigma_k^{(\mp)}$ have to be identically zero. Thus, with respect to the splitting of $\Sigma_{k+1}$ the maps $\sigma$ for different $\delta$ and $\varepsilon$ form a Spin($k$)-equivariant map $\Sigma_{k+1} \to \Sigma_{k+1}$ that can be written as

$$\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} b_1 + b_2 \\ b_1 - b_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

for suitable $b_i \in \mathbb{C}$. Summarizing, for $k$ even, $\sigma$ maps $\tau \mapsto a_1 \tau + a_2 e_{k+1} \cdot \tau$ with $a_i \in \mathbb{C}$. \[\square\]

Then using Lemma 4.3 we obtain immediately

**Corollary 5.2.** Let $F : \Sigma_{k+1} \to C^\infty(S^k, \Sigma_{k+1}^\varepsilon |_{\mathbb{R}^{k+1}})$ be a Spin($k+1$)-equivariant map. Let $\psi_0 \in \Sigma_{k+1}$ and $\varphi = F \psi_0$. Then $(D^{\delta k})^2 \varphi = k^2 \varphi$.

We say that $\varphi$ is in the spherical mode $k^{2 \varepsilon}$, and thus $\varphi$ is in the mode of lowest energy on the sphere.

Now we want to carry over the last result to $M^{m,k}$. In the following $p_0 \in \mathbb{H}^{k+1}_c$ denotes again the fixed point of the Spin($k+1$)-action, and let $y_0, y \in N$.

**Lemma 5.3.** Let $F : \Sigma_{k+1}^\varepsilon \times N |_{(p_0, y_0)} \to C^\infty(S^k, \Sigma_{k+1}^\varepsilon \times \mathbb{R}^{k+1} |_{(r, y)})$ be a Spin($k+1$)-equivariant map. Let $\psi_0 \in \Sigma_{k+1}^\varepsilon \times N |_{(p_0, y_0)}$ and $\varphi = F \psi_0$. Then $(D^{\delta k})^2 \varphi = k^2 \varphi$.

**Proof.** Note that the composition $I_c := I_{c,1} \circ I_c$ where $I_c$ is defined as in (6) maps the spinor bundle over $(\mathbb{H}^{k+1} \setminus \{p_0\}) \times N$ to the spinor bundle over $(\mathbb{R}^{k+1} \setminus \{0\}) \times N$. This map preserves the intrinsic connection $\nabla^{\text{int}}$ and uniquely extends int o $p_0 \equiv 0$. Via pullback we then obtain a Spin($k+1$)-equivariant vector space isomorphism

$$C^\infty(\{r\} \times S^k, \Sigma_{k+1}^\varepsilon \times \mathbb{R}^{k+1} |_{(r, x)}) \overset{\partial_r}{\longrightarrow} C^\infty(\{r\} \times S^k, \Sigma_{k+1}^\varepsilon \times \mathbb{R}^{k+1} |_{(r, x)})$$

Moreover, we can write in the sense of Spin($k+1$)-modules $\Sigma_{k+1}^\varepsilon \times N |_{(x, y)} \cong \oplus_m \cong \Sigma_{k+1} \otimes V$ if $k$ is even or $\Sigma_{k+1}^\varepsilon \times N |_{(x, y)} \cong \Sigma_{k+1}^{(+)} \otimes V^{(+)} \oplus \Sigma_{k+1}^{(-)} \otimes V^{(-)}$ if $k$ is odd, where $V^{c} := \text{Hom}_{\text{Spin}(k+1)}(\Sigma_{k+1}^{c}, \Sigma_{k+1} \otimes \mathbb{R}^{k+1})$ is a vector space which is independent of $x \in \mathbb{R}^{k+1}$.

Let now $k$ be odd. Then any $\alpha \in (V^{c})^*$ defines a map $\Sigma_{k+1}^\varepsilon \times N |_{(x, y)} \to \Sigma_{k+1}^\varepsilon$. Let $A : \Sigma_{k+1}^\varepsilon \to \Sigma_{k+1}^\varepsilon \times N |_{(p_0, y_0)}$ be a Spin($k+1$)-equivariant map. By composition we obtain for fixed $A$, $\alpha$ and $\delta, \varepsilon \in \{+, -\}$ a Spin($k+1$)-equivariant map

$$\Sigma_{k+1}^\varepsilon \overset{\alpha}{\longrightarrow} \Sigma_{k+1}^\varepsilon \times \mathbb{R}^{k+1} |_{(p_0, y_0)} \overset{F}{\longrightarrow} C^\infty(S^k, \Sigma_{k+1}^\varepsilon \times \mathbb{R}^{k+1} |_{(r, x)})$$

$$\overset{\partial_r}{\longrightarrow} C^\infty(S^k, \Sigma_{k+1}^\varepsilon \times \mathbb{R}^{k+1} |_{(r, x)}) \cong C^\infty(S^k, \Sigma_{k+1}^\varepsilon \otimes \mathbb{R}^{k+1}) \overset{\alpha}{\longrightarrow} C^\infty(S^k, \Sigma_{k+1}^\varepsilon \otimes V) \overset{\partial_r}{\longrightarrow} C^\infty(S^k, \Sigma_{k+1}^\varepsilon \otimes V)$$

Let now $k$ be even. Then the argumentation is analogous to the one above when replacing $V^{c}$ by $V$ and $\Sigma_{k+1}^\varepsilon$ by $\Sigma_{k+1}$.

Then the Lemma follows from Corollary 5.2 together with the identification by $I_{r, y}$. \[\square\]

**Corollary 5.4.** Let $G(q, p)$ be the Green function of the operator $D - \mu$, $\mu \notin \text{Spec}_{L^2}(D)$. Let $q = (r, x, y) \in \mathbb{M}^{m,k}$ be the polar coordinates when using $p_0$ as the origin, $r > 0$. Let $\psi_0 \in \Sigma_{\text{even}, k} \times N |_{(p_0, y_0)}$, $y_0 \in N$. Set $\varphi(q) := G(q, (p_0, y_0)) \psi_0$. Then

$$(D^{\delta k})^2 \varphi |_{(r, x)} = \frac{k^2}{4} \varphi |_{(r, x)}.$$
Proof. Now we consider the Green function of the shifted Dirac operator $D - \mu$ for $\mu \not\in \text{Spec}_{L^2}^{M^{m,k}}(D)$. We define
\[
G(\cdot, (p_0, y_0)) : \Sigma_{H_c^{k+1} \times N \setminus (p_0, y_0)} \to \Gamma(\Sigma_{H_c^{k+1} \times N \setminus (p_0, y_0)}).
\]
It follows from the definition of $G$, in particular from its uniqueness, and from the equivariance of $D$ under $\text{Spin}(k + 1)$ that $G(\cdot, (p_0, y_0))$ is $\text{Spin}(k + 1)$-equivariant as well. In particular, $G(\cdot, (p_0, y_0))|_{\Sigma_{H_c^{k+1} \times N}}$ is a $\text{Spin}(k + 1)$-equivariant map as considered in Lemma 5.3. Thus, together with Lemma 5.3, the corollary follows.

6. Decay estimates for a fixed mode

Let $\mu \not\in \text{Spec}_{L^2}^{M^{m,k}}(D)$. Then, by Theorem 3.6, there exists a unique Green function for $D - \mu$. The goal of this section is to estimate the decay of this Green function at infinity. For that, let $y = (p_0, y_N) \in \mathbb{H}_c^{k+1} \times N$ and $\psi_0 \in \Sigma_{M_c} y$ be fixed. Set $\varphi(x) := G(x, y)\psi_0$. The Definition of the Green function, cf. [2], implies that $\varphi$ is an $L^2$-eigenspinor of $D$ to the eigenvalue $\mu$ outside a neighbourhood of $y$. Moreover, by Corollary 5.4, we know that $\varphi$ is in the spherical mode $\frac{\lambda^2}{4}$. Before starting to estimate the decay we give the following Remark:

Remark 6.1. The $L^2$-spectrum of the square of the Dirac operator on the product space $M_1 \times M_2$ is given by the set theoretic sum \{\lambda_1^2 + \lambda_2^2 | \lambda_1^2 \in \text{Spec}_{L^2}^{M_1}((D^N)^2)\}. This is seen immediately by [1] and the spectral theorem.

The $L^2$-spectrum of the Dirac operator on the hyperbolic space, and thus also on $\mathbb{H}_c^{k+1}$, is the whole real line, cf. [10]. Let $\lambda_0, \lambda_0 \geq 0$, be the smallest eigenvalue of $(D^N)^2$. Then the above together with Lemma 2.8 implies for $M^{m,k}$ that
\[
\text{Spec}_{L^2}(D^2) = [\lambda_0^2, \infty).
\]
Together with Lemma 2.11 and Example 2.12,
\[
\text{Spec}_{L^2}(D) = (-\infty, -\lambda_0] \cup [\lambda_0, \infty).
\]
The complement of this spectrum is denoted by $I_{\lambda_0} := (\mathbb{C} \setminus \mathbb{R}) \cup (-\lambda_0, \lambda_0)$.

Now we decompose the space of spinors restricted to $\{r_1\} \times \mathbb{S}^k \times N$ into complex subspaces of minimal dimensions which are invariant under $D^N, \partial_r, D^{\mathbb{S}^k}$. Such spaces have a basis of the form $\psi, \partial_r \cdot \psi, P\psi, \text{ and } \partial_r \cdot P\psi$, where $\psi$ satisfies $D^N \psi = \lambda \psi$, $(D^{\mathbb{S}^k})^2 \psi = \rho^2 \psi$, $\rho \in \mathbb{R}^+ \setminus \{0\}$, and $P := D^{\mathbb{S}^k} / \rho$. All these operations commute with parallel transport in $r$-direction, so by applying parallel transport in $r$-direction we obtain spinors $\psi, \partial_r \cdot \psi, P\psi, \text{ and } \partial_r \cdot P\psi$ on $\mathbb{R}^+ \times \mathbb{S}^k \times N$ with similar relations, and the space of all spinors of the form
\[
\varphi = \varphi_1(r)\psi + \varphi_2(r)\partial_r \cdot \psi + \varphi_3(r)P\psi + \varphi_4(r)\partial_r \cdot P\psi
\]
is preserved under the Dirac operator $D$ on $M^{m,k}_c$ because of (4). Then the operators discussed above restricted to such a minimal subspace are represented by the matrices, cp. Lemma 4.2.

$$D^N = \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & -\lambda & 0 & 0 \\
0 & 0 & -\lambda & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix}, \quad D^{\mathbb{S}^k} = \begin{pmatrix}
0 & 0 & \rho & 0 \\
0 & 0 & 0 & -\rho \\
\rho & 0 & 0 & 0 \\
0 & -\rho & 0 & 0
\end{pmatrix}, \quad \partial_r \cdot = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}$$

Proposition 6.2. Assume that $\varphi$ is an $L^2$-solution to the equation $D\varphi = \mu\varphi$, $\mu \in I_{\lambda_0}$ on $(M^{m,k}_c)_{>r_0} := (\mathbb{H}_c^{k+1} \setminus B_{r_0}(p_0)) \times N$. Assume that $\varphi$ has the form given in (8) with parameters $\rho$ and $\lambda$. Let $\kappa$ satisfy $\kappa^2 = \lambda^2 - \mu^2$, $\text{Re}\kappa \geq 0$. Then $\text{Re}\kappa > 0$. Moreover, let $\kappa_0^2 = \lambda_0^2 - \mu^2$. If $\text{Re}\kappa_0 > 0$, then there is an positive constant $C$ and $r_1$ such that
\[
|\varphi(x)| \leq C\|\varphi\|_{L^2((M^{m,k}_c)_{>r_0})} e^{(-ck/2 - \text{Re}\kappa_0)d(x_1, p_0)} \text{ for all } x = (x_1, x_2) \in (\mathbb{H}_c^{k+1} \setminus B_{r_1}(p_0)) \times N
\]
where $C$ is a constant that only depends on $c, k, r_1, \lambda_0, \mu$ and $\rho$ but not on $\lambda$. For $r = 0$ an analogous estimate holds when replacing $e^{(-ck/2)d(x_1, p_0)}$ by $r^{-k/2}$ where $r = d(x_1, p_0)$.  

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Thus the solution extends to $t^\infty$. This estimate yields explicit asymptotic control for $\varphi$. So by [4] the following equation is equivalent to $D\varphi = \mu \varphi$:

$$0 = \begin{pmatrix} \lambda - \mu & -k & 0 & 0 \\ 2k & \frac{1}{2} \frac{\coth_c r - h}{\sinh_c r} & 0 & 0 \\ 0 & 0 & -\lambda - \mu & \frac{1}{2} \coth_c r \\ 0 & -\frac{\rho}{\sinh_c r} & -\frac{\rho}{\sinh_c r} & \frac{1}{2} \coth_c r \end{pmatrix} \Phi(r) + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Phi'(r).$$

Thus using $I$ for the identity matrix and setting

$$A := \begin{pmatrix} 0 & \lambda + \mu & 0 & 0 \\ 0 & -\lambda - \mu & 0 & 0 \\ 0 & 0 & 0 & -\lambda + \mu \\ 0 & 0 & -\lambda - \mu & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

we obtain

$$\Phi'(r) = \left(A - \frac{k\coth_c r}{2}I + \frac{\rho}{\sinh_c r}B\right) \Phi(r).$$

We start with the case $c \neq 0$: We now substitute $t = e^{-cr}$, $\Phi(t) = \Phi(-c^{-1}\log t)$. Then

$$\frac{d\Phi}{dt} = \left(-\frac{1}{ct}A + \frac{k(1 + t^2)}{2(t - t^3)}I + \frac{2k\rho}{t^2 - 1}\right) \Phi(t).$$

Such singular ordinary differential equations are well understood, see [13] Chap. 4, Sec. 1-3. In particular, $t = 0$ is a singular point of first kind, and [13] Chap. 4 Thm 2.1 yields that $t = 0$ is a so-called “regular singular point”, and the associated theory applies. However, in our situation it is more efficient to analyse the equation directly.

We set $h(t) := (\log t - \log(t + 1) - \log(1 - t))k/2$, then $h'(t) = \frac{k(1 + t^2)}{2(t - t^3)}$. We define

$$\tilde{\Phi}(t) := e^{-h(t)}t^{A/c} \Phi(t),$$

and we calculate

$$\frac{d\tilde{\Phi}}{dt} = -\frac{2k\rho}{1 - t^2} \Phi.$$

As $B$ anticommutes with $A$, we have $t^{A/c}Bt^{-A/c} = t^{2A/c}B$, and as $B$ is an isometry of $\mathbb{C}^4$, we see that

$$\|t^{A/c}Bt^{-A/c}\| = t^{2|\Re \kappa|/c}$$

where $\|\cdot\|$ denotes the operator norm and where

$$\kappa_\pm := \pm \sqrt{\lambda^2 - \mu^2}.$$

are the (complex) eigenvalues of $A$. It follows that for $0 < t < 1/2$

$$\left|\frac{d}{dt} \log \|\tilde{\Phi}(t)\|\right| \leq \left|\frac{d\tilde{\Phi}}{\tilde{\Phi}}\right| \leq \frac{2k\rho}{1 - t^2} \|t^{A/c}Bt^{-A/c}\| \leq 3\rho t^{2|\Re \kappa|/c}.$$

Thus the solution extends to $t = 0$, and

$$|\tilde{\Phi}(0)|e^{-3\rho t^{2|\Re \kappa|/c}} \leq |\tilde{\Phi}(t)| \leq |\tilde{\Phi}(0)|e^{3\rho t^{2|\Re \kappa|/c}}.$$

This estimate yields explicit asymptotic control for $\Phi(t)$, and thus for $\varphi$. Namely, assume $cr_0 \geq 1 > \log 2$, there are two fundamental solutions $\varphi_\pm$ of $D\varphi = \mu \varphi_\pm$ such that $\Phi_\pm(0)$ is an eigenvector of $A$ to the eigenvalue $\kappa_\pm$ and such that

$$e^{-3\rho e^{-2|\Re \kappa|/c}} r^{e\Re \kappa} e^{h(e^{-cr})} \leq \frac{|\varphi_\pm(x)|}{|\Phi_\pm(0)|} \leq e^{3\rho e^{-2|\Re \kappa|/c}} r^{e\Re \kappa} e^{h(e^{-cr})} \quad r := d(x_1, p_0) > r_0.$$
This implies that for every $\delta \in (0, 1)$ there is $\bar{r}_0$ such that
\begin{equation}
(1 - \delta)e^{-(ck/2 + \text{Re} \kappa_\pm)r} \leq \frac{\varphi_\pm(x)}{\Phi_\pm(0)} \leq (1 + \delta)e^{-(ck/2 + \text{Re} \kappa_\pm)r} \quad r := d(x_1, p_0) > \bar{r}_0. \tag{9}
\end{equation}
From
\begin{equation}
\int_{\bar{r}_0}^\infty \left| \Phi(r) \right|^2 (\sinh_r r)^k dr \leq \frac{\left\| \varphi \right\|_{L^2}^2}{\text{vol}(\mathbb{S}^k \times N)}
\end{equation}
and the left inequality of (9) we see that $\varphi_\pm$ is in $L^2((\mathbb{M}_c^{m,k})_{> \bar{r}_0})$ if and only of $\text{Re} \kappa_\pm < 0$. In the following we call this $\kappa_\pm$ just $\kappa_\lambda$ and also replace the $\pm$ index by $\lambda$ in all other occurrences. We note that $|\text{Re} \kappa \lambda|$ is increasing in $|\lambda|$. Thus, $\delta$ and $\bar{r}_0$ from above can be chosen independent on $\lambda$.

Next, we multiply the first inequality of (9) by $|\hat{\Phi}_\lambda(0)|$ and then integrate its square:
\begin{equation}
\frac{\left\| \varphi \right\|_{L^2}^2}{\text{vol}(\mathbb{S}^k \times N)} \geq (1 - \delta)^2 |\hat{\Phi}_\lambda(0)|^2 \int_{\hat{r}_0}^\infty e^{(-ck + 2\text{Re} \kappa_\lambda)r}(\sinh_r r)^k dr.
\end{equation}
Hence, we obtain an upper bound
\begin{equation}
|\hat{\Phi}_\lambda(0)|^2 \leq C_1^2 (1 - \delta)^{-2} \left\| \varphi \right\|_{L^2((\mathbb{M}_c^{m,k})_{> \bar{r}_0})}^2 (e^{2\text{Re} \kappa_\lambda \bar{r}_0}) -1
\end{equation}
where $C_1$ is a constant independent on $\lambda$.

Using this again with the right inequality of (9) we get for all $x$ with $r = \text{dist}(x, p_0) > \bar{r}_0$ that
\begin{equation}
|\varphi(x)| \leq \frac{1 + \delta}{1 - \delta} C_1 \left\| \varphi \right\|_{L^2((\mathbb{M}_c^{m,k})_{> \bar{r}_0})} e^{(-ck/2 + \text{Re} \kappa_\lambda)r} \left( \frac{e^{2\text{Re} \kappa_\lambda \bar{r}_0}}{-2\text{Re} \kappa_\lambda} \right)^{-1/2}
\end{equation}
\begin{equation}
\leq C_1 (-2\text{Re} \kappa_\lambda)^{1/2} \left\| \varphi \right\|_{L^2((\mathbb{M}_c^{m,k})_{> \bar{r}_0})} e^{-ckr/2 + \text{Re} \kappa_\lambda(r - \bar{r}_0)}. \tag{10}
\end{equation}
For $r > \bar{r}_0$ we see that $(-2\text{Re} \kappa_\lambda) e^{2\text{Re} \kappa_\lambda(r - \bar{r}_0)}$ is monotonically decreasing in $|\text{Re} \kappa_\lambda|$, and we obtain from (10)
\begin{equation}
|\varphi(x)| \leq C_1 (-2\text{Re} \kappa_\lambda)^{1/2} \left\| \varphi \right\|_{L^2((\mathbb{M}_c^{m,k})_{> \bar{r}_0})} e^{-ckr/2 + \text{Re} \kappa_\lambda(r - \bar{r}_0)}
\end{equation}
\begin{equation}
\leq C \left\| \varphi \right\|_{L^2((\mathbb{M}_c^{m,k})_{> \bar{r}_0})} e^{(-ck/2 + \text{Re} \kappa_\lambda)r}
\end{equation}
for all $x$ with $r = \text{dist}(x, p_0) > \bar{r}_0$. Here, $C$ can be chosen such that it only depends on $c$, $k$, $\bar{r}_0$, $\lambda_0$, $\mu$ and $\rho$ but not on $\lambda$. Note the $\kappa$ in the claim is simply $-\kappa_{\lambda_0}$.

It remains the case $c = 0$.
\begin{equation}
\Phi'(r) = \left( A - \frac{k}{2r} 1 + \frac{\rho}{r} B \right) \Phi(r).
\end{equation}
Set $\hat{\Phi}(r) = r^k e^{-Ar} \Phi(r)$. Then,
\begin{equation}
\hat{\Phi}'(r) = \frac{c}{r} e^{-Ar} B e^{Ar} \hat{\Phi} = \frac{c}{r} e^{-2Ar} B \hat{\Phi}.
\end{equation}
Then we can proceed as above and obtain the claim. \hfill \Box

In order to estimate the decay of $\varphi(x) = G(x, y) \psi_0$, $\psi_0 \in \Sigma_{\mathbb{M}_c^{m,k}}$ at infinity we will decompose $\varphi$ into its modes in $\mathbb{S}^k$ and $N$ direction, respectively. Lemma 5.2 provides an estimate of the decay of each mode which is independent of the mode in direction of $N$. Moreover, from Corollary 5.4 we know that $\varphi$ has spherical mode $\frac{e^k}{k}$. Thus, we obtain a decay estimate for $\varphi$:

**Lemma 6.3.** Let $\mu \not\in \text{Spec}_{1/2}(D)$, and let $G$ be the unique Green function of $D - \mu$. We set $M_\mu := \{ x \in \mathbb{M}_c \mid \text{dist}(x, N^y) = 0 \}$ where $N^y = \{ p_0 \} \times N$ and $y = (p_0, y_N) \in \mathbb{H}_c^{k+1} \times N$. Let $\kappa$ satisfy $\kappa^2 = \lambda_0^2 - \mu^2$ and $\text{Re} \kappa \geq 0$. Then for all $\varepsilon > 0$ and $\bar{r}_0$ sufficiently large there is a constant $C > 0$ independent on $y$ such that
\begin{equation}
\int_{M_\mu(r)} |G(x, y)|^2 dx \leq Ce^{-2rR\text{Re} \kappa} \quad \text{for all } r > \bar{r}_0.
\end{equation}
Proof. Let $\psi_0 \in \Sigma_{\psi}$. Set $\varphi(x) := G(x,y)\psi_0$. Then, $\varphi$ decomposes into a sum of spinors $\varphi \rho^2$, $\lambda, \nu$ of the form (8) with $(D^N)^2 \varphi \rho^2, \lambda, \nu = \rho^2 \varphi \rho^2, \lambda, \nu$ and $D^N \varphi \rho^2, \lambda, \nu = \lambda \varphi \rho^2, \lambda, \nu$, respectively, and as the multiplicities of the combined eigenspaces might be larger than one the index $\nu$ runs through a basis. By Corollary 5.4 $\rho^2$ may only take the value $\frac{k^2}{4}$. Thus, $\int_{M_0(r)} |\varphi(x)|^2 dx = \sum_{\lambda, \nu} \|\varphi \rho^2, \lambda, \nu\|_{L^2(M_0(r))}^2$. Together with Proposition 6.2 we obtain for $c \neq 0$

\[
\int_{M_0(r)} |\varphi(x)|^2 dx \leq \sum_{\lambda, \nu} C_1 \|\varphi \rho^2, \lambda, \nu\|_{L^2(M_0(r))}^2 e^{(-c k - 2 \Re \kappa) r} \sinh^k(r)
\]

\[
\leq C_1 e^{-2 \Re \kappa} \sum_{\lambda, \nu} \|\varphi \rho^2, \lambda, \nu\|_{L^2(M_0(r))}^2
\]

\[
\leq C_1 e^{-2 \Re \kappa} \|\varphi\|_{L^2(M_0(r))}^2.
\]

The case $c = 0$ follows analogously. \qed

7. Decomposition of the Green function

We decompose the Green function $G$ of the shifted Dirac operator $D - \mu$ on $M = \mathbb{M}^{m,k}$ into a singular part and a smoothing operator. Both operators will be shown to be bounded operators from $L^p$ to $L^p$ for all $p \in [1, \infty]$. At first we choose a smooth cut-off function $\chi: \mathbb{R} \to [0, 1]$ with supp $\chi \subset [-R, R]$ and $\chi_{|_{(-R, R/2, R)}} \equiv 1$. Let $\rho: M \times M \to [0, 1]$ be given by $\rho(x,y) = \chi(\text{dist}_{\text{H}}(\pi_B(x), \pi_B(y)))$.

Let now $G_1(x,y) := \rho(x,y)G(x,y)$ and $G_2(x,y) := G(x,y) - G_1(x,y)$.

Then $G_2$ is zero on a neighbourhood of the diagonal, and thus smooth everywhere. The singular part is only contained in $G_1$.

Proposition 7.1. Let $M = \mathbb{M}^{m,k}_c$ and $G_1$ be as defined above. Then, for all $1 \leq p \leq \infty$ the map $P_1: \varphi \mapsto \int_M G_1(., y) \varphi(y) dy$ defines a bounded operator from $L^p$ to $L^p$.

Proof. We start with a smooth spinor $\varphi$ compactly supported in $B_{2R}(0) \times N \subset \mathbb{M}^{m,k}$. For such a $\varphi$ the spinor $P_1 \varphi$ is supported in $B_{3R}(0) \times N \subset M$. We embed $B_{3R}(0)$ isometrically into a closed Riemannian manifold $M_R$. Let $M_R \times N$. The metric on $M_R$ can be chosen such that $D^{M_R \times N} - \mu$ is invertible, cf. Proposition C.1. The norm of $(D^{M_R \times N} - \mu)^{-1}$: $L^p \to L^p$ is denoted by $C_R(p)$.

For $p < \infty$ we estimate

\[
\int_M |P_1 \varphi|^p dx = \left(\int_M \left(\int_M G_1(x,y) \varphi(y) dy\right)^p dx\right) \leq \int_{B_{2R}(0) \times N} \left(\int_{B_{2R}(0) \times N} G(x,y) \varphi(y) dy\right)^p dx \leq \int_{M_R \times N} \left| \left(D^{M_R \times N} - \mu\right)^{-1} \varphi\right|^p dx \leq C_R(p)^p \int_{M_R \times N} \|\varphi\|_{L^p}^p dx = C_R(p)^p \|\varphi\|_{L^p}^p.
\]

Next we want to consider arbitrary $\varphi \in L^p(M, \Sigma_M)$, $p < \infty$. Then $C_c^\infty(M, \Sigma_M)$ is dense in $L^p(M, \Sigma_M)$, and it suffices to consider $\varphi \in C_c^\infty(M, \Sigma_M)$. Choose points $(x_i)_{i \in I} \subset \mathbb{H}^{k+1}$ as in Lemma 2.1. Then $(B_{2R}(x_i) \times N)_{i \in I}$ and $(B_{3R}(x_i) \times N)_{i \in I}$ both cover $\mathbb{M}^{m,k}$ uniformly locally finite. We denote the multiplicity of the second cover by $L$ and choose a partition of unity $\eta_i$ subordinated to $(B_{2R}(x_i) \times N)_{i \in I}$. Let $\varphi = \sum \varphi_i$, where $\varphi_i = \eta_i \varphi \in C_c^\infty(B_{2R}(x_i) \times N, \Sigma_M)$. Hence, $P_1 \varphi_i \in C_c^\infty(B_{3R}(x_i) \times N, \Sigma_M)$. Moreover, let $f_i: M \to M$ be given by $f_i = (\text{Id}, f_i)$ where $f_i$ is an isometry of $\mathbb{H}^{k+1}$.
we now turn to the off-diagonal part \( G_2 \).

Note that \( H^{k+1} \) is homogeneous for all \( c \). In particular, the representation of the metric in polar coordinates \( -dr^2 + \sinh^2(r)\sigma_{ij}^k \) (cf. Section 2.2) is independent of the chosen origin of the polar coordinates on \( H^{k+1} \). We set \( M_y(r) := \{ x \in \mathbb{M}^{m,k} \mid \text{dist}(x, N^y) = r \} \) where \( N^y = \{ y_1 \} \times N \) where \( y = (y_1, y_2) \in \mathbb{H}^{k+1} \times N \). Then, the volume \( \text{vol}(M_y(r)) = f(r)^{k} \text{vol}(N) \text{vol}(S^k) = \sinh^{\frac{k}{2}}(r) \text{vol}(N) \text{vol}(S^k) \) is independent of \( y \). We will subsequently leave out the \( y \) in the notation and write \( \text{vol}(M(r)) \).

**Proposition 7.2.** Using the notations from above, assume that there are constants \( C, \rho > 0 \) with

\[
\int_{M_y(r)} |G_2(x,y)|^2 dx \leq C e^{-2\rho r} \quad \text{for all } r > 0.
\]

Let \( p = 1 \) and \( p = \infty \). Then, for \( \rho > \frac{c_k}{2} \) the operator \( P_2: \varphi \mapsto \int_M G_2(.,y)\varphi(y) dy \) from \( L^p \) to \( L^p \) is bounded.

**Proof.** We start with \( p = 1 \) and estimate for \( \varphi \in C_c^\infty(M,\Sigma_M) \)

\[
\int_M |(P_2\varphi)(x)| dx \leq \int_M \int_{M} |G_2(x,y)||\varphi(y)| dy dx = \int_M \left( \int_{M} |G_2(x,y)| dx \right) |\varphi(y)| dy
\]

\[
= \int_M \left( \int_{\mathbb{R}_+} \int_{M_y(r)} |G_2(x,y)| dx dr \right) |\varphi(y)| dy
\]

\[
\leq \int_M \left( \int_{\mathbb{R}_+} \text{vol}(M(r))^\frac{1}{2} \left( \int_{M_y(r)} |G_2(x,y)|^2 dx \right)^{\frac{1}{2}} \right) |\varphi(y)| dy
\]

\[
\leq C' \int_{r \geq r_0} \sinh^{\frac{k}{2}}(r) e^{-\rho r} dr \| \varphi \|_{L^1}.
\]

where \( \hat{x} \) is the angular part and \( r \) the radial part of \( x \).

For \( \rho > \frac{c_k}{2} \) the integral \( \int_{r \geq r_0} \sinh^{\frac{k}{2}}(r) e^{-\rho r} dr \) is bounded. Hence, \( P_2: L^1 \rightarrow L^1 \) is invertible.
Next, we consider the other case $p = \infty$. Then for $\varphi \in L^\infty(M, \Sigma_M)$

$$
| (P_2 \varphi)(x) | \leq \int_{\frac{c}{2}}^{\frac{c}{2}} \int_{M_x(r)} |G_2(x, y)||\varphi(y)|| \, dy \, dr
\leq \int_{\frac{c}{2}}^{\frac{c}{2}} \sup_{M_x(r)} |\varphi| \left( \int_{M_x(r)} |G_2(x, y)|| \, dy \right) \, dr
\leq \|\varphi\|_{L^\infty} \int_{\frac{c}{2}}^{\frac{c}{2}} \|G_2(x, y)\|_{L^2(M_x(r))} \text{vol}(M(r))^{\frac{1}{2}} \, dr
\leq C \|\varphi\|_{L^\infty} \int_{\frac{c}{2}}^{\frac{c}{2}} e^{-\rho r} \, \sinh^2(r) \, dr \leq \tilde{C} \|\varphi\|_{\infty}.
$$

where for $\rho > \frac{c}{k}$ the last inequality follows as above. Thus, $\|P_2 \varphi\|_{\infty} \leq \tilde{C} \|\varphi\|_{\infty}$.

8. $\sigma_p$ contains the $L^p$-spectrum on $\mathbb{M}^{m,k}_c$

In this section we prove one direction of Theorem 1.1:

**Proposition 8.1.** Let $p \in [1, \infty]$. Let $\lambda_0^-, \lambda_0^+ \geq 0$, be the lowest eigenvalue of the Dirac square on the closed Riemannian spin manifold $N$. The $L^p$-spectrum of the Dirac operator on $\mathbb{M}^{m,k}_c$ is a subset of

$$
\sigma_p := \left\{ \mu \in \mathbb{C} \left| \mu^2 = \lambda_0^2 + \kappa^2, |\text{Im}\, \kappa| \leq \frac{ck}{p - \frac{1}{2}} \right. \right\}.
$$

**Proof.** We will show that $D - \mu : H^p \subset L^p \rightarrow L^p$ has a bounded inverse for all $\mu \in \mathbb{C} \setminus \sigma_p$. Fix $\mu \in \mathbb{C} \setminus \sigma_p$, and let $\kappa \in \mathbb{C}$ such that $\mu^2 = \lambda_0^2 + \kappa^2$. For $p = 2$, the lemma follows from Remark 8.2.

Let now $p \in [1, \infty)$ and $\mu \notin \sigma_1 = \sigma_\infty$. Then $\mu \notin \sigma_2$ and $(D - \mu) : H^2(\mathbb{M}^{m,k}_c) \rightarrow L^2(\mathbb{M}^{m,k}_c)$ has a bounded inverse given by $P_\mu : \varphi \mapsto \int_{\mathbb{M}_c} G_\mu(x, y) \varphi(y) \, dy$. By Proposition 7.1 and Lemma 6.3 the operator $P_\mu : L^p \rightarrow L^p$ is bounded for $|\text{Im}\, \kappa| > ck \left| \frac{1}{p} - \frac{1}{2} \right| = c_2$. Hence, the $L^1$- and the $L^\infty$-spectrum of $D$ on $\mathbb{M}^{m,k}_c$ has to be contained in $\sigma_1 = \sigma_\infty$.

First we deal with the case that $\text{Im}\, \kappa > 0$. For $p \in [1, 2]$ we use the Stein Interpolation Theorem 2.3. Fix $\varepsilon > 0$ and $y_0 \in \mathbb{R}$. We set $h(z) := \mu(z)^2 := \lambda_0^2 + \kappa(z)^2 := \lambda_0^2 + (y_0 + \frac{c}{k}iz + i\varepsilon)^2$ and $A_z = (D^2 - h(z))^{-1}$. By Remark 6.1 the operators

$$
A_w + i y = \left( D^2 - \left( \lambda_0^2 + \left( y_0 - \frac{c}{k} y + i \left( \frac{ck}{2} w + \varepsilon \right) \right)^2 \right) \right)^{-1},
$$

for $0 \leq w \leq 1$ and $y \in \mathbb{R}$, are bounded as operators from $L^2$ to $L^2$. Furthermore

$$
A_{1 + iy} = \left( D^2 - \left( \lambda_0^2 + \left( y_0 - \frac{c}{k} y + i \left( \frac{ck}{2} + \varepsilon \right) \right)^2 \right) \right)^{-1}
$$

is bounded from $L^1$ to $L^1$ as seen above. Thus - as required to apply the Stein interpolation theorem - $A_y$ and $A_{1 + iy}$ are bounded operators from $L^1 \cap L^2$ to $L^1 \cap L^2$. Let now $\varphi \in L^1 \cap L^2$ and $\psi \in L^\infty \cap L^2$. Set $S := \{ z \in \mathbb{C} \mid 0 \leq \text{Re} \, z \leq 1 \}$. We define $b_{\varphi, \psi}\left( z \right) = (A_z \varphi, \psi)$. The map $b_{\varphi, \psi}$ is analytic in the interior of $S$, since the resolvent is, see Lemma B.5. Moreover, $|b_{\varphi, \psi}\left( z \right)| \leq \|A_z\| \|\varphi\|_{L^2} \|\psi\|_{L^2} \leq \left( \max_{0 \leq \text{Re} \, z \leq 1} \|A_z\| \right) \|\varphi\|_{L^2} \|\psi\|_{L^2}$ where $\|A_z\|$ denotes the operator norm for $A_z : L^2 \rightarrow L^2$. Thus, $b_{\varphi, \psi}\left( z \right)$ is uniformly bounded and continuous on $S := \{ z \in \mathbb{C} \mid 0 \leq \text{Re} \, z \leq 1 \}$. Thus, we can apply Theorem 2.3 and obtain for $t \in (0, 1)$
and \( p = \frac{2}{1+\varepsilon} \) that 
\[
A_t = \left( D^2 - h \left( \frac{2}{p} - 1 \right) \right)^{-1} = \left( D^2 - \left( \lambda_0^2 + \left( y_0 + c k i \left( \frac{1}{2} - \frac{1}{2} \right) + i \varepsilon \right)^2 \right) \right)^{-1}
\]
is bounded from \( L^p \) to \( L^p \).

In the case \( \text{Im} \mu < 0 \) we set analogously \( A_z = (D^2 - g(z))^{-1} \) for \( g(z) = \lambda_0^2 + (y_0 - \frac{d}{2}iz - i\varepsilon)^2 \) and obtain that 
\[
A_t = \left( D^2 - g \left( \frac{2}{p} - 1 \right) \right)^{-1}
\]
is bounded from \( L^p \) to \( L^p \). Since \( y_0 \in \mathbb{R} \) and \( \varepsilon > 0 \) can be chosen arbitrarily, we get for all \( \mu \in \mathbb{C} \setminus \sigma_p \) that \( \mu^2 \) is not in the \( L^p \)-spectrum of \( D^2 \).

Using Lemma B.8 the claim follows for \( p \in [1, 2] \) and with Lemma B.3(i) for \( p \in [2, \infty) \). \( \square \)

9. Construction of Test Spinors on \( \mathbb{H}^{k+1} \)

In this section we determine the Dirac \( L^p \)-spectrum of the hyperbolic space. The general case for \( \mathcal{M}_c \) is given in the next section.

**Proposition 9.1.** Let \( p \in [1, \infty] \). The \( L^p \)-spectrum of the Dirac operator \( D \) on the hyperbolic space \( \mathbb{H}^{k+1} \) is given by the set

\[
\sigma^p_{H} := \left\{ \mu \in \mathbb{C} \mid | \text{Im} \mu | \leq k \left\lfloor \frac{1}{p} - \frac{1}{2} \right\rfloor \right\}.
\]

**Proof.** From Proposition 8.1 we know that the \( L^p \)-spectrum is contained in \( \sigma^p_{H} \). Thus, it remains to show that each element \( \mu \) of \( \sigma^p_{H} \) is contained in the \( L^p \)-spectrum of \( D \). For that we start with a similar ansatz as was used in [16, Lemma 7] for the Laplacian.

Let the hyperbolic space \( \mathbb{H}^{k+1} \) be modelled by the space \( \{(y, x_1, \ldots, x_k) \mid y > 0\} \) equipped with the metric \( g = y^{-2}(dx_1^2 + \ldots + dx_k^2 + dy^2) \). We set \( e_i = y\frac{\partial}{\partial x_i} = y\partial_i \) for \( i = 1, \ldots, k \) and \( e_y = y\frac{\partial}{\partial y} = y\partial_y \). Then, \( (e_y, e_1, \ldots, e_k) \) forms an orthonormal basis, which can assumed to be positively oriented. Further we have \( [e_y, e_i] = e_i = -[e_i, e_y] \). All other commutators vanish. Then, \(-\Gamma^i_y = \Gamma^i_{yi} = 1\) and all other Christoffel symbols vanish. The orthonormal frame \( (e_y, e_1, \ldots, e_k) \) can be lifted to the spin structure \( \vartheta : P_{\text{Spin}}(\mathbb{H}^{k+1}) \to P_{SO}(\mathbb{H}^{k+1}) \), namely we choose a map \( E : \mathbb{H}^{k+1} \to P_{\text{Spin}}(\mathbb{H}^{k+1}) \) with \( \vartheta(E) = (e_y, e_1, \ldots, e_k) \). A spinor is by definition a section of the associated bundle \( \Sigma_{\mathbb{H}^{k+1}} = \text{Spin}(\mathbb{H}^{k+1}) \times_{\text{Spin}} \Sigma_{k+1} \), so every spinor can be written as \( x \mapsto [E(x), \varphi(x)] \) for a function \( \varphi : \mathbb{H}^{k+1} \to \Sigma_{k+1} \). Hence, identifying \( (e_y, e_1, \ldots, e_k) \) with the standard basis of \( \mathbb{R}^{k+1} \) we obtain [11 (4.8)], [3, Lemma 4.1]

\[
\nabla_{e_i}[E, \varphi] = [E, \partial_{e_i} \varphi + \frac{1}{2} e_i \cdot e_y \cdot \varphi]; \quad \nabla_{e_y}[E, \varphi] = [E, \partial_{e_y} \varphi]
\]

and

\[
D[E, \varphi] = [E, \sum_{i=1}^{k} e_i \cdot \partial_{e_i} \varphi + e_y \cdot \partial_{e_y} \varphi - \frac{k}{2} e_y \cdot \varphi]
\]

\[
= [E, \sum_{i=1}^{k} ye_i \cdot \partial_{e_i} \varphi + ye_y \cdot \partial_{e_y} \varphi - \frac{k}{2} e_y \cdot \varphi]. \quad (11)
\]

Let \( \psi_0 \in \Sigma_{k+1} \) be a unit-length eigenvector of the Clifford multiplication with the vector \( e_y = (1, 0, \ldots, 0)^t \in \mathbb{R}^{k+1} \) to the eigenvalue \( \pm i \), i.e. \( e_y \cdot \psi_0 = \pm i\psi_0 \). Set \( \varphi_n(x, y) = b(x)e_n(\log y)^n\psi_0 \) where \( a \in \mathbb{C}, b(x) \) is any compactly supported function on \( \mathbb{R}^k \), and where \( e_n : \mathbb{R} \to \mathbb{R} \) is chosen to be a smooth cut-off function compactly supported on \( (-4n, -n) \), \( e_n|_{[-3n, -2n]} \equiv 1 \) and \( |e_n'| \leq 2/n \). Then for \( p \in [1, \infty) \) one estimates \( \|e_n'\|_p / \|e_n\|_p \leq C n^{-p} \to 0 \) as \( n \to \infty \). For \( p = \infty \) we have \( \|e_n'\|_\infty / \|e_n\|_\infty \leq 2/n \to 0 \) as \( n \to \infty \). Then we set
\(\Phi_n := [E, \varphi_n]\) and obtain

\[
(D - \mu)\Phi_n = \left[ E, yc_n(\log y) y^\alpha \sum_{i=1}^k (\partial_i b) e_i \cdot \psi_0 \pm b(x)c'_n(\log y) y^\alpha i\psi_0 \right.
\]

\[
+ b(x)c_n(\log y)(\pm i\alpha \mp \frac{k}{2} - \mu)y^\alpha \psi_0 \right]
\]

(12)

In the following we will use the notation \((X \cdot \cdot) \in \text{End}(\Sigma_{k+1})\) for the Clifford multiplication by \(X \in \mathbb{R}^{k+1}\), and obviously its operator norm \(\|(X \cdot \cdot)\|\) equals to the usual norm of \(X\).

Let \(\mu = s \pm ik \left( \frac{1}{p} - \frac{1}{2} \right), s \in \mathbb{R}\). We choose \(z = \log y\) and \(\alpha = \frac{k}{2} \mp \mu = \frac{k}{p} \mp is\). Thus, the last summand in (12) vanishes and \(p\text{Re} \alpha = k\). Then, for \(p \in [1, \infty)\) we have

\[
\| (D - \mu)\Phi_n \|_p \leq C e^{-n} + \frac{\int_{-\infty}^0 |c_n(z)|^p dz}{\int_{-n}^0 |c_n(z)|^p dz} \rightarrow 0
\]

where the last inequality uses

\[
\int_{-\infty}^0 |c_n(z)|^p e^{zp} dz = \int_{-4n}^0 |c_n(z)|^p e^{zp} dz \leq e^{-np} \int_{-4n}^0 |c_n(z)|^p dz = e^{-np} \int_{-\infty}^0 |c_n(z)|^p dz.
\]

For \(p = \infty\) we have \(\mu = s \pm i\frac{k}{2}, \alpha = \mp is\) and the estimate above is done analogously.

Summarizing, we have shown that \(\partial_{\sigma^H_{p}}\), the boundary of \(\sigma^H_{p}\), is a subset of the Dirac \(L^p\)-spectrum for \(\mathbb{H}^{k+1}\) for \(p \in [1, \infty)\). Note that \(\sigma^H_{s} = \bigcup_{2 \geq t \geq s} \partial_{\sigma^H_{t}}\) for \(s < 2\) and \(\sigma^H_{s} = \bigcup_{2 \leq r \leq s} \partial_{\sigma^H_{r}}\) for \(s > 2\), respectively. Thus, using the Riesz-Thorin interpolation theorem we see that \(\sigma^H_{p}\) is a subset of the \(L^p\)-spectrum of \(D\) on \(\mathbb{H}^{k+1}\) for \(p \in [1, \infty]\).

\[\square\]

**Remark 9.2.** From (11) we obtain

\[
D^2[E, \varphi] = [E, \sum_{i,j} y^2 e_i \cdot e_j \cdot \partial_i \partial_j \varphi + \sum_{i} y^2 e_i \cdot e_y \cdot \partial_i \varphi - \frac{k}{2} \sum_{i} e_i \cdot e_y \cdot \partial_i \varphi
\]

\[
+ \sum_{i} y^2 e_y \cdot e_i \cdot \partial_y \varphi + \sum_{i} ye_y \cdot e_i \cdot \partial_i \varphi - y^2 \partial_y \varphi - y \partial_y \varphi + y \frac{k}{2} \partial_y \varphi
\]

\[
- y \frac{k}{2} \sum_{i} e_y \cdot e_i \cdot \partial_i \varphi + y \frac{k}{2} \partial_y \varphi - \frac{k^2}{4} \varphi
\]

\[
= [E, -y^2 \sum_{i} \partial_i^2 \varphi - y^2 \partial_y^2 \varphi + y(k-1) \partial_y \varphi + \sum_{i} ye_y \cdot e_i \cdot \partial_i \varphi - \frac{k^2}{4} \varphi].
\]
We use $\mu$ and $\varphi_n = b(x)c_n(log y)\psi_0$ of the last proposition with $b$, $\alpha$, $c_n$ and $\psi_0$ as therein. For $c_n$ we require additionally $|c''_n| \leq 8n^{-2}$. Hence, $||c''_n||_p/||c_n||_p \to 0$ as $n \to \infty$ for $p \in [1, \infty]$. Then we have

$$(D^2 - \mu^2)[E, \varphi_n] = \left[ E, \left( -y^2 c_n(log y) y^\alpha \sum_i \partial_i^2 b - y^2 b \partial^2_i (c_n(log y) y^\alpha) + y(k-1) b \partial_i(c_n(log y) y^\alpha) \right) \right]$$

and by analogous estimates as in Proposition 9.1 we have $||(D^2 - \mu^2)[E, \varphi_n]||_p/||[E, \varphi_n]||_p \to 0$ as $n \to \infty$.

Remark 9.3. Note that while the $L^2$-spectrum of the hyperbolic space only consists of continuous spectrum, this is no longer true for the $L^p$-spectrum for $p \neq 2$ as can be seen by considering $0 \in \sigma_{reg}^p$:

We view the hyperbolic space $(\mathbb{H}^{k+1}, g_\mathbb{H})$ modelled on the unit ball $B_1(0) \subset \mathbb{R}^{k+1}$ of the Euclidean space and equipped with the metric $g_\mathbb{H} = f^2 g_E$ where $f(x) = \frac{1}{1-|x|^2}$ and $|\cdot|$ denotes the Euclidean norm. Take a constant spinor $\psi$ on $B_1(0)$ normalized such that $||\psi||_{L^2(B_1(0), g_E)} = 1$. Then $D^{g_E}\psi = 0$. Using the identification of spinors of conformal metrics set $\varphi := f^{-\frac{k}{2}} \psi$. Then $D^{g}\varphi = 0$ and $||\varphi||_{L^p(g_\mathbb{H})} = \int_{B_1(0)} f^{k+1-\frac{k}{2}p} |\psi|^p d\text{vol}_{g_E}$. Thus, $\varphi$ is an $L^p$-harmonic spinor if and only if $\int_{B_1(0)} (1 - |x|^2)^{-k-\frac{k}{2}+\frac{k}{2}p} d\text{vol}_{g_E} < \infty$, i.e., if and only if $\int_{B_1(0)} (1 - |x|^2)^{-k-\frac{k}{2}+\frac{k}{2}p} d\text{vol}_{g_E} < \infty$. This is true precisely if $p > 2$. Thus, for all $p > 2$ the $L^p$-kernel of the Dirac operator on $(\mathbb{H}^{k+1}, g_\mathbb{H})$ is nontrivial.

10. THE $L^p$-SPECTRUM ON $\mathbb{M}^{m,k}_{c}$ CONTAINS $\sigma_p$

In this section we complete the proof of Theorem 1.1. In Proposition 9.1 it was shown that the $L^p$-spectrum on $\mathbb{M}^{m,k}_{c}$ is contained in $\sigma_p$. Thus, the converse remains to be shown. The case $N = \{y\}$ was solved in Proposition 9.1. Recall that by Lemma B.11 and Example B.12 the Dirac $L^p$-spectrum on $\mathbb{M}^{m,k}_{c}$ is point symmetric, i.e., it is symmetric with respect to the reflection $\lambda \mapsto -\lambda$.

Let now $\mu \in \partial \sigma_p$ with $\mu^2 = \lambda_0^2 + \kappa^2$, $|\Im \kappa| = ck \left| \frac{1}{2} - \frac{1}{2} \right|$ be given. By Proposition 9.1 and scaling, we see that $\kappa$ is in the spectrum of the Dirac operator of $\mathbb{H}^{k+1}$. Then, by Lemma B.8 $\kappa^2$ is in the $L^p$-spectrum of $(D^{\mathbb{H}^{k+1}})^2$, and by Remark 9.2 there is a sequence $\psi_i \in \Gamma(\Sigma^{\mathbb{H}^{k+1}})$ with $||(D^{\mathbb{H}^{k+1}})^2 - \kappa^2||_{L^p(\mathbb{H}^{k+1})} \to 0$ while $||\psi_i||_{L^p(\mathbb{H}^{k+1})} = 1$. Moreover, by Remark B.7 there is a $\psi \in \Gamma(\Sigma_N)$ with $||\psi||_{L^p(\Sigma_N)} = 1$ and $(D^{N})^2 \psi = \lambda_0^2 \psi$. Assume that at least one of the dimensions of $N$ and $\mathbb{H}^{k+1}$ is even. Then $\Sigma_{\mathbb{M}} = \Sigma_{\mathbb{H}^{k+1}} \otimes \Sigma_N$ and by (1) we have $D^2 = (D^{\mathbb{H}^{k+1}})^2 + (D^{N})^2$. We set $\varphi_i = \psi_i \otimes \psi$. Then

$$||D^2 - \mu^2||_{\varphi_i} = ||\psi_i \otimes \left( (D^{N})^2 - \lambda_0^2 \right) \psi + \left( (D^{\mathbb{H}^{k+1}})^2 - \kappa^2 \right) \psi \otimes \psi||_p$$

$$= ||\left( (D^{\mathbb{H}^{k+1}})^2 - \kappa^2 \right) \psi \otimes \psi||_p \to 0.$$
Thus, $\mu^2$ is in the $L^p$-spectrum of $D^2$. By the point symmetry of the spectrum and by Lemma [B.8] both $\mu$ and $-\mu$ are in the $L^p$-spectrum of $D$. Similarly we obtain the result if both the dimensions of $N$ and $\mathbb{H}^{k+1}_c$ are odd by setting $\varphi_i := \psi_i \otimes (\psi, \psi)$ in notation of Section 2.5.

Up to now we have shown that all $\mu \in \partial \sigma_p$ are in the $L^p$-spectrum of the Dirac operator on $\mathbb{M}_c$. Following the same arguments as in the last lines of the proof of Proposition [G.1] the proof of Theorem [H.1] is completed.

**Remark 10.1.** From Theorem [I.1] and Lemma [B.8] we can immediately read of the proof of Theorem 1.1 is completed.

Let us compare the $L^p$-spectrum for $D^2$ on $\mathbb{M}_c^{k+1}$, $\mathbb{H}^{k+1}$ ($c = 1$ and $\lambda_0 = 0$)

\[ s \mapsto -k^2 \left( \frac{1}{p} - \frac{1}{2} \right)^2 + s^2 + 2is\kappa \left( \frac{1}{2} - \frac{1}{p} \right), \]

with the one of the Laplacian on functions whose $L^p$-spectrum is given by the closed parabolic region bounded by [16] (1.5)

\[ s \mapsto k^2 \left( \frac{1}{p} - \frac{1}{2} \right) + s^2 + 2is\kappa \left( \frac{1}{2} - \frac{1}{p} \right). \]

Up to a shift in the real direction this is the same spectrum. However the qualitative difference is that for $p \neq 2$ the spectrum of $D^2$ contains negative real numbers, in contrast to the Laplacian.

**Appendix A. Function spaces**

We want to recall some analytical facts which are helpful to define spinorial function spaces on manifolds.

Let $(M^n, g)$ be an $n$-dimensional Riemannian spin manifold with Dirac operator $D$. A distributional spinor (or distribution with spinor values) is a linear map $C_c^\infty(M, \Sigma M) \to \mathbb{C}$ with the usual continuity properties of distributions. Any spinor with regularity $L^1_{loc}$ defines a distributional spinor by using the standard $L^2$-scalar product on spinors.

Then $D\varphi$ can be defined in the sense of distributions. Let $H^1(M, \Sigma M)$ be the set of distributional spinors $\varphi$, such that $\varphi$ and $D\varphi$ are in $L^s$, $s \in [1, \infty]$. Equipped with the norm $\|\varphi\|_{H^1} := \|\varphi\|_s + \|D\varphi\|_s$, this is a Banach space. This norm is the graph norm of $D$ viewed as an operator in $L^s$ to $L^s$.

**Lemma A.1.** Let $1 \leq s < \infty$. $C_c^\infty(M, \Sigma M)$ is dense in $H^1_s(M, \Sigma M)$.

**Proof.** Assume that $\varphi \in H^1_s(M, \Sigma M)$, $s < \infty$, is given. For a given point $p \in M$ and for any $\bar{R} > 0$ one can find a compactly supported smooth function $\eta_R: M \to [0, 1]$ such that $\eta_R \equiv 1$ on $B_{\bar{R}}(p)$ and such that $|\nabla \eta_R| \leq R^{-1}$. Then one easily sees $\lim_{R \to \infty} \|\varphi - \eta_R \varphi\|_s = 0$. Further we calculate

\[ \|D(\varphi - \eta_R \varphi)\|_s \leq \|\nabla \eta_R \cdot \varphi\|_s + \|(1 - \eta_R) D\varphi\|_s \to 0 \quad \text{as} \quad R \to \infty. \]

Thus the elements with compact support are dense in $H^1_s(M, \Sigma M)$. Now if $\psi \in H^1_s(M, \Sigma M)$ has compact support, it follows from standard results that it can be approximated by smooth compactly supported spinors.

Thus, for $s < \infty$, $H^1_s(M, \Sigma M)$ is equal to the completion of $C_c^\infty(M, \Sigma M)$ with respect to the graph norm of $D: L^s \to L^s$.

**Lemma A.2.** Let $1 < s < \infty$. On manifolds with bounded geometry, the $H^1_s$-norm is equivalent to the norm $\|\varphi\|_s + \|\nabla \varphi\|_s$. 23
The proof of the lemma relies on local elliptic estimates which follow from the Calderon-Zygmund inequality, e.g. [18] Theorem 9.9, see also [2] Lemma 3.2.2 for the geometric adaptation.

**Appendix B. General notes on the L_p-spectrum**

In this section we collect general facts on the L_p-spectrum of the Dirac operator. Unless stated otherwise, we only assume that (M, g) is complete.

Let \( D: \mathcal{H}_1^p(M, \Sigma_M) = \text{dom} \, D \subset L^2(M, \Sigma_M) \to L^2(M, \Sigma_M) \) be the classical Dirac operator on L^2-spinors. The set of compactly supported spinors \( C_c^\infty(M, \Sigma_M) \) is a core of \( D \), i.e., \( D \) is the closure of \( D|_{C_c^\infty(M, \Sigma_M)} \) w.r.t. the graph norm \( H^2_p \). If we consider the restriction \( D|_{C_c^\infty(M, \Sigma_M)} \) and complete it w.r.t. the graph norm \( \| \varphi \|_{H^2_p} = \| \varphi \|_s + \| D\varphi \|_s \) for \( 1 \leq s < \infty \), then we obtain for each \( s \) a closed Dirac operator \( D_s: \mathcal{H}_1^p = \text{dom} \, D_s \subset L^s \to L^s \).

For \( s = \infty \) we define \( D_\infty: H^\infty \to L^\infty \), \( \psi \to D_\infty \psi \) by \( (D_\infty \varphi, \psi) = (\varphi, D_\infty \psi) \) for all \( \varphi \in C_c^\infty(M, \Sigma_M) \). Then \( D_\infty \) is a closed, continuous extension of \( D|_{C_c^\infty(M, \Sigma_M)} \) but \( C_c^\infty(M, \Sigma_M) \) is in general no longer a core for this operator. Note that in contrast to that, in the standard literature for L^p-theory of the Laplacian, e.g. [16], the operator for \( s = \infty \) is directly defined to be as the adjoint operator for \( s = 1 \). For \( s < \infty \) one can define \( D_s \) distributional as well and the resulting operator coincides with the definition given above as will be seen in Lemma [B.1].

Next, we can examine the adjoint of the operator \( D_s: L^s \to L^s \) with respect to the duality pairing \( \langle \cdot, \cdot \rangle: L^s \times (L^s)^* \to \mathbb{C} \) whose restriction to compactly supported spinors coincides with the hermitian L^2-product. We use the convention that this pairing is antilinear in the second component. The adjoint \( D_s^* \) is an operator in \((L^s)^*\). For \( 1 \leq s < \infty \) and \( s^{-1} + (s^*)^{-1} = 1 \), \((L^s)^* = L^{s^*}\) whereas \((L^\infty)^* \) is larger than \( L^1 \). From the formal self-adjointness of \( D \) we see, that \( D_s|_{C_c^\infty(M, \Sigma_M)} = D_s^*|_{C_c^\infty(M, \Sigma_M)} \). Moreover, we have

**Lemma B.1.** For all \( \varphi \in H^s_1 \) and \( \psi \in H^{s^*}_1 \), \( 1 \leq s \leq \infty \), we have
\[
\langle D_s \varphi, \psi \rangle = \langle \varphi, D_s^* \psi \rangle.
\]

**Proof.** For \( 1 < s < \infty \), let \( \varphi_i, \psi_j \in C_c^\infty(M, \Sigma_M) \) with \( \varphi_i \to \varphi \) in \( H^s_1 \) and \( \psi_j \to \psi \) in \( H^{s^*}_1 \). Then,
\[
\int_M \langle D_s \varphi_i, \psi_j \rangle dvol_g = \int_M \langle D_s \varphi_i, \psi \rangle dvol_g = \int_M \langle \varphi_i, D_s^* \psi \rangle dvol_g = \int_M \langle \varphi, D_s^* \psi \rangle dvol_g
\]
as \( i, j \to \infty \).

Let now \( s = 1 \). For \( \varphi \in C_c^\infty(M, \Sigma_M) \) the equality follows from the distributional definition of \( D_\infty \). The rest follows since \( C_c^\infty(M, \Sigma_M) \) is dense in \( H^1_1 \). The remaining case \( s = \infty \) just follows from the last one by interchanging \( s \) and \( s^* \). \( \square \)

**Lemma B.2.** For all \( 1 \leq s < \infty \) the operators \( D_s^* \) and \( D_s^* \) coincide.

**Proof.** For \( \psi \in H^{s^*}_1 \) Lemma B.1 yields \( \langle D_s \varphi, \psi \rangle = \langle \varphi, D_s^* \psi \rangle \) for all \( \varphi \in H^s_1 \) = dom \( D_s \). This implies \( \psi \in \text{dom} \, D_s^* \) and \( D_j^* \psi = D_s^* \psi \). Hence, \( H^{s^*}_1 \subset \text{dom} \, D_s^* \) and \( D_s^* \psi = D_s^* \psi \). It remains to show that \( \text{dom} \, D_s^* \subset L^{s^*} \). Let \( \psi \in \text{dom} \, D_s^* \subset (L^s)^* = L^{s^*} \). Then there is a \( \rho \in L^{s^*} \) such that for all \( \varphi \in \text{dom} \, D_s \) it holds \( \langle D_s \varphi, \psi \rangle = \langle \varphi, \rho \rangle \). In particular, this is true for all \( \varphi \in C_c^\infty(M, \Sigma_M) \). In other words \( D_s^* \psi = \rho \) in the sense of distributions. Thus, \( \psi \in H^{s^*}_1 \). \( \square \)

Since \( \varphi \in H^s_1 \cap H^{s^*}_1 \) implies \( D_s \varphi = D_{s^*} \varphi \) we often denote all those Dirac operators in the following just by \( D \).

Moreover, a closed operator \( P: \text{dom} \, P \subset V_1 \to V_2 \) between Banach spaces \( V_i \), and with dense domain \( \text{dom} \, P \), will be called invertible if there exists a bounded inverse \( P^{-1}: V_2 \to V_1 \). We will use the phrase “\( P \) has a bounded inverse” synonymously.

**Lemma B.3.** Let \( 1 \leq s < \infty \).
(i) If \( \mathcal{P} \) is in the \( L^s \)-spectrum of the Dirac operator where \( (s^*)^{-1} + s^{-1} = 1 \), then \( \mu \) is in its \( L^s \)-spectrum.

(ii) Let \( D_\ast - \mu \) be invertible. Then, \( (D_\ast - \mu)^{-1} = ((D_\ast - \mu)^{-1})^* \) and \( \| (D_\ast - \mu)^{-1} \| = \| (D_\ast - \mu)^{-1} \| \).

**Proof.** We prove this for \( \mu = 0 \). For arbitrary \( \mu \) this is done analogously.

Assume that 0 is not in the \( L^s \)-spectrum of \( D \), i.e., it has a bounded inverse \( E = D^{-1} : L^s \to L^s \) with range \( \text{ran} \ E = H^1_1 \). Let \( \varphi \in L^s \). Since \( E \) is bounded, \( f : L^s \to \mathbb{C}, \rho \mapsto (E\rho, \varphi) \) is a bounded functional and, thus, \( f \) is in the dual space of \( L^s \), i.e., there is \( \psi \in L^s \) with \( (\rho, \psi) = f(\rho) = (E\rho, \varphi) \) for all \( \rho \in L^s \). Hence, \( \varphi \in \text{dom} \ E^* \), i.e., \( \text{dom} \ E^* = L^s \).

Now we can estimate for all \( \varphi \in H^1_1 \) and all \( \psi \in \text{dom} \ E^* \) that \( (D\varphi, E^*\psi) = (\text{Ed}\varphi, \psi) = (\varphi, \psi) \) which implies \( E^*\psi \in \text{dom} \ D^* \) and \( D^*E^*\psi = \psi \). Thus, \( \text{ran} \ D^* = L^s \) and \( D^*E^* = \text{Id} : L^s \to L^s \).

If \( \rho \in L^s \) and \( \varphi \in \text{dom} \ D^* \), we get \( (\rho, E^*D^*\varphi) = (E\rho, D^*\varphi) = (DE\rho, \varphi) = (\rho, \varphi) \). Hence, \( E^*D^* = \text{Id} : \text{dom} \ D^* \to \text{dom} \ D^* \). Together with the corresponding statement from above this gives that \( (D^{-1})^* = (D^*)^{-1} \). Thus, 0 is not in the \( L^s \)-spectrum of \( D \). This proves (i) and the first claim of (ii). The operator norm of an operator and its adjoint coincide, see [22] Thm VI.2. Thus, the equality of the operator norms follows.

**Corollary B.4.** If \( D : H^1_q \to L^q \) has a bounded inverse for some \( q \in (1, \infty) \). Then as an operator from \( H^1_q \to L^q \) it has a bounded inverse for all \( s \in [q_1, q_2] \) where \( q_1 = \min\{q, q^*\} \), \( q_2 = \max\{q, q^*\} \), and \( (q^*)^{-1} + q^{-1} = 1 \). In particular, the \( L^2 \)-spectrum of \( D \) is a subset of the \( L^q \)-spectrum.

**Proof.** This Lemma follows directly from the Riesz-Thorin Interpolation Theorem [22] (using \( D = C^\infty_c(M, \Sigma_M) \)) and Lemma B.3.

**Lemma B.5.** Let \( 1 \leq s \leq \infty \). Let \( R_s = \mathbb{C} \setminus \text{Spec}_{L^s}(D) \) be the resolvent set of \( D : L^s \to L^s \). Then, the resolvent

\[
\mu \in R_s \mapsto (D - \mu)^{-1} \in \mathcal{B}(L^s)
\]

is analytic, i.e., the map is locally given by a convergent power series with coefficients in \( \mathcal{B}(L^s) \). Here, \( \mathcal{B}(L^s) \) denotes the set of bounded operators from \( L^s \) to itself.

**Proof.** The proof is done similar as in the case of bounded operators [24] Satz 23.4]: Choose \( \mu_0 \in R_s \) and \( \mu \in \mathbb{C} \) such that \( |\mu - \mu_0| < \| (D - \mu_0)^{-1} \|^{-1} \). Then, one calculates that \( D - \mu \) is invertible as well, see the proof of [24] Lemma 23.2. Here we used the fact the operator \( (D - \mu)^{-1} \) and \( (D - \mu_0)^{-1} \) have the common core \( C^\infty_c(M, \Sigma_M) \). Then,

\[
(D - \mu)^{-1} = \sum_{n=0}^{\infty} ((D - \mu_0)^{-1})^{n+1} (\mu - \mu_0)^n.
\]

For rounding up our presentation we will next add a lemma not needed in our context but maybe helpful to other applications.

**Lemma B.6.**

(1) The operator \( D : H^1_q \subset L^s \to L^s \), \( s \in [1, \infty] \), is an invertible map onto its image if and only if there is a constant \( C > 0 \) with \( \|D\varphi\|_s \geq C\|\varphi\|_s \) for all \( \varphi \in H^1_q \).

(2) Under the above conditions the image \( D(H^1_q) \) is closed.

(3) Let \( s^{-1} + (s^*)^{-1} = 1 \), \( s < \infty \), and assume the conditions from above. Then \( D \) is surjective if and only if there is a \( C > 0 \) with \( \|D\varphi\|_{s^*} \geq C\|\varphi\|_{s^*} \) for all \( \varphi \in H^1_q \).

**Proof.** (1) The proof is straightforward.

(2) The operator \( D : H^1_q \to D(H^1_q) \), where the latter space is equipped with the \( L^s \)-norm, is a bijective bounded linear map. Hence, \( D(H^1_q) \) is a complete subspace of \( L^s \) and thus closed.

(3) Suppose that \( D(H^1_q) \) is a proper subspace of \( L^s \). Due to Hahn-Banach there is a non-zero
continuous functional ψ: L^s → C vanishing on D(∂_1^*)}. We interpret ψ as an element in L^s* using the Riesz representation theorem, i.e. ψ ∈ L^s* is orthogonal on D(∂_1^*). Then, ψ ∈ dom (D_λ)^*, and we even have D_λ^* ψ = 0. Hence, by Lemma B.2, ψ ∈ H_1^*. This contradicts the estimate.

Now assume that D is surjective. Then there is a bounded operator D^{-1}: L^s → L^s, inverse to D. Thus (D^{-1})^*: L^s* → L^s* is bounded as well, and (D^{-1})^* is the inverse of D^*: H_1^* → L^s. The fact that the latter map has a bounded inverse is equivalent to the existence of a constant C > 0 with \|D_λ\|_s^* ≤ C\|ψ\|_s^*.

\[\square\]

**Remark B.7.** The L^s-spectrum of the Dirac operator D on a closed manifold (M^m, γ) is independent of s. We sketch the proof: Let ϕ be an L^2-eigenvalue of D. Then regularity theory implies that ϕ ∈ C^∞(M, SM) and, hence, ϕ ∈ L^s for all 1 ≤ s ≤ ∞. In particular, Spec_L^M(D) ⊂ Spec_{LM}^M(D). Let now µ ∈ Spec_{LM}^M(D), i.e., (D − µ)^{-1}: L^2 → L^2 is bounded. Let G(x, y) be the unique Green function of D − µ, see Proposition 3.2. Then, \(\int_M |G(., y)|^2 dy\) is bounded uniformly in y. Hölder’s inequality implies that also \(\int_M |G(., y)|^2 dy\) is bounded uniformly in y. Hence, (D − µ)^{-1}: L^1 → L^1 is a bounded operator. Then interpolation gives that (D − µ)^{-1}: L^s → L^s is bounded for all 1 ≤ s ≤ 2. Because of Spec_L^M(D) ⊂ R the same is true for (D − µ)^{-1}: L^s → L^s, and by using Lemma B.3 we get that (D − µ)^{-1}: L^s → L^s is bounded for all 2 < s < ∞. It remains s = ∞: Let r > m. Then by the Sobolev Embedding Theorem H_1^r → L^∞ is bounded. Moreover, by the discussion above and using the fact that H_1^r carries the graph norm of D we know that (D − µ)^{-1}: L^1 → H_1^r is bounded for µ ∈ Spec_{LM}^M(D) the Hölder inequality gives that

\[(D − µ)^{-1}: L^∞ → L^r → H_1^r → L^∞\]

is bounded.

**Lemma B.8.** Let 1 ≤ s ≤ ∞, and let Spec_{LM}^M(D) ⊄ C. Then the complex number µ^2 is in the L^s-spectrum of D^2 if and only if µ or −µ is in the L^s-spectrum of D.

**Proof.** We start with the “only if” part. So assume that both µ and −µ are not in the L^s-spectrum of D. Then we have bounded operators (D − µ)^{-1}: L^s → L^s and (D + µ)^{-1}: L^s → L^s. It is then easy to verify that (D − µ)^{-1} ∘ (D + µ)^{-1}: L^s → L^s is a bounded inverse of D^2 − µ^2 = (D + µ) ∘ (D − µ). Thus µ^2 is not in the L^s-spectrum of D^2.

In order to prove the “if” statement, we assume that µ^2 is not in the spectrum of D^2. Then D^2 − µ^2 has a bounded inverse P := (D^2 − µ^2)^{-1}: L^s → L^s. Let ψ ∈ P(L^s). Then ψ ∈ L^s and D^2ψ ∈ L^s. Next we will show that this implies D_λ ψ ∈ L^s. For that we choose λ ∉ Spec_{LM}^M(D). Then D_λ ψ = (D − λ)^{-1} (D^2 − λ^2) ψ − λ ψ, and hence D_λ ψ ∈ L^s. Thus, P(L^s) ⊂ H_1^s. Hence Q_1 := (D + µ) ∘ P is a bounded operator with dom Q_1 = L^s, and one easily checks that this a right inverse to (D + µ). Similarly, one shows that Q_2 := P ∘ (D + µ) is a left inverse of (D − µ). A priori Q_2 is only defined on H_1^s, but using Q_1 = Q_1 ∘ (D + µ) ∘ Q_2 = Q_2 it is clear that Q_2 and Q_1 coincide on H_1^s. So the integral kernels of Q_1 and Q_2 have to coincide, so Q_1 is a left and right inverse of (D ± µ) and thus ±µ is not in the spectrum of D.

\[\square\]

**Remark B.9.** In the case 1 < s < ∞ and M of bounded geometry, one can also prove that Spec_{LM}^M(D) = C implies Spec_{LM}^M(D^2) = C: As in the proof of the “if” statement from above one has to show that D_λ ψ ∈ L^s. This can be proven using regularity theory on manifolds of bounded geometry.

**Lemma B.10** (Pointwise symmetries). Let 1 ≤ s ≤ ∞. Let (M, γ) be an m-dimensional Riemannian spin manifold.

(i) m ≡ 0 mod 2: The number µ is in the L^s-spectrum of D if and only if −µ is in the L^s-spectrum of D if and only if µ is in the L^s-spectrum of D.
(ii) $m \equiv 1 \mod 4$: The number $\mu$ is in the $L^s$-spectrum of $D$ if and only if $-\bar{\mu}$ is in the $L^s$-spectrum of $D$.

(iii) $m \equiv 3 \mod 4$: The number $\mu$ is in the $L^s$-spectrum of $D$ if and only if $\bar{\mu}$ is in the $L^s$-spectrum of $D$.

Proof. By [17, Prop. p. 31] we have a map $\alpha: \Sigma_m \to \Sigma_m$ that is

- a Spin$(m)$-equivariant real structure that anticommutes with Clifford multiplication if $m \equiv 0, 1 \mod 8$.
- a Spin$(m)$-equivariant quaternionic structure that commutes with Clifford multiplication if $m \equiv 2, 3 \mod 8$.
- a Spin$(m)$-equivariant quaternionic structure that anticommutes with Clifford multiplication if $m \equiv 4, 5 \mod 8$.
- a Spin$(m)$-equivariant real structure that commutes with Clifford multiplication if $m \equiv 6, 7 \mod 8$.

Note that by definition real structure means that $\alpha^2 = \text{Id}$ and $\alpha(iv) = -i\alpha(v)$. Moreover, quaternionic structure means that $\alpha^2 = -\text{Id}$ and $\alpha(iv) = -i\alpha(v)$.

Due to the Spin$(m)$-equivariance $\alpha$ induces a fiber preserving map $\tilde{\alpha}$ on the spinor bundle with the same properties as above. Thus,

$$(D - \mu) \circ \tilde{\alpha}(\varphi) = \begin{cases} \tilde{\alpha} \circ (D - \bar{\mu})(\varphi) & m \equiv 0, 1 \mod 4 \\ \tilde{\alpha} \circ (D - \bar{\mu})(\varphi) & m \equiv 2, 3 \mod 4. \end{cases}$$

Thus, if $\mu$ is in the $L^s$-spectrum of $D$ then $-\bar{\mu}$ (resp. $\bar{\mu}$) in the $L^s$-spectrum of $D$ for $m \equiv 0, 1$ (resp. 2, 3) mod 4. This gives (ii) and (iii).

If $m$ is even, then $D(\omega_M \cdot \varphi) = -\omega_M \cdot D\varphi$. Thus, the spectrum is symmetric when reflected on the imaginary axis. Together with the symmetries from above, (i) follows. \qed

Lemma B.11 (Orientation reversing isometry). Let $1 \leq s \leq \infty$. Assume there is an orientation reversing isometry $f: M^n \to M^m$ that “lifts” to the spin structure as described in the proof. Then $\mu$ is in the $L^s$-spectrum of $D$ if and only if $-\mu$ is in the $L^s$-spectrum of $D$.

Proof. The proof follows the lines of [3, Appendix A]. In this reference, $f$ is required to be a reflection at a hyperplane of $M$. But this doesn’t change the part we need: We lift $f$ to the bundle $P_{SO(m)}M$ of oriented orthonormal frames by mapping the frame $E = (e_1, \ldots, e_m)$ to $f_*E = (-df(e_1), df(e_2), \ldots, df(e_m))$, so $f_*: P_{SO(m)}M \to P_{SO(m)}M$. Since $f$ is an orientation reversing isometry, $f_*(\mathcal{E}A) = f_*(\mathcal{E})JA$ for all $A \in SO(m)$ where $J = \text{diag}(-1, 1, 1, \ldots, 1)$. The map $f$ is assumed to lift to the spin structure, i.e., there is a lift $f_*: P_{Spin(m)}(M) \to P_{Spin(m)}(M)$ with $\vartheta \circ f_* = f_* \circ \vartheta$ where $\vartheta$ denotes the double covering $\vartheta: P_{Spin(m)}(M) \to P_{SO(m)}(M)$. By [3, Lemma A.1 and Lemma A.4], $f$ then lifts to a map $f_\sharp: \Sigma_M \to \Sigma_M$ on the spinor bundle which fulfills $f_\sharp(D\varphi) = -D(f_\sharp\varphi)$. \qed

Example B.12.

(i) Let $M^{n+1}$ be a Riemannian spin manifold with a spin structure $\vartheta$ as above. Assume that up to isomorphism this is the unique spin structure on $M$. Let $f: M \to M$ be an orientation reversing isometry. By pulling back the double covering $P_{Spin}M \to P_{SO}M$ by $f_*$ we obtain the double covering $f^*\vartheta: f^*P_{Spin}M \to P_{SO}M$. We then turn $f^*P_{Spin}M$ into a Spin$(n+1)$-principal bundle by conjugating the action of Spin$(n+1)$ on $P_{Spin}M$ with Clifford multiplication with $c_0$. Then $f^*\vartheta$ is a spin structure on $M$. Thus an isomorphism from $\vartheta$ to $f^*\vartheta$ yields a map $f_\sharp$ as above.

(ii) Consider the map $f = f_1 \times \text{id}: M^m = \mathbb{H}^{k+1} \times N^m \to M^m \times \mathbb{H}^{k+1}$ where $f_1$ is an orientation reversing isometry as in (i). Then, $f$ is again an orientation reversing isometry. Using $P_{SO}(\mathbb{H}^c \times N) = (P_{SO}(\mathbb{H}^{k+1}) \times P_{SO}(N)) \times \xi SO(m)$ where $\xi: SO(k+1) \times SO(n) \to SO(m)$
is the standard embedding and using the analogous description for $P_{\text{Spin}}(\mathbb{H}_c \times N)$, see Section 2.5, one see that also $f$ lifts to the spin structure.

Appendix C. Dirac eigenvalues of generic metrics

Proposition C.1. Let $(M, g)$ be a closed, connected Riemannian spin manifold, let $\mu \in \mathbb{R}$. Let $U \subset M$ be a nonempty open subset. In case that $\mu = 0$, assume additionally that the $\alpha$-genus of $M$ is zero. Then, there is a metric $\tilde{g}$ on $M$ with $\tilde{g} = g$ on $M \setminus U$ and $\ker (D_{\tilde{g}} - \mu) = \{0\}$.

Proof. For $\mu = 0$, the proposition follows from [3, Theorem 1.1]. For $\mu \neq 0$, the proof is a direct consequence of the following lemma. \hfill \Box

Lemma C.2. Let $(M, g)$ be a closed, connected Riemannian spin manifold, let $\mu \in \mathbb{R} \setminus \{0\}$, and let $U \subset M$ be a nonempty open subset. Then there is a function $f \in C^\infty(M, \mathbb{R}^+)$ with $f|_{M \setminus U} \equiv 1$ such that $\ker (D_{\tilde{g}} - \mu) = \{0\}$.

Proof. Choose $f \in C^\infty(M, \mathbb{R}^+)$ with $f|_{M \setminus U} \equiv 1$ such that $d = \dim (E_{f, \mu} := \ker (D_{\tilde{g}} - \mu))$ is minimal. Assume $d > 0$, and set $g_0 = fg$. For $\alpha \in C^\infty(M)$ with supp $\alpha \subset U$ and $t$ close to 0 we define $g_t := (1 + \alpha t)fg$. Then by [3] there are real analytic functions $\mu_1, \ldots, \mu_d : (\varepsilon, \varepsilon) \to \mathbb{R}$ with $\mu_0(0) = 0$ such that $\text{Spec}_{E_t}(D^{\alpha}) \cap (\mu - \delta, \mu + \delta) = \{\mu_1(t), \ldots, \mu_d(t)\}$ including multiplicities. It is shown in [3] that there is an orthonormal basis $(\psi^{(1)}, \ldots, \psi^{(d)})$ of $E_{f, \mu}$ depending on the choice of $\alpha$ such that

$$\frac{d}{dt}|_{t=0}\mu_i(t) = -\frac{1}{2} \int_M \langle \alpha g_0, Q_{\psi^{(i)}} \rangle \text{dvol}_{g_0}$$

where

$$Q_{\psi}(X, Y) = \frac{1}{2} \text{Re} \langle X, \nabla_Y \psi + Y, \nabla_X \psi \rangle.$$ 

Thus,

$$\langle g_0, Q_{\psi^{(i)}} \rangle = \sum_r \langle e_r, \nabla_{e_r} \psi^{(i)}, \psi^{(i)} \rangle = \mu \langle \psi^{(i)}, \psi^{(i)} \rangle^2.$$ 

As $d$ is minimal, we see that $\frac{d}{dt}|_{t=0}\mu_i(t) = 0$, and thus for all $\alpha$ as above

$$-\frac{1}{2} \int_M \alpha \mu \sum_{i=1}^d |\psi^{(i)}|^2 \text{dvol}_{g_0} = 0.$$ 

Note that $\varphi := \sum_{i=1}^d |\psi^{(i)}|^2 \in C^\infty(M)$ does not depend on the choice of $\alpha$. This can be seen by direct calculation with base change matrices or alternatively by observing that $\varphi$ is the pointwise trace of the integral kernel of the projection to $E_{f, \mu}$. With $\mu \neq 0$ this implies that $\varphi$ and thus all $\psi^{(i)}$ vanish on $U$. The unique continuation principle implies then $\psi^{(i)} \equiv 0$ which gives a contradiction. \hfill \Box

References


