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On Modeling Risk Shocks*

Abstract

Within the context of a financial accelerator model, we model time-varying uncertainty (i.e. risk shocks) through the use of a mixture Normal model with time variation in the weights applied to the underlying distributions characterizing entrepreneur productivity. Specifically, we model capital producers (i.e. the entrepreneurs) as either low-risk (relatively small second moment for productivity) and high-risk (relatively large second moment for productivity) and the fraction of both types is time-varying. We show that a small change in the fraction of risky types (a change from 1% to 2% of the population) can result in a large quantitative effect or a risk shock relative to standard models. The bankruptcy rate and the risk premium in the economy are very sensitive to a change in the composition of agents and is countercyclical.

- *JEL Classification: E22, E32*
- Keywords: *agency costs, credit channel, time-varying uncertainty, mixture models.*

Victor Dorofeenko
Department of Economics and Finance
Institute for Advanced Studies
Josefstaetterstr. 39
A-1080 Vienna, Austria

Gabriel S. Lee (Corresponding Author)
Department of Economics
University of Regensburg
Universtitaetstrasse 31
93053 Regensburg, Germany
And
Institute for Advanced Studies
Josefstaetterstr. 39
A-1080 Vienna, Austria

Kevin D. Salyer
Department of Economics
University of California
Davis, CA 95616

Johannes Strobel
University of Regensburg
Universitaetsstr. 31, 93053 Regensburg, Germany
johannes.strobel@ur.de,+ 49 941 943 5063
<u>Contact Information:</u>
Lee: 49.941.943.5060; E-mail: gabriel.lee@ur.de
Salyer: (530) 752 8359; E-mail: kdsalyer@ucdavis.edu

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1 Introduction

A better understanding of the effects of time varying uncertainty, motivated in no small part by recent economic events as well as improvements in numerical methods, has become the goal of much recent research in macroeconomics. An example of this literature is Christiano, Motto, and Rostagno (2014) in which they find that changes in productivity uncertainty, which the authors characterize as risk shocks, are the dominant source of shocks in the Euro area and, in the U.S., are second only to an aggregate technology shock in accounting for business cycle volatility.[1] Another example is that of Bloom, Floetotto, Jaimovich, Saporta and Terry (2012) in which, using a different empirical strategy, they also demonstrate the countercyclical nature of risk shocks; the authors then construct a model with heterogeneous firms consistent with many of the observed features. Unlike the aforementioned papers, Jurado, Ludvingson and Ng (2015), however, show their estimate of time varying macroeconomic uncertainty occurs less frequently and delivers quantitatively unimportant uncertainty episodes than stated by other existing uncertainty proxies. But, Jurado et al (2015) also show that when these infrequent uncertainty events do occur then they are large, more persistent, and are more correlated with real activity.

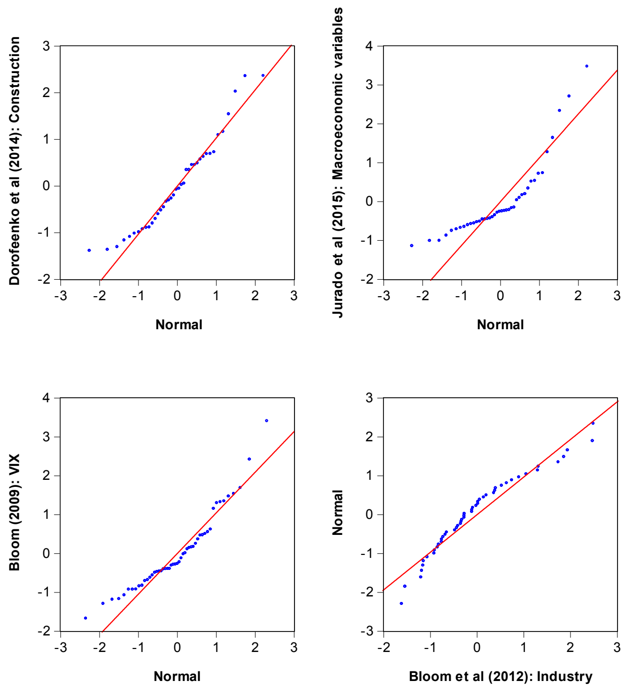
While this burgeoning literature on time-varying uncertainty explores different amplification and propagation mechanisms, a common theme is that a risk shock is characterized as a second moment shock, i.e. as a mean-preserving spread in some distribution (typically that describing firms' productivity shocks). Figure 1, however, shows the Quantile-Quantile plots of four different uncertainty (risk) shocks[2] with clear skewness and kurtosis effects.

In this paper, we thus present a richer characterization of risk (uncertainty) by introducing two types of firms (low and high risk) so that the aggregate cumulative density function for

[1] Some other papers which examine the effects of time varying uncertainty are: Bloom (2009) identifies 17 uncertain periods using the VIX. Arellano, Bai, and Kehoe (2012) examine time varying uncertainty with a focus on the recent financial crisis. Dorofeenko, Lee, and Salyer, (2014) examine the effects of risk shocks in a model of housing production. Using firm level data, Chugh (2014) finds that the quantitative effects of risk shocks, when model is estimated using micro data, to be quite small. Bachman and Bayer (2012) also use firm level data and find that the quantitative effects of risk shocks are small.

[2] Four uncertainty measures are Bloom (2009) for VIX, Bloom et al (2012) for firm productivity, Dorofeenko et al (2014) for construction industry, and Jurado et al (2015) for macroeconomic variables.

Figure 1: Quantile-Quantile Plots for Uncertainty Measures vs Normal Distribution (2001:4-2014:1)



firm productivity shocks is a mixture density. Consequently, we demonstrate that not only the second moment but also the third (skewness) and the fourth (kurtosis) moments play a significant role in explaining the large quantitative effects of risk shocks.[3] In doing so, our objective is to further support the existing evidence that changes in uncertainty are quantitatively important (sometimes) and to provide some basis (a model) for how this time varying uncertainty comes about.

Our model framework is that of Dorofeenko, Lee, and Salyer (2008)— henceforth, DLS — who analyze the importance of risk shocks in the financial accelerator model of Carlstrom and Fuerst (1997). But we depart from the DLS framework by adding additional heterogeneity in the entrepreneur section and introducing time-varying uncertainty by treating the fraction of agent

[3] There are a few recent papers which analyse the higher moment effects on aggregate output. For example, Atalay and Drautzburg (2015) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) look at the contribution of industries' shocks to aggregate output and employment tail risks. Guvenen, Karahan, Ozkan and Song (2015), analyse the higher moment effects of earning shocks.

types as time-varying. The main result of this analysis is that a small absolute change in the fraction of risky agents (a change from 1% of the population to 2%) has large quantitative effects for firm bankruptcy and the risk premium associated with bank loans. In addition, as in DLS, an increase in the number of risky agents results in a countercyclical bankruptcy rate as observed in the data in contrast to the behavior in the original Carlstrom and Fuerst (1997) model.

2 Model

As our model departs from the DLS framework by adding heterogeneity in the entrepreneur section, the model description presented here will be brief but nevertheless self-contained. (A complete description of the model is provided in the Appendix.)

The model is a variant of a standard RBC model in which an additional production sector is added. This sector produces capital using a technology which transforms investment into capital. In a standard RBC framework, this conversion is always one-to-one; in the Carlstrom and Fuerst (1997) "investment model" framework, the production technology is subject to technology shocks. (The aggregate production technology is also subject to technology shocks as is standard.) Specifically, letting i_t denote investment, the production function for new capital is given by the linear relationship $\omega_t i_t$ where ω_t is the technology shock. The capital production sector is owned by risk-neutral entrepreneurs who finance their production via loans from a risk neutral financial intermediation sector - this lending channel is characterized by a loan contract with a fixed interest rate. (Both capital production and the loans are intra-period.) If a capital producing firm realizes a low technology shock , the firm will declare bankruptcy and the financial intermediary will take over production; this activity is subject to monitoring costs which reduces aggregate capital production.

To this scenario, unlike in the DLS model, we introduce heterogeneity by assuming that there exists two types of entrepreneurs in the economy: a risky type with $c.d.f.$for the technology shock affecting capital production given by $\Phi(\omega;\sigma_1)$ and a normal (i.e. low-risk) entrepreneur with *c.d.f.*for the technology shock affecting capital production given by $\Phi(\omega;\sigma_2)$ where $\sigma_1 > \sigma_2$ and σ_i denotes the standard deviation for entrepreneur i. For both types of entrepreneurs, it is assumed that ω is normally distributed with the mean of the technology shock equal to unity; i.e. $E(\omega) = 1$. Uncertainty is introduced by assuming that the probability of being a risky type, denoted p_t, is a random variable where $p_t \in (0,1)$. (Given the assumption that there are a continuum of entrepreneurs distributed on the unit interval, p_t also denotes the fraction of risky entrepreneurs.) It is assumed that banks and entrepreneurs observe p_t but entrepreneurs do not know their type; this is observed when the value of ω_t is revealed. As a consequence, lending contracts can not be written conditional on σ_i. (Aside from the distribution of technology shocks, all entrepreneurs are identical and, as in Carlstrom and Fuerst (1997), are assumed to be risk neutral with preferences over consumption and leisure.)

This description implies that the *c.d.f.* relevant for both lenders (i.e. banks) and borrowers (i.e. entrepreneurs) is a mixture model given by:

$$\Phi_m(\omega, p_t) = p_t \Phi(\omega;\sigma_1) + (1-p_t)\Phi(\omega;\sigma_2) \tag{1}$$

An appealing feature of a mixture model is that the linear combination of two normal distributions results in a distribution with much greater kurtosis and skewness[4] than implied by normality.[5] In previous research, risk shocks have been characterized solely in terms of second moments[6] but the

[4] The explicit formulas for skewness and kurtosis of the mixture model are as follows: Skewness:

$$\frac{p\left(\sigma_1^2+3\right)\sigma_1^4+(1-p)\left(\sigma_2^2+3\right)\sigma_2^4}{\left(p\sigma_1^2+(1-p)\sigma_2^2\right)^{3/2}}$$

Kurtosis:

$$\frac{\begin{array}{c} p\left(\sigma_1^8+6\sigma_1^6+15\sigma_1^4+16\sigma_1^2+3\right)\sigma_1^4+ \\ (1-p)\left(\sigma_2^8+6\sigma_2^6+15\sigma_2^4+16\sigma_2^2+3\right)\sigma_2^4 \end{array}}{\left(p\sigma_1^2+(1-p)\sigma_2^2\right)^{2}}-3$$

[5] It is well-known in the finance literature that financial returns exhibit more extreme observations than would be predictedby the normal or Gaussian distribution: that is, financial returns have fat tails. Since Mandelbrot (1963) highlighted the issue, the most if not all the VaR (value at risk) analysis employ various distributions that display fat tails.

[6] Some of the proposed measures of uncertainty in the literature are the financial market uncertainty by Bloom (2009), the policy uncertainty measure by Baker, Bloom, and Davis (2013), and the macro uncertainty measure by

mixture model, as presented here, provides a richer characterization of increased risk. Moreover, even though we employ linearization methods to solve the model, increased risk in the mixture model can produce relatively large quantitative responses in the model.

As stated above, all other features of the Carlstrom and Fuerst (1997) model are the same. In particular, the timing of events within a time-period is as follows:

1. The exogenous state vector of technology shocks and fraction of risky agents, denoted (θ_t, p_t), is realized.

2. Firms hire inputs of labor and capital from households and entrepreneurs and produce output via an aggregate production function.

3. Households make their labor, consumption and savings/investment decisions. The household transfers q_t consumption goods to the banking sector for each unit of investment.

4. With the savings resources from households, the banking sector provide loans to entrepreneurs via the optimal financial contract. The contract is defined by the size of the loan (i_t) and a cutoff level of productivity for the entrepreneurs technology shock, $\bar{\omega}_t$.

5. Entrepreneurs use their net worth and loans from the banking sector as inputs into their capital-creation technology.

6. The idiosyncratic technology shock of each entrepreneur is realized. If $\omega_{j,t} \geq \bar{\omega}_t$ the entrepreneur is solvent and the loan from the bank is repaid; otherwise the entrepreneur declares bankruptcy and production is monitored by the bank at a cost of μi_t.

7. Entrepreneurs that are solvent make consumption choices; these in part determine their net worth for the next period.

We now focus on the lending contract and the role of time varying uncertainty.

Jurado, Ludvigson, and Ng (2015). See for example Bloom (2014) for the recent literature overview.

2.1 Optimal Financial Contract

The optimal financial contract between entrepreneur and lender is described by Carlstrom and Fuerst (1997). But for expository purposes as well as to explain our approach in addressing the effect of greater risk on equilibrium, we briefly outline the model. In deriving the optimal contract, both entrepreneurs and lenders take the price of capital, q_t, and net worth, n_t, as given.

As described above, the entrepreneur has access to a stochastic technology that transforms i_t units of consumption into $\omega_t i_t$ units of capital. In Carlstrom and Fuerst (1997), the technology shock ω_t was assumed to be distributed as $i.i.d.$ with $E(\omega_t) = 1$. In contrast, here the relevant $c.d.f.$for both lender and entrepreneur, $\Phi_m(\omega, p_t)$, is given by the mixture model as defined in eq. (1); we denote the associated $p.d.f.$ as $\phi_m(\omega, p_t)$. It is assumed that the fraction of risky agents, p_t, following process [7]:

$$p_t = p_0 \exp(u_t) \tag{2}$$

where $u_t = \rho_u u_{t-1} + \epsilon_{u,t}, \epsilon_{u,t} \tilde{} N(0, \sigma_u)$. To keep the value of p_t in the interval $[0, 1]$, the distribution of shocks should be truncated from above $u_t \le u_m$. One can show that u_m then should satisfy the inequality $u_m \le \log(p_0^{-1})$. The unconditional mean of the fraction of risky agents is given by $\bar{p}$. While the current value of p_t is known by all agents, the realization of ω_t is privately observed by the entrepreneur; banks can observe the realization at a cost of μi_t units of consumption.

The entrepreneur enters period t with one unit of labor endowment and z_t units of capital. Labor is supplied inelastically while capital is rented to firms at the rental rate r_t, hence income in the period is $w_t + r_t z_t$. This income along with remaining capital determines net worth (denoted as n_t and denominated in units of consumption) at time t:

$$n_t = w_t + z_t (r_t + q_t (1 - \delta)) \tag{3}$$

[7] For the calibration exercise, we scale u_t so that 1% increase in u_t translates into 1% increase in p_t.

With a positive net worth, the entrepreneur borrows $(i_t - n_t)$ consumption goods and agrees to pay back $\left(1+r^k\right)(i_t - n_t)$ capital goods to the lender, where r_t^k is the interest rate on loans. Thus, the entrepreneur defaults on the loan if his realization of output is less then the re-payment, i.e.

$$\omega_t < \frac{\left(1+r_t^k\right)(i_t - n_t)}{i_t} \equiv \bar{\omega}_t \tag{4}$$

The optimal borrowing contract is given by the pair $(i_t, \bar{\omega}_t)$ that maximizes entrepreneur's return subject to the lender's willingness to participate (all rents go to the entrepreneur). The optimum is determined by the solution to:

$$\max_{\{i,\bar{\omega}\}} q_t i_t f_m(\bar{\omega}_t, p_t) \text{ subject to } q_t i_t g_m(\bar{\omega}_t, p_t) \geq (i_t - n_t)$$

where

$$f_m(\bar{\omega}_t, p_t) = p_t \int_{\bar{\omega}_t}^{\infty} \omega\phi(\omega;\sigma_1)\,d\omega + (1-p_t)\int_{\bar{\omega}_t}^{\infty} \omega\phi(\omega;\sigma_2)\,d\omega - [1-\Phi_m(\bar{\omega}_t, p_t)]\,\bar{\omega}_t \tag{5}$$

which can be interpreted as the fraction of the expected net capital output received by the entrepreneur. Note that the first two terms represent the expected fraction of capital received by the entrepreneur if solvent while the last term represents the interest payment weighted by the probability of being solvent (where the relevant $c.d.f.$is $\Phi_m(\omega, p_t)$.) The function $g_m(\bar{\omega}_t, p_t)$ is the corresponding fraction of the expected net capital output received by the lender and is given by:

$$g_m(\bar{\omega}_t, p_t) = p_t \int_{-\infty}^{\bar{\omega}_t} \omega\phi(\omega;\sigma_1)\,d\omega + (1-p_t)\int_{-\infty}^{\bar{\omega}_t} \omega\phi(\omega;\sigma_2)\,d\omega + [1-\Phi_m(\bar{\omega}_t, p_t)]\,\bar{\omega}_t - \Phi_m(\bar{\omega}_t, p_t)\,\mu \tag{6}$$

The last term represents the expected fraction of capital lost due to monitoring costs. Specifically, note that $f_m(\bar{\omega}_t, p_t) + g_m(\bar{\omega}_t, p_t) = 1 - \Phi_m(\bar{\omega}_t, p_t)\,\mu$: the right hand side is the average amount of capital produced per unit of investment. This is split between entrepreneurs and lenders while monitoring costs reduce net capital production.

The necessary conditions for the optimal contract problem are

$$\frac{\partial(.)}{\partial\bar{\omega}}: q_t i_t \frac{\partial f_m(\bar{\omega}_t, p_t)}{\partial\bar{\omega}} = -\lambda_t q_t i_t \frac{\partial g_m(\bar{\omega}_t, p_t)}{\partial\bar{\omega}} \tag{7}$$

where λ_t is the shadow price associated with the lender's resources.

The second necessary condition is:

$$\frac{\partial(.)}{\partial i_t}: q_t f_m(\bar{\omega}_t, p_t) = -\lambda_t\left[1 - q_t g_m(\bar{\omega}_t, p_t)\right] \tag{8}$$

Solving for q using the first order conditions, we have:

$$\begin{aligned} q_t^{-1} &= \left[(f_m(\bar{\omega}_t, p_t) + g_m(\bar{\omega}_t, p_t)) + \frac{\phi_m(\bar{\omega}_t, p_t)\,\mu f_m(\bar{\omega}_t, p_t)}{\frac{\partial f_m(\bar{\omega}_t, p_t)}{\partial\bar{\omega}}}\right] \\ &= \left[1 - \Phi_m(\bar{\omega}_t, p_t)\,\mu + \frac{\phi_m(\bar{\omega}_t, p_t)\,\mu f_m(\bar{\omega}_t, p_t)}{\frac{\partial f_m(\bar{\omega}_t, p_t)}{\partial\bar{\omega}}}\right] \\ &\equiv \left[1 - D(\bar{\omega}_t, p_t)\right] \end{aligned} \tag{9}$$

where $D(\bar{\omega}_t, p_t)$ can be thought of as the total default costs.

It is straightforward to show that equation (9) defines an implicit function $\bar{\omega}(q_t, p_t)$ that is increasing in q_t. Also note that, in equilibrium, the price of capital, q_t, differs from unity due to the presence of the credit market frictions. (Note that $\frac{\partial f_m(\bar{\omega}_t, p_t)}{\partial\bar{\omega}} = \Phi_m(\bar{\omega}_t, p_t) - 1 < 0$.)

The incentive compatibility constraint implies

$$i_t = \frac{1}{(1 - q_t g(\bar{\omega}_t, p_t))} n_t \tag{10}$$

Equation (10) implies that investment is linear in net worth and defines a function that represents the amount of consumption goods placed in to the capital technology: $i(q_t, n_t, p_t)$. The fact that the function is linear implies that the aggregate investment function is well defined.

The effect of an increase in uncertainty on investment in this model can be understood by first turning to eq. (9). Under the assumption that the price of capital is unchanged, this implies that the costs of default, represented in the function $D(\bar{\omega}_t, p_t)$, must also be unchanged. While more general, an increase in p_t is similar to a mean-preserving spread in the distribution for ω_t, this implies that $\bar{\omega}_t$ must fall to keep $D(\bar{\omega}_t, p_t)$ constant. It is shown in Dorofeenko, Lee, Salyer (2008) through an approximation analysis that $g_m(\cdot) \approx \bar{\omega}_t$ so that the fall in $\bar{\omega}_t$ results in a fall in the expected capital return to entrepreneurs. Using the incentive compatibility constraint,

$$q_t g_m(\bar{\omega}_t, p_t) = 1 - \frac{n_t}{i_t}$$

the fall in the left-hand side induces a fall in i_t. As demonstrated below, the quantitative response of investment due to an increase in the fraction of risky agents is high relative to a simpler model in which a risk shock is associated with a mean preserving spread (a change in the second moment only).

2.2 Equilibrium

Equilibrium in the economy is represented by market clearing in the labor and goods markets. (A complete description of the economy is provided in the Appendix.) Letting (H_t, H_t^e) denote the aggregate labor supply of, respectively households and entrepreneurs, we have

$$H_t = (1 - \eta) l_t \tag{11}$$

where l_t denotes labor supply of households and η denotes the fraction of entrepreneurs in the economy.

$$H_t^e = \eta \tag{12}$$

Goods market equilibrium is represented by

$$C_t + I_t = Y_t \tag{13}$$

where $C_t = (1-\eta)\, c_t + \eta c_t^e$ and $I_t = \eta i_t$. (Note upper case variables denotes aggregate quantities while lower case denote per-capita quantities.)

The law of motion of aggregate capital is given by:

$$K_{t+1} = (1-\delta)\, K_t + I_t \left[1 - \Phi_m\left(\bar{\omega}_t; p_t\right)\mu\right] \tag{14}$$

A competitive equilibrium is defined by the decision rules for (aggregate capital, entrepreneurs capital, household labor, entrepreneur's labor, the price of capital, entrepreneur's net worth, investment, the cutoff productivity level, household consumption, and entrepreneur's consumption) given by the vector: $\{K_{t+1}, Z_{t+1}, H_t, H_t^e, q_t, n_t, i_t, \bar{\omega}_t, c_t, c_t^e\}$ where these decision rules are stationary functions of $\{K_t, Z_t, \theta_t, p_t\}$ and satisfy the following equations:

$$
\begin{aligned}
\nu c_t &= \alpha_H \frac{Y_t}{H_t} && (15)\\
\frac{q_t}{c_t} &= \beta E_t \left\{ \frac{1}{c_{t+1}} \left(q_{t+1}(1-\delta) + \alpha_K \frac{Y_{t+1}}{K_{t+1}} \right) \right\} && (16)\\
q_t &= \left\{ 1 - \Phi_m\left(\bar{\omega}_t, p_t\right)\mu + \frac{\phi_m\left(\bar{\omega}; p_t\right)\mu f_m\left(\bar{\omega}_t, p_t\right)}{\frac{\partial f_m(\bar{\omega}_t, p_t)}{\partial \bar{\omega}}} \right\}^{-1} && (17)\\
i_t &= \frac{1}{\left(1 - q_t g_m\left(\bar{\omega}_t, p_t\right)\right)} n_t && (18)\\
q_t &= \beta\gamma E_t \left\{ \left(q_{t+1}(1-\delta) + \alpha_K \frac{Y_{t+1}}{K_{t+1}} \right) \left(\frac{q_{t+1} f_m\left(\bar{\omega}_t, p_t\right)}{\left(1 - q_{t+1} g_m\left(\bar{\omega}_t, p_t\right)\right)} \right) \right\} && (19)\\
n_t &= \alpha_{H^e} \frac{Y_t}{H_t^e} + Z_t \left(q_t (1-\delta) + \alpha_K \frac{Y_t}{K_t} \right) && (20)\\
Z_{t+1} &= \eta n_t \left\{ \frac{f_m\left(\bar{\omega}_t, p_t\right)}{1 - q_t g_m\left(\bar{\omega}_t, p_t\right)} \right\} - \eta \frac{c_t^e}{q_t} && (21)\\
\theta_{t+1} &= \theta_t^{\rho_\theta} \xi_{t+1} \text{ where } \xi_t \sim i.i.d. \text{ with } E\left(\xi_t\right) = 1 && (22)\\
p_t &= p_0 \exp\left(u_t\right) \text{ where } u_t = \rho_u u_{t-1} + \epsilon_{u,t}, \epsilon_{u,t}\tilde{\ } N\left(0, \sigma_u\right) && (23)
\end{aligned}
$$

The first equation represents the labor-leisure choice for households while the second equation is the necessary condition associated with household's savings decision. The third and fourth equation are from the optimal lending contract while the fifth equation is the necessary condition associated with entrepreneur's savings decision. The sixth equation is the determination of net worth while the seventh gives the evolution of entrepreneur's capital. (The evolution of aggregate capital is given in eq. (14)). The final two equations represent the laws of motion for the aggregate technology and fraction of risky agents respectively.

3 Equilibrium Characteristics

3.1 Calibration

For this analysis, we use, to a large extent, the parameters employed in Carlstrom and Fuerst's (1997) and DLS analysis. Specifically, the following parameter values are used:

Table 1: Parameter Values

β	α	δ	ρ_θ	μ
0.99	0.36	0.02	0.95	0.25

Agents discount factor, the depreciation rate and capital's share are fairly standard in RBC analysis. In addition, the aggregate technology shock is assumed to follow a standard AR(1) process with autoregressive parameter of 0.95. The remaining parameter, μ, represents the monitoring costs associated with bankruptcy. This value, as noted by Carlstrom and Fuerst (1997) is relatively prudent given estimates of bankruptcy costs (which range from 20% (Altman (1984) to 36% (Alderson and Betker (1995) of firm assets).

In order to calibrate the parameters of the mixture model (i.e.. $(\sigma_1, \sigma_2, \bar{p})$, we choose the parameters so that the implied steady-state bankruptcy rate and annual risk premium on loans(both terms are defined below) are consistent with U.S. data. In particular we use the following values from Carlstrom and Fuerst (1997): a bankruptcy rate of 0.974% (per quarter) and an annual risk premium of 187 basis points. We assume throughout that the average fraction of risky agents ($\bar{p}$) is equal to 1%.

Let the steady-state bankruptcy rate be denoted as $\overline{br}$ and $\hat{\omega}$ denote the steady-state level of $\bar{\omega}_t$. The steady-state bankruptcy rate is given by the mixture *c.d.f.* evaluated at the cutoff productivity level. That is, we require:

$$\Phi_m\left(\hat{\omega};\bar{p}\right)=\overline{br}=0.00974 \tag{24}$$

All loans in the economy are intra-period so the (gross) risk-free rate is, by definition, equal to unity. Hence, the gross interest rate (defined in terms of consumption goods) associated with the bank loan can also be thought of as the risk premium on bank loans. We denote this risk premium as ζ; in steady-state, it is given by (see eq.(4)):

$$\zeta=\bar{q}\hat{\omega}\frac{\bar{\imath}}{\bar{\imath}-\bar{n}} \tag{25}$$

But the incentive compatibility constraint (eq.(18)) implies that, in steady-state:

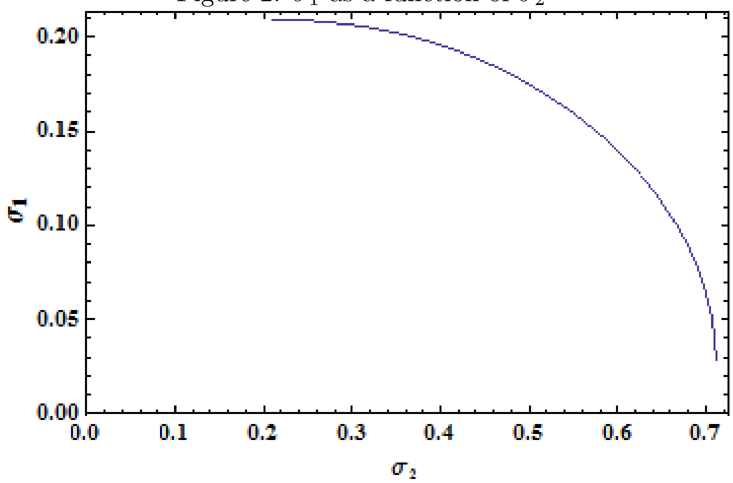
$$\frac{\bar{n}}{\bar{\imath}}=1-\bar{q}g\left(\hat{\omega},\bar{p}\right) \tag{26}$$

Using this in eq. (25) yields the second restriction (the frequency of the model is assumed to be quarterly):

$$\zeta=\frac{\hat{\omega}}{g\left(\hat{\omega},\bar{p}\right)}=\frac{0.0187}{4} \tag{27}$$

With $\bar{p}$ assumed to be 1%, these two equations define an implicit function $\sigma_2\left(\sigma_1\right)$. Numerically solving for this function results in the relationship depicted in Figure 2.Note that in the original Carlstrom and Fuerst (1997) model, the standard deviation of technology shocks was calibrated (using the same empirical strategy given above) to be equal to 0.207. That explains the starting

Figure 2: σ_1 as a function of σ_2

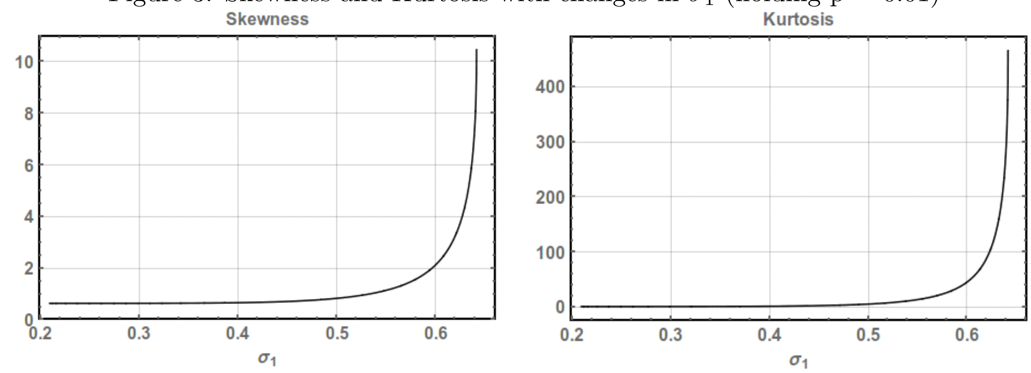


value for the graph in which $\sigma_1 = \sigma_2 = 0.207$. Then as the standard deviation of the risky agents increases, the low risk agents' standard deviation must be adjusted downward so that the model remains consistent with the observed bankruptcy rate and risk premium on loans. Figure 3 shows the corresponding skewness and excess kurtosis values for the mixed Normal distribution $\Phi_m(\omega, p_t)$ when σ_1 changes. Both skewness and kurtosis remain close to the values for a Normal distribution until σ_1 reaches 0.6. At the limit value for $\sigma_1 = 0.643$, the value for kurtosis changes dramatically to 465.2: these large changes both in skewness and kurtosis are the base for the empirical results that we obtain in this paper.

For our empirical analysis, we examine two economies that differ in the degree of riskiness of the two types of agents. In Economy 1, we assume that the standard deviation of the high risk agents is roughly 2.5 times that of the low risk agents. Specifically, it is assumed that $\sigma_1 = 0.52$ implying that $\sigma_2 = 0.191$.[8] In Economy 2, we examine a more extreme case in which standard deviation of risky agent $\sigma_1 = 0.642$ while the low risk agents' standard deviation $\sigma_2 = 0.052$. That is, the high risk agents' standard deviation is roughly 12 times larger than that of the low risk agents. While this is dramatic, it is important to keep in mind that the high risk agents are only

[8] As $\sigma_1 = 0.52$ leads to the kurtosis value of 6.96, which is close to the value of 6.75 that has been reported in Jurado, *et al* (2015), we take $\sigma_1 = 0.52$ to represent Economy 1.

Figure 3: Skewness and Kurtosis with changes in σ_1 (holding p = 0.01)



Corresponding (σ_1: skewness)
(0.207:0.64), (0.52:0.87), (0.643:10.4)

Corresponding (σ_1: Kurtosis)
(0.207: 0.73), (0.52:6.96), (0.643:465.2)

1% of the population of entrepreneurs.

The effect of greater uncertainty as represented by economy II ($\sigma_1 = 0.642, \sigma_2 = 0.052$) in the capital production sector is seen in Table 2. (All values in Table 2 are percentage changes relative to the Carlstrom and Fuerst economy. i.e. $\sigma_1 = \sigma_2 = 0.207$). Consistent with the partial equilibrium analysis presented earlier, a mean-preserving spread in entrepreneur's shock causes the price of capital (q) to increase and steady-state capital to fall. This also implies a decrease in consumption, a slight increase in steady-state labor, and a fall in steady-state output.

Table 2: Steady-State Effects of Greater Uncertainty ($\sigma_1 = 0.642, \sigma_2 = 0.052$)

(comparison to Carlstrom & Fuerst Economy: $\sigma_1 = \sigma_2 = 0.207$)

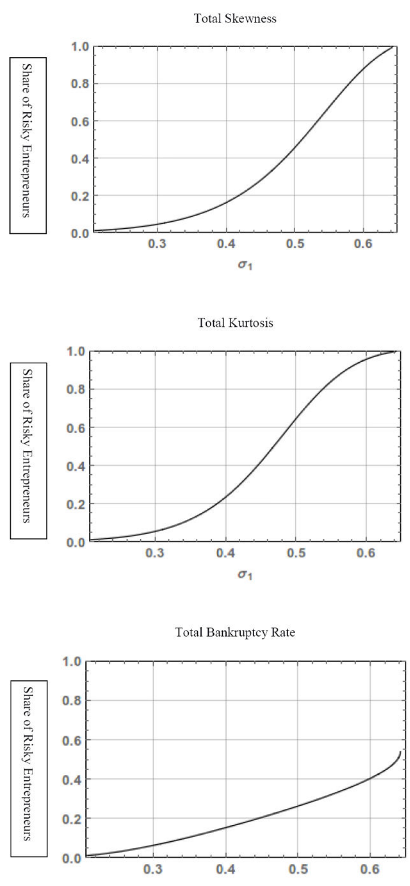
variable	Economy II
c	-0.64
k	-1.79
h	0.00
y	-0.64
q	1.13

3.2 Cyclical Behavior

As described in Section 2, eqs. (15) through (23) determine the equilibrium properties of the economy. To analyze the cyclical properties of the economy, we linearize (i.e. take a first-order Taylor series expansion) of these equations around the steady-state values and express all terms as percentage deviations from steady-state values. This numerical approximation method is standard in quantitative macroeconomics. What is not standard in this model is that the higher moments of technology shocks affecting the capital production sector for both risky and non-risky entrepreneurs, in particular the kurtosis of the mixture distribution, will influence equilibrium behavior and, therefore, the equilibrium policy rules. Linearizing the equilibrium conditions around the steady-state typically imposes certainty equivalence so that only the first moment matters. In this model, however, the higher moments of the entrepreneur's technology shock continue to influence the economy through their role in determining lending activities and, in particular, the nature of the lending contract. Linearizing the system of equilibrium conditions does not eliminate that role in this economy and, hence, we think that this is an attractive feature of the model.

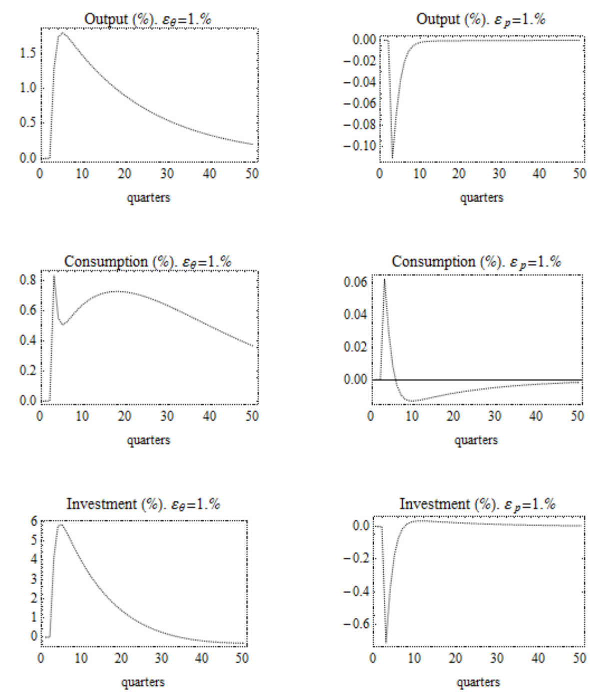
In Table 3, a few key second moments of the economies are reported along with the corresponding moments in the data. (In producing the artificial data, both economies were subject to an aggregate technology shock as well as the stochastic variation in the composition of risky agents.) Note that the behavior of the real side of the economy, as represented in the second moments, is fairly invariant to the composition of the risky agents. Hence, the aggregate technology shock continues to be the primary determinant of the real sector of the economy. However, as seen below, the financial sector variables are highly sensitive to the changes in uncertainty that includes both skewness and kurtosis in the economy. These effects can first be seen in Figure 4 that shows the share of risky entrepreneur contributing to the total bankruptcy rate in the economy: When risky entrepreneurs' σ take the value of 0.6 then the total bankruptcy rate in the economy is caused by 40% of risky entrepreneurs, which is 0.4% (since p is set at 1%)of the total population.

Figure 4: Share of Risky Entrepreneurs for Total Skewness, Kurtosis and Bankruptcy Rate with changes in σ_1 (holding p = 0.01)



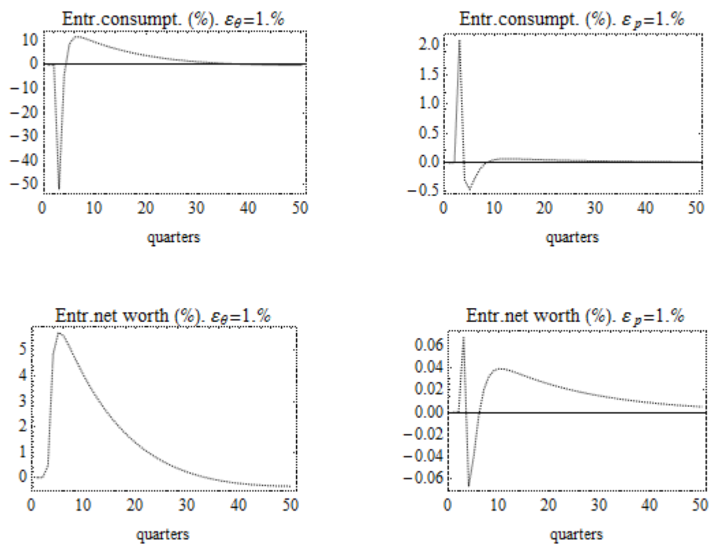
The impulse response functions in Figures 5 to 7 also confirm the aforementioned effects to which we now turn. We first examine aggregate output, household consumption and investment; the impulse response functions for a 1% innovation to the technology shock and the percentage of risky agents are given in Figure 5. The responses to technology shocks are well understood while

Figure 5: Response of Output, Consumption and Investment to 1% changes in Aggregate technology ($\sigma = 0.207$) and the fraction of risky entrepreneurs (1% to 2%) for risky entrepreneur's variance ($\sigma = 0.52$)



the responses to greater uncertainty are best viewed through the lens of the returns to investment. With greater uncertainty as represented by an increase in the fraction of risky entrepreneurs, the bankruptcy rate increases in the economy (see Figure 7), which implies that agency costs increase. This implies that the rate of return on investment for the economy therefore falls. Households, in response, reduce investment and increase consumption and leisure. The latter response causes output to fall. While these effects are qualitatively interesting, it is clear that, quantitatively, the effects of changes in uncertainty are relatively small compare to the effects of the aggregate technology shock. Figure 6 presents the response of the entrepreneur's consumption and net worth. Again, the quantitative impact of the technology shock is much larger than that of a change in the number of risky agents.

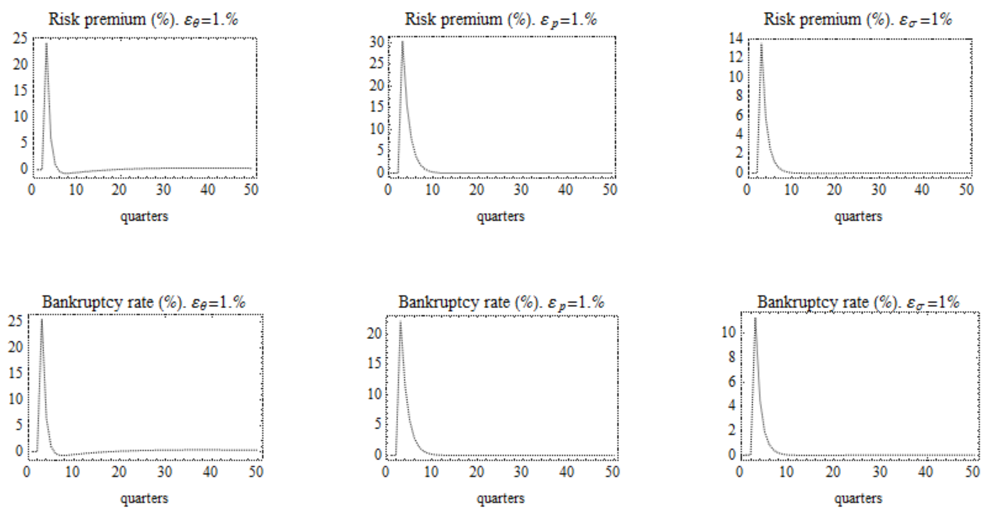
Figure 6: Response of Entrepreneur Consumption and Entrepreneur Net Worth to 1% changes in Aggregate technology ($\sigma = 0.207$) and the fraction of risky entrepreneurs (1% to 2%) for risky entrepreneur's variance ($\sigma = 0.52$)



This small quantitative effects on aggregate variables are, however, overturned for the lending channel variables as seen in Figure 7 which presents the responses of the risk premium and the bankruptcy rate to a 1% change in the aggregate technology shock and an absolute 1% change in the percentage of risky entrepreneurs shock (i.e. from 1% of the population to 2%). For the sake of comparison, we also present the response to a standard risk shock as studied in DLS (and is the typical characterization in the risk shocks literature) in which there is 1% change in the standard deviation of the distribution for the entrepreneur's technology shock.[9] As seen in Figure

[9] For the σ_1 values for $\varepsilon_\theta, \varepsilon_p$, and ε_σ are 0.207, 0.52 and 0.207 respectively.

Figure 7: Response of Risk Premium and Bankruptcy Rate to 1% changes in Aggregate technology (σ = 0.207), the fraction of risky entrepreneurs (1% to 2%) for risky entrepreneur's variance (σ = 0.52), and uncertainty in DLS for entrepreneur's variance (σ = 0.207)



7, time variation in the composition of risky and non-risky agents has the most dramatic effect on the risk premium associated with bank loans and the bankruptcy rate. For instance, when the standard deviation of the risky agents' technology shock is little over two times larger than that in the basic (i.e. Carlstrom and Fuerst (1997)) model (σ_1 = 0.52 vs σ_1 = 0.207), the risk premium on bank loans increases by roughly 30 basis points when the percentage of risky agents increases from 1% to 2%; in the standard model, a 1% risk shock leads to roughly 14 basis point increase. Moreover, the effect of the percentage of risky agents increases from 1% to 2% dominates the effects of aggregate technology shock (i.e. an increase of roughly 30 percent compares to 24 percent). Similarly, in the same economy the change in the bankruptcy rate due to an increase in risky agents is roughly 2 times that caused by a risk shock in the standard model and almost same as that caused by a 1% change in the technology shock. We view these relatively large quantitative responses as suggestive that further investigation of our modeling of risk shocks is merited.

Table 3: Business Cycle Characteristics[10]

shocks[11]	σ_y	Volatility relative to y: c	h	i	k	Correlation with y: c	h	i	k
Economy 1($\sigma_1 = 0.52$)	0.046	0.52	0.74	3.16	0.76	0.69	0.87	0.93	0.40
Economy 2($\sigma_1 = 0.64$)	0.046	0.53	0.73	3.19	0.80	0.71	0.86	0.93	0.40
shocks: only θ_t (tech shock)									
Economy 1($\sigma_1 = 0.52$)	0.046	0.51	0.74	3.17	0.76	0.69	0.87	0.93	0.39
Economy 2($\sigma_1 = 0.64$)	0.046	0.52	0.73	3.20	0.75	0.71	0.86	0.93	0.38
US data[12]	2.04	0.47	0.91	4.03	0.38	0.78	0.86	0.87	-0.07

4 Conclusion

The analysis presented here uses standard solution methods (i.e. linearizing around the steady-state) but exploits features of the Carlstrom and Fuerst (1997) agency cost model of business cycles so that time varying uncertainty can be analyzed. Our measure of time varying uncertainty is different than that of DLS as we introduce additional heterogeneity in the entrepreneur section and treating the fraction of agent types as time-varying so that the effects of time-varying uncertainty manifest through bank lending and investment activity. While development of more general solution methods that capture second moments effects is encouraged, we think that the intuitive nature of this model and its standard solution method make it an attractive environment to study the effects of time-varying uncertainty.

Our primary findings fall into two broad categories. First, as in DLS, we also demonstrate that time varying uncertainty results in countercyclical bankruptcy rates - a finding which is consistent

[10] For this comparative analysis, the standard deviation of the innovation to both shocks was assumed to be 0.007. This figure is typical for total factor productivity shocks but whether this is a good figure for shocks to the second moments is an open question. We also assumed that both shocks exhibit high persistence with an autocorrelation of 0.95 for θ_t and 0.90 for σ_ω.

[11] Both the aggregate production technology shock (θ_t) and the shock to the fraction of risky firms (u_t) are included in both economies.

[12] The US figures are from 1947q1 to 2009q4.

with the data and opposite the result in Carlstrom and Fuerst (1997). Second, we show that the uncertainty affects both quantitatively and qualitatively the behavior of the economy. More specifically, the quantitatively impact of an increase is significantly more than that of an aggregate technology shock for the risk premium and bankruptcy rate. Quantitative effects of changes in uncertainty (even with the skewness and kurtosis), however, on the aggregate variables are still small. Although we believe that the characterization of uncertainty shocks (i.e., second moments or rare catastrophic events) and the development of richer theoretical models which introduce more non-linearities in the equations defining equilibrium is in need for future research, we also believe that our measure of uncertainty does shed light into the quantitative issues that have been discussed in the recent literature.

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5 Appendix:

5.1 Model Description

5.1.1 Households

The representative household is infinitely lived and has expected utility over consumption c_t and leisure $1 - l_t$ with functional form given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln\left(c_t\right) + \nu\left(1 - l_t\right)\right] \tag{28}$$

where E_0 denotes the conditional expectation operator on time zero information, $\beta \in (0, 1)\,, \nu > 0$, and l_t is time t labor. The household supplies labor, l_t, and rents its accumulated capital stock, k_t, to firms at the market clearing real wage, w_t, and rental rate r_t, respectively, thus earning a total income of $w_t l_t + r_t k_t$. The household then purchases consumption good from firms at price of one (i.e. consumption is the numeraire), and purchases new capital, i_t, at a price of q_t. Consequently, the household's budget constraint is

$$w_t l_t + r_t k_t \geq c_t + q_t i_t \tag{29}$$

The law of motion for households' capital stock is standard:

$$k_{t+1} = (1-\delta)\,k_t + i_t \tag{30}$$

where $\delta \in (0,1)$ is the depreciation rate on capital.

The necessary conditions associated with the maximization problem include the standard labor-leisure condition and the intertemporal efficiency condition associated with investment. Given the functional form for preferences, these are:

$$\nu c_t = w_t \tag{31}$$

$$\frac{q_t}{c_t} = \beta E_t \left(\frac{q_{t+1}\,(1-\delta) + r_{t+1}}{c_{t+1}} \right) \tag{32}$$

5.1.2 Firms

The economy's output is produced by firms using Cobb-Douglas technology[13]

$$Y_t = \theta_t K_t^{\alpha_K} H_t^{\alpha_H} \left(H_t^e\right)^{\alpha_{H^e}} \tag{33}$$

where Y_t represents the aggregate output, θ_t denotes the aggregate technology shock, K_t denotes the aggregate capital stock, H_t denotes the aggregate household labor supply, H_t^e denotes the aggregate supply of entrepreneurial labor, and $\alpha_K + \alpha_H + \alpha_{H^e} = 1$.[14]

The profit maximizing representative firm's first order conditions are given by the factor mar-

[13] Note that we denote aggregate variables with upper case while lower case represents per-capita values. Prices are also lower case.

[14] As in Carlstrom and Fuerst, we assume that the entrepreneur's labor share is small, in particular, $\alpha_{H^e} = 0.0001$. The inclusion of entrepreneurs' labor into the aggregate production function serves as a technical device so that entrepreneurs' net worth is always positive, even when insolvent.

ket's condition that wage and rental rates are equal to their respective marginal productivities:

$$
\begin{aligned}
w_t &= \alpha_H \frac{Y_t}{H_t} && (34)\\
r_t &= \alpha_K \frac{Y_t}{K_t} && (35)\\
w_t^e &= \alpha_{H^e} \frac{Y_t}{H_t^e} && (36)
\end{aligned}
$$

where w_t^e denotes the wage rate for entrepreneurial labor.

5.1.3 Entrepreneurs

A risk neutral representative entrepreneur's course of action is as follows. To finance his project at period t, he borrows resources from the Capital Mutual Fund according to the optimal financial contract. The entire borrowed resources, along with his total net worth at period t, are then invested into his capital creation project. If the representative entrepreneur is solvent after observing his own technology shock, he then makes his consumption decision; otherwise, he declares bankruptcy and production is monitored (at a cost) by the Capital Mutual Fund.

5.2 Entrepreneur's Consumption Choice

To rule out self-financing by the entrepreneur (i.e. which would eliminate the presence of agency costs), it is assumed that the entrepreneur discounts the future at a faster rate than the household. This is represented by following expected utility function:

$$E_0 \sum_{t=0}^{\infty} (\beta\gamma)^t c_t^e \tag{37}$$

where c_t^e denotes entrepreneur's consumption at date t, and $\gamma \in (0,1)$. This new parameter, γ, will be chosen so that it offsets the steady-state internal rate of return to entrepreneurs' investment.

At the end of the period, the entrepreneur finances consumption out of the returns from the investment project implying that the law of motion for the entrepreneur's capital stock is:

$$z_{t+1} = n_t \left\{ \frac{f_m\left(\bar{\omega}_t, p_t\right)}{1 - q_t g_m\left(\bar{\omega}_t, p_t\right)} \right\} - \frac{c_t^e}{q_t} \tag{38}$$

Note that the expected return to internal fund is $\frac{q_t f_m(\bar{\omega};\sigma_{\omega,t}) i_t}{n_t}$; that is, the net worth of size n_t is leveraged into a project of size i_t, entrepreneurs keep the share of the capital produced and capital is priced at q_t consumption goods. Since these are intra-period loans, the opportunity cost is 1.[15]

Consequently, the representative entrepreneur maximizes his expected utility function in equation (37) over consumption and capital subject to the law of motion for capital, equation (38), and the definition of net worth given in equation (3). The resulting Euler equation is as follows:

$$q_t = \beta\gamma E_t \left\{ \left(q_{t+1}\left(1-\delta\right) + r_{t+1}\right) \left(\frac{q_{t+1} f_m\left(\bar{\omega}_t, p_t\right)}{\left(1 - q_{t+1} g_m\left(\bar{\omega}_t, p_t\right)\right)} \right) \right\}$$

5.3 Financial Intermediaries

The Capital Mutual Funds (CMFs) act as risk-neutral financial intermediaries who earn no profit and produce neither consumption nor capital goods. There is a clear role for the CMF in this economy since, through pooling, all aggregate uncertainty of capital production can be eliminated. The CMF receives capital from three sources: entrepreneurs sell un-depreciated capital in advance of the loan, after the loan, the CMF receives the newly created capital through loan repayment and through monitoring of insolvent firms, and, finally, those entrepreneur's that are still solvent, sell some of their capital to the CMF to finance current period consumption. This capital is then sold at the price of q_t units of consumption to households for their investment plans.

[15] As noted above, we require in steady-state $1 = \gamma \frac{q_t f_m(\bar{\omega}_t)}{(1 - q_t g_m(\bar{\omega}_t))}$.

5.4 Steady-state conditions in the Carlstrom and Fuerst Agency Cost Model

We first present the equilibrium conditions and express these in scaled (by the fraction of entrepreneurs in the economy) terms. Then the equations are analyzed for steady-state implications. As in the text, upper case variables denote aggregate wide while lower case represent household variables. Preferences and technology are:

$$\begin{aligned} U\left(\tilde{c}, 1-l\right) &= \ln \tilde{c} + \nu \left(1-l\right) \\ Y &= \theta K^{\alpha} \left[\left(1-\eta\right) l\right]^{1-\alpha-\phi} \eta^{\phi} \end{aligned}$$

Where η denotes the fraction of entrepreneurs in the economy and θ is the technology shock. Note that aggregate household labor is $L = \left(1-\eta\right) l$ while entrepreneurs inelastically supply one unit of labor. We assume that the share of entrepreneur's labor is approximately zero so that the production function is simply

$$Y = \theta K^{\alpha} \left[\left(1-\eta\right) l\right]^{1-\alpha}$$

This assumption implies that entrepreneurs receive no wage income (see eq. (9) in C&F.

There are nine equilibrium conditions:

The resource constraint

$$\left(1-\eta\right) \tilde{c}_t + \eta c_t^e + \eta i_t = Y_t = \theta_t K_t^{\alpha} \left[\left(1-\eta\right) l_t\right]^{1-\alpha} \tag{39}$$

Let $c = \frac{(1-\eta)\tilde{c}}{\eta}$, $h = \frac{(1-\eta)}{\eta} l$, and $k_t = \frac{K_t}{\eta}$ then eq(39) can be written as:

$$c_t + c_t^e + i_t = \theta_t k_t^{\alpha} h_t^{1-\alpha} \tag{40}$$

Household's intratemporal efficiency condition

$$\tilde{c}_t = \frac{(1-\alpha)}{\nu} K_t^{\alpha} \left[(1-\eta)\, l_t\right]^{-\alpha}$$

Defining $\nu_0 = \frac{\eta}{1-\eta}\nu$, this can be expressed as:

$$\nu_0 c_t = (1-\alpha)\, k_t^{\alpha} h_t^{-\alpha} \tag{41}$$

Law of motion of aggregate capital stock

$$K_{t+1} = (1-\delta)\, K_t + \eta i_t \left[1 - \Phi_m\left(\bar{\omega}_t, p_t\right)\mu\right]$$

Dividing by η yields the scaled version:

$$k_{t+1} = (1-\delta)\, k_t + i_t \left[1 - \Phi_m\left(\bar{\omega}_t, p_t\right)\mu\right] \tag{42}$$

Household's intertemporal efficiency condition

$$q_t \frac{1}{\tilde{c}_t} = \beta E_t \left\{ \frac{1}{\tilde{c}_{t+1}} \left[q_{t+1}\,(1-\delta) + \theta_{t+1} \alpha K_{t+1}^{\alpha-1} \left[(1-\eta)\, l_{t+1}\right]^{1-\alpha} \right] \right\}$$

Dividing both sides by $\frac{1-\eta}{\eta}$ and scaling the inputs by η yields:

$$q_t \frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left[q_{t+1}\,(1-\delta) + \theta_{t+1} \alpha k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} \right] \right\} \tag{43}$$

The conditions from the financial contract are already in scaled form:

Contract efficiency condition

$$q_t = \frac{1}{1 - \Phi_m\left(\bar{\omega}_t, p_t\right)\mu + \frac{\phi_m(\bar{\omega};p_t)\mu f_m(\bar{\omega}_t,p_t)}{\frac{\partial f_m(\bar{\omega}_t,p_t)}{\partial \bar{\omega}}}} \tag{44}$$

Contract incentive compatibility constraint

$$\frac{i_t}{n_t} = \frac{1}{1 - q_t g_m\left(\bar{\omega}; p_t\right)} \tag{45}$$

Where n_t is entrepreneur's net worth.

Determination of net worth

$$\eta n_t = Z_t\left[q_t\left(1-\delta\right) + \theta_t K_t^{\alpha-1}\left[\left(1-\eta\right)l_t\right]^{1-\alpha}\right]$$

or, in scaled terms:

$$n_t = z_t\left[q_t\left(1-\delta\right) + \theta_t k_t^{\alpha-1} h_t^{1-\alpha}\right] \tag{46}$$

Note that z_t denotes (scaled) entrepreneur's capital.

Law of motion of entrepreneur's capital

$$Z_{t+1} = \eta n_t\left\{\frac{f_m\left(\bar{\omega}; p_t\right)}{1 - q_t g_m\left(\bar{\omega}; p_t\right)}\right\} - \eta\frac{c_t^e}{q_t}$$

Or, dividing by η

$$z_{t+1} = n_t\left\{\frac{f_m\left(\bar{\omega}; p_t\right)}{1 - q_t g_m\left(\bar{\omega}; p_t\right)}\right\} - \frac{c_t^e}{q_t} \tag{47}$$

Entrepreneur's intertemporal efficiency condition

$$q_t = \gamma\beta E_t \left\{ \left[q_{t+1}(1-\delta) + \theta_{t+1}\alpha K_{t+1}^{\alpha-1}\left[(1-\eta)\, l_{t+1}\right]^{1-\alpha} \right] \left(\frac{q_{t+1} f_m(\bar{\omega}; p_t)}{1 - q_{t+1} g_m(\bar{\omega}; p_t)} \right) \right\}$$

Or, in scaled terms:

$$q_t = \gamma\beta E_t \left\{ \left[q_{t+1}(1-\delta) + \theta_{t+1}\alpha k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} \right] \left(\frac{q_{t+1} f_m(\bar{\omega}; p_t)}{1 - q_{t+1} g_m(\bar{\omega}; p_t)} \right) \right\} \tag{48}$$

5.5 Definition of Steady-state

Steady-state is defined by time-invariant quantities:

$$c_t = \hat{c}, c_t^e = \hat{c}^e, k_t = \hat{k}, \bar{\omega}_t = \hat{\omega}, h_t = \hat{h}, q_t = \hat{q}, z_t = \hat{z}, n_t = \hat{n}, i_t = \hat{\imath}$$

So there are nine unknowns. While we have nine equilibrium conditions, the two intertemporal efficiency conditions become identical in steady-state since C&F impose the condition that the internal rate of return to entrepreneur is offset by their additional discount factor:

$$\gamma \left(\frac{\hat{q} f_m(\hat{\omega}, \bar{p})}{1 - \hat{q} g_m(\hat{\omega}, \bar{p})} \right) = 1 \tag{49}$$

This results in an indeterminacy - but there is a block recursiveness of the model due to the calibration exercise. In particular, we demonstrate that the risk premium and bankruptcy rate determine $(\hat{\omega}, \sigma)$ - these in turn determine the steady-state price of capital. From eq.(43)we have:

$$\hat{q} = \frac{\alpha\beta}{1 - \beta(1-\delta)} \hat{k}^{\alpha-1} \hat{h}^{1-\alpha} = \frac{\alpha\beta}{1 - \beta(1-\delta)} \frac{\hat{y}}{\hat{k}} \tag{50}$$

From eq.(41)we have:

$$\hat{h} = \frac{1-\alpha}{\nu_0}\frac{\hat{k}^{\alpha}\hat{h}^{1-\alpha}}{\hat{c}} = \frac{1-\alpha}{\nu_0}\frac{\hat{y}}{\hat{c}} \tag{51}$$

From eq.(42)we have:

$$\hat{k} = \frac{1-\Phi_m\left(\hat{\omega};\bar{p}\right)\mu}{\delta}\hat{\imath} \tag{52}$$

Note that these three equations are normally (i.e. in a typical RBC framework) used to find steady-state $\left(\hat{k},\hat{h},\hat{c}\right)$ - because $\hat{q}=1$. Here since the price of capital is endogenous, we have four unknowns. From eq. (46)and eq. (43)we have

$$\hat{n} = \hat{z}\left(\hat{q}\left(1-\delta\right)+\alpha\frac{\hat{y}}{\hat{k}}\right) = \hat{z}\frac{\hat{q}}{\beta} \tag{53}$$

From eq. (47)and the restriction on the entrepreneur's additional discount factor (eq. (49)), we have

$$\hat{z} = \hat{n}\frac{1}{\hat{q}\gamma} - \frac{\hat{c}^e}{\hat{q}} \tag{54}$$

Combining eqs. (53)and (54) yields:

$$\frac{\hat{c}^e}{\hat{n}} = \frac{1}{\gamma} - \beta \tag{55}$$

We have the two conditions from the financial contract

$$\hat{q} = \frac{1}{1-\Phi_m\left(\hat{\omega};\bar{p}\right)\mu+\phi_m\left(\hat{\omega};\bar{p}\right)\mu\frac{f_m(\hat{\omega},\bar{p})}{\frac{\partial f_m(\hat{\omega},\bar{p})}{\partial\hat{\omega}}}} \tag{56}$$

And

$$\hat{\imath} = \frac{1}{1-\hat{q}\left(1-\Phi_m\left(\hat{\omega};\bar{p}\right)\mu-f_m\left(\hat{\omega},\bar{p}\right)\right)}\hat{n} \tag{57}$$

Finally, we have the resource constraint:

$$\hat{c} + \hat{c}^e + \hat{\imath} = \hat{k}^{\alpha} \hat{h}^{1-\alpha} \tag{58}$$

The eight equations $(50), (51), (52), (53), (54), (56), (57), (58)$ are insufficient to find the nine unknowns. However, the risk premium, denoted as ζ, is defined by the following

$$\hat{q}\hat{\omega}\frac{\hat{\imath}}{\hat{\imath} - \hat{n}} = \zeta \tag{59}$$

But we also know (from eq.(57) that

$$\frac{\hat{n}}{\hat{\imath}} = 1 - \hat{q} g_m (\hat{\omega}, \bar{p})$$

Rearranging eq.(59) yields:

$$\frac{\hat{q}\hat{\omega}}{\zeta} = 1 - \frac{\hat{n}}{\hat{\imath}}$$

substituting from the previous expression yields

$$\hat{\omega} = \zeta g_m (\hat{\omega}, \bar{p}) \tag{60}$$

Let br = bankruptcy rate – this observable also provides another condition on the distribution. That is, we require:

$$\Phi_m (\hat{\omega}; \bar{p}) = br \tag{61}$$

The two equations eq.(60) and eq. (61) can be solved for the two unknowns - $(\hat{\omega}, \sigma)$. Note that the price of capital in steady-state, is a function of $(\hat{\omega}, \sigma)$ as determined by eq. (56). The other preference parameter, γ is then determined by eq. (49). Once this is determined, the remaining unknowns: $\left(\hat{c}, \hat{c}^e, \hat{h}, \hat{\imath}, \hat{k}, \hat{z}, \hat{n}\right)$ are determined by eqs. $(50), (51), (52), (53), (55), (57), (58)$.

Finally, we note that the parameter η does not play a role in the characteristics of equilibrium and, in particular, the behavior of aggregate consumption. This can be seen by first defining aggregate consumption:

$$(1-\eta)\,\tilde{c}_t + \eta c_t^e = C_t^A$$

Dividing by η and using the earlier definitions:

$$c_t + c_t^e = c_t^A \tag{62}$$

Since the policy rules for household and entrepreneurial consumption are defined as the percentage deviations from steady-state, aggregate consumption will be similarly defined (and note that since $c_t^A = \frac{1}{\eta} C_t^A$, percentage deviations of aggregate consumption and scaled aggregate consumption are identical). Using an asterisk to denote percentage deviations from steady-state, we have:

$$\frac{\hat{c}}{\hat{c}+\hat{c}^e} c_t^* + \frac{\hat{c}^e}{\hat{c}+\hat{c}^e} c_t^{e*} = c_t^{A*} \tag{63}$$

It is this equation that is used to analyze the cyclical properties of aggregate consumption.