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Slices of motivic Landweber spectra

by

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Abstract

In this paper we show that a conjecture of Voevodsky about the slices of the motivic cobordism spectrum implies a statement about the slices of motivic Landweber spectra. Over perfect fields these slices are given by the coefficients of the corresponding topological Landweber spectrum and the motivic Eilenberg MacLane spectrum. We also prove a cohomological version of Landweber exactness which applies to the compact objects of the stable motivic homotopy category.

Key Words: motivic Landweber spectra, slice filtration, algebraic cobordism spectrum.

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Contents

1. Introduction

In this paper we show that a conjecture of Voevodsky concerning the slices of the motivic Thom spectrum MGL [15] implies a general statement about slices of motivic Landweber spectra.

Voevodsky's conjecture on the slices of MGL is part of a circle of conjectures introduced in [15]. Among other things these conjectures make predictions for the slices of the motivic Eilenberg MacLane spectrum, the algebraic K-theory spectrum KGL, the algebraic cobordism spectrum MGL and the sphere spectrum.

We refer to these conjectures as the slice conjectures.

For the definition of the slices of a motivic spectrum we refer the reader to section (3).

A proof of Voevodsky's slice conjecture for MGL over fields of characterisic 0, modulo an unpublished result announced by Hopkins and Morel, is given in [11].

We point out that Voevodsky's slice conjectures use the motivic Eilenberg-MacLane spectrum, in particular the conjectures say that the zeroth slice of MGL is the motivic Eilenberg MacLane spectrum.

In our discussion we formulate a conjecture about all slices of MGL relative to the zeroth slice, see assumption (3.1). This conjecture is implied by Voevodsky's slice conjecture for MGL. Voevodsky's conjecture is equivalent to our conjecture together with the statement that the zeroth slice of MGL is the motivic Eilenberg MacLane spectrum.

Our main theorem (6.1) gives then the computation of the slices of a Landweber spectrum E modelled on the Landweber exact MU_*-module M_* (for Landweber spectra we refer the reader to section (5)) in terms of the coefficients M_* and the zeroth slice of MGL, if assumption (3.1) is fulfilled.

The zeroth slice of the sphere spectrum is known to be motivic cohomology over perfect fields (see [13] for fields of characteristic 0 and [5] for perfect fields). By [11, Corollary 1.3] the zeroth slices of the sphere spectrum and MGL agree.

In [5] Levine gives an unconditional computation of the slices of the algebraic K-theory spectrum KGL over perfect fields yielding (shifted) motivic Eilenberg MacLane spectra.

In [15] it is suggested that a Conner-Floyd type isomorphism

$$\mathsf{KGL}_{**}(X) \cong \mathsf{MGL}_{**}(X) \otimes_{\mathsf{MU}_*} \mathbf{Z}[u,u^{-1}]$$

for homotopy algebraic K-theory could yield a proof of the slice conjecture for KGL, assuming the slice conjecture for MGL.

Using the Conner-Floyd isomorphism for homotopy algebraic K-theory established in [12] our result gives a positive answer to the strategy suggested in

[15], see corollary (6.2), since the Conner-Floyd isomorphism says that KGL is the Landweber spectrum for the coefficients $\mathbf{Z}[u,u^{-1}]$.

The proof of the main result consists of two steps. In the first we show that the statement holds for Landweber exact spectra of the form $\mathsf{MGL} \wedge \mathsf{E}$, where E is also a Landweber spectrum. The main ingredient is a topological result about the projective dimension of the MU-homology of an even topological Landweber spectrum, [3].

In the second step we use a cosimplicial resolution of the given Landweber spectrum in terms of spectra of the form appearing in the first step.

In a last paragraph we show that the argument used here also shows that cohomological Landweber exactness holds for all compact spectra, not only for the strongly dualizable ones as shown in [7].

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2. Conventions

We fix a base scheme S, Noetherian and of finite Krull dimension, and denote the stable $\mathbb{A}^1$-homotopy category over S by $\mathsf{SH}(S)$. As in [7] we let $\mathsf{SH}(S)_{\mathcal{T}}$ be the full localizing triangulated category spanned by all Tate spheres $S^{p,q} = S_s^{\wedge(p-q)} \wedge \mathbb{G}_m^{\wedge q}$, also known as the category of cellular spectra.

We let $\mathbf{1}$ be the motivic sphere spectrum and Sm/S the category of smooth schemes of finite type over S.

We let $\Sigma^{p,q} = S^{p,q} \wedge \_$ be the shift functor in p simplicial and q Tate directions. Moreover we set $\Sigma_T^i = \Sigma^{2i,i}$.

For a pointed simplicial presheaf X on Sm/S we denote by $\Sigma_T^\infty X \in \mathsf{SH}(S)$ the suspension spectrum of X.

Generalizing the notion of rigid homotopy groups of a spectrum E given in [15] (i.e. $\pi_{p,q}^{\mathrm{rig}}(\mathsf{E}) = \pi_{p,q} s_q \mathsf{E}$, $s_q \mathsf{E}$ the q-th slice of E, see the next paragraph) we set $\pi_{p,q}^{\mathrm{rig}}(X_\bullet) = \pi_{p,q}(X_q)$ for an object $X_\bullet \in \mathsf{SH}(S)^{\mathbf{Z}}$.

We set $\pi_p^{\mathrm{rig}}(X_\bullet) := \pi_{2p,p}^{\mathrm{rig}}(X_\bullet)$. Note that for $\mathsf{E} \in \mathsf{SH}(S)$ we have natural maps

$$\pi_{p,q}\mathsf{E} \to \pi_{p,q} s_q \mathsf{E} = \pi_{p,q}^{\mathrm{rig}}(\mathsf{E}).$$

3. Preliminaries on the slice filtration

As in [15] we denote the slices of a spectrum $\mathsf{E} \in \mathsf{SH}(S)$ by $s_q(\mathsf{E}) \in \mathsf{SH}(S)$.

We recall the definition of the q-th slice of a spectrum E.

We denote by $\mathsf{SH}(S)^{\mathrm{eff}} \subset \mathsf{SH}(S)$ the full localizing triangulated subcategory generated by the $\Sigma^{i,j}\Sigma_T^\infty X_+$, $i \in \mathbf{Z}$, $j \in \mathbf{N}$, $X \in \mathsf{Sm}/S$.

Let $i_q : \Sigma_T^q \mathsf{SH}(S)^{\mathrm{eff}} \to \mathsf{SH}(S)$ be the inclusion. Since all triangulated categories appearing here are compactly generated and i_q commutes with sums it follows by a result of Neeman [8] that i_q has a right adjoint. We denote it by r_q. Let $f_q = i_q \circ r_q$. There are natural maps of functors $f_{q+1} \to f_q$. Then $s_q\mathsf{E}$ is defined to be the cofiber of the natural map $f_{q+1}\mathsf{E} \to f_q\mathsf{E}$. We thus have an exact triangle

$$f_{q+1}\mathsf{E} \to f_q\mathsf{E} \to s_q\mathsf{E} \to f_{q+1}\mathsf{E}[1].$$

The functor

$$s_* : \mathsf{SH}(S) \to \mathsf{SH}(S)^{\mathbf{Z}}$$

has good multiplicative properties, for a general treatment of that using the theory of model categories see [10]. It is in particular shown in loc. cit. that the functor s_* preserves certain module objects in a highly structured sense. For most of the paper we use these statements on the level of homotopy, see [10, page iv, (5)].

In [15, p. 5] it is observed that there are natural maps $s_i(\mathsf{E}) \wedge s_j(\mathsf{F}) \to s_{i+j}(\mathsf{E} \wedge \mathsf{F})$ (the map in loc. cit. is written after taking sums over all i resp. j). Assembling these maps in a graded way gives natural maps in $\mathsf{SH}(S)^{\mathbf{Z}}$

$$\alpha_{\mathsf{E},\mathsf{F}} : s_*(\mathsf{E}) \wedge s_*(\mathsf{F}) \to s_*(\mathsf{E} \wedge \mathsf{F}),$$

where the $\wedge$-product in $\mathsf{SH}(S)^{\mathbf{Z}}$ is defined using the $\wedge$-product in $\mathsf{SH}(S)$ and taking sums of $\wedge$-products of a fixed total degree.

Indeed the $\alpha_{\mathsf{E},\mathsf{F}}$ assemble to give s_* the structure of a lax tensor functor by the following argument:

The slice $s_0(\mathsf{E})$ for an effective spectrum $\mathsf{E} \in \mathsf{SH}(S)^{\mathrm{eff}}$ can be obtained by a left Bousfield localization of the triangulated category $\mathsf{SH}(S)^{\mathrm{eff}}$ along the subcategory $\Sigma_T \mathsf{SH}(S)^{\mathrm{eff}}$, see [10] for a model category version of this. In detail the functor s_0 restricted to effective objects is the composition

$$\mathsf{SH}(S)^{\mathrm{eff}} \to \mathsf{SH}(S)^{\mathrm{eff}}/\Sigma_T \mathsf{SH}(S)^{\mathrm{eff}} \to \mathsf{SH}(S)^{\mathrm{eff}},$$

where the first arrow is the quotient map and the second arrow the right adjoint to the quotient map which exhibits the quotient as a full subcategory of the first category. For the existence of the quotients see e.g. [4, par. 5.6]. Now $\Sigma_T \mathsf{SH}(S)^{\mathrm{eff}}$ is a tensor ideal of $\mathsf{SH}(S)^{\mathrm{eff}}$, therefore there is an induced $\wedge$-product on the quotient $\mathsf{SH}(S)^{\mathrm{eff}}/\Sigma_T \mathsf{SH}(S)^{\mathrm{eff}}$ and the quotient map $\mathsf{SH}(S)^{\mathrm{eff}} \to \mathsf{SH}(S)^{\mathrm{eff}}/\Sigma_T \mathsf{SH}(S)^{\mathrm{eff}}$ is a tensor functor. Thus the right adjoint is a lax tensor functor, which gives

$s_0\colon \mathsf{SH}(S)^{\mathrm{eff}} \to \mathsf{SH}(S)^{\mathrm{eff}}$ the structure of a lax tensor functor. By applying suitable shifts Σ^i_T this construction gives the functor $s_*\colon \mathsf{SH}(S) \to \mathsf{SH}(S)^{\mathbf{Z}}$ the structure of a lax tensor functor.

In the whole paper we will denote the spectrum $s_0(\mathsf{MGL})$ by H. By the above it is a ring spectrum and using the effectivity of MGL ([11, Corollary 3.2]) it comes with a ring map $\mathsf{MGL} \to \mathsf{H}$.

We make the following assumption:

Assumption 3.1 *We have* $s_i(\mathsf{MGL}) \cong \Sigma^i_T \mathsf{H} \otimes \pi_{2i}(\mathsf{MU})$ *in* $\mathsf{SH}(S)$ *compatible with the homomorphism* $\mathsf{MU}_* \to \mathsf{MGL}_{**}$ *as in [15, Conjecture 5].*

The assumption implies that shifted slices $\Sigma^{0,-i} s_i \mathsf{M}$ of a cellular MGL-module M are in the localizing triangulated subcategory of $\mathsf{SH}(S)$ generated by H. We call such spectra *strictly* H*-cellular*. We call a module $X_\bullet \in \mathsf{SH}(S)^{\mathbf{Z}}$ stricly H-cellular if for all i the module $\Sigma^{0,-i} X_i$ is strictly H-cellular.

4. Remarks on phantom maps

Throughout the paper we will use the notion of *phantom map*. We recall that in a compactly generated triangulated category with sums a map between two objects is called *phantom* if it induces the zero map between the represented cohomology theories on compact objects.

If the triangulated category has a compatible tensor product and if every compact object is strongly dualizable then this is the same that the corresponding homology transformation on the whole category, or equivalently on the compact objects, is zero.

This is the case e.g. for the categories $\mathsf{SH}(S)_{\mathcal{T}}$, $\mathsf{SH}(S)_{\mathcal{D}}$ (the last category is spanned by strongly dualizable objects, see [7, par. 4]).

Let $f : T \to S$ be a map between base schemes. Let g be the right adjoint to the pullback functor $f^*\colon \mathsf{SH}(S)_{\mathcal{T}} \to \mathsf{SH}(T)_{\mathcal{T}}$. Then g is a $\mathsf{SH}(S)_{\mathcal{T}}$-module functor (compare [7, Lemma 4.7]). Let $F\colon \mathsf{E} \to \mathsf{F}$ be a phantom map in $\mathsf{SH}(S)_{\mathcal{T}}$. Then g applied to $f^*\mathsf{E} \wedge K \to f^*\mathsf{F} \wedge K$ yields $\mathsf{E} \wedge g(K) \to \mathsf{F} \wedge g(K)$. It follows that $f^*\colon \mathsf{SH}(S)_{\mathcal{T}} \to \mathsf{SH}(T)_{\mathcal{T}}$ preserves phantom maps. A similar argument shows that $\mathsf{SH}(S)_{\mathcal{T}} \hookrightarrow \mathsf{SH}(S)_{\mathcal{D}}$ preserves phantoms.

Let F be as above. Let X be a smooth S-scheme and $f : X \to S$ the structure morphism. Using $[X] = f_! \mathbf{1}_X$ and the adjunction between $f_!$ and f^* one also sees that the transformation $\mathrm{Hom}([X], F)$ is zero. It is not clear to the author if F is necessarily phantom in $\mathsf{SH}(S)$.

5. Landweber spectra

We recall briefly some results from [7] which we shall need in this paper.

The motivic Thom spectrum MGL is a commutative monoid in $\mathsf{SH}(S)$. By the construction of [9, 2.1] there is a strictly commutative model as symmetric T-spectrum, T the Tate object $\mathbf{A}^1/(\mathbf{A}^1 \setminus \{0\})$.

We let BAb be the abelian category of bigraded abelian groups.

Recall the definition of a Landweber exact MU_*-module from [3, Def. 2.6]: for any prime p there is a certain regular sequence $v_0^{(p)} = p, v_1^{(p)}, \dots$ in MU_* where $v_n^{(p)}$ has degree $2(p^n-1)$. An MU_*-module M is called Landweber exact if $(v_0^{(p)}, v_1^{(p)}, \dots)$ is a regular sequence on M for every prime p. Note that in particular M is torsion free as an abelian group.

For a Landweber exact MU_*-module M_* (which we always consider to be evenly graded in the usual topological grading, but we adopt the convention that we regrade by dividing by 2) one looks at the functor

$$\begin{array}{rcl} \mathsf{SH}(S) & \to & \mathsf{BAb} \\ X & \mapsto & \mathsf{MGL}_{**}(X) \otimes_{\mathsf{MU}_*} M_*. \end{array}$$

Here MU_* and M_* are considered as bigraded (more precisely Adams graded graded) abelian groups via the diagonal $\mathbf{Z}(2,1)$ (see [7, before Theorem 7.3]). By [7, Proposition 7.7 and Theorem 8.7] this functor is a homology theory on $\mathsf{SH}(S)$ and representable by a cellular (or Tate-) spectrum E.

We recall how one constructs the representing spectrum E. First there is a base change formula for spectra representing Landweber homology theories: Let $f: T \to S$ be a morphism of Noetherian base schemes of finite Krull dimension and E a spectrum representing the Landweber theory corresponding to a Landweber exact MU_*-module M_* over S, then $\mathbb{L} f^* \mathsf{E}$ represents the Landweber theory on M_* over T [7, Proposition 8.5].

Next recall that $\mathsf{SH}(\mathbf{Z})_{\mathcal{T}}$ is a Brown category. We refer to the beginning of section 8 in [7] where Brown categories are discussed. This implies that the Landweber theory corresponding to M_* is representable by a Tate spectrum in $\mathsf{SH}(\mathbf{Z})_{\mathcal{T}}$. Pulling this spectrum back to a given base scheme S gives thus a spectrum representing the Landweber theory.

A refined version of the above statement gives a representing object as highly structured MGL-module.

Let $\mathcal{D}_{\mathsf{MGL}}$ be the derived category of (highly structured) MGL-modules. Then the functor

$$\begin{array}{rcl} \mathcal{D}_{\mathsf{MGL}} & \to & \mathsf{BAb} \\ X & \mapsto & X_{**} \otimes_{\mathsf{MU}_*} M_*. \end{array}$$

is a homology theory and representable by a cellular MGL-module.

Let $\mathcal{D}_{\mathsf{MGL},\mathcal{T}} \subset \mathcal{D}_{\mathsf{MGL}}$ be the subcategory of cellular MGL-modules.

Then again $\mathcal{D}_{\mathsf{MGL}_{\mathbf{Z}},\mathcal{T}}$ is a Brown category [7, Proof of Proposition 8.9], and the representing spectrum for a given Landweber theory on MGL-modules is constructed as in the absolute case above.

We remark that in [7] it is not established that $\mathcal{D}_{\mathsf{MGL}_S,\mathcal{T}}$ is Brown for arbitrary base schemes S. It is Brown if S can be covered by affines which are spectra of countable rings, this is the MGL-analog of [7, Lemma 8.3].

6. Slices of Landweber exact theories

The main theorem of this text is

Theorem 6.1 *Suppose assumption (3.1) is fulfilled. Let M_* be a Landweber exact MU_*-module and let $\mathsf{E}_{\mathbf{Z}}$ be the corresponding Landweber exact motivic spectrum in $\mathsf{SH}(\mathbf{Z})_{\mathcal{T}}$ given by [7, Theorem 8.7]. Let E be its pullback to S. Then $s_i(\mathsf{E}) \cong \Sigma^i_T \mathsf{H} \otimes M_i$ (here M_i is the $2i$-th homotopy group of the corresponding topological Landweber spectrum) compatible with the homomorphism $M_* \to \mathsf{E}_{**}$.*

The proof of the theorem will be given in sections (6.1) and (7).

In the above $\mathsf{H} \otimes A$ for a torsion free abelian group A is the spectrum $\mathsf{H} \otimes (\mathsf{S}^{\mathrm{Top}} \otimes A)$, where the first $\otimes$ is the exterior action of the stable topological homotopy category and $\mathsf{S}^{\mathrm{Top}} \otimes A$ is the sphere spectrum with A-coefficients, i.e. a spectrum representing the homology theory $X \mapsto X_0 \otimes A$ on the topological stable homotopy category. $\mathsf{S}^{\mathrm{Top}} \otimes A$ is well defined up to possible phantom maps.

Corollary 6.2 *Suppose assumption (3.1) is fulfilled. Then $s_i(\mathsf{KGL}) \cong \Sigma^i_T \mathsf{H}$ compatible with the natural map $\mathbf{Z} \to \pi_{2i,i}\mathsf{KGL}$.*

Proof: The spectrum KGL is Landweber exact for the MU_*-algebra $\mathbf{Z}[u,u^{-1}]$ classifying the mutliplicative formal group law over $\mathbf{Z}[u,u^{-1}]$, see [12, Theorem 1.2]. The result follows from Theorem 6.1. □

Lemma 6.3 *Let R be a motivic ring spectrum (i.e. a commutative monoid in $\mathsf{SH}(S)$), A a torsion free abelian group, M a R-module and $\varphi\colon A \to \pi_{0,0}M$ a map. Then there is a map $R \otimes A \to M$ which is an R-module map and which induces φ via $A \to \pi_{0,0}(R \otimes A) \to \pi_{0,0}M$. Moreover it is well defined up to phantoms in $\mathsf{SH}(S)$.*

Proof: First note that $\mathbf{1} \otimes A$ has such a universal property by using the adjunction $\mathsf{SH} \to \mathsf{SH}(S)$, SH the topological stable homotopy category and the corresponding universal property of $\mathsf{S}^{\mathrm{Top}} \otimes A$. Tensoring the resulting map $\mathbf{1} \otimes A \to M$ with R and composing with the module structure map gives the required map. It is unique up to phantoms since on the level of cohomology theories on compacts it is well defined. □

6.1. *Slices of Landweber spectra of the form* MGL ∧ E

One idea is to use resolutions of the MU_*module M_* by free or projective MU_*-modules. Let M_* be the coefficients of a Landweber spectrum $\mathsf{MU} \wedge \mathsf{E}^{\mathrm{Top}}$ for $\mathsf{E}^{\mathrm{Top}}$ also Landweber. Here we induce the MU_*-module structure from the first factor in $\mathsf{MU} \wedge \mathsf{E}^{\mathrm{Top}}$. We let $\mathsf{E}_{\mathbf{Z}}$ be the $\mathsf{MGL}_{\mathbf{Z}}$-module representing the theory for the module $\mathsf{E}^{\mathrm{Top}}_*$. Hence $\mathsf{MGL}_{\mathbf{Z}} \wedge \mathsf{E}_{\mathbf{Z}}$ represents the theory corresponding to M_*.

By [3, 2.12 and 2.16] there exists a 2-term resolution of M_* by projective MU_*-modules

$$0 \to P_* \xrightarrow{\phi} Q_* \to M_* \to 0, \tag{1}$$

where P_* and Q_* come by construction as retracts of free MU_*-modules (see [1, Lemma 4.6] which is cited in the proof of [3, 2.14]), say of $\bigoplus_i \mathsf{MU}_*(n_i)$ and $\bigoplus_j \mathsf{MU}_*(m_j)$.

As MU_*-module M_* is flat, see [3, 2.12 and 2.16]. We shall not need this fact in this paragraph, it will become relevant in the last paragraph where we also give a short proof. Note that this does not violate the fact that a general Landweber exact MU_*-module is not flat, since we are considering coefficients of the particular shape $M_* = (\mathsf{MU} \wedge \mathsf{E}^{\mathrm{Top}})_*$.

For any Landweber exact MU_*-module N_* (in particular for any projective MU_*-module) we denote by h_{N_*} the corresponding homology theory on $\mathcal{D}_{\mathsf{MGL}_{\mathbf{Z}}}$ given by $X \mapsto (X_* \otimes_{\mathsf{MU}_*} N_*)_0$. Any MU_*-module map between such modules gives rise to a transformation betweeen the homology theories.

Hence we get the sequence

$$0 \to h_{P_*} \to h_{Q_*} \to h_{M_*} \to 0 \tag{2}$$

of homology theories on $\mathcal{D}_{\mathsf{MGL}_{\mathbf{Z}}}$.

We claim this sequence is short exact. We have to prove that for any $X \in \mathcal{D}_{\mathsf{MGL}_{\mathbf{Z}}}$ the map $h_{P_*}(X) \to h_{Q_*}(X)$ is a monomorphism. By the claimed flatness of M_* this follows from the long exact Tor-sequence, but we can also give the following proof which avoids the flatness of M_*: By Landweber exactness M_* is flat as a quasi coherent sheaf over the moduli stack of formal groups with trivialized constant vector fields. Moreover by [7, Proof of Proposition 7.9] X also gives rise to a quasi coherent sheaf on this moduli stack and we can perform the tensor product over this stack. Thus again the long exact Tor-sequence for the sheaves on the stack shows the claim.

Now lift $h_\phi\colon h_{P_*} \to h_{Q_*}$ to a map between cellular $\mathsf{MGL}_{\mathbf{Z}}$-modules $\Phi\colon \mathsf{M}_P \to \mathsf{M}_Q$, i.e. Φ represents the homology transformation h_ϕ and M_P and M_Q are chosen to represent the homology theories h_{P_*} and h_{Q_*}. (P_* and Q_* are projective so this is easy, one can also invoke that $\mathcal{D}_{\mathsf{MGL}_{\mathbf{Z}},\mathcal{T}}$ is a Brown category, see [7, Proof of Proposition 8.9], or that P_* and Q_* are Landweber exact. Note also that we are dealing with homology theories that have values in abelian groups and not graded or bigraded abelian groups, for details and the facts about representability see [7, sections 4 and 8].)

Let $\mathsf{C}_{\mathbf{Z}}$ be the cofiber of Φ. Since h_ϕ is a monomorphism the sequence of homology theories associated to the exact triangle

$$\mathsf{M}_P \to \mathsf{M}_Q \to \mathsf{C}_{\mathbf{Z}} \to \mathsf{M}_P[1] \tag{3}$$

is isomorphic to the sequence (2), in particular the homology theory associated to $\mathsf{C}_{\mathbf{Z}}$ is canonically isomorphic to h_{M_*}. Hence $\mathsf{C}_{\mathbf{Z}}$ is isomorphic to $\mathsf{MGL}_{\mathbf{Z}} \wedge \mathsf{E}_{\mathbf{Z}}$ since $\mathcal{D}_{\mathsf{MGL}_{\mathbf{Z}},\mathcal{T}}$ is a Brown category.

We now look at the triangle

$$s_*(\mathsf{M}_{P,S}) \to s_*(\mathsf{M}_{Q,S}) \to s_*(\mathsf{C}_S) \to s_*(\mathsf{M}_{P,S})[1] \tag{4}$$

in $\mathsf{SH}(S)^{\mathbf{Z}}$, $\mathsf{M}_{P,S}$, $\mathsf{M}_{Q,S}$, C_S the pullbacks of M_P, M_Q, $\mathsf{C}_{\mathbf{Z}}$ to S.

Since we have maps $P_* \to \mathsf{M}_{P,S,*}$, $Q_* \to \mathsf{M}_{Q,S,*}$, $M_* \to \mathsf{C}_{S,*}$ we get maps

$$P_* \to \pi_*^{\mathrm{rig}} s_*(\mathsf{M}_{P,S}),$$

likewise for Q_* and M_*. These are MU_*-module maps (s_*X has the structure of an $s_*(\mathsf{MGL})$-module for X a MGL-module).

For a MU_*-module N_* which is torsion free as abelian group we informally denote by $s_*(\mathsf{MGL}) \otimes_{\mathsf{MU}_*} N_*$ the module in $\mathsf{SH}(S)^{\mathbf{Z}}$ which has $\Sigma_T^q \mathsf{H} \otimes N_q$ in the q-th component, similarly for maps between such MU_*-modules. By Lemma (6.3) the module $s_*(\mathsf{MGL}) \otimes_{\mathsf{MU}_*} N_*$ has the weak universal property that for a given map of MU_*-modules $\phi\colon N_* \to \pi_*^{\mathrm{rig}} s_*(\mathsf{N}')$, N' a MGL-module, there is an induced map

$$s_*(\mathsf{MGL}) \otimes_{\mathsf{MU}_*} N_* \to s_*(\mathsf{N}'),$$

compatible with ϕ unique up to possible phantoms.

Thus we get maps

$$\psi_P : s_*(\mathsf{MGL}) \otimes_{\mathsf{MU}_*} P_* \to s_*(\mathsf{M}_{P,S}),$$

similarly ψ_Q and ψ_M for Q_* and M_*. The maps ψ_P and ψ_Q are isomorphisms by assumption (3.1) and since P_* and Q_* are retracts of free MU_*-modules. Via these isomorphisms the map

$$s_*(\mathsf{M}_{P,S}) \to s_*(\mathsf{M}_{Q,S})$$

represents the map

$$s_*(\mathsf{MGL}) \otimes_{\mathsf{MU}_*} (P_* \to Q_*).$$

We claim the cofiber of the last map is $s_*(\mathsf{MGL}) \otimes_{\mathsf{MU}_*} M_*$. For that we show that the cofibers C_q of the maps $\mathsf{S}^{\mathrm{Top}} \otimes (P_q \to Q_q)$ in the topological stable homotopy category are the $S^{\mathrm{Top}} \otimes M_q$. Since M_* is torsion free, i.e. flat as an abelian group, the long exact Tor-sequence shows that the maps $\mathsf{S}^{\mathrm{Top}} \otimes (P_q \to Q_q)$ yield monomorphisms of cohomology theories on compacts. It follows that $\mathsf{S}^{\mathrm{Top}} \otimes P_q \to S^{\mathrm{Top}} \otimes Q_q \to C_q$ gives a short exact sequence of cohomology theories on compacts. We deduce that $S^{\mathrm{Top}} \otimes M_q$ and C_q define canonically isomorphic cohomology theories on compacts.

This shows that the map $\psi_M : s_*(\mathsf{MGL}) \otimes_{\mathsf{MU}_*} M_* \to s_*(\mathsf{C}_S)$ is an isomorphism. This is the content of the following proposition.

Proposition 6.4 *Theorem (6.1) holds for Landweber spectra of the form* $\mathsf{MGL} \wedge \mathsf{E}$ *for* E *Landweber.*

Remark 6.5 Consider the base change of the boundary map $\mathsf{C}_{\mathbf{Z}} \to M_P[1]$ of the triangle (3) to the spectrum S of a subfield of $\mathbf{C}$. It is phantom in $\mathcal{D}_{\mathsf{MGL}_S,\mathcal{T}}$ and hence also in $\mathsf{SH}(S)_{\mathcal{T}}$ (Lemma (8.1)) since the corresponding homology theories yield a short exact sequence. In general it is non-trivial since after topological realization we recover the original sequence $P_* \to Q_* \to M_*$ as coefficients, and M_* is in general not projective.

7. Cosimplicial resolutions

In this section we prove theorem (6.1).

Let $\triangle$ be the simplicial category, $\triangle_*$ the category of the ordered *pointed* sets $[n]_* = \{0,\ldots,n\} \coprod \{*\}$ for $n \in \{-1,0,1,\ldots\}$ pointed by $*$ and order preserving pointed maps. Here the element $*$ is considered as the biggest element in the ordered set$[n]_*$. An extension of a cosimplicial diagram to $\triangle_*$ corresponds to a 'contraction' to the value at $[-1]_*$. For example the homotopy limit of a cosimplicial diagram which is the restriction of a $\triangle_*$-diagram in a model category is weakly equivalent to the value at $[-1]_*$. We shall only need the following strict version of the assertion.

Lemma 7.1 *Let $\psi_\bullet : A_\bullet \to B_\bullet$ be a map between $\triangle_*$-diagrams in a category. Suppose $\psi_\bullet$ is an isomorphism on the objects $[i]_*$ of $\triangle_*$ for $i \geq 0$. Then ψ_{-1} is also an isomorphism.*

Proof: Recall the notion of a split coequalizer diagram [2, Definition I.5.3]. Dually there is the notion of a split equalizer diagram. As remarked in loc. cit. a split coequalizer furnishes in particular a coequalizer, dually a split equalizer furnishes in particular an equalizer.

We let $g\colon A_{-1} \to A_0$, $h\colon A_0 \to A_{-1}$ be the maps induced by the unique maps in $\triangle_*$, $f,e\colon A_0 \to A_1$ the maps induced by the maps $[0]_* \to [1]_*$ which send 0 to 0 resp. 1, $k\colon A_1 \to A_0$ the map induced by the map $[1]_* \to [0]_*$ sending 0 to 0 and 1 to $*$. It is easily seen that these maps furnish a split equalizer. Hence A_{-1} is the limit of $A_\bullet|_\triangle$, likewise for $B_\bullet$. The result follows. □

We remark that in the application of Lemma (7.1) the reader can assume that the value category is additive.

Let us fix a Landweber exact MGL-module E giving rise to a Landweber homology theory for the MU_*-module M_*. Let $\mathsf{E}^{\mathrm{Top}}$ be the topological Landweber spectrum.

Since MGL is a monoid in $\mathsf{SH}(S)$ it gives naturally rise to a cosimplicial spectrum $\mathsf{MGL}^{\wedge\bullet}$ in $\mathsf{SH}(S)$ where the coface maps are given by units and the codegenracy maps are given by multiplication maps, see [15, p. 6] where this cosimplicial spectrum is denoted by $N(\mathsf{MGL})$. Smashing with E we obtain a cosimplicial resolution $\mathsf{MGL}^{\wedge\bullet} \wedge \mathsf{E}$ of E. It extends to a functor $\triangle_* \to \mathcal{D}_{\mathsf{MGL}}$ using the MGL-module structure on E. The wedge $\mathsf{MGL}^{\wedge i} \wedge \mathsf{E}$ is regarded as MGL-module via the last factor E. Note that if $X \in \mathsf{SH}(S)$ then $X \wedge \mathsf{E}$ is canonically an object in $\mathcal{D}_{\mathsf{MGL}}$.

Since $\mathsf{MGL}^{\wedge i} \wedge \mathsf{E}$ is Landweber exact for the coefficients $(\mathsf{MU}^{\wedge i} \wedge \mathsf{E}^{\mathrm{Top}})_*$ we have natural maps

$$\pi_{2j}(\mathsf{MU}^{\wedge i} \wedge \mathsf{E}^{\mathrm{Top}}) \to \pi_{2j,j}(\mathsf{MGL}^{\wedge i} \wedge \mathsf{E}) \to \pi_j^{\mathrm{rig}} s_*(\mathsf{MGL}^{\wedge i} \wedge \mathsf{E})$$

which induce maps

$$\Sigma_T^j \mathsf{H} \otimes \pi_{2j}(\mathsf{MU}^{\wedge i} \wedge \mathsf{E}^{\mathrm{Top}}) \to s_j(\mathsf{MGL}^{\wedge i} \wedge \mathsf{E})$$

which are unique up to possible phantoms.

These maps are also functorial in i up to possible phantom maps. More precisely we have a $\triangle_*$-diagram $\Sigma_T^j \mathsf{H} \otimes \pi_{2j}(\mathsf{MU}^{\wedge\bullet} \wedge \mathsf{E}^{\mathrm{Top}})$ in $\mathsf{SH}(S)$ modulo phantoms and a transformation of $\triangle_*$-diagrams

$$\Sigma_T^j \mathsf{H} \otimes \pi_{2j}(\mathsf{MU}^{\wedge\bullet} \wedge \mathsf{E}^{\mathrm{Top}}) \to s_j(\mathsf{MGL}^{\wedge\bullet} \wedge \mathsf{E}),$$

again well defined up to possible phantoms.

This induces a transformation of diagrams of cohomology theories defined on compact objects of $\mathsf{SH}(S)$

$$\mathrm{Hom}(-, \Sigma_T^j \mathsf{H} \otimes \pi_{2j}(\mathsf{MU}^{\wedge\bullet} \wedge \mathsf{E}^{\mathrm{Top}})) =$$

$$\mathrm{Hom}(-,\Sigma^j_T\mathsf{H})\otimes\pi_{2j}(\mathsf{MU}^{\wedge\bullet}\wedge\mathsf{E}^{\mathrm{Top}})\to\mathrm{Hom}(-,s_j(\mathsf{MGL}^{\wedge\bullet}\wedge\mathsf{E})).$$

By Proposition 6.4 we know that this is an isomorphism on the subcategory of $\triangle_*$ spanned by the objects $\{[0]_*,[1]_*,\ldots\}$. By Lemma (7.1) it follows that it is also an isomorphism on $[-1]_*$. Using the fact that $\mathsf{SH}(S)$ is compactly generated with compact generators the $\Sigma^{i,j}\Sigma^\infty_T U_+$, $U\in\mathsf{Sm}/S$, $i,j\in\mathbf{Z}$, [14, Proposition 5.5], Theorem (6.1) follows.

Remark 7.2 One can try to streamline the argument in the second step by showing that H can be realized as an E_∞-algebra. First note that s_0 can be obtained by colocalization along all $\{\Sigma^{p,q}\Sigma^\infty_+ X\,|\,q\geq 0\}$ and then localization along the maps $\mathcal{S}=\{\Sigma^{p,q}\Sigma^\infty_+ X\to 0\,|\,q>0\}$. There is the problem that the colocalization might not be cofibrantly generated, hence we cannot apply the techniques available to persue the further localization. Instead one looks at the full ∞-subcategory of the ∞-category associated to the semimodel category of E_∞-ring spectra whose underlying objects are effective. This is presentable in the sense of [6] an thus one should be able to find a left proper combinatorial model. Then one can directly localize this model category of effective E_∞-ring spectra along the free E_∞-maps generated by $\mathcal{S}$. Alternatively one can try to localize the ∞-category directly. A local model with respect to this localization conjecturally yields H as an E_∞-algebra under MGL.

Having this one can form the derived category of H-modules $\mathcal{D}_\mathsf{H}$ and using in the arguments of this paragraph that a map between strictly H-cellular objects in $\mathcal{D}_\mathsf{H}$ (with the definition of being strictly H-cellular altered to be generated by H inside $\mathcal{D}_\mathsf{H}$) is an isomorphism if it is so on the $\pi_{i,0}$, $i\in\mathbf{Z}$.

8. Cohomological Landweber Exactness

We start again with a topological evenly graded Landweber spectrum $\mathsf{E}^{\mathrm{Top}}$ and let $M_*=\mathsf{E}^{\mathrm{Top}}_*$ be the coefficients. Let $\mathsf{E}\in\mathcal{D}_{\mathsf{MGL}}$ be the corresponding Landweber module. It is well defined up to phantoms in $\mathcal{D}_{\mathsf{MGL},\mathcal{T}}$. We also denote by E the underlying spectrum in $\mathsf{SH}(S)_\mathcal{T}$ with the MGL-module structure in $\mathsf{SH}(S)$.

Lemma 8.1 *The functor $v\colon\mathcal{D}_{\mathsf{MGL},\mathcal{T}}\to\mathsf{SH}(S)_\mathcal{T}$ preserves phantom maps.*

Proof: For $X\in\mathsf{SH}(S)_{\mathcal{T},f}$ and $E\in\mathcal{D}_{\mathsf{MGL},\mathcal{T}}$ we have $\mathrm{Hom}(X,vE)=\mathrm{Hom}(\mathsf{MGL}\wedge X,E)$. □

We want to exhibit a natural map

$$\alpha_{M_*,X}\colon\mathsf{MGL}^{**}X\otimes_{\mathsf{MU}^*}M^*\to\mathsf{E}^{**}X$$

for any $X\in\mathsf{SH}(S)$. As usual $M^*=M_{-*}$.

Therefore let $a \in \mathsf{MGL}^{p,q}X$ and $b \in M^i$. By smashing the map $a\colon \Sigma^{-p,-q}X \to \mathsf{MGL}$ with E and applying the module structure map we get a map $\Sigma^{-p,-q}X \wedge \mathsf{E} \to \mathsf{E}$. Composing with $b\colon \mathbf{1}^{-2i,-i} \to \mathsf{E}$ we get a map $\Sigma^{-2i-p,-i-q}X \to \mathsf{E}$. This defines the map $\alpha_{M_*,X}$.

Let N_* be other Landweber coefficients and $M_* \to N_*$ a MU_*-map. Let F be the motivic spectrum corresponding to N_* derived from a MGL-module and $f : \mathsf{E} \to \mathsf{F}$ be a map of MGL-modules in $\mathsf{SH}(S)$ corresponding to $M_* \to N_*$. It is unique up to possible phantoms in $\mathsf{SH}(S)_{\mathcal{T}}$.

From the definition of $\alpha_{M_*,X}$ and $\alpha_{N_*,X}$ it follows that these maps are natural in $M_* \to N_*$ and f.

It follows that we get a transformation of $\triangle_*$-diagrams

$$\alpha_{(\mathsf{E}^{\mathrm{Top}} \wedge \mathsf{MU}^{\wedge\bullet})_*,X} : \mathsf{MGL}^{**}X \otimes_{\mathsf{MU}^*} (\mathsf{E}^{\mathrm{Top}} \wedge \mathsf{MU}^{\wedge\bullet})^* \to (\mathsf{E} \wedge \mathsf{MGL}^{\wedge\bullet})^{**}X.$$

Lemma 8.2 $\alpha_{(\mathsf{E}^{\mathrm{Top}} \wedge \mathsf{MU}^{\wedge\bullet})_*,X}$ *is an isomorphism for compact X and $\bullet > 0$.*

Proof: Clearly it is sufficient to prove the statement for $\bullet = 1$. Let $N_* = (\mathsf{E}^{\mathrm{Top}} \wedge \mathsf{MU})_*$ be the coefficients of $\mathsf{E} \wedge \mathsf{MGL}$. Here we view N_* as MU_*-module via the last factor. As already remarked N_* is flat as MU_*-module. This can be seen by considering M_* as flat quasi coherent sheaf on the moduli stack of formal groups with trivialized constant vector fields. Then N_* is just the pullback of this sheaf to $\mathrm{Spec}(\mathsf{MU}_*)$.

Let

$$0 \to P_* \xrightarrow{\phi} Q_* \to N_* \to 0,$$

be a resolution by projective MU_*-modules as in section (6.1).

Then

$$0 \to \mathsf{MGL}^{**}X \otimes_{\mathsf{MU}^*} P_* \to \mathsf{MGL}^{**}X \otimes_{\mathsf{MU}^*} Q_* \to \mathsf{MGL}^{**}X \otimes_{\mathsf{MU}^*} N_*$$

is again exact by the flatness of N_*. Moreover $\alpha_{P_*,\_}$, $\alpha_{Q_*,\_}$ are easily seen to be isomorphisms on compacts. Thus the map induced by ϕ on the targets of these maps is injective on compacts. Since this is part of the long exact cohomology sequence for the triangle corresponding to the resolution we deduce that the target of $\alpha_{N_*,\_}$ is the cokernel of the above injection on compacts. This proves the claim. □

Corollary 8.3 $\alpha_{M_*,\_}$ *is an isomorphism between cohomology theories defined on compact objects.*

We also deduce the following uniqueness statement:

Corollary 8.4 *The phantom maps in $\mathsf{SH}(S)_{\mathcal{T}}$ coming from $\mathcal{D}_{\mathsf{MGL}_{\mathbf{Z}},\mathcal{T}}$ up to which the Landweber spectrum E is well-defined are also phantom in $\mathsf{SH}(S)$.*

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