Analysis of diffuse interface models for two-phase flows with and without surfactants

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Abstract

We consider two diffuse interface models for two-phase flows with and without surfactants. The model without surfactants is a thermodynamically consistent diffuse interface model for two-phase flows with different densities in a bounded domain. For this model we prove existence and uniqueness of strong solutions for a short time in the case of two or three space dimensions. We linearize the system of partial differential equations, split it into a linear and nonlinear part where the nonlinear part is Lipschitz continuous and apply the Banach fixed-point theorem, which yields the existence of a unique strong solution for a short time.
The model with surfactants extends the first model to the case where surfactants are soluble in both phases. We prove existence of weak solutions in a bounded domain for two or three space dimensions. To this end, we use a semi-implicit time discretization and prove existence of weak solutions for the time-discrete problem by applying the Leray-Schauder principle. Then we pass to the limit in two approximation steps using appropriate compactness results and showing that every weak solution of this phase field model satisfies an energy estimate. Moreover, we study the sharp interface limit of this model for the case $\rho \equiv 1$ via the method of formally matched asymptotic expansions. In this way we recover the corresponding sharp interface model.

Contents

1 Introduction

In recent years, several diffuse interface models have been developed to describe the behaviour of two-phase flows with or without surfactants. In the present work we consider one model without surfactants and one model where surfactants are soluble in both phases. The model without surfactants is a thermodynamically consistent diffuse interface model for two-phase flows of two incompressible fluids with different densities developed by Abels, Garcke and Grün in [AGG12]. Thermodynamically consistent means that the model satisfies local entropy or free energy inequalities and therefore fulfills the second law of thermodynamics. The second model extends the model without surfactants to the case where surfactants are soluble in both fluids. The word surfactant is a blend of surface, active and agent as it is a compound that affects the surface by accumulating on the interface. Surfactants consist of a hydrophilic head and a hydrophobic tail and they reduce the surface tension of fluid interfaces. This ability is used in industrial and domestic applications, e.g. in detergents, where surfactants increase solubility of grease and dirt particles by reducing the surface tension of the interface between the water and the particles. Moreover, surfactants are used in biochemistry, photography, firefighting, biological systems and many other applications. The model we study is thermodynamically consistent and it is a variant of the models developed by Garcke, Lam and Stinner in [GLS14]. Note that we assume that the surfactants are soluble in both phases. This leads to an exchange of the surfactants between the interface and the bulk driven by adsorption and desorption. Before we present the different models and discuss which results already exist, we first of all want to explain what a diffuse interface model is. Moreover, we give a short overview which other phase field models have been derived in recent years to describe the behaviour of two-phase flows.

The classical models for two-phase flows in fluid dynamics are the sharp interface models. In these models we consider two bulk phases for the fluids in a bounded domain $\Omega \subseteq \mathbb{R}^d$ with $d = 2, 3$. These bulk phases are separated by an interface, which is a $(d-1)$-dimensional surface. But these models fail when the topology of the surface develops singularities and therefore they can not describe processes such as merging and reconnection of several parts of the fluid interfaces. They do not consider the possibility of mixing in a narrow area near the interface and thus exclude partially miscible fluids.

Therefore, the diffuse interface models or also called phase field models were developed. In these models one allows for partial mixing in a thin interfacial region. To this end, we introduce an order parameter φ in the diffuse interface model, which represents the volume fraction difference of both fluids. It takes values close to -1 in the second phase and $+1$ in the first phase and it changes its value very fast from -1 to $+1$ in an interfacial region which is called the diffuse interface. Moreover, we introduce another parameter $\varepsilon > 0$, which we assume to be very small and which is related to the "thickness" of the diffuse interface.

For $\varepsilon \to 0$ we would like to recover the sharp interface model on which the diffuse interface model is based. To this end, we use the method of formally matched asymptotic expansions which yields in the limit, when ε converges to 0, the corresponding sharp interface model. But note that the results by this method are only formal. In the method of formally matched asymptotic expansions we construct two expansions, where we assume that one expansion is valid away from the interface and the other expansion is valid near the interface. Since both expansions are solutions to the same problem, we expect that there exists a region where both solutions are valid, i.e., the asymptotic expansion in the bulk region has to match with the expansion in the interfacial region. As a result we obtain the sharp interface model of the corresponding diffuse interface model. The phase field model which is related to a certain sharp interface model is in general not uniquely determined, cf. [LLRV09].

Diffuse interface models have gained popularity to describe two-phase flows for theoretical studies as well as a tool for numerical simulations. The standard diffuse interface model for two-phase flows is called "model H" and it was developed by Hohenberg and Halperin in 1977, cf. [HH77]. For another derivation we refer to Gurtin, Polignone and Viñals, cf. [GPV96]. The "model H" is valid for two incompressible, viscous Newtonian fluids with identical densities and is given by the Navier-Stokes/Cahn-Hilliard system

$$\begin{aligned}\rho\partial_t\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \nabla p &= -\sigma\varepsilon\operatorname{div}(\nabla\varphi\otimes\nabla\varphi),\\ \operatorname{div}\,\mathbf{v} &= 0,\\ \partial_t\varphi + \mathbf{v}\cdot\nabla\varphi &= \operatorname{div}(m\nabla\mu),\\ \mu &= \sigma\varepsilon^{-1}W'(\varphi) - \sigma\varepsilon\Delta\varphi,\end{aligned}$$

where we use a similar notation as in [AGG12] and [ADG13], i.e., $\mathbf{v}$ is the mean velocity of both fluids, $D\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$ and the constant ρ is the density of the mixture. Moreover, the pressure is denoted by p and φ is the order parameter which is related to the concentration difference of both fluids. Furthermore, $\eta(\varphi) > 0$ is the viscosity of the mixture, W is the homogeneous free energy density and $m = m(\varphi) \geq 0$ is the mobility coefficient, which models the diffusion of both fluids. The chemical potential is denoted by μ, the constant σ is the surface tension coefficient related to the energy density on the surface and $\varepsilon > 0$ is the parameter associated to the "thickness" of the interfacial region. This model consists of the momentum equation for a divergence-free velocity field together with the convective Cahn–Hilliard equation for the order parameter. Gurtin, Polignone and Viñals proved that the "model H" is thermodynamically consistent, cf. [GPV96]. For the free energy density W there exist several choices which lead to different analytical difficulties. One possibility is to choose W as a smooth double-well potential, e.g. $W(\varphi) = C(1-\varphi^2)^2$ for a constant $C > 0$. Its main features are that it is defined on $\mathbb{R}$ with $W(\pm 1) = W'(\pm 1) = 0$, $W''(\pm 1) > 0$ and $W(\varphi) > 0$ for $\varphi \neq \pm 1$. But this free energy density allows W to attain values

outside the interval $[-1,1]$. Therefore, another approach proposed by Cahn and Hilliard in [CH58] is to choose W as a logarithmic free energy, e.g. $W(\varphi) = \frac{\theta}{2}((1+\varphi)\ln(1+\varphi)+(1-\varphi)\ln(1-\varphi)) - \frac{\theta_\varphi}{2}\varphi^2$ for $\varphi \in [-1,1]$ and $\theta, \theta_\varphi > 0$, cf. (1.1) in [ADG13]. Another possibility is to define W as a double obstacle potential, e.g. $W(\varphi) = \frac{1}{2}(1-\varphi^2) + I_{[-1,1]}(\varphi)$, where $I_{[-1,1]} = 0$ for $\varphi \in [-1,1]$ and $I_{[-1,1]} = +\infty$ for $\varphi \notin [-1,1]$. For $\Omega = \mathbb{R}^2$, constant viscosity η and a suitable double-well potential W, Starovoĭtov proved the existence of strong solutions, cf. [Sta97]. If $\Omega \subseteq \mathbb{R}^d$, $d = 2,3$, is a periodical channel and W is a suitable double-well potential, the existence of global weak solutions and the existence of unique strong solutions for short time was obtained by Boyer in [Boy99]. For a class of singular free energy densities, Abels proved the existence of weak solutions for the space dimensions $d = 2,3$ and the existence of strong solutions globally in time for two space dimensions and locally in time for three space dimensions in [Abe09b].
But the "model H" assumes constant densities for the fluids and the mixture. For different densities, there have been several approaches to develop appropriate diffuse interface models, e.g. by Lowengrub and Truskinovsky [LT98] and Ding, Spelt and Shu [DSS07]. The model proposed in [LT98] is thermodynamically consistent and extends the "model H" since it allows different densities. But the model leads to a velocity field which is not divergence-free. The model is given by

$$
\begin{aligned}
\rho\partial_t \mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \operatorname{div}\mathbf{S}(\varphi, D\mathbf{v}) + \nabla p &= -\sigma\varepsilon \operatorname{div}(\rho\nabla\varphi\otimes\nabla\varphi),\\
\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) &= 0,\\
\rho\partial_t\varphi + \rho\mathbf{v}\cdot\nabla\varphi &= \operatorname{div}(m(\varphi)\nabla\mu),\\
\mu &= -\rho^{-2}\frac{\partial\rho}{\partial\varphi}p + \frac{\sigma}{\varepsilon}W'(\varphi) - \frac{\sigma\varepsilon}{\rho}\operatorname{div}(\rho\nabla\varphi),
\end{aligned}
$$

where $\rho = \rho(\varphi)$, $\mathbf{S}(\varphi, D\mathbf{v}) = 2\eta(\varphi)D\mathbf{v} + \lambda(\varphi)\operatorname{div}(\mathbf{v})\mathbf{I}$ is the viscous part of the stress tensor and $\lambda(\varphi)$ is the bulk viscosity coefficient. One problem which arises with this model is that the velocity field $\mathbf{v}$ is not divergence-free. Hence, the standard solution concepts are not applicable. Moreover, the pressure p also appears in the equation for μ, i.e., the coupling of the system is stronger. Abels proved the existence of weak solutions in [Abe09a] and the existence of strong solutions locally in time in [Abe12b]. The model proposed by Ding, Spelt and Shu in [DSS07] is given by the equations of the "model H", but with a variable density $\rho = \rho(\varphi)$. It is unknown if the model is thermodynamically consistent. For more results, we refer to [AGG12] and [ADG13].

The model proposed in [AGG12] is a thermodynamically consistent, frame indifferent diffuse interface model for two-phase flows with different densities in a bounded domain in two or three space dimensions without surfactants. If the mobility is positive or if it converges to 0 slower than linearly with respect to ε, then the convergence of weak solutions to solutions of a corresponding sharp interface model was rigorously shown by Abels and Lengeler in [AL14]. The existence of weak solutions for this model was proven by Abels, Depner and Garcke in [ADG13]. The model consists of the following equations

$$
\begin{aligned}
\partial_t(\rho\mathbf{v})+\operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) + \operatorname{div}\left(\mathbf{v}\otimes\frac{\tilde{\rho}_1-\tilde{\rho}_2}{2}m(\varphi)\nabla(\frac{1}{\varepsilon}W'(\varphi)-\varepsilon\Delta\varphi)\right)+\nabla p& \\
=\operatorname{div}(-\varepsilon\nabla\varphi\otimes\nabla\varphi)+\operatorname{div}(2\eta(\varphi)D\mathbf{v}) &\quad \text{in } Q_T,\\
\operatorname{div}(\mathbf{v})=0 &\quad \text{in } Q_T,\\
\partial_t^{\bullet}\varphi=\operatorname{div}\left(m(\varphi)\nabla\mu\right) &\quad \text{in } Q_T,\\
\mu=-\varepsilon\Delta\varphi+\frac{1}{\varepsilon}W'(\varphi) &\quad \text{in } Q_T,
\end{aligned}
$$

together with the initial and boundary values

$$
\begin{aligned}
\mathbf{v}_{|\partial\Omega}=\partial_n\mu_{|\partial\Omega}=\partial_n\varphi_{|\partial\Omega}=0 &\qquad \text{on } (0,T)\times\partial\Omega,\\
\varphi(0)=\varphi_0, \mathbf{v}(0)=\mathbf{v}_0 &\qquad \text{in } \Omega.
\end{aligned}
$$

In these equations, the notation is the same as before, i.e., $\mathbf{v}$ is the mean velocity, φ is the order parameter for the difference of the volume fractions of both fluids, p is the pressure and W is the homogeneous free energy density. Furthermore, $\partial_n = n\cdot\nabla$, where n denotes the exterior normal at $\partial\Omega$ and $Q_T = \Omega\times(0,T)$ for a sufficiently smooth bounded domain $\Omega\subseteq\mathbb{R}^d$, $d=2,3$. Moreover, $D\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T)$ and $\partial_t^{\bullet}\varphi = \partial_t\varphi+\mathbf{v}\cdot\nabla\varphi$ is the material time derivative. The equations describe the momentum equation, the incompressibility condition and the process of phase separation. In Chapter 5 we prove the existence of a unique strong solution for short time in two or three space dimensions.

In [GLS14], Garcke, Lam and Stinner developed several mathematical models which describe the behaviour of surfactants in two-phase flows evolving in time. In these models, surfactants are soluble in possibly both fluids. The model we consider in this work assumes instantaneous adsorption and is two-sided, i.e., the surfactant is soluble in both phases. This model is related to the model denoted by "model C" in [GLS14] and it extends the model in [AGG12] to the case where surfactants are soluble in both fluids. "Model A" assumes dynamic adsorption and "model B" assumes instantaneous adsorption which is one-sided. The model we study leads to a system of partial differential equations given by

$$\begin{aligned}
\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes(\rho\mathbf{v}+\tilde{\mathbf{J}})) + \nabla p - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) - \frac{R\mathbf{v}}{2}& \\
= \operatorname{div}(-\varepsilon\nabla\varphi\otimes\nabla\varphi) &\qquad \text{in } Q_T, \qquad (1.1)\\
\operatorname{div}\,\mathbf{v} = 0 &\qquad \text{in } Q_T, \qquad (1.2)\\
\partial_t^\bullet\left(\frac{1}{\varepsilon}f(q)W(\varphi)+g(q)\right) = \operatorname{div}\left(m(\varphi,q)\nabla q\right) &\qquad \text{in } Q_T, \qquad (1.3)\\
\partial_t^\bullet\varphi = \operatorname{div}(\tilde{m}(\varphi)\nabla\mu) &\qquad \text{in } Q_T, \qquad (1.4)\\
-\varepsilon\Delta\varphi + h(q)\frac{1}{\varepsilon}W'(\varphi) = \mu &\qquad \text{in } Q_T, \qquad (1.5)
\end{aligned}$$

with initial values

$$\mathbf{v}_{|t=0} = \mathbf{v}_0, \quad \varphi_{|t=0} = \varphi_0, \quad \left(\frac{1}{\varepsilon}f(q)W(\varphi)+g(q)\right)_{|t=0} = \frac{1}{\varepsilon}f(q_0)W(\varphi_0)+g(q_0) \quad \text{in } \Omega, \tag{1.6}$$

and boundary conditions

$$\mathbf{v}_{|\partial\Omega} = 0, \qquad \partial_n\varphi_{|\partial\Omega} = 0, \qquad \partial_n q_{|\partial\Omega} = 0, \qquad \partial_n\mu_{|\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{1.7}$$

where

$$\tilde{\mathbf{J}} = \frac{\partial\rho(\varphi)}{\partial\varphi}(-\tilde{m}(\varphi)\nabla\mu), \qquad R = -\nabla\frac{\partial\rho(\varphi)}{\partial\varphi}\cdot(\tilde{m}(\varphi)\nabla\mu).$$

Moreover, we set

$$d(q) = h(q) + f(q)q$$

for all $q \in \mathbb{R}$ and demand

$$d'(q) = f'(q)q.$$

In this model, the notation is the same as in the “model H”. The mobility coefficient for the diffusion of both fluids is denoted by $\tilde{m}(\varphi)$ and $m(\varphi, q)$ is the mobility for the diffusion of the surfactant. Moreover, q denotes the chemical potential of the surfactant, $h(q)$ is related to the surface tension and $\frac{1}{\varepsilon}f(q)W(\varphi)+g(q)$ is the free energy density, where $\frac{1}{\varepsilon}f(q)W(\varphi)$ is the part related to mixing of both phases and $g(q)$ is the part of the free energy density related to the surfactant concentration. The flux of the fluid density is denoted by $\tilde{\mathbf{J}}$. For the density ρ we assume $\rho \in C_b^2(\mathbb{R})$ such that

$$\rho = \rho(\varphi) = \frac{\tilde{\rho}_1+\tilde{\rho}_2}{2} + \frac{\tilde{\rho}_2-\tilde{\rho}_1}{2}\varphi \qquad \text{if } \varphi\in[-1,1] \tag{1.8}$$

and $\inf\limits_{s\in\mathbb{R}}\rho(s) > 0$. Note that we can not define ρ by (1.8) on $\mathbb{R}$ since then ρ can be negative. This is the case because we suppose W to be non-singular. More precisely, we assume W to be a suitable smooth double-well potential. Hence in the analysis we are not able to conclue $\varphi(x) \in [-1,1]$ for every $x \in \Omega$. Therefore, we define ρ in such a way that it is defined as in (1.8) for every φ in the physically meaningful interval and we demand $\inf\limits_{s\in\mathbb{R}}\rho(s) > 0$ so that $\rho(\varphi)$ is always positive. Note that $\frac{\partial\rho(\varphi)}{\partial\varphi}$ is only constant for $\varphi \in [-1,1]$. Hence, the continuity equation

$$\partial_t\rho(\varphi) + \mathrm{div}(\rho(\varphi)\mathbf{v} + \tilde{\mathbf{J}}) = 0$$

is only satisfied where ρ is defined by (1.8). For ρ defined as above we obtain the modified continuity equation

$$\partial_t\rho(\varphi) + \mathrm{div}(\rho(\varphi)\mathbf{v} + \tilde{\mathbf{J}}) = R,$$

where the modified equation reduces to the continuity equation in the case $\varphi \in [-1,1]$. In the modified equation, R denotes an additional source term and in the momentum equation (1.1) the term $\frac{R\mathbf{v}}{2}$ describes the change in the kinetic energy due to the source term R. Note that this additional source term R also appears in [AB15], where Abels and Breit proved the existence of weak solutions for a diffuse interface model for two-phase flows without surfactants and two non-Newtonian viscous, incompressible fluids of power-law type and different densities, i.e., one can choose $\mathbf{S}(\varphi, D\mathbf{v}) = 2\eta(\varphi)|D\mathbf{v}|^{p-2}D\mathbf{v}$ for some $p > 1$, in the case that Ω is a bounded and sufficiently smooth domain in two or three space dimensions.
Equation (1.1) is the momentum equation derived by the balance of forces according to Newton's Law and the equation $\mathrm{div}(\mathbf{v}) = 0$, cf. (1.2), is the incompressibility condition. The Cahn-Hilliard equation is given by (1.4) and describes the process of phase separation, i.e., it describes the process when the fluids separate in pure phases. Equation (1.5) describes the chemical potential μ and (1.3) is the mass balance equation for the surfactant.
In Chapter 3 we prove existence of weak solutions for (1.1) - (1.5) together with the initial and boundary conditions. To this end, we have to generalize methods used in [ADG13], where the existence of weak solutions for the model without surfactants was proven. In the first step of the existence result for weak solutions, we use a semi-implicit time discretization and insert the additional terms $\delta\Delta^2\mathbf{v}$ and $\delta\partial_t\varphi$. For this time-discrete problem, we show existence of weak solutions by using the Leray-Schauder principle. Then we construct piecewise linear interpolants, pass to the limit $N \to \infty$ and prove the existence of weak solutions for the time dependent case and $\delta > 0$ by applying appropriate compactness results. Finally, we study the case $\delta \to 0$ and prove that every weak solution of the diffuse interface model satisfies an energy estimate and is therefore thermodynamically consistent.

In Chapter 4 we study the sharp interface limit of (1.1) - (1.5) for the case of constant density ρ via the method of formally matched asymptotic expansions and derive an energy estimate. Moreover, we identify the relation between the sharp interface model derived in [GLS14] and the sharp interface model which we recover from the diffuse interface model (1.1) - (1.5) for constant density ρ. From this phase field model we recover the following sharp interface model

$$\begin{aligned}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p &= \operatorname{div}\left(2\eta^{(i)} D\mathbf{v}\right) && \text{in } \Omega^{(i)}(t),\\
\operatorname{div}(\mathbf{v}) &= 0 && \text{in } \Omega^{(i)}(t),\\
\partial_t g(q) + \nabla g(q) \cdot \mathbf{v} &= \Delta q && \text{in } \Omega^{(i)}(t),\\
[p]_-^+ \nu - 2[\eta D\mathbf{v}]_-^+ \nu - \kappa \sigma(c^\Gamma(q))\nu &= \nabla_\Gamma \left(\sigma(c^\Gamma(q))\right) && \text{on } \Gamma(t),\\
-\mathcal{V} + \mathbf{v} \cdot \nu &= 0 && \text{on } \Gamma(t),\\
\partial_t^\bullet c^\Gamma(q) + c^\Gamma(q)\operatorname{div}_\Gamma \mathbf{v} - \operatorname{div}_\Gamma \left(M_\Gamma(q)\nabla_\Gamma q\right) &= [\nabla q \cdot \nu]_-^+ && \text{on } \Gamma(t),\\
[\mathbf{v}]_-^+ = [q]_-^+ = [\mathbf{v} \cdot \nu]_-^+ &= 0 && \text{on } \Gamma(t).
\end{aligned}$$

In these equations, $\Omega^{(i)}(t)$ denotes the bulk phase of fluid i for $i = 1, 2$. Moreover, $\Gamma(t)$ denotes the evolving interface between the two bulk phases and $[\cdot]_-^+$ denotes the jump of a quantity across the interface $\Gamma(t)$ from bulk $\Omega^{(1)}(t)$ to $\Omega^{(2)}(t)$. Furthermore, κ is the mean curvature of $\Gamma(t)$, i.e., the sum of the principal curvatures κ_i, and $|\mathcal{S}|$ is the spectral norm of the Weingarten map $d\nu_{\gamma(t,s)}$, where ν is the unit normal on $\Gamma(t)$ pointing into phase 2. The local parametrization of $\Gamma(t)$ is given by $\hat{\gamma}$ and $\mathcal{V} = \partial_t \hat{\gamma} \cdot \nu$ is the normal velocity for the parametrization of the evolving hypersurface $\Gamma(t)$. The surface gradient and surface divergence on $\Gamma(t)$ are denoted by ∇_Γ and $\operatorname{div}_\Gamma$. Note that the sharp interface model considered in [BP10] and [BPS05] is most similar to the model which we recover from the diffuse interface model by the method of formally matched asymptotic expansions, see also [GLS14].

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2 Mathematical Background

In this chapter we introduce the notation which we will use in the following and present some useful results. To this end, we introduce several function spaces, e.g. the Sobolev spaces, Besov spaces, Bessel potential spaces and Banach valued Sobolev spaces, and cite some basic results and embedding properties for these spaces. Moreover, we introduce the real interpolation method and state several results from interpolation theory.

2.1 Notation

Let $\mathbf{u} : \Omega \to \mathbb{R}^d$ be a vector field, i.e., $\mathbf{u}(x) = (u_1(x), ..., u_d(x))$ for every $x \in \Omega$, where $\Omega \subseteq \mathbb{R}^d$ and $d = 2, 3$. For such a vector field $\mathbf{u}$ we use a similar notation as in [Soh01]. We define $\partial_j := \partial_{x_j} = \frac{\partial}{\partial x_j}$ for every $j = 1, ..., d$ and $\nabla := (\partial_{x_1}, ..., \partial_{x_d})^T$. Moreover, we define

$$\begin{aligned}
\operatorname{div} \mathbf{u} &:= \nabla \cdot \mathbf{u} := \partial_{x_1} u_1 + ... + \partial_{x_d} u_d \in \mathbb{R}, \\
\Delta \mathbf{u} &:= (\partial_{x_1}^2 + ... + \partial_{x_d}^2) u = (\Delta u_1, ..., \Delta u_d)^T \in \mathbb{R}^d, \\
\nabla \mathbf{u} &:= (\partial_{x_j} u_k)_{j,k=1}^d \in \mathbb{R}^{d \times d}, \\
\mathbf{u} \otimes \mathbf{u} &:= (u_i u_j)_{i,j=1}^d \in \mathbb{R}^{d \times d}
\end{aligned}$$

and

$$\mathbf{u} \cdot \nabla \mathbf{u} := (\mathbf{u} \cdot \nabla) \mathbf{u} := (u_1 \partial_{x_1} u_k + ... + u_d \partial_{x_d} u_k)_{k=1}^d \in \mathbb{R}^d.$$

For a suitable matrix field $M : \Omega \to \mathbb{R}^{d \times d}$, i.e., $M(x) = (M_{ij}(x))_{i,j=1}^d$, we define

$$\operatorname{div} M := (\partial_{x_1} M_{k1} + ... + \partial_{x_d} M_{kd})_{k=1}^d \in \mathbb{R}^d,$$

i.e., div applies to the rows of M, which are in $\mathbb{R}^d$. In particular, this implies for $\mathbf{u} \otimes \mathbf{w}$

$$\operatorname{div}(\mathbf{u} \otimes \mathbf{w}) = (\partial_{x_1}(u_k w_1) + ... + \partial_{x_d}(u_k w_d))_{k=1}^d = \partial_{x_1}(w_1 \mathbf{u}) + ... + \partial_{x_d}(w_d \mathbf{u}),$$

where $\mathbf{u}, \mathbf{w} : \Omega \to \mathbb{R}^d$ are vector fields. This leads to

$$\begin{aligned}
\operatorname{div}(\mathbf{u} \otimes \rho \mathbf{w}) &= \partial_{x_1}(\rho w_1 \mathbf{u}) + ... + \partial_{x_d}(\rho w_d \mathbf{u}) \\
&= (\partial_{x_1}(\rho w_1) + ... + \partial_{x_d}(\rho w_d))\, \mathbf{u} + \rho w_1 \partial_{x_1} \mathbf{u} + ... + \rho w_d \partial_{x_d} \mathbf{u} \\
&= \operatorname{div}(\rho \mathbf{w}) \mathbf{u} + \rho \mathbf{w} \cdot (\nabla \mathbf{u}) \qquad (2.1)
\end{aligned}$$

for vector fields $\mathbf{u}, \mathbf{w} : \Omega \to \mathbb{R}^d$.

The natural numbers without 0 are denoted by $\mathbb{N}$ and we define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a Banach space X we denote its dual space by X' and its duality product by

$$\langle x', x \rangle_{X',X} = \langle x', x \rangle = x'(x)$$

for every $x' \in X'$ and $x \in X$. If H is a Hilbert space, its inner product is denoted by $(\cdot,\cdot)_H$ or $(\cdot,\cdot)$ if the space is obvious from the context. For $R > 0$ and $x_0 \in X$, $B_R(x_0)$ or $B_R^X(x_0)$ denote the open ball around x_0 in X with radius R. If X compactly embeds into Y, this is denoted by $X \hookrightarrow\hookrightarrow Y$.

2.2 Functional Analysis

In this section we present some definitions and basic results from functional analysis, which we will need to prove the existence results. For a normed vector space $(X, ||\cdot||)$, we denote the strong convergence of a sequence $(x_k)_{k\in\mathbb{N}} \subseteq X$ to $x \in X$ by $x_k \to x$, i.e., for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $||x_k - x|| < \varepsilon$ for all $k \geq N$. If a sequence $(x_k)_{k\in\mathbb{N}}$ converges weakly to $x \in X$, i.e., it holds $x'(x_k) \to x'(x)$ for every $x' \in X'$, this is denoted by $x_k \rightharpoonup x$ in X. Now let X, Y be two Banach spaces. Then $\mathcal{L}(X,Y) := \{A : X \to Y :\ \text{A is linear and bounded}\}$ and $\mathcal{L}(X) := \mathcal{L}(X,X)$. To prove existence of weak solutions for (1.1) - (1.7), we will use a semi-implicit time discretization for the equations. For the time-discrete problem, we will need to solve linear elliptic equations of second order. Therefore, the first result we want to remember is the Lax-Milgram theorem, which we will use to prove existence of weak solutions for the elliptic operators. We use the version from [RR04a].

Theorem 2.1. *(Lax-Milgram)*
Let H be a Hilbert space and $B : H \times H \to \mathbb{R}$ be a bilinear mapping such that

$$\begin{aligned} &i)\ |B(x,y)| \leq c_1||x||_H||y||_H && \text{for all } x,y \in H,\\ &ii)\ B(x,x) \geq c_2||x||_H^2 && \text{for all } x \in H \end{aligned}$$

for some constants $c_1, c_2 > 0$. Then for every $f \in H'$ there exists a unique $y \in H$ such that

$$B(x,y) = f(x) \qquad \text{for every } x \in H.$$

Moreover, there exists a constant $C > 0$ independent of f such that

$$||y||_H \leq C||f||_{H'}.$$

The proof can be found in [RR04a, Theorem 9.14]. As already mentioned, we will discretize (1.1) - (1.7) with respect to the time-variable t. This time-discrete system will be solved with the Leray-Schauder principle.

Theorem 2.2. *(Leray-Schauder principle)*
Let X be a Banach space over $\mathbb{K}$ and $A : X \to X$ a compact operator. Suppose that there exists a number $r > 0$ such that if u is a solution of

$$u = tAu, \qquad u \in X, \qquad 0 \leq t < 1,$$

then it holds $||u||_X \leq r$. *Then the equation*

$$u = Au, \qquad u \in X,$$

has a solution.

The proof can be found in [Zei95, Theorem 1.D.]. We proceed with another result from functional analysis. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence which converges weakly to x in a Banach space X. Moreover, we assume that X embeds continously into another Banach space Y. Then we can already conclude $x_k \rightharpoonup x$ in Y by the following lemma.

Lemma 2.3. *Let* X, Y *be Banach spaces,* $T \in \mathcal{L}(X;Y)$ *and* $(x_k)_{k\in\mathbb{N}} \subseteq X$ *such that* $x_k \rightharpoonup x$ *in* X*. Then it holds* $Tx_k \rightharpoonup Tx$ *in* Y*.*

Proof. Let $y' \in Y'$ be arbitrary. Since $T' \in \mathcal{L}(Y';X')$, it holds

$$y'(Tx_k) = (T'y')(x_k) \rightharpoonup (Ty')(x) = y'(Tx).$$

$\square$

To get a parabolic PDE in the unknown q, we will need some definitions for monotone operators. Therefore, we use the definition in [Zei90, Definition 25.2].

Definition 2.4. *Let* X *be a real Banach space and* $A : X \to X'$*. Then*

i) A is called monotone iff

$$\langle Au - Av, u - v\rangle_{X',X} \geq 0 \qquad \text{for all } u, v \in X.$$

ii) A is called strictly monotone iff

$$\langle Au - Av, u - v\rangle_{X',X} > 0 \qquad \text{for all } u, v \in X \text{ with } u \neq v$$

iii) A is called strongly monotone iff there is a constant $C > 0$ *such that*

$$\langle Au - Av, u - v\rangle_{X',X} \geq C||u - v||_X^2 \qquad \text{for all } u, v \in X.$$

In the existence proof of strong solutions for the model without surfactants we will apply Theorem 5.8, which uses the terms subgradient and subdifferential. Therefore, we introduce these terms here, where we use the definition in [Zei90, Definition 32.11].

Definition 2.5. *Let* X *be a Banach space and* $f : X \to [-\infty, +\infty]$*. The functional* $u' \in X'$ *is called subgradient of* f *at the point* u*, if it holds* $f(u) \neq \pm\infty$ *and*

$$f(v) \geq f(u) + \langle u', v - u\rangle_{X',X}$$

holds for all $v \in X$*. The set of all subgradients of* f *at* u *is called the subdifferential at* u *and is denoted by* $\partial f(u)$*. If no subgradients exist, then we set* $\partial f(u) = \emptyset$*. If it holds* $f(u) = \pm\infty$*, then* $\partial f(u) = \emptyset$*.*

2.3 Function Spaces

In this section we introduce the function spaces which we will use in this work and present some results concerning these spaces. For the definition of these spaces, basic knowledge about measure theory and the Lebesgue integral is needed. For an introduction to measure theory, the construction of the Lebesgue integral and essential properties of it we refer to the literature, e.g. [Els05] and [For09].

A domain Ω in $\mathbb{R}^d$ with $d \geq 1$ is an open, non-empty and connected set $\Omega \subseteq \mathbb{R}^d$. For a measurable set $M \subseteq \mathbb{R}^d$ and $1 \leq p \leq \infty$ we denote by $L^p(M)$ the usual space of all measurable functions $f : M \to \mathbb{R}$ such that $||f||_{L^p(M)} < \infty$, where

$$||f||_{L^p(M)} := \begin{cases} \left(\int\limits_M |f(x)|^p \, dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}\limits_{x \in M} |f(x)| & \text{if } p = \infty. \end{cases}$$

If $M = (a, b)$ is an interval in $\mathbb{R}$ we write $L^p(a, b)$. For a domain $\Omega \subseteq \mathbb{R}^d$, we denote by $W_p^k(\Omega)$ with $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$ the usual L^p-Sobolev space of order k. More precisely it is defined as

$$W_p^k(\Omega) := \{f \in L^p(\Omega) : \ \partial_x^\alpha f \in L^p(\Omega) \text{ for every } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq k\}$$

equipped with the norm

$$||f||_{W_p^k(\Omega)} := \sum_{|\alpha| \leq k} ||\partial_x^\alpha f||_{L^p(\Omega)},$$

where $\partial_x f$ is the weak derivative of f with respect to x. The space of all $\varphi \in C^\infty(\Omega)$ with compact support in Ω is denoted by $C_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$. Furthermore, we define

$$W_{p,0}^k(\Omega) := \overline{C_0^\infty(\Omega)}^{||\cdot||_{W_p^k(\Omega)}}, \quad W_p^{-k}(\Omega) := (W_{p',0}^k(\Omega))', \quad W_{p,0}^{-k}(\Omega) := (W_{p'}^k(\Omega))',$$

where p' is the dual Sobolev exponent to p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. The definition of these spaces can be found in many books, e.g. in [Eva10] and [AF03]. In these definitions we always assumed $k \in \mathbb{N}_0$. But it is also possible to define Sobolev spaces for non-integer k. These spaces are called Sobolev-Slobodeckij spaces. So let $s > 0$ and $s \notin \mathbb{N}$ such that $s = \lfloor s \rfloor + \theta$ with $\lfloor s \rfloor \in \mathbb{N}_0$ and $\theta \in (0, 1)$. Then we define the Sobolev-Slobodeckij space with order s in the same way as in [Tri10], i.e.,

$$\begin{aligned} W_p^s(\Omega) &:= \{f \in W_p^{\lfloor s \rfloor}(\Omega) : \ ||f||_{W_p^s(\Omega)} < \infty\}, \\ ||f||_{W_p^s(\Omega)} &:= ||f||_{W_p^{\lfloor s \rfloor}(\Omega)} + \sum_{|\alpha| = \lfloor s \rfloor} \left(\int\limits_\Omega \int\limits_\Omega \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{d+\theta p}} \, dx \, dy \right)^{\frac{1}{p}}. \end{aligned}$$

Since we always consider the velocity field $\mathbf{v}$ to be divergence-free, i.e., $\operatorname{div}(\mathbf{v}) = 0$, we define

$$\begin{aligned} C^\infty_{0,\sigma}(\Omega) &:= \{\boldsymbol{\varphi} \in C^\infty_0(\Omega)^d : \ \operatorname{div}(\boldsymbol{\varphi}) = 0\}, \\ L^2_\sigma(\Omega) &:= \overline{C^\infty_{0,\sigma}(\Omega)}^{||\cdot||_{L^2(\Omega)}} \subseteq L^2(\Omega)^d. \end{aligned}$$

Note that for simplicity we also write $||\mathbf{v}||_{L^2(\Omega)}$ instead of $||\mathbf{v}||_{L^2(\Omega)^d}$ for $\mathbf{v} \in L^2(\Omega)^d$. If $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain, then it holds

$$L^2_\sigma(\Omega) = \{\mathbf{v} \in L^2(\Omega)^d : \ \operatorname{div}(\mathbf{v}) = 0, \ n \cdot \mathbf{v}_{|\partial\Omega} = 0\}, \tag{2.2}$$

where n is the outer unit normal for Ω and $n \cdot \mathbf{v}_{|\partial\Omega}$ is the generalized trace, i.e.,

$$n \cdot \mathbf{v}_{|\partial\Omega} = \langle \cdot, n \cdot \mathbf{v}\rangle_{\partial\Omega} \in W_2^{-\frac{1}{2}}(\partial\Omega) = W_2^{\frac{1}{2}}(\partial\Omega)',$$

where

$$\begin{aligned} W^\alpha_p(\partial\Omega) &:= \{f \in L^p(\partial\Omega) : \ ||f||_{W^\alpha_p(\partial\Omega)} < \infty\}, \\ ||f||_{W^\alpha_p(\partial\Omega)} &:= \left(||f||^p_{L^p(\partial\Omega)} + \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{|f(s_1) - f(s_2)|^p}{|s_1 - s_2|^{d-1+\alpha p}} \, ds_1 \, ds_2 \right)^{\frac{1}{p}}, \end{aligned}$$

for $\alpha \in (0,1)$, cf. (3.6.8) in [Soh01, Chapter I, Section 3.6], and

$$\begin{aligned} L^p(\partial\Omega) &:= \{f : \partial\Omega \to \mathbb{R} : \ f \text{ is measurable and } ||f||_{L^p(\partial\Omega)} < \infty\}, \\ ||f||_{L^p(\partial\Omega)} &:= \left(\int\limits_{\partial\Omega} |f(s)|^p \, ds \right)^{\frac{1}{p}}, \end{aligned}$$

cf. (3.4.4) in [Soh01, Chapter I, Section 3.4]. For more details and a proof of (2.2), we refer to [Soh01, Chapter II, Lemma 2.5.3]. Furthermore, if $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a bounded Lipschitz domain, then we get the characterizations

$$\begin{aligned} W^k_p(\Omega) &= \overline{C^\infty(\overline{\Omega})}^{||\cdot||_{W^k_p(\Omega)}}, \\ W^1_{p,0}(\Omega) &= \{f \in W^1_p(\Omega) : \ f_{|\partial\Omega} = 0\}, \end{aligned}$$

where $f_{|\partial\Omega} := \operatorname{tr}_{\partial\Omega} f$ for $f \in W^1_p(\Omega)$ and $\operatorname{tr}_{\partial\Omega} : W^1_p(\Omega) \to W_p^{1-\frac{1}{p}}(\partial\Omega)$ is the trace operator such that $\operatorname{tr}_{\partial\Omega}\varphi = \varphi_{|\partial\Omega}$ for all $\varphi \in C^\infty(\overline{\Omega})$. The existence of such a trace operator directly follows from [Neč67, Chapter 2, Theorem 5.5 and Theorem 5.7], also see [Soh01, Chapter II, Section 1.2]. The proofs for both characterizations can be found in [Neč67, Chapter 2, Theorem 3.1 and Theorem 4.10] and in [Leo09, Theorem 15.29], cf. [Soh01, Chapter II, Section 1.2].

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$, which is also called the space of all rapidly decreasing smooth functions, is defined by

$$\mathcal{S}(\mathbb{R}^d) := \{\varphi \in C^\infty(\mathbb{R}^d) : \ \forall \alpha, \beta \in \mathbb{N}_0^d : \ \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta \varphi(x)| < \infty\}.$$

Its dual space is denoted by $\mathcal{S}'(\mathbb{R}^d) := (\mathcal{S}(\mathbb{R}^d))'$ and is also called the space of tempered distributions. Now, let $s \in \mathbb{R}$ and $1 < p < \infty$. Then we define the Bessel potential space $H_p^s(\mathbb{R}^d)$ as e.g. in [Abe12a] by

$$\begin{aligned} H_p^s(\mathbb{R}^d) &:= \{f \in \mathcal{S}'(\mathbb{R}^d) : \ \langle D_x \rangle^s f \in L^p(\mathbb{R}^d)\}, \\ ||f||_{H_p^s(\mathbb{R}^d)} &:= ||\langle D_x \rangle^s f||_{L^p(\mathbb{R}^d)}, \end{aligned}$$

where $\langle D_x \rangle^s f = \mathcal{F}^{-1}[\langle \xi \rangle^s \hat{f}]$ for all $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. Here $\hat{f} = \mathcal{F}[f]$ and $\mathcal{F}^{-1}[f]$ are the Fourier transform and the Fourier inverse for $f \in \mathcal{S}'(\mathbb{R}^d)$. For the definitions and basic results about Fourier transformation and tempered distributions, we refer to [Abe12a, Chapter 2]. If $\Omega \subseteq \mathbb{R}^d$ is an non-empty open set in $\mathbb{R}^d$ such that there exists a continuous linear extension operator $E : W_2^s(\Omega) \to W_2^s(\mathbb{R}^d)$ with $Eu_{|\Omega} = u$ for all $u \in W_2^s(\Omega)$, then it holds $H^s(\Omega) = W_2^s(\Omega)$ for all $s \geq 0$, where $H^s(\mathbb{R}^d) := H_2^s(\mathbb{R}^d)$, cf. [McL00, Theorem 3.18].
For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we introduce the Besov spaces $B_{pq}^s(\mathbb{R}^d)$ and define them analogously as in [Abe12a] by

$$\begin{aligned} B_{pq}^s(\mathbb{R}^d) &:= \{f \in \mathcal{S}'(\mathbb{R}^d) : \ ||f||_{B_{pq}^s(\mathbb{R}^d)} < \infty\}, \\ ||f||_{B_{pq}^s(\mathbb{R}^d)} &:= \begin{cases} \left(\sum\limits_{j=0}^{\infty} 2^{sjq} ||\varphi_j(D_x) f||_{L^p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup\limits_{j \in \mathbb{N}_0} 2^{sj} ||\varphi_j(D_x) f||_{L^p(\mathbb{R}^d)} & \text{if } q = \infty, \end{cases} \end{aligned}$$

where $(\varphi_j)_{j \in \mathbb{N}_0} \subseteq C_0^\infty(\mathbb{R}^d)$ is a partition of unity on $\mathbb{R}^d$ such that $\operatorname{supp}(\varphi_0) \subseteq B_2(0)$, $\operatorname{supp}(\varphi_j) \subseteq \{\xi \in \mathbb{R}^d : \ 2^{-j-1} \leq |\xi| \leq 2^{j+1}\}$ for $j \in \mathbb{N}$ and $\varphi_j(D_x) f := \mathcal{F}^{-1}[\varphi_j(\xi) \hat{f}(\xi)]$. Details about the construction of such a partition of unity can be found in [Abe12a, Section 5.4]. From [Abe12a, Corollary 6.13] it follows $H^s(\mathbb{R}^d) = B_{22}^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$.

Since we study diffuse interface models for a bounded domain $\Omega \subseteq \mathbb{R}^d$ with $d = 2, 3$, we need some restriction of the previous definitions for Besov spaces and Bessel potential spaces on the domain Ω. To this end, let $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with $C^{0,1}$-boundary, $s \geq 0$ and $1 \leq p, q \leq \infty$. Then we define as in [Tri10] and [Tri92]

$$B_{pq}^s(\Omega) := \{f \in \mathcal{D}'(\Omega) : \ \exists g \in B_{pq}^s(\mathbb{R}^d) \text{ with } g_{|\Omega} = f\}$$

equipped with the norm

$$||f||_{B^s_{pq}(\Omega)} := \inf_{g \in B^s_{pq}(\mathbb{R}^d), g_{|\Omega} = f} ||g||_{B^s_{pq}(\mathbb{R}^d)}.$$

Analogously we define

$$\begin{aligned} H^s_p(\Omega) &:= \{f \in \mathcal{D}'(\Omega) : \ \exists g \in H^s_p(\mathbb{R}^d) \text{ with } g_{|\Omega} = f\}, \\ ||f||_{H^s_p(\Omega)} &:= \inf_{g \in H^s_p(\mathbb{R}^d), g_{|\Omega} = f} ||g||_{H^s_p(\mathbb{R}^d)}. \end{aligned}$$

From these definitions for the Bessel potential spaces and Besov spaces restricted to Ω together with $H^s(\mathbb{R}^d) = B^s_{22}(\mathbb{R}^d)$ for all $s \in \mathbb{R}$, it directly follows $H^s(\Omega) = B^s_{22}(\Omega)$ for all $s \geq 0$.

Now that we have introduced all these spaces, we are also interested in how Sobolev spaces, Sobolev-Slobodeckij spaces, Besov spaces and Bessel potential spaces are related. First of all we note that for $p = q = \infty$ and $s > 0$ we have the Hölder-Zygmund spaces $C^s_*(\mathbb{R}^d) := B^s_{\infty\infty}(\mathbb{R}^d)$, where $C^s_*(\mathbb{R}^d) = C^s(\mathbb{R}^d)$ in the case of $0 < s < 1$, cf. [Abe12a, Theorem 6.1 and Remark 6.4.1]. Moreover, we use the identity

$$W^s_p(\mathbb{R}^d) = \begin{cases} H^s_p(\mathbb{R}^d), & \text{if } s \in \mathbb{N}_0, \\ B^s_{pp}(\mathbb{R}^d), & \text{if } s > 0 \text{ and } s \notin \mathbb{N}_0 \end{cases}$$

from [Tri78, Section 2.3]. Furthermore, it holds

$$B^s_{p,\min(p,2)}(\mathbb{R}^d) \subseteq H^s_p(\mathbb{R}^d) \subseteq B^s_{p,\max(p,2)}(\mathbb{R}^d), \tag{2.3}$$

which implies in the case $p = 2$

$$B^s_{22}(\mathbb{R}^d) = H^s_2(\mathbb{R}^d) \tag{2.4}$$

for every $s \in \mathbb{R}$, cf. [BL76, Theorem 6.4.4], [Abe12a, Corollary 6.13] or [Tri78, Section 2.3.3].

For a Banach space X and a measurable set $M \subseteq \mathbb{R}^d$, we define the Banach space-valued L^p-functions as in [Yos74]. We denote by $L^p(M; X)$ the set of all strongly measurable functions $f : M \to X$, which are p-integrable, i.e., $||f||_{L^p(M;X)} < \infty$, where

$$||f||_{L^p(M;X)} := || \, ||f(\cdot)||_X ||_{L^p(M)} = \begin{cases} \left(\int\limits_M ||f(x)||^p_X \, dx \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x \in M} ||f(x)||_X & \text{if } p = \infty. \end{cases}$$

If $M = (a,b)$ is an interval in $\mathbb{R}$, then we write $L^p(a,b;X)$. The space $L^p_{loc}([0,\infty);X)$ is defined as the set of all measurable functions f such that $f \in L^p(0,T;X)$ for all $T > 0$. Moreover, we denote by

$$\begin{aligned} &L^p_{uloc}([0,\infty);X) := \{f : [0,\infty) \to X \text{ strongly measurable} : \ ||f||_{L^p_{uloc}([0,\infty);X)} < \infty\}, \\ &||f||_{L^p_{uloc}([0,\infty);X)} := \sup_{t \geq 0} ||f||_{L^p(t,t+1;X)}. \end{aligned}$$

For $T < \infty$, we define $L^p_{uloc}([0,T);X) := L^p(0,T;X)$. For an open set $\Omega \subseteq \mathbb{R}^d$, the L^p-Sobolev space of order $k \in \mathbb{N}_0$ and values in X is defined as

$$W^k_p(\Omega;X) := \{f \in L^p(\Omega;X) : \ \partial^\alpha_x f \in L^p(\Omega;X) \text{ for every } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq k\}$$

equipped with the norm

$$||f||_{W^k_p(\Omega;X)} := \begin{cases} \left(\sum\limits_{|\alpha| \leq k} ||\partial^\alpha_x f||^p_{L^p(\Omega;X)} \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max\limits_{|\alpha| \leq k} ||\partial^\alpha_x f||_{L^\infty(\Omega;X)} & \text{if } p = \infty, \end{cases}$$

where $\partial^\alpha_x f$ has to be understood in the sense of distributions with values in X.

Moreover, we define for $k \in \mathbb{N}$

$$\begin{aligned} &C^k(\Omega;X) := \{f : \Omega \to X : \ f \text{ is k-times continuously differentiable}\}, \\ &C^k(\overline{\Omega};X) := \{f \in C^k(\Omega;X) : \ \partial^\alpha_x f \text{ has continuous extension on } \overline{\Omega} \text{ for all } |\alpha| \leq k\}, \\ &C^k_b(\overline{\Omega};X) := \{f \in C^k(\overline{\Omega};X) : \ \partial^\alpha_x f \text{ are bounded for all } |\alpha| \leq k\}, \end{aligned}$$

where we equip $C^k_b(\overline{\Omega};X)$ with the norm

$$||f||_{C^k_b(\overline{\Omega};X)} = \max_{|\alpha| \leq k} \sup_{x \in \overline{\Omega}} ||\partial^\alpha_x f(x)||_X.$$

Moreover, we define

$$\begin{aligned} &C^\infty(\Omega;X) := \bigcap_{k \in \mathbb{N}} C^k(\Omega;X), \quad C^\infty(\overline{\Omega};X) := \bigcap_{k \in \mathbb{N}} C^k(\overline{\Omega};X), \\ &C^\infty_b(\overline{\Omega};X) := \bigcap_{k \in \mathbb{N}} C^k_b(\overline{\Omega};X). \end{aligned}$$

For $\alpha \in (0,1]$ we denote by $C^{0,\alpha}(\overline{\Omega};X)$ the Hölder continuous functions defined by

$$\begin{aligned} C^{0,\alpha}(\overline{\Omega};X) &:= \{f \in C^0_b(\overline{\Omega};X) : \ ||f||_{C^{0,\alpha}(\overline{\Omega};X)} < \infty\}, \\ ||f||_{C^{0,\alpha}(\overline{\Omega};X)} &:= ||f||_{C^0_b(\overline{\Omega};X)} + \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{||f(x) - f(y)||_X}{|x-y|^\alpha}. \end{aligned}$$

Note that for $\alpha \in (0,1)$ we also write $C^{\alpha}(\overline{\Omega};X)$. For $\alpha = 1$ we get the set of all Lipschitz continuous functions. For the special case $X = \mathbb{R}$, one can show that $C_b^k(\overline{\Omega};X)$ and $C^{0,\alpha}(\overline{\Omega};X)$ are complete and therefore Banach spaces. The proofs can be found in [Alt06, Section 1.5 and Section 1.6]. More details about integration and differentiation of functions with values in Banach spaces can be found in [Růž04] and [Yos74].
Now let $I = [0,T]$ for $0 < T < \infty$ or $I = [0,\infty)$ for $T = \infty$. Then

$$\begin{aligned} BC(I;X) &:= C_b^0(I;X), \\ BUC(I;X) &:= \{f \in BC(I;X) : \ f \text{ is uniformly continuous}\} \end{aligned}$$

are the Banach spaces of all bounded and continuous functions $f : I \to X$ and its subspace of all bounded and uniformly continuous functions.

2.4 Basic Results about Sobolev Spaces

In this section we collect some results about the spaces we have already introduced. We start with the generalized Hölder's inequality.

Theorem 2.6. *(Generalized Hölder's inequality)*
Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $1 \leq p_1, ..., p_k \leq \infty$ and $q \in [1,\infty]$ such that $\frac{1}{p_1} + ... + \frac{1}{p_k} = \frac{1}{q}$, where $\frac{1}{\infty} := 0$. Moreover, let $u_i \in L^{p_i}(\Omega)$ for $i = 1, ..., k$. Then it holds

$$\left|\left|\prod_{i=1}^k u_i\right|\right|_{L^q(\Omega)} \leq \prod_{i=1}^k ||u_i||_{L^{p_i}(\Omega)}.$$

The proof can be found in most books about functional analysis, e.g. in [Alt06, Lemma 1.16] and for the case $k = 2$ in [Els05, Chapter 6, Theorem 1.5], where the case for $k > 2$ follows by induction. In the analysis for the existence of weak solutions of (1.1) - (1.5), we often have estimates for the derivatives of a function u in the $L^2(\Omega)$-norm together with estimates for its mean value. Then we can use the following result to estimate u in the H^1-norm.

Theorem 2.7. *(Poincaré inequality with mean value)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary. Then there exists a constant $C > 0$ such that

$$||u||_{L^2(\Omega)} \leq C \left(\sum_{j=1}^d ||\partial_{x_j} u||_{L^2(\Omega)} + \left| \int_\Omega u(x)\,dx \right| \right)$$

for all $u \in H^1(\Omega)$.

For a proof we refer to (1.21) in [Neč67, Chapter 1]. Next we present two results about convergence in L^p-spaces and pointwise convergence.

Theorem 2.8. *Let (M,μ) be a measure space, $1 \leq p \leq \infty$ and $f_k \to f$ in $L^p(M,\mu)$ as $k \to \infty$. Then there exists a subsequence $(f_{k_j})_{j\in\mathbb{N}}$ such that $f_{k_j}(x) \to f(x)$ a.e. as $j \to \infty$.*

The proof of this theorem can be found in [Alt06, Lemma 1.20].

Theorem 2.9. *Let (M,μ) be a measure space, $1 < p \leq \infty$ and $(f_k)_{k\in\mathbb{N}} \subseteq L^p(M,\mu)$ be a bounded sequence such that $f_k(x) \to f(x)$ a.e. as $k \to \infty$. Then it holds $f_k \to f$ in $L^q(M,\mu)$ for all $1 \leq q < p$ and $k \to \infty$.*

This statement follows from [Els05, Corollary 5.5]. In the proofs for the existence results, we often have to estimate terms like $\eta(\varphi)$, $m(\varphi,q)$, $\tilde{m}(\varphi)$, $\rho(\varphi)$, $f(q)$ and so on. Hence, we need to know in which L^p-spaces these compositions are bounded and if these compositions are continuous in the sense that $f(u_k) \to f(u)$ in a certain L^q-space if it holds $u_k \to u$ in an appropriate L^p-space. This question is answered by the next theorem.

Theorem 2.10. *Let $\mathbf{u} : G \subseteq \mathbb{R}^d \to \mathbb{R}^n$ and $f : G \times \mathbb{R}^n \to \mathbb{R}$, where $G \subseteq \mathbb{R}^d$ is an arbitrary domain. Moreover, we assume that f satisfies*

i) Carathéodory-Condition:

$$\begin{aligned} f(\cdot,\eta) : x \mapsto f(x,\eta) \qquad & \text{is measureable on } G \text{ for all } \eta \in \mathbb{R}^n, \\ f(x,\cdot) : \eta \mapsto f(x,\eta) \qquad & \text{is continuous on } \mathbb{R}^n \text{ for almost every } x \in G. \end{aligned}$$

ii) Growth condition:

$$|f(x,\eta)| \leq |a(x)| + b\sum_{i=1}^{n} |\eta^i|^{\frac{p_i}{q}}$$

for a constant $b > 0$, $a \in L^q(G)$, $1 \leq p_i, q < \infty$ and $i = 1, ..., n$.

Then the Nemyckii-operator $F : \prod_{i=1}^{n} L^{p_i}(G) \to L^q(G)$ defined by

$$(F\mathbf{u})(x) := f(x,\mathbf{u}(x)) \qquad \text{for all } \mathbf{u} \in \prod_{i=1}^{n} L^{p_i}(G)$$

is continuous and bounded. Furthermore, there exists a constant $c > 0$ such that

$$||Fu||_{L^q(G)} \leq c\left(||a||_{L^q(G)} + \sum_{i=1}^{n} ||u^i||_{L^{p_i}(G)}^{\frac{p_i}{q}} \right)$$

for all $u \in \prod_{i=1}^{n} L^{p_i}(G)$.

The proof of this theorem can be found in [Růž04, Chapter 3, Lemma 1.19]. Moreover, we also want to estimate compositions of a continuous function and a L^p-function in an appropriate Sobolev space $W_p^m(\Omega)$. To this end, we need a characterization for terms of the form $\partial_{x_j} F(u)$ in $\mathcal{D}'(\Omega)$.

Lemma 2.11. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, $F \in C^1(\mathbb{R})$ and $u \in W_p^1(\Omega)$, $1 \leq p < \infty$. Moreover, let $(u_k)_{k\in\mathbb{N}} \subseteq C^\infty(\overline{\Omega})$ be a sequence with $u_k \to u$ in $W_p^1(\Omega)$ such that $(F(u_k))_{k\in\mathbb{N}}$ and $(F'(u_k))_{k\in\mathbb{N}}$ are bounded in $L^r(\Omega)$, where $1 < r \leq \infty$ satisfies $\frac{1}{r} + \frac{1}{p} \leq 1$. Then we can conclude*

$$\partial_{x_j} F(u) = F'(u)\partial_{x_j} u \qquad \text{in } \mathcal{D}'(\Omega) \text{ for all } j = 1, ..., d.$$

Proof. Due to Theorem 2.8 there exists a suitable subsequence of $(u_k)_{k\in\mathbb{N}}$, which we denote by $(u_k)_{k\in\mathbb{N}}$ again, such that

$$u_k(x) \to u(x) \qquad \text{a.e. in } \Omega.$$

Thus it holds $F(u_k(x)) \to F(u(x))$ a.e. in Ω. Since $(F(u_k))_{k\in\mathbb{N}}$ and $(F'(u_k))_{k\in\mathbb{N}}$ are bounded in $L^r(\Omega)$ by assumption for $1 < r \leq \infty$ with $\frac{1}{r} + \frac{1}{p} \leq 1$, Theorem 2.9 yields

$$F(u_k) \to F(u) \text{ and } F'(u_k) \to F'(u) \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < r.$$

Altogether we can conclude

$$\begin{aligned}
\langle \partial_{x_j} F(u), \psi \rangle &= -\int_\Omega F(u)\partial_{x_j}\psi \, dx = -\lim_{k\to\infty} \int_\Omega F(u_k)\partial_{x_j}\psi \, dx \\
&= \lim_{k\to\infty} \int_\Omega \partial_{x_j} F(u_k)\psi \, dx = \lim_{k\to\infty} \int_\Omega F'(u_k)\partial_{x_j} u_k \psi \, dx \\
&= \int_\Omega F'(u)\partial_{x_j} u\psi \, dx = \langle F'(u)\partial_{x_j} u, \psi \rangle
\end{aligned}$$

for every $\psi \in C_0^\infty(\Omega)$. □

For the proof that the term $\frac{1}{\varepsilon} f(q)W(\varphi) + g(q)$ satisfies the initial condition $\frac{1}{\varepsilon} f(q_0)W(\varphi_0) + g(q_0)$ we use the following lemma.

Lemma 2.12. *Let $(u_k)_{k\in\mathbb{N}} \subseteq C([0,T]; H^{-1}(\Omega))$ be a sequence such that $u_k \rightharpoonup u$ in $C([0,T]; H^{-1}(\Omega))$ for $k \to \infty$. Then it holds $u_k(0) \rightharpoonup u(0)$ in $H^{-1}(\Omega)$.*

Proof. Let $\varphi \in H_0^1(\Omega) \cong H^{-1}(\Omega)'$. Then we define $F_\varphi \in C([0,T]; H^{-1}(\Omega))'$ by

$$\langle F_\varphi, u \rangle_{V',V} \;:=\; \langle u(0), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$

for all $u \in V := C([0,T]; H^{-1}(\Omega))$. Since $u_k \rightharpoonup u$ in $C([0,T]; H^{-1}(\Omega))$ we can conclude

$$\langle u_k(0), \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle F_\varphi, u_k\rangle_{V',V} \to \langle F_\varphi, u\rangle_{V',V} = \langle u(0), \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$

for all $\varphi \in H_0^1(\Omega)$. □

For Banach space-valued L^p-spaces we use the following theorem to identify its dual spaces.

Theorem 2.13. *Let X be a reflexive and separable Banach space, S an interval in $\mathbb{R}$ and $1 < p < \infty$. Then for every $f \in (L^p(S;X))'$ there exists a unique representation of the form*

$$f(u) = \int_S \langle v(s), u(s)\rangle_{X',X}\, ds \qquad \text{for all } u \in L^p(S;X),$$

where it holds $v \in L^{p'}(S;X')$ with $\frac{1}{p} + \frac{1}{p'} = 1$. The assignment $f \to v$ for $f \in (L^p(S;X))'$ is linear and it holds

$$||f||_{(L^p(S;X))'} = ||v||_{L^{p'}(S;X')}.$$

The proof of this theorem can be found in [GGZ74, Chapter IV, Section 1, Theorem 1.14].

The next result will be helpful to get estimates for the velocity field $\mathbf{v}$.

Theorem 2.14. *(Korn's inequality)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded and connected domain with C^2-boundary $\partial\Omega$ and $\varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Then there exist constants c_0, c_1, c_2, which depend on Ω, such that

$$\int_\Omega \sum_{i,j=1}^d |\varepsilon_{ij}(\mathbf{u})|^2\, dx \geq c_0 \int_\Omega (|\mathbf{u}|^2 + |D\mathbf{u}|^2)\, dx$$

for all $\mathbf{u} \in H_0^1(\Omega)^d$ and

$$\int_\Omega \sum_{i,j=1}^d |\varepsilon_{ij}(\mathbf{u})|^2\, dx + c_1 \int_\Omega |\mathbf{u}|^2\, dx \geq c_2 \int_\Omega |D\mathbf{u}|^2\, dx$$

for all $\mathbf{u} \in H^1(\Omega)^d$.

For a proof of this theorem we refer to [Zei88, Theorem 62.F]. If $\partial\Omega$ is a smooth boundary, the proof can be found in [EGK08, Theorem 6.14]. For the general case $1 < p < \infty$ and $\mathbf{u} \in W_{p,0}^1(\Omega)^d$, we refer to [Rou05, Theorem 1.33].

2.5 Embedding Results

Up to this point we have introduced several function spaces like Sobolev spaces, Besov spaces and Bessel potential spaces. Now we are interested in how these spaces are related and which conditions have to be satisfied such that a certain function space embeds into another function space. In the first result of this section, we study this question in the case of Sobolev spaces.
Note that most results on Sobolev spaces, Bessel potential spaces, Besov spaces and interpolation theory are only stated for the case $\Omega = \mathbb{R}^d$. But if Ω is a bounded domain in $\mathbb{R}^d$ with $C^{0,1}$-boundary, then there exists a bounded and linear extension operator $E_\Omega : W_p^m(\Omega) \to W_p^m(\mathbb{R}^d)$ for all $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, which satisfies $E_\Omega f_{|\Omega} = f$ for all $f \in W_p^m(\Omega)$, cf. [Ste70, Chapter VI, Section 3.2]. Moreover, this operator extends to $E_\Omega : H^s(\Omega) \to H^s(\mathbb{R}^d)$. Hence, all results are also valid for a bounded domain $\Omega \subseteq \mathbb{R}^d$ with $C^{0,1}$-boundary.

Theorem 2.15. *(Sobolev embedding theorem)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary, $k, m \in \mathbb{N}$ such that $k < m$ and $1 \leq p \leq \infty$, $1 \leq q < \infty$, $\alpha \in (0,1)$. Then it holds

$$W_p^m(\Omega) \hookrightarrow W_q^k(\Omega)$$

if $m - \frac{d}{p} \geq k - \frac{d}{q}$ with continuous embedding and

$$W_p^m(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$$

if $m - \frac{d}{p} \geq k + \alpha$ with continuous embedding.

The first embedding result can be found in [Tri78, Section 4.6.1]. Both embeddings follow inductively from [Eva10, Section 5.6, Theorem 6]. Note that this theorem is also true for Banach-valued functions. Furthermore, we want to know under which conditions the Sobolev embeddings are compact. Here we obtain the following theorem.

Theorem 2.16. *(Compact Sobolev embedding theorem)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary, $m, k \in \mathbb{N}_0$ such that $k < m$ and $1 \leq p, q < \infty$. If it holds $m - \frac{d}{p} > k - \frac{d}{q}$, then the embedding $W_p^m(\Omega) \subseteq W_q^k(\Omega)$ is compact.

This compactness result follows from the Rellich-Kondrachov compactness theorem, cf. [Eva10, Section 5.7, Theorem 1]. Moreover, we are also interested under which assumptions the multiplication of two Sobolev functions respectively the composition of a continuous function with a Sobolev function is a Sobolev function again and in which Sobolev spaces the product respectively composition is bounded. Therefore, we present the following theorems.

Theorem 2.17. *(Multiplication of Sobolev functions)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary, $m \in \mathbb{N}_0$, and $1 \leq p, q, r \leq \infty$ such that $r \leq \min(p, q)$ and $2m - \frac{d}{p} - \frac{d}{q} > m - \frac{d}{r}$. Then there exists a constant $C > 0$ such that $fg \in W_r^m(\Omega)$ and

$$||fg||_{W_r^m(\Omega)} \leq C||f||_{W_p^m(\Omega)}||g||_{W_q^m(\Omega)}$$

for all $f \in W_p^m(\Omega)$, $g \in W_q^m(\Omega)$.

For a proof of this theorem, we refer to [RS96, Theorem 4.5.2], where we need to choose $s_1 = s_2 = m$ and where we need to use the identity $F_{p,2}^s(\mathbb{R}^d) = W_p^s(\mathbb{R}^d)$ for $1 \leq p < \infty$ and $s \in \mathbb{N}_0$, cf. (1.4) in [Tri06]. Here we denote by $F_{p,q}^s(\mathbb{R}^d)$ the Triebel-Lizorkin spaces , where $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For a definition of these spaces, we also refer to [RS96]. For the composition of a continuous function with a Sobolev function we have:

Theorem 2.18. *(Composition with Sobolev functions)*
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary, $m, N \in \mathbb{N}$ and $1 \leq p < \infty$ such that $m - \frac{d}{p} > 0$. Then for all $f \in C^m(\mathbb{R}^N)$ and every $R > 0$ there exists a constant $C > 0$ such that for all $\mathbf{u} \in W_p^m(\Omega)^N$ with $||\mathbf{u}||_{W_p^m(\Omega)^N} \leq R$, it holds $f(\mathbf{u}) \in W_p^m(\Omega)$ and $||f(\mathbf{u})||_{W_p^m(\Omega)} \leq C$.
Moreover, if $f \in C^{m+1}(\mathbb{R}^N)$, then for all $R > 0$ there exists a constant $L > 0$ such that

$$||f(\mathbf{u}) - f(\mathbf{v})||_{W_p^m(\Omega)} \leq L||\mathbf{u} - \mathbf{v}||_{W_p^m(\Omega)^N}$$

for all $\mathbf{u}, \mathbf{v} \in W_p^m(\Omega)^N$ with $||\mathbf{u}||_{W_p^m(\Omega)^N}, ||\mathbf{v}||_{W_p^m(\Omega)^N} \leq R$.

Proof. The first part follows from [RS96, Chapter 5, Theorem 1 and Lemma]. For the second part, let $\mathbf{u}, \mathbf{v} \in W_p^m(\Omega)^N$ be given with $||\mathbf{u}||_{W_p^m(\Omega)^N}, ||\mathbf{v}||_{W_p^m(\Omega)^N} \leq R$. Then we define $G(\mathbf{u}(x), \mathbf{v}(x)) := \int\limits_0^1 Df(t\mathbf{u}(x) + (1-t)\mathbf{v}(x)dt$. Hence, it holds

$$f(\mathbf{u}(x)) - f(\mathbf{v}(x)) = G(\mathbf{u}(x), \mathbf{v}(x)) \cdot (\mathbf{u}(x) - \mathbf{v}(x)),$$

where $G \in C^m(\mathbb{R}^N \times \mathbb{R}^N)$ for $f \in C^{m+1}(\mathbb{R}^N)$. Thus the first part of this theorem yields the existence of a constant $C > 0$ such that $||G(\mathbf{u}, \mathbf{v})||_{W_p^m(\Omega} \leq C$. Therefore, we can apply the estimate in Theorem 2.17, which shows the statment. $\square$

For negative s and $k \in \mathbb{N}$, we already introduced the notation $(W_p^k(\Omega))' := W_{p',0}^{-k}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. For Bessel potential spaces and Besov spaces we have the following result.

Lemma 2.19. *If $1 \le p < \infty$ we have $(H_p^s(\mathbb{R}^d))' = H_{p'}^{-s}(\mathbb{R}^d)$ for $s \in \mathbb{R}$. If, in addition, $1 \le q < \infty$, we also have $(B_{pq}^s(\mathbb{R}^d))' = B_{p'q'}^{-s}(\mathbb{R}^d)$ for $s \in \mathbb{R}$.*

This statement is proven in [BL76, Corollary 6.2.8]. In particular this implies for $s \in \mathbb{Z}$ the identity $W_p^s(\mathbb{R}^d) = H_p^s(\mathbb{R}^d)$ and therefore

$$(W_p^s(\mathbb{R}^d))' = (H_p^s(\mathbb{R}^d))' = H_{p'}^{-s}(\mathbb{R}^d) = W_{p'}^{-s}(\mathbb{R}^d).$$

Furthermore, we are also interested in which other spaces Besov and Bessel potential spaces embed. A useful result is the following theorem.

Theorem 2.20. *Assume $s, s_1, s_2, s_3 \in \mathbb{R}$ such that $s - \frac{d}{p} = s_1 - \frac{d}{p_1}$. Then the following embeddings hold*

$$\begin{aligned}
&B_{pq}^s(\mathbb{R}^d) \subseteq B_{p_1q_1}^{s_1}(\mathbb{R}^d), && \text{if } 1 \le p \le p_1 \le \infty,\ 1 \le q \le q_1 \le \infty,\\
&B_{pq}^{s_3}(\mathbb{R}^d) \subseteq B_{pq}^{s_2}(\mathbb{R}^d), && \text{if } 1 \le p, q \le \infty,\ s_2 < s_3,\\
&H_p^s(\mathbb{R}^d) \subseteq H_{p_1}^{s_1}(\mathbb{R}^d), && \text{if } 1 < p \le p_1 < \infty,\ s, s_1 \in \mathbb{R}.
\end{aligned}$$

The proof can be found in [BL76, Theorem 6.2.4] and [BL76, Theorem 6.5.1].

At the beginning of this chapter we introduced Banach space-valued Sobolev spaces. Thus we are also interested in embedding results for these spaces.

Lemma 2.21. *Let $0 < T < \infty$ and (V, H, V') be a Gelfand triple, i.e., V, H are separable and real Hilbert spaces such that there is a continuous embedding $i : V \to H$ with $\overline{i(V)} = H$. Then it holds*

$$L^2(0, T; V) \cap W_2^1(0, T; V') \subseteq C([0, T]; H)$$

with continuous embedding. Furthermore for all $f \in L^2(0, T; V) \cap W_2^1(0, T; V')$ and $0 \le s \le t \le T$ it holds

$$\frac{1}{2}||f(t)||_H^2 = \int_s^t \langle \partial_t f(\tau), f(\tau)\rangle_{V',V}\, d\tau + \frac{1}{2}||f(s)||_H^2.$$

For the proof we refer to [Rou05, Lemma 7.3]. Analogously to the real-valued case, we want to know under which assumptions a Banach space-valued Sobolev space embeds into a continuous Banach space-valued function space. Here we have the following result.

Lemma 2.22. *Let X be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $1 \le p < \infty$, $k \in \mathbb{N}$ and $0 < T < \infty$. Then it holds*

$$W_p^k((0, T); X) \subseteq C^{k-1}([0, T]; X)$$

with continuous embedding. Moreover, it holds

$$u(t) = u(s) + \int_s^t \partial_\tau u(\tau)\,d\tau \qquad \text{in } X \text{ for all } 0 \leq s, t \leq T$$

for all $u \in W_p^1(0,T;X)$.

This lemma is proven in [Rou05, Theorem 7.1].

Lemma 2.23. *Let* X, Y *be two Banach spaces such that* $Y \hookrightarrow X$ *and* $X' \hookrightarrow Y'$ *densely. Then* $L^\infty(I;Y) \cap BUC(I;X) \hookrightarrow BC_w(I;Y)$*, where* $I = [0,T]$ *with* $0 < T < \infty$ *or* $I = [0,\infty)$.

For a proof we refer to [Abe09a]. The following lemma is not an embedding result. But since its proof is based on the embedding in Lemma 2.22 we also state it in this section. In the proof of the existence of weak solutions for a diffuse interface model with soluble surfactants in two-phase flows, we use a semi-implicit time discretization and then construct interpolant functions for the time-dependent case. Hence, we need to estimate terms of the form $f_k(t+h) - f_k(t)$ and show that its norm converges to 0 in a certain Banach space as h converges to 0. More precisely, we get the following lemma.

Lemma 2.24. *Let* B *be a Banach space and* $(f_k)_{k\in\mathbb{N}} \subseteq W_p^1(0,T;B)$ *such that* $(\partial_t f_k)_{k\in\mathbb{N}} \subseteq L^p(0,T;B)$ *is a bounded sequence. Then there exists a constant* $C > 0$ *such that*

$$\sup_{t\in[0,T-h]} ||f_k(t+h) - f_k(t)||_B \leq C h^{\frac{1}{p'}}$$

for all $k \in \mathbb{N}$*, where* p *and* p' *satisfy* $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Since $(\partial_t f_k)_{k\in\mathbb{N}} \subseteq L^p(0,T;B)$ is bounded, there exists a constant $C > 0$ such that $||\partial_t f_k||_{L^p(0,T;B)} \leq C$ for all $k \in \mathbb{N}$. Moreover, it holds

$$f_k(t+h) - f_k(t) = \int_t^{t+h} \partial_\tau f_k(\tau)\,d\tau \qquad \text{in } B$$

for all $t \in [0, T-h]$ according to Lemma 2.22. Therefore, we can conclude

$$\begin{aligned}||f_k(t+h) - f_k(t)||_B &\leq \int_t^{t+h} ||\partial_\tau f_k(\tau)||_B\,d\tau \leq \left(\int_t^{t+h} ||\partial_\tau f_k(\tau)||_B^p\,d\tau\right)^{\frac{1}{p}} \left(\int_t^{t+h} 1^{p'}\,d\tau\right)^{\frac{1}{p'}} \\ &\leq C h^{\frac{1}{p'}}.\end{aligned}$$

Taking the supremum over all $t \in [0, T-h]$ yields the statement. □

2.6 Real Interpolation

In this section we give a short introduction in interpolation theory based on [Abe16], [Lun09] and [BL76]. Roughly speaking interpolation theory is the study and construction of Banach spaces X, which are "between" two Banach spaces X_0 and X_1 in such a way that every linear operator T, which is bounded on X_0 and X_1, is also bounded on X. To this end, we need some definitions that constrain which Banach spaces X_0 and X_1 we will bring together.

Let X_0 and X_1 be two Banach spaces. Then the pair (X_0, X_1) is called compatible or admissible, if there is a Hausdorff topological vector space Z such that X_0 and X_1 continuously embed in Z. We also call (X_0, X_1) an interpolation couple.
Let (X_0, X_1) be a compatible pair of Banach spaces. Then $X_0 \cap X_1$ and $X_0 + X_1$ normed by

$$\begin{aligned}
||x||_{X_0 \cap X_1} &:= \max\{||x||_{X_0}, ||x||_{X_1}\},\\
||x||_{X_0 + X_1} &:= \inf_{\substack{x = x_0 + x_1,\\ x_0 \in X_0, x_1 \in X_1}} (||x_0||_{X_0} + ||x_1||_{X_1})
\end{aligned}$$

are Banach spaces.

Let (X_0, X_1) and (Y_0, Y_1) be two admissible pairs of Banach spaces and let X, Y be Banach spaces. We call X intermediate space with respect to (X_0, X_1), if it holds

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$$

with continuous embeddings. Moreover, X and Y are called interpolation spaces with respect to (X_0, X_1) and (Y_0, Y_1) if X and Y are intermediate spaces with respect to (X_0, X_1) and (Y_0, Y_1) respectively, and if

$$T \in \mathcal{L}(X_j, Y_j),\ j = 0, 1 \qquad \Rightarrow \qquad T_{|X} \in \mathcal{L}(X, Y)$$

for all $T : X_0 + X_1 \to Y_0 + Y_1$ linear. X is called interpolation space with respect to (X_0, X_1) if the previous conditions hold with $X = Y$ and $(X_0, X_1) = (Y_0, Y_1)$.

There exist several kind of interpolation spaces, e.g. the real and the complex interpolation spaces. In this work we will only use the real interpolation spaces which are defined by the following.

Definition 2.25. *Let (X, Y) be a real or complex interpolation couple. For $\theta \in (0, 1)$ and $1 \leq p \leq \infty$ we define the real interpolation space*

$$(X_0, X_1)_{\theta,p} := \Big\{x \in X_0 + X_1 :\ t \mapsto t^{-\theta} K(t, x, X_0, X_1) \in L^p_*(0, \infty)\Big\},$$

equipped with the norm

$$||x||_{(X_0,X_1)_{\theta,p}} := ||t^{-\theta} K(t, x, X_0, X_1)||_{L^p_*(0,\infty)} \qquad \textit{for all } x \in (X_0, X_1)_{\theta,p},$$

where

$$K(t,x,X_0,X_1) := \inf_{x=a+b,\ a\in X_0,\ b\in X_1} (||a||_{X_0} + t||b||_{X_1})$$

and $L^p_*(0,\infty)$ *is the Lebesgue space* L^p *with respect to the measure* $\frac{dt}{t}$*, i.e.,* $L^p_*(I)$ *is the space of the real or complex valued* L^p*-functions in* I*, where* I *is an interval contained in* $(0,\infty)$ *and* $1 \le p \le \infty$*, equipped with the norm*

$$||f||_{L^p_*(I)} := \begin{cases} \left(\int\limits_0^\infty |f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}} & \text{if } p<\infty,\\ \operatorname*{ess\,sup}_{t\in I}|f(t)| & \text{if } p=\infty. \end{cases}$$

It can be shown that $(X_0,X_1)_{\theta,p}$ is an intermediate space with respect to (X_0,X_1) for every $\theta\in(0,1)$, $1\le p\le\infty$. It can even be verified that it holds

$$X_0\cap X_1 \hookrightarrow (X_0,X_1)_{\theta,p_1} \hookrightarrow (X_0,X_1)_{\theta,p_2} \hookrightarrow X+Y$$

for all $\theta\in(0,1)$ and $1\le p_1\le p_2\le\infty$. It remains to show that it is an interpolation space. This is shown in the next theorem.

Theorem 2.26. *Let* (X_0,X_1) *and* (Y_0,Y_1) *be interpolation couples and* $T\in\mathcal{L}(X_0,Y_0)\cap\mathcal{L}(X_1,Y_1)$*. Then it holds* $T\in\mathcal{L}((X_0,X_1)_{\theta,p},(Y_0,Y_1)_{\theta,p})$ *for every* $\theta\in(0,1)$ *and* $1\le p\le\infty$*. Moreover, it holds*

$$||T||_{\mathcal{L}((X_0,X_1)_{\theta,p},(Y_0,Y_1)_{\theta,p}} \le ||T||^{1-\theta}_{\mathcal{L}(X_0,Y_0)}||T||^{\theta}_{\mathcal{L}(X_1,Y_1)}.$$

For a proof of this theorem, cf. [Lun09, Theorem 1.6]. Due to the next result we can estimate the norm of a real interpolation space by the norms of the spaces of its interpolation couple.

Lemma 2.27. *Let* (X_0,X_1) *be an interpolation couple of Banach spaces. For* $0<\theta<1$ *and* $1\le p\le\infty$ *there is a constant* $c(\theta,p)>0$ *such that*

$$||u||_{(X_0,X_1)_{\theta,p}} \le c(\theta,p)||u||^{1-\theta}_{X_0}||u||^{\theta}_{X_1} \tag{2.5}$$

for all $u\in X_0\cap X_1$*.*

For a proof of this lemma, cf. [Lun09, Corollary 1.7]. The definition of real interpolation spaces is quite abstract and it is not obvious how the interpolation spaces look like. In the case that the interpolation couple consists of two Besov spaces or of two Bessel potential spaces, we can identify the interpolation spaces due to the following theorem.

Theorem 2.28. *Let $s_0 \neq s_1 \in \mathbb{R}$, $0 < \theta < 1$ and $1 \leq q_0, q_1 \leq \infty$. Then for every $1 \leq q \leq \infty$ we have*

$$(B^{s_0}_{pq_0}(\mathbb{R}^d), B^{s_1}_{pq_1}(\mathbb{R}^d))_{\theta,q} = B^s_{pq}(\mathbb{R}^d).$$

Moreover, we have

$$(H^{s_0}_p(\mathbb{R}^d), H^{s_1}_p(\mathbb{R}^d))_{\theta,q} = B^s_{pq}(\mathbb{R}^d)$$

for any $1 \leq p, q \leq \infty$, where $s = (1-\theta)s_0 + \theta s_1$.

The proof can be found in [BL76, Theorem 6.2.4 and Theorem 6.4.5]. The following lemma yields another important result that helps us to identify real interpolation spaces in the case that the interpolation couple consists of two L^p-spaces.

Lemma 2.29. *Let (Ω, μ) be a σ-finite measure space and $1 \leq q_0 < q < q_1 \leq \infty$ and $0 < \theta < 1$ such that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then it holds*

$$(L^{q_0}(\Omega), L^{q_1}(\Omega))_{\theta,q} = L^q(\Omega).$$

For a proof, cf. [Lun09, Example 1.27].

Theorem 2.30. *Let X_0, X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$ densely. Then for all $1 \leq p < \infty$*

$$X_T := W^1_p(0,T;X_0) \cap L^p(0,T;X_1) \hookrightarrow BUC(0,T;(X_0,X_1)_{1-\frac{1}{p},p})$$

continuously. Moreover, for the trace map

$$\gamma : X_T \to X_0, \qquad u \mapsto u(0)$$

it holds

$$\gamma X_T = (X_0, X_1)_{1-\frac{1}{p},p}.$$

The proof of this result can be found in [Ama95, Chapter III, Theorem 4.10.2]. The next lemma is not a result from interpolation theory. But it will be important for proving an embedding result based on the previous results in interpolation theory.

Lemma 2.31. *Let X, X_0, X_1 be Banach spaces such that $X_0 \cap X_1 \subseteq X$ and*

$$||x||_X \leq M||x||^{1-\theta}_{X_0}||x||^{\theta}_{X_1}$$

for all $x \in X_0 \cap X_1$, where $\theta \in [0,1]$ and $M > 0$. Then it holds

$$||f||_{L^p(0,T;X)} \leq M||f||^{1-\theta}_{L^{p_0}(0,T;X_0)}||f||^{\theta}_{L^{p_1}(0,T;X_1)} \tag{2.6}$$

for all $f \in L^{p_0}(0,T;X_0) \cap L^{p_1}(0,T;X_1)$, where $1 \leq p, p_0, p_1 \leq \infty$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. Due to the generalized Hölder's inequality, cf. Theorem 2.6, it holds

$$||\tilde{f}\tilde{g}||_{L^p(0,T)} \leq ||\tilde{f}||_{L^{\frac{p_0}{1-\theta}}(0,T)} ||\tilde{g}||_{L^{\frac{p_1}{\theta}}(0,T)}$$

for all $\tilde{f} \in L^{\frac{p_0}{1-\theta}}(0,T)$ and $\tilde{g} \in L^{\frac{p_1}{\theta}}(0,T)$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Applying this estimate to $||x||_X \leq M||x||_{X_0}^{1-\theta}||x||_{X_1}^{\theta}$ yields the statement. □

With the previous results we can conclude the following embedding theorem.

Theorem 2.32. *Let* (Ω, μ) *be a* σ*-finite measure space,* $1 \leq q_0 < q < q_1 \leq \infty$ *and* $\theta \in (0,1)$ *such that* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$*. Then it holds*

$$L^{p_0}(0,T;L^{q_0}(\Omega)) \cap L^{p_1}(0,T;L^{q_1}(\Omega)) \hookrightarrow L^p(0,T;L^q(\Omega)), \tag{2.7}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $1 \leq p_0, p_1 \leq \infty$.

Proof. From Lemma 2.29 it follows

$$L^q(\Omega) = (L^{q_0}(\Omega), L^{q_1}(\Omega))_{\theta,q}.$$

Thus Lemma 2.27 yields

$$||f||_{L^q(\Omega)} \leq c(\theta,q)||f||_{L^{q_0}(\Omega)}^{1-\theta}||f||_{L^{q_1}(\Omega)}^{\theta}$$

for a constant $c(\theta,q) > 0$. Applying Lemma 2.31 to this estimate yields the statement. □

2.7 Compactness Results

In the model (1.1) - (1.5) we have terms like $f(q)$ and $m(\varphi,q)$. In the existence proof for weak solutions we will study linear interpolants $(\mathbf{v}^N, \varphi^N, \mu^N, q^N)$ with $N \in \mathbb{N}$ and we will pass to the limit $N \to \infty$. Hence, we want to have $q^N(t,x) \to q(t,x)$ and $\varphi^N(t,x) \to \varphi(t,x)$ a.e. in $(0,T) \times \Omega$ for some q, φ, which we will specify later. Therefore, we want to show $q^N \to q$ and $\varphi^N \to \varphi$ in appropriate Banach spaces. To this end, we will need the following compactness results.

Theorem 2.33. *(Aubin-Lions)*
Let X_0, X, X_1 *be some Banach spaces such that* $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$*. Moreover, let* $1 \leq q \leq \infty$, $1 < p < \infty$ *and* X_0, X_1 *be reflexive and* $W := \left\{u \in L^p(0,T;X_0) | \ \frac{du}{dt} \in L^q(0,T;X_1)\right\}$ *with* $0 < T < \infty$*. Then it holds* $W \subseteq L^p(0,T;X)$ *with compact embedding.*

The proof for $q > 1$ can be found in [Lio69]. If $q = 1$, cf. [Sim87].

Due to the Aubin-Lions lemma we will be able to prove the strong convergence of φ^N and $\mathbf{v}^N$ in appropriate Banach spaces since the equations provide estimates for $\partial_t \mathbf{v}^N$ and $\partial_t \varphi^N$. For q^N we only have estimates for $\partial_t(f(q^N)W(\varphi^N) + g(q^N)$. Hence, we will not use the Aubin-Lions lemma to prove compactness but a result by Simon.

Theorem 2.34. *Let $X \subseteq B \subseteq Y$ with compact imbedding $X \hookrightarrow B$, where X, B and Y are Banach spaces. Furthermore, let $1 \leq p \leq \infty$ and*

i) F is bounded in $L^p(0, T; X)$,

ii) $||\tau_h f - f||_{L^p(0,T-h;Y)} \to 0$ as $h \to 0$, uniformly for $f \in F$,

where $(\tau_h f)(t) := f(t + h)$ for $h > 0$. Then F is relatively compact in $L^p(0, T; B)$ (and in $C(0, T; B)$ if $p = \infty$).

The proof of this theorem can be found in [Sim87, Theorem 5]. When we prove compactness of $(q^N)_{N\in\mathbb{N}}$ in $L^2(0, T; L^2(\Omega))$, we show that $(q^N)_{N\in\mathbb{N}}$ fulfills the assumptions of Theorem 2.34. For the proof of condition ii) we use the following result.

Lemma 2.35. *Let $\bar{t} = mh$ for $m \in \mathbb{N}$ and $h = \frac{1}{N}$ for $N \in \mathbb{N}$. Moreover, let H be a Hilbert space and let $(u^N)_{N\in\mathbb{N}}$ be step functions defined by $u^N(t) = u_k$ for $t \in [(k - 1)h, kh)$, $k \in \mathbb{N}_0$, where it holds $(u_k)_{k\in\mathbb{N}_0} \subseteq H$. If*

$$\int_0^{\bar{t}-s} e(u^N(t + s), u^N(t))\,dt \leq C\omega(s)$$

for $s > 0$ which are multiple of h, then this inequality holds for any real $s > 0$. Here ω is a concave function and $e : H \times H \to \mathbb{R}$ is continuous.

For a proof of this lemma, we refer to [Alt12, Lemma 9.1]

3 Existence of Weak Solutions for a Diffuse Interface Model with Soluble Surfactants

In this chapter we prove the existence of weak solutions for the surfactant model (1.1) - (1.5). To this end, this chapter is divided into four parts. In the first part, we do some formal calculations since we want to derive some assumptions on the functions g, h, f and so on. In particular, we want to derive some growth conditions which we will need for the analysis. Moreover, we want that the weak solutions satisfy an energy estimate and therefore we study under which assumptions we are able to get such an estimate.
In the second part of this chapter, we use a semi-implicit time discretization for the system of partial differential equations and add the terms $\delta\Delta^2\mathbf{v}$ and $\delta\partial_t\varphi$ so that we can conclude $\mathbf{v} \in L^2(0,T;H^2(\Omega)^d)$ and $\partial_t\varphi \in L^2(0,T;L^2(\Omega))$ in the case $\delta > 0$. We solve this time-discrete problem by applying the Leray-Schauder principle on appropriate operators and show that the time-discrete weak solutions satisfy a time-discrete version of the energy estimate.
In the third section of this chapter, we construct interpolant functions with the help of the time-discrete weak solutions and pass to the limit $N \to \infty$. We show compactness for $\mathbf{v}^N$, φ^N and q^N in appropriate Banach spaces and the convergence to the initial values. Moreover, we show that these interpolant functions converge to a weak solution of the system of partial differential equations (1.1) - (1.5) with the additional terms $\delta\Delta^2\mathbf{v}$ and $\delta\partial_t\varphi$ and that they satisfy an energy estimate.
In the final part of this chapter, we pass to the limit for $\delta \to 0$ and show that the weak solutions from the third section, which depend on $\delta > 0$, converge to a weak solution of (1.1) - (1.5) which satisfies the initial and boundary conditions together with an energy estimate.

3.1 Preliminary Results

In this section we assume that all functions are smooth enough, i.e., all appearing derivatives exist and are continuous. Thus the following calculations are just formal, but they give us a first idea in which function spaces the solutions will be bounded. Moreover, we can not expect that there will exist solutions for any arbitrary choice of $f, h, g, W, m, \tilde{m}$ and so on. Therefore we will derive some assumptions for f, h, g, W and so on such that there holds an energy estimate. Hence, we will be able to derive boundedness of the solutions in certain function spaces. But note that all calculations in this section are just formal since we assume that the solutions are smooth.

But before we proceed with deriving the energy estimate and the assumptions for the functions, we first of all need to reformulate the system (1.1) - (1.5) since the reformulated equations have some advantages for the analysis in contrast to the original problem. From the definitions of $\tilde{\mathbf{J}}$ and R together with (1.4) we obtain the continuity equation

$$
\begin{aligned}
\partial_t\rho(\varphi) + \mathbf{v}\cdot\nabla\rho(\varphi) &= \frac{\partial\rho}{\partial\varphi}(\varphi)\partial_t\varphi + \mathbf{v}\cdot\left(\frac{\partial\rho}{\partial\varphi}(\varphi)\nabla\varphi\right) = \frac{\partial\rho}{\partial\varphi}(\varphi)\partial_t^\bullet\varphi \\
&= \frac{\partial\rho}{\partial\varphi}(\varphi)\operatorname{div}(\tilde{m}(\varphi)\nabla\mu) \\
&= -\operatorname{div}\left(\frac{\partial\rho(\varphi)}{\partial\varphi}(-\tilde{m}(\varphi)\nabla\mu)\right) - \nabla\frac{\partial\rho(\varphi)}{\partial\varphi}\cdot(\tilde{m}(\varphi)\nabla\mu) \\
&= -\operatorname{div}\tilde{\mathbf{J}} + R.
\end{aligned}
\tag{3.1}
$$

In Section 3.2 and Section 3.3 we use this as the definition of R, i.e., we use

$$
R = \partial_t\rho(\varphi) + \mathbf{v}\cdot\nabla\rho(\varphi) + \operatorname{div}\tilde{\mathbf{J}}. \tag{3.2}
$$

Hence, we will need some estimate for $\partial_t\varphi$. Therefore, we will add the term $\delta\partial_t\varphi$ in one of the equations. In the next step we reformulate equation (1.1). For the right-hand side of (1.1) we can calculate

$$
\begin{aligned}
\operatorname{div}(-\varepsilon\nabla\varphi\otimes\nabla\varphi) &= -\varepsilon\left(\partial_{x_1}(\partial_{x_1}\varphi\nabla\varphi) + \ldots + \partial_{x_d}(\partial_{x_d}\varphi\nabla\varphi)\right) \\
&= -\varepsilon\partial_{x_1}\partial_{x_1}\varphi\nabla\varphi - \varepsilon\partial_{x_1}\varphi\partial_{x_1}(\nabla\varphi) - \ldots - \varepsilon\partial_{x_d}\partial_{x_d}\varphi\nabla\varphi - \varepsilon\partial_{x_d}\varphi\partial_{x_d}(\nabla\varphi) \\
&= -\frac{1}{2}\nabla(\varepsilon|\nabla\varphi|^2) - \varepsilon\Delta\varphi\nabla\varphi \\
&= -\frac{1}{2}\nabla(\varepsilon|\nabla\varphi|^2) + \left(\mu - \frac{h(q)}{\varepsilon}W'(\varphi)\right)\nabla\varphi,
\end{aligned}
\tag{3.3}
$$

where we used (1.5) to replace $\Delta\varphi$ in the last step. Thus we can use this equation on the right-hand side of (1.1) and get the equivalent equation

$$
\begin{aligned}
\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes(\rho\mathbf{v}+\tilde{\mathbf{J}})) + \nabla p - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) - \frac{R\mathbf{v}}{2}& \\
= -\frac{1}{2}\nabla(\varepsilon|\nabla\varphi|^2) + \left(\mu - \frac{h(q)}{\varepsilon}W'(\varphi)\right)\nabla\varphi & \qquad \text{in } Q_T.
\end{aligned}
$$

But equation (1.1) can also be reformulated in another way. To this end, we use (3.2) and get

$$
\begin{aligned}
\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes\rho\mathbf{v} + \mathbf{v}\otimes\tilde{\mathbf{J}}) - R\frac{\mathbf{v}}{2} &= \partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes\rho\mathbf{v} + \mathbf{v}\otimes\tilde{\mathbf{J}}) \\
&\quad - (\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) + \operatorname{div}\tilde{\mathbf{J}})\mathbf{v} + R\frac{\mathbf{v}}{2} \\
&= \rho\partial_t\mathbf{v} + (\rho\mathbf{v} + \tilde{\mathbf{J}})\cdot\nabla\mathbf{v} + R\frac{\mathbf{v}}{2}.
\end{aligned}
$$

In the last step we used (2.1) together with the fact that $\mathbf{v}$ is a divergence-free vector field, i.e., div $\mathbf{v} = 0$ and $\operatorname{div}(\rho\mathbf{v}) = \mathbf{v}\cdot\nabla\rho$. Hence, (1.1) can equivalently be written as

$$
\rho\partial_t\mathbf{v} + (\rho\mathbf{v} + \tilde{\mathbf{J}})\cdot\nabla\mathbf{v} + \nabla p - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \frac{R\mathbf{v}}{2} = \operatorname{div}(-\varepsilon\nabla\varphi\otimes\nabla\varphi) \quad \text{in } Q_T.
$$

3.1.1 Formal Derivation of the Energy Inequality

Multiplying equation (1.1) with $\mathbf{v}$ and integrating over the domain Ω yields

$$
\begin{aligned}
0 =& \int_\Omega \partial_t\rho|\mathbf{v}|^2 + \rho\partial_t\mathbf{v}\cdot\mathbf{v}\,dx + \int_\Omega ((\tilde{\mathbf{J}}\cdot\nabla)\mathbf{v} + (\operatorname{div}\tilde{\mathbf{J}})\mathbf{v})\cdot\mathbf{v}\,dx + \int_\Omega 2\eta(\varphi)|D\mathbf{v}|^2\,dx \\
&- \int_\Omega \partial_t\rho\frac{|\mathbf{v}|^2}{2} + \frac{(\mathbf{v}\cdot\nabla\rho)|\mathbf{v}|^2}{2} + (\operatorname{div}\tilde{\mathbf{J}})\frac{|\mathbf{v}|^2}{2}\,dx + \int_\Omega \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{v}\,dx \\
&+ \int_\Omega \left(-\mu + \frac{h(q)}{\varepsilon}W'(\varphi)\right)\nabla\varphi\cdot\mathbf{v}\,dx,
\end{aligned}
$$

where we used (3.2) for R. Reordering the terms yields

$$
\begin{aligned}
0 =& \int_\Omega \frac{1}{2}\partial_t\rho|\mathbf{v}|^2 + \frac{1}{2}\rho\partial_t|\mathbf{v}|^2\,dx + \int_\Omega \left((\operatorname{div}\tilde{\mathbf{J}})\frac{\mathbf{v}}{2} + (\tilde{\mathbf{J}}\cdot\nabla)\mathbf{v}\right)\cdot\mathbf{v}\,dx + \int_\Omega 2\eta(\varphi)|D\mathbf{v}|^2\,dx \\
&+ \int_\Omega \left(\operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) - (\nabla\rho\cdot\mathbf{v})\frac{\mathbf{v}}{2}\right)\cdot\mathbf{v}\,dx + \int_\Omega \left(-\mu + \frac{h(q)}{\varepsilon}W'(\varphi)\right)\nabla\varphi\cdot\mathbf{v}\,dx.
\end{aligned}
$$

Since it holds

$$
\int_\Omega \left((\operatorname{div}\tilde{\mathbf{J}})\frac{\mathbf{v}}{2} + \left(\tilde{\mathbf{J}}\cdot\nabla\right)\mathbf{v}\right)\cdot\mathbf{v}\,dx = \int_\Omega \operatorname{div}\left(\tilde{\mathbf{J}}\frac{|\mathbf{v}|^2}{2}\right)dx = 0
$$

and

$$
\begin{aligned}
\int_\Omega \left(\operatorname{div}(\mathbf{v}\otimes\rho\mathbf{v}) - (\nabla\rho\cdot\mathbf{v})\frac{\mathbf{v}}{2}\right)\cdot\mathbf{v}\,dx &= \int_\Omega \left(\operatorname{div}(\mathbf{v}\otimes\rho\mathbf{v}) - \operatorname{div}(\rho\mathbf{v})\frac{\mathbf{v}}{2}\right)\cdot\mathbf{v}\,dx \\
&= \int_\Omega \left(\operatorname{div}(\rho\mathbf{v})|\mathbf{v}|^2 + (\rho\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{v} - \operatorname{div}(\rho\mathbf{v})\frac{|\mathbf{v}|^2}{2}\right)dx \\
&= \int_\Omega \left(\operatorname{div}(\rho\mathbf{v})|\mathbf{v}|^2 + \rho\mathbf{v}\cdot\nabla\left(\frac{|\mathbf{v}|^2}{2}\right) - \operatorname{div}(\rho\mathbf{v})\frac{|\mathbf{v}|^2}{2}\right)dx \\
&= \int_\Omega \operatorname{div}\left(\rho\mathbf{v}\frac{|\mathbf{v}|^2}{2}\right)dx = 0,
\end{aligned}
$$

we can simplify the previous equation to

$$
0 = \frac{d}{dt}\int_\Omega \frac{1}{2}\left(\rho|\mathbf{v}|^2\right)dx + \int_\Omega 2\eta(\varphi)|D\mathbf{v}|^2\,dx + \int_\Omega \left(-\mu + \frac{h(q)}{\varepsilon}W'(\varphi)\right)\nabla\varphi\cdot\mathbf{v}\,dx.
$$

Multiplying (1.3) with q, (1.4) with μ and (1.5) with $(-\partial_t\varphi)$ and integrating over the domain Ω yields

$$
\begin{aligned}
0 =& \int_\Omega \frac{1}{\varepsilon} f'(q)W(\varphi)\left(\partial_t q + \mathbf{v}\cdot\nabla q\right) q\,dx + \frac{1}{\varepsilon}\int_\Omega f(q)qW'(\varphi)\left(\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi\right) dx \\
&+ \int_\Omega g'(q)q\left(\partial_t q + \mathbf{v}\cdot\nabla q\right) dx + \int_\Omega m(\varphi,q)|\nabla q|^2\,dx, \\
0 =& \int_\Omega \partial_t\varphi\mu\,dx + \int_\Omega (\mathbf{v}\cdot\nabla\varphi)\mu\,dx + \int_\Omega \tilde{m}(\varphi)|\nabla\mu|^2\,dx, \\
0 =& \frac{d}{dt}\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2\,dx + \int_\Omega h(q)\partial_t W(\varphi)\frac{1}{\varepsilon}\,dx - \int_\Omega \partial_t\varphi\mu\,dx.
\end{aligned}
$$

Altogether we get

$$
\begin{aligned}
0 =& \frac{d}{dt}\int_\Omega \frac{1}{2}\rho|\mathbf{v}|^2\,dx + \int_\Omega 2\eta(\varphi)|D\mathbf{v}|^2\,dx + \int_\Omega \left(\frac{h(q)}{\varepsilon}W'(\varphi)\right)\nabla\varphi\cdot\mathbf{v}\,dx \\
&+ \int_\Omega \frac{1}{\varepsilon} f'(q)W(\varphi)\left(\partial_t q + \mathbf{v}\cdot\nabla q\right) q\,dx + \frac{1}{\varepsilon}\int_\Omega f(q)qW'(\varphi)\left(\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi\right) dx \\
&+ \int_\Omega m(\varphi,q)|\nabla q|^2\,dx + \int_\Omega g'(q)q\left(\partial_t q + \mathbf{v}\cdot\nabla q\right) dx \\
&+ \int_\Omega \tilde{m}(\varphi)|\nabla\mu|^2\,dx + \frac{d}{dt}\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2\,dx + \int_\Omega h(q)\partial_t W(\varphi)\frac{1}{\varepsilon}\,dx.
\end{aligned}
\tag{3.4}
$$

Moreover, we postulate that the total energy density $e(\mathbf{v},\varphi,\nabla\varphi,q)$ is given by

$$
e(\mathbf{v},\varphi,\nabla\varphi,q) := \frac{1}{2}\rho|\mathbf{v}|^2 + \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{d(q)}{\varepsilon}W(\varphi) + G(q) \tag{3.5}
$$

for some functions d and G which we will specify later. In (3.5), $\frac{1}{2}\rho|\mathbf{v}|^2$ is the kinetic energy density while the other terms are the Helmholtz free energy density. Note that in the limit for ε tending to 0, it holds $\nabla\varphi = 0$ in the bulk phases since φ is the difference of the volume fractions of both fluids and therefore it holds $\varphi \equiv \pm 1$ in the bulk. Moreover, we assume that W is a potential of double-well type, i.e., $W(\pm 1) = W'(\pm 1) = 0$ and $W(\varphi) > 0$ if $\varphi \notin \{-1, 1\}$. Thus the term $\frac{1}{\varepsilon}d(q)W(\varphi)$ is also 0 in the bulk phases as $\varepsilon \to 0$. Altogehter, $\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{d(q)}{\varepsilon}W(\varphi)$ can be considered as approximation of the free energy density on the surface while $G(q)$ is the free energy density in the bulk associated to the bulk surfactant. For the total energy density we demand

$$
\frac{d}{dt}\int_\Omega e(\mathbf{v},\varphi,\nabla\varphi,q)\,dx + \int_\Omega \left(m(\varphi,q)|\nabla q|^2 + \tilde{m}(\varphi)|\nabla\mu|^2 + 2\eta(\varphi)|D\mathbf{v}|^2\right) dx = 0, \tag{3.6}
$$

where the dissipation $\mathcal{D}$ is defined by

$$-\mathcal{D} := -\int_\Omega m(\varphi, q)|\nabla q|^2\,dx - \int_\Omega \tilde{m}(\varphi)|\nabla \mu|^2\,dx - \int_\Omega 2\eta(\varphi)|D\mathbf{v}|^2\,dx \le 0$$

since we will assume $m, \tilde{m}$ and η to be strictly positive functions. Therefore, it follows

$$\begin{aligned}0 =& \frac{d}{dt}\int_\Omega \frac{1}{2}\rho|\mathbf{v}|^2\,dx + \frac{d}{dt}\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2\,dx + \frac{d}{dt}\int_\Omega \frac{d(q)}{\varepsilon}W(\varphi)\,dx + \int_\Omega G'(q)\partial_t q\,dx \\ &+ \int_\Omega m(\varphi, q)|\nabla q|^2\,dx + \int_\Omega \tilde{m}(\varphi)|\nabla\mu|^2\,dx + \int_\Omega 2\eta(\varphi)|D\mathbf{v}|^2\,dx. \qquad (3.7)\end{aligned}$$

Now we compare equation (3.7) with equation (3.4). Here we want to derive some characterizations for G' and d. On the one hand we want to have

$$G'(q)\partial_t q = g'(q)q(\partial_t q + \mathbf{v}\cdot\nabla q).$$

Therefore, we assume $G'(q) = g'(q)q$ for all $q \in \mathbb{R}$. Then it holds

$$\begin{aligned}\int_\Omega G'(q)\partial_t q\,dx &= \int_\Omega g'(q)q\partial_t q\,dx = \int_\Omega g'(q)q\partial_t q\,dx - \int_\Omega g(q)\nabla q\cdot\mathbf{v}\,dx \\ &= \int_\Omega g'(q)q\partial_t q\,dx + \int_\Omega \nabla(g(q))\cdot(q\mathbf{v})\,dx \\ &= \int_\Omega g'(q)q\left(\partial_t q + \mathbf{v}\cdot\nabla q\right)\,dx, \qquad (3.8)\end{aligned}$$

where we used

$$\int_\Omega g(q)\nabla q\cdot\mathbf{v}\,dx = \int_\Omega \nabla\tilde{G}(q)\cdot\mathbf{v}\,dx = 0$$

for an antiderivative $\tilde{G}$ of g. Moreover, when we compare equation (3.7) with equation (3.4), then on the other hand, we want that it holds

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{d(q)}{\varepsilon}W(\varphi)\,dx =& \int_\Omega \frac{h(q)}{\varepsilon}W'(\varphi)(\nabla\varphi\cdot\mathbf{v})\,dx + \int_\Omega \frac{1}{\varepsilon}f'(q)W(\varphi)\left(\partial_t q + \mathbf{v}\cdot\nabla q\right)q\,dx \\ &+ \frac{1}{\varepsilon}\int_\Omega f(q)W'(\varphi)\left(\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi\right)q\,dx + \int_\Omega h(q)\partial_t W(\varphi)\frac{1}{\varepsilon}\,dx.\end{aligned}$$

Using integration by parts and the fact that the velocity $\mathbf{v}$ is divergence-free we can calculate

$$
\begin{aligned}
\frac{d}{dt}\int\limits_{\Omega}\frac{d(q)}{\varepsilon}W(\varphi)\,dx &= \int\limits_{\Omega}\nabla\left(\frac{h(q)}{\varepsilon}W(\varphi)\right)\cdot\mathbf{v}\,dx + \int\limits_{\Omega}\frac{1}{\varepsilon}f'(q)W(\varphi)\partial_t q\; q\,dx \\
&\quad + \int\limits_{\Omega}\frac{1}{\varepsilon}f(q)W'(\varphi)\partial_t\varphi\; q\,dx + \int\limits_{\Omega}\frac{1}{\varepsilon}\nabla\left(f(q)W(\varphi)\right)\cdot(q\mathbf{v})\,dx \\
&\quad - \int\limits_{\Omega}\frac{h'(q)}{\varepsilon}W(\varphi)(\nabla q\cdot\mathbf{v})\,dx + \int\limits_{\Omega}\frac{h(q)}{\varepsilon}\partial_t W(\varphi)\,dx \\
&= \int\limits_{\Omega}\frac{1}{\varepsilon}f'(q)qW(\varphi)\partial_t q\,dx + \int\limits_{\Omega}\frac{1}{\varepsilon}\left(f(q)q+h(q)\right)\partial_t W(\varphi)\,dx \\
&\quad - \int\limits_{\Omega}\frac{1}{\varepsilon}(f(q)W(\varphi))(\nabla q\cdot\mathbf{v})\,dx - \int\limits_{\Omega}\frac{h'(q)}{\varepsilon}W(\varphi)(\nabla q\cdot\mathbf{v})\,dx.
\end{aligned}
$$

As the left-hand side is independent of $\mathbf{v}$ we demand

$$f(q) = -h'(q) \tag{3.9}$$

for all $q \in \mathbb{R}$. Then it is ensured that the right-hand side is also independent of $\mathbf{v}$ and the equation simplifies to

$$\frac{d}{dt}\int\limits_{\Omega}\frac{d(q)}{\varepsilon}W(\varphi)\,dx = \int\limits_{\Omega}\frac{1}{\varepsilon}f'(q)qW(\varphi)\partial_t q\,dx + \int\limits_{\Omega}\frac{1}{\varepsilon}\left(f(q)q+h(q)\right)\partial_t W(\varphi)\,dx.$$

Hence, we also demand that the following relations hold:

$$f'(q)q = d'(q), \tag{3.10}$$

$$d(q) = h(q) + f(q)q. \tag{3.11}$$

In equation (3.8) we already assumed

$$G'(q) = g'(q)q. \tag{3.12}$$

Now we want to verify if all three equations (3.9), (3.10) and (3.11) can be satisfied at the same time or if there is a redundant equation or even a contradiction. To this end, we differentiate (3.11) with respect to q and then we use (3.10). This yields

$$d'(q) = h'(q) + f'(q)q + f(q) = h'(q) + d'(q) + f(q).$$

Thus (3.9) holds. This means that (3.9) is in fact redundant as we can deduce it from (3.10) and (3.11).

The calculations above showed that (3.10) and (3.11) are sufficient assumptions such that equation (3.6) holds. The total energy of the system in a domain Ω at time t is defined as the integral of the total energy density over the domain Ω, i.e.,

$$\begin{aligned} E_{tot}(\mathbf{v},\varphi,\nabla\varphi,q) &:= \int_\Omega e(\mathbf{v},\varphi,\nabla\varphi,q)\,dx \\ &= \int_\Omega \left(\frac{1}{2}\rho(\varphi)|\mathbf{v}|^2 + \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{d(q)}{\varepsilon}W(\varphi) + G(q)\right) dx, \end{aligned} \tag{3.13}$$

where we neglect the time-dependence for the sake of clarity. Due to equation (3.6) and as we will assume that $m, \tilde{m}$ and η are strictly positive functions, we can conclude that the total energy is maximal at $t = 0$. Thus there exists a constant $M > 0$ such that

$$E_{tot}(\mathbf{v}_0,\varphi_0,\nabla\varphi_0,q) = \left(\int_\Omega e(\mathbf{v},\varphi,\nabla\varphi,q)\,dx\right)_{|t=0} \leq M$$

and

$$E_{tot}(\mathbf{v},\varphi,\nabla\varphi,q)_{|t=T} \leq E_{tot}(\mathbf{v}_0,\varphi_0,\nabla\varphi_0,q_0) \leq M$$

for all $T \geq 0$. Integrating equation (3.6) from s to t with $0 \leq s < t$ yields the energy estimate

$$\begin{aligned} E_{tot}(\mathbf{v}(t),\varphi(t),\nabla\varphi(t),q(t)) + \int_s^t \int_\Omega \left(m(\varphi,q)|\nabla q|^2 + \tilde{m}(\varphi)|\nabla\mu|^2 + 2\eta(\varphi)|D\mathbf{v}|^2\right) dx\,d\tau \\ \leq E_{tot}(\mathbf{v}(s),\varphi(s),\nabla\varphi(s),q(s)). \end{aligned} \tag{3.14}$$

3.1.2 Assumptions on the Equations

When we have a look at equation (1.3), we note that this equation is a parabolic PDE with the form

$$\partial_t a(\varphi,q) - \operatorname{div}(m(\varphi,q)\nabla q) = 0.$$

A formal calculation yields

$$\partial_t q - \frac{1}{\partial_2 a(\varphi,q)}\operatorname{div}(m(\varphi,q)\nabla q) + \frac{\partial_1 a(\varphi,q)}{\partial_2 a(\varphi,q)}\partial_t\varphi = 0.$$

Since we will assume that m is a strictly positive function, we demand

$$\partial_2 a(\varphi,q) > 0$$

for all $\varphi, q \in \mathbb{R}$ to get a well-posed parabolic PDE. According to (1.3), $a(\varphi, q)$ is given by $\frac{1}{\varepsilon} f(q)W(\varphi) + g(q)$. Hence, we demand that the mapping

$$q \mapsto \frac{1}{\varepsilon} f(q)W(\varphi) + g(q)$$

is strongly monotone for all $\varphi \in \mathbb{R}$, where we want that the constant C does not depend on φ, i.e., there exists a constant $C > 0$ such that

$$\left(\frac{1}{\varepsilon} f(q_1)W(\varphi) + g(q_1) - \frac{1}{\varepsilon} f(q_2)W(\varphi) - g(q_2)\right)(q_1 - q_2) \geq C|q_1 - q_2|^2$$

for every $\varphi, q_1, q_2 \in \mathbb{R}$, cf. Definition 2.4. Therefore, we make the following assumption.

Assumption 3.1. *The function $f \in C^\infty(\mathbb{R})$ is monotone increasing and $G \in C^2(\mathbb{R})$ is strictly convex. Moreover, it holds*

$$G'(q) \begin{cases} < c_0 q & \text{if } q < 0 \\ = 0 & \text{if } q = 0 \\ > c_0 q & \text{if } q > 0 \end{cases} .$$

for a constant $c_0 > 0$.

From this assumption we can conclude $G''(q) > 0$ for all $q \in \mathbb{R}$. Since we demand that for G and g relation (3.12) must hold, i.e., $G'(q) = g'(q)q$ for all $q \in \mathbb{R}$, we proceed with the following assumption for g.

Assumption 3.2. *The function $g \in C^2(\mathbb{R})$ is stongly monotone, i.e., there exists a constant $C > 0$ such that*

$$(g(a) - g(b))(a - b) \geq C|a - b|^2 \qquad \text{for all } a, b \in \mathbb{R}.$$

Moreover, g fulfills

$$G'(q) = g'(q)q$$

for every $q \in \mathbb{R}$.

We note that g' is continuous since we have

$$g'(h) = \frac{G'(h)}{h} = \frac{G'(h) - G'(0)}{h} \to G''(0)$$

as $h \to 0$. Here we used $G \in C^2(\mathbb{R})$ and $G'(0) = 0$ due to Assumption 3.1.

From these assumptions it follows $g'(q) \geq c_0 > 0$ for all $q \in \mathbb{R}$, where c_0 is the constant from Assumption 3.1. Hence, the assumption that g is strongly monotone is satisfied as we have

$$(g(a) - g(b))(a - b) = \int\limits_b^a g'(x)\,dx(a - b) \geq c_0(a - b)^2.$$

Note that the constant c_0 does not depend on φ. Later we will assume that the function h is concave. This implies that h' is monotonically decreasing. As it holds $f(q) = -h'(q)$ for every $q \in \mathbb{R}$, cf. (3.9), this implies that f is monotone and therefore the mapping

$$q \mapsto \frac{1}{\varepsilon} f(q)W(\varphi) + g(q)$$

is strongly monotone as it is the sum of a strongly monotone and a monotone operator and since we will assume W to be non-negative. This ensures the parabolicity of equation (1.3) with respect to q. Moreover, we can conclude that there exists a constant $C > 0$, which does not depend on φ, such that

$$\left| \frac{1}{\varepsilon} f(q_1)W(\varphi) + g(q_1) - \frac{1}{\varepsilon} f(q_2)W(\varphi) - g(q_2) \right| \geq C\,|q_1 - q_2|$$

for all $\varphi, q_1, q_2 \in \mathbb{R}$.

For the analysis of (1.1) - (1.5) we will need some growth condition for g as we can not expect that there will exist solutions for any arbitrary choice of g. Thus we make the following assumption.

Assumption 3.3. *There exists a constant $C > 0$ such that*

$$|G(q)| \leq C(|q|^2 + 1), \qquad |G'(q)| \leq C(|q| + 1)$$

for all $q \in \mathbb{R}$.

Note that Assumption 3.1 implies that there exists a constant $C > 0$ such that

$$G(q) \geq C(|q|^2 - 1) \tag{3.15}$$

for all $q \in \mathbb{R}$. We will use inequality (3.15) to estimate the mean value of q_{k+1} in the time-discrete approximation of (1.1) - (1.5), cf. the calculations after (3.48). Due to Assumption 3.2 it holds

$$g(s) = \int\limits_\varepsilon^s \frac{G'(s)}{s}\,ds + g(\varepsilon)$$

for all $0 < \varepsilon < s$. Thus Assumption 3.3 together with Assumption 3.2 yield that there exists a constant $C > 0$ such that

$$g(s) \leq C(|s| + 1) \tag{3.16}$$

for all $0 < \varepsilon < s$. Analogously this holds for all $s < -\varepsilon < 0$. Since g is continuous this growth condition for g holds for all $s \in \mathbb{R}$.

The energy inequality (3.14) does not yield any estimate for the solutions if its terms can be negative or if they are not bounded from below. Thus we have to guarantee that all terms are positive or at least bounded from below. Moreover, for the analysis of the system we need to know in which L^p-space the term $h(q)$ is integrable and we need some assumptions on the domain Ω. Therefore, we postulate:

Assumption 3.4. *The functions d, f, h, W, η and m are smooth, i.e., they are in $C^\infty(\mathbb{R})$, and $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with C^2-boundary. Moreover, there exist some constants $0 < c_1 < c_2 < \infty$ and a constant $c_0 > 0$ such that*

$$\begin{aligned} &d(q), \eta(q) > c_0, && W(\varphi) \geq 0, && d(q) = h(q) + f(q)q, \\ &f'(q)q = d'(q), && c_1 \leq m(\varphi, q), \tilde{m}(\varphi) \leq c_2 \end{aligned}$$

for all $q, \varphi \in \mathbb{R}$. Furthermore, h is concave and there exist constants $q_{min}, q_{max} \in \mathbb{R}$ with $q_{min} < q_{max}$ such that

$$d(q) \equiv const.$$

for all $q \notin [q_{min}, q_{max}]$.

From this assumption it follows $d'(q) = 0$ for all $q \notin [q_{min}, q_{max}]$. This implies $f'(q) = 0$ for all $q \notin [q_{min}, q_{max}]$ because of $f'(q)q = d'(q)$ and therefore f is constant there. Hence, there exists a constant $C > 0$ such that

$$|h(q)| \leq C(|q| + 1) \tag{3.17}$$

since h is linear outside the interval $[q_{min}, q_{max}]$ due to $h(q) = d(q) - f(q)q$. Note that we demand $d(q) = h(q) + f(q)q$ and $d'(q) = f'(q)q$, cf. (3.10) and (3.11), since these identities are sufficient for the derivation of the energy estimate. Moreover, we demand $d(q) > c_0 > 0$ for all $q \in \mathbb{R}$ since we want that the part $\frac{1}{\varepsilon}d(q)W(\varphi)$ of the free energy density is positive.
Furthermore, we note that from the previous assumptions it follows that G is bounded from below since $G'(q) > 0$ for all $q > 0$ and $G'(q) < 0$ for all $q < 0$. Thus we do not need to assume that G is a positive function. In particular we note that the growth condition (3.15) for G holds.

3.1.3 Expected Function Spaces for the Weak Solutions

In this section we want to derive in which function spaces the weak solutions will be bounded. Remember that all calculations are just formal. Note that when we write that a function is in a certain space X, we mean that the function is bounded in this space, e.g. $\varphi \in L^\infty(0,\infty;H^1(\Omega))$ means that all weak solutions φ are bounded in $L^\infty(0,\infty;H^1(\Omega))$.

From equation (1.4) it follows

$$\frac{d}{dt}\int\limits_\Omega \varphi dx = \int\limits_\Omega \partial_t\varphi dx = \int\limits_\Omega \operatorname{div}(\tilde{m}(\varphi)\nabla\mu)\,dx - \int\limits_\Omega \mathbf{v}\cdot\nabla\varphi dx = \int\limits_{\partial\Omega} \tilde{m}(\varphi)(n\cdot\nabla\mu)\,dx = 0$$

since μ fulfills the Neumann boundary condition. Hence, the mean value $\frac{1}{|\Omega|}\int\limits_\Omega \varphi dx$ is independent of t. This yields

$$\frac{1}{|\Omega|}\int\limits_\Omega \varphi(t,x)\,dx = \frac{1}{|\Omega|}\int\limits_\Omega \varphi_0(x)\,dx$$

for all $0 \le t < \infty$, where $\varphi_0(x) = \varphi(0,x)$ for all $x \in \Omega$. Since the mean value of φ is constant and with the energy inequality (3.14), the Poincaré inequality with mean value, cf. Theorem 2.7, yields $\varphi(t) \in H^1(\Omega)$ for all $0 \le t < \infty$ together with the estimate

$$||\varphi(t)||_{H^1(\Omega)} \le C$$

for a constant $C > 0$ depending on M, ε and φ_0. Since this constant is independent of t, we can conclude

$$\varphi \in L^\infty(0,\infty;H^1(\Omega)).$$

Moreover, we can deduce from the energy inequality

$$\mathbf{v} \in L^\infty(0,\infty;L^2_\sigma(\Omega)) \cap L^2(0,\infty;H^1_0(\Omega)^d) \hookrightarrow L^4(0,\infty;L^3(\Omega)^d),$$

where we will prove the embedding above in a later section, cf. (3.87) below. Furthermore, the energy inequality (3.14) provides

$$\nabla q \in L^2(0,\infty;L^2(\Omega)).$$

Due to the growth condition (3.15) for G and the boundedness of G because of the energy inequality, we can conclude $q \in L^\infty(0,\infty;L^2(\Omega))$. Hence, we can apply Theorem 2.7 again and obtain

$$q \in L^2_{uloc}([0,\infty);H^1(\Omega)) \cap L^\infty(0,\infty;L^2(\Omega)).$$

From the energy inequality (3.14) it follows

$$\nabla\mu \in L^2(0,\infty;L^2(\Omega)).$$

Testing (1.5) with the constant 1 yields

$$\int_\Omega \mu dx = \int_\Omega h(q)\frac{1}{\varepsilon}W'(\varphi)dx.$$

At this point, we need some growth condition for W'. Therefore, we make the following assumption.

Assumption 3.5. *There exist some constants $C_1, C_2, C_3 > 0$ such that*

$$|W(a)| \le C_1(|a|^3+1), \qquad |W'(a)| \le C_1(|a|^2+1), \qquad W(a) \ge C_2|a| - C_3$$

for all $a \in \mathbb{R}$. If it holds $\frac{\partial\rho(\varphi)}{\partial\varphi} \not\equiv const$ then there exists a constant $C > 0$ and $0 < s < 1$ such that

$$|W'(a)| \le C(|a|^s + 1)$$

for all $a \in \mathbb{R}$.

From the growth conditions for h, cf. (3.17), and for W' and due to the boundedness of q in $L^2_{uloc}([0,\infty);H^1(\Omega)) \hookrightarrow L^2_{uloc}([0,\infty);L^6(\Omega))$ and the boundedness of φ in $L^\infty(0,\infty;L^6(\Omega))$, it follows $h(q) \in L^2_{uloc}([0,\infty);L^6(\Omega))$ and $W'(\varphi) \in L^\infty(0,\infty;L^3(\Omega))$. Thus we can deduce that $h(q)W'(\varphi)$ is bounded in $L^2_{uloc}([0,\infty);L^2(\Omega))$. Using the equation above for the mean value of μ together with Theorem 2.7 again yields

$$\mu \in L^2_{uloc}([0,\infty);H^1(\Omega)).$$

Moreover, we get from $h(q)W'(\varphi) \in L^2_{uloc}([0,\infty);L^2(\Omega))$, $\mu \in L^2(0,\infty;H^1(\Omega))$ and (1.5) with elliptic regularity theory for Neumann boundary condition

$$\varphi \in L^2_{uloc}([0,\infty);H^2(\Omega)).$$

This regularity will be proven in detail in Section 3.3. Now we have a look at equation (1.4), which provides

$$\partial_t\varphi = \operatorname{div}(\tilde{m}(\varphi)\nabla\mu) - \nabla\varphi\cdot\mathbf{v}.$$

From $\mu \in L^2_{uloc}([0,\infty);H^1(\Omega))$ we obtain that $\operatorname{div}(\nabla\mu)$ is bounded in $L^2_{uloc}([0,\infty);H_0^{-1}(\Omega))$. Due to the boundedness of $\nabla\varphi$ in $L^2_{uloc}([0,\infty);L^6(\Omega))$ and $\mathbf{v}$ in $L^\infty(0,\infty;L^2(\Omega))$ we can conclude that $\nabla\varphi\cdot\mathbf{v}$ is bounded in $L^2_{uloc}([0,\infty);L^{\frac{3}{2}}(\Omega))$ and therefore it is also bounded in $L^2_{uloc}([0,\infty);H_0^{-1}(\Omega))$. Altogether we obtain

$$\varphi \in L^2_{uloc}([0,\infty);H^2(\Omega)) \cap L^\infty(0,\infty;H^1(\Omega)) \cap W^1_{2,uloc}([0,\infty);H_0^{-1}(\Omega)).$$

Note that the essential part to prove the boundedness of the functions $\mathbf{v},\varphi,\mu$ and q in the corresponding function spaces was that for weak solutions of the system (1.1) - (1.5) the energy inequality (3.14) holds.

3.2 Semi-Implicit Time Discretization

In this section we formulate an appropriate time discretization of the diffuse interface model with surfactants (1.1) - (1.5) with the additional terms $\delta\Delta^2\mathbf{v}$ in (1.1) and $\delta\partial_t\varphi$ in (1.5) and prove the existence of weak solutions for these equations by using the Leray-Schauder principle. Then in the next section we use these solutions to construct linear interpolant functions and prove the convergence to a weak solution with the additional terms $\delta\Delta^2\mathbf{v}$ and $\delta\partial_t\varphi$ for $N\to\infty$. This method was also applied in [ADG13] to prove existence of weak solutions for the model without surfactants. Since some terms are the same as in [ADG13], we can discuss them in a similar way. Here we additionally have to study the limit $\delta\to 0$ in the final part of this chapter. Moreover, we show that the weak solutions for the time-discrete model as well as the weak solutions for (1.1) - (1.5) satisfy suitable energy estimates.

3.2.1 Definition of the Time-Discrete Problem and the Existence Result of Weak Solutions

For the proof of the existence of weak solutions, we use a semi-implicit time discretization. To this end, we set $h=\frac{1}{N}$ for $N\in\mathbb{N}$. Moreover, let $\mathbf{v}_k\in L^2_\sigma(\Omega)$, $\varphi_k\in H^2_n(\Omega)$ and $q_k\in L^2(\Omega)$ be given and let

$$\tilde{\mathbf{J}}_{k+1} = -\frac{\partial\rho(\varphi)}{\partial\varphi}_{|\varphi=\varphi_k}\tilde{m}(\varphi_k)\nabla\mu_{k+1}, \tag{3.18}$$

$$R_{k+1} = \frac{\rho(\varphi_{k+1})-\rho(\varphi_k)}{h} + \operatorname{div}(\rho(\varphi_k)\mathbf{v}_{k+1}+\tilde{\mathbf{J}}_{k+1}). \tag{3.19}$$

Note that in the case of matched densities, i.e., $\rho\equiv const$ and $\tilde{\mathbf{J}}_{k+1}\equiv R_{k+1}\equiv 0$, the model reduces to the case of matched densities.
We determine $(\mathbf{v}_{k+1},\varphi_{k+1},\mu_{k+1},q_{k+1})$ as solution of the nonlinear system

$$\begin{aligned}
0 &= -\frac{\rho_{k+1}\mathbf{v}_{k+1}-\rho_k\mathbf{v}_k}{h} - \operatorname{div}(\rho_k\mathbf{v}_{k+1}\otimes\mathbf{v}_{k+1}) - \operatorname{div}(\mathbf{v}_{k+1}\otimes\tilde{\mathbf{J}}_{k+1}) \\
&\quad + \operatorname{div}\left(2\eta(\varphi_k)D\mathbf{v}_{k+1}\right) - \nabla p_{k+1} + \frac{R_{k+1}\mathbf{v}_{k+1}}{2} \\
&\quad + \left(\mu_{k+1} - \frac{h(q_{k+1})}{\varepsilon}W'(\varphi_k)\right)\nabla\varphi_k - \delta\Delta^2\mathbf{v}_{k+1}, \qquad (3.20)\\
\operatorname{div}(\mathbf{v}_{k+1}) &= 0, \qquad (3.21)\\
\operatorname{div}\left(m(\varphi_k,q_k)\nabla q_{k+1}\right) &= \frac{1}{\varepsilon}\left(\frac{f(q_{k+1})-f(q_k)}{h}W(\varphi_k) + f(q_{k+1})\frac{W(\varphi_{k+1})-W(\varphi_k)}{h}\right) \\
&\quad + \frac{g(q_{k+1})-g(q_k)}{h} + \nabla\left(\frac{1}{\varepsilon}f(q_{k+1})W(\varphi_k)+g(q_{k+1})\right)\cdot\mathbf{v}_{k+1}, \qquad (3.22)
\end{aligned}$$

$$0 = \frac{\varphi_{k+1} - \varphi_k}{h} + \nabla\varphi_k \cdot \mathbf{v}_{k+1} - \operatorname{div}(\tilde{m}(\varphi_k)\nabla\mu_{k+1}), \tag{3.23}$$

$$\mu_{k+1} - \delta\frac{\varphi_{k+1} - \varphi_k}{h} = -\varepsilon\Delta\varphi_{k+1} + h(q_{k+1})\frac{1}{\varepsilon}H(\varphi_{k+1}, \varphi_k), \tag{3.24}$$

with boundary conditions

$$\mathbf{v}_{k+1_{|\partial\Omega}} = \Delta\mathbf{v}_{k+1_{|\partial\Omega}} = \partial_n\varphi_{k+1_{|\partial\Omega}} = \partial_n\mu_{k+1_{|\partial\Omega}} = \partial_n q_{k+1_{|\partial\Omega}} = 0, \tag{3.25}$$

where $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$H(a, b) := \begin{cases} \frac{W(a)-W(b)}{a-b} & \text{if } a \neq b, \\ W'(b) & \text{if } a = b. \end{cases}$$

Note that in (3.24) we inserted the term $\delta\frac{\varphi_{k+1}-\varphi_k}{h}$ since for the analysis we will need an estimate for the discrete time-derivative of φ. Moreover, we added the term $\delta\Delta^2\mathbf{v}_{k+1}$ in (3.20) since we will need boundedness of $\mathbf{v}^N$ in $L^2(0, T; H^2(\Omega)^d)$ to prove $\left\langle \frac{R^N\mathbf{v}^N}{2}, \boldsymbol{\psi} \right\rangle \to \left\langle \frac{R\mathbf{v}}{2}, \boldsymbol{\psi} \right\rangle$ for every $\boldsymbol{\psi} \in C_0^\infty(0, T; H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$ as $N \to \infty$. But later we will show that the set of weak solutions $(\mathbf{v}^\delta, \varphi^\delta, \mu^\delta, q^\delta)_{\delta>0}$ converges to a weak solution $(\mathbf{v}, \varphi, \mu, q)$ of the initial problem (1.1) - (1.5) as $\delta \to 0$ for a suitable subsequence.
It remains to define a weak solution for the time-discrete problem (3.20) - (3.25).

Definition 3.6. *(Weak solution of the time-discrete problem)*
We call

$$(\mathbf{v}_{k+1}, \varphi_{k+1}, \mu_{k+1}, q_{k+1}) \in \left(H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega)\right) \times H_n^2(\Omega) \times H_n^2(\Omega) \times H^1(\Omega)$$

a weak solution of (3.20) - (3.25) for given initial datas $\mathbf{v}_k \in L_\sigma^2(\Omega)$, $\varphi_k \in H_n^2(\Omega)$ and $q_k \in L^2(\Omega)$ if it holds

$$\begin{aligned}
&\int_\Omega \frac{\rho_{k+1}\mathbf{v}_{k+1} - \rho_k\mathbf{v}_k}{h} \cdot \boldsymbol{\psi}\, dx + \int_\Omega \operatorname{div}(\rho_k\mathbf{v}_{k+1} \otimes \mathbf{v}_{k+1}) \cdot \boldsymbol{\psi}\, dx + \int_\Omega 2\eta(\varphi_k)D\mathbf{v}_{k+1} : D\boldsymbol{\psi}\, dx \\
&\quad - \int_\Omega (\tilde{\mathbf{J}}_{k+1} \otimes \mathbf{v}_{k+1}) : \nabla\boldsymbol{\psi}\, dx - \left\langle R_{k+1}\frac{\mathbf{v}_{k+1}}{2}, \boldsymbol{\psi} \right\rangle + \delta\int_\Omega \Delta\mathbf{v}_{k+1}\Delta\boldsymbol{\psi}\, dx \\
&= \int_\Omega \left(\mu_{k+1} - \frac{h(q_{k+1})}{\varepsilon}W'(\varphi_k)\right)\nabla\varphi_k \cdot \boldsymbol{\psi}\, dx
\end{aligned} \tag{3.26}$$

for all $\boldsymbol{\psi} \in H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega)$ and

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{\varepsilon}f(q_{k+1})W(\varphi_k) + g(q_{k+1})\right)\mathbf{v}_{k+1} \cdot \nabla\phi dx = \int_\Omega m(\varphi_k, q_k)\nabla q_{k+1} \cdot \nabla\phi dx \\
&\quad + \frac{1}{h}\int_\Omega \left(\frac{f(q_{k+1})W(\varphi_{k+1})}{\varepsilon} + g(q_{k+1}) - \frac{f(q_k)W(\varphi_k)}{\varepsilon} - g(q_k)\right)\phi dx,
\end{aligned} \tag{3.27}$$

$$0 = \int_\Omega \tilde{m}(\varphi_k)\nabla\mu_{k+1}\cdot\nabla\phi\, dx + \int_\Omega \frac{\varphi_{k+1}-\varphi_k}{h}\phi\, dx + \int_\Omega (\nabla\varphi_k\cdot\mathbf{v}_{k+1})\phi\, dx, \tag{3.28}$$

$$\begin{aligned}\int_\Omega \mu_{k+1}\phi\, dx &= \int_\Omega \varepsilon\nabla\varphi_{k+1}\cdot\nabla\phi\, dx + \int_\Omega h(q_{k+1})\frac{1}{\varepsilon}H(\varphi_{k+1},\varphi_k)\phi\, dx\\ &\quad + \delta\int_\Omega \frac{\varphi_{k+1}-\varphi_k}{h}\phi\, dx\end{aligned} \tag{3.29}$$

for all $\phi \in H^1(\Omega)$, where we define for $\boldsymbol{\psi} \in H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$

$$\langle R_{k+1}\mathbf{v}_{k+1},\boldsymbol{\psi}\rangle := \int_\Omega \frac{\rho_{k+1}-\rho_k}{h}\mathbf{v}_{k+1}\cdot\boldsymbol{\psi}\, dx - \int_\Omega \left(\rho_k\mathbf{v}_{k+1} + \tilde{\mathbf{J}}_{k+1}\right)\cdot\nabla(\mathbf{v}_{k+1}\cdot\boldsymbol{\psi})\, dx.$$

Note that by definition it holds $\mu_{k+1},\ \varphi_{k+1} \in H^2_n(\Omega)$ for a weak solution of the time-discrete problem. As a consequence, we are able to reformulate (3.20) since $(\operatorname{div}(\tilde{\mathbf{J}}_{k+1})\mathbf{v}_{k+1},\boldsymbol{\psi})_\Omega$ is then well-defined for every $\boldsymbol{\psi} \in H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$. For the reformulation we use $\operatorname{div}(\mathbf{v}_{k+1}\otimes\tilde{\mathbf{J}}_{k+1}) = \operatorname{div}(\tilde{\mathbf{J}}_{k+1})\mathbf{v}_{k+1} + (\tilde{\mathbf{J}}_{k+1}\cdot\nabla)\mathbf{v}_{k+1}$. Then we can deduce

$$\begin{aligned}&\int_\Omega \Bigg(\frac{\rho_{k+1}\mathbf{v}_{k+1}-\rho_k\mathbf{v}_k}{h} + \operatorname{div}(\rho_k\mathbf{v}_{k+1}\otimes\mathbf{v}_{k+1}) + \operatorname{div}(\tilde{\mathbf{J}}_{k+1})\mathbf{v}_{k+1}\\ &\qquad +(\tilde{\mathbf{J}}_{k+1}\cdot\nabla)\mathbf{v}_{k+1}\Bigg)\cdot\boldsymbol{\psi}\, dx + \int_\Omega 2\eta(\varphi_k)D\mathbf{v}_{k+1} : D\boldsymbol{\psi}\, dx + \delta\int_\Omega \Delta\mathbf{v}_{k+1}\Delta\boldsymbol{\psi}\, dx\\ &\qquad -\int_\Omega\left(\frac{\rho(\varphi_{k+1})-\rho(\varphi_k)}{h} + \operatorname{div}\left(\rho(\varphi_k)\mathbf{v}_{k+1} + \tilde{\mathbf{J}}_{k+1}\right)\right)\frac{\mathbf{v}_{k+1}}{2}\cdot\boldsymbol{\psi}\, dx\\ &\quad = \int_\Omega\left(\mu_{k+1} - \frac{h(q_{k+1})}{\varepsilon}W'(\varphi_k)\right)\nabla\varphi_k\cdot\boldsymbol{\psi}\, dx\end{aligned}$$

for all $\boldsymbol{\psi} \in H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$. This yields

$$\begin{aligned}&\int_\Omega\left(\frac{\rho_{k+1}\mathbf{v}_{k+1}-\rho_k\mathbf{v}_k}{h} + \operatorname{div}(\rho_k\mathbf{v}_{k+1}\otimes\mathbf{v}_{k+1})\right)\cdot\boldsymbol{\psi}\, dx + \int_\Omega 2\eta(\varphi_k)D\mathbf{v}_{k+1} : D\boldsymbol{\psi}\, dx\\ &\quad + \int_\Omega\left(\operatorname{div}\tilde{\mathbf{J}}_{k+1} - \frac{\rho_{k+1}-\rho_k}{h} - \mathbf{v}_{k+1}\cdot\nabla\rho_k\right)\frac{\mathbf{v}_{k+1}}{2}\cdot\boldsymbol{\psi}\, dx + \int_\Omega (\tilde{\mathbf{J}}_{k+1}\cdot\nabla)\mathbf{v}_{k+1}\cdot\boldsymbol{\psi}\, dx\\ &\quad + \delta\int_\Omega \Delta\mathbf{v}_{k+1}\cdot\Delta\boldsymbol{\psi}\, dx\\ &= \int_\Omega\left(\mu_{k+1} - \frac{h(q_{k+1})}{\varepsilon}W'(\varphi_k)\right)\nabla\varphi_k\cdot\boldsymbol{\psi}\, dx.\end{aligned} \tag{3.30}$$

Furthermore, we can simplify equation (3.22) to

$$\begin{aligned}\operatorname{div}\left(m(\varphi_k, q_k)\nabla q_{k+1}\right) =&\frac{1}{h}\left(\frac{f(q_{k+1})W(\varphi_{k+1})}{\varepsilon} + g(q_{k+1}) - \frac{f(q_k)W(\varphi_k)}{\varepsilon} - g(q_k)\right)\\ &+ \nabla\left(\frac{1}{\varepsilon}f(q_{k+1})W(\varphi_{k+1}) + g(q_{k+1})\right)\cdot \mathbf{v}_{k+1}.\end{aligned}$$

But in the following we prefer the formulation (3.22) since this one has some advantages for the analysis of the system as we will see later.
Moreover, we note $H(a,b)(a-b) = W(a) - W(b)$ for every $a, b \in \mathbb{R}$. Therefore, we do not have to distinguish the case if $a = b$ or $a \neq b$ since in the analysis we will only study terms of the form $H(a,b)(a-b)$.

The following theorem yields the existence of weak solutions for the time-discrete problem (3.20) - (3.25) for given appropriate initial values.

Theorem 3.7. *(Existence of weak solutions for the time-discrete problem)*
Let the assumptions from Section 3.1 hold and $\mathbf{v}_k \in L^2_\sigma(\Omega)$, $\varphi_k \in H^2_n(\Omega)$ and $q_k \in L^2(\Omega)$ be given. Then there exist $\mathbf{v}_{k+1} \in H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, $\varphi_{k+1} \in H^2_n(\Omega)$, $\mu_{k+1} \in H^2_n(\Omega)$ and $q_{k+1} \in H^1(\Omega)$ solving (3.20) - (3.25) in the sense of Definition 3.6. Moreover, the discrete energy estimate

$$\begin{aligned}&E_{tot}(\mathbf{v}_{k+1}, \varphi_{k+1}, \nabla\varphi_{k+1}, q_{k+1}) + \int_\Omega \frac{\rho_k|\mathbf{v}_{k+1} - \mathbf{v}_k|^2}{2}\,dx + h\int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2\,dx\\ &+ h\int_\Omega m(\varphi_k, q_k)|\nabla q_{k+1}|^2\,dx + h\int_\Omega \tilde{m}(\varphi_k)|\nabla\mu_{k+1}|^2\,dx + \varepsilon\int_\Omega \frac{|\nabla\varphi_{k+1} - \nabla\varphi_k|^2}{2}\,dx\\ &+ \delta\int_\Omega \frac{|\varphi_{k+1} - \varphi_k|^2}{h}\,dx + \delta h\int_\Omega |\Delta\mathbf{v}_{k+1}|^2\,dx \leq E_{tot}(\mathbf{v}_k, \varphi_k, \nabla\varphi_k, q_k)\end{aligned} \tag{3.31}$$

is satisfied.

The proof is done in a similar way as the proof in [ADG13, Lemma 4.2] for the model without surfactants. In this model, we additionally have soluble surfactants in both fluids, which leads to an extra equation that models the mass balance of the surfactant. Moreover, the other equations are a bit different to the ones in [ADG13], but can often be treated in a similar way.
First of all, we prove the discrete energy estimate (3.31). Remember that the total energy $E_{tot}(\mathbf{v}, \varphi, \nabla\varphi, q)$ is defined by

$$E_{tot}(\mathbf{v}, \varphi, \nabla\varphi, q) := \int_\Omega \left(\frac{1}{2}\rho(\varphi)|\mathbf{v}|^2 + \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{d(q)}{\varepsilon}W(\varphi) + G(q)\right)dx,$$

cf. (3.13). Afterwards the existence of weak solutions for the time-discrete problem is shown with the help of the Leray-Schauder principle, cf. Theorem 2.2.

Proof. (Proof of Theorem 3.7)
The proof is divided into two parts. In the first part, we assume that a weak solution in the sense of Definition 3.6 exists and prove that it satisfies the energy estimate (3.31). In the second part of the proof, we use the Leray-Schauder principle to prove the existence of weak solutions.

3.2.2 The Energy Inequality for Weak Solutions of the Time-Discrete Problem

First of all we start with the proof that the energy estimate (3.31) holds for any weak solution $(\mathbf{v}_{k+1}, \varphi_{k+1}, \mu_{k+1}, q_{k+1})$ solving (3.20) - (3.24) in the sense of Definition 3.6. To this end, we test equation (3.26) with $\mathbf{v}_{k+1}$, (3.27) with q_{k+1}, (3.28) with μ_{k+1} and (3.29) with $\frac{\varphi_{k+1}-\varphi_k}{h}$. Hence, we obtain

$$\begin{aligned}
0 = &\int\limits_\Omega \frac{\rho_{k+1}\mathbf{v}_{k+1} - \rho_k \mathbf{v}_k}{h} \cdot \mathbf{v}_{k+1}\, dx + \int\limits_\Omega \operatorname{div}(\rho_k \mathbf{v}_{k+1} \otimes \mathbf{v}_{k+1}) \cdot \mathbf{v}_{k+1}\, dx \\
&+ \int\limits_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2\, dx - \int\limits_\Omega \left(\tilde{\mathbf{J}}_{k+1} \otimes \mathbf{v}_{k+1}\right) : \nabla \mathbf{v}_{k+1}\, dx \\
&- \frac{1}{2}\int\limits_\Omega \frac{\rho_{k+1} - \rho_k}{h}|\mathbf{v}_{k+1}|^2\, dx + \frac{1}{2}\int\limits_\Omega \left(\rho_k \mathbf{v}_{k+1} + \tilde{\mathbf{J}}_{k+1}\right) \cdot \nabla(\mathbf{v}_{k+1} \cdot \mathbf{v}_{k+1})\, dx \\
&+ \int\limits_\Omega \left(\frac{h(q_{k+1})}{\varepsilon} W'(\varphi_k) - \mu_{k+1}\right) \nabla\varphi_k \cdot \mathbf{v}_{k+1}\, dx + \delta \int\limits_\Omega |\Delta \mathbf{v}_{k+1}|^2\, dx,
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
0 = &\int\limits_\Omega m(\varphi_k, q_k)|\nabla q_{k+1}|^2\, dx + \int\limits_\Omega \frac{(g(q_{k+1}) - g(q_k))\, q_{k+1}}{h}\, dx \\
&+ \int\limits_\Omega \frac{1}{\varepsilon}\left(\frac{f(q_{k+1}) - f(q_k)}{h} W(\varphi_k) + f(q_{k+1})\frac{W(\varphi_{k+1}) - W(\varphi_k)}{h}\right) q_{k+1}\, dx \\
&- \int\limits_\Omega \left(\frac{1}{\varepsilon} f(q_{k+1})W(\varphi_k) + g(q_{k+1})\right) \mathbf{v}_{k+1} \cdot \nabla q_{k+1}\, dx,
\end{aligned} \tag{3.33}$$

$$0 = \int\limits_\Omega \tilde{m}(\varphi_k)|\nabla \mu_{k+1}|^2\, dx + \int\limits_\Omega \frac{\varphi_{k+1} - \varphi_k}{h} \cdot \mu_{k+1}\, dx + \int\limits_\Omega (\nabla\varphi_k \cdot \mathbf{v}_{k+1})\mu_{k+1}\, dx, \tag{3.34}$$

$$\begin{aligned}
0 = &-\int\limits_\Omega \mu_{k+1} \cdot \frac{\varphi_{k+1} - \varphi_k}{h}\, dx + \int\limits_\Omega \varepsilon \frac{\nabla\varphi_{k+1} \cdot (\nabla\varphi_{k+1} - \nabla\varphi_k)}{h}\, dx \\
&+ \int\limits_\Omega h(q_{k+1})\frac{1}{\varepsilon}\frac{W(\varphi_{k+1}) - W(\varphi_k)}{h}\, dx + \delta \int\limits_\Omega \frac{|\varphi_{k+1} - \varphi_k|^2}{h^2}\, dx.
\end{aligned} \tag{3.35}$$

For the derivation of (3.35) we used

$$\int_\Omega h(q_{k+1})\frac{1}{\varepsilon}H(\varphi_{k+1},\varphi_k)\frac{(\varphi_{k+1}-\varphi_k)}{h}\,dx = \int_\Omega h(q_{k+1})\frac{W(\varphi_{k+1})-W(\varphi_k)}{\varepsilon h}\,dx,$$

since it holds $H(a,b)(a-b) = W(a)-W(b)$ for every $a,b\in\mathbb{R}$.
In the following we simplify the equations (3.32) - (3.35) step by step and finally we combine them in such a way that we obtain the energy inequality which we wanted to show. One important tool is

$$\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) = \frac{|\mathbf{a}|^2}{2}-\frac{|\mathbf{b}|^2}{2}+\frac{|\mathbf{a}-\mathbf{b}|^2}{2} \qquad \text{for } \mathbf{a},\mathbf{b}\in\mathbb{R}^d. \tag{3.36}$$

Analogously as in [ADG13] this yields for the first term in (3.32)

$$\begin{aligned}(\rho_{k+1}\mathbf{v}_{k+1}-\rho_k\mathbf{v}_k)\cdot\mathbf{v}_{k+1} &= (\rho_{k+1}-\rho_k)|\mathbf{v}_{k+1}|^2+\rho_k(\mathbf{v}_{k+1}-\mathbf{v}_k)\cdot\mathbf{v}_{k+1}\\ &= (\rho_{k+1}-\rho_k)|\mathbf{v}_{k+1}|^2+\rho_k\left(\frac{|\mathbf{v}_{k+1}^2|}{2}-\frac{|\mathbf{v}_k|^2}{2}\right)+\rho_k\frac{|\mathbf{v}_{k+1}-\mathbf{v}_k|^2}{2}\\ &= \left(\rho_{k+1}\frac{|\mathbf{v}_{k+1}|^2}{2}-\rho_k\frac{|\mathbf{v}_k|^2}{2}\right)+(\rho_{k+1}-\rho_k)\frac{|\mathbf{v}_{k+1}|^2}{2}+\rho_k\frac{|\mathbf{v}_{k+1}-\mathbf{v}_k|^2}{2}.\end{aligned}$$

Moreover, we can calculate

$$\begin{aligned}&\frac{1}{2}\int_\Omega \tilde{\mathbf{J}}_{k+1}\cdot\nabla|\mathbf{v}_{k+1}|^2\,dx-\int_\Omega\left(\tilde{\mathbf{J}}_{k+1}\otimes\mathbf{v}_{k+1}\right):\nabla\mathbf{v}_{k+1}\,dx\\ &=\frac{1}{2}\int_\Omega \tilde{\mathbf{J}}_{k+1}\cdot\nabla|\mathbf{v}_{k+1}|^2\,dx-\int_\Omega\sum_{i,j=1}^d(\tilde{\mathbf{J}}_{k+1})_i(\mathbf{v}_{k+1})_j\partial_{x_i}(\mathbf{v}_{k+1})_j\,dx\\ &=\frac{1}{2}\int_\Omega \tilde{\mathbf{J}}_{k+1}\cdot\nabla|\mathbf{v}_{k+1}|^2\,dx-\int_\Omega\tilde{\mathbf{J}}_{k+1}\cdot\nabla\frac{|\mathbf{v}_{k+1}|^2}{2}\,dx=0\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\left(\operatorname{div}(\rho_k\mathbf{v}_{k+1}\otimes\mathbf{v}_{k+1})\cdot\mathbf{v}_{k+1}+\rho_k\mathbf{v}_{k+1}\cdot\nabla\frac{|\mathbf{v}_{k+1}|^2}{2}\right)dx\\ &=\int_\Omega\left(\operatorname{div}(\rho_k\mathbf{v}_{k+1})|\mathbf{v}_{k+1}|^2+(\rho_k\mathbf{v}_{k+1}\cdot\nabla\mathbf{v}_{k+1})\cdot\mathbf{v}_{k+1}-\operatorname{div}(\rho_k\mathbf{v}_{k+1})\frac{|\mathbf{v}_{k+1}|^2}{2}\right)dx\\ &=\int_\Omega\left(\operatorname{div}(\rho_k\mathbf{v}_{k+1})|\mathbf{v}_{k+1}|^2+\rho_k\mathbf{v}_{k+1}\cdot\nabla\left(\frac{|\mathbf{v}_{k+1}|^2}{2}\right)-\operatorname{div}(\rho_k\mathbf{v}_{k+1})\frac{|\mathbf{v}_{k+1}|^2}{2}\right)dx\\ &=\int_\Omega\operatorname{div}\left(\rho_k\mathbf{v}_{k+1}\frac{|\mathbf{v}_{k+1}|^2}{2}\right)dx=0,\end{aligned}$$

where we used integration by parts, (2.1) in the first step and $\operatorname{div}(\mathbf{v}_{k+1}) = 0$. Moreover, we used that for arbitrary $\mathbf{a}, \mathbf{b} \in H_0^1(\Omega)^d$ it holds

$$\begin{aligned}
(\mathbf{a} \cdot \nabla \mathbf{b}) \cdot \mathbf{b} &= \big(\mathbf{a} \cdot \big(\nabla b_1 \quad \nabla b_2 \quad \dots \quad \nabla b_d\big)\big)\, \mathbf{b} = \begin{pmatrix} \mathbf{a} \cdot \nabla b_1 \\ \mathbf{a} \cdot \nabla b_2 \\ \dots \\ \mathbf{a} \cdot \nabla b_d \end{pmatrix} \mathbf{b} \\
&= (\mathbf{a} \cdot \nabla b_1) b_1 + (\mathbf{a} \cdot \nabla b_2) b_2 + \dots + (\mathbf{a} \cdot \nabla b_d) b_d \\
&= \mathbf{a} \cdot \frac{\nabla b_1^2}{2} + \mathbf{a} \cdot \frac{\nabla b_2^2}{2} + \dots + \mathbf{a} \cdot \frac{\nabla b_d^2}{2} = \mathbf{a} \cdot \frac{\nabla |\mathbf{b}|^2}{2}.
\end{aligned}$$

Hence, we get for (3.26) tested with $\mathbf{v}_{k+1}$

$$\begin{aligned}
0 = &\int_\Omega \frac{\rho_{k+1} |\mathbf{v}_{k+1}|^2 - \rho_k |\mathbf{v}_k|^2}{2h} \, dx + \int_\Omega \rho_k \frac{|\mathbf{v}_{k+1} - \mathbf{v}_k|^2}{2h} \, dx + \int_\Omega 2\eta(\varphi_k) |D\mathbf{v}_{k+1}|^2 \, dx \\
&- \int_\Omega \mu_{k+1} \nabla \varphi_k \cdot \mathbf{v}_{k+1} \, dx + \int_\Omega \frac{h(q_{k+1})}{\varepsilon} W'(\varphi_k) \nabla \varphi_k \cdot \mathbf{v}_{k+1} \, dx + \delta \int_\Omega |\Delta \mathbf{v}_{k+1}|^2 \, dx.
\end{aligned}$$

Using integration by parts and $-h'(q) = f(q)$ for all $q \in \mathbb{R}$ yields

$$\begin{aligned}
0 = &\int_\Omega \frac{\rho_{k+1} |\mathbf{v}_{k+1}|^2 - \rho_k |\mathbf{v}_k|^2}{2h} \, dx + \int_\Omega \rho_k \frac{|\mathbf{v}_{k+1} - \mathbf{v}_k|^2}{2h} \, dx + \int_\Omega 2\eta(\varphi_k) |D\mathbf{v}_{k+1}|^2 \, dx \\
&- \int_\Omega \mu_{k+1} \nabla \varphi_k \cdot \mathbf{v}_{k+1} \, dx + \int_\Omega \frac{f(q_{k+1})}{\varepsilon} W(\varphi_k) \nabla q_{k+1} \cdot \mathbf{v}_{k+1} \, dx \\
&+ \delta \int_\Omega |\Delta \mathbf{v}_{k+1}|^2 \, dx. \qquad (3.37)
\end{aligned}$$

Now we have a look at equation (3.33). Since we assume h to be a concave function, cf. Assumption 3.4, it holds

$$h'(q_k)(q_{k+1} - q_k) \geq h(q_{k+1}) - h(q_k).$$

Using this inequality and the identity $f(q) = -h'(q)$ for all $q \in \mathbb{R}$, cf. (3.9), we calculate

$$\begin{aligned}
(f(q_{k+1}) - f(q_k)) q_{k+1} &= f(q_{k+1}) q_{k+1} - f(q_k) q_k + f(q_k)(q_k - q_{k+1}) \\
&= f(q_{k+1}) q_{k+1} - f(q_k) q_k + h'(q_k)(q_{k+1} - q_k) \\
&\geq f(q_{k+1}) q_{k+1} - f(q_k) q_k + h(q_{k+1}) - h(q_k).
\end{aligned}$$

Moreover, we can show

$$(g(q_2) - g(q_1))\, q_2 \geq G(q_2) - G(q_1) \qquad \text{for all } q_1, q_2 \in \mathbb{R}.$$

To prove this inequality we first of all assume $q_1 \leq q_2$. Then we obtain

$$(g(q_2) - g(q_1))\, q_2 = \left(\int_{q_1}^{q_2} g'(z)\, dz \right) q_2 = \int_{q_1}^{q_2} g'(z) q_2\, dz.$$

Due to $g'(z) > 0$ for all $z \in \mathbb{R}$, cf. Assumption 3.2, and $z \leq q_2$ for all $z \in [q_1, q_2]$, we can proceed

$$\int_{q_1}^{q_2} g'(z) q_2\, dz \geq \int_{q_1}^{q_2} g'(z) z\, dz = \int_{q_1}^{q_2} G'(z)\, dz = G(q_2) - G(q_1).$$

Finally, we also have to consider the case $q_1 \geq q_2$. In this case it analogously holds

$$(g(q_2) - g(q_1))\, q_2 = \left(\int_{q_2}^{q_1} -g'(z)\, dz \right) q_2 \geq \int_{q_2}^{q_1} -g'(z) z\, dz = G(q_2) - G(q_1).$$

Using these calculations in (3.33) yields

$$\begin{aligned} 0 \geq & \int_\Omega m(\varphi_k, q_k) |\nabla q_{k+1}|^2\, dx + \int_\Omega \frac{G(q_{k+1}) - G(q_k)}{h}\, dx \\ & + \int_\Omega f(q_{k+1}) q_{k+1} \frac{W(\varphi_{k+1}) - W(\varphi_k)}{\varepsilon h}\, dx \\ & + \int_\Omega \frac{1}{\varepsilon h} W(\varphi_k) \left(f(q_{k+1}) q_{k+1} - f(q_k) q_k + h(q_{k+1}) - h(q_k) \right) dx \\ & - \int_\Omega \left(\frac{1}{\varepsilon} f(q_{k+1}) W(\varphi_k) + g(q_{k+1}) \right) \mathbf{v}_{k+1} \cdot \nabla q_{k+1}\, dx. \end{aligned} \tag{3.38}$$

Since equation (3.34) remains unchanged, we can skip to (3.35). Here we use (3.36) to get

$$\begin{aligned} 0 = & - \int_\Omega \mu_{k+1} \frac{\varphi_{k+1} - \varphi_k}{h}\, dx + \int_\Omega \frac{\varepsilon}{h} \left(\frac{|\nabla \varphi_{k+1}|^2}{2} - \frac{|\nabla \varphi_k|^2}{2} + \frac{|\nabla \varphi_{k+1} - \nabla \varphi_k|^2}{2} \right) dx \\ & + \int_\Omega h(q_{k+1}) \frac{1}{\varepsilon} \frac{W(\varphi_{k+1}) - W(\varphi_k)}{h}\, dx + \delta \int_\Omega \frac{|\varphi_{k+1} - \varphi_k|^2}{h^2}\, dx. \end{aligned} \tag{3.39}$$

From the identities (3.34), (3.37), (3.38) and (3.39) we obtain

$$
\begin{aligned}
0 \geq \frac{1}{h} \int_\Omega \frac{\rho_{k+1}|\mathbf{v}_{k+1}|^2}{2} - \frac{\rho_k|\mathbf{v}_k|^2}{2} + \frac{\rho_k|\mathbf{v}_{k+1}-\mathbf{v}_k|^2}{2} \, dx + \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2 \, dx \\
+ \int_\Omega m(\varphi_k, q_k)|\nabla q_{k+1}|^2 \, dx + \frac{1}{h}\int_\Omega G(q_{k+1}) - G(q_k) \, dx + \delta \int_\Omega \frac{|\varphi_{k+1}-\varphi_k|^2}{h^2} \, dx \\
+ \int_\Omega \frac{1}{\varepsilon h} W(\varphi_k) \left(f(q_{k+1})q_{k+1} - f(q_k)q_k + h(q_{k+1}) - h(q_k) \right) dx + \delta \int_\Omega |\Delta \mathbf{v}_{k+1}|^2 \, dx \\
+ \int_\Omega \frac{1}{\varepsilon} f(q_{k+1})q_{k+1} \frac{W(\varphi_{k+1}) - W(\varphi_k)}{h} \, dx + \int_\Omega h(q_{k+1}) \frac{1}{\varepsilon} \frac{W(\varphi_{k+1}) - W(\varphi_k)}{h} \, dx \\
+ \int_\Omega \tilde{m}(\varphi_k)|\nabla \mu_{k+1}|^2 \, dx + \int_\Omega \frac{\varepsilon}{h} \left(\frac{|\nabla\varphi_{k+1}|^2}{2} - \frac{|\nabla\varphi_k|^2}{2} + \frac{|\nabla\varphi_{k+1} - \nabla\varphi_k|^2}{2} \right) dx,
\end{aligned}
$$

where we used $\int_\Omega g(q_{k+1})\nabla q_{k+1} \cdot \mathbf{v}_{k+1} \, dx = 0$ as we have already seen in Section 3.1.1. From the identity $d(q) = h(q) + f(q)q$, cf. (3.11), we can conclude

$$
\begin{aligned}
\int_\Omega \frac{1}{\varepsilon h} W(\varphi_k) \Big(f(q_{k+1})q_{k+1} - f(q_k)q_k + h(q_{k+1}) - h(q_k) \Big) \, dx \\
+ \int_\Omega \frac{1}{\varepsilon} f(q_{k+1})q_{k+1} \frac{W(\varphi_{k+1}) - W(\varphi_k)}{h} \, dx + \int_\Omega h(q_{k+1}) \frac{1}{\varepsilon} \frac{W(\varphi_{k+1}) - W(\varphi_k)}{h} \, dx \\
= \int_\Omega \frac{W(\varphi_{k+1})d(q_{k+1})}{\varepsilon h} \, dx - \int_\Omega \frac{W(\varphi_k)d(q_k)}{\varepsilon h} \, dx.
\end{aligned}
$$

Inserting this identity in the previous estimate and reordering the different terms implies

$$
\begin{aligned}
\frac{1}{h}\int_\Omega \frac{\rho_k|\mathbf{v}_k|^2}{2} \, dx + \frac{1}{h}\int_\Omega G(q_k) \, dx + \frac{\varepsilon}{h}\int_\Omega \frac{|\nabla\varphi_k|^2}{2} \, dx + \int_\Omega \frac{W(\varphi_k)d(q_k)}{\varepsilon h} \, dx \\
\geq \frac{1}{h}\int_\Omega \frac{\rho_{k+1}|\mathbf{v}_{k+1}|^2}{2} + \frac{\rho_k|\mathbf{v}_{k+1}-\mathbf{v}_k|^2}{2} \, dx + \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2 \, dx + \frac{1}{h}\int_\Omega G(q_{k+1}) \, dx \\
+ \int_\Omega m(\varphi_k, q_k)|\nabla q_{k+1}|^2 \, dx + \int_\Omega \frac{W(\varphi_{k+1})d(q_{k+1})}{\varepsilon h} \, dx + \int_\Omega \tilde{m}(\varphi_k)|\nabla \mu_{k+1}|^2 \, dx \\
+ \frac{\varepsilon}{h}\int_\Omega \frac{|\nabla\varphi_{k+1}|^2}{2} + \frac{|\nabla\varphi_{k+1} - \nabla\varphi_k|^2}{2} \, dx + \delta\int_\Omega \frac{|\varphi_{k+1}-\varphi_k|^2}{h^2} \, dx + \delta\int_\Omega |\Delta\mathbf{v}_{k+1}|^2 \, dx.
\end{aligned}
$$

Hence, we obtain the discrete energy estimate

$$
\begin{aligned}
&E_{tot}(\mathbf{v}_{k+1}, \varphi_{k+1}, \nabla\varphi_{k+1}, q_{k+1}) + \int\limits_\Omega \frac{\rho_k|\mathbf{v}_{k+1} - \mathbf{v}_k|^2}{2}\, dx + h\int\limits_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2\, dx \\
&+ h\int\limits_\Omega m(\varphi_k, q_k)|\nabla q_{k+1}|^2\, dx + h\int\limits_\Omega \tilde{m}(\varphi_k)|\nabla\mu_{k+1}|^2\, dx + \varepsilon\int\limits_\Omega \frac{|\nabla\varphi_{k+1} - \nabla\varphi_k|^2}{2}\, dx \\
&+ \delta\int\limits_\Omega \frac{|\varphi_{k+1} - \varphi_k|^2}{h}\, dx + \delta h\int\limits_\Omega |\Delta\mathbf{v}_{k+1}|^2\, dx \le E_{tot}(\mathbf{v}_k, \varphi_k, \nabla\varphi_k, q_k),
\end{aligned}
$$

where $E_{tot}(\mathbf{v}, \varphi, \nabla\varphi, q)$ is defined as in (3.13).

3.2.3 Existence Proof of Weak Solutions for the Time-Discrete Problem

In this section we prove the existence of weak solutions for the time-discrete problem (3.20) - (3.24). To this end, we define two operators $\mathcal{L}_k, \mathcal{F}_k : X \to Y$ and apply the Leray-Schauder principle, where

$$
\begin{aligned}
X &:= V(\Omega) \times H^1(\Omega) \times H^2_n(\Omega) \times H^1(\Omega), \\
Y &:= V'(\Omega) \times H^{-1}_0(\Omega) \times L^2(\Omega) \times H^{-1}_0(\Omega).
\end{aligned}
$$

Here we set $V(\Omega) := H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, $V'(\Omega)$ is its dual space and $H^{-1}_0(\Omega) := (H^1(\Omega))'$. For $\mathbf{w}_{k+1} := (\mathbf{v}_{k+1}, q_{k+1}, \mu_{k+1}, \varphi_{k+1}) \in X$ we define the operator $\mathcal{L}_k : X \to Y$ by

$$
\mathcal{L}_k(\mathbf{w}_{k+1}) = \begin{pmatrix} \mathcal{A}(\varphi_k)\mathbf{v}_{k+1} \\ \operatorname{div}_N(m(\varphi_k, q_k)\nabla q_{k+1}) - \int\limits_\Omega q_{k+1}\, dx \\ \operatorname{div}(\tilde{m}(\varphi_k)\nabla\mu_{k+1}) - \int\limits_\Omega \mu_{k+1}\, dx \\ \varepsilon\Delta_N\varphi_{k+1} - \int\limits_\Omega \varphi_{k+1}\, dx \end{pmatrix},
$$

where $\mathcal{A}(\varphi_k) : V(\Omega) \to V'(\Omega)$ is given by

$$
\langle \mathcal{A}(\varphi_k)\mathbf{v}_{k+1}, \boldsymbol{\psi}\rangle := -\int\limits_\Omega 2\eta(\varphi_k)D\mathbf{v}_{k+1} : D\boldsymbol{\psi}\, dx - \delta\int\limits_\Omega \Delta\mathbf{v}_{k+1}\Delta\boldsymbol{\psi}\, dx
$$

for all $\boldsymbol{\psi} \in V(\Omega)$ and $\operatorname{div}_N : L^2(\Omega)^d \to H^{-1}_0(\Omega)$ and $\Delta_N : H^1(\Omega) \to H^{-1}_0(\Omega)$ are defined by

$$
\begin{aligned}
\langle \operatorname{div}_N \mathbf{f}, \phi\rangle &:= -\int\limits_\Omega \mathbf{f}\cdot\nabla\phi\, dx, \\
\langle \Delta_N\varphi, \phi\rangle &:= -\int\limits_\Omega \nabla\varphi\cdot\nabla\phi\, dx
\end{aligned}
$$

for all $\mathbf{f} \in L^2(\Omega)^d$, $\varphi \in H^1(\Omega)$ and $\phi \in H^1(\Omega)$. Moreover, we define for $\mathbf{w}_{k+1} = (\mathbf{v}_{k+1}, q_{k+1}, \mu_{k+1}, \varphi_{k+1}) \in X$ the operator $\mathcal{F}_k : X \to Y$ by

$$\mathcal{F}_k(\mathbf{w}_{k+1}) = \begin{pmatrix} \frac{\rho_{k+1}\mathbf{v}_{k+1}-\rho_k\mathbf{v}_k}{h} + \left(\operatorname{div}\tilde{\mathbf{J}}_{k+1} - \frac{\rho_{k+1}-\rho_k}{h} - \mathbf{v}_{k+1}\cdot\nabla\rho_k\right)\frac{\mathbf{v}_{k+1}}{2} \\ +(\tilde{\mathbf{J}}_{k+1}\cdot\nabla)\mathbf{v}_{k+1} - \left(\mu_{k+1} - \frac{h(q_{k+1})}{\varepsilon}W'(\varphi_k)\right)\nabla\varphi_k \\ +\operatorname{div}(\rho_k\mathbf{v}_{k+1}\otimes\mathbf{v}_{k+1}) \\ \\ \frac{1}{\varepsilon}\left(\frac{f(q_{k+1})-f(q_k)}{h}W(\varphi_k) + f(q_{k+1})\frac{W(\varphi_{k+1})-W(\varphi_k)}{h}\right) + \frac{g(q_{k+1})-g(q_k)}{h} \\ +\nabla\left(\frac{1}{\varepsilon}f(q_{k+1})W(\varphi_k) + g(q_{k+1})\right)\cdot\mathbf{v}_{k+1} - \int_\Omega q_{k+1}\,dx \\ \\ \frac{\varphi_{k+1}-\varphi_k}{h} + \nabla\varphi_k\cdot\mathbf{v}_{k+1} - \int_\Omega \mu_{k+1}\,dx \\ \\ h(q_{k+1})\frac{1}{\varepsilon}H(\varphi_{k+1},\varphi_k) - \mu_{k+1} + \delta\frac{\varphi_{k+1}-\varphi_k}{h} - \int_\Omega \varphi_{k+1}\,dx \end{pmatrix},$$

where $\tilde{\mathbf{J}}_{k+1}$ is defined as in (3.18). Note that in the definition of the operators $\mathcal{L}_k, \mathcal{F}_k : X \to Y$ we used equation (3.30), which is (3.20) in the weak sense. With these definitions it holds

$$\mathcal{L}_k(\mathbf{w}_{k+1}) - \mathcal{F}_k(\mathbf{w}_{k+1}) = 0 \qquad \text{in } Y$$

if and only if $\mathbf{w}_{k+1} = (\mathbf{v}_{k+1}, q_{k+1}, \mu_{k+1}, \varphi_{k+1}) \in X$ is a weak solution of (3.20) - (3.24). Now we want to show that the operator $\mathcal{L}_k : X \to Y$ is invertible with bounded inverse. To this end, let $g_0 \in V'(\Omega)$, $g_1, g_3 \in H_0^{-1}(\Omega)$ and $g_2 \in L^2(\Omega)$ be given. Then we need to study the elliptic equations

$$\begin{aligned} \mathcal{A}(\varphi_k)\mathbf{v}_{k+1} &= g_0 && \text{in } V'(\Omega), \\ \operatorname{div}_N(m(\varphi_k, q_k)\nabla q_{k+1}) - \int_\Omega q_{k+1}\,dx &= g_1 && \text{in } H_0^{-1}(\Omega), \\ \operatorname{div}_N(\tilde{m}(\varphi_k)\nabla\mu_{k+1}) - \int_\Omega \mu_{k+1}\,dx &= g_2 && \text{in } H_0^{-1}(\Omega), \\ \varepsilon\Delta_N\varphi_{k+1} - \int_\Omega \varphi_{k+1}\,dx &= g_3 && \text{in } H_0^{-1}(\Omega). \end{aligned}$$

The Lax-Milgram theorem, cf. Theorem 2.1, yields the existence of unique weak solutions $\mathbf{v}_{k+1} \in H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L^2_\sigma(\Omega)$ and φ_{k+1}, μ_{k+1}, $q_{k+1} \in H^1(\Omega)$.
But for μ_{k+1} it remains to show that it is in $H^2_n(\Omega)$. Therefore we want to use a bootstrapping argument as in [ADG13]. We note that μ_{k+1} is also a weak solution

of the equation

$$\begin{aligned} \Delta\mu_{k+1} &= (\tilde{m}(\varphi_k))^{-1}\left(-\nabla(\tilde{m}(\varphi_k))\cdot\nabla\mu_{k+1} + \int_\Omega \mu_{k+1}dx + g_2\right) && \text{in } \Omega, \\ \partial_n\mu_{k+1|\partial\Omega} &= 0 && \text{on } \partial\Omega \end{aligned}$$

for some $g_2 \in L^2(\Omega)$. Due to $\tilde{m}(\varphi_k) \in H^2(\Omega)$ we get $\nabla(\tilde{m}(\varphi_k)) \in H^1(\Omega) \hookrightarrow L^6(\Omega)$. From $\nabla\mu_{k+1} \in L^2(\Omega)$ it follows $\nabla(\tilde{m}(\varphi_k))\cdot\nabla\mu_{k+1} \in L^{\frac{3}{2}}(\Omega)$. Since we know from Assumption 3.4 that the mobility $\tilde{m}$ fulfills the estimate $0 < \frac{1}{\tilde{m}(\varphi_k)} \leq \frac{1}{c_1}$ for a constant $c_1 > 0$, it holds for some $\tilde{g}_2 \in L^{\frac{3}{2}}(\Omega)$

$$\Delta\mu_{k+1} = \tilde{g}_2 \quad \in L^{\frac{3}{2}}(\Omega)$$

in the weak sense with Neumann boundary condition. Therefore, elliptic regularity theory yields $\mu_{k+1} \in W^2_{\frac{3}{2}}(\Omega)$, cf. [Lun95, Theorem 3.1.2 and Theorem 3.1.3]. This implies $\nabla\mu_{k+1} \in W^1_{\frac{3}{2}}(\Omega) \hookrightarrow L^3(\Omega)$ for $d = 2,3$, cf. Theorem 2.15. Since it holds $\nabla(\tilde{m}(\varphi_k)) \in L^6(\Omega)$, we can conclude $\nabla(\tilde{m}(\varphi_k))\cdot\nabla\mu_{k+1} \in L^2(\Omega)$. Altogether this yields

$$\Delta\mu_{k+1} = \tilde{g}_2 \quad \in L^2(\Omega)$$

in the weak sense with Neumann boundary condition. Therefore, elliptic regularity theory yields $\mu_{k+1} \in H^2_n(\Omega)$ together with the estimate

$$||\mu_{k+1}||_{H^2(\Omega)} \leq C\left(||\mu_{k+1}||_{H^1(\Omega)} + ||\tilde{g}_2||_{L^2(\Omega)}\right). \tag{3.40}$$

Moreover, the Lax-Milgram theorem provides that $\mathcal{L}_k : X \to Y$ is invertible with bounded inverse

$$\mathcal{L}_k^{-1} : Y \to X.$$

The next step is to observe that $\mathcal{F}_k : X \to Y$ is a compact operator. To this end, we introduce the Banach space

$$\tilde{Y} := L^{\frac{4}{3}}_\sigma(\Omega) \times L^{\frac{4}{3}}(\Omega) \times W^1_{\frac{3}{2}}(\Omega) \times L^2(\Omega).$$

We can conclude that $\mathcal{F}_k : X \to \tilde{Y}$ is continuous and bounded, i.e., it maps sets which are bounded in X into sets which are bounded in $\tilde{Y}$. We start with the proof that $\mathcal{F}_k$ is bounded. Afterwards we prove the continuity of $\mathcal{F}_k : X \to \tilde{Y}$. Note that $\mathbf{v}_{k+1} \in H^1(\Omega)^d \cap L^2_\sigma(\Omega)$ would be sufficient to prove the boundedness and continuity of $\mathcal{F}_k : X \to \tilde{Y}$. Therefore, we do all estimates in the following for

$\mathbf{v}_{k+1} \in H^1(\Omega)^d \cap L^2_\sigma(\Omega)$ although it even holds $\mathbf{v}_{k+1} \in H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$. For $\mathbf{w}_{k+1} = (\mathbf{v}_{k+1}, q_{k+1}, \mu_{k+1}, \varphi_{k+1}) \in X$ it holds

$$\begin{aligned}
||\rho_{k+1}\mathbf{v}_{k+1}||_{L^{\frac{4}{3}}(\Omega)} &\leq C||\mathbf{v}_{k+1}||_{H^1(\Omega)}, \\
||\mathrm{div}(\rho_k \mathbf{v}_{k+1} \otimes \mathbf{v}_{k+1})||_{L^{\frac{4}{3}}(\Omega)} &\leq C_k||\mathbf{v}_{k+1}||^2_{H^1(\Omega)}, \\
||(\mathrm{div}\tilde{\mathbf{J}}_{k+1})\mathbf{v}_{k+1}||_{L^{\frac{4}{3}}(\Omega)} &\leq C_k||\mathbf{v}_{k+1}||_{H^1(\Omega)}||\mu_{k+1}||_{H^2(\Omega)}, \\
||h(q_{k+1})W'(\varphi_k)\nabla\varphi_k||_{L^{\frac{4}{3}}(\Omega)} &\leq C_k\left(||q_{k+1}||_{H^1(\Omega)} + 1\right), \\
||f(q_{k+1})W(\varphi_{k+1})||_{L^{\frac{4}{3}}(\Omega)} &\leq C\left(||\varphi_{k+1}||^3_{H^1(\Omega)} + 1\right), \\
||\nabla(f(q_{k+1})W(\varphi_k)) \cdot \mathbf{v}_{k+1}||_{L^{\frac{4}{3}}(\Omega)} &\leq C_k||q_{k+1}||_{H^1(\Omega)}||\mathbf{v}_{k+1}||_{H^1(\Omega)}, \\
||\nabla g(q_{k+1}) \cdot \mathbf{v}_{k+1}||_{L^{\frac{4}{3}}(\Omega)} &\leq C||q_{k+1}||_{H^1(\Omega)}||\mathbf{v}_{k+1}||_{H^1(\Omega)}, \\
||f(q_{k+1})W(\varphi_k)||_{L^{\frac{4}{3}}(\Omega)} &\leq C_k, \\
||\nabla\varphi_k \cdot \mathbf{v}_{k+1}||_{W^1_{\frac{3}{2}}(\Omega)} &\leq C_k||\mathbf{v}_{k+1}||_{H^1(\Omega)}, \\
||h(q_{k+1})H(\varphi_{k+1}, \varphi_k)||_{L^2(\Omega)} &\leq C_k\left(||q_{k+1}||_{L^6(\Omega)} + 1\right)\left(||\varphi_{k+1}||^2_{L^6(\Omega)} + 1\right).
\end{aligned}$$

To prove these estimates, we use that f is bounded together with the growth conditions $|h(q)| \leq C(|q| + 1)$, $|W(q)| \leq C(|q|^3 + 1)$, $|W'(q)| \leq C(|q|^2 + 1)$ and $|G'(q)| \leq C(|q| + 1)$ for all $q \in \mathbb{R}$ and a constant $C > 0$. Moreover, we use the identities $f = -h'$, $G'(q) = g'(q)q$ and the fact that f is constant outside an interval $[q_{min}, q_{max}]$. More precisely:

i) Since ρ is a bounded function we get

$$||\rho_{k+1}\mathbf{v}_{k+1}||_{L^{\frac{4}{3}}(\Omega)} \leq C||\mathbf{v}_{k+1}||_{H^1(\Omega)}.$$

ii) Here we have to estimate terms of the form $\rho_k\partial_l(\mathbf{v}_{k+1})_i(\mathbf{v}_{k+1})_j$ and terms of the form $\partial_l\rho_k(\mathbf{v}_{k+1})_i(\mathbf{v}_{k+1})_j$ in $L^{\frac{4}{3}}(\Omega)$. For the first kind of terms we note $\rho_k \in L^\infty(\Omega)$, $\nabla\mathbf{v}_{k+1} \in L^2(\Omega)$ and $\mathbf{v}_{k+1} \in H^1(\Omega) \hookrightarrow L^6(\Omega)$, which implies the boundedness of the product in $L^{\frac{3}{2}}(\Omega) \hookrightarrow L^{\frac{4}{3}}(\Omega)$. For the other kind of terms we use $\nabla\rho(\varphi_k) \in L^6(\Omega)$ and $\mathbf{v}_{k+1} \in L^6(\Omega)$ to conclude the boundedness of the product in $L^2(\Omega) \hookrightarrow L^{\frac{4}{3}}(\Omega)$. Here we used that for $\varphi_k \in H^2(\Omega)$, Theorem 2.18 implies $\rho(\varphi_k) \in H^2(\Omega) \hookrightarrow W^1_6(\Omega)$.

iii) The velocity field $\mathbf{v}_{k+1}$ is bounded in $H^1(\Omega) \hookrightarrow L^6(\Omega)$. Moreover, $\tilde{\mathbf{J}}_{k+1}$ is defined by $\tilde{\mathbf{J}}_{k+1} = -\rho'(\varphi_k)\tilde{m}(\varphi_k)\nabla\mu_{k+1}$, cf. (3.18). Since φ_k is in $H^2(\Omega)$, the Theorem about the composition with Sobolev functions, cf. Theorem 2.18, yields that $\rho'(\varphi_k)$ and $\tilde{m}(\varphi_k)$ are bounded in $H^2(\Omega)$. Thus the Theorem about the multiplication of Sobolev functions, cf. Theorem 2.17, implies $\rho'(\varphi_k)\tilde{m}(\varphi_k) \in H^2(\Omega) \hookrightarrow W^1_6(\Omega)$. As it holds $\nabla\mu_{k+1} \in H^1(\Omega)$, Theorem 2.17 provides $\tilde{\mathbf{J}}_{k+1} = -\rho'(\varphi_k)\tilde{m}(\varphi_k)\nabla\mu_{k+1} \in W^1_r(\Omega)$ with $1 \leq r \leq 2$ for $d = 2, 3$ and therefore $\operatorname{div}(\tilde{\mathbf{J}}_{k+1}) \in L^2(\Omega)$. Hence it follows that the product $\operatorname{div}(\tilde{\mathbf{J}}_{k+1})\mathbf{v}_{k+1}$ is bounded in $L^{\frac{3}{2}}(\Omega) \hookrightarrow L^{\frac{4}{3}}(\Omega)$.

iv) From the growth conditions cited before we can even conclude that this term is bounded in $L^2(\Omega)$ and therefore in $L^{\frac{4}{3}}(\Omega)$. Using Hölder's inequality we obtain

$$\begin{aligned}||h(q_{k+1})W'(\varphi_k)\nabla\varphi_k||_{L^2(\Omega)} &\leq C||h(q_{k+1})||_{L^6(\Omega)}||W'(\varphi_k)||_{L^6(\Omega)}||\nabla\varphi_k||_{L^6(\Omega)} \\ &\leq C_k\left(||q_{k+1}||_{H^1(\Omega)} + 1\right)\end{aligned}$$

as we have $q_{k+1} \in H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $\varphi_k \in H^2(\Omega) \hookrightarrow L^{12}(\Omega)$.

v) Since f is a bounded function, $f(q_{k+1})$ is in $L^\infty(\Omega)$. From the growth condition on W together with $\varphi_{k+1} \in H^1(\Omega) \hookrightarrow L^6(\Omega)$ we get that $W(\varphi_{k+1})$ is in $L^2(\Omega)$ and therefore in $L^{\frac{4}{3}}(\Omega)$.

vi) Now we want to estimate $\nabla(f(q_{k+1})W(\varphi_k)) \cdot \mathbf{v}_{k+1}$ in the $L^{\frac{4}{3}}$-norm. Since we know $\mathbf{v}_{k+1} \in L^6(\Omega)$ we need to show that $\nabla(f(q_{k+1})W(\varphi_k))$ is bounded in $L^{\frac{12}{7}}(\Omega)$, i.e., $f(q_{k+1})W(\varphi_k)$ is bounded in $W^1_{\frac{12}{7}}(\Omega)$.
First of all we can conclude the boundedness of $W(\varphi_k)$ in $H^2(\Omega) \hookrightarrow W^1_6(\Omega)$ due to $\varphi_k \in H^2(\Omega)$ and Theorem 2.18. Now we want to use Theorem 2.17 for the multiplication of two Sobolev functions. The theorem yields that if $f(q_{k+1})$ is bounded in $W^1_{\frac{12}{7}}(\Omega)$, then $f(q_{k+1})W(\varphi_k)$ is also bounded in $W^1_{\frac{12}{7}}(\Omega)$ and this would yield the estimate which we wanted to show.
So it remains to show the boundedness of $f(q_{k+1})$ in $W^1_{\frac{12}{7}}(\Omega)$. To this end, let $(q_\varepsilon)_{\varepsilon>0} \subseteq C^\infty(\overline{\Omega})$ be a sequence such that $q_\varepsilon \to q_{k+1}$ in $H^1(\Omega)$ as $\varepsilon \to 0$. In particular we obtain a subsequence, which we denote by $(q_\varepsilon)_{\varepsilon>0}$ again, such that $q_\varepsilon(x) \to q_{k+1}(x)$ for a.e. $x \in \Omega$. Thus it holds $f(q_\varepsilon(x)) \to f(q_{k+1}(x))$ for a.e. $x \in \Omega$. Since f is a bounded function, Theorem 2.9 yields $f(q_\varepsilon) \to f(q_{k+1})$ in $L^p(\Omega)$ for every $1 \leq p < \infty$. Since f' is also a bounded function, we can analogously show $f'(q_\varepsilon) \to f'(q_{k+1})$ in $L^p(\Omega)$ for every $1 \leq p < \infty$.
Altogether we can conclude $\langle\nabla f(q_{k+1}), \varphi\rangle = \int_\Omega f'(q_{k+1})\nabla q_{k+1} \cdot \varphi dx$ for all $\varphi \in C_0^\infty(\Omega)^d$, cf. Lemma 2.11. Since we know that f' is bounded and $q_{k+1} \in H^1(\Omega)$, we get that $f'(q_{k+1})\nabla q_{k+1}$ is bounded in $L^2(\Omega)$. Thus we have shown the boundedness of $\nabla f(q_{k+1})$ in $L^2(\Omega) \hookrightarrow L^{\frac{12}{7}}(\Omega)$ and therefore the estimate we wanted to prove.

vii) To prove the boundedness of $\nabla g(q_{k+1}) \cdot \mathbf{v}_{k+1}$ in the $L^{\frac{4}{3}}$-norm we use the same ideas as in the previous estimate. By definition we have $G'(q) = g'(q)q$ for all $q \in \mathbb{R}$, cf. Assumption 3.2. Due to $|G'(q)| \leq C(|q| + 1)$, cf. Assumption 3.3, we can conclude that there exists a constant $C > 0$ such that $|g'(q)| \leq C$ for all $q \in \mathbb{R}$.

Let $(q_\varepsilon)_{\varepsilon>0} \subseteq C^\infty(\overline{\Omega})$ be a sequence as before such that $q_\varepsilon \to q_{k+1}$ in $H^1(\Omega)$ and $q_\varepsilon(x) \to q_{k+1}(x)$ for a.e. $x \in \Omega$ as $\varepsilon \to 0$. Then Theorem 2.9 yields

$$\begin{aligned} g(q_\varepsilon) &\to g(q_{k+1}) && \text{in } L^p(\Omega) \text{ for all } 1 \le p < 6, \\ g'(q_\varepsilon) &\to g'(q_{k+1}) && \text{in } L^p(\Omega) \text{ for all } 1 \le p < \infty. \end{aligned}$$

Thus we can conclude $\langle \nabla g(q_{k+1}), \varphi \rangle = \int_\Omega g'(q_{k+1}) \nabla q_{k+1} \varphi dx$ for all $\varphi \in C_0^\infty(\Omega)$, cf. Lemma 2.11. Due to the boundedness of g' we get that $g'(q_{k+1})\nabla q_{k+1}$ is bounded in $L^2(\Omega)$, i.e., $\nabla g(q_{k+1})$ is bounded in $L^2(\Omega) \hookrightarrow L^{\frac{12}{7}}(\Omega)$.
Since we have $\mathbf{v}_{k+1} \in L^6(\Omega)$, we can deduce the boundedness of $\nabla g(q_{k+1}) \cdot \mathbf{v}_{k+1}$ in $L^{\frac{4}{3}}(\Omega)$.

viii) Since φ_k is in $H_n^2(\Omega)$, this boundedness can be shown analogously to estimate *v)*.

ix) Since we need to estimate $\nabla \varphi_k \cdot \mathbf{v}_{k+1}$ in $W^1_{\frac{3}{2}}(\Omega)$, we need to study terms of the form $\partial_j \varphi_k \partial_i (\mathbf{v}_{k+1})_l$ and $\partial_i \partial_j \varphi_k (\mathbf{v}_{k+1})_i$. Due to the boundedness of $\partial_j \varphi_k$ in $L^6(\Omega)$ and the boundedness of $\partial_i (\mathbf{v}_{k+1})_l$ in $L^2(\Omega)$, its product is bounded in $L^{\frac{3}{2}}(\Omega)$. For the second kind of terms we note that $\partial_i \partial_j \varphi_k$ is bounded in $L^2(\Omega)$ and $(\mathbf{v}_{k+1})_i$ is bounded in $L^6(\Omega)$. Hence, this product is also bounded in $L^{\frac{3}{2}}(\Omega)$, which implies the boundedness of $\nabla \varphi_k \cdot \mathbf{v}_{k+1}$ in $W^1_{\frac{3}{2}}(\Omega)$.

x) First of all we note that due to $q_{k+1} \in H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the growth condition $|h(q_{k+1})| \le C(|q_{k+1}| + 1)$, cf. (3.17), it holds $h(q_{k+1}) \in L^6(\Omega)$. For the other term we can in general derive the estimate

$$|H(a,b)| \le C(|a|^2 + |b|^2 + 1) \tag{3.41}$$

for all $a, b \in \mathbb{R}$. This can be seen as follow:
Let $a \neq b$. W.l.o.g. we assume $a < b$. Then it holds

$$|H(a,b)| = \left| \frac{W(a) - W(b)}{a - b} \right| \le C(|a|^2 + |b|^2 + 1),$$

where we used

$$|W(a) - W(b)| = \left| \int_b^a W'(s) dx \right| = |(a-b)W'(\xi)| \le |a-b| C(|a|^2 + |b|^2 + 1)$$

for $\xi \in [a,b]$ and where we used the growth condition for W'. Dividing this estimate by $|a-b|$ yields the statement.
For the case $a = b$ estimate (3.41) is obvious due to the growth condition for W'.

As it holds $\varphi_k, \varphi_{k+1} \in H^1(\Omega) \hookrightarrow L^6(\Omega)$ we can conclude from (3.41)

$$||H(\varphi_{k+1}, \varphi_k)||_{L^3(\Omega)} \leq C \left(||\varphi_k||^2_{L^6(\Omega)} + ||\varphi_{k+1}||^2_{L^6(\Omega)} + 1 \right).$$

Finally, Hölder's inequality yields

$$||h(q_{k+1})H(\varphi_{k+1}, \varphi_k)||_{L^2(\Omega)} \leq C||h(q_{k+1})||_{L^6(\Omega)} \left(||\varphi_k||^2_{L^6(\Omega)} + ||\varphi_{k+1}||^2_{L^6(\Omega)} + 1 \right).$$

Using the previous estimate for $h(q_{k+1})$ in the L^6-norm yields the statement.

For all the other terms it is quite obvious that they are bounded in $L^2(\Omega)$, $W^1_{\frac{3}{2}}(\Omega)$ and in $L^{\frac{4}{3}}(\Omega)$, respectively. It still remains to prove the continuity of $\mathcal{F}_k : X \to \tilde{Y}$. Here we only need to study the nonlinear terms since for the linear terms continuity follows from boundedness. In particular, we need to study the terms $\rho(\varphi_{k+1})$, $h(q_{k+1})$, $f(q_{k+1})$, $W(\varphi_{k+1})$, $g(q_{k+1})$ and $H(\varphi_{k+1}, \varphi_k)$. We use Theorem 2.10 about Nemyckii operators and show that all conditions are satisfied. First of all, we note that the Carathéodory-Condition is satisfied for all terms since we assume all functions to be at least in $C^0(\mathbb{R})$. It remains to show the growth condition

$$|\hat{f}(x, \hat{\eta})| \leq |a(x)| + b|\hat{\eta}|^{\frac{p}{q}}$$

for a constant $b > 0$, $a \in L^q(\Omega)$ and suitable $1 \leq p, q < \infty$. Then Theorem 2.10 implies that $\hat{F} : L^p(\Omega) \to L^q(\Omega)$ defined by $(\hat{F}u)(x) := \hat{f}(x, u(x))$ is continuous.
Consequently, $f : L^6(\Omega) \to L^q(\Omega)$ is continuous for every $1 \leq q < \infty$ since f is a bounded function. Analogously, we can show that $\rho : L^6(\Omega) \to L^q(\Omega)$ is continuous for every $1 \leq q < \infty$. Due to the growth condition $|W(\hat{\eta})| \leq C(|\hat{\eta}|^3 + 1)$ we obtain that $W : L^6(\Omega) \to L^2(\Omega)$ is continuous. Furthermore, the growth conditions $|h(\hat{\eta})| \leq C(|\hat{\eta}| + 1)$ and $|g(\hat{\eta})| \leq C(|\hat{\eta}| + 1)$ imply that $h, g : L^6(\Omega) \to L^6(\Omega)$ are continuous.
From the growth condition (3.41) for H we get that $H(\cdot, \varphi_k) : L^6(\Omega) \to L^3(\Omega)$ is continuous. Hence the continuity for most terms of $\mathcal{F}_k : X \to \tilde{Y}$ can be shown with similar arguments as for the boundedness. We only want to study the continuity for the terms $\nabla(f(q_{k+1})W(\varphi_k)) \cdot \mathbf{v}_{k+1}$ and $\nabla g(q_{k+1}) \cdot \mathbf{v}_{k+1}$ in detail. To this end, we need to study terms of the form $f'(q_{k+1})\nabla q_{k+1} \cdot \mathbf{v}_{k+1}$ and $g'(q_{k+1})\nabla q_{k+1} \cdot \mathbf{v}_{k+1}$. Since f' and g' are bounded, we can conclude $f', g' : L^6(\Omega) \to L^q(\Omega)$ are continuous for every $1 \leq q < \infty$. This implies that the mappings $(q_{k+1} \mapsto \nabla(f(q_{k+1})W(\varphi_k)) \cdot \mathbf{v}_{k+1})$ and $(q_{k+1} \mapsto \nabla g(q_{k+1}) \cdot \mathbf{v}_{k+1})$ are continuous from $H^1(\Omega) \hookrightarrow L^6(\Omega)$ to $L^{\frac{3}{2}-s}(\Omega)$ for every $0 < s \leq \frac{1}{2}$. In particular, these mappings are continuous from $H^1(\Omega)$ to $L^{\frac{4}{3}}(\Omega)$.

Thus we have shown that $\mathcal{F}_k : X \to \tilde{Y}$ is a continuous and bounded operator. Since the embeddings $H^1(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega)$ and $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ are compact, cf.

Theorem 2.16, we obtain

$$L^{\frac{4}{3}}(\Omega) \cong (L^4(\Omega))' \hookrightarrow\hookrightarrow H_0^{-1}(\Omega),$$
$$L^2(\Omega) \cong (L^2(\Omega))' \hookrightarrow\hookrightarrow H_0^{-1}(\Omega).$$

Moreover, Theorem 2.16 yields

$$W^1_{\frac{3}{2}}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega).$$

Hence it holds $\tilde{Y} \hookrightarrow\hookrightarrow Y$ compactly and therefore we can conclude that

$$\mathcal{F}_k : X \to Y \qquad \text{is a compact operator.}$$

In the following we want to apply the Leray-Schauder principle analogously as in [ADG13]. We already noted that $\mathbf{w}_{k+1} \in X$ is a weak solution of (3.20) - (3.24) if and only if

$$\mathcal{L}_k(\mathbf{w}_{k+1}) - \mathcal{F}_k(\mathbf{w}_{k+1}) = 0 \qquad \text{in } Y.$$

This is equivalent to

$$\mathbf{g}_{k+1} - \mathcal{F}_k \circ \mathcal{L}_k^{-1}(\mathbf{g}_{k+1}) = 0 \qquad \text{in } Y \quad \text{ for } \mathbf{g}_{k+1} := \mathcal{L}_k(\mathbf{w}_{k+1}). \tag{3.42}$$

Since $\mathcal{F}_k$ is compact and $\mathcal{L}_k^{-1}$ is continuous for every $k \in \mathbb{N}$, we note that the composition $\mathcal{K}_k := \mathcal{F}_k \circ \mathcal{L}_k^{-1} : Y \to Y$ is also a compact operator. Moreover, we note that proving the existence of a weak solution for (3.20) - (3.24) is equivalent to proving the existence of a fixed-point of $\mathcal{K}_k$ because of (3.42), i.e., we look for $\mathbf{g}_{k+1} \in Y$ such that

$$\mathbf{g}_{k+1} - \mathcal{K}_k(\mathbf{g}_{k+1}) = 0 \quad \text{in } Y \quad \Leftrightarrow \quad \mathbf{g}_{k+1} = \mathcal{K}_k(\mathbf{g}_{k+1}) \quad \text{in } Y.$$

For the proof of the existence of such a fixed-point we want to apply the Leray-Schauder principle, cf. Theorem 2.2. In particular, we need to prove

$$\text{There exists } r_{k+1} > 0 \text{ such that if } \mathbf{g}_{k+1} \in Y \text{ solves } \mathbf{g}_{k+1} = \lambda \mathcal{K}_k \mathbf{g}_{k+1} \text{ with } 0 \le \lambda < 1, \\ \text{then it holds } ||\mathbf{g}_{k+1}||_Y \le r_{k+1}. \tag{3.43}$$

To this end, we consider $\mathbf{g}_{k+1} \in Y$ and $0 \le \lambda < 1$ such that $\mathbf{g}_{k+1} = \lambda \mathcal{K}_k \mathbf{g}_{k+1}$. As it holds $\mathbf{w}_{k+1} = \mathcal{L}_k^{-1}(\mathbf{g}_{k+1}) \in X$ we can conclude

$$\mathbf{g}_{k+1} = \lambda \mathcal{K}_k(\mathbf{g}_{k+1}) \text{ in } Y \quad \Leftrightarrow \quad \mathcal{L}_k(\mathbf{w}_{k+1}) - \lambda \mathcal{F}_k(\mathbf{w}_{k+1}) = 0 \text{ in } Y.$$

The last equation is equivalent to

$$
\begin{aligned}
&\lambda \int_\Omega \left(\frac{\rho_{k+1}\mathbf{v}_{k+1} - \rho_k \mathbf{v}_k}{h} + \operatorname{div}(\rho_k \mathbf{v}_{k+1} \otimes \mathbf{v}_{k+1}) \right) \cdot \boldsymbol{\psi}\, dx + \lambda \int_\Omega (\tilde{\mathbf{J}}_{k+1} \cdot \nabla)\mathbf{v}_{k+1} \cdot \boldsymbol{\psi}\, dx \\
&\quad + \lambda \int_\Omega \left(\operatorname{div}\tilde{\mathbf{J}}_{k+1} - \frac{\rho_{k+1} - \rho_k}{h} - \mathbf{v}_{k+1} \cdot \nabla \rho_k \right) \frac{\mathbf{v}_{k+1}}{2} \cdot \boldsymbol{\psi}\, dx \\
&\quad - \lambda \int_\Omega \left(\mu_{k+1} - \frac{h(q_{k+1})}{\varepsilon} W'(\varphi_k) \right) \nabla \varphi_k \cdot \boldsymbol{\psi}\, dx \\
&= - \int_\Omega 2\eta(\varphi_k) D\mathbf{v}_{k+1} : D\boldsymbol{\psi}\, dx - \delta \int_\Omega \Delta \mathbf{v}_{k+1} \Delta \boldsymbol{\psi}\, dx
\end{aligned} \tag{3.44}
$$

for all $\boldsymbol{\psi} \in H^2(\Omega) \cap H_0^1(\Omega)^d \cap L^2_\sigma(\Omega)$ and

$$
\begin{aligned}
\lambda \int_\Omega \left(\frac{1}{\varepsilon} f(q_{k+1}) W(\varphi_k) + g(q_{k+1}) \right) \mathbf{v}_{k+1} \cdot \nabla \phi\, dx &= \int_\Omega m(\varphi_k, q_k) \nabla q_{k+1} \cdot \nabla \phi\, dx \\
&\quad + (1-\lambda) \int_\Omega \int_\Omega q_{k+1}\, dy \phi\, dx \\
&\quad + \frac{\lambda}{h} \int_\Omega \left(\frac{f(q_{k+1}) W(\varphi_{k+1})}{\varepsilon} + g(q_{k+1}) - \frac{f(q_k) W(\varphi_k)}{\varepsilon} - g(q_k) \right) \phi\, dx,
\end{aligned} \tag{3.45}
$$

$$
\begin{aligned}
0 = \int_\Omega \tilde{m}(\varphi_k) \nabla \mu_{k+1} \cdot \nabla \phi\, dx + \lambda \int_\Omega \frac{\varphi_{k+1} - \varphi_k}{h} \phi\, dx + \lambda \int_\Omega (\nabla \varphi_k \cdot \mathbf{v}_{k+1}) \phi\, dx \\
+ (1-\lambda) \int_\Omega \int_\Omega \mu_{k+1}\, dy \phi\, dx,
\end{aligned} \tag{3.46}
$$

$$
\begin{aligned}
\lambda \int_\Omega \mu_{k+1} \phi\, dx = \int_\Omega \varepsilon \nabla \varphi_{k+1} \cdot \nabla \phi\, dx + \lambda \int_\Omega h(q_{k+1}) \frac{1}{\varepsilon} H(\varphi_{k+1}, \varphi_k) \phi\, dx \\
+ \lambda \delta \int_\Omega \frac{\varphi_{k+1} - \varphi_k}{h} \phi\, dx + (1-\lambda) \int_\Omega \int_\Omega \varphi_{k+1}\, dy \phi\, dx
\end{aligned} \tag{3.47}
$$

for all $\phi \in H^1(\Omega)$.

Now we need to estimate $\mathbf{w}_{k+1} = \mathcal{L}_k^{-1}(\mathbf{g}_{k+1}) = (\mathbf{v}_{k+1}, \varphi_{k+1}, \mu_{k+1}, q_{k+1})$ in X. Then the estimate for $\mathbf{g}_{k+1}$ in the norm of Y follows from the compactness, in particular from the boundedness, of $\mathcal{F}_k : X \to Y$. To get this estimate of $\mathbf{g}_{k+1} = \mathcal{L}_k(\mathbf{w}_{k+1})$ in Y, we use that $\mathbf{g}_{k+1} - \lambda \mathcal{F}_k \circ \mathcal{L}_k^{-1}(\mathbf{g}_{k+1}) = 0$ implies $\mathbf{g}_{k+1} = \lambda \mathcal{F}_k(\mathbf{w}_{k+1})$ and the

fact that $\mathcal{F}_k : X \to Y$ maps bounded sets into bounded sets, which holds due to the estimates above for $\mathcal{F}_k$.
Thus (3.43) is fulfilled and the Leray-Schauder principle yields the existence of $\mathbf{g}_{k+1} \in Y$ such that $\mathbf{g}_{k+1} - \mathcal{K}_k(\mathbf{g}_{k+1}) = 0$, which is equivalent to $\mathcal{L}_k(\mathbf{w}_{k+1}) - \mathcal{F}_k(\mathbf{w}_{k+1}) = 0$, where $\mathbf{w}_{k+1} = \mathcal{L}_k^{-1}(\mathbf{g}_{k+1})$.
To this end, we test equation (3.44) with $\mathbf{v}_{k+1}$, (3.45) with q_{k+1}, (3.46) with μ_{k+1} and (3.47) with $\frac{\varphi_{k+1}-\varphi_k}{h}$. Then we obtain with similar calculations as before

$$
\begin{aligned}
0 \geq {} & \frac{\lambda}{h} \int_\Omega \frac{\rho_{k+1}|\mathbf{v}_{k+1}|^2}{2} - \frac{\rho_k|\mathbf{v}_k|^2}{2} + \frac{\rho_k|\mathbf{v}_{k+1} - \mathbf{v}_k|^2}{2} \, dx + \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2 \, dx \\
& + \int_\Omega m(\varphi_k, q_k)|\nabla q_{k+1}|^2 \, dx + \lambda \int_\Omega \frac{1}{h}(G(q_{k+1}) - G(q_k)) \, dx + (1-\lambda) \left(\int_\Omega q_{k+1} \, dx \right)^2 \\
& + \int_\Omega \tilde{m}(\varphi_k)|\nabla \mu_{k+1}|^2 \, dx + \lambda \int_\Omega \frac{W(\varphi_{k+1})d(q_{k+1})}{\varepsilon h} \, dx - \lambda \int_\Omega \frac{W(\varphi_k)d(q_k)}{\varepsilon h} \, dx \\
& + (1-\lambda) \left(\int_\Omega \mu_{k+1} \, dx \right)^2 + \int_\Omega \frac{\varepsilon}{h} \left(\frac{|\nabla \varphi_{k+1}|^2}{2} - \frac{|\nabla \varphi_k|^2}{2} \right) dx \\
& + (1-\lambda) \left(\int_\Omega \varphi_{k+1} \, dx \right)^2 + \lambda\delta \int_\Omega \frac{|\varphi_{k+1} - \varphi_k|^2}{h^2} \, dx + \delta \int_\Omega |\Delta \mathbf{v}_{k+1}|^2 \, dx.
\end{aligned}
$$

Thus we get the estimate

$$
\begin{aligned}
& h \int_\Omega m(\varphi_k, q_k)|\nabla q_{k+1}|^2 \, dx + \lambda \int_\Omega G(q_{k+1}) \, dx + h(1-\lambda) \left(\int_\Omega q_{k+1} \, dx \right)^2 \\
& + h \int_\Omega \tilde{m}(\varphi_k)|\nabla \mu_{k+1}|^2 \, dx + h(1-\lambda) \left(\int_\Omega \mu_{k+1} \, dx \right)^2 + \int_\Omega \varepsilon \frac{|\nabla \varphi_{k+1}|^2}{2} \, dx \\
& + h(1-\lambda) \left(\int_\Omega \varphi_{k+1} \, dx \right)^2 + \lambda \int_\Omega \frac{W(\varphi_{k+1})d(q_{k+1})}{\varepsilon} \, dx \\
& + h \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2 \, dx + \delta h \int_\Omega |\Delta \mathbf{v}_{k+1}|^2 \, dx \\
\leq {} & \int_\Omega G(q_k) \, dx + \int_\Omega \varepsilon \frac{|\nabla \varphi_k|^2}{2} \, dx + \frac{1}{\varepsilon} \int_\Omega W(\varphi_k)d(q_k) \, dx + \int_\Omega \frac{\rho_k|\mathbf{v}_k|^2}{2} \, dx,
\end{aligned}
$$

where we estimated every λ on the right-hand side by 1 and omitted the non-negative terms

$$\lambda \int_\Omega \frac{\rho_{k+1}|\mathbf{v}_{k+1}|^2}{2} dx, \ \lambda \int_\Omega \frac{\rho_k |\mathbf{v}_{k+1} - \mathbf{v}_k|^2}{2} dx \text{ and } \lambda\delta \int_\Omega \frac{|\varphi_{k+1} - \varphi_k|^2}{h} dx.$$

In this estimate we can conclude that the right-hand side is bounded. In detail: Due to the growth condition $|G(q_k)| \leq C(|q_k|^2 + 1)$, cf. Assumption 3.3, and $q_k \in L^2(\Omega)$, the term $\int_\Omega G(q_k)\,dx$ is finite. Since φ_k is in $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $\mathbf{v}_k$ is in $H^1(\Omega)$, the second and fourth integral are obviously bounded. For the term $\frac{1}{\varepsilon}\int_\Omega W(\varphi_k)d(q_k)\,dx$ we use the growth condition $|W(\varphi_k)| \leq C\,(|\varphi_k|^3 + 1)$, cf. Assumption 3.5, and the boundedness of the function d, cf. Assumption 3.4, to see that it is also bounded. Thus we can summarize the previous estimate to

$$\begin{aligned}
&h \int_\Omega m(\varphi_k, q_k)|\nabla q_{k+1}|^2\,dx + \lambda \int_\Omega G(q_{k+1})\,dx + h(1-\lambda)\left(\int_\Omega q_{k+1}\,dx\right)^2 \\
&+ h \int_\Omega \tilde{m}(\varphi_k)|\nabla \mu_{k+1}|^2\,dx + h(1-\lambda)\left(\int_\Omega \mu_{k+1}\,dx\right)^2 + \int_\Omega \varepsilon \frac{|\nabla \varphi_{k+1}|^2}{2}\,dx \\
&+ h(1-\lambda)\left(\int_\Omega \varphi_{k+1}\,dx\right)^2 + \lambda \int_\Omega \frac{W(\varphi_{k+1})d(q_{k+1})}{\varepsilon}\,dx \\
&+ h \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2\,dx + \delta h \int_\Omega |\Delta \mathbf{v}_{k+1}|^2\,dx \leq C_k
\end{aligned} \tag{3.48}$$

for a constant $C_k > 0$ which only depends on $\mathbf{v}_k, \varphi_k$ and q_k from the previous iteration.

To obtain an estimate of the H^1-norm of q_{k+1}, we distinguish two cases. If $\lambda \in [\frac{1}{2}, 1]$, then we get from (3.48)

$$\int_\Omega (|q_{k+1}| - 1) \leq C \int_\Omega G(q_{k+1})\,dx \leq \frac{C}{\lambda} C_k,$$

where we used $G(q) \geq c(|q|^2 - 1)$ for all $q \in \mathbb{R}$ and a constant $c > 0$, cf. (3.15). But this estimate yields that the mean value of q_{k+1} is bounded in the case $\lambda \in [\frac{1}{2}, 1]$. For $\lambda \in [0, \frac{1}{2})$ we can use (3.48) to estimate the mean value of q_{k+1} by $|\int_\Omega q_{k+1}\,dx| \leq C_k$.

Thus we get the estimate

$$
\begin{aligned}
&||q_{k+1}||^2_{H^1(\Omega)} + h\int_\Omega \tilde{m}(\varphi_k)|\nabla\mu_{k+1}|^2\,dx + h(1-\lambda)\left(\int_\Omega \mu_{k+1}\,dx\right)^2 + \int_\Omega \varepsilon\frac{|\nabla\varphi_{k+1}|^2}{2}\,dx \\
&+ (1-\lambda)\left(\int_\Omega \varphi_{k+1}\,dx\right)^2 + \lambda\int_\Omega \frac{W(\varphi_{k+1})d(q_{k+1})}{\varepsilon}\,dx \\
&+ h\int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2\,dx + \delta h\int_\Omega |\Delta\mathbf{v}_{k+1}|^2\,dx \le C_k, \qquad (3.49)
\end{aligned}
$$

where we used the Poincaré inequality with mean value, cf. Theorem 2.7. To get an estimate of the H^1-norm of φ_{k+1}, we need to estimate its mean value since (3.49) already yields an L^2-estimate of $\nabla\varphi_{k+1}$. For $\lambda \in [0, \frac{1}{2})$, we use (3.49) directly to get $|\int_\Omega \varphi_{k+1}\,dx| \le C_k$. In the case of $\lambda \in [\frac{1}{2}, 1]$ we use

$$
C\frac{\lambda}{\varepsilon}\int_\Omega (|\varphi_{k+1}| - 1)\,dx \le \lambda\int_\Omega \frac{W(\varphi_{k+1})d(q_{k+1})}{\varepsilon}\,dx, \qquad (3.50)
$$

where we used Assumption 3.5 and the lower bound of d and W. This estimate yields the boundedness of the mean value of φ_{k+1} for $\lambda \in [\frac{1}{2}, 1]$. Altogether it follows from estimate (3.49)

$$
\begin{aligned}
&||q_{k+1}||^2_{H^1(\Omega)} + ||\varphi_{k+1}||^2_{H^1(\Omega)} + h\int_\Omega \tilde{m}(\varphi_k)|\nabla\mu_{k+1}|^2\,dx + h(1-\lambda)\left(\int_\Omega \mu_{k+1}\,dx\right)^2 \\
&+ h\int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2\,dx + \delta h\int_\Omega |\Delta\mathbf{v}_{k+1}|^2\,dx \le C_k. \qquad (3.51)
\end{aligned}
$$

To estimate the chemical potential μ_{k+1} in the H^2-norm, we need to estimate its mean value again. Its mean value has to be bounded by a constant independent of λ. For $\lambda \in [0, \frac{1}{2})$, we use (3.51) directly to get $|\int_\Omega \mu_{k+1}\,dx| \le C_k$. For $\lambda \in [\frac{1}{2}, 1)$, we test (3.47) with constant 1 and obtain

$$
\begin{aligned}
\lambda\int_\Omega \mu_{k+1}\,dx &= \lambda\int_\Omega \frac{1}{\varepsilon}h(q_{k+1})H(\varphi_{k+1},\varphi_k)\,dx + (1-\lambda)|\Omega|\int_\Omega \varphi_{k+1}\,dx \\
&+ \lambda\delta\int_\Omega \frac{\varphi_{k+1}-\varphi_k}{h}\,dx.
\end{aligned}
$$

Since we already know $||q_{k+1}||_{H^1(\Omega)}$, $||\varphi_{k+1}||_{H^1(\Omega)} \leq C_k$ from estimate (3.51), we can conclude

$$\lambda \int_\Omega \mu_{k+1} dx \leq C_k$$

for a constant $C_k > 0$ independent of λ, where we used (3.41) to estimate the term $H(\varphi_{k+1}, \varphi_k)$. Moreover, we used $h(q) \leq C(|q|+1)$ for all $q \in \mathbb{R}$ and a constant $C > 0$, cf. (3.17), and estimated $(1-\lambda)$ and λ by 1 on the right-hand side. This means that we can improve (3.51) to

$$\begin{aligned} &||q_{k+1}||^2_{H^1(\Omega)} + ||\varphi_{k+1}||^2_{H^1(\Omega)} + ||\mu_{k+1}||^2_{H^1(\Omega)} + h \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2 dx \\ &+ \delta h \int_\Omega |\Delta \mathbf{v}_{k+1}|^2 dx \leq C_k. \end{aligned} \tag{3.52}$$

Now we want to derive an estimate for the chemical potentail μ_{k+1} in the H^2-norm analogously as in (3.40). From equation (3.46) it follows

$$\operatorname{div}(\tilde{m}(\varphi_k)\mu_{k+1}) = \lambda \tilde{g}_2 + (1-\lambda) \int_\Omega \mu_{k+1} dx,$$

where $\tilde{g}_2 = \frac{\varphi_{k+1}-\varphi_k}{h} + \nabla\varphi_k \cdot \mathbf{v}_{k+1}$ is bounded in $L^2(\Omega)$. Analogously as in the derivation of (3.40) we can derive

$$\Delta\mu_{k+1} = (\tilde{m}(\varphi_k))^{-1} \left(-\nabla(\tilde{m}(\varphi_k)) \cdot \nabla\mu_{k+1} + (1-\lambda) \int_\Omega \mu_{k+1} dx + \lambda\tilde{g}_2 \right) \quad \text{in } \Omega.$$

With the same statements as in the derivation of (3.40) we can conclude that the right-hand side is bounded in $L^{\frac{3}{2}}(\Omega)$. With the same bootstrapping method as before it follows $\mu_{k+1} \in H^2_n(\Omega)$ together with the estimate

$$\begin{aligned} ||\mu_{k+1}||_{H^2(\Omega)} &\leq C\Big(||(\tilde{m}(\varphi_k))^{-1}(-\nabla(\tilde{m}(\varphi_k)) \cdot \nabla\mu_{k+1} + (1-\lambda) \int_\Omega \mu_{k+1} dx + \lambda\tilde{g}_2)||_{L^2(\Omega)} \\ &\quad + ||\mu_{k+1}||_{H^1(\Omega)} \Big) \\ &\leq C_k \left(||\mu_{k+1}||_{H^1(\Omega)} + (1-\lambda)||\mu_{k+1}||_{H^1(\Omega)} + \lambda||\tilde{g}_2||_{L^2(\Omega)} \right) \\ &\leq C_k \left(||\mu_{k+1}||_{H^1(\Omega)} + ||\tilde{g}_2||_{L^2(\Omega)} \right), \end{aligned}$$

where $C_k > 0$ does not depend on λ. Hence, we also get an estimate of the H^2-norm of the chemical potential μ_{k+1} and can conclude

$$||q_{k+1}||^2_{H^1(\Omega)} + ||\varphi_{k+1}||^2_{H^1(\Omega)} + ||\mu_{k+1}||^2_{H^2(\Omega)} + h\int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2\,dx$$
$$+ \delta h \int_\Omega |\Delta \mathbf{v}_{k+1}|^2\,dx \leq C_k.$$

Now we use Korn's inequality, cf. Theorem 2.14, for $\mathbf{v}_{k+1} \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$ to get

$$||\mathbf{v}_{k+1}||^2_{H^2(\Omega)} + ||q_{k+1}||^2_{H^1(\Omega)} + ||\mu_{k+1}||^2_{H^2(\Omega)} + ||\varphi_{k+1}||^2_{H^1(\Omega)} \leq C_k. \tag{3.53}$$

This means that $\mathbf{w}_{k+1} = \mathcal{L}_k^{-1}(\mathbf{g}_{k+1}) = (\mathbf{v}_{k+1}, \varphi_{k+1}, \mu_{k+1}, q_{k+1})$ is bounded in the norm of X. Thus (3.43) is fulfilled and the Leray-Schauder principle yields the existence of $\mathbf{g}_{k+1} \in Y$ such that $\mathbf{g}_{k+1} - \mathcal{K}_k(\mathbf{g}_{k+1}) = 0$, which is equivalent to $\mathcal{L}_k(\mathbf{w}_{k+1}) - \mathcal{F}_k(\mathbf{w}_{k+1}) = 0$, where $\mathbf{w}_{k+1} = \mathcal{L}_k^{-1}(\mathbf{g}_{k+1})$.

Finally, we need to show higher regularity for φ_{k+1}. As it holds $\mathcal{L}_k(\mathbf{w}_{k+1}) = \mathcal{F}_k(\mathbf{w}_{k+1})$ with $\mathbf{w}_{k+1} = (\mathbf{v}_{k+1}, q_{k+1}, \mu_{k+1}, \varphi_{k+1})$, we can conclude

$$\varepsilon \Delta_N \varphi_{k+1} = -\mu_{k+1} + h(q_{k+1})\frac{1}{\varepsilon}H(\varphi_{k+1}, \varphi_k) + \delta\frac{\varphi_{k+1} - \varphi_k}{h} \qquad \text{in } H_0^{-1}(\Omega),$$

where the right-hand side is bounded in the L^2-norm as we have already shown. Thus elliptic regularity theory yields $\varphi_{k+1} \in H^2_n(\Omega)$, cf. [Lun95, Theorem 3.1.2 and Theorem 3.1.3].
Hence there exists a weak solution for the time-discrete problem (3.20) - (3.24) in the sense of Definition 3.6, which fulfills the discrete energy estimate (3.31). ☐

3.3 Existence of Weak Solutions for the Surfactant Model in the Case $\delta > 0$

In the previous section we proved the existence of weak solutions for the time-discrete problem (3.20) - (3.24). In this section we prove the existence of weak solutions for the system

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v} \otimes (\rho\mathbf{v} + \tilde{\mathbf{J}})) + \nabla p - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) - \frac{R\mathbf{v}}{2} + \delta\Delta^2\mathbf{v}$$
$$= \operatorname{div}(-\varepsilon\nabla\varphi \otimes \nabla\varphi) \qquad \text{in } Q_T, \tag{3.54}$$
$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } Q_T, \tag{3.55}$$
$$\partial_t^\bullet\left(\frac{1}{\varepsilon}f(q)W(\varphi) + g(q)\right) = \operatorname{div}\left(m(\varphi, q)\nabla q\right) \qquad \text{in } Q_T, \tag{3.56}$$
$$\partial_t^\bullet\varphi = \operatorname{div}(\tilde{m}(\varphi)\nabla\mu) \qquad \text{in } Q_T, \tag{3.57}$$
$$-\varepsilon\Delta\varphi + h(q)\frac{1}{\varepsilon}W'(\varphi) + \delta\partial_t\varphi = \mu \qquad \text{in } Q_T, \tag{3.58}$$

together with the initial and boundary conditions (1.6) - (1.7). Moreover, we show that the weak solutions of (3.54) - (3.58) satisfy an energy estimate. Note that in contrast to the equations (1.1) - (1.7) we have the additional terms $\delta\Delta^2\mathbf{v}$ in equation (3.54) and $\delta\partial_t\varphi$ in equation (3.58). In the last section of this chapter we will pass to the limit $\delta \to 0$ and show that the weak solutions for the model (3.54) - (3.58) together with the initial and boundary conditions converge to a weak solution of (1.1) - (1.7) and satisfy the energy estimate (3.14) for all $s \leq t < T$ and almost all $0 \leq s < T$ including $s = 0$.
But first of all we define what we mean with a weak solution for the equations (3.54) - (3.58) together with initial and boundary conditions (1.6) - (1.7).

Definition 3.8. *(Weak solution in the case $\delta > 0$)*
Let $T \in (0, \infty)$, $\delta > 0$ and $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $\varphi_0 \in H^2(\Omega)$, $q_0 \in L^2(\Omega)$ be given. We call $(\mathbf{v}, \varphi, \mu, q)$ with the properties

$$\begin{aligned}
&\mathbf{v} \in L^2(0,T; H^2(\Omega)^d \cap H^1_0(\Omega)^d) \cap L^\infty(0,T; L^2_\sigma(\Omega)),\\
&\varphi \in L^\infty(0,T; H^1(\Omega)) \cap W^1_2(0,T; L^2(\Omega)),\\
&\mu \in L^2(0,T; H^1(\Omega)),\\
&q \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))
\end{aligned}$$

a weak solution of (3.54) - (3.58) together with initial and boundary conditions (1.6) - (1.7), if the following equations are satisfied.

$$\begin{aligned}
&-\int_0^T\int_\Omega \rho\mathbf{v}\cdot\partial_t\boldsymbol{\psi}\,dx\,dt + \int_0^T\int_\Omega (\rho\mathbf{v}\otimes\mathbf{v}) : \nabla\boldsymbol{\psi}\,dx\,dt + \int_0^T\int_\Omega 2\eta(\varphi)D\mathbf{v} : D\boldsymbol{\psi}\,dx\,dt\\
&-\int_0^T\int_\Omega (\tilde{\mathbf{J}}\otimes\mathbf{v}) : \nabla\boldsymbol{\psi}\,dx\,dt - \left\langle \frac{R\mathbf{v}}{2}, \boldsymbol{\psi}\right\rangle + \delta\int_0^T\int_\Omega \Delta\mathbf{v}\cdot\Delta\boldsymbol{\psi}\,dx\,dt\\
&= \int_0^T\int_\Omega \left(\mu - \frac{h(q)}{\varepsilon}W'(\varphi)\right)\nabla\varphi\cdot\boldsymbol{\psi}\,dx\,dt
\end{aligned} \tag{3.59}$$

for all $\boldsymbol{\psi} \in C^\infty_0(0,T; H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega))$ and

$$\begin{aligned}
\int_0^T\int_\Omega m(\varphi,q)\nabla q\cdot\nabla\phi\,dx\,dt &= \int_0^T\int_\Omega (f(q)W(\varphi) + g(q))\,\partial_t\phi\,dx\,dt\\
&+ \int_0^T \left(\frac{1}{\varepsilon}f(q)W(\varphi) + g(q)\right)\mathbf{v}\cdot\nabla\phi\,dx\,dt,
\end{aligned} \tag{3.60}$$

$$\int_0^T\int_\Omega \tilde{m}(\varphi)\nabla\mu\cdot\nabla\phi\,dx\,dt = \int_0^T\int_\Omega \varphi\partial_t\phi\,dx\,dt - \int_0^T\int_\Omega (\nabla\varphi\cdot\mathbf{v})\phi\,dx\,dt, \tag{3.61}$$

$$\int_0^T \int_\Omega \mu \phi \, dx\, dt = \int_0^T \int_\Omega \varepsilon \nabla \varphi \cdot \nabla \phi \, dx\, dt + \int_0^T \int_\Omega \frac{1}{\varepsilon} h(q) W'(\varphi) \phi \, dx\, dt + \delta \int_0^T \int_\Omega \partial_t \varphi \phi \, dx\, dt \tag{3.62}$$

for all $\phi \in C_0^\infty(0,T; C^1(\overline{\Omega}))$. Moreover, the energy inequality

$$\int_s^t \int_\Omega (m(\varphi,q)|\nabla q|^2 + \tilde{m}(\varphi)|\nabla \mu|^2 + 2\eta(\varphi)|D\mathbf{v}|^2 + \delta|\Delta \mathbf{v}|^2 + \delta|\partial_t \varphi|^2)\, dx\, d\tau + E_{tot}(\mathbf{v}(t), \varphi(t), \nabla\varphi(t), q(t)) \leq E_{tot}(\mathbf{v}(s), \varphi(s), \nabla\varphi(s), q(s)) \tag{3.63}$$

has to hold for all $t \in [s,T)$ and almost all $s \in [0,T)$ including $s = 0$. In (3.59) we define for $\boldsymbol{\psi} \in C_0^\infty(0,T; H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$

$$\left\langle \frac{R\mathbf{v}}{2}, \boldsymbol{\psi} \right\rangle := \int_0^T \int_\Omega \partial_t \rho(\varphi) \frac{\mathbf{v}}{2} \cdot \boldsymbol{\psi} \, dx\, dt - \frac{1}{2} \int_0^T \int_\Omega \left(\rho(\varphi)\mathbf{v} + \tilde{\mathbf{J}}\right) \cdot \nabla(\mathbf{v} \cdot \boldsymbol{\psi})\, dx\, dt. \tag{3.64}$$

In this section we construct solutions for the time-dependent problem via interpolants which depend on the time-step h. Then we pass to the limit $h \to 0$ resp. $N \to \infty$ and show convergence in appropriate Banach space-valued Sobolev spaces. In the end we will obtain the following result about the existence of weak solutions in the case $\delta > 0$.

Theorem 3.9. *(Existence of weak solutions for $\delta > 0$)*
Let the assumptions from Section 3.1 hold and $\mathbf{v}_0 \in L_\sigma^2(\Omega)$, $\varphi_0 \in H_n^2(\Omega)$ and $q_0 \in L^2(\Omega)$ be given. Then there exists a weak solution $(\mathbf{v}, \varphi, \mu, q)$ in the sense of Definition 3.8. Moreover, it holds $\varphi \in L^2(0,T; H^2(\Omega))$.

Proof. We start with fixed $N \in \mathbb{N}$ and set $h = \frac{1}{N}$. Moreover, we have the initial values

$$(\mathbf{v}_0, \varphi_0, q_0) \in L_\sigma^2(\Omega) \times H^2(\Omega) \times L^2(\Omega).$$

Then Theorem 3.7 iteratively yields the existence of weak solutions

$$(\mathbf{v}_{k+1}, q_{k+1}, \mu_{k+1}, \varphi_{k+1}) \in \left(H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega)\right) \times H^1(\Omega) \times H_n^2(\Omega) \times H_n^2(\Omega)$$

for the time-discrete problem (3.20) - (3.24).

For the existence proof we use the same notation as in [ADG13], i.e., we define so-called interpolant functions $f^N(t)$ on $[-h, \infty)$ by $f^N(t) = f_k$ for $t \in [(k-1)h, kh)$,

where $k \in \mathbb{N}_0$ and $f_k \in \{\mathbf{v}_k, \varphi_k, q_k\}$, respectively $\mu^N(t)$ on $[0,\infty)$ by $\mu^N(t) = \mu_k$ for $t \in [(k-1)h, kh)$, where $k \in \mathbb{N}$. Note that μ^N is defined on $[0,\infty)$ while $\mathbf{v}^N$, φ^N and q^N are defined on $[-h,\infty)$ as we have no initial value μ_0.
With these definitions it holds $f^N((k-1)h) = f_k$, $f^N(kh) = f_{k+1}$ and $f^N(t) = f_{k+1}$ for $t \in [kh,(k+1)h)$ for $f^N \in \{\mathbf{v}^N, \varphi^N, q^N\}$, $k \in \mathbb{N}_0$, and $\mu^N((k-1)h) = \mu_k$, $\mu^N(kh) = \mu_{k+1}$ for $k \in \mathbb{N}$. Furthermore, we use the abbreviations

$$\begin{aligned}
(\Delta_h^+ f)(t) &:= f(t+h) - f(t), & (\Delta_h^- f)(t) &:= f(t) - f(t-h),\\
\partial_{t,h}^+ f(t) &:= \frac{1}{h}(\Delta_h^+ f)(t), & \partial_{t,h}^- f(t) &:= \frac{1}{h}(\Delta_h^- f)(t),\\
f_h(t) &:= (\tau_h^* f)(t) = f(t-h), & f_{h+}(t) &:= f(t+h)
\end{aligned}$$

and set

$$\begin{aligned}
\rho^N &:= \rho(\varphi^N), \qquad \rho_h^N := \rho(\varphi_h^N),\\
\tilde{\mathbf{J}}^N &:= -\rho'(\varphi_h^N)\tilde{m}(\varphi_h^N)\nabla\mu^N,\\
R^N &:= \partial_{t,h}^-\rho^N + \operatorname{div}\left(\rho_h^N\mathbf{v}^N + \tilde{\mathbf{J}}^N\right).
\end{aligned}$$

Now we need to determine which equations are solved by the interpolant functions $\mathbf{v}^N$, q^N, μ^N, φ^N and in which function spaces they are bounded. To this end, we choose an arbitrary $\boldsymbol{\psi} \in C_0^\infty((0,\infty); H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$ and set $\tilde{\boldsymbol{\psi}}_{\boldsymbol{k}} := \int\limits_{kh}^{(k+1)h} \boldsymbol{\psi}\, dt$ as test function in (3.26). Then we sum over $k \in \mathbb{N}_0$. Exemplarily we do this for one simple term. The other terms can be derived in the same way. It holds

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left(\frac{\rho_{k+1}\mathbf{v}_{k+1} - \rho_k\mathbf{v}_k}{h}, \tilde{\boldsymbol{\psi}}_{\boldsymbol{k}}\right)_{L^2(\Omega)} = \sum_{k=0}^{\infty}\int\limits_{\Omega}\left(\frac{\rho_{k+1}\mathbf{v}_{k+1} - \rho_k\mathbf{v}_k}{h}\cdot\int\limits_{kh}^{(k+1)h}\boldsymbol{\psi}\, dt\right) dx\\
&= \sum_{k=0}^{\infty}\int\limits_{\Omega}\int\limits_{kh}^{(k+1)h}\frac{\rho_{k+1}\mathbf{v}_{k+1} - \rho_k\mathbf{v}_k}{h}\cdot\boldsymbol{\psi}\, dt\, dx\\
&= \sum_{k=0}^{\infty}\int\limits_{\Omega}\int\limits_{kh}^{(k+1)h}\frac{\rho^N(t)\mathbf{v}^N(t) - \rho^N(t-h)\mathbf{v}^N(t-h)}{h}\cdot\boldsymbol{\psi}\, dt\, dx\\
&= \sum_{k=0}^{\infty}\int\limits_{\Omega}\int\limits_{kh}^{(k+1)h}\partial_{t,h}^-(\rho^N\mathbf{v}^N)\cdot\boldsymbol{\psi}\, dt\, dx = \int\limits_0^\infty\int\limits_{\Omega}\partial_{t,h}^-(\rho^N\mathbf{v}^N)\cdot\boldsymbol{\psi}\, dx\, dt.
\end{aligned}$$

Doing this analogously for all the other terms in (3.26) and using the previous abbreviations yields

$$\int_0^\infty \int_\Omega \partial_{t,h}^-(\rho^N \mathbf{v}^N) \cdot \boldsymbol{\psi}\, dx\, dt + \int_0^\infty \int_\Omega (\rho_h^N \mathbf{v}^N \otimes \mathbf{v}^N) : \nabla \boldsymbol{\psi}\, dx\, dt$$
$$+ \int_0^\infty \int_\Omega 2\eta(\varphi_h^N) D\mathbf{v}^N : D\boldsymbol{\psi}\, dx\, dt - \int_0^\infty \int_\Omega \left(\tilde{\mathbf{J}}^N \otimes \mathbf{v}^N \right) : \nabla \boldsymbol{\psi}\, dx\, dt - \left\langle \frac{R^N \mathbf{v}^N}{2}, \boldsymbol{\psi} \right\rangle$$
$$+ \delta \int_0^\infty \int_\Omega \Delta \mathbf{v} \Delta \boldsymbol{\psi}\, dx\, dt = \int_0^\infty \int_\Omega \left(\mu^N - \frac{h(q^N)}{\varepsilon} W'(\varphi_h^N) \right) \nabla \varphi_h^N \cdot \boldsymbol{\psi}\, dx\, dt \tag{3.65}$$

for all $\boldsymbol{\psi} \in C_0^\infty((0,\infty); H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$, where $\left\langle \frac{R^N \mathbf{v}^N}{2}, \boldsymbol{\psi} \right\rangle$ is defined analogously as in (3.64), i.e.,

$$\left\langle \frac{R^N \mathbf{v}^N}{2}, \boldsymbol{\psi} \right\rangle := \frac{1}{2} \int_0^\infty \int_\Omega \frac{\rho^N - \rho_h^N}{h} \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt$$
$$- \frac{1}{2} \int_0^\infty \int_\Omega \left(\rho_h^N \mathbf{v}^N + \tilde{\mathbf{J}}^N \right) \cdot \nabla(\mathbf{v}^N \cdot \boldsymbol{\psi})\, dx\, dt.$$

Now let $\phi \in C_0^\infty((0,\infty); C^1(\overline{\Omega}))$ be arbitrary. We set $\tilde{\phi} := \int_{kh}^{(k+1)h} \phi\, dt$ as test function in (3.27) - (3.29) and then sum over $k \in \mathbb{N}_0$. Using the same techniques as before we get

$$- \int_0^\infty \int_\Omega m(\varphi_h^N, q_h^N) \nabla q^N \cdot \nabla \phi\, dx\, dt = \int_0^\infty \int_\Omega \partial_{t,h}^- \left(f(q^N) W(\varphi^N) + g(q^N) \right) \phi\, dx\, dt,$$
$$- \int_0^\infty \int_\Omega \left(\frac{1}{\varepsilon} f(q^N) W(\varphi_h^N) + g(q^N) \right) \mathbf{v}^N \cdot \nabla \phi\, dx\, dt, \tag{3.66}$$

$$- \int_0^\infty \int_\Omega \tilde{m}(\varphi_h^N) \nabla \mu^N \cdot \nabla \phi\, dx\, dt = \int_0^\infty \int_\Omega \partial_{t,h}^- \varphi^N \phi\, dx\, dt + \int_0^\infty \int_\Omega \nabla \varphi_h^N \cdot \mathbf{v}^N \phi\, dx\, dt, \tag{3.67}$$

$$\int_0^\infty \int_\Omega \mu^N \phi\, dx\, dt = \int_0^\infty \int_\Omega \varepsilon \nabla \varphi^N \cdot \nabla \phi\, dx\, dt + \int_0^\infty \int_\Omega h(q^N) \frac{1}{\varepsilon} H(\varphi^N, \varphi_h^N) \phi\, dx\, dt$$
$$+ \delta \int_0^\infty \int_\Omega \partial_{t,h}^- \varphi^N \phi\, dx\, dt \tag{3.68}$$

for all $\phi \in C_0^\infty((0,\infty); C^1(\overline{\Omega}))$. Remember that by definition it holds

$$H(\varphi^N(t), \varphi^N(t-h)) = \begin{cases} W'(\varphi^N(t-h)) & \text{if } \varphi^N(t) = \varphi^N(t-h), \\ \frac{W(\varphi^N(t)) - W(\varphi^N(t-h))}{\varphi^N(t) - \varphi^N(t-h)} & \text{if } \varphi^N(t) \neq \varphi^N(t-h) \end{cases}$$

and that we have already proven the estimate

$$|H(a,b)| \leq C(|a|^2 + |b|^2 + 1)$$

for all $a, b \in \mathbb{R}$, cf. (3.41).

Now we want to derive from the time-discrete energy estimate the energy inequality for the interpolant functions $\mathbf{v}^N, q^N, \mu^N$ and φ^N. This can be done analogously as in [ADG13], i.e., we define the piecewise linear interpolant $E^N(t)$ of $E_{tot}(\mathbf{v}_k, \varphi_k, \nabla\varphi_k, q_k)$ at $t_k = kh$ by

$$E^N(t) := \frac{(k+1)h - t}{h} E_{tot}(\mathbf{v}_k, \varphi_k, \nabla\varphi_k, q_k) + \frac{t - kh}{h} E_{tot}(\mathbf{v}_{k+1}, \varphi_{k+1}, \nabla\varphi_{k+1}, q_{k+1})$$

for $t \in [kh, (k+1)h)$. Moreover, we define for all $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}_0$

$$D^N(t) := \int_\Omega m(\varphi_k, q_k)\, |\nabla q_{k+1}|^2\, dx + \int_\Omega \tilde{m}(\varphi_k) |\nabla \mu_{k+1}|^2 dx + \int_\Omega 2\eta(\varphi_k) |D\mathbf{v}_{k+1}|^2 dx$$
$$+ \delta \int_\Omega |\Delta \mathbf{v}_{k+1}|^2 dx + \delta \int_\Omega \frac{|\varphi_{k+1} - \varphi_k|^2}{h^2} dx.$$

Thus the time-discrete energy estimate (3.31) yields

$$-\frac{d}{dt} E^N(t) = \frac{E_{tot}(\mathbf{v}_k, \varphi_k, \nabla\varphi_k, q_k) - E_{tot}(\mathbf{v}_{k+1}, \varphi_{k+1}, \nabla\varphi_{k+1}, q_{k+1})}{h} \geq D^N(t) \quad (3.69)$$

for all $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}_0$. We multiply this inequality by $\tau \in W_1^1(0,\infty)$ with $\tau \geq 0$, integrate over $(0,\infty)$ with respect to t and use integration by parts. Then it follows

$$E_{tot}(\mathbf{v}_0, \varphi_0^N, \nabla\varphi_0^N, q_0)\tau(0) + \int_0^\infty E^N(t)\tau'(t)\, dt \geq \int_0^\infty D^N(t)\tau(t)\, dt. \quad (3.70)$$

We will need this estimate for the derivation of the time-continuous energy estimate as $N \to \infty$. When we integrate (3.69) we get the energy estimate for the interpolant functions $\mathbf{v}^N, q^N, \mu^N$ and φ^N given by

$$E_{tot}\left(\mathbf{v}_h^N(t), \varphi_h^N(t), \nabla\varphi_h^N(t), q_h^N(t)\right) + \int_s^t \int_\Omega \Big(m(\varphi_h^N, q_h^N)|\nabla q^N|^2$$
$$+ \tilde{m}(\varphi_h^N)|\nabla\mu^N|^2 + 2\eta(\varphi_h^N)|D\mathbf{v}^N|^2 + \delta|\Delta\mathbf{v}^N|^2 + \delta\left|\partial_{t,h}^- \varphi^N\right|^2 \Big)\, dx\, d\tau$$
$$\leq E_{tot}\left(\mathbf{v}_h^N(s), \varphi_h^N(s), \nabla\varphi_h^N(s), q_h^N(s)\right) \quad (3.71)$$

for all $0 \leq s \leq t < \infty$ with $s, t \in h\mathbb{N}_0$, where we used the definition of E^N. Since it holds $\mathbf{v}_h^N(0) = \mathbf{v}_0$, $q_h^N(0) = q_0$ and $\varphi_h^N(0) = \varphi_0$ for all $N \in \mathbb{N}$, we can conclude that

$$\left(E_{tot}(\mathbf{v}_h^N(0), \varphi_h^N(0), \nabla\varphi_h^N(0), q_h^N(0))\right)_{N\in\mathbb{N}} \text{ is bounded in } \mathbb{R}.$$

The boundedness of $(E_{tot}(\mathbf{v}_h^N(0), \varphi_h^N(0), \nabla\varphi_h^N(0), q_h^N(0)))_{N\in\mathbb{N}}$ implies:

$$\begin{aligned}
&i) && (\mathbf{v}^N)_{N\in\mathbb{N}} \text{ is bounded in } L^\infty(0,\infty; L^2(\Omega)^d),\\
&ii) && (\mathbf{v}^N)_{N\in\mathbb{N}} \text{ is bounded in } L^2(0,\infty; H^2(\Omega)^d),\\
&iii) && (\nabla q^N)_{N\in\mathbb{N}} \text{ is bounded in } L^2(0,\infty; L^2(\Omega)^d),\\
&iv) && (\nabla \mu^N)_{N\in\mathbb{N}} \text{ is bounded in } L^2(0,\infty; L^2(\Omega)^d),\\
&v) && (\nabla \varphi^N)_{N\in\mathbb{N}} \text{ is bounded in } L^\infty(0,\infty; L^2(\Omega)^d),\\
&vi) && (W(\varphi^N))_{N\in\mathbb{N}} \text{ is bounded in } L^\infty(0,\infty; L^1(\Omega)),\\
&vii) && (G(q^N))_{N\in\mathbb{N}} \text{ is bounded in } L^\infty(0,\infty; L^1(\Omega)),\\
&viii) && (\partial_{t,h}^- \varphi^N)_{N\in\mathbb{N}} \text{ is bounded in } L^2(0,\infty; L^2(\Omega)).
\end{aligned} \tag{3.72}$$

From the growth condition $G(q^N) \geq C(|q^N|^2 - 1)$, cf. (3.15), it follows that $(q^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty(0,\infty; L^2(\Omega))$. Together with iii) this yields

$$(q^N)_{N\in\mathbb{N}} \text{ is bounded in } L^2_{uloc}([0,\infty); H^1(\Omega)) \cap L^\infty(0,\infty; L^2(\Omega)).$$

Since $(\mathbf{v}^N(t), \varphi^N(t), \mu^N(t), q^N(t))_{N\in\mathbb{N}}$ solves the discrete problem (3.20) - (3.24) for any fixed $t \in (0,\infty)$, we can conclude from (3.23) by testing with constant 1

$$0 = \frac{1}{h}\int_\Omega \varphi^N(t)\,dx - \frac{1}{h}\int_\Omega \varphi_h^N(t)\,dx. \tag{3.73}$$

But this means that the mean values $(\varphi^N(t))_{N\in\mathbb{N}}$ are constant for a.e. $t \in (0,\infty)$ and for any fixed $N \in \mathbb{N}$. Thus we can conclude that they are also bounded independently of N since it holds

$$\left|\int_\Omega \varphi^N(t)\,dx\right| = \left|\int_\Omega \varphi_0\,dx\right|$$

for all $t \in [0,\infty)$. Hence, we can deduce the existence of a constant $C > 0$ independent of N such that

$$\left|\int_\Omega \varphi^N(t)\,dx\right| \leq C \qquad \text{for all } t \in [0,\infty),\ N \in \mathbb{N}.$$

In particular, it holds

$$\int_0^T \left|\int_\Omega \varphi^N\,dx\right| dt \leq C(T) \qquad \text{for all } 0 < T < \infty$$

for a monotone function $C : \mathbb{R}^+ \to \mathbb{R}^+$. Due to these estimates and the boundedness of $(\nabla\varphi^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,\infty;L^2(\Omega)^d)$, we can conclude that

$$(\varphi^N)_{N\in\mathbb{N}} \text{ is bounded in } L^\infty(0,\infty;H^1(\Omega)).$$

Testing equation (3.24) with constant 1, using the growth condition for the function h and using estimate (3.41) for H, we can deduce

$$\begin{aligned}\left|\int_\Omega \mu^N(t)\,dx\right| &= \left|\int_\Omega h(q^N(t))H(\varphi^N(t),\varphi^N(t-h))\,dx\right| \\ &\le C\int_\Omega \left(|q^N(t)|+1\right)\left(|\varphi^N(t)|^2+|\varphi^N(t-h)|^2+1\right)dx\end{aligned}$$

for a.e. $t \in (0,\infty)$. Since it holds $\varphi^N(-h) = \varphi_0 \in H^2(\Omega)$, we can deduce that $(\varphi_h^N)_{N\in\mathbb{N}}$ is also bounded in $L^\infty(0,\infty;L^6(\Omega))$. Due to the boundedness of φ^N in $L^\infty(0,\infty;L^6(\Omega))$, q^N in $L^2_{uloc}([0,\infty);L^6(\Omega))$ and the boundedness of $(\nabla\mu^N)_{N\in\mathbb{N}}$ in $L^2(0,\infty;L^2(\Omega))$ the previous estimate implies that

$$(\mu^N)_{N\in\mathbb{N}} \text{ is bounded in } L^2_{uloc}([0,\infty);H^1(\Omega)).$$

Due to these bounds we can conclude that for every $0 < T < \infty$ there exists a suitable subsequence, which we denote by $(\mathbf{v}^N, q^N, \mu^N, \varphi^N)_{N\in\mathbb{N}}$ again, such that

i) $\mathbf{v}^N \rightharpoonup \mathbf{v}$ in $L^2(0,\infty;H^2(\Omega)^d)$,

ii) $\mathbf{v}^N \rightharpoonup^* \mathbf{v}$ in $L^\infty(0,\infty;L^2(\Omega)^d) \cong (L^1(0,\infty;L^2(\Omega)^d))'$,

iii) $q^N \rightharpoonup q$ in $L^2(0,T;H^1(\Omega))$,

iv) $q^N \rightharpoonup^* q$ in $L^\infty(0,\infty;L^2(\Omega)) \cong (L^1(0,\infty;L^2(\Omega)))'$,

v) $\varphi^N \rightharpoonup^* \varphi$ in $L^\infty(0,\infty;H^1(\Omega)) \cong (L^1(0,\infty;H^1(\Omega)))'$,

vi) $\mu^N \rightharpoonup \mu$ in $L^2(0,T;H^1(\Omega))$.

Remark 3.10. *For most calculations $\mathbf{v}^N \rightharpoonup \mathbf{v}$ in $L^2(0,\infty;H^1(\Omega)^d)$ is sufficient. We only need the weak convergence in $L^2(0,\infty;H^2(\Omega)^d)$ when we pass to the limit $N\to\infty$ in the term $\left\langle \frac{R^N\mathbf{v}^N}{2}, \boldsymbol{\psi}\right\rangle$ for $\boldsymbol{\psi} \in C_0^\infty(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L^2_\sigma(\Omega))$. Therefore, we do all the other calculations and estimates for $\mathbf{v}^N \rightharpoonup \mathbf{v}$ in $L^2(0,\infty;H^1(\Omega)^d)$.*

Note that in the following we often pass to suitable subsequences $N_k \to \infty$, which we always denote by $(\mathbf{v}^N, q^N, \mu^N, \varphi^N)_{N\in\mathbb{N}}$ again.

3.3.1 Compactness of φ^N and Convergence of its Initial Values

First of all we want to show $\varphi(0) = \varphi_0$ in an appropriate Banach space, where φ is the weak-* limit as before. To this end, we denote by $\tilde{\varphi}^N$ the piecewise linear interpolant of $\varphi^N(t_k)$, where $t_k = kh$, $k \in \mathbb{N}_0$, i.e.,

$$\tilde{\varphi}^N(t) = \frac{(k+1)h - t}{h}\varphi^N(t-h) + \frac{t-kh}{h}\varphi^N(t)$$

for $t \in [kh, (k+1)h)$. Then it follows

$$\begin{aligned}
\partial_t\tilde{\varphi}^N(t) &= -\frac{1}{h}\varphi^N(t-h) + \frac{1}{h}\varphi^N(t) = \partial_{t,h}^-\varphi^N, \\
\tilde{\varphi}^N(t) - \varphi^N(t) &= \frac{((k+1)h-t)\varphi^N(t-h) + (t-(k+1)h)\varphi^N(t)}{h} \\
&= (-(k+1)h+t)\frac{\varphi^N(t) - \varphi^N(t-h)}{h} \\
&= (-(k+1)h+t)\partial_{t,h}^-\varphi^N(t) \qquad (3.74)
\end{aligned}$$

for $t \in [kh, (k+1)h)$. Using both equations yields

$$||\tilde{\varphi}^N(t) - \varphi^N(t)||_{H^{-1}(\Omega)} \leq h||\partial_t\tilde{\varphi}^N(t)||_{H^{-1}(\Omega)} \qquad (3.75)$$

for every $t \in [0,\infty)$, where we used that there exists $k \in \mathbb{N}_0$ such that $t \in [kh, (k+1)h)$ and therefore $|(t-(k+1)h)| \leq h = \frac{1}{N} \leq 1$.

In the following we want to use the Aubin-Lions lemma, cf. Theorem 2.33. From equation (3.67) we get that $(\partial_t\tilde{\varphi}^N)_{N\in\mathbb{N}} \subseteq L^2(0,\infty;H^{-1}(\Omega))$ is bounded since $(\nabla\mu^N)_{N\in\mathbb{N}}$ is bounded in $L^2(0,\infty;L^2(\Omega)^d)$ as we have seen before and $\nabla\varphi_h^N \cdot \mathbf{v}^N$ is bounded in $L^2(0,\infty;L^{\frac{3}{2}}(\Omega))$ since $\nabla\varphi_h^N$ is bounded in $L^\infty(0,\infty;L^2(\Omega)^d)$ and $\mathbf{v}^N$ is bounded in $L^2(0,\infty;L^6(\Omega)^d)$. Moreover, we can conclude that $(\tilde{\varphi}^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty(0,\infty;H^1(\Omega))$ since $(\varphi^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty(0,\infty;H^1(\Omega))$. Thus we can apply the Aubin-Lions lemma, which yields the relative compactness of $\{\tilde{\varphi}^N : N \in \mathbb{N}\}$ in $L^p(0,T;L^2(\Omega))$ for every $0 < T < \infty$ and $1 \leq p < \infty$. In particular this implies the strong convergence

$$\tilde{\varphi}^N \to \tilde{\varphi} \qquad \text{in } L^p(0,T;L^2(\Omega))$$

for all $0 < T < \infty$, $1 \leq p < \infty$ and for $\tilde{\varphi} \in L^\infty(0,\infty;L^2(\Omega))$. Here it holds $\tilde{\varphi} \in L^\infty(0,\infty;L^2(\Omega))$ due to the following arguments:
Since we know that $(\tilde{\varphi}^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty(0,\infty;H^1(\Omega))$ there exists $R > 0$ such that $||\tilde{\varphi}^N(t)||_{L^2(\Omega)} \leq R$ for all $N \in \mathbb{N}$ and a.e. $t \in (0,\infty)$. Moreover, we obtain from $\tilde{\varphi}^N \to \tilde{\varphi}$ in $L^p(0,T;L^2(\Omega))$ the existence of a subsequence such that $\tilde{\varphi}^N(t) \to \tilde{\varphi}(t)$ in $L^2(\Omega)$ for a.e. $t \in (0,T)$. Hence, we can conclude $||\tilde{\varphi}(t)||_{L^2(\Omega)} \leq R$ for a.e. $t \in (0,T)$. Moreover, lower semi-continuity of norms

implies $||\tilde{\varphi}(t)||_{L^2(\Omega)} \leq \liminf_{N\to\infty} ||\tilde{\varphi}^N(t)||_{L^2(\Omega)} \leq R$ for a.e. $t \in (0,T)$ and therefore $||\tilde{\varphi}||_{L^\infty(0,T;L^2(\Omega))} \leq \liminf_{N\to\infty} ||\tilde{\varphi}^N||_{L^\infty(0,T;L^2(\Omega))} < \infty$. Since these arguments hold for every $T > 0$, this implies the statement.
From $\tilde{\varphi}^N \to \tilde{\varphi}$ in $L^p(0,T;L^2(\Omega))$ we can deduce the existence of a subsequence such that $\tilde{\varphi}^N \to \tilde{\varphi}$ pointwise a.e. in $(0,\infty)\times\Omega$. Furthermore, estimate (3.75) yields

$$\tilde{\varphi}^N - \varphi^N \to 0 \qquad \text{in } L^2(0,\infty;H^{-1}(\Omega)),$$

since $(\partial_t \tilde{\varphi}^N)_{N\in\mathbb{N}} \subseteq L^2(0,\infty;H^{-1}(\Omega))$ is bounded and $h \to 0$ for $N \to \infty$. This implies $\tilde{\varphi} = \varphi$ together with

$$\tilde{\varphi}^N \to \varphi \qquad \text{in } L^2(0,T;L^2(\Omega))$$

for every $0 < T < \infty$. Here we could conclude $\tilde{\varphi} = \varphi$ since the weak-$*$ convergence $\varphi^N \rightharpoonup^* \varphi$ in $L^\infty(0,\infty;H^1(\Omega)) \cong (L^1(0,\infty;H^1(\Omega))'$ means

$$\langle \psi, \varphi^N \rangle_{L^1(0,\infty;H^1(\Omega)),(L^1(0,\infty;H^1(\Omega)))'} \to \langle \psi, \varphi \rangle_{L^1(0,\infty;H^1(\Omega)),(L^1(0,\infty;H^1(\Omega)))'}$$

for every $\psi \in L^1(0,\infty;H^1(\Omega))$. But this implies the convergence also for every $\psi \in L^1(0,T;H^1(\Omega))$ since we can extend every $\psi \in L^1(0,T;H^1(\Omega))$ on (T,∞) by 0. Hence, we get

$$\varphi^N \rightharpoonup \varphi \qquad \text{in } L^2(0,T;H^1(\Omega)). \tag{3.76}$$

Moreover, we can conclude

$$\varphi^N \rightharpoonup \varphi \qquad \text{in } L^2(0,T;H_0^{-1}(\Omega)).$$

As we already know $\tilde{\varphi}^N \to \tilde{\varphi}$ in $L^2(0,T;L^2(\Omega)) \hookrightarrow L^2(0,T;H_0^{-1}(\Omega))$ and $\tilde{\varphi}^N - \varphi^N \to 0$ in $L^2(0,\infty;H_0^{-1}(\Omega))$, we can conclude

$$\varphi - \tilde{\varphi} = \operatorname*{w-lim}_{N\to\infty} \varphi^N - \tilde{\varphi}^N = 0 \qquad \text{in } L^2(0,T;H_0^{-1}(\Omega)).$$

This implies $\varphi = \tilde{\varphi}$ in $L^2(0,T;H_0^{-1}(\Omega))$ for every $0 < T < \infty$ and therefore $\varphi(t) = \tilde{\varphi}(t)$ for a.e. $t \in (0,\infty)$. Since we know that $\tilde{\varphi}$ is in $L^\infty(0,\infty;L^2(\Omega))$ and $\tilde{\varphi}^N \to \tilde{\varphi}$ in $L^p(0,T;L^2(\Omega))$ for all $0 < T < \infty$, $1 \leq p < \infty$, this also holds for φ.

Due to Lemma 2.21 it holds

$$W_2^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega)) \hookrightarrow C([0,T];L^2(\Omega)).$$

Since $(\tilde{\varphi}^N)_{N\in\mathbb{N}}$ is bounded in $W_2^1(0,\infty;H^{-1}(\Omega)) \cap L^\infty(0,\infty;H^1(\Omega))$, there is a subsequence such that

$$\tilde{\varphi}^N \rightharpoonup \varphi \qquad \text{in } W_2^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1(\Omega))$$

for every $0 < T < \infty$. From Lemma 2.3 it follows

$$\tilde{\varphi}^N \rightharpoonup \varphi \qquad \text{in } C([0,T];L^2(\Omega))$$

for every $0 < T < \infty$. As the mapping

$$\begin{aligned} \mathrm{tr}_{t=0} : C([0,T];L^2(\Omega)) &\to L^2(\Omega), \\ f &\mapsto f(0) \end{aligned}$$

is linear and continuous, it is also weakly continuous. Thus we can conclude $\tilde{\varphi}^N(0) \rightharpoonup \varphi(0)$ in $L^2(\Omega)$, where it holds $\tilde{\varphi}^N(0) = \varphi_0$. Hence we can deduce $\varphi(0) = \varphi_0$ in $L^2(\Omega)$.

3.3.2 Higher Regularity of φ^N

To prove higher regularity of φ^N and φ_h^N, respectively, we use equation (3.68), which yields

$$\varepsilon\Delta\varphi^N = \frac{1}{\varepsilon}h(q^N)H(\varphi^N,\varphi_h^N) - \mu^N + \delta\partial_{t,h}^-\varphi^N =: f_1^N$$

in the weak sense. As it holds $q^N \in L^2_{uloc}([0,\infty);H^1(\Omega))$ and $\varphi^N \in L^\infty(0,\infty;H^1(\Omega))$, we can conclude that $h(q^N)H(\varphi^N,\varphi_h^N)$ is bounded in $L^2_{uloc}([0,\infty);L^2(\Omega))$, where we used estimate (3.41) and $\varphi_h^N \in L^\infty(0,\infty;H^1(\Omega))$.
Moreover, we know that $\partial_{t,h}^-\varphi^N$ is bounded in $L^2((0,T)\times\Omega)$, cf. (3.72) viii). Using standard elliptic regularity theory with Neumann boundary condition yields $\varphi^N(t) \in H^2(\Omega)$ for a.e. $t \in (0,\infty)$ together with the estimate

$$||\varphi^N(t)||_{H^2(\Omega)} \le C\left(||\varphi^N(t)||_{H^1(\Omega)} + ||f_1^N(t)||_{L^2(\Omega)}\right) \qquad \text{for a.e. } t \in (0,\infty).$$

As it holds $f_1^N \in L^2_{uloc}([0,\infty);L^2(\Omega))$, the estimate above implies

$$\varphi^N, \varphi_h^N \in L^2_{uloc}([0,\infty);H^2(\Omega)). \tag{3.77}$$

Due to the boundedness of $(\varphi_h^N)_{N\in\mathbb{N}}$ in $L^2_{uloc}([0,\infty);H^2(\Omega)) \cap L^\infty(0,\infty;H^1(\Omega))$, we interpolate the spaces $H^1(\Omega)$ and $H^2(\Omega)$ to get the boundedness of $(\varphi_h^N)_{N\in\mathbb{N}}$ in an "intermediate space". From Theorem 2.28 it follows

$$(H^2(\mathbb{R}^d), H^1(\mathbb{R}^d))_{\frac{1}{2},1} = B^{\frac{3}{2}}_{21}(\mathbb{R}^d)$$

for all $d \in \mathbb{N}$. In the case $d = 3$, Theorem 2.20 yields $B^{\frac{3}{2}}_{21}(\mathbb{R}^d) \hookrightarrow B^0_{\infty 1}(\mathbb{R}^d)$. In the case $d = 2$, Theorem 2.20 also provides $B^{\frac{3}{2}}_{21}(\mathbb{R}^d) \hookrightarrow B^1_{21}(\mathbb{R}^d) \hookrightarrow B^0_{\infty 1}(\mathbb{R}^d)$. Due to (2.3) we get in both cases $B^0_{\infty 1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. Together with Lemma 2.27 we obtain the estimate

$$||f||_{L^\infty(\mathbb{R}^d)} \le C||f||^{\frac{1}{2}}_{H^1(\mathbb{R}^d)}||f||^{\frac{1}{2}}_{H^2(\mathbb{R}^d)}$$

for all $f \in H^2(\mathbb{R}^d)$ and $d \leq 3$. Since there exists a continuous extension operator $E : H^k(\Omega) \to H^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}_0$, cf. [Ste70, Chapter VI, Section 3.2], we can also show

$$||f||_{L^\infty(\Omega)} \leq C||f||_{H^1(\Omega)}^{\frac{1}{2}}||f||_{H^2(\Omega)}^{\frac{1}{2}}$$

for all $f \in H^2(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ with $d \leq 3$, and for a constant $C > 0$. As we know that $(\varphi_h^N)_{N\in\mathbb{N}}$ is bounded in $L^2_{uloc}([0,\infty); H^2(\Omega)) \cap L^\infty(0,\infty; H^1(\Omega))$, we can use the estimate above to conclude that

$$(\varphi_h^N)_{N\in\mathbb{N}} \text{ is bounded in } L^4(0,T; L^\infty(\Omega)) \tag{3.78}$$

for every $0 < T < \infty$. Thus Theorem 2.32 yields that

$$(\varphi_h^N)_{N\in\mathbb{N}} \text{ is bounded in } L^4(0,T; L^\infty(\Omega)) \cap L^\infty(0,T; L^6(\Omega)) \hookrightarrow L^{12}(0,T; L^9(\Omega)).$$

Note that φ_h^N is even bounded in $L^4_{uloc}([0,\infty); L^\infty(\Omega))$. But since we are only interested in estimates for every $0 < T < \infty$, we focus on the boundedness in spaces with finite T from now on. Moreover, Lemma 2.27 and Theorem 2.28 yield $(H^1(\Omega), H^{-1}(\Omega))_{\frac{1}{2},2} = L^2(\Omega)$ together with the estimate

$$\begin{aligned} ||\tilde{\varphi}^N(t) - \varphi^N(t)||_{L^2(\Omega)} &\leq C||\tilde{\varphi}^N(t) - \varphi^N(t)||_{H^{-1}(\Omega)}^{\frac{1}{2}}||\tilde{\varphi}^N(t) - \varphi^N(t)||_{H^1(\Omega)}^{\frac{1}{2}} \\ &\leq Ch||\partial_t\tilde{\varphi}^N(t)||_{H^{-1}(\Omega)}^{\frac{1}{2}}||\tilde{\varphi}^N(t) - \varphi^N(t)||_{H^1(\Omega)}^{\frac{1}{2}}, \end{aligned}$$

where we used (3.75). Due to the boundedness of $\partial_t\tilde{\varphi}^N$ in $L^2(0,\infty; H^{-1}(\Omega))$ and of $\tilde{\varphi}^N, \varphi^N$ in $L^2_{uloc}([0,\infty); H^1(\Omega))$ we can conclude with the estimate above

$$\varphi^N \to \varphi \qquad \text{in } L^2(0,T; L^2(\Omega)) \tag{3.79}$$

for every $0 < T < \infty$ since $\tilde{\varphi}^N$ and φ^N must converge to the same limit. Due to Theorem 2.28 we can conclude

$$(H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))_{\frac{1}{2},2} = B^1_{22}(\mathbb{R}^d).$$

Since we know $B^1_{22}(\mathbb{R}^d) = H^1(\mathbb{R}^d)$, cf. (2.4), and $\varphi^N(t) \in H^2(\Omega)$ for a.e. $t \in (0,\infty)$, we can deduce with Lemma 2.27

$$||\varphi^N(t) - \varphi(t)||_{H^1(\Omega)} \leq C||\varphi^N(t) - \varphi(t)||_{H^2(\Omega)}^{\frac{1}{2}}||\varphi^N(t) - \varphi(t)||_{L^2(\Omega)}^{\frac{1}{2}}. \tag{3.80}$$

As it holds that $(\varphi^N)_{N\in\mathbb{N}}$ is bounded in $L^2(0,T; H^2(\Omega))$, there exists a subsequence such that $\varphi^N \rightharpoonup \overline{\varphi}$ in $L^2(0,T; H^2(\Omega))$. But since we have already proven that $\varphi^N \rightharpoonup^* \varphi$ in $L^\infty(0,\infty; H^1(\Omega))$ implies $\varphi^N \rightharpoonup \varphi$ in $L^2(0,T; H^1(\Omega))$, cf. (3.76), we can deduce $\varphi = \overline{\varphi}$. Thus it holds $\varphi \in L^2(0,T; H^2(\Omega))$. Hence, the first term on the right-hand side in estimate (3.80) is bounded. Since $\varphi^N \to \varphi$ in $L^2(0,T; L^2(\Omega))$ for every $0 < T < \infty$ this estimate yields

$$\varphi^N \to \varphi \qquad \text{in } L^2(0,T; H^1(\Omega)) \tag{3.81}$$

for every $0 < T < \infty$.

3.3.3 Compactness of q^N

In the following we prove the strong convergence of $(q^N)_{N\in\mathbb{N}}$ in $L^2(0,T;L^2(\Omega))$ for a suitable subsequence. To this end, we show that $\{q^N : N \in \mathbb{N}\}$ fulfills the assumptions of Simon's theorem, cf. Theorem 2.34, where $X = H^1(\Omega)$, $B = L^2(\Omega)$ and $Y = L^2(\Omega)$. Then we can conclude that $\{q^N : N \in \mathbb{N}\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$.
First of all we note that $(q^N)_{N\in\mathbb{N}}$ is bounded in $L^2_{uloc}([0,\infty);H^1(\Omega))$ as we concluded from (3.72). Thus condition i) in Theorem 2.34 is satisfied for every $0 < T < \infty$. It remains to show condition ii). To this end, we prove

$$\left(\int_0^{T-s} ||q^N(t+s) - q^N(t)||^2_{L^2(\Omega)}\,dt\right)^{\frac{1}{2}} \leq C(T)s^{\frac{1}{4}} \tag{3.82}$$

for all $s = \tilde{m}h$ with $\tilde{m} \in \mathbb{N}$ and a constant $C(T) > 0$ independent of s and $N \in \mathbb{N}$. Then Lemma 2.35 yields

$$\left(\int_0^{T-\lambda} ||q^N(t+\lambda) - q^N(t)||^2_{L^2(\Omega)}\,dt\right)^{\frac{1}{2}} \leq C\lambda^{\frac{1}{4}} \tag{3.83}$$

for any $\lambda > 0$ and a constant $C > 0$ independent of λ and N. This proves condition ii) in Theorem 2.34 and therefore shows that $\{q^N : N \in \mathbb{N}\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$ for every $0 < T < \infty$.

In the following let $s = \tilde{m}h$ be given for $\tilde{m} \in \mathbb{N}$ and $h = \frac{1}{N}$. Moreover, we define

$$\tilde{F}(\varphi^N, q^N) := \frac{1}{\varepsilon} f(q^N)W(\varphi^N) + g(q^N), \qquad \tilde{f}(t) := \tilde{F}(\varphi^N(t), q^N(t)).$$

Since g is strongly monotone and f is monotone, it holds that $\tilde{F}(\varphi^N, \cdot)$ is strongly monotone and therefore there exists a constant $C > 0$ such that

$$\left|\tilde{F}(\varphi_k, q_{k+\tilde{m}}) - \tilde{F}(\varphi_k, q_k)\right| \geq C\,|q_{k+\tilde{m}} - q_k| \geq C\left|q^N(t+s) - q^N(t)\right|$$

for every $t \in [kh, (k+1)h)$. Multiplying these inequalities with $|q^N(t+s) - q^N(t)|$, integrating from $(k-1)h$ to kh with respect to t, integrating over the domain Ω and summing over $k = 1, ..., TN - m$ for $T \in \mathbb{N}$ yields

$$
\begin{aligned}
C \sum_{k=1}^{TN-\tilde{m}} \int_\Omega \int_{(k-1)h}^{kh} |q^N(t+s) - q^N(t)|^2 \,dt\,dx &\leq C \sum_{k=1}^{TN-\tilde{m}} \int_\Omega \int_{(k-1)h}^{kh} |q_{k+\tilde{m}} - q_k|^2 \,dt\,dx \\
&\leq \sum_{k=1}^{TN-\tilde{m}} \int_\Omega \int_{(k-1)h}^{kh} (\tilde{F}(\varphi_k, q_{k+\tilde{m}}) - \tilde{F}(\varphi_k, q_k))(q_{k+\tilde{m}} - q_k) \,dt\,dx \\
&\leq \sum_{k=1}^{TN-\tilde{m}} \int_\Omega \int_{(k-1)h}^{kh} (\tilde{F}(\varphi_k, q_{k+\tilde{m}}) - \tilde{F}(\varphi_{k+\tilde{m}}, q_{k+\tilde{m}}))(q_{k+\tilde{m}} - q_k) \,dt\,dx \\
&\quad + \sum_{k=1}^{TN-\tilde{m}} \int_\Omega \int_{(k-1)h}^{kh} (\tilde{F}(\varphi_{k+\tilde{m}}, q_{k+\tilde{m}}) - \tilde{F}(\varphi_k, q_k))(q_{k+\tilde{m}} - q_k) \,dt\,dx, \qquad (3.84)
\end{aligned}
$$

where we omitted the modulus since the product is always positive due to the strong monotonicity of $\tilde{F}(\varphi, \cdot)$.

We estimate both summands in (3.84) separately. Since f is a bounded function, we can conclude for the first summand

$$
\begin{aligned}
&\sum_{k=1}^{TN-\tilde{m}} \int_\Omega \int_{(k-1)h}^{kh} (\tilde{F}(\varphi_k, q_{k+\tilde{m}}) - \tilde{F}(\varphi_{k+\tilde{m}}, q_{k+\tilde{m}}))(q_{k+\tilde{m}} - q_k) \,dt\,dx \\
&= \int_0^{T-s} \int_\Omega \left(\tilde{F}(\varphi^N, q^N_{s+}) - \tilde{F}(\varphi^N_{s+}, q^N_{s+}) \right) \left(q^N_{s+} - q^N \right) \,dx\,dt \\
&\leq C \int_0^{T-s} \int_\Omega \left| W(\varphi^N) - W(\varphi^N_{s+}) \right| \, \left| q^N_{s+} - q^N \right| \,dx\,dt. \\
&\leq C \int_0^{T-s} \int_\Omega \left| \left(\varphi^N_{s+} - \varphi^N \right) \left(|\varphi^N|^2 + |\varphi^N_{s+}|^2 + 1 \right) \right| \, \left| q^N_{s+} - q^N \right| \,dx\,dt \\
&\leq C \int_0^{T-s} \left|\left| \varphi^N_{s+} - \varphi^N \right|\right|_{L^2(\Omega)} \left|\left| |\varphi^N|^2 + |\varphi^N_{s+}|^2 + 1 \right|\right|_{L^3(\Omega)} \left|\left| q^N_{s+} - q^N \right|\right|_{L^6(\Omega)} \,dt \\
&\leq C(T) s^{\frac{1}{4}},
\end{aligned}
$$

where we used $\varphi^N \in L^\infty(0, \infty; L^6(\Omega))$, q^N_{s+}, $q^N \in L^2(0, T-s; L^6(\Omega))$ and

$$
\sup_{0 \leq t \leq T-s} ||\varphi^N(t+s) - \varphi^N(t)||_{L^2(\Omega)} \leq C(T) s^{\frac{1}{4}} \qquad \text{a.e. in } (0, \infty) \qquad (3.85)
$$

for every $0 < T < \infty$ and a constant $C(T) > 0$ depending on T. The latter inquality (3.85) is not obvious. Therefore, we prove it in the following:
Let $t \in [0, T-s)$ be given. Then there exists $k \in \mathbb{N}$ such that $t \in [kh, (k+1)h)$ and therefore $t + s \in [(k+\tilde{m})h, (k+\tilde{m}+1)h)$. Due to $\varphi^N(t) = \varphi_{k+\tilde{m}+1}$ for $t \in [(k+\tilde{m})h, (k+\tilde{m}+1)h)$, which holds by definition, we can conclude

$$\begin{aligned}\varphi^N(t+s) - \varphi^N(t) = \varphi_{k+\tilde{m}+1} - \varphi_{k+1} &= \tilde{\varphi}^N((k+\tilde{m}+2)h) - \tilde{\varphi}^N((k+2)h) \\ &= \tilde{\varphi}^N(\tilde{t}+s) - \tilde{\varphi}^N(\tilde{t}),\end{aligned}$$

where $\tilde{t} := (k+2)h$. Since we have already proven that $(\tilde{\varphi}^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty(0,\infty; H^1(\Omega)) \cap W_2^1(0,\infty; H^{-1}(\Omega))$, Lemma 2.24 provides

$$\sup_{t\in[0,T-s)} ||\tilde{\varphi}^N(\tilde{t}+s) - \tilde{\varphi}^N(\tilde{t})||_{H^{-1}(\Omega)} \le Cs^{\frac{1}{2}}$$

for every $0 < T < \infty$ and a constant $C(T) > 0$. From Theorem 2.28 it follows $(H^1(\Omega), H^{-1}(\Omega))_{\frac{1}{2},2} = L^2(\Omega)$ and due to the estimate from Lemma 2.27 we can deduce

$$\sup_{0\le t\le T-s} ||\tilde{\varphi}^N(\tilde{t}+s) - \tilde{\varphi}^N(\tilde{t})||_{L^2(\Omega)} \le Cs^{\frac{1}{4}}.$$

Since $0 < T < \infty$ can be chosen arbitrary, we have shown inequality (3.85).

To estimate the second term in (3.84), we use equation (3.22). Moreover, we set $l(t) = \lfloor \frac{t}{h} \rfloor$ and $\tilde{t}(t) := h\lfloor \frac{t}{h} \rfloor$, i.e., it holds $\tilde{t}(t) = t_k$ for $t \in [kh, (k+1)h)$ and $k = 0, ..., TN - 1 - \tilde{m}$ for $N \in \mathbb{N}$. Then we obtain

$$\begin{aligned}&\sum_{k=1}^{TN-\tilde{m}} \int_\Omega \int_{(k-1)h}^{kh} (\tilde{F}(\varphi_{k+\tilde{m}}, q_{k+\tilde{m}}) - \tilde{F}(\varphi_k, q_k))(q_{k+\tilde{m}} - q_k)\, dt\, dx \\ &= \sum_{k=1}^{TN-\tilde{m}} \int_\Omega \int_{(k-1)h}^{kh} \left(\sum_{j=1}^{\tilde{m}} \tilde{f}(\tilde{t}(t) + jh) - \tilde{f}(\tilde{t}(t) + (j-1)h) \right) (q_{k+\tilde{m}} - q_k)\, dt\, dx \\ &= \int_0^{T-s} \int_\Omega \left(\sum_{j=1}^{\tilde{m}} \tilde{f}(\tilde{t}(t) + jh) - \tilde{f}(\tilde{t}(t) + (j-1)h) \right) (q_{k+\tilde{m}} - q_k)\, dx\, dt,\end{aligned}$$

where it holds

$$
\begin{aligned}
&\sum_{j=1}^{\tilde{m}} \tilde{f}(\tilde{t}(t)+jh) - \tilde{f}(\tilde{t}(t)+(j-1)h) \\
&= h \sum_{j=l}^{l+\tilde{m}-1} \operatorname{div}\left(m(\varphi_j, q_j)\nabla q_{j+1}\right) - \nabla\left(\frac{1}{\varepsilon} f(q_{j+1}) W(\varphi_j) + g(q_{j+1})\right) \cdot \mathbf{v}_{j+1} \\
&= \int_{\tilde{t}(t)}^{\tilde{t}(t)+s} \operatorname{div}\left(m(\varphi^N(\tau), q^N(\tau))\nabla q^N(\tau+h)\right) d\tau \\
&\quad - \int_{\tilde{t}(t)}^{\tilde{t}(t)+s} \nabla g(q^N(\tau+h)) \cdot \mathbf{v}^N(\tau+h) d\tau \\
&\quad - \int_{\tilde{t}(t)}^{\tilde{t}(t)+s} \operatorname{div}\left(\frac{1}{\varepsilon} f(q^N(\tau+h)) W(\varphi^N(\tau)) \mathbf{v}^N(\tau+h)\right) d\tau
\end{aligned}
$$

in $H_0^{-1}(\Omega)$. We use this identity in (3.84) instead of the second summand and get

$$
\begin{aligned}
&\int_0^{T-s} \int_\Omega \left(\tilde{F}(\varphi_{s+}^N, q_{s+}^N) - \tilde{F}(\varphi^N, q^N)\right) \left(q_{s+}^N - q^N\right) dx\, dt \\
&\leq \int_0^{T-s} \int_\Omega \int_{\tilde{t}(t)}^{\tilde{t}(t)+s} \left|m(\varphi^N(\tau), q^N(\tau))\nabla q^N(\tau+h)\right| d\tau |\nabla q_{s+}^N - \nabla q^N| dx\, dt \\
&\quad - \int_0^{T-s} \int_\Omega \int_{\tilde{t}(t)}^{\tilde{t}(t)+s} \left|g(q^N(\tau+h)\mathbf{v}^N(\tau+h)\right| d\tau |\nabla q_{s+}^N - \nabla q^N| dx\, dt \\
&\quad - \int_0^{T-s} \int_\Omega \int_{\tilde{t}(t)}^{\tilde{t}(t)+s} \left|\frac{1}{\varepsilon} f(q^N(\tau+h)) W(\varphi^N(\tau)) \mathbf{v}^N(\tau+h)\right| d\tau |\nabla q_{s+}^N - \nabla q^N| dx\, dt.
\end{aligned}
$$

Now we estimate these three terms separately. For the first term we obtain

$$\begin{aligned}
&\int_0^{T-s}\int_\Omega\int_{\tilde{t}(t)}^{\tilde{t}(t)+s} \left|m(\varphi^N(\tau),q^N(\tau))\nabla q^N(\tau+h)\right| d\tau \left|\nabla q^N_{s+}-\nabla q^N\right| dx\,dt \\
&= \int_0^{T-s}\int_{\tilde{t}(t)}^{\tilde{t}(t)+s}\int_\Omega \left|m(\varphi^N(\tau),q^N(\tau))\nabla q^N(\tau+h)\right| \left|\nabla q^N_{s+}-\nabla q^N\right| dx\,d\tau\,dt \\
&\le C\int_0^{T-s}\int_{\tilde{t}(t)}^{\tilde{t}(t)+s} ||\nabla q^N(\tau+h)||_{L^2(\Omega)}\,d\tau ||\nabla q^N(t+s)-\nabla q^N(t)||_{L^2(\Omega)}\,dt \\
&\le C\int_0^{T-s} s^{\frac{1}{2}}||q^N||_{L^2(0,T;H^1(\Omega))}||\nabla q^N(t+s)-\nabla q^N(t)||_{L^2(\Omega)}\,dt \\
&\le Cs^{\frac{1}{2}}||q^N||_{L^2(0,T;H^1(\Omega))}||\nabla q^N_{s+}-\nabla q^N||_{L^2(0,T;L^2(\Omega))} \\
&\le C(T)s^{\frac{1}{2}}
\end{aligned}$$

for a constant $C(T) > 0$, where we used $0 < c_0 < m(\varphi,q) < c_1$ and the boundedness of $(q^N)_{N\in\mathbb{N}}$ in $L^2_{uloc}([0,\infty);H^1(\Omega))$. For the third term we use the boundedness of f and the growth condition for W. Then we can estimate

$$\begin{aligned}
&\int_0^{T-s}\int_\Omega\int_{\tilde{t}(t)}^{\tilde{t}(t)+s} \left|\frac{1}{\varepsilon}f(q^N(\tau+h))W(\varphi^N(\tau))\mathbf{v}^N(\tau+h)\right| d\tau \left|\nabla q^N_{s+}-\nabla q^N\right| dx\,dt \\
&= \int_0^{T-s}\int_{\tilde{t}(t)}^{\tilde{t}(t)+s}\int_\Omega \left|\frac{1}{\varepsilon}f(q^N(\tau+h))W(\varphi^N(\tau))\mathbf{v}^N(\tau+h)\right| \left|\nabla q^N_{s+}-\nabla q^N\right| dx\,d\tau\,dt \\
&\le C\int_0^{T-s}\int_{\tilde{t}(t)}^{\tilde{t}(t)+s} ||W(\varphi^N(\tau))||_{L^3(\Omega)}||\mathbf{v}^N(\tau+h)||_{L^6(\Omega)}\,d\tau||\nabla q^N_{s+}-\nabla q^N||_{L^2(\Omega)}\,dt \\
&\le C\int_0^{T-s}\int_{\tilde{t}(t)}^{\tilde{t}(t)+s} ||(\varphi^N(\tau)^3+1)||_{L^3(\Omega)}||\mathbf{v}^N(\tau+h)||_{L^6(\Omega)}\,d\tau||\nabla q^N_{s+}-\nabla q^N||_{L^2(\Omega)}\,dt \\
&\le C\int_0^{T-s} s^{\frac{1}{4}}(||\varphi^N||^3_{L^{12}(0,T;L^9(\Omega))}+1)||\mathbf{v}^N||_{L^2(0,T;L^6(\Omega))}||\nabla q^N(t+s)-\nabla q^N(t)||_{L^2(\Omega)}\,dt \\
&\le Cs^{\frac{1}{4}}(||\varphi^N||^3_{L^{12}(0,T;L^9(\Omega))}+1)||\mathbf{v}^N||_{L^2(0,T;L^6(\Omega))}||\nabla q^N_{s+}-\nabla q^N||_{L^2(0,T;L^2(\Omega))} \\
&\le C(T)s^{\frac{1}{4}}.
\end{aligned}$$

Analogously, we can estimate the second term by

$$\int_0^{T-s}\int_\Omega\int_{\tilde{t}(t)}^{\tilde{t}(t)+s} \left|g(q^N(\tau+h)\mathbf{v}^N(\tau+h)\right| d\tau \left|\nabla q^N_{s+} - \nabla q^N\right| dx\,dt$$
$$= \int_0^{T-s}\int_{\tilde{t}(t)}^{\tilde{t}(t)+s}\int_\Omega \left|g(q^N(\tau+h)\mathbf{v}^N(\tau+h)\right| \left|\nabla q^N_{s+} - \nabla q^N\right| dx\,d\tau\,dt$$
$$\leq \int_0^{T-s}\int_{\tilde{t}(t)}^{\tilde{t}(t)+s} ||g(q^N(\tau+h)||_{L^6(\Omega)}||\mathbf{v}^N(\tau+h)||_{L^3(\Omega)}\,d\tau||\nabla q^N(t+s) - \nabla q^N(t)||_{L^2(\Omega)}\,dt$$
$$\leq \int_0^{T-s} s^{\frac{1}{4}}||g(q^N)||_{L^2(0,T;L^6(\Omega))}||\mathbf{v}^N||_{L^4(0,T;L^3(\Omega))}||\nabla q^N(t+s) - \nabla q^N(t)||_{L^2(\Omega)}\,dt$$
$$\leq C(T)s^{\frac{1}{4}}.$$

Using these estimates in (3.84) yields that there exists a constant $C(T) > 0$ such that (3.82) holds. Hence, Lemma 2.35 implies that (3.83) holds for every $\lambda > 0$ and therefore it follows from Theorem 2.34 that $\{q^N : N \in \mathbb{N}\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$. Moreover, we can conclude that there exists $\tilde{q} \in L^2_{loc}([0,\infty);L^2(\Omega))$ such that

$$q^N \to \tilde{q} \qquad \text{in } L^2(0,T;L^2(\Omega))$$

for every $0 < T < \infty$. Since the weak and the strong limit have to coincide, we deduce $\tilde{q} = q$. In particular it holds for a subsequence

$$q^N(t,x) \to q(t,x) \qquad \text{a.e. in } (0,\infty)\times\Omega.$$

3.3.4 Convergence to the Initial Value of $\frac{1}{\varepsilon}f(q)W(\varphi) + g(q)$

In the next step we want to prove the convergence of $\left(\frac{1}{\varepsilon}f(q^N)W(\varphi^N) + g(q^N)\right)_{|t=0}$ to the initial value $\frac{1}{\varepsilon}f(q_0)W(\varphi_0) + g(q_0)$ as $N \to \infty$. To this end, we define $g^N := f(q^N)W(\varphi^N) + g(q^N)$. Moreover, let $\tilde{g}^N$ be the piecewise linear interpolant of $f(q^N(t_k))W(\varphi^N(t_k)) + g(q^N(t_k))$, where $t_k = kh$, $k \in \mathbb{N}_0$, i.e.,

$$\tilde{g}^N(t) := \frac{(k+1)h - t}{h}\left(f(q^N_h)W(\varphi^N_h) + g(q^N_h)\right) + \frac{t - kh}{h}\left(f(q^N)W(\varphi^N) + g(q^N)\right)$$

for $t \in [kh,(k+1)h)$. Then it holds

$$\partial_t\tilde{g}^N = -\frac{1}{h}\left(f(q^N_h)W(\varphi^N_h) + g(q^N_h)\right) + \left(\frac{1}{h}f(q^N)W(\varphi^N) + g(q^N)\right) = \partial^-_{t,h}g^N.$$

In the following we want to show boundedness of

$$\operatorname{div}(m(\varphi_h^N, q_h^N)\nabla q^N) - \nabla\left(\frac{1}{\varepsilon}f(q^N)W(\varphi_h^N) + g(q^N)\right)\cdot \mathbf{v}^N = \partial_t \tilde{g}^N \tag{3.86}$$

in $(L^4(0,T;H^1(\Omega)))'$, i.e.,

$$\begin{aligned}\int_0^T\int_\Omega \partial_t\tilde{g}^N\phi\,dx\,dt = &-\int_0^T\int_\Omega m(\varphi_h^N, q_h^N)\nabla q^N\cdot\nabla\phi\,dx\,dt\\ &+\int_0^T\int_\Omega\left(\frac{1}{\varepsilon}f(q^N)W(\varphi_h^N)+g(q^N)\right)\mathbf{v}^N\cdot\nabla\phi\,dx\,dt\end{aligned}$$

for all $\phi \in L^4(0,T;H^1(\Omega))$. First of all we can deduce that $(\operatorname{div}(m(\varphi_h^N, q_h^N)\nabla q^N))_{N\in\mathbb{N}}$ is bounded in $(L^4(0,T;H^1(\Omega)))'$ since $(q^N)_{N\in\mathbb{N}}$ is bounded in $L^2_{uloc}([0,\infty);H^1(\Omega))$, cf. (3.72), and since there exist some constants $0 < c_0 < c_1 < \infty$ such that $c_0 < m(\varphi_h^N, q_h^N) < c_2$.
Now we use (3.78) to show that the remaining term $\nabla\left(\frac{1}{\varepsilon}f(q^N)W(\varphi_h^N)+g(q^N)\right)\cdot\mathbf{v}^N$ is also bounded in $(L^4(0,T;H^1(\Omega)))' \cong L^{\frac{4}{3}}(0,T;H^{-1}(\Omega))$, cf. Theorem 2.13. So let $\phi \in L^4(0,T;H^1(\Omega))$ be given. Then it holds

$$\begin{aligned}&\left\langle\nabla\left(\frac{1}{\varepsilon}f(q^N)W(\varphi_h^N)+g(q^N)\right)\cdot\mathbf{v}^N,\phi\right\rangle_{(L^4(0,T;H^1(\Omega)))',L^4(0,T;H^1(\Omega))}\\ &\quad= -\int_0^T\int_\Omega\left(\left(\frac{1}{\varepsilon}f(q^N)W(\varphi_h^N)+g(q^N)\right)\mathbf{v}^N\right)\cdot\nabla\phi\,dx\,dt.\end{aligned}$$

Using the growth conditions for W and g and the fact that φ_h^N is bounded in $L^4(0,T;L^\infty(\Omega))$ for every $0 < T < \infty$, we can use Hölder's inequality and the boundedness of the function f to get

$$\begin{aligned}&-\int_0^T\int_\Omega\left(\left(\frac{1}{\varepsilon}f(q^N)W(\varphi_h^N)+g(q^N)\right)\mathbf{v}^N\right)\cdot\nabla\phi\,dx\,dt\\ &\qquad\leq C\left(||\varphi_h^N||^3_{L^4(0,T;L^\infty(\Omega))}+1\right)||\mathbf{v}^N||_{L^\infty(0,T;L^2(\Omega))}||\phi||_{L^4(0,T;H^1(\Omega))}\\ &\qquad\quad+\left(||q^N||_{L^2(0,T;H^1(\Omega))}+1\right)||\mathbf{v}^N||_{L^4(0,T;L^3(\Omega))}||\phi||_{L^4(0,T;H^1(\Omega))}.\end{aligned}$$

From all the norms on the right-hand side we know that they are bounded except the norm of $\mathbf{v}^N$ in $L^4(0,T;L^3(\Omega))$, which we still have to show. But as we already know that $\mathbf{v}^N$ is bounded in $L^\infty(0,\infty;L^2(\Omega))\cap L^2(0,\infty;H^1(\Omega))$ and $H^1(\Omega)\hookrightarrow L^6(\Omega)$ for $d\leq 3$, we can apply Theorem 2.32 with $\theta = \frac{1}{2}$ and obtain

$$\mathbf{v}^N \in L^\infty(0,\infty;L^2_\sigma(\Omega))\cap L^2(0,\infty;H^1(\Omega)^d)\hookrightarrow L^4(0,\infty;L^3(\Omega)^d). \tag{3.87}$$

Altogether, we have shown that the set $\left(\partial_t \tilde{g}^N\right)_{N\in\mathbb{N}}$ is bounded in $(L^4(0,T;H^1(\Omega)))' \cong L^{\frac{4}{3}}(0,T;H_0^{-1}(\Omega))$. Using the boundedness of f and the growth conditions for W and g, we can conclude that $\left(\tilde{g}^N\right)_{N\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(\Omega))$ and therefore in $L^{\frac{4}{3}}(0,T;H^{-1}(\Omega))$. Thus we obtain

$$\left(\tilde{g}^N\right)_{N\in\mathbb{N}} \subseteq W^1_{\frac{4}{3}}(0,T;H^{-1}(\Omega)) \text{ is bounded.}$$

As a consequence there exists a subsequence of $\left(\tilde{g}^N\right)_{N\in\mathbb{N}}$, which we denote by $\left(\tilde{g}^N\right)_{N\in\mathbb{N}}$ again, such that

$$\tilde{g}^N \rightharpoonup \tilde{g} \qquad \text{in } W^1_{\frac{4}{3}}(0,T;H^{-1}(\Omega)).$$

Due to Lemma 2.22 it holds $W^1_{\frac{4}{3}}(0,T;H^{-1}(\Omega)) \hookrightarrow C([0,T];H^{-1}(\Omega))$ for every $0<T<\infty$. Thus Lemma 2.3 yields

$$\tilde{g}^N \rightharpoonup \tilde{g} \qquad \text{in } C([0,T];H^{-1}(\Omega))$$

for every $0<T<\infty$ and therefore Lemma 2.12 shows

$$\tilde{g}^N(0) \rightharpoonup \tilde{g}(0) \qquad \text{in } H^{-1}(\Omega),$$

where it holds

$$\tilde{g}^N(0) = f(q^N(0-h))W(\varphi^N(0-h)) + g(q^N(0-h)) = f(q_0)W(\varphi_0) + g(q_0).$$

Due to the boundedness of f and the growth condition for W we can conclude that $\left(\tilde{g}^N(0)\right)_{N\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. We get

$$\tilde{g}^N(0)(x) \to f(q_0(x))W(\varphi_0(x)) + g(q_0(x)) \qquad \text{a.e. in } \Omega.$$

Therefore, Theorem 2.9 yields

$$\tilde{g}^N(0) \to f(q_0)W(\varphi_0) + g(q_0) \qquad \text{in } L^s(\Omega)$$

for all $1 \le s < 2$. This implies

$$\tilde{g}(0) = f(q_0)W(\varphi_0) + g(q_0) \qquad \text{in } H^{-1}(\Omega).$$

With the same calculations as for $\tilde{\varphi}^N$ we can conclude

$$\tilde{g}^N(t) - g^N(t) = (-(k+1)h + t)\partial^-_{t,h} g^N(t).$$

Using $\partial^-_{t,h} g^N(t) = \partial_t \tilde{g}^N$ yields

$$||\tilde{g}^N(t) - g^N(t)||_{H^{-1}(\Omega)} \le h||\partial_t \tilde{g}^N(t)||_{H^{-1}(\Omega)} \tag{3.88}$$

for every $t \in (0, \infty)$, where we used that there exists $k \in \mathbb{N}_0$ such that $t \in [kh, (k+1)h)$ and therefore $|(t - (k+1)h)| \leq h = \frac{1}{N} \leq 1$.
Since $\partial_t \tilde{g}^N$ is bounded in $L^{\frac{4}{3}}(0, T; H^{-1}(\Omega))$ and $h \to 0$ for $N \to \infty$, this yields

$$\tilde{g}^N - g^N \to 0 \qquad \text{in } L^{\frac{4}{3}}(0, T; H^{-1}(\Omega)).$$

As $\left(g^N\right)_{N \in \mathbb{N}}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$, there exists $\hat{g} \in L^{\frac{4}{3}}(0, T; H^{-1}(\Omega))$ such that

$$g^N \rightharpoonup \hat{g} \qquad \text{in } L^{\frac{4}{3}}(0, T; H^{-1}(\Omega))$$

and therefore $\tilde{g}^N \rightharpoonup \hat{g}$ in $L^{\frac{4}{3}}(0, T; H^{-1}(\Omega))$. Due to $\tilde{g}^N \rightharpoonup \tilde{g}$ in $W^1_{\frac{4}{3}}(0, T; H^{-1}(\Omega))$, this implies $\hat{g} = \tilde{g}$ in $W^1_{\frac{4}{3}}(0, T; H^{-1}(\Omega)) \hookrightarrow C([0, T]; H^{-1}(\Omega))$ for every $0 < T < \infty$. Hence, it holds

$$\hat{g}(0) = \tilde{g}(0) = f(q_0)W(\varphi_0) + g(q_0) \qquad \text{in } H^{-1}(\Omega).$$

Since the right-hand side is in $L^2(\Omega)$, it even holds $\hat{g}(0) = \tilde{g}(0) \in L^2(\Omega)$.

3.3.5 Compactness of $\mathbf{v}^N$ and Convergence of its Initial Values

Now we want to show that $\mathbf{v}$ attains its initial value $\mathbf{v}_0$ and that it holds $\mathbf{v}^N \to \mathbf{v}$ in $L^2(0, T; L^2(\Omega)^d)$ for all $0 < T < \infty$ as $N \to \infty$. Thus we can deduce the pointwise convergence a.e. in $(0, T) \times \Omega$. To this end, let $\widetilde{\rho \mathbf{v}}^N$ be the piecewise linear interpolant of $(\rho^N \mathbf{v}^N)(t_k)$, where $t_k = kh$, $k \in \mathbb{N}_0$, i.e.,

$$\widetilde{\rho \mathbf{v}}^N(t) = \frac{(k+1)h - t}{h}(\rho^N \mathbf{v}^N)(t - h) + \frac{t - kh}{h}(\rho^N \mathbf{v}^N)(t)$$

for $t \in [kh, (k+1)h)$. As before we can show $\partial_t(\widetilde{\rho \mathbf{v}}^N) = \partial^-_{t,h}(\rho^N \mathbf{v}^N)$. Moreover, we consider the function space

$$G(\Omega) := \{\mathbf{f} \in L^2(\Omega)^d : \ \exists p \in L^2(\Omega) : \ \mathbf{f} = \nabla p\}.$$

Then for every $\mathbf{f} \in L^2(\Omega)^d$ there exists a unique decomposition

$$\mathbf{f} = \mathbf{f}_0 + \nabla p \tag{3.89}$$

such that $\langle \mathbf{f}_0, \nabla p \rangle_{L^2(\Omega)} = 0$, where $\nabla p \in G(\Omega)$, $\mathbf{f}_0 \in L^2_\sigma(\Omega)$ and $\langle \mathbf{g}, \mathbf{h} \rangle_{L^2(\Omega)} := \int_\Omega \mathbf{g} \cdot \mathbf{h} dx$ for all $\mathbf{g}, \mathbf{h} \in L^2(\Omega)^d$, cf. [Soh01, Chapter II, Lemma 2.5.1]. Here $p \in W^1_2(\Omega) \cap L^2_{(0)}(\Omega)$ is the solution of the weak Neumann problem

$$(\nabla p, \nabla \varphi)_{L^2(\Omega)} = (\mathbf{f}, \nabla \varphi)_{L^2(\Omega)} \qquad \text{for all } \varphi \in C^\infty(\overline{\Omega}).$$

Decomposition (3.89) is called the Helmholtz decomposition of $\mathbf{f}$. [Soh01, Chapter II, Lemma 2.5.1] leads to the definition of the bounded linear operator

$$\begin{aligned}\mathbb{P}_\sigma : L^2(\Omega)^d &\to L^2_\sigma(\Omega),\\ \mathbf{f} &\mapsto \mathbb{P}_\sigma \mathbf{f} := \mathbf{f}_0\end{aligned}$$

with $\mathbf{f}_0$ as in (3.89). The projection $\mathbb{P}_\sigma$ is called the Helmholtz projection of $L^2(\Omega)^d$ onto $L^2_\sigma(\Omega)$. From (3.65) it follows that $\partial_t(\mathbb{P}_\sigma(\widetilde{\rho \mathbf{v}}^N))$ is bounded in $L^1(0,T;H^{-2}(\Omega)^d)$ due to the following bounds:

$$\begin{aligned}\rho_h^N \mathbf{v}^N \otimes \mathbf{v}^N &\text{ is bounded in } L^2(0,T;L^{\frac{3}{2}}(\Omega)^{d\times d}),\\ \mathbf{v}^N \otimes \tilde{\mathbf{J}}^N &\text{ is bounded in } L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega)),\\ \mu^N \nabla \varphi_h^N &\text{ is bounded in } L^2(0,T;L^{\frac{3}{2}}(\Omega)^d),\\ \frac{h(q^N)}{\varepsilon} W'(\varphi_h^N)\nabla\varphi_h^N &\text{ is bounded in } L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega)^d),\\ 2\eta(\varphi_h^N) D\mathbf{v}^N &\text{ is bounded in } L^2(0,T;L^2(\Omega)^{d\times d}).\end{aligned}$$

Moreover, it remains to prove the boundedness of $\langle R^N \mathbf{v}^N, \boldsymbol{\psi}\rangle$ for all $\boldsymbol{\psi} \in L^\infty(0,T;W^1_6(\Omega)^d)$. But we postpone this estimate and show the bounds above in detail.

i) As it holds $\mathbf{v}^N \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;L^6(\Omega)^d)$ and $\rho_h^N \in L^\infty(Q_T)$, we can conclude that $\mathbf{v}_k^N \mathbf{v}_l^N$ is bounded in $L^2(0,T;L^{\frac{3}{2}}(\Omega)^d)$ for all $k,l = 1,...,d$.

ii) By definition of $\tilde{\mathbf{J}}^N$ we need to study products of the form $\mathbf{v}_k^N \rho'(\varphi_h^N)\tilde{m}(\varphi_h^N)\partial_{x_l}\mu^N$ for $k,l = 1,...,d$, where it holds $\rho'(\varphi_h^N), \tilde{m}(\varphi_h^N) \in L^\infty(Q_T)$. Due to $\nabla\mu^N \in L^2(0,T;L^2(\Omega))$ and since we have the embedding $\mathbf{v}^N \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1(\Omega)^d) \hookrightarrow L^4(0,T;L^3(\Omega)^d)$, cf. (3.87), this implies the statement.

iii) From $\mu^N \in L^2(0,T;L^6(\Omega))$ and $\nabla\varphi_h^N \in L^\infty(0,T;L^2(\Omega)^d)$ we can conclude $\mu^N\nabla\varphi_h^N \in L^2(0,T;L^{\frac{3}{2}}(\Omega)^d)$.

iv) Due to the growth conditions for W' and h we get the estimate

$$\left|\frac{h(q^N)}{\varepsilon}W'(\varphi_h^N)\nabla\varphi_h^N\right| \le \frac{C}{\varepsilon}\left(|q^N|+1\right)\left(|\varphi_h^N|^2+1\right)\left|\nabla\varphi_h^N\right| \quad \text{ in } (0,T)\times\Omega.$$

First of all we have $q^N \in L^2(0,T;L^6(\Omega))$. Moreover, we use Theorem 2.32 with $\theta = \frac{1}{2}$ to obtain

$$\varphi^N \in L^4(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;L^6(\Omega)) \hookrightarrow L^8(0,T;L^{12}(\Omega)),$$

where $\varphi^N \in L^4(0,T;L^\infty(\Omega))$ has been shown in (3.78). Finally, we use $\nabla\varphi_h^N \in L^\infty(0,T;L^2(\Omega)^d)$. Hence, the latter embedding yields the boundedness.

v) As it holds $\mathbf{v}^N \in L^2(0,T;H^1(\Omega)^d)$ and η is a bounded function, this bound is obvious.

Altogether we have shown that all these terms are bounded in $L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega))$. In particular this implies that they are bounded in $L^1(0,T;L^{\frac{6}{5}}(\Omega))$ and therefore we can allow for these terms in (3.65) for test functions $\boldsymbol{\psi} \in L^1(0,T;W_6^1(\Omega)^d)$. Finally, we need to study the terms $\langle \frac{R^N \mathbf{v}^N}{2}, \boldsymbol{\psi}\rangle$ and $\delta\langle \Delta \mathbf{v}^N, \Delta\boldsymbol{\psi}\rangle_{L^2(Q_T)}$. To this end, we need to show $\partial_{t,h}^- \varphi^N \rightharpoonup \partial_t \varphi$ in $L^2(0,T;L^2(\Omega))$.

Remark 3.11. *Note that up to this point of the proof we did not use the boundedness of $(\partial_{t,h}^- \varphi^N)_{N\in\mathbb{N}}$ and $(\Delta \mathbf{v}^N)_{N\in\mathbb{N}}$ in $L^2(0,\infty;L^2(\Omega))$, cf. (3.72) viii) and (3.72) ii). Remember that the solution also depends on δ and in the final step of the proof we will pass to the limit $\delta \to 0$. Hence, we will not be able to conclude that $(\partial_t\varphi^\delta)_{\delta>0}$ and $(\Delta\mathbf{v}^\delta)_{\delta>0}$ are bounded in $L^2(0,\infty;L^2(\Omega))$. But the following estimate together with the convergence of $\langle \frac{R^N\mathbf{v}^N}{2}, \boldsymbol{\psi}\rangle$ to $\langle \frac{R\mathbf{v}}{2}, \boldsymbol{\psi}\rangle$ are the only points of the proof where we need the boundedness of $(\partial_{t,h}^- \varphi^N)_{N\in\mathbb{N}}$ and $(\Delta\mathbf{v}^N)_{N\in\mathbb{N}}$ in $L^2(0,\infty;L^2(\Omega))$. To prove the convergence of $\langle \frac{R^\delta \mathbf{v}^\delta}{2}, \boldsymbol{\psi}\rangle$ as $\delta \to 0$, we will not need boundedness of $(\partial_t\varphi^\delta)_{\delta>0}$ and $(\Delta\mathbf{v}^\delta)_{\delta>0}$ in $L^2(0,\infty;L^2(\Omega))$ since we will use the identity $R^\delta = -\nabla\frac{\partial\rho(\varphi^\delta)}{\partial\varphi^\delta}\cdot(\tilde{m}(\varphi^\delta)\nabla\mu^\delta)$. All the other estimates and convergence results can be derived analogously for the solutions $(\mathbf{v}^\delta, \varphi^\delta, \mu^\delta, q^\delta)$.*

From (3.72) viii) it follows that there exists a subsequence of $(\varphi^N)_{N\in\mathbb{N}}$ such that $\partial_{t,h}^-\varphi^N \rightharpoonup \Phi$ in $L^2((0,T);L^2(\Omega))$. Now we want to show $\Phi = \partial_t\varphi$. To this end, let $\psi \in C_0^\infty((0,T)\times\Omega)$ be given. Since Φ is the weak limit of $\partial_{t,h}^-\varphi^N$ in $L^2((0,T);L^2(\Omega))$ it holds on the one hand

$$\lim_{N\to\infty} \langle \partial_{t,h}^-\varphi^N, \psi\rangle = \langle \Phi, \psi\rangle.$$

But on the other hand it also holds

$$\begin{aligned}\lim_{N\to\infty} \langle \partial_{t,h}^-\varphi^N, \psi\rangle &= \lim_{N\to\infty} \int_0^T\int_\Omega \partial_{t,h}^-\varphi^N \psi \, dx\, dt = \lim_{N\to\infty} -\int_0^T\int_\Omega \varphi^N \partial_{t,h}^+\psi\, dx\, dt \\ &= -\int_0^T\int_\Omega \varphi\partial_t\psi\, dx\, dt = \langle \partial_t\varphi, \psi\rangle\end{aligned}$$

for all $\psi \in C_0^\infty((0,T)\times\Omega)$. Hence, it follows $\Phi = \partial_t\varphi$ and therefore we can conclude

$$\partial_{t,h}^-\varphi^N \rightharpoonup \partial_t\varphi \qquad \text{in } L^2((0,T)\times\Omega). \tag{3.90}$$

Now we can use this to estimate the term $\langle \frac{R^N \mathbf{v}^N}{2}, \boldsymbol{\psi} \rangle$. It holds

$$\begin{aligned}
\left\langle \frac{R^N \mathbf{v}^N}{2}, \boldsymbol{\psi} \right\rangle &= \frac{1}{2} \int_0^T \int_\Omega \frac{\rho^N - \rho_h^N}{h} \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt - \frac{1}{2} \int_0^T \int_\Omega (\rho_h^N \mathbf{v}^N + \tilde{\mathbf{J}}^N) \cdot \nabla (\mathbf{v}^N \cdot \boldsymbol{\psi})\, dx\, dt \\
&= \frac{1}{2} \int_0^T \int_\Omega \frac{\rho^N - \rho_h^N}{h} \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt - \frac{1}{2} \int_0^T \int_\Omega \left(\rho_h^N \mathbf{v}^N\right) \cdot \nabla \left(\mathbf{v}^N \cdot \boldsymbol{\psi}\right) dx\, dt \\
&+ \frac{1}{2} \int_0^T \int_\Omega \left(\rho'(\varphi_h^N) \tilde{m}(\varphi_h^N) \nabla \mu^N\right) \cdot \nabla \left(\mathbf{v}^N \cdot \boldsymbol{\psi}\right) dx\, dt \qquad (3.91)
\end{aligned}$$

for $\boldsymbol{\psi} \in L^\infty(0,T; W_6^1(\Omega)^d)$. The most interesting terms are the second and the third one. For $\mathbf{v}^N(t) \in H^1(\Omega)^d$ and $\boldsymbol{\psi}(t) \in W_6^1(\Omega)^d$ for a.e. $t \in (0,T)$, the theorem about the multiplication of Sobolev functions, cf. Theorem 2.17, yields $(\mathbf{v}^N \cdot \boldsymbol{\psi})(t) \in W_2^1(\Omega)$ for a.e. $t \in (0,T)$ together with the estimate

$$||(\mathbf{v}^N \cdot \boldsymbol{\psi})(t)||_{W_2^1(\Omega)} \leq ||\mathbf{v}^N(t)||_{H^1(\Omega)} ||\boldsymbol{\psi}(t)||_{W_6^1(\Omega)}.$$

As it holds $\mathbf{v}^N \in L^2(0,T; H^1(\Omega)^d)$ and $\boldsymbol{\psi} \in L^\infty(0,T; W_6^1(\Omega)^d)$, the estimate above implies $\left(\mathbf{v}^N \cdot \boldsymbol{\psi}\right) \in L^2(0,T; W_r^1(\Omega))$ for $1 \leq r \leq 2$. In particular we obtain $\left(\mathbf{v}^N \cdot \boldsymbol{\psi}\right) \in L^2(0,T; H^1(\Omega))$ and therefore $\nabla \left(\mathbf{v}^N \cdot \boldsymbol{\psi}\right) \in L^2(0,T; L^2(\Omega))$. Now we use this to show that all integrals in (3.91) are bounded.

i) The first integral is bounded due to the boundedness of $\frac{\rho^N - \rho_h^N}{h}$ in $L^2(Q_T)$ and the boundedness of $\mathbf{v}^N \in L^2(0,T; L^6(\Omega)^d)$ and $\boldsymbol{\psi} \in L^\infty(0,T; W_6^1(\Omega)^d)$. We have already proven the last two statements. For the boundedness of $\frac{\rho^N - \rho_h^N}{h}$ in $L^2(Q_T)$ we show

$$\frac{\rho^N - \rho_h^N}{h} \rightharpoonup \rho'(\varphi) \partial_t \varphi \qquad \text{in } L^2(Q_T), \qquad (3.92)$$

which implies the boundedness in $L^2(Q_T)$, cf. [Wer08, Corollary IV.2.3]. To this end, we set

$$M(a,b) := \begin{cases} \rho'(a), & \text{if } a = b, \\ \frac{\rho(a) - \rho(b)}{a-b}, & \text{if } a \neq b. \end{cases}$$

Since ρ is differentiable, we can conclude $\lim_{b \to a} M(a,b) = M(a,a) = \rho'(a)$. Moreover, it holds $\rho(a) - \rho(b) = M(a,b)(a-b)$ for all $a, b \in \mathbb{R}$. Due to the boundedness of $M(\varphi^N, \varphi_h^N)$ in $L^\infty(Q_T)$ and $\varphi^N(t,x) \to \varphi(t,x)$ a.e. in Q_T we obtain $M(\varphi^N, \varphi_h^N) u \to \rho'(\varphi) u$ in $L^2(Q_T)$ for every $u \in L^2(Q_T)$. Together with $\frac{\varphi^N - \varphi_h^N}{h} = \partial_{t,h}^- \varphi^N \rightharpoonup \partial_t \varphi$ in $L^2(Q_T)$ we can deduce

$$\int_{Q_T} \frac{\rho^N - \rho_h^N}{h} u\, d(x,t) = \int_{Q_T} M(\varphi^N, \varphi_h^N) \frac{\varphi^N - \varphi_h^N}{h} u\, d(x,t) \to \int_{Q_T} \rho'(\varphi) \partial_t \varphi u\, d(x,t)$$

for every $u \in L^2(Q_T)$ as $N \to \infty$, which shows the statement.

ii) The boundedness of the second integral follows from $\rho_h^N \in L^\infty(Q_T)$ together with $\mathbf{v}^N \in L^2(0,T;L^6(\Omega))$ and $\nabla\left(\mathbf{v}^N \cdot \boldsymbol{\psi}\right) \in L^2(Q_T)$.

iii) For the last integral we know $\rho'(\varphi_h^N), \tilde{m}(\varphi_h^N) \in L^\infty(Q_T)$. Moreover, $\nabla\mu^N$ is bounded in $L^2(Q_T)$ and $\nabla\left(\mathbf{v}^N \cdot \boldsymbol{\psi}\right) \in L^2(Q_T)$. Hence, the product is in $L^1(Q_T)$ and therefore the functional is bounded in $L^1(0,T;W_6^1(\Omega)')$.

For the remaining term $\delta \int\limits_{Q_T} \Delta\mathbf{v}^N \cdot \Delta\boldsymbol{\psi}\, d(x,t)$ we need to choose $\boldsymbol{\psi} \in L^2(0,T;V(\Omega))$, where we set

$$V(\Omega) := H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega).$$

Thus we can allow for all terms in (3.65) for test functions $\boldsymbol{\psi} \in L^\infty(0,T;H^2(\Omega)^d)$. As we know $\tilde{\mathbf{v}}^N \in L^2(0,T;H^1(\Omega)^d)$ and $\mathbb{P}_\sigma \in \mathcal{L}(H^1(\Omega))$, we can finally conclude

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) \text{ is bounded in } L^2(0,T;H^1(\Omega)^d) \cap W_1^1(0,T;V(\Omega)'). \tag{3.93}$$

Note that when we pass to the limit $\delta \to 0$, we use that the energy estimate yields the boundedness of the term $\delta \int\limits_{Q_T} |\Delta\mathbf{v}^\delta|^2\, d(x,t)$. Hence, we can conclude that $\delta^{\frac{1}{2}}\Delta\mathbf{v}^\delta$ is bounded in $L^2(0,T;L^2(\Omega))$ and therefore $\delta\Delta^2\mathbf{v}^\delta = \delta^{\frac{1}{2}}\Delta\delta^{\frac{1}{2}}\Delta\mathbf{v}^\delta \to 0$ in $L^2(0,T;H^{-2}(\Omega))$ as $\delta \to 0$. Thus we can also show (3.93) when we study the case $\delta \to 0$.

Using Aubin-Lions, cf. Theorem 2.33, with $X_0 = H^1(\Omega)^d \cap L_\sigma^2(\Omega)$, $X = L_\sigma^2(\Omega)$ and $X_1 = V(\Omega)'$, $p = 2$ and $q = 1$, we get

$$L^2(0,T;H^1(\Omega)^d \cap L_\sigma^2(\Omega)) \cap W_1^1(0,T;V(\Omega)') \hookrightarrow\hookrightarrow L^2(0,T;L_\sigma^2(\Omega)) \text{ is compact.}$$

In particular there exists $\boldsymbol{\omega} \in L^2(0,T;L_\sigma^2(\Omega))$ and a subsequence such that

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) \to \boldsymbol{\omega} \text{ in } L^2(0,T;L_\sigma^2(\Omega)) \tag{3.94}$$

for all $0 < T < \infty$. As we know that $(\widetilde{\rho\mathbf{v}}^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,\infty;L^2(\Omega)^d)$ is bounded, we can even deduce $\boldsymbol{\omega} \in L^\infty(0,\infty;L^2(\Omega)^d)$. This can be proven analogously as for $\tilde{\varphi}$ before. Furthermore, we can deduce $\widetilde{\rho\mathbf{v}}^N \rightharpoonup \rho\mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$, what we will prove in the following. To this end, we remember

$$\widetilde{\rho\mathbf{v}}^N(t) := (\rho^N\mathbf{v}^N)(t) + (-(k+1)h + t)\frac{(\rho^N\mathbf{v}^N)(t) - (\rho^N\mathbf{v}^N)(t-h)}{h}$$

for all $t \in [kh, (k+1)h)$. So let $\boldsymbol{\psi} \in C_0^\infty(0,T;V(\Omega))$ be given. Then it holds

$$\begin{aligned}
\int_0^T \int_\Omega \widetilde{\rho \mathbf{v}}^N \cdot \boldsymbol{\psi}\, dx\, dt &= \int_0^T \int_\Omega \rho^N \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt \\
&\quad + \int_0^T \int_\Omega (-(k(t)+1)h + t) \frac{(\rho^N \mathbf{v}^N)(t) - (\rho^N \mathbf{v}^N)(t-h)}{h} \cdot \boldsymbol{\psi}(t)\, dx\, dt \\
&= \int_0^T \int_\Omega \rho^N \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt + \int_0^T \int_\Omega \frac{(-(k(t)+1)h + t)}{h} \rho^N \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt \\
&\quad - \int_0^T \int_\Omega \frac{-(k(t)+1)h + t}{h} (\rho^N \mathbf{v}^N)(t-h) \cdot \boldsymbol{\psi}(t)\, dx\, dt \\
&= \int_0^T \int_\Omega \rho^N \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt + \int_0^T \int_\Omega \frac{(-(k(t)+1)h + t)}{h} \rho^N \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt \\
&\quad - \int_{-h}^{T-h} \int_\Omega \frac{-(k(t)+1)h + t}{h} \rho^N \mathbf{v}^N(t) \cdot \boldsymbol{\psi}(t+h)\, dx\, dt
\end{aligned}$$

for $k(t) = \left\lfloor \frac{t}{h} \right\rfloor$. For the last integral it is not obvious that the prefactor $\frac{-(k(t)+1)h+t}{h}$ stays the same although we substitute t by $t+h$. But when we substitute t by $t+h$ we also have to substitute $-(k(t)+1)h$ by $-(k(t)+2)h$. Note that therefore we allow $k(t) \in \mathbb{N}_0 \cup \{-1\}$ since we integrate from $-h$ to $T-h$ now. Splitting the last two integrals and combining the appropriate terms yields

$$\begin{aligned}
\int_0^T \int_\Omega \widetilde{\rho \mathbf{v}}^N \cdot \boldsymbol{\psi}\, dx\, dt &= \int_0^{T-h} \int_\Omega (-(k(t)+1)h + t) \rho^N(t) \mathbf{v}^N(t) \cdot \frac{\boldsymbol{\psi}(t) - \boldsymbol{\psi}(t+h)}{h}\, dx\, dt \\
&\quad + \int_{T-h}^T \int_\Omega \frac{(-(k(t)+1)h + t)}{h} \rho^N \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt \\
&\quad - \int_{-h}^0 \int_\Omega \frac{-(k(t)+1)h + t}{h} \rho^N(t) \mathbf{v}^N(t) \cdot \boldsymbol{\psi}(t+h)\, dx\, dt + \int_0^T \int_\Omega \rho^N \mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt.
\end{aligned}$$

Since $\boldsymbol{\psi} \in C_0^\infty(0,T;V(\Omega))$ has compact support, the second and third integral vanish as $N \to \infty$, respectively $h \to 0$. Moreover, the first integral also vanishes as it holds

i) $|-(k(t)+1)h + t| \le h \to 0$ as $N \to \infty$.

ii) $(\rho^N \mathbf{v}^N)_{N\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(\Omega)^d)$.

iii) $\frac{\boldsymbol{\psi}(t)-\boldsymbol{\psi}(t+h)}{h} \to \partial_t \boldsymbol{\psi}(t)$ since it holds $\boldsymbol{\psi} \in C_0^\infty(0,T;V(\Omega))$.

As we already know $\mathbf{v}^N \rightharpoonup \mathbf{v}$ in $L^2(0,T;H^1(\Omega)^d)$ and $\rho(\varphi^N(t,x)) \to \rho(\varphi(t,x))$ a.e. in $(0,T)\times\Omega$ due to $\varphi^N \to \varphi$ in $L^2(0,T;H^1(\Omega))$, cf. (3.81), we can conclude

$$\int_0^T \int_\Omega \widetilde{\rho\mathbf{v}}^N \cdot \boldsymbol{\psi}\, dx\, dt \to \int_0^T \int_\Omega \rho\mathbf{v} \cdot \boldsymbol{\psi}\, dx\, dt$$

for $N \to \infty$. Since $C_0^\infty((0,T)\times\Omega)^d$ is dense in $L^2(0,T;L^2(\Omega)^d)$ we get

$$\widetilde{\rho\mathbf{v}}^N \rightharpoonup \rho\mathbf{v} \qquad \text{in } L^2(0,T;L^2(\Omega)^d).$$

Since $\mathbb{P}_\sigma : L^2(0,T;L^2(\Omega)^d) \to L^2(0,T;L^2_\sigma(\Omega))$ is linear and bounded, it is also weakly continuous, cf. Lemma 2.3, i.e.,

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) \rightharpoonup \mathbb{P}_\sigma(\rho\mathbf{v}) = \boldsymbol{\omega}. \tag{3.95}$$

Due to $\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) \to \boldsymbol{\omega}$ in $L^2(0,T;L^2(\Omega)^d)$, cf. (3.94), this implies

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) \to \mathbb{P}_\sigma(\rho\mathbf{v}) \qquad \text{in } L^2(0,T;L^2(\Omega)^d).$$

Now we can use this to prove the strong convergence $\mathbf{v}^N \to \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$. To this end, we calculate

$$\int_0^T \int_\Omega \rho^N |\mathbf{v}^N|^2\, dx\, dt = \int_0^T \int_\Omega \mathbb{P}_\sigma(\rho^N\mathbf{v}^N)\cdot\mathbf{v}^N\, dx\, dt \to \int_0^T \int_\Omega \mathbb{P}_\sigma(\rho\mathbf{v})\cdot\mathbf{v} = \int_0^T \int_\Omega \rho|\mathbf{v}|^2\, dx\, dt, \tag{3.96}$$

where we used $\mathbb{P}_\sigma(\rho^N\mathbf{v}^N) \to \mathbb{P}_\sigma(\rho\mathbf{v})$ in $L^2(0,T;L^2(\Omega)^d)$, $\mathbf{v}^N \rightharpoonup \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$ and the fact that $\mathbf{v}^N$ and $\mathbf{v}$ are divergence-free, more precisely the part of the Helmholtz decomposition that can be written as $\nabla\tilde{p}$ for $\tilde{p} \in L^2_{loc}(\Omega)$ vanishes when testing with $\mathbf{v}^N$ or $\mathbf{v}$. In the following we prove the strong convergence of $\mathbb{P}_\sigma(\rho^N\mathbf{v}^N)$ to $\mathbb{P}_\sigma(\rho\mathbf{v})$ in $L^2(0,T;L^2(\Omega)^d)$. To this end, we use

$$||\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) - \mathbb{P}_\sigma(\rho^N\mathbf{v}^N)||_{L^1(0,T;V(\Omega)')} \leq h||\partial_t\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)||_{L^1(0,T;V(\Omega)')} \to 0,$$

as $N\to\infty$, cf. (3.74). Since $\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)$ and $\mathbb{P}_\sigma(\rho^N\mathbf{v}^N)$ are bounded in $L^\infty(0,T;L^2(\Omega)^d)$ we can conclude

$$\begin{aligned}
&||\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) - \mathbb{P}_\sigma(\rho^N\mathbf{v}^N)||_{L^2(0,T;(H^1_0(\Omega)^d\cap L^2_\sigma(\Omega))')} \\
&\quad \leq C||\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) - \mathbb{P}_\sigma(\rho^N\mathbf{v}^N)||^{\frac12}_{L^\infty(0,T;L^2(\Omega))}||\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) - \mathbb{P}_\sigma(\rho^N\mathbf{v}^N)||^{\frac12}_{L^1(0,T;V(\Omega)')},
\end{aligned}$$

where we used the interpolation $(L^2_\sigma(\Omega), V(\Omega))_{\frac{1}{2},2} = H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, cf. [Abe07, Lemma 5.2.7], which implies by duality $(L^2_\sigma(\Omega), V(\Omega)')_{\frac{1}{2},2} = (H^1_0(\Omega)^d \cap L^2_\sigma(\Omega))'$, cf. [Abe16, Theorem 2.43]. Here we note that the first term on the right-hand side is bounded while the second term converges to 0 as $N \to \infty$. Hence, we obtain

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) - \mathbb{P}_\sigma(\rho^N \mathbf{v}^N) \to 0 \qquad \text{in } L^2(0,T;(H^1_0(\Omega)^d \cap L^2_\sigma(\Omega))')$$

as $N \to \infty$. Furthermore, we use that for every $u \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$ we have the estimate $||u||^2_{L^2(\Omega)} = (u,u)_{L^2(\Omega)} \le ||u||_{(H^1_0(\Omega)^d \cap L^2_\sigma(\Omega))'} ||u||_{H^1(\Omega)}$. With this estimate we get

$$\begin{aligned} &||\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)(t) - \mathbb{P}_\sigma(\rho^N \mathbf{v}^N)(t)||_{L^2(\Omega)} \\ &\quad \le C||\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)(t) - \mathbb{P}_\sigma(\rho^N \mathbf{v}^N)(t)||^{\frac{1}{2}}_{H^1(\Omega)} ||\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)(t) - \mathbb{P}_\sigma(\rho^N \mathbf{v}^N)(t)||^{\frac{1}{2}}_{(H^1_0(\Omega)^d \cap L^2_\sigma(\Omega))'} \end{aligned}$$

for a.e. $t \in (0,T)$. Together with the boundedness of $\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)$, $\mathbb{P}_\sigma(\rho^N \mathbf{v}^N)$ in $L^2(0,T;H^1_\sigma(\Omega))$, cf. (3.93), we can conclude

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) - \mathbb{P}_\sigma(\rho^N \mathbf{v}^N) \to 0 \qquad \text{in } L^2(0,T;L^2(\Omega)^d)$$

as $N \to \infty$. Since we already know $\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) \to \mathbb{P}_\sigma(\rho\mathbf{v})$ in $L^2(0,T;L^2(\Omega)^d)$ this implies

$$\mathbb{P}_\sigma(\rho^N \mathbf{v}^N) \to \mathbb{P}_\sigma(\rho\mathbf{v}) \qquad \text{in } L^2(0,T;L^2(\Omega)^d),$$

what we wanted to show. From equation (3.96) it follows $(\rho^N)^{\frac{1}{2}}\mathbf{v}^N \to (\rho)^{\frac{1}{2}}\mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$. Moreover, we know $\rho^N(t,x) \to \rho(t,x)$ a.e. in $(0,T)\times\Omega$ due to $\varphi^N \to \varphi$ in $L^2(0,T;H^1(\Omega))$. Altogether this implies the strong convergence of $\mathbf{v}^N$ to $\mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$ as it holds

$$\mathbf{v}^N = (\rho^N)^{-\frac{1}{2}}((\rho^N)^{\frac{1}{2}}\mathbf{v}^N) \to \mathbf{v} \qquad \text{in } L^2(0,T;L^2(\Omega)^d). \tag{3.97}$$

This implies $\mathbf{v}^N(t,x) \to \mathbf{v}(t,x)$ a.e. in $(0,T)\times\Omega$ for a suitable subsequence. Note that in [Lio96, Section 2.1] and [ADG13] similar arguments were used to prove the strong convergence for the velocity $\mathbf{v}^N$.
Due to $\mathbf{v}^N \in L^2(0,T;H^1(\Omega)^d) \hookrightarrow L^2(0,T;L^6(\Omega)^d)$ for $d=2,3$ we can also show

$$\mathbf{v}^N \to \mathbf{v} \qquad \text{in } L^2(0,T;L^{6-\varepsilon}(\Omega)^d) \text{ for every } 0<\varepsilon\le 5.$$

Moreover, we can conclude

$$\mathbf{v}^N \to \mathbf{v} \qquad \text{in } L^q(0,T;L^2(\Omega)^d) \text{ for every } 1 \le q < \infty$$

since we can estimate

$$\begin{aligned} \int_0^T ||\mathbf{v}^N - \mathbf{v}||^q_{L^2(\Omega)}\,dt &\le \int_0^T ||\mathbf{v}^N - \mathbf{v}||^{q-2}_{L^2(\Omega)} ||\mathbf{v}^N - \mathbf{v}||^2_{L^2(\Omega)}\,dt \\ &\le ||\mathbf{v}^N - \mathbf{v}||^{q-2}_{L^\infty(0,T;L^2(\Omega))} ||\mathbf{v}^N - \mathbf{v}||_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

From Theorem 2.28 it follows

$$(H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))_{\frac{1}{4},1} = B^{\frac{3}{2}}_{21}(\mathbb{R}^d)$$

for all $d \in \mathbb{N}$. In the case $d = 2, 3$, Theorem 2.20 yields $B^{\frac{3}{2}}_{21}(\mathbb{R}^d) \hookrightarrow B^0_{\infty 1}(\mathbb{R}^d)$. Due to (2.3) it holds $B^0_{\infty 1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ and therefore Lemma 2.27 implies

$$||f||_{L^\infty(\mathbb{R}^d)} \leq C||f||^{\frac{3}{4}}_{H^2(\mathbb{R}^d)}||f||^{\frac{1}{4}}_{L^2(\mathbb{R}^d)}$$

for all $f \in H^2(\mathbb{R}^d)$ and $d = 2, 3$. Since there exists an extension operator $E : H^k(\Omega) \to H^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}_0$, cf. [Ste70, Chapter VI, Section 3.2], we can conclude

$$||\mathbf{v}^N(t) - \mathbf{v}(t)||_{L^\infty(\Omega)} \leq C||\mathbf{v}^N(t) - \mathbf{v}(t)||^{\frac{3}{4}}_{H^2(\Omega)}||\mathbf{v}^N(t) - \mathbf{v}(t)||^{\frac{1}{4}}_{L^2(\Omega)}$$

for a.e. $t \in (0, T)$. Hence, it holds

$$\mathbf{v}^N \to \mathbf{v} \qquad \text{in } L^q(0, T; L^\infty(\Omega)^d) \text{ for every } 1 \leq q < \frac{8}{3}, \tag{3.98}$$

where we used $\mathbf{v}^N \to \mathbf{v}$ in $L^p(0, T; L^2(\Omega)^d)$ for every $1 \leq p < \infty$ and the boundedness of $\mathbf{v}^N$ and $\mathbf{v}$ in $L^2(0, T; H^2(\Omega)^d)$. Note that when we pass to the limit $\delta \to 0$, we can not conclude $\mathbf{v}^\delta \to \mathbf{v}$ in $L^q(0, T; L^\infty(\Omega)^d)$ for all $1 \leq q < \frac{8}{3}$ since in this case we can not deduce $\mathbf{v}^\delta, \mathbf{v} \in L^2(0, T; H^2(\Omega)^d)$.

In the following we show that $\mathbf{v}$ satisfies the initial condition $\mathbf{v}_{|t=0} = \mathbf{v}_0$. This is proven with the same arguments as in [ADG13]. Nevertheless, we also write this proof in detail for the sake of completeness since we refer to it in the last section when we study the limit $\delta \to 0$. We remember the following bounds:

$$\begin{aligned} &\mathbb{P}_\sigma(\widetilde{\rho \mathbf{v}}^N) \text{ is bounded in } W^1_1(0, T; H^{-2}(\Omega)) \hookrightarrow C([0, T]; H^{-2}(\Omega)), \\ &\mathbb{P}_\sigma(\widetilde{\rho \mathbf{v}}^N) \text{ is bounded in } L^\infty(0, T; L^2(\Omega)^d), \end{aligned}$$

where the first embedding follows from Lemma 2.22. Due to these bounds we were already able to conclude $\mathbb{P}_\sigma(\widetilde{\rho \mathbf{v}}^N) \to \boldsymbol{\omega}$ in $L^2(0, T; L^2(\Omega)^d)$, cf. (3.94). Moreover, we get

$$\boldsymbol{\omega} \in BC_w([0, T]; L^2(\Omega)^d) \tag{3.99}$$

due to Lemma 2.23. Before we proceed with the proof we have a look at the auxiliary problem

$$\begin{aligned} -\mathrm{div}\left(\frac{1}{\rho(t)}\nabla p(t)\right) &= \mathrm{div}\left(\frac{1}{\rho(t)}\boldsymbol{\omega}(t)\right) && \text{in } \Omega, \\ \nabla p(t) \cdot n &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.100}$$

with $p \in L^2(0,T;H^1_{(0)}(\Omega))$ and $t \in (0,T)$. Its weak formulation is given by

$$-\int_\Omega \frac{1}{\rho(t)} \nabla p(t) \cdot \nabla \phi dx = \int_\Omega \frac{1}{\rho(t)} \boldsymbol{\omega}(t) \cdot \nabla \phi dx \tag{3.101}$$

for all $\phi \in H^1(\Omega)$ and a.e. $t \in [0,T]$. The Lax-Milgram theorem, cf. Theorem 2.1, yields the existence of a unique solution together with the estimate

$$||\nabla p(t)||_{L^2(\Omega)} \leq C||\boldsymbol{\omega}(t)||_{L^2(\Omega)}.$$

Note that the constant C does not depend on t and that we used that there exist constants $c_1, c_2 > 0$ such that $c_1 \leq \rho \leq c_2$. From this estimate we can conclude $\nabla p \in BC_w([0,T];L^2(\Omega)^d)$ by the following arguments:
Let $(t_n)_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence such that $t_n \to t_0$ for given $t_0 \in [0,T]$. Since $\boldsymbol{\omega}$ is in $BC_w([0,T];L^2(\Omega)^d)$ we know that $\boldsymbol{\omega}(t_n)$ is bounded in $L^2(\Omega)$ and due to the estimate above this is also true for $\nabla p(t_n)$. Thus there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ such that $\nabla p(t_{n_k}) \rightharpoonup \tilde{\mathbf{q}}$ in $L^2(\Omega)^d$ for some $\tilde{\mathbf{q}} \in L^2(\Omega)$, more precisely

$$-\int_\Omega \nabla p(t_{n_k}) \cdot \nabla \phi dx \to -\int_\Omega \tilde{\mathbf{q}} \cdot \nabla \phi dx$$

as $k \to \infty$. But as we know that $\{\nabla q : \ q \in H^1_{(0)}(\Omega)\}$ is a closed subspace in $L^2(\Omega)^d$ we can conclude that there exists $q \in H^1_{(0)}(\Omega)$ such that $\tilde{\mathbf{q}} = \nabla q$. Moreover, it holds

$$-\int_\Omega \nabla p(t_{n_k}) \cdot \nabla \phi dx = \int_\Omega \boldsymbol{\omega}(t_{n_k}) \cdot \nabla \phi dx \to \int_\Omega \boldsymbol{\omega}(t_0) \cdot \nabla \phi dx = -\int_\Omega \nabla p(t_0) \cdot \nabla \phi dx,$$

where we used that $\boldsymbol{\omega}$ is in $BC_w([0,T];L^2(\Omega)^d)$. From the fact that the problem (3.101) has a unique solution it follows $\nabla q = \nabla p(t_0)$. Since $(\nabla p(t_{n_k}))_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)^d$ and these arguments are true for any weakly convergent subsequence, we can even conclude $\nabla p(t_n) \rightharpoonup \nabla p(t_0)$, cf. [Růž04, Chapter III, Lemma 0.3]. This implies

$$\nabla p \in BC_w([0,T];L^2(\Omega)^d). \tag{3.102}$$

In (3.95) we already showed $\boldsymbol{\omega} = \mathbb{P}_\sigma(\rho\mathbf{v})$ in $L^2(0,T;L^2(\Omega)^d)$ for every $0 < T < \infty$. This implies

$$\boldsymbol{\omega}(t) = \mathbb{P}_\sigma(\rho(t)\mathbf{v}(t)) \qquad \text{a.e. in } (0,T).$$

Therefore, the Helmholtz decomposition (3.89) yields

$$\boldsymbol{\omega}(t) = \rho(t)\mathbf{v}(t) - \nabla\tilde{p}(t) \qquad \text{a.e. in } (0,T), \tag{3.103}$$

where $\tilde{p}(t)$ is a weak solution of the auxiliary problem

$$\begin{aligned} \operatorname{div}(\nabla\tilde{p}(t)) &= \operatorname{div}(\rho(t)\mathbf{v}(t)) && \text{in } \Omega, \\ \nabla\tilde{p}(t)\cdot n &= 0 && \text{on } \partial\Omega. \end{aligned}$$

When we divide (3.103) by $\rho(t)$, we obtain

$$\mathbf{v}(t) = \frac{1}{\rho(t)}\boldsymbol{\omega}(t) + \frac{1}{\rho(t)}\nabla\tilde{p}(t) \qquad \text{a.e. in } (0,T).$$

Multiplying this with $\nabla\psi$ for $\psi \in H^1(\Omega)$ and integrating over the domain Ω yields

$$\int_\Omega \frac{1}{\rho(t)}\boldsymbol{\omega}(t)\cdot\nabla\psi\,dx = -\int_\Omega \frac{1}{\rho(t)}\nabla\tilde{p}(t)\cdot\nabla\psi\,dx \qquad \text{for all } \psi \in H^1(\Omega)$$

for a.e. $t \in (0,T)$ since $\mathbf{v}$ is a divergence-free vector field. But this is the weak formulation (3.101) of the auxiliary problem (3.100). Thus we get $\nabla\tilde{p}(t) = \nabla p(t)$ for a.e. $t \in (0,T)$ as the solution of this problem is unique.
Redefining $\nabla\tilde{p}$ on an appropriate measure zero set yields $\nabla\tilde{p} \in BC_w([0,T];L^2(\Omega)^d)$ since this property is true for ∇p, cf. (3.102).
Using the Helmholtz decomposition of $\mathbf{v}$, cf. (3.103), and redefining $\mathbf{v}$ on a measure zero set with the redefined $\nabla\tilde{p}$ yields $\mathbf{v} \in BC_w([0,T];L^2(\Omega)^d)$ since $\nabla\tilde{p}$ and $\boldsymbol{\omega}$ have this property, cf. (3.99), and since we know $\rho \in C([0,T];L^2(\Omega))$ together with the lower bound $\rho \geq c > 0$.
Now we can use this to show that $\mathbf{v}$ attains the initial value $\mathbf{v}_0$. As it holds that $\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)$ is bounded in $W_1^1(0,T);V(\Omega)') \hookrightarrow C([0,T];V(\Omega)')$, cf. (3.93), and as the map $\operatorname{tr}_{t=0} : C([0,T];V(\Omega)') \to V(\Omega)'$ given by $f \mapsto f(0)$ is linear and continuous and therefore weakly continuous, we can conclude

$$\int_\Omega \mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)(0)\cdot\boldsymbol{\psi}\,dx \to \int_\Omega \mathbb{P}_\sigma(\rho_0\mathbf{v}(0))\cdot\boldsymbol{\psi}\,dx = \int_\Omega \rho_0\mathbf{v}(0)\cdot\boldsymbol{\psi}\,dx$$

for every $\boldsymbol{\psi} \in C_{0,\sigma}^\infty(\Omega)$. But by the definition of $\widetilde{\rho\mathbf{v}}^N$ we get

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)_{|t=0} = \mathbb{P}_\sigma(\rho(\varphi^N(-h))\mathbf{v}^N(-h)) = \mathbb{P}_\sigma(\rho_0\mathbf{v}_0).$$

Using this in the convergence above yields

$$\int_\Omega \rho_0\mathbf{v}_0\cdot\boldsymbol{\psi}\,dx = \int_\Omega \rho_0\mathbf{v}(0)\cdot\boldsymbol{\psi}\,dx \tag{3.104}$$

for all $\boldsymbol{\psi} \in C_{0,\sigma}^\infty(\Omega)$. Now we set $\boldsymbol{\psi} := \mathbf{v}_0 - \mathbf{v}(0)$. Then it holds $\boldsymbol{\psi} \in L^2_\sigma(\Omega)$. Hence, we can approximate $\boldsymbol{\psi}$ by a sequence $(\boldsymbol{\psi}_k)_{k\in\mathbb{N}} \subseteq C_{0,\sigma}^\infty(\Omega)$ since $C_{0,\sigma}^\infty(\Omega)$ is dense in $L^2_\sigma(\Omega)$. Therefore, we can conclude that (3.104) also holds for $\boldsymbol{\psi} = \mathbf{v}_0 - \mathbf{v}(0)$. But this implies

$$\int_\Omega \rho_0|\mathbf{v}_0 - \mathbf{v}(0)|^2\,dx = 0.$$

Since it holds $\rho_0 \geq c > 0$, it follows $\mathbf{v}(0) = \mathbf{v}_0$ in $L^2(\Omega)^d$.

3.3.6 Convergence of the Interpolant Functions to a Weak Solution in the Case $\delta > 0$

It remains to show that $(\mathbf{v}, \varphi, \mu, q)$ is a weak solution for the equations (3.54) - (3.58) in the sense of Definition 3.8. To this end, we pass to the limit $N \to \infty$ in the equations (3.65) - (3.68).

We start with (3.65). The first term can be rewritten as

$$\int_0^T \int_\Omega \partial_{t,h}^-(\rho^N \mathbf{v}^N) \cdot \boldsymbol{\psi}\, dx\, dt = \frac{1}{h}\int_0^T \int_\Omega (\rho^N \mathbf{v}^N)(t) \cdot \boldsymbol{\psi}\, dx\, dt - \frac{1}{h}\int_0^T \int_\Omega (\rho^N \mathbf{v}^N)(t-h) \cdot \boldsymbol{\psi}\, dx\, dt$$

$$= \frac{1}{h}\int_0^T \int_\Omega (\rho^N \mathbf{v}^N)(t) \cdot \boldsymbol{\psi}\, dx\, dt - \frac{1}{h}\int_{-h}^T \int_\Omega (\rho^N \mathbf{v}^N)(t) \cdot \boldsymbol{\psi}(t+h)\, dx\, dt$$

$$= -\int_0^T \int_\Omega (\rho^N \mathbf{v}^N) \cdot \partial_{t,h}^+ \boldsymbol{\psi}\, dx\, dt - \frac{1}{h}\int_{-h}^0 \int_\Omega (\rho^N \mathbf{v}^N)(t) \cdot \boldsymbol{\psi}(t+h)\, dx\, dt$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T; H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$. Since $\mathbf{v}^N \to \mathbf{v}$ in $L^2(0,T; L^2(\Omega)^d)$, cf. (3.97), and $\rho^N(t,x) \to \rho(t,x)$ a.e. in Q_T we get

$$-\int_0^T \int_\Omega (\rho^N \mathbf{v}^N) \cdot \partial_{t,h}^+ \boldsymbol{\psi}\, dx\, dt \to -\int_0^T \int_\Omega \rho \mathbf{v} \cdot \partial_t \boldsymbol{\psi}\, dx\, dt$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T; H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$. As we know that $\boldsymbol{\psi}$ has compact support in $(0,T)$, we can conclude

$$-\frac{1}{h}\int_{-h}^0 \int_\Omega (\rho^N \mathbf{v}^N)(t) \cdot \boldsymbol{\psi}(t+h)\, dx\, dt \to 0$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T; H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$. Altogether we get for all $\boldsymbol{\psi} \in C_0^\infty(0,T; H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$

$$\int_0^T \int_\Omega \partial_{t,h}^-(\rho^N \mathbf{v}^N) \cdot \boldsymbol{\psi}\, dx\, dt \to -\int_0^T \int_\Omega \rho \mathbf{v} \cdot \partial_t \boldsymbol{\psi}\, dx\, dt.$$

Now we pass to the limit $N \to \infty$ in the second term of (3.65). Here we use $\rho^N(t,x) \to \rho(t,x)$ a.e. in Q_T and $\mathbf{v}_h^N \to \mathbf{v}$ in $L^2(0,T; L^2(\Omega)^d)$. Then we obtain

$$\int_0^T \int_\Omega (\rho_h^N \mathbf{v}^N \otimes \mathbf{v}^N) : \nabla \boldsymbol{\psi}\, dx\, dt \to \int_0^T \int_\Omega (\rho \mathbf{v} \otimes \mathbf{v}) : \nabla \boldsymbol{\psi}\, dx\, dt$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$. So now let us proceed with the other terms of (3.65). First of all we have

$$\int_0^T \int_\Omega \mu^N \nabla\varphi_h^N \cdot \boldsymbol{\psi}\, dx\, dt \to \int_0^T \int_\Omega \mu \nabla\varphi \cdot \boldsymbol{\psi}\, dx\, dt$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$, where we used $\mu^N \rightharpoonup \mu$ in $L^2(0,T;H^1(\Omega))$ and

$$\varphi_h^N \to \varphi \text{ in } L^p(0,T;H^1(\Omega)) \text{ for all } 1 \le p < \infty,$$
$$\varphi^N \to \varphi \text{ in } L^p(0,T;H^1(\Omega)) \text{ for all } 1 \le p < \infty.$$

We prove these convergences in detail. It holds

$$||\nabla\varphi_h^N - \nabla\varphi||_{L^2(Q_T)} \le ||\nabla\varphi_h^N - \nabla\varphi_h||_{L^2(Q_T)} + ||\nabla\varphi_h - \nabla\varphi||_{L^2(Q_T)},$$

where the first term converges to 0 as $N \to \infty$ since it holds $\varphi^N \to \varphi$ in $L^2(0,T;H^1(\Omega))$. Furthermore, the second term also converges to 0 as $N \to \infty$ due to

$$\int_0^T ||f(t+h) - f(t)||_X^p\, dt \to 0 \qquad \text{as } h \to 0 \text{ for every } f \in L^p(0,T+\varepsilon;X)$$

and for every $\varepsilon > 0$ and Banach space X. This yields $\varphi_h^N \to \varphi$ in $L^2(0,T;H^1(\Omega))$. From Lemma 2.31 it follows

$$||\nabla\varphi^N - \nabla\varphi||_{L^p(0,T;L^2(\Omega))} \le ||\nabla\varphi^N - \nabla\varphi||_{L^\infty(0,T;L^2(\Omega))}^{1-\theta} ||\nabla\varphi^N - \nabla\varphi||_{L^2(0,T;L^2(\Omega))}^{\theta}$$

for every $\theta \in (0,1)$ and $\frac{1}{p} = \frac{\theta}{2}$. Since $\nabla\varphi^N$, $\nabla\varphi$ are bounded in $L^\infty(0,T;L^2(\Omega)^d)$ and $\nabla\varphi^N \to \nabla\varphi$ in $L^2(0,T;L^2(\Omega)^d)$, we obtain $\varphi^N \to \varphi$ in $L^p(0,T;H^1(\Omega))$ for every $1 \le p < \infty$. Analogously it follows $\varphi_h^N \to \varphi$ in $L^p(0,T;H^1(\Omega))$ for every $1 \le p < \infty$.

Now we pass to the limit $N \to \infty$ in the next term in (3.65). Due to the growth conditions for W' and $\varphi_h^N \in L^\infty(0,\infty;L^6(\Omega))$ we get that $W'(\varphi_h^N)$ is bounded in $L^\infty(0,\infty;L^3(\Omega))$. Furthermore, the growth condition for h and the boundedness of q^N in $L^2_{uloc}([0,\infty);L^6(\Omega))$ yield that $h(q^N)$ is bounded in $L^2_{uloc}([0,\infty);L^6(\Omega))$. Finally, we know that

$$\nabla\varphi_h^N \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)^d) \cap L^2_{uloc}([0,\infty);H^1(\Omega)^d) \hookrightarrow L^4(0,T;L^3(\Omega)^d).$$

Altogether this yields that

$$h(q^N)W'(\varphi_h^N)\nabla\varphi_h^N \qquad \text{is bounded in } L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega)^d).$$

Moreover, we know $\varphi_h^N(t,x) \to \varphi(t,x)$ and $\nabla\varphi_h^N(t,x) \to \nabla\varphi(t,x)$ a.e. in $(0,T)\times\Omega$ as $N \to \infty$ for a suitable subsequence since it holds $\varphi_h^N \to \varphi$ in $L^2(0,T;H^1(\Omega))$. Hence, we can deduce

$$h(q^N)\frac{1}{\varepsilon}W'(\varphi_h^N)\nabla\varphi_h^N \to h(q)\frac{1}{\varepsilon}W'(\varphi)\nabla\varphi \qquad \text{a.e. in } (0,T)\times\Omega$$

as $N \to \infty$ since the strong convergence of q^N to q in $L^2(Q_T)$ implies $q^N(t,x) \to q(t,x)$ a.e. in $(0,T)\times\Omega$ for a suitable subsequence. But as this term is bounded in $L^{\frac{6}{5}}(Q_T)$ and converges a.e. as $N \to \infty$, we can use Lemma 2.9 and obtain

$$h(q^N)\frac{1}{\varepsilon}W'(\varphi_h^N)\nabla\varphi_h^N \to h(q)\frac{1}{\varepsilon}W'(\varphi)\nabla\varphi \quad \text{in } L^p((0,T)\times\Omega)^d,\ 1 \le p < \frac{6}{5},$$

for $0 < T < \infty$. In particular, this convergence holds in $L^1((0,T)\times\Omega))$. Altogether this yields

$$\int_0^T\int_\Omega \frac{h(q^N)}{\varepsilon}W'(\varphi_h^N)\nabla\varphi_h^N\cdot\boldsymbol{\psi}\,dx\,dt \to \int_0^T\int_\Omega \frac{h(q)}{\varepsilon}W'(\varphi)\nabla\varphi\cdot\boldsymbol{\psi}\,dx\,dt$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$. Next we need to study

$$\begin{aligned}
\left\langle \frac{R^N\mathbf{v}^N}{2}, \boldsymbol{\psi} \right\rangle &= \frac{1}{2}\int_0^T\int_\Omega \frac{\rho^N-\rho_h^N}{h}\mathbf{v}^N\cdot\boldsymbol{\psi}\,dx\,dt - \frac{1}{2}\int_0^T\int_\Omega \left(\rho_h^N\mathbf{v}^N+\tilde{\mathbf{J}}\right)\cdot\nabla(\mathbf{v}^N\cdot\boldsymbol{\psi})\,dx\,dt \\
&= \frac{1}{2}\int_0^T\int_\Omega \frac{\rho^N-\rho_h^N}{h}\mathbf{v}^N\cdot\boldsymbol{\psi}\,dx\,dt - \frac{1}{2}\int_0^T\int_\Omega (\rho_h^N\mathbf{v}^N)\cdot\nabla(\mathbf{v}^N\cdot\boldsymbol{\psi})\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^T\int_\Omega (\rho'(\varphi_h^N)\tilde{m}(\varphi_h^N)\nabla\mu^N)\cdot\nabla(\mathbf{v}^N\cdot\boldsymbol{\psi})\,dx\,dt.
\end{aligned}$$

The second term converges to $\int_0^T\int_\Omega(\rho\mathbf{v})\cdot\nabla(\mathbf{v}\cdot\boldsymbol{\psi})\,dx\,dt$ as we know $\rho_h^N\mathbf{v}^N \to \rho\mathbf{v}$ in

$L^2(Q_T)$ and $\nabla \mathbf{v}^N \rightharpoonup \nabla \mathbf{v}$ in $L^2(Q_T)$. For the third term we get

$$\int_0^T \int_\Omega (\rho'(\varphi_h^N)\tilde{m}(\varphi_h^N)\nabla\mu^N) \cdot \nabla(\mathbf{v}^N \cdot \boldsymbol{\psi})\,dx\,dt = \left\langle -\mathrm{div}(\tilde{m}(\varphi_h^N)\nabla\mu^N), \rho'(\varphi_h^N)(\mathbf{v}^N \cdot \boldsymbol{\psi})\right\rangle$$
$$- \int_0^T \int_\Omega \tilde{m}(\varphi_h^N)\nabla(\rho'(\varphi_h^N)) \cdot \nabla\mu^N(\mathbf{v}^N \cdot \boldsymbol{\psi})\,dx\,dt$$
$$= -\int_0^T \int_\Omega \rho'(\varphi_h^N)\partial_{t,h}^-\varphi^N(\mathbf{v}^N \cdot \boldsymbol{\psi})\,dx\,dt - \int_0^T \int_\Omega \rho'(\varphi_h^N)(\nabla\varphi_h^N \cdot \mathbf{v}^N)(\mathbf{v}^N \cdot \boldsymbol{\psi})\,dx\,dt$$
$$- \int_0^T \int_\Omega \tilde{m}(\varphi_h^N)\nabla(\rho'(\varphi_h^N)) \cdot \nabla\mu^N(\mathbf{v}^N \cdot \boldsymbol{\psi})\,dx\,dt,$$

where we used equation (3.67). But these integrals converge to

$$-\int_0^T \int_\Omega \rho'(\varphi)\partial_t\varphi(\mathbf{v} \cdot \boldsymbol{\psi})\,dx\,dt - \int_0^T \int_\Omega \rho'(\varphi)(\nabla\varphi \cdot \mathbf{v})(\mathbf{v} \cdot \boldsymbol{\psi})\,dx\,dt$$
$$- \int_0^T \int_\Omega \tilde{m}(\varphi)\nabla(\rho'(\varphi)) \cdot \nabla\mu(\mathbf{v} \cdot \boldsymbol{\psi})\,dx\,dt.$$

Here we used for the first term $\partial_{t,h}^-\varphi^N \rightharpoonup \partial_t\varphi$ in $L^2(Q_T)$ and $\rho'(\varphi_h^N)\mathbf{v}^N \to \rho'(\varphi)\mathbf{v}$ in $L^2(Q_T)$. For the second term, the convergence follows from $\rho'(\varphi_h^N)\nabla\varphi_h^N \to \rho'(\varphi)\nabla\varphi$ in $L^p(0,T;L^2(\Omega)^d)$ for all $1 \le p < \infty$ and $\mathbf{v}^N \otimes \mathbf{v}^N \to \mathbf{v} \otimes \mathbf{v}$ in $L^q(0,T;L^2(\Omega)^{d\times d})$ for every $1 \le q < \frac{4}{3}$, which we will prove in the following.
From Lemma 2.29 we obtain $(L^2(\Omega), L^6(\Omega))_{\frac{3}{4},4} = L^4(\Omega)$ together with the estimate

$$||\mathbf{v}^N(t) - \mathbf{v}(t)||_{L^4(\Omega)} \le C||\mathbf{v}^N(t) - \mathbf{v}(t)||_{L^2(\Omega)}^{\frac{1}{4}}||\mathbf{v}^N(t) - \mathbf{v}(t)||_{H^1(\Omega)}^{\frac{3}{4}}$$

for a.e. $t \in (0,T)$. Hence, Lemma 2.31 yields

$$||\mathbf{v}^N - \mathbf{v}||_{L^2(0,T;L^4(\Omega))} \le C||\mathbf{v}^N - \mathbf{v}||_{L^2(0,T;L^2(\Omega))}^{\frac{1}{4}}||\mathbf{v}^N - \mathbf{v}||_{L^2(0,T;H^1(\Omega))}^{\frac{3}{4}}.$$

Due to the boundedness of $\mathbf{v}^N, \mathbf{v}$ in $L^2(0,T;H^1(\Omega)^d)$ and $\mathbf{v}^N \to \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$ we can deduce

$$\mathbf{v}^N \to \mathbf{v} \qquad \text{in } L^2(0,T;L^4(\Omega)^d).$$

Moreover, Lemma 2.31 implies

$$\mathbf{v}^N \to \mathbf{v} \qquad \text{in } L^q(0,T;L^2(\Omega)^d) \text{ for all } 1 \le q < \infty$$

since $\mathbf{v}^N \to \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$ and $\mathbf{v}^N, \mathbf{v}$ are bounded in $L^\infty(0,T;L^2(\Omega)^d)$. Thus we can apply Lemma 2.31 again and obtain

$$||\mathbf{v}^N - \mathbf{v}||_{L^p(0,T;L^4(\Omega))} \leq C||\mathbf{v}^N - \mathbf{v}||^{\frac{1}{4}}_{L^q(0,T;L^2(\Omega))}||\mathbf{v}^N - \mathbf{v}||^{\frac{3}{4}}_{L^2(0,T;H^1(\Omega))}$$

for every $1 \leq q < \infty$ and $\frac{1}{p} = \frac{1}{4q} + \frac{3}{8}$. Since it holds $\mathbf{v}^N \to \mathbf{v}$ in $L^q(0,T;L^2(\Omega)^d)$ for every $1 \leq q < \infty$, it follows

$$\mathbf{v}^N \to \mathbf{v} \qquad \text{in } L^p(0,T;L^4(\Omega)^d) \text{ for all } 1 \leq p < \frac{8}{3}.$$

Furthermore, we can estimate for a.e. $t \in (0,T)$

$$\begin{aligned} ||\mathbf{v}^N \otimes \mathbf{v}^N - \mathbf{v} \otimes \mathbf{v}||_{L^2(\Omega)} &\leq ||\mathbf{v}^N \otimes (\mathbf{v}^N - \mathbf{v})||_{L^2(\Omega)} + ||\mathbf{v} \otimes (\mathbf{v}^N - \mathbf{v})||_{L^2(\Omega)} \\ &\leq ||\mathbf{v}^N||_{L^4(\Omega)}||\mathbf{v}^N - \mathbf{v}||_{L^4(\Omega)} + ||\mathbf{v}||_{L^4(\Omega)}||\mathbf{v}^N - \mathbf{v}||_{L^4(\Omega)}, \end{aligned}$$

where we omitted $t \in (0,T)$ for the sake of clarity. Due to $\mathbf{v}^N \to \mathbf{v}$ in $L^p(0,T;L^4(\Omega)^d)$ for every $1 \leq p < \frac{8}{3}$, this estimate implies

$$\mathbf{v}^N \otimes \mathbf{v}^N \to \mathbf{v} \otimes \mathbf{v} \qquad \text{in } L^q(0,T;L^2(\Omega)^{d\times d}) \text{ for every } 1 \leq q < \frac{4}{3}.$$

For the convergence of the third term of $\left\langle \frac{R^N \mathbf{v}^N}{2}, \boldsymbol{\psi} \right\rangle$ we still have to prove

$$\int_0^T \int_\Omega \tilde{m}(\varphi_h^N)\nabla(\rho'(\varphi_h^N)) \cdot \nabla\mu^N(\mathbf{v}^N \cdot \boldsymbol{\psi})\, dx\, dt \to \int_0^T \int_\Omega \tilde{m}(\varphi)\nabla(\rho'(\varphi)) \cdot \nabla\mu(\mathbf{v} \cdot \boldsymbol{\psi})\, dx\, dt$$

for every $\boldsymbol{\psi} \in C_0^\infty(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L^2_\sigma(\Omega))$ as $N \to \infty$, i.e., we need to study terms of the form $\rho''(\varphi_h^N)\partial_j\varphi_h^N\tilde{m}(\varphi_h^N)\partial_j\mu^N\mathbf{v}_k^N\boldsymbol{\psi}_k$. This convergence follows from $\nabla\varphi^N \to \nabla\varphi$ in $L^p(0,T;L^2(\Omega)^d)$ for every $1 \leq p < \infty$, $\nabla\mu^N \rightharpoonup \nabla\mu$ in $L^2(0,T;L^2(\Omega)^d)$ and $\mathbf{v}^N \to \mathbf{v}$ in $L^q(0,T;L^\infty(\Omega)^d)$ for every $1 \leq q < \frac{8}{3}$, cf. (3.98).

Hence, it remains to study the first term of $\left\langle \frac{R^N \mathbf{v}^N}{2}, \boldsymbol{\psi} \right\rangle$. But since we have already proven $\frac{\rho^N - \rho_h^N}{h} \rightharpoonup \rho'(\varphi)\partial_t\varphi$ in $L^2(Q_T)$, cf. (3.92), and as it holds $\mathbf{v}^N \to \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$, we can conclude

$$\int_0^T \int_\Omega \frac{\rho^N - \rho_h^N}{h}\mathbf{v}^N \cdot \boldsymbol{\psi}\, dx\, dt \to \int_0^T \int_\Omega \rho'(\varphi)\partial_t\varphi(\mathbf{v} \cdot \boldsymbol{\psi})\, dx\, dt$$

as $N \to \infty$. For the next term of (3.65) we get

$$\begin{aligned} \int_0^T \int_\Omega \left(\tilde{\mathbf{J}} \otimes \mathbf{v}^N\right) : \nabla\boldsymbol{\psi}\, dx\, dt &= -\int_0^T \int_\Omega \left((\rho'(\varphi_h^N)\tilde{m}(\varphi_h^N)\nabla\mu^N) \otimes \mathbf{v}^N\right) : \nabla\boldsymbol{\psi}\, dx\, dt \\ \to -\int_0^T \int_\Omega ((\rho'(\varphi)\tilde{m}(\varphi)\nabla\mu) \otimes \mathbf{v}) : \nabla\boldsymbol{\psi}\, dx\, dt &= \int_0^T \int_\Omega (\tilde{\mathbf{J}} \otimes \mathbf{v}) : \nabla\boldsymbol{\psi} dx\, dt, \end{aligned}$$

where we used $\mathbf{v}^N \to \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$, $\nabla\mu^N \rightharpoonup \nabla\mu$ in $L^2(0,T;L^2(\Omega))$, $\rho'(\varphi_h^N(t,x)) \to \rho'(\varphi(t,x))$ and $\tilde{m}(\varphi_h^N(t,x)) \to \tilde{m}(\varphi(t,x))$ a.e. in $(0,T)\times\Omega$.

Altogether we obtain

$$\left\langle \frac{R^N\mathbf{v}^N}{2}, \boldsymbol{\psi} \right\rangle \to \left\langle \frac{R\mathbf{v}}{2}, \boldsymbol{\psi} \right\rangle$$

as $N\to\infty$ for all $\boldsymbol{\psi} \in C_0^\infty(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$, where $\left\langle \frac{R\mathbf{v}}{2}, \boldsymbol{\psi} \right\rangle$ is defined as in (3.64).

For the additional term $\delta\Delta^2\mathbf{v}^N$ the convergence

$$\delta \int_0^T \int_\Omega \Delta\mathbf{v}^N \Delta\boldsymbol{\psi}\, dx\, dt \to \delta \int_0^T \int_\Omega \Delta\mathbf{v}\Delta\boldsymbol{\psi}\, dx\, dt$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$ as $N\to\infty$ follows from $\mathbf{v}^N \rightharpoonup \mathbf{v}$ in $L^2(0,T;H^2(\Omega)^d)$.

Now we pass to the limit $N\to\infty$ in the last remaining term of (3.65). Since η is bounded from above and below and as we know $\varphi_h^N(t,x) \to \varphi(t,x)$ a.e. in $(0,T)\times\Omega$, we can also conclude $\eta(\varphi_h^N(t,x)) \to \eta(\varphi(t,x))$ a.e. in $(0,T)\times\Omega$. Thus Lemma 2.9 yields

$$\eta(\varphi_h^N) \to \eta(\varphi) \qquad \text{in } L^p((0,T)\times\Omega) \text{ for all } 1\le p<\infty,\ 0<T<\infty.$$

This convergence together with $D\mathbf{v}^N \rightharpoonup D\mathbf{v}$ in $L^2(0,\infty;L^2(\Omega)^{d\times d})$ implies

$$-\int_0^T \int_\Omega 2\eta(\varphi_h^N) D\mathbf{v}^N : D\boldsymbol{\psi}\, dx\, dt \to -\int_0^T \int_\Omega 2\eta(\varphi) D\mathbf{v} : D\boldsymbol{\psi}\, dx\, dt$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega))$. Hence, we have shown (3.59).

Now we pass to the limit $N\to\infty$ in equation (3.66). To show the convergence for the left-hand side of (3.66) we define

$$(F_1\boldsymbol{u})(x) := f_1(x,\boldsymbol{u}(x)),$$

where

$$f_1(x,\boldsymbol{\eta}) := m(\eta^1,\eta^2)$$

for $\boldsymbol{\eta} = (\eta^1,\eta^2) \in \mathbb{R}^2$ and $x\in\Omega$. Then it holds

$$|f_1(x,\boldsymbol{\eta})| \le C_1 \le C_1 + C_2(|\eta^1|^{\frac{2}{p}} + |\eta^2|^{\frac{2}{p}})$$

for some constants $C_1, C_2 > 0$ and every $1 \leq p < \infty$ since m is bounded from below and above. But this shows that $F_1 : L^2(Q_T)^2 \to L^p(Q_T)$, $\boldsymbol{\eta} \mapsto m(\eta^1, \eta^2)$ is a continuous and bounded mapping for every $1 \leq p < \infty$, cf. Theorem 2.10. Since it holds $q_h^N \to q$ in $L^2(0,T;L^2(\Omega))$ and $\varphi_h^N \to \varphi$ in $L^p(0,T;H^1(\Omega))$ for every $1 \leq p < \infty$, we can deduce

$$m(\varphi_h^N, q_h^N) \to m(\varphi, q) \qquad \text{in } L^p(0,T;L^p(\Omega))$$

for every $1 \leq p < \infty$. Hence, we can conclude

$$-\int_0^T \int_\Omega m(\varphi_h^N, q_h^N) \nabla q^N \cdot \nabla \phi \, dx \, dt \to -\int_0^T \int_\Omega m(\varphi, q) \nabla q \cdot \nabla \phi \, dx \, dt$$

for all $\phi \in C_0^\infty((0,T);C^1(\overline{\Omega}))$, where we used $\nabla q^N \rightharpoonup \nabla q$ in $L^2(0,T;L^2(\Omega))$. For the first term on the right-hand side of (3.66) we can conclude as before

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_{t,h}^- \left(f(q^N) W(\varphi^N) + g(q^N) \right) \phi \, dx \, dt \\
&= -\int_0^T \int_\Omega \left(f(q^N) W(\varphi^N) + g(q^N) \right) \partial_{t,h}^+ \phi \, dx \, dt \\
&\quad - \frac{1}{h} \int_{-h}^0 \int_\Omega \left(f(q^N(t)) W(\varphi^N(t)) + g(q^N(t)) \right) \phi(t+h) \, dx \, dt
\end{aligned}$$

for all $\phi \in C_0^\infty((0,T);C^1(\overline{\Omega}))$. By definition q^N and φ^N are piecewise constant and it holds $\varphi^N(t) = \varphi_0$, $q^N(t) = q_0$ for all $t \in [-h, 0)$. Using this for the last integral we can summarize

$$\begin{aligned}
\int_0^T \int_\Omega \partial_{t,h}^- \left(f(q^N) W(\varphi^N) + g(q^N) \right) \phi \, dx \, dt &= -\int_0^T \int_\Omega \left(f(q^N) W(\varphi^N) + g(q^N) \right) \partial_{t,h}^+ \phi \, dx \, dt \\
&\quad - \int_\Omega \left(f(q_0) W(\varphi_0) + g(q_0) \right) \frac{1}{h} \int_{-h}^0 \phi(t+h) \, dt \, dx.
\end{aligned}$$

Due to the strong convergence of $(\varphi^N)_{N \in \mathbb{N}}$ and $(q^N)_{N \in \mathbb{N}}$ in $L^2(0,T;L^2(\Omega))$, there exist subsequences such that $\varphi^N(t,x) \to \varphi(t,x)$ and $q^N(t,x) \to q(t,x)$ a.e. in $(0,T) \times \Omega$ as $N \to \infty$. This yields

$$f(q^N(t,x)) W(\varphi^N(t,x)) + g(q^N(t,x)) \to f(q(t,x)) W(\varphi(t,x)) + g(q(t,x))$$

a.e. in $(0,T) \times \Omega$ as $N \to \infty$. Using the growth conditions $|W(s)| \leq C(|s|^3 + 1)$ and $|g(s)| \leq C(|s| + 1)$ for all $s \in \mathbb{R}$, the boundedness of the function f and the fact that $(\varphi^N)_{N \in \mathbb{N}}$ is bounded in $L^2_{uloc}([0,\infty); L^6(\Omega))$, we can conclude that

$$\left(f(q^N)W(\varphi^N) + g(q^N)\right)_{N \in \mathbb{N}} \qquad \text{is bounded in } L^2((0,T) \times \Omega).$$

Hence, Lemma 2.9 yields

$$f(q^N)W(\varphi^N) + g(q^N) \to f(q)W(\varphi) + g(q) \quad \text{in } L^p((0,T) \times \Omega),\ 1 \leq p < 2. \tag{3.105}$$

Moreover, we know

$$\partial^+_{t,h}\phi \to \partial_t \phi, \qquad \frac{1}{h}\int_{-h}^{0} \phi(t+h)\,dt \to \phi(0) = 0$$

as $h \to 0$ since ϕ is in $C_0^\infty(0,T; C^1(\overline{\Omega}))$. Using all these convergences we can finally conclude

$$\int_0^T \int_\Omega \partial^-_{t,h}\left(f(q^N)W(\varphi^N) + g(q^N)\right)\phi\,dx\,dt \to -\int_0^T \int_\Omega \left(f(q)W(\varphi) + g(q)\right)\partial_t \phi\,dx\,dt.$$

For the second summand on the right-hand side of (3.66) we also use (3.105) together with $\mathbf{v}^N \to \mathbf{v}$ in $L^p(0,T; L^4(\Omega)^d)$ for every $1 \leq p < \frac{8}{3}$. Then we get

$$\begin{aligned}\int_0^T \int_\Omega &\left(\frac{1}{\varepsilon} f(q^N)W(\varphi^N_h) + g(q^N)\right)\mathbf{v}^N \cdot \nabla\phi\,dx\,dt \\ &\to \int_0^T \int_\Omega \left(\frac{1}{\varepsilon} f(q)W(\varphi) + g(q)\right)\mathbf{v} \cdot \nabla\phi\,dx\,dt\end{aligned}$$

for all $\phi \in C_0^\infty(0,T; C^1(\overline{\Omega}))$. This shows (3.60).

For the first term on the right-hand side of (3.67) the calculations for the back-differences are the same as before so that we can directly state

$$\int_0^T \int_\Omega \partial^-_{t,h}\varphi^N \phi\,dx\,dt \to -\int_0^T \int_\Omega \varphi \partial_t \phi\,dx\,dt$$

for all $\phi \in C_0^\infty(0,T; C^1(\overline{\Omega}))$ as $N \to \infty$. For the second term we have the convergence

$$\int_0^T \int_\Omega (\nabla\varphi^N_h \cdot \mathbf{v}^N)\phi\,dx\,dt \to \int_0^T \int_\Omega (\nabla\varphi \cdot \mathbf{v})\phi\,dx\,dt$$

for all $\phi \in C_0^\infty(0,T;C^1(\overline{\Omega}))$ due to $\varphi_h^N \to \varphi$ in $L^p(0,T;H^1(\Omega))$ for all $1 \le p < \infty$ and $\mathbf{v}^N \to \mathbf{v}$ in $L^2((0,T)\times\Omega)$. The convergence for the left-hand side of (3.67) follows from $\tilde{m}(\varphi_h^N) \to \tilde{m}(\varphi)$ in $L^p(Q_T)$ for every $1 \le p < \infty$ and $\nabla\mu^N \rightharpoonup \nabla\mu$ in $L^2(0,T;L^2(\Omega)^d)$. Hence, it holds (3.61).

In (3.68) the first two terms converge due to $\mu^N \rightharpoonup \mu$ in $L^2(0,T;H^1(\Omega))$ and $\nabla\varphi^N \to \nabla\varphi$ in $L^p(0,T;L^2(\Omega)^d)$ for all $1 \le p < \infty$. So it remains to show the convergence for the last two terms. Therefore, we study the term $H(\varphi^N,\varphi_h^N)$. From estimate (3.41) and the growth condition for h we get

$$\left|h(q^N)\frac{1}{\varepsilon}H(\varphi^N,\varphi_h^N)\right| \le \frac{C}{\varepsilon}(|q^N|+1)\left(|\varphi^N(t)|^2+|\varphi^N(t-h)|^2+1\right),$$

for a.e. $(t,x)\in(0,\infty)\times\Omega$. From this estimate together with $q^N \in L^2_{uloc}([0,\infty);L^6(\Omega))$ and $\varphi^N,\varphi_h^N \in L^4(0,T;L^\infty(\Omega))$, cf. (3.78), it follows that $h(q^N)\frac{1}{\varepsilon}H(\varphi^N,\varphi_h^N)$ is bounded in $L^{\frac{4}{3}}(0,T;L^6(\Omega))$. Furthermore, we can conclude

$$h(q^N)\frac{1}{\varepsilon}H(\varphi^N,\varphi_h^N) = \begin{cases} h(q^N)\frac{1}{\varepsilon}W'(\varphi^N(t-h)) & \text{if } \varphi^N(t)=\varphi^N(t-h),\\ h(q^N)\frac{1}{\varepsilon}W'(\xi_{t,t-h}(x)) & \text{if } \varphi^N(t)\neq\varphi^N(t-h),\end{cases}$$

where $\xi_{t,t-h}(x) \in [\varphi^N(t-h,x),\varphi^N(t,x)] = [\varphi_k(x),\varphi_{k+1}(x)]$ for $k\in\mathbb{N}_0$ such that $t\in[kh,(k+1)h)$ and where we assumed w.l.o.g. $\varphi_k(x)<\varphi_{k+1}(x)$.
Here the second case could be derived as follows: For $a\neq b$ we have

$$H(a,b)=\frac{W(a)-W(b)}{a-b}=W'(\xi),$$

where $\xi\in[a,b]$. Since it holds $\varphi^N(t,x)\to\varphi(t,x)$ a.e and $\varphi^N(t-h,x)\to\varphi(t,x)$ a.e. and therefore $\xi_{t,t-h}(x)\to\varphi(t,x)$ a.e. as $N\to\infty$ we get

$$h(q^N)\frac{1}{\varepsilon}H(\varphi^N,\varphi_h^N)_{|(t,x)} \to h(q(t,x))\frac{1}{\varepsilon}W'(\varphi(t,x)) \qquad \text{a.e. in } (0,T)\times\Omega$$

as $N\to\infty$. But as this term is bounded in $L^{\frac{4}{3}}((0,T)\times\Omega)$ and converges a.e. as $N\to\infty$, we can use Lemma 2.9 and obtain

$$h(q^N)\frac{1}{\varepsilon}H(\varphi^N,\varphi_h^N)\to h(q)\frac{1}{\varepsilon}W'(\varphi) \quad \text{in } L^p((0,T)\times\Omega),\ 1\le p<\frac{4}{3},\ 0<T<\infty.$$

In particular this convergence holds for $p=1$ and therefore we get

$$\int_0^T\int_\Omega h(q^N)\frac{1}{\varepsilon}H(\varphi^N,\varphi_h^N)\phi\,dx\,dt \to \int_0^T\int_\Omega h(q)\frac{1}{\varepsilon}W'(\varphi)\phi\,dx\,dt$$

for all $\phi\in C_0^\infty(0,T;C^1(\overline{\Omega}))$. For the last remaining term we use $\partial_{t,h}^-\varphi^N \rightharpoonup \partial_t\varphi$ in $L^2(0,T;L^2(\Omega))$, cf (3.90). Thus we can finally conclude that (3.62) holds.

3.3.7 The Energy Inequality for $\delta > 0$

In the last step in the existence proof for $\delta > 0$ we prove the energy inequality (3.63). This part of the proof can be done analogously as in [ADG13] again, i.e., we want to apply Lemma 3.12. Due to $\mathbf{v}^N \to \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$ and $\varphi^N \to \varphi$ in $L^2(0,T;H^1(\Omega))$ for every $0 < T < \infty$, cf. (3.81), we can deduce $\mathbf{v}^N(t) \to \mathbf{v}(t)$ in $L^2(\Omega)$ and $\varphi^N(t) \to \varphi(t)$ in $H^1(\Omega)$ for a.e. $t \in (0,T)$. Thus we can show

$$E^N(t) \to E_{tot}(\mathbf{v}(t), \varphi(t), \nabla\varphi(t), q(t)) \qquad \text{for a.e. } t \in (0,T),$$

where E_{tot} is defined by

$$E_{tot}(\mathbf{v}, \varphi, \nabla\varphi, q) = \int_\Omega \left(\frac{1}{2}\rho(\varphi)|\mathbf{v}|^2 + \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{d(q)}{\varepsilon}W(\varphi) + G(q) \right) dx.$$

Since norms are lower semicontinuous and as it holds $\varphi^N(t,x) \to \varphi(t,x)$ and $q^N(t,x) \to q(t,x)$ a.e. in $(0,T) \times \Omega$, we can deduce that the inequality

$$\liminf_{N\to\infty} \int_0^T \tilde{D}^N(t)\tau(t)\,dt \geq \int_0^T \tilde{D}(t)\tau(t)\,dt$$

holds for all $\tau \in W_1^1(0,T)$ with $\tau \geq 0$ and $\tau(T)$, where $\tilde{D}$ is defined by

$$\begin{aligned} \tilde{D}(t) := {} & \int_\Omega m(\varphi, q)\,|\nabla q|^2\,dx + \int_\Omega \tilde{m}(\varphi)|\nabla\mu|^2\,dx + \int_\Omega 2\eta(\varphi)|D\mathbf{v}|^2\,dx \\ & + \delta\int_\Omega |\Delta\mathbf{v}|^2\,dx + \delta\int_\Omega |\partial_t\varphi|^2\,dx. \end{aligned}$$

Note that (3.70) is also true when we integrate with respect to t from 0 to T instead of 0 to ∞. So we replace ∞ by T in (3.70). Then it holds in the limit $N \to \infty$

$$E_{tot}(\mathbf{v}_0, \varphi_0, \nabla\varphi_0, q_0)\tau(0) + \int_0^T E_{tot}(\mathbf{v}(t), \varphi(t), \nabla\varphi(t), q(t))\tau'(t)\,dt \geq \int_0^T \tilde{D}(t)\tau(t)\,dt$$

for all $\tau \in W_1^1(0,T)$ with $\tau \geq 0$ and $\tau(T) = 0$. Thus we can apply Lemma 3.12, which yields the energy estimate (3.63), i.e.,

$$\begin{aligned} & \int_s^t \int_\Omega (m(\varphi,q)|\nabla q|^2 + \tilde{m}(\varphi)|\nabla\mu|^2 + 2\eta(\varphi)|D\mathbf{v}|^2 + \delta|\Delta\mathbf{v}|^2 + \delta|\partial_t\varphi|^2)\,dx\,d\tau \\ & \qquad + E_{tot}(\mathbf{v}(t), \varphi(t), \nabla\varphi(t), q(t)) \leq E_{tot}(\mathbf{v}(s), \varphi(s), \nabla\varphi(s), q(s)) \end{aligned}$$

for all $s \leq t \leq T$ and almost all $0 \leq s < T$ including $s = 0$. $\square$

For the last step in the proof for the energy estimate we used the following lemma.

Lemma 3.12. *Let* $E : [0,T) \to [0,\infty)$, $0 < T \leq \infty$, *be a lower semi-continuous function and let* $D : (0,T) \to [0,\infty)$ *be an integrable function. Then*

$$E(0)\varphi(0) + \int_0^T E(t)\varphi'(t)\,dx \geq \int_0^T D(t)\varphi(t)\,dt$$

holds for all $\varphi \in W_1^1(0,T)$ *with* $\varphi(T) = 0$ *and* $\varphi \geq 0$ *if and only if*

$$E(t) + \int_s^t D(\tau)\,d\tau \leq E(s)$$

holds for all $s \leq t < T$ *and almost all* $0 \leq s < T$ *including* $s = 0$.

For a proof of this lemma we refer to [Abe09a, Lemma 4.3].

3.4 Existence of Weak Solutions in the Case $\delta \to 0$

In the previous section we have proven that for every $0 < T < \infty$ and every $\delta > 0$ there exists a weak solution $(\mathbf{v}^\delta, \varphi^\delta, \mu^\delta, q^\delta)$ of (3.54) - (3.58) in the sense of Definition 3.8, cf. Theorem 3.9, which depends on δ and which satisfies

$$\begin{aligned}
\mathbf{v}^\delta &\in L^2(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d) \cap L^\infty(0,T;L^2_\sigma(\Omega)),\\
\varphi^\delta &\in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap W_2^1(0,T;L^2(\Omega)),\\
\mu^\delta &\in L^2(0,T;H^1(\Omega)),\\
q^\delta &\in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)).
\end{aligned}$$

But the weak solution and the energy inequality still depend on $\delta > 0$ since we inserted the additional terms $\delta\Delta^2\mathbf{v}$ in (3.54) and $\delta\partial_t\varphi$ in (3.58). Hence, in the final step of the existence proof it remains to pass to the limit $\delta \to 0$.

Note that in contrast to the definition of a weak solution in the case $\delta > 0$, the terms $\delta\partial_t\varphi$ and $\delta\Delta^2\mathbf{v}$ vanish and we replace the term $\left\langle \frac{R\mathbf{v}}{2}, \boldsymbol{\psi}\right\rangle$ by $\left\langle \frac{\tilde{R}\mathbf{v}}{2}, \boldsymbol{\psi}\right\rangle$ for $\boldsymbol{\psi} \in C_0^\infty(0,T;H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L^2_\sigma(\Omega))$ and $\tilde{R} = -\nabla\left(\frac{\partial\rho(\varphi)}{\partial\varphi}\right) \cdot (\tilde{m}(\varphi)\nabla\mu)$. In the previous calculations we used R and not $\tilde{R}$ because if we had used $\tilde{R}$ we had not been able to derive the discrete energy estimate and therefore no time continuous energy estimate in the case $\delta > 0$. But as we had to use R instead of $\tilde{R}$ to get an energy estimate, we also needed the additional terms $\delta\Delta^2\mathbf{v}$ and $\delta\partial_t\varphi$ to conclude

$\langle R^N \mathbf{v}^N, \boldsymbol{\psi}\rangle \to \langle R\mathbf{v}, \boldsymbol{\psi}\rangle$ for every $\boldsymbol{\psi} \in C_0^\infty(0,T; H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L^2_\sigma(\Omega))$ as $N \to \infty$. When we pass to the limit $\delta \to 0$ to get a weak solution for (1.1) - (1.7), this implies that we can not estimate the terms $\partial_t\varphi$ and $\Delta\mathbf{v}$ in $L^2(0,T;L^2(\Omega))$ since the energy estimate does not yield an estimate for these terms anymore. Therefore, we now need to replace R by $\tilde{R}$. Moreover, the energy estimate (3.63) with the δ-terms stays valid and we have an energy estimate which we had not gotten if we had started the proof with $\tilde{R}$ instead of R. But when we use $\tilde{R}$ we need to estimate terms of the form $\rho''(\varphi^\delta)\partial_j\varphi^\delta\tilde{m}(\varphi^\delta)\partial_j\mu^\delta\mathbf{v}_k^\delta\boldsymbol{\psi}_k$. Therefore, we demanded in Assumption 3.5 that if it holds $\frac{\partial\rho(\varphi)}{\partial\varphi} \neq const$ then there exists a constant $C > 0$ and $0 < s < 1$ such that

$$|W'(a)| \leq C(|a|^s + 1) \qquad \text{for every } a \in \mathbb{R}.$$

In the following we assume w.l.o.g. $\int\limits_\Omega \varphi dx = 0$. Moreover, we will need to split the equation

$$\begin{aligned}
\delta\partial_t\varphi^\delta - \Delta\varphi^\delta &= h(q^\delta)W'(\varphi^\delta) + \mu^\delta && \text{in } (0,T)\times\Omega, && (3.106)\\
\partial_n\varphi^\delta_{|\partial\Omega} &= 0 && \text{on } (0,T)\times\partial\Omega, && (3.107)\\
\varphi^\delta_{|t=0} &= \varphi_0 && \text{in } \Omega. && (3.108)
\end{aligned}$$

To this end, we consider the problem

$$\begin{aligned}
\delta\partial_t\varphi_1^\delta - \Delta\varphi_1^\delta &= P_0(h(q^\delta)W'(\varphi^\delta) + \Delta\varphi_0) && \text{in } (0,T)\times\Omega, && (3.109)\\
\delta\partial_t\varphi_2^\delta - \Delta\varphi_2^\delta &= P_0(\mu^\delta) && \text{in } (0,T)\times\Omega, && (3.110)\\
\partial_n\varphi^\delta_{1|\partial\Omega} = \partial_n\varphi^\delta_{2|\partial\Omega} &= 0 && \text{on } (0,T)\times\partial\Omega, && (3.111)\\
\varphi^\delta_{1|t=0} = \varphi^\delta_{2|t=0} &= 0 && \text{in } \Omega, && (3.112)\\
\int\limits_\Omega \varphi_1^\delta dx = \int\limits_\Omega \varphi_2^\delta dx &= 0, && && (3.113)
\end{aligned}$$

where we define

$$P_0 f := f - \frac{1}{|\Omega|}\int\limits_\Omega f\,dx$$

for every $f \in L^1(\Omega)$. We note that, if φ_1^δ and φ_2^δ are solutions of (3.109) - (3.113), then $\varphi^\delta = \varphi_1^\delta + \varphi_2^\delta + \varphi_0$ is a solution of (3.106) - (3.108), where we used

$$P_0(h(q^\delta)W'(\varphi^\delta) + \mu^\delta) = h(q^\delta)W'(\varphi^\delta) + \mu^\delta$$

due to

$$\int\limits_\Omega h(q^\delta)W'(\varphi^\delta) + \mu^\delta\,dx = \delta\frac{d}{dt}\int\limits_\Omega \varphi^\delta\,dx - \int\limits_\Omega \Delta\varphi^\delta\,dx = 0.$$

To solve the equations (3.109) - (3.113) for φ_1^δ and φ_2^δ, we use the following lemma.

Lemma 3.13. *Let $\delta > 0$ and $f \in L^p(0,T;L^q_{(0)}(\Omega))$ be given for $1 < p < \infty$, $2 \leq q \leq \infty$ and $0 < T < \infty$. Then there exists a solution $\varphi \in L^p(0,T;W^2_q(\Omega))$ of*

$$\begin{aligned} \delta\partial_t\varphi - \Delta\varphi &= f && \textit{in } (0,T)\times\Omega,\\ \partial_n\varphi_{|\partial\Omega} &= 0 && \textit{on } (0,T)\times\partial\Omega,\\ \varphi_{|t=0} &= 0 && \textit{in } \Omega, \end{aligned}$$

which can be estimated by

$$||\varphi||_{L^p(0,T;W^2_q(\Omega))} \leq C||f||_{L^p(0,T;L^q(\Omega))}$$

for a constant $C > 0$ independent of $\delta > 0$.

Proof. We extend f on (T,∞) by

$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \in (0,T),\\ 0 & \text{else.} \end{cases}$$

Then it holds $\tilde{f} \in L^p(0,\infty;L^q_{(0)}(\Omega))$. We consider the problem

$$\begin{aligned} \delta\partial_t\tilde{\varphi} - \Delta\tilde{\varphi} &= \tilde{f} && \text{in } (0,\infty)\times\Omega,\\ \partial_n\tilde{\varphi}_{|\partial\Omega} &= 0 && \text{on } (0,\infty)\times\partial\Omega,\\ \tilde{\varphi}_{|t=0} &= 0 && \text{in } \Omega. \end{aligned}$$

Moreover, we define $\psi_\delta(t) := \tilde{\varphi}(\delta t)$ and $\tilde{f}_\delta := \tilde{f}(\delta t)$ and rewrite the problem above as

$$\begin{aligned} \partial_t\psi_\delta - \Delta\psi_\delta &= \tilde{f}_\delta && \text{in } (0,\infty)\times\Omega,\\ \partial_n\psi_{\delta|\partial\Omega} &= 0 && \text{on } (0,\infty)\times\partial\Omega,\\ \psi_{\delta|t=0} &= 0 && \text{in } \Omega. \end{aligned}$$

The existence of a unique solution $\psi_\delta \in L^p(0,T;W^2_q(\Omega))$ for every $0 < T < \infty$ follows analogously as in Lemma 5.15 below from [DHP03, Theorem 8.2].
Due to [Dor93, Theorem 2.4] we obtain $\psi_\delta \in L^p(0,\infty;W^2_q(\Omega))$ together with the estimate

$$||\psi_\delta||_{L^p(0,\infty;W^2_q(\Omega))} \leq C||\tilde{f}_\delta||_{L^p(0,\infty;L^q(\Omega))} \tag{3.114}$$

for a constant $C > 0$ independent of δ. Here we used that $\sigma(\Delta_N) \subseteq (-\infty,0)$ implies that Δ_N has negative exponential type, cf. [RR04a, Theorem 12.33], where

$$\Delta_N : \mathcal{D}(\Delta_N) = W^2_{q,N}(\Omega) \cap L^q_{(0)}(\Omega) \subseteq L^q_{(0)}(\Omega) \to L^q_{(0)}(\Omega)$$

is the Neumann-Laplace operator and $\sigma(\Delta_N) \subseteq (-\infty,0)$ holds because of the following arguments:

Since the embedding $W^2_{q,N}(\Omega) \cap L^q_{(0)}(\Omega) \to L^q_{(0)}(\Omega)$ is compact, we can deduce that $(\lambda - \Delta_N)^{-1} : L^q_{(0)}(\Omega) \to L^q_{(0)}(\Omega)$ is a compact operator for every $\lambda \in \sigma(\Delta_N)$. Hence, the spectral theorem for compact operators implies $\sigma(\Delta_N) = \sigma_p(\Delta_N)$. Now we can conclude that, if it holds $\lambda \in \sigma_p(\Delta_N)$, then there exists $u \in W^2_{q,N}(\Omega) \cap L^q_{(0)}(\Omega)$ with $u \not\equiv 0$ such that

$$\begin{aligned} \lambda u - \Delta_N u &= 0 && \text{in } \Omega, \\ \partial_n u_{|\partial\Omega} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Testing this equation with the complex conjugate $\overline{u}$ we obtain

$$\lambda ||u||^2_{L^2(\Omega)} + ||\nabla u||^2_{L^2(\Omega)} = 0.$$

Since the Poincaré inequality with mean value yields $||\nabla u||^2_{L^2(\Omega)} \geq c_0 ||u||^2_{H^1(\Omega)} > 0$ for a constant $c_0 > 0$, cf. Theorem 2.7, it holds $\lambda \in (-\infty, 0)$ and thus we can conclude $\sigma(\Delta_N) \subseteq (-\infty, 0)$.

A change of variables in (3.114) implies

$$||\tilde{\varphi}||_{L^p(0,\infty;W^2_q(\Omega))} \leq C||\tilde{f}||_{L^p(0,\infty;L^q(\Omega))}$$

and therefore

$$||\varphi||_{L^p(0,T;W^2_q(\Omega))} \leq C||f||_{L^p(0,T;L^q(\Omega))}$$

for a constant $C > 0$ independent of δ. □

It still remains to define what we mean with a weak solution for (1.1) - (1.7).

Definition 3.14. *Let $T \in (0,\infty)$ and $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $\varphi_0 \in H^2_n(\Omega)$, $q_0 \in L^2(\Omega)$ be given. We call $(\mathbf{v}, \varphi, \mu, q)$ with the properties*

$$\begin{aligned} \mathbf{v} &\in L^2(0,T;H^1_0(\Omega)^d) \cap L^\infty(0,T;L^2_\sigma(\Omega)), \\ \varphi &\in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\ \mu &\in L^2(0,T;H^1(\Omega)), \\ q &\in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \end{aligned}$$

a weak solution of (1.1) - (1.7) if the following equations are satisfied:

$$\begin{aligned} &-\int_0^T\int_\Omega \rho\mathbf{v}\cdot\partial_t\boldsymbol{\psi}\,dx\,dt + \int_0^T\int_\Omega (\rho\mathbf{v}\otimes\mathbf{v}) : \nabla\boldsymbol{\psi}\,dx\,dt + \int_0^T\int_\Omega 2\eta(\varphi)D\mathbf{v} : D\boldsymbol{\psi}\,dx\,dt \\ &-\int_0^T\int_\Omega (\tilde{\mathbf{J}}\otimes\mathbf{v})\cdot\nabla\boldsymbol{\psi}\,dx\,dt - \left\langle \frac{\tilde{R}\mathbf{v}}{2}, \psi \right\rangle = \int_0^T\int_\Omega \left(\mu - \frac{h(q)}{\varepsilon}W'(\varphi)\right)\nabla\varphi\cdot\boldsymbol{\psi}\,dx\,dt \end{aligned}$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T; C_{0,\sigma}^\infty(\Omega))$ *and*

$$\int_0^T \int_\Omega m(\varphi, q)\nabla q \cdot \nabla \phi dx\, dt = \int_0^T \int_\Omega \left(f(q)W(\varphi) + g(q)\right) \partial_t \phi dx\, dt + \int_0^T \left(\frac{1}{\varepsilon} f(q)W(\varphi) + g(q)\right) \mathbf{v} \cdot \nabla \phi dx\, dt,$$

$$\int_0^T \int_\Omega \tilde{m}(\varphi)\nabla \mu \cdot \nabla \phi dx\, dt = \int_0^T \int_\Omega \varphi \partial_t \phi dx\, dt - \int_0^T \int_\Omega \nabla \varphi \cdot \mathbf{v}\ \phi dx\, dt,$$

$$\int_0^T \int_\Omega \mu \phi dx\, dt = \int_0^T \int_\Omega \varepsilon \nabla \varphi \cdot \nabla \phi dx\, dt + \int_0^T \int_\Omega \frac{1}{\varepsilon} h(q) W'(\varphi)\phi dx\, dt$$

for all $\phi \in C_0^\infty(0,T; C^1(\overline{\Omega}))$. *Moreover, the energy inequality (3.14) has to hold for all* $t \in [s,T)$ *and almost all* $s \in [0,T)$ *including* $s = 0$.

Then we get the main result of this chapter.

Theorem 3.15. *(Existence of weak solutions)*
Let the assumptions from Section 3.1 hold. Moreover, let $0 < T < \infty$ *and* $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $\varphi_0 \in H^2_n(\Omega)$ *and* $q_0 \in L^2(\Omega)$ *be given. Then there exists a weak solution* $(\mathbf{v}, \varphi, \mu, q)$ *in the sense of Definition 3.14.*

Proof. From Theorem 3.9 we get the existence of weak solutions $(\mathbf{v}^\delta, \varphi^\delta, \mu^\delta, q^\delta)$ in the sense of Definition 3.8 for every $\delta > 0$ together with the energy estimate

$$\int_s^t \int_\Omega (m(\varphi^\delta, q^\delta)|\nabla q^\delta|^2 + \tilde{m}(\varphi^\delta)|\nabla \mu^\delta|^2 + 2\eta(\varphi^\delta)|D\mathbf{v}^\delta|^2 + \delta|\Delta \mathbf{v}^\delta|^2 + \delta|\partial_t \varphi^\delta|^2)\, dx\, d\tau + E_{tot}(\mathbf{v}^\delta(t), \varphi^\delta(t), \nabla \varphi^\delta(t), q^\delta(t)) \le E_{tot}(\mathbf{v}(s), \varphi(s), \nabla \varphi(s), q(s))$$

for all $s \le t < T$ and almost all $0 \le s < T$ including $s = 0$. Moreover, it holds $\varphi^\delta \in L^2(0,T; H^2(\Omega))$ for every $\delta > 0$. Analogously as in (3.72) we can derive the following bounds from the energy inequality:

$$\begin{aligned}
&i) && (\mathbf{v}^\delta)_{\delta>0} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)^d),\\
&ii) && (\mathbf{v}^\delta)_{\delta>0} \text{ is bounded in } L^2(0,T;H^1(\Omega)^d),\\
&iii) && (\nabla q^\delta)_{\delta>0} \text{ is bounded in } L^2(0,T;L^2(\Omega)^d),\\
&iv) && (\nabla \mu^\delta)_{\delta>0} \text{ is bounded in } L^2(0,T;L^2(\Omega)^d),\\
&v) && (\nabla \varphi^\delta)_{\delta>0} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)^d),\\
&vi) && (W(\varphi^\delta))_{\delta>0} \text{ is bounded in } L^\infty(0,T;L^1(\Omega)),\\
&vii) && (G(q^\delta))_{\delta>0} \text{ is bounded in } L^\infty(0,T;L^1(\Omega)).
\end{aligned} \tag{3.115}$$

Since R also depends on δ, we write R^δ instead of R. Due to (3.1) we get

$$\begin{aligned}
\left\langle \frac{R^\delta \mathbf{v}^\delta}{2}, \boldsymbol{\psi} \right\rangle &= \int_0^T \int_\Omega \partial_t \rho(\varphi^\delta) \frac{\mathbf{v}^\delta}{2} \cdot \boldsymbol{\psi} \, dx\, dt - \frac{1}{2} \int_0^T \int_\Omega \left(\rho(\varphi^\delta)\mathbf{v}^\delta + \tilde{\mathbf{J}} \right) \cdot \nabla(\mathbf{v}^\delta \cdot \boldsymbol{\psi}) \, dx\, dt \\
&= \frac{1}{2} \left((\partial_t \rho(\varphi^\delta) + \operatorname{div}\tilde{\mathbf{J}} + \mathbf{v}^\delta \cdot \nabla \rho(\varphi^\delta))\mathbf{v}^\delta, \boldsymbol{\psi} \right)_{L^2(Q_T)} \\
&= \frac{1}{2} \left\langle \tilde{R}^\delta \mathbf{v}^\delta, \boldsymbol{\psi} \right\rangle
\end{aligned}$$

with

$$\tilde{R}^\delta = -\nabla \frac{\partial \rho(\varphi^\delta)}{\partial \varphi^\delta} \cdot (\tilde{m}(\varphi^\delta) \nabla \mu^\delta).$$

From now on we use $\tilde{R}^\delta$ instead of R^δ. We proceed with the existence proof for $\delta \to 0$. This proof can be done analogously to the proof of Theorem 3.9, where a lot of calculations simplify since we do not have to distinguish between the interpolants $(\mathbf{v}^N, \varphi^N, \mu^N, q^N)$ and $(\tilde{\mathbf{v}}^N, \tilde{\varphi}^N, \tilde{\mu}^N, \tilde{q}^N)$ for $N \in \mathbb{N}$.
As in the proof of Theorem 3.9 we can also conclude that the mean value of $\varphi^\delta(t)$ is constant and the mean value of $\mu^\delta(t)$ is bounded for a.e. $t \in (0,\infty)$. Therefore, we also get subsequences

i) $\mathbf{v}^\delta \rightharpoonup \mathbf{v}$ in $L^2(0,\infty;H^1(\Omega)^d)$,

ii) $\mathbf{v}^\delta \rightharpoonup^* \mathbf{v}$ in $L^\infty(0,\infty;L^2(\Omega)^d) \cong (L^1(0,\infty;L^2(\Omega)^d))'$,

iii) $q^\delta \rightharpoonup q$ in $L^2(0,T;H^1(\Omega))$,

iv) $q^\delta \rightharpoonup^* q$ in $L^\infty(0,\infty;L^2(\Omega)) \cong (L^1(0,\infty;L^2(\Omega)))'$,

v) $\varphi^\delta \rightharpoonup^* \varphi$ in $L^\infty(0,\infty;H^1(\Omega)) \cong (L^1(0,\infty;H^1(\Omega)))'$,

vi) $\mu^\delta \rightharpoonup \mu$ in $L^2(0,T;H^1(\Omega))$,

for all $T \in (0,\infty)$. Note that in contrast to Section 3.3, where we passed to the limit $N \to \infty$, we do not obtain $\mathbf{v}^\delta \rightharpoonup \mathbf{v}$ in $L^2(0,\infty; H^2(\Omega)^d)$ and $\partial_t\varphi^\delta \rightharpoonup \partial_t\varphi$ in $L^2(0,T;L^2(\Omega))$ as $\delta \to 0$.
The next step in the proof of Theorem 3.9 was to show compactness of $(\varphi^N)_{N\in\mathbb{N}}$ in $L^2(0,T;H^1(\Omega))$ together with higher regularity. The proof for $(\varphi^\delta)_{\delta>0}$ is similar to the proof for $(\tilde{\varphi}^N)_{N\in\mathbb{N}}$, where we can also use the Aubin-Lions lemma and do not have to distinguish between φ^N and $\tilde{\varphi}^N$. Analogously, we can prove compactness of $(q^\delta)_{\delta>0}$ in $L^2(0,T;L^2(\Omega))$.
Most calculations for the compactness of $(\mathbf{v}^\delta)_{\delta>0}$ can also be done analogously as in Section 3.3.5 for the compactness of $(\mathbf{v}^N)_{N\in\mathbb{N}}$. Therefore, we noted that all calculations until Remark 3.11 were independent of $\mathbf{v}^N \in L^2(0,T;H^2(\Omega)^d)$ and $\partial_t\tilde{\varphi}^N \in L^2(0,T;L^2(\Omega))$ and hence all conclusions also hold when we pass to the limit $\delta \to 0$.
But when we want to show the boundedness of $\partial_t(\mathbb{P}_\sigma(\rho^\delta\mathbf{v}^\delta))$ in $L^1(0,T;H^{-2}(\Omega)^d)$, we have to estimate (3.91) in another way since we use $\tilde{R}^\delta$ instead of R^δ. So let $\boldsymbol{\psi} \in L^\infty(0,T;H^2(\Omega)^d)$ be given. We have to show

$$\left|\left\langle \frac{\tilde{R}^\delta\mathbf{v}^\delta}{2}, \boldsymbol{\psi}\right\rangle\right| = \left|\int_0^T\int_\Omega \nabla\frac{\partial\rho(\varphi^\delta)}{\partial\varphi^\delta}\cdot(\tilde{m}(\varphi^\delta)\nabla\mu^\delta)\mathbf{v}^\delta\cdot\boldsymbol{\psi}\,dx\,dt\right| < C||\boldsymbol{\psi}||_{L^\infty(0,T;H^2(\Omega))} \tag{3.116}$$

for every $\boldsymbol{\psi} \in L^\infty(0,T;H^2(\Omega)^d)$ and a constant $C>0$ independent of $\delta>0$. Hence, we have to estimate terms of the form $\rho''(\varphi^\delta)\partial_j\varphi^\delta\tilde{m}(\varphi^\delta)\partial_j\mu^\delta\mathbf{v}_k^\delta\boldsymbol{\psi}_k$. To this end, we consider $\varphi^\delta = \varphi_1^\delta + \varphi_2^\delta + \varphi_0$, where φ_1^δ and φ_2^δ are the solutions of

$$\begin{aligned}
\delta\partial_t\varphi_1^\delta - \Delta\varphi_1^\delta &= P_0(h(q^\delta)W'(\varphi^\delta)+\Delta\varphi_0) && \text{in } (0,T)\times\Omega,\\
\delta\partial_t\varphi_2^\delta - \Delta\varphi_2^\delta &= P_0(\mu^\delta) && \text{in } (0,T)\times\Omega,\\
\partial_n\varphi^\delta_{1|\partial\Omega} = \partial_n\varphi^\delta_{2|\partial\Omega} &= 0 && \text{on } (0,T)\times\partial\Omega,\\
\varphi^\delta_{1|t=0} = \varphi^\delta_{2|t=0} &= 0 && \text{in } \Omega,\\
\int_\Omega\varphi_1^\delta\,dx = \int_\Omega\varphi_2^\delta\,dx &= 0.
\end{aligned}$$

Due to the growth condition $|h(s)| \le C(|s|+1)$ for every $s\in\mathbb{R}$ and the boundedness of q^δ in $L^\infty(0,T;L^2(\Omega))$, we can conclude that $h(q^\delta)$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Furthermore, the growth condition $|W'(a)| \le C(|a|^s+1)$ for every $a\in\mathbb{R}$ and fixed $0<s<1$ yields the boundedness of $W'(\varphi^\delta)$ in $L^\infty(0,T;L^{6+s_1}(\Omega))$, where $s_1>0$ depends on s. Therefore, $h(q^\delta)W'(\varphi^\delta)$ is bounded in $L^\infty(0,T;L^{\frac{3}{2}+s_2}(\Omega))$, where $s_2>0$ depends on s. Together with the boundedness of $\Delta\varphi_0$ in $L^\infty(0,T;L^2(\Omega))$, Lemma 3.13 yields that φ_1^δ is bounded in $L^p(0,T;W^2_{\frac{3}{2}+s_2}(\Omega))$ for every $1\le p<\infty$. Hence, it holds that

$$\partial_j\varphi_1^\delta \text{ is bounded in } L^p(0,T;W^1_{\frac{3}{2}+s_2}(\Omega)) \hookrightarrow L^p(0,T;L^{3+s_3}(\Omega))$$

for every $1 \leq p < \infty$ and $j = 1, ..., d$, where $s_3 > 0$ depends on s. Moreover, Lemma 3.13 implies the boundedness of φ_2^δ in $L^2(0,T;W_6^2(\Omega))$ and therefore

$$\partial_j \varphi_2^\delta \text{ is bounded in } L^2(0,T;W_6^1(\Omega)) \hookrightarrow L^2(0,T;L^\infty(\Omega))$$

for $j = 1, ..., d$. Furthermore, Theorem 2.32 yields that

$$\mathbf{v}^\delta \text{ is bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;L^6(\Omega)^d) \hookrightarrow L^{2+\varepsilon_1}(0,T;L^{6-\varepsilon_2}(\Omega)^d)$$

for some $\varepsilon_1, \varepsilon_2 > 0$. Altogether, the boundedness of $\partial_j \mu^\delta$ in $L^2(0,T;L^2(\Omega))$, $\mathbf{v}^\delta$ in $L^{2+\varepsilon_1}(0,T;L^{6-\varepsilon_2}(\Omega)^d)$ and $\partial_j \varphi_1^\delta$ in $L^p(0,T;L^{3+s_3}(\Omega))$ for every $1 \leq p < \infty$ and some $\varepsilon_1, \varepsilon_2, s_3 > 0$ yields

$$\left| \int_0^T \int_\Omega \rho''(\varphi^\delta) \partial_j \varphi_1^\delta \tilde{m}(\varphi^\delta) \partial_j \mu^\delta \mathbf{v}_k^\delta \boldsymbol{\psi}_k \, dx \, dt \right| < C ||\boldsymbol{\psi}||_{L^\infty(0,T;H^2(\Omega))}$$

for every $j, k = 1, ..., d$, a constant $C > 0$ independent of δ and $\boldsymbol{\psi} \in L^\infty(0,T;H^2(\Omega)^d)$ since $\tilde{m}(\varphi^\delta)$ and $\rho''(\varphi^\delta)$ are bounded in $L^\infty(Q_T)$. Analogously, the boundedness of $\partial_j \varphi_2^\delta$ in $L^2(0,T;L^\infty(\Omega))$, $\partial_j \mu^\delta$ in $L^2(0,T;L^2(\Omega))$ and $\mathbf{v}^\delta$ in $L^\infty(0,T;L^2_\sigma(\Omega))$ implies

$$\left| \int_0^T \int_\Omega \rho''(\varphi^\delta) \partial_j \varphi_2^\delta \tilde{m}(\varphi^\delta) \partial_j \mu^\delta \mathbf{v}_k^\delta \boldsymbol{\psi}_k \, dx \, dt \right| < C ||\boldsymbol{\psi}||_{L^\infty(0,T;H^2(\Omega))}$$

for every $j, k = 1, ..., d$, a constant $C > 0$ independent of δ and $\boldsymbol{\psi} \in L^\infty(0,T;H^2(\Omega)^d)$. Both estimates together prove (3.116). Hence, we can conclude that $\mathbb{P}_\sigma(\rho^\delta \mathbf{v}^\delta)$ is bounded in $L^2(0,T;H^1(\Omega)^d) \cap W_1^1(0,T;H^{-2}(\Omega)^d)$, which embeds compactly into $L^2(0,T;L^2(\Omega)^d)$ due to Aubin-Lions. Thus we can show $\mathbb{P}_\sigma(\rho^\delta \mathbf{v}^\delta) \to \mathbb{P}_\sigma(\rho \mathbf{v})$ in $L^2(0,T;L^2(\Omega)^d)$ and therefore

$$\int_0^T \int_\Omega \rho^\delta |\mathbf{v}^\delta|^2 \, dx \, dt = \int_0^T \int_\Omega \mathbb{P}_\sigma(\rho^\delta \mathbf{v}^\delta) \cdot \mathbf{v}^\delta \, dx \, dt \to \int_0^T \mathbb{P}_\sigma(\rho \mathbf{v}) \cdot \mathbf{v} = \int_0^T \int_\Omega \rho |\mathbf{v}|^2 \, dx \, dt$$

as $\delta \to 0$, which implies $(\rho^\delta)^{\frac{1}{2}} \mathbf{v}^\delta \to \rho^{\frac{1}{2}} \mathbf{v}$ in $L^2(Q_T)$. This yields

$$\mathbf{v}^\delta = (\rho^\delta)^{\frac{1}{2}} (\rho^\delta)^{\frac{1}{2}} \mathbf{v}^\delta \to \mathbf{v} \qquad \text{in } L^2(0,T;L^2(\Omega)^d).$$

Note that we can also prove

$$\mathbf{v}^\delta \to \mathbf{v} \qquad \text{in } L^2(0,T;L^{6-\varepsilon}(\Omega)^d) \text{ for every } 0 < \varepsilon \leq 5,$$

but now we are not able to show $\mathbf{v}^\delta \to \mathbf{v}$ in $L^q(0,T;L^\infty(\Omega)^d)$ for all $1 \leq q < \frac{8}{3}$ again, cf. (3.98), since we can not conclude that $\mathbf{v}^\delta$ is bounded in $L^2(0,T;H^2(\Omega)^d)$.

The next steps in the proof are similar to the ones in the proof of Theorem 3.9, where many calculations simplify since we do not have to distinguish between two interpolant functions.
The only remaining part which is very different to the proof of Theorem 3.9 is where we want to show

$$\left\langle \frac{\tilde{R}^\delta \mathbf{v}^\delta}{2}, \boldsymbol{\psi} \right\rangle \to \left\langle \frac{R\mathbf{v}}{2}, \boldsymbol{\psi} \right\rangle \tag{3.117}$$

and

$$\delta \int_0^T \int_\Omega \Delta \mathbf{v}^\delta \Delta \boldsymbol{\psi} \, dx \, dt \to 0 \tag{3.118}$$

as $\delta \to 0$ for all $\boldsymbol{\psi} \in C_0^\infty(0,T; C_{0,\sigma}^\infty(\Omega))$. We already proved that $\left\langle \frac{\tilde{R}^\delta \mathbf{v}^\delta}{2}, \cdot \right\rangle$ is bounded in $L^1(0,T; H^{-2}(\Omega)^d)$, cf. (3.116). For the proof of (3.117), we consider as before $\varphi^\delta = \varphi_1^\delta + \varphi_2^\delta + \varphi_0$, where φ_1^δ and φ_2^δ are the solutions of

$$\begin{aligned}
\delta \partial_t \varphi_1^\delta - \Delta \varphi_1^\delta &= P_0(h(q^\delta) W'(\varphi^\delta) + \Delta \varphi_0) && \text{in } (0,T) \times \Omega, \\
\delta \partial_t \varphi_2^\delta - \Delta \varphi_2^\delta &= P_0(\mu^\delta) && \text{in } (0,T) \times \Omega, \\
\partial_n \varphi^\delta_{1|\partial\Omega} = \partial_n \varphi^\delta_{2|\partial\Omega} &= 0 && \text{on } (0,T) \times \partial\Omega, \\
\varphi^\delta_{1|t=0} = \varphi^\delta_{2|t=0} &= 0 && \text{in } \Omega, \\
\int_\Omega \varphi_1^\delta \, dx = \int_\Omega \varphi_2^\delta \, dx &= 0.
\end{aligned}$$

We can prove the convergence (3.117) for the term $\rho''(\varphi^\delta) \partial_j \varphi_1^\delta \tilde{m}(\varphi^\delta) \partial_j \mu^\delta \mathbf{v}_k^\delta \boldsymbol{\psi}_k$ by the following arguments:

i) It holds $\mathbf{v}^\delta \to \mathbf{v}$ in $L^2(0,T; L_\sigma^{6-\varepsilon}(\Omega))$ as $\delta \to 0$ for every $0 < \varepsilon \leq 5$.

ii) Since $(\mathbf{v}^\delta)_{\delta>0}, \mathbf{v}$ is bounded in $L^\infty(0,T; L^2_\sigma(\Omega))$ and $\mathbf{v}^\delta \to \mathbf{v}$ in $L^2(0,T; L^{6-\varepsilon}(\Omega)^d)$ for every $0 < \varepsilon \leq 5$, we can conclude with Theorem 2.32

$$\mathbf{v}^\delta \to \mathbf{v} \qquad \text{in } L^{2+\varepsilon_1}(0,T; L^{6-\varepsilon_2}(\Omega)^d),$$

where $\varepsilon_1, \varepsilon_2 > 0$ depend on $\varepsilon > 0$ and it holds $\varepsilon_i \to 0$ as $\varepsilon \to 0$ for $i = 1, 2$.

iii) We already know that $h(q^\delta)$ is bounded in $L^\infty(0,T; L^2(\Omega))$. Moreover, the growth condition $|W'(a)| \leq C(|a|^s + 1)$ implies the boundedness of $W'(\varphi^\delta)$ in $L^\infty(0,T; L^{6+s_1}(\Omega))$, where $s_1 > 0$ depends on s. Hence, we can conclude that $h(q^\delta) W'(\varphi^\delta) + \Delta \varphi_0$ is bounded in $L^\infty(0,T; L^{\frac{3}{2}+s_2}(\Omega))$, where $s_2 > 0$ depends on s_1 and therefore on s. Thus Lemma 3.13 yields for every $1 \leq q < \infty$ the estimate

$$||\varphi_1^\delta||_{L^q(0,T;W^2_{\frac{3}{2}+s_2}(\Omega))} \leq C||P_0(h(q^\delta)W'(\varphi^\delta) + \Delta\varphi_0)||_{L^q(0,T;L^{\frac{3}{2}+s_2}(\Omega))}.$$

Hence, $\partial_j\varphi_1^\delta$ is bounded in $L^q(0,T;W^1_{\frac{3}{2}+s_2}(\Omega)) \hookrightarrow L^q(0,T;L^{3+s_3}(\Omega))$ for every $1 \leq q < \infty$, where $s_3 > 0$ depends on s. Due to its boundedness and since it holds $\partial_j\varphi_1^\delta(t,x) \to \partial_j\varphi_1(t,x)$ a.e. in $(0,T)\times\Omega$, Theorem 2.9 yields

$$\partial_j\varphi_1^\delta \to \partial_j\varphi_1 \qquad \text{in } L^q(0,T;L^{3+s_4}(\Omega)) \text{ for all } 1 \leq q < \infty,\ j = 1,...,d$$

as $\delta \to 0$, where $0 < s_4 < s_3$ depends on s.

iv) From the previous steps we know

$$\begin{aligned}
\partial_j\varphi_1^\delta &\to \partial_j\varphi_1 && \text{in } L^q(0,T;L^{3+s_4}(\Omega)) \text{ for all } 1 \leq q < \infty,\ j = 1,...,d,\\
\mathbf{v}^\delta &\to \mathbf{v} && \text{in } L^{2+\varepsilon_1}(0,T;L^{6-\varepsilon_2}_\sigma(\Omega)),\\
\partial_j\mu^\delta &\rightharpoonup \partial_j\mu && \text{in } L^2(0,T;L^2(\Omega)) \text{ for all } j = 1,...,d,
\end{aligned}$$

where $s_4 > 0$ depends on $0 < s < 1$ and $\varepsilon_1, \varepsilon_2 > 0$ depend on $0 < \varepsilon \leq 5$ and it holds $\varepsilon_i \to 0$ for $i = 1,2$ as $\varepsilon \to 0$. Since $0 < s < 1$ from the growth condition for W' is fixed, $s_4 > 0$ is also fixed. Hence, we need to choose $\varepsilon_2 > 0$ and therefore $\varepsilon > 0$ so small that

$$\frac{1}{3+s_4} + \frac{1}{6-\varepsilon_2} + \frac{1}{2} \leq 1.$$

When we have chosen $\varepsilon > 0$ small enough, we also get $\varepsilon_1 > 0$. Then we have to choose $1 \leq q < \infty$ sufficiently large such that

$$\frac{1}{q} + \frac{1}{2+\varepsilon_1} + \frac{1}{2} = 1.$$

Hence, we can pass to the limit

$$\int_0^T\int_\Omega \rho''(\varphi^\delta)\partial_j\varphi_1^\delta\tilde{m}(\varphi^\delta)\partial_j\mu^\delta\mathbf{v}_k^\delta\boldsymbol{\psi}_k\,dx\,dt \to \int_0^T\int_\Omega \rho''(\varphi)\partial_j\varphi_1\tilde{m}(\varphi)\partial_j\mu\mathbf{v}_k\boldsymbol{\psi}_k\,dx\,dt$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T;C^\infty_{0,\sigma}(\Omega))$ as $\delta \to 0$.

Now we have to do similar estimates for φ_2^δ. We already know that $\partial_j\varphi_2^\delta$ is bounded in $L^2(0,T;W^1_6(\Omega))$ for $j = 1,...,d$. Moreover, we can conclude that $\partial_j\varphi_2^\delta$ is bounded in $L^\infty(0,T;L^2(\Omega))$ since it holds $\partial_j\varphi_2^\delta = \partial_j\varphi^\delta - \partial_j\varphi_1^\delta - \partial_j\varphi_0$ and this property holds for all terms on the right-hand side. From [AF03, Theorem 5.9] it follows

$$||\varphi_2^\delta(t,\cdot)||_{L^\infty(\Omega)} \leq C||\varphi_2^\delta(t,\cdot)||^\theta_{W^1_6(\Omega)}||\varphi_2^\delta(t,\cdot)||^{1-\theta}_{L^2(\Omega)}$$

for a.e. $t \in (0,T)$, where $\theta = \frac{3d}{2d+6}$, i.e., $\theta = \frac{3}{5}$ for $d=2$ and $\theta = \frac{3}{4}$ in the case $d=3$. In both cases we get that

$$\partial_j \varphi_2^\delta \text{ is bounded in } L^2(0,T;W_6^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^{\frac{8}{3}}(0,T;L^\infty(\Omega)).$$

Therefore, we can prove the convergence (3.117) for the remaining term $\rho''(\varphi^\delta)\partial_j\varphi_2^\delta \tilde{m}(\varphi^\delta)\partial_j\mu^\delta \mathbf{v}_k^\delta \boldsymbol{\psi}_k$:

i) We have already shown that $(\partial_j\varphi_2^\delta)_{\delta>0}$ is bounded in $L^{\frac{8}{3}}(0,T;L^\infty(\Omega))$ with $j=1,...,d$. Moreover, $\partial_j\varphi_2^\delta$ converges strongly in $L^2(0,T;L^2(\Omega))$ since this holds for $\partial_j\varphi^\delta$ and $\partial_j\varphi_1^\delta$. Hence, Theorem 2.32 yields that for every $1 \leq q_1 < \infty$ there exists $\varepsilon_1 > 0$ such that

$$\partial_j\varphi_2^\delta \to \partial_j\varphi_2 \qquad \text{in } L^{\frac{8}{3}-\varepsilon_1}(0,T;L^{q_1}(\Omega)),\ j=1,...,d,$$

where $\varepsilon_1 \to 0$ as $q_1 \to \infty$.

ii) Analogously as in the previous step, it follows from the boundedness of $\mathbf{v}^\delta$ in $L^\infty(0,T;L^2_\sigma(\Omega))$ and $\mathbf{v}^\delta \to \mathbf{v}$ in $L^2(0,T;L^{6-\varepsilon}(\Omega)^d)$ for every $0<\varepsilon\leq 5$ that for every $1 \leq q_2 < \infty$ there exists $\varepsilon_2 > 0$ such that

$$\mathbf{v}^\delta \to \mathbf{v} \qquad \text{in } L^{q_2}(0,T;L^{2+\varepsilon_2}(\Omega)^d),\ j=1,...,d,$$

where it holds $\varepsilon_2 \to 0$ as $q_2 \to \infty$.

iii) Since we know

$$\begin{aligned}
\partial_j\varphi_2^\delta &\to \partial_j\varphi_2 && \text{in } L^{\frac{8}{3}-\varepsilon_1}(0,T;L^{q_1}(\Omega)) \text{ for all } 1\leq q_1<\infty,\ j=1,...,d,\\
\mathbf{v}^\delta &\to \mathbf{v} && \text{in } L^{q_2}(0,T;L^{2+\varepsilon_2}(\Omega)^d) \text{ for all } 1\leq q_2<\infty,\\
\partial_j\mu^\delta &\rightharpoonup \partial_j\mu && \text{in } L^2(0,T;L^2(\Omega)) \text{ for all } j=1,...,d,
\end{aligned}$$

where $\varepsilon_i \to 0$ as $q_i \to \infty$, $i=1,2$, we can choose q_1 and q_2 in such a way that

$$\int_0^T\int_\Omega \rho''(\varphi^\delta)\partial_j\varphi_2^\delta\tilde{m}(\varphi^\delta)\partial_j\mu^\delta\mathbf{v}_k^\delta\boldsymbol{\psi}_k\,dx\,dt \to \int_0^T\int_\Omega \rho''(\varphi)\partial_j\varphi_2\tilde{m}(\varphi)\partial_j\mu\mathbf{v}_k\boldsymbol{\psi}_k\,dx\,dt$$

for all $\boldsymbol{\psi} \in C_0^\infty(0,T;C_{0,\sigma}^\infty(\Omega))$ as $\delta \to 0$.

Thus we have shown (3.117) and it remains to prove (3.118). Here we use that the energy estimate yields the boundedness of $\delta^{\frac{1}{2}}\Delta\mathbf{v}^\delta$ in $L^2(0,T;L^2(\Omega))$. Since it holds $\delta\Delta^2\mathbf{v}^\delta = \delta^{\frac{1}{2}}\Delta(\delta^{\frac{1}{2}}\Delta\mathbf{v}^\delta)$, we can conclude $\delta\Delta^2\mathbf{v}^\delta \to 0$ in $L^2(0,T;H^{-2}(\Omega))$, which shows (3.118).

Finally, we need to prove the energy estimate (3.14) for all $s \leq t < T$ and almost all $0 \leq s < T$ including $s=0$. Analogously as in the proof for $\delta > 0$ and in

[ADG13], one can show $E_{tot}(\mathbf{v}^\delta(t), \varphi^\delta(t), \nabla\varphi^\delta(t), q^\delta(t)) \to E_{tot}(\mathbf{v}(t), \varphi(t), \nabla\varphi(t), q(t))$ for a.e. $t \in (0,T)$ and since norms are lower semicontinuous and $\varphi^\delta(t,x) \to \varphi(t,x)$, $q^\delta(t,x) \to q(t,x)$ a.e. in $(0,T) \times \Omega$ we get

$$\liminf_{\delta\to 0} \int_0^T D^\delta(t)\tau(t)\,dt \geq \int_0^T D(t)\tau(t)\,dt$$

for all $\tau \in W_1^1(0,T)$ with $\tau \geq 0$ and $\tau(T) = 0$, where D^δ and D are defined by

$$\begin{aligned} D^\delta(t) := &\int_\Omega m(\varphi^\delta, q^\delta) \left|\nabla q^\delta\right|^2 dx + \int_\Omega \tilde{m}(\varphi^\delta)|\nabla\mu^\delta|^2\,dx + \int_\Omega 2\eta(\varphi^\delta)|D\mathbf{v}^\delta|^2\,dx \\ &+ \delta \int_\Omega |\Delta\mathbf{v}^\delta|^2\,dx + \delta \int_\Omega |\partial_t\varphi^\delta|^2\,dx, \\ D(t) := &\int_\Omega m(\varphi, q) \left|\nabla q\right|^2 dx + \int_\Omega \tilde{m}(\varphi)|\nabla\mu|^2\,dx + \int_\Omega 2\eta(\varphi)|D\mathbf{v}^\delta|^2\,dx. \end{aligned}$$

Due to the energy estimate in the case $\delta > 0$, cf. (3.63), we can deduce

$$E_{tot}(\mathbf{v}_0, \varphi_0, \nabla\varphi_0, q_0)\tau(0) + \int_0^T E_{tot}(\mathbf{v}^\delta(t), \varphi^\delta(t), \nabla\varphi^\delta(t), q^\delta(t))\tau'(t)\,dt \geq \int_0^T D^\delta(t)\tau(t)\,dt.$$

For the limit $\delta \to 0$ it follows

$$E_{tot}(\mathbf{v}_0, \varphi_0, \nabla\varphi_0, q_0)\tau(0) + \int_0^T E_{tot}(\mathbf{v}(t), \varphi(t), \nabla\varphi(t), q(t))\tau'(t)\,dt \geq \int_0^T D(t)\tau(t)\,dt$$

for all $\tau \in W_1^1(0,T)$ with $\tau \geq 0$ and $\tau(T) = 0$. Hence, we can apply Lemma 3.12, which yields the energy estimate (3.14) for all $s \leq t < T$ and almost all $0 \leq s < T$ including $s = 0$. $\square$

4 Sharp Interface Asymptotics for the Surfactant Model

In the following we give a short introduction to the method of formally matched asymptotic expansions and apply this method to the diffuse interface model (1.1) - (1.5) for $\rho \equiv 1$ so that we recover the corresponding sharp interface model, i.e., we study the sharp interface limit of the equations

$$\begin{aligned} \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) + \nabla p - \operatorname{div}(2\eta(\varphi) D\mathbf{v}) &= \operatorname{div}(-\varepsilon \nabla\varphi \otimes \nabla\varphi) && \text{in } Q_T, && (4.1) \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } Q_T, && (4.2) \\ \partial_t^\bullet \left(\frac{1}{\varepsilon} f(q) W(\varphi) + g(q) \right) &= \operatorname{div}\left(m(\varphi, q) \nabla q \right) && \text{in } Q_T, && (4.3) \\ \partial_t^\bullet \varphi &= \operatorname{div}(\tilde{m}(\varphi) \nabla \mu) && \text{in } Q_T, && (4.4) \\ -\varepsilon \Delta\varphi + h(q) \frac{1}{\varepsilon} W'(\varphi) &= \mu && \text{in } Q_T. && (4.5) \end{aligned}$$

In Chapter 3 we proved the existence of weak solutions $(\mathbf{v}_\varepsilon, q_\varepsilon, \mu_\varepsilon, \varphi_\varepsilon)_{\varepsilon>0}$ in the sense of Definition 3.14 depending on $\varepsilon > 0$, cf. Theorem 3.15, where the density $\rho(\varphi)$ was not necessarily constant. In this chapter we assume $\rho \equiv 1$ and that the solutions are sufficiently smooth. Moreover, we assume that there exist two different asymptotic expansions in powers of ε, one which is valid in the bulk and another one near the interface. In the following we call the asymptotic expansion in the bulk region the outer expansion and the other one the inner expansion. Since we assume that both expansions are solutions to the same problem, there has to exist a narrow region near the interface where both expansions are valid, i.e., both expansions have to match in this region. This leads to the so-called matching conditions for the inner and outer expansion.

In the first section of this chapter, we introduce the notation which we will use for the asymptotic analysis and we make further assumptions on m, $\tilde{m}$ and h. In the second section we derive the outer expansion for the diffuse interface model (4.1) - (4.5). Before we continue with the inner expansion, we introduce new coordinates in the inner region near the interface and deduce the matching conditions for the inner and outer expansion, cf. Section 4.3 and Section 4.4. In Section 4.5 we use the results from the previous sections to obtain the inner expansion of the phase field model (4.1) - (4.5) and to recover the corresponding sharp interface model. In the last section we identify the sharp interface model from Section 4.5 with one of the sharp interface models in [GLS14] and show that an energy estimate holds. For more details about this method we refer to [EGK08] and [Hol13].

4.1 Preliminaries for the Matched Asymptotics

In this section we state the assumptions on h and the mobilities m and $\tilde{m}$ and we introduce the notation which we need to describe how fast terms depending on the parameter ε converge to 0 as $\varepsilon \to 0$. To this end, we recall the Landau notation. Let $f, g : (0, \infty) \to \mathbb{R}$ be given. In the following, we define $\mathcal{O}(g)$, $o(g)$ and $\mathcal{O}_s(g)$ as in [Hol13, Section 1.3].

i) $f = \mathcal{O}(g)$ as $\varepsilon \searrow 0$ if there are constants $\delta, \kappa > 0$ such that

$$|f(\varepsilon)| \leq \kappa |g(\varepsilon)| \qquad \text{for all } 0 < \varepsilon < \delta.$$

ii) $f = o(g)$ as $\varepsilon \searrow 0$ if for every $\kappa > 0$ there is $\delta > 0$ such that

$$|f(\varepsilon)| \leq \kappa |g(\varepsilon)| \qquad \text{for all } 0 < \varepsilon < \delta.$$

iii) $f = \mathcal{O}_s(g)$ as $\varepsilon \searrow 0$ iff $f = \mathcal{O}(g)$ and $f \neq o(g)$ as $\varepsilon \searrow 0$.

Note that in the following we will omit the subscript s, i.e., we write $\mathcal{O}(1)$ instead of $\mathcal{O}_s(1)$. We explain this notation by the following example. Consider the equation

$$\varepsilon \partial_t \varphi + \varphi = \varepsilon \partial_{x_1} \mu + \Delta \mu.$$

Then we refer to $\varphi = \Delta\mu$ as the $\mathcal{O}(1)$ equation and to $\partial_t \varphi = \partial_{x_1}\mu$ as the $\mathcal{O}(\varepsilon)$ equation although by definition of the order symbols, every term in the previous equation would be $\mathcal{O}(1)$.

In the diffuse interface model (4.1) - (4.5) we have the mobilities $m(\varphi, q)$ and $\tilde{m}(\varphi)$. For these mobilities, we make the following assumptions.

Assumption 4.1. *There exist constants $c_0, c_1 > 0$ such that $c_0 < m(\varphi, q), \tilde{m}(\varphi) < c_1$ and $h(q) > c_0$ for all $\varphi, q \in \mathbb{R}$. Moreover, there exist functions M, m_0 and K such that*

$$\begin{aligned} m(\varphi, q) &= \frac{1}{\varepsilon} M(\varphi) K(q) + 1, \\ \tilde{m}(\varphi) &= \varepsilon m_0(\varphi) \end{aligned}$$

for all $\varphi, q \in \mathbb{R}$, where it holds $0 < K(q) < C$ for a constant C and M satisfies

$$M(\varphi) = W(\varphi)$$

for all $\varphi \in \mathbb{R}$ with $|\varphi| \leq 3$.

Note that this assumption implies $M(\pm 1) = M'(\pm 1) = 0$ and $M(\varphi) > 0$ for all $|\varphi| < 1$.

4.2 Outer Expansions

We assume that in the bulk phases, i.e., in the subsets of Ω where it holds $\varphi(x) \approx \pm 1$, the solutions of the phase field model (4.1) - (4.5) can be expanded in powers of ε, i.e.,

$$\begin{aligned} \mathbf{v}_\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k, & \varphi_\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k \varphi_k, & \mu_\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k \mu_k, \\ q_\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k q_k, & p_\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k p_k, \end{aligned} \tag{4.6}$$

where $\mathbf{v}_\varepsilon : (0,T) \times \Omega \to \mathbb{R}^d$, $\varphi_\varepsilon, \mu_\varepsilon, q_\varepsilon, p_\varepsilon : (0,T) \times \Omega \to \mathbb{R}$ for all $\varepsilon \in (0,\delta)$ and $\delta > 0$ sufficiently small. We start the outer expansion with equation (4.2). To this end, we substitute (4.6) into (4.2). Hence, we get

$$\operatorname{div}\left(\sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k\right) = 0$$

and therefore $\mathbf{v}_k = 0$ for every $k \in \mathbb{N}_0$. Next, we insert (4.6) into (4.4) and obtain

$$\partial_t\left(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k\right) + \nabla\left(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k\right) \cdot \left(\sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k\right) = \operatorname{div}\left(\varepsilon m_0\left(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k\right) \nabla\left(\sum_{k=0}^{\infty} \varepsilon^k \mu_k\right)\right).$$

Here the term $m_0(\sum_{k=0}^{\infty} \varepsilon \varphi_k)$ appears. For such terms we use the Taylor expansion. So let $f \in C^\infty(\mathbb{R})$ be given. Then we have for an arbitrary expansion $\psi_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k \psi_k$ the Taylor series

$$f\left(\sum_{k=0}^{\infty} \varepsilon^k \psi_k\right) = f(\psi_0) + \sum_{i=1}^{\infty}\left(\frac{f^{(i)}(\psi_0)}{i!}\left(\sum_{k=1}^{\infty} \varepsilon^k \psi_k\right)^i\right). \tag{4.7}$$

Since ε can be expected to be very small as we study the behaviour of the equations for $\varepsilon \to 0$, the most important terms are the ones with the lowest power in ε, which is denoted as the leading order expansion. Using the Taylor series for m_0 in the outer expansion of (4.4), the leading order expansion is given by

$$\partial_t \varphi_0 + \nabla \varphi_0 \cdot \mathbf{v}_0 = 0 \qquad\qquad (4.4), \mathcal{O}(1),$$

where the notation (4.4), $\mathcal{O}(1)$ means that this is the outer expansion of equation (4.4) with terms in the power of $\varepsilon^0 = 1$. Analogously, (4.4), $\mathcal{O}(\varepsilon)$ denotes the outer expansion of equation (4.4) with all the terms in the power of ε. From equation

(4.4),$\mathcal{O}(1)$ we want to deduce $\varphi_0 = \pm 1$. We note that (4.4),$\mathcal{O}(1)$ can also be written as

$$(*)\begin{cases} \partial_t \varphi_0(t,x) + \sum_{i=1}^{d} (\mathbf{v}_0(t,x))_i \partial_{x_i} \varphi_0(t,x) = 0 & \text{for all } (t,x) \in (0,\infty) \times \Omega, \\ \varphi_0(0,x) := \pm 1 & \text{for all } x \in \Omega. \end{cases}$$

The method of characteristics yields the existence of a unique solution φ_0. Moreover, the characteristic equations are given by

$$\begin{cases} \frac{d}{d\tau}\begin{pmatrix} t(\tau) \\ x(\tau) \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{v}_0(t(\tau), x(\tau)) \end{pmatrix}, \qquad \begin{pmatrix} t(0) \\ x(0) \end{pmatrix} = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}, \\ z'(\tau) = 0, \qquad z(0) = \pm 1, \end{cases}$$

where $z(\tau) := \varphi_0(t(\tau), x(\tau))$. Thus we get

$$t(\tau) = \tau, \qquad z(\tau) = \pm 1$$

for all τ. This implies

$$\varphi_0 = \pm 1. \tag{4.8}$$

This is what we would physically expect since φ models the volume fraction difference of both liquids, i.e., $\varphi = \varphi^2 - \varphi^1$, where φ^1 is the volume fraction of liquid 1 and φ^2 is the volume fraction of liquid 2. Thus in the bulk phases one volume fraction has to equal 1 while the other one vanishes. Therefore, φ should be ± 1 in the bulk phases. Since φ_0 is the term of the outer expansion of φ with the lowest power in ε, it is physically meaningful that φ_0 is ± 1 in the bulk.

In the next step we determine the outer expansion of (4.5). It holds

$$\sum_{k=0}^{\infty} \varepsilon^k \mu_k = -\varepsilon \Delta\Big(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k\Big) + h\Big(\sum_{k=0}^{\infty} \varepsilon^k q_k\Big) \frac{1}{\varepsilon} W'\Big(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k\Big).$$

Approximating the last terms with two Taylor series yields

$$\begin{aligned} \sum_{k=0}^{\infty} \varepsilon^k \mu_k = &- \varepsilon \Delta\Big(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k\Big) + \frac{1}{\varepsilon} \left\{ h(q_0) + \sum_{i=1}^{\infty} \frac{h^{(i)}(q_0)}{i!} \Big(\sum_{k=1}^{\infty} \varepsilon^k q_k\Big)^i \right\} \\ &\cdot \left\{ W'(\varphi_0) + \sum_{i=1}^{\infty} \frac{W^{(i+1)}(\varphi_0)}{i!} \Big(\sum_{k=1}^{\infty} \varepsilon^k \varphi_k\Big)^i \right\}. \end{aligned}$$

Hence, the leading order expansion is given by

$$\frac{1}{\varepsilon} h(q_0) W'(\varphi_0) = 0 \qquad\qquad (4.5), \mathcal{O}(\varepsilon^{-1}).$$

Since we have already shown $\varphi_0 = \pm 1$, it holds $W'(\varphi_0) = 0$. Thus we need to study (4.5),$\mathcal{O}(1)$. We obtain

$$\mu_0 = h(q_0)W''(\varphi_0)\varphi_1 + h'(q_0)q_1W'(\varphi_0) = h(q_0)W''(\varphi_0)\varphi_1 \qquad (4.5), \mathcal{O}(1)$$

due to $W'(\varphi_0) = 0$.

We proceed with equation (4.3). Substituting the outer expansions (4.6) into (4.3) yields

$$\begin{aligned} 0 = \partial_t &\left(\frac{1}{\varepsilon} f(\sum_{k=0}^{\infty} \varepsilon^k q_k) W(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k) + g(\sum_{k=0}^{\infty} \varepsilon^k q_k) \right) \\ &+ \nabla \left(\frac{1}{\varepsilon} f(\sum_{k=0}^{\infty} \varepsilon^k q_k) W(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k) + g(\sum_{k=0}^{\infty} \varepsilon^k q_k) \right) \cdot \left(\sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k \right) \\ &- \operatorname{div} \left(\frac{1}{\varepsilon} M(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k) K(\sum_{k=0}^{\infty} \varepsilon^k q_k) \nabla (\sum_{k=0}^{\infty} \varepsilon^k q_k) \right) - \Delta \left(\sum_{k=0}^{\infty} \varepsilon^k q_k \right). \end{aligned}$$

We apply the Taylor series (4.7) as before, where we only need to take into account the term with no power in ε to derive the equation with the lowest power in ε, i.e.,

$$\begin{aligned} 0 =& \partial_t \left(\frac{1}{\varepsilon} f(q_0) W(\varphi_0) \right) + \frac{1}{\varepsilon} \nabla \left(f(q_0) W(\varphi_0) \right) \cdot \mathbf{v}_0 \\ &- \operatorname{div} \left(\frac{1}{\varepsilon} M(\varphi_0) K(q_0) \nabla q_0 \right) \qquad (4.3), \mathcal{O}(\varepsilon^{-1}). \end{aligned}$$

But as it holds $W(\varphi_0) = M(\varphi_0) = 0$ for $\varphi_0 = \pm 1$, we have a look at the next order terms in (4.3), which we calculate with the Taylor series again. For the sake of clarity we study the terms separately. For the term $\partial_t(\frac{1}{\varepsilon} f(q) W(\varphi))$ we obtain

$$\partial_t \left(\frac{1}{\varepsilon} \Big\{ f(q_0) + \sum_{i=1}^{\infty} \Big(\frac{f^{(i)}(q_0)}{i!} (\sum_{k=1}^{\infty} \varepsilon^k q_k)^i \Big) \Big\} \Big\{ W(\varphi_0) + \sum_{i=1}^{\infty} \Big(\frac{W^{(i)}(\varphi_0)}{i!} (\sum_{k=1}^{\infty} \varepsilon^k \varphi_k)^i \Big) \Big\} \right).$$

Analogously, the term $\partial_t g(q)$ yields

$$\partial_t \left(g(q_0) + \sum_{i=1}^{\infty} \Big(\frac{g^{(i)}(q_0)}{i!} (\sum_{k=1}^{\infty} \varepsilon^k q_k)^i \Big) \right).$$

For the summand $\nabla(\frac{1}{\varepsilon} f(q) W(\varphi)) \cdot \mathbf{v}$ we get

$$\begin{aligned} &\nabla \left(\frac{1}{\varepsilon} \Big\{ f(q_0) + \sum_{i=1}^{\infty} \Big(\frac{f^{(i)}(q_0)}{i!} (\sum_{k=1}^{\infty} \varepsilon^k q_k)^i \Big) \Big\} \Big\{ W(\varphi_0) + \sum_{i=1}^{\infty} \Big(\frac{W^{(i)}(\varphi_0)}{i!} (\sum_{k=1}^{\infty} \varepsilon^k \varphi_k)^i \Big) \Big\} \right) \\ &\cdot \left(\sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k \right) \end{aligned}$$

and for the term $\nabla g(q) \cdot \mathbf{v}$ we obtain

$$\left(g(q_0) + \sum_{i=1}^{\infty} \Big(\frac{g^{(i)}(q_0)}{i!} (\sum_{k=1}^{\infty} \varepsilon^k q_k)^i \Big) \right) \cdot \left(\sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k \right).$$

Finally, the last term $\operatorname{div}(\frac{1}{\varepsilon} M(\varphi) K(q) \nabla q) + \Delta q$ yields for the $\mathcal{O}(1)$ expansion Δq_0, where the other terms vanish due to $M(\varphi_0) = M'(\varphi_0) = 0$. Comparing the terms with the second lowest power in ε, that is ε^0, we obtain the (4.3),$\mathcal{O}(1)$ equation

$$\begin{aligned} 0 = {} & \partial_t \left(f(q_0) W'(\varphi_0) \varphi_1 + f'(q_0) q_1 W(\varphi_0) + g(q_0) \right) \\ & + \nabla \left(f(q_0) W'(\varphi_0) \varphi_1 + f'(q_0) q_1 W(\varphi_0) + g(q_0) \right) \cdot \mathbf{v}_0 \\ & + \nabla \left(f(q_0) W(\varphi_0) \right) \cdot \mathbf{v}_1 - \Delta q_0 \qquad (4.3), \mathcal{O}(1). \end{aligned}$$

Since we already know $\varphi_0 = \pm 1$ this equation simplifies to

$$\partial_t g(q_0) + \nabla g(q_0) \cdot \mathbf{v}_0 = \Delta q_0 \qquad (4.3), \mathcal{O}(1).$$

Now we turn to equation (4.1). Inserting the outer expansions (4.6) into (4.1) yields

$$\begin{aligned} & \partial_t \left(\sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k \right) + \left(\sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k \right) \cdot \nabla \left(\sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k \right) + \nabla \left(\sum_{k=0}^{\infty} \varepsilon^k p_k \right) \\ & - \operatorname{div} \left(2\eta (\sum_{k=0}^{\infty} \varepsilon^k \varphi_k) D \left(\sum_{k=0}^{\infty} \varepsilon^k \mathbf{v}_k \right) \right) \\ = {} & -\frac{1}{2} \nabla \left(\varepsilon |\nabla \sum_{k=0}^{\infty} \varepsilon^k \varphi_k|^2 \right) + \left(\sum_{k=0}^{\infty} \varepsilon^k \mu_k - \frac{1}{\varepsilon} h(\sum_{k=0}^{\infty} \varepsilon^k q_k) W'(\sum_{k=0}^{\infty} \varepsilon^k \varphi_k) \right) \nabla \sum_{k=0}^{\infty} \varepsilon^k \varphi_k. \end{aligned}$$

We use the Taylor series (4.7) again. Hence, the leading order terms are given by

$$0 = -\frac{1}{\varepsilon} h(q_0) W'(\varphi_0) \nabla \varphi_0. \qquad (4.1), \mathcal{O}(\varepsilon^{-1}).$$

Since it holds $\varphi_0 = \pm 1$, we have a look at the next order equations, i.e., (4.1),$\mathcal{O}(1)$. We obtain

$$\begin{aligned} \partial_t \mathbf{v}_0 + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \nabla p_0 - \operatorname{div} \left(2\eta^{(i)} D \mathbf{v}_0 \right) = {} & \mu_0 \nabla \varphi_0 - h(q_0) W'(\varphi_0) \nabla \varphi_1 \qquad (4.1), \mathcal{O}(1) \\ & - h(q_0) W''(\varphi_0) \varphi_1 \nabla \varphi_0 - h'(q_0) q_1 W'(\varphi_0) \nabla \varphi_0, \end{aligned}$$

where $\eta^{(i)} = \eta(\varphi_0)$, i.e., $\eta^{(1)} = \eta(-1)$ is the viscosity in fluid 1 and $\eta^{(2)} = \eta(1)$ is the viscosity in the phase related to fluid 2. Since it holds $W'(\varphi_0) = 0$, we can conclude

$$\partial_t \mathbf{v}_0 + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \nabla p_0 - \operatorname{div} \left(2\eta^{(i)} D \mathbf{v}_0 \right) = 0 \qquad (4.1), \mathcal{O}(1).$$

Altogether the following equations have to hold in the bulk phases $\Omega^{(i)}(t)$, $i = 1, 2$,

$$\begin{aligned} \partial_t \mathbf{v}_0 + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \nabla p_0 - \operatorname{div}\left(\eta^{(i)} D\mathbf{v}_0\right) &= 0 && \text{in } \Omega^{(i)}(t), \\ \operatorname{div}(\mathbf{v}_0) &= 0 && \text{in } \Omega^{(i)}(t), \\ \partial_t g(q_0) + \nabla g(q_0) \cdot \mathbf{v}_0 &= \Delta q_0 && \text{in } \Omega^{(i)}(t), \end{aligned}$$

where $\Omega^{(i)}(t)$ denotes the bulk phase of fluid i, i.e., $\Omega^{(1)}(t) = \{x \in \Omega : \ \varphi_0 = -1\}$ and $\Omega^{(2)}(t) = \{x \in \Omega : \ \varphi_0 = 1\}$.

4.3 New Coordinates in the Inner Region

In the previous section we derived the equations which have to hold in the bulk phases as ε tends to 0. To this end, we assumed that the solutions are sufficiently smooth and that there exist asymptotic expansions in powers of ε for these solutions. Thus we were able to derive $\varphi_0 = \pm 1$ in the bulk. But this implies that there has to exist an interfacial region where φ_0 changes its value very fast from -1 to $+1$. Since we want to study the behaviour of the equations (4.1) - (4.5) in this region near the interface, we introduce new coordinates in the interfacial region. Moreover, we assume that there exists an inner expansion for the solutions which is valid near the interface.
Note that shorter versions of the following calculations can be found in [AGG12, Section 4.3 and Appendix], [ALS15, Section 3.2] and [EGK08, Section 7.9].

We assume that the limit of $(\Gamma^\varepsilon(t))_{\varepsilon>0}$ exists as $\varepsilon \to 0$, e.g. with respect to the Hausdorff distance, where

$$\Gamma^\varepsilon(t) := \{x \in \Omega : \ \varphi_\varepsilon(t, x) = 0\}.$$

We denote this limit set by $\Gamma := \Gamma^0 := (\Gamma(t))_{t \geq 0}$ and expect it to be a smoothly evolving interface between the two bulk phases which does not intersect with $\partial\Omega$. Note that we also assume that $\Gamma^\varepsilon(t)$ and $\partial\Omega$ do not intersect for every $\varepsilon > 0$. Moreover, we introduce new coordinates which we expect to be valid near Γ. The time interval is denoted by $I \subseteq \mathbb{R}$ and $U \subseteq \mathbb{R}^{d-1}$ denotes the spatial domain. Then we define a local parametrization of Γ by

$$\hat{\gamma} : I \times U \to \mathbb{R}^d.$$

Furthermore, we denote the unit normal to $\Gamma(t)$ by ν . It is pointing into the second phase, i.e., the phase where $\varphi_0 = 1$. For a point x close to $\hat{\gamma}(I \times U)$ we consider $d(t, x)$ to be the signed distance function to the interface $\Gamma(t)$, where it holds $d(t, x) > 0$ if $x \in \Omega^{(2)}(t) = \{x \in \Omega : \ \varphi(x, t) > 0\}$. Hence, we can introduce a local parametrization of $I \times \mathbb{R}^d$ in an interfacial region near $\hat{\gamma}(I \times U)$. To this end, we rescale the distance function d by ε^{-1}, i.e., we set $z(t, x) := \frac{d(t,x)}{\varepsilon}$, and define

$$\begin{aligned} &G^\varepsilon : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^d, \\ &G^\varepsilon(t, s, z) := (t, \hat{\gamma}(t, s) + \varepsilon z \nu(t, s)), \end{aligned}$$

where it holds $s \in U \subseteq \mathbb{R}^{d-1}$. Here we used that for every $t \in \mathbb{R}$ there exists $\delta(t) > 0$ such that every $x \in \Omega$ with $d(t,x) < \delta(t)$ has a unique representation $\hat{\gamma}(t,s) + \varepsilon z \nu(t,s)$. The normal velocity for the parametrization of the evolving hypersurface Γ^0 is denoted by $\mathcal{V} = \partial_t \hat{\gamma} \cdot \nu$. With these definitions it holds

$$D_{(t,s,z)} G^\varepsilon(t,s,z) = \begin{pmatrix} 1 & 0 & 0 \\ \partial_t \hat{\gamma} + \varepsilon z \partial_t \nu & \partial_s \hat{\gamma} + \varepsilon z \partial_s \nu & \varepsilon \nu \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}.$$

Hence, $D_{(t,s,z)} G^\varepsilon(t,s,z)$ can be written as

$$D_{(t,s,z)} G^\varepsilon(t,s,z) = \begin{pmatrix} 1 & 0 & ... & 0 \\ \mathbf{u} & & A & \end{pmatrix}$$

for a vector $\mathbf{u} \in \mathbb{R}^d$ and a matrix $A \in \mathbb{R}^{d\times d}$ given by

$$A = \begin{pmatrix} \partial_{s_1} \hat{\gamma} + \varepsilon z \partial_{s_1} \nu & ... & \partial_{s_{d-1}} \hat{\gamma} + \varepsilon z \partial_{s_{d-1}} \nu & \varepsilon \nu \end{pmatrix} \in \mathbb{R}^{d\times d}.$$

Since $\partial_{s_1} \hat{\gamma}, ..., \partial_{s_{d-1}} \hat{\gamma}$ is a basis of $T_{(t,x)}\Gamma(t)$ for fixed $t \in \mathbb{R}$ and ν is the unit normal, we can conclude that $\partial_{s_1} \hat{\gamma}, ..., \partial_{s_{d-1}} \hat{\gamma}, \nu$ is a local basis at the interface $\Gamma(t)$. Denoting the principal curvatures by κ_i, we can choose the parametrization $\hat{\gamma}$ in such a way that

$$\partial_{s_i} \nu(t,s) = d\nu_{\hat{\gamma}(t,s)}(\partial_{s_i} \hat{\gamma}) = -\kappa_i \partial_{s_i} \hat{\gamma}, \tag{4.9}$$

since the Weingarten map $d\nu_{\hat{\gamma}(t,s)} : T_{\hat{\gamma}(t,s)}\Gamma(t) \to T_{\hat{\gamma}(t,s)}\Gamma(t)$ is self-adjoint and therefore there exists an orthonormal basis such that the identity (4.9) holds.
Hence, we define $\tau_i := \partial_{s_i} \hat{\gamma}$ for every $i = 1, ..., d-1$. Then the matrix A can be written as

$$A = \begin{pmatrix} (1 - \varepsilon z \kappa_1)\tau_1 & ... & (1 - \varepsilon z \kappa_{d-1})\tau_{d-1} & \varepsilon \nu \end{pmatrix} \in \mathbb{R}^{d\times d}.$$

The vectors $\tau_1, ..., \tau_{d-1}, \nu$ are a local orthonormal basis at $\Gamma(t)$. We define the metric tensor in the new coordinates by

$$\begin{aligned} g_{ij} &:= (\partial_{s_i} \hat{\gamma} + \varepsilon z \partial_{s_i} \nu) \cdot (\partial_{s_j} \hat{\gamma} + \varepsilon z \partial_{s_j} \nu), && i,j = 1, ..., d-1, \\ g_{id} &:= g_{di} = (\partial_{s_i} \hat{\gamma} + \varepsilon z \partial_{s_i} \nu) \cdot \varepsilon \nu = 0, && i = 1, ..., d-1, \\ g_{dd} &:= \varepsilon \nu \cdot \varepsilon \nu = \varepsilon^2. \end{aligned}$$

Moreover, we can make a coordinate transformation such that

$$G := A^T \cdot A = \begin{pmatrix} (1 - \kappa_1 \varepsilon z)^2 & 0 & ... & 0 & 0 \\ 0 & (1 - \kappa_2 \varepsilon z)^2 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & (1 - \kappa_{d-1} \varepsilon z)^2 & 0 \\ 0 & 0 & ... & 0 & \varepsilon^2 \end{pmatrix}.$$

Note that we call this new matrix G since this is the metric tensor in the new coordinates. The inverse matrix of $G = (g_{ij})_{i,j=1}^d$ is then given by

$$G^{-1} = (g^{ij})_{i,j=1}^d = \begin{pmatrix} \frac{1}{(1-\kappa_1\varepsilon z)^2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{(1-\kappa_2\varepsilon z)^2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{(1-\kappa_{d-1}\varepsilon z)^2} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\varepsilon^2} \end{pmatrix}.$$

Due to

$$D_{(t,s,z)}G^\varepsilon(t,s,z) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \mathbf{u} & & A & \end{pmatrix},$$

we can deduce

$$(D_{(t,s,z)}G^\varepsilon(t,s,z))^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \hat{\mathbf{w}} & & B & \end{pmatrix},$$

where $B = G^{-1}A^T$ and

$$\hat{\mathbf{w}} = -B\mathbf{u} = -G^{-1}A^T(\partial_t\hat{\gamma} + \varepsilon z\partial_t\nu).$$

Therefore, it holds

$$\hat{\mathbf{w}} = \begin{pmatrix} -\frac{(A^T\partial_t\hat{\gamma})_1}{(1-\kappa_1\varepsilon z)^2} - \frac{\varepsilon z(A^T\partial_t\nu)_1}{(1-\kappa_1\varepsilon z)^2} \\ \vdots \\ -\frac{(A^T\partial_t\hat{\gamma})_{d-1}}{(1-\kappa_{d-1}\varepsilon z)^2} - \frac{\varepsilon z(A^T\partial_t\nu)_{d-1}}{(1-\kappa_{d-1}\varepsilon z)^2} \\ -\frac{(A^T\partial_t\hat{\gamma})_d}{\varepsilon^2} - \frac{z(A^T\partial_t\nu)_d}{\varepsilon} \end{pmatrix} = \begin{pmatrix} -\frac{\tau_1\cdot(\partial_t\hat{\gamma}+\varepsilon z\partial_t\nu)}{1-\kappa_1\varepsilon z} \\ \vdots \\ -\frac{\tau_{d-1}\cdot(\partial_t\hat{\gamma}+\varepsilon z\partial_t\nu)}{1-\kappa_{d-1}\varepsilon z} \\ -\frac{\nu\cdot(\partial_t\hat{\gamma}+\varepsilon z\partial_t\nu)}{\varepsilon} \end{pmatrix}.$$

Now let $t \in \mathbb{R}$ and $x \in \Omega$ be given such that $(G^\varepsilon)^{-1}(t,x) = (t, s(t,x), z(t,x))$ for $s \in \mathbb{R}^{d-1}$ and $z \in \mathbb{R}$, where $(G^\varepsilon)^{-1}$ is the inverse function of G^ε. Then we can conclude that $D_{(t,x)}(G^\varepsilon)^{-1}(t,x)$ reads as

$$(D_{(t,s,z)}G^\varepsilon(t,s,z))^{-1} = D_{(t,x)}(G^\varepsilon)^{-1}(t,x) = \begin{pmatrix} \partial_t t(t,x) & \nabla_x t(t,x) \\ \partial_t s_1(t,x) & \nabla_x s_1(t,x) \\ \vdots & \vdots \\ \partial_t s_{d-1}(t,x) & \nabla_x s_{d-1}(t,x) \\ \partial_t z(t,x) & \nabla_x z(t,x) \end{pmatrix}.$$

With these calculations we are able to derive some formulas for the transformation of a function defined in the outer variables to a function depending on the inner variables. To this end, we need to determine $\partial_t z(t,x)$. Due to $z(t,x) = \frac{d(t,x)}{\varepsilon}$ we can deduce

$$\partial_t z(t,x) = \frac{\partial_t d(t,x)}{\varepsilon} = \frac{1}{\varepsilon}\frac{\partial_t d(t,x)}{|\nabla d(t,x)|} = -\frac{1}{\varepsilon}\mathcal{V},$$

where we used $|\nabla d(t,x)| \equiv 1$ and $\mathcal{V}(t,x) = \frac{\partial_t d(t,x)}{|\nabla d(t,x)|}$, cf. (2.6) and (2.20) in [DDE05]. Note that it would not have been necessary to use $z(t,x) = \frac{d(t,x)}{\varepsilon}$. Due to the calculations above we know

$$\partial_t z(t,x) = -\frac{(A^T\partial_t\hat{\gamma})_d}{\varepsilon^2} - \frac{z(A^T\partial_t\nu)_d}{\varepsilon} = -\frac{\partial_t\hat{\gamma}\cdot\nu}{\varepsilon} - z\partial_t\nu\cdot\nu = -\frac{1}{\varepsilon}\mathcal{V}.$$

Now we define for a scalar function $b(t,x)$ the function $\tilde{b}(t,s,z)$ in the new coordinates via

$$\tilde{b}(t,s(t,x),z(t,x)) := b(t,x).$$

We remember that for a function $\tilde{b} : \Gamma \to \mathbb{R}$ defined on a surface $\Gamma \subseteq \mathbb{R}^d$ with $\hat{\gamma} : \mathbb{R}^{d-1} \to \mathbb{R}^d$ being a local parametrization of Γ with $\hat{\gamma}(u) = p$, the surface gradient of $\tilde{b}$ in $p = \hat{\gamma}(u)$ is defined by

$$\nabla_\Gamma \tilde{b}(p) := \sum_{i,j=1}^{d-1} g^{ij}(u)\partial_i(\tilde{b}\circ\hat{\gamma})(u)\partial_j\hat{\gamma}(u), \tag{4.10}$$

cf. [Dep10, Remark 2.22], where g denotes the metric and the coefficients g_{ij} are defined by $g(\tau_i,\tau_j)$ for a basis $\{\tau_1,...,\tau_d\}$ and (g^{ij}) are the coefficients of the inverse matrix of $G := (g_{ij})_{i,j=1}^d$. For a vector quantity $\mathbf{j}(t,x)$ we define $\tilde{\mathbf{j}}(t,s,z)$ in the new coordinates via

$$\tilde{\mathbf{j}}(t,s(t,x),z(t,x)) := \mathbf{j}(t,x).$$

For such a vector quantity the divergence on a surface Γ in $p = \gamma(u)$ is given by

$$\mathrm{div}_\Gamma \tilde{\mathbf{j}}(p) := \sum_{i,j=1}^{d-1} g^{ij}(u)\partial_i(\tilde{\mathbf{j}}\circ\hat{\gamma})(u)\cdot\partial_j\hat{\gamma}(u),$$

cf. [Dep10, Remark 2.24]. Note that we need to distinguish between the surface gradient $\nabla_\Gamma\tilde{b}$ on the hypersurface $\Gamma(t) := \{\hat{\gamma}(t,s) : \; s\in U\}$ and the surface gradient $\nabla_{\Gamma_{\varepsilon z}}\tilde{b}$ on the hypersurfaces $\Gamma_{\varepsilon z}(t) := \{\hat{\gamma}(t,s) + \varepsilon z\nu(t,s) : \; s \in U\}$ for $t \in I$ and arbitrary but fixed $\varepsilon > 0$, $z \in \mathbb{R}$. In our case formula (4.10) simplifies to

$$\nabla_\Gamma \tilde{b}(p) = \sum_{i=1}^{d-1} g^{ii}(u)\partial_i(b\circ\hat{\gamma})(u)\partial_i\hat{\gamma}(u),$$

$$\nabla_{\Gamma_{\varepsilon z}} \tilde{b}(p) = \sum_{i=1}^{d-1} g^{ii}(u)\partial_i(b\circ\hat{\gamma})(u)\partial_i\hat{\gamma}(u),$$

since the local basis $\partial_{s_1}\hat{\gamma}, ..., \partial_{s_{d-1}}\hat{\gamma}, \nu$ is orthonormal and for the second formula the local basis $\partial_{s_1}\hat{\gamma} + \varepsilon z \partial_{z_1}\nu, ..., \partial_{s_{d-1}}\hat{\gamma} + \varepsilon z \partial_{z_{d-1}}\nu, \nu$ is orthogonal. Note that in the formulas above the parametrizations $\hat{\gamma}$ and the coefficients g^{ii} are not the same. In the first formula $\hat{\gamma}$ is the parametrization of Γ, while in the second formula $\hat{\gamma}(t,s) + \varepsilon z \nu(t,s)$ is the parametrization of $\Gamma_{\varepsilon z}$.

With these considerations we are able to derive the identity

$$\begin{aligned}
\frac{d}{dt}b(t,x) &= \frac{d}{dt}\tilde{b}(t, s(t,x), z(t,x)) \\
&= \partial_t \tilde{b} + \nabla_s \tilde{b} \cdot \partial_t s + \partial_z \tilde{b} \partial_t z \\
&= \partial_t \tilde{b} - \sum_{i=1}^{d-1} \partial_{s_i}\tilde{b} \left(\frac{(A^T \partial_t \hat{\gamma})_i}{(1-\kappa_i \varepsilon z)^2} + \frac{\varepsilon z (A^T \partial_t \nu)_i}{(1-\kappa_i \varepsilon z)^2} \right) + \partial_z \tilde{b} \partial_t z \\
&= \partial_t \tilde{b} - \sum_{i=1}^{d-1} \partial_{s_i}\tilde{b} \left(\frac{(\partial_{s_i}\hat{\gamma} + \varepsilon z \partial_{s_i}\nu) \cdot \partial_t \hat{\gamma}}{(1-\kappa_i \varepsilon z)^2} + \frac{\varepsilon z (\partial_{s_i}\hat{\gamma} + \varepsilon z \partial_{s_i}\nu) \cdot \partial_t \nu}{(1-\kappa_i \varepsilon z)^2} \right) - \partial_z \tilde{b} \frac{1}{\varepsilon}\mathcal{V} \\
&= -\frac{1}{\varepsilon}\mathcal{V}\partial_z \tilde{b} + \partial_t \tilde{b} - \nabla_{\Gamma_{\varepsilon z}}\tilde{b} \cdot \partial_t \hat{\gamma} + \varepsilon z \nabla_{\Gamma_{\varepsilon z}}\tilde{b} \cdot \partial_t \nu.
\end{aligned} \tag{4.11}$$

With respect to the spatial variables we obtain

$$\begin{aligned}
\nabla_x b(t,x) = \nabla_x \tilde{b}(t, s(t,x), z(t,x)) &= \sum_{i=1}^{d-1} \left(\partial_{s_i}\tilde{b}\ \nabla_x s_i \right) + \partial_z \tilde{b} \nabla_x z(t,x) \\
&= \sum_{i=1}^{d-1} \left(\frac{\partial_{s_i}\tilde{b}}{(1-\kappa_i \varepsilon z)^2} (\partial_{s_i}\hat{\gamma} + \varepsilon z \partial_{s_i}\nu) \right) + \partial_z \tilde{b} \frac{\varepsilon \nu}{\varepsilon^2} \\
&= \frac{1}{\varepsilon}\partial_z \tilde{b} \nu + \nabla_{\Gamma_{\varepsilon z}}\tilde{b}.
\end{aligned} \tag{4.12}$$

This formula was also derived in [ALS15, Lemma 3.1]. Note that in the formulas (4.11) and (4.12) the surface gradient is with respect to the hypersurface $\Gamma_{\varepsilon z}$. But for the inner expansion we need the surface gradient on Γ. Hence, we need to show some relation between these two gradients. More precisely, it holds

$$\begin{aligned}
\nabla_{\Gamma_{\varepsilon z}}\tilde{b} &= \sum_{i=1}^{d-1} \left(\frac{\partial_{s_i}\tilde{b}}{(1-\kappa_i \varepsilon z)^2} (\partial_{s_i}\hat{\gamma} - \varepsilon z \kappa_i \partial_{s_i}\hat{\gamma}) \right) \\
&= \sum_{i=1}^{d-1} (1 + \kappa_i \varepsilon z + \text{h.o.t.}) \partial_{s_i}\tilde{b} \partial_{s_i}\hat{\gamma} = (1 + \text{h.o.t.}) \nabla_\Gamma \tilde{b},
\end{aligned} \tag{4.13}$$

where we used the Taylor series for $\frac{1}{1-\kappa_i \varepsilon z} = 1 + \kappa_i \varepsilon z + \text{h.o.t.}$ and $\partial_{s_i}\nu = -\kappa_i \partial_{s_i}\hat{\gamma}$ for $i = 1, ..., d-1$, cf. (4.9). Thus it follows from (4.12)

$$\nabla_x b = \frac{1}{\varepsilon}\partial_z \tilde{b} \nu + (1 + \text{h.o.t.}) \nabla_\Gamma \tilde{b}, \tag{4.14}$$

where $(1+\text{h.o.t.})\nabla_\Gamma \tilde{b}$ implies that the higher order terms in ε are in the tangential direction of Γ. Using the abbreviation $\partial_t^\circ \tilde{b} := \partial_t \tilde{b} - \nabla_\Gamma \tilde{b} \cdot \partial_t \hat{\gamma}$ and neglecting the terms to the power ε and higher order, we get from (4.11) the identity

$$\frac{d}{dt} b(t,x) = -\frac{1}{\varepsilon} \mathcal{V} \partial_z \tilde{b} + \partial_t^\circ \tilde{b} + \text{h.o.t.}. \tag{4.15}$$

For a vector quantity $\mathbf{j}(t,x)$ written in the new coordinates via

$$\tilde{\mathbf{j}}(t, s(t,x), z(t,x)) := \mathbf{j}(t,x),$$

we obtain with analogous calculations

$$\begin{aligned}
\operatorname{div}(\mathbf{j}(t,x)) &= \sum_{i=1}^{d} \partial_{x_i} (\tilde{\mathbf{j}}(t, s(t,x), z(t,x))_i \\
&= \sum_{i=1}^{d} \left(\sum_{l=1}^{d-1} (\partial_{s_l} \tilde{\mathbf{j}})_i (\partial_{x_i} s_l(t,x)) + (\partial_z \tilde{\mathbf{j}})_i \partial_{x_i} z(t,x) \right) \\
&= \sum_{i=1}^{d-1} \partial_{s_i} \tilde{\mathbf{j}} \cdot \frac{(\partial_{s_i} \hat{\gamma} + \varepsilon z \partial_{s_i} \nu)}{(1-\kappa_i \varepsilon z)^2} + \partial_z \tilde{\mathbf{j}} \cdot \frac{\varepsilon \nu}{\varepsilon^2} \\
&= \frac{1}{\varepsilon} \partial_z \tilde{\mathbf{j}} \cdot \nu + \operatorname{div}_{\Gamma_{\varepsilon z}} \tilde{\mathbf{j}},
\end{aligned}$$

where $\operatorname{div}_{\Gamma_{\varepsilon z}} \tilde{\mathbf{j}}$ is the divergence of $\tilde{\mathbf{j}}$ on $\Gamma_{\varepsilon z}$. We want to express the divergence in terms depending on the interface Γ again and not on the shifted interface $\Gamma_{\varepsilon z}$. Therefore, we can use the Taylor series analogously as in the derivation of (4.13) and obtain

$$\begin{aligned}
\operatorname{div}_{\Gamma_{\varepsilon z}} \tilde{\mathbf{j}} &= \sum_{i=1}^{d-1} \partial_{s_i} \tilde{\mathbf{j}} \cdot \frac{(\partial_{s_i} \hat{\gamma} + \varepsilon z \partial_{s_i} \nu)}{(1-\kappa_i \varepsilon z)^2} = \sum_{i=1}^{d-1} \partial_{s_i} \tilde{\mathbf{j}} \cdot \frac{(1-\kappa_i \varepsilon z) \partial_{s_i} \hat{\gamma}}{(1-\kappa_i \varepsilon z)^2} \\
&= \sum_{i=1}^{d-1} (1 + \kappa_i \varepsilon z + \text{h.o.t.}) \partial_{s_i} \tilde{\mathbf{j}} \cdot \partial_{s_i} \hat{\gamma} = (1 + \text{h.o.t.}) \operatorname{div}_\Gamma \tilde{\mathbf{j}},
\end{aligned}$$

where we used (4.9) again. Altogether, this yields the formula

$$\operatorname{div}(\mathbf{j}(t,x)) = \frac{1}{\varepsilon} \partial_z \tilde{\mathbf{j}} \cdot \nu + \operatorname{div}_\Gamma \tilde{\mathbf{j}} + \text{h.o.t.}. \tag{4.16}$$

Finally, we want to deduce a formula for the Laplace. To this end, we can use the results we have already proven. For the derivation of such a formula, we start with the formulas for $\Gamma_{\varepsilon z}$. Hence, we do all the calculations for the terms on the shifted interface $\Gamma_{\varepsilon z}$ and at the end we use how the formulas on the interface $\Gamma_{\varepsilon z}$ transfer to terms on the interface Γ. We obtain

$$\begin{aligned}\Delta_x b(t,s) &= \mathrm{div}_x(\nabla_x b(t,x)) = \mathrm{div}_x(\nabla_x \tilde{b}(t,s(t,x),z(t,x)))\\ &= \mathrm{div}_x\left(\frac{1}{\varepsilon}\partial_z\tilde{b}\nu + \nabla_{\Gamma_{\varepsilon z}}\tilde{b}\right)\\ &= \frac{1}{\varepsilon}\partial_z\left(\frac{1}{\varepsilon}\partial_z\tilde{b}\nu + \nabla_{\Gamma_{\varepsilon z}}\tilde{b}\right)\cdot\nu + \mathrm{div}_{\Gamma_{\varepsilon z}}\left(\frac{1}{\varepsilon}\partial_z\tilde{b}\nu + \nabla_{\Gamma_{\varepsilon z}}\tilde{b}\right)\\ &= \frac{1}{\varepsilon^2}\partial_{zz}\tilde{b}\nu\cdot\nu + \frac{1}{\varepsilon^2}\partial_z\tilde{b}\partial_z\nu\cdot\nu + \frac{1}{\varepsilon}(\partial_z\nabla_{\Gamma_{\varepsilon z}}\tilde{b})\cdot\nu\\ &\quad + \frac{1}{\varepsilon}(\nabla_{\Gamma_{\varepsilon z}}(\partial_z\tilde{b}))\cdot\nu + \frac{1}{\varepsilon}\partial_z\tilde{b}\mathrm{div}_{\Gamma_{\varepsilon z}}\nu + \mathrm{div}_{\Gamma_{\varepsilon z}}(\nabla_{\Gamma_{\varepsilon z}}\tilde{b}). \end{aligned}\tag{4.17}$$

Now we need to study the terms in (4.17) separately. Since ν is the unit normal to $\Gamma_{\varepsilon z}$, we obtain $\nu\cdot\nu = 1$ and therefore $\partial_z\nu\cdot\nu = 0$. Using these identities in the first two terms of (4.17) implies

$$\frac{1}{\varepsilon^2}\partial_{zz}\tilde{b}(\nu\cdot\nu) = \frac{1}{\varepsilon^2}\partial_{zz}\tilde{b}, \qquad \frac{1}{\varepsilon^2}\partial_z\tilde{b}(\partial_z\nu\cdot\nu) = 0.$$

Since ν is the unit normal to $\Gamma_{\varepsilon z}$ and $\nabla_{\Gamma_{\varepsilon z}}\tilde{b}$ is in the tangent space of the shifted interface $\Gamma_{\varepsilon z}$, we can conclude $\nabla_{\Gamma_{\varepsilon z}}\tilde{b}\cdot\nu = 0$ and therefore $\partial_z(\nabla_{\Gamma_{\varepsilon z}}\tilde{b}\cdot\nu) = 0$. But this implies

$$\partial_z(\nabla_{\Gamma_{\varepsilon z}}\tilde{b})\cdot\nu + \nabla_{\Gamma_{\varepsilon z}}\tilde{b}\cdot\partial_z\nu = 0.$$

Moreover, it holds $\partial_z\nu = 0$ and therefore we get for the third term in (4.17)

$$(\partial_z\nabla_{\Gamma_{\varepsilon z}}\tilde{b})\cdot\nu = 0.$$

The fourth term in (4.17) is also 0. Therefore, (4.17) simplifies to

$$\Delta_x b(t,s) = \frac{1}{\varepsilon^2}\partial_{zz}\tilde{b} + \frac{1}{\varepsilon}\partial_z\tilde{b}\mathrm{div}_{\Gamma_{\varepsilon z}}\nu + \mathrm{div}_{\Gamma_{\varepsilon z}}(\nabla_{\Gamma_{\varepsilon z}}\tilde{b}).$$

As it holds

$$\mathrm{div}_x\nu = \frac{1}{\varepsilon}\partial_z\nu\cdot\nu + \mathrm{div}_{\Gamma_{\varepsilon z}}\nu = \mathrm{div}_{\Gamma_{\varepsilon z}}\nu$$

and $\nu = \nabla_x d$, we can conclude

$$\Delta_x b(t,s) = \frac{1}{\varepsilon^2}\partial_{zz}\tilde{b} + \frac{1}{\varepsilon}\partial_z\tilde{b}(\Delta_x d) + \Delta_{\Gamma_{\varepsilon z}}\tilde{b}.$$

Now we need to express $\Delta_{\Gamma_{\varepsilon z}}$ in terms of Δ_Γ. We calculate

$$\begin{aligned}
\Delta_{\Gamma_{\varepsilon z}}\tilde{b} &= \operatorname{div}_{\Gamma_{\varepsilon z}}(\nabla_{\Gamma_{\varepsilon z}}\tilde{b}) = \operatorname{div}_{\Gamma_{\varepsilon z}}\left(\sum_{i=1}^{d-1}\left(\frac{\partial_{s_i}\tilde{b}}{(1-\kappa_i\varepsilon z)}\partial_{s_i}\hat{\gamma}\right)\right)\\
&= \sum_{k=1}^{d-1}\frac{1}{1-\kappa_k\varepsilon z}\partial_{s_k}\left(\sum_{i=1}^{d-1}\left(\frac{\partial_{s_i}\tilde{b}}{(1-\kappa_i\varepsilon z)}\partial_{s_i}\hat{\gamma}\right)\right)\cdot\partial_{s_k}\hat{\gamma}\\
&= \sum_{i=1}^{d-1}(1+\text{h.o.t.})\partial_{s_i}\tilde{b}\partial_{s_i}\hat{\gamma}\cdot\partial_{s_i}\hat{\gamma}\\
&= \Delta_\Gamma\tilde{b} + \text{h.o.t.},
\end{aligned}$$

where we used Taylor series again and the fact that $(\partial_{s_i}\hat{\gamma})$ is an orthonormal basis of Γ. Furthermore, we need to identify $\Delta_x d$. To this end, we use [GT01, Lemma 14.17], which yields

$$D^2 d = \begin{pmatrix}
\frac{-\kappa_1}{1-\kappa_1 d} & 0 & \dots & 0 & 0\\
0 & \frac{-\kappa_2}{1-\kappa_2 d} & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \dots & \frac{-\kappa_{d-1}}{1-\kappa_{d-1} d} & 0\\
0 & 0 & \dots & 0 & 0
\end{pmatrix}.$$

Moreover, we use the definition of z, i.e., $z(t,x) = \frac{d(t,x)}{\varepsilon}$, and the Taylor series $\frac{-\kappa_i}{1-\kappa_i\varepsilon z} = -\kappa_i - \varepsilon\kappa_i^2 z + \text{h.o.t.}$. Then we get

$$\Delta_x d = \sum_{i=1}^{d-1}\frac{-\kappa_i}{1-\kappa_i d} = \frac{-\kappa_i}{1-\varepsilon\kappa_i z} = -\sum_{i=1}^{d-1}\kappa_i - \sum_{i=1}^{d-1}\varepsilon\kappa_i^2 z + \text{h.o.t.}.$$

Denoting by κ the mean curvature, i.e., the sum of the principal curvatures κ_i, and by $|\mathcal{S}|$ the spectral norm of the Weingarten map $d\nu_{\gamma(t,s)}$, i.e., the l_2-norm of $(\kappa_1, ..., \kappa_{d-1})$, we can conclude

$$\Delta_x d = -\kappa - \varepsilon z|\mathcal{S}|^2 + \text{h.o.t.}.$$

Using these identities for $\Delta_{\Gamma_{\varepsilon z}}\tilde{b}$ and $\Delta_x d$ we can derive

$$\Delta_x b = \frac{1}{\varepsilon^2}\partial_{zz}\tilde{b} - \frac{1}{\varepsilon}(\kappa + \varepsilon z|\mathcal{S}|^2)\partial_z\tilde{b} + \Delta_\Gamma\tilde{b} + \text{h.o.t.}. \tag{4.18}$$

4.4 Matching Conditions

In the previous sections we derived the outer expansion for the diffuse interface model (4.1) - (4.5) and introduced new variables in an interfacial region. In this section, we assume that there exists an asymptotic expansion of the solutions in powers of ε with respect to the inner variables, i.e., we assume

$$\begin{aligned}
\mathbf{v}^\varepsilon(t,x) &= \mathbf{V}^\varepsilon(t,s(t,x),z(t,x)),\\
p^\varepsilon(t,x) &= P^\varepsilon(t,s(t,x),z(t,x)),\\
\varphi^\varepsilon(t,x) &= \Phi^\varepsilon(t,s(t,x),z(t,x)),\\
\mu^\varepsilon(t,x) &= M^\varepsilon(t,s(t,x),z(t,x)),\\
q^\varepsilon(t,x) &= Q^\varepsilon(t,s(t,x),z(t,x)),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{V}^\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k \mathbf{V}_k, & P^\varepsilon &= \sum_{k=-1}^{\infty} \varepsilon^k P_k, & \Phi^\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k \Phi_k,\\
M^\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k M_k, & Q^\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k Q_k.
\end{aligned} \tag{4.19}$$

Since the inner and the outer expansions are expansions of the same functions, we assume that there exists a narrow region near the interface where both expansions are valid, i.e., they have to match. This leads to the so-called matching conditions which have to be satisfied at the interface $\Gamma(t)$ and which we will derive in the following for the diffuse interface model (4.1) - (4.5) analogously as in [EGK08, Chapter 7.9]. Moreover, a short derivation of the matching conditions together with an introduction to the variable transformation in inner variables can also be found in [GS06, Appendix].

In the following all calculations are done for the functions φ_k and Φ_k from the outer and inner expansion, respectively , but the calculations are also valid for $\mathbf{v}_k, \mathbf{V}_k$, μ_k, M_k and q_k, Q_k.
First of all we introduce the new variable $r := z\varepsilon = d(t,x)$. Then we define functions $\hat{\varphi}_k$ in the new coordinates via

$$\hat{\varphi}_k(t,s(t,x),r(t,x)) := \varphi_k(t,x), \qquad k = 0,1,2,\dots .$$

Note that $r(t,x)$ describes the unscaled distance to the interface $\Gamma(t)$ since $z(t,x)$ was defined as the scaled distance which we used for the inner expansion, i.e., $z(t,x) = \frac{d(t,x)}{\varepsilon}$.
For the derivation of the outer expansion we assumed that the solutions are sufficiently smooth. For the derivation of the matching conditions, we additionally assume that the functions φ_k of the outer expansion (4.6) can be expanded smoothly

on Γ^0 from both regions $\Omega^{(1)}(t)$ and $\Omega^{(2)}(t)$. Due to Taylor-expansion nearby $r = 0$ we obtain

$$\hat{\varphi}_k(t,s,r) = \hat{\varphi}_k(t,s,0+) + \partial_r\hat{\varphi}_k(t,s,0+)r + \frac{1}{2}\partial_{rr}\hat{\varphi}_k(t,s,0+)r^2 + ..., \tag{4.20}$$

$$\hat{\varphi}_k(t,s,r) = \hat{\varphi}_k(t,s,0-) + \partial_r\hat{\varphi}_k(t,s,0-)r + \frac{1}{2}\partial_{rr}\hat{\varphi}_k(t,s,0-)r^2 + ..., \tag{4.21}$$

for small $r > 0$ and $r < 0$, respectively, where we define $\hat{\varphi}_k(t,s,0+) := \lim\limits_{\delta\searrow 0}\hat{\varphi}_k(t,s,\delta)$ and $\hat{\varphi}_k(t,s,0-) := \lim\limits_{\delta\nearrow 0}\hat{\varphi}_k(t,s,\delta)$.

For the functions $\Phi_k(t,s,z)$ of the inner expansion (4.19) we are interested in their behaviour for $z \to \pm\infty$. We assume that for every $k \in \mathbb{N}_0$ there exists suitable $n_k \in \mathbb{N}$ such that

$$\Phi_k(t,s,z) \approx \Phi^+_{k,0}(t,s) + \Phi^+_{k,1}(t,s)z + ... + \Phi^+_{k,n_k}(t,s)z^{n_k} \qquad \text{for } z \to \infty,$$
$$\Phi_k(t,s,z) \approx \Phi^-_{k,0}(t,s) + \Phi^-_{k,1}(t,s)z + ... + \Phi^-_{k,n_k}(t,s)z^{n_k} \qquad \text{for } z \to -\infty.$$

Since the inner and the outer expansions are solutions to the same problem, we can assume that there exists a narrow region near the interface where both solutions have to match. In particular this implies that the values of Q, Φ, M and $\mathbf{V}$ as $z \to \infty$ have to coincide with the values of $\hat{q}, \hat{\varphi}, \hat{\mu}$ and $\hat{\mathbf{v}}$ as $r \to 0$.
For the derivation of the matching condition we introduce an intermediate variable $\tilde{r}$ given by $\tilde{r} := \frac{r}{\varepsilon^\alpha} = \varepsilon^{1-\alpha}z$, which is valid in the narrow region near the interface where both expansions match. Therefore, we assume $0 < \alpha < 1$ since this intermediate variable is located in the region where the inner and outer expansion are valid. Then we study the limit $\varepsilon \to 0$ for fixed $\tilde{r}$, which implies $r \to 0$ and $z \to \infty$.
Moreover, we need to change the variables for the outer expansions from r to $\tilde{r}$. As it holds $r = \varepsilon^\alpha\tilde{r}$ we obtain

$$\hat{\varphi}^\varepsilon(r) := \hat{\varphi}(\varepsilon, r) := \sum_{k=0}^{\infty}\varepsilon^k\hat{\varphi}_k(r)$$

at $r = \varepsilon^\alpha\tilde{r}$ with $\tilde{r} > 0$, i.e.,

$$\begin{aligned}\hat{\varphi}^\varepsilon(\varepsilon^\alpha\tilde{r}) =& \hat{\varphi}_0(0+) + \varepsilon^\alpha\partial_r\hat{\varphi}_0(0+)\tilde{r} + \frac{1}{2}\varepsilon^{2\alpha}\partial_{rr}\hat{\varphi}_0(0+)\tilde{r}^2 + \mathcal{O}(\varepsilon^{3\alpha})\\ &+ \varepsilon\hat{\varphi}_1(0+) + \varepsilon^{1+\alpha}\partial_r\hat{\varphi}_1(0+)\tilde{r} + \mathcal{O}(\varepsilon^{1+2\alpha})\\ &+ \varepsilon^2\hat{\varphi}_2(0+) + \mathcal{O}(\varepsilon^{2+\alpha}),\end{aligned} \tag{4.22}$$

where we used the Taylor expansion (4.20) for $r > 0$ small enough. Note that the right-hand side of (4.22) now depends on $\tilde{r}$ instead of r.
Analogously we change the variables for the inner expansion from z to $\tilde{r}$. As it holds $z = \frac{r}{\varepsilon} = \varepsilon^{\alpha-1}\tilde{r}$, we obtain

$$\hat{\Phi}^\varepsilon(z) := \hat{\Phi}(\varepsilon, z) := \sum_{k=0}^{\infty}\varepsilon^k\Phi_k(z) = \sum_{k=0}^{\infty}\left\{\varepsilon^k\sum_{j=0}^{n_k}\Phi^+_{k,j}z^j\right\}$$

at $z = \varepsilon^{\alpha-1}\tilde{r}$, i.e.,

$$\begin{aligned}\hat{\Phi}^\varepsilon(\varepsilon^{\alpha-1}\tilde{r}) =&\Phi^+_{0,0} + \varepsilon^{\alpha-1}\Phi^+_{0,1}\tilde{r} + ... + \varepsilon^{n_0(\alpha-1)}\Phi^+_{0,n_0}\tilde{r}^{n_0} + \\ &+ \varepsilon\Phi^+_{1,0} + \varepsilon^{\alpha}\Phi^+_{1,1}\tilde{r} + ... + \varepsilon^{1+n_1(\alpha-1)}\Phi^+_{1,n_1}\tilde{r}^{n_1} \\ &+ \varepsilon^2\Phi^+_{2,0} + \varepsilon^{1+\alpha}\Phi^+_{2,1}\tilde{r} + \varepsilon^{2\alpha}\Phi^+_{2,2}\tilde{r}^2 + ... + \varepsilon^{2+n_2(\alpha-1)}\Phi^+_{2,n_2}\tilde{r}^{n_2} + ..., \end{aligned} \tag{4.23}$$

where the right-hand side now depends on $\tilde{r}$ instead of z. Hence, the expansions (4.22) and (4.23) match if $\hat{\varphi}^\varepsilon$ and $\hat{\Phi}^\varepsilon$ coincide in all terms where they have the same power in ε and $\tilde{r}$. It follows

$$\begin{aligned}&\Phi^+_{0,0} = \hat{\varphi}_0(0+), \quad \Phi^+_{0,i} = 0 && \text{for } 1 \le i \le n_0, \\ &\hat{\Phi}^+_{1,0} = \hat{\varphi}_1(0+), \quad \Phi^+_{1,1} = \partial_r\hat{\varphi}_0(0+), \quad \Phi^+_{1,i} = 0 && \text{for } 2 \le i \le n_1, \\ &\hat{\Phi}^+_{2,0} = \hat{\varphi}_2(0+), \quad \Phi^+_{2,1} = \partial_r\hat{\varphi}_1(0+), \quad \Phi^+_{2,2} = \frac{1}{2}\partial_{rr}\hat{\varphi}_0(0+), \quad \Phi^+_{2,i} = 0 && \text{for } 3 \le i \le n_2.\end{aligned}$$

Every x near the interface $\Gamma(t)$ can be described in terms of r and s by $x(r,s) = \hat{\gamma}(s) + r\nu(\hat{\gamma}(s))$. So let $x = x(0,s) \in \Gamma(t)$ be given and let $x\pm$ be the limit as $x_n \to x$ in $\Omega^{(2)}(t)$ resp. in $\Omega^{(1)}(t)$. Then it holds

$$\begin{aligned}\partial_r\hat{\varphi}_0(0\pm) = \partial_r\hat{\varphi}_0(t,s,r)_{|r=0\pm} = \partial_r\varphi_0(t,x(r,s))_{|r=0\pm} = \partial_r\varphi_0(t,\hat{\gamma}(s) + r\nu(\hat{\gamma}(s)))_{|r=0\pm} \\ = \nabla\varphi_0(t,x\pm)\cdot\nu(x\pm).\end{aligned}$$

Analogous calculations yield

$$\partial_{rr}\hat{\varphi}_0(0\pm) = (\nu(x\pm)\cdot\nabla)(\nu(x\pm)\cdot\nabla)\varphi_0(t,x\pm).$$

Moreover, we have

$$\begin{aligned}&\Phi_1(t,s,z) \approx \Phi^\pm_{1,0}(t,s) + \Phi^\pm_{1,1}(t,s)z && \text{for } z \to \pm\infty, \\ &\Phi_2(t,s,z) \approx \Phi^\pm_{2,0}(t,s) + \Phi^\pm_{2,1}(t,s)z + \Phi^\pm_{2,2}(t,s)z^2 && \text{for } z \to \pm\infty,\end{aligned}$$

since the terms with higher powers in z are 0. Hence, we can derive the following matching conditions at $x = \hat{\gamma}(s)$

$$\Phi_0(z,s) = \Phi^\pm_{0,0}(s) = \hat{\varphi}_0(0\pm) = \varphi_0(x\pm), \tag{4.24}$$

$$\Phi_1(z) = \hat{\varphi}_1(0\pm) + \partial_r\hat{\varphi}_0(0\pm)z = \varphi_1(x\pm) + (\nabla\varphi_0(x\pm)\cdot\nu)z, \tag{4.25}$$

$$\partial_z\Phi_1(z,s) = \Phi^\pm_{1,1}(t,s) \approx \nabla\varphi_0(x\pm)\cdot\nu, \tag{4.26}$$

$$\partial_z\Phi_0(z,s) = 0, \tag{4.27}$$

$$\Phi_2(z,s) = \varphi_2(x\pm) + (\nabla\varphi_1(x\pm)\cdot\nu)z + \frac{1}{2}(\nu\cdot\nabla)(\nu\cdot\nabla)\varphi_0(x\pm)z^2, \tag{4.28}$$

$$\partial_z\Phi_2(z,s) = \nabla\varphi_1(x\pm)\cdot\nu + (\nu\cdot\nabla)(\nu\cdot\nabla)\varphi_0(x\pm)z \tag{4.29}$$

as $z \to \pm\infty$.

4.5 Inner Expansions

Analogously as in Section 4.2 we insert the asymptotic expansion (4.19) for the inner variables into the equations for the diffuse interface model (4.1) - (4.5), apply the formulas from Section 4.3 for functions defined with respect to the inner variables and compare the terms with the same power in ε. Due to the matching conditions (4.24) - (4.29) from Section 4.4 we are then able to derive the equations on the interface $\Gamma(t)$ for the corresponding sharp interface model.

At the beginning we derive the asymptotic expansion for every equation and then have a look at the different order terms, where the leading order term is the one with the lowest power in ε again. Since higher order terms appear in every inner expansion, we omit the term +h.o.t. for the sake of clarity.

Equation (4.1):
For the first terms we do the calculations in detail. From the first two terms on the left-hand side we obtain

$$\begin{aligned}
\partial_t^\bullet \mathbf{v} = &-\frac{1}{\varepsilon}\mathcal{V}\partial_z\left(\sum_{k=0}^{\infty}\varepsilon^k\mathbf{V}_k\right)+\partial_t\left(\sum_{k=0}^{\infty}\varepsilon^k\mathbf{V}_k\right)-\nabla_\Gamma\left(\sum_{k=0}^{\infty}\varepsilon^k\mathbf{V}_k\right)\partial_t\hat{\gamma}\\
&+\left(\nabla_\Gamma\left(\sum_{k=0}^{\infty}\varepsilon^k\mathbf{V}_k\right)+\frac{1}{\varepsilon}\partial_z\left(\sum_{k=0}^{\infty}\varepsilon^k\mathbf{V}_k\right)\nu\right)\cdot\left(\sum_{k=0}^{\infty}\varepsilon^k\mathbf{V}_k\right),\\
\nabla p = &\left(\nabla_\Gamma\left(\sum_{k=-1}^{\infty}\varepsilon^k P_k\right)+\frac{1}{\varepsilon}\partial_z\left(\sum_{k=-1}^{\infty}\varepsilon^k P_k\right)\nu\right).
\end{aligned}$$

In the next step we use $D\mathbf{v}=\frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T)$. Moreover, we have to take into account that the velocity field $\mathbf{v}$ is a vector field. Thus we obtain

$$\nabla_x\mathbf{v}=\nabla_\Gamma\mathbf{V}^\varepsilon+\frac{1}{\varepsilon}\partial_z\mathbf{V}^\varepsilon\otimes\nu.$$

Using this identity we can deduce

$$\begin{aligned}
D\mathbf{v}&=\frac{1}{2}\left(\left(\nabla_\Gamma\mathbf{V}^\varepsilon+\frac{1}{\varepsilon}\partial_z\mathbf{V}^\varepsilon\otimes\nu\right)+\left(\nabla_\Gamma\mathbf{V}^\varepsilon+\frac{1}{\varepsilon}\partial_z\mathbf{V}^\varepsilon\otimes\nu\right)^T\right)\\
&=\frac{1}{2\varepsilon}\left(\partial_z\mathbf{V}^\varepsilon\otimes\nu+\nu\otimes\partial_z\mathbf{V}^\varepsilon\right)+\frac{1}{2}\left(\nabla_\Gamma\mathbf{V}^\varepsilon+(\nabla_\Gamma\mathbf{V}^\varepsilon)^T\right),
\end{aligned}$$

where we used $(\mathbf{A}\otimes\mathbf{B})^T=\mathbf{B}\otimes\mathbf{A}$ for all $\mathbf{A},\mathbf{B}\in\mathbb{R}^d$. Moreover, we introduce the notation $\mathcal{E}(M)=\frac{1}{2}(M+M^T)$ for a matrix M. Then we can conclude with the previous calculations and formula (4.16) for the calculation of the divergence of a vector-valued quantity applied on every column of the matrix $2\eta(\varphi)D\mathbf{v}$

$$\begin{aligned}
\operatorname{div}(2\eta(\varphi)D\mathbf{v}) &= \frac{1}{\varepsilon}\partial_z\left(2\eta(\Phi^\varepsilon)\left\{\frac{1}{\varepsilon}\mathcal{E}(\partial_z\mathbf{V}^\varepsilon\otimes\nu)+\mathcal{E}(\nabla_\Gamma\mathbf{V}^\varepsilon)\right\}\right)\cdot\nu \\
&\quad+\operatorname{div}_\Gamma\left(2\eta(\Phi^\varepsilon)\left\{\frac{1}{\varepsilon}\mathcal{E}(\partial_z\mathbf{V}^\varepsilon\otimes\nu)+\mathcal{E}(\nabla_\Gamma\mathbf{V}^\varepsilon)\right\}\right) \\
&= \frac{1}{\varepsilon^2}\partial_z(2\eta(\Phi^\varepsilon)\mathcal{E}(\partial_z\mathbf{V}^\varepsilon\otimes\nu)\cdot\nu)+\frac{1}{\varepsilon}\partial_z(2\eta(\Phi^\varepsilon)\mathcal{E}(\nabla_\Gamma\mathbf{V}^\varepsilon)\cdot\nu) \\
&\quad+\frac{1}{\varepsilon}\operatorname{div}_\Gamma(2\eta(\Phi^\varepsilon)\mathcal{E}(\partial_z\mathbf{V}^\varepsilon\otimes\nu))+\operatorname{div}_\Gamma(2\eta(\Phi^\varepsilon)\mathcal{E}(\nabla_\Gamma\mathbf{V}^\varepsilon))+\text{ h.o.t..}
\end{aligned}$$

For the right-hand side of (4.1) we get

$$\begin{aligned}
\nabla\varphi\otimes\nabla\varphi &= \left(\frac{1}{\varepsilon}\partial_z\Phi^\varepsilon\nu+\nabla_\Gamma\Phi^\varepsilon\right)\otimes\left(\frac{1}{\varepsilon}\partial_z\Phi^\varepsilon\nu+\nabla_\Gamma\Phi^\varepsilon\right) \\
&= \frac{1}{\varepsilon^2}(\partial_z\Phi^\varepsilon)^2\nu\otimes\nu+\frac{1}{\varepsilon}\partial_z\Phi^\varepsilon(\nabla_\Gamma\Phi^\varepsilon\otimes\nu+\nu\otimes\nabla_\Gamma\Phi^\varepsilon)+\nabla_\Gamma\Phi^\varepsilon\otimes\nabla_\Gamma\Phi^\varepsilon.
\end{aligned}$$

Applying (4.16) to this yields

$$\begin{aligned}
\operatorname{div}(-\varepsilon\nabla\varphi\otimes\nabla\varphi) = &-\frac{1}{\varepsilon^2}\partial_z((\partial_z\Phi^\varepsilon)^2\nu)-\frac{1}{\varepsilon}\partial_z(\partial_z\Phi^\varepsilon\nabla_\Gamma\Phi^\varepsilon)-\frac{1}{\varepsilon}\operatorname{div}_\Gamma((\partial_z\Phi^\varepsilon)^2\nu\otimes\nu) \\
&-\operatorname{div}_\Gamma(\partial_z\Phi^\varepsilon(\nu\otimes\nabla_\Gamma\Phi^\varepsilon+\nabla_\Gamma\Phi^\varepsilon\otimes\nu))+\text{ h.o.t..}
\end{aligned}$$

Here we used that $\nu(t,s)$ does not depend on z, i.e., $\partial_z\nu=0$, and that we can write

$$(\mathbf{A}\otimes\mathbf{B})\cdot\mathbf{C}=(\mathbf{A}\cdot\mathbf{B}^T)\cdot\mathbf{C}=\mathbf{A}(\mathbf{B}^T\cdot\mathbf{C})=\mathbf{A}\langle\mathbf{B},\mathbf{C}\rangle$$

for every $\mathbf{A},\mathbf{B},\mathbf{C}\in\mathbb{R}^d$. Thus we could calculate in the equation above

$$\frac{1}{\varepsilon^2}\partial_z((\partial_z\Phi^\varepsilon)^2(\nu\otimes\nu)\cdot\nu)=\frac{1}{\varepsilon^2}\partial_z((\partial_z\Phi^\varepsilon)^2\nu).$$

Some other terms in the equation above were calculated in a similar way. Moreover, we used

$$\langle\nu,\nabla_\Gamma\Phi^\varepsilon\rangle=0$$

since the unit normal ν is orthogonal to the tangent space.

Equation (4.2):
Using formula (4.16) for the divergence of a vector field we obtain

$$\operatorname{div}(\mathbf{v})=\frac{1}{\varepsilon}\partial_z\mathbf{V}^\varepsilon\cdot\nu+\operatorname{div}_\Gamma\mathbf{V}^\varepsilon=0.$$

Equation (4.3):
Now we do the same calculations for equation (4.3). Here we get for the right-hand side of the equation

$$
\begin{aligned}
\operatorname{div}(m(\varphi,q)\nabla q) &= \operatorname{div}\left(\frac{1}{\varepsilon}M(\varphi)K(q)\nabla q\right) + \Delta q \\
&= \operatorname{div}\left(\frac{1}{\varepsilon}M(\Phi^\varepsilon)K(Q^\varepsilon)\left\{\frac{1}{\varepsilon}\partial_z Q^\varepsilon \nu + \nabla_\Gamma Q^\varepsilon\right\}\right) \\
&\quad + \frac{1}{\varepsilon^2}\partial_{zz}Q^\varepsilon - \left(\frac{1}{\varepsilon}\kappa + z|\mathcal{S}|^2\right)\partial_z Q^\varepsilon + \Delta_\Gamma Q^\varepsilon \\
&= \frac{1}{\varepsilon}\partial_z\left(\frac{1}{\varepsilon}M(\Phi^\varepsilon)K(Q^\varepsilon)\left\{\frac{1}{\varepsilon}\partial_z Q^\varepsilon \nu + \nabla_\Gamma Q^\varepsilon\right\}\right)\cdot \nu \\
&\quad + \operatorname{div}_\Gamma\left(\frac{1}{\varepsilon}M(\Phi^\varepsilon)K(Q^\varepsilon)\left\{\frac{1}{\varepsilon}\partial_z Q^\varepsilon \nu + \nabla_\Gamma Q^\varepsilon\right\}\right) \\
&\quad + \frac{1}{\varepsilon^2}\partial_{zz}Q^\varepsilon - \left(\frac{1}{\varepsilon}\kappa + z|\mathcal{S}|^2\right)\partial_z Q^\varepsilon + \Delta_\Gamma Q^\varepsilon.
\end{aligned}
$$

Moreover, we get for the left-hand side in (4.3)

$$
\begin{aligned}
\partial_t^\bullet\left(\frac{1}{\varepsilon}f(q)W(\varphi)+g(q)\right) &= \partial_t\left(\frac{1}{\varepsilon}f(q)W(\varphi)+g(q)\right) + \nabla\left(\frac{1}{\varepsilon}f(q)W(\varphi)+g(q)\right)\cdot\mathbf{v} \\
&= -\frac{1}{\varepsilon}\mathcal{V}\partial_z\left(\frac{1}{\varepsilon}f(Q^\varepsilon)W(\Phi^\varepsilon)+g(Q^\varepsilon)\right) + \partial_t^\circ\left(\frac{1}{\varepsilon}f(Q^\varepsilon)W(\Phi^\varepsilon)+g(Q^\varepsilon)\right) \\
&\quad + \left\{\frac{1}{\varepsilon}\partial_z\left(\frac{1}{\varepsilon}f(Q^\varepsilon)W(\Phi^\varepsilon)+g(Q^\varepsilon)\right)\nu + \nabla_\Gamma\left(\frac{1}{\varepsilon}f(Q^\varepsilon)W(\Phi^\varepsilon)+g(Q^\varepsilon)\right)\right\}\cdot\mathbf{V}^\varepsilon.
\end{aligned}
$$

Equation (4.4):
We continue with equation (4.4). First of all we study the left-hand side of (4.4). Using the identities (4.15) and (4.14) and the inner expansions of φ and $\mathbf{v}$ in the inner variables, cf. (4.19), we obtain for the terms on the left-hand side

$$
\begin{aligned}
\partial_t\varphi &= -\frac{1}{\varepsilon}\mathcal{V}\partial_z\Phi^\varepsilon + \partial_t^\circ\Phi^\varepsilon = -\frac{1}{\varepsilon}\mathcal{V}\partial_z\Phi^\varepsilon + \partial_t\Phi^\varepsilon - \nabla_\Gamma\Phi^\varepsilon\cdot\partial_t\hat{\gamma}, \\
\nabla\varphi\cdot\mathbf{v} &= \left(\frac{1}{\varepsilon}\partial_z\Phi^\varepsilon\nu + \nabla_\Gamma\Phi^\varepsilon\right)\cdot\mathbf{V}^\varepsilon.
\end{aligned}
$$

From (4.18) and the inner expansion of μ in the inner variables, cf. (4.19), we obtain for the right-hand side

$$
\begin{aligned}
\operatorname{div}(\tilde{m}(\varphi)\nabla\mu) = \operatorname{div}(\varepsilon m_0(\varphi)\nabla\mu) &= \operatorname{div}\left(\varepsilon m_0(\Phi^\varepsilon)\left(\frac{1}{\varepsilon}\partial_z M^\varepsilon\nu + \nabla_\Gamma M^\varepsilon\right)\right) \\
&= \frac{1}{\varepsilon}\partial_z\left(m_0(\Phi^\varepsilon)\partial_z M^\varepsilon\nu\right)\cdot\nu + \partial_z\left(m_0(\Phi^\varepsilon)\nabla_\Gamma M^\varepsilon\right) \\
&\quad + \operatorname{div}_\Gamma\left(m_0(\Phi^\varepsilon)\partial_z M^\varepsilon\nu\right) + \varepsilon\operatorname{div}_\Gamma\left(m_0(\Phi^\varepsilon)\nabla_\Gamma M^\varepsilon\right).
\end{aligned}
$$

Equation (4.5):
The equation for the inner expansion is given by

$$M^\varepsilon = -\varepsilon\left(\frac{1}{\varepsilon^2}\partial_{zz}\Phi^\varepsilon - \frac{1}{\varepsilon}\kappa\partial_z\Phi^\varepsilon - |\mathcal{S}|^2 z\partial_z\Phi^\varepsilon + \Delta_\Gamma\Phi^\varepsilon\right) + \frac{1}{\varepsilon}h(Q^\varepsilon)W'(\Phi^\varepsilon).$$

4.5.1 Leading Order Terms

In the previous section we derived the asymptotic expansions for the equations (4.1) - (4.5). Now we study the leading order terms for every equation. We start with the leading order terms for (4.2).

Leading order of (4.2):
Replacing $\mathbf{V}^\varepsilon$ by its inner expansion (4.19), the leading order term is given by

$$\frac{1}{\varepsilon}\partial_z\mathbf{V}_0\cdot\nu = 0 \qquad\qquad \mathcal{O}(\varepsilon^{-1}), (4.2).$$

Since $\nu = \nu(\gamma(s(x)))$ does not depend on z, we can deduce

$$\partial_z\mathbf{V}_0\cdot\nu = \partial_z(\mathbf{V}_0\cdot\nu) = 0. \tag{4.30}$$

Integrating this equation formally from $-\infty$ to $+\infty$ and using the matching conditions yields

$$[\mathbf{v}_0\cdot\nu]_-^+ = 0, \tag{4.31}$$

where $[\mathbf{v}_0\cdot\nu]_-^+$ is the jump on the interface, i.e., $[\mathbf{v}_0\cdot\nu]_-^+ = (\mathbf{v}_0\cdot\nu)(x+) - (\mathbf{v}_0\cdot\nu)(x-)$.

Leading order of (4.1):
We get the leading order equation

$$-\frac{1}{\varepsilon^2}\partial_z(2\eta(\Phi_0)\mathcal{E}(\partial_z\mathbf{V}_0\otimes\nu)\nu) + \frac{1}{\varepsilon^2}(\partial_z P_{-1})\nu = -\frac{1}{\varepsilon^2}\partial_z((\partial_z\Phi_0)^2\nu) \quad (4.1), \mathcal{O}(\varepsilon^{-2}).$$

Using the definition of $\mathcal{E}$ together with (4.30), we can simplify the first term on the left-hand side of this equation and obtain

$$\begin{aligned}\partial_z(2\eta(\Phi_0)\mathcal{E}(\partial_z\mathbf{V}_0\otimes\nu)\cdot\nu) &= \partial_z\left(2\eta(\Phi_0)\left(\frac{1}{2}\partial_z\mathbf{V}_0 + \frac{1}{2}(\nu^T\partial_z\mathbf{V}_0)\nu\right)\right)\\ &= \partial_z(\eta(\Phi_0)\partial_z\mathbf{V}_0).\end{aligned}$$

Hence, we can write

$$-\partial_z(\eta(\Phi_0)\partial_z\mathbf{V}_0) + (\partial_z P_{-1})\nu = -\partial_z((\partial_z\Phi_0)^2\nu). \tag{4.32}$$

Now we multiply (4.32) with ν, use (4.30) and the fact that ν does not depend on z, to derive

$$\partial_z P_{-1} = -\partial_z((\partial_z \Phi_0)^2). \tag{4.33}$$

Thus (4.32) implies

$$\partial_z(\eta(\Phi_0)\partial_z \mathbf{V}_0) = 0.$$

Therefore, we get that $\eta(\Phi_0)\partial_z \mathbf{V}_0$ is constant with respect to z. From the matching conditions for $\partial_z \mathbf{V}_0$ and Φ_0 we deduce

$$\partial_z \mathbf{V}_0 = 0 \tag{4.34}$$

and therefore it follows from the matching condition $\lim\limits_{z\to\pm\infty} \mathbf{V}_0(z,s) = \mathbf{v}_0(x\pm)$

$$[\mathbf{v}_0]_-^+ = 0, \tag{4.35}$$

i.e., there is no jump for the velocity field on the interface.

Leading order of (4.3):
The leading order terms of the inner expansion of (4.3) are given by

$$\frac{1}{\varepsilon^3}\partial_z(M(\Phi_0)K(Q_0)\partial_z Q_0) = 0 \qquad\qquad (4.3), \mathcal{O}(\varepsilon^{-3}).$$

Integrating formally from $-\infty$ to z and applying the matching condition $\lim\limits_{z\to\pm\infty} \Phi_0(z,s) = \varphi_0(x\pm)$ yields

$$\begin{aligned} 0 = \int\limits_{-\infty}^{z} \partial_z(M(\Phi_0)K(Q_0)\partial_z Q_0)\, dz &= M(\Phi_0)K(Q_0)\partial_z Q_0 - M(\varphi_0)K(q_0)\partial_z q_0 \\ &= M(\Phi_0)K(Q_0)\partial_z Q_0, \end{aligned}$$

since $M(\varphi_0) = M(\pm 1) = 0$ by Assumption 4.1. Hence,

$$\partial_z Q_0 = 0 \qquad \text{whenever } |\Phi_0| < 1 \tag{4.36}$$

due to the assumptions on M and K. Integrating from $-\infty$ to ∞ with respect to z and matching implies

$$[q_0]_-^+ = 0 \tag{4.37}$$

whenever $|\Phi_0| < 3$. Later, we will derive $-1 < \Phi_0 < 1$. Hence, the condition $|\Phi_0| < 3$ is no restraint and therefore we will neglect it from now on.

Leading order of (4.4):
The leading order of (4.4) is ε^{-1} and is given by

$$-\frac{1}{\varepsilon}\mathcal{V}\partial_z\Phi_0 + \frac{1}{\varepsilon}\partial_z\Phi_0\nu\cdot\mathbf{V}_0 = \frac{1}{\varepsilon}\partial_z(m_0(\Phi_0)\partial_z M_0) \qquad (4.4), \mathcal{O}(\varepsilon^{-1}).$$

Using (4.30), we get

$$-\frac{1}{\varepsilon}\mathcal{V}\partial_z\Phi_0 + \frac{1}{\varepsilon}\partial_z(\Phi_0\nu\cdot\mathbf{V}_0) = \frac{1}{\varepsilon}\partial_z(m_0(\Phi_0)\partial_z M_0).$$

Integrating this equation formally from $-\infty$ to $+\infty$ yields

$$-2\mathcal{V} + 2\mathbf{v}_0\cdot\nu = 0 \tag{4.38}$$

due to the matching condition $\lim_{z\to\pm\infty}\partial_z M_0(z,s) = 0$. Moreover, we already know $\partial_z\mathbf{V}_0 = 0$, cf (4.34). From the matching condition $\lim_{z\to\pm\infty}\mathbf{V}_0 = \mathbf{v}_0$ it follows

$$\mathbf{V}_0 = \mathbf{v}_0 \tag{4.39}$$

for all z. Hence, the equation above together with (4.38) implies

$$0 = \partial_z((\mathbf{v}_0\cdot\nu - \mathcal{V})\Phi_0) = \partial_z(m_0(\Phi_0)\partial_z M_0).$$

We integrate this equation formally from $-\infty$ to $+\infty$ and use the matching condition $\lim_{z\to\pm\infty}\partial_z M_0 = 0$. Then we obtain

$$\partial_z M_0 = 0. \tag{4.40}$$

Integrating from $-\infty$ to $+\infty$ again and using the matching condition $\lim_{z\to\pm\infty} M_0(z,s) = \mu_0(x\pm)$ finally yields

$$[\mu]_-^+ = 0. \tag{4.41}$$

Leading order of (4.5):
With analogous calculations as before, in particular with the Taylor series for $W'(\Phi^\varepsilon)$ and $h(Q^\varepsilon)$, we obtain the leading order equation

$$0 = -\frac{1}{\varepsilon}\partial_{zz}\Phi_0 + \frac{1}{\varepsilon}h(Q_0)W'(\Phi_0) \qquad (4.5), \mathcal{O}(\varepsilon^{-1}).$$

From this equation we can deduce $|\Phi_0(z)| < 1$ for all $z \in \mathbb{R}$ by the following arguments:
Due to the matching conditions it holds $\lim_{z\to\pm\infty}\Phi_0(z) = \pm 1$ and by assumption it holds $\Phi_0(0) = 0$. We assume that $|\Phi_0(z)| \not< 1$ for all $z \in \mathbb{R}$. Then we distinguish four cases:

i) There exists $\tilde{z} \in \mathbb{R}$ such that $\min_{z\in\mathbb{R}} \Phi_0(z) = \Phi_0(\tilde{z}) < -1$.
In this case it holds $\partial_{zz}\Phi_0(\tilde{z}) \geq 0$ and $W'(\Phi_0(\tilde{z})) < 0$. From (4.5), $\mathcal{O}(\varepsilon^{-1})$ and $h(Q_0) > 0$ it follows the contradiction

$$0 = -\frac{1}{\varepsilon}\partial_{zz}\Phi_0(\tilde{z}) + \frac{1}{\varepsilon}h(Q_0)W'(\Phi_0(\tilde{z})) < 0.$$

ii) There exists $\tilde{z} \in \mathbb{R}$ such that $\max_{z\in\mathbb{R}} \Phi_0(z) = \Phi_0(\tilde{z}) > 1$.
Due to $\partial_{zz}\Phi_0(\tilde{z}) \leq 0$ and $W'(\Phi_0(\tilde{z})) > 0$ the contradiction follows analogously as in *i)*.

iii) There exists $\tilde{z} \in \mathbb{R}$ such that $\min_{z\in\mathbb{R}} \Phi_0(z) = \Phi_0(\tilde{z}) = -1$.
In this case we consider the initial value problem

$$\begin{aligned} \frac{1}{\varepsilon}\partial_{zz}\Phi(z) &= \frac{1}{\varepsilon}h(Q_0)W'(\Phi(z)) \\ \Phi(\tilde{z}) &= -1, \\ \partial_z\Phi(\tilde{z}) &= 0. \end{aligned}$$

Then Φ_0 and $\tilde{\Phi} \equiv -1$ solve the same initial value problem of second order. From the uniqueness of the solution it follows $\Phi_0(z) \equiv \tilde{\Phi}(z) \equiv -1$ for all $z \in \mathbb{R}$, which is a contradiction to $\lim_{z\to\infty} \Phi_0(z) = 1$.

iv) There exists $\tilde{z} \in \mathbb{R}$ such that $\max_{z\in\mathbb{R}} \Phi_0(z) = \Phi_0(\tilde{z}) = 1$.
In this case the contradiction can be shown analogously as in *iii)*.

Hence, we have shown

$$|\Phi_0(z)| < 1 \qquad \text{for all } z \in \mathbb{R}$$

and therefore Q_0 does not depend on z, cf. (4.36).

Now we consider a function $\hat{\Phi} : \mathbb{R} \to [-1, 1]$ solving the nonlinear ordinary differential equation

$$\hat{\Phi}''(z) = W'(\hat{\Phi}(z)), \qquad \lim_{z\to\pm\infty} \hat{\Phi}(z) = \pm 1, \qquad \hat{\Phi}(0) = 0. \tag{4.42}$$

The existence of a unique $\hat{\Phi}$ solving (4.42) such that $\hat{\Phi}(z) \in [-1, 1]$ and $\hat{\Phi}'(z) > 0$ for all $z \in \mathbb{R}$ was proven in [Sch13, Lemma 2.6.1]. Moreover, we set

$$\hat{\Phi}_0(t, s, z) := \Phi_0(t, s, \frac{z}{\sqrt{h(Q_0(t, s))}}).$$

Then $\hat{\Phi}_0(t, s, \cdot)$ solves

$$\hat{\Phi}_0''(z) = W'(\hat{\Phi}_0(z)), \qquad \lim_{z\to\pm\infty} \hat{\Phi}_0(z) = \pm 1, \qquad \hat{\Phi}_0(0) = 0.$$

Due to the uniqueness of the solution it follows

$$\hat{\Phi}(z) = \hat{\Phi}_0(z) = \Phi_0(t, s, \frac{z}{\sqrt{h(Q_0(t,s))}})$$

and therefore

$$\Phi_0(t, s, z) = \hat{\Phi}(\sqrt{h(Q_0(t,s))}z). \tag{4.43}$$

For $z = 0$ it holds $\Phi_0(t, s, 0) = \hat{\Phi}(0) = 0$ and we obtain the interface Γ as the set where Φ_0 is 0. In the next step we multiply (4.5),$\mathcal{O}(\varepsilon^{-1})$ with $\partial_z \Phi_0$ and formally integrate from $-\infty$ to z. Then we can conclude

$$\begin{aligned} \frac{1}{2}|\partial_z \Phi_0|^2 &= \int_{-\infty}^{z} \partial_{zz}\Phi_0 \partial_z \Phi_0 \, dz = h(Q_0(t,s)) \int_{-\infty}^{z} W'(\Phi_0)\partial_z \Phi_0 \, dz \\ &= h(Q_0(s,t))W(\Phi_0(t,s,z)). \end{aligned} \tag{4.44}$$

We integrate this equation from $-\infty$ to $+\infty$ and get

$$\int_{-\infty}^{\infty} |\partial_z \Phi_0|^2 dz = 2h(Q_0(t,s)) \int_{-\infty}^{\infty} W(\Phi_0)\, dz = 2h(Q_0(t,s)) \int_{-\infty}^{\infty} W(\hat{\Phi}(\sqrt{h(Q_0(t,s))}z))\, dz.$$

Introducing the variable transformation

$$\hat{z} := \sqrt{h(Q_0(t,s))}z, \qquad \frac{dz}{d\hat{z}} = (h(Q_0(t,s)))^{-\frac{1}{2}},$$

we proceed with

$$\begin{aligned} \int_{-\infty}^{\infty} |\partial_z \Phi_0|^2 dz &= 2h(Q_0(t,s)) \int_{-\infty}^{\infty} W(\hat{\Phi}(\sqrt{h(Q_0(t,s))}z))\, dz \\ &= 2\sqrt{h(Q_0(t,s))} \int_{-\infty}^{\infty} W(\hat{\Phi}(\hat{z}))\, d\hat{z}. \end{aligned}$$

For the following calculations we define

$$K_W := \left(2 \int_{-\infty}^{\infty} W(\hat{\Phi}(z))\, dz \right)^{-1}. \tag{4.45}$$

Hence, we can conclude

$$\int_{-\infty}^{\infty} |\partial_z \Phi_0|^2 dz = \sqrt{h(Q_0(t,s)} K_W^{-1} \tag{4.46}$$

and with the calculations above it follows

$$\int_{-\infty}^{\infty} 2W(\Phi_0)\,dz = (h(Q_0(t,s)))^{-1} \int_{-\infty}^{\infty} |\partial_z \Phi_0|^2\,dz = \frac{K_W^{-1}}{\sqrt{h(Q_0(t,s))}}. \tag{4.47}$$

4.5.2 Second Order Terms

In this section we study the second order terms of the inner expansion for the diffuse interface model (4.1) - (4.5), i.e., we study the terms with the second lowest power in ε.

Second order of (4.2):
The second order terms of (4.2) are given by

$$\partial_z \mathbf{V}_1 \cdot \nu + \mathrm{div}_\Gamma \mathbf{V}_0 = 0 \qquad\qquad (4.2), \mathcal{O}(1).$$

Second order of (4.4):
To second order we have

$$\frac{1}{\varepsilon}\left(-\mathcal{V} + \mathbf{V}_0 \cdot \nu\right) \partial_z \Phi_0 = \frac{1}{\varepsilon} \partial_{zz} M_1 \qquad\qquad (4.4), \mathcal{O}(\varepsilon^{-1}),$$

where we used $\partial_z M_0 = 0$, cf. (4.40). Due to (4.38) and (4.39) it holds

$$\partial_{zz} M_1 = 0.$$

Integrating formally from $-\infty$ to $+\infty$ and matching implies

$$[\nabla \mu_0 \cdot \nu]_-^+ = 0. \tag{4.48}$$

Second order of (4.3):
The second order terms are given by

$$\begin{aligned}
&-\frac{1}{\varepsilon^2}\mathcal{V}\partial_z(f(Q_0)W(\Phi_0)) + \frac{1}{\varepsilon^2}\partial_z(f(Q_0)W(\Phi_0))\nu \cdot \mathbf{V}_0 \\
&\quad= \frac{1}{\varepsilon^2}\partial_z(\partial_z Q_0) + \frac{1}{\varepsilon^2}\partial_z(M(\Phi_0)K(Q_0)\nabla_\Gamma Q_0) \cdot \nu + \frac{1}{\varepsilon^2}\partial_z(M(\Phi_0)K(Q_0)\partial_z Q_1) \\
&\qquad + \frac{1}{\varepsilon^2}\partial_z(M(\Phi_0)K'(Q_0)Q_1\partial_z Q_0) + \frac{1}{\varepsilon^2}\partial_z(M'(\Phi_0)\Phi_1 K(Q_0)\partial_z Q_0) \\
&\qquad + \frac{1}{\varepsilon^2}\mathrm{div}_\Gamma(M(\Phi_0)K(Q_0)\partial_z Q_0 \nu) \qquad\qquad (4.3), \mathcal{O}(\varepsilon^{-2}),
\end{aligned}$$

where we used the Taylor series for $M(\Phi^\varepsilon)$ and $K(Q^\varepsilon)$, i.e.,

$$\frac{1}{\varepsilon}M(\Phi^\varepsilon)K(Q^\varepsilon) = \frac{1}{\varepsilon}(M(\Phi_0) + \varepsilon M'(\Phi_0)\Phi_1 + \text{ h.o.t.})(K(Q_0) + \varepsilon K'(Q_0)Q_1 + \text{ h.o.t.}).$$

Using $\partial_z Q_0 = 0$, cf. (4.36), and $\nabla_\Gamma Q_0 \cdot \nu = 0$, we can simplify (4.3),$\mathcal{O}(\varepsilon^{-2})$ to

$$(\nu \cdot \mathbf{V}_0 - \mathcal{V})\partial_z(f(Q_0)W(\Phi_0)) = \partial_z(M(\Phi_0)K(Q_0)\partial_z Q_1).$$

Due to (4.38) and (4.39) the terms on the left-hand side vanish. Integrating formally from $-\infty$ to z we can deduce

$$M(\Phi_0)K(Q_0)\partial_z Q_1 = 0.$$

From the fact that $M(\Phi_0) > 0$ for all $|\Phi_0| < 1$ and $K(q) > 0$ for all $q \in \mathbb{R}$, cf. Assumption 4.1, it follows

$$\partial_z Q_1 = 0. \tag{4.49}$$

Second order of (4.5):
To zeroth order, (4.5) yields

$$M_0 = -\partial_{zz}\Phi_1 + \kappa\partial_z\Phi_0 + h'(Q_0)Q_1 W'(\Phi_0) + h(Q_0)W''(\Phi_0)\Phi_1. \qquad (4.5), \mathcal{O}(1)$$

Multiplying (4.5),$\mathcal{O}(1)$ by $\partial_z\Phi_0$ and integrating from $-\infty$ to ∞ implies

$$\begin{aligned}\int_{-\infty}^{\infty} M_0\partial_z\Phi_0\,dz = \int_{-\infty}^{\infty} -\partial_{zz}\Phi_1\partial_z\Phi_0\,dz + \int_{-\infty}^{\infty} \kappa(\partial_z\Phi_0)^2\,dz + \int_{-\infty}^{\infty} h'(Q_0)Q_1W'(\Phi_0)\partial_z\Phi_0\,dz \\ + \int_{-\infty}^{\infty} h(Q_0)W''(\Phi_0)\Phi_1\partial_z\Phi_0\,dz.\end{aligned} \tag{4.50}$$

Using $\partial_z M_0 = 0$, cf. (4.40), and matching we can deduce for the left-hand side of this equation

$$\int_{-\infty}^{\infty} M_0\partial_z\Phi_0\,dz = \int_{-\infty}^{\infty} \partial_z(M_0\Phi_0)\,dz = [M_0\Phi_0]_{-\infty}^{+\infty} = 2\mu_0.$$

For the right-hand side of (4.50) we get

$$
\begin{aligned}
&\int_{-\infty}^{\infty} -\partial_{zz}\Phi_1\partial_z\Phi_0 + \kappa(\partial_z\Phi_0)^2 + h'(Q_0)Q_1W'(\Phi_0)\partial_z\Phi_0 + h(Q_0)W''(\Phi_0)\Phi_1\partial_z\Phi_0\,dz \\
&\quad = \int_{-\infty}^{\infty} -\partial_z(\partial_z\Phi_1\partial_z\Phi_0) + \partial_z\Phi_1\partial_{zz}\Phi_0\,dz + \int_{-\infty}^{\infty} \kappa(\partial_z\Phi_0)^2\,dz + \int_{-\infty}^{\infty} h'(Q_0)Q_1\partial_z W(\Phi_0)\,dz \\
&\qquad + \int_{-\infty}^{\infty} h(Q_0)\Phi_1\partial_z W'(\Phi_0)\,dz \\
&\quad = [-\partial_z\Phi_1\partial_z\Phi_0]_{-\infty}^{+\infty} + \int_{-\infty}^{\infty} \partial_z\Phi_1\partial_{zz}\Phi_0\,dz + \int_{-\infty}^{\infty} \kappa(\partial_z\Phi_0)^2\,dz + \int_{-\infty}^{\infty} \partial_z(h(Q_0)W'(\Phi_0)\Phi_1)\,dz \\
&\qquad - \int_{-\infty}^{\infty} h(Q_0)W'(\Phi_0)\partial_z\Phi_1\,dz + \int_{-\infty}^{\infty} h'(Q_0)Q_1\partial_z W(\Phi_0)\,dz.
\end{aligned}
$$

From (4.5),$\mathcal{O}(\varepsilon^{-1})$ we could conclude $-\partial_{zz}\Phi_0 + h(Q_0)W'(\Phi_0) = 0$. Moreover, the jump term is 0 due to the matching conditions. Hence, the remaining terms on the right-hand side simplify to

$$
\int_{-\infty}^{\infty} \kappa(\partial_z\Phi_0)^2\,dz + \int_{-\infty}^{\infty} \partial_z(h(Q_0)W'(\Phi_0)\Phi_1)\,dz + \int_{-\infty}^{\infty} h'(Q_0)Q_1\partial_z W(\Phi_0)\,dz. \tag{4.51}
$$

From the matching conditions it follows that the second term vanishes. As it holds $\partial_z Q_0 = \partial_z Q_1 = 0$, cf. (4.36) and (4.49), the last term also vanishes since

$$
\int_{-\infty}^{\infty} h'(Q_0)Q_1\partial_z W(\Phi_0)\,dz = \int_{-\infty}^{\infty} \partial_z(h'(Q_0)Q_1W(\Phi_0))\,dz = [h'(Q_0)Q_1W(\Phi_0)]_{-\infty}^{+\infty} = 0,
$$

where we assumed that $Q_1(z)$ grows at most polynomially as $z \to \pm\infty$. For the first term in (4.51) we use (4.46). Thus equation (4.50) yields

$$
\mu_0 = \frac{\kappa}{2}\sqrt{h(q_0)}K_W^{-1},
$$

which is the solvability condition for Φ_1.

Second order of (4.1):
The second order terms of (4.1) are the ones with ε^{-1}. Due to $\partial_z \mathbf{V}_0 = 0$ we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\nabla_\Gamma P_{-1} + \frac{1}{\varepsilon}(\partial_z P_0)\nu - \frac{1}{\varepsilon}\partial_z(2\eta(\Phi_0)\mathcal{E}(\partial_z \mathbf{V}_1 \otimes \nu)\cdot\nu) - \frac{1}{\varepsilon}\partial_z(2\eta(\Phi_0)\mathcal{E}(\nabla_\Gamma \mathbf{V}_0))\cdot\nu \\
&\qquad = -\frac{1}{\varepsilon}2\partial_z(\partial_z\Phi_0\partial_z\Phi_1\nu) - \frac{1}{\varepsilon}\partial_z(\partial_z\Phi_0\nabla_\Gamma\Phi_0) - \frac{1}{\varepsilon}\mathrm{div}_\Gamma((\partial_z\Phi_0)^2\nu\otimes\nu).
\end{aligned} \tag{(4.1), $\mathcal{O}(\varepsilon^{-1})$}$$

From (4.33) and (4.44) it follows

$$P_{-1} = -|\partial_z\Phi_0|^2 + C = -2h(Q_0)W(\Phi_0) + C$$

for a constant $C \in \mathbb{R}$. Hence, (4.1),$\mathcal{O}(\varepsilon^{-1})$ can be rewritten to

$$\begin{aligned}
0 = &- 2\nabla_\Gamma(h(Q_0)W(\Phi_0)) + (\partial_z P_0)\nu - \partial_z(2\eta(\Phi_0)\mathcal{E}(\partial_z\mathbf{V}_1\otimes\nu)\nu) \\
&- \partial_z(2\eta(\Phi_0)\mathcal{E}(\nabla_\Gamma\mathbf{V}_0))\nu + 2\partial_z(\partial_z\Phi_0\partial_z\Phi_1\nu) + \partial_z(\partial_z\Phi_0\nabla_\Gamma\Phi_0) \\
&+ \mathrm{div}_\Gamma((\partial_z\Phi_0)^2\nu\otimes\nu).
\end{aligned} \tag{4.52}$$

We integrate this equation from $-\infty$ to ∞ with respect to z and apply the matching conditions. Then we study the different terms of (4.52) separately:

i) For the first term we obtain

$$2\int_{-\infty}^{\infty}\nabla_\Gamma(h(Q_0)W(\Phi_0))\,dz = \nabla_\Gamma\left(h(Q_0)\int_{-\infty}^{\infty}2W(\Phi_0)\,dz\right) = \nabla_\Gamma\left(\sqrt{h(Q_0)}K_W^{-1}\right),$$

where we used (4.47).

ii) Since ν does not depend on z, the second term yields

$$\int_{-\infty}^{\infty}(\partial_z P_0)\nu\,dz = [p_0]_-^+\nu.$$

iii) For the next calculation we have to study two term. Hence,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\partial_z(2\eta(\Phi_0)\mathcal{E}(\partial_z\mathbf{V}_1\otimes\nu)\nu)\,dz + \int_{-\infty}^{\infty}\partial_z(2\eta(\Phi_0)\mathcal{E}(\nabla_\Gamma\mathbf{V}_0))\nu\,dz \\
&\quad = 2\left[\eta(\Phi_0)\mathcal{E}(\partial_z\mathbf{V}_1\otimes\nu)\right]_{-\infty}^{+\infty}\nu + \left[2\eta(\Phi_0)\mathcal{E}(\nabla_\Gamma\mathbf{V}_0)\right]_{-\infty}^{+\infty}\nu \\
&\quad = 2\left[\eta^{(i)}\mathcal{E}((\nabla\mathbf{v}_0^T\nu)\otimes\nu)\right]_-^+\nu + \left[\eta^{(i)}(\nabla_\Gamma\mathbf{v}_0 + \nabla_\Gamma\mathbf{v}_0^T)\right]_-^+\nu \\
&\quad = \left[\eta^{(i)}\left((\nabla\mathbf{v}_0^T\nu)\nu^T + \nu(\nu^T\nabla\mathbf{v}_0)\right)\right]_-^+\nu + \left[\eta^{(i)}(\nabla_\Gamma\mathbf{v}_0 + \nabla\mathbf{v}_0^T)\right]_-^+\nu \\
&\quad = \left[\eta^{(i)}\left(\nu(\nu^T\nabla\mathbf{v}_0) + \nabla_\Gamma\mathbf{v}_0\right)\right]_-^+\nu + \left[\eta^{(i)}\left((\nabla\mathbf{v}_0^T\nu)\nu^T + \nabla_\Gamma\mathbf{v}_0^T\right)\right]_-^+\nu \\
&\quad = [\eta^{(i)}\nabla\mathbf{v}_0]_-^+\nu + [\eta^{(i)}\nabla\mathbf{v}_0^T]_-^+\nu \\
&\quad = 2\left[\eta^{(i)}D(\mathbf{v}_0)\right]_-^+\nu,
\end{aligned}$$

where we used the matching condition $\lim\limits_{z\to\pm\infty}\partial_z \mathbf{V}_1 = \nabla \mathbf{v}_0(x\pm)^T \nu$ for vector fields.

iv) For the next integral we use $\lim\limits_{z\to\pm\infty}\partial_z\Phi_0(z) = 0$. Hence, we can conclude

$$\int\limits_{-\infty}^{\infty} 2\partial_z(\partial_z\Phi_0\partial_z\Phi_1\nu)\,dz = 0.$$

v) Using the definition of Φ_0, cf. (4.43), together with (4.44), we obtain

$$\int\limits_{-\infty}^{\infty} \partial_z(\partial_z\Phi_0\nabla_\Gamma\Phi_0)\,dz = [\sqrt{2h(Q_0)W(\Phi_0)}\nabla_\Gamma\hat{\Phi}(\sqrt{h(Q_0)}z)]_{-\infty}^{+\infty} = 0,$$

where we used $W(\pm 1) = 0$ by matching.

vi) For the last integral we use (4.44) and the identity

$$\mathrm{div}_\Gamma(\mathbf{u}\otimes\mathbf{v}) = (\mathrm{div}_\Gamma\mathbf{u})\mathbf{v} + \mathbf{u}\cdot(\nabla_\Gamma\mathbf{v})$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^d$. Moreover, we use

$$\mathrm{div}_\Gamma\nu = \sum_{i=1}^{d-1}\partial_{s_i}\nu\cdot\partial_{s_i}\hat{\gamma} = \sum_{i=1}^{d-1} -\kappa_i\partial_{s_i}\hat{\gamma}\cdot\partial_{s_i}\hat{\gamma} = -\kappa.$$

Then we obtain

$$\begin{aligned}\int\limits_{-\infty}^{\infty} \mathrm{div}_\Gamma((\partial_z\Phi_0)^2\nu\otimes\nu)\,dz &= \int\limits_{-\infty}^{\infty} \mathrm{div}_\Gamma(2h(Q_0)W(\Phi_0)\nu\otimes\nu)\,dz \\ &= \int\limits_{-\infty}^{\infty} (\nabla_\Gamma(2h(Q_0)W(\Phi_0)\nu))\cdot\nu + 2h(Q_0)W(\Phi_0)\nu(\mathrm{div}_\Gamma\nu)\,dz \\ &= -\kappa\nu h(Q_0)\int\limits_{-\infty}^{\infty} 2W(\Phi_0)\,dz = -\kappa\nu\sqrt{h(Q_0)}K_W^{-1}.\end{aligned}$$

Altogether we obtain from (4.52)

$$[p_0]_-^+\nu - 2[\eta^{(i)}D(\mathbf{v}_0)]_-^+\nu = \nabla_\Gamma\left(\sqrt{h(Q_0)}K_W^{-1}\right) + \kappa\sqrt{h(Q_0)}K_W^{-1}\nu. \tag{4.53}$$

4.5.3 Third Order Terms

Third order of (4.3):
The third order terms for equation (4.3) are the ones with ε^{-1}. First of all we study the right-hand side of the equation and define

$$\mathbf{J} := \left(\frac{1}{\varepsilon}M(\varphi)K(q) + 1\right)\nabla q.$$

From the previous calculations we already know that the outer and inner expansion of $\mathbf{J}$ are given by

$$\mathbf{J}^{bulk} = \sum_{k=-2}^{\infty} \varepsilon^k \mathbf{J}_k^{bulk}, \qquad \mathbf{J}^{int} = \sum_{k=-2}^{\infty} \varepsilon^k \mathbf{J}_k^{int},$$

where it holds

$$\begin{aligned}
\mathbf{J}_{-2}^{bulk} &= 0,\\
\mathbf{J}_{-1}^{bulk} &= M(\varphi_0)K(q_0)\nabla q_0 = 0,\\
\mathbf{J}_{0}^{bulk} &= M(\varphi_0)K'(q_0)q_1 + M'(\varphi_0)\varphi_1 K(q_0)\nabla q_0 + M(\varphi_0)K(q_0)\nabla q_1 + \nabla q_0 = \nabla q_0
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{J}_{-2}^{int} &= M(\Phi_0)K(Q_0)\partial_z \nu = 0,\\
\mathbf{J}_{-1}^{int} &= \partial_z Q_0 \nu + \left(M'(\Phi_0)\Phi_1 K(Q_0) + M(\Phi_0)K'(Q_0)Q_1\right)\partial_z Q_0 \nu\\
&\quad + M(\Phi_0)K(Q_0)\left(\partial_z Q_1 \nu + \nabla_\Gamma Q_0\right)\\
&= M(\Phi_0)K(Q_0)\nabla_\Gamma Q_0.
\end{aligned}$$

Note that we do not calculate $\mathbf{J}_0^{int}$ in detail since we do not need its explicit form. Instead, we use the following matching condition for $\mathbf{J}_0^{int}$:

$$\mathbf{J}_0^{int}(t,s,z) = (\mathbf{J}_0^{bulk})^{\pm}(t,x) + \nabla(\mathbf{J}_{-1}^{bulk})^{\pm}(t,x)\cdot \nu z = (\mathbf{J}_0^{bulk})^{\pm}(t,x) \qquad \text{as } z \to \pm\infty.$$

This matching condition together with other matching conditions for $\mathbf{J}^{int}$ can be found in [GLS14], cf. (4.23) - (4.25), and in [GS06]. Since $\mathbf{J}_{-2}^{int}$ is equal to 0, we can conclude that the third order terms which we get from the term $\mathrm{div}(\mathbf{J})$ are given by

$$\frac{1}{\varepsilon}\partial_z \mathbf{J}_0^{int}\cdot \nu + \frac{1}{\varepsilon}\mathrm{div}_\Gamma \mathbf{J}_{-1}^{int}.$$

We formally integrate from $-\infty$ to $+\infty$ with respect to z. For the first term, we use the matching condition for $\mathbf{J}_0^{int}$ to obtain

$$\int_{-\infty}^{\infty} \partial_z \mathbf{J}_0^{int}\cdot \nu \, dz = [\nabla q_0 \cdot \nu]_-^+.$$

Using $M(\varphi) = W(\varphi)$ for all $|\varphi| \leq 3$ together with $|\Phi_0| < 1$ and the fact that Q_0 does not depend on z, we obtain for the second integral

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{div}_\Gamma \mathbf{J}_{-1}^{int} dz &= \int_{-\infty}^{\infty} \mathrm{div}_\Gamma (M(\Phi_0) K(Q_0) \nabla_\Gamma Q_0)\, dz \\
&= \mathrm{div}_\Gamma \left(K(q_0) \nabla_\Gamma q_0 \int_{-\infty}^{\infty} W(\Phi_0)\, dz \right) \\
&= \mathrm{div}_\Gamma \left(K(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \nabla_\Gamma q_0 \right),
\end{aligned}$$

since we already calculated the integral in (4.47).

Now we study the left-hand side of the third order equation. We use the Taylor series as before to derive

$$f(Q^\varepsilon) W(\Phi^\varepsilon) = f(Q_0) W(\Phi_0) + \varepsilon (f(Q_0) W'(\Phi_0) \Phi_1 + W(\Phi_0) f'(Q_0) Q_1) + \text{ h.o.t. } .$$

Hence, the left-hand side of the equation to third order yields

$$\begin{aligned}
&-\mathcal{V} \partial_z \left(f(Q_0) W'(\Phi_0) \Phi_1 + W(\Phi_0) f'(Q_0) Q_1 + g(Q_0) \right) + \partial_t^\circ (f(Q_0) W(\Phi_0)) \\
&\quad + \partial_z (f(Q_0) W(\Phi_0) \nu \cdot \mathbf{V}_1 + \partial_z ((f(Q_0) W'(\Phi_0) \Phi_1 + W(\Phi_0) f'(Q_0) Q_1) \nu) \cdot \mathbf{V}_0 \\
&\quad + \nabla_\Gamma (f(Q_0) W(\Phi_0)) \cdot \mathbf{V}_0.
\end{aligned}$$

Integrating from $-\infty$ to $+\infty$ with respect to z, using $W(\pm 1) = W'(\pm 1) = 0$ and the fact that $\mathbf{V}_0$ does not depend on z, i.e., $\partial_z \mathbf{V}_0 = 0$, we deduce

$$\begin{aligned}
-\mathcal{V}[g(q_0)]_-^+ &+ \int_{-\infty}^{\infty} \left\{ \partial_t (f(Q_0) W(\Phi_0)) - \nabla_\Gamma (f(Q_0) W(\Phi_0)) \cdot \partial_t \hat{\gamma} \right\} dz \\
&+ \int_{-\infty}^{\infty} \partial_z (f(Q_0) W(\Phi_0) \nu) \cdot \mathbf{V}_1 dz + \int_{-\infty}^{\infty} \nabla_\Gamma (f(Q_0) W(\Phi_0)) \cdot \mathbf{V}_0 dz.
\end{aligned}$$

Now we study these terms separately.

i) Since g is continuous and the jump of q_0 across the interface is 0, i.e., $[q_0]_-^+ = 0$, we can deduce

$$-\mathcal{V}[g(q_0)]_-^+ = 0.$$

ii) Taking into account that Q_0 is independent of z and that q_0 is a function depending on t and x, we obtain for the second term

$$\int_{-\infty}^{\infty} \partial_t (f(Q_0)W(\Phi_0))\, dz = \partial_t \left(f(Q_0) \int_{-\infty}^{\infty} W(\Phi_0)\, dz \right)$$
$$= \partial_t \left(f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \right) + \nabla \left(f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \right) \cdot \partial_t \hat{\gamma}.$$

iii) Since $\hat{\gamma}$ does not depend on z as it is the parametrization of the interface and therefore only depends on the variables t and s, the third term gives

$$\int_{-\infty}^{\infty} \nabla_\Gamma (f(Q_0)W(\Phi_0)) \cdot \partial_t \hat{\gamma}\, dz = \nabla_\Gamma \left(\int_{-\infty}^{\infty} (f(Q_0)W(\Phi_0))\, dz \right) \cdot \partial_t \hat{\gamma}$$
$$= \nabla_\Gamma \left(f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \right) \cdot \partial_t \hat{\gamma}.$$

iv) Integration by parts yields for the fourth term

$$\int_{-\infty}^{\infty} \partial_z (f(Q_0)W(\Phi_0)\nu) \cdot \mathbf{V}_1\, dz$$
$$= \int_{-\infty}^{\infty} \partial_z (f(Q_0)W(\Phi_0)\nu \cdot \mathbf{V}_1) - f(Q_0)W(\Phi_0)\nu \cdot \partial_z \mathbf{V}_1\, dz$$
$$= \int_{-\infty}^{\infty} f(Q_0)W(\Phi_0) \mathrm{div}_\Gamma \mathbf{V}_0\, dz = f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \mathrm{div}_\Gamma \mathbf{v}_0,$$

where we used (4.2),$\mathcal{O}(1)$ and the fact that $\mathbf{V}_0$ does not depend on z. Moreover, we assumed that $\mathbf{V}_1$ grows at most polynomially as $z \to \pm\infty$.

v) Since $\mathbf{V}_0$ and Q_0 are independent of z, we obtain for the last integral

$$\int_{-\infty}^{\infty} \nabla_\Gamma (f(Q_0)W(\Phi_0)) \cdot \mathbf{V}_0\, dz = \nabla_\Gamma \left(f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \right) \cdot \mathbf{v}_0.$$

Altogether we get the following sharp interface equation on the interface $\Gamma(t)$:

$$\begin{aligned}
&\partial_t \left(f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \right) + \nabla \left(f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \right) \cdot \partial_t \hat{\gamma} - \nabla_\Gamma \left(f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \right) \cdot \partial_t \hat{\gamma} \\
&+ f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \operatorname{div}_\Gamma \mathbf{v}_0 + \nabla_\Gamma \left(f(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \right) \cdot \mathbf{v}_0 \\
&= \operatorname{div}_\Gamma \left(K(q_0) \frac{K_W^{-1}}{2\sqrt{h(q_0)}} \nabla_\Gamma q_0 \right) + [\nabla q \cdot \nu]_-^+. \qquad (4.54)
\end{aligned}$$

4.6 The Sharp Interface Model for the Surfactant Model

In the previous sections we derived the sharp interface model for the surfactant model by using the method of formally matched asymptotic expansions. From now on, we omit the index 0 in the expansions, i.e., we write $\mathbf{v}$, p and q instead of $\mathbf{v}_0$, p_0 and q_0. Hence, we recover the following sharp interface model from the phase field model (4.1) - (4.5), resp. (1.1) - (1.5) with $\rho \equiv 1$:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \operatorname{div}\left(2\eta^{(i)} D\mathbf{v}\right) \qquad \text{in } \Omega^{(i)}(t), \qquad (4.55)$$

$$\operatorname{div}(\mathbf{v}) = 0 \qquad \text{in } \Omega^{(i)}(t), \qquad (4.56)$$

$$\partial_t g(q) + \nabla g(q) \cdot \mathbf{v} = \Delta q \qquad \text{in } \Omega^{(i)}(t), \qquad (4.57)$$

$$[p]_-^+ \nu - 2[\eta^{(i)} D\mathbf{v}]_-^+ \nu - \kappa \sqrt{h(q)} K_W^{-1} \nu = \nabla_\Gamma \left(\sqrt{h(q)} K_W^{-1} \right) \qquad \text{on } \Gamma(t),$$

$$-\mathcal{V} + \mathbf{v} \cdot \nu = 0 \qquad \text{on } \Gamma(t),$$

$$[\mathbf{v}]_-^+ = [q]_-^+ = [\mathbf{v} \cdot \nu]_-^+ = 0 \qquad \text{on } \Gamma(t),$$

$$\begin{aligned}
&\partial_t \left(f(q) \frac{K_W^{-1}}{2\sqrt{h(q)}} \right) + \nabla \left(f(q) \frac{K_W^{-1}}{2\sqrt{h(q)}} \right) \cdot \partial_t \hat{\gamma} - \nabla_\Gamma \left(f(q) \frac{K_W^{-1}}{2\sqrt{h(q)}} \right) \cdot \partial_t \hat{\gamma} \\
&+ f(q) \frac{K_W^{-1}}{2\sqrt{h(q)}} \operatorname{div}_\Gamma \mathbf{v} + \nabla_\Gamma \left(f(q) \frac{K_W^{-1}}{2\sqrt{h(q)}} \right) \cdot \mathbf{v} \\
&= \operatorname{div}_\Gamma \left(K(q) \frac{K_W^{-1}}{2\sqrt{h(q)}} \nabla_\Gamma q \right) + [\nabla q \cdot \nu]_-^+ \qquad \text{on } \Gamma(t).
\end{aligned}$$

In this section we identify the relation between the sharp interface model derived in [GLS14] and the sharp interface model which we obtained from (4.1) - (4.5), resp. (1.1) - (1.5) with constant mass density $\rho \equiv 1$. Furthermore, we derive an energy estimate similar to (3.6). To this end, we define

$$\begin{aligned}
c^\Gamma(q) &:= -h'(q)\frac{K_W^{-1}}{2\sqrt{h(q)}} = -K_W^{-1}\frac{d}{dq}\sqrt{h(q)},\\
\sigma(c^\Gamma(q)) &:= \sqrt{h(q)}K_W^{-1},\\
\gamma(c^\Gamma(q)) &:= c^\Gamma(q)q + \sigma(c^\Gamma(q)),\\
M_\Gamma(q) &:= K(q)\frac{K_W^{-1}}{2\sqrt{h(q)}}.
\end{aligned}$$

Analogously as in [GLS14], c^Γ denotes the density of the surfactant on the interface Γ and $\sigma(c^\Gamma)$ is the surface tension which depends on the surfactant density c^Γ. Moreover, $\gamma(c^\Gamma)$ is the free energy density and M_Γ denotes the mobility of the surfactants on the interface. Note that the identity $\gamma(c^\Gamma(q)) = c^\Gamma(q)q + \sigma(c^\Gamma(q))$ is a fundamental identity in chemical thermodynamics relating the surface tension σ, the density c^Γ and the chemical potential, cf. [GW06].
Since it holds $f(q) = -h'(q)$ for every $q \in \mathbb{R}$, we can deduce for the equations on the interface $\Gamma(t)$

$$[p]_-^+\nu - 2[\eta^{(i)}D\mathbf{v}]_-^+\nu - \kappa\sigma(c^\Gamma(q))\nu = \nabla_\Gamma\left(\sigma(c^\Gamma(q))\right) \qquad \text{on } \Gamma(t), \tag{4.58}$$

$$-\mathcal{V} + \mathbf{v}\cdot\nu = 0 \qquad \text{on } \Gamma(t), \tag{4.59}$$

$$\partial_t^\bullet c^\Gamma(q) + c^\Gamma(q)\mathrm{div}_\Gamma\mathbf{v} - \mathrm{div}_\Gamma\left(M_\Gamma(q)\nabla_\Gamma q\right) = [\nabla q\cdot\nu]_-^+ \qquad \text{on } \Gamma(t), \tag{4.60}$$

$$[\mathbf{v}]_-^+ = [q]_-^+ = [\mathbf{v}\cdot\nu]_-^+ = 0 \qquad \text{on } \Gamma(t), \tag{4.61}$$

where $\partial_t^\bullet(\cdot) = \partial_t(\cdot) + \nabla(\cdot)\cdot\mathbf{v}$ is the material time derivative. For the derivation of (4.60) we used that for the quantity $c^\Gamma(q) := f(q)\frac{K_W^{-1}}{2\sqrt{h(q)}}$ it holds

$$\begin{aligned}
\partial_t^\bullet c^\Gamma(q) &= \partial_t c^\Gamma(q) + \nabla c^\Gamma(q)\cdot\mathbf{v}\\
&= \partial_t c^\Gamma(q) + (\nabla c^\Gamma(q)\cdot\nu)\mathbf{v}\cdot\nu + \nabla_\Gamma c^\Gamma(q)\cdot\mathbf{v}\\
&= \partial_t c^\Gamma(q) + \nabla c^\Gamma(q)\cdot\partial_t\hat{\gamma} - \nabla_\Gamma c^\Gamma(q)\cdot\partial_t\hat{\gamma} + \nabla_\Gamma c^\Gamma(q)\cdot\mathbf{v}
\end{aligned}$$

due to

$$(\nabla c^\Gamma(q)\cdot\nu)\mathbf{v}\cdot\nu = (\nabla c^\Gamma(q)\cdot\nu)\partial_t\hat{\gamma}\cdot\nu = \nabla c^\Gamma(q)\cdot\partial_t\hat{\gamma} - \nabla_\Gamma c^\Gamma(q)\cdot\partial_t\hat{\gamma}$$

and $\mathbf{v}\cdot\nu = \mathcal{V} = \partial_t\hat{\gamma}\cdot\nu$.

Now we want to derive an energy estimate for this sharp interface model. To this end, we start with some calculations which we will need in the following. With the previous definitions we can deduce

$$\begin{aligned}
&\frac{d}{dq}\sigma(c^\Gamma(q)) = \frac{d}{dq}\left(\sqrt{h(q)}K_W^{-1}\right) = -c^\Gamma(q),\\
&\frac{d}{dq}\gamma(c^\Gamma(q)) = c^\Gamma(q) + q\frac{d}{dq}c^\Gamma(q) + \frac{d}{dq}\sigma(c^\Gamma(q)) = q\frac{d}{dq}c^\Gamma(q).
\end{aligned}$$

The last identity implies $\gamma'(c^\Gamma(q)) = q$. From the first identity it follows

$$\frac{d}{dq}\sigma(c^\Gamma(q)) = \sigma'(c^\Gamma(q))\frac{d}{dq}c^\Gamma(q) = -c^\Gamma(q) < 0$$

if $f(q) > 0$ because then it holds $c^\Gamma(q) = \frac{f(q)K_W^{-1}}{2\sqrt{h(q)}} > 0$. Therefore, we can conclude

$$\sigma'(c^\Gamma(q)) < 0$$

due to

$$\frac{d}{dq}c^\Gamma(q) = -\frac{\sqrt{h(q)}}{2h(q)}K_W^{-1}\left(h''(q) - \frac{(h'(q))^2}{2h(q)}\right) > 0,$$

where we used $h(q) > 0$ and $h''(q) < 0$. The fact that $\sigma'(c^\Gamma(q)) < 0$ is physically meaningful as the surface tension $\sigma(c^\Gamma(q))$ decreases if the concentration $c^\Gamma(q)$ of the surfactant on the interface increases. Moreover, we can verify

$$q\partial_t^\bullet c^\Gamma(q) = q\partial_t^\bullet q\left(\frac{d}{dq}c^\Gamma(q)\right) = \partial_t^\bullet q\frac{d}{dq}\gamma(c^\Gamma(q)) = \partial_t^\bullet\left(\gamma(c^\Gamma(q))\right). \tag{4.62}$$

We use these identities to derive the energy estimate for the sharp interface model. We want to recover an energy estimate similar to (3.6). To this end, we want to prove

$$\frac{d}{dt}\left(\int\limits_\Omega \frac{|\mathbf{v}|^2}{2} + G(q)\,dx + \int\limits_{\Gamma(t)}\gamma(c^\Gamma(q))\,d\mathcal{H}^{d-1}\right) \leq 0.$$

For the sake of clarity we study all three terms separately. The transport theorem in [EGK08, Theorem 7.3] yields for the first term

$$\frac{d}{dt}\int\limits_\Omega \frac{|\mathbf{v}|^2}{2}dx = \sum_{i=1}^2\int\limits_{\Omega^{(i)}(t)}\mathbf{v}\cdot\partial_t\mathbf{v}\,dx - \int\limits_{\Gamma(t)}\left[\frac{|\mathbf{v}|^2}{2}\right]_-^+\mathcal{V}\,d\mathcal{H}^{d-1} = \sum_{i=1}^2\int\limits_{\Omega^{(i)}(t)}\mathbf{v}\cdot\partial_t\mathbf{v}\,dx,$$

where we used the jump condition $[\mathbf{v}]_-^+ = 0$ on $\Gamma(t)$. From (4.55) it follows

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega \frac{|\mathbf{v}|^2}{2}dx &= \sum_{i=1}^2\int\limits_{\Omega^{(i)}(t)}\mathbf{v}\cdot\left(\operatorname{div}(2\eta^{(i)}D\mathbf{v}) - \nabla p - \mathbf{v}\cdot\nabla\mathbf{v}\right)dx\\ &= \sum_{i=1}^2\int\limits_{\Omega^{(i)}(t)}\mathbf{v}\cdot\left(\operatorname{div}(2\eta^{(i)}D\mathbf{v}) - \nabla p - \frac{1}{2}\operatorname{div}(\mathbf{v}\otimes\mathbf{v})\right)dx\\ &= -\sum_{i=1}^2\int\limits_{\Omega^{(i)}(t)}2\eta^{(i)}|D\mathbf{v}|^2dx + \int\limits_{\Gamma(t)}\mathbf{v}\cdot\left([p]_-^+\nu - [2\eta^{(i)}D\mathbf{v}]_-^+\nu\right)d\mathcal{H}^{d-1}.\end{aligned}$$

In this equation we have two jump terms across the interface $\Gamma(t)$. We derive one jump term in detail since the other one can be derived analogously. It holds

$$\int_{\Omega^{(1)}(t)} \mathbf{v} \cdot \nabla p = - \int_{\Omega^{(1)}(t)} \operatorname{div}(\mathbf{v}) p \, dx + \int_{\Gamma(t)} (\mathbf{v} \cdot \nu) p_1 \, d\mathcal{H}^{d-1} = \int_{\Gamma(t)} (\mathbf{v} \cdot \nu) p_1 \, d\mathcal{H}^{d-1},$$

$$\begin{aligned}\int_{\Omega^{(2)}(t)} \mathbf{v} \cdot \nabla p &= - \int_{\Omega^{(2)}(t)} \operatorname{div}(\mathbf{v}) p \, dx - \int_{\Gamma(t)} (\mathbf{v} \cdot \nu) p_2 \, d\mathcal{H}^{d-1} + \int_{\partial\Omega} (\mathbf{v} \cdot \hat{\nu}) p \, d\mathcal{H}^{d-1} \\ &= - \int_{\Gamma(t)} (\mathbf{v} \cdot \nu) p_2 \, d\mathcal{H}^{d-1},\end{aligned}$$

where we denote by ν the unit normal to the interface $\Gamma(t)$, which is pointing into $\Omega^{(2)}(t)$. Moreover, $\hat{\nu}$ denotes the unit normal to $\partial\Omega$ and

$$p_1(x) := \lim_{t \searrow 0} p(x - t\nu), \qquad p_2(x) := \lim_{t \searrow 0} p(x + t\nu)$$

for every $x \in \Gamma(t)$. This yields the jump term

$$\begin{aligned}\int_{\Omega^{(1)}(t)} \mathbf{v} \cdot \nabla p + \int_{\Omega^{(2)}(t)} \mathbf{v} \cdot \nabla p &= \int_{\Gamma(t)} (\mathbf{v} \cdot \nu) p_1 \, d\mathcal{H}^{d-1} - \int_{\Gamma(t)} (\mathbf{v} \cdot \nu) p_2 \, d\mathcal{H}^{d-1} \\ &= - \int_{\Gamma(t)} [p]_-^+ (\mathbf{v} \cdot \nu) \, dx.\end{aligned}$$

In the following calculations we derive several jump terms across the interface $\Gamma(t)$. But we will not present these calculations in detail since they are similar to the ones above. So we proceed with the estimate. Due to equation (4.58) we can conclude

$$\begin{aligned}\frac{d}{dt} \int_{\Omega} \frac{|\mathbf{v}|^2}{2} dx &= - \sum_{i=1}^{2} \int_{\Omega^{(i)}(t)} 2\eta^{(i)} |D\mathbf{v}|^2 dx + \int_{\Gamma(t)} \mathbf{v} \cdot \left(\kappa\sigma\nu + \nabla_{\Gamma(t)}\sigma \right) d\mathcal{H}^{d-1} \\ &= - \sum_{i=1}^{2} \int_{\Omega^{(i)}(t)} 2\eta^{(i)} |D\mathbf{v}|^2 dx + \int_{\Gamma(t)} \kappa\sigma(\mathbf{v} \cdot \nu) \, d\mathcal{H}^{d-1} - \int_{\Gamma(t)} \kappa\sigma(\mathbf{v} \cdot \nu) \, d\mathcal{H}^{d-1} \\ &\quad - \int_{\Gamma(t)} \sigma \operatorname{div}_{\Gamma(t)}(\mathbf{v}) \, d\mathcal{H}^{d-1} \\ &= - \sum_{i=1}^{2} \int_{\Omega^{(i)}(t)} 2\eta^{(i)} |D\mathbf{v}|^2 dx - \int_{\Gamma(t)} \sigma \operatorname{div}_{\Gamma(t)}(\mathbf{v}) \, d\mathcal{H}^{d-1},\end{aligned}$$

where we used [DE13, Theorem 2.10] for the integration by parts on the interface $\Gamma(t)$.

For the next term we apply the transport theorem in [EGK08, Theorem 7.3] again. Furthermore, we use $G'(q) = g'(q)q$ and (4.57). Then we get

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega} G(q)\,dx &= \sum_{i=1}^{2}\int_{\Omega^{(i)}(t)} \partial_t G(q)\,dx - \int_{\Gamma(t)} [G(q)]_-^+\,\mathcal{V}\,d\mathcal{H}^{d-1} = \sum_{i=1}^{2}\int_{\Omega^{(i)}(t)} G'(q)\partial_t q\,dx \\
&= \sum_{i=1}^{2}\int_{\Omega^{(i)}(t)} g'(q)q\partial_t q\,dx = \sum_{i=1}^{2}\int_{\Omega^{(i)}(t)} q\partial_t g(q)\,dx = \sum_{i=1}^{2}\int_{\Omega^{(i)}(t)} q\left(\Delta q - \nabla g(q)\cdot\mathbf{v}\right)dx \\
&= \sum_{i=1}^{2}\int_{\Omega^{(i)}(t)} \nabla G(q)\cdot\mathbf{v} - |\nabla q|^2\,dx - \int_{\Gamma(t)} [\nabla q]_-^+\nu q\,d\mathcal{H}^{d-1} \\
&= -\sum_{i=1}^{2}\int_{\Omega^{(i)}(t)} |\nabla q|^2\,dx - \int_{\Gamma(t)} [\nabla q]_-^+\nu q\,d\mathcal{H}^{d-1}.
\end{aligned}$$

In the first step, we used that G is a continuous function. Hence, the jump condition $[q]_-^+ = 0$ on $\Gamma(t)$ also implies $[G(q)]_-^+ = 0$.
Finally, we estimate the last term. Here we can not apply the transport theorem in [EGK08, Theorem 7.3] again since we integrate over the interface $\Gamma(t)$ and not over $\Omega^{(i)}(t)$ with $i = 1, 2$. Instead, we use the transport theorem in [DE13, Theorem 5.1], which yields

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma(t)} \gamma(c^\Gamma(q))\,d\mathcal{H}^{d-1} &= \int_{\Gamma(t)} \partial_t^\bullet \gamma(c^\Gamma(q))\,d\mathcal{H}^{d-1} + \int_{\Gamma(t)} \gamma(c^\Gamma(q))\mathrm{div}_{\Gamma(t)}\mathbf{v}\,d\mathcal{H}^{d-1} \\
&= \int_{\Gamma(t)} \gamma'(c^\Gamma(q))\partial_t^\bullet c^\Gamma(q)\,d\mathcal{H}^{d-1} + \int_{\Gamma(t)} \gamma(c^\Gamma(q))\mathrm{div}_{\Gamma(t)}\mathbf{v}\,d\mathcal{H}^{d-1},
\end{aligned}$$

where we used (4.62) together with $\gamma'(c^\Gamma(q)) = q$. Due to equation (4.60) we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma(t)} \gamma(c^{\Gamma(t)}(q))\,dx &= \int_{\Gamma(t)} \gamma'(c^\Gamma(q))\big([\nabla q]_-^+\nu + \mathrm{div}_{\Gamma(t)}(M_{\Gamma(t)}(q)\nabla_{\Gamma(t)} q \\
&\qquad - c^{\Gamma(t)}(q)\mathrm{div}_{\Gamma(t)}\mathbf{v}\big)\,d\mathcal{H}^{d-1} + \int_{\Gamma(t)} \gamma(c^\Gamma(q))\mathrm{div}_{\Gamma(t)}\mathbf{v}\,d\mathcal{H}^{d-1} \\
&= \int_{\Gamma(t)} \gamma'(c^\Gamma(q))\big([\nabla q]_-^+\nu - c^{\Gamma(t)}(q)\mathrm{div}_{\Gamma(t)}\mathbf{v}\big)\,d\mathcal{H}^{d-1} \\
&\qquad - \int_{\Gamma(t)} M_{\Gamma(t)}(q)|\nabla_{\Gamma(t)} q|^2\,d\mathcal{H}^{d-1} + \int_{\Gamma(t)} \gamma(c^\Gamma(q))\mathrm{div}_{\Gamma(t)}\mathbf{v}\,d\mathcal{H}^{d-1}.
\end{aligned}$$

Hence, we have estimated all three terms. Altogether, this implies the energy estimate

$$
\begin{aligned}
&\frac{d}{dt}\left(\int\limits_{\Omega} \frac{|\mathbf{v}|^2}{2} + G(q)\,dx + \int\limits_{\Gamma(t)} \gamma(c^{\Gamma(t)}(q))\,d\mathcal{H}^{d-1}\right) \\
&\quad = -\int\limits_{\Gamma(t)} M_{\Gamma(t)}(q)|\nabla_{\Gamma(t)}q|^2\,d\mathcal{H}^{d-1} - \sum_{i=1}^{2}\int\limits_{\Omega^{(i)}(t)} \left(2\eta|D\mathbf{v}|^2 + |\nabla q|^2\right)dx \le 0,
\end{aligned}
$$

where we used the identity $\gamma(c^\Gamma(q)) = c^\Gamma(q)q + \sigma(c^\Gamma(q))$.

5 Existence of Strong Solutions Locally in Time for a Diffuse Interface Model for Two-Phase Flows of Incompressible Fluids with Different Densities

In this chapter we study a thermodynamically consistent, diffuse interface model for two-phase flows with different densities in a bounded domain in two or three space dimensions derived in [AGG12]. This model is given by the following governing equations

$$
\begin{aligned}
\partial_t(\rho\mathbf{v})+\operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) + \operatorname{div}\left(\mathbf{v}\otimes\frac{\tilde\rho_1-\tilde\rho_2}{2}m(\varphi)\nabla(\frac{1}{\varepsilon}W'(\varphi)-\varepsilon\Delta\varphi)\right)& \\
= \operatorname{div}(-\varepsilon\nabla\varphi\otimes\nabla\varphi)+\operatorname{div}(2\eta(\varphi)D\mathbf{v})-\nabla p & \qquad \text{in } Q_T, \quad (5.1)\\
\operatorname{div}(\mathbf{v}) = 0 & \qquad \text{in } Q_T, \quad (5.2)\\
\partial_t^{\bullet}\varphi = \operatorname{div}(m(\varphi)\nabla\mu) & \qquad \text{in } Q_T, \quad (5.3)\\
\mu = -\varepsilon\Delta\varphi+\frac{1}{\varepsilon}W'(\varphi) & \qquad \text{in } Q_T, \quad (5.4)
\end{aligned}
$$

together with the initial and boundary values

$$
\begin{aligned}
\mathbf{v}_{|\partial\Omega} = \partial_n\varphi_{|\partial\Omega} = \partial_n\mu_{|\partial\Omega} = 0 & \qquad \text{on } (0,T)\times\partial\Omega, \quad (5.5)\\
\varphi(0)=\varphi_0, \mathbf{v}(0)=\mathbf{v}_0 & \qquad \text{in } \Omega. \quad (5.6)
\end{aligned}
$$

Here $Q_T = (0,T)\times\Omega$, where $\Omega\subseteq\mathbb{R}^d$, $d=2,3$, is a bounded domain with C^4-boundary. In [AGG12] it is shown that the first equation is equivalent to

$$
\rho\partial_t\mathbf{v}+\left(\left(\rho\mathbf{v}+\frac{\tilde\rho_1-\tilde\rho_2}{2}m(\varphi)\,\nabla\,(\frac{1}{\varepsilon}W'(\varphi)-\varepsilon\Delta\varphi)\right)\cdot\nabla\right)\mathbf{v}+\nabla p-\operatorname{div}(2\eta(\varphi)D\mathbf{v}) = -\varepsilon\Delta\varphi\nabla\varphi.
$$

For this coupled system the existence of weak solutions was proven by Abels, Depner and Garcke in [ADG13].
In this chapter we prove the existence of a unique strong solution $(\mathbf{v},\varphi)\in X_T$ for small $T>0$, where the space X_T will be specified later. The idea for the proof is to linearize the highest order terms in the equations above at the initial data and then to split the equations in a linear and a nonlinear part such that

$$
\mathcal{L}(\mathbf{v},\varphi)=\mathcal{F}(\mathbf{v},\varphi),
$$

where we still have to specify in which sense this equation has to hold. In particular linearizing only the highest order terms means that we do not linearize all terms of the equations. To linearize it formally at the initial data we replace $\mathbf{v}$, p and φ by $\mathbf{v}_0+\varepsilon\mathbf{v}$, $p_0+\varepsilon p$ and $\varphi_0+\varepsilon\varphi$ and then differentiate with respect to ε at $\varepsilon=0$. In

(5.1) and the equivalent equation in [AGG12], the highest order terms with respect to t and x are $\rho\partial_t\mathbf{v}$, $\operatorname{div}(2\eta(\varphi)D\mathbf{v})$ and ∇p. Hence the linearizations are given by

$$\begin{aligned}
\frac{d}{d\varepsilon}\left(\rho(\varphi_0+\varepsilon\varphi)\partial_t(\mathbf{v}_0+\varepsilon\mathbf{v})\right)_{|\varepsilon=0} &= \rho'(\varphi_0)\varphi\partial_t\mathbf{v}_0+\rho(\varphi_0)\partial_t\mathbf{v} \\
&= \rho_0\partial_t\mathbf{v}, \\
\frac{d}{d\varepsilon}\left(\operatorname{div}(2\eta(\varphi_0+\varepsilon\varphi)D(\mathbf{v}_0+\varepsilon\mathbf{v}))\right)_{|\varepsilon=0} &= \operatorname{div}(2\eta'(\varphi_0)\varphi D\mathbf{v}_0)+\operatorname{div}(2\eta(\varphi_0)D\mathbf{v}), \\
\frac{d}{d\varepsilon}\nabla(p_0+\varepsilon p)_{|\varepsilon=0} &= \nabla p,
\end{aligned}$$

where $\rho_0 := \rho(\varphi_0)$ and $\rho_0' := \rho'(\varphi_0)$. Moreover, we omit the term $\operatorname{div}(2\eta'(\varphi_0)\varphi D\mathbf{v}_0)$ in the second linearization since it is of lower order. For the last equation we get the linearization

$$\begin{aligned}
\frac{d}{d\tilde{\varepsilon}}\operatorname{div}(m(\varphi_0+\tilde{\varepsilon}\varphi)\nabla(-\varepsilon\Delta(\varphi_0+\tilde{\varepsilon}\varphi)))_{|\tilde{\varepsilon}=0} =& -\varepsilon\operatorname{div}(m'(\varphi_0)\varphi\nabla\Delta\varphi_0) \\
&-\varepsilon\operatorname{div}(m(\varphi_0)\nabla\Delta\varphi).
\end{aligned}$$

We can omit the first term since it is of lower order. The second term can formally be reformulated to

$$-\varepsilon\operatorname{div}(m(\varphi_0)\nabla\Delta\varphi) = -\varepsilon m'(\varphi_0)\nabla\varphi_0\cdot\nabla\Delta\varphi-\varepsilon m(\varphi_0)\Delta(\Delta\varphi).$$

Here the first summand is of lower order again. Hence, the linearization is given by $-\varepsilon m(\varphi_0)\Delta^2\varphi$. Due to these linearizations we define the linear operator $\mathcal{L}: X_T\to Y_T$ by

$$\mathcal{L}(\mathbf{v},\varphi) = \begin{pmatrix} \mathbb{P}_\sigma(\rho_0\partial_t\mathbf{v})-\mathbb{P}_\sigma(\operatorname{div}(2\eta(\varphi_0)D\mathbf{v})) \\ \partial_t\varphi+\varepsilon m(\varphi_0)\Delta^2\varphi \end{pmatrix},$$

where $\mathcal{L}$ consists of the principal part of the linearizations, i.e., of the terms of the highest order. Furthermore, we define the nonlinear operator $\mathcal{F}: X_T\to Y_T$ by

$$\mathcal{F}(\mathbf{v},\varphi) = \begin{pmatrix} \mathbb{P}_\sigma F_1(\mathbf{v},\varphi) \\ -\nabla\varphi\cdot\mathbf{v}+\operatorname{div}(\frac{1}{\varepsilon}m(\varphi)\nabla W'(\varphi))+\varepsilon m(\varphi_0)\Delta^2\varphi-\varepsilon\operatorname{div}(m(\varphi)\nabla\Delta\varphi) \end{pmatrix},$$

where

$$\begin{aligned}
F_1(\mathbf{v},\varphi) =& (\rho_0-\rho)\partial_t\mathbf{v}-\operatorname{div}(2\eta(\varphi_0)D\mathbf{v})+\operatorname{div}(2\eta(\varphi)D\mathbf{v})-\varepsilon\Delta\varphi\nabla\varphi \\
&-\left(\left(\rho\mathbf{v}+\frac{\tilde{\rho}_1-\tilde{\rho}_2}{2}m(\varphi)\nabla(\frac{1}{\varepsilon}W'(\varphi)-\varepsilon\Delta\varphi)\right)\cdot\nabla\right)\mathbf{v}.
\end{aligned}$$

It still remains to define the spaces X_T and Y_T. To this end, we set

$$\begin{aligned}
Z_T^1 &:= L^2(0,T;H^2(\Omega)^d\cap H_0^1(\Omega)^d)\cap W_2^1(0,T;L_\sigma^2(\Omega)), \\
Z_T^2 &:= L^p(0,T;W_{p,N}^4(\Omega))\cap W_p^1(0,T;L^p(\Omega))
\end{aligned}$$

with $4 < p < 6$ and

$$W^4_{p,N}(\Omega) := \{\varphi \in W^4_p(\Omega) |\ \partial_n \varphi = \partial_n(\Delta\varphi) = 0\}.$$

We equip Z^1_T and Z^2_T with the norms $||\cdot||'_{Z^1_T}$ and $||\cdot||'_{Z^2_T}$ defined by

$$\begin{aligned}
||\mathbf{v}||'_{Z^1_T} &:= ||\mathbf{v}'||_{L^2(0,T;L^2(\Omega))} + ||\mathbf{v}||_{L^2(0,T;H^2(\Omega))} + ||\mathbf{v}(0)||_{(L^2(\Omega),H^2(\Omega))_{\frac{1}{2},2}}, \\
||\varphi||'_{Z^2_T} &:= ||\varphi'||_{L^p(0,T;L^p(\Omega))} + ||\varphi||_{L^p(0,T;W^4_{p,N}(\Omega))} + ||\varphi(0)||_{(L^p(\Omega),W^4_p(\Omega))_{1-\frac{1}{p},p}}.
\end{aligned} \tag{5.7}$$

We use these norms since they guarantee that for all embeddings we will study later the embedding constant C does not depend on T, cf. Lemma 5.2. But first of all we have to show that the norms $||\cdot||'_{Z^1_T}$ and $||\cdot||'_{Z^2_T}$ are equivalent to the norms $||\cdot||_{Z^1_T}$ and $||\cdot||_{Z^2_T}$. Moreover, we need to show that $\mathbf{v}(0) \in (L^2(\Omega), H^2(\Omega))_{\frac{1}{2},2}$ and $\varphi(0) \in (L^p(\Omega), W^4_p(\Omega))_{1-\frac{1}{p},p}$ are well-defined.

Lemma 5.1. *For every $0 < T < \infty$, the norms $||\cdot||'_{Z^1_T}$ and $||\cdot||'_{Z^2_T}$ are equivalent to the norms $||\cdot||_{Z^1_T}$ and $||\cdot||_{Z^2_T}$, i.e., there exist constants $c(T), C(T) > 0$ depending on T such that*

$$c(T)||f||'_{Z^i_T} \leq ||f||_{Z^i_T} \leq C(T)||f||'_{Z^i_T}$$

for every $f \in Z^i_T$, $i = 1, 2$.

Proof. The second inequality is obvious since the norm $||\cdot||'_{Z^i_T}$ consists of an extra term which not appears in the norm $||\cdot||_{Z^i_T}$, $i = 1, 2$.
Thus it remains to show the first inequality. Due to Theorem 2.28 and Theorem 2.30 we can deduce

$$\begin{aligned}
Z^1_T &\hookrightarrow BUC([0,T];(L^2(\Omega),H^2(\Omega))_{1-\frac{1}{2},2}) = BUC([0,T];H^1(\Omega)), \\
Z^2_T &\hookrightarrow BUC([0,T];(L^p(\Omega),W^4_p(\Omega))_{1-\frac{1}{p},p}) = BUC([0,T];W_p^{4-\frac{4}{p}}(\Omega))
\end{aligned}$$

together with the estimates

$$\begin{aligned}
\sup_{t\in[0,T]} ||\mathbf{v}(t)||_{H^1(\Omega)} &\leq C_1(T)||\mathbf{v}||_{Z^1_T}, \\
\sup_{t\in[0,T]} ||\varphi(t)||_{W_p^{4-\frac{4}{p}}(\Omega)} &\leq C_2(T)||\varphi||_{Z^2_T},
\end{aligned}$$

for some constants $C_1(T), C_2(T) > 0$. Hence, the norms $||\cdot||'_{Z^i_T}$, $i = 1, 2$, are well-defined as we can conclude $\mathbf{v}(0) \in (L^2(\Omega), H^2(\Omega))_{\frac{1}{2},2}$ for $\mathbf{v} \in Z^1_T$ and $\varphi(0) \in (W^4_p(\Omega), L^p(\Omega))_{1-\frac{1}{p},p}$ for $\varphi \in Z^2_T$. Moreover, these estimates imply that the first inequality in this lemma holds for $i = 1, 2$. $\square$

Due to Lemma 5.1, Z_T^1 and Z_T^2 are Banach spaces with respect to the norms $||\cdot||'_{Z_T^1}$, resp. $||\cdot||'_{Z_T^2}$. Equipped with these norms we are able to prove that there exists a unique strong solution locally in time for (5.1) - (5.6). From now on we write $||\cdot||_{Z_T^i}$, but we mean the equivalent norm $||\cdot||'_{Z_T^i}$ for $i=1,2$.

In Lemma 5.1 we proved that both norms are equivalent. In the next lemma we show that the embedding constants are independent of T.

Lemma 5.2. *Let $0 < T_0 < \infty$ be given and X_0, X_1 be some Banach spaces such that $X_1 \hookrightarrow X_0$ densely. For every $0 < T < \frac{T_0}{2}$ we define*

$$X_T := L^p(0,T;X_1) \cap W_p^1(0,T;X_0),$$

where $1 \leq p < \infty$. Then there exists an extension operator $E : X_T \to X_{T_0}$ and some constant $C > 0$ independent of T such that $Eu_{|(0,T)} = u$ in X_T and

$$||Eu||_{X_{T_0}} \leq C||u||_{X_T}$$

for every $u \in X_T$ and every $0 < T < \frac{T_0}{2}$. Moreover, there exists a constant $\tilde{C}(T_0) > 0$ independent of T such that

$$||u||_{BUC([0,T];(X_0,X_1)_{1-\frac{1}{p},p})} \leq \tilde{C}(T_0)||u||_{X_T}$$

for every $u \in X_T$ and every $0 < T < \frac{T_0}{2}$.

Note that in this lemma we already use the notation from above, i.e., we write $||\cdot||_{X_{T_{(0)}}}$ but always mean the norm $||\cdot||'_{X_{T_{(0)}}}$.

Proof. First of all we prove the first inequality. To this end, let $u \in X_T$ be given. Then we distinguish two cases:
1$^{\text{st}}$ case: $u(0) = 0$
In this case we define

$$(Eu)(t) := \begin{cases} u(t) & \text{if } t \in [0,T], \\ u(2T-t) & \text{if } t \in (T,2T], \\ 0 & \text{if } t \in (2T,T_0]. \end{cases}$$

Then it holds $Eu \in X_{T_0}$ together with the estimate

$$||Eu||_{X_{T_0}} \leq 2||u||_{X_T}.$$

2$^{\text{nd}}$ case: $u(0) \not\equiv 0$
Because of $u \in X_T \hookrightarrow BUC([0,T];(X_0,X_1)_{1-\frac{1}{p},p})$, cf. Theorem 2.30, we can deduce

$u_0 := u(0) \in (X_0, X_1)_{1-\frac{1}{p},p}$. Moreover, Theorem 2.30 also implies the existence of some $\tilde{u} \in X_{T_0}$ with $\tilde{u}(0) = u_0$ and

$$||\tilde{u}||_{X_{T_0}} \leq C||u_0||_{(X_0,X_1)_{1-\frac{1}{p},p}}$$

for a constant $C > 0$. For $\omega := u - \tilde{u}$ it holds $\omega(0) = 0$. Thus we define the extension operator E by

$$(Eu)(t) := (E_1\omega)(t) + \tilde{u}(t)$$

for every $t \in [0, T_0]$, where E_1 is the extension operator from the first case. Hence, we get

$$\begin{aligned}
||Eu||_{X_{T_0}} &\leq ||E_1\omega||_{X_{T_0}} + ||\tilde{u}||_{X_{T_0}} \leq C(||\omega||_{X_T} + ||u_0||_{(X_0,X_1)_{1-\frac{1}{p},p}}) \\
&\leq C(||u||_{X_T} + ||\tilde{u}||_{X_T} + ||u_0||_{(X_0,X_1)_{1-\frac{1}{p},p}}) \\
&\leq C(||u||_{X_T} + ||\tilde{u}||_{X_{T_0}} + ||u_0||_{(X_0,X_1)_{1-\frac{1}{p},p}}) \\
&\leq C(||u||_{X_T} + ||u_0||_{(X_0,X_1)_{1-\frac{1}{p},p}}) \leq C||u||_{X_T}
\end{aligned}$$

for every $u \in X_T$ and $0 < T < \frac{T_0}{2}$.

It remains to prove the second inequality in the lemma. So let $u \in X_T$ be given. Then Theorem 2.30 implies $u \in BUC([0,T]; (X_0, X_1)_{1-\frac{1}{p},p})$ and we can estimate

$$\begin{aligned}
||u||_{BUC([0,T];(X_0,X_1)_{1-\frac{1}{p},p})} &\leq ||Eu||_{BUC([0,T_0];(X_0,X_1)_{1-\frac{1}{p},p})} \leq C(T_0)||Eu||_{X_{T_0}} \\
&\leq \tilde{C}(T_0)||u||_{X_T}
\end{aligned}$$

for every $u \in X_T$ and every $0 < T < \frac{T_0}{2}$. $\square$

This lemma shows that, if we equip X_T with the new norm, then the embedding constant for the embedding $X_T \hookrightarrow BUC([0,T]; (X_0, X_1)_{1-\frac{1}{p},p})$ does not depend on T. This will be useful for the existence proof since we want to control the embedding constant for small T, i.e., we want to avoid $C \to \infty$ for $T \to 0$.

The last preparation before we can start with the existence proof is the definition of the function spaces $X_T := X_T^1 \times X_T^2$ and Y_T by

$$\begin{aligned}
X_T^1 &:= \{\mathbf{v} \in Z_T^1 | \ \mathbf{v}_{|t=0} = \mathbf{v}_0\}, \\
X_T^2 &:= \{\varphi \in Z_T^2 | \ \varphi_{|t=0} = \varphi_0\}, \\
Y_T &:= Y_T^1 \times Y_T^2 := L^2(0,T; L^2_\sigma(\Omega)) \times L^p(0,T; L^p(\Omega)),
\end{aligned}$$

where

$$\mathbf{v}_0 \in (L^2_\sigma(\Omega), H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega))_{\frac{1}{2},2} = H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$$

and

$$\varphi_0 \in (L^p(\Omega), W^4_{p,N}(\Omega))_{1-\frac{1}{p},p}$$

are the initial values from (5.6). Note that in the space X^2_T we have to ensure that $\varphi_{|t=0} = \varphi_0 \in [-1,1]$ since we will use this property to show the Lipschitz continuity of $\mathcal{F} : X_T \to Y_T$ in Proposition 5.7. Moreover, we note that X_T is not a vector space due to the condition $\varphi_{|t=0} = \varphi_0$. It is only an affine linear subspace of $Z_T := Z^1_T \times Z^2_T$.

5.1 Existence Proof

In this section we state the main theorem on the short time existence of strong solutions for the coupled system (5.1) - (5.6) and then prove this theorem. Note that in the proof we will use several results which we will prove later. For the sake of clarity we always refer to these results.
For the analysis it is necessary to estimate terms like $\eta(\varphi)$, $m(\varphi)$, $\rho(\varphi)$ and $W'(\varphi)$. Hence, we make the following assumptions which we assume to hold in the whole chapter.

Assumption 5.3.

i) *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^4-boundary and $d = 2, 3$.*

ii) *Let $\eta \in C^4_b(\mathbb{R})$ such that $\eta(s) \geq s_0 > 0$ for every $s \in \mathbb{R}$ and some $s_0 > 0$.*

iii) *The mobility $m \in C^4_b(\Omega)$ is Lipschitz continuous and it holds $m \geq m_0 > 0$.*

iv) *The density ρ is given by*

$$\rho = \rho(\varphi) = \frac{\tilde{\rho}_1 + \tilde{\rho}_2}{2} + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\varphi \qquad \text{for all } \varphi \in \mathbb{R}.$$

v) *The double-well potential W is twice continuously differentiable.*

With these assumptions we get the main existence result about short time existence of strong solutions for (5.1) - (5.6).

Theorem 5.4. *(Existence of strong solutions)*
Let Ω, η, m, ρ and W be as in Assumption 5.3. Moreover, let $\mathbf{v}_0 \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$ and $\varphi_0 \in (L^p(\Omega), W^4_{p,N}(\Omega))_{1-\frac{1}{p},p}$ be given with $4 < p < 6$. Then there exists $T > 0$ such that (5.1) - (5.6) has a unique strong solution

$$\begin{aligned} &\mathbf{v} \in W^1_2(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^2(\Omega)^d \cap H^1_0(\Omega)^d), \\ &\varphi \in W^1_p(0,T; L^p(\Omega)) \cap L^p(0,T; W^4_{p,N}(\Omega)). \end{aligned}$$

Before we start with the proof of Theorem 5.4, we make some remarks on the space for φ_0.

Remark 5.5. *For Lemma 5.11 it is necessary that φ_0 is in $W^1_r(\Omega)$ for $r > d \geq 2$. But due to the identity $(L^p(\Omega), W^4_p(\Omega))_{1-\frac{1}{p},p} = W_p^{4-\frac{4}{p}}(\Omega)$ with $4 < p < 6$ this condition is satisfied. Moreover, the assumption that φ_0 is in $(L^p(\Omega), W^4_{p,N}(\Omega))_{1-\frac{1}{p},p}$ is necessary for Lemma 5.15*

Proof. (Proof of Theorem 5.4)
First of all we note that (5.1) - (5.4) is equivalent to

$$\begin{aligned} & \mathcal{L}(\mathbf{v},\varphi) = \mathcal{F}(\mathbf{v},\varphi) && \text{in } Y_T, \\ \Leftrightarrow \quad & (\mathbf{v},\varphi) = \mathcal{L}^{-1} \circ \mathcal{F}(\mathbf{v},\varphi) && \text{in } X_T. \end{aligned} \tag{5.8}$$

The fact that $\mathcal{L}$ is invertible will be proven later. Equation (5.8) implies that we have rewritten the system to a fixed-point equation which we want to solve by using the Banach fixed-point theorem.

To this end, we consider some $(\tilde{\mathbf{v}}, \tilde{\varphi}) \in X_T$ and define

$$M := ||\mathcal{L}^{-1} \circ \mathcal{F}(\tilde{\mathbf{v}}, \tilde{\varphi})||_{X_T} < \infty.$$

Now let $R > 0$ be given such that $(\tilde{\mathbf{v}}, \tilde{\varphi}) \in B_R^{X_T}(0)$ and $R > 2M$. Then it follows from Proposition 5.7 that there exists a constant $C = C(T,R) > 0$ such that

$$||\mathcal{F}(\mathbf{v}_1,\varphi_1) - \mathcal{F}(\mathbf{v}_2,\varphi_2)||_{Y_T} \leq C(T,R)||(\mathbf{v}_1,\varphi_1) - (\mathbf{v}_2,\varphi_2)||_{X_T}$$

for all $(\mathbf{v}_i, \varphi_i) \in X_T$ with $||(\mathbf{v}_i,\varphi_i)||_{X_T} \leq R$, $j = 1,2$, where it holds $C(T,R) \to 0$ as $T \to 0$. Furthermore, we choose T so small that

$$||\mathcal{L}^{-1}||_{\mathcal{L}(Y_T,X_T)} C(T,R) < \frac{1}{2}.$$

Here we have to ensure that $||\mathcal{L}^{-1}||_{\mathcal{L}(Y_T,X_T)}$ does not converge to $+\infty$ as $T \to 0$. But since Lemma 5.13 and Lemma 5.16 below yield $||\mathcal{L}^{-1}||_{\mathcal{L}(Y_T,X_T)} < C(T_0)$ for every $0 < T < T_0$ and for a constant that does not depend on T, this is not the case and we can choose $T > 0$ in such a way that the previous estimate holds. Note that T depends on R and in general T has to become smaller the larger we choose R.
Since we want to apply the Banach fixed-point theorem on $B_R^{X_T}(0) \subseteq X_T$ as we only consider functions $(\mathbf{v},\varphi) \in X_T$ which satisfy $||(\mathbf{v},\varphi)||_{X_T} \leq R$, we have to show that $\mathcal{L}^{-1} \circ \mathcal{F}$ maps from $B_R^{X_T}(0)$ to $B_R^{X_T}(0)$.

From the considerations above we know that there exists $(\tilde{\mathbf{v}}, \tilde{\varphi}) \in B_R^{X_T}(0)$ such that

$$||\mathcal{L}^{-1} \circ \mathcal{F}(\tilde{\mathbf{v}}, \tilde{\varphi})||_{X_T} = M < \frac{R}{2}. \tag{5.9}$$

Then a direct calculation shows

$$\begin{aligned}||\mathcal{L}^{-1} \circ \mathcal{F}(\mathbf{v}, \varphi)||_{X_T} &\leq ||\mathcal{L}^{-1} \circ \mathcal{F}(\mathbf{v}, \varphi) - \mathcal{L}^{-1} \circ \mathcal{F}(\tilde{\mathbf{v}}, \tilde{\varphi})||_{X_T} + ||\mathcal{L}^{-1} \circ \mathcal{F}(\tilde{\mathbf{v}}, \tilde{\varphi})||_{X_T} \\ &< ||\mathcal{L}^{-1}||_{\mathcal{L}(Y_T, X_T)} ||\mathcal{F}(\mathbf{v}, \varphi) - \mathcal{F}(\tilde{\mathbf{v}}, \tilde{\varphi})||_{Y_T} + \frac{R}{2} \\ &\leq ||\mathcal{L}^{-1}||_{\mathcal{L}(Y_T, X_T)} C(R,T) ||(\mathbf{v}, \varphi) - (\tilde{\mathbf{v}}, \tilde{\varphi})||_{X_T} + \frac{R}{2} < R\end{aligned}$$

for every $(\mathbf{v}, \varphi) \in \overline{B_R^{X_T}(0)}$, where we used the estimate for the Lipschitz continuity of $\mathcal{F}$. This shows that $\mathcal{L}^{-1} \circ \mathcal{F}(\mathbf{v}, \varphi)$ is in $\overline{B_R^{X_T}(0)}$ for every $(\mathbf{v}, \varphi) \in \overline{B_R^{X_T}(0)}$, i.e.,

$$\mathcal{L}^{-1} \circ \mathcal{F} : \overline{B_R^{X_T}(0)} \to \overline{B_R^{X_T}(0)}.$$

For applying the Banach fixed-point theorem it remains to show that the mapping $\mathcal{L}^{-1} \circ F : B_R^{X_T}(0) \to B_R^{X_T}(0)$ is a contraction. To this end, let $(\mathbf{v}_i, \varphi_i) \in B_R^{X_T}(0)$ be given for $i = 1, 2$. Then it holds

$$\begin{aligned}&||\mathcal{L}^{-1} \circ \mathcal{F}(\mathbf{v}_1, \varphi_1) - \mathcal{L}^{-1} \circ \mathcal{F}(\mathbf{v}_2, \varphi_2)||_{X_T} \\ &\quad \leq ||\mathcal{L}^{-1}||_{\mathcal{L}(Y_T, X_T)} C(R,T) ||(\mathbf{v}_1, \varphi_1) - (\mathbf{v}_2, \varphi_2)||_{X_T} \\ &\quad < \frac{1}{2} ||(\mathbf{v}_1, \varphi_1) - (\mathbf{v}_2, \varphi_2)||_{X_T},\end{aligned}$$

which shows the statement. Hence, the Banach fixed-point theorem can be applied and yields some $(\hat{\mathbf{v}}, \hat{\varphi}) \in B_R^{X_T}(0) \subseteq X_T$ such that the fixed-point equation (5.8) holds, which implies that $(\hat{\mathbf{v}}, \hat{\varphi})$ is a strong solution for the equations (5.1) - (5.4). $\square$

Note that this method was already used e.g. in [ADL16] to prove the existence of a strong solution for the Beris–Edwards model for nematic liquid crystals.
In the proof of Theorem 5.4 we used that $\mathcal{F} : X_T \to Y_T$ is Lipschitz continuous and that the Lipschitz constant C converges to 0 as $T \to 0$. Moreover, we used that $\mathcal{L} : X_T \to Y_T$ is invertible. We will prove these statements in the remaining parts of this chapter. Since the equations of $\mathcal{L}$ are decoupled, we are able to solve both equations separately. We will start with the first one and show that for every right-hand side $\mathbf{f} \in Y_T^1$ there exists a unique solution $\mathbf{v}$ in X_T^1. Afterwards we study the second equation and also prove the existence of a unique solution $\varphi \in X_T^2$ for every right-hand side $f \in Y_T^2$, which shows the statement and therefore completes the existence proof of Theorem 5.4.
But first of all we need to make some preparations for the analysis of these proofs. Therefore, we collect the most important results in the following section.

5.2 Preparations for the Analysis

Before we continue we study in which Banach spaces $\mathbf{v}$, φ, $\nabla\varphi$, $m(\varphi)$ and so on are bounded. To this end, we use some interpolation results from Chapter 2.

Note that in the definition of X_T^2, p has to be larger than 4 because we will need to estimate terms like $\nabla\Delta\varphi \cdot \nabla \mathbf{v}$, where $p = 2$ is not sufficient for the analysis and therefore we need to choose $p > 2$. But for most terms in the analysis $p = 2$ would be sufficient and $4 < p < 6$ would not be necessary. Nevertheless, for consistency all calculations are done for the case $4 < p < 6$.

Interpolation spaces for v

Due to Theorem 2.28 and Theorem 2.30 it holds

$$\mathbf{v} \in X_T^1 \hookrightarrow BUC([0,T]; B_{22}^1(\Omega)) = BUC([0,T]; H^1(\Omega)), \tag{5.10}$$

where we used $B_{22}^s(\Omega) = H_2^s(\Omega)$ for every $s \in \mathbb{R}$, cf. (2.4). In particular this implies

$$\nabla \mathbf{v} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; L^6(\Omega)) \hookrightarrow L^{\frac{8}{3}}(0,T; L^4(\Omega)), \tag{5.11}$$
$$\nabla \mathbf{v} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; L^6(\Omega)) \hookrightarrow L^4(0,T; L^3(\Omega)), \tag{5.12}$$

where we applied Theorem 2.32 with $\theta = \frac{3}{4}$ and $\theta = \frac{1}{2}$.

Interpolation spaces for $\boldsymbol{\varphi}$

Let $\varphi \in X_T^2$ be given. From Theorem 2.28 and Theorem 2.30 it follows

$$\varphi \in L^p(0,T; W_{p,N}^4(\Omega)) \cap W_p^1(0,T; L^p(\Omega)) \hookrightarrow BUC([0,T]; W_p^{4-\frac{4}{p}}(\Omega)). \tag{5.13}$$

This implies

$$\nabla\Delta\varphi \in BUC([0,T]; W_p^{1-\frac{4}{p}}(\Omega)) \tag{5.14}$$

since it holds $p > 4$. Note that when we write "φ is bounded in Z" for some function space Z, we mean that the set of all functions $\{\varphi \in X_T^2 : \ ||\varphi||_{X_T^2} \le R\}$ is bounded in Z in such a way that the upper bound only depends on R and not on T, i.e., there exists $C(R) > 0$ such that $||\varphi||_Z \le C(R)$ for every $\varphi \in X_T^2$ with $||\varphi||_{X_T^2} \le R$.

The following lemma will imply a lot of embeddings for the further analysis.

Lemma 5.6. *Let $X_0 \subseteq Y \subseteq X_1$ be some Banach spaces such that*

$$||x||_Y \le C||x||_{X_0}^{1-\theta}||x||_{X_1}^{\theta}$$

for every $x \in X_0$ and a constant $C > 0$, where $\theta \in (0,1)$. Then it holds

$$C^{0,\alpha}([0,T]; X_1) \cap L^\infty(0,T; X_0) \hookrightarrow C^{0,\alpha\theta}([0,T]; Y).$$

In addition, if $f \in X := C^{0,\alpha}([0,T]; X_1) \cap L^\infty(0,T; X_0)$ such that $||f||_X \le R$, then the embedding constant only depends on R.

Proof. Let $f \in C^{0,\alpha}([0,T];X_1) \cap L^\infty(0,T;X_0)$ be given. Then it holds

$$\begin{aligned} ||f(t) - f(s)||_Y &\leq C||f(t) - f(s)||_{X_0}^{1-\theta} ||f(t) - f(s)||_{X_1}^{\theta} \\ &\leq C||f||_{L^\infty(0,T;X_0)}^{1-\theta} |t-s|^{\alpha\theta} ||f||_{C^{0,\alpha}([0,T];X_1)}^{\theta} \\ &\leq C|t-s|^{\alpha\theta} \end{aligned}$$

for a.e. $t \in (0,T)$, where we used

$$\sup_{0 \leq t \neq s \leq T} \frac{||f(t) - f(s)||_{X_1}}{|t-s|^\alpha} \leq ||f||_{C^{0,\alpha}([0,T];X_1)} < C. \tag{5.15}$$

If it holds $f \in X$ such that $||f||_X \leq R$, then in the last inequality the constant only depends on R. □

We can use this lemma to prove further embeddings. We apply Theorem 2.15 for Banach-valued Sobolev functions and obtain

$$\varphi \in W_p^1(0,T;L^p(\Omega)) \hookrightarrow C^{0,1-\frac{1}{p}}([0,T];L^p(\Omega)).$$

Moreover, we already know $\varphi \in BUC([0,T];W_p^{4-\frac{4}{p}}(\Omega))$. Due to (2.3) and $4 < p < 6$ it holds $L^p(\Omega) \hookrightarrow B_{pp}^0(\Omega)$. By choosing $\theta := \frac{\frac{4}{p}-1}{\frac{4}{p}-4}$ in Theorem 2.28 we obtain

$$(B_{pp}^{4-\frac{4}{p}}(\Omega), B_{pp}^0(\Omega))_{\theta,2} = B_{p2}^3(\Omega) \hookrightarrow H_p^3(\Omega) = W_p^3(\Omega)$$

together with the estimate

$$||\varphi(t)||_{W_p^3(\Omega)} \leq C||\varphi(t)||_{W_p^{4-\frac{4}{p}}(\Omega)}^{1-\theta} ||\varphi(t)||_{L^p(\Omega)}^{\theta}$$

for every $t \in [0,T]$. Hence, Lemma 5.6 implies

$$\varphi \in C^{0,1-\frac{1}{p}}([0,T];L^p(\Omega)) \cap C([0,T];W_p^{4-\frac{4}{p}}(\Omega)) \hookrightarrow C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega)). \tag{5.16}$$

Since it holds $W_p^3(\Omega) \hookrightarrow C^2(\overline{\Omega})$ for $d = 2,3$ and $4 < p < 6$, cf. Theorem 2.15, we obtain that

$$\varphi \text{ is bounded in } C([0,T];C^2(\overline{\Omega})). \tag{5.17}$$

Interpolation spaces for $\eta(\varphi)$, $m(\varphi)$, $W'(\varphi)$, $m(\varphi_0)$ and $\nabla\varphi$

In the nonlinear operator $\mathcal{F} : X_T \to Y_T$ the terms $\eta(\varphi)$, $\eta(\varphi_0)$, $m(\varphi)$, $m(\varphi_0)$ and $W'(\varphi)$ appear. Hence, we need to know in which spaces these terms are bounded in

the sense that there is a constant $C(R) > 0$, which does not depend on T, such that the norms of these terms in a certain Banach space are bounded by $C(R)$ for every $(\mathbf{v}, \varphi) \in X_T$ with $||(\mathbf{v}, \varphi)||_{X_T} \leq R$.
Due to (5.16) and because the embedding constant only depends on R, it holds

$$||\varphi(t)||_{W_p^3(\Omega)} \leq C(R)$$

for every $t \in [0, T]$ and $\varphi \in X_T^2$ with $||\varphi||_{X_T^2} \leq R$. Hence, the theorem for the composition with Sobolev functions, cf. Theorem 2.18, yields

$$||f(\varphi(t))||_{W_p^3(\Omega)}, ||f(\varphi_0)||_{W_p^3(\Omega)}, ||W'(\varphi(t))||_{W_p^3(\Omega)} \leq C(R)$$

for every $t \in [0, T]$ and every $\varphi \in X_T^2$ with $||\varphi||_{X_T^2} \leq R$, where $f \in \{\eta, m\}$. Thus

$$f(\varphi), f(\varphi_0), W'(\varphi) \text{ are bounded in } L^\infty(0, T; W_p^3(\Omega)) \text{ for } f \in \{\eta, m\}. \tag{5.18}$$

Moreover, Theorem 2.18 yields the existence of $L > 0$ such that

$$||f(\varphi_1(t)) - f(\varphi_2(t))||_{W_p^3(\Omega)} \leq L||\varphi_1(t) - \varphi_2(t)||_{W_p^3(\Omega)} \tag{5.19}$$

for every $t \in [0, T]$, $\varphi_1, \varphi_2 \in X_T^2$ and $f \in \{\eta, m, W'\}$.

In the next step, we want to show that $f(\varphi)$ is bounded in X_T^2 and therefore the same embeddings hold as for φ, where $f \in \{\eta, m, W'\}$. Note that from now on until the end of the proof of the interpolation result for $f(\varphi)$, we always use some general $f \in C_b^4(\mathbb{R})$. But all these embeddings are valid for $f \in \{\eta, m, W'\}$.
We want to prove that if it holds $\varphi \in X_T^2$ with $||\varphi||_{X_T^2} \leq R$, then there exists a constant $C(R) > 0$ such that $||f(\varphi)||_{X_T^2} \leq C(R)$. To this end, let $\varphi \in X_T^2$ be given with $||\varphi||_{X_T^2} \leq R$. Since we already know $\varphi \in C([0, T]; C^2(\overline{\Omega}))$, cf. (5.17), we can conclude

$$||\varphi(t)||_{C^2(\overline{\Omega})} \leq C(R)$$

for all $t \in [0, T]$. Hence, it holds $f(\varphi(t)) \in C^2(\overline{\Omega})$ for every $t \in [0, T]$ and

$$\nabla f((\varphi(t)) = f'(\varphi(t))\nabla\varphi(t).$$

Due to (5.18), $f'(\varphi)$ is bounded in $L^\infty(0, T; W_p^3(\Omega))$. In particular, this implies $||f'(\varphi(t))||_{W_p^3(\Omega)} \leq C(R)$ for a.e. $t \in (0, T)$ and a constant $C(R) > 0$. Since it holds $\varphi \in L^p(0, T; W_p^4(\Omega))$, it follows $\nabla\varphi(t) \in W_p^3(\Omega)$ for a.e. $t \in (0, T)$. Note that this does not imply that there exists a constant C such that $\nabla\varphi(t)$ is bounded in $W_p^3(\Omega)$ by this constant for every $\varphi \in B_R^{X_T^2}$ and a.e. $t \in (0, T)$. But the theorem about the multiplication of Sobolev functions, cf. Theorem 2.17, yields $f'(\varphi(t))\nabla\varphi(t) \in W_p^3(\Omega)$ for a.e. $t \in (0, T)$ together with the estimate

$$||\nabla f(\varphi(t))||_{W_p^3(\Omega)} = ||f'(\varphi(t))\nabla\varphi(t)||_{W_p^3(\Omega)} \leq C||f'(\varphi(t))||_{W_p^3(\Omega)}||\nabla\varphi(t)||_{W_p^3(\Omega)}$$

for a.e. $t \in (0,T)$ and every $\varphi \in X_T^2$ with $||\varphi||_{X_T^2} \leq R$. Since $f'(\varphi)$ is bounded in $L^\infty(0,T;W_p^3(\Omega))$ and $\nabla\varphi$ is bounded in $L^p(0,T;W_p^3(\Omega))$, the estimate above implies the boundedness of $\nabla f(\varphi)$ in $L^p(0,T;W_p^3(\Omega))$, i.e., there exists $C(R) > 0$ such that

$$||\nabla f(\varphi)||_{L^p(0,T;W_p^3(\Omega))} \leq C(R) \qquad \text{for all } \varphi \in X_T^2 \text{ with } ||\varphi||_{X_T^2} \leq R.$$

Altogether this implies that

$$f(\varphi) \text{ is bounded in } L^p(0,T;W_p^4(\Omega)).$$

Analogously we can conclude from the boundedness of φ in $W_p^1(0,T;L^p(\Omega))$ that $f(\varphi)$ is also bounded in $W_p^1(0,T;L^p(\Omega))$. More precisely, Lemma 2.11 yields

$$\frac{d}{dt}f(\varphi(t)) = f'(\varphi(t))\partial_t\varphi(t).$$

Using the boundedness of $f'(\varphi)$ in $C^0(\overline{Q}_T)$ together with the boundedness of $\partial_t\varphi$ in $L^p(0,T;L^p(\Omega))$ we get that $f(\varphi)$ is bounded in $W_p^1(0,T;L^p(\Omega))$. Thus the same interpolation result holds as in (5.16), i.e.,

$$f(\varphi) \text{ is bounded in } C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega)), \tag{5.20}$$

where $\theta := \frac{\frac{4}{p}-1}{\frac{4}{p}-4}$.

5.3 Lipschitz Continuity of $\mathcal{F}$

In the proof for the existence of a unique strong solution for short time, cf. Theorem 5.4, we used that the operator $\mathcal{F} : X_T \to Y_T$ is Lipschitz continuous, which we have not proven yet. Using the notation from before we get the following result for the operator $\mathcal{F} : X_T \to Y_T$.

Proposition 5.7. *Let the Assumptions 5.3 hold and φ_0 be given as in Theorem 5.4. Then there is a constant $C(T,R) > 0$ such that*

$$||\mathcal{F}(\mathbf{v}_1,\varphi_1) - \mathcal{F}(\mathbf{v}_2,\varphi_2)||_{Y_T} \leq C(T,R)||(\mathbf{v}_1 - \mathbf{v}_2, \varphi_1 - \varphi_2)||_{X_T} \tag{5.21}$$

for all $(\mathbf{v}_i,\varphi_i) \in X_T$ with $||(\mathbf{v}_i,\varphi_i)||_{X_T} \leq R$ and $i = 1,2$. Moreover, it holds $C(T,R) \to 0$ as $T \to 0$.

Proof. Let $(\mathbf{v}_i,\varphi_i) \in X_T$ with $||(\mathbf{v}_i,\varphi_i)||_{X_T} \leq R$, $i = 1,2$, be given. Then it holds

$$\begin{aligned}||\mathcal{F}(\mathbf{v}_1,\varphi_1) - \mathcal{F}(\mathbf{v}_2,\varphi_2)||_{Y_T} &= ||\mathbb{P}_\sigma(F_1(\mathbf{v}_1,\varphi_1) - F_1(\mathbf{v}_2,\varphi_2))||_{L^2(Q_T)} \\ &\quad + ||(\nabla\varphi_2\cdot\mathbf{v}_2 - \nabla\varphi_1\cdot\mathbf{v}_1) + \frac{1}{\varepsilon}\mathrm{div}(m(\varphi_1)\nabla W'(\varphi_1) - m(\varphi_2)\nabla W'(\varphi_2)) \\ &\qquad + \varepsilon m(\varphi_0)\Delta^2(\varphi_1 - \varphi_2) + \varepsilon\mathrm{div}(m(\varphi_2)\nabla\Delta\varphi_2 - m(\varphi_1)\nabla\Delta\varphi_2)||_{L^p(Q_T)}.\end{aligned} \tag{5.22}$$

For the sake of clarity we study both summands in (5.22) separately and begin with the first one. Remember that the operator F_1 is defined by

$$\begin{aligned} F_1(\mathbf{v},\varphi) =& \rho_0\partial_t\mathbf{v} - \rho\partial_t\mathbf{v} - \operatorname{div}(2\eta(\varphi_0)D\mathbf{v}) + \operatorname{div}(2\eta(\varphi)D\mathbf{v}) - \varepsilon\Delta\varphi\nabla\varphi \\ & - \left(\left(\rho\mathbf{v} + \frac{\tilde{\rho}_1 - \tilde{\rho}_2}{2} m(\varphi)\nabla(\frac{1}{\varepsilon}W'(\varphi) - \varepsilon\Delta\varphi)\right)\cdot\nabla\right)\mathbf{v} \end{aligned}$$

and that it holds $||\mathbb{P}_\sigma||_{\mathcal{L}(L^2(\Omega)^d, L^2_\sigma(\Omega))} \leq 1$ for the Helmholtz projection $\mathbb{P}_\sigma$. We estimate $||\mathbb{P}_\sigma(F_1(\mathbf{v}_1,\varphi_1) - F_1(\mathbf{v}_2,\varphi_2))||_{L^2(Q_T)}$:

i) For the first two terms we can calculate

$$\begin{aligned} &||\rho_0\partial_t\mathbf{v}_1 - \rho(\varphi_1)\partial_t\mathbf{v}_1 - \rho_0\partial_t\mathbf{v}_2 + \rho(\varphi_2)\partial_t\mathbf{v}_2||_{L^2(Q_T)} \\ &\quad \leq ||(\rho_0 - \rho(\varphi_1))\partial_t(\mathbf{v}_1 - \mathbf{v}_2)||_{L^2(Q_T)} + ||(\rho(\varphi_1) - \rho(\varphi_2))\partial_t\mathbf{v}_2||_{L^2(Q_T)}. \end{aligned}$$

Since it holds $\partial_t\mathbf{v}_i \in L^2(0,T;L^2_\sigma(\Omega))$, $i = 1,2$, we need to estimate every ρ-term in the L^∞-norm. To this end, we use that ρ is affine linear and

$$\varphi_i \text{ is bounded in } C^{0,(1-\frac{1}{p})\theta}([0,T];W^3_p(\Omega)) \hookrightarrow C^{0,(1-\frac{1}{p})\theta}([0,T];C^2(\overline{\Omega}))$$

for $i = 1,2$ and $\theta = \frac{\frac{4}{p}-1}{\frac{4}{p}-4}$, cf. (5.16). Then we obtain for the first summand

$$\begin{aligned} ||(\rho_0 - \rho(\varphi_1))\partial_t(\mathbf{v}_1 - \mathbf{v}_2)||_{L^2(Q_T)} &\leq ||\rho(\varphi_0) - \rho(\varphi_1)||_{L^\infty(Q_T)}||\partial_t(\mathbf{v}_1 - \mathbf{v}_2)||_{L^2(Q_T)} \\ &\leq C\sup_{t\in[0,T]}||\varphi_1(0) - \varphi_1(t)||_{L^\infty(\Omega)}||\mathbf{v}_1 - \mathbf{v}_2||_{X^1_T} \\ &\leq CT^{(1-\frac{1}{p})\theta}||\varphi_1||_{C^{0,(1-\frac{1}{p})\theta}([0,T];C^2(\overline{\Omega}))}||\mathbf{v}_1 - \mathbf{v}_2||_{X^1_T} \\ &\leq CRT^{(1-\frac{1}{p})\theta}||\mathbf{v}_1 - \mathbf{v}_2||_{X^1_T}, \end{aligned}$$

where we used (5.15) in the penultimate step. Analogously the second term can be estimated by

$$\begin{aligned} &||(\rho(\varphi_1) - \rho(\varphi_2))\partial_t\mathbf{v}_2||_{L^2(Q_T)} \leq ||\rho(\varphi_1) - \rho(\varphi_2)||_{L^\infty(Q_T)}||\mathbf{v}_2||_{X^1_T} \\ &\quad \leq C\sup_{t\in[0,T]}||(\varphi_1(t) - \varphi_2(t)) - (\varphi_1(0) - \varphi_2(0))||_{L^\infty(\Omega)}||\mathbf{v}_2||_{X^1_T} \\ &\quad \leq CRT^{(1-\frac{1}{p})\theta}||\varphi_1 - \varphi_2||_{C^{0,(1-\frac{1}{p})\theta}([0,T];C^2(\overline{\Omega}))} \\ &\quad \leq CRT^{(1-\frac{1}{p})\theta}||\varphi_1 - \varphi_2||_{X^2_T}. \end{aligned}$$

Here we used the fact that $\varphi_1(0) = \varphi_0 = \varphi_2(0)$ for $\varphi_i \in X^2_T$, $i = 1,2$.

ii) The next term of $||\mathbb{P}_\sigma(F_1(\mathbf{v}_1,\varphi_1) - F_1(\mathbf{v}_2,\varphi_2))||_{L^2(Q_T)}$ is given by

$$\begin{aligned} &||(\operatorname{div}(2\eta(\varphi_0)D\mathbf{v}_2) - \operatorname{div}(2\eta(\varphi_0)D\mathbf{v}_1)) + (\operatorname{div}(2\eta(\varphi_1)D\mathbf{v}_1) - \operatorname{div}(2\eta(\varphi_2)D\mathbf{v}_2))||_{Y^1_T} \\ &\quad \leq ||\operatorname{div}(2(\eta(\varphi_0) - \eta(\varphi_1))(D\mathbf{v}_2 - D\mathbf{v}_1))||_{Y^1_T} + ||\operatorname{div}(2((\eta(\varphi_1) - \eta(\varphi_2))D\mathbf{v}_2))||_{Y^1_T}. \end{aligned}$$

In the next step we apply the divergence on the $\eta(\varphi_i)$- and $D\mathbf{v}_i$-terms and for the sake of clarity we study both terms in the previous inequality separately. For the first one we use $\eta(\varphi) \in C^{0,(1-\frac{1}{p})\theta}([0,T]; W_p^3(\Omega))$ with $\theta = \frac{\frac{4}{p}-1}{\frac{4}{p}-4}$, cf. (5.20), to obtain

$$
\begin{aligned}
&||\mathrm{div}(2(\eta(\varphi_0) - \eta(\varphi_1))(D\mathbf{v}_2 - D\mathbf{v}_1))||_{Y_T^1} \\
&\quad \le ||2\nabla(\eta(\varphi_0) - \eta(\varphi_1)) \cdot (D\mathbf{v}_2 - D\mathbf{v}_1)||_{Y_T^1} + ||2(\eta(\varphi_0) - \eta(\varphi_1))\Delta(\mathbf{v}_2 - \mathbf{v}_1)||_{Y_T^1} \\
&\quad \le C \sup_{t\in[0,T]} ||\nabla\eta(\varphi_1(0)) - \nabla\eta(\varphi_1(t))||_{C^1(\overline{\Omega})} ||D\mathbf{v}_2 - D\mathbf{v}_1||_{L^2(0,T;H^1(\Omega))} \\
&\qquad + C \sup_{t\in(0,T)} ||\eta(\varphi_1(0)) - \eta(\varphi_1(t))||_{C^2(\overline{\Omega})} ||\Delta(\mathbf{v}_2 - \mathbf{v}_1)||_{L^2(0,T;L^2(\Omega))} \\
&\quad \le CT^{(1-\frac{1}{p})\theta} ||\nabla\eta(\varphi_1)||_{C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^2(\Omega))} ||\mathbf{v}_1 - \mathbf{v}_2||_{X_T^1} \\
&\qquad + CT^{(1-\frac{1}{p})\theta} ||\eta(\varphi_1)||_{C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega))} ||\mathbf{v}_1 - \mathbf{v}_2||_{X_T^1} \\
&\quad \le CR\left(T^{(1-\frac{1}{p})\theta} + T^{(1-\frac{1}{p})\theta}\right) ||\mathbf{v}_1 - \mathbf{v}_2||_{X_T^1}.
\end{aligned}
$$

Analogously as before we can estimate the second summand by

$$
\begin{aligned}
&||\mathrm{div}(2((\eta(\varphi_1) - \eta(\varphi_2))D\mathbf{v}_2))||_{Y_T^1} \\
&\quad \le 2||\eta'(\varphi_1)(\nabla\varphi_1 - \nabla\varphi_2) \cdot D\mathbf{v}_2||_{Y_T^1} + 2||(\eta'(\varphi_1) - \eta'(\varphi_2))\nabla\varphi_2 \cdot D\mathbf{v}_2||_{Y_T^1} \\
&\qquad + 2||(\eta(\varphi_1) - \eta(\varphi_2))\Delta\mathbf{v}_2||_{Y_T^1}.
\end{aligned}
$$

For the sake of clarity we study these three terms separately again. In the following we will use inequality (5.15) several times and $\eta \in C_b^4(\mathbb{R})$. Hence, it holds

$$
\begin{aligned}
&||\eta'(\varphi_1)(\nabla\varphi_1 - \nabla\varphi_2) \cdot D\mathbf{v}_2||_{Y_T^1} \le C(R) \left|\left| ||D\mathbf{v}_2||_{L^2(\Omega)} ||\nabla\varphi_1 - \nabla\varphi_2||_{C^1(\overline{\Omega})} \right|\right|_{L^2(0,T)} \\
&\quad \le C(R) \sup_{t\in[0,T]} ||\nabla(\varphi_1(t) - \varphi_2(t)) - \nabla(\varphi_1(0) - \varphi_2(0))||_{C^1(\overline{\Omega})} ||D\mathbf{v}_2||_{L^2(0,T;L^2(\Omega))} \\
&\quad \le C(R) T^{(1-\frac{1}{p})\theta} ||\nabla\varphi_1 - \nabla\varphi_2||_{C^{(1-\frac{1}{p})\theta}([0,T];W_p^2(\Omega))} ||\mathbf{v}_2||_{X_T^1} \\
&\quad \le C(R) T^{(1-\frac{1}{p})\theta} ||\varphi_1 - \varphi_2||_{X_T^2},
\end{aligned}
$$

where we used in the first step that $\eta'(\varphi)$ is bounded in $C([0,T]; C^2(\overline{\Omega}))$. Furthermore, (5.19) together with

$$
\varphi \in C^{0,(1-\frac{1}{p})\theta}([0,T]; W_p^3(\Omega)) \hookrightarrow C([0,T]; C^2(\overline{\Omega}))
$$

implies

$$\begin{aligned}
&||(\eta'(\varphi_1)-\eta'(\varphi_2))\nabla\varphi_2\cdot D\mathbf{v}_2||_{Y_T^1}\\
&\quad\le \sup_{t\in[0,T]}||\eta'(\varphi_1)-\eta'(\varphi_2)||_{W_p^3(\Omega)}||\nabla\varphi_2||_{C([0,T];C^1(\overline{\Omega}))}||D\mathbf{v}_2||_{L^2(Q_T)}\\
&\quad\le C(R)\sup_{t\in[0,T]}||\varphi_1(t)-\varphi_2(t)||_{W_p^3(\Omega)}\\
&\quad\le C(R)\sup_{t\in(0,T)}||(\varphi_1(t)-\varphi_2(t))-(\varphi_1(0)-\varphi_2(0))||_{W_p^3(\Omega)}\\
&\quad\le C(R)T^{(1-\frac{1}{p})\theta}||\varphi_1-\varphi_2||_{X_T^2}.
\end{aligned}$$

Analogously to the second summand we can estimate the third one by

$$||(\eta(\varphi_1)-\eta(\varphi_2))\Delta\mathbf{v}_2||_{Y_T}\le C(R)T^{(1-\frac{1}{p})\theta}||\varphi_1-\varphi_2||_{X_T^2},$$

which shows the statement for ii).

iii) For the third term we obtain

$$\begin{aligned}
&||\rho(\varphi_2)\mathbf{v}_2\cdot\nabla\mathbf{v}_2-\rho(\varphi_1)\mathbf{v}_1\cdot\nabla\mathbf{v}_1||_{Y_T^1}\\
&\quad\le ||(\rho(\varphi_2)-\rho(\varphi_1))\mathbf{v}_2\cdot\nabla\mathbf{v}_2||_{Y_T^1}+||\rho(\varphi_1)(\mathbf{v}_2\cdot\nabla\mathbf{v}_2-\mathbf{v}_1\cdot\nabla\mathbf{v}_1)||_{Y_T^1}\\
&\quad\le ||(\rho(\varphi_2)-\rho(\varphi_1))\mathbf{v}_2\cdot\nabla\mathbf{v}_2||_{Y_T^1}+||\rho(\varphi_1)(\mathbf{v}_2-\mathbf{v}_1)\cdot\nabla\mathbf{v}_2||_{Y_T^1}\\
&\qquad+||\rho(\varphi_1)\mathbf{v}_1\cdot(\nabla\mathbf{v}_2-\nabla\mathbf{v}_1))||_{Y_T^1}.
\end{aligned}$$

We estimate these three terms separately again. For the first term we use that $\mathbf{v}_2$ is bounded in $L^\infty(0,T;L^6(\Omega))$, cf. (5.10), and $\nabla\mathbf{v}_2$ is bounded in $L^2(0,T;L^6(\Omega))$ together with (5.19). Thus

$$\begin{aligned}
&||(\rho(\varphi_2)-\rho(\varphi_1))\mathbf{v}_2\cdot\nabla\mathbf{v}_2||_{Y_T^1}\\
&\quad\le C(R)T^{(1-\frac{1}{p})\theta}||\varphi_2-\varphi_1||_{C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega))}||\mathbf{v}_2||_{L^\infty(0,T;L^6(\Omega))}||\nabla\mathbf{v}_2||_{L^2(0,T;L^6(\Omega))}\\
&\quad\le C(R)T^{(1-\frac{1}{p})\theta}||\varphi_2-\varphi_1||_{X_T^1}.
\end{aligned}$$

For the second term we use $\rho(\varphi_1)\in C([0,T];C^2(\overline{\Omega}))$, $\mathbf{v}_i\in L^\infty(0,T;L^6(\Omega))$ and $\nabla\mathbf{v}_2\in L^4(0,T;L^3(\Omega))$, cf. (5.10) and (5.12), $i=1,2$. Hence,

$$\begin{aligned}
||\rho(\varphi_1)(\mathbf{v}_2-\mathbf{v}_1)\cdot\nabla\mathbf{v}_2||_{Y_T^1}&\le C(R)T^{\frac{1}{4}}||\mathbf{v}_1-\mathbf{v}_2||_{L^\infty(0,T;L^6(\Omega))}||\nabla\mathbf{v}_2||_{L^4(0,T;L^3(\Omega))}\\
&\le C(R)T^{\frac{1}{4}}||\mathbf{v}_1-\mathbf{v}_2||_{X_T^1}.
\end{aligned}$$

For the third term we analogously use the same function spaces. This implies

$$\begin{aligned}
||\rho(\varphi_1)\mathbf{v}_1\cdot(\nabla\mathbf{v}_2-\nabla\mathbf{v}_1))||_{Y_T}&\le C(R)T^{\frac{1}{4}}||\nabla\mathbf{v}_1-\nabla\mathbf{v}_2||_{L^4(0,T;L^3(\Omega))}\\
&\le C(R)T^{\frac{1}{4}}||\mathbf{v}_1-\mathbf{v}_2||_{X_T^1}.
\end{aligned}$$

iv) In the first step of the next term we use that the prefactor $\frac{\tilde{\rho}_1-\tilde{\rho}_2}{2}$ is a constant. Hence,

$$\begin{aligned}&\left|\left|\frac{\tilde{\rho}_1-\tilde{\rho}_2}{2}m(\varphi_1)\nabla(\Delta\varphi_1)\cdot\nabla\mathbf{v}_1-\frac{\tilde{\rho}_1-\tilde{\rho}_2}{2}m(\varphi_2)\nabla(\Delta\varphi_2)\cdot\nabla\mathbf{v}_2\right|\right|_{Y_T^1}\\&\quad\leq C\Big(||m(\varphi_1)\nabla(\Delta\varphi_1)\cdot(\nabla\mathbf{v}_1-\nabla\mathbf{v}_2)||_{Y_T^1}\\&\qquad+||m(\varphi_1)(\nabla(\Delta\varphi_1)-\nabla(\Delta\varphi_2))\cdot\nabla\mathbf{v}_2||_{Y_T^1}\\&\qquad+||(m(\varphi_1)-m(\varphi_2))\nabla(\Delta\varphi_2)\cdot\nabla\mathbf{v}_2||_{Y_T^1}\Big).\end{aligned}$$

For the sake of clarity we study all three terms separately again. In the following we use $\nabla\Delta\varphi_i\in L^\infty(0,T;L^4(\Omega))$, cf. (5.16), $\nabla\mathbf{v}_i\in L^{\frac{8}{3}}(0,T;L^4(\Omega))$, cf. (5.11), for $i=1,2$, and $m(\varphi_1)\in C([0,T];C^2(\overline{\Omega}))$. Altogether this implies

$$\begin{aligned}&||m(\varphi_1)\nabla(\Delta\varphi_1)\cdot(\nabla\mathbf{v}_1-\nabla\mathbf{v}_2)||_{Y_T^1}\\&\quad\leq CT^{\frac{1}{8}}||\nabla\Delta\varphi_1||_{L^\infty(0,T;L^4(\Omega))}||\nabla\mathbf{v}_1-\nabla\mathbf{v}_2||_{L^{\frac{8}{3}}(0,T;L^4(\Omega))}\\&\quad\leq C(R)T^{\frac{1}{8}}||\mathbf{v}_1-\mathbf{v}_2||_{X_T^1}.\end{aligned}$$

Analogously the second summand yields

$$||m(\varphi_1)(\nabla(\Delta\varphi_1)-\nabla(\Delta\varphi_2))\cdot\nabla\mathbf{v}_2||_{Y_T^1}\leq C(R)T^{\frac{1}{8}}||\varphi_1-\varphi_2||_{X_T^2}.$$

For the last term we use $m(\varphi_i)\in C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega))\hookrightarrow C^0([0,T];C^2(\overline{\Omega}))$ together with (5.19) and obtain

$$\begin{aligned}&||(m(\varphi_1)-m(\varphi_2))\nabla(\Delta\varphi_2)\cdot\nabla\mathbf{v}_2||_{Y_T^1}\\&\quad\leq C(R)T^{\frac{1}{8}}||\varphi_1(t)-\varphi_2(t)||_{C^0([0,T];C^2(\overline{\Omega}))}||\nabla\Delta\varphi_2||_{L^\infty(0,T;L^4(\Omega)}||\nabla\mathbf{v}_2||_{L^{\frac{8}{3}}(0,T;L^4(\Omega))}\\&\quad\leq C(R)T^{\frac{1}{8}}||\varphi_1-\varphi_2||_{X_T^2}.\end{aligned}$$

v) The next term has the same structure as the one before.

$$\begin{aligned}&\left|\left|\frac{\tilde{\rho}_1-\tilde{\rho}_2}{2}m(\varphi_1)\nabla(W'(\varphi_1))\cdot\nabla\mathbf{v}_1-\frac{\tilde{\rho}_1-\tilde{\rho}_2}{2}m(\varphi_2)\nabla(W'(\varphi_2))\cdot\nabla\mathbf{v}_2\right|\right|_{Y_T^1}\\&\leq C\Big(||m(\varphi_1)\nabla W'(\varphi_1)\cdot(\nabla\mathbf{v}_1-\nabla\mathbf{v}_2)||_{Y_T^1}\\&\quad+||m(\varphi_1)(\nabla W'(\varphi_1)-\nabla W'(\varphi_2))\cdot\nabla\mathbf{v}_2||_{Y_T^1}\\&\quad+||(m(\varphi_1)-m(\varphi_2))\nabla W'(\varphi_2)\cdot\nabla\mathbf{v}_2||_{Y_T^1}\Big).\end{aligned}\tag{5.23}$$

For $\nabla\mathbf{v}_i$, $i=1,2$, we use its boundedness in $L^4(0,T;L^3(\Omega))$, cf. (5.12). Moreover, we know $\nabla W'(\varphi)\in C([0,T];W_p^{3-\frac{4}{p}}(\Omega))$ and $m(\varphi)\in C([0,T];C^2(\overline{\Omega}))$ for

$\varphi \in B_R^{X_T^2}$. Using all these bounds we can estimate the three terms in (5.23) separately. For the first term we obtain

$$\begin{aligned}||m(\varphi_1)\nabla W'(\varphi_1)\cdot(\nabla \mathbf{v}_1 - \nabla \mathbf{v}_2)||_{Y_T^1} &\le C(R)T^{\frac{1}{4}}||\nabla \mathbf{v}_1 - \nabla \mathbf{v}_2||_{L^4(0,T;L^3(\Omega))}\\ &\le C(R)T^{\frac{1}{4}}||\mathbf{v}_1 - \mathbf{v}_2||_{X_T^1}.\end{aligned}$$

For the second summand in (5.23) we have to estimate the difference $\nabla W'(\varphi_1) - \nabla W'(\varphi_2)$ in an appropriate way. To this end, we use (5.16), (5.19) and $W_p^2(\Omega) \hookrightarrow C^1(\overline{\Omega})$. Moreover, we use $\nabla \mathbf{v}_2 \in L^4(0,T;L^3(\Omega))$, cf. (5.12), and $m(\varphi) \in C([0,T];C^2(\overline{\Omega}))$. Then it follows

$$\begin{aligned}&||m(\varphi_1)(\nabla W'(\varphi_1) - \nabla W'(\varphi_2))\cdot \nabla \mathbf{v}_2||_{Y_T^1}\\ &\quad\le C(R)T^{\frac{1}{4}}\sup_{t\in[0,T]}||\nabla W'(\varphi_1(t)) - \nabla W'(\varphi_2(t))||_{W_p^2(\Omega)}\\ &\quad\le C(R)T^{\frac{1}{4}}\sup_{t\in[0,T]}||\varphi_1(t) - \varphi_2(t)||_{W_p^3(\Omega)}\\ &\quad\le C(R)T^{\frac{1}{4}+(1-\frac{1}{p})\theta}||\varphi_1 - \varphi_2||_{X_T^2}.\end{aligned}$$

So it remains to estimate the third term of (5.23). As before we get

$$\begin{aligned}&||(m(\varphi_1) - m(\varphi_2))\nabla W'(\varphi_2)\cdot\nabla \mathbf{v}_2||_{Y_T^1}\\ &\quad\le C(R)T^{\frac{1}{4}+(1-\frac{1}{p})\theta}||\varphi_1 - \varphi_2||_{C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega))}\\ &\qquad ||\nabla W'(\varphi_2)||_{BUC([0,T];C^1(\overline{\Omega}))}||\nabla \mathbf{v}_2||_{L^4(0,T;L^3(\Omega))}\\ &\quad\le C(R)T^{\frac{1}{4}+(1-\frac{1}{p})\theta}||\varphi_1 - \varphi_2||_{X_T^2},\end{aligned}$$

which completes the estimate for (5.23).

vi) Finally, we study the last term of $||\mathbb{P}_\sigma(F_1(\mathbf{v}_1,\varphi_1) - F_1(\mathbf{v}_2,\varphi_2))||_{L^2(Q_T)}$. It holds

$$||\Delta\varphi_2\nabla\varphi_2 - \Delta\varphi_1\nabla\varphi_1||_{Y_T} \le ||\Delta\varphi_2(\nabla\varphi_2 - \nabla\varphi_1)||_{Y_T} + ||(\Delta\varphi_2 - \Delta\varphi_1)\nabla\varphi_1||_{Y_T}.$$

Using $\Delta\varphi_i \in C([0,T];C^0(\overline{\Omega}))$ and $\nabla\varphi_i \in C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^2(\Omega))$, $i = 1,2$, cf. (5.16), the first term can be estimated by

$$\begin{aligned}||\Delta\varphi_2(\nabla\varphi_2 - \nabla\varphi_1)||_{Y_T^1} &\le C(R)T^{\frac{1}{2}+(1-\frac{1}{p})\theta)}||\nabla\varphi_1 - \nabla\varphi_2||_{C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^2(\Omega))}\\ &\le C(R)T^{\frac{1}{2}+(1-\frac{1}{p})\theta)}||\varphi_1 - \varphi_2||_{X_T^2}.\end{aligned}$$

Analogously the second term can be estimated by

$$||(\Delta\varphi_2 - \Delta\varphi_1)\nabla\varphi_1||_{Y_T} \le C(R)T^{\frac{1}{2}+(1-\frac{1}{p})\theta)}||\varphi_1 - \varphi_2||_{X_T^2}.$$

Hence, we obtain

$$||\mathbb{P}_\sigma(F_1(\mathbf{v}_1,\varphi_1) - F_1(\mathbf{v}_2,\varphi_2))||_{L^2(Q_T)} \leq C(R,T)||(\mathbf{v}_1 - \mathbf{v}_2),(\varphi_1-\varphi_2)||_{X_T}$$

for a constant $C(R,T) > 0$ such that $C(R,T) \to 0$ as $T \to 0$.

Remember that we study the nonlinear operator $\mathcal{F} : X_T \to Y_T$ given by

$$\mathcal{F}(\mathbf{v},\varphi) = \begin{pmatrix} \mathbb{P}_\sigma F_1(\mathbf{v},\varphi) \\ -\nabla\varphi\cdot\mathbf{v} + \operatorname{div}(\frac{1}{\varepsilon}m(\varphi)\nabla W'(\varphi)) + \varepsilon m(\varphi_0)\Delta^2\varphi - \varepsilon\operatorname{div}(m(\varphi)\nabla\Delta\varphi) \end{pmatrix}$$

and we want to show its Lipschitz continuity such that (5.21) holds. We already showed its Lipschitz continuity for the first part. Now we continue to study the second one. This part has to be estimated in $L^p(0,T;L^p(\Omega))$ for $4 < p < 6$.
For the analysis we use the boundedness of $\nabla\varphi$ in $C([0,T];C^1(\overline{\Omega}))$ and of $\mathbf{v}$ in $L^\infty(0,T;L^6(\Omega))$. Then it holds

$$\begin{aligned}
&||(\nabla\varphi_1\cdot\mathbf{v}_1 - \nabla\varphi_2\cdot\mathbf{v}_2)||_{L^p(Q_T)} \\
&\quad\leq ||\nabla\varphi_1\cdot(\mathbf{v}_1-\mathbf{v}_2)||_{L^p(Q_T)} + ||(\nabla\varphi_1 - \nabla\varphi_2)\cdot\mathbf{v}_2||_{L^p(Q_T)} \\
&\quad\leq T^{\frac{1}{p}}||\nabla\varphi_1||_{L^\infty(Q_T)}||\mathbf{v}_1-\mathbf{v}_2||_{L^\infty(0,T;L^6(\Omega))} \\
&\qquad + T^{\frac{1}{p}}||\nabla\varphi_1-\nabla\varphi_2||_{L^\infty(Q_T)}||\mathbf{v}_2||_{L^\infty(0,T;L^6(\Omega))} \\
&\quad\leq T^{\frac{1}{p}}R||\mathbf{v}_1-\mathbf{v}_2||_{X_T^1} + T^{\frac{1}{p}}R||\varphi_1-\varphi_2||_{X_T^2}.
\end{aligned}$$

Next we study the term $\operatorname{div}(m(\varphi)\nabla W'(\varphi))$. We use the boundedness of $f(\varphi)$ in $C([0,T];C^2(\overline{\Omega})) \cap C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega))$ for $f \in \{m, W'\}$ and $\varphi \in X_T^2$ with $||\varphi||_{X_T^2} \leq R$. Then it holds

$$\begin{aligned}
&||\operatorname{div}(m(\varphi_1)\nabla W'(\varphi_1)) - \operatorname{div}(m(\varphi_2)\nabla W'(\varphi_2))||_{Y_T^2} \\
&\quad\leq C(R)||m(\varphi_1)\nabla W'(\varphi_1)) - m(\varphi_2)\nabla W'(\varphi_2)||_{L^p(0,T;W_p^1(\Omega))} \\
&\quad\leq C(R)T^{\frac{1}{p}}\sup_{t\in[0,T]}||m(\varphi_1(t)) - m(\varphi_2(t))||_{W_p^3(\Omega)}||\nabla W'(\varphi_1)||_{C([0,T];C^1(\overline{\Omega}))} \\
&\qquad + C(R)T^{\frac{1}{p}}||m(\varphi_2)||_{C([0,T];C^2(\overline{\Omega}))}\sup_{t\in[0,T]}||W'(\varphi_1(t)) - W'(\varphi_2(t))||_{W_p^3(\Omega)} \\
&\quad\leq C(R)T^{\frac{1}{p}}\left(\sup_{t\in[0,T]}||\varphi_1(t)-\varphi_2(t)||_{W_p^3(\Omega)} + \sup_{t\in[0,T]}||\varphi_1(t)-\varphi_2(t)||_{W_p^3(\Omega)}\right) \\
&\quad\leq C(R)T^{\frac{1}{p}}\sup_{t\in[0,T]}||(\varphi_1(t)-\varphi_2(t)) - (\varphi_1(0)-\varphi_2(0))||_{W_p^3(\Omega)} \\
&\quad\leq C(R)T^{\frac{1}{p}+(1-\frac{1}{p})\theta}||\varphi_1-\varphi_2||_{C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega))} \\
&\quad\leq C(R)T^{\frac{1}{p}+(1-\frac{1}{p})\theta}||\varphi_1-\varphi_2||_{X_T^2}.
\end{aligned}$$

Here we also used $\varphi_1(0) = \varphi_2(0) = \varphi_0$ for $\varphi_1, \varphi_2 \in X_T^2$ and (5.19).

Now there remain two terms which we need to study together for the proof of the Lipschitz continuity. Due to the boundedness of $m(\varphi)$ in $BUC([0,T]; W_p^3(\Omega))$ and of $\nabla\Delta\varphi$ in $L^p(0,T; W_p^1(\Omega))$, the theorem for the multiplication of Sobolev functions, cf. Theorem 2.17, yields the boundedness of $m(\varphi)\nabla\Delta\varphi$ in $L^p(0,T; W_p^1(\Omega))$. Hence, this term is well-defined in the $L^p(Q_T)$-norm. We omit the prefactor ε for both terms again and estimate

$$
\begin{aligned}
&||m(\varphi_0)\Delta^2\varphi_1 - m(\varphi_0)\Delta^2\varphi_2 + \operatorname{div}(m(\varphi_2)\nabla\Delta\varphi_2) - \operatorname{div}(m(\varphi_1)\nabla\Delta\varphi_1)||_{L^p(Q_T)} \\
&= ||(m(\varphi_0) - m(\varphi_1))(\Delta^2\varphi_1 - \Delta^2\varphi_2) + m(\varphi_1)\Delta^2\varphi_1 - m(\varphi_1)\Delta^2\varphi_2 + \nabla m(\varphi_2)\cdot\nabla\Delta\varphi_2 \\
&\quad + m(\varphi_2)\Delta^2\varphi_2 - \nabla m(\varphi_1)\cdot\nabla\Delta\varphi_1 - m(\varphi_1)\Delta^2\varphi_1||_{L^p(Q_T)} \\
&\le ||(m(\varphi_1(0) - m(\varphi_1))(\Delta^2\varphi_1 - \Delta^2\varphi_2)||_{L^p(Q_T)} + ||(m(\varphi_2) - m(\varphi_1))\Delta^2\varphi_2||_{L^p(Q_T)} \\
&\quad + ||\nabla m(\varphi_2)\cdot\nabla\Delta\varphi_2 - \nabla m(\varphi_1)\cdot\nabla\Delta\varphi_1||_{L^p(Q_T)}
\end{aligned} \tag{5.24}
$$

For the sake of clarity, we study these three terms separately again. Due to the boundedness of $m(\varphi_1)$ in $C^{0,(1-\frac{1}{p})\theta}([0,T]; W_p^3(\Omega))$ we obtain for the first term

$$
\begin{aligned}
&||(m(\varphi_1(0) - m(\varphi_1))(\Delta^2\varphi_1 - \Delta^2\varphi_2)||_{L^p(Q_T)} \\
&\quad \le \sup_{t\in(0,T)} ||m(\varphi_1(0)) - m(\varphi_1(t))||_{C^0(\overline{\Omega})}||\Delta^2\varphi_1 - \Delta^2\varphi_2||_{L^p(Q_T)} \\
&\quad \le C(R)T^{(1-\frac{1}{p})\theta}||m(\varphi_1)||_{C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega))}||\varphi_1 - \varphi_2||_{X_T^2}.
\end{aligned}
$$

Since $m(\varphi_1)$ is bounded in $C^{0,(1-\frac{1}{p})\theta}([0,T]; W_p^3(\Omega))$, we can estimate the second term in (5.24) by

$$
\begin{aligned}
||(m(\varphi_2) - m(\varphi_1))\Delta^2\varphi_2||_{L^p(Q_T)} &\le \sup_{t\in(0,T)} ||m(\varphi_2(t)) - m(\varphi_1(t))||_{C^2(\overline{\Omega})}||\Delta^2\varphi_2||_{L^p(Q_T)} \\
&\le C(R) \sup_{t\in(0,T)} ||m(\varphi_2(t)) - m(\varphi_1(t))||_{W_p^3(\Omega)} \\
&\le C(R) \sup_{t\in(0,T)} ||(\varphi_2(t) - \varphi_1(t)) - (\varphi_2(0) - \varphi_1(0))||_{W_p^3(\Omega)} \\
&\le C(R)T^{(1-\frac{1}{p})\theta}||\varphi_1 - \varphi_2||_{C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\overline{\Omega}))},
\end{aligned}
$$

where we used (5.19) again in the penultimate step. Finally, we study the last term in (5.24). Here we get

$$
\begin{aligned}
&||\nabla m(\varphi_2)\cdot\nabla\Delta\varphi_2 - \nabla m(\varphi_1)\cdot\nabla\Delta\varphi_1||_{L^p(Q_T)} \\
&\quad \le ||(\nabla m(\varphi_2) - \nabla m(\varphi_1))\cdot\nabla\Delta\varphi_2||_{L^p(Q_T)} \\
&\qquad + ||\nabla m(\varphi_1)\cdot(\nabla\Delta\varphi_2 - \nabla\Delta\varphi_1)||_{L^p(Q_T)}.
\end{aligned} \tag{5.25}
$$

Since $\nabla m(\varphi_1)$ is bounded in $C([0,T];C^1(\overline{\Omega}))$ and $\nabla\Delta\varphi_i$ is bounded in $C([0,T];L^p(\Omega))$ for $i=1,2$, we can estimate the second summand by

$$\begin{aligned}
&||\nabla m(\varphi_1)\cdot(\nabla\Delta\varphi_2-\nabla\Delta\varphi_1)||_{L^p(Q_T)}\\
&\qquad\le C(R)T^{\frac{1}{p}}||\nabla m(\varphi_1)||_{C([0,T];C^1(\overline{\Omega}))}||\nabla\Delta\varphi_1-\nabla\Delta\varphi_2||_{C([0,T];L^p(\Omega))}\\
&\qquad\le C(R)T^{\frac{1}{p}}||\varphi_1-\varphi_2||_{X_T^2}.
\end{aligned}$$

Thus it remains to estimate the first term of (5.25). Here we get

$$\begin{aligned}
&||(\nabla m(\varphi_2)-\nabla m(\varphi_1))\cdot\nabla\Delta\varphi_2||_{L^p(Q_T)}\\
&\qquad\le C(R)T^{\frac{1}{p}}\sup_{t\in[0,T]}||\nabla m(\varphi_2(t))-\nabla m(\varphi_1(t))||_{C^0(\overline{\Omega})}||\nabla\Delta\varphi_2||_{C([0,T];L^p(\Omega))}\\
&\qquad\le C(R)T^{\frac{1}{p}}\sup_{t\in[0,T]}||m(\varphi_2(t))-m(\varphi_1(t))||_{W_p^3(\Omega)}||\varphi_2||_{C([0,T];W_p^3(\Omega))}\\
&\qquad\le C(R)T^{\frac{1}{p}}\sup_{t\in[0,T]}||\varphi_1(t)-\varphi_2(t)||_{W_p^3(\Omega)}\\
&\qquad\le C(R)T^{\frac{1}{p}+(1-\frac{1}{p})\theta}||\varphi_1-\varphi_2||_{C^{0,(1-\frac{1}{p})\theta}([0,T];W_p^3(\Omega))}.
\end{aligned}$$

Hence, (5.25) is Lipschitz continuous and therefore also the second part of $\mathcal{F}$ is Lipschitz continuous. Together with the Lipschitz continuity of the first part of $\mathcal{F}$ we have shown

$$||\mathcal{F}(\mathbf{v}_1,\varphi_1)-\mathcal{F}(\mathbf{v}_2,\varphi_2)||_{Y_T}\le C(T,R)||(\mathbf{v}_1-\mathbf{v}_2,\varphi_1-\varphi_2)||_{X_T}$$

for all $(\mathbf{v}_i,\varphi_i)\in X_T$ with $||(\mathbf{v}_i,\varphi_i)||_{X_T}\le R$, $i=1,2$, and a constant $C(T,R)>0$ such that $C(T,R)\to 0$ as $T\to 0$.

$\square$

5.4 Existence and Continuity of $\mathcal{L}^{-1}$ (first part)

To complete the proof it remains to show the existence of $(\tilde{\mathbf{v}},\tilde{\varphi})\in X_T$ such that (5.9) holds and to prove that $\mathcal{L}:X_T\to Y_T$ is invertible with uniformly bounded inverse, i.e., there exists a constant $C>0$ which does not depend on T such that $||\mathcal{L}^{-1}||_{\mathcal{L}(Y_T,X_T)}\le C$. Hence, we remember that the linear operator $\mathcal{L}:X_T\to Y_T$ is defined by

$$\mathcal{L}(\mathbf{v},\varphi)=\begin{pmatrix}\mathbb{P}_\sigma(\rho_0\partial_t\mathbf{v})-\mathbb{P}_\sigma(\operatorname{div}(2\eta(\varphi_0)D\mathbf{v}))\\ \partial_t\varphi+\varepsilon m(\varphi_0)\Delta^2\varphi\end{pmatrix}.$$

We note that the first part only depends on $\mathbf{v}$ while the second part only depends on φ. Thus both equations can be solved separately.

To show the existence of a unique solution $\mathbf{v}$ for every right-hand side $\mathbf{f}$ in the first equation we use the following theorem from [Sho97].

Theorem 5.8. *Let the linear, symmetric and monotone operator $\mathcal{B}$ be given from the real vector space E to its algebraic dual E', and let E'_b be the Hilbert space which is the dual of E with the seminorm*

$$|x|_b = \mathcal{B}x(x)^{\frac{1}{2}}, \qquad x \in E.$$

Let $A \subseteq E \times E'_b$ be a relation with domain $D = \{x \in E : A(x) \neq \emptyset\}$. Let A be the subdifferential, $\partial\varphi$, of a convex lower-semi-continuous function $\varphi : E_b \to [0, \infty]$ with $\varphi(0) = 0$. Then for each u_0 in the E_b-closure of $\mathrm{dom}(\varphi)$ and each $f \in L^2(0, T; E'_b)$ there is a solution $u : [0, T] \to E$ with $\mathcal{B}u \in C([0, T], E'_b)$ of

$$\frac{d}{dt}(\mathcal{B}u(t)) + A(u(t)) \ni f(t), \qquad 0 < t < T,$$

with

$$\varphi \circ u \in L^1(0, T), \sqrt{t}\frac{d}{dt}\mathcal{B}u(\cdot) \in L^2(0, T; E'_b), u(t) \in D, \ \text{a.e. } t \in [0, T],$$

and $\mathcal{B}u(0) = \mathcal{B}u_0$. If in addition $u_0 \in \mathrm{dom}(\varphi)$, then

$$\varphi \circ u \in L^\infty(0, T), \qquad \frac{d}{dt}\mathcal{B}u \in L^2(0, T; E'_b).$$

The proof of this theorem can be found in [Sho97, Chapter IV, Theorem 6.1].

So we have to specify what E, E'_b, φ and so on are in the problem we study and show that the conditions of Theorem 5.8 are fulfilled. Then Theorem 5.8 yields the existence of a solution. More precisely, we obtain the following lemma.

Lemma 5.9. *Let Assumption 5.3 hold. Then for every $\mathbf{v}_0 \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, $f \in L^2(0, T; L^2_\sigma(\Omega))$, $\varphi_0 \in W^1_r(\Omega)$, $r > d \geq 2$, and every $0 < T < \infty$ there exists a unique solution*

$$\mathbf{v} \in W^1_2(0, T; L^2_\sigma(\Omega)) \cap L^\infty(0, T; H^1_0(\Omega)^d)$$

such that

$$\mathbb{P}_\sigma(\rho_0\partial_t\mathbf{v}) - \mathbb{P}_\sigma(\mathrm{div}(2\eta(\varphi_0)D\mathbf{v})) = f \qquad \text{in } Q_T, \tag{5.26}$$
$$\mathrm{div}(\mathbf{v}) = 0 \qquad \text{in } Q_T, \tag{5.27}$$
$$\mathbf{v}_{|\partial\Omega} = 0 \qquad \text{on } (0, T) \times \partial\Omega, \tag{5.28}$$
$$\mathbf{v}(0) = \mathbf{v}_0 \qquad \text{in } \Omega \tag{5.29}$$

for a.e. (t, x) in $(0, T) \times \Omega$, where $\mathbf{v}(t) \in H^2(\Omega)^d$ for a.e. $t \in (0, T)$.

Proof. Since we want to solve (5.26) - (5.29) with Theorem 5.8, we have to define

$$\mathcal{B}u := \mathbb{P}_\sigma(\rho_0 u)$$

for $u \in E$, where we still need to specify the real vector space E. But as we want to have $\frac{d}{dt}\mathcal{B}u \in L^2(0,T;L^2_\sigma(\Omega))$, the dual space E'_b has to coincide with $L^2_\sigma(\Omega)$. But this can be realized by choosing $E = L^2_\sigma(\Omega)$. Then $E'_b \cong L^2_\sigma(\Omega)$ and with the notation in Theorem 5.8 we get the Hilbert space E'_b equipped with the seminorm

$$\begin{aligned}|\mathbf{u}|_b = \mathcal{B}\mathbf{u}(\mathbf{u})^{\frac{1}{2}} &= \left(\int\limits_\Omega \mathbb{P}_\sigma(\rho_0\mathbf{u})\cdot\mathbf{u}dx\right)^{\frac{1}{2}} = \left(\int\limits_\Omega \rho_0\mathbf{u}\cdot\mathbb{P}_\sigma\mathbf{u}dx\right)^{\frac{1}{2}}\\ &= \left(\int\limits_\Omega \rho_0|\mathbf{u}|^2dx\right)^{\frac{1}{2}} \cong ||\mathbf{u}||_{L^2(\Omega)}.\end{aligned}$$

Thus we obtain $E'_b \cong L^2_\sigma(\Omega) = E_b$. Moreover, we define $A : \mathcal{D}(A) \to L^2_\sigma(\Omega)' \cong L^2_\sigma(\Omega)$ by

$$(A\mathbf{u})(\mathbf{v}) := \begin{cases} -\int\limits_\Omega \mathbb{P}_\sigma\mathrm{div}(2\eta(\varphi_0)D\mathbf{u})\cdot\mathbf{v}dx & \text{if } \mathbf{u}\in\mathrm{dom}(A)\\ \emptyset & \text{if } \mathbf{u}\notin\mathrm{dom}(A).\end{cases} \tag{5.30}$$

for every $\mathbf{v} \in L^2_\sigma(\Omega)$ and $\mathcal{D}(A) = H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$. Thus we get for the relation $\mathcal{A}$ defined by $\mathcal{A} := \{(\mathbf{u},\mathbf{v}) :\ \mathbf{v} = A\mathbf{u},\ \mathbf{u}\in\mathcal{D}(A)\}$ the following inclusions

$$\mathcal{A} = \{(\mathbf{u}, -\mathbb{P}_\sigma\mathrm{div}(2\eta(\varphi_0)D\mathbf{u}) : \mathbf{u}\in H^2(\Omega)^d\cap H^1_0(\Omega)^d\cap L^2_\sigma(\Omega)\} \subseteq E\times E'_b,$$

since the term $\mathbb{P}_\sigma\mathrm{div}(2\eta(\varphi_0)D\mathbf{u})$ is in $L^2_\sigma(\Omega)' \cong L^2_\sigma(\Omega)$. Now we define $\psi : L^2_\sigma(\Omega) \to [0,+\infty]$ by

$$\psi(\mathbf{u}) := \begin{cases}\int\limits_\Omega \eta(\varphi_0)D\mathbf{u} : D\mathbf{u}\ dx & \text{if } \mathbf{u}\in H^1_0(\Omega)^d\cap L^2_\sigma(\Omega) = \mathrm{dom}(\psi),\\ +\infty & \text{else.}\end{cases} \tag{5.31}$$

We note $\psi(0) = 0$ and $\mathbf{v}_0$ is in the L^2-closure of $\mathrm{dom}(\psi)$, i.e., in $L^2_\sigma(\Omega)$. Hence, it remains to show that ψ is convex and lower-semi-continuous and that A is the subdifferential of ψ. Then we can apply Theorem 5.8. But the first two properties are obvious. Thus it remains to show the subdifferential property, which is satisfied by Lemma 5.10 below. Hence, we are able to apply Theorem 5.8 which yields the existence. Moreover, the initial condition is also fulfilled as Theorem 5.8 yields

$$\mathbb{P}_\sigma(\rho_0\mathbf{v}(0)) = \mathcal{B}\mathbf{v}(0) = \mathcal{B}\mathbf{v}_0 = \mathbb{P}_\sigma(\rho_0\mathbf{v}_0) \qquad \text{in } L^2(\Omega).$$

In particular we can conclude

$$0 = \int_\Omega \mathbb{P}_\sigma(\rho_0\mathbf{v}(0) - \rho_0\mathbf{v}_0)\cdot\boldsymbol{\psi}\,dx = \int_\Omega (\rho_0\mathbf{v}(0) - \rho_0\mathbf{v}_0)\cdot\boldsymbol{\psi}\,dx$$

for every $\boldsymbol{\psi} \in C^\infty_{0,\sigma}(\Omega)$. By approximation this identity also holds for $\boldsymbol{\psi} := \mathbf{v}(0) - \mathbf{v}_0 \in L^2_\sigma(\Omega)$ and we get

$$\int_\Omega \rho_0|\mathbf{v}(0) - \mathbf{v}_0|^2\,dx = 0.$$

This implies $\mathbf{v}(0) = \mathbf{v}_0$ in $L^2_\sigma(\Omega)$.

For the uniqueness we consider $\mathbf{v}_1, \mathbf{v}_2 \in W^1_2(0,T;L^2_\sigma(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)\cap L^2_\sigma(\Omega))$ such that (5.26) holds for a.e. $(t,x) \in (0,T)\times\Omega$. Then $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$ solves the homogeneous equation a.e. in $(0,T)\times\Omega$. Testing this homogeneous equation with $\mathbf{v}$ we get

$$\int_\Omega \frac{1}{2}\rho_0\mathbf{v}^2_{|t=T}\,dx + \int_0^T\int_\Omega 2\eta(\varphi_0)D\mathbf{v} : D\mathbf{v}\,dx\,dt = 0.$$

Hence, it follows $\mathbf{v} \equiv 0$ and therefore $\mathbf{v}_1 = \mathbf{v}_2$, which yields the uniqueness. $\square$

In the proof above we used that the mapping A coincides with the subdifferential $\partial\varphi$. More precisely, we have the following lemma.

Lemma 5.10. *Let $\Omega \subseteq \mathbb{R}^d$, $d = 2,3$, be an open domain and $\psi : L^2_\sigma(\Omega) \to [0,+\infty]$ be given as in (5.31). Moreover, we consider $A : L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ to be given as in (5.30). Then it holds*

i) $\mathcal{D}(\partial\psi) = \mathcal{D}(A)$.

ii) $\partial\psi(u) = \{Au\}$ *for all* $u \in \mathcal{D}(A)$.

Proof. Remember that

$$\mathcal{D}(\partial\psi) = \{\mathbf{v} \in L^2_\sigma(\Omega) :\ \partial\psi(\mathbf{v}) \neq \emptyset\}$$

and $\mathcal{D}(A) = H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$ by definition.

1$^{\text{st}}$ part: $\mathcal{D}(A) \subseteq \mathcal{D}(\partial\psi)$ and $Au \in \partial\psi(u)$ for every $u \in \mathcal{D}(A)$.
To show the first part of the proof let $\mathbf{u} \in \mathcal{D}(A)$ be given. If it holds $\mathbf{v} \in L^2_\sigma(\Omega)$ but $\mathbf{v} \notin H^1_0(\Omega)^d$, then the inequality

$$\langle A\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle_{L^2(\Omega)} \leq \psi(\mathbf{v}) - \psi(\mathbf{u})$$

is satisfied since it holds $\psi(\mathbf{v}) = +\infty$ in this case by definition.
So let $\mathbf{v} \in H_0^1(\Omega)^d \cap L^2_\sigma(\Omega)$. Then it holds

$$\begin{aligned}
\langle A\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle_{L^2(\Omega)} &= -\int_\Omega \mathbb{P}_\sigma \operatorname{div}(2\eta(\varphi_0)D\mathbf{u}) \cdot (\mathbf{v} - \mathbf{u})\, dx \\
&= \int_\Omega 2\eta(\varphi_0)D\mathbf{u} : D\mathbf{v}\, dx - \int_\Omega 2\eta(\varphi_0)D\mathbf{u} : D\mathbf{u}\, dx \\
&\le \int_\Omega \eta(\varphi_0)|D\mathbf{u}|^2\, dx + \int_\Omega \eta(\varphi_0)|D\mathbf{v}|^2\, dx - 2\int_\Omega \eta(\varphi_0)|D\mathbf{u}|^2\, dx \\
&= \psi(\mathbf{v}) - \psi(\mathbf{u})
\end{aligned}$$

for every $\mathbf{v} \in H_0^1(\Omega)^d \cap L^2_\sigma(\Omega)$. This implies that $A\mathbf{u}$ is a subgradient of ψ at $\mathbf{u}$, i.e., $A\mathbf{u} \in \partial\psi(\mathbf{u})$, and $\partial\psi(\mathbf{u}) \neq \emptyset$, i.e., $\mathbf{u} \in \mathcal{D}(A) \subseteq \mathcal{D}(\partial\psi)$. Hence, we have shown the first part of the proof.

2nd part: $\mathcal{D}(\partial\psi) \subseteq \mathcal{D}(A)$ and $\partial\psi(\mathbf{u}) = \{A\mathbf{u}\}$.
Let $\mathbf{u} \in \mathcal{D}(\partial\psi) \subseteq \operatorname{dom}(\psi) \subseteq H_0^1(\Omega)^d \cap L^2_\sigma(\Omega)$ be given, where the first inclusion follows from the definition of $\mathcal{D}(\partial\psi)$ and $\operatorname{dom}(\psi)$, cf. Definition 2.5. Then by definition there exists $\mathbf{w} \in \partial\psi(u) \subseteq \mathcal{P}(L^2_\sigma(\Omega))$ such that

$$\psi(\mathbf{u}) - \psi(\mathbf{v}) \le \langle \mathbf{w}, \mathbf{u} - \mathbf{v}\rangle_{L^2(\Omega)} \tag{5.32}$$

for every $\mathbf{v} \in L^2_\sigma(\Omega)$. Now we choose $\mathbf{v} := \mathbf{u} + t\tilde{\mathbf{w}}$ for some $\tilde{\mathbf{w}} \in H_0^1(\Omega)^d \cap L^2_\sigma(\Omega)$ and $t > 0$. Then inequality (5.32) yields

$$\begin{aligned}
\psi(\mathbf{u}) - \psi(\mathbf{v}) &= \int_\Omega \eta(\varphi_0)D\mathbf{u} : D\mathbf{u}\, dx - \int_\Omega \eta(\varphi_0)D(\mathbf{u} + t\tilde{\mathbf{w}}) : D(\mathbf{u} + t\tilde{\mathbf{w}})\, dx \\
&= -2t\int_\Omega \eta(\varphi_0)D\mathbf{u} : D\tilde{\mathbf{w}}\, dx - t^2\int_\Omega \eta(\varphi_0)D\mathbf{u} : D\tilde{\mathbf{w}}\, dx \\
&\le -t\int_\Omega \mathbf{w} \cdot \tilde{\mathbf{w}}\, dx.
\end{aligned}$$

Dividing this inequality by $-t < 0$ and passing to the limit $t \searrow 0$ yields

$$\int_\Omega \mathbf{w} \cdot \tilde{\mathbf{w}}\, dx \le \int_\Omega 2\eta(\varphi_0)D\mathbf{u} : D\tilde{\mathbf{w}}\, dx.$$

When we replace $\tilde{\mathbf{w}}$ by $-\tilde{\mathbf{w}}$ we can conclude

$$\int_\Omega \mathbf{w} \cdot \tilde{\mathbf{w}}\, dx \ge \int_\Omega 2\eta(\varphi_0)D\mathbf{u} : D\tilde{\mathbf{w}}\, dx.$$

Thus it follows

$$\int_\Omega \mathbf{w} \cdot \tilde{\mathbf{w}} \, dx = \int_\Omega 2\eta(\varphi_0) D\mathbf{u} : D\tilde{\mathbf{w}} \, dx \tag{5.33}$$

for every $\tilde{\mathbf{w}} \in H_0^1(\Omega)^d \cap L_\sigma^2(\Omega)$. Since we assumed $\mathbf{w} \in L_\sigma^2(\Omega)$, we can apply Lemma 5.11 below which yields $\mathbf{u} \in H^2(\Omega)^d \cap H_0^1(\Omega)^d \cap L_\sigma^2(\Omega)$. Using this regularity in (5.33) we can conclude

$$\int_\Omega \mathbf{w} \cdot \tilde{\mathbf{w}} \, dx = \int_\Omega 2\eta(\varphi_0) D\mathbf{u} : D\tilde{\mathbf{w}} \, dx = -\int_\Omega \mathbb{P}_\sigma \mathrm{div}(2\eta(\varphi_0) D\mathbf{u}) \cdot \tilde{\mathbf{w}} \, dx$$

for every $\tilde{\mathbf{w}} \in H_0^1(\Omega)^d \cap L_\sigma^2(\Omega)$. Therefore, we obtain $\mathbf{w} = -\mathbb{P}_\sigma \mathrm{div}(2\eta(\varphi_0) D\mathbf{u}) = A\mathbf{u}$ in $L^2(\Omega)$, i.e., $\mathbf{u} \in \mathcal{D}(A)$ and $\partial\psi(\mathbf{u}) = \{A\mathbf{u}\}$. $\square$

For the regularity of the Stokes system with variable viscosity we used the following lemma.

Lemma 5.11. *Let $\eta \in C^2(\mathbb{R})$ be such that $\eta(s) \geq s_0 > 0$ for all $s \in \mathbb{R}$ and some $s_0 > 0$, $\varphi_0 \in W_r^1(\Omega)$, $r > d \geq 2$, with $||\varphi_0||_{W_r^1(\Omega)} \leq R$, and let $\mathbf{u} \in H_0^1(\Omega)^d \cap L_\sigma^2(\Omega)$ be a solution of*

$$\langle 2\eta(\varphi_0) D\mathbf{u}, D\tilde{\boldsymbol{w}} \rangle_{L^2(\Omega)} = \langle \mathbf{w}, \tilde{\boldsymbol{w}} \rangle_{L^2(\Omega)} \qquad \text{for all } \tilde{\boldsymbol{w}} \in C_{0,\sigma}^\infty(\Omega),$$

where $\mathbf{w} \in L^2(\Omega)^d$. Then it holds $\mathbf{u} \in H^2(\Omega)^d$ and

$$||\mathbf{u}||_{H^2(\Omega)} \leq C(R) ||\mathbf{w}||_{L^2(\Omega)},$$

where $C(R)$ only depends on Ω, η, $r > d$, and $R > 0$.

The proof can be found in [Abe09b, Lemma 4].

Lemma 5.9 yielded $\mathbf{v} \in W_2^1(0,T; L_\sigma^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega)^d)$. But as we want to show that $\mathbf{v}$ is in X_T^1, it remains to show $\mathbf{v} \in L^2(0,T; H^2(\Omega)^d)$. To this end, we also use Lemma 5.11 above.

Lemma 5.12. *For the unique solution $\mathbf{v} \in W_2^1(0,T; L_\sigma^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega)^d)$ of (5.26) - (5.29) from Lemma 5.9 it holds*

$$\mathbf{v} \in L^2(0,T; H^2(\Omega)^d).$$

Proof. Let $\mathbf{v} \in W_2^1(0,T; L_\sigma^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega)^d)$ be the unique solution of (5.26) from Lemma 5.9, i.e.,

$$A(\mathbf{v}(t)) = \mathbf{f}(t) - \frac{d}{dt}(\mathcal{B}\mathbf{v}(t)) = \mathbf{f}(t) - \mathbb{P}_\sigma(\rho_0 \partial_t \mathbf{v}(t)), \qquad 0 < t < T.$$

Since the right-hand side is not the empty set, we get by definition of A

$$\mathbb{P}_\sigma(\operatorname{div}(2\eta(\varphi_0)D\mathbf{v}(t))) = \mathbb{P}_\sigma(\rho_0\partial_t\mathbf{v}(t)) - \mathbf{f}(t), \qquad 0 < t < T,$$

for given $\mathbf{f} \in L^2(0,T;L^2_\sigma(\Omega))$. From $\partial_t\mathbf{v} \in L^2(0,T;L^2_\sigma(\Omega))$ it follows

$$\langle 2\eta(\varphi_0)D\mathbf{v}(t), D\mathbf{w}\rangle_{L^2(\Omega)} = \langle \rho_0\partial_t\mathbf{v}(t) - \mathbf{f}(t), \mathbf{w}\rangle \qquad \text{for every } \mathbf{w} \in C^\infty_{0,\sigma}(\Omega)$$

and a.e. $t \in (0,T)$. Hence, we can apply Lemma 5.11 and obtain

$$||\mathbf{v}(t)||_{H^2(\Omega)} \le C(R)||\rho_0\partial_t\mathbf{v}(t) - \mathbf{f}(t)||_{L^2(\Omega)} \le C(R)\left(||\rho_0\partial_t\mathbf{v}(t)||_{L^2(\Omega)} + ||\mathbf{f}(t)||_{L^2(\Omega)}\right)$$

for a.e. $t \in (0,T)$. Since the right-hand side of this inequality is bounded in $L^2(0,T)$, this shows the lemma. $\square$

We still need to ensure that $||\mathcal{L}^{-1}||_{\mathcal{L}(Y_T,X_T)}$ remains bounded. This is shown in the next lemma.

Lemma 5.13. *Let the assumptions of Lemma 5.9 hold and* $0 < T_0 < \infty$ *be given. Then*

$$||\mathcal{L}^{-1}_{1,T}||_{\mathcal{L}(Y^1_T,X^1_T)} \le ||\mathcal{L}^{-1}_{1,T_0}||_{\mathcal{L}(Y^1_{T_0},X^1_{T_0})} < \infty \qquad \textit{for all } 0 < T < T_0.$$

Proof. Let $0 < T < T_0$ be given. Lemma 5.9 together with Lemma 5.12 yields that the operator $\mathcal{L}_{1,T} : X_T \to Y_T$ is invertible for every $0 < T < \infty$ and every given $\mathbf{f} \in L^2(0,T;L^2_\sigma(\Omega))$, $\varphi_0 \in W^1_r(\Omega)$, $\mathbf{v}_0 \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$. Then we define $\tilde{\mathbf{f}} \in L^2(0,T_0,L^2_\sigma(\Omega))$ by

$$\tilde{\mathbf{f}}(t) := \begin{cases} \mathbf{f}(t) & \text{if } t \in (0,T], \\ 0 & \text{if } t \in (T,T_0). \end{cases}$$

Due to Lemma 5.9 together with Lemma 5.12 there exists a unique solution $\tilde{\mathbf{v}} \in X^1_{T_0}$ of

$$\begin{aligned} \mathbb{P}_\sigma(\rho_0\partial_t\tilde{\mathbf{v}}) - \mathbb{P}_\sigma(\operatorname{div}(2\eta(\varphi_0)D\tilde{\mathbf{v}})) &= \tilde{\mathbf{f}} && \text{in } Q_{T_0}, \\ \operatorname{div}(\tilde{\mathbf{v}}) &= 0 && \text{in } Q_{T_0}, \\ \tilde{\mathbf{v}}_{|\partial\Omega} &= 0 && \text{on } (0,T_0)\times\partial\Omega, \\ \tilde{\mathbf{v}}(0) &= \mathbf{v}_0 && \text{in } \Omega. \end{aligned}$$

So let $\mathbf{v} \in X^1_T$ be the solution of the previous equations with T_0 replaced by T. Then $\tilde{\mathbf{v}}$ and $\mathbf{v}$ solve these equations on $(0,T)\times\Omega$. Since the solution is unique, we can deduce $\tilde{\mathbf{v}}_{|(0,T)} = \mathbf{v}$. Hence,

$$\begin{aligned} ||\mathcal{L}^{-1}_{1,T}(\mathbf{f})||_{X^1_T} = ||\mathbf{v}||_{X^1_T} \le ||\tilde{\mathbf{v}}||_{X^1_{T_0}} &= ||\mathcal{L}^{-1}_{1,T_0}(\tilde{\mathbf{f}})||_{X^1_{T_0}} \\ &\le ||\mathcal{L}^{-1}_{1,T_0}||_{\mathcal{L}(Y^1_{T_0},X^1_{T_0})}||\tilde{\mathbf{f}}||_{Y^1_{T_0}} \\ &= ||\mathcal{L}^{-1}_{1,T_0}||_{\mathcal{L}(Y^1_{T_0},X^1_{T_0})}||\mathbf{f}||_{Y^1_T}, \end{aligned}$$

which shows the statement since it holds $||\mathcal{L}^{-1}_{1,T_0}||_{\mathcal{L}(Y^1_{T_0},X^1_{T_0})} < \infty$ by the bounded inverse theorem, cf. [Rud73, Corollary 2.12]. $\square$

5.5 Existence and Continuity of $\mathcal{L}^{-1}$ (second part)

Finally, it remains to show that there exists a solution for the second equation of $\mathcal{L}$, i.e., of $\partial_t \varphi + \varepsilon m(\varphi_0)\Delta^2 \varphi = f$ for some f, where we will later specify f and what we mean with solution. First of all we cite the assumptions and the result which we will use from [DHP03].

We consider a partial differential operator

$$\mathcal{A}(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$$

and the boundary differential operators

$$\mathcal{B}_j(x, D) = \sum_{|\beta| \leq m_j} b_{j\beta}(x) D^\beta,$$

where we assume that $G \subseteq \mathbb{R}^d$ is an open and connected set with compact C^{2m}-boundary ∂G and $m_j < 2m$ for every $j = 1, ..., m$. Moreover, we assume that the following conditions are satisfied:

i) Smoothness conditions:

(a) $a_\alpha \in C_l(\overline{G}, \mathcal{L}(E))$ for each $|\alpha| = 2m$,

(b) $a_\alpha \in [L_\infty + L_{r_k}](G, \mathcal{L}(E))$ for each $|\alpha| = k < 2m$ with $r_k \geq p$ and $2m - k > \frac{d-1}{r_k}$

(c) $b_{j\beta} \in C^{2m-m_j}(\partial G, \mathcal{L}(E))$ for each j, β ,

ii) Ellipticity conditions:
There exists $\phi_\mathcal{A} \in [0, \pi)$ such that the following assertions hold.

(a) The principal symbol

$$\mathcal{A}_\#(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha$$

is parameter-elliptic with angle of ellipticity $< \phi_\mathcal{A}$ for each $x \in \overline{G} \cup \{\infty\}$.

(b) (Lopatinskii-Shapiro Condition) Set

$$\mathcal{B}_{j\#}(x, D) := \sum_{|\beta| = m_j} b_{j\beta}(x) D^\beta$$

and $B_\# := (B_{1\#}, ..., B_{m\#})$. For each $x_0 \in \partial G$ we write the boundary value problem $(\mathcal{A}_\#(x_0, D), B_\#(x_0, D))$ in local coordinates corresponding to x_0. Then the ODE problem in $\mathbb{R}_+$

$$\begin{aligned} (\lambda + \mathcal{A}_\#(x_0, \xi', D_d))v(y) &= 0, & y > 0, \\ \mathcal{B}_{j\#}(x_0, \xi', D_d)v(0) &= h_j, & j = 1, ..., m \end{aligned}$$

has a unique solution $v \in C_b^0(\mathbb{R}_+; E)$ for each $(h_1, ..., h_m) \in E^m$ and each $\lambda \in \overline{\Sigma}_{\pi-\phi_{\mathcal{A}}}$ and $\xi' \in \mathbb{R}^{d-1}$ with $|\xi'| + |\lambda| \neq 0$.

In this notation $\mathcal{L}(E) := \mathcal{L}(E, E)$ is the set of all bounded linear operators from a Banach space E to E. Moreover, we call a Banach space E of class $\mathcal{HT}$ if the Hilbert transform is bounded on $L^p(\mathbb{R}; E)$ for some $p \in (1, \infty)$.
A $\mathcal{L}(E)$-valued polynomial $\mathcal{A}(\xi)$ is called parameter-elliptic if there is an angle $\phi \in [0, \pi)$ such that the spectrum $\sigma(\mathcal{A}(\xi))$ of $\mathcal{A}(\xi)$ in $\mathcal{L}(E)$ satisfies

$$\sigma(\mathcal{A}(\xi)) \subseteq \Sigma_\phi \qquad \text{for all } \xi \in \mathbb{R}^d \text{ with } |\xi| = 1, \tag{5.34}$$

where $\Sigma_\phi \subseteq \mathbb{C}$ denotes the open sector with vertex 0 and opening angle 2ϕ, which is symmetric with respect to the positive halfaxis $\mathbb{R}_+$, i.e.,

$$\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} : \ |\arg(\lambda)| < \phi\}.$$

Then we call

$$\phi_{\mathcal{A}} := \inf\{\phi : \ (5.34) \text{ holds}\} = \sup_{|\xi|=1} |\arg \sigma(\mathcal{A}(\xi))|$$

the angle of ellipticity of $\mathcal{A}$, cf. [DHP03, Definition 5.1]. Now we have introduced all notations and assumptions which are necessary for the following theorem.

Theorem 5.14. *Let E be a Banach space of class $\mathcal{HT}$, $n, m \in \mathbb{N}$ and $1 < p < \infty$. Let G be a domain in $\mathbb{R}^d$ with compact C^{2m}-boundary ∂G. Suppose that for $\phi_{\mathcal{A}} \in [0, \pi)$ the boundary value problem $(\mathcal{A}, \mathcal{B}_1, ...\mathcal{B}_m)$ satisfies the smoothness and ellipticity conditions (i) and (ii) above.*
Let A_B denote the realization of $\mathcal{A}(x, D)$ in $X = L^p(G; E)$ with domain

$$D(A_B) = \{u \in H_p^{2m}(G; E) : \ B_j(x, D)u = 0, \ j = 1, ..., m\}.$$

Then for each $\phi > \phi_{\mathcal{A}}$ there is $\mu_\phi \geq 0$ such that $\mu_\phi + A_B$ is $\mathcal{R}$-sectorial with $\phi_{\mu_\phi + A_B} \leq \phi$. In particular, if $\phi_{\mathcal{A}} < \frac{\pi}{2}$ then the parabolic initial-boundary value problem

$$\begin{aligned} \partial_t u + A_B u + \mu_\phi u = f, \qquad t > 0, \\ u(0) = 0 \end{aligned}$$

has the property of maximal regularity in $L^q(\mathbb{R}_+; L^p(G; E))$ for each $q \in (1, \infty)$.

The proof of this theorem can be found in [DHP03, Theorem 8.2].

We use Theorem 5.14 to show the second part of the existence proof of $\mathcal{L}^{-1}$. More precisely, we get the following lemma.

Lemma 5.15. *Let Assumption 5.3 hold and* $\varphi_0 \in (L^p(\Omega), W^4_{p,N}(\Omega))_{1-\frac{1}{p},p}$, $f \in L^p(0,T;L^p(\Omega))$ *with* $4 < p < 6$ *be given. Then for every* $0 < T < \infty$ *there exists*

$$\varphi \in L^p(0,T;W^4_{p,N}(\Omega)) \cap \{u \in W^1_p(0,T;L^p(\Omega)) : \ u_{|t=0} = \varphi_0\}$$

such that

$$\begin{aligned} \partial_t\varphi + \varepsilon m(\varphi_0)\Delta^2\varphi &= f && \text{in } (0,T)\times\Omega, && (5.35)\\ \partial_n\varphi_{|\partial\Omega} &= 0 && \text{on } (0,T)\times\partial\Omega, && (5.36)\\ \partial_n\Delta\varphi_{|\partial\Omega} &= 0 && \text{on } (0,T)\times\partial\Omega, && (5.37)\\ \varphi(0) &= \varphi_0 && \text{in } \{0\}\times\Omega. && (5.38) \end{aligned}$$

Proof. We set $E = \mathbb{R}$ and note that $\mathbb{R}$ is of class $\mathcal{HT}$. Moreover, it holds $m = 2$ and we need to define

$$\mathcal{A}(x,D) := \varepsilon m(\varphi_0)\sum_{i,j=1}^{d}\partial^2_{ii}\partial^2_{jj}, \qquad \mathcal{B}_1(x,D) := \sum_{j=1}^{d}\nu_j(x)\partial_j,$$
$$\mathcal{B}_2(x,D) := \sum_{i,j=1}^{d}\nu_i(x)\partial_i\partial^2_{jj},$$

where $\nu(x) = (\nu_1(x),...,\nu_d(x))^T$ is the exterior normal at $x \in \partial\Omega$. Note that the coefficients are $\mathcal{B}(\mathbb{R})$-valued since they are constants, e.g. the coefficient ε has to be understood as a mapping $\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $z \mapsto \varepsilon z$.
Then $(\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2)$ satisfies the smoothness condition i) due to the embedding $m(\varphi_0) \in W^3_p(\Omega) \hookrightarrow C^2(\overline{\Omega})$, cf. (5.18). Next we study the ellipticity condition ii). We have the principal symbol

$$\mathcal{A}_\#(x,\xi) = \varepsilon m(\varphi_0)\sum_{i,j=1}^{d}\xi_i^2\xi_j^2 = \varepsilon m(\varphi_0)|\xi|^4$$

for every $\xi \in \mathbb{R}^d$. This polynomial has to be understood to be $\mathcal{B}(\mathbb{R})$-valued as above, i.e., as a mapping

$$\mathcal{A}_\#(x,\xi) : \mathbb{R} \to \mathbb{R}, \qquad z \mapsto \left(\varepsilon m(\varphi_0)\sum_{i,j=1}^{d}\xi_i^2\xi_j^2\right)z,$$

for every fixed $\xi \in \mathbb{R}^d$. In particular we obtain that $\varepsilon m(\varphi_0)|\xi|^4$ is the only eigenvalue of this mapping. This eigenvalue is a positive real number since it holds $\varepsilon, m(\varphi_0(x_0)) > 0$ for every $x_0 \in \partial\Omega$. Hence, the principal symbol $\mathcal{A}_\#(x,\xi)$ is parameter-elliptic with angle of ellipticity $\phi = 0$.

Next we show that the Lopatinskii-Shapiro Condition ii) (b), is satisfied. To this end, we have to show that for each $x_0 \in \partial\Omega$ the ODE problem in $\mathbb{R}_+$

$$\begin{aligned}\left(\lambda + \varepsilon m(\varphi_0(x_0))|\xi'|^4 - 2\varepsilon m(\varphi_0(x_0))|\xi'|^2\partial_d^2 + \varepsilon m(\varphi_0(x_0))\partial_d^4\right) v(y) = 0, \qquad y > 0,\\ \partial_d v(0) = h_1,\\ \partial_d\left(|\xi'|^2 + \partial_d^2\right) v(0) = h_2,\end{aligned} \tag{5.39}$$

has a unique solution $v \in C_b^0(\mathbb{R}_+)$ for each $h_1, h_2 \in \mathbb{R}$ and each $\lambda \in \overline{\Sigma}_{\pi-\phi_{\mathcal{A}}}$ and $\xi' \in \mathbb{R}^{d-1}$. From the assumption we know $|\xi'| + |\lambda| \neq 0$, where $\lambda \in \overline{\Sigma}_\pi = \mathbb{C}$ and $\xi' \in \mathbb{R}^{d-1}$. Hence, we distinguish two cases:

1st case: $\lambda = 0$: If $\lambda = 0$, we obtain from $|\xi'| + |\lambda| \neq 0$ that $|\xi'| \neq 0$ and (5.39) simplifies to

$$\varepsilon m(\varphi_0(x_0))\left(|\xi'|^4 - 2|\xi'|^2\partial_d^2 + \partial_d^4\right) v(y) = 0, \qquad y > 0$$

with the same boundary conditions. Therefore, we study the polynomial

$$P(T) = T^4 - 2|\xi'|^2T^2 + |\xi'|^4 = (T^2 - |\xi'|^2)^2.$$

Hence, $|\xi'|$ and $-|\xi'|$ are two zeros of this polynomial with multiplicity two. From standard ODE theory, cf. [For06, Chapter 15, Theorem 2], it follows that a fundamental solution to this ODE is given by

$$y_d e^{|\xi'|y_d}, \quad e^{|\xi'|y_d}, \quad y_d e^{-|\xi'|y_d}, \quad e^{-|\xi'|y_d}.$$

Now we note

$$\begin{aligned}\lim_{y_d\to\infty} y_d e^{|\xi'|y_d} &= +\infty, & \lim_{y_d\to\infty} e^{|\xi'|y_d} &= +\infty,\\ \lim_{y_d\to\infty} y_d e^{-|\xi'|y_d} &= 0, & \lim_{y_d\to\infty} e^{-|\xi'|y_d} &= 0.\end{aligned}$$

Since we look for a unique solution v in $C_b^0(\mathbb{R}_+)$, this implies

$$\operatorname{span}\{y_d e^{|\xi'|y_d}, e^{|\xi'|y_d}, y_d e^{-|\xi'|y_d}, e^{-|\xi'|y_d}\} \cap C_b^0(\mathbb{R}_+) = \operatorname{span}\{y_d e^{-|\xi'|y_d}, e^{-|\xi'|y_d}\}.$$

Hence, there exist c_1, c_2 such that

$$v(y_d) = c_1 y_d e^{-|\xi'|y_d} + c_2 e^{-|\xi'|y_d}.$$

Moreover, we have the boundary conditions

$$\begin{aligned}
\partial_d v(0) &= c_1 - |\xi'|c_2 = h_1,\\
\partial_d(|\xi'|^2 + \partial_d^2)v(0) &= \partial_d(c_1|\xi'|^2 y_d e^{-|\xi'|y_d} + |\xi'|^2 c_2 e^{-|\xi'|y_d})_{|y_d=0}\\
&\quad + \partial_d^3(c_1 y_d e^{-|\xi'|y_d} + c_2 e^{-|\xi'|y_d})_{|y_d=0}\\
&= c_1|\xi'|^2 - |\xi'|^3 c_2 + \partial_d^2(c_1 e^{-|\xi'|y_d} - c_1 y_d|\xi'|e^{-|\xi'|y_d} - c_2|\xi'|e^{-|\xi'|y_d})_{|y_d=0}\\
&= c_1|\xi'|^2 - |\xi'|^3 c_2 + \partial_d\Big(c_2|\xi'|^2 e^{-|\xi'|y_d}\\
&\quad -c_1|\xi'|e^{-|\xi'|y_d} - c_1|\xi'|e^{-|\xi'|y_d} + c_1 y_d|\xi'|^2 e^{-|\xi'|y_d}\Big)_{|y_d=0}\\
&= c_1|\xi'|^2 - |\xi'|^3 c_2 + c_1|\xi'|^2 - c_2|\xi'|^3 + c_1|\xi'|^2 + c_1|\xi'|^2\\
&= 4c_1|\xi'|^2 - 2c_2|\xi'|^3 = h_2.
\end{aligned}$$

Since it holds $|\xi'| \neq 0$, we have two linear independent conditions for the two unknowns c_1 and c_2. Hence, the solution $v \in C_b^0(\mathbb{R}_+)$ is unique, which shows the Lopatinskii-Shapiro Condition for the case $\lambda = 0$.

2nd case: $\lambda \neq 0$: We define $a := \lambda + \varepsilon m(\varphi_0(x_0))|\xi'|^4$ and $b := \varepsilon m(\varphi_0(x_0))|\xi'|^2$. Hence, (5.39) simplifies to

$$\left(\varepsilon m(\varphi_0(x_0))\partial_d^4 - 2b\partial_d^2 + a\right) v(y) = 0.$$

Its characteristic polynomial is given by $\varepsilon m(\varphi_0(x_0))\alpha^4 - 2b\alpha^2 + a$. Defining $z := \alpha^2$ this polynomial simplifies to

$$\varepsilon m(\varphi_0(x_0))z^2 - 2bz + a = 0.$$

Its solution is given by

$$\begin{aligned}
z_{1,2} &= \frac{2b \pm \sqrt{4b^2 - 4\varepsilon m(\varphi_0(x_0))a}}{2\varepsilon m(\varphi_0(x_0))} = \frac{b \pm \sqrt{b^2 - \varepsilon m(\varphi_0(x_0))a}}{\varepsilon m(\varphi_0(x_0))}\\
&= |\xi'|^2 \pm \frac{1}{\varepsilon m(\varphi_0(x_0))}\sqrt{\varepsilon^2 m(\varphi_0(x_0))^2|\xi'|^4 - \varepsilon m(\varphi_0(x_0))\lambda - \varepsilon^2 m(\varphi_0(x_0))^2|\xi'|^4}\\
&= |\xi'|^2 \pm i\sqrt{\frac{\lambda}{\varepsilon m(\varphi_0(x_0))}}.
\end{aligned}$$

Before we can continue calculating $\alpha_1, ..., \alpha_4$ from these solutions, we need to study where z_1 and z_2 are on the Gaussian plane. From $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ it follows $\frac{\lambda}{\varepsilon m(\varphi_0(x_0))} \in \mathbb{C} \setminus (-\infty, 0]$ for every $\varepsilon \in \mathbb{R}$ and every $x_0 \in \partial\Omega$. Therefore, we can deduce $\sqrt{\frac{\lambda}{\varepsilon m(\varphi_0(x_0))}} \in \Sigma_{\frac{\pi}{2}}$ and thus $i\sqrt{\frac{\lambda}{\varepsilon m(\varphi_0(x_0))}} \in \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $-i\sqrt{\frac{\lambda}{\varepsilon m(\varphi_0(x_0))}} \in \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$, respectively. Adding a positive number yields

$$\begin{aligned} z_1 &= \left(|\xi'|^2 - i\sqrt{\frac{\lambda}{\varepsilon m(\varphi_0(x_0))}} \right) \in \{z \in \mathbb{C} :\ \mathrm{Im}(z) > 0\}, \\ z_2 &= \left(|\xi'|^2 - i\sqrt{\frac{\lambda}{\varepsilon m(\varphi_0(x_0))}} \right) \in \{z \in \mathbb{C} :\ \mathrm{Im}(z) < 0\}. \end{aligned}$$

Hence, it holds $z_{1,2} \in \mathbb{C} \setminus (-\infty, 0]$ and therefore $\sqrt{z_{1,2}} \in \Sigma_{\frac{\pi}{2}}$ is well-defined. From $\alpha_{1,2} = \sqrt{z_{1,2}} \in \Sigma_{\frac{\pi}{2}}$ and $\alpha_{3,4} = -\sqrt{z_{1,2}} \in \{z \in \mathbb{C} :\ \mathrm{Re}(z) < 0\}$ it follows $\mathrm{Re}(\alpha_{3,4}) < 0$. Note that it holds $\alpha_1 \neq \alpha_2$ and $\alpha_3 \neq \alpha_4$ for $\lambda \neq 0$.
From standard ODE theory we know that a fundamental solution to this ODE is given by

$$\{e^{\alpha_j y_d} :\ j = 1, 2, 3, 4\}.$$

Due to $\mathrm{Re}(\alpha_{1,2}) > 0$ and $\mathrm{Re}(\alpha_{3,4}) < 0$ we can conclude

$$\begin{aligned} &\lim_{y_d \to \infty} e^{\alpha_j y_d} = +\infty && \text{for } j = 1, 2, \\ &\lim_{y_d \to \infty} e^{\alpha_j y_d} = 0 && \text{for } j = 3, 4. \end{aligned}$$

Since we look for a unique solution v in $C_b^0(\mathbb{R}_+)$, this implies

$$\mathrm{span}\{e^{\alpha_j y_d} :\ j = 1, 2, 3, 4\} \cap C_b^0(\mathbb{R}_+) = \mathrm{span}\{e^{\alpha_j y_d} :\ j = 3, 4\}.$$

Thus there exist c_1, c_2 such that

$$v(y_d) = c_1 e^{\alpha_3 y_d} + c_2 e^{\alpha_4 y_d}.$$

Moreover, we have the boundary conditions

$$\begin{aligned} \partial_d v(0) &= c_1\alpha_3 + c_2\alpha_4 = h_1, \\ \partial_d(|\xi'|^2 + \partial_d^2)v(0) &= |\xi'|^2(c_1\alpha_3 + c_2\alpha_4) + c_1\alpha_3^3 + c_2\alpha_4^3 = h_2. \end{aligned}$$

This means that we have two linear independent conditions for the two unknowns c_1 and c_2. Hence, the solution $v \in C_b^0(\mathbb{R}_+)$ is unique, which shows the Lopatinskii-Shapiro Condition.

Now we are able to apply Theorem 5.14, which yields the existence of some $\mu_\phi \geq 0$ such that the initial-boundary value problem

$$\begin{aligned} \partial_t \tilde{\varphi} + \varepsilon\Delta(\Delta\tilde{\varphi}) + \mu_\phi \tilde{\varphi} &= \tilde{f} && \text{in } (0, \infty) \times \Omega, && (5.40) \\ \partial_n \tilde{\varphi}_{|\partial\Omega} &= 0 && \text{on } (0, \infty) \times \partial\Omega, && (5.41) \\ \partial_n \Delta\tilde{\varphi}_{|\partial\Omega} &= 0 && \text{on } (0, \infty) \times \partial\Omega, && (5.42) \\ \tilde{\varphi}(0) &= 0 && \text{in } \{0\} \times \Omega && (5.43) \end{aligned}$$

has a solution with the property of maximal regularity in $L^p(0,\infty;L^p(\Omega))$.

In the next step we choose $\varphi_0 \in (L^p(\Omega),\mathcal{D}(A_B))_{1-\frac{1}{p},p}$, where

$$\mathcal{D}(A_B) = \{u \in W^4_p(\Omega) : \ \partial_n u_{|\partial\Omega} = \partial_n \Delta u_{|\partial\Omega} = 0\} = W^4_{p,N}(\Omega),$$

since there exists a continuous extension operator

$$E : (X_0, X_1)_{1-\frac{1}{p},p} \to W^1_p(0,\infty;X_0) \cap L^p(0,\infty;X_1)$$

such that $E\varphi_{0|t=0} = \varphi_0$, cf. [Ama95, Chapter III, Theorem 4.10.2]. From [Gri67, Theorem 8.1'] it follows

$$(L^p(\Omega),\mathcal{D}(A_B))_{1-\frac{1}{p},p} = \left\{u \in W_p^{4-\frac{4}{p}}(\Omega) : \ \partial_n u_{|\partial\Omega} = \partial_n \Delta u_{|\partial\Omega} = 0\right\}.$$

So let $\psi \in W^1_p(0,\infty;L^p(\Omega)) \cap L^p(0,\infty;W^4_{p,N}(\Omega))$ be the extension of φ_0, i.e., $\psi := E\varphi$. Then we define

$$\tilde{f} := \hat{f} - \partial_t \psi - \varepsilon\Delta(\Delta\psi) - \mu_\phi \psi \in L^p(0,\infty;L^p(\Omega)),$$

where μ_ϕ is given as in (5.40), f is extended on (T,∞) by 0 and

$$\hat{f} := \exp(-\mu_\phi t) f \in L^p(0,\infty;L^p(\Omega)).$$

Then there exists a solution $\hat{\varphi}$ of (5.40) - (5.43) for $\tilde{f}$ given as above and we know that this solution has the property of maximal L^p-regularity, i.e.,

$$\hat{\varphi} \in W^1_p(0,\infty;L^p(\Omega)) \cap L^p(0,\infty;W^4_{p,N}(\Omega)),$$

where the boundary conditions are satisfied since it holds $A_B\hat{\varphi} \in L^p(0,\infty;L^p(\Omega))$ and therefore we can conclude $\hat{\varphi}(t) \in D(A_B)$ for a.e. $t \in (0,\infty)$ with $D(A_B)$ given as in Theorem 5.14. We set $u := \hat{\varphi} + \psi$. Then u solves

$$\begin{aligned}
\partial_t u + \varepsilon\Delta(\Delta u) + \mu_\phi u &= \hat{f} && \text{in } (0,\infty)\times\Omega,\\
\partial_n u_{|\partial\Omega} &= 0 && \text{on } (0,\infty)\times\partial\Omega,\\
\partial\Delta u_{|\partial\Omega} &= 0 && \text{on } (0,\infty)\times\partial\Omega,\\
u(0) &= \varphi_0 && \text{in } \{0\}\times\Omega,
\end{aligned}$$

due to $\hat{\varphi}(0) = 0$ and $\psi(0) = E\varphi_0(0) = \varphi_0$. Moreover, the boundary values hold since $\hat{\varphi}$ and ψ fulfill them. Finally, we set $\varphi := e^{\mu_\phi t} u_{|[0,T]}$. Hence, it solves

$$\begin{aligned}
\hat{f} &= \partial_t u + \varepsilon\Delta(\Delta u) + \mu_\phi u\\
&= -\mu_\phi e^{-\mu_\phi t}\varphi + e^{-\mu_\phi t}\partial_t\varphi + \varepsilon\Delta(\Delta e^{-\mu_\phi t}\varphi) + \mu_\phi e^{-\mu_\phi t}\varphi\\
&= e^{-\mu_\phi t}(\partial_t\varphi + \varepsilon\Delta(\Delta\varphi)).
\end{aligned}$$

Using the definition of $\hat{f}$ we can conclude that $\varphi \in W_p^1(0,T;L^p(\Omega)) \cap L^p(0,T;W^4_{p,N}(\Omega))$ is a solution of the initial-boundary value problem (5.35) - (5.38), i.e., of

$$\begin{aligned} \partial_t\varphi + \varepsilon\Delta(\Delta\varphi) &= f && \text{in } (0,T)\times\Omega, \\ \partial_n\varphi_{|\partial\Omega} &= 0 && \text{on } (0,T)\times\partial\Omega, \\ \partial_n\Delta\varphi_{|\partial\Omega} &= 0 && \text{on } (0,T)\times\partial\Omega, \\ \varphi(0) &= \varphi_0 && \text{in } \{0\}\times\Omega. \end{aligned}$$

□

Analogously to the previous part we need to ensure that $||\mathcal{L}^{-1}||_{\mathcal{L}(Y_T,X_T)}$ remains bounded.

Lemma 5.16. *Let the assumptions of Lemma 5.15 hold and* $0 < T_0 < \infty$ *be given. Then*

$$||\mathcal{L}_{2,T}^{-1}||_{\mathcal{L}(Y_T^2,X_T^2)} \leq ||\mathcal{L}_{2,T_0}^{-1}||_{\mathcal{L}(Y_{T_0}^2,X_{T_0}^2)} < \infty \qquad \textit{for all } 0 < T < T_0.$$

This lemma can be proven analogously to Lemma 5.13.

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