

## On sharp interface limits for diffuse interface models for two-phase flows

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We discuss the sharp interface limit of a diffuse interface model for a two-phase flow of two partly miscible viscous Newtonian fluids of different densities, when a certain parameter  $\varepsilon > 0$  related to the interface thickness tends to zero. In the case that the mobility stays positive or tends to zero slower than linearly in  $\varepsilon$  we will prove that weak solutions tend to varifold solutions of a corresponding sharp interface model. But, if the mobility tends to zero faster than  $\varepsilon^3$  we will show that certain radially symmetric solutions tend to functions, which will not satisfy the Young-Laplace law at the interface in the limit.

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### 1. Introduction

The present contribution is devoted to the study of the relations between so-called diffuse and sharp interface models for the flow of two viscous incompressible Newtonian fluids. Such two-phase flows play a fundamental role in many fluid dynamical applications in physics, chemistry, biology, and the engineering sciences. There are two basic types of models namely the (classical) sharp interface models, where the interface  $\Gamma(t)$  between the fluids is modeled as a (sufficiently smooth) surface and so-called diffuse interface models, where the “sharp” interface  $\Gamma(t)$  is replaced by an interfacial region, where a suitable order parameter (e.g., the difference of volume fractions) varies smoothly, but with a large gradient between two distinguished values (e.g.,  $\pm 1$  for the difference of volume fractions). Then the natural question arises how diffuse and sharp interface models are related if a suitable parameter  $\varepsilon > 0$ , which is related to the width of the diffuse interface, tends to zero. There are several results on this question, which are based on formally matched asymptotics calculations. But so far there are very few mathematically rigorous convergence results.

More precisely, we study throughout the paper the sharp interface limit of the following diffuse interface model:

$$\rho \partial_t \mathbf{v} + \left( \rho \mathbf{v} + \frac{\partial \rho}{\partial c} \mathbf{J} \right) \cdot \nabla \mathbf{v} - \operatorname{div}(v(c) D \mathbf{v}) + \nabla p = -\varepsilon \operatorname{div}(a(c) \nabla c \otimes \nabla c) \text{ in } Q, \quad (1.1)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q, \quad (1.2)$$

$$\partial_t c + \mathbf{v} \cdot \nabla c = \operatorname{div}(m_\varepsilon(c) \nabla \mu) \quad \text{in } Q, \quad (1.3)$$

$$\mu = \varepsilon^{-1} f'(c) - \varepsilon \Delta c \quad \text{in } Q, \quad (1.4)$$

$$\mathbf{v}|_{\partial\Omega} = 0 \quad \text{on } S, \quad (1.5)$$

$$\mathbf{n}_{\partial\Omega} \cdot \nabla c|_{\partial\Omega} = \mathbf{n}_{\partial\Omega} \cdot \nabla \mu|_{\partial\Omega} = 0 \quad \text{on } S, \quad (1.6)$$

$$(\mathbf{v}, c)|_{t=0} = (\mathbf{v}_0, c_0) \quad \text{in } \Omega, \quad (1.7)$$

where  $Q = \Omega \times (0, \infty)$ ,  $S = \partial\Omega \times (0, \infty)$ ,  $\Omega \subseteq \mathbb{R}^n$  is a suitable domain, and  $\mathbf{J} = -m_\varepsilon(c) \nabla \mu$ . Here  $c = c_2 - c_1$  is the volume fraction difference of the fluids,  $\rho = \rho(c)$  is the density of the fluid mixture, depending explicitly on  $c$  through  $\rho(c) = \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} c - \frac{\tilde{\rho}_1 + \tilde{\rho}_2}{2}$ , where  $\tilde{\rho}_j$  is the specific density of fluid  $j = 1, 2$ , and  $f$  is a suitable “double-well potential”, e.g.,  $f(c) = \frac{1}{8}(1 - c^2)^2$ . Precise assumptions will be made below. Moreover,  $\varepsilon > 0$  is a small parameter related to the interface thickness,  $\mu$  is the so-called chemical potential,  $m_\varepsilon(c) > 0$  a mobility coefficient related to the strength of diffusion in the mixture and  $a(c)$  is a coefficient in front of the  $|\nabla c|^2$ -term in the free energy of the system. Finally,  $\mathbf{n}_{\partial\Omega}$  denotes the exterior normal of  $\partial\Omega$ . The model was derived by A., Garcke, and Grün [5]. In the case  $\rho(c) \equiv \text{const.}$  it coincides with the so-called “Model H” in Hohenberg and Halperin [14], cf. also Gurtin et al. [13]. Existence of weak solutions for this system in the case of a bounded, sufficiently smooth domain  $\Omega$  and for a suitable class of singular free energy densities  $f$  was proved by A., Depner, and Garcke [4]. We refer to the latter article for further references concerning analytic results for this diffuse interface model in the case  $\rho(c) \equiv \text{const.}$  and related models.

In [5] the sharp interface limit  $\varepsilon \rightarrow 0$  was discussed with the method of formally matched asymptotics. It was shown that for the scaling  $m_\varepsilon(c) \equiv \tilde{m} \varepsilon^\alpha$  with  $\alpha = 0, 1$ ,  $\tilde{m} > 0$ , solutions of the system (1.1)–(1.5) converges to solutions of

$$\rho^\pm \partial_t \mathbf{v} + (\rho^\pm \mathbf{v} + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \mathbf{J}) \cdot \nabla \mathbf{v} - \operatorname{div} \mathbb{T}^\pm(\mathbf{v}, p) = 0 \quad \text{in } \Omega^\pm(t), t > 0, \quad (1.8)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega^\pm(t), t > 0, \quad (1.9)$$

$$m_0 \Delta \mu = 0 \quad \text{in } \Omega^\pm(t), t > 0, \quad (1.10)$$

$$-\mathbf{n} \cdot [\mathbb{T}(\mathbf{v}, p)] = \sigma H \mathbf{n} \quad \text{on } \Gamma(t), t > 0, \quad (1.11)$$

$$V - \mathbf{n} \cdot \mathbf{v}|_{\Gamma(t)} = -[\frac{m_0}{2} \mathbf{n} \cdot \nabla \mu] \quad \text{on } \Gamma(t), t > 0, \quad (1.12)$$

$$\mu|_{\Gamma(t)} = \sigma H \quad \text{on } \Gamma(t), t > 0, \quad (1.13)$$

with  $\mathbf{J} = -m_0 \nabla \mu$ . Here  $\mathbf{n}$  denotes the unit normal of  $\Gamma(t)$  that points inside  $\Omega^+(t)$  and  $V$  and  $H$  the normal velocity and scalar mean curvature of  $\Gamma(t)$  with respect to  $\mathbf{n}$ . Moreover, by  $[\cdot]$  we denote the jump of a quantity across the interface in direction of  $\mathbf{n}$ , i.e.,  $[f](x) = \lim_{h \rightarrow 0} (f(x + h\mathbf{n}) - f(x - h\mathbf{n}))$  for  $x \in \Gamma(t)$ . Furthermore,  $\sigma$  is a surface tension coefficient determined uniquely by  $f$  and  $m_0 = \tilde{m}$  if  $\alpha = 0$  and  $m_0 = 0$  if  $\alpha = 1$  is a mobility constant. Implicitly it is assumed that  $\mathbf{v}, \mu$  do not jump across  $\Gamma(t)$ , i.e.,

$$[\mathbf{v}] = [\mu] = 0 \quad \text{on } \Gamma(t), t > 0.$$

In the following we close the system with the boundary and initial conditions

$$\mathbf{v}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega, t > 0, \quad (1.14)$$

$$\mathbf{n}_{\partial\Omega} \cdot m_\varepsilon(c) \nabla \mu|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega, t > 0, \quad (1.15)$$

$$\Omega^+(0) = \Omega_0^+, \tag{1.16}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \tag{1.17}$$

where  $\mathbf{v}_0, \Omega_0^+$  are given initial data satisfying  $\partial\Omega_0^+ \cap \partial\Omega = \emptyset$ . Equations (1.8)–(1.9) describe the conservation of linear momentum and mass in both fluids, and (1.11) is the balance of forces at the boundary. The equations for  $\mathbf{v}$  are complemented by the non-slip condition (1.14) at the boundary of  $\Omega$ . The conditions (1.10), (1.15) describe together with (1.12) a continuity equation for the mass of the phases, and (1.13) relates the chemical potential  $\mu$  to the  $L^2$ -gradient of the surface area, which is given by the mean curvature of the interface.

We note that in the case  $\alpha = 1$ , i.e.,  $m_0 = 0$ , (1.12) describes the usual kinematic condition that the interface is transported by the flow of the surrounding fluids and (1.8)–(1.17) reduces to the classical model of a two-phase Navier–Stokes flow. Existence of strong solutions locally in time was first proved by Denisova and Solonnikov [11]. We refer to Prüss and Simonett [19] and Köhne et al. [15] for more recent results and further references. Existence of generalized solutions globally in times was shown by Plotnikov [18] and A. [1, 2]. On the other hand, if  $\alpha = 0, m_0 > 0$ , respectively, the equations (1.10), (1.13), (1.15) are a variant of the Mullins–Sekerka flow of a family of interfaces with an additional convection term  $\mathbf{n} \cdot \mathbf{v}|_{\Gamma(t)}$ . In the case  $\bar{\rho}_1 = \bar{\rho}_2$  existence of weak solutions for large times and general initial data was proved by A. and Röger [6] and existence of strong solutions locally in time and stability of spherical droplets was proved by A. and Wilke [8].

In the following we address the following question: Under which assumptions on the behavior of  $m_\varepsilon(c)$  as  $\varepsilon \rightarrow 0$  do weak solutions of (1.1)–(1.7) converge to weak/generalized solutions of (1.8)–(1.17)? In this paper we provide a partial answer to that question. If one assumes e.g.  $m_\varepsilon(c) = \tilde{m}\varepsilon^\alpha$ , the results in the following will show that convergence holds true in the case  $\alpha \in [0, 1)$ . More precisely, we will show that weak solutions of (1.1)–(1.7) converge to so-called varifold solutions of (1.8)–(1.17), which are defined in the spirit of Chen [10]. But in the case  $\alpha \in (3, \infty)$  we will construct radially symmetric solutions of (1.1)–(1.4) in the domain  $\Omega = \{x \in \mathbb{R} : 1 < |x| < M\}$  with suitable inflow and outflow boundary conditions, which do not converge to a solution of (1.8)–(1.13). In particular, the pressure  $p$  in the limit  $\varepsilon \rightarrow 0$  satisfies

$$[p] = \sigma\kappa(t)H \quad \text{on } \Gamma(t) = \partial B_{R(t)}(0),$$

where  $R(t), \kappa(t) \rightarrow_{t \rightarrow \infty} \infty$  and  $\mathbf{v}$  is independent of  $t$  and smooth in  $\Omega$ . This shows that the Young–Laplace law (1.11) is not satisfied. We note that these results are consistent with the numerical studies of Jacqmin, where a scaling of the mobility as  $m_\varepsilon(c) = \tilde{m}\varepsilon^\alpha$  with  $\alpha \in [1, 2)$  was proposed and considered.

The structure of the article is as follows: First we introduce some notation and preliminary results in Section 2. Then we prove our main result on convergence of weak solutions of (1.1)–(1.7) to varifold solutions of (1.8)–(1.17) in the case that the mobility  $m_\varepsilon(c)$  tends to zero as  $\varepsilon \rightarrow 0$  slower than linearly in Section 3. Finally, in Section 4, we consider certain radially symmetric solutions of (1.1)–(1.7) and show that these do not converge to a solution of (1.8)–(1.13) if the mobility tends to zero too fast as  $\varepsilon \rightarrow 0$ .

## 2. Notation and preliminaries

Let  $X$  be a locally compact separable metric space and let  $C_0(X; \mathbb{R}^N)$  by the closure of compactly supported continuous functions  $f: X \rightarrow \mathbb{R}^N, N \in \mathbb{N}$ , in the supremum norm. Moreover, denote by

$\mathfrak{M}(X; \mathbb{R}^N)$  the space of all finite  $\mathbb{R}^N$ -valued Radon measures,  $\mathfrak{M}(X) := \mathfrak{M}(X; \mathbb{R})$ . Then by Riesz representation theorem  $\mathfrak{M}(X; \mathbb{R}^N) = C_0(X; \mathbb{R}^N)'$ , cf., e.g., Ambrosio et al. [9, Theorem 1.54]. Given  $\lambda \in \mathfrak{M}(X; \mathbb{R}^N)$  we denote by  $|\lambda|$  the total variation measure defined by

$$|\lambda|(A) = \sup \left\{ \sum_{k=0}^{\infty} |\lambda(A_k)| : A_k \in \mathfrak{B}(X) \text{ pairwise disjoint, } A = \bigcup_{k=0}^{\infty} A_k \right\}$$

for every  $A \in \mathfrak{B}(X)$ , where  $\mathfrak{B}(X)$  denotes the  $\sigma$ -algebra of Borel sets of  $X$ . Moreover,  $\frac{\lambda}{|\lambda|} : X \rightarrow \mathbb{R}^N$  denotes the Radon-Nikodym derivative of  $\lambda$  with respect to  $|\lambda|$ . The restriction of a measure  $\mu$  to a  $\mu$ -measurable set  $A$  is denoted by  $(\mu \lfloor A)(B) = \mu(A \cap B)$ . Furthermore, the  $s$ -dimensional Hausdorff measure on  $\mathbb{R}^d$ ,  $0 \leq s \leq d$ , is denoted by  $\mathcal{H}^s$ . Recall that

$$BV(U) = \{f \in L^1(U) : \nabla f \in \mathfrak{M}(U; \mathbb{R}^d)\}$$

$$\|f\|_{BV(U)} = \|f\|_{L^1(U)} + \|\nabla f\|_{\mathfrak{M}(U; \mathbb{R}^d)},$$

where  $\nabla f$  denotes the distributional derivative and  $U \subseteq \mathbb{R}^n$  is an open set. Moreover,  $BV(U; \{0, 1\})$  denotes the set of all  $\mathfrak{X} \in BV(U)$  such that  $\mathfrak{X}(x) \in \{0, 1\}$  for almost all  $x \in U$ .

A set  $E \subseteq U$  is said to have finite perimeter in  $U$  if  $\mathfrak{X}_E \in BV(U)$ . By the structure theorem of sets of finite perimeter  $|\nabla \mathfrak{X}_E| = \mathcal{H}^{d-1} \lfloor \partial^* E$ , where  $\partial^* E$  is the so-called reduced boundary of  $E$  and for all  $\varphi \in C_0(U, \mathbb{R}^d)$

$$-\langle \nabla \mathfrak{X}_E, \varphi \rangle = - \int_{\partial^* E} \varphi \cdot \mathbf{n}_E \, d\mathcal{H}^{d-1},$$

where  $\mathbf{n}_E(x) = \frac{\nabla \mathfrak{X}_E}{|\nabla \mathfrak{X}_E|}$ , cf., e.g., [9]. Note that, if  $E$  is a domain with  $C^1$ -boundary, then  $\partial^* E = \partial E$  and  $\mathbf{n}_E$  coincides with the interior unit normal.

As usual the space of smooth and compactly supported functions in an open set  $U$  is denoted by  $C_0^\infty(U)$ . Moreover,  $C^\infty(\overline{U})$  denotes the set of all smooth functions  $f : U \rightarrow \mathbb{C}$  such that all derivatives have continuous extensions on  $\overline{U}$ . For  $0 < T \leq \infty$ , we denote by  $L^p_{loc}([0, T]; X)$ ,  $1 \leq p \leq \infty$ , the space of all strongly measurable  $f : (0, T) \rightarrow X$  such that  $f \in L^p(0, T'; X)$  for all  $0 < T' < T$ . Here  $L^p(M)$  and  $L^p(M; X)$  denote the standard Lebesgue spaces for scalar and  $X$ -valued functions, respectively. Furthermore,  $C_{0,\sigma}^\infty(\Omega) = \{\varphi \in C_0^\infty(\Omega)^d : \operatorname{div} \varphi = 0\}$  and

$$L^2_\sigma(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{L^2(\Omega)}.$$

If  $Y = X'$  is a dual space and  $Q \subseteq \mathbb{R}^N$  is open, then  $L_{\omega^*}^\infty(Q; Y)$  denotes the space of all functions  $\nu : Q \rightarrow Y$  that are weakly- $*$  measurable and essentially bounded, i.e.,

$$x \mapsto \langle \nu_x, F(x, \cdot) \rangle_{X', X}$$

is measurable for each  $F \in L^1(Q; X)$  and

$$\|\nu\|_{L_{\omega^*}^\infty(Q; Y)} := \operatorname{ess\,sup}_{x \in Q} \|\nu_x\|_Y < \infty.$$

Moreover, we note that there is a separable Banach space  $X$  such that  $X' = BV(\Omega)$ , cf. [9]. As a consequence [12] we obtain that  $L_{\omega^*}^\infty(0, T; BV(\Omega)) = (L^1(0, T; X))^*$  and that uniformly bounded sets in  $L_{\omega^*}^\infty(0, T; BV(\Omega))$  are weakly\*-precompact.

### 3. Sharp interface limit

In this section we discuss the relation between (1.8)–(1.17) and its diffuse interface analogue (1.1)–(1.7).

ASSUMPTION 3.1 We assume that the domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$  is bounded and smooth. Furthermore, we assume that there exist constants  $c_0, C_0 > 0$  such that

- $f \in C^3(\mathbb{R})$ ,  $f(c) \geq 0$ ,  $f(c) = 0$  if and only if  $c = -1, 1$ , and  $f''(c) \geq c_0|c|^{p-2}$  if  $|c| \geq 1 - c_0$  for some constant  $p \geq 3$
- $\rho, a, v \in C^1(\mathbb{R})$  with  $c_0 \leq \rho, a, v \leq C_0$  and

$$\rho(c) = \frac{\tilde{\rho}_2 + \tilde{\rho}_1}{2} + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}c$$

for  $c \in [-1, 1]$

- $m_\varepsilon, m_0 \in C^1(\mathbb{R})$ ,  $0 \leq m_\varepsilon, m_0 \leq C_0$ ,  $m_\varepsilon \rightarrow_{\varepsilon \rightarrow 0} m_0$  in  $C^1(\mathbb{R})$ , and either  $m_0 \geq c_0$  or  $m_0 \equiv 0$ . If  $m_0 \equiv 0$ , then  $m_\varepsilon \geq \bar{m}_\varepsilon$  for constants  $\bar{m}_\varepsilon > 0$  with  $\bar{m}_\varepsilon \rightarrow_{\varepsilon \rightarrow 0} 0$ .

The stronger assumption  $p \geq 3$  (compared to  $p > 2$  in [10]) is needed here for the uniform estimate of  $\mathbf{v}_\varepsilon \cdot \nabla c_\varepsilon = \operatorname{div}(\mathbf{v}_\varepsilon c_\varepsilon)$  in  $L^2(0, T; H^{-1}(\Omega))$ . A possible choice for the homogeneous free energy density is  $f(s) = (s^2 - 1)^2$ . Moreover, let  $\sigma = \int_{-1}^1 \sqrt{f(s)/2} ds$  and  $A(s) = \int_0^s \sqrt{a(\tau)} d\tau$ .

Now, let us consider the energy identities corresponding to our two systems. We recall that every sufficiently smooth solution of the Navier–Stokes/Mullins–Sekerka system (1.8)–(1.17) satisfies

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega} \rho(c) |\mathbf{v}|^2 dx + \sigma \frac{d}{dt} \mathcal{H}^{d-1}(\Gamma) = - \int_{\Omega} v(c) |D\mathbf{v}|^2 dx - \int_{\Omega} m_0(c) |\nabla \mu|^2 dx, \quad (3.1)$$

where  $c(t, x) = -1 + 2\chi_{\Omega^+(t)}(x)$ . On the other hand, every sufficiently smooth solution of (1.1)–(1.7) satisfies

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega} \rho(c) |\mathbf{v}|^2 dx + \frac{d}{dt} \mathcal{E}_\varepsilon(c) = - \int_{\Omega} v(c) |D\mathbf{v}|^2 dx - \int_{\Omega} m_\varepsilon(c) |\nabla \mu|^2 dx, \quad (3.2)$$

where

$$\mathcal{E}_\varepsilon(c) = \int_{\Omega} \left( \varepsilon \frac{|\nabla A(c)|^2}{2} + \frac{f(c)}{\varepsilon} \right) dx$$

is the free energy. Moreover, by Modica and Mortola [17] or Modica [16], for  $A(c) = c$ , we have

$$\mathcal{E}_\varepsilon \rightarrow_{\varepsilon \rightarrow 0} \mathcal{P} \quad \text{w.r.t. } L^1\text{-}\Gamma\text{-convergence,}$$

where

$$\mathcal{P}(u) = \begin{cases} \sigma \mathcal{H}^{d-1}(\partial^* E) & \text{if } u = -1 + 2\chi_E \text{ and } E \text{ has finite perimeter,} \\ +\infty & \text{else.} \end{cases}$$

Here,  $\partial^* E$  denotes the reduced boundary. Note that  $\partial^* E = \partial E$  if  $E$  is a sufficiently regular domain. Therefore, we see that the energy identity (3.1) is formally identical to the sharp interface limit of the energy identity (3.2) of the diffuse interface model (1.1)–(1.7).

We will now adapt the arguments of Chen [10], see also A. and Röger [6], to show that, as  $\varepsilon \rightarrow 0$ , solutions of the diffuse interface model (1.1)–(1.7) converge to *varifold solutions* of the system (1.8)–(1.17). Let  $Q = \Omega \times (0, \infty)$  and  $G_{d-1} := S^{d-1}/\sim$  where  $\mathbf{v}_0 \sim \mathbf{v}_1$  for  $\mathbf{v}_0, \mathbf{v}_1 \in S^{d-1}$  iff  $\mathbf{v}_0 = \pm \mathbf{v}_1$  and  $S^{d-1}$  is the unit sphere in  $\mathbb{R}^d$ .

DEFINITION 3.2 Let  $\mathbf{v}_0 \in L^2_\sigma(\Omega)$  and  $E_0 \subset \Omega$  be a set of finite perimeter. Then  $(\mathbf{v}, E, \mu, V)$  if  $m_0 > 0$  and  $(\mathbf{v}, E, V)$  else is called a *varifold solution* of (1.8)–(1.17) with initial values  $(\mathbf{v}_0, E_0)$  if the following conditions are satisfied:

1.  $\mathbf{v} \in L^2((0, \infty); H^1(\Omega)^d) \cap L^\infty((0, \infty); L^2_\sigma(\Omega))$ ;  $\mu \in L^2_{loc}([0, \infty); H^1(\Omega))$ ,  $\nabla \mu \in L^2((0, \infty); L^2(\Omega)^d)$  if  $m_0 > 0$ .
2.  $E = \bigcup_{t \geq 0} E_t \times \{t\}$  is a measurable subset of  $\Omega \times [0, \infty)$  such that  $\chi_E \in C([0, \infty); L^1(\Omega)) \cap L^\infty_{\omega^*}((0, \infty); BV(\Omega))$  and  $|E_t| = |E_0|$  for all  $t \geq 0$ .
3.  $V$  is a Radon measure on  $\overline{\Omega} \times G_{d-1} \times (0, \infty)$  such that  $V = V^t dt$  where  $V^t$  is a Radon measure on  $\overline{\Omega} \times G_{d-1}$  for almost all  $t \in (0, \infty)$ , i.e., a general varifold in  $\overline{\Omega}$ . Moreover, for almost all  $t \in (0, \infty)$   $V^t$  has the representation

$$\int_{\overline{\Omega} \times G_{d-1}} \psi(x, \mathbf{p}) dV^t(x, \mathbf{p}) = \sum_{i=1}^d \int_{\overline{\Omega}} b_i^t(x) \psi(x, \mathbf{p}_i^t(x)) d\lambda^t(x) \quad (3.3)$$

for all  $\psi \in C(\overline{\Omega} \times G_{d-1})$ . Here, for almost all  $t \in (0, \infty)$ ,  $\lambda^t$  is a Radon measure on  $\overline{\Omega}$ , and the  $\lambda^t$ -measurable functions  $b_i^t, \mathbf{p}_i^t$  are  $\mathbb{R}$ - and  $G_{d-1}$ -valued, respectively, such that

$$0 \leq b_i^t \leq 1, \quad \sum_{i=1}^d b_i^t \geq 1, \quad \sum_{i=1}^d \mathbf{p}_i^t \otimes \mathbf{p}_i^t = I, \quad \lambda^t\text{-a.e.}$$

and

$$\frac{|\nabla \chi_{E_t}|}{\lambda^t} \leq \frac{1}{2\sigma}.$$

4. For  $c := -1 + 2\chi_E$ ,  $\mathbf{J} := -m_0(c)\nabla \mu$  if  $m_0 > 0$  and  $\mathbf{J} := 0$  else as well as  $\tilde{\mathbf{J}} := \frac{\partial \rho}{\partial c}(c)\mathbf{J}$  we have

$$\begin{aligned} \int_Q \left( -\rho(c)\mathbf{v} \cdot \partial_t \boldsymbol{\varphi} - \mathbf{v} \otimes (\rho(c)\mathbf{v} + \tilde{\mathbf{J}}) : \nabla \boldsymbol{\varphi} + v(c)D\mathbf{v} : D\boldsymbol{\varphi} \right) d(x, t) \\ - \int_{\Omega} \rho(c|_{t=0}) \mathbf{v}_0 \cdot \boldsymbol{\varphi}|_{t=0} dx = - \int_0^\infty \langle \delta V^t, \boldsymbol{\varphi} \rangle dt \end{aligned} \quad (3.4)$$

for all  $\boldsymbol{\varphi} \in C_0^\infty([0, \infty); C_{0,\sigma}^\infty(\Omega))$  and

$$2 \int_E \partial_t \psi + \operatorname{div}(\psi \mathbf{v}) d(x, t) + \int_Q \mathbf{J} \cdot \nabla \psi d(x, t) + \int_{E_0} \psi|_{t=0} dx = 0 \quad (3.5)$$

for all  $\psi \in C_0^\infty([0, \infty) \times \overline{\Omega})$ . Here

$$\langle \delta V^t, \boldsymbol{\varphi} \rangle := \int_{\overline{\Omega} \times G_{d-1}} (I - \mathbf{p} \otimes \mathbf{p}) : \nabla \boldsymbol{\varphi} d(x, \mathbf{p}) \quad \text{for all } \boldsymbol{\varphi} \in C^\infty(\overline{\Omega}; \mathbb{R}^d).$$

Furthermore, if  $m_0 > 0$  we have

$$2 \int_{E_t} \operatorname{div}(\mu \boldsymbol{\eta}) dx = \langle \delta V^t, \boldsymbol{\eta} \rangle \quad (3.6)$$

for all  $\boldsymbol{\eta} \in C_0^1(\Omega; \mathbb{R}^d)$  and almost all  $t \in (0, \infty)$ .

5. Finally, for almost all  $0 < s < t < \infty$

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \rho(c(t)) |\mathbf{v}(t)|^2 dx + \lambda^t (\overline{\Omega}) + \int_s^t \int_{\Omega} (v(c) |D\mathbf{v}|^2 - \mathbf{J} \cdot \nabla \mu) d(x, \tau) \\ \leq \frac{1}{2} \int_{\Omega} \rho(c(s)) |\mathbf{v}(s)|^2 dx + \lambda^s (\overline{\Omega}). \end{aligned} \quad (3.7)$$

We define the free energy density by

$$e_{\varepsilon}(c) = \varepsilon \frac{|\nabla A(c)|^2}{2} + \frac{f(c)}{\varepsilon}.$$

In [3] the existence of global weak solutions is shown for a class of singular free energies. We note that this proof can be easily carried over to the present situation with only minor modifications and even some simplifications since  $f$  is non-singular. Throughout this paper we will use the definition of weak solutions in [3]. By this definition we have

$$\begin{aligned} \mathbf{v}_{\varepsilon} &\in BC_{\omega}([0, \infty); L^2_{\sigma}(\Omega)) \cap L^2((0, \infty); H^1(\Omega)^d), \\ c_{\varepsilon} &\in BC_{\omega}([0, \infty); H^1(\Omega)) \cap L^2_{loc}([0, \infty); H^2(\Omega)), f(c_{\varepsilon}) \in L^2_{loc}([0, \infty); L^2(\Omega)), \\ \mu &\in L^2_{loc}([0, \infty); L^2(\Omega)), \nabla \mu \in L^2([0, \infty); L^2(\Omega)^d), \end{aligned}$$

and

$$\begin{aligned} \int_Q -\rho(c_{\varepsilon}) \mathbf{v}_{\varepsilon} \cdot \partial_t \boldsymbol{\varphi} - \mathbf{v}_{\varepsilon} \otimes (\rho(c_{\varepsilon}) \mathbf{v}_{\varepsilon} + \tilde{\mathbf{J}}_{\varepsilon}) : \nabla \boldsymbol{\varphi} + v(c_{\varepsilon}) D\mathbf{v}_{\varepsilon} : D\boldsymbol{\varphi} d(x, t) \\ - \int_Q \rho(c_{0,\varepsilon}) \mathbf{v}_{0,\varepsilon} \cdot \boldsymbol{\varphi}|_{t=0} dx = \int_Q \varepsilon a(c_{\varepsilon}) \nabla c_{\varepsilon} \otimes \nabla c_{\varepsilon} : \nabla \boldsymbol{\varphi} d(x, t) \end{aligned} \quad (3.8)$$

for  $\mathbf{J}_{\varepsilon} := -m_{\varepsilon}(c_{\varepsilon}) \nabla \mu_{\varepsilon}$ ,  $\tilde{\mathbf{J}}_{\varepsilon} := \frac{\partial \rho}{\partial c}(c_{\varepsilon}) \mathbf{J}_{\varepsilon}$ , and all  $\boldsymbol{\varphi} \in C_0^{\infty}([0, \infty); C_{0,\sigma}^{\infty}(\Omega))$ , as well as

$$\int_Q c_{\varepsilon} (\partial_t \psi + \operatorname{div}(\psi \mathbf{v}_{\varepsilon})) d(x, t) + \int_{\Omega} c_{0,\varepsilon} \psi|_{t=0} dx = \int_Q m_{\varepsilon}(c_{\varepsilon}) \nabla \mu_{\varepsilon} \cdot \nabla \psi d(x, t) \quad (3.9)$$

for all  $\psi \in C_0^{\infty}([0, \infty) \times \overline{\Omega})$ , and

$$\mu_{\varepsilon} = \frac{f'(c_{\varepsilon})}{\varepsilon} + \varepsilon a'(c_{\varepsilon}) \frac{|\nabla c_{\varepsilon}|^2}{2} - \varepsilon \operatorname{div}(a(c_{\varepsilon}) \nabla c_{\varepsilon}) \quad \text{a.e. in } Q, \quad (3.10)$$

$$\mathbf{n}_{\partial\Omega} \cdot \nabla c_{\varepsilon} = 0 \quad \text{a.e. on } (0, \infty) \times \partial\Omega. \quad (3.11)$$

Moreover, we have

$$\begin{aligned} \int_{\Omega} \frac{\rho(c_{\varepsilon}(t)) |\mathbf{v}_{\varepsilon}(t)|^2}{2} dx + \mathcal{E}_{\varepsilon}(c_{\varepsilon}(t)) + \int_s^t \int_{\Omega} v(c_{\varepsilon}) |D\mathbf{v}_{\varepsilon}|^2 dx d\tau \\ + \int_s^t \int_{\Omega} m_{\varepsilon}(c_{\varepsilon}) |\nabla \mu_{\varepsilon}|^2 d(x, t) \leq \int_{\Omega} \frac{\rho(c_{\varepsilon}(s)) |\mathbf{v}_{\varepsilon}(s)|^2}{2} dx + \mathcal{E}_{\varepsilon}(c_{\varepsilon}(s)) \end{aligned} \quad (3.12)$$

for all  $t \geq s$  and almost every  $s \geq 0$  including  $s = 0$ .

**THEOREM 3.3** For all  $\varepsilon \in (0, 1]$ , let initial data  $(\mathbf{v}_{0,\varepsilon}, c_{0,\varepsilon}) \in L^2_\sigma(\Omega) \times H^1(\Omega)$  be given such that  $\frac{1}{|\Omega|} \int_\Omega c_{0,\varepsilon} dx = \bar{c} \in (-1, 1)$  and

$$\int_\Omega \frac{|\mathbf{v}_{0,\varepsilon}|^2}{2} dx + \mathcal{E}_\varepsilon(c_{0,\varepsilon}) \leq R \quad (3.13)$$

for some  $R > 0$ . Furthermore, let  $(\mathbf{v}_\varepsilon, c_\varepsilon, \mu_\varepsilon)$  be weak solutions of (1.1)–(1.7) in the interval  $[0, \infty)$ . Then there exists a sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$ , converging to 0 as  $k \rightarrow \infty$ , such that the following assertions are true.

1. There are  $\mathbf{v} \in L^2((0, \infty); H^1(\Omega)^d) \cap L^\infty((0, \infty); L^2_\sigma(\Omega)^d)$ ,  $\mathbf{v}_0 \in L^2_\sigma(\Omega)$  such that, as  $k \rightarrow \infty$ ,

$$\mathbf{v}_{\varepsilon_k} \rightharpoonup \mathbf{v} \quad \text{in } L^2((0, \infty); H^1(\Omega)^d), \quad (3.14)$$

$$\mathbf{v}_{\varepsilon_k} \rightarrow \mathbf{v} \quad \text{in } L^2_{loc}([0, \infty); L^2_\sigma(\Omega)), \quad (3.15)$$

$$\mathbf{v}_{0,\varepsilon_k} \rightharpoonup \mathbf{v}_0 \quad \text{in } L^2_\sigma(\Omega). \quad (3.16)$$

If  $m_0 > 0$ , there exists a  $\mu \in L^2_{loc}([0, \infty); H^1(\Omega))$  with  $\nabla \mu \in L^2((0, \infty); L^2(\Omega)^d)$  and such that

$$\mu_{\varepsilon_k} \rightharpoonup \mu \quad \text{in } L^2_{loc}([0, \infty); H^1(\Omega)). \quad (3.17)$$

2. There are measurable sets  $E \subset \Omega \times [0, \infty)$  and  $E_0 \subset \Omega$  such that, as  $k \rightarrow \infty$ ,

$$c_{\varepsilon_k} \rightarrow -1 + 2\chi_E \quad \text{a.e. in } \Omega \times (0, \infty) \text{ and in } C^{\frac{1}{2}}_{loc}([0, \infty); L^2(\Omega)) \quad (3.18)$$

$$c_{0,\varepsilon_k} \rightarrow -1 + 2\chi_{E_0} \quad \text{a.e. in } \Omega. \quad (3.19)$$

In particular, we have  $\chi_E|_{t=0} = \chi_{E_0}$  in  $L^2(\Omega)$ .

3. There exist Radon measures  $\lambda$  and  $\lambda_{ij}$ ,  $i, j = 1, \dots, d$  on  $\overline{\Omega} \times [0, \infty)$  such that for every  $T > 0$ ,  $i, j = 1, \dots, d$ , as  $k \rightarrow \infty$ ,

$$e_{\varepsilon_k}(c_{\varepsilon_k}) dx dt \rightharpoonup^* \lambda \quad \text{in } \mathfrak{M}(\overline{\Omega} \times [0, T]), \quad (3.20)$$

$$\varepsilon_k a(c_{\varepsilon_k}) \partial_{x_i} c_{\varepsilon_k} \partial_{x_j} c_{\varepsilon_k} dx dt \rightharpoonup^* \lambda_{ij} \quad \text{in } \mathfrak{M}(\overline{\Omega} \times [0, T]). \quad (3.21)$$

4. There exists a Radon measure  $V = V^t dt$  on  $\overline{\Omega} \times G_{d-1} \times (0, \infty)$  such that  $(\mathbf{v}, E, \mu, V)$  if  $m_0 > 0$  and  $(\mathbf{v}, E, V)$  else is a varifold solution of (1.8)–(1.17) in the sense of Definition 3.2 with initial values  $(\mathbf{v}_0, E_0)$  and  $\sigma = \int_{-1}^1 \sqrt{f(s)/2} ds$ . Furthermore,

$$\int_0^T \langle \delta V^t, \eta \rangle dt = \int_0^T \int_\Omega \nabla \eta : \left( d\lambda I - (d\lambda_{ij})_{i,j=1}^d \right) dt \quad (3.22)$$

for all  $\eta \in C^1_0(\Omega \times [0, T]; \mathbb{R}^d)$ .

5. If  $\mathbf{v}_{0,\varepsilon_k} \rightharpoonup \mathbf{v}_0$  in  $L^2_\sigma(\Omega)$  and  $\mathcal{E}_\varepsilon(c_{0,\varepsilon}) \rightarrow 2\sigma |\nabla \chi_{E_0}|(\Omega)$  as  $k \rightarrow \infty$ , then (3.7) holds for almost all  $t \in (0, \infty)$ ,  $s = 0$ , and  $\lambda^0(\overline{\Omega})$  replaced by  $2\sigma |\nabla \chi_{E_0}|(\Omega)$ .

By (3.12) and the assumptions on the initial data we obtain

$$\int_\Omega \frac{\rho(c_\varepsilon(t)) |\mathbf{v}_\varepsilon(t)|^2}{2} dx + \mathcal{E}_\varepsilon(c_\varepsilon(t)) + \int_0^t \int_\Omega v(c_\varepsilon) |D\mathbf{v}_\varepsilon|^2 + m_\varepsilon(c_\varepsilon) |\nabla \mu_\varepsilon|^2 dx dt \leq R \quad (3.23)$$



for all  $t \geq 0$ .

From this estimate, Korn's inequality, and (3.13) we deduce that there exists a sequence  $\varepsilon_k \searrow 0$  as  $k \rightarrow \infty$  such that (3.14), (3.16), (3.17), (3.20), and (3.21) hold. Using the assumptions on  $f$ , we further deduce that

$$\int_{\Omega} |c_{\varepsilon}(t)|^p dx \leq C(1 + R), \tag{3.24}$$

$$\int_{\Omega} (|c_{\varepsilon}(t)| - 1)^2 dx \leq C\varepsilon R \tag{3.25}$$

for all  $t \geq 0$ . In particular, for (3.25) we used that  $f(c) \geq C(|c| - 1)^2$  for all  $c \in \mathbb{R}$  and some constant  $C > 0$  which follows from the positivity of  $f''(\pm 1)$  and the  $p$ -growth of  $f$  for large  $c$ . With the definitions (cf. [10])

$$W(c) = \int_{-1}^c \sqrt{2\tilde{f}(s)} ds, \text{ where } \tilde{f}(s) = \min(f(s), 1 + |s|^2),$$

and

$$w_{\varepsilon}(x, t) = W(c_{\varepsilon}(x, t)),$$

the functions  $w_{\varepsilon}$  are uniformly bounded in  $L^{\infty}((0, \infty); BV(\Omega))$  since

$$\int_{\Omega} |\nabla w_{\varepsilon}(x, t)| dx = \int_{\Omega} \sqrt{2\tilde{f}(c_{\varepsilon}(x, t))} |\nabla c_{\varepsilon}(x, t)| dx \leq \int_{\Omega} e_{\varepsilon}(c_{\varepsilon}(x, t)) dx \leq R. \tag{3.26}$$

Moreover, note that by the assumptions on  $f$ , there exist constants  $C_0, C_1 > 0$  such that for all  $c_0, c_1 \in \mathbb{R}$

$$C_0|c_0 - c_1|^2 \leq |W(c_0) - W(c_1)| \leq C_1|c_0 - c_1|(1 + |c_0| + |c_1|). \tag{3.27}$$

Here, for the first inequality we used again that  $f(s) \geq C(|s| - 1)^2$  for all  $s \in \mathbb{R}$ .

LEMMA 3.4 There exists a constant  $C > 0$  such that

$$\|w_{\varepsilon}\|_{C^{\frac{1}{8}}([0, \infty); L^1(\Omega))} + \|c_{\varepsilon}\|_{C^{\frac{1}{8}}([0, \infty); L^2(\Omega))} \leq C.$$

*Proof.* The proof is a modification of [10, Proof of Lemma 3.2]. Therefore, we only give a brief presentation. For sufficiently small  $\eta > 0$ ,  $x \in \Omega$ , and  $t \geq 0$  let

$$c_{\varepsilon}^{\eta}(x, t) = \int_{B_1} \omega(y) c_{\varepsilon}(x - \eta y, t) dy,$$

where  $\omega$  is a standard mollifying kernel and  $c_{\varepsilon}$  is extended to a small neighborhood of  $\Omega$  as in [10, Proof of Lemma 3.2]. Then, there exist constants  $C, C' > 0$  such that

$$\|\nabla c_{\varepsilon}^{\eta}(t)\|_{L^2(\Omega)} \leq C\eta^{-1} \|c_{\varepsilon}(t)\|_{L^2(\Omega)} \leq C'\eta^{-1} \tag{3.28}$$

$$\|c_{\varepsilon}^{\eta}(t) - c_{\varepsilon}(t)\|_{L^2(\Omega)}^2 \leq C\eta \|\nabla w_{\varepsilon}(t)\|_{L^1(\Omega)} \leq C'\eta \tag{3.29}$$

for all sufficiently small  $\eta > 0$ , cf. [10, Proof of Lemma 3.2]. From (3.28) and (3.9) we deduce that for all  $0 \leq \tau < t < \infty$  such that  $|t - \tau| \leq 1$

$$\begin{aligned} & \int_{\Omega} (c_{\varepsilon}(x, t) - c_{\varepsilon}(x, \tau))(c_{\varepsilon}^{\eta}(x, t) - c_{\varepsilon}^{\eta}(x, \tau)) dx \\ &= - \int_{\tau}^t \int_{\Omega} (m_{\varepsilon}(c_{\varepsilon}(x, s)) \nabla \mu_{\varepsilon}(x, s) - \mathbf{v}_{\varepsilon}(x, s) c_{\varepsilon}(x, s)) \cdot (\nabla c_{\varepsilon}^{\eta}(x, t) - \nabla c_{\varepsilon}^{\eta}(x, \tau)) d(x, s) \\ &\leq C(R)(t - \tau)^{\frac{1}{2}} \sup_{s \in (\tau, t)} \|\nabla c_{\varepsilon}^{\eta}(s)\|_{L^2(\Omega)} \leq C(R)\eta^{-1}(t - \tau)^{\frac{1}{2}}. \end{aligned} \quad (3.30)$$

Here, we used the fact that for all  $\tau, t$  as above we have

$$\|m_{\varepsilon}(c_{\varepsilon}) \nabla \mu_{\varepsilon} - \mathbf{v}_{\varepsilon} c_{\varepsilon}\|_{L^2(\Omega \times (\tau, t))} \leq C(R),$$

since the sequences  $(\mathbf{v}_{\varepsilon}) \subset L^2((0, \infty); L^6(\Omega))$  and  $(c_{\varepsilon}) \subset L^{\infty}((0, \infty); L^3(\Omega))$  are bounded due to the assumptions  $d \leq 3$  and  $p \geq 3$ . Now, combining (3.30), (3.29) and using Hölder's and Young's inequality we conclude that for  $\eta, \tau$ , and  $t$  as above we have

$$\|c_{\varepsilon}(t) - c_{\varepsilon}(\tau)\|_{L^2(\Omega)}^2 \leq C(\eta + \eta^{-1}|t - \tau|^{\frac{1}{2}}).$$

Choosing  $\eta = (t - \tau)^{\frac{1}{4}}$  for sufficiently small  $t - \tau$  we conclude the claim concerning  $c_{\varepsilon}$ . Using (3.27), one derives the claim concerning  $w_{\varepsilon}$  as in [10].  $\square$

**REMARK 3.5** It is possible to understand the proof of Lemma 3.4 from a more general point of view. From (3.9) and (3.23) we easily deduce that the distributional time-derivative of  $(c_{\varepsilon})$  is uniformly bounded in  $L^2((0, \infty), H^{-1}(\Omega))$ . In particular,  $(c_{\varepsilon})$  is uniformly bounded in  $C^{1/2}([0, \infty), H^{-1}(\Omega))$ . On the other hand, the computations leading to (3.29) show that  $(c_{\varepsilon})$  is uniformly bounded in  $L^{\infty}((0, \infty); B_{2\infty}^{1/3}(\Omega))$ . This follows from  $B_{2\infty}^{1/3}(\Omega) = (L^2(\Omega), H^1(\Omega))_{1/3, \infty}$  and the definition of the real interpolation spaces with the aid of the  $K$ -method. By interpolation, we obtain uniform boundedness in  $C^{1/8}([0, \infty), L^2(\Omega))$ .

The proof of the following lemma is literally the same as the proof of [10, Lemma 3.3].

**LEMMA 3.6** There exists a sub-sequence (again denoted by  $\varepsilon_k$ ) and a measurable set  $E \subset \Omega \times [0, \infty)$  such that, as  $k \rightarrow \infty$ ,

$$\begin{aligned} w_{\varepsilon_k} &\rightarrow 2\sigma\chi_E & \text{a.e. in } \Omega \times (0, \infty) \text{ and in } C_{loc}^{\frac{1}{2}}([0, \infty); L^1(\Omega)) \\ c_{\varepsilon_k} &\rightarrow -1 + 2\chi_E & \text{a.e. in } \Omega \times (0, \infty) \text{ and in } C_{loc}^{\frac{1}{2}}([0, \infty); L^2(\Omega)) \end{aligned}$$

Moreover,  $\chi_E \in L_{\omega^*}^{\infty}((0, \infty); BV(\Omega)) \cap C^{\frac{1}{4}}([0, \infty); L^1(\Omega))$  and for all  $t \geq 0$  we have  $|E_t| = |E_0| = \frac{1+\bar{c}}{2}|\Omega|$ .

**LEMMA 3.7** There exist constants  $C, \varepsilon_0 > 0$  such that

$$\|\mu_{\varepsilon}(t)\|_{H^1(\Omega)} \leq C(\mathfrak{E}_{\varepsilon}(c_{\varepsilon}(t)) + \|\nabla \mu_{\varepsilon}(t)\|_{L^2(\Omega)}) \quad (3.31)$$

for almost all  $t > 0$  and  $0 < \varepsilon \leq \varepsilon_0$ . Using  $m_{\varepsilon} \geq \bar{m}_{\varepsilon}$  we deduce from (3.31) and (3.23) that

$$\bar{m}_{\varepsilon} \int_0^T \|\mu_{\varepsilon}(t)\|_{L^2(\Omega)}^2 dt \leq C(R, T) \quad \text{for all } 0 < T < \infty. \quad (3.32)$$

*Proof.* Let us suppress the time variable. Due to Poincaré's inequality it suffices to control the average of  $\mu_\varepsilon$ . Equation (3.10) can be written in the form

$$\mu_\varepsilon = \frac{f'(c_\varepsilon)}{\varepsilon} - \varepsilon \sqrt{a(c_\varepsilon)} \Delta A(c_\varepsilon). \quad (3.33)$$

Multiplying by  $\boldsymbol{\eta} \cdot \nabla c_\varepsilon$  for  $\boldsymbol{\eta} \in C^1(\overline{\Omega}; \mathbb{R}^d)$ , integrating over  $\Omega$ , and integrating by parts yields

$$\int_{\Omega} \boldsymbol{\eta} \cdot \nabla c_\varepsilon \mu_\varepsilon \, dx = - \int_{\Omega} \nabla \boldsymbol{\eta} : (e_\varepsilon(c_\varepsilon) I - \varepsilon \nabla A(c_\varepsilon) \otimes \nabla A(c_\varepsilon)) \, dx + \int_{\partial\Omega} e_\varepsilon(c_\varepsilon) \boldsymbol{\eta} \cdot \mathbf{n}_{\partial\Omega} \, d\mathcal{H}^{d-1}. \quad (3.34)$$

Now we can proceed exactly as in the proof of [10, Lemma 3.4].  $\square$

LEMMA 3.8 There exists a sub-sequence (again denoted by  $\varepsilon_k$ ) such that, as  $k \rightarrow \infty$ ,

$$\begin{aligned} \mathbf{v}_{\varepsilon_k} &\rightarrow \mathbf{v} && \text{in } L^2_{loc}([0, \infty); L^2_\sigma(\Omega)) \\ \mathbf{v}_{\varepsilon_k}(t) &\rightarrow \mathbf{v}(t) && \text{in } L^2(\Omega) \text{ for almost every } t > 0. \end{aligned}$$

Furthermore, there exists a measurable, non-increasing function  $\mathfrak{E}(t)$ ,  $t > 0$ , such that for almost all  $t > 0$

$$\mathfrak{E}_{\varepsilon_k}(c_{\varepsilon_k}(t)) \rightarrow \mathfrak{E}(t) \quad \text{and} \quad |\nabla \chi_{E_t}|(\Omega) \leq \frac{1}{2\sigma} \mathfrak{E}(t) \leq \frac{1}{2\sigma} R. \quad (3.35)$$

*Proof.* Let us fix some  $T > 0$  and let  $P_\sigma : L^2(\Omega)^d \rightarrow L^2_\sigma(\Omega)$  denote the Helmholtz projection. In order to prove the claim concerning  $\mathbf{v}_{\varepsilon_k}$  it suffices to show that for a sub-sequence we have

$$P_\sigma(\rho(c_{\varepsilon_k}) \mathbf{v}_{\varepsilon_k}) \xrightarrow{k \rightarrow \infty} P_\sigma(\rho(c) \mathbf{v}) \quad \text{in} \quad L^2\left((0, T); (L^2_\sigma(\Omega) \cap H^1(\Omega)^d)'\right) \quad (3.36)$$

since then

$$\begin{aligned} \int_0^T \int_{\Omega} \rho(c_{\varepsilon_k}) |\mathbf{v}_{\varepsilon_k}|^2 \, dx \, dt &= \int_0^T \int_{\Omega} P_\sigma(\rho(c_{\varepsilon_k}) \mathbf{v}_{\varepsilon_k}) \cdot \mathbf{v}_{\varepsilon_k} \, dx \, dt \\ &\xrightarrow{k \rightarrow \infty} \int_0^T \int_{\Omega} P_\sigma(\rho(c) \mathbf{v}) \cdot \mathbf{v} \, dx \, dt = \int_0^T \int_{\Omega} \rho(c) |\mathbf{v}|^2 \, dx \, dt, \end{aligned}$$

and from this convergence, the strong convergence of  $(c_{\varepsilon_k})$ , and the strict positivity of  $\rho$  we easily deduce the claim, cf. [3]. But (3.36) follows from the Aubin-Lions lemma by noting that, firstly,

$$L^2_\sigma(\Omega) \hookrightarrow L^2_\sigma(\Omega) \cap H^1(\Omega)^d \hookrightarrow (L^2_\sigma(\Omega) \cap W^{1,\infty}(\Omega))'$$

and that, secondly, the distributional time-derivative of  $(P_\sigma(\rho(c_{\varepsilon_k}) \mathbf{v}_{\varepsilon_k}))$  is uniformly bounded in  $L^{8/7}((0, T); (L^2_\sigma(\Omega) \cap W^{1,\infty}(\Omega))')$ . This last bound follows by estimating each term in (3.8). We have (abbreviating  $L^p((0, T); L^q(\Omega))$  by  $L^p L^q$ )

$$\begin{aligned} \|\rho(c_\varepsilon) \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon\|_{L^2 L^{3/2}} &\leq \|\rho(c_\varepsilon) \mathbf{v}_\varepsilon\|_{L^\infty L^2} \|\mathbf{v}_\varepsilon\|_{L^2 L^6}, \\ \|\mathbf{v}_\varepsilon \otimes \tilde{\mathbf{J}}_\varepsilon\|_{L^{8/7} L^{4/3}} &\leq \|\mathbf{v}_\varepsilon \otimes \tilde{\mathbf{J}}_\varepsilon\|_{L^1 L^{3/2}}^{3/4} \|\mathbf{v}_\varepsilon \otimes \tilde{\mathbf{J}}_\varepsilon\|_{L^2 L^1}^{1/4} \\ &\leq C \|\mathbf{v}_\varepsilon\|_{L^2 L^6}^{3/4} \|\mathbf{v}_\varepsilon\|_{L^\infty L^2}^{1/4} \|m(c_\varepsilon) |\nabla \mu_\varepsilon|^2\|_{L^1 L^1}, \\ \|\nu(c_\varepsilon) D \mathbf{v}_\varepsilon\|_{L^2 L^2} &\leq C \|D \mathbf{v}_\varepsilon\|_{L^2 L^2}, \\ \|\varepsilon a(c_\varepsilon) \nabla c_\varepsilon \otimes \nabla c_\varepsilon\|_{L^\infty L^1} &\leq C \|\varepsilon |\nabla A(c_\varepsilon)|^2\|_{L^\infty L^1}. \end{aligned}$$

Concerning the remaining claims we note that the total energies

$$\mathfrak{E}_\varepsilon^{tot}(t) := \frac{1}{2} \|v_\varepsilon(t)\|_{L^2(\Omega)}^2 + \mathfrak{E}_\varepsilon(c_\varepsilon(t)), \quad t \geq 0,$$

form a sequence of bounded, non-increasing functions and that  $\mathbf{v}_{\varepsilon_k}(t) \rightarrow_{k \rightarrow \infty} \mathbf{v}(t)$  for almost all  $t > 0$  in  $L^2(\Omega)$ . Now, we can proceed exactly as in the proof of [10, Lemma 3.3].  $\square$

Finally, we define the discrepancy function by

$$\xi^\varepsilon(c_\varepsilon) := \frac{\varepsilon}{2} |\nabla A(c_\varepsilon)|^2 - \frac{1}{\varepsilon} f(c_\varepsilon).$$

**THEOREM 3.9** For all sufficiently small  $\eta > 0$  there exists a constant  $C(\eta)$  such that for all sufficiently small  $\varepsilon > 0$  (the maximal  $\varepsilon$  may depend on  $\eta$ ) we have

$$\int_0^T \int_\Omega (\xi^\varepsilon(c_\varepsilon))^+ d(x, t) \leq \eta \int_0^T \int_\Omega e_\varepsilon(c_\varepsilon) d(x, t) + \varepsilon C(\eta) \int_0^T \int_\Omega |\mu_\varepsilon|^2 d(x, t).$$

Combining this estimate with the assumption  $\varepsilon/\overline{m}_\varepsilon \rightarrow_{\varepsilon \rightarrow 0} 0$  and (3.32) we deduce that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega (\xi^\varepsilon(c_\varepsilon))^+ d(x, t) = 0 \quad \text{for all } 0 < T < \infty.$$

*Proof.* The proof is based on the elliptic equation (3.33) which can be written in the form

$$\mu_\varepsilon a(A(c_\varepsilon))^{-1/2} = \frac{(f \circ A^{-1})'(A(c_\varepsilon))}{\varepsilon} - \varepsilon \Delta A(c_\varepsilon).$$

Let  $c_\pm := A(\pm 1)$ ,  $B(c) := c \frac{c_+ - c_-}{2} + \frac{c_+ + c_-}{2}$ , and  $\tilde{f}(c) := f(A^{-1}(B(c)))/(B')^2$  for  $c \in \mathbb{R}$ . Then  $\tilde{f}$  fulfills Assumption 3.1, and for  $\tilde{c}_\varepsilon := B^{-1}(A(c_\varepsilon))$  we have

$$\mu_\varepsilon a(A(c_\varepsilon))^{-1/2} (B')^{-1} = \frac{\tilde{f}'(\tilde{c}_\varepsilon)}{\varepsilon} - \varepsilon \Delta \tilde{c}_\varepsilon.$$

Since the function  $a(A(c_\varepsilon))^{-1/2} (B')^{-1}$  is uniformly bounded, [10, Theorem 3.6] yields

$$\int_0^T \int_\Omega (\tilde{\xi}^\varepsilon(c_\varepsilon))^+ d(x, t) \leq \eta \int_0^T \int_\Omega \tilde{e}_\varepsilon(c_\varepsilon) d(x, t) + \varepsilon C(\eta) \int_0^T \int_\Omega |\mu_\varepsilon|^2 d(x, t) \quad (3.37)$$

where

$$\begin{aligned} \tilde{\xi}^\varepsilon(\tilde{c}_\varepsilon) &:= \frac{\varepsilon}{2} |\nabla \tilde{c}_\varepsilon|^2 - \frac{1}{\varepsilon} \tilde{f}'(\tilde{c}_\varepsilon) = \xi^\varepsilon(c_\varepsilon)/(B')^2 \\ \tilde{e}^\varepsilon(\tilde{c}_\varepsilon) &:= \frac{\varepsilon}{2} |\nabla \tilde{c}_\varepsilon|^2 + \frac{1}{\varepsilon} \tilde{f}(\tilde{c}_\varepsilon) = e^\varepsilon(c_\varepsilon)/(B')^2. \end{aligned}$$

This proves the claim.  $\square$

Using the previous statements, we can now easily finish the proof of Theorem 3.3 by the arguments of [10, Section 3.5]. To be more precise, item 1 follows from (3.13) and Lemmas 3.7 and 3.8. Item 2 follows from Lemma 3.6 and the energy inequality (3.23). Item 3 follows from (3.23) as well. Furthermore, we note that  $\lambda = \lambda^t dt$  for Radon measures  $\lambda^t$  on  $\overline{\Omega}$  since

$$\lambda(A \times I) \leq \lambda(\overline{\Omega} \times I) = \lim_{k \rightarrow \infty} \int_I \mathcal{E}_{\varepsilon_k}(\tau) d\tau \leq |I|R$$

for any measurable  $A \subseteq \overline{\Omega}, I \subseteq [0, \infty)$ . Similarly, we also get  $\lambda^t(\overline{\Omega}) = \mathcal{E}(t)$  for almost all  $t \in (0, \infty)$  due to (3.35). From (3.12) we deduce that

$$\begin{aligned} \lambda^t(\overline{\Omega}) &= \lim_{k \rightarrow \infty} \mathcal{E}_{\varepsilon_k}(c_{\varepsilon_k}(t)) \\ &\leq -\liminf_{k \rightarrow \infty} \int_s^t \int_{\Omega} (v(c_{\varepsilon_k})|D\mathbf{v}_{\varepsilon_k}|^2 + m_{\varepsilon_k}(c_{\varepsilon_k})|\nabla\mu_{\varepsilon_k}|^2) d(x, \tau) \\ &\quad + \lim_{k \rightarrow \infty} \left( \mathcal{E}_{\varepsilon_k}(c_{\varepsilon_k}(s)) + \frac{1}{2} \int_{\Omega} \rho(c_{\varepsilon_k}(s))|\mathbf{v}_{\varepsilon_k}(s)|^2 dx - \frac{1}{2} \int_{\Omega} \rho(c_{\varepsilon_k}(t))|\mathbf{v}_{\varepsilon_k}(t)|^2 dx \right) \\ &\leq -\int_s^t \int_{\Omega} (v(c)|D\mathbf{v}|^2 dx - \mathbf{J} \cdot \nabla\mu) d(x, \tau) + \lambda^s(\overline{\Omega}) \\ &\quad + \frac{1}{2} \int_{\Omega} \rho(c(s))|\mathbf{v}(s)|^2 dx - \frac{1}{2} \int_{\Omega} \rho(c(t))|\mathbf{v}(t)|^2 dx \end{aligned}$$

for almost all  $0 < s < t < \infty$  where  $c := -1 + 2\chi_E$ . This is (3.7). Item 5 follows similarly. We can proceed as in [10, Section 3.5] to construct the varifold  $V$ . Therefore, we only give a sketch. We deduce from Theorem 3.9 that for all  $\eta_0, \eta_1 \in C^1(\overline{\Omega}; \mathbb{R}^d)$  and all  $0 < T < \infty$

$$\int_0^T \int_{\overline{\Omega}} \eta_0 \otimes \eta_1 : (d\lambda_{ij}) \leq \int_0^T \int_{\overline{\Omega}} |\eta_0| |\eta_1| d\lambda.$$

This proves the existence of  $\lambda$ -measurable  $\mathbb{R}$ -valued, non-negative functions  $\gamma_i$  and  $\lambda$ -measurable unit vector fields  $\mathbf{v}_i, i = 1, \dots, d$ , such that

$$(\lambda_{ij}) = \sum_{i=1}^d \gamma_i \mathbf{v}_i \otimes \mathbf{v}_i \lambda \quad \text{and} \quad \sum_{i=1}^d \gamma_i \leq 1, \quad \sum_{i=1}^d \mathbf{v}_i \otimes \mathbf{v}_i = I \quad \lambda\text{-a.e.}$$

We denote the equivalence class of  $\mathbf{v}_i(x, t)$  in  $G_{d-1}$  by  $\mathbf{p}_i^t(x)$ , define the functions  $b_i^t$  by

$$b_i^t(x) := \gamma_i(x, t) + \frac{1}{d-1} \left( 1 - \sum_{i=1}^d \gamma_i(x, t) \right)$$

and define the varifold  $V$  as in (3.3). Then item 3 in Definition 3.2 follows taking into account (3.35). Furthermore, in the case  $m_0 > 0$  we infer from (3.34) that

$$\begin{aligned} \int_{\Omega} 2\chi_{E_t} \operatorname{div}(\mu\eta) dx &= \int_{\Omega} \nabla\eta : (d\lambda I - (d\lambda_{ij})_{i,j=1}^d) = \int_{\Omega} \nabla\eta : \sum_{i=1}^d b_i^t (I - \mathbf{p}_i^t \otimes \mathbf{p}_i^t) d\lambda \\ &= \langle \delta V^t, \eta \rangle \end{aligned}$$

for all  $\eta \in C_0^1(\overline{\Omega}; \mathbb{R}^d)$  and almost all  $t \in (0, \infty)$ . This is (3.6). Furthermore, these calculations prove (3.22). Similarly, (3.4) and (3.5) follow from (3.8) and (3.9), respectively, where one uses that

$$\int_Q \varepsilon a(c_\varepsilon) \nabla c_\varepsilon \otimes \nabla c_\varepsilon : \nabla \varphi \, d(x, t) = \int_Q \varphi \cdot \nabla c_\varepsilon \mu_\varepsilon \, d(x, t) \xrightarrow{\varepsilon \rightarrow 0} \langle \delta V^t, \varphi \rangle$$

for all  $\varphi \in C^\infty([0, \infty); C_0^\infty(\Omega))$ . This proves item 4 in Definition 3.2. Finally, Item 2 in Definition 3.2 follows from Lemma 3.6. This concludes the proof of Theorem 3.3.

In the radially symmetric case we can prove a stronger statement concerning the discrepancy measure.

**THEOREM 3.10** Let  $\Omega = B_1(0)$ , and assume that the solutions  $(\mathbf{v}_\varepsilon, c_\varepsilon, \mu_\varepsilon)$  are radially symmetric. Assume, furthermore, that  $A(c) = c$  for all  $c \in \mathbb{R}$ , and that the constants  $\overline{m}_\varepsilon$  in the Assumptions 3.1 satisfy

$$\varepsilon^{\frac{1}{d-1}} / \overline{m}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0. \tag{3.38}$$

Then, for all  $T > 0$ , we have

$$\lim_{\varepsilon \searrow 0} \int_0^T \int_\Omega |\xi^\varepsilon(c_\varepsilon)| \, d(x, t) = 0.$$

For the proof we need the following result from [10, Lemma 4.4].

**LEMMA 3.11** There exist positive constants  $C_0$  and  $\eta_0$  such that for every  $\eta \in [0, \eta_0]$ ,  $\varepsilon \in (0, 1]$ , and every  $(u^\varepsilon, v^\varepsilon) \in H^2(\Omega) \times L^2(\Omega)$  such that

$$v^\varepsilon = -\varepsilon \Delta u^\varepsilon + \varepsilon^{-1} f'(u^\varepsilon), \quad \mathbf{n}_{\partial\Omega} \cdot \nabla u^\varepsilon|_{\partial\Omega} = 0$$

we have

$$\int_{\{x \in \Omega : u^\varepsilon|_{x \geq 1-\eta}\}} \left( e^\varepsilon(u^\varepsilon) + \varepsilon^{-1} (f'(u^\varepsilon))^2 \right) \leq C_0 \eta \int_{\{x \in \Omega : |u^\varepsilon| \leq 1-\eta\}} \varepsilon |\nabla u^\varepsilon|^2 \, dx + C_0 \varepsilon \int_\Omega |v^\varepsilon|^2 \, dx \tag{3.39}$$

*Proof of Theorem 3.10:* We can show exactly like in [10, Proof of Theorem 5.1] that there exists a constant  $C > 0$  such that for almost all  $t > 0$  we have

$$\int_{B_\delta} e_\varepsilon(c_\varepsilon(t)) \, dx \leq C \delta M^\varepsilon(t) \quad \text{for all } \delta \in (0, 1), \tag{3.40}$$

$$|\xi^\varepsilon(c_\varepsilon(r, t)) + \mu_\varepsilon(r, t) c_\varepsilon(r, t)| \leq C r^{1-d} M^\varepsilon(t) \quad \text{for all } r \in (0, 1). \tag{3.41}$$

Here, we use the notation  $r = |x|$  and

$$M^\varepsilon(t) := 1 + \|\mu_\varepsilon(t)\|_{H^1(\Omega)} + \varepsilon \|\mu_\varepsilon(t)\|_{H^1(\Omega)}^2.$$

From (3.41) we deduce that for small  $\delta, \eta > 0$

$$\begin{aligned} \int_\Omega |\xi^\varepsilon(c_\varepsilon(t))| \, dx &\leq \int_{B_\delta \cup \{|c_\varepsilon(t)| \geq 1-\eta\}} e_\varepsilon(c_\varepsilon(t)) \, dx + \int_{\Omega \cap \{r > \delta, |c_\varepsilon(t)| < 1-\eta\}} |\mu_\varepsilon(t)| (1-\eta) \, dx \\ &\quad + C M^\varepsilon(t) \int_{\Omega \cap \{r > \delta, |c_\varepsilon(t)| < 1-\eta\}} r^{1-d} \, dx. \end{aligned}$$

Using (3.40) and (3.39), the first integral on the right hand side may be estimated by

$$C\delta M^\varepsilon(t) + C'\eta\mathcal{E}_\varepsilon(t) + C'\varepsilon\|\mu_\varepsilon(t)\|_{L^2(\Omega)}^2.$$

By (3.25), the second integral is dominated by

$$\|\mu_\varepsilon(t)\|_{L^2(\Omega)} \left\{ |c_\varepsilon(t)| \geq 1 - \eta \right\}^{1/2} \leq C''(\eta)M^\varepsilon(t)\varepsilon^{1/2}.$$

Finally, using (3.39) again, the third integral is smaller than  $C'(\eta)M^\varepsilon(t)\delta^{1-d}\varepsilon$ . Summing up, we have

$$\int_{\Omega} |\xi^\varepsilon(c_\varepsilon(t))| dx \leq C'\eta\mathcal{E}_\varepsilon(t) + C'\varepsilon\|\mu_\varepsilon(t)\|_{L^2(\Omega)}^2 + C''(\eta)M^\varepsilon(t)(\varepsilon^{1/2} + \delta^{1-d}\varepsilon + \delta).$$

Integrating this estimate from 0 to  $T$  and choosing  $\eta$  small, the first term on the right hand side gets arbitrarily small. Choosing then  $\delta = \varepsilon^{1/(2d-2)}$  and  $\varepsilon$  small the other two terms get arbitrarily small, too. While this is obvious for the second term, concerning the third term we remark that it takes the form

$$C''(\eta) \int_0^T M^\varepsilon(t) dt (\varepsilon^{1/2} + \varepsilon^{1/(2d-2)}) = o(1) \quad \text{as } \varepsilon \rightarrow 0$$

due to (3.38). □

#### 4. Nonconvergence

In this section we show that solutions of (1.1)–(1.4) do not converge in general to solutions of (1.8)–(1.12) if  $m_\varepsilon(c) = \tilde{m}\varepsilon^\alpha$  for some  $\alpha > 3$  or  $m_\varepsilon(c) \equiv 0$ , which corresponds to the case “ $\alpha = \infty$ ”. More precisely, we will determine radially symmetric solutions which converge as  $\varepsilon \rightarrow 0$  to a solution, which does not satisfy (1.11). Moreover, for these solutions the discrepancy measure  $\xi_\varepsilon(c_\varepsilon)$  does not vanish in the limit  $\varepsilon \rightarrow 0$ .

For simplicity of the following presentation we assume that  $v(c) \equiv v$ ,  $\rho(c) \equiv \rho$ . We will construct radially symmetric solutions of the form

$$\mathbf{v}(x, t) = u(r, t)\mathbf{e}_r, \quad p(x, t) = \tilde{p}_\varepsilon(r, t), \quad c(x, t) = \tilde{c}_\varepsilon(r, t), \quad \mu(x, t) = \tilde{\mu}_\varepsilon(r, t), \quad (4.1)$$

where  $r = |x|$ ,  $\mathbf{e}_r = \frac{x}{|x|}$ . If  $(\mathbf{v}, p, c, \mu)$  are of this form, (1.1)–(1.4) reduce to

$$\begin{aligned} \rho\partial_t u + \rho u\partial_r u - v\frac{1}{r^{n-1}}\partial_r(r^{n-1}\partial_r u) \\ + \partial_r \tilde{p}_\varepsilon = -\varepsilon\frac{n-1}{r}|\partial_r \tilde{c}_\varepsilon|^2 - \varepsilon\partial_r|\partial_r \tilde{c}_\varepsilon|^2 \end{aligned} \quad (4.2)$$

$$\partial_r(r^{n-1}u) = 0 \quad (4.3)$$

$$\partial_t \tilde{c}_\varepsilon + u\partial_r \tilde{c}_\varepsilon = m_0\varepsilon^\alpha\frac{1}{r^{n-1}}\partial_r(r^{n-1}\partial_r \tilde{\mu}_\varepsilon) \quad (4.4)$$

$$\tilde{\mu}_\varepsilon = -\varepsilon\frac{1}{r^{n-1}}\partial_r(r^{n-1}\partial_r \tilde{c}_\varepsilon) + \varepsilon^{-1}f'(\tilde{c}_\varepsilon). \quad (4.5)$$

Here we have used

$$\begin{aligned} -\varepsilon \operatorname{div}(\nabla c \otimes \nabla c) &= -\varepsilon \operatorname{div}(|\partial_r \tilde{c}_\varepsilon|^2 \mathbf{e}_r \otimes \mathbf{e}_r) \\ &= -\varepsilon(n-1)\frac{1}{r}|\partial_r \tilde{c}_\varepsilon|^2 \mathbf{e}_r - \varepsilon\partial_r|\partial_r \tilde{c}_\varepsilon|^2 \mathbf{e}_r \end{aligned}$$

since

$$\nabla \mathbf{e}_r = \frac{1}{r}(I - \mathbf{e}_r \otimes \mathbf{e}_r), \quad \operatorname{div} \mathbf{e}_r = \frac{n-1}{r}.$$

We note that because of (4.3)  $u(r, t) \equiv ar^{-n+1}\mathbf{e}_r$  for some  $a \in \mathbb{R}$ , which will be determined by the boundary conditions in the following. Hence we can solve (4.4)–(4.5) together with suitable boundary conditions and  $c|_{t=0} = c_{0,\varepsilon}$  independently and use (4.2) afterwards to determine  $\tilde{p}_\varepsilon$ .

#### 4.1 Nonconvergence in the case $\alpha = \infty$

First we consider the case  $m_0 = 0$  (resp. “ $\alpha = \infty$ ”). In this case we consider (4.2)–(4.5) in the domain  $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$  together with the inflow boundary condition

$$u(1, t) = a \quad \text{for all } t > 0, \quad (4.6)$$

$$\tilde{c}_\varepsilon(1, t) = 1 \quad \text{for all } t > 0 \quad (4.7)$$

for some  $a > 0$  and the initial values

$$(u, \tilde{c})|_{t=0} = \left( \frac{a}{r^{n-1}}, c_{0,\varepsilon} \right).$$

Here (4.3) and (4.6) already determine  $u$  uniquely as

$$u(r, t) = \frac{a}{r^{n-1}} \quad \text{for all } r \geq 1, t > 0. \quad (4.8)$$

Moreover, we choose

$$\tilde{c}_{0,\varepsilon}(r) = \theta \left( \frac{r - r_0}{\varepsilon} \right) \quad \text{for all } r \geq 1 \quad (4.9)$$

for some  $r_0 > 1$ , where

$$\theta \in C^\infty(\mathbb{R}) \quad \text{such that} \quad \theta(s) = \begin{cases} 1 & \text{if } s < -\delta, \\ -1 & \text{if } s > \delta \end{cases} \quad (4.10)$$

and  $\delta \in (0, r_0 - 1)$  and  $\varepsilon \in (0, 1]$ . Hence  $\tilde{c}$  is a solution of the transport equation

$$\begin{aligned} \partial_t \tilde{c}_\varepsilon(r, t) + \frac{a}{r^{n-1}} \partial_r \tilde{c}_\varepsilon(r, t) &= 0 & \text{for } r > 1, t > 0, \\ \tilde{c}_\varepsilon(1, t) &= 1 & \text{for } t > 0, \end{aligned}$$

which can be calculated with the method of characteristics. The solution for the initial condition above is

$$\tilde{c}_\varepsilon^\infty(r, t) := \tilde{c}_\varepsilon(r, t) = \begin{cases} c_{0,\varepsilon}(\sqrt[n]{r^n - ant}) & \text{if } r^n \geq ant, \\ 1 & \text{if } r^n < ant. \end{cases} \quad (4.11)$$

By the construction we have

$$\tilde{c}_\varepsilon(r, t) \xrightarrow{\varepsilon \rightarrow 0} \begin{cases} -1 & \text{if } r > R(t), \\ 1 & \text{if } r < R(t), \end{cases} \quad (4.12)$$

where  $R(t) = \sqrt[n]{r_0^n + nat}$  is the radius of the level set  $\{c_\varepsilon(x, t) = 0\} = \partial B_{R(t)}(0)$ .



In order to determine  $\tilde{p}_\varepsilon$  we use that (4.2) and (4.8) imply

$$\begin{aligned}\partial_r \tilde{p}_\varepsilon &= -\varepsilon \frac{n-1}{r} |\partial_r \tilde{c}_\varepsilon|^2 - \varepsilon \partial_r |\partial_r \tilde{c}_\varepsilon|^2 + \nu \frac{1}{r^{n-1}} \partial_r (r^{n-1} \partial_r u) - \rho u \partial_r u \\ &= -\varepsilon \frac{n-1}{r} |\partial_r \tilde{c}_\varepsilon|^2 - \varepsilon \partial_r |\partial_r \tilde{c}_\varepsilon|^2 + \partial_r \left( \frac{a(n-1)}{2n+2} r^{-2n+2} - \frac{\nu a(n-1)}{n} r^{-n} \right).\end{aligned}\quad (4.13)$$

Now we decompose  $\tilde{p}_\varepsilon = p_{1,\varepsilon} + p_{2,\varepsilon} + p_3$  such that

$$\partial_r p_{1,\varepsilon}(r) = -\varepsilon(n-1) \frac{1}{r} |\partial_r \tilde{c}_\varepsilon(r)|^2, \quad p_{2,\varepsilon}(r) = -\varepsilon |\partial_r \tilde{c}_\varepsilon(r)|^2 \quad \text{for all } r > 1.$$

Hence up to a constant

$$p_3 = \frac{a(n-1)}{2n+2} r^{-2n+2} - \frac{\nu a(n-1)}{n} r^{-n}.$$

Because of the explicit form of  $\tilde{p}_\varepsilon$  and

$$\partial_r \tilde{c}_\varepsilon(r, t) = -\frac{1}{\varepsilon} \theta' \left( \frac{\sqrt[n]{r^n - ant} - r_0}{\varepsilon} \right) r^{n-1} (r^n - ant)^{\frac{1}{n}-1}, \quad (4.14)$$

it is easy to observe that

$$\tilde{p}_\varepsilon(r) \rightarrow_{\varepsilon \rightarrow 0} \tilde{p}_0(r) \quad \text{for all } r \neq R(t)$$

for some smooth  $\tilde{p}_0: (1, M) \setminus \{R(t)\} \rightarrow \mathbb{R}$ . Now we consider

$$[p_{j,\varepsilon}]_{R(t),\delta} := p_j(R(t) + \delta) - p_j(R(t) - \delta) = \int_{R(t)-\delta}^{R(t)+\delta} \partial_r p_j(s, t) ds,$$

which converges as  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow 0$  (in that order) to several contributions of  $[\tilde{p}_0]$  at  $R(t)$ . For  $j = 2$  we have that

$$[p_{2,\varepsilon}]_{R(t),\delta} = -\varepsilon |\partial_r \tilde{c}_\varepsilon(R(t) + \delta)|^2 + \varepsilon |\partial_r \tilde{c}_\varepsilon(R(t) - \delta)|^2 = 0$$

if  $\varepsilon < \delta$ . Hence

$$\lim_{\varepsilon \rightarrow 0} [p_{2,\varepsilon}]_{R(t),\delta} = 0.$$

Moreover, since  $p_3$  is independent of  $\varepsilon$  and continuous, we have

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} [p_3]_{R(t),\delta} = 0.$$

Finally, using (4.14) we obtain

$$\begin{aligned}[\tilde{p}_0]_{R(t),\delta} &= \lim_{\varepsilon \rightarrow 0} [\tilde{p}_\varepsilon]_{R(t),\delta} = \lim_{\varepsilon \rightarrow 0} [p_{1,\varepsilon}]_{R(t),\delta} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{n-1}{\varepsilon} \int_{R(t)-\delta}^{R(t)+\delta} \left| \theta' \left( \frac{\sqrt[n]{r^n - ant} - r_0}{\varepsilon} \right) \right|^2 r^{2n-3} (r^n - ant)^{\frac{2}{n}-2} dr \\ &= \sigma(n-1) r^{2n-3} (r^n - ant)^{\frac{2}{n}-2} \Big|_{r=R(t)} \\ &= \sigma \left( \frac{R(t)}{r_0} \right)^{2n-2} \frac{n-1}{R(t)},\end{aligned}\quad (4.15)$$

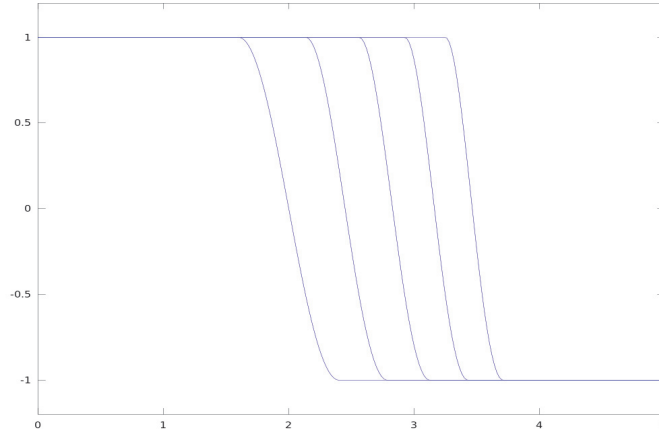


FIG. 1. Plot of  $\tilde{c}_\varepsilon^\infty$  for  $t = 0, 1, 2, 3, 4$  (from left to right) with  $a = 1, \varepsilon = 0.4, r_0 = 2, n = 2$

where  $\sigma := \int_{\mathbb{R}} |\theta'(s)|^2 ds$ . Here  $R(t) > r_0$  for all  $t > 0$  and  $R(t) \rightarrow_{t \rightarrow \infty} \infty$ . The exact solution of the classical sharp interface model, i.e., (1.8)–(1.17) with  $m_0 = 0$  and  $\Omega_0^+ = B_{r_0}(0) \setminus \overline{B_1(0)}$ ,  $\mathbf{v}_0 = ar^{1-n} \frac{x}{|x|}$ , is given by

$$\mathbf{v}(x, t) = ar^{1-n} \frac{x}{|x|}, \quad \Omega^+(t) = B_{R(t)}(0) \setminus \overline{B_1(0)},$$

where  $R(t) = \sqrt[n]{r_0^n + ant}$  as before,  $p: \Omega \times (0, T) \rightarrow \mathbb{R}$  is constant in  $\Omega^\pm(t)$  such that

$$[p](x, t) = \sigma \frac{n-1}{R(t)} \quad \text{on} \quad \partial\Omega^+(t) = \Gamma(t). \tag{4.16}$$

Hence the pressure  $\tilde{p}$  of the limit solution as  $\varepsilon \rightarrow 0$  differs from the solution of the sharp interface (4.16) by a time dependent factor  $\left(\frac{R(t)}{r_0}\right)^{2n-2} > 1$ , which corresponds to an increased surface tension coefficient that even increases strictly in time.

REMARK 4.1 From the explicit solution (4.11) one observes

$$\partial_r \tilde{c}_\varepsilon(R(t), t) = -\frac{1}{\varepsilon} \theta'(0) \left(\frac{R(t)}{r_0}\right)^{n-1}.$$

Hence  $|\partial_r \tilde{c}|$  increases at the diffuse interface “ $r \approx R(t)$ ” as  $t$  increases, cf. Figure 1.

Finally, we determine the limit of the discrepancy measure:

$$\begin{aligned} \int_{\Omega} \xi_{\varepsilon}(c_{\varepsilon}^{\infty}) \varphi \, dx &= \int_1^M \left( \varepsilon \frac{|\partial_r \tilde{c}_{\varepsilon}^{\infty}(r)|^2}{2} - \frac{f(\tilde{c}_{\varepsilon}^{\infty}(r))}{\varepsilon} \right) \tilde{\varphi}(r) r^{n-1} \, dr \\ &= \int_1^M \frac{1}{\varepsilon} \left| \theta' \left( \frac{\sqrt[n]{r^n - ant} - r_0}{\varepsilon} \right) \right|^2 \left( \frac{r^{n-1}}{(r^n - ant)^{1-\frac{1}{n}}} \right) \tilde{\varphi}(r) r^{n-1} \, dr \\ &\quad - \int_1^M \frac{1}{\varepsilon} f \left( \theta \left( \frac{\sqrt[n]{r^n - ant} - r_0}{\varepsilon} \right) \right) \tilde{\varphi}(r) r^{n-1} \, dr \\ &\rightarrow_{\varepsilon \rightarrow 0} (\sigma \kappa(t) - \tilde{\sigma}) \int_{\partial B_{R(t)}} \varphi(x) \, dx \end{aligned}$$

for all  $\varphi \in C_0^{\infty}(\Omega)$  where  $\tilde{\sigma} := \int_{\mathbb{R}} f(\theta(s)) \, ds$ ,  $\sigma = \int_{\mathbb{R}} |\theta'(s)|^2 \, ds$  as before, and

$$\tilde{\varphi}(r) = \int_{\partial B_r(0)} \varphi(x) \, dx \quad \text{for all } r \in (1, M).$$

Hence

$$\xi_{\varepsilon}(c_{\varepsilon}^{\infty}) \rightarrow_{\varepsilon \rightarrow 0} (\sigma \kappa(t) - \tilde{\sigma}) \delta_{\partial B_{R(t)}} \quad \text{in } \mathfrak{D}'(\Omega) \tag{4.17}$$

since  $\kappa(t)$  is strictly increasing in  $t > 0$ , we have  $\sigma \kappa(t) - \tilde{\sigma} \neq 0$  for all  $t > 0$  except possibly one.

#### 4.2 Nonconvergence in the case $3 < \alpha < \infty$

Based on the solution for the extreme case “ $\alpha = \infty$ ” from the previous section, we will prove essentially the same result in the case  $3 < \alpha < \infty$ . In order to avoid technical difficulties with the unboundedness of  $\{x \in \mathbb{R}^n : |x| > 1\}$ , we will consider (4.2)–(4.5) in

$$\Omega_M = \{x \in \mathbb{R}^n : 1 < |x| < M\},$$

where  $M > r_0 > 1$  is arbitrary, together with

$$u_{\varepsilon}(1, t) = a, \quad c_{\varepsilon}(1, t) = 1 \quad \text{for all } t \in (0, T), \tag{4.18}$$

$$u_{\varepsilon}(M, t) = \frac{a}{M^{n-1}}, \quad c_{\varepsilon}(M, t) = -1 \quad \text{for all } t \in (0, T). \tag{4.19}$$

DEFINITION 4.2 (Weak Solutions) Let

$$H_{(0)}^1 = \left\{ u \in C^0([1, M]) : r^{(n-1)/2} \partial_r u \in L^2(1, M), \int_1^M u(r) r^{n-1} \, dr = 0 \right\}$$

be equipped with the inner product

$$(u, v)_{H_{(0)}^1} := \int_1^{\infty} \partial_r u(r) \partial_r v(r) r^{n-1} \, dr$$

for all  $u, v \in H_{(0)}^1$ . We embed  $L_{(0)}^2(1, M) \hookrightarrow H_{(0)}^{-1} := (H_{(0)}^1)'$  by identifying  $u \in L^2(1, M)$  with

$$\langle u, \varphi \rangle_{H_{(0)}^{-1}, H_{(0)}^1} := \int_1^M u(r) \varphi(r) r^{n-1} \, dr \quad \text{for all } \varphi \in H_{(0)}^1.$$

We call  $(\tilde{c}_\varepsilon, \tilde{\mu}_\varepsilon)$  a weak solution of (4.4)–(4.5) together with (4.18), (4.19) and  $\tilde{c}_\varepsilon|_{t=0} = \tilde{c}_{0,\varepsilon}$  if

$$\begin{aligned} \tilde{c}_\varepsilon - \chi &\in C([0, T]; H_0^1(1, M)) \cap L^2(0, T; H^3(1, M)), \\ \partial_t \tilde{c}_\varepsilon &\in L^2(0, T; H_{(0)}^{-1}(1, M)), \quad \tilde{\mu}_\varepsilon \in L^2(0, T; H^1(1, M)), \end{aligned}$$

where  $\chi \in C^\infty([1, M])$  with  $\chi(1) = 1, \chi(M) = -1$  and

$$\langle \partial_t \tilde{c}_\varepsilon(t), \varphi \rangle_{H_{(0)}^{-1}, H_{(0)}^1} + \int_1^M ar^{-n+1} \partial_r \tilde{c}_\varepsilon(r, t) \varphi(r) r^{n-1} dr = -m_0 \varepsilon^\alpha \int_1^M \partial_r \mu_\varepsilon \partial_r \varphi r^{n-1} dr$$

for almost every  $t \in (0, T)$  and for all  $\varphi \in H_{(0)}^1(1, M)$ , (4.5) is satisfied point-wise almost everywhere, and  $c|_{t=0} = c_{0,\varepsilon}$  in  $H^1(1, M)$ .

Existence of weak solutions can be proved by standard methods. E.g. it follows from [7, Theorem 3.1] applied to  $H_1 = H_{(0)}^1, H_0 = H_{(0)}^{-1}$ ,

$$\begin{aligned} \varphi(u) &= \int_1^M \left( \varepsilon \frac{|\partial_r u(r)|^2}{2} + \varepsilon^{-1} f_0(u(r)) \right) r^{n-1} dr \\ \langle \mathcal{B}(v), w \rangle_{H_{(0)}^{-1}, H_{(0)}^1} &= m_0 \varepsilon^{\alpha-1} \beta \int_1^M \partial_r v(r) \partial_r w(r) r^{n-1} dr - \int_1^M a \partial_r v(r) w(r) dr \end{aligned}$$

for all  $v, w \in H_1$  and  $u \in \text{dom}(\varphi) := \{v \in H^1(1, M) : \frac{1}{M-1} \int_1^M v(r) r^{n-1} dr = m\}$ , where  $m := \int_1^M \tilde{c}_{0,\varepsilon}(r) r^{n-1} dr, f_0(s) := f(s) - \frac{\beta}{2} s^2$  for all  $s \in \mathbb{R}$  and  $\beta := \inf_{s \in \mathbb{R}} f''(s)$ . Then  $f_0$  and  $\varphi$  are convex and the subgradient  $\mathcal{Q} = \partial\varphi$  taken with respect to  $H_{(0)}^{-1}$  satisfies

$$\begin{aligned} \langle \mathcal{Q}(u), w \rangle_{H_{(0)}^{-1}, H_{(0)}^1} &= m_0 \varepsilon^{\alpha+1} \int_1^M \partial_r (r^{-n+1} \partial_r (r^{n-1} \partial_r u)) \partial_r w r^{n-1} dr + m_0 \varepsilon^{\alpha-1} \int_1^M f'_0(u(r)) r^{n-1} dr \end{aligned}$$

for all  $u \in \mathcal{D}(\partial\varphi) = \{v \in H^3(1, M) : \frac{1}{M-1} \int_1^M v(r) r^{n-1} dr = m\}$ . Then it is easy to verify that all conditions of [7, Theorem 3.1] are satisfied.

Finally, if  $(\tilde{c}_\varepsilon, \mu_\varepsilon)$  is a weak solution as above, we can choose  $\varphi = \mu_\varepsilon - \tilde{\mu}_\varepsilon$  with  $\tilde{\mu}_\varepsilon = \int_1^M \mu_\varepsilon(r) r^{n-1} dr$  in the weak formulation of the convective Cahn-Hilliard equation and obtain the energy identity

$$\begin{aligned} &\int_1^M \left( \varepsilon |\partial_r \tilde{c}_\varepsilon(r, t)|^2 + \varepsilon^{-1} f(\tilde{c}_\varepsilon(r, t)) \right) r^{n-1} dr \\ &+ \int_0^t \int_1^M m_0 \varepsilon^\alpha |\nabla \mu_\varepsilon(r, \tau)|^2 r^{n-1} dr d\tau = \int_1^M \left( \varepsilon |\partial_r \tilde{c}_{0,\varepsilon}(r)|^2 + \varepsilon^{-1} f(\tilde{c}_{0,\varepsilon}(r)) \right) r^{n-1} dr \end{aligned} \tag{4.20}$$

for all  $t \in (0, T)$ .

**THEOREM 4.3** Let  $\kappa > 3, r_0 \in (1, M), 0 < \delta < \min(r_0 - 1, M - r_0)$  and  $T > 0$  such that  $R(T) = \sqrt[n]{r_0^n + naT} < M - \delta, \Omega = \{x \in \mathbb{R}^n : 1 < |x| < M\}, \tilde{c}_{0,\varepsilon}$  and  $\theta$  be as in (4.9)–(4.10),

and let  $(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon)$  be the radially symmetric solutions of the form (4.1) of (1.1)–(1.5), (1.7) and boundary conditions (4.6)–(4.7),  $\mathbf{n} \cdot \nabla \mu_\varepsilon|_{\partial\Omega} = 0$ . Then

$$\mathbf{v}_\varepsilon \equiv ar^{-n+1}\mathbf{e}_r$$

and

$$\begin{aligned} c_\varepsilon &\xrightarrow{\varepsilon \rightarrow 0} 2\chi_{B_{R(t)}(0)} - 1 && \text{for every } x \in \Omega \setminus \partial B_{R(t)}(0), t \in (0, T), \\ p_\varepsilon &\xrightarrow{\varepsilon \rightarrow 0} p && \text{in } \mathfrak{D}'(\Omega \times (0, T)), \end{aligned}$$

where  $R(t) = \sqrt[n]{r_0^n + nat}$ ,  $p \in \mathfrak{D}'(\Omega \times (0, T))$  coincides with a function that is continuous in  $x \in \Omega \setminus \partial B_{R(t)}(0)$  for every  $t \in (0, T)$ , and

$$[p] = \kappa(t)\sigma H \quad \text{on } \Gamma(t) = \partial B_{R(t)}(0) \quad \text{for all } t \in (0, T)$$

and  $1 < \kappa(t) = \left(\frac{R(t)}{r_0}\right)^{2n-2} \rightarrow_{t \rightarrow \infty} \infty$ . Moreover,

$$\xi_\varepsilon(c_\varepsilon) = \frac{\varepsilon|\nabla c_\varepsilon|^2}{2} - \frac{f(c_\varepsilon)}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} (\sigma\kappa(t) - \tilde{\sigma})\delta_{\partial B_{R(t)}} \quad \text{in } \mathfrak{D}'(\Omega \times (0, T)), \quad (4.21)$$

where  $\sigma = \int_{\mathbb{R}} |\theta'(s)|^2 ds$ ,  $\tilde{\sigma} = \int_{\mathbb{R}} f(\theta(s)) ds$ .

*Proof.* First of all, we show that  $\|\tilde{c}_\varepsilon\|_{L^\infty((1,M) \times (0,T))}$  is uniformly bounded. To this end let  $W(c)$  be as in Section 3. Then as in (3.26)

$$\int_1^M |\partial_r W(\tilde{c}_\varepsilon(r,t))| r^{n-1} dr \leq C \int_1^M \left( \varepsilon \frac{|\partial_r \tilde{c}_{0,\varepsilon}(r)|^2}{2} + \frac{f(\tilde{c}_{0,\varepsilon}(r))}{\varepsilon} \right) r^{n-1} dr \leq C'$$

by (4.20) and the choice of the initial data. Hence

$$\sup_{0 < t < T, 0 < \varepsilon < 1} \|W(\tilde{c}_\varepsilon(t))\|_{L^\infty(1,M)} \leq \sup_{0 < t < T, 0 < \varepsilon < 1} \|\partial_r W(\tilde{c}_\varepsilon(t))\|_{L^1(1,M)} \leq C'$$

due to  $\tilde{c}_\varepsilon(t, M) = 1$ , which implies

$$\sup_{0 < \varepsilon < 1} \|\tilde{c}_\varepsilon\|_{L^\infty((1,M) \times (0,T))} \leq \tilde{M} \quad (4.22)$$

for some  $\tilde{M} > 0$  due to (3.27). Now let  $d_\varepsilon := \tilde{c}_\varepsilon - \tilde{c}_\varepsilon^\infty$ , where  $\tilde{c}_\varepsilon^\infty$  is as in (4.11). First we will show

$$\int_0^T \varepsilon \|\partial_r d_\varepsilon(t)\|_{L^2}^2 dt \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where  $L^2 = L^2((1, M); r^{n-1} dr)$ . To this end we use that

$$\langle \partial_t d_\varepsilon(t), \varphi \rangle + m_0 \varepsilon^{\alpha+1} \int_1^M \tilde{\Delta} d_\varepsilon(r,t) \tilde{\Delta} \varphi(r) r^{n-1} dr = \int_1^M g_\varepsilon(r,t) \tilde{\Delta} \varphi(r) r^{n-1} dr \quad (4.23)$$

for all  $\varphi \in H^1_{(0)}$  and almost every  $t \in (0, T)$ , where

$$\tilde{\Delta} u(r) = r^{-n+1} \partial_r (r^{n-1} \partial_r u(r)) \quad \text{for all } u \in H^2(1, M)$$

and

$$g_\varepsilon(r, t) = m_0 \varepsilon^{\alpha+1} \tilde{\Delta} c_\varepsilon^\infty(r, t) + m_0 \varepsilon^{\alpha-1} f'(\tilde{c}_\varepsilon(r, t)).$$

Moreover,

$$\|g_\varepsilon(t)\|_{L^2} \leq m_0 \varepsilon^{\alpha+1} \|\tilde{\Delta} \tilde{c}_\varepsilon^\infty(t)\|_{L^2} + m_0 \varepsilon^{\alpha-1} \|f'(\tilde{c}_\varepsilon(t))\|_{L^2} \leq C(T) m_0 \varepsilon^{\alpha-\frac{1}{2}}$$

where  $L^2 = L^2(1, M; r^{n-1} dr)$  and we have used that

$$\|\tilde{c}_\varepsilon^\infty(t)\|_{H^2(1, M)} \leq C(T) \varepsilon^{-\frac{3}{2}}$$

and

$$\varepsilon^{-1} \|f'(c_\varepsilon(t))\|_{L^2}^2 \leq C E_\varepsilon(c_\varepsilon(t)) \leq C'$$

due to  $|f'(s)|^2 \leq C(\tilde{M}) f(s)$  for all  $s \in [-\tilde{M}, \tilde{M}]$  and (4.22). Hence, choosing  $\varphi = d_\varepsilon(t)$  in (4.23) and integrating in time, we conclude

$$\sup_{0 \leq t \leq T} \|d_\varepsilon(t)\|_{L^2}^2 + m_0 \varepsilon^{\alpha+1} \int_0^T \|\tilde{\Delta} d_\varepsilon(t)\|_{L^2}^2 dt \leq \int_0^T \varepsilon^{-\frac{\alpha+1}{2}} \|g_\varepsilon(t)\|_{L^2} \varepsilon^{\frac{\alpha+1}{2}} \|\tilde{\Delta} d_\varepsilon(t)\|_{L^2} dt. \quad (4.24)$$

Using the Cauchy–Schwarz and Young’s inequality, we obtain

$$\sup_{0 \leq t \leq T} \|d_\varepsilon(t)\|_{L^2}^2 + m_0 \varepsilon^{\alpha+1} \int_0^T \|\tilde{\Delta} d_\varepsilon(t)\|_{L^2}^2 dt \leq C \int_0^T \varepsilon^{-\alpha-1} \|g_\varepsilon(t)\|_{L^2}^2 dt \leq C(T) \varepsilon^{\alpha-2}.$$

Combining this estimate with

$$\|\partial_r v\|_{L^2}^2 \leq C \|v\|_{L^2} \|v\|_{H^2} \leq C' \|v\|_{L^2} \|\tilde{\Delta} v\|_{L^2} \quad \text{for all } v \in H^2(1, M) \cap H_0^1(1, M),$$

we conclude

$$\int_0^T \varepsilon \|\partial_r d_\varepsilon(t)\|_{L^2}^2 dt \leq C \varepsilon^{\frac{\alpha-1}{2}} \varepsilon^{-1} = C \varepsilon^{\frac{\alpha-3}{2}} \rightarrow_{\varepsilon \rightarrow 0} 0 \quad (4.25)$$

since  $\alpha > 3$ .

In order to determine  $\tilde{p}_\varepsilon$  we use again (4.13) and decompose  $\tilde{p}_\varepsilon = p_{1,\varepsilon} + p_{2,\varepsilon} + p_3$  similarly as before, where

$$\partial_r p_{1,\varepsilon} = -\varepsilon(n-1) \frac{1}{r} |\partial_r \tilde{c}_\varepsilon|^2, \quad p_{2,\varepsilon} = -\varepsilon \partial_r |\partial_r \tilde{c}_\varepsilon|^2.$$

Hence up to a constant

$$p_3 = \frac{a(n-1)}{2n+2} r^{-2n+2} - \frac{va(n-1)}{n} r^{-n}$$

as before. Moreover, let

$$\partial_r p_{1,\varepsilon}^\infty = -\varepsilon(n-1) \frac{1}{r} |\partial_r \tilde{c}_\varepsilon^\infty|^2, \quad p_{2,\varepsilon}^\infty = -\varepsilon \partial_r |\partial_r \tilde{c}_\varepsilon^\infty|^2$$

be the corresponding parts of the pressure for the case  $\alpha = \infty$ . Then (4.25) implies that

$$\begin{aligned} \partial_r p_{1,\varepsilon} - \partial_r p_{1,\varepsilon}^\infty &\rightarrow_{\varepsilon \rightarrow 0} 0 && \text{in } L^1((1, M) \times (0, T)), \\ \partial_r p_{2,\varepsilon} - \partial_r p_{2,\varepsilon}^\infty &\rightarrow_{\varepsilon \rightarrow 0} 0 && \text{in } \mathfrak{D}'((1, M) \times (0, T)). \end{aligned}$$

Since  $p_3$  is independent of  $\varepsilon$  and the same as in the case  $\alpha = \infty$ , we conclude that

$$\tilde{p}_0 := \lim_{\varepsilon \rightarrow 0} \tilde{p}_\varepsilon = \lim_{\varepsilon \rightarrow 0} \tilde{p}_\varepsilon^\infty = p_0^\infty \quad \text{in } \mathfrak{D}'((1, M) \times (0, T)),$$

where  $\tilde{p}_\varepsilon^\infty$  is the pressure in the case  $\alpha = \infty$  and  $\tilde{p}_0^\infty$  is its limit as  $\varepsilon \rightarrow 0$ . Therefore

$$[\tilde{p}_0] = \sigma \left( \frac{R(t)}{r_0} \right)^{2n-2} \frac{n-1}{R(t)},$$

where  $\sigma := \int_{\mathbb{R}} |\theta'(s)|^2 ds$ , by the result of the previous section.

Finally, it remains to prove (4.21). First of all, because of (4.25), we conclude

$$\lim_{\varepsilon \rightarrow 0} \left( \varepsilon \frac{|\nabla c_\varepsilon^\infty|^2}{2} - \varepsilon \frac{|\nabla c_\varepsilon|^2}{2} \right) = 0 \quad \text{in } L^1(\Omega \times (0, T)),$$

where  $c_\varepsilon^\infty(x, t) = \tilde{c}_\varepsilon^\infty(|x|, t)$ . Moreover, using (4.22) and a Taylor expansion of  $f(\tilde{c}_\varepsilon(r, t))$  around  $\tilde{c}_\varepsilon^\infty(r, t)$ , we conclude

$$\begin{aligned} & \int_0^T \int_1^M \left| \frac{f(\tilde{c}_\varepsilon^\infty(r, t))}{\varepsilon} - \frac{f(\tilde{c}_\varepsilon(r, t))}{\varepsilon} \right| dr dt \\ & \leq \int_0^T \int_1^M \left| \frac{f'(\tilde{c}_\varepsilon^\infty(r, t))d_\varepsilon(r, t)}{\varepsilon} \right| dr dt + C \int_0^T \int_1^M \left| \frac{d_\varepsilon(r, t)^2}{\varepsilon} \right| dr dt \\ & \leq C(M, T)\varepsilon^{-1} \left( \|d_\varepsilon\|_{L^2(\Omega \times (0, T))}^2 + \|d_\varepsilon\|_{L^2(\Omega \times (0, T))} \|f'(\tilde{c}_\varepsilon^\infty)\|_{L^2(\Omega \times (0, T))} \right) \\ & \leq C'(M, T) \left( \varepsilon^{\alpha-3} + \varepsilon^{\frac{\alpha-3}{2}} \right) \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$

since  $\|\tilde{c}_\varepsilon\|_{L^\infty(\Omega \times (0, T))}$  and  $\varepsilon^{\frac{1}{2}} \|f'(\tilde{c}_\varepsilon^\infty)\|_{L^2(\Omega \times (0, T))}$  are uniformly bounded in  $\varepsilon \in (0, 1)$  due to  $|f'(c_\varepsilon^\infty)|^2 \leq C f(c_\varepsilon^\infty)$ . Altogether we obtain

$$\lim_{\varepsilon \rightarrow 0} (\xi_\varepsilon(c_\varepsilon^\infty) - \xi_\varepsilon(c_\varepsilon)) = 0 \quad \text{in } L^1(\Omega \times (0, T)),$$

which implies (4.21) due to (4.17). □

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