Arithmetic Divisors on Products of Curves over non-Archimedean Fields

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
an der Fakultät für Mathematik
der Universität Regensburg

Vorgelegt von Philipp Vollmer aus Herford
bei Prof. Dr. Klaus Künnemann

Regensburg im Jahre 2016

Arithmetic Divisors on Products of Curves over non-Archimedean Fields

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
an der Fakultät für Mathematik
der Universität Regensburg

Vorgelegt von Philipp Vollmer aus Herford
bei Prof. Dr. Klaus Künnemann

Regensburg im Jahre 2016

Das Promotionsgesuch wurde eingereicht am 18.5.2016
Die Arbeit wurde angeleitet von Prof. Dr. Klaus Künnemann

Prüfungsausschuß:

Vorsitzender:	Prof. Dr. Helmut Abels
Erster Gutachter:	Prof. Dr. Klaus Künnemann
Zweiter Gutachter:	Prof. Dr. Walter Gubler
Weiterer Prüfer:	Prof. Dr. Felix Finster

Contents

CHAPTER 1

Introduction

Let k be a field which is complete with respect to a non-trivial discrete non-Archimedean absolute value $|\cdot|$ and let k° be its valuation ring, $k^{\circ\circ}$ its maximal ideal, π a uniformiser, and set $b = |\pi|$. Let $\widetilde{k} = k^\circ/k^{\circ\circ}$ be the residue field of k. We require $\widetilde{k}$ to be algebraically closed. Fields with that property are classified by [**Ser79**, §4, Thm. 2, §5, Thm. 3].

We denote by $\mathbb{K}$ the completion of an algebraic closure of k equipped with the unique absolute value which extends the given absolute value on k. This is an algebraically closed complete field by [**BGR84**, 3.4, Prop. 3]. We denote by $\mathbb{K}^\circ$ the valuation ring and by $\mathbb{K}^{\circ\circ}$ the maximal ideal of $\mathbb{K}^\circ$. We will consider the unique totally ramified extensions k_n of k of degree $n \geq 1$ as canonically embedded into $\mathbb{K}$ and equipped with the absolute value induced from that of $\mathbb{K}$. Let π_n be a uniformiser of k_n°. Note that by our choice of an absolute value on k_n the absolute of π_n is $b^{1/n}$. We denote by η, s the generic point and the special point of $\operatorname{Spec}(k^\circ)$ respectively. For a scheme $\mathfrak{B}$ over k° we denote by $\mathfrak{B}_\eta$ and $\mathfrak{B}_s$ the generic and the special fibre respectively.

Let X be a geometrically integral, projective variety over k. We set $X_{\mathbb{K}} := X \otimes_k \mathbb{K}$. We denote by $X_{\mathbb{K}}^{\mathrm{an}}$ the analytification of $X_{\mathbb{K}}$ in the sense of Berkovich (cf. [**Ber90**, Thm. 3.4.1]).

For $q \in \mathbb{Q}$ let $D = qD'$ be a $\mathbb{Q}$-Cartier divisor on X such that D' is an integral Cartier divisor. Then a *Green's function* for D is a continuous function on the complement of $\operatorname{supp}(D'_{\mathbb{K}})^{\mathrm{an}}$ in $X_{\mathbb{K}}^{\mathrm{an}}$ which has logarithmic poles along $D_{\mathbb{K}}^{\mathrm{an}}$ i.e., a function

$$g\colon (X_{\mathbb{K}})^{\mathrm{an}} \setminus \operatorname{supp}(D'_{\mathbb{K}})^{\mathrm{an}} \longrightarrow \mathbb{R},$$

such that if $(f_i)_{i\in I}$ are local equations for D' on an open covering $(U_i)_{i\in I}$ of X, for each $i \in I$ the function

$$g + q\log_b |f_i|^2\colon U_{i,\mathbb{K}}^{\mathrm{an}} \setminus \operatorname{supp}(D'_{\mathbb{K}})^{\mathrm{an}} \longrightarrow \mathbb{R}$$

extends to a continuous function on each $U_{i,\mathbb{K}}^{\mathrm{an}}$. Here, we consider $\operatorname{supp}(D'_{\mathbb{K}})$ as equipped with the reduced closed subscheme structure in $X_{\mathbb{K}}$. A pair of a $\mathbb{Q}$-Cartier divisor D and a Green's function g for D will be called *arithmetic divisor* and written $D + g$. The set of arithmetic divisors forms a $\mathbb{Q}$-vector space.

An important class of Green's functions arises from models. Let $\mathfrak{X}$ be a *model* of X over k_n°, that is a projective flat scheme over k_n° with a fixed isomorphism $\mathfrak{X} \otimes_{k_n^\circ} k_n \cong X \otimes_k k_n$, and let $\mathfrak{D}'$ be a *model* of an integral Cartier divisor D' on X, that is a Cartier divisor $\mathfrak{D}'$ on $\mathfrak{X}$ satisfying $\mathfrak{D}'|_\eta = D'$. We denote by $\pi_{\mathfrak{X}} : (X_{\mathbb{K}})^{\mathrm{an}} \to \mathfrak{X} \otimes \widetilde{\mathbb{K}}$ the reduction map which is given on $\mathbb{K}$-valued points by reduction modulo $\mathbb{K}^{\circ\circ}$ of suitable projective coordinates in $\mathbb{K}^\circ$. For any $x \in X_{\mathbb{K}}^{\mathrm{an}}$ let f be a local equation for $\mathfrak{D}' \otimes \mathbb{K}^\circ$ around $\pi_{\mathfrak{X}}(x)$ in $\mathfrak{X} \otimes \mathbb{K}^\circ$. Then the *model function* $g_{\mathfrak{X},\mathfrak{D}'}$ is defined by

$$g_{\mathfrak{X},\mathfrak{D}'}(x) = -\log_b |f(x)|^2.$$

Clearly the definition of model functions can be extended to $\mathbb{Q}$-Cartier divisors on $\mathfrak{X}$.

Let g, h be two Green's functions for the divisor D. Then $g - h$ is a Green's function for the trivial Cartier divisor and we can define the *distance* of g and h by

$$d(g,h) = \sup_{x \in X_{\mathbb{K}}^{\mathrm{an}}} |(g-h)(x)|,$$

and this number is finite as X was assumed to be projective. Hence d is a metric on the space of Green's functions for D. We say that an arithmetic divisor $D + g$ is *semipositive* if g is the limit with respect to the distance d of a sequence of model functions $(g_{\mathfrak{X}_l, \mathfrak{D}_l})_{l \in \mathbb{N}}$ where each $\mathfrak{D}_l$ is a vertically nef $\mathbb{Q}$-Cartier divisor on a model $\mathfrak{X}_l$, that is, the degree of $\mathscr{O}(\mathfrak{D}_l)$ restricted to any complete curve in the special fibre of $\mathfrak{X}_l$ is non-negative. We say that an arithmetic divisor is *DSP* if it can be written as a difference of semipositive ones.

It is an interesting question which arithmetic divisors are DSP because for a d-dimensional prime cycle Z of X we can define the local height

$$\lambda_{(D_0+g_0),\dots,(D_d+g_d)}(Z)$$

with respect to DSP arithmetic divisors $D_0+g_0, \dots, D_d+g_d$ with $\operatorname{supp} D_0 \cap \ldots \cap \operatorname{supp} D_d \cap Z = \emptyset$ after [**Zha95**] and [**Gub03**] (cf. Ch. 4). This local height extends the local intersection numbers of Cartier divisors on models and, in fact, interesting local height functions such as canonical local height functions on abelian varieties in the case of bad reduction arise as heights with respect to DSP Green's functions in a natural way. In general it is difficult to give an explicit description of the class of DSP arithmetic divisors, but in the case of the self product of a curve we can achieve a result.

Let X be a geometrically integral smooth projective curve over k and let $\mathfrak{X}$ be a projective regular strictly semi-stable model of X over k°. By the theory of Berkovich skeleta (cf. [**Ber99**]) the self-product of the geometric realisation of the dual graph $|\Gamma(\mathfrak{X})|^2$ of the special fibre naturally embeds into the Berkovich analytification $(X \times_k X)_{\mathbb{K}}^{\mathrm{an}}$ of the self product $(X \times_k X)_{\mathbb{K}}$ and the image $S(\mathfrak{X} \times_{k^\circ} \mathfrak{X})$ under this embedding is a strong deformation retract via a retraction map called τ. Let $g : S(\mathfrak{X} \times_{k^\circ} \mathfrak{X}) \to \mathbb{R}$ be a continuous function. Then one can show that $g \circ \tau$ is a Green's function for the trivial divisor and we are interested in the question when $0 + g \circ \tau$ is a DSP arithmetic divisor. If $\mathfrak{X}$ is smooth over k° then $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$ is smooth and the skeleton $S(\mathfrak{X} \times_{k^\circ} \mathfrak{X})$ is a point. Otherwise the topological space $S(\mathfrak{X} \times_{k^\circ} \mathfrak{X})$ has a natural cover by charts isomorphic to $[0,1]^2$ which are unique up to a choice of an order on the set of irreducible components of the special fibre of $\mathfrak{X}$ and which we will refer to as *canonical charts*. The following result states that if the restriction of g to each of this chart has reasonable regularity, then $g \circ \tau$ is a DSP Green's function:

Theorem A (Cor. 3.7.4)
If the restriction of g to each chart of $S(\mathfrak{X} \times_{k^\circ} \mathfrak{X})$ is of class C^2 then $0 + g \circ \tau$ is a DSP arithmetic divisor.

In the proof of the theorem we construct a series of projective models $(\mathfrak{B}_n)$ of $X_{k_n} \times_{k_n} X_{k_n}$ independent of the function g and Cartier divisors $\mathfrak{D}_n$ on each $\mathfrak{B}_n$ such that $g_{\mathfrak{B}_n, g_n} \to g \circ \tau$ for $n \to \infty$. We want to relate the corollary to the following result of Kolb (cf. [**Kol16a**, Theorem 3.32]), which has been stated before in a similar way for different models by Zhang (cf. [**Zha10**, Prop. 3.3.1, 3.4.1]):

Theorem (Zhang,Kolb)
Assume that g_0, g_1, g_2 are functions on $S(\mathfrak{X} \times_{k^\circ} \mathfrak{X})$ such that the restriction to each canonical chart is of class C^2. Denote by $\mathfrak{D}_{in}$ the Cartier divisors on $\mathfrak{B}_n$ from above such that

$g_{\mathfrak{B}_n,\mathfrak{D}_{in}} \to g_i \circ \tau$. *Then the limit of the series of local intersection numbers*

$$\left(\frac{1}{n^3}\left\langle \mathfrak{D}_{0,n}, \mathfrak{D}_{1,n}, \mathfrak{D}_{2,n}\right\rangle\right)_{n\in\mathbb{N}}$$

exists for $n \to \infty$ *and equals*

$$\int_{S(\mathfrak{X}\times_{k^\circ}\mathfrak{X})} \partial_x f_0 \partial_y f_1 \partial_{xy} f_2 + \text{ permutations.} \tag{ZF}$$

By x *and* y *we mean local coordinates in the canonical charts.*

The context of this result is as follows. In [**GS95**] Gross and Schoen have constructed a *modified diagonal cycle* Δ' on the triple product $X \times_K X \times_K X$ of a curve X over a global field K, which is a 1-dimensional cycle and is homologically trivial. They define the height pairing $\langle \Delta', \Delta' \rangle$ in the sense of Beilinson–Bloch of Δ' with itself. Later in [**Zha10**] Zhang relates this number with the self intersection ω_a^2 of the admissibly metrised dualising sheaf ω_a of X in the following shape.

$$\omega_a^2 = \langle \Delta', \Delta' \rangle + \text{ local contributions}$$

The local contributions are (up to a factor) integrals of the form (ZF) for a special choice of the functions f_i. These terms come from harmonic analysis on graphs and these been shown to be positive in [**Cin11**]. Assuming $\langle \Delta', \Delta' \rangle \geq 0$ which is presently known in the case of $K = \mathbb{C}(T)$ this yields $\omega_a^2 > 0$. By [**Zha93**] this is in turn known to be equivalent to Bogomolov's conjecture on the distribution of points of small height on X.

We are now confronted with two quantities. One quantity is the local height of $X \times_k X$ with respect to the DSP arithmetic divisors $0 + g_i$ for $i \in \{0, 1, 2\}$. The other quantity is the number (ZF) occuring as a limit of local intersection numbers. In fact these two numbers coincide which is not clear neither from the existence of the limit of the intersection numbers, nor from the property that $g \circ \tau$ is DSP. Kolb ([**Kol16a**, Ex. 3.36]) gives a counterexample of a DSP arithmetic divisor on $\mathbb{P}^1 \times_k \mathbb{P}^1$ where the limit of intersection numbers does indeed depend on the way the Green's function of this divisor is approximated. It was first observed by Zhang in [**Zha95**] that the height is independend of the approximating sequence whenever this sequence satisfies certain positivity conditions i.e., semipositivity using an important result of Kleiman [**Kle66**].

In fact in our situation we achieve the following result:

Corollary (Cor. 4.3.2)
The formula (ZF) computes the local height of $X \times_k X$ *with respect to* $0 + f_0 \circ \tau, 0 + f_1 \circ \tau, 0 + f_2 \circ \tau$.

The main ingredient to the proof of this corollary is Theorem A. Fix $i \in \{0, 1, 2\}$. It is known from algebraic geometry, that each divisor $\mathfrak{D}_{in}$ can be written as the difference of ample ones, say $\mathfrak{D}_{in} = \mathfrak{D}'_{in} - \mathfrak{D}''_{in}$. The problem is, that this decomposition is not canonical in general and one has to achieve that the sequences of Green's functions $g_{\mathfrak{B}_n,\mathfrak{D}'_{in}}$ or $g_{\mathfrak{B}_n,\mathfrak{D}''_{in}}$ converge for $n \to \infty$. In our work we can show that if f_i satisfies some convexity condition CC, there is an ample divisor $\mathfrak{M}$ on[1] each $\mathfrak{B}_n$ and a positive constant $C \in \mathbb{N}$ such that $C \cdot \mathfrak{M} + \mathfrak{D}_{in}$ is vertically nef for all n. In fact, $\mathfrak{M}$ can be chosen rather freely. The prove relies on explicit computations of intersection numbers of curves in the special fibre of $\mathfrak{B}_n$. Now by construction $f_i \circ \tau = g_{\mathfrak{B},\mathfrak{M}} + f_i \circ \tau - g_{\mathfrak{B},\mathfrak{M}}$ is DSP and the associated model functions $g_{\mathfrak{B}_n,\mathfrak{M}+\mathfrak{D}_{in}}$ converge for $n \to \infty$. This convergence is the essential ingredient to

[1] to be precise: There are maps $\mathfrak{B}_n \to \mathfrak{B}_1$ for each n and $\mathfrak{M}$ is the pull-back of a fixed line bundle on $\mathfrak{B}_1$.

prove the formula for the local height. From this point on the statement of the theorem follows formally. To extend the results to arbitrary C^2 functions we show that every such function can be written as the difference of CC functions which is a purely analytic task.

One application of the explicit formula lies in solving partial differential equations on $(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}$. Let Y be an arbitrary projective variety of dimension d over k. Let $D_i + f_i$ be DSP arithmetic divisors for $i \in \{1, \ldots, d\}$ and set D_0 as the trivial divisor. The rule

$$f_0 \mapsto \lambda_{(D+f_i)_{i\in\{0,\ldots,d\}}}(Y)$$

which assigns to each DSP Green's function f_0 for the trivial divisor the local height of Y with respect to $D_0 + f_0, \ldots, D_d + f_d$ determines a measure on $Y^{\mathrm{an}}_{\mathbb{K}}$, the Chambert-Loir measure, which we will denote by

$$c_1(\mathscr{O}(D_1), \|\cdot\|_{f_1}) \wedge \ldots \wedge c_1(\mathscr{O}(D_d), \|\cdot\|_{f_d}).$$

By solving partial differential equations on $(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}$ we mean the following: Given a measure μ we look for f_1, f_2, f such that

$$\mu = f \cdot c_1(\mathscr{O}_{X\times_k X}, \|\cdot\|_{f_1\circ\tau}) \wedge c_1(\mathscr{O}_{X\times_k X}, \|\cdot\|_{f_2\circ\tau})$$

where f, f_1, f_2 may be subject to constraints. In our work we focus on two kinds of partial differential equations: Monge–Ampère type and Poisson type differential equations. The first kind of differential equation has been investigated in [**YZ10**], [**Liu11**] and [**BFJ15**]. The second kind has been proposed by Shou-Wu Zhang in [**Zha15**]. It was observed in [**Liu11**] that explicit formulae for local heights can be used to solve differential equations by reducing to differential equations on real manifolds. In our case, the explicit formulae for the local height allow us to reduce our differential equations to the real Monge-Ampère equation and a linear elliptic equation of second order respectively.

Theorem B
[2] *Let $f : |\Gamma|^2 \to \mathbb{R}$ be a smooth function such that the support of f is contained in a canonical square $S = [0,1]^2$ and has empty intersection with the topological boundary of S in $\mathbb{R}^2$. Let $g\colon |\Gamma|^2 \to \mathbb{R}$ be a smooth function. Assume that $u : S \to \mathbb{R}$ is a smooth solution of the partial differential equation*

$$f(\partial_{xx}g\partial_{yy}g - 2(\partial_{xy}g)^2) = \partial_{xx}g\partial_{yy}u + \partial_{yy}g\partial_{xx}u - 2\partial_{xy}g\partial_{xy}u$$

supported in the interior of S. Then $u \circ \tau$ is a solution for the Poisson problem i.e.,

$$c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{u\circ\tau}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g\circ\tau}) =$$
$$f \cdot c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g\circ\tau}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g\circ\tau})$$

holds.

Another aspect of the functions $f_i \circ \tau$ is the following: Let Y be a good $\mathbb{K}$-analytic space. In [**CLD12**] Chambert-Loir and Ducros have developed a theory of smooth differential forms and currents of type (p, q) on Y. Roughly the idea is locally regarding the Berkovich space as a tropical variety and then using the differential forms from tropical geometry developed in [**Lag12**]. Chambert-Loir and Ducros have developed a theory of *plurisubharmonic* metrics on line bundles or psh for short in this context of differential forms. There is also a way of integration (n, n)-forms on n-dimensional Berkovich spaces. If a metric can be written as the limit of psh metrics, then the say it is *psh-approximable.* If a metric is the quotient of two psh-approximable metrics, then they call it *approximable.*

[2] On request of the reviewers the statement of Theorem B has been changed.

For a psh approximable metric $\|\cdot\|$ on a line bundle L of Y there is a well defined Chern-current $c_1(L, \|\cdot\|)$. Moreover, products of such Chern currents can be obtained using an analogue of Bedford-Taylor theory.

Let Y be a projective variety over k. Any arithmetic divisor $D+g$ defines a metric $\|\cdot\|_g$ on $\mathscr{O}(D)$. Now we can compare the different notions of positivity of metrics as follows.

Theorem C
Let Y be the analytification of a projective variety of dimension d. Then

(i) If $D+g$ is semipositive then the metric $\|\cdot\|_g$ is a psh-approximable metric on $\mathscr{O}(D)^{\mathrm{an}}_{\mathbb{K}}$,
(ii) if $D_1+g_1, \ldots, D_d+g_d$ are semipositive then the Chambert-Loir measure associated to these DSP arithmetic divisors coincides with the measure associated to the wedge product of the Chern currents $c_1(\mathscr{O}(D_1), \|\cdot\|_{g_i \circ \tau})$ for $i \in \{1, \ldots, d\}$.

Point (i) is due to Chambert-Loir and Ducros in case the metrics come from models $\mathfrak{D}$ where some power of $\mathscr{O}(\mathfrak{D})$ is a globally generated line bundle. We generalise this result to the semipositive case using a result from [**BFJ16**] which in turn we had to generalise to the non-discretely valued case. Then (ii) follows from purely measure-theoretic arguments.

We turn back to the case of the self product of a curve. Now our approximation theorem implies that $f_i \circ \tau$ determines an approximable metric on the trivial line bundle and we can evaluate the integral

$$\int_{(X \times_k X)^{\mathrm{an}}} f_0 \circ \tau \cdot d'd''(f_1 \circ \tau) \wedge d'd''(f_2 \circ \tau) \tag{DF}$$

explicitly which is the integral of f_0 against the product of the Chern currents $c_1(\mathscr{O}_{X \times X}, \|\cdot\|_{f_i \circ \tau})$ for $i \in \{1, 2\}$. Using the previous theorem we see that this is the local height of $X \times X$ with respect to $f_0 \circ \tau$, $f_1 \circ \tau$, and $f_2 \circ \tau$. Assume now that f_i is a smooth function on each canonical square for all $i \in \{0, 1, 2\}$. We can show (cf. Thm. 5.4.2) that each $f_i \circ \tau$ is a smooth function in the sense of [**CLD12**] on the dense open subset Q of $(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}$ which is the preimage of all topological interiors of charts $|\Gamma|^2$ isomorphic to $[0,1]^2$ under the retraction τ.

Theorem D (Thm. 5.5.2 (i))
We have the following explicit formula for the integral

$$\int_Q f_0 \circ \tau \cdot d'd''(f_1 \circ \tau) \wedge d'd''(f_2 \circ \tau) = \int_{S(\mathfrak{X} \times_{k^\circ} \mathfrak{X})} f_0 \partial_{xx} f_1 \partial_{yy} f_2 - \int_{S(\mathfrak{X} \times_{k^\circ} \mathfrak{X})} f_0 \partial_{xy} f_1 \partial_{xy} f_2, \tag{SP}$$

We also show that if all $f_i \circ \tau$ are smooth functions in the sense of Chambert-Loir and Ducros then this is in fact the integral over the whole analytification. The essential ingredient for the proof of our theorem is the computation of the *canonical calibration*, i.e., data for the integration of $(2,2)$-forms coming from tropical geometry. Here the interplay with the geometry of the Berkovich space is essential: For example, the formula would not hold true if $\widetilde{k}$ was not algebraically closed.

Comparing (DF) and (SP) we get a decomposition of the local height into a *smooth contribution* which is an integral over Q and a *singular contribution*, which is an integral with respect to a measure supported in the complement of Q.

The structure of this text is a follows: In Chapter 2 the necessary definitions and constructions are given. Chapter 3 is devoted to our approximation theorem: In Section

3.1 we discuss the analysis of a suitable class of functions on the geometric realisation of products of graphs. In Section 3.2 we construct the models which we will use for showing the DSP property of arithmetic divisors and investigate the structure of the special fibre. In Section 3.7 we will show the DSP property for a class arithmetic divisors on products of curves. To illustrate the ideas of the proof we do the baby case of a curve in Section 3.6. Finally we compare our result with existing results, one due to Liu (cf. [**Liu11**]) and one due to Burgos, Philippon, and Sombra (cf. [**BGPS14**]).

Chapter 4 is devoted to applications: We prove our results on local heights of $X \times_k X$. Here we also give the application of our approximation theorem to differential equations of Monge-Ampère type and Poisson type.

Chapter 5 is devoted to the theory of smooth differential forms by [**CLD12**] and Monge-Ampère measures. Here we prove our comparison results between the smooth setting and the setting of DSP arithmetic divisors.

Acknowledgements: I would like to express my gratitude to my supervisor Klaus Künnemann for inspiring discussions, many of the ideas contained in this thesis, patience, and guidance during my PhD. I would also like to thank my co advisor Walter Gubler for many discussions and ideas, and for encouraging me to pursue ideas. I want to thank Johannes Kolb for explaining his thesis to me which was the starting point for my thesis. I am very grateful to Christian Dahlhausen, Julius Hertel, Philipp Jell, Florent Martin, and Jascha Smacka for numerous inspiring discussions. I want to thank Ruben Jakob and Nikolai Nowaczyk for answering me questions to the fourth chapter. I acknowledge support from the DFG graduate school GRK 1692 Curvature, Cycles, and Cohomology in Regensburg.

Moreover, I would like to thank my parents Brigitte and Werner Vollmer for supporting me during my studies in mathematics for the last 8 years and I want to thank my companion Magdalena Kärcher for support and consolation during the last years.

CHAPTER 2

Definitions and Constructions

2.1. Notations/Conventions

The natural numbers are assumed to contain zero. For two sets we write $A \subset B$ if every element of A is contained in B. In particular, we allow equality. By an *order* on a set S we will mean a total order. For a subset $Z \subset \mathbb{R}^n$ we denote by $\mathrm{convhull}(Z)$ the convex hull of Z in $\mathbb{R}^n$. A *variety* is an integral separated scheme of finite type over a field K. If $Z \subset X$ is a closed subscheme of a scheme X then we denote by Z_{red} the closed subscheme of X which has the unique reduced closed subscheme structure of Z. Unless otherwise stated a divisor on a scheme will always be a $\mathbb{Q}$-Cartier divisor. We write divisors additively and reserve the multiplicative notation for line bundles. Let R be a ring. By $X \times_R Y$ we mean $X \times_{\operatorname{Spec} R} Y$ for two R-schemes X and Y. If S/R is a ring extension we denote by X_S the base change $X \times_R \operatorname{Spec} S$. We say that a morphism $\varphi\colon X \to Y$ of schemes is *projective* if it factors into a closed immersion $X \to Y \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{P}^m_{\mathbb{Z}}$ and the projection to Y for some $m \geq 1$. This kind of morphism is sometimes called H-projective. We say that a line bundle $\mathscr{L}$ is *very ample relative to* Y if there exists an immersion $i\colon X \hookrightarrow \mathbb{P}^m \times_{\operatorname{Spec}\mathbb{Z}} Y$ such that $i^*\mathscr{O}(1) \cong \mathscr{L}$. An immersion is an open immersion followed by a closed immersion.

Let $\mathbb{K}$ be a field which is complete with respect to a non-Archimedean absolute value and $f : X \to Y$ be a morphism of $\mathbb{K}$-analytic spaces in the sense of Berkovich. Then we denote by $\mathrm{Int}(X/Y)$ the interior of f and by $\partial(X/Y)$ the boundary.

2.2. Berkovich Analytification

We recall the definition of the analytic space associated to an admissible formal scheme as in [**Ber94**, §1]: Let $\mathbb{K}$ be a field which is complete with respect to a complete non-trivial non-Archimedean absolute value $|\cdot|$. Let $\mathbb{K}^\circ$ be the valuation ring, $\mathbb{K}^{\circ\circ}$ the maximal ideal and $\widetilde{\mathbb{K}}$ be the residue field. Let η, s be the generic and the special points of $\operatorname{Spec}\mathbb{K}^\circ$ respectively. We say that a scheme $\mathfrak{X}$ over $\mathbb{K}^\circ$ is *vertical* if the the image of the canonical map $\mathfrak{X} \to \operatorname{Spec}(\mathbb{K}^\circ)$ is the special point. By an *admissible algebra* we mean a $\mathbb{K}^\circ$-algebra isomorphic to a $\mathbb{K}^\circ$-torsion-free quotient of $\mathbb{K}^\circ\langle x_1, \ldots, x_n\rangle$ by an ideal I. We denote by $\operatorname{Spf}(A)$ the formal spectrum of an admissible algebra A. An *admissible formal scheme* is a formal $\mathbb{K}^\circ$-scheme which admits a locally finite cover by $\operatorname{Spf}(A)$ for admissible $\mathbb{K}^\circ$-algebras A.

The analytic space associated to $\operatorname{Spf}(A)$ of an admissible formal algebra A is $\mathscr{M}(A \otimes_{\mathbb{K}^\circ} \mathbb{K})$, where $\mathscr{M}(B)$ is the Berkovich spectrum of an affinoid algebra B as defined in [**Ber90**, 1.2]. The *special fibre* over $\mathbb{K}^\circ$ is defined as $\operatorname{Spec}(A \otimes_{\mathbb{K}^\circ} \widetilde{\mathbb{K}})$. These definitions globalise yielding the analytification $\mathfrak{X}^{\mathrm{an}}$ and the special fibre $\mathfrak{X}_s$ of $\mathfrak{X}$ for an admissible formal scheme $\mathfrak{X}$. This construction is functorial: For a $\mathbb{K}^\circ$-morphism $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ between admissible formal schemes we get a morphism $\varphi^{\mathrm{an}}\colon \mathfrak{X}^{\mathrm{an}} \to \mathfrak{Y}^{\mathrm{an}}$ between the associated analytic spaces. Furthermore, there is a reduction map

$$\pi_{\mathfrak{X}}\colon \mathfrak{X}^{\mathrm{an}} \to \mathfrak{X}_s$$

which is surjective by [**Ber90**, §2.4].

Let X be a $\mathbb{K}$-analytic space. A *formal model* of X is a proper admissible formal scheme $\mathfrak{X}$ with a distinguished isomorphism $\mathfrak{X}^{\mathrm{an}} \cong X$. Let X be a separated scheme of finite type over $\mathbb{K}$. In virtue of [**Ber90**, Thm. 3.4.1] we associate to X the analytification X^{an}. This construction is also functorial. So, for a separated algebraic scheme $\mathfrak{B}$ of finite presentation over $\mathbb{K}^\circ$ we have two ways of associating a Berkovich analytic space with it: First we can take the generic fibre of $\mathfrak{B}$ and take the associated analytic space $(\mathfrak{B}_\eta)^{\mathrm{an}}$. We will denote this analytic space by $\mathfrak{B}^{\mathrm{an}}$. On the other hand we can consider the analytic space $\widehat{\mathfrak{B}}^{\mathrm{an}}$ associated to the completion $\widehat{\mathfrak{B}}$ of $\mathfrak{B}$ along the special fibre. There is always a functorial inclusion $\widehat{\mathfrak{B}}^{\mathrm{an}} \hookrightarrow \mathfrak{B}^{\mathrm{an}}$ which is an isomorphism if $\mathfrak{B}$ is proper (cf. [**Thu07**, Prop. 1.10]). We will be in the following setting repeatedly.

Assumption 2.2.1
Let k be a field which is complete with respect to a non-trivial non-Archimedean discrete absolute value. Let k° be the valuation ring of k and π be a uniformiser. Set $b = |\pi|^{-1}$. Let $\widetilde{k}$ be the residue field. Assume that $\widetilde{k}$ is algebraically closed. Let k_n be the totally ramified extension of k of degree n. We denote by $\mathbb{K}$ the completion of an algebraic closure of k, $\mathbb{K}^\circ$ the valuation ring of $\mathbb{K}$, and by $\mathbb{K}^{\circ\circ}$ the maximal ideal of $\mathbb{K}^\circ$. We consider k_n as embedded into $\mathbb{K}$ and equipped with the induced absolute value.

2.3. Arithmetic Divisors

We will introduce the language of arithmetic divisors on projective varieties over a non-Archimedean valued field which is partially a generalisation of the definition given in [**Mor16**].

We will be in the situation of Assumption 2.2.1.

Convention 2.3.1
Let $\mathfrak{B}$ be a scheme over k_n°. Unless otherwise stated, by the generic and special fibre we mean the generic resp. special fibre over the generic and special point of k_n°. Under abuse of notation we will denote them by $\mathfrak{B}_\eta$ and $\mathfrak{B}_s$ respectively.

Definition 2.3.2
Let X be a geometrically integral, projective variety over k.

(i) For $d \in \mathbb{Q}$ let $D = qD'$ be a $\mathbb{Q}$-Cartier divisor on X such that D' is an integral Cartier divisor. Then a *Green's function* for D is a continuous function

$$g : (X_{\mathbb{K}})^{\mathrm{an}} \smallsetminus \operatorname{supp}(D'_{\mathbb{K}})^{\mathrm{an}} \to \mathbb{R}$$

such that if $(f_i)_{i\in I}$ are local equations for D' on an open covering $(U_i)_{i\in I}$ of X, for each $i \in I$ the function

$$g + q \log_b |f_i|^2 : U_{i,\mathbb{K}}^{\mathrm{an}} \smallsetminus \operatorname{supp}(D'_{\mathbb{K}})^{\mathrm{an}}_{\mathrm{red}} \to \mathbb{R}$$

extends to a continuous function on each $U_{i,\mathbb{K}}^{\mathrm{an}}$.

(ii) A pair of a $\mathbb{Q}$-Cartier divisor D and a D-Green function g will be called *arithmetic divisor* and written $D + g$. The set of arithmetic divisors has naturally the structure of a $\mathbb{Q}$-vector space.

(iii) Let D' be a $\mathbb{Z}$-Cartier divisor on X. Let $\mathfrak{X}$ be a *model* of X over k_n°, that is a projective flat scheme over k_n° with a distinguished isomorphism $\mathfrak{X} \otimes_{k_n^\circ} k_n \cong X \otimes_k k_n$, and let $\mathfrak{D}'$ be a *model* of a Cartier divisor D' on X, that is a $\mathbb{Q}$-Cartier divisor $\mathfrak{D}'$ on $\mathfrak{X}$ and an isomorphism $\mathfrak{D}'|_\eta$ compatible with the isomorphism $\mathfrak{X} \otimes_{k_n^\circ} k_n \cong X \otimes_k k_n$. Assume that $\mathfrak{D}'$ is a $\mathbb{Z}$-Cartier divisor. We denote by $\pi_{\mathfrak{X}} \colon (X_{\mathbb{K}})^{\mathrm{an}} \to \mathfrak{X} \otimes \widetilde{\mathbb{K}}$ the reduction map. For any $x \in X_{\mathbb{K}}^{\mathrm{an}}$ let f be a local equation for $\mathfrak{D}$ around $\pi_{\mathfrak{X}}(x)$. Then the *model Green's function* $g_{\mathfrak{X},\mathfrak{D}'}$ is given by

$$g_{\mathfrak{X},\mathfrak{D}'}(x) = -\log_b |f(x)|^2.$$

Clearly, the definition of model functions can be extended to $\mathbb{Q}$-Cartier divisors on $\mathfrak{X}$ and X respectively. For the rôle of b in the definition of $g_{\mathfrak{X},\mathfrak{D}'}$ cf. Rem. 3.3.1 later on.

(iv) Let g, h be two D-Green's functions. Then $g - h$ is a Green's function for the trivial Cartier divisor and we can define the *distance* of g and h by

$$d(g,h) = \sup_{x \in X_{\mathbb{K}}^{\mathrm{an}}} |(g-h)(x)|.$$

The space $X_{\mathbb{K}}^{\mathrm{an}}$ is compact, as X was assumed projective. Hence d attains only finite values and so defines a distance on the space of Green's functions for D, indeed.

(v) We say that $D + g$ is *semipositive* if g is the limit with respect to the distance d of model functions $g_{\mathfrak{X}_l,\mathfrak{D}_l}$ where each $\mathfrak{D}_l$ is a *vertically nef* $\mathbb{Q}$-Cartier divisor on a model $\mathfrak{X}_l$, that is, the degree of $\mathscr{O}(\mathfrak{D}_l)$ restricted to any complete curve in the special fibre of $\mathfrak{X}_l$ is non-negative.

(vi) We say that an arithmetic divisor is a *difference of semipositive* arithmetic divisors or *DSP* for short if it can be written as a difference of semipositive ones.

CHAPTER 3

The Approximation Theorem

3.1. Analysis on Products of Graphs

In this section we develop the analysis which will be used to prove our approximation theorem. Namely we will define a notion of positivity of real valued functions on the self product of the geometric realisation of a graph and will show that a large class of differentiable functions can be written as a difference of this kind of positive functions. These positive functions functions will give rise to DSP Green's functions for the trivial divisor. In the following let Γ be a finite graph in the sense of Def. A.4. In particular every edge of Γ has an orientation.

Definition 3.1.1
We define the *valence* of Γ, denoted by $\mathrm{val}(\Gamma)$, as the maximal number of edges connected to a vertex.

Definition 3.1.2
Let Γ be a graph. Assume that Γ has no loop edges. The orientation on the edges of Γ provides us with an atlas of canonical charts on the self product $|\Gamma|^2$ of the geometric realisation of Γ (cf. Appendix A) each of them isomorphic to $[0,1]^2$. Choose coordinates x, y of $[0,1]^2$. By dividing $[0,1]^2$ into the standard simplices $\{x \leq y\}$ and $\{x \geq y\}$ we define the *canonical two-simplices* of $|\Gamma|^2$.

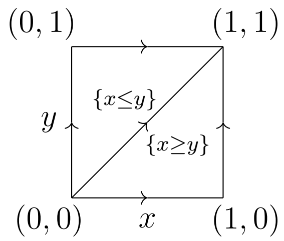


FIGURE 3.1.1. A canonical chart with the canonical two-simplices

(i) The relative boundary of $[0,1]^2$ is the topological boundary of $[0,1]^2$ in $\mathbb{R}^2$. The *relative boundary* of $|\Gamma|^2$ is the union of all relative boundaries of canonical charts. It will be denoted by $\mathrm{relbd}(|\Gamma|^2)$. The *relative interior* is the complement of $\mathrm{relbd}(|\Gamma|^2)$ in $|\Gamma|^2$ and will be denoted by $\mathrm{relint}(|\Gamma|^2)$.
(ii) Let Z be a closed subset of $\mathbb{R}^n$ for $n \geq 1$. We say that a real valued function $f\colon Z \to \mathbb{R}$ is of class C^k on Z if there exists an open set $U \supset Z$ such that f extends to a $C^k(U)$-function. We say that f is *smooth* if there exists a neighbourhood U of Z in $\mathbb{R}^n$ such that f extends to a smooth function on U.
(iii) Let $f\colon |\Gamma|^2 \to \mathbb{R}$ be a continuous function. Let $k \in \mathbb{N} \cup \{\infty\}$ be a number. We say f is a C^k *function* on $|\Gamma|^2$ if the restriction of f to every chart in the atlas associated to $|\Gamma|^2$ is a C^k function on $[0,1]^2$. Then, we say that f *is* C^k *on the squares* and denote the space of these functions by $C^k_\square(\Gamma^2)$. If moreover, we demand the restriction of this

function to each canonical square to attain rational values at rational points then we write $f \in C^k_{\square,\mathbb{Q}}(\Gamma^2)$.

(iv) If f is a C^k-function on $|\Gamma|^2$ and $i \in \{1,2\}$ we can define differential operators ∂_{x_i} for $i_1,\dots,i_n \in \{1,2\}$ as follows. Let Z be a chart of $|\Gamma|^2$ and consider f as a function on $[0,1]^2$. Then $\partial_{x_i} f$ is the usual derivative with respect to the i-th coordinate on the relative interior of $[0,1]^2$. This extends to a function $\partial_{x_{i_1}\cdots x_{i_n}} f$ on the relative interior of $|\Gamma|^2$. Accordingly we define higher partial derivatives.

(v) Similarly we introduce the space $C^k_\Delta(\Gamma^2)$ of functions which are C^k on the canonical two-simplices of $|\Gamma|^2$ as the space of continuous functions on $|\Gamma|^2$ whose restrictions to the canonical 2-simplices are C^k. For such functions it makes sense to define the partial derivatives on the union of the relative interiors of the canonical two simplices. If we demand that in each chart the partial derivatives in x and y direction up to the k-th order attain rational values at rational points then we write $C^k_{\Delta,\mathbb{Q}}(\Gamma^2)$.

(vi) Let $f\colon [0,1]^2 \to \mathbb{R}$ be a function. We say that f satisfies the convexity condition CC if

(a) It is Lipschitz continuous on each square.

(b) For every $x, y \in [0,1]$ and every $\varepsilon > 0$ such that $x+\varepsilon \in [0,1]$ and $y + 2\varepsilon \in [0,1]$ the inequality

$$g(x+2\varepsilon, y+\varepsilon) - g(x+\varepsilon, y) - g(x+\varepsilon, y+\varepsilon) + g(x,y) \geq 0 \tag{CC1}$$

holds. For every $x, y \in [0,1]$ and every $\varepsilon > 0$ such that $x+2\varepsilon \in [0,1]$ and $y+\varepsilon \in [0,1]$ the inequality

$$g(x+\varepsilon, y+2\varepsilon) - g(x, y+\varepsilon) - g(x+\varepsilon, y+\varepsilon) + g(x,y) \geq 0 \tag{CC2}$$

holds. For every $x, y \in [0,1]$ and every $\varepsilon > 0$ such that $x+\varepsilon$ and $y+\varepsilon$ lie in $[0,1]$ we demand that

$$g(x+\varepsilon, y) + g(x, y+\varepsilon) - g(x,y) - g(x+\varepsilon, y+\varepsilon) \geq 0 \tag{CC3}$$

holds. We will speak of *CC-functions* for short for functions which satisfy the convexity condition CC.

(vii) Let $f\colon |\Gamma|^2 \to \mathbb{R}$ be a function. We say that f satisfies CC if the restriction of f to every chart of the atlas associated to Γ^2 is a function which satisfies CC.

Remark 3.1.3

(i) The property of a function of beeing CC on a graph depends on the orientation of the edges of a graph. A priori it is not clear how the property of beeing CC behaves under refinement of graphs.

(ii) Not every convex function on $[0,1]^2$ is a CC function. For example $x^2 + xy + y^2$ is convex but not CC.

Definition 3.1.4

If $f\colon |\Gamma|^2 \to \mathbb{R}$ is a continuous function we can define the integral $\int_{|\Gamma|^2} f$: For every canonical chart $Z \cong [0,1]^2$ we define $\int_Z f = \int_{[0,1]^2} f$ with respect to the standard Lebesgue measure of total mass one. We define $\int_{|\Gamma|^2} f$ as the sum over all $\int_Z f$ for all canonical charts Z of $|\Gamma|^2$. Let $f\colon \operatorname{relbd}(|\Gamma|^2) \to \mathbb{R}$ be a continuous function on the boundary. Let Z be a canonical chart of $|\Gamma|^2$. Then we define $\int_{\partial Z} f$ as the integral of f over the topological boundary of Z in $\mathbb{R}^2$. The space ∂Z is the union of four copies of the unit interval on which we have the standard Lebesgue measure each. We define

$$\int_{\operatorname{relbd}(|\Gamma|^2)} f = \sum_Z \int_{\partial Z} f,$$

where the Z runs over all canonical charts of $|\Gamma|^2$.

Remark 3.1.5
Every CC-function $f\colon [0,1]^2 \to \mathbb{R}$ satisfies

$$f(x,y) + f(x+2\varepsilon, y) - 2f(x+\varepsilon, y) \geq 0 \tag{3.1.1}$$

and

$$f(x,y) + f(x, y+2\varepsilon) - 2f(x, y+\varepsilon) \geq 0. \tag{3.1.2}$$

Indeed

$$f(x,y) + f(x+2\varepsilon, y+\varepsilon) - f(x+\varepsilon, y+\varepsilon) - f(x+\varepsilon, y) \geq 0$$

and

$$-f(x+\varepsilon, y) - f(x+2\varepsilon, y+\varepsilon) + f(x+\varepsilon, y+\varepsilon) + f(x+2\varepsilon, y) \geq 0$$

holds and adding these two inequalities we get (3.1.1) and similarly we get (3.1.2). Furthermore, it can be shown that the conditions (3.1.1), (3.1.2), and (CC3) jointly imply (CC1) and (CC2). However, the condition CC has the advantage that it can be checked "locally"[1] on each square, which will be useful in the proof of Prop. 3.1.13.

For later applications we are interested in functions on the self-product of a graph which can be written as differences of CC-functions. It can be shown that C^2-functions on a closed subset of $\mathbb{R}^n$ can be written as a difference of convex functions. In analogy to this result we will show that $C^2_{\square,\mathbb{Q}}$-functions on the self product of a graph after a suitable subdivision of the graph can be written as the difference of two CC functions. In the remainder of the section we will prove the following theorem which is the crucial ingredient in our approximation theorem. The theorem is divided in two parts: The first claims that every C^2 function on the self product of the S-subdivision in the sense of Def. A.9 of a finite graph can be written as a difference of CC-functions. Moreover, the second allows non-differentiability along the diagonals. Although the first claim follows from the second and the ideas of the proof are essentially the same, the second proof is much more technical due to the techniques of smoothening which are applied.

In the sequel we will assume Γ is a finite graph such that

(i) Γ has more than one vertex and at least one edge,
(ii) Γ has no double edges that is, for two vertices v_1, v_2 there is at most one edge linking v_1 and v_2,
(iii) Γ has no loop edges.

Theorem 3.1.6
Assume that Γ be the S-subdivision of a finite graph. Let $f\colon |\Gamma|^2 \to \mathbb{R}$ be a continuous function.

(i) If $f \in C^2_\square(\Gamma^2)$ then it is the difference of two CC-functions g, h in $C^2_\square(\Gamma^2)$. If furthermore $f \in C^2_{\square,\mathbb{Q}}(\Gamma^2)$ then g and h can be chosen in $C^2_{\square,\mathbb{Q}}(\Gamma^2)$,
(ii) if $f \in C^4_\Delta(\Gamma^2)$ then it is the difference of two CC-functions in $C^2_\Delta(\Gamma^2)$. If furthermore $f \in C^4_{\Delta,\mathbb{Q}}(\Gamma^2)$ then g and h can be chosen in $C^2_{\Delta,\mathbb{Q}}(\Gamma^2)$.

Proof. We define an auxillary function $a\colon |\Gamma|^2 \to \mathbb{R}$. Let x, y be local coordinates in one of the canonical charts of Γ. Then we define

$$a(x,y) = x^2 - xy + y^2. \tag{3.1.3}$$

[1] i.e. in an appropriate G-topology on $|\Gamma|^2$

In Figure 3.1.2 we have depicted all eight possible ways how the canonical charts of $|\Gamma|^2$ can intersect a fixed chart, which is filled in grey. The arrows at the diagonals indicate the diagonal from $(0,0)$ to $(1,1)$ in the canonical charts. Then it is easy to see that a is well-defined. This special structure of the canonical charts of $|\Gamma|^2$ is due to to the fact that Γ is the S-subdivision of some graph (cf. Rem. A.11 and A.8).

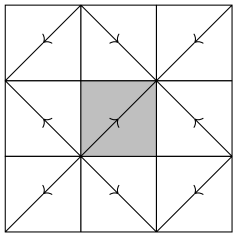


FIGURE 3.1.2. All eight ways how charts in an S-subdivision can intersect. The arrow is the diagonal from $(0,0)$ to $(1,1)$ in each canonical chart

Let x, y be local coordinates in a canonical chart and $\varepsilon > 0$. Then we have the following equalities.

$$a(x+2\varepsilon, y+\varepsilon) - a(x+\varepsilon, y) - a(x+\varepsilon, y+\varepsilon) + a(x,y) = \varepsilon^2, \tag{3.1.4}$$

$$a(x+\varepsilon, y+2\varepsilon) - a(x, y+\varepsilon) - a(x+\varepsilon, y+\varepsilon) + a(x,y) = \varepsilon^2, \tag{3.1.5}$$

$$a(x+\varepsilon, y) + a(x, y+\varepsilon) - a(x,y) - a(x+\varepsilon, y+\varepsilon) = \varepsilon^2. \tag{3.1.6}$$

This shows that a is a CC function. Accordingly, whenever the failure of a function to be CC is at least of quadratic order for $\varepsilon \to 0$, we can use positive multiples of a (which are again CC) to make it CC.

We begin with the proof of (i): We restrict f and a to a canonical square which we identify with $[0,1]^2$ and introduce coordinates x, y. We will now show that there is a real constant $M \geq 0$ such that $M \cdot a + f$ satisfies CC. As $|\Gamma|^2$ can be covered by finitely many canonical charts we can choose M large enough that this holds for every square. Then Ma is CC and $Ma + f$ is CC hence $f = Ma + f - Ma$ is difference of two CC functions and we have shown our claim.

Note the following lemma:

Lemma 3.1.7
Let $f\colon [0,b] \to \mathbb{R}$ be a continuous function and let f be twice continuously differentiable on $(0,b)$. Then we have the estimate

$$\left| \frac{1}{2}\left[f(0) + f(b)\right] - f(b/2) \right| \leq \frac{1}{4} b^2 \|f''\|_\infty$$

where $\|\cdot\|_\infty$ is the sup*-norm on $C^0([0,1])$.*

PROOF. We compute

$$\left| \frac{f(0) + f(b) - 2f(b/2)}{(b/2)^2} \right| = \left| \frac{f'(\zeta) - f'(\xi)}{b/2} \right|$$

for suitable $\zeta \in (b/2, b)$ and $\xi \in (0, b/2)$ by the mean value theorem. Then

$$\left| \frac{f'(\zeta) - f'(\xi)}{b/2} \right| \leq \left| \frac{f'(\zeta) - f'(\xi)}{\frac{1}{2}(\xi - \zeta)} \right|$$

which equals

$$2|f''(\eta)|$$

for a suitable $\eta \in (0, b)$ again by the mean value theorem, hence the claim. □

Applying Lemma 3.1.7 to the restriction of f to the lines convhull($\{(x+\varepsilon, y), (x+\varepsilon, y+\varepsilon)\}$) and convhull($\{(x+2\varepsilon, y+\varepsilon), (x, y)\}$) respectively we conclude

$$-f(x+\varepsilon, y) - f(x+\varepsilon, y+\varepsilon) \geq -C\varepsilon^2 - 2f(x+\varepsilon, y+\frac{\varepsilon}{2})$$

and

$$\begin{aligned} & f(x+2\varepsilon, y+\varepsilon) - f(x+\varepsilon, y) - f(x+\varepsilon, y+\varepsilon) + f(x, y) \\ \geq & -C\varepsilon^2 - 2\cdot f(x+\varepsilon, y+\frac{\varepsilon}{2}) + f(x+2\varepsilon, y+\varepsilon) + f(x, y) \geq -C\varepsilon^2 - 5\varepsilon^2 \cdot C \end{aligned}$$

for $C = \frac{1}{4}\|f''\|_\infty$. So (3.1.4) shows that we can choose $M > 0$ such that $Ma + f$ satisfies (CC1). The argument for (CC2) follows by symmetry as the condition (CC2) is the same as (CC1) with x and y interchanged. Likewise we see that there exists a $C > 0$ such that

$$-f(x, y) - f(x+\varepsilon, y+\varepsilon) \geq -C\varepsilon^2 - 2f(x+\frac{\varepsilon}{2}, y+\frac{\varepsilon}{2}).$$

Increasing C we get

$$f(x+\varepsilon, y) + f(x, y+\varepsilon) - 2f(x+\frac{\varepsilon}{2}, y+\frac{\varepsilon}{2}) \geq -C\varepsilon^2$$

and finally

$$f(x+\varepsilon, y) + f(x, y+\varepsilon) - f(x, y) - f(x+\varepsilon, y+\varepsilon) \geq -2C\varepsilon^2.$$

Accordingly we can increase M to achieve (CC3) for $Ma + f$. Now the proof of (i) is complete.

For the proof of (ii) we define another auxilliary function b defined in canonical coordinates x, y by

$$b(x, y) = |x - y|.$$

We have the equalities

$$0 = b(x+2\varepsilon, y+\varepsilon) - b(x+\varepsilon, y) - b(x+\varepsilon, y+\varepsilon) + b(x, y)$$

and

$$0 = b(x+\varepsilon, y+2\varepsilon) - b(x, y+\varepsilon) - b(x+\varepsilon, y+\varepsilon) + b(x, y)$$

and

$$b(x+\varepsilon, y) + b(x, y+\varepsilon) - b(x, y) - b(x+\varepsilon, y+\varepsilon) = \begin{cases} 0 & \text{for } |x-y| > \varepsilon \\ 2\varepsilon - 2|x-y| & \text{for } |x-y| \leq \varepsilon \end{cases}. \tag{3.1.7}$$

Which in particular shows, that b is a CC-function.

Definition 3.1.8
Let $f\colon U \to \mathbb{R}$ be a function on an open subset of $\mathbb{R}^n$ for some $n \geq 1$. Let $v \in \mathbb{R}^n$ be a vector. We will use the following notation for one-sided partial derivatives

$$\partial_v^+ f(x) = \lim_{t \searrow 0} \frac{f(x+tv) - f(x)}{|t|}, \qquad \partial_v^- f(x) = \lim_{t \searrow 0} \frac{f(x-tv) - f(x)}{|t|}.$$

and similarly define higher one-sided derivatives.

The following lemma will be used for the regularisation of functions on $[0,1]^2$ with singularities along the diagonal with functions which are differences of CC functions.

Lemma 3.1.9
Let $g\colon [0,1]^2 \to \mathbb{R}$ be a continuous function which is of class C^2 on the sets $\{x \geq y\}$ and $\{x \leq y\}$ respectively. We define:

$$k_g(x,y) = \begin{cases} \frac{1}{2}(y-x)\partial^+_{(1,-1)}g(\frac{1}{2}(x+y),\frac{1}{2}(x+y)) & \text{if } x \leq y, y \neq 0, x \neq 1 \\ \frac{1}{2}(x-y)\partial^-_{(1,-1)}g(\frac{1}{2}(x+y),\frac{1}{2}(x+y)) & \text{if } x > y, x \neq 0, y \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then

(i) *The function $g + k_g$ is of class C^1 on $[0,1]^2$.*
(ii) *There exists an $M > 0$ such that $k_g + M(a+b)$ is CC on $[0,1]^2$.*

Proof.

(i) The function $g+k_g$ is C^1 at all points except possibly at the diagonal. By construction

$$\partial^+_{(1,-1)}k_g(z,z) = \lim_{t\searrow 0}\frac{2t\partial^-_{(1,-1)}g(z,z)-0}{2t} = \partial^-_{(1,-1)}g(z,z)$$

and

$$\partial^-_{(1,-1)}k_g(z,z) = \lim_{t\searrow 0}\frac{2t\partial^+_{(1,-1)}g(z,z)-0}{2t} = \partial^+_{(1,-1)}g(z,z)$$

hold for all $z \in [0,1]$. So we have the equality

$$\partial^+_{(1,-1)}(g+k_g) = \partial^+_{(1,-1)}g + \partial^-_{(1,-1)}g = \partial^-_{(1,-1)}g + \partial^+_{(1,-1)}g = \partial^-_{(1,-1)}(g+k_g).$$

The two-sided partial derivatives $\partial_{(-1,1)}$ of k_g and g exist everywhere in $[0,1]^2$ and are continuous. Now the gradient of $g+k_g$ on the triangle $\{x \geq y\} \subset [0,1]^2$ agrees with the gradient on the triangle $\{x \leq y\}$. As we assumed that g is differentiable on each of these triangles, this implies, that $g+k_g$ is differentiable on $[0,1]^2$ and hence the first part of the lemma.

(ii) As g is C^2 on the closed subsets $\{x \geq y\}$ and $\{x \leq y\}$ of $[0,1]^2$ we can choose $M > 0$ such that $M \cdot a + k_g$ satisfies (CC1), (CC2), and (CC3) whenever all points appearing in these inequalities are contained in either one of the subsets $\{x \leq y\}$ or $\{x \geq y\}$ of $[0,1]^2$. This can be seen by extending the restriction of the function k_g to a C^2 function on the upper (resp. lower triangle) to the whole square and applying the first part of the theorem. We see that all points in the inequality (CC1) are contained in the same two-simplex of $[0,1]^2$ if and only if $|x-y| \geq \varepsilon$. Indeed, assume by symmetry that $y \geq x$ holds. Then for every $\varepsilon > 0$

$$y \geq x \wedge y \geq x+\varepsilon \wedge y+\varepsilon \geq \varepsilon + 2\varepsilon \wedge y+\varepsilon \geq x+\varepsilon \Leftrightarrow y \geq x+\varepsilon$$

holds.

Let $\varepsilon > 0$ be a positive number. Now choose x, y with $|x-y| \leq \varepsilon$ to treat the cases where possibly not all coordinates in the inequalities (CC1), (CC2), and (CC3) lie in the same triangle of $[0,1]^2$. By symmetry assume $y \geq x$. Then

$$\begin{aligned} &|k_g(x,y) - k_g(x+\varepsilon, y+\varepsilon)| \\ &= \frac{1}{2}|x-y| \cdot \left|\partial^+_{(1,-1)}g(\frac{1}{2}(x+y),\frac{1}{2}(x+y)) - \partial^+_{(1,-1)}g(\frac{1}{2}(x+y)+\varepsilon,\frac{1}{2}(x+y)+\varepsilon)\right| \\ &\leq \varepsilon^2 C, \end{aligned}$$

where C only depends on the Lipschitz constants of the first partial derivatives of g which are finite by the mean value theorem. Now if $y \geq x$ and $|x-y| \leq \varepsilon$ then $y+\varepsilon \leq x+2\varepsilon$ and $y \leq x+\varepsilon$. Hence

$$\begin{aligned}
&|k_g(x+\varepsilon, y) - k_g(x+2\varepsilon, y+\varepsilon)| = \\
&\frac{1}{2}|x-y| \cdot \Big| \partial^-_{(1,-1)} g(\frac{1}{2}(x+y+\varepsilon), \frac{1}{2}(x+y+\varepsilon)) \\
&-\partial^-_{(1,-1)} g(\frac{1}{2}(x+y+3\varepsilon)+\varepsilon, \frac{1}{2}(x+y+3\varepsilon)) \Big| \\
&\leq \varepsilon^2 C,
\end{aligned}$$

holds for a constant C only depending on g again by the Lipschitz continuity of the first partial derivatives. So after possibly enlarging M we can arrange that (CC1) and (CC2) are satisfied for $M \cdot a + k_g$ on the whole of $[0,1]^2$ for all $\varepsilon > 0$.

It remains to show that we can achieve (CC3). Note that (CC3) is satisfied for $M \cdot a + k_g$ whenever $|x-y| \geq \varepsilon$. So let x, y be such that $|x-y| \leq \varepsilon$. We can assume that $x \leq y$ by symmetry. Then the point (x,y) is contained in the closed segment $L \subset [0,1]^2$ from

$$(\underbrace{\frac{1}{2}(x+y)-\varepsilon}_{=:z_1}, \underbrace{\frac{1}{2}(x+y)+\varepsilon}_{=:z_2})$$

to

$$(\underbrace{\frac{1}{2}(x+y)}_{=:z_1'}, \underbrace{\frac{1}{2}(x+y)}_{=:z_2'}).$$

We define $k_N'(z,w) = N(a(z,w)+b(z,w)) + k_g(z,w)$ for any number $N \geq 0$ and set

$$k_N(z,w) := k_N'(z+\varepsilon, w) + k_N'(z, w+\varepsilon) - k_N'(z,w) - k_N'(z+\varepsilon, w+\varepsilon)$$

for $(z,w) \in [0,1]^2$ such that $z+\varepsilon \leq 1$ and $w+\varepsilon \leq 1$. The restriction of k_N to L is affine for any $N \geq 0$ as ε is fixed. So to check $k_N(x,y) \geq 0$ it suffices to check $k_N(z_1,z_2) \geq 0$ and $k_N(z_1', z_2') \geq 0$. Note that for $N = M$ we have $k_N(z_1,z_2) \geq 0$. By (3.1.6) and (3.1.7) we have the following equality.

$$k_N(z_1', z_2') = 2N\varepsilon + N\varepsilon^2 + \frac{\varepsilon}{2} \cdot \partial^-_{(1,-1)} g(z_1' + \frac{\varepsilon}{2}, z_2' + \frac{\varepsilon}{2}) + \frac{\varepsilon}{2} \cdot \partial^+_{(1,-1)} g(z_1' + \frac{\varepsilon}{2}, z_2' + \frac{\varepsilon}{2})$$

As the restrictions of the derivatives to the diagonal are bounded functions we may enlarge N such that $k_N(z_1', z_2') \geq 0$ for all ε and we are done. Enlarging M such that $M \geq N$, we are finished with the proof of the lemma.

□

Lemma 3.1.10
Assume that the function $g\colon [0,1]^2 \to \mathbb{R}$ is C^1 on $[0,1]^2$ and C^4 on the closed sets $\{x \geq y\}$ and $\{x \leq y\}$. Then we define

$$k_g^2(x,y) = \begin{cases} \frac{1}{8}(x-y)^2 \partial^{2,+}_{(1,-1)} g(\frac{1}{2}(x+y), \frac{1}{2}(x+y)) & \text{for } x \leq y \\ \frac{1}{8}(x-y)^2 \partial^{2,-}_{(1,-1)} g(\frac{1}{2}(x+y), \frac{1}{2}(x+y)) & \text{for } x > y \end{cases}$$

Then:

(i) The function $g + k_g^2(x,y)$ is C^2 on $[0,1]^2$,
(ii) there is an $M > 0$ such that $M \cdot (a+b) + k_g^2$ is CC.

PROOF.

(i) We define functions on $[0,1]^2$ as follows.

$$k^+(x,y) = \frac{1}{8}(x-y)^2 \partial^{2,+}_{(1,-1)} g(\frac{1}{2}(x+y), \frac{1}{2}(x+y)),$$
$$k^-(x,y) = \frac{1}{8}(x-y)^2 \partial^{2,-}_{(1,-1)} g(\frac{1}{2}(x+y), \frac{1}{2}(x+y)).$$

As k^+ and k^- are products of C^2-functions on $[0,1]^2$ they are again C^2-functions on $[0,1]^2$. In particular k_g^2 is a C^2 function on the sets $\{x \leq y\}$ and $\{x \geq y\}$ respectively. For any $z \in [0,1]$ we have the equality

$$\partial^+_{(1,-1)} k_g^2(z,z) = \partial^+_{(1,-1)} k^+(z,z) = \lim_{t \searrow 0} \frac{1}{t}\left(\frac{1}{8} 4t^2 \partial^{2,-}_{(1,-1)} g(z,z)\right) = 0$$

and accordingly

$$\partial^-_{(1,-1)} k_g^2(z,z) = 0.$$

We compute the second single-sided derivatives of k_g^2 at the diagonal. We do this by computing the two-sided derivative of k^+ and k^- respectively using difference quotients for the second derivatives. We have

$$\partial^{2,+}_{(1,-1)} k_g^2(z,z) = \lim_{t \searrow 0} \frac{1}{t^2}\left(k^-(z+t, z-t) - 2k^-(z,z) + k^-(z-t, z+t)\right) =$$
$$\lim_{t \searrow 0} \frac{1}{t^2}\left(\frac{1}{8} 4t^2 \partial^{2,-}_{(1,-1)} g(z,z) - 0 + \frac{1}{8} 4t^2 \partial^{2,-}_{(1,-1)} g(z,z)\right) = \partial^{2,-}_{(1,-1)} g(z,z)$$

and likewise

$$\partial^{2,-}_{(1,-1)} k_g^2(z,z) = \partial^{2,+}_{(1,-1)} g(z,z).$$

Hence

$$\partial^+_{(1,-1)}(g + k_g^2) = \partial^+_{(1,-1)} g,$$
$$\partial^-_{(1,-1)}(g + k_g^2) = \partial^-_{(1,-1)} g,$$
$$\partial^{2,+}_{(1,-1)}(g + k_g^2) = \partial^{2,+}_{(1,-1)} g + \partial^{2,-}_{(1,-1)} g,$$
$$\partial^{2,-}_{(1,-1)}(g + k_g^2) = \partial^{2,-}_{(1,-1)} g + \partial^{2,+}_{(1,-1)} g.$$

at (z,z) for $z \in [0,1]$. We conclude that $g + k_g^2$ is C^2 on $[0,1]^2$.

(ii) We have the estimate

$$|k_g^2(z,w)| \leq C|z-w|^2 \tag{3.1.8}$$

for all $(z,w) \in [0,1]^2$ and a real constant $C > 0$ only depending on g using the boundedness of the partial derivatives and the definition of k_g^2. We can find an M such that $Ma + k_g^2$ is CC: Choose $(x,y) \in [0,1]^2$ and $\varepsilon > 0$. As k_g^2 is C^2 on the two-simplices we find an $M > 0$ such that $Ma + k_g^2$ satisfies (CC1), (CC2), and (CC3) whenever $|x-y| \geq 2\varepsilon$. If $|z-w| \leq 2\varepsilon$ then by 3.1.8 we have

$$|k_g^2(z,w)| \leq C\varepsilon^2$$

for some bigger C. So we conclude

$$|k_g^2(x,y) - k_g^2(x+\varepsilon, y) - k_g^2(x+\varepsilon, y+\varepsilon) + k_g^2(x+2\varepsilon, y+\varepsilon)| \leq 4C\varepsilon^2,$$
$$|k_g^2(x,y) - k_g^2(x, y+\varepsilon) - k_g^2(x+\varepsilon, y+\varepsilon) + k_g^2(x+\varepsilon, y+2\varepsilon)| \leq 4C\varepsilon^2,$$
$$|k_g^2(x+\varepsilon, y) + k_g^2(x, y+\varepsilon) - k_g^2(x,y) - k_g^2(x+\varepsilon, y+\varepsilon)| \leq 4C\varepsilon^2.$$

Hence using (3.1.4), (3.1.5), and (3.1.6) we find a bigger $M > 0$ such that $Ma + k_g$ is CC on $[0,1]^2$.

□

After these preparations we can finish the proof of Theorem 3.1.6 (ii). We claim that a $C^2_\Delta(\Gamma^2)$ function c exists which satisfies CC and such that $f + c$ satisfies CC. Let S be a set of squares (possibly empty) of Γ^2 and c be a CC function in $C^2_\Delta(\Gamma^2)$ such that the restriction of $f + c$ to each square of S is CC. Let T' be any square not contained in S. Using the notations from Lemma 3.1.9 and Lemma 3.1.10, the function $f + k_f + k_f^2$ is C^2 on T'. Hence by the proof of the first part of the theorem there is an $M > 0$ such that

$$M(a + b) + f + k_f + k_f^2$$

is CC on T'. Moreover, by Lemmas 3.1.9 and 3.1.10 after enlarging M, the functions $k_f + M(a + b)$ and $k_f^2 + M(a + b)$ are CC on T', hence after enlarging M, also $k_f + k_f^2 + M(a + b)$. In virtue of the following Lem. 3.1.11 extend k_f and k_f^2 to functions which are C^2 on each square different from T'. Then by (i) we find that $M(a + b) + k_f + k_f^2$ is CC on the whole of $|\Gamma|^2$ for some bigger M. We get that

- $M(a + b) + k_f + k_f^2 + f$ is CC on T',
- $M(a + b) + k_f + k_f^2$ is CC on $|\Gamma|^2$,
- $f + c$ is CC on all squares of S,
- c is CC on $|\Gamma|^2$.

Hence $M(a + b) + k_f + k_f^2 + f + c$ is CC on all squares of S. Moreover, we see that $M(a + b) + k_f + k_f^2 + f$ is CC on T' and c is CC on T'. Hence $M(a + b) + k_f + k_f^2 + f + c$ is CC on $S \cup \{T'\}$ and if we replace c by $M(a + b) + k_f + k_f^2 + c$ we have reached our goal. Iterating the whole process of enlarging S, we can find a function c as desired. The rationality claim is clear by construction. This finishes the proof of (ii). □

Lemma 3.1.11
Let f be a function on a square S of $|\Gamma|^2$ which is C^2 on the canonical two-simplices and attains rational values at rational points. Then f admits a continuous extension to $|\Gamma|^2$ which has rational values at rational points and is C^2 on all squares different from S.

Proof. Let T be a square different from S. We define the extension $\widetilde{f}$ of f to T. If T does not intersect S then we set $\widetilde{f} = 0$ on T. Let T intersect S in a line l of T. Let p be the orthogonal projection of T on that line and $d(x)$ be the Euclidean distance of a point $x \in T$ to l. Then we set

$$\widetilde{f}(x) = \begin{cases} (1 - 16 \cdot d(x)^4) f(p(x)) & \text{if } d(x) \le \frac{1}{2}, \\ 0 & \text{else.} \end{cases}$$

Let T intersect S in a point s. Denote by $d(x)$ the distance of x to s. Then we define

$$\widetilde{f}(x) = \begin{cases} (1 - 16 \cdot d(x)^4) f(s) & \text{if } d(x) \le \frac{1}{2}, \\ 0 & \text{else.} \end{cases}$$

As Γ has no double edges this is well-defined. Taking the distance to the fourth power causes the second derivatives of $\widetilde{f}$ to vanish in the direction of the normal to l. □

We close this section with a discussion of the rationality assumptions appearing in this section. Let Γ be the S-subdivision of a graph.

Definition 3.1.12
Let f be a real valued continuous function on a compact metric space (X, d). We define the Lipschitz-norm by

$$\|f\|_{0,1} := \sup_{x \in X} |f(x)| + \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right| .$$

We want to prove the following assertion:

Proposition 3.1.13
Assume that $g\colon |\Gamma|^2 \to \mathbb{R}$ is CC. Then g is the Lipschitz limit of $C^2_{\Delta,\mathbb{Q}}(\Gamma^2)$ functions which are CC.

The proof will be split up in several parts.

Proposition 3.1.14
Let Γ be a graph. We denote by Γ' the n-th BHM-subdivision (cf. Def.A.12) of Γ. Let g be any Lipschitz continuous function on $|\Gamma'|^2$. Then the following conditions are equivalent.

(i) The function g is CC on $|\Gamma|^2$,
(ii) the function g satisfies the inequalities (CC1), (CC2), *and* (CC3) *in every square of $|\Gamma|^2$ for rational values of x, y, ε,*
(iii) The following conditions are satisfied:
  - *(a) The function g satisfies the inequality (CC3) in every square of $|\Gamma'|^2$ for rational values of x, y, ε,*
  - *(b) the function satisfies the inequality (CC2) for rational values of x, y, ε in each union of two charts Z and Z' of $|\Gamma'|^2$ contained in the same chart of $|\Gamma|^2$ and intersecting in the lines $\{x = 0\}$ in the chart Z and $\{x = 1\}$ in the chart Z' respectively,*
  - *(c) the inequality (CC1) holds in each union of two charts Z and Z' of $|\Gamma'|^2$ contained in the same chart of $|\Gamma|^2$ and intersecting in the lines $\{y = 0\}$ in the chart Z and $\{y = 1\}$ in the chart Z' respectively for rational values of x, y, ε.*

PROOF. The implication (ii)⇒(i) follows from the continuity of g and the continuity of the expressions appearing in the inequalities. As the implications (i)⇒(ii)⇒(iii) are trivial it remains to show that (iii) implies (ii). So assume that (iii) holds. We show the case of inequality (CC1). We may assume that Γ is the connected standard graph B_1 with two vertices and one edge.

We choose a canonical chart of $|\Gamma|^2$ which we canonically identify with $[0, 1]^2$. Let x, y be the coordinates. We assume that $x, y, \varepsilon \in \mathbb{Q}$ are given such that such that (CC1) holds and that the coordinates appearing in (CC1) lie in the chosen canonical chart. By rationality of the coordinates and by the assumption (iii) we find an $m \geq 1$ such that:

(i) $\frac{m}{\varepsilon} \in \mathbb{N}$,
(ii) for all natural numbers s, k with $0 \leq k \leq \frac{2m}{\varepsilon}$ and $0 \leq s + 1 \leq \frac{m}{\varepsilon}$ the points $(\frac{k}{m}\varepsilon, \frac{s}{m}\varepsilon)$ and $(\frac{k}{m}\varepsilon, \frac{s+1}{m}\varepsilon)$ lie in the same chart of $|\Gamma'|^2$,
(iii) there exist natural numbers s, k with $0 \leq k \leq \frac{2m}{\varepsilon}$ and $0 \leq s \leq \frac{m}{\varepsilon}$ such that $(x, y) = (\frac{k}{m}\varepsilon, \frac{s}{m}\varepsilon)$.

Then by assumption for all natural numbers s, k with $0 \leq k \leq \frac{2m}{\varepsilon}$ and $0 \leq s \leq \frac{m}{\varepsilon}$

$$0 \leq g(\frac{k}{m}\varepsilon, \frac{s}{m}\varepsilon) + g(\frac{k+2}{m}\varepsilon, \frac{s+1}{m}\varepsilon) - g(\frac{k+1}{m}\varepsilon, \frac{s}{m}\varepsilon) - g(\frac{k+1}{m}\varepsilon, \frac{s+1}{m}\varepsilon)$$

holds. We want to show

$$0 \leq g(x,y) + g(x + 2\varepsilon + \frac{2}{m}, y + \varepsilon + \frac{1}{m}) - g(x + \varepsilon + \frac{1}{m}, y) - g(x + \varepsilon + \frac{1}{m}, y + \varepsilon + \frac{1}{m}).$$

This yields (CC1) on all squares of $|\Gamma|^2$ because then for a given value of $x, y, \varepsilon \in \mathbb{Q}$ such that (CC1) holds we can enlarge ε arbitrarily in steps of $\frac{1}{m}$ as long the coordinates appearing in (CC1) lie in the same chart of $|\Gamma|^2$.

After rescaling we have reduced to the following statement: For $m \geq 1$ an integer let $g \colon [0, 3m]^2 \to \mathbb{R}$ be a function. Assume that

$$g(0,0) + g(2m, m) - g(m,m) - g(m, 0) \geq 0 \tag{3.1.9}$$

holds. Assume that for every choice of natural numbers k and l with $0 \leq k, l \leq m$ and $k + 2 \leq 3m$ and $l + 1 \leq 3m$ the inequality

$$g(k,l) + g(k+2, l+1) - g(k+1, l+1) - g(k+1, l) \geq 0$$

holds. Then

$$g(0,0) + g(2m+2, m+1) - g(m+1, m+1) - g(m+1, 0) \geq 0.$$

We will now prove this claim. Adding

$$g(m,m) + g(m+2, m+1) - g(m+1, m+1) - g(m+1, m) \geq 0$$

to (3.1.9) yields

$$g(0,0) + g(2m,m) - g(m,0) - g(m+1,m+1) - g(m+1,m) + g(m+2,m+1) \geq 0$$

and adding

$$g(m+l, m) + g(m+l+2, m+1) - g(m+l+1, m+1) - g(m+l+1, m) \geq 0$$

for all $1 \leq l \leq m-1$ yields

$$g(0,0) - g(m,0) - g(m+1,m+1) + g(2m+1, m+1) \geq 0. \tag{3.1.10}$$

We add

$$g(m,0) + g(m+2,1) - g(m+1,1) - g(m+1,0) \geq 0$$

to (3.1.10) and get

$$\begin{aligned} g(0,0) - g(m+1,m+1) + g(2m+1,m+1) \\ +g(m+2,1) - g(m+1,1) - g(m+1,0) \geq 0. \end{aligned} \tag{3.1.11}$$

and adding the inequalities

$$g(m+l, l) + g(m+l+2, l+1) - g(m+l+1, l+1) - g(m+l+1, l) \geq 0$$

to (3.1.11) for all $1 \leq l \leq m$ finally yields

$$0 \leq g(0,0) + g(2m+2, m+1) - g(m+1, m+1) - g(m+1, 0),$$

as desired.

The case of (CC2) follows by symmetry.

We will now show the case of (CC3). We make the same reduction as in the case of (CC1). Hence we have to show that if

$$0 \leq g(m,0) + g(0,m) - g(m,m) - g(0,0) \tag{3.1.12}$$

for a function $g\colon [0, m+1] \to \mathbb{R}$ and

$$0 \leq g(l+1,k) + g(l,k+1) - g(l+1,k+1) - g(l,k) \tag{3.1.13}$$

for all $0 \leq l, k \leq m+1$ then

$$0 \leq g(m+1,0) + g(0,m+1) - g(0,0) - g(m+1,m+1)$$

holds. We add (3.1.13) for $l = m$ and all $0 \leq k \leq m-1$ to (3.1.12) to get

$$0 \leq g(m+1,0) + g(0,m) - g(0,0) - g(m+1,m) \tag{3.1.14}$$

and adding (3.1.13) to (3.1.14) for $k = m$ and all $0 \leq l \leq m$ we get

$$0 \leq g(m+1,0) + g(0,m+1) - g(0,0) - g(m+1,m+1)$$

as desired. □

Next, we will prove the following lemma.

Lemma 3.1.15
Let g_n be the unique function on $|\Gamma'|^2$ which equals g on the 0-simplices and is affine on the 2-simplices.

(i) Let S be the union of two squares of $|\Gamma'|^2$ intersecting in the lines $\{x = 0\}$ and $\{x = 1\}$ respectively. We canonically identify S with $[0,2] \times [0,1]$. If g satisfies

$$g(0,0) + g(2,1) - g(1,1) - g(1,0) \geq 0$$

for all such unions then g_n satisfies (CC1) on $|\Gamma|^2$.

(ii) Let S be the union of two squares of $|\Gamma'|^2$ intersecting in the lines $\{y = 0\}$ and $\{y = 1\}$ respectively. We canonically identify S with $[0,1] \times [0,2]$. If g satisfies

$$g(0,0) + g(1,2) - g(1,1) - g(0,1) \geq 0$$

for all such unions then g_n satisfies (CC2) on $|\Gamma|^2$.

(iii) Let S be a square of $|\Gamma'|^2$ intersecting in the lines $\{y = 0\}$ and $\{y = 1\}$ respectively. We canonically identify S with $[0,1]^2$. If g satisfies

$$g(1,0) + g(0,1) - g(1,1) - g(0,0) \geq 0$$

for all such squares then g_n satisfies (CC3) on $|\Gamma|^2$.

(iv) If g satisfies CC then g_n satisfies CC.

Proof. The statement (iv) follows from (i), (ii), and (iii). We start with the proof of (i). In virtue of Prop. 3.1.14 we can check this condition locally on the union of two squares of $|\Gamma'|^2$ intersecting in a vertical line and which we identify with $S = [0,2] \times [0,1]$. We have reduced to the following statement: Let g be a function which is affine on the canonical two-simplices of S. Assume that

$$g(0,0) + g(2,1) - g(1,1) - g(1,0) \geq 0 \tag{3.1.15}$$

holds. We claim that g satisfies (CC1) on S. The map

$$S(x,y,\varepsilon) = g(x,y) + g(x+2\varepsilon,y) - g(x+\varepsilon,y+\varepsilon) - g(x+\varepsilon,y)$$

is affine on each of the following subsets of the set T of all $(x, y, \varepsilon) \in [0,2]^3$ such that $x + 2\varepsilon \in [0,2], y + \varepsilon \in [0,1]$.

$$\text{convhull}\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right), \text{convhull}\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),$$

$$\text{convhull}\left(\begin{pmatrix}0\\0\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\right), \text{convhull}\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right),$$

$$\text{convhull}\left(\begin{pmatrix}0\\0\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right), \text{convhull}\left(\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right),$$

$$\text{convhull}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right), \text{convhull}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}1\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix}\right),$$

$$\text{convhull}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\right).$$

Here convhull denotes the convex hull of a finite subset of points of $\mathbb{R}^3$. As the function S is affine on each of the preceeding convex sets it suffices to check the positivity on the extremal points. If $\frac{1}{2}$ is a coordinate of an extremal point then all of the points (x, y), $(x + \varepsilon, y + \varepsilon)$, $(x + \varepsilon, y)$, and $(x + 2\varepsilon, y + \varepsilon)$ are contained in the same 2-simplex in $[0,2] \times [0,1]$. The function g is affine on the 2-simplices and S vanishes for this value of (x, y, ε) by affinity. For all other extremal points the positivity of S follows from (3.1.15). It is easily seen that these sets indeed cover T. The statement (ii) follows by symmetry.

The case (iii) follows using the same argument: The expression

$$R(x, y, \varepsilon) = g(x + \varepsilon, y) + g(x, y + \varepsilon) - g(x, y) - g(x + \varepsilon, y + \varepsilon)$$

is affine on each convex subset of

$$H = \left\{(x, y, \varepsilon) \in [0,1]^3 \,\middle|\, (x + \varepsilon \leq 1) \wedge (y + \varepsilon \leq 1)\right\}$$

given in the following list.

$$\text{convhull}\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\right), \text{convhull}\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}\right),$$

$$\text{convhull}\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\right), \text{convhull}\left(\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix},\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\right).$$

The non-negativity of piecewise affine functions can be checked on the extremal points of convex sets. If $\frac{1}{2}$ appears as a coordinate, then all of the points (x, y), $(x+\varepsilon, y+\varepsilon)$, $(x+\varepsilon, y)$, and $(x, y+\varepsilon)$ lie in the same 2-simplex in $[0,1]^2$. But the function g is affine there, and it is easily seen that this causes R to vanish. If $\frac{1}{2}$ does not appear as a coordinate the non negativity of R can be immediately deduced from the fact that g satisfies (CC3). This finishes the proof. □

Now we can prove Prop. 3.1.13.

Proof. Let g_n be the n-th lattice approximation of g. Using a diagonal argument it suffices to show that g_n is the Lipschitz limit of CC functions with rational values at rational points. Let $m \geq 1$ be a natural number. Choose a function c_m which is affine on the 2-simplices, $|c_m| \leq \frac{1}{4mn^2}$, and such that $g_n + c_m$ has rational values at rational points.

Let a be the auxiliary function from (3.1.3) and let a_n be the n-lattice approximation. Then by Lemma 3.1.15 the conditions of CC can be checked on the corners of squares and unions of squares of $|\Gamma'|^2$ identified with $[0,2] \times [0,1]$, $[0,1] \times [0,2]$, and $[0,1]^2$ respectively. Here we get

$$|c_m(0,0) - c_m(1,0) - c_m(1,1) + c_m(2,1)| \leq \frac{1}{mn^2},$$
$$|c_m(0,0) - c_m(0,1) - c_m(1,1) + c_m(1,2)| \leq \frac{1}{mn^2},$$
$$|c_m(1,0) + c_m(0,1) - c_m(1,1) - c_m(0,0)| \leq \frac{1}{mn^2},$$

and Equations (3.1.4),(3.1.5), and (3.1.6) yield

$$a_n(0,0) - a_n(1,0) - a_n(1,1) + a_n(2,1) = \frac{1}{n^2},$$
$$a_n(0,0) - a_n(0,1) - a_n(1,1) + a_n(1,2) = \frac{1}{n^2},$$
$$a_n(1,0) + a_n(0,1) - a_n(1,1) - a_n(0,0) = \frac{1}{n^2}.$$

Hence the function $\frac{1}{m}a_n + c_m$ is CC and accordingly the function

$$g_n = \lim_{m\to\infty} (g_n + \frac{1}{m}a + c_m)$$

is the Lipschitz limit of CC functions. □

3.2. Semi-Stable Models and their Berkovich Skeleta

In general, models are difficult to understand and so are the induced Green's functions. However, they are easier to understand when the special fibre satisfies some additional conditions. In the case of regular strictly semi-stable models Green's functions for the trivial divisor arising from these models are more accessible.

We will be in the situation of Assumption 2.2.1. Let X be a smooth projective geometrically irreducible scheme over k and let $\mathfrak{B}$ be a regular strictly semi-stable model over k°. Let $g\colon |R(\mathfrak{B})| \to \mathbb{R}$ be a real valued function on the geometric realisation of $R(\mathfrak{B})$. Let τ be the deformation retract $\tau\colon (X \times_k X)^{\mathrm{an}}_{\mathbb{K}} \to |R(\mathfrak{B})|$ from Thm. B.6 (v) determined by the model $\mathfrak{B}$. As every continuous function on the $(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}$ is a Green's function for the trivial divisor the function $g \circ \tau$ is a Green's function for the trivial divisor.

Proposition 3.2.1
Let $\mathfrak{B}$ be a proper regular strictly semi-stable scheme over k°. Let g be a function on $|R(\mathfrak{B})|$ which is affine on the simplices of $R(\mathfrak{B})$ and assumes rational values at the vertices. By Prop. B.16 the function g corresponds to a Cartier divisor $\mathfrak{D}$ on $\mathfrak{X}$ which is trivial on the generic fibre. Then we have

$$g_{\mathfrak{B},\mathfrak{D}} = 2 \cdot g \circ \tau,$$

where we choose b as a base for the logarithm in the definition of the Green's function on the left hand side.

Proof. Let $x \in \widehat{\mathfrak{B}}^{\mathrm{an}}$ be a point. Let $\mathfrak{U}'$ be an open neighbourhood of $\pi_{\mathfrak{B}}(x)$ in $\mathfrak{B}$, where $\pi_{\mathfrak{B}}$ is the reduction map, and let $\mathfrak{U}$ be the base change $\mathfrak{U}'_{\mathbb{K}^\circ}$. By linearity of both sides of the equation we can assume that $\mathfrak{D}$ is an irreducible component of the special fibre. Moreover, after shrinking $\mathfrak{U}'$ we can assume that $\operatorname{Spec} A = \mathfrak{U} \subset \mathfrak{B}_{\mathbb{K}^\circ}$ is étale over

$$L := \mathbb{K}^\circ[T_1, \dots, T_n]/(T_1 \cdots T_p - \pi)$$

for $n, p \in \mathbb{N}$ and we have an isomorphism of reduction sets $R(\mathfrak{U}) \to R(L)$ by Prop. B.3. Let f be a local equation of $\mathfrak{D}$ on $\mathfrak{U}'$ around $\pi_{\mathfrak{B}}(x)$. If f is a unit in $\mathscr{O}(\mathfrak{U})$ then $g_{\mathfrak{B},\mathfrak{D}}(x) = 0$ and the restriction of g to $|R(\mathfrak{U}_i)|$ is zero, hence $g \circ \tau$ is also zero. So let f be no unit. After a permutation of coordinates and multiplication with a unit we can assume that T_1 maps to $f \in \mathscr{O}_{\mathfrak{U}}$. By definition of $g_{\mathfrak{B},\mathfrak{D}}$ we have

$$g_{\mathfrak{B},\mathfrak{D}}(x) = -\log_b |f(x)|^2.$$

On the other hand by the construction of the retraction map (cf. Thm. B.6 (v)) $g \circ \tau$ is given by

$$g(-\log_b |f_1|, \dots, -\log_b |f_p|),$$

where $f_j \in \mathscr{O}_{\mathfrak{U}}$ is the pull back of T_j for all $j \in \{1, \dots, p\}$.

By definition, the function g representing $\mathfrak{D}$ is the piecewise affine function taking the value 1 at the point of $|R(\mathfrak{U})|$ corresponding to the component $f_1 = 0$ and 0 at all other vertices. Consider $|R(\mathfrak{U})|$ which is the image of the skeleton of $\mathfrak{U}$ under $(-\log_b |f_1|, \dots, -\log_b |f_p|)$ by Thm. B.6 (v):

$$|R(\mathfrak{U})| = \left\{ x \in \mathbb{R}^p \;\middle|\; \sum_{i=1}^{p} x_i = 1, x_i \in [0,1] \right\}.$$

The point corresponding to the component $f = 0$ is given by the point $(1, 0, \dots, 0)$ of $|R(\mathfrak{U})|$ and the restriction of the function g to $|R(\mathfrak{U})|$ is given by the first coordinate function x_1 as the function g is uniquely determined by the property of beeing 0 on all vertices different from $f = 0$ and affine on all simplices. Hence we get the claimed equality

$$2g \circ \tau = -\log_b |f|^2.$$

This finishes the proof. □

3.3. Gross–Schoen Semi-Stable Models

Remark 3.3.1
Let $\mathfrak{B}$ be a proper regular strictly semistable scheme over k_n and $\mathfrak{D}$ be a Cartier divisor with support in the special fibre of $\mathfrak{B}$. Set $b_n = |\pi_n|^{-1}$ and let $g_{\mathfrak{B},\mathfrak{D}}$ be the model Green's function with respect to the b_n-logarithm and $g'_{\mathfrak{B},\mathfrak{D}}$ be the model Green's function with respect to the b-logarithm. Then by the definition of model functions we have $g'_{\mathfrak{B},\mathfrak{D}} = n \cdot g_{\mathfrak{B},\mathfrak{D}}$.

Choose any real valued function g on $|R(\mathfrak{B})|$. Then $g \circ \tau$ is a Green's function for the trivial divisor as every continuous function on $\widehat{\mathfrak{B}}^{\mathrm{an}}$ is a Green's function for the trivial divisor. We propose the following strategy to show the DSP property of $g \circ \tau$ for g with sufficient regularity:

(i) Find a series of models $\mathfrak{B}_n$ such that $R(\mathfrak{B}_n)$ is naturally a subdivision of $R(\mathfrak{B})$.
(ii) Consider the divisors g_n which are *lattice approximations* of g via the correspondence of Prop. B.16.

(iii) Choose a suitable series of semipositive divisors $\mathfrak{D}_n$ such that $g_{\mathfrak{B}_n,\mathfrak{D}_n} + g_n$ is semipositive and the sequence of model functions associated to $\mathfrak{D}_n$ and g_n converges uniformly.

If we set $h = \lim_{n\to\infty} g_{\mathfrak{B}_n,\mathfrak{D}_n}$ then we have $g \circ \tau = h + g \circ \tau - h$, hence $g \circ \tau$ is DSP. We will follow this strategy for the self product of a curve.

In [**GS95**] Gross and Schoen provide techniques to resolve the singularities of the product of regular strictly semi-stable curves. Kolb generalises these techniques to construct models with good properties for the d-fold self product of a projective curve (cf. [**Kol13**]).

We will be in the situation of Assumption 2.2.1.

We have the following theorem, which is due to Deligne and Mumford ([**DM69**, Lemma 1.12],[**Kol16b**, Satz 3.1]) which realises the first part of our strategy.

Theorem 3.3.2
Let $\mathfrak{X}$ be a regular strictly semi-stable model over k° of the smooth geometrically irreducible projective curve X. Fix an order of the irreducible components of the special fibre of $\mathfrak{X}$. Let k_n be the totally ramified extension of k of degree $n \geq 1$ and k_n° its valuation ring.

Then for every singular point of $\mathfrak{X} \otimes k_n^\circ$ we need to blow up at most $\lfloor n/2 \rfloor$ times to get a regular point. Doing this for all singular points we get a regular strictly semi-stable curve $\mathfrak{X}_n$ over k_n° with generic fibre $X \otimes k_n$. Then there is a an order on the set of the irreducible components of the special fibre of $\mathfrak{X}_n$ such that

(i) the order is compatible with the order on the irreducible components of $\mathfrak{X}_s$ in the following sense: Let $\gamma\colon \mathfrak{X}_n \to \mathfrak{X}$ be the blow-up morphism. If $C \leq C'$ are irreducible components of the special fibre of $\mathfrak{X}_n$ and $\gamma(C)$ and $\gamma(C')$ are irreducible components of the special fibre of $\mathfrak{X}$ then $\gamma(C) \leq \gamma(C')$,
(ii) there is a natural isomorphism of simplicial sets

$$R(\mathfrak{X}_n) \to \mathrm{sd}_{\mathrm{BHM},n}(R(\mathfrak{X})).$$

Moreover, there is an order on the set of irreducible components of $\mathfrak{X}_{n,s}$ such that the same holds with the S-subdivision in place of the BHM-subdivision.

The next theorem by Kolb ([**Kol16b**, Theorem 3.3]) is based on the desingularisation of products by Gross and Schoen ([**GS95**, Prop. 6.3, 6.11], see also [**Har01**, Prop. 2.1]). It shows that we can find good models of products of curves.

Theorem 3.3.3
Let $\mathfrak{X}$ be a regular strictly semi-stable curve over k° and fix an order of the irreducible components of the special fibre. Let $d \geq 1$ be a natural number. We succesively blow up the irreducible components of the special fibre of the d-fold self product of $\mathfrak{X}$ over k° in the lexicographic order. We denote the resulting scheme by $\mathfrak{B}$. Then $\mathfrak{B}$ is regular strictly semi-stable and naturally $R(\mathfrak{B})$ is the d-fold product of $R(\mathfrak{X})$ together with the maps of simplicial sets $\mathrm{pr}_{i,*}\colon R(\mathfrak{B}) \to R(\mathfrak{X})$ induced by the natural projection maps $\mathrm{pr}_i\colon \mathfrak{B} \to \mathfrak{X}$ to the i-th factor for $i \in \{1, \ldots, d\}$ (cf. Prop. B.13) as projection maps.

Our main tool will be intersection theory with supports.

Definition 3.3.4 (Intersection theory with supports)
Let $\mathfrak{B}$ be a separated equidimensional scheme of dimension n and of finite type over k°. Let Y be an equidimensional closed subscheme. We denote by $\mathrm{CH}^p_Y(\mathfrak{B})$ the Chow group of cycles of codimension p with support in Y in the sense of [**Sou92**, Ch. 2] (see also

[**Kol16b**, Kap. 4.1]). Let Z be an equidimensional closed subscheme. If $\mathfrak{B}$ is a regular scheme we have an intersection pairing

$$(\text{-} \cdot_{\mathfrak{B}} \text{-})\colon \mathrm{CH}^1_Z(\mathfrak{B}) \times \mathrm{CH}^p_Y(\mathfrak{B}) \to \mathrm{CH}^{p+1}_{Z\cap Y}(\mathfrak{B}).$$

If Z is equidimensional of codimension d in $\mathfrak{B}$, then the natural map

$$\mathrm{CH}^l_Z(\mathfrak{B}) \to \mathrm{CH}^{l-d}(Z) \tag{3.3.1}$$

is an isomorphism. We have proper push-forward of cycles and pull-back of divisor classes, satisfying the projection formula (cf. Prop. [**Kol13**, Prop. 4.13]).

If $\mathfrak{B}$ is proper over k° and $Y \subset \mathfrak{B}_s$ then we have a push-forward to $\mathrm{CH}^n_Y(\mathfrak{B}) \to \mathrm{CH}^1_{\{s\}}(\operatorname{Spec} k^\circ)$. The latter group admits a degree homomorphism to $\mathbb{Z}$ mapping the class of $\{s\}$ to 1. This defines a degree map $\deg_Y\colon \mathrm{CH}^n_Y(\mathfrak{B}) \to \mathbb{Z}$.

Definition 3.3.5
Let $\mathfrak{B}$ be a scheme of finite type over k°.

(i) A *stratum of the special fibre* is an irreducible component of the intersection of a set of irreducible components of the special fibre.
(ii) We say that $\mathfrak{B}$ satisfies the irreducibility condition IC if every non-empty intersection of irreducible components of the special fibre is again irreducible.

For the rest of the section let $\mathfrak{X}$ be a proper regular strictly semi-stable curve. Choose an order on the set of irreducible components of $\mathfrak{X}_s$. Let $\mathfrak{B}$ be the scheme from Thm. 3.3.3 for $d = 2$. We assume that $\mathfrak{X}$ satisfies the irreducibility condition IC. If F is an irreducible component of the special fibre of $\mathfrak{B}$ then we also have an intersection product on that component. The intersection product with supports on $\mathfrak{B}$ and the intersection product on F are related by following proposition.

Proposition 3.3.6
(i) Let F be an irreducible component of the special fibre of $\mathfrak{B}$. Let $\mathfrak{D}$ be a Cartier divisor on $\mathfrak{B}$ and $Z \in \mathrm{CH}^2_F(\mathfrak{B})$. Let F' be any Cartier divisor. Denote by $\cdot_F$ the intersection product and by $\deg_F$ the degree map for zero-dimensional cycles on the smooth surface F. Then

$$\deg_F((F \cdot_{\mathfrak{B}} F') \cdot_F Z) = \deg_{\mathfrak{B}_s}(F' \cdot_{\mathfrak{B}} Z)$$

(ii) Let K be an irreducible component of the special fibre of $\mathfrak{X}$ and let $\mathfrak{D}$ be a Cartier divisor on $\mathfrak{X}$. Then

$$\deg_{\mathfrak{X}_s}(\mathfrak{D} \cdot K) = \deg_K(\mathscr{O}(\mathfrak{D})|_K)$$

holds.

Proof.

(i) We can assume that Z is a prime cycle. We denote by $i_Z\colon Z \hookrightarrow F$, $i_F\colon F \hookrightarrow \mathfrak{B}$ the closed embeddings of Z and F. Then by construction of the degree maps it suffices to show that

$$(F \cdot_{\mathfrak{B}} F') \cdot_F Z = F' \cdot_{\mathfrak{B}} Z$$

holds in $\mathrm{CH}^2(Z)$. But this follows as the left hand side is

$$c_1(i_Z^* i_F^* \mathscr{O}(F'))$$

as well as the right hand side. Here c_1 denotes the first Chern class of a line bundle.
(ii) This follows like the first part.

□

We have a moving lemma in the special fibre of $\mathfrak{B}$ which allows us to compute intersections of strata of the special fibre.

Theorem 3.3.7
Let $i \in \{1, \dots, d\}$ be a number. We denote by pr_i the natural map from $\mathfrak{B}$ to the i-th factor. Let K, K' be components of the special fibre of $\mathfrak{B}$ and let $\mathrm{pr}_i K \neq \mathrm{pr}_i K'$ for some i. Then the cycle

$$\sum_{\substack{D \in (\mathfrak{B}_s)^{(0)} \\ \mathrm{pr}_i(D) = \mathrm{pr}_i(K)}} K \cdot_{\mathfrak{B}} K' \cdot_{\mathfrak{B}} D$$

is zero in $\mathrm{CH}^3_{\mathfrak{B}_s}(\mathfrak{B})$.

PROOF. Cf. [**Kol16b**, Prop. 4.11]. □

Definition 3.3.8
Let K be an irreducible component of $\mathfrak{B}_s$. We make the following definitions:

(i) We define $H(K)$ as the set of irreducible components H' of $\mathfrak{B}_s$ with $H' \cap K \neq \emptyset$ and $\mathrm{pr}_1(H') \neq \mathrm{pr}_1(K)$ and $\mathrm{pr}_2(H') = \mathrm{pr}_2(K)$.
(ii) We define $V(K)$ as the set of irreducible components H' of $\mathfrak{B}_s$ with $H' \cap K \neq \emptyset$ and $\mathrm{pr}_2(H') \neq \mathrm{pr}_2(K)$ and $\mathrm{pr}_1(H') = \mathrm{pr}_1(K)$.
(iii) We define $D(K)$ as the set of irreducible components H' of $\mathfrak{B}_s$ with $H' \cap K \neq \emptyset$ and $\mathrm{pr}_i H' \neq \mathrm{pr}_i K$ for all $i \in \{1, 2\}$. For each $H \in D(K)$ the set $\gamma(H \cap K)$ is a single closed point as we asked $\mathfrak{X}$ to fulfill IC. We denote by $P(K)$ the finite set of points $p = \gamma(K \cap H) \in K'$ for $H \in D(K)$.

Remark 3.3.9
(i) We can determine the maximal cardinality of the sets H, V, and D: We have $|H(F)| \leq \mathrm{val}(R(\mathfrak{X}))$, $|V(F)| \leq \mathrm{val}(R(\mathfrak{X}))$, and $|D(F)| \leq \mathrm{val}(R(\mathfrak{X}))^2$.
(ii) The points $P(K)$ are the points (p, q) of $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$, where $p, q \in \mathfrak{X}_s$ are double points of the special fibre.

We have the following reformulation of Thm. 3.3.7 for $d = 2$:

Proposition 3.3.10
Let F be an irreducible component of the special fibre of $\mathfrak{B}$.

(i) If $K \in H(F)$ then

$$-K \cdot_{\mathfrak{B}} F \cdot_{\mathfrak{B}} F = \sum_{K' \in V(F)} K \cdot_{\mathfrak{B}} F \cdot_{\mathfrak{B}} K'$$

holds in $\mathrm{CH}^3_{\mathfrak{B}_s}(\mathfrak{B})$.

(ii) If $K \in V(F)$ then

$$-K \cdot_{\mathfrak{B}} F \cdot_{\mathfrak{B}} F = \sum_{K' \in H(F)} K \cdot_{\mathfrak{B}} F \cdot_{\mathfrak{B}} K'$$

holds in $\mathrm{CH}^3_{\mathfrak{B}_s}(\mathfrak{B})$.

We also have the following proposition:

Proposition 3.3.11
We have the equality

$$\sum_{K\in(\mathfrak{B}_s)^{(0)}} K\cdot_{\mathfrak{B}} Z = 0$$

in $\mathrm{CH}^{1+l}_{\mathfrak{B}_s}(\mathfrak{B})$ *for every* $Z\in \mathrm{CH}^{l}_{\mathfrak{B}_s}(\mathfrak{B})$.

Proof. In principal, this is [**Kol16b**, Prop. 4.10]. Note that as the special fibre of $\mathfrak{B}$ is reduced we have

$$\sum_{K\in(\mathfrak{B}_s)^{(0)}} K = \mathrm{div}(\pi) = 0$$

in $\mathrm{CH}^1(\mathfrak{B})$. Hence

$$\sum_{K\in(\mathfrak{B}_s)^{(0)}} K\cdot_{\mathfrak{B}} Z = 0$$

in $\mathrm{CH}^{1+l}_{\mathfrak{B}_s}(\mathfrak{B})$ for every $Z\in \mathrm{CH}^{l}_{\mathfrak{B}_s}(\mathfrak{B})$. □

3.4. Curves in the Special Fibre of Gross–Schoen Models

In order to investigate positivity properties of Cartier divisors on regular strictly semi-stable models we want to compute intersection numbers of divisors with curves supported in the special fibre. When these curves are strata of the special fibre (in the sense of Def. 3.3.5(i)) these numbers can be computed using the purely combinatorial relations of Theorem 3.3.7. In the case of arbitrary curves we will find relations which allow us to investigate positivity of divisors. These relations however will be in terms of the special geometry of the irreducible components of the special fibre of the Gross–Schoen Models.

We will be in the situation of Assumption 2.2.1. Let $\mathfrak{X}$ be a (not necessarily proper) regular strictly semi-stable curve over k°, X its generic fibre. We assume that $\mathfrak{X}$ satisfies the condition IC from Def. 3.3.5. Fix an order on the set of irreducible components of the special fibre of $\mathfrak{X}$. We blow up the components of the special fibre of $\mathfrak{X}\times_{k^\circ}\mathfrak{X}$ in the lexicographic order as in Thm. 3.3.3. Let $\mathfrak{B}$ be the resulting model of $X\times_k X$ and $\gamma\colon \mathfrak{B}\to\mathfrak{X}\times_{k^\circ}\mathfrak{X}$ the natural morphism. For $i\in\{1,2\}$ we have natural maps $\mathrm{pr}_i\colon\mathfrak{B}\to\mathfrak{X}$ arising from the two projections $\mathfrak{X}\times_{k^\circ}\mathfrak{X}\to\mathfrak{X}$.

We shall need the following invariant of a point on a curve.

Definition 3.4.1
Let K be an algebraically closed field. Let S be a smooth integral surface and C be an integral curve on S, let p be a closed point of S, and let f be a local equation for C around p in S. We denote by $\mathfrak{m}_p$ be the maximal ideal of the stalk $\mathscr{O}_{X,p}$. Then we define the multiplicity $\mu_C(p)$ of p in C as the biggest number $r\geq 0$ such that $f\in\mathfrak{m}_p^r$.

Remark 3.4.2
If $p\in C$ the number $\mu_C(p)$ is a local invariant of C at p (cf. [**Ful98**, Ex. 4.3.9]).

Proposition 3.4.3
Let C, S, p *be as in Def. 3.4.1 and let* $\pi\colon\widetilde{S}\to S$ *be the blow-up of* S *at* p *and* E *be the exceptional divisor. Let* $\widetilde{C}$ *be the strict transform of* C *and let* r *be the multiplicity of* C *at* p. *Then the following holds.*

(i) *We have the equality of Cartier divisors* $\pi^*C=\widetilde{C}+r\cdot E$.

(ii) If D is an effective Cartier divisor intersecting C properly and $p \in D \cap C$, we have the inequality $D \cdot C \geq r \geq 0$.

PROOF. For (i) claim see [**Har77**, Prop. V 3.6] and (ii) follows from [**Ful98**, Cor. 12.4]. □

We first investigate the impact of the morphism $\gamma\colon \mathfrak{B} \to \mathfrak{X} \times_{k^\circ} \mathfrak{X}$ to irreducible components of the special fibre of $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$. Let K be an irreducible component of the special fibre of $\mathfrak{B}$ and set $K' = \gamma(K)_{\mathrm{red}}$. In the following proposition we will see that K' is an irreducible component of the special fibre of $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$. We denote by $\gamma|_K\colon K \to K'$ the restriction of the morphism γ to K.

Proposition 3.4.4
The closed subset K' of the special fibre of $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$ is an irreducible component of the special fibre. The map $\gamma|_K\colon K \to K'$ induced by the map $\gamma\colon \mathfrak{B} \to \mathfrak{X} \times_{k^\circ} \mathfrak{X}$ is the blow-up of K' at the set of points $P(K)$ defined in Def. 3.3.8.

PROOF. First, we show the claim for $\mathfrak{X} = \operatorname{Spec} k^\circ[x,y]/(xy - \pi)$. The map $\gamma\colon \mathfrak{B} \to \mathfrak{X} \times_{k^\circ} \mathfrak{X}$ is a priori given by a sequence of blow-ups. We define the following irreducible components of the special fibre of $\mathfrak{X}$

$$K_1 = V(x), K_2 = V(y)$$

and choose the order $K_1 \leq K_2$ on these components.

We set

$$S := k^\circ[x_1, y_1, x_2, y_2]/(x_1y_1 - \pi, x_2y_2 - \pi)$$

i.e., $\operatorname{Spec}(S) = \mathfrak{X} \times_{k^\circ} \mathfrak{X}$. We define the following irreducible components of the special fibre of $\operatorname{Spec} S$.

$$\begin{aligned} C'_1 &= V(x_1, x_2) = K_1 \times_{\tilde{k}} K_1, \\ C'_2 &= V(x_1, y_2) = K_1 \times_{\tilde{k}} K_2, \\ C'_3 &= V(y_1, x_2) = K_2 \times_{\tilde{k}} K_1, \\ C'_4 &= V(y_1, y_2) = K_2 \times_{\tilde{k}} K_2. \end{aligned}$$

We choose the lexicographic order on the set of irreducible components of the special fibre of $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$ i.e.,

$$C'_1 \leq C'_2 \leq C'_3 \leq C'_4,$$

and we investigate the scheme $\mathfrak{B}$ associated to this order in virtue of Thm. 3.3.3.

First we investigate the blow-up $\mathfrak{B}'$ of $\operatorname{Spec} S$ at the irreducible component C'_1. This is the first step in the chain of blow-ups $\mathfrak{B} \to \mathfrak{X} \times_{k^\circ} \mathfrak{X}$. By [**Liu02**, Lemma 8.1.4] the scheme $\mathfrak{B}'$ can be covered by the spectra of the two k°-algebras

$$R_i = k^\circ[x_1, y_1, x_2, y_2]/(x_1y_1 - \pi, x_2y_2 - \pi)[x_i/x_j]$$

for $i \in \{1,2\}$. Here $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$. For each $i \in \{1,2\}$ the morphism from $\operatorname{Spec} k^\circ[x,y,z]/(xyz - \pi) \to \operatorname{Spec} R_i$ defined by the map of k°-algebras

$$x \mapsto x_j, y \mapsto \frac{x_i}{x_j}, z \mapsto y_i$$

is an isomorphism with inverse map defined by the map of k°-algebras

$$x_i \mapsto yx, y_i \mapsto z, x_j \mapsto x, y_j \mapsto yz.$$

By Thm. B.11 the blown-up scheme $\mathfrak{B}'$ is already regular strictly semi-stable and in particular the irreducible components of the special fibre of $\mathfrak{B}'$ are already Cartier divisors. Further blow-ups in irreducible components of the special fibre have no effect, so $\mathfrak{B} = \mathfrak{B}'$.

By Thm. 3.3.3 the reduction set of $\mathfrak{B}$ is the product of $R(\mathfrak{X})$ with itsself and the projections are induced by the projections $\mathfrak{B} \to \mathfrak{X} \times_{k^\circ} \mathfrak{X} \to \mathfrak{X}$ in virtue of Prop. B.13.

Using the description of Rem. A.8 the reduction set has the following form:

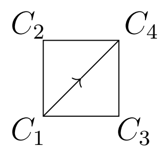


FIGURE 3.4.1. The Reduction set of $\mathfrak{B}$

Here C_i is the strict transform of C_i' for all $i \in \{1, \ldots, 4\}$.

We see that $P(C_1) = P(C_4) = \{V(x_1, y_1, x_2, y_2)\}$ and $P(C_2) = P(C_3) = \emptyset$.

As C_4' is not contained in the center of the blow-up $\gamma\colon \mathfrak{B} \to \mathfrak{X} \times_{k^\circ} \mathfrak{X}$, namely C_1', the induced map $\gamma|_{C_4}\colon C_4 \to C_4'$ is the blow-up of C_4' at $C_1' \cap C_4' = V(x_1, x_2, y_1, y_2)$ by [**Liu02**, Cor. 8.1.17]. Hence the claim holds for $K = C_4$. Moreover, we see that the restrictions of C_1' to C_2' and C_3' respectively are Cartier divisors and hence the maps $\gamma\colon C_i \to C_i'$ are isomorphisms for $i \in \{2, 3\}$. Hence the claim holds for $i \in \{2, 4\}$.

It remains to investigate the impact of the blow-up on the irreducible component C_1' of $(\mathfrak{X} \times_{k^\circ} \mathfrak{X})_s$.

The restriction of $\gamma|_{C_1}\colon C_1 \to C_1'$ is given in the charts $\operatorname{Spec} R_i \cap C_1$ by the natural map

$$\widetilde{k}[y_1, y_2] \to \widetilde{k}[y_1, y_2][x_i/x_j]$$

for $i \in \{1, 2\}$. But for each i we have the relation $x_i/x_j = y_j/y_i$, hence $\gamma|_{C_1}\colon C_1 \to C_1'$ is the blow-up of C_1' at the point of C_1' given by the ideal (x_1, y_1, x_2, y_2) of S, so our claim also holds for C_1' and we have shown our claim for our particular choice of a regular strictly semi-stable scheme $\mathfrak{X}$.

Now we consider a general regular strictly semi-stable curve $\mathfrak{X}$ following the argument of [**Kol16b**, Lemma 3.5]. The property of beeing a blow-up is local on the target. A priori K' is not an irreducible component of the special fibre, so we assume that K' is an irreducible component of the special fibre of $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$ containing the image of K under γ. Let $\mathfrak{X}$ be a regular strictly semi-stable curve and let $x \in \mathfrak{X} \times_{k^\circ} \mathfrak{X}$ be a closed point in the special fibre contained in the irreducible component K' of the special fibre. If x is a regular point, then K' will be a Cartier divisor in a neighbourhood U of x and the map $K \cap \gamma^{-1}(U) \to K' \cap U$ is an isomorphism. In particular the image of K under γ is an irreducible component of the special fibre which then has to equal K'. By Rem. 3.3.9 (ii) the points $P(K)$ will not lie in U.

So let x be a singular closed point of $(\mathfrak{X} \times_{k^\circ} \mathfrak{X})_s$. Then $x = (z, w)$ for two double points z and w in the special fibre of $\mathfrak{X}$ by [**Kol13**, 2.4]. By Prop. B.3 we can choose $\mathfrak{U}, \mathfrak{V} \subset \mathfrak{X}$ with $z \in \mathfrak{U}$ and $w \in \mathfrak{V}$ which admit étale maps to $\mathfrak{S} = \operatorname{Spec} k^\circ[x, y]/(xy - \pi)$ and induce bijections of the reduction sets $R(\mathfrak{U}) \to R(\mathfrak{S})$ and $R(\mathfrak{V}) \to R(\mathfrak{S})$ respectively in virtue of Prop. B.7. Then $f\colon \mathfrak{U} \times_{k^\circ} \mathfrak{V} \to \mathfrak{S} \times_{k^\circ} \mathfrak{S}$ is an étale map.

As any étale map is flat we can use the argument of of [**Kol16b**, Lemma 3.6] to show that we get a cartesian diagram

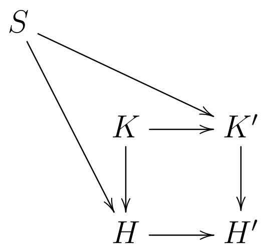
$$\begin{array}{ccc} \mathfrak{B} & \longrightarrow & \mathfrak{U} \times_{k^\circ} \mathfrak{V} \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow[\widetilde{g}]{} & \mathfrak{S} \times_{k^\circ} \mathfrak{S}, \end{array}$$

where M is the blow-up of $M' := \mathfrak{S} \times_{k^\circ} \mathfrak{S}$ at the components of the special fibre as above and $\mathfrak{B}$ is the successive blow-up of $W' := \mathfrak{U} \times_{k^\circ} \mathfrak{V}$ at the components of the special fibre. This yields the following cartesian diagram of the special fibres.

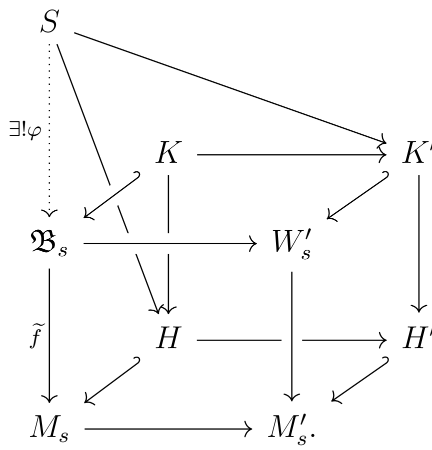
$$\begin{array}{ccc} \mathfrak{B}_s & \longrightarrow & (\mathfrak{U} \times_{k^\circ} \mathfrak{V})_s \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ M_s & \xrightarrow[\widetilde{g}]{} & (\mathfrak{S} \times_{k^\circ} \mathfrak{S})_s \end{array} \tag{3.4.1}$$

As all special fibres have the same number of irreducible components and the restriction of γ to the special fibre is surjective there cannot be irreducible components of $\mathfrak{B}_s$ which don't map to an irreducible component of the special fibre under γ. So the map γ maps irreducible components of the special fibre of $\mathfrak{B}$ to irreducible components of the special fibre of $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$.

Recall that K is the unique irreducible component of the special fibre of W' mapping to K' under γ. Set $H = \widetilde{f}(K)_{\mathrm{red}}$ and $H' = \widetilde{g}(H)_{\mathrm{red}}$, $p = f(x)$. Assume we have a scheme $S/\widetilde{k}$ and morphisms $S \to K'$ and $S \to H$ such that the diagram



commutes. We consider the diagram



Then as (3.4.1) is cartesian the dotted arrow φ exists uniquely. By [**GW10**, Cor. 11.49], $\widetilde{f}^*(M_s) = \widetilde{f}^{-1}(M_s)$ as Cartier divisors in M. As $\mathfrak{B}_s = \widetilde{f}^*(M_s)$ is reduced, $\widetilde{f}^*(H) = \widetilde{f}^{-1}(H)$ is reduced. As K is reduced, K is the scheme theoretic preimage of H under $\widetilde{f}$, by uniqueness of the reduced closed scheme structure. Hence, the map φ uniquely factors over K by the universal property of the preimage. But as φ was arbitrary this means that the diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \downarrow & & \downarrow \\ H & \longrightarrow & H' \end{array} \tag{3.4.2}$$

is cartesian. We know that $H \to H'$ is the blow-up of H' at the point $p \in P(H)$ of H'. Note that by assumption the maps between the sets of strata of the special fibres of $\mathfrak{U}, \mathfrak{V}$, and $\mathfrak{S}$ are isomorphisms and hence $\widetilde{g}^{-1}(p) = \{x\}$. As the diagram (3.4.2) is cartesian and by base change the map $K' \to H'$ is flat this implies that K is the blow-up of K' at the point $\widetilde{g}^{-1}(p) = \{x\} = P(K)$ by [**GW10**, Thm. 13.91]. This is what we wanted to show. This proves the claim for a general choice of $\mathfrak{X}$ and thus we have established the proposition. ☐

We make the following assumptions:

Assumption 3.4.5
(i) Assume that X is a projective geometrically integral smooth curve and that $\mathfrak{X}$ is a projective regular strictly semi-stable model of X. Let $\mathfrak{B}$ be the regular strictly semi-stable model of $X \times_k X$ associated to the regular strictly semi-stable model $\mathfrak{X}$ and an order on the irreducible components of the special fibre of $\mathfrak{X}$ in virtue of Thm. 3.3.3. Let $\gamma \colon \mathfrak{B} \to \mathfrak{X} \times_{k^\circ} \mathfrak{X}$ be the natural morphism.
(ii) Let F be an irreducible component of the special fibre of $\mathfrak{B}$. Under γ the component F will map to an irreducible component $F' := \gamma(F)_{\mathrm{red}}$ of the special fibre of $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$ by the preceeding proposition. Let Z be an integral curve in F which does not map to a point under γ. Set $Z' := \gamma(Z)_{\mathrm{red}}$. As F' is smooth, Z' will be a Cartier divisor in F'. The restriction $\gamma|_F : F \to F'$ of γ to the irreducible component F is dominant and we can pull back Cartier divisors along $\gamma|_F : F \to F'$. We denote by $\widehat{Z'}$ the Cartier divisor $(\gamma|_F)^*(Z')$ in F. This is the total transform of Z' under the blow-up $\gamma|_F \colon F \to F'$.

Corollary 3.4.6
(i) For each $H \in D(F)$ the divisor $H \cdot_{\mathfrak{B}} F \in \mathrm{CH}^1(F)$ is the exceptional divisor $\gamma^{-1}(\gamma(H \cap F))$.
(ii) We have the equality of Cartier divisors

$$Z = \widehat{Z'} - \sum_{H \in D(F)} r_H (H \cdot_{\mathfrak{B}} F) \tag{3.4.3}$$

in F, where for each $H \in D(F)$ the number r_H is the multiplicity of Z' in the closed point $\gamma(H \cap F) \in P(F)$ and $D(K)$ is defined as in Def. 3.3.8.
(iii) Let D' be a Cartier divisor in F'. Let $H \in D(F)$ be an irreducible component. Then $\deg_F((\gamma|_F)^*(D') \cdot_F (H \cdot_{\mathfrak{B}} F)) = 0$ holds.
(iv) Let W be a curve in F. Define $\widehat{W'} = (\gamma|_F)^*(\gamma(W)_{\mathrm{red}})$. Then we have

$$\deg_F(\widehat{W'} \cdot_F \widehat{Z'}) = \deg_F(W \cdot_F \widehat{Z'}).$$

(v) Let D be any Cartier divisor on $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$. Then

$$\deg_{\mathfrak{B}_s}(\gamma^* D \cdot_{\mathfrak{B}} Z) = \deg_{\mathfrak{B}_s}(\gamma^* D \cdot_{\mathfrak{B}} \widehat{Z'}).$$

Proof.

(i) For each $H \in D(F)$ by Prop. 3.4.4 the supports of $H \cdot_{\mathfrak{B}} F$ and the exceptional divisor E associated to $\gamma(F \cap H)$ agree. Both supports are irreducible. $H \cdot_{\mathfrak{B}} F$ is reduced, as H and F intersect with multiplicity one by [**Kol16b**, Prop. 4.8], and the exceptional divisor E is reduced, as F is the blow-up of F' at smooth points. Hence, $H \cdot_{\mathfrak{B}} F$ and E agree as Cartier divisors.

(ii) The right side of (3.4.3) is the strict transform $\widetilde{Z'}$ of Z' under $\gamma|_F$ by Prop. 3.4.3 using (i). We claim that $\widetilde{Z'} = Z$. We can choose an open subset $U \subset \mathfrak{B}$ such that $\gamma|_{Z \cap U}$ is an isomorphism to $Z' \cap \gamma(U)$. We have an open immersion $\widetilde{Z'} \cap U \hookrightarrow Z$ and by irreducibility of Z we have equality.

(iii) By (i) for every $H \in D(F)$ the divisor $H \cdot_{\mathfrak{B}} F$ is an exceptional divisor in F of the blow-up $\gamma|_F \colon F \to F'$, which means $\deg_F((H \cdot_{\mathfrak{B}} F) \cdot_F \gamma^* D') = 0$ for every Cartier divisor D' in F'.

(iv) We have the equality of Cartier divisors

$$Z = \widehat{Z'} - \sum_{H \in D(F)} r_H \cdot (H \cdot_{\mathfrak{B}} F)$$

in F, where r_H are the multiplicities of Z' in $\gamma(E \cap F)$. Analogous we get

$$W = \widehat{W'} - \sum_{H \in D(F)} s_H \cdot (H \cdot_{\mathfrak{B}} F)$$

for integers s_H. As $\deg_F(\widehat{Z'} \cdot_F (H \cdot_{\mathfrak{B}} F)) = 0$ by (iv), we get the claim.

(v) Again we write

$$Z = \widehat{Z'} - \sum_{H \in D(F)} r_H \cdot (H \cdot_{\mathfrak{B}} F).$$

We note that by Prop. 3.3.6(i)

$$\deg_{\mathfrak{B}_s}(\gamma^* D \cdot Z) = \deg_F(((\gamma^* D) \cdot_{\mathfrak{B}} F) \cdot_F Z)$$

We have the natural commutative diagram

$$\begin{array}{ccc} F' & \xrightarrow{\;h\;} & \mathfrak{X} \times_{k^\circ} \mathfrak{X} \\ {\scriptstyle \gamma|_F}\uparrow & & \uparrow{\scriptstyle \gamma} \\ F & \xrightarrow[\;i\;]{} & \mathfrak{B} \end{array}$$

and see

$$(\gamma|_F)^*(h^* D) = i^* \gamma^* D$$

so

$$\deg_{\mathfrak{B}_s}(\gamma^* D \cdot_{\mathfrak{B}} Z) = \deg_F((\gamma|_F)^*(D|_{F'}) \cdot_F Z)$$

holds. But for every $H \in D(F)$ we have

$$\deg_F\left((\gamma_F)^*(D|_F) \cdot_F (H \cdot_{\mathfrak{B}} F)\right) = 0$$

by the second point and hence the claim. □

We describe the structure of irreducible components of the special fibre of $\mathfrak{B}_s$.

Proposition 3.4.7

(i) All irreducible components of the special fibre of $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$ are of the form $K_1 \times_{\tilde{k}} K_2$ for irreducible components K_1, K_2 of the special fibre of $\mathfrak{X}$.

(ii) *Assume $F' = K_1 \times_{\widetilde{k}} K_2$ for irreducible components K_1, K_2 of the special fibre of $\mathfrak{X}$. We have $A \in V(F)$ if and only if $\gamma(A) = K_1 \times_{\widetilde{k}} K_2'$ and $K_2 \cap K_2'$ is a closed point. Likewise $A \in H(F)$ if and only if $\gamma(A) = K_1' \times_{\widetilde{k}} K_2$ and $K_1 \cap K_1'$ is a closed point.*

(iii) *For $i \in \{1, 2\}$ denote by $\mathrm{pr}_i|_F \to K_i$ the restriction of pr_i to F. If $A \in H(F)$ then*

$$(\gamma|_F)^*(\gamma(A \cdot_{\mathfrak{B}} F)_{\mathrm{red}}) = (\mathrm{pr}_1|_F)^*(\{p\})$$

for a closed point p of the special fibre of $\mathfrak{X}$. Likewise, if $A \in V(F)$ then

$$(\gamma|_F)^*(\gamma(A \cdot_{\mathfrak{B}} F)_{\mathrm{red}}) = (\mathrm{pr}_2|_F)^*(\{q\})$$

for a closed point q of the special fibre of $\mathfrak{X}$.

(iv) *If $H \in D(F)$ then $\gamma(H) = K_1' \times_{\widetilde{k}} K_2'$ for $K_1' \cap K_1 = \{p\}$ and $K_2 \cap K_2' = \{q\}$ for irreducible components K_1', K_2' of the special fibre of $\mathfrak{X}$ and closed points p, q of the special fibre of $\mathfrak{X}$.*

PROOF.

(i) This follows from the fact that $\widetilde{k}$ is algebraically closed and hence each irreducible component of $\mathfrak{X}_s$ is geometrically irreducible.

(ii) This follows as the projection maps $\mathrm{pr}_i \colon \mathfrak{B} \to \mathfrak{X}$ are defined as the composition $\mathfrak{B} \to \mathfrak{X} \times_{k^\circ} \mathfrak{X} \to \mathfrak{X}$.

(iii) By the previous we have the equalities

$$(\gamma|_F)^{-1}(\gamma(A \cdot_{\mathfrak{B}} F)_{\mathrm{red}}) = (\mathrm{pr}_1|_F)^{-1}(\{p\})$$

and

$$(\gamma|_F)^{-1}(\gamma(A \cdot_{\mathfrak{B}} F)_{\mathrm{red}}) = (\mathrm{pr}_2|_F)^{-1}(\{q\})$$

as subschemes of F for some points p, q. But as $F \to F'$ is dominant, [**GW10**, Cor. 11.49] implies (iii).

(iv) This follows from the definition of the projection maps. □

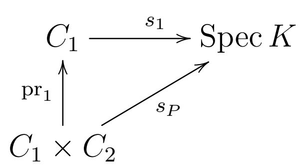
Proposition 3.4.8

Let C_1 and C_2 be two integral, proper non-singular curves over an algebraically closed field K. For closed points $p, q \in C_1$ consider the divisors $\{p\} \times C_2$ and $\{q\} \times C_2$ on $P := C_1 \times C_2$. Then for any divisor D on $C_1 \times C_2$ the equality

$$\deg_{C_1 \times C_2}(D \cdot_{C_1 \times C_2} (\{p\} \times C_2)) = \deg_{C_1 \times C_2}(D \cdot_{C_1 \times C_2} (\{q\} \times C_2))$$

holds.

PROOF. We consider the following commutative diagram.



Here s_1, s_2, s_P are the structure morphisms of C_1, C_2 and P to $\operatorname{Spec} K$. Then

$$s_{P*}(D \cdot (\{p\} \times C_2 - \{q\} \times C_2)) =$$
$$s_{1*}\mathrm{pr}_{1*}((\mathrm{pr}_1^*(\{p\}) - \mathrm{pr}_1^*(\{q\})) \cdot D) = s_{1*}[\{p\} \cdot \mathrm{pr}_{1*}(D)] - s_{1*}[\{q\} \cdot \mathrm{pr}_{1*}(D)]$$

holds by the projection formula. Now $\mathrm{pr}_{1*}(D)$ equals $n \cdot [C_1]$ in $\mathrm{CH}^0(C_1)$ for some $n \in \mathbb{Z}$. Hence $s_{1*}[\{p\} \cdot \mathrm{pr}_{1*}(D)] = n s_{1*}\{p\}$ and $s_{1*}[\{q\} \cdot \mathrm{pr}_{1*}(D)] = n s_{1*}\{q\}$, but $n s_{1*}\{q\} =$

$ns_{1*}\{p\} = n$ by the construction of the degree map and the fact that K is algebraically closed. So

$$\deg_{C_1\times C_2}(D \cdot_{C_1\times C_2} (\{p\} \times C_2)) = n = \deg_{C_1\times C_2}(D \cdot_{C_1\times C_2} (\{q\} \times C_2))$$

and hence the claim. □

Corollary 3.4.9
Assume that $A, B \in H(F)$ or $A, B \in V(F)$ holds. Then we have the following equality:

$$\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} \widehat{Z'}) = \deg_{\mathfrak{B}_s}(B \cdot_{\mathfrak{B}} \widehat{Z'})$$

Proof. We only treat the case $A, B \in H(F)$, the case $A, B \in V(F)$ follows by symmetry. By Prop. 3.3.6(i)

$$\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} \widehat{Z'}) = \deg_F((A \cdot_{\mathfrak{B}} F) \cdot_F \widehat{Z'}).$$

We denote by W' the Cartier divisor $\gamma(A \cap F)_{\text{red}}$ in F' and set $\widehat{W'} = (\gamma|_F)^*(W')$. Then by Cor. 3.4.6 (iv) we see

$$\deg_F((A \cdot_{\mathfrak{B}} F) \cdot_F \widehat{Z'}) = \deg_F(\widehat{W'} \cdot_F \widehat{Z'}).$$

Now our claim follows from Prop. 3.4.8 using that $\widehat{W'} = (\mathrm{pr}_2|_F)^*(\{p\})$ for a closed point p of $\mathfrak{X}_s$ by Prop. 3.4.7 (iii). □

Proposition 3.4.10
Assume that Z is not a stratum of the special fibre (Def. 3.3.5(i)). Let $p \in P(F)$ be a point and r_p the multiplicity of Z' in p. Let $K \in H(F) \cup V(F)$ be arbitrary. Then

$$0 \leq r_p \leq \deg_{\mathfrak{B}_s}(K \cdot \widehat{Z'})$$

Proof. By symmetry of the argument we restrict to the case where $K \in H(F)$. By assumption there is a component $H' \in D(F)$ with $\gamma(H' \cap F) = p$. By Prop. 3.4.7 (iv) the irreducible component F is of the form $K_1 \times_{\tilde{k}} K_2$ for two irreducible components of the special fibre of $\mathfrak{X}$ and $\gamma(H')$ is of the form $K_1' \times_{\tilde{k}} K_2'$ for two irreducible components K_1', K_2' of the special fibre of $\mathfrak{X}$ with $K_1' \cap K_1 = \{q\}$ and $K_2' \cap K_2 = \{r\}$ for closed points q and r of the special fibre of $\mathfrak{X}$. By Prop. 3.4.7 (ii) the unique irreducible component A of the special fibre of $\mathfrak{B}$ mapping to $K_1' \times K_2$ will be an element of $H(F)$. Hence by Cor. 3.4.9 we will have

$$\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} \widehat{Z'}) = \deg_{\mathfrak{B}_s}(K \cdot_{\mathfrak{B}} \widehat{Z'}).$$

and hence we can assume that H and K intersect non-trivially.

As Z is not a stratum of the special fibre, we get that $0 \leq \deg_{\mathfrak{B}_s}(K \cdot_{\mathfrak{B}} Z)$. We use Prop. 3.4.6 (ii) and Prop. 3.3.6(i) to get

$$0 \leq \deg_{\mathfrak{B}_s}(K \cdot_{\mathfrak{B}} Z) = \deg_F((K \cdot_{\mathfrak{B}} F) \cdot_F \widehat{Z'}) - \sum_{H \in D(F)} r_H,$$

where for each H the number r_H is the multiplicity of Z' in the point $\gamma(F \cap H)$ of $P(F)$.

So we get

$$0 \leq \sum_{H \in D(F)} r_H \leq \deg_F((K \cdot_{\mathfrak{B}} F) \cdot_F \widehat{Z'}) = \deg_{\mathfrak{B}_s}(K \cdot_{\mathfrak{B}} \widehat{Z'})$$

and in particular $0 \leq r_{H'} \leq \deg_{\mathfrak{B}_s}(K \cdot_{\mathfrak{B}} \widehat{Z'})$ as desired. □

Proposition 3.4.11
Let A be an irreducible component of $\mathfrak{B}_s$ different from F and assume that Z is not a stratum of the special fibre.

(i) Assume that $A \in D(F)$ holds. Then for any $B \in V(F) \cup H(F)$ we have

$$|\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} Z)| \leq \deg_{\mathfrak{B}_s}(B \cdot_{\mathfrak{B}} \widehat{Z}')$$

(ii) Assume that $A \in H(F) \cup V(F)$ holds. Then

$$\left|\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} Z)\right| \leq (\mathrm{val}(R(\mathfrak{X}))^2 + 1)|\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} \widehat{Z})|.$$

Proof.

(i) Note that by Prop. 3.4.3 and Prop. 3.3.6(i) the number $\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} Z)$ is the multiplicity of Z' in the point $\gamma(A \cdot_{\mathfrak{B}} F)$. Now the claim follows from Prop. 3.4.10.

(ii) By Prop. 3.4.6 we have

$$\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} Z) = \deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} \widehat{Z}') - \sum_{H \in D(F)} r_H \cdot \deg_{\mathfrak{B}_s}(H \cdot_{\mathfrak{B}} F \cdot_{\mathfrak{B}} A).$$

There are at most $\mathrm{val}(R(\mathfrak{X}))^2$ elements in $D(F)$ hence by the first part

$$\left|\sum_{H \in D(F)} r_H\right| \leq \mathrm{val}(R(\mathfrak{X}))^2 \deg_{\mathfrak{B}_s}(A \cdot \widehat{Z}')$$

holds and hence the claim.

□

Proposition 3.4.12
Let K, K' be different components of the special fibre of $\mathfrak{B}$ and assume there exists an $i \in \{1, 2\}$ such that $\mathrm{pr}_i K = \mathrm{pr}_i K'$. Then

$$\sum_{A \in H(K) \cup V(K) \cup D(K)} |\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} K \cdot_{\mathfrak{B}} K')| \leq 4\,\mathrm{val}(R(\mathfrak{X})) + \mathrm{val}(R(\mathfrak{X}))^2$$

Proof. By symmetry of the argument we can restrict to the case $i = 2$. We can assume that K and K' intersect non-trivially, because otherwise the claim is clear. Let $W = (H(K) \cup V(K) \cup D(K)) \smallsetminus \{K, K'\}$. We know that $|H(K)| \leq \mathrm{val}(R(\mathfrak{X}))$, $|V(K)| \leq \mathrm{val}(R(\mathfrak{X}))$, and $|D(K)| \leq \mathrm{val}(R(\mathfrak{X}))^2$ holds by Rem. 3.3.9 (i). Hence, using the condition IC, we get

$$\sum_{A \in W} |\deg_{\mathfrak{B}_s}(A \cdot_{\mathfrak{B}} K \cdot_{\mathfrak{B}} K')| \leq 2 \cdot \mathrm{val}(R(\mathfrak{X})) + \mathrm{val}(R(\mathfrak{X}))^2.$$

By Prop. 3.3.10 in $\mathrm{CH}^3_{\mathfrak{B}_s}(\mathfrak{B})$ we have

$$K \cdot_{\mathfrak{B}} K' \cdot_{\mathfrak{B}} K = -\left(\sum_{A \in V(K)} K \cdot_{\mathfrak{B}} K' \cdot_{\mathfrak{B}} A\right) \in \mathrm{CH}^3_{\mathfrak{B}_s}(\mathfrak{B})$$

and

$$K \cdot_{\mathfrak{B}} K' \cdot_{\mathfrak{B}} K' = -\left(\sum_{A \in V(K')} K \cdot_{\mathfrak{B}} K' \cdot_{\mathfrak{B}} A\right) \in \mathrm{CH}^3_{\mathfrak{B}_s}(\mathfrak{B})$$

so

$$|\deg_{\mathfrak{B}_s}(K \cdot_{\mathfrak{B}} K' \cdot_{\mathfrak{B}} K')| + |\deg_{\mathfrak{B}_s}(K \cdot_{\mathfrak{B}} K \cdot_{\mathfrak{B}} K')| \leq 2\,\mathrm{val}(R(\mathfrak{X}))$$

and the claim follows. □

We will make the following assumptions:

Assumption 3.4.13
(i) Let $n \geq 2$ be an integer. We denote by $\mathfrak{X}_n$ the scheme from Thm. 3.3.2. We can put an order $\leq$ on the irreducible components of $\mathfrak{X}_{n,s}$ such that $R(\mathfrak{X}_n)$ is the n-fold BHM-subdivision of $R(\mathfrak{X})$. Denote by $c_n\colon \mathfrak{X} \otimes k_n^\circ \to \mathfrak{X}$ the base change morphism and by $\beta_n\colon \mathfrak{X}_n \to \mathfrak{X} \otimes k_n^\circ$ the natural morphism.

Let $\mathfrak{B}_n$ be the regular strictly semi-stable scheme associated to $\mathfrak{X}_n$ for $d = 2$ from Thm. 3.3.3 with respect to $\leq$ and denote by γ_n the morphism $\mathfrak{B}_n \to \mathfrak{X}_n \times_{k_n^\circ} \mathfrak{X}_n$. For $i \in \{1, 2\}$ we denote by $\mathrm{pr}_{i,n}$ the projection maps $\mathfrak{B}_n \to \mathfrak{X}_n \times_{k_n^\circ} \mathfrak{X}_n \to \mathfrak{X}_n$ and by pr_i be the projections $\mathfrak{B}_n \to \mathfrak{X}$. We choose a very ample integral divisor $\mathfrak{D}'$ on $\mathfrak{X}$ and set $\mathfrak{D}_n = \mathrm{pr}_1^*\mathfrak{D}' + \mathrm{pr}_2^*\mathfrak{D}'$ on $\mathfrak{B}_n$. Recall that by convention the *special fibre of* $\mathfrak{B}_n$ is the fibre over the special point of k_n° and will be denoted by $\mathfrak{B}_{n,s}$ under abuse of notation.
(ii) Let Z be an integral curve contained in the special fibre of $\mathfrak{B}_n$. Assume that Z is not a stratum of the special fibre of $\mathfrak{B}_n$ and that Z is contained in the irreducible component F of the special fibre. Then $Z' := \gamma_n(Z)_{\mathrm{red}}$ will be a curve in the irreducible component F' of $\mathfrak{X}_n \times_{k_n^\circ} \mathfrak{X}_n$ which is the image of F under γ_n. We denote by $\widehat{Z}'$ the pull-back $(\gamma_n|_F)^*(Z') \in \mathrm{CH}^1(F)$.

Proposition 3.4.14
Let K_1 be an irreducible component of $\mathfrak{X}_{n,s}$ which maps onto an irreducible component L_1' of $(\mathfrak{X} \otimes k_n^\circ)_s$ under β_n.

(i) The divisor $c_n^*\mathfrak{D}'$ is very ample on $\mathfrak{X} \otimes k_n^\circ$.
(ii) The induced map $K_1 \to L_1'$ is an isomorphism.
(iii) The line bundle $\mathscr{O}(\beta_n^* c_n^* \mathfrak{D}')|_{K_1}$ is very ample.
(iv) If $p \in K_1$ is a closed point then the line bundle $\mathscr{O}(\beta_n^* c_n^* \mathfrak{D}')|_{K_1} \otimes \mathscr{O}(\{p\})^{-1}$ has non-negative degree on K_1.

Proof.

(i) Let $\mathscr{O}(\mathfrak{D}')$ be isomorphic to $i^*\mathscr{O}(1)$ for an embedding $i\colon \mathfrak{X} \to \mathbb{P}^m_{k^\circ}$ via $\varphi\colon \mathscr{O}(\mathfrak{D}') \widetilde{\to} i^*\mathscr{O}(1)$. Let $w\colon \mathfrak{X} \otimes k_n^\circ \to \mathbb{P}^m_{k_n^\circ}$ be the embedding induced by i. Then $\varphi \otimes k_n^\circ$ is an isomorphism between $w^*\mathscr{O}(1) = \mathscr{O}_{\mathbb{P}^m_{k_n^\circ}}(1)$ and $\mathscr{O}(c_n^*\mathfrak{D}')$, hence $c_n^*\mathfrak{D}'$ is very ample.
(ii) We endow L_1' with the induced reduced subscheme structure. Let S be the singular locus of $\mathfrak{X} \otimes k_n^\circ$. As L_1' is not contained in S, the induced map $\beta_n|_{K_1}\colon K_1 \to L_1'$ is the blow-up of L_1' at $L_1' \cap S$ by [**Liu02**, Cor. 8.1.17]. The scheme $\mathfrak{X}$ is regular strictly semi-stable, hence the irreducible components of the special fibre are smooth over $\widetilde{k}$. As $\widetilde{k}$ is algebraically closed this is also is true for the irreducible components of $(\mathfrak{X} \otimes k_n)_s$. This means that L_1' is smooth over $\widetilde{k}_n$. Hence the map $\beta_n|_{K_1}$ is an isomorphism.
(iii) By the previous the map $\beta_n|_{K_1}$ is an isomorphism and the pull back of a very ample line bundle under an isomorphism is very ample.
(iv) The line bundle $\mathscr{O}(\beta_n^* c_n^* \mathfrak{D}')|_{K_1}$ is very ample, hence has strictly positive degree on K_1. The degree of a closed point p with the reduced closed subscheme structure on the smooth curve K_1 is always one, as $\widetilde{k}_n$ algebraically closed, and hence the claim. □

Proposition 3.4.15
Assume that there exists an $i \in \{1, 2\}$ such that $\beta_n \circ \mathrm{pr}_{i,n}$ maps F to an irreducible

component of the special fibre of $\mathfrak{X} \otimes k_n^\circ$*. Furthermore, assume that* Z *is not a statum of the special fibre. Let* K' *be an irreducible component of the special fibre of* $\mathfrak{B}_n$*.*

Assume that either $i = 1$ *and* $K' \in H(F)$ *or* $i = 2$ *and* $K' \in V(F)$*.*

Then

$$0 \leq \deg_{\mathfrak{B}_{n,s}}(K' \cdot_{\mathfrak{B}_n} \widehat{Z'}) \leq \deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n \cdot_{\mathfrak{B}_n} Z).$$

PROOF. We only show the claim for $i = 1$ and $K' \in H(F)$ the case $i = 2$ and $K' \in V(F)$ follows by symmetry. We have

$$0 \leq \deg_{\mathfrak{B}_{n,s}}(K' \cdot_{\mathfrak{B}_n} \widehat{Z'})$$

as K' intersects each irreducible component of $\widehat{Z'}$ properly: These are given by the strict transform of Z' which equals Z and was by assumption no stratum and propably exceptional divisors of F and by Prop. 3.4.6 (i) the exceptional divisors are of the form $(L \cdot_{\mathfrak{B}_n} F)$ for F, and those intersect K' properly as $K' \in H(F)$. Note that

$$\deg_{\mathfrak{B}_{n,s}}(K' \cdot_{\mathfrak{B}_n} \widehat{Z'}) = \deg_F((F \cdot_{\mathfrak{B}_n} K') \cdot_F \widehat{Z'})$$

by Prop. 3.3.6(i).

Now F maps surjectively on a irreducible component of the special fibre of $\mathfrak{X}_n \times_{k_n^\circ} \mathfrak{X}_n$ under γ_n. This component is of the form $K_1 \times_{\widetilde{k_n}} K_2$ for two irreducible components of the special fibre of $\mathfrak{X}_n$ by Prop. 3.4.7 (i). Moreover, by the hypothesis $K' \in H(F)$, the component K' maps to a product of the form $K_1' \times_{\widetilde{k_n}} K_2$ with $K_1 \cap K_1' = \{p\}$ for a closed point p of K_1 by Prop. 3.4.7 (ii). We set

$$W = (\gamma_n|_F)^*(\gamma_n(F \cdot_{\mathfrak{B}_n} K')_{\mathrm{red}}) \in \mathrm{CH}^1(F).$$

Then

$$W = (\mathrm{pr}_{1,n}|_F)^*(\{p\})$$

holds by Prop. 3.4.7 (iii). We have

$$\deg_F((F \cdot_{\mathfrak{B}_n} K') \cdot_F \widehat{Z'}) = \deg_F(W \cdot_F \widehat{Z'})$$

by Cor. 3.4.6 (iv).

Now $\mathrm{pr}_2^* \mathfrak{D}'$ is a vertically nef divisor. Hence

$$\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n \cdot_{\mathfrak{B}_n} \widehat{Z'}) = \deg_{\mathfrak{B}_{n,s}}((\mathrm{pr}_1^* \mathfrak{D}' + \mathrm{pr}_2^* \mathfrak{D}') \cdot_{\mathfrak{B}_n} \widehat{Z'}) \geq \deg_{\mathfrak{B}_{n,s}}((\mathrm{pr}_1^* \mathfrak{D}') \cdot_{\mathfrak{B}_n} \widehat{Z'}).$$

Now by the projection formula

$$\deg_{\mathfrak{B}_{n,s}}((\mathrm{pr}_1^* \mathfrak{D}') \cdot_{\mathfrak{B}_n} \widehat{Z'}) = \deg_{\mathfrak{X}_{n,s}}(\beta_n^* c_n^* \mathfrak{D}' \cdot_{\mathfrak{X}_n} \mathrm{pr}_{1,n,*}(\widehat{Z'})).$$

Now $\mathrm{pr}_{1,n,*}(\widehat{Z'})$ is $s \cdot K_1$ for some $s \geq 0$. So

$$\deg_{\mathfrak{X}_{n,s}}(\beta_n^* c_n^* \mathfrak{D}' \cdot_{\mathfrak{X}_n} \mathrm{pr}_{1,n,*}(\widehat{Z'})) = s \cdot \deg_{\mathfrak{X}_{n,s}}(\beta_n^* c_n^* \mathfrak{D}' \cdot_{\mathfrak{X}_n} K_1)$$

and by Prop. 3.3.6(ii) this equals

$$s \cdot \deg_{K_1}((\beta_n^* c_n^* \mathfrak{D}')|_{K_1}).$$

We compute

$$\deg_F((\mathrm{pr}_{1,n}|_F)^*(\{p\}) \cdot_F \widehat{Z'}) = \deg_{K_1}(\{p\} \cdot_{K_1} (\mathrm{pr}_{1,n,*}|_F)(\widehat{Z'})) = s \cdot \deg_{K_1}(\{p\})$$

by the projection formula. We have

$$\deg_{K_1}((\beta_n^* c_n^* \mathfrak{D}')|_{K_1} - \{p\}) \geq 0$$

by Prop. 3.4.14(iv) and this yields

$$s \cdot \deg_{K_1}(((\beta_n^* c_n^* \mathfrak{D}')|_{K_1} - \{p\})) \geq 0$$

hence the desired inequality. □

Proposition 3.4.16

Let $i \in \{1,2\}$ be a number. Let K and K' be irreducible components of the special fibre of $\mathfrak{B}_n$. Assume that either $K' \in H(K)$ and $\mathrm{pr}_2(K)$ is an irreducible component of $\mathfrak{X}_s$ or $K' \in V(K)$ and $\mathrm{pr}_1(K)$ is an irreducible component of $\mathfrak{X}_s$. Then

$$\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n \cdot_{\mathfrak{B}} K \cdot_{\mathfrak{B}} K') \geq 1.$$

Proof. We only consider the case $K' \in H(K)$ the other case follows by symmetry. Recall that every irreducible component of $(\mathfrak{X}_n \times_{k_n^\circ} \mathfrak{X}_n)_s$ is of the form $L_1 \times_{\widetilde{k_n}} L_2$ for irreducible components L_j of $\mathfrak{X}_{n,s}$ by Prop. 3.4.7 (i) for $j \in \{1,2\}$.

Now γ_n maps K to $K_1 \times_{\widetilde{k_n}} K_2 \subset \mathfrak{X}_n \times_{k_n^\circ} \mathfrak{X}_n$ and maps K' to $K_1' \times_{\widetilde{k_n}} K_2$ with $K_1 \cap K_1' = \{p\}$ for a closed point p of K_1 for irreducible components K_1, K_1', K_2 of the special fibre of $\mathfrak{X}_n$ by Prop. 3.4.7 (ii). Then $\gamma_n(K \cap K') = \{p\} \times K_2$ holds so $\mathrm{pr}_{2,n}(K \cap K')$ is an irreducible component of the special fibre of $\mathfrak{X}_n$, namely K_2.

By definition of $\mathfrak{D}_n$ we have

$$\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K') = \deg_{\mathfrak{B}_{n,s}}((\mathrm{pr}_{1,n}^* \beta_n^* c_n^* \mathfrak{D}' + \mathrm{pr}_{2,n}^* \beta_n^* c_n^* \mathfrak{D}') \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K'). \quad (3.4.4)$$

Note that $\mathrm{pr}_{1,n}^* \beta_n^* c_n^* \mathfrak{D}'$ is vertically nef. Hence the right hand side of (3.4.4) is bigger or equal than

$$\deg_{\mathfrak{B}_{n,s}}((\mathrm{pr}_{2,n}^* \beta_n^* c_n^* \mathfrak{D}') \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K').$$

We use the projection formula and see that this expression equals

$$\deg_{\mathfrak{X}_{n,s}}(\beta_n^* c_n^* \mathfrak{D}' \cdot_{\mathfrak{X}_n} \mathrm{pr}_{2,n,*}(K \cdot_{\mathfrak{B}_n} K'))$$

As $\mathrm{pr}_{2,n*}(K \cdot_{\mathfrak{B}_n} K')$ is an irreducible component of the special fibre of $\mathfrak{X}_n$ we have $\mathrm{pr}_{2,n,*}(K \cdot_{\mathfrak{B}_n} K') = s \cdot K_2$ for some $s \geq 1$. Hence

$$\deg_{\mathfrak{X}_{n,s}}(\beta_n^* c_n^* \mathfrak{D}' \cdot_{\mathfrak{X}_n} \mathrm{pr}_{2,n,*}(K \cdot_{\mathfrak{B}_n} K')) = s \cdot \deg_{\mathfrak{X}_{n,s}}(\beta_n^* c_n^* \mathfrak{D}' \cdot_{\mathfrak{X}_n} K_2)$$

and this equals

$$s \cdot \deg_{K_2}(\beta_n^* c_n^* \mathfrak{D}'|_{K_2}) \quad (3.4.5)$$

by Prop. 3.3.6(ii). We can apply Prop. 3.4.14(iii) as $\mathrm{pr}_2(K)$ is an irreducible component of the special fibre of $\mathfrak{X}$, and see that the line bundle $\mathscr{O}(\beta_n^* c_n^* \mathfrak{D}')|_{K_2}$ is very ample. Hence $\deg_{K_2}(\beta_n^* c_n^* \mathfrak{D}'|_{K_2})$ is ≥ 1, which means that (3.4.5) is ≥ 1. This proves our claim. □

Proposition 3.4.17

Let F be a component of the special fibre of $\mathfrak{B}_n$.

(i) If $\mathrm{pr}_1(F)$ is a point then $H(F)$ consists of two irreducible components contained in the same canonical chart of $R(\mathfrak{B})$ as F,
(ii) If $\mathrm{pr}_2(F)$ is a point then $V(F)$ consists of two irreducible components contained in the same canonical chart of $R(\mathfrak{B})$ as F.

Proof. We treat only the first case, the second follows by symmetry. Let F map on $K_1 \times_{\widetilde{k}} K_2$ under γ_n, where K_1 and K_2 are irreducible components of $\mathfrak{X}_n$. Then K_1 maps to a point under the map $\mathfrak{X}_n \to \mathfrak{X}$. Using the description of the special fibre of $\mathfrak{X}_n$ in [**DM69**, Lemma 1.12] this means that $\mathrm{pr}_{1,n}(F)$ corresponds to an element of a chain of rational curves connecting two strict transforms C_1, C_2 of irreducible components of $\mathfrak{X}_s$. This means there are exactly two irreducible components L, L' of $\mathfrak{X}_{n,s}$ which intersect K_1 and that the point corresponding to the irreducible component $\mathrm{pr}_{1,n}(F)$ is in the interior of the line connecting the points corresponding to C_1 and C_2. But this means that $H(F)$ consists

FIGURE 3.4.2. An edge of the reduction set of $\mathfrak{X}_n$. The thick irreducible components map to a component of $\mathfrak{X}_s$, the others don't.

of the two irreducible components of $\mathfrak{B}_{n,s}$ mapping to $L \times K_2$ and $L' \times K_2$ respectively under γ_n. Furthermore, these components are contained in the canonical chart of $R(\mathfrak{B})$ containing the two components mapping to $C_1 \times_{\widetilde{k_n}} K_2$ and $C_2 \times_{\widetilde{k_n}} K_2$ respectively. ☐

Lemma 3.4.18
Assume that

$$g = \sum_{A \in \mathfrak{B}_{n,s}^{(0)}} g(A) \cdot A$$

is a divisor with support in the special fibre of $\mathfrak{B}_n$. In virtue of Prop. B.16 consider g as a function on $|R(\mathfrak{B}_n)|$. We canonically identify each square of $|R(\mathfrak{B}_n)|$ with $[0,1]^2$. Let $L \in \mathbb{Q}$ be a global Lipschitz constant of g. Then we can establish the following estimates.

(i) Assume that F surjects onto an irreducible component of $\mathfrak{X}_s$ under the map $\mathrm{pr}_1 \colon \mathfrak{B}_{n,s} \to \mathfrak{X}$. If $L \in \mathbb{Q}$ is a global Lipschitz constant of g then

$$\left| \deg_{\mathfrak{B}_{n,s}} \left(\frac{n}{2} \sum_{K \in H(F)} (g(K) - g(F)) K \cdot_{\mathfrak{B}_n} Z \right) \right| \leq$$
$$L \cdot \mathrm{val}(R(\mathfrak{X})) \cdot (\mathrm{val}(R(\mathfrak{X}))^2 + 1) \deg_{\mathfrak{B}_{n,s}} (\mathfrak{D}_n \cdot_{\mathfrak{B}_n} Z).$$

holds.

(ii) Assume that F surjects to an irreducible component of $\mathfrak{X}_s$ under the map $\mathrm{pr}_2 \colon \mathfrak{B}_{n,s} \to \mathfrak{X}$. If $L \in \mathbb{Q}$ is a global Lipschitz constant of g then

$$\frac{n}{2} \left| \deg_{\mathfrak{B}_{n,s}} \left(\sum_{K \in V(F)} (g(K) - g(F)) K \cdot_{\mathfrak{B}_n} Z \right) \right| \leq$$
$$L \cdot \mathrm{val}(R(\mathfrak{X})) \cdot (\mathrm{val}(R(\mathfrak{X}))^2 + 1) \deg_{\mathfrak{B}_{n,s}} (\mathfrak{D}_n \cdot_{\mathfrak{B}_n} Z).$$

(iii) Assume that F surjects on an irreducible component of $\mathfrak{X}_s$ under one of the maps $\mathrm{pr}_i \colon \mathfrak{B}_{n,s} \to \mathfrak{X}$ for an $i \in \{1, 2\}$. If $L \in \mathbb{Q}$ is a global Lipschitz constant of g then

$$\frac{n}{2} \left| \sum_{K \in D(F)} |g(K) - g(F)| K \cdot_{\mathfrak{B}_n} Z \right| \leq$$
$$2 \cdot L \cdot \mathrm{val}(R(\mathfrak{X}))^2 \cdot (\mathrm{val}(R(\mathfrak{X}))^2 + 1) \deg_{\mathfrak{B}_{n,s}} (\mathfrak{D}_n \cdot_{\mathfrak{B}_n} Z).$$

PROOF. We start with (i): We apply Prop. 3.4.11 (ii) and see for every $K \in H(F)$ we have

$$|\deg_{\mathfrak{B}_{n,s}} (K \cdot_{\mathfrak{B}_n} Z)| \leq \deg_{\mathfrak{B}_{n,s}} (K \cdot_{\mathfrak{B}_n} \widehat{Z'}) \cdot (\mathrm{val}(R(\mathfrak{X}))^2 + 1), \tag{3.4.6}$$

and furthermore by Prop. 3.4.15

$$|\deg_{\mathfrak{B}_{n,s}} (K \cdot_{\mathfrak{B}_n} \widehat{Z'})| \leq \deg_{\mathfrak{B}_{n,s}} (\mathfrak{D}_n \cdot_{\mathfrak{B}_n} Z). \tag{3.4.7}$$

Now let $L \in \mathbb{Q}$ be a global Lipschitz constant of g, then

$$\left|\deg_{\mathfrak{B}_{n,s}}\left(\frac{n}{2}\sum_{K\in H(F)}(g(K)-g(F))K\cdot_{\mathfrak{B}_n} Z\right)\right| \leq L\cdot \sum_{K\in H(F)} |\deg_{\mathfrak{B}_{n,s}}(K\cdot_{\mathfrak{B}_n} Z)|,$$

and by (3.4.6) and (3.4.7) this is bounded from above by

$$L\cdot \mathrm{val}(R(\mathfrak{X}))\cdot(\mathrm{val}(R(\mathfrak{X}))^2+1)\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n\cdot Z)$$

using Rem. 3.3.9 (i). The case of (ii) follows by symmetry.

Now we prove (iii). By symmetry of the argument we may assume that pr_1 maps F to an irreducible component of $\mathfrak{X}_s$. Then using Prop. 3.4.11 (i) and Prop. 3.4.15 we see that for every $K' \in H(F)$ and $K \in D(F)$

$$0 \leq \deg_{\mathfrak{B}_{n,s}}(K\cdot Z) \leq \deg_{\mathfrak{B}_{n,s}}(K'\cdot_{\mathfrak{B}_n}\widehat{Z}') \leq \deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n\cdot Z).$$

Hence using Rem. 3.3.9 (i) we get the desired estimate

$$\frac{n}{2}\left|\sum_{K\in D(F)}|g(K)-g(F)|K\cdot_{\mathfrak{B}_n} Z\right| \leq$$
$$2\cdot L\cdot \mathrm{val}(R(\mathfrak{X}))^2\cdot(\mathrm{val}(R(\mathfrak{X}))^2+1)\deg(\mathfrak{D}_n\cdot Z)$$

as before using that the diagonal of $[0,1]^2$ has length $\sqrt{2}\leq 2$. □

3.5. Berkovich Skeleta of Gross–Schoen Models

In this section we will investigate skeleta of Gross–Schoen/Kolb models and how they behave under base change. We will be in the situation of Assumption 2.2.1.

Let $\mathfrak{X}$ be a regular strictly semi-stable curve over k° and $n \geq 1$ be a natural number. Equip the set of irreducible components of $\mathfrak{X}_s$ with an order. In this section we will not assume that IC (Def. 3.3.5(ii)) holds for $\mathfrak{X}$. Denote by $\mathfrak{X}_n$ the model from Thm. 3.3.2 corresponding to the n-th BHM-subdivision of $R(\mathfrak{X})$ and by $\mathfrak{B}_n$ the model of $(X\otimes k_n)\times_{k_n}(X\otimes k_n)$ constructed from $\mathfrak{X}_n$ using Thm. 3.3.3.

In the following we denote for any strictly poly-stable scheme $\mathfrak{W}$ over $\mathbb{K}^\circ$ by $S(\mathfrak{W})$ the skeleton of $\widehat{\mathfrak{W}}^{\mathrm{an}}$ associated to $\mathfrak{W}$ and by $\tau_{\mathfrak{W}}$ the retraction to $S(\mathfrak{W})$ (cf. App. B). Our aim is to prove the following proposition:

Proposition 3.5.1
There exists a k°-morphism $\varphi\colon \mathfrak{B}_n \to \mathfrak{B}$ such that

(i) The diagram

$$\begin{array}{ccc} \mathfrak{B} & \longrightarrow & \mathfrak{X}\times_{k^\circ}\mathfrak{X} \\ {\scriptstyle\varphi}\uparrow & & \uparrow \\ \mathfrak{B}_n & \longrightarrow & \mathfrak{X}_n\times_{k_n^\circ}\mathfrak{X}_n \end{array} \tag{3.5.1}$$

commutes, where $\mathfrak{X}_n\times_{k_n^\circ}\mathfrak{X}_n \to \mathfrak{X}\times_{k^\circ}\mathfrak{X}$ is the product map induced by the map $\mathfrak{X}_n\to\mathfrak{X}$.

(ii) The map $\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}$ maps the skeleton of $\widehat{\mathfrak{B}_{n,\mathbb{K}^\circ}}^{\mathrm{an}}$ associated to $\mathfrak{B}_{n,\mathbb{K}^\circ}$ into the skeleton of $\widehat{\mathfrak{B}_{\mathbb{K}^\circ}}^{\mathrm{an}}$ associated to $\mathfrak{B}_{\mathbb{K}^\circ}$.

(iii) The diagram

$$\begin{array}{ccc}
\widehat{\mathfrak{B}_{n,\mathbb{K}^\circ}}^{\mathrm{an}} & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}} & \widehat{\mathfrak{B}_{\mathbb{K}^\circ}}^{\mathrm{an}} \\
\Big\downarrow{\scriptstyle \tau_{\mathfrak{B}_{n,\mathbb{K}^\circ}}} & & \Big\downarrow{\scriptstyle \tau_{\mathfrak{B}_{\mathbb{K}^\circ}}} \\
S(\mathfrak{B}_{n,\mathbb{K}^\circ}) & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathfrak{B}_{n,\mathbb{K}^\circ})}} & S(\mathfrak{B}_{\mathbb{K}^\circ}) \\
\Big\downarrow{\scriptstyle \mathrm{Thm.}\, B.6\ (v)} & & \Big\downarrow{\scriptstyle \mathrm{Thm.}\, B.6\ (v)} \\
|R(\mathfrak{B}_{n,\mathbb{K}^\circ})| & \xrightarrow[(A.5)]{} & |R(\mathfrak{B}_{\mathbb{K}^\circ})|
\end{array}$$

commutes, the lower arrow is the isomorphism (A.5) induced by the subdivision, and the restriction of $\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}$ to the skeleton is an isomorphism.

We will need some preliminary results in order to prove Prop. 3.5.1.

Lemma 3.5.2
Let φ be the canonical morphism $\mathfrak{X}_n \to \mathfrak{X}$. Then $\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}$ maps the skeleton of $\widehat{\mathfrak{X}_{n,\mathbb{K}^\circ}}^{\mathrm{an}}$ associated to $\mathfrak{X}_{n,\mathbb{K}^\circ}$ to the skeleton of $\widehat{\mathfrak{X}_{\mathbb{K}^\circ}}^{\mathrm{an}}$ associated to $\mathfrak{X}_{\mathbb{K}^\circ}$ and the diagram

$$\begin{array}{ccc}
\widehat{\mathfrak{X}_{n,\mathbb{K}^\circ}}^{\mathrm{an}} & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}} & \widehat{\mathfrak{X}_{\mathbb{K}^\circ}}^{\mathrm{an}} \\
\Big\downarrow{\scriptstyle \tau_{\mathfrak{X}_{n,\mathbb{K}^\circ}}} & & \Big\downarrow{\scriptstyle \tau_{\mathfrak{X}_{\mathbb{K}^\circ}}} \\
S(\mathfrak{X}_{n,\mathbb{K}^\circ}) & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathfrak{X}_{n,\mathbb{K}^\circ})}} & S(\mathfrak{X}_{\mathbb{K}^\circ}) \\
\Big\downarrow{\scriptstyle \mathrm{Thm.}\, B.6\ (v)} & & \Big\downarrow{\scriptstyle \mathrm{Thm.}\, B.6\ (v)} \\
|R(\mathfrak{X}_{n,\mathbb{K}^\circ})| & \xrightarrow[(A.5)]{} & |R(\mathfrak{X}_{\mathbb{K}^\circ})|
\end{array}$$

commutes. Moreover, the lower arrow is the isomorphism from (A.5).

Proof. Set $\mathfrak{S}' = \operatorname{Spec} k_n^\circ[x,y]/(xy - \pi_n^n)$ and $\mathfrak{S} = \mathfrak{S}' \otimes \mathbb{K}^\circ$. Let $\mathfrak{S}'_n$ be the scheme associated to $\mathfrak{S}'$ from Thm. 3.3.2 and set $\mathfrak{S}_n = \mathfrak{S}'_n \otimes \mathbb{K}^\circ$. We will use the explict description of the desingularisation in the proof of [**Kol13**, Lemma 2.7] to show that there is a commutative diagram

$$\begin{array}{ccc}
\widehat{\mathfrak{S}_n}^{\mathrm{an}} & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}} & \widehat{\mathfrak{S}}^{\mathrm{an}} \\
\Big\downarrow{\scriptstyle \tau_{\mathfrak{S}_n}} & & \Big\downarrow{\scriptstyle \tau_{\mathfrak{S}}} \\
S(\mathfrak{S}_n) & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathfrak{S}_n)}} & S(\mathfrak{S}) \\
\Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \psi} \\
|R(\mathfrak{S}_n)| & \xrightarrow[\xi]{} & |R(\mathfrak{S})|
\end{array}, \qquad (3.5.2)$$

where μ and ψ are the isomorphisms from Thm. B.6 (v) and φ is the natural morphism $\mathfrak{S}_n \to \mathfrak{S}$. We claim that ξ is the natural homeomorphism from the n-fold subdivision (cf. Prop. A.13). We set $A = \mathbb{K}^\circ[x,y]/(xy - \pi_n^n)$. By the proof of [**Kol13**, Lemma 2.7] we can cover M by charts $\mathfrak{U}_i = \operatorname{Spec} A_i$ for $i \in \{1, \ldots, n\}$, where $A_i = \mathbb{K}^\circ[x,y]/(xy - \pi_n)$ and the morphism $\mathfrak{U}_i \to \mathfrak{S}$ is induced by

$$x \mapsto x\pi_n^{n-i}$$
$$y \mapsto y\pi_n^{i-1}.$$

Denote by $\widehat{A}$ and $\widehat{A_i}$ the $\mathbb{K}^{\circ\circ}$-adic completions of A and A_i respectively. Then the diagram

$$\begin{CD} \mathscr{M}(\widehat{A_i} \otimes_{\mathbb{K}^\circ} \mathbb{K}) @>>> \mathscr{M}(\widehat{A} \otimes_{\mathbb{K}^\circ} \mathbb{K}) \\ @V{n\cdot(-\log|x|,-\log|y|)}VV @VV{(-\log|x|,-\log|y|)}V \\ |R(\mathrm{Spec}(A_i))| @>>{\xi}> |R(\mathrm{Spec}(A))| \end{CD} \tag{3.5.3}$$

commutes, where ξ is the map

$$\begin{aligned} \{x, y \in [0,1] | x + y = 1\} \to \{x, y \in [0,1] | x + y = 1\} : \\ (x, y) \mapsto \frac{1}{n}(x, y) + \left(\frac{n-i}{n}, \frac{i-1}{n}\right). \end{aligned} \tag{3.5.4}$$

We can consider the commutative diagram

$$\begin{CD} S(\mathrm{Spec}(A_i)) @>{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathrm{Spec}(A_i))}}>> \mathscr{M}(\widehat{A} \otimes \mathbb{K}) \\ @VVV @VV{\psi}V \\ |R(\mathrm{Spec}(A_i))| @>>> |R(\mathrm{Spec}(A))|. \end{CD}$$

Now the restriction of the isomorphism ψ between the skeleton and the reduction set from Thm. B.6 (v) to the skeleton is an isomorphism, in particular injective. Hence, by commutativity of the diagram, the image of $\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathrm{Spec}(A_i))}$ lies in the skeleton of $\mathscr{M}(\widehat{A} \otimes_{\mathbb{K}^\circ} \mathbb{K})$ associated to $\mathrm{Spec}(A)$. As the retraction maps as well as the isomorphisms between the skeleton and the reduction set commute with the colimit over all i, we get the commutative diagram (3.5.2). From the explicit description in (3.5.4) we see that the lower arrow is indeed the homeomorphism coming from the BHM-n-subdivision, as the homeomorphism commutes with colimits and hence is determined on the simplices of $R(\mathfrak{S}'_n)$.

Now let $\mathfrak{X}$ be an arbitrary proper regular strictly semi-stable curve and $\mathfrak{U}$ open in $\mathfrak{X}$. If $\mathfrak{U}$ is smooth over $\mathbb{K}^\circ$ then the statement is trivial because the special fibre of $\mathfrak{U}$ does not contain double points. Otherwise we can choose $\mathfrak{U}$ which is étale over $\mathfrak{S}'$, elementary, and such that the map $\mathfrak{U} \to \mathfrak{S}'$ induces an isomorphism of reduction sets by Prop. B.3. As the retractions and the isomorphism between the skeleton and the reduction set, as well as the homeomorphism from Prop. A.13 commute with direct limits, we can assume $\mathfrak{X} = \mathfrak{U}$. Let $\mathfrak{U}'$ be the scheme from Thm. 3.3.2 associated to $\mathfrak{U}$ and $\theta \colon \mathfrak{U}' \to \mathfrak{U}$ the natural morphism. We get a cartesian diagram

$$\begin{CD} \mathfrak{U}'_{\mathbb{K}^\circ} @>{\theta_{\mathbb{K}^\circ}}>> \mathfrak{U}_{\mathbb{K}^\circ} \\ @V{\alpha}VV @VV{\beta}V \\ \mathfrak{S}_n @>>{\chi}> \mathfrak{S}. \end{CD}$$

This gives a diagram

$$\begin{array}{ccccccc}
S(\mathfrak{U}'_{\mathbb{K}^\circ}) & \xleftarrow{\tau_{\mathfrak{U}'_{\mathbb{K}^\circ}}} & \widehat{\mathfrak{U}'}^{\mathrm{an}}_{\mathbb{K}^\circ} & \xrightarrow{\widehat{\theta}^{\mathrm{an}}_{\mathbb{K}^\circ}} & \widehat{\mathfrak{U}}^{\mathrm{an}}_{\mathbb{K}^\circ} & \xrightarrow{\tau_{\mathfrak{U}_{\mathbb{K}^\circ}}} & S(\mathfrak{U}_{\mathbb{K}^\circ}) \\
 & & \downarrow{\scriptstyle \widehat{\alpha}^{\mathrm{an}}_{\mathbb{K}^\circ}} & & \downarrow{\scriptstyle \widehat{\beta}^{\mathrm{an}}_{\mathbb{K}^\circ}} & & \\
{\scriptstyle \widehat{\alpha}^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathfrak{U}'_{\mathbb{K}^\circ})}}\downarrow & & \widehat{\mathfrak{S}_n}^{\mathrm{an}} & \xrightarrow[\widehat{\chi}^{\mathrm{an}}_{\mathbb{K}^\circ}]{} & \widehat{\mathfrak{S}}^{\mathrm{an}} & & \downarrow{\scriptstyle \widehat{\beta}^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathfrak{U}_{\mathbb{K}^\circ})}} \\
 & \swarrow{\scriptstyle \tau_{\mathfrak{S}_n}} & & & & \searrow{\scriptstyle \tau_{\mathfrak{S}}} & \\
S(\mathfrak{S}_n) & & & \xrightarrow[\chi^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathfrak{S}_n)}]{} & & & S(\mathfrak{S}).
\end{array}$$

The diagram is indeed commutative by Thm. B.6 (ii). Diagram chasing shows that $\theta^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathfrak{U}'_{\mathbb{K}^\circ})}$ is an isomorphism onto $S(\mathfrak{U}_{\mathbb{K}^\circ})$. The induced isomorphism between $|R(\mathfrak{U}'_{\mathbb{K}^\circ})|$ and $|R(\mathfrak{U}_{\mathbb{K}^\circ})|$ is the homeomorphism from the n-subdivision which again follows from the description of this homeomorphism on 1-simplices, so we have finished the proof. $\square$

Lemma 3.5.3
We have a commutative diagram

$$\begin{array}{ccc}
\widehat{\mathfrak{B}}^{\mathrm{an}}_{\mathbb{K}^\circ} & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}} & {(\widehat{\mathfrak{X} \times_{k^\circ} \mathfrak{X}})^{\mathrm{an}}}_{\mathbb{K}^\circ} \\
{\scriptstyle \tau_{\mathfrak{B}_{\mathbb{K}^\circ}}}\downarrow & & \downarrow{\scriptstyle \tau_{(\mathfrak{X} \times_{k^\circ} \mathfrak{X})_{\mathbb{K}^\circ}}} \\
S(\mathfrak{B}_{\mathbb{K}^\circ}) & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}|_{S(\mathfrak{B}_{\mathbb{K}^\circ})}} & S((\mathfrak{X} \times_{k^\circ} \mathfrak{X})_{\mathbb{K}^\circ}) \\
\downarrow & & \downarrow \\
|R(\mathfrak{B}_{\mathbb{K}^\circ})| & \xrightarrow{\psi} & |R((\mathfrak{X} \times_{k^\circ} \mathfrak{X})_{\mathbb{K}^\circ})|,
\end{array}$$

where φ is the natural morphism $\mathfrak{B} \to \mathfrak{X} \times_{k^\circ} \mathfrak{X}$ and ψ is the natural homeomorphism $|R(\mathfrak{B})| = |R(\mathfrak{X}) \times R(\mathfrak{X})| \to |R(\mathfrak{X} \times_{k^\circ} \mathfrak{X})| = |R(\mathfrak{X})| \times |R(\mathfrak{X})|$.

Proof. Noting that the case where $\mathfrak{X}$ is smooth is trivial, we reduce to the case $\mathfrak{X} = \operatorname{Spec} \mathbb{K}^\circ[x, y]/(xy - \pi)$ using the same arguments as in the proof of Lemma 3.5.2. So consider

$$A := \mathbb{K}^\circ[x_1, y_1, x_2, y_2]/(x_1 y_1 - \pi, x_2 y_2 - \pi),$$

i.e. $\operatorname{Spec} A = \mathfrak{X}_{\mathbb{K}^\circ} \times_{\mathbb{K}^\circ} \mathfrak{X}_{\mathbb{K}^\circ}$.

As in the proof of Prop. 3.4.4 the blow-up $\mathfrak{B}'$ of $\operatorname{Spec} A$ at the subscheme $V(x_1, y_1)$ can be covered by $\operatorname{Spec} A_i$ where

$$A_i = \mathbb{K}^\circ[x, y, z]/(xyz - \pi)$$

for $i \in \{1, 2\}$ and $\operatorname{Spec} A_i \to \operatorname{Spec} A$ is induced by

$$x_i \mapsto yx, \qquad y_i \mapsto z, \qquad x_j \mapsto x, \qquad y_j \mapsto yz,$$

where $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$. As in the proof of Prop. 3.4.4 we see that $\mathfrak{B}'$ is already the base-change to $\mathbb{K}^\circ$ of the scheme from Thm. 3.3.3 associated to $\mathfrak{X}$ for $d = 2$.

Let $\widehat{A_i}$ and $\widehat{A}$ be the $\mathbb{K}^{\circ\circ}$-adic completions of A_i and A respectively. Consider the natural commutative diagram

$$\begin{array}{ccc} \mathscr{M}(\widehat{A_i} \otimes_{\mathbb{K}^\circ} \mathbb{K}) & \longrightarrow & \mathscr{M}(\widehat{A} \otimes_{\mathbb{K}^\circ} \mathbb{K}) \\ \downarrow & & \downarrow \\ |R(\mathrm{Spec}(A_i))| & \xrightarrow[\psi]{} & |R(\mathrm{Spec}(A))|. \end{array}$$

where ψ is the map

$$\left((x,y,z) \mapsto \begin{cases} (y+x, z, x, y+z) & \text{if } i=1 \\ (x, y+z, y+x, z) & \text{if } i=2 \end{cases} \right),$$

from

$$|\Delta_2| = \{(x,y,z) \in [0,1]^3 | x+y+z = 1\}$$

to

$$|R(\mathrm{Spec}\, A)| = |\Delta_1| \times |\Delta_1| = \{(x,y,z,w) \in [0,1]^4 | x+y=1, z+w=1\}.$$

So $|R(\mathrm{Spec}\, A_i)|$ gets mapped to the lower left and upper right triangle of the square $|R(\mathrm{Spec}\, A)|$ for $i=1$ and $i=2$ respectively and the claim follows. □

Proof. (of Prop. 3.5.1) First we note from [**Kol16a**, Theorem 2.18] that there exists a morphism such that (3.5.1) commutes. Repeatedly applying 3.5.2 and Lemma 3.5.3 yields the following commutative diagram.

$$\begin{array}{ccccccc} \widehat{\mathfrak{B}_{n,\mathbb{K}^\circ}}^{\mathrm{an}} & & \longrightarrow & & & & \widehat{\mathfrak{B}_{\mathbb{K}^\circ}}^{\mathrm{an}} \\ \downarrow & \searrow & & & & \swarrow & \downarrow \\ & & \widehat{\mathfrak{X}_{n,\mathbb{K}^\circ} \times_{\mathbb{K}^\circ} \mathfrak{X}_{n,\mathbb{K}^\circ}}^{\mathrm{an}} & \longrightarrow & \widehat{\mathfrak{X}_{\mathbb{K}^\circ} \times_{\mathbb{K}^\circ} \mathfrak{X}_{\mathbb{K}^\circ}}^{\mathrm{an}} & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S(\mathfrak{B}_{n,\mathbb{K}^\circ}) & \longrightarrow & S(\mathfrak{X}_{n,\mathbb{K}^\circ} \times_{\mathbb{K}^\circ} \mathfrak{X}_{n,\mathbb{K}^\circ}) & \longrightarrow & S(\mathfrak{X}_{\mathbb{K}^\circ} \times_{\mathbb{K}^\circ} \mathfrak{X}_{\mathbb{K}^\circ}) & \longleftarrow & S(\mathfrak{B}_{\mathbb{K}^\circ}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ |R(\mathfrak{B}_{n,\mathbb{K}^\circ}| & \longrightarrow & |R(\mathfrak{X}_{n,\mathbb{K}^\circ} \times_{\mathbb{K}^\circ} \mathfrak{X}_{n,\mathbb{K}^\circ})| & \longrightarrow & |R(\mathfrak{X}_{\mathbb{K}^\circ} \times_{\mathbb{K}^\circ} X_{\mathbb{K}^\circ})| & \longleftarrow & |R(\mathfrak{B}_{\mathbb{K}^\circ}| \end{array}$$

As the homeomorphism coming from the BHM-n subdivision commutes with finite products and direct limits, the morphism $|R(\mathfrak{B}_n)| \to |R(\mathfrak{B})|$ is indeed the homeomorphism from the subdivision. This proves the proposition. □

Let $\mathfrak{X}'$ be the model over k_2° corresponding to the S-subdivision of $R(\mathfrak{X})$ and $\mathfrak{B}'$ the product model corresponding to $\mathfrak{X}'$.

Proposition 3.5.4
There exists a morphism of k°-schemes $\varphi\colon \mathfrak{B}' \to \mathfrak{B}$ such that

(i) The diagram

$$\begin{array}{ccc} \mathfrak{B} & \longrightarrow & \mathfrak{X} \times_{k^\circ} \mathfrak{X} \\ \uparrow & & \uparrow \\ \mathfrak{B}' & \longrightarrow & \mathfrak{X}' \times_{k_2^\circ} \mathfrak{X}' \end{array}$$

commutes.

(ii) The restriction of $\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}^\circ}$ to the skeleton of $\widehat{\mathfrak{B}'_{\mathbb{K}^\circ}}^{\mathrm{an}}$ associated to $\mathfrak{B}'_{\mathbb{K}^\circ}$ maps into the skeleton of $\widehat{\mathfrak{B}_{\mathbb{K}^\circ}}^{\mathrm{an}}$ associated to $\mathfrak{B}_{\mathbb{K}^\circ}$. The diagram

$$\begin{array}{ccc}
\widehat{\mathfrak{B}}'^{\,\mathrm{an}}_{\mathbb{K}^\circ} & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}}} & \widehat{\mathfrak{B}}^{\mathrm{an}}_{\mathbb{K}^\circ} \\
\downarrow \tau & & \downarrow \tau \\
S(\mathfrak{B}'_{\mathbb{K}^\circ}) & \xrightarrow{\widehat{\varphi}^{\mathrm{an}}_{\mathbb{K}}|_{S(\mathfrak{B}'_{\mathbb{K}^\circ})}} & S(\mathfrak{B}_{\mathbb{K}^\circ}) \\
\downarrow & & \downarrow \\
|R(\mathfrak{B}'_{\mathbb{K}^\circ})| & \xrightarrow{(A.4)} & |R(\mathfrak{B}_{\mathbb{K}^\circ})|
\end{array}$$

commutes and the lower arrow is the isomorphism from Prop. A.10.

Proof. Once the existence of the map φ is established the proof goes exactly as the proof of Prop. 3.5.1.

In the proof of the existence of φ in [**Kol16a**, Theorem 2.18] all arguments used for the BHM-subdivision also work for the S-subdivision. Hence the statement of Prop. 3.5.1 also holds for S-subdivisions. □

3.6. The Case of a Curve

Before passing to the proof of our approximation theorem we want to discuss the case of a curve briefly to illustrate the ideas of the proof. All results about curves will later follow from the case of a self product of a curve (cf. Cor. 3.7.6 and Cor. 3.7.7), so the presentation will be very brief. We will be in the situation of Assumption 2.2.1. Let X be a geometrically integral smooth projective curve over the field k. We assume that X possesses a projective regular strictly semi-stable model $\mathfrak{X}$ over the valuation ring k°. For simplicity we restrict to the case

$$\mathfrak{X} = \operatorname{Proj} k^\circ[x, y, t]/(xy - t^2\pi),$$

that is $X_{\mathbb{K}} = \mathbb{P}^1_{\mathbb{K}}$ but the arguments are valid for the general case.

Let $|R(\mathfrak{X})| = [0, 1]$ be the geometric realisation of the reduction set of $\mathfrak{X}$ and $\tau : X^{\mathrm{an}}_{\mathbb{K}} \to |R(\mathfrak{X})|$ be the retraction map determined by $\mathfrak{X}$. Then any continuous function $g : |R(\mathfrak{X})| \to \mathbb{R}$ defines a Green's function $g \circ \tau$ for the trivial divisor. We are interested in the question when $g \circ \tau$ is DSP.

Firstly if g is piecewise affine with rational values at points in $\mathbb{Q} \cap [0, 1]$ it is DSP by algebraic geometry: We can choose $n \in \mathbb{N}$, such that all points where g changes slope are multiples of $\frac{1}{n}$. Now the reduction set of $\mathfrak{X}_n$ as constructed in Thm. 3.3.2 is the n-fold BHM-subdivision of $R(\mathfrak{X})$. Hence $\frac{n}{2} \cdot g$ is affine on the edges of $R(\mathfrak{X}_n)$, hence belongs to a Cartier divisor $\mathfrak{D}$ on $\mathfrak{X}_n$ with support in the special fibre by Prop. B.16. Moreover, it satisfies

$$g_{\mathfrak{X}_n, \mathfrak{D}} = g \circ \tau$$

by Prop. 3.2.1 and Rem. 3.3.1. Now, as $\mathfrak{X}$ is projective, there is an ample line bundle $\mathfrak{A}$ on $\mathfrak{X}_n$ such that $\mathfrak{A} + \mathfrak{D}$ is ample. Hence $g \circ \tau = g_{\mathfrak{X}_n, \mathfrak{A}+\mathfrak{D}} - g_{\mathfrak{X}_n, \mathfrak{A}}$ is a DSP Green's function for the trivial Cartier divisor.

It is interesting how to construct explicit examples of functions g which are DSP but not piecewise affine. The following construction is in a sense, a geometric approach to this question. Choose a very ample integral divisor $\mathfrak{D}$ on $\mathfrak{X}$. The special fibre of $\mathfrak{X}_{2^n}$ has $2^n + 1$

components and the reduction set looks as follows: Here we numbered the components of

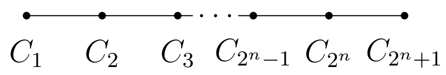


FIGURE 3.6.1. The reduction set of $\mathfrak{X}_{2^n}$

the special fibre $C_1, \ldots, C_{2^n+1}$ as indicated. Note the following lemma.

Lemma 3.6.1
The scheme $\mathfrak{X}_{2^n}$ is the scheme obtained from $\mathfrak{X}_{2^{n-1}}$ in virtue of Thm. 3.3.2.

PROOF. This can be immediately checked explicitly in charts. □

In virtue of the lemma we can recursively define $\mathfrak{D}_{2^n} = 2^n \mathfrak{D}_{2^{n-1}} - E$ where E is the exceptional divisor of the blow-up $\mathfrak{X}_{2^n} \to \mathfrak{X}_{2^{n-1}}$ and $\mathfrak{D}_{2^0} = \mathfrak{D}$. Then Lemma C.7 implies that $\mathfrak{D}_{2^n}$ is very ample for every $n \geq 1$.

Now fix $n \geq 1$. We define

$$g_n := g_{\mathfrak{X}_{2^n}, 2^{-\frac{1}{2}(n^2+n)}\mathfrak{D}_{2^n}}$$

and

$$\widetilde{h}_n := g_n - g_{\mathfrak{X},\mathfrak{D}}.$$

Unwinding the definition of $\mathfrak{D}_{2^n}$ and using Gauß' summation formula one sees that

$$(\mathfrak{D}_{2^n})_\eta = 2^{\frac{1}{2}(n+1)n}(\mathfrak{D})_\eta$$

and hence that $\widetilde{h}_n$ is trivial on the generic fibre.

By Prop. B.16 and Prop. 3.2.1 the function $\widetilde{h}_n$ factors through the skeleton of $\mathfrak{X}_{2^n}$ as $\widetilde{h}_n = h_n \circ \tau$, where $\tau \colon X_{\mathbb{K}}^{\mathrm{an}} \to S(\mathfrak{X}_{2^n})$ is the retraction and h_n is a piecewise affine function on $|R(\mathfrak{X}_n)|$.

Now consider the function $f(x) = 2x^2 - 2x$ on $[0,1]$. We compute the difference of the 2^n-th lattice approximation and the 2^{n-1}-th lattice approximation at every point $x_i = \frac{2i+1}{2^n} \in [0,1]$ for $i = 0, \ldots 2^n - 2$. Set $y_i = \frac{2i}{2^n} \in [0,1]$ for $i = 0, \ldots, 2^{n-1}$.

We have

$$\frac{1}{2}\left(f(y_i) + f(y_{i+1})\right) - f(x_i) = 2^{-2n} \tag{3.6.1}$$

for $i = 0, \ldots, 2^n - 2$. So the two lattice approximations agree on the points $\frac{2i}{2^n}$, and differ by 2^{-2n} on the points $\frac{2i+1}{2^n}$. We see that

$$\mathfrak{D}_{2^n} - (2^n)\mathfrak{D}_{2^{n-1}} = \sum_{i=1}^{2^{n-1}} -C_{2i-1},$$

so

$$h_n - 2h_{n-1} = \sum_{i=1}^{2^{n-1}} -2^n C_{2i-1},$$

hence

$$2^{-n}h_n - 2^{-(n-1)}h_{n-1} = \sum_{i=1}^{2^{n-1}} -2^{2n} C_{2i-1}. \tag{3.6.2}$$

We conclude from (3.6.1) and (3.6.2) that under the correspondence of Prop. B.16 each $2^{-n}h_n$ is the piecewise affine function which agrees with f on the vertices of $R(\mathfrak{X}_{2^n})$ respectively. This is illustrated in the following picture:

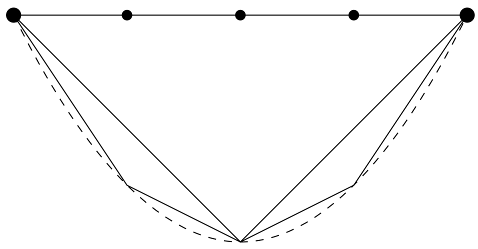


FIGURE 3.6.2. The Cartier divisors $2 \cdot h_2$ and $\frac{1}{2}h_4$ considered as functions on the reduction set.

This means that $f \circ \tau$ is a DSP arithmetic divisor.

The convexity of the limit function f is not a coincident and this is manifest in the following theorem:

Theorem 3.6.2
Set $\mathfrak{X} = \operatorname{Proj} k^\circ[x,y,t]/(xy - t^2\pi)$ and for each $n \geq 1$ let $\mathfrak{X}_n$ be the model of Thm. 3.3.2. Let $g\colon |R(\mathfrak{X})| = [0,1] \to \mathbb{R}$ be a function which is convex, Lipschitz continuous, and attains rational values at rational points. Let g_n be the divisor on $\mathfrak{X}_n$ which corresponds to the function g_n which satisfies $g(p) = \frac{n}{2}g_n(p)$ on any vertex p of $R(\mathfrak{X}_n)$ under the correspondence of Prop. B.16. Choose a very ample $\mathbb{Z}$-divisor $\mathfrak{D}$ on $\mathfrak{X}$. Then there is a number $C > 0$ such that

$$C \cdot L \cdot \mathfrak{D} + g_n$$

is vertically nef. Here $L \in \mathbb{Q}$ is a global Lipschitz constant for g.

Using the fact that any C^2 function on the unit interval can be written as the difference of two convex functions we get the following corollary.

Corollary 3.6.3
Let $g\colon |R(\mathfrak{X})| \to \mathbb{R}$ be a function which is C^2 on each edge and has rational values at rational points. Then $g \circ \tau$ is a DSP Green's function.

PROOF. (of Thm. 3.6.2) As Thm. 3.6.2 will be a corollary (Cor. 3.7.6) of Thm. 3.7.1 we will only briefly explain the main idea of the proof. So let g_n be defined as in Thm. 3.6.2 for each n. Now fix an $n \geq 1$. Intersection theory shows for $2 \leq j \leq n$ that

$$C_i \cdot C_j = \begin{cases} 1 & \text{if } i = j+1 \\ -2 & \text{if } i = j \\ 0 & \text{if } i \neq j+1, j-1, j. \end{cases}$$

All irreducible curves in the special fibre of $\mathfrak{X}_n$ appear among the C_i. Hence if $Z = C_j$ for $2 \leq j \leq n$. This yields

$$g_n \cdot Z = \frac{n}{2}\left[g\left(\frac{j-1}{n+1}\right) + g\left(\frac{j+1}{n+1}\right) - 2\left(\frac{j}{n+1}\right)\right]$$

which is indeed non-negative by the convexity of g. The cases $Z = C_i$ for $i = 1, n+1$ are treated using the fact that $\mathfrak{D}$ is ample on $\mathfrak{X}$, the Lipschitz condition, and the projection formula. □

3.7. Proof of the Approximation Theorem

We make Assumption 2.2.1 on the fields k, k_n, and $\mathbb{K}$. Let X be a smooth geometrically integral projective curve and assume that X has a a projective regular strictly semi-stable model of $\mathfrak{X}$ over k°. Fix an order on the set of irreducible components of $\mathfrak{X}_s$. Let $\mathfrak{B}$ the model of $X \times_k X$ constructed in Thm. 3.3.3. Then $R(\mathfrak{B}) = R(\mathfrak{X})^2$. Let $\tau : X_{\mathbb{K}}^{\mathrm{an}} \times_{\mathbb{K}} X_{\mathbb{K}}^{\mathrm{an}} \to |R(\mathfrak{B})|$ be the retraction map associated to $\mathfrak{B}$. We will prove the following theorem:

Theorem 3.7.1
If $g \in C^0_{\mathbb{Q}}(R(\mathfrak{B}))$ is a CC-function in the sense of Def. 3.1.2 (vii). Then $0 + g \circ \tau$ is a DSP arithmetic divisor.

Proof. We will first assume that $\mathfrak{X}$ satisfies IC (Def. 3.3.5). We begin by constructing for each $n \in \mathbb{N}$ a model $\mathfrak{B}_n$ of $(X \otimes k_n) \times_{k_n} (X \otimes k_n)$ over the valuation ring k_n°. By Thm. 3.3.2 we can realise the n-BHM-subdivision of $R(\mathfrak{X})$ as the reduction set of the scheme $\mathfrak{X}_n$. Moreover, Thm. 3.3.3 provides us with a model $\mathfrak{B}_n$ over k_n° of $(X \otimes k_n) \times_{k_n} (X \otimes k_n)$ such that naturally $R(\mathfrak{B}_n) = R(\mathfrak{X}_n)^2$ holds. As naturally $|R(\mathfrak{X}_n)| \cong |R(\mathfrak{X})|$ we get a distinquished isomorphism $|R(\mathfrak{B}_n)| \cong |R(\mathfrak{B})|$ for every $n \geq 1$. By Prop. 3.5.1 these identifications are compatible with the retraction maps and the identifications respect the property of g beeing CC on the canonical squares. Recall that by convention the special fibre $\mathfrak{B}_{n,s}$ of $\mathfrak{B}_n$ is the fibre over the special point of k_n°. We identify irreducible components of $\mathfrak{B}_{n,s}$ with the corresponding points in $|R(\mathfrak{B}_n)|$.

For $i \in \{1, 2\}$ we define projection maps $\mathrm{pr}_{i,n} \colon \mathfrak{B}_n \to \mathfrak{X}_n$ as the compositions

$$\mathfrak{B}_n \to \mathfrak{X}_n \times_{k_n^\circ} \mathfrak{X}_n \to \mathfrak{X}_n$$

and $\mathrm{pr}_i \colon \mathfrak{B}_n \to \mathfrak{X}$ as the composition

$$\mathfrak{B}_n \to \mathfrak{X}_n \times_{k_n^\circ} \mathfrak{X}_n \to \mathfrak{X}_n \to \mathfrak{X}.$$

We denote by γ_n the map $\mathfrak{B}_n \to \mathfrak{X}_n \times_{k_n^\circ} \mathfrak{X}_n$. We define a series of Cartier divisors with support in the special fibre g_n on the models $\mathfrak{B}_n$: For each $n \geq 1$ we define $\widetilde{g_n}$ to be the unique function on $|R(\mathfrak{B}_n)|$ which equals $\frac{n}{2} \cdot g$ on all vertices of $R(\mathfrak{B}_n)$ and is affine on its simplices. Under the correspondence of Prop. B.16 this function corresponds to a Cartier divisor g_n on $\mathfrak{B}_n$ with support in the special fibre. Choose a very ample $\mathbb{Z}$-divisor $\mathfrak{D}'$ on $\mathfrak{X}$. We consider for each $n \geq 1$ the Cartier divisors $\mathfrak{D}_n = \mathrm{pr}_1^*\mathfrak{D}' + \mathrm{pr}_2^*\mathfrak{D}'$ on $\mathfrak{B}_n$.

Lemma 3.7.2
Define $g_{\mathfrak{B}_n, g_n}$ with respect to the b-logarithm (cf. Rem. 3.3.1). Then we have uniform convergence

$$g_{\mathfrak{B}_n, g_n} = g_n \circ \tau \to g \circ \tau \qquad \textit{for } n \to \infty$$

on $X_{\mathbb{K}}^{\mathrm{an}} \times_{\mathbb{K}} X_{\mathbb{K}}^{\mathrm{an}}$.

Proof. By continuity of g on the compact set $|R(\mathfrak{B})|$ the series of functions $\frac{2}{n}\widetilde{g}_n$ on $|R(\mathfrak{B})|$ converges uniformly to g. Using Prop. 3.2.1 and Rem. 3.3.1 we get $g_{\mathfrak{B}_n, g_n} = \frac{2}{n}\widetilde{g}_n \circ \tau \to g \circ \tau$ for $n \to \infty$. □

The main step in the proof of the theorem will be proving the following lemma.

Lemma 3.7.3
Assume that $\mathfrak{X}$ satisfies IC. Then there is a constant M depending only on the reduction

set of $\mathfrak{X}$ *such that for each Lipschitz constant* $L \in \mathbb{Q}$ *of* g *the divisor*

$$L \cdot M \cdot \mathfrak{D}_n + g_n$$

is a vertically nef divisor on $\mathfrak{B}_n$.

The lemma implies the theorem because

$$0 + g = LM \cdot \mathfrak{D}_\eta + LM \cdot g_{\mathfrak{B},\mathfrak{D}} + g - (LM \cdot \mathfrak{D}_\eta + LM \cdot g_{\mathfrak{B},\mathfrak{D}})$$

then is DSP.

In order to show Lemma 3.7.3 fix $n \geq 1$ and choose any integral curve C in the special fibre of $\mathfrak{B}_n$. We compute $\deg_{\mathfrak{B}_{n,s}}(g_n \cdot_{\mathfrak{B}_n} C)$, and adjust M accordingly.

We have to deal with three kinds of curves C.

Case I) Strata $K \cap K'$ for components K, K' of the special fibre of $\mathfrak{B}_n$ with $\mathrm{pr}_{i,n}(K) = \mathrm{pr}_{i,n}(K')$ for some $i \in \{1, 2\}$,
Case II) Strata $K \cap K'$ for two components K, K' of the special fibre of $\mathfrak{B}_n$ with $\mathrm{pr}_{i,n}(K) \neq \mathrm{pr}_{i,n}(K')$ for all $i \in \{1, 2\}$.
Case III) Curves that are not strata of the special fibre of $\mathfrak{B}_n$.

We will treat these cases step by step.

First, we investigate Case (I). We restrict to the case $C = K \cdot_{\mathfrak{B}_n} K'$ with $\mathrm{pr}_{2,n} K = \mathrm{pr}_{2,n} K'$. The case $\mathrm{pr}_{1,n} K = \mathrm{pr}_{1,n} K'$ follows by symmetry.

Now there are two cases.

Case A) The set $\mathrm{pr}_2 K = \mathrm{pr}_2 K'$ is a point in the special fibre of $\mathfrak{X}$,
Case B) The set $\mathrm{pr}_2 K = \mathrm{pr}_2 K'$ is an irreducible component of the special fibre of $\mathfrak{X}$.

In case A) by Prop. 3.4.17 there exist exactly two components $A, B \in V(K)$ and $A', B' \in V(K')$ respectively, where $V(K)$ and $H(K)$ are defined as in Def. 3.3.8 and the points in $|R(\mathfrak{B})|$ associated to A, A' and B, B' lie in the same canonical chart of $|R(\mathfrak{B})|$ as K and K' respectively. *Caution:* The projection maps used to define H, V, D are the maps $\mathrm{pr}_{i,n}$, not pr_i for $i \in \{1, 2\}$. Note that K and K' lie in the same canonical chart of $R(\mathfrak{B})$. Using the description of $R(\mathfrak{X}_n)$ in Rem. A.14 we see that the non-degenerate 2-simplices containing degenerating to the 0-simplices K, K', A, A', B, B' are the simplices spanned by

- B, A', and K,
- K, K', and A',
- K, K', and A,
- and A, K', and B',

(see Fig. 3.7.1).

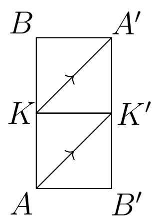


FIGURE 3.7.1. C is a "horizontal" intersection of irreducible components.

So we see among the components of the special fibre of $\mathfrak{B}_n$ different from K and K' only A and A' satisfy $\deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} K' \cdot_{\mathfrak{B}_n} A) \neq 0$ and $\deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} K' \cdot_{\mathfrak{B}_n} A') \neq 0$ respectively. Prop. 3.3.10 implies $-1 = \deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} K' \cdot_{\mathfrak{B}_n} K)$ and similarly

$$\deg_{\mathfrak{B}_{n,s}}(K' \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K') = \deg_{\mathfrak{B}_{n,s}}(-A \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K') = -1.$$

Hence we have

$$\deg_{\mathfrak{B}_{n,s}}(g_n \cdot_{\mathfrak{B}_n} C) = \frac{n}{2}\left[g(A) - g(K) - g(K') + g(A')\right] \tag{3.7.1}$$

and (3.7.1) is positive in virtue of (CC2) in Definition 3.1.2 (vi) using

$$\begin{aligned} \varepsilon &= \frac{1}{n}, \\ A &= (x,y), \\ K &= (x, y+\varepsilon), \\ K' &= (x+\varepsilon, y+\varepsilon), \\ A' &= (x+\varepsilon, y+2\varepsilon), \end{aligned}$$

and the fact that all points are contained in the same canonical chart of $|R(\mathfrak{B})|$.

In case B) we consider

$$\frac{n}{2}\left|\deg_{\mathfrak{B}_{n,s}}\left(\sum_{A \in H(K)\cup V(K)\cup D(K)} A \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K' \left(g(A) - g(K)\right)\right)\right|. \tag{3.7.2}$$

By Prop. 3.4.12 we have

$$\sum_{A \in H(K)\cup V(K)\cup D(K)} |\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K')| \leq 4\,\mathrm{val}(R(\mathfrak{X})) + \mathrm{val}(R(\mathfrak{X}))^2.$$

Moreover, $\mathrm{pr}_2 K = \mathrm{pr}_2 K'$ and K maps to an irreducible component of $\mathfrak{X}_s$ under the map pr_2, so we apply Prop. 3.4.16 and get

$$\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K') \geq 1.$$

Now

$$|\deg(g_n \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K')| = \left|\deg_{\mathfrak{B}_{n,s}}\left(K \cdot_{\mathfrak{B}_n} K' \cdot_{\mathfrak{B}_n} \frac{n}{2}\left[g(K)\cdot K + \sum_{A \in H(K)\cup V(K)\cup D(K)} g(A)\cdot A\right]\right)\right|$$

and this equals (3.7.2) by Prop. 3.3.11. If $L \in \mathbb{Q}$ is a global Lipschitz constant of g then the expression (3.7.2) and hence $|\deg(g_n \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K')|$ can be bounded from above by

$$\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K') \cdot L \cdot (4\,\mathrm{val}(R(\mathfrak{X})) + \mathrm{val}(R(\mathfrak{X}))^2).$$

by the triangle inequality. Hence for

$$M > (4\,\mathrm{val}(R(\mathfrak{X})) + \mathrm{val}(R(\mathfrak{X}))^2)$$

we have $\deg_{\mathfrak{B}_{n,s}}((LM \cdot \mathfrak{D}_n + g_n) \cdot_{\mathfrak{B}_{n,s}} K \cdot_{\mathfrak{B}_n} K') \geq 0$.

Now we treat case (II). The reduction set $R(\mathfrak{B})$ is the product of the simplicial set $R(\mathfrak{X})$ with itself there are exactly two irreducible components A and D of $\mathfrak{B}_{n,s}$ different from K and K' which satisfy $\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K') \neq 0$ and $\deg_{\mathfrak{B}_{n,s}}(D \cdot_{\mathfrak{B}_n} K \cdot_{\mathfrak{B}_n} K') \neq 0$ and the points corresponding to A and D lie in the same canonical chart of $R(\mathfrak{B})$ (cf.

Rem. A.8). Then Prop. 3.3.10 yields $\deg(K \cdot_{\mathfrak{B}_n} K' \cdot_{\mathfrak{B}_n} K) = -1 = \deg(K' \cdot_{\mathfrak{B}_n} K' \cdot_{\mathfrak{B}_n} K)$. Hence

$$\deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} K' \cdot_{\mathfrak{B}_n} g_n) = \frac{n}{2}\left[f(A) - f(K) + f(D) - f(K')\right]. \tag{3.7.3}$$

We assume $K \le K'$ where "$\le$" denotes the lexicographic order on the irreducible components of $\mathfrak{B}_{n,s}$ induced by the order on the set $\mathfrak{X}_{n,s}^{(0)}$:

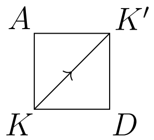


FIGURE 3.7.2. C is a "diagonal" intersection of components.

By (CC3) in Definition 3.1.2 (vi) the expression (3.7.3) is non-negative using

$$\begin{aligned} K &= (x, y) \\ K' &= (x+\varepsilon, y+\varepsilon) \\ A &= (x, y+\varepsilon) \\ D &= (x+\varepsilon, y). \end{aligned}$$

We treat case (III), that means we want to investigate $\deg_{\mathfrak{B}_{n,s}}(g_n \cdot_{\mathfrak{B}_n} Z)$ for an integral curve Z which is not an intersection of two different irreducible components of the special fibre of $\mathfrak{B}_n$. We use the notation $\widehat{Z}'$ for the Cartier divisor $(\gamma|_F)^*(\gamma(Z)_{\text{red}}))$ in F (cf. Ass. 3.4.13 (ii)). We want to compute

$$\deg_{\mathfrak{B}_{n,s}}(g_n \cdot_{\mathfrak{B}_n} Z) = \deg_{\mathfrak{B}_{n,s}}\left[\frac{n}{2}\left(\sum_{K\in(\mathfrak{B}_n)^{(0)}} g(K)\cdot K\right) \cdot_{\mathfrak{B}_n} Z\right]. \tag{3.7.4}$$

By irreducibility, Z will be contained in an irreducible component F of the special fibre of $\mathfrak{B}_n$. We apply Prop. 3.3.11 to (3.7.4) and get

$$\deg_{\mathfrak{B}_{n,s}}(g_n \cdot_{\mathfrak{B}_n} Z) = \deg_{\mathfrak{B}_{n,s}}\left[\frac{n}{2}\left(\sum_{\substack{K\in(\mathfrak{B}_{n,s})^{(0)} \\ K\neq F}} (g(K)-g(F))K\right) \cdot_{\mathfrak{B}_n} Z\right]. \tag{3.7.5}$$

Only irreducible components K of $\mathfrak{B}_{n,s}$ satisfying $K \cap F \neq \emptyset$ as subsets of $\mathfrak{B}_{n,s}$ will contribute to this expression and following the notation of from Def. 3.3.8 these are given by the elements of $H(F)$, the elements of $V(F)$, the elements $D(F)$, and F itself. Then we can restate (3.7.5) as

$$\deg_{\mathfrak{B}_{n,s}}(g_n \cdot_{\mathfrak{B}_n} Z) = \frac{n}{2}\deg_{\mathfrak{B}_{n,s}}\left(\sum_{K\in H(F)} (g(K)-g(F))K \cdot_{\mathfrak{B}_n} Z + \right.$$
$$\left.\sum_{K\in V(F)} (g(K)-g(F))K \cdot_{\mathfrak{B}_n} Z + \sum_{K\in D(F)} (g(K)-g(F))K \cdot_{\mathfrak{B}_n} Z\right).$$

We will treat three cases:

Case a) The component F maps to irreducible components of $\mathfrak{X}_s$ under both pr_1 and pr_2,
Case b) the component F maps to an irreducible component of $\mathfrak{X}_s$ under pr_1 and to a point under pr_2 or vice versa,
Case c) the component F maps to a point under both pr_1 and pr_2.

We now treat Case a) i.e., assume that F surjects on irreducible components of $\mathfrak{X}$ under pr_1 and under pr_2. Then using Lemma 3.4.18 we see that the absolute value of (3.7.4) is bounded by $L \cdot M \cdot \deg(\mathfrak{D}_n \cdot Z)$ with

$$M > 4 \cdot \mathrm{val}(R(\mathfrak{X}))^2 \cdot (\mathrm{val}(R(\mathfrak{X}))^2 + 1),$$

and $L \in \mathbb{Q}$ a global Lipschitz constant of g. This means that we have

$$\deg_{\mathfrak{B}_{n,s}}((L \cdot M \cdot \mathfrak{D}_n + g_n) \cdot_{\mathfrak{B}_n} Z) \geq 0.$$

Now we treat Case b). By symmetry we can restrict to the case that F maps to an irreducible component of $\mathfrak{X}_s$ under pr_1 and to a point under pr_2. Note that in this case, by Prop. 3.4.17, we have $\#V(F) = 2$. Let $V(F) = \{A, A'\}$. We again know that the points of $|R(\mathfrak{B})|$ corresponding to A and A' lie in the same chart of $R(\mathfrak{B})$. By Cor. 3.4.9 we have

$$\deg_{\mathfrak{B}_{n,s}}(A \cdot \widehat{Z'}) = \deg_{\mathfrak{B}_{n,s}}(A' \cdot_{\mathfrak{B}_n} \widehat{Z'}). \tag{3.7.6}$$

Moreover, by Prop. 3.4.6 we have

$$Z = \widehat{Z'} - \sum_{K \in D(F)} r_K (K \cdot_{\mathfrak{B}_n} F) \in \mathrm{CH}^2_{\mathfrak{B}_{n,s}}(\mathfrak{B}_n), \tag{3.7.7}$$

where for each $K \in D(F)$ the number r_K is the multiplicity of Z' in the point $\gamma_n(F \cap K)$.

Then we see using (3.7.7) and then (3.7.6):

$$
\begin{aligned}
\deg_{\mathfrak{B}_{n,s}}(g_n \cdot_{\mathfrak{B}_n} Z) =& \frac{n}{2}\Bigg(\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} Z)\,(g(F)-g(A)) \\
&+ \deg_{\mathfrak{B}_{n,s}}(A' \cdot_{\mathfrak{B}_n} Z)\,(g(F)-g(A')) \\
&+ \sum_{K\in H(F)} \deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} Z)\,(g(K)-g(F)) \\
&+ \sum_{K\in D(F)} \deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} Z)\,(g(K)-g(F)) \Bigg) \\
=& \frac{n}{2}\Bigg(\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} \widehat{Z}')(g(A)+g(A')-2g(F)) \\
&- \sum_{K\in D(F)} r_K \deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} F \cdot_{\mathfrak{B}_n} A)(g(A)-g(F)) \\
&- \sum_{K\in D(F)} r_K \deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} F \cdot_{\mathfrak{B}_n} A')(g(A')-g(F)) \\
&+ \sum_{K\in H(F)} \deg_{\mathfrak{B}_{n,s}}(K \cdot Z)\cdot(g(K)-g(F)) \\
&+ \sum_{K\in D(F)} (g(K)-g(F))\cdot \deg_{\mathfrak{B}_{n,s}}(K \cdot Z) \Bigg).
\end{aligned}
$$

The expression

$$\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} \widehat{Z}')(g(A)+g(A')-2g(F))$$

is non-negative by equation (3.1.2) in Rem. 3.1.5 with $\varepsilon = \frac{1}{n}$,$A=(x,y)$, $B=(x,y+2\varepsilon)$, $F=(x,y+\varepsilon)$, so it suffices to bound the absolute value of

$$
\begin{aligned}
\frac{n}{2}\Bigg(&- \sum_{K\in D(F)} r_K \deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} F \cdot_{\mathfrak{B}_n} A)(g(A)-g(F)) \\
&- \sum_{K\in D(F)} r_K \deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} F \cdot_{\mathfrak{B}_n} A')(g(A')-g(F)) \\
&+ \sum_{K\in H(F)} \deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} Z)\cdot(g(K)-g(F)) \\
&+ \sum_{K\in D(F)} (g(K)-g(F))\cdot \deg_{\mathfrak{B}_{n,s}}(K \cdot_{\mathfrak{B}_n} Z) \Bigg)
\end{aligned}
$$

from above by $M \cdot L \cdot \deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n \cdot_{\mathfrak{B}_n} Z)$ for M only depending on $R(\mathfrak{X})$ and $L \in \mathbb{Q}$ a global Lipschitz constant of g and to prove $\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n \cdot_{\mathfrak{B}_n} Z) \geq 0$.

From Lemma 3.4.18 (i) we get

$$\frac{n}{2}\sum_{K\in H(F)}|g(K)-g(F)|\cdot \deg_{\mathfrak{B}_{n,s}}(K\cdot Z)$$
$$\leq L\cdot \mathrm{val}(R(\mathfrak{X}))\cdot(\mathrm{val}(R(\mathfrak{X}))^2+1)\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n\cdot_{\mathfrak{B}_n} Z)$$

where $L\in\mathbb{Q}$ is a global Lipschitz constant. Moreover, from Lemma. 3.4.18 (iii) we get

$$\frac{n}{2}\sum_{K\in D(F)}|g(K)-g(F)|\cdot \deg_{\mathfrak{B}_{n,s}}(K\cdot_{\mathfrak{B}_n} Z)$$
$$\leq 2\cdot L\cdot \mathrm{val}(R(\mathfrak{X}))^2\cdot(\mathrm{val}(R(\mathfrak{X}))^2+1)\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n\cdot_{\mathfrak{B}_n} Z). \qquad (3.7.8)$$

Let $H\in H(F)$ be an irreducible component. Then by Prop. 3.4.10 we have

$$0\leq r_K\leq \deg_{\mathfrak{B}_{n,s}}(H\cdot_{\mathfrak{B}_n}\widehat{Z'}).$$

Moreover, by Prop. 3.4.15 we have

$$\deg_{\mathfrak{B}_{n,s}}(H\cdot_{\mathfrak{B}_n}\widehat{Z'})\leq \deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n\cdot Z).$$

Note that by the assumption IC, three different irreducible components of the special fibre of $\mathfrak{B}_n$ either intersect in a point with multiplicity one or don't intersect. Hence

$$\frac{n}{2}\sum_{K\in D(F)} r_K\deg_{\mathfrak{B}_{n,s}}(K\cdot_{\mathfrak{B}_n}F\cdot_{\mathfrak{B}_n}A)|g(A)-g(F)|\leq 2\cdot L\cdot\mathrm{val}(R(\mathfrak{X}))^2\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n\cdot_{\mathfrak{B}_n}Z)$$

and

$$\frac{n}{2}\sum_{K\in D(F)} r_K\deg_{\mathfrak{B}_{n,s}}(K\cdot_{\mathfrak{B}_n}F\cdot_{\mathfrak{B}_n}A')|g(A')-g(F)|\leq 2\cdot L\cdot\mathrm{val}(R(\mathfrak{X}))^2\deg_{\mathfrak{B}_{n,s}}(\mathfrak{D}_n\cdot_{\mathfrak{B}_n}Z).$$

So if we choose

$$M>8\cdot\mathrm{val}(R(\mathfrak{X}))^2\cdot(\mathrm{val}(R(\mathfrak{X}))^2+1)$$

we have

$$\deg_{\mathfrak{B}_{n,s}}((L\cdot M\cdot\mathfrak{D}_n+g_n)\cdot Z)\geq 0.$$

Now it remains to check case c) where F maps to a point of $\mathfrak{X}_s$ under both projections pr_i for $i\in\{1,2\}$. By Prop. 3.4.17 we have the following picture in $|R(\mathfrak{B}_n)|$ drawing all 0-simplices having a common non-degenerate 2-simplex with F:

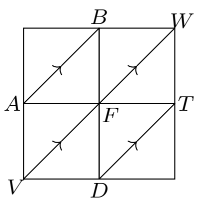


FIGURE 3.7.3. Intersection with $Z\subset F$ where $\mathrm{pr}_1(F)$ and $\mathrm{pr}_2(F)$ are points.

Note that all depicted points lie in the same canonical chart of $R(\mathfrak{B})$ by Prop. 3.4.17. Then Cor. 3.4.6 (ii) tells us that

$$Z=\widehat{Z'}-r_V(F\cdot_{\mathfrak{B}_n}V)-r_W(F\cdot_{\mathfrak{B}_n}W)$$

holds as Cartier divisors in F, where r_V is the multiplicity of Z' in the point $\gamma_n(V \cap F)$ and r_W is the multiplicity of Z' in the point $\gamma_n(W \cap F)$. Now

$$\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} F \cdot_{\mathfrak{B}_n} W) = 0,$$
$$\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} F \cdot_{\mathfrak{B}_n} V) = 1.$$

This yields

$$\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} Z) = \deg_F((A \cdot_{\mathfrak{B}_n} F) \cdot_F \widehat{Z'}) - r_V.$$

Likewise for the other components:

$$\begin{aligned}\deg_{\mathfrak{B}_{n,s}}(B \cdot_{\mathfrak{B}_n} Z) &= \deg_F((B \cdot_{\mathfrak{B}_n} F) \cdot_F \widehat{Z'}) - r_W\\ \deg_{\mathfrak{B}_{n,s}}(T \cdot_{\mathfrak{B}_n} Z) &= \deg_F((T \cdot_{\mathfrak{B}_n} F) \cdot_F \widehat{Z'}) - r_W\\ \deg_{\mathfrak{B}_{n,s}}(D \cdot_{\mathfrak{B}_n} Z) &= \deg_F((D \cdot_{\mathfrak{B}_n} F) \cdot_F \widehat{Z'}) - r_V.\end{aligned}$$

By Cor. 3.4.9 we have

$$\deg_F((A \cdot_{\mathfrak{B}_n} F) \cdot_F \widehat{Z'}) = \deg_F((T \cdot_{\mathfrak{B}_n} F) \cdot_F \widehat{Z'})$$

and

$$\deg_F((B \cdot_{\mathfrak{B}_n} F) \cdot_F \widehat{Z'}) = \deg_F((D \cdot_{\mathfrak{B}_n} F) \cdot_F \widehat{Z'}).$$

Putting everything together we get

$$\begin{aligned}\deg_{\mathfrak{B}_{n,s}}(g_n \cdot_{\mathfrak{B}_n} Z) =& \frac{n}{2}\Big[\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} \widehat{Z'})(g(A) + g(T) - 2g(F))\\ &- r_V \cdot (g(A) + g(D) - 2g(F)) + r_V \cdot (g(V) - g(F))\\ &+ \deg_{\mathfrak{B}_{n,s}}(B \cdot_{\mathfrak{B}_n} \widehat{Z'})(g(B) + g(D) - 2g(F))\\ &-r_W \cdot (g(B) + g(T) - 2g(F)) + r_W \cdot (g(W) - g(F))\Big].\end{aligned}$$

We claim that $\deg_{\mathfrak{B}_{n,s}}(g_n \cdot_{\mathfrak{B}_n} Z) \geq 0$ holds. We will only treat the term

$$\begin{aligned}&\frac{n}{2}\Big[\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} \widehat{Z'})(g(A) + g(T) - 2g(F))\\ &-r_V \cdot (g(A) + g(D) - 2g(F)) + r_V \cdot (g(V) - g(F))\Big],\end{aligned} \tag{3.7.9}$$

the non-negativity of

$$\begin{aligned}&\frac{n}{2}\deg_{\mathfrak{B}_{n,s}}(B \cdot_{\mathfrak{B}_n} \widehat{Z'})(g(B) + g(D) - 2g(F))\\ &-r_W \cdot (g(B) + g(T) - 2g(F)) + r_W \cdot (g(W) - g(F))\end{aligned}$$

follows by symmetry. By Prop. 3.4.10 we have

$$0 \leq r_V \leq \deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} \widehat{Z'}),$$

so the case $\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} \widehat{Z'}) = 0$ is trivial. Assume that $\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} \widehat{Z'}) \neq 0$ holds. We divide (3.7.9) by

$$\frac{n}{2}\deg_{\mathfrak{B}_{n,s}}(A \cdot_{\mathfrak{B}_n} \widehat{Z'})$$

and hence have to show that

$$g(A) + g(T) - 2g(F) - \lambda\,(g(A) + g(D) - 2g(F)) + \lambda\,(g(V) - g(F)) \tag{3.7.10}$$

is non-negative for all $\lambda \in [0,1]$. As the expression is affine in the variable λ it suffices to show this for all $\lambda \in \{0,1\}$. If $\lambda = 0$, then (3.7.10) becomes

$$g(A) + g(T) - 2g(F),$$

which is non-negative by Equation (3.1.2) in Rem. 3.1.5 with $A =: (x,y)$, $F =: (x+\varepsilon, y)$, and $T =: (x+2\varepsilon, y)$. If $\lambda = 1$ we get

$$g(T) - g(D) - g(F) + g(V) \tag{3.7.11}$$

for (3.7.10). If we set $V =: (x,y)$,$T =: (x+2\varepsilon, y+\varepsilon$, $F =: (x+\varepsilon, y+\varepsilon)$, and $D =: (x+\varepsilon, y)$ then (3.7.11) becomes

$$g(x+2\varepsilon, y+\varepsilon) - g(x+\varepsilon, y) - g(x+\varepsilon, y+\varepsilon) + g(x,y),$$

and this is positive in virtue of (CC1) in Definition 3.1.2 (vi). This finishes Case c).

As we always could choose M only depending on $R(\mathfrak{X})$ such that

$$\deg_{\mathfrak{B}_{n,s}}((LM\mathfrak{D}_n + g_n) \cdot_{\mathfrak{B}_n} Z) \geq 0$$

for every complete integral curve Z in the special fibre of $\mathfrak{B}_n$, we have finished the proof of Lemma 3.7.3 and hence of Thm. 3.7.1 in the case that $\mathfrak{X}$ satisfies IC.

We remove the irreducibility hypothesis IC as follows: It is easy to see that the 2-BHM-subdivision of a graph does not have multiple edges. Hence $\mathfrak{X}_2$ satisfies IC. This implies that $\mathfrak{B}_2$ satisfies IC. The pull back of g to the skeleton of $\mathfrak{B}_2$ will satisfy the conditions of Lemma 3.7.3 and we have proven the theorem in full generality. □

Using the analysis developed in Ch. 3.1 we get the following corollary:

Corollary 3.7.4
Let $g \in C^2_{\square,\mathbb{Q}}(R(\mathfrak{B}))$ be a function on $|R(\mathfrak{B})|$. Then $0 + g \circ \tau$ is an DSP arithmetic divisor.

Proof. We consider the model $\mathfrak{X}_2$ over k_2° and can put an order on the irreducible components of the special fibre of $\mathfrak{X}_2$ such that $R(\mathfrak{X}_2)$ is the S-subdivision of $R(\mathfrak{X})$. Denote by $\mathfrak{B}'$ the model of the product $(X \times_k X) \otimes k_2$ corresponding the lexicographic order on the irreducible components of the special fibre of $\mathfrak{X}_2$. Note that by Prop. 3.5.4 the retraction maps of $(X^{\mathrm{an}}_{\mathbb{K}}) \times_{\mathbb{K}} (X^{\mathrm{an}}_{\mathbb{K}})$ to $|R(\mathfrak{B})|$ and $|R(\mathfrak{B}')|$ respectively coincide and hence g induces the same Green's function for the zero divisor on $(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}$ either considered as function on $|R(\mathfrak{B})|$ or considered as function on $|R(\mathfrak{B}')|$. Moreover, pulling back g to $|R(\mathfrak{B}')|$ doesn't alter the differentiability of g. Hence we can assume that $R(\mathfrak{X})$ is itself the S-subdivision of a simplicial set. By Thm. 3.1.6 (i) the function g is the difference of two CC-functions h in $C^0_{\square,\mathbb{Q}}(\Gamma^2)$. By Thm. 3.7.1 $(0, h \circ \tau)$ and $(0, l \circ \tau)$ are DSP, hence $(0, g \circ \tau)$ is DSP. □

Similarly we can prove the following corollary.

Corollary 3.7.5
Let $g \in C^4_{\Delta,\mathbb{Q}}(R(\mathfrak{B}))$ be a function on $R(\mathfrak{B})$. Then $0 + g$ is a DSP-arithmetic divisor.

Proof. Apply Thm. 3.1.6 (ii) and argue as in Cor. 3.7.4. □

We show how to derive the result for Green's functions on curves.

Corollary 3.7.6
Let $g\colon |R(\mathfrak{X})| \to \mathbb{R}$ be a function which is convex on each edge and which has rational values at rational points. Then $g \circ \tau$ is a DSP Green's function on X.

PROOF. Choose a very ample divisor $\mathfrak{D}$ on $\mathfrak{X}$. We see that the pull back $\widetilde{g}$ of g to $|R(\mathfrak{B})|$ via the first projection is a CC function, because fixing x, it is constant in y-direction. The proof of Thm. 3.7.1 we see that $C\,\mathrm{pr}_1^*\mathfrak{D} + \widetilde{g}_n$ is vertically nef for some constant $C > 0$. Note we didn't need to add $\mathrm{pr}_2^*\mathfrak{D}$ because $\widetilde{g}$ is constant in y direction: This means that terms of the form $\sum_{K\in V(F)}(g(K) - g(F))\dots$ always vanish and an estimate of the form (3.7.8) can be established whenever $\mathrm{pr}_i(F)$ is an irreducible component for *any* $i \in \{1, 2\}$.

But by [**Laz04**, Ex. 1.4.3 ii)] $C\,\mathrm{pr}_1^*\mathfrak{D} + \widetilde{g}_n$ beeing vertically nef means that $C\mathfrak{D} + g_n$ is semipositive, as desired. □

We also get the following corollary:

Corollary 3.7.7
If $g\colon |R(\mathfrak{X})| \to \mathbb{R}$ is the difference of two convex functions with rational values at rational points, then $g \circ \tau$ is DSP. In particular every function which is C^2 on the canonical 1-simplices of $|R(\mathfrak{X})|$ induces a DSP Green's function.

PROOF. The first part is clear. The second follows from the fact that for every C^2-function f on $[0, 1]$ we can find an $M > 0$ such that $Mx^2 + f(x)$ is a convex function. □

Remark 3.7.8
Lemma 3.7.3 together with Prop. 3.1.13 implies that Lipschitz limits of CC-functions induce DSP Green's functions. Hence, one sees that the rationality claims on the functions in Thm 3.7.1, Cor. 3.7.4, Cor. 3.7.5, Cor. 3.7.6, and Cor. 3.7.7 can be dropped.

3.8. Comparison to Similar Results

In this section we first compare our result with a result by Liu (cf. [**Liu11**]) for totally degenerate abelian varieties over $\mathbb{C}_p$. For the rest of this section we assume that k equals the completion of the maximal unramified extension of $\mathbb{Q}_p$ for a prime number p. Let π be a uniformiser and $b = |\pi|^{-1}$. We denote by $\mathbb{K} = \mathbb{C}_p$.

First we consider $T = (\mathbb{G}_m^d)$. We have a natural evaluation map

$$\tau : T^{\mathrm{an}} \to \mathbb{R}^d, \qquad x \mapsto (-\log_b |x_1|, \dots, -\log_b |x_d|)$$

This map has a continuous section i which maps a tuple $(x_1, \dots, x_d)$ to the corresponding Gauß point.

A *complete lattice* in T^{an} is an analytic subgroup M which maps bijectively to a lattice Λ in $\mathbb{R}^d$ under τ. We say that a lattice is *rational* if the corresponding lattice in $\mathbb{R}^d$ is rational.

Definition 3.8.1
A totally degenerate abelian variety is an abelian variety A/k such that $A^{\mathrm{an}} = T^{\mathrm{an}}/M$ for a complete rational lattice M in an algebraic torus $T = \mathbb{G}_m^d$. Set $\Lambda = \tau(M)$. By hypothesis we have maps

$$\tau : A^{\mathrm{an}} \to \mathbb{R}^d/\Lambda$$

and

$$i_A : \mathbb{R}^d/\Lambda \to A^{\mathrm{an}}$$

induced by the maps τ and i from above.

We have the following result:

Theorem 3.8.2
Let A be a totally degenerate abelian variety over $\mathbb{K}$ and $A^{\mathrm{an}} = T/M$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a Λ-periodic C^2 function. Then $g \circ \tau$ is a DSP-Green's function for the trivial Cartier divisor.

Proof. Cf. [**Liu11**, Cor. 3.10]. □

Now we want to compare this result with our result. The totally degenerate abelian varieties which are the self product of a curve are exactly products of two Tate curves. This can for example be seen using an argument involving the Albanese variety. So let X be a Tate elliptic curve over k. We assume that X possesses a regular strictly semi-stable Mumford model $\mathfrak{X}$ over k°. The analytification of X is of the form $\mathbb{G}_m^{1,\mathrm{an}}/M$ for a complete lattice M. Then $X \times_k X$ is totally degenerate and its analytification is of the form $\mathbb{G}_m^{2,\mathrm{an}}/(M^2)$. Let Λ be the lattice in $\mathbb{R}^2$ which is the image of M^2 under the natural map $\tau\colon X^{\mathrm{an}} \times_k X^{\mathrm{an}} \to \mathbb{R}^2$ and let $i\colon \mathbb{R}^2 \to X^{\mathrm{an}} \times_k X^{\mathrm{an}}$ be the inclusion. We put an order on the set of irreducible components of $\mathfrak{X}$ and form the associated model $\mathfrak{B}$ of X^2 using Thm 3.3.3. Then by [**Gub07**, Ex. 9.8] the set $i(\mathbb{R}^2/\Lambda^2)$ agrees with the skeleton $S(\mathfrak{B})$. Let $\tau_{\mathfrak{B}}$ the retraction map to the skeleton associated to $\mathfrak{B}$. Then $\tau_{\mathfrak{B}}$ and $i \circ \tau$ agree. So every function as in the statement of Thm. 3.8.2 is in particular C^2 on the canonical squares of $S(\mathfrak{B})$. In our special case we get a generalisation of Thm. 3.8.2 in the following sense:

(i) The functions considered need to be $C^2_{\Delta,\mathbb{Q}}$, or $C^2_{\square,\mathbb{Q}}$.
(ii) We also allow models of X which are not Mumford models, that is, its skeleta are not necessarily real tori.

Remark 3.8.3
We can also recover the formula in [**Liu11**, Thm 4.3] from Thm. 4.3.2. As in the proof of Lemma 5.5.4 one can transform Zhang's formula into the limit formula in [**Liu11**, Thm 4.3] by repeatedly applying integration by parts. By the Λ-periodicity of the functions in question, all boundary integrals cancel.

We now come to the discussion of toric varieties. The following result is due to Burgos, Philippon, and Sombra ([**BGPS14**]). Let k be any field which is complete with respect to a non-Archimedean non-trivial discrete absolute value. Let Σ be a complete fan in $\mathbb{R}^n$ and X_Σ be the associated toric variety over k. Then we have the following theorem:

Theorem 3.8.4
Every function on $\mathbb{R}^n$ which is a difference of convex bounded continuous functions defines a DSP Green's function for 0 on X_Σ.

Proof. See [**BGPS14**, 4.7.6]. □

Using the classification of toric smooth toric surfaces (cf. [**CLS11**, Thm. 10.4.3]) one sees that the only toric variety in the form of $C \times_k C$ for a smooth projective curve is $\mathbb{P}^1 \times_k \mathbb{P}^1$. We define

$$\mathfrak{X} = \operatorname{Proj} k^\circ[x, y, t]/(xy - t^2\pi),$$

so $\mathfrak{X} \otimes \mathbb{K}$ is $\mathbb{P}^1_{\mathbb{K}}$ and use Thm. 3.3.3 to form a model $\mathfrak{B}$ so that $\mathfrak{B} \otimes \mathbb{K} = (\mathbb{P}^1 \times_k \mathbb{P}^1)_{\mathbb{K}}$. The skeleton S of $\mathfrak{B}$ can naturally be identified with a subspace of the *variety with corners* V in the sense of [**BGPS14**] and the restrictions of the retraction map $\tau : X^{\mathrm{an},2}_{\mathbb{K}} \to S$ to the skeleton and the tropicalisation map $X^{\mathrm{an}}_{\mathbb{K}} \times_{\mathbb{K}} X^{\mathrm{an}}_{\mathbb{K}} \to V$ to $\tau^{-1}(S)$ agree. Every

$C^2_{\Delta,\mathbb{Q}}$-function and $C^2_{\square,\mathbb{Q}}$-function on S with support in the interior of S defines a bounded function on V. Here Thm. 3.8.4 implies our result.

CHAPTER 4

Applications

4.1. Formal Metrics and Local Heights

We survey the theory of formal metrics and local heights and closely follow [**Gub07**, Ch. 3]. Let k be a field complete with respect to a (not necessarily discrete) non-trivial non-Archimedean absolute value $|\cdot|$. Denote by $\mathbb{K}$ the completion of an algebraic closure of k equipped with the unique absolute value which extends $|\cdot|$. Let X be a k-analytic space and let L be a line bundle on X i.e., a sheaf of $\mathscr{O}_X$-modules which is locally trivial for the usual topology on X. A *metric* on L is given by the following data: For each open set $U \subset X$ and each section $s \in \Gamma(L, U)$ we have a continuous map

$$\|s(\cdot)\| \colon U \to \mathbb{R}_{\geq 0}$$

such that $\|f \cdot s\| = |f| \cdot \|s\|$ for each $f \in \Gamma(\mathscr{O}_X, U)$ and which vanishes at a point x of U if and only if s vanishes at x.

Let X be a proper variety over $\mathbb{K}$ and L be a line bundle on X. Whenever there is no harm of confusion we will mean a metric on L^{an} whenever we speak of a metric on L. A metric $\|\cdot\|$ on L is said to be *formal* if there is a formal $\mathbb{K}^\circ$-model $\mathfrak{X}$ of X^{an} i.e., $\mathfrak{X}^{\mathrm{an}} = X^{\mathrm{an}}$, and a line bundle $\mathfrak{L}$ on $\mathfrak{X}$ such that $\mathfrak{L}^{\mathrm{an}} \cong L^{\mathrm{an}}$ and this isomorphism is compatible with the chosen isomorphism $\mathfrak{X}^{\mathrm{an}} \cong X^{\mathrm{an}}$ such that for every formal trivialisation $\mathfrak{U}$ of $\mathfrak{L}$ and for every $s \in \Gamma(\mathfrak{U}, \mathfrak{L})$ mapped to $\gamma \in \mathscr{O}(\mathfrak{U})$ under this trivialisation we have $|\gamma(x)| = \|s(x)\|$. A formal metric is called *semipositive* iff in the definition above $\mathfrak{L}_s$ is a numerically effective line bundle on $\mathfrak{X}_s$. A metrised line bundle $(L, \|\cdot\|)$ has a *root of a formal metric* if some positive tensor power is formally metrised. A root of a formal metric is *vertically nef* if some positive power is a vertically nef formally metrised line bundle.

We denote by $\mathfrak{g}^+$ the set of isometry classes of vertically nef roots of formally metrised line bundles on X. This is a monoid under the tensor product of metrised line bundles. On the space of metrics for a fixed line bundle L on X we have the distance function

$$d(\|\cdot\|, \|\cdot\|') = \sup_{x \in X^{\mathrm{an}}} (|\|s(x)\|/\|s(x)\|'|).$$

where s is a non-zero rational section of L^{an}. The function d defines a distance on the space of metrics on L, as X was assumed proper.

A *metrised pseudo divisor* $\widehat{D}$ is a quadruple $(L, \|\cdot\|, Y, s)$ where L is a line bundle on X, $\|\cdot\|$ is a metric on L^{an}, Y is a closed subset of X, and s is a nowhere vanishing section of L on $X \setminus Y$. The *support* $\operatorname{supp} \widehat{D}$ of $\widehat{D}$ is Y. We set $\mathscr{O}(\widehat{D}) := L$. If $\varphi \colon X' \to X$ is a morphism of algebraic varieties we denote by $\varphi^* \widehat{D} = (\varphi^* L, \varphi^*(\|\cdot\|), \varphi^{-1}(Y), \varphi^* s)$ the pull back.

For a t-dimensional cycle Z on X and formally metrised pseudo divisors

$$\widehat{D_0}, \ldots, \widehat{D_t}$$

such that

$$\mathrm{supp}(\widehat{D_0}) \cap \ldots \cap \mathrm{supp}(\widehat{D_t}) \cap Z = \emptyset$$

we define the local height $\lambda_{\widehat{D_0},\ldots,\widehat{D_t}}(Z)$ as the intersection number in the sense of [**Gub98**] on a joint formal $\mathbb{K}^\circ$-model. When k is a discretely valued field and all models are algebraisable this is the intersection number in the sense of an intersection product with supports (cf. Def. 3.3.4) by [**Gub98**, Rem. 6.6]. This construction extends in the evident way to pseudo divisors metrised by roots of formal metrics.

Definition 4.1.1
The *completion* $\widehat{\mathfrak{g}^+}$ of $\mathfrak{g}^+$ is the set of isometry classes of line bundles $(L, \|\cdot\|)$ on X with the following property: For all $n \in \mathbb{N}$ there is a proper surjective morphism $\varphi_n \colon X_n \to X$ and a root of a semipositive formal metric $\|\cdot\|_n$ on $\varphi_n^*(L^{\mathrm{an}})$ such that $d(\|\cdot\|_n, \varphi_n^*(\|\cdot\|)) \to 0$ for $n \to \infty$. We call the elements of $\widehat{\mathfrak{g}^+}$ *semipositively metrised line bundles.* We define: $\widehat{\mathfrak{g}} := \widehat{\mathfrak{g}^+} - \widehat{\mathfrak{g}^+}$. We call elements in this group DSP-metrised line bundles.

Theorem 4.1.2
The local height of formally metrised line bundles uniquely extends to a local height function for DSP-metrised pseudo-divisors such that the following properties hold:

(i) If $\widehat{D_0}, \ldots, \widehat{D_t}$ are formally metrised pseudo divisors then λ coincides with the definition of the local height for formally metrised pseudo divisors,
(ii) the function λ is multilinear and commutative with respect to the variables $\widehat{D_i}$ and linear with respect to Z,
(iii) for a proper morphism $\varphi\colon X' \to X$ we have

$$\lambda_{\varphi^*(\widehat{D_0}),\ldots,\varphi^*(\widehat{D_0})}(Z') = \lambda_{\widehat{D_0},\ldots,\widehat{D_t}}(\varphi_* Z'),$$

(iv) assume that $\widehat{D_0}, \ldots, \widehat{D_t}$ are $\widehat{\mathfrak{g}^+}$-metrised pseudo divisors. Let $\lambda(Z)$ be the local height of Z with respect to $\widehat{D_0}, \ldots, \widehat{D_t}$ and $\lambda'(Z)$ be the local height of Z with respect to $\widehat{D_0}', \widehat{D_1}, \ldots, \widehat{D_t}$ where we resplaced the metric $\|\cdot\|_0$ of $\widehat{D_0}$ by the metric $\|\cdot\|_0'$. Assume that the metrics on $\mathscr{O}(\widehat{D_i})$ are semipositive for $i = 1, \ldots, t$. Then

$$|\lambda(Z) - \lambda'(Z)| \le d(\|\cdot\|, \|\cdot\|') \deg_{\mathscr{O}(D_1),\ldots,\mathscr{O}(D_t)}(Z).$$

Proof. See [**Gub02**, Prop. 5.1.8]. □

Remark 4.1.3
(i) We show how to compute the local height of metrised pseudo divisors which are limits of vertically nef formally metrised pseudo divisors. It suffices to compute λ for $\widehat{D_0}, \ldots, \widehat{D_t}$ for $\widehat{\mathfrak{g}}^+$ metrics and then to extend this by multilinearity.

Assume that for $i \in \{0, \ldots, t\}$ the metrised pseudo-divisors $\widehat{D}_{in}$ are formally metrised with metrics $\|\cdot\|_{in}$ converging to $\|\cdot\|_i$ in the sense of Def. 4.1.1. As in the proof of [**Gub02**, Prop. 5.1.8] we can uniformly in i choose for each n a variety Z_n admitting a proper surjective morphism $\varphi_n \colon Z_n \to Z$ such that for each $i \in \{1, \ldots, t\}$

$$d(\|\cdot\|_{in}, \varphi_n^* \|\cdot\|_i) < \varepsilon$$

holds, where $\|\cdot\|_{in}$ are vertically nef formal metrics. We define

$$\lambda_n(Z) = \frac{1}{[K(Z_n) : K(Z)]} \lambda_{\widehat{D}_{0n},\ldots,\widehat{D}_{tn}}(Z_n).$$

Then, by Thm. 4.1.2(iv) the sequence $(\lambda_n(Z))_{n\in\mathbb{N}}$ is a Cauchy sequence and we get

$$\lambda_{\widehat{D}_0,\ldots,\widehat{D}_t}(Z) = \lim_{n\to\infty} \lambda_n(Z).$$

(ii) We can still define the local height if $\|\cdot\|_0$ is an arbitrary continuous metric on $\mathscr{O}_{X^{\mathrm{an}}}$ and $\widehat{D}_t$ are $\widehat{\mathfrak{g}}$-metrised pseudo divisors for $t \in \{1,\ldots,t\}$. By [**Gub07**, Prop. 3.3], the metric $\|\cdot\|_0$ is the uniform limit of a sequence formal metrics $(\|\cdot\|_n)_{n\in\mathbb{N}}$. We define the pseudo divisors $\widehat{D_0} = (\mathscr{O}_{X^{\mathrm{an}}}, \|\cdot\|_0, \emptyset, 1)$ and $\widehat{D_{0n}} = (\mathscr{O}_{X^{\mathrm{an}}}, \|\cdot\|_n, \emptyset, 1)$.

Then we set

$$\lambda_{\widehat{D_0},\ldots,\widehat{D_t}}(Z) = \lim_{n\to\infty} \lambda_{\widehat{D_{0n}},\widehat{D_1},\ldots,\widehat{D_t}}(Z).$$

This quantity only depends on the metrics $\|\cdot\|_i$ for $i = 0,\ldots,t$ by [**Gub07**, 3.7].

(iii) Let $g\colon X^{\mathrm{an}} \to \mathbb{R}$ be a continuous function. Then we define a metric $\|\cdot\|_g$ on the trivial line bundle as follows: We take the global 1-section of $\mathscr{O}_{X^{\mathrm{an}}}$ and set $\|1\|_g(x) = \exp(-g(x))$.

Definition 4.1.4
For each $i \in \{1,\ldots,d\}$ let $\|\cdot\|_i$ be a DSP-metric and we choose sections s_i of L_i such that

$$\cap_i \operatorname{supp}(s_i) = \emptyset.$$

Then we consider the metrised pseudo divisors $\widehat{D_i} = (L_i, \|\cdot\|_i, \operatorname{supp} s_i, s_i)$. Then there exists a unique measure μ such that if for each continuous function $f\colon X^{\mathrm{an}} \to \mathbb{R}$ we have

$$\int f d\mu = \lambda_{(\mathscr{O}_X, \|\cdot\|_f, \emptyset, 1), \widehat{D_1},\ldots,\widehat{D_d}}(X).$$

We call this measure the *Chambert-Loir measure*. This is a Radon measure and only depends on the metrics $\|\cdot\|_i$ by [**Gub07**, 3.7] and not on the choice of sections. We denote this measure by $c_1(L_1, \|\cdot\|_1) \wedge \ldots \wedge c_1(L_d, \|\cdot\|_d)$.

4.2. Local Heights for Arithmetic Divisors

We will be in the situation of Assumption 2.2.1. Our goal is to define the local height for DSP arithmetic divisors.

Definition 4.2.1 (Local height for model arithmetic divisors)
Let X be n-dimensional projective variety over k for some $n \geq 1$. Let Z be a t-dimensional subvariety of X for some $t \geq 1$. Let

$$D_0 + g_0, \ldots, D_t + g_t$$

be arithmetic divisors where for $i \in \{0,\ldots,t\}$ the functions g_i are model Green's functions defined by models $(\mathfrak{X}_i, \mathfrak{D}_i)$ over $k_{n_i}^\circ$ for various integers $n_i \geq 1$. Set $n = \prod_{i=0}^t n_i$. We assume that

$$\operatorname{supp} D_{0,k_n} \cap \ldots \cap \operatorname{supp} D_{t,k_n} \cap Z_{k_n} = \emptyset.$$

Denote by $\mathfrak{R}$ the scheme-theoretic closure of the image of the diagonal map

$$\Delta : X \otimes k_n \to (\mathfrak{X}_0 \otimes_{k_n^\circ} k_n^\circ) \times_{k_n^\circ} \ldots \times_{k_n^\circ} (\mathfrak{X}_t \otimes k_n^\circ).$$

We have canonical projection maps $\varphi_i\colon \mathfrak{R} \to \mathfrak{X}_i$ for all $i \in \{0,\ldots,t\}$. We define the *local height* of Z with respect to the family $(D_i, g_i)_{i\in\{0,\ldots,t\}}$ as

$$\frac{1}{n^t} \deg_{\mathfrak{Z}_s}(\varphi_0^*\mathfrak{D}_0 \cdot \ldots \cdot \varphi_t^*\mathfrak{D}_t),$$

where $\mathfrak{Z}$ is the closure of Z in $\mathfrak{R}$.

To justify the name *local height* we want to relate this to the local heights after Gubler (cf. Ch. 4.1). Let D be an integral Cartier divisor. Every arithmetic divisor $D+g$ gives rise to a metrised pseudo divisor $\widehat{D+g} = (\mathscr{O}(D), \|\cdot\|_g, \operatorname{supp} D, s_D)$. Here s_D is the canonical rational section of $\mathscr{O}(D)$ and we construct the metric $\|\cdot\|_g$ as follows: We choose an $x \in X_{\mathbb{K}}^{\mathrm{an}}$. Let f be a local equation for D on an open set U such that $x \in U_{\mathbb{K}}^{\mathrm{an}}$. We set

$$\left\| \frac{1}{f} \right\|_g = \frac{1}{|f(x)|^2} \cdot \exp(\ln b \cdot g(x)),$$

where $b = |\pi|$. Note that prescribing the value on $\frac{1}{f}$ completely determines the metric on U. This construction extends to $\mathbb{Q}$-divisors using roots of metrics. If $D+g$ is a model arithmetic divisor corresponding to a model $\mathfrak{X}$ of X and a model $\mathfrak{D}$ of D then we denote by $\widehat{\mathfrak{X}}$ the completion of $\mathfrak{X}$ along the special fibre and $\widehat{\mathfrak{D}}$ is the completion of $\mathfrak{D}$. Then $\widehat{D+g}$ is the formally metrised pseudo divisor induced by the formal model $(\widehat{\mathfrak{X}}, \widehat{\mathfrak{D}})$.

We note the following properties:

Remark 4.2.2

(i) If $D+g$ is semipositive, then $\widehat{D+g}$ is semipositively metrised pseudo divisor.

(ii) For model arithmetic divisors $D_0+g_0, \dots, D_t+g_t$ and a t-dimensional cycle Z, we have

$$\lambda_{D_0+g_0,\dots,D_t+g_t}(Z) = \lambda_{\widehat{D_0+g_0},\dots,\widehat{D_t+g_t}}(Z),$$

where $\lambda_{\widehat{D_0+g_0},\dots,\widehat{D_t+g_t}}(Z)$ is the *local height* with respect to the formally metrised pseudo divisors $\widehat{D_i+g_i}$ for $i = 0, \dots, t$.

(iii) If $g_n \to g$ for $n \to \infty$ uniformly then $\|\cdot\|_{g_n} \to \|\cdot\|_g$ for $n \to \infty$.

Proof. The first two statements follow from [**Gub98**, Rem. 6.6]. The third goes as follows: The function $g - g_n$ is a Green's function for the trivial divisor and by the definition of the convergence of the g_n, for every $\varepsilon > 0$ we have an n such that

$$d(g_n, g) = \sup_{x \in X_{\mathbb{K}}^{\mathrm{an}}} |(g_n - g)(x)| < \varepsilon.$$

Let $x \in X_{\mathbb{K}}^{\mathrm{an}}$ and f be a local equation for X in U such that $x \in U_{\mathbb{K}}^{\mathrm{an}}$. Then by definition of $\|\cdot\|_g$ and $\|\cdot\|_{g_n}$ and the distance of norms

$$\left| \frac{\log(\|\cdot\|_{g_n})}{\log(\|\cdot\|_g)} \right| = \left| \log \frac{\frac{1}{|f(x)|^2} \exp(\ln b \cdot g(x))}{\frac{1}{|f(x)|^2} \exp(\ln b \cdot g_n(x))} \right| = |g(x) - g_n(x)| < \varepsilon$$

holds. As this estimate does not depend on f neither on U we have $d(|\cdot\|_{g_n}, \|\cdot\|_g) \to 0$ uniformly in n as claimed. □

Definition 4.2.3

Let X be a projective variety over k and Z be a t-dimensional prime cycle. Let $D_0, \dots, D_t$ be Cartier divisors such that

$$\operatorname{supp} D_0 \cap \ldots \cap \operatorname{supp} D_t \cap Z = \emptyset$$

and let $g_0, \dots, g_t$ be DSP Green's functions for $D_0, \dots, D_t$. In virtue of Rem. 4.2.2 we can define the height of Z with respect to the DSP arithmetic divisors $D_i + g_i$ as

$$\lambda(Z) := \lambda_{\widehat{D_0+g_0},\dots,\widehat{D_t+g_t}}(Z).$$

Remark 4.2.4
Let $D_i + g_i$ for all $i \in \{0, \dots, t\}$ be arithmetic Divisors on a projective variety Y over k and Z be a t-dimensional cycle. Assume that there exist semipositive arithmetic divisors $D_i + g_i'$ and $D_i'' + g_i''$ such that $D_i + g_i = D_i' + g_i' - (D_i'' + g_i'')$ for every i and

$$\bigcap_i (\operatorname{supp} D_i' \cup \operatorname{supp} D_i'') \cap Z = \emptyset. \tag{4.2.1}$$

Assume there are models $\mathfrak{Y}_n$ of $Y \otimes k_{d(n)}$ over $k_{d(n)}^\circ$ for integers $d(n) \geq 1$ and vertically nef Cartier divisors $\mathfrak{D}_{in}'$ and $\mathfrak{D}_{in}''$ on models $\mathfrak{Y}_n$ such that

$$g_{\mathfrak{Y}_n, \mathfrak{D}_{in}'} \to g_i'$$
$$g_{\mathfrak{Y}_n, \mathfrak{D}_{in}''} \to g_i''$$

for $n \to \infty$. Then by Rem. 4.2.2 (ii) and Rem. 4.1.3 (i)

$$\lambda_{(D_i' + g_{\mathfrak{Y}_n, \mathfrak{D}_{in}'} - (D_i'' + g_{\mathfrak{Y}_n, \mathfrak{D}_{in}''}))_{i=0,\dots,t}}(Z) \to \lambda_{(D_i + g_i)_{i=0,\dots,t}}(Z)$$

for $n \to \infty$.

4.3. Zhang's/Kolb's Formulae for Local Heights

We will make Assumptions 2.2.1 and 3.4.13 (i).

Choose three functions f_0, f_1, f_2 in $C^0_{\mathbb{Q}}(R(\mathfrak{B}_1))$. Then we can define lattice approximations f_{in} for $i \in \{0, 1, 2\}$ and $n \in \mathbb{N}$ which are affine on the simplices of $R(\mathfrak{B}_n)$ and equal $\frac{n}{2} f_i$ on the 0-simplices as in the proof of Cor. 3.7.4. In virtue of Prop. B.16 the functions f_{in} correspond to Cartier divisors with support in the special fibre of $\mathfrak{B}_n$ which we also denote by f_{in}. Denote by

$$\langle \text{-}, \text{-}, \text{-} \rangle_{\text{alg}} = \frac{1}{n^3} \deg_{\mathfrak{B}_{n,s}} (\text{-} \cdot_{\mathfrak{B}_n} \text{-} \cdot_{\mathfrak{B}_n} \text{-})$$

the normalised local intersection pairing on each $\mathfrak{B}_n$ (cf. Def. 3.3.4).

Kolb proves the following theorem, which has been proven for a different class of models $\mathfrak{B}_n$ by Zhang (cf. [**Zha10**, Prop. 3.3.1, 3.4.1]).

Theorem 4.3.1 (Zhang,Kolb)
Assume that $f_0, f_1, f_2 \in C^2_{\square,\mathbb{Q}}(R(\mathfrak{B}))$ are functions. Then the limit of the sequence

$$(\langle f_{0,n}, f_{1,n}, f_{2,n} \rangle_{\text{alg}})_{n \in \mathbb{N}} \tag{4.3.1}$$

exists for $n \to \infty$ and equals

$$\langle f_0, f_1, f_2 \rangle_{\text{an}} := \int_{|R(\mathfrak{B})|} \partial_x f_0 \partial_y f_1 \partial_{xy} f_2 + \textit{ permutations.} \tag{4.3.2}$$

Proof. For a proof we refer to [**Kol16a**, Theorem 3.32]. Note that for dimension two the hypothesis on the degree of differentiability of the functions in the proof can be relaxed to C^2. □

Then we have the following formula for the height of $X \times_k X$ with respect to the f_i in case the f_i are of type $C^2_{\square,\mathbb{Q}}(R(\mathfrak{B}))$ for all $i \in \{0, 1, 2\}$.

Corollary 4.3.2
If the f_i are $C^2_{\square,\mathbb{Q}}$, the formula (4.3.2) computes the local height of $X \times_k X$ with respect to the arithmetic divisors $0 + f_0$, $0 + f_1$ and $0 + f_2$.

Proof. After passing to a subdivision of $R(\mathfrak{X})$ in virtue of Thm. 3.1.6 (i) for all $i \in \{0,1,2\}$ we choose f_i' and f_i'' which satisfy CC on the geometric realisation of the reduction set $|R(\mathfrak{B})|$ and such that $f_i = f_i' - f_i''$. By multilinearity of the local height and the formula we can assume that f_i itself satisfies CC. Recall that the f_{in}-s were defined above as the lattice approximations. For all $i \in \{0,1,2\}$ we choose very ample Cartier divisors $\mathfrak{D}_i'$ on $\mathfrak{X}$ such that

$$\bigcap_{i=0}^{2} \operatorname{supp}(\mathfrak{D}_i'|_\eta) = \emptyset.$$

Then we set for each $i \in \{0,1,2\}$

$$\mathfrak{D}_i = \operatorname{pr}_1^* \mathfrak{D}_i' + \operatorname{pr}_2^* \mathfrak{D}_i'.$$

By Lemma 3.7.3 we can assume $\mathfrak{X}$ to fulfill condition IC and replace all $\mathfrak{D}_i$ by sufficiently high multiples such that $\mathfrak{D}_i + f_{in} \circ \tau$ is are vertically nef Cartier divisors on $\mathfrak{B}_n$ for all $i \in \{0,1,2\}$ and all $n \in \mathbb{N}$. Then Rem. 4.2.4 tells us that

$$\Big(\lambda_{\mathfrak{D}_i|_\eta + g_{\mathfrak{B}_n,\mathfrak{D}_i} + f_{in}\circ\tau - (\mathfrak{D}_i|_\eta + g_{\mathfrak{B}_n,\mathfrak{D}_i})}(X \times_k X)\Big)_{n\in\mathbb{N}} = (\lambda_{g_{\mathfrak{B}_n}, f_{in}}(X \times_k X))_{n\in\mathbb{N}} = (\langle f_{0n}, f_{1n}, f_{2n}\rangle_{\mathrm{alg}})_{n\in\mathbb{N}}$$

converges to

$$\lambda_{f_0\circ\tau, f_1\circ\tau, f_2\circ\tau}(X \times_k X)$$

for $n \to \infty$. On the other hand side the limit is given by (4.3.2) by Theorem 4.3.1 which proves the claim. □

Assume for the rest of the section that f_0, f_1, f_2 are functions of class $C^2_{\Delta,\mathbb{Q}}(R(\mathfrak{B}))$. Such functions are possibly not differentiable at the diagonals of the canonical charts and we set

$$\delta(f_i)(x,x) = \frac{\partial^+}{\partial x} f_i(x,x) - \frac{\partial^-}{\partial x} f_i(x,x).$$

We define the *regular* contribution of $\langle f_0, f_1, f_2\rangle$ to be

$$\langle f_0, f_1, f_2\rangle_{\mathrm{reg}} = \int_{|R(\mathfrak{B})| \smallsetminus D} \partial_x f_0 \partial_y f_1 \partial_{xy} f_2 + \text{ permutations},$$

where D is the the union of all diagonals $\{x = y\}$ of the canonical charts. We define the *singular* contribution of $\langle f_0, f_1, f_2\rangle$ as

$$\langle f_0, f_1, f_2\rangle_{\mathrm{sg}} = \int_D \partial_x(f_0)\partial_y(f_1)\delta(f_2) + \text{ permutations} - \int_D \frac{1}{2}\delta(f_0)\delta(f_1)\delta(f_2)$$

We set

$$\langle f_0, f_1, f_2\rangle_{\mathrm{an}} = \langle f_0, f_1, f_2\rangle_{\mathrm{reg}} + \langle f_0, f_1, f_2\rangle_{\mathrm{sg}},$$

which is consistent with the previous definition as for functions which are differentiable on the squares, the singular contribution vanishes. Using Cor. 3.7.5 and [**Kol13**, Bem. 5.33] we get the following theorem:

Theorem 4.3.3
Let $f_0, f_1, f_2 \in C^4_{\Delta,\mathbb{Q}}$ be functions. Then we can compute the local height of $X \times_k X$ with respect to the f_i as

$$\langle f_0, f_1, f_2\rangle_{\mathrm{an}} = \langle f_0, f_1, f_2\rangle_{\mathrm{reg}} + \langle f_0, f_1, f_2\rangle_{\mathrm{sg}}. \tag{4.3.3}$$

PROOF. We can use the same argument as in Cor. 4.3.2. Note that in [**Kol13**, Bem. 5.33] again the hypothesis can be relaxed so that the statement of the theorem holds for the chosen degree of differentiability. □

Remark 4.3.4
We can drop the rationality conditions in Thm. 4.3.3 and Cor. 4.3.2.

PROOF. We can assume that $R(\mathfrak{X})$ is the S-subdivision of some graph and that $\mathfrak{B}$ satisfies CC. So we can assume that f_0, f_1, f_2 are functions in $C^2_\Delta(R(\mathfrak{B}))$ which are CC-functions.

We denote by $a\colon |R(\mathfrak{B})| \to \mathbb{R}$ the function which in local charts is given by

$$a(x,y) = x^2 - xy + y^2.$$

Denote for each natural number $m \geq 1$ and each $i \in \{0,1,2\}$ by f_{im} and a_m the m-th lattice approximations on $|R(\mathfrak{B}_m)|$ of f_i and a respectively. Choose c_{imn} piecewise affine on $R(\mathfrak{B}_m)$ such that $|c_{imn}| \leq \frac{1}{4m^2n}$ and such that $f_{im} + c_{imn} \in C^0_{\Delta,\mathbb{Q}}(\mathfrak{B}_m)$.

In the proof of Prop. 3.1.13 we have seen that for all $i \in \{0,1,2\}$ the function $f_{im} + c_{imn} + \frac{1}{n}a_m$ is CC for all $n \geq 1$. Using Rem. 4.2.4 we see that the limit of the sequence

$$\left(\lambda_{f_{0m}+c_{0mn}+\frac{1}{n}a_m, f_{1m}+c_{1mn}+\frac{1}{n}a_m, f_{2m}+c_{2mn}+\frac{1}{n}a_m}(X \times_k X)\right)_{n\in\mathbb{N}}$$

for $n \to \infty$ is

$$\lambda_{f_{0m}, f_{1m}, f_{2m}}(X \times_k X),$$

the height of $X \times_k X$ with respect to the DSP Green's functions f_{im} for $i \in \{0,1,2\}$.

The one-sided derivatives of $\frac{1}{n}a_m$ and c_{imn} tend to zero uniformly for $n \to \infty$. So using the formula in Thm. 4.3.3 we see that

$$\begin{aligned} &\left\langle f_{0m} + c_{0mn} + \frac{1}{n}a_m, f_{1m} + c_{1mn} + \frac{1}{n}a_m, f_{2m} + c_{2mn} + \frac{1}{n}a_m \right\rangle_{\text{alg}} \\ = &\left\langle f_{0m} + c_{0mn} + \frac{1}{n}a_m, f_{1m} + c_{1mn} + \frac{1}{n}a_m, f_{2m} + c_{2mn} + \frac{1}{n}a_m \right\rangle_{\text{an}} \end{aligned}$$

which equals

$$\lambda_{f_{0m}+c_{0mn}+\frac{1}{n}a_m, f_{1m}+c_{1mn}+\frac{1}{n}a_m, f_{2m}+c_{2mn}+\frac{1}{n}a_m}(X \times_k X)$$

by Thm. 4.3.3 converges to

$$\langle f_{0m}, f_{1m}, f_{2m}\rangle_{\text{an}}$$

for $n \to \infty$ using the dominated convergence theorem.

So

$$\lambda_{f_{0m}, f_{1m}, f_{2m}}(X \times_k X) = \langle f_{0m}, f_{1m}, f_{2m}\rangle_{\text{an}}.$$

Now the analytic argument in the proof of [**Kol16a**, Thm. 3.32] shows that

$$\langle f_{0m}, f_{1m}, f_{2m}\rangle_{\text{an}} \to \langle f_0, f_1, f_2\rangle_{\text{an}} \qquad \text{for } m \to \infty.$$

We set $f_{imn} := f_{im} + \frac{1}{n}c_{imn} + \frac{1}{n}a_m$. We can choose uniformly convergent diagonal sequences $(f_{imm})_{m\in\mathbb{N}}$ for each $i \in \{0,1,2\}$ such that

$$\langle f_{0mm}, f_{1mm}, f_{2mm}\rangle_{\text{an}} \to \langle f_0, f_1, f_2\rangle_{\text{an}} \qquad \text{for } m \to \infty.$$

and

$$\langle f_{0mm}, f_{1mm}, f_{2mm}\rangle_{\text{an}} \to \lambda_{f_0\circ\tau, f_1\circ\tau, f_2\circ\tau}(X \times_k X) \qquad \text{for } m \to \infty.$$

Hence

$$\lambda_{f_0\circ\tau, f_1\circ\tau, f_2\circ\tau}(X \times_k X) = \langle f_0, f_1, f_2\rangle_{\mathrm{an}}$$

and this is what we claimed. □

Remark 4.3.5
Theorem 4.3.3 gives an explanation for the application of Formula (4.3.3) in the proof of [**Zha10**, Cor. 1.3.2]). Here, Zhang implicitly uses that (4.3.3) computes a local height. This can for example be justified by Thm. 4.3.3.

4.4. A Monge–Ampère Type Differential Equation

We will make Assumption 2.2.1.

Let X be a smooth, geometrically integral, projective curve over k. Let g_1, g_2 be DSP Green's functions on $X \times_k X$ for the trivial Cartier divisor. Note that by Def. 4.1.4 they define a signed measure

$$c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g_1}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g_2})$$

of total mass zero on $(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}$ the *Chambert-Loir measure.*

Let μ be any signed measure on $(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}$ of total mass zero. We ask the question, whether there is a DSP Green's function g for the trivial divisor such that

$$\mu = c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g}).$$

Similarly as in [**Liu11**] we give an approach to answering the question reducing it to the solvability of a Monge-Ampère equation of mixed type in dimension two. The strategy is as follows:

(i) Find analytic expressions in g computing the integral

$$\int f d c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g})$$

of a dense set S of functions f (e.g. smooth functions),

(ii) try to find g such that

$$\int f d\mu = \int f d c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g})$$

for all $f \in S$ by solving an associated real differential equation,

(iii) if g is a solution of the problem, show that g in fact gives rise to a DSP metric using an approximation argument.

In our case the associated real differential equation in question is the classical real Monge-Ampére equation in dimension two.

Definition 4.4.1
Let $\Omega \subset [0,1]^2$ be a bounded open subset. Let $f \colon \overline{\Omega} \to \mathbb{R}$ be a C^2-function. We denote by $D^2u(x)$ the Hessian of $u : \Omega \to \mathbb{R}$ at a point $x \in \Omega$. The classical *real Monge-Ampère problem* is the solution of

$$\det D^2u(x) = f(x), \qquad u = 0 \text{ on } \partial\Omega \tag{4.4.1}$$

for a function u.

Remark 4.4.2
In [**Liu11**] the Monge-Ampère equation in question is elliptic, in particular in (4.4.1) the function f is positive. However, in our situation only explicit formulae for heights are available only for Monge-Ampère measures coming from metrics on the trivial line bundle. Hence the measures in question will always have total mass zero and therefore the associated real Monge-Ampère equation will always be of mixed type i.e., f changes sign, unless the measure is trivial. Little is known about solution of Monge-Ampère equations of mixed type. For references however, cf. [**HK13**] for a discussion of equations of this type.

We will make Assumption 3.4.5 (i). Let λ be the Lebesgue measure on $S(\mathfrak{B})$ having mass one on each canonical square of $S(\mathfrak{B})$.

Theorem 4.4.3
Let $\mu = fd\lambda$ be a signed measure on $(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}$ which is supported in the interior of a canonical square S of $S(\mathfrak{B})$ which we canonically identify with $[0,1]^2$ for a smooth function f on S. Let $u \in C^3$ be a solution of

$$\det D^2 u(x) = f(x).$$

with support contained in the interior of $[0,1]^2$. [1] *Then*

$$\mu = c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{u\circ\tau}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{u\circ\tau}), \tag{4.4.2}$$

i.e., u is a solution of the non-Archimedean Monge-Ampère equation

Proof. By hypothesis, μ is determined by the equation

$$\int_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}} f d\mu = \int_{S(\mathfrak{B})} fg d\lambda \tag{4.4.3}$$

for every continuous function f on $(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}$ and a fixed smooth function g on $S(\mathfrak{B})$ supported in the interior of the square S. The smooth functions with support in the interior of S are dense in the continuous functions with support in the interior of S. Hence μ is determined by the property that (4.4.3) holds for all smooth functions f with support in the interior of S.

By hypothesis, u satisfies

$$\partial_{xx}u\partial_{yy}u - (\partial_{xy}u)^2 = g$$

with $u = 0$ on $\partial[0,1]^2$.

We will use Lemma 5.5.4 which will be essential in Ch. 5 and get

$$\int_S gh - (\partial_x h\partial_y u\partial_{xy}u + \partial_y h\partial_x u\partial_{xy}u + \partial_{xy}h\partial_x u\partial_y u) = 0$$

for all smooth functions h with support strictly contained in the interior of S. On the other hand side $u\circ\tau$ is DSP, by Cor. 3.7.4. So the measure

$$c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{u\circ\tau}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{u\circ\tau})$$

is defined and by Cor. 4.3.2 is uniquely determined by the property that

$$\int fc_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{u\circ\tau}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{u\circ\tau}) = \int_{|R(\mathfrak{B})|} \partial_x f\partial_y u\partial_{xy}u + \text{ permutations}$$

[1] The hypotheses of Theorem 4.4.3 have been changed on request of the reviewers.

for every smooth function f on S. But we can restrict to functions which have support in the interior of S, because u has that property. Hence u is a solution of the differential equation (4.4.2). □

4.5. A Poisson Type Differential Equation

We will make Assumption 2.2.1. Let X be a smooth, geometrically integral, projective variety over k of dimension n.

Definition 4.5.1
Let D be a Cartier divisor on X and g be a DSP Green's function for D. Let g be a DSP Green's function for D on X^{an}. Let f be a DSP Green's function for the trivial divisor on X^{an}. The *Poisson problem* for g, f is finding a DSP Green's function u for the trivial divisor such that

$$c_1(\mathscr{O}_X, \|\cdot\|_u) \wedge c_1(\mathscr{O}(D), \|\cdot\|_g)^{n-1} = f \cdot c_1(\mathscr{O}(D), \|\cdot\|_g)^n$$

holds as measures on X^{an}. Informally speaking if $\omega = c_1(\mathscr{O}(D), \|\cdot\|_g)$, then we look for a solution of $dd^c u \omega^{n-1} = f\omega^n$.

As in Assumption 3.4.5 (i) we choose a regular strictly semi-stable model $\mathfrak{B}$ over k° of $X \times_k X$. Then we have the following result:

Theorem 4.5.2
[2] *Let $f\colon |R(\mathfrak{B})| \to \mathbb{R}$ be a smooth function such that the support of f is contained in a canonical square S of $R(\mathfrak{B})$ and has empty intersection with the relative boundary of $R(\mathfrak{B})$. Let $g\colon |R(\mathfrak{B})| \to \mathbb{R}$ be a smooth function. Assume that $u\colon S \to \mathbb{R}$ is a smooth solution of the partial differential equation*

$$f(\partial_{xx} g \partial_{yy} g - 2(\partial_{xy} g)^2) = \partial_{xx} g \partial_{yy} u + \partial_{yy} g \partial_{xx} u - 2\partial_{xy} g \partial_{xy} u$$

supported in the interior of S. Then $u \circ \tau$ is a solution for the Poisson problem i.e.

$$\begin{aligned} &c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{u\circ\tau}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g\circ\tau}) = \\ &f \cdot c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g\circ\tau}) \wedge c_1(\mathscr{O}_{(X\times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \|\cdot\|_{g\circ\tau}) \end{aligned} \tag{4.5.1}$$

holds.

Proof. We use the same strategy as in Section 4.4. By hypothesis we have a smooth solution $u : S \to \mathbb{R}$ of the partial differential equation

$$f(\partial_{xx} g \partial_{yy} g - 2(\partial_{xy} g)^2) = \partial_{xx} g \partial_{yy} u + \partial_{yy} g \partial_{xx} u - 2\partial_{xy} g \partial_{xy} u$$

which has support in the interior of S.

For any smooth function h on S we have

$$\int_S h \cdot f(\partial_{xx} g \partial_{yy} g - 2(\partial_{xy} g)^2) = \int_S h \cdot (\partial_{xx} g \partial_{yy} u + \partial_{yy} g \partial_{xx} u - 2\partial_{xy} g \partial_{xy} u).$$

Using Lem. 5.5.4 and using that f and u are supported in the interior of S we see

$$\int_S \partial_x (h \cdot f) \partial_y g \partial_{xy} g + \text{permutations} = \int_S \partial_x h \partial_y g \partial_{xy} u + \text{permutations}. \tag{4.5.2}$$

[2]The hypotheses and the proof of Theorem 4.5.2 have been changed on request of the reviewers.

But by Cor. 3.7.4 and Rem. 3.1.13 the functions $u \circ \tau$, $f \circ \tau$, and $g \circ \tau$ are DSP Green's functions for the trivial divisor. Moreover, by Cor. 4.3.2 and Rem. 4.3.4 the left hand side of (3.1.13) equals

$$\int h \cdot f c_1(\mathscr{O}_{(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \| \cdot \|_{g \circ \tau}) \wedge c_1(\mathscr{O}_{(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \| \cdot \|_{g \circ \tau}),$$

and the right hand side equals

$$\int h \cdot c_1(\mathscr{O}_{(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \| \cdot \|_{u \circ \tau}) \wedge c_1(\mathscr{O}_{(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \| \cdot \|_{g \circ \tau}).$$

Note that the measures $c_1(\mathscr{O}_{(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \| \cdot \|_{u \circ \tau}) \wedge c_1(\mathscr{O}_{(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \| \cdot \|_{g \circ \tau})$ and $c_1(\mathscr{O}_{(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \| \cdot \|_{g \circ \tau}) \wedge c_1(\mathscr{O}_{(X \times_k X)^{\mathrm{an}}_{\mathbb{K}}}, \| \cdot \|_{u \circ \tau})$ are supported in S. The smooth functions on S are dense in the continuous functions. As we could choose h as an arbitrary smooth function on S we get (4.5.1) i.e., u solves the Poisson problem. $\square$

CHAPTER 5

Differential forms on Berkovich Spaces and Monge–Ampère Measures

5.1. Differential Forms on Berkovich spaces

We recall the definition of differential forms on Berkovich spaces after Lagerberg (cf. [**Lag12**]) and Chambert-Loir and Ducros (cf. [**CLD12**]).

Definition 5.1.1
Let V be a real affine space of dimension r, $\vec{V}$ the corresponding vector space, and $\vec{V}^*$ its dual.

(i) Let U be an open subset of V. A *superform* of type (p,q) on U is an element in

$$A^{p,q}(U) = C^\infty(U) \otimes_{\mathbb{R}} \bigwedge^p \vec{V}^* \otimes_{\mathbb{R}} \bigwedge^q \vec{V}^*.$$

This defines a sheaf $A^{p,q}$ on V.

(ii) We have differential operators on the sheaves $A^{p,q}$ given by $d' = D \otimes \mathrm{id}$ and $d'' = (-1)^p \mathrm{id} \otimes D$ on $A^{p,q}$ where D is the usual exterior derivative. Moreover, we have a map $J\colon A^{p,q} \to A^{q,p}$ flipping the factors. Note that given a form $\mu \in \bigwedge^q \vec{V}$ we can contract $\omega \in A^{p,q}(U)$ with respect to the q-component. We denote the contraction by $\langle \omega, \mu \rangle \in A^{p,0}(U)$.

(iii) An *orientation* for V is an orientation for $\vec{V}$. The set of orientations will be denoted by $\mathrm{Or}(V)$.

(iv) We denote by $|\bigwedge^r \vec{V}|$ the set

$$\mathrm{Or}(V) \times \bigwedge^r \vec{V}$$

modulo the equivalence relation $(o,\lambda) \cong (-o,-\lambda)$. A class $[(o,\lambda)]$ will be called a *vector volume*. If $f\colon V \to W$ is an affine isomorphism of affine spaces of dimension r, then we have an induced map $f_*\colon |\bigwedge^r V| \to |\bigwedge^r W|$, which is defined in the obvious way.

(v) Note that $\mathbb{R}^r$ has a canonical vector volume: It is given by the class of

$$((e_1,\dots,e_r), e_1 \wedge \dots \wedge e_r),$$

where $(e_i)_{i=1,\dots,r}$ is the standard basis of $\mathbb{R}^r$.

(vi) Let $\mu = [(o,\lambda)]$ be a vector volume in V and U be an open subset of V. Then for $\omega \in A^{r,r}(U)$ we can define $\int_U \langle \omega, \mu \rangle$ as $\int_U \langle \omega, \lambda \rangle$ where U carries the orientation induced by the orientation o, whenever this integral makes sense. The integral is independent of the choice of a representative for $[(o,\lambda)]$.

(vii) A *polyhedral complex* $\mathscr{C}$ in V is a finite collection of convex polyhedra satisfying the following properties:
(a) If σ is a polyhedron of $\mathscr{C}$ and τ a face of σ, then $\tau \in \mathscr{C}$.
(b) The intersection of two polyhedra in $\mathscr{C}$ is a face of both.
The *support* $|\mathscr{C}|$ of $\mathscr{C}$ is the set theoretic union of all polyhedra of $\mathscr{C}$. Every polyhedron S of dimension l spans an l-dimensional affine subspace $\mathbb{A}_S$ of V with associated linear space $\mathbb{L}_S$.

(viii) Let $\mathscr{C}$ be a polyhedral complex and $P \in \mathscr{C}$ be a polyhedron. The *relative interior* of P is the topological interior of P in $\mathbb{A}_P$.

(ix) We define a *superform* α of type (p, q) on an open subset U of $|\mathscr{C}|$ is given by the equivalence class of elements $\alpha' \in A^{p,q}(\Omega)$, where Ω is an open subset of V with $\Omega \cap |\mathscr{C}| = U$ modulo the following equivalence relation: Two superforms α', α'' are the same if their restrictions to each polyhedron of $\mathscr{C}$ agree.

(x) A *calibration* of a polyhedral complex $\mathscr{C}$ of pure dimension n is a map which assigns to every polyhedron C of dimension n an element μ_C in $|\bigwedge^n \mathbb{L}_C|$. A *calibrated polyhedral complex* is a polyhedral complex $\mathscr{C}$ together with a calibration.

(xi) We define the integration of superforms of type (n, n) on an open subset U of a calibrated polytope $(\mathscr{C}, \mu)$. We denote by $\mathscr{C}_n$ the set of n-dimensional polyhedra of $\mathscr{C}$. We set

$$\int_U \omega = \sum_{C \in \mathscr{C}_n} \int_{U \cap C} \langle \omega, \mu_C \rangle .$$

Now we can introduce differential forms on Berkovich spaces. Let $\mathbb{K}$ be a field which is complete with respect to a non-trivial, non-Archimedean absolute value $|\cdot|$. Let X be a $\mathbb{K}$-analytic space.

Let T be a split algebraic torus of rank r over $\mathbb{K}$ i.e., a $\mathbb{K}$-scheme isomorphic to $\mathbb{G}_m^r = \operatorname{Spec}(\mathbb{K}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}])$ where X_i are indeterminates for $i \in \{1, \ldots, n\}$. Denote by $X(T)$ the abelian group of maps of analytic groups from T^{an} to $\mathbb{G}_m^{\mathrm{an}}$. We denote by T_{trop} the $\mathbb{R}$-vector space $\operatorname{Hom}(X(T), \mathbb{R})$.

We define a map $\operatorname{trop}\colon T^{\mathrm{an}} \to T_{\mathrm{trop}}$: Let $t \in T^{\mathrm{an}}$ be a point. The element of T_{trop} which assigns to $\varphi \in X(T)$ the value $-\log|\varphi(t)|$ is called $\operatorname{trop}(t)$.

If $f\colon X \to T^{\mathrm{an}}$ is a map of $\mathbb{K}$-analytic spaces then the map $\operatorname{trop} \circ f$ is called f_{trop}. We have a distinguished isomorphism of $(\mathbb{G}_m^n)_{\mathrm{trop}}$ with $\mathbb{R}^r$ given by mapping x to

$$(-\log|X_1(x)|, \ldots, -\log|X_r(x)|).$$

A map $f\colon X \to T$ of a k-analytic space X to a split torus T will be called a *moment map*.

Definition 5.1.2

(i) For a tuple $x = (x_1, \ldots, x_r) \in \mathbb{R}^r_{>0}$ the *Gauß norm* on $\mathbb{K}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ is given by

$$\sum_{\nu \in \mathbb{Z}^r} a_\nu X^\nu \mapsto \max_{\nu \in \mathbb{Z}^r} |a_\nu| x^\nu,$$

where by x^ν we mean $X^\nu(x)$.

(ii) The map $\operatorname{trop}\colon \mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{R}^n$ has a canonical section: It assigns to each tuple $\underline{x} \in \mathbb{R}^r$ the Gauß-norm on $\mathbb{K}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ corresponding to the tuple $(\exp(-x_1), \ldots, exp(-x_r))$. The image of this section will be denoted by S. By [**CLD12**, 2.2.1] if $\varphi\colon \mathbb{G}_m^{r,\mathrm{an}} \to T^{\mathrm{an}}$ is any isomorphism of the analytifictation of a split algebraic torus T with $\mathbb{G}_m^{r,\mathrm{an}}$ then $\varphi(S)$ does not depend on φ and will be called the *topological skeleton* $T^{\mathrm{an}}_{\mathrm{sq}}$ of T.

We will state the Bieri-Groves theorem:

Theorem 5.1.3 (Bieri,Groves,Ducros)
Let X be a $\mathbb{K}$-analytic space of dimension n and $f\colon X \to T^{\mathrm{an}}$ a map to the analytification of a split algebraic torus T. Then $f_{\mathrm{trop}}(X)$ is a polyhedral complex in T_{trop} of dimension at most n. Moreover, the image of the boundary $\partial(X/\mathscr{M}(\mathbb{K}))$ of X is contained in a polyhedron of dimension at most $n-1$.

Proof. This particular version is proven in [**Duc12**, Thm.3.2]. Cf. also ([**BG86**]) and [**Gub07**, Prop. 5.4]. □

In the sequel let X be a $\mathbb{K}$-analytic space of pure dimension n.

Definition 5.1.4
A *tropical chart* on X is given by a map $f\colon X \to T^{\mathrm{an}}$ to the analytifiction a split algebraic torus T and a compact polyhedron P in $T^{\mathrm{an}}_{\mathrm{trop}}$ which contains $f_{\mathrm{trop}}(X)$. A *map of tropical* charts $(f\colon X \to T^{\mathrm{an}}, P)$ and $(f'\colon X \to T'^{\,\mathrm{an}}, P')$ is given by a morphism of tori $q\colon T^{\mathrm{an}} \to T'^{\,\mathrm{an}}$ such that $f' = q \circ f$ and $q_{\mathrm{trop}}(P) \subset P'$.

Definition 5.1.5
Let U be an open subset of X.

(i) We define the *presheaf* $A^{p,q}_{\mathrm{pre}}$ of *smooth differential forms* of type (p,q) as follows: For an open subset X of U the group $A^{p,q}_{\mathrm{pre}}(U)$ is the colimit over all $A^{p,q}(P)$ where $(f\colon U \to X, P)$ runs over all tropical charts.
(ii) We define the *sheaf* $A^{p,q}$ *of smooth differential forms* of type (p,q) as the sheaf associated to $A^{p,q}_{\mathrm{pre}}$. We say that an element $x \in A^{p,q}$ is *tropicalised* on an open set U of X if it is in the image of the natural map

$$f^*\colon A^{p,q}(P) \to A^{p,q}(U)$$

for a tropical chart $(f\colon U \to T, P)$.
(iii) We denote by d' and d'' the differential operators on $A^{p,q}$ induced by the respective operators on superforms on polyhedra.

Remark 5.1.6
By [**CLD12**, 3.1.3] the forms of type $(0,0)$ can be identified with continuous functions $f\colon X \to \mathbb{R}$ that are locally of the form

$$\varphi \circ (-\log|f_1|, \ldots, -\log|f_s|)$$

for invertible analytic functions f_i with $i \in \{1, \ldots, s\}$ and a smooth function $\varphi\colon U \to \mathbb{R}$ on an open subset U of $\mathbb{R}^s$.

By the Bieri-Groves theorem, Thm. 5.1.3, the tropicalisation $f_{\mathrm{trop}}(X)$ of a map $f\colon X \to T$ from a $\mathbb{K}$-analytic space X of dimension n to a split torus T in T_{trop} of rank r is the support of a polyhedral complex $\mathscr{C}$. After [**CLD12**, §3.5] this polyhedral complex has a canonical nowhere vanishing calibration which we will describe in the sequel:

Definition 5.1.7
Let $C \in \mathscr{C}$ be an n-dimensional polyhedron of $f_{\mathrm{trop}}(X)$. Choose a morphism of tori $q\colon T \to \mathbb{G}_m^n$ such that the restriction to $\mathbb{A}_C$ of

$$q_{\mathrm{trop}}\colon T_{\mathrm{trop}} \to (\mathbb{G}_m^n)_{\mathrm{trop}} = \mathbb{R}^n$$

is an isomorphism. Let $\sigma\colon \mathbb{R}^n \to T_{\mathrm{trop}}$ be the unique section of q_{trop} whose image is $\mathbb{A}_C$. By Thm. 5.1.3 the set $(q \circ f)_{\mathrm{trop}}(\partial X)$ is contained in a polyhedron of dimension $\leq n-1$. So after a refinement of $\mathscr{C}$ we can assume that for every $D \in \mathscr{C}$ the image of the relative interior of D under q_{trop} has empty intersection with $(q \circ f)_{\mathrm{trop}}(\partial X)$. By [**CLD12**, 2.2.5] the tropicalisation map $T \to T_{\mathrm{trop}}$ is a homeomorphism when restricted to the topological skeleton of T (Def. 5.1.2 (ii)). Hence we can identify D with the corresponding subset of the skeleton of T. After [**CLD12**, §2.4] the map f is finite and flat over every point of D° of some positive degree d_D (cf. Def. D.2). Let $v = ((e_1, \ldots, e_n), e_1 \wedge \ldots \wedge e_n)$ be the canonical vector volume of $\mathbb{R}^n$ (cf. Def. 5.1.1 (v)). Set

$$\mu_D = d_D \cdot \sigma_*(v).$$

We will define the integration of (n,n)-forms on a Berkovich space X of dimension n.

Definition 5.1.8
An *atlas of integration* for an (n,n)-form ω is a family $(X_i, \lambda_i, Y_i)_{i\in I}$ of triples where

- I is an index set,
- (X_i) is a family of relatively compact opens in X covering $\mathrm{supp}(\omega)$,
- (λ_i) is a smooth partition of unity associated to (X_i),
- Y_i is an analytic neighbourhood of $\mathrm{supp}(\lambda_i\omega|_{X_i})$ in X_i and $\lambda_i\omega|_{Y_i}$ is tropicalised on Y_i.

If an atlas of integration exists we set

$$\int_X \omega = \sum_i \int_{Y_i} \lambda_i\omega$$

if the expression is defined. The expressions on the right side are explained as follows: $\lambda_i\omega$ is tropical on Y_i, hence it is the image of $\omega' \in A^{n,n}(P)$ for a tropical chart $(f\colon Y_i \to T, P)$. We restrict ω' to $f_{\mathrm{trop}}(Y_i)$. We denote by μ the canonical calibration of $f_{\mathrm{trop}}(Y_i)$ and set

$$\int_{Y_i} \lambda_i\omega := \int_{f_{\mathrm{trop}}(Y_i)} \langle \omega', \mu\rangle$$

if the expression is defined.

Definition 5.1.9
Let V be an affine space of dimension n and let X be a $\mathbb{K}$-analytic space.

(i) Let α be a superform on V of type (p,p) on an open subset U. We say that α is *strongly positive* if there exist superforms $\alpha_{j,s}$ of type $(1,0)$ and non-negative functions f_s such that

$$\alpha = \sum_s f_s \alpha_{1,s} \wedge J(\alpha_{1,s}) \wedge \ldots \wedge \alpha_{p,s} \wedge J(\alpha_{p,s}).$$

(ii) Let P be a polyhedron. We say that a superform α is *strongly positive* if there is a polyhedral decomposition of P such that the restriction of α to each polyhedron is strongly positive.
(iii) A form α of type (p,p) on X is *strongly positive* if every point has an open neighbourhood which admits a moment map $f : U \to T$ such that α is tropical on U, say $\alpha = f^*\omega$, and ω is a strongly positive form on $f_{\mathrm{trop}}(U)$.
(iv) A smooth function $f\colon X \to \mathbb{R}$ is *plurisubharmonic* (psh) if $d'd''f$ is a strongly positive $(1,1)$-form.
(v) A continuous function $f\colon X \to \mathbb{R}$ is *globally psh-approximable* or just psh-approximable if it is the uniform limit of smooth psh functions. We say that f is *approximable* if it can be written as the difference of psh-approximable functions.

Let L be a line bundle on a proper $\mathbb{K}$-analytic space X. We say that a smooth metric $\|\cdot\|$ on L is *plurisubharmonic* if for each local invertible section s the function $-\log\|s\|$ is a plurisubharmonic function.

We have a distance on the space of continuous metrics on L: For two metrics $\|\cdot\|$, $\|\cdot\|'$ we define

$$d(\|\cdot\|,\|\cdot\|') = \sup_{x\in X} |\log\|s(x)\| - \log\|s(x)\|'|$$

where $s \neq 0$ is any rational section of L.

Similarly we say that a metric $\|\cdot\|$ is psh-approximable if it is the uniform limit of smooth psh-metrics with respect to d. We say it is approximable if there exist metrics $\|\cdot\|_1$, $\|\cdot\|_2$ on line bundles L_1 and L_2 respectively such that

$$(L,\|\cdot\|) = (L_1,\|\cdot\|_1) \otimes (L_2,\|\cdot\|_2)^{-1},$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are psh-appriximable metrics. [1]

Assume that $\|\cdot\|$ is a smooth metric on the line bundle L. We can define the first Chern form $c_1(L,\|\cdot\|)$ as follows: Pick a point x and a section s which is invertible in a neighbourhood U of x. Then we set

$$c_1(L,\|\cdot\|)|_U = d'd''(-\log\|s\|).$$

This glues together to a smooth form $c_1(L,\|\cdot\|)$ on the whole of X.

Remark 5.1.10
Let X be a proper $\mathbb{K}$-analytic space of pure dimension d. If $\omega \in A^{d,d}(X)$ is a smooth form then there exists a unique Radon measure μ on X such that for every function $f \in A^{0,0}(X)$ we have $\int f d\mu = \int f\cdot\omega$.

Proof. See [**CLD12**, 5.4.6]. □

Proposition 5.1.11
Let $L_1,\dots,L_d$ be line bundles on a proper $\mathbb{K}$-analytic space X of dimension d. Assume that for every $i \in \{1,\dots,d\}$ the line bundle L_i carries the metric $\|\cdot\|_i$ which is the uniform limit of a family of smooth psh metrics $(\|\cdot\|_{in})_{n\in\mathbb{N}}$. Then the sequence of Radon measures

$$(c_1(L_1,\|\cdot\|_{1n})\wedge\dots\wedge c_1(L_d,\|\cdot\|_{dn}))_{n\in\mathbb{N}}$$

converges weakly. We denote by

$$c_1(L_1,\|\cdot\|_1)\wedge\dots\wedge c_1(L_d,\|\cdot\|_d)$$

the limit. This is a Radon measure which we will call Monge–Ampère measure.

Proof. Let $(U_j)_{j\in J}$ be an open cover of X which admits invertible sections s_{ij} of L_i on U_j for all i and j. By compactness of X we can choose J finite. Now by [**CLD12**, 5.6.5] there for each $j \in J$ there exists a unique measure

$$d'd''(-\log\|s_{1j}\|_1)\wedge\dots\wedge d'd''(-\log\|s_{dj}\|_d)$$

such that if $f\colon U_j \to \mathbb{R}$ is a smooth function with compact support then

$$\int_{U_j} f\cdot d'd''(-\log\|s_{1j}\|_1)\wedge\dots\wedge d'd''(-\log\|s_{dj}\|_d) =$$
$$\lim_{n\to\infty}\int_{U_j} f\cdot d'd''(-\log\|s_{1j}\|_{1n})\wedge\dots\wedge d'd''(-\log\|s_{dj}\|_{dn}).$$

As $-\log\|s_{ij}\otimes s_{il}^{-1}\|_i$ equals $-\log|h|$ for some invertible function h on $U_j\cap U_l$ for all i,j,l the form

$$d'd''(-\log\|s_{ij}\|) - d'd''(-\log\|s_{il}\|)$$

[1] The definition of approximable metrics has been completed on request of the reviewers.

vanishes on $U_j \cap U_l$. Hence

$$d'd''(-\log \|s_{1j}\|_1) \wedge \ldots \wedge d'd''(-\log \|s_{dj}\|_d)$$

glues together to a well-defined measure

$$c_1(L_1, \|\cdot\|_1) \wedge \ldots \wedge c_1(L_d, \|\cdot\|_d).$$

We want to show the statement about weak convergence. We choose a partition of unity $(\lambda_j)_{j\in J}$ subordinated to the cover. Let f be an arbitrary smooth function on X. Then $\lambda_j f$ has compact support inside U_j and by construction we have

$$\int_X f \cdot c_1(L_1, \|\cdot\|_1) \wedge \ldots \wedge c_1(L_d, \|\cdot\|_d) =$$
$$\sum_j \int_{U_j} \lambda_j f \cdot d'd''(-\log \|s_{1j}\|) \wedge \ldots \wedge d'd''(-\log \|s_{dj}\|) =$$
$$\sum_j \lim_{n\to\infty} \int_{U_j} \lambda_j f \cdot d'd''(-\log \|s_{1j}\|_{1n}) \wedge \ldots \wedge d'd''(-\log \|s_{dj}\|_{dn}) =$$
$$\lim_{n\to\infty} \int_X f \cdot c_1(L_1, \|\cdot\|_{1n}) \wedge \ldots \wedge c_1(L_d, \|\cdot\|_{dn})$$

hence the weak convergence. □

The following theorem connects Chambert-Loir measures (cf. Def. 4.1.4) and Monge–Ampère measures.

Theorem 5.1.12
[2] *Let X be a projective variety over $\mathbb{K}$ of dimension $d \geq 0$. Let $L_1, \ldots, L_d$ be formally metrised line bundles. Then*

(i) The metrics on L_i are approximable.
(ii) The Monge–Ampère measure coincides with the Chambert-Loir measure.

PROOF. Assertion (i) is [**CLD12**, 6.3.5]. [**CLD12**, Thm. 6.9.3] shows that the measure conincides with the measure defined in [**CL06**, Def. 2.4] and [**Gub07**, Prop. 3.11] in turn shows that this in fact coincides with our definition of Chambert-Loir-measure, so (ii) follows. □

5.2. PSH-Metrics and Monge–Ampère Measures

We will assume that $\mathbb{K}$ is a field which is complete with respect to a non-trivial, non-Archimedean absolute value. Denote by $\mathbb{K}^\circ$ its valuation ring. We will prove the following assertion connecting semipositive algebraic metrics with psh metrics.

Proposition 5.2.1
Let X be a projective variety of dimension d over $\mathbb{K}$ and L be an ample line bundle on X. Let $\|\cdot\|$ be a metric on L and assume that there exists a family of projective models $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ and semipositive line bundles $\mathfrak{L}_n$ on each $\mathfrak{X}_n$ such that $\|\cdot\|_{\mathfrak{L}_n} \to \|\cdot\|$ for $n \to \infty$. Then $\|\cdot\|$ is globally psh-approximable.

PROOF. By Prop. C.5 each $\|\cdot\|_n$ is the limit of model metrics induced by ample models $\mathfrak{L}_{ns}$. By [**CLD12**, Prop. 6.3.2] every $\|\cdot\|_{\mathfrak{L}_{ns}}$ is the limit of smooth psh metrics $\|\cdot\|_{nsl}$, as the pull-back of smooth psh metrics is smooth psh again using [**CLD12**, 3.1.7].

[2] The statement of Theorem 5.1.12 has been changed on request of the reviewers.

Fix $n, k \in \mathbb{N}$. We pass to a subsequence $(\|\cdot\|_{nsl})_{l\in N}$ such that $d(\|\cdot\|_{\mathfrak{L}_{nsl}}, \|\cdot\|_{ns}) < \frac{1}{l}$. Now fix $n \in \mathbb{N}$. We pass to a subsequence $(\|\cdot\|_{\mathfrak{L}_{ns}})_{s\in\mathbb{N}}$ such that $d(\|\cdot\|_{\mathfrak{L}_{ns}}, \|\cdot\|_{n}) < \frac{1}{s}$. Then we pass to a subsequence to achieve that $d(\|\cdot\|_{\mathfrak{L}_n}, \|\cdot\|) < 1/n$. Now the diagonal sequence $(\|\cdot\|_{nnn}\|)_{n\in\mathbb{N}}$ satisfies $d(\|\cdot\|_{nnn}, \|\cdot\|) < 3/n$. We have found a sequence $\|\cdot\|_n$ which converges uniformly to $\|\cdot\|$, hence the claim. □

Let $L_1, \ldots, L_d$ be ample[3] line bundles on X. For each $i \in \{1, \ldots, d\}$ let $\|\cdot\|_i$ be a semipositive metric and assume that each $\|\cdot\|_i$ is the limit of a series of vertically nef model metrics $(\|\cdot\|_{\mathfrak{L}_{im}})_{m\in\mathbb{N}}$ coming from projective models of X. We can define the associated Chambert-Loir measure μ^{CL} (Def. 4.1.4). By Prop. 5.2.1 each $\|\cdot\|_i$ is a psh approximable metric and we can define the Monge–Ampère measure μ^{MA}.

Theorem 5.2.2
We have the equality of measures $\mu^{\mathrm{MA}} = \mu^{\mathrm{CL}}$.

Proof. As both measures are Radon measures on X^{an} by [**CLD12**, Prop. 5.4.6] and [**Gub07**, 3.12] we can achieve the equality by showing that for every continuous function $f\colon X^{\mathrm{an}} \to \mathbb{R}$ the equality

$$\int f d\mu^{\mathrm{MA}} = \int f d\mu^{\mathrm{CL}}$$

holds. As every continuous function is the uniform limit of smooth functions by [**CLD12**, Prop. 5.4.6] it suffices to restrict to smooth functions $f\colon X^{\mathrm{an}} \to \mathbb{R}$.

Fix a smooth function $f\colon X^{\mathrm{an}} \to \mathbb{R}$. For every $i \in \{1, \ldots, d\}$ we assumed that $\|\cdot\|_i$ is the uniform limit of a series of model metrics $(\|\cdot\|_{\mathfrak{L}_{im}})_{m\in\mathbb{N}}$. We denote for each $m \in \mathbb{N}$ by μ_m^{CL} the Chambert-Loir measure associated to the model metrics $\|\cdot\|_{\mathfrak{L}_{im}}$ for $i \in \{1, \ldots, d\}$. Then by [**Gub07**, Prop. 3.12] we have weak convergence of measures $\mu_m^{\mathrm{CL}} \to \mu^{\mathrm{CL}}$ for $m \to \infty$. For each $i \in \{1, \ldots, d\}$ we pass to a subsequence of $(\|\cdot\|_{\mathfrak{L}_{im}})_{m\in\mathbb{N}}$ to achieve

$$\left| \int f d\mu_m^{\mathrm{CL}} - \int f d\mu^{\mathrm{CL}} \right| < \frac{1}{m}$$

and $d(\|\cdot\|_{\mathfrak{L}_{im}}, \|\cdot\|_i) < \frac{1}{m}$. Now by Prop. 5.2.1 each $\|\cdot\|_{\mathfrak{L}_{im}}$ is the limit of a series of smooth psh metrics $(\|\cdot\|_{iml})_{l\in\mathbb{N}}$. We denote by μ_{ml}^{MA} the corresponding Monge–Ampère measures. By Thm. 5.1.11 for each $m \in \mathbb{N}$ the sequence of real numbers $(\int f d\mu_{ml}^{\mathrm{MA}})_{l\in\mathbb{N}}$ converges and from Thm. 5.1.12 it follows that for each m the limit is $\int f d\mu_m^{\mathrm{CL}}$. For each $l \in \mathbb{N}$ we pass to a subsequence of $(\|\cdot\|_{iml})_{l\in\mathbb{N}}$ such that

$$\left| \int f d\mu_m^{\mathrm{CL}} - \int f d\mu_{ml}^{\mathrm{MA}} \right| < \frac{1}{l}$$

and

$$d(\|\cdot\|_{iml}, \|\cdot\|_{im}) < \frac{1}{l}$$

holds after renumbering the sequence for all $i \in \{1, \ldots, d\}$.

Now in particular we have

$$d(\|\cdot\|_{inn}, \|\cdot\|_i) < \frac{2}{n}$$

[3]The hypothesis "ample" has been added on request of the reviewers.

for all $n \geq 1$. Using Prop. 5.1.11 we can pass to a subsequence of $(\|\cdot\|_{inn})_{n\in\mathbb{N}}$ such that

$$\left|\int f d\mu_{nn}^{\mathrm{MA}} - \int f d\mu^{\mathrm{MA}}\right| < \frac{1}{n}.$$

Then for all $n \geq 1$ we have

$$\left|\int f d\mu^{\mathrm{CL}} - \int f d\mu^{\mathrm{MA}}\right| \leq \left|\int f d\mu^{\mathrm{CL}} - \int f d\mu_n^{\mathrm{CL}}\right| + \left|\int f d\mu_n^{\mathrm{CL}} - \int f d\mu_{nn}^{\mathrm{MA}}\right| + \left|\int f d\mu_{nn}^{\mathrm{MA}} - \int f d\mu^{\mathrm{MA}}\right| < \frac{3}{n}$$

so

$$\int f d\mu^{\mathrm{CL}} = \int f d\mu^{\mathrm{MA}}$$

which we wanted to prove. □

5.3. The Canonical Calibration for a Poly-Stable Scheme

Let $\mathbb{K}$ be a field which is complete with respect to a non-trivial non-Archimedean absolute value $|\cdot|$. Let $\mathbb{K}^\circ$ be the valuation ring and $\mathbb{K}^{\circ\circ}$ be the maximal ideal of $\mathbb{K}^\circ$ and $\widetilde{\mathbb{K}} = \mathbb{K}^\circ/\mathbb{K}^{\circ\circ}$ the residue field. We make the following assumption.

Assumption 5.3.1
Let $\mathfrak{U}$ be an affine scheme of dimension $m+1$ locally of finite presentation over $\mathbb{K}^\circ$ such that there exists an étale morphism

$$\varphi : \mathfrak{U} \to \mathfrak{S}(\underline{n}, \underline{p}, \underline{\alpha}) =: \mathfrak{S}$$

to a standard poly-stable scheme $\mathfrak{S}$ associated to tuples $\underline{n}, \underline{p}$, and $\underline{\alpha}$ (cf. Def. B.1 (i)), and such that the induced map between the sets of strata $\varphi_*\colon \mathrm{str}(\mathfrak{U}) \to \mathrm{str}(\mathfrak{S})$ from cf. Def. B.1 (iv) is an isomorphism. We denote by f_{ij} the pull-backs of the functions T_{ij} on $\mathfrak{S}$ to $\mathscr{O}_{\mathfrak{U}}(\mathfrak{U})$. We set $n = \sum p_i$. By projection to the first $p_i - 1$ factors for each $i \in \{1, \ldots, r\}$ we get an analytic map to the analytification of a split torus

$$\psi\colon \widehat{\mathfrak{S}}^{\mathrm{an}} \to \mathbb{G}_m^{n-r,\mathrm{an}} =: T^{\mathrm{an}}.$$

We set $t := (\psi \circ \widehat{\varphi}^{\mathrm{an}})_{\mathrm{trop}}$ and $S := t(\widehat{\mathfrak{U}}^{\mathrm{an}})$. We will identify $T^{\mathrm{an}}_{\mathrm{trop}}$ with $\mathbb{R}^{n-r}$ in the canonical way.

Our aim is to describe the tropicalisation S.

Theorem 5.3.2

(i) *The set $S \subset \mathbb{R}^{n-r}$ is given by*

$$S = (-\log|\alpha_1| \cdot |\Delta_{p_1-1}|) \times \ldots \times (-\log|\alpha_r| \cdot |\Delta_{p_r-1}|),$$

where for $m \geq 1$ the space $|\Delta_m|$ is given by

$$\left\{(x_1, \ldots, x_m) \in [0,1]^m \,\middle|\, \sum_{i=1}^m x_i \leq 1\right\}.$$

(ii) *The restriction of the map $(\psi \circ \widehat{\varphi}^{\mathrm{an}})_{\mathrm{trop}}$ to $S(\mathfrak{U})$ is a homeomorphism onto its image: It is the homeomorphism between $S(\mathfrak{U})$ and $|R(\mathfrak{U})|$ from Thm. B.6 (v).*

(iii) *If $S(\mathfrak{U})$ is m-dimensional the image of the relative Berkovich boundary $\partial(\widehat{\mathfrak{U}}^{\mathrm{an}}/\mathscr{M}(\mathbb{K}))$ under t is contained in the topological boundary of S in $\mathbb{R}^{n-r}$.*

(iv) *Assume that* $\widetilde{\mathbb{K}}$ *is algebraically closed and assume that* $S(\mathfrak{U})$ *is* m*-dimensional. For each factor* $\mathbb{R}^{p_i-1}$ *of* $T^{\mathrm{an}}_{\mathrm{trop}}$ *for* $i \in \{1, \dots, r\}$ *we denote by* e_{il} *the* l*-th element of the standard basis for* $l \in \{1, \dots, p_i - 1\}$*. We consider the polyhedral decomposition* $\mathscr{D}$ *of* S *containing only* S *itself and its proper faces. Then the canonical calibration (cf. Def. 5.1.7) of* $\mathscr{D}$ *is given by assigning to* S *the class of*

$$\Big((e_{11}, \dots, e_{r,p_r-1}), e_{11} \wedge \dots \wedge e_{r,p_r-1}\Big) \in \Big|\bigwedge\nolimits^{n-r} \mathbb{R}^{n-r}\Big|.$$

Proof. By construction the map $(\psi \circ \widehat{\varphi}^{\mathrm{an}})_{\mathrm{trop}}$ is given by mapping $x \in \widehat{\mathfrak{U}}^{\mathrm{an}}$ to

$$(-\log|f_{1,1}(x)|, \dots, -\log|f_{1,p_1-1}(x)|, \dots, -\log|f_{r,1}(x)|, \dots, -\log|f_{r,p_r-1}(x)|) \in \mathbb{R}^{n-p}. \tag{5.3.1}$$

We consider the map from $\widehat{\mathfrak{U}}^{\mathrm{an}}$ to $|\Delta|_{p_1-1} \times \dots \times |\Delta_{p_r-1}|$ which is given by assigning to x the point

$$(-\log|f_{1,1}(x)|/-\log|\alpha_1|, \dots, -\log|f_{1,p_1}(x)|/-\log|\alpha_1|, \dots,$$
$$-\log|f_{r,1}(x)|/-\log|\alpha_r|, \dots, -\log|f_{r,p_r}(x)|/-\log|\alpha_r|).$$

By Thm. B.6 (v), under our assumptions if we restrict this map to the skeleton of $\widehat{\mathfrak{U}}^{\mathrm{an}}$ associated to $\mathfrak{U}$, then it becomes a homeomorphism onto its image. From this it follows that (5.3.1) is a homeomorphism onto its image, and hence (i) and (ii) follow.

We prove (iii): First we assume $\varphi = \mathrm{id}$, hence $f_{ij} = T_{ij}$. This hypothesis will be removed in the final step of the proof. The topological boundary of S in $\mathbb{R}^{n-r}$ is given by

$$\partial S = \Bigg\{ (x_{11}, \dots, x_{r,p_r-1}) \in [0, -\log|\alpha_1|]^{p_1-1} \times \dots \times [0, -\log|\alpha_r|]^{p_r-1} \;\Bigg|\; \forall i : \sum_{j=1}^{p_i-1} x_{ij} \le -\log|\alpha_i| \wedge \left(\exists l : \sum_{m=1}^{p_l-1} x_{lm} = -\log|\alpha_l| \;\; \vee \exists l \exists m : x_{ml} = 0\right)\Bigg\}.$$

Let $x \in \partial(\widehat{\mathfrak{U}}^{\mathrm{an}}/\mathscr{M}(\mathbb{K}))$ be a point in the boundary. If $S(\mathfrak{U})$ is m-dimensional, then the unique maximal stratum of the special fibre of $\mathfrak{U}$ is a point. Hence, $V(T_{ij})_{ij}$ is a point in the special fibre. Let $\pi\colon \widehat{\mathfrak{U}}^{\mathrm{an}} \to \mathfrak{U}_s$ be the reduction map. By Prop. D.3 the set $\mathrm{Int}(\widehat{\mathfrak{U}}^{\mathrm{an}}/\mathscr{M}(\mathbb{K}))$ is given by the preimage of the set of closed points of $\mathfrak{U}_s$ under π and by [**Ber90**, Lem. 2.4.1] we have

$$\pi^{-1}(V((T_{ij})_{ij})) = \bigcap_{i,j} \left\{ y \in \widehat{\mathfrak{U}}^{\mathrm{an}} \;\middle|\; |T_{ij}(y)| < 1 \right\} \subset \mathrm{Int}(\widehat{\mathfrak{U}}^{\mathrm{an}}/\mathscr{M}(\mathbb{K})). \tag{5.3.2}$$

But as x is a point in the boundary, there must be indices $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n_i\} = \{1, \dots, p_i\}$ (recall that $S(\mathfrak{U})$ was assumed m-dimensional) such that $|T_{ij}(x)| = 1$, but this means that $-\log|T_{ij}(x)| = 0$. As

$$\sum_{l=1}^{p_i} -\log|T_{il}(x)| = -\log|\alpha_i|$$

we conclude that $\psi_{\mathrm{trop}}(x) \in \partial S$ holds and hence the claim.

We prove (iv): The polyhedral complex $\mathscr{D}$ has only one top-dimensional cell, namely S. As the dimension of S equals $n - r$, we can choose $q = \sigma = \mathrm{id}$ using the notation from Def. 5.1.7. By (iii) we have empty intersection of the interior of S with $(\psi \circ \widehat{\varphi}^{\mathrm{an}})_{\mathrm{trop}}(\partial(\widehat{\mathfrak{U}}^{\mathrm{an}}/\mathscr{M}(\mathbb{K})))$, so we need not pass to a subdivision of $\mathscr{D}$. We denote by

$T_{\mathrm{sq}}^{\mathrm{an}}$ the topological skeleton of T^{an} (cf. Def. 5.1.2 (ii)). The natural map $\chi\colon T_{\mathrm{sq}}^{\mathrm{an}} \to \mathbb{R}^{n-p}$ is a homeomorphism by [**CLD12**, 2.2.5]. We denote by $\mathrm{Int}(S)$ the topological interior of S in $\mathbb{R}^{n-p}$. By [**CLD12**, §2.4] $(\psi \circ \widehat{\varphi}^{\mathrm{an}})$ is finite and flat of some constant degree $d \geq 1$ over each point of $\chi^{-1}(\mathrm{Int}(S))$ (cf. Def. D.2).

We claim that $d = 1$ holds: We have assumed that $S(\mathfrak{U})$ is of dimension m. Hence, the maximal strata of the special fibres of $\mathfrak{U}$ and $\mathfrak{S}$ are each zero-dimensional, hence closed points z and w. As ψ is an isomorphism onto its image, it suffices to show that for any point $x \in \psi^{-1}(\chi^{-1}(\mathrm{Int}(S)))$ the degree of $\widehat{\varphi}^{\mathrm{an}}$ above x is one. By (5.3.2) the reduction of x will be the point w. The induced map between the residue fields of z and w is an isomorphism, as $\widetilde{\mathbb{K}}$ is algebraically closed. Moreover, by assumption, w has only one preimage under φ_s. By [**Ber99**, Lemma 4.4] $\pi^{-1}(\{z\}) \to \pi^{-1}(\{w\})$ is an isomorphism. Hence the degree of $\widehat{\varphi}^{\mathrm{an}}$ is one i.e., $d = 1$. By construction, the canonical calibration will be the class of

$$\Big((e_{11}, \dots, e_{r,p_r-1}), e_{11} \wedge \dots \wedge e_{r,p_r-1}\Big)$$

as desired.

It remains to remove the hypothesis $\varphi = \mathrm{id}$ in (iii). From [**Ber90**, Prop. 2.5.8 iii)] we see that

$$\varphi(\partial(\widehat{\mathfrak{U}}^{\mathrm{an}}/\mathscr{M}(\mathbb{K}))) \subset \varphi(\partial(\widehat{\mathfrak{U}}^{\mathrm{an}}/\widehat{\mathfrak{S}}^{\mathrm{an}})) \cup \partial(\widehat{\mathfrak{S}}/\mathscr{M}(\mathbb{K}))$$

Let $p \in \partial(\widehat{\mathfrak{U}}^{\mathrm{an}}/\widehat{\mathfrak{S}}^{\mathrm{an}})$ and let χ_p be the associated character and write $\widehat{\mathfrak{U}}^{\mathrm{an}} = \mathscr{M}(\mathfrak{B})$, $\widehat{\mathfrak{S}}^{\mathrm{an}} = \mathscr{M}(\mathfrak{A})$. Then by [**Ber90**, Prop. 2.5.2] $\mathfrak{B}/\ker(\chi_p)$ is not finite over $\mathfrak{A}/\ker(\chi_{\varphi^{-1}(p)})$. In particular φ is not an isomorphism in any neighbourhood of p. But $\pi^{-1}(\{z\}) \to \pi^{-1}(\{w\})$ is an isomorphism, hence $\varphi(p) \notin \pi^{-1}(\{w\})$. In particular there must be indices i, j such that $|f_{ij}(p)| = 1$, which means that $t(p)$ is contained in the boundary of S as we conclude as in the case $\varphi = \mathrm{id}$. So we see that $\varphi(\partial(\widehat{\mathfrak{U}}^{\mathrm{an}}/\mathscr{M}(\mathbb{K})))$ is contained in $\partial(\widehat{\mathfrak{S}}^{\mathrm{an}}/\mathscr{M}(\mathbb{K}))$ or $t(p)$ is in the boundary of S. Hence we have removed the hypothesis $\varphi = \mathrm{id}$ and proved (iii) in full generality. □

5.4. Functions on the Skeleton and Smooth Functions

We keep Assumption 5.3.1 and make the following observation.

Proposition 5.4.1
Let $f : S \to \mathbb{R}$ be a smooth function. Then $f \circ t$ is a smooth function.

Proof. This follows from the description of smooth functions in Rem. 5.1.6. □

This result generalises as follows to strictly poly-stable schemes over $\mathbb{K}^\circ$.

Theorem 5.4.2
Let $\mathfrak{B}$ be a quasicompact strictly poly-stable scheme over $\mathbb{K}^\circ$. Let $(\mathfrak{U}_i)_{i\in I}$ be a cover by open subschemes satisfying Assumption 5.3.1. Let $f\colon |R(\mathfrak{B})| \to \mathbb{R}$ be a continuous function such that the restriction to each polysimplex $|R(\mathfrak{U}_i)|$ for $i \in I$ is a smooth function. We define

$$U = \widehat{\mathfrak{B}}^{\mathrm{an}} \setminus (\cup_i \partial(\widehat{\mathfrak{U}}_i^{\mathrm{an}}/\mathscr{M}(\mathbb{K}))).$$

Let $\tau\colon \widehat{\mathfrak{B}}^{\mathrm{an}} \to |R(\mathfrak{B})|$ be the retraction map. Then $f \circ \tau|_U$ is a smooth function.

PROOF. For each $i \in I$ the set $\partial(\widehat{\mathfrak{U}_i}^{\mathrm{an}}/\mathscr{M}(\mathbb{K}))$ is closed in $\widehat{\mathfrak{U}_i}^{\mathrm{an}}$ and hence by quasicompactness in $\widehat{\mathfrak{B}}^{\mathrm{an}}$. Now beeing a smooth function is local on $\widehat{\mathfrak{B}}^{\mathrm{an}}$. So it suffices to show that $f \circ \tau|_{\mathrm{Int}(\widehat{\mathfrak{U}}^{\mathrm{an}}/\mathscr{M}(\mathbb{K}))}$ is smooth. But as $\tau|_{\widehat{\mathfrak{U}}^{\mathrm{an}}}$ equals t composed with an affine transformation by Thm. 5.3.2 (ii), this result follows from Prop. 5.4.1. □

5.5. Computation of the Monge–Ampère Measure on $X \times_k X$

Throughout the following section we will be in the situation of Assumption 2.2.1.

Assumption 5.5.1
Let $\mathfrak{X}$ be a projective regular strictly semi-stable model over k° of a smooth, projective, and geometrically irreducible curve X. In virtue of Prop. B.7 we can find a finite covering $(\mathfrak{U}_r)_{r\in T}$ of $\mathfrak{X}$ satisfying the following conditions:

(i) Each $\mathfrak{U}_r$ is affine,
(ii) we have maps $\varphi_r\colon \mathfrak{U}_r \to \mathfrak{S}_r$ to standard poly-stable schemes $\mathfrak{S}_r$ which induce bijections $\varphi_{r,*}\colon \mathrm{str}(\mathfrak{U}_r) \to \mathrm{str}(\mathfrak{S}_r)$,
(iii) if $S(\mathfrak{U}_r)$ is one dimensional then $r \neq u$ implies $S(\mathfrak{U}_r) \neq S(\mathfrak{U}_u)$.

Then $\mathfrak{X} \times_{k^\circ} \mathfrak{X}$ is covered by the family $(\mathfrak{U}_r \times \mathfrak{U}_l)_{(r,l)\in T^2}$, hence is a strictly poly-stable model of $X \times_k X$. Set $T^2 =: J$. For each $(r,l) = j \in J$ we set $\mathfrak{B}_j = (\mathfrak{U}_r \times_{k^\circ} \mathfrak{U}_l)_{\mathbb{K}^\circ}$. We choose an order on the set of irreducible components of the special fibre of $\mathfrak{X}$. This provides us with a simplicial set $R(\mathfrak{X})$ and provides us with canonical charts of the self product of the geometric realisation of the reduction set $|R(\mathfrak{X})|^2$ isomorphic to $[0,1]^2$ and we have the notions of differentiation and integration of Def. 3.1.4. By Prop. B.5 we can naturally identify $|R(\mathfrak{X})|^2$ with $|R(\mathfrak{X} \times_{k^\circ} \mathfrak{X})|$ and by Rem. B.8 with $|R((\mathfrak{X} \times_{k^\circ} \mathfrak{X})_{\mathbb{K}^\circ})|$. Let $\tau\colon (X \times_k X)^{\mathrm{an}}_{\mathbb{K}} \to |R((\mathfrak{X} \times_k \mathfrak{X})_{\mathbb{K}^\circ})|$ be the retraction to $|R((\mathfrak{X} \times_k \mathfrak{X})_{\mathbb{K}^\circ})|$ (cf. Thm. B.4, B.6) and we denote by $t_j\colon \widehat{\mathfrak{B}_j}^{\mathrm{an}} \to |R(\mathfrak{B}_j)|$ the restriction of τ to $\widehat{\mathfrak{B}_j}^{\mathrm{an}}$ for $j \in J$. By Thm. 5.3.2 (ii) t_j equals t from Assumption 5.3.1.

Theorem 5.5.2
Let f_i be smooth functions on $H := |R(\mathfrak{X}\times_{k^\circ}\mathfrak{X})|$ for each $i \in \{0,1,2\}$. Set $P = (X\times_k X)^{\mathrm{an}}_{\mathbb{K}}$ and $Q = P\smallsetminus\tau^{-1}(\mathrm{relbd}(R(\mathfrak{X}) \times R(\mathfrak{X}))$ and $\partial H = \mathrm{relbd}(R(\mathfrak{X}) \times R(\mathfrak{X}))$.

(i) For each $i \in \{0,1,2\}$ the functions $f_i \circ \tau$ are smooth on Q. In addition we have the following equality:

$$\int_Q (f_0 \circ \tau) \wedge d'd''(f_1 \circ \tau) \wedge d'd''(f_2 \circ \tau)$$
$$-\int_{\partial H} \partial_y f_0 \partial_y f_1 \partial_x f_2 d\nu_x + \int_{\partial H} f_0 \partial_y f_1 \partial_{xx} f_2 d\nu_y - \int_{\partial H} f_0 \partial_y f_1 \partial_{xy} f_2 d\nu_x$$
$$-\int_{\partial H} \partial_x f_0 \partial_x f_1 \partial_y f_2 d\nu_y + \int_{\partial H} f_0 \partial_x f_1 \partial_{yy} f_2 d\nu_x - \int_{\partial H} f_0 \partial_x f_1 \partial_{xy} f_2 d\nu_y =$$
$$\int_H \partial_x f_0 \partial_y f_1 \partial_{xy} f_2 + \int_H \partial_{xy} f_0 \partial_y f_1 \partial_x f_2 + \int_H \partial_y f_0 \partial_{xy} f_1 \partial_x f_2$$
$$+\int_H \partial_y f_0 \partial_x f_1 \partial_{xy} f_2 + \int_H \partial_{xy} f_0 \partial_x f_1 \partial_y f_2 + \int_H \partial_x f_0 \partial_{xy} f_1 \partial_y f_2, \tag{5.5.1}$$

where $d\nu_x$ and $d\nu_y$ denote the x and y components respectively of the surface outward unit normal of H.

(ii) Assume that for each $i \in \{0,1,2\}$ the function $f_i \circ \tau$ is smooth. Then we have

$$\int_P (f_0 \circ \tau) \wedge d'd''(f_1 \circ \tau) \wedge d'd''(f_2 \circ \tau) = \int_Q (f_0 \circ \tau) \wedge d'd''(f_1 \circ \tau) \wedge d'd''(f_2 \circ \tau).$$

PROOF. We begin with the proof of (i): By Thm. 5.3.2 (iii), if $|R(\mathfrak{B}_j)|$ is 2-dimensional then $Q \cap \widehat{\mathfrak{B}_j}^{\mathrm{an}}$ is contained in the set U which is defined as the union of all the interiors $\{\mathrm{Int}(\widehat{\mathfrak{B}_j}^{\mathrm{an}}/\mathscr{M}(\mathbb{K}))|j \in J\}$. If $|R(\mathfrak{B}_j)|$ is one or zero dimensional then $\tau(\widehat{\mathfrak{B}_j}^{\mathrm{an}})$ is a one or zero-dimensional face respectively of a 2-dimensional skeleton $|R(\mathfrak{B}_m)| \cong [0,1]^2$ for some m by Prop. B.9. So we get $Q \subset U$ and clearly Q is open by continuity of τ. In particular for each $i \in \{0,1,2\}$ the function $f_i \circ \tau$ is smooth on Q by Thm. 5.4.2. For each $j \in J$ and each $i \in \{0,1,2\}$ the function f_i is smooth on $\widehat{\mathfrak{B}_j}^{\mathrm{an}}$ by Prop. 5.4.1 and hence

$$(f_0 \circ \tau) \wedge d'd''(f_1 \circ t_j) \wedge d'd''(f_2 \circ t_j)$$

is a smooth form on $\widehat{\mathfrak{B}_j}^{\mathrm{an}}$ and hence integrable. In particular it is integrable on $\widehat{\mathfrak{B}_j}^{\mathrm{an}} \cap Q$ and as this holds for all $j \in J$ the form

$$\omega := (f_0 \circ \tau) \wedge d'd''(f_1 \circ \tau) \wedge d'd''(f_2 \circ \tau) \in A^{2,2}(Q)$$

is integrable on Q by finiteness of J.

For each $j \in J$ set $Q_j := Q \cap \widehat{\mathfrak{B}_j}^{\mathrm{an}}$. Note that $\mathrm{relint}(R(\mathfrak{X}) \times R(\mathfrak{X}))$ is the disjoint union of the relative interiors of $(R(\mathfrak{B}_j))_{j \in J}$. Hence Q is the disjoint union of the family $(Q_l)_{j \in J}$ of open subsets of Q. Using Assumption 5.5.1 iii) which guarantees that the family $(\mathrm{id}|_{Q_j})_{j \in J}$ is a partition of unity we see that

$$(Q_j, Q_j, \mathrm{id}|_{Q_j})_{j \in J}$$

is an atlas of integration for $\omega|_Q$.

Fix $j \in J$. By Prop. 5.4.1 and the construction of t_j we see that $\omega|_{Q_j}$ is tropical on Q_j and by [**CLD12**, 3.8.5] we have

$$\int_{Q_j} \omega|_{Q_j} = \int_{\mathrm{relint}(V)} f_0 \cdot d'd''(f_1) \wedge d'd''(f_2)$$

for $V = |R(\mathfrak{B}_j)|$. As all f_i are smooth functions on $\mathrm{relint}(V)$ we have

$$\int_{\mathrm{relint}(V)} f_0 \cdot d'd''(f_1) \wedge d'd''(f_2) = \int_V f_0 \cdot d'd''(f_1) \wedge d'd''(f_2)$$

as the relative boundary has Lebesgue measure zero. Only the integrals where V is two-dimensional can be non-zero, because $(2,2)$-forms on one-dimensional polyhedra are zero. So we assume that V is two-dimensional. We canonically identify V with $[0,1]^2$ and introduce coordinates x, y.

By Thm. 5.3.2 (iv) the canonical calibration is given by

$$1 \cdot |x \wedge y|.$$

Now we can evaluate the integral. For $i \in \{1,2\}$ we compute

$$d'd'' f_i = \partial_{xx} f_i d'x \wedge d''x + \partial_{xy} f_i d'y \wedge d''x + \partial_{xy} f_i d''x \wedge d''y + \partial_{yy} f_i d'y \wedge d''y$$

hence

$$\begin{aligned} d'd''f_1 \wedge d'd''f_2 =& \partial_{yy}f_1\partial_{xx}f_2 d'x \wedge d''x \wedge d'y \wedge d''y + \partial_{xx}f_1\partial_{yy}f_2 d'x \wedge d''x \wedge d'y \wedge d''y \\ &- \partial_{xy}f_1\partial_{xy}f_2 d'd'' \wedge d'y \wedge d''y - \partial_{xy}f_1\partial_{xy}f_2 d'x \wedge d''x \wedge d'y \wedge d''y \end{aligned}$$

and so contracting $d'x \wedge d''x \wedge d'y \wedge d''y$ by the canonical calibration we get

$$\begin{aligned} &\langle f_0 \wedge d'd''(f_1) \wedge d'd''(f_2), |x \wedge y| \rangle = \\ &(f_0\partial_{xx}f_1\partial_{yy}f_2 + f_0 f_0 \partial_{yy}f_1\partial_{xx}f_2 - f_0\partial_{xy}f_1\partial_{xy}f_2 - f_0\partial_{xy}f_1\partial_{xy}f_2)\, dx \wedge dy, \end{aligned} \tag{5.5.2}$$

hence

$$\begin{aligned} &\int_{Q_j} f_0 \wedge d'd''(f_1 \circ t_j) \wedge d'd''(f_2 \circ t_j) = \\ &\int_V f_0\partial_{xx}f_1\partial_{yy}f_2 + \int_V f_0\partial_{yy}f_1\partial_{xx}f_2 - \\ &\int_V f_0\partial_{xy}f_1\partial_{xy}f_2 - \int_V f_0\partial_{xy}f_1\partial_{xy}f_2, \end{aligned} \tag{5.5.3}$$

and the following Lemma 5.5.4 now implies (i) using that by definition of the integral we have

$$\int_Q \omega|_Q = \sum_j \int_{Q_j} \omega|_{Q_j}.$$

Now we will prove (ii). Consider a chart $\widehat{\mathfrak{B}_j}^{\mathrm{an}}$ for $j \in J$. Let Δ be any d-dimensional polyhedron in $|R(\mathfrak{B}_j)|$. By [**Gub07**, Prop. 5.4] the closed domain $U_\Delta := t_j^{-1}(\Delta)$ in $\widehat{\mathfrak{B}_j}^{\mathrm{an}}$ has dimension d. But this means by Thm. 5.1.3 that the tropicalisation of U_Δ will be contained in a d dimensional polyhedron in any tropical chart. So for each tropical chart, $(U_\Delta)_{\mathrm{trop}}$ will be a Lebesgue null-set if $d \leq 1$. So we can discard closed sets of the form U_Δ for d-dimensional polyhedra Δ with $d \leq 1$ when integrating a $(2,2)$-form. In particular we can discard $\tau^{-1}(\mathrm{relbd}(R(\mathfrak{X}) \times R(\mathfrak{X})))$ when integrating and hence for every $\beta \in A^{2,2}(P)$ we have

$$\int_Q \beta|_Q = \int_P \beta$$

and the claim follows.

□

Using our results about the computation of local heights on products of curves we get the following theorem.

Theorem 5.5.3

(i) For each $i \in \{0,1,2\}$ the function $f_i \circ \tau$ is approximable in the sense of Def. 5.1.9 (v) and we have the following equality.

$$\begin{aligned} &\int_P (f_0 \circ \tau) \wedge d'd''(f_1 \circ \tau) \wedge d'd''(f_2 \circ \tau) = \\ &\int_H \partial_x f_0\partial_y f_1\partial_{xy}f_2 + \int_H \partial_{xy}f_0\partial_y f_1\partial_x f_2 + \int_H \partial_y f_0\partial_{xy}f_1\partial_x f_2 \\ &+ \int_H \partial_y f_0\partial_x f_1\partial_{xy}f_2 + \int_H \partial_{xy}f_0\partial_x f_1\partial_y f_2 + \int_H \partial_x f_0\partial_{xy}f_1\partial_y f_2. \end{aligned} \tag{5.5.4}$$

(ii) We have a decomposition

$$\int_P (f_0 \circ \tau) \wedge d'd''(f_1 \circ \tau) \wedge d'd''(f_2 \circ \tau)$$
$$= \int_Q (f_0 \circ \tau) \wedge d'd''(f_1 \circ \tau) \wedge d'd''(f_2 \circ \tau)$$
$$- \int_{\partial H} \partial_y f_0 \partial_y f_1 \partial_x f_2 d\nu_x + \int_{\partial H} f_0 \partial_y f_1 \partial_{xx} f_2 d\nu_y - \int_{\partial H} f_0 \partial_y f_1 \partial_{xy} f_2 d\nu_x$$
$$- \int_{\partial H} \partial_x f_0 \partial_x f_1 \partial_y f_2 d\nu_y + \int_{\partial H} f_0 \partial_x f_1 \partial_{yy} f_2 d\nu_x - \int_{\partial H} f_0 \partial_x f_1 \partial_{xy} f_2 d\nu_y.$$

(iii) Assume that for each $i \in \{0, 1, 2\}$ the function $f_i \circ \tau$ is smooth. Then

$$- \int_{\partial H} \partial_y f_0 \partial_y f_1 \partial_x f_2 d\nu_x + \int_{\partial H} f_0 \partial_y f_1 \partial_{xx} f_2 d\nu_y - \int_{\partial H} f_0 \partial_y f_1 \partial_{xy} f_2 d\nu_x$$
$$- \int_{\partial H} \partial_x f_0 \partial_x f_1 \partial_y f_2 d\nu_y + \int_{\partial H} f_0 \partial_x f_1 \partial_{yy} f_2 d\nu_x - \int_{\partial H} f_0 \partial_x f_1 \partial_{xy} f_2 d\nu_y = 0.$$

PROOF. We will prove (i). By Cor. 3.7.4 the arithmetic divisors $0 + f_i \circ \tau$ are DSP for $i \in \{0, 1, 2\}$. By Thm. 5.2.2 the left hand side of (5.5.4) is given by the integral $\int (f_0 \circ \tau) d\mu^{\mathrm{CL}}$ of $f_0 \circ \tau$ against the Chambert-Loir measure associated to the metrics $\|\cdot\|_{f_1 \circ \tau}$ and $\|\cdot\|_{f_2 \circ \tau}$ on the trivial line bundle. By Cor. 4.3.2 this integral is precisely given by the right hand side of (5.5.4).

Now (ii) and (iii) follow formally from Thm. 5.5.2 (i) and Thm. 5.5.2 (ii). □

Lemma 5.5.4
Let f_0, f_1, f_2 be functions on $S := [0, 1]^2$ such that $f_0 \in C^2(S)$ and $f_1, f_2 \in C^3(S)$. Then

$$\int_S f_0 \partial_{xx} f_1 \partial_{yy} f_2 + \int_S f_0 \partial_{yy} f_1 \partial_{xx} f_2 - \int_S f_0 \partial_{xy} f_1 \partial_{xy} f_2 - \int_S f_0 \partial_{xy} f_1 \partial_{xy} f_2$$
$$= \int_S \partial_x f_0 \partial_y f_1 \partial_{xy} f_2 + \int_S \partial_{xy} f_0 \partial_y f_1 \partial_x f_2 + \int_S \partial_y f_0 \partial_{xy} f_1 \partial_x f_2$$
$$+ \int_S \partial_y f_0 \partial_x f_1 \partial_{xy} f_2 + \int_S \partial_{xy} f_0 \partial_x f_1 \partial_y f_2 + \int_S \partial_x f_0 \partial_{xy} f_1 \partial_y f_2$$
$$- \int_{\partial S} \partial_y f_0 \partial_y f_1 \partial_x f_2 d\nu_x + \int_{\partial S} f_0 \partial_y f_1 \partial_{xx} f_2 d\nu_y - \int_{\partial S} f_0 \partial_y f_1 \partial_{xy} f_2 d\nu_x$$
$$- \int_{\partial S} \partial_x f_0 \partial_x f_1 \partial_y f_2 d\nu_y + \int_{\partial S} f_0 \partial_x f_1 \partial_{yy} f_2 d\nu_x - \int_{\partial S} f_0 \partial_x f_1 \partial_{xy} f_2 d\nu_y,$$

where $d\nu_x$ and $d\nu_y$ denote the x and y components respectively of the surface outward unit normal of S.

PROOF. We will repeatedly use the following integration by parts formula which follows from Fubini's theorem and the fundamental theorem of calculus. Let $S = [0, 1]^2$ and x, y be coordinates. Let f, g be $C^1(S)$ functions. Then

$$\int_S \partial_x f g = - \int_S f \partial_x g + \int_{\partial S} f g d\nu_x.$$

We compute

$$\int_S f_0 \partial_{xy} f_1 \partial_{xy} f_2 = -\int_S \partial_x f_0 \partial_y f_1 \partial_{xy} f_2$$
$$-\int_S f_0 \partial_y f_1 \partial_{xxy} f_2 + \int_{\partial S} f_0 \partial_y f_1 \partial_{xy} f_2 d\nu_x \tag{5.5.5}$$

using integration by parts. Furthermore,

$$\int_S f_0 \partial_y f_1 \partial_{xxy} f_2 = -\int_S \partial_y f_0 \partial_{xx} f_2 \partial_y f_1$$
$$-\int_S f_0 \partial_{yy} f_1 \partial_{xx} f_2 + \int_{\partial S} f_0 \partial_y f_1 \partial_{xx} f_2 d\nu_y \tag{5.5.6}$$

again using integration by parts and

$$\int \partial_y f_0 \partial_y f_1 \partial_{xx} f_2 = -\int \partial_{xy} f_0 \partial_y f_1 \partial_x f_2$$
$$-\int \partial_y f_0 \partial_x f_2 \partial_{xy} f_1 + \int_{\partial S} \partial_y f_0 \partial_y f_1 \partial_x f_2 d\nu_x. \tag{5.5.7}$$

Putting (5.5.6), (5.5.7) in (5.5.5) we get

$$-\int f_0 \partial_{xy} f_1 \partial_{xy} f_2 = \int \partial_x f_0 \partial_y f_1 \partial_{xy} f_2 + \int \partial_{xy} f_0 \partial_y f_1 \partial_x f_2$$
$$+\int \partial_y f_0 \partial_{xy} f_1 \partial_x f_2 - \int_{\partial S} \partial_y f_0 \partial_y f_1 \partial_x f_2 d\nu_x. - \int f_0 \partial_{yy} f_1 \partial_{xx} f_2$$
$$+\int_{\partial S} f_0 \partial_y f_1 \partial_{xx} f_2 d\nu_y - \int_{\partial S} f_0 \partial_y f_1 \partial_{xy} f_2 d\nu_x \tag{5.5.8}$$

and by changing the roles of x and y

$$-\int f_0 \partial_{xy} f_1 \partial_{xy} f_2 = \int \partial_y f_0 \partial_x f_1 \partial_{xy} f_2 + \int \partial_{xy} f_0 \partial_x f_1 \partial_y f_2$$
$$+\int \partial_x f_0 \partial_{xy} f_1 \partial_y f_2 - \int_{\partial S} \partial_x f_0 \partial_x f_1 \partial_y f_2 d\nu_y - \int f_0 \partial_{xx} f_1 \partial_{yy} f_2$$
$$+\int_{\partial S} f_0 \partial_x f_1 \partial_{yy} f_2 d\nu_x - \int_{\partial S} f_0 \partial_x f_1 \partial_{xy} f_2 d\nu_y \tag{5.5.9}$$

and putting (5.5.8) and (5.5.9) in

$$\int_S f_0 \partial_{xx} f_1 \partial_{yy} f_2 + \int_S f_0 \partial_{yy} f_1 \partial_{xx} f_2$$

and

$$\int_S f_0 \partial_{xy} f_1 \partial_{xy} f_2 - \int_S f_0 \partial_{xy} f_1 \partial_{xy} f_2$$

respectively we get

$$\int_S f_0\partial_{xx}f_1\partial_{yy}f_2 + \int_S f_0\partial_{yy}f_1\partial_{xx}f_2 -$$
$$\int_S f_0\partial_{xy}f_1\partial_{xy}f_2 - \int_S f_0\partial_{xy}f_1\partial_{xy}f_2$$
$$=$$
$$\int_S f_0\partial_{xx}f_1\partial_{yy}f_2 + \int_S f_0\partial_{yy}f_1\partial_{xx}f_2 + \int_S \partial_x f_0\partial_y f_1\partial_{xy}f_2 + \int_S \partial_{xy}f_0\partial_y f_1\partial_x f_2$$
$$+ \int_S \partial_y f_0\partial_{xy}f_1\partial_x f_2 - \int_S f_0\partial_{yy}f_1\partial_{xx}f_2 + \int_S \partial_y f_0\partial_x f_1\partial_{xy}f_2$$
$$+ \int_S \partial_{xy}f_0\partial_x f_1\partial_y f_2 + \int_S \partial_x f_0\partial_{xy}f_1\partial_y f_2 - \int_S f_0\partial_{yy}f_1 f_2\partial_{xx}$$
$$- \int_{\partial S} \partial_y f_0\partial_y f_1\partial_x f_2 d\nu_x + \int_{\partial S} f_0\partial_y f_1\partial_{xx}f_2 d\nu_y - \int_{\partial S} f_0\partial_y f_1\partial_{xy}f_2 d\nu_x$$
$$- \int_{\partial S} \partial_x f_0\partial_x f_1\partial_y f_2 d\nu_y + \int_{\partial S} f_0\partial_x f_1\partial_{yy}f_2 d\nu_x - \int_{\partial S} f_0\partial_x f_1\partial_{xy}f_2 d\nu_y$$

which equals

$$\int \partial_x f_0\partial_y f_1\partial_{xy}f_2 + \int \partial_{xy}f_0\partial_y f_1\partial_x f_2 + \int \partial_y f_0\partial_{xy}f_1\partial_x f_2$$
$$+ \int \partial_y f_0\partial_x f_1\partial_{xy}f_2 + \int \partial_{xy}f_0\partial_x f_1\partial_y f_2 + \int \partial_x f_0\partial_{xy}f_1\partial_y f_2$$
$$- \int_{\partial S} \partial_y f_0\partial_y f_1\partial_x f_2 d\nu_x + \int_{\partial S} f_0\partial_y f_1\partial_{xx}f_2 d\nu_y - \int_{\partial S} f_0\partial_y f_1\partial_{xy}f_2 d\nu_x$$
$$- \int_{\partial S} \partial_x f_0\partial_x f_1\partial_y f_2 d\nu_y + \int_{\partial S} f_0\partial_x f_1\partial_{yy}f_2 d\nu_x - \int_{\partial S} f_0\partial_x f_1\partial_{xy}f_2 d\nu_y$$

as desired. □

APPENDIX A

Simplicial Sets and Graphs

Let Δ be the category of the finite sets $[n] := \{0, \ldots, n\}$ with non-decreasing maps as morphisms. For $i \in \{0, \ldots, n\}$ we denote by $d_i : [n-1] \to [n]$ the face maps which miss the i-s element of $[n]$ and for $i \in \{0, \ldots, n-1\}$ we denote by $s_i : [n] \to [n-1]$ the degeneracy maps which are defined as follows

$$s_i(k) = \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i. \end{cases}$$

A *simplicial set* is a contravariant functor from Δ to the category of sets. We denote by $\Delta^\circ\mathscr{E}ns$ the category of simplicial sets with natural transformations as morphisms. We denote by $\Delta_n := \mathrm{Hom}(\cdot, [n])$ the standard n-simplex. Let S be a simplicial set. We introduce the notation $S_n := S([n])$ for the set of n-simplices of S. We say a simplex $s \in S_n$ is *degenerate* if it is in the image of one of the maps $s_i : S_{n-1} \to S_n$. If s is not degenerate it is called *non-denegerate*. We define the *dimension* of S to be

$$\dim(S) = \sup\{n \in \mathbb{N} \mid S_n \text{ contains a non-degenerate simplex}\}.$$

Proposition A.1
Let $\mathscr{C}$ be a small category and $P\colon \mathscr{C} \to \mathscr{E}ns$ be a contravariant functor to the category of sets. Then P is the colimit of representable functors.

Proof. Cf. [**Awo06**, Prop. 8.10]. □

Definition A.2
Using the proposition and by demanding that $|\Delta_n|$ is the standard topological n-simplex and that the functor $|?|$ commutes with colimits we get the geometric realisation functor $|?|\colon \Delta^\circ \to \mathrm{Top}$.

For any natural number $n \geq 1$ we define the n-truncated category Δ^n as the full subcategory of Δ only containing the objects $[0], \ldots, [n]$ and we define a n-truncated simplicial set to be a contravariant functor from Δ^n to the category of sets. We define $\Delta^{n,\circ}\mathscr{E}ns$ as the category of n-truncated simplicial sets. Prop. A.1 applies to n-truncated simplicial sets as well and we can define a geometric realisation functor $|?|\colon \Delta^{n,\circ}\mathscr{E}ns \to \mathscr{T}op$. The inclusion of categories $\Delta^n \hookrightarrow \Delta$ defines a functor $T\colon \Delta^\circ\mathscr{E}ns \to \Delta^{n,\circ}\mathscr{E}ns$. Moreover, we get a functor $A\colon \Delta^{n,\circ}\mathscr{E}ns \to \Delta^\circ\mathscr{E}ns$ as follows: For an n-truncated simplicial set Y we set

$$A(Y)_m = \operatorname*{colim}_{\substack{k \leq n \\ [m] \to [k]}} Y_k.$$

In words $A(Y)_k$ is the same as Y_k for $k \leq n$ and for $k > n$ and it is filled with degenerate k-simplices.

Proposition A.3
Let $n \geq 1$ be a natural number.

(i) *The functor A is left adjoint to T. The unit of the adjunction is an isomorphism* $\mathrm{id} \to T \circ A$.

(ii) *The restriction of the counit* $A \circ T \to \mathrm{id}$ *of this adjunction to the full subcategory of $\Delta^\circ\mathscr{E}ns$ consisting of simplicial sets of dimension of at most n is an isomorphism.*

(iii) *We have equality $A \circ |?| = |?| \circ A$.*

Proof.

(i) This is proven in [**GJ99**, Ch. V.1.].

(ii) We want to prove that the counit is an isomorphism when restricted to the category of simplicial sets of dimension of at most n. To achieve this we show that for each $m \geq 0$ and each n-dimensional simplicial set Y the natural map

$$TA(Y)_m \to Y_m \tag{A.1}$$

is bijective. We show surjectivity: If $m \leq n$ then the index category has an initial object, namely $\mathrm{id}\colon [m] \to [m]$, hence the left hand side equals Y_m and the map (A.1) is the identity.

Assume that $m > n$ holds and x be an element of Y_m. As Y is at most n-dimensional, there is a composition of degeneration maps $s\colon [m] \to [n]$ and an n-simplex a such that $s(a) = x$. Now the diagram

$$\begin{array}{ccc} (TA(Y))_n = Y_n & \longrightarrow & Y_n \\ \downarrow s & & \downarrow s \\ TA(Y)_m & \longrightarrow & Y_m \end{array}$$

commutes and the top arrow is the identity by the previous. So x is in the image of $TA(Y)_m \to Y_m$ as desired.

We show injectivity: Now choose representatives $x \in (Y_k, a)$ for $a\colon [m] \to [k]$ of an element z and $y \in (Y_l, b)$ for a map $b\colon [m] \to [l]$ of an element w which map to the same element in Y_m under the map (A.1). But this map is induced by a and b respectively and the mapping of x and y to the same element in Y_m is exactly the condition of z and w to be equal in the colimit. Hence the map (A.1) is injective.

(iii) Equivalences of categories commute with colimits and the respective geometric realisation functors also commute with colimits. As any n-truncated simplicial set is the colimit of l-simplices for $l \leq n$ it suffices to show the statement for such simplices. But here the statement is clear as $A \circ |?|$ as well as $|?| \circ A$ are the standard topological l-simplices.

□

Definition A.4

(i) A *graph* consists of the following data:

- a set V of *vertices*,
- a family E of tuples (a, b) for $a, b \in V$ indexed over an index set I, the *edges*.

(ii) A *morphism* $\varphi\colon \Gamma \to \Gamma'$ of two graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ is given by a map $\psi\colon V \to V'$ such that if $(a, b) \in E$ then $(\psi(a), \psi(b))$ is a member of E' or $\psi(a) = \psi(b)$. We denote by $\mathscr{G}r$ the category of graphs.

(iii) We denote by B_1 the following graph: It consists of two vertices a, b and one edge (a, b).

(iv) Every graph can be written as the colimit of its subgraphs isomorphic to B_1 and points. Hence there is a geometric realisation functor $|?|$ which is uniquely determined by asking that it associated to B_1 the unit interval $[0,1]$ and that it commutes with colimits of graphs.

(v) Every graph $\Gamma = (V, E)$ defines a 1-truncated simplicial set X as follows: We define X_0 as the set of vertices. The set X_1 is the index set I of E plus for every $v \in V$ an element e_v. We define the face maps d_0 and d_1 as follows: We require $d_0((a,b)) = a$ for an edge (a,b) and $d_1((a,b)) = b$, and $d_0(e_v) = d_1(e_v) = v$. The degeneracy map s_0 is asked to map every vertex v to the vertex e_v. This determines a functor $F' \colon \mathscr{G}r \to \Delta^{1,\circ}\mathscr{E}ns$ to the category of simplicial sets. We compose this functor with the functor A from above and get a functor $F \colon \mathscr{G}r \to \Delta^\circ\mathscr{E}ns$.

(vi) A graph is said to be *finite* if its set of vertices is finite and if the family of edges is finite.

Proposition A.5
The functor F is an equivalence of categories between the category of graphs and the category of simplicial sets of dimension at most 1*. The diagram of functors*

$$\begin{array}{ccc} \mathscr{G}r & \longrightarrow & \Delta^\circ\mathscr{E}ns \\ & {\scriptstyle |?|}\searrow \quad \swarrow{\scriptstyle |?|} & \\ & \mathscr{T}op & \end{array} \tag{A.2}$$

commutes.

Proof. By Prop. A.3 it suffices to show that the functor F' to the category of 1-truncated simplicial sets is an isomorphism. We define an inverse H as follows: Let S be a 1-truncated simplicial set. Let V be the set of 0-simplices and I be the set of non-degenerate 1-simplices. Then we define a family E of edges as follows: If $k \in I$ then we set $e_k = (d_0(k), d_1(k))$. We set $E = (e_k)_{k\in I}$. Then $H(S) := (V, E)$. We immediately get $H \circ F' = \mathrm{id}$ from the construction. It suffices to show the commutativity of the diagram (A.2) only for the graph B_1 and the point graph. But here the statements are immediately clear. □

Let $d \geq 1$ be number and let Γ be any graph without loop edges. We denote by Γ^d the d-fold self-product of Γ in the category of simplicial sets and by $|\Gamma^d|$ its geometric realisation. Then $|\Gamma^d| = |\Gamma|^d$. Let $e_1, \dots, e_d$ be edges of Γ. We consider the edges as subgraphs of Γ isomorphic to B_1. Then there is a canonical isomorphism of the topological space $|e_1 \times \ldots \times e_d|$ with $[0,1]^d$ induced by the canonical isomorphisms $|e_i| \cong [0,1]$ for $i = 1, \dots, d$. We have a canonical inclusion map of topological spaces

$$|e_1 \times \ldots \times e_d| \hookrightarrow |\Gamma^d|. \tag{A.3}$$

Denote by E the family of edges of Γ. Then we have a family of tuples $(Z_{\underline{e}}, \varphi_{\underline{e}})_{\underline{e}\in E^d}$ where $\underline{e} \in E^d$ and $Z_{\underline{e}}$ is a closed topological subspace of $|\Gamma^d|$ and $\varphi_{\underline{e}} \colon Z_{\underline{e}} \to [0,1]^d$ is an isomorphism of topological spaces: If $\underline{e} = (e_1, \dots, e_d)$ then $Z_{\underline{e}}$ is the image of $|e_1 \times \ldots \times e_d|$ in $|\Gamma^d|$ under the map (A.3) and $\varphi_{\underline{e}}$ is the induced isomorphism $Z_{\underline{e}} \to [0,1]^d$.

Definition A.6
We call the family $(Z_{\underline{e}}, \varphi_{\underline{e}})_{\underline{e}\in E^d}$ the *atlas associated to* Γ^d. For a fixed $\underline{e}$ we will call the pair $(Z_{\underline{e}}, \varphi_{\underline{e}})$ a *chart* of the atlas. The sets $Z_{\underline{e}}$ will be referred to as *canonical squares* or just *squares*.

Definition A.7
Let $f \colon |\Gamma^d| \to \mathbb{R}$ be a function. Let E be the set of edges of Γ. We consider the atlas associated to Γ^d. Let $\underline{e} = (e_1, \dots, e_d)$ be an element of E^d. Denote by $(Z_{\underline{e}}, \varphi_{\underline{e}})$ the

corresponding chart. Note that $\varphi_{\underline{e}}\colon Z_{\underline{e}} \to [0,1]^d$ is an isomorphism. By *restriction of* f *to* $(Z_{\underline{e}}, \varphi_{\underline{e}})$ we mean the map $f \circ \varphi_{\underline{e}}^{-1}\colon [0,1]^d \to \mathbb{R}$.

Remark A.8

We want to compute the non-degenerate simplices of $\Delta[1]^d$ for an integer $d \geq 1$. The category of simplicial sets products and coproducts can be constructed component-wise. In particular if $F, G\colon \Delta^\circ \to \mathscr{E}ns$ are functors, then $(F \times G)([n]) = F([n]) \times G([n])$. By [**Kol16b**, Cor. A 7], we have a canonical isomorphism

$$\Delta[1]^d \cong \mathrm{Hom}(\cdot, [1]^d),$$

where $[1]^d$ is considered as a partially ordered set with respect to the lexicographic order. Then an n-simplex is non-degenerate on the left hand side, if and only if it corresponds to a strictly increasing map on the right hand side.

Definition A.9

We give the definition of the subdivision of a simplicial set after Segal ([**Seg73**, Appendix 1]) which we will call *S-subdivision*. We denote by $\mathrm{sd}_S^\Delta\colon \Delta \to \Delta$ the convariant functor which is determined by the following properties.

- $\mathrm{sd}_S^\Delta([n]) = [2n+1]$,
- if $\alpha\colon [n] \to [m]$ is a morphism then

$$\mathrm{sd}_S^\Delta(\alpha)(k) = \begin{cases} m - \alpha(n-k) & \text{if } 0 \leq k \leq n \\ \alpha(k-n-1) + m + 1 & \text{if } n+1 \leq k \leq 2n+1. \end{cases}$$

We define $\mathrm{sd}_S\colon \Delta^\circ\mathscr{E}ns \to \Delta^\circ\mathscr{E}ns$ by $S \mapsto S \circ \mathrm{sd}_S^\Delta$.

Proposition A.10

Denote by $\mathrm{sd}_S(\cdot)$ *the S-subdivision functor. Then*

(i) *The functor* $\mathrm{sd}_S(\cdot)$ *commutes with limits and colimits,*
(ii) *We have a functorial isomorphism* $|?| \circ \mathrm{sd}_S(\cdot) \cong |?|$ *which is affine on the simplices of* $\mathrm{sd}_S(\cdot)$.

Proof.

(i) This follows as colimits and limits in the category of simplicial sets can be given by the rule

$$(\lim_{i \in I} Y_i)_m = \lim_{i \in I} (Y_i)_m$$

and the same holds for colimits.
(ii) This is [**Seg73**, Prop. A.1].

□

Remark A.11

We will describe the impact of S-subdivision on graphs. More precisely, if F is the functor from the category of graphs to the category of one-dimensional simplicial sets, we describe the functor $F^{-1}\,\mathrm{sd}_S(F(\cdot))$. The graph $G := F^{-1}\,\mathrm{sd}_S(FB_1)$ will have three vertices a, b, c. We demand (a, b) and (c, b) to be the edges of G. The natural homeomorphism $|\,\mathrm{sd}_S(B_1)| \to |I|$ maps a to 0, b to 1/2, and c to 1. The definition of $\mathrm{sd}_S(B_1)$ and the isomorphism

$$|\,\mathrm{sd}_S(B_1)| \to |B_1| \tag{A.4}$$

extends to arbitrary graphs as every graph is the colimit of points and graphs B_1.

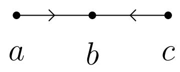


FIGURE A.1. The S-subdivision of B_1.
For details cf. [**Ber09**, Ex. 5.2.6]

Definition A.12
Let $n \geq 1$ be a natural number. We recall the definition of the *BHM-subdivision* after [**BHM93**]. Let $r \geq 1$ be an integer. Denote by sd_r^Δ the functor $\Delta \to \Delta$ which is given by $[n] \mapsto [(n+1) \cdot r - 1]$ on the objects and for every non-decreasing map $\alpha\colon [n] \to [m]$ given by

$$\mathrm{sd}_r^\Delta(\alpha)(ar + b) = ar + \alpha(b)$$

for $0 \leq b < r$. Then we define the r-fold subdivision functor $\mathrm{sd}_r(\cdot)\colon \Delta^\circ\mathscr{E}ns \to \Delta^\circ\mathscr{E}ns$ by $S \mapsto S \circ \mathrm{sd}_r^\Delta$.

Proposition A.13
Denote by $\mathrm{sd}_{\mathrm{BHM},r}(\cdot)$ *the BHM-subdivision functor for* $r \geq 1$. *Then*

(i) The functor $\mathrm{sd}_r(\cdot)$ *commutes with limits and colimits,*
(ii) the functor $\mathrm{sd}_r(\cdot)$ *induces a distinguished isomorphism of functors* $|?| \circ \mathrm{sd}_r(\cdot) \to |?|$ *such that for each simplicial set the restriction of this isomorphism to each simplex of* $\mathrm{sd}_r(S)$ *is affine.*

PROOF.

(i) Cf. [**Kol13**, Prop. A.21].
(ii) Cf. [**BHM93**, Lem. 1.1].

□

Remark A.14
We describe the impact of the subdivision operator on graphs which is enough for our purposes. We first describe the effect on the standard graph B_1. The resulting graph $\mathrm{sd}_r(B_1)$ will have $r+1$ vertices $v_1, \ldots v_{r+1}$ and we demand (v_i, v_{i+1}) to be an edge of $\mathrm{sd}_r(B_1)$. The natural homeomorphism $|\mathrm{sd}_r(B_1)| \to |B_1|$ maps v_i to $\frac{i-1}{r} \in [0,1]$ for all $i \in \{1, \ldots, n+1\}$. This description extends to arbitrary graphs as every graph can written as the colimit of copies of B_1 and points and we get a natural homeomorphism

$$|\mathrm{sd}_r(\Gamma)| \to |\Gamma| \tag{A.5}$$

for every graph Γ.

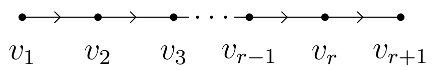


FIGURE A.2. The r-BHM subdivision of B_1

Cf. also [**Kol13**, Beispiel A.22].

APPENDIX B

Skeleta of Poly-Stable Schemes

Let $\mathbb{K}$ be a field which is complete with respect to a non-trivial non-Archimedean absolute value $|\cdot|$, which is not necessarily discrete. We denote by $\mathbb{K}^\circ$ the valuation ring and by $\mathbb{K}^{\circ\circ}$ the maximal ideal. We set $\widetilde{\mathbb{K}} = \mathbb{K}^\circ/\mathbb{K}^{\circ\circ}$. We keep the notations of Ch. 2.

Definition B.1
(i) Let $r \geq 0$ be a non-negative number and $\underline{n} = (n_1, \ldots, n_r)$, $\underline{p} = (p_1, \ldots, p_r)$ be tuples of non-negative numbers with $p_i \leq n_i$ for all $i \in \{1, \ldots, r\}$ and let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ be a tuple of elements from $\mathbb{K}^{\circ\circ} \smallsetminus \{0\}$. We define the *standard poly-stable scheme* $\mathfrak{S}(\underline{n}, \underline{p}, \underline{\alpha})$ as

$$\operatorname{Spec}\left(\mathbb{K}^\circ[T_{11}, \ldots, T_{1n_1}, \ldots, T_{r1}, \ldots, T_{rn_r}]/_{(T_{11}\cdots T_{1p_1} - \alpha_1, \ldots, T_{r1}\cdots T_{rp_r} - \alpha_r)}\right).$$

(ii) A *strictly poly-stable scheme* is an integral scheme locally of finite presentation over $\mathbb{K}^\circ$ such that every point x in the special fibre has a neighbourhood which is étale over a standard poly-stable scheme $\mathfrak{S}(\underline{n}, \underline{p}, \underline{\alpha})$ for suitable parameters $\underline{n}, \underline{p}, \underline{\alpha}$ which depend on x.

(iii) A morphism of $\mathbb{K}^\circ$-schemes $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ is called *trivially poly-stable* if every point in the special fibre of $\mathfrak{X}$ possesses a neighbourhood $\mathfrak{U}$ such that there exists a strictly poly-stable scheme $\mathfrak{Z}$ and a commutative diagram

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{\ \varphi|_{\mathfrak{U}}\ } & \mathfrak{Y}\,, \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\chi} & \\ & \mathfrak{Y} \times \mathfrak{Z} & \end{array}$$

where ψ is étale and χ is the projection. We denote by $s\mathscr{P}st^{tps}$ the category of strictly polystable schemes together with trivially poly-stable $\mathbb{K}^\circ$-morphisms of schemes.

(iv) Let $\mathfrak{X}$ be a strictly poly-stable scheme. A *stratum* of the special fibre is an irreducible component of an intersection

$$C_1 \cap \ldots \cap C_p$$

of irreducible components C_i of the special fibre $\mathfrak{X}_s$ of $\mathfrak{X}$. The set of strata of the special fibre of $\mathfrak{X}$ defines a partially ordered set $\operatorname{str}(\mathfrak{X})$ by demanding that for $s, t \in \operatorname{str}(\mathfrak{X})$ we have $s \leq t$ if $s \supset t$. *Caution: Note the reversed order of the inclusion.*

(v) If $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ is a $\mathbb{K}^\circ$-morphism between strictly poly-stable schemes then we define a map $\varphi_*\colon \operatorname{str}(\mathfrak{X}) \to \operatorname{str}(\mathfrak{Y})$ by the rule

$$s \mapsto \overline{s}.$$

(vi) We say that a strictly poly-stable scheme $\mathfrak{X}$ is *elementary*, if $\mathfrak{X}$ posses a unique maximal stratum of the special fibre.

The following proposition is immediate from the definitions.

Proposition B.2

(i) Let $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ be a trivially poly-stable morphism. Then φ is flat in a neighbourhood of the generic point of each irreducible component of the special fibre.

(ii) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be strictly poly-stable schemes and let $p\colon \mathfrak{X} \times_{\mathbb{K}^\circ} \mathfrak{Y} \to \mathfrak{X}$ be the projection. Then p is trivially poly-stable.

Proposition B.3

Let $\mathfrak{X}$ be a strictly poly-stable scheme and $x \in \mathfrak{X}_s$ be a point. Then x possesses a neighbourhood $\mathfrak{U}$ which is elementary and admits an étale map $\varphi\colon \mathfrak{U} \to \mathfrak{S}$ to a standard poly-stable scheme $\mathfrak{S}$ such that $\varphi_*\colon \operatorname{str}(\mathfrak{X}) \to \operatorname{str}(\mathfrak{S})$ is an isomorphism of partially ordered sets.

Proof. Cf. the remark after [**Ber99**, Cor. 2.11] to see that every point possesses an elementary neighbourhood. By definition of strictly poly-stable schemes there is an étale map $\varphi\colon \mathfrak{U} \to \mathfrak{S}$ to a standard poly-stable scheme $\mathfrak{S}$. Deleting all irreducible components of $\mathfrak{S}$ which have empty intersection with the image of φ yields an open subscheme $\mathfrak{U}'$ of $\mathfrak{S}$. This is an open subscheme of a standard poly-stable scheme $\mathfrak{S}'$ such that $\mathfrak{U} \to \mathfrak{U}' \to \mathfrak{S}'$ has the desired property. □

Every strictly poly-stable scheme $\mathfrak{B}$ over $\mathbb{K}^\circ$ comes with an associated polysimplicial set $R(\mathfrak{B})$ which reflects the combinatorial structure of the special fibre. However, as we will not make use of this structure, we will only state the properties of the geometric realisation of $R(\mathfrak{B})$.

Theorem B.4

There is a unique functor $|R(\cdot)|\colon s\mathscr{P}st^{tps} \to \mathscr{T}op$ such that:

(i) If $\mathfrak{X}$ is an elementary strictly poly-stable scheme étale over a standard poly-stable scheme $\mathfrak{S}$ associated to numbers $n_1, \dots, n_r$ and $p_1, \dots, p_r$ and the map $\mathfrak{X} \to \mathfrak{S}$ induces a bijection of partially ordered set of the sets of strata of the special fibres. Then

$$|R(\mathfrak{X})| = |\Delta_{p_1-1}| \times \ldots \times |\Delta_{p_r-1}|,$$

where $|\Delta_s|$ is the topological s-simplex for $s \geq 0$.

(ii) For a strictly poly-stable scheme $\mathfrak{X}$ the space $|R(\mathfrak{X})|$ is the direct limit over all $|R(\mathfrak{X})|$ for $\mathfrak{X}$ as in (i).

Proof. The uniqueness is clear by Prop. B.3. In [**Ber99**, §3] the functor $|R(\cdot)|$ is defined as the functor which maps a strictly poly-stable scheme $\mathfrak{X}$ to the geometric realisation of the polysimplicial reduction set of $\mathfrak{X}$. That this functor indeed satisfies the desired properties follows from [**Ber99**, Step. 15 of pf. of Thm. 5.1] and the construction of $|R(\cdot)|$ for elementary strictly poly-stable schemes (cf. the remark after [**Ber99**, Cor. 3.7]). □

Proposition B.5

Assume that $\widetilde{\mathbb{K}}$ is algebraically closed. Then for two strictly poly-stable schemes $\mathfrak{X}$ and $\mathfrak{Y}$ we have a natural isormophism $|R(\mathfrak{X} \times_{\mathbb{K}^\circ} \mathfrak{Y}| \cong |R(\mathfrak{X})| \times |R(\mathfrak{Y}))|$.

Proof. Let $\mathfrak{B}$ be an arbitrary strictly poly-stable scheme. We consider $\operatorname{str}(\mathfrak{B})$ as a category. We denote by $N(\mathfrak{B})$ the simplicial nerve of the category $\operatorname{str}(\mathfrak{B})$. This defines a functor $N(\cdot)$ from $s\mathscr{P}st^{tps}$ to the category of topological spaces. Then by [**Ber99**, Lemma 3.10] we have an isomorphism of functors $|N(\cdot)| \cong |R(\cdot)|$. But as the residue field $\widetilde{\mathbb{K}}$ is algebraically closed we have $N(\mathfrak{X} \times_{\mathbb{K}^\circ} \mathfrak{Y}) = N(\mathfrak{X}) \times N(\mathfrak{Y})$ and hence $|R(\mathfrak{X} \times_{\mathbb{K}^\circ} \mathfrak{Y})| = |R(\mathfrak{X})| \times |R(\mathfrak{Y}))|$. □

The topological space $|R(\mathfrak{B})|$ of a strictly poly-stable scheme $\mathfrak{B}$ can be realised as a closed subspace $S(\mathfrak{B})$ of $\widehat{\mathfrak{B}}^{\mathrm{an}}$, the *skeleton*.

Theorem B.6 (Berkovich)
Let $\mathfrak{X}$ be a strictly poly-stable scheme. Then there is a closed subset $S(\mathfrak{B})$ of $\widehat{\mathfrak{X}}^{\mathrm{an}}$, the skeleton*, such that the following holds:*

(i) If $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ is a trivially poly-stable morphism of strictly poly-stable schemes, then $\widehat{\varphi}^{\mathrm{an}}(S(\mathfrak{X})) \subset S(\mathfrak{Y})$.
(ii) There is a deformation retract $\tau\colon \widehat{\mathfrak{X}}^{\mathrm{an}} \to S(\mathfrak{X})$. If $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ is a trivially poly-stable morphism of strictly poly-stable schemes then

$$\widehat{\varphi}^{\mathrm{an}} \circ \tau_{\mathfrak{X}} = \tau_{\mathfrak{Y}} \circ \widehat{\varphi}^{\mathrm{an}}.$$

(iii) The formation of the skeleton defines a functor from the category $s\mathscr{P}st^{tps}$ to the category of topological spaces in the natural way.
(iv) The space $S(\mathfrak{X})$ is the direct limit over all $S(\mathfrak{U})$, where $\mathfrak{U} \subset \mathfrak{X}$ are open subschemes of $\mathfrak{X}$ which are elementary and admit a map to a standard poly-stable scheme $\mathfrak{S}(\underline{n}, \underline{p}, \underline{\alpha})$ which induces a bijection of partially ordered sets of the set of strata of the special fibre.
(v) There is a unique isomorphism of functors

$$\Phi\colon S(\cdot) \to |R(\cdot)|$$

enjoying the following property:
Let $\mathfrak{U}$ be a strictly poly-stable scheme as in Thm. B.6 (iv).
Denote by f_{ij} the pull-backs of the functions T_{ij}. Then the isomorphism

$$S(\mathfrak{U}) \to |R(\mathfrak{U})| = |\Delta_{p_1-1}| \times \ldots \times |\Delta_{p_r-1}|$$

is given by the map

$$\begin{aligned} x \mapsto (&-\log|f_{11}(x)|/-\log|\alpha_1|, \ldots, \\ &-\log|f_{1p_1}(x)|/-\log|\alpha_1|, \\ &-\log|f_{21}(x)|/-\log|\alpha_2|, \ldots, \\ &-\log|f_{rp_r}(x)|/-\log|\alpha_r|). \end{aligned}$$

Proof.

(i) This follows from [**Ber99**, Thm. 5.2. vii)].
(ii) This follows [**Ber99**, Thm. 5.2. vii)].
(iii) This follows from (i) and (ii).
(iv) This follows from Step 15 in the proof of [**Ber99**, Thm. 5.2. vii)].
(v) This follows from Step 1 of the proof of [**Ber99**, Thm. 5.2. vii)] together with Step 6 a). □

Proposition B.7
Let $\mathfrak{X}$ be a strictly poly-stable scheme of dimension $n+1$. Then there exists a convering $(\mathfrak{U}_i)_{i\in I}$ by open subschemes and étale maps maps $\varphi_i\colon \mathfrak{U}_i \to \mathfrak{S}_i$ to schemes $\mathfrak{S}_i = \mathfrak{S}(\underline{n}, \underline{p}, \underline{\alpha})$ such that

(i) $\mathrm{str}(\mathfrak{U}_i) \to \mathrm{str}(\mathfrak{S}_i)$ is an isomorphism of partially ordered sets,
(ii) If $i \neq j$ and $S(\mathfrak{U}_i)$ is n-dimensional then $S(\mathfrak{U}_i) \neq S(\mathfrak{U}_j)$.

Moreover, if $\mathfrak{X}$ is projective, we can assume that each member of the family $(\mathfrak{U}_i)_{i\in I}$ is affine and that the covering is finite.

Proof. In virtue of Prop. B.3 we choose a covering $(\mathfrak{U}_i)_{i\in I}$ of $\mathfrak{X}$ consisting of elementary open sets admitting étale maps

$$\varphi_i\colon \mathfrak{U}_i \to \mathfrak{S}_i.$$

for standard schemes $\mathfrak{S}_i = \mathfrak{S}(\underline{n_i}, \underline{p_i}, \underline{a_i})$ and satisfying (i). We can assume that each zero-dimensional stratum is contained in exactly one $\mathfrak{U}_i$. This enforces (ii). By [**Liu02**, Prop. 3.3.36] every finite set of point is contained in a common affine. So we replace each $\mathfrak{U}_i$ by an affine subset containing all generic points of strata of the special fibre of $\mathfrak{U}_i$ and add open subsets to the family $(\mathfrak{U}_i)_{i\in I}$ for points of $\mathfrak{X}$ not covered by the family. As the zero dimensional strata are closed points, we can assume that the new members don't contain these strata, hence the family $(\mathfrak{U}_i)_{i\in I}$ still satisfies (ii). □

Remark B.8
Assume that the residue field of $\mathbb{K}$ is algebraically closed. Denote by $\mathbb{L}$ the completion of the algebraic closure of $\mathbb{K}$. Then the residue field extension $\widetilde{\mathbb{L}}/\widetilde{\mathbb{K}}$ is trivial. Hence, arguing as in Prop. B.5 we get

$$|R(\mathfrak{B})| = |N(\mathfrak{B})| = |N(\mathfrak{B}_{\mathbb{K}^\circ}| = |R(\mathfrak{B}_{\mathbb{K}^\circ})|.$$

Proposition B.9
Let $\mathfrak{B}$ be a strictly poly-stable scheme. Let $\mathfrak{U}_1 \to \mathfrak{S}$ and $\mathfrak{U}_2 \to \mathfrak{S}'$ be étale morphisms from open subschemes of $\mathfrak{B}$ inducing bijections of the sets of strata. Assume that all generic points of strata of $\mathfrak{U}_1$ are contained in $\mathfrak{U}_2$. Then $S(\mathfrak{U}_1)$ is a face of $S(\mathfrak{U}_2)$.

Proof. Clearly we can assume $\mathfrak{U}_1 \subset \mathfrak{U}_2$ and as $\mathfrak{U}_1 \to \mathfrak{U}_2$ is étale we have a natural commutative diagram

$$\begin{array}{ccc} S(\mathfrak{U}_1) & \longrightarrow & S(\mathfrak{U}_2) \\ \downarrow & & \downarrow \\ |N(\mathrm{str}(\mathfrak{U}_1))| & \longrightarrow & |N(\mathrm{str}(\mathfrak{U}_2))| \end{array}$$

by Thm B.6 (v). But $|N(\mathrm{str}(\mathfrak{U}_1))|$ is a face of $|N(\mathrm{str}(\mathfrak{U}_2))|$ and hence the claim. □

Now assume that k is a field which is complete with respect to a discrete non-trivial non-Archimedean absolute value $|\cdot|$. Let π be a uniformiser and set $b = |\pi|^{-1}$. Almost all schemes we will be dealing with will lie in the smaller class of so called regular strictly semi-stable schemes. This smaller class has the advantage that its members are regular schemes and have an intersection theory of divisors with support in the special fibre which is determined combinatorially.

Definition B.10

(i) A *regular strictly semi-stable scheme* is a flat integral separated k°-scheme $\mathfrak{B}$ of finite type with the following properties:
- The generic fiber is smooth and geometrically irreducible,
- The special fiber is reduced,
- Every irreducible component of the special fiber $\mathfrak{B}_s$ is a Cartier divisor on $\mathfrak{B}$,
- The scheme theoretic intersection of any pairwise distinct components

$$C_1, \dots, C_m$$

of the special fibre is smooth of codimension m in $\mathfrak{B}$ or empty.

(ii) A *regular strictly semi-stable model* of a smooth projective geometrically irreducible variety B is a *proper* regular strictly semi-stable scheme $\mathfrak{B}$ with generic fibre B.

The following characterisation of semi-stability is due to de Jong and Hartl.

Theorem B.11
Let $\mathfrak{B}$ be a irreducible scheme of finite type over k°. Then $\mathfrak{B}$ is regular strictly semi-stable if and only if

- *The generic fibre is smooth,*
- *The special fibre is reduced,*
- *Every closed point x of the special fibre has a neighbourhood which is étale over*

$$k^\circ[x_1, \ldots, x_n]/(x_1 \cdots x_p - \pi)$$

for n and p non-negative numbers with $p \leq n$ possibly depending on x.

Proof. Cf. [**DJ96**, 2.16],[**Har01**, Prop. 1.3]. □

Every regular strictly semi-stable scheme is in particular strictly poly-stable. Let $\mathfrak{B}$ be a regular strictly semi-stable scheme. We equip the set of irreducible components of the special fibre with a total order and define the *reduction set* as in [**Kol16b**, Def. 2.6]:

Definition B.12
Let $\mathfrak{X}$ be a regular strictly semistable scheme. Fix an order on the set $\mathfrak{X}_s^{(0)}$ of irreducible components of the special fibre of $\mathfrak{X}$. Let $\beta\colon [n] \to \mathfrak{X}_s^{(0)}$ be a non-decreasing map. We set

$$[\beta] = \beta(0) \cap \ldots \cap \beta(n)$$

and denote by $[\beta]^{(0)}$ the set of irreducible components of $[\beta]$. For every $f\colon [n] \to [m]$ in the simplex category Δ we define a map $f_\beta\colon [\beta]^{(0)} \to [\beta \circ f]^{(0)}$ by mapping each irreducible component C to the irreducible component which contains C. We set

$$R(\mathfrak{X}) := \coprod_{\beta \in \mathrm{Hom}([n], \mathfrak{X}_s^{(0)})} [\beta]^{(0)}$$

and for $f\colon [n] \to [m]$ we set

$$R(\mathfrak{X})(f) := \coprod_{\beta \in \mathrm{Hom}([m], \mathfrak{X}_s^{(0)})} f_\beta.$$

Then $R(\mathfrak{X})$ is a simplicial set.

Note the following fact.

Proposition B.13
Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism between regular strictly semi-stable schemes which is flat in a neighbourhood of the generic point of each irreducible component of the special fibre of $\mathfrak{X}$. Fix an order on the set of irreducible components of $\mathfrak{X}_s$ and $\mathfrak{Y}_s$ respectively. Assume that the rule

$$f_* = (C \mapsto \overline{f(C)})$$

for every irreducible component C of the special fibre of $\mathfrak{X}$ defines a non-decreasing map. Then f_* determines a morphism $f_*\colon R(\mathfrak{X}) \to R(\mathfrak{Y})$ between the simplicial reduction sets.

Proof. See [**Kol13**, Prop. 1.17 ii)]. □

Definition B.14
We denote by $s\mathscr{S}st^{tps}$ the following category: Its objects are pairs $(\mathfrak{X}, \leq)$ where $\mathfrak{X}$ is a strictly semi-stable scheme and $\leq$ is an order on the set of irreducible components of the special fibre. The maps are given by trivially poly-stable maps which respect the order on the irreducible components of the special fibre. Then $\mathfrak{X} \mapsto R(\mathfrak{X})$ defines a functor $R(\cdot)$, the *reduction set functor*, from $s\mathscr{S}st^{tps}$ to $\Delta^\circ\mathscr{E}ns$ in virtue of Prop. B.13. We have a natural forgetful functor F from $s\mathscr{S}st^{\mathrm{tps}}$ to $s\mathscr{P}st^{\mathrm{tps}}$ mapping a pair $(\mathfrak{X}, \leq)$ to $\mathfrak{X}$.

Theorem B.15
Let $R(\cdot)\colon s\mathscr{S}st^{\mathrm{tps}} \to \Delta^\circ\mathscr{E}ns$ be the reduction set functor. Then the following is true.

- *(i) Let $\mathfrak{B}$ be a regular strictly semi-stable k°-scheme. The simplical set $R(\mathfrak{B})$ is the direct limit over all $R(\mathfrak{U})$ where $\mathfrak{U}$ are open subschemes of $\mathfrak{B}$ which are elementary and such that there exists an étale map to*

$$k^\circ[x_1, \ldots, x_n]/(x_1 \cdots x_r - \pi),$$

which induces a bijection of the partially ordered sets of strata of the special fibre. If $\mathfrak{U}$ is such a subset then $R(\mathfrak{X}) = \Delta[p-1]$.
- *(ii) Let $F\colon s\mathscr{S}st^{tps} \to s\mathscr{P}st^{tps}$ be the forgetful functor. If we denote by $|?|$ the geometric realisation functor of simplicial sets then*

$$|?| \circ R = |R(\cdot)| \circ F.$$

PROOF.

(i) This is [**Kol13**, Prop. 1.18].
(ii) This follows from the first point and the unicity properties of $|R(\cdot)|$. □

Let $\mathfrak{B}$ be any regular strictly semi-stable scheme over k°. We denote by $\mathrm{CaDiv}_{\mathfrak{B}_s}(\mathfrak{B})_{\mathbb{Q}}$ the group of $\mathbb{Q}$-Cartier divisors with support in the special fibre. Let $\mathfrak{D} \in \mathrm{CaDiv}_{\mathfrak{B}_s}(\mathfrak{B})_{\mathbb{Q}} = \mathrm{CH}^0(\mathfrak{B}_s)_{\mathbb{Q}}$ be a Cartier divisor with support in the special fibre. Then, as the latter group is free, there is a unique representation

$$\mathfrak{D} = \sum_{i=1}^{m} q_i C_i,$$

where $q_i \in \mathbb{Q}$ and C_i for $i = 1, \ldots, m$ are the irreducible components of the special fibre of $\mathfrak{B}$. The components C_i are naturally identified with points in the geometric realization $|R(\mathfrak{B})|$ of the simplicial set $R(\mathfrak{B})$. Requiring $f_{\mathfrak{D}}(C_i) = q_i$ and $f_{\mathfrak{D}}$ to be affine on each simplex of $R(\mathfrak{B})$ we get a real valued function in the space

$$C^{\mathrm{aff}}_{\Delta,\mathbb{Q}}(R(\mathfrak{B}))$$

of functions on $|R(\mathfrak{B})|$ which have values in $\mathbb{Q}$ on the vertices of $R(\mathfrak{B})$ and are affine on each simplex. Clearly any such function gives a Cartier divisor with support in the special fibre. We summarise this in the following proposition:

Proposition B.16
The correspondence

$$\mathfrak{D} \mapsto f_{\mathfrak{D}}$$

is an isomorphism between $\mathrm{CaDiv}_{\mathfrak{B}_s}(\mathfrak{B})_{\mathbb{Q}}$ *and* $C^{\mathrm{aff}}_{\Delta,\mathbb{Q}}(R(\mathfrak{B}))$.

APPENDIX C

Algebraic Geometry over Valuation Rings

Let $\mathbb{K}$ be field with a non-trivial, non-Archimedean absolute value. Denote by $\mathbb{K}^\circ$ the valuation ring, by $\mathbb{K}^{\circ\circ}$ the maximal ideal of $\mathbb{K}^\circ$, and by $\widetilde{\mathbb{K}}$ the residue field. Let X be a projective variety over $\mathbb{K}$ and let $\mathfrak{X}$ and $\mathfrak{Y}$ be finitely presented, projective models of X over $\mathbb{K}^\circ$.

Proposition C.1
Let $\varphi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of models. Then it is a blow-up at a vertical finitely presented ideal sheaf.

Proof. As $\mathfrak{Y}$ is separated the morphism φ is projective by [**DG67**, II, Prop. 5.5.5]. The schemes $\mathfrak{X}$ and $\mathfrak{Y}$ are defined by finitely generated homogenous ideals I and J in $\mathbb{K}^\circ[T_1, \ldots, T_n]$ and $\mathbb{K}^\circ[T_1, \ldots, T_m]$ respectively. We denote by A the smallest subring of $\mathbb{K}^\circ$ which contains all coefficients of I and J. This is a noetherian ring. Then I and J are by construction defined over A, say by ideals I' and J'. We denote by $\mathfrak{X}'$ and $\mathfrak{Y}'$ the subschemes of $\mathbb{P}^n_A$ and $\mathbb{P}^m_A$ defined by I' and J'. Now, locally on each standard affine of $\mathbb{P}^m$, the map $\mathfrak{X} \to \mathfrak{Y}$ is given by finitely many polynomials. We adjoin their coefficients to A. This guarantees that φ is defined over A and provides us the following cartesian diagram of schemes.

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathfrak{Y} \\ \downarrow & & \downarrow p \\ \mathfrak{X}' & \longrightarrow & \mathfrak{Y}'. \end{array} \tag{C.1}$$

Now we have a commutative diagram of function fields

$$\begin{array}{ccc} K(\mathfrak{Y}) & \longrightarrow & K(\mathfrak{X}) \\ \uparrow & & \uparrow \\ K(\mathfrak{Y}') & \longrightarrow & K(\mathfrak{X}') \end{array}$$

where by assumption $K(\mathfrak{Y}) \to K(\mathfrak{X})$ is an isomorphism. Now, the extension of fields $K(\mathfrak{X}')/K(\mathfrak{Y}')$ is generated by finitely many elements $f_1, \ldots, f_r$. We can think of $f_1, \ldots, f_r$ as polynomials over $\mathbb{K}^\circ$ and as $K(\mathfrak{Y}) \to K(\mathfrak{X})$ is an isomorphism we can adjoin their coefficients to A to achieve that the diagram (C.1) is still cartesian and the map $\mathfrak{X}' \to \mathfrak{Y}'$ is birational and projective. Now by [**Liu02**, Thm. 8.1.24], this means that it is a blow-up at a closed subscheme Z defined by a finitely generated ideal $\mathfrak{I}$. Clearly, $\mathfrak{X}' \otimes \mathbb{K} = X$, hence $\mathfrak{X} \to \mathfrak{X}'$ is dominant as the natural map $\mathfrak{X}' \otimes \mathbb{K} \to \mathfrak{X}' \otimes Q(A)$ is. By [**DG67**, IV, 21.4.5], we can pull back Cartier divisors along $\mathfrak{X} \to \mathfrak{X}'$. For an effective Cartier divisor D, the Cartier divisor associated with the scheme theoretic preimage of D coincides with the pull-back of the Cartier divisor associated to D. Now we can apply the argument of the proof of [**GW10**, Prop. 13.91 (2)] to see that $\mathfrak{X} \to \mathfrak{Y}$ is the blow-up of $\mathfrak{Y}$ at $p^{-1}(Z)$, which is defined by a finitely generated ideal, namely $\mathfrak{I} \otimes \mathbb{K}^\circ$. □

Proposition C.2
Let $\pi\colon \mathfrak{X} \to \mathfrak{Y}$ be a blow-up of projective schemes of finite presentation over $\mathbb{K}^\circ$ at a finitely presented ideal sheaf $\mathfrak{I}$. Denote by $\mathscr{O}(1)$ the canonical line bundle of $\mathfrak{X}$ and let $\mathfrak{A}$ be an ample $\mathbb{Q}$-line bundle on $\mathfrak{Y}$. Then for $m \gg 0$ the line bundle $m \cdot \pi^*\mathfrak{A} \otimes \mathscr{O}(1)$ is an ample $\mathbb{Q}$-line bundle.

Proof. This is [**Har77**, II, Prop. 7.10]. □

Definition C.3
Let $\mathfrak{X}$ be a model of X. Let $\mathfrak{L}$ be a $\mathbb{Q}$-line bundle on $\mathfrak{X}$. A *vertical modification* is a $\mathbb{Q}$-line bundle of the form $\mathfrak{L} \otimes \mathfrak{L}'$ for a $\mathbb{Q}$-line bundle with $\mathfrak{L}'|_X = \mathscr{O}_X$.

Proposition C.4
Let $\mathfrak{X}$ be a projective model of X of finite presentation over $\mathbb{K}^\circ$ and $\mathfrak{L}$ be a $\mathbb{Q}$-line bundle. Assume that $L := \mathfrak{L}|_X$ is an ample $\mathbb{Q}$-line bundle. Let $\mathfrak{X}' \to \mathfrak{X}$ be a morphism of projective models of X. Then there is a blow-up $\mathfrak{X}''$ of $\mathfrak{X}'$ at a finitely presented vertical ideal sheaf such that L extends to an ample $\mathbb{Q}$-line bundle on $\mathfrak{X}''$.

Proof. Let $\mathfrak{X}'''$ be a model of X such that there exists a very ample $\mathbb{Q}$-line bundle $\mathfrak{M}$ such that $\mathfrak{M}|_X = L$. We choose $\mathfrak{X}''$ as the closure of the diagonal map

$$\Delta\colon X \times X \to \mathfrak{X}''' \times \mathfrak{X}'.$$

First, the morphism $\mathfrak{X}'' \to \mathfrak{X}'''$ is a blow-up at a finitely generated ideal by Prop. C.1. Hence, by Prop. C.2 a vertical modification of the pull back of $\mathfrak{M}$ extends is an ample line bundle on $\mathfrak{X}''$. The morphism $\mathfrak{X}'' \to \mathfrak{X}'$ is projective and birational, hence a vertical blow-up at a finitely generated ideal. This finishes the proof. □

The next proposition is a generalisation of [**BFJ16**, Prop. 5.2]:

Proposition C.5
Let $\mathfrak{L}$ be semipositive $\mathbb{Q}$-line bundle on a projective model of X of finite presentation over $\mathbb{K}^\circ$. Assume that $\mathfrak{L}|_X$ is ample. Then for every $\varepsilon > 0$, there exists a $\mathbb{Q}$-line bundle $\mathfrak{D}$ which is trivial on the generic fibre such that for each $m \geq 1$ the line bundle $\mathfrak{D}^{\otimes m}$ is an integral line bundle and for each rational section $s \neq 0$ of $\mathfrak{D}^{\otimes m}$ the inequality

$$|\log \|s(x)\|_{\mathfrak{D}}| \leq \varepsilon^m$$

holds for all $x \in \mathfrak{X}^{\mathrm{an}}$ and $\mathfrak{L} \otimes \mathfrak{D}$ is ample. Here $\|\cdot\|_{\mathfrak{D}}$ is the norm defined in Ch. 4.1.

Proof. The proof is the same as [**BFJ16**, Prop. 5.2] but with [**BFJ16**, Cor. 1.5] replaced by Prop. C.4. □

The following Lemma is [**Har77**, Ex. II 5.12 b)].

Lemma C.6
Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes. Let $\mathscr{L}$ be a line bundle on X very ample relative Y and $\mathscr{M}$ be a line bundle on Y very ample relative to Z. Then $\mathscr{L} \otimes f^*\mathscr{M}$ is very ample relative to Z.

Proof. If $i\colon X \to \mathbb{P}^n \times Y$ and $j\colon Y \to \mathbb{P}^m \times Z$ are immersions such that $\mathscr{L} = i^*\mathscr{O}(1)$ and $\mathscr{M} = j^*\mathscr{O}(1)$, we have the natural maps

$$h = X \to \mathbb{P}^n \times Y \to \mathbb{P}^n \times \mathbb{P}^m \times Z \to \mathbb{P}^N \times Z$$

for $N = nm + n + m$, where the last arrow is induced by the Segre embedding. Then $\mathscr{L} \otimes f^*\mathscr{M} = h^*\mathscr{O}(1)$ and we have shown our claim. □

Lemma C.7
Assume that $\widetilde{\mathbb{K}}$ is algebraically closed. Let $\mathfrak{Y}$ be a projective scheme over $\mathbb{K}^\circ$ and $\mathfrak{D}$ be a very ample integral divisor. Let $p \in \mathfrak{Y}_s$ be a reduced closed point of the special fibre and let $\pi\colon \widetilde{\mathfrak{Y}} \to \mathfrak{Y}$ be the blow-up of $\mathfrak{Y}$ in p and E the exceptional divisor. Then $2\pi^*\mathfrak{D} - E$ is very ample.

Proof. Set $\mathfrak{L} = \mathscr{O}(\mathfrak{D})$ and let $\mathscr{I}_p$ be the ideal sheaf of p. Let $i\colon \mathfrak{Y} \to \mathbb{P}^n_{\mathbb{K}^\circ}$ be an embedding such that $i^*\mathscr{O}(1) \cong \mathfrak{L}$. By the projection formula, $i_*(\mathfrak{L}^2 \otimes \mathscr{I}_p) = \mathscr{O}(2) \otimes i_*(\mathscr{I}_P)$ and the sheaves $i_*(\mathfrak{L}^2 \otimes \mathscr{I}_p)$ and $\mathscr{L}^2 \otimes \mathscr{I}_p$ have the same global sections. Hence we have reduced to the case $\mathfrak{X} = \mathbb{P}^n_{k^\circ}$ and $\mathfrak{L} = \mathscr{O}(1)$. As $\widetilde{\mathbb{K}}$ is algebraically closed $\mathscr{I}_p$ is the ideal sheaf on $\mathbb{P}^n_{\widetilde{\mathbb{K}}}$ associated to the homogenous ideal $(x_0, \ldots, x_n)$ of $\widetilde{\mathbb{K}}[x_0, \ldots, x_n]$ after an $\mathbb{K}^\circ$-automorphism of $\mathbb{P}^n_{\mathbb{K}^\circ}$ by Hilbert's Nullstellensatz. So $\mathscr{I}_p \otimes \mathscr{O}(1)$ is generated by its global sections. As in the proof of [**Har77**, Prop. II 7.10 b)] we see that $\mathfrak{D} - E$ is very ample relative to π. Furthermore, $\mathfrak{D}$ is very ample on $\mathfrak{Y}$. Hence by Lem. C.6 the Divisor $\mathfrak{D} + \mathfrak{D} - E$ is very ample on $\widetilde{\mathfrak{Y}}$. □

APPENDIX D

Berkovich Analytic Geometry

Assume that $\mathbb{K}$ is a field which is complete with respect to a non-trivial non-Archimedean absolute value.

Proposition D.1
Let $f : X \to Y$ be finite flat morphism of good $\mathbb{K}$-analytic spaces. Then $f_*(\mathscr{O}_X)$ is locally free of finite rank.

Proof. The statement is local on the target, so we can assume Y to be affinoid (by goodness of Y). By finiteness X will also be affinoid. By [**Ber90**, Prop. 3.3.5] $f_*(\mathscr{O}_X)$ is coherent and by using the projection formula we see that $f_*(\mathscr{O}_X)$ is flat. Using the remark after [**Ber90**, Cor. 2.3.2] we see that flatness and beeing coherent implies that $f_*(\mathscr{O}_X)$ is locally free using [**Har77**, III, §9, Prop. 9.2 (e)]. □

Definition D.2
Let $f : X \to Y$ be a finite flat morphism of connected $\mathbb{K}$-analytic spaces. By Prop. D.1 $f_*(\mathscr{O}_X)$ is locally free of finite rank which we define to be the *degree* of f (according to [**Ber93**, Rem. 6.3.1 iii)]). Let $y \in Y$ be a point. We say that f is finite of degree d above y if there exists an analytic neighbourhood Z of y in Y such that the induced map $f^{-1}(Z) \to Z$ is finite of degree d (according to [**CLD12**, 2.4.2]).

The following proposition is well-known (e.g. [**Tem10**, Ex. 4.3.2.5]), we state it anyway due to the lack of a suitable reference. For a map $\varphi\colon \mathscr{A} \to \mathscr{B}$ of strictly $\mathbb{K}$-affinoid algebras we denote by $\widetilde{\varphi}\colon \widetilde{\mathscr{A}} \to \widetilde{\mathscr{B}}$ the induced map between the canonical reductions.

Proposition D.3
Let $\mathscr{M}(\mathscr{A})$ be a strictly $\mathbb{K}$-analytic space. Then $\mathrm{Int}(\mathscr{M}(\mathscr{A})/\mathscr{M}(\mathbb{K}))$ is the preimage of the closed points of $\mathrm{Spec}(\widetilde{\mathscr{A}})$ under the reduction map.

Proof. We denote by $\varphi\colon \mathscr{M}(\mathscr{A}) \to \mathscr{M}(\mathbb{K})$ the natural map. Let x be a point in $\mathscr{M}(\mathscr{A})$ and χ_x be the corresponding character. Then by [**Ber90**, Prop. 2.5.2] the point is an inner point if and only if $\widetilde{\varphi}(\widetilde{\mathscr{A}})$ is integral over $\widetilde{\mathbb{K}}$ i.e., $\widetilde{\varphi}(\widetilde{\mathscr{A}})$ is a field. But this happens if and only if $\ker(\widetilde{\chi_x})$ is a maximal ideal of $\mathrm{Spec}(\widetilde{\mathscr{A}})$ that is, the reduction of x is a closed point. □

Table of Symbols

Index

Bibliography

[Awo06] Steve Awodey. *Category theory*, volume 49 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 2006, http://dx.doi.org/10.1093/acprof:oso/9780198568612.001.0001.

[Ber90] Vladimir.G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. Mathematical Surveys and Monographs. American Mathematical Society, 1990.

[Ber93] Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.*, (78):5–161 (1994), 1993.

[Ber94] Vladimir G. Berkovich. Vanishing cycles for formal schemes. *Invent. Math.*, 115(3):539–571, 1994.

[Ber99] Vladimir G. Berkovich. Smooth p-adic analytic spaces are locally contractible. *Invent. Math.*, 137(1):1–84, 1999.

[Ber09] Knut Berg. Edgewise subdivision and simple maps, 2009, `http://folk.uio.no/rognes/theses/berg-master.pdf`.

[BFJ15] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Solution to a non-Archimedean Monge-Ampère equation. *J. Amer. Math. Soc.*, 28(3):617–667, 2015, `http://dx.doi.org/10.1090/S0894-0347-2014-00806-7`.

[BFJ16] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Singular semipositive metrics in non-Archimedean geometry. *J. Algebraic Geom.*, 25(1):77–139, 2016.

[BG86] Robert Bieri and John R. J. Groves. A rigidity property for the set of all characters induced by valuations. *Trans. Amer. Math. Soc.*, 294(2):425–434, 1986.

[BGPS14] Jose Ignacio Burgos Gil, Patrice Philippon, and Martin Sombra. Arithmetic geometry of toric varieties. metrics, measures, and heights. *Astérisque*, 360:vi+222, 2014.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984, `http://dx.doi.org/10.1007/978-3-642-52229-1`. A systematic approach to rigid analytic geometry.

[BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.

[Cin11] Zubeyir Cinkir. Zhang's conjecture and the effective Bogomolov conjecture over function fields. *Invent. Math.*, 183(3):517–562, 2011, `http://dx.doi.org/10.1007/s00222-010-0282-7`.

[CL06] Antoine Chambert-Loir. Mésures et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.*, 595:215–235, 2006, `http://dx.doi.org/10.1515/CRELLE.2006.049`.

[CLD12] Antoine Chambert-Loir and Antoine Ducros. Formes différentielles réelles et courants sur les espaces de Berkovich, 2012, arXiv:1204.6277v1.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[DG67] Jean Dieudonné and Alexander Grothendieck. Éléments de géométrie algébrique. *Publications Mathématiques de l'IHÉS*, 4, 8, 11, 17, 20, 24, 28, 32, 1961–1967.

[DJ96] A. J. De Jong. Smoothness, semi-stability, and alterations. *Publications Mathématiques de l'IHÉS*, 83:51–93, 1996.

[DM69] Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. *Publications Mathématiques de l'IHÉS*, 36:75–109, 1969.

[Duc12] Antoine Ducros. Espaces de Berkovich, polytopes, squelettes et théorie des modèles. *Confluentes Math.*, 4(4):1250007, 57, 2012.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, second edition, 1998, `http://dx.doi.org/10.1007/978-1-4612-1700-8`.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999, `http://dx.doi.org/10.1007/978-3-0348-8707-6`.

[GS95] B. H. Gross and C. Schoen. The modified diagonal cycle on the triple product of a pointed curve. *Ann. Inst. Fourier (Grenoble)*, 45(3):649–679, 1995.

[Gub98] Walter Gubler. Local heights of subvarieties over non-Archimedean fields. *J. Reine Angew. Math.*, 498:61–113, 1998.

[Gub02] Walter Gubler. Basic properties of heights of subvarieties. *Habilitationsschrift*, 2002.

[Gub03] Walter Gubler. Local and canonical heights of subvarieties. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 2(4):711–760, 2003.

[Gub07] Walter Gubler. Tropical varieties for non-Archimedean analytic spaces. *Invent. Math.*, 169(2):321–376, 2007.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010, `http://dx.doi.org/10.1007/978-3-8348-9722-0`. Schemes with examples and exercises.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Har01] Urs T. Hartl. Semi-stability and base change. *Archiv der Mathematik*, 77(3):215–221, 2001.

[HK13] Qing Han and Marcus Khuri. Smooth solutions to a class of mixed type Monge-Ampère equations. *Calc. Var. Partial Differential Equations*, 47(3-4):825–867, 2013, `http://dx.doi.org/10.1007/s00526-012-0547-7`.

[Kle66] Steven L. Kleiman. Toward a numerical theory of ampleness. *Ann. of Math. (2)*, 84:293–344, 1966.

[Kol13] Johannes Kolb. Lokale Schnitttheorie an nicht-archimedischen Stellen semistabiler Kurven, Dissertation, Regensburg, 2013, http://epub.uni-regensburg.de/28447/.

[Kol16a] Johannes Kolb. An analytic description of local intersection numbers at non-Archimedean places for products of semi-stable curves. *Annales de la faculté des sciences de Toulouse*, 25(1):19–63, 2016.

[Kol16b] Johannes Kolb. A simplicial calculus for local intersection numbers at non-archimedian places on products of semi-stable curves. *Abh. Math. Semin. Univ. Hambg.*, 86(1):97–132, 2016.

[Lag12] Aron Lagerberg. Super currents and tropical geometry. *Math. Z.*, 270(3-4):1011–1050, 2012.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004, http://dx.doi.org/10.1007/978-3-642-18808-4. Classical setting: line bundles and linear series.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Liu11] Yifeng Liu. A non-Archimedean analogue of the Calabi-Yau theorem for totally degenerate abelian varieties. *J. Differential Geom.*, 89(1):87–110, 2011, http://projecteuclid.org/euclid.jdg/1324476752.

[Mor16] Atsushi Moriwaki. Adelic divisors on arithmetic varieties. *Memoirs of the American Mathematical Society*, 242(1144), 2016.

[Seg73] Graeme Segal. Configuration-spaces and iterated loop-spaces. *Invent. Math.*, 21:213–221, 1973.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[Sou92] Christophe Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992, http://dx.doi.org/10.1017/CBO9780511623950. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[Tem10] Michael Temkin. Introduction to berkovich analytic spaces, 2010, arXiv:1010.2235.

[Thu07] Amaury Thuillier. Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d'homotopie de certains schémas formels. *Manuscripta Math.*, 123(4):381–451, 2007, http://dx.doi.org/10.1007/s00229-007-0094-2.

[YZ10] Xinyi Yuan and Shou-Wu Zhang. Calabi-yau theorem and algebraic dynamics, 2010, http://www.math.columbia.edu/~szhang/papers/cyds.pdf.

[Zha93] Shou-Wu Zhang. Admissible pairing on a curve. *Invent. Math.*, 112(1):171–193, 1993, http://dx.doi.org/10.1007/BF01232429.

[Zha95] Shou-Wu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995.

[Zha10] Shou-Wu Zhang. Gross-Schoen cycles and dualising sheaves. *Invent. Math.*, 179(1):1–73, 2010.

[Zha15] Shou-Wu Zhang. On non-Archimedean Poisson's equation. *To appear in: Oberwolfach Reports*, (4), 2015.