Real-valued differential forms on Berkovich analytic spaces and their cohomology

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
der Fakultät für Mathematik
der Universität Regensburg

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im Jahr 2016

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Chapter 1

Introduction

Differential forms are a fundamental tool in differential and complex geometry. For a smooth manifold X the de Rham complex $(\mathcal{A}^\bullet, d)$ of smooth differential forms carries a lot of information about the geometry of the space X. The Poincaré lemma shows that the de Rham complex is an exact complex of sheaves. It is a fundamental result of de Rham from the 1930s that the cohomology of the complex of global sections $(\mathcal{A}^\bullet(X), d)$ agrees with the singular cohomology of X with real coefficients. This in particular shows that the cohomology of this complex is an invariant of the topological space X, independent of the differentiable structure.

If X is indeed a complex manifold, and we consider complex valued differential forms, we have canonical decompositions $\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$ and $d = \partial + \overline{\partial}$. The cohomology of the complex $(\mathcal{A}^{p,\bullet}, \overline{\partial})$ is called the Dolbeault cohomology of X. By Dolbeault's theorem from the 1950s, there is a canonical isomorphism between the cohomology of $(\mathcal{A}^{p,\bullet}(X), \overline{\partial})$ and the sheaf cohomology $H^q(X, \Omega^p)$, where Ω^p denotes the sheaf of holomorphic differentials of degree p on X.

We now want to work over a non-archimedean field and use analytic spaces in the sense of Berkovich [Ber90] instead of complex manifolds. In their fundamental preprint [CLD12] Chambert–Loir and Ducros introduced real-valued differential forms on Berkovich analytic spaces. Their idea is to map subsets of the analytic space into analytic tori and then formally pull back superforms, as defined by Lagerberg [Lag12], along tropicalization maps. They obtain a sheaf of bigraded bidifferential algebras $(\mathcal{A}^{\bullet,\bullet}, d', d'')$ on the analytic space. They show that analogous statements of fundamental properties from complex pluri-potential theory hold for these forms. For example they prove analogues of the Poincaré–Lelong-formula and establish a variant of the Bedford–Taylor approach to define products of $(1,1)$-currents which satisfy suitable positivity conditions.

Another aspect of differential forms on complex manifolds is their use in Arakelov theory to calculate intersection numbers at infinite places. Gubler and Künnemann have used an extended class of differential forms (so called δ-forms) to give an analytic description of local heights also at finite places.

The purpose of this thesis is to study the properties of differential forms in Berkovich spaces, introduced by Chambert-Loir and Ducros, with respect to cohomology. The fundamental result will be the Poincaré lemma (Theorem 3.4.3), which, as the classical

Poincaré lemma, is a statement about local exactness of closed forms.

Chambert–Loir and Ducros defined differential forms for arbitrary Berkovich analytic spaces. We will mostly work in the algebraic situation, by which we mean that our analytic space is the analytification X^{an} in the sense of Berkovich of an algebraic variety X over a algebraically closed, non-archimedean field K. We have a fine sheaf of bigraded bidifferential algebras $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$ on X^{an}. This should be thought of as an analogue of the sheaf of (p,q)-differential forms with differential operators ∂ and $\overline{\partial}$ on a complex manifold. The sheaf $\mathcal{A}^{\bullet,\bullet}$ has many analogous properties, such as canonical integration of (n,n)-forms with compact support and Stokes' theorem. For our cohomological considerations we will most of the time fix q and focus on the complex $(\mathcal{A}^{\bullet,q}, d')$. We denote the cohomology of the global sections of this complex by $H_{d'}^{p,q}(X)$. We have the following main exactness property:

Theorem 1 (d'-Poincaré lemma on X^{an})**.** Let X be a variety over K and $V \subset X^{\mathrm{an}}$ an open subset. Let $x \in V$ and $\alpha \in \mathcal{A}_X^{p,q}(V)$ with $p > 0$ and $d'\alpha = 0$. Then there exists an open subset $W \subset V$ with $x \in W$ and $\beta \in \mathcal{A}_X^{p-1,q}(W)$ such that $d'\beta = \alpha|_W$.

This leads, as in Dolbeault's theorem, to an identification of the cohomology of the complex $(\mathcal{A}^{\bullet,q}, d')$ with the sheaf cohomology of a certain sheaf $\mathcal{L}_X^q$ on X (cf. Corollary 3.4.6). Note that the analogy with the theory of complex manifolds fails here a little bit, since $\mathcal{L}_X^0 = \underline{\mathbb{R}}$, where $\underline{\mathbb{R}}$ is the constant sheaf with stalks $\mathbb{R}$. Thus the kernel of d' consists only of locally constant functions, which is not true for the kernel of $\overline{\partial}$, which consists of holomorphic functions.

We obtain an identification of the cohomology of the complex $(\mathcal{A}^{\bullet,0}, d')$ with the singular cohomology of X with real coefficients. Using results by Hrushovski and Loeser on the homotopy type of Berkovich spaces associated with quasi-projective varieties, we then obtain as a consequence the following result:

Theorem 2. Let X be a variety over K. Then the real vector space $H_{d'}^{p,0}(X^{\mathrm{an}})$ is finite dimensional for all p.

Differential forms on Berkovich spaces are locally given by superforms, in the sense of Lagerberg [Lag12], on polyhedral complexes. These are then formally pulled back to the analytic space via the tropicalization procedure, as we will explain in Chapter 3. Therefore our strategy for proving Theorem 1 is to consider first the theory on polyhedral complexes (cf. Chapter 2). We will use the results from Chapter 2 and consider the local situation on analytic spaces in Subsection 3.4.1 and the global situation in Subsection 3.4.2. We will also define new types of charts, which we can use to define this formal pullback. Previously (cf. [CLD12, Gub13a]), tropical charts were always obtained by maps from open subsets to tori. We show that the same sheaves of differential forms are obtained when we use maps of open subsets into affine space. Gubler's approach uses canonical embeddings of suitable (i.e. very affine) open subsets of X into tori. If K is trivially valued, then it is not possible to extract enough local information about X^{an} from tropicalizations of embeddings of open subsets of X into tori (cf. Example 3.3.1). However, replacing tori by affine space fixes this problem (cf. Lemma 3.3.2) and allows us to make a natural generalization of Gubler's approach to the case where the base field is trivially valued (cf. Section 3.3). Furthermore, if the variety X has enough toric embeddings (cf. 3.2.32), we can use global embeddings into toric varieties to define differential forms on X^{an}. In particular, if X is projective and normal, then any differential form on X^{an} is globally given by one differential form on a tropical space (cf. Corollary 3.2.54). This global definition is helpful in concrete examples and will hopefully be useful in future studies of the cohomology of these forms.

Superforms were introduced by Lagerberg in [Lag12]. They are analogues on $\mathbb{R}^r$ of real-valued (p,q)-differential forms on complex manifolds. Their restrictions to supports of polyhedral complexes were first studied systematically in [CLD12] (see also [Gub13a]). As introduced in [JSS15] the definitions of superforms on supports of polyhedral complexes in $\mathbb{R}^r$ can be generalized to supports of polyhedral complexes in $\mathbb{T}^r = [-\infty, \infty)^r$ and to spaces which are locally modeled on such spaces. These spaces will be called polyhedral spaces (cf. Definition 2.1.54). For a polyhedral space X, we obtain a sheaf of bigraded bidifferential algebras $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$ on X.

In Chapter 2 we give the definitions of $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$ and study its cohomology. The fundamental result is:

Theorem 3 (d'-Poincaré lemma for polyhedral complexes)**.** Let X be a polyhedral subspace in $\mathbb{R}^r$ and $\Omega \subset X$ a polyhedrally star shaped open subset with center z. Let $\alpha \in \mathcal{A}^{p,q}(\Omega)$ with $p > 0$ and $d'\alpha = 0$. Then there exists $\beta \in \mathcal{A}^{p-1,q}(\Omega)$ such that $d'\beta = \alpha$.

Theorem 3 was proven by the author in [Jel16]. The proof is inspired by the proof of the classical Poincaré lemma. However, due to the very nature of Lagerberg's superforms the natural pullback morphism is only defined along affine maps. An important new tool is the introduction of a pullback of superforms along C^∞ maps. Here the direct definition fails to commute with the differential operator d' (cf. Remark 2.2.4). We thus need a different kind of pullback which commutes with d' (cf. Definition 2.2.6).

For polyhedral spaces, Theorem 3 has the following consequences:

Theorem 4 (Poincaré lemma for polyhedral spaces)**.** Let X be a polyhedral space and $U \subset X$ an open subset. Let $\alpha \in \mathcal{A}^{p,q}(U)$ with $p > 0$ and $d'\alpha = 0$. Then for every $x \in U$ there exists an open subset $V \subset X$ with $x \in V$ and a superform $\beta \in \mathcal{A}^{p-1,q}(V)$ such that $d'\beta = \alpha|_V$.

Note that, conversely to the case of polyhedral subspaces of $\mathbb{R}^r$, Theorem 4 does not give any acyclic domains, since the set V can depend on the form α. If we put some regularity assumptions on X however, we can obtain acyclic domains. (For the definition of regular at infinity cf. Definition 2.1.54 and for the definition of basic open cf. Definition 2.2.23).

Theorem 5. If X is regular at infinity and U is basic open, we can choose $V = U$ in Theorem 4. So any closed form on U is exact.

The first part of the Theorem was derived from Theorem 3 by Kristin Shaw, Jascha Smacka and myself in [JSS15]. Theorem 5 is also contained there, though not explicitly stated. It is a consequence of [JSS15, Theorem 3.18 & Proposition 3.10]. We will however give a direct proof here (cf. Theorem 2.2.27).

Another important property of the cohomology of differential forms on complex manifolds is Poincaré duality. For superforms, if we make further assumptions on the polyhedral space we indeed obtain canonical integration of (n,n)-forms and Stokes' theorem. Therefore we obtain a natural pairing on cohomology and the associated map

$$\mathrm{PD} : H^{p,q}_{d'}(X) \to H^{n-p,n-q}_{d',c}(X)^*,$$

which we call the Poincaré duality map. For this to be an isomorphism we have to put additional conditions on the polyhedral space X. Just the map PD being defined is not enough, as can be seen in Example 2.2.40. We require X to be locally modeled on the Bergman fan of a matroid. In that case we will call X a tropical manifold. We have the following Theorem:

Theorem 6 (Poincaré duality for tropical manifolds)**.** Let X be an n-dimensional tropical manifold. Then the Poincaré duality map is an isomorphism for all p, q.

Theorem 6 was also shown by Kristin Shaw, Jascha Smacka and the author in [JSS15]. The key tools are the characterization of matroidal fans via tropical modifications (cf. Construction 2.2.42), which is due to Shaw [Sha13, Proposition 2.25], and the invariance of cohomology under closed tropical modifications (cf. Definition 2.2.43). We then show how to control the change in cohomology produced by open tropical modifications to obtain the result for matroidal fans. To pass to general tropical manifolds, we use a sheaf theoretic argument.

We will also consider a different approach to a theory of $(1,1)$-forms in Berkovich geometry by Boucksom, Favre and Jonsson [BFJ16, BFJ15], which they used to solve a non-archimedean Monge–Ampère equation. Their approach uses line bundles, models and model metrics instead of forms in the sense of [CLD12, Gub13a]. We extend one of their statements, the dd^c-lemma. This was shown in [BFJ16, Theorem 4.3] for smooth projective varieties over a discretely valued field of residue characteristic zero. We work over an algebraically closed, complete field with valued group $\mathbb{Q}$ and require the variety to be merely normal and proper. We do not put any assumption on the residue characteristic. For definitions of the terms in the statement, cf. Chapter 4.

Theorem 7 (dd^c-lemma)**.** Let X be a normal and proper variety. Then the sequence

$$0 \to \mathbb{R} \to \mathcal{D}(X)_{\mathbb{R}} \stackrel{dd^c}{\to} \mathcal{Z}^{1,1}(X) \to N^1(X) \to 0$$

is exact.

We now outline the organization of this thesis. In Chapter 2 we consider superforms on polyhedral spaces and their cohomology. In Section 2.1 we recall the definition of superforms on $\mathbb{R}^r$ due to Lagerberg [Lag12] and their restriction to supports of polyhedral complexes in $\mathbb{R}^r$ (cf. [CLD12, Gub13a]). We then give the extension of these forms to supports of polyhedral complexes in $\mathbb{T}^r$ and polyhedral spaces as in [JSS15].

In Section 2.2 we study the cohomology defined by these forms. We first prove the Poincaré lemma for polyhedral complexes in $\mathbb{R}^r$ as in [Jel16]. As said above, the crucial tool is the introduction of a pullback along C^∞ maps that still commutes with the differential operator d'. This enables us to prove a homotopy formula (Theorem 2.2.9). We can then use the pullback along the contraction to construct preimages for closed forms (cf. proof of Theorem 2.2.15). Then we consider consequences of this result for polyhedral spaces, as in [JSS15, Section 3.2]. Afterwards we show finite dimensionality of the cohomology for polyhedral subspaces. These results generalize the results for polyhedral complexes in [Jel16, Section 3.3 & 3.4]. The last subsection is devoted to the proof of Poincaré duality for tropical manifolds as in [JSS15].

In Chapter 3 we consider differential forms on Berkovich analytic spaces. For the case where the analytic space is the analytification X^{an} of an algebraic variety X, we recall Gubler's approach. It uses tropicalization of canonical embeddings of very affine open subsets U into tori $\mathbb{G}_m^r$ and superforms on polyhedral complexes in $\mathbb{R}^r$. We choose to work with this approach instead of the one by Chambert–Loir and Ducros, which works for general analytic space, for a variety of reasons. First of all the theory uses only analytifications of algebraic varieties and maps between algebraic varieties. This makes the presentation both easier and shorter. Another reason is that our studies in Section 2.2 are using techniques and properties from tropical geometry, and this is easier to utilize in Gubler's approach. A third reason is the introduction of new approaches in Subsection 3.2.2. One of our approaches uses tropicalization of embeddings of affine open subsets U into $\mathbb{A}^r$ and superforms on polyhedral complexes in $\mathbb{T}^r$. This approach is the most general, since it also works over trivially valued fields, as we explain in Section 3.3 and we do not make any extra assumptions on the variety. Another approach uses tropicalization of embeddings of X into toric varieties and superforms on polyhedral spaces. This approach produces a lot more global charts (cf. Corollary 3.2.54), which the author believes to be useful in the future study of the cohomology of differential forms. While the author believes that the approach which uses embeddings into $\mathbb{A}^r$ can be made work for general analytic spaces, we can not expect to be able to globally embed general analytic spaces into analytifications of toric varieties. This is another reason to work in the algebraic setup. We show that all approaches yield the same theory of real valued differential forms in Subsection 3.2.3.

In Section 3.4 we study the cohomology defined by differential forms on Berkovich spaces. We first prove the Poincaré lemma. Afterwards we again show finiteness results. These are tied to existence of skeleta for the analytic space.

In Chapter 4 we show a version of the dd^c-lemma in another approach to forms on Berkovich spaces, namely the one taken by Boucksom, Favre and Jonsson. Theorem 4.2.7 is an extension of [BFJ16, Theorem 4.3].

We now explain how this thesis intersects with the papers [Jel16] and [JSS15]. In Subsections 2.1.2 to 2.1.4 we follow the presentation of [JSS15] very closely. The exception are some extensions of the theory in $\mathbb{R}^r$ to polyhedral subspaces in $\mathbb{T}^r$ resp. polyhedral spaces which were not needed there, thus were not considered by the authors at that time. Subsection 2.2.1 is precisely [Jel16, Section 2] with some changes in notations, to fit with the rest of the thesis. Subsection 2.2.2 is then again a part of [JSS15]. We give a new and direct proof for Theorem 2.2.27, but the statement is already know from [JSS15, Proposition 3.10 & Theorem 3.18], though not explicitly stated. Also the proof of Proposition 2.2.25 can be shortened here a little bit, because one has Theorem 2.2.9 at hand. Subsection 2.2.3 contains the logical extension of [Jel16, Section 3.2 - Section 3.4] to the world of polyhedral subspaces in $\mathbb{T}^r \times \mathbb{R}^s$, when only the case of subspaces of $\mathbb{R}^s$ was treated before. The techniques are completely analogous to the ones used in [Jel16]. Subsection 2.2.4 is directly from [JSS15, Subsection 4.2]. Subsection 3.4.1 and Theorem 3.4.9 are again results from [Jel16]. Note that in this thesis we choose to work with $\mathbb{A}$-tropical charts, to also include the case where our field in trivially valued.

Acknowledgements. There are many people whom I would like to thank for helping me do my research and writing this thesis. There is first of all my advisor, Klaus Künnemann, who suggested this topic to me and was helpful in general, especially at times when I was not making progress as I had hoped. Also I would like to thank my secondary advisor Walter Gubler, who was open for questions all the time and invited me to an Oberwolfach Workshop.

There are a lot more people I want to thank here, namely:

- José-Ignacio Burgos Gil and Florent Martin for many helpful discussions.
- Simon Flossmann for also organizing the wonderful workshop 'Student's workshop on Tropical and Non-Archimedian Geometry' and for inviting many guests from tropical geometry to give talks in Regensburg.
- Johann Haas, who, despite not being directly on topic, was always open for discussion and thus helped me produce a lot of new ideas and get a better understanding of things I already knew.
- Kristin Shaw for coming to Regensburg and giving the series of lectures which inspired the paper [JSS15] and for her efforts during the writing of this paper.
- Jascha Smacka for his ideas and contributions to the paper [JSS15].
- Philipp Vollmer always telling me about good conferences to go to and organizing the wonderful workshop 'Student's workshop on Tropical and Non-Archimedian Geometry' at Universität Regensburg in August 2015.
- Veronika Wanner for proofreading drafts of this work and asking very good questions about it.
- Andreas Weber for being a very helpful and fun office colleague, especially showing me all the bureaucratic stuff in the beginning of my PhD.
- The anonymous referee of the paper [Jel16], who provided a very careful review and helpful comments.

The last thank you goes to the collaborative research center "SFB 1085: Higher Invariants" for both its financial support as well as the great activities and talks enabled by it.

Chapter 2

Superforms and their cohomology

In this chapter we consider superforms on polyhedral spaces. In Section 2.1 we give the definitions and prove basic properties such as functoriality, integration and Stokes' theorem. In Section 2.2 we study the cohomology defined by these superforms. We prove versions of the Poincaré lemma and show finite dimensionality and Poincaré duality for the cohomology under suitable conditions.

2.1 Superforms

The goal of this section is to introduce superforms on polyhedral spaces. Polyhedral spaces are spaces which are equipped with an atlas to supports of polyhedral complexes. Superforms are analogues of differential forms, which take into account both the smooth structure of the polyhedra as well as the linear structures of the polyhedra. We recall the definition of superforms on open subsets of $\mathbb{R}^r$ in Subsection 2.1.1 and give the extension to open subsets of a partial compactification, $\mathbb{T}^r = [-\infty, \infty)^r$, in Subsection 2.1.2. We then give the definition of these forms on supports of polyhedral complexes. In Subsection 2.1.4, we consider polyhedral spaces.

Superforms were originally introduced by Lagerberg in [Lag12] for open subsets of $\mathbb{R}^r$. Restrictions of these forms to supports of polyhedral complexes were introduced by Chambert-Loir and Ducros in [CLD12] (see also [Gub13a]). The extensions to subspaces of $\mathbb{T}^r$ and polyhedral spaces were introduced in [JSS15].

2.1.1 Lagerberg's Superforms

We recall the definitions and basic properties of superforms, as introduced by Lagerberg in [Lag12]. These are bigraded differential forms, which are analogues on $\mathbb{R}^r$ of (p, q)-differential forms on complex manifolds.

Definition 2.1.1. i) For an open subset $U \subset \mathbb{R}^r$ denote by $\mathcal{A}^p(U)$ the space of smooth real differential forms of degree p. The space of *superforms of bidegree* (p, q) on U is defined as

$$\mathcal{A}^{p,q}(U) := \mathcal{A}^p(U) \otimes_{C^\infty(U)} \mathcal{A}^q(U) = \mathcal{A}^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} = C^\infty(U) \otimes_{\mathbb{R}} \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*}.$$

If we choose a basis $x_1, \dots, x_r$ of $\mathbb{R}^r$ we can formally write a superform $\alpha \in \mathcal{A}^{p,q}(U)$ as

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J,$$

where $I = \{i_1, \dots i_p\}$ and $J = \{j_1, \dots j_q\}$ are ordered subsets of $\{1, \dots, r\}$, $\alpha_{IJ} \in C^\infty(U)$ are smooth functions and

$$d'x_I \wedge d''x_J := (dx_{i_1} \wedge \dots \wedge dx_{i_p}) \otimes_{\mathbb{R}} (dx_{j_1} \wedge \dots \wedge dx_{j_q}). \tag{2.1}$$

ii) There is a wedge product

$$\begin{aligned} \mathcal{A}^{p,q}(U) \times \mathcal{A}^{p',q'}(U) &\to \mathcal{A}^{p+p',q+q'}(U) \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta, \end{aligned}$$

which is, up to sign, induced by the usual wedge product and is in coordinates given by

$$\begin{aligned} (\alpha_{KL} d'x_K \wedge d''x_L) \wedge (\beta_{K'L'} d'x_{K'} \wedge d''x_{L'}) :=& \alpha_{KL}\beta_{K'L'} d'x_K \wedge d''x_L \wedge d'x_{K'} \wedge d''x_{L'} \\ :=& (-1)^{p'q} \alpha_{KL}\beta_{K'L'} d'x_K \wedge d'x_{K'} \wedge d''x_L \wedge d''x_{L'}. \end{aligned}$$

Note that this fits with (2.1) in the sense that both meanings of $d'x_I \wedge d''x_J$ agree.

iii) There is a *differential operator*

$$d' \colon \mathcal{A}^{p,q}(U) = \mathcal{A}^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} \to \mathcal{A}^{p+1}(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} = \mathcal{A}^{p+1,q}(U)$$

which is given by $D \otimes \mathrm{id}$, where D is the usual exterior derivative. We also have $\mathcal{A}^{p,q}(U) = \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^q(U)$ and can take the derivative in the second component. We put a sign on this operator and define $d'' := (-1)^p \mathrm{id} \otimes D$. In coordinates we have

$$d'\left(\sum_{IJ} \alpha_{IJ} d'x_I \wedge d''x_J\right) = \sum_{IJ}\sum_{i=1}^r \frac{\partial \alpha_{IJ}}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J$$

and

$$\begin{aligned} d''\left(\sum_{IJ} \alpha_{IJ} d'x_I \wedge d''x_J\right) &= (-1)^p \sum_{IJ}\sum_{i=1}^r \frac{\partial \alpha_{IJ}}{\partial x_i} d'x_I \wedge d''x_i \wedge d''x_J \\ &= \sum_{IJ}\sum_{i=1}^r \frac{\partial \alpha_{IJ}}{\partial x_i} d''x_i \wedge d'x_I \wedge d''x_J. \end{aligned}$$

We further define $d := d' + d''$. The sign in d'' is such that d' and d'' anti-commute and hence d is a differential.

Remark 2.1.2. For all p, q the functor

$$U \mapsto \mathcal{A}^{p,q}(U)$$

defines a sheaf on $\mathbb{R}^r$. We obtain a sheaf of bigraded differential algebras $(\mathcal{A}^{\bullet,\bullet}, d', d'')$ on $\mathbb{R}^r$. Since our convention is that d' and d'' anti-commute, we also obtain the sheaf of graded differential algebras $(\mathcal{A}^{\bullet}, d)$ where $d = d' + d''$ and $\mathcal{A}^k := \bigoplus_{p+q=k} \mathcal{A}^{p,q}$. We further have $\mathcal{A}^{p,q} = 0$ if $\max(p, q) > r$.

Note further that $\mathcal{A}^{0,0}(U) = C^\infty(U)$.

Remark 2.1.3. It follows from the usual Poincaré lemma that columns and rows of the double complex $(\mathcal{A}^{\bullet,\bullet}, d', d'')$ of sheaves are exact in positive degrees [Lag12, Proposition 2.4]. This is not true for the total complex (cf. Remark 2.2.16).

Definition 2.1.4. Switching the factors in $\mathcal{A}^{p,q}(U) := \mathcal{A}^p(U) \otimes_{C^\infty(U)} \mathcal{A}^q(U)$ (or, equivalently, the last two factors in $C^\infty(U) \otimes_\mathbb{R} \Lambda^p \mathbb{R}^{r*} \otimes_\mathbb{R} \Lambda^q \mathbb{R}^{r*}$) induces an isomorphism $\mathcal{A}^{p,q}(U) \simeq \mathcal{A}^{q,p}(U)$. We define

$$\begin{aligned} J \colon \mathcal{A}^{p,q}(U) &\to \mathcal{A}^{q,p}(U) \\ \alpha = \sum_{IJ} \alpha_{IJ} d'x_I \wedge d''x_J &\mapsto \sum_{IJ} \alpha_{IJ} d''x_I \wedge d'x_J = (-1)^{pq} \sum_{IJ} \alpha_{IJ} d'x_J \wedge d''x_I. \end{aligned}$$

Remark 2.1.5. Let $F \colon \mathbb{R}^{r'} \to \mathbb{R}^r$ be an affine map and $U' \subset \mathbb{R}^{r'}$ and $U \subset \mathbb{R}^r$ open subsets such that $F(U') \subset U$. Then there is a well defined *pullback morphism* $F^* \colon \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p,q}(U')$ that commutes with the differentials d', d'' and d, the operator J and the wedge product.

Definition 2.1.6. Let U be an open subset of $\mathbb{R}^r$ and $\alpha \in \mathcal{A}^{p,q}(U)$. The *support* of α is its support in the sense of sheaves, thus the set of points $x \in U$ which do not have a neighborhood U_x such that $\alpha|_{U_x} = 0$. It is denoted by $\mathrm{supp}(\alpha)$. A superform is said to have *compact support* if its support is a compact set. The space of (p, q)-superforms on U which have compact support is denoted by $\mathcal{A}_c^{p,q}(U)$.

Definition 2.1.7. Let $U \subset \mathbb{R}^r$ be an open subset and $\alpha \in \mathcal{A}_c^{r,r}(U)$. We choose a basis $x_1, \dots, x_r$ of $\mathbb{Z}^r$. We can write

$$\alpha = f_\alpha d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_r \wedge d''x_r.$$

Note that $f_\alpha \in C_c^\infty(U)$ is independent of the choice of the *integral* basis $x_1, \dots, x_r$. We then define

$$\int_U \alpha := \int_U f_\alpha,$$

where the integral on the right is taken with respect to the volume defined by the lattice $\mathbb{Z}^r \subset \mathbb{R}^r$.

Remark 2.1.8. This definition is written up a little differently from the ones in [CLD12, 1.3] and [Gub13a, 2.4] (which are themselves a little different from each other), but easily seen to be equivalent.

Proposition 2.1.9. *Let $F\colon \mathbb{R}^r \to \mathbb{R}^r$ be a linear map. Let U be an open subset of $\mathbb{R}^r$ and $\alpha \in \mathcal{A}_c^{r,r}(U)$. Then we have*

$$\int_{F^{-1}(U)} F^*(\alpha) = |\det F| \int_U \alpha.$$

Proof. [Lag12, Equation (2.3)] □

2.1.2 Superforms on tropical affine space

We now extend the definition of superforms from open subsets of $\mathbb{R}^r$ to open subsets of a partial compactification, namely $\mathbb{T}^r = [-\infty, \infty)^r$. We introduce an extension of affine maps to these spaces and a pullback of superforms along these maps. All this was introduced by Shaw, Smacka and the author in [JSS15].

Definition 2.1.10. Let $\mathbb{T} = [-\infty, \infty)$ and equip it with the topology of a half open interval. Then $\mathbb{T}^r$ is equipped with the product topology. We write $[r] := \{1, \dots, r\}$.

The *sedentarity* of a point $x = (x_1, \dots, x_r) \in \mathbb{T}^r$ is the subset $\operatorname{sed}(x) \subset [r]$ consisting of indices i such that $x_i = -\infty$.

For $I \subset [r]$ set

$$\mathbb{R}^r_I = \{x \in \mathbb{T}^r \mid \operatorname{sed}(x) = I\} \qquad \text{and}$$
$$\mathbb{T}^r_I = \{x \in \mathbb{T}^r \mid \operatorname{sed}(x) \supset I\}.$$

Clearly we have $\mathbb{R}^r_I \cong \mathbb{R}^{r-|I|}$ and $\mathbb{T}^r_I \cong \mathbb{T}^{r-|I|}$. As a convention throughout, for a subset $S \subset \mathbb{T}^r$ we denote $S_I := S \cap \mathbb{R}^r_I$. Note that this is slightly inconsistent with $\mathbb{T}^r_I$, but there we already have the notation $\mathbb{R}^r_I$ so this will not cause any confusion.

Moreover, for $J \subset I$ there is a canonical projection $\pi_{IJ}\colon \mathbb{R}^r_J \to \mathbb{R}^r_I$. Coordinate-wise the map π_{IJ} sends x_i to $-\infty$ if $i \in I$ and to x_i otherwise.

Definition 2.1.11. Let $U \subset \mathbb{T}^r$ be an open subset. A (p,q)-*superform* α on U is given by a collection of superforms $(\alpha_I)_{I \subset [r]}$ such that

i) $\alpha_I \in \mathcal{A}^{p,q}(U_I)$ for all I,

ii) for each point $x \in U \subset \mathbb{T}^r$ of sedentarity I, there exists a neighborhood U_x of x contained in U such that for each $J \subset I$ the projection satisfies $\pi_{IJ}(U_{x,J}) = U_{x,I}$ and $\pi^*_{IJ}(\alpha_I|_{U_{x,I}}) = \alpha_J|_{U_{x,J}}$.

We denote the space of (p,q)-superforms on U by $\mathcal{A}^{p,q}(U)$.

Condition *ii)* of Definition 2.1.11 will be referred to as the *condition of compatibility.* Suppose $U \subset \mathbb{T}^r$ is an open set whose points have a unique maximal sedentarity I and $\alpha \in \mathcal{A}^{p,q}(U)$. If for each $J \subset I$ we have $\pi^*_{IJ}\alpha_I = \alpha_J$, then we say that α *is determined by α_I on U*. Notice that the condition of compatibility implies that each $x \in U$ has an open neighborhood U_x such that $\alpha|_{U_x}$ is determined by $(\alpha|_{U_x})_{\operatorname{sed}(x)}$ on U_x.

Definition 2.1.12. Let $D \in \{d', d'', d\}$. For $U \subset \mathbb{T}^r$ an open subset and $\alpha = (\alpha_I)_I \in \mathcal{A}^{p,q}(U)$ a superform, we define $D\alpha$ to be given by the collection $(D\alpha_I)_I$.

We also define $J\alpha$ to be given by the collection $(J\alpha_I)_I$. If also $\beta = (\beta_I)_I \in \mathcal{A}^{p',q'}(U)$, then we define $\alpha \wedge \beta := (\alpha_I \wedge \beta_I)_I \in \mathcal{A}^{p+p',q+q'}(U)$.

Since all these constructions commute with pullback along the projections π_{IJ} we indeed obtain elements of $\mathcal{A}^{\bullet,\bullet}(U)$ again.

Lemma 2.1.13. *The functor*

$$U \mapsto \mathcal{A}^{p,q}(U)$$

defines a sheaf on $\mathbb{T}^r$*.*

Proof. This is a direct consequence of the corresponding statement for $\mathbb{R}^r$ and the fact that the condition of compatibility is local. □

Remark 2.1.14. Let U be an open subset of $\mathbb{T}^r$. An element of $\mathcal{A}^{0,0}(U)$ is precisely given by a function $f\colon U \to \mathbb{R}$ such that the restriction to each U_I for $I \subset [r]$ is smooth and the condition of compatibility is satisfied. We will thus call elements of $\mathcal{A}^{0,0}(U)$ *smooth functions* and sometimes write $C^\infty(U) := \mathcal{A}^{0,0}(U)$.

We will now show that the sheaf C^∞ has partitions of unity. In the next lemma we use upper indexing to avoid confusion with the notation for the sedentarity of sets.

Lemma 2.1.15. *Let* $U \subset \mathbb{T}^r$ *be an open subset and* $(U^l)_{l\in L}$ *an open cover of* U*. Then there exist a countable, locally finite cover* $(V^k)_{k\in K}$ *of* U*, a collection of non-negative smooth functions* $(f^k : V^k \to \mathbb{R})_{k\in K}$ *with compact support and a map* $s : K \to L$ *such that* $V^k \subset U^{s(k)}$ *for every* $k \in K$*, and* $\sum_{k\in K} f^k \equiv 1$*. Such a family is called a partition of unity subordinated to the cover* $(U^l)_{l\in L}$*.*

Proof. We first show that for any $x \in \mathbb{T}^r$ and any open neighborhood $x \in V$ there exists a non-negative function $f \in \mathcal{A}^{0,0}(\mathbb{T}^r)$ and a neighborhood $V' \subset V$ of x such that $f|_{V'} \equiv 1$ and $\operatorname{supp}(f) \subset V$ is compact. This is clear if $r = 1$. Otherwise, a basis of open neighborhoods of x is given by products of open sets in $\mathbb{T}^1$, thus we may assume V to be of that form. Then taking functions f^i on neighborhoods of x_i on $\mathbb{T}^1$ with the above property for every $i \in [r]$ and defining $f(x_1, \dots, x_r) = \prod f^i(x_i)$ gives the desired function.

The general theorem then follows from standard arguments, see for instance the proof in [War83, Theorem 1.11]. □

Corollary 2.1.16. *The sheaves* $\mathcal{A}^{p,q}$ *are fine sheaves.*

Proof. For $\mathcal{A}^{0,0}$ this is just Lemma 2.1.15. In general this follows from the fact that the sheaves $\mathcal{A}^{p,q}$ are $\mathcal{A}^{0,0}$–modules via the wedge product. □

Remark 2.1.17. Let $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ be an affine map and let M_F denote the matrix representing the linear part of F. Let I be the set of $i \in [r']$ such that the i-th column of M_F has only non-negative entries. Then F can be extended to a map

$$F\colon \left(\bigcup_{J \subset I} \mathbb{R}^{r'}_J \right) \to \mathbb{T}^r$$

by continuity, (equivalently, using the usual $-\infty$-conventions for arithmetic). The extended map is also denoted by F. Note that $\bigcup_{J\subset I} \mathbb{R}^{r'}_J \simeq \mathbb{T}^{|I|} \times \mathbb{R}^{r'-|I|}$.

Definition 2.1.18. Let $U' \subset \mathbb{T}^{r'}$ be an open subset, then a map $F\colon U' \to \mathbb{T}^r$, which is the restriction to U' of a map arising as above is called an *extended affine map*. Note that this only makes sense once we have $\operatorname{sed}(x) \subset I$ for all $x \in U'$, where I is defined as above. An extended affine map is called an *integral extended affine map*, if it is the extension of an integral affine map $\mathbb{R}^{r'} \to \mathbb{R}^r$, i.e. its linear part is induced by a map of the standard lattices $\mathbb{Z}^{r'} \to \mathbb{Z}^r$.

Definition 2.1.19. Let $U' \subset \mathbb{T}^{r'}$ be an open subset and $F\colon U' \to \mathbb{T}^r$ be an extended affine map. Let $U \subset \mathbb{T}^r$ be an open subset such that $F(U') \subset U$. Define

$$\begin{aligned} F\colon \{\text{sedentarities of points in } U'\} &\to \{S \subset [r]\} \\ I' &\mapsto \operatorname{sed}(F(x)) \text{ for some and then every } x \in \mathbb{R}^{r'}_{I'}. \end{aligned}$$

Note that this map respects inclusions. Then F induces an affine map $F_{I'}\colon \mathbb{R}^{r'}_{I'} \to \mathbb{R}^r_{F(I')}$ with $F_{I'}(U'_{I'}) \subset U_{F(I')}$ for all $I' \subset [r']$. The *pullback of the superform* $\alpha = (\alpha_I)_I$ *along* F is the collection of superforms $F^*(\alpha) := (F^*_{I'}(\alpha_{F(I')}))_{I'}$. The next lemma shows that this defines a superform on U' and thus we have a pullback map $F^*\colon \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p,q}(U')$.

Lemma 2.1.20. *The pullback of a (p,q)-superform α on $U \subset \mathbb{T}^r$ along an extended affine map $F\colon U' \to U$ is a (p,q)-superform on $U' \subset \mathbb{T}^{r'}$. Furthermore this pullback is functorial, commutes with the wedge product and the differential operators.*

Proof. We have to verify the condition of compatibility. For $J' \subset I'$ we have $F(J') \subset F(I')$ and $F_{I'} \circ \pi_{I',J'} = \pi_{F(I')F(J')} \circ F_{J'}$. Thus if $\alpha \in \mathcal{A}^{p,q}(U)$ is determined by $\alpha_{F(I')}$ on U_x, we have

$$\begin{aligned} \pi^*_{I'J'}(F^*(\alpha)_{I'}) &= \pi^*_{I'J'} F^*_{I'}(\alpha_{F(I')}) \\ &= F^*_{J'}(\pi^*_{F(I')F(J')}(\alpha_{F(I')})) \\ &= F^*_{J'}(\alpha_{F(J')}) \\ &= (F^*(\alpha))_{J'}, \end{aligned}$$

which shows that $F^*(\alpha)$ is determined by $F^*(\alpha)_{I'}$ on $F^{-1}(U_x)$. This shows the required compatibility. The rest of the statement follows directly from the fact that this is true for the pullback on open subsets of $\mathbb{R}^r$. ∎

2.1.3 Superforms on polyhedral subspaces

We now consider the restriction of superforms to polyhedral subspaces, which are supports of polyhedral complexes. We give the definition of pullbacks along extended affine maps between these spaces and also define a canonical integration of (n, n)-superforms over weighted $\mathbb{R}$-rational polyhedral complexes. For polyhedral complexes in $\mathbb{R}^r$ these were originally introduced by Chambert–Loir and Ducros in [CLD12]. Since we work more with tropical properties, our presentation is closer to the one in [Gub13a]. The extension to polyhedral complexes in $\mathbb{T}^r$ (and therefore also polyhedral complexes in $\mathbb{T}^r \times \mathbb{R}^s$) was given in [JSS15].

Definition 2.1.21. A *polyhedron* in $\mathbb{R}^r$ is a subset defined by a finite system of affine (non-strict) inequalities. A *face* of a polyhedron σ is a polyhedron which is obtained by turning some of the defining inequalities of σ into equalities. For conventions of convex geometry we follow [Gub13b, Appendix A].

A *polyhedron* in $\mathbb{T}^r \times \mathbb{R}^s$ is the closure of a polyhedron in $\mathbb{R}^r_I \times \mathbb{R}^s \cong \mathbb{R}^{r-|I|+s} \subset \mathbb{T}^r \times \mathbb{R}^s$ for some $I \subset [r]$. A *face* of a polyhedron σ in $\mathbb{T}^r \times \mathbb{R}^s$ is the closure of a face of $\sigma \cap \mathbb{R}^r_J$ for some $J \subset [r]$. A *polyhedral complex* $\mathcal{C}$ in $\mathbb{T}^r \times \mathbb{R}^s$ is a finite set of polyhedra in $\mathbb{T}^r \times \mathbb{R}^s$, satisfying the following properties:

i) For a polyhedron $\sigma \in \mathcal{C}$, if τ is a face of σ (denoted $\tau \prec \sigma$) we have $\tau \in \mathcal{C}$.

ii) For two polyhedra $\sigma, \tau \in \mathcal{C}$ the intersection $\sigma \cap \tau$ is a face of both σ and τ.

The maximal polyhedra, with respect to inclusion, are called *facets*. The *dimension* of a polyhedral complex is the maximal dimension among its polyhedra. The polyhedral complex is called *pure dimensional* if all maximal polyhedral have the same dimension. We denote by $\mathcal{C}_k$ the set of polyhedra of dimension k. The support of a polyhedral complex $\mathcal{C}$ is the union of all its polyhedra and is denoted by $|\mathcal{C}|$. If $X = |\mathcal{C}|$, then X is called a *polyhedral subspace* of $\mathbb{T}^r \times \mathbb{R}^s$ and $\mathcal{C}$ is called a *polyhedral structure* on X.

The relative interior of a polyhedron σ is denoted by $\mathring{\sigma}$. The *sedentarity of* σ is defined as the sedentarity of the points in $\mathring{\sigma}$ and denoted $\operatorname{sed}(\sigma)$. We use the notation $\mathcal{C}_I$ for the polyhedral complex in $\mathbb{R}^r_I \times \mathbb{R}^s$ obtained by intersecting all polyhedra of $\mathcal{C}$ with $\mathbb{R}^r_I \times \mathbb{R}^s$. Note that $|\mathcal{C}_I| = |\mathcal{C}|_I$.

A polyhedron σ spans an affine space $\mathbb{A}(\sigma) \subset \mathbb{R}^r_{\operatorname{sed}(\sigma)} \times \mathbb{R}^s$ and we denote by $\mathbb{L}(\sigma)$ the corresponding linear space.

Remark 2.1.22. By a *polyhedral subspace* we always mean a polyhedral subspace of some $\mathbb{T}^r \times \mathbb{R}^s$. Many of our constructions will concern open subsets of polyhedral subspaces. Since $\mathbb{T}^r \times \mathbb{R}^s$ is an open subset of $\mathbb{T}^{r+s}$, we can then always assume that these polyhedral subspaces are subspaces of $\mathbb{T}^r$. Note also that $\mathbb{T}^r \times \mathbb{R}^s$ is indeed isomorphic to the polyhedral subspace $\{(y_1, \ldots, y_r, x_1, -x_1, \ldots, x_n, -x_n) \in \mathbb{T}^{r+2s} | y_i \in \mathbb{T}, x_i \in \mathbb{R}\}$ of $\mathbb{T}^{r+2s}$.

Definition 2.1.23. A polyhedron $\sigma \subset \mathbb{R}^r$ is called $\mathbb{R}$-rational if it can be defined by equations of the form $\langle . , v \rangle \geq c$ with v having integer coefficients and $c \in \mathbb{R}$. A polyhedron $\sigma \subset \mathbb{T}^r \times \mathbb{R}^s$ is called $\mathbb{R}$-rational if it is the closure of an $\mathbb{R}$-rational polyhedron in some $\mathbb{R}^r_I \times \mathbb{R}^s$. A polyhedral complex $\mathcal{C}$ in $\mathbb{T}^r \times \mathbb{R}^s$ is called $\mathbb{R}$-*rational* if all it polyhedra are $\mathbb{R}$-rational polyhedra. Then for any polyhedron σ there is a canonical lattice of full rank $\mathbb{Z}(\sigma) \subset \mathbb{L}(\sigma)$. A polyhedral subspace X is called $\mathbb{R}$-*rational* if it is the support of an $\mathbb{R}$-rational polyhedral complex $\mathcal{C}$. In this case we will call $\mathcal{C}$ an $\mathbb{R}$-*rational polyhedral structure* on X.

Remark 2.1.24. Let X be an $\mathbb{R}$-rational polyhedral subspace in $\mathbb{R}^r$. Then its closure in $\mathbb{T}^r$ is $\mathbb{R}$-rational.

Definition 2.1.25. A *weighted polyhedral complex* $\mathcal{C}$ is a pure dimensional polyhedral complex equipped with integer valued weights on its top dimensional facets. For a facet σ we will always write m_σ for its weight. The *support* of a weighted polyhedral complex $\mathcal{C}$ of dimension n is defined as $|\,\mathcal{C}\,| = \bigcup_{\sigma \in \mathcal{C}_n : m_\sigma \neq 0} \sigma$.

A *weighted polyhedral subspace* X is a pure dimensional polyhedral subspace $\mathcal{C}$ with $|\,\mathcal{C}\,| = X$ up to common refinement preserving the weights. We call a weighted polyhedral complex $\mathcal{C}$ representing X a *weighted polyhedral structure* on X.

Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{R}^r$. Let $\sigma \in \mathcal{C}_n$ and τ a codimension 1 face of σ. We denote by $\nu_{\tau,\sigma} \in \mathbb{Z}(\sigma)$ a representative of the unique generator of $\mathbb{Z}(\sigma)/\mathbb{Z}(\tau)$ which points inside of σ.

If $\mathcal{C}$ is a polyhedral complex in $\mathbb{T}^r \times \mathbb{R}^s$ we define $\nu_{\tau,\sigma}$ to be zero if τ and σ have different sedentarities and $\nu_{\tau,\sigma} := \nu_{\tau_I,\sigma_I}$ if τ and σ are both of sedentarity I.

Definition 2.1.26. Let $\mathcal{C}$ be an n-dimensional weighted $\mathbb{R}$-rational polyhedral complex and $\tau \in \mathcal{C}_{n-1}$. Then $\mathcal{C}$ is said to be *balanced at* τ if we have

$$\sum_{\sigma : \tau \prec \sigma} m_\sigma \nu_{\tau,\sigma} \in \mathbb{Z}(\tau). \tag{2.2}$$

$\mathcal{C}$ is said to fulfill the *balancing condition* (or *is balanced*) if it is balanced at every face of codimension 1.

We say that a weighted $\mathbb{R}$-rational polyhedral subspace X is *balanced* if it has a weighted $\mathbb{R}$-rational polyhedral structure which is balanced. If that is the case we also call X a *tropical cycle*. If the integer weights of X are all positive, then X is a *tropical variety*.

Remark 2.1.27. We will later see that if X is a tropical cycle then any weighted $\mathbb{R}$-rational polyhedral structure on X satisfies the balancing condition (cf. Remark 2.1.50).

Definition 2.1.28. Let X_1 and X_2 be n-dimensional weighted polyhedral subspaces of $\mathbb{T}^r \times \mathbb{R}^s$. Let $\mathcal{C}$ be a polyhedral complex such that there exist sets of weights m_1 and m_2 such that $(\mathcal{C}, m_i)$ is a weighted polyhedral structure on X_i for $i = 1, 2$. Then define $X_1 + X_2$ to be the weighted polyhedral subspace represented by $(\mathcal{C}, m_1 + m_2)$.

Remark 2.1.29. This defines the structure of an abelian group on the set of n-dimensional weighted polyhedral subspaces of $\mathbb{T}^r \times \mathbb{R}^s$, with the neutral element being the empty set. The n-dimensional weighted $\mathbb{R}$-rational polyhedral subspaces and tropical cycles of dimension n form subgroups. The tropical subvarieties form a submonoid.

Lemma 2.1.30. *Let $\mathcal{C}$ be a weighted $\mathbb{R}$-rational polyhedral complex in $\mathbb{T}^r \times \mathbb{R}^s$. Let $I_1, \ldots, I_k \subset [r]$ be the collection of sets such that there exist maximal polyhedra of these sedentarities in $\mathcal{C}$. Let $\mathcal{C}_i$ be the weighted $\mathbb{R}$-rational polyhedral complex in $\mathbb{R}^r_{I_i} \times \mathbb{R}^s$ whose maximal polyhedra are the maximal ones of $\mathcal{C}$ which are contained in $\mathbb{R}^r_{I_i} \times \mathbb{R}^s$ and whose weights are inherited. Write $X := |\,\mathcal{C}\,|$ and $X_i := |\,\mathcal{C}_i\,|$. Then $X = \bigcup \overline{X_i}$, $X = \sum \overline{X_i}$ and $\mathcal{C}$ is balanced if and only if all the $\mathcal{C}_i$ are balanced.*

Proof. Every maximal polyhedron of $\mathcal{C}$ is the closure of one which is contained in one of the $\mathcal{C}_i$. Since the support is the union of the maximal polyhedra, we have $X = \bigcup \overline{X_i}$. Since the weights are inherited and there is no cancellation we also have $X = \sum \overline{X_i}$. Let $\tau \in \mathcal{C}$ be a codimension 1 face. Let I denote its sedentarity. Note that by definition of $\nu_{\tau,\sigma}$ we can run the sum in (2.2) only over maximal faces of same sedentarity. Thus the sum is either empty or $I = I_i$ for some i, then it is precisely the condition for τ being balanced in $\mathcal{C}_i$. □

Definition 2.1.31. Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{T}^r \times \mathbb{R}^s$ and $\sigma \in \mathcal{C}$. Let $x \in \sigma$ and $I := \mathrm{sed}(x)$. Then $\sigma_I := \sigma \cap (\mathbb{R}^r_I \times \mathbb{R}^s)$ is a polyhedron in $\mathbb{R}^r_I \times \mathbb{R}^s$. Define *the tangent space of σ at x* to be $\mathbb{L}(\sigma, x) := \mathbb{L}(\sigma_I) \subset \mathbb{R}^r_I \times \mathbb{R}^s$, where $\mathbb{L}(\sigma_I)$ is the tangent space to σ_I at any point in its relative interior.

Let $\sigma \in \mathcal{C}$ and write $I := \mathrm{sed}(\sigma)$. The p-th *multitangent* and *multicotangent space* of $\mathcal{C}$ at σ are the vector subspaces respectively

$$\mathbf{F}_p(\sigma) = \sum_{\tau \in \mathcal{C}_I : \sigma \prec \tau} \Lambda^p \mathbb{L}(\tau) \subset \Lambda^p(\mathbb{R}^r_I \oplus \mathbb{R}^s) \qquad \text{and} \qquad \mathbf{F}^p(\sigma) = \left(\sum_{\tau \in \mathcal{C}_I : \sigma \prec \tau} \Lambda^p \mathbb{L}(\tau) \right)^* .$$

For $x \in |\,\mathcal{C}\,|$ we take the unique $\sigma_x \in \mathcal{C}$, such that $x \in \mathring{\sigma}_x$. Then we define

$$\mathbf{F}_p(x) := \mathbf{F}_p(\sigma_x) \qquad \text{and} \qquad \mathbf{F}^p(x) := \mathbf{F}^p(\sigma_x).$$

Note that these are invariant under subdivision, thus well defined for a polyhedral subspace X.

Definition 2.1.32. The evaluation of a (p,q)-superform α at a collection of vectors $v_1, \ldots, v_p, w_1, \ldots, w_q \in \mathbb{L}(\sigma, x)$ is denoted $\langle \alpha_I(x); v_1 \ldots, v_p, w_1, \ldots, w_q \rangle$.

Let $U \subset \mathbb{T}^r$ be an open subset and $\alpha = (\alpha_I)_I \in \mathcal{A}^{p,q}(U)$. For $v \in \mathbb{R}^r$ and $s \in [p+q]$ we define *the contraction of α with v in the s-th component* $\langle \alpha, v \rangle_s$ to be given by the collection $(\langle \alpha_I, \pi_{I,\emptyset}(v) \rangle_s)_I$. Here $\langle \alpha_I, \pi_{I,\emptyset}(v) \rangle_s \in \mathcal{A}^{p-1,q}(U_I)$ (resp. $\mathcal{A}^{p,q-1}$ if $s > p$) is the contraction of α_I with $\pi_{I,\emptyset}(v)$ in the sense of multilinear forms (cf. [Gub13a, 2.6]). We obtain a well defined form in $\mathcal{A}^{p-1,q}(U)$ (resp. $\mathcal{A}^{p,q-1}(U)$ if $s > p$).

Next we consider the restriction of bigraded superforms to polyhedral complexes in $\mathbb{T}^r$.

Definition 2.1.33. Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{T}^r \times \mathbb{R}^s$ and $\Omega \subset |\,\mathcal{C}\,|$ an open subset. A *superform of bidegree (p,q) on Ω* is given by an open subset $U \subset \mathbb{T}^{r+s}$ (or equivalently $\mathbb{T}^r \times \mathbb{R}^s$) such that $U \cap |\,\mathcal{C}\,| = \Omega$ and a superform $\alpha = (\alpha_I)_{I \subset [r]} \in \mathcal{A}^{p,q}(U)$.

Two such pairs (U, α) and (U', α') are equivalent if for any $\sigma \in \mathcal{C}$, any $x \in \Omega \cap \sigma$ of sedentarity I and all tangent vectors $v_1, \ldots, v_p, w_1, \ldots, w_q \in \mathbb{L}(\sigma, x)$ we have

$$\langle \alpha_I(x); v_1 \ldots, v_p, w_1, \ldots, w_q \rangle = \langle \alpha'_I(x); v_1 \ldots, v_p, w_1, \ldots, w_q \rangle.$$

Let $\mathcal{A}^{p,q}_{\mathcal{C}}(\Omega)$ denote the set of equivalence classes of pairs (U, α) as above.

Remark 2.1.34. It is easy to see that two forms $\alpha \in \mathcal{A}^{p,q}(U)$ and $\alpha' \in \mathcal{A}^{p,q}(U')$ are equivalent if and only if for each $x \in \Omega$, $v \in \mathbf{F}_p(x)$ and $w \in \mathbf{F}_q(x)$ we have $\langle \alpha(x), v, w \rangle = \langle \alpha'(x), v, w \rangle$. This shows that $\mathcal{A}^{p,q}(\Omega)$ is independent of the underlying polyhedral complex $\mathcal{C}$ and only depends on Ω as an open subset of $|\,\mathcal{C}\,|$, not on the polyhedral structure. Thus for $X = |\,\mathcal{C}\,|$ the definition $\mathcal{A}^{p,q}_X(\Omega) := \mathcal{A}^{p,q}_{\mathcal{C}}(\Omega)$ is well defined.

Remark 2.1.35. By definition we have that α and α' define the same superform on Ω if and only if for all $I \subset [r]$ the superforms α_I and α'_I define the same superform on Ω_I. Moreover, to determine if two superforms are equivalent when restricted to Ω, it is enough to consider only points in the relative interior of facets.

Definition 2.1.36. For $D \in \{d', d'', d\}$ and $\alpha \in \mathcal{A}^{p,q}(\Omega)$ which is given by $\alpha' \in \mathcal{A}^{p,q}(U)$ we define $D\alpha$ to be given by $D\alpha'$ and $J\alpha$ to be given by $J\alpha'$. If further $\beta \in \mathcal{A}^{p',q'}(\Omega)$ is given by $\beta' \in \mathcal{A}^{p',q'}(U')$ then we define $\alpha \wedge \beta$ to be given by $\alpha'|_{U \cap U'} \wedge \beta'|_{U \cap U'}$. These definitions are independent of the choices of α' resp. β'.

Remark 2.1.37. For a polyhedral space X we obtain a bigraded bidifferential algebra $(\mathcal{A}^{\bullet,\bullet}_X, d', d'')$ and the total graded differential algebra $(\mathcal{A}^{\bullet}, d)$ of sheaves on X. We will later prove that the rows and columns of the double complex $(\mathcal{A}^{\bullet,\bullet}_X, d', d'')$ are exact in positive degrees. We will also show some partial results for the total complex.

Lemma 2.1.38. *For a polyhedral subspace X, the functor*

$$\Omega \mapsto \mathcal{A}^{p,q}_X(\Omega)$$

is a sheaf on X. Furthermore, this sheaf is fine, hence soft and acyclic (with respect to both the functor of global sections and the functor of global sections with compact support).

Proof. We rely on the existence of partitions of unity. Let (Ω^l) be a collection of open sets and suppose that we have superforms $\alpha^l \in \mathcal{A}^{p,q}(\Omega^l)$ which agree on the intersections of the Ω^l's and are the restrictions to X of superforms $\beta^l \in \mathcal{A}^{p,q}(U^l)$. We take a partition of unity $(f^k)_{k \in K}$ subordinate to the cover $(U^l)_{l \in L}$. This exists by Lemma 2.1.15. By definition there is a map $s : K \to L$, so that if $s(k) = l$, then $\operatorname{supp} f^k \subset U^l$. Thus $\beta = \sum\limits_{l \in L} \sum\limits_{k : s(k) = l} f^k \beta^l$ defines a superform on the union $\bigcup_l \Omega^l$. Moreover, for fixed l_0 we have

$$\begin{aligned}
\beta|_{\Omega^{l_0}} &= \sum_{l \in L} \sum_{k : s(k) = l} f^k|_{\Omega^{l_0}} \beta^l|_{\Omega^{l_0}} = \sum_{l \in L} \sum_{k : s(k) = l} f^k|_{\Omega^{l_0}} \alpha^l|_{\Omega^{l_0}} \\
&= \sum_{l \in L} \sum_{k : s(k) = l} f^k|_{\Omega^{l_0}} \alpha^{l_0} = (\sum_{k \in K} f^k|_{\Omega^{l_0}}) \alpha^{l_0} = \alpha^{l_0}.
\end{aligned}$$

Therefore, the superform given by β restricted to $\bigcup_l \Omega^l$ gives the gluing of the superforms α^l above. The fact that $\mathcal{A}^{0,0}$ is fine follows from Lemma 2.1.15 and the sheaves $\mathcal{A}^{p,q}$ are also fine since they are $\mathcal{A}^{0,0}$-modules via the wedge product. Softness and acyclicity for global sections follows from [Wel80, Chapter II, Proposition 3.5 & Theorem 3.11] respectively and acyclicity for sections with compact support follows from [Ive86, III, Theorem 2.7]. □

Definition 2.1.39. The *support* of a superform α is its support in the sense of sheaves, thus it consists of the points x which do not have a neighborhood Ω_x such that $\alpha|_{\Omega_x} = 0$. The space of (p, q)-superforms with compact support on U is denoted $\mathcal{A}_c^{p,q}(U)$.

Lemma 2.1.40. *Let X be a polyhedral subspace of $\mathbb{T}^r \times \mathbb{R}^s$ and Ω an open subset. Let $\alpha = (\alpha_I)_I \in \mathcal{A}^{p,q}(\Omega)$. Then we have* $\operatorname{supp} \alpha = \bigcup_I \operatorname{supp} \alpha_I$.

Proof. Let $x \in \Omega_I$. If $x \notin \operatorname{supp}(\alpha_I)$, then there exists a neighborhood U of x in Ω_I such that $\alpha_I|_U = 0$. By the condition of compatibility, we find a neighborhood V of x in X such that $\alpha|_V$ is determined by $\alpha|_{V_I}$ on V and such that $V_I \subset U$. Then $\alpha|_V = 0$. This shows $\operatorname{supp}(\alpha) \subset \bigcup_I \operatorname{supp}(\alpha_I)$. Then other inclusion is obvious, thus we have equality. □

Definition 2.1.41. Let X' resp. X be polyhedral subspaces in $\mathbb{T}^{r'}$ resp. $\mathbb{T}^r$. Let Ω' be an open subsets of X'. An *extended affine map* $F\colon \Omega' \to \mathbb{T}^r$ is given by the restriction of an extended affine map $U \to \mathbb{T}^r$, for an open subset $U \subset \mathbb{T}^{r'}$, to Ω'.

Lemma 2.1.42. *Let X and X' be polyhedral subspaces and let $\Omega \subset X$ and $\Omega' \subset X'$ be open subsets. If $F\colon \Omega' \to \Omega$ is an extended affine map, then there exists a well defined pullback $F^* : \mathcal{A}^{p,q}(\Omega) \to \mathcal{A}^{p,q}(\Omega')$, which is induced by the pullback in Definition 2.1.19. Moreover, the pullback is functorial and commutes with the differential operators, the operator J and the wedge product.*

Proof. Let $\alpha \in \mathcal{A}^{p,q}(\Omega)$, then there exist open subsets $U' \subset \mathbb{T}^{r'}$ and $U \subset \mathbb{T}^r$ such that α is defined by some $\beta \in \mathcal{A}^{p,q}(U)$, $F(U') \subset U$ and $U' \cap X' = \Omega'$. Now the pullback $F^*(\beta) \in \mathcal{A}^{p,q}(U')$ defines a superform on Ω'. Set this to be $F^*(\alpha)$. To see that this is independent of the choice of β we suppose that γ is another superform on an open set defining α on Ω. After intersecting their respective domains of definition, we may assume that β and γ are defined on the same open set U. Since $\beta|_\Omega = \gamma|_\Omega$ we have that $\beta|_{\Omega_{F(I')}} = \gamma|_{\Omega'_{F(I')}}$ for all $I' \subset [r']$. Since the pullback via affine maps between vector spaces is well defined on polyhedral complexes [Gub13a, 3.2], we have $F^*_{I'}(\beta)|_{\Omega'_{I'}} = F^*_{I'}(\gamma)|_{\Omega'_{I'}}$ for all $I' \subset [r']$ and therefore $F^*(\beta)|_{\Omega'} = F^*(\gamma)|_{\Omega'}$. Thus the pullback is well defined. The last two statements are direct consequences of the definition of pullbacks of forms along extended affine maps and the fact that pullback by affine maps is functorial and commutes with the differential operators, the operator J and the wedge product. □

Definition 2.1.43. i) Let $\alpha \in \mathcal{A}_c^{n,n}(\sigma)$ for $\sigma \subset \mathbb{R}^r$ an $\mathbb{R}$-rational polyhedron of dimension n. Choose a basis $x_1, \ldots, x_n$ of $\mathbb{Z}(\sigma)$. Then α can be written as

$$f_\alpha d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_n \wedge d''x_n$$

for $f_\alpha \in \mathcal{A}_c^{0,0}(\sigma)$. Note that, since this is an integral basis, f_α is independent of the choice of $x_1, \ldots, x_n$. Then the *integral of α over σ* is

$$\int_\sigma \alpha := \int_\sigma f_\alpha,$$

where the integral on the right is taken with respect to the volume defined by the lattice $\mathbb{Z}(\sigma) \subset \mathbb{L}(\sigma)$.

ii) For $\beta \in \mathcal{A}_c^{n,n-1}(\sigma)$ the *boundary integral of β over $\partial\sigma$* is

$$\int_{\partial\sigma} \beta = \sum_{\tau \prec \sigma} \int_\tau \langle \beta; \nu_{\tau,\sigma} \rangle_n,$$

where the sum runs over the codimension 1 faces of σ and on the right hand side we use the integral of the $(n-1, n-1)$-form $\langle \beta; \nu_{\tau,\sigma} \rangle_n$ over the $(n-1)$-dimensional $\mathbb{R}$-rational polyhedron τ as defined in *i*).

Analogously for $\gamma \in \mathcal{A}_c^{n-1,n}(\sigma)$ we define

$$\int_{\partial\sigma} \gamma = \sum_{\tau \prec \sigma} \int_\tau \langle \gamma; \nu_{\tau,\sigma} \rangle_{2n-1}.$$

Lemma 2.1.44. *Let X be a polyhedral subspace of dimension n in $\mathbb{T}^r$ which is the closure of a polyhedral subspace X_I of dimension n in $\mathbb{R}_I^r$. If $\alpha \in \mathcal{A}_c^{p,q}(X)$ is such that $\max(p, q) = n$, then $\alpha_I \in \mathcal{A}^{p,q}(X_I)$ has compact support and for each $J \supsetneq I$ we have $\alpha_J = 0$.*

Proof. For $J \neq I$ we have $\dim X_J < n$. Thus $\operatorname{supp}(\alpha_J) = \emptyset$ for $J \neq I$. Then Lemma 2.1.40 shows $\operatorname{supp}(\alpha) = \operatorname{supp}(\alpha_I)$. □

Definition 2.1.45. Let X be an n-dimensional weighted $\mathbb{R}$-rational polyhedral subspace of $\mathbb{T}^r \times \mathbb{R}^s$. Let $\mathcal{C}$ be a weighted $\mathbb{R}$-rational polyhedral structure on X. For $\sigma \in \mathcal{C}$ we denote by σ' the unique polyhedron which lies in $\mathbb{R}_I^r \times \mathbb{R}^s$ for some $I \subset [r]$ and whose closure is σ. Note that by Lemma 2.1.44 we have $\alpha|_{\sigma'} \in \mathcal{A}_c^{n,n}(\sigma')$ for any $\alpha \in \mathcal{A}_c^{n,n}(X)$ and any $\sigma \in \mathcal{C}_n$.

i) We define for $\alpha \in \mathcal{A}_c^{n,n}(X)$ and $\sigma \in \mathcal{C}_n$:

$$\int_\sigma \alpha := \int_{\sigma'} \alpha|_{\sigma'}$$

ii) For $\beta \in \mathcal{A}_c^{n,n-1}(\sigma)$ the *boundary integral of* β *over* $\partial\sigma$ is

$$\int_{\partial\sigma} \beta = \sum_{\tau \prec \sigma} \int_{\tau} \langle \beta ; \nu_{\tau,\sigma} \rangle_n ,$$

where the sum runs over all codimension 1 faces τ of σ. Analogously for $\gamma \in \mathcal{A}_c^{n-1,n}(\sigma)$ we define

$$\int_{\partial\sigma} \gamma = \sum_{\tau \prec \sigma} \int_{\tau} \langle \gamma ; \nu_{\tau,\sigma} \rangle_{2n-1} .$$

iii) We now define for $\alpha \in \mathcal{A}_c^{n,n}$ the *integral of* α *over* $\mathcal{C}$ by

$$\int_{\mathcal{C}} \alpha := \sum_{\sigma \in \mathcal{C}_n} m_\sigma \int_{\sigma} \alpha .$$

This definition is invariant under weight preserving $\mathbb{R}$-rational subdivision. We therefore define

$$\int_X \alpha := \int_{\mathcal{C}} \alpha$$

and this is independent of our polyhedral structure $\mathcal{C}$.

iv) For $\beta \in \mathcal{A}_c^{n-1,n}(X)$ (resp. $\mathcal{A}_c^{n,n-1}(X)$), we also define *the integral of* β *over the boundary of* $\mathcal{C}$ by

$$\int_{\partial\mathcal{C}} \beta = \sum_{\sigma \in \mathcal{C}_n} m_\sigma \int_{\partial\sigma} \beta .$$

We also define

$$\int_{\partial X} \beta := \int_{\partial\mathcal{C}} \beta .$$

It follows from Stokes' theorem 2.1.49 that this is independent of the weighted $\mathbb{R}$-rational polyhedral structure $\mathcal{C}$ on X (cf. Remark 2.1.50).

v) Let $\alpha \in \mathcal{A}_c^{n,n}(\Omega)$ for Ω an open subset of an n-dimensional weighted $\mathbb{R}$-rational polyhedral subspace X. Then α can be extended by zero to a form $\alpha \in \mathcal{A}_c^{n,n}(X)$. Hence we can integrate superforms with compact support on open subsets of a polyhedral subspace over the entire polyhedral subspace.

Remark 2.1.46. Integration is linear both in the form that is integrated as well as the n-dimensional weighted $\mathbb{R}$-rational polyhedral space over which it is integrated.

Proposition 2.1.47. *Let $\sigma \subset \mathbb{R}^{r'}$ be an $\mathbb{R}$-rational polyhedron of dimension n. Let $F : \mathbb{R}^{r'} \to \mathbb{R}^r$ be an integral affine map such that* $\dim(F(\sigma)) = n$. *Let $\alpha \in \mathcal{A}_c^{n,n}(F(\sigma))$. Then we have*

$$\int_{F(\sigma)} \alpha = [\mathbb{Z}(\sigma) : \mathbb{Z}(F(\sigma)] \int_{\sigma} F^*\alpha.$$

Proof. We may assume that $r = r' = n$ and thus $\mathbb{L}(\sigma) = \mathbb{R}^{r'}$ and $\mathbb{L}(F(\sigma)) = \mathbb{R}^r$. Then this is true for the same reason as the case of open subsets, cf. [Lag12, Equation (2.3)]. Note therefore that $[\mathbb{Z}(\sigma) : \mathbb{Z}(F(\sigma))] = |\det(\mathbb{L}(F)|_{\mathbb{L}(\sigma)})|$ when the right hand side is calculated with respect to integral bases of $\mathbb{L}(\sigma)$ resp. $\mathbb{L}(F(\sigma))$. □

Remark 2.1.48. Let X be an n-dimensional weighted $\mathbb{R}$-rational polyhedral subspace of $\mathbb{T}^r \times \mathbb{R}^s$. Let $I_1, \ldots, I_r \subset [r]$ as in Lemma 2.1.30. It is a direct consequence of the definitions that for $\alpha \in \mathcal{A}_c^{p,q}(X)$ we have

$$\int_X \alpha = \sum_{i=1}^{k} \int_{X_{I_i}} \alpha|_{X_{I_i}},$$

where the X_{I_i} inherit the weights from X.

Theorem 2.1.49 (Stokes' theorem)**.** *Let $\mathcal{C}$ be an n-dimensional weighted $\mathbb{R}$-rational polyhedral complex, $\beta \in \mathcal{A}_c^{n,n-1}(|\,\mathcal{C}\,|)$ and $\gamma \in \mathcal{A}_c^{n-1,n}(|\,\mathcal{C}\,|)$. Then we have*

$$\int_{\partial \mathcal{C}} \beta = \int_{\mathcal{C}} d''\beta \qquad \text{and} \qquad \int_{\partial \mathcal{C}} \gamma = \int_{\mathcal{C}} d'\gamma. \tag{2.3}$$

We further have that $\mathcal{C}$ is balanced if and only if one (and then all) of the four terms in 2.3 vanishes for all $\beta \in \mathcal{A}_c^{n,n-1}(|\,\mathcal{C}\,|)$ resp. $\gamma \in \mathcal{A}_c^{n-1,n}(|\,\mathcal{C}\,|)$.

Proof. This follows from [Gub13a, Proposition 3.5] and [Gub13a, Proposition 3.8]. □

Remark 2.1.50. Note that for $\beta \in \mathcal{A}_c^{n,n-1}(X)$ the term $\int_X d''\beta$ does not depend on the polyhedral structure on a given weighted $\mathbb{R}$-rational polyhedral subspace and thus neither does $\int_{\partial X} \beta$. Further, if there exists a balanced weighted $\mathbb{R}$-rational polyhedral structure on X, then $0 = \int_X d''\beta = \int_{\mathcal{C}} d''\beta$ for all $\beta \in \mathcal{A}^{n,n-1}(X)$ and any polyhedral structure $\mathcal{C}$ on X. Thus $\mathcal{C}$ is then balanced by Stokes's theorem 2.1.49.

Definition 2.1.51. Let X be an n-dimensional weighted $\mathbb{R}$-rational polyhedral subspace of $\mathbb{R}^{r'}$ and $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ an integral affine map. We want to define the push-forward F_*X as a weighted $\mathbb{R}$-rational polyhedral subspace of $\mathbb{R}^r$. For details we refer to [AR10, §7]. We find a weighted rational polyhedral structure $\mathcal{C}$ on X such that

$$F_*(\mathcal{C}) := \{F(\tau) | \tau \prec \sigma, \sigma \in \mathcal{C}_n \text{ and } \dim(F(\sigma)) = n\}$$

is a polyhedral complex in $\mathbb{R}^r$. For $\rho \in F_*(\mathcal{C})_n$ we define

$$m_\rho := \sum_{\sigma \in \mathcal{C}_n, F(\sigma) = \rho} [\mathbb{Z}(\rho) : F(\mathbb{Z}(\sigma))] \cdot m_\sigma.$$

Then we define F_*X to be the weighted $\mathbb{R}$-rational polyhedral subspace defined by $F_*(\mathcal{C})$ with these weights.

Proposition 2.1.52 (Tropical Projection Formula). *Let X be a weighted $\mathbb{R}$-rational n-dimensional polyhedral subspace of $\mathbb{R}^{r'}$ and $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ an integral affine map. Let $\alpha \in \mathcal{A}_c^{n,n}(F_*X)$. Then we have*

$$\int_{F_*X} \alpha = \int_X F^*(\alpha).$$

Proof. This follows from the corresponding formula for polyhedra (Proposition 2.1.47), as is shown in [Gub13a, Propostion 3.10]. □

Corollary 2.1.53. *Let X be a tropical cycle in $\mathbb{R}^{r'}$ and $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ an integral affine map. Then F_*X is a tropical cycle. The same is true for tropical variety.*

Proof. This follows from the Projection formula and Stokes' theorem. For tropical variety we notice further that if the weights of X are all positive, so are the ones of F_*X. □

2.1.4 Superforms on polyhedral spaces

This subsection defines superforms on polyhedral spaces. These are spaces equipped with an atlas of charts to polyhedral subspaces in $\mathbb{T}^r$, with coordinate changes given by extended affine maps. Our definitions are generalizations of the definition of tropical spaces given in, for example, [Mik06, MZ14, BIMS15]. We do not require our polyhedral subspaces to be $\mathbb{R}$-rational, also the transition maps are required only to be extended affine maps, not integral affine. We also remove the finite type condition on the charts (cf. [MZ14, Definition 1.2]).

As in the case of polyhedral subspaces we again introduce a canonical integration for (n, n)-superforms on weighted $\mathbb{R}$-rational polyhedral spaces, show an analogue of Stokes' theorem and that integration of superforms can detect whether a weighted $\mathbb{R}$-rational polyhedral space is a tropical space (cf. Definition 2.1.55).

All this was introduced in [JSS15].

We also introduce the notion of morphism of polyhedral spaces and show that it induces a pullback of superforms. At the end we show that for a compact connected effective tropical space all smooth functions in the kernel of $d'd''$ are constant.

Definition 2.1.54. A *polyhedral space* X is a paracompact, second countable Hausdorff topological space with an atlas of charts $A = (\varphi_i : U_i \to \Omega_i \subset X_i)_{i\in I}$ such that:

i) The U_i are open subsets which cover X, the Ω_i are open subsets of X_i, which are polyhedral subspaces, and $\varphi_i : U_i \to \Omega_i$ is a homeomorphism for all i;

ii) For all $i, j \in I$ the transition map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to X_i$$

is an extended affine map.

As usual, we identify two atlases if their union is an atlas.

A polyhedral space X is *regular at infinity* if there is an atlas such that $X_i = \mathbb{T}^{r_i} \times Y_i \subset \mathbb{T}^{r_i} \times \mathbb{R}^{s_i}$, where Y_i is a polyhedral subspace of $\mathbb{R}^{s_i}$ for every chart φ_i. Throughout, if we require that our polyhedral space is regular at infinity, we will always assume that all charts we use have this property.

The *dimension* of X is the maximal dimension among polyhedra which intersect the Ω_i. The polyhedral space is *pure dimensional* if the dimension of the maximal, with respect to inclusion, polyhedra intersecting the open sets $\Omega_i \subset X_i$ is constant.

Definition 2.1.55. A polyhedral space is called $\mathbb{R}$-*rational* if all targets of its charts are $\mathbb{R}$-rational polyhedral subspaces and the transition maps are integral extended affine maps. It is called *weighted* if the targets of all of its charts are weighted polyhedral subspaces and the transition maps are weight preserving.

A weighted $\mathbb{R}$-rational polyhedral space is called *tropical space* if all targets of its charts are tropical cycles. It is called *effective tropical space* if the targets are tropical varieties.

Definition 2.1.56. Let X and Y be polyhedral spaces with atlases $A = (\varphi_i : U_i \to \Omega_i \subset X_i)_{i \in I}$ and $B = (\varphi_j : U_j \to \Omega_j \subset Y_j)_{j \in J}$. A continuous map $F \colon X \to Y$ is called a *morphism of polyhedral spaces* if for all $i \in I, j \in J$ we have that

$$\varphi_j \circ F \circ \varphi_i^{-1}$$

is an extended affine map.

If X and Y are $\mathbb{R}$-rational the morphism is called *integral* if these maps are integral extended affine maps.

Definition 2.1.57. Let X be a polyhedral space with atlas $A = (\varphi_i : U_i \to \Omega_i \subset X_i)_{i \in I}$. Define *the sheaf* $\mathcal{A}_X^{p,q}$ *of* (p,q)*-superforms on* X to be the gluing of the sheaves $\mathcal{A}_{U_i}^{p,q}$, which is the pullback of the sheaf $\mathcal{A}_{\Omega_i}^{p,q}$ via φ_i. The pullback of forms along the charts φ_i is well defined and functorial, so this gives a well defined sheaf of superforms on X. We again denote the sections with compact support by $\mathcal{A}_c^{p,q}(X)$.

Since the pullback commutes with the differentials d', d'', d, the operator J and the wedge product, these are well defined on $\mathcal{A}_X^{p,q}$.

Definition 2.1.58. Let X be a polyhedral space with atlas $A = (\varphi_i : U_i \to \Omega_i \subset X_i)_{i \in I}$. We define

$$\mathbf{F}_p(x) := \mathbf{F}_p(\varphi_i(x)) \qquad \text{and} \qquad \mathbf{F}^p(x) := \mathbf{F}^p(\varphi_i(x))$$

for some i such that $x \in U_i$. This is well defined since coordinate changes are invertible extended affine maps.

Remark 2.1.59. For a polyhedral space X we obtain a bigraded bidifferential algebra $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$ and the total graded differential algebra $(\mathcal{A}_X^{\bullet}, d)$ of sheaves on X. We will later prove that the rows and columns of the double complex $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$ are exact in positive degree. We will also show some partial results for the total complex.

The sheaves $\mathcal{A}^{p,q}$ are again fine, hence soft and acyclic.

Remark 2.1.60. Let $F\colon X \to Y$ be a morphism of polyhedral spaces. Then by using the pullback on the chart domains and gluing we obtain a pullback morphism

$$F^* : \mathcal{A}_Y^{p,q} \to f_* \mathcal{A}_X^{p,q} .$$

This commutes with the differentials d', d'', d, with the operator J and with the wedge product.

We will now extend the definition of integration from polyhedral subspaces to polyhedral spaces. We do this as it is done in the theory of manifolds, namely using partitions of unity.

Definition 2.1.61. Let X be an n-dimensional weighted $\mathbb{R}$-rational polyhedral space with atlas $(\varphi_i : U_i \to \Omega_i \subset X_i)_{i\in I}$. Let $\alpha \in \mathcal{A}_c^{n,n}(X)$ and $(f_j)_{j\in J}$ be a partition of unity with functions in $\mathcal{A}_c^{0,0}$ subordinate to the cover (U_i) as in Lemma 2.1.15. Then we have

$$\alpha = \sum_{j\in J} f_j \alpha,$$

which is a finite sum. Define $\alpha_j \in \mathcal{A}_c^{n,n}(\Omega_i)$ the superform corresponding to $f_j\alpha \in \mathcal{A}_c^{n,n}(U_i)$. As mentioned the superform α_j can be extended to a superform in $\mathcal{A}_c^{n,n}(X_i)$. Then the integral of α over X is

$$\int_X \alpha := \sum_{j\in J} \int_{X_i} \alpha_j,$$

with the integral on the right as defined in Definition 2.1.45.

We also define for $\beta \in \mathcal{A}_c^{n,n-1}(X)$ (resp. $\mathcal{A}_c^{n-1,n}(X)$) the *boundary integral* by

$$\int_{\partial X} \beta := \sum_{j\in J} \int_{\partial X_i} \beta_j.$$

We have the following lemma, which implies, by the same arguments which are used for integration of differential forms on manifolds, that the integral defined above is independent of the choice of partition of unity.

Lemma 2.1.62. *Let X and X' be weighted $\mathbb{R}$-rational polyhedral subspaces of dimension n and let $\Omega \subset X$ and $\Omega' \subset X'$ be open subsets. Let $F\colon \Omega' \to \Omega$ and $G : \Omega \to \Omega'$ be extended integral affine maps such that $F \circ G = \mathrm{id}_\Omega$, $G \circ F = \mathrm{id}_{\Omega'}$ and F and G preserve weights. Then for $\alpha \in \mathcal{A}_c^{n,n}(\Omega)$ we have $\int_X \alpha = \int_{X'} F^*\alpha$.*

Further, for $\beta \in \mathcal{A}_c^{n,n-1}(\Omega)$ (resp. $\beta \in \mathcal{A}_c^{n-1,n}(\Omega)$) we have $\int_{\partial X} \beta = \int_{\partial X'} F^*(\beta)$.

Proof. Using Remark 2.1.48 we may assume that X and X' are polyhedral subspaces of $\mathbb{R}^r$ resp. $\mathbb{R}^{r'}$. After translation we may assume that both Ω and Ω' contain zero and that F and G are linear maps. Replacing $\mathbb{R}^{r'}$ by the linear hull of Ω' and $\mathbb{R}^r$ by the linear hull of Ω we may assume that F and G are mutually inverse automorphisms of $\mathbb{Z}^r$. Replacing X by $F(X')$ we may further assume that $F \circ G = \mathrm{id}_{X'}$ and $G \circ F = \mathrm{id}_X$. Now we can assume $\Omega' = X'$ and $\Omega = X$. We now have $F_*(X') = X$ and thus the result follows directly from the tropical projection formula Proposition 2.1.52.

The second part follows from the first and Stokes theorem 2.1.49. □

Theorem 2.1.63 (Stokes' theorem for tropical spaces)**.** *Let X be a weighted $\mathbb{R}$-rational n-dimensional polyhedral space. Then for $\beta \in \mathcal{A}_c^{n,n-1}(X)$ and $\gamma \in \mathcal{A}_c^{n-1,n}(X)$ we have*

$$\int_{\partial X} \beta = \int_X d''\beta \qquad \text{and} \qquad \int_{\partial X} \gamma = \int_X d'\gamma. \tag{2.4}$$

Further, X is a tropical space if and only if one (and then all) of the four terms in (2.4) vanishes for all $\beta \in \mathcal{A}_c^{n,n-1}(X)$ resp. $\gamma \in \mathcal{A}_c^{n-1,n}(X)$.

Proof. Follows from Stokes' theorem 2.1.49. □

In the rest of this section we will state some basic properties of differential forms on polyhedral spaces which we will use later.

Lemma 2.1.64. *Let X be a polyhedral space. Let $\alpha \in \mathcal{A}^{0,k}(X)$ such that $d'\alpha = 0$. Then $d''\alpha = 0$ and $d\alpha = 0$.*

Proof. It is sufficient to check this in a chart, thus X being an open subset of the support of a polyhedral complex. There it is sufficient to check this after restriction to a polyhedron. Let $\sigma \in \mathcal{C}$ and let $v_1, \dots, v_r$ be a basis of $\mathbb{L}(\sigma)$. Then $\alpha|_\sigma = \sum_{|J|=k} \alpha_J d''v_J$ and $d'\alpha|_\sigma = 0$ if and only if $\frac{\partial \alpha_J}{\partial v_i} = 0$ for all i, J. But then also $d''\alpha|_\sigma = 0$. Since $d'\alpha = 0$ and $d''\alpha = 0$ we have $d\alpha = 0$. □

Lemma 2.1.65. *Let X be a polyhedral space. The operator $J : \mathcal{A}^{p,q} \to \mathcal{A}^{q,p}$ has the following properties:*

i) $J^2 = \mathrm{id}$

ii) $d'J = Jd''$

iii) $d''J = Jd'$

Proof. *i*) is obvious from the construction. *ii*) and *iii*) are shown in [CLD12, Lemme 1.2.10]. □

Proposition 2.1.66. *Let X be an effective tropical space. Let $f \in C^\infty(X)$ such that $d'd''f = 0$. If f has a local maximum at $x \in X$, then f is locally constant at x.*

Proof. We may reduce to the case where X is a tropical variety in $\mathbb{T}^r$ and f is defined on an open subset $\Omega \subset X$. By Lemma 2.1.30, X is then the union of closures of tropical varieties contained in some (possibly different) $\mathbb{T}_I^r$. By argueing seperately for each I we may assume that X is a closure of one of the $\mathbb{T}_I^r$. Since $\mathbb{T}_I^r \cong \mathbb{T}^{r-|I|}$ we may assume $I = \emptyset$. After shrinking Ω we may assume that f is determined by $f_{\mathrm{sed}(x)}$ on Ω and that x is a maximum of f on Ω. Since $X = \overline{X_\emptyset}$ there exists a point $y \in \Omega$ such that $\mathrm{sed}(y) = \emptyset$ and $\pi_{\mathrm{sed}(x)}(y) = x$. Since f is determined by $f_{\mathrm{sed}(x)}$ we have $f(y) = f(x)$. Thus y is also a maximum of f on Ω. Further, this yields that f is constant on a neighborhood Ω' of y, then f is constant on $\pi_{\mathrm{sed}(x)}^{-1}(\pi_{\mathrm{sed}(x)}(\Omega')) \cap \Omega$, which is an open neighborhood of x. Thus replacing y by x and Ω by $\Omega_\emptyset$ we may assume that X is a polyhedral subspace of $\mathbb{R}^r$.

Choose a $\mathbb{R}$-rational weighted polyhedral structure $\mathcal{C}$ on X, which then satisfies the balancing condition by Remark 2.1.50. Since $d'd''f = 0$ we have that the restriction of f to every polyhedron $\sigma \in \mathcal{C}$ is affine. After translation and replacing f by $f - f(x)$ we may assume that $x = 0$ and $f(0) = 0$. Let f be given by a smooth function $F \in C^\infty(U)$ for $U \subset \mathbb{R}^r$ an open subset. We may assume, after shrinking U and Ω that U is an open ball around the origin which only intersects polyhedra which contain the origin. Then f is on Ω also given by the total differential $DF(0)$. We show that for every maximal $\sigma \in \Omega$ we have $\sigma \subset \ker(DF)$, then we are done. Suppose there exists σ_0 such that $\sigma_0 \notin \ker(DF)$. Then, after possible subdivision, $\tau := \sigma_0 \cap \ker(DF)$ is a proper face of σ_0. We thus have $DF(\nu_{\tau,\sigma_0}) \neq 0$ and since f has a local maximum at 0 we have indeed $DF(\nu_{\tau,\sigma_0}) < 0$. Since $\tau \subset \ker(DF)$ the balancing condition tells us that

$$\sum_{\sigma : \tau \prec \sigma} m_\sigma DF(\nu_{\tau,\sigma}) = 0$$

which is a contradiction, since $m_\sigma > 0$ and $DF(\nu_{\tau,\sigma}) \leq 0$ for all σ and $DF(\nu_{\tau,\sigma_0}) < 0$. □

Corollary 2.1.67. *Let X be an effective tropical space, which is connected and compact. Let $f \in C^\infty(X)$ such that $d'd''f = 0$. Then f is constant.*

Proof. Since X is compact there exists a global maximum x. Then $f^{-1}(\{f(x)\})$ is both closed and open by Proposition 2.1.66 and thus equals X. □

2.2 Cohomology of superforms

In this section, we study the cohomology defined by superforms on polyhedral spaces. To do this, we prove in Subsection 2.2.1 a Poincaré lemma for polyhedral subspaces of $\mathbb{R}^r$. We give the consequences of this for polyhedral spaces in Subsection 2.2.2. Afterwards we show finiteness results for the cohomology in Subsection 2.2.3. In the last Subsection 2.2.4 we show that for tropical manifolds, which is a special class of polyhedral spaces, the cohomology of superforms satisfies Poincaré duality.

In Subsection 2.2.1 we show results from [Jel16]. Subsection 2.2.2 and 2.2.4 consist of results from [JSS15]. Subsection 2.2.3 gives the natural extension of the results on finite dimensionality for polyhedral subspaces of $\mathbb{R}^r$ from [Jel16] to polyhedral subspaces in $\mathbb{T}^r \times \mathbb{R}^s$.

Definition 2.2.1. Let X be a polyhedral space. The *Dolbeault cohomology of superforms* is defined as $H^{p,q}_{d'}(X) := H^q(\mathcal{A}^{p,\bullet}_X(X), d')$ and the *Dolbeault cohomology of superforms with compact support* is defined as $H^{p,q}_{d',c}(X) := H^q(\mathcal{A}^{p,\bullet}_{X,c}(X), d')$.

Lemma 2.2.2. *Let X be a polyhedral space. The operator J induces isomorphisms*

$$J : H^{p,q}_{d'}(X) \simeq H^{q,p}_{d''}(X) \qquad \textit{and} \qquad H^{p,q}_{d',c}(X) \simeq H^{q,p}_{d'',c}(X)$$

for all p, q.

Proof. This is a direct consequence of Lemma 2.1.65, which shows $J\ker(d') = \ker(d''J)$ and $J\operatorname{im}(d') = \operatorname{im}(Jd'')$. □

With this lemma in mind we will often only talk about the cohomology with respect to the operator d' and implicitly mean that analogous statements are always true for d''.

2.2.1 The Poincaré lemma for polyhedral subspaces in $\mathbb{R}^r$

In this subsection we prove a local exactness result for the complex of superforms on polyhedral subspaces in $\mathbb{R}^r$. The proof is a variant of the proof of the classical Poincaré lemma. The crucial tool we need to introduce is a pullback along a contraction of a star shaped set to its center. This map can not be affine. We therefore define a pullback of superforms along C^∞ maps (cf. Definition 2.2.6).

The contents of this subsection were published in [Jel16].

Lemma 2.2.3 (Chain Homotopy Lemma)**.** *Let X be a polyhedral subspace in $\mathbb{R}^r$ and $\Omega \subset X$ an open subset. Let $B = [0,1] \subset \mathbb{R}$ be the closed unit interval and for $i = 0, 1$*

$$\iota_i : \Omega \to \Omega \times \{i\} \subset \Omega \times B$$

the inclusions. Then for all $p \in \{0, \ldots, n+1\}$ and $q \in \{0, \ldots, n\}$ there exists a linear map

$$K' : \mathcal{A}^{p,q}(\Omega \times B) \to \mathcal{A}^{p-1,q}(\Omega), \tag{2.5}$$

such that

$$d'K' + K'd' = \iota_1^* - \iota_0^*. \tag{2.6}$$

Proof. The proof is a variant of the classical chain homotopy lemma for ordinary differential forms. Observe first that $X \times B$ is a polyhedral subspace in $\mathbb{R}^r \times \mathbb{R}$ and hence it makes sense to talk about superforms on $\Omega \times B$. Let $\alpha \in \mathcal{A}^{p,q}(\Omega \times B)$ be given by $\beta \in \mathcal{A}^{p,q}(V \times B')$ for some open set $V \subset \mathbb{R}^r$ and some open interval B' such that $B \subset B' \subset \mathbb{R}$. Let $x_1, \ldots, x_r$ be a basis of $\mathbb{R}^r$ and denote by t the coordinate of B. We write

$$\begin{aligned}\beta = &\sum_{|I|=p,|J|=q} a_{IJ} d'x_I \wedge d''x_J \\ &+ \sum_{|I|=p-1,|J|=q} b_{IJ} d't \wedge d'x_I \wedge d''x_J \\ &+ \sum_{|I|=p,|J|=q-1} e_{IJ} d'x_I \wedge d''t \wedge d''x_J \\ &+ \sum_{|I|=p-1,|J|=q-1} g_{IJ} d't \wedge d'x_I \wedge d''t \wedge d''x_J.\end{aligned} \tag{2.7}$$

Then we define

$$\begin{aligned} K' : \mathcal{A}^{p,q}(V \times B) &\to \mathcal{A}^{p-1,q}(V) \\ \beta &\mapsto \sum_{|I|=p-1,|J|=q} c_{IJ} d'x_I \wedge d''x_J \\ \text{with } c_{IJ}(x) :&= \int_0^1 b_{IJ}(x,t)dt. \end{aligned}$$

We show that this definition is independent of the choice of the basis $x_1, \dots, x_r$. Let therefore $y_1, \dots, y_r$ be another basis. First of all we notice that the decomposition into the four summands as in (2.7) is not affected by our base change. We further notice that

$$d'x_I \wedge d''x_J = \sum_{|I'|=|I|,|J'|=|J|} \lambda_{I,I'} \lambda_{J,J'} d'y_{I'} \wedge d''y_{J'},$$

where $\lambda_{I,I'}$ is the determinant of the $I \times I'$ minor of the base change matrix from $x_1, \dots, x_r$ to $y_1, \dots, y_r$ and similar for J and J'. Now we have

$$b_{IJ} d't \wedge d'x_I \wedge d''x_J = b_{IJ} \sum_{I',J'} \lambda_{I,I'} \lambda_{J,J'} d't \wedge d'y_{I'} \wedge d''y_{J'}$$

and this term is mapped under K' to

$$\begin{aligned} &\sum_{I',J'} \left(\int_0^1 \lambda_{I,I'} \lambda_{J,J'} b_{IJ} dt \right) d'y_{I'} \wedge d''y_{J'} \\ &= \left(\int_0^1 b_{IJ} dt \right) \sum_{I',J'} \lambda_{I,I'} \lambda_{J,J'} d'y_{I'} \wedge d''y_{J'} \\ &= \left(\int_0^1 b_{IJ} dt \right) d'x_I \wedge d''x_J, \end{aligned}$$

which shows the independence of the choice of the basis.

Given V and B' we have the diagram

$$\begin{array}{ccc} \mathcal{A}^{p,q}(V \times B') & \xrightarrow{K'} & \mathcal{A}^{p-1,q}(V) \\ \downarrow & & \downarrow \\ \mathcal{A}^{p,q}(\Omega \times B) & \dashrightarrow & \mathcal{A}^{p-1,q}(\Omega). \end{array}$$

To get a well defined map on the bottom that makes this diagram commutative, we fix a polyhedral structure $\mathcal{C}$ on X and we show that $\beta|_{\sigma \times B} = 0$ for all $\sigma \in \mathcal{C}$ implies $K'(\beta)|_\sigma = 0$ for all $\sigma \in \mathcal{C}$. Let therefore σ be a maximal polyhedron in $\mathcal{C}$ and $W = V \cap \sigma$.

It suffices to show that if $\beta|_{W\times B} = 0$, then $K'(\beta)|_W = 0$. By what we did above we may choose a basis as we like. Let therefore $x_1, \dots, x_m$ be a basis of $\mathbb{L}(\sigma)$ and $x_{m+1}, \dots, x_r$ a basis of a complement. Then from $\beta|_{W\times B} = 0$ we get $b_{IJ}|_{W\times B} = 0$ for all $I, J \subset \{1, \dots, m\}$. This means however that $c_{IJ}|_W = 0$ for all $I, J \subset \{1, \dots, m\}$. From that we get $K'(\beta)|_W = 0$. Hence setting $K'(\alpha) := K'(\beta)$ is independent of the choice of the form β by which α is given. It is also independent of the choice of V and B'. This gives a well defined map

$$K' : \mathcal{A}^{p,q}(\Omega \times B) \to \mathcal{A}^{p-1,q}(\Omega)$$

as required in (2.5). We will now show that (2.6) holds. It is enough to check that

$$d'K'\beta + K'd'\beta = \iota_1^*\beta - \iota_0^*\beta$$

holds for every $\beta \in \mathcal{A}^{p,q}(V \times B')$, where V is an open subset of $\mathbb{R}^r$ and B' is an open interval such that $B \subset B' \subset \mathbb{R}$. It suffices to check the following four cases:

i) $\beta = a_{IJ} d'x_I \wedge d''x_J$:
We have $K'(\beta) = 0$ and

$$\begin{aligned} K'(d'(\beta)) &= K'\left(\frac{\partial a_{IJ}}{\partial t} d't \wedge d'x_I \wedge d''x_J\right) \\ &\quad + \sum_{i=1}^r K'\left(\frac{\partial a_{IJ}}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J\right) \\ &= \left(\int_0^1 \frac{\partial a_{IJ}}{\partial t} dt\right) d'x_I \wedge d''x_J \\ &= (a_{IJ}(.,1) - a_{IJ}(.,0)) d'x_I \wedge d''x_J \\ &= \iota_1^*(\beta) - \iota_0^*(\beta). \end{aligned}$$

ii) $\beta = b_{IJ} d't \wedge d'x_I \wedge d''x_J$:
We have $\iota_1^*(\beta) = \iota_0^*(\beta) = 0$, since the pullback of $d't$ is zero. We further have

$$d'K'(\beta) = \sum_{i=1}^r \left(\int_0^1 \frac{\partial b_{IJ}}{\partial x_i} dt\right) d'x_i \wedge d'x_I \wedge d''x_J$$

and

$$\begin{aligned} K'd'(\beta) &= \sum_{i=1}^r K'\left(\frac{\partial b_{IJ}}{\partial x_i} d'x_i \wedge d't \wedge d'x_I \wedge d''x_J\right) \\ &= -\sum_{i=1}^r K'\left(\frac{\partial b_{IJ}}{\partial x_i} d't \wedge d'x_i \wedge d'x_I \wedge d''x_J\right) \\ &= -\sum_{i=1}^r \left(\int_0^1 \frac{\partial b_{IJ}}{\partial x_i} dt\right) d'x_i \wedge d'x_I \wedge d''x_J. \end{aligned}$$

iii) $\beta = e_{IJ} d'x_I \wedge d''t \wedge d''x_J$:
Similarly to ii), the pullbacks are zero. Since both β and $d'\beta$ have a factor $d''t$ by definition they are sent to 0 by K'.

iv) $\beta = g_{IJ} d't \wedge d'x_I \wedge d''t \wedge d''x_J$:
Same as iii).

Adding up these parts we have proven that (2.6) holds on V. Now if $\alpha \in \mathcal{A}^{p,q}(\Omega \times B)$ is given by $\beta \in \mathcal{A}^{p,q}(V \times B')$ then the equation holds for α simply because it holds for β. □

In the classical proof of the Poincaré lemma for star shaped subsets U of $\mathbb{R}^n$ the idea is to pull back differential forms via a contraction of U to its center. This contraction is however not an affine map. So we will introduce in Definition 2.2.6 a pullback for superforms along C^∞-maps that still commutes with d' (as we will see in 2.2.8). This will be a crucial ingredient in our proof of the Poincaré lemma for superforms. The following example shows that the direct approach does not work.

Remark 2.2.4. Given a C^∞-map $F\colon V' \to V$, where V' resp. V are open subsets of $\mathbb{R}^{r'}$ resp. $\mathbb{R}^r$ we can define a naive pullback

$$F^* : \mathcal{A}^{p,q}(V) = \mathcal{A}^p(V) \otimes \mathcal{A}^q(V) \to \mathcal{A}^p(V') \otimes \mathcal{A}^q(V') = \mathcal{A}^{p,q}(V'),$$

which is just given by the tensor products of the usual pullback of differential forms. This pullback however does not commute with the differential d' in general, as can be seen in the following example. Let $V' = \mathbb{R}^2$, $V = \mathbb{R}$ and $F(x, y) = xy$. Denote the coordinate on $\mathbb{R}$ by t. Then we have $d'F^*(d''t) = d'(xd''y+yd''x) = d'x\wedge d''y+d'y\wedge d''x \neq 0$, however $d'(d''t) = 0$ and thus $F^*(d'd''t) = 0$.

The reason for this is that the definition of this pullback uses the presentation $\mathcal{A}^{p,q} = \mathcal{A}^p \otimes \mathcal{A}^q$, while the definition of d' uses the presentation $\mathcal{A}^{p,q} = \mathcal{A}^p \otimes \Lambda^q \mathbb{R}^{r*}$ and thus these two are not compatible. We would therefore like to define a pullback which uses the presentation $\mathcal{A}^{p,q} = \mathcal{A}^p \otimes \Lambda^q \mathbb{R}^{r*}$.

Lemma 2.2.5. *Let X be a polyhedral subspace in $\mathbb{R}^r$, $\Omega \subset X$ an open subset and $W \subset \mathbb{R}^r$ an open subset such that $\Omega = W \cap X$. Then the restriction map $\mathcal{A}^{p,q}(W) \to \mathcal{A}^{p,q}(\Omega)$ is surjective. In particular, we may assume that any form $\alpha \in \mathcal{A}^{p,q}(\Omega)$ is given by a form on W.*

Proof. Let $\alpha \in \mathcal{A}^{p,q}(\Omega)$ be given by $\beta \in \mathcal{A}^{p,q}(V)$. Then α is also given by $\beta|_{V\cap W}$, hence we may assume $V \subset W$. Notice that Ω is a closed subset of W. Choose a function $f \in C^\infty(W)$ such that $f|_\Omega \equiv 1$ and $\operatorname{supp}_W f \subset V$. Then α is given by $f|_V \beta$ and this can be extended by zero to a form in $\mathcal{A}^{p,q}(W)$. □

Definition 2.2.6 (C^∞-pullback of (p, q)-forms)**.** We define a pullback for superforms on open subsets $V \subset \mathbb{R}^r$ and, under certain conditions, for superforms on polyhedral subspaces.

i) Let $V' \subset \mathbb{R}^{r'}$ and $V \subset \mathbb{R}^r$ be open subsets. Let $F = (s_F, L_F)$ be a pair of maps such that $s_F : V' \to V$ is a C^∞-map and $L_F : \mathbb{R}^{r'} \to \mathbb{R}^r$ is linear. We define

$$F^* := s_F^* \otimes L_F^* \colon \mathcal{A}^{p,q}(V) = \mathcal{A}^p(V) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} \\ \to \mathcal{A}^p(V') \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r'^*} = \mathcal{A}^{p,q}(V').$$

Explicitly, if $\beta \in \mathcal{A}^{p,q}(V)$ we have

$$\langle F^*(\beta)(x); v_1, \ldots, v_p, w_1, \ldots, w_q \rangle \\ = \langle \beta(s_F(x)); d(s_F)_x(v_1), \ldots, d(s_F)_x(v_p), L_F(w_1), \ldots, L_F(w_q) \rangle$$

for all $x \in V'$ and $v_i, w_i \in \mathbb{R}^{r'}$, where $d(s_F)_x$ denotes the differential of s_F at x.

ii) Let X' and X be polyhedral subspaces in $\mathbb{R}^{r'}$ and $\mathbb{R}^r$ respectively. Let $\Omega' \subset X'$ and $\Omega \subset X$ be open subsets and V' resp. V be open neighborhoods of Ω' resp. Ω in $\mathbb{R}^{r'}$ resp. $\mathbb{R}^r$. Let $s_F : V' \to V$ be a C^∞-map and $L_F : \mathbb{R}^{r'} \to \mathbb{R}^r$ a linear map such that $s_F(\Omega') \subset \Omega$. The pair $F = (s_F, L_F)$ is said to *allow a pullback from* Ω *to* Ω' if there exist open subsets W of V and W' of V' such that $W \cap |\,\mathcal{C}\,| = \Omega$, $W' \cap |\,\mathcal{C}'\,| = \Omega'$, $s_F(W') \subset W$ and for all $\beta \in \mathcal{A}^{p,q}(W)$ such that $\beta|_\Omega = 0$ we have $F^*(\beta)|_{\Omega'} = 0$. In that case, for a form $\alpha \in \mathcal{A}^{p,q}(\Omega)$ we choose $\beta \in \mathcal{A}^{p,q}(W)$ by which α is given (which is possible by Lemma 2.2.5) and we define $F^*(\alpha) \in \mathcal{A}^{p,q}(\Omega')$ to be given by $F^*(\beta) \in \mathcal{A}^{p,q}(W')$. The form $F^*(\alpha) \in \mathcal{A}^{p,q}(\Omega')$ is then independent of the choice of W, W' and β, as will be shown in the next Lemma.

Lemma 2.2.7. *The definition of $F^*(\alpha)$ above is independent of the choice of W, W' and β.*

Proof. The independence of β is simply due to the property that F^* respects forms that restrict to zero. Now if both W_1, W_1', β_1 and W_2, W_2', β_2 have the properties required in the definition above, then by Lemma 2.2.5 we can choose a form $\delta \in \mathcal{A}^{p,q}(W_1 \cup W_2)$ such that $\delta|_\Omega = \alpha$. By independence of the form, we have

$$F^*(\beta_1)|_{\Omega'} = F^*(\delta|_{W_1})|_{\Omega'} = F^*(\delta)|_{W_1'}|_{\Omega'} = F^*(\delta)|_{\Omega'}$$

and the same works for $F^*(\beta_2)|_{\Omega'}$, which proves exactly the independence we wanted to show. □

Remark 2.2.8. i) The pullback between open subsets of vector spaces commutes with taking d' since both use the presentation $\mathcal{A}^{p,q}(V) = \mathcal{A}^p(V) \otimes \Lambda^q \mathbb{R}^{r*}$. We have $F^* = s_F^* \otimes L_F^*$ and $d' = D \otimes \mathrm{id}$ and s_F^* and D commute. If F allows a pullback, then the pullback F^* between open subsets of the polyhedral subspaces commutes with d' since both F^* and d' are defined via restriction.

ii) The pullback is functorial in the following sense: Let X, X' and X'' be polyhedral subspaces in $\mathbb{R}^r$, $\Omega \subset X$, $\Omega' \subset X'$ and $\Omega'' \subset X''$ open subsets and $V \subset \mathbb{R}^r$ resp. $V' \subset \mathbb{R}^{r'}$ resp. $V'' \subset \mathbb{R}^{r''}$ open neighborhoods of Ω resp. Ω' resp. Ω''. Let further $F = (s_F, L_F)$ and $G = (s_G, L_G)$ be pairs of maps such that $s_F : V' \to V$ and $s_G : V'' \to V'$ are C^∞-maps, $L_F : \mathbb{R}^{r'} \to \mathbb{R}^r$ and $L_G : \mathbb{R}^{r''} \to \mathbb{R}^{r'}$ are linear maps and $s_F(\Omega') \subset \Omega$ and

$s_G(\Omega'') \subset \Omega'$. If both F resp. G allow a pullback from Ω to Ω' resp. Ω' to Ω'' and we define $F \circ G := (s_F \circ s_G, L_F \circ L_G)$ then $F \circ G$ allows a pullback from Ω to Ω'' and we have $(F \circ G)^* = G^* \circ F^*$.

iii) Let $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ be an affine map and denote by $\mathbb{L}_F := F - F(0)$ the associated linear map. Then the pullback via F in the sense of Remark 2.1.5 is the pullback via $(F, \mathbb{L}_F)$ in the sense of Definition 2.2.6 above.

Theorem 2.2.9 (Homotopy Formula)**.** *Let V be an open subset of $\mathbb{R}^r$. Let further $s_F\colon V \to V$ be a C^∞-map and $L_F := \mathrm{id}$. Let $s_G\colon V \times \mathbb{R} \to V$ such that $s_G(.,0) = s_F$ and $s_G(.,1) = \mathrm{id}$. Let $L_G = \mathrm{pr}_1\colon \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^r$ be the projection to the first factor. Denote by F^* respectively G^* the pullback from V to V respectively to $V \times \mathbb{R}$ via pairs $F := (s_F, L_F)$ respectively $G := (s_G, L_G)$. Then for $\alpha \in \mathcal{A}^{p,q}(V)$ we have*

$$\alpha - F^*\alpha = d'K'G^*\alpha + K'G^*d'\alpha \tag{2.8}$$

for any operator K' satisfying the equality (2.6) of Lemma 2.2.3.

Proof. We calculate

$$\begin{aligned}
\mathrm{id}^* - F^* &= (G \circ \iota_1)^* - (G \circ \iota_0)^* \\
&= \iota_1^* \circ G^* - \iota_0^* \circ G^* \\
&= (\iota_1^* - \iota_0^*) \circ G^* \\
&= (K'd' + d'K')G^* \\
&= K'd'G^* + d'K'G^* \\
&\overset{(2.2.8)}{=} K'G^*d' + d'K'G^*,
\end{aligned}$$

where we denote by ι_i the pair $(\iota_i, \mathbb{L}_{\iota_i})$. Now putting in α and using $\mathrm{id}^*(\alpha) = \alpha$ gives the desired result. □

Remark 2.2.10. If Ω is an open subset of X for some polyhedral subspace X in $\mathbb{R}^r$ and F resp. G allows a pullback from Ω to Ω resp. $\Omega \times B$, where $B = [0,1]$ is the closed unit interval, then (2.8) also holds for $\alpha \in \mathcal{A}^{p,q}(\Omega)$ since all operators are defined via restriction.

Definition 2.2.11. Let X be a polyhedral subspace in $\mathbb{R}^r$. An open subset Ω of X is called *polyhedrally star shaped* with center z if there is a polyhedral subspace Y of X such that Ω is an open subset of Y and a polyhedral structure $\mathcal{C}$ on Y such that for all maximal $\sigma \in \mathcal{C}$ the set $\sigma \cap \Omega$ is star shaped with center z in the sense that for all $x \in \sigma \cap \Omega$ and for all $t \in [0,1]$ the point $z + t(x - z)$ is contained in $\sigma \cap \Omega$.

Remark 2.2.12. It is obvious that if $\Omega \subset X$ is a polyhedrally star shaped open subset with center z, then Ω is also star shaped with center z. The converse is not true however: Take $X = [-1,1] \times [-1,1] \cup \{0\} \times [1,2] \cup [1,2] \times \{0\} \subset \mathbb{R}^2$. Then X is star shaped but not polyhedrally star shaped.

Lemma 2.2.13. *Let X' and X be polyhedral subspaces in $\mathbb{R}^{r'}$ and $\mathbb{R}^r$ respectively. Let $\Omega' \subset X'$ and $\Omega \subset X$ be open subsets and V' resp. V open neighborhoods of Ω' resp. Ω in $\mathbb{R}^{r'}$ resp. $\mathbb{R}^r$. Let $s_F\colon V' \to V$ be a C^∞-map and $L_F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ a linear map such that $s_F(\Omega') \subset \Omega$. Suppose there exist polyhedral structures $\mathcal{C}'$ on X' and $\mathcal{C}$ on X such that for all maximal $\sigma' \in \mathcal{C}'$ there exists a maximal $\sigma \in \mathcal{C}$ such that we have*

(a) $\forall x \in \sigma' \cap \Omega', s_F(x) \in \sigma$ *and*

(b) $\forall w \in \mathbb{L}(\sigma'), L_F(w) \in \mathbb{L}(\sigma)$.

Then $F := (s_F, L_F)$ allows a pullback from Ω to Ω'.

Proof. Let $W \subset V$ be an open subset such that $W \cap |\mathcal{C}| = \Omega$ and let $\beta \in \mathcal{A}^{p,q}(W)$. For F to allow a pullback we have to show that if $\beta|_\sigma = 0$ for all maximal polyhedra $\sigma \in \mathcal{C}$ then $(F^*\beta)|_{\sigma'} = 0$ for all maximal $\sigma' \in \mathcal{C}'$.
Let $\sigma' \in \mathcal{C}'$ be a maximal polyhedron and $\sigma \in \mathcal{C}$ the maximal polyhedron such that σ and σ' satisfy conditions (a) and (b). We then have that

(c) $\forall x \in \sigma' \cap \Omega',\ \ \forall v \in \mathbb{L}(\sigma'), d(s_F)_x(v) \in \mathbb{L}(\sigma)$,

due to condition (a).
For $\sigma \in \mathcal{C}$ the fact that $\beta|_\sigma = 0$ just means

$$\langle \beta(x); v_1, \ldots, v_p, w_1, \ldots, w_q \rangle = 0$$

for all $x \in \sigma \cap \Omega, v_i, w_i \in \mathbb{L}(\sigma)$. But then we have

$$\begin{aligned}&\langle F^*(\beta)(x); v_1, \ldots, v_p, w_1, \ldots, w_q \rangle \\ =&\langle \beta(s_F(x)); d(s_F)_x(v_1), \ldots, d(s_F)_x(v_p), L_F(w_1), \ldots, L_F(w_q) \rangle = 0\end{aligned}$$

for all $x \in \sigma' \cap \Omega', v_i, w_i \in \mathbb{L}(\sigma')$ by conditions (a), (b) and (c). Hence $F^*(\beta)|_{\sigma'} = 0$. This shows that if $\beta|_\sigma = 0$ for all $\sigma \in \mathcal{C}$ then $F^*(\beta)|_{\sigma'} = 0$ for all maximal and hence all $\sigma' \in \mathcal{C}'$. Thus F allows a pullback from Ω to Ω'. □

Proposition 2.2.14. *Let X be a polyhedral subspace in $\mathbb{R}^r$ and $\Omega \subset X$ a polyhedrally star shaped open subset with center z. Let*

$$\begin{aligned}s_G\colon \mathbb{R}^r \times \mathbb{R} &\to \mathbb{R}^r \\ (x,t) &\mapsto z + t(x-z)\end{aligned}$$

be the contraction of Ω to its center and $L_G\colon \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^r$ the projection to the first factor. Then $G := (s_G, L_G)$ allows a pullback from Ω to $\Omega \times B$, where B denotes the closed unit interval

Proof. We show that G fulfills the conditions required in Lemma 2.2.13. Since Ω is polyhedrally star shaped we know that there exists a polyhedral structure $\mathcal{C}$ of X such that $\sigma \cap \Omega$ is star shaped with center z for all maximal $\sigma \in \mathcal{C}$. We take $\mathcal{C}'$ to be the polyhedral complex whose maximal polyhedra are of the form $\sigma \times B$ for $\sigma \in \mathcal{C}$ a maximal polyhedron. Let $\sigma' = \sigma \times B \in \mathcal{C}'$ be such a maximal polyhedron. For $(x,t) \in \sigma'$ we have $s_G(x,t) \in \sigma$ because $\sigma \cap \Omega$ is star shaped with center z. Since it is obvious that $L_G(\mathbb{L}(\sigma')) \subset \mathbb{L}(\sigma)$, G allows a pullback from Ω to $\Omega \times B$ by Lemma 2.2.13. □

Theorem 2.2.15 (d'-Poincaré lemma for polyhedral complexes)**.** *Let X be a polyhedral subspace in $\mathbb{R}^r$ and $\Omega \subset X$ a polyhedrally star shaped open subset with center z. Let $\alpha \in \mathcal{A}^{p,q}(\Omega)$ with $p > 0$ and $d'\alpha = 0$. Then there exists $\beta \in \mathcal{A}^{p-1,q}(\Omega)$ such that $d'\beta = \alpha$.*

Proof. Let s_F be the constant map to the center z of Ω and $L_F = \mathrm{id}$. Let further s_G be the contraction of Ω to the center and $L_G = \mathrm{pr}_1$ (as in Proposition 2.2.14). It is easy to check that both F and G have the properties required to use the homotopy forumla (Theorem 2.2.9 and Remaerk 2.2.10). Since s_F is constant and L_F is the identity we also see that F has the properties of Lemma 2.2.13 and hence allows a pullback from Ω to Ω. By Proposition 2.2.14 we know that G also allows a pullback. Now since $\alpha \in \mathcal{A}^{p,q}(\Omega)$ with $p > 0$ we have $F^*\alpha = 0$ (since s_F is a constant map). Together with our assumption $d'\alpha = 0$, Theorem 2.2.9 yields

$$\alpha = d'(K'G^*\alpha),$$

which proves the theorem. □

Remark 2.2.16. The corresponding statement from this section are all true for d'', always using the identification J to switch between d' and d''.

We further note that we can not hope for a similar statement of the Poincaré lemma with respect to the operator d. This is due to the fact that any exact 1-form is J-invariant, and this does not need to be true for closed 1-forms. We have the following partial result.

Corollary 2.2.17. *Let X be a polyhedral subspace in $\mathbb{R}^r$ and $\Omega \subset X$ a polyhedrally star shaped open subset. Let $\alpha \in \mathcal{A}^k(\Omega)$ be a d-closed form. Then there exists $\beta \in \mathcal{A}^{k-1}(\Omega)$ such that $\alpha - d\beta \in \mathcal{A}^{0,k}(\Omega)$ and such that $\alpha - d\beta$ is d', d'' and d-closed. If $k > \dim X$ then α is d-exact.*

Proof. Write $\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_k$ with $\alpha_i \in \mathcal{A}^{k-i,i}(\Omega)$. Then the decomposition of $d\alpha \in \mathcal{A}^{k+1}(\Omega) = \bigoplus_{p+q=k+1} \mathcal{A}^{p,q}(\Omega)$ is given by

$$d\alpha = d'\alpha_0 + (d''\alpha_0 + d'\alpha_1) + \cdots + (d''\alpha_{k-1} + d'\alpha_k) + d''\alpha_k.$$

Since those terms have different bidegrees each of them is zero. Therefore the statement is trivially true if $k = 0$ and we may from now on assume $k > 0$.

We construct inductively for $i = 0, \ldots, k-1$ forms $\beta_i \in \mathcal{A}^{k-i-1,i}(\Omega)$ such that $\beta_{-1} = 0$ and $d'\beta_i = \alpha_i - d''\beta_{i-1}$. Note therefore that $\alpha_i - d''\beta_{i-1}$ is d'-closed for $i = 0, \ldots, k$, since this is immediate for $i = 0$ and for $i = 1, \ldots, k$ we have

$$\begin{aligned} d'(\alpha_i - d''\beta_{i-1}) &= d'\alpha_i - d'd''\beta_{i-1} \\ &= d'\alpha_i + d''d'\beta_{i-1} \\ &= d'\alpha_i + d''\alpha_{i-1} - d''d''\beta_{i-2} \\ &= d'\alpha_i + d''\alpha_{i-1} = 0. \end{aligned}$$

Hence given β_{i-1}, Theorem 2.2.15 gives us $\beta_i \in \mathcal{A}^{k-i-1,i}(\Omega)$ such that $d'\beta_i = \alpha_i - d''\beta_{i-1}$ for $i = 0, \dots, k-1$. We define $\beta := \sum_{i=0}^{k-1} \beta_i \in \mathcal{A}^{k-1}(\Omega)$. Then we have

$$\alpha - d\beta = \sum_{i=0}^{k-1} (\alpha_i - d''\beta_{i-1} - d'\beta_i) + \alpha_k - d''\beta_{k-1} = \alpha_k - d''\beta_{k-1} \in \mathcal{A}^{0,k}(\Omega).$$

As shown above we have that $\alpha_k - d''\beta_{k-1}$ is d' closed, thus $\alpha - d\beta$ is. Since it is also d-closed, it is d''-closed. If $k > \dim X$, then $\mathcal{A}^{0,k}(\Omega) = 0$ and hence $\alpha = d\beta$. □

2.2.2 Cohomology of polyhedral spaces: locally

In this subsection, we study the local behavior of the Dolbeault cohomology of superforms. We will give the consequences of the Poincaré lemma 2.2.15 for polyhedral spaces and calculate $H^{0,p}_{d'}(\Omega)$ for a basic open set Ω. All this was originally done in [JSS15]. Note that Theorem 2.2.27 is not explicitly stated there, but follows from [JSS15, Proposition 3.10 & Theorem 3.18].

Theorem 2.2.18 (Poincaré lemma for polyhedral spaces)**.** *Let X be a polyhedral space and $U \subset X$ an open subset. Let $\alpha \in \mathcal{A}^{p,q}(U)$ with $p > 0$ and $d'\alpha = 0$. Then for every $x \in U$ there exists an open subset $V \subset X$ with $x \in V$ and a superform $\beta \in \mathcal{A}^{p-1,q}(V)$ such that $d'\beta = \alpha|_V$.*

Proof. After shrinking U, we may assume that there is a chart $\varphi : U \to \Omega$ for Ω an open subset of the support of a polyhedral complex $\mathcal{C}$ in $\mathbb{T}^r$. Since this question is purely local, we may prove the statement for an open subset $\Omega \subset |\mathcal{C}|$ for a polyhedral complex $\mathcal{C}$ in $\mathbb{T}^r$. If $\mathrm{sed}(x) = \emptyset$ then this is shown in Theorem 2.2.15.

For the general case, let $I = \mathrm{sed}(x)$ and after possibly shrinking Ω we may assume that I is the unique maximal sedentarity among points in Ω and α is determined by α_I on Ω. After possibly shrinking Ω again, by the case $I = \emptyset$, we have $\beta_I \in \mathcal{A}^{p,q}(\Omega_I)$ such that $d'\beta_I = \alpha_I$. For each $J \subset I$, set $\beta_J = \pi^*_{IJ}\beta_I$, then this determines a superform $\beta \in \mathcal{A}^{p,q}(\Omega)$ and since the pullback commutes with d', we have $d'\beta_J = \alpha_J$, hence β has the required property and the theorem is proven. □

Remark 2.2.19. Note that this Poincaré lemma is weaker than the one polyhedral subspaces of $\mathbb{R}^r$, since it does not give any acyclic domains. We will later (cf. Theorem 2.2.27) see that in the case where X is regular at infinity we indeed have a basis of open sets which is acyclic. This was also already contained in [JSS15], using tropical cohomology [JSS15, Proposition 3.10 & Theorem 3.18], but we give a direct proof here.

Corollary 2.2.20. *Let X be a polyhedral space. Then all rows and columns of the double complex $(\mathcal{A}^{p,q}_X, d', d'')$ of sheaves on X are exact in positive degrees.*

Definition 2.2.21. For X a polyhedral space and $q \in \mathbb{N}$ we define the sheaf

$$\mathcal{L}^q_X := \ker(d' : \mathcal{A}^{0,q}_X \to \mathcal{A}^{1,q}_X).$$

Again we omit the subscript X on $\mathcal{L}^q_X$ if the space X is clear from context.

Corollary 2.2.22. *For a polyhedral space X and all $q \in \mathbb{N}$, the complex*

$$0 \to \mathcal{L}^q \to \mathcal{A}^{0,q} \xrightarrow{d'} \mathcal{A}^{1,q} \xrightarrow{d'} \mathcal{A}^{2,q} \to \dots$$

of sheaves on X is exact. Furthermore, it is an acyclic resolution, we thus have canonical isomorphisms

$$H^p(X, \mathcal{L}^q) \cong H^{p,q}_{d'}(X) \qquad \textit{and} \qquad H^q_c(X, \mathcal{L}^q) \cong H^{p,q}_{d',c}(X).$$

Proof. We noted in Remark 2.1.59 that the sheaves $\mathcal{A}^{p,q}$ are fine. Exactness is a direct consequence of Theorem 2.2.18 and Definition 2.2.21. □

We will now define the notion of *basic open subsets*, which was introduced in [JSS15]. These play the role of well-behaved small open subsets. We will see in Theorem 2.2.27 that, if X is regular at infinity, they are acyclic for the cohomology of $(\mathcal{A}^{\bullet,q}_X, d')$.

Definition 2.2.23. A subset $\Delta \subset \mathbb{T}^r$ is an *open cube* if it is a product of intervals which are either (a_i, b_i) or $[-\infty, c_i)$ for $a_i \in \mathbb{T}$, $b_i, c_i \in \mathbb{R} \cup \{\infty\}$.

For a polyhedral complex $\mathcal{C}$ in $\mathbb{T}^r$, a subset Ω of $|\,\mathcal{C}\,|$ is called a *basic open* subset if there exists an open cube $\Delta \subset \mathbb{T}^r$ such that $\Omega = |\,\mathcal{C}\,| \cap \Delta$ and such that the set of polyhedra of $\mathcal{C}$ intersecting Ω has a unique minimal element. Note that the sedentarity of the minimal polyhedron of Ω is the maximal sedentarity among points in Ω.

Let X be a polyhedral space with atlas $(\varphi_i : U_i \to \Omega_i \subset X_i)$, such that for each i we have a fixed polyhedral structure $\mathcal{C}_i$ on X_i. Then we say that an open subset U is a *basic open* subset (with respect to these structures) if there exists a chart $\varphi : U_i \to X_i$ such that $U \subset U_i$ and $\varphi(U)$ is a basic open subset of $|\,\mathcal{C}_i\,|$.

Lemma 2.2.24. *Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{T}^r$, then the basic open sets form a basis of the topology of $|\,\mathcal{C}\,|$. Further, if Ω is a basic open subset of a polyhedral complex $|\,\mathcal{C}\,|$ of sedentarity I, then Ω_I is a basic open subset of the polyhedral complex $|\,\mathcal{C}_I\,|$ in $\mathbb{R}^r_I$.*

Proof. Basic open sets form a basis of the topology of $|\,\mathcal{C}\,|$ since open cubes form a basis of the topology of $\mathbb{T}^r$. For the second statement, we have that $\Omega_I = |\,\mathcal{C}_I\,| \cap \Delta_I$ and the minimal polyhedron of Ω_I is the same as the one of Ω, so the lemma is proven. □

Proposition 2.2.25. *Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{T}^r$ and Ω be a basic open set of $|\,\mathcal{C}\,|$ with minimal polyhedron σ. Then we have*

$$\mathcal{L}^q(\Omega) = \mathbf{F}^q(\sigma).$$

Proof. Let $I := \operatorname{sed}(\sigma)$. We start with the case $I = \emptyset$, thus $\Omega \subset \mathbb{R}^r$. Then Ω is polyhedrally star shaped with center any point in the relative interior of σ. As in the proof of Theorem 2.2.15 we let $F = (s_F, L_F)$ with s_F the constant map to the center and L_F the identity on $\mathbb{R}^r$. Let further as in Proposition 2.2.14 $G = (s_G, L_G)$ be the pair with $s_G \colon \Omega \times \mathbb{R} \to \Omega$ the contraction of Ω to its center and $L_G \colon \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^r$ the projection to the first factor. Let $\alpha \in \mathcal{A}^{0,q}(\Omega)$ be a d'-closed form. Then equation (2.8) from Theorem 2.2.9 shows that $\alpha = F^*(\alpha)$.

There is a natural map $\mathbf{F}^q(\sigma) \to \mathcal{L}^q(\Omega) \subset \mathcal{A}^{0,q}(\Omega)$ and this is clearly injective. To show surjectivity choose $v_1, \ldots, v_k$ such that each $v_i \in \Lambda^q \mathbb{L}(\tau)$ for some τ and $v_1, \ldots, v_k$ is a basis of $\mathbf{F}^q(\sigma)$ and extend this to a basis $v_1, \ldots, v_k, v_{k+1}, \ldots, v_s$ of $\Lambda^q \mathbb{R}^r$. Then we have that $F^*(\alpha)$ is given by a form $\sum_{i=1}^{s} c_i d' v_i$, with $c_i \in \mathbb{R}$, by construction of F. Since for $i \in \{k+1, \ldots, s\}$ we have $d' v_i|_\Omega = 0$, we see that $F^*(\alpha)$ (and thus α) are given by $\sum_{i=1}^{k} c_i d' v_i$ and this is clearly in $\mathbf{F}^q(\sigma)$.

For the general case $I \neq \emptyset$, first we apply the above argument to Ω_I which is a basic open subset of the polyhedral complex $\mathcal{C}_I$ by Lemma 2.2.24. Writing $X = |\,\mathcal{C}\,|$ and $X_I = |\,\mathcal{C}_I\,|$ we obtain

$$\mathcal{L}^q_{X_I}(\Omega_I) = \left(\sum_{\tau \in \mathcal{C}_I : \sigma \prec \tau} \Lambda^q \mathbb{L}(\tau) \right)^* .$$

Thus we only have to show

$$\mathcal{L}^q_{X_I}(\Omega_I) \cong \mathcal{L}^q_X(\Omega).$$

Using the pullbacks of the projection maps define

$$\begin{aligned} \mathcal{L}^q_{X_I}(\Omega_I) &\to \mathcal{L}^q_X(\Omega) \\ \alpha_I &\mapsto (\pi^*_{IJ} \alpha_I)_{J \subset I}. \end{aligned}$$

This is clearly well defined and injective, we thus have to show surjectivity. More precisely, for $\alpha \in \mathcal{L}^q_X(\Omega)$, it remains to show that $\alpha_J|_{\Omega_J \cap \tau} = \pi^*_{IJ}(\alpha_I|_{\Omega_I \cap \tau})$ for all $J \subset I$ and τ such that $\sigma \prec \tau$. By the condition of compatibility for α there exists a neighborhood Ω_x of x such that

$$\alpha_J|_{\Omega_{x,J}} = \pi^*_{IJ}(\alpha_I|_{\Omega_{x,I}}),$$

hence in particular

$$\alpha_J|_{\Omega_{x,J} \cap \tau} = \pi^*_{IJ}(\alpha_I|_{\Omega_{x,I} \cap \tau}).$$

Since $\Omega_I \cap \tau$ is connected, the restriction $\mathcal{L}^q_{X_I}(\Omega_I \cap \tau) \to \mathcal{L}^q_{X_I}(\Omega_{x,I} \cap \tau)$ is injective and the same for J, thus we have

$$\alpha_J|_{\Omega_J \cap \tau} = (\pi^*_{IJ} \alpha_I|_{\Omega_I \cap \tau}).$$

□

Lemma 2.2.26. *Let X be a polyhedral subspace of $\mathbb{T}^r \times \mathbb{R}^s$. Let U be an open subset of X which contains the point $(-\infty, \ldots, -\infty, 0, \ldots, 0)$. Let U' be an open neighborhood of $U_{\{r\}}$ in U. Write $V = U \setminus U_{\{r\}}$ and $V' = V \cap U'$. Then if restriction induces an isomorphism $H^{p,q}_{d'}(V) \simeq H^{p,q}_{d'}(V')$ for all p, q it also induces an isomorphism $H^{p,q}_{d'}(U) \simeq H^{p,q}_{d'}(U')$ for all p, q.*

Proof. We have $U' \cup V = U$ and $U' \cap V = V'$. Now the Lemma is a direct consequence of the Mayer-Vietoris-sequence A.1.1. □

Theorem 2.2.27. *Let X be a polyhedral space which is regular at infinity and $U \subset X$ be a basic open subset. Then*

$$H^{p,q}_{d'}(U) = 0$$

unless $p = 0$.

Proof. We may assume that X is a polyhedral subspace of $\mathbb{T}^s$. We do induction on $|\operatorname{sed}(U)|$, where $\operatorname{sed}(U) = \emptyset$ follows from Theorem 2.2.15. Let $r := |\operatorname{sed}(U)|$. We may assume that $X = \mathbb{T}^r \times Y$ for a polyhedral fan Y and U being a basic open neighborhood of the point $(-\infty, \dots, -\infty, 0, \dots, 0)$.

Let $\alpha \in \mathcal{A}^{p,q}(U)$ be a d'-closed form. Let U' be a basic open neighborhood of $U_{\{r\}}$ on which α is determined by $\alpha_{\{r\}}$. Since $U_{\{r\}}$ is a basic open subset of $\mathbb{T}^r_{\{r\}} \times Y \simeq \mathbb{T}^{r-1} \times Y$ there exists, by induction hypothesis, $\beta_{\{r\}} \in \mathcal{A}^{p-1,q}(U_{\{r\}})$ such that $d'\beta_{\{r\}} = \alpha_{\{r\}}$. Defining $\beta := \pi^*_{\{r\}}\beta_{\{r\}} \in \mathcal{A}^{p-1,q}(U')$ we have $d'\beta = \alpha$. Thus $[\alpha]|_{U'} = 0$ in $H^{p,q}_{d'}(U')$.

Now defining $V = U \setminus U_{\{r\}}$ and $V' = U' \cap V$, we find that V' and V are basic open with minimal face $(-\infty, \dots, -\infty, \mathbb{R}, 0, \dots, 0)$. We conclude by Proposition 2.2.25 and the induction hypothesis that the restriction

$$H^{p,q}_{d'}(V) \simeq H^{p,q}_{d'}(V')$$

is an isomorphism for all p, q. Thus by Lemma 2.2.26, restriction induces an isomorphism

$$H^{p,q}_{d'}(U) \simeq H^{p,q}_{d'}(U')$$

for all p, q. This in turn shows $[\alpha] = 0 \in H^{p,q}_{d'}(U)$. Since α was arbitrary, this proves the claim. □

Corollary 2.2.28. *Let X be a polyhedral space which is regular at infinity. Let $\Omega \subset X$ be a basic open subset. Let $\alpha \in \mathcal{A}^k(\Omega)$ be a d-closed form. Then there exists $\beta \in \mathcal{A}^{k-1}(\Omega)$ such that $\alpha - d\beta \in \mathcal{A}^{0,k}(\Omega)$ and such that $\alpha - d\beta$ is d', d'' and d-closed. If $k > \dim X$ then α is d-exact.*

Proof. This works exactly like the proof of Corollary 2.2.17 using Theorem 2.2.27 instead of Theorem 2.2.15. □

2.2.3 Finite dimensionality

In this subsection, we show that the Dolbeault cohomology of a polyhedral subspace X of $\mathbb{T}^r \times \mathbb{R}^s$ is finite dimensional if the space is regular at infinity. As a consequence we also obtain that the cohomology of the total complex $H^k_d(X)$ is finite dimensional. To do that we will use the standard tool of good covers and the Mayer-Vietoris sequences from Appendix A. These results were already contained in [Jel16] in the case of polyhedral subspaces in $\mathbb{R}^r$. We use the same techniques here.

Definition 2.2.29. Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{T}^r \times \mathbb{R}^s$ and $\sigma \in \mathcal{C}$. We denote by $\mathring{\sigma}$ the *relative interior* of σ, which is just σ without its proper faces. We define the *polyhedral star of* σ to be

$$\Omega_\sigma := \bigcup_{\tau \in \mathcal{C}, \sigma \prec \tau} \mathring{\tau}.$$

Lemma 2.2.30. *Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{T}^r \times \mathbb{R}^s$. For $\sigma \in \mathcal{C}$ the polyhedral star Ω_σ of σ is an open neighborhood of $\mathring{\sigma}$ in $|\,\mathcal{C}\,|$.*

Proof. Since $\sigma \prec \sigma$, we have $\mathring{\sigma} \subset \Omega_\sigma$. Let $z \in \Omega_\sigma$. Let B be an open neighborhood of z in $\mathbb{T}^r \times \mathbb{R}^s$ that only intersects polyhedra in $\mathcal{C}$ that contain z. Then we have $B \cap |\,\mathcal{C}\,| \subset \bigcup_{\tau : z \in \tau} \mathring{\tau}$ and since $z \in \Omega_\sigma$ there exists some $\nu \in \mathcal{C}$ such that $z \in \mathring{\nu}$ and $\sigma \prec \nu$. Now if $z \in \tau$, then $z \in \nu \cap \tau$, which is a face of both. But since $z \in \mathring{\nu}$ this can not be a proper face of ν, hence $\nu \cap \tau = \nu$. Thus we have $\nu \prec \tau$ and by transitivity $\sigma \prec \tau$. We have shown $\{\tau \in \mathcal{C}\,|z \in \tau\} \subset \{\tau \in \mathcal{C}\,|\sigma \prec \tau\}$. This shows in turn that $\bigcup_{\tau : z \in \tau} \mathring{\tau} \subset \bigcup_{\tau : \sigma \prec \tau} \mathring{\tau}$ and thus $B \cap |\,\mathcal{C}\,| \subset \bigcup_{\tau : \sigma \prec \tau} \mathring{\tau} = \Omega_\sigma$. Hence for every point $z \in \Omega_\sigma$, the set Ω_σ contains an open neighborhood of z in $|\,\mathcal{C}\,|$, which shows that $\Omega_\sigma \subset |\,\mathcal{C}\,|$ is an open set. □

Lemma 2.2.31. *Let $\tau_1, \ldots, \tau_n \in \mathcal{C}$. Then the set of polyhedra in $\mathcal{C}$ which contain all τ_i is either empty or has a unique minimal (i.e. smallest) element $\sigma_{\tau_1 \ldots \tau_n}$. Further we have $\bigcap_{i=1}^{n} \Omega_{\tau_i} = \Omega_{\sigma_{\tau_1 \ldots \tau_n}}$.*

Proof. The first assertion is clear since the set of polyhedra which contain all τ_i is closed under intersection. The second part is straight from the definition, since

$$\bigcap_{i=1}^{n} \Omega_{\tau_i} = \bigcup_{\nu : \tau_i \prec \nu \forall i} \mathring{\nu} = \bigcup_{\nu : \sigma_{\tau_1, \ldots, \tau_n} \prec \nu} \mathring{\nu} = \Omega_{\sigma_{\tau_1, \ldots, \tau_n}}.$$

□

Lemma 2.2.32. *Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{R}^r$. Let $\sigma \in \mathcal{C}$ and $z \in \mathring{\sigma}$. Then Ω_σ is polyhedrally star shaped with respect to z.*

Proof. Let $\mathcal{D}$ be the polyhedral complex whose maximal polyhedra are the maximal ones in $\mathcal{C}$ that contain σ. Let $\tau \in \mathcal{D}$ be maximal and $y \in \tau \cap \Omega_\sigma$. Then there exists ν such that $y \in \mathring{\nu}$ and $\sigma \prec \nu \prec \tau$. Then $[y, z) \subset \mathring{\nu}$ and hence $[y, z] \subset \mathring{\nu} \cup \mathring{\sigma} \subset \Omega_\sigma \cap \tau$. This just means that $\tau \cap \Omega_\sigma$ is star shaped, hence Ω_σ is polyhedrally star shaped. □

Proposition 2.2.33. *Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{T}^r \times \mathbb{R}^s$ such that $|\,\mathcal{C}\,|$ is regular at infinity (i.e. locally isomorphic to $\mathbb{T}^{r'} \times Y$ for Y a polyhedral subspace of some $\mathbb{R}^{s'}$). Let σ in $\mathcal{C}$. Then*

$$H^{p,q}_{d'}(\Omega_\sigma) = 0 \text{ for } p > 0 \text{ and } H^{0,q}_{d'}(\Omega_\sigma) = \mathbf{F}^q(\sigma).$$

Proof. In the case $p = 0$, this follows exactly like in the proof of Proposition 2.2.25. For the case $p > 0$, we may assume that σ contains $(-\infty, \dots, -\infty, 0, \dots, 0)$. We do induction on r, with $r = 0$ being just the Poincaré lemma 2.2.15 and Lemma 2.2.32. The proof now works exactly as the one of Theorem 2.2.27, replacing U by Ω_σ. Note that $(\Omega_\sigma)_{\{r\}}$ is the polyhedral star of $\sigma_{\{r\}}$ in $\mathcal{C}_{\{r\}}$ and V and V' have minimal face $(-\infty, \dots, -\infty, \mathbb{R}) \times \sigma$. □

Theorem 2.2.34. *Let X be a polyhedral subspace of $\mathbb{T}^r \times \mathbb{R}^s$, which is regular at infinity. Then $H^{p,q}_{d'}(X)$ is finite dimensional for all $p, q \in \mathbb{N}_0$.*

Proof. Fix a polyhedral structure $\mathcal{C}$ on X. Let $\tau_1, \dots, \tau_k$ be the minimal polyhedra of $\mathcal{C}$. We claim that the family $(\Omega_{\tau_i})_{i=1,\dots,k}$ is a good cover of X. Let therefore $z \in X$. Then z is in the relative interior of some polyhedron σ and there is τ_i such that $\tau_i \prec \sigma$. This means however that $z \in \Omega_{\tau_i}$. Hence we have a cover and Lemma 2.2.33 precisely shows that this is a good cover, as defined in Definition A.2.1. Now Lemma A.2.2 shows our result. □

Corollary 2.2.35. *Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{T}^r \times \mathbb{R}^s$ such that $|\,\mathcal{C}\,|$ is regular at infinity. Let $\sigma \in \mathcal{C}$. Then there is a surjective map $H^{0,k}_{d'}(\Omega_\sigma) \twoheadrightarrow H^k_d(\Omega_\sigma)$. In particular $H^k_d(\Omega_\sigma)$ is finite dimensional for all $k \in \mathbb{N}_0$.*

Proof. By Lemma 2.1.64 the inclusion $A^{0,k}(\Omega) \hookrightarrow A^k(\Omega)$ induces $H^{0,k}_{d'}(\Omega) \to H^k_d(\Omega)$ (note that $H^{0,k}_{d'} = \ker(d'_{0,k})$). Now Corollary 2.2.28 shows the surjectivity and Proposition 2.2.33 shows that $H^{0,k}_{d'}(\Omega_\sigma)$ is finite dimensional, hence $H^k_d(\Omega_\sigma)$ is. □

Theorem 2.2.36. *Let X be a polyhedral subspace of $\mathbb{T}^r \times \mathbb{R}^s$ which is regular at infinity. Then $H^k_d(X)$ is finite dimensional for all $k \in \mathbb{N}_0$.*

Proof. Fix a polyhedral structure $\mathcal{C}$ on X. Let $\tau_1, \dots, \tau_s$ be the minimal polyhedra in $\mathcal{C}$. Again, as in the proof of Theorem 2.2.34, $(\Omega_{\tau_i})_{i=1,\dots,s}$ is a cover of X. By Corollary 2.2.35 and Lemma 2.2.33, this is a reasonable cover. Hence A.2.2 shows our result. □

2.2.4 Poincaré duality

In this subsection, we prove Poincaré duality for tropical manifolds. These are polyhedral spaces which are locally modeled on Bergman fans of matroids. The key tool is a recursive description of matroidal fans using tropical modifications, which was shown by Shaw in [Sha13, Proposition 2.25]. We then use exact sequences to compare the cohomology groups and the cohomology groups with compact support for different matroidal fans and derive Poincaré duality for matroidal fans inductively. We pass to tropical manifolds via a sheaf theoretic argument.

The presentation here is essentially the same as in [JSS15, Section 4]. Note that we use the identification of tropical cohomology with the cohomology of superforms from [JSS15, Theorem 3.15]. For that reason we will work with the cohomology with respect to the operator d''. The statements are of course equivalently true for the operator d', but for the operator d'' the indexing is the same as tropical cohomology. We also write $H^{p,q}$ for $H^{p,q}_{d''}$.

Definition 2.2.37. Let X be a tropical space of dimension n. We define

$$\begin{aligned} \mathrm{PD}\colon \mathcal{A}^{p,q}(X) &\to \mathcal{A}_c^{n-p,n-q}(X)^*, \\ \alpha &\mapsto \left(\beta \mapsto \varepsilon \int_X \alpha \wedge \beta\right) \end{aligned}$$

where $\mathcal{A}_c^{n-p,n-q}(X)^* := \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}_c^{n-p,n-q}(X), \mathbb{R})$ denotes the (non-topological) dual vector space of $\mathcal{A}_c^{n-p,n-q}(X)$ and $\varepsilon = (-1)^{p+q/2}$ if q is even and $\varepsilon = (-1)^{(q+1)/2}$ if q is odd.

Remark 2.2.38. By Stokes' theorem 2.1.63, we have for $\alpha \in \mathcal{A}^{p,q}(X)$ and $\beta \in \mathcal{A}_c^{n-p,n-q-1}(X)$ the following

$$0 = \int_X d''(\alpha \wedge \beta) = \int_X d''\alpha \wedge \beta + \int_X (-1)^{p+q} \alpha \wedge d''\beta.$$

Thus

$$\int_X d''\alpha \wedge \beta = (-1)^{p+q+1} \int_X \alpha \wedge d''\beta.$$

Therefore our choice of ε implies that we have a morphism of complexes

$$\mathrm{PD}\colon \mathcal{A}^{p,\bullet}(X) \to \mathcal{A}_c^{n-p,n-\bullet}(X)^*,$$

where the complex $\mathcal{A}_c^{n-p,n-\bullet}(X)^*$ is equipped with the dual maps d''^* as the differential. We now get a map in cohomology

$$\mathrm{PD}\colon H_{d''}^{p,q}(X) \to H_{d'',c}^{n-p,n-q}(X)^*,$$

since we have

$$H^q(\mathcal{A}_c^{n-p,n-\bullet}(X)^*, d''^*) = (H_q(\mathcal{A}_c^{n-p,n-\bullet}(X), d''))^* = H_{d'',c}^{n-p,n-q}(X)^*.$$

Definition 2.2.39. Let X be an n-dimensional tropical space. We say that X *has Poincaré duality* (PD) if the Poincaré duality map

$$\mathrm{PD}\colon H^{p,q}(X) \to H_c^{n-p,n-q}(X)^*$$

is an isomorphism for all p, q.

Before we show that tropical manifolds (cf. Definition 2.2.41 below) have Poincaré duality, we give an example that shows that not all tropical spaces do.

Example 2.2.40. Let X be the union of the coordinate axes in $\mathbb{R}^2$. Then X is connected, thus $H^{0,0}(X) \simeq \mathbb{R}$. However a form of positive degree on X is just given by a pair for forms, one on each of the axes. It is thus easy to see that we have $H_c^{1,1}(X) \simeq \mathbb{R}^2$.

Tropical manifolds are tropical spaces locally modeled on matroidal fans. Matroids are a combinatorial abstraction of the notion of independence in mathematics. See [Oxl11] for a comprehensive introduction to the theory of matroids. Every matroid has a representation as a fan tropical cycle (cf. [Stu02]). A way of explicitly constructing this fan can be found in [Sha13, Section 2.4] (or [FS05, AK06]). Here matroidal fans are always equipped with weights 1 on all facets.

Definition 2.2.41. A *tropical manifold* is a tropical space X of dimension n such that there is an atlas $A = (\varphi_i \colon U_i \to \Omega_i \subset X_i)$, where all X_i are of the form $\mathbb{T}^{r_i} \times V_i$ for matroidal fans V_i of dimension $n - r_i$ in $\mathbb{R}^{s_i}$.

Construction 2.2.42. We now describe the operation of tropical modification (see [BIMS15] for a more detailed introduction). Let $W \subset \mathbb{R}^{r-1}$ be a tropical variety and $P \colon \mathbb{R}^{r-1} \to \mathbb{R}$ a piecewise integral affine function. The graph $\Gamma_P(W) \subset \mathbb{R}^r$ is an $\mathbb{R}$-rational polyhedral complex, which inherits weights from the weights of W. In general, this graph is not a tropical cycle, since it does not satisfy the balancing condition, because P is only piecewise linear. However, the graph $\Gamma_P(W)$ can be completed to a tropical cycle V in a canonical way; at a codimension 1 face E of $\Gamma_P(W)$ not satisfying the balancing condition attach a facet to E generated by the direction $-e_r$. Then this facet can be equipped with a unique integer weight so that the resulting polyhedral complex is now balanced at E. Applying this procedure at all codimension one facets of $\Gamma_P(W)$ produces a tropical cycle V. Notice that there is a map $\delta \colon V \to W$ induced by the linear projection. The divisor of the piecewise integer affine function P restricted to W is a tropical cycle $\mathrm{div}_W(P) \subset W$ which is supported on the points $w \in W$ such that $\delta^{-1}(w)$ is a half-line. The weights on the facets of $\mathrm{div}_W(P)$ are inherited from the weights of V.

Definition 2.2.43. Let $W \subset \mathbb{R}^{r-1}$ be a tropical cycle and $P \colon \mathbb{R}^r \to \mathbb{R}$ a piecewise integer affine function, then the *open tropical modification* of W along P is the map $\delta \colon V \to W$ where $V \subset \mathbb{R}^r$ is the tropical cycle described above. A *closed tropical modification* is a map $\overline{\delta} \colon \overline{V} \to W$ where $\overline{V} \subset \mathbb{R}^{r-1} \times \mathbb{T}$ is the closure of V and $\overline{\delta}$ is the extension of δ.

A *matroidal tropical modification* is a modification where V, W, and $\mathrm{div}_W(P)$ are all matroidal fans.

Remark 2.2.44. Note that for a closed tropical modification $\overline{\delta} \colon \overline{V} \to W$ with divisor D, we have that $\overline{\delta}|_{\overline{V}_r} \colon \overline{V}_r \to D$ identifies the subspace $\overline{V}_r = \{x \in V \mid x_r = -\infty\}$ with D. We thus may also consider D as a subspace of $\overline{V}$.

We begin by showing that matroidal fans in $\mathbb{R}^r$ have Poincaré duality. To do this we use an alternative recursive description of matroidal fans via tropical modifications. In the language of matroids, the operation of tropical modification is related to deletions and contractions.

It follows from the next proposition that for any matroidal fan $V \subset \mathbb{R}^r$ of dimension n there is a sequence of open matroidal tropical modifications $V \to W_1 \to \cdots \to W_{r-n} = \mathbb{R}^n$.

Proposition 2.2.45. *Let $V \subsetneq \mathbb{R}^r$ be a matroidal fan, then there is a coordinate direction e_i such that the linear projection $\delta\colon \mathbb{R}^r \to \mathbb{R}^{r-1}$ with kernel generated by e_i is a matroidal tropical modification $\delta : V \to W$ along a piecewise integer affine function P, i.e. $W \subset \mathbb{R}^{r-1}$ and $D = \operatorname{div}_W(P) \subset \mathbb{R}^{r-1}$ are matroidal fans.*

Proof. [Sha13, Proposition 2.25] □

Tropical cohomology is invariant under closed tropical modifications [Sha15, Theorem 4.13]. The next lemma checks that these isomorphisms also apply to cohomology with compact support and that they are compatible with the PD map.

Proposition 2.2.46. *Let $\delta\colon \overline{V} \to W$ be a closed tropical modification of matroidal fans where $W \subset \mathbb{R}^{r-1}$ and $\overline{V} \subset \mathbb{R}^{r-1} \times \mathbb{T}$. Then there are isomorphisms*

$$\delta^*\colon H^{p,q}(W) \to H^{p,q}(\overline{V}) \qquad \text{and} \qquad \delta^*\colon H^{p,q}_c(W) \to H^{p,q}_c(\overline{V}),$$

which are induced by the pullback of superforms and are compatible with the Poincaré duality map.

Proof. The fact that δ^* is an isomorphism for tropical cohomology is shown in [Sha15, Theorem 4.13]. By Proposition [JSS15, Proposition 3.20] this also applies to $H^{p,q}$. The same arguments as used in [Sha15, Theorem 4.13] work for cohomology with compact support, since δ and the homotopy used there are proper maps. Thus again by [JSS15, Proposition 3.20], this also applies to $H^{p,q}_c$.

To show that the isomorphism δ^* is compatible with the Poincaré duality map, it suffices to show that for $\omega \in \mathcal{A}^{n,n}_c(W)$ we have

$$\int_W \omega = \int_{\overline{V}} \delta^*(\omega),$$

since the wedge product is compatible with the pullback. The fan W is the push-forward of $\overline{V} \cap \mathbb{R}^r$ along δ in the sense of polyhedral subspaces (cf. Definition 2.1.51) and then the result follows from the projection formula 2.1.52 because the support of $\delta^{-1}(\omega)$ is contained in $\overline{V} \cap \mathbb{R}^r$ by Lemma 2.1.44. □

The next statements relate the cohomology groups of the matroidal fans appearing in an open tropical modification by exact sequences.

Proposition 2.2.47. *Let Ω be an open subset of a polyhedral subspace in $\mathbb{T}^r$. Let $i \in [r]$ and write $D = \Omega \cap \mathbb{T}^r_i$ and $U := \Omega \setminus D$. Then there exists a long exact sequence in cohomology with compact support*

$$\ldots \to H^{p,q-1}_c(D) \to H^{p,q}_c(U) \to H^{p,q}_c(\Omega) \to H^{p,q}_c(D) \to H^{p,q+1}_c(U) \to \ldots$$

Proof. Note first that U and Ω are polyhedral spaces via the inclusion into $\mathbb{T}^r$. Further D is a polyhedral space via the inclusion into $\mathbb{T}^r_i$. We claim that the natural sequence of complexes

$$0 \to \mathcal{A}^{p,\bullet}_{\Omega,c}(U) \to \mathcal{A}^{p,\bullet}_{\Omega,c}(\Omega) \to \mathcal{A}^{p,\bullet}_{D,c}(D) \to 0$$

is exact. By the condition of compatibility, if a superform restricts to 0 on D, then it must be 0 on a neighborhood of D. This shows exactness in the middle of the short exact sequence. Both surjectivity of the last map and injectivity of the first map are clear. The result then follows by the long exact cohomology sequence. □

If $\delta\colon V \to W$ is a tropical modification with divisor D, upon writing $U := V$ and $\Omega := \overline{V}$ we can apply Proposition 2.2.47, where we identify D with the subspace of $\overline{V}$ given by $\overline{V} \cap \mathbb{T}^r_r$, as explained in Remark 2.2.44. Together with Proposition 2.2.46, we can relate the cohomology with compact support of V with the ones of W and D.

Lemma 2.2.48. *Let $V \subset \mathbb{R}^r$ be a matroidal fan of dimension n. Then for all p*

$$H^{p,q}_c(V) = 0 \text{ if } q \neq n.$$

Proof. The lemma is proven by induction on r, which is the dimension of the surrounding space. When $r = 0$ the assertion is clear. We now argue from $r-1$ to r: If $r = n$, we are in the case $V = \mathbb{R}^n$, since $\mathbb{R}^n$ is the only matroidal fan of dimension n in $\mathbb{R}^n$. Then we have $H^{p,q}_c(\mathbb{R}^n) = \Lambda^p\mathbb{R}^{n*} \otimes H^q_c(\mathbb{R}^n)$, where $H^q_c(\mathbb{R}^n)$ denotes the usual de Rham cohomology with compact support of $\mathbb{R}^n$. We have $H^q_c(\mathbb{R}^n) = 0$ unless $q = n$, thus we have the statement in this case. Otherwise $r > n$ and we can apply Proposition 2.2.45. Thus there exists a matroidal fan W and a tropical modification $\delta\colon V \to W$ whose divisor $D \subset W$ is a matroidal fan. Now by the induction assumption, $H^{p,q}_c(D) = 0$ unless $q = n-1$ and $H^{p,q}(W) = 0$ unless $q = n$. Applying the long exact sequence from Proposition 2.2.47 and replacing $H^{p,q}_c(\overline{V})$ with $H^{p,q}(W)$ by Proposition 2.2.46 we have

$$\ldots \to H^{p,q-1}_c(D) \to H^{p,q}_c(V) \to H^{p,q}_c(W) \to \ldots,$$

thus we obtain that $H^{p,q}_c(V) = 0$ if $q \neq n$ and the lemma is proven. □

The following short exact sequence involving the $(p,0)$-cohomology groups of matroidal fans is a consequence of a short exact sequence for Orlik-Solomon algebras of matroids [OT92]. For the translation to our setting see [Sha11, Lemma 2.2.7] and [Zha13]. Recall the contraction of superforms we defined in Definition 2.1.32.

Lemma 2.2.49. *Let $V \subset \mathbb{R}^{r+1}$ and $W \subset \mathbb{R}^r$ be matroidal fans and $\delta\colon V \to W$ be an open tropical modification along a divisor $D \subset W$ which is a matroidal fan, then*

$$0 \longrightarrow H^{p,0}(W) \longrightarrow H^{p,0}(V) \xrightarrow{\langle \cdot\, ; e_i\rangle_p} H^{p-1,0}(D) \longrightarrow 0$$

is an exact sequence.

It follows from [Sha11] that the map $H^{p,0}(V) \to H^{p-1,0}(D)$ in the exact sequence above is induced by the contraction $\langle \cdot\, ; e_i\rangle_p$ with e_i in the p-th component, where the vector e_i generates the kernel of the linear projection giving the map $\delta\colon V \to W$. Note that this map is not induced by a map on the level of forms. However for a closed $(p,0)$-form α, the form $\langle \alpha; e_i\rangle_p \in \mathcal{L}^{p-1,0}(V)$ is then the restriction of a unique form in $\mathcal{L}^{p-1,0}(\overline{V})$ and we can restrict this form to D, since we can identify D with a subspace of $\overline{V}$. This is easy to see once we use the identifications from Proposition 2.2.25.

Lemma 2.2.50. *Let $\delta\colon V \to W$ be an open tropical modification of matroidal fans $V \subset \mathbb{R}^{r+1}$ and $W \subset \mathbb{R}^r$ along a divisor $D \subset W$ which is a matroidal fan. Then the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{p,0}(W) & \longrightarrow & H^{p,0}(V) & \xrightarrow{\langle\cdot\,;e_i\rangle_p} & H^{p-1,0}(D) & \longrightarrow & 0 \\ & & \downarrow \mathrm{PD} & & \downarrow \mathrm{PD} & & \downarrow (-1)^{n-1}\,\mathrm{PD} & & \\ 0 & \longrightarrow & H_c^{n-p,n}(W)^* & \longrightarrow & H_c^{n-p,n}(V)^* & \xrightarrow{g^*} & H_c^{n-p,n-1}(D)^* & \longrightarrow & 0, \end{array}$$

which is obtained by the exact sequences in Proposition 2.2.47 and Lemma 2.2.49, is commutative.

Proof. Note that by Proposition 2.2.46 the statement is equivalent to the one we obtain when replacing W by $\overline{V}$. Then the fact that the first square commutes is immediate. The map $g\colon H_c^{n-p,n-1}(D) \to H_c^{n-p,n}(V)$ is the boundary operator in a long exact cohomology sequence. We recall its construction: For a closed superform $\beta \in \mathcal{A}_c^{n-p,n-1}(D)$, take any lift $l(\beta) \in \mathcal{A}_c^{n-p,n-1}(\overline{V})$ such that $l(\beta)|_D = \beta$. Then $d''(l(\beta))$ restricts to 0 on D and thus is a superform with compact support on V. Then $g(\beta)$ is given by the class of $d''(l(\beta))$ in $H_c^{n-p,n}(V)$. As usual this does not depend on the choice of $l(\beta)$. We have to show that for all closed forms $\alpha \in \mathcal{A}^{p,0}(V)$ and $\beta \in \mathcal{A}_c^{n-p,n-1}(V)$ we have

$$(-1)^p \int_V \alpha \wedge d''(l(\beta)) = (-1)^{n+p} \int_D \langle \alpha; e_i\rangle_p \wedge \beta$$

for some lift $l(\beta)$, where e_i is the coordinate direction of the modification. Let P be the piecewise affine function of the modification and $P' = P - 1$. The graph of P' divides V into two polyhedral complexes, one living above the graph and the other one below, which we denote by $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Equip all facets of both polyhedral complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ with weight 1. Note that $\delta(|\,\mathcal{C}_2\,|) \subset D$. We find a lift $l(\beta) \in \mathcal{A}_c^{n-p,n-1}(\overline{V})$ such that $l(\beta)|_{|\,\mathcal{C}_2\,|} = (\delta|_{|\,\mathcal{C}_2\,|})^*(\beta)$. Then we have

$$\int_V \alpha \wedge d''(l(\beta)) = \int_{\mathcal{C}_1} \alpha \wedge d''(l(\beta)) + \int_{\mathcal{C}_2} \alpha \wedge d''(l(\beta)) = \int_{\mathcal{C}_1} \alpha \wedge d''(l(\beta)).$$

By Stokes' theorem 2.1.49 and the Leibniz rule we have

$$\int_{\mathcal{C}_1} \alpha \wedge d''(l(\beta)) = (-1)^p \int_{\partial\,\mathcal{C}_1} \alpha \wedge l(\beta).$$

It follows from the proof of [Gub13a, Theorem 3.8] that the boundary integral of $\alpha \wedge l(\beta)$ over balanced codimension 1 faces vanishes. We further have that the unbalanced faces of $\mathcal{C}_1$ are precisely the ones in the polyhedral subspace $D' := \mathcal{C}_1 \cap \Gamma_{P'}(W) = \mathcal{C}_2 \cap \Gamma_{P'}(W)$. The facets of the polyhedral subspace D' are equipped with weight 1. Thus we obtain

$$\int_{\partial\,\mathcal{C}_1} \alpha \wedge l(\beta) = \int_{D'} \langle \alpha \wedge l(\beta); e_i\rangle_n.$$

Since $l(\beta)|_{D'} = (\delta|_{D'})^*(\beta)$ and $\delta(e_i) = 0$, we have that $\langle l(\beta); e_i\rangle_p|_{D'} = 0$ and therefore

$$\int_{D'} \langle \alpha \wedge l(\beta), e_i\rangle_n = (-1)^{n-p} \int_{D'} \langle \alpha \wedge l(\beta); e_i\rangle_p = (-1)^{n-p} \int_{D'} \langle \alpha; e_i\rangle_p \wedge l(\beta).$$

Altogether, we obtain

$$\int_V \alpha \wedge d''(l(\beta)) = (-1)^n \int_{D'} \langle \alpha; e_i\rangle_p \wedge l(\beta).$$

Denote by $F\colon W \to \Gamma_{P'}(W)$ the map into the graph of the function P'. Then there exist polyhedral structures $\mathcal{D}$ on D and $\mathcal{D}'$ on D', such that for each facet σ of $\mathcal{D}$ the restriction $F|_\sigma$ is linear, the image $F(\sigma)$ is a facet of $\mathcal{D}'$ and each facet of $\mathcal{D}'$ is of this form. Then the inverse of $F|_\sigma$ is given by $\delta|_{\sigma'}$. Thus $\delta|_{\sigma'}$ is an isomorphism of $\mathbb{R}$-rational polyhedra for each $\sigma' \in \mathcal{D}'$. Since we have that δ^* preserves $\langle \alpha; e_i\rangle_p$ and $(\delta|_{D'})^*\beta = l(\beta)|_{D'}$ we obtain

$$\int_{D'} \langle \alpha; e_i\rangle_p \wedge l(\beta) = \int_D \langle \alpha; e_i\rangle_p \wedge \beta,$$

which concludes the proof. □

Proposition 2.2.51. *Let $V \subset \mathbb{R}^r$ be a matroidal fan, then V has Poincaré duality.*

Proof. Let n be the dimension of V. We perform induction on r. The base case $r = 0$ is obvious.
For the induction step, we have two cases, these being $n = r$ and $n < r$. If $n = r$, then $V = \mathbb{R}^n$ and we have

$$H^{p,q}(\mathbb{R}^n) = \Lambda^p \mathbb{R}^{n*} \otimes H^q(\mathbb{R}^n) \quad \text{and} \quad H_c^{p,q}(\mathbb{R}^n) = \Lambda^p \mathbb{R}^{n*} \otimes H_c^q(\mathbb{R}^n),$$

where H^q, respectively H_c^q, denote the usual de Rham cohomology. Thus we know $H^{p,q}(\mathbb{R}^n) = 0$ and $H_c^{n-p,n-q}(\mathbb{R}^n) = 0$ unless $q = 0$. Otherwise $H^{p,0}(\mathbb{R}^n) = \Lambda^p \mathbb{R}^{n*}$ and $H_c^{n-p,n}(\mathbb{R}^n) = \Lambda^{n-p}\mathbb{R}^{n*}$ and the PD map is just $(-1)^p$ times the map induced by the canonical pairing $\Lambda^p\mathbb{R}^{n*} \times \Lambda^{n-p}\mathbb{R}^{n*} \to \Lambda^n \mathbb{R}^{n*} \cong \mathbb{R}$. Since this pairing is non-degenerate the PD map is an isomorphism.

If $n < r$ then by Theorem 2.2.27 and Lemma 2.2.48 the only non-trivial case to check is when $q = 0$. In other words, that PD: $H^{p,0}(V) \to H_c^{n-p,n}(V)^*$ is an isomorphism. Consider an open tropical modification $\delta\colon V \to W$ along a divisor D where $D, W \subset \mathbb{R}^{r-1}$ are matroidal fans. Now D and W have PD by the induction hypothesis, so that in the commutative diagram from Lemma 2.2.50 the vertical arrows on the left and right are isomorphisms. By the five lemma we obtain PD for $V \subset \mathbb{R}^r$ and the proposition is proven. □

The next two lemmas help to prove Proposition 2.2.54, which is analogous to Proposition 2.2.51 but for spaces of the form $V \times \mathbb{T}^r$ where V is a matroidal fan. We will relate the cohomologies of $V \times \mathbb{T}^r$, $V \times \mathbb{R}^r$ and V by way of an exact sequence.

Lemma 2.2.52. *Let $Y = V \times \mathbb{T}^r$ for a polyhedral fan V. Then we have a short exact sequence*

$$0 \longrightarrow H^{p,0}(Y \times \mathbb{T}) \longrightarrow H^{p,0}(Y \times \mathbb{R}) \xrightarrow{\langle\ \cdot\ ;e_i\rangle_p} H^{p-1,0}(Y) \longrightarrow 0$$

where e_i is the coordinate of $\mathbb{R}$ in $Y \times \mathbb{R}$.

Proof. We use the explicit calculation in Proposition 2.2.25. First this shows that none of the cohomology groups in the statement change when we replace Y by V, thus we assume $Y = V$. After identifying $\mathcal{L}^p$ with $H^{p,0}$ and undualizing we have to show that

$$0 \to \sum_{\sigma \in Y} \Lambda^{p-1}\mathbb{L}(\sigma) \stackrel{\wedge e_i}{\to} \sum_{\sigma \in Y} \Lambda^p \mathbb{L}(\sigma \times \mathbb{R}) \to \sum_{\sigma \in Y} \Lambda^p \mathbb{L}(\sigma) \to 0 \tag{2.9}$$

is exact. The first map is clearly injective and the last map is clearly surjective. For exactness in the middle notice that the composition of the maps is certainly zero and that any element $v \in \sum_{\sigma \in Y} \Lambda^p \mathbb{L}(\sigma \times \mathbb{R})$ can be written as $v = \sum_{\sigma \in Y} v_\sigma + \left(\sum_{\sigma \in Y} v'_\sigma\right) \wedge e_i$ for $v_\sigma \in \Lambda^p \mathbb{L}(\sigma)$ and $v'_\sigma \in \Lambda^{p-1}\mathbb{L}(\sigma)$. Now if v maps to zero, it is of the form $\left(\sum_{\sigma \in Y} v'_\sigma\right) \wedge e_i$ and thus in the image of $\wedge e_i$. This proves exactness. □

Lemma 2.2.53. *Let $Y = V \times \mathbb{T}^r$ of dimension n, where $V \subset \mathbb{R}^s$ is a matroidal fan, then the following diagram*

$$\begin{array}{ccccc} H^{p,0}(Y \times \mathbb{T}) & \longrightarrow & H^{p,0}(Y \times \mathbb{R}) & \xrightarrow{\langle\ \cdot\ ;e_i\rangle_p} & H^{p-1,0}(Y) \\ \downarrow \text{PD} & & \downarrow \text{PD} & & \downarrow (-1)^n \text{PD} \\ H_c^{n-p+1,n+1}(Y \times \mathbb{T})^* & \longrightarrow & H_c^{n-p+1,n+1}(Y \times \mathbb{R})^* & \xrightarrow{g^*} & H_c^{n-p+1,n}(Y)^*, \end{array}$$

which is obtained by the sequences in Proposition 2.2.47 and Lemma 2.2.52, commutes.

Proof. The proof follows exactly along the lines of the proof of the commutativity of the diagram in Lemma 2.2.50 for tropical modifications with Y replacing D, $Y \times \mathbb{R}$ replacing V and $Y \times \mathbb{T}$ replacing $\overline{V}$ and P' being any constant function. □

Proposition 2.2.54. *Let $Y = V \times \mathbb{T}^r$ for a matroidal fan $V \subset \mathbb{R}^s$. Then Y has Poincaré duality.*

Proof. We do induction on r with $r = 0$ being Proposition 2.2.51. For the induction step we have to show that if Y has PD then $Y \times \mathbb{T}$ also has PD. Since Y is a basic open subset, $H^{p,q}(Y) = 0$ unless $q = 0$ by Theorem 2.2.27. Since Y has PD, we have $H_c^{p,q}(Y) = 0$ unless $q = n = \dim(Y)$. Note also that $Y \times \mathbb{R} = V \times \mathbb{R} \times \mathbb{T}^r$ and so this space has PD. We therefore also have $H_c^{p,q}(Y \times \mathbb{R}) = 0$ unless $q = n+1 = \dim(Y \times \mathbb{R})$. Now the sequence from Proposition 2.2.47 yields that $H_c^{p,q}(Y \times \mathbb{T}) = 0$ if $q \neq n, n+1$ and that

$$0 \to H_c^{p,n}(Y \times \mathbb{T}) \to H_c^{p,n}(Y) \xrightarrow{f} H_c^{p,n+1}(Y \times \mathbb{R}) \to H_c^{p,n+1}(Y \times \mathbb{T}) \to 0 \tag{2.10}$$

is exact. By the commutativity of the second square of the diagram in Lemma 2.2.53, the map f is up so sign the dual map to $\langle\,\cdot\,;e_i\rangle_p$, once we use PD for Y and $Y\times\mathbb{R}$ to identify $H^{p,0}(Y\times\mathbb{R})\cong H_c^{n-p+1,n+1}(Y\times\mathbb{R})^*$ and $H^{p-1,0}(Y)\cong H_c^{n-p+1,n}(Y)^*$. Now $\langle\,\cdot\,;e_i\rangle_p$ is known to be surjective by Lemma 2.2.52, thus f is injective and we have $H_c^{p,q}(Y\times\mathbb{T})=0$ unless $q=n+1$. Since $Y\times\mathbb{T}$ is a basic open subset, we also know $H^{p,q}(Y\times\mathbb{T})=0$ unless $q=0$ by Theorem 2.2.27, thus we only have to consider PD : $H^{p,0}(Y\times\mathbb{T})\to H_c^{n+1-p,n+1}(Y\times\mathbb{T})$. Note that this is precisely the first vertical map in the diagram in Lemma 2.2.53 and that the respective first horizontal maps are injective by Lemma 2.2.52 and the sequence (2.10). Since the other vertical maps are isomorphisms, this shows that $Y\times\mathbb{T}$ has PD. □

Before we complete the proof of Theorem 2.2.58, we make the following observation.

Remark 2.2.55. By Proposition 2.2.25 and Theorem 2.2.27 we have $H^{p,q}(V\times\mathbb{T}^r)=H^{p,q}(V)$. Since Poincaré duality holds for these spaces, we further have $H_c^{p,q}(V)=H_c^{p+r,q+r}(V\times\mathbb{T}^r)$. This is the same behavior that taking the product with $\mathbb{C}$ exhibits for classical Dolbeault cohomology.

The following technical lemma allows us to deduce Poincaré duality for basic open subsets of matroidal fans.

Lemma 2.2.56. *Let $Y=V\times\mathbb{T}^r\subset\mathbb{R}^s\times\mathbb{T}^r$ for a matroidal fan $V\subset\mathbb{R}^s$ and Ω a basic open neighborhood of $(0,\dots,0,-\infty,\dots,-\infty)$. Then there are canonical isomorphisms*

$$H^{p,q}(Y)\to H^{p,q}(\Omega)\ \textit{and}$$
$$H_c^{p,q}(\Omega)\to H_c^{p,q}(Y)$$

which are induced by restriction and inclusion of superforms. In particular Ω has PD.

Proof. For $H^{p,q}$ this follows immediately from the explicit calculation we did in Proposition 2.2.25. For cohomology with compact support, we first see that there is a homeomorphism between U and Y which respects strata and polyhedra. This induces an isomorphism of compactly supported tropical cohomology and thus we already know $H_c^{p,q}(\Omega)\cong H_c^{p,q}(Y)$ by [JSS15, Theorem 3.18]. By PD for Y we furthermore know that these cohomology groups are finite dimensional, thus it is sufficient to show that inclusion of superforms induces a surjective map on cohomology. Again by PD for Y this is trivial if $q\neq n:=\dim Y$. We choose a basis $\alpha_1,\dots,\alpha_k$ of $H^{p,0}(Y)$. By PD for Y there exist $\omega_1,\dots,\omega_k\in H_c^{n-p,n}(Y)$ such that $\int_Y\alpha_i\wedge\omega_j=\delta_{ij}$ and for surjectivity of $H_c^{n-p,n}(\Omega)\to H_c^{n-p,n}(Y)$ it is sufficient to show that there exist $\beta_1,\dots,\beta_k\in H_c^{n-p,n}(\Omega)$ such that $\int_Y\alpha_i\wedge\beta_j\neq0$ if and only if $i=j$. Let B be the union of the supports of all ω_i and $C\in\mathbb{R}_{>0}$ and $v\in\mathbb{R}^r$ such that $B\subset C\cdot\Omega+v$. Define F to be the extended affine map given by $w\mapsto C\cdot w+v$ and set $\beta_i:=F^*(\omega_i)\in\mathcal{A}_c^{p,q}(\Omega)$ for all i. Since $\alpha_i\in\mathcal{L}^p(Y)$ we have $F^*(\alpha_i)=C^p\alpha_i$ and thus we have

$$\int_Y\alpha_i\wedge\beta_j=\int_Y\alpha_i\wedge F^*(\omega_j)=C^{-p}\int_Y F^*(\alpha_i\wedge\omega_j)=C^{n-p}\delta_{ij},$$

where the last equality is given by the transformation formula Proposition 2.1.9. This proves surjectivity and thus the lemma. □

Lemma 2.2.57. *Let V be a matroidal fan in $\mathbb{R}^s$, $Y = V \times \mathbb{T}^r$ and Ω a basic open subset of Y. Then Ω has* PD.

Proof. If $I \subset [r]$ is the maximal sedentarity among points of Ω, then Ω is a basic open subset of $V \times \mathbb{T}^{|I|} \times \mathbb{R}^{r-|I|}$ of maximal sedentarity. Let x be a point in the relative interior of the minimal face of the basic open set Ω. The polyhedral star $\Omega_x(V)$ of any point in a matroidal fan is again a matroidal fan, see [AK06, Proposition 2]. Applying this fact to the fan $V \times \mathbb{R}^{r-|I|}$ we obtain that, after translation of x to the origin, Ω is a basic open neighborhood of $(0, \dots, 0, -\infty, \dots, -\infty)$ in the matroidal fan $\Omega_x(V) \times \mathbb{T}^{|I|}$. Thus we are now in the situation of Lemma 2.2.56, which shows that Ω has PD. □

Theorem 2.2.58. *Let X be an n-dimensional tropical manifold. Then the Poincaré duality map is an isomorphism for all p, q.*

Proof. We write $\mathcal{A}_c^{p,q*}$ for the presheaf $U \mapsto \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}_c^{p,q}(U), \mathbb{R})$. Then $\mathcal{A}_c^{p,q*}$ is a sheaf, since $\mathcal{A}^{p,q}$ is fine. Furthermore, $\mathcal{A}_c^{p,q*}$ is a flasque sheaf, since for $U' \subset U$ the inclusion $\mathcal{A}_c^{p,q}(U') \to \mathcal{A}_c^{p,q}(U)$ is injective. We then obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}^p & \longrightarrow & \mathcal{A}^{p,0} & \xrightarrow{d''} & \mathcal{A}^{p,1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{PD}} & & \downarrow{\scriptstyle \mathrm{PD}} & & \\
0 & \longrightarrow & \mathcal{L}^p & \longrightarrow & \mathcal{A}_c^{n-p,n*} & \xrightarrow{d''^*} & \mathcal{A}_c^{n-p,n-1*} & \longrightarrow & \dots
\end{array}$$

and we have

$$H^q(\mathcal{A}_c^{n-p,n-\bullet *}(U), d''^*) = (H_q(\mathcal{A}_c^{n-p,n-\bullet}(U), d''))^* = H_c^{n-p,n-q}(U)^*.$$

If we consider the sections of this diagram over a basic open subset, then the first row is exact by Theorem 2.2.27. By Lemma 2.2.57 the second row is then also exact. This shows that both rows are exact sequences of sheaves on X. Thus we have a commutative diagram of acyclic resolutions of $\mathcal{L}^p$, thus PD induces isomorphisms on the cohomology of the complexes of global sections, which precisely means that X has PD. □

Corollary 2.2.59. *Let X be a compact tropical manifold of dimension n. Then*

$$\mathrm{PD} : H^{p,q}(X) \to H^{n-p,n-q}(X)^*$$

is an isomorphism for all p, q.

Chapter 3

Differential forms on Berkovich analytic spaces and their cohomology

In this chapter, we consider real-valued differential forms on Berkovich spaces. These are defined by locally using tropicalizations and the theory of superforms on polyhedral (sub)spaces. We first recall the respective tropicalization maps in Section 3.1. In Section 3.2, consider different approaches to define these differential forms and show that they all produce the same forms. In Subsection 3.2.1, we recall Gubler's approach to these forms using algebraic moment maps and canonical tropical charts. In Subsection 3.2.2, we then define new approaches, using $\mathbb{A}$- resp. T-moment maps. We do this with the help of the theory of differential forms on polyhedral subspaces of $\mathbb{T}^r$ resp. polyhedral spaces, which we developed in Chapter 2. In Section 3.3, we show that the new approaches also work when K is trivially valued. In the last Section 3.4, we study the cohomology defined by these forms.

In this chapter, K is an algebraically closed field which is complete with respect to a non-archimedean absolute value. In Section 3.2, we assume that this abolsute value is non-trivial. In Section 3.3 we explain the trivially valued case.

A variety is an irreducible reduced separated K-scheme of finite type. For a variety X we denote by X^{an} the associated analytic space in the sense of Berkovich. For a morphism $F\colon X \to Y$ we denote by $F^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ its analytification (cf. [Ber90]).

3.1 Tropicalizations

In this section we recall properties of tropicalizations. We consider classical tropicalization of tori as well as tropicalization of affine space and extended tropicalization of toric varieties in the sense of Payne (cf. [Pay09]). We relate these notions with polyhedral (sub)spaces considered in Chapter 2.

Definition 3.1.1. Let Z be a closed subvariety of $\mathbb{G}_m^r = \mathrm{Spec}(K[T_1^{\pm 1}, \dots, T_r^{\pm 1}])$. The space $\mathbb{G}_m^{r,\mathrm{an}}$ is then the set of multiplicative seminorms on $K[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$ extending the absolute value on K. We denote by $\mathrm{Trop}(Z)$ the image of Z^{an} under the tropicalization map

$$\begin{aligned}\mathrm{trop}: \mathbb{G}_m^{r,\mathrm{an}} &\to \mathbb{R}^r\\ |.|_x &\mapsto (\log|T_1|_x \ldots, \log|T_r|_x).\end{aligned}$$

Note that we differ in sign from e.g. [Gub13a, Gub13b, Pay09] here, because of the following extension: Let Z be a closed subvariety of $\mathbb{A}^r = \mathrm{Spec}(K[T_1, \ldots, T_r])$. Then we define $\mathrm{Trop}(Z)$ to be the image of Z^{an} under the tropicalization map

$$\begin{aligned}\mathrm{trop}: \mathbb{A}^{r,\mathrm{an}} &\to \mathbb{T}^r\\ |.|_x &\mapsto (\log|T_1|_x, \ldots, \log|T_r|_x).\end{aligned}$$

Lemma 3.1.2. *Let $Z \subset \mathbb{A}^r$ be a closed subvariety. Then* $\mathrm{trop}: Z^{\mathrm{an}} \to \mathrm{Trop}(Z)$ *is a continuous proper map of topological spaces and* $\mathrm{Trop}(Z)$ *is a closed subset of* $\mathbb{T}^r$*. If K is non trivially valued, then we have*

$$\mathrm{Trop}(Z) = \overline{\mathrm{trop}(Z^{\mathrm{an}}(K))} = \overline{\mathrm{trop}(Z(K))}.$$

The corresponding statements for subvarieties of $\mathbb{G}_m^r$ are well known and their proofs work very similarly.

Proof. Continuity follows directly from the definition of the Berkovich topology. Any compact subset of $\mathbb{T}^r$ is contained in a product of intervals $[a_i, b_i]$, with $a_i, b_i \in \mathbb{T}$. The preimage of this product of intervals is by definition a Laurent domain in $\mathbb{A}^{r,\mathrm{an}}$, thus affinoid and therefore compact. This shows that $\mathrm{trop}: \mathbb{A}^{r,\mathrm{an}} \to \mathbb{T}^r$ is proper. Now since Z^{an} is a closed subset of $\mathbb{A}^r$, the restriction $\mathrm{trop}: Z^{\mathrm{an}} \to \mathbb{T}^r$ is proper. It thus has closed image, which shows that $\mathrm{Trop}(Z)$ is closed. Further $\mathrm{trop}: Z^{\mathrm{an}} \to \mathrm{Trop}(Z)$ is proper.

If K is non-trivially valued, then $Z^{\mathrm{an}}(K) = Z(K)$ is dense in Z^{an}. This shows the last statement. □

Theorem 3.1.3 (Bieri-Groves)**.** *Let Z be a closed subvariety of $\mathbb{G}_m^r$ of dimension n. Then* $\mathrm{Trop}(Z)$ *is an n-dimensional $\mathbb{R}$-rational polyhedral subspace of* $\mathbb{R}^r$*.*

Proof. This was originally proven by Bieri and Groves. See [Gub13b, Theorem 3.3] for a nice summary of proofs in different setups and generalities. □

Lemma 3.1.4. *Let Z be a closed subvariety of $\mathbb{A}^r$ such that $Z' := \mathbb{G}_m^r \cap Z$ is non-empty. Then* $\mathrm{Trop}(Z)$ *is the closure in $\mathbb{T}^r$ of* $\mathrm{Trop}(Z') \subset \mathbb{R}^r$*.*

Proof. By irreducibility of Z we have $\overline{Z'} = Z$ and thus $\overline{Z'^{\mathrm{an}}} = Z^{\mathrm{an}}$. Therefore $\mathrm{Trop}(Z) = \mathrm{trop}(Z^{\mathrm{an}}) = \mathrm{trop}(\overline{Z'^{\mathrm{an}}}) \subset \overline{\mathrm{Trop}(Z')} = \overline{\mathrm{Trop}(Z')}$. Since further $\mathrm{Trop}(Z)$ is closed by Lemma 3.1.2 and contains $\mathrm{Trop}(Z')$ we have equality. □

Corollary 3.1.5. *Let Z be a closed subvariety of $\mathbb{A}^r$ of dimension n. Then* $\mathrm{Trop}(Z)$ *is an n-dimensional $\mathbb{R}$-rational polyhedral subspace of* $\mathbb{T}^r$*.*

Proof. Let $I \subset [r]$ be the set of indices such that $T_i|_Z \equiv 0$. Then Z is a closed subvariety of the vanishing locus $V((T_i)_{i\in I}) \simeq \mathbb{A}^{r-|I|}$. After replacing $\mathbb{A}^r$ by this we may assume that that $Z' := Z \cap \mathbb{G}_m^r$ is non-empty. Then $\mathrm{Trop}(Z) = \overline{\mathrm{Trop}(Z')}$ by Lemma 3.1.4. Since $\mathrm{Trop}(Z')$ is a polyhedral space in $\mathbb{R}^r$ by Theorem 3.1.3, $\mathrm{Trop}(Z)$ is a polyhedral space in $\mathbb{T}^r$. Rationality now follows from rationality of $\mathrm{Trop}(Z')$ and Remark 2.1.24. □

Definition 3.1.6. Let N be a free abelian group of rank n, M its dual and denote by $N_\mathbb{R}$ resp. $M_\mathbb{R}$ the respective scalar extensions to $\mathbb{R}$. A rational cone $\sigma \in N_\mathbb{R}$ is a polyhedron defined by equations of the form $\varphi(\,.\,) \geq 0$ with $\varphi \in M$, which does not contain a positive dimensional linear subspace. A rational fan Δ in $N_\mathbb{R}$ is a polyhedral complex all of whose polyhedra are rational cones. For $\sigma \in \Delta$ we define the monoid $S_\sigma := \{\varphi \in M \mid \varphi(v) \geq 0 \text{ for all } v \in \sigma\}$. We denote by $U_\sigma := \mathrm{Spec}(K[S_\sigma])$. For $\tau \prec \sigma$ we obtain open immersions $U_\tau \to U_\sigma$. We define the toric variety Y_Δ to be the gluing of the $(U_\sigma)_{\sigma\in\Delta}$ along these open immersions. For an introduction to toric varieties, see for example [Ful93].

Remark 3.1.7. The toric variety Y_Δ comes with an open immersion $T \to Y_\Delta$, where $T = \mathrm{Spec}(K[M])$ and a T-action which extended the group action of T on itself by translation. In fact any normal variety with such an immersion and action arises by the above described procedure ([CLS11, Corollary 3.1.8]). This was shown by Sumihiro.

Choosing a basis of N gives an identification $N \simeq \mathbb{Z}^r \simeq M$ and $T \simeq \mathbb{G}_m^r$.

Definition 3.1.8. Let $\Delta \subset N_\mathbb{R}$ be a fan and Y_Δ be the corresponding toric variety. Payne defined in [Pay09] a tropicalization map

$$\mathrm{trop} : Y_\Delta^{\mathrm{an}} \to \mathrm{Trop}(Y_\Delta)$$

to the topological space $\mathrm{Trop}(Y_\Delta)$ as follows:

For $\sigma \in \Delta$, write $N(\sigma) := N_\mathbb{R}/\operatorname{span}(\sigma)$. Then as a set we have

$$\mathrm{Trop}(Y_\Delta) = \coprod_{\sigma\in\Delta} N(\sigma).$$

The topology of $\mathrm{Trop}(Y)$ is given in the following way:

For $\sigma \in \Delta$ we consider the affine toric subvariety U_σ. By the definition above we have $\mathrm{Trop}(U_\sigma) = \coprod_{\tau\prec\sigma} N_\tau$. This is naturally identified with $\mathrm{Hom}_{\mathrm{Monoids}}(S_\sigma, \mathbb{T})$ (for details, cf. [Pay09, Section 3]). We give $\mathrm{Trop}(U_\sigma)$ the subspace topology of $\mathbb{T}^{S_\sigma}$. For $\tau \prec \sigma$, the space $\mathrm{Hom}(S_\tau, \mathbb{T})$ is naturally identified with the open subspace of $\mathrm{Hom}_{\mathrm{Monoids}}(S_\sigma, \mathbb{T})$ of maps which map $\tau^\perp \cap M$ to $\mathbb{R}$. We then define the topology of $\mathrm{Trop}(Y_\Delta)$ to be the one obtained by gluing along these identifications.

For Z a closed subvariety of Y_Δ we define $\mathrm{Trop}(Z)$ to be the image of Z^{an} under $\mathrm{trop} : Y_\Delta^{\mathrm{an}} \to \mathrm{Trop}(Y_\Delta)$.

Lemma 3.1.9. *Let Y_Δ be a toric variety of dimension n with corresponding fan Δ. Then the tropicalization* $\mathrm{Trop}(Y_\Delta)$ *of Y_Δ has a canonical structure as an n-dimensional $\mathbb{R}$-rational polyhedral space.*

Proof. Let $\sigma \in \Delta$ and choose a finite generator set B_σ of the monoid S_σ. As noted in [Pay09, Remark 3.1] this gives rise to an identification $\operatorname{Hom}(S_\sigma, \mathbb{T})$ with a subspace of $\mathbb{T}^r$ and thus to an embedding $\psi_{B_\sigma} : \operatorname{Trop}(U_\sigma) \hookrightarrow \mathbb{T}^r$. Further B_σ also gives rise to a closed immersion $\varphi_{B_\sigma} : U_\sigma \to \mathbb{A}^r$. The diagram

$$\begin{array}{ccc} U_\sigma^{\mathrm{an}} & \xrightarrow{\varphi_{B_\sigma}^{\mathrm{an}}} & \mathbb{A}^{r,\mathrm{an}} \\ \downarrow & & \downarrow \scriptstyle{\mathrm{trop}} \\ \operatorname{Trop}(U_\sigma) & \xrightarrow{\psi_{B_\sigma}} & \mathbb{T}^r \end{array} \tag{3.1}$$

commutes and thus the image $\psi_{B_\sigma}(\operatorname{Trop}(U_\sigma)) \subset \mathbb{T}^r$ agrees with $\operatorname{Trop}(\varphi_{B_\sigma}(U_\sigma))$ in the sense of Definition 3.1.1. We conclude by Corollary 3.1.5 that this is indeed a tropical subspace.

Let M be a set of pairs (σ, B_σ) such that

i) $\sigma \in \Delta$ and B_σ is a finite generating set of S_σ.

ii) Each maximal $\sigma \in \Delta$ appears in at least one pair in M.

We define an atlas for $\operatorname{Trop}(Y_\Delta)$ by

$$A_M = \Big(\psi_{B_\sigma} : \operatorname{Trop}(U_\sigma) \to \operatorname{Trop}(\varphi_{B_\sigma}(U_\sigma)) \subset \mathbb{T}^{|B_\sigma|}\Big)_{(\sigma, B_\sigma)\in M}. \tag{3.2}$$

The domains of these cover $\operatorname{Trop}(Y_\Delta)$. Further, Definition 2.1.54 *i)* is satisfied by our discussion above. We show that all these are indeed atlases and are equivalent by showing that their union is an atlas, for which we need to show that coordinate changes $\psi_{B_\tau} \circ \psi_{B_\sigma}^{-1}$ for $\sigma, \tau \in \Delta$ are extended affine maps.

We do this in two cases, first assuming that $\sigma = \tau$ and just varying the generator sets and then assuming $\tau \prec \sigma$ and choosing special generating sets. The general case then follows.

If $\sigma = \tau$, then the matrix of $\psi_{B_{\sigma,1}} \circ \psi_{B_{\sigma,2}}^{-1}$ is just given by the presentation of the elements of $B_{\sigma,1}$ as a linear combination of elements of $B_{\sigma,2}$. Since these are generating sets of monoids, all entries are positive.

If $\tau \prec \sigma$, we choose B_σ in such a way that a subset $\{v_1, \ldots, v_k\}$ generates $\tau^\perp \cap M$. Then we take $B_\tau := B_\sigma \cup \{-v_1, \ldots, -v_k\}$. The map $\psi_{B_\tau} \circ \psi_{B_\sigma}^{-1}$ is again just given by writing the elements of B_σ as a linear combination of elements of B_τ and this again only has positive entries. For $\psi_{B_\sigma} \circ \psi_{B_\tau}^{-1}$ we write the elements of B_τ as a linear combination of elements of B_σ. Denote the corresponding matrix by D. By construction of B_τ, the only entries of D are on the diagonal and they are all 1 or -1. Further, if $b_{ii} = -1$, then $\operatorname{Trop}(\varphi_{B_\tau}(U_\tau))$ does not contain any points whose sedentarity contains i. Thus D defines a well defined extended affine map $\operatorname{Trop}(\varphi_{B_\sigma}(U_\tau)) \to \operatorname{Trop}(\varphi_{B_\tau}(U_\tau))$. $\square$

Corollary 3.1.10. *Let Z be an n-dimensional closed subvariety of a toric variety Y_Δ. Then the canonical structure of* $\operatorname{Trop}(Y_\Delta)$ *as a polyhedral space induces a canonical structure for* $\operatorname{Trop}(Z)$ *as a polyhedral space of dimension n.*

Proof. Restricting the charts of the atlas constructed in Lemma 3.1.9 to $\mathrm{Trop}(U_\sigma \cap Z) = \mathrm{Trop}(U_\sigma) \cap \mathrm{Trop}(Z)$ defines an atlas for $\mathrm{Trop}(Z)$ by Corollary 3.1.5. □

Lemma 3.1.11. *Let Z be a closed subvariety of a toric variety Y_Δ. The tropicalization map* $\mathrm{trop} : Z^{\mathrm{an}} \to \mathrm{Trop}(Z)$ *is a proper map of topological spaces.*

Proof. Since Z^{an} is a closed subvariety of Y_Δ^{an}, it is enough to show this for $Z = Y_\Delta$. The tropicalization map of Y_Δ is just the gluing of the tropicalization maps for open affine toric subvarieties U_σ. Since properness is a local property, we may thus assume $Y_\Delta = U_\sigma$ is affine. Now any finite generating set B_σ gives a closed immersion $\varphi_{B_\sigma} : U_\sigma \hookrightarrow \mathbb{A}^r$ and an embedding $\psi_{B_\sigma} : \mathrm{Trop}(U_\sigma) \hookrightarrow \mathbb{T}^r$. The commutativity of diagram (3.1) and properness of $\mathrm{trop} : \mathbb{A}^r \to \mathbb{T}^r$ show the result. □

Remark 3.1.12. Let Z be a closed irreducible subset of a toric variety Y_Δ that meets the dense torus $\mathbb{G}_m^r$. Note that by construction of the atlas for $\mathrm{Trop}(Z)$ resp. $\mathrm{Trop}(Y_\Delta)$ the set $\mathrm{Trop}(\mathbb{G}_m^r \cap Z)$ is a dense open subset of $\mathrm{Trop}(Z)$ which admits a chart to $\mathbb{R}^r$.

Remark 3.1.13. Let L/M be an extension of non-archimedean, algebraically closed complete fields. Let Z be a closed subvariety of $\mathbb{G}_{m,M}^r$ and denote by $Z_L := Z \times \mathrm{Spec}(L)$ the base change to L. Then Z_L is a subvariety of $\mathbb{G}_{m,L}^r$. By [Pay09, Section 6, Appendix] we have $\mathrm{Trop}(Z) = \mathrm{Trop}(Z_L)$

Definition 3.1.14. Let Z be a closed subvariety of $\mathbb{G}_m^r$. We define weights on $\mathrm{Trop}(Z)$ as follows: If K is non-trivially valued, let $x \in \mathrm{Trop}(Z)$ and $t \in Z^{\mathrm{an}}(K) = Z(K)$ such that $\mathrm{trop}(t) = x$. Since Z is integral, so is $t^{-1}Z$ and thus its Zariski closure (with the induced reduced structure) and its scheme theoretic closure in $\mathbb{G}_{m,K^\circ}^r$ agree. We write $\mathrm{in}_x(Z)$ for the special fiber of this closure. We then define $m(x) := \sum m_W$, where the sum goes over the irreducible components W of $\mathrm{in}_x(Z)$ and m_W is the multiplicity of the component W.

If K is trivially valued we choose an extension L/K such that L is non-trivally valued, complete and algebraically closed. We then define m on $\mathrm{Trop}(Z) = \mathrm{Trop}(Z_L)$ as above, using L as the ground field.

This procedure defines m for a dense subset of $\mathrm{Trop}(Z)$. It turns out that there is a polyhedral structure $\mathcal{C}$ on $\mathrm{Trop}(Z)$ such that $m(x)$ is constant over $\mathring{\sigma}$ for all maximal $\sigma \in \mathcal{C}$. Thus $\mathrm{Trop}(Z)$ is a weighted $\mathbb{R}$-rational polyhedral subspace in $\mathbb{R}^r$. We refer to [Gub13b, Section 5 & Section 13] for details on these constructions.

Let Z be a closed subvariety of $\mathbb{A}^r$. Then we observed in Corollary 3.1.5 that $\mathrm{Trop}(Z)$ is the closure of the tropicalization $\mathrm{Trop}(Z')$ of the intersection $Z' = Z \cap \mathbb{G}_m^{r'}$ for some torus $\mathbb{G}_m^{r'}$. We take on $\mathrm{Trop}(Z')$ a weighted polyhedral structure $\mathcal{C}$ as constructed above. This induces a weighted polyhedral structure on $\mathrm{Trop}(Z)$, where the maximal polyhedra are the closures in $\mathbb{T}^r$ of the maximal ones in $\mathcal{C}$ and we assign the same weight to the closure. This make $\mathrm{Trop}(Z)$ into a weighted polyhedral subspace of $\mathbb{T}^r$.

Let Z be a closed subvariety of a toric variety Y_Δ. Assume first that Z meets the dense torus. Then by Remark 3.1.12 there is a chart to $\mathbb{R}^r$ with dense domain. We define the weights in that chart as for the torus case. This determines the weights in all other charts uniquely, if we want our transition maps to preserve weights.

In general, let $Y_\Delta = \coprod Y_i$ be the decomposition of Y_Δ into torus orbits. Since Z is irreducible, it is contained in $\overline{Y_i}$ for some i such that $Y_i \cap Z$ is non-empty. Then Z is a closed subscheme of $\overline{Y_i}$ and meets the dense torus Y_i. We thus reduced to the case where Z meets the dense torus.

Theorem 3.1.15. *Let Z be a closed subvariety of $\mathbb{G}_m^r$ of dimension n. Then* $\mathrm{Trop}(Z)$ *is a tropical variety in $\mathbb{R}^r$ of dimension n.*

Proof. The case of trivial valuation was shown in [ST08, Corollary 3.8]. The general case is derived from this in [Gub13b, Theorem 13.11]. □

Corollary 3.1.16. *Let Z be a closed subvariety of $\mathbb{A}^r$ of dimension n. Then* $\mathrm{Trop}(Z)$ *is a tropical variety in $\mathbb{T}^r$ of dimension n.*

Proof. We may assume that $Z' = Z \cap \mathbb{G}_m^r$ is non-empty. Then $\mathrm{Trop}(Z')$ is a tropical variety by 3.1.15. The result follows them from Lemma 2.1.30 and Lemma 3.1.4 □

Corollary 3.1.17. *Let Z be a closed subvariety of a toric variety Y_Δ. Then* $\mathrm{Trop}(Z)$ *is an effective tropical space of dimension n.*

Proof. We use the atlas constructed in Lemma 3.1.9, with the notations introduced there. We have to show that for $\sigma \in \Delta$ and B_σ a finite generator set of S_σ, the image of $\psi_{B_\sigma}(\mathrm{trop}(Z^{\mathrm{an}} \cap U_\sigma^{\mathrm{an}})) \subset \mathbb{T}^r$ is a tropical variety with the weights as defined above. We already saw in the proof of Lemma 3.1.9 that this equals $\mathrm{Trop}(\varphi_{B_\sigma}^{\mathrm{an}}(Z^{\mathrm{an}} \cap U_\sigma^{\mathrm{an}}))$ as a set and it is a direct consequence of Definition 3.1.14 that both viewpoints produce the same weights . Now $\mathrm{Trop}(\varphi_{B_\sigma}^{\mathrm{an}}(Z^{\mathrm{an}} \cap U_\sigma^{\mathrm{an}}))$ is a tropical variety by Corollary 3.1.16, which proves the result. □

The following result gives an application of the results from Subsection 2.2.3 to the cohomology of tropicalizations of closed subvarieties of tori.

Proposition 3.1.18. *Let Z be a closed subvariety of a toric variety Y_Δ. Assume that* $\mathrm{Trop}(Z)$ *is regular at infinity. Then* $H_{d'}^{p,q}(\mathrm{Trop}(Z))$ *and* $H_d^k(\mathrm{Trop}(Z))$ *are finite dimensional for all $p, q, k \in \mathbb{N}_0$.*

Proof. Let $\sigma \in \Delta$. Since $\mathrm{Trop}(Z)$ is regular at infinity, so is $\mathrm{Trop}(Z \cap U_\sigma)$, and since this is a polyhedral subspace of some $\mathbb{T}^r$, we have that the cohomology groups of $\mathrm{Trop}(Z \cap U_\sigma)$ are finite dimensional by Theorem 2.2.34 and Theorem 2.2.36. Thus $(\mathrm{Trop}(Z \cap U_\sigma))_{\sigma \in \Delta}$ is a reasonable cover of $\mathrm{Trop}(Z)$ and Lemma A.2.2 shows our result. □

Definition 3.1.19. Let $Z = \sum_{i=1}^{k} \lambda_i Z_i$ be an algebraic cycle on $\mathbb{G}_m^r$. Then we define the tropical cycle $\mathrm{Trop}(Z) := \sum_{i=1}^{k} \lambda_i \,\mathrm{Trop}(Z_i)$.

Definition 3.1.20. Let $F\colon X \to Y$ be a morphism of algebraic varieties. The *degree of F* is defined as $\deg(F) := [K(X) : K(F(X))]$ if this is finite and 0 otherwise. We define the *push forward of X along F* to be the cycle $F_*(X) := \deg(F)\overline{F(X)}$.

The following Theorem is originally due to Sturmfels and Tevelev (cf. [ST08]) in the case of trivial valuation and was generalized by Baker, Payne and Rabinoff to the case of non-trivial valuation.

Theorem 3.1.21 (Sturmfels-Tevelev-multiplicity-formula)**.** *Let Z be a closed subvariety of $\mathbb{G}_m^{r'}$. Let $\psi\colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$ be a morphism of algebraic groups. Then*

$$\operatorname{Trop}(\psi)_*(\operatorname{Trop}(Z)) = \operatorname{Trop}(\psi_*(Z)).$$

Proof. See [Gub13b, Theorem 13.17], where this particular version is derived from [BPR16, Corollary 7.3]. □

3.2 Differential forms on Berkovich spaces

In this section, we recall the definition of differential forms on the analytification of an algebraic variety by Gubler (cf. [Gub13a]). We then give new and analogous definitions using the different types of tropicalizations from Section 3.1 and the theory of superforms on polyhedral spaces as introduced in Chapter 2. We show that all these produce the same sheaves and thus we can always use the most suitable definition when studying differential forms on the Berkovich analytification of an algebraic variety.

Chambert-Loir and Ducros (cf. [CLD12]) define sheaves of differential forms for general Berkovich analytic spaces. Gubler shows in [Gub13a, Section 7] that his approach is equivalent to theirs in the algebraic case.

In this section, we require K to be algebraically closed, complete and non-trivally valued. As before, X is a variety over K.

3.2.1 Approach using canonical tropical charts

In this subsection, we recall the algebraic approach to differential forms on the analytification of an algebraic variety from [Gub13a]. We give all the definitions and properties we will use and give most proofs, for the convenience of the reader. The approach by Chambert-Loir and Ducros in [CLD12] works in higher generality, but gives the same forms in our setup ([Gub13a, Section 7]). We choose to introduce Gubler's approach since we will give other approaches which work only in the algebraic case anyway.

Definition 3.2.1. Let X be a variety and U an open subset. Then a *moment map* of U is a map $\varphi\colon U \to \mathbb{G}_m^r$ for some r. The *tropicalization of* φ is

$$\varphi_{\text{trop}} := \operatorname{trop} \circ \varphi^{\text{an}}\colon U^{\text{an}} \to \mathbb{R}^r.$$

Let $U' \subset U$ be another open subset and $\varphi'\colon U' \to \mathbb{G}_m^{r'}$ be another moment map. We say that φ' *refines* φ if there exists a morphism of tori $\psi\colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$ (by which we mean a group homomorphism composed with a multiplicative translation), such that $\varphi = \psi \circ \varphi'$.

Remark 3.2.2. If a moment map $\varphi'\colon U' \to X$ refines $\varphi\colon U \to X$, then the map $\psi\colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$ induces a map between character lattices and thus an affine map $\operatorname{Trop}(\psi)\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ such that $\varphi_{\text{trop}} = \operatorname{Trop}(\psi) \circ \varphi'_{\text{trop}}$. Note that ψ and thus $\operatorname{Trop}(\psi)$ are not unique, but the restrictions to $\varphi'(U')$ resp. $\operatorname{trop}(\varphi'(U'))$ are.

If $\varphi_i\colon U_i \to \mathbb{G}_m^{r_i}$ are finitely many moment maps, then $\varphi := \varphi_1 \times \cdots \times \varphi_n\colon \bigcap_{i=1}^n U_i \to \mathbb{G}_m^{\sum r_i}$ is a moment map which refines all φ_i. In fact, every moment map refining all φ_i also refines φ.

Definition 3.2.3. Let X be a variety and U an open affine subset. Then U has a *canonical moment map*, which is constructed as follows: Denote $M_U := \mathcal{O}(U)^\times / K^\times$. Then M_U is a finitely generated free abelian group (cf. [Sam66, Lemme 1]). We choose representatives $\varphi_1, \dots, \varphi_r$ of a basis. We obtain a map

$$K[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \to \mathcal{O}_X(U),$$
$$T_i \mapsto \varphi_i$$

which leads to a map $\varphi_U \colon U \to \mathbb{G}_m^r$, which is canonical up to multiplicative translation and coordinate change. Note that the canonical moment map refines all moment maps of U by construction.

Remark 3.2.4. Let X be a variety and U an open subset. Then the following properties are equivalent

i) The K-algebra $\mathcal{O}_X(U)$ is generated by $\mathcal{O}_X(U)^\times$.

ii) The canonical moment map φ_U is a closed immersion.

iii) U has a closed immersion to $\mathbb{G}_m^s$ for some s.

Definition 3.2.5. Let X be a variety. An open subset U is called *very affine* if it satisfies the equivalent properties of Remark 3.2.4. In this case, we define $\operatorname{Trop}(U)$ as the image of $\varphi_{U,\mathrm{trop}} \colon U^{\mathrm{an}} \to \mathbb{R}^r$. Note that this is well defined up to integral affine translation.

Remark 3.2.6. Let $F \colon X' \to X$ be a morphism of algebraic varieties and let $U' \subset X'$ and $U \subset X$ be affine open subsets such that $F(U') \subset U$. Then the map $\mathcal{O}_X(U) \to \mathcal{O}_{X'}(U')$ induces an affine homomorphism $M_U \to M_{U'}$ which induces a morphism $\psi_{U,U'} \colon \operatorname{Spec}(K[M_{U'}]) \to \operatorname{Spec}(K[M_U])$ such that $\psi_{U,U'} \circ \varphi_{U'} = \varphi_U \circ F$.

Definition 3.2.7. Let X be a variety. A *canonical tropical chart* (V, φ_U) is given by an open set V of X^{an} and the canonical moment map φ_U of a very affine open subset U of X, such that $V \subset U^{\mathrm{an}}$ and $V = \varphi_{U,\mathrm{trop}}^{-1}(\Omega)$ for Ω an open subset of $\operatorname{Trop}(U)$. Another canonical tropical chart $(V', \varphi_{U'})$ is called a *canonical tropical subchart* of (V, φ_U) if $V' \subset V$ and $U' \subset U$.

Remark 3.2.8. Let X be a variety and $U' \subset U$ be open subsets. Then Remark 3.2.6 shows that there exists a map $\psi_{U,U'}$ such that $\psi_{U,U'} \circ \varphi_{U'} = \varphi_U$, which precisely means that φ_U' refines φ_U. Therefore by Remark 3.2.2 we have that $\varphi_{U,\mathrm{trop}} = \operatorname{Trop}(\psi_{U,U'}) \circ \varphi_{U',\mathrm{trop}}$. Thus $\operatorname{Trop}(\psi_{U,U'})$ restricts to a map

$$\operatorname{Trop}(\psi_{U,U'}) \colon \operatorname{Trop}(U') \to \operatorname{Trop}(U).$$

We conclude from [Gub13a, Lemma 4.9] that this map is surjective.

Further, if $(V', \varphi_{U'})$ is a subchart of (V, φ_U), then we have that $\operatorname{Trop}(\psi_{U,U'})(V') \subset V$.

Proposition 3.2.9. *Let X be a variety. Then we have the following properties:*

i) For every open subset $W \subset X^{\mathrm{an}}$ and every $x \in W$ there exists a canonical tropical chart (V, φ_U) such that $x \in V$ and $V \subset W$. Further, V can be chosen in such a way that $\varphi_{U,\mathrm{trop}}(V)$ is relatively compact in $\operatorname{Trop}(U)$.

ii) For canonical tropical charts (V, φ_U) and $(V', \varphi_{U'})$ the pair $(V \cap V', \varphi_{U \cap U'})$ is a canonical subchart of both.

iii) If (V, φ_U) is a canonical tropical chart and U' is a very affine open subset of X such that $V \subset U'^{\mathrm{an}} \subset U^{\mathrm{an}}$, then $(V, \varphi_{U'})$ is a canonical tropical chart (thus in particular a canonical tropical subchart of (V, φ_U)).

Proof. [Gub13a, Proposition 4.16]. □

Definition 3.2.10. Let X be a variety and $V \subset X^{\mathrm{an}}$ an open subset. A *differential form of bidegree (p, q) on V* is given by a family $(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}$ such that:

i) For all $i \in I$ the pair (V_i, φ_{U_i}) is a canonical tropical chart and $\bigcup_{i\in I} V_i = V$.

ii) For all $i \in I$ we have $\alpha_i \in \mathcal{A}^{p,q}_{\mathrm{Trop}_\varphi(U_i)}(\varphi_{U_i,\mathrm{trop}}(V_i))$.

iii) The α_i agree on intersections in the sense that for all $i, j \in I$, we have tropical subcharts $(V_{ijl}, \varphi_{U_{ijl}})$ of $(V_i \cap V_j, \varphi_{U_i\cap U_j})$ such that $V_i \cap V_j = \bigcup V_{ijl}$ and

$$\mathrm{Trop}(\psi_{U_i,U_{ijl}})^*(\alpha_i) = \mathrm{Trop}(\psi_{U_j,U_{ijl}})^*(\alpha_j) \in \mathcal{A}^{p,q}(\varphi_{U_{ijl},\mathrm{trop}}(V_{ijl}))$$

for all l.

Another such family $(V'_j, \varphi_{U'_j}, \beta_j)_{j\in J}$ defines the same form if there is a common refinement of the covers of V by canonical tropical charts such that the affine pullbacks to the refined cover agree. The space of these forms is denoted by $\mathcal{A}^{p,q}_X(V)$ or $\mathcal{A}^{p,q}(V)$ if there is no confusion about X.

Let $V' \subset V$ and $\alpha \in \mathcal{A}^{p,q}(V)$ given by $(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}$. We can cover V' by subcharts $(W_{ij}, \varphi_{U_{ij}})$ of the canonical tropical charts (V_i, φ_{U_i}) and then define $\alpha|_{V'}$ to be given by $(W_{ij}, \varphi_{U_{ij}}, \mathrm{Trop}(\psi_{U_i,U_{ij}})^*\alpha_i)_{ij}$. Note that this is independent of the chosen presentation of α.

Proposition 3.2.11. *The functor*

$$V \mapsto \mathcal{A}^{p,q}_X(V)$$

is a sheaf on X^{an} which we denote by $\mathcal{A}^{p,q}_X$.

Proof. This is a direct consequence of the local nature of Definition 3.2.10. □

Lemma 3.2.12. *Let $\alpha \in \mathcal{A}^{p,q}_X(V)$ be given by one canonical tropical chart (V, φ_U, α') and assume that $\alpha = 0$. Then $\alpha' = 0$.*

Proof. The equality $\alpha = 0$ means that there exist canonical tropical subcharts (V_i, φ_{U_i}) such that $V = \bigcup_{i\in I} V_i$ and such that $\mathrm{Trop}(\psi_{U,U_i})^*\alpha' = 0$. We argue locally, fixing a point $x \in \mathrm{Trop}(U)$. We choose Ω' a relatively compact open neighborhood of x in Ω and $V' := \varphi^{-1}_{U,\mathrm{trop}}(\Omega')$ to be its preimage. Since the tropicalization map is proper, V' is relatively compact and thus covered by finitely many V_i. Thus replacing V by V' and V_i by $V \cap V_i$ we may assume that I is finite.

We choose polyhedral structures $\mathcal{C}$ resp. $\mathcal{C}_i$ on $\mathrm{Trop}(U)$ resp. $\mathrm{Trop}(U_i)$ such that for each $\sigma_i \in \mathcal{C}_i$ there exists $\sigma \in \mathcal{C}$ such that $\mathrm{Trop}(\psi_{U,U_i})(\sigma_i) = \sigma$. We fix $\sigma \in \mathcal{C}$ and want to show $\alpha'|_{\Omega\cap\sigma} = 0$. Let $(\sigma_{ij} \in \mathcal{C}_i)_{ij}$ be the collection of polyhedra such that $\mathrm{Trop}(\psi_{U,U_i})(\sigma_{ij}) = \sigma$. For all i, j, the induced linear map $\mathbb{L}(\sigma_{ij}) \to \mathbb{L}(\sigma)$ is surjective, hence open. Thus $\Omega \cap \sigma = \bigcup_{ij} \mathrm{Trop}(\psi_{U,U_i})(\sigma_{ij} \cap \Omega_i)$ is an open cover. However, $\alpha'|_{\mathrm{Trop}(\psi_{U,U_i})(\sigma_{ij}\cap\Omega_i)} = 0$, since $\mathrm{Trop}(\psi_{U,U_i})^*\alpha'|_{\sigma_{ij}\cap\Omega_i} = 0$. □

Corollary 3.2.13. *Let* $\alpha_1 \in \mathcal{A}^{p,q}(V_1)$ *and* $\alpha_2 \in \mathcal{A}^{p,q}(V_2)$ *be given by single charts* $(V_i, \varphi_{U_i}, \alpha'_i)$. *If* $\alpha_1|_{V_1\cap V_2} = \alpha_2|_{V_1\cap V_2}$, *then* $\mathrm{Trop}(\psi_{U_1,U_1\cap U_2})^*\alpha_1 = \mathrm{Trop}(\psi_{U_2,U_1\cap U_2})^*\alpha_2$.

Remark 3.2.14. Corollary 3.2.13 shows that in Definition 3.2.10 *iii*) we may require

$$\mathrm{Trop}(\psi_{U_i,U_i\cap U_j})^*(\alpha_i) = \mathrm{Trop}(\psi_{U_j,U_i\cap U_j})^*(\alpha_j) \in \mathcal{A}^{p,q}(\mathrm{Trop}_{U_i\cap U_j}(V_i \cap V_j)).$$

This is for example done in [Jel16, Definition 4.3 iii)].

Definition 3.2.15. Let $\alpha \in \mathcal{A}^{p,q}(V)$ be given by $(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}$. Then for $D \in \{d', d'', d\}$ we define $D\alpha$ to be given by the family $(V_i, \varphi_{U_i}, D\alpha_i)_{i\in I}$ and $J\alpha$ to be given by $(V_i, \varphi_{U_i}, J\alpha_i)_{i\in I}$. If further $\beta \in \mathcal{A}^{p',q'}(V)$ is given by $(V_i, \varphi_{U_i}, \beta_i)$ then we define $\alpha \wedge \beta$ to be given by $(V_i, \varphi_{U_i}, \alpha_i \wedge \beta_i)_i$. Note here that we may assume β to be given on the same cover as α after pulling back to a common refinement. Since all these constructions commute with affine maps of the polyhedral complexes, these all define elements of $\mathcal{A}^{\bullet,\bullet}$ again and are independent of the respective presentations.

Definition 3.2.16. We obtain a sheaf of bigraded bidifferential algebras $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$ on X^{an}. We also obtain the sheaf of total graded differential algebras $(\mathcal{A}_X^{\bullet}, d)$, where $\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$. We will see in Theorem 3.4.3 that the rows and columns of the double complex $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$ are exact in positive degree.

We denote by $\mathcal{A}_{X,c}^{p,q}(V)$ the sections of $\mathcal{A}_X^{p,q}(V)$ which have compact support (in the sense of sheaves) in V.

We will sometimes write $C^\infty(V) := \mathcal{A}_X^{0,0}(V)$ and call elements of $\mathcal{A}^{0,0}$ smooth functions. This is justified since every $(0,0)$-form defines a continuous function on X. That is, if $f \in \mathcal{A}^{0,0}(V)$ is given by a family $(V_i, \varphi_i, f_i)_{i\in I}$, then defining $f(x) := f_i \circ \varphi_{i,\mathrm{trop}}(x)$ for $x \in U_i^{\mathrm{an}}$ yields the function.

Lemma 3.2.17. *The sheaves* $\mathcal{A}_X^{p,q}$ *are fine sheaves. They are thus soft and hence acyclic with respect to both the global section functor and the functor of global section with compact support.*

Proof. The fact that $\mathcal{A}_X^{0,0}$ is fine follows from [Gub13a, 5.10]. The general case follows since $\mathcal{A}_X^{p,q}$ is an $\mathcal{A}_X^{0,0}$-module via the wedge product. Softness and acyclicity for global sections follows from [Wel80, Chapter II, Proposition 3.5 & Theorem 3.11] respectively and acyclicity for sections with compact support follows from [Ive86, III, Theorem 2.7]. □

Remark 3.2.18. Let $F\colon X \to Y$ be a morphism of algebraic varieties. Let $\alpha \in \mathcal{A}_Y^{p,q}(W)$ be given by $(W_i, \varphi_{Z_i}, \alpha_i)_i$. Let $U_i := F^{-1}(Z_i)$ and $V_i := F^{\mathrm{an},-1}(W_i)$. Let U_{ij} be very affine open subsets such that $U_i = \bigcup U_{ij}$ and write $V_{ij} := V_i \cap U_{ij}^{\mathrm{an}}$. Since $F(U_{ij}) \subset Z_i$ we obtain maps $\mathrm{Trop}(\psi_{Z_i,U_{ij}})$ by Remark 3.2.6. Now $(V_{ij}, \varphi_{U_{ij}})$ are canonical tropical charts and we define $F^*(\alpha) \in \mathcal{A}^{p,q}(F^{\mathrm{an},-1}(W))$ to be given by $(V_{ij}, \varphi_{U_{ij}}, \mathrm{Trop}(\psi_{W_i,U_{ij}})^*\alpha_i)_{ij}$. This is independent of the chosen presentation for α.

We obtain a pullback morphism

$$F^*\colon \mathcal{A}_Y^{p,q} \to F_*^{\mathrm{an}}\, \mathcal{A}_X^{p,q}$$

which commutes with the differentials d', d'', d and the operator J.

Proposition 3.2.19. *Let X be a variety of dimension n with generic point η. Denote by $F\colon X^{\mathrm{an}} \to X$ the analytification map. Let $\alpha \in \mathcal{A}^{p,q}(X)$ such that* $\max(p,q) = n$*. Then* $\operatorname{supp}(\alpha) \subset F^{-1}(\{\eta\})$*. In particular,* $\operatorname{supp}(\alpha)$ *is contained in the analytification of every open subset of X.*

Proof. This follows from [Gub13a, Corollary 5.12]. □

Lemma 3.2.20. *Let X be a variety of dimension n. Let $\alpha \in \mathcal{A}_c^{p,q}(X)$ such that* $\max(p,q) = n$*. Then there exists $U \subset X$ very affine such that* $\operatorname{supp}(\alpha) \subset U^{\mathrm{an}}$*, $\alpha|_{U^{\mathrm{an}}}$ is given by $(U^{\mathrm{an}}, \varphi_U, \alpha_U)$ and $\alpha_U \in \mathcal{A}_c^{p,q}(\mathrm{Trop}(U))$. Further, if U has this property then any non-empty very affine open subset of U has this property.*

Proof. This follows from [Gub13a, Proposition 5.13 & Lemma 5.15 (a)]. We repeat the argument for the convenience of the reader and because we want to apply it later in similar situations. Since the support of α is compact, there exists $V \subset X$ open such that $\operatorname{supp}\alpha \subset V$ and $\alpha|_V$ is defined by a finite family $(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}$. Thus $\bigcap_{i\in I} U_i$ as a non-empty, very affine open subset of X. Denote by U any non-empty very affine open subset of $\bigcap_{i\in I} U_i$. Define $V_i' := V_i \cap U^{\mathrm{an}}$ and $V' := \bigcup_{i\in I} V_i'$. For each i the canonical tropical chart (V_i', φ_U) is a subchart of (V_i, φ_{U_i}). Denote by $\alpha_i' := \mathrm{Trop}(\psi_{U_i,U})^*\alpha_i$ and $\Omega_i := \varphi_{U,\mathrm{trop}}(V_i')$. Then $\alpha|_{V_i'\cap V_j'}$ is given by both $(V_i' \cap V_j', \varphi_U, \alpha_i'|_{\Omega_i\cap\Omega_j})$ and $(V_i' \cap V_j', \varphi_U, \alpha_j'|_{\Omega_i\cap\Omega_j})$. By Lemma 3.2.12, the forms α_i' and α_j' agree on $\Omega_i \cap \Omega_j$, thus glue to give a form $\alpha' \in \mathcal{A}^{p,q}(\varphi_{U,\mathrm{trop}}(V))$.

By Proposition 3.2.19, we have $\operatorname{supp}(\alpha) \subset U^{\mathrm{an}}$. Thus $\operatorname{supp}(\alpha) \subset U^{\mathrm{an}} \cap V = V'$. It follows from Lemma 3.2.12 that $\operatorname{supp}(\alpha') = \varphi_{U,\mathrm{trop}}(\operatorname{supp}(\alpha))$ (cf. [CLD12, Corollaire 3.2.3]). Thus $\operatorname{supp}(\alpha')$ is compact. Extending α' by zero to $\alpha_U \in \mathcal{A}^{p,q}_{\mathrm{Trop}(U),c}(\mathrm{Trop}(U))$, we conclude the proof. □

Definition 3.2.21. Let X be a variety of dimension n. Let $\alpha \in \mathcal{A}_c^{p,q}(X)$ such that $\max(p,q) = n$. Then any canonical tropical chart $(U^{\mathrm{an}}, \varphi_U)$ with the property above is called a *canonical tropical chart of integration for* α. We denote by $\alpha_U \in \mathcal{A}_c^{p,q}(\mathrm{Trop}(U))$ the unique form such that α is given by $(U^{\mathrm{an}}, \varphi_U, \alpha_U)$.

Definition 3.2.22. Let X be a variety of dimension n and $\alpha \in \mathcal{A}_c^{n,n}(X)$. Then we define

$$\int_X \alpha := \int_{\mathrm{Trop}(U)} \alpha_U$$

for some tropical chart of integration $(U^{\mathrm{an}}, \varphi_U)$.

Proposition 3.2.23. *The definition of the integral in Definition 3.2.22 is independent of the choice of the canonical tropical chart of integration.*

Proof. Let (U, φ_U) and $(U', \varphi_{U'})$ be two canonical tropical charts of integration for a form α. After replacing U' by $U \cap U'$ we may assume $U' \subset U$. Then there exists a torus equivariant map $\psi_{U,U'}\colon T_{U'} \to T_U$ such that $\varphi_U = \psi_{U,U'} \circ \varphi_{U'}$. To simplify the notation we will write $\psi := \psi_{U,U'}$. After a translation we may assume ψ to be a morphism of algebraic groups. Also, by Lemma 3.2.12 we have $\mathrm{Trop}(\psi)^*(\alpha_U) = \alpha_{U'}$. By the Sturmfels-Tevelev-multiplicity-formula 3.1.21, we have

$$\mathrm{Trop}(\psi)_*(\mathrm{Trop}(\varphi_{U'}(U'))) = \mathrm{Trop}(\psi_*((\varphi_{U'}(U'))).$$

Since both φ_U and $\varphi_{U'}$ are closed immersions, $\psi_*(\varphi_{U'}(U')) = \psi(\varphi_{U'}(U')) = \varphi_U(U')$. Since U' is dense in U, we have $\mathrm{Trop}(\varphi_U(U')) = \mathrm{Trop}(\varphi_U(U))$ by [Gub13a, Lemma 4.9] and thus

$$\mathrm{Trop}(\psi)_* \,\mathrm{Trop}(U') = \mathrm{Trop}(U).$$

The result now follows from the tropical projection formula 2.1.52. □

Theorem 3.2.24 (Stokes' Theorem)**.** *Let X be a variety of dimension n. For $\beta \in \mathcal{A}_c^{n,n-1}(X)$ and $\gamma \in \mathcal{A}_c^{n-1,n}(X)$ we have*

$$\int_X d''\beta = 0 \qquad \text{and} \qquad \int_X d'\gamma = 0.$$

Proof. This is a direct consequence of Stokes' Theorem 2.1.63 and the fact that $\mathrm{Trop}(U)$ is a tropical variety (Theorem 3.1.15). □

The next result was already shown by Gubler and Künnemann in [GK14] for a generalization of our forms, called δ-forms. We still give a proof in our special case.

Proposition 3.2.25. *Let $F\colon X \to Y$ be a proper dominant morphism of varieties of the same dimension n. Then for $\alpha \in \mathcal{A}_c^{n,n}(Y)$ we have*

$$\int_X F^*(\alpha) = \deg(F) \int_Y \alpha.$$

Proof. Let $W \subset Y$ be a very affine open subset such that $(W^{\mathrm{an}}, \varphi_W, \alpha_W)$ is a canonical tropical chart of integration for α. After replacing Y by W and X by $F^{-1}(W)$ we may assume that Y is very affine and admits a very affine chart of integration for α. Let $U \subset X$ be very affine and write $\alpha_U = \mathrm{Trop}(\psi_{Y,U})^*\alpha_Y$, where $\psi_{Y,U}$ is the map from Remark 3.2.6. Then (U, φ_U, α_U) is a tropical chart of integration for $F^*\alpha$. By the Sturmfels-Tevelev multiplicity formula 3.1.21 we have

$$(\mathrm{Trop}(\psi_{Y,U}))_*(\mathrm{Trop}(\varphi_{U,*}(U))) = \mathrm{Trop}(\psi_{Y,U,*}(\varphi_{U,*}(U)).$$

Since $\psi_{Y,U} \circ \varphi_U = \varphi_Y \circ F$ the latter term equals $\mathrm{Trop}(\varphi_{Y,*}(F_*(U)))$. Since F is dominant we have $F_*(U) = \deg(F) \cdot Y$. Thus

$$(\mathrm{Trop}(\psi_{Y,U}))_*(\mathrm{Trop}(\varphi_{U,*}(U))) = \deg(F) \cdot \mathrm{Trop}(\varphi_{Y,*}(Y)).$$

Since φ_U and φ_Y are both closed immersions, this is precisely

$$(\mathrm{Trop}(\psi_{Y,U}))_*(\mathrm{Trop}(U)) = \deg(F) \cdot \mathrm{Trop}(Y).$$

Now

$$\int_X F^*(\alpha) = \int_{\mathrm{Trop}(U)} \alpha_U = \int_{\mathrm{Trop}(\psi_{Y,U})_*(\mathrm{Trop}(\varphi(U)))} \alpha_Y = \int_{\deg(F)\cdot\mathrm{Trop}(Y)} \alpha_Y = \deg(F) \int_Y \alpha,$$

where the second equality is the tropical projection formula 2.1.52. □

3.2.2 A new approach using $\mathbb{G}$-, $\mathbb{A}$- and T-tropical charts

In this subsection, we give different approaches to defining differential forms on analytification of algebraic varieties. We replace the canonical moment maps from Subsection 3.2.1 by arbitrary closed immersions into tori resp. affine spaces. When the variety X has enough toric embeddings (cf. 3.2.32) we also give an approach which uses closed immersions of X (not of open subsets of X) into toric varieties. Note that the approach using $\mathbb{A}$- and T-tropical charts (cf. Definition 3.2.27) is new and relies on the theory of differential forms on polyhedral spaces, as introduced in [JSS15] (cf. Chapter 2). The approach using $\mathbb{G}$-moment maps is already implicit in [Gub13a]. We will see in Subsection 3.2.3 that all these produce the same sheaves of forms on X^{an}.

Definition 3.2.26. Let X be a variety and $U \subset X$ an open affine subset. For a closed immersion $\varphi\colon U \hookrightarrow \mathbb{A}^r$ resp. $\varphi\colon U \hookrightarrow \mathbb{G}_m^r$ we denote by $\varphi_{\mathrm{trop}} := \mathrm{trop} \circ \varphi^{\mathrm{an}}\colon U^{\mathrm{an}} \to \mathbb{T}^r$ resp. $\varphi_{\mathrm{trop}} := \mathrm{trop} \circ \varphi^{\mathrm{an}}\colon U^{\mathrm{an}} \to \mathbb{R}^r$ the respective composition with the tropicalization map. We denote by $\mathrm{Trop}_\varphi(U)$ the image of φ_{trop}.

Let $U' \subset U$ be another open subset. An immersion $\varphi'\colon U' \to \mathbb{A}^{r'}$ resp. $\varphi'\colon U' \to \mathbb{G}_m^{r'}$ is called a *refinement of* φ, if there exists a torus equivariant morphism $\psi\colon \mathbb{A}^{r'} \to \mathbb{A}^r$ resp. $\psi\colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$ such that

$$\begin{array}{ccc} U' & \xrightarrow{\varphi'} & \mathbb{A}^{r'} \\ \downarrow & & \downarrow \psi \\ U & \xrightarrow{\varphi} & \mathbb{A}^r \end{array} \qquad \text{resp.} \qquad \begin{array}{ccc} U' & \xrightarrow{\varphi'} & \mathbb{G}_m^{r'} \\ \downarrow & & \downarrow \psi \\ U & \xrightarrow{\varphi} & \mathbb{G}_m^r \end{array}$$

commutes. The map ψ induces an integral (extended) linear map

$$\mathrm{Trop}(\psi_{\varphi,\varphi'})\colon \mathrm{Trop}_{\varphi'}(U') \to \mathrm{Trop}_\varphi(U).$$

Let $\varphi\colon X \hookrightarrow Y_\Delta$ be a closed immersion of X into a toric variety Y_Δ. Denote by $\varphi_{\mathrm{trop}} := \mathrm{trop} \circ \varphi^{\mathrm{an}}\colon X^{\mathrm{an}} \to \mathrm{Trop}(Y_\Delta)$. Again we denote by $\mathrm{Trop}_\varphi(U)$ the image of φ_{trop}.

Another closed immersion $\varphi'\colon X \to Y_{\Delta'}$ is called a *refinement of* φ, if there exists a torus equivariant map $\psi\colon Y_{\Delta'} \to Y_\Delta$ such that

$$\begin{array}{ccc} & & Y_{\Delta'} \\ & \overset{\varphi'}{\nearrow} & \downarrow \psi \\ X & \xrightarrow{\varphi} & Y_\Delta \end{array}$$

commutes. In that case we get an induced morphism of polyhedral spaces

$$\mathrm{Trop}(\psi_{\varphi,\varphi'})\colon \mathrm{Trop}_{\varphi'}(X) \to \mathrm{Trop}_\varphi(X).$$

Definition 3.2.27. Let X be a variety. An $\mathbb{A}$-*tropical chart* resp. $\mathbb{G}$-*tropical chart* is given by a pair (V, φ), where $V \subset X^{\mathrm{an}}$ is an open subset and $\varphi\colon U \hookrightarrow \mathbb{A}^r$ resp. $\varphi\colon U \hookrightarrow \mathbb{G}_m^r$ is a closed immersion of an affine open subset U of X such that $V = \varphi_{\mathrm{trop}}^{-1}(\Omega)$ for an open subset $\Omega \subset \mathrm{Trop}_\varphi(U)$.

Another $\mathbb{A}$- resp. $\mathbb{G}$-tropical chart (V', φ') is called an $\mathbb{A}$- resp. $\mathbb{G}$-*tropical subchart of* (V, φ) if φ' is a refinement of φ and $V' \subset V$.

A *T-tropical chart* is a pair (V, φ) where $V \subset X^{\mathrm{an}}$ is an open subset, $\varphi\colon X \hookrightarrow Y_\Delta$ is a closed immersion of X into a toric variety Y_Δ and $V = \varphi_{\mathrm{trop}}^{-1}(\Omega)$ where Ω is an open subset of $\mathrm{Trop}_\varphi(X)$.

Another T-tropical chart (V', φ') is called a *T-tropical subchart of* (V, φ) if φ' is a refinement of φ and $V' \subset V$.

Remark 3.2.28. Note that in contrast to canonical tropical charts, the choice of U does not fix φ. Then same set U can be the domain of different maps $\varphi_i\colon U \hookrightarrow \mathbb{A}^{r_i}$ for tropical charts (V_i, φ_i).

Remark 3.2.29. Let $(V_1, \varphi_1), \dots, (V_n, \varphi_n)$ be finitely many $\mathbb{A}$ resp. $\mathbb{G}$ resp. T-tropical charts. Then $(V_1 \cap \dots \cap V_n, \varphi_1 \times \dots \times \varphi_n)$ is a common subchart. Here we use the intersection of the domains of the φ_i as the domain of their product. Indeed, any common subchart is also a subchart of this chart.

Remark 3.2.30. Let (V, φ) be a $\mathbb{G}$-tropical chart, where $\varphi\colon U \hookrightarrow \mathbb{G}_m^r$. Let $\psi\colon \mathbb{G}^r \to \mathbb{A}^{2r}$ be the closed immersion given by

$$K[T_1, \dots, T_{2r}] \twoheadrightarrow K[S_1^{\pm 1}, \dots, S_r^{\pm 1}]$$
$$T_i \mapsto \begin{cases} S_{i/2} & \text{if } i \text{ is even} \\ S_{(i+1)/2}^{-1} & \text{if } i \text{ is odd.} \end{cases}$$

Then $\varphi_{\mathbb{A}} := \psi \circ \varphi\colon U \hookrightarrow \mathbb{A}^{2r}$ defines a closed immersion and $(V, \varphi_{\mathbb{A}})$ is an $\mathbb{A}$-tropical chart. Further, we have an isomorphism

$$\mathrm{Trop}(\psi)\colon\ \mathrm{Trop}_\varphi(U) \to \mathrm{Trop}_{\varphi_{\mathbb{A}}}(U)$$

Note that this construction is compatible with refinements.

Lemma 3.2.31. *Let X be a variety. Then*

i) any closed immersion $\varphi\colon X \to \mathbb{A}^r$ has a refinement $\varphi'\colon X \to \mathbb{A}^{r'}$ which meets the dense torus $\mathbb{G}_m^{r'}$.

ii) any closed immersion $\varphi\colon X \to Y_\Delta$ has a refinement $\varphi'\colon X \to Y_{\Delta'}$ which meets the dense torus $\mathbb{G}_m^{r'}$.

Proof. If $\varphi(X)$ does not meet $\mathbb{G}_m^{r'}$, it is contained in the union of the coordinate hyperplanes. Since it is irreducible, it is thus contained in one coordinate hyperplane $H \cong \mathbb{A}^{r-1}$. Arguing inductively we find r' such that φ factors through $\mathbb{A}^{r'}$ via the inclusion $\mathbb{A}^{r'} \hookrightarrow \mathbb{A}^r$. This is then indeed a refinement.

For the toric case we argue similarly. Let $Y_\Delta = \coprod Y_i$ be the decomposition into torus strata. Let i be such that Y_i is maximal with the property $Y_i \cap \varphi(X) \neq \emptyset$. Then again by irreducibility $\varphi(X)$ is contained in $\overline{Y_i}$ which is a toric variety with dense torus Y_i. We find again that φ factors through $\overline{Y_i}$ and this is a refinement via the inclusion $\overline{Y_i} \hookrightarrow Y$. ☐

Condition (†) **3.2.32.** The variety X is normal and for any two points there exists an open affine subvariety containing both.

Remark 3.2.33. By Włodarczyk's Embedding Theorem, Condition (†) is equivalent to X being normal and admitting a closed immersion into a toric variety (cf. [Wło93]). Observe also that it is satisfied by any quasi-projective normal variety.

We will use this condition to make sure that X has enough closed immersions into toric varieties. More precisely, the property we need is captured in the next lemma. When using T-tropical charts we will always require X to satisfy this condition.

Lemma 3.2.34. *Let X be a variety which satisfies* (†). *Let (V, φ) be an $\mathbb{A}$-tropical chart, where $\varphi \colon U \hookrightarrow \mathbb{A}^r$. There exists a closed immersion $\varphi' \colon X \hookrightarrow Y_\Delta$ for a toric variety Y_Δ such that:*

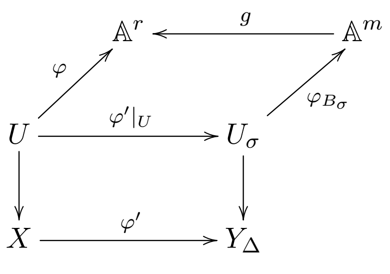
i) The pair (V, φ') is a T-tropical chart.

ii) The set U is the preimage of an affine toric subvariety U_σ of Y_Δ.

iii) For every finite generating set B_σ of σ, the pair $(V, \varphi_{B_\sigma} \circ \varphi'|_U)$ is an $\mathbb{A}$-tropical subchart of (V, φ).

iv) The map $(\varphi_{B_\sigma} \circ \varphi'|_U)_{\mathrm{trop}}$ is a chart (in the sense of polyhedral spaces) for the polyhedral space $\mathrm{Trop}_{\varphi'}(X)$.

Proof. We use [FGP14, Theorem 4.2] with $j = 1$, $U_1 = U$ and $R_1 = \{f_1, \ldots, f_r\}$ which are the functions which define φ. This yields a closed immersion $\varphi' \colon X \hookrightarrow Y_\Delta$ such that U is the preimage of some U_σ and the f_i are preimages of elements α_i of $S_\sigma \subset K[S_\sigma]$. Choosing a generating set $B_\sigma = \{\psi_1, \ldots, \psi_m\}$ of S_σ, we find the following commutative diagram



Here the map g is given by mapping the i-th coordinate function T_i of $\mathbb{A}^r$ to $\prod X_j^{\lambda_{ij}}$, where X_j is the j-th coordinate function of $\mathbb{A}^m$ and $\alpha_i = \sum \lambda_{ij} \psi_j$. Now the commutativity of the upper diagram shows that $(V, \varphi_{B_\sigma} \circ \varphi'|_U)$ is an $\mathbb{A}$-tropical subchart of (V, φ). The fact that $(\varphi_{B_\sigma} \circ \varphi'|_U)_{\mathrm{trop}}$ is a chart for the polyhedral space $\mathrm{Trop}_{\varphi'}(X)$ is now a direct consequence of the construction of the atlas for $\mathrm{Trop}_{\varphi'}(X)$ in Lemma 3.1.9 and Corollary 3.1.10. □

Lemma 3.2.35. *Let S be $\mathbb{A}$, $\mathbb{G}$ or T. If S is T, then we require X to satisfy* (†). *We have the following:*

i) For every open subset $W \subset X^{\mathrm{an}}$ and every $x \in W$ there exists an S-tropical chart (V, φ) such that $x \in V$ and $V \subset W$. Further, V can be chosen in such a way that $\varphi_{\mathrm{trop}}(V)$ is relatively compact in $\mathrm{Trop}(U)$.

ii) For S-tropical charts (V_1, φ_1) and (V_2, φ_2) the pair $(V_1 \cap V_2, \varphi_1 \times \varphi_2)$ is a subchart of both.

iii) Let (V, φ) be a tropical chart and $W \subset V$ an open subset. Then there exist refinements (V_i, φ_i) of (V, φ) by S-tropical charts such that $W = \bigcup V_i$.

Proof. Property *iii)* is a direct consequence of *i)* and *ii)*, and property *ii)* is immediate.

For *i)*, we notice that for $\mathbb{G}$ this is a consequence of the case for canonical tropical charts (Proposition 3.2.9). For $\mathbb{A}$ this then follows because every $\mathbb{G}$-tropical chart induces an $\mathbb{A}$-tropical chart with same domain (Remark 3.2.30). For T this then follows since after passing to a subchart, every $\mathbb{A}$-tropical chart induces a T-tropical chart by Lemma 3.2.34. □

Definition 3.2.36. Let X be a variety and V an open subset of X^{an}. Let S be $\mathbb{A}$, $\mathbb{G}$, or T. If S is T, we furthermore require X to satisfy (†). An element of $\mathcal{A}^{p,q}_{X,S}(V)$ is given by a family $(V_i, \varphi_i, \alpha_i)_{i \in I}$ such that

i) For all $i \in I$ the pair (V_i, φ_i) is an S-tropical chart and $\bigcup_{i \in I} V_i = V$.

ii) For all $i \in I$ we have $\alpha_i \in \mathcal{A}^{p,q}_{\mathrm{Trop}_{\varphi_i}(U_i)}(\varphi_{i,\mathrm{trop}}(V_i))$.

iii) The α_i agree on intersections in the sense that for all $i, j \in I$, we have subcharts (V_{ijl}, φ_{ijl}) of $(V_i \cap V_j, \varphi_i \times \varphi_j)$ such that $V_i \cap V_j = \bigcup_l V_{ijl}$ and

$$\mathrm{Trop}(\psi_{U_i,U_{ijl}})^*(\alpha_i) = \mathrm{Trop}(\psi_{U_j,U_{ijl}})^*(\alpha_j) \in \mathcal{A}^{p,q}(\varphi_{ijl,\mathrm{trop}}(V_{ijl}))$$

for all l.

Another such family $(V'_j, \varphi_{U'_j}, \beta_j)_{j \in J}$ defines the same form if there is a cover of V by common S-tropical subcharts such that the pullbacks to the refined cover agree.

Let $V' \subset V$ and $\alpha \in \mathcal{A}^{p,q}_{X,S}(V)$ given by $(V_i, \varphi_i, \alpha_i)$. By Lemma 3.2.35, we can cover V' by S-tropical subcharts $(W_{ij}, \varphi_{U_{ij}})$ of the S-tropical charts (V_i, φ_i) and then define $\alpha|_{V'}$ to be given by $(W_{ij}, \varphi_{ij}, \mathrm{Trop}(\psi_{U_i,U_{ij}})^* \alpha_i)_{ij}$. Note that this is independent of the chosen presentation of α.

Lemma 3.2.37. *Let X be a variety and S be $\mathbb{A}$, $\mathbb{G}$ or T. If S is T, we furthermore require X to satisfy (†). The functor*

$$V \mapsto \mathcal{A}^{p,q}_{X,S}(V)$$

is a sheaf on X^{an} which we denote by $\mathcal{A}^{p,q}_{X,S}$.

Proof. This is a direct consequence of the local nature of Definition 3.2.36 □

Definition 3.2.38. Let X be a variety and S be $\mathbb{A}$, $\mathbb{G}$ or T. If S is T, then we further require that X satisfies (†). Let $\alpha \in \mathcal{A}^{p,q}(V)$ be given by $(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}$, where (V_i, φ_i) are S-tropical charts. Then for $D \in \{d', d'', d\}$ we define $D\alpha$ to be given by $(V_i, \varphi_{U_i}, D\alpha_i)_{i\in I}$ and $J\alpha$ to be given by $(V_i, \varphi_{U_i}, J\alpha_i)_{i\in I}$. If further $\beta \in \mathcal{A}^{p',q'}(V)$ is given by $(V_i, \varphi_{U_i}, \beta_i)$, then we define $\alpha \wedge \beta$ to be given by $(V_i, \varphi_{U_i}, \alpha_i \wedge \beta_i)_i$. Note here that we may assume β to be given on the same cover as α after pulling back to a common refinement. Since all these constructions commute with the maps $\mathrm{Trop}(\psi_{\varphi,\varphi'})$ for subcharts, these all define elements of $\mathcal{A}^{\bullet,\bullet}_{X,S}$ and are independent of the respective presentations.

Definition 3.2.39. We obtain a sheaf of bigraded bidifferential algebras $(\mathcal{A}^{\bullet,\bullet}_{X,S}, d', d'')$ on X^{an}. We will see in Theorem 3.2.41 that these are canonically isomorphic to $(\mathcal{A}^{\bullet,\bullet}_{X}, d', d'')$. We also obtain the sheaf of total differential algebras $(\mathcal{A}^{\bullet}_{X,S}, d)$ where $\mathcal{A}^{k}_{X,S} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_{X,S}$.

We denote by $\mathcal{A}^{p,q}_{X,S,c}(V)$ the sections of $\mathcal{A}^{p,q}_{X,S}(V)$ which have compact support (in the sense of sheaves) in V.

Remark 3.2.40. Let $F\colon X \to Y$ be a morphism of varieties. Let Z_i be an affine open subset of Y and let U_{ij} be an affine open subset of $F^{-1}(Z_i)$. Let $\varphi_i\colon Z_i \hookrightarrow \mathbb{A}^r$ be a closed immersion. Then there exists $s \geq r$, a closed immersion $\varphi'_{ij}\colon U_{ij} \hookrightarrow \mathbb{A}^s$ such that

$$\begin{array}{ccc} \mathbb{A}^s & \xrightarrow{\pi_{ij}} & \mathbb{A}^r \\ \uparrow{\scriptstyle \varphi'_{ij}} & & \uparrow{\scriptstyle \varphi_i} \\ U_{ij} & \xrightarrow{F} & Z_i \end{array} \tag{3.3}$$

commutes, where $\pi\colon \mathbb{A}^s \to \mathbb{A}^r$ is the projection to a subset of coordinates. If (V, φ) is a tropical chart, then so is $(F^{\mathrm{an},-1}(V) \cap U^{\mathrm{an}}, \varphi')$.

For general X and Y, let V be an open subset of Y^{an} and $\alpha \in \mathcal{A}^{p,q}_{Y,\mathbb{A}}(Y)$ be given by $(V_i, \varphi_i, \alpha_i)$ for $\varphi_i\colon Z_i \hookrightarrow \mathbb{A}^{r_i}$ and $\alpha_i \in \mathcal{A}^{p,q}_{\mathrm{Trop}_{\varphi_i}}(\Omega_i)$. For each i we cover $F^{-1}(U_i)$ by open affine subsets U_{ij}. For each i, j we chose φ'_{ij} as in (3.3) and define $F^*(\alpha) \in \mathcal{A}^{p,q}_{X,\mathbb{A}}(F^{\mathrm{an},-1}(V))$ to be given by $(F^{\mathrm{an},-1}(V) \cap U_{ij}, \varphi'_{ij}, \mathrm{Trop}(\pi_{ij})^*(\alpha_i))$. Since diagrams of the form (3.3) are compatible with taking products of the maps φ_i resp. φ_{ij}, it is straightforward to check that this is independent of the presentation of α, the choice of U_{ij} and the choice of φ_{ij} (as long as they satisfy (3.3)).

This define a sheaf homomorphism

$$F^*\colon \mathcal{A}^{p,q}_{Y,\mathbb{A}} \to F^{\mathrm{an}}_*\, \mathcal{A}^{p,q}_{X,\mathbb{A}}$$

which commutes with the differentials, the wedge product and the operator J.

3.2.3 Comparison of approaches

In this subsection, we compare the four approaches which we introduced in Subsections 3.2.1 and 3.2.2. We show that all these produce canonically isomorphic sheaves on X^{an}. We also show that there is a natural notion of integration in all formalisms and that they all produce the same integral. The isomorphism of the sheaves $\mathcal{A}^{p,q}$ and $\mathcal{A}^{p,q}_{\mathbb{G}}$ is already implicit in [Gub13a], while the other comparisons are new. At the end, we give an application of the approach using T-moment maps by showing that on a proper variety any smooth function in the kernel of $d'd''$ is constant.

Theorem 3.2.41. *Let X be a variety. There are canonical isomorphisms*

$$\mathcal{A}^{p,q}_X \simeq \mathcal{A}^{p,q}_{X,\mathbb{G}} \simeq \mathcal{A}^{p,q}_{X,\mathbb{A}}$$

for all p, q. If X satisfies (†), then we even have

$$\mathcal{A}^{p,q}_X \simeq \mathcal{A}^{p,q}_{X,\mathbb{G}} \simeq \mathcal{A}^{p,q}_{X,\mathbb{A}} \simeq \mathcal{A}^{p,q}_{X,T}\,.$$

Furthermore, all these isomorphisms are compatible with d', d'', d, J and the wedge product.

Lemma 3.2.42. *Let S be $\mathbb{G}$ or $\mathbb{A}$. Let $\alpha \in \mathcal{A}^{p,q}_{X,S}(V)$ be given by a single S-tropical chart (V, φ_U, α'). Then $\alpha = 0$ if and only if $\alpha' = 0$.*

Proof. Works exactly as the proof of Lemma 3.2.12. □

Remark 3.2.43. We want to point out the following difference between the approach by Gubler [Gub13a] (and also the one by Chambert – Loir and Ducros [CLD12]) and the approaches from Subsection 3.2.2. Canonical moment maps are only defined up to an automorphism of algebraic groups composed with a linear translation on $\mathbb{G}^r_m$. This corresponds to not fixing a basis of $\mathbb{G}^r_m$ resp. $\mathbb{R}^r$. Since the partial compactification $\mathbb{T}^r$ of $\mathbb{R}^r$ is not independent of the chosen basis, we fix coordinates on $\mathbb{A}^r$ resp. $\mathbb{T}^r$ when extending the tropicalization map to these spaces.

We use the definition of $\mathcal{A}^{p,q}_{\mathbb{G}}$ as a bridge between $\mathcal{A}^{p,q}$ and $\mathcal{A}^{p,q}_{\mathbb{A}}$ in the sense that we still use charts to $\mathbb{G}^r_m$, but we only consider torus equivariant transition maps (resulting in linear maps on the tropical level) and we always fix bases of $\mathbb{G}^r_m$. To get from canonical tropical charts to $\mathbb{G}$-tropical charts, we will therefore have to choose a basis for each chart (cf. proof of Proposition 3.2.45). Lemma 3.2.44 will ensure that this choice does not make any difference.

Lemma 3.2.44. *Let $\varphi\colon U \hookrightarrow \mathbb{G}^r_m$ be a closed immersion. Let $\psi\colon \mathbb{G}^r_m \to \mathbb{G}^r_m$ be given by an automorphism of algebraic groups composed with a multiplicative translation. Let $V = \varphi^{-1}_{\mathrm{trop}}(\Omega)$ for Ω an open subset of* $\mathrm{Trop}_\varphi(U)$*. For $\alpha \in \mathcal{A}^{p,q}(\Omega)$, both $(V, \varphi, \mathrm{Trop}(\psi)^*\alpha)$ and $(V, \psi\circ\varphi, \alpha)$ define the same form in $\mathcal{A}^{p,q}_{\mathbb{G}}(V)$.*

Proof. We consider the product $\mathbb{G}^{2r}_m = \mathbb{G}^r_m \times \mathbb{G}^r_m$ with projection maps π_1 and π_2. Then $(V, \varphi \times (\psi \circ \varphi))$ is a $\mathbb{G}$-tropical subchart of both (V, φ) and $(V, \psi \circ \varphi)$. Thus we have to show that $\mathrm{Trop}(\pi_2)^*\alpha = \mathrm{Trop}(\pi_1)^*\,\mathrm{Trop}(\psi)^*\alpha$ on $\mathrm{Trop}(\pi_2)^{-1}(\Omega)$. This follows directly since π_2 and $\psi \circ \pi_1$ agree on $(\varphi \times (\psi \circ \varphi))(U)$. □

Proposition 3.2.45. *There is a canonical isomorphism*

$$\Psi_{\mathbb{G}}\colon \mathcal{A}^{p,q}_X \xrightarrow{\sim} \mathcal{A}^{p,q}_{X,\mathbb{G}}\,.$$

Proof. Let $\alpha \in \mathcal{A}_X^{p,q}(V)$ be given by a family $(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}$. Recall that φ_{U_i} is only fixed up to automorphism of algebraic groups and multiplicative translation. We now fix for each i a closed immersion φ_i which represents φ_{U_i}. Now by Lemma 3.2.44 the family $(V, \varphi_i, \alpha_i)_{i\in I}$ defines an element of $\mathcal{A}_{X,\mathbb{G}}^{p,q}(V)$ which is independent of the choices of the φ_i.

For injectivity and surjectivity we argue locally. Let $\alpha \in \mathcal{A}_X^{p,q}(V)$ locally be given by (V, φ, α'). Then $\Psi_{\mathbb{G}}(\alpha)$ is given by (V, φ, α'). Thus if $\Psi_{\mathbb{G}}(\alpha) = 0$ by Lemma 3.2.42 we have that $\alpha' = 0$ which in turn implies $\alpha = 0$.

To see that it is surjective, let $\alpha \in \mathcal{A}_{X,\mathbb{G}}^{p,q}(V)$ locally be given by (V, φ, α') with $\varphi : U \to \mathbb{G}_m^s$ a closed immersion. Denote by φ_U the canonical moment map of U. Then (V, φ_U) is a subchart of (V, φ). Thus α is also given by $(V, \varphi_U, \mathrm{Trop}(\psi_{\varphi,\varphi_U})^*\alpha')$ and this is clearly in the image of $\Psi_{\mathbb{G}}$. □

Proposition 3.2.46. *There is a canonical isomorphism*

$$\Psi_{\mathbb{G},\mathbb{A}} \colon \mathcal{A}_{X,\mathbb{G}}^{p,q} \xrightarrow{\sim} \mathcal{A}_{X,\mathbb{A}}^{p,q}\,.$$

Proof. We first define the map $\Psi_{\mathbb{G},\mathbb{A}} \colon \mathcal{A}_{X,\mathbb{G}}^{p,q} \to \mathcal{A}_{X,\mathbb{A}}^{p,q}$. Let $\alpha \in \mathcal{A}_{\mathbb{G}}^{p,q}(V)$ be given by $(V_i, \varphi_i, \alpha_i)_{i\in I}$. We define $\Psi_{\mathbb{G},\mathbb{A}}(\alpha)$ to be given by $(V_i, \varphi_{i,\mathbb{A}}, \alpha_i)_{i\in I}$, with $\varphi_{i,\mathbb{A}}$ as defined in Remark 3.2.30. Note that we identify $\mathrm{Trop}_{\varphi_i}(U)$ with $\mathrm{Trop}_{\varphi_{i,\mathbb{A}}}(U)$ via the isomorphism from Remark 3.2.30 and thus write α' both times.

We again argue locally to show that this is indeed an isomorphism. Injectivity follows again easily from Lemma 3.2.42.

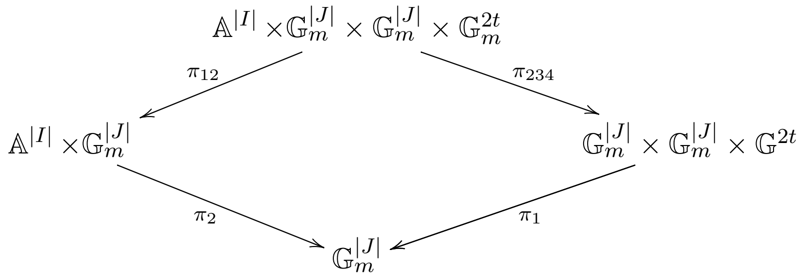
For surjectivity assume that $\alpha \in \mathcal{A}_{X,\mathbb{A}}^{p,q}(V)$ is locally at $x \in X^{\mathrm{an}}$ given by (V, φ, α'). Write $z := \varphi_{\mathrm{trop}}(x)$. Let $f_1, \dots, f_r \in \mathcal{O}_X(U)$ which define $\varphi \colon U \to \mathbb{A}^r$. Let $I := \mathrm{sed}(z) \subset [r]$ and J its complement. After shrinking V we may assume that $V = \varphi_{\mathrm{trop}}^{-1}(\Omega)$ for $\Omega \subset \mathrm{Trop}_{\varphi}(U)$ an open subset such that α' is determined on Ω by α'_I. Let $U' := \bigcap_{j\in J} D(f_j) \subset U$ and $\varphi' \colon U' \hookrightarrow \mathbb{A}^r \times \mathbb{A}^{|J|}$ be the immersion given by $f_1, \dots, f_r, (f_j^{-1})_{j\in J}$. Then (V, φ') is a subchart of (V, φ) via the projection to the first factor of $\mathbb{A}^r \times \mathbb{A}^{|J|}$. Replacing U by U' and φ by φ' we may thus assume that $f_j \in \mathcal{O}_X^{\times}(U)$ for all $j \in J$.

Let $g_1, \dots, g_t$ such that $((f_j)_{j\in J}, g_1, \dots, g_t) \in \mathcal{O}_X(U)^{\times}$ generates $\mathcal{O}_X(U)^{\times}/K^{\times}$. This defines a closed immersion $\varphi' : U \to \mathbb{G}^s$. Denote by $\pi : \mathbb{G}_m^s \to \mathbb{G}_m^{|J|}$ the projection.

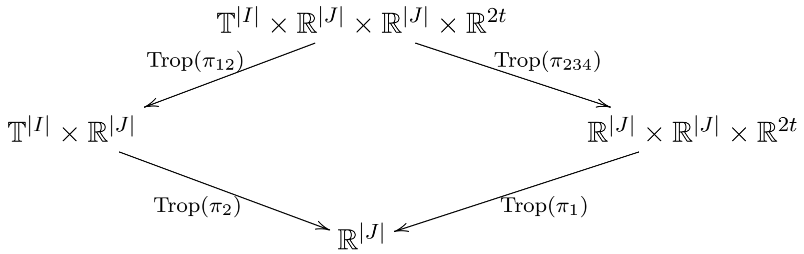
We now define $\beta \in \mathcal{A}_{X,\mathbb{G}}^{p,q}(V)$ to be given by $(V, \varphi', \mathrm{Trop}(\pi)^*(\alpha'_I))$. By definition of $\Psi_{\mathbb{G},\mathbb{A}}$, the form $\Psi_{\mathbb{G},\mathbb{A}}(\beta) \in \mathcal{A}_{X,\mathbb{G}}^{p,q}(V)$ is then given by $(V, \varphi'_{\mathbb{A}}, \mathrm{Trop}(\pi)^*(\alpha'_I))$. We will denote throughout by π_{ij} the projection to the i-th and j-th component. Now if we write $\mathbb{A}^r = \mathbb{A}^{|I|} \times \mathbb{A}^{|J|}$ and $\mathbb{A}^{2s} = \mathbb{A}^{2|J|} \times \mathbb{A}^{2t}$ we obtain the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathbb{A}^{|I|} \times \mathbb{A}^{|J|} \times \mathbb{A}^{|J|} \times \mathbb{A}^{2t} & & \\
 & \swarrow^{\pi_{12}} & \uparrow{\scriptstyle \varphi_1} & \searrow^{\pi_{234}} & \\
\mathbb{A}^{|I|} \times \mathbb{A}^{|J|} & \xleftarrow{\ \varphi\ } & U & \xrightarrow{\ \varphi'_{\mathbb{A}}\ } & \mathbb{A}^{2|J|} \times \mathbb{A}^{2t}\,.
\end{array}$$

This shows that φ_1 is a common refinement of φ and $\varphi'_{\mathbb{A}}$. If we pull back the forms $\operatorname{Trop}(\pi)^*(\alpha'_I)$ resp. α' to $\varphi_{1,\mathrm{trop}}(V)$, we see that they agree there since

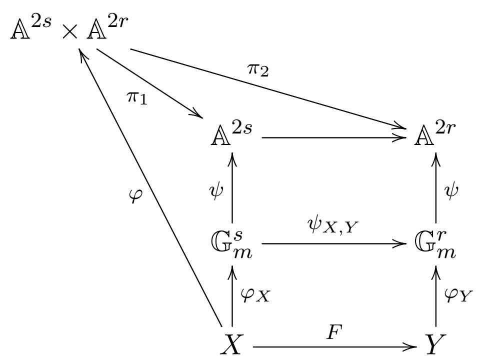


induces a commutative diagram



of tropicalizations. Thus both $\operatorname{Trop}(\pi_{234})^*(\operatorname{Trop}(\pi)^*(\alpha'_I))$ and $\operatorname{Trop}(\pi_{12})^*(\alpha')$ are just pullbacks of α'_I, thus agree. This shows $\Psi_{\mathbb{G},\mathbb{A}}(\beta) = \alpha \in \mathcal{A}^{p,q}_{X,\mathbb{A}}(V)$ which was what we wanted to show. □

Remark 3.2.47. The pullbacks on $\mathcal{A}^{p,q}_{X,\mathbb{A}}$ resp. $\mathcal{A}^{p,q}_X$ as explained in Remark 3.2.40 resp. Remark 3.2.18 commute with the isomorphism $\Psi_{\mathbb{A},\mathbb{G}} \circ \Psi_{\mathbb{G}}$. We may check this when $F\colon X \to Y$ is a morphism of very affine varieties and $\alpha \in \mathcal{A}^{p,q}_X$ is given by one canonical tropical chart. We have the following commutative diagram, where ψ is the map introduced in Remark 3.2.30:



Note here that ψ is an isomorphism onto its image and torus equivariant, thus induces an isomorphism on tropicalizations. Denote by $\operatorname{Trop}(\psi)_*$ the pullback by its inverse. Now $F^* \circ (\Psi_{\mathbb{A},\mathbb{G}} \circ \Psi_{\mathbb{G}})$ is obtained by using $\operatorname{Trop}(\pi_2)^* \circ \operatorname{Trop}(\psi)_*$ while $\Psi_{\mathbb{A},\mathbb{G}} \circ \Psi_{\mathbb{G}} \circ F^*$ is obtained by $\operatorname{Trop}(\psi)_* \circ \operatorname{Trop}(\psi_{X,Y})^*$. Now since φ is a refinement of $\psi \circ \varphi_X$ via π_1, we obtain the result.

Lemma 3.2.48. *Let U_σ be an affine toric variety and $\varphi\colon X \to U_\sigma$ a closed immersion. Fix a finite generating set B of S_σ. Denote by $\varphi_B\colon U_\sigma \hookrightarrow \mathbb{A}^r$ the corresponding closed immersion. Let $V \subset X^{\mathrm{an}}$ be an open subset. Then there is a natural correspondence*

$$\{\alpha \in \mathcal{A}^{p,q}_{X,\mathbb{A}}(V) \text{ given by } (V, \varphi_B \circ \varphi)\} \leftrightarrow \{\alpha \in \mathcal{A}^{p,q}_{X,T}(V) \text{ given by } (V, \varphi)\}.$$

Proof. This is a direct consequence of the fact that for all B the map $\psi_{\sigma,B}$ (in the notation of the proof of 3.1.9) is a chart for the polyhedral space $\mathrm{Trop}(U_\sigma)$. □

Proposition 3.2.49. *Let X be a variety which satisfies (†). We have a canonical isomorphism*

$$\Psi_{T,\mathbb{A}}\colon \mathcal{A}^{p,q}_{X,T} \to \mathcal{A}^{p,q}_{X,\mathbb{A}}\,.$$

Proof. We give the construction of $\Psi_{T,\mathbb{A}}$. Let $\alpha \in \mathcal{A}^{p,q}_{X,T}(V)$ be given by T-tropical charts $(V_i, \varphi_i, \alpha_i)$, where $\varphi_i\colon X \to Y_{\Delta_i}$ are closed immersions into toric varieties. After passing to subcharts we may assume that $\varphi_i^{\mathrm{an}}(V_i) \subset U_{\sigma_i}^{\mathrm{an}}$ for $\sigma_i \in \Delta_i$. Denote by X_i the preimage $\varphi_i^{-1}(U_{\sigma_i})$. For each i, choose a finite generating set B_{σ_i} of S_{σ_i}. Then the form in $\mathcal{A}^{p,q}_{X_i,T}(V_i)$ defined by $(V_i, \varphi_{B_{\sigma_i}} \circ \varphi_i|_{X_i}, \alpha_i)$ defines a unique form $\alpha_{V_i} \in \mathcal{A}^{p,q}_{X_i,\mathbb{A}}(V_i) = \mathcal{A}^{p,q}_{X,\mathbb{A}}(V_i)$, which is independet of the choice of B_{σ_i}, by Lemma 3.2.48. This construction is compatible with subcharts. Thus the forms α_{V_i} glue to a form $\Psi_{T,\mathbb{A}}(\alpha) \in \mathcal{A}^{p,q}_{X,\mathbb{A}}(V)$ and this form is independent of the chosen presentation.

For injectivity and surjectivity we again argue locally and can thus assume the forms to be given by one tropical chart (V, φ). We may also assume that V is mapped into the analytification of an affine toric subvariety. Then $\Psi_{T,\mathbb{A}}(\alpha)$ is also given by one chart and injectivity follows directly from Lemma 3.2.42.

To prove surjectivity let locally at $x \in X^{\mathrm{an}}$ a form $\alpha \in \mathcal{A}^{p,q}_{X,\mathbb{A}}(V)$ be given by an $\mathbb{A}$-tropical chart (V, φ, α'). After possibly passing to a refinement, by Lemma 3.2.34 we may assume this is the restriction of a T-tropical chart (V, φ'). We define $\beta \in \mathcal{A}^{p,q}_{X,T}(V)$ by (V, φ', α'). It is clear from the construction that $\Psi_{T,\mathbb{A}}(\beta) = \alpha$. □

Proof of Theorem 3.2.41. The isomorphisms are given by $\Psi_{\mathbb{G}}$, $\Psi_{\mathbb{G},\mathbb{A}}$ and $\Psi_{T,A}$. The fact that these commute with d', d'', d, J and the wedge product is a direct consequence of the definitions. □

Corollary 3.2.50. *Let (V, φ) be a T-tropical chart and let $\alpha \in \mathcal{A}^{p,q}_{X,T}(V)$ be given by (V, φ, α'). Then if $\alpha = 0$ we have $\alpha' = 0$.*

Proof. Let $\varphi\colon X \hookrightarrow Y_\Delta$ and $\Omega = \varphi_{\mathrm{trop}}(V)$. Denote for $\sigma \in \Delta$ by U_σ the corresponding affine toric variety. We further denote $X_\sigma := \varphi^{-1}(U_\sigma)$, $V_\sigma := V \cap U_\sigma^{\mathrm{an}}$ and $\Omega_\sigma := \varphi_{\mathrm{trop}}(V_\sigma)$. Then $\alpha|_{V_\sigma}$ is given by $(V_\sigma, \varphi, \alpha'|_{\Omega_\sigma})$. Thus $\Psi_{T,\mathbb{A}}(\alpha)|_{V_\sigma}$ is given by the correspondence from Lemma 3.2.48, which in turn shows $\alpha'|_{\Omega_\sigma} = 0$ by Lemma 3.2.42. Since the Ω_σ cover Ω, we obtain $\alpha' = 0$. □

Corollary 3.2.51. *Let S be $\mathbb{A}$, $\mathbb{G}$ or T. Let X be a variety, which we assume to satisfy (†) if S is T. Let $\alpha_1 \in \mathcal{A}^{p,q}_{X,S}(V_1)$ and $\alpha_2 \in \mathcal{A}^{p,q}_{X,S}(V_2)$ be given by single S-tropical charts $(V_i, \varphi_i, \alpha_i')$ for $i = 1, 2$. If $\alpha_1|_{V_1 \cap V_2} = \alpha_2|_{V_1 \cap V_2}$, then* $\mathrm{Trop}(\psi_{\varphi_1, \varphi_1 \times \varphi_2})^* \alpha_1' = \mathrm{Trop}(\psi_{\varphi_2, \varphi_1 \times \varphi_2})^* \alpha_2'$.

Remark 3.2.52. Corollary 3.2.51 shows that in Definition 3.2.36 *iii)* we may require

$$\mathrm{Trop}(\psi_{\varphi_i,\varphi_i\times\varphi_j})^*(\alpha_i) = \mathrm{Trop}(\psi_{\varphi_j,\varphi_i\times\varphi_j})^*(\alpha_j) \in \mathcal{A}^{p,q}((\varphi_i \times \varphi_j)_{\mathrm{trop}}(V_i \cap V_j)).$$

Also, Lemma 3.2.31 shows that we can restrict to $\mathbb{A}$- resp. T-tropical charts (V, φ) where the image of φ meets the dense torus.

Proposition 3.2.53. *Let X be a variety and $V \subset X^{\mathrm{an}}$ be an open subset. Suppose $\alpha \in \mathcal{A}^{p,q}_{X,T}(V)$ can be defined by a finite familiy $(V_i, \varphi_i, \alpha_i)_{i\in I}$. Then α can be defined using a single T-tropical chart.*

Proof. The proof works like the one of Lemma 3.2.20. Note that we do not need to intersect the domains of our closed immersions, since all closed immersions are defined on all of X.

Let $\varphi_i\colon X_i \hookrightarrow Y_{\Delta_i}$ and let $\varphi := \varphi_1\times\cdots\times\varphi_k\colon X \hookrightarrow Y_{\Delta_1}\times\cdots\times Y_{\Delta_k}$. Then φ refines all the φ_i. By Remark 3.2.52 the forms $\mathrm{Trop}(\psi_{\varphi_i,\varphi})^*(\alpha_i)$ agree on intersection and thus define a form $\alpha' \in \mathcal{A}^{p,q}(\bigcup \mathrm{Trop}(\pi_i)^{-1}(\Omega_i))$, where $\pi : Y_{\Delta_1}\times\cdots\times Y_{\Delta_k} \to Y_{\Delta_i}$ is the projection to the i-th component. By construction we also have $\varphi_{\mathrm{trop}}^{-1}(\bigcup \mathrm{Trop}(\pi_i)^{-1}(\Omega_i)) = V$. Then (V, φ, α') defines α. □

Corollary 3.2.54. *Let X be a variety satisfying (†) and $\alpha \in \mathcal{A}^{p,q}_{T,c}(X^{\mathrm{an}})$. Then there exists a closed immersion $\varphi\colon X \to Y_\Delta$ for a toric variety Y_Δ such that α is given by $(X^{\mathrm{an}}, \varphi, \alpha')$ for $\alpha' \in \mathcal{A}^{p,q}_{\mathrm{Trop}_\varphi(X),c}(\mathrm{Trop}_\varphi(X))$.*

Proof. Since α has compact support, there exists an open subset V of X^{an}, which contains the support of α and such that $\alpha|_V$ is given by finitely many T-tropical charts. Then $\alpha|_V$ is given by one T-tropical chart (V, φ_i, α_1) by Proposition 3.2.53. The fact that α_1 has compact support follows as in the proof of Lemma 3.2.20 from Corollary 3.2.50. Now extending α_1 by zero to all of $\mathrm{Trop}_\varphi(X)$ gives the desired form α'. □

Definition 3.2.55. Let S be $\mathbb{A}$ or $\mathbb{G}$. Let X be a variety of dimension n. Let $\alpha \in \mathcal{A}^{p,q}_{X,S,c}(X)$ such that $\max(p,q) = n$. If S is $\mathbb{A}$ or $\mathbb{G}$, then any S-tropical chart $(U^{\mathrm{an}}, \varphi)$, where $\varphi : U \to S^r$, with the property that $\alpha|_{U^{\mathrm{an}}}$ is given by $(U^{\mathrm{an}}, \varphi, \alpha_U)$ for some $\alpha_U \in \mathcal{A}^{p,q}_{\mathrm{Trop}_\varphi(U),c}(\mathrm{Trop}_\varphi(U))$ is called an *S-tropical chart of integration for α*. A *T-tropical chart of integration* is a T-tropical chart $(X^{\mathrm{an}}, \varphi)$ such that α is given by $(X^{\mathrm{an}}, \varphi, \alpha_X)$ for some $\alpha_X \in \mathcal{A}^{p,q}_{\mathrm{Trop}_\varphi(X),c}(\mathrm{Trop}_\varphi(X))$.

For existence of tropical charts of integration, the following proposition is needed. It follows from [Gub13a, Corollary 5.12] (resp. Proposition 3.2.19). The proof there relies on a deep result by Ducros, namely [Duc12, Theorem 3.2]. We give a self-contained proof using the isomorphism $\mathcal{A}^{p,q}_X \simeq \mathcal{A}^{p,q}_{X,\mathbb{A}}$.

Proposition 3.2.56. *Let X be a variety of dimension n with generic point η. Let $\alpha \in \mathcal{A}^{p,q}(V)$ for an open subset $V \subset X^{\mathrm{an}}$. Let $F\colon X^{\mathrm{an}} \to X$ be the analytification map. Let $x \in X^{\mathrm{an}}$ such that $F(x) \neq \eta$. Then if $\max(p,q) = n$, we have $x \notin \mathrm{supp}(\alpha)$. In particular, $\mathrm{supp}(\alpha)$ is contained in the analytification of every non-empty open subset of X.*

Proof. We use the isomorphism $\mathcal{A}_X^{p,q} \simeq \mathcal{A}_{X,\mathbb{A}}^{p,q}$. Let α locally at x be given by (V, φ, α') where φ meets the dense torus of $\mathbb{A}^r$. Let φ be given by $f_1, \ldots, f_r \in \mathcal{O}_X(U)$. Choose $f \in \mathcal{O}_X(U)$ such that $f(F(x)) = 0$, but f is not identically zero. Let $\varphi' : U \to \mathbb{A}^{r+1}$ be the closed immersion given by $f, f_1, \ldots, f_r$. Then φ' is a refinement of φ via the projection $\pi : \mathbb{A}^{r+1} \to \mathbb{A}^r$. Thus α is also given by $(V, \varphi', \mathrm{Trop}(\pi)^*\alpha')$. Further, φ' still meets the dense torus. We write $W = \varphi'(U) \cap \mathbb{G}_m^{r+1}$. Then we have $\mathrm{Trop}_{\varphi'}(U) = \overline{\mathrm{Trop}(W)} \subset \mathbb{T}^{r+1}$. By construction we have $\mathrm{sed}(\varphi'_{\mathrm{trop}}(x)) \neq \emptyset$. Thus by Lemma 2.1.44 and Lemma 2.1.40 the point $\varphi'_{\mathrm{trop}}(x)$ is not in the support of $\mathrm{Trop}(\pi)^*(\alpha')$. This in turn shows $x \notin \mathrm{supp}(\alpha)$. □

Lemma 3.2.57. *Let S be $\mathbb{A}$, $\mathbb{G}$ or T. Let X be a variety of dimension n, which we require to satisfy (†) if S is T. Let $\alpha \in \mathcal{A}_{X,S,c}^{p,q}(X)$ such that $\max(p,q) = n$. Then there exists a tropical chart of integration for α. Further, if $(U^{\mathrm{an}}, \varphi)$ is an S-tropical chart of integration and $(U'^{\mathrm{an}}, \varphi')$ is an S-tropical subchart, then $(U'^{\mathrm{an}}, \varphi')$ is an S-tropical chart of integration. (Note here that if S is T, then $U' = U = X$.)*

Proof. For $\mathbb{A}$- and $\mathbb{G}$-tropical charts, the arguments from Lemma 3.2.20 work, replacing Lemma 3.2.12 and Proposition 3.2.19 by Lemma 3.2.42 and Proposition 3.2.56. For T-tropical charts, this follows directly from Corollary 3.2.54. □

Definition 3.2.58. Let S be $\mathbb{A}$, $\mathbb{G}$ or T. Let X be a variety of dimension n, which we require to satisfy (†) if S is T. Let $\alpha \in \mathcal{A}_{X,S,c}^{n,n}(X^{\mathrm{an}})$. Then we define

$$\int_{X,S} \alpha := \int_{\mathrm{Trop}_{\varphi}(U)} \alpha_U$$

for $(U^{\mathrm{an}}, \varphi_U)$ an S-tropical chart of integration.

Lemma 3.2.59. *This definition is well defined, compatible with the isomorphisms $\Psi_{\mathbb{G}}, \Psi_{\mathbb{G},\mathbb{A}}$ and $\Psi_{T,\mathbb{A}}$ and satisfies Stokes' theorem.*

Proof. We start with the case when S is $\mathbb{G}$. Then the proof of Proposition 3.2.23 works, replacing canonical tropical charts by $\mathbb{G}$-tropical charts. Thus $\int_{X,\mathbb{G}}$ is well defined. Since any canonical chart of integration is also a $\mathbb{G}$-tropical chart of integration, we get compatibility with $\Psi_{\mathbb{G}}$.

Now let S be $\mathbb{A}$ or T. Let $(U^{\mathrm{an}}, \varphi)$ be an S-tropical chart of integration (with $U = X$ if S is T). Then $\varphi \colon U \hookrightarrow A$, where A is $\mathbb{A}^r$ if S is $\mathbb{A}$ and A is a toric variety otherwise. We may assume that φ meets the dense torus $\mathbb{G}_m^r$. Write $U' := \varphi^{-1}(\mathbb{G}_m^r)$ and $\varphi' := \varphi|_{U'}$. If we denote for $\alpha \in \mathcal{A}_{X,S,c}^{p,q}(X^{\mathrm{an}})$ by $\Psi(\alpha) \in \mathcal{A}_{X,\mathbb{G},c}^{p,q}(X^{\mathrm{an}})$ the corresponding form, then $(U'^{\mathrm{an}}, \varphi', \alpha_U|_{\mathrm{Trop}_{\varphi'}(U')})$ defines $\Psi(\alpha)|_{U'}$. Since $\mathrm{Trop}'_{\varphi}(U')$ is a dense subset of $\mathrm{Trop}_{\varphi}(U)$ which contains precisely the points which map to points of empty sedentarity in each chart, we have $\mathrm{supp}(\alpha_U) \subset \mathrm{Trop}_{\varphi'}(U')$ by Lemma 2.1.44. Thus $(U'^{\mathrm{an}}, \varphi', \alpha_U|_{\mathrm{Trop}_{\varphi'}(U')})$ is a $\mathbb{G}$-tropical chart of integration for $\Psi(\alpha)$. We conclude from the case of $\mathbb{G}$-tropical charts that

$$\int_{X,S} \alpha = \int_{\mathrm{Trop}_{\varphi}(U)} \alpha_U = \int_{\mathrm{Trop}_{\varphi'}(U')} \alpha_U|_{\mathrm{Trop}_{\varphi'}(U')} = \int_{X,\mathbb{G}} \Psi(\alpha) = \int_X \Psi(\alpha).$$

This shows that the integral is compatible with the respective isomorphisms. It is thus also independent of the chosen chart of integration. Stokes' theorem follows immediately, either from the compatibility and Theorem 3.2.24, or simply from Theorem 2.1.63. □

Remark 3.2.60. Since we have shown that the sheaves $\mathcal{A}_X^{p,q}$, $\mathcal{A}_{X,\mathbb{A}}^{p,q}$ and $\mathcal{A}_{X,\mathbb{G}}^{p,q}$ are all isomorphic and these isomorphisms commute with all constructions we consider, we will not distinguish them anymore, always using what is most suitable for the situation. We will do the same with $\mathcal{A}_{X,T}^{p,q}$ if X satisfies (†).

We give an application of the new approaches in the next theorem. The proof uses crucially the fact that in the case of a proper variety satisfying (†) any differential form (and thus any smooth function) is given by *one* form defined on a *compact* tropical space. We hope that this behavior enables more results on the cohomology of these forms in the future.

Theorem 3.2.61. *Let X be a proper variety. Let $f \in C^\infty(X^{\mathrm{an}})$ such that $d'd''f = 0$. Then f is constant.*

Proof. We first assume that X is projective and normal. Then X satisfies condition (†). Since X^{an} is compact we may, by Corollary 3.2.54, assume that f is given by one T-tropical chart (V, φ, g). Then $d'd''g = 0$ by Corollary 3.2.50. Thus g is constant by Corollary 2.1.67, which in turn shows that f is constant.

If X is not normal, denote by $F \colon X' \to X$ its normalization. This is a finite map, thus X' is still projective [Har77, Chapter II, Exercise 3.8]. Then $d'd''F^*(f) = F^*(d'd''f) = 0$, thus $F^*(f)$ is constant. Since F is surjective this implies that f is constant.

For the general case, by Chow's lemma [Har77, Chapter II, Exercise 4.10] we find a surjective morphism $F \colon X' \to X$ such that X' is a projective variety. Then $d'd''F^*(f) = F^*(d'd''f) = 0$, thus $F^*(f)$ is constant. Since F is surjective this shows that f is constant. □

3.3 Fields with trivial valuation

In this subsection, we consider the case where K is trivially valued. Gubler's formalism which we introduced in Section 3.2.1 does not work in this case, as we will see in Example 3.3.1. However the previously established approaches, namely using $\mathbb{A}$- and T-tropical charts work. We will show this and explain how the previously established results generalize to this setting.

In this section, K will be an algebraically closed trivially valued field and X will be a variety over K.

Example 3.3.1. Let $X = \mathbb{A}_K^1 = \operatorname{Spec} K[T]$. Let y be the origin. We claim that the open subset $V := \{x \in \mathbb{A}_K^{1,\mathrm{an}} \mid |T(x)| < 1/2\}$ does not contain an open neigborhood of y which is the preimage of an open set under φ_{trop} for a closed immersion $\varphi \colon U \hookrightarrow \mathbb{G}_m^r$.

Let U be a very affine open subset of X. Let z be a rational point in U^{an}. Then $|f(z)| = 1$ for all $f \in \mathcal{O}_X(U)^\times$. Thus for any open neighborhood $\Omega \subset \mathbb{R}^r$ of $0 \in \mathbb{R}^r$, the preimage $\varphi_{\text{trop}}^{-1}(\Omega)$ contains all rational points in U. However, y is the only rational point contained in V. Thus there is no canonical tropical chart around y which is contained in V.

Lemma 3.3.2. *$\mathbb{A}$-tropical charts form a basis of the topology of X^{an}.*

Proof. We may assume $X = \text{Spec}(A)$ is affine. By definition, a basis of the topology of X^{an} is given by set of the form

$$V = \{x \in X^{\text{an}} | b_i < |f_i(x)| < c_i, i = 1, \dots, r\}$$

for functions $f_1, \dots, f_r \in A$ and elements $b_i \in \mathbb{R}$ and $c_i \in \mathbb{R}_{>0}$. We take $g_1, \dots, g_s$ such that $f_1, \dots, f_r, g_1, \dots, g_s$ generate A as a K-algebra and denote by φ the corresponding closed immersion. Then V is precisely the preimage of the product of intervals of the form $[-\infty, \log(c_i))$, $(\log(b_i), \log(c_i))$ and $[-\infty, \infty)$ in $\mathbb{T}^{r+s}$. Thus (V, φ) is a tropical chart. □

Definition 3.3.3. We define the sheaf $\mathcal{A}^{p,q}_{X,\mathbb{A}}$ in the same way we defined it in Definition 3.2.36. If X satisfies (†), we also define $\mathcal{A}^{p,q}_{X,T}$ in the same way we defined it there.

For a morphism F of varieties, we also define the pullback F^* as in Remark 3.2.40.

Proposition 3.3.4. *Let X be a variety which satisfies (†). Then the sheaves $\mathcal{A}^{p,q}_{X,\mathbb{A}}$ and $\mathcal{A}^{p,q}_{X,T}$ are isomorphic.*

Proof. Works exactly like the proof of Proposition 3.2.49. □

Remark 3.3.5. The statements of Lemma 3.2.42, 3.2.50 - 3.2.57 and 3.2.61 remain true for $\mathbb{A}$ and T even if K is trivially valued.

Also, as in Definition 3.2.16, $(0,0)$-forms define functions on X^{an}. We will again call these functions smooth functions and sometimes write $C^\infty = \mathcal{A}^{0,0}_X$.

Lemma 3.3.6. *The sheaves $\mathcal{A}^{p,q}_X$ are fine, hence soft and acyclic.*

Proof. Again, as in the proof of Lemma 2.1.15, it suffices to show that for each open subset V of X^{an} and $x \in V$ there exists $f \in C^\infty(V)$ such that f has compact support in V and $f \equiv 1$ on a neighborhood V' of x. After shrinking V we may assume there exists an $\mathbb{A}$-tropical chart (V, φ). Since the sheaf $\mathcal{A}^{0,0}_{\text{Trop}_\varphi(U)}$ is fine, there exists $g \in \mathcal{A}^{0,0}_{\text{Trop}_\varphi(U),c}(\varphi_{\text{trop}}(V))$ such that $g|_\Omega \equiv 1$ for a neighborhood Ω of $\varphi_{\text{trop}}(x)$. Then the smooth function defined by (V, φ, g) has the desired property, since the tropicalization map is proper by Lemma 3.1.2. Everything else then follows as in the proof of Lemma 3.2.17. □

Remark 3.3.7. Let L/M be an extension of non-archimedean, algebraically closed complete fields. Let Y be a variety over M and denote by $Y_L := Y \times \text{Spec}(L)$ the base change to L. Then Y_L is a variety over L by [Har77, Exercise 3.15]. Let $p\colon Y_L \to Y$ be the canonical map. Let $\alpha \in \mathcal{A}^{p,q}_Y(V)$ be given by $(V_i, \varphi_i, \alpha_i)$. Since tropicalization is invariant under base field extension (cf. [Pay09, Section 6 Appendix]), we can define $\alpha_L \in \mathcal{A}^{p,q}_{Y_L}(p^{\text{an},-1}(V))$ to be given by $(p^{\text{an},-1}(V_i), \varphi_{i,L}, \alpha_i)$.

We can now define integration in the same way as in Definition 3.2.58. To see that this is well defined we reduce to the non-trivially valued situation. Let L be a complete, algebraically closed, non-trivially valued, non-archimedean extension of K. Let $\alpha \in \mathcal{A}_c^{n,n}(X^{\mathrm{an}})$ and let $\alpha_L \in \mathcal{A}_c^{n,n}(X_L^{\mathrm{an}})$ the corresponding form on X_L^{an}. Then the base change of any $\mathbb{A}$-tropical chart of integration of α is an $\mathbb{A}$-tropical chart of integration for α_L, thus we see that

$$\int_{X^{\mathrm{an}}} \alpha = \int_{X_L^{\mathrm{an}}} \alpha_L.$$

Then applying Lemma 3.2.59 shows that the right hand side of this equation is well defined, thus so is the left hand side.

We also again have Stokes' theorem.

3.4 Cohomological results

In this section, we study the cohomology defined by the differential forms which we introduced in the last sections. We start by proving a Poincaré lemma and deriving consequences. Afterwards, we show finite dimensionality of the cohomology in certain (bi-)degrees.

All these results were obtained by the author with the results from Section 3.4.1 and Theorem 3.4.9 being published in [Jel16], for the case where K is non-trivially valued.

Definition 3.4.1. Let X be a variety over K. We denote by

$$H^{p,q}_{d'}(X^{\mathrm{an}}) = H^p(\mathcal{A}^{\bullet,q}_X(X^{\mathrm{an}}), d') \text{ and } H^{p,q}_{d',c}(X^{\mathrm{an}}) = H^p(\mathcal{A}^{\bullet,q}_{X,c}(X^{\mathrm{an}}), d')$$

the cohomology respectively cohomology with compact support defined by differential forms on X^{an}.

Lemma 3.4.2. *Let X be a variety. The operator $J : \mathcal{A}^{p,q}_X \to \mathcal{A}^{q,p}_X$, as defined in Definition 3.2.38, has the following properties*

i) $J^2 = \mathrm{id}$,

ii) $d'J = Jd''$,

iii) $d''J = Jd'$.

Further, J induces isomorphisms

$$J : H^{p,q}_{d'}(X) \simeq H^{q,p}_{d''}(X)$$

for all p, q.

Proof. The properties $i) - iii)$ follow directly from Lemma 2.1.65. The isomorphism then follows because $J\ker(d') = \ker(d''J)$ and $J\,\mathrm{im}(d') = \mathrm{im}(Jd'')$. □

3.4.1 The Poincaré lemma

In this subsection, we prove the Poincaré lemma, a local exactness result for the complex of differential forms on X^{an}. As a consequence we obtain that certain cohomology groups, namely the groups $H^{p,0}_{d'}(X^{\mathrm{an}})$, agree with singular cohomology. We also outline a generalization to general analytic spaces in the sense of Berkovich.

In this subsection, K will be an algebraically closed field which is complete with respect to a non-archimedean absolute value and X will be a variety over K of dimension n. We work with $\mathbb{A}$-tropical charts, since these work for arbitrary varieties both in the trivially and non-trivially valued case. The proofs work with the other tropical charts as well.

All these results were published in [Jel16] for the case where K is non-trivially valued.

Theorem 3.4.3 (d'-Poincaré lemma on X^{an})**.** *Let X be a variety and $V \subset X^{\mathrm{an}}$ an open subset. Let $x \in V$ and $\alpha \in \mathcal{A}_X^{p,q}(V)$ with $p > 0$ and $d'\alpha = 0$. Then there exists an open $W \subset V$ with $x \in W$ and $\beta \in \mathcal{A}_X^{p-1,q}(W)$ such that $d'\beta = \alpha|_W$.*

Proof. Let α be given by a family $(V_i, \varphi_i, \alpha_i)_{i \in I}$ where (V_i, φ_i) are $\mathbb{A}$-tropical charts, $\alpha_i \in \mathcal{A}^{p,q}(\Omega_i)$ and $\Omega_i := \varphi_{i,\mathrm{trop}}(V_i)$ is an open subset of $\mathrm{Trop}_{\varphi_i}(U_i)$. Choose i such that $x \in V_i$ and let $z := \varphi_{i,\mathrm{trop}}(x)$. Then $\alpha|_{V_i}$ is given by the single chart $(V_i, \varphi_i, \alpha_i)$ and $d'\alpha|_{V_i}$ is given by $(V_i, \varphi_i, d'\alpha_i)$. Since $d'\alpha|_{V_i} = 0$ and it is given by a single chart, we know that $d'\alpha_i = 0$ by Lemma 3.2.42. By Theorem 2.2.18 there exists a neighborhood Ω' of z and $\beta' \in \mathcal{A}^{p-1,q}_{\mathrm{Trop}_{\varphi_i}(U_i)}(\Omega')$ such that $d'\beta = \alpha_i|_{\Omega'}$. We define $W := \varphi_{i,\mathrm{trop}}^{-1}(\Omega')$. The form $\beta \in \mathcal{A}_X^{p-1,q}(W)$ given by (W, φ_i, β') now has the desired property. □

Definition 3.4.4. Let X be a variety. We define the sheaf

$$\mathcal{L}_X^q := \ker(d' : \mathcal{A}_X^{0,q} \to \mathcal{A}_X^{1,q}).$$

Lemma 3.4.5. *We have a canonical isomorphism*

$$\underline{\mathbb{R}} \simeq \mathcal{L}_X^0,$$

where $\underline{\mathbb{R}}$ denotes the constant sheaf with stalks $\mathbb{R}$.

Proof. As noted in Definition 3.2.16 every element of $\mathcal{A}^{0,0}(V)$ defines a continuous function on V. By construction, we have that $d'f = 0$ if and only if f is locally constant since the corresponding statement is true on polyhedral spaces. This proves the lemma. □

Corollary 3.4.6. *The complex*

$$0 \to \mathcal{L}_X^q \to \mathcal{A}_X^{0,q} \xrightarrow{d'} \mathcal{A}_X^{1,q} \xrightarrow{d'} \dots \xrightarrow{d'} \mathcal{A}_X^{n,q} \to 0 \tag{3.4}$$

of sheaves on X^{an} is exact. Furthermore, it is an acyclic resolution, we thus have canonical isomorphisms

$$H^q(X^{\mathrm{an}}, \mathcal{L}^q) \cong H^{p,q}_{d'}(X^{\mathrm{an}}) \qquad \text{and} \qquad H^q_c(X^{\mathrm{an}}, \mathcal{L}^q) \cong H^{p,q}_{d',c}(X^{\mathrm{an}}).$$

In particular, we have isomorphisms

$$H^q_{\mathrm{sing}}(X^{\mathrm{an}}) \cong H^q(X^{\mathrm{an}}, \underline{\mathbb{R}}) \cong H^{p,0}_{d'}(X^{\mathrm{an}})$$
$$\text{and} \qquad H^q_{\mathrm{sing},c}(X^{\mathrm{an}}) \cong H^q_c(X^{\mathrm{an}}, \underline{\mathbb{R}}) \cong H^{p,0}_{d',c}(X^{\mathrm{an}}).$$

Proof. Exactness of (3.4) is a direct consequence of the definition of $\mathcal{L}_X^q$ and Theorem 3.4.3. The second statement follows from Lemma 3.2.17 resp. 3.3.6. The third statement is then a direct consequence of Lemma 3.4.5 and [Bre97, Chapter III, Theorem 1.1]. Note therefore that X^{an} is indeed paracompact, Hausdorff and locally compact. □

Proposition 3.4.7. *Let X be a variety and $V \subset X^{\mathrm{an}}$ an open subset. Let $\alpha \in \mathcal{A}^k(V)$ such that $d\alpha = 0$. Then for $x \in V$ there exists an open neighborhood W of x in V and a form $\beta \in \mathcal{A}^{k-1}(W)$ such that $\alpha|_W - d\beta \in \mathcal{A}^{0,k}(W)$ and such that $\alpha|_W - d\beta$ is closed under d, d' and d''. If $k > \dim(X)$ then $\alpha|_W$ is d-exact.*

Proof. The proof works the same as the proof of Corollary 2.2.17, using Theorem 3.4.3, instead of Theorem 2.2.15. We have to shrink the our open subset during each step, but since only finitely many steps are needed, this is not a problem. □

Theorem 3.4.8. *Let X be a Berkovich analytic space of dimension n. Let $\mathcal{A}^{p,q}$ be the sheaf of differential (p,q)-forms on X as introduced by Chambert-Loir and Ducros in [CLD12]. Then for all $q \in \{0, \dots, n\}$ the complex*

$$0 \to \mathcal{A}^{0,q} \xrightarrow{d'} \mathcal{A}^{1,q} \xrightarrow{d'} \dots \xrightarrow{d'} \mathcal{A}^{n,q} \to 0$$

of sheaves on X is exact in positive degrees. Further, the complex

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^{0,0} \xrightarrow{d'} \mathcal{A}^{1,0} \xrightarrow{d'} \dots \xrightarrow{d'} \mathcal{A}^{n,0} \to 0$$

of sheaves on X is exact.
If X is a good analytic space which is Hausdorff and paracompact, then the cohomology of the complex

$$0 \to \mathcal{A}^{0,0}(X) \xrightarrow{d'} \mathcal{A}^{1,0}(X) \xrightarrow{d'} \dots \xrightarrow{d'} \mathcal{A}^{n,0}(X) \to 0$$

is equal to the sheaf cohomology $H^*(X, \underline{\mathbb{R}})$ of the constant sheaf $\underline{\mathbb{R}}$, which is isomorphic to the singular cohomology $H^*_{\mathrm{sing}}(X, \mathbb{R})$.

Proof. Using [CLD12, Lemme 3.2.2] the same arguments as used in the proof of Theorem 3.4.3 work since forms in the sense of [CLD12] are also locally given by forms on polyhedral complexes. If X is good, Hausdorff and paracompact, then [CLD12, Proposition 3.3.6] shows that there are partitions of unity and the arguments in the proof of Corollary 3.4.6 work. □

3.4.2 Finite dimensionality of cohomology

In this subsection, we show finite dimensionality results for the cohomology defined by differential forms. The main result, which also appeared in [Jel16] is Theorem 3.4.9 and closely tied to existence of skeleta thanks to a result by Hrushovski and Loeser in [HL12]. We give some additional consequences which follow from results previously shown in this thesis.

Theorem 3.4.9. *Let X be a variety. Then the real vector space $H^{p,0}_{d'}(X^{\mathrm{an}})$ is finite dimensional for all p.*

Proof. We show that any cover by affine open subvarieties is a reasonable cover. Then the result follows from Lemma A.2.2. Since X is separated, the intersection of affine subschemes is again affine, thus we only have to show that $H^{p,0}_{d'}(X^{\mathrm{an}})$ is finite dimensional for X affine. By [HL12, Theorem 13.2.1] there exists a strong deformation retraction of X^{an} to a finite simplicial complex S. Finite simplicial complexes have finite dimensional singular cohomology as is certainly well known from algebraic topology (cf. any standard book on algebraic topology, e.g. [Hat02, Theorems 2.27 & 3.2]). As we have $H^p_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{R}) = H^p_{\mathrm{sing}}(S, \mathbb{R})$ by homotopy invariance and $H^{p,0}_{d'}(X^{\mathrm{an}}) = H^p_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{R})$ by Corollary 3.4.6 the result follows. □

Proposition 3.4.10. *Let $F\colon X \to Y$ be a birational morphism of varieties of dimension n. Then for all p, q such that $\max(p, q) = n$, the pullback $F^*\colon \mathcal{A}^{p,q}_c(Y^{\mathrm{an}}) \to \mathcal{A}^{p,q}_c(X^{\mathrm{an}})$ is well defined and an isomorphism. In particular*

$$F^*\colon H^{p,n}_{d',c}(Y^{\mathrm{an}}) \to H^{p,n}_{d',c}(X^{\mathrm{an}})$$

is an isomorphism.

Proof. We first show that if $W \subset Y$ is an open subset and F is the inclusion, thus F^* being just the restriction. Let $\alpha \in \mathcal{A}^{p,q}_c(Y^{\mathrm{an}})$. Then $\mathrm{supp}_{X^{\mathrm{an}}}(\alpha) = \mathrm{supp}_{W^{\mathrm{an}}}(\alpha|_{W^{\mathrm{an}}})$ by Proposition 3.2.56, thus $\alpha|_{W^{\mathrm{an}}} \in \mathcal{A}^{p,q}_c(W^{\mathrm{an}})$. For $\beta \in \mathcal{A}^{p,q}_c(W^{\mathrm{an}})$ we denote by $\beta_0 \in \mathcal{A}^{p,q}_c(Y^{\mathrm{an}})$ the extension of β by zero. It is easy to see that since the support is always contained in W^{an} we have $(\alpha|_{W^{\mathrm{an}}})_0 = \alpha$ and $(\beta_0)|_{W^{\mathrm{an}}} = \beta$, thus restriction defines an isomorphism.

In general we let $W \subset Y$ and $U \subset X$ be open subsets such that $F\colon U \to W$ is an isomorphism. Then

$$\mathcal{A}^{p,q}_c(Y^{\mathrm{an}}) \simeq \mathcal{A}^{p,q}_c(W^{\mathrm{an}}) \simeq \mathcal{A}^{p,q}_c(U^{\mathrm{an}}) \simeq \mathcal{A}^{p,q}_c(X^{\mathrm{an}}),$$

where the first and third isomorphism are due to the case considered before and the second is due to F being an isomorphism between W and U. □

Proposition 3.4.11. *Let X be a variety of dimension n. Then there exists for all p a homomorphism*

$$H^{0,p}_{d'}(X^{\mathrm{an}}) \to H^{0,p}_{d''}(X^{\mathrm{an}}) \simeq H^{p,0}_{d'}(X^{\mathrm{an}}),$$

where this first map is induced by the identity on $\mathcal{A}^{0,p}$ and the second by J. If X is proper, then this map is injective if $p = 0, 1, n$.

Proof. Let $\alpha \in \mathcal{A}^{0,p}(X^{\mathrm{an}})$ such that $d'\alpha = 0$. Then by Lemma 2.1.64 we have $d''\alpha = 0$ and since the map $\alpha \mapsto [\alpha]_{d'}$ is injective we can define

$$\begin{aligned} H^{0,p}_{d'}(X^{\mathrm{an}}) &\to H^{0,p}_{d''}(X^{\mathrm{an}}) \simeq H^{p,0}_{d'}(X^{\mathrm{an}}) \\ [\alpha]_{d'} &\mapsto [\alpha]_{d''} \mapsto [J\alpha]_{d'}. \end{aligned}$$

The second map is an isomorphism by Lemma 3.4.2. The fact that if $p = 0$ the first map is also an isomorphism is obvious.

Let $p = 1$ and suppose $[\alpha]_{d''} = 0$. This means that there exists $f \in C^\infty(X)$ such that $d''f = \alpha$. Then $d'd''f = d'\alpha = 0$, thus by Theorem 3.2.61 we have that f is constant. We therefore have $\alpha = d''f = 0$ which shows injectivity.

Let $p = n$ and suppose $[J\alpha]_{d'} = 0$. This means that there exists $\beta \in \mathcal{A}^{n-1,0}$ such that $d'\beta = J\alpha$. Then we have $d'(\beta \wedge \alpha) = d'\beta \wedge \alpha + (-1)^{p-1}\beta \wedge d'\alpha = d'\beta \wedge \alpha = J\alpha \wedge \alpha$. By Stokes' theorem (Lemma 3.2.59), we have that $\int J\alpha \wedge \alpha = 0$.

Let $(U^{\mathrm{an}}, \varphi, \alpha')$ be an $\mathbb{A}$-tropical chart of integration for α. Then $(U^{\mathrm{an}}, \varphi, J\alpha' \wedge \alpha')$ is one for $J\alpha \wedge \alpha$. Thus we have $\int_{\mathrm{Trop}_\varphi(U)} J\alpha' \wedge \alpha' = 0$. Let $\mathcal{C}$ be a weighted $\mathbb{R}$-rational polyhedral structure on $\mathrm{Trop}(U)$ and $\sigma \in \mathcal{C}_n$. Then $\alpha'|_\sigma = f_\sigma d''x_{[n]}$ for $f_\sigma \in C^\infty(\sigma)$ and $x_1, \ldots, x_n$ a basis of $\mathbb{Z}(\sigma)$, where we denote $[n] := \{1, \ldots, n\}$. Then $J\alpha'|_\sigma = f_\sigma d'x_{[n]}$ and $J\alpha' \wedge \alpha'|_\sigma = f_\sigma^2 d'x_{[n]} \wedge d''x_{[n]}$. Therefore $(-1)^{n(n-1)/2} \int_\sigma J\alpha' \wedge \alpha' \geq 0$ with equality only if $f_\sigma = 0$ and thus $\alpha'|_\sigma = 0$. Summing over all $\sigma \in \mathcal{C}_n$ we obtain that $\int_{\mathrm{Trop}(U)} J\alpha' \wedge \alpha' = 0$ only if $\alpha' = 0$, which is what we wanted to show. □

Lemma 3.4.12. *Let X be a variety. Then there is an exact sequence*

$$0 \to H_{d'}^{0,1}(X^{\mathrm{an}}) \to H_d^1(X^{\mathrm{an}}) \to H_{d'}^{1,0}(X^{\mathrm{an}}),$$

whose maps are induced by the canonical inclusion and projection on the level of forms.

Proof. It is easy to see that

$$\mathcal{A}^{0,1}(X^{\mathrm{an}}) \to \mathcal{A}^1(X^{\mathrm{an}}) \to \mathcal{A}^{1,0}(X^{\mathrm{an}})$$

induces well defined maps on cohomology.

Let $\alpha \in \mathcal{A}^{0,1}(X^{\mathrm{an}})$ such that there exists $f \in C^\infty(X^{\mathrm{an}})$ with $df = \alpha$. This means $d''f = \alpha$ and $d'f = 0$. Thus $\alpha = d''f = d''Jf = Jd'f = 0$. Therefore $H_{d'}^{0,1}(X^{\mathrm{an}}) \hookrightarrow H_d^1(X^{\mathrm{an}})$.

Let $\alpha \in \mathcal{A}^1(X^{\mathrm{an}})$ and write $\alpha = \alpha_0 + \alpha_1$ with $\alpha_0 \in \mathcal{A}^{0,1}(X^{\mathrm{an}})$ and $\alpha_1 \in \mathcal{A}^{1,0}(X^{\mathrm{an}})$. Then $d\alpha = 0$ translates to $d'\alpha_0 = -d''\alpha_1$. Assume further that there exists $f \in C^\infty(X^{\mathrm{an}})$ such that $d'f = \alpha_1$. Then $[\alpha]_d = [\alpha - df]_d = [\alpha_0 - d''f]_d$. Since $d'(\alpha_0 - d''f) = d'\alpha_0 + d''d'f = d'\alpha_0 + d''\alpha_1 = 0$ this shows that

$$H_{d'}^{0,1}(X^{\mathrm{an}}) \to H_d^1(X^{\mathrm{an}}) \to H_{d'}^{1,0}(X^{\mathrm{an}})$$

is exact, which completes the proof. □

Corollary 3.4.13. *Let X be a proper variety. Then the real vector spaces $H_{d'}^{0,1}(X^{\mathrm{an}})$, $H_{d'}^{0,n}(X^{\mathrm{an}})$ and $H_d^1(X^{\mathrm{an}})$ are finite dimensional.*

Proof. The first and second claim are direct consequences of Theorem 3.4.9 and Proposition 3.4.11. The third claim is then a direct consequence of Lemma 3.4.12. □

Chapter 4

Approach to (1,1)-forms via line bundles and models

In this chapter we consider an alternate approach to forms on analytifications of algebraic varieties. This approach was used by Boucksom, Favre and Jonsson in their papers [BFJ16, BFJ15] to solve a non-archimedean Monge-Ampère equation. It uses line bundles and model metrics instead of superforms on tropicalizations. We recall their definitions of metrics, model functions and model metrics. We then prove an analogue of the dd^c-lemma (Theorem 4.2.7). For smooth projective varieties X over fields K, which have residue characteristic zero, this was already done in [BFJ16, Theorem 4.3]. We extend this to varieties which are normal and proper and to any residue characteristic. We choose to work over an algebraically closed field, while Boucksom, Favre and Jonsson work over a discretely valued field.

In this chapter, K will be an algebraically closed field which is complete with respect to a non-trivial absolute value $|\,.\,|$. We will assume that the value group $\log |K^\times|$ equals $\mathbb{Q}$. We denote by K° the valuation ring of K and by $S := \operatorname{Spec}(K^\circ)$ its spectrum. We do not impose any conditions on the characteristic of K or its residue field. X will always be a variety over K, which we assume to be proper and normal.

4.1 Models and model functions

Here we recall the definitions of models, model functions, metrics and model metrics.

Definition 4.1.1. Let L be a line bundle on X. A *metric* $\|\ \|$ on L^{an} associates with each section $s \in \Gamma(U, L)$ on a Zariski open subset U of X a function $\|s\| : U^{\mathrm{an}} \to [0, \infty)$ such that $\|f \cdot s\| = |f| \cdot \|s\|$ holds for each $f \in \mathcal{O}_X(U)$. This function is further non-zero if the section is non-vanishing.

Definition 4.1.2. Let L be a line bundle on the proper variety X. A model for X is a proper flat S-scheme $\mathcal{X}$ with a fixed isomorphism $\mathcal{X} \times \operatorname{Spec}(K) \simeq X$. We will use this to identify X as a subscheme of $\mathcal{X}$. A model of (X, L) is a pair $(\mathcal{X}, \mathscr{L})$ such that $\mathcal{X}$ is a model of X and $\mathscr{L}$ is a line bundle on X and a fixed isomorphism $\mathscr{L}|_X \simeq L$.

For a model $\mathcal{X}$ we denote by $\mathcal{X}_s := \mathcal{X} \otimes_{K^\circ} \tilde{K}$ its special fiber. There is a canonical *reduction map* $\operatorname{red} : X^{\mathrm{an}} \to \mathcal{X}_s$.

Lemma 4.1.3. *Let L be a line bundle on the proper variety X. Then there exists a model $(\mathscr{X}, \mathscr{L})$ of (X, L).*

Proof. This follows from Nagata's compactification theorem ([Voj07, Theorem 4.1 & Theorem 5.7]) and noetherian approximation. □

Definition 4.1.4. Let $(\mathscr{X}, \mathscr{L})$ be a model of $(X, L^{\otimes m})$ for some $m \in \mathbb{N}_{>0}$. There is a unique metric $\| \ \|_{\mathscr{L}}$ on L^{an} such that the following holds: Given a frame t of $\mathscr{L}$ over some open subset $\mathcal{U}$ of $\mathscr{X}$ and a section s of L over $U = X \cap \mathcal{U}$ such that $s^{\otimes m} = ht$ for some regular function h on U, we have $\|s\| = \sqrt[m]{|h|}$ on $U^{\mathrm{an}} \cap \mathrm{red}^{-1}(\mathcal{U}_s)$. A metric on L^{an} which arises this way is called a *model metric determined on* $\mathscr{X}$. A metric is called a *model metric* if it is determined on some model $\mathscr{X}$ of X.

Definition 4.1.5. Let $\mathcal{O}_X$ denote the trivial line bundle on the proper K-variety X. Each model metric $\| \ \|$ on $\mathcal{O}_X$ induces a continuous real-valued function $f = -\log \|1\|$ on X^{an}. The space of *model functions*

$$\mathcal{D}(X) = \{f \colon X^{\mathrm{an}} \to \mathbb{R} \mid f = -\log \|1\| \text{ for a model metric } \| \ \| \text{ on } \mathcal{O}_X\}$$

has a natural structure of a $\mathbb{Q}$-vector space. We write $\mathcal{D}(X)_{\mathbb{R}} = \mathcal{D}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $\mathscr{X}$ be a model of the variety X. We say that a model function $f = -\log \|1\|$ is *determined on* $\mathscr{X}$ if the model metric $\| \ \|$ is determined on $\mathscr{X}$.

4.2 Closed $(1,1)$-forms and the dd^c-lemma

In this subsection we define the terms which are relevant to state and prove the dd^c-lemma (Theorem 4.2.7). The result is an extension of [BFJ16, Theorem 4.3], which requires the variety to be smooth and projective and the field to have residue characteristic zero. We choose to work over an algebraically closed field, thus K° resp. S will not be noetherian.

Definition 4.2.1. Let X be a proper variety over a field K. A line bundle L on X is called *numerically effective (nef)* if $L \cdot C \geq 0$ for all closed curves C in X. It is called *numerically trivial* if $L \cdot C = 0$ for all such curves C.

Let $\mathscr{X}$ be a proper S-scheme. A line bundle $\mathscr{L}$ on $\mathscr{X}$ is called *numerically effective (nef)* if its restriction to each fiber of $\mathscr{X} \to S$ is nef. It is called *numerically trivial* if its restriction to each fiber of $\mathscr{X} \to S$ is numerically trivial.

Lemma 4.2.2. *Let $\mathscr{X}$ be a proper S scheme. Then a line bundle $\mathscr{L}$ is nef if and only if $\mathscr{L}|_{\mathscr{X}_s}$ is nef and the same holds for numerically trivial.*

Proof. For nef this can be shown as in [BFJ16, Lemma 1.2] and for numerically trivial this follows since both $\mathscr{L}$ and $\mathscr{L}^{\otimes -1}$ are nef. □

Definition 4.2.3. We denote by $\mathrm{Pic}(X)_{\mathbb{Q}} := \mathrm{Pic}(X) \otimes \mathbb{Q}$ resp. $\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}} := \mathrm{Pic}(\mathscr{X}) \otimes \mathbb{Q}$ the respective scalar extensions.

We define the space $N^1(X)_{\mathbb{Q}}$ resp. $N^1(X)$ to be the quotient of $\mathrm{Pic}(X)_{\mathbb{Q}}$ resp. $\mathrm{Pic}(X)_{\mathbb{R}}$ by the resp. space generated by numerically trivial line bundles.

Let $\mathcal{X}$ be a proper S scheme. The space $N^1(\mathcal{X}/S)_{\mathbb{Q}}$ resp. $N^1(\mathcal{X}/S)$ is then defined as $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ resp. $\mathrm{Pic}(\mathcal{X})_{\mathbb{R}}$ modulo the resp. subspace generated by numerically trivial line bundles.

Intersection numbers with curves are still well defined on $N^1(\mathcal{X}/S)$ resp. $N^1(X)$. An element $[\mathscr{L}] \in N^1(\mathcal{X}/S)$ resp. $N^1(X)$ is called *nef* if $\mathscr{L} \cdot C \geq 0$ for all closed curves C contained in a fiber of $\mathcal{X} \to S$ resp. X.

The *space of closed* $(1,1)$*-forms on* X is defined as the direct limit

$$\mathcal{Z}^{1,1}(X) := \varinjlim_{\mathcal{X} \in \mathcal{M}_X} N^1(\mathcal{X}/S),$$

where $\mathcal{M}_X$ denotes the space of isomorphism classes of models of X. As with model functions, we say that $\theta \in \mathcal{Z}^{1,1}(X)$ is *determined* on a model $\mathcal{X}$ if it is given by an element in $N^1(\mathcal{X}/S)$.

Remark 4.2.4. Let $\mathcal{X}$ be a model of X and $\mathscr{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ such that $\mathscr{L}|_X = \mathcal{O}_X \in \mathrm{Pic}(X)_{\mathbb{Q}}$. Then $\mathscr{L}$ determines a unique model function $\varphi \in \mathcal{D}(X)$. This model function is obtained by choosing q such that $q\mathscr{L} \in \mathrm{Pic}(X)$ and then taking $q\mathscr{L}$ as the model and q as the integer in Definition 4.1.4. We say that φ is *determined* on $\mathcal{X}$ by $\mathscr{L}$.

Definition 4.2.5. Denote by $\widehat{\mathrm{Pic}}(X)$ the group of isometry classes of line bundles on X endowed with a model metric and by $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ its scalar extension to $\mathbb{Q}$. There are natural maps

$$\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \tilde{\to} \varinjlim_{\mathcal{X} \in \mathcal{M}_X} \mathrm{Pic}(\mathcal{X})_{\mathbb{Q}} \to \mathcal{Z}^{1,1}(X),$$

where the first map is an isomorphism. The *curvature form* of a metrized line bundle $(L, \| \ \|) \in \widehat{\mathrm{Pic}}(X)$ is its image under the composition of these maps and is denoted by $c_1(L, \| \ \|) \in \mathcal{Z}^{1,1}(X)$.

By definition, any model function $\varphi \in \mathcal{D}(X)$ is determined on some model $\mathcal{X}$ by some $\mathscr{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ such that $\mathscr{L}|_X = \mathcal{O}_X$. We set $dd^c\varphi$ to be the closed $(1,1)$-form determined by the numerical class of $\mathscr{L}$ in $N^1(\mathcal{X}/S)$. This then defines a natural linear map

$$dd^c : \mathcal{D}(X)_{\mathbb{R}} \to \mathcal{Z}^{1,1}(X).$$

Further we have the restriction maps $N^1(\mathcal{X}/S) \to N^1(X)$ which induce a linear map

$$\{.\} : \mathcal{Z}^{1,1}(X) \to N^1(X).$$

For $\theta \in \mathcal{Z}^{1,1}(X)$ we call its image $\{\theta\}$ the *de Rham class of* θ.

Remark 4.2.6. It is a direct consequence of the definitions that for $\lambda \in \mathcal{D}(X)$ we have that $dd^c\lambda = c_1(\mathcal{O}_X, \| \ \|_{\mathrm{triv}} e^{-\lambda}) \in \mathcal{Z}^{1,1}(X)$.

We will now proceed to prove an analogue of the dd^c-lemma.

Theorem 4.2.7. *Let X be a normal and proper variety. The sequence*

$$0 \to \mathbb{R} \to \mathcal{D}(X)_{\mathbb{R}} \stackrel{dd^c}{\to} \mathcal{Z}^{1,1}(X) \to N^1(X) \to 0$$

is exact.

Remark 4.2.8. If we denote by $\mathcal{Z}^{1,1}(X)_{\mathbb{Q}} := \varinjlim_{\mathcal{X}\in\mathcal{M}_X} N^1(\mathcal{X}/S)_{\mathbb{Q}}$, then both dd^c and the de Rham class are defined over $\mathbb{Q}$ in the sense that dd^c maps $\mathcal{D}(X)$ into $\mathcal{Z}^{1,1}(X)_{\mathbb{Q}}$ and the de Rham class of an element in $\mathcal{Z}^{1,1}(X)_{\mathbb{Q}}$ is in $N^1(X)_{\mathbb{Q}}$.

Let $\pi \in K$ of valuation 1. Then for any model $\mathcal{X}$ of X we have that $\mathcal{O}_{\mathcal{X}}(\pi^{-1})$ is a model of $\mathcal{O}_X$ and the corresponding model functions is the constant function with value 1. Thus constant rational valued functions are model functions.

Lemma 4.2.9. *Let $\lambda \in \mathcal{D}(X)$ be a model function such that $dd^c\lambda = 0$. Then λ is a constant rationally valued function.*

Proof. The equality $dd^c\lambda = 0$ means that λ is determined by $\mathcal{M} \in \mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ for a model $\mathcal{X}$ of X such that $\mathcal{M}|_X = \mathcal{O}_X$ and such that $\mathcal{M}$ is numerically trivial. After replacing λ by a multiple, we may assume $\mathcal{M} \in \mathrm{Pic}(X)$.

We first consider the case where X is projective. Let L be an ample line bundle on X. By passing to a higher model $\mathcal{X}'$ and replacing $\mathcal{X}$ by $\mathcal{X}'$ and $\mathcal{M}$ by $\varphi^*\mathcal{M}$ we may assume L to have a model $\mathscr{L}$ on $\mathcal{X}$. Since $\mathcal{M}|_X$ is trivial, the intersection number $\mathcal{M}^2 \cdot \mathscr{L}^{n-1}$ is well defined. Using that $\mathcal{M}$ is numerically trivial, we deduce that $\mathcal{M}^2 \cdot \mathscr{L}^{n-1} = 0$ and hence by [YZ16, Theorem 2.1] $\mathcal{M}$ is a pullback from K, which just means that λ is constant. Since the value group of K is $\mathbb{Q}$, this also means that $\lambda \in \mathbb{Q}$.

Now we get rid of projectiveness: By Chow's lemma, there exists a surjective birational morphism $f : X' \to X$ such that X' is projective. If λ is determined by $\mathcal{M}$ on some model $\mathcal{X}$ of X, then we can find a model $\mathcal{X}'$ of X' such that f extends to a map $\mathcal{X}' \to \mathcal{X}$. Then by definition, $f^*(\lambda)$ is the model function determined on $\mathcal{X}'$ by $f^*(\mathcal{M})$. By projection formula we have $dd^c(f^*(\lambda)) = 0$. Thus we have that $f^*(\lambda)$ is constant and in $\mathbb{Q}$, which means that, by surjectivity, λ is constant and in $\mathbb{Q}$. □

Lemma 4.2.10. *Let X be a normal and proper variety and $L \in \mathrm{Pic}^0(X)$. Then there exists an abelian variety A, a morphism $h : X \to A$ and a line bundle $L' \in \mathrm{Pic}^0(A)$ such that $L = h^*(L')$.*

Proof. We first assume that X is smooth. Fix $x_0 \in X(K)$. Then there exists a Poincaré class $\mathcal{P} \in \mathrm{Pic}^0(X \times B)$, where $B := \mathrm{Pic}^0(X)$ is the Picard variety of X, with the following universal property: Let T be a variety over K and $c \in \mathrm{Pic}(X \times T)$ such that

i) $c|_{X\times t} \in \mathrm{Pic}^0(X)$ $\forall t \in T(K)$ and

ii) $c|_{x_0\times T} = 0$.

Then there exists a unique morphism $\varphi : T \to \mathrm{Pic}^0(X)$ such that $c = (\mathrm{id} \times\varphi)^*\mathcal{P}$.

Write $A := \mathrm{Pic}^0(B)$. Since B is an abelian variety, we choose $0 \in B(K)$ as the base point and obtain a Poincaré class $\mathcal{P}_B \in \mathrm{Pic}^0(A \times B)$. We want to apply the universal property of $\mathcal{P}_B$ to $\mathcal{P}$. For $x \in X(K)$ we have that $\mathcal{P}|_{x\times B} \in \mathrm{Pic}^0(B)$ since $\mathcal{P}|_{x_0\times B} = 0$. Further we have $\mathcal{P}|_{X\times 0} = 0$. Hence conditions *i)* and *ii)* are fulfilled and we get a unique morphism $\psi : X \to A$ such that $\mathcal{P} = (\psi \times \mathrm{id})^*\mathcal{P}_B$. Now for $L \in \mathrm{Pic}^0(X)$, we have

$$L = \mathcal{P}|_{X\times L} = (\psi \times \mathrm{id})^*(\mathcal{P}_B)|_{X\times L} = \psi^*(\mathcal{P}_B|_{A\times L}).$$

These arguments generalize to X normal and proper using [FGI$^+$05, Remark 9.5.25], which shows that $B := (\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$ and $(\mathrm{Pic}^0_{B/K})$ exist and are abelian varieties. Further [FGI$^+$05, Exercise 9.4.3] guarantees the existence of a Poincaré class on $\mathrm{Pic}_{X/K}$ resp. $\mathrm{Pic}_{\mathrm{Pic}_{X/K}/K}$ and their restrictions to $\mathrm{Pic}^0_{X/K}$ resp. $\mathrm{Pic}^0_{B/K}$ have the correct universal properties. Hence the theorem is true for X normal and projective. In these arguments the projectiveness of X is only required in [FGI$^+$05, 9.5.25]. However [FGI$^+$05, 9.5.6] says that [FGI$^+$05, 9.5.3 & 9.5.4] are also true for proper X. Since these are the only arguments which require projectiveness in the proof of [FGI$^+$05, 9.5.25], we get the statement also for X proper and normal. □

Lemma 4.2.11. *Let X be a normal and proper variety and $L \in \mathrm{Pic}^0(X)$. Then there exists $\rho \in \mathbb{N}$ and a numerically trivial model $\mathcal{M}$ for $L^{\otimes\rho}$.*

Proof. In the case where X is an abelian variety, this follows from [Gub10, Example 3.7], since the canonical metric is given by a numerically trivial line bundle $\mathcal{M}$. In the general case, we use a morphism $\varphi : X \to A$ with the property that there exists $N \in \mathrm{Pic}^0(A)$ such that $\varphi^*(N) = L$. This is possible by Lemma 4.2.10. If we choose models $\mathscr{X}$ and $\mathcal{A}$ of X and A such that there exists a numerically trivial model $\mathcal{N} \in \mathrm{Pic}(\mathcal{A})_{\mathbb{Q}}$ for $N^{\otimes\rho}$, then after a blowup in the special fiber of $\mathscr{X}$, φ extends to a morphism $\psi : \mathscr{X} \to \mathcal{A}$. If we define $\mathcal{M} := \psi^*(\mathcal{N})$ then $\mathcal{M}$ is a model of $L^{\otimes\rho}$ and numerically trivial by the projection formula. □

Lemma 4.2.12. *Let $\omega \in \mathcal{Z}^{1,1}(X)_{\mathbb{Q}}$ such that $\{\omega\} = 0$. Then there exists $\lambda \in \mathcal{D}(X)$ such that $dd^c\lambda = \omega$.*

Proof. Since $\{\omega\} = 0$ there exists a model $\mathscr{X}$ and $\mathscr{L} \in \mathrm{Pic}(X)_{\mathbb{Q}}$ on $\mathscr{X}$ such that $\mathscr{L}$ determines $\omega \in \mathcal{Z}^{1,1}(X)$ and $L := \mathscr{L}|_X$ is numerically trivial. After replacing ω by a multiple, we may, by [Kle66, Chapter II, §2, Corollary 1 (i)], assume $L \in \mathrm{Pic}^0(X)$. By Lemma 4.2.11 we get a model $\mathcal{M}$ of $L^{\otimes\rho}$ which is numerically trivial. Again after replacing ω by $\rho\omega$, we may assume $\rho = 1$. Since $\mathcal{M}$ is numerically trivial, its class in $\mathcal{Z}^{1,1}(X)_{\mathbb{Q}}$ is trivial and hence ω is also determined by $\mathscr{L} \otimes \mathcal{M}^{-1}$. We further have $\mathscr{L} \otimes \mathcal{M}^{-1}|_X = \mathcal{O}_X$ and hence $\mathscr{L} \otimes \mathcal{M}^{-1}$ induces a metric $\| \|'$ on $\mathcal{O}_X$. We have $\| \|' = \| \|_{\mathrm{triv}} e^{-\lambda}$ for λ the model function determined by $\mathscr{L} \otimes \mathcal{M}^{-1}$. This however just means $dd^c\lambda = \omega$. □

Proof of Theorem 4.2.7. Lemmas 4.1.3, 4.2.9 and 4.2.12 show exactness of

$$0 \to \mathbb{Q} \to \mathcal{D}(X) \stackrel{dd^c}{\to} \mathcal{Z}^{1,1}(X)_{\mathbb{Q}} \to N^1(X)_{\mathbb{Q}} \to 0.$$

The theorem is then obtained by tensoring with $\mathbb{R}$. □

Appendix A

Good covers and Mayer-Vietoris-Sequences

In this section X is a paracompact, Hausdorff, locally compact topological space and $(\mathcal{A}^\bullet, D)$ is a complex of fine sheaves on X. We will apply this when X is either a polyhedral space or a Berkovich analytic space. For $U \subset X$ an open subset we write $H^k(U)$ for the k-th cohomology group of $(\mathcal{A}^\bullet(U), D)$. Then $(\mathcal{A}^\bullet, D)$ will be either $(\mathcal{A}_X^{\bullet,q}, d')$ or $(\mathcal{A}_d^\bullet, d)$. The statements of Theorem A.1.1 and Lemma A.2.2 are special cases of theorems which are certainly well known. We choose to present them here with short proofs for the convenience of the reader.

A.1 Mayer-Vietoris-Sequences

Theorem A.1.1 (Mayer-Vietoris-Sequence)**.** *Let U, U_1, U_2 be open subsets of X such that $U = U_1 \cup U_2$. Let further $U_{12} := U_1 \cap U_2$. Then there exists a long exact sequence*

$$\cdots \to H^{k-1}(U_1) \oplus H^{k-1}(U_2) \to H^{k-1}(U_{12}) \to H^k(U) \to H^k(U_1) \oplus H^k(U_2) \to \dots$$

Proof. A partition of unity argument shows that the sequence

$$0 \to \mathcal{A}^\bullet(U) \to \mathcal{A}^\bullet(U_1) \oplus \mathcal{A}^\bullet(U_2) \to \mathcal{A}^\bullet(U_{12}) \to 0$$

is exact. The result is then obtained by the long exact cohomology sequence. □

Theorem A.1.2 (Mayer-Vietoris-Sequence compact support)**.** *Let U, U_1, U_2 open subsets of X such that $U = U_1 \cup U_2$. Let further $U_{12} := U_1 \cap U_2$. Then there exists a long exact sequence*

$$\cdots \to H_c^k(U_1) \oplus H_c^k(U_2) \to H_c^k(U) \to H_c^{k+1}(U_{12}) \to H_c^{k+1}(U_1) \oplus H_c^{k+1}(U_2) \to \dots$$

Proof. The sequence

$$0 \to \mathcal{A}_c^\bullet(U_{12}) \to \mathcal{A}_c^\bullet(U_1) \oplus \mathcal{A}_c^\bullet(U_2) \to \mathcal{A}_c^\bullet(U) \to 0$$

is exact and the result again follows by the long exact cohomology sequence. □

A.2 Good Covers

Definition A.2.1. Let $U \subset X$ be an open subset. An open cover $(U_i)_{i \in I}$ of U is called a *reasonable cover for* $(\mathcal{A}^\bullet, D)$ if I is finite and for all $n \in \mathbb{N}_{>0}$ and for all $\iota_1, \ldots, \iota_n \in I$ we have that $H^k(\bigcap_{i=1}^{n} U_{\iota_i})$ is finite dimensional for all $k \in \mathbb{N}_0$. It is called a *good cover* if further $H^k(\bigcap_{i=1}^{n} U_{\iota_i}) = 0$ for all $k > 0$.

Lemma A.2.2. *Let $U \subset X$ be an open subset and $(U_i)_{i=1,\ldots,m}$ be a reasonable cover of U for $(\mathcal{A}^\bullet, D)$. Then $H^k(U)$ is a finite dimensional real vector space for all $k \in \mathbb{N}_0$. If $(U_i)_{i=1,\ldots,m}$ is a good cover, then we further have $H^k(U) = 0$ if $k \geq m$.*

Proof. We use induction on m with $m = 1$ being just Definition A.2.1 in both the reasonable and the good case. Now let $m \geq 2$. Let $U' := \bigcup_{i=1}^{m-1} U_i$ and for all $i = 1, \ldots, m-1$ let $U_i' := U_i \cap U_m$. Then $(U_i)_{i=1,\ldots,m-1}$ is a reasonable cover of U' and $(U_i')_{i=1,\ldots,m-1}$ is a reasonable cover of $U' \cap U_m$. The Mayer-Vietoris-Sequence A.1.1 shows that for all k the complex

$$H^{k-1}(U' \cap U_m) \to H^k(U) \to H^k(U') \oplus H^k(U_m) \tag{A.1}$$

is exact. By induction hypothesis both $H^k(U' \cap U_m)$ and $H^k(U')$ are finite dimensional and by definition so is $H^k(U_m)$. Then by exactness of (A.1), $H^k(U)$ is finite dimensional.

If $(U_i)_{i=1,\ldots,m}$ is a good cover, then so are $(U_i)_{i=1,\ldots,m-1}$ and $(U_i')_{i=1,\ldots,m-1}$. So for $k \geq m$, by induction hypothesis $H^k(U') = 0$ (since $k \geq m-1$) and $H^{k-1}(U' \cap U_m) = 0$ (since $k - 1 \geq m - 1$). Since then also $k \geq 2$ we further have $H^k(U_m) = 0$ and (A.1) becomes

$$0 \to H^k(U) \to 0 \oplus 0,$$

which shows $H^k(U) = 0$. □

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Bibliography

[AK06] Federico Ardila and Caroline Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Comb. Theory Ser. B*, 96(1):38–49, 2006.

[AR10] Lars Allermann and Johannes Rau. First steps in tropical intersection theory. *Math. Z.*, 264(3):633–670, 2010.

[Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[BFJ15] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Solution to a non-Archimedean Monge-Ampère equation. *J. Amer. Math. Soc.*, 28(3):617–667, 2015.

[BFJ16] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Singular semipositive metrics in non-Archimedean geometry. *J. Algebraic Geom.*, 25(1):77–139, 2016.

[BIMS15] Erwan Brugallé, Ilia Itenberg, Grigory Mikhalkin, and Kristin Shaw. Brief introduction to tropical geometry. In *Gökova Geometry-Topology conference*, 2015. To appear in Proceedings of 21st Gökova Geometry-Topology Conference.

[BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *Algebr. Geom.*, 3(1):63–105, 2016.

[Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[CLD12] Antoine Chambert-Loir and Antoine Ducros. Formes différentielles réelles et courants sur les espaces de Berkovich. 2012. `http://arxiv.org/abs/1204.6277`.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[Duc12] Antoine Ducros. Espaces de Berkovich, polytopes, squelettes et théorie des modèles. *Confluentes Math.*, 4(4):1250007, 57, 2012.

[FGI+05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[FGP14] Tyler Foster, Philipp Gross, and Sam Payne. Limits of tropicalizations. *Israel J. Math.*, 201(2):835–846, 2014.

[FS05] Eva Maria Feichtner and Bernd Sturmfels. Matroid polytopes, nested sets and Bergman fans. *Port. Math. (N.S.)*, 62(4):437–468, 2005.

[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[GK14] Walter Gubler and Klaus Künnemann. A tropical approach to non-archimedean Arakelov theory. 2014. `http://arxiv.org/abs/1406.7637`.

[Gub10] Walter Gubler. Non-Archimedean canonical measures on abelian varieties. *Compos. Math.*, 146(3):683–730, 2010.

[Gub13a] Walter Gubler. Forms and currents on the analytification of an algebraic variety (after Chambert-Loir and Ducros). 2013. `http://arxiv.org/abs/1303.7364`.

[Gub13b] Walter Gubler. A guide to tropicalizations. In *Algebraic and combinatorial aspects of tropical geometry*, volume 589 of *Contemp. Math.*, pages 125–189. Amer. Math. Soc., Providence, RI, 2013.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HL12] Ehud Hrushovski and François Loeser. Non-archimedean tame topology and stably dominated types. 2012. `http://arxiv.org/abs/1009.0252`.

[Ive86] Birger Iversen. *Cohomology of sheaves*. Universitext. Springer-Verlag, Berlin, 1986.

[Jel16] Philipp Jell. A Poincaré lemma for real-valued differential forms on Berkovich spaces. *Math. Z.*, 282(3-4):1149–1167, 2016.

[JSS15] Philipp Jell, Kristin Shaw, and Jascha Smacka. Superforms, tropical cohomology and Poincaré duality. 2015. `http://arxiv.org/abs/1512.07409`.

[Kle66] Steven L. Kleiman. Toward a numerical theory of ampleness. *Ann. of Math. (2)*, 84:293–344, 1966.

[Lag12] Aron Lagerberg. Super currents and tropical geometry. *Math. Z.*, 270(3-4):1011–1050, 2012.

[Mik06] Grigory Mikhalkin. Tropical geometry and its applications. In *International Congress of Mathematicians. Vol. II*, pages 827–852. Eur. Math. Soc., Zürich, 2006.

[MZ14] Grigory Mikhalkin and Ilia Zharkov. Tropical eigenwave and intermediate Jacobians. In *Homological mirror symmetry and tropical geometry*, volume 15 of *Lect. Notes Unione Mat. Ital.*, pages 309–349. Springer, Cham, 2014.

[OT92] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes.* Springer Verlag, 1992.

[Oxl11] James Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, second edition, 2011.

[Pay09] Sam Payne. Analytification is the limit of all tropicalizations. *Math. Res. Lett.*, 16(3):543–556, 2009.

[Sam66] Pierre Samuel. À propos du théorème des unités. *Bull. Sci. Math. (2)*, 90:89–96, 1966.

[Sha11] Kristin Shaw. Tropical intersection theory and surfaces, 2011. PhD Thesis, available at http://www.math.toronto.edu/shawkm/.

[Sha13] Kristin Shaw. A tropical intersection product in matroidal fans. *SIAM J. Discrete Math.*, 27(1):459–491, 2013.

[Sha15] Kristin Shaw. Tropical surfaces. 2015. `http://arxiv.org/abs/1506.07407`.

[ST08] Bernd Sturmfels and Jenia Tevelev. Elimination theory for tropical varieties. *Math. Res. Lett.*, 15(3):543–562, 2008.

[Stu02] Bernd Sturmfels. *Solving systems of polynomial equations*, volume 97 of *CBMS Regional Conference Series in Mathematics.* American Mathematical Society, Providence, RI., 2002.

[Voj07] Paul Vojta. Nagata's embedding theorem. `arXiv:0706.1907`, 2007.

[War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[Wel80] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, second edition, 1980.

[Wło93] Jarosław Włodarczyk. Embeddings in toric varieties and prevarieties. *J. Algebraic Geom.*, 2(4):705–726, 1993.

[YZ16] Xinyi Yuan and Shou-Wu Zhang. The arithmetic hodge index theorem for adelic line bundles. *Math. Ann.*, 2016. `http://dx.doi.org/10.1007/s00208-016-1414-1`.

[Zha13] Ilia Zharkov. The Orlik-Solomon algebra and the Bergman fan of a matroid. *J. Gökova Geom. Topol. GGT*, 7:25–31, 2013.