Yamabe Constants of Collapsing Riemannian Submersions

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Abstract

Let $\pi : (M^{n+k}, g) \to (B^n, \hat{g})$ be a surjective Riemannian submersion, where M and B are assumed to be closed, $\dim M \geq 3$, and the scalar curvature $\mathrm{scal}_{g^\perp}$ of every fibre F_b, $b \in B$, with respect to the induced metric $g^\perp$ is positive. We consider the metric $r^2\hat{g}$ on B and rescale g on the horizontal subspaces accordingly to obtain a Riemannian submersion

$$\pi : (M, g_{r^2}) \to (B, r^2\hat{g}).$$

Then the limit of the Yamabe constants of (M, g_{r^2}) exists and

$$\lim_{r\to\infty} Y(M, [g_{r^2}]) = \inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]).$$

If M is a smooth manifold of dimension $\dim M \geq 3$ that is the total space of a smooth fibre bundle with fibre F carrying a Riemannian metric g_F such that $\mathrm{scal}_{g_F} > 0$ and structure group $G = \mathrm{Isom}(F, g_F)$, we obtain a lower bound for the Yamabe invariant of M by

$$Y(M) \geq Y(\mathbb{R}^n \times F, [g_{\mathrm{eucl}} \oplus g_F]).$$

Zusammenfassung

Es sei $\pi : (M^{n+k}, g) \to (B^n, \hat{g})$ eine surjektive Riemannsche Submersion, wobei wir annehmen, dass M and B kompakte Mannigfaltigkeiten ohne Rand sind und $\dim M \geq 3$. Außerdem sei die Skalarkrümmung $\mathrm{scal}_{g^\perp}$ jeder Faser F_b, $b \in B$, bezüglich der induzierten Metrik $g^\perp$ positiv. Wir betrachten die Metrik $r^2\hat{g}$ auf B und erhalten nach entsprechender Reskalierung von g auf den Horizontalräumen eine Riemannsche Submersion

$$\pi : (M, g_{r^2}) \to (B, r^2\hat{g}).$$

Dann existiert der Grenzwert der Yamabe-Konstanten von (M, g_{r^2}), und es gilt

$$\lim_{r\to\infty} Y(M, [g_{r^2}]) = \inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]).$$

Ist nun M eine glatte Mannigfaltigkeit der Dimension $\dim M \geq 3$, die der Totalraum eines glatten Faserbündels ist, dessen Fasertyp F eine Riemannsche Metrik g_F trägt, sodass $\mathrm{scal}_{g_F} > 0$ und die Strukturgruppe G gleich $\mathrm{Isom}(F, g_F)$ ist, so erhalten wir eine untere Schranke für die Yamabe-Invariante von M vermöge

$$Y(M) \geq Y(\mathbb{R}^n \times F, [g_{\mathrm{eucl}} \oplus g_F]).$$

Contents

Chapter 1

Overview

1.1 The Yamabe Constant of a Conformal Manifold

Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$. The famous Yamabe problem asks whether there exists a Riemannian metric $\bar{g}$ conformal to g with constant scalar curvature.

This question was answered affirmatively by Aubin [Au], Schoen [Sch] and Trudinger [Tr].

Writing $\bar{g} = f^{p-2} \cdot g$ with $p = p_n = \frac{2n}{n-2}$ and $f \in C^\infty(M, \mathbb{R}_{>0})$ and using the transformation rules for conformal changes one finds that

$$\mathrm{scal}_{\bar{g}} = s$$

if and only if

$$Y^g(f) := \Delta^g f + \frac{n-2}{4(n-1)} \cdot \mathrm{scal}_g \cdot f = \frac{n-2}{4(n-1)} \cdot s \cdot f^{p-1},$$

where Y^g is called conformal Laplacian.

It turns out that the nonlinear PDE $Y^g(f) = \lambda \cdot f^{p-1}$ is the Euler-Lagrange equation of the Yamabe functional

$$Q(\bar{g}) := \frac{\int_M \mathrm{scal}_{\bar{g}} \, d\mathrm{vol}_{\bar{g}}}{\left(\int_M d\mathrm{vol}_{\bar{g}}\right)^{2/p}},$$

where $\bar{g}$ varies in the conformal class $[g]$.
Writing again $\bar{g} = f^{p-2} \cdot g$ for some function $f \in C^\infty(M, \mathbb{R}_{>0})$ and setting $a = a_n = \frac{n-2}{4(n-1)}$, one finds

$$Q(\bar{g}) = Q_g(f) := \frac{\int_M \left(\frac{1}{a} \|\nabla^g f\|_g^2 + \mathrm{scal}_g \cdot f^2\right) d\mathrm{vol}_g}{\|f\|^2_{L^p(M,g)}}.$$

We define the *Yamabe constant* $Y(M,[g])$ of $[g]$ as

$$Y(M,[g]) := \inf_{\bar{g}\in[g]} Q(\bar{g}) = \inf\left\{Q_g(f) \mid f \in C^\infty(M,\mathbb{R}_{>0})\right\}.$$

Aubin showed (see Lemma 3.4 in [L-P]) that

$$Y(M,[g]) \leq Y(S^n,[g_{\mathrm{sph}}])$$

where g_{sph} is the standard metric on the sphere $S^n \subset \mathbb{R}^{n+1}$.

It turns out that

$$Y(M,[g]) = \inf\left\{Q_g(f) \mid f \in C^\infty(M)\setminus\{0\}\right\},$$

which motivates the following

Definition 1.1. *Let (E^n,g) be a not necessarily compact Riemannian manifold without boundary of dimension $n \geq 3$. Then we define its Yamabe constant as*

$$Y(E,[g]) = \inf\left\{Q_g(f) \mid f \in C_0^\infty(E)\setminus\{0\}\right\}.$$

We note that $Y(E,[g])$ is indeed an invariant of the conformal class, since for all $h \in C^\infty(E,\mathbb{R}_{>0})$ we have

$$Q_g(hf) = Q_{h^{p-2}g}(f).$$

Applying an approximation argument one can show that

$$Y(S^n,[g_{\mathrm{sph}}]) = \inf\left\{Q_{g_{\mathrm{sph}}}(f) \mid f \in \mathcal{F}\right\}$$

where

$$\mathcal{F} := \left\{f \in C^\infty(S^n)\setminus\{0\} \mid f|_{B_\rho(q)} = 0 \text{ for a } \rho > 0\right\}$$

with $q \in S^n$ fixed.
Using stereographic projection one deduces that

$$\begin{aligned} Y(S^n,[g_{\mathrm{sph}}]) &= Y(\mathbb{R}^n,[g_{\mathrm{eucl}}]) \\ &= \frac{1}{a}\cdot\inf\left\{\frac{\|\nabla\varphi\|_{L^2}^2}{\|\varphi\|_{L^p}^2}\,\middle|\,\varphi \in C_0^\infty(\mathbb{R}^n)\right\}. \end{aligned}$$

In other words, computing $Y(S^n,[g_{\mathrm{sph}}])$ is equivalent to finding the best constant in the Sobolev inequality, which is realized by a family of spherically symmetric functions (see e.g. the appendix to chapter V in [S-Y2]).
It follows that

$$Y(S^n,[g_{\mathrm{sph}}]) = n(n-1)\mathrm{vol}(S^n)^{2/n}.$$

Moreover, using Obata's lemma we have that the Yamabe functional on (S^n,g_{sph}) is minimized by constant multiples of the standard metric and its images under conformal diffeomorphisms. These are the only metrics conformal to the standard on on S^n that have constant scalar curvature.

We remark that Aubin's lemma above carries over to noncompact manifolds (M^n, g) of dimension $n \geq 3$ without boundary.

Considering Riemannian products, Akutagawa, Florit and Petean proved in [A-F-P] the following

Theorem 1.2. *Let (M^m, g) be a closed Riemannian manifold of dimension $m \geq 2$ with positive scalar curvature* $\mathrm{scal}_g > 0$ *and (N^n, h) any closed Riemannian manifold. Then*

$$\lim_{r\to\infty} Y(M \times N, [g \oplus r^2 h]) = Y(M \times \mathbb{R}^n, [g \oplus g_{\mathrm{eucl}}]).$$

We briefly sketch the proof of Theorem 1.2. Due to the compactness of M one obtains an r_0 and a constant $c > 0$ such that $\mathrm{scal}_{g\oplus r^2 h} > c$ for all $r > r_0$ and hence

$$Y(M \times N, [g \oplus r^2 h]) > 0 \quad \text{for all } r_0 > 0$$

by Remark 3.1. Moreover, $(M \times \mathbb{R}^n, g \oplus g_{\mathrm{eucl}})$ being a complete Riemannian manifold with strictly positive injectivity radius and bounded sectional curvature, by Theorem 2.21 in [Au] there is a continuous embedding

$$W^{1,2}(M \times \mathbb{R}^n, g \oplus g_{\mathrm{eucl}}) \hookrightarrow L^p(M \times \mathbb{R}^n, g \oplus g_{\mathrm{eucl}}),$$

which then yields the key observation

$$Y(M \times \mathbb{R}^n, [g \oplus g_{\mathrm{eucl}}]) > 0.$$

Now one considers normal coordinates with respect to the rescaled metric $r^2 h$ on N and uses a linear isometry to identify balls $B^n_{r\varepsilon}(0) \subset \mathbb{R}^n$ with balls $B^h_\varepsilon(0) = B^{r^2 h}_{r\varepsilon}(0) = V \subset T_q N$, where $q \in N$ and $\varepsilon > 0$ is sufficiently small such that uniform estimates in r can be made between the euclidean metric on $B^n_{r\varepsilon}(0)$ and $r^2 h$ in normal coordinates on

$$\exp^h_q(V) = \exp^{r^2 h}_q(V).$$

Given $\delta > 0$ there exists an $\varepsilon > 0$ such that

$$(1+\delta)^{-n/2} < \sqrt{\det((r^2 h)_{ij}(x))} < (1+\delta)^{n/2},$$

$$\frac{1}{1+\delta}\|\eta\|^2 < \sum_{ij=1}^n (r^2 h)_{ij}\eta_i\eta_j < (1+\delta)\|\eta\|^2,$$

$$\frac{1}{1+\delta}\|\eta\|^2 < \sum_{ij=1}^n (r^2 h)^{ij}\eta_i\eta_j < (1+\delta)\|\eta\|^2$$

for all $x \in B^n_{r\varepsilon}(0)$ and $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$.
This allows to compare test functions in $C_0^\infty(M \times \mathbb{R}^n)$ with test functions in $C^\infty(M \times N)$.
One proceeds by showing

$$\limsup_{r\to\infty} Y(M \times N, [g \oplus r^2 h]) \leq Y(M \times \mathbb{R}^n, [g \oplus g_{\mathrm{eucl}}])$$

and

$$Y(M \times \mathbb{R}^n, [g \oplus g_{\mathrm{eucl}}]) \leq \liminf_{r\to\infty} Y(M \times N, [g \oplus r^2 h]),$$

which yields the theorem.

We significantly generalize the theorem above and replace Riemannian products by surjective Riemannian submersions such that all fibres have positive scalar curvature with respect to the induced metric.

Given a surjective Riemannian submersion

$$(M, g) \to (B, \hat{g}),$$

where $\dim M \geq 3$, we have for any $p \in M$ a $g-$orthogonal decomposition

$$T_pM = T_pF_b \oplus \mathcal{H}_p,$$

where $b = \pi(p)$, $F_b = \pi^{-1}(b)$.
Using that $d\pi_p : \mathcal{H}_p \to T_bB$ is a linear isometry we may replace for $r > 0$ the metric g on the horizontal subspaces $\mathcal{H}_p$ by the pullback of $r^2\hat{g}$ under $d\pi$ to obtain a metric g_{r^2} on M such that

$$\pi : (M, g_{r^2}) \to (B, r^2\hat{g})$$

is a Riemannian submersion.

We investigate $Y(M, [g_{r^2}])$ for $r \to \infty$ while assuming that all fibres of $\pi : M \to B$ have positive scalar curvature with respect to the metric induced by g. It turns out that the limit exists. More precisely, we have

Theorem 1.3. *Let $\pi : (M^{n+k}, g) \to (B^n, \hat{g})$ be a surjective Riemannian submersion, where M and B are assumed to be closed,* $\dim M \geq 3$, *and the scalar curvature* $\mathrm{scal}_{g^\perp}$ *of every fibre F_b, $b \in B$, with respect to the induced metric $g^\perp$ is positive. Considering the Riemannian submersion*

$$\pi : (M, g_{r^2}) \to (B, r^2\hat{g})$$

we have

$$\lim_{r\to\infty} Y(M, [g_{r^2}]) = \inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]).$$

The proof of the theorem above is much more involved than the product case, but follows a similar pattern taking into account that all the arguments therein are local in nature.

We prove the inequalities

$$\limsup_{r\to\infty} Y(M,[g_{r^2}]) \le \inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^{\perp}])$$

and

$$\inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^{\perp})] \le \liminf_{r\to\infty} Y(M,[g_{r^2}])$$

in Proposition 3.4 and Proposition 3.5, respectively.

As we will see in Corollary 2.18, there exists an $r_0 > 0$ such that

$$\mathrm{scal}_{g_{r^2}} > 0 \quad \text{for all} \quad r > r_0,$$

which yields

$$Y(M,[g_{r^2}]) > 0.$$

As above in the product case we have

$$Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^{\perp}]) > 0 \quad \text{for all } b \in B.$$

Following Herrmann (Theorem 9.42 in [Be]) we use that horizontal lifts of geodesics in B are geodesics in M, choose $\varepsilon > 0$ such that

$$\exp_b^{\hat g} = \exp_b^{r^2\hat g} =: \exp_b : T_bB \supset U := B_\varepsilon^{\hat g}(0) = B_{r\varepsilon}^{r^2\hat g}(0) \to \exp_b(U) =: V$$

is a diffeomorphism and construct in section 2.4.2 a local trivialization

$$\Psi : V \times F_b \to \pi^{-1}(V)$$

in a neighbourhood of $b \in B$ by lifting geodesics.

Using Ψ and $r^2\hat g-$normal coordinates centered at $b \in V$ we identify test functions on $\mathbb{R}^n \times F_b$ with test functions on $\pi^{-1}(V)$ for sufficiently large r. Vice versa, given a test function f on M, we find due to the compactness of B finitely many b_i and associated trivializations $\Psi_i : V_i \times F_{b_i} \to \pi^{-1}(V_i)$. After choosing a partition of unity $\{\eta_i\}$ subordinated to $\{V_i\}$ we are able to identify $\eta_i \cdot f$ with a test function on $\mathbb{R}^n \times F_{b_i}$.

In order to prove the claimed inequalities we have to compare the Yamabe functionals on $V \times F_b$ with respect to the product metric $r^2\hat g \oplus g^{\perp}$ and the induced metric $\Psi^* g_{r^2}$, respectively.

As a key observation we recognize in Lemma 2.29 that

$$r^2\hat{g} \oplus g^{\perp}\Big|_{(b,p)} = \Psi^* g_{r^2}|_{(b,p)}$$

for any $r > 0$ and $p \in F_b$.

Now we choose $\varepsilon > 0$ sufficiently small and normal coordinates near p on F_b and near b on B to obtain estimates for the local representation of the metrics $r^2\hat{g} \oplus g^{\perp}$ and $\Psi^* g_{r^2}$.

This allows us in section 2.4.4 to compare the volume elements and the gradients of test functions on such "admissible" trivializations $V \times F_b$ with respect to the metrics $r^2\hat{g} \oplus g^{\perp}$ and $\Psi^* g_{r^2}$.

1.2 The Yamabe Invariant of a Manifold

Definition 1.4. *We define the Yamabe invariant of a smooth manifold M of $\dim M \geq 3$ as*

$$Y(M) := \sup_g Y(M, [g]),$$

where the supremum is taken over all Riemannian metrics g on M.

One reason why one is interested in the Yamabe invariant is that a smooth manifold of dimension $n \geq 3$ admits a metric of positive scalar curvature if and only if $Y(M) > 0$.

Due to Aubin one has an upper bound

$$Y(M) \leq Y(S^n) = Y(S^n, [g_{\mathrm{sph}}]).$$

In dimension $n \geq 5$ it is an open question whether there is a closed manifold satisfying $Y(M) \neq 0$ and $Y(M) \neq Y(S^n)$, but one expects that many such manifolds exist.

Concerning lower bounds for the Yamabe invariant, Ammann, Dahl and Humbert showed

Theorem 1.5 (Corollary 1.4 in [A-D-H1])**.** *If N is obtained from a closed $n-$dimensional manifold M by $k-$dimensional surgery, $k \leq n-3$, then*

$$Y(N) \geq \min\{\Lambda_{n,k}, Y(M)\},$$

where $\Lambda_{n,k}$ is a positive number that depends only on n and k. In addition, $\Lambda_{n,0} = Y(S^n)$.

This theorem generalizes previous results by Gromov-Lawson [G-L] and Schoen-Yau [S-Y1], Kobayashi [Ko], Petean [Pe1] and Petean-Yun [P-Y].

It allows several applications by using methods from bordism theory, see section 1.4 in [A-D-H1]. For these applications it is important to have explicit lower bounds for the Yamabe invariant of $\mathbb{H}P^2$-bundles. Having this in mind we study the following situation:

Suppose we have a smooth fibre bundle $\pi : M \to B$ whith fibre F carrying a Riemannian metric g_F and structure group $G = \mathrm{Isom}(F, g_F)$. Given a metric $\hat{g}$ on B we apply Lemma 2.35 and find a Riemannian metric g on M such that

$$\pi : (M, g) \to (B, \hat{g})$$

is a *Riemannian submersion* with all fibres $(F_b, g_b^{\perp})$ being *isometric* to (F, g_F).

By Theorem 1.3 we obtain

Corollary 1.6. *Let M be a smooth manifold of dimension $\dim M \geq 3$ and suppose that M is the total space of a smooth fibre bundle with fibre F carrying a Riemannian metric g_F such that $\mathrm{scal}_{g_F} > 0$ and structure group G equal to $\mathrm{Isom}(F, g_F)$. Then a lower bound for the Yamabe invariant of M is given by*

$$Y(M) \geq Y(\mathbb{R}^n \times F, [g_{\mathrm{eucl}} \oplus g_F])$$

where n is the dimension of the base space.

As mentioned above this corollary is particularly interesting if F is the quaternionic projective plane $\mathbb{H}P^2$ equipped with its standard metric $g_{\mathbb{H}P^2}$. Note that PSp(3) acts by isometries on $\mathbb{H}P^2$. Stolz proved

Theorem 1.7 (Theorem B in [St])**.** *Let M be a compact spin manifold of dimension $n \geq 5$. We assume that the index $\alpha(M) \in \mathrm{KO}_n(pt)$ vanishes. Then M is spin-bordant to the total space of an $\mathbb{H}P^2$-bundle over a base B such that the structure group is* PSp(3).

In the following we assume that $n \geq 11$. Let M_0 be the total space of a bundle with fibre $\mathbb{H}P^2$ and structure group PSp(3) over a base B of dimension $n - 8$. Applying Corollary 1.6 yields

$$Y(M_0) \geq Y(\mathbb{H}P^2 \times \mathbb{R}^{n-8}, [g_{\mathbb{H}P^2} \oplus g_{\mathrm{eucl}}]).$$

Ammann, Dahl, Humbert [Theorem 2.3 in [A-D-H2]] estimated

$$\begin{aligned} &Y(\mathbb{H}P^2 \times \mathbb{R}^{n-8}, [g_{\mathbb{H}P^2} \oplus g_{\mathrm{eucl}}]) \geq \\ \geq\ &\frac{n/a_n}{(8/a_8)^{8/n}((n-8)/a_{n-8})^{(n-8)/n}} \cdot Y(\mathbb{H}P^2, [g_{\mathbb{H}P^2}])^{8/n} \cdot Y(S^{n-8}, [g_{\mathrm{sph}}])^{(n-8)/n}. \end{aligned}$$

Using Corollary 1.6 they [Proposition 6.8 in [A-D-H3]] were able to prove that

$$Y(M_0) \geq \frac{n}{a_n} \left(\frac{3^6 \cdot 2^{18}}{7^8 \cdot 5^2} \cdot \pi^8 \right)^{1/n} \nu_{n-8}^{1/n},$$

where $a_n = \frac{n-2}{4(n-1)}$ and $\nu_j = \left(\frac{Y(S^j,[g_{\rm sph}])}{j/a_j}\right)^j$.

Now suppose that M is a compact, simply connected spin manifold with $\alpha(M) = 0$. Then M is spin-bordant to the total space M_0 of an $\mathbb{H}P^2$-bundle with structure group PSp(3) by Theorem 1.7. Moreover, M can be obtained from M_0 by performing a sequence of surgeries of dimensions $0, \ldots, n-3$. Consequently, we can estimate

$$\begin{aligned} Y(M) &\geq \min\{\Lambda_{n,1}, \ldots, \Lambda_{n,n-3}, Y(M_0)\} \\ &\geq \min\left\{\Lambda_{n,1}, \ldots, \Lambda_{n,n-3}, Y(\mathbb{H}P^2 \times \mathbb{R}^{n-8}, [g_{\mathbb{H}P^2} \oplus g_{\rm eucl}])\right\}. \end{aligned}$$

(Compare also Proposition 6.9 in [A-D-H3].)

This application is currently the most important one of this PhD thesis.

For sake of completeness we also comment on dimension $n \leq 10$.

Considering $n \in \{9, 10\}$ we first note

Lemma 1.8 (Lemma 5.5 in [A-D-H4]). *Let M be a compact 2-connected spin manifold of dimension $n \in \{9, 10\}$, which has $\alpha(M) = 0$. Then M is obtained from S^9 oder $\mathbb{H}P^2 \times S^1$ (for n=9) or from S^{10} or $\mathbb{H}P^2 \times S^1 \times S^1$ (for n=10) by a sequence of surgeries of dimensions $k \in \{0, 1, \ldots, n-4\}$. All these surgeries are compatible with orientation and spin structure.*

Let $s_1 := Y(\mathbb{H}P^2 \times S^1)$ and $s_2 := Y(\mathbb{H}P^2 \times S^1 \times S^1)$.

Ammann, Dahl und Humbert showed

Corollary 1.9 (Corollary 5.6 in [A-D-H4]). *Let M be a $2-$connected compact spin manifold of dimension $n = 9$ or $n = 10$ with $\alpha(M) = 0$. Then*

$$Y(M^{n=9}) \geq \min\{\Lambda_{9,1}, \Lambda_{9,2}, \Lambda_{9,3}, \Lambda_{9,4}, \Lambda_{9,5}, s_1\} > 109.2$$

and

$$Y(M^{n=10}) \geq \{\Lambda_{10,1}, \Lambda_{10,2}, \Lambda_{10,3}, \Lambda_{10,4}, \Lambda_{10,5}, \Lambda_{10,6}, s_2\} \geq 97.3.$$

By [Theorem 1.2 in [Pe2]] we have

$$s_1 \geq Y(\mathbb{H}P^2 \times \mathbb{R}, [g_{\mathbb{H}P^2} \oplus g_{\rm eucl}]) \geq 0.9370 \cdot Y(S^9, [g_{\rm sph}]) = 138.57... > 109.2,$$

and using [Example after Theorem 1.7 in [P-R]] it follows

$$s_2 \geq Y(\mathbb{H}P^2 \times \mathbb{R}^2, [g_{\mathbb{H}P^2} \oplus g_{\rm eucl}]) \geq 0.59 \cdot Y(S^{10}, [g_{\rm sph}]) > 97.3 < \Lambda_{10,1}.$$

In dimension $n = 8$ the only $\mathbb{H}P^2$-bundles in the sense of Stolz are compact manifolds the connected components of which are diffeomorphic to

$\mathbb{H}P^2$. Stolz' $\mathbb{H}P^2$-bundles have to carry an orientation, and the diffeomorphism may either be orientation preserving or orientation reversing. Therefore we have

$$Y(M) \geq Y(\mathbb{H}P^2, [g_{\mathbb{H}P^2}]).$$

We note that $Y(\mathbb{H}P^2, [g_{\mathbb{H}P^2}])$ can be computed explicitly, since $(\mathbb{H}P^2, g_{\mathbb{H}P^2})$ is an Einstein manifold and Obata's lemma applies.

In dimension $n \in \{5, 6, 7\}$ the total space of the $\mathbb{H}P^2$-bundle in Theorem 1.7 is the empty set, thus the phenomena discussed in our PhD thesis do not play a major role in this case.

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Chapter 2

Riemannian Submersions

We will give a self-contained introduction with complete proofs to Riemannian submersions

$$\pi : (M, g) \to (B, \hat{g}) ,$$

which are always assumed to be **surjective**.
The main reference is chapter 9, sections A-F in [Be] and chapter II, section 6 in [Sa].
After dealing with some very basic concepts concerning horizontal and vertical vector fields we introduce in section 2.1 the tensors A and T which are obstructions to the horizontal distribution to be integrable and the fibres to be totally geodesic, respectively.
In section 2.2 we prove O'Neill's formulas for curvature, which relate the curvature tensors of M, B and the fibres, and compute afterwards the sectional, Ricci and scalar curvature of M. We discuss in section 2.3 how the curvatures change if we rescale the metric on B and the horizontal subspaces accordingly to obtain a Riemannian submersion

$$\pi : (M, g_{r^2}) \to \left(B, r^2 \hat{g}\right) .$$

We remark that in contrast to [Be] we vary the metric on the horizontal subspaces and not on the vertical subspaces. The formula for $\mathrm{scal}_{g_{r^2}}$ in Proposition 2.17 shows that for large r the scalar curvature of the fibres dominates, which yields in Corollary 2.18 a metric of positive scalar curvature on M provided that M and B are compact and $\mathrm{scal}_{g^\perp} > 0$ for all fibres.
In section 2.4 we investigate lifting properties of curves and prove that geodesics on B have unique lifts to horizontal geodesics on M if B and M are compact, which yields a local product structure. After adjusting the trivialization neighbourhood V near $b \in B$ we are able to compare the product metric $r^2 \hat{g} \oplus g^\perp$ with the metric induced by g on $V \times F_b$, which will be crucial for the estimates in chapter 3. Afterwards we explain how to construct Riemannian submersions with isometric fibres from fibre bundles having fibre (F, g_F) and structure group $\mathrm{Isom}(F, g_F)$.

We end the chapter by proving a generalization of Fubini's theorem for Riemannian submersions. This section is based on section II.5 in [Sa].

2.1 Preliminaries

Let

$$\pi : (M, g) \to (B, \hat{g})$$

be a submersion between Riemannian manifolds M^{n+k}, B^n with Levi-Civita connections ∇ and $\hat{\nabla}$. Due to the implicit function theorem we find for any $p \in M$ an $\varepsilon > 0$ and charts $\varphi : p \in U \to \varphi(U) = (-\varepsilon, \varepsilon)^{n+k}$ and $\psi : b = \pi(p) \in V \to \psi(V) = (-\varepsilon, \varepsilon)^n$ such that $\pi(U) \subset V$ and

$$\psi \circ \pi \circ \varphi^{-1}(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}) \mapsto (x_1, \dots, x_n).$$

We note that every fibre $F_b = \pi^{-1}(b) \subset M$ is a $k-$dimensional submanifold of M with induced metric $g^\perp$. A chart is given by the composition

$$\begin{aligned} \varphi_b : F_b \cap U &\xrightarrow{\varphi} (-\varepsilon, \varepsilon)^{n+k} \to (-\varepsilon, \varepsilon)^k \\ q &\mapsto (\psi(b), y_1, \dots, y_k) \mapsto (y_1, \dots, y_k). \end{aligned}$$

The tangent subspace to F_b in T_pM is the *vertical subspace* $\mathcal{V}_p = T_pF_b$ at p, whereas the *horizontal subspace* at p is the orthogonal complement $\mathcal{H}_p$ to $\mathcal{V}_p$ in T_pM, the elements of which are called *vertical* and *horizontal vectors,* respectively. Given $v \in T_pM$ we have a unique decompostion $v = v^\top + v^\perp$ with $v^\top \in \mathcal{H}_p$ and $v^\perp \in \mathcal{V}_p$. As the union of these spaces we obtain the *vertical distribution* $\mathcal{V}$ and the *horizontal distribution* $\mathcal{H}$.

In the following $\pi : M \to B$ will be a *Riemannian submersion,* i. e. the induced isomorphism

$$d\pi_p : \mathcal{H}_p \to T_{\pi(p)}M$$

is a *Riemannian isometry* for every $p \in M$, so that the length of horizontal vectors is preserved. It follows that every vector field $\hat{X}$ on B has a unique smooth horizontal lift X.

We call a vector field X on M *basic* if there exists a vector field $\hat{X}$ on B such that $d\pi_pX_p = \hat{X}_{\pi(p)}$ for every $p \in M$. In other words the vector fields X and $\hat{X}$ are $\pi-$related.

We make the following elementary observations:

1. Let X and Y be basic vector fields which induce $\hat{X} = d\pi(X)$ and $\hat{Y} = d\pi(Y)$. Then we have $[\hat{X}, \hat{Y}] = d\pi[X, Y] = d\pi([X, Y]^\top)$, and $[X, Y]^\top$ is the horizontal lift of $[\hat{X}, \hat{Y}]$.

2. For a basic X and vertical U we obtain $d\pi([X, U]^\top) = 0$, so $[X, U]$ is vertical.

3. If X and Y are basic and U is vertical, then $Ug(X, Y) = 0$, because the inner product is constant along the fibres.

Whereas the vertical distribution $\mathcal{V}$ is integrable in the sense of Frobenius, the horizontal distribution $\mathcal{H}$ need not be. In fact, we have

Lemma 2.1 (Proposition 9.24 in [Be]). *For any horizontal vector fields X and Y the equality*

$$(\nabla_x Y)^\perp = \frac{1}{2}[X, Y]^\perp$$

holds.

Proof. We observe that the expressions $(\nabla_x Y)^\perp$ and $\frac{1}{2}[X, Y]^\perp$ are tensorial in X and Y, so we may assume X and Y to be basic. For any vertical U Koszul's formula yields

$$\begin{aligned} 2g\left((\nabla_X Y)^\perp, U\right) &= 2g(\nabla_X Y, U) \\ &= Xg(Y, U) + Yg(U, X) - Ug(X, Y) \\ &\quad + g([X, Y], U) - g([Y, U], X) + g([U, X], Y) \\ &= g\left([X, Y]^\perp, U\right) \end{aligned}$$

and the formula is proven. □

Following O'Neill we embed $(X, Y) \mapsto (\nabla_x Y)^\perp$ into a tensor field A of type (2,1) on M.

Definition 2.2 ((9.20) in [Be]). *For vector fields E and F on M we set*

$$A_E F := \left(\nabla_{E^\top} F^\perp\right)^\top + \left(\nabla_{E^\top} F^\top\right)^\perp.$$

As a basic observation we remark that A_X is alternating, since

$$\begin{aligned} g(A_X E, F) &= g\left(\nabla_X E^\perp, F^\top\right) + g\left(\nabla_X E^\top, F^\perp\right) \\ &= -g\left(E^\perp, (\nabla_X F^\top)^\perp\right) - g\left(E^\top, (\nabla_X F^\perp)^\top\right) \\ &= -g\left(E, (\nabla_X F^\top)^\perp\right) - g\left(E, (\nabla_X F^\perp)^\top\right) \\ &= -g(E, A_X F). \end{aligned}$$

Furthermore

$$A_X Y = \frac{1}{2}[X, Y]^\perp = -\frac{1}{2}[Y, X]^\perp = -A_Y X.$$

Lemma 2.3. *The horizontal distribution $\mathcal{H}$ is integrable if and only if $A \equiv 0$.*

Proof. If $A \equiv 0$, then $[X,Y]^\perp = 2A_XY = 0$, and $[X,Y]$ is horizontal. Assuming $\mathcal{H}$ to be integrable we obtain

$$0 = g\left(\frac{1}{2}[X,Y]^\perp, U\right) = g(A_XY, U) = -g(Y, A_XU)$$

for any horizontal X, Y and vertical U. Consequently, $A_XU = 0$, and it follows $A \equiv 0$. □

The Levi-Civita connection ∇ on (M,g) induces the Levi-Civita connection $\nabla^\perp$ on each fibre given by

$$\nabla^\perp_U V = (\nabla_U V)^\perp .$$

We embed the horizontal part $(\nabla_U V)^\top$, i. e. the second fundamental form of the fibres, into a tensor field T of type (2,1) on M.

Definition 2.4 ((9.17) in [Be]). *For vector fields E and F on M we set*

$$T_EF := \left(\nabla_{E^\perp} F^\perp\right)^\top + \left(\nabla_{E^\perp} F^\top\right)^\perp .$$

We remark that T_U is alternating, as

$$\begin{aligned} g(T_UE, F) &= g\left(\nabla_U E^\perp, F^\top\right) + g\left(\nabla_U E^\top, F^\perp\right) \\ &= -g\left(E^\perp, (\nabla_U F^\top)^\perp\right) - g\left(E^\top, (\nabla_U F^\perp)^\top\right) \\ &= -g\left(E, (\nabla_U F^\top)^\perp\right) - g\left(E, (\nabla_U F^\perp)^\top\right) \\ &= -g(E, T_UF) . \end{aligned}$$

Moreover

$$T_UV - T_VU = (\nabla_U V - \nabla_V U)^\top = [U,V]^\top = 0.$$

It follows

$$T_UV = T_VU.$$

Since T_UV is the second fundamental form of the fibres, we obtain

Lemma 2.5. *Each fibre is totally geodesic if and only if $T \equiv 0$.*

Proof. Firstly, we assume that T vanishes identically. Let $\gamma : (-\varepsilon, \varepsilon) \to F_b$ be a geodesic in some fibre $(F_b, g^\perp, \nabla^\perp)$ and $t_0 \in (-\varepsilon, \varepsilon)$. Using Lemma 2.22 below and the fact that a Riemannian submersion is locally a projection we obtain a vertical extension U of $\gamma'(t_0) \in \mathcal{V}_{\gamma(t_0)}$ in a neighbourhood $\mathcal{U} \subset M$ such that $\nabla^\perp_U U = 0$ in $\mathcal{U} \cap F_b$. An integral curve

$$\eta : (-\varepsilon, \varepsilon) \supset (t_0 - \varepsilon', t_0 + \varepsilon') \to \mathcal{U}$$

of U with $\eta(t_0) = \gamma(t_0)$ is then a geodesic in $(F_b, g^\perp, \nabla^\perp)$ and

$$\eta(t) = \gamma(t) \quad \text{for all} \quad t \in (t_0 - \varepsilon', t_0 + \varepsilon').$$

Furthermore, $T \equiv 0$ implies $\nabla_U U = \nabla^\perp_U U + T_U U = 0$ in $\mathcal{U} \cap F_b$. Consequently, η and hence γ is a geodesic in (M, g, ∇) through $\gamma(t_0)$. In other words, $(F_b, g^\perp)$ is a totally geodesic submanifold of (M, g).

Conversely, let U, V be vertical and X horizontal. Then

$$g(T_U X, V) = -g(X, T_U V)$$

and

$$T_{U+V}(U+V) = T_U U + T_V V + T_U V + T_V U = T_U U + T_V V + 2 \cdot T_U V.$$

Therefore, it suffices to show $T_U U = 0$ for all vertical vector fields U if the fibres are totally geodesic. Let $p \in M$ and $U_p \in \mathcal{V}_p$. Applying Lemma 2.22 as above we find a vertical extension U of U_p in a neighbourhood $\mathcal{U} \subset M$ such that $\nabla^\perp_U U = 0$ in $\mathcal{U} \cap F_{\pi(p)}$. An integral curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{U}$ of U with $\gamma(0) = p$ is then a geodesic of (M, g) and takes values in the fibre $F_{\pi(p)}$, since $\gamma'(0) = U_p$. It follows $(\nabla_U U)_p = 0$ and

$$(T_U U)_p = (\nabla_U U)_p - \left(\nabla^\perp_U U\right)_p = 0.$$

□

Lemma 2.6 (9.32 in [Be]). *For an arbitrary vector field E on M, vertical U, V and horizontal X, Y we have*

$$\begin{aligned} g((\nabla_E A)(X, Y), U) &= -g((\nabla_E A)(Y, X), U) \\ g((\nabla_E T)(U, V), X) &= g((\nabla_E T)(V, U), X). \end{aligned}$$

Proof. We use $A_X Y = -A_Y X$, which implies $\nabla_E(A_X Y) = -\nabla_E(A_Y X)$ and

$$A_{\nabla_E X} Y = -A_Y (\nabla_E X)^\top = -A_Y(\nabla_E X) + \left(\nabla_Y (\nabla_E X)^\perp\right)^\top.$$

So we obtain

$$g(A_{\nabla_E X} Y, U) = -g(A_Y(\nabla_E X), U)$$

and

$$g(A_{\nabla_E Y} X, U) = -g(A_X(\nabla_E Y), U),$$

respectively. Similarly,

$$T_U V = T_V U \quad \text{yields} \quad \nabla_E(T_U V) = \nabla_E(T_V U)$$

and

$$T_{\nabla_E U} V = T_V (\nabla_E U)^{\perp} = T_V (\nabla_E U) - \nabla_V \left((\nabla_E U)^{\top} \right)^{\perp}.$$

As a result,

$$g(T_{\nabla_E U} V, X) = g(T_V (\nabla_E U), X) \text{ and } g(T_{\nabla_E V} U, X) = g(T_U (\nabla_E V), X),$$

respectively. □

We conclude the section with

Lemma 2.7 ((6.5) in [Sa])**.** *Let $\hat{X}$ and $\hat{Y}$ be vector fields on B with horizontal lifts X and Y on M. Then $(\nabla_X Y)^{\top}$ is the horizontal lift of $\hat{\nabla}_{\hat{X}} \hat{Y}$.*

Proof. By Koszul's formula we have

$$\begin{aligned} 2\hat{g}\left(\hat{\nabla}_{\hat{X}}\hat{Y}, \hat{Z}\right) &= \hat{X}\hat{g}\left(\hat{Y}, \hat{Z}\right) + \hat{Y}\hat{g}\left(\hat{Z}, \hat{X}\right) - \hat{Z}\hat{g}\left(\hat{X}, \hat{Y}\right) \\ &= +\hat{g}\left(\left[\hat{X}, \hat{Y}\right], \hat{Z}\right) - \hat{g}\left(\left[\hat{Y}, \hat{Z}\right], \hat{X}\right) + \hat{g}\left(\left[\hat{Z}, \hat{X}\right], \hat{Y}\right). \end{aligned}$$

Since $d\pi_p : \mathcal{H}_p \to T_{\pi(p)} M$ is a *Riemannian isometry* for every $p \in M$, we obtain

$$X_p g(Y, Z) = \hat{X}_{\pi(p)} \hat{g}\left(\hat{Y}, \hat{Z}\right),$$

$$Y_p g(Z, X) = \hat{Y}_{\pi(p)} \hat{g}\left(\hat{Z}, \hat{X}\right)$$

and

$$Z_p g(X, Y) = \hat{Z}_{\pi(p)} \hat{g}\left(\hat{X}, \hat{Y}\right).$$

Moreover, $\left[\hat{X}, \hat{Y}\right] = d\pi\left([X, Y]^{\top}\right)$ implies

$$\hat{g}_{\pi(p)}\left(\left[\hat{X}, \hat{Y}\right], \hat{Z}\right) = g_p([X, Y], Z).$$

Analogously,

$$\hat{g}_{\pi(p)}\left(\left[\hat{Y}, \hat{Z}\right], \hat{X}\right) = g_p([Y, Z], X)$$

and

$$\hat{g}_{\pi(p)}\left(\left[\hat{Z}, \hat{X}\right], \hat{Y}\right) = g_p([Z, X], Y).$$

It follows $\hat{g}_{\pi(p)}\left(\hat{\nabla}_{\hat{X}}\hat{Y}, \hat{Z}\right) = g_p\left((\nabla_X Y)^{\top}, Z\right) = \hat{g}_{\pi(p)}\left(d\pi_p (\nabla_X Y)_p^{\top}, \hat{Z}_{\pi(p)}\right)$ and consequently $\left(\hat{\nabla}_{\hat{X}}\hat{Y}\right)_{\pi(p)} = d\pi_p (\nabla_X Y)_p^{\top}$. □

2.2 O'Neill's Formulas for Curvature

Let $R, \hat{R}$ and $R^{\perp}$ the curvature tensors corresponding to (M, g, ∇), $\left(B, \hat{g}, \hat{\nabla}\right)$ and the fibres $\left(F_b, g^{\perp}, \nabla^{\perp}\right)$, respectively. We give complete proofs for O'Neill's formula for curvature (cf. Proposition 6.2 in [Sa])

Formula 1. *Let U, V, W and W' be vertical. Then*

$$\begin{aligned} g\left(R\left(U,V\right)W,W'\right) &= g\left(R^{\perp}\left(U,V\right)W,W'\right) \\ &\quad + g\left(T_U W, T_V W'\right) - g\left(T_V W, T_U W'\right) \end{aligned}$$

Proof. By definition we have

$$R^{\perp}\left(U,V\right)W = \nabla^{\perp}_U\left(\nabla^{\perp}_V W\right) - \nabla^{\perp}_V\left(\nabla^{\perp}_U W\right) - \nabla^{\perp}_{[U,V]}W.$$

Since U, V and W are vertical, we obtain $\nabla^{\perp}_U\left(\nabla^{\perp}_V W\right) = \left(\nabla_U\left(\nabla_V W\right)^{\perp}\right)^{\perp}$, $\nabla^{\perp}_V\left(\nabla^{\perp}_U W\right) = \left(\nabla_V\left(\nabla_U W\right)^{\perp}\right)^{\perp}$ and $\nabla^{\perp}_{[U,V]}W = \left(\nabla_{[U,V]}W\right)^{\perp}$.
It follows

$$\begin{aligned} g\left(R^{\perp}\left(U,V\right)W,W'\right) &= g\left(\nabla_U\left(\nabla_V W\right)^{\perp} - \nabla_V\left(\nabla_U W\right)^{\perp} - \nabla_{[U,V]}W, W'\right) \\ &= g\left(R\left(U,V\right)W,W'\right) - g\left(\nabla_U\left(\nabla_V W\right)^{\top}, W'\right) \\ &\quad + g(\nabla_V(\nabla_U W)^{\top}, W') \\ &= g\left(R\left(U,V\right)W,W'\right) + g\left(\left(\nabla_V W\right)^{\top}, \left(\nabla_U W'\right)^{\top}\right) \\ &\quad - g\left(\left(\nabla_U W\right)^{\top}, \left(\nabla_V W'\right)^{\top}\right) \\ &= g\left(R\left(U,V\right)W,X\right) + g\left(T_V W, T_U W'\right) \\ &\quad - g\left(T_U W, T_V W'\right). \end{aligned}$$

□

Formula 2. *Let U, V, W be vertical and X be horizontal. Then*

$$g\left(R\left(U,V\right)W,X\right) = g\left(\left(\nabla_U T\right)\left(V,W\right),X\right) - g\left(\left(\nabla_V T\right)\left(U,W\right),X\right)$$

Proof. With

$$\left(\nabla_U T\right)\left(V,W\right) = \nabla_U\left(T_V W\right) - T_{\nabla_U V}W - T_V\left(\nabla_U W\right)$$

and

$$\left(\nabla_V T\right)\left(U,W\right) = \nabla_V\left(T_U W\right) - T_{\nabla_V U}W - T_U\left(\nabla_V W\right)$$

we get $(\nabla_U T)(V,W) - (\nabla_V T)(U,W) =$

$$\begin{aligned} &= \nabla_U (\nabla_V W)^{\top} - \nabla_V (\nabla_U W)^{\top} - T_{[U,V]} W \\ &\quad - \left(\nabla_V (\nabla_U W)^{\perp}\right)^{\top} - \left(\nabla_V (\nabla_U W)^{\top}\right)^{\perp} \\ &\quad + \left(\nabla_U (\nabla_V W)^{\perp}\right)^{\top} - \left(\nabla_U (\nabla_V W)^{\top}\right)^{\perp}. \end{aligned}$$

Hence, $g((\nabla_U T)(V,W) - (\nabla_V T)(U,W), X) =$

$$\begin{aligned} &= g\left(\nabla_U (\nabla_V W)^{\top}, X\right) - g\left(\nabla_V (\nabla_U W)^{\top}, X\right) - g\left(\nabla_{[U,V]} W, X\right) \\ &\quad -g\left(\nabla_V (\nabla_U W)^{\perp}, X\right) + g\left(\nabla_U (\nabla_V W)^{\perp}, X\right) \\ &= g\left(\nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]}, X\right) \\ &= g\left(R(U,V)W, X\right). \end{aligned}$$

□

Formula 3. *Let U, V be vertical and X, Y be horizontal. Then*

$$\begin{aligned} g(R(U,X)Y,V) &= g((\nabla_X T)(U,V), Y) - g(T_U X, T_V Y) \\ &\quad + g((\nabla_U A)(X,Y), V) + g(A_X U, A_Y V). \end{aligned}$$

Proof. Since we are dealing with tensors, we may assume X and Y to be basic vector fields.
We calculate $g((\nabla_X T)(U,V), Y) =$

$$\begin{aligned} &= g\left(\nabla_X (\nabla_U V)^{\top}, Y\right) - g\left(T_{\nabla_X U} V, Y\right) - g\left(\nabla_U (\nabla_X V)^{\perp}, Y\right) \\ &= g(R(X,U)V,Y) - g\left(\nabla_X (\nabla_U V)^{\perp}, Y\right) - g\left(T_{\nabla_X U} V, Y\right) \\ &\quad + g(\nabla_U (A_X V), Y) + g\left(\nabla_{[X,U]} V, Y\right) \end{aligned}$$

Using $[X,U]^{\top} = 0$ and $[X,U] = \nabla_X U - \nabla_U X$ we obtain

$$g\left(\nabla_{[X,U]} V, Y\right) = g\left(T_{\nabla_X U} V, Y\right) - g\left(T_{\nabla_U X} V, Y\right)$$

and consequently

$$\begin{aligned} g((\nabla_X T)(U,V), Y) &= g(R(X,U)V,Y) - g\left(\nabla_X (\nabla_U V)^{\perp}, Y\right) \\ &\quad + g(\nabla_U (A_X V), Y) - g\left(T_{\nabla_U X} V, Y\right). \end{aligned}$$

Furthermore,

$$g\left(T_{\nabla_U X} V, Y\right) = g\left(T_V (\nabla_U X)^{\perp}, Y\right) = -g(T_V Y, T_U X)$$

and

$$\begin{aligned} g\left(\nabla_X \left(\nabla_U V\right)^{\perp}, Y\right) &= -g\left(\nabla_U V, A_X Y\right) \\ &= -Ug\left(V, A_X Y\right) + g\left(V, \nabla_U \left(A_X Y\right)\right) \end{aligned}$$

together with

$$\begin{aligned} g\left(\nabla_U \left(A_X V\right), Y\right) &= Ug\left(A_X V, Y\right) - g\left(A_X V, \nabla U Y\right) \\ &= -Ug\left(A_X Y, V\right) + g\left(A_X \left(\nabla_U Y\right), V\right) \end{aligned}$$

imply

$$\begin{aligned} g\left(R\left(X,U\right)V,Y\right) &= g\left(\left(\nabla_X T\right)\left(U,V\right),Y\right) - g\left(T_V Y, T_U X\right) \\ &\quad + g\left(V, \nabla_U \left(A_X Y\right)\right) - g\left(A_X \left(\nabla_U Y\right), V\right) \\ &= g\left(\left(\nabla_X T\right)\left(U,V\right),Y\right) - g\left(T_V Y, T_U X\right) \\ &\quad + g\left(\left(\nabla_U A\right)\left(X,Y\right),V\right) + g\left(A_{\nabla_U X} Y, V\right). \end{aligned}$$

Finally,

$$\begin{aligned} g\left(A_{\nabla_U X} Y, V\right) &= g\left(A_{\left(\nabla_U X\right)^{\top}} Y, V\right) = g\left(A_{\left(\nabla_X U\right)^{\top}} Y, V\right) \\ &= -g\left(A_Y \left(\nabla_X U\right)^{\top}, V\right) = g\left(A_Y V, A_X U\right) \end{aligned}$$

yields

$$\begin{aligned} g\left(R\left(X,U\right)V,Y\right) &= g\left(R\left(U,X\right)Y,V\right) \\ &= g\left(\left(\nabla_X T\right)\left(U,V\right),Y\right) - g\left(T_U X, T_V Y\right) \\ &\quad + g\left(\left(\nabla_U A\right)\left(X,Y\right),V\right) + g\left(A_X U, A_Y V\right). \end{aligned}$$

□

Formula 4. *Let U, V be vertical and X, Y horizontal. Then*

$$\begin{aligned} g\left(R\left(U,V\right)X,Y\right) &= g\left(\left(\nabla_V A\right)\left(X,Y\right),U\right) - g\left(\left(\nabla_U A\right)\left(X,Y\right),V\right) \\ &\quad + g\left(A_X V, A_Y U\right) - g\left(A_X U, A_Y V\right) \\ &\quad + g\left(T_U X, T_V Y\right) - g\left(T_V X, T_U Y\right) \end{aligned}$$

and

Proof. We may assume X and Y to be basic, so $[V,X]^{\top} = [U,Y]^{\top} = 0$ and consequently

$$\begin{aligned} g\left(A_{\left(\nabla_V X\right)^{\top}} Y, U\right) &= g\left(A_{\left(\nabla_X V\right)^{\top}} Y, U\right) = -g\left(A_Y \left(\nabla_X V\right)^{\top}, U\right) \\ &= g\left(A_Y U, A_X V\right). \end{aligned}$$

It follows $g\left(\left(\nabla_V A\right)(X,Y),U\right)=$

$$\begin{aligned}
&= g\left(\nabla_V\left(A_X Y\right)-A_X\left(\nabla_V Y\right),U\right)-g\left(A_Y U,A_X V\right)\\
&= g\left(R\left(V,X\right)Y,U\right)-g\left(\nabla_V\left(\nabla_X Y\right)^{\top},U\right)+g\left(\nabla_X\left(\nabla_V Y\right)^{\perp},U\right)\\
&\quad -g\left(A_Y U,A_X V\right)+g\left(\nabla_{[V,X]}Y,U\right)
\end{aligned}$$

and $g\left(\left(\nabla_V A\right)(X,Y),U\right)=g\left(\left(\nabla_U A\right)(X,Y),V\right)=$

$$\begin{aligned}
&= g\left(R\left(V,X\right)Y,U\right)+g\left(R\left(Y,V\right)X,U\right)-g\left(\nabla_V\left(\nabla_X Y\right)^{\top},U\right)\\
&\quad -g\left(A_Y U,A_X V\right)+g\left(\nabla_X\left(\nabla_V Y\right)^{\perp},U\right)+g\left(A_Y V,A_X U\right)\\
&\quad +g\left(\nabla_U\left(\nabla_X Y\right)^{\top},V\right)-g\left(\nabla_X\left(\nabla_U Y\right)^{\perp},V\right)\\
&\quad +g\left(\nabla_{[V,X]}Y,U\right)-g\left(\nabla_{[U,X]}Y,V\right).
\end{aligned}$$

Moreover, $g\left(\nabla_X\left(\nabla_V Y\right)^{\perp},U\right)=$

$$\begin{aligned}
&= Xg\left(T_V Y,U\right)-g\left(T_V Y,\left(\nabla_X U\right)^{\top}\right)\\
&= g\left(T_V Y,U\right)-g\left(T_V Y,T_U X\right)+g\left(T_V Y,[U,X]^{\perp}\right)\\
&= Xg\left(T_V Y,U\right)-g\left(T_V Y,T_U X\right)+g\left(\nabla_{[U,X]}Y,V\right),
\end{aligned}$$

where we used

$$\begin{aligned}
g\left(T_V Y,[U,X]^{\perp}\right) &= -g\left(T_V\,[U,X]^{\perp},Y\right)=-g\left(T_{[U,X]^{\perp}}V,Y\right)\\
&= g\left(T_{[U,X]^{\perp}}Y,V\right)=g\left(\nabla_{[U,X]}Y,V\right).
\end{aligned}$$

Analogously,

$$g\left(\nabla_X\left(\nabla_U Y\right)^{\perp},V\right)=Xg\left(T_U Y,V\right)-g\left(T_U Y,T_V X\right)+g\left(\nabla_{[V,X]}Y,U\right).$$

We calculate

$$Xg\left(T_V Y,U\right)-Xg\left(T_U Y,V\right)=Xg\left(T_U V-T_V U,Y\right)=0$$

and

$$\begin{aligned}
g\left(\nabla_U\left(\nabla_X Y\right)^{\top},V\right) &= g\left(T_U\left(\nabla_X Y\right)^{\top},V\right)=-g\left(T_U V,\left(\nabla_X Y\right)^{\top}\right)\\
&= -g\left(T_V U,\left(\nabla_X Y\right)^{\top}\right)=g\left(T_V\left(\nabla_X Y\right)^{\top},U\right)\\
&= g\left(\nabla_V\left(\nabla_X Y\right)^{\top},U\right).
\end{aligned}$$

Finally, we use Bianchi's identity

$$\begin{aligned} g\left(R\left(V,X\right)Y,U\right)+g\left(R\left(Y,V\right)X,U\right) &= -g\left(R\left(X,Y\right)V,U\right) \\ &= g\left(R\left(U,V\right)X,Y\right) \end{aligned}$$

and obtain the claimed formula. □

Formula 5. *Let X,Y,Z be horizontal and U vertical. Then*

$$\begin{aligned} g\left(R\left(X,Y\right)Z,U\right) &= -g\left(\left(\nabla_Z A\right)\left(X,Y\right),U\right)-g\left(T_U Z,A_X Y\right) \\ &\quad +g\left(T_U Y,A_Z X\right)+g\left(T_U X,A_Y Z\right) \end{aligned}$$

Proof. Let $p\in M$. We apply Corollary 2.21 below and choose extensions of $d\pi_p X_p=\hat{X}_{\pi(p)}$ and $d\pi_p Y_p=\hat{Y}_{\pi(p)}$ to local vector fields $\hat{X}$ and $\hat{Y}$ on B such that

$$\left(\hat{\nabla}_{\hat{X}}\hat{Y}\right)_{\pi(p)}=\left(\hat{\nabla}_{\hat{Y}}\hat{X}\right)_{\pi(p)}=0.$$

We may assume that X and Y are the horizontal lifts of $\hat{X}$ and $\hat{Y}$ near p. Since $\left(\nabla_X Y\right)^{\top}$ is the horizontal lift of $\hat{\nabla}_{\hat{X}}\hat{Y}$, it follows $\left(\nabla_X Y\right)_p^{\top}=0$. In other words, we can choose local extensions X,Y and Z of X_p,Y_p and Z_p such that

$$\left(\nabla_X Y\right)_p^{\top}=\left(\nabla_Y X\right)_p^{\top}=\left(\nabla_X Z\right)_p^{\top}=\left(\nabla_Z X\right)_p^{\top}=\left(\nabla_Z Y\right)_p^{\top}=\left(\nabla_Y Z\right)_p^{\top}=0.$$

It follows

$$g_p\left(A_Z\left(\nabla_X Y\right),U\right)=g_p\left(\nabla_Z\left(\nabla_X Y\right)^{\top},U\right)=-g_p\left(\left(\nabla_X Y\right)^{\top},\nabla_Z U\right)=0.$$

Combining with $\left(A_{\nabla_Z X}Y\right)_p=A_{\left(\nabla_Z X\right)_p^{\top}}Y_p=0$ and

$$g_p\left(A_X\left(\nabla_Z Y\right),U\right)=g_p\left(A_{X_p}\left(\nabla_Z Y\right)_p^{\top},U_p\right)=0$$

we have

$$g_p\left(\left(\nabla_Z A\right)\left(X,Y\right),U\right)=g_p\left(\nabla_Z\left(A_X Y\right),U\right).$$

Using $\left(A_Y Z\right)_p=\frac{1}{2}\left(\left(A_Y Z\right)_p-\left(A_Z Y\right)_p\right)=\frac{1}{2}\left(\left(\nabla_Y Z\right)_p-\left(\nabla_Z Y\right)_p\right)$ we get

$$\begin{aligned} g_p\left(\nabla_X\left(A_Y Z\right),U\right) &= X_p g\left(A_Y Z,U\right)-g_p\left(A_Y Z,\nabla_X U\right) \\ &= \frac{1}{2}g\left(\nabla_Y Z-\nabla_Z Y,U\right)-\frac{1}{2}g_p\left(\nabla_Y Z-\nabla_Z Y,\nabla_X U\right) \end{aligned}$$

and similar equalities for $g_p\left(\left(\nabla_Y A\right)\left(Z,X\right),U\right)$ and $g_p\left(\left(\nabla_Z A\right)\left(X,Y\right),U\right)$.

As a consequence,

$$g_p\left(\left(\nabla_X A\right)(Y,Z),U\right)+g_p\left(\left(\nabla_Y A\right)(Z,X),U\right)+g_p\left(\left(\nabla_Z A\right)(X,Y),U\right)=$$

$$\begin{aligned} =\; & \frac{1}{2}g_p\left(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z,U\right)+\frac{1}{2}g_p\left(\nabla_Z\nabla_X Y-\nabla_X\nabla_Z Y,U\right) \\ & +\frac{1}{2}g_p\left(\nabla_Y\nabla_Z X-\nabla_Z\nabla_Y X,U\right). \end{aligned}$$

Now, $[X,Y]_p=[X,Y]_p^\perp$, $[X,Z]_p=[X,Z]_p^\perp$ and $[Y,Z]_p=[Y,Z]_p^\perp$ imply

$$\begin{aligned} g_p\left(\nabla_{[X,Y]}Z,U\right) &= g_p\left(T_{[X,Y]_p^\perp}Z_p,U_p\right)=-g_p\left(T_{[X,Y]_p^\perp}U_p,Z_p\right) \\ &= -g_p\left(T_{U_p}[X,Y]_p^\perp,Z_p\right)=g_p\left(T_{U_p}Z_p,[X,Y]_p^\perp\right) \\ &= 2g_p\left(T_U Z,A_X Y\right), \end{aligned}$$

as well as

$$g_p\left(\nabla_{[Z,X]}Y,U\right)=2g_p\left(T_U Y,A_Z X\right)$$

and

$$g_p\left(\nabla_{[Y,Z]}X,U\right)=2g_p\left(T_U X,A_Y Z\right).$$

Applying Bianchi's identity yields

$$\begin{aligned} & g_p\left(\left(\nabla_X A\right)(Y,Z),U\right)+g_p\left(\left(\nabla_Y A\right)(Z,X),U\right)+g_p\left(\left(\nabla_Z A\right)(X,Y),U\right)= \\ & \qquad =g_p\left(T_U Z,A_X Y\right)+g_p\left(T_U Y,A_Z X\right)+g_p\left(T_U X,A_Y Z\right). \end{aligned}$$

Finally, we conclude

$$\begin{aligned} g_p\left(R(X,Y)Z,U\right) &= g_p\left(\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,U\right) \\ &= X_p g\left(A_Y Z,U\right)-g_p\left(A_Y Z,\nabla_X U\right)-Y_p g\left(A_X Z,U\right) \\ &\quad +g_p\left(A_X Z,\nabla_Y U\right)-g_p\left(\nabla_{[X,Y]}Z,U\right) \\ &= g_p\left(\nabla_X\left(A_Y Z\right)-\nabla_Y\left(A_X Z\right),U\right)-2g_p\left(T_U Z,A_X Y\right) \\ &= -g_p\left(\left(\nabla_Z A\right)(X,Y),U\right)-g_p\left(T_U Z,A_X Y\right) \\ &\quad +g_p\left(T_U Y,A_Z X\right)+g_p\left(T_U X,A_Y Z\right). \end{aligned}$$

□

Formula 6. *Let X,Y,Z and Z' be horizontal. Then*

$$\begin{aligned} g\left(R(X,Y)Z,Z'\right) &= g\left(R^\top(X,Y)Z,Z'\right)+2g\left(A_X Y,A_Z Z'\right) \\ &\quad +g\left(A_X Z,A_Y Z'\right)-g\left(A_Y Z,A_X Z'\right), \end{aligned}$$

where $R^\top(X_p,Y_p)Z_p$ denotes the horizontal lift of $\hat{R}(d\pi_p X_p,d\pi_p Y_p)d\pi_p Z_p$ at every $p\in M$.

Proof. We may assume X, Y, Z and Z' to be basic. Since $(\nabla_Y Z)_p^\top$ is the horizontal lift of $\left(\hat{\nabla}_{d\pi(Y)} d\pi(Z)\right)_{\pi(p)}$ and $d\pi$ preserves the inner product of horizontal vectors, we have

$$\begin{aligned} g_p\left(\nabla_X (\nabla_Y Z)^\top, Z'\right) &= X_p g\left((\nabla_Y Z)^\top, Z'\right) - g_p\left((\nabla_Y Z)^\top, (\nabla_X Z')^\top\right) \\ &= d\pi_p X_p \hat{g}\left(\hat{\nabla}_{d\pi(Y)} d\pi(Z), d\pi(Z')\right) \\ &\quad -\hat{g}_{\pi(p)}\left(\hat{\nabla}_{d\pi(Y)} d\pi(Z), \hat{\nabla}_{d\pi(X)} d\pi(Z')\right) \\ &= \hat{g}_{\pi(p)}\left(\hat{\nabla}_{d\pi(X)} \hat{\nabla}_{d\pi(Y)} d\pi(Z), d\pi(Z')\right) \end{aligned}$$

and similarly

$$g_p\left(\nabla_Y (\nabla_X Z)^\top, Z'\right) = \hat{g}_{\pi(p)}\left(\hat{\nabla}_{d\pi(Y)} \hat{\nabla}_{d\pi(X)} d\pi(Z), d\pi(Z')\right).$$

Moreover,

$$\begin{aligned} g\left(\nabla_X (\nabla_Y Z)^\perp, Z'\right) &= -g\left((\nabla_Y Z)^\perp, (\nabla_X Z')^\perp\right) \\ &= -g\left(A_Y Z, A_X Z'\right) \end{aligned}$$

and

$$g\left(\nabla_Y (\nabla_X Z)^\perp, Z'\right) = -g\left(A_X Z, A_Y Z'\right).$$

Furthermore,

$$\begin{aligned} g_p\left(\nabla_{[X,Y]} Z, Z'\right) &= g_p\left(\left(\nabla_{[X,Y]^T} Z\right)^\top, Z'\right) + g_p\left(\nabla_{[X,Y]^\perp} Z, Z'\right) \\ &= \hat{g}_{\pi(p)}\left(\hat{\nabla}_{[d\pi(X), d\pi(Y)]} d\pi(Z), d\pi(Z')\right) + g_p\left(\nabla_{[X,Y]^\perp} Z, Z'\right). \end{aligned}$$

Finally, $[X, Y]^\perp = 2 \cdot A_X Y = (\nabla_X Y)^\perp$ implies

$$\begin{aligned} g\left(\nabla_{[X,Y]^\perp} Z, Z'\right) &= 2g\left(\left(\nabla_{(\nabla_X Y)^\perp} Z\right)^\top, Z'\right) \\ &= 2g\left(\left(\nabla_Z (\nabla_X Y)^\perp\right)^\top, Z'\right) \\ &= 2g\left(A_Z (\nabla_X Y)^\perp, Z'\right) \\ &= -2g\left(A_Z Z', A_X Y\right), \end{aligned}$$

where we used $\left[(\nabla_X Y)^\perp, Z\right]^\top = 0$. Putting the terms together yields the claimed equality. □

In particular, we have for the sectional curvatures K, $K^{\perp}$ and $\hat{K}$ corresponding to ∇, $\nabla^{\perp}$ and $\hat{\nabla}$, respectively,

Proposition 2.8 (Corollary 6.3 in [Sa])**.** *Let U_p, V_p and X_p, Y_p be unit vertical and horizontal vectors, respectively, such that $U_p \perp V_p$ and $X_p \perp Y_p$, and let denote $d\pi_p X_p = \hat{X}_{\pi(p)}, d\pi_p Y_p = \hat{Y}_{\pi(p)}$. Then*

$$\begin{aligned} K(U_p, V_p) &= K^{\perp}(U_p, V_p) + \|T_{U_p} V_p\|^2 - g_p\left(T_{U_p} U_p, T_{V_p} V_p\right), \\ K(X_p, U_p) &= g_p\left(\left(\nabla_{X_p} T\right)(U_p, U_p), X_p\right) - \|T_{U_p} X_p\|^2 + \|A_{X_p} U_p\|^2, \\ K(X_p, Y_p) &= \hat{K}\left(\hat{X}_p, \hat{Y}_p\right) - 3\|A_{X_p} Y_p\|^2. \end{aligned}$$

Proof. This is a direct consequence of the formulas above taking into account that $A_X X = 0$ and $g((\nabla_U A)(X, X), U) = 0$. □

In the following let $(X_i)_i$ and $(U_j)_j$ be local orthonormal frames spanning $\mathcal{H}$ and $\mathcal{V}$, respectively.

Definition 2.9 ((9.34) in [Be])**.** *We define the mean curvature vector field by*

$$N := \sum_j T_{U_j} U_j.$$

We note that N is horizontal as $T_{U_j} U_j = \left(\nabla_{U_j} U_j\right)^{\top}$ and can be described as vector-valued trace of the second fundamental form of the fibres. Indeed, N is independent of the choice of the vertical orthonormal frame $(U_j)_j$. First we compute

$$\begin{aligned} \sum_j T_{U_j} U_j &= \sum_i g(N, X_i) X_i = \sum_{i,j}\left(\left(\nabla_{U_j} U_j\right)^{\top}, X_i\right) X_i \\ &= \sum_{i,j} g\left(\nabla_{U_j} U_j, X_i\right) X_i = -\sum_{i,j}\left(U_j, \nabla_{U_j} X_i\right) X_i \\ &= -\sum_{i,j} g\left(U_j, \left(\nabla_{U_j} X_i\right)^{\perp}\right) X_i = -\sum_{i,j}\left(U_j, T_{U_j} X_i\right) X_i. \end{aligned}$$

Now let $(U'_k)_k$ be another orthonormal vertical frame. Then

$$\begin{aligned} T_{U_j} X_i &= \sum_k g\left(T_{U_j} X_i, U'_k\right) U'_k = -\sum_k g\left(T_{U_j} U'_k, X_i\right) U'_k \\ &= -\sum_k g\left(T_{U'_k} U_j, X_i\right) U'_k = \sum_k g\left(T_{U'_k} X_i, U_j\right) U'_k. \end{aligned}$$

It follows

$$\begin{aligned}
\sum_j T_{U_j} U_j &= -\sum_{i,j,k} g\left(U_j, U_k'\right)\left(T_{U_k'} X_i, U_j\right) X_i \\
&= -\sum_{i,k} g\left(\sum_j g\left(T_{U_k'} X_i, U_j\right) U_j, U_k'\right) X_i \\
&= -\sum_{i,k} g\left(T_{U_k'} X_i, U_k'\right) X_i \\
&= \sum_k T_{U_k'} U_k'.
\end{aligned}$$

Lemma 2.10. *Let $p \in M$, $x \in \mathcal{H}_p$, $e \in T_pM$ and $(u_j)_j$ be an orthonormal basis of $\mathcal{V}_p$. Then we have*

$$\sum_j g\left(\left(\nabla_e T\right)\left(u_j, u_j\right), x\right) = g\left(\nabla_e N, x\right).$$

Proof. We choose a continuation $(U_j)_j$ of $(u_j)_j$ to a local orthonormal frame spanning $\mathcal{V}$ in a neighbourhood of p.
We compute $g\left(\nabla_e N, x\right) =$

$$\begin{aligned}
&= \sum_j g\left(\left(\nabla_e T\right)(u_j, u_j), x\right) + \sum_j g\left(T_{(\nabla_e U_j)^\perp} u_j, x\right) + \sum_j \left(T_{u_j}\left(\nabla_e U_j\right)^\perp, x\right) \\
&= \sum_j g\left(\left(\nabla_e T\right)(u_j, u_j), x\right) + 2 \cdot \sum_j g\left(T_{(\nabla_e U_j)^\perp} u_j, x\right).
\end{aligned}$$

We write $\left(\nabla_e U_j\right)^\perp = \sum_k \alpha_{jk}(e) u_k$ and note that $\alpha_{jk}(e) = -\alpha_{kj}(e)$. It follows

$$\sum_j g\left(T_{(\nabla_e U_j)^\perp} u_j, x\right) = \sum_{j,k} \alpha_{jk}(e)\, g\left(T_{u_k} u_j, x\right) = -\sum_{j,k} \alpha_{kj}(e)\, g\left(T_{u_j} u_k, x\right) = 0$$

and consequently,

$$\sum_j g\left(\left(\nabla_e T\right)\left(u_j, u_j\right), x\right) = g\left(\nabla_e N, x\right).$$

□

To deduce the Ricci and scalar curvature of (M, g) it will be convenient to use the following notation.

Definition 2.11 (cf. 9.33 in [Be]). *For vertical U, V and horizontal X, Y we set*

$$
\begin{aligned}
g(A_X, A_Y) &= \sum_i g(A_X X_i, A_Y X_i) = \sum_j g(A_X U_j, A_Y U_j), && (2.1)\\
g(A_X, T_U) &= \sum_i g(A_X X_i, T_U X_i) = \sum_j g(A_X U_j, T_U U_j), && (2.2)\\
g(T_U, T_V) &= \sum_i g(T_U X_i, T_V X_i) = \sum_j g(T_U U_j, T_V U_j), && (2.3)\\
g(AU, AV) &= \sum_i g(A_{X_i} U, A_{X_i} V), && (2.4)\\
g(TX, TY) &= \sum_j g\left(T_{U_j} X, T_{U_j} Y\right). && (2.5)
\end{aligned}
$$

It remains to show that these expressions are well-defined.

Proof. For (2.1) we note that

$$
\begin{aligned}
\sum_i g(A_X X_i, A_Y X_i) &= \sum_i g\left(\sum_j g(A_X X_i, U_j) U_j, \sum_k g(A_Y X_i, U_k) U_k\right)\\
&= \sum_{i,j} g(A_X X_i, U_j)\, g(A_Y X_i, U_j)\\
&= \sum_j g\left(A_X U_j, \sum_i g(A_Y U_j, X_i) X_i\right)\\
&= \sum_j g(A_X U_j, A_Y U_j).
\end{aligned}
$$

(2.2) follows from

$$
\begin{aligned}
\sum_i g(A_X X_i, T_U X_i) &= \sum_i g\left(\sum_j g(A_X X_i, U_j) U_j, \sum_k g(T_U X_i, U_k) U_k\right)\\
&= \sum_{i,j} g(A_X X_i, U_j)\, g(T_U X_i, U_j)\\
&= \sum_{i,j} g(A_X U_j, X_i)\, g(T_U U_j, X_i)\\
&= \sum_j g\left(A_X U_j, \sum_i g(T_U U_k, X_i) X_i\right)\\
&= \sum_j g(A_X U_j, T_U U_j).
\end{aligned}
$$

Similarly, we obtain (2.3) from

$$\begin{aligned}
\sum_i g\left(T_U X_i, T_V X_i\right) &= \sum_i g\left(\sum_k g\left(T_U X_i, U_k\right) U_k, \sum_j g\left(T_V X_i, U_j\right) U_j\right) \\
&= \sum_{i,j} g\left(T_U X_i, U_j\right) g\left(T_V X_i, U_j\right) \\
&= \sum_{i,j} g\left(T_U U_j, X_i\right) g\left(T_V U_j, X_i\right) \\
&= \sum_j g\left(T_U U_j, \sum_i g\left(T_V U_j, X_i\right) X_i\right) \\
&= \sum_j g\left(T_U U_j, T_V U_j\right).
\end{aligned}$$

Now let $(X'_l)_l$ and $(U'_k)_k$ two other local orthonormal frames spanning $\mathcal{H}$ and $\mathcal{V}$, respectively. Then we have

$$\begin{aligned}
\sum_i g\left(A_{X_i} U, A_{X_i} V\right) &= \sum_i g\left(\sum_l g\left(A_{X_i} U, X'_l\right) X'_l, \sum_r g\left(A_{X_i} U, X'_r\right) X'_r\right) \\
&= \sum_{i,l} g\left(A_{X_i} U, X'_l\right) g\left(A_{X_i} V, X'_l\right) \\
&= \sum_{i,l} g\left(A_{X_i} X'_l, U\right) g\left(A_{X_i} X'_l, V\right) \\
&= \sum_{i,l} g\left(A_{X'_l} X_i, U\right) g\left(A_{X'_l} X_i, V\right) \\
&= \sum_{i,l} g\left(A_{X'_l} U, X_i\right) g\left(A_{X'_l} V, X_i\right) \\
&= \sum_l g\left(A_{X'_l} U, \sum_i g\left(A_{X'_l} V, X_i\right) X_i\right) \\
&= \sum_l g\left(A_{X'_l} U, A_{X'_l} V\right)
\end{aligned}$$

and analogously

$$\begin{aligned}
\sum_j g\left(T_{U_j}X, T_{U_j}Y\right) &= \sum_j g\left(\sum_k g\left(T_{U_j}X, U_k'\right)U_k', \sum_s g\left(T_{U_j}Y, U_s'\right)U_s'\right) \\
&= \sum_{k,j} g\left(T_{U_j}X, U_k'\right) g\left(T_{U_j}Y, U_k'\right) \\
&= \sum_{k,j} g\left(T_{U_k'}X, U_j\right) g\left(T_{U_k'}Y, U_j\right) \\
&= \sum_k g\left(T_{U_k'}X, \sum_j g\left(T_{U_k'}Y, U_j\right)U_j\right) \\
&= \sum_k g\left(T_{U_k'}X, T_{U_k'}Y\right).
\end{aligned}$$

□

Definition 2.12 ((9.33h) in [Be]). *We define the symmetric 2-tensor field* $\tilde{\delta}T$ *on* $\mathcal{V}$ *by*

$$\tilde{\delta}T(U,V) = \sum_i g\left(\left(\nabla_{X_i}T\right)(U,V), X_i\right).$$

We remark that $\tilde{\delta}T(U,V)$ is well-defined as the trace of the mapping

$$X \mapsto \left(\left(\nabla_X T\right)(U,V)\right)^{\top}.$$

For later use we compute

$$\begin{aligned}
\sum_j \tilde{\delta}T(U_j,U_j) &= \sum_{i,j} g\left(\nabla_{X_i}\left(T_{U_j}U_j\right), X_i\right) - \sum_{i,j} g\left(T_{\nabla_{X_i}U_j}U_j, X_i\right) \\
&\quad - \sum_{i,j} g\left(T_{U_j}\left(\nabla_{X_i}U_j\right), X_i\right) \\
&= \sum_i g\left(\nabla_{X_i}N, X_i\right) - \sum_{i,j} g\left(T_{U_j}\left(\nabla_{X_i}U_j\right)^{\perp}, X_i\right) \\
&\quad + \sum_{i,j} g\left(T_{U_j}X_i, \nabla_{X_i}U_j\right) \\
&= \sum_i g\left(\nabla_{X_i}N, X_i\right)
\end{aligned}$$

as in the proof of Lemma 2.10. Finally, we have

$$\|T\|^2 := \sum_j g\left(T_{U_j}, T_{U_j}\right) = \sum_i g\left(TX_i, TX_i\right)$$

and

$$\|A\|^2 := \sum_i g\left(A_{X_i}, A_{X_i}\right) = \sum_j g\left(AU_j, AU_j\right).$$

Proposition 2.13 (9.36 in [Be]). *Let X, Y be horizontal, U, V be vertical and as above $(X_i)_i$ and $(U_j)_j$ local orthonormal frames spanning $\mathcal{H}$ and $\mathcal{V}$, respectively. Then the Ricci curvature of (M, g) is given by*

$$\begin{aligned}
\mathrm{ric}(U,V) &= \mathrm{ric}^{\perp}(U,V) + g(AU,AV) - g(T_U V, N) + \tilde{\delta} T(U,V), \\
\mathrm{ric}(X,Y) &= \widehat{\mathrm{ric}}\,(d\pi(X), d\pi(Y)) - 2g(A_X, A_Y) - g(TX, TY) \\
&\quad + \sum_j g\left((\nabla_X T)(U_j, U_j), Y\right) + \sum_j g\left(\left(\nabla_{U_j} A\right)(X,Y), U_j\right), \\
\mathrm{ric}(X,U) &= \sum_i g\left((\nabla_{X_i} A)(X_i, X), U\right) - \sum_j g\left(\left(\nabla_{U_j} T\right)(U, U_j), X\right) \\
&\quad + g(\nabla_U N, X) - 2g(A_X, T_U),
\end{aligned}$$

where $\widehat{\mathrm{ric}}$ denotes the Ricci curvature of $(B, \hat{g})$ and $\mathrm{ric}^{\perp}$ the Ricci curvature of $(F_b, g^{\perp})$.

Proof. First we note that

$$\begin{aligned}
\sum_i g(R(X_i,U)V, X_i) &= \sum_i g(R(U,X_i)X_i, V) \\
&= \sum_i g((\nabla_{X_i} T)(U,V), X_i) - \sum_i g(T_U X_i, T_V X_i) \\
&\quad + \sum_i g((\nabla_U A)(X_i, X_i), V) + \sum_i g(A_{X_i} U, A_{X_i} V) \\
&= \tilde{\delta} T(U,V) - g(T_U, T_V) + g(AU, AV).
\end{aligned}$$

Using then

$$\begin{aligned}
\sum_j g(R(U_j,U)V, U_j) &= \sum_j g\left(R^{\perp}(U_j,U)V, U_j\right) \\
&\quad + \sum_j g\left(T_{U_j} V, T_U U_j\right) - \sum_j g\left(T_U V, T_{U_j} U_j\right) \\
&= \mathrm{ric}^{\perp}(U,V) + g(T_V, T_U) - g(T_U V, N)
\end{aligned}$$

yields

$$\begin{aligned}
\mathrm{ric}(U,V) &= \sum_i g(R(X_i,U)V, X_i) + \sum_j g(R(U_j,U)V, U_j) \\
&= \mathrm{ric}^{\perp}(U,V) + g(AU,AV) - g(T_U V, N) + \tilde{\delta} T(U,V).
\end{aligned}$$

Secondly, we have $\sum_j g\left(R\left(U_j,X\right)Y,U_j\right) =$

$$\begin{aligned}
&= \sum_j g\left(\left(\nabla_X T\right)\left(U_j.U_j\right),Y\right) - \sum_j g\left(T_{U_j}X, T_{U_j}Y\right) \\
&\quad + \sum_j g\left(\left(\nabla_{U_j}A\right)(X,Y),U_j\right) + \sum_j g\left(A_X U_j, A_Y U_j\right) \\
&= \sum_j g\left(\left(\nabla_X T\right)\left(U_j.U_j\right),Y\right) + \sum_j g\left(\left(\nabla_{U_j}A\right)(X,Y),U_j\right) \\
&\quad - g(TX,TY) + g\left(A_X, A_Y\right)
\end{aligned}$$

and $\sum_i g\left(R\left(X_i,X\right)Y,X_i\right) =$

$$\begin{aligned}
&= \sum_i g\left(R^\top\left(X_i,X\right)Y,X_i\right) + \sum_i g\left(A_{X_i}Y, A_X X_i\right) \\
&\quad + 2\sum_i g\left(A_{X_i}X, A_Y X_i\right) - \sum_i g\left(A_X Y, A_{X_i}X_i\right) \\
&= \widehat{\mathrm{ric}}\left(d\pi(X), d\pi(Y)\right) - 3g\left(A_X, A_Y\right),
\end{aligned}$$

where we denote by $R^\top\left(X_i,X\right)Y$ the horizontal lift of

$$\hat{R}\left(d\pi(X_i), d\pi(X)\right)d\pi(Y)$$

and take into account that $\left(d\pi(X_i)\right)_i$ is a local orthonormal frame on $(B,\hat{g})$. It follows

$$\begin{aligned}
\mathrm{ric}(X,Y) &= \sum_i g\left(R\left(X_i,X\right)Y,X_i\right) + \sum_j g\left(R\left(U_j,X\right)Y,U_j\right) \\
&= \widehat{\mathrm{ric}}\left(d\pi(X), d\pi(Y)\right) - 2g\left(A_X, A_Y\right) - g\left(TX,TY\right) \\
&\quad + \sum_j g\left(\left(\nabla_X T\right)\left(U_j,U_j\right),Y\right) + \sum_j g\left(\left(\nabla_{U_j}A\right)(X,Y),U_j\right).
\end{aligned}$$

For the remaining equality we note

$$\begin{aligned}
g\left(R\left(X_i,X\right)U,X_i\right) &= -g\left(R\left(X_i,X\right)X_i,U\right) \\
&= g\left(\left(\nabla_{X_i}A\right)\left(X_i,X\right),U\right) + g\left(T_U X_i, A_{X_i}X\right) \\
&\quad - g\left(T_U X, A_{X_i}X_i\right) - g\left(T_U X_i, A_X X_i\right) \\
&= g\left(\left(\nabla_{X_i}A\right)\left(X_i,X\right),U\right) - 2g\left(A_X X_i, T_U X_i\right)
\end{aligned}$$

and

$$\begin{aligned}
g\left(R\left(U_j,X\right)U,U_j\right) &= g\left(R\left(U,U_j\right)U_j,X\right) \\
&= g\left(\left(\nabla_U T\right)\left(U_j,U_j\right),X\right) - g\left(\left(\nabla_{U_j}T\right)\left(U,U_j\right),X\right),
\end{aligned}$$

which implies

$$\begin{aligned}\mathrm{ric}(X,U) &= \sum_i g\left(R\left(X_i,X\right)U,X_i\right)+\sum_j g\left(R\left(U_j,X\right)U,U_j\right)\\ &= \sum_i g\left(\left(\nabla_{X_i}A\right)\left(X_i,X\right),U\right)-\sum_j g\left(\left(\nabla_{U_j}T\right)\left(U,U_j\right),X\right)\\ &\quad +g\left(\nabla_U N,X\right)-2g\left(A_X,T_U\right).\end{aligned}$$

□

Finally we combine

$$\sum_j \mathrm{ric}\left(U_j,U_j\right)=\mathrm{scal}^{\perp}+\|A\|^2-\|N\|^2+\sum_j \tilde{\delta}T\left(U_j,U_j\right)$$

and

$$\sum_j \mathrm{ric}\left(X_i,X_i\right)=\mathrm{scal}_{B,\hat{g}}\circ\pi-2\|A\|^2-\|T\|^2+\sum_j \tilde{\delta}T\left(U_j,U_j\right)$$

and obtain

Proposition 2.14 (9.37 in [Be]). *The scalar curvature of (M,g) is given by*

$$\mathrm{scal}_{M,g}=\mathrm{scal}_{B,\hat{g}}\circ\pi+\mathrm{scal}^{\perp}-\|A\|^2-\|T\|^2-\|N\|^2+2\cdot\sum_i g\left(\nabla_{X_i}N,X_i\right),$$

where $\mathrm{scal}^{\perp}=\mathrm{scal}_{g^{\perp}}$ is the scalar curvature of the fibres with respect to the induced metric.

2.3 Rescaling the Metric

Let $\pi:(M,g)\to(B,\hat{g})$ be a Riemannian submersion with induced metric $g^{\perp}$ on the fibres. For every $p\in M$ the isomorphism $d\pi_p:\mathcal{H}_p\to T_{\pi(p)}B$ is an isometry. We consider the rescaled metric $r^2\hat{g}$, $r>0$, pull it back via π on $\mathcal{H}$ and obtain a metric g_{r^2} on M satisfying

$$\begin{aligned}g_{r^2}\left(X_p,Y_p\right) &= r^2\hat{g}\left(d\pi_p X_p,d\pi_p Y_p\right),\\ g_{r^2}\left(X_p,U_p\right) &= g\left(X_p,U_p\right)=0,\\ g_{r^2}\left(U_p,V_p\right) &= g\left(U_p,V_p\right).\end{aligned}$$

Using Koszul's formula we recognize that $\hat{\nabla}^{r^2}=\hat{\nabla}$, where $\hat{\nabla}^{r^2}$ denotes the Levi-Civita connection on $(B,r^2\hat{g})$. Consequently,

$$\exp_b^{r^2\hat{g}}=\exp_b^{\hat{g}}=:\exp_b\quad\text{for all}\quad b\in B.$$

By construction, $\pi : (M, g_{r^2}) \to \left(B, r^2\hat{g}\right)$ is a Riemannian submersion with the same vertical and horizontal distributions $\mathcal{V}$ and $\mathcal{H}$, respectively, as $\pi : (M, g) \to (B, \hat{g})$. The metric g_{r^2} induces the Levi-Civita connection ∇^{r^2} on M and associated rescaled tensor fields A^{r^2} and T^{r^2}.

Lemma 2.15. *For vertical vector fields U, V and horizontal vector fields X, Y we have*

$$\begin{aligned} A^{r^2}_X Y &= A_X Y, \\ A^{r^2}_X U &= \frac{1}{r^2} A_X U, \\ T^{r^2}_U V &= \frac{1}{r^2} T_U V, \\ T^{r^2}_U X &= T_U X. \end{aligned}$$

Proof. The first equality is clear, since $A^{r^2}_X Y = \frac{1}{2}\left[X, Y\right]^{\perp} = A_X Y$. To prove the others we use Koszul's formula. For an arbitrary horizontal vector field Z we have

$$\begin{aligned} 2g\left(\left(\nabla_X U\right)^{\top}, Z\right) &= 2g\left(\nabla_X U, Z\right) \\ &= Xg\left(U, Z\right) + Ug\left(X, U\right) - Zg\left(X, U\right) \\ &\quad + g\left(\left[X, U\right], Z\right) - g\left(\left[U, Z\right], X\right) + g\left(\left[X, Z\right], U\right) \\ &= g\left(\left[X, Z\right]^{\perp}, U\right) = g_{r^2}\left(\left[X, Z\right]^{\perp}, U\right) \\ &= 2g_{r^2}\left(\left(\nabla^{r^2}_X U\right)^{\top}, Z\right) = 2r^2 g\left(\left(\nabla^{r^2}_X U\right)^{\top}, Z\right), \end{aligned}$$

which implies

$$A^{r^2}_X Y = \left(\nabla^{r^2}_X U\right)^{\top} = \frac{1}{r^2}\left(\nabla_X U\right)^{\top} = \frac{1}{r^2} A_X U.$$

Since

$$\begin{aligned} 2g\left(\left(\nabla_U V\right)^{\top}, X\right) &= 2g\left(\nabla_U V, X\right) \\ &= Ug\left(V, X\right) + Vg\left(U, X\right) - Xg\left(U, V\right) \\ &\quad + g\left(\left[U, V\right], X\right) - g\left(\left[V, X\right], U\right) + g\left(\left[U, X\right], V\right) \\ &= -Xg\left(U, V\right) - g\left(\left[V, X\right], U\right) + g\left(\left[U, X\right], V\right) \\ &= -Xg_{r^2}\left(U, V\right) - g_{r^2}\left(\left[V, X\right], U\right) + g_{r^2}\left(\left[U, X\right], V\right) \\ &= 2g_{r^2}\left(\left(\nabla^{r^2}_U V\right)^{\top}, X\right) = 2r^2 g\left(\left(\nabla^{r^2}_U V\right)^{\top}, X\right) \end{aligned}$$

holds for every horizontal vector field X, we obtain

$$T^{r^2}_U V = \left(\nabla^{r^2}_U V\right)^{\top} = \frac{1}{r^2}\left(\nabla_U V\right)^{\top} = \frac{1}{r^2} T_U V.$$

Finally,

$$\begin{aligned}
2g\left((\nabla_U X)^{\perp}, V\right) &= 2g\left(\nabla_U X, V\right)\\
&= Ug\left(X, V\right) + Xg\left(U, V\right) - Vg\left(U, X\right)\\
&\quad +g\left([U, X], V\right) - g\left([X, V], U\right) + g\left([U, V], X\right)\\
&= Xg\left(U, V\right) + g\left([U, X], V\right) - g\left([X, V], U\right)\\
&= Xg_{r^2}\left(U, V\right) + g_{r^2}\left([U, X], V\right) - g_{r^2}\left([X, V], U\right)\\
&= 2g_{r^2}\left(\nabla^{r^2}_U X, V\right) = 2g\left(\left(\nabla^{r^2}_U X\right)^{\perp}, V\right)
\end{aligned}$$

yields

$$T^{r^2}_U X = \left(\nabla^{r^2}_U X\right)^{\perp} = (\nabla_U X)^{\perp} = T_U X.$$

□

In addition we have

Lemma 2.16. *Let U be vertical and X, Y be horizontal. Then*

$$\left(\nabla^{r^2}_X U\right)^{\perp} = (\nabla_X U)^{\perp} \quad \text{and} \quad \left(\nabla^{r^2}_X Y\right)^{\top} = (\nabla_X Y)^{\top}.$$

Proof. For the first equality we note

$$\left(\nabla^{r^2}_X U\right)^{\perp} = [X, U]^{\perp} + T^{r^2}_U X = [X, U]^{\perp} + T_U X = (\nabla_X U)^{\perp},$$

and for the other we apply Koszul's formula once again

$$\begin{aligned}
2g_{r^2}\left(\nabla^{r^2}_X Y, Z\right) &= Xg_{r^2}\left(Y, Z\right) + Yg_{r^2}\left(X, Z\right) - Zg_{r^2}\left(X, Y\right)\\
&\quad +g_{r^2}\left([X, Y], Z\right) - g_{r^2}\left([Y, Z], X\right) + g_{r^2}\left([X, Z], Y\right)\\
&= r^2 Xg\left(Y, Z\right) + r^2 Yg\left(X, Z\right) - r^2 Zg\left(X, Y\right)\\
&\quad +r^2 g\left([X, Y]^{\top}, Z\right) - r^2 g\left([Y, Z]^{\top}, X\right)\\
&\quad +r^2 g\left([X, Z]^{\top}, Y\right)\\
&= 2r^2 g\left(\nabla_X Y, Z\right),
\end{aligned}$$

which implies $\left(\nabla^{r^2}_X Y\right)^{\top} = (\nabla_X Y)^{\top}$. □

Now we consider a $g-$orthonormal basis $\{(X_i)_p\}$ and $\{(U_j)_p\}$ of $\mathcal{H}_p$ and $\mathcal{V}_p$, respectively, and get an $g_{r^2}-$orthonormal basis $\{\frac{1}{r}(X_i)_p\}$ and $\{(U_j)_p\}$ of $\mathcal{H}_p$ and $\mathcal{V}_p$, respectively. It follows

$$N^{r^2} = \sum_j T^{r^2}_{U_j} U_j = \frac{1}{r^2}\sum_j T_{U_j} U_j = \frac{1}{r^2} N.$$

and

$$\|N^{r^2}\|^2_{g_{r^2}} = g_{r^2}\left(\frac{1}{r^2}N, \frac{1}{r^2}N\right) = \frac{1}{r^2}\|N\|^2_g.$$

Furthermore,

$$\begin{aligned}
\|A^{r^2}\|^2_{g_{r^2}} &= \sum_{i,j} g_{r^2}\left(A^{r^2}_{\frac{1}{r}X_i}U_j, A^{r^2}_{\frac{1}{r}X_i}U_j\right) \\
&= \sum_{i,j} r^2 g\left(\frac{1}{r^2}A_{\frac{1}{r}X_i}U_j, \frac{1}{r^2}A_{\frac{1}{r}X_i}U_j\right) \\
&= \frac{1}{r^4}\sum_{i,j} g\left(A_{X_i}U_j, A_{X_i}U_j\right) \\
&= \frac{1}{r^4}\|A\|^2_g
\end{aligned}$$

and

$$\begin{aligned}
\|T^{r^2}\|^2_{g_{r^2}} &= \sum_{i,j} g_{r^2}\left(T^{r^2}_{U_j}\left(\frac{1}{r}X_i\right), T^{r^2}_{U_j}\left(\frac{1}{r}X_i\right)\right) \\
&= \sum_{i,j} g\left(\frac{1}{r}T^{r^2}_{U_j}X_i, \frac{1}{r}T^{r^2}_{U_j}X_i\right) \\
&= \frac{1}{r^2}\sum_{i,j} g\left(T_{U_j}X_i, T_{U_j}X_i\right) \\
&= \frac{1}{r^2}\|T\|^2_g.
\end{aligned}$$

Moreover,

$$\begin{aligned}
g_{r^2}\left(T^{r^2}_{U_j}\left(\frac{1}{r}X_i\right), \nabla^{r^2}_{\frac{1}{r}X_i}U_j\right) &= g\left(\frac{1}{r}T^{r^2}_{U_j}X_i, \left(\nabla^{r^2}_{\frac{1}{r}X_i}U_j\right)^{\perp}\right) \\
&= \frac{1}{r^2}g\left(T_{U_j}X_i, (\nabla_{X_i}U_j)^{\perp}\right) \\
&= \frac{1}{r^2}g\left(T_{U_j}X_i, \nabla_{X_i}U_j\right)
\end{aligned}$$

and

$$\begin{aligned}
g_{r^2}\left(\nabla^{r^2}_{\frac{1}{r}X_i}N^{r^2}, \frac{1}{r}X_i\right) &= r^2 g\left(\left(\nabla^{r^2}_{\frac{1}{r}X_i}\frac{1}{r^2}N\right)^{\top}, \frac{1}{r}X_i\right) \\
&= g\left(\left(\nabla_{X_i}\frac{1}{r^2}N\right)^{\top}, X_i\right) \\
&= \frac{1}{r^2}g\left(\nabla_{X_i}N, X_i\right)
\end{aligned}$$

yield

$$\begin{aligned}\sum_j \tilde{\delta}T^{r^2}(U_j,U_j) &= \sum_i g_{r^2}\left(\nabla^{r^2}_{\frac{1}{r}X_i}N^{r^2}, \frac{1}{r}X_i\right) \\ &\quad +2\sum_{i,j} g_{r^2}\left(T^{r^2}_{U_j}\left(\frac{1}{r}X_i\right), \nabla^{r^2}_{\frac{1}{r}X_i}U_j\right) \\ &= \frac{1}{r^2}\sum_i g\left(\nabla_{X_i}N, X_i\right) + \frac{2}{r^2}\sum_{i,j} g\left(T_{U_j}X_i, \nabla_{X_i}U_j\right) \\ &= \frac{1}{r^2}\sum_j \tilde{\delta}T\left(U_j,U_j\right).\end{aligned}$$

Proposition 2.17. *The scalar curvature of (M, g_{r^2}) is given by*

$$\begin{aligned}\mathrm{scal}_{M,g_{r^2}} &= \frac{1}{r^2}\cdot \mathrm{scal}_{B,\hat{g}}\circ\pi + \mathrm{scal}^{\perp} - \frac{1}{r^4}\|A\|_g^2 - \frac{1}{r^2}\|T\|_g^2 \\ &\quad -\frac{1}{r^2}\|N\|_g^2 + \frac{2}{r^2}\cdot\sum_i g\left(\nabla_{X_i}N, X_i\right).\end{aligned}$$

As an immediate application we remark

Corollary 2.18. *Let (M,g) and $(B,\hat{g})$ be closed Riemannian manifolds. Suppose that there is a Riemannian submersion $\pi : (M,g) \to (B,\hat{g})$ such that the scalar curvature of every fibre with respect to the induced metric is positive, then M carries a metric of positive scalar curvature.*

2.4 Local Trivializations and Induced Metrics

Let $\pi : (M,g) \to (B,\hat{g})$ be a Riemannian submersion, where M and B are assumed to be closed. We will show that geodesics in $(B,\hat{g})$ lift to unique horizontal geodesics in (M,g), which allows us to construct local trivializations.

2.4.1 Lifting Properties

We begin with an elementary observation.

Lemma 2.19. *Let $I \subset \mathbb{R}$ be an interval and $\hat{\gamma} : I \to B$ a regular smooth curve, $q \in \pi^{-1}(\hat{\gamma}(t_0))$ for some $t_0 \in I$ and suppose there exists a horizontal lift of $\hat{\gamma}$, i.e. a smooth curve $\gamma : I \to M$ such that $\pi\circ\gamma = \hat{\gamma}$ and $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for all $t \in I$. Then γ is unique.*

Proof. Let $\gamma_1, \gamma_2 : I \to M$ be horizontal lifts of $\hat{\gamma}$ with $\gamma_1(t_0) = \gamma_2(t_0)$. We consider the subset

$$J = \{t \in I | \gamma_1(t) = \gamma_2(t)\},$$

which contains t_0, and show that J is open and closed in I. Let $s \in J$. Then there exists a vector field $\hat{X}$ in a neighbourhood U of $\hat{\gamma}(s)$ in B and an $\varepsilon > 0$ such that $\hat{X} \circ \hat{\gamma}(t) = \hat{\gamma}'(t)$ for all $t \in I \cap (s-\varepsilon, s+\varepsilon)$. Let X be the unique horizontal lift of $\hat{X}$ defined on $\pi^{-1}(U) \subset M$. It follows

$$d\pi_{\gamma_i(t)} X_{\gamma_i(t)} = \hat{X}_{(\pi\circ\gamma_i)(t)} = \hat{X}_{\hat{\gamma}(t)} = \hat{\gamma}'(t) = (\pi \circ \gamma_i)'(t) = d\pi_{\gamma_i(t)}\gamma_i'(t)$$

and consequently $X_{\gamma_i(t)} = \gamma_i'(t)$ for all $t \in I \cap (s-\varepsilon, s+\varepsilon)$ and $i = 1, 2$. I.e.

$$I \cap (s-\varepsilon, s+\varepsilon) \ni t \mapsto \gamma_1(t) \quad \text{and} \quad I \cap (s-\varepsilon, s+\varepsilon) \ni t \mapsto \gamma_2(t)$$

are integral curves of X satisfying $\gamma_1(s) = \gamma_2(s)$. It follows $\gamma_1(t) = \gamma_2(t)$ for all $t \in I \cap (s-\varepsilon, s+\varepsilon)$ and $I \cap (s-\varepsilon, s+\varepsilon) \subset J$. In other words, J is an open subset of I. Finally, J is closed in I since γ_1 and γ_2 are continuous. □

To investigate the lifting properties of curves it will be convenient to use local vector fields with prescribed properties.

Lemma 2.20. *Let (M^m, g) be a Riemannian manifold with induced Levi-Civita connection ∇. For any $p \in M$ and $v \in T_pM$ there exists a vector field X in a neighbourhood of p such that $X_p = v$ and $\nabla_w X = 0$ for all $w \in T_pM$.*

Proof. We choose an orthonormal basis of T_pM and identify T_pM with $\mathbb{R}^m$. Then $\exp_p : B_\varepsilon(0) \to B_\varepsilon(p)$ is a diffeomorphism for a suitable $\varepsilon > 0$. Since

$$[0,1] \times B_\varepsilon(0) \ni (t, w) \mapsto \exp_p(tw)$$

is smooth, parallel transport along the geodesic $[0,1] \ni t \mapsto \exp_p(tw)$ yields a smooth map

$$[0,1] \times B_\varepsilon(0) \ni (t, w) \mapsto \mathcal{P}_w(t) \quad \text{with} \quad \mathcal{P}_w(0) = v$$

and a vector field X on $B_\varepsilon(p)$ defined by

$$X_q = \mathcal{P}_{\exp_p^{-1}(q)}(1) \quad \text{for all} \quad q \in B_\varepsilon(p).$$

For $t \in [0,1]$ and $w \in B_\varepsilon(0)$ we remark that $[0,1] \in s \mapsto V_1(s) = \mathcal{P}_w(ts)$ and $[0,1] \ni s \mapsto \mathcal{P}_{tw}(s)$ are parallel along the geodesic $s \mapsto \exp_p(stw)$ and satisfy $V_1(0) = v = V_2(0)$. It follows $\mathcal{P}_{tw}(s) = \mathcal{P}_w(ts)$ and

$$X_{\exp_p(tw)} = \mathcal{P}_{tw}(1) = \mathcal{P}_w(t).$$

Consequently, $\nabla_w X = 0$ for all $w \in T_pM$. □

As an immediate application of the construction above we consider an orthonormal basis $(e_1, \ldots, e_m)$ of T_pM. Let $\varepsilon > 0$ be small enough such that $\exp_p : T_p(M) \supset B_\varepsilon(0) \to B_\varepsilon(p)$ is a diffeomorphism. On $B_\varepsilon(p)$ we find vector fields E_i with $E_i(p) = e_i$ and $(\nabla E_i)_p = 0$ for all $1 \le i \le m$. Since the vector fields $E_1, \ldots, E_m$ are parallel along geodesics $t \mapsto \exp_p(tw)$ with $w \in B_\varepsilon(0) \subset T_pM$, we obtain that $(E_1(q), \ldots, E_m(q))$ is an orthonormal basis of T_qM for all $q \in B_\varepsilon(p)$. Hence we have proven

Corollary 2.21. *Let (M^m, g, ∇) be a Riemannian manifold and $p \in M$. Then there exists a (geodesic) frame $(E_1, \ldots, E_m)$ in a neighbourhood U of p, such that $E_1, \ldots, E_m$ are orthonormal vector fields on U and*

$$\left(\nabla_{E_i} E_j\right)(p) = 0$$

for all $1 \leq i, j \leq m$.

Lemma 2.22. *Let (M^m, g, ∇) be a Riemannian manifold. For $p \in M$ and $0 \neq v \in T_pM$ there exists a vector field X in a neighbourhood U of p such that $X_p = v$ and $\left(\nabla_X X\right)(q) = 0$ for all $q \in U$.*

Proof. We choose an open subset $U \subset M$ containing p together with an $\varepsilon > 0$ such that $\exp_q : B_\varepsilon(0) \to B_\varepsilon(q)$ is a diffeomorphism satisfying

$$U \subset \exp_q\left(B_\varepsilon(0)\right) \quad \text{for every} \quad q \in U,$$

i. e. a totally normal neighbourhood of p. It follows that any two points $q_1, q_2 \in U$ can be joined by a unique minimizing geodesic of length $< \varepsilon$. W.l.o.g. we may assume $\|v\| < \varepsilon$ and $p_0 = \exp_p(v) \in U$. Let $V \subset U$ be open containing p but not p_0. We have $0 < \|\exp_{p_0}^{-1}(q)\| < \varepsilon$ for $q \in V$ and define a smooth vector field X on V by

$$X_q = -\frac{\|\exp_{p_0}^{-1}(p)\|}{\|\exp_{p_0}^{-1}(q)\|} \cdot \left.\frac{d}{dt}\right|_{t=1} \exp_{p_0}\left(t \cdot \exp_{p_0}^{-1}(q)\right).$$

Since U is totally normal, we have

$$[0,1] \ni t \mapsto \exp_{p_0}\left(t \exp_{p_0}^{-1}(p)\right) = \exp_p\left((1-t)v\right)$$

and hence $X_p = v$. For $s > 0$ such that $\|s \cdot \exp_{p_0}^{-1}(q)\| < \varepsilon$ we compute

$$\begin{aligned} X_{\exp_{p_0}\left(s \cdot \exp_{p_0}^{-1}(q)\right)} &= -\frac{\|\exp_{p_0}^{-1}(p)\|}{\|s \cdot \exp_{p_0}^{-1}(q)\|} \cdot \left.\frac{d}{dt}\right|_{t=1} \exp_{p_0}\left(t \cdot s \cdot \exp_{p_0}^{-1}(q)\right) \\ &= -\frac{\|\exp_{p_0}^{-1}(p)\|}{\|\exp_{p_0}^{-1}(q)\|} \cdot \left.\frac{d}{dt}\right|_{t=s} \exp_{p_0}\left(t \cdot \exp_{p_0}^{-1}(q)\right). \end{aligned}$$

Since $s \mapsto \gamma(s) = \exp_{p_0}\left(s \cdot \exp_{p_0}^{-1}(q)\right)$ is a geodesic through q at $s = 1$ with

$$\gamma'(1) = -\frac{\|\exp_{p_0}^{-1}(q)\|}{\|\exp_{p_0}^{-1}(p)\|} \cdot X_q$$

we deduce $\left(\nabla_X X\right)_q = -\frac{\|\exp_{p_0}^{-1}(p)\|}{\|\exp_{p_0}^{-1}(q)\|} \cdot \nabla_{\gamma'(1)} X = 0.$ □

In order to show that every regular curve $\hat{\gamma} : I \to B$, where $I \subset \mathbb{R}$ is an interval, has a horizontal lift $\gamma : I \to M$ which passes through an arbitrary point $p \in \pi^{-1}\left(\hat{\gamma}(t_0)\right)$ with $t_0 \in I$ we need as preparation

Lemma 2.23. *Let U and W be open subsets of a smooth manifold M such that the closure $\bar{W}$ of W is a compact subset of U. We consider a vector field X on U and an integral curve $\gamma : (a, b) \to W$. Then there exists an $\varepsilon > 0$ and an extension to an integral curve $\gamma : (a - \varepsilon, b + \varepsilon) \to U$.*

Proof. We choose a Riemannian metric g on M. Since $\bar{W} \subset M$ is compact, there exists an $\alpha > 0$ such that $\|X_q\| < \alpha$ for all $q \in \bar{W}$. With $X_{\gamma(t)} = \gamma'(t)$ we obtain

$$d(\gamma(t_2), \gamma(t_1)) \leq \int_{t_1}^{t_2} \|X_{\gamma(t)}\| dt \leq \alpha |t_1 - t_2| \quad \text{for all} \quad t_1, t_2 \in (a, b).$$

Now we consider a sequence $(t_j)_j$ in (a, b) with $t_j \to b$. Using the compactness of $\bar{W}$ again we find a subsequence (t_{j_k}) and a $p \in \bar{W}$ such that $\gamma(t_{j_k}) \to p$. Let s_k be another sequence in (a, b) with $s_k \to b$. Given $\eta > 0$ there exists $k_0 \in \mathbb{N}$ satisfying $|s_k - t_{j_k}| < \frac{\eta}{2\alpha}$ and $d(p, \gamma(t_{j_k})) < \frac{\eta}{2}$ for all $k \geq k_0$. It follows

$$d(p, \gamma(s_k)) \leq d(p, \gamma(t_{j_k})) + d(\gamma(t_{j_k}), \gamma(s_k))$$

for all $k \geq k_0$. Using a similar argument with b replaced by a we get a continuous extension $\gamma : [a, b] \to \bar{W} \subset U$. For each component γ^i of γ in local coordinates near $\gamma(b)$ and $h > 0$ such that $b - h > 0$, by the mean-value theorem there exists $\xi_h^i \in (b - h, b)$ with

$$\frac{\gamma^i(b) - \gamma^i(b - h)}{h} = \left(\gamma^i\right)' \left(\xi_h^i\right) = X^i_{\gamma(\xi_h^i)} \to X^i_{\gamma(b)} \quad \text{for} \quad h \to 0.$$

An analogous argument works for a, and as a consequence $\gamma \in C^1([a, b], U)$. Finally, the existence and uniqueness of integral curves of X through $\gamma(a)$ and $\gamma(b)$ yields an extension to an integral curve $\gamma : (a - \varepsilon, b + \varepsilon) \to U$ for a suitable $\varepsilon > 0$. □

Now we can prove

Proposition 2.24. *Let $I \subset \mathbb{R}$ an interval and $\hat{\gamma} : I \to B$ be a regular curve. For any $p \in M$ such that $\hat{\gamma}(t_0) = \pi(p)$ with $t_0 \in I$ there exists a unique horizontal lift $\gamma : I \to M$ of $\hat{\gamma}$ satisfying $\gamma(t_0) = p$.*

Proof. Uniqueness is clear by the Lemma 2.19. It remains to show the existence of a suitable horizontal lift. To begin with we consider the case of a closed intervall $I = [a, b]$ and $t_0 = a$. For every $t \in [a, b]$ we find a neighbourhood $U(t)$ of $\hat{\gamma}(t)$, an $\varepsilon(t) > 0$ such that $\hat{\gamma}((t - \varepsilon(t), t + \varepsilon(t)) \cap [a, b]) \subset U(t)$ and a vector field $\hat{X}^t$ ond $U(t)$ satisfying

$$\hat{X}^t_{\hat{\gamma}(s)} = \hat{\gamma}'(s) \quad \text{for all} \quad s \in (t - \varepsilon(t), t + \varepsilon(t)) \cap [a, b].$$

By the compactness of $[a, b]$ and a Lebesgue-number argument we find a partition $a = t_0 < t_1 < \ldots < t_{k+1} = b$ together with vector fields $\hat{X}_i$

defined on neighbourhoods U_i of $\hat{\gamma}([t_i, t_{i+1}])$ such that $\hat{X}_i \circ \hat{\gamma}(t) = \hat{\gamma}'(t)$ for all $t \in [t_i, t_{i+1}]$ and $i = 0, \ldots, k$. The vector fields $\hat{X}_0, \ldots, \hat{X}_k$ have unique horizontal lifts $X_0, \ldots, X_k$ defined on $\pi^{-1}(U_0), \ldots, \pi^{-1}(U_k)$, respectively. Let $\gamma_0 : [a, s] \to \pi^{-1}(U_0)$ be the integral curve of X_0 with $\gamma_0(a) = p$, where $a \leq s \leq t_1$. Then we have

$$(\pi \circ \gamma_0)'(t) = d\pi_{\gamma_0(t)}\gamma_0'(t) = d\pi_{\gamma_0(t)}\left(X_0 \circ \gamma_0(t)\right) = \hat{X}_0 \circ (\pi \circ \gamma_0)(t)$$

for all $a \leq t \leq s$. I.e. $\pi \circ \gamma_0$ is an integral curve of $\hat{X}_0$ such that

$$\pi \circ \gamma_0(a) = \pi(p) = \hat{\gamma}(a)$$

and consequently

$$\pi \circ \gamma_0(t) = \hat{\gamma}(t) \quad \text{for all} \quad a \leq t \leq s.$$

In other words, $\gamma_0 : [a, s] \to \pi^{-1}(U_0)$ is the horizontal lift of $\hat{\gamma} : [a, s] \to U_0$. Now let s_1 be the supremum of all $s \in [a, t_1]$ such that γ_0 is defined on $[a, s]$. We are going to show that $s_1 = t_1$. Suppose on the contrary that $s_1 < t_1$. Then there exists a neighbourhood W_0 of $\hat{\gamma}([a, s_1])$ with compact closure in U_0. It follows that $\gamma_0([a, s_1)) \subset \pi^{-1}(W_0)$, where $\pi^{-1}(W_0)$ has compact closure in $\pi^{-1}(U_0)$. Now Lemma 2.23 yields an $\varepsilon > 0$ such that the integral curve γ_0 is actually defined on $[a, s_1 + \varepsilon)$ in contradiction to the choice of s_1. Applying Lemma 2.23 once again we obtain that γ_0 is defined on $[a, t_1 + \varepsilon_1]$ for an $\varepsilon_1 > 0$.
Now suppose we have a horizontal lift $\gamma : [a, t_i + \varepsilon_i) \to M$ of $\hat{\gamma}$ for some $1 \leq i \leq k$ and $0 < \varepsilon_i < t_{i+1} - t_i$. In particular, $\gamma : [t_i, t_i + \varepsilon_i) \to M$ is an integral curve of X_i. Analogously to the construction of γ_0 above we find as a horizontal lift of $\hat{\gamma}$ the integral curve $\gamma_i : [t_i, t_{i+1} + \varepsilon_{i+1}] \cap [a, b] \to M$ of X_i with $\gamma_i(t_i) = \gamma(t_i)$ for a suitable $\varepsilon_{i+1} > 0$. Therefore, $\gamma|_{[t_i, t_i + \varepsilon_i]} = \gamma_i|_{[t_i, t_i + \varepsilon_i]}$ and we can extend γ to a horizontal lift of $\hat{\gamma}$ defined on $[a, t_{i+1} + \varepsilon_{i+1}] \cap [0, 1]$. Now we are done by induction. A similar argument works for $t_0 = b$. If $a < t_0 < b$, there exist horizontal lifts $\gamma_1 : [a, t_0] \to M$ and $\gamma_2 : [t_0, b] \to M$ such that $\gamma_1(t_0) = p = \gamma_2(t_0)$. We find a vector field $\hat{X}$ on a neighbourhood U of $\pi(p)$ and an $\varepsilon > 0$ with $\hat{\gamma}'(t) = \hat{X}_{\hat{\gamma}(t)}$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Then $\gamma_1 : (t_0 - \varepsilon', t_0) \to M$ and $\gamma_2 : (t_0, t_0 + \varepsilon') \to M$ are integral curves of the horizontal lift X of $\hat{X}$ for a suitable $\varepsilon' > 0$. By Lemma 2.23 we can choose $\varepsilon' > 0$ such that we have extensions to integral curves

$$\tilde{\gamma}_1 : (t_0 - \varepsilon', t_0 + \varepsilon') \to M \quad \text{and} \quad \tilde{\gamma}_2 : (t_0 - \varepsilon', t_0 + \varepsilon') \to M$$

of X. Since $\pi \circ \tilde{\gamma}_1$ and $\pi \circ \tilde{\gamma}_2$ are integral curves of X with

$$\pi \circ \tilde{\gamma}_1(t_0) = \hat{\gamma}(t_0) = \pi \circ \tilde{\gamma}_2(t_0),$$

we obtain $\tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ as unique horizontal lift of $\hat{\gamma}$ through p. It follows that

$$\begin{aligned} \gamma : [a,b] &\to M \\ \gamma(t) &= \begin{cases} \gamma_1(t) & \text{if } a \leq t \leq t_0, \\ \gamma_2(t) & \text{if } t_0 \leq t \leq b. \end{cases} \end{aligned}$$

is the horizontal lift of $\hat{\gamma}$ with $\gamma(t_0) = p$.
For the general case let $t \in I$. We take an interval $[a,b] \subset I$ such that $t, t_0 \in [a,b]$ and set $\gamma(t) = \tilde{\gamma}(t)$, where $\tilde{\gamma} : [a,b] \to M$ is the horizontal lift of $\hat{\gamma} : [a,b] \to B$ with $\tilde{\gamma}(t_0) = p$. By the uniqueness of horizontal lifts γ is well-defined. □

We use Proposition 2.24 and investigate the lifting properties of geodesics.

Lemma 2.25 (9.44 in [Be]). *Consider a geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ being horizontal at $p = \gamma(0)$. Then γ is horizontal everywhere and*

$$\hat{\gamma} = \pi \circ \gamma : (-\varepsilon, \varepsilon) \to B$$

is a geodesic.

Proof. We consider the set

$$I = \{t \in (-\varepsilon, \varepsilon) | \gamma'(t) \in \mathcal{H}_{\gamma(t)}\} \ni 0.$$

We take a $t_0 \in I$ and set $\gamma(t_0) = p_0$. By Lemma 2.22 we find a vector field $\hat{X}$ in a neighbourhood $U \subset B$ of $\pi(p_0)$ such that $\hat{X}_{\pi(p_0)} = (\pi \circ \gamma)'(t_0)$ and $\hat{\nabla}_{\hat{X}} \hat{X} = 0$ in U. Let X be the horizontal lift of $\hat{X}$ defined on $\pi^{-1}(U)$. Then $X_{p_0} = \gamma'(t_0)$ and $(\nabla_X X)^\top = 0$ by Lemma 2.7. Let

$$\eta : (-\varepsilon, \varepsilon) \supset (t_0 - \varepsilon', t_0 + \varepsilon') \to M$$

be an integral curve of X with $\eta(t_0) = p_0$ and $\eta'(t_0) = X_{p_0} = \gamma'(t_0)$. Then $\nabla_X X = (\nabla_X X)^\top + A_X X = 0$ implies that η is a geodesic. Hence,

$$\gamma(t) = \eta(t) \quad \text{for all} \quad t \in (t_0 - \varepsilon', t_0 + \varepsilon')$$

and

$$\gamma'(t) \in \mathcal{H}_{\gamma(t)} \quad \text{for all} \quad t \in (t_0 - \varepsilon', t_0 + \varepsilon').$$

It follows that $I \subset (-\varepsilon, \varepsilon)$ is open. Since γ is smooth, we can use a local frame spanning $\mathcal{H}$ to see that $I \subset (-\varepsilon, \varepsilon)$ is closed. Consequently, γ is horizontal everywhere.
To prove that $\hat{\gamma} = \pi \circ \gamma : (-\varepsilon, \varepsilon) \to B$ is a geodesic we take $t_0 \in (-\varepsilon, \varepsilon)$ and set $\gamma(t_0) = p_0$. Similar to the argument above there exists a vector field $\hat{X}$

in a neighbourhood U of $\pi(p_0)$ such that $\hat{X}_{\pi(p_0)} = (\pi \circ \gamma)'(t_0)$ and $\hat{\nabla}_{\hat{X}}\hat{X} = 0$ in U. Then there exists an $\varepsilon' > 0$ such that

$$\gamma : (-\varepsilon, \varepsilon) \supset (t_0 - \varepsilon', t_0 + \varepsilon') \to M$$

is the integral curve of the horizontal lift X of $\hat{X}$ through p_0. It follows that

$$\hat{\gamma} : (t_0 - \varepsilon', t_0 + \varepsilon') \to B$$

is the integral curve of $\hat{X}$ through $\pi(p_0)$, i.e. a geodesic since $\hat{\nabla}_{\hat{X}}\hat{X} = 0$. □

The other way round we are able to lift geodesics.

Lemma 2.26. *Let $\hat{\gamma} : (-\varepsilon, \varepsilon) \to B$ be geodesic and $p \in \pi^{-1}(\hat{\gamma}(0))$. Then there exists a unique horizontal geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ which lifts $\hat{\gamma}$.*

Proof. By Proposition 2.24 we find a unique horizontal lift $\gamma : (-\varepsilon, \varepsilon) \to M$ of $\hat{\gamma}$ with $\gamma(0) = p$. For any $t \in (-\varepsilon, \varepsilon)$ we use Lemma 2.22 to obtain a vector field $\hat{X}$ defined in a neighbourhood U of $\hat{\gamma}(0)$ such that $\hat{X}_{\hat{\gamma}(t_0)} = \hat{\gamma}'(t_0)$ and $\hat{\nabla}_{\hat{X}}\hat{X} = 0$ in U. Integral curves of $\hat{X}$ are geodesics, hence

$$\hat{X}_{\hat{\gamma}(t)} = \hat{\gamma}'(t) \quad \text{for all} \quad t \in (t_0 - \varepsilon', t_0 + \varepsilon')$$

with a suitable $\varepsilon' > 0$. Let X be the horizontal lift of $\hat{X}$. By Lemma 2.7 we have $(\nabla_X X)^\top = 0$ and $\nabla_X X = (\nabla_X X)^\top + A_X X = 0$. An integral curve

$$\eta : (t_0 - \varepsilon', t_0 + \varepsilon') \supset (t_0 - \varepsilon'', t_0 + \varepsilon'') \to M$$

of X with $\eta(t_0) = \gamma(t_0)$ is both a geodesic and the horizontal lift of

$$\hat{\gamma} : (t_0 - \varepsilon'', t_0 + \varepsilon'') \to B.$$

It follows $\eta(t) = \gamma(t)$ for all $t \in (t_0 - \varepsilon'', t_0 + \varepsilon'')$, and γ is geodesic at t_0. □

2.4.2 Local Trivializations

Proposition 2.24 allows us to identify fibres using paths in B joining the base points.

Definition 2.27. *Let $\hat{\gamma} : [0,1] \to B$ be a regular curve. We define the induced fibre-diffeomorphism $\tau_{\hat{\gamma}}$ as the mapping*

$$\begin{array}{rcl} \tau_{\hat{\gamma}} : F_{\hat{\gamma}(0)} & \to & F_{\hat{\gamma}(1)} \\ p & \mapsto & \gamma_p(1), \end{array}$$

where $\gamma_p : [0,1] \to M$ is the unique horizontal lift of $\hat{\gamma}$ with $\gamma_p(0) = p$.

We have to show that $\tau_{\hat{\gamma}} : F_{\hat{\gamma}(0)} \to F_{\hat{\gamma}(1)}$ is in fact a diffeomorphism.

Proof. We set

$$\begin{aligned} \hat\gamma^- : [0,1] &\to B \\ t &\mapsto \hat\gamma(1-t). \end{aligned}$$

Then $\tau_{\hat\gamma^-} : F_{\hat\gamma(1)} \to F_{\hat\gamma(0)}$ is the inverse of $\tau_{\hat\gamma} : F_{\hat\gamma(0)} \to F_{\hat\gamma(1)}$ since

$$[0,1] \ni t \mapsto (\gamma_p)^-(t) = \gamma_p(1-t)$$

is the horizontal lift of $\hat\gamma^-$ with $(\gamma_p)^-(0) = \gamma_p(1)$ and $(\gamma_p)^-(1) = \gamma_p(0)$. The smoothness of $\tau_{\hat\gamma} : F_{\hat\gamma(0)} \to F_{\hat\gamma(1)}$ follows immediately as γ_p is locally the solution of an ordinary differential equation which depends smoothly on the initial data, i.e. $p \in F_{\hat\gamma(0)}$. □

Given $b \in B$ we find an $\varepsilon > 0$ such that the exponential map

$$\exp_b^{\hat g} : B_\varepsilon^{\hat g}(0) \to B_\varepsilon^{\hat g}(b)$$

is a diffeomorphism. Consequently, we can choose for $b' \in B_\varepsilon^{\hat g}(b)$ the unique geodesic of length less than ε to join b and b' and identify $F_{b'}$ with F_b. This gives rise to a local trivialisation

$$\Phi : \pi^{-1}\left(B_\varepsilon^{\hat g}(b)\right) \to B_\varepsilon^{\hat g}(b) \times F_b$$

such that $\mathrm{pr}_1(\Phi(q)) = \pi(q)$ as follows.
Let $q \in \pi^{-1}\left(B_\varepsilon^{\hat g}(b)\right)$. Then

$$\begin{aligned} \hat\gamma : [0,1] &\to B_\varepsilon^{\hat g}(b) \\ t &\mapsto \exp_b^{\hat g}\left(t \cdot \left(\exp_b^{\hat g}\right)^{-1}(\pi(q))\right) \end{aligned}$$

is the unique geodesic of length less than ε which joins b and $\pi(q) \in B_\varepsilon^{\hat g}(b)$. The horizontal lift of $\hat\gamma^-$ which starts at q is by Lemma 2.26 the geodesic

$$\begin{aligned} \gamma_q^- : [0,1] &\to \pi^{-1}\left(B_\varepsilon^{\hat g}(b)\right) \\ t &\mapsto \exp_q\left(t \cdot d\pi_q|_{\mathcal{H}_q}^{-1}\left(\left(\hat\gamma^-\right)'(0)\right)\right) \end{aligned}$$

where

$$\left(\hat\gamma^-\right)'(0) = -\hat\gamma'(1) = -\left(d\exp_b^{\hat g}\right)_{\left(\exp_b^{\hat g}\right)^{-1}(\pi(q))}\left(\exp_b^{\hat g}\right)^{-1}(\pi(q)).$$

This defines a smooth map

$$\begin{aligned} \Phi : \pi^{-1}\left(B_\varepsilon^{\hat g}(b)\right) &\to B_\varepsilon^{\hat g}(b) \times F_b \\ q &\mapsto \left(\pi(q), \gamma_q^-(1)\right). \end{aligned}$$

Conversely, let $(b', p) \in B^{\hat{g}}_{\varepsilon}(b) \times F_b$. We consider the geodesic

$$\begin{aligned} \hat{\gamma} : [0,1] &\to B^{\hat{g}}_{\varepsilon}(b) \\ t &\mapsto \exp^{\hat{g}}_b \left(t \cdot \left(\exp^{\hat{g}}_b\right)^{-1}(b') \right) \end{aligned}$$

and its horizontal lift

$$\begin{aligned} \gamma_p : [0,1] &\to \pi^{-1}\left(B^{\hat{g}}_{\varepsilon}(b)\right) \\ t &\mapsto \exp_p \left(t \cdot d\pi_p|^{-1}_{\mathcal{H}_p} \left(\hat{\gamma}'(0)\right) \right) \end{aligned}$$

with $\pi\left(\gamma_p(1)\right) = b'$. We obtain a smooth map

$$\begin{aligned} \Psi : B^{\hat{g}}_{\varepsilon}(b) \times F_b &\to \pi^{-1}\left(B^{\hat{g}}_{\epsilon}(b)\right) \\ (b', p) &\mapsto \gamma_p(1), \end{aligned}$$

which is the inverse of $\Phi : \pi^{-1}\left(B^{\hat{g}}_{\varepsilon}(b)\right) \to B^{\hat{g}}_{\varepsilon}(b) \times F_b$.

Proposition 2.28 (cf. 9.40 and 9.42 in [Be])**.** *Let $b \in B$ and $\varepsilon > 0$ such that $\exp_b : B^{\hat{g}}_{\varepsilon}(0) \to B^{\hat{g}}_{\varepsilon}(b)$ is a diffeomorphism. Then $\pi^{-1}\left(B^{\hat{g}}_{\varepsilon}(b)\right)$ is a tubular $\varepsilon-$neighbourhood of the fibre F_b, and the trivialization Φ as constructed above yields Fermi-coordinates.*

Proof. Let $q \in \pi^{-1}\left(B^{\hat{g}}_{\varepsilon}(b)\right)$. We consider a sequence $(q_i)_{i\in\mathbb{N}}$ in F_b such that

$$d(q_i, q) \to \inf_{p \in F_b} d(p, q),$$

where d denotes the Riemannian distance on (M, g). By compactness of M and hence F_b we may assume $q_i \to p_0 \in F_b$. Due to the Hopf-Rinow theorem there exists a minimizing geodesic $\eta : [0,1] \to M$ with $\eta(0) = p_0$ and $\eta(1) = q$. Let $\delta > 0$ and $U \subset M$ be a totally normal $\delta-$neighbourhood of p_0. We choose $t_0 \in (0, 1]$ such that $\eta(t_0) \in U$. Consequently,

$$\exp_{\eta(t_0)} : T_{\eta(t_0)}M \supset B_\delta(0) \to \exp_{\eta(t_0)}\left(B_\delta(0)\right)$$

is a diffeomorphism and $U \subset \exp_{\eta(t_0)}\left(B_\delta(0)\right)$.
Let $v \in \mathcal{V}_{p_0} = T_{p_0}F_b$ and $\beta : (-1,1) \to F_b \cap U$ be a smooth curve with $\beta(0) = p_0$ and $\beta'(0) = v$. Then

$$\begin{aligned} [0, t_0] \times (-1,1) &\to U \\ (t, s) &\mapsto \eta_s(t) = \exp_{\eta(t_0)}\left(\left(1 - \frac{t}{t_0}\right) \exp^{-1}_{\eta(t_0)}(\beta(s)) \right) \end{aligned}$$

defines a smooth variation of $\eta_0 = \eta : [0, t_0] \to U$ with $\eta_s(0) = \beta(s)$ and $\eta_s(t_0) = \eta(t_0)$ for all $s \in (-1, 1)$. We compute

$$\begin{aligned}\left.\frac{\partial}{\partial s}\right|_{s=0} \sqrt{g\left(\eta_s'(t), \eta_s'(t)\right)} &= \frac{1}{\|\eta'(t)\|}\, g\left(\left.\frac{\nabla}{\partial s}\right|_{s=0} \eta_s'(t), \eta'(t)\right) \\ &= \frac{1}{\|\eta'(t)\|}\, g\left(\frac{\nabla}{\partial t} \left.\frac{\partial}{\partial s}\right|_{s=0} \eta_s'(t), \eta'(t)\right) \\ &= \frac{1}{\|\eta'(t)\|}\, \frac{\partial}{\partial t} g\left(\left.\frac{\partial}{\partial s}\right|_{s=0} \eta_s'(t), \eta'(t)\right)\end{aligned}$$

and obtain for the first variation of the length of the curves η_s that

$$0 = \left.\frac{d}{ds}\right|_{s=0} L(\eta_s) = \int_0^{t_0} \left.\frac{\partial}{\partial s}\right|_{s=0} \sqrt{g\left(\eta_s'(t), \eta_s'(t)\right)} dt = -\frac{1}{\|\eta'(0)\|}\, g\left(v, \eta'(0)\right).$$

In other words, the geodesic $\eta : [0, 1] \to M$ is horizontal at $t = 0$ and hence everywhere by Lemma 2.25. We note that the geodesic $\hat{\eta} = \pi \circ \eta : [0, 1] \to B$ has the same length as η, since $d\pi$ preserves the lengths of horizontal vectors. Taking in mind that $\exp_b : B_\varepsilon^{\hat{g}}(0) \to B_\varepsilon^{\hat{g}}(b)$ is a diffeomorphism it follows that

$$\hat{\eta}(t) = \exp_b^{\hat{g}}\left((1-t)\left(\exp_b^{\hat{g}}\right)^{-1}(\pi(q))\right) \quad \text{for all} \quad t \in [0, 1].$$

Otherwise the horizontal lift of $[0, 1] \ni t \mapsto \exp_b^{\hat{g}}\left((1-t)\left(\exp_b^{\hat{g}}\right)^{-1}(\pi(q))\right)$ at q was a curve shorter than η between q and F_b. By construction we have

$$\Phi(q) = (\pi(q), p_0).$$

□

Now we consider the rescaled metric g_{r^2} and the Riemannian submersion $\pi : (M, g_{r^2}) \to \left(B, r^2\hat{g}\right)$. Since

$$\exp_b^{r^2\hat{g}} = \exp_b^{\hat{g}} =: \exp_b \quad \text{for all} \quad b \in B,$$

it follows that the geodesics in $(B, \hat{g})$ and $(B, r^2\hat{g})$ coincide as well as their unique lifts to horizontal geodesics in (M, g) and (M, g_{r^2}).
Let $b \in B$ and $\varepsilon > 0$ such that the exponential map

$$\exp_b : T_bB \supset U := B_\varepsilon^{\hat{g}}(0) = B_{r\varepsilon}^{r^2\hat{g}}(0) \to \exp_b(U) =: V$$

is a diffeomorphism. The induced local trivialization

$$\Psi_r : V \times F_b \to \pi^{-1}(V)$$

is then independent of $r > 0$.

2.4.3 Induced Metrics and Estimates in Normal Coordinates

We choose a local trivialization $\Psi_r = \Psi : V \times F_b \to \pi^{-1}(V)$ as above and compare on $V \times F_b$ the product metric $r^2\hat{g} \oplus g^\perp$ with the pullback metric $\Psi^* g_{r^2}$ for varying $r > 0$.
As a first observation we remark

Lemma 2.29. *For any $r > 0$ and $p \in F_b$ we have*

$$r^2\hat{g} \oplus g^\perp\Big|_{(b,p)} = \Psi^* g_{r^2}|_{(b,p)} .$$

Proof. Let $(w, v) \in T_bB \times T_pF_b \cong T_{(b,p)}(V \times F_b)$.
We choose a curve $c : (-\varepsilon, \varepsilon) \to F_b$ such that $c(0) = p$ and $c'(0) = v$. Then

$$\Psi(b, c(t)) = c(t) \quad \text{for all} \quad t \in (-\varepsilon, \varepsilon).$$

We represent w by the geodesic $t \mapsto \exp_b(tw) \subset B$ and consider its horizontal lift γ with $\gamma(0) = p$. Since $\pi : (M, g_{r^2}) \to \big(B, r^2\hat{g}\big)$ is a Riemannian submersion, we obtain

$$g_{r^2}(\gamma'(0), \gamma'(0)) = r^2\hat{g}(w, w)$$

and

$$\Psi(\exp_b(tw), p) = \gamma(t)$$

for all sufficiently small t. We calculate

$$d\Psi_{(b,p)}(w, v) = \frac{d}{dt}\Big|_{t=0} \Psi(\exp_b(tw), c(t)) = \gamma'(0) + c'(0)$$

and conclude

$$(\Psi^* g_{r^2})_{(b,p)}((w, v), (w, v)) = r^2\hat{g}(w, w) + g^\perp(v, v).$$

□

In the following we fix $(b, p) \in V \times F_b = \exp_b\left(B^{r^2\hat{g}}_{r\varepsilon}(0)\right) \times F_b$.
W.l.o.g. we may assume that

$$\exp_p^{g^\perp} : B^{g^\perp}_\varepsilon(0) \to \exp_p^{g^\perp}\left(B^{g^\perp}_\varepsilon(0)\right) \subset F_b$$

is a diffeomeorphism. We choose an orthonormal basis $(v_1, \ldots, v_k)$ of T_pF_b w.r.t. $g^\perp$ and an orthonormal basis $(w_1, \ldots, w_n)$ of T_bB w.r.t. $\hat{g}$ to identify $\mathbb{R}^k \cong T_pF_b$ and $\mathbb{R}^n \cong T_bB$ via

$$\iota_k : \sum_{i=1}^k \lambda_i e_i \mapsto \sum_{i=1}^k \lambda_i v_i \quad \text{and} \quad \iota_n : \sum_{j=1}^n \mu_j e_j \mapsto \sum_{j=1}^n \mu_j w_j.$$

and obtain normal coordinates

$$\varphi^{g^\perp} := \exp_p^{g^\perp} \circ \iota_k : B_\varepsilon^k(0) \subset \mathbb{R}^k \to \exp_p^{g^\perp}\left(B_\varepsilon^{g^\perp}(0)\right) \subset F_b$$

on F_b w.r.t. $g^\perp$ and

$$\varphi^{\hat g} := \exp_b \circ \iota_n : B_\varepsilon^n(0) \subset \mathbb{R}^n \to \exp_b\left(B_\varepsilon^{\hat g}(0)\right) = V \subset B$$

on B w.r.t. $\hat g$.
Since $(\frac{1}{r}w_1, \ldots, \frac{1}{r}w_n)$ is an orthonormal basis of T_bB w.r.t. $r^2\hat g$, we identify $\mathbb{R}^n \cong T_bB$ using

$$\iota_n^{(r)} : \sum_{j=1}^n \mu_j e_j \to \sum_{j=1}^n \frac{\mu_j}{r} w_j,$$

which yields normal coordinates

$$\varphi^{r^2\hat g} := \exp_b \circ \iota_n^{(r)} : B_{r\varepsilon}^n(0) \subset \mathbb{R}^n \to \exp_b\left(B_{r\varepsilon}^{r^2\hat g}(0)\right) = V \subset B$$

on B w.r.t. $r^2\hat g$.
In summary, we obtain local parametrizations

$$\varphi^{r^2\hat g} \times \varphi^{g^\perp} : B_{r\varepsilon}^n(0) \times B_\varepsilon^k(0) \to V \times F_b$$

near $(b,p) \in V \times F_b$.
For $(x,y) \in B_{r\varepsilon}^n(0) \times B_\varepsilon^k(0)$ and $1 \le i,j \le n+k$ we set

$$b_{ij}^r(x,y) := \Psi^* g_{r^2}\left(d(\varphi^{r^2\hat g} \times \varphi^{g^\perp})_{(x,y)} e_i, d(\varphi^{r^2\hat g} \times \varphi^{g^\perp})_{(x,y)} e_j\right)$$

and

$$c_{ij}^r(x,y) := r^2\hat g \oplus g^\perp\left(d(\varphi^{r^2\hat g} \times \varphi^{g^\perp})_{(x,y)} e_i, d(\varphi^{r^2\hat g} \times \varphi^{g^\perp})_{(x,y)} e_j\right).$$

For $x \in B_{r\varepsilon}^n(0)$ and $1 \le i,j \le n$ we define

$$d_{ij}^r(x) := r^2\hat g\left(d(\varphi^{r^2\hat g})_x e_i, d(\varphi^{r^2\hat g})_x e_j\right).$$

By $(d\exp_b)_0 = \mathrm{id}_{T_bB}$ and $(d\exp_p^{g^\perp})_0 = \mathrm{id}_{T_pF_b}$ and Lemma 2.29 it follows

$$b_{ij}^r(0,0) = \delta_{ij} = c_{ij}^r(0,0) \quad \text{for all } 1 \le i,j \le n+k$$

and

$$d_{ij}^r(0) = \delta_{ij} \quad \text{for all } 1 \le i,j \le n,$$

independent of $r > 0$.

In order to make estimates for the eigenvalues of the symmetric matrices $B^r(x,y)$, $C^r(x,y)$ and $D^r(x)$ with entries $b_{ij}^r(x,y)$, $c_{ij}^r(x,y)$ and $d_{ij}^r(x)$ we make the following elementary observation.

Lemma 2.30. *Let $W \subset \mathbb{R}^m$ be open, $m \geq 2$, $0 \in W$ and $A : W \to \mathbb{R}^{m \times m}$ a continuous map such that $A(0) = \mathrm{Id}$ and $(A(x))^\top = A(x)$ for all $x \in W$. For any $\delta > 0$ there exists an $\eta > 0$ such that $B_\eta(0) \subset W$, and for all $x \in B_\eta(0)$ the eigenvalues of $A(x)$ and $A(x)^{-1}$ lie in the interval $\left(\frac{1}{1+\delta}, 1+\delta\right)$.*

Proof. We consider

$$\begin{aligned} f : W \times S^{m-1} &\to \mathbb{R} \\ (x, v) &\mapsto \langle A(x)v, v\rangle. \end{aligned}$$

Then, f is continuous and $f(0, v) = \langle v, v\rangle = 1$. Let $\delta > 0$. Given $v \in S^{m-1}$, we find an open neighbourhood $B_{\eta_v}(0) \times U_v$ of $(0, v)$ in $W \times S^{m-1}$ such that

$$\frac{1}{1+\delta} < f(x, w) < 1 + \delta \quad \text{for all} \quad (x, w) \in B_{\eta_v}(0) \times U_v.$$

Since S^{m-1} is compact, there exists an $\eta > 0$ satisfying

$$\frac{1}{1+\delta} < f(x, w) < 1 + \delta \quad \text{for all} \quad (x, w) \in B_\eta(0) \times S^{m-1}.$$

In particular, we have the following estimate

$$\frac{1}{1+\delta} < \min_{\|v\|=1} \langle A(x)v, v\rangle \leq \max_{\|v\|=1} \langle A(x)v, v\rangle < 1 + \delta$$

and conclude that all eigenvalues $\lambda_1(x), \ldots, \lambda_n(x)$ of $A(x)$ lie in the interval $\left(\frac{1}{1+\delta}, 1+\delta\right)$ for all $x \in B_\eta(0)$. The same is true for the eigenvalues $\mu_1(x), \ldots, \mu_n(x)$ of $A(x)^{-1}$, since $\mu_i(x) = \frac{1}{\lambda_i(x)}$ for $1 \leq i \leq n$ and $x \in B_\eta(0)$ by symmetry of $A(x)$. □

Now we turn back to B^r, C^r and D^r. We write $B := B^1$, $C := C^1$ and $D := D^1$.
If $x \in B^n_{r\varepsilon}(0)$ and $w_0 \in \mathbb{R}^n$, one readily checks that

$$d(\varphi^{r^2\hat{g}})_x w_0 = \frac{1}{r}\, d(\varphi^{\hat{g}})_{\frac{1}{r}x} w_0.$$

Given $(x, y) \in B^n_{r\varepsilon}(0) \times B^k_\varepsilon(0)$ and $u = (w, v) \in \mathbb{R}^{n+k} \cong \mathbb{R}^n \times \mathbb{R}^k$, it follows

$$d(\varphi^{r^2\hat{g}} \times \varphi^{g^\perp})_{(x,y)}(w, 0) = \frac{1}{r} d(\varphi^{\hat{g}} \times \varphi^{g^\perp})_{(\frac{1}{r}x, y)}(w, 0)$$

and

$$d(\varphi^{r^2\hat{g}} \times \varphi^{g^\perp})_{(x,y)}(0, v) = d(\varphi^{\hat{g}} \times \varphi^{g^\perp})_{(\frac{1}{r}x, y)}(0, v).$$

We conclude that

$$\langle D^r(x) w_0, w_0\rangle = r^2 \hat{g}\left(d(\varphi^{r^2\hat{g}})_x w_0, d(\varphi^{r^2\hat{g}})_x w_0\right) = \left\langle D\left(\frac{1}{r}x\right) w_0, w_0\right\rangle$$

and $\langle C^r(x,y)u,u\rangle =$

$$\begin{aligned}
&= r^2\hat{g}\oplus g^{\perp}\left(d(\varphi^{r^2\hat{g}}\times\varphi^{g^{\perp}})_{(x,y)}u, d(\varphi^{r^2\hat{g}}\times\varphi^{g^{\perp}})_{(x,y)}u\right)\\
&= \frac{1}{r^2}\left(r^2\hat{g}\oplus g^{\perp}\right)\left(d(\varphi^{\hat{g}}\times\varphi^{g^{\perp}})_{(\frac{1}{r}x,y)}(w,0), d(\varphi^{\hat{g}}\times\varphi^{g^{\perp}})_{(\frac{1}{r}x,y)}(w,0)\right)\\
&\quad + r^2\hat{g}\oplus g^{\perp}\left(d(\varphi^{\hat{g}}\times\varphi^{g^{\perp}})_{(\frac{1}{r}x,y)}(0,v), d(\varphi^{\hat{g}}\times\varphi^{g^{\perp}})_{(\frac{1}{r}x,y)}(0,v)\right)\\
&= \hat{g}\oplus g^{\perp}\left(d(\varphi^{\hat{g}}\times\varphi^{g^{\perp}})_{(\frac{1}{r}x,y)}u, d(\varphi^{\hat{g}}\times\varphi^{g^{\perp}})_{(\frac{1}{r}x,y)}u\right)\\
&= \left\langle C\left(\frac{1}{r}x,y\right)u,u\right\rangle.
\end{aligned}$$

Given $\delta>0$, we find $0<\varepsilon_0<\varepsilon$ such that

$$\frac{1}{1+\delta}<\langle D(x)w_0,w_0\rangle<1+\delta \quad \text{for all } x\in B^n_{\varepsilon_0}(0) \text{ and } w_0\in S^n$$

and analogously

$$\frac{1}{1+\delta}<\langle C(x,y)u,u\rangle<1+\delta$$

for all $(x,y)\in B^n_{\varepsilon_0}(0)\times B^k_{\varepsilon_0}(0)$ and $u\in S^{n+k-1}$. Consequently,

$$\frac{1}{1+\delta}<\langle D^r(x)w_0,w_0\rangle<1+\delta \quad \text{for all } x\in B^n_{r\varepsilon_0}(0) \text{ and } w_0\in S^n$$

and

$$\frac{1}{1+\delta}<\langle C^r(x,y)u,u\rangle<1+\delta$$

for all $(x,y)\in B^n_{r\varepsilon_0}(0)\times B^k_{\varepsilon_0}(0)$ and $u\in S^{n+k-1}$.
Hence, the eigenvalues of

$$D^r(x),\ (D^r(x))^{-1} \quad \text{for } x\in B^n_{r\varepsilon_0}(0)$$

and of

$$C^r(x,y),\ (C^r(x,y))^{-1} \quad \text{for } (x,y)\in B^n_{r\varepsilon_0}(0)\times B^k_{\varepsilon_0}(0)$$

lie in $\left(\frac{1}{1+\delta},1+\delta\right)$.
In particular, we have the following estimates

$$\begin{aligned}
(1+\delta)^{-\frac{n+k}{2}} &< \sqrt{\det C^r(x,y)} < (1+\delta)^{\frac{n+k}{2}},\\
(1+\delta)^{-\frac{n}{2}} &< \sqrt{\det D^r(x)} < (1+\delta)^{\frac{n}{2}},\\
\frac{1}{1+\delta}\|u\|^2 &< \langle C^r(x,y)u,u\rangle < (1+\delta)\|u\|^2,\\
\frac{1}{1+\delta}\|w\|^2 &< \langle D^r(x)w,w\rangle < (1+\delta)\|w\|^2,\\
\frac{1}{1+\delta}\|u\|^2 &< \left\langle (C^r(x,y))^{-1}u,u\right\rangle < (1+\delta)\|u\|^2,\\
\frac{1}{1+\delta}\|w\|^2 &< \left\langle (D^r(x))^{-1}w,w\right\rangle < (1+\delta)\|w\|^2
\end{aligned}$$

for all $x \in B^n_{r\varepsilon_0}(0)$, $(x,y) \in B^n_{r\varepsilon_0}(0) \times B^k_{\varepsilon_0}(0)$, $w \in \mathbb{R}^n$ and $u \in \mathbb{R}^{n+k}$.
The discussion of B^r is a bit more involved.
We calculate $\langle B^r(x,y)u,u\rangle =$

$$\begin{aligned}
&= \Psi^* g_{r^2}\left(d(\varphi^{r^2\hat{g}} \times \varphi^{g^\perp})_{(x,y)}u, d(\varphi^{r^2\hat{g}} \times \varphi^{g^\perp})_{(x,y)}u\right)\\
&= \frac{1}{r^2}\cdot g_{r^2}\left(d(\Psi\circ(\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(\frac{1}{r}x,y)}(w,0), d(\Psi\circ(\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(\frac{1}{r}x,y)}(w,0)\right)\\
&\quad + \frac{2}{r}\cdot g_{r^2}\left(d(\Psi\circ(\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(\frac{1}{r}x,y)}(w,0), d(\Psi\circ(\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(\frac{1}{r}x,y)}(0,v)\right)\\
&\quad + g_{r^2}\left(d(\Psi\circ(\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(\frac{1}{r}x,y)}(0,v), d(\Psi\circ(\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(\frac{1}{r}x,y)}(0,v)\right).
\end{aligned}$$

We write

$$\tilde{w} = d(\Psi\circ(\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(\frac{1}{r}x,y)}(w,0)$$

and

$$\tilde{v} = d(\Psi\circ(\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(\frac{1}{r}x,y)}(0,v).$$

We use that $\tilde{v}$ is vertical and calculate $\langle B^r(x,y)u,u\rangle =$

$$\begin{aligned}
&= g\left(\tilde{w}^\top, \tilde{w}^\top\right) + \frac{1}{r^2}\cdot g\left(\tilde{w}^\perp, \tilde{w}^\perp\right) + \frac{2}{r}\cdot g\left(\tilde{w}^\perp, \tilde{v}\right) + g(\tilde{v},\tilde{v})\\
&= \left\langle B\left(\frac{1}{r}x,y\right)u,u\right\rangle + \left(\frac{1}{r^2}-1\right) g\left(\tilde{w}^\perp, \tilde{w}^\perp\right) + 2\cdot\left(\frac{1}{r}-1\right) g\left(\tilde{w}^\perp, \tilde{v}\right).
\end{aligned}$$

For a given $\delta > 0$, there exists $0 < \varepsilon_1 < \varepsilon$ such that

$$\frac{1}{1+\delta} < \langle B(x,y)u,u\rangle < 1+\delta$$

for all $(x,y) \in B^n_{\varepsilon_1}(0) \times B^k_{\varepsilon_1}(0)$ and $u \in S^{n+k-1}$.
Hence, we find $0 < \varepsilon_2 < \varepsilon_1$ and $c, d > 0$ with

$$\frac{1}{1+\delta} < c \leq \langle B(x,y)u,u\rangle \leq d < 1+\delta$$

for all $(x,y) \in \overline{B^n_{\varepsilon_2}(0)} \times \overline{B^k_{\varepsilon_2}(0)}$ and $u \in S^{n+k-1}$.
Consequently,

$$\frac{1}{1+\delta} < c \leq \left\langle B\left(\frac{1}{r}x,y\right)u,u\right\rangle \leq d < 1+\delta$$

for all $(x,y) \in \overline{B^n_{r\varepsilon_2}(0)} \times \overline{B^k_{\varepsilon_2}(0)}$ and $u \in S^{n+k-1}$.

As above we write $u = (w, v) \in \mathbb{R}^{n+k} \cong \mathbb{R}^n \times \mathbb{R}^k$.
We consider the smooth maps

$$f, h : \overline{B^n_{\varepsilon_2}(0)} \times \overline{B^k_{\varepsilon_2}(0)} \times S^{n+k-1} \to \mathbb{R}$$

defined by

$$f(x,y,u) = g\left(\left(d(\Psi \circ (\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(x,y)}(w,0)\right)^\perp, \left(d(\Psi \circ (\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(x,y)}(w,0)\right)^\perp\right).$$

and

$$h(x,y,u) = g\left(\left(d(\Psi \circ (\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(x,y)}(w,0)\right)^\perp, d(\Psi \circ (\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(x,y)}(0,v)\right).$$

By definition of Ψ we have

$$\left(d(\Psi \circ (\varphi^{\hat{g}} \times \varphi^{g^\perp}))_{(0,0)}(w,0)\right)^\perp = 0.$$

Due to the compactness of S^{n+k-1} we find $0 < \varepsilon_3 < \varepsilon_2$ such that

$$\max\{f(x,y,u), |h(x,y,u)|\} < \frac{1}{6} \cdot \min\left\{c - \frac{1}{1+\delta}, 1 + \delta - d\right\}$$

for all $(x,y) \in \overline{B^n_{\varepsilon_3}(0)} \times \overline{B^k_{\varepsilon_3}(0)}$ and $u \in S^{n+k-1}$. Hence,

$$\max\left\{f\left(\frac{1}{r}x, y, u\right), \left|h\left(\frac{1}{r}x, y, u\right)\right|\right\} < \frac{1}{6} \cdot \min\left\{c - \frac{1}{1+\delta}, 1 + \delta - d\right\}$$

for all $(x,y) \in \overline{B^n_{r\varepsilon_3}(0)} \times \overline{B^k_{\varepsilon_3}(0)}$ and $u \in S^{n+k-1}$.
Consequently, there exists an $r_0 > 0$ such that

$$\left|\langle B^r(x,y)u, u\rangle - \left\langle B\left(\frac{1}{r}x, y\right)u, u\right\rangle\right| < \frac{1}{2} \cdot \min\left\{c - \frac{1}{1+\delta}, 1 + \delta - d\right\}$$

for all $r > r_0$, $(x,y) \in \overline{B^n_{r\varepsilon_3}(0)} \times \overline{B^k_{\varepsilon_3}(0)}$ and $u \in S^{n+k-1}$.
It follows that

$$\frac{1}{1+\delta} < \langle B^r(x,y)u, u\rangle < 1 + \delta$$

for all $r > r_0$, $(x,y) \in \overline{B^n_{r\varepsilon_3}(0)} \times \overline{B^k_{\varepsilon_3}(0)}$ and $u \in S^{n+k-1}$.
In particular, the eigenvalues of $B^r(x,y)$ and $(B^r(x,y))^{-1}$ with such r and (x,y) lie in the interval $\left(\frac{1}{1+\delta}, 1+\delta\right)$.
As a result, we obtain estimates

$$(1+\delta)^{-\frac{n+k}{2}} < \sqrt{\det B^r(x,y)} < (1+\delta)^{\frac{n+k}{2}},$$

$$\frac{1}{1+\delta}\|u\|^2 < \langle B^r(x,y)u, u\rangle < (1+\delta)\|u\|^2,$$

$$\frac{1}{1+\delta}\|u\|^2 < \left\langle (B^r(x,y))^{-1}u, u\right\rangle < (1+\delta)\|u\|^2$$

for all $r > r_0$, $(x,y) \in \overline{B^n_{r\varepsilon_3}(0)} \times \overline{B^k_{\varepsilon_3}(0)}$ and $u \in S^{n+k-1}$.

2.4.4 Admissible Trivializations

Let $b \in B$ and $\varepsilon > 0$ such that $\exp_b : B_\varepsilon^{\hat{g}}(0) \to \exp_b(B_\varepsilon^{\hat{g}}(0)) =: V$ is a diffeomorphism, which gives rise to a local trivialization

$$\Psi : V \times F_b \to \pi^{-1}(V).$$

Given $\delta > 0$, using the compactness of F_b and shrinking ε we find finitely many points $p_1, \dots, p_m \in F_b$, $\varepsilon_1, \dots, \varepsilon_m > 0$ and $r_0 > 0$ such that

$$\exp_{p_l} : B_{\varepsilon_l}^{g^\perp}(0) \to \exp_{p_l}(B_{\varepsilon_l}^{g^\perp}(0)) \text{ for all } 1 \le l \le m$$

are diffeomorphisms,

$$F_b = \bigcup_{l=1}^{m} \exp_{p_l}(B_{\varepsilon_l}^{g^\perp}(0)),$$

and in local parametrizations

$$\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp} : B_{r\varepsilon}^n(0) \times B_{\varepsilon_l}^k(0) \to V \times \exp_{p_l}(B_{\varepsilon_l}^{g^\perp}(0))$$

near $(b, p_l) \in V \times F_b$ for all $1 \le l \le m$ and $r > r_0$ the following estimates are hold:

$$\begin{aligned}
(1+\delta)^{-\frac{n+k}{2}} &< \sqrt{\det B_l^r(x,y)} < (1+\delta)^{\frac{n+k}{2}}, \\
(1+\delta)^{-\frac{n+k}{2}} &< \sqrt{\det C_l^r(x,y)} < (1+\delta)^{\frac{n+k}{2}}, \\
\frac{1}{1+\delta}\|u\|^2 &< \langle B_l^r(x,y)u, u\rangle < (1+\delta)\|u\|^2, \\
\frac{1}{1+\delta}\|u\|^2 &< \langle C_l^r(x,y)u, u\rangle < (1+\delta)\|u\|^2, \\
\frac{1}{1+\delta}\|u\|^2 &< \left\langle (B_l^r(x,y))^{-1}u, u\right\rangle < (1+\delta)\|u\|^2, \\
\frac{1}{1+\delta}\|u\|^2 &< \left\langle (C_l^r(x,y))^{-1}u, u\right\rangle < (1+\delta)\|u\|^2,
\end{aligned}$$

where $(x, y) \in B_{r\varepsilon}^n(0) \times B_{\varepsilon_l}^k(0)$, $u = (u_1, \dots, u_{n+k}) \in \mathbb{R}^{n+k}$ and

$$B_l^r(x,y) = ((b_l^r)_{ij}(x,y))_{ij}, \quad C_l^r(x,y) = ((c_l^r)_{ij}(x,y))_{ij}$$

are defined by

$$(b_l^r)_{ij}(x,y) := \Psi^* g_{r^2}\left(d(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp})_{(x,y)}e_i, d(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp})_{(x,y)}e_j\right)$$

and

$$(c_l^r)_{ij}(x,y) := r^2\hat{g} \oplus g^\perp\left(d(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp})_{(x,y)}e_i, d(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp})_{(x,y)}e_j\right)$$

We obtain

$$\sqrt{\det B_l^r(x,y)} < (1+\delta)^{n+k}\sqrt{\det C_l^r(x,y)}$$

and

$$\sqrt{\det C_l^r(x,y)} < (1+\delta)^{n+k}\sqrt{\det B_l^r(x,y)}.$$

Let $f \in C^\infty(V \times F_b)$. We can compare for all $r > r_0$ the gradient of f w.r.t. $\Psi^* g_{r^2}$ and $r^2\hat{g} \oplus g^\perp$ with the gradient of $f \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right)$ w.r.t. the Euclidean norm on $\mathbb{R}^{n+k}$ by

$$\left\|\nabla^{\Psi^* g_{r^2}} f\right\|^2_{\Psi^* g_{r^2}} \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right) < (1+\delta)\left\|\nabla\left(f \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right)\right)\right\|^2,$$

$$\left\|\nabla\left(f \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right)\right)\right\|^2 < (1+\delta)\left\|\nabla^{\Psi^* g_{r^2}} f\right\|^2_{\Psi^* g_{r^2}} \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right)$$

and

$$\left\|\nabla^{r^2\hat{g}\oplus g^\perp} f\right\|^2_{r^2\hat{g}\oplus g^\perp} \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right) < (1+\delta)\left\|\nabla\left(f \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right)\right)\right\|^2,$$

$$\left\|\nabla\left(f \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right)\right)\right\|^2 < (1+\delta)\left\|\nabla^{r^2\hat{g}\oplus g^\perp} f\right\|^2_{r^2\hat{g}\oplus g^\perp} \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right),$$

where $\left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right)$ is defined on $B^n_{r\varepsilon}(0) \times B^k_{\varepsilon_l}(0)$.

Combining, we have

$$\left\|\nabla^{\Psi^* g_{r^2}} f\right\|^2_{\Psi^* g_{r^2}} \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right) < (1+\delta)^2\left\|\nabla^{r^2\hat{g}\oplus g^\perp} f\right\|^2_{r^2\hat{g}\oplus g^\perp} \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right)$$

and

$$\left\|\nabla^{r^2\hat{g}\oplus g^\perp} f\right\|^2_{r^2\hat{g}\oplus g^\perp} \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right) < (1+\delta)^2\left\|\nabla^{\Psi^* g_{r^2}} f\right\|^2_{\Psi^* g_{r^2}} \circ \left(\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp}\right).$$

We take ε small enough, so that

$$(1+\delta)^{-\frac{n}{2}} < \sqrt{\det D^r(x)} < (1+\delta)^{\frac{n}{2}},$$
$$\frac{1}{1+\delta}\|w\|^2 < \langle D^r(x)w, w\rangle < (1+\delta)\|w\|^2,$$
$$\frac{1}{1+\delta}\|w\|^2 < \left\langle (D^r(x)^{-1}w, w\right\rangle < (1+\delta)\|w\|^2.$$

for all $x \in B^n_{r\varepsilon}(0)$ and $w \in \mathbb{R}^n$, where $D^r(x) = (d^r_{ij}(x))$ and

$$d^r_{ij}(x) := r^2\hat{g}\left(d(\varphi^{r^2\hat{g}})_x e_i, d(\varphi^{r^2\hat{g}})_x e_j\right) \quad \text{for } 1 \le i,j \le n.$$

We deduce that

$$\left\|\nabla^{r^2\hat{g}\oplus g^\perp} f\right\|^2_{r^2\hat{g}\oplus g^\perp} \circ \left(\varphi^{r^2\hat{g}} \times \mathrm{id}\right) < (1+\delta)\left\|\nabla^{g_{\mathrm{eucl}}\oplus g^\perp}\left(f \circ \left(\varphi^{r^2\hat{g}} \times \mathrm{id}\right)\right)\right\|^2_{g_{\mathrm{eucl}}\oplus g^\perp}.$$

Definition 2.31. *Given $\delta > 0$ and $b \in B$, a local trivialization*

$$\Psi : \exp_b(B^{\hat{g}}_{\varepsilon}(0)) \times F_b \to \pi^{-1}(\exp_b(B^{\hat{g}}_{\varepsilon}(0)))$$

together with local parametrizations

$$\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp} : B^n_{r\varepsilon}(0) \times B^k_{\varepsilon_l}(0) \to \exp_b(B^{\hat{g}}_{\varepsilon}(0)) \times \exp_{p_l}(B^{g^\perp}_{\varepsilon_l}(0)), \quad 1 \le l \le m,$$

such that

$$F_b = \bigcup_{l=1}^{m} \exp_{p_l}(B^{g^\perp}_{\varepsilon_l}(0))$$

satisfying all the local estimates above for B^r, C^r, and D^r for a suitable $r > r_0$ is called **admissible** *for δ.*

2.4.5 Riemannian Submersions with Totally Geodesic Fibres

As above we consider a Riemannian submersion $\pi : (M, g) \to (B, \hat{g})$, where M and B are assumed to be closed. The fibre-diffeomorphisms provide a necessary and sufficient condition for the fibres to be totally geodesic.

Proposition 2.32 (9.56 in [Be])**.** *Suppose that all fibres of the Riemannian submersion $\pi : (M, g) \to (B, \hat{g})$ are totally geodesic with respect to the induced metric. Then the fibre-diffeomorphisms $\tau_{\hat{\gamma}} : F_{\hat{\gamma}(0)} \to F_{\hat{\gamma}(1)}$ are isometries for every regular curve $\hat{\gamma} : [0,1] \to B$.*

Proof. Let U and V be vertical vector fields defined in a neighbourhood of $p \in F_{\hat{\gamma}(0)}$. Since $F_{\hat{\gamma}(0)}$ is totally geodesic, the second fundamental form vanishes, i.e.

$$(\nabla_U V)^\top = 0 = (\nabla_V U)^\top .$$

For a basic vector field X we obtain $0 = g(X, \nabla_U V) = -g(\nabla_U X, V)$ and similarly $g(\nabla_V X, U) = 0$. It follows

$$\mathcal{L}_X g(U, V) = g(\nabla_U X, V) + g(\nabla_V X, U) = 0.$$

Because the composition of isometries is again an isometry, we may assume without loss of generality that there exists a vector field $\hat{X}$ on B such that

$$\hat{X} \circ \hat{\gamma}(t) = \hat{\gamma}'(t) \quad \text{for all} \quad t \in [0,1].$$

We consider the horizontal lift X of $\hat{X}$ together with its flow

$$\begin{aligned} \varphi : \mathbb{R} \times F_{\hat{\gamma}(0)} &\to M \\ (t, q) &\mapsto \varphi_t(q) \end{aligned}$$

and note that $\pi(\varphi_t(q)) = \hat{\gamma}(t)$ for all $t \in [0,1]$. Then $\varphi_1 : F_{\hat{\gamma}(0)} \to F_{\hat{\gamma}(1)}$ is the induced fibre-diffeomorphism $\tau_{\hat{\gamma}}$.
We use

$$0 = \mathcal{L}_X g(U_q, V_q) = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* g(U_q, V_q) \quad \text{and} \quad \varphi_{t+s} = \varphi_t \circ \varphi_s$$

and obtain

$$\left.\frac{d}{dt}\right|_{t=0} \varphi_t^* g(U_q, V_q) = 0$$

for all $t_0 \in [0,1]$ and $U_q, V_q \in \mathcal{V}_q$, $q \in F_{\hat{\gamma}(0)}$, where we extend locally U_q and V_q to vertical vector fields.
It follows that

$$[0,1] \ni t \mapsto (\varphi_t^* g)(U_q, V_q)$$

is constant and

$$\begin{aligned} g(U_q, V_q) &= (\varphi_0^* g)(U_q, V_q) \\ &= (\varphi_1^* g)(U_q, V_q) \\ &= g\left((d\tau_{\hat{\gamma}})_q U_q, (d\tau_{\hat{\gamma}})_q U_q\right). \end{aligned}$$

Consequently, $\tau_{\hat{\gamma}} : F_{\hat{\gamma}(0)} \to F_{\hat{\gamma}(1)}$ is an isometry. □

The converse is also true.

Proposition 2.33. *Suppose that the fibre-diffeomorphisms*

$$\tau_{\hat{\gamma}} : F_{\hat{\gamma}(0)} \to F_{\hat{\gamma}(1)}$$

are isometries for every regular curve $\hat{\gamma} : [0,1] \to B$. Then the fibres are totally geodesic with respect to the induced metric.

Proof. We have to show that the second fundamental form of the fibres vanishes. Let $U_q, V_q \in \mathcal{V}_q$ and $X_q \in \mathcal{H}_q$. We extend $d\pi_q X_q$ to a vector field $\hat{X}$ on B such that its horizontal lift X yields an extension of X_q to a basic vector field. We consider the integral curve $\hat{\gamma} : \mathbb{R} \to B$ of $\hat{X}$ with $\hat{\gamma}(0) = \pi(q)$. Let

$$\begin{aligned} \varphi : \mathbb{R} \times F_{\hat{\gamma}(0)} &\to M \\ (t, q) &\mapsto \varphi_t(q) \end{aligned}$$

the flow of X. Then $\pi(\varphi_t(q)) = \hat{\gamma}(t)$, and $\varphi_t : F_{\hat{\gamma}(0)} \to F_{\hat{\gamma}(t)}$ is the fibre-diffeomorphism induced by $\hat{\gamma}|_{[0,t]}$ for $t \geq 0$. I.e. φ_t is an isometry for every $t \geq 0$. Hence, $[0,\infty) \ni t \mapsto (\varphi_t^* g)(U_q, V_q) = g(U_q, V_q)$ and

$$0 = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* g(U_q, V_q) = \mathcal{L}_X g(U_q, V_q).$$

We choose local extensions of U_q and V_q to vertical vector fields and obtain

$$0 = \mathcal{L}_X g(U_q, V_q) = g_q(\nabla_U X, V) + g_q(\nabla_V X, U).$$

Consequently,

$$\begin{aligned} -g_q(X, \nabla_U V) &= g_q(\nabla_U X, V) = -g_q(\nabla_V X, U) \\ &= g_q(X, \nabla_V U) = g_q\left(X, (\nabla_V U)^\top\right) \\ &= g_q\left(X, (\nabla_U V)^\top\right), \end{aligned}$$

where we used that the second fundamental form on $F_{\hat\gamma(0)}$ is symmetric,

$$(\nabla_U V)_q^\top = (\nabla_V U)_q^\top.$$

It follows that $g_q\left(X_q, (\nabla_U V)_q^\top\right) = 0$ for any horizontal X_q. Thus we have $(\nabla_U V)_q^\top = 0$, i.e. the second fundamental form on $F_{\hat\gamma(0)}$ vanishes and $F_{\hat\gamma(0)}$ is totally geodesic. □

2.4.6 Fibre Bundles

We consider now a smooth fibre bundle

$$(\pi : M \to B; F)$$

with fibre F and suppose that $(B, \hat g)$ and (F, g_F) are Riemannian manifolds. Near any point $b \in B$ there exists an open neighbourhood $b \in U \subset B$ and a bundle chart, i.e. a diffeomorphism

$$\Phi_U : \pi^{-1}(U) \to U \times F$$

such that $\mathrm{pr}_1 \circ \Phi_U = \pi$. This induces a diffeomorphism

$$\Phi_{U,p} := \mathrm{pr}_2 \circ \Phi_U|_{F_b} : F_b \to F,$$

where $F_b = \pi^{-1}(b)$ and pr_i denotes the projection onto the i-th factor of $U \times F$. Let $\mathcal{U} = \{(U_i, \Phi_{U_i})\}$ be a bundle atlas, i.e. $M = \bigcup_{i\in I} U_i$. On intersecting neighbourhoods U_i and U_j we have transition maps

$$\Phi_{U_i} \circ \Phi_{U_k}^{-1} : (U_i \cap U_k) \times F \to (U_i \cap U_k) \times F$$

which yield maps

$$\begin{aligned} \Phi_{ik} : U_i \cap U_k &\to \mathrm{Diff}(F) \\ b &\to \Phi_{U_i,b} \circ \Phi_{U_k,b}^{-1}, \end{aligned}$$

where Diff(F) denotes the group of diffeomorphisms of F.
In particular, we have for all $b \in U_i \cap U_k$ and $v \in F$ that

$$\Phi_{U_i} \circ \Phi_{U_k}^{-1}(b, v) = (b, \Phi_{ik}(b)(v)).$$

Also the cocycle relations are hold, i.e.

$$\Phi_{ik}(b) \circ \Phi_{kj}(b) = \Phi_{ij}(b) \quad \text{for all} \quad b \in U_i \cap U_j \cap U_k$$

and

$$\Phi_{ii}(b) = \mathrm{id}_F \quad \text{for all} \quad b \in U_i.$$

Definition 2.34 (9.47 in [Be])**.** *We say that a fibre bundle $(\pi : M \to B; F)$ has structure group G if there exists a bundle atlas $\mathcal{U} = \{(U_i, \Phi_{U_i})\}$ such that all maps Φ_{ik} are in G.*

In the following we assume that the structure group G is the isometry group Isom(F, g_F).

Let $b \in B$. If $b \in U_k \cap U_i$, then

$$\Phi_{U_i,b}^* \, g_F = \Phi_{U_i,b}^* (\Phi_{ki}(b))^* \, g_F = \Phi_{U_k,b}^* \, g_F.$$

Hence, we have on each fibre F_b a well-defined metric

$$g_b^{\perp} := \Phi_{U_i,b}^* \, g_F \quad \text{if} \quad b \in U_i.$$

We choose a background metric h on M and decompose

$$T_pM = T_pF_{\pi(p)} \oplus \mathcal{H}_p \quad \text{for all} \quad p \in M,$$

where $\mathcal{H}_p$ denotes the orthogonal complement of $T_pF_{\pi(p)}$ in T_pM with respect to the metric h. We note that

$$d\pi_p : \mathcal{H}_p \to T_{\pi(b)}B$$

is an isomorphism. Consequently, we may define a metric g on M by

$$g_p(v, w) := g_{\pi(p)}^{\perp}(v_1, w_1) + \hat{g}_{\pi(p)}(d\pi_p(v_2), d\pi_p(w_2))$$

for all $p \in M$ and $v = (v_1, v_2)$, $w = (w_1, w_2) \in T_pM = T_pF_{\pi(p)} \oplus \mathcal{H}_p$.
As a consequence we have

Lemma 2.35. *Let $(\pi : M \to B; F)$ be a smooth fibre bundle with the fibre F carrying a Riemannian metric g_F and structure group $G = \mathrm{Isom}(F, g_F)$. Given a metric $\hat{g}$ on B there exists a Riemannian metric on M such that*

$$\pi : (M, g) \to (B, \hat{g})$$

is a Riemannian submersion whith all fibres $(F_b, g_b^{\perp})$ being isometric to (F, g_F).

Actually, we have even more. Due to Vilms (Theorem 3.5 in [Vi] and 9.59 in [Be]) there exists a metric g which yields totally geodesic fibres.

2.5 Integration

We generalize Fubini's theorem to the case of Riemannian submersions. Let

$$\pi : \left(M^{n+k}, g\right) \to (B^n, \hat{g})$$

be a surjective Riemannian submersion. For any $p \in M$ we choose an $\varepsilon > 0$ and charts $\varphi : p \in U \to \varphi(U) = (-\varepsilon, \varepsilon)^{n+k}$ and $\psi : b = \pi(p) \in V \to \psi(V) = (-\varepsilon, \varepsilon)^n$ such that $\pi(U) \subset V$ and

$$\psi \circ \pi \circ \varphi^{-1}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) \mapsto (x_1, \ldots, x_n)$$

with associated coordinate vector fields

$$\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n+k}}\right) \text{ on } U \quad \text{and} \quad \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) \text{ on } V$$

which satisfy

$$d\pi_q \left(\left.\frac{\partial}{\partial x^i}\right|_q \right) = \begin{cases} \left.\frac{\partial}{\partial x^i}\right|_{\pi(q)} & \text{if } \ 1 \le i \le n \\ 0 & \text{if } \ n+1 \le i \le n+k \end{cases}$$

for all $q \in U$. Let $(e_1, \ldots, e_{n+k})$ be an orthonormal basis of T_pM such that

$$\{e_1, \ldots, e_n\} \subset \mathcal{H}_p \quad \text{and} \quad \{e_{n+1}, \ldots, e_{n+k}\} \subset \mathcal{V}_p.$$

We identify T_pM via $(e_1, \ldots, e_{n+k})$ with $\mathbb{R}^{n+k}$. For $v_1, \ldots, v_{n+k} \in T_pM$ we have $v_i = \sum_{j=1}^{n+k} g(v_i, e_j) e_j$ and consequently

$$\begin{aligned} \det(v_1, \ldots, v_{n+k}) &= \sqrt{\det\left((v_1, \ldots, v_{n+k})^\top (v_1, \ldots, v_{n+k})\right)} \\ &= \sqrt{\det\left(\sum_{k=1}^{n+k} g(v_i, e_k) g(v_j, e_k)\right)_{i,j}} \\ &= \sqrt{\det\left(g\left(v_i, \sum_{k=1}^{n+k} g(v_j, e_k) e_k\right)\right)_{i,j}} \\ &= \sqrt{\det(g(v_i, v_j))_{i,j}}. \end{aligned}$$

For $1 \le i \le n$ let

$$\xi_i = \left.\frac{\partial}{\partial x^i}\right|_p - \sum_{j=n+1}^{n+k} g\left(\left.\frac{\partial}{\partial x^i}\right|_p, e_j\right) e_j = \left(\left.\frac{\partial}{\partial x^i}\right|_p\right)^\top \in \mathcal{H}_p$$

be the horizontal part of $\left.\frac{\partial}{\partial x^i}\right|_p$. We identify $\mathcal{H}_p \cong \mathbb{R}^n$ and $\mathcal{V}_p \cong \mathbb{R}^k$ via $(e_1, \ldots, e_k)$ and $(e_{k+1}, \ldots, e_{k+n})$, respectively.

Since $d\pi_p : \mathcal{H}_p \to T_{\pi(p)}B$ is an isometry, ξ_i is the horizontal lift of $\frac{\partial}{\partial x^i}\big|_{\pi(p)}$ and $(d\pi_p e_i)_{1\leq i\leq n}$ forms an orthonormal basis of $T_{\pi(p)}B$, which yields an identification $T_{\pi(p)}B \cong \mathbb{R}^n$.

We obtain $\sqrt{\det\left(g\left(\frac{\partial}{\partial x^i}\big|_p, \frac{\partial}{\partial x^j}\big|_p\right)\right)_{i,j}} =$

$$\begin{aligned}
&= \det_{T_pM}\left(\frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p, \frac{\partial}{\partial x^{n+1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^{n+k}}\bigg|_p\right)\\
&= \det_{T_pM}\left(\xi_1, \ldots, \xi_n, \frac{\partial}{\partial x^{n+1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^{n+k}}\bigg|_p\right)\\
&= \det_{\mathcal{H}_p}(\xi_1, \ldots, \xi_n) \cdot \det_{\mathcal{V}_p}\left(\frac{\partial}{\partial x^{n+1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^{n+k}}\bigg|_p\right)\\
&= \det_{T_{\pi(p)}B}\left(\frac{\partial}{\partial x^1}\bigg|_{\pi(p)}, \ldots, \frac{\partial}{\partial x^n}\bigg|_{\pi(p)}\right) \cdot \det_{\mathcal{V}_p}\left(\frac{\partial}{\partial x^{n+1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^{n+k}}\bigg|_p\right),
\end{aligned}$$

i.e.

$$\sqrt{\det(g_{ij}(\varphi(p)))_{i,j}} = \sqrt{\det(\hat{g}_{ij}(\psi(\pi(p))))_{i,j}} \cdot \sqrt{\det(g^{\perp}_{ij}(\varphi(p)))_{i,j}}.$$

Let $f \in C_0(M)$. We set $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_k) \in (-\varepsilon, \varepsilon)^{n+k}$ and $\psi^{-1}(x) = b$. A calculation shows

$$\begin{aligned}
&\int_{(-\varepsilon,\varepsilon)^{n+k}} f \circ \varphi^{-1}(x, y) \cdot \sqrt{\det(g_{ij}(x, y))_{i,j}}\, d(x, y)\\
&= \int_{(-\varepsilon,\varepsilon)^{n+k}} f \circ \varphi^{-1}(x, y) \cdot \sqrt{\det(\hat{g}_{ij}(x))_{i,j}} \cdot \sqrt{\det(g^{\perp}_{ij}(y))_{i,j}}\, d(x, y)\\
&= \int_{(-\varepsilon,\varepsilon)^{n}} \left(\int_{(-\varepsilon,\varepsilon)^{k}} f \circ \varphi^{-1}(x, y) \cdot \sqrt{\det(g^{\perp}_{ij}(y))_{i,j}}\, dy\right) \sqrt{\det(\hat{g}_{ij}(x))_{i,j}}\, dx\\
&= \int_V \left(\int_{(-\varepsilon,\varepsilon)^{k}} (f \circ \varphi^{-1})(\psi(b), y) \cdot \sqrt{\det(g^{\perp}_{ij}(y))_{i,j}}\, dy\right) d\mathrm{vol}_{\hat{g}}.
\end{aligned}$$

Since a chart of $F_b \cap U$ is given by the composition

$$\begin{aligned}
\varphi_b : F_b \cap U &\xrightarrow{\varphi} (-\varepsilon, \varepsilon)^{n+k} \to (-\varepsilon, \varepsilon)^k\\
q &\mapsto (\psi(b), y_1, \ldots, y_k) \mapsto (y_1, \ldots, y_k),
\end{aligned}$$

we conclude

$$\int_U f\, d\mathrm{vol}_g = \int_V \left(\int_{F_b\cap U} f|_{F_b\cap U}\, d\mathrm{vol}_{g^{\perp}}\right) d\mathrm{vol}_{\hat{g}}.$$

Using coverings $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ and $(V_\alpha, \psi_\alpha)_{\alpha\in A}$ of M and B, respectively, consisting of charts as above such that each map $\psi_\alpha \circ \pi \circ \varphi^{-1}$ is a projection $(x_1, \ldots, x_{n+k}) \mapsto (x_1, \ldots, x_n)$, together with a partition of unity $(\rho_\alpha)_{\alpha\in A}$ subordinate to $(U_\alpha)_{\alpha\in A}$ we obtain

Proposition 2.36 (Theorem 5.6 in [Sa]). *Let* $\pi : \big(M^{n+k}, g\big) \to (B^n, \hat{g})$ *be a surjective Riemannian submersion. Then for any* $f \in C_0(M)$ *we have*

$$\int_M f\ dvol_g = \int_B \left(\int_{F_b} f|_{F_b}\ dvol_{g^\perp} \right) dvol_{\hat{g}}.$$

Considering the rescaled metric $r^2\hat{g}$ *and* $\pi : (M, g_{r^2}) \to (B, r^2\hat{g})$ *we obtain*

$$\int_M f\ dvol_{g_{r^2}} = \int_B \left(\int_{F_b} f|_{F_b}\ dvol_{g^\perp} \right) dvol_{r^2\hat{g}}.$$

As a special case we have Fubini's theorem

Corollary 2.37. *Let* $(M \times N, g \oplus h)$ *be a Riemannian product and* $f \in C_0(M)$*. Then for every* $p \in M$ *the function* $\bar{f} : M \ni p \mapsto \int_N f(p, q)\, \mathrm{dvol}_h$ *is integrable and*

$$\int_{M\times N} f\, \mathrm{dvol}_{g\oplus h} = \int_M \bar{f}\, \mathrm{dvol}_g.$$

Chapter 3

Collapsing Riemannian Submersions

We recall in section 3.1 some basic facts concerning the Yamabe constant as already mentioned in the overview in chapter 1. In particular, we prove that $Y(M,[g]) > 0$ and $Y(M \times \mathbb{R}^m, [g \oplus g_{\mathrm{eucl}}]) > 0$ provided that $\mathrm{scal}_g > 0$. In section 3.2 we give the proof of our main theorem (Theorem 1.3) making use of the local estimates for the product metric and the induced metric on admissible trivializations as developed in chapter 2.

3.1 The Yamabe Constant

Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$. As in chapter 1 we consider the normalized total scalar curvature functional

$$Q(\bar{g}) := \frac{\int_M \mathrm{scal}_{\bar{g}} \, d\mathrm{vol}_{\bar{g}}}{\left(\int_M d\mathrm{vol}_{\bar{g}}\right)^{2/p}},$$

where $p = p_n = \frac{2n}{n-2}$ and $\bar{g}$ varies in the conformal class $[g]$. Writing

$$\bar{g} = f^{p-2} \cdot g$$

for some function $f \in C^\infty(M, \mathbb{R}_{>0})$ and setting $a = a_n = \frac{n-2}{4(n-1)}$, we find

$$Q(\bar{g}) = Q_g(f) := \frac{\int_M \left(\frac{1}{a}\|\nabla^g f\|_g^2 + \mathrm{scal}_g \cdot f^2\right) d\mathrm{vol}_g}{\|f\|^2_{L^p(M,g)}}.$$

We define the *Yamabe constant* $Y(M,[g])$ of $[g]$ as

$$Y(M,[g]) := \inf_{\bar{g} \in [g]} Q(\bar{g}) = \inf \left\{ Q_g(f) \mid f \in C^\infty(M, \mathbb{R}_{>0}) \right\}.$$

Remark 3.1. *Let* (M^n, g) *be a closed Riemannian manifold of dimension* $n \geq 3$ *such that* $\mathrm{scal}_g > 0$. *Then* $Y(M, [g]) > 0$.

Proof. Since M is compact, there exists a constant $c_1 > 0$ such that

$$\mathrm{scal}_g \geq c_1$$

and hence a constant $c_2 > 0$ with

$$\int_M \left(\frac{1}{a} \|\nabla^g f\|_g^2 + \mathrm{scal}_g \cdot f^2 \right) d\mathrm{vol}_g \geq c_2 \, \|f\|^2_{H^{1,2}(M,g)} \, .$$

Due to the Sobolev embedding theorem we find a constant $c_3 > 0$ such that

$$\|f\|^2_{L^p(M,g)} \leq c_3 \cdot \|f\|^2_{H^{1,2}(M,g)}.$$

Consequently,

$$Q_g(f) \geq \frac{c_2}{c_3} \quad \text{and} \quad Y(M, [g]) \geq \frac{c_2}{c_3} > 0.$$

□

Motivated by

$$Y(M, [g]) = \inf \{Q_g(f) \mid f \in C^\infty(M) \setminus \{0\}\}$$

we define the Yamabe constant of a not necessarily compact Riemannian manifold (E^n, g) of dimension $n \geq 3$ without boundary as

$$Y(E, [g]) = \inf \{Q_g(f) \mid f \in C_0^\infty(E) \setminus \{0\}\} \, .$$

In analogy to Remark 3.1 above we have

Lemma 3.2. *Let* (M^n, g) *a closed Riemannian manifold with* $\mathrm{scal}_g > 0$ *and* $m \in \mathbb{N}$ *such that* $m + n \geq 3$. *Then*

$$Y(M \times \mathbb{R}^m, [g \oplus g_{\mathrm{eucl}}]) > 0.$$

Proof. We first note that there exists a constant $c > 0$ such that

$$\mathrm{scal}_{g \oplus g_{\mathrm{eucl}}}(p, q) = \mathrm{scal}_g(p) \geq c.$$

Since $(M \times \mathbb{R}^m, [g \oplus g_{\mathrm{eucl}}])$ is a complete Riemannian manifold with strictly positive injectivity radius and bounded sectional curvature, due to Theorem 2.21 in [Au] there is a continuous embedding

$$H^{1,2}(M \times \mathbb{R}^m, [g \oplus g_{\mathrm{eucl}}]) \hookrightarrow L^p(M \times \mathbb{R}^m, [g \oplus g_{\mathrm{eucl}}]).$$

The claim follows by the same pattern as in the proof of Remark 3.1. □

3.2 Collapsing Riemannian Submersions

We turn now to the proof of

Theorem 1.3 *Let $\pi : (M^{n+k}, g) \to (B^n, \hat{g})$ be a Riemannian submersion, where M and B are assumed to be closed,* $\dim M \geq 3$, *and the scalar curvature* $\mathrm{scal}_{g^\perp}$ *of every fibre F_b, $b \in B$, with respect to the induced metric is positive. Considering the Riemannian submersion $\pi : (M, g_{r^2}) \to (B, r^2\hat{g})$ we have*

$$\lim_{r\to\infty} Y(M, [g_{r^2}]) = \inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]).$$

We begin with an elementary remark.

Lemma 3.3. *Let $\rho > 0$, $b \in B$ and $B^n_\rho(0) \subset \mathbb{R}^n$. Then we have*

$$\lim_{\rho\to\infty} Y(B^n_\rho(0) \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]) = Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]).$$

Proof. Given $0 < \rho_1 < \rho_2$ we have inclusions

$$C^\infty_0(B^n_{\rho_1}(0) \times F_b) \subset C^\infty_0(B^n_{\rho_2}(0) \times F_b) \subset C^\infty_0(\mathbb{R}^n \times F_b)$$

and consequently

$$Y(B^n_{\rho_1}(0)\times F_b, [g_{\mathrm{eucl}}\oplus g^\perp]) \geq Y(B^n_{\rho_2}(0)\times F_b, [g_{\mathrm{eucl}}\oplus g^\perp]) \geq Y(\mathbb{R}^n\times F_b, [g_{\mathrm{eucl}}\oplus g^\perp]).$$

Let $\varepsilon > 0$. Then there exists an $f \in C^\infty_0(\mathbb{R}^n \times F_b)$ such that

$$Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]) \leq Q_{g_{\mathrm{eucl}}\oplus g^\perp}(f) < Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]) + \varepsilon.$$

Since f is compactly supported, we find a $\rho > 0$ with

$$f \in C^\infty_0(B^n_\rho(0) \times F_b).$$

It follows

$$Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]) \leq Y(B^n_\rho(0) \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]) \leq Q_{g_{\mathrm{eucl}}\oplus g^\perp}(f)$$

and

$$\lim_{\rho\to\infty} Y(B^n_\rho(0) \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]) = Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]).$$

□

Proposition 3.4. *For any $b \in B$ we have*

$$\limsup_{r\to\infty} Y(M,[g_{r^2}]) \leq Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^{\perp}]).$$

Proof. We consider a test function $f \in C_0^\infty(\mathbb{R}^n \times F_b)$. Then there exists $\rho > 0$ such that $f \in C_0^\infty(B_\rho^n(0) \times F_b)$. Given $\delta > 0$ we take an admissible trivialization of $\pi : (M, g_{r^2}) \to (B, r^2\hat{g})$ near b, i.e. a local trivialization

$$\Psi : V \times F_b = \exp_b\left(B_\varepsilon^{\hat{g}}(0)\right) \times F_b \to \pi^{-1}\left(\exp_b\left(B_\varepsilon^{\hat{g}}(0)\right)\right)$$

together with local parametrizations

$$\varphi^{r^2\hat{g}} \times \varphi_l^{g^\perp} : B_{r\varepsilon}^n(0) \times B_{\varepsilon_l}^k(0) \to \exp_b\left(B_\varepsilon^{\hat{g}}(0)\right) \times \exp_{p_l}\left(B_{\varepsilon_l}^{g^\perp}(0)\right)$$

and r_0 as in Definition 2.31. We choose a partition of unity $\{\lambda_l\}_{l=1,\dots,m}$ subordinated to the cover $\left\{\exp_{p_l}\left(B_{\varepsilon_l}^{g^\perp}(0)\right)\right\}_{l=1,\dots,m}$ of F_b.
W.l.o.g. $r_0\varepsilon > \rho$, which implies

$$f \in C_0^\infty(B_{r\varepsilon}^n(0) \times F_b) \quad \text{for all } r > r_0.$$

Then,

$$f \circ \left(\left(\varphi^{r^2\hat{g}}\right)^{-1} \times \mathrm{id}\right) \in C_0^\infty(V \times F_b).$$

We set

$$f_r := f \circ \left(\left(\varphi^{r^2\hat{g}}\right)^{-1} \times \mathrm{id}\right) \circ \Psi^{-1} \in C^\infty(\pi^{-1}(V)).$$

We recall from Proposition 2.17 that

$$\begin{aligned}\mathrm{scal}_{g_{r^2}} &= \frac{1}{r^2}\cdot \mathrm{scal}_{\hat{g}} \circ \pi + \mathrm{scal}_{g^\perp} - \frac{1}{r^4}\|A\|_g^2 - \frac{1}{r^2}\|T\|_g^2 - \frac{1}{r^2}\|N\|_g^2 \\ &\quad + \frac{2}{r^2}\sum_i g\left(\nabla_{X_i} N, X_i\right).\end{aligned}$$

Since $\mathrm{scal}_{g^\perp} > 0$, we may assume by compactness of M and B that

$$\mathrm{scal}_{g_{r^2}} > K \quad \text{for } r > r_0$$

and some constant $K > 0$. Then by Remark 3.1,

$$Y(M, [g_{r^2}]) > 0 \quad \text{for } r > r_0.$$

By definition of Ψ we have

$$\mathrm{scal}_{g^\perp}(\Psi(b,p)) = \mathrm{scal}_{g^\perp}(p) > 0 \quad \text{for all } p \in F_b.$$

Using the compactness of F_b we can shrink ε and find constants c and d such that

$$\frac{1}{1+\delta} < c \leq \frac{\mathrm{scal}_{g^\perp}(\Psi(b',p))}{\mathrm{scal}_{g^\perp}(p)} \leq d < 1+\delta \quad \text{for all } (b',p) \in V \times F_b.$$

From the compactness of M, B and F_b it follows for sufficiently large r, w.l.o.g. $r > r_0$, that

$$\frac{|(\mathrm{scal}_{g_{r^2}} - \mathrm{scal}_{g^\perp})(\Psi(b',p))|}{\mathrm{scal}_{g^\perp}(p)} < \frac{1}{2} \cdot \min\left\{1+\delta-d, , c - \frac{1}{1+\delta}\right\}$$

for all $(b',p) \in V \times F_b$. As a consequence, we have

$$\frac{1}{1+\delta} < \frac{\mathrm{scal}_{g_{r^2}}(\Psi(b',p))}{\mathrm{scal}_{g^\perp}(p)} < 1+\delta \quad \text{for all } (b',p) \in V \times F_b \text{ and } r > r_0.$$

We estimate

$$\begin{aligned}
\|f\|_p^2 &:= \|f\|^2_{L^p(B^n_{r\varepsilon}(0)\times F_b, g_{\mathrm{eucl}}\oplus g^\perp)} \\
&= \left(\int_{B^n_{r\varepsilon}(0)\times F_b} |f|^p \, d\mathrm{vol}_{g_{\mathrm{eucl}}\oplus g^\perp}\right)^{2/p} \\
&= \left(\int_{F_b}\left(\int_{B^n_{r\varepsilon}(0)} |f|^p \cdot (1+\delta)^{n/2} \cdot (1+\delta)^{-n/2}\, dx\right) d\mathrm{vol}_{g^\perp}\right)^{2/p} \\
&\leq (1+\delta)^{n/p}\left(\int_{F_b}\left(\int_{B^n_{r\varepsilon}(0)} |f|^p \cdot \sqrt{\det D^r(x)}\, dx\right) d\mathrm{vol}_{g^\perp}\right)^{2/p} \\
&= (1+\delta)^{n/p}\left(\int_{V\times F_b}\left|f \circ \left(\left(\varphi^{r^2\hat{g}}\right)^{-1} \times \mathrm{id}\right)\right|^p d\mathrm{vol}_{r^2\hat{g}\oplus g^\perp}\right)^{2/p}.
\end{aligned}$$

Hence, $\|f\|_p^2 \leq$

$$\leq (1+\delta)^{n/p}\left(\sum_{l=1}^m \int_{V\times \exp_{p_l}\left(B^{g^\perp}_{\varepsilon_l}(0)\right)} \lambda_l \cdot \left|f \circ \left(\left(\varphi^{r^2\hat{g}}\right)^{-1} \times \mathrm{id}\right)\right|^p d\mathrm{vol}_{r^2\hat{g}\oplus g^\perp}\right)^{2/p}.$$

Using

$$\sqrt{\det C^r(x,y)} < (1+\delta)^{n+k}\sqrt{\det B^r(x,y)} \quad \text{for } (x,y) \in B^n_{r\varepsilon}(0) \times B^k_{\varepsilon_l}(0)$$

we obtain

$$\int_{V\times \exp_{p_l}\left(B^{g^\perp}_{\varepsilon_l}(0)\right)} \lambda_l \cdot \left|f \circ \left(\left(\varphi^{r^2\hat{g}}\right)^{-1} \times \mathrm{id}\right)\right|^p d\mathrm{vol}_{r^2\hat{g}\oplus g^\perp} \leq$$

$$\leq (1+\delta)^{n+k} \int_{V\times \exp_{p_l}\left(B^{g^\perp}_{\varepsilon_l}(0)\right)} \lambda_l \cdot \left| f \circ \left(\left(\varphi^{r^2\hat{g}} \right)^{-1} \times \mathrm{id} \right) \right|^p \, d\mathrm{vol}_{\Psi^* g_{r^2}}$$

and consequently,

$$\|f\|_p^2 \leq (1+\delta)^{(3n+2k)/p} \left(\int_{\pi^{-1}(V)} |f_r|^p \, d\mathrm{vol}_{g_{r^2}} \right)^{2/p}.$$

From

$$Y(M,[g_{r^2}]) \leq \frac{\int_{\pi^{-1}(V)} \left(\frac{1}{a_{n+k}} \|\nabla^{g_{r^2}} f_r\|^2_{g_{r^2}} + \mathrm{scal}_{g_{r^2}} \cdot f_r^2 \right) d\mathrm{vol}_{g_{r^2}}}{\left(\int_{\pi^{-1}(V)} |f_r|^p \right)^{2/p}}$$

it follows that

$$\|f\|_p^2 \leq \frac{(1+\delta)^{(3n+2k)/p}}{Y(M,[g_{r^2}])} \cdot \int_{\pi^{-1}(V)} \left(\frac{1}{a_{n+k}} \|\nabla^{g_{r^2}} f_r\|^2_{g_{r^2}} + \mathrm{scal}_{g_{r^2}} \cdot f_r^2 \right) d\mathrm{vol}_{g_{r^2}}.$$

We obtain

$$\int_{\pi^{-1}(V)} \frac{1}{a_{n+k}} \|\nabla^{g_{r^2}} f_r\|^2_{g_{r^2}} \, d\mathrm{vol}_{g_{r^2}} =$$

$$\begin{aligned}
&= \int_{V\times F_b} \frac{1}{a_{n+k}} \left\| \nabla^{\Psi^* g_{r^2}} f \circ \left(\left(\varphi^{r^2\hat{g}} \right)^{-1} \times \mathrm{id} \right) \right\|^2_{\Psi^* g_{r^2}} d\mathrm{vol}_{\Psi^* g_{r^2}} \\
&\leq (1+\delta)^{n+k+2} \int_{V\times F_b} \frac{1}{a_{n+k}} \left\| \nabla^{r^2\hat{g}\oplus g^\perp} f \circ \left(\left(\varphi^{r^2\hat{g}} \right)^{-1} \times \mathrm{id} \right) \right\|^2_{r^2\hat{g}\oplus g^\perp} d\mathrm{vol}_{r^2\hat{g}\oplus g^\perp} \\
&\leq (1+\delta)^{3+n+n/2+k} \int_{B^n_\rho(0)\times F_b} \frac{1}{a_{n+k}} \left\| \nabla^{g_{\mathrm{eucl}}\oplus g^\perp} f \right\|^2_{g_{\mathrm{eucl}}\oplus g^\perp} d\mathrm{vol}_{g_{\mathrm{eucl}}\oplus g^\perp},
\end{aligned}$$

once again using the partition of unity and the local estimates for $B^r_l(x,y)$, $C^r_l(x,y)$ and $D^r(x)$.
Finally, we consider

$$\int_{\pi^{-1}(V)} \mathrm{scal}_{g_{r^2}} \cdot f_r^2 \, d\mathrm{vol}_{g_{r^2}} =$$

$$\begin{aligned}
&= \int_{V\times F_b} \left(\mathrm{scal}_{g_{r^2}} \circ \Psi \right) \cdot \left(f \circ \left(\left(\varphi^{r^2\hat{g}} \right)^{-1} \times \mathrm{id} \right) \right)^2 d\mathrm{vol}_{\Psi^* g_{r^2}} \\
&\leq (1+\delta)^{n+k+1} \int_{V\times F_b} \mathrm{scal}_{g^\perp} \cdot \left(f \circ \left(\left(\varphi^{r^2\hat{g}} \right)^{-1} \times \mathrm{id} \right) \right)^2 d\mathrm{vol}_{r^2\hat{g}\oplus g^\perp} \\
&\leq (1+\delta)^{n+n/2+k+1} \int_{B^n_\rho(0)\times F_b} \mathrm{scal}_{g^\perp} \cdot f^2 \, d\mathrm{vol}_{g_{\mathrm{eucl}}\oplus g^\perp}.
\end{aligned}$$

On the whole we obtain $\|f\|_p^2 \leq$

$$\leq \frac{(1+\delta)^{\alpha(n,k)}}{Y(M,[g_{r^2}])} \cdot \int_{B_\rho^n(0)\times F_b} \left(\frac{1}{a_{n+k}} \left\| \nabla^{g_{\mathrm{eucl}}\oplus g^\perp} f \right\|^2_{g_{\mathrm{eucl}}\oplus g^\perp} + \mathrm{scal}_{g^\perp} \cdot f^2 \right) d\mathrm{vol}_{g_{\mathrm{eucl}}\oplus g^\perp},$$

where $\alpha(n,k) = \frac{3n+2k}{p} + (4+3n+2k)$ and $r > r_0$.
Since $f \in C_0^\infty(B_\rho^n(0) \times F_b)$ was arbitrary, we have

$$Y(M,[g_{r^2}]) \leq (1+\delta)^{\alpha(n,k)} \cdot Y(B_\rho^n(0) \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]) \quad \text{for all } r > r_0$$

and consequently

$$\limsup_{r\to\infty} Y(M,[g_{r^2}]) \leq (1+\delta)^{\alpha(n,k)} \cdot Y(B_\rho^n(0) \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]).$$

Tending $\delta \to 0$ and $\rho \to \infty$ the claimed inequality

$$\limsup_{r\to\infty} Y(M,[g_{r^2}]) \leq Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp])$$

follows. □

To conclude the theorem we prove

Proposition 3.5.

$$\liminf_{r\to\infty} Y(M,[g_{r^2}]) \geq \inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\mathrm{eucl}} \oplus g^\perp]).$$

Proof. Given $\delta > 0$ we find finitely many points $b_1, \ldots, b_{n(\delta)} \in B$ and $\varepsilon_1, \ldots, \varepsilon_{n(\delta)} > 0$ together with admissible local trivializations

$$\Psi^j : V_j \times F_{b_j} = \exp_{b_j}\left(B_\varepsilon^{\hat g}(0)\right) \times F_{b_j} \to \pi^{-1}\left(\exp_{b_j}\left(B_\varepsilon^{\hat g}(0)\right)\right) \quad 1 \leq j \leq n(\delta)$$

and r_0 with local estimates for $r > r_0$ as in Definition 2.31.
We choose a partition of unity $\{\eta_j = \chi_j^2\}_{j=1,\ldots,n(\delta)}$ subordinated to the cover $\{V_j\}_{j=1,\ldots,n(\delta)}$. Since $\operatorname{supp}\chi_j \subset V_j$ is compact, we find a constant $K > 0$ such that

$$\left\|\nabla^{\hat g}\chi_j\right\|_{\hat g} \leq K$$

and consequently

$$\left\|\nabla^{r^2\hat g}\chi_j\right\|_{r^2\hat g} \leq \frac{K}{r}$$

for all $j = 1, \ldots, n(\delta)$.
We may assume by compactness of F_b and shrinking ε_j that there are constants c and d such that

$$\frac{1}{1+\delta} < c \leq \frac{\mathrm{scal}_{g^\perp}(\Psi^j(b',p))}{\mathrm{scal}_{g^\perp}(p)} \leq d < 1+\delta \quad \text{for all } (b',p) \in V_j \times F_{b_j}$$

as in the proof of Proposition 3.4. From the compactness of M, B and F_b it follows for sufficiently large r, w.l.o.g. $r > r_0$, that

$$\frac{|-\frac{K^2\cdot n(\delta)}{r^2\cdot a_{n+k}} + (\mathrm{scal}_{g_{r^2}} - \mathrm{scal}_{g^\perp})(\Psi^j(b',p))|}{\mathrm{scal}_{g^\perp}(p)} < \frac{1}{2}\cdot \min\left\{1+\delta-d,, c-\frac{1}{1+\delta}\right\}$$

for all $(b',p) \in V_j \times F_{b_j}$. As a consequence, we have

$$\frac{1}{1+\delta} < \frac{\mathrm{scal}_{g_{r^2}}(\Psi^j(b',p)) - \frac{K^2\cdot n(\delta)}{r^2\cdot a_{n+k}}}{\mathrm{scal}_{g^\perp}(p)} < 1+\delta$$

for all $(b',p) \in V_j \times F_{b_j}$ and $r > r_0$ and $j = 1, \ldots, n(\delta)$. In the following we always assume $r > r_0$.
Let $F \in C^\infty(M)$. We calculate

$$\begin{aligned}\|F\|^2_{L^p(M,g_{r^2})} &= \left\|F^2\right\|_{L^{p/2}(M,g_{r^2})} = \left\|\sum_{j=1}^{n(\delta)}(\chi_j\circ\pi)^2\cdot F^2\right\|_{L^{p/2}(M,g_{r^2})}\\ &\le \sum_{j=1}^{n(\delta)}\left(\int_{\pi^{-1}(V_j)}|(\chi_j\circ\pi)\cdot F|^p\,d\mathrm{vol}_{g_{r^2}}\right)^{2/p}.\end{aligned}$$

We write $F_j := (\chi_j\circ\pi)\cdot F$ and estimate

$$\begin{aligned}\left(\int_{\pi^{-1}(V_j)}|F_j|^p\,d\mathrm{vol}_{g_{r^2}}\right)^{2/p} &= \left(\int_{V_j\times F_{b_j}}|F_j\circ\Psi^j|^p\,d\mathrm{vol}_{(\Psi^j)^*g_{r^2}}\right)^{2/p}\\ \le\ (1+\delta)^{2(n+k)/p}&\left(\int_{V_j\times F_{b_j}}|F_j\circ\Psi^j|^p\,d\mathrm{vol}_{r^2\hat g\oplus g^\perp}\right)^{2/p}\\ \le\ (1+\delta)^{(3n+2k)/p}&\left(\int_{B^n_{r\varepsilon_j}(0)\times F_{b_j}}\left|(F_j\circ\Psi^j)\circ\left(\varphi_j^{r^2\hat g}\times\mathrm{id}\right)\right|^p d\mathrm{vol}_{g_{\mathrm{eucl}}\oplus g^\perp}\right)^{2/p}.\end{aligned}$$

We set $f_j := (F_j\circ\Psi^j)\circ\left(\varphi_j^{r^2\hat g}\times\mathrm{id}\right) \in C_0^\infty(\mathbb{R}^n\times F_{b_j})$ and

$$Y_0^\delta := \min_{1\le j\le n(\delta)} Y_0^{j,\delta},$$

where $Y_0^{j,\delta} := Y(\mathbb{R}^n\times F_{b_j}, [g_{\mathrm{eucl}}\oplus g^\perp]) > 0$ by Lemma 3.2.
Hence,

$$\begin{aligned}&\left(\int_{B^n_{r\varepsilon_j}(0)\times F_{b_j}}|f_j|^p d\mathrm{vol}_{g_{\mathrm{eucl}}\oplus g^\perp}\right)^{2/p}\\ \le\ &\frac{1}{Y_0^\delta}\int_{B^n_{r\varepsilon_j}(0)\times F_{b_j}}\left(\frac{1}{a_{n+k}}\left\|\nabla^{g_{\mathrm{eucl}}\oplus g^\perp}f_j\right\|^2_{g_{\mathrm{eucl}}\oplus g^\perp} + \mathrm{scal}_{g^\perp}\cdot f_j^2\right)d\mathrm{vol}_{g_{\mathrm{eucl}}\oplus g^\perp}.\end{aligned}$$

We estimate further

$$\begin{aligned}
& \int_{B^n_{r\varepsilon_j}(0)\times F_{b_j}} \left\|\nabla^{g_{\mathrm{eucl}}\oplus g^\perp} f_j\right\|^2_{g_{\mathrm{eucl}}\oplus g^\perp} d\mathrm{vol}_{g_{\mathrm{eucl}}\oplus g^\perp} \\
\leq\ & (1+\delta)^{1+n/2}\int_{V_j\times F_{b_j}} \left\|\nabla^{r^2\hat{g}\oplus g^\perp}((\chi_j\circ\pi)\cdot F)\circ\Psi^j\right\|^2_{r^2\hat{g}\oplus g^\perp} d\mathrm{vol}_{r^2\hat{g}\oplus g^\perp} \\
\leq\ & (1+\delta)^{3+n+n/2+k}\int_{V_j\times F_{b_j}} \left\|\nabla^{\Psi^* g_{r^2}}((\chi_j\circ\pi)\cdot F)\circ\Psi^j\right\|^2_{(\Psi^j)^* g_{r^2}} d\mathrm{vol}_{(\Psi^j)^* g_{r^2}} \\
=\ & (1+\delta)^{3+n+n/2+k}\int_{\pi^{-1}(V_j)} \|\nabla^{g_{r^2}}(\chi_j\circ\pi)\cdot F\|^2_{g_{r^2}} d\mathrm{vol}_{g_{r^2}}.
\end{aligned}$$

Partial integration yields

$$\begin{aligned}
& \int_{\pi^{-1}(V_j)} \|\nabla^{g_{r^2}}(\chi_j\circ\pi)\cdot F\|^2_{g_{r^2}} d\mathrm{vol}_{g_{r^2}} \\
=\ & \int_{\pi^{-1}(V_j)} (\chi_j\circ\pi)^2\cdot F\cdot\Delta^{g_{r^2}}F + F^2\cdot\|\nabla^{g_{r^2}}(\chi_j\circ\pi)\|^2_{g_{r^2}} d\mathrm{vol}_{g_{r^2}} \\
=\ & \int_{\pi^{-1}(V_j)} (\chi_j\circ\pi)^2\cdot F\cdot\Delta^{g_{r^2}}F + F^2\cdot\left\|\nabla^{r^2\hat{g}}\chi_j\right\|^2_{r^2\hat{g}}\circ\pi\, d\mathrm{vol}_{g_{r^2}} \\
\leq\ & \int_{\pi^{-1}(V_j)} (\chi_j\circ\pi)^2\cdot F\cdot\Delta^{g_{r^2}}F + F^2\cdot\frac{K^2}{r^2}\, d\mathrm{vol}_{g_{r^2}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \int_{B^n_{r\varepsilon_j}(0)\times F_{b_j}} \mathrm{scal}_{g^\perp}\cdot f_j^2\, d\mathrm{vol}_{g_{\mathrm{eucl}}\oplus g^\perp} \\
\leq\ & (1+\delta)^{n/2}\int_{V_j\times F_{b_j}} \mathrm{scal}_{g^\perp}\cdot((\chi_j\circ\pi)\cdot F)^2\circ\Psi^j\, d\mathrm{vol}_{r^2\hat{g}\oplus g^\perp} \\
\leq\ & (1+\delta)^{n+n/2+k+1}\int_{V_j\times F_{b_j}} \left(\mathrm{scal}_{g_{r^2}}\circ\Psi^j - \frac{K^2\cdot n(\delta)}{r^2\cdot a_{n+k}}\right)\cdot((\chi_j\circ\pi)\cdot F)^2\circ\Psi^j\, d\mathrm{vol}_{(\Psi^j)^* g_{r^2}} \\
\leq\ & (1+\delta)^{n+n/2+k+1}\int_{\pi^{-1}(V_j)} \left(\mathrm{scal}_{g_{r^2}} - \frac{K^2\cdot n(\delta)}{r^2\cdot a_{n+k}}\right)\cdot((\chi_j\circ\pi)\cdot F)^2\, d\mathrm{vol}_{g_{r^2}}.
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
& \|F\|^2_{L^p(M,g_{r^2})} \leq \\
\leq\ & \frac{(1+\delta)^{\beta(n,k)}}{Y_0^\delta}\int_M \frac{1}{a_{n+k}}F\cdot\Delta^{g_{r^2}}F + \mathrm{scal}_{g_{r^2}}\cdot F^2\, d\mathrm{vol}_{g_{r^2}} \\
=\ & \frac{(1+\delta)^{\beta(n,k)}}{Y_0^\delta}\int_M \frac{1}{a_{n+k}}\|\nabla^{g_{r^2}}F\|^2_{g_{r^2}} + \mathrm{scal}_{g_{r^2}}\cdot F^2\, d\mathrm{vol}_{g_{r^2}}
\end{aligned}$$

with $\beta(n,k) = 3 + n + n/2 + k + (3n+2k)/p$.
Consequently,

$$Y_0^\delta \leq (1+\delta)^{\beta(n,k)} \cdot Y(M, [g_{r^2}])$$

and

$$\inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\text{eucl}} \oplus g^\perp]) \leq (1+\delta)^{\beta(n,k)} \cdot \liminf_{r\to\infty} Y(M, [g_{r^2}]),$$

which yields for $\delta \to 0$ the claimed inequality

$$\inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\text{eucl}} \oplus g^\perp]) \leq \liminf_{r\to\infty} Y(M, [g_{r^2}]),$$

□

Combining Proposition 3.4 and Proposition 3.5 we obtain

$$\limsup_{r\to\infty} Y(M, [g_{r^2}]) \leq \inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\text{eucl}} \oplus g^\perp]) \leq \liminf_{r\to\infty} Y(M, [g_{r^2}])$$

and consequently,

$$\lim_{r\to\infty} Y(M, [g_{r^2}]) = \inf_{b\in B} Y(\mathbb{R}^n \times F_b, [g_{\text{eucl}} \oplus g^\perp])$$

which proves Theorem 1.3.

Bibliography

[A-D-H1] B. Ammann, M. Dahl, E. Humbert, *Smooth Yamabe invariant and surgery,* J. Diff. Geom. **94**, 1-58 (2013)

[A-D-H2] B. Ammann, M. Dahl, E. Humbert, *The conformal Yamabe constant of product manifolds,* Proc. Amer. Math. Soc. **141**, 295-307 (2013)

[A-D-H3] B. Ammann, M. Dahl, E. Humbert, *Square-integrability of solutions of the Yamabe equation,* Comm. Anal. Geom. **21**, 891-916 (2013)

[A-D-H4] B. Ammann, M. Dahl, E. Humbert, *Low-dimensional surgery and the Yamabe invariant,* J. Math. Soc. Japan **67**, 159-182 (2015)

[A-F-P] K. Akutagawa, L. A. Florit, J. Petean, *On Yamabe constants of Riemannian products,* Comm. Anal. Geom. **15**, 947-969 (2007)

[Au] T. Aubin, *Some nonlinear problems in Riemannian geometry,* Springer Monographs in Mathematics, Springer-Verlag, Berlin (1998)

[Be] A. L. Besse, *Einstein Manifolds,* Classics in Mathematics. Springer-Verlag, Berlin, (2008)

[G-L] M. Gromov, H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **111**, 423–434 (1980)

[Ko] O. Kobayashi, *Scalar curvature of a metric with unit volume*, Math. Ann. **279**, 253–265, (1987)

[L-P] J. M. Lee, T. H. Parker, *The Yamabe problem,* Bull. Amer. Math. Soc. (N.S.) **17**, 33-91 (1987)

[Pe1] J. Petean, *The Yamabe invariant of simply connected manifolds*, J. Reine Angew. Math. **523**, 225–231 (2000)

[Pe2] J. Petean, *Isoperimetric regions in spherical cones and Yamabe constants of $M \times S^1$*, Geom. Dedicata **143**, 37-48 (2009)

[P-R] J. Petean, J. M. Ruiz, *On the Yamabe constants of $S^2 \times \mathbb{R}^3$ and $S^3 \times \mathbb{R}^2$,* Diff. Geom. & Appl. **31**, 308-319 (2013)

[P-Y] J. Petean, G. Yun, *Surgery and the Yamabe invariant*, Geom. Funct. Anal. **9**, 1189–1199 (1999)

[Sa] T. Sakai, *Riemannian Geometry,*Translations of mathematical monographs, vol. 149, AMS, Providence, RI, (1996)

[Sch] R.Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature,* J. Diff. Geom. **20**, 479-495 (1984)

[S-Y1] R. Schoen, S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28**, 159–183 (1979)

[S-Y2] R. Schoen, S.-T- Yau, *Lectures on differential geometry,* Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, (1994)

[St] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. **136**, 511-540 (1963).

[Tr] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds,* Ann. Scuola Norm. Sup. Pisa **22**, 265-274 (1968)

[Vi] J. Vilms, *Totally geodesic maps,* J. Diff. Geom. 4, 73-79 (1970)