Universität Regensburg
Mathematik

Linear stability of the non-extreme Kerr black hole

Felix Finster and Joel Smoller

Preprint Nr. 05/2016

LINEAR STABILITY OF THE NON-EXTREME KERR BLACK HOLE

FELIX FINSTER AND JOEL SMOLLER

JUNE 2016

Abstract. It is proven that for smooth initial data with compact support outside the event horizon, the solution of every azimuthal mode of the Teukolsky equation for general spin decays pointwise in time.

Contents

F.F. is supported in part by the Deutsche Forschungsgemeinschaft.
J.S. is supported in part by the National Science Foundation, Grant No. DMS-1105189.

1. Introduction

In this paper we prove linear stability of the non-extreme Kerr black hole under perturbations by gravitational and electromagnetic waves. More precisely, we consider the initial value problem for the Teukolsky equation of general spin $s \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$ for smooth initial data supported outside the event horizon. Thus, rewriting the equation as a first-order system in time, we analyze the solution for initial data $\Psi_0 \in C^\infty((r_1,\infty) \times S^2, \mathbb{C}^2)$ (where r_1 is the event horizon, and the two components of Ψ_0 describe the Teukolsky wave function and its first time derivative at time zero; for details see Section 2). We decompose the initial data into a Fourier series of azimuthal modes,

$$\Psi_0(r,\vartheta,\varphi) = \sum_{k\in\mathbb{Z}} e^{-ik\varphi}\, \Psi_0^{(k)}(r,\vartheta)\,. \tag{1.1}$$

Since the Kerr geometry is axisymmetric, the Teukolsky equation decouples into separate equations for each mode. Therefore, the solution of the Cauchy problem with initial data Ψ_0 is obtained by solving the Cauchy problem for each mode and taking the sum of the resulting solutions. With this in mind, we here restrict attention to the Cauchy problem for a single mode, i.e.

$$\Psi(0,r,\vartheta,\varphi) = e^{-ik\varphi}\, \Psi_0^{(k)}(r,\vartheta) \in C^\infty\big((r_1,\infty)\times S^2, \mathbb{C}^2\big)\,. \tag{1.2}$$

We derive an integral representation of the solution which involves the fundamental solutions of the ODEs arising in the separation of variables. Moreover, we prove the following pointwise decay result:

Theorem 1.1. *Consider a non-extreme Kerr black hole of mass M and angular momentum aM with $M^2 > a^2 > 0$. Then for any $s \geq \frac{1}{2}$, the solution Ψ of the Teukolsky equation with initial data of the form* (1.2) *decays to zero in $L^\infty_{\text{loc}}((r_1,\infty)\times S^2)$.*

This theorem establishes in the dynamical setting that the non-extreme Kerr black hole is linearly stable.

In general terms, the problem of linear stability of black holes can be stated mathematically as the question whether solutions of massless linear wave equations in the Kerr geometry decay in time. The different types of equations are characterized systematically in the Newman-Penrose formalism by their spin, taking the possible values $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. From the physical point of view, the most interesting cases are $s = 1$ (Maxwell field) and $s = 2$ (gravitational waves). The case $s = 0$ of scalar waves is a major mathematical simplification. The black hole stability problem has a long history and has been studied by many authors. For brevity, we only mention a few recent results and refer for the broader context to [13] and the references therein. Despite many results for scalar waves in the Kerr geometry (see for example [11, 12, 30, 29, 25, 7]) and for higher spin waves in spherically symmetric space-times (see for example [17, 28, 3, 6]), only few results are known for higher spin waves in the Kerr geometry. There are results for the Dirac field [10] and for the Maxwell field [26, 2, 1], all of which use the specific structure of the respective equations. Also, we would like to mention recent nonlinear stability results in the related Kerr-De Sitter geometry [22]. A general framework for analyzing the equations of arbitrary spin in the Kerr geometry goes back to Teukolsky [32] who showed that the massless equations for any spin can be rewritten as a single wave equation for a complex scalar field ϕ, referred to as the Teukolsky equation (sometimes also called the Teukolsky master equation). If $s \neq 0$, the coefficients of the Teukolsky equation are *complex*.

The Teukolsky equation has the remarkable property that it can be separated into a coupled system of a radial and an angular ODE (for details see for example the textbook [5]). The only known stability result in the Teukolsky framework was obtained by Whiting [33], who proved that the Teukolsky equation does not admit solutions which decay both at spatial infinity and at the event horizon and increase exponentially in time. This so-called *mode stability* result is also a key ingredient to our analysis of the long-time dynamics of solutions of the Cauchy problem.

We now outline our method of proof. We first bring the Teukolsky equation into the Hamiltonian form by employing the ansatz

$$\Psi = \begin{pmatrix} \Phi \\ i\partial_t \Phi \end{pmatrix}$$

and writing the equation as

$$i\partial_t \Psi = H \Psi \,,$$

where H is a second-order spatial differential operator. Using suitable PDE estimates, we show that the resolvent $R_\omega := (H - \omega)^{-1}$ exists if ω lies outside a strip enclosing the real axis (see Lemma 4.1). We then derive an integral representation for the solution of the Cauchy problem which involves a Cauchy-type contour integral over the resolvent (see Theorem 5.1). Next, we decompose the resolvent on the contour into an infinite sum of angular modes (see Theorem 7.1). These angular modes arise from our earlier paper [21] where we derive a spectral decomposition of the angular operator into invariant subspaces. After employing this angular mode decomposition, Whiting's mode stability [33] makes it possible to move the contour integrals of the separated resolvent onto the real axis.

At this point two major problems remain: to show that the separated resolvents have no poles on the real axis, and to control the infinite sum of angular modes uniformly in time. In order to resolve these problems, we write the radial part of the separated Teukolsky equation in Sturm-Liouville form

$$\Big(-\frac{d^2}{du^2} + V \Big) X = 0 \,.$$

By a careful analysis of the potential V and of the solutions of this ODE, we show that we can approximate X in different regions by WKB, Airy, and parabolic cylinder functions, with rigorous error estimates. Here we rely crucially on our previous work on special functions [19] and on the ODE estimates developed in [18, 20]. These results also give rise to corresponding estimates for the separated resolvent (see Proposition 10.11).

For smooth initial data supported outside the event horizon, we thus obtain an integral representation for the solution Ψ of the Cauchy problem for the Teukolsky equation involving an infinite sum of angular modes (see Theorem 12.1). We prove that this infinite sum, for large n is uniformly small, and that the remaining finite sum decays using the Riemann-Lebesgue lemma (see Corollary 12.2). This gives the above theorem.

2. Preliminaries

We consider the Kerr metric in Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $r > 0$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$ (see for example [5]). Then the line element takes the form

$$\begin{aligned} ds^2 &= g_{jk}\, dx^j x^k \\ &= \frac{\Delta}{U}\,(dt - a\, \sin^2\vartheta\, d\varphi)^2 - U\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) - \frac{\sin^2\vartheta}{U}\,(a\, dt - (r^2+a^2)\, d\varphi)^2\,, \end{aligned}$$

where

$$U(r,\vartheta) = r^2 + a^2\, \cos^2\vartheta \qquad \text{and} \qquad \Delta(r) = r^2 - 2Mr + a^2\,. \tag{2.1}$$

Here the parameters M and aM denote the mass and the angular momentum of the black hole, respectively. We shall restrict attention to the *non-extreme case* with non-zero angular momentum, i.e. $M^2 > a^2 > 0$. In this case, the function Δ has two distinct zeros,

$$r_0 = M - \sqrt{M^2 - a^2} \qquad \text{and} \qquad r_1 = M + \sqrt{M^2-a^2}\,, \tag{2.2}$$

corresponding to the Cauchy and the event horizon, respectively. We shall consider only the region $r > r_1$ *outside the event horizon*, and thus $\Delta > 0$.

Our starting point is the Teukolsky equation in the form given by Whiting [33]

$$\begin{aligned} \Bigg(\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} - \frac{1}{\Delta}\left\{(r^2+a^2)\,\frac{\partial}{\partial t} + a\,\frac{\partial}{\partial\varphi} - (r-M)\,s\right\}^2 - 4s\,(r + ia\cos\vartheta)\,\frac{\partial}{\partial t} \\ + \frac{\partial}{\partial\cos\vartheta}\,\sin^2\vartheta\,\frac{\partial}{\partial\cos\vartheta} + \frac{1}{\sin^2\vartheta}\left\{a\sin^2\vartheta\,\frac{\partial}{\partial t} + \frac{\partial}{\partial\varphi} + is\cos\vartheta\right\}^2\Bigg)\phi = 0\,. \end{aligned} \tag{2.3}$$

We restrict attention to a fixed φ-mode. Thus for a given $k \in \mathbb{Z}$ we make the ansatz

$$\phi(t,r,\vartheta,\varphi) = e^{-ik\varphi}\, R(t,r,\vartheta)\,. \tag{2.4}$$

Moreover, we introduce the Regge-Wheeler coordinate $u \in \mathbb{R}$ by

$$\frac{du}{dr} = \frac{r^2+a^2}{\Delta}\,, \qquad \frac{\partial}{\partial r} = \frac{r^2+a^2}{\Delta}\,\frac{\partial}{\partial u} \tag{2.5}$$

and introduce the new function Φ by

$$\Phi(t,u,\vartheta) = \sqrt{r^2+a^2}\, R(t,r,\vartheta)\,. \tag{2.6}$$

Using the transformation

$$\begin{aligned} \frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} &= \frac{r^2+a^2}{\Delta}\,\frac{\partial}{\partial u}(r^2+a^2)\frac{\partial}{\partial u} \\ &= \frac{r^2+a^2}{\Delta}\,\sqrt{r^2+a^2}\left(\frac{\partial^2}{\partial u^2}\,\sqrt{r^2+a^2} - \left(\partial_u^2\sqrt{r^2+a^2}\right)\right), \end{aligned}$$

we find that

$$\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r}R = \frac{1}{\sqrt{r^2+a^2}}\,\frac{(r^2+a^2)^2}{\Delta}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial_u^2\sqrt{r^2+a^2}}{\sqrt{r^2+a^2}}\right)\Phi\,.$$

Then the Teukolsky equation takes the form

$$T\Phi = 0\,, \tag{2.7}$$

where

$$T = \frac{(r^2+a^2)^2}{\Delta} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}} \right)$$
$$- \frac{1}{\Delta} \left\{ (r^2+a^2) \frac{\partial}{\partial t} - iak - (r-M)\, s \right\}^2 - 4s\,(r + ia\cos\vartheta) \frac{\partial}{\partial t}$$
$$+ \frac{\partial}{\partial \cos\vartheta} \sin^2\vartheta \frac{\partial}{\partial \cos\vartheta} + \frac{1}{\sin^2\vartheta} \left\{ a \sin^2\vartheta \frac{\partial}{\partial t} - ik + is\cos\vartheta \right\}^2 .$$

3. Hamiltonian Formulation

In order to write the Teukolsky equation (2.7) in the Hamiltonian form, we make the ansatz

$$\Psi = \begin{pmatrix} \Phi \\ i\partial_t \Phi \end{pmatrix} . \tag{3.1}$$

Then the equation takes the form

$$i\,\partial_t \Psi = H\,\Psi , \tag{3.2}$$

where H is the Hamiltonian

$$H = \begin{pmatrix} 0 & 1 \\ A & \beta \end{pmatrix} , \tag{3.3}$$

whose matrix entries are the operators

$$\rho = r^2 + a^2 - a^2 \sin^2\vartheta \, \frac{\Delta}{r^2+a^2} \tag{3.4}$$

$$A = \frac{r^2+a^2}{\rho} \left(-\frac{\partial^2}{\partial u^2} + \frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}} \right) \tag{3.5}$$

$$+ \frac{\Delta}{\rho\,(r^2+a^2)} \left(-\frac{\partial}{\partial \cos\vartheta} \sin^2\vartheta \frac{\partial}{\partial\cos\vartheta} + \frac{(-k + s\cos\vartheta)^2}{\sin^2\vartheta} \right) \tag{3.6}$$

$$- \frac{\big(ak + i(M-r)s\big)^2}{\rho\,(r^2+a^2)} \tag{3.7}$$

$$\beta = \frac{2}{\rho} \left[-\Big(ak + i(M-r)s\Big) + \Big(ak - 2irs + as\cos\vartheta\Big) \frac{\Delta}{r^2+a^2} \right] . \tag{3.8}$$

As the domain of definition of H we choose the smooth wave functions which are compactly supported outside the event horizon. Thus, working with the Regge-Wheeler coordinate u throughout (see (2.5)), we choose

$$\mathcal{D}(H) = C_0^\infty(\mathbb{R} \times S^2, \mathbb{C}^4) . \tag{3.9}$$

We remark that in the limiting case $a \searrow 0$, the above Hamiltonian reduces to that in [17, Section 4]. In the case $s = 0$, on the other hand, our Hamiltonian coincides with that in [11, eq. (2.25)], except for the factor $\sqrt{r^2+a^2}$ in the transformation (2.6).

The next step is to introduce a scalar product. Our starting point is the bilinear form

$$<\Psi, \tilde{\Psi}> = \int_{-\infty}^{\infty} \frac{\rho}{r^2+a^2}\, du \int_{-1}^{1} d\cos\vartheta \, \langle \Psi, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \tilde{\Psi} \rangle_{\mathbb{C}^2} . \tag{3.10}$$

This bilinear form has a similar structure as the "energy scalar product" used for example in Minkowski space. In our setting, however, this bilinear form does not have the symmetry property $\overline{<\Psi_1, \Psi_2>} =<\Psi_2, \Psi_1>$ (because the term (3.7) is complex) and is therefore certainly not positive definite. Our strategy is to modify (3.10) in such a way that it becomes symmetric and positive definite. We first verify the sign of the zero order term in (3.5).

Lemma 3.1. *Outside the event horizon, the zero order term in* (3.5) *is non-negative, i.e.*

$$\frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}} \geq 0 \qquad \text{for all } r > r_1 \, .$$

Proof. By direct computation, one finds that

$$\frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}} = \frac{\Delta}{(r^2+a^2)^4}\, f(r) \qquad \text{with} \qquad f(r) := a^4 - 4a^2 M r + a^2 r^2 + 2 M r^3 \, .$$

We want to show that the function f is non-zero outside the event horizon. To this end, we first note that its derivative

$$f'(r) = -4a^2 M + 2a^2 r + 6 M r^2$$

is obviously monotone increasing. Therefore, a direct computation using (2.2) gives

$$f'(r) \geq f'(r_1) = \left(12 M^3 - 8a^2 M\right) + \left(12 M^2 - 2a^2\right) \sqrt{M^2 - a^2} \geq 0 \, ,$$

where in the last step we used that $a < M$. We conclude that f is monotone increasing. Therefore,

$$f(r) \geq f(r_1) = 8M^2 \left(M^2 - a^2\right) + \left(8M^3 - 4a^2 M\right) \sqrt{M^2 - a^2} \geq 0 \, ,$$

where we again used (2.2) together with the inequality $a < M$. This concludes the proof. ◻

In view of this lemma, the term (3.5) gives a positive contribution to the bilinear form (3.10). Obviously, the same is true for the term (3.6). In order to get rid of the troublesome complex term (3.7) we set

$$\delta = 1 + \frac{\left(ak + i(M-r)s\right)^2}{\rho\,(r^2+a^2)}$$

and introduce the scalar product $(.,.)$ by

$$(\Psi, \tilde{\Psi}) = \int_{-\infty}^{\infty} \frac{\rho}{r^2+a^2}\, du \int_{-1}^{1} d\cos\vartheta \, \langle \Psi, \begin{pmatrix} A+\delta & 0 \\ 0 & 1 \end{pmatrix} \tilde{\Psi} \rangle_{\mathbb{C}^2} \, . \tag{3.11}$$

Taking the completion of the domain (3.9) gives rise to a Hilbert space $(\mathcal{H}, (.,.))$. Using that $A + \delta \geq \mathbb{1}$ and that the weight factor in (3.11) written as

$$\frac{\rho}{r^2+a^2} = 1 - \frac{\Delta\, a^2 \sin^2\vartheta}{(r^2+a^2)^2}$$

is clearly bounded uniformly from above and below in u and ϑ, the corresponding Hilbert space norm $\|.\|$ is equivalent to the Sobolev norm on $(H^{1,2} \oplus L^2)(\mathbb{R} \times S^2, \mathbb{C}^4)$.

4. Resolvent Estimates

Obviously, the Hamiltonian H is not symmetric on $(\mathcal{H}, (.|.))$. But, as is verified by direct computation, we obtain a symmetric operator by modifying the Hamiltonian to

$$H_+ = \begin{pmatrix} 0 & 1 \\ A+\delta & \operatorname{Re}\beta \end{pmatrix} ,$$

where we again choose the domain (3.9). The difference of H and H_+ is a bounded operator. Namely,

$$\begin{aligned} \|(H-H_+)\Psi\| &= \left\| \begin{pmatrix} 0 & 0 \\ -\delta & i\operatorname{Im}\beta \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \right\| = \big\| -\delta\Psi_1 + i \operatorname{Im}\beta\, \Psi_2 \big\|_{L^2} \\ &\leq \sup_{\mathbb{R}\times S^2} \big(|\delta| + |\operatorname{Im}\beta|\big) \Big(\|\Psi_1\|_{L^2} + \|\Psi_2\|_{L^2} \Big) \leq \sup_{\mathbb{R}\times S^2} \big(|\delta| + |\operatorname{Im}\beta|\big)\, \|\Psi\| , \end{aligned}$$

implying that

$$\|(H-H_+)\| \leq c := \sup_{\mathbb{R}\times S^2} \big(|\delta| + |\operatorname{Im}\beta|\big) . \tag{4.1}$$

Now we use a method similar as in [11, Lemma 4.1] to obtain resolvent estimates.

Lemma 4.1. *For every ω with*

$$|\operatorname{Im}\omega| > c , \tag{4.2}$$

the resolvent $R_\omega = (H-\omega)^{-1}$ exists and is bounded by

$$\|R_\omega\| \leq \frac{1}{|\operatorname{Im}\omega| - c} . \tag{4.3}$$

Proof. The inequality (4.1) gives rise to the bound

$$\begin{aligned} \|H - H^*\| &= \big\|(H-H_+) - (H-H_+)^*\big\| \\ &\leq \|H-H_+\| + \|(H-H_+)^*\| = 2\,\|H-H_+\| \leq 2c . \end{aligned}$$

It follows that for every normalized vector $\Psi \in \mathcal{D}(H)$,

$$\begin{aligned} \|(H-\omega)\Psi\| &\geq \big|\big(\Psi, (H-\omega)\Psi\big)\big| \geq \big|\operatorname{Im}\big(\Psi, (H-\omega)\Psi\big)\big| \\ &\geq |\operatorname{Im}\omega|\,\|\Psi\|^2 - \frac{1}{2}\,\big|\big(\Psi, (H-H^*)\Psi\big)\big| \\ &\geq \big(|\operatorname{Im}\omega| - c\big)\,\|\Psi\|^2 . \end{aligned} \tag{4.4}$$

It follows that the operator $(H-\omega)$ is injective.

In order to show that this operator is also surjective, we first note that the estimate (4.4) implies that the image of $(H-\omega)$ is a closed subspace of $\mathcal{H}$. Therefore, it suffices to show that the image of the operator $H-\omega$ is dense in $\mathcal{H}$. If this were not the case, there would exist a non-zero vector $\hat{\Psi} \in \mathcal{H}$ such that

$$\big((H-\omega)\Psi, \hat{\Psi}\big) = 0 \qquad \text{for all } \Psi \in \mathcal{D}(H) .$$

In other words, $\hat{\Psi}$ would be a weak solution of the adjoint equation $(H^* - \overline{\omega})\hat{\Psi} = 0$. By the regularity theorem for elliptic operators on manifolds (cf. [31, Chapter 5, Theorem 1.3]), every weak solution of this equation is a solution in the strong sense (note that the operator A is uniformly elliptic). On the other hand, repeating the estimate (4.4) with H replaced by H^* and ω replaced by $\overline{\omega}$, we get the inequality

$$\big\|(H^* - \overline{\omega})\hat{\Psi}\big\| \geq \big(|\operatorname{Im}\omega| - c\big)\,\big\|\hat{\Psi}\big\|^2 .$$

This is a contradiction.

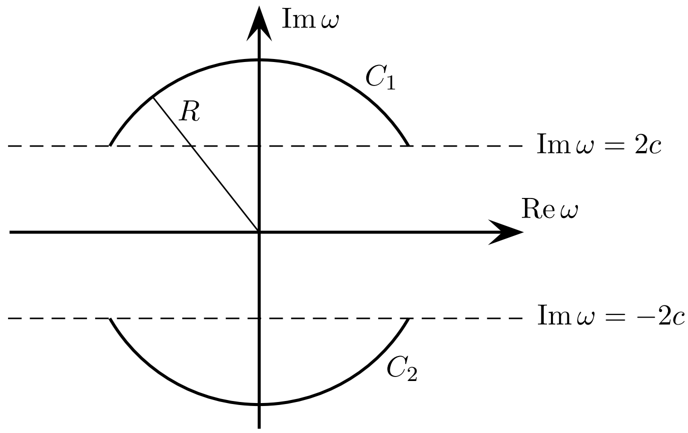

FIGURE 1. The contour C_R.

The above arguments show that the resolvent R_ω exists. Applying the inequality (4.4) for $\Psi = R_\omega \Phi$ gives the estimate (4.3). This concludes the proof. □

5. Contour Integrals and Completeness

For given $R > 0$ we consider the two contours C_1 and C_2 in the complex ω-plane defined by

$$C_1 = \partial B_R(0) \cap \{\mathrm{Im}\,\omega > 2c\}\,, \qquad C_2 = \partial B_R(0) \cap \{\mathrm{Im}\,\omega < -2c\}\,,$$

both taken with positive orientation (see Figure 1). We set $C_R = C_1 \cup C_2$. We can now state the following completeness result. The proof uses similar methods as in [17, Section 7] and is based on an idea which we learned from A. Bachelot [4, Proof of Theorem 2.12].

Theorem 5.1. *For every $\Psi \in \mathcal{D}(H)$, we have the representation*

$$\Psi = -\frac{1}{2\pi i} \lim_{R\to\infty} \int_{C_R} (R_\omega \Psi)\, d\omega\,. \tag{5.1}$$

Proof. For given ω satisfying the inequality (4.2) and $\Psi \in \mathcal{D}(\mathcal{H})$, we know that

$$R_\omega\,(H-\omega)\,\Psi = \Psi\,. \tag{5.2}$$

Solving for $R_\omega \Psi$ gives the identity

$$R_\omega \Psi = -\frac{\Psi}{\omega} + \frac{1}{\omega}\, R_\omega(H\Psi)\,.$$

Clearly, this identity also holds for Ψ replaced by $H\Psi$. This makes it possible to iterate the identity to obtain

$$R_\omega \Psi = -\frac{\Psi}{\omega} - \frac{1}{\omega^2}\,(H\Psi) + \frac{1}{\omega^2}\, R_\omega\big(H^2\Psi\big)\,. \tag{5.3}$$

Integrating ω over the contour C_R gives the estimate

$$\left\| \int_{C_R} \Big(R_\omega \Psi + \frac{\Psi}{\omega}\Big)\, d\omega \right\| \leq \Big(\|H\Psi\| + \big\|R_\omega(H^2\Psi)\big\| \Big) \int_{C_R} \frac{d|\omega|}{|\omega^2|}$$

Using the resolvent estimate (4.3) and noting that the length of the contour grows only linearly in R, one sees that the right side tends to zero as $R \to \infty$. Hence

$$\lim_{R\to\infty} \int_{C_R} (R_\omega \Psi)\, d\omega = -\Psi \lim_{R\to\infty} \int_{C_R} \frac{d\omega}{\omega} = -2\pi i\, \Psi\,,$$

where the last step can be verified by computing the integral explicitly or by using the estimate

$$\left| \oint_{\partial B_R(0)} \frac{d\omega}{\omega} - \int_{C_R} \frac{d\omega}{\omega} \right| \leq \frac{12\, c}{R} \overset{R\to\infty}{\longrightarrow} 0\,.$$

This concludes the proof. □

The integral representation in Theorem 5.1 has the disadvantage that the integrand decays at infinity only like $1/|\omega|$, making it impossible to work with unbounded contours (because these would not converge in the Hilbert space). In order to avoid this problem, we use the method introduced in [17, Section 7] to subtract counter terms which do not change the value of the contour integral. We thus obtain the following result.

Theorem 5.2. *Choosing C as the contour*

$$C = \big\{ \omega \;\big|\; \mathrm{Im}\, \omega = 2c \big\} \cup \big\{ \omega \;\big|\; \mathrm{Im}\, \omega = -2c \big\} \tag{5.4}$$

with counter-clockwise orientation, the following completeness relation holds for every $\Psi \in \mathcal{D}(H)$,

$$\Psi = -\frac{1}{2\pi i} \int_C \Big(R_\omega \Psi + \frac{\Psi}{\omega + 3ic} \Big)\, d\omega\,. \tag{5.5}$$

Moreover, the Cauchy problem for the Teukolsky equation with initial data $\Psi|_{t=0} = \Psi_0 \in \mathcal{D}(H)$ has a unique solution given by

$$\Psi(t) = -\frac{1}{2\pi i} \int_C e^{-i\omega t} \Big(R_\omega \Psi_0 + \frac{\Psi_0}{\omega + 3ic} \Big)\, d\omega\,. \tag{5.6}$$

Proof. Since the resolvent is holomorphic in the region $\{ |\,\mathrm{Im}\, \omega| > c \}$, we may continuously deform the contour C_R in (5.1) for any R. In particular, we may deform the contours to new contours $\tilde{C}_R$ which all lie inside the region $|\,\mathrm{Im}\, \omega| < 3c$. Then the function $1/(\omega + 3ic)$, having its poles outside this region, does not contribute to the contour integral in the the limit $R \to \infty$, i.e.

$$\Psi = -\frac{1}{2\pi i} \lim_{R\to\infty} \int_{\tilde{C}_R} \Big(R_\omega \Psi + \frac{\Psi}{\omega + 3ic} \Big)\, d\omega\,.$$

Iterating (5.3) and expanding, one finds

$$R_\omega \Psi = -\frac{\Psi}{\omega} - \frac{H\Psi}{\omega^2} + \frac{R_\omega(H^2\Psi)}{\omega^2} = -\frac{\Psi}{\omega} + \mathcal{O}\big(\omega^{-2}\big) = -\frac{\Psi}{\omega + 3ic} + \mathcal{O}\big(\omega^{-2}\big)\,.$$

According to (5.3), the norm of the integrand decays quadratically for large $|\omega|$. Therefore, in the limit $R \to \infty$ the integrals converges to the unbounded contour integral (5.5).

In order to prove (5.6), we insert one more counter term (which again does not change the value of the contour integral) to obtain

$$\Psi_0 = -\frac{1}{2\pi i} \int_{C_R} \Big(R_\omega \Psi_0 + \frac{\Psi_0}{\omega + 3ic} + \frac{(H + 3ic)\Psi_0}{(\omega + 3ic)^2} \Big)\, d\omega\,. \tag{5.7}$$

A direct computation using the identity

$$R_\omega \Psi_0 = -\frac{\Psi_0}{\omega} - \frac{H\Psi_0}{\omega^2} - \frac{H^2\Psi_0}{\omega^3} + \frac{R_\omega(H^3\Psi_0)}{\omega^3}$$

(obtained again by iterating (5.3)) shows that the integrand in (5.7) decays even cubically for large $|\omega|$. Clearly, the additional counter term can also be inserted in (5.6), so that the function $\Psi(t)$ as defined by (5.6) takes the form

$$\Psi(t) = -\frac{1}{2\pi i} \int_{C_R} e^{-i\omega t} \Big(R_\omega \Psi_0 + \frac{\Psi_0}{\omega + 3ic} + \frac{(H+3ic)\Psi_0}{(\omega+3ic)^2} \Big)\, d\omega \,. \tag{5.8}$$

Setting $t = 0$ and using (5.7), we find that $\Psi(t) = \Psi_0$, showing that the initial conditions are satisfied. It remains to show that $\Psi(t)$ satisfies the Teukolsky equation $(i\partial_t - H)\Psi(t) = 0$. Using that the integrand in (5.8) decays cubically for large ω and that the time derivative generates a factor ω, Lebesgue's dominated convergence theorem allows us to interchange the differential operator with the integration. We thus obtain

$$\begin{aligned} (i\partial_t - H)\Psi(t) &= \frac{1}{2\pi i} \int_{C_R} e^{-i\omega t}\, (H-\omega) \Big(R_\omega \Psi_0 + \frac{\Psi_0}{\omega+3ic} + \frac{(H+3ic)\Psi_0}{(\omega+3ic)^2} \Big)\, d\omega \\ &= \frac{1}{2\pi i} \int_{C_R} e^{-i\omega t}\, \Big(\Psi_0 + \frac{(H-\omega)\Psi_0}{\omega+3ic} + \frac{(H-\omega)(H+3ic)\Psi_0}{(\omega+3ic)^2} \Big)\, d\omega = 0 \,, \end{aligned}$$

because no poles are enclosed by the contour. This shows that $\Psi(t)$ as given by (5.6) really is a solution of the Cauchy problem. Uniqueness follows immediately because the Teukolsky equation is hyperbolic. □

We now derive alternative integral representations for the solution of the Cauchy problem, which will be useful for our estimates.

Corollary 5.3. *For any integer $p \geq 1$, the solution of the Cauchy problem for the Teukolsky equation with initial data $\Psi|_{t=0} = \Psi_0 \in \mathcal{D}(H)$ has the representation*

$$\Psi(t) = -\frac{1}{2\pi i} \int_C e^{-i\omega t}\, \frac{1}{(\omega+3ic)^p}\, \Big(R_\omega \left(H+3ic\right)^p \Psi_0 \Big)\, d\omega \,. \tag{5.9}$$

Proof. Rewriting (5.2) as

$$R_\omega \Psi = -\frac{1}{\omega+3ic}\, \Psi + \frac{1}{\omega+3ic}\, R_\omega \left(H+3ic\right) \Psi \,,$$

we can iterate similar to (5.3) to obtain

$$R_\omega \Psi = -\frac{1}{\omega+3ic}\, \Psi - \frac{(H+3ic)\, \Psi}{(\omega+3ic)^2} - \cdots - \frac{(H+3ic)^{p-1}}{(\omega+3ic)^p}\, \Psi \tag{5.10}$$

$$+ \frac{1}{(\omega+3ic)^p}\, R_\omega \left(H+3ic\right)^p \Psi \,. \tag{5.11}$$

Using this identity in (5.6), the first summand in (5.10) cancels. For all the other summands in (5.10), one can compute the integral in (5.6) with residues to obtain zero. Therefore, only the summand (5.11) remains, giving the result. □

For negative times, the integral representation of the solution of the Cauchy problem can be further simplified. This is the representation which we will use in the remainder of this paper.

Corollary 5.4. *For negative times, the solution of the Cauchy problem for the Teukolsky equation with initial data* $\Psi|_{t=0} = \Psi_0 \in \mathcal{D}(H)$ *has the integral representation*

$$\Psi(t) = -\frac{1}{2\pi i} \int_{\mathbb{R}-2ic} e^{-i\omega t} \left(R_\omega \Psi_0 + \frac{\Psi_0}{\omega + 3ic} \right) d\omega \qquad (\textit{if } t < 0)\,.$$

Moreover, for any integer $p \geq 1$,

$$\Psi(t) = -\frac{1}{2\pi i} \int_{\mathbb{R}-2ic} e^{-i\omega t}\, \frac{1}{(\omega+3ic)^p} \left(R_\omega \left(H + 3ic\right)^p \Psi_0 \right) d\omega\,. \tag{5.12}$$

Proof. Starting from (5.6) and (5.9), we take the limit where the upper part of the contour $\mathbb{R} + 2ic$ is deformed towards $\mathrm{Im}\,\omega \to \infty$, making use of the fact that the factor $e^{-i\omega t}$ decays exponentially in this limit. □

6. Separation of Variables and Jost Solutions

Let $\omega \in \mathbb{C}$. We make the separation ansatz

$$\Phi(t, u, \vartheta) = e^{-i\omega t}\, X(u)\, Y(\vartheta)\,.$$

This gives rise to the coupled system of ODEs

$$\mathcal{R}_\omega X(r) = -\lambda X(r)\,, \qquad \mathcal{A}_\omega Y(\vartheta) = \lambda Y(\vartheta)\,, \tag{6.1}$$

where the radial operator $\mathcal{R}_\omega$ and the angular operator $\mathcal{A}_\omega$ are given by

$$\begin{aligned}\mathcal{R}_\omega &= -\frac{(r^2+a^2)^2}{\Delta} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}} \right) \\ &\qquad + \frac{1}{\Delta} \Big(-i\omega\,(r^2+a^2) - iak - (r-M)\,s \Big)^2 + 4sr + 4k\,a\omega \end{aligned}\tag{6.2}$$

$$\begin{aligned}\mathcal{A}_\omega &= 4sa\omega \cos\vartheta - 4k\,a\omega \\ &\qquad - \frac{\partial}{\partial \cos\vartheta}\, \sin^2\vartheta\, \frac{\partial}{\partial\cos\vartheta} - \frac{1}{\sin^2\vartheta} \Big(-ia\omega \sin^2\vartheta - ik + is\cos\vartheta \Big)^2 \\ &= -\frac{\partial}{\partial \cos\vartheta}\, \sin^2\vartheta\, \frac{\partial}{\partial\cos\vartheta} - \frac{1}{\sin^2\vartheta} \Big(ia\omega \sin^2\vartheta - ik + is\cos\vartheta \Big)^2 \\ &= -\frac{\partial}{\partial \cos\vartheta}\, \sin^2\vartheta\, \frac{\partial}{\partial\cos\vartheta} + \frac{1}{\sin^2\vartheta} \Big(-a\omega \sin^2\vartheta + k - s\cos\vartheta \Big)^2 . \end{aligned}\tag{6.3}$$

The operator $\mathcal{A}_\omega$ coincides with the angular Teukolsky operator as studied in [20, 21] (with the aspherical parameter $\Omega = -a\omega$). Note that in order to get this agreement, we added and subtracted the constant $4ka\omega$.

The radial equation can be written as the Sturm-Liouville equation

$$\left(-\frac{d^2}{du^2} + V \right) X = 0\,, \tag{6.4}$$

where V is the potential

$$\begin{aligned} V(u) &= \frac{\lambda\,\Delta}{(r^2+a^2)^2} + \frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}} \\ &\qquad + \frac{4\omega\,\Delta}{(r^2+a^2)^2}\, \big(ak - irs\big) - \Big(\omega + \frac{ak - i(r-M)\,s}{r^2+a^2}\Big)^2 . \end{aligned}\tag{6.5}$$

Near spatial infinity, the potential has the asymptotics

$$V(u) = -\omega^2 - \frac{2is\omega}{u} + \mathcal{O}\left(\frac{\log u}{u^2}\right) \qquad \text{if } u \to \infty\,. \tag{6.6}$$

Likewise, near the event horizon, we have the asymptotics

$$V(u) = -\Omega^2 + \mathcal{O}\big(e^{\gamma u}\big) \qquad \text{if } u \to -\infty\,, \tag{6.7}$$

where

$$\Omega := \omega + \frac{ak - i(r_1 - M)\,s}{r_1^2 + a^2}\,, \qquad \gamma := \frac{r_1 - r_0}{r_1^2 + a^2} \tag{6.8}$$

(and r_0 and r_1 are again the horizons (2.2)). These asymptotics are exactly the same as in the Schwarzschild geometry, if at the event horizon we replace ω by $\omega + (ak)/(r_1^2 + a^2)$. Therefore, we can proceed exactly as in [17, Section 3] to construct Jost solutions $\acute{\phi}_\pm$ and $\grave{\phi}_\pm$ with the following properties (see also [12, Section 3] or [8]).

Theorem 6.1. *For every ω in the domain*

$$D_- := \left\{\omega \,\Big|\, \operatorname{Im}\omega < \frac{(r_1 - M)s}{r_1^2 + a^2} + \frac{\gamma}{2}\right\}$$

there is a solution $\acute{\phi}_-$ of (6.4) having the asymptotics

$$\lim_{u\to-\infty} e^{-i\Omega u}\,\acute{\phi}_-(u) = 1\,, \qquad \lim_{u\to-\infty}\Big(e^{-i\Omega u}\,\acute{\phi}_-(u)\Big)' = 0\,. \tag{6.9}$$

These solutions can be chosen to form a holomorphic family, in the sense that for every $u \in \mathbb{R}$, the function $\acute{\phi}_-(u)$ is holomorphic in $\omega \in D_-$. Similarly, on the domain

$$D_+ := \left\{\omega \,\Big|\, \operatorname{Im}\omega > \frac{(r_1 - M)s}{r_1^2 + a^2} - \frac{\gamma}{2}\right\}$$

there is a holomorphic family of solutions $\acute{\phi}_+$ of (6.4) with the asymptotics

$$\lim_{u\to-\infty} e^{i\Omega u}\,\acute{\phi}_+(u) = 1\,, \qquad \lim_{u\to-\infty}\Big(e^{i\Omega u}\,\acute{\phi}_+(u)\Big)' = 0\,.$$

Theorem 6.2. *On the domain $E_+ := \{\omega \,|\, \omega \neq 0$ and $\operatorname{Im}\omega \geq 0\}$, there is a family of solutions $\grave{\phi}_+(u)$ of (6.4), holomorphic in the interior of E_+, having the asymptotics*

$$\lim_{u\to\infty} u^s\,e^{-i\omega u}\,\grave{\phi}_+(u) = 1\,, \qquad \lim_{u\to\infty}\Big(u^s\,e^{-i\omega u}\,\grave{\phi}_+(u)\Big)' = 0\,.$$

Likewise, on the domain $E_- := \{\omega \,|\, \omega \neq 0$ and $\operatorname{Im}\omega \leq 0\}$, there is a family of solutions $\grave{\phi}_-(u)$ of (6.4), holomorphic in the interior of E_-, with the asymptotics

$$\lim_{u\to\infty} u^{-s}\,e^{i\omega u}\,\grave{\phi}_-(u) = 1\,, \qquad \lim_{u\to\infty}\Big(u^{-s}\,e^{i\omega u}\,\grave{\phi}_-(u)\Big)' = 0\,. \tag{6.10}$$

7. Separation of the Resolvent

We return to the solution of the Cauchy problem given in Corollary 5.4. Possibly by increasing c, we can arrange that

$$c > \frac{(r_1 - M)s}{r_1^2 + a^2}\,.$$

Choosing the Jost function as

$$\acute{\phi} = \acute{\phi}_- \qquad \text{and} \qquad \grave{\phi} = \grave{\phi}_+\,,$$

these functions all decay exponentially in their asymptotic ends.

We now apply the results of [21] on the spectral decomposition of the angular operator. According to [21, Theorem 1.1], for any ω in the strip U defined by

$$U := \big\{\omega \in \mathbb{C} \text{ with } |\text{Im}\,\omega| < 3c\big\}$$

(and c as in (5.4) so that the contour C lies inside U), there is a family $(Q_n^\omega)_{n\in\mathbb{N}\cup\{0\}}$ of idempotent operators whose images are invariant subspaces of the angular operator $\mathcal{A}_\omega$ with the following properties:

(i) The Q_n^ω are complete in the sense that

$$\sum_{n=0}^\infty Q_n^\omega = \mathbb{1} \tag{7.1}$$

with strong convergence of the series.

(ii) The Q_n^ω have rank at most N, where N can be chosen uniformly in ω.

(iii) The Q_n^ω are uniformly bounded in $L^2(S^2)$, i.e. there is a constant c_2 such that

$$\|Q_n^\omega\| \leq c_2 \qquad \text{for all } n \in \mathbb{N}\cup\{0\} \text{ and } \omega \in U\,. \tag{7.2}$$

We now choose ω on the contour C and let $n \in \mathbb{N}\cup\{0\}$. We point out that the range of the operator Q_n^ω need not be an eigenspace of $\mathcal{A}_\omega$ because there might be Jordan chains. However, since the length of the Jordan chains is bounded by N, we can write $\mathcal{A}_\omega$ on the invariant subspace as

$$\mathcal{A}_\omega\, Q_n^\omega = (\lambda\,\mathbb{1} + \mathcal{N})\,Q_n^\omega\,, \tag{7.3}$$

where $\mathcal{N}$ is a nilpotent operator with $\mathcal{N}^N = 0$. Let us consider the Teukolsky equation (2.7) on the invariant subspace. Using (7.3), the resulting equation is obtained from the radial equation (6.4) and (6.5) if the separation constant λ is replaced by the operators $\lambda\,\mathbb{1} + \mathcal{N}$. This gives the equation

$$\left(-\frac{d^2}{du^2} + V + \frac{\Delta}{(r^2+a^2)^2}\,\mathcal{N}\right) X(u) = 0\,,$$

where $X(u)$ is now vector-valued, taking values in the invariant subspace. We want to construct a Green's function $g_\omega(u,v)$ of this equation, defined by

$$\left(-\frac{d^2}{du^2} + V + \frac{\Delta}{(r^2+a^2)^2}\,\mathcal{N}\right) g_\omega(u,v) = \delta(u-v)\,. \tag{7.4}$$

If the nilpotent operator $\mathcal{N}$ is absent, this Green's function is given just as in [17, Section 4] by a function which we now denote by $s_\omega(u,v)$,

$$s_\omega(u,v) = \frac{1}{w(\acute{\phi},\grave{\phi})} \times \begin{cases} \acute{\phi}(u)\,\grave{\phi}(v) & \text{if } v \geq u \\ \grave{\phi}(u)\,\acute{\phi}(v) & \text{if } v < u\,. \end{cases} \tag{7.5}$$

Namely, a straightforward computation yields

$$\left(-\frac{d^2}{du^2} + V(u)\right) s_\omega(u,v) = \delta(u-v)\,.$$

We also regard s_ω as an operator with corresponding integral kernel $s_\omega(u,v)$. Then we can multiply (7.4) by s_ω to obtain the operator equation

$$\left(\mathbb{1} + s_\omega\,\frac{\Delta}{(r^2+a^2)^2}\,\mathcal{N}\right) g_\omega = s_\omega\,.$$

Since $\mathcal{N}$ is nilpotent, this equation can be solved by a finite Neumann series,

$$g_\omega = \sum_{l=0}^N \Big(- s_\omega \, \frac{\Delta}{(r^2+a^2)^2} \, \mathcal{N} \Big)^l \, s_\omega \,. \tag{7.6}$$

The existence of powers of s_ω is proved exactly as in [11, Lemma 5.2].

After these preparations, we can now decompose the resolvent into angular modes:

Theorem 7.1. *For any ω on the contour C, (5.4), the resolvent $R_\omega = (H-\omega)^{-1}$ has the representation*

$$R_\omega = \sum_{n=0}^\infty R_{\omega,n} \, Q^\omega_n \,,$$

where the operators $R_{\omega,n}$ can be written as integral operators,

$$\big(R_{\omega,n} \Psi\big)(u, \vartheta) = \int_{-\infty}^\infty \frac{\rho(v, \vartheta)}{r(v)^2+a^2} \, \mathfrak{R}_{\omega,n}(u,v) \, \Psi(v, \vartheta) \, dv \,, \tag{7.7}$$

with integral kernels given by

$$\mathfrak{R}_{\omega,n}(u,v) = \frac{r^2+a^2}{\rho} \, \delta(u-v) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + g_\omega(u,v) \, \begin{pmatrix} \omega - \beta(v) & 1 \\ \omega \big(\omega - \beta(v) \big) & \omega \end{pmatrix} . \tag{7.8}$$

Proof. It is obvious from (3.3) that

$$(H - \omega) \, \frac{r^2+a^2}{\rho} \, \delta(u-v) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{r^2+a^2}{\rho} \, \delta(u-v) \begin{pmatrix} 1 & 0 \\ \beta(v) - \omega & 0 \end{pmatrix} . \tag{7.9}$$

We next compute the operator product

$$(H_u - \omega) \, g_\omega(u,v) \, \begin{pmatrix} \omega - \beta(v) & 1 \\ \omega \big(\omega - \beta(v) \big) & \omega \end{pmatrix} Q^\omega_n \,, \tag{7.10}$$

where the index u at the Hamiltonian clarifies that its derivatives act on the variable u. If $u \neq v$, we know from (7.4) that $g_\omega(u,v)$ is a solution of the radial Teukolsky equation. Moreover, using the fact that the second row in the matrix is ω times the first row, one sees that every column of this matrix is of the form (3.1). This implies that (7.10) vanishes. If $u=v$, the only additional contribution is obtained when the operator ∂_u^2 contained in H acts on g. In view of (3.3) and (3.5), we obtain

$$\begin{aligned}
&(H_u - \omega) \, g_\omega(u,v) \, \begin{pmatrix} \omega - \beta(v) & 1 \\ \omega \big(\omega - \beta(v) \big) & \omega \end{pmatrix} Q^\omega_n \\
&\qquad = \frac{r^2+a^2}{\rho} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \delta(u-v) \, \begin{pmatrix} \omega - \beta(v) & 1 \\ \omega \big(\omega - \beta(v) \big) & \omega \end{pmatrix} Q^\omega_n \\
&\qquad = \frac{r^2+a^2}{\rho} \, \delta(u-v) \, \begin{pmatrix} 0 & 0 \\ \omega - \beta(v) & 1 \end{pmatrix} Q^\omega_n \,.
\end{aligned}$$

Adding (7.9) and using (7.8), we conclude that

$$(H_u - \omega) \, \mathfrak{R}_{\omega,n}(u,v) = \frac{r^2+a^2}{\rho} \, \delta(u-v) \, Q^\omega_n \,.$$

Summing over n and using the completeness relation (7.1) gives the result. ☐

8. Contour Deformations

Using the result of Theorem 7.1 in the integral representations of Corollary 5.4, we obtain an integral over an infinite sum of angular modes. Since it is a-priori not clear whether the integration and summation can be interchanged, our method is to first analyze the partial sums defined by

$$\Psi^N(t) = -\frac{1}{2\pi i} \sum_{n=0}^N \int_{\mathbb{R}-2ic} e^{-i\omega t} \left(R_{\omega,n}\, Q_n^\omega\, \Psi_0 + Q_n^\omega\, \frac{\Psi_0}{\omega+3ic} \right) d\omega \tag{8.1}$$

$$\Psi^{N,p}(t) = -\frac{1}{2\pi i} \sum_{n=0}^N \int_{\mathbb{R}-2ic} \frac{e^{-i\omega t}}{(\omega+3ic)^p} \left(R_{\omega,n}\, Q_n^\omega\, (H+3ic)^p\, \Psi_0 \right) d\omega \tag{8.2}$$

(where again $p \geq 1$ and $t \leq 0$). After getting suitable estimates, we will be able to prove that the limit $N \to \infty$ of the partial sums exists, both with the summation inside and outside the integral (see Section 10.5).

We now use Whiting's mode stability result [33] to move the contour for the partial sums up to the real axis:

Lemma 8.1. *For any $\Psi_0 \in \mathcal{D}(H)$ and any integer $p \geq 1$, the partial sums* (8.1) *and* (8.2) *can be written for any $t \leq 0$ as*

$$\Psi^N(t) = -\frac{1}{2\pi i} \sum_{n=0}^N \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}-i\varepsilon} e^{-i\omega t} \left(R_{\omega,n}\, Q_n^\omega\, \Psi_0 + Q_n^\omega\, \frac{\Psi_0}{\omega+3ic} \right) d\omega \tag{8.3}$$

$$\Psi^{N,p}(t) = -\frac{1}{2\pi i} \sum_{n=0}^N \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}-i\varepsilon} \frac{e^{-i\omega t}}{(\omega+3ic)^p} \left(R_{\omega,n}\, Q_n^\omega\, (H+3ic)^p\, \Psi_0 \right) d\omega\,. \tag{8.4}$$

Proof. We first verify that the above integrands decay so fast near $\omega = \pm\infty$ that the integrals (8.1), (8.2) and (8.3), (8.4) converge. To this end, given n, we need to analyze the asymptotics for large ω. In this asymptotic region, the angular eigenvalue λ_n scales like $|\lambda_n| \lesssim |\omega|$ (see Lemma A.3 in the appendix). Therefore, for large ω the summand $-\omega^2$ dominates all the other terms in (6.5), so that the potential goes over to a constant potential. In this limiting case, the solutions $\acute{\phi}$ and $\grave{\phi}$ go over to plane waves. By direct computation, one verifies that in this limiting case, the kernels $s(u,v)$ and $g_\omega(u,v)$ in (7.5) and (7.6) are bounded. Therefore, these kernels are bounded uniformly in ω. This makes it possible to estimate the integral in (7.7) by

$$\Big| \big(R_{\omega,n} \Psi_0\big)(u,\vartheta) \Big| \leq \sup_{v \in \text{supp}\, \Psi_0} \big|\mathfrak{R}_{\omega,n}(u,v)\big| \int_{\text{supp}\, \Psi_0} \frac{\rho(v,\vartheta)}{r(v)^2+a^2}\, |\Psi_0(v,\vartheta)|\, dv$$

to obtain an estimate of the form

$$\big\| \big(R_{\omega,n}\, Q_n^\omega\, \Psi_0\big)(u) \big\|_{L^2(S^2)} \leq C(n, u_\infty, \Psi_0)\, \big(1+|\omega|^2\big)\,, \tag{8.5}$$

valid for all $u < u_\infty$ (for any given parameter $u_\infty \in \mathbb{R}$). Next, we can use the lower bound for $|\lambda|$ in Lemma A.3 to generate factors of $1/|\omega|$,

$$\begin{aligned} \big\|(Q_n^\omega\, \Psi_0)(u)\big\|_{L^2(S^2)} &\lesssim \frac{1}{\inf |\lambda|^q}\, \big\|(Q_n^\omega\, A_\omega^q \Psi_0)(u)\big\|_{L^2(S^2)} \\ &\lesssim \frac{c(n)^q}{(1+|\omega|)^q}\, \big\|(Q_n^\omega\, A_\omega^q \Psi_0)(u)\big\|_{L^2(S^2)}\,, \end{aligned} \tag{8.6}$$

giving the desired decay for large $|\omega|$.

Since the angular spectral projection operators Q_n^ω as well as the Jost solutions are holomorphic in ω, the integrand in the above contour integrals clearly is meromorphic in ω. If there were poles, the resulting contributions to the contour integral would be mode solutions, in contradiction to [33]. We conclude that the integrands in (8.1) and (8.2) are holomorphic in the strip $-2c < \mathrm{Im}\,\omega < 0$. This makes it possible to deform the contours, giving the result. □

9. Estimates of the Potential

We now enter the derivation of the general ODE estimates. Recall that our equations involve the parameters k, s, ω and λ. We always keep k and s fixed. The parameters ω and λ, however, may vary in a certain parameter range to be specified later on, and we must make sure that our estimates are uniform in these parameters. In order to keep track of the dependence on ω and λ, we use the same conventions and notation as in [21]. Namely, we adopt the convention that

$$\text{all constants are independent of } \omega \text{ and } \lambda$$

(but they may depend on k and s). Moreover, in order to have a compact and clear notation, we always denote constants which may be increased during our constructions by capital letters $\mathcal{C}_1, \mathcal{C}_2, \ldots$. However, constants with small letters $\mathfrak{c}_1, \mathfrak{c}_2, \ldots$ are determined at the beginning and are fixed throughout. We use the symbol

$$\lesssim \cdots \qquad \text{for} \qquad \leq \mathfrak{c} \cdots$$

with a constant $\mathfrak{c}$ which is independent of the capital constants $\mathcal{C}_l$ (and may thus be fixed right away, without the need to increase it later on).

When increasing the constants $\mathcal{C}_l$, we must keep track of the mutual dependences of these constants. We adopt the convention that the constant $\mathcal{C}_l$ may depend on all previous constants $\mathcal{C}_1, \ldots, \mathcal{C}_{l-1}$, but is independent of the subsequent constants $\mathcal{C}_{l+1}, \ldots$. In particular, we may choose the capital constants such that $\mathcal{C}_1 \ll \mathcal{C}_2 \ll \cdots$. This dependence of the constants implies that increasing $\mathcal{C}_l$ may also make it necessary to increase the subsequent constants $\mathcal{C}_{l+1}, \mathcal{C}_{l+2}, \ldots$. For brevity, when we write "possibly after increasing $\mathcal{C}_l$" we implicitly mean that the subsequent constants $\mathcal{C}_{l+1}, \mathcal{C}_{l+2}, \ldots$ are also suitably increased.

9.1. Different Cases and Regions.

In Lemma 8.1 we could deform the integration contours up to the real axis. With this in mind, it suffices to consider the case that ω is real. Then the potential in the angular Teukolsky operator is real. Consequently, its eigenvalues λ_n are also real. They have the properties as worked out in [21, Section 7]. Moreover, the imaginary part of Ω as defined by (6.8) is a negative constant,

$$\mathrm{Im}\,\Omega = -\varpi \qquad \text{with} \qquad \varpi := \frac{(r_1 - M)\, s}{r_1^2 + a^2} > 0\,. \tag{9.1}$$

In preparation for proving convergence of our sum of contour integrals (see Section 10.5), we need to estimate the behavior of the fundamental solutions of the Sturm-Liouville equation (6.4) for large λ and $|\omega|$. With this mind, we now restrict attention to the parameter range

$$\boxed{\omega^2 \geq \mathcal{C}_6 \qquad \text{and} \qquad \lambda \geq \mathcal{C}_7\,.} \tag{9.2}$$

Expanding the potential in (6.5), we obtain

$$\begin{aligned} V = -\omega^2 &+ \frac{\lambda\,\Delta}{(r^2+a^2)^2} - \frac{2\omega a k}{r^2+a^2}\left(1 - \frac{2\Delta}{r^2+a^2}\right) \\ &- \frac{2i\omega s}{r^2+a^2}\left(M - r + \frac{2r\Delta}{r^2+a^2}\right) + \mathcal{O}\big(\omega^0\big) + \mathcal{O}\big(\lambda^0\big)\,. \end{aligned} \tag{9.3}$$

From this formula one sees in particular that the potential is almost constant if ω is large. This implies that the WKB conditions are satisfied, as is quantified in the next lemma.

Lemma 9.1. *Assume that*

$$\lambda < \mathcal{C}_5\,|\omega|$$

for a given constant $\mathcal{C}_5$. Then for any $\varepsilon > 0$, we can arrange by choosing the constants $\mathcal{C}_6$ and $\mathcal{C}_7$ sufficiently large that

$$\frac{|V'(u)|}{|V(u)|^{\frac{3}{2}}},\, \frac{|V''(u)|}{|V(u)|^2} \leq \varepsilon \qquad \text{for all } u \in \mathbb{R}\,,$$

uniformly in ω and λ.

Proof. From the form of the potential (9.3) it is obvious that

$$\big|V + \omega^2\big| \lesssim \lambda + |\omega| \qquad \text{and} \qquad \big|V'\big|, \big|V''\big| \lesssim \lambda + |\omega|\,. \tag{9.4}$$

As a consequence,

$$\begin{aligned} \frac{|V'|}{|V|^{\frac{3}{2}}} &\lesssim \frac{\lambda + |\omega|}{|\omega|^3} \leq \frac{\mathcal{C}_5 + 1}{\omega^2} \leq \frac{\mathcal{C}_5+1}{\mathcal{C}_6} \\ \frac{|V''|}{|V|^2} &\lesssim \frac{\lambda + |\omega|}{|\omega|^4} \leq \frac{\mathcal{C}_5 + 1}{|\omega|^3} \leq \frac{\mathcal{C}_5+1}{\mathcal{C}_6^{\frac{3}{2}}}\,. \end{aligned}$$

This concludes the proof. □

In view of this result, in what follows we may assume that

$$\boxed{\lambda \geq \mathcal{C}_5\,|\omega|\,,} \tag{9.5}$$

because otherwise the potential is nearly constant and can be treated easily with the WKB approximation. Then, choosing $\mathcal{C}_5$ sufficiently large, we can sometimes work with the simpler approximation

$$V = -\omega^2 + \frac{\lambda\,\Delta}{(r^2+a^2)^2} + \mathcal{O}\big(\omega\big) + \mathcal{O}\big(\lambda^0\big)\,. \tag{9.6}$$

Discussing the form of the approximate potential immediately gives the following result:

Lemma 9.2. *For any ω and λ in the range* (9.2) *and* (9.5)*, we have the bounds*

$$\big|\operatorname{Im} V\big|, \big|\operatorname{Im} V'\big|, \big|\operatorname{Im} V''\big| \lesssim |\omega|\,. \tag{9.7}$$

Moreover, for sufficiently large $\mathcal{C}_5$, the real part of the potential has a unique maximum at a point $u_{\max}$ with $\frac{12}{5}\,m \leq r(u_{\max}) \leq 3m$. The maximal value is bounded by

$$\operatorname{Re} V(u_{\max}) \lesssim \lambda\,.$$

The function $\mathrm{Re}\, V$ *is concave near near* $u_{\max}$. *More quantitatively, there is a constant* $\mathfrak{c}$ *such that*

$$\frac{\lambda}{\mathfrak{c}} \leq -\mathrm{Re}\, V''(u) \leq \mathfrak{c}\lambda \qquad \text{on} \qquad \Big[u_{\max} - \frac{1}{2}, u_{\max} + \frac{1}{2}\Big] \,. \tag{9.8}$$

Proof. We substitute the formula for Δ, (2.1), into (9.6) and compute the first and and second u-derivatives with the help of the formula (see also (2.5))

$$\frac{\partial}{\partial u} = \frac{\Delta}{r^2 + a^2} \, \frac{\partial}{\partial r} \,.$$

Then the result follows by a direct computation. □

The value of the real part of the potential at the point $u_{\max}$ distinguishes different *cases*:

$$\begin{cases} \text{WKB case} & \text{if } \mathrm{Re}\, V(u_{\max}) < -\mathfrak{C}_4 \sqrt{\lambda} \\ \text{parabolic cylinder (PC) case} & \text{if } -\mathfrak{C}_4 \sqrt{\lambda} \leq \mathrm{Re}\, V(u_{\max}) < \mathfrak{C}_4 \sqrt{\lambda} \\ \text{Airy case} & \text{if } \mathrm{Re}\, V(u_{\max}) \geq \mathfrak{C}_4 \sqrt{\lambda}\,. \end{cases} \tag{9.9}$$

In each of these above cases, we estimate the solution by considering different *regions*, which we now introduce. To this end, we work with the zeros of the function $\mathrm{Re}\, V$ characterized in the next lemma.

Lemma 9.3. *By increasing the constant* $\mathfrak{C}_5$ *in* (9.5) *we can arrange that whenever* $\mathrm{Re}\, V(u_{\max}) > 0$, *there are unique points* $u_0^L, u_0^R \in \mathbb{R}$ *with*

$$\mathrm{Re}\, V(u_0^L) = 0 = \mathrm{Re}\, V(u_0^R) \qquad \text{and} \qquad u_0^L \leq u_{\max} \leq u_0^R \,.$$

Proof. Since ω^2 enters the potential (6.5) only via a constant, for large $\mathfrak{C}_5$ the derivative $\mathrm{Re}\, V'$ is dominated by the term $\lambda \partial_u (\Delta/(r^2+a^2)^2)$. An asymptotic expansion shows that $\mathrm{Re}\, V'$ is positive near $u = -\infty$ and negative near $u = -\infty$. Moreover, the function $\Delta/(r^2+a^2)^2$ is monotone increasing up to a turning point where its second derivative is negative, and from then on is monotone decreasing. This gives the result. □

If $\mathrm{Re}\, V(u_{\max}) > 0$, we denote the zeros of $\mathrm{Re}\, V$ by u_0^L and u_0^R, i.e.

$$\mathrm{Re}\, V(u_0^L) = 0 = \mathrm{Re}\, V(u_0^R) \,, \qquad u_0^L \leq u_{\max} \leq u_0^R \,.$$

If $\mathrm{Re}\, V(u_{\max}) < 0$ (as is always true in the WKB case and may be true in the PC case), we set $u_0^{L/R} = u_{\max}$. We introduce

$$u_-^L = \begin{cases} u_0^L & \text{in the WKB case} \\ u_0^L - \mathfrak{C}_3 (\mathfrak{C}_1 \mathfrak{C}_4)^{-\frac{1}{6}} \, |\omega|^{-\frac{1}{2}} & \text{in the PC case} \\ u_0^L - \mathfrak{C}_3 \max \Big(|\omega|^{-\frac{2}{3}}, \big(\mathfrak{C}_1 \,\mathrm{Re}\, V(u_{\max})\big)^{-\frac{1}{6}} \, |\omega|^{-\frac{1}{3}} \Big) & \text{in the Airy case} \end{cases} \tag{9.10}$$

$$u_-^R = \begin{cases} u_0^R & \text{in the WKB case} \\ u_0^R + \mathfrak{C}_3 (\mathfrak{C}_1 \mathfrak{C}_4)^{-\frac{1}{6}} \, |\omega|^{-\frac{1}{2}} & \text{in the PC case} \\ u_0^R + \mathfrak{C}_3 \, \lambda^{\frac{1}{6}} \, |\omega|^{-\frac{1}{3}} & \\ \qquad\quad \times \max \Big(|\omega|^{-\frac{2}{3}}, \big(\mathfrak{C}_1 \,\mathrm{Re}\, V(u_{\max})\big)^{-\frac{1}{6}} \, |\omega|^{-\frac{1}{3}} \Big) & \text{in the Airy case}\,. \end{cases} \tag{9.11}$$

Moreover, in the Airy case we set

$$u_+^L = u_0^L + \mathcal{C}_3 \max\Big(|\omega|^{-\frac{2}{3}}, \big(\mathcal{C}_1 \,\mathrm{Re}\, V(u_{\max})\big)^{-\frac{1}{6}} \,|\omega|^{-\frac{1}{3}} \Big) \tag{9.12}$$

$$u_+^R = u_0^R - \mathcal{C}_3\, \lambda^{\frac{1}{6}}\, |\omega|^{-\frac{1}{3}} \,\max\Big(|\omega|^{-\frac{2}{3}}, \big(\mathcal{C}_1 \,\mathrm{Re}\, V(u_{\max})\big)^{-\frac{1}{6}} \,|\omega|^{-\frac{1}{3}} \Big) \,. \tag{9.13}$$

We thus obtain the following regions:

$$\begin{cases} \text{WKB regions} & (-\infty, u_-^L),\; (u_-^R, \infty) & \text{in all cases} \\ \text{PC region} & (u_-^L, u_-^R) & \text{in the PC case} \\ \text{Airy regions} & (u_-^L, u_+^L),\; (u_+^R, u_-^R) & \text{in the Airy case} \\ \text{WKB region with } \mathrm{Re}\, V > 0 & (u_+^L, u_+^R) & \text{in the Airy case}\,. \end{cases} \tag{9.14}$$

9.2. **Estimates in the WKB Regions.** In this section we shall prove the following results:

Proposition 9.4. *For any $\varepsilon > 0$, we can arrange by choosing the constants $\mathcal{C}_1, \ldots, \mathcal{C}_4$ sufficiently large that for all ω and λ in the range* (9.2),

$$\frac{|V'|}{|V|^{\frac{3}{2}}}, \frac{|V''|}{|V|^2} \leq \varepsilon \qquad \textit{on } (-\infty, u_-^L)\,.$$

Proposition 9.5. *For any $\varepsilon > 0$, we can arrange by choosing the constants $\mathcal{C}_1, \ldots, \mathcal{C}_4$ sufficiently large that for all ω and λ in the range* (9.2),

$$\frac{|V'|}{|V|^{\frac{3}{2}}}, \frac{|V''|}{|V|^2} \leq \varepsilon \qquad \textit{on } (u_-^R, \infty)\,.$$

Proof of Propositions 9.4 and 9.5 in the WKB case in (9.9). From (9.6) and (9.5), it is obvious that

$$|V'|, |V''| \lesssim \lambda\,.$$

Therefore, using (9.9), we find that

$$\frac{|V'|}{|V|^{\frac{3}{2}}} \lesssim \frac{1}{\mathcal{C}_4^{\frac{3}{2}}\, \sqrt{\lambda}} \qquad \text{and} \qquad \frac{|V''|}{|V|^2} \lesssim \frac{1}{\mathcal{C}_4^2}\,.$$

This concludes the proof. ☐

It remains to prove Propositions 9.4 and 9.5 in the Airy and PC cases in (9.9). We distinguish the two cases

$$\textbf{(a)} \quad \omega^2 > \mathcal{C}_1 \,\mathrm{Re}\, V(u_{\max}) \qquad \text{and} \qquad \textbf{(b)} \quad \omega^2 \leq \mathcal{C}_1 \,\mathrm{Re}\, V(u_{\max})\,. \tag{9.15}$$

We begin with a preparatory lemma.

Lemma 9.6. *In the Airy and PC cases,*

$$\lambda \gtrsim \omega^2\,. \tag{9.16}$$

Moreover,

$$\lambda \lesssim \omega^2 \qquad \textit{in case } \textbf{(a)} \textit{ and in the PC case}\,. \tag{9.17}$$

Proof. From (9.9) and the form of the potential (9.3), we obtain

$$-\mathscr{C}_4 \sqrt{\lambda} \leq \mathrm{Re}\, V(u_{\max}) \leq -\omega^2 + \mathfrak{c}\lambda\,.$$

Applying (9.2) and increasing $\mathscr{C}_7$, we obtain (9.16).

In case **(a)**, we know that

$$\omega^2 > \mathscr{C}_1 \,\mathrm{Re}\, V(u_{\max}) \geq \mathscr{C} \Big(-\omega^2 + \frac{\lambda}{\mathfrak{c}} \Big),$$

implying that

$$\lambda \lesssim \Big(1 + \frac{1}{\mathscr{C}_1} \Big) \omega^2 \lesssim \omega^2\,.$$

This gives (9.17) in case **(a)**.

Finally, in the PC case, we know from (9.9) that

$$\mathscr{C}_4 \sqrt{\lambda} > \mathrm{Re}\, V(u_{\max}) \geq -\omega^2 + \frac{\lambda}{\mathfrak{c}}\,.$$

Choosing the constant $\mathscr{C}_7$ in (9.2) sufficiently large, we again obtain the inequality in (9.17). This concludes the proof. □

The next lemma gives an alternative characterization of the cases in (9.15).

Lemma 9.7. *The maximum of* $\mathrm{Re}\, V$ *satisfies the bounds*

$$\left\{ \begin{array}{ll} \mathrm{Re}\, V(u_{\max}) \lesssim \dfrac{\lambda}{\mathscr{C}_1} & \text{in case } \textbf{(a)} \\[2ex] \mathrm{Re}\, V(u_{\max}) \gtrsim \dfrac{\lambda}{\mathscr{C}_1} & \text{in case } \textbf{(b)}\,. \end{array} \right. \tag{9.18}$$

Moreover,

$$\left\{ \begin{array}{ll} \big(u_{\max} - u_0^L\big),\, \big(u_0^R - u_{\max}\big) \lesssim \dfrac{1}{\sqrt{\mathscr{C}_1}} & \text{in case } \textbf{(a)} \\[2ex] \big(u_{\max} - u_0^L\big),\, \big(u_0^R - u_{\max}\big) \gtrsim \dfrac{1}{\sqrt{\mathscr{C}_1}} & \text{in case } \textbf{(b)}\,. \end{array} \right. \tag{9.19}$$

Proof. In case **(a)**, it follows that

$$\frac{\mathrm{Re}\, V(u_{\max})}{\lambda} < \frac{\omega^2}{\mathscr{C}_1\, \lambda} \lesssim \frac{1}{\mathscr{C}_1}\,,$$

where in the last step we applied Lemma 9.6. Likewise, in case **(b)**, from (9.15) and (9.6) we have the alternative estimates

$$\mathrm{Re}\, V(u_{\max}) \gtrsim \frac{\omega^2}{\mathscr{C}_1} \qquad \text{and} \qquad \mathrm{Re}\, V(u_{\max}) \gtrsim -\omega^2 + \frac{\lambda}{\mathfrak{c}}\,.$$

Multiplying the first inequality by $\mathscr{C}_1$ and adding them, we obtain

$$(1 + \mathscr{C}_1)\, \mathrm{Re}\, V(u_{\max}) \gtrsim \frac{\lambda}{\mathfrak{c}}\,,$$

giving (9.18).

In order to derive (9.19), we approximate the function $\mathrm{Re}\, V$ near its maximum by a parabola. More precisely, according to (9.8)

$$-\frac{\mathfrak{c}\lambda}{2}\, (u - u_{\max})^2 \leq \mathrm{Re}\, V(u) - \mathrm{Re}\, V(u_{\max}) \leq -\frac{\lambda}{2\mathfrak{c}}\, (u - u_{\max})^2 \qquad \text{if } |u - u_{\max}| \leq \frac{1}{2}\,.$$

In case **(a)** we can arrange by increasing $\mathcal{C}_1$ that $\mathrm{Re}\, V$ has a zero u_0^L in a neighborhood of $u_{\max}$, as is made precise in (9.19). Likewise, in case **(b)** we find that the function $\mathrm{Re}\, V$ has no zero in a $1/\mathcal{C}_1$-neighborhood of $u_{\max}$, implying (9.19). □

For the remaining estimates, we consider the regions $(-\infty, u_-^L)$ and (u_-^R, ∞) separately. We begin with the region $(-\infty, u_-^L)$ as considered in Proposition 9.4. We treat the two cases in (9.15) after each other.

Proof of Proposition 9.4 in case **(a)**. In this case, equation (9.10) reduces to

$$u_-^L = u_0^L - \mathcal{C}_3 \big(\mathcal{C}_1 \, \mathrm{Re}\, V(u_{\max})\big)^{-\frac{1}{6}} \, |\omega|^{-\frac{1}{3}} \,. \tag{9.20}$$

Moreover, by integrating (9.8), we obtain for any $u < u_-^L$,

$$-\,\mathrm{Re}\, V(u) = -\,\mathrm{Re}\, V(u) + \mathrm{Re}\, V(u_0^L) \geq \frac{\lambda}{2\mathfrak{c}} \Big((u-u_{\max})^2 - (u_0^L - u_{\max})^2 \Big) \,.$$

Setting $v = u_{\max} - u$ and $v_0 = u_{\max} - u_0^L$, we obtain

$$-\,\mathrm{Re}\, V(u) \geq \frac{\lambda}{2\mathfrak{c}} \left(v^2 - v_0^2\right) = \frac{\lambda}{2\mathfrak{c}} \,(v-v_0)(v+v_0) \tag{9.21}$$

$$\big|\, \mathrm{Re}\, V'(u) \big| \leq \mathfrak{c} \lambda \, v \,. \tag{9.22}$$

Using (9.20), we conclude that

$$\frac{|\,\mathrm{Re}\, V'(u)|^2}{|V(u)|^3} \lesssim \frac{\lambda^2 \, v^2}{\lambda^3 \, (v-v_0)^3 \, (v+v_0)^3} \leq \frac{\big(\mathcal{C}_1 \, \mathrm{Re}\, V(u_{\max})\big)^{\frac{1}{2}} \, |\omega| \, v^2}{\lambda \, \mathcal{C}_3^3 \, (v+v_0)^3} \,.$$

Next, we can estimate $\mathrm{Re}\, V(u_{\max})$ by

$$0 = \mathrm{Re}\, \big(V(u_0^L)\big) \geq \mathrm{Re}\, V(u_{\max}) - \frac{\lambda}{2\mathfrak{c}} \; v_0^2 \qquad \text{and thus} \qquad \mathrm{Re}\, V(u_{\max}) \lesssim \lambda \, v_0^2 \,. \tag{9.23}$$

We thus obtain

$$\frac{|\,\mathrm{Re}\, V'(u)|^2}{|V(u)|^3} \lesssim \frac{\sqrt{\mathcal{C}_1}}{\mathcal{C}_3^3} \; \frac{|\omega|}{\sqrt{\lambda}} \; \frac{v_0 \, v^2}{(v+v_0)^3} \lesssim \frac{\sqrt{\mathcal{C}_1}}{\mathcal{C}_3^3} \,,$$

where in the last step we used (9.16). The imaginary part of V' can be handled similarly. Namely, from (9.7) and (9.16), we know that

$$\frac{|\,\mathrm{Im}\, V'(u)|^2}{|V(u)|^3} \lesssim \frac{\lambda}{\lambda^3 \, (v-v_0)^3 \, (v+v_0)^3} \leq \frac{1}{\lambda^2 \, (v-v_0)^{\frac{9}{2}} \, (v+v_0)^{\frac{3}{2}}} \,.$$

Applying again (9.20) and (9.23), we obtain

$$\begin{aligned} \frac{|\,\mathrm{Im}\, V'(u)|^2}{|V(u)|^3} &\lesssim \frac{\big(\mathcal{C}_1 \, \mathrm{Re}\, V(u_{\max})\big)^{\frac{3}{4}} |\omega|^{\frac{3}{2}}}{\lambda^2 \, \mathcal{C}_3^{\frac{9}{2}} \, (v+v_0)^{\frac{3}{2}}} \lesssim \frac{\big(\mathcal{C}_1 \lambda \, v_0^2\big)^{\frac{3}{4}} |\omega|^{\frac{3}{2}}}{\lambda^2 \, \mathcal{C}_3^{\frac{9}{2}} \, (v+v_0)^{\frac{3}{2}}} \\ &\lesssim \frac{\big(\mathcal{C}_1 \lambda\big)^{\frac{3}{4}} |\omega|^{\frac{3}{2}}}{\lambda^2 \, \mathcal{C}_3^{\frac{9}{2}}} \overset{(9.16)}{\lesssim} \frac{\mathcal{C}_1^{\frac{3}{4}}}{\sqrt{\lambda}\, \mathcal{C}_3^{\frac{9}{2}}} \leq \frac{\mathcal{C}_1^{\frac{3}{4}}}{\sqrt{\mathcal{C}_7}\, \mathcal{C}_3^{\frac{9}{2}}} \,. \end{aligned}$$

This concludes the proof of the term involving the first derivatives.

The second derivatives can be handled similarly as follows,

$$\big|V''(u)\big| \leq \big|\operatorname{Re} V''(u)\big| + \big|\operatorname{Im} V''(u)\big| \overset{(9.7),(9.8)}{\leq} \mathfrak{c}\big(\lambda + |\omega|\big) \overset{(9.5)}{\lesssim} \lambda$$

$$\frac{|V''(u)|}{|V(u)|^2} \lesssim \frac{\lambda}{\lambda^2\,(v-v_0)^2\,(v+v_0)^2} \leq \frac{1}{\lambda\,(v-v_0)^3\,(v+v_0)}$$

$$\overset{(9.20)}{\leq} \frac{\big(\mathfrak{C}_1 \operatorname{Re} V(u_{\max})\big)^{\frac{1}{2}}\,|\omega|}{\lambda\,\mathfrak{C}_3^3\,(v+v_0)} \overset{(9.23)}{\lesssim} \frac{\big(\mathfrak{C}_1 \lambda v_0^2\big)^{\frac{1}{2}}\,|\omega|}{\lambda\,\mathfrak{C}_3^3\,(v+v_0)} \overset{(9.16)}{\lesssim} \frac{\mathfrak{C}_1^{\frac{1}{2}}}{\mathfrak{C}_3^3}\,\frac{v_0}{v+v_0} \leq \frac{\mathfrak{C}_1^{\frac{1}{2}}}{\mathfrak{C}_3^3}\,.$$

This concludes the proof. □

Proof of Proposition 9.4 in case **(b)**. In this case, equation (9.10) reduces to

$$u_-^L = u_0^L - \mathfrak{C}_2\,|\omega|^{-\frac{2}{3}}\,. \tag{9.24}$$

Expanding the potential (9.6) similar to (6.7), one sees that for sufficiently large λ, our potential satisfies the inequalities

$$\big|V'(u)\big| \leq \mathfrak{c}\,\lambda\,e^{\gamma u} \qquad \text{on } (-\infty, u_{\max}) \tag{9.25}$$

$$\big|V''(u)\big| \leq \mathfrak{c}\,\lambda\,e^{\gamma u} \qquad \text{on } (-\infty, u_{\max}) \tag{9.26}$$

$$\operatorname{Re} V'(u) \geq \frac{\lambda}{\mathfrak{C}_2}\,e^{\gamma u} \qquad \text{on } \Big(-\infty, u_{\max} - \frac{1}{\mathfrak{C}_1}\Big) \tag{9.27}$$

for a suitable choice of the constants $\mathfrak{c}, \mathfrak{C}_2 > 1$. Using (6.7), (9.25) and (9.27), it follows that

$$\operatorname{Re}\big(\Omega^2\big) = \int_{-\infty}^{u_0^L} \operatorname{Re} V' \leq \mathfrak{c}\lambda \int_{-\infty}^{u_0^L} e^{\gamma v}\,dv = \frac{\mathfrak{c}\lambda}{\gamma}\,e^{\gamma u_0^L}$$

$$\operatorname{Re}\big(\Omega^2\big) = \int_{-\infty}^{u_0^L} \operatorname{Re} V' \geq \frac{\lambda}{\mathfrak{C}_2} \int_{-\infty}^{u_0^L-1} e^{\gamma v}\,dv = \frac{\lambda e^{-\gamma}}{\mathfrak{C}_2 \gamma}\,e^{\gamma u_0^L}$$

and thus

$$\frac{\lambda e^{-\gamma}}{\mathfrak{C}_2 \gamma}\,e^{\gamma u_0^L} \leq \operatorname{Re}\big(\Omega^2\big) \leq \frac{\mathfrak{c}\lambda}{\gamma}\,e^{\gamma u_0^L}\,.$$

According to (6.8), the imaginary part of Ω is uniformly bounded. Combining this fact with the first inequality in (9.2), we obtain

$$\frac{\lambda e^{-\gamma}}{2\mathfrak{C}_2 \gamma}\,e^{\gamma u_0^L} \leq \omega^2 \leq \frac{2\mathfrak{c}\lambda}{\gamma}\,e^{\gamma u_0^L}\,. \tag{9.28}$$

Combining the above inequalities, for any $u < u_0^L$ we obtain

$$\begin{aligned} -\operatorname{Re} V(u) &= \int_u^{u_0^L} \operatorname{Re} V' \geq \frac{\lambda}{\mathfrak{C}_2} \int_u^{u_0^L} e^{\gamma v}\,dv = \frac{\lambda}{\mathfrak{C}_2 \gamma}\,\big(e^{\gamma u_0^L} - e^{\gamma u}\big) \\ &= \frac{\lambda}{\mathfrak{C}_2 \gamma} e^{\gamma u_0^L}\,\big(1 - e^{\gamma(u-u_0^L)}\big) \geq \frac{1}{2\mathfrak{c}\,\mathfrak{C}_2}\,\big(1 - e^{\gamma(u-u_0^L)}\big)\,\omega^2\,, \end{aligned}$$

where in the last step we applied (9.28). Using (9.24), we conclude that for any $u < u_-^L$ and for sufficiently large $|\omega|$,

$$-\operatorname{Re} V(u) \geq \frac{1}{2\mathfrak{c}\,\mathfrak{C}_2}\,\frac{\gamma}{2}\,\mathfrak{C}_3\,|\omega|^{-\frac{2}{3}}\,\omega^2 \gtrsim \frac{\gamma\,\mathfrak{C}_3}{\mathfrak{c}\,\mathfrak{C}_2}\,|\omega|^{\frac{4}{3}}\,. \tag{9.29}$$

It follows that

$$\frac{|V'|}{|V|^{\frac{3}{2}}} \leq \frac{\mathfrak{c}\,\lambda\, e^{\gamma u}}{|V|^{\frac{3}{2}}} \leq \frac{\mathfrak{c}\,\lambda\, e^{\gamma u_0^L}}{|V|^{\frac{3}{2}}} \lesssim \frac{\mathfrak{c}\, \mathscr{C}_2 \gamma\, |\omega|^2}{|V|^{\frac{3}{2}}} \lesssim \mathfrak{c}\, \mathscr{C}_2 \gamma \left(\frac{\mathfrak{c}\, \mathscr{C}_2}{\gamma\, \mathscr{C}_3} \right)^{\frac{3}{2}}$$
$$\frac{|V''|}{|V|^2} \leq \frac{\mathfrak{c}\,\lambda\, e^{\gamma u_0^L}}{|V|^2} \lesssim \mathfrak{c}\, \mathscr{C}_2 \gamma \left(\frac{\mathfrak{c}\, \mathscr{C}_2}{\gamma\, \mathscr{C}_3} \right)^2 |\omega|^{-\frac{2}{3}} \,.$$

Obviously, the right side of these inequalities can be made arbitrarily small by increasing $\mathscr{C}_3$. □

It remains to consider the region (u_-^R, ∞) as considered in Proposition 9.5. We again treat the two cases in (9.15) after each other.

Proof of Proposition 9.5 in case **(a)**. In this case, equation (9.11) reduces to

$$u_-^R = u_0^R + \mathscr{C}_3 \big(\mathscr{C}_1 \operatorname{Re} V(u_{\max}) \big)^{-\frac{1}{6}}\, \lambda^{\frac{1}{6}}\, |\omega|^{-\frac{2}{3}} \,.$$

Using Lemma 9.6, we know that $\lambda^{\frac{1}{6}} \eqsim \omega^{\frac{1}{3}}$, so that

$$u_-^R \eqsim u_0^R + \mathscr{C}_3 \big(\mathscr{C}_1 \operatorname{Re} V(u_{\max}) \big)^{-\frac{1}{6}}\, |\omega|^{-\frac{1}{3}} \,.$$

Therefore, the identity (9.20) again holds up to a uniform constant. This makes it possible to proceed just as in the proof of Proposition 9.4 after (9.20). □

Proof of Proposition 9.5 in case **(b)**. In this case, equation (9.11) reduces to

$$u_-^R = u_0^R + \mathscr{C}_3\, \lambda^{\frac{1}{6}}\, |\omega|^{-1} \,. \tag{9.30}$$

Using the form of the potential (6.5) and (6.6), we obtain the estimates

$$\operatorname{Re} V + \omega^2 \eqsim \frac{\tilde{\lambda}}{u^2} \qquad \text{on } (u_{\max}, \infty) \tag{9.31}$$
$$\Longrightarrow \;\; u_0^R \eqsim \frac{\tilde{\lambda}^{\frac{1}{2}}}{|\omega|} \tag{9.32}$$
$$-\operatorname{Re} V'(u) \lesssim \frac{\tilde{\lambda}}{u^3} \qquad \text{on } (u_{\max}, \infty) \tag{9.33}$$
$$\big| V''(u) \big| \lesssim \frac{\tilde{\lambda}}{u^4} \qquad \text{on } (u_{\max}, \infty) \tag{9.34}$$
$$-\operatorname{Re} V'(u) \gtrsim \frac{\tilde{\lambda}}{\mathscr{C}_2 u^3} \qquad \text{on } \Big(u_{\max} + \frac{1}{\mathscr{C}_1}, \infty \Big) \,, \tag{9.35}$$

where we introduced the abbreviation

$$\tilde{\lambda} := \lambda + s^2 + 2ak\omega \overset{(9.5)}{\eqsim} \lambda \,.$$

Setting $u_0 = u_0^R$, for any $u > u_-^R$ we obtain the estimates

$$\mathrm{Re}\, V(u) - \mathrm{Re}\, V(u_0) \lesssim -\frac{\tilde{\lambda}}{\mathcal{C}_2} \int_{u_0}^u \frac{1}{u^3}\, du = \frac{\tilde{\lambda}}{\mathcal{C}_2} \left(\frac{1}{u^2} - \frac{1}{u_0^2} \right) = \frac{\tilde{\lambda}}{\mathcal{C}_2}\, \frac{u_0^2 - u^2}{u^2\, u_0^2} \tag{9.36}$$

$$\Longrightarrow \; |\, \mathrm{Re}\, V(u)| \geq \frac{\tilde{\lambda}}{\mathcal{C}_2}\, \frac{u^2 - u_0^2}{u^2\, u_0^2} \tag{9.37}$$

$$\begin{aligned} \frac{|\, \mathrm{Re}\, V'|}{|\, \mathrm{Re}\, V|^{\frac{3}{2}}} &\lesssim \frac{\tilde{\lambda}}{u^3}\, \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{3}{2}}}\, \frac{u^3\, u_0^3}{(u^2 - u_0^2)^{\frac{3}{2}}} = \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}}\, \frac{u_0^3}{(u^2 - u_0^2)^{\frac{3}{2}}} = \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}}\, \frac{u_0^3}{(u-u_0)^{\frac{3}{2}} (u+u_0)^{\frac{3}{2}}} \\ &\leq \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}}\, \frac{u_0^{\frac{3}{2}}}{(u-u_0)^{\frac{3}{2}}} \overset{(9.32),(9.30)}{\lesssim} \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}}\, \frac{\tilde{\lambda}^{\frac{3}{4}}}{|\omega|^{\frac{3}{2}}}\, \frac{|\omega|^{\frac{3}{2}}}{\mathcal{C}_3^{\frac{3}{2}}\, \lambda^{\frac{1}{4}}} \lesssim \frac{\mathcal{C}_2^{\frac{3}{2}}}{\mathcal{C}_3^{\frac{3}{2}}}\,. \end{aligned} \tag{9.38}$$

In order to estimate the imaginary part, we again use the form of the potential (6.5) and (6.6) to obtain

$$|\, \mathrm{Im}\, V(u)| \lesssim \frac{|\omega|}{u}\,, \qquad |\, \mathrm{Im}\, V'(u)| \lesssim \frac{|\omega|}{u^2} \tag{9.39}$$

$$\Longrightarrow \; \frac{|\, \mathrm{Im}\, V'(u)|}{|\, \mathrm{Re}\, V(u)|^{\frac{3}{2}}} \lesssim \frac{|\omega|}{u^2}\, \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{3}{2}}}\, \frac{u^3\, u_0^3}{(u^2 - u_0^2)^{\frac{3}{2}}} = \frac{\mathcal{C}_2^{\frac{3}{2}}\, |\omega|}{\tilde{\lambda}^{\frac{3}{2}}}\, \frac{u\, u_0^3}{(u-u_0)^{\frac{3}{2}} (u+u_0)^{\frac{3}{2}}} \tag{9.40}$$

$$\leq \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}}\, \frac{u\, u_0^{\frac{3}{2}}}{(u-u_0)^{\frac{3}{2}}} \overset{(9.32),(9.30)}{\lesssim} \frac{\mathcal{C}_2^{\frac{3}{2}}}{\lambda}\, \frac{\lambda^{\frac{3}{4}}}{|\omega|^{\frac{3}{2}}}\, \frac{|\omega|^{\frac{3}{2}}}{\mathcal{C}_3^{\frac{3}{2}}\, \lambda^{\frac{1}{4}}}\, u \leq \frac{\mathcal{C}_2^{\frac{3}{2}}}{\mathcal{C}_3^{\frac{3}{2}}}\, \frac{u}{\sqrt{\lambda}}\,. \tag{9.41}$$

This gives the desired estimate provided that $u \leq \mathfrak{c}$ (for a constant $\mathfrak{c} > 0$ which is independent of the parameters λ and ω). For large u, on the other hand, we know that

$$|\, \mathrm{Im}\, V(u)| \eqsim \frac{|\omega|}{u} \qquad \text{for } u \geq \mathfrak{c} \tag{9.42}$$

$$\Longrightarrow \; \frac{|\, \mathrm{Im}\, V'(u)|}{|V(u)|^{\frac{3}{2}}} \leq \frac{|\, \mathrm{Im}\, V'(u)|}{|\, \mathrm{Im}\, V(u)|^{\frac{3}{2}}} \lesssim \frac{|\omega|}{u^2}\, \frac{u^{\frac{3}{2}}}{|\omega|^{\frac{3}{2}}} \leq \frac{1}{|\omega|^{\frac{1}{2}}\, u_0^{\frac{1}{2}}} \overset{(9.32)}{\eqsim} \frac{1}{\tilde{\lambda}^{\frac{1}{4}}}\,. \tag{9.43}$$

The second derivatives are estimated similarly:

$$|\, \mathrm{Re}\, V''| \eqsim \frac{\tilde{\lambda}}{u^4} \tag{9.44}$$

$$\Longrightarrow \; \frac{|\, \mathrm{Re}\, V''|}{|\, \mathrm{Re}\, V|^2} \overset{(9.37)}{\lesssim} \frac{\tilde{\lambda}}{u^4}\, \frac{\mathcal{C}_2^2}{\tilde{\lambda}^2}\, \frac{u^4\, u_0^4}{(u^2 - u_0^2)^2} \leq \frac{\mathcal{C}_2^2\, u_0^2}{\tilde{\lambda}\, (u-u_0)^2} \overset{(9.32),(9.30)}{\lesssim} \frac{\mathcal{C}_2^2}{\tilde{\lambda}}\, \frac{\tilde{\lambda}}{\omega^2}\, \frac{\omega^2}{\mathcal{C}_3^2\, \lambda^{\frac{1}{3}}} = \frac{\mathcal{C}_2^2}{\mathcal{C}_3^2\, \lambda^{\frac{1}{3}}}$$

$$|\, \mathrm{Im}\, V''| \eqsim \frac{|\omega|}{u^3} \tag{9.45}$$

$$\Longrightarrow \; \frac{|\, \mathrm{Im}\, V''|}{|\, \mathrm{Re}\, V|^2} \lesssim \frac{|\omega|}{u^3}\, \frac{\mathcal{C}_2^2}{\tilde{\lambda}^2}\, \frac{u^4\, u_0^4}{(u^2 - u_0^2)^2} \overset{(9.16)}{\leq} \frac{\mathcal{C}_2^2\, u\, u_0^2}{\tilde{\lambda}^{\frac{3}{2}}\, (u-u_0)^2} \overset{(9.32),(9.30)}{\lesssim} \frac{\mathcal{C}_2^2}{\mathcal{C}_3^2\, \lambda^{\frac{1}{3}}}\, \frac{u}{\sqrt{\lambda}}\,, \tag{9.46}$$

giving the desired estimate if $u \leq \mathfrak{c}$. On the other hand, if $u \geq \mathfrak{c}$, we again use (9.42) to obtain

$$\frac{|\, \mathrm{Im}\, V''|}{|\, \mathrm{Im}\, V|^2} \lesssim \frac{|\omega|}{u^3}\, \frac{u^2}{\omega^2} \leq \frac{1}{|\omega|\, u_0} \overset{(9.32)}{\eqsim} \frac{1}{|\omega|^{\frac{1}{2}}\, \tilde{\lambda}^{\frac{1}{4}}}\,.$$

This concludes the proof. ∎

9.3. **Estimates in the WKB Region with** $\text{Re}\, V > 0$**.** In this section we shall prove the following results:

Proposition 9.8. *For any $\varepsilon > 0$, we can arrange by choosing the constants $\mathfrak{C}_1, \ldots, \mathfrak{C}_4$ sufficiently large that for all Ω and λ in the range* (9.2) *where we are in the Airy case (see* (9.14)*), the following WKB estimates hold:*

$$\frac{|V'|}{|V|^{\frac{3}{2}}}, \frac{|V''|}{|V|^2} \leq \varepsilon \qquad \textit{on } (u_+^L, u_+^R)\,.$$

For the proof, we again consider the cases **(a)** and **(b)** in (9.15) after each other.

Proof of Proposition 9.8 in case **(a)**. We proceed similar as in the proofs of Propositions 9.4 and 9.5 in case **(a)**. It suffices to consider the region $(u_{\max}, u_+^R)$, because on the interval $(u_+^L, u_{\max})$ the proof is the same with obvious changes. In case **(a)**, equation (9.13) reduces to

$$u_+^R = u_0^R - \mathfrak{C}_3\, \lambda^{\frac{1}{6}}\, |\omega|^{-\frac{2}{3}} \big(\mathfrak{C}_1 \,\text{Re}\, V(u_{\max})\big)^{-\frac{1}{6}}\,. \tag{9.47}$$

Moreover, in view of the inequality (9.19), on the interval $(u_{\max}, u_+^R)$ the second derivative of $\text{Re}\, V$ satisfies the inequalities in (9.8). Hence, setting $v = u - u_{\max}$ and $v_0 = u_0^R - u_{\max}$, we obtain for all $u \in (u_{\max}, u_-^R)$

$$\begin{aligned}
\frac{\lambda v}{\mathfrak{c}} &\leq -\,\text{Re}\, V' \leq \mathfrak{c} \lambda v \\
\text{Re}\, V(u) = \text{Re}\, V(u) - \text{Re}\, V(u_0^R) &= -\int_u^{u_0^R} \text{Re}\, V'(u) \leq \mathfrak{c}\lambda\big(v_0^2 - v^2\big) \\
\Longrightarrow\; \frac{\lambda}{\mathfrak{c}}\big(v_0^2 - v^2\big) &\leq \text{Re}\, V(u) \leq \mathfrak{c}\lambda\big(v_0^2 - v^2\big)\,.
\end{aligned}$$

Combining these estimates with (9.47), we obtain

$$\frac{|\,\text{Re}\, V'(u)|^2}{|V(u)|^3} \lesssim \frac{\lambda^2\, v^2}{\lambda^3\,(v_0 - v)^3\,(v_0 + v)^3} \leq \frac{\big(\mathfrak{C}_1 \,\text{Re}\, V(u_{\max})\big)^{\frac{1}{2}}\, \omega^2\, v^2}{\lambda^{\frac{3}{2}}\, \mathfrak{C}_3^3\, (v_0 + v)^3}\,.$$

Next, we can estimate $\text{Re}\, V(u_{\max})$ by

$$0 = \text{Re}\,\big(V(u_0^L)\big) \geq \text{Re}\, V(u_{\max}) - \frac{\lambda}{2\mathfrak{c}}\; v_0^2\,,$$

implying that

$$\text{Re}\, V(u_{\max}) \lesssim \lambda\, v_0^2\,. \tag{9.48}$$

We thus obtain

$$\frac{|\,\text{Re}\, V'(u)|^2}{|V(u)|^3} \lesssim \frac{\sqrt{\mathfrak{C}_1}}{\mathfrak{C}_3^3}\, \frac{\omega^2}{\lambda}\, \frac{v_0\, v^2}{(v + v_0)^3} \lesssim \frac{\sqrt{\mathfrak{C}_1}}{\mathfrak{C}_3^3}\,,$$

where in the last step we used (9.16). The imaginary part of V' can be handled similar as in the proof of Proposition 9.4. Namely, from (9.7) and (9.16), we know that

$$\frac{|\,\text{Im}\, V'(u)|^2}{|V(u)|^3} \lesssim \frac{\lambda}{\lambda^3\,(v_0 - v)^3\,(v_0 + v)^3} \leq \frac{1}{\lambda^2\,(v_0 - v)^{\frac{9}{2}}\,(v_0 + v)^{\frac{3}{2}}}\,.$$

Applying again (9.47) and (9.48), we obtain

$$\frac{|\operatorname{Im} V'(u)|^2}{|V(u)|^3} \lesssim \frac{\big(\mathcal{C}_1 \operatorname{Re} V(u_{\max})\big)^{\frac{3}{4}}\, |\omega|^3}{\lambda^2\, \mathcal{C}_3^{\frac{9}{2}}\, \lambda^{\frac{3}{4}}\, (v+v_0)^{\frac{3}{2}}} \lesssim \frac{\big(\mathcal{C}_1 \lambda\, v_0^2\big)^{\frac{3}{4}} |\omega|^3}{\lambda^{\frac{11}{4}}\, \mathcal{C}_3^{\frac{9}{2}}\, (v+v_0)^{\frac{3}{2}}}$$
$$\lesssim \frac{\big(\mathcal{C}_1 \lambda\big)^{\frac{3}{4}} |\omega|^3}{\lambda^{\frac{11}{4}}\, \mathcal{C}_3^{\frac{9}{2}}} \overset{(9.16)}{\lesssim} \frac{\mathcal{C}_1^{\frac{3}{4}}}{\sqrt{\lambda}\, \mathcal{C}_3^{\frac{9}{2}}} \leq \frac{\mathcal{C}_1^{\frac{3}{4}}}{\sqrt{\mathcal{C}_7}\, \mathcal{C}_3^{\frac{9}{2}}} \,.$$

This concludes the proof for the first derivatives.

The second derivatives are estimated similarly by

$$\big|V''(u)\big| \leq \big|\operatorname{Re} V''(u)\big| + \big|\operatorname{Im} V''(u)\big| \overset{(9.7),(9.8)}{\leq} \mathfrak{c}\, \big(\lambda + 1 + |\omega|\big) \overset{(9.5)}{\lesssim} \lambda$$
$$\frac{|V''(u)|}{|V(u)|^2} \lesssim \frac{\lambda}{\lambda^2\, (v_0-v)^2\, (v_0+v)^2} \leq \frac{1}{\lambda\, (v_0-v)^3\, (v_0+v)}$$
$$\leq \frac{\big(\mathcal{C}_1 \operatorname{Re} V(u_{\max})\big)^{\frac{1}{2}}\, |\omega|^2}{\lambda^{\frac{3}{2}}\, \mathcal{C}_3^3\, (v_0+v)} \overset{(9.48)}{\lesssim} \frac{\big(\mathcal{C}_1 \lambda v_0^2\big)^{\frac{1}{2}}\, |\omega|^2}{\lambda^{\frac{3}{2}}\, \mathcal{C}_3^3\, (v+v_0)} \overset{(9.16)}{\lesssim} \frac{\mathcal{C}_1^{\frac{1}{2}}}{\mathcal{C}_3^3}\, \frac{v_0}{v+v_0} \leq \frac{\mathcal{C}_1^{\frac{1}{2}}}{\mathcal{C}_3^3} \,.$$

This concludes the proof. □

Proof of Proposition 9.8 in case **(b)**. In this case, the identities (9.12) and (9.13) simplify to

$$u_+^L = u_0^L + \mathcal{C}_3\, |\omega|^{-\frac{2}{3}} \,, \qquad u_+^R = u_0^R - \mathcal{C}_3\, \lambda^{\frac{1}{6}}\, |\omega|^{-1} \,. \tag{9.49}$$

We first consider the interval $[u_{\max} - \mathcal{C}_1^{-1}, u_{\max} + \mathcal{C}_1^{-1}]$ (according to (9.19) and (9.49), this interval is contained in (u_+^L, u_+^R)). Combining the estimate (9.18) with the upper bound in (9.8), we know that on $[u_{\max} - \mathcal{C}_1^{-1}, u_{\max} + \mathcal{C}_1^{-1}]$,

$$\operatorname{Re} V(u) \geq \frac{\lambda}{\mathcal{C}_1} - \frac{\mathfrak{c}}{2}\, \lambda\, (u - u_{\max})^2 \geq \frac{\lambda}{\mathcal{C}_1} - \frac{\mathfrak{c}\, \lambda}{2\mathcal{C}_1^2} \geq \frac{\lambda}{2\mathcal{C}_1} \,,$$

where in the last step we increased $\mathcal{C}_1$. Moreover, from the right of (9.4) we know that $\big|V'\big|, \big|V''\big| \lesssim \lambda + |\omega|$. Hence

$$\frac{|V'(u)|^2}{|V(u)|^3} \lesssim \mathcal{C}_1^3\, \frac{(\lambda + |\omega|)^2}{\lambda^3} \overset{(9.5)}{\lesssim} \frac{\mathcal{C}_1^3}{\lambda} \overset{(9.2)}{\lesssim} \frac{\mathcal{C}_1^3}{\mathcal{C}_7}$$
$$\frac{|V''(u)|}{|V(u)|^2} \lesssim \mathcal{C}_1^2\, \frac{\lambda + |\omega|}{\lambda^2} \overset{(9.5)}{\lesssim} \frac{\mathcal{C}_1^2}{\lambda} \overset{(9.2)}{\lesssim} \frac{\mathcal{C}_1^2}{\mathcal{C}_7} \,.$$

Choosing $\mathcal{C}_7$ sufficiently large, we obtain the result.

It remains to consider the regions $(u_+^L, u_{\max} - \mathcal{C}_1^{-1})$ and $(u_{\max} + \mathcal{C}_1^{-1}, u_+^R)$. In these regions, we can proceed similar as in the proofs of Propositions 9.4 and 9.5 in case **(b)**. Namely, on the interval $(u_+^L, u_{\max} - \mathcal{C}_1^{-1})$ the inequalities (9.25)–(9.27) and (9.28) hold. As a consequence,

$$\operatorname{Re} V(u) = \int_{u_0^L}^u \operatorname{Re} V' \geq \frac{\lambda}{\mathcal{C}_2} \int_{u_0^L}^u e^{\gamma v}\, dv = \frac{\lambda}{\mathcal{C}_2 \gamma}\, \big(e^{\gamma u} - e^{\gamma u_0^L}\big) \tag{9.50}$$
$$\Longrightarrow\; \frac{|V'(u)|^2}{|V(u)|^3} \overset{(9.25)}{\lesssim} \frac{\mathcal{C}_2^3}{\lambda}\, \frac{e^{2\gamma u}}{\big(e^{\gamma u} - e^{\gamma u_0^L}\big)^3} \,.$$

By computing its u-derivative, one sees that the last fraction is monotone decreasing in u. Hence

$$\frac{|V'(u)|^2}{|V(u)|^3} \lesssim \frac{\mathcal{C}_2^3}{\lambda} \frac{e^{2\gamma u_+^L}}{\left(e^{\gamma u_+^L} - e^{\gamma u_0^L}\right)^3} \leq \frac{\mathcal{C}_2^3}{\lambda} \frac{e^{2\gamma u_+^L}}{e^{3\gamma u_0^L}\left(\gamma\left(u_+^L - u_0^L\right)\right)^3},$$

where in the last step we used the mean value inequality. Applying (9.28) and (9.49), we obtain

$$\frac{|V'(u)|^2}{|V(u)|^3} \lesssim \frac{\mathcal{C}_2^3}{\lambda} \frac{e^{2\gamma(u_+^L - u_0^L)}}{e^{\gamma u_0^L}(u_+^L - u_0^L)^3} \lesssim \frac{\mathcal{C}_2^3}{\lambda} \frac{\lambda}{\omega^2\left(\mathcal{C}_3\,|\omega|^{-\frac{2}{3}}\right)^3} = \frac{\mathcal{C}_2^3}{\mathcal{C}_3},$$

which can be made arbitrarily small by increasing $\mathcal{C}_3$ (note that, in view of (9.49), the factor $e^{2\gamma(u_+^L - u_0^L)}$ is uniformly bounded). The second derivatives can be estimated similarly as follows. First, using (9.26) and (9.50),

$$\frac{|V''(u)|}{|V(u)|^2} \lesssim \frac{\mathcal{C}_2^2}{\lambda} \frac{e^{\gamma u}}{\left(e^{\gamma u} - e^{\gamma u_0^L}\right)^2}.$$

This is again monotone decreasing, implying that

$$\begin{aligned} \frac{|V''(u)|}{|V(u)|^2} &\lesssim \frac{\mathcal{C}_2^2}{\lambda} \frac{e^{\gamma u_+^L}}{\left(e^{\gamma u_+^L} - e^{\gamma u_0^L}\right)^2} \lesssim \frac{\mathcal{C}_2^2}{\lambda} \frac{e^{\gamma(u_+^L - u_0^L)}}{e^{\gamma u_0^L}(u_+^L - u_0^L)^2} \\ &\lesssim \frac{\mathcal{C}_2^2}{\lambda} \frac{\lambda}{\omega^2\left(\mathcal{C}_3\,|\omega|^{-\frac{2}{3}}\right)^2} \lesssim \frac{\mathcal{C}_2^2}{\mathcal{C}_3^2}\,\omega^{-\frac{2}{3}}, \end{aligned}$$

where in the last line we again applied (9.28) and (9.49).

In the remaining region $(u_{\max} + \mathcal{C}_1^{-1}, u_+^R)$, we can again use the estimates (9.31)–(9.35). Again omitting the index R, for any $u \in (u_{\max} + \mathcal{C}_1^{-1}, u_+^R)$ we obtain

$$\operatorname{Re} V(u) = \operatorname{Re} V(u) - \operatorname{Re} V(u_0) = -\int_u^{u_0} \operatorname{Re} V \overset{(9.35)}{\gtrsim} \frac{\tilde{\lambda}}{\mathcal{C}_2}\left(\frac{1}{u^2} - \frac{1}{u_0^2}\right) = \frac{\tilde{\lambda}}{\mathcal{C}_2}\frac{u_0^2 - u^2}{u^2\,u_0^2}$$

$$\begin{aligned} \frac{|\operatorname{Re} V'(u)|}{|\operatorname{Re} V(u)|^{\frac{3}{2}}} &\lesssim \frac{\tilde{\lambda}}{u^3}\,\frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{3}{2}}}\,\frac{u^3\,u_0^3}{(u_0^2 - u^2)^{\frac{3}{2}}} = \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}}\,\frac{u_0^3}{(u_0^2 - u^2)^{\frac{3}{2}}} = \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}}\,\frac{u_0^3}{(u_0 - u)^{\frac{3}{2}}(u_0 + u)^{\frac{3}{2}}} \\ &\leq \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}}\,\frac{u_0^{\frac{3}{2}}}{(u_0 - u)^{\frac{3}{2}}} \overset{(9.32),(9.30)}{\lesssim} \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}}\,\frac{\tilde{\lambda}^{\frac{3}{4}}}{|\omega|^{\frac{3}{2}}}\,\frac{|\omega|^{\frac{3}{2}}}{\mathcal{C}_3^{\frac{3}{2}}\,\lambda^{\frac{1}{4}}} \lesssim \frac{\mathcal{C}_2^{\frac{3}{2}}}{\mathcal{C}_3^{\frac{3}{2}}}. \end{aligned}$$

For the imaginary part, we again use the estimate (9.39) to obtain

$$\begin{aligned} \frac{|\operatorname{Im} V'(u)|}{|\operatorname{Re} V(u)|^{\frac{3}{2}}} &\lesssim \frac{|\omega|}{u^2}\,\frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{3}{2}}}\,\frac{u^3\,u_0^3}{(u_0^2 - u^2)^{\frac{3}{2}}} = \frac{\mathcal{C}_2^{\frac{3}{2}}\,|\omega|}{\tilde{\lambda}^{\frac{3}{2}}}\,\frac{u\,u_0^3}{(u_0^2 - u^2)^{\frac{3}{2}}} \\ &\overset{(9.16)}{\lesssim} \frac{\mathcal{C}_2^{\frac{3}{2}}}{\tilde{\lambda}}\,\frac{u\,u_0^3}{(u_0 - u)^{\frac{3}{2}}(u_0 + u)^{\frac{3}{2}}} \lesssim \frac{\mathcal{C}_2^{\frac{3}{2}}}{\mathcal{C}_3^{\frac{3}{2}}}\,\frac{u}{\sqrt{\lambda}}. \end{aligned}$$

This gives the desired estimate if $u \leq \mathfrak{c}$. On the other hand, if $u > \mathfrak{c}$ we again apply (9.42). This gives

$$\frac{|\operatorname{Im} V'(u)|}{|\operatorname{Im} V(u)|^{\frac{3}{2}}} \lesssim \frac{|\omega|}{u^2}\,\frac{u^{\frac{3}{2}}}{|\omega|^{\frac{3}{2}}} \lesssim \frac{1}{|\omega|^{\frac{1}{2}}}.$$

The second derivatives are estimated similarly using (9.44) and (9.45),

$$\frac{|\operatorname{Re} V''|}{|\operatorname{Re} V|^2} \lesssim \frac{\tilde{\lambda}}{u^4}\, \frac{\mathfrak{C}_2^2}{\tilde{\lambda}^2}\, \frac{u^4\, u_0^4}{(u_0^2-u^2)^2} \leq \frac{\mathfrak{C}_2^2\, u_0^2}{\tilde{\lambda}\,(u_0-u)^2} \overset{(9.32),(9.30)}{\lesssim} \frac{\mathfrak{C}_2^2}{\tilde{\lambda}}\, \frac{\tilde{\lambda}}{\omega^2}\, \frac{\omega^2}{\mathfrak{C}_3^2\, \lambda^{\frac{1}{3}}} = \frac{\mathfrak{C}_2^2}{\mathfrak{C}_3^2\, \lambda^{\frac{1}{3}}}$$

$$\frac{|\operatorname{Im} V''|}{|\operatorname{Re} V|^2} \lesssim \frac{|\omega|}{u^3}\, \frac{\mathfrak{C}_2^2}{\tilde{\lambda}^2}\, \frac{u^4\, u_0^4}{(u_0^2-u^2)^2} \leq \frac{\mathfrak{C}_2^2\, u\, u_0^2}{\tilde{\lambda}^{\frac{3}{2}}\,(u_0-u)^2} \overset{(9.32),(9.30)}{\lesssim} \frac{\mathfrak{C}_2^2}{\mathfrak{C}_3^2\, \lambda^{\frac{1}{3}}}\, \frac{u}{\sqrt{\lambda}}\,,$$

giving the desired estimate if $u \leq \mathfrak{c}$. On the other hand, if $u \geq \mathfrak{c}$, we again use (9.42) to obtain

$$\frac{|\operatorname{Im} V''|}{|\operatorname{Im} V|^2} \lesssim \frac{|\omega|}{u^3}\, \frac{u^2}{\omega^2} \lesssim \frac{1}{|\omega|}\,.$$

This concludes the proof. □

9.4. **Estimates in the Airy Regions.**

Lemma 9.9. *In the Airy case, one can arrange by suitably increasing the constants* $\mathfrak{C}_1, \ldots, \mathfrak{C}_4$ *that*

$$\sup_{[u_-^L, u_+^L]} |V|\, (u_+^L - u_-^L)^2 \lesssim \mathfrak{C}_4\,.$$

Proof. We consider the two cases in (9.15) separately. In case **(a)**, we know from (9.19) and (9.10) that the interval $[u_-^L, u_+^L]$ is contained in the interval $[u_{\max} - \frac{1}{2}, u_{\max} + \frac{1}{2}]$. Therefore, we can integrate the inequality (9.8) to conclude that for any $u \in [u_-^L, u_+^L]$,

$$\operatorname{Re} V(u_{\max}) \gtrsim \lambda \left(u_{\max} - u_0^L\right)^2 \tag{9.51}$$

$$|\operatorname{Re} V(u)| \lesssim \left|\operatorname{Re} V'(u_0^L)\right| \left(u_+^L - u_-^L\right) + \lambda \left(u_+^L - u_-^L\right)^2 \tag{9.52}$$

and thus

$$|\operatorname{Re} V(u)| \left(u_+^L - u_-^L\right)^2 \lesssim \left|\operatorname{Re} V'(u_0^L)\right| \left(u_+^L - u_-^L\right)^3 + \lambda \left(u_+^L - u_-^L\right)^4 . \tag{9.53}$$

Moreover, similar to (9.20), the identities (9.10) and (9.12) imply that

$$u_+^L - u_-^L = 2\, \mathfrak{C}_3 \big(\mathfrak{C}_1 \operatorname{Re} V(u_{\max})\big)^{-\frac{1}{6}}\, |\omega|^{-\frac{1}{3}}\,.$$

Using this equation in (9.53), we obtain two contributions, which can be estimated as follows,

$$\begin{aligned}
\left|\operatorname{Re} V'(u_0^L)\right| \left(u_+^L - u_-^L\right)^3 &\overset{(9.22)}{\lesssim} \lambda\, v_0\, \mathfrak{C}_3^3 \big(\mathfrak{C}_1 \operatorname{Re} V(u_{\max})\big)^{-\frac{1}{2}}\, |\omega|^{-1} \\
&\overset{(9.51)}{\lesssim} \lambda\, v_0\, \mathfrak{C}_3^3 \big(\mathfrak{C}_1\, \lambda\, v_0^2\big)^{-\frac{1}{2}}\, |\omega|^{-1} \\
&= \mathfrak{C}_3^3\, \mathfrak{C}_1^{-\frac{1}{2}}\, \lambda^{\frac{1}{2}}\, |\omega|^{-1} \overset{(9.17)}{\lesssim} \mathfrak{C}_3^3\, \mathfrak{C}_1^{-\frac{1}{2}} \\
\lambda \left(u_+^L - u_-^L\right)^4 &\lesssim \lambda\, \mathfrak{C}_3^4\, \mathfrak{C}_1^{-\frac{2}{3}}\, \operatorname{Re} V(u_{\max})^{-\frac{2}{3}}\, |\omega|^{-\frac{4}{3}} \\
&\overset{(9.9)}{\leq} \mathfrak{C}_3^4\, \mathfrak{C}_1^{-\frac{2}{3}}\, \mathfrak{C}_4^{-\frac{2}{3}}\, \lambda^{\frac{2}{3}}\, |\omega|^{-\frac{4}{3}} \overset{(9.17)}{\lesssim} \mathfrak{C}_3^4\, \mathfrak{C}_1^{-\frac{2}{3}}\, \mathfrak{C}_4^{-\frac{2}{3}}\,,
\end{aligned}$$

where we used the abbreviation $v_0 := u_{\max} - u_0^L$. The imaginary part of the potential is estimated similarly with the help of (9.7),

$$\sup_{[u_-^L, u_+^L]} |\operatorname{Im} V| \left(u_+^L - u_-^L\right)^2 \lesssim |\omega| \left(u_+^L - u_-^L\right)^2 \lesssim |\omega|\, \mathcal{C}_3^2\, \mathcal{C}_1^{-\frac{1}{3}}\, \operatorname{Re} V(u_{\max})^{-\frac{1}{3}}\, |\omega|^{-\frac{2}{3}}$$

$$\overset{(9.9)}{\leq} \mathcal{C}_3^2\, \mathcal{C}_1^{-\frac{1}{3}}\, \mathcal{C}_4^{-\frac{1}{3}}\, \lambda^{-\frac{1}{6}}\, |\omega|^{\frac{1}{3}} \overset{(9.16)}{\lesssim} \mathcal{C}_3^2\, \mathcal{C}_1^{-\frac{1}{3}}\, \mathcal{C}_4^{-\frac{1}{3}}\,.$$

This can be made arbitrarily small by increasing $\mathcal{C}_4$. This completes the proof in case **(a)**.

In case **(b)**, similar to (9.24), the identities (9.10) and (9.12) imply that

$$u_+^L - u_-^L = 2\, \mathcal{C}_2\, |\omega|^{-\frac{2}{3}}\,. \tag{9.54}$$

We integrate the inequality (9.25) to obtain for any $u \in [u_-^L, u_+^L]$

$$|\operatorname{Re} V(u)| \lesssim \lambda\, e^{\gamma u_0^L} \left(u_+^L - u_-^L\right) \overset{(9.28)}{\lesssim} \mathcal{C}_2\, \omega^2 \left(u_+^L - u_-^L\right)$$

$$\Longrightarrow \quad |\operatorname{Re} V(u)| \left(u_+^L - u_-^L\right)^2 \lesssim \mathcal{C}_2\, \omega^2 \left(u_+^L - u_-^L\right)^3 \overset{(9.54)}{\lesssim} \mathcal{C}_2^4\,.$$

This concludes the proof. □

Lemma 9.10. *In the Airy case, one can arrange by suitably increasing the constants* $\mathcal{C}_1, \ldots, \mathcal{C}_4$ *that*

$$\sup_{[u_+^R, u_-^R]} |V|\, (u_-^R - u_+^R)^2 \lesssim \mathcal{C}_3^3\,.$$

Proof. We consider the two cases in (9.15) separately. In case **(a)**, we can proceed exactly as in the proof of Lemma 9.9. In the remaining case **(b)**, similar to (9.30), the identities (9.10) and (9.12) imply that

$$u_-^R - u_+^R = 2\, \mathcal{C}_3\, \lambda^{\frac{1}{6}}\, |\omega|^{-1}\,. \tag{9.55}$$

Integrating (9.33), we obtain for any $u \in [u_+^R - u_-^R]$ that

$$|\operatorname{Re} V(u)| \lesssim \frac{\tilde{\lambda}}{\left(u_0^R\right)^3} \left(u_-^R - u_+^R\right) \overset{(9.32)}{\lesssim} \frac{\omega^3}{\tilde{\lambda}^{\frac{1}{2}}} \left(u_-^R - u_+^R\right)$$

$$\Longrightarrow \quad |\operatorname{Re} V(u)|\, (u_-^R - u_+^R)^2 \lesssim \frac{\omega^3}{\tilde{\lambda}^{\frac{1}{2}}} \left(u_-^R - u_+^R\right)^3 \overset{(9.55)}{\lesssim} \mathcal{C}_3^3\,.$$

Similarly, using (9.39),

$$|\operatorname{Im} V(u)|\, (u_-^R - u_+^R)^2 \lesssim \frac{|\omega|}{u_0} (u_-^R - u_+^R)^2 \overset{(9.32)}{\lesssim} \frac{|\omega|^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}} (u_-^R - u_+^R)^2$$

$$\overset{(9.55)}{\lesssim} \frac{|\omega|^{\frac{3}{2}}}{\tilde{\lambda}^{\frac{1}{2}}}\, \mathcal{C}_3^2\, \lambda^{\frac{1}{3}}\, \omega^{-2} \lesssim \mathcal{C}_3^2\, \lambda^{-\frac{1}{6}}\, |\omega|^{-\frac{1}{2}}\,.$$

This concludes the proof. □

9.5. **Estimates in the Parabolic Cylinder Region.**

Lemma 9.11. *In the PC case, one can arrange by suitably increasing the constants* $\mathcal{C}_1, \ldots, \mathcal{C}_4$ *that*

$$\sup_{[u_-^L, u_-^R]} |V| \, (u_-^R - u_-^L)^2 \lesssim \mathcal{C}_4^2 \, .$$

Proof. We first estimate $u_-^R - u_-^L$. In the case $\mathrm{Re}\, V(u_{\max}) > 0$, we know that $\mathrm{Re}\, V(u_0^L)$ vanishes. Hence, using (9.9) and again integrating (9.8), we obtain

$$\lambda \, (u_{\max} - u_0^L)^2 \lesssim \mathrm{Re}\, V(u_{\max}) \leq \mathcal{C}_4 \sqrt{\lambda}$$

and thus

$$(u_{\max} - u_0^L) \lesssim \frac{\sqrt{\mathcal{C}_4}}{\lambda^{\frac{1}{4}}} \, .$$

Now we can use (9.10) to obtain

$$(u_{\max} - u_-^L) \lesssim \frac{\sqrt{\mathcal{C}_4}}{\lambda^{\frac{1}{4}}} + \mathcal{C}_3 (\mathcal{C}_1 \mathcal{C}_4)^{-\frac{1}{6}} \, |\omega|^{-\frac{1}{2}} \overset{(9.17)}{\lesssim} \frac{\sqrt{\mathcal{C}_4}}{\lambda^{\frac{1}{4}}} \, .$$

In the case $\mathrm{Re}\, V(u_{\max}) > 0$, on the other hand, we know that $u_0^L = u_{\max}$, so that the last estimate again holds. Repeating this estimate for $u_-^R - u_{\max}$, we conclude that

$$(u_-^R - u_-^L) \lesssim \frac{\sqrt{\mathcal{C}_4}}{\lambda^{\frac{1}{4}}} \, .$$

Combining the last inequality with (9.8) and (9.9), we obtain for any $u \in [u_-^L, u_-^R]$ the estimate

$$\begin{aligned} |V(u)| \, \big(u_-^R - u_-^L\big)^2 &\lesssim |V(u_{\max})| \, \big(u_-^R - u_-^L\big)^2 + \sup_{[u_-^L, u_-^R]} |V''| \, \big(u_-^R - u_-^L\big)^4 \\ &\lesssim \mathcal{C}_4 \sqrt{\lambda} \, \big(u_-^R - u_-^L\big)^2 + \lambda \, \big(u_-^R - u_-^L\big)^4 \lesssim \mathcal{C}_4^2 \, . \end{aligned}$$

This concludes the proof. □

9.6. **Estimates of the Zeros of** $\mathrm{Im}\, V$**.** For the T-estimates introduced in [20, Section 3.2], the sign of $\mathrm{Im}\, V$ is of particular importance. More precisely, if y is in the upper half plane and $\mathrm{Im}\, V > 0$ (and similarly if y is in the lower half plane and $\mathrm{Im}\, V < 0$), then we can use these estimates setting $g \equiv 0$ (the "good" sign). If, however, the imaginary part of V has the opposite "bad" sign, then the estimates apply only if $|\mathrm{Im}\, V|$ is small in a quite restrictive sense. In the next lemma, we identify the regions where $\mathrm{Im}\, V$ has a "good" sign and estimate $\mathrm{Im}\, V$ in the regions where the sign is "bad."

Lemma 9.12. *By choosing the constants* $\mathcal{C}_6$ *and* $\mathcal{C}_7$ *in* (9.2) *sufficiently large, one can arrange that*

$$\left\{ \begin{array}{ll} \omega \; \mathrm{Im}\, V(u) \geq 0 & \text{on } \big(-\infty, u_{\max} - \mathcal{C}_1^{-\frac{1}{2}}\big] \\ \omega \; \mathrm{Im}\, V(u) \leq 0 & \text{on } \big(u_{\max} + \mathcal{C}_1^{-\frac{1}{2}}, \infty\big) \, . \end{array} \right. \tag{9.56}$$

Moreover, on the remaining intervals, the function $\mathrm{Im}\, V$ *satisfies the inequalities*

$$\left\{ \begin{array}{ll} \omega \; \mathrm{Im}\, V(u) \gtrsim -\omega & \text{on } \big(u_{\max} - \mathcal{C}_1^{-\frac{1}{2}}, u_{\max}\big] \\ \omega \; \mathrm{Im}\, V(u) \lesssim \omega & \text{on } \big[u_{\max}, u_{\max} + \mathcal{C}_1^{-\frac{1}{2}}\big) \, . \end{array} \right. \tag{9.57}$$

Proof. Using the form of the potential (6.5), one obtains the expansions

$$\omega \,\mathrm{Im}\, V = \frac{2\omega^2}{(r^2+a^2)^2} \Big(r^2\,(r-3M) - a^2\,(r+M) \Big) + \mathcal{O}\big(\omega^0\big) \tag{9.58}$$

$$\begin{aligned} \frac{d}{dr} \,\mathrm{Re}\, V &= \lambda\, \frac{d}{dr} \frac{\Delta}{(r^2+a^2)^2} + \mathcal{O}\big(\lambda^0\big) + \mathcal{O}(\omega) \\ &= \frac{2\lambda}{(r^2+a^2)^3} \Big(r^2\,(r-3M) - a^2\,(r+M) \Big) + \mathcal{O}\big(\lambda^0\big) + \mathcal{O}(\omega)\,. \end{aligned} \tag{9.59}$$

Comparing these formulas, one sees that the leading contributions to both functions have the same sign as the factor $r^2(r-3M) - a^2\,(r+M)$. In order to control the error term, we denote the zero of the function $r^2(r-3M) - a^2\,(r+M)$ by u_{f}. Then the zero u_{im} of $\mathrm{Im}\, V$ and the zero $u_{\max}$ of the function $\partial_r \,\mathrm{Re}\, V$ are given by

$$u_{\mathrm{im}} = u_{\mathrm{f}} + \mathcal{O}\Big(\frac{1}{\omega^2}\Big) \qquad \text{and} \qquad u_{\max} = u_{\mathrm{f}} + \mathcal{O}\Big(\frac{\omega}{\lambda}\Big)\,.$$

This proves (9.56). In order to derive (9.57), we need to take into account two contributions: First, the error term in (9.58), which is uniformly bounded and thus unproblematic. Second, we need to take into account that the deviations of the zeros u_{im} and $u_{\max}$ from u_{f} may have the effect that the leading contribution to $\omega \,\mathrm{Im}\, V$ in (9.58) has the wrong sign. This contribution scales like

$$|\,\mathrm{Im}\, V'|\; \big|u_{\mathrm{im}} - u_{\max}\big| \lesssim |\omega| \left(\mathcal{O}\Big(\frac{1}{\omega^2}\Big) + \mathcal{O}\Big(\frac{\omega}{\lambda}\Big) \right) = \mathcal{O}(1)\,,$$

where in the last step we used (9.2) and (9.16). This concludes the proof. □

10. Invariant Region Estimates

10.1. **Estimates of $\acute{\phi}$.** We again restrict attention to the parameter range (9.2). We consider the solutions $\acute{\phi}_-$ constructed in Theorem 6.1. For ease in notation, we shall omit the index $-$. We denote the corresponding solution of the Riccati equation by

$$\acute{y}(u) := \frac{\acute{\phi}'(u)}{\acute{\phi}(u)}\,.$$

According to Proposition 9.4, the WKB approximation holds on the interval $(-\infty, u_-^L)$, meaning that

$$\acute{\phi} \;\approx\; \frac{1}{\sqrt[4]{-V}}\, \exp\left(\pm i \int^u \sqrt{-V} \right) \qquad \text{and} \qquad \acute{y} \;\approx\; \pm i\,\sqrt{-V} - \frac{V'}{4V} \tag{10.1}$$

with an arbitrarily small error, where the sign is chosen such that

$$\lim_{u \to -\infty} \pm\sqrt{-V} = \Omega\,.$$

Moreover, the integration constant is chosen in agreement with the normalization convention in (6.9). The goal of this section is to estimate the solution $\acute{\phi}$ all the way to $u = u_{\max}$.

We begin with the parabolic cylinder case:

Lemma 10.1. *In the parabolic cylinder case, there is a constant $\mathcal{C}_9$ such that on the interval $[u_-^L, u_{\max}]$, the solutions $\acute{y}$ and $\acute{\phi}$ are bounded in terms of its values at u_-^L by*

$$|\acute{y}(u)| \leq \mathcal{C}_9 \left|\acute{y}(u_-^L)\right| \tag{10.2}$$

$$\operatorname{Im} \acute{y}(u) \geq \frac{\operatorname{Im} \acute{y}(u_-^L)}{\mathcal{C}_9} \tag{10.3}$$

$$\frac{\left|\acute{\phi}(u_-^L)\right|}{\mathcal{C}_9} \leq \left|\acute{\phi}(u)\right| \leq \mathcal{C}_9 \left|\acute{\phi}(u_-^L)\right| . \tag{10.4}$$

Proof. Our strategy is to estimate y using the T-method as introduced in [20, Section 3.2]. More precisely, we shall apply [20, Theorem 3.3] for a suitable function g. Moreover, setting

$$\nu = \sup_{[u_-^L, u_{\max}]} |V| ,$$

we choose

$$\alpha = \sqrt{2\nu} \qquad \text{and} \qquad \tilde{\beta} = 0 . \tag{10.5}$$

Then $\tilde{V}$ and U are given by

$$\tilde{V} = \alpha^2 = 2\nu \qquad \text{and} \qquad U = \operatorname{Re} V - \alpha^2 \leq -\nu . \tag{10.6}$$

Moreover, the error terms $E_1, \ldots, E_4$ are bounded by

$$\begin{aligned} |E_1| &\lesssim \frac{1}{\sqrt{\nu}} \,|\operatorname{Re} V - \operatorname{Re} \tilde{V}| + \frac{\operatorname{Re} V'}{\nu} \lesssim \sqrt{\nu} + \frac{\operatorname{Re} V'}{\nu} \\ E_2 = 0 \,, &\qquad |E_3| + |E_4| \lesssim \frac{|\operatorname{Im} V|}{\sqrt{\nu}} \,\big(1 + g\big) . \end{aligned} \tag{10.7}$$

The integral over the error term E_1 can estimated by

$$\int_{u_-^L}^{u_{\max}} |E_1| \lesssim \sqrt{\nu}\, (u_{\max} - u_-^L) \bigg(1 + \sup_{[u_-^L, u_{\max}]} \frac{|\operatorname{Re} V'|}{\nu^{\frac{3}{2}}} \bigg).$$

The factor $\sqrt{\nu}\,(u_{\max} - u_-^L)$ was estimated in Lemma 9.11. The factor $\operatorname{Re} V'|/\nu^{\frac{3}{2}}$, on the other hand, is bounded at the left end point u_-^L because of the WKB estimate in Proposition 9.4. This bound can be extended to the interval $(u_-^L, u_{\max})$ by using that $\operatorname{Re} V'$ is monotone decreasing according to (9.8), i.e.

$$0 \leq \operatorname{Re} V'(u) \leq \operatorname{Re} V'(u_-^L) \qquad \text{for all } u \in [u_-^L, u_{\max}] . \tag{10.8}$$

From Lemma 9.12 we know that $\operatorname{Im} V$ and $\operatorname{Im} \acute{y}$ have the same signs, except for the error terms in (9.57). We choose

$$g(u) = \left\{ \begin{array}{cl} 0 & \text{if } \omega \operatorname{Im} V \geq 0 \\ \sqrt{\lambda} & \text{if } \omega \operatorname{Im} V < 0 . \end{array} \right.$$

Then, using (9.57) together with the estimate

$$\nu \eqsim \lambda \left(u_{\max} - u_-^L\right)^2 ,$$

we obtain

$$\int_{u_-^L}^{u_{\max}} |E_3| + |E_4| \lesssim \sqrt{\lambda} \int_{u_-^L}^{u_-^R} \frac{1}{\sqrt{\lambda}\, \left(u_{\max} - u_-^L\right)} \lesssim 1 .$$

Combining the above estimates, we conclude that

$$\int_{u_-^L}^{u_{\max}} |E_1| + |E_2| + |E_3| + |E_4| \lesssim \mathfrak{C}_8 \,.$$

As a consequence, the function T is bounded by $e^{\mathfrak{C}_8}$. It follows that the inequality

$$g \geq T - 1 \qquad \text{if } \operatorname{Im} V < 0$$

is satisfied for large λ.

Having verified the hypothesis of [20, Theorem 3.3], we can apply this theorem to obtain (10.2) and (10.3). The inequality (10.4) is derived as follows. At u_-^L, this inequality clearly holds in view of the WKB approximation (10.1). Expressing $\acute{\phi}(u)$ as

$$\phi(u) = \phi\big(u_-^L\big)\, \exp\bigg(\int_{u_-^L}^{u} y \bigg),$$

it remains to show that the integral in the exponent is uniformly bounded. To this end, we use (10.2) to obtain the estimate

$$\int_{u_-^L}^{u_{\max}} |y| \leq \mathfrak{C}_9 \,\big|\acute{y}(u_-^L)\big|\, \big(u_{\max} - u_-^L\big) \lesssim \mathfrak{C}_9 \sqrt{|V(u_-^L)|}\, \big(u_{\max} - u_-^L\big) \lesssim \mathfrak{C}_9 \mathfrak{C}_4 \,,$$

where we employed the WKB approximation at u_-^L and applied Lemma 9.11. This concludes the proof. ☐

In the remaining Airy case, we need to consider the Airy region and the WKB region with $\operatorname{Re} V > 0$. We begin with the Airy region.

Lemma 10.2. *In the Airy case, there is a constant $\mathfrak{C}_9$ such that on the interval $[u_-^L, u_+^L]$, the solutions $\acute{y}$ and $\acute{\phi}$ are bounded in terms of its values at u_-^L by*

$$|\acute{y}(u)| \leq \mathfrak{C}_9 \,\big|\acute{y}(u_-^L)\big|$$

$$\operatorname{Im} \acute{y}(u) \geq \frac{\operatorname{Im} \acute{y}(u_-^L)}{\mathfrak{C}_9}$$

$$\frac{\big|\acute{\phi}(u_-^L)\big|}{\mathfrak{C}_9} \leq \big|\acute{\phi}(u)\big| \leq \mathfrak{C}_9 \,\big|\acute{\phi}(u_-^L)\big| \,.$$

Proof. Our strategy is to estimate y using the T-method as introduced in [20, Section 3.2]. More precisely, we shall apply [20, Theorem 3.3] for a suitable function g. Moreover, setting

$$\nu = \sup_{[u_-^L, u_+^L]} |V| \,,$$

we again choose α and $\tilde{\beta}$ according to (10.5). Then $\tilde{V}$ and U are again given by (10.6). Moreover, the error terms $E_1, \ldots, E_4$ can again be estimated as in (10.7). The integral over the error term E_1 can estimated by

$$\int_{u_-^L}^{u_+^L} |E_1| \lesssim \sqrt{\nu}\, \big(u_+^L - u_-^L\big) \bigg(1 + \sup_{[u_-^L, u_+^L]} \frac{|\operatorname{Re} V'|}{\nu^{\frac{3}{2}}} \bigg). \tag{10.9}$$

The factor $\sqrt{\nu}\, (u_+^L - u_-^L)$ was estimated in Lemma 9.9. The factor $\operatorname{Re} V'|/\nu^{\frac{3}{2}}$, on the other hand, is bounded at the left end point u_-^L because of the WKB estimate in Proposition 9.4. This bound can be extended to the interval (u_-^L, u_+^L) again by using the estimate (10.8).

In order to control the error terms E_3 and E_4, we again distinguish the two cases in (9.15). In case **(b)**, we know from Lemma 9.12 that $\operatorname{Im} V$ and $\operatorname{Im} \acute{y}$ have the same signs, making it possible to choose $g \equiv 0$. Hence

$$\int_{u_-^L}^{u_+^L} |E_3| + |E_4| \lesssim \int_{u_-^L}^{u_+^L} \frac{|\operatorname{Im} V|}{\sqrt{\nu}} \overset{(9.7)}{\lesssim} \frac{|\omega|}{\sqrt{\nu}} \left(u_+^L - u_-^L\right) = \frac{|\omega|}{\nu} \sqrt{\nu} \left(u_+^L - u_-^L\right) .$$

the factor $\sqrt{\nu}\,(u_+^L - u_-^L)$ was estimated in Lemma 9.9. Moreover, the factor ν can be estimated with the help of (9.29) by

$$\nu \geq \frac{\gamma\, \mathscr{C}_3}{\mathfrak{c}\, \mathscr{C}_2}\, |\omega|^{\frac{4}{3}} .$$

It follows that

$$\int_{u_-^L}^{u_+^L} |E_3| + |E_4| \lesssim \frac{\mathfrak{c}\, \mathscr{C}_2}{\gamma\, \mathscr{C}_3} \frac{|\omega|}{|\omega|^{\frac{4}{3}}} \lesssim \frac{\mathscr{C}_2}{\mathscr{C}_3}\, |\omega|^{-\frac{1}{3}} ,$$

which can be made arbitrarily small by increasing $\mathscr{C}_3$.

The remaining case **(a)** is more subtle. We begin by proving the inequality

$$-\operatorname{Re} V\big(u_-^L\big) \gtrsim \mathscr{C}_3^{\frac{3}{2}}\, \mathscr{C}_1^{-\frac{1}{4}}\, \sqrt{\lambda} . \tag{10.10}$$

To this end, we integrate (9.8) to obtain

$$\operatorname{Re} V(u_{\max}) \eqsim \lambda\, v_0^2 .$$

As a consequence, using (9.10), (9.12) and (9.17), we obtain

$$\begin{aligned} v_0 - v_- &= \mathscr{C}_3 \big(\mathscr{C}_1 \operatorname{Re} V(u_{\max})\big)^{-\frac{1}{6}}\, |\omega|^{-\frac{1}{3}} \\ &\eqsim \mathscr{C}_3\, \mathscr{C}_1^{-\frac{1}{6}}\, \lambda^{-\frac{1}{6}}\, |v_0|^{-\frac{1}{3}}\, |\omega|^{-\frac{1}{3}} \eqsim \mathscr{C}_3\, \mathscr{C}_1^{-\frac{1}{6}}\, |v_0|^{-\frac{1}{3}}\, \lambda^{-\frac{1}{3}} \end{aligned}$$

and thus

$$(v_0 - v_-)\, |v_0|^{\frac{1}{3}} \eqsim \mathscr{C}_3\, \mathscr{C}_1^{-\frac{1}{6}}\, \lambda^{-\frac{1}{3}} .$$

Hence

$$\begin{aligned} -\operatorname{Re} V(v_-) &\eqsim \lambda\, |v_0 - v_-|\, |v_0 + v_-| \geq \lambda\, |v_0 - v_-|^{\frac{3}{2}}\, |v_0 + v_-|^{\frac{1}{2}} \\ &\eqsim \lambda \left((v_0 - v_-)\, |v_0|^{\frac{1}{3}} \right)^{\frac{3}{2}} \eqsim \lambda\, \mathscr{C}_3^{\frac{3}{2}}\, \mathscr{C}_1^{-\frac{1}{4}}\, \lambda^{-\frac{1}{2}} = \mathscr{C}_3^{\frac{3}{2}}\, \mathscr{C}_1^{-\frac{1}{4}}\, \sqrt{\lambda} , \end{aligned}$$

giving (10.10).

Next, we know from Lemma 9.12 that $\operatorname{Im} V$ and $\operatorname{Im} \acute{y}$ have the same signs, except for the error terms in (9.57). We choose

$$g(u) = \left\{ \begin{array}{cl} 0 & \text{if } \omega\ \operatorname{Im} V \geq 0 \\ \sqrt{\lambda} & \text{if } \omega\ \operatorname{Im} V < 0 . \end{array} \right.$$

Then, using (9.57) and (10.7), we obtain

$$\int_{u_-^L}^{u_+^L} |E_3| + |E_4| \lesssim \sqrt{\nu}\, (u_+^L - u_-^L) \left(\sup_{[u_-^L, u_+^L]} \frac{|\operatorname{Im} V|}{\nu} + \frac{\sqrt{\lambda}}{\nu} \right) . \tag{10.11}$$

The prefactor $\sqrt{\nu}\,(u_+^L - u_-^L)$ was estimated in Lemma 9.9. Moreover, using (10.10), we find that the factor $\sqrt{\lambda}/\nu$ in (10.11) is bounded by a constant. In order to estimate the term $|\operatorname{Im} V|/\nu$, we compare the equations (9.58) and (9.59) to obtain

$$\frac{|\operatorname{Im} V|}{\nu} \lesssim \frac{|\omega|}{\lambda} \, \frac{|\operatorname{Re} V'|}{\nu} \lesssim \frac{|\omega|}{\lambda} \, \sqrt{\nu} \, \frac{|\operatorname{Re} V'|}{\nu^{\frac{3}{2}}} \overset{(9.16)}{\lesssim} \frac{\sqrt{\nu}}{\sqrt{\lambda}} \, \frac{|\operatorname{Re} V'|}{\nu^{\frac{3}{2}}} \,.$$

The factor ν/λ can be estimated with the help of (9.7) and (6.5) by

$$\begin{aligned} \frac{\nu}{\lambda} &\leq \frac{1}{\lambda} \sup_{[u_-^L, u_+^L]} |\operatorname{Im} V| + \frac{u_+^L - u_-^L}{\lambda} \sup_{[u_-^L, u_+^L]} |\operatorname{Re} V'| \\ &\lesssim \frac{|\omega|}{\lambda} + \frac{\lambda + |\omega|}{\lambda} \left(u_+^L - u_-^L \right) , \end{aligned}$$

which is uniformly bounded in view of (9.16) and (9.10). In order to estimate the remaining factor $|\operatorname{Re} V'|/\nu^{\frac{3}{2}}$, we first note that this factor is uniformly bounded at u_-^L because of the WKB approximation (see Lemma 9.4). In order to extend this inequality to $u \in [u_-^L, u_+^L]$, we make use of the monotonicity (see (9.19), (9.10) and (9.8))

$$0 \leq \operatorname{Re}' V(u) \leq \operatorname{Re} V(u_-^L) \,.$$

We conclude that (10.11) is uniformly bounded.

Combining the above estimates, we obtain in case **(a)**,

$$\int_{u_-^L}^{u_+^L} |E_1| + |E_2| + |E_3| + |E_4| \lesssim \mathcal{C}_8 \,.$$

As a consequence, the function T is bounded by $e^{\mathcal{C}_8}$. It follows that the inequality

$$g \geq T - 1 \qquad \text{if } \operatorname{Im} V < 0$$

is satisfied for large λ.

Having verified the hypothesis of [20, Theorem 3.3], we can proceed exactly as in the proof of Lemma 10.1. This concludes the proof. □

Lemma 10.3. *There are constants* $\mathcal{C}_{10}$ *and* $c = c(\lambda, \omega)$ *such that in the WKB region with* $\operatorname{Re} V > 0$ *the following inequality holds on the interval* $[u_+^L, u_{\max}]$*,*

$$\frac{|\acute{\phi}(u)|}{\mathcal{C}_{10}} \leq \frac{c(\lambda, \omega)}{(\operatorname{Re} V(u))^{\frac{1}{4}}} \, e^{\int_{u_+^L}^u \operatorname{Re} \sqrt{V}} \leq \mathcal{C}_{10} \, |\acute{\phi}(u)| \,.$$

Proof. In Proposition 9.8 it was shown that the WKB conditions are satisfied on the interval $[u_+, u_{\max}]$. Thus the solution is well-approximated by the WKB solution

$$\acute{\phi} \approx \frac{1}{(\operatorname{Re} V)^{\frac{1}{4}}} \left(C_1 \, e^{\int_{u_+^L}^u \sqrt{V}} + C_2 \, e^{-\int_{u_+^L}^u \sqrt{V}} \right), \tag{10.12}$$

with error terms which are under control in view of the estimates in [20]. Note that one of the fundamental solutions in (10.12) is exponentially increasing, whereas the other is exponentially decaying.

The estimate of Proposition 9.4 implies that the WKB approximation (10.1) holds at u_-^L with an arbitrarily small error. Moreover, the estimates of Lemma 10.2 give uniform control of the solution on the interval $[u_-^L, u_+^L]$. As a consequence, the coefficient C_1 of the exponentially increasing fundamental solution in (10.12) is bounded away from zero. This gives the result. □

10.2. **Estimates of $\grave{\phi}$.** We again restrict attention to the parameter range (9.2). We consider the solutions $\grave{\phi}_-$ constructed in Theorem 6.2. For ease in notation, we shall omit the index $-$. We denote the corresponding solution of the Riccati equation by

$$\grave{y}(u) := \frac{\grave{\phi}'(u)}{\grave{\phi}(u)} .$$

According to Proposition 9.5, the WKB approximation holds on the interval (u_-^R, ∞), meaning that

$$\grave{\phi} \approx \frac{1}{\sqrt[4]{-V}} \exp\left(\pm i \int^u \sqrt{-V} \right) \qquad \text{and} \qquad \grave{y} \approx \pm i \sqrt{-V} - \frac{V'}{4V} \tag{10.13}$$

with an arbitrarily small error, where the sign is chosen such that

$$\lim_{u \to \infty} \pm\sqrt{-V} = -\omega .$$

Moreover, the integration constant is chosen in agreement with the normalization convention in (6.10). The goal of this section is to estimate the solution $\grave{\phi}$ backwards in u all the way to $u = u_{\max}$.

We again begin with the parabolic cylinder case:

Lemma 10.4. *In the parabolic cylinder case, there is a constant $\mathcal{C}_9$ such that on the interval $[u_{\max}, u_-^R]$, the solutions $\grave{y}$ and $\grave{\phi}$ are bounded in terms of its values at u_-^R by*

$$|\grave{y}(u)| \le \mathcal{C}_9 \left|\grave{y}(u_-^R)\right|$$
$$\text{Im}\, \grave{y}(u) \ge \frac{\text{Im}\, \grave{y}(u_-^R)}{\mathcal{C}_9}$$
$$\frac{\left|\grave{\phi}(u_-^R)\right|}{\mathcal{C}_9} \le \left|\grave{\phi}(u)\right| \le \mathcal{C}_9 \left|\grave{\phi}(u_-^R)\right| .$$

Proof. Follows exactly as in the proof of Lemma 10.1. □

In the remaining Airy case, we again need to consider the Airy region and the WKB region with $\text{Re}\, V > 0$:

Lemma 10.5. *In the Airy case, there is a constant $\mathcal{C}_9$ such that on the interval $[u_+^R, u_-^R]$, the solutions $\grave{y}$ and $\grave{\phi}$ are bounded in terms of its values at u_-^R by*

$$|\grave{y}(u)| \le \mathcal{C}_9 \left|\grave{y}(u_-^R)\right|$$
$$\text{Im}\, \grave{y}(u) \ge \frac{\text{Im}\, \grave{y}(u_-^R)}{\mathcal{C}_9}$$
$$\frac{\left|\grave{\phi}(u_-^R)\right|}{\mathcal{C}_9} \le \left|\grave{\phi}(u)\right| \le \mathcal{C}_9 \left|\grave{\phi}(u_-^R)\right| .$$

Proof. Our strategy is to estimate y using the T-method in [20, Theorem 3.3]. We set

$$\nu = \sup_{[u_+^R, u_-^R]} |V|$$

and choose α and $\tilde{\beta}$ again according to (10.5). The function g will be specified below. Then $\tilde{V}$ and U are again given by (10.6). Moreover, the error terms $E_1, \ldots, E_4$ are again estimated by (10.7). The integral over the error term E_1 can be estimated exactly as explained after (10.9).

In order to control the error terms E_3 and E_4, we again distinguish the two cases in (9.15). In case **(a)**, we can proceed exactly as in the proof of Lemma 10.2. In the remaining case **(b)**, we know from Lemma 9.12 that $\mathrm{Im}\, V$ and $\mathrm{Im}\, \grave{y}$ have opposite signs, making it possible to choose $g \equiv 0$. Next, from (9.37), we know that

$$\nu \gtrsim \frac{\lambda}{\mathfrak{C}_1} \, \frac{\big(u_-^R - u_0^R\big)\big(u_-^R + u_0^R\big)}{\big(u_-^R\big)^2 \big(u_0^R\big)^2} \overset{(9.32),(9.30)}{\gtrsim} \frac{\lambda}{\mathfrak{C}_1} \, \frac{u_-^R - u_0^R}{\big(u_0^R\big)^3} \,.$$

It follows that

$$\begin{aligned} \int_{u_+^R}^{u_-^R} \frac{|\mathrm{Im}\, V|}{\sqrt{\nu}} &\overset{(9.39)}{\lesssim} |\omega| \, \frac{\sqrt{\mathfrak{C}_1}}{\sqrt{\lambda}} \, \sqrt{u_0^R} \, \sqrt{u_-^R - u_+^R} \\ &\overset{(9.32),(9.30),(9.49)}{\lesssim} |\omega| \, \frac{\sqrt{\mathfrak{C}_1}}{\sqrt{\lambda}} \, \frac{\lambda^{\frac{1}{4}}}{|\omega|^{\frac{1}{2}}} \, \sqrt{\mathfrak{C}_3} \, \lambda^{\frac{1}{12}} \, |\omega|^{-\frac{1}{2}} = \sqrt{\mathfrak{C}_1 \mathfrak{C}_3} \, \lambda^{-\frac{1}{6}} \,, \end{aligned}$$

which can be made arbitrarily small by increasing $\mathfrak{C}_6$. This concludes the proof. ☐

Lemma 10.6. *There are constants $\mathfrak{C}_{10}$ and $c = c(\lambda, \omega)$ such that in the WKB region region with $\mathrm{Re}\, V > 0$ the following inequality holds on the interval $[u_{\max}, u_+^R]$,*

$$\frac{|\grave{\phi}(u)|}{\mathfrak{C}_{10}} \gtrsim \frac{c(\lambda, \omega)}{(\mathrm{Re}\, V(u))^{\frac{1}{4}}} \; e^{-\int_{u_+^R}^u \mathrm{Re} \sqrt{V}} \le \mathfrak{C}_{10} \, |\grave{\phi}(u)| \,.$$

Proof. We proceed similar as in Lemma 10.3. According to Proposition 9.8 we know that on the interval $[u_{\max}, u_+^R]$ the WKB approximation

$$\grave{\phi} \approx \frac{1}{(\mathrm{Re}\, V)^{\frac{1}{4}}} \left(C_1 \, e^{\int_{u_+^R}^u \sqrt{V}} + C_2 \, e^{-\int_{u_+^R}^u \sqrt{V}} \right)$$

holds with an arbitrarily small error. By combining the results of Proposition 9.5 and Lemma 10.5, we conclude that the coefficient C_2 of the exponentially decaying fundamental solution is bounded away from zero. ☐

10.3. **Estimates for Bounded ω and Large λ.** In Section 9 we restricted attention to the case that $|\omega|$ is large (see (9.2)). We now consider the complimentary region that ω is in a bounded set, but again for large λ. We exclude the case $\omega = 0$, which will be considered separately in Section 10.4. We thus consider the parameter range

$$0 \neq \omega^2 < \mathfrak{C}_6 \qquad \text{and} \qquad \lambda \geq \mathfrak{C}_7 \,. \tag{10.14}$$

Choosing $\mathfrak{C}_7$ sufficiently large, the potential looks qualitatively as in the Airy case in Section 9. The real part of the potential is negative both at $u = \pm\infty$ with the asymptotics (6.6) and (6.7). Since the summand involving λ in (6.4) is non-negative, for large λ the real part of the potential will be non-negative on an interval (u_0^L, u_0^R) whose size tends to infinity as $\lambda \to \infty$. We now work out the resulting estimates in detail. The only major change compared to the estimates in Section 9 and Sections 10.1 and 10.2 is that for large negative u we must approximate $\acute{\phi}$ by Bessel functions and must derive suitable error estimates.

Clearly, the real part of the potential again has a unique maximum $u_{\max}$. We begin with the estimates in the region $(u_{\max}, \infty)$. Expanding the potential (6.5) gives (see also (9.31) and (6.6))

$$\mathrm{Re}\, V = -\omega^2 + \frac{\tilde{\lambda}}{u^2} + \mathcal{O}\big(u^{-3}\big) \tag{10.15}$$

$$\mathrm{Im}\, V = -\frac{2s\omega}{u} + \mathcal{O}\big(u^{-2}\big) \,. \tag{10.16}$$

The connection to the situation of large ω considered in Section 9 can be understood directly from the following scaling argument. Suppose that $\phi(u)$ is a solution of the Sturm-Liouville equation (6.4). Then, introducing the new variable $\tilde{u} = \omega u$ and the function $\tilde{\phi}(\tilde{u}) = \phi(u)$, we obtain

$$\frac{d^2}{d\tilde{u}^2} \tilde{\phi}(\tilde{u}) = \frac{1}{\omega^2}\, \phi''(u) = \frac{1}{\omega^2}\, V(u)\, \phi(u) = \tilde{V}(\tilde{u})\, \tilde{\phi}(\tilde{u}) \,,$$

where the new potential $\tilde{V}(\tilde{u})$ has the asymptotics

$$\mathrm{Re}\, \tilde{V}(\tilde{u}) = \frac{1}{\omega^2}\, \mathrm{Re}\, V\Big(\frac{\tilde{u}}{\omega}\Big) = -1 + \frac{\tilde{\lambda}}{\tilde{u}^2} + \mathcal{O}\big(\tilde{u}^{-3}\big)$$
$$\mathrm{Im}\, \tilde{V}(\tilde{u}) = -\frac{2s}{\tilde{u}} + \mathcal{O}\big(\tilde{u}^{-2}\big) \,.$$

This means that the potential looks just as before, but with ω replaced by one. Hence the above scaling argument makes it possible to change ω arbitrarily. This explains why the methods in Section 9 and Section 10.2 again apply. The situation is even a bit easier because we are in case **(b)** in (9.15). For clarity, we summarize these estimates: According to (10.15), the function $\mathrm{Re}\, V$ has a unique zero for large u,

$$\mathrm{Re}\, V(u_0^R) = 0 \,, \qquad u_0^R = \frac{\omega^2}{\tilde{\lambda}} + \mathcal{O}\big(\lambda^{-2}\big) \,.$$

As in (9.30) and (9.49) we set

$$u_\pm^R = u_0^R \mp \mathcal{C}_3\, \lambda^{\frac{1}{6}}\, |\omega|^{-1} \,.$$

Then the results of Proposition 9.5 and Lemma 9.10 remain true for all ω and λ in the range (10.14). As a consequence, the fundamental solution $\grave{\phi}$ satisfies on the interval (u_-^R, ∞) the WKB approximation (10.13), again with an arbitrarily small error. Moreover, the behavior on the interval (u_+^R, u_-^R) can be estimated as in Lemma 10.5. Finally, on the interval $(u_{\max}, u_-^R)$ one can estimate the solution exactly as in Lemma 10.6.

We come to the estimates in the region $(-\infty, u_{\max})$. Near $u = -\infty$, the potential has the asymptotic form (see also (6.7))

$$V(u) = -\Omega^2 + c_1 \big(\lambda + \nu\big)\, e^{\gamma u} + c_2\, \lambda\, e^{2\gamma u} + \mathcal{O}\big(e^{2\gamma u}\big) \,, \tag{10.17}$$

where c_1 is the positive constant

$$c_1 = \frac{r_1 - r_0}{a^2 + r_1^2} \,,$$

c_2 is a real constant, and $\nu = \nu(\omega)$ is a linear polynomial in ω with complex coefficients. The real part of the potential again has a unique zero for large negative u,

$$\mathrm{Re}\, V(u_0^L) = 0 \,, \qquad u_0^L = \frac{1}{\gamma}\, \log\bigg(\frac{\Omega^2}{c_1\, (\lambda + \nu)}\bigg) \Big(1 + \mathcal{O}\big(\lambda^{-1}\big)\Big) \,.$$

Obviously, u_0^L tends to $-\infty$ as $\lambda \to \infty$. In order to see the basic difference to the estimates in Section 9, let us consider the situation that $u \ll u_0^L$ and $\Omega = 0$. Then $V \approx c_1\big(\lambda+\nu\big)\, e^{\gamma u}$. Evaluating the expressions in the WKB conditions, we obtain

$$\frac{|V'|}{|V|^{\frac{3}{2}}} \approx \frac{c_2}{(\lambda+\nu)^{\frac{1}{2}}}\, e^{-\frac{\gamma u}{2}}\,, \qquad \frac{|V''|}{|V|^2} \approx \frac{c_3}{\lambda+\nu}\, e^{-\gamma u}\,.$$

Since the exponential factors increase exponentially as $u \to -\infty$, the WKB conditions fail if $u \lesssim -\log\lambda$. In particular, in the limiting case $\Omega = 0$, the WKB approximation does not apply near $u = -\infty$. This is why the methods in Section 9 and Section 10.1 no longer apply. Instead, we use the κ-method introduced in [20, Section 3.3] to obtain the following result:

Proposition 10.7. *There are constants $\mathfrak{c}_0, \mathfrak{c}_1 > 0$ such that for all ω and λ in the range* (10.14)*, the solution $\acute{y}(u)$ in Theorem 6.1 satisfies the estimate*

$$\operatorname{Re} \acute{y}(u) \geq \frac{\sqrt{\lambda}}{\mathfrak{c}_1}\, e^{\frac{\gamma u}{2}} - \mathfrak{c}_1 \tag{10.18}$$

for all u in the interval

$$u \in (-\infty, u_{\min}] \qquad \textit{with} \qquad u_{\min} = -\frac{\log\lambda}{2\gamma} - \mathfrak{c}_0\,. \tag{10.19}$$

Before coming to the proof of this proposition, we explain its significance and work out an application. To this end, we evaluate the inequality (10.18) at $u = u_{\min}$. Then

$$\lambda\, e^{\gamma u} = \lambda\, e^{-\mathfrak{c}_0}\, e^{-\frac{\log\lambda}{2}} = \frac{\sqrt{\lambda}}{e^{\mathfrak{c}_0}}\,, \tag{10.20}$$

which can be made arbitrarily large by increasing λ (note that the factor 1/2 in (10.19) is essential). As a consequence, at $u_{\min}$ the summand $\lambda\, e^{\gamma u}$ dominates the potential (10.17). This also implies that we are in the WKB regime. The inequality (10.18) shows that in this regime, the solution $\acute{y}$ has a large real part, meaning that $\acute{\phi}$ is well-approximated by the exponentially *increasing* fundamental solution. This gives rise to the following result:

Lemma 10.8. *There are constants $\mathfrak{C}_{10}$ and $c = c(\lambda,\omega)$ such that on the interval $(u_{\min}, u_{\max})$ the following inequalities hold*

$$\frac{|\acute{\phi}(u)|}{\mathfrak{C}_{10}} \leq \frac{c(\lambda,\omega)}{(\operatorname{Re} V(u))^{\frac{1}{4}}}\, e^{-\int_{-\infty}^u \operatorname{Re}\sqrt{V}} \leq \mathfrak{C}_{10}\, |\acute{\phi}(u)|\,,$$

for all ω and λ in the range (10.14)*.*

Proof. We proceed similar as in the proof of Lemma 10.3. As shown after (10.20), the WKB conditions are satisfied at $u_{\min}$. Moreover, as in Proposition 9.8 one verifies that the WKB conditions are also satisfied on the interval $(u_{\min}, u_{\max})$. Thus on this interval, the WKB approximation

$$\acute{\phi} \approx \frac{1}{(\operatorname{Re} V)^{\frac{1}{4}}} \left(C_1\, e^{\int_{u_+^R}^u \sqrt{V}} + C_2\, e^{-\int_{u_+^R}^u \sqrt{V}} \right)$$

holds with an error which can be made arbitrarily small by increasing λ. The inequality (10.18) implies that the coefficient C_2 of the exponentially increasing fundamental solution is bounded away from zero. ◻

The remainder of this section is devoted to the proof of Proposition 10.7. According to (9.1), we know that $\mathrm{Im}\,\Omega < 0$. Moreover, it suffices to consider the case

$$\mathrm{Re}\,\Omega \geq 0\,,$$

because the case $\mathrm{Re}\,\Omega < 0$ can be treated in exactly the same way by considering the complex conjugate equation. Then the solution $\acute{\phi}$ with the asymptotics (6.9) starts at $u = -\infty$ in the upper half plane. But, depending on the sign of the imaginary part of the parameter ν in (10.17), its imaginary part could change signs. Thus there might be $u_{\mathrm{flip}} \in (-\infty, u_{\min})$ such that

$$\mathrm{Im}\,y\big|_{(-\infty, u_{\mathrm{flip}})} \geq 0 \qquad \text{and} \qquad \mathrm{Im}\,y\big|_{(u_{\mathrm{flip}}, u_{\min})} \leq 0\,. \tag{10.21}$$

In order to treat all possible cases at once, in the case that $\mathrm{Im}\,y$ does not change signs, we again work with (10.21) but choose $u_{\mathrm{flip}} = u_{\min}$.

We choose the approximate potential as

$$\tilde{V}(u) = -\tilde{\Omega}^2 + c_1\big(\lambda + \tilde{\nu}\big)\, e^{\gamma u} \qquad \text{with} \qquad \tilde{\nu} := i\,|\,\mathrm{Im}\,\nu| + i \tag{10.22}$$

and $\tilde{\Omega} \in \mathbb{C}$ to be determined below. The corresponding Sturm-Liouville equation $\tilde{\phi}''(u) = \tilde{V}(u)\,\tilde{\phi}(u)$ can be solved explicitly in terms of Bessel functions. Similar to (6.9), we want to arrange the asymptotic behavior

$$\tilde{\phi}(u) \sim e^{i\tilde{\Omega} u} \qquad \text{as } u \to -\infty\,.$$

This gives the unique solution

$$\tilde{\phi}(u) = e^{\frac{i\tilde{\Omega}}{\gamma}\left(-\log\left(c_1(\lambda+\tilde{\nu})\right) + 2\log\gamma\right)}\, \Gamma\Big(1 + \frac{2i\tilde{\Omega}}{\gamma}\Big)\, I_{\frac{2i\tilde{\Omega}}{\gamma}}\Big(\frac{2}{\gamma}\,\sqrt{c_1(\lambda+\tilde{\nu})\,e^{\gamma u}}\Big)\,. \tag{10.23}$$

This solution is well-defined and regular. It is well-behaved in the limit $\tilde{\Omega} \to 0$. The corresponding solution of the Riccati equation

$$\tilde{y} := \frac{\tilde{\phi}'(u)}{\tilde{\phi}(u)}$$

is also well-defined and smooth and has the asymptotics

$$\lim_{u \to -\infty} \tilde{y}(u) = i\tilde{\Omega}\,. \tag{10.24}$$

Next, we choose

$$\mathrm{Re}\,\tilde{\Omega} = (1+\delta)\ \mathrm{Re}\,\Omega \geq 0 \qquad \text{and} \qquad \mathrm{Im}\,\tilde{\Omega} = \frac{\mathrm{Im}\,\Omega}{1+\delta} < 0 \tag{10.25}$$

for a parameter $\delta > 0$ which later on we will choose sufficiently small. Using these inequalities in (10.24), one sees that the solution $\tilde{y}$ starts at $u = -\infty$ in the upper half plane. Moreover, as $\mathrm{Im}\,\tilde{V} > 0$ (see (10.22) and again (10.25)), we conclude that $\tilde{y}$ stays in the upper half plane,

$$\mathrm{Im}\,\tilde{y} \geq 0 \qquad \text{for all } u \in \mathbb{R}\,.$$

In order to get more detailed information on $\tilde{y}$, it is useful to again consider the WKB approximation and expand it for large λ,

$$\tilde{\phi}_{\text{WKB}} := \frac{1}{\sqrt[4]{\tilde{V}}}\, e^{\int^u \sqrt{\tilde{V}}} \tag{10.26}$$

$$\begin{aligned}
\tilde{y}_{\text{WKB}} &:= \frac{\tilde{\phi}'_{\text{WKB}}}{\tilde{\phi}_{\text{WKB}}} = \sqrt{\tilde{V}} - \frac{\tilde{V}'}{4\tilde{V}} \\
&= \sqrt{-\tilde{\Omega}^2 + c_1\big(\lambda+\tilde{\nu}\big)\, e^{\gamma u}} - \frac{1}{4}\, \frac{c_1\big(\lambda+\tilde{\nu}\big)\,\gamma\, e^{\gamma u}}{-\tilde{\Omega}^2 + c_1\big(\lambda+\tilde{\nu}\big)\, e^{\gamma u}} \\
&= \sqrt{c_1 \lambda}\, e^{\frac{\gamma u}{2}} - \frac{\gamma}{4} + \frac{i\mathcal{K}}{\sqrt{\lambda}}\, e^{\frac{\gamma u}{2}} + \mathcal{O}\big(\lambda^{-\frac{1}{2}}\big)
\end{aligned} \tag{10.27}$$

$$\text{Re}\,\big(\tilde{y}_{\text{WKB}}\big)\;\text{Im}\,\big(\tilde{y}_{\text{WKB}}\big) = \frac{1}{2}\;\text{Im}\,\big(\tilde{y}_{\text{WKB}}^2\big) \approx \frac{1}{2}\;\text{Im}\,\big(\tilde{V}\big)\,. \tag{10.28}$$

Clearly, this WKB approximation only applies if $\lambda e^{\gamma u} \gg 1$. Even in this regime, it is not at all obvious that $\tilde{y}_{\text{WKB}}$ approximates $\tilde{y}$. Namely, the function $\tilde{\phi}$ is in general a linear combination of the WKB solution in (10.26) and the other, exponentially decaying WKB solution. As a consequence, the function $\tilde{y}$ could have a different form. It turns out that, using the explicit form of (10.23), the function $\tilde{y}_{\text{WKB}}$ does describe the qualitative behavior of the solution correctly, as is made precise in the following lemma.

Lemma 10.9. *There are constants $\mathfrak{c}_2$, $\mathfrak{c}_3$, $\mathfrak{c}_4$ such that for all ω and λ in the range* (10.14)*, the functions*

$$\alpha := \text{Re}\,\tilde{y} \qquad \textit{and} \qquad \tilde{\beta} := \text{Im}\,\tilde{y} \tag{10.29}$$

satisfy on the interval $(-\infty, u_{\min}]$ (with $u_{\min}$ according to (10.19)*) the inequalities*

$$\frac{\sqrt{\lambda}}{\mathfrak{c}_2}\, e^{\frac{\gamma u}{2}} \leq \alpha \tag{10.30}$$

$$\frac{1}{\mathfrak{c}_3\, \alpha} \leq \tilde{\beta} \leq \mathfrak{c}_3 \tag{10.31}$$

$$\left|\frac{\tilde{\beta}'}{\tilde{\beta}}\right| \leq \mathfrak{c}_4\, e^{\frac{\gamma u}{2}}\,. \tag{10.32}$$

Proof. Writing the potential $\tilde{V}$, (10.22) as

$$\tilde{V}(u) = -\tilde{\Omega}^2 + c_1\left(1 + \frac{\tilde{\nu}}{\lambda}\right) e^{\gamma u + \log \lambda} = -\tilde{\Omega}^2 + \tilde{b}\, e^{\gamma v}$$

with

$$v := u + \frac{\log \lambda}{\gamma} \qquad \text{and} \qquad \tilde{b} := c_1\left(1 + \frac{\tilde{\nu}}{\lambda}\right), \tag{10.33}$$

one can arrange that the problem depends on two parameters $\tilde{\Omega}$ and $\tilde{b}$ which both lie in a bounded set. Indeed, by increasing the constant $\mathcal{C}_7$ in (10.14) one can even arrange that $\tilde{b}$ is arbitrarily close to c_1. Therefore, it suffices to analyze the perturbation of a one-parameter problem.

In the variable v, the fundamental solution $\tilde{\phi}$, (10.23), becomes

$$\tilde{\phi}(v) \sim I_{\frac{2i\tilde{\Omega}}{\gamma}}\left(\frac{2}{\gamma}\,\sqrt{\tilde{b}\, e^{\gamma v}}\right).$$

In the limit $v \to -\infty$, the argument of the Bessel function tends to zero. Using the power expansion of the Bessel functions (see [27, eq. (10.25.2)]), one obtains

$$\begin{aligned}\tilde{\phi}(v) &\sim e^{i\tilde{\Omega}v}\,\Big(a_0 + a_1\, e^{\gamma v}\Big) + \mathcal{O}\big(e^{2\gamma v}\big) \\ \tilde{\phi}'(v) &\sim i\tilde{\Omega}\,\tilde{\phi}(v) + e^{i\tilde{\Omega}v}\,\gamma a_1\, e^{\gamma v} + \mathcal{O}\big(e^{2\gamma v}\big) \\ \tilde{y}(v) &= i\tilde{\Omega} + \frac{\gamma a_1}{a_0}\, e^{\gamma v} + \mathcal{O}\big(e^{2\gamma v}\big)\,,\end{aligned}$$

where the complex coefficients a_0 and a_1 depend on $\tilde{\Omega}$ and $\tilde{b}$. These coefficients are given in terms of the gamma function, and one verifies explicitly that they are non-zero. This shows that the relations (10.30)–(10.32) hold for sufficiently small and negative v.

For large v, the Bessel function goes over to the exponentially increasing WKB approximation,

$$\tilde{\phi}(v) \approx \frac{c}{\sqrt[4]{\tilde{V}(v)}}\,\exp\bigg(\int^v \sqrt{\tilde{V}}\bigg) \tag{10.34}$$

$$\begin{aligned}\tilde{y}(v) &\approx \sqrt{\tilde{V}} - \frac{\tilde{V}'}{4\tilde{V}} = \sqrt{\tilde{b}\,e^{\gamma v} - \tilde{\Omega}^2} - \frac{\gamma \tilde{b}\,e^{\gamma v}}{4\,\big(\tilde{b}\,e^{\gamma v} - \tilde{\Omega}^2\big)} \\ &\approx \sqrt{\tilde{b}}\,e^{\frac{\gamma v}{2}} - \frac{\tilde{\Omega}^2}{2\sqrt{\tilde{b}}}\,e^{-\frac{\gamma v}{2}} - \frac{1}{4} - \frac{\tilde{\Omega}^2}{4b}\,e^{-\gamma v}\,,\end{aligned} \tag{10.35}$$

where $\approx$ means that we neglect higher orders in $e^{-\gamma v}$. Taking the real part of the last equation, we immediately obtain

$$\alpha \approx e^{\frac{\gamma v}{2}}\,, \tag{10.36}$$

giving in particular the lower bound (10.30). Taking the imaginary part of (10.35), we can make use of the fact that, according to (10.33), the parameter $\tilde{b}$ is real up to an error of order $1/\lambda$, so that

$$\tilde{\beta} \approx e^{\frac{\gamma v}{2}}\,\mathcal{O}\Big(\frac{1}{\lambda}\Big) - \mathrm{Im}\left(\frac{\tilde{\Omega}^2}{2\sqrt{\tilde{b}}}\right) e^{-\frac{\gamma v}{2}} - \frac{1}{4} - \mathrm{Im}\left(\frac{\tilde{\Omega}^2}{4b}\right)\,e^{-\gamma v}\,. \tag{10.37}$$

Using (10.20), we obtain at $v_{\min} := u_{\min} + (\log\lambda)/\gamma$ that

$$e^{\frac{\gamma v_{\min}}{2}}\,\mathcal{O}\Big(\frac{1}{\lambda}\Big) = \mathcal{O}\big(\lambda^{-\frac{3}{4}}\big)\,, \qquad e^{-\frac{\gamma v_{\min}}{2}} = \mathcal{O}\big(\lambda^{-\frac{1}{4}}\big)\,, \qquad e^{-\gamma v_{\min}} = \mathcal{O}\big(\lambda^{-\frac{1}{2}}\big)\,,$$

showing that the second summand in (10.37) dominates. Combining this estimate with the upper bound (10.36), we conclude that also (10.31) and (10.32) hold at $v = v_{\min}$.

Since $\mathrm{Im}\,\tilde{V} > 0$, we know furthermore that $\tilde{\beta}$ remains strictly positive. Using the validity of the inequalities (10.30)–(10.32) asymptotically as $v \to -\infty$ and at $v_{\min}$, we conclude that for every $\tilde{\Omega}$ and $\tilde{\beta}$, there are constants $\mathfrak{c}_2$, $\mathfrak{c}_3$ and $\mathfrak{c}_4$ such that (10.30)–(10.32) hold for all $v \in (-\infty, v_{\min}]$. Since the constants can be chosen continuously in the parameters $\tilde{\Omega}$ and $\tilde{\beta}$, it follows that the constants can be chosen uniformly for the parameters in any compact set. This concludes the proof. □

Comparing (10.17) and (10.22), we obtain

$$\begin{aligned}
\mathrm{Re}(V-\tilde V) &= -\,\mathrm{Re}\,\big(\Omega^2-\tilde\Omega^2\big)+\mathcal{O}\big(e^{\gamma u}\big)+\mathcal{O}\big(\lambda e^{2\gamma u}\big)\\
&\overset{(10.25)}{=} \big(2\delta+\delta^2\big)\left(\mathrm{Re}^2\,\Omega+\frac{\mathrm{Im}^2\,\Omega}{(1+\delta)^2}\right)+\mathcal{O}\big(e^{\gamma u}\big)+\mathcal{O}\big(\lambda e^{2\gamma u}\big)>0 \qquad (10.38)\\
\mathrm{Im}(V-\tilde V) &= -\,\mathrm{Im}\,\big(\Omega^2-\tilde\Omega^2\big)+c_1\,\mathrm{Im}\,\big(\nu-\tilde\nu\big)\,e^{\gamma u}+\mathcal{O}\big(e^{2\gamma u}\big)\\
&\overset{(10.25)}{=} c_1\,\mathrm{Im}\,\big(\nu-\tilde\nu\big)\,e^{\gamma u}+\mathcal{O}\big(e^{2\gamma u}\big)\\
&\overset{(10.22)}{=} c_1\big(\mathrm{Im}\,\nu-|\,\mathrm{Im}\,\nu|-1\big)\,e^{\gamma u}+\mathcal{O}\big(e^{2\gamma u}\big)<0\,, \qquad (10.39)
\end{aligned}$$

where the inequalities hold for sufficiently small $u_{\min}=u_{\min}(\delta)$.

We first consider the region $(-\infty, u_{\text{flip}})$ where $\mathrm{Im}\,y \geq 0$. Let us choose the function κ. Recall that this function is defined by

$$\kappa(u)=\frac{g(u)}{\sigma(u)}+\frac{1}{\sigma}\int_{-\infty}^{u}\sigma\,\mathrm{Im}(V-\tilde V)\,, \qquad (10.40)$$

where g can be any monotone increasing function. Since $\mathrm{Im}(V-\tilde V)<0$, we may choose $g(u)$ as

$$g(u)=-\int_{-\infty}^{u}\sigma\,\mathrm{Im}(V-\tilde V)\,,$$

to obtain

$$\kappa\equiv 0\,.$$

Then [20, Lemma 3.4] simplifies to

$$\kappa-R=\frac{-\,\mathrm{Re}(V-\tilde V)}{2\tilde\beta}\,.$$

As a consequence, the formula for the determinator [20, eqn (3.27)] can be rewritten as follows,

$$\begin{aligned}
\mathfrak{D} &= 2\alpha\,\mathrm{Re}(V-\tilde V)+\frac{1}{2}\,\mathrm{Re}(V-\tilde V)'+\tilde\beta\,\mathrm{Im}(V-\tilde V)+(\kappa-R)\,\mathrm{Im}\,V \qquad (10.41)\\
&= \left(2\alpha-\frac{\mathrm{Im}\,V}{2\tilde\beta}\right)\mathrm{Re}(V-\tilde V)+\mathcal{O}\big(e^{\gamma u}\big)+\mathcal{O}\big(\lambda e^{2\gamma u}\big)\,, \qquad (10.42)
\end{aligned}$$

where in the last line we used (10.31) and (10.39). Next,

$$\begin{aligned}
2\alpha-\frac{\mathrm{Im}\,V}{2\tilde\beta} &= \alpha+\frac{2\alpha\tilde\beta-\mathrm{Im}\,\tilde V}{2\tilde\beta}-\frac{\mathrm{Im}(V-\tilde V)}{2\tilde\beta}\overset{(10.39)}{\geq}\alpha+\frac{2\alpha\tilde\beta-\mathrm{Im}\,\tilde V}{2\tilde\beta}\\
&= \alpha+\frac{\mathrm{Im}\,\big(\tilde y^2-\mathrm{Im}\,\tilde V\big)}{2\tilde\beta}=\alpha-\frac{\mathrm{Im}\,\tilde y'}{2\tilde\beta}=\alpha-\frac{\tilde\beta'}{2\tilde\beta}\geq\frac{\sqrt{\lambda}}{2\mathfrak{c}_2}\,e^{\frac{\gamma u}{2}}\,,
\end{aligned}$$

where in the last step we applied (10.30) and (10.32) and increased the constant $\mathcal{C}_7$ in (10.14). Using (10.38), we conclude that

$$\mathfrak{D}=\frac{\sqrt{\lambda}}{\mathfrak{c}}\,e^{\frac{\gamma u}{2}}\Big(1+\mathcal{O}\big(e^{\gamma u}\big)+\mathcal{O}\big(\lambda\,e^{2\gamma u}\big)\Big)+\mathcal{O}\big(e^{\gamma u}\big)+\mathcal{O}\big(\lambda e^{2\gamma u}\big)$$

with a positive constant $\mathfrak{c}=\mathfrak{c}(\delta)$. By choosing $\mathfrak{c}_0$ in (10.19) sufficiently large, we can arrange that all the error terms are small on the interval $(-\infty, u_{\text{flip}})$. Thus the determinator is positive. We conclude that the invariant region estimate in [20, Proposition 3.5] applies, giving the estimate (10.18) on the interval $(u_{\text{flip}}, u_{\min})$.

It remains to consider the interval $(u_{\rm flip}, u_{\rm min})$. Since we want to apply again [20, Proposition 3.5], it is most convenient to take the complex conjugate of the equation. This corresponds to the replacements

$$\mathrm{Im}\, V \to -\mathrm{Im}\, V\,, \quad \mathrm{Im}\, \tilde{V} \to -\mathrm{Im}\, \tilde{V}\,, \quad \mathrm{Im}\, y \to -\mathrm{Im}\, y\,, \quad \mathrm{Im}\, \tilde{y} \to -\mathrm{Im}\, \tilde{y}\,, \ldots\,.$$

Then the solution $\tilde{y}$ is again in the upper half plane. The invariant circle is reflected at the real axis (corresponding to the transformation $\beta \to -\beta$). The only difference compared to the above analysis is that the factor $\mathrm{Im}(V - \tilde{V})$ in (10.40) is now positive, so that the integral in (10.40) is increasing. Therefore, we now choose $g \equiv 0$, implying that $\kappa \geq 0$. Using the formula for $\kappa - R$ in [20, Lemma 3.4], the last summand in (10.41) can be estimated by

$$\begin{aligned}(\kappa - R)\, \mathrm{Im}\, V &= \frac{\kappa^2 - \mathrm{Re}(V - \tilde{V})}{2\,(\tilde{\beta} + \kappa)}\, \mathrm{Im}\, V = \frac{\kappa^2}{2\,(\tilde{\beta} + \kappa)}\, \mathrm{Im}\, V - \frac{\mathrm{Re}(V - \tilde{V})}{2\,(\tilde{\beta} + \kappa)}\, \mathrm{Im}\, V \\ &\geq -\frac{\mathrm{Re}(V - \tilde{V})}{2\,(\tilde{\beta} + \kappa)}\, \mathrm{Im}\, V \geq -\frac{\mathrm{Re}(V - \tilde{V})}{2\tilde{\beta}}\, \mathrm{Im}\, V\,,\end{aligned}$$

where in the last line we used the fact that $\mathrm{Im}\, V \geq 0$ (otherwise the solution y would not have crossed the real axis), and that $\mathrm{Re}(V - \tilde{V})$ is positive according to (10.38). Thus we have estimated the determinator by the expression in (10.42), making it possible to proceed just as on the interval $(-\infty, u_{\rm flip})$ above. Note that, estimating the integral in (10.40) using (10.39) for $g \equiv 0$, one sees that

$$0 \leq \kappa \leq \mathfrak{c} \qquad \text{on } (-\infty, u_{\rm min}) \tag{10.43}$$

(where $\mathfrak{c}$ is again a constant which is uniform in ω and λ in the range (10.14); note that this inequality is trivial on the interval $(-\infty, u_{\rm flip})$ where $\kappa \equiv 0$).

In the above arguments we concluded that [20, Proposition 3.5] applies. It follows that the solution $\acute{y}$ lies inside the circle with center $m = \alpha + i\beta$ and radius R, where α as defined in (10.29) and

$$R + \beta = \tilde{\beta} + \kappa\,, \qquad R - \beta = \frac{U}{R + \beta}\,. \tag{10.44}$$

Here the function U is given by (see [20, eqns (3.3) and (3.17)],

$$U := \mathrm{Re}\, V - \alpha^2 - \alpha' = \mathrm{Re}(V - \tilde{V}) - \tilde{\beta}^2\,. \tag{10.45}$$

Let us analyze what our estimate means for the radius. Combining (10.44) and (10.45), we obtain

$$2R = \big(\tilde{\beta} + \kappa\big) + \frac{U}{\tilde{\beta} + \kappa} = \big(\tilde{\beta} + \kappa\big) + \frac{\mathrm{Re}(V - \tilde{V})}{\tilde{\beta} + \kappa} - \frac{\tilde{\beta}^2}{\tilde{\beta} + \kappa} \leq \big(\tilde{\beta} + \kappa\big) + \frac{\mathrm{Re}(V - \tilde{V})}{\tilde{\beta} + \kappa}\,,$$

where we used that the summand $\tilde{\beta} + \kappa$ is non-negative according to (10.31) and (10.43). Next, we know from (10.38) that the term $\mathrm{Re}(V - \tilde{V})$ is uniformly bounded and can be made arbitrarily small by decreasing δ. Also using that $\tilde{\beta}$ and κ are both positive (see again (10.31) and (10.43)), we conclude that

$$2R \leq \tilde{\beta} + \kappa + \frac{\mathfrak{c}\,\delta}{\tilde{\beta}}\,.$$

Applying the estimates (10.31) and (10.43), we conclude that

$$R \leq \mathfrak{c}\Big(1 + \frac{\delta}{\tilde{\beta}}\Big) \leq \mathfrak{c}\big(1 + \mathfrak{c}_3\, \delta\, \alpha\big) \,. \tag{10.46}$$

The fact that $\acute{y}$ lies inside the invariant circle gives the inequality $\mathrm{Re}\, \acute{y} \geq \alpha - R$. Combining this inequality with (10.46), we obtain

$$\mathrm{Re}\, \acute{y} \geq \big(1 - \mathfrak{c}\, \mathfrak{c}_3\, \delta\big)\alpha - \mathfrak{c} \,.$$

We choose δ so small that $\mathfrak{c}\mathfrak{c}_3\delta < 1/2$. Using (10.30) gives the inequality (10.18). This concludes the proof of Proposition 10.7.

10.4. **The Limit** $\omega \to 0$**.** In the construction of the Jost solution $\grave{\phi} := \grave{\phi}_-$ in Theorem 6.2 as well as in all the previous estimates of $\grave{\phi}$ we always assumed that $\omega \neq 0$. We now analyze the behavior of these Jost solutions in the limit $\omega \to 0$, coming from the lower half plane $\mathrm{Im}\, \omega < 0$. Before beginning, we point out that if λ is sufficiently large, the asymptotics for small ω is obtained immediately by taking the limit $\omega \to 0$ in the estimates of Sections 10.2 and 10.3. This can be understood directly by analyzing the WKB conditions: For $\omega = 0$, the asymptotics of the potential in (10.15) and (10.16) simplifies to

$$V(u) = \frac{\tilde{\lambda}}{u^2} + \mathcal{O}\big(u^{-3}\big) \,.$$

Hence

$$\frac{|V'|}{|V|^{\frac{3}{2}}} = \frac{2}{\sqrt{|\tilde{\lambda}|}} \Big(1 + \mathcal{O}\big(u^{-1}\big)\Big) \qquad \text{and} \qquad \frac{|V''|}{|V|^2} = \frac{6}{|\tilde{\lambda}|} \Big(1 + \mathcal{O}\big(u^{-1}\big)\Big) \,,$$

showing that the WKB conditions are satisfied for large λ near infinity. Combining this result with the estimates in Section 9.3, one finds that for $\omega = 0$ and large λ, the solution $\grave{\phi}$ is well-approximated by the WKB solution. Consequently, the behavior for small ω and large λ can be described simply by perturbing this WKB solution.

If λ is not large, we can use methods and results in [17]. For self-consistency, we now restate these results in our setting and outline the proofs.

Lemma 10.10. *Setting*

$$\sigma = \frac{1}{2}\left(\sqrt{1 + 4\lambda + 4s^2 + 8ak\omega} - 1\right), \tag{10.47}$$

the following limit exists,

$$\lim_{\omega \to 0,\ \mathrm{Im}\, \omega \leq 0,\ \omega \neq 0} \omega^{s+\sigma} \grave{\phi} = \grave{\phi}_0 \,. \tag{10.48}$$

The limit function $\grave{\phi}_0$ is a solution of the Sturm-Liouville equation (6.4) *for $\omega = 0$ and has the asymptotics*

$$\lim_{u \to \infty} \Big(u^\sigma\ \grave{\phi}_0\Big) = \frac{(-4)^{-\frac{\sigma}{4}}\, \Gamma(2\sigma + 2)}{(2i)^s\, \Gamma(\sigma + 1 - s)} \,. \tag{10.49}$$

Proof. We proceed as in the proof of [17, Lemma 10.10]. Again working in the r-coordinate and writing the radial equation as

$$-\frac{d^2}{dr^2}\psi(r) + \mathcal{V}(r)\, \psi(r) = 0 \,, \tag{10.50}$$

the potential $\mathcal{V}$ has the following asymptotics near infinity:

$$\mathcal{V}(r) = -\omega^2 - 2\,\frac{is\omega + M\omega^2}{r} + \frac{\lambda + s^2 + 2ak\omega - 2iMs\omega - 12M^2\,\omega^2}{r^2} + \mathcal{O}(r^{-3})\,. \quad (10.51)$$

(this differs from the potential in [17, eqn (8.6)] only by the summand $(2ak\omega)/r^2$). Dropping the error term, the equation (10.50) can be solved explicitly in terms of Whittaker functions. Satisfying the correct asymptotics at infinity (6.10), one obtains the unique solution

$$\tilde{\phi}(r) = \frac{r}{\sqrt{\Delta}}\,(2i\omega)^{-s+2iM\omega}\,W_{\kappa,\mu}\big(2i\omega r\big)\,,$$

where the parameters κ and μ are given by

$$\kappa = s - 2i\omega M \qquad \text{and} \qquad \mu = \frac{1}{2}\,\sqrt{1 + 4\lambda + 4s^2 + 8ak\omega - 8iMs\omega - 48\,M^2\omega^2}\,.$$

For small ω, this solution has the asymptotics

$$\tilde{\phi}(r) = \frac{r}{\sqrt{\Delta}}\,\omega^{-s-\sigma} r^{-\sigma}\,\frac{(-4)^{-\frac{\sigma}{4}}\,\Gamma(2\sigma+2)}{(2i)^s\Gamma(\sigma+1-s)}$$

with σ as in (10.47). This function obviously satisfies (10.48) and (10.49).

The error term in (10.51) can be treated exactly as in the proof of [17, Lemma 8.1] by a Jost iteration, taking the solution $\tilde{\phi}$ as the starting point. □

10.5. **Estimates of the Large Angular Modes.** Combining the estimates of Sections 10.1–10.4, we obtain the following a-priori estimate:

Proposition 10.11. *For any $u_\infty > 0$, there is a constant $\mathcal{C}_{11} > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, the kernels of the Green's functions s and of the operator g,* (7.5) *and* (7.6)*, satisfy for all $\omega \in \mathbb{R}$ the bound*

$$\left|e^{-\varpi u} s(u,u')\right|, \left|e^{-\varpi u} g(u,u')\right| \leq \mathcal{C}_{11} \qquad \text{for all } u < u_\infty \text{ and } -u_\infty < u' < u_\infty\,. \quad (10.52)$$

Before coming to the proof, we point out that the exponential factor $e^{-\varpi u}$ compensates for the exponential decay as $u \to \infty$ of the fundamental solution $\acute{\phi}(u)$ contained in $g(u,u')$ (see (7.6), (7.5) (6.9) and (9.1)). In order to verify that this exponential factor really controls the asymptotics uniformly in λ and ω, we need to estimate the absolute value of the exponential in the WKB solution (10.1). This is done in the next lemma.

Lemma 10.12. *There is a constant $\mathcal{C}_4$ such that the following estimate holds in the WKB region $(-\infty, u_L)$ for all λ and ω in the range* (9.2)*,*

$$\int_{-\infty}^{u_-^L} \Big(-\varpi \mp \operatorname{Im}\sqrt{-V}\Big) < \mathcal{C}_4\,.$$

Proof. We begin with the PC and Airy cases. We again consider the cases **(a)** and **(b)** in (9.15) after each other. We begin with case **(b)**. In the Airy case, we know from (9.7) and (9.29) that on the interval $(-\infty, u_-^L)$ the inequalities

$$|\operatorname{Im} V| \lesssim |\omega| \qquad \text{and} \qquad -\operatorname{Re} V \gtrsim \frac{\mathcal{C}_3}{\mathcal{C}_2}\,|\omega|^{\frac{4}{3}} \quad (10.53)$$

hold. In the PC case, by combining (9.19) (in case **(b)**) with (9.8) and (9.9), we find that $-\operatorname{Re} V \gtrsim \omega^2$, so that (10.53) again holds. As a consequence, the real part of V dominates its imaginary part, giving rise to the expansion

$$\operatorname{Im}\sqrt{-V} = -\frac{\operatorname{Im} V}{2\sqrt{-\operatorname{Re} V}} \Big(1 + \mathcal{O}\big(|\omega|^{-\frac{1}{3}}\big)\Big) .$$

Using the asymptotic form of the potential (10.17), we obtain the expansions

$$\begin{aligned}
\sqrt{-V} &= \Omega - \frac{c_1}{2\Omega}(\lambda+\nu)\, e^{\gamma u} \Big(1 + \mathcal{O}\big(e^{\gamma u}\big)\Big) \\
\operatorname{Re}\sqrt{-V} &= \operatorname{Re}\Omega - \frac{c_1}{2|\Omega|^2}\, \operatorname{Re}\Omega\, \lambda\, e^{\gamma u} \Big(1 + \mathcal{O}\big(\lambda^{-1}\big) + \mathcal{O}\big(e^{\gamma u}\big)\Big) \\
\operatorname{Im} V &= -2\operatorname{Re}(\Omega)\, \operatorname{Im}(\Omega) + c_1\, \operatorname{Im}\nu\, e^{\gamma u} + \mathcal{O}\big(e^{2\gamma u}\big)
\end{aligned}$$

and thus

$$\operatorname{Im}\sqrt{-V} = \operatorname{Im}\Omega + \Big(-\frac{c_1\, e^{\gamma u}}{2\operatorname{Re}\Omega}\, \operatorname{Im}\nu + \frac{c_1\, e^{\gamma u}}{2|\Omega|^2}\, \lambda\Big)\Big(1 + \mathcal{O}\big(\lambda^{-1}\big) + \mathcal{O}\big(|\omega|^{-\frac{1}{3}}\big) + \mathcal{O}\big(e^{\gamma u}\big)\Big) .$$

Using the notation (9.1), we obtain the estimate

$$\Big| -\varpi + \operatorname{Im}\sqrt{-V}\Big| \lesssim \Big(\frac{|\operatorname{Im}\nu|}{|\omega|} + \frac{\lambda}{\omega^2}\Big)\, e^{\gamma u} \lesssim \Big(1 + \frac{\lambda}{|\omega|}\Big)\, e^{\gamma u} ,$$

where in the last step we used that the function ν in (10.17) is a linear polynomial in ω. Applying (9.5) gives the result.

In case **(a)**, the last estimates apply without changes in the region $(-\infty, u_{\max} - \mathcal{C}^{-\frac{1}{2}})$ away from the maximum of $\operatorname{Re} V$. On the interval $(u_{\max} - \mathcal{C}^{-\frac{1}{2}}, u_-^L)$, on the other hand, we know from (9.8) that $|\operatorname{Re} V''| \simeq \lambda$. Therefore, the WKB inequality for the second derivative in (9.4) implies that $|V| \gtrsim \sqrt{\lambda/\varepsilon} \eqsim |\omega|/\sqrt{\varepsilon}$ (where in the last step we used Lemma 9.6). Combining this inequality with (9.7), we see that for sufficiently small ε, the real part of the potential again dominates its imaginary part, implying that

$$\Big| -\varpi + \operatorname{Im}\sqrt{-V}\Big| \lesssim 1 + \frac{|\operatorname{Im} V|}{\sqrt{|\operatorname{Re} V|}} \qquad \text{on } \big(u_{\max} - \mathcal{C}^{-\frac{1}{2}}, u_-^L\big) . \tag{10.54}$$

The inequalities (9.57) in Lemma 9.12 show that $|\operatorname{Im} V(u_{\max})| \lesssim 1$, and integrating (9.7), we obtain the bound

$$|\operatorname{Im} V(u)| \lesssim 1 + |\omega|\, \big|u - u_{\max}\big| .$$

Moreover, integrating (9.8), we know that

$$|\operatorname{Re} V| \gtrsim \lambda\, \big(u - u_{\max}\big)^2 .$$

Hence

$$\begin{aligned}
\int_{u_{\max}-\mathcal{C}^{-\frac{1}{2}}}^{u_-^L} \frac{|\operatorname{Im} V|}{\sqrt{|\operatorname{Re} V|}} &\lesssim \int_{u_{\max}-\mathcal{C}^{-\frac{1}{2}}}^{u_-^L} \Big(\frac{1}{\sqrt{\lambda}\, |u - u_{\max}|} + \frac{|\omega|}{\sqrt{\lambda}}\Big) \\
&\lesssim \frac{|\log(u_{\max} - u_-^L)|}{\sqrt{\lambda}} + \frac{|\omega|}{\sqrt{\lambda}} ,
\end{aligned} \tag{10.55}$$

which is uniformly bounded in view of (9.10), (9.17) and (9.2). This concludes the proof in case **(a)**.

In the remaining WKB case, we use the following monotonicity argument: We increase λ until $\operatorname{Re} V(u_{\max}) = -\mathcal{C}_4\sqrt{\lambda}$. Then we are in the PC case (see (9.9)), where the above method applies. When decreasing λ, the real part of the potential increases, whereas its imaginary part remains unchanged. Therefore, the inequality (10.54) remains valid, and the integral (10.55) decreases. This concludes the proof. □

Proof of Proposition 10.11. Let us go through the different cases, beginning with the parameter range that both $|\omega|$ and λ are large (9.2): First, according to Proposition 9.4, on the interval $(-\infty, u_-^L)$ the fundamental solution $\acute{\phi}$ is approximated by the WKB solution in (10.1), up to an arbitrarily small error. From Lemma 10.12 we conclude that the asymptotics of $|\acute{\phi}(u)|$ is controlled by the exponential $e^{\varpi u}$, uniformly in λ and ω.

We next proceed by analyzing the different cases in (9.9). In the WKB case, the WKB approximation applies on the whole interval $(-\infty, u_{\max})$. Likewise, on the interval $(u_{\max}, \infty)$ also the fundamental solution $\grave{\phi}$ is well-approximated by the WKB solution. Moreover, the fundamental solutions $\acute{y}$ and $\grave{y}$ lie in different half planes (see (6.9) and (6.10)). This implies that

$$\big|s(u,u')\big|, \big|g(u,u')\big| \lesssim \frac{e^{\varpi u}}{|\omega|} \,. \tag{10.56}$$

Next, in the parabolic cylinder case, the estimates of Lemmas 10.1 and 10.4 show that (10.56) again holds. Finally, in the Airy case, the estimates of Lemmas 10.2, 10.3, 10.5 and 10.6 imply that $\acute{\phi}$ is increasing exponentially in the WKB region with $\mathrm{Re}\, V > 0$, whereas $\grave{\phi}$ is exponentially decaying in this region. Hence $\big|s(u,u')\big|$ and $\big|g(u,u')\big|$ decay for large λ, uniformly in ω. This concludes the proof in the parameter range (9.2).

If $\omega \neq 0$ is in a bounded set and λ is large (10.14), the estimates in Section 10.3 show that $\acute{\phi}$ and $\grave{\phi}$ behave again just as described for the Airy case. Moreover, as by rescaling one can arrange a compact parameter range (as explained after (10.33)), it is obvious that the exponential $e^{-\varpi u}$ in (10.52) again controls the behavior as $u \to -\infty$ uniformly in all parameters. Finally, in Section 10.4 it is shown that the fundamental solution $\grave{\phi}$ as well as the Wronskian are continuous at $\omega = 0$, and that the Wronskian is non-zero in the limit. This concludes the proof. □

The estimate of Proposition 10.11 gives us uniform control of the large angular modes:

Proposition 10.13. *For sufficiently large p, the following estimate holds for all $u < u_\infty$,*

$$\frac{1}{|\omega + 3ic|^p} \left\| \Big(R_{\omega,n}\, Q^\omega_n \left(H + 3ic\right)^p \Psi_0 \Big)\!(u) \right\|_{L^2(S^2)} \leq \frac{c(u_\infty, \Psi_0)}{(n+1)^2\, (1+|\omega|)^2} \,.$$

Proof. Using Proposition 10.11, similar to (8.5) we obtain the estimate

$$\left\| \Big(R_{\omega,n}\, Q^\omega_n \left(H + 3ic\right)^p \Psi_0 \Big)(u) \right\|_{L^2(S^2)} \leq C(u_\infty, \Psi_0)\, (1 + \omega^2 + \lambda_n) \,.$$

Using the method in (8.6), one can generate factors of $1/\lambda_n$,

$$\frac{1}{|\omega + 3ic|^p} \left\| \Big(R_{\omega,n}\, Q^\omega_n \left(H + 3ic\right)^p \Psi_0 \Big)\!(u) \right\|_{L^2(S^2)} \leq C(u_\infty, \Psi_0)\, \frac{1 + \omega^2 + \lambda_n}{(1+|\omega|)^p\, \lambda_n^q} \,.$$

Since the operator $\mathcal{A}_\omega$ involves ω at most quadratically (see (6.3)), we obtain the estimate

$$\frac{1}{|\omega+3ic|^p} \left\| \Big(R_{\omega,n}\, Q_n^\omega \left(H+3ic\right)^p \Psi_0 \Big)(u) \right\|_{L^2(S^2)} \leq C(u_\infty, \Psi_0)\, \frac{1+\omega^2+\lambda_n}{(1+|\omega|)^{p-2q}\, \lambda_n^q} \,.$$

Choosing p sufficiently large and estimating the eigenvalues λ_n from below with the help of Proposition A.2, we obtain the result. ☐

Corollary 10.14. *For sufficiently large p, the solution of the Cauchy problem for the Teukolsky equation with initial data $\Psi|_{t=0} = \Psi_0 \in \mathcal{D}(H)$ can be written for any $t<0$ as*

$$\Psi(t) = -\frac{1}{2\pi i} \sum_{n=0}^\infty \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}-i\varepsilon} \frac{e^{-i\omega t}}{(\omega+3ic)^p} \Big(R_{\omega,n}\, Q_n^\omega \left(H+3ic\right)^p \Psi_0 \Big)\, d\omega \,.$$

Here the series converges absolutely in the sense that for any $\varepsilon>0$, there is N such that for all $t<0$ and all $u<u_\infty$,

$$\sum_{n=N}^\infty \left\| \lim_{\varepsilon \searrow 0} \left(\int_{\mathbb{R}-i\varepsilon} \frac{e^{-i\omega t}}{(\omega+3ic)^p} \Big(R_{\omega,n}\, Q_n^\omega \left(H+3ic\right)^p \Psi_0 \Big)\, d\omega \right)(u) \right\|_{L^2(S^2)} < \varepsilon \,.$$

Proof. Our starting point is the integral representation (5.12) in Corollary 5.4. Separating the resolvent (Theorem 7.1), applying the estimates of the large angular modes of Proposition 10.13 and deforming the contours (Lemma 8.1), we obtain the result. ☐

11. Ruling out Radiant Modes

In the integral representation of Corollary 10.14, we know that all integrands are holomorphic for ω in the lower half plane, making it possible to move the contour arbitrarily close to the real axis. However, our analysis so far does not rule out the possibility that the integrands might have poles on the real axis. We refer to such poles as *radiant modes*. In this section we rule out radiant modes.

11.1. Ruling out Radiant Modes at $\omega = 0$.

For $\omega=0$, the potential (6.5) simplifies to

$$\begin{aligned} V(u) &= \frac{\lambda\,\Delta}{(r^2+a^2)^2} + \frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}} - \Big(\frac{ak - i(r-M)\,s}{r^2+a^2} \Big)^2 \\ &= \frac{\lambda\,\Delta}{(r^2+a^2)^2} + \frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}} + \frac{(r-M)^2\, s^2 - a^2 k^2}{(r^2+a^2)^2} + 2is\, \frac{ak\,(r-M)}{(r^2+a^2)^2} \,. \end{aligned}$$

In particular, one sees that the imaginary part of V has a fixed sign,

$$\operatorname{Im} V(u) = 2s\, \frac{ak\,(r-M)}{(r^2+a^2)^2} > 0 \,. \tag{11.1}$$

Lemma 11.1. *For every angular mode, the kernels of the Green's functions s and of the operator g, (7.5) and (7.6), are uniformly bounded in a neighborhood of $\omega=0$ (here again $\operatorname{Im}\omega \leq 0$).*

Proof. In view of the continuity results of Theorem 6.1 and Lemma 10.10, it remains to show that choosing $\omega=0$, the Wronskian $w(\acute{\phi}, \grave{\phi}_0)$ (with $\grave{\phi}_0$ as in (10.48)) is nonzero. Assume conversely that this Wronskian were zero. Then the solutions $\acute{\phi}$, $\grave{\phi}_0$ are multiples of each other. Thus there is a non-trivial solution ϕ of the Sturm-Liouville equation (6.4) which decays both as $u \to \pm\infty$. More precisely, this solution decays exponentially as $u \to -\infty$ (see (6.9), keeping in mind that Ω in (6.8) has a negative imaginary part), whereas it decays polynomially as $u \to \infty$ (see (10.49)). In particular, the solution is in $L^2(\mathbb{R})$.

We now make use of an observation made previously in [21, Section 9]. Multiplying the differential equation for ϕ by $\overline{\phi}$ and integrating, we obtain

$$0 = \int_0^\pi \overline{\phi} \left(-\frac{d^2}{du^2} + V \right) \phi \overset{(\star)}{=} \int_0^\pi \overline{\left(-\frac{d^2}{du^2} + \overline{V} \right) \phi}\, \phi = \int_0^\pi (V - \overline{V})\, \overline{\phi} \phi\,,$$

where in $(\star)$ we integrated by parts and used the decay properties of ϕ to conclude that the boundary terms vanish. We thus obtain the relation

$$\int_0^\pi \operatorname{Im} V\, |\phi|^2 = 0\,.$$

Using (11.1), we conclude that ϕ must vanish identically, a contradiction. □

11.2. **A Causality Argument.** In the following proposition we show that the separated resolvent has no poles on the real axis. The method makes use of finite speed of propagation and is an improvement of the method first developed for the scalar wave equation in [12, Section 7].

Proposition 11.2. *For any $n \in \mathbb{N}_0$, the separated resolvent $R_{\omega,n}$, (7.7), is holomorphic in the lower half plane $\{\operatorname{Im} \omega < 0\}$. Moreover, it is continuous up to the real axis, i.e. the limit*

$$R^-_{\omega,n} \Psi := \lim_{\varepsilon \searrow 0} \left(R_{\omega - i\varepsilon, n} \Psi \right) \qquad \text{exists for all } \omega \in \mathbb{R}\,.$$

Proof. Let $\omega_0 \in \mathbb{R}$. We want to show that $R_{\omega,n}$ is continuous at ω_0. In the case $\omega_0 = 0$, the result follows immediately from Lemma 11.1. In the remaining case $\omega_0 \neq 0$, for test functions $\eta_1, \eta_2 \in C^\infty_0(\mathbb{R})$ and a real parameter L we set

$$\Phi_L(u, \vartheta, \varphi) = \eta_1(u + 2L)\, e^{-ik\varphi}\, e^{2i\Omega_0 L}\, \Theta_{\omega_0,n}(\vartheta) \tag{11.2}$$

$$\Phi_{\text{test}}(u, \vartheta, \varphi) = \frac{r^2 + a^2}{\rho}\, \eta_2(u)\, e^{-ik\varphi}\, \Theta_{\omega_0,n}(\vartheta)\,, \tag{11.3}$$

where $\Theta_{\omega_0,n}$ is an eigenfunction of the angular operator $\mathcal{A}_{\omega_0}$ corresponding to the eigenvalue λ_n. Moreover, we set

$$\Psi_L = \begin{pmatrix} \Phi_L \\ 0 \end{pmatrix} \qquad \text{and} \qquad \Psi_{\text{test}} = \begin{pmatrix} 0 \\ \Phi_{\text{test}} \end{pmatrix}.$$

Finally, we let $\Psi_{L,t}$ be the solution of the Cauchy problem with initial data $\Psi_{L,0} = \Psi_L$. Then, due to finite propagation speed, it follows that for sufficiently large L, the functions Ψ^t_L and Ψ_{test} have disjoint supports if $t \in [-L, 0]$. Hence for any power $r \in \mathbb{N}$,

$$0 = \frac{1}{L} \int_{-L}^0 e^{i\omega_0 t} \left\langle (H - \omega_0)^r\, \Psi^t_L, \Psi_{\text{test}} \right\rangle_{L^2(\mathbb{R}\times S^2)} dt\,. \tag{11.4}$$

Corollary 10.14 yields the integral representation

$$\begin{aligned} &(H - \omega_0)^r \Psi^t_L \\ &= -\frac{1}{2\pi i} \sum_{n'=0}^\infty \lim_{\varepsilon \searrow 0} \int_{\mathbb{R} - i\varepsilon} e^{-i\omega t}\, \frac{(\omega - \omega_0)^r}{(\omega + 3ic)^p} \left(R_{\omega,n'}\, Q^\omega_{n'}\, (H + 3ic)^p\, \Psi_L \right) d\omega\,, \end{aligned}$$

where the infinite sum over n' converges absolutely if for any given $r \in \mathbb{N}$ we choose p sufficiently large. Using this representation in (11.4) and introducing the short notation

$$\Xi_{L,\omega} = \frac{(\omega-\omega_0)^r}{(\omega+3ic)^p}\, \big(H+3ic\big)^p\, \Psi_L\,,$$

we obtain

$$\begin{aligned} 0 &= \sum_{n'=0}^\infty \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}-i\varepsilon} \big\langle R_{\omega,n'}\, Q^\omega_{n'}\, \Xi_{L,\omega}, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)}\, \frac{i}{L} \int_{-L}^0 e^{-i(\omega-\omega_0)t}\, dt \\ &= \sum_{n'=0}^\infty \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}-i\varepsilon} \frac{e^{i(\omega-\omega_0)L}-1}{(\omega-\omega_0)L}\, \big\langle R_{\omega,n'}\, Q^\omega_{n'}\, \Xi_{L,\omega}, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)}\,. \end{aligned} \tag{11.5}$$

Let $\delta>0$. According to Corollary 10.14, we know that for sufficiently large N,

$$\sum_{n'=N}^\infty \Big| \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}-i\varepsilon} \frac{e^{i(\omega-\omega_0)L}-1}{(\omega-\omega_0)L}\, \big\langle R_{\omega,n'}\, Q^\omega_{n'}\, \Xi_{L,\omega}, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \Big| < \delta\,,$$

uniformly for large L. Moreover, using Proposition 10.13, we may choose $\omega_{\max} > 2\,|\omega_0|$ such that

$$\sum_{n=0}^N \Big| \Big(\int_{-\infty}^{-\omega_{\max}} + \int_{\omega_{\max}}^\infty \Big) \frac{e^{i(\omega-\omega_0)L}-1}{(\omega-\omega_0)L}\, \big\langle R_{\omega,n'}\, Q^\omega_{n'}\, \Xi_{L,\omega}, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \Big| < \delta\,,$$

again uniformly in L. Using these estimates in (11.5), we conclude that

$$\Big| \sum_{n'=0}^N \lim_{\varepsilon \searrow 0} \int_{-\omega_{\max}-i\varepsilon}^{\omega_{\max}-i\varepsilon} \frac{e^{i(\omega-\omega_0)L}-1}{(\omega-\omega_0)L}\, \big\langle R_{\omega,n'}\, Q^\omega_{n'}\, \Xi_{L,\omega}, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \Big| < 2\delta\,, \tag{11.6}$$

uniformly for large L.

In order to estimate the remaining integrals, we iteratively apply the identity

$$\begin{aligned} &\frac{1}{(\omega+3ic)^q}\, R_{\omega,n'}\, Q^\omega_{n'}\, (H+3ic)^q \\ &= \frac{1}{(\omega+3ic)^q}\, R_{\omega,n'}\, Q^\omega_{n'} \Big((H-\omega) + (\omega+3ic) \Big) (H+3ic)^{q-1} \\ &= \frac{1}{(\omega+3ic)^q}\, Q^\omega_{n'} + \frac{1}{(\omega+3ic)^{q-1}}\, R_{\omega,n'}\, Q^\omega_{n'}\, (H+3ic)^{q-1}\,. \end{aligned} \tag{11.7}$$

Using this relation in (11.6), the first summand in (11.7) gives rise to integrals of the form

$$\int_{-\omega_{\max}-i\varepsilon}^{\omega_{\max}-i\varepsilon} \frac{e^{i(\omega-\omega_0)L}-1}{(\omega-\omega_0)L}\, \frac{(\omega-\omega_0)^r}{(\omega+3ic)^q}\, \big\langle Q^\omega_{n'}\, \Psi_L, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)}\,.$$

As the integrand is holomorphic in a neighborhood of ω_0, we may deform the contour into the upper half plane (keeping the end points fixed) such that $|\omega-\omega_0| > \omega_{\max}/2$ along the contour. Taking the limit $\varepsilon \searrow 0$, we obtain the integral along a contour Γ which joins the points $-\omega_{\max}$ and $\omega_{\max}$ and lies in the upper half plane (see Figure 2). Then the bounds $|e^{i(\omega-\omega_0)L}| \leq 1$ and $|\omega-\omega_0| > \omega_{\max}/2$ show that the integral tends to zero in the limit $L \to \infty$. Therefore, for large L only the second summand in (11.7) must be taken into account. We conclude that for large L,

$$\Big| \sum_{n'=0}^N \lim_{\varepsilon \searrow 0} \int_{-\omega_{\max}-i\varepsilon}^{\omega_{\max}-i\varepsilon} \frac{e^{i(\omega-\omega_0)L}-1}{(\omega-\omega_0)L}\, (\omega-\omega_0)^r\, \big\langle R_{\omega,n'}\, Q^\omega_{n'}\, \Psi_L, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \Big| < 3\delta. \tag{11.8}$$

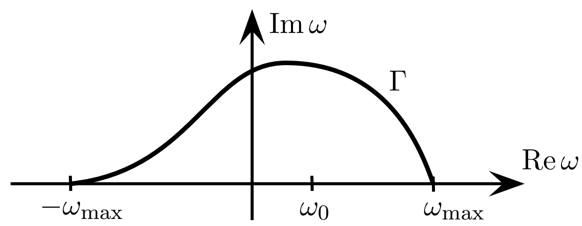

FIGURE 2. The contour Γ.

We proceed indirectly. Let us assume that the separated resolvents $R_{\omega,n'}$ have poles at ω_0. Since the poles of meromorphic functions are isolated, there is a small neighborhood of ω_0 on the real axis where the resolvent has no other poles. Moreover, we may choose n such that the pole of $R_{\omega,n}$ has a pole of order $q \geq 1$, and that for all $n' \neq n$, the separated resolvents $R_{\omega,n'}$ have a pole of order at most q. By choosing $r = q-1$, we can arrange that the integrand in (11.8) for $n' = n$ has a pole of order one, whereas all the integrands for $n' \neq n$ have a pole of order at most one.

For all modes n' with $n' \neq n$, we can make use of the fact that Ψ_L is an eigenfunction of the angular operator $\mathcal{A}_{\omega_0}$ corresponding to the eigenvalue λ_n (see (11.2)). As a consequence, $Q^{\omega_0}_{n'} \Psi_L = 0$, and thus

$$\big| \big\langle R_{\omega,n'}\, Q^\omega_{n'}\, \Psi_L, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \big| = \big| \big\langle R_{\omega,n'} \left(Q^\omega_{n'} - Q^{\omega_0}_{n'} \right) \Psi_L, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \big| \,.$$

Next, since ω_0 is real, the angular operator $\mathcal{A}_{\omega_0}$ is self-adjoint and has simple eigenvalues (for details see [21, Section 7]). Therefore, the operator $Q^\omega_{n'} - Q^{\omega_0}_{n'}$ is given linearly in $(\omega - \omega_0)$ by a standard first order perturbation calculation without degeneracies. We thus obtain the estimate

$$\big| \big\langle R_{\omega,n'}\, Q^\omega_{n'}\, \Psi_L, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \big| \leq c\big(\eta_1, \eta_2, \omega_0\big)\, \big|\omega - \omega_0\big| \,.$$

Using this estimate in (11.8), the factor $|\omega - \omega_0|$ has the effect that the integrand is bounded near $\omega = \omega_0$. Due to the factor $1/L$ in (11.8), the corresponding summand in (11.8) tends to zero as $L \to \infty$. We conclude that for sufficiently large L,

$$\left| \int_{-\omega_{\max}-i\varepsilon}^{\omega_{\max}-i\varepsilon} \frac{e^{i(\omega-\omega_0)L} - 1}{(\omega-\omega_0)L}\; (\omega-\omega_0)^{q-1} \, \big\langle R_{\omega,n}\, Q^\omega_n\, \Psi_L, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \right| < 4\delta \,. \tag{11.9}$$

Thus it remains to analyze the angular mode n: The integrand can be simplified with the relations

$$\begin{aligned}
&\big\langle R_{\omega,n}\, Q^\omega_n\, \Psi_L, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \\
&= \langle Q^\omega_n\, \Theta_{\omega_0,n}, \Theta_{\omega_0,n} \rangle_{L^2(S^2)} \int_{-\infty}^\infty du \int_{-\infty}^\infty dv\; \overline{\Phi_L(u)}\; \big(\mathfrak{R}_{\omega,n}(u,v)\big)^1_2\, \Phi_{\text{test}}(v) \\
&= \langle Q^\omega_n\, \Theta_{\omega_0,n}, \Theta_{\omega_0,n} \rangle_{L^2(S^2)} \int_{-\infty}^\infty du \int_{-\infty}^\infty dv\; \overline{\Phi_L(u)}\; g_\omega(u,v)\; \Phi_{\text{test}}(v) \,,
\end{aligned}$$

where in the last step we used the explicit form of the kernel $\mathfrak{R}_{\omega,n}$ in (7.8). Since ω_0 is real, the angular operator $\mathcal{A}_{\omega_0}$ is self-adjoint and has no degeneracies. A standard perturbation argument implies that if $|\omega - \omega_0|$ is sufficiently small, the operator $\mathcal{A}_\omega$ is diagonalizable. Therefore, the nilpotent matrix $\mathcal{N}$ in (7.6) vanishes, so that $g_\omega = s_\omega$

with s_ω given by (7.5). We conclude that

$$\begin{aligned}
&\big\langle R_{\omega,n}\, Q^\omega_n\, \Psi_L, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \\
&\quad = \frac{\langle Q^\omega_n\, \Theta_{\omega_0,n}, \Theta_{\omega_0,n}\rangle_{L^2(S^2)}}{w(\acute{\phi}, \grave{\phi})} \bigg(\int_{-\infty}^\infty \overline{\Phi_L(u)}\, \acute{\phi}(u)\, du \bigg) \bigg(\int_{-\infty}^\infty \Phi_{\text{test}}(v)\, \grave{\phi}(v)\, dv \bigg) \,.
\end{aligned}$$

Using this relation in (11.9), the integral can be computed with residues. Since the Wronskian is assumed to have a zero of order q, we obtain

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \int_{-\omega_{\max}-i\varepsilon}^{\omega_{\max}-i\varepsilon} \frac{e^{i(\omega-\omega_0)L}-1}{(\omega-\omega_0)L}\, (\omega-\omega_0)^{q-1} \big\langle R_{\omega,n}\, Q^\omega_n\, \Psi_L, \Psi_{\text{test}} \big\rangle_{L^2(\mathbb{R}\times S^2)} \\
&= \bigg(\lim_{L\to\infty} \int_{-\infty}^\infty \overline{\Phi_L(u)}\, \acute{\phi}(u)\, du \bigg) \bigg(\int_{-\infty}^\infty \Phi_{\text{test}}(v)\, \grave{\phi}(v)\, dv \bigg)\, \langle \Theta_{\omega_0,n}, \Theta_{\omega_0,n}\rangle_{L^2(S^2)} \\
&\qquad \times (-i\pi)\, \text{Res}_{\omega_0} \bigg(\frac{e^{i(\omega-\omega_0)L}-1}{(\omega-\omega_0)L}\, \frac{(\omega-\omega_0)^{q-1}}{w(\acute{\phi}, \grave{\phi})} \bigg) + \mathcal{O}\big(L^{-1}\big) \,.
\end{aligned}$$

Clearly, the residue is non-zero. Moreover, the limit $L \to \infty$ of the first integral exists in view of the asymptotics of the fundamental solution (6.9). By choosing the test functions η_1 and η_2 in (11.2) and (11.3), we can clearly arrange that this limit as well as the second integral are non-zero. We conclude that the integral in (11.9) has a non-zero limit as $\varepsilon \searrow 0$. Since δ can be chosen arbitrarily small, we obtain a contradiction. This concludes the proof. □

12. Integral Representation and Proof of Decay

We now consider the Cauchy problem for the Teukolsky equation (2.3) with smooth and compactly supported initial data $\Psi_0 = (\Phi|_{t=0}, \partial_t \Phi|_{t=0}) \in C^\infty(\mathbb{R}\times S^2, \mathbb{C}^2)$ (we always work in the Regge-Wheeler variable $u \in \mathbb{R}$ (see (2.5)) and the function $\Phi := \sqrt{r^2+a^2}\, \phi$ (see (2.6)). We decompose the initial data into a Fourier series of azimuthal modes (cf. (2.4)),

$$\Psi_0(u,\vartheta,\varphi) = \sum_{k\in\mathbb{Z}} e^{-ik\varphi}\, \Psi_0^{(k)}(u,\vartheta) \,.$$

By linearity, the solution Cauchy problem for Ψ_0 is obtained by solving the Cauchy problem for each azimuthal mode k and taking the sum of all the resulting solutions. In the next theorem an integral representation for the solution of each azimuthal mode is given and it is shown that the solution decays pointwise.

Theorem 12.1. *For any $k \in \mathbb{Z}$ there is a parameter $p > 0$ such that for any $t > 0$, the solution of the Cauchy problem for the Teukolsky equation with initial data*

$$\Psi|_{t=0} = e^{-ik\varphi}\, \Psi_0^{(k)}(r,\vartheta) \qquad \text{with} \qquad \Psi_0^{(k)} \in C^\infty(\mathbb{R}\times S^2, \mathbb{C}^2)$$

has the integral representation

$$\begin{aligned}
&\Psi(t,u,\vartheta,\varphi) \\
&\quad = -\frac{1}{2\pi i}\, e^{-ik\varphi} \sum_{n=0}^\infty \int_{-\infty}^\infty \frac{e^{-i\omega t}}{(\omega+3ic)^p} \Big(R^-_{\omega,n}\, Q^\omega_n \big(H+3ic\big)^p\, \Psi_0^{(k)}\Big)(u,\vartheta)\, d\omega \,.
\end{aligned} \tag{12.1}$$

Moreover, the integrals in (12.1) *all exist in the Lebesgue sense. Furthermore, for every* $\varepsilon > 0$ *and* $u_\infty \in \mathbb{R}$*, there is* N *such that for all* $u < u_\infty$*,*

$$\sum_{n=N}^\infty \int_{-\infty}^\infty \left\| \frac{1}{(\omega+3ic)^p} \left(R^-_{\omega,n}\, Q^\omega_n \,(H+3ic)^p\, \Psi_0^{(k)} \right)(u) \right\|_{L^2(S^2)} d\omega < \varepsilon \,. \tag{12.2}$$

Proof. Starting from the result of Corollary 10.14, we apply Proposition 11.2 to move the contour up to the real axis. □

Corollary 12.2. *For every* $k \in \mathbb{Z}$*, the solution of the Cauchy problem for the Teukolsky equation with initial data* $\Psi|_{t=0} = \Psi_0^{(k)} \in \mathcal{D}(H)$ *decays pointwise, i.e.*

$$\lim_{t \to -\infty} \Psi(t,u,\vartheta,\varphi) = 0 \quad \textit{in } L^\infty_{\text{loc}}(\mathbb{R} \times S^2)\,.$$

Proof. Given $\varepsilon > 0$, we choose N such that (12.2) holds. For each of the angular modes $n = 0, \ldots, N-1$, the Riemann-Lebesgue lemma gives pointwise decay as $t \to -\infty$, locally uniformly in the spatial variables. We conclude that $\Psi(t)$ decays in $L^2_{\text{loc}}(\mathbb{R} \times S^2, \mathbb{C}^2)$. Differentiating the equation with respect to t, we conclude that all time derivatives $\partial_t^q \Psi(t)$ decay in $L^2_{\text{loc}}(\mathbb{R} \times S^2, \mathbb{C}^2)$. Using the Teukolsky equation (3.2) and applying the Sobolev embedding theorem, we obtain pointwise decay in L^∞_{loc}. □

For clarity, we point out that, applying the Teukolsky-Starobinsky identities (see for example [5]), one also gets decay of all other components of the spin s wave. Applying the above corollary to the lowest component of the spin wave with reversed time direction, one also gets decay of the Teukolsky solution Ψ in the limit $t \to +\infty$. In the case $s = 2$ of gravitational waves, the corresponding metric perturbations can be constructed as explained in [24].

13. Concluding Remarks

We close with a few remarks. We first point out that the integral representation of Theorem 12.1 is a suitable starting point for a detailed analysis of dynamics of the solutions of the Teukolsky equation. In particular, one can study decay rates (similar as worked out for massive Dirac waves in [12]) and derive uniform energy estimates outside the ergosphere (similar as for scalar waves in [16]). Moreover, using the methods in [14], one could analyze superradiance phenomena for wave packets in the time-dependent setting.

We finally comment on the limitations of our methods. First, we do not aim for minimal regularity assumptions on the initial data, and we do not analyze decay in weighted Sobolev spaces. Also, we do not study to which extent our estimates are uniform in the support of the initial data. Moreover, we do not consider whether our estimates are uniform in the azimuthal separation constant k, and we do not analyze the convergence and decay properties of the infinite series of azimuthal modes. Indeed, the analysis of the infinite sum of azimuthal modes is closely related to the analysis of optimal regularity. Namely, for smooth initial data, the coefficients of the Fourier series (1.1) clearly decay rapidly in k, so that the convergence of the k-series is not an issue. The question of whether this rapid decay in k also holds for later times is intimately linked to the question of whether the regularity of the solution (as quantified by suitable weighted Sobolev norms) is preserved under the time evolution. As just mentioned, such regularity questions are not addressed in this paper. In order to attack these important open problems, it seems a promising strategy to us to combine our methods and results with techniques of microlocal analysis as used in [22] to study the high frequency behavior in the related Kerr-De Sitter geometry.

Appendix A. Some Estimates of the Angular Eigenvalues

As in [21, Section 2] we rewrite the angular equation in (6.1) as the eigenvalue equation

$$H\phi = \lambda \phi \,, \tag{A.1}$$

where H has the form of a one-dimensional Hamiltonian

$$H = -\frac{d^2}{du^2} + W$$

with the complex potential

$$\begin{aligned} W &= -\frac{1}{4}\,\frac{\cos^2 u}{\sin^2 u} - \frac{1}{2} + \frac{1}{\sin^2 u} (\Omega \sin^2 u + k - s \cos u)^2 \\ &= \Omega^2 \,\sin^2 u + \left(k^2 + s^2 - \frac{1}{4}\right) \frac{1}{\sin^2 u} \;+\; 2\Omega k - s^2 - \frac{1}{4} \\ &\quad - 2s\Omega \cos u - 2sk\, \frac{\cos u}{\sin^2 u} \end{aligned} \tag{A.2}$$

and $u = \vartheta$, $\Omega := -a\omega$.

We begin with estimates for real Ω. Then, as explained in detail in [21, Section 5 and 7], the Hamiltonian has non-degenerate eigenvalues $\lambda_0 < \lambda_1 < \cdots$.

Lemma A.1. *There is a constant $c > 0$ such that*

$$\lambda_n \geq \frac{1}{c} \left(1 + |\Omega|\right) \qquad \textit{for all } \Omega \in \mathbb{R}\,.$$

Proof. It clearly suffices to prove the inequality for the lowest eigenvalues λ_0. In the formulation as an eigenvalue equation for the partial differential operator on the sphere (see [21, eqn (1.1)]), the angular operator is a sum of two positive operators. Hence its spectrum is clearly non-negative, so that $\lambda_0 \geq 0$. Assume that the statement of the lemma is false. Then there is a sequence $(\Omega^\ell)_{\ell \in \mathbb{N}}$ with $|\Omega^\ell| \to \infty$, so that the corresponding eigenvalues λ_0^ℓ satisfy the relation

$$\frac{\lambda_0^\ell}{|\Omega^\ell|} \to 0\,.$$

Let us derive a contradiction. Due to the summand $\Omega^2 \sin^2 u$, the potential is positive except possibly at a neighborhood of $u = 0$ or π. Near the pole at $u = 0$, the potential has the asymptotic form (see [21, eqn (11.18)]),

$$\begin{aligned} V(u) &= \frac{\Lambda}{u^2} + \Omega^2\, u^2 - 2s\Omega - \mu + \mathcal{O}\big(|\Omega| u^2\big) + \mathcal{O}\big(|\Omega|^2\, u^4\big) \\ \Lambda &= (k-s)^2 - \frac{1}{4} \\ \mu &= \lambda_0 - 2\Omega k + s^2 + \frac{1}{4}\,. \end{aligned}$$

Introducing the new variable $\tilde{u} = \sqrt{|\Omega|} u$, the eigenvalue equation (A.1) becomes

$$\left(-\frac{d^2}{d\tilde{u}^2} + \tilde{V}(\tilde{u}) \right) \phi = 0$$

with the new potential

$$\begin{aligned}\tilde{V} &= \frac{\Lambda}{\tilde{u}^2} - \frac{2s\Omega+\mu}{|\Omega|} + \tilde{u}^2 + \mathcal{O}\Big(\frac{\tilde{u}^2}{|\Omega|}\Big) + \mathcal{O}\Big(\frac{\tilde{u}^4}{|\Omega|}\Big) \\ &= -\frac{1}{4\tilde{u}^2} + \frac{(k-s)^2}{\tilde{u}^2} \pm 2(k-s) + \tilde{u}^2 - \frac{\lambda_0}{|\Omega|} + \mathcal{O}\Big(\frac{\tilde{u}^2}{|\Omega|}\Big) + \mathcal{O}\Big(\frac{\tilde{u}^4}{|\Omega|}\Big)\,,\end{aligned}$$

where the plus and minus signs correspond to the cases $\Omega > 0$ and $\Omega < 0$, respectively. Hence the potential can be regarded as a perturbation of the potential

$$\tilde{V}_{\text{asy}} := -\frac{1}{4\tilde{u}^2} + \frac{(k-s)^2}{\tilde{u}^2} \pm 2(k-s) + \tilde{u}^2\,.$$

For this potential, the fundamental solution with the same asymptotics as $\acute{\phi}$ is given explicitly in terms of generalized Laguerre polynomials (see [27, §18.8.1]),

$$\phi_{\text{asy}}(\tilde{u}) = c\, e^{\frac{\tilde{u}^2}{2}}\, \tilde{u}^{\frac{1}{2}\pm(k-s)}\, L^{\frac{1}{2}\pm(k-s)}_{-\frac{1}{2}}\big(-\tilde{u}^2\big)\,.$$

Considering the asymptotics for large $\tilde{u}$, one sees that this function is strictly monotone increasing for large $\tilde{u}$. A perturbation argument shows that the same is true for the eigenfunction ϕ if ℓ is sufficiently large (this perturbation argument could be carried out in a straightforward way for example by performing a Jost iteration, taking ϕ_{asy} as the unperturbed solution; for details see [12, Section 3] or [8]). This implies that for any sufficiently large c and sufficiently large $|\Omega|$,

$$\phi'\big|_{\tilde{u}=c} > 0\,.$$

Repeating the above argument at the pole at $u = -\pi$, we conclude the the eigenfunction ϕ has the properties that for any sufficiently large c and sufficiently large $|\Omega|$,

$$\phi'\big(c\,|\Omega|^{-\frac{1}{2}}\big) > 0 \qquad \text{and} \qquad \phi'\big(\pi - c\,|\Omega|^{-\frac{1}{2}}\big) < 0\,.$$

Moreover, the potential is positive on the interval $(c\,|\Omega|^{-\frac{1}{2}}, \pi - c\,|\Omega|^{-\frac{1}{2}})$, implying that ϕ is convex on this interval (see for example [15, Section 5]). This is a contradiction. ☐

Proposition A.2. *There is a constant c such that the eigenvalues λ_n satisfy the inequalities*

$$\frac{(n+1)^2}{c} \leq \lambda_n \leq c\,|\Omega|\,(n+1)^2 \qquad \text{for all } n \in \mathbb{N}_0 \text{ and } \Omega \in \mathbb{R}\,. \tag{A.3}$$

Proof. We shall apply [21, Corollary 7.6], which states that if for given λ_n, we choose two intervals $I_L, I_R \subset (0, \pi)$ such that potential V is non-negative on the complement of these intervals, and if we choose any two solutions $\acute{y}$ and $\grave{y}$ of the Riccati equation on the intervals I_L respectively I_R which lie in the upper half plane $\{\text{Im}\, y > 0\}$, then

$$\pi\,(n-2) \;<\; \int_{I_L} \text{Im}\,\acute{y} + \int_{I_R} \text{Im}\,\grave{y} \;\leq\; \pi\,(n+2)\,.$$

We choose the intervals as $I_L = \big(0, \min(u_+^L, u_{\max})\big)$ and $I_R = \big(\max(u_+^R, u_{\max}), \pi\big)$. Then obviously $\text{Re}\, V \geq 0$ on the complement of these intervals.

Thus our task is to estimate the integral of $\text{Im}\,\acute{y}$ over the interval I_L (and similarly for the integral of $\text{Im}\,\grave{y}$ over I_R). Here we want to use the results of the detailed estimates of the Riccati solutions near the poles and in the WKB and Airy regions carried out in [19, 20, 21]. These estimates apply in the parameter range (see [21, Section 10.1], keeping in mind that now the potential is real)

$$|\Omega| \geq \mathcal{C}_4 \tag{A.4}$$

$$\lambda \geq \mathcal{C}_5 \, |\Omega| \, . \tag{A.5}$$

Let us argue why we may restrict attention to this parameter range. First, for Ω in a compact set, the inequalities (A.3) follow immediately from a continuity argument and Weyl's asymptotics (see [21, Section 7.3]). Therefore, we may restrict attention to large $|\Omega|$, (A.4). Next, for proving the upper bound in (A.3), it is clearly no restriction to assume that (A.5) holds. For the lower bound, we can argue as follows: Given $N \in \mathbb{N}$, for the first N eigenvalues, the lower bound in (A.3) follows immediately from Lemma A.1. On the other hand, by choosing N sufficiently large, we can apply [21, Proposition 7.7] to conclude that

$$\lambda_n \geq \mathcal{C}_5 \, |\Omega| \qquad \text{for all } n \geq N \, .$$

Therefore, we may indeed restrict attention to the parameter range (A.4) and (A.5).

The detailed estimates of the Riccati solutions near the poles and in the WKB and Airy regions carried out in [19, 20, 21] show that

$$\frac{1}{\mathcal{C}} \int_{u_\ell^L}^{u_r^L} \sqrt{-V} \leq \int_{I_L} \operatorname{Im} \acute{y} \leq \mathcal{C} \int_{u_\ell^L}^{u_r^L} \sqrt{-V}$$

(and similarly for the integral of $\operatorname{Im} \grave{y}$ over I_R). Thus it remains to estimate the integral of $\sqrt{-V}$. In order to get upper estimates, we restrict attention to the interval (u_ℓ, u_0) near the pole, where (see [21, eqns (10.8), (11.12) and (11.19)])

$$u_\ell = \frac{\mathcal{C}_1}{\sqrt{\operatorname{Re} \lambda}} \qquad \text{and} \qquad u_0 = \Lambda^{\frac{1}{4}} \, |\Omega|^{-\frac{1}{2}} + \mathcal{O}\big(|\Omega|^{-\frac{3}{2}}\big) \, .$$

On this interval, the potential can be estimated by (see [21, eqns (11.14) and (11.18)])

$$V(u) \leq \frac{\mathfrak{c}}{u^2} - \lambda \leq -\frac{\lambda}{2} \, ,$$

where in the last step we increased the constant $\mathcal{C}_1$. Hence

$$n \gtrsim \int_{u_\ell}^{u_r} \sqrt{-V} \gtrsim \sqrt{\lambda} \, (u^0 - u_\ell) \gtrsim \left(\frac{\lambda_n}{|\Omega|} \right)^{\frac{1}{2}} \, .$$

We thus obtain the estimate

$$\lambda_n \lesssim |\Omega| \, n^2 \, ,$$

proving the upper bound in (A.3).

In order to obtain simple lower bounds for the eigenvalues, we make use of the fact that away from the pole region, the potential is bounded by

$$V\big|_{(u_0, \pi)} \gtrsim -\lambda \, .$$

Hence

$$n \lesssim \int_{u_\ell}^{u_r} \sqrt{-V} = \int_{u_\ell}^{u_0} \sqrt{-V} + \int_{u_0}^{u_r} \sqrt{-V} \lesssim \left(\frac{\lambda_n}{|\Omega|} \right)^{\frac{1}{2}} + \sqrt{\lambda_n} \, ,$$

giving rise to the estimate

$$\lambda_n \gtrsim n^2 \, .$$

This concludes the proof. □

Lemma A.3. *For any given $c_1 > 0$, we let $U \subset \mathbb{C}$ be the region*

$$|\mathrm{Im}\,\Omega| < c_1 \,, \qquad |\Omega| \geq \mathcal{C}_4 \,. \tag{A.6}$$

Then for any $n \in \mathbb{N}_0$, there is a constant $c = c(n, c_1, \mathcal{C}_4) > 0$ such all spectral points λ of the angular operator $\mathcal{A}_\omega$ restricted to the image of the operator Q^ω_n are in the range

$$\frac{|\Omega|}{c} \leq |\lambda| \leq c\,|\Omega| \qquad \text{for all } \Omega \in U \,. \tag{A.7}$$

Proof. If Ω is real, the inequalities (A.7) were already derived in Lemma A.1 and Proposition A.2. In order to extend these results to complex Ω, similar as in [21, Section 16] we consider the homotopy

$$W_\tau = \tau\, W[\Omega] + (1-\tau)\, W[\mathrm{Re}\,\Omega] \tag{A.8}$$

with $\tau \in [0,1]$ (where the argument in the square brackets is the respective value for the parameter Ω in (A.2)) Then for $\tau = 0$, the potential W_τ is real, and the results of Lemma A.1 and Proposition A.2 apply.

In [21, Section 16] a similar homotopy was considered, and the eigenvalues were traced in detail. However, as we will become clear below, these estimates are not good enough for getting the lower bound in (A.6) for the first N spectral points, making it necessary to slightly refine the method. In preparation, we now explain an alternative method for tracking the eigenvalues. Apart from giving a different point of view, this method has the advantage that it can be refined to get the required control of the first N spectral points.

For families of self-adjoint operators, the change of the eigenvalues can be estimated in terms of the sup-norm of the perturbation, i.e. (see for example [23])

$$\big|\Delta \lambda_n\big| \leq \|\Delta W\| \,. \tag{A.9}$$

This inequality is not necessarily true for non-selfadjoint operators. But it holds in our setting up to a uniform constant, if we make use of the fact that the spectral operators Q^ω_n are uniformly bounded (7.2). In order to make the argument precise, we first consider a non-degenerate eigenspace, in which case the eigenvalue $\lambda_n(\tau)$ depends smoothly on τ. Differentiating the eigenvalue equation

$$\big(H(\tau) - \lambda_n(\tau)\big)\, \phi_n(\tau) = 0$$

with respect to τ, we obtain

$$\big(H - \lambda_n\big)\, \dot{\phi}_n = \big(\dot{H} - \dot{\lambda}_n\big)\, \phi_n$$

(where we omitted the argument τ, and the dot denotes the τ-derivative). Multiplying by Q^ω_n, the left side vanishes, and thus

$$0 = Q^\omega_n \,\big(\dot{H} - \dot{\lambda}_n\big)\, \phi_n = Q^\omega_n\, \dot{H} \phi_n - \dot{\lambda}_n\, \phi_n \,. \tag{A.10}$$

Taking the norm and using (7.2), we obtain the estimate

$$\big|\dot{\lambda}_n\big| \leq c_2\, \big\|\dot{H}\big\| = c_2\, \big\|\dot{W}_\tau\big\| \,. \tag{A.11}$$

Integrating this inequality, we obtain (A.9), up to the constant c_2,

$$\big|\Delta \lambda_n\big| \leq c_2\, \|\Delta W\| \,. \tag{A.12}$$

In the general case with degeneracies, the situation is more involved, because the eigenvalues no longer depend smoothly on τ. But the spectrum is still continuous in τ. Moreover, the eigenvalues depend smoothly on τ except at the points where degeneracies form and the dimensions of the invariant subspaces change. Therefore, the inequality (A.11) shows that the total variation of the change of the spectral points can be estimated by a constant times $\|\Delta W\|$.

Using the assumption (A.6) in (A.2), one sees that for our homotopy (A.8),

$$\|\Delta W\| \equiv \|W_1 - W_0\| \lesssim |\Omega| \,.$$

Using this inequality in (A.12), we obtain

$$\big|\lambda_n[\Omega] - \lambda_n[\operatorname{Re}\Omega]\big| \lesssim |\Omega| \,. \tag{A.13}$$

Combining this estimate with the upper bound in (A.3), we obtain the upper bound in (A.7). Moreover, the lower bound in (A.3) gives the lower bound in (A.7), but only if n is sufficiently large and Ω not too large.

In order to derive the lower bound in (A.3), we make use of the fact that for large $|\Omega|$, the eigenfunction is localized mainly near the poles, where the imaginary part of W is bounded uniformly in $|\Omega|$. This is made precise by the following estimate: We choose a test function $\eta \in C^\infty((0,\pi))$ taking values in the interval $[0,1]$ with the properties

$$\operatorname{supp}\eta \subset \Big[|\Omega|^{-\frac{1}{2}}, \pi - |\Omega|^{-\frac{1}{2}}\Big] \qquad \text{and} \qquad \eta\big|_{\big[2\,|\Omega|^{-\frac{1}{2}},\, \pi - 2\,|\Omega|^{-\frac{1}{2}}\big]} \equiv 1 \,. \tag{A.14}$$

Clearly, η can be chosen such that $\sup_{[0,\pi]} |\eta''| \lesssim |\Omega|$. Integrating by parts, we obtain

$$2\int_0^\pi \eta \,\operatorname{Re}V\, |\phi|^2\, du = \int_0^\pi \eta\left(\overline{\phi}\phi'' + \overline{\phi''}\phi\right) du = \int_0^\pi \left(\eta''\,|\phi|^2 - |\phi'|^2\right) du \,,$$

giving rise to the inequality

$$\int_0^\pi \eta\,\operatorname{Re}V\,|\phi|^2\,du \lesssim |\Omega|\,\|\phi\|_{L^2} \,.$$

Using (A.2) as well as the upper bound in (A.7), we conclude that

$$\operatorname{Re}\left(\Omega^2\right)\int_0^\pi \eta\,|\phi|^2\,\sin^2 u\,du \lesssim |\Omega|\,\|\phi\|_{L^2} \tag{A.15}$$

(note that, in view of (A.14), the summands with poles in (A.2) are pointwise bounded on the support of η by a constant times $|\Omega|$). Moreover, using that the function $\sin u$ has zeros at $u=0$ and $u=\pi$, we also have

$$\operatorname{Re}\left(\Omega^2\right)\int_0^\pi (1-\eta)\,|\phi|^2\,\sin^2 u\,du \lesssim |\Omega|\,\|\phi\|_{L^2}$$

Adding this estimate to (A.15), we obtain the inequality

$$\int_0^\pi |\phi|^2\,\sin^2 u\,du \lesssim \frac{\|\phi\|_{L^2}}{|\Omega|} \,. \tag{A.16}$$

We now multiply our flow equation (A.10) by $\overline{\phi}$ and integrate. Omitting the index n of the wave function ϕ_n, this gives the estimate

$$\big|\dot\lambda_n\big|\|\phi\|_{L^2} \leq \big\|Q_n^\omega\big\|\,\big\|\dot W_\tau\phi\big\|_{L^2} \,. \tag{A.17}$$

Next, we estimate the last factor as follows,

$$\big\|\dot W_\tau \phi\big\|^2_{L^2} = \int_0^\pi \big|\dot W_\tau\big|^2\,|\phi|^2\,du \leq \big\|\dot W_\tau\big\|_\infty \int_0^\pi \big|\dot W_\tau\big|\,|\phi|^2\,du \lesssim |\Omega| \int_0^\pi \big|\dot W_\tau\big|\,|\phi|^2\,du \,.$$

Again using the explicit form of the potential (A.2), one sees that

$$\big|\dot W_\tau - 2i\Omega\,\sin^2 u\big| \lesssim 1 \,.$$

We thus obtain

$$\big\|\dot{W}_\tau \phi\big\|^2_{L^2} \lesssim |\Omega|\, \|\phi\|^2_{L^2} + |\Omega|^2 \int_0^\pi \sin^2 u\, |\phi|^2\, du \overset{(A.16)}{\lesssim} |\Omega|\, \|\phi\|^2_{L^2}\,.$$

Using this inequality in (A.17), we get

$$\big|\dot{\lambda}_n\big| \lesssim \sqrt{|\Omega|}\, \big\|Q^\omega_n\big\| \lesssim \sqrt{|\Omega|}\,,$$

where in the last step we again used (7.2). We thus obtain the following improvement of (A.13),

$$\big|\lambda_n[\Omega] - \lambda_n[\operatorname{Re}\Omega]\big| \lesssim \sqrt{|\Omega|}\,.$$

Combining this estimate with the result of Lemma A.1 gives the lower bound in (A.7). This concludes the proof. □

Acknowledgments: We would like to thank Yunlong Zang for helpful comments on the manuscript. F.F. is grateful to the Center of Mathematical Sciences and Applications at Harvard University for hospitality and support.

References

[1] L. Andersson, T. Bäckdahl, and P. Blue, *A new tensorial conservation law for Maxwell fields on the Kerr background*, arXiv:1412.2960 [gr-qc] (2014).

[2] L. Andersson and P. Blue, *Uniform energy bound and asymptotics for the Maxwell field on a slowly rotating Kerr black hole exterior*, arXiv:1310.2664 [math.AP], J. Hyperbolic Differ. Equ. **12** (2015), no. 4, 689–743.

[3] ———, *Decay of solutions to the Maxwell equation on the Schwarzschild background*, arXiv:1501.04641 [math.AP], Class. Quantum Grav. **33** (2016), 085010.

[4] A. Bachelot, *Superradiance and scattering of the charged Klein-Gordon field by a step-like electrostatic potential*, J. Math. Pures Appl. (9) **83** (2004), no. 10, 1179–1239.

[5] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford Classic Texts in the Physical Sciences, The Clarendon Press Oxford University Press, New York, 1998.

[6] M. Dafermos, G. Holzegel, and I. Rodnianski, *The linear stability of the Schwarzschild solution to gravitational perturbations*, arXiv:1601.06467 [gr-qc] (2016).

[7] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman, *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case* $|a| < M$, arXiv:1402.7034 [gr-qc], Ann. of Math. (2) **183** (2016), no. 3, 787–913.

[8] V. de Alfaro and T. Regge, *Potential scattering*, North-Holland Publishing Company, Amsterdam, 1965.

[9] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau, *Decay rates and probability estimates for massive Dirac particles in the Kerr-Newman black hole geometry*, arXiv:gr-qc/0107094, Comm. Math. Phys. **230** (2002), no. 2, 201–244.

[10] ———, *The long-time dynamics of Dirac particles in the Kerr-Newman black hole geometry*, arXiv:gr-qc/0005088, Adv. Theor. Math. Phys. **7** (2003), no. 1, 25–52.

[11] ———, *An integral spectral representation of the propagator for the wave equation in the Kerr geometry*, gr-qc/0310024, Comm. Math. Phys. **260** (2005), no. 2, 257–298.

[12] ———, *Decay of solutions of the wave equation in the Kerr geometry*, gr-qc/0504047, Comm. Math. Phys. **264** (2006), no. 2, 465–503.

[13] ———, *Linear waves in the Kerr geometry: a mathematical voyage to black hole physics*, arXiv:0801.1423 [math-ph], Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 635–659.

[14] ———, *A rigorous treatment of energy extraction from a rotating black hole*, arXiv:gr-qc/0701018, Comm. Math. Phys. **287** (2009), no. 3, 829–847.

[15] F. Finster and H. Schmid, *Spectral estimates and non-selfadjoint perturbations of spheroidal wave operators*, J. Reine Angew. Math. **601** (2006), 71–107.

[16] F. Finster and J. Smoller, *A time-independent energy estimate for outgoing scalar waves in the Kerr geometry*, arXiv:0707.2290 [math-ph], J. Hyperbolic Differ. Equ. **5** (2008), no. 1, 221–255.

[17] ______, *Decay of solutions of the Teukolsky equation for higher spin in the Schwarzschild geometry*, arXiv:gr-qc/0607046, Adv. Theor. Math. Phys. **13** (2009), no. 1, 71–110.
[18] ______, *Error estimates for approximate solutions of the Riccati equation with real or complex potentials*, arXiv:0807.4406 [math-ph], Arch. Ration. Mech. Anal. **197** (2010), no. 3, 985–1009.
[19] ______, *Absence of zeros and asymptotic error estimates for Airy and parabolic cylinder functions*, arXiv:1207.6861 [math.CA], Commun. Math. Sci. **12** (2014), no. 1, 175–200.
[20] ______, *Refined error estimates for the Riccati equation with applications to the angular Teukolsky equation*, arXiv:1307.6470 [math.CA], Methods Appl. Anal. **22** (2015), no. 1, 67–100.
[21] ______, *A spectral representation for spin-weighted spheroidal wave operators with complex aspherical parameter*, arXiv:1507.05756 [math-ph], Methods Appl. Anal. **23** (2016), no. 1, 35–118.
[22] P. Hintz and A. Vasy, *The global non-linear stability of the Kerr-de Sitter family of black holes*, arXiv:1606.04014 [math.DG] (2016).
[23] T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
[24] C.O. Lousto and B.F. Whiting, *Reconstruction of black hole metric perturbations from the Weyl curvature*, arXiv:gr-qc/0203061, Phys. Rev. D (3) **66** (2002), no. 2, 024026, 7.
[25] J. Metcalfe, D. Tataru, and M. Tohaneanu, *Price's law on nonstationary space-times*, arXiv:1104.5437 [math.AP], Adv. Math. **230** (2012), no. 3, 995–1028.
[26] ______, *Pointwise decay for the Maxwell field on black hole space-times*, arXiv:1411.3693 [math.AP] (2014).
[27] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, and C.W. Clark (eds.), *Digital Library of Mathematical Functions*, National Institute of Standards and Technology from http://dlmf.nist.gov/ (Version 1.0.11; Release date 2016-06-08), Washington, DC, 2016.
[28] J. Sterbenz and D. Tataru, *Local energy decay for Maxwell fields Part I: Spherically symmetric black-hole backgrounds*, arXiv:1305.5261 [math.AP], Int. Math. Res. Not. IMRN (2015), no. 11, 3298–3342.
[29] D. Tataru, *Local decay of waves on asymptotically flat stationary space-times*, arXiv:0910.5290 [math.AP], Amer. J. Math. **135** (2013), no. 2, 361–401.
[30] D. Tataru and M. Tohaneanu, *A local energy estimate on Kerr black hole backgrounds*, arXiv:0810.5766 [math.AP], Int. Math. Res. Not. IMRN (2011), no. 2, 248–292.
[31] M.E. Taylor, *Partial Differential Equations. I*, Applied Mathematical Sciences, vol. 115, Springer-Verlag, New York, 1996.
[32] S.A. Teukolsky, *Perturbations of a rotating black hole I. Fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations*, Astrophys. J. **185** (1973), 635–647.
[33] B.F. Whiting, *Mode stability of the Kerr black hole*, J. Math. Phys. **30** (1989), no. 6, 1301–1305.

Fakultät für Mathematik, Universität Regensburg, D-93040 Regensburg, Germany
E-mail address: `Felix.Finster@mathematik.uni-regensburg.de`

Mathematics Department, The University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: `smoller@umich.edu`