Algebraic K-Theory of ∞-Operads

by

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Abstract

The theory of dendroidal sets has been developed to serve as a combinatorial model for homotopy coherent operads, see [MW07, CM13b]. An ∞-operad is a dendroidal set D satisfying certain lifting conditions.

In this paper we give a definition of K-groups $K_n(D)$ for a dendroidal set D. These groups generalize the K-theory of symmetric monoidal (resp. permutative) categories and algebraic K-theory of rings. We establish some useful properties like invariance under the appropriate equivalences and long exact sequences which allow us to compute these groups in some examples. Using results from [Heu11b] and [BN12] we show that the K-theory groups of D can be realized as homotopy groups of a K-theory spectrum $\mathcal{K}(D)$.

1. Introduction

Operads are an important tool in modern mathematics, especially in topology and algebra [MSS02]. Throughout this paper we use the term 'operad' for what should really be called a 'coloured, symmetric operad' or maybe even better a 'symmetric multicategory'. In order to make clear what is meant let us briefly recall that an operad P in this sense is given by a set of colours $\{a,b,c,...\}$, sets of operations $P(a_1,...,a_n;b)$ equipped with Σ_n-actions, and composition maps. Clearly classical non-coloured operads (i.e. coloured operads which have only one colour) are an important special case. But there is another class of examples which sit at the other end of the spectrum of coloured operads; namely small symmetric monoidal categories C, which we consider as operads with colours the objects of C and operations $C(a_1,...,a_n;b) := \mathrm{Hom}_C(a_1 \otimes ... \otimes a_n;b)$.

The idea is that ∞-operads are both higher categorical and homotopy coherent versions of ordinary operads. There are several ways of making this idea precise. The easiest model, which has been successfully used in topology for a long time [BV73, May99], are topologically enriched operads, i.e. the sets of operations $P(a_1,...,a_n;c)$ are replaced by topological spaces. But there are other models which are technically more convenient. One model has been given by Lurie [Lur12,

Section 2] and another one by Moerdijk and Weiss [MW07]. We will restrict our attention in this paper to the latter model which goes by the name of dendroidal sets. But all models are (at least conjecturally) equivalent, so the results hold independently and should in principle have proofs in all settings. We review the theory of dendroidal sets in Section 2.

In this paper we introduce abelian groups $K_n(D)$ for an ∞-operad D which we call the K-theory groups of D. The zeroth group $K_0(D)$ can be defined very explicitly using generators and relations, see Section 3. For the higher groups we have to make use of homotopy theoretic methods. More precisely we use a model structure on the category of dendroidal sets which was introduced in [BN12]. By means of this model structure we can define for every dendroidal set D a 'derived underlying space' whose homotopy groups are the groups $K_n(D)$ (Section 4).

We show that these groups are invariant under equivalences of ∞-operads and that they admit long exact sequences coming from cofibre sequences of ∞-operads. Using these properties one can already compute the K-groups for basic cases. For example, for (a dendroidal version of) an E_∞-operad we show that $K_n(E_\infty) = 0$ for all n. Another easy example of an ∞-operad is η, the trivial operad without higher operations; for this ∞-operad we show that the K-groups are given by the stable homotopy groups of spheres. More generally, we treat the example $\Omega[T]$, the ∞-operad associated to a tree T, and show that $K_n\big(\Omega[T]\big) \cong \bigoplus_{\ell(T)} \pi_n^S$, where $\ell(T)$ is the number of leaves of the tree T and π_n^S are the stable homotopy groups of spheres (Corollary 5.8).

It has been sketched by Heuts [Heu11b] how to associate an infinite loop space, i.e. a connective spectrum, to a dendroidal set D. We use a slight variant of his construction to define what we call the algebraic K-theory spectrum $\mathcal{K}(D)$ of an ∞-operad (Section 5). We show that the homotopy groups of this spectrum agree with our K-theory groups $K_n(D)$ (Theorem 5.5). In some cases we can identify this spectrum. For example every simplicial set X gives rise to a dendroidal set $i_!X$. For this case we can show that the associated spectrum is the suspension spectrum of the geometric realization of X. The main result of [BN12] even implies that the functor $D \mapsto \mathcal{K}(D)$ induces an equivalence between a suitable localization of the category of ∞-operads and the homotopy category of connective spectra. In particular all connective spectra arise as $\mathcal{K}(D)$ for some ∞-operad D.

Finally we want to explain why we have decided to call these invariants K-groups (and K-theory spectra). Recall that, by definition, algebraic K-theory of a ring R is computed using its category of finitely generated projective modules (or some related space like $BGL(R)$). There are several equivalent variants that produce the

K-theory groups $K_n(R)$ and a K-theory spectrum $\mathcal{K}(R)$ from this category. The whole theory was initiated by Quillen [Qui73], but see also [Tho82]. We have already explained in the first paragraph that a symmetric monoidal category can be considered as an operad and thus also as an ∞-operad. For the groupoid of finitely generated projective modules over a ring R this ∞-operad is denoted by $N_d\text{Proj}_R$. We then show in Theorem 6.1 that the K-theory of this ∞-operad is equivalent to the algebraic K-theory of the ring R:

$$K_n(N_d\text{Proj}_R) \cong K_n(R) \quad \text{and} \quad \mathcal{K}(N_d\text{Proj}_R) \cong \mathcal{K}(R) \, .$$

In this sense our K-theory generalizes the algebraic K-theory of rings (resp. symmetric monoidal categories) and therefore deserves to be called K-theory. Moreover, it is shown in [Heu11a] that not only ordinary symmetric monoidal categories can be seen as dendroidal sets, but also symmetric monoidal ∞-categories, i.e. E_∞-algebras in the ∞-category of ∞-categories (modeled by the Joyal model structure on simplicial sets). In this sense the K-theory of dendroidal sets contains as a special case the K-theory of symmetric monoidal ∞-categories.

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2. Preliminaries about dendroidal sets

In this section we briefly recall the theory of dendroidal sets as discussed in [MW07, MW09, CM13a, CM13b, CM11]. Dendroidal sets are a generalization of simplicial sets. Recall that simplicial sets are presheaves on the category Δ of finite, linearly ordered sets, i.e. $\text{sSet} = [\Delta^{op}, \text{Set}]$. Dendroidal sets are similarly a presheaf category on an extension of the simplicial category Δ to the dendroidal category Ω.

The category Ω is defined as follows. The objects are given by finite, rooted trees; that is, graphs with no loops equipped with a distinguished outer edge called the root and a (possibly empty) set of outer edges not containing the root called leaves. As an example consider the tree:

$$T = \qquad \qquad \qquad \qquad \qquad \qquad (1)$$

We also allow the tree which only consist of a single edge without vertices. For this tree the root is also a leaf, in contrast to the requirement that the leaves are

distinct from the root. In particular, this tree should be carefully distinguished from the other tree with one edge

$$T = \bullet{\mid}_a$$

which has no leaves.

Every finite, rooted tree T generates a coloured symmetric operad $\Omega(T)$ as follows. The set of colours is given by the edges of T. In Example (1) this is the set $\{a,b,c,d,e\}$. The operations are freely generated by the vertices of T. In Example (1) there is one generating operation $v \in \Omega(T)(a,b;c)$ and another one $w \in \Omega(T)(c,d;e)$. These operations of course generate other operations such as $w \circ_c v \in \Omega(T)(a,b,d;e)$, the permutations $\sigma v \in \Omega(T)(b,a;c)$, $\sigma w \in \Omega(T)(d,c;e)$, and the six possible permutations of $w \circ_c v$.

Now we can complete the definition of the category Ω, whose objects are all finite rooted trees T, by specifying the morphisms. A morphism $T \to S$ in Ω is a morphism of coloured symmetric operads $\Omega(T) \to \Omega(S)$ (for the definition of operad morphisms see e.g. [MW07, Section 2]). Thus, the category Ω is, by definition, a full subcategory of the category of coloured operads. Examples of morphisms are shown in the next picture:
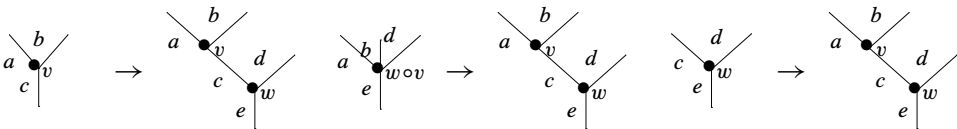
Here the labeling of the edges and vertices in the domain trees indicates to which edges and vertices in the target trees they are sent. The depicted morphisms are all examples of a special class of morphisms in Ω called *face maps*. There are two types of face maps in Ω. The first type are the outer face maps, which are obtained by chopping off an outer vertex of a tree T. The first and the third morphism in the picture are outer face maps. The second type of face maps are the inner face maps, which are obtained by contracting an inner edge of T. An example is the morphism in the middle of the picture which is obtained by contracting the edge c. Every tree has a set of *outer face maps* which are labeled by outer vertices v and a set of *inner face maps* which are labeled by inner edges e.

The category dSet of dendroidal sets is defined as the presheaf category on Ω, i.e.

$$\text{dSet} := [\Omega^{op}, \text{Set}].$$

For a dendroidal set D we denote the value on the tree T by D_T and call it the set of T-dendrices. The dendroidal set represented by a tree T is denoted by $\Omega[T]$. In

particular for the tree with one edge and no vertex we set $\eta := \Omega[\ |\ ]$. The Yoneda lemma shows that we have $D_T \cong \mathrm{Hom}(\Omega[T], D)$.

There is a fully faithful embedding of the simplex category Δ into Ω by considering finite linear ordered sets as linear trees. More precisely this inclusion maps the object $\{0, 1, ..., n\} = [n] \in \Delta$ to the tree

$$
L_n = \quad
\begin{array}{c}
a_0 \\
a_1 \\
\vdots \\
a_n
\end{array}
$$
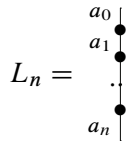
This inclusion $\Delta \subset \Omega$ induces an adjunction

$$
i_! : \ \mathrm{sSet} \ \rightleftarrows \ \mathrm{dSet} : i^*
$$

with fully faithful left adjoint $i_!$ (there is also a further right adjoint i_* which does not play a role in this paper). The functor i^* is given by restriction to linear trees and the functor $i_!$ is extension by zero, i.e. the dendroidal set $i_! X$ agrees with X on linear trees and is empty otherwise.

The inclusion of Ω into the category of coloured, symmetric operads induces a fully faithful functor $N_d : \mathrm{Oper} \to \mathrm{dSet}$ called the *dendroidal nerve*. Concretely the dendroidal nerve of an operad P is given by $N_d(P)_T := \mathrm{Hom}(\Omega(T), P)$. By definition of Ω and the Yoneda lemma we have $N_d(\Omega(T)) = \Omega[T]$. A particularly important case of the dendroidal nerve is when the operad P comes from a (small) symmetric monoidal category C. Recall that the operad associated to C has as colours the objects of C and as operations $\mathrm{Hom}(c_1, ..., c_n; c) = \mathrm{Hom}_C(c_1 \otimes ... \otimes c_n; c)$. By abuse of notation we call the nerve of this operad $N_d(C)$ as well. This assignment defines a fully faithful inclusion

$$
\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \to \mathrm{dSet}
$$

where $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$ denotes the category of (small) symmetric monoidal categories and lax monoidal functors. Here lax monoidal for a functor F means that the structure morphisms $F(a) \otimes F(b) \to F(a \otimes b)$ and $\mathbb{1} \to F(\mathbb{1})$ are not necessarily invertible.

As described above, every tree T has a set of subobjects called faces, which are obtained by either contracting an inner edge (these are the inner faces) or by truncating outer edges (these are the outer faces). These faces are used to define dendroidal *horns* and *boundaries* in a way that generalizes horns and boundaries in

simplicial sets. The boundary

$$\partial\Omega[T] \subset \Omega[T]$$

is a subobject of $\Omega[T]$ which is the union of all inner and outer faces of T. The horns of T carry a label a which can be either an inner edge or an outer vertex. Then the horn

$$\Lambda^a[T] \subset \Omega[T]$$

is defined as the union of all faces of T except the face labelled by a. We call the horn inner if a is an inner edge and outer if a is an outer vertex. The reader can easily convince himself that for the case $T = L_n$ these horns reduce to the horns of the simplex $\Delta[n]$.

There is one case which deserves special attention, namely the case of trees with exactly one vertex. These trees are called *corollas*. More concretely the n-corolla is given by
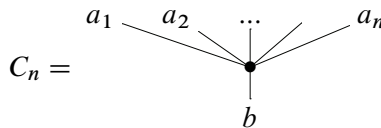
We observe that C_n has $n+1$ faces given by the inclusion of the colours $a_1,...,a_n,b$. All faces are outer. Consequently C_n also has $n+1$ outer horns which are inclusions of n-disjoint unions of η.

Definition 2.1 A dendroidal set D is called *inner Kan* or an ∞-*operad* if D admits fillers for all inner horns, i.e. for each morphism $\Lambda^e[T] \to D$ with e an inner edge there is a morphism $\Omega[T] \to D$ that renders the diagram
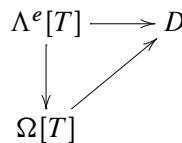
commutative. A dendroidal set is called *fully Kan* if it admits fillers for all horns.

The two classes of inner Kan and fully Kan dendroidal sets are very important in the theory of dendroidal sets and for the rest of the paper. Thus we make some easy remarks such that the reader gets a feeling for these classes:

Remark 2.2   • Let X be a simplicial set. Then $i_! X$ is an ∞-operad if and only if X is an ∞-category (in the sense of Boardman-Vogt, Joyal, and Lurie). Moreover for every ∞-operad D the underlying simplicial set $i^* D$ is an ∞-category.

- For a fully Kan dendroidal set D the underlying simplicial set i^*D is a Kan complex, but for a non-empty simplicial set X the dendroidal set $i_!X$ is never fully Kan since there are no fillers for corolla horns.

- For every (coloured, symmetric) operad P the dendroidal nerve $N_d P$ is an ∞-operad. This shows that ordinary operads are a special case of ∞-operads. In particular all representable dendroidal sets $\Omega[T]$ are ∞-operads.

- The dendroidal nerve $N_d P$ is fully Kan if and only if P comes from a grouplike symmetric monoidal groupoid C. These are also called Picard-groupoids. In [BN12] it has been shown that fully Kan dendroidal sets model Picard-∞-groupoids. Therefore the last statement shows that Picard-groupoids are a special case of Picard-∞-groupoids.

The foundational result in the theory of ∞-operads is that there is a model structure on the category of dendroidal sets with fibrant objects given by ∞-operads. This model structure is a generalization of the the Joyal model structure on simplicial sets. In order to state the result properly we have to introduce the class of *normal monomorphisms* of dendroidal sets. This class is defined as the smallest class of morphisms in dSet that contains the boundary inclusions of trees and that is closed under pushouts, retracts and transfinite compositions. One can also give an easy concrete description (see [CM13b, 2.3]) but we will not need this description here.

Theorem 2.3 (Cisinski-Moerdijk) *There is a model structure on the category of dendroidal sets with cofibrations given by normal monomorphisms and fibrant objects given by ∞-operads.*

Note that a model structure is uniquely determined by its class of cofibrations and the fibrant objects. Thus the above result can be read as an existence statement. The weak equivalences in the Cisinski-Moerdijk model structure are called *operadic equivalences*. Their definition is not quite explicit, though one can give explicit criteria for a morphisms between ∞-operads to be an operadic equivalence, see [CM13a, Theorem 3.5 & Theorem 3.11]. A similar model structure exists for fully Kan dendroidal sets [BN12]:

Theorem 2.4 *There is a model structure on the category of dendroidal sets with cofibrations given by normal monomorphisms and fibrant objects given by fully Kan dendroidal sets.*

We call this second model structure the *stable* model structure and the weak equivalences *stable equivalences*. The stable model structure is a left Bousfield localization of the Cisinski-Moerdijk model structure, i.e. every operadic equivalence

is also a stable equivalence. Again stable equivalences are defined very indirectly, but we will give a more direct criterion in Proposition 4.3.

The existence of the stable model structure also implies that for every dendroidal set D we can choose a functorial fibrant replacement $D \to D_K$ with D_K being fully Kan. This can for example be done using Quillen's small objects argument by iteratively gluing in fillers for horns. This fibrant replacement will play an important role in our definition of the K-theory groups of D.

Finally we want to remark that the adjunction $i_! : \text{sSet} \rightleftarrows \text{dSet} : i^*$ becomes a Quillen adjunction in the following two cases:

- For the Kan-Quillen model structure on sSet and the stable model structure on dSet.

- For the Joyal model structure on sSet and the Cisinski-Moerdijk model structure on dSet.

3. K_0 of dendroidal sets

In this section we want to define an abelian group $K_0(D)$ for each dendroidal sets D which generalizes the group $K_0(C)$ for a symmetric monoidal category C. The latter group $K_0(C)$ is defined as the group completion of the abelian monoid $\pi_0(C)$.

Note that there are two possible meanings for $\pi_0(C)$ of a symmetric monoidal category C. It can either be the set of isomorphism classes in C or the connected components in NC. In the case that C is a groupoid the two choices agree. Here we want $\pi_0(C)$ to mean the connected components. In particular to recover the classical K-theory of a ring R we have to compute $K_0(\text{Proj}_R)$ where Proj_R denotes the maximal subgroupoid inside the category of finitely generated projective R-modules.

Definition 3.1 Let D be a dendroidal set. We define $K_0(D)$ as the abelian group freely generated by the elements $x \in D_{L_0}$ (i.e. morphisms $\eta \to D$) subject to the relations

$$x_1 + ... + x_n = x$$

whenever there is a corolla $\Omega[C_n] \to D$ with ingoing faces $x_1,...,x_n$ and outgoing face x. For a 0-corolla $\Omega[C_0] \to D$ the left hand sum is understood to be 0. A map $f : D \to D'$ of dendroidal sets induces a morphism $f_* : K_0(D) \to K_0(D')$ of abelian groups by applying f_{L_0}.

Note that the relations we impose in the definition of $K_0(D)$ also include the possibility of the 1-corolla $C_1 = L_1$. Hence if two elements $x, y \in D_{L_0}$ are equal in

$\pi_0(i^*(D))$, i.e. there is a chain of edges in the underlying simplicial set connecting them, then they are also equal in $K_0(D)$. Therefore we have a well defined map (of sets)

$$\lambda_D: \quad \pi_0(i^*D)) \longrightarrow K_0(D).$$

Proposition 3.2   *1. If $D = i_!X$ for a simplicial set X, then $K_0(D)$ is freely generated by $\pi_0(i^*D) = \pi_0(X)$.*

   2. If $D = N_dC$ for a symmetric monoidal category C, then $\pi_0(i^*D) = \pi_0(NC)$ is an abelian monoid and λ_D exhibits $K_0(D)$ as the group completion of $\pi_0(NC)$. In particular we have $K_0(D) = K_0(C)$.

   3. If D is fully Kan, then λ_D is a bijection.

Proof: 1) For a simplicial set X there are no n-corollas $C_n \to i_!(X)$ except for $n = 1$. Hence $K_0(i_!(X))$ is the free group generated by the vertices of X where two vertices are identified if there is an edge connecting them. The usual description of $\pi_0(X)$ then proves the statement.

2) The existence of the monoid structure on $\pi_0(NC)$ and the fact that λ_D is a group homomorphism are immediate from the definition and the fact that an n-corolla in $N_d(C)$ is exactly a morphisms $c_1 \otimes ... \otimes c_n \to c$ in C. The claim then follows from the fact that the group completion of an abelian monoid A can be described as the free group generated by objects $a \in A$ subject to the relations $a_1 + ... + a_n = a$ whenever this holds in A.

3) We want to explicitly construct an inverse to λ_D. But as a first step we endow $\pi_0(i^*D)$ with the structure of an abelian group. Therefore for two elements $a,b \in (i^*D)_0 = D_{L_0}$ we choose a 2-corolla $\Omega[C_2] \to D$ with inputs a and b and output c. Such a corolla exists since D is fully Kan. Then we set $[a] + [b] := [c]$.

We have to show that this addition is well defined. So first assume that there is another $a' \in D_{L_0}$ with $[a'] = [a]$ in $\pi_0(i^*D)$. This means that there is an edge $L_1 \to D$ connecting a' and a. We look at the tree

$$T = \quad \begin{array}{c} a' \\ \bullet \\ a \diagdown \diagup b \\ \bullet \\ c \end{array}$$
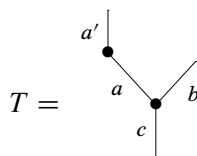
The two maps $\Omega[C_2] \to D$ and $L_1 \to D$ chosen above determine a map from the inner horn $\Lambda^a[T] \to D$. Since D is fully Kan we can fill this horn and obtain a

map $\Omega[T] \to D$. In particular there is a 2-corolla $\Omega[C_2] \to D$ with inputs a',b and output c. Thus we have $[a'] + [b] = [c] = [a] + [b]$. This shows that the addition does not depend on the representatives of $[a]$ and by symmetry also for $[b]$.

It remains to check that the addition does not depend on the choice of 2-corolla $\Omega[C_2] \to D$ with inputs a and b. Therefore assume we have another 2-corolla in D with inputs a,b and output c'. Then consider the tree

$$T = \qquad \begin{array}{c} a \diagdown \ \diagup b \\ \bullet \\ c \\ | \\ \bullet \\ c' \\ | \end{array}$$
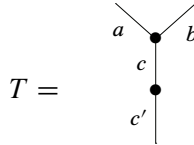
As before we fill this tree at the outer horn of the binary vertex. This yields an edge between c and c' and thus shows that the addition is well-defined.

Before we show that $\pi_0(i^*D)$ together with the addition is really an abelian group we need another preparatory fact: in $\pi_0(i^*D)$ the equality $\big((...((a_1 + a_2) + a_3) + ...) + a_n\big) = b$ holds precisely if there is an n-corolla $\Omega[C_n] \to D$ with inputs $a_1,...,a_n$ and output b. We show this by induction. For $n = 2$ this is the definition. Assume it holds for n. Then the claim for $n + 1$ follows straightforward by looking at horns of the tree obtained by grafting an n-corolla onto a 2-corolla.

Now we need to show that $\pi_0(i^*D)$ together with the addition operation just defined is in fact an abelian group. The fact that the multiplication is abelian is automatic by the fact that we consider symmetric operads (resp. non-planar trees). So we need to show that it is associative, there are inverses and units. This follows be filling the root horns $\eta_a \sqcup \eta_b \to \Omega[C_2]$ and $\emptyset \to \Omega[C_0]$ and we leave the details to the reader.

Altogether we have shown that $\pi_0(i^*D)$ admits the structure of an abelian group. By definition of the two group structures the morphism $\lambda_D : \pi_0(i^*D) \to K_0(D)$ is a group homomorphism. An inverse is now induced by the map $D_{L_0} \to \pi_0(i^*D)$ and the fact that $K_0(D)$ is freely generated by D_{L_0} subject to relations which, as shown above, hold in $\pi_0(D)$. □

Example 3.3   • For a tree T let ℓ_T denote the set of leaves of T. Then we have

$$K_0\big(\Omega[T]\big) \cong \mathbb{Z}\langle \ell_T \rangle \,,$$

i.e. K_0 is the free abelian group generated by the set of leaves of T.

• For a map $f : \Omega[S] \to \Omega[T]$ in Ω we have for each edge $e \in \ell_S$ the subset $\ell_{f(e)} \subset \ell_T$ of leaves over the edge $f(e)$. The induced map

$$K_0(S) \cong \mathbb{Z}\langle \ell_S \rangle \to K_0(T) \cong \mathbb{Z}\langle \ell_T \rangle$$

is then the map that sends the generator $e \in \mathbb{Z}\langle \ell_S \rangle$ to the element $e_1 + \dots + e_k \in \mathbb{Z}\langle \ell_T \rangle$ where $\{e_1, \dots, e_k\} = \ell_{f(e)}$ is the set of leaves over $f(e)$.

Lemma 3.4 *1. The functor $K_0 : dSet \to$ AbGr is a left adjoint and thus preserves all colimits.*

2. *For a horn inclusion $\Lambda^a[T] \to \Omega[T]$ the induced morphism*
$$K_0\big(\Lambda^a[T]\big) \to K_0\big(\Omega[T]\big)$$
is an isomorphism.

Proof: 1) K_0 is left adjoint to the inclusion functor $i : $ AbGr $\to$ dSet which can be described as follows. Let A be an abelian group. We consider it as a discrete symmetric monoidal category (i.e. without non trivial morphisms) then we consider it as an operad (as described in the previous section) and take the dendroidal nerve. Explicitly we obtain $i(A)_T = A\langle \ell_T \rangle$. Then it is easy to see that K_0 is left adjoint to i.

2) If the tree T has more than two vertices one easily verifies that the inclusion $\Lambda^a[T] \to \Omega[T]$ induces a bijection when evaluated on L_0 and on C_n. Thus it clearly induces an isomorphism on K_0. Therefore it remains to check the claim for horns of trees with one or two vertices. For one vertex the tree is a corolla and the horn is a disjoint union of η's. Then the verification of the statement is straightforward using Example 3.3. Thus only the case of trees with two vertices remains.

Such trees can all be obtained by grafting an n-corolla C_n for $n \geq 0$ on top of a k-corolla for $k \geq 1$. We call this tree $C_{n,k}$.

$$C_{n,k} =$$
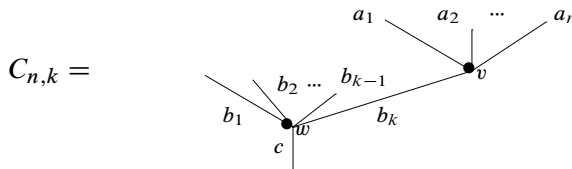
There are three possible horns and applying the definitions yields the following groups:

$$K_0(\Lambda^{b_k}[C_{n,k}]) = \frac{\mathbb{Z}\langle a_1, \dots, a_n, b_1, \dots, b_k, c \rangle}{a_1 + \dots + a_n = b_k, \; b_1 + \dots + b_k = c}$$

$$K_0(\Lambda^{v}[C_{n,k}]) = \frac{\mathbb{Z}\langle a_1, \dots, a_n, b_1, \dots, b_k, c \rangle}{a_1 + \dots + a_n = b_k, \; b_1 + \dots + b_{k-1} + a_1 + \dots + a_n = c}$$

$$K_0(\Lambda^{w}[C_{n,k}]) = \frac{\mathbb{Z}\langle a_1, \dots, a_n, b_1, \dots, b_k, c \rangle}{b_1 + \dots + b_k = c, \; b_1 + \dots + b_{k-1} + a_1 + \dots + a_n = c}$$

Clearly these groups are all isomorphic to $K_0(\Omega[C_{n,k}]) \cong \mathbb{Z}\langle a_1,...,a_n,b_1,...,b_{k-1}\rangle$.
<div style="text-align: right">□</div>

Proposition 3.5 *For a stable equivalence $f : D \to D'$ the induced morphism $f_* : K_0(D) \to K_0(D')$ is an isomorphism. In particular this holds also for operadic equivalences.*

Proof: We first show that for D a dendroidal set and D_K the fibrant replacement obtained by Quillen's small object argument the induced morphism $K_0(D) \to K_0(D_K)$ is an isomorphism. Therefore remember that D_K is built as the directed colimit

$$D_K = \varinjlim(D_0 \to D_1 \to D_2 \to ...)$$

where $D_0 = D$ and each D_{n+1} is obtained by attaching trees along horns to D_n. Since K_0 is left adjoint by Lemma 3.4 we have

$$K_0(D_K) \cong \varinjlim(K_0(D_0) \to K_0(D_1) \to K_0(D_2) \to ...).$$

Hence it suffices to check that each group homomorphism $K_0(D_n) \to K_0(D_{n+1})$ is an isomorphism. To see this note that D_{n+1} is build as a pushout of the form

$$
\begin{array}{ccc}
\bigsqcup \Lambda^a[T] & \longrightarrow & D_n \\
\downarrow & & \downarrow \\
\bigsqcup \Omega[T] & \longrightarrow & D_{n+1}
\end{array}
$$

Applying K_0 yields a pushout diagram

$$
\begin{array}{ccc}
\bigoplus K_0(\Lambda^a[T]) & \longrightarrow & K_0(D_n) \\
\downarrow & & \downarrow \\
\bigoplus K_0(\Omega[T]) & \longrightarrow & K_0(D_{n+1})
\end{array}
$$

where the left vertical morphism is, by Lemma 3.4, an isomorphism. Thus the right vertical morphism is an isomorphism too. Altogether this shows that the morphism $K_0(D) \to K_0(D_K)$ is an isomorphism of groups.

Now assume we have an arbitrary stable equivalence $f : D \to D'$ of dendroidal sets. Applying the fibrant replacement described above to both objects yields a weak equivalence $f_K : D_K \to D'_K$. By the above argument it remains only to check that the induced morphism $(f_K)_* : K_0(D_K) \to K_0(D'_K)$ is an isomorphism. By the fact that i^* is right Quillen we know that f_K induces a weak equivalence of simplicial sets $i^*(D_K) \to i^*(D'_K)$. Together with Proposition 3.2 (3) this shows

that $(f_K)_* : K_0(D_K) \to K_0(D'_K)$ is an isomorphism, hence $f_* : K_0(D) \to K_0(D')$ is too.

Finally the statement about operadic equivalences follows since each operadic equivalence is a stable equivalence. ☐

4. Higher K-groups of dendroidal sets

In this section we want to define higher K-groups of a dendroidal set D. Note that by the results of the last section we can compute $K_0(D)$ for a dendroidal set D always as follows: choose a fully Kan replacement D_K of D. Then by Proposition 3.5 the inclusion $D \to D_K$ induces an isomorphism $K_0(D) \xrightarrow{\sim} K_0(D_K)$. But by Proposition 3.2(3) we can compute $K_0(D_K)$ as the connected components of the underlying space $i^*(D_K)$. The idea for the higher K-groups is to generalize this procedure.

Definition 4.1 Let D be a dendroidal set and D_K be a fully Kan replacement. We define

$$K_n(D) := \pi_n(i^* D_K).$$

The space $i^* D_K$ is also called the *derived underlying space* of D.

Remark 4.2   • There is a subtlety involved in the above definition, namely the choice of basepoint in $i^* D_K$ to compute the higher homotopy groups. It turns out that the choice of basepoint is inessential since $i^* D_K$ is an infinite loop space as we will see later.

• The fact that $i^* D_K$ is an infinite loop space also implies that all the $K_n(D)$ are abelian groups (especially K_0 and K_1).

• Note that in order to turn K_n into functors dSet $\to$ AbGr we have to make functorial choices of fibrant replacements. This can, e.g., be done using Quillen's small object argument. This also solves the problem of basepoints, since then the fibrant replacement D_K has a distinguished morphism $\Omega[C_0] \to D_K$ coming from gluing in $\Omega[C_0]$ along its outer horn $\emptyset$. After applying i^* this leads to a (functorial) choice of basepoint $\Delta[0] \to i^* D_K$.

Proposition 4.3   *1. Definition 3.1 and 4.1 of K_0 agree, i.e the groups are canonically isomorphic.*

2. The higher K-groups are well-defined, i.e. for two fully Kan replacements D_K and D'_K of D there is an isomorphism $\pi_n(i^* D_K) \xrightarrow{\sim} \pi_n(i^* D'_K)$.

3. *For a stable equivalence between dendroidal sets the induced morphisms on K-groups are isomorphisms. In particular, for stably equivalent dendroidal sets the K-groups are isomorphic.*

   Conversely if for a morphism $f : D \to D'$ the induced maps $f_* : K_n(D) \to K_n(D')$ are isomorphisms then f_* is a stable equivalence.

Proof: 1) This follows directly from our remarks preceding Definition 4.1.

2) The two fibrant replacements D_K and D'_K are related by the chain $D_K \leftarrow D \to D'_K$ of weak equivalences. As a lift in the diagram

$$
\begin{array}{ccc}
D & \longrightarrow & D'_K \\
\downarrow & & \downarrow \\
D_K & \dashrightarrow & *
\end{array}
$$

we find a morphism $f : D_K \to D'_K$ which is by the 2-out-of-3 property a stable equivalence. By the fact that i^* is right Quillen this implies that $i^* f : i^* D_K \to i^* D'_K$ is a weak equivalence of simplicial set. Therefore $f_* : \pi_n(i^* D_K) \to \pi_n(i^* D'_K)$ is an isomorphism.

3) The morphism $f : D \to D'$ induces a morphism between fibrant replacements $f_K : D_K \to D_{K'}$. Then f is a stable equivalence if and only if f_K is a stable equivalence. By [BN12, Theorem 4.2 (3)] this is equivalent to the fact that $i^* f_K : i^* D_K \to i^* D_{K'}$ is a weak equivalence of simplicial sets. And this last condition is equivalent to the fact that the induced morphisms on homotopy groups, which are the K-groups of D and D', are isomorphisms.   □

Corollary 4.4 *For an operadic equivalence $f : D \to D'$ of dendroidal sets, the induced morphisms $f_* : K_n(D) \to K_n(D')$ are isomorphisms. Hence the K-groups are an invariant of ∞-operads.*

Proof: This follows from the fact, that the stable model structure is a left Bousfield localization of the Moerdijk-Cisinski model structure. So the weak operadic equivalences are also stable equivalences.   □

Example 4.5 Let $\mathcal{E}_\infty$ in dSet be a cofibrant resolution of the point $* \in$ dSet. This is the dendroidal version of an E_∞-operad. Then we have $K_n(\mathcal{E}_\infty) \cong K_n(*) \cong 0$ for all n.

There are also dendroidal versions $\mathcal{E}_k$ of the little k-disks operad. One can show that we also have $K_n(\mathcal{E}_k) \cong 0$ for all n. We will not do this here, since it is most easily deduced using monoidal properties of K-theory which will be investigated elsewhere.

Proposition 4.6 *Let $D = \varinjlim D_i$ be a filtered colimit of dendroidal sets D_i. Then we have*

$$K_n(D) \cong \varinjlim K_n(D_i).$$

Proof: We will use the fact that there is an endofunctor $T : \text{dSet} \to \text{dSet}$ with the property that when applied to a dendroidal set D it produces a fibrant replacement $T(D)$ and furthermore T preserves filtered colimits. To see that such a T exists one can use the fact that dSet is combinatorial and use a general existence result for accessible fibrant replacement functors (see, e.g., [Lur09, Proposition A.1.2.5]) or use [Nik11, Proposition 2.23] for an explicit construction.

Now for a given filtered colimit $D = \varinjlim D_i$ we obtain a replacement $T(D) = \varinjlim T(D_i)$. Then note that the functor $i^* : \text{dSet} \to \text{sSet}$ is not only right adjoint to $i_!$ but also left adjoint to a functor i_*. Hence it also preserves filtered colimits. Therefore we have $i^* T(D) \cong \varinjlim i^* T(D_i)$ which implies

$$K_n(D) = \pi_n\big(i^* T(D)\big) \cong \varinjlim \pi_n\big(i^* T(D_i)\big) \cong \varinjlim K_n(D_i).$$

$\square$

There are two model structures on the category of dendroidal sets: the stable model structure and the operadic (Cisinski-Moerdijk) model structure. The fact that the stable model structure is a left Bousfield localization of the operadic model structure implies that an operadic cofibre sequence is also a stable cofibre sequence. In the following by cofibre sequence we mean cofibre sequence in the stable model structure and thereby we also cover the case of cofibre sequences in the Cisinski-Moerdijk model structure. The prototypical example is induced by a normal inclusion of a dendroidal subset:

$$D_0 \hookrightarrow D \twoheadrightarrow D/D_0.$$

Proposition 4.7 *For a cofibre sequence $X \to Y \to Z$ in dSet we obtain a fibre sequence $i^* X_K \to i^* Y_K \to i^* Z_K$ of spaces and thus a long exact sequence*

$$
\begin{array}{ccccc}
\longrightarrow K_n(X) & \longrightarrow K_n(Y) & \longrightarrow K_n(Z) \\
\longrightarrow K_{n-1}(X) & \longrightarrow \cdots & \longrightarrow K_1(Z) \\
\longrightarrow K_0(X) & \longrightarrow K_0(Y) & \longrightarrow K_0(Z) \longrightarrow 0
\end{array}
$$

of K-theory groups.

Proof: The first assertion is Corollary 5.5. in [BN12] and the second is just the long exact sequence of homotopy groups.   □

Example 4.8 Let D be an arbitrary dendroidal set. Then the sequence $\emptyset \to D \to D$ is a cofibre sequence. Thus the long exact sequence implies that $K_n(\emptyset) = 0$ for all n. This in particular implies that the morphism $\emptyset \to *$ is a stable equivalence and the homotopy category is pointed.

5. The K-theory spectrum

In [BN12] it was shown that the category dSet together with the stable model structure is Quillen equivalent to the category of connective spectra. The proof was based on results of [Heu11a, Heu11b]. We do not want to go into the details of the construction here. We only briefly note a few facts and refer to Appendix 7 for more details. First let E_∞ denote the Barratt-Eccles operad, which is a simplicial E_∞-operad. The category of algebras for this operad is then denoted by E_∞-$\mathcal{S}$paces and carries a model structure induced from the model structure on simplicial sets. There is a functor

$$\widetilde{St} : \mathrm{dSet} \to E_\infty\text{-}\mathcal{S}\mathrm{paces}$$

that is left Quillen and has the property that it maps operadic equivalences to weak homotopy equivalences of E_∞-spaces. Moreover it induces after localization on both sides an equivalence of the stable homotopy category of dendroidal sets to the homotopy-category of grouplike E_∞-spaces. In particular $\widetilde{St}$ sends stable equivalences to group-completion equivalences. The definition and properties of the functor $\widetilde{St}$ can be found in Appendix 7 but the reader does not need to know the details of the construction. They are just needed for the following lemma and the rest is deduced by abstract reasoning.

Lemma 5.1 *The functor $\widetilde{St}$ sends the dendroidal set $\eta = i_! \Delta[0]$ to an E_∞-space which is weakly homotopy equivalent to the free E_∞-space on one generator. This E_∞-space can be described as the nerve of the category of finite sets with isomorphisms and tensor product given by disjoint union.*

Proof: We show this by explicitly computing the functor. By definition, η gets send to $\widetilde{St}(\eta) = St_{E_\infty}(\eta \times \mathcal{E}_\infty)$ as described in the Appendix. The first thing we use is that η is cofibrant as a dendroidal set, and thus admits a morphism $\eta \to \mathcal{E}_\infty$ (for the choices we have made in the Appendix this morphism is actually unique). Thus there is a morphism $\eta \to \eta \times \mathcal{E}_\infty$ over $\mathcal{E}_\infty$ which is an operadic equivalence . Therefore we have that $St_{E_\infty}(\eta) \simeq St(\eta \times \mathcal{E}_\infty)$ and it only remains to compute $St(\eta)$. Using the definition of the straightening (see [Heu11a] or Appendix 7) it is easy to see that $St_{E_\infty}(\eta)$ is the free E_∞-space on one generator. Finally we note

that for E_∞ the Barratt-Eccles Operad we immediately get

$$\mathrm{Fr}_{E_\infty}(\Delta[0]) \cong \bigsqcup_{n \in \mathbb{N}} B\Sigma_n \cong N(\mathrm{FinSet})$$

using the usual formula for free algebras over operads.   □

Lemma 5.2   *1. For $X \in sSet$ there is a natural isomorphisms $\widetilde{St}(i_! X) \cong$ $\mathrm{Fr}_{E_\infty}(X)$ in the homotopy category of E_∞-spaces (with weak homotopy equivalences inverted).*

   2. For $D \in dSet$ there is a natural isomorphism in the homotopy category of simplicial sets from the derived underlying space $i^* D_K$ to the underlying simplicial set $\Omega B(\widetilde{St} D)$ of the group-completion of the E_∞-space $\widetilde{St} D$.

Proof: For the proof of (1) we will use the fact that the homotopy category of simplicial sets is the universal homotopy theory on a point. A more precise statement using model categories is that left Quillen functors from sSet to any model category M are fully determined on the point (see [Dug01, Proposition 2.3 and Example 2.4]). This means that evaluation on the point induces an equivalence between the homotopy category of left Quillen functors sSet → M and the homotopy category of M. In the theory of ∞-categories the universal property is that left adjoint functors from the ∞-categories of simplicial sets are determined on the point (see [Lur09, Theorem 5.1.5.6]). This means likewise that there is an equivalence between the ∞-category of left adjoint functors (in the ∞-sense) from the ∞-category of simplicial sets to any presentable ∞-category $\mathcal{C}$ and $\mathcal{C}$.

   Using one of these two statements we see that the two left Quillen functors sSet → E_∞-$\mathcal{S}$paces given by

$$X \mapsto \mathrm{Fr}_{E_\infty}(X) \qquad \text{and} \qquad X \mapsto \widetilde{St}(i_! X)$$

are isomorphic in the homotopy category if they agree on the point. But this is the assertion of Lemma 5.1.

   In order to prove the second statement we note that the first part of the lemma implies that $\widetilde{St}(i_! X)$ and $\mathrm{Fr}_{E_\infty}(X)$ are also equivalent in the group-completion model structure on E_∞-$\mathcal{S}$paces. In other words the diagram of homotopy categories

$$
\begin{array}{ccc}
 & \mathcal{H}o(\mathrm{sSet}) & \\
{\scriptstyle i_!}\swarrow & & \searrow{\scriptstyle \mathrm{Fr}} \\
\mathcal{H}o(\mathrm{dSet}_{stab}) & \xrightarrow[\widetilde{St}]{} & \mathcal{H}o(E_\infty\text{-}\mathcal{S}\mathrm{paces}_{grp})
\end{array}
$$

commutes (up to a natural isomorphism which we suppress). Replacing $\widetilde{St}$ by its inverse we get an isomorphism of functors $i_! \cong \widetilde{St}^{-1} \circ \mathrm{Fr}_{E_\infty}$ on the respective

homotopy categories. Thus there is an induced isomorphism for the right adjoint functors $Ri^* \cong RU^* \circ \widetilde{St}$:

$$
\begin{array}{ccc}
 & \mathcal{H}o(\text{sSet}) & \\
{}^{Ri^*}\nearrow & & \nwarrow^{RU^*} \\
\mathcal{H}o(\text{dSet}_{stab}) \xrightarrow{\quad \widetilde{St} \quad} & & \mathcal{H}o(E_\infty\text{-}\mathcal{S}\text{paces}_{grp})
\end{array}
$$

Here Ri^* is the right derived functor of the underlying space, i.e. given by $i^* D_K$ for a dendroidal set D. The functor RU^* is the right derived functor for the underlying space of an E_∞-space. Since we are dealing with the group completion model structure the functor RU^* is given by ΩBX for an E_∞-space X (or another model of the group completion). Together this shows that $i^* D_k$ and the space $\Omega B \widetilde{St}(D)$ are naturally equivalent in the homotopy category of simplicial sets. $\square$

Remark 5.3 The natural isomorphisms constructed above are actually slightly more structured than just transformations on the homotopy category. The proof shows, mutatis mutandis, that they are transformations of ∞-functors between ∞-categories.

We have already mentioned that fully Kan dendroidal sets correspond to grouplike E_∞-spaces (this is the main result in [BN12]). On the other hand it is well-known that grouplike E_∞-spaces are essentially the same thing as connective spectra. By means of a delooping machine we can define a functor

$$B^\infty : E_\infty\text{-}\mathcal{S}\text{paces} \to \mathcal{S}\text{p}$$

where $\mathcal{S}$p is the category of spectra. The functor B^∞ can be chosen such that it sends group completion equivalences of E_∞-spaces (which are weak homotopy equivalence after group completion) to stable equivalences and induces after localization the desired equivalence of grouplike E_∞-spaces to connective spectra. We do not want to fix a specific choice of delooping machine or model of spectra but refer the reader to the extensive literature on the topic [May74, May77, MT78]. It is just important to note that the spectrum $B^\infty X$ comes with a natural morphism

$$X \to \Omega^\infty B^\infty X$$

which is a weak homotopy equivalence if X is grouplike. It follows that this morphism is a group-completion if X is not grouplike.

Definition 5.4 Let D be a dendroidal set. We define the *K-theory* spectrum of D as

$$\mathcal{K}(D) := B^\infty \widetilde{St}(D).$$

This assignment defines a functor $\mathcal{K} : \mathrm{dSet} \to \mathcal{S}\mathrm{p}$ that preserves stable equivalences and induces an equivalence between the stable homotopy categories of dendroidal sets and connective spectra (see [BN12, Theorem 5.4].)

Theorem 5.5 *1. For a dendroidal set D the derived underlying space i^*D_K (Definition 4.1) is naturally homotopy equivalent to $\Omega^\infty \mathcal{K}(D)$. In particular i^*D_K is an infinite loop space for each dendroidal set D. Notably all $K_n(D)$ are abelian groups and we have*

$$K_n(D) \cong \pi_n(\mathcal{K}(D)).$$

2. For a simplicial set X the spectrum $\mathcal{K}(i_! X)$ is weakly equivalent to the suspension spectrum $\Sigma_+^\infty X$.

Proof: The first statement follows from Lemma 5.2 and the fact that $\Omega^\infty \mathcal{K}(D) = \Omega^\infty B^\infty \widetilde{St}(D)$ is homotopy equivalent to the underlying space of the group completion of $\widetilde{St}(D)$ by the properties of the delooping machine B^∞.

For the second statement note that the first assertion means that we have an isomorphism of functors $\Omega^\infty \circ \mathcal{K} \cong Ri^*$ on the homotopy categories. Since $\mathcal{K}$ is an equivalence (when restricted to connective spectra) we also have an equivalence $\Omega^\infty \cong Ri^* \circ \mathcal{K}^{-1}$. Therefore we get an equivalence of left adjoint functors $\Sigma_+^\infty \cong \mathcal{K} \circ i_!$ where we have used that $\mathcal{K}$ is left adjoint to $\mathcal{K}^{-1}$. □

Corollary 5.6 *We have $\mathcal{K}(\eta) \simeq \mathbb{S}$, where $\mathbb{S}$ denotes the sphere spectrum. Thus $K_n(\eta) \cong \pi_n^{\mathcal{S}}$ where $\pi_n^{\mathcal{S}}$ is the n-th stable homotopy group of spheres.*

Now we have computed the K-theory for the object $\eta \in \mathrm{dSet}$ corresponding to the simplest tree $L_0 = |$. Our next goal is to compute it for the objects $\Omega[T]$ corresponding to arbitrary trees T. Therefore we need the following lemma.

Lemma 5.7 *The morphism*

$$\bigsqcup_{\ell(T)} \eta \longrightarrow \Omega[T]$$

is a stable equivalence of dendroidal sets. Here $\ell(T)$ denotes the set of leaves of T and the morphism is given by the associated morphisms $\eta \to \Omega[T]$ for each leaf.

Proof: We prove this lemma using a result of [CM13a]. For each tree we define the *Segal core*

$$Sc[T] := \bigcup_v \Omega[C_{n(v)}] \subset \Omega[T]$$

where the union is over all the vertices of T, and $n(v)$ is the number of input edges at v. For $\Omega[T] = \eta$ we put $Sc[T] := \eta$. Then the inclusion $Sc[T] \to \Omega[T]$ is a weak operadic equivalence [CM13a, Proposition 2.4].

Our morphism obviously factors through the Segal core:

$$\bigsqcup_{\ell(T)} \eta \longrightarrow Sc[T] \longrightarrow \Omega[T].$$

Thus we have to show that the left morphism is a stable equivalence (note that the right hand morphism is also a stable equivalence since it is a weak operadic equivalence). We do this by induction over the number N of vertices of T. For $N = 1$ we have $Sc[T] = \Omega[T] = \Omega[C_n]$ and therefore the morphism is the inclusion of leaf colours in the corolla which is an outer horn and hence a stable equivalence.

Now assume the statement is true for $N \in \mathbb{N}$ and T has $N + 1$ vertices. We pick the vertex v_0 at the root with input colours $a_1, ..., a_n$. Moreover we denote by $T_1, ..., T_n$ the trees that sit over its leaves (possibly $T_i = \eta$ if there are not further vertices over a_i). Then we have a pushout diagram

$$
\begin{array}{ccc}
\bigsqcup_{i=1}^{n} \eta & \longrightarrow & \Omega[C_n] \\
{\scriptstyle \sqcup a_i} \downarrow & & \downarrow {\scriptstyle v_0} \\
\bigsqcup_{i=1}^{n} Sc[T_i] & \longrightarrow & Sc[T]
\end{array}
$$

in dSet. The upper horizontal morphism is a trivial stable cofibration and therefore also the lower horizontal morphism. Thus in the factorization

$$\bigsqcup_{\ell(T)} \eta \longrightarrow \bigsqcup_{i=1}^{n} Sc[T_i] \longrightarrow Sc[T]$$

the left hand morphism is a stable equivalence by the induction hypothesis and the right hand morphism is a stable equivalence as shown above. $\square$

Corollary 5.8 *We have weak equivalences*

$$\mathcal{K}(\Omega[T]) \simeq \bigvee_{\ell(T)} \mathbb{S} \simeq (\mathbb{S})^{\times \ell(T)}$$

and thus $K_n(\Omega[T]) \cong \bigoplus_{\ell(T)} \pi_n^{\mathcal{S}}$ where $\ell(T)$ is the set of leaves of T and $\pi_n^{\mathcal{S}}$ are the stable homotopy groups of spheres.

Proof: By Lemma 5.7 and the fact that $\mathcal{K}$ is invariant under stable equivalences we only need to show that $\mathcal{K}(\bigsqcup \eta) \simeq \bigvee \mathbb{S}$. This is true by Corollary 5.6 together with the fact that $\mathcal{K}$ is an equivalence of homotopy-categories and therefore preserves coproducts (which are computed in connective spectra as in spectra by the wedge sum). $\square$

6. Comparison with ordinary algebraic K-Theory

In this section we want to show how algebraic K-theory of dendroidal sets generalizes classical algebraic K-theory of rings.

Therefore recall that by definition algebraic K-theory of a ring R is computed using its groupoid of finitely generated projective modules (or some related space like $BGl(R)$). There are several equivalent variants to produce a spectrum $\mathcal{K}(R)$ from this category. Let us recall an easy 'group-completion' variant here.

The variant we want to describe does not only work for the category of finitely generated projective modules over a ring but more generally for an arbitrary symmetric monoidal category C. The first step is to take the nerve NC which is an E_∞-space and then apply a delooping machine to get a K-theory spectrum $\mathcal{K}(C) := B^\infty NC$ (see e.g. [Tho82]). The main purpose of this section is to prove the following theorem:

Theorem 6.1 *For a symmetric monoidal groupoid C we have a weak equivalence of spectra*

$$\mathcal{K}(N_d C) \simeq \mathcal{K}(C)$$

and therefore also $K_n(N_d C) \cong K_n(C)$.

Remark 6.2 We restrict ourselves here to the case of symmetric monoidal groupoids but the theorem holds for arbitrary symmetric monoidal categories C. However groupoids are the most important case if we are interested in algebraic K-theory of rings. The reason for our restriction is that the proof of the theorem as it is stated here is relatively easy and formal whereas in the case of arbitrary symmetric monoidal categories one has to use explicit models and the calculation becomes long and technical.

Corollary 6.3 *Let R be a ring and $Proj_R$ denote the groupoid of finitely generated, projective R-modules. Then we have*

$$\mathcal{K}(N_d Proj_R) \simeq \mathcal{K}(R)$$

and therefore $K_n(N_d Proj_R) = K_n(R)$ where $K_n(R)$ are the algebraic K-theory groups of R.

Lemma 6.4 *Let $\mathcal{SymMonGrpd}$ denote the category of small symmetric monoidal groupoids (which is really a 2-category). Then the functors*

$$N : \mathcal{SymMonGrpd} \to E_\infty\text{-}\mathcal{Spaces} \quad and \quad N_d : \mathcal{SymMonGrpd} \to d\mathcal{Set}_{cov}$$

both admit left adjoint functors at the level of homotopy-categories (resp. ∞-categories). Here we take the covariant model structure on dendroidal sets.

Proof: We explicitly describe both left adjoint functors. The first is well known and the second is a variant of a functor considered by Moerdijk-Weiss [MW07, section 4].

The left adjoint of $N : \mathcal{H}o(\mathcal{S}ym\mathcal{M}on\mathcal{G}rpd) \to \mathcal{H}o(E_\infty\text{-}\mathcal{S}paces)$ is the '1-truncation' functor τ. Let X be an E_∞-space, i.e. a simplicial set with the structure of an algebra over the Barratt-Eccles operad. Then $\tau(X)$ is the free groupoid generated by 0-simplices of X as objects and 1-simplices of X as morphisms subject to the relations generated by 2-simplices of X. The groupoid $\tau(X)$ inherits the structure of a symmetric monoidal category given on generators by the E_∞-structure of X. The assignment $X \mapsto \tau(X)$ even defines a functor

$$\tau : E_\infty\text{-}\mathcal{S}paces \to \mathcal{S}ym\mathcal{M}on\mathcal{G}rpd$$

which sends weak homotopy equivalences of E_∞-spaces to equivalences of symmetric monoidal groupoids and which is left adjoint to N as functors of ∞-categories, hence also on the level of homotopy categories.

The left adjoint of $N_d : Ho(\mathcal{S}ym\mathcal{M}on\mathcal{G}rpd) \to Ho(\text{dSet}_{cov})$ can be described very similarly. For a dendroidal set D it is given by the free symmetric monoidal groupoid generated by the objects of D and for each n-corolla from $d_0,...,d_n$ to d there is an isomorphism $d_0 \otimes ... \otimes d_n \to d$. The relations are given by corollas with two vertices. This is just the symmetric monoidal groupoid version of the functor τ_d described in [MW07, section 4]. We denote this functor by

$$\tau_d^\otimes : \text{dSet} \to \mathcal{S}ym\mathcal{M}on\mathcal{G}rpd$$

It is easy to see that this functor descends to the level of infinity categories. By construction it is clear that the functor is then an adjoint on the level of ∞-categories.

□

Proof of Theorem 6.1.: The proof is based on the fact that both K-theory spectra are defined similarly. The K-theory spectrum $\mathcal{K}(N_d C)$ is defined as the spectrum associated to the E_∞-space $\widetilde{St}(N_d C)$ and the K-theory spectrum of C as the spectrum associated to the E_∞-space NC. Thus we only have to show that for a given symmetric monodical groupoid there is a (natural) equivalence of E_∞-spaces $\widetilde{St}(N_d C) \cong NC$. In other words we want to show that the diagram

$$\mathcal{H}o(\mathcal{S}ym\mathcal{M}on\mathcal{G}rpd) \tag{2}$$

$$\mathcal{H}o(\text{dSet}_{cov}) \xrightarrow[\quad\widetilde{St}\quad]{} \mathcal{H}o(E_\infty\text{-}\mathcal{S}paces)$$

(with N_d and N)

commutes. Here $\mathcal{S}ym\mathcal{M}on\mathcal{G}rpd$ denotes the category of symmetric monoidal groupoids and dSet_{cov} indicates that we are working with the covariant model

structure on dendroidal sets. In this case the lower horizontal functor is an equivalence.

In order to show the commutativity of the above diagram we use the fact that all functors in this diagram admit left adjoints. The left adjoints of the upper two functors are described in the last lemma and the left adjoint of $\widetilde{St}$ is the inverse $\widetilde{St}^{-1}$. The commutativity of the above diagram (2) translates into a natural equivalence $\tau_d^\otimes \circ \widetilde{St}^{-1} \cong \tau$ of left adjoint functors

$$\mathcal{H}o(\mathcal{S}ym\mathcal{M}on\mathcal{G}\mathrm{rpd}) \tag{3}$$

$$\mathcal{H}o(\mathrm{dSet}_{cov}) \xleftarrow{\quad \widetilde{St}^{-1} \quad} \mathcal{H}o(E_\infty\text{-}\mathcal{S}\mathrm{paces})$$

with $\tau_d^\otimes$ and τ the diagonal maps.

As a next step we want to use the universal property of the ∞-category of E_∞-spaces established in [GGN13]: it is the free presentable, preadditive ∞-category on one generator. Let us explain what this means. First an ∞-category $\mathcal{D}$ is called preadditive if finite products and coproducts agree, more precisely if it is pointed and the canonical map from the coproduct to the product is an equivalence. Let $\mathcal{D}$ be a presentable, preadditive ∞-category. Then the universal property of E_∞-$\mathcal{S}$paces is that there is an equivalence of ∞-categories

$$\mathrm{Fun}^L(E_\infty\text{-}\mathcal{S}\mathrm{paces}, \mathcal{D}) \simeq \mathcal{D}$$

where Fun^L denotes the category of left adjoint functors. The equivalence is given by evaluation on the free E_∞-algebra on one generator $\mathrm{F} := \mathrm{Fr}_{E_\infty}(\Delta[0])$. In particular if we have two left adjoint functors $A, B : E_\infty\text{-}\mathcal{S}\mathrm{paces} \to \mathcal{D}$ then they are naturally equivalent if and only if there is an equivalence of $A(\mathrm{F}) \to B(\mathrm{F})$ in $\mathcal{D}$.

In our case we first observe that $\mathcal{S}ym\mathcal{M}on\mathcal{G}\mathrm{rpd}$ is clearly preadditive. Thus in order to compare the left adjoint functors $\tau_d^\otimes \circ \widetilde{St}^{-1}$ and τ we only have to show that they assign equivalent symmetric monoidal groupoids to the free E_∞-algebra on the point. Now the right functor τ is easy to evaluate since $\mathrm{F} = \mathrm{Fr}_{E_\infty}(\Delta[0])$ is the nerve of the category of finite sets with isomorphisms. Thus $\tau(\mathrm{F})$ is the category of finite sets with isomorphisms and tensor product given by disjoint union. Observe that this is also the free symmetric monoidal category on one generator.

The functor $\widetilde{St}^{-1}$ evaluated on F is given by η as shown in Lemma 5.1. Thus $\tau_d \circ \widetilde{St}^{-1}$ of F is given by the free symmetric monoidal category on one generator and hence equivalent to $\tau(F)$. Together we have shown that the two functors $\tau_d \circ \widetilde{St}^{-1}$ and τ agree on F and together with the universal property of E_∞-$\mathcal{S}$paces this completes the proof.

$\square$

7. Appendix: the straightening functor of Heuts

In this appendix we want to collect some facts about the dendroidal straightening functor which has been defined and studied by Heuts [Heu11a]. We do not describe the most general instance of the straightening functor, but only the variant which is used in this paper. First we have to fix some notation. By E_∞ we denote the Barratt-Eccles operad. This is a simplicially enriched operad with one colour and the simplicial set of operations is given by

$$E_\infty(n) = E\Sigma_n = \Sigma_n // \Sigma_n \ .$$

Here Σ_n is the permutation group on n letters and $\Sigma_n // \Sigma_n$ is the simplicial set given as the nerve of the action groupoid of Σ_n on itself by right multiplication.

Now there is an adjunction between dendroidal sets and simplicially enriched operads

$$hc\tau_d : \text{dSet} \rightleftarrows \text{sOper} : hcN_d \qquad (4)$$

which was introduced in [MW07] and studied in much more detail in [CM13b]. The left adjoint is completely determined on representables since dSet is a presheaf category. The definition of $hc\tau_d$ is

$$hc\tau_d(\Omega[T]) := W\Omega(T)$$

where W is the Boardman-Vogt resolution of operads [BV73]. Let us describe the simplicially enriched operad $W\Omega(T)$ explicitly here: Its colours are the edges of T. For edges $c_1,...,c_n,c$ the simplicial set of operations $W\Omega(T)(c_1,...,c_n;c)$ is given by

$$W\Omega(T)(c_1,...,c_n;c) = \Delta[1]^{i(V)}$$

if there is a maximal subtree V of T with leaves $c_1,...,c_n$ and root c. In this case $i(V)$ denotes the number of inner edges of the uniquely determined subtree V. If there is no such subtree we set $W\Omega(T)(c_1,...,c_n;c) = \emptyset$. The composition in $W\Omega(T)$ is given by grafting trees, assigning length 1 to the newly arising inner edges. See Remark 7.3 of [MW09] for a more detailed description of the composition.

There is a distinguished algebra A_T for the operad $W\Omega(T)$ which will play an important role in the definition of the straightening later. The value of A_T on the colour c (which is an edge of T) is given by:

$$A_T(c) := \Delta[1]^{i(c)}$$

where $i(c)$ is the number of edges over c in T. The structure maps of A_T as an $W\Omega(T)$-algebra

$$W\Omega(T)(c_1,...,c_n;c) \times A_T(c_1) \times ... \times A_T(c_n) \to A_T(c)$$

are given by grafting trees, assigning length 1 to the newly arising inner edges $c_1,...,c_n$.

In fact Moerdijk and Cisinski have shown that the adjunction (4) is a Quillen equivalence between the Cisinski-Moerdijk model structure on dSet and an appropriate model structure on simplicially enriched, coloured operads. We now apply the homotopy coherent dendroidal nerve to the Barratt-Eccles operad and obtain a dendroidal set

$$\mathcal{E}_\infty := hcN_d(E_\infty).$$

Let dSet/$\mathcal{E}_\infty$ denote the category of dendroidal sets over $\mathcal{E}_\infty$. The category of E_∞-$\mathcal{S}$paces is the category of algebras for the Barratt-Eccles operad in the category of simplicial sets (so space here means simplicial set).

We are now ready to give the definition of the straightening functor

$$St_{E_\infty} : \text{dSet}/\mathcal{E}_\infty \to E_\infty\text{-}\mathcal{S}\text{paces} .$$

The original definition of the straightening functor (which is a significantly more general variant) is given in [Heu11a, Section 2.2]. First we remark that the category dSet/$\mathcal{E}_\infty$ is freely generated under colimits by objects of the form $\Omega[T] \xrightarrow{s} \mathcal{E}_\infty$, where T is a tree and s and arbitrary morphism. We will define St_{E_∞} for those objects and then left Kan extend it to the whole category dSet/$\mathcal{E}_\infty$.

In order to define $St_{\mathcal{E}_\infty}(\Omega[T] \xrightarrow{s} \mathcal{E}_\infty)$ we use the fact that by adjunction the morphism $s : \Omega[T] \to \mathcal{E}_\infty$ uniquely determines a morphism $\tilde{s} : W\Omega[T] \to E_\infty$. This morphism $\tilde{s}$ then induces an adjunction

$$\tilde{s}_! : W\Omega(T)\text{-}\mathcal{S}\text{paces} \rightleftarrows E_\infty\text{-}\mathcal{S}\text{paces} : \tilde{s}^*$$

where the right adjoint $\tilde{s}^*$ is given by pullback along $\tilde{s}$. Finally we can define

$$St_{E_\infty}(\Omega[T] \xrightarrow{s} \mathcal{E}_\infty) := \tilde{s}_!(A_T).$$

where A_T is the $W\Omega(T)$-algebra defined above. Together this defines the desired functor St_{E_∞} by left Kan extension.

The straightening functor as defined above has nice homotopical properties (see the next proposition). But before we describe these properties, we want to get rid of the overcategory dSet/$\mathcal{E}_\infty$. There is an easy way to do this, by considering the functor dSet $\to$ dSet/$\mathcal{E}_\infty$ which is defined by $D \mapsto D \times \mathcal{E}_\infty$. It turns out that this functor has both adjoints. Replacing a dendroidal set D by the dendroidal set $D \times \mathcal{E}_\infty$ amounts to cofibrant replacement. This is basically because $\mathcal{E}_\infty$ is

a cofibrant resolution of the point. Thus homotopically the category dSet/$\mathcal{E}_\infty$ is equivalent to dSet itself. Therefore we define the functor $\widetilde{St}$ as the composition dSet $\to$ dSet/$\mathcal{E}_\infty$ $\to$ E_∞-$\mathcal{S}$paces, concretely $\widetilde{St}(D) := St_{E_\infty}(D \times \mathcal{E}_\infty)$.

Proposition 7.1 (Heuts)   *1. The functors*

$$\widetilde{St} : dSet \to E_\infty\text{-}\mathcal{S}paces \qquad and \qquad St_{E_\infty} : dSet/\mathcal{E}_\infty \to E_\infty\text{-}\mathcal{S}paces$$

are left Quillen with respect to the Moerdijk-Cisinski model structure and the usual model structure on E_∞-$\mathcal{S}$paces (in which the fibrations and weak equivalences are taken to be those of the underlying simplicial sets). Moreover both functors send operadic equivalences to weak homotopy equivalences.

2. *There are further model structures on dSet and dSet/$\mathcal{E}_\infty$ called the covariant model structures. For these model structure the above functors are Quillen equivalences.*

Variant 7.2 There are variants of the straightening functor for different choices of E_∞-operads than the Barratt-Eccles operad. More precisely if E is any simplicial E_∞-operad which is cofibrant, then there is a corresponding straightening functor

$$St_E : dSet/hcN_d(E) \to E\text{-spaces}$$

Conversely for any choice of cofibrant resolution of the point $\mathcal{E} \to *$ in dendroidal sets, there is a straightening functor

$$St : dSet/\mathcal{E} \to hc\tau_d(E)\text{-spaces}$$

These functors are defined with the same formulas as above and are all essentially equivalent to the variant using the Barratt-Eccles operad.

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