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The continuum limit of a fermion system involving leptons and quarks: Strong, electroweak and gravitational interactions

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THE CONTINUUM LIMIT OF A FERMION SYSTEM INVOLVING LEPTONS AND QUARKS: STRONG, ELECTROWEAK AND GRAVITATIONAL INTERACTIONS

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Abstract. The causal action principle is analyzed for a system of relativistic fermions composed of massive Dirac particles and neutrinos. In the continuum limit, we obtain an effective interaction described by classical gravity as well as the strong and electroweak gauge fields of the standard model.

Contents

1. Introduction

This is the third paper in the series [3, 4] in which the continuum limit of causal fermion systems is worked out. Here we consider a system which is composed of seven massive sectors and one neutrino sector, each containing three generations of particles. Analyzing the Euler-Lagrange equations of the causal action principle in the continuum limit, we obtain a unification of gravity with the strong and electroweak forces of the standard model.

More precisely, we obtain three main results. The first main result is the so-called *spontaneous block formation* (see Theorem 3.2), stating that the eight sectors form pairs, so-called blocks. The block involving the neutrinos can be regarded as the leptons, whereas the three other blocks correspond to the quarks (in the three colors). The index distinguishing the two sectors within each block can be identified with the isospin. The interaction can be described effectively by $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$-gauge potentials acting on the blocks and on the isospin index. In this way, one recovers precisely the gauge potentials of the standard model together with their correct coupling to the fermions.

Our second main result is to derive the *effective field equations* corresponding to the gauge fields. Theorem 4.1 gives the general structure of the electroweak theory of the standard model after spontaneous symmetry breaking, but the masses and coupling constants involve more free parameters than in the standard model. In Theorem 4.2 it is shown that one gets precise agreement with the electroweak theory if one imposes three additional relations between the free parameters. Finally, in Proposition 4.3 it is shown that these three additional relations hold in the limit when the mass of the top quark is much larger than the lepton masses. We thus obtain agreement with the strong and electroweak theory up to small corrections. These corrections are discussed, and some of them are specified quantitatively.

Our third main result is to derive the gravitational interaction and the *Einstein equations* (see Theorem 4.4).

We point out that the continuum limit gives the correspondence to the standard model and to general relativity on the level of second-quantized fermion fields coupled to classical bosonic fields. For the connection to second-quantized bosonic fields we refer to [5, 6]. We also point out that we do not consider a Higgs field. This is why we get the correspondence to the standard model after spontaneous symmetry breaking without the Higgs field (i.e. for a constant Higgs potential). But in Section 5 it is explained that the Higgs potential can possibly be identified with scalar potentials in the Dirac equation.

2. Preliminaries

In this section we repeat constructions used in the previous papers [3, 4] and adapt them to the system of Dirac seas to be considered here.

2.1. The Fermionic Projector and its Perturbation Expansion. We want to extend the analysis in [3, 4] to a system involving quarks. Exactly as explained in [2, §5.1], the quarks are described by additional sectors of the fermionic projector. More precisely, we describe the vacuum similar to [4, eq. (1.4)] by the fermionic projector

$$P(x,y) = P^N(x,y) \oplus P^C(x,y) \,, \tag{2.1}$$

where the *charged component* P^C is formed as the direct sum of seven identical sectors, each consisting of a sum of three Dirac seas,

$$P^C(x,y) = \bigoplus_{a=1}^{7} \sum_{\beta=1}^{3} P_{m_\beta}(x,y) \,, \tag{2.2}$$

where m_β are the masses of the fermions and P_m is the distribution

$$P_m(x,y) = \int \frac{d^4k}{(2\pi)^4}\, (\not{k}+m)\, \delta(k^2-m^2)\, \Theta(-k^0)\, e^{-ik(x-y)} \,.$$

Every massive sector has the form as considered in [3, eq. 3.1]. For the *neutrino sector* P^N we choose the ansatz of massive neutrinos (cf. [4, eq. (1.8)])

$$P^N(x,y) = \sum_{\beta=1}^{3} P_{\tilde{m}_\beta}(x,y) \,. \tag{2.3}$$

The neutrino masses $\tilde{m}_\beta \geq 0$ will in general be different from the masses m_β in the charged sector. For a discussion of this ansatz we also refer to [4, §2.4], where the alternative ansatz of chiral neutrinos is ruled out.

We introduce an *ultraviolet regularization* on the length scale ε. The regularized vacuum fermionic projector is denoted by P^ε. We again use the formalism of the continuum limit as developed in [2, Chapter 4] and described in [3, Section 5]. In the neutrino sector, we work exactly as in [4, § 2.5] with a non-trivial regularization by right-handed high-energy states.

In order to describe an interacting system, we proceed exactly as described in [2, §2.3], [3, Section 4] and [4, §2.6]. We first introduce the *auxiliary fermionic projector* by

$$P^{\text{aux}} = P^N_{\text{aux}} \oplus P^C_{\text{aux}} \,,$$

where

$$P^N_{\text{aux}} = \Big(\bigoplus_{\beta=1}^{3} P_{\tilde{m}_\beta} \Big) \oplus 0 \qquad \text{and} \qquad P^C_{\text{aux}} = \bigoplus_{a=1}^{7} \bigoplus_{\beta=1}^{3} P_{m_\beta} \,. \tag{2.4}$$

Note that P^{aux} is composed of 25 direct summands, four in the neutrino and 21 in the charged sector. The fourth direct summand of P^N has the purpose of describing the right-handed high-energy states. Moreover, we introduce the *chiral asymmetry matrix* X and the *mass matrix* Y by (cf. [4, eqs. (2.42) and (2.43)])

$$X = (\mathbb{1}_{\mathbb{C}^3} \oplus \tau_{\text{reg}}\, \chi_R) \oplus \bigoplus_{a=1}^{7} \mathbb{1}_{\mathbb{C}^3}$$

$$mY = \text{diag}\big(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, 0\big) \oplus \bigoplus_{a=1}^{7} \text{diag}\big(m_1, m_2, m_3\big) \,,$$

where m is an arbitrary mass parameter. Here $\tau_{\text{reg}} \in (0,1]$ is a dimensionless parameter for which we always assume the scaling

$$\tau_{\text{reg}} = (m\varepsilon)^{p_{\text{reg}}} \qquad \text{with} \qquad 0 < p_{\text{reg}} < 2 \,.$$

This allows us to rewrite the vacuum fermionic projector as

$$P^{\text{aux}} = Xt = tX^* \qquad \text{with} \qquad t := \bigoplus_{\beta=1}^{25} P_{mY^\beta_\beta} \,. \tag{2.5}$$

Now t is a solution of the Dirac equation

$$(i\partial\!\!\!/ - mY)\, t = 0 \,.$$

In order to introduce the interaction, we insert an operator $\mathcal{B}$ into the Dirac equation,

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\, \tilde{t} = 0 \,. \tag{2.6}$$

The causal perturbation theory (see [2, §2.2] and [7]) defines $\tilde{t}$ in terms of a unique perturbation series. The *light-cone expansion* (see [2, §2.5] and the references therein) is a method for analyzing the singularities of $\tilde{t}$ near the light cone. This gives a representation of $\tilde{t}$ of the form

$$\tilde{t}(x,y) = \sum_{n=-1}^{\infty} \sum_k m^{p_k} (\text{nested bounded line integrals}) \times T^{(n)}(x,y) + P^{\text{le}}(x,y) + P^{\text{he}}(x,y) \,,$$

where $P^{\text{le}}(x,y)$ and $P^{\text{he}}(x,y)$ are smooth to every order in perturbation theory. For the resulting light-cone expansion to involve only *bounded* line integrals, we need to assume the *causality compatibility condition*

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\, X = X^* \,(i\partial\!\!\!/ + \mathcal{B} - mY) \qquad \text{for all } \tau_{\text{reg}} \in (0,1] \,. \tag{2.7}$$

Then the auxiliary fermionic projector of the sea states P^{sea} is obtained similar to (2.5) by multiplication with the chiral asymmetry matrix.

As in [4, §2.6], we built the regularization into the formulas of the light-cone expansion by the formal replacements

$$m^p\, T^{(n)} \to m^p\, T^{(n)}_{[p]} \,,$$
$$\tau_{\text{reg}}\, T^{(n)} \to \tau_{\text{reg}} \sum_{k=0}^{\infty} \frac{1}{k!}\, \frac{1}{\delta^{2k}}\, T^{(k+n)}_{[R,2n]} \,.$$

Moreover, we introduce particles and anti-particles by occupying additional states or by removing states from the sea, i.e.

$$P^{\text{aux}}(x,y) = P^{\text{sea}}(x,y) - \frac{1}{2\pi} \sum_{k=1}^{n_{\text{p}}} \Psi_k(x) \overline{\Psi_k(y)} + \frac{1}{2\pi} \sum_{l=1}^{n_{\text{a}}} \Phi_l(x) \overline{\Phi_l(y)}$$

(for the normalization of the particle and anti-particle states we refer to [2, §2.8], [3, §4.3] and [7]). Finally, we introduce the regularized fermionic projector P by forming the *sectorial projection* (see also [2, §2.3], [3, eq. (4.3)] or [4, eq. (2.52)]),

$$(P)^i_j = \sum_{\alpha,\beta} (\tilde{P}^{\text{aux}})^{(i,\alpha)}_{(j,\beta)} \,, \tag{2.8}$$

where $i,j \in \{1,\ldots,8\}$ is the sector index, and the indices α and β run over the corresponding generations (i.e., $\alpha \in \{1,\ldots 4\}$ if $i=1$ and $\alpha \in \{1,2,3\}$ if $i=2,\ldots,8$). We again indicate the sectorial projection of the mass matrices by accents (see [2, §7.1], [3, eq. (5.2)] or [4, eq. (2.53)]),

$$\hat{Y} = \sum_\alpha Y^\alpha_\alpha \,, \qquad \acute{Y} Y \cdots \grave{Y} = \sum_{\alpha,\beta,\gamma_1,\ldots,\gamma_{p-1}} Y^\alpha_{\gamma_1} \cdots Y^{\gamma_1}_{\gamma_2} \cdots Y^{\gamma_{p-1}}_\beta \,.$$

As in [4], we need assumptions on the regularization. Namely, again setting

$$L^{(n)}_{[p]} = T^{(n)}_{[p]} + \frac{1}{3}\, \tau_{\text{reg}}\, T^{(n)}_{[R,p]} \,,$$

we impose the following regularization conditions (see [4, eqs. (5.36), (5.38) and (8.2)])

$$T^{(0)}_{[0]} T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} = 0 \qquad \text{in a weak evaluation on the light cone} \tag{2.9}$$

$$\big|L^{(n)}_{[0]}\big| = \big|T^{(n)}_{[0]}\big| \big(1 + \mathcal{O}\big((m\varepsilon)^{2p_{\text{reg}}}\big)\big) \qquad \text{for } n=0,-1 \text{ pointwise} \,. \tag{2.10}$$

Here by *weak evaluation* we mean that we multiply by a test function η and integrate, staying away from the origin. More precisely, we use the weak evaluation formula (for details see [3, §5.1])

$$\int_{|\vec{\xi}|-\varepsilon}^{|\vec{\xi}|+\varepsilon} dt \; \eta(t,\vec{\xi}) \; \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ \; \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ \; \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} = \eta(|\vec{\xi}|,\vec{\xi}) \; \frac{c_{\text{reg}}}{(i|\vec{\xi}|)^L} \; \frac{\log^k(\varepsilon|\vec{\xi}|)}{\varepsilon^{L-1}} \;, \tag{2.11}$$

which holds up to

$$\text{(higher orders in } \varepsilon/\ell_{\text{macro}} \text{ and } \varepsilon/|\vec{\xi}|) \,.$$

Here L is the degree defined by $\deg T^{(n)}_\circ = 1-n$, and c_{reg} is a so-called *regularization parameter* (for details see again [2, §4.5] or [3, §5.1]). In (2.10) by *pointwise* we mean that if we multiply $T^{(n)}_{[p]} - L^{(n)}_{[p]}$ by any simple fraction and evaluate weakly on the light cone, we get zero up to an error of the specified order. We remark that (2.10) could be replaced by a finite number of equations to be satisfied in a weak evaluation on the light cone. But in order to keep our analysis reasonably simple, we always work with the easier pointwise conditions (2.10).

2.2. **Chiral Gauge Potentials and Gauge Phases.** Similar as in [3, §6.2] and [4, §3.2] we consider chiral gauge potentials. Thus the operator $\mathcal{B}$ in the Dirac equation (2.6) is chosen as

$$\mathcal{B} = \chi_L \, \not{A}_R + \chi_R \, \not{A}_L \,, \tag{2.12}$$

where A^j_L and A^j_R are Hermitian 25×25-matrices acting on the sectors. A-priori, the chiral gauge potentials can be chosen according to the gauge group

$$\text{U}(25)_L \times \text{U}(25)_R \,.$$

This gauge group is too large for mathematical and physical reasons. First, exactly as in [4, §3.2], the causality compatibility condition (2.7) inhibits that non-trivial high-energy contributions are mixed with the Dirac seas, giving rise to the smaller gauge group

$$\text{U}(24)_L \times \text{U}(24)_R \times \text{U}(1)_R \,, \tag{2.13}$$

(where the U(24) acts on the first three direct summands of P^N_{aux} and on the 21 direct summands in P^M_{aux} in (2.4)). Similar as described in [3, §6.2] and [4, §3.2], to degree five the gauge potentials describe generalized phase transformations of the left- and right-handed components of the fermionic projector,

$$P^{\text{aux}}(x,y) \to \big(\chi_L \, U_L(x,y) + \chi_R \, U_R(x,y)\big) P^{\text{aux}}(x,y) + (\deg < 2) \,, \tag{2.14}$$

where the unitary operators U_c are ordered exponentials (for details see [2, §2.5] or [1, Section 2.2]),

$$U_c = \text{Pexp}\Big(-i \int_x^y A^j_c \, \xi_j \Big) \,.$$

The fermionic projector is obtained from (2.14) by forming the sectorial projection (2.8). Summing over the generation indices has the effect that wave functions are superimposed which may involve different gauge phases. In other words, the sectorial projection in general involves relative gauge phases. In order to simplify the form of the gauge potentials, we now argue that such relative gauge phases should be absent. In fact, if such relative phases occurred, the different Dirac seas forming the fermionic projector would get out of phase, implying that all relations for the fermionic projector would have to be satisfied for each Dirac sea separately. This would give rise to many additional constraints for the regularization, which seem impossible to satisfy. We remark that a similar argument is given in [4, §3.2]. Moreover, the physical picture is similar for the gravitational field, where it was argued in [4, §4.6] that the metric tensor must be independent of the isospin index.

The simplest method to avoid such relative phases would be to choose gauge potentials which do not depend on the generation index, i.e.

$$(A_L)^{(i,\alpha)}_{(j,\beta)} = (A^{\text{sec}}_L)^i_j \, \delta^\alpha_\beta \tag{2.15}$$

(where the superscript "sec" clarifies that the potential only carries sector indices). In order to be compatible with the $U(1)_R$-subgroup in (2.13) acting on the right-handed high-energy states in the neutrino sector, we need to choose the potentials in (2.15) corresponding to the gauge group

$$\mathrm{U}(8)_L \times \mathrm{U}(1)_R \times \mathrm{U}(7)_R \,, \tag{2.16}$$

where the U(7) acts on the seven direct summands in (2.2) but is trivial on the neutrinos (2.3). The ansatz (2.15) can be slightly generalized by allowing for unitary transformations in each sector. This leads to the ansatz

$$\mathcal{B} = \chi_R \, U^{\text{mix}}_L \slashed{A}^{\text{sec}}_L (U^{\text{mix}}_L)^* + \chi_L \, U^{\text{mix}}_R \slashed{A}^{\text{sec}}_R (U^{\text{mix}}_R)^* \,, \tag{2.17}$$

where the potentials A^{sec}_c are again of the form (2.15), and the matrices U^{mix}_c are constant unitary matrices which are diagonal in the sector index,

$$(U^{\text{mix}}_c)^{(i,\alpha)}_{(j,\beta)} = \delta^i_j \, (U^i_c)^\alpha_\beta \qquad \text{with} \qquad U^i_c \in \mathrm{U}(3) \,. \tag{2.18}$$

Thus we allow for a different mixing matrix for every sector. Also, the mixing matrices may be different for the left- and right-handed components of the spinors. The fact that the mixing matrices are constant could be justified by using arguments similar to those worked out for two sectors in [4, Lemma 3.1]. Here we do not enter such arguments again but simply take (2.17) as our ansatz for the chiral gauge potentials. It seems the most general ansatz which avoids relative phases when forming the sectorial projection. Specializing the chiral gauge fields to the ansatz (2.17), the matrices U_c in (2.14) become

$$U_c = U^{\text{mix}}_c \; \mathrm{Pexp}\Big(-i \int_x^y A^j_c \, \xi_j \Big) (U^{\text{mix}}_c)^* \,. \tag{2.19}$$

2.3. **The Microlocal Chiral Transformation.** Exactly as in [3, §7.10] and [4, §4.4], our method is to compensate the logarithmic singularities of the current and mass terms by a microlocal chiral transformation. To this end, one considers a Dirac equation of the form

$$(U^{-1})^* (i\slashed{\partial} + \mathcal{B} - mY) \, U^{-1} \tilde{P}^{\text{aux}} = 0 \,, \tag{2.20}$$

where U is an integral operator with an integral kernel $U(x,y)$, which we write in the the microlocal form

$$U(x,y) = \int \frac{d^4k}{(2\pi)^4} \, U\Big(k, \frac{x+y}{2}\Big) \, e^{-ik(x-y)} \,,$$

where $U(k,z)$ is a chiral transformation

$$U(k,z) = \mathbb{1} + \frac{i}{\sqrt{\Omega}} \, Z(k,z) \qquad \text{with} \qquad Z(z) = \chi_L \, L^j(k,z) \gamma_j + \chi_R \, R^j(k,z) \gamma_j \,. \tag{2.21}$$

Writing the Dirac equation (2.20) in the form (2.6) with a nonlocal operator $\mathcal{B}$, the perturbative methods of §2.1 again apply.

More specifically, the matrices L and R in (2.21) are chosen such that the matrices $L[k,x] := \acute{L}_j(k,x)\, k^j$ and $R[k,x] := \acute{L}_j(k,x)\, k^j$ satisfy the conditions

$$L[k,x]\, L[k,x]^* = R[k,x]\, R[k,x]^* = \mathfrak{c}_0(k,x)\, \mathbb{1}_{\mathbb{C}^2} \tag{2.22}$$

$$L[k,x]\, m^2 Y^2\, L[k,x]^* = \frac{\Omega}{2}\, v_L(x) + \mathfrak{c}_2(k,x)\, \mathbb{1}_{\mathbb{C}^2} \tag{2.23}$$

$$R[k,x]\, m^2 Y^2\, R[k,x]^* = \frac{\Omega}{2}\, v_R(x) + \mathfrak{c}_2(k,x)\, \mathbb{1}_{\mathbb{C}^2}\,, \tag{2.24}$$

where $\mathfrak{c}_0$ and $\mathfrak{c}_2$ are real parameters, and $\Omega = |k^0|$ denotes the frequency of the four-momentum k. The vector fields v_L and v_R are the currents or potentials which multiply the logarithmic singularities to be compensated.

Writing the Dirac equation (2.20) raises the question how the potential $\mathcal{B}$ is to be chosen. The most obvious procedure would be to choose $\mathcal{B}$ equal to the chiral potentials in (2.12). However, as shown in [3, §7.11] and [4, §4.5], this is not the correct choice, intuitively speaking because the microlocal chiral transformation in (2.20) has contributions which flip the chirality, making it necessary to also modify the potentials in the Dirac operator. We decompose $\mathcal{D}$ into its even and odd components,

$$\mathcal{D} = \mathcal{D}_{\text{odd}} + \mathcal{D}_{\text{even}}\,,$$

where

$$\mathcal{D}_{\text{odd}} = \chi_L \mathcal{D} \chi_R + \chi_R \mathcal{D} \chi_L \qquad \text{and} \qquad \mathcal{D}_{\text{even}} = \chi_L \mathcal{D} \chi_L + \chi_R \mathcal{D} \chi_R\,.$$

In [3, §7.11] we flipped the chirality of the gauge fields in $\mathcal{D}_{\text{even}}$. As will become clear below, here we need more freedom to modify the gauge potentials in $\mathcal{D}_{\text{even}}$. To this end, we now replace the gauge fields in $\mathcal{D}_{\text{even}}$ by new gauge fields $A^{\text{even}}_{L/R}$ to be determined later,

$$\mathcal{D}^{\text{flip}}_{\text{even}} = \sum_{c=L/R} \chi_c\, (U^{-1})^* \big(i\partial\!\!\!/_x + \chi_L A\!\!\!/^{\text{even}}_R + \chi_R A\!\!\!/^{\text{even}}_L - mY\big) U^{-1}\, \chi_c\,. \tag{2.25}$$

We replace the Dirac equation (2.20) by

$$\big(\mathcal{D}_{\text{odd}} + \mathcal{D}^{\text{flip}}_{\text{even}}\big)\, \tilde{P}^{\text{aux}} = 0\,.$$

2.4. The Causal Action Principle.

We again consider the causal action principle introduced in [2]. The action is

$$\mathcal{S}[P] = \iint_{M\times M} \mathcal{L}[A_{xy}]\, d^4x\, d^4y$$

with the Lagrangian

$$\mathcal{L}[A_{xy}] = |A_{xy}^2| - \frac{1}{32}\, |A_{xy}|^2\,,$$

where $A_{xy} = P(x,y)\, P(y,x)$ denotes the closed chain and $|A| = \sum_{i=1}^8 |\lambda_i|$ is the spectral weight. As shown in [3, §5.2], the Euler-Lagrange equations in the continuum limit can be written as

$$Q(x,y) = 0 \quad \text{if evaluated weakly on the light cone}\,, \tag{2.26}$$

where $Q(x,y)$ is defined as follows. Similar as explained in [4, §5.1], we count the eigenvalues of the the closed chain A_{xy} with algebraic multiplicities and denote them by λ^{xy}_{ncs}, where $n \in \{1, \ldots, 8\}$, $c \in \{L, R\}$ and $s \in \{+, -\}$. The corresponding spectral projectors are denoted by F^{xy}_{ncs}. Then $Q(x,y)$ is given by

$$\begin{aligned} Q(x,y) &= \frac{1}{2} \sum_{ncs} \frac{\partial \mathcal{L}}{\partial \lambda^{xy}_{ncs}} \, F^{xy}_{ncs} \, P(x,y) \\ &= \sum_{n,c,s} \Big[|\lambda^{xy}_{ncs}| - \frac{1}{8} \sum_{n',c',s'} |\lambda^{xy}_{n'c's'}| \Big] \, \frac{\overline{\lambda^{xy}_{ncs}}}{|\lambda^{xy}_{ncs}|} \, F^{xy}_{ncs} \, P(x,y) \, . \end{aligned} \tag{2.27}$$

The equation (2.26) is satisfied in the vacuum (see [3, §6.1] and [4, §3.1]). When evaluating the EL equations in the interacting situation, it will in most cases be sufficient to consider (2.26) for perturbations of the eigenvalues,

$$0 = \Delta Q(x,y) := \sum_{n,c,s} \Big[\Delta |\lambda^{xy}_{ncs}| - \frac{1}{8} \sum_{n',c',s'} \Delta |\lambda^{xy}_{n'c's'}| \Big] \, \frac{\overline{\lambda^{xy}_{ncs}}}{|\lambda^{xy}_{ncs}|} \, F^{xy}_{ncs} \, P(x,y) \, . \tag{2.28}$$

3. Spontaneous Block Formation

The goal of this section is to derive constraints for the form of the admissible gauge fields. The arguments are similar in style to those in [2, Chapter 7]. However, as a main difference, we here consider the effect of the sectorial projection and the mixing of the generations, whereas in [2, Chapter 7] the contributions of higher order in a mass expansion (which are of lower degree on the light cone) were analyzed. The analysis given here supersedes the arguments in [2, Chapter 7], which with the present knowledge must be regarded as being preliminary.

3.1. **The Statement of Spontaneous Block Formation.** Analyzing the EL equations to degree five and degree four on the light cone gives rise to a number of equations which involve the chiral potentials without derivatives. These equations clearly do not describe a dynamics of the potentials and fields, but merely pose constraints for the structure of the possible interactions. We refer to these equations as the *algebraic constraints* for the gauge potentials. The algebraic constraints trigger a mechanism where the eight sectors form pairs, the so-called *blocks*. Describing the interaction within and among the four blocks by chiral gauge fields gives rise to precisely the gauge groups and couplings in the standard model.

In order to introduce a convenient notation, we denote chiral potentials of the form (2.17) which satisfy all the algebraic constraints as *admissible*. Since ordered exponentials of the chiral potentials appear (see for example (2.14) and (2.19)), it seems necessary for mathematical consistency to consider a set of admissible chiral gauge potentials which forms a Lie algebra, the so-called *dynamical gauge algebra* $\mathfrak{g}$. More precisely, the commutator of two elements $\mathcal{A} = (A_L, A_R)$ and $\tilde{\mathcal{A}} = (\tilde{A}_L, \tilde{A}_R)$ in $\mathfrak{g}$ is defined by

$$[\mathcal{A}, \tilde{\mathcal{A}}] = \Big([A_L, \tilde{A}_L], \; [A_R, \tilde{A}_R] \Big)$$

(where the brackets $[.,.]$ is the commutator of symmetric 8×8-matrices; note that the mixing matrices in (2.17) drop out of all commutators). The assumption that $\mathfrak{g}$ is a Lie algebra is the implication $\mathcal{A}, \tilde{\mathcal{A}} \in \mathfrak{g} \Rightarrow i[\mathcal{A}, \tilde{\mathcal{A}}] \in \mathfrak{g}$. The corresponding Lie group will be a Lie subgroup of the gauge group (2.16). We denote this Lie group by $\mathcal{G} \subset \mathrm{U}(8)_L \times \mathrm{U}(1)_R \times \mathrm{U}(7)_R$ and refer to it as the *dynamical gauge group*.

The potentials in the dynamical gauge algebra should be regarded as describing the physical interactions of the system. In order to understand the algebraic constraints, we clearly want to find *all* the potentials which satisfy the algebraic constraints. Therefore, we always choose $\mathcal{G}$ *maximal* in the sense that $\mathcal{G}$ has no Lie group extension extension $\tilde{\mathcal{G}}$ with $\mathcal{G} \subsetneqq \tilde{\mathcal{G}} \subset \mathrm{U}(8)_L \times \mathrm{U}(1)_R \times \mathrm{U}(7)_R$ which is also generated by admissible chiral potentials.

We begin with the following definition.

Definition 3.1. *An admissible chiral potential $\mathcal{A} = (A_L, A_R) \in \mathfrak{g}$ is a* **free gauge potential** *if it has the following properties:*

(a) *The potential is vectorial: $A_L = A_R =: A$.*
(b) *The potential does not depend on the generation index: $A^{(i,\alpha)}_{(j,\beta)} = \delta^\alpha_\beta \, (A^{\mathrm{sec}})^i_j$.*
(c) *The potential commutes with the mass matrix: $[A, mY] = 0$.*

The Lie group generated by all free gauge potentials is referred to as the **free gauge group** $\mathcal{G}_{\mathrm{free}} \subset \mathcal{G}$.

Since the conditions (a)–(c) are linear and invariant under forming the Lie bracket, $\mathcal{G}_{\mathrm{free}}$ is indeed a Lie subgroup of $\mathcal{G}$.

A free gauge potential has the desirable property that it corresponds to a gauge symmetry of the system (because it describes isometries of the spin spaces). As a consequence, the mass terms vanish, implying that the corresponding bosonic fields are necessarily massless. Moreover, chiral potentials with the above properties (a)–(c) satisfy all algebraic constraints (see §3.2–§3.4 below) and are thus admissible.

Here is the main result of this section:

Theorem 3.2. (spontaneous block formation) *Consider the setting introduced in §2.1 and assume that the following conditions hold:*

(i) *The admissible gauge potentials involve non-abelian left- or right-handed gauge potentials.*
(ii) *The mixing matrices U^{mix}_c in (2.17) are chosen such that the dimension of the free gauge group is maximal.*

Then the effective gauge group is given by

$$\mathcal{G} = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)\,. \tag{3.1}$$

By relabeling the massive sectors and performing constant phase transformations of the wave functions, we can arrange that the corresponding gauge potentials $A^{\mathrm{em}} \in \mathfrak{u}(1)$, $W \in \mathfrak{su}(2)$ and $G \in \mathfrak{su}(3)$ enter the operator $\mathcal{B}$ in the Dirac equation (2.6) as follows,

$$\mathcal{B}[A^{\mathrm{em}}] = \not{A}^{\mathrm{em}} \,\mathrm{diag}\Big(0, -1, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\Big) \tag{3.2}$$

$$\mathcal{B}[W] = \chi_R \left(\not{W}_{\mathrm{MNS}} \oplus \not{W}_{\mathrm{CKM}} \oplus \not{W}_{\mathrm{CKM}} \oplus \not{W}_{\mathrm{CKM}} \right) \tag{3.3}$$

$$\mathcal{B}[G] = \big(\mathbb{1} \oplus \not{G} \big) \otimes \mathbb{1}_{\mathbb{C}^2}\,, \tag{3.4}$$

where

$$W_{\mathrm{MNS}} = \begin{pmatrix} (W)^{11} & (W)^{12}\, U^*_{\mathrm{MNS}} \\ (W)^{21}\, U_{\mathrm{MNS}} & (W)^{22} \end{pmatrix}, \qquad W_{\mathrm{CKM}} = \begin{pmatrix} (W)^{11} & (W)^{12}\, U^*_{\mathrm{CKM}} \\ (W)^{21}\, U_{\mathrm{CKM}} & (W)^{22} \end{pmatrix},$$

and $U_{\mathrm{MNS}}, U_{\mathrm{CKM}} \in \mathrm{U}(3)$ are fixed unitary matrices. If one of these matrices is non-trivial, the other is also non-trivial and

$$\hat{U}_{\mathrm{MNS}} = \hat{U}_{\mathrm{CKM}}\,. \tag{3.5}$$

If the masses of the charged leptons and neutrinos (2.2) *and* (2.3) *are different in the sense that*

$$\sum_{\beta=1}^{3} m_\beta^2 \neq \sum_{\beta=1}^{3} \tilde{m}_\beta^2 \,, \tag{3.6}$$

then the mixing matrices are necessarily non-trivial,

$$U_{\text{MNS}}, U_{\text{CKM}} \neq \mathbb{1}_{\mathbb{C}^3} \,. \tag{3.7}$$

To clarify the notation, we first note that we always identify $\mathfrak{u}(n)$ with the Hermitian $n \times n$-matrices, and $\mathfrak{su}(n)$ are the corresponding traceless matrices. Next, the diagonal matrix in (3.2) acts on the eight sectors. The potential in (3.3) only couples to the left-handed component of the spinors. Each of the four direct summands acts on two sectors (i.e. W_{MNS} on the first and second sector, the next summand W_{CKM} on the third and fourth sector, etc.). In (3.4) the direct sum $\mathbb{1} + \not{G}$ is a 4×4 matrix acting on pairs of sectors as indicated by the factor $\mathbb{1}_{\mathbb{C}^2}$ (i.e. the first column acts on the first and second sector, the second column on the third and fourth sector, etc.).

The specific form of the potentials in the above theorem can be understood as a mechanism where the sectors form pairs, referred to as *blocks*. Thus the first two sectors form the first block (referred to as the *lepton block*), the third and fourth sectors form the second block (referred to as the first *quark block*), and so on. The potentials in (3.3) are the same in each of the four blocks, except for the mixing matrices U_{MNS} and U_{CKM} which may be different in the lepton and in the quark blocks. The potentials in (3.4) describe an interaction of the three quark blocks. Clearly, the potentials A^{em} and G correspond to the electromagnetic and the strong potentials in the standard model. The potential W corresponds to the gauge potentials of the weak isospin. The reduction from the large gauge group (2.16) to its subgroup (3.1) and to gauge potentials of the specific form (3.2)–(3.4) can be regarded as a spontaneous breaking of the gauge symmetry. We refer to this effect as the *spontaneous block formation*.

We point out that without any additional assumptions (like (i) and (ii) above), the dynamical gauge group will not be uniquely determined. This is due to the fact that the algebraic constraints are nonlinear, and therefore these constraints will in general be satisfied by different Lie algebras. Thus in general, there will be a finite (typically small) number of possible dynamical gauge groups, leaving the freedom to choose one of them as being the "physical" one. The above assumptions (i) and (ii) give a way to single out a unique dynamical gauge group, corresponding to the choice which we consider to be physically relevant. Clearly, this procedure can be criticized as not deriving the structure of the physical interactions purely from the causal action principle and the form of the vacuum. But at least, the choice of the dynamical gauge group is *global* in space-time, i.e. it is to be made once and forever. Moreover, our procedure clarifies the following points:

- The gauge groups and couplings of the gauge fields to the fermion as used in the standard model follow uniquely from general assumptions on the interaction, which do not involve any specific characteristics of the groups or of the couplings.
- The gauge groups of the standard model are maximal in the sense that no additional chiral potentials are admissible. Thus we get an explanation why there are *not more* physical gauge fields than those in the standard model.

As an example of a dynamical gauge group which we do not consider as being physically relevant, one could choose $\mathcal{G}_{\text{free}}$ as the Lie group U(7) acting on the 7 massive sectors. Forming $\mathcal{G}$ as a maximal extension gives a dynamical gauge group where the corresponding left- and right-handed gauge potentials are all abelian. This explains why an assumption like (i) above is needed.

We remark that the specific form of assumption (i) is a major simplification of our analysis, because it makes it possible to disregard the situation that there are non-abelian admissible potentials, but that every such potential is a mixture of a left- and right-handed component. We expect that assumption (i) could be weakened by refining our methods, but we leave this as a problem for future research.

The remainder of this section is devoted to the proof of Theorem 3.2. We first work out all the constraints for the gauge potentials (§3.2–§3.4) and then combine our findings to infer the theorem (§3.5).

3.2. **The Sectorial Projection of the Chiral Gauge Phases.**

Similar as explained in [4, Section 3], we shall now analyze the effect of the gauge phases in the EL equations to degree five on the light cone. Combining (2.14), (2.19) and (2.8), the closed chain is computed by (see also [4, §3.2])

$$\chi_L A_{xy} = \chi_L \, \hat{U}_L \, \hat{U}_R^* \, A_{xy}^{\text{vac}} + (\deg < 3) \,. \tag{3.8}$$

Here A_{xy}^{vac} is the closed chain in the vacuum. In it diagonal in the sector index and has the form (cf. [4, §3.1])

$$\chi_L A_{xy}^{\text{vac}} = \begin{cases} \dfrac{3}{4}\, \chi_L \left(3 \not{\xi} T^{(-1)}_{[0]} \overline{\not{\xi} T^{(-1)}_{[0]}} + \tau_{\text{reg}} \, \not{\xi} T^{(-1)}_{[0]} \overline{\not{\xi} T^{(-1)}_{[R,0]}} \right) & \text{on the neutrino sector} \\ \frac{3}{4}\, \chi_L \, 3 \not{\xi} T^{(-1)}_{[0]} \overline{\not{\xi} T^{(-1)}_{[0]}} & \text{on the massive sectors}\,, \end{cases}$$

up to contributions of the form $\not{\xi}\, (\deg < 3) + (\deg < 2)$. In [4, §3.2] the size of τ is discussed, leading to the two cases **(i)** and **(ii)** (see [4, eq. (3.36)]). For brevity, we here only consider case **(i)**, noting that case **(ii)** can be treated exactly as in [4, §3.2], without gaining any insight of importance for what follows. Thus we assume that τ is so small that the factor $T^{(-1)}_{[R,0]}$ may be disregarded, so that the closed chain of the vacuum simplifies to

$$\chi_L A_{xy}^{\text{vac}} = \frac{9}{4}\, \chi_L \, \not{\xi} T^{(-1)}_{[0]} \overline{\not{\xi} T^{(-1)}_{[0]}} \,. \tag{3.9}$$

In order to satisfy the EL equations to degree five, the non-trivial eigenvalues of the matrix (3.8) must all have the same absolute value. Since the matrix (3.9) commutes with the matrices $\hat{U}_L$ and $\hat{U}_R^*$, the eigenvalues of the closed chain are simply the products of the eigenvalues of $\chi_L A_{xy}^{\text{vac}}$ and the eigenvalues of $\hat{U}_L \hat{U}_R^*$. Since the nontrivial eigenvalues of $\chi_L A_{xy}^{\text{vac}}$ must form a complex conjugate pair, the EL equations to degree five are satisfied if and only if

$$\text{the eigenvalues } \hat{U}_L \, \hat{U}_R^* \text{ all have the same absolute value.}$$

This leads to constraints for the gauge potentials, which we now work out.

In preparation, we introduce a convenient notation. Our goal is to determine the dynamical gauge group $\mathcal{G}$. At the moment, we only know that it should be a Lie subgroup of the group in (2.16). The admissible chiral gauge potentials are vectors in the corresponding Lie algebra $\mathfrak{g} = T_e\mathcal{G}$. More precisely, in view of (2.17), these chiral potentials have the form

$$\mathfrak{g} \ni \mathcal{A} = (A_L, A_R) \qquad \text{and} \qquad A_c = U_c^{\text{mix}} A_c^{\text{sec}} (U_c^{\text{mix}})^* ,$$

where A_c^{sec} are Hermitian 8×8-matrices acting on the sectors. Moreover, the matrix A_R does not mix the first with the other 7 sectors, i.e.

$$A_R = \begin{pmatrix} (A_R)^1_1 & 0 & \cdots & 0 \\ 0 & (A_R)^2_2 & \cdots & (A_R)^2_8 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (A_R)^8_2 & \cdots & (A_R)^8_8 \end{pmatrix} . \tag{3.10}$$

Lemma 3.3. *Assume that for any $(U_L, U_R) \in \mathcal{G}$, the eigenvalues of the matrix $\hat{U}_L \hat{U}_R^*$ all have the same absolute value. Then for any $\mathcal{A} = (A_L, A_R) \in \mathfrak{g}$ there is a real number $c(\mathcal{A})$ such that the matrix*

$$\acute{A}_L \grave{A}_L + \acute{A}_R \grave{A}_R - \hat{A}_L^2 - \hat{A}_R^2 - c(\mathcal{A}) \, \mathbb{1}_{\mathbb{C}^8} \tag{3.11}$$

vanishes on all the eigenspaces of the matrix $\hat{A}_L - \hat{A}_R$.

Proof. For simplicity, we only consider the situation that the group element (U_L, U_R) is in a neighborhood of the identity $e \in \mathcal{G}$. Then, since $\mathcal{G}$ is assumed to be a Lie group, we can represent the group element with the exponential map,

$$U_c = \exp(-iA_c) = \mathbb{1} - iA_c - \frac{1}{2}\, A_c^2 + \mathcal{O}(\mathcal{A}^3) .$$

Forming the sectorial projection, we obtain

$$\hat{U}_c = \exp(-iA_c) = \mathbb{1} - i\hat{A}_c - \frac{1}{2}\, \acute{A}_c \grave{A}_c + \mathcal{O}(\mathcal{A}^3) .$$

The effect of the sectorial projection becomes clearer when comparing with the unitary matrix obtained by exponentiating the sectorial projection of A_c,

$$\exp(-i\hat{A}_c) = \mathbb{1} - i\hat{A}_c - \frac{1}{2}\, \hat{A}_c^2 + \mathcal{O}(\mathcal{A}^3) .$$

This gives

$$\begin{aligned} \hat{U}_c &= \exp(-i\hat{A}_c) + \frac{1}{2} \Big(\hat{A}_c^2 - \acute{A}_c \grave{A}_c \Big) + \mathcal{O}(\mathcal{A}^3) \\ &= \exp(-i\hat{A}_c) \left(\mathbb{1} + \frac{1}{2} \big(\hat{A}_c^2 - \acute{A}_c \grave{A}_c \big) \right) + \mathcal{O}(\mathcal{A}^3) , \end{aligned}$$

showing that $\hat{U}_c$ is unitary up to a contribution to second order which is Hermitian. As a consequence,

$$\hat{U}_L \hat{U}_R^* = \exp(-i\hat{A}_L) \left\{ \mathbb{1} + \frac{1}{2} \big(\hat{A}_L^2 - \acute{A}_L \grave{A}_L + \hat{A}_R^2 - \acute{A}_R \grave{A}_R \big) \right\} \exp(i\hat{A}_R) + \mathcal{O}(\mathcal{A}^3). \tag{3.12}$$

The curly brackets enclose a Hermitian matrix. Moreover, to the considered second order in $\mathcal{A}$, the curly brackets can be commuted to the left or right. This shows that the matrix $\hat{U}_L \hat{U}_R^*$ is normal (i.e. it commutes with its adjoint). Therefore, the eigenvalues can be computed with a standard perturbation calculation with degeneracies. To first order in $\mathcal{A}$, we need to diagonalize the matrix $A_L - A_R$. The exponentials in (3.12) are unitary and thus only change the eigenvalues by a phase. Therefore, the change of the absolute values of the eigenvalues is described by a first order perturbation calculation for the matrix in the curly brackets. This gives the result. □

The condition (3.11) arising from this lemma is difficult to analyze because the eigenspaces of the matrix $\hat{A}_L - \hat{A}_R$ are unknown and depend on the potential in a complicated non-linear way. A good strategy for satisfying the conditions for all $\mathcal{A} \in \mathfrak{g}$ is to demand that the matrix in (3.11) vanishes identically, i.e.

$$\acute{A}_L \grave{A}_L + \acute{A}_R \grave{A}_R - \hat{A}_L^2 - \hat{A}_R^2 = c(\mathcal{A})\, \mathbb{1}_{\mathbb{C}^8} \,. \tag{3.13}$$

Clearly, this is a stronger condition than (3.11). But by perturbing the potentials in $\mathfrak{g}$, one could also get information on the matrix elements of (3.13) which mix different eigenspaces of $\hat{A}_L - \hat{A}_R$, suggesting that the assumptions of Lemma 3.3 even imply that (3.13) holds. Making this argument precise would make it necessary to study third order perturbations. In order to keep our analysis reasonably simple, we shall not enter higher oder perturbation theory. Instead, in what follows we take (3.13) as a necessary condition which all admissible potentials $\mathcal{A} = (A_L, A_R) \in \mathfrak{g}$ must satisfy.

Let us reformulate (3.13) in a convenient notation. First, we let $\breve{\pi} : \mathbb{C}^3 \to \mathbb{C}^3$ be the orthogonal projection onto the subspace spanned by the vector $(1, 1, 1)$. We introduce the vector space

$$T := \mathbb{C}^8 \times \mathbb{C}^3$$

of vectors carrying a sector and a generation index. We also consider $\breve{\pi}$ as an operator on T which acts on the second factor (i.e. on the generation index). Then the sectorial projections in (3.13) can be written as

$$\sum_{c=L,R} \breve{\pi} A_c (\mathbb{1} - \breve{\pi}) A_c \breve{\pi} = c(\mathcal{A})\, \mathbb{1}_T \,. \tag{3.14}$$

We introduce the subspaces $I := \breve{\pi}(T)$ and $J := (\mathbb{1} - \breve{\pi})(T)$; they are 8- respectively 16-dimensional. Moreover, we introduce the operators

$$B_c = (\mathbb{1} - \breve{\pi}) A_c \breve{\pi} \;:\; I \to J \,. \tag{3.15}$$

Combining the left- and right-handed matrices,

$$B := \begin{pmatrix} B_L \\ B_R \end{pmatrix} : I \to K := J \oplus J \,, \tag{3.16}$$

we can write the condition (3.14) as

$$\langle Bu | Bu \rangle = c(\mathcal{A})\, \|u\|^2 \qquad \text{for all } u \in I \tag{3.17}$$

(where the scalar product and the norm refer to the canonical scalar products on K and I, respectively). In other words, the matrix B must be a multiple of an isometry. We denote the possible values of B by $\mathcal{B}$,

$$\mathcal{B} := \left\{ \begin{pmatrix} (1 - \breve{\pi}) A_L \breve{\pi} \\ (1 - \breve{\pi}) A_R \breve{\pi} \end{pmatrix} : I \to K \quad \text{with} \quad \mathcal{A} \in \mathfrak{g} \right\}. \tag{3.18}$$

Then $\mathcal{B}$ is a real vector space of matrices. The condition (3.17) must hold on the whole vector space,

$$\langle Bu | Bu \rangle = c(B)\, \|u\|^2 \qquad \text{for all } B \in \mathcal{B} \text{ and } u \in I \,. \tag{3.19}$$

The analysis of (3.19) bears some similarity to the "uniform splitting lemma" used in [2, Lemma 7.1.3]. In fact, if $\mathcal{B}$ were a complex vector space, we could polarize (3.19) to conclude that

$$\langle Bu | B'u \rangle = c(B, B')\, \|u\|^2 \qquad \text{for all } B, B' \in \mathcal{B} \text{ and } u \in I\,,$$

making it possible to apply [2, Lemma 7.1.3]. However, there is the subtle complication that $\mathcal{B}$ is only a *real* vector space, implying that the above polarization is in general wrong. This makes it necessary to modify the method such that we work purely with real vector spaces. To this end, we consider I and K as real vector spaces, for clarity denoted by a subscript $\mathbb{R}$. These vector spaces have the real dimensions 16 respectively 64. On $I_\mathbb{R}$ and $K_\mathbb{R}$ we introduce the scalar product

$$\langle .|. \rangle_\mathbb{R} := \mathrm{Re} \langle .|. \rangle\,.$$

We encode the complex structure in a real linear operator $\mathbb{I}$ acting on $I_\mathbb{R}$ and $K_\mathbb{R}$ with the properties

$$\mathbb{I}^* = -\mathbb{I} \qquad \text{and} \qquad \mathbb{I}^2 = -\mathbb{1}\,.$$

Next, we let $\mathrm{Re}\, I$ be the subspace of I formed of all vectors with real components. We also consider $\mathrm{Re}\, I$ as an 8-dimensional subspace of $I_\mathbb{R}$. Moreover, we let $\mathrm{Re} : I_\mathbb{R} \to \mathrm{Re}\, I$ be the orthogonal projection to the real part. By restricting to $\mathrm{Re}\, I$, every operator $B \in \mathcal{B}$ gives rise to a mapping

$$B_\mathbb{R} := B|_{\mathrm{Re}\, I} \;:\; \mathrm{Re}\, I \to K_\mathbb{R}\,.$$

Note that the operator $B_\mathbb{R}$ is represented by a 64×8-matrix. Knowing $B_\mathbb{R}$, we can uniquely reconstruct the corresponding B by "complexifying" according to

$$Bu = B\, \mathrm{Re}\, u - \mathbb{I} B\, \mathrm{Re}(\mathbb{I} u)\,.$$

Lemma 3.4. *There is an isometry $V : K_\mathbb{R} \to K_\mathbb{R}$ and a basis $B_1, \ldots B_L$ of $\mathcal{B}$ (with $L \geq 0$) such that the matrices $(B_\ell)_\mathbb{R}$ have the representation*

$$(B_\ell)_\mathbb{R} = V\, M_\ell$$

with operators $M_\ell : I_\mathbb{R} \to K_\mathbb{R}$ which in the canonical bases have the block matrix representation

$$M_1 = \begin{pmatrix} \mathbb{1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ \mathbb{1} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \;\ldots\;, \quad M_L = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \mathbb{1} \\ 0 \end{pmatrix}.$$

Here the upper L matrix entries are 8×8-matrices, whereas the lowest matrix entry is a $(64 - 8L) \times 8$-matrix.

Proof. We rewrite (3.19) in real vector spaces as

$$\langle B_\mathbb{R} u | B_\mathbb{R} u \rangle_\mathbb{R} = c(B)\, \|u\|_R^2 \qquad \text{for all } B \in \mathcal{B} \text{ and } u \in \mathrm{Re}\, I\,.$$

Using the symmetry of the real scalar product, polarization gives

$$\langle B_\mathbb{R} u | B'_\mathbb{R} u \rangle_\mathbb{R} = c(B, B')\, \langle u | v \rangle_\mathbb{R}^2 \qquad \text{for all } B, B' \in \mathcal{B} \text{ and } u \in \mathrm{Re}\, I\,. \tag{3.20}$$

Now we can proceed as in the proof of [2, Lemma 7.1.3]: Let $(e_1, \ldots, e_8)$ be the canonical basis of $\mathrm{Re}\, I$. We introduce the subspaces

$$E_i = \mathrm{span}\{B_\mathbb{R} e_i \text{ with } B \in \mathcal{B}\} \subset K_\mathbb{R}$$

as well as the mappings

$$\kappa_i \;:\; \mathcal{B} \to E_i\;, \quad B \mapsto B_\mathbb{R} e_i\;.$$

The property (3.20) implies that for all $B, B' \in \mathcal{B}$,

$$\langle B_\mathbb{R} e_i | B'_\mathbb{R} e_j \rangle_\mathbb{R} = c(B, B')\, \delta_{ij}\;. \tag{3.21}$$

If $i \neq j$, this relation shows that the subspaces $(E_i)_{i=1,\ldots,p_1}$ are orthogonal. Moreover, in the case $i = j$, (3.21) yields that the scalar products $\langle \kappa_i(B') | \kappa_i(B') \rangle_\mathbb{R}$ are independent of i. Thus the mappings κ_i are isometrically equivalent, and so we can arrange by an isometry V that the κ_i have the matrix representations

$$\kappa_1 = \begin{pmatrix} \kappa \\ \vdots \\ 0 \\ 0 \end{pmatrix}\;,\;\ldots\;,\quad \kappa_L = \begin{pmatrix} 0 \\ \vdots \\ \kappa \\ 0 \end{pmatrix}\;,$$

where $\kappa : \mathcal{B} \to \mathbb{R}^8$.

Finally, we choose a basis $B_1, \ldots B_L$ of $\mathcal{B}$ such that $\kappa(B_1) = (1, \ldots, 0)$, $\kappa(B_2) = (0, 1, \ldots, 0)$, etc. This gives the result. ☐

Counting dimensions, this lemma shows in particular that the dimension of $\mathcal{B}$ is at most 8. In our applications we need the following refined counting of dimensions.

Corollary 3.5. *Assume that the images of the matrices $B_1, \ldots, B_L : I \to K$ span an N-dimensional subspace of K. Then the dimension of $\mathcal{B}$ is bounded from above by*

$$L \leq \frac{N}{4}\;. \tag{3.22}$$

Proof. Note that the real dimension of the image of $(B_\ell)|_\mathbb{R} : I_\mathbb{R} \to K_\mathbb{R}$ is twice the complex dimension of $B_\ell : I \to K$. ☐

3.3. **The Bilinear Logarithmic Terms.** In [4, §5.1], the left-handed component of the bilinear logarithmic terms quadratic in the mass matrices are computed by (see [4, eq. (5.21)])

$$\begin{aligned} B_L := &-\frac{m^2}{4} \Big\{ A_R^\text{even}[\xi], \big(A_L[\xi] (z_1)\, YY - 2Y A_R[\xi]\, Y + YY A_L[\xi] \big) \Big\}\, T^{(1)} \\ &+ \frac{m^2}{8} \Big(A_L[\xi]^2 YY + 2 A_L[\xi] YY A_L[\xi] + YY A_L[\xi]^2 \xi_k) \Big)\, T^{(1)} \\ &- \frac{m^2}{2}\, Y A_R[\xi]^2\, Y\, T^{(1)}\;. \end{aligned} \tag{3.23}$$

The right-handed component is obtained similarly by the replacements $L \leftrightarrow R$. Exactly as shown in [4, Lemma 5.5], the EL equations in the continuum limit are satisfied only if the matrices $\hat{B}_L$ and $\hat{B}_R$ coincide and are multiples of the matrix $\acute{Y} \grave{Y}$.

Let us specify the potentials A_c^even in (2.25). Exactly as shown in [4, §4.5], the shear contributions vanish only if, in a suitable basis, the matrices A_L^even coincides with A_R and A_R^even coincides with A_L, up to the choice of the mixing matrices. More precisely, in order to introduce A_L^even, we we let $\mathfrak{e}_{i\alpha}(k, x)$ with $i \in \{1, \ldots, 8\}$ and $\alpha \in \{1, 2, 3\}$ be an orthonormal basis of $\mathbb{C}^{8 \times 3}$ such that the vectors $\mathfrak{e}_{i1}$ are multiples of the eight columns of the matrix $L[k,x]^*$ (note that these column vectors are orthogonal according to (2.22)). In this basis, the potential A_L^{even} is defined by

$$A_L^{\text{even}} = V_R A_R^{\text{sec}} V_R^* \,, \tag{3.24}$$

where A_R^{sec} is the potential in (2.17) (in the standard basis), and V_R are unitary matrices which are diagonal in the sector index,

$$(V_R)^{(i,\alpha)}_{(j,\beta)}(x) = \delta^i_j \, (V^i_R)^\alpha_\beta(x) \qquad \text{with} \qquad V^i_c(x) \in \mathrm{U}(3) \,.$$

This is analogous to (2.17) and (2.18), with the only difference that different mixing matrices V^i_c appear, which may even depend on the space-time point x. In order to introduce A_R^{even}, one chooses similarly a basis $\mathfrak{e}_{i\alpha}(k,x)$ such that the vectors $\mathfrak{e}_{i1}$ are multiples of the eight columns of the matrix $R[k,x]^*$, and in this basis one sets

$$A_R^{\text{even}} = V_L A_L^{\text{sec}} V_L^* \tag{3.25}$$

with a sector-diagonal unitary matrix $V_L(x)$. We point out that the construction of the potentials $A_{L/R}^{\text{even}}$ depends on the momentum k of the microlocal chiral transformation. As a consequence, these potentials are non-local operators (for details see the discussion in [4, §4.5]).

When using (3.24) and (3.25) in (3.23), the freedom in choosing the matrices V^i_c gives many free parameters to modify B_L and B_R, making the situation rather complicated. In order to derive necessary conditions, it suffices to consider particular choices for the potentials for which the matrices V^i_c do not come into play. One possibility is to assume that $\mathfrak{g}$ contains a right-handed potential $\mathcal{A} = (0, A_R) \in \mathfrak{g}$. Then A_R^{even} and A_L vanish, so that

$$B_L = -\frac{m^2}{2} \, \acute{Y} A_R[\xi]^2 \, \grave{Y} \, T^{(1)} \,. \tag{3.26}$$

This must be a multiple of the matrix $\acute{Y}\grave{Y}$. Proceeding similarly for left-handed potentials gives the following result.

Lemma 3.6. *Suppose that $\mathcal{A} = (A_L, 0) \in \mathfrak{g}$ (or $\mathcal{A} = (0, A_R) \in \mathfrak{g}$) is a left-handed (respectively right-handed) admissible gauge potential. Then the matrix $A_L[\xi]^2$ (respectively $A_R^2[\xi]$) is a multiple of the identity matrix at every space-time point and for all directions ξ.*

The next lemma gives additional information on left-handed or right-handed admissible gauge potentials. For notational simplicity, we only state the result for left-handed potentials.

Lemma 3.7. *Suppose that $\mathcal{A} = (A_L, 0) \in \mathfrak{g}$ does not depend on the generation index, i.e.*

$$(A_L)^{(i,\alpha)}_{(j,\beta)} = \delta^\alpha_\beta \, (A^{\text{sec}})^i_j \,. \tag{3.27}$$

Then

$$\acute{Y} \, A_L[\xi]^2 \, \grave{Y} = \acute{A}_L[\xi] \, Y^2 \, \grave{A}_L[\xi] \,.$$

Proof. According to (3.27), we may compute B_L according to (3.23) with $A_R^{\text{even}} = A_L$. Then

$$B_L = -\frac{m^2}{8} \left(A_L[\xi]^2 \, Y^2 + 2 A_L[\xi] \, Y^2 \, A_L[\xi] + Y^2 \, A_L[\xi]^2 \right) T^{(1)}.$$

This matrix must coincide with B_R, which is computed similar to (3.26) by

$$B_R = -\frac{m^2}{2}\, Y A_L[\xi]^2\, Y\, T^{(1)} \,.$$

Applying Lemma 3.6, the matrix $A_L[\xi]^2$ is a multiple of the identity and thus commutes with Y. This gives the result. □

3.4. **The Field Tensor Terms.** The methods in [4, §5.2] also apply to the present situation of eight sectors. In particular, [4, Proposition 5.8] can be restated as follows:

Proposition 3.8. *Taking into account the contributions by the field tensor terms, the EL equations to degree four can be satisfied only if the regularization satisfies the conditions* (2.10) *and* (2.9). *If no further regularization conditions are imposed, then the chiral potentials must satisfy at all space-time points the conditions*

$$\mathrm{Tr}(\mathfrak{I}_1 A_R) = 0 \qquad \text{and} \qquad \mathrm{Tr}(A_L + A_R) = 0 \,, \tag{3.28}$$

where $\mathfrak{I}_1$ is the projection on the neutrino sector. If conversely (2.10), (2.9) *and* (3.28) *are satisfied, then the field tensor terms do not contribute to the EL equations of degree four.*

3.5. **Proof of Spontaneous Block Formation.** Instead of working with gauge groups, it will usually be more convenient to consider the corresponding Lie algebras. This is no restriction, because the corresponding Lie groups can then be recovered by exponentiation. When forming the Lie algebra of the product of groups, this gives rise to the direct sum of the algebras, like for example

$$T_e\big(\mathrm{U}(8)_L \times \mathrm{U}(1)_R \times \mathrm{U}(7)_R\big) = \mathfrak{u}(8)_L \oplus \mathfrak{u}(1)_R \oplus \mathfrak{u}(7)_R \,.$$

The proof is given in several steps, which are organized in separate paragraphs.

3.5.1. *Left-handed $\mathfrak{su}(2)$-potentials.* We now evaluate our assumption (i) that $\mathfrak{g}$ should contain left- or right-handed non-abelian potentials.

We first note that $\mathfrak{g}$ cannot contain right-handed potentials:

Lemma 3.9. *The dynamical gauge algebra $\mathfrak{g}$ does not contain potentials of the form $(0, A_R)$ with $A_R \neq 0$.*

Proof. Assume conversely that $\mathcal{A} = (0, A_R) \in \mathfrak{g}$ is a non-trivial admissible right-handed potential. It would follow from Lemma 3.6 that A^2 is a multiple of the identity. On the other hand, combining (3.10) with the fact that the right-handed potential vanishes on the neutrino sector (see the first equation in (3.28)), we find that A_R must be of the form

$$A_R = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & (A_R)^2_2 & \cdots & (A_R)^2_8 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (A_R)^8_2 & \cdots & (A_R)^8_8 \end{pmatrix} . \tag{3.29}$$

As a consequence, A_R^2 cannot be a multiple of the identity, a contradiction. □

Thus it remains to consider the case that $\mathfrak{g}$ contains non-abelian left-handed potentials. The left-handed potentials form a Lie subalgebra of $\mathfrak{g}$,

$$\mathfrak{g}_L := \big\{ \mathcal{A} = (A_L, 0) \in \mathfrak{g} \big\} \subset \mathfrak{g} \,. \tag{3.30}$$

Again applying Lemma 3.6, we know that every $\mathcal{A} = (A_L, 0) \in \mathfrak{g}$ has the property that A^2 is a multiple of the identity. The following general lemma gives an upper bound for the dimension of $\mathfrak{g}_L$.

Lemma 3.10. *Let $\mathfrak{h} \subset \mathfrak{su}(N)$ be a Lie algebra with the additional property that*

$$A^2 \sim \mathbb{1}_{\mathbb{C}^N} \qquad \text{for all } A \in \mathfrak{h} \,. \tag{3.31}$$

Then $\mathfrak{h}$ is isomorphic to a subalgebra of $\mathfrak{su}(2)$.

Proof. Polarizing (3.31), we find that for all $A, A' \in \mathfrak{h}$,

$$\big\{A, A'\big\} = k(A, A')\, \mathbb{1}_{\mathbb{C}^N}$$

with a bilinear form $k : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$. Since the square of a non-zero Hermitian matrix is positive semi-definite and non-zero, we conclude that k is positive definite and thus defines a scalar product on $\mathfrak{h}$. Hence $\mathfrak{h}$ generates a Clifford algebra $\mathcal{C}\ell(\mathfrak{h}, \mathbb{R})$. Since $\mathfrak{h}$ is also a Lie algebra, the commutator of two elements in $\mathfrak{h}$ is again an element of $\mathfrak{h}$. This means for the Clifford algebra that the bilinear covariants $[u, v]$ with $u, v \in \mathfrak{h}$ all multiples of the generators of the Clifford algebra. This in turn implies that the dimension of the Clifford algebra is at most three (for details see the classification of Clifford algebras in [8]). Moreover, $\mathfrak{h}$ is a Lie algebra isomorphic to a subalgebra of $\mathfrak{su}(2)$. ☐

Since every Lie subalgebra of $\mathfrak{su}(2)$ is abelian we immediately obtain the following result.

Corollary 3.11. *The left-handed dynamical gauge group $\mathfrak{g}_L$,* (3.30)*, is Lie algebra isomorphic to $\mathfrak{su}(2)$.*

We now write $\mathfrak{g}_L$ more explicitly as matrices.

Lemma 3.12. *There is a unitary matrix $V \in \mathrm{U}(8)$ (acting on the generations) and a basis $(A_{L,\alpha})_{\alpha=1,2,3}$ of $\mathfrak{g}_L$ such that*

$$A_{L,\alpha} = U_L^{\mathrm{mix}}\, V \begin{pmatrix} \sigma_\alpha & 0 & 0 & 0 \\ 0 & \sigma_\alpha & 0 & 0 \\ 0 & 0 & \sigma_\alpha & 0 \\ 0 & 0 & 0 & \sigma_\alpha \end{pmatrix} V^*\, (U_L^{\mathrm{mix}})^* \,, \tag{3.32}$$

where σ^α are the Pauli matrices, and U_L^{mix} is the matrix in (2.18)*.*

Proof. Using (2.17), we can pull out the mixing matrices and work with 8×8-matrices. Since $\mathfrak{g}_L$ is Lie algebra isomorphic to $\mathfrak{su}(2)$, it can be regarded as a representation of $\mathfrak{su}(2)$ on $\mathbb{C}^8$. We decompose this representation into irreducible components. The fact that the matrix $(A_{L,\alpha})^2$ is a multiple of the identity implies that every irreducible component is the fundamental representation (because all the other irreducible representations are not generators of a Clifford algebra). This gives the result. ☐

3.5.2. *Arranging the free gauge group of maximal dimension.* We denote the *commutant* of g_L by g'_L,

$$g'_L = \{\mathcal{A}' \in \mathfrak{u}(8)_L \oplus \mathfrak{u}(1)_R \oplus \mathfrak{u}(7)_R \text{ with } [\mathcal{A}, \mathcal{A}'] = 0 \quad \forall \mathcal{A} \in \mathfrak{g}_L\} \,.$$

Lemma 3.13. *The dynamical gauge algebra is contained in the direct sum*

$$\mathfrak{g} \subset \mathfrak{g}_L \oplus \mathfrak{g}'_L \,.$$

Proof. For any $\mathcal{A} = (A_L, 0) \in \mathfrak{g}_L$ and $\tilde{\mathcal{A}} = (\tilde{A}_L, \tilde{A}_R) \in \mathfrak{g}$, the commutator is left-handed

$$[\mathcal{A}, \tilde{\mathcal{A}}] = \big([A_L, \tilde{A}_L, 0], 0\big) .$$

Therefore, this commutator must be an element of $\mathfrak{g}_L$. In this way, every $\tilde{\mathcal{A}} \in \mathfrak{g}$ gives rise to a linear endomorphism of $\mathfrak{g}_L$. In view of the structure equations for $\mathfrak{su}(2)$ (which in the usual notation with Pauli matrices can be written as $[\sigma_\alpha, \sigma_\beta] = i\varepsilon_{\alpha\beta\gamma}\, \sigma_\gamma$ with ε the totally anti-symmetric Levi-Civita symbol), this endomorphism of $\mathfrak{g}_L$ can be realized uniquely as an inner endomorphism, i.e. there is a unique $\hat{\mathcal{A}} = (\hat{A}_L, 0) \in \mathfrak{g}_L$ with

$$[\mathcal{A}, \tilde{\mathcal{A}}] = [\mathcal{A}, \hat{\mathcal{A}}] \qquad \text{for all } \mathcal{A} \in \mathfrak{g}_L .$$

As a consequence, the potential $\mathcal{A}^c := \tilde{\mathcal{A}} - \hat{\mathcal{A}}$ lies in the commutant $\mathfrak{g}_L'$. We thus obtain a unique decomposition

$$\tilde{\mathcal{A}} = \hat{\mathcal{A}} + \mathcal{A}^c \qquad \text{with} \qquad \hat{\mathcal{A}} \in \mathfrak{g}_L \text{ and } \mathcal{A}^c \in \mathfrak{g}_L' .$$

This conclues the proof. □

In particular, this lemma gives information on the left-handed component of $\mathfrak{g}$, denoted by

$$\mathrm{pr}_L\, \mathfrak{g} = \{A_L \text{ with } (A_L, A_R) \in \mathfrak{g}\} \subset \mathfrak{u}(8) .$$

Note that the projection $\mathrm{pr}_L\, \mathfrak{g}$ is a Lie algebra which contains $\mathfrak{g}_L$.

Lemma 3.14. *The left-handed component of the dynamical gauge algebra satisfies the inclusion*

$$\mathrm{pr}_L\, \mathfrak{g} \subset \mathfrak{g}_L \oplus \mathfrak{u}(4) . \tag{3.33}$$

Here the elements of $\mathfrak{u}(4)$ *come with the matrix representation*

$$U_L^{\mathrm{mix}}\, V \left(A \otimes \mathbb{1}_{\mathbb{C}^2}\right) V^* \, (U_L^{\mathrm{mix}})^* , \qquad A \in \mathfrak{u}(4) , \tag{3.34}$$

where the factor $\mathbb{1}_{\mathbb{C}^2}$ *acts on the block matrix entries in* (3.32).

Proof. The commutant of the matrices in (3.32) is computed to be all matrices whose block matrix entries are the identity. Taking a unitary transformation gives the result. □

Let us consider what this lemma tells us about the possible form of the free gauge algebra. Since $\mathfrak{g}$ does not contain right-handed gauge potentials (see Lemma 3.9), the left-handed component of the free gauge potentials is disjoint from $\mathfrak{g}_L$. Hence, using (3.33),

$$\mathrm{pr}_L\, \mathfrak{g}_{\mathrm{free}} \subset \mathfrak{u}(4) , \tag{3.35}$$

where the potentials in $\mathfrak{u}(4)$ are again of the form (3.34), plus possibly vectors of $\mathfrak{g}_L$. The right-handed component of $\mathfrak{g}_{\mathrm{free}}$, on the other hand, must be of the form (3.29). Combining these findings gives the following result.

Lemma 3.15. *Choosing the mixing matrices such that the free gauge group has the maximal dimension, we obtain*

$$\mathfrak{g}_{\mathrm{free}} = \mathfrak{u}(1) \times \mathfrak{su}(3) .$$

In a suitable global gauge, the gauge potentials in $\mathfrak{g}_L$ (see (3.30) *and Lemma 3.12) have a basis* $(A_{L,\alpha})_{\alpha=1,2,3}$ *with*

$$A_{L,\alpha} = U_L^{\mathrm{mix}} \begin{pmatrix} \sigma_\alpha & 0 & 0 & 0 \\ 0 & \sigma_\alpha & 0 & 0 \\ 0 & 0 & \sigma_\alpha & 0 \\ 0 & 0 & 0 & \sigma_\alpha \end{pmatrix} (U_L^{\mathrm{mix}})^* \,, \tag{3.36}$$

where the mixing matrix U_L is a diagonal matrix on the sectors of the form

$$U_L^{\mathrm{mix}} = \mathrm{diag}(U_1, U_2, \mathbb{1}, U_{\mathrm{CKM}}, \mathbb{1}, U_{\mathrm{CKM}}, \mathbb{1}, U_{\mathrm{CKM}}) \tag{3.37}$$

with unitary matrices $U_1, U_2, U_{\mathrm{CKM}} \in \mathrm{U}(3)$. *The free* $\mathfrak{u}(1)$*- and* $\mathfrak{su}(3)$*-potentials, on the other hand, have the respective matrix representations*

$$A_L = A_R = B\, \mathrm{diag}\Big(0, -1, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\Big) \qquad \text{with } B \in \mathfrak{u}(1) = \mathbb{R} \tag{3.38}$$

$$A_L = A_R = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \otimes \mathbb{1}_{\mathbb{C}^2} \qquad \text{with } C \in \mathfrak{su}(3)\,. \tag{3.39}$$

Proof. The free gauge potentials are those vectorial potentials which are compatible with both (3.35) and (3.29). The zeros in (3.29) imply that at least one of the rows and columns of $\mathfrak{u}(4)$ must be zero. But it is possible to realize the subgroup $\mathfrak{u}(3)$ by considering the potentials of the form (3.39) (but with $C \in \mathfrak{u}(3)$). In order to get consistency, we must make sure that all mixing matrices drop out of (3.34). This forces us to choose the mixing matrix of the form

$$U_L^{\mathrm{mix}} = \mathrm{diag}(U_1, U_2, U_3, U_4, U_3, U_4, U_3, U_4)$$

with four unitary matrices $U_j \in \mathrm{U}(3)$. Since a joint unitary transformations of all sectors has no influence on the potentials in (2.17), it is no loss of generality to choose $U_3 = \mathbb{1}$. This gives (3.37). In order to satisfy the second relation in (3.28), the matrix C must be trace-free. This gives (3.39).

In order to find all the remaining vectorial potentials which are compatible with both (3.35) and (3.29), one must keep in mind that the left-handed component may be formed of linear combinations of (3.34) and the potentials in $\mathfrak{g}_L$ of the form (3.36). In order for the first row and column to vanish, the only possibility is to form the linear combinations of $A_{L,3}$ and $\mathbb{1} \in \mathfrak{u}(4)$

$$B\, \mathrm{diag}(0, -1, 0, -1, 0, -1, 0, -1)\,.$$

In view of the second relation in (3.28), we must remove the trace by adding a multiple of the potential (3.39) with $C = \mathbb{1}_{\mathbb{C}^3}$. This gives (3.38) and concludes the proof. ☐

3.5.3. *Proof that $\mathfrak{g}$ is maximal.* So far we constructed the dynamical gauge algebra

$$\mathfrak{g}_L \oplus \mathfrak{g}_{\mathrm{free}} \simeq \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(3)\,.$$

Let us show that this dynamical gauge algebra is maximal. To this end, assume that there is a chiral potential $\mathcal{A}^{\mathrm{new}} = (A_L^{\mathrm{new}}, A_R^{\mathrm{new}}) \in \mathfrak{u}(8) \oplus \mathfrak{u}(7)$ which is an element of $\mathfrak{g}$ but not of $\mathfrak{g}_L \oplus \mathfrak{g}_{\mathrm{free}}$,

$$\mathcal{A}^{\mathrm{new}} \in \mathfrak{g} \supsetneq \mathfrak{g}_L \oplus \mathfrak{g}_{\mathrm{free}}\,.$$

We want to deduce a contradiction. Since right-handed potentials were excluded in Lemma 3.9, we can assume that $A_L^{\mathrm{new}} \neq 0$. According to Lemma 3.14, it suffices to consider the potentials in $\mathfrak{g}_L \oplus \mathfrak{u}(4)$ (where the $\mathfrak{u}(4)$-potentials are represented similar to (3.39) as $\mathbb{C} \times \mathbb{1}_{\mathbb{C}^2}$ with $C \in \mathfrak{u}(4)$). Moreover, by adding suitable potentials in $\mathfrak{g}$ and using the freedom to conjugate with exponentials of potentials in $\mathfrak{g}_{\text{free}}$, we can arrange that A_L^{new} is of the form

$$A_L^{\text{new}} = \alpha \begin{pmatrix} \mathbb{1}_{\mathbb{C}^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \beta\, U_L^{\text{mix}} \begin{pmatrix} 0 & \mathbb{1}_{\mathbb{C}^2} & 0 & 0 \\ \mathbb{1}_{\mathbb{C}^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (U_L^{\text{mix}})^*$$

with real parameters α and β. By subtracting multiples of the potentials $A_{L,3}$ and the $\mathfrak{u}(1)$-potential, we can even arrange that A_L^{new} is of the form

$$A_L^{\text{new}} = \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbb{1}_{\mathbb{C}^2} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{\mathbb{C}^2} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{\mathbb{C}^2} \end{pmatrix} + \beta\, U_L^{\text{mix}} \begin{pmatrix} 0 & \mathbb{1}_{\mathbb{C}^2} & 0 & 0 \\ \mathbb{1}_{\mathbb{C}^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (U_L^{\text{mix}})^* . \tag{3.40}$$

The corresponding right-handed component A_R^{new} is unknown, except that it must be of the form (3.29). In particular, A_R^{new} might involve a non-trivial mixing matrix.

Our strategy is to evaluate off-diagonal matrix elements of $\hat{B}_R$ (see (3.23) with L and R exchanged) for specific choices of the potential. The vanishing of these matrix elements gives us constraints for $\mathcal{A}^{\text{new}}$ which in turn imply that the parameters α and β in (3.40) must vanish. We begin with the parameter β.

Lemma 3.16. *If $\mathfrak{g}$ contains a potential $\mathcal{A} = (A_L, A_R)$ with A_L of the form (3.40) with $\beta \neq 0$, then necessarily*

$$U_1 = \mathbb{1}_{\mathbb{C}^3} \qquad \text{and} \qquad U_2 U_{\text{CKM}}^* = \mathbb{1}_{\mathbb{C}^3} . \tag{3.41}$$

Proof. Using the symmetries of the free gauge group, we can transform A_L^{new} from (3.40) to the matrix

$$A_L = \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbb{1}_{\mathbb{C}^2} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{\mathbb{C}^2} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{\mathbb{C}^2} \end{pmatrix} + \beta\, U_L^{\text{mix}} M (U_L^{\text{mix}})^* , \tag{3.42}$$

where M can be any of the six matrices

$$\begin{pmatrix} 0 & \mathbb{1} & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbb{1} & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -i\mathbb{1} & 0 & 0 \\ i\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & -i\mathbb{1} & 0 \\ 0 & 0 & 0 & 0 \\ i\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & -i\mathbb{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\mathbb{1} & 0 & 0 & 0 \end{pmatrix}.$$

If (3.41) is violated, then the corresponding matrices $(1 - \check{\pi}) A_L \check{\pi}$ are obviously non-trivial and linearly independent. As a consequence, the vector space $\mathcal{B}$ (see (3.18)) has dimension at least six. If the mixing matrix U_{CKM} is non-trivial, the operators B corresponding to the left-handed potentials $A_{L,1}$ and $A_{L,2}$ in (3.36) are also non-zero, increasing the dimension of $\mathcal{B}$ to at least eight. We now show that these dimensions contradict the upper bound of Corollary 3.5. We treat the cases separately when U_{CKM} is trivial and non-trivial.

In the case that the matrix U_{CKM} is non-trivial, the form of the right-handed component of the potentials (3.29) implies that the first row cannot contribute to the operators B. As a consequence, the dimension N in Corollary 3.5 is at most $2 \times (8+7) = 30$. Hence (3.22) yields that the dimension of $\mathcal{B}$ is at most 7, a contradiction.

In the case that the matrix U_{CKM} is trivial, the mixing of the generations comes about only as a consequence of the matrices U_1 and U_2 in (3.37). Hence the matrices B_L (see (3.15)) are of the general form

$$(1-\check{\pi}) \begin{pmatrix} 0 & \star U_1 U_2^* & \star U_1 & \star U_1 & \star U_1 & \star U_1 & \star U_1 & \star U_1 \\ \star U_2 U_1^* & 0 & \star U_2 & \star U_2 & \star U_2 & \star U_2 & \star U_2 & \star U_2 \\ \star U_1^* & \star U_2^* & 0 & 0 & 0 & 0 & 0 & 0 \\ \star U_1^* & \star U_2^* & 0 & 0 & 0 & 0 & 0 & 0 \\ \star U_1^* & \star U_2^* & 0 & 0 & 0 & 0 & 0 & 0 \\ \star U_1^* & \star U_2^* & 0 & 0 & 0 & 0 & 0 & 0 \\ \star U_1^* & \star U_2^* & 0 & 0 & 0 & 0 & 0 & 0 \\ \star U_1^* & \star U_2^* & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \check{\pi} ,$$

where the stars $\star$ denote complex factors. Evaluating more specifically the six possible choices for the matrix M in (3.42), one immediately verifies that the span of the images of the corresponding matrices B_L equals 8 (note that the terms involving α drop out, and that the first two rows have a six-dimensional image, whereas the other six rows have a two-dimensional image). Hence the dimension N in Corollary 3.5 is at most $8 + 2 \times 7 = 22$. The inequality (3.22) implies that the dimension of $\mathcal{B}$ is at most 5. This is again a contradiction. □

Lemma 3.17. *Assume that the dynamical gauge algebra* $\mathfrak{g}$ *contains a potential* $\mathcal{A}^{\text{new}} = (A_L^{\text{new}}, A_R^{\text{new}}) \in \mathfrak{u}(8) \oplus \mathfrak{u}(7)$ *with* A_L^{new} *of the form* (3.40). *Then* β *vanishes.*

Proof. For a parameter $\varepsilon > 0$, we consider the family of potentials

$$\mathcal{A} = \mathcal{A}^{\text{old}} + \varepsilon\, \mathcal{A}^{\text{new}} \qquad \text{with} \qquad \mathcal{A}^{\text{old}} = (A_L^{\text{old}} = A_{L,3}, 0)$$

and $A_{L,3}$ as in (3.36). We compute the terms linear in ε. Moreover, we consider the matrix entry of B_R in the third row and first column (where we again consider B_R as an 8×8-matrix on the sectors). Using that A_L^{old} is sector-diagonal and the matrix component $(A_R)^1_3$ vanishes, the matrix A_R drops out. Similarly, the matrix A_L^{even} drops out (see (3.24)), which also has the desirable effect that the corresponding mixing matrix V_R does not enter. We thus obtain

$$\begin{aligned} (B_R)^3_1 &= -\frac{m^2}{2}\, \varepsilon \Big(Y \big(A_L^{\text{old}} A_L^{\text{new}} + A_L^{\text{new}} A_L^{\text{old}} \big)\, Y \Big)^3_1\, T^{(1)} + \mathcal{O}(\varepsilon^2) \\ &= -\frac{m^2}{2}\, \varepsilon\, Y^3_3 \Big((A_L^{\text{old}})^3_3\, (A_L^{\text{new}})^3_1 + (A_L^{\text{new}})^3_1\, (A_L^{\text{old}})^1_1 \Big)\, Y^1_1\, T^{(1)} + \mathcal{O}(\varepsilon^2) \\ &= m^2\, \varepsilon\, Y^3_3 (A_L^{\text{new}})^3_1\, Y^1_1\, T^{(1)} + \mathcal{O}(\varepsilon^2) \\ &= m^2 \beta\, \varepsilon\, Y^3_3\, U_1^* Y^1_1\, T^{(1)} + \mathcal{O}(\varepsilon^2) , \end{aligned}$$

where in the last step we used (3.40) together with the form of the mixing matrix (3.37). Taking the sectorial projection and applying Lemma 3.16 gives the result. □

The argument which shows that α vanishes is somewhat different:

Lemma 3.18. *Assume that the dynamical gauge algebra* $\mathfrak{g}$ *contains a potential* $\mathcal{A}^{\text{new}} = (A_L^{\text{new}}, A_R^{\text{new}}) \in \mathfrak{u}(8) \oplus \mathfrak{u}(7)$ *with* A_L^{new} *of the form* (3.40) *and* $\beta = 0$. *Then* α *vanishes.*

Proof. Assume conversely that there is an admissible potential $\mathcal{A} = (A_L, A_R) \in \mathfrak{g}$ with

$$A_L = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{\mathbb{C}^3} \end{pmatrix} \otimes \mathbb{1}_{\mathbb{C}^2} \,.$$

This potential involves no mixing matrices. Thus we may compute B_L according to (3.23) with $A_R^{\text{even}} = A_L$. Since the left-handed component is sector-diagonal, we obtain

$$B_L = -\frac{m^2}{2}\, Y\, (A_L - A_R)^2\, Y \,.$$

The sectorial projection of this matrix must be a multiple of $\acute{Y}\grave{Y}$. In view of (3.29), this implies that $(A_L - A_R)^2 = 0$, and thus $A_L = A_R$. But the resulting potential violates the second equation in (3.28) and is thus not admissible. This is a contradiction. □

The previous lemmas show that the parameters α and β in (3.40) are both zero. But then A_L^{new} vanishes, a contradiction. We conclude that $\mathfrak{g} = \mathfrak{g}_L \oplus \mathfrak{g}_{\text{free}}$ is maximal.

3.5.4. *Non-triviality of the mixing matrices.* We now analyze the properties of the mixing matrices U_{MNS} and U_{CKM}. Suppose that one of these matrices is non-trivial. Then the matrix B (see (3.16)) corresponding to the left-handed potentials $A_{L,1}$ and $A_{L,2}$ in (3.36) is non-zero. The representation of Lemma 3.4 yields in particular that the column of the matrix B all have the same norm. This implies that

$$\big\|(\mathbb{1} - \check{\pi})U_{\text{MNS}}\check{\pi}\big\| = \big\|(\mathbb{1} - \check{\pi})U_{\text{CKM}}\check{\pi}\big\| \,.$$

This shows that if one of the matrices U_{MNS} and U_{CKM} is non-trivial, the other is also non-trivial. Moreover, by a global phase transformation, we can arrange that the relation (3.5) holds.

In the next lemma we show that (3.6) implies (3.7).

Lemma 3.19. *Suppose that the masses of the charged leptons and neutrinos are different in the sense* (3.6). *Then the mixing matrices U_{MNS} and U_{CKM} are non-trivial.*

Proof. Assume conversely that the potentials in $\mathfrak{g}_L$ do not involve mixing matrices. Then for any $\mathcal{A} \in \mathfrak{g}_L$, we can compute B_L according to (3.23) with $A_R^{\text{even}} = A_L$. Considering the first sector and taking the sectorial projection, we obtain

$$\mathfrak{I}_1\, \hat{B}_L\, \mathfrak{I}_1 = -\frac{m^2}{4}\, \mathfrak{I}_1 \Big(\acute{Y}\grave{Y}\, A_L[\xi]^2 + A_L[\xi]\, \acute{Y}\grave{Y}\, A_L[\xi] \Big)\, \mathfrak{I}_1 T^{(1)} \,,$$

where $\mathfrak{I}_1$ denotes the projection on the neutrino sector. Since the right-handed component of $\mathcal{A}$ vanishes, we can also compute B_R with $A_L^{\text{even}} = A_R$. This gives

$$\mathfrak{I}_1\, \hat{B}_R\, \mathfrak{I}_1 = -\frac{m^2}{2}\, \mathfrak{I}_1\, \acute{Y}\grave{Y}\, A_L[\xi]^2\, \mathfrak{I}_1 T^{(1)} \,.$$

We conclude that

$$\begin{aligned} \mathfrak{I}_1 \big(\hat{B}_L - \hat{B}_R\big)\, \mathfrak{I}_1 &= \frac{m^2}{4}\, \mathfrak{I}_1 \Big(\acute{Y}\grave{Y}\, A_L[\xi]^2 - A_L[\xi]\, \acute{Y}\grave{Y}\, A_L[\xi] \Big)\, \mathfrak{I}_1\, T^{(1)} \\ &= \frac{m^2}{4}\, \mathfrak{I}_1 A_L[\xi]^2\, \mathfrak{I}_1\, T^{(1)} \sum_{\alpha=1}^{3} \big(\tilde{m}_\alpha^2 - m_\alpha^2\big) \,. \end{aligned}$$

This is non-zero for the potentials $A_{L,1}$ and $A_{L,2}$ in (3.36), a contradiction. □

3.5.5. *Proof that $\mathfrak{g}$ is admissible.* We have shown that, under the assumptions of Theorem 3.2, the dynamical gauge potentials are necessarily of the form (3.2)–(3.4). It remains to show that these potentials are indeed admissible in the sense that they satisfy all algebraic constraints. This can be done explicitly as follows: In order to study the sectorial projection of the gauge phases, one can use the fact that the phases of the free gauge potentials (3.2) and (3.4) drop out of the closed chain and thus do not enter the EL equations. Therefore, it suffices to consider the weak potentials (3.3). Since the weak potentials are block-diagonal, one can consider the lepton block and the charged blocks separately. For the lepton block, the computations were carried out in [4, §3.2]. In the charged blocks, the computation is even easier because there are no shear contribution; it reduces to applying [4, Lemma 3.2]. For the bilinear logarithmic terms one can again use that the free gauge potentials drop out. Therefore, one can analyze the neutrino block and the charged blocks exactly as in [4, §5.1]. For the field tensor terms, the relevant contributions are linear in the field. Therefore, one may choose a basis on the sectors where the field tensor is diagonal and use the computations and results of [4, §5.2].

This completes the proof of Theorem 3.2.

4. The Effective Action

In this section we rewrite the EL equations in the continuum limit (2.28) as the Euler-Lagrange equations of an effective action. We adapt the methods introduced in [4]. This adaptation is straightforward because the dynamical gauge potentials as obtained in Theorem 3.2 either act separately within each block (the weak and electromagnetic gauge potentials), or else they mix identical blocks (the strong gauge potentials). This makes it possible to analyze the blocks separately. For each block, we can proceed just as in [4, Sections 6 and 7].

4.1. **The General Strategy.** Our goal is to rewrite (2.28) as the Euler-Lagrange equations of an effective action of the form

$$\mathcal{S}_{\text{eff}} = \int_M \left(\mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{EH}} \right) \sqrt{-\deg g}\, d^4x \,, \tag{4.1}$$

involving a Dirac Lagrangian (which describes the coupling of the Dirac wave functions to the gauge fields and gravity), a Yang-Mills-type Lagrangian for the gauge fields and the Einstein-Hilbert action. Moreover, the Dirac wave functions should satisfy the Dirac equation (see (2.6))

$$(i\partial\!\!\!/ + \mathcal{B} - mY)\Psi = 0 \,. \tag{4.2}$$

Exactly as explained in [4, 6.2], one must take into account that the Dirac equation (4.2) holds for the auxiliary fermionic projector (without taking the sectorial projection), whereas the current and the energy-momentum tensor which is to be obtained by varying the Dirac Lagrangian in (4.1) involve a sectorial projection. This leads us to choose the effective Dirac Lagrangian as

$$\mathcal{L}_{\text{Dirac}} = \text{Re} \left(\overline{\psi}\, 3\acute{\pi}_\tau (i\partial\!\!\!/ + \acute{\pi}\mathcal{B}\acute{\pi} - mY)\psi \right) , \tag{4.3}$$

where the operator $\acute{\pi}_\tau$ has the form

$$\acute{\pi}_\tau := (1 + \tau\chi_L \mathfrak{I}_1)\, \acute{\pi} \qquad \text{with } \tau \in \mathbb{R} \,, \tag{4.4}$$

and $\mathfrak{I}_1$ is again the projection on the neutrino sector. The sectorial projections in (4.3) are needed in order to get the correct coupling of the Dirac wave functions to the bosonic fields. The parameter τ in (4.4) gives us the freedom to modify the coupling of the right-handed component of the neutrinos to the gravitational field.

Our goal is to choose the Lagrangians $\mathcal{L}_{\text{YM}}$ and $\mathcal{L}_{\text{EH}}$ such that their first variation is compatible with (2.28). In order to treat the gauge fields, one first rewrites ΔQ as

$$\Delta Q(x,y) = \frac{i}{2} \sum_{n,c} \text{Tr}_{\mathbb{C}^2}\big(I_n\, \mathcal{Q}_c\big)\, I_n\, \chi_c\, \not{\xi}\;,$$

and represents the factors $\mathcal{Q}_c$ by

$$\mathcal{Q}_L := \mathcal{K}_L - \frac{1}{4}\; \text{Tr}_{\mathbb{C}^2}\big(\mathcal{K}_L + \mathcal{K}_R\big)\; \mathbb{1}_{\mathbb{C}^2}$$

(and similarly for $\mathcal{Q}_R$), where the matrices $\mathcal{K}_c$ are defined by

$$\text{Tr}_{\mathbb{C}^2}\left(I_n\, \mathcal{K}_c\right) = \frac{\Delta|\lambda^{xy}_{nc-}|}{|\lambda_-|}\; 3^3\, T^{(0)}_{[0]} T^{(-1)}_{[0]}\, \overline{T^{(-1)}_{[0]}} + (\deg < 4)\;.$$

In [4], the matrices $\mathcal{K}_c$ are computed in the neutrino block, and these computations apply just as well to the quark blocks. Next, in order to treat the tensor indices properly, one writes $\mathcal{K}_c$ as

$$\mathcal{K}_c = i\xi_k\, \mathfrak{J}^k_c \;+\; (\deg < 4) + o\Big(|\vec{\xi}|^{-3}\Big) \tag{4.5}$$

and sets

$$\mathcal{Q}^k_c := \mathcal{K}^k_c - \frac{1}{4}\; \text{Tr}_{\mathbb{C}^2}\big(\mathcal{K}^k_L + \mathcal{K}^k_R\big)\; \mathbb{1}_{\mathbb{C}^2}\;.$$

The Lagrangian $\mathcal{L}_{\text{YM}}$ must be chosen such as to satisfy the conditions

$$K(\varepsilon,\xi)\; \frac{\delta}{\delta \mathcal{A}} \Big(\mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{YM}}\Big) = \text{Tr}_{\mathbb{C}^8}\left(\mathcal{Q}^k_L[\hat{\mathfrak{J}}, \mathcal{A}]\; (\delta \hat{A}_R)_k + \mathcal{Q}^k_R[\hat{\mathfrak{J}}, \mathcal{A}]\; (\delta \hat{A}_L)_k\right)\,, \tag{4.6}$$

where K is a constant and $\delta\mathcal{A} = (\delta A_L, \delta A_R) \in \mathfrak{g}$ is a dynamical gauge potential. Here the square brackets $[\hat{\mathfrak{J}}, \mathcal{A}]$ clarify the dependence on the chiral potentials and on the sectorial projection of the Dirac current.

In order to treat the gravitational field, we rewrite the trace component of ΔQ as

$$\text{Tr}_{\mathbb{C}^{8\times 4}}\left(\Delta Q\; \not{u}\right) = i\xi_j u^j\; \mathcal{Q}^{kl}[\hat{T}, g]\, \xi_k \xi_l\;.$$

Our task is to satisfy the relation

$$iK(\varepsilon,\xi)\; \frac{\delta}{\delta g}\Big(\, (\mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{curv}})\, \sqrt{-\deg g}\Big) = \mathcal{Q}^{kl}[\hat{T}, g]\; \delta g_{kl}\;. \tag{4.7}$$

Once we have arranged (4.6) and (4.7), we may consider (4.1) as the effective action in the continuum limit. Varying the chiral potentials in $\mathfrak{g}$ gives the bosonic field equations, whereas varying the metric gives the equations for gravity. We point out that the variation of the effective action must always be performed under the constraint that the Dirac equation (4.2) holds. As explained in [4, §6.3], this gives rise to the so-called *sectorial corrections* to the field equations. Since these corrections are computed exactly as in [4], we do not enter the calculations again. Instead, we restrict attention to deriving the effective action and to discussing our findings.

4.2. **The Effective Lagrangian for Chiral Gauge Fields.**

4.2.1. *General structure of the effective Lagrangian.* We begin with a general result on the structure of the effective Lagrangian for the gauge fields. The connection to the Lagrangian of the standard model will be explained in Section 4.2.2 below.

Theorem 4.1. *Denoting the dynamical gauge potentials as in Theorem 3.2 and decomposing the weak potentials as $W = \sum_{\alpha=1}^3 W^\alpha \sigma^\alpha \in \mathfrak{su}(2)$, the EL equations in the continuum limit are of variational form (4.6), where the effective Lagrangians is of the form (4.3) and*

$$\mathcal{L}_{\text{YM}} = c_1 \left(\text{Tr}_{\mathbb{C}^3} \left((\partial_j G)(\partial^j G) \right) + \frac{4}{3} (\partial_j A^{\text{em}})(\partial^j A^{\text{em}}) \right) \tag{4.8}$$

$$+ c_2 \left((\partial_j W^1)(\partial^j W^1) + (\partial_j W^2)(\partial^j W^2) \right) + c_3 \, (\partial_j W^3)(\partial^j W^3) \tag{4.9}$$

$$+ c_4 \, (\partial_j A^{\text{em}})(\partial^j W^3) + M_1^2 \Big(W^1 W^1 + W^2 W^2 \Big) + M_3^2 \, W^3 W^3 \,. \tag{4.10}$$

Here c_1, c_2, c_3, c_4 and M_1, M_3 are parameters which depend on the regularization.

Proof. The matrix-valued vector fields $\mathfrak{J}_c$ in (4.5) were computed in [4, Section §4.3]. Combining (2.9) with the integration-by-parts rule

$$0 = \nabla \left(T^{(0)}_{[0]} T^{(0)}_{[0]} \overline{T^{(0)}_{[0]}} \right) = 2\, T^{(0)}_{[0]} T^{(-1)}_{[0]} \overline{T^{(0)}_{[0]}} + T^{(0)}_{[0]} T^{(0)}_{[0]} \overline{T^{(-1)}_{[0]}} \,,$$

one sees that the following simple fraction vanishes,

$$K_2 := \frac{3}{4} \, \frac{1}{\overline{\overline{T^{(0)}_{[0]}}}} \Big[T^{(0)}_{[0]} T^{(0)}_{[0]} \, \overline{T^{(-1)}_{[0]} T^{(0)}_{[0]}} - c.c. \Big] = 0 \,.$$

As a consequence, one obtains

$$\mathfrak{J}^k_L = \hat{J}^k_R \, K_1 + \jmath^k_R \, K_3 \tag{4.11}$$

$$- 3m^2 \left(\acute{A}^k_L Y \grave{Y} + \acute{Y} Y \grave{A}^k_L \right) K_4 \tag{4.12}$$

$$+ m^2 \left(\hat{A}^k_L \, \acute{Y} \grave{Y} + \acute{Y} \grave{Y} \, \hat{A}^k_L \right) K_4 \tag{4.13}$$

$$- 3m^2 \left(\acute{A}^k_R Y \grave{Y} - 2 \acute{Y} A^k_L \, \grave{Y} + \acute{Y} Y \grave{A}^k_R \right) K_5 \tag{4.14}$$

$$- 6m^2 \left(\acute{A}^k_L \grave{Y} \, \hat{Y} + \hat{Y} \, \acute{Y} \grave{A}^k_L \right) K_6 \tag{4.15}$$

$$+ 6m^2 \left(\hat{Y} \acute{A}^k_L \grave{Y} + \acute{Y} \grave{A}^k_L \, \hat{Y} \right) K_7 \tag{4.16}$$

$$+ m^2 \left(\hat{A}^k_L \hat{Y} \hat{Y} + 2 \hat{Y} \hat{A}^k_R \hat{Y} + \hat{Y} \hat{Y} \hat{A}^k_L \right) K_6 \tag{4.17}$$

$$- m^2 \left(\hat{A}^k_R \hat{Y} \hat{Y} + 2 \hat{Y} \hat{A}^k_L \hat{Y} + \hat{Y} \hat{Y} \hat{A}^k_R \right) K_7 \,, \tag{4.18}$$

with the simple fractions $K_1, \ldots, K_7$ as given in [4, eqs (4.22)–(4.29)] (and $\mathfrak{J}^k_R$ is obtained by the obvious replacements $L \leftrightarrow R$). For the Dirac current, we thus obtain

$$\text{Tr}_{\mathbb{C}^8} \left(\mathcal{Q}^k_L[\hat{\partial}, \mathcal{A}] \, (\delta \hat{A}_R)_k + \mathcal{Q}^k_R[\hat{\partial}, \mathcal{A}] \, (\delta \hat{A}_L)_k \right) \asymp K_1 \, \text{Tr}_{\mathbb{C}^8} \left(J^k_R \, (\delta \hat{A}_R)_k + J^k_L \, (\delta \hat{A}_L)_k \right) .$$

This is compatible with (4.6) and the variation of the Dirac Lagrangian (4.3) (for fixed wave functions) if we choose

$$K(\varepsilon, \xi) = 3 \, K_1 \,.$$

For the bosonic current and mass terms, one must compensate the logarithmic poles on the light cone by a microlocal chiral transformation, just as described in [4, §4.4]. For the free gauge potentials $(A^{\text{em}}, G) \in \mathfrak{u}(1) \oplus \mathfrak{su}(3)$, the mass terms vanish. A direct computation gives

$$\begin{aligned}
&\mathrm{Tr}_{\mathbb{C}^8}\big(\mathcal{Q}^k_L[\hat{\jmath}, \mathcal{A}]\,(\delta \hat{A}_R)_k + \mathcal{Q}^k_R[\hat{\jmath}, \mathcal{A}]\,(\delta \hat{A}_L)_k\big) \\
&\qquad \asymp K_{19}\; \mathrm{Tr}_{\mathbb{C}^8}\Big(j^k[A^{\text{em}}]\,\delta A^{\text{em}} + j^k[G]\,\delta G\Big)
\end{aligned}$$

for a suitable simple fraction K_{19}, where $j[A]^k = \partial^k_j A^j - \Box A^k$ is the bosonic current. If only the potential W is considered, we can compute the right side of (4.6) exactly as in [4, Section 7] to obtain

$$\begin{aligned}
&\mathrm{Tr}_{\mathbb{C}^8}\big(\mathcal{Q}^k_L[\hat{\jmath}, \mathcal{A}]\,(\delta \hat{A}_R)_k + \mathcal{Q}^k_R[\hat{\jmath}, \mathcal{A}]\,(\delta \hat{A}_L)_k\big) \\
&\quad \asymp K_{20}\,\Big(j^k[W^1]\,(\delta W^1)_k + j^k[W^2]\,(\delta W^2)_k\Big) + K_{21}\, j^k[W^3]\,(\delta W^3)_k \\
&\qquad + m^2 K_{22}\,\Big((W^1)^k\,(\delta W^1)_k + (W^2)^k\,(\delta W^2)_k\Big) + m^2 K_{23}\,(W^3)^k\,(\delta W^3)_k
\end{aligned}$$

for suitable simple fractions K_ℓ. Finally, we must take into account cross terms of A^{em} and W^3. These have the form

$$\begin{aligned}
&\mathrm{Tr}_{\mathbb{C}^8}\big(\mathcal{Q}^k_L[\hat{\jmath}, \mathcal{A}]\,(\delta \hat{A}_R)_k + \mathcal{Q}^k_R[\hat{\jmath}, \mathcal{A}]\,(\delta \hat{A}_L)_k\big) \\
&\quad \asymp K_{24}\, j^k[W^3]\,(\delta A^{\text{em}})_k + K_{25}\, j^k[A^{\text{em}}]\,(\delta W^3)_k \qquad (4.19)\\
&\qquad + m^2 K_{26}\,(W^3)^k\,(\delta A^{\text{em}})_k + m^2 K_{27}\,(A^{\text{em}})^k\,(W^3)_k\,. \qquad (4.20)
\end{aligned}$$

Let us consider the terms (4.19). In order to be compatible with the variational ansatz (4.6), we must impose that

$$K_{24} = K_{25}\,. \tag{4.21}$$

This relation is automatically satisfied if we use the form of the current terms in (4.11). However, one should keep in mind that K_3 has a logarithmic pole which must be compensated by a microlocal chiral transformation. We thus obtain the condition for the microlocal chiral transformation that it should preserve (4.21).

Moreover, the factors K_{26} and K_{27} vanish, as we now explain. First, the potential A^{em} does not contribute to the mass terms, implying that K_{27} is zero. Moreover, direct inspection of the contributions (4.12)–(4.18) shows that for a sector-diagonal potential which does not involve a mixing matrix, the mass terms depends only on the combination $A_L - A_R$. This also implies that

$$\mathcal{Q}^k_L = -\mathcal{Q}^k_R\,. \tag{4.22}$$

On the other hand, for a variation by an electromagnetic potential, $\delta A_L = \delta A_R$. Therefore, the right side of (4.6) vanishes by symmetry, implying that K_{26} is zero. Similar as explained above for (4.21), the microlocal chiral transformation must be performed in such a way that (4.22) is respected.

Combining all the terms gives the result. □

4.2.2. *Correspondence to electroweak theory.* Let us discuss the form of the effective Lagrangian obtained in Theorem 4.1. The first summand in (4.8) is precisely the Lagrangian of the strong interaction in the standard model. The second summand (4.8) is the Lagrangian of the electromagnetic field. One difference to the standard model is that the coupling constants of the strong and electromagnetic fields are not independent, but are related to each other by an algebraic relation. In order to understand this relation, one should keep in mind that the masses and coupling constants appearing in Theorem 4.1 should be regarded as the "naked" constants, which coincide with the physical constants only at certain energy scale which can be thought of as being very large (like for example the Planck energy). Thus a relation for the "naked" constants does not mean that this relation should be valid also for the physical constants. This situation is indeed very similar to that in grand unified theories (GUTs); we refer the reader for example to the textbook [10]. The terms in (4.9) and (4.10) have a similarity to the Lagrangian of the weak potential after spontaneous symmetry breaking. Indeed, one obtains complete agreement for specific values of the constants:

Theorem 4.2. *Assume that the parameters in the effective Lagrangian of Theorem 4.1 satisfy the conditions*

$$c_2 = c_3 = c_4 \qquad \text{and} \qquad M_1 = M_3 \,. \tag{4.23}$$

Then the effective Lagrangian coincides with the Lagrangian of the standard model after spontaneous symmetry breaking excluding the Higgs field. The coupling constants of the strong and weak gauge potentials as well of weak hypercharge are given by

$$g_{\text{strong}} = \frac{1}{2\sqrt{c_1}} \,, \qquad g_{\text{weak}} = \frac{2}{\sqrt{c_2}} \,, \qquad g_{\text{hyp}} = \frac{1}{2} \left(\frac{16}{3}\, c_1 - c_2 \right)^{-\frac{1}{2}} \,.$$

Under the assumptions of this theorem, one can introduce the Z and $W^\pm$-potentials by forming the usual linear combinations of the weak potential and the potential of weak hypercharge. The masses m_Z and m_W of the corresponding gauge bosons are related to each other by

$$m_Z = \frac{m_W}{\cos \Theta_W} \,,$$

where the Weinberg angle Θ_W is given as usual by

$$\cos \Theta_W = \frac{g_{\text{weak}}}{\sqrt{g_{\text{weak}}^2 + g_{\text{hyp}}^2}} \,.$$

Proof. So far, we parametrized the isospin diagonal electroweak potentials by the electromagnetic potential A^{em} and the weak potential W^3. The standard model, however, is usually formulated instead in terms of the potential of weak hypercharge A^{hyp} and the weak potential. Since the transformation from one parametrization to the other also change the weak potential, we denote the weak potential in the parametrization of the standard model by an additional tilde. Then the potentials are related by

$$A^{\text{em}} = 2\, A^{\text{hyp}} \,, \qquad W^3 = \tilde{W}^3 - A^{\text{hyp}} \,.$$

Using these relations in (4.8)–(4.10), the relevant part of the Lagrangian transforms to

$$\begin{aligned}\mathcal{L}_{\text{YM}} \asymp\;& \frac{16}{3}\, c_1 (\partial_j A^{\text{hyp}})(\partial^j A^{\text{hyp}}) \\ &+ c_3\, (\partial_j A^{\text{hyp}})(\partial^j A^{\text{hyp}}) - 2c_3\, (\partial_j A^{\text{hyp}})(\partial^j \tilde{W}^3) + c_3\, (\partial_j \tilde{W}^3)(\partial^j \tilde{W}^3) \\ &+ 2c_4\, (\partial_j A^{\text{hyp}})(\partial^j \tilde{W}^3) - 2c_4\, (\partial_j A^{\text{hyp}})(\partial^j A^{\text{hyp}}) \\ &+ M_3^2\, (\tilde{W}^3 - A^{\text{hyp}})(\tilde{W}^3 - A^{\text{hyp}})\end{aligned}$$

and thus

$$\begin{aligned}\mathcal{L}_{\text{YM}} \asymp\;& \Big(\frac{16}{3}\, c_1 + c_3 - 2c_4\Big)(\partial_j A^{\text{hyp}})(\partial^j A^{\text{hyp}}) + c_3\, (\partial_j \tilde{W}^3)(\partial^j \tilde{W}^3) \\ &- 2(c_3 - c_4)\, (\partial_j A^{\text{hyp}})(\partial^j \tilde{W}^3) + M_3^2\, (\tilde{W}^3 - A^{\text{hyp}})(\tilde{W}^3 - A^{\text{hyp}})\,.\end{aligned} \tag{4.24}$$

This differs from the Lagrangian of the standard model in that the kinetic term of the standard model does not involve the cross terms $\sim (\partial_j A^{\text{hyp}})(\partial^j \tilde{W}^3)$. But using the equation $c_3 = c_4$ in (4.23), this cross term vanishes. We thus obtain for the full Lagrangian

$$\mathcal{L}_{\text{YM}} = c_1\, \text{Tr}_{\mathbb{C}^3}\big((\partial_j G)(\partial^j G)\big) + \Big(\frac{16}{3}\, c_1 - c_3\Big)(\partial_j A^{\text{hyp}})(\partial^j A^{\text{hyp}}) \tag{4.25}$$

$$+ c_2\, \Big((\partial_j W^1)(\partial^j W^1) + (\partial_j W^2)(\partial^j W^2)\Big) + c_3\, (\partial_j \tilde{W}^3)(\partial^j \tilde{W}^3) \tag{4.26}$$

$$+ M_1^2 \Big(W^1 W^1 + W^2 W^2\Big) + M_3^2\, (\tilde{W}^3 - A^{\text{hyp}})(\tilde{W}^3 - A^{\text{hyp}})\,. \tag{4.27}$$

The constants in front of the quadratic derivative terms can be absorbed into the coupling constants by rescaling the potentials. To this end, we introduce the coupling constants

$$g_{\text{strong}} = \frac{1}{2\sqrt{c_1}}\,, \quad g_{\text{hyp}} = \frac{1}{2}\left(\frac{16}{3}\, c_1 - c_3\right)^{-\frac{1}{2}}\,, \quad g_2 = \frac{1}{2\,\sqrt{c_2}}\,, \qquad g_3 = \frac{1}{2\,\sqrt{c_3}}\,.$$

Rescaling the potentials according to

$$G \to g_{\text{strong}}\, G\,, \quad A^{\text{hyp}} \to g_{\text{hyp}}\, A^{\text{hyp}}\,, \quad W^{1/2} \to g_2\, W^{1/2}\,, \quad W^3 \to g_3\, W^3\,, \tag{4.28}$$

the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\text{YM}} =\;& \frac{1}{4}\, \text{Tr}_{\mathbb{C}^3}\big((\partial_j G)(\partial^j G)\big) + \frac{1}{4}\, (\partial_j A^{\text{hyp}})(\partial^j A^{\text{hyp}}) \\ &+ \frac{1}{4}\, \Big((\partial_j W^1)(\partial^j W^1) + (\partial_j W^2)(\partial^j W^2) + (\partial_j \tilde{W}^3)(\partial^j \tilde{W}^3)\Big) \\ &+ M_1^2\, g_2^2 \Big(W^1 W^1 + W^2 W^2\Big) + M_3^2\, (g_3 \tilde{W}^3 - g_{\text{hyp}} A^{\text{hyp}})(g_3 \tilde{W}^3 - g_{\text{hyp}} A^{\text{hyp}})\,.\end{aligned}$$

Now the kinetic term of the Lagrangian looks just as in the standard model. Clearly, the rescaling of the potentials (4.28) must also be performed in the Dirac Lagrangian 4.3. This amounts to inserting coupling constants into the gauge covariant derivative, which thus becomes

$$D_j = \partial_j - i g_{\text{strong}}\, G_j - i g_{\text{hyp}}\, A_j^{\text{hyp}}\, \mathcal{Y} - i g_2\, \chi_L \big(W_j^1 \sigma^1_{\text{iso}} + W_j^2 \sigma^2_{\text{iso}}\big) - i g_3\, \chi_L W_j^3 \sigma^3_{\text{iso}}\,,$$

where $\sigma^\alpha_{\text{iso}}$ are the Pauli matrices acting on isospin, and $\mathcal{Y}$ is the generator of the weak hypercharge,

$$\mathcal{Y} = \chi_L \,\text{diag}\Big(-1, -1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\Big) + \chi_R \,\text{diag}\Big(0, -2, \frac{4}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}\Big) \,.$$

In the standard model, there is only one coupling constant for the $\mathfrak{su}(2)$-potentials. This leads us to impose the equation $c_2 = c_3$ in (4.23). Then $g_2 = g_3 =: g_{\text{weak}}$. The last relation in (4.23) is needed in order for the mass matrix after spontaneous symmetry breaking to be compatible with (4.27) (see for example [9, Section 20.2]). This concludes the proof. □

4.2.3. *Additional relations between the regularization parameters.* The remaining important question is whether the relations (4.23) hold for suitable regularizations of the fermionic projector. Do they always hold? Or are there in general deviations?

The general answer is that the relations (4.23) do not need to hold in general. But as will be specified in Proposition 4.3 below, the relations (4.23) do hold in the limiting cases when the masses of the leptons are much larger than the masses of the neutrinos, and the mass of the top quark is much larger than the mass of the leptons. Therefore, using the hierarchy of the fermion masses of the standard model, we obtain agreement with the standard model. In view of the experimental observations

$$\frac{m_{\nu_\tau}^2}{m_\tau^2} \lesssim 8 \times 10^{-5} \qquad \text{and} \qquad \frac{m_\tau^2}{m_{\text{top}}^2} \approx 10^{-4} \,, \tag{4.29}$$

it seems that our limiting case should be an excellent approximation. But for general regularizations, we expect deviations for the masses and coupling constants of electroweak theory of the order (4.29). Unfortunately, since at the moment we do not have detailed information on how the microscopic structure of the physical regularization is, we cannot make a prediction for the deviations.

Proposition 4.3. *Assume that the all the mass parameters in* (2.2) *and* (2.3) *are dominated by the mass of the heaviest charged fermion, i.e.*

$$m_3 \gg m_1, m_2 \tag{4.30}$$

and

$$m_3 \gg \tilde{m}_1, \tilde{m}_2, \tilde{m}_3 \,. \tag{4.31}$$

Moreover, assume that the physical (=renormalized) mass of the top quark is much larger than that of the leptons,

$$m_{\text{top}} \gg m_e, m_\nu, m_\tau \,. \tag{4.32}$$

Then the parameters in the effective Lagrangian of Theorem 4.1 satisfies the relations (4.23) *up to relative errors of the order*

$$\frac{m_1^2 + m_2^2}{m_3^2} \,, \qquad \frac{\tilde{m}_1^2 + \tilde{m}_2^2 + \tilde{m}_3}{m_3^2} \qquad \textit{and} \qquad \frac{m_e^2 + m_\nu^2 + m_\tau^2}{m_{\text{top}}^2} \,.$$

The remainder of this section is devoted to the derivation of this proposition. Our derivation will not be a mathematical proof. Instead, we are content with explaining the involved approximations in the non-rigorous style common in theoretical physics.

We begin by noting that the term involving the bosonic currents in (4.11) contributes to the right side of (4.6) by

$$
\begin{aligned}
\mathrm{Tr}_{\mathbb{C}^8}\big(\mathcal{Q}^k_L[\hat{\jmath},\mathcal{A}]\,(\delta\hat{A}_R)_k + \mathcal{Q}^k_R[\hat{\jmath},\mathcal{A}]\,(\delta\hat{A}_L)_k\big) &\asymp K_3\;\mathrm{Tr}_{\mathbb{C}^8}\Big(\hat{\jmath}^k_R\,(\delta\hat{A}_R)_k + \hat{\jmath}^k_L\,(\delta\hat{A}_L)_k\Big) \\
&= \frac{16}{3}\,K_3\,9\,j[A_{\mathrm{em}}]^k\,(\delta A_{\mathrm{em}})_k + 4\,K_3\,9\,\Big(j[A_{\mathrm{em}}]^k\,(\delta W^3)_k + j[W^3]^k\,(\delta A_{\mathrm{em}})_k\Big) \\
&\quad + 8\,K_3\Big(9\,j[W^3]^k\,(\delta W_3)_k + j[\hat{W}^1]^k\,(\delta\hat{W}_1)_k + j[\hat{W}^2]^k\,(\delta\hat{W}_2)_k\Big)
\end{aligned}
$$

(the factors of 9 come up whenever we leave out the sectorial projection). This is of variational form, leading us to choose

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YM}} \asymp \frac{8}{3}\,K_3\,9\,(\partial_j A_{\mathrm{em}})\,(\partial^j A_{\mathrm{em}}) + 4\,K_3\,9\,\big(\partial_j A_{\mathrm{em}})(\partial^j W^3)\big) \\
+ 4\,K_3\Big(9\,(\partial_j W^3)(\partial^j W_3) + (\partial_j \hat{W}^1)(\partial_j \hat{W}^1) + (\partial_j \hat{W}^2)(\partial_j \hat{W}^2)\Big)\,.
\end{aligned}
\tag{4.33}
$$

This is of the general form of Theorem 4.1, but with $c_3 = c_4$. Thus one of the relations in (4.23) is automatically satisfied. Moreover, the coupling constants c_1 and c_3 are related by

$$
c_1 = \frac{c_3}{2}\,. \tag{4.34}
$$

The relation $c_2 = c_3$ is violated because of the sectorial projection of the mixing matrix. However, keeping in mind that the Dirac Lagrangian (4.3) as well as the mass terms also involve sectorial projections, these sectorial projections indeed play no role. This will be explained at the end of this section. If we disregard the sectorial projection for the moment, the relation $c_2 = c_3$ is also satisfied. We conclude that the structure of how the bosonic currents enter the EL equations in the continuum limit is consistent with the relations on the left of (4.23). Moreover, one has the additional relation (4.34).

The subtle point is that K_3 has a logarithmic pole which must be compensated by a microlocal chiral transformation. Thus in order to decide if the relations on the left of (4.23) or the relation (4.34) remain valid, we need to analyze whether the microlocal chiral transformation respects these relations. This is not easy to tell because the analysis in [4, §4.4] depends in a complicated way on the ratios of the fermion masses. Moreover, the parameters $\mathfrak{c}_0$ and $\mathfrak{c}_2$ were not determined explicitly. But at least, we can analyze the behavior of the microlocal chiral transformation if we make use of the mass hierarchies, as we now explain.

Before beginning, we need to adapt our method of compensating the logarithmic poles to the construction of the effective Lagrangian in (4.6). Recall that when introducing the microlocal chiral transformation in [3, §7.9 and §7.10] and [4, §4.4], we always compensated *all* the logarithmic poles. However, in the construction of the effective Lagrangian as introduced in [4, §6.1], we argued that the EL equations in the continuum limit (2.28) should be satisfied only in the "directions parallel to the bosonic degrees of freedom." This is implemented mathematically by the fact that (4.6) involves testing with a dynamical gauge potential $\delta\mathcal{A} \in \mathfrak{g}$. As a consequence, it is no longer necessary to compensate the logarithmic poles completely. It suffices to arrange that the logarithmic poles drop out of (4.6). More precisely, the contributions (j_L, j_R) involving logarithmic poles which remain after the microlocal chiral transformation must satisfy the condition

$$
\mathrm{Tr}_{\mathbb{C}^8}(j_L\,\delta A_L + j_R\,\delta A_R) = 0 \qquad \forall\,\delta\mathcal{A} \in \mathfrak{g}\,. \tag{4.35}
$$

In order to express this condition in a convenient way, we introduce the real vector space

$$\mathfrak{S}_8 := \text{Symm}(\mathbb{C}^8) \oplus \text{Symm}(\mathbb{C}^8) ,$$

where $\text{Symm}(\mathbb{C}^8)$ denotes the Hermitian 8×8-matrices. Moreover, we introduce the bilinear form

$$\langle .,. \rangle_{\mathfrak{S}_8} \,:\, \mathfrak{S}_8 \times \mathfrak{g} \to \mathbb{C} , \qquad \langle (j_L, j_R), \mathcal{A} \rangle_{\mathfrak{S}_8} = \text{Tr}_{\mathbb{C}^8}(j_L A_L + j_R A_R) . \tag{4.36}$$

Then (4.35) can be expressed by saying that the logarithmic contribution must be orthogonal to $\mathfrak{g}$ with respect to the bilinear form (4.36).

We begin by considering *sector-diagonal transformations*. The microlocal chiral transformation is worked out explicitly in [4, Example 4.5]. The transformation involves the eigenvalues $\mu_1, \ldots, \mu_4$ of the matrix $S_0^{-1} S_2$. In the lepton block, these eigenvalues are given by (see also the equation before Proposition 7.7 in [3])

$$\mu_{1/2} = \frac{1}{3} \left(\tilde{m}_1^2 + \tilde{m}_2^2 + \tilde{m}_3^2 \mp \sqrt{\tilde{m}_1^4 + \tilde{m}_2^4 + \tilde{m}_3^4 - \tilde{m}_1^2\, \tilde{m}_2^2 - \tilde{m}_2^2\, \tilde{m}_3^2 - \tilde{m}_1^2\, \tilde{m}_3^2} \right)$$
$$\mu_{3/4} = \frac{1}{3} \left(m_1^2 + m_2^2 + m_3^2 \mp \sqrt{m_1^4 + m_2^4 + m_3^4 - m_1^2\, m_2^2 - m_2^2\, m_3^2 - m_1^2\, m_3^2} \right) .$$

In the quark blocks, one has similarly the eigenvalues $\mu_{3/4}$, both with multiplicity two. As explained in [4, Example 4.5], the amplitude κ of the microlocal chiral transformation in each sector can vary in the range (see [4, eq. (4.63)])

$$\mathfrak{c}_0\, \mu_1 \leq \kappa \leq \mathfrak{c}_0\, \mu_2 \qquad \text{and} \qquad \mathfrak{c}_0\, \mu_3 \leq \kappa \leq \mathfrak{c}_0\, \mu_4 . \tag{4.37}$$

The general strategy is to compensate the logarithmic poles choosing $\mathfrak{c}_0$ as small as possible. The eigenvalues $\mu_1, \ldots, \mu_4$ scale like the masses squared. Therefore, if the neutrino masses are much smaller than the masses of the lepton and quarks (4.31), then the microlocal chiral transformation has no effect in the neutrino sector. Let us assume in addition that one of the masses of the charged leptons dominates (4.30). Then

$$\mu_3 = \mathcal{O}\Big(\frac{m_1^2 + m_2^2}{m_3^2} \Big) , \qquad \mu_4 = \frac{2}{3}\, m_3^2 + \mathcal{O}\Big(\frac{m_1^2 + m_2^2}{m_3^2} \Big). \tag{4.38}$$

As a consequence, the inequalities in (4.37) reduce to the interval

$$0 \leq \kappa \leq \frac{2}{3}\, \mathfrak{c}_0\, m_3^2 . \tag{4.39}$$

We conclude that for a sector-diagonal transformation, our freedom in choosing the microlocal chiral transformation reduces to selecting for the left- and right-handed component of every charged sector a parameter κ in the range (4.39). We denote these parameters by κ_{ac} with $a \in \{2, \ldots, 8\}$ and $c \in \{L, R\}$. In order to minimize $\mathfrak{c}_0$, the best strategy is to choose every parameter κ_{ac} at one of the boundary points of the interval, i.e.

$$\kappa_{ac} = 0 \qquad \text{or} \qquad \kappa_{ac} = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2 \tag{4.40}$$

(with errors as specified in (4.38)). Let us try this strategy for the current corresponding to A^{hyp}. As this current is sector-diagonal, testing in (4.6) gives zero if $\delta \mathcal{A}$ is the potential $\mathcal{A}[\delta W^1]$ or $\mathcal{A}[\delta W^2]$. Moreover, this current is invariant under the action of the strong SU(3), implying that (4.6) also vanishes if $\delta \mathcal{A}$ is a strong potential. Therefore, it suffices to consider the cases that $\delta \mathcal{A}$ is $\mathcal{A}[\delta A^{\text{hyp}}]$ or $\mathcal{A}[\delta \tilde{W}^3]$ (the tilde again clarifies that we parametrize the potentials by $(A^{\text{hyp}}, \tilde{W}^3)$). The terms with logarithmic poles generated by the current of weak hypercharge are collinear to $\mathcal{A}[\delta A^{\rm hyp}]$ and orthogonal to $\mathcal{A}[\delta \tilde{W}^3]$ (with respect to the bilinear form (4.36)). Thus we need to make sure that the logarithmic pole is compensated when testing with $\mathcal{A}[\delta A^{\rm hyp}]$, but that we get no contribution when testing with $\mathcal{A}[\delta \tilde{W}^3]$. This can be arranged by the two choices

$$(\kappa_{aL}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,0,0,0,0,0,0) \qquad \text{and} \qquad (\kappa_{aR}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,1,0,1,0,1,0)$$

or alternatively

$$(\kappa_{aL}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,0,0,0,0,0,0) \qquad \text{and} \qquad (\kappa_{aR}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,0,1,0,1,0,1)\;.$$

Indeed, since the contributions generated by the microlocal chiral transformation have a definite sign (see [4, eq. (4.49)]), we need both cases, depending on whether the bosonic current is future or past directed. By adjusting $\mathfrak{c}_0$, we can arrange that the contributions involving logarithmic poles satisfy the condition (4.35), and thus drop out of (4.6).

The next step is to compute the corresponding smooth contributions generated by the microlocal chiral transformation. Again using that the largest mass dominates (4.30), the contribution by the microlocal chiral transformation is simply given by the corresponding Dirac sea, i.e.

$$P(x,y) \sim \log|m^2 \xi^2| + c + i\pi\, \Theta(\xi^2)\, \epsilon(\xi^0) \tag{4.41}$$

with a numerical constant c (see [2, §2.5] or [3, §4.4]). Therefore, the smooth contribution is explicit. It is proportional to the original contribution involving the logarithmic pole. This is very useful because we conclude that the current term after compensating the logarithmic pole is again orthogonal to $\mathcal{A}(\delta \tilde{W}^3)$ (with respect to the bilinear form (4.36)). This means that in the kinetic term of the resulting Lagrangian, there is no cross term of $A^{\rm hyp}$ and $\tilde{W}^3$. Comparing with (4.24), this gives precisely the relation $c_1 = c_3$. We conclude that the logarithmic pole of weak of the bosonic current corresponding to weak hypercharge is compensated such that the relation $c_1 = c_3$ is preserved (up to error terms as mentioned above).

We now proceed similarly for the current corresponding to $\tilde{W}^3$. Thus we want to choose parameters κ_{ac} of the form (4.40) which respect the strong SU(3)-symmetry, such that the logarithmic poles of the current are removed, but the resulting contribution is orthogonal to $\mathcal{A}[\delta A^{\rm hyp}]$. A short computation shows that the only two solutions are

$$\begin{aligned} (\kappa_{aL}) &= \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,1,1,0,1,0,1,0)\;, \quad (\kappa_{aR}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,0,0,0,0,0,0) \\ (\kappa_{aL}) &= \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,1,0,1,0,1,0,1)\;, \quad (\kappa_{aR}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,0,0,0,0,0,0)\;. \end{aligned} \tag{4.42}$$

Note that these ansätze have a contribution in the charged lepton sector. As will be explained below, this leads to difficulties. The only method for avoiding these difficulties is to give up (4.40) and to allow for the parameters κ_{ac} to take values in the interior of the interval (4.39). This makes it possible to choose the parameters κ_{ac} such that they vanish in the lepton block. Namely, a direct computation gives the solutions

$$\begin{aligned} (\kappa_{aL}) &= \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,a,0,a,0,a,0)\;, \quad (\kappa_{aR}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,b,c,b,c,b,c) \\ (\kappa_{aL}) &= \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,0,a,0,a,0,a)\;, \quad (\kappa_{aR}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,b,c,b,c,b,c)\;, \end{aligned} \tag{4.43}$$

where the parameters a, b, c are to be chosen such that

$$a + 4b - 2c = 0\;, \qquad 0 \leq a, b, c \leq 1 \qquad \text{and} \qquad \max(a,b,c) = 1\;.$$

Let us explain the consequence of these different solutions. In the case (4.42), the relation (4.41) again holds. This implies that the relation (4.34) will hold after removing the logarithmic poles. In the case (4.43), however, the relation (4.41) no longer holds, because all three Dirac seas contribute substantially to the microlocal chiral transformation. This makes the situation much more complicated, and we do want to not enter the details here. For our purposes, it suffices to make the following remarks. First, the parameters κ_{ac} must necessarily be chosen in accordance to the relation $c_1 = c_3$, because otherwise (4.6) could not be satisfied, and the EL equations in the continuum limit would no longer be of variational form. Moreover, since (4.41) is violated, the relation (4.34) will no longer hold after the logarithmic poles have been removed. This makes it necessary to treat c_1 and c_3 as independent effective parameters, giving rise to independent effective coupling constants g_{hyp} and g_{weak}.

We next consider *non-sectordiagonal transformations*. Since the ansatz (4.43) only affects the quark blocks, it can immediately be generalized to non-sectordiagonal transformations. Namely, since the microlocal transformation can be performed independently for the two chiral components, it suffices to consider for example the left-handed component. Then one can use an U(2)-transformation to diagonalize the logarithmic contribution. Using the degeneracy of the masses in each block, this U(2)-transformation can also be performed for the local chiral transformation by

$$L[k] \to U\, L[k]\, U^* \qquad \text{with } U \in \mathrm{U}(2)\;.$$

In this way, the constructions and results of [4, Example 4.5] can also be used for the non-sectordiagonal transformations in the quark blocks. This implies in particular that the constant c_2 in the dynamical term of the gauge fields W^1 and W^2 in (4.9) coincides with the corresponding constant c_3 for the gauge field W^3. We point out that this U(2)-transformation cannot be used in the lepton block because the masses of the neutrinos are different from those of the charged leptons. In particular, it is not clear if and how the ansatz (4.42) can be generalized to non-sectordiagonal transformations.

Next, we need to analyze the *mass terms*. This is considerably more complicated because we must analyze the contributions (4.12)–(4.18). The only contribution with logarithmic pole is the term (4.14). For the W^3-potential, we can compensate the logarithm as explained above, choosing for example

$$\begin{aligned} (\kappa_{aL}) &= \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, \Big(0,0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0\Big)\;, \quad (\kappa_{aR}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,0,1,0,1,0,1) \\ (\kappa_{aL}) &= \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, (0,0,0,1,0,1,0,1)\;, \qquad (\kappa_{aR}) = \frac{2}{3}\, \mathfrak{c}_0\, m_3^2\, \Big(0,0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0\Big)\;. \end{aligned}$$

The resulting contribution is orthogonal to the electromagnetic component, implying that the parameter K_{27} in (4.20) again vanishes. Hence we only need to take into account the contributions where the mass terms are tested by the left-handed weak potentials. In view of (4.6), it thus suffices to consider $\mathfrak{J}_R$. Moreover, as the mass terms vanish identically for free gauge fields, it suffices to consider (4.12)–(4.18) for a left-handed weak potential. Hence the relevant contribution by the mass terms reduces to

$$\mathfrak{J}^k_R \asymp -3m^2 \left(\acute{A}^k_L Y \grave{Y} + \acute{Y} Y \grave{A}^k_L \right) K_5 \tag{4.44}$$

$$+ 2m^2\, \hat{Y} \hat{A}^k_L \hat{Y}\, K_6 - m^2 \left(\hat{A}^k_L \hat{Y} \hat{Y} + \hat{Y} \hat{Y} \hat{A}^k_L \right) K_7 \,. \tag{4.45}$$

The following argument shows that the contribution (4.44) drops out of the effective EL equations: In view of the hierarchy (4.30), the logarithmic pole of the mass term is of the form (4.41). Since the contribution by the microlocal chiral transformation is of the same form, it cancels the contribution by (4.44) including the smooth contributions. As a result, (4.44) drops out of the effective EL equations.

The remaining contribution (4.45) has the following structure. In the three quark blocks, the factors $\hat{Y}$ are constants, so that the mass term can be written as $c\hat{W}$. In the lepton block, however, the fact that the neutrino masses are different from the masses of the charged leptons implies that the mass terms for W^3 have a different structure than those for W^1 and W^2. This implies that the constants M_1 and M_3 in (4.10) will in general be different.

We now give an argument which shows that M_1 and M_3 coincide in the limiting case (4.32) when the quark masses are much larger than the lepton masses. This argument will also explain why the solution (4.42) must be dismissed, leaving us with the ansatz (4.43) for the microlocal chiral transformation. Our argument makes use of the concept that the masses m_β in (2.1) are the "naked" masses, and that these masses are modified by the self-interaction to the physical masses. Having this concept in mind, it is a natural idea that the physical mass of the gauge bosons should again be described by (4.45) if only the masses of in the mass matrix mY are replaced by the physical fermion masses. This idea is motivated by the renormalization program which states that for a renormalizable theory the self-interaction describes a shift of the masses and coupling constants but leaves the structure of the interaction unchanged. However, it must be said that the renormalization of the fermionic projector is work in progress (see [7]). If we take the results of the normalization program for granted and combine them with the mass hierarchy (4.32), then we conclude that all the contributions involving the fermion masses are much smaller in the lepton block than in the quark blocks. In particular, in the ansatz (4.42) we must replace the sequences $0, 1, \ldots$ by $0, \delta, \ldots$ with $\delta \ll 1$. But then the resulting contribution is no longer orthogonal to $\mathcal{A}[\delta A^{\text{hyp}}]$. Therefore, the ansätze (4.42) must be dismissed. For the mass terms in (4.45), we conclude that the main contribution comes from the quark sectors, giving rise to an effective mass Lagrangian of the form

$$M^2 \Big(\hat{W}^1 \hat{W}^1 + \hat{W}^2 \hat{W}^2 + 9\, W^3 W^3 \Big) \tag{4.46}$$

which involves only one mass parameter.

It remains to analyze the effect of the sectorial projection of the potentials W^1 and W^2. For notational simplicity, we only consider the potential W^1. By inspecting (4.3), (4.33) and (4.46), one sees that only the sectorial projection of the potential W^1 enters. Thus varying $\hat{W}^1$, one sees that the rest mass of the bosonic field remained unchanged if all the sectorial projections were left out. Moreover, varying the Dirac Lagrangian as explained in [4, §6.3], one sees that the coupling to the Dirac particles has the same form as without the sectorial projection, except for the sectorial corrections mentioned after (4.7). This explains the last equation in (4.23) and thus establishes Proposition 4.3.

4.3. **The Effective Lagrangian for Gravity.**

Theorem 4.4. *Assume that the parameters δ and p_{reg} satisfy the scaling*

$$\varepsilon \ll \delta \ll \frac{1}{m}\,(m\varepsilon)^{\frac{p_{\text{reg}}}{2}}\,,$$

and that the regularization satisfies the conditions (2.10)*. Then the EL equations in the continuum limit* (2.27) *can be expressed in terms of the effective action* (4.1) *with the Einstein-Hilbert action*

$$\mathcal{L}_{\text{EH}} = \frac{1}{\kappa(\varepsilon,\delta)}\,(R + 2\Lambda)$$

(where R denotes scalar curvature and $\Lambda \in \mathbb{R}$ is the cosmological constant). Here the gravitational constant κ is given by

$$\kappa = \frac{\delta^2}{\tau_{\text{reg}}}\,\frac{K_{17}}{K_{18}}\,,$$

where K_{17} and K_{18} are the simple fractions

$$K_{17} = -K_{16}\left(1 - \frac{L^{(0)}_{[0]}}{T^{(0)}_{[0]}}\right) \qquad \text{and} \qquad K_{18} = \frac{1}{2}\,K_8\left(1 - \frac{L^{(0)}_{[0]}}{T^{(0)}_{[0]}}\right)$$

(which are both to be evaluated weakly on the light cone (2.11)*). The parameter τ in the Dirac Lagrangian* (4.3) *is determined to have the value $\tau = -16$.*

Proof. One proceeds exactly as in [4, Section 8]. The variation of the matrices $\mathcal{Q}^{kl}$ is computed as in [4, Lemmas 8.1 and 8.2]. In order to satisfy (4.7) one must choose $\tau = -16$. Then the result follows immediately. ☐

5. The Higgs Field

As explained in [3, §8.5], the Higgs potential of the standard model can be identified with suitable scalar/pseudoscalar potentials in the Dirac equation. As shown in [3, Lemma B.1], the contributions by the pseudoscalar potentials to the fermionic projector drop out of the EL equations. The scalar potentials, on the other hand, contribute to the EL equations to degree three on the light cone. As the detailed computations are rather involved, we postpone the analysis of these contributions to a future publication.

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