University of Regensburg
Department of Mathematics

Master's Thesis in Arithmetic Geometry

Harmonic Functions on the Berkovich Projective Line

Author: Veronika Wanner
Supervisor: Prof. Dr. Walter Gubler
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Zusammenfassung

In dieser Arbeit werden harmonische Funktionen auf der Projektiven Berkovich Geraden $\mathbb{P}^1_{\mathrm{Berk}}$ eingeführt und analoge Resultate zu denjenigen der klassischen Potentialtheorie gezeigt. Des Weiteren wird eine Verbindung zu den glatten Funktionen im Sinne der Theorie der Formen und Ströme auf Berkovichräumen hergestellt.

In den ersten vier Kapiteln der Masterarbeit widmen wir uns, wie bereits angekündigt, der Theorie der harmonischen Funktionen auf $\mathbb{P}^1_{\mathrm{Berk}}$, wobei wir [BR] von Matthew Baker and Robert Rumely folgen. Die Konstruktion der Projektiven Berkovich Geraden $\mathbb{P}^1_{\mathrm{Berk}}$ über einem algebraisch abgeschlossenen und bezüglich eines nicht-trivialen nicht-archimedischen Absolutbetrages vollständigen Körpers K wird beschrieben und wesentliche Eigenschaften, wie deren Baumstruktur, werden gegeben. Mit Hilfe dieser Struktur kann ein Laplace Operator definiert werden, welcher es ermöglicht harmonische Funktionen wie gewohnt als dessen Lösungen zu definieren. Analoge Resultate zu denen der klassischen Potentialtheorie wie beispielsweise das Maximum Prinzip, die Poisson Formel oder das Harnack Prinzip werden bewiesen.

Nachdem harmonische Funktionen auf $\mathbb{P}^1_{\mathrm{Berk}}$ in den ersten Kapiteln eingehend studiert wurden, werden im letzten Kapitel glatte Funktionen auf der Analytifizierung einer beliebigen algebraischen Varietät X über K definiert. Diese wurden ursprünglich von Antoine Chambert-Loir und Antoine Ducros in [CD] als reellwertige $(0,0)$-Differentialformen auf Berkovich anaytischen Räumen eingeführt. Im Falle einer algebraischen Varietät X über K können Differentialformen auf der Analytifizierung X^{an} mit Hilfsmitteln der tropischen Geometrie definiert werden. Hierbei folgen wir Walter Gubler in seiner Veröffentlichung [Gu13]. Wir erhalten Differentialoperatoren d' und d'', wobei wir besonders den Kern der Komposition $d'd''$ auf den glatten Funktionen betrachten werden. Glatte Funktionen im Kern von $d'd''$ können durch die Prägarbe $\log|\mathcal{O}_X^\times|$ charakterisiert werden, was uns ermöglicht eine Verbindung zu den harmonischen Funktionen auf $\mathbb{P}^1_{\mathrm{Berk}}$ herzustellen. Es kann gezeigt werden, dass eine reellwertige Funktion auf einer offenen Teilmenge W von $\mathbb{P}^1_{\mathrm{Berk}}$ genau dann harmonisch ist falls sie sich als eine Linearkombination von Funktionen der Form $\log|\mathcal{O}_{\mathbb{P}^1_K}(W)^\times|$ schreiben lässt. Dies impliziert die Übereinstimmung des Vektorraumes der harmonischen Funktionen auf W mit dem Unterraum $\ker d'd''$ des Verkotrraumes der glatten Funktionen auf W. Folglich ist jeden harmonische Funktion auf einer offenen Teilmenge W von $\mathbb{P}^1_{\mathrm{Berk}}$ glatt.

Um auch im allgemeinen Fall, d.h. im Falle einer beliebigen glatten algebraischen Kurve X über K, eine Verbindung geben zu können, führen wir die Garbe der harmonische Funktionen $\mathcal{H}_X$ im Sinne von Amaury Thuillier in [Th] ein. Insbesondere wird gezeigt, dass Thuillier's Definition diejenige von Baker und Rumely fortsetzt und $\ker d'd''$ eine Untergarbe von $\mathcal{H}_X$ ist. Es werden zwei explizite Bedingungen gegeben in denen die Garbe $\mathcal{H}_X$ gänzlich mit $\ker d'd''$ übereinstimmt. Darüber hinaus wird eine Kurve konstruiert, sodass wir eine harmonische Funktion auf einer offenen Teilmenge von X^{an} finden können welche sich nicht im Kern des Operators $d'd''$ befindet. Da wir zeigen,

dass glatte und zugleich harmonische Funktionen bereits im Kern von $d'd''$ liegen, können wir die Frage ob jede harmonische Funktion notwendigerweise glatt ist abschließend mit einem Nein beantworten.

Contents

1 Introduction

Potential theory is a very old area of mathematics and originates in the 18th century. One can say that the foundation was laid by Joseph-Louis Lagrange and Pierre-Simon Laplace. Lagrange discovered that gravitational forces derive from a function which was later called the potential function by George Green. A few years later, Laplace showed that in a mass free region this function satisfies the partial differential equation which is today known as the Laplacian equation. Todays classical potential theory still contains the study of solutions of Laplace's equation which are called *harmonic functions*. Potential theory, and so in particular the theory of harmonic functions, can be extended to non-archimedean analytic geometry. This is for example done by Matthew Baker and Robert Rumely in [BR] and by Amaury Thuillier in [Th]. In [BR], Baker and Rumely give an approach to potential theory on the non-archimedean projective line, and Thuillier develops in [Th] a non-archimedean potential theory for general curves. In this Master's thesis, we first follow [BR] and elaborate on the theory of harmonic functions on the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}}$, including the construction of $\mathbb{P}^1_{\mathrm{Berk}}$ and the definition of a Laplacian operator. We will see that there are analogues of the main results from complex potential theory, where we refer to the book [Ra] by Thomas Ransford for the classical results. However there are some statements in the classical theory which are not considered in this context. For example the property that every harmonic function on an open subset of $\mathbb{C}$ is smooth (cf. [Ra, Corollary 1.1.4]). This statement raises the question if there is a suitable definition of smoothness and an analogue statement in the non-archimedean potential theory. Antoine Chambert-Loir and Antoine Ducros introduced real-valued differential forms on Berkovich analytic spaces in their preprint [CD]. They define smooth functions as differential forms of bidegree $(0, 0)$. In the algebraic situation, i.e. the Berkovich analytic space is the analytification of an algebraic variety, this theory is summarized and compared with tropical algebraic geometry by Walter Gubler in his paper [Gu13]. For the introduction of the theory of smooth functions we will follow his approach. The link between harmonic functions and smoothness in the sense of [CD] resp. [Gu13] is the main purpose of this thesis.

We now outline the contents of this Master's thesis and emphasize the main results. In the first three chapter we do potential theory on the Berkovich projective line after [BR]. In these chapters we work over an algebraically closed field K which is complete with respect to a non-trivial non-archimedean absolute value $|\ |$. In Chapter 2 we explain the construction and the natural tree structure of the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk}}$. In Section 2.1, we therefore recall the definition of $\mathcal{D}(0, 1)$ and state Berkovich's classification theorem of the points contained in the Berkovich unit disc (Theorem 2.1.6). This says that every $x \in \mathcal{D}(0,1)$ corresponds to a sequence of nested closed discs $(D(a_i, r_i))$. The nature of the intersection of these discs leads to a classification of points into four different types. $\mathcal{D}(0,1)$ has a tree structure which is explained in Section 2.2.

In Section 2.3, we describe the construction of the topological space $\mathbb{P}^1_{\text{Berk}}$ and state some fundamental properties. One obtains $\mathbb{P}^1_{\text{Berk}}$ by glueing together two copies of $\mathcal{D}(0,1)$ along a common annulus but there is also another way to construct $\mathbb{P}^1_{\text{Berk}}$. Let $\mathbb{A}^1_{\text{Berk}}$ denote the Berkovich spectrum of $K[T]$, then $\mathbb{P}^1_{\text{Berk}}$ can be seen as the one-point compactification of the locally compact Hausdorff space $\mathbb{A}^1_{\text{Berk}}$, i.e. $\mathbb{P}^1_{\text{Berk}} = \mathbb{A}^1_{\text{Berk}} \sqcup \{\infty\}$. The extra point ∞ is regarded as a point of type I, i.e. a point in K. In particular, we will see that $\mathbb{P}^1_{\text{Berk}}$ is uniquely path-connected and it is homeomorphic to the inverse limit over all finite subgraphs Γ contained in $\mathbb{P}^1_{\text{Berk}}$. Here a *finite subgraph* Γ of $\mathbb{P}^1_{\text{Berk}}$ is the union of the unique paths between a finite set of points of type II, III or IV.

In Chapter 3 we will see that this structure enables us to construct a measure-valued Laplacian operator on a class of functions $f\colon U \to \mathbb{R} \cup \{\pm\infty\}$ for a *domain* (i.e. open and connected) $U \subset \mathbb{P}^1_{\text{Berk}}$. Section 3.1 includes a description of the whole development and construction of the Laplacian operator. If Γ is a finite subgraph of $\mathbb{P}^1_{\text{Berk}}$ (or more generally a metrized graph), Baker and Rumely give in [BR] an extension Δ_Γ of Zhang's Laplacian operator, introduced in [Zh], to the space of functions of 'bounded differential variation' on Γ. We will denote this space by $\text{BDV}(\Gamma)$. $\Delta_\Gamma(f)$ is a finite signed Borel measure of total mass zero on Γ for each function $f \in \text{BDV}(\Gamma)$. Further, for every finite signed Borel measure μ of total mass zero on Γ there is a function $f \in \text{BDV}(\Gamma)$ such that $\mu = \Delta_\Gamma(f)$. Defining $\text{BDV}(U)$ for a domain U of $\mathbb{P}^1_{\text{Berk}}$ as the class of functions $f\colon U \to \mathbb{R} \cup \{\pm\infty\}$ satisfying

- $f|_\Gamma \in \text{BDV}(\Gamma)$ for every finite subgraph $\Gamma \subset U$, and
- $|\Delta_\Gamma(f)|(\Gamma) \leq B(f)$ for a constant $B(f)$ for every finite subgraph $\Gamma \subset U$,

the collection of measures $\{\Delta_\Gamma(f)\}$ is a coherent system for every $f \in \text{BDV}(U)$. This leads to a unique Borel measure $\Delta_{\overline{U}}(f)$ of total mass zero on the subset $\overline{U}$ of the inverse limit space $\mathbb{P}^1_{\text{Berk}}$. This Borel measure $\Delta_{\overline{U}}(f)$ is called the *complete Laplacian* and its restriction to U is called the *Laplacian*.

In Section 3.2, we give concrete examples of functions contained in the vector space $\text{BDV}(\mathbb{P}^1_{\text{Berk}})$ and calculate their Laplacians. Clearly every constant function $f\colon \mathbb{P}^1_{\text{Berk}} \to \mathbb{R}$ is contained in $\text{BDV}(\mathbb{P}^1_{\text{Berk}})$ and $\Delta_{\mathbb{P}^1_{\text{Berk}}}(f) = 0$. For more complicated examples we introduce the *Hsia kernel* $\delta(x,y)_\infty$ for $x, y \in \mathbb{A}^1_{\text{Berk}}$. By Berkovich's classification theorem $x, y \in \mathbb{A}^1_{\text{Berk}}$ correspond to sequences of nested discs $D(a_i, r_i)$ and $D(b_i, s_i)$, then $\delta(x,y)_\infty := \lim_{i\to\infty} \max(r_i, s_i, |a_i - b_i|)$. Baker and Rumely introduced the Hsia kernel as the fundamental kernel for potential theory on the Berkovich line inspired by a function defined by Liang-Chung Hsia in [Hs]. Further, we define the *generalized Hsia kernel* $\delta(x,y)_\zeta$ with respect to an arbitrary point $\zeta \in \mathbb{P}^1_{\text{Berk}}$ and show that the function $f(x) := -\log_v(\delta(x,y)_\zeta)$ belongs to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ with $\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) = \delta_y - \delta_\zeta$ for fixed $y, \zeta \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$. We use the notation $\log_v$ for the logarithm to the base q_v, where $q_v > 1$ is a fixed real number chosen so that $\log_v |\ |$ is a normalized valuation on K. Moreover, we derive from the last equation the following version of *Poincaré-Lelong formula*: If $f(x) := -\log_v([g]_x)$ on $\mathbb{P}^1_{\mathrm{Berk}}$ for a $g \in K(T)^\times$ and $\mathrm{div}(g) = \sum_{i=1}^m n_i(a_i)$, the function belongs to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and we have

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) = \sum_{i=1}^m n_i \delta_{a_i}.$$

In Chapter 4 we introduce the theory of harmonic functions on $\mathbb{P}^1_{\mathrm{Berk}}$ and give analogues of the main results in the classical potential theory where we again follow [BR]. In Section 4.1, we define *harmonic* functions on open subsets of $\mathbb{P}^1_{\mathrm{Berk}}$ as real-valued functions which are locally *strongly harmonic*, i.e. for every point x we can find an open and connected neighborhood U of x such that the function f is continuous, belongs to $\mathrm{BDV}(U)$ and $\Delta_{\overline{U}}(f)$ is supported on ∂U. Among others we instance the function $f(x) := -\log_v([g]_x)$ on $\mathbb{P}^1_{\mathrm{Berk}}$ for a $g \in K(T)^\times$ with $\mathrm{div}(g) = \sum_{i=1}^m n_i(a_i)$ as a (strongly) harmonic function on a domain U in $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{a_1, \dots, a_m\}$. Further, we state fundamental properties of (strongly) harmonic functions. Additionally to the properties stated in [BR, §7.1], we say something about the behavior of a function $f \colon U \to \mathbb{R}$ with Laplacian $\Delta_U(f) = 0$ on finite subgraphs $\Gamma \subset U$ for a domain U with $|\partial U| < \infty$.

In Section 4.2, we study the *main dendrite* of an open subset U of $\mathbb{P}^1_{\mathrm{Berk}}$ as the points contained in the interior of paths between two boundary points of U. We will see that the behavior of a harmonic function on U is determined by its values on the main dendrite. The knowledge about the main dendrite enables us to compare the terms harmonic and strongly harmonic for a function defined on a domain. We show that every harmonic function in $\mathrm{BDV}(U)$ is already strongly harmonic which is not explicitly stated in [BR]. Further, we give a concrete function which is harmonic but not strongly harmonic.

In Section 4.3, we formulate and prove an analogue of the Maximum Principle, i.e. that every harmonic function on a domain U of $\mathbb{P}^1_{\mathrm{Berk}}$ which attains a minimum or a maximum value on U must be constant. Further, we give a strengthening of it called the Strong Maximum Principle. In Section 4.4, we consider domains with a finite boundary of points of type II, III or IV called *finite-dendrite domains*. First, we see that every harmonic function on such a domain U belongs to $\mathrm{BDV}(U)$ (and so is strongly harmonic on U) and has a continuous extension to ∂U. In particular, we give the additional description $f = f_0 \circ r_\Gamma$ on U where Γ is the main dendrite and its endpoints, r_Γ is a retraction map (cf. Definition 3.1.17) and f_0 a piecewise affine function on Γ. This description of f will help us to connect the two definitions of harmonic functions ([BR] and [Th]). Further, we give an analogue of the Poisson formula on finite-dendrite domains, i.e. the values of a harmonic function on U are recaptured only from the knowledge of f on ∂U. With the help of Poisson formula we will see that the Dirichlet and the Neumann problem are solvable on finite-dendrite domains. Additionally to [BR], we consider the formula explicitly in the case of a *strict simple domain*, i.e. a finite-dendrite domain whose boundary points are all of type II.

Corollary (4.4.9). *If V is a strict simple domain with $\partial V = \{x_1, \ldots, x_m\}$ and f a harmonic function on V, then there exist $c_0, \ldots c_m \in \mathbb{R}$ and $a_1, \ldots, a_m \in K$ not contained in $\overline{V}$ such that*

$$f(x) = c_0 - \sum_{i=1}^{m} c_i \log_v([T - a_i]_x)$$

for all $x \in \overline{V}$.

There is also another version of the Poisson formula (following from the first one) which is stated in Section 4.5 and implies that the Equilibrium measure (cf. Definition 4.3.3) and the Poisson-Jensen measure (cf. Definition 4.5.1) coincide. In Section 4.6, we see another implication of the Poisson formula, an analogue of uniform convergence: If $f_1, f_2, \ldots$ is a sequence of harmonic functions on an open subset U of $\mathbb{P}^1_{\mathrm{Berk}}$ which converge pointwise to a function $f: U \to \mathbb{R}$, then f is harmonic on U, and the f_i converge uniformly to f on compact subsets of U. With the help of uniform convergence one can formulate an analogue of Harnack's Principle which is stated in Section 4.7. If $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ is a domain and $f_1, f_2, \ldots$ a sequence of harmonic functions on U with $f_1 \leq f_2 \leq \ldots$, Harnack's Principle says that the sequence either converges locally uniformly to ∞, or converges locally uniformly to an harmonic function on the domain. Note that we do not require that $0 \leq f_1 \leq f_2 \leq \ldots$ as in [BR].

In Chapter 5 we introduce smooth functions and try to link them with harmonic functions. In Section 5.1 we start with the definition of (p, q)-superforms on open subsets of $\mathbb{R}^r$ together with linear differential operators d' and d'' which were introduced by Lagerberg in [La]. Further, we consider their restriction to supports of polyhedral complexes in $\mathbb{R}^r$ (cf. [CD] and [Gu13]). Afterwards, we will recall Gubler's approach to define differential forms of bidegree (p, q) on the analytification X^{an} of an algebraic variety X over an algebraically closed field endowed with a non-trivial complete non-archimedean absolute value $|\ |$ (cf. [Gu13]). If X is such a n-dimensional algebraic variety, we can cover X by very affine open sets U, i.e. sets U which have a closed immersion to a multiplicative torus $\mathbb{G}^r_m$. Then there is a map $\mathrm{trop}_U : U^{\mathrm{an}} \to \mathbb{R}^r$ such that $\mathrm{Trop}(U) := \mathrm{trop}_U(U^{\mathrm{an}})$ is the support of a polyhedral complex of pure dimension n. We define superforms on U^{an} as a formal pullback of forms on $\mathrm{Trop}(U)$ (for details cf. Definition 5.1.20). Therefore, we obtain for every open subset W of X^{an} a real vector space $A^{p,q}(W)$ of differential forms of bidegree (p, q) and differential operators $d' : A^{p,q}(W) \to A^{p+1,q}(W)$ and $d'' : A^{p,q}(W) \to A^{p,q+1}(W)$. We will see that a differential form of bidegree $(0, 0)$ is a well-defined continuous function $f : W \to \mathbb{R}$, and hence *smooth* functions can be defined as $(0, 0)$-differential forms. We will denote the vector space $A^{0,0}(W)$ by $C^\infty(W)$.

In Section 5.2, it is shown that the function $\log|f| : W \to \mathbb{R}$ is smooth and satisfies

$d'd''(\log|f|) = 0$ if W is an open subset of X^{an} and $f \in \mathcal{O}_{X^{\mathrm{an}}}(W)^\times$. Further, we have the following main characterization:

Theorem (5.2.4). *Let W be an open subset of X^{an}. A function $f\colon W \to \mathbb{R}$ belongs to the kernel of $d'd''\colon C^\infty(W) \to A^{1,1}(W)$ if and only if for every $x \in W$ there is an open neighborhood V of x in W and an open subset U of X with $V \subset U^{\mathrm{an}}$ such that*

$$f = \sum_{i=1}^{r} \lambda_i \log|f_i|$$

on V for $f_1, \ldots, f_r \in \mathcal{O}_X(U)^\times$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$.

In Section 5.3, we give similar characterizations of harmonic functions. If $X = \mathbb{P}^1_K$ over an algebraically closed field K, we can formulate the following description of harmonic functions by using the above stated Poincaré-Lelong formula and the Poisson formula for strict simple domains.

Theorem (5.3.1). *Let W be an open subset of $\mathbb{P}^1_{\mathrm{Berk}}$, then f is harmonic on W if and only if for every $x \in W$ there is an open neighborhood V of x in W and an open subset U of $\mathbb{P}^1_K$ with $V \subset U^{\mathrm{an}}$ such that*

$$f = \sum_{i=1}^{r} \lambda_i \log|f_i|$$

on V where $f_1, \ldots, f_r \in \mathcal{O}_{\mathbb{P}^1_K}(U)^\times$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$.

Hence, we get the link between harmonic and smooth functions on X^{an} if $X = \mathbb{P}^1_K$.

Corollary (5.3.2). *A function f is harmonic on an open subset W of $\mathbb{P}^1_{\mathrm{Berk}}$ if and only if f is smooth on W and $d'd''f = 0$.*

In the general case, we introduce Thuillier's approach to harmonic functions from [Th], show that his definition is an extension to the one made in Chapter 4 (cf. Proposition 5.3.14) and state the following Theorem by Thuillier (cf. [Th, Théorème 2.3.21]) including an elaboration on the proof. Thuillier works over a field k which is complete with respect to a non-trivial non-archimedean absolute value $|\ |$. Note that k is not required to be algebraically closed.

Theorem (5.3.17). *Let X be a smooth strictly k-analytic curve and $\mathcal{F}_X$ be the sheaf of $\mathbb{R}$-vector spaces generated by the germes of functions $\log|f|$ where $f \in \mathcal{O}_X^\times$. Then $\mathcal{F}_X$ is a subsheaf of $\mathcal{H}_X$ and $\mathcal{H}_X/\mathcal{F}_X$ is zero if one of the following conditions is satisfied:*

i) The residue field $\tilde{k}$ is algebraic over a finite field.

ii) The curve $X\widehat{\otimes}_k\widehat{k^a}$ is locally isomorphic to $\mathbb{P}^1_{\mathrm{Berk}}$ over $\widehat{k^a}$ where $\widehat{k^a}$ is the completion of the algebraic closure of k.

The definition of a *strictly k-analytic curve* is given in Definition 5.3.3. For instance the analytification of an algebraic curve over k is a strictly k-analytic curve.

If K is an algebraically closed field endowed with a non-trivial complete non-archimedean absolute value and X is a smooth algebraic curve over K, X^{an} is a strictly k-analytic smooth curve. Further, $\mathcal{F}_{X^{\mathrm{an}}}$ coincides with $\ker d'd''$ by Theorem 5.2.4. Hence, Thuillier's theorem delivers two explicit conditions in which a function is harmonic if and only if it is smooth and belongs to $\ker d'd''$. In particular, one can see that Thuillier's theorem leads to the same result if $X = \mathbb{P}^1_K$ as Theorem 5.3.1.

Corollary (5.3.21). *If X is a smooth algebraic curve over K and*

i) $\widetilde{K}$ is algebraic over a finite field, or

ii) X^{an} is locally isomorphic to $\mathbb{P}^1_{\mathrm{Berk}}$,

a function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} is harmonic if and only if it is smooth and $d'd''f = 0$.

By the proof of Theorem 5.3.17 one can construct a smooth algebraic curve X over K such that $\mathcal{H}_{X^{\mathrm{an}}}/\mathcal{F}_{X^{\mathrm{an}}}$ is nonzero (cf. Corollary 5.3.23). Thus, there is a harmonic function which is not contained in $\ker d'd''$. However, we cannot yet say if the function is smooth or not. To give finally an answer to the question if every harmonic function on an open subset W of X^{an} is smooth, we state a further Theorem:

Theorem (5.3.22). *Every smooth function $f\colon W \to \mathbb{R}$ which is harmonic satisfies $d'd''f = 0$.*

Altogether, we have the following conclusion:

Corollary (5.3.23). *Harmonic functions are not smooth in general, i.e. there is a smooth curve X over K and a harmonic function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} which is not smooth.*

Terminology

If A and B are two sets with $A \subset B$, then A may be equal to B. We denote the complement of A in B by $B \backslash A$. The zero is included in $\mathbb{N}$. Further, all rings and algebras are with 1. For a ring R we use the notation $R^\times$ for the group of multiplicative units. If K is a field, then $| \, |$ denotes a non-trivial non-archimedean absolute value on K. We write $|K^\times|$ for its value group. The completion of K with respect to $| \, |$ is denoted by $\widehat{K}$ and an algebraic closure of K by K^a. A variety over a field is an irreducible separated reduced scheme of finite type.

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2 The Berkovich projective line

In this section, we will explain the structure and the topology of $\mathbb{P}^1_{\text{Berk}}$. We start by studying the *Berkovich unit disc* $\mathcal{D}(0,1)$ and its properties in Section 2.1. In particular, we will see that there is a classification of points in $\mathcal{D}(0,1)$. Further, in Section 2.2 we recall the basic knowledge about $\mathbb{R}$-trees and show that the Berkovich unit disc carries a tree structure. By glueing two copies of the Berkovich unit disc together, one can construct the *Berkovich projective line* $\mathbb{P}^1_{\text{Berk}}$ (see Section 2.3). Hence, this construction leads to a tree structure on $\mathbb{P}^1_{\text{Berk}}$ as well. This property makes it possible to define a Laplacian operator on $\mathbb{P}^1_{\text{Berk}}$ (see Chapter 3) and do potential theory on it. In this Chapter we fix an algebraically closed field K which is complete with respect to a non-trivial non-archimedean absolute value $|\ |$, e.g. $K = \mathbb{C}_p$. Let $a \in K$ and $r \geq 0$, then we use the notions $D(a,r) := \{b \in K|\ |b-a| \leq r\}$ and $D(a,r)^- := \{b \in K|\ |b-a| < r\}$ for the closed respectively the open ball.

2.1 The definition and structure of the Berkovich unit disc

For a real number $R > 0$, let

$$K\langle R^{-1}T\rangle = \{\sum_{k=0}^{\infty} a_k T^k |\ a_k \in K,\ \lim_{n\to\infty} R^k |a_k| = 0\}$$

be the ring of formal power series on K converging on $D(0,R)$. $K\langle R^{-1}T\rangle$ is complete under the norm $\|\ \|_R$ defined by $\|f\|_R := \max_{k\geq 0} R^k|a_k|$ for $f = \sum_{k=0}^{\infty} a_k T^k$ with $\lim_{k\to\infty} R^k|a_k| = 0$. If $R = 1$, we call $\|\ \|_R$ the *Gauß norm* and use the notion $\|\ \|$. We will define the Berkovich unit disc $\mathcal{D}(0,1)$ as the Berkovich spectrum of the Banach algebra $\mathcal{A} := K\langle T\rangle$.

Definition 2.1.1. A map $|\ |_x \colon \mathcal{A} \to \mathbb{R}_{\geq 0}$ is called a *bounded multiplicative seminorm on $\mathcal{A}$* if it satisfies

i) $|0|_x = 0$ and $|1|_x = 1$,

ii) $|f+g|_x \leq |f|_x + |g|_x$,

iii) $|fg|_x = |f|_x \cdot |g|_x$,

for all $f, g \in \mathcal{A}$ and there is a constant C_x such that $|f|_x \leq C_x \|f\|$ for each $f \in \mathcal{A}$.

Lemma 2.1.2. *Let $|\ |_x$ be a bounded multiplicative seminorm on $\mathcal{A}$, then* *i)* $|f|_x \leq \|f\|$,

ii) $|c|_x = |c|$ *for all* $c \in K$,

iii)
$$|f+g|_x \leq \max(|f|_x, |g|_x),$$
and
$$|f+g|_x = \max(|f|_x, |g|_x)$$
if $|f|_x \neq |g|_x$.

Proof. See [BR, Lemma 1.1]. □

Definition 2.1.3. We call the Berkovich spectrum of $\mathcal{A}$, i.e. the set of all bounded multiplicative seminorms on $\mathcal{A}$, the *Berkovich unit disc* and write $\mathcal{D}(0,1)$. The topology on $\mathcal{D}(0,1)$ is taken to be the weakest topology such that the function

$$\mathcal{D}(0,1) \to \mathbb{R}_{\geq 0}, x \mapsto |f|_x$$

is continuous for all $f \in \mathcal{A}$.

Remark. The topology is generated by the sets of the form

$$U(f,\alpha) := \{x \in \mathcal{D}(0,1) | \ |f|_x < \alpha\}$$

and

$$V(f,\alpha) := \{x \in \mathcal{D}(0,1) | \ |f|_x > \alpha\}$$

with $f \in \mathcal{A}$ and $\alpha \in \mathbb{R}_{\geq 0}$. This topology makes $\mathcal{D}(0,1)$ into a compact Hausdorff space (cf. [BR, Theorem C.3]).

At next, we will see that the points in $\mathcal{D}(0,1)$ can be classified in four different types of points.

Definition 2.1.4. Let $a \in D(0,1) \subset K$ and $r \in (0,1]$, then we define

$$|f|_a := |f(a)| \text{ and } |f|_{D(a,r)} := \sup_{b \in D(a,r)} |f(b)|$$

for $f \in \mathcal{A}$. If $a_i \in D(0,1) \subset K$ and $r_i \in (0,1]$ for $i \geq 1$ and $x = (D(a_i, r_i))_{i=1,2,\ldots}$ is a sequence of nested discs, then

$$|f|_x := \inf_{i \geq 1} |f|_{D(a_i,r_i)}$$

for all $f \in \mathcal{A}$.

Remark 2.1.5. The Maximum Modulus Principle in non-archimedean analysis ([BGR, Propositions 5.1.4/2 and 5.1.4/3]) says if $D(a,r) \subset D(0,1)$ and $f = \sum_{n \geq 0} a_n (T-a)^n$

in $K\langle T\rangle$, then $|f|_{D(a,r)} = \max_{n\geq 0}(|a_n|r^n)$. Hence, one can verify that $|f|_{D(a,r)}$ is multiplicative. Indeed, each of the three defined maps on $\mathcal{D}(0,1)$ are bounded multiplicative seminorms on $\mathcal{A}$ (cf. [BR, §1.2 p.3]).

Theorem 2.1.6 (Berkovich's Classification Theorem)**.** *For every $x \in \mathcal{D}(0,1)$ one can find a sequence of nested discs $D(a_1, r_1) \supset D(a_2, r_2) \supset \ldots$ such that*

$$|f|_x = \inf_{i\geq 1} |f|_{D(a_i,r_i)}.$$

Moreover, if the sequence has a non-empty intersection, then $\bigcap_{i\geq 1} D(a_i, r_i) = D(a, r)$ for $r \geq 0$ and $a \in K$, and

$$|f|_x = |f|_{D(a,r)}.$$

Proof. See [BR, Theorem 1.2]. □

Corollary 2.1.7. *The points of $\mathcal{D}(0,1)$ can be classified in the following four types: Let $x \in \mathcal{D}(0,1)$ and $\{D(a_i, r_i)\}$ be its corresponding sequence of nested discs.*

- Type I*:* *If $\inf_i r_i = 0$, we call x a point of type I.*
  Since K is complete, $\bigcap_{i\geq 1} D(a_i, r_i) = a \in D(0,1)$, and so $|\ |_x = |\ |_a$.
- Type II: *If $r := \inf_i r_i > 0$ and $r \in |K^\times|$, we call x a point of type II.*
  Then $\bigcap_{i\geq 1} D(a_i, r_i) = D(a, r) \neq \emptyset$ for some $a \in D(0,1) \subset K$, and $|\ |_x = |\ |_{D(a,r)}$.
- Type III: *If $r := \inf_i r_i > 0$ and $r \notin |K^\times|$, we call x a point of type III.*
  Then $\bigcap_{i\geq 1} D(a_i, r_i) = D(a, r) \neq \emptyset$ for some $a \in D(0,1) \subset K$, and $|\ |_x = |\ |_{D(a,r)}$.
- Type IV: *If the sequence has an empty intersection, we call x a point of type IV.*
  Then necessarily $\inf_i r_i > 0$.

2.2 ℝ-trees and the tree structure of $\mathcal{D}(0,1)$

First, we repeat the definition of a rooted ℝ-tree and a parametrized ℝ-tree. Afterwards, we give a one-to-one correspondence between the two definitions. With the help of Berkovich's classification theorem one can show that the Berkovich unit disc $\mathcal{D}(0,1)$ is a parametrized ℝ-tree, and so a rooted ℝ-tree. This tree structure on $\mathcal{D}(0,1)$ implies that $\mathbb{P}^1_{\mathrm{Berk}}$ is *profinite* ℝ*-tree*, i.e. an inverse limit of finite ℝ-trees, which leads directly to the construction of a Laplacian operator (cf. Chapter 3).

Definition 2.2.1. Let (X, d) be a metric space.

i) A *geodesic segment* is the image of an isometric embedding $[a, b] \to X$ for a real interval $[a, b]$.

ii) An *arc* is an injective continuous map $\iota\colon [a, b] \to X$.

iii) (X, d) is an $\mathbb{R}$-*tree* if every two points $x \neq y \in X$ are joined by a unique arc, i.e. there is an arc $\iota\colon [a, b] \to X$ such that $\iota(a) = x$ and $\iota(b) = y$. $\alpha([a, b])$ is a geodesic segment.

iv) A *rooted tree* is a triple (X, d, ζ) consisting of an $\mathbb{R}$-tree (X, d) and a point $\zeta \in X$, which is called the *root*.

v) Let (X, d) be an $\mathbb{R}$-tree. A point $x \in X$ is called *ordinary* if $X \backslash \{x\}$ has two connected components. It is called a *branch point* if $X \backslash \{x\}$ has more than two connected components. And we call x an *end point* if $X \backslash \{x\}$ has only one connected component.

vi) A *finite* $\mathbb{R}$-*tree* is an $\mathbb{R}$-tree which is compact and has only finitely many branch points.

Definition 2.2.2. A *parametrized* $\mathbb{R}$-*tree* is a partially ordered set $(X, \geq)$ with a function $\alpha\colon X \to \mathbb{R}_{\geq 0}$ satisfying

i) X contains a unique maximal element ζ.

ii) $S_x := \{z \in X |\ z \geq x\}$ is totally ordered for all $x \in X$.

iii) $\alpha(\zeta) = 0$.

iv) $\alpha(x) \geq \alpha(y)$ for all $x, y \in X$ with $x \leq y$.

v) The restriction of α to any full totally ordered subset of X gives a bijection onto a real interval, where a totally ordered subset S is called *full* if $x \leq z \leq y$ with $x, y \in S$ implies $z \in S$.

Proposition 2.2.3. *There is a one-to-one correspondence between rooted* $\mathbb{R}$*-trees and parametrized* $\mathbb{R}$*-trees.*

Proofsketch. If (X, d, ζ) is a rooted $\mathbb{R}$-tree, one can show that $x \geq y$ iff y is contained in the unique geodesic segment $[x, \zeta]$ defines a partial order on X and $\alpha(x) := d(x, \zeta)$ is a function such that $(X, \geq)$ is a parametrized $\mathbb{R}$-tree. Conversely, if $(X, \geq)$ is a parametrized $\mathbb{R}$-tree and α the required function on X, then we can set $\zeta := \max(X)$ and

$$d(x, y) := \alpha(x) + \alpha(y) - 2\alpha(x \vee y)$$

defines a metric on X, where

$$x \vee y := \alpha^{-1}(\inf(\alpha(S_x \cap S_y)) \cap S_x.$$

In particular, $x \vee y$ satisfies $x \vee y \geq x$, $x \vee y \geq y$, and $z \geq x$ and $z \geq y$ imply $z \geq x \vee y$. □

2.2.4. Considering an $\mathbb{R}$-tree (X, d), X is equipped with the topology induced by the metric d, which we will call the *strong topology* of X. But we can also define a weaker topology on X, which is given as follows:

For a $p \in X$, we define an equivalence relation on $X\backslash\{p\}$ by $x \sim y$ iff

$$[p,x]\backslash\{p\} \cap [y,p]\backslash\{p\} \neq \emptyset$$

for the unique geodesic segments $[p,x]$ and $[y,p]$. Let $T_p(X)$ denote the set of equivalence classes for this relation and define $\mathcal{B}_p(\vec{v})$ as the set of points in an equivalence class $\vec{v}$. Then we call the topology induced by the sets $\mathcal{B}_p(\vec{v})$ for $p \in X$, $\vec{v} \in T_p(X)$ the *weak topology* of X.

Now, we will use Berkovich's classification theorem to show that $\mathcal{D}(0,1)$ is homeomorphic to an $\mathbb{R}$-tree endowed with its weak topology. One can define a partial order on $\mathcal{D}(0,1)$ by $x \leq y$ iff $|f|_x \leq |f|_y$ for all $f \in \mathcal{A}$. Hence, the unique maximal element is the bounded multiplicative seminorm $|\ |_{D(0,1)}$ which we will call the *Gauss point* and it is denoted by ζ_{Gauss}. The minimal points under this partial order are the points of type I and IV (cf. [BR, Corollary 1.11]).

Definition 2.2.5. Let $x \in \mathcal{D}(0,1)$, then there exists a sequence of nested discs $(D(a_i, r_i))$ such that $|\ |_x = \inf_{i \geq 1} |\ |_{D(a_i,r_i)}$ by the Berkovich's Classification Theorem. We define the *diameter of* x as

$$\operatorname{diam}(x) := \lim_{i \to \infty} r_i.$$

2.2.6. One can verify that the function

$$\alpha := 1 - \operatorname{diam}(x)$$

makes $\mathcal{D}(0,1)$ into a parametrized $\mathbb{R}$-tree (cf. [BR, §1.4 p.11]). By Proposition 2.2.3, the metric

$$d(x,y) := 2\operatorname{diam}(x \vee y) - \operatorname{diam}(x) - \operatorname{diam}(y),$$

where $x \vee y$ denotes the least upper bound of x and y, makes $\mathcal{D}(0,1)$ into an $\mathbb{R}$-tree. The endpoints are given by the points of type I and IV, the ordinary points are the points of type III and the branch points coincide with the points of type II (cf. [BR, §1.4 p.12]).

Definition 2.2.7. The metric d is called the *small metric.* Note that the topology induced by the small metric is not the same as the Berkovich topology. On $\mathcal{D}(0,1)\backslash D(0,1)$ we can define the *big distance* or *logarithmic distance*

$$\rho(x,y) := 2\log_v(\operatorname{diam}(x \vee y)) - \log_v(\operatorname{diam}(x)) - \log_v(\operatorname{diam}(y)).$$

Proposition 2.2.8. *$\mathcal{D}(0,1)$ with its Berkovich topology is homeomorphic to the $\mathbb{R}$-tree $(\mathcal{D}(0,1), d)$ with its weak topology. Further, the metric ρ makes $\mathcal{D}(0,1)\backslash D(0,1)$ into an $\mathbb{R}$-tree.*

Proof. See [BR, Proposition 1.13] and [BR, Proposition 1.15]. □

Corollary 2.2.9. *The space $\mathcal{D}(0,1)$ is uniquely path-connected.*

Proof. This is a direct consequence of the previous proposition and the fact that an $\mathbb{R}$-tree is uniquely path-connected in its weak topology by [BR, Corollary B.20]. $\square$

2.3 The construction of $\mathbb{P}^1_{\mathrm{Berk}}$

In this section, we construct $\mathbb{P}^1_{\mathrm{Berk}}$ by glueing together two copies of $\mathcal{D}(0,1)$, which was defined in the previous section, along the following common annulus. Consider the Berkovich spectrum of $K\langle T, T^{-1}\rangle := \{\sum_{n\in\mathbb{Z}} a_n T^n |\ \lim_{|n|\to\infty} |a_n| = 0\}$ which we denote by S^1. Then the points of $S^1 \subset \mathcal{D}(0,1)$ of type I are the points $a \in K$ such that $|a| = 1$, the points of type II and III are corresponding to discs $D(a,r)$ with $|a| = 1$ and the points of type IV are corresponding to a sequence $(D(a_i, r_i))$ with $|a_i| = 1$ for all $i \geq 1$. With the help of the involution $\iota\colon S^1 \to S^1$ given by

$$|g(T)|_{\iota(x)} := |g(1/T)|_x$$

one can glue two copies of $\mathcal{D}(0,1)$ together, i.e.

$$E \coprod E'/(z \in E \sim \iota(z) \in E'),$$

where $E := \mathcal{D}(0,1)$ and $E' := \mathcal{D}(0,1)$. There is also a further way to construct $\mathbb{P}^1_{\mathrm{Berk}}$. Let $\mathbb{A}^1_{\mathrm{Berk}}$ denote the Berkovich spectrum of $K[T]$, i.e. $\mathbb{A}^1_{\mathrm{Berk}}$ is the set of all multiplicative seminorms on $K[T]$ extending $|\ |$ endowed with the weakest topology such that $x \mapsto |f|_x$ is continuous for all $f \in K[T]$. Note that $\mathbb{A}^1_{\mathrm{Berk}}$ is homeomorphic to the union $\bigcup_{R>0} \mathcal{D}(0,R)$ (cf. [BR, §2.1 Equ.(2.1)]). Hence, Berkovich's classification theorem can be extended to $\mathbb{A}^1_{\mathrm{Berk}}$ (cf. [BR, Theorem 2.2]). $\mathbb{P}^1_{\mathrm{Berk}}$ can be seen as the one-point compactification of the locally compact Hausdorff space $\mathbb{A}^1_{\mathrm{Berk}}$, i.e.

$$\mathbb{P}^1_{\mathrm{Berk}} = \mathbb{A}^1_{\mathrm{Berk}} \sqcup \{\infty\}.$$

The extra point ∞ is regarded as a point of type I. Identifying $\mathbb{P}^1_{\mathrm{Berk}}$ with $\mathbb{A}^1_{\mathrm{Berk}} \cup \{\infty\}$, we view the open and closed Berkovich disc

$$\mathcal{D}(a,r)^- := \{x \in \mathbb{A}^1_{\mathrm{Berk}} \mid |T-a|_x < r\}$$

$$\mathcal{D}(a,r) := \{x \in \mathbb{A}^1_{\mathrm{Berk}} \mid |T-a|_x \leq r\}$$

as subsets of $\mathbb{P}^1_{\mathrm{Berk}}$.

Lemma 2.3.1. *If the intersection of two open balls $\mathcal{D}(a,r)^-$ and $\mathcal{D}(b,s)^-$ is non-empty, one of them contains the other.*

Proof. Assume that $r \leq s$, and let x be an element in the intersection. Since $|T-b|_a = |a-b|$ and

$$|a-b| = |a-b|_x \leq \max(|T-a|_x, |T-b|_x) < s,$$

a is contained in $\mathcal{D}(b,s)^-$. For each $y \in \mathcal{D}(a,r)^-$ we have

$$|T-b|_y = |T-a+a-b|_y \leq \max(|T-a|_y, |a-b|_y) < s,$$

i.e. $y \in \mathcal{D}(b,s)^-$. □

Proposition 2.3.2. *Identifying $\mathbb{P}^1_{\text{Berk}}$ with $\mathbb{A}^1_{\text{Berk}} \cup \{\infty\}$, a basis for the open sets of $\mathbb{P}^1_{\text{Berk}}$ is given by the sets of the form*

$$\mathcal{D}(a,r)^-, \ \mathcal{D}(a,r)^- \backslash \bigcup_{i=1}^{N} \mathcal{D}(a_i,r_i), \ \text{and} \ \mathbb{P}^1_{\text{Berk}} \backslash \bigcup_{i=1}^{N} \mathcal{D}(a_i,r_i),$$

where $a, a_i \in K$ and $r, r_i > 0$. If desired, one can require that the r, r_i belong to $|K^\times|$.

Proof. See [BR, Proposition 2.7]. □

The constructed topological space $\mathbb{P}^1_{\text{Berk}}$ satisfies the following properties:

Proposition 2.3.3. *i) $\mathbb{P}^1_{\text{Berk}}$ is a compact Hausdorff topological space.*

ii) Both $\mathbb{P}^1(K)$ and $\mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}^1(K)$ are dense in $\mathbb{P}^1_{\text{Berk}}$.

iii) $\mathbb{P}^1_{\text{Berk}}$ is uniquely path-connected.

Proof. See [BR, Proposition 2.6], [BR, Lemma 2.9] and [BR, Lemma 2.10]. □

Furthermore, we will see that $\mathbb{P}^1_{\text{Berk}}$ and $\mathbb{H}_{\text{Berk}} := \mathbb{P}^1_{\text{Berk}} \backslash \mathbb{P}(K)$ have a tree structure as well.

Definition 2.3.4. i) $\Gamma_S \subset \mathbb{P}^1_{\text{Berk}}$ is called a *finite subgraph*, if there is a finite set $S \subset \mathbb{H}_{\text{Berk}}$ such that

$$\Gamma_S = \bigcup_{x,y \in S} [x,y],$$

where $[x,y]$ denotes the unique path between x and y.

ii) By a *vertex set* for Γ_S, we mean a finite set of points S such that $\Gamma_S \backslash S$ is a union of open intervals where each of them has two distinct endpoints in Γ_S.

iii) Let Γ', Γ be two finite subgraphs of $\mathbb{P}^1_{\text{Berk}}$ such that $\Gamma' \subset \Gamma$. Then we have a *retraction map* $r_{\Gamma,\Gamma'} \colon \Gamma \to \Gamma'$, where $r(x)$ is given by the first point of the path $[x,p]$ in Γ' for a fixed point $p \in \Gamma'$.

iv) We define the *path distance metric* ρ on $\mathbb{H}_{\text{Berk}}$ by $\rho(x,y) := \rho(x,y)$ if $x, y \in E$ or $x, y \in E'$, and $\rho(x,y) := \rho(x, \zeta_{\text{Gauss}}) + \rho(y, \zeta_{\text{Gauss}})$ if not.

Remark. Every finite subgraph endowed with the induced path distance metric ρ is a finite $\mathbb{R}$-tree. Moreover, [BR, Proposition 2.29] states that $\mathbb{H}_{\text{Berk}}$ is a complete metric space under ρ. In the next section we will define the retraction map in a more general way and we will see that the retraction map is well-defined, i.e. it is independent of the fixed point $p \in \Gamma'$. Further,

$$r_{\Gamma,\Gamma''} = r_{\Gamma',\Gamma''} \circ r_{\Gamma,\Gamma'}$$

for finite subgraphs $\Gamma'' \subset \Gamma' \subset \Gamma$.

Proposition 2.3.5. *There is a canonical homeomorphism*

$$\mathbb{P}^1_{\mathrm{Berk}} \simeq \varprojlim_{\Gamma \in \mathcal{F}} \Gamma,$$

where $\mathcal{F}$ is the set of all finite subgraphs in $\mathbb{P}^1_{\mathrm{Berk}}$ and $\mathbb{P}^1_{\mathrm{Berk}}$ is equipped with the Berkovich topology.

Proof. See [BR, Theorem 2.21]. □

3 The Laplacian on the Berkovich projective line

The goal of this chapter is to define a measure-valued Laplacian operator on a class of functions $f\colon U \to \mathbb{R} \cup \{\pm\infty\}$ for a *domain* (i.e. open and connected) $U \subset \mathbb{P}^1_{\text{Berk}}$. We will see that the profinite $\mathbb{R}$-tree structure on $\mathbb{P}^1_{\text{Berk}}$ (cf. Proposition 2.3.5) leads directly to the construction of such a Laplacian operator. More precisely, we give in Section 3.1 an extension Δ_Γ of the Laplacian introduced by [Zh] on the class of continuous, piecewise $\mathcal{C}^2$ functions to a larger class of continuous functions called $\text{BDV}(\Gamma)$ for a finite subgraph $\Gamma \subset \mathbb{P}^1_{\text{Berk}}$ (after [BR]). This class is characterized by the fact that for every Borel measure μ of total mass zero on Γ there is a function $f \in \text{BDV}(\Gamma)$ such that $\mu = \Delta_\Gamma(f)$. We define a class of continuous functions $f\colon \mathbb{P}^1_{\text{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$, denoted by $\text{BDV}(\mathbb{P}^1_{\text{Berk}})$, such that the collection of measures $\{\Delta_\Gamma(f)\}$ is a coherent system for every $f \in \text{BDV}(\mathbb{P}^1_{\text{Berk}})$. This leads to a unique Borel measure $\Delta(f)$ of total mass zero on the inverse limit space $\mathbb{P}^1_{\text{Berk}}$. In Section 3.2, we give explicit examples of functions contained in $\text{BDV}(\mathbb{P}^1_{\text{Berk}})$ and determine their Laplacians. Next to some natural examples, we will define the *Hsia kernel* which leads to further examples of functions in $\text{BDV}(\mathbb{P}^1_{\text{Berk}})$. In particular, the function $f(x) := -\log_v([g]_x)$ for $g \in K(T)^\times$ can be verified to be contained in $\text{BDV}(\mathbb{P}^1_{\text{Berk}})$. We will calculate that $\Delta(f) = \sum_{i=1}^m n_i \delta_{a_i}$ if $\text{div}(g) = \sum_{i=1}^m n_i (a_i)$, which is known as the *Poincaré-Lelong formula*.

3.1 Construction and properties of the Laplacian on a subdomain of $\mathbb{P}^1_{\text{Berk}}$

In this section, we first give a definition of the Laplacian operator on the mentioned classes of continuous functions on finite subgraphs of $\mathbb{P}^1_{\text{Berk}}$, which were defined in the previous chapter. For the construction we follow [BR]. Different to [BR] we just define the Laplacian operator for finite subgraphs of $\mathbb{P}^1_{\text{Berk}}$ instead for general metrized graphs (cf. [BR, Chapter 3]). However, the definitions and the constructions are totally the same and also hold for metrized graphs.

Definition 3.1.1. i) An injective length-preserving continuous map $\gamma\colon [0, L] \to \Gamma$ is called an *isometric path*, and we say that γ *emanates from* p and *terminates at* q, if $\gamma(0) = p$ and $\gamma(L) = q$.

ii) Two isometric paths emanating from p are said to be *equivalent* if they share a common initial segment.

iii) For each $p \in \Gamma$ we define the *projectivized tangent space* at p as the set of equivalence classes of isometric paths in Γ emanating from p, and we write $T_p(\Gamma)$.

Remark. There is a bijection between $T_p(\Gamma)$ and the 'edges' of Γ emanating from p. We will associate to each element of $T_p(\Gamma)$ a formal unit tangent vector $\vec{v}$, and write $p + t\vec{v}$ instead of $\gamma(t)$ for a representative path γ.

Definition 3.1.2. If $f : \Gamma \to \mathbb{R}$ is a function, and $\vec{v}$ is a unit tangent vector at p, then we define the *(one-sided) derivative of f in the direction $\vec{v}$* to be

$$d_{\vec{v}}f(p) = \lim_{t\to 0^+} \frac{f(p + t\vec{v}) - f(p)}{t} = \lim_{t\to 0^+} \frac{f(\gamma(t)) - f(p)}{t}$$

provided the limit exists as a finite number.

Definition 3.1.3. i) A function $f : \Gamma \to \mathbb{R}$ is called *piecewise affine*, if there is a vertex set X_f for Γ such that f is affine on each edge in $\Gamma\backslash X_f$ with respect to an arclength parametrization of that edge. Let $\mathrm{CPA}(\Gamma)$ be the space of continuous, piecewise affine real-valued functions on Γ.

ii) Since $d_{\vec{v}}f(p)$ is defined for every $f \in \mathrm{CPA}(\Gamma)$ for all $p \in \Gamma$ and $\vec{v} \in T_p(\Gamma)$ we can introduce a Laplacian operator on $\mathrm{CPA}(\Gamma)$ like Chinburg and Rumely did in [CR]:

$$\Delta(f) := \sum_{p\in\Gamma} \left(- \sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}}f(p) \right) \delta_p,$$

where δ_p is the Dirac unit measure at p. This Laplacian is a map from $\mathrm{CPA}(\Gamma)$ to the space of discrete signed measures on Γ.

This Laplacian can be extended on larger classes of functions:

Definition 3.1.4. We call a function $f : \Gamma \to \mathbb{R}$ *piecewise $\mathcal{C}^2$*, if there is a vertex set X_f such that $f'' \in \mathcal{C}^2(\Gamma\backslash X_f)$. Let $\mathrm{Zh}(\Gamma)$ be the space of continuous, piecewise $\mathcal{C}^2$ functions whose one-sided directional derivatives $d_{\vec{v}}f(p)$ exists for all $p \in \Gamma$ and $\vec{v} \in T_p(\Gamma)$.

Zhang has defined in [Zh] the following Laplacian on $\mathrm{Zh}(\Gamma)$

$$\Delta_{\mathrm{Zh}}(f) := -f''(x)dx + \sum_{p\in\Gamma}(- \sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}}f(p))\delta_p(x),$$

where $f''(x)$ is taken relative to the arclength parametrization on each segment in the complement of an appropriate vertex set X_f for Γ, i.e. $f''(x) = \frac{d^2}{dt^2} f(p + t\vec{v})$ for $x = p + t\vec{v} \in \Gamma\backslash X_f$.

Let $\mathcal{A} := \mathcal{A}(\Gamma)$ be the Boolean algebra of subsets of Γ generated by the connected open sets, i.e. each subset $S \subset \Gamma$ is in $\mathcal{A}$ iff S is a finite disjoint union of sets isometric to open, half-open or closed intervals, where isolated points are considered as degenerate closed intervals. For this Boolean algebra we have the following Mass Formula:

Let $\mathrm{In}(p,S) := \{\vec{v} \in T_p(\Gamma) | p+t\vec{v} \in S$ for all sufficiently small $t > 0\}$ the *inward-directed unit vectors at* p and $\mathrm{Out}(p,S) := T_p(\Gamma) \backslash \mathrm{In}(p,S)$ the *outward-directed unit vectors at* p.

Lemma 3.1.5 (Mass Formula). *Let S be a set in the Boolean Algebra $\mathcal{A}$. Then for each $f \in \mathrm{Zh}(\Gamma)$*

$$\Delta_{\mathrm{Zh}}(f)(S) = \sum_{p \in \partial S, p \notin S} \sum_{\vec{v} \in \mathrm{In}(p,S)} d_{\vec{v}} f(p) - \sum_{p \in \partial S, p \in S} \sum_{\vec{v} \in \mathrm{Out}(p,S)} d_{\vec{v}} f(p).$$

Proof. In [BR, Lemma 3.4] they give a Mass Formula for sets in $\mathcal{A}(\Gamma)$ which are a finite union of closed intervals. It is easy to see that this Mass Formula can be extended on $\mathcal{A}$ in the stated way. Let $S \in \mathcal{A}(\Gamma)$, i.e. S can be written as a finite disjoint union of points, open, closed and half-open intervals. Moreover, we can write $S = S \backslash E \cup E$, where E is a finite disjoint union of points and closed intervals such that $S \backslash E$ is a finite disjoint union of open intervals. Applying [BR, Lemma 3.4] to E, we get

$$\Delta_{\mathrm{Zh}}(f)(E) = - \sum_{p \in \partial E} \sum_{\vec{v} \in \mathrm{Out}(p,E)} d_{\vec{v}} f(p).$$

As Γ is also a finite union of closed intervals, the lemma states that

$$\Delta_{\mathrm{Zh}}(f)(\Gamma) = - \sum_{p \in \partial \Gamma} \sum_{\vec{v} \in \mathrm{Out}(p,\Gamma)} d_{\vec{v}} f(p) = 0,$$

because $\partial \Gamma = \emptyset$. Setting $U := S \backslash E$, then $\Gamma \backslash U$ is a finite disjoint union of points and closed intervals, so we also can apply [BR, Lemma 3.4] to $\Gamma \backslash U$:

$$\Delta_{\mathrm{Zh}}(f)(\Gamma \backslash U) = - \sum_{p \in \partial U} \sum_{\vec{v} \in \mathrm{In}(p,U)} d_{\vec{v}} f(p),$$

where we have additionally used that $\mathrm{Out}(p, \Gamma \backslash U) = \mathrm{In}(p,U)$. Since $\partial E = \partial S \cap S$ and $\partial U = \partial S \cap \Gamma \backslash S$, the three equations above imply

$$\begin{aligned} \Delta_{\mathrm{Zh}}(f)(S) &= \Delta_{\mathrm{Zh}}(f)(E) - \Delta_{\mathrm{Zh}}(f)(\Gamma \backslash U) \\ &= \sum_{p \in \partial S, p \notin S} \sum_{\vec{v} \in \mathrm{In}(p,S)} d_{\vec{v}} f(p) - \sum_{p \in \partial S, p \in S} \sum_{\vec{v} \in \mathrm{Out}(p,S)} d_{\vec{v}} f(p). \end{aligned}$$

□

The Mass Formula is the start point to extend the Laplacian on $\mathrm{Zh}(\Gamma)$ to a even larger class of functions which is called $\mathrm{BDV}(\Gamma)$.

Definition 3.1.6. i) Let $\mathcal{D}(\Gamma)$ be the class of all functions on the finite subgraph Γ whose one-sided derivatives exist everywhere, i.e.

$$\mathcal{D}(\Gamma) := \{f\colon \Gamma \to \mathbb{R} |\ d_{\vec{v}} f(p) \text{ exists for each } p \in \Gamma \text{ and } \vec{v} \in T_p(\Gamma)\}.$$

ii) For $f \in \mathcal{D}(\Gamma)$ we define a finitely additive set function m_f on $\mathcal{A}$

$$m_f(S) := \sum_{p\in\partial S, p\notin S} \sum_{\vec{v}\in \mathrm{In}(p,S)} d_{\vec{v}} f(p) - \sum_{p\in\partial S, p\in S} \sum_{\vec{v}\in \mathrm{Out}(p,S)} d_{\vec{v}} f(p).$$

iii) We will say that $f \in \mathcal{D}(\Gamma)$ is of *bounded differential variation,* and write $f \in \mathrm{BDV}(\Gamma)$, if there is a constant $B > 0$ such that for any countable collection $\mathcal{F}$ of pairwise disjoint sets in $\mathcal{A}$,

$$\sum_{S_i \in \mathcal{F}} |m_f(S_i)| \le B.$$

Remark 3.1.7. i) Since $\partial\emptyset = \partial\Gamma = \emptyset$,

$$m_f(\emptyset) = m_f(\Gamma) = 0$$

for all $f \in \mathcal{D}(\Gamma)$. Consequently, we have

$$m_f(\Gamma\backslash S) = -m_f(S).$$

ii) Consider a set S in $\mathcal{A}$. If S is open,

$$m_f(S) = \sum_{p\in\partial S, p\notin S} \sum_{\vec{v}\in \mathrm{In}(p,S)} d_{\vec{v}} f(p).$$

If S is closed,

$$m_f(S) = -\sum_{p\in\partial S, p\in S} \sum_{\vec{v}\in \mathrm{Out}(p,S)} d_{\vec{v}} f(p).$$

And in the case that $S = \{p\}$,

$$m_f(\{p\}) = -\sum_{\vec{v}\in T(\Gamma)} d_{\vec{v}} f(p).$$

iii) If $f_1, f_2 \in \mathcal{D}(\Gamma)$ and $c_1, c_2 \in \mathbb{R}$, then

$$m_{c_1 f_1 + c_2 f_2} = c_1 m_{f_1} + c_2 m_{f_2}.$$

iv) $\mathrm{BDV}(\Gamma)$ is a linear subspace of $\mathcal{D}(\Gamma)$.

Theorem 3.1.8. *If $f \in \mathrm{BDV}(\Gamma)$, then the finitely additive set function m_f extends uniquely to a finite signed Borel measure m_f^* of total mass zero on Γ.*

Proof. The existence of the extension is proved in [BR, Theorem 3.6]. For the uniqueness, we assume that μ is another finite signed Borel measure extending m_f^* of total mass zero on Γ. Since μ and $\Delta(f)$ are extending the set function m_f, the measures coincide on the Boolean algebra $\mathcal{A}(\Gamma)$ which is generated by the connected open subsets of Γ. In particular, we have the identity $\mu(\Gamma) = 0 = \Delta(f)(\Gamma)$. Without loss of generality, we can assume that $\Gamma = [a, b]$ is a closed interval. Consider an open subset T of Γ, then T is the union of at most countably many open intervals in $[a, b]$, i.e. $\mathcal{A}(\Gamma)$ generates the Borel σ-algebra on Γ. Thus, the Theorem of Uniqueness of Measures (cf. [El, Eindeutigkeitssatz 5.6]) states that μ and $\Delta(f)$ have to coincide. □

Definition 3.1.9. For every $f \in \mathrm{BDV}(\Gamma)$, we define the Laplacian $\Delta(f)$ to be the measure from Theorem 3.1.8:

$$\Delta(f) := m_f^*.$$

Remark 3.1.10. Conversely, if ν is a finite signed Borel measure of total mass zero on Γ, then there exists a function $h \in \mathrm{BDV}(\Gamma)$ such that $\Delta(h) = \nu$. This function h is unique up to addition of a real constant.

Proof. See [BR, Corollary 3.12] and [BR, Proposition 3.14 (B)]. □

Lemma 3.1.11. *Let $f, g \in \mathrm{BDV}(\Gamma)$ and $\alpha, \beta \in \mathbb{R}$, then*

$$\Delta(\alpha f + \beta g) = \alpha\Delta(f) + \beta\Delta(g).$$

Proof. By construction, $\Delta(\alpha f+\beta g)$ extends the set function $m_{\alpha f+\beta g}$ and $\alpha\Delta(f)+\beta\Delta(g)$ extends $\alpha m_f + \beta m_g$. Due to $m_{\alpha f+\beta g} = \alpha m_f + \beta m_g$, the uniqueness of the extension in Theorem 3.1.8 implies the equality. □

Lemma 3.1.12. $\mathrm{Zh}(\Gamma)$ *is a subset of* $\mathrm{BDV}(\Gamma)$*, and for each* $f \in \mathrm{Zh}(\Gamma) \subset \mathrm{BDV}(\Gamma)$

$$\Delta(f) = \Delta_{\mathrm{Zh}}(f).$$

Proof. Let $f \in \mathrm{Zh}(\Gamma)$ and X_f be a vertex set such that $f \in \mathcal{C}^2(\Gamma\backslash X_f)$. By the definition of $\mathrm{Zh}(\Gamma)$, every function in $\mathrm{Zh}(\Gamma)$ belongs to $\mathcal{D}(\Gamma)$, so the directional derivative exist as real numbers in every point $p \in X_f$. One can show that this implies $f'' \in \mathrm{L}^1(\Gamma\backslash X_f, dx)$. Consider a countable family $\mathcal{F}$ of pairwise disjoint sets $S_i \in \mathcal{A}(\Gamma)$. By the Mass Formula $m_f(S_i) = \Delta_{\mathrm{Zh}}(f)(S_i)$ and $S_i\backslash X_f$ is a again a union of $n_i \in \mathbb{N}$ disjoint open, closed and half-open intervals. Let a_{ij}, b_{ij} for $j = 1, \ldots, n_i$ be the endpoints of these segments. Thus,

$$m_f(S_i) = -\sum_{j=1}^{n_i} \int_{a_{ij}}^{b_{ij}} f''(x)\ dx + \sum_{p\in S_i\cap X_f} \sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}} f(p).$$

In sum, we have

$$\begin{aligned}\sum_{S_i\in\mathcal{F}}|m_f(S_i)| &= \sum_{S_i\in\mathcal{F}}|-\sum_{j=1}^{n_i}\int_{a_{ij}}^{b_{ij}} f''(x)\ dx + \sum_{p\in S_i\cap X_f}\ \sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}}f(p)| \\ &\le \sum_{S_i\in\mathcal{F}}\sum_{j=1}^{n_i}\int_{a_{ij}}^{b_{ij}} |f''(x)|\ dx + \sum_{p\in S_i\cap X_f}\ \sum_{\vec{v}\in T_p(\Gamma)} |d_{\vec{v}}f(p)| \\ &\le \int\limits_{\Gamma\backslash X_f} |f''(x)|dx + \sum_{p\in X_f}|m_f(\{p\})| < \infty.\end{aligned}$$

Therefore, $f \in \mathrm{BDV}(\Gamma)$ with $B(f) := \int_{\Gamma\backslash X_f}|f''(x)|dx + \sum\limits_{p\in X_f}|m_f(\{p\})|$, and consequently the finite signed Borel measure $\Delta(f)$ on Γ exists. Further, we show that $\Delta_{\mathrm{Zh}}(f)$ extends m_f on $\mathcal{A}(\Gamma)$. It suffices to consider an isolated point $p \in \Gamma$ and an open interval (c, d) contained in an edge of $\Gamma\backslash X_f$. Clearly, $\Delta_{\mathrm{Zh}}(f)(\{p\}) = m_f(\{p\})$ is true for all $p \in \Gamma$. And,

$$\begin{aligned}\Delta_{\mathrm{Zh}}(f)((c,d)) &= -\int_c^d f''(x)\ dx \\ &= f'(c) - f'(d) \\ &= m_f((c,d)).\end{aligned}$$

Since the extensio of m_f to a finite signed Borel measure on Γ of total mass zero is unique by Theorem 3.1.8, we have the identity $\Delta(f) = \Delta_{\mathrm{Zh}}(f)$. □

This definition makes it possible to define a Laplacian operator on the Berkovich projective line in the following way.

Definition 3.1.13. A *domain* U in $\mathbb{P}^1_{\mathrm{Berk}}$ is a non-empty connected open subset of $\mathbb{P}^1_{\mathrm{Berk}}$. We call a domain V *simple* if ∂V is a non-empty finite set $\{x_1, \ldots, x_m\} \subset \mathbb{H}_{\mathrm{Berk}}$, where each x_i is of type II or III. A *strict simple domain* is a simple domain whose boundary points are all of type II.

Remark 3.1.14.
i) Different to [BR], we will say that U_0 is a *subdomain* of an open set U if U_0 is a domain and $U_0 \subset U$. If we require $\overline{U_0} \subset U$, this is stated additionally.

ii) The strict simple domains (resp. simple domains) form a basis for the Berkovich topology on $\mathbb{P}^1_{\mathrm{Berk}}$ (cf. [BR, §2.6 p.42]). Since $\mathbb{P}^1_{\mathrm{Berk}}$ is a compact Hausdorff space, the compact subsets are just the closed subsets. The closures of the simple domains form a fundamental system of the closed, and so the compact neighborhoods for the Berkovich topology on $\mathbb{P}^1_{\mathrm{Berk}}$.

Definition 3.1.15. Let $\mathcal{C}(\overline{U})$ be the space of continuous functions $f\colon \overline{U} \to \mathbb{R}$, where $\overline{U}$ is the closure of a domain $U \subset \mathbb{P}^1_{\mathrm{Berk}}$.

Recall from §2.3 that a finite subgraph of $\mathbb{P}^1_{\mathrm{Berk}}$ is the union of the unique paths between a finite set of points in $\mathbb{H}_{\mathrm{Berk}}$.

Lemma 3.1.16. *Let* Γ *be a finite subgraph of* $\mathbb{P}^1_{\mathrm{Berk}}$. *Then*

i) Γ *is a closed subset of* $\mathbb{P}^1_{\mathrm{Berk}}$.

ii) The metric topology on Γ *coincides with the relative (i.e. subspace) topology induced from* $\mathbb{P}^1_{\mathrm{Berk}}$.

Proof. See [BR, Lemma 5.2]. □

Now we generalize the retraction map, which was only defined for finite subgraphs in §2.3, to every non-empty connected closed subset $E \subset \mathbb{P}^1_{\mathrm{Berk}}$ equipped with the relative topology. Recall that a set is connected under the topology of $\mathbb{P}^1_{\mathrm{Berk}}$ if and only if it is uniquely path-connected.

Definition 3.1.17. We define the *retraction map* $r_E\colon \mathbb{P}^1_{\mathrm{Berk}} \to E$ by setting $r_E(x)$ as the first point p in E on the path from x to a point $p_0 \in E$.

Remark. By construction, we have $r_E(x) = x$ for all $x \in E$.

Lemma 3.1.18. *The map* $r_E\colon \mathbb{P}^1_{\mathrm{Berk}} \to E$ *is well-defined.*

Proof. Since $\mathbb{P}^1_{\mathrm{Berk}}$ is path-connected, $r_E(x)$ exists for each $x \in \mathbb{P}^1_{\mathrm{Berk}}$. Furthermore, we have to show that the definition is independent of p_0. Let $x \notin E$, p_0' another fixed point in E and p' the first point in E on the path from x to p_0'. In the case of $p \neq p'$, there is a path from $p\prime$ to p in E, since E is path-connected. But we also get a path from p' to p which is not contained in E, by going from p' to x and from x to p. This contradicts the fact that $\mathbb{P}^1_{\mathrm{Berk}}$ is uniquely path-connected. □

Lemma 3.1.19. *For each non-empty closed connected subset* $E \subset \mathbb{P}^1_{\mathrm{Berk}}$*, the retraction map* $r_E\colon \mathbb{P}^1_{\mathrm{Berk}} \to E$ *is continuous.*

Proof. See [BR, Lemma 5.3]. □

3.1.20. If $E_1 \subset E_2 \subset \mathbb{P}^1_{\mathrm{Berk}}$ are two non-empty connected closed subsets, the retraction map r_{E_1} induces a retraction map $r_{E_2,E_1}\colon E_2 \to E_1$ such that

$$r_{E_1}(x) = r_{E_2,E_1}(r_{E_2}(x))$$

for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$. If E_1 and E_2 have the relative topology, r_{E_2,E_1} is continuous as well.

Let $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a domain, and for each finite subgraph $\Gamma \subset U$ we consider a finite signed Borel measure μ_Γ on Γ.

Definition 3.1.21. A system of measures $\{\mu_\Gamma\}$ on the finite subgraphs of U is called *coherent* if

i) For each pair of finite subgraphs Γ_1, Γ_2 of U with $\Gamma_1 \subset \Gamma_2$ we have

$$(r_{\Gamma_2,\Gamma_1})_*(\mu_{\Gamma_2}) = \mu_{\Gamma_1}.$$

ii) There is a constant B such that for each finite subgraph $\Gamma \subset U$

$$|\mu_\Gamma|(\Gamma) \leq B.$$

3.1.22. For any two graphs Γ_1, Γ_2 there is a unique minimal finite subgraph Γ_3 containing Γ_1 and Γ_2. Hence the collection of finite subgraps $\Gamma \subset U$ forms a directed set under containment. For every finite signed Borel measure μ on $\overline{U}$, the system of measures $\{\mu_\Gamma\}$ on the finite subgraphs of U given by

$$\mu_\Gamma := (r_{\overline{U},\Gamma})_*(\mu)$$

for each $\Gamma \subset U$ is coherent.

There is a 1-1 correspondence between finite signed Borel measures μ on $\overline{U}$, and coherent systems of finite signed Borel measures on finite subgraphs of U:

Proposition 3.1.23. *If $\{\mu_\Gamma\}$ is a coherent system of measures in U, the map*

$$\Lambda(F) = \varinjlim_{\Gamma} \int_\Gamma F(x) d\mu_\Gamma(x)$$

defines a bounded linear functional on $\mathcal{C}(\overline{U})$, and there is a unique Borel measure μ on $\overline{U}$ such that

$$\Lambda(F) = \int_{\overline{U}} F(x) d\mu(x)$$

for each $F \in \mathcal{C}(\overline{U})$. This measure is characterized by the fact that

$$(r_{\overline{U},\Gamma})_*(\mu) = \mu_\Gamma$$

for each finite subgraph $\Gamma \subset U$.

In particular, if μ_0 is a finite signed Borel measure on $\overline{U}$, and we put $\mu_\Gamma = (r_{\overline{U},\Gamma})_*(\mu_0)$ for each finite subgraph $\Gamma \subset U$, then μ_0 is the unique measure associated to the coherent system $\{\mu_\Gamma\}$ by the construction above.

Proof. See [BR, Proposition 5.10]. □

Using coherent systems of measures and the Laplacian on metrized graphs, we are able to construct a measure-valued Laplacian operator on a suitable class of functions $f\colon U \to \mathbb{R}\cup\{\pm\infty\}$.

Definition 3.1.24. Let $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a domain. We will say that a function $f\colon U \to \mathbb{R}\cup\{\pm\infty\}$ is of *bounded differential variation* on U, and write $f \in \mathrm{BDV}(U)$, if

i) $f|_\Gamma \in \mathrm{BDV}(\Gamma)$ for each finite subgraph $\Gamma \subset U$, and

ii) there is a constant $B(f)$ such that for each finite subgraph $\Gamma \subset U$,

$$|\Delta_\Gamma(f)|(\Gamma) \leq B(f).$$

Remark. Due to $\Gamma \subset U \cap \mathbb{H}_{\mathrm{Berk}}$ for every finite subgraph $\Gamma \subset U$, there is nothing required on the behavior of f on $U \cap \mathbb{P}^1(K)$ by this definition, and so f may be undefined at some points of $U \cap \mathbb{P}^1(K)$. We will use the notation $\mathcal{C}(\overline{U}) \cap \mathrm{BDV}(U)$ for the space of functions $f \in \mathcal{C}(\overline{U})$ whose restrictions to U belong to $\mathrm{BDV}(U)$.

Proposition 3.1.25. *If $f \in \mathrm{BDV}(U)$, the system of measures $\{\Delta_\Gamma(f)\}_{\Gamma\subset U}$ is coherent.*

Proof. Let Γ_1, Γ_2 be a pair of finite subgraphs of U with $\Gamma_1 \subset \Gamma_2$. Since Γ_2 can be obtained by sequentially attaching a finite number of edges to Γ_1, it suffices to consider the case where $\Gamma_2 = \Gamma_1 \cup T$ for an attached segment T at a point $p \in \Gamma_1$. As a segment, T is a finite subgraph as well. We have to show that for every Borel subset $e \subset \Gamma_1$

$$\Delta_{\Gamma_2}(f)(r^{-1}_{\Gamma_2,\Gamma_1}(e)) = \Delta_{\Gamma_1}(f)(e).$$

At first, we consider the case that $e \subset \Gamma_1\backslash\{p\}$. Due to $r_{\Gamma_2,\Gamma_1}(q) = p \notin e$ for all $q \in \Gamma_2\backslash\Gamma_1$, we have $r^{-1}_{\Gamma_2,\Gamma_1}(e) \subset \Gamma_1$. Since r_{Γ_2,Γ_1} is the identity on Γ_1, $r^{-1}_{\Gamma_2,\Gamma_1}(e) = e$. Thus,

$$\Delta_{\Gamma_2}(f)(r^{-1}_{\Gamma_2,\Gamma_1}(e)) = \Delta_{\Gamma_2}(f)(e) = \Delta_{\Gamma_1}(f)(e),$$

where the last identity follows by the definition of the Laplacian on finite subgraphs. Due to the additivity of the Laplacian, it remains to consider the Borel set $\{p\} \subset \Gamma_1$. We know that $r_{\Gamma_2,\Gamma_1}(q) = p$ for all $q \in T$ and $r_{\Gamma_2,\Gamma_1}|_{\Gamma_1} = \mathrm{id}_{\Gamma_1}$. Due to $\Gamma_2 = \Gamma_1 \cup T$,

$$r^{-1}_{\Gamma_2,\Gamma_1}(\{p\}) = T.$$

Since T is just a closed interval, we have seen in Remark 3.1.7 that

$$\Delta_{\Gamma_2}(f)(T) = m_f(T) = -\sum_{\vec{v}\in\mathrm{Out}(p,T)} d_{\vec{v}}(f)(p),$$

where m_f is the set function on the Boolean algebra $\mathcal{A}(\Gamma_2)$. By $\mathrm{Out}(p,T) = T_p(\Gamma_1)$ and the two foregoing equations,

$$\Delta_{\Gamma_2}(f)(r^{-1}_{\Gamma_2,\Gamma_1}(\{p\})) = -\sum_{\vec{v}\in T_p(\Gamma_1)} d_{\vec{v}}(f)(p) = \Delta_{\Gamma_1}(f)(p).$$

By the definition of the space $\mathrm{BDV}(U)$, there is a constant B such that $|\Delta_\Gamma(f)|(\Gamma) \leq B$ for all finite subgraphs $\Gamma \subset U$, and hence $\{\Delta_\Gamma(f)\}_{\Gamma\subset U}$ is coherent. □

Definition 3.1.26. Let $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a domain and $f \in \mathrm{BDV}(U)$.

i) We define the *complete Laplacian*

$$\Delta_{\overline{U}}(f)$$

as the unique finite signed Borel measure on $\overline{U}$ associated to the coherent system $\{\Delta_\Gamma(f)\}_{\Gamma\subset U}$ from Proposition 3.1.23, characterized by the property that

$$(r_{\overline{U},\Gamma})_*(\Delta_{\overline{U}}(f)) = \Delta_\Gamma(f)$$

for each finite subgraph $\Gamma \subset U$.

ii) We call the restriction of $\Delta_{\overline{U}}(f)$ to U the *Laplacian*

$$\Delta_U(f) := \Delta_{\overline{U}(f)}|_U.$$

iii) The *Boundary Derivative*

$$\Delta_{\partial U}(f) := \Delta_{\overline{U}}(f)|_{\partial U}$$

is the restriction of $\Delta_{\overline{U}}(f)$ to ∂U.

Remark 3.1.27. i) $\Delta_U(f)$ is the Borel meaure on $\overline{U}$ with

$$\Delta_U(f)(S) = \Delta_{\overline{U}}(f)(S \cap U)$$

for each Borel set $S \subset \overline{U}$.

ii) By construction,

$$\Delta_{\overline{U}}(f) = \Delta_U(f) + \Delta_{\partial U}(f).$$

iii) When $U = \mathbb{P}^1_{\mathrm{Berk}}$, we have

$$\Delta_{\overline{U}}(f) = \Delta_U(f),$$

and we will write $\Delta(f)$ for $\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f)$.

Before we will see some examples, we give some important properties of the (complete) Laplacian:

Lemma 3.1.28. *Let $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ be domain, $f, g \in \mathrm{BDV}(U)$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$\Delta_{\overline{U}}(\alpha f + \beta g) = \alpha\Delta_{\overline{U}}(f) + \beta\Delta_{\overline{U}}(g).$$

Proof. By the definition of the complete Laplacian,

$$\Delta_\Gamma(f) = (r_{\overline{U},\Gamma})_*(\Delta_{\overline{U}}(f))$$

and

$$\Delta_\Gamma(g) = (r_{\overline{U},\Gamma})_*(\Delta_{\overline{U}}(g))$$

are satisfied for every finite subgraph $\Gamma \subset U$. Since the Laplacian Δ_Γ is linear by Lemma 3.1.11, it follows that

$$\begin{aligned}\Delta_\Gamma(\alpha f + \beta g) &= \alpha \Delta_\Gamma(f) + \beta \Delta_\Gamma(g)\\ &= \alpha(r_{\overline{U},\Gamma})_*(\Delta_{\overline{U}}(f)) + \beta(r_{\overline{U},\Gamma})_*(\Delta_{\overline{U}}(g))\\ &= (r_{\overline{U},\Gamma})_*(\alpha \cdot \Delta_{\overline{U}}(f) + \beta \cdot \Delta_{\overline{U}}(g))\end{aligned}$$

for every finite subgraph $\Gamma \subset U$. Due to the uniqueness in Proposition 3.1.23, the Laplacians have to coincide. □

Proposition 3.1.29. *Suppose* $U_1 \subset U_2 \subset \mathbb{P}^1_{\mathrm{Berk}}$ *are domains, and* $f \in \mathrm{BDV}(U_2)$*. Then* $f|_{U_1} \in \mathrm{BDV}(U_1)$ *and*

$$\Delta_{U_1}(f) = \Delta_{U_2}(f)|_{U_1}, \quad \Delta_{\overline{U_1}}(f) = (r_{\overline{U_2},\overline{U_1}})_*(\Delta_{\overline{U_2}}(f)).$$

Proof. See [BR, Proposition 5.26]. □

Proposition 3.1.30. *Let* $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ *be a domain, and let* $V_1, \dots, V_r \subset U$ *be subdomains such that* $U = \bigcup_{i=1}^r V_i$*. Then for any function* f*, we have* $f \in \mathrm{BDV}(U)$ *iff* $f|_{V_i} \in \mathrm{BDV}(V_i)$ *for all* $i = 1, \dots, r$*. Moreover, in the latter case, for each* $i = 1, \dots, r$

$$\Delta_{V_i}(f) = \Delta_U(f)|_{V_i}, \quad \Delta_{\overline{V_i}}(f) = (r_{\overline{U},\overline{V_i}})_*(\Delta_{\overline{U}}(f)).$$

Proof. See [BR, Proposition 5.27]. □

3.2 Examples and the Hsia kernel

In this section, we give explicit examples of functions having bounded differential variation and calculate their Laplacians. Next to the obvious one of a constant function we consider the composition $f \circ r_\Gamma$ for a finite subgraph Γ and a function $f_0 \in \mathrm{BDV}(\Gamma)$. For further examples we will define the *Hsia kernel*. The Hsia kernel $\delta(x, y)_\infty$ for $x, y \in \mathbb{A}^1_{\mathrm{Berk}}$ extends the usual distance $|x - y|$ on K and the function $-\log_v(\delta(x, y)_\infty)$ is a generalization of the usual potential theory kernel $-\log_v(|x - y|)$ on K. We will also give a definition for an analogous kernel $\delta(x, y)_\zeta$ for an arbitrary $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$ which is called the *generalized Hsia kernel*. The functions of the form $f(x) = -\log_v(\delta(x, y)_\zeta)$

for fixed $y, \zeta \in \mathbb{P}^1_{\mathrm{Berk}}$ belong to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and they make it possible to verify a version of the *Poincaré-Lelong formula* at the end of this chapter. Moreover, these functions are used for the analogue *Poisson formula* in Chapter 4.

Example 3.2.1. If $f(x) = C$ on $\mathbb{P}^1_{\mathrm{Berk}}$ for a constant $C \in \mathbb{R}$, then $f \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and

$$\Delta(f) = 0.$$

Proof. Clearly, $f \in \mathrm{Zh}(\Gamma) \subset \mathrm{BDV}(\Gamma)$ and

$$\Delta(f) = \Delta_{\mathrm{Zh}}(f) = -f''(x)dx + \sum_{p\in\Gamma}(-\sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}}f(p))\delta_p(x) = 0$$

for each finite subgraph $\Gamma \subset \mathbb{P}^1_{\mathrm{Berk}}$. Due to the uniqueness in Proposition 3.1.23, we have $\Delta(f) = 0$. □

Example 3.2.2. If $f = f_0 \circ r_{\Gamma_0}$ for a finite subgraph Γ_0 of $\mathbb{P}^1_{\mathrm{Berk}}$ and $f_0 \in \mathrm{BDV}(\Gamma_0)$, then $f \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and

$$\Delta(f) = \Delta_{\Gamma_0}(f_0).$$

Proof. First, we show that f is a function in $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$. We can determine the one-sided derivative of f for all $p \in \mathbb{P}^1_{\mathrm{Berk}}$ and all directions $\vec{v}$. In the case of $p \in \Gamma_0$ and $\vec{v} \in T_p(\Gamma_0)$, we have $d_{\vec{v}}f(p) = d_{\vec{v}}f_0(p)$. If $p \in \mathbb{P}^1_{\mathrm{Berk}}$ and $\vec{v} \notin T_p(\Gamma_0)$, one can calculate that $d_{\vec{v}}f(p) = 0$. The case that $p \notin \Gamma_0$ and $\vec{v} \in T_p(\Gamma_0)$ is not possible, because $p = \lim_{t\to 0^+}\gamma(t)$, γ is continuous and Γ_0 is closed in $\mathbb{P}^1_{\mathrm{Berk}}$. Hence, $f \in \mathcal{D}(\mathbb{P}^1_{\mathrm{Berk}})$. Furthermore, they imply for each finite subgraph Γ of $\mathbb{P}^1_{\mathrm{Berk}}$ and $S \in \mathcal{A}(\Gamma)$

$$\begin{aligned} m_f(S) &= \sum_{p\in\partial S, p\notin S}\ \sum_{\vec{v}\in \mathrm{In}(p,S)} d_{\vec{v}}f(p) - \sum_{p\in\partial S, p\in S}\ \sum_{\vec{v}\in \mathrm{Out}(p,S)} d_{\vec{v}}f(p) \\ &= \sum_{p\in\partial S\cap\Gamma_0, p\notin S}\ \sum_{\vec{v}\in \mathrm{In}(p,S)\cap T_p(\Gamma_0)} d_{\vec{v}}f_0(p) - \sum_{p\in\partial S\cap\Gamma_0, p\in S}\ \sum_{\vec{v}\in \mathrm{Out}(p,S)\cap T_p(\Gamma_0)} d_{\vec{v}}f_0(p). \end{aligned}$$

In the formula above, we may write $\mathrm{In}(p, S \cap \Gamma_0)$ instead of $\mathrm{In}(p, S) \cap T_p(\Gamma_0)$ and $\mathrm{Out}(p, S \cap \Gamma_0)$ instead of $\mathrm{Out}(p, S) \cap T_p(\Gamma_0)$. Moreover, we will see that we can replace $\partial S \cap \Gamma_0$ by $\partial(S \cap \Gamma_0)$. Let $p \in \partial S \cap \Gamma_0$ and $p \notin S$ such that $\mathrm{In}(p, S \cap \Gamma_0) \neq \emptyset$. Hence, there is an $\vec{v} \in \mathrm{In}(p, S \cap \Gamma_0)$ such that for a representative γ we have $\gamma(t) \in S \cap \Gamma_0$ for all sufficiently small $t > 0$, and so $p = \lim\limits_{t\to 0^+}\gamma(t) \in \overline{S \cap \Gamma_0}$. Since we have required that $p \notin S \cap \Gamma_0$, p belongs to $\partial(S \cap \Gamma_0)$. Otherwise, if $p \in \partial(S \cap \Gamma_0) \subset \Gamma_0$ and $p \notin S$ such that $\mathrm{In}(p, S \cap \Gamma_0) \neq \emptyset$, then there is a continuous map $\gamma\colon [0, L] \to \Gamma_0$ such that $\gamma(t) \in S$ for all sufficiently small $t > 0$. Due to $p = \lim\limits_{t\to 0^+}\gamma(t) \in \overline{S} \subset \Gamma$, $p \in \partial S \cap \Gamma_0$.

Similar arguments show the same for the sum over $\mathrm{Out}(p, S) \cap T_p(\Gamma_0)$ respectively $\mathrm{Out}(p, S \cap \Gamma_0)$.

Consequently, $m_f(S) = m_{f_0}(S \cap \Gamma_0)$, where $S \cap \Gamma_0 \in \mathcal{A}(\Gamma_0)$. Since $f_0 \in \mathrm{BDV}(\Gamma_0)$, there is a constant B such that

$$\sum_{S_i \in \mathcal{F}} |m_f(S_i)| = \sum_{S_i \in \mathcal{F}} |m_{f_0}(S_i \cap \Gamma_0)| \leq B$$

for any countable collection $\mathcal{F}$ of pairwise disjoint sets in $\mathcal{A}(\Gamma)$. Therefore, $f|_\Gamma$ is a function in $\mathrm{BDV}(\Gamma)$ for each finite subgraph Γ of $\mathbb{P}^1_{\mathrm{Berk}}$. It remains to show the existence of a constant $B(f)$ such that $|\Delta_\Gamma(f)|(\Gamma) \leq B(f)$ for every finite subgraph Γ. For each finite subgraph Γ containing Γ_0, $f|_\Gamma$ is constant on branches off Γ_0, and $f|_{\Gamma_0} = (f_0 \circ r_{\Gamma_0})|_{\Gamma_0} = f_0$, and so

$$\Delta_\Gamma(f) = \Delta_{\Gamma_0}(f) = \Delta_{\Gamma_0}(f_0)$$

by the definition of the Laplacian on finite subgraphs (Theorem 3.1.8). Due to $f_0 \in \mathrm{BDV}(\Gamma_0)$, there is a constant $B(f_0)$ such that

$$|\Delta_\Gamma(f)|(\Gamma) = |\Delta_{\Gamma_0}(f_0)|(\Gamma_0) \leq B(f_0) =: B(f).$$

So we have proved that f is a function in $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$. Proposition 3.1.23 states that $\Delta(f) = \Delta_{\Gamma_0}(f_0)$. □

For further examples we will introduce the (generalized) Hsia kernel.

Definition 3.2.3. i) For $x \in \mathbb{A}^1_{\mathrm{Berk}}$ corresponding to a sequence of nested discs $\{D(a_i, r_i)\}$, $\mathrm{diam}_\infty(x) := \lim_{i \to \infty} r_i$ is called the *diameter* of x.

ii) If $x, y \in \mathbb{A}^1_{\mathrm{Berk}}$, let $x \vee_\infty y$ be the point where $[x, \infty]$ and $[y, \infty]$ first meet. We define the *Hsia kernel*

$$\delta(x, y)_\infty := \mathrm{diam}_\infty(x \vee_\infty y).$$

Remark. i) If x corresponds to a sequence of nested discs $\{D(a_i, r_i)\}$ and y to $\{D(b_i, s_i)\}$, then

$$\delta(x, y)_\infty = \lim_{i \to \infty} \max(r_i, s_i, |a_i - b_i|).$$

Clearly, $\delta(x, x)_\infty = \mathrm{diam}_\infty(x)$ for each $x \in \mathbb{A}^1_{\mathrm{Berk}}$, and $\delta(x, y)_\infty = |x - y|$ for all $x, y \in \mathbb{A}^1(K)$. If x, y are of type I,II or III, corresponding to $D(a, r)$ and $D(b, s)$,

$$\delta(x, y)_\infty = \max(r, s, |a - b|) = \sup_{z \in D(a,r), w \in D(b,s)} |z - w|.$$

ii) The definitions above can be extended to $\mathbb{P}^1_{\text{Berk}} \backslash \mathcal{D}(0,1) \cong \mathcal{D}(0,1)^-$ by setting

$$\operatorname{diam}_\infty(x) := \operatorname{diam}_\infty(\psi(x)),$$

where ψ is the homeomorphism from Chapter 2, which maps t to $1/t$ for all $t \in \mathbb{P}^1(K)\backslash D(0,1)$.

Definition 3.2.4. i) Let Γ be a finite subgraph. For fixed $y, z \in \Gamma$, [BR, §3.3] tells us that there is a unique function $j_z(x,y) \in \operatorname{CPA}(\Gamma)$ on Γ such that

$$\Delta_x(j_z(x,y)) = \delta_y(x) - \delta_z(x) \text{ and } j_z(z,y) = 0$$

for all $x \in \Gamma$. We call $j_z(x,y)$ the *potential kernel*.

ii) The metric $\rho\colon \mathbb{H}_{\text{Berk}} \to \mathbb{R}_{\geq 0}$ defined by

$$\rho(x,y) := 2\log_v(\operatorname{diam}_\infty(x \vee_\infty y)) - \log_v(\operatorname{diam}_\infty(x)) - \log_v(\operatorname{diam}_\infty(y)),$$

is called the *path metric*. We call the topology introduced by this metric the *strong topology* of $\mathbb{H}_{\text{Berk}}$.

3.2.5. Let Γ be a finite subgraph, so $\Gamma \subset \mathbb{H}_{\text{Berk}}$. For $x, y, z \in \Gamma$, let $w := w_z(x,y)$ be the point where the path from x to z and the path from y to z first meet. One can show that

$$j_z(x,y) = p(z,w).$$

For a fixed $z \in \mathbb{H}_{\text{Berk}}$ we write $j_z(x,y)_\Gamma$ for the potential kernel on Γ where Γ vary over finite subgraphs of $\mathbb{P}^1_{\text{Berk}}$ containing z. The functions $\{j_z(x,y)\}_\Gamma$ coher to give a well-defined function $j_z(x,y)$ on $\mathbb{H}_{\text{Berk}} \times \mathbb{H}_{\text{Berk}}$ (cf. [BR, §4.2]). Further, we can extend j_z to $\mathbb{P}^1_{\text{Berk}} \times \mathbb{P}^1_{\text{Berk}}$ by

$$j_z(x,y) := \begin{cases} j_z(r_\Gamma(x), r_\Gamma(y))_\Gamma & \text{if } (x,y) \notin \operatorname{Diag}(K), \\ \infty & \text{if } (x,y) \in \operatorname{Diag}(K), \end{cases}$$

where Γ is any finite subgraph containing z and $w_z(x,y)$. Explicitly, for $x, y \in \mathbb{P}^1(K)$ with $x \neq y$ we have $j_z(x,y) = \rho(z, w_z(x,y))$. If $z, \zeta \in \mathbb{H}_{\text{Berk}}$, then

$$j_\zeta(x,y) = j_z(x,y) - j_z(x,\zeta) - j_z(\zeta,y) + j_z(\zeta,\zeta).$$

Proposition 3.2.6 (Retraction Formula)**.** *Let Γ be a finite subgraph of $\mathbb{P}^1_{\text{Berk}}$ and $z, x \in \Gamma$. Then for any $y \in \mathbb{P}^1_{\text{Berk}}$*

$$j_z(x,y) = j_z(x, r_\Gamma(y))_\Gamma.$$

Proof. Since $x, z \in \Gamma$, the path $[x,z]$ lies in Γ. Clearly, $w_z(x,y) \in [x,z] \subset \Gamma$, and so $j_z(x,y) = j_z(x, r_\Gamma(y))_\Gamma$. $\square$

Definition 3.2.7. i) We call the function $\|\cdot,\cdot\|\colon \mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}} \to [0,1]$

$$\|x,y\| := q_v^{-j_{\zeta_{\mathrm{Gauss}}}(x,y)}$$

the *spherical kernel*.

ii) For a fixed $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$, we define the *generalized Hsia kernel* to be the function $\mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{\infty\}$ given by

$$\delta(x,y)_\zeta := \frac{\|x,y\|}{\|x,\zeta\|\,\|y,\zeta\|}$$

if $\zeta \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(K)$, and

$$\delta(x,y)_\zeta = \begin{cases} \frac{\|x,y\|}{\|x,\zeta\|\|y,\zeta\|} & \text{if } x,y \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\} \\ \infty & \text{if } x = \zeta \text{ or } y = \zeta \end{cases}$$

if $\zeta \in \mathbb{P}^1(K)$.

Remark. i) Since $\|x,y\| = 0$ if and only if $x = y \in \mathbb{P}^1(K)$, the generalized Hsia kernel is well-defined in both cases.

ii) The generalized Hsia kernel has the following geometric interpretation (cf. [BR, §4.4]): Let $x,y \in \mathbb{P}^1_{\mathrm{Berk}}$ and $w := x \vee_\zeta y$, i.e. the point where the paths $[x,\zeta]$ and $[y,\zeta]$ first meet. Then

$$\delta(x,y)_\zeta = \mathrm{diam}_\zeta(w),$$

where $\mathrm{diam}_\zeta(x) := \delta(x,x)_\zeta$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$. One has the identitiy

$$\mathrm{diam}_\zeta(x) = \frac{1}{\|\zeta,\zeta\|} \cdot q_v^{-\rho(x,\zeta)}$$

for each $\zeta \in \mathbb{H}_{\mathrm{Berk}}$ and $x \in \mathbb{P}^1_{\mathrm{Berk}}$ (cf. [BR, §4.4 Equation (4.32)]).

Proposition 3.2.8. *Fix $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$.*

i) The generalized Hsia kernel $\delta(x,y)_\zeta\colon \mathbb{P}^1_{\mathrm{Berk}} \times \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ is nonnegative, symmetric, upper semicontinuous as a function of two variables and continuous in each variable separately. We have $\delta(x,y)_\zeta = 0$ if and only if $x = y \in \mathbb{P}^1(K)$.

ii) If $\zeta \in \mathbb{H}_{\mathrm{Berk}}$, then $\delta(x,y)_\zeta$ is bounded and valued in $[0, 1/\|\zeta,\zeta\|]$.

iii) If $\zeta \in \mathbb{P}^1(K)$, then $\delta(x,y)_\zeta$ is unbounded and $\delta(x,y)_\zeta = \infty$ if and only if $x = \zeta$ or $y = \zeta$.

iv) For all $x,y,z \in \mathbb{P}^1_{\mathrm{Berk}}$,

$$\delta(x,y)_\zeta \leq \max(\delta(x,z)_\zeta, \delta(y,z)_\zeta,$$

with equality if $\delta(x,z)_\zeta \neq \delta(y,z)_\zeta$.

v) For each $a \in \mathbb{P}^1_{\text{Berk}}$ *and* $r > 0$*, the 'open ball'*

$$\mathcal{B}(a,r)^-_\zeta := \{x \in \mathbb{P}^1_{\text{Berk}} | \ \delta(x,z)_\zeta < r\}$$

is connected and open in the Berkovich topology. It is empty if $r \leq \operatorname{diam}_\zeta(a)$*, and coincides with an open ball* $\mathcal{B}(b,r)^-_\zeta$ *for some* $b \in \mathbb{P}^1(K)$ *if* $r > \operatorname{diam}_\zeta(a)$*. Likewise, the 'closed ball'*

$$\mathcal{B}(a,r)_\zeta := \{x \in \mathbb{P}^1_{\text{Berk}} | \ \delta(x,z)_\zeta \leq r\}$$

is connected and closed in the Berkovich topology. It is empty if $r < \operatorname{diam}_\zeta(a)$*, and coincides with* $\mathcal{B}(b,r)_\zeta$ *for some* $b \in \mathbb{P}^1(K)$ *if* $r > \operatorname{diam}_\zeta(a)$ *or if* $r = \operatorname{diam}_\zeta(a)$ *and* a *is of type II or III. If* $r = \operatorname{diam}_\zeta(a)$ *and* a *is of type I or IV, then* $\mathcal{B}(a,r)_\zeta = \{a\}$*.*

Proof. See [BR, Proposition 4.10]. □

Example 3.2.9. We fix $y, \zeta \in \mathbb{P}^1_{\text{Berk}}$, and consider the function $f\colon \mathbb{P}^1_{\text{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ defined by $f(x) = -\log_v(\delta(x,y)_\zeta)$. One can show, that $f \in \operatorname{BDV}(\mathbb{P}^1_{\text{Berk}})$ and

$$\Delta(-\log_v(\delta(x,y)_\zeta)) = \delta_y(x) - \delta_\zeta(x).$$

Proof. Let Γ be a finite subgraph of $\mathbb{P}^1_{\text{Berk}}$. By Remark 3.2.5, we may assume that the Gauss point ζ_{Gauss} is contained in Γ. Set $\tilde{y} := r_\Gamma(y)$ and $\tilde{\zeta} := r_\Gamma(\zeta)$. By the definition of the generalized Hsia kernel and by Proposition 3.2.6,

$$\begin{aligned} -\log_v(\delta(x,y)_\zeta) &= j_{\zeta_{\text{Gauss}}}(x,y) - j_{\zeta_{\text{Gauss}}}(x,\zeta) - j_{\zeta_{\text{Gauss}}}(y,\zeta) \\ &= j_{\zeta_{\text{Gauss}}}(x,\tilde{y}) - j_{\zeta_{\text{Gauss}}}(x,\tilde{\zeta}) - j_{\zeta_{\text{Gauss}}}(y,\zeta) \end{aligned}$$

for all $x \in \Gamma$. Since the potential kernel on Γ for fixed $\tilde{y}$ respectively $\tilde{\zeta}$ belongs to $\operatorname{CPA}(\Gamma)$, $f|_\Gamma \in \operatorname{BDV}(\Gamma)$. Due to the definition of the potential kernel,

$$\begin{aligned} \Delta_\Gamma(f) &= \Delta_\Gamma(j_{\zeta_{\text{Gauss}}}(\cdot,\tilde{y})) - \Delta_\Gamma(j_{\zeta_{\text{Gauss}}}(\cdot,\tilde{\zeta})) - \Delta_\Gamma(j_{\zeta_{\text{Gauss}}}(y,\zeta)) \\ &= \delta_{\tilde{y}} - \delta_{\zeta_{\text{Gauss}}} - (\delta_{\tilde{\zeta}} - \delta_{\zeta_{\text{Gauss}}}) - 0 \\ &= \delta_{\tilde{y}} - \delta_{\tilde{\zeta}}. \end{aligned}$$

Therefore, $|\Delta_\Gamma(f)|(\Gamma) = (\delta_{\tilde{y}} + \delta_{\tilde{\zeta}})(\Gamma) = 2 < \infty$, and so f belongs to $\operatorname{BDV}(\mathbb{P}^1_{\text{Berk}})$. Since $\Delta_\Gamma(f) = \delta_{\tilde{y}} - \delta_{\tilde{\zeta}} = r_{\Gamma *}(\delta_y - \delta_\zeta)$ for each Γ, we get $\Delta(f) = \delta_y - \delta_\zeta$ by Definition 3.1.26. □

Example 3.2.10 (Poincaré-Lelong formula)**.** Let $0 \neq g \in K(T)$ with $\operatorname{div}(g) = \sum_{i=1}^m n_i(a_i)$. We consider the function $f\colon \mathbb{P}^1_{\text{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ defined by $f(x) := -\log_v([g]_x)$. Then

$f \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and

$$\Delta(-\log_v([g]_x) = \sum_{i=1}^{m} n_i \delta_{a_i}(x).$$

Proof. Let $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$ be disjoint from the support of $\mathrm{div}(g)$, i. e. $\zeta \notin \{a_1, \ldots, a_m\}$. The decomposition formula for the generalized Hsia kernel (cf. [BR, Corollary 4.14]) tells us that there is a constant C_ζ such that

$$[g]_x = C_\zeta \cdot \prod_{i=1}^{m} \delta(x, a_i)_\zeta^{n_i}.$$

Hence,

$$f(x) = -\log_v(C_\zeta) + \sum_{i=1}^{m} -n_i \log_v(\delta(x, a_i)_\zeta).$$

Due to Example 3.2.9, f is a function in the vector space $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$, and

$$\begin{aligned}\Delta(f) &= \Delta(\log_v(C_\zeta)) + \sum_{i=1}^{m} n_i \Delta(-\log_v(\delta(x, a_i)_\zeta)) \\ &= 0 + \sum_{i=1}^{m} n_i \delta_{a_i}(x) - \sum_{i=1}^{m} n_i \delta_\zeta(x) \\ &= \sum_{i=1}^{m} n_i \delta_{a_i}(x),\end{aligned}$$

where last equation is true because of $\sum_{i=1}^{m} n_i = 0$. □

Example 3.2.11 (The potential function)**.** Let ν be a finite signed Borel measure on $\mathbb{P}^1_{\mathrm{Berk}}$. We define the *potential function* in the following way: If $\zeta \in \mathbb{H}_{\mathrm{Berk}}$ or $\zeta \notin \mathrm{supp}(\nu)$, we set

$$u_\nu(x, \zeta) := \int -\log_v(\delta(x, y)_\zeta) d\nu(y).$$

If $\zeta \in \mathbb{P}^1(K) \cap \mathrm{supp}(\nu)$, then the potential function is defined by

$$u_\nu(x, \zeta) := u_\nu(x, \zeta_{\mathrm{Gauss}}) + \nu(\mathbb{P}^1_{\mathrm{Berk}}) \log_v(\|x, \zeta\|).$$

One can show that $u_\nu(x, \zeta) \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and

$$\Delta(u_\nu(x, \zeta)) = \nu - \nu(\mathbb{P}^1_{\mathrm{Berk}}) \delta_\zeta(x).$$

Proof. Let Γ be any finite subgraph containing ζ_{Gauss}. By the Retraction formula, Proposition 3.2.6, one have the identity $j_{\zeta_{\mathrm{Gauss}}}(x,y) = j_{\zeta_{\mathrm{Gauss}}}(x, r_\Gamma(y))$ for all $y \in \mathbb{P}^1_{\mathrm{Berk}}$ and $x \in \Gamma$.

At first, we consider the case that $\zeta \in \mathbb{H}_{\mathrm{Berk}}$ or $\zeta \notin \mathrm{supp}(\nu)$. If $\zeta \in \mathbb{H}_{\mathrm{Berk}}$, we can enlarge Γ such that $\zeta \in \Gamma$. By [BR, Proposition 3.3], $j_{\zeta_{\mathrm{Gauss}}}(\cdot,\zeta)\colon \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R}$ is bounded, and so

$$C_\zeta := \int j_{\zeta_{\mathrm{Gauss}}}(y,\zeta) d\nu(y) < \infty.$$

If $\zeta \notin \mathrm{supp}(\nu)$, then $j_{\zeta_{\mathrm{Gauss}}}(\cdot,\zeta)$ is real valued on $\mathrm{supp}(\nu)$, because $j_{\zeta_{\mathrm{Gauss}}}(y,z) \notin \mathbb{R}$ iff $y = z \in \mathbb{P}^1(K)$. Furthermore, $j_{\zeta_{\mathrm{Gauss}}}(\cdot,\zeta)$ is continuous on the compact set $\mathrm{supp}(\nu)$ by [BR, Proposition 3.3]. Consequently, $j_{\zeta_{\mathrm{Gauss}}}(\cdot,\zeta)$ is bounded on $\mathrm{supp}(\nu)$, and so we can set

$$C_\zeta := \int j_{\zeta_{\mathrm{Gauss}}}(y,\zeta) d\nu(y) < \infty$$

as well. For all $x \in \Gamma$, we have

$$\begin{aligned} u_\nu(x,\zeta) &= \int -\log_v(\delta(x,y)_\zeta) d\nu(y) \\ &= \int (j_{\zeta_{\mathrm{Gauss}}}(x,y) - j_{\zeta_{\mathrm{Gauss}}}(x,\zeta) - j_{\zeta_{\mathrm{Gauss}}}(y,\zeta)) d\nu(y) \\ &= \int j_{\zeta_{\mathrm{Gauss}}}(x, r_\Gamma(y)) d\nu(y) - \int j_{\zeta_{\mathrm{Gauss}}}(x, r_\Gamma(\zeta)) d\nu(y) - \int j_{\zeta_{\mathrm{Gauss}}}(y,\zeta) d\nu(y) \\ &= \int_\Gamma j_{\zeta_{\mathrm{Gauss}}}(x,t) d(r_{\Gamma *}(\nu))(t) - \nu(\mathbb{P}^1_{\mathrm{Berk}}) j_{\zeta_{\mathrm{Gauss}}}(x, r_\Gamma(\zeta)) - C_\zeta. \end{aligned}$$

Now, consider $\zeta \in \mathbb{P}^1(K) \cap \mathrm{supp}(\nu)$. Since $\|y, \zeta_{\mathrm{Gauss}}\| = q_v^{-j_{\zeta_{\mathrm{Gauss}}}(y,\zeta_{\mathrm{Gauss}})} = 1$ for every $y \in \mathbb{P}^1_{\mathrm{Berk}}$, the Retraction formula implies $-\log_v(\delta(x,y)_{\zeta_{\mathrm{Gauss}}}) = j_{\zeta_{\mathrm{Gauss}}}(x, r_\Gamma(y))$ for all $x \in \Gamma$. Hence,

$$\begin{aligned} u_\nu(x,\zeta) &= u_\nu(x,\zeta_{\mathrm{Gauss}}) + \nu(\mathbb{P}^1_{\mathrm{Berk}}) \log_v(\|x,\zeta\|) \\ &= \int j_{\zeta_{\mathrm{Gauss}}}(x, r_\Gamma(y)) d\nu(y) - \nu(\mathbb{P}^1_{\mathrm{Berk}}) j_{\zeta_{\mathrm{Gauss}}}(x, r_\Gamma(\zeta)) \\ &= \int_\Gamma j_{\zeta_{\mathrm{Gauss}}}(x,t) d(r_{\Gamma *}\nu)(t) - \nu(\mathbb{P}^1_{\mathrm{Berk}}) j_{\zeta_{\mathrm{Gauss}}}(x, r_\Gamma(\zeta)). \end{aligned}$$

Thus, we can calculate the Laplacian jointly for both cases. By [BR, Proposition 3.11], $h(x) := \int_\Gamma -\log_v(\delta(x,t)_{\zeta_{\mathrm{Gauss}}}) d(r_{\Gamma *}\nu)(t) \in \mathrm{BDV}(\Gamma)$ and

$$\Delta_\Gamma(h) = r_{\Gamma *}\nu - (r_{\Gamma *}\nu)(\Gamma)\delta_{\zeta_{\mathrm{Gauss}}}.$$

We already know that $\tilde{h}(x) := j_{\zeta_{\mathrm{Gauss}}}(x, r_\Gamma(\zeta)) \in \mathrm{CPA}(\Gamma)$ and $\Delta_\Gamma(\tilde{h}) = \delta_{r_\Gamma(\zeta)} - \delta_{\zeta_{\mathrm{Gauss}}}$.

Together, we get $u_\nu(x,\zeta) \in \mathrm{BDV}(\Gamma)$ and

$$\begin{aligned}\Delta_\Gamma(u_\nu(x,\zeta)) &= \Delta_\Gamma(h) - \nu(\mathbb{P}^1_{\mathrm{Berk}})\Delta_\Gamma(\tilde{h}) \\ &= r_{\Gamma*}\nu - r_{\Gamma*}\nu(\Gamma)\delta_{\zeta_{\mathrm{Gauss}}} - \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_{r_\Gamma(\zeta)} + \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_{\zeta_{\mathrm{Gauss}}} \\ &= r_{\Gamma*}\nu - \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_{\zeta_{\mathrm{Gauss}}} - \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_{r_\Gamma(\zeta)} + \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_{\zeta_{\mathrm{Gauss}}} \\ &= r_{\Gamma*}(\nu - \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_\zeta).\end{aligned}$$

The potential function $u_\nu(x,\zeta)$ is a function in $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ by the inequality

$$\begin{aligned}|\Delta_\Gamma(u_\nu(x,\zeta))|(\Gamma) &= |r_{\Gamma*}(\nu - \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_\zeta)|(\Gamma) \\ &\leq |\nu - \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_\zeta|(r_\Gamma^{-1}(\Gamma)) \\ &= |\nu - \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_\zeta|(\mathbb{P}^1_{\mathrm{Berk}}) \\ &< \infty.\end{aligned}$$

Proposition 3.1.23 and Proposition 3.1.25 state that $\Delta(u_\nu(x,\zeta)) = \nu - \nu(\mathbb{P}^1_{\mathrm{Berk}})\delta_\zeta(x)$. □

Example 3.2.12. In particular, if ν is a probability measure on $\mathbb{P}^1_{\mathrm{Berk}}$, we define the potential function for $\zeta \in \mathbb{H}_{\mathrm{Berk}}$ or $\zeta \notin \mathrm{supp}(\nu)$ by

$$u_\nu(x,\zeta) := \int -\log_v(\delta(x,y)_\zeta)d\nu(y),$$

and for $\zeta \in \mathbb{P}^1(K) \cap \mathrm{supp}(\nu)$ by

$$u_\nu(x,\zeta) := u_\nu(x,\zeta_{\mathrm{Gauss}}) + \log_v(\|x,\zeta\|).$$

Then $u_\nu(x,\zeta) \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and

$$\Delta(u_\nu(x,\zeta)) = \nu - \delta_\zeta(x).$$

Moreover, if $\zeta \in \mathbb{H}_{\mathrm{Berk}}$ or $\zeta \notin \mathrm{supp}(\nu)$, there is a constant C_ζ such that for all $z \in \mathbb{P}^1_{\mathrm{Berk}}$

$$u_\nu(x,\zeta) = u_\nu(x,\zeta_{\mathrm{Gauss}}) + \log_v(\|x,\zeta\|) - C_\zeta.$$

Proof. This example is just a special case of Example 3.2.11, so it remains to show the last statement. Let ζ be as required and $x \in \mathbb{P}^1_{\mathrm{Berk}}$, then by the calculation in Example

3.2.11 and the fact that $\nu(\mathbb{P}^1_{\mathrm{Berk}}) = 1$, we have

$$\begin{aligned} u_\nu(x,\zeta) &= \int j_{\zeta_{\mathrm{Gauss}}}(x,y)d\nu(y) - j_{\zeta_{\mathrm{Gauss}}}(x,\zeta) - C_\zeta \\ &= \int -\log_v(\delta(x,y)_{\zeta_{\mathrm{Gauss}}})d\nu(y) + \log_v(q_v^{-j_{\zeta_{\mathrm{Gauss}}}(x,\zeta)}) - C_\zeta \\ &= u_\nu(x,\zeta_{\mathrm{Gauss}}) + \log_v(\|x,\zeta\|) - C_\zeta. \end{aligned}$$

□

4 Harmonic functions

The classical potential theory is including the study of harmonic functions. Baker and Rumely developed the theory of harmonic functions on $\mathbb{P}^1_{\text{Berk}}$ and established analogues of the main results of this theory in [BR]. In [Th] this theory is developed in a more general way extending the definition made in this chapter. One considers a general smooth strictly k-analytic curve X instead of just $\mathbb{P}^1_{\text{Berk}}$. We will give a short introduction to this theory in Chapter 5 and we will verify that both definitions actually coincide. In this Chapter we elaborate on the theory of harmonic functions on $\mathbb{P}^1_{\text{Berk}}$ from [BR] and try to extend it with some new or slightly modified statements. In particular, we are interested in the connection between the terms *strongly harmonic* and *harmonic*, which are defined at the beginning of Section 4.1. Next to the definitions, we give examples and fundamental properties of (strongly) harmonic functions. In Section 4.2, we will introduce the *main dendrite* of a domain. We will see that the values of a harmonic function on this $\mathbb{R}$-tree determine the behavior of the function on the whole domain. In the Sections 4.3 to 4.7, we prove analogues of the Maximum Principle (§4.3), the Poisson formula (§4.4 and §4.5), Uniform convergence (§4.6) and Harnack's Principle (§4.7). Most of the mentioned main results are true in the general case. If so, we will give a reference in [Th], and refer for a precise definition of a harmonic function on X to Section 5.3.

4.1 Harmonic functions

In the last chapter we have seen the existence of a Laplacian for a function f of bounded differential variation. Hence, we can define harmonic functions similarly to the definition in the classical potential theory. These definitions are followed by some examples related to those in Chapter 3. Afterwards, we will give some nice fundamental properties which are needed for the proofs of the main theorems and propositions which are stated in §4.3-4.7. In particular, we study the behavior of a function $f\colon U \to \mathbb{R}$ with Laplacian $\Delta_U(f) = 0$ on finite subgraphs $\Gamma \subset U$ for a domain U satisfying $|\partial U| < \infty$.

Definition 4.1.1. i) If U is a domain, a function $f\colon U \to \mathbb{R}$ is called *strongly harmonic* on U if it is continuous on U, belongs to $\mathrm{BDV}(U)$, and satisfies

$$\Delta_U(f) = 0.$$

ii) If U is an arbitrary open set, then $f\colon U \to \mathbb{R}$ is called *harmonic* on U if for each

$x \in U$ there is a domain $V_x \subset U$ with $x \in V_x$ such that f is strongly harmonic on V_x.

At the end of Section 4.2, we give an example of a function f on a domain U which is harmonic but not strongly harmonic on U.

4.1.2. Since the Laplacian operator Δ_Γ is linear by Lemma 3.1.28, the function $a \cdot f + b \cdot g$ is harmonic (resp. strongly harmonic) on V for any harmonic (resp. strongly harmonic) functions f and g on V and $a, b \in \mathbb{R}$. We denote the space of harmonic functions on U by $\mathcal{H}(U)$.

Example 4.1.3. Let $U \subset \mathbb{P}^1_{\text{Berk}}$ be a domain and $f \colon U \to \mathbb{R}$ given by $f \equiv C$ on U for a constant $C \in \mathbb{R}$. Then f is strongly harmonic on U.

Proof. The function f is clearly continuous, $f \in \mathrm{BDV}(U)$ and $\Delta(f) = 0$ by Example 3.2.1. Using Proposition 3.1.29 with $U_1 = U$ and $U_2 = \mathbb{P}^1_{\text{Berk}}$, we get

$$\Delta_U(f) = \Delta(f)|_U = 0.$$

Hence, f is strongly harmonic on U. □

Example 4.1.4. Fix $y, \zeta \in \mathbb{P}^1_{\text{Berk}}$ such that $\zeta \notin \mathbb{P}^1(K)$ or $y \neq \zeta$. Then the function $f \colon \mathbb{P}^1_{\text{Berk}} \backslash \{y, \zeta\} \to \mathbb{R}$ given by $f(x) := -\log_v(\delta(x, y)_\zeta)$ is strongly harmonic on each connected component of $\mathbb{P}^1_{\text{Berk}} \backslash \{y, \zeta\}$.

Proof. Let U be a connected component of $\mathbb{P}^1_{\text{Berk}} \backslash \{y, \zeta\}$, then U is a domain. Since the generalized Hsia kernel is continuous in every $x \in U$ by [BR, Proposition 4.1], f is continuous as well. In Example 3.2.9, we have seen that $f \in \mathrm{BDV}(U)$ and $\Delta(f) = \delta_y - \delta_\zeta$. Due to $U \subset \mathbb{P}^1_{\text{Berk}} \backslash \{y, \zeta\}$,

$$\Delta_U(f) = \Delta(f)|_U = (\delta_y - \delta_\zeta)|_U = 0$$

by Proposition 3.1.29. □

Example 4.1.5. If $0 \neq g \in K(T)$ and $\mathrm{div}(g) = \sum_{i=1}^m n_i(a_i)$, then $f(x) = -\log_v([g]_x)$ is strongly harmonic on $\mathbb{P}^1_{\text{Berk}} \backslash \{a_1, \ldots, a_m\}$.

Proof. Due to $a_1, \ldots, a_m \in K$, i.e. they are of type I, $U = \mathbb{P}^1_{\text{Berk}} \backslash \{a_1, \ldots, a_m\}$ is connected. Clearly, U is open as well, so U is by definition a domain. In Example 3.2.10, one has seen that $f \in \mathrm{BDV}(U)$ and

$$f(x) = -\log_v(C_\zeta) + \sum_{i=1}^m -n_i \log_v(\delta(x, a_i)_\zeta).$$

Consequently, f is also continuous by [BR, Proposition 4.1]. We have also calculated that $\Delta(f) = \sum_{i=1}^{m} n_i \delta_{a_i}$, and so

$$\Delta_U(f) = \Delta(f)|_U = (\sum_{i=1}^{m} n_i \delta_{a_i})|_U = 0$$

since $U = \mathbb{P}^1_{\mathrm{Berk}} \backslash \{a_1, \dots, a_m\}$. Therefore, f is strongly harmonic on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{a_1, \dots, a_m\}$. □

Remark. In Proposition 5.3.15, we will see an analogue statement in the general case.

Example 4.1.6. Let ν be a probability measure on $\mathbb{P}^1_{\mathrm{Berk}}$ and $\zeta \notin \mathrm{supp}(\nu)$, then the potential function $u_\nu(z, \zeta) := \int -\log_v(\delta(x, y)_\zeta)\ d\nu(y)$ is strongly harmonic on each connected component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash (\mathrm{supp}(\nu) \cup \{\zeta\})$.

Proof. By Proposition 3.2.8, the generalized Hsia kernel $\delta(x, y)_\zeta$ is continuous in the variable x on $\mathbb{P}^1_{\mathrm{Berk}} \backslash (\mathrm{supp}(\nu) \cup \{\zeta\})$. Therefore, $u_\nu(\cdot, \zeta)$ is continuous on $\mathbb{P}^1_{\mathrm{Berk}} \backslash (\mathrm{supp}(\nu) \cup \{\zeta\})$ as well. Let U be a connected component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash (\mathrm{supp}(\nu) \cup \{\zeta\})$, then U is a domain. By Example 3.2.12, we know that $u_\nu(\cdot, \zeta) \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and $\Delta(u_\nu(\cdot, \zeta)) = (\nu - \delta_\zeta)$. Hence, $u_\nu(\cdot, \zeta)|_U \in \mathrm{BDV}(U)$ and

$$\Delta_U(u_\nu(\cdot, \zeta)) = (\nu - \delta_\zeta)|_U = 0$$

since $U \cap \mathrm{supp}(\nu) = \emptyset$ and $U \cap \{\zeta\} = \emptyset$. Thus, f is strongly harmonic on U. □

Next we will see some properties of (strongly) harmonic functions, which are used to prove important theorems in following sections.

Lemma 4.1.7. *i) If $U_1 \subset U_2$ are domains, and f is strongly harmonic on U_2, then f is strongly harmonic on U_1.*

ii) If f is harmonic on an open set V, and U is a subdomain of V with $\overline{U} \subset V$, then f is strongly harmonic on U.

iii) If f is harmonic on V and $E \subset V$ is compact and connected, there is a subdomain $U \subset V$ containing E such that f is strongly harmonic on U.

Proof. For i), let f be strongly harmonic on U_2. Since f is continuous on U_2, it is continuous on $U_1 \subset U_2$. By Proposition 3.1.29, $f \in \mathrm{BDV}(U_2)$ implies $f \in \mathrm{BDV}(U_1)$ and

$$\Delta_{U_1}(f) = \Delta_{U_2}(f)|_{U_1} = 0.$$

Therefore, f is strongly harmonic on U_1.

For ii), we consider a harmonic function f on an open set V and a subdomain U of V such that $\overline{U} \subset V$. Therefore, there is a domain $U_x \subset V$ for each $x \in \overline{U}$ such that f is strongly harmonic on U_x and $x \in U_x$. Since $\mathbb{P}^1_{\mathrm{Berk}}$ is compact by Proposition 2.3.3, the closed subset $\overline{U}$ is compact as well. Therefore, $\overline{U} \subset \bigcup_{x\in\overline{U}} U_x$ implies that there are $U_{x_1}, \ldots, U_{x_m}$ such that $\overline{U} \subset \bigcup_{i=1}^m U_{x_i} =: W$. Clearly, W is open as the union of open sets. Since $\overline{U}$ is connected, we know that $\bigcap_{i=1}^m U_{x_i} \neq \emptyset$. Therefore, W is connected as a union of non-disjoint connected sets. Thus, W is a domain, and we can apply Proposition 3.1.30. So we get $f|_W \in \mathrm{BDV}(W)$ and

$$\Delta_W(f)|_{U_{x_i}} = \Delta_{U_{x_i}}(f) = 0$$

for each $i = 1, \ldots, m$. Hence, $\Delta_W(f) = 0$. Proposition 3.1.29 implies $f|_U \in \mathrm{BDV}(U)$ and $\Delta_U(f) = \Delta_W(f)|_U = 0$ for our domain $U \subset W$. Since f is continuous on U_{x_i} for every $i = 1, \ldots, m$ and $U \subset \bigcup_{i=1}^m U_{x_i}$, f is continuous on U and so strongly harmonic on U.

For iii), let f be harmonic on V and $E \subset V$ a compact and connected subset. For each $x \in E \subset V$, there is a domain $U_x \subset V$ such that $x \in U_x$ and f is strongly harmonic on U_x. Since $E \subset \bigcup_{x\in E} U_x$ and E is compact, $E \subset \bigcup_{i=1}^m U_{x_i} =: U$. As above, U is connected and f is strongly harmonic on U. □

A direct consequence of part i) of the lemma above is the following:

Corollary 4.1.8. *If f is harmonic on an open set $U \subset \mathbb{P}^1_{\mathrm{Berk}}$, then f is harmonic on each open subset V of U.*

Proof. Consider $x \in V \subset U$. Since f is harmonic on U, there is a domain $U_x \subset U$ containing x such that f is strongly harmonic on U_x. Let V_x be the connected component in the open set $U_x \cap V$ containing x. Then V_x is a domain in U_x, and therefore f is strongly harmonic on V_x by Lemma 4.1.7 i). □

Lemma 4.1.9. *Let V be a domain with a finite number of boundary points $\{x_1, \ldots, x_m\}$ and h a strongly harmonic function on V.*

i) The function h belongs to $\mathrm{CPA}(\Gamma)$ for every finite subgraph $\Gamma \subset V$.

ii) If Γ is a finite subgraph of V satisfying $r_{\overline{V},\Gamma}(\{x_1, \ldots, x_m\}) \subset \partial\Gamma$,

$$\sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}} h(p) = 0$$

for every $p \in \Gamma \backslash \partial\Gamma$.

Proof. For i), we set $y_i := r_{\overline{V},\Gamma}(x_i)$. Since h is strongly harmonic on V, h belongs to

BDV(Γ) and $\Delta_{\overline{V}}(h) = \Delta_{\partial V}(h)$. Hence,

$$\begin{aligned}\Delta_\Gamma(h) &= (r_{\overline{V},\Gamma})_*(\Delta_{\overline{V}}(h)) \\ &= (r_{\overline{V},\Gamma})_*(\Delta_{\partial V}(h)) \\ &= \sum_{i=1}^{m} c_i \cdot \delta_{y_i},\end{aligned}$$

where $c_i := \Delta_{\partial V}(h)(x_i)$. By [BR, Corollary 3.9], we get $h \in \mathrm{CPA}(\Gamma)$.

For ii), Remark 3.1.7 and the definition of $\Delta_{\overline{V}}(h)$ state

$$\sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}} h(p) = -\Delta_\Gamma(h)(p) = -\Delta_{\overline{V}}(h)(r^{-1}_{\overline{V},\Gamma}(p)).$$

The requirements imply $r^{-1}_{\overline{V},\Gamma}(p) \subset V$, and so

$$\sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}} h(p) = -\Delta_V(h)(r^{-1}_{\overline{V},\Gamma}(p)) = 0.$$

□

4.2 Harmonic functions and the main dendrite

The behavior of a harmonic function on a domain U is controlled by its behavior on a special subset which is called *main dendrite* and is closely related to the skeleton in [Th]. This subset is defined below, and some properties of it are stated afterwards. In particular, the main dendrite is an $\mathbb{R}$-tree. Further, we get from the proof of this property (cf. [BR, Proposition 7.10]) a countable exhaustion of any domain different from $\mathbb{P}^1_{\mathrm{Berk}}$ by subdomains on which a harmonic function is strongly harmonic. This leads to the fact that every harmonic function on a domain of bounded differential variation is actually strongly harmonic. The main result of this section is that every harmonic function is determined by its values on the main dendrite. This knowledge enables us to give an example of a harmonic function which is not strongly harmonic at the end of this section.

Definition 4.2.1. If U is a domain, the *main dendrite* $D = D(U) \subset U$ is the set of all $x \in U$ belonging to paths between two boundary points $y, z \in \partial U$.

Remark. The main dendrite is empty iff $|\partial U| \in \{0, 1\}$. Clearly, if $|\partial U| \in \{0, 1\}$, $D(U) = \emptyset$. If $|\partial U| \geq 2$, there are at least two different points $y, z \in \partial U$. Since U is connected, the unique path from y to z is contained in $\overline{U}$. So there are points belonging to the path from y and z which are contained in U. $|\partial U| \in \{0, 1\}$ for a domain iff $U = \mathbb{P}^1_{\mathrm{Berk}}$ or U is a connected component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}$ for some $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$.

Lemma 4.2.2. *Let $W \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a domain, $x \in W$ and $y \in \mathbb{P}^1_{\mathrm{Berk}} \backslash W$. Then the unique path Γ from x to y contains some boundary point of W.*

Proof. Set $W' := \mathbb{P}^1_{\mathrm{Berk}} \backslash \overline{W}$. Supposing $\Gamma \cap \partial W = \emptyset$, we have $\Gamma \cap W' = \Gamma \cap \mathbb{P}^1_{\mathrm{Berk}} \backslash W$. Hence, $y \in \Gamma \cap W'$ and $(\Gamma \cap W) \cap (\Gamma \cap W') = \emptyset$. We also know that $x \in \Gamma \cap W$. Thus, the two sets $\Gamma \cap W'$ and $\Gamma \cap W$ are non-empty relatively open disjoint subsets with $\Gamma = (\Gamma \cap W) \cup (\Gamma \cap W')$. This contradicts the fact that Γ is connected. □

Proposition 4.2.3. *Let U be a domain in $\mathbb{P}^1_{\mathrm{Berk}}$ and D be the main dendrite of U. If D is non-empty, then*

i) D is finitely branched at every point.

ii) D is a countable union of finite $\mathbb{R}$-trees, whose boundary points are all of type II.

Proof. See [BR, Proposition 7.10]. □

We have defined in Section 2.1 a strict simple domain as a domain with only finitely many boundary points which are all of type II. The proof of the last proposition implies that every domain $U \neq \mathbb{P}^1_{\mathrm{Berk}}$ can be exhausted by a sequence of such domains:

Corollary 4.2.4. *If $U \neq \mathbb{P}^1_{\mathrm{Berk}}$ is a domain in $\mathbb{P}^1_{\mathrm{Berk}}$, then U can be exhausted by a sequence $W_1 \subset W_2 \subset \cdots$ of strict simple domains with $\overline{W_n} \subset W_{n+1} \subset U$ for each n.*

Proof. See [BR, Corollary 7.11]. □

Corollary 4.2.5. *Let $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a domain and f harmonic on U. If $f \in \mathrm{BDV}(U)$, then f is already strongly harmonic on U.*

Proof. Due to $f \in \mathrm{BDV}(U)$, the Laplacian $\Delta_U(f)$ exists, and it remains to show that $\Delta_U(f) = 0$. At first, we consider the case that $U = \mathbb{P}^1_{\mathrm{Berk}}$. If T is a Borel measurable set in $\mathbb{P}^1_{\mathrm{Berk}}$, the compact set $\overline{T} \subset \mathbb{P}^1_{\mathrm{Berk}}$, and so T, can be covered by finitely many subdomains $U_{x_1}, \ldots, U_{x_m}$, where f is strongly harmonic on each U_{x_i} for a point $x_i \in \mathbb{P}^1_{\mathrm{Berk}}$. By Lemma 3.1.29 $\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f)|_{U_{x_i}} = \Delta_{U_{x_i}}(f)$ for each $i = 1, \ldots, m$, and so $\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f)(T) = 0$. Thus, f is strongly harmonic on $U = \mathbb{P}^1_{\mathrm{Berk}}$.

If $U \neq \mathbb{P}^1_{\mathrm{Berk}}$, we will use Corollary 4.2.4 to verify $\Delta_U(f) = 0$. Let $(W_n)_{n \geq 1}$ be the exhaustion from the corollary. Then $U = \bigcup_{n=1}^{\infty} W_n$, $W_n \subset W_{n+1}$ and $\overline{W_n} \subset U$ for all $n \geq 1$. It follows directly from the σ-additivity of $\Delta_U(f)$ that $\Delta_U(f)$ is continuous from below. Since f is strongly harmonic on W_n for each $n \geq 1$ by Lemma 4.1.7 ii), this implies

$$\begin{aligned}\Delta_U(f)(T) &= \Delta_U(f)(\cup_{n\geq 1} T \cap W_n)\\ &= \lim_{n\to\infty} \Delta_U(f)(T\cap W_n)\\ &= \lim_{n\to\infty} \Delta_{W_n}(f)(T\cap W_n)\\ &= 0\end{aligned}$$

for every Borel measurable subset T of U. □

In the following, we will see the connection between the main dendrite and harmonic functions:

Proposition 4.2.6. *Let $U \subset \mathbb{P}^1_{\text{Berk}}$ be a domain and D the main dendrite of U. If D is empty, every harmonic function on U is constant. If D is non-empty, every harmonic function f on U is constant along every path leading away from D.*

Proof. At first we consider the case that $D \neq \emptyset$. We fix a $y_0 \in D$ and let x be a point in $U\backslash D$. We denote the first point of the path $[x, y_0]$ in D by w, and we show that $f(x) = f(w)$. Let V be the connected component of $\mathbb{P}^1_{\text{Berk}}\backslash\{w\}$ which contains x. V is connected and open with $\partial V = \{w\} \subset U$, i.e. V is a domain, and $\overline{V} \subset U$. Applying Lemma 4.1.7 ii), f is strongly harmonic on V, and so $\Delta_V(f) = 0$. [BR, Proposition 5.25] implies $\Delta_{\overline{V}}(f)(\{w\}) = -\Delta_V(f)(V) = 0$, i.e. $\Delta_{\partial V}(f) = 0$. Thus, $\Delta_{\overline{V}}(f) = \Delta_V(f) + \Delta_{\partial V}(f) = 0$. [BR, Lemma 5.14] tells us, that in that case f is constant on $V \cap \mathbb{H}_{\text{Berk}}$. We know that $\mathbb{H}_{\text{Berk}}$ is dense in $\mathbb{P}^1_{\text{Berk}}$, f is continuous on U and $\overline{V} = V \cup \{w\} \subset U$. Therefore, f is constant on $\overline{V}$ with $f(x) = f(w)$.

If $D = \emptyset$, then U is either $\mathbb{P}^1_{\text{Berk}}$ or a connected component of $\mathbb{P}^1_{\text{Berk}}\backslash\{\zeta\}$ for some $\zeta \in \mathbb{P}^1_{\text{Berk}}$. We fix an element $w \in U$, and consider an arbitrary $x \in U$. There is a disc V containing x and w such that $\overline{V} \subset U$ because of the description of U. Then V has a unique boundary point, and we can prove the claim as we did it in the first case. □

Remark 4.2.7. Let f be a harmonic function on a domain U.

i) There are only finitely many tangent directions at every point $x \in U$ where f is nonconstant. This is a direct consequence of Proposition 4.2.3 and 4.2.6.

ii) The function f is locally constant outside the main dendrite for the weak topology which we have defined in Remark 4.2.7 by Proposition 4.2.6

We now give an example of a function f on a domain U which is harmonic but not strongly harmonic on U. By Corollary 4.2.5, our function must not be contained in $\text{BDV}(U)$.

Example 4.2.8. Let $K = \mathbb{C}_p$, and fix coordinates such that $\mathbb{P}^1_{\text{Berk}} = \mathbb{A}^1_{\text{Berk}} \cup \{\infty\}$. At first, we verify that the set $U := \mathbb{P}^1_{\text{Berk}}\backslash\mathbb{Z}_p$ is open by showing $\mathbb{Z}_p$ is closed. Since $\mathbb{P}^1_{\text{Berk}}$

is a Hausdorff space, it suffices to prove that $\mathbb{Z}_p$ is compact respective to the subspace topology of $\mathbb{P}^1_{\text{Berk}}$. Let $\mathbb{Z}_p = \bigcup_{i \in I} U_i$ for $U_i \subset \mathbb{Z}_p$ open in the Berkovich topology. As the Berkovich topology is the weakest topology on $\mathbb{Z}_p$ such that the map $\mathbb{Z}_p \to \mathbb{R}_{\geq 0}$ given by $x \mapsto |f(x)|_p$ is continuous for all $f \in \mathbb{C}_p[T]$ and polynomials are continuous in the $p-$adic topology of $\mathbb{Z}_p$, the sets U_i are also open in this finer topology. $\mathbb{Z}_p$ is compact in the $p-$adic topology, so there is a finite number of the sets U_i covering $\mathbb{Z}_p$. Thus, $\mathbb{Z}_p$ is compact in the Berkovich topology, too. Due to $\mathbb{Z}_p \subset \mathbb{C}_p = \mathbb{P}^1(K)$, U is also connected, and so U is a domain. By Proposition 4.2.6, it suffices to describe the function f on the main dendrite D of U. So we try to describe D such that we can define f properly on D. At first, we will show that D is a rooted $\mathbb{R}$-tree whose root is the Gauss point ζ_{Gauss}. As $|x|_p \leq 1$ for every point $x \in \mathbb{Z}_p$, we can see the main dendrite D as an $\mathbb{R}$-tree contained in $\mathcal{D}(0,1) \backslash D(0,1)$ which is an $\mathbb{R}$-tree respective to the metric

$$\rho(x,y) = 2\log_p(\text{diam}(x \vee y)) - \log_p(\text{diam}(x)) - \log_p(\text{diam}(y))$$

by Proposition 2.2.8. Since $0, 1 \in \mathbb{Z}_p = \partial U$ and $|1-0|_p = 1$, the first point where $[0, \zeta_{\text{Gauss}}]$ and $[1, \zeta_{\text{Gauss}}]$ meet is the Gauss point. Thus, ζ_{Gauss} has to be contained in D, and so ζ_{Gauss} is a root of D. Next, we determine all branches extending down from ζ_{Gauss}. Because each point $x \in \mathbb{Z}_p$ with $|x|_p < 1$ is contained in the same branch off ζ_{Gauss} as 0, it suffices to consider the points in $\mathbb{Z}_p^\times$. Let $x, y \in \mathbb{Z}_p^\times$, then we can write $x = \sum_{i=0}^\infty a_i p^i$ and $y = \sum_{i=0}^\infty b_i p^i$ where $a_i, b_i \in \{1, \ldots, p-1\}$. If $a_0 = b_0$, we have $|x-y|_p < 1$, i.e. x and y are on the same branch. If $a_0 \neq b_0$, then $|x-y|_p = 1$, and so they are on different ones. Hence, there are p different branches extending down from ζ_{Gauss}. Each other node of D is corresponding to the disc $D(a, p^{-n})$ for $a, n \in \mathbb{Z}$ with $n \geq 1$ and $0 \leq a \leq p^n - 1$. One can see ζ_{Gauss} as the case where $n = 0$, and so $a = 0$. Consider an arbitrary node $D(a, p^{-n})$. One can show that there are branches extending down from the node $D(a, p^{-n})$ to the nodes $D(a + k \cdot p, p^{-(n+1)})$ with $k \in \{0, \ldots, p-1\}$. Since $1/p^{n+1} < 1/p^n$ and

$$|a + k \cdot p^n - a|_p = |k \cdot p^n|_p = 1/p^n,$$

we have $D(a + k \cdot p, p^{-(n+1)}) \subsetneq D(a, p^{-n})$. Furthermore, two such nodes $D(a + k \cdot p, p^{-(n+1)})$ are on different branches since

$$|a + k \cdot p^n - (a + k' \cdot p^n)|_p = |k - k'|_p \cdot |p^n|_p = 1/p^n \geq 1/p^{n+1}$$

for $k, k' \in \{0, \ldots, p-1\}, k \neq k'$. Since we know that every node is of that form, there are clearly no other branches extending down off $D(a, p^{-n})$. Thus, there are p branches extending down from each node. Let x be the point corresponding to $D(a, p^{-n})$ and y

to $D(a + k \cdot p, p^{-(n+1)})$ with $k \in \{0, \ldots, p-1\}$. Then

$$\begin{aligned}
\rho(x, y) &= 2\log_p(\operatorname{diam}(x \vee y)) - \log_p(\operatorname{diam}(x)) - \log_p(\operatorname{diam}(y)) \\
&= 2\log_p(\operatorname{diam}(x)) - \log_p(\operatorname{diam}(x)) - \log_p(\operatorname{diam}(y)) \\
&= \log_p(\operatorname{diam}(x)) - \log_p(\operatorname{diam}(y)) \\
&= \log_p(p^{-n}) - \log_p(p^{-(n+1)}) \\
&= -n + n + 1 = 1,
\end{aligned}$$

i.e. that each edge has length 1. Now we are able to give a proper description of f on D. Set $f(\zeta_{\text{Gauss}}) = 0$ and define f recursively. Let z_a be a node on which $f(z_a)$ has been already defined. Let N_a denote the slope of f on the edge entering z_a from above, and if $z_a = \zeta_{\text{Gauss}}$, we put $N_a = 0$. We have seen above, that there are p edges extending down from z_a. We choose two distinguished edges, and let $f(z)$ have the slope $N_a + 1$ on one and -1 on the other one until the next node. On the remaining $p - 2$ edges, we set $f(z) = f(z_a)$ until the next node. By construction, f is continuous and locally piecewise linear. Furthermore, the sum of the slopes of f on the edges leading away from each node is 0, so f is harmonic on U (we can extend f from D to U properly by $f(x) := f(w)$ where w is the first point of $[x, \zeta_{\text{Gauss}}]$ in D for each $x \in U$).

However, f is not strongly harmonic on U. By the definition of f, there are edges of D with arbitrarily large slopes of f. Let Γ be an edge of $D \subset U$ with slope m_Γ, then $|\Delta_\Gamma(f)|(\Gamma) = 2|m_\Gamma|$. Hence, f cannot be contained in $\mathrm{BDV}(U)$, and so f cannot be strongly harmonic on U.

4.3 The Maximum Principle

In the classical theory, a harmonic function on a domain D in $\mathbb{C}$ does not achieve a maximum or a minimum within the domain ([Ra, Theorem 1.1.8]). This property is called the Maximum Principle. We will prove the analogue for harmonic functions on domains of $\mathbb{P}^1_{\text{Berk}}$ and give a reference in [Th] for the formulation in the case of an arbitrary smooth strictly k-analytic curve. Further we give a strengthening which is called the Strong Maximum Principle. Note that the formulation of the Strong Maximum Principle differs slightly from the one in [BR]. Afterwards, the Riemann Extension Theorem and the uniqueness of the Equilibrium measure are deduced from this strengthening.

Theorem 4.3.1 (Maximum Principle). *Let $U \subset \mathbb{P}^1_{\text{Berk}}$ be a domain and f a harmonic function on U.*

i) If f is nonconstant on U, f does not achieve a maximum or a minimum value on U.

ii) The inequality $\limsup_{x\to\partial U} f(x) \leq M$ *implies*

$$f(x) \leq M \textit{ for all } x \in U.$$

Respectively, if $\liminf_{x\to\partial U} f(x) \geq m$ *is satisfied, we have*

$$f(x) \geq m \textit{ for all } x \in U.$$

Proof. If f is harmonic, $-f$ is harmonic as well. Since

$$\min(f) = -\max(-f) \text{ and } \lim \inf_{x\to\partial U} f(x) = -\lim \sup_{x\to\partial U} (-f(x)),$$

it suffices to consider the case of a maximum in i) respectively $\limsup_{x\to\partial U} f(x)$ in ii).

We prove i) by contradiction, so suppose that f is achieving a maximum at a point $x \in U$. By definition, f is strongly harmonic on a subdomain V of U containing x. At first, we will show that f is constant on V, and subsequently we will conclude that f is constant on U. Without loss of generality, we may assume that the main dendrite D of V is non-empty because otherwise f is constant on V by Proposition 4.2.6. Let T be the branch off of D containing x, and let w be the point where T attaches to D. Then $w \in D$, and by Proposition 4.2.6 $f(w) = f(x)$. Thus, f is achieving the maximum in D. Let $\Gamma \subset D$ be a finite subgraph with w in its interior. Because of the definition of the main dendrite, we have the identity

$$r_{\overline{V},\Gamma}(\partial V) = \{z \in D|\ z \text{ endpoint of } \Gamma\} =: E. \tag{4.1}$$

If $z \in r_{\overline{V},\Gamma}(\partial V)$, there is a $y \in \partial V$ such that $r_{\overline{V},\Gamma}(y) = z$, i.e. the first point of the path $[y, w]$ in Γ is z, and so z is an endpoint of Γ. If $z \in E \subset D$, there are $y, v \in \partial V$ such that z is contained in the path $[y, v]$. Since $z \in E$, $r_{\overline{V},\Gamma}(y) = z$ or $r_{\overline{V},\Gamma}(v) = z$.

Since f is strongly harmonic on V, $\Delta_{\overline{V}}(f)$ is supported on ∂V. By Equation (4.1), we know that $r^{-1}_{\overline{V},\Gamma}(\{\Gamma\backslash E\}) \subset V$. Hence, $\Delta_\Gamma(f) = (r_{\overline{V},\Gamma})_*(\Delta_{\overline{V}}(f))$ implies

$$\operatorname{supp}(\Delta_\Gamma(f)) \subset E. \tag{4.2}$$

Since Γ is a finite subgraph, E is finite. Thus, $\Delta_\Gamma(f)$ is a discrete measure on Γ. By [BR, Corollary 3.9], $f|_\Gamma$ therefore belongs to CPA(Γ). We will show that Γ coincides with the connected component of $\{z \in \Gamma | f(z) = f(w)\}$ containing w which is denoted by Γ_w. Suppose $\Gamma_w \neq \Gamma$, then we can find a boundary point p of Γ_w in Γ which is not contained in E. This point $p \in \Gamma_w$ satisfies $f(p) = f(w) = f(x)$, i.e. $f(p)$ is maximal. Hence,

$$d_{\vec{v}} f(p) = \lim_{t\to 0} \frac{f(p + t\vec{v}) - f(p)}{t} \leq 0$$

for all tangent vectors $\vec{v} \in T_p(\Gamma)$. The point p is a boundary point of Γ_w, so f is nonconstant near p. We can find therefore a tangent vector $v \in T_p(\Gamma)$ such that

$d_{\vec{v}}f(p) < 0$ for our piecewise affine function f on Γ. Thus,

$$\Delta_\Gamma(f)(p) = -\sum_{\vec{v}\in T_p(\Gamma)} d_{\vec{v}}f(p) > 0.$$

By Equation (4.2), p has to be contained in E, what is not possible by the choice of p. So $\Gamma = \Gamma_w$. Because Γ can be taken arbitrary large, f is constant on D. By Proposition 4.2.6, f is constant on V.

With this result, we can conclude easily that f is also constant on U. We consider the set $W := \{z \in U | f(z) = f(x)\}$. This set is non-empty, because x is contained in it. Since f is continuous on U and $W = f^{-1}(f(x))$, this set is closed. W is also open, because for every $z_0 \in W$ we have seen above that there is an open neighborhood $V_{z_0} \subset W$ of z_0. We know that U is connected as a domain, and so the non-empty open and closed set W has to coincide with U, i.e. f is constant on U.

For ii), we consider the function $f^\sharp \colon \overline{U} \to \mathbb{R}$ defined by

$$f^\sharp(x) := \begin{cases} f(x) & \text{for } x \in U, \\ \limsup_{y\to x, y\in U} f(y) & \text{for } x \in \partial U. \end{cases}$$

Since f is continuous on U, the defined function $f^\sharp$ is upper semicontinuous by construction. $\mathbb{P}^1_{\mathrm{Berk}}$ is compact by Proposition 2.3.3 i), so $\overline{U}$ is compact. Therefore, the upper semicontinuous function $f^\sharp$ is achieving a maximal value in $\overline{U}$. By i), we know that this maximum has to be achieved on ∂U. Since we have required that $f^\sharp(x) = \limsup_{y\to x} f(y) \leq M$ for every $x \in \partial U$, $f(x) = f^\sharp(x) \leq M$ for all $x \in U$. □

Corollary 4.3.2. *If U is an open set and $f\colon U \to \mathbb{R}$ harmonic, then f achieves a local extremum in a point $x \in U$ if and only if f is locally constant in x.*

Proof. Let $V \subset U$ be a neighborhood of x such that f has an extremum in x on V. The connected component V_0 of V containing x is a domain, f is harmonic on V_0 and f achieves an extremum on V_0. Hence, f has to be constant on V_0 by the Maximum Principle. The other direction is obvious. □

Remark. If X is a smooth strictly k-analytic curve, then we have the same statement as in the corollary above in [Th, Proposition 3.1.1].

In the following, we see an important strengthening of the Maximum Principle. One can show that in some cases, sets of capacity 0 in ∂U can be ignored. Before we will state and prove the Stong Maximum Principle, we will define capacity and prove some lemmata.

Definition 4.3.3. Fix $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}$, and let e be a compact subset of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}$.

i) Let $\mathbb{P}(e)$ be the collection of all probability measures ν on $\mathbb{P}^1_{\text{Berk}}$ with $\text{supp}(\nu) \subset e$. For a given $\nu \in \mathbb{P}(e)$ we define the *energy integral*

$$I_\zeta(\nu) := \int\int_{e\times e} -\log_v \delta(x,y)_\zeta d\nu(x)d\nu(y),$$

where $\delta(x,y)_\zeta$ is the generalized Hsia kernel which was defined in Definition 3.2.7.

ii) We call

$$V_\zeta(e) := \inf_{\nu\in\mathbb{P}(e)} I_\zeta(\nu)$$

the *Robin constant.*

iii) The *logarithmic capacity* of e relative to ζ is defined by

$$\gamma_\zeta(e) := q_v^{-V_\zeta(e)}.$$

For example, if $K = \mathbb{C}_p$ we can take $q_v = p$. If $H \subset \mathbb{P}^1_{\text{Berk}}$ is an arbitrary set, we define the logarithmic capacity as

$$\gamma_\zeta(H) := \sup_{e\subset H \textit{ compact}} \gamma_\zeta(e).$$

iv) We call a probability measure μ supported on e with $I_\zeta(\mu) = V_\zeta(e)$ *Equilibrium measure* for e with respect to ζ. If $\gamma_\zeta(e) > 0$, [BR, Proposition 6.6] states the existence of such a probability measure μ. Later on, we will give a proof of the uniqueness in Corollary 4.3.10.

Remark. The capacity of a set e with respect to a $\zeta \in \mathbb{P}^1_{\text{Berk}}\backslash e$ is 0 if and only if the capacity of a set e is 0 to any $\zeta \in \mathbb{P}^1_{\text{Berk}}\backslash e$ (cf. [BR, Proposition 6.1]). Hence, in the following we will just say that a set has capacity 0 if $\gamma_\zeta(e) = 0$ for any $\zeta \in \mathbb{P}^1_{\text{Berk}}\backslash e$.

Lemma 4.3.4. *Let* $e := \{a_1, \ldots, a_n\} \subset \mathbb{P}^1(K)$*, then* e *has capacity* 0*.*

Proof. By the definition of $\mathbb{P}(e)$, every measure $\mu \in \mathbb{P}(e)$ is supported on $e = \{a_1, \ldots, a_m\}$. Hence, we can write $\mu = \sum_{i=1}^m c_i\delta_{a_i}$ for $c_i \in \mathbb{R}$. For any $\zeta \in \mathbb{P}^1_{\text{Berk}}\backslash\{a_1, \ldots, a_m\}$ we have

$$I_\zeta(\mu) = \sum_{i=1}^m -c_i^2 \cdot \log_v(\delta(a_i,a_i)_\zeta) = \infty$$

since $\delta(a_i,a_i)_\zeta = 0$ by Proposition 3.2.8. Thus, $V_\zeta(e) = \infty$, and finally $\gamma_\zeta(e) = 0$. □

Lemma 4.3.5. *If* e *has capacity* 0*, then* e *is contained in* $\mathbb{P}^1(K)$*.*

Proof. Suppose there is an element $a \in e \cap \mathbb{H}_{\text{Berk}}$. Then the Dirac measure δ_a is a measure in $\mathbb{P}(e)$, and

$$I_\zeta(\delta_a) = -\log_v \delta(a,a)_\zeta = -\log_v(\text{diam}_\zeta(a)) < \infty$$

for any $\zeta \notin e$. Thus, $V_\zeta(e) < \infty$, i.e. $\gamma_\zeta(e) > 0$, for any $\zeta \notin e$, contradicting that e has capacity 0. Consequently, every point of e has to be of type I. □

We need the following lemma to prove a strengthening of the Maximum Principle.

Lemma 4.3.6. *Let $e \subset \mathbb{P}^1_{\text{Berk}}$ be a compact set of capacity 0 and $\zeta \notin e$. Then there is a $\nu \in \mathbb{P}(e)$ such that*

$$\lim_{x \to y} u_\nu(x, \zeta) = \infty$$

for all $y \in e$. A function with this property is called an Evans function.

Proof. See [BR, Lemma 7.18]. □

Theorem 4.3.7 (Strong Maximum Principle)**.** *Let $U \subset \mathbb{P}^1_{\text{Berk}}$ be a domain and f a harmonic function on U.*

i) If f is bounded above on U, and $\limsup_{x\to z} f(x) \leq M$ is satisfied for all $z \in \partial U \backslash e$, where $e \subsetneq \partial U$ is of capacity 0, then

$$f(x) \leq M \text{ for all } x \in U.$$

ii) If f is bounded below on U, and $\liminf_{x\to z} f(x) \geq m$ is satisfied for all $z \in \partial U \backslash e$, where $e \subsetneq \partial U$ is of capacity 0, then

$$f(x) \geq m \text{ for all } x \in U.$$

Proof. As in the proof of Theorem 4.3.1, it suffices to deal with the claim in i). If f is constant with $f \equiv c$ on U and there is at least one $z \in \partial U$ such that $\limsup_{x\to z} f(x) \leq M$, we get

$$f(y) = c = \limsup_{x \to z} f(x) \leq M$$

for each $y \in U$. So we may assume that f is nonconstant on U. We will show the claim by contradiction. Suppose that there is a function f as required and there exists an element $x_0 \in U$ such that $f(x_0) > M$.

Since f is nonconstant, there is a $\zeta \in U$ such that $f(\zeta) \neq f(x_0)$. If $f(\zeta) > f(x_0)$, then we just interchange x_0 and ζ. Hence, the domain U contains two points x_0 and ζ satisfying $f(\zeta) < f(x_0)$ and $M < f(x_0)$. Therefore, we can fix a $M_1 > M$ such that

$$f(x_0) > M_1 > f(\zeta).$$

In the next step, we will construct a suitable compact set e_1 to which Lemma 4.3.6 can be applied. We define

$$W := \{x \in U | \; f(x) > M_1\} \text{ and } W' := \{x \in U | \; f(x) < M_1\}.$$

Since f is continuous, the sets $W = f^{-1}((M_1, \infty))$ and $W' = f^{-1}((-\infty, M_1))$ are open. By definition, one can see that $\zeta \in W'$, $x_0 \in W$ and $W' \cap \overline{W} = \emptyset$. Let V be the connected component of W containing x_0. Then V is open and connected, i.e. $V \subset U$ is a domain.

We will show that $e_1 := \partial V \cap \partial U \neq \emptyset$. Suppose that the intersection is empty, i.e. $\overline{V} \subset U$. We therefore can find for each $y \in \partial V$ a neighborhood $U_y \subset U$ of y with $U_y \cap V \neq \emptyset$ and $U_y \cap U \backslash V \neq \emptyset$. We can actually find a neighborhood, for example a connected one, such that $U_y \cap V \neq \emptyset$ and $U_y \cap U \backslash W \neq \emptyset$, because V is a connected component of W. The points contained in $U_y \cap V$ satisfy $f(z) > M_1$ and the points $U_y \cap U \backslash W$ satisfy $f(z) \leq M_1$. Since f is continuous on U, $f(y) = M_1$. Hence, $\limsup_{x \to \partial V} f(x) = M_1$. The Maximum Principle implies $f(x) \leq M_1$ for all $x \in V$. This contradicts our supposition $f(x_0) > M_1$, because $x_0 \in V$ by the construction of V. Consequently, $e_1 = \partial V \cap \partial U \neq \emptyset$.

Next, we verify that e_1 has capacity 0. The closed subset e_1 is compact. Further, every $z \in e_1 = \partial V \cap \partial U$ is clearly contained in the boundary of U and satisfies $\limsup_{x \to z} f(x) \geq M_1 > M$. Hence, e_1 has to be a subset of e. By the definition of capacity,

$$\gamma_\zeta(e_1) \leq \gamma_\zeta(e) = 0.$$

Additionally, $\zeta \notin e_1 \subset \partial U$, because $\zeta \in U$, so we can apply Lemma 4.3.6 to e_1 and ζ. Lemma 4.3.6 states the existence of an Evans function h for e_1 with respect to ζ, or more specifically there is a probability measure ν such that $\mathrm{supp}(\nu) \subset e_1$ and for all $y \in e_1$

$$\lim_{x \to y} h(x) = \infty, \tag{4.3}$$

where $h(x) := u_\nu(x, \zeta)$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}}$.

Now we will define a harmonic function with the help of h on V such that we can apply the Maximum Principle 4.3.1. Then we will get a contradiction to our supposition $f(x_0) > M$. First, we show that h is harmonic on V. We know that $\zeta \notin V$. Furthermore, $V \cap \mathrm{supp}(\nu)$ is empty, because $\mathrm{supp}(\nu) \cap V \subset e_1 \cap V = \emptyset$. Therefore, $V \subset \mathbb{P}^1_{\mathrm{Berk}} \backslash (\mathrm{supp}(\nu) \cup \{\zeta\})$, and so h is harmonic on V by Example 4.1.6. [BR, Proposition 6.12] tells us that $h(x) := u_\nu(x, \zeta)$ is lower semicontinuous on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}$, and so especially on $\overline{V}$ which does not contain ζ. Since $\overline{V}$ is compact, h is bounded below, so there is a constant $B > 0$ such that

$$h(x) \geq -B \tag{4.4}$$

for all $x \in \overline{V}$. For $\eta > 0$ we define the function $f_\eta(x) := f(x) - \eta h(x)$. Since f and h are harmonic on V, each f_η is harmonic on V as well. We have required that f is bounded above on U, so (4.3) implies

$$\limsup_{x\to y} f_\eta(x) = \limsup_{x\to y}(f(x) - \eta h(x)) = -\infty \tag{4.5}$$

for all $y \in e_1$. Our function f is continuous in y and satisfies $f(y) = M_1$ for each $y \in \partial V \cap U$. Thus, we have the inequality

$$\limsup_{x\to y} f_\eta(x) = \limsup_{x\to y}(f(x) - \eta h(x)) \leq M_1 + \eta B \tag{4.6}$$

for each $y \in \partial V \cap U$ by Equation (4.4). Because of the disjoint union $\partial V = e_1 \dot\cup (\partial V \cap U)$, (4.5) and (4.6) state that

$$\limsup_{x\to y} f_\eta \leq M_1 + \eta B$$

for all $y \in \partial V$. Since f_η is harmonic on V, the Maximum Principle says that

$$f_\eta(x) \leq M_1 + \eta B$$

for all $x \in V$. Consequently, we get the following inequality

$$\begin{aligned} f(x) &= f_\eta(x) + \eta h(x) \\ &\leq M_1 + \eta B + \eta h(x) \\ &= M_1 + \eta(B + h(x)) \end{aligned}$$

on V. Letting $\eta \to 0$, we have $f(x) \leq M_1$ for all $x \in V$. This contradicts our supposition $f(x_0) > M_1$, because $x_0 \in V$ by the definition of V. Hence, $f(x) \leq M$ for all $x \in U$. □

Two nice consequences of the Strong Maximum Principle are the Riemann Extension Theorem and the uniqueness of the Equilibrium measure.

Corollary 4.3.8 (Riemann Extension Theorem)**.** *Let U be a domain and $e \subset U$ be a compact set of capacity 0. Then every bounded harmonic function $f\colon U\backslash e \to \mathbb{R}$ can be extended uniquely to a harmonic function on U.*

Proof. Since $\mathbb{P}^1_{\text{Berk}}$ is a Hausdorff space, the compact subset e is closed, and so $U\backslash e$ is open. We have seen in Lemma 4.3.5, that having capacity 0 implies $e \subset \mathbb{P}^1(K)$. By definition, U is connected as a domain. Therefore, $U\backslash e$ is connected as well, i.e. $U\backslash e$ is actually a domain. To extend $f\colon U\backslash e \to \mathbb{R}$ properly, we will show that for each $a \in e$ there is a neighborhood of a in U on which f is constant. We consider an arbitrary point $a \in e \subset \mathbb{P}^1(K)$. Since a is a point of type I and U is open, we can find a $r \in \mathbb{R}_{>0}$ such that $\mathcal{D}(a, r) \subset U$. Then the ball $\mathcal{B} := \mathcal{D}(a, r)^-$ is open and connected, $\overline{\mathcal{B}} = \mathcal{D}(a, r)$, and $\mathcal{B}$ has a unique boundary point z in $\mathbb{H}_{\text{Berk}}$. More precisely, z is the point in $\mathbb{P}^1_{\text{Berk}}$ corresponding to the disc $D(a, r)$.

We consider the set $V := \mathcal{B}\backslash e$. Then

$$\partial V \cap \mathbb{P}^1_{\mathrm{Berk}}\backslash e = \partial\mathcal{B} = \{z\}. \tag{4.7}$$

Our strategy is to apply Theorem 4.3.7 to $f|_V$. Note that V is a domain, by the same reasons that $U\backslash e$ is a domain. Additionally, f is harmonic and bounded on $V = \mathcal{B}\backslash e \subset U\backslash e$, because we have required that for f on $U\backslash e$. In particular, f is continuous in $z \in U\backslash e$. Thus,

$$\lim_{x\to z, x\in V} f(x) = f(z). \tag{4.8}$$

Set $e' := e \cap \partial V$, then e' is also a compact set of capacity 0. By Equation (4.7), $\partial V\backslash e' = \{z\}$. Equation (4.8) and the Strong Maximum Principle (Theorem 4.3.7 i) and ii)) imply $f(x) = f(z)$ for all $x \in V$. By setting $f(x) = f(z)$ for all $x \in e \cap \mathcal{B}$, we have $f(x) = f(a)$ for all $x \in \overline{\mathcal{B}}$. Since such two balls are either disjoint, or they coincide (cf. Lemma 2.3.1), f is well-defined on the domain U.

By Example 4.1.3, f is strongly harmonic on $\mathcal{B}$ as a constant function on $\mathcal{B}$. We have required that f is harmonic on $U\backslash e$, so f is harmonic on $U = \mathcal{B} \cup U\backslash e$.

By the construction of the extension, one can see that it has to be unique, but we also can verify that explicitly. Let h be a harmonic function on U such that $h \equiv f$ on $U\backslash e$. If $a \in e$, we have seen above that there is a $r \in \mathbb{R}_{>0}$ such that for $\mathcal{B} := \mathcal{D}(a,r)^-$ we have $\overline{\mathcal{B}} \subset U$. Since $h-f$ is harmonic on U, $h-f$ is also harmonic on the domain $\mathcal{B} \subset U$ which has only one boundary point. Hence, the main dendrite of $\mathcal{B}$ is empty, and so the harmonic function $h-f$ is constant on $\mathcal{B}$ by Proposition 4.2.6. Due to $e \subset \mathbb{P}^1(K)$, the set $\mathcal{B}\backslash e$ cannot be empty which means that $(h-f)(a) = 0$. Thus, $h \equiv f$ on U. □

Corollary 4.3.9. *Let $\{a_1, \ldots, a_m\} \subset \mathbb{P}^1(K)$. Then every bounded harmonic function on $\mathbb{P}^1_{\mathrm{Berk}}\backslash\{a_1, \ldots, a_m\}$, or on $\mathcal{B}(a,r)^-_\zeta\backslash\{a_1, \ldots, a_m\}$ for some open ball $\mathcal{B}(a,r)^-_\zeta$, is constant.*

Proof. Let U be $\mathbb{P}^1_{\mathrm{Berk}}$ or an open ball $\mathcal{B}(a,r)^-_\zeta$ and f a bounded harmonic function on $U\backslash\{a_1, \ldots, a_m\}$. Clearly, the set $e := \{a_1, \ldots, a_m\}$ is closed, and so e is compact. By Lemma 4.3.4, e has capacity 0. So we can extend the function f to a harmonic function on U by the Riemann Extension Theorem (Corollary 4.3.8). Since $|\partial U| \leq 1$, the main dendrite D of U is empty in both cases. By Proposition 4.2.6, the only harmonic function on U are the constant ones. In particular, f is constant on $\mathbb{P}^1_{\mathrm{Berk}}\backslash\{a_1, \ldots, a_m\}$, respectively on $\mathcal{B}(a,r)^-_\zeta\backslash\{a_1, \ldots, a_m\}$. □

By the previous results, we can show that the Equilibrium measure, which we have defined in 4.3.3 iv) is unique.

Corollary 4.3.10. *Let $E \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a compact set with positive capacity, and let $\zeta \in \mathbb{P}^1_{\mathrm{Berk}}\backslash E$. Then the Equilibrium measure μ_ζ of E with respect to ζ is unique.*

Proof. Suppose μ_1 and μ_2 are two Equilibrium measures for E with respect to ζ, i.e.

$$I_\zeta(\mu_1) = I_\zeta(\mu_2) = V_\zeta(E) < \infty.$$

Since μ_i is supported on E, the potential function $u_i(x) := u_{\mu_i}(x, \zeta)$ for $i = 1, 2$ is well-defined on $\mathbb{P}^1_{\text{Berk}}$, continuous on $\mathbb{P}^1_{\text{Berk}} \backslash \text{supp}(\mu_i)$, and achieves its minimum at $x = \zeta$ by [BR, Proposition 6.12]. Furthermore, the Frostman's Theorem [BR, Theorem 6.18] states that $u_i(z) \leq V_\zeta(E) < \infty$ for all $z \in \mathbb{P}^1_{\text{Berk}} \backslash \{\zeta\}$, and so u_i is bounded above by $V_\zeta(E)$ for all $z \in \mathbb{P}^1_{\text{Berk}}$. Additionally, there is a F_σ set $f_i \subset E$ of capacity zero such that $u_i(z) = V_\zeta(E)$ for all $z \in E \backslash f_i$ and u_i is continuous on $E \backslash f_i$. We have seen in Example 3.2.11 that

$$u_i \in \text{BDV}(\mathbb{P}^1_{\text{Berk}}) \text{ and } \Delta_{\mathbb{P}^1_{\text{Berk}}}(u_i) = \mu_i - \delta_\zeta. \tag{4.9}$$

Let U be the connected component of $\mathbb{P}^1_{\text{Berk}} \backslash E$ containing ζ. Then U is open and connected, and so a domain. By [BR, Proposition 6.8], the measures μ_i are supported on $\partial U \subset \overline{U} \cap E$. Hence, $\mu_i \in \mathbb{P}(\partial U)$, where ∂U is compact because it is closed. Due to $\zeta \notin U$, we can consider $V_\zeta(\partial U)$, and

$$V_\zeta(\partial U) = \inf_{\nu \in \mathbb{P}(\partial U)} I_\zeta(\nu) \leq I_\zeta(\mu_i) < \infty.$$

Thus, ∂U has positive capacity.

We consider two cases. First, we will assume that $\zeta \in \mathbb{H}_{\text{Berk}}$, and afterwards $\zeta \in \mathbb{P}^1(K)$. So let $\zeta \in \mathbb{H}_{\text{Berk}}$, then the functions u_i are bounded below by [BR, Proposition 6.12]. We have already seen that they are bounded above as well. Hence, $u \colon \mathbb{P}^1_{\text{Berk}} \to \mathbb{R}$ with $u(x) := u_1(x) - u_2(x)$ is well-defined and bounded. Furthermore, u is continuous on U, because the potential functions u_i are continuous on $\mathbb{P}^1_{\text{Berk}} \backslash (\text{supp}(\mu_i))$ for $i = 1, 2$, and U is contained in $\mathbb{P}^1_{\text{Berk}} \backslash (\text{supp}(\mu_1) \cup \text{supp}(\mu_2))$. By Equation (4.9), we know that $u \in \text{BDV}(\mathbb{P}^1_{\text{Berk}})$, and so $u \in \text{BDV}(U)$, and

$$\Delta_{\mathbb{P}^1_{\text{Berk}}}(u) = \Delta_{\mathbb{P}^1_{\text{Berk}}}(u_1) - \Delta_{\mathbb{P}^1_{\text{Berk}}}(u_2) = (\mu_1 - \delta_\zeta) - (\mu_2 - \delta_\zeta) = \mu_1 - \mu_2.$$

Since $\text{supp}(\mu_i) \subset \partial U \subset \overline{U}$ by [BR, Proposition 6.8] and the retraction map $r_{\mathbb{P}^1_{\text{Berk}}, \overline{U}}$ fixes $\overline{U}$, it follows that

$$\Delta_{\overline{U}}(u) = (r_{\mathbb{P}^1_{\text{Berk}}, \overline{U}})_*(\Delta_{\mathbb{P}^1_{\text{Berk}}}(u)) = \Delta_{\mathbb{P}^1_{\text{Berk}}}(u) = \mu_1 - \mu_2 \tag{4.10}$$

by Proposition 3.1.29.

Consequently, it remains to show $\Delta_{\overline{U}}(u) = 0$ to prove $\mu_1 = \mu_2$.

To do that, we will apply the Strong Maximum Principle. We have already mentioned that u is continuous on U and contained in $\text{BDV}(U)$, so u is strongly harmonic on U since $\text{supp}(\Delta_{\overline{U}}(u)) = \text{supp}(\mu_1 - \mu_2) \subset \partial U$. Furthermore, we know that u is bounded on U. Let $f := f_1 \cup f_2$, then f has capacity 0, because

$$\gamma_\zeta(f) = \gamma_\zeta(f_1 \cup f_2) = \gamma_\zeta(f_1) = 0$$

by [BR, Corollary 6.21]. Since ∂U has positive capacity, $\partial U \backslash f$ cannot be empty. Consider an element $z \in \partial U \backslash f \subset E \backslash f \subset E \backslash f_i$, then u is continuous in z, and

$$u(z) = u_1(z) - u_2(z) = V_\zeta(E) - V_\zeta(E) = 0$$

by the Frostman's Theorem. Hence, $\lim_{x \to z,\ x \in U} u(x) = 0$. We can apply the Strong Maximum Principle (Theorem 4.3.7) to u two times, and we get that $u \equiv 0$ on U. Thus, $\Delta_{\overline{U}}(u) = 0$ by [BR, Lemma 5.24].

Now, let $\zeta \in \mathbb{P}^1(K)$. Since $u_i(\zeta) = -\infty$ for $i = 1, 2$, $u(\zeta) = u_1(\zeta) - u_2(\zeta) = -\infty + \infty$ is undefined. Hence, we consider the function $u\colon \mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\} \to \mathbb{R}$ with $u(x) = u_1(x) - u_2(x)$. By [BR, Proposition 6.12] and [BR, Proposition 6.18], u_i is lower semicontinuous on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}$, so particularly $u_i(x) \neq -\infty$ for all $x \in \mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}$, and u_i is bounded above on $\mathbb{P}^1_{\mathrm{Berk}}$. Hence, u is well-defined on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}$. Since U is a domain and ζ is of type I, $U \backslash \{\zeta\}$ is a domain as well. Again, we try to apply the Strong Maximum Principle. Differently to the first case, we will apply it to u defined on the domain $U \backslash \{\zeta\}$. The function u is continuous on $U \backslash \{\zeta\}$, because the potential functions u_i are continuous on $\mathbb{P}^1_{\mathrm{Berk}} \backslash \mathrm{supp}(\mu_i)$ and $U \backslash \{\zeta\}$ is contained in $\mathbb{P}^1_{\mathrm{Berk}} \backslash (\mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2))$. By [BR, Proposition 6.12], there exists an open neighborhood V of ζ such that $u_i(z) = \log_v(\|z, \zeta\|)$ for $i = 1, 2$. Thus, $u \equiv 0$ on $V \backslash \{\zeta\}$. Since $\mathbb{P}^1_{\mathrm{Berk}} \backslash V$ is compact, the lower semicontinuous functions u_i are bounded below on $\mathbb{P}^1_{\mathrm{Berk}} \backslash V$. By Frostman's Theorem, the functions u_i are bounded above on $\mathbb{P}^1_{\mathrm{Berk}} \backslash V \subset \mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}$, and so u is bounded on $\mathbb{P}^1_{\mathrm{Berk}} \backslash V$. Thus, u is bounded on $U \backslash \{\zeta\}$.

By Equation (4.9), $u \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\})$, and so particularly $u \in \mathrm{BDV}(U \backslash \{\zeta\})$. Further,

$$\begin{aligned}
\Delta_{\overline{\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}}}(u) &= \Delta_{\overline{\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}}}(u_1) - \Delta_{\overline{\mathbb{P}^1_{\mathrm{Berk}} \backslash \{\zeta\}}}(u_2) \\
&= \Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(u_1) - \Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(u_2) \\
&= \mu_1 - \mu_2.
\end{aligned}$$

By the same arguments as in the first case and $\overline{U \backslash \{\zeta\}} = \overline{U}$, we have

$$\Delta_{\overline{U \backslash \{\zeta\}}}(u) = \Delta_{\overline{U}}(u) = \mu_1 - \mu_2,$$

and $\mathrm{supp}(\Delta_{\overline{U \backslash \{\zeta\}}}(u)) \subset \partial U$. Thus, u is strongly harmonic on $U \backslash \{\zeta\}$. Again, we verify $\mu_1 = \mu_2$ by showing $\Delta_{\overline{U}}(u) = 0$.

Consider an element $z \in \partial U \backslash f$, which exists by the same reasons as above. We know that u is continuous in z and $u(z) = 0$, and so $\lim_{x\to z,\ x\in U} u(x) = 0$. Additionally,

$$\begin{aligned}\lim_{x\to\zeta,\ x\in U} u(x) &= \lim_{x\to\zeta,\ x\in U} (u_1(x) - u_2(x)) \\ &= \lim_{x\to\zeta,\ x\in U} (u_1(x) - \log_v(\|x,\zeta\|) + \log_v(\|x,\zeta\|) - u_2(x)) \\ &= 0\end{aligned}$$

by [BR, Proposition 6.12]. Together, we have

$$\lim_{x\to z,\ x\in U} u(x) = 0$$

for each $z \in \partial(U\backslash\{\zeta\})\backslash f = \partial U\backslash f \cup \{\zeta\}$. Applying the Strong Maximum Principle to u on $U\backslash\{\zeta\}$ and the exceptional set $f \subset \partial(U\backslash\{\zeta\})$ of capacity 0, $u = 0$ on $U\backslash\{\zeta\}$. Thus,

$$0 = \Delta_{\overline{U\backslash\{\zeta\}}}(u) = \Delta_{\overline{U}}(u) = \mu_1 - \mu_2$$

by [BR, Lemma 5.24]. Hence, $\mu_1 = \mu_2$ is also true in the second case. □

4.4 Poisson Formula and the Dirichlet and the Neumann Problem

Let D be a domain in $\mathbb{C}$ and $\phi\colon \partial D \to \mathbb{R}$ a continuous function, then the *Dirichlet problem* (cf. [Ra, Definition 1.2.1]) is to find a harmonic function h on D such that

$$\lim_{z\to\zeta} h(z) = \phi(\zeta)$$

for each $\zeta \in \partial D$. The Dirichlet problem can be uniquely solved if D is an open disc with the help of the *Poisson formula* (cf. [Ra, Theorem 1.2.2] and [Ra, Theorem 1.2.4]). The Poisson formula in the classical theory says, if a function f is harmonic on an open disc $\mathcal{D} = \{z \in \mathbb{C}|\ |z - z_0| > r\} \subset \mathbb{C}$ of radius r and centered in z_0, and can be extended continuously to the closure of this disc $\overline{\mathcal{D}}$, then for any $z \in \overline{D}$ the value $f(z)$ can be recaptured only from knowledge of f on $\partial\mathcal{D}$ (cf. [Ra, Corollary 1.2.6]). The Dirichlet problem is not generally solvable for domains in $\mathbb{C}$ (cf. [Ra, §4.1 p.85]), but if D is a simply connected domain in the Riemann sphere $\mathbb{C}_\infty$ such that $\mathbb{C}_\infty\backslash D$ contains at least two points there exists a unique solution (cf. [Ra, Corollary 4.1.8] and [Ra, Theorem 4.2.1]). In this section, we like to generalize the Poisson formula for a special class of domains in $\mathbb{P}^1_{\text{Berk}}$ and show that the Dirichlet problem is uniquely solvable on these domains as well. Furthermore, we will formulate the Neumann problem for these domains, and we will see that the solvability is a consequence of the unique solution of the Dirichlet problem and the Poisson formula.

If $U \subset \mathbb{P}^1_{\text{Bek}}$ is a domain with $\partial U = \{x_1, \ldots, x_m\}$, then we have the two problems which we have mentioned above:

Dirichlet Problem. Given $A_1, \dots, A_m \in \mathbb{R}$, there is a continuous function $f\colon \overline{U} \to \mathbb{R}$ which is harmonic on U and satisfies

$$f(x_i) = A_i$$

for all $i = 1, \dots, m$.

Neumann Problem. For given $c_1, \dots, c_m \in \mathbb{R}$ with $\sum_{i=1}^m c_i = 0$, there exists a continuous function $f\colon \overline{U} \to \mathbb{R}$ which is harmonic on U and

$$\Delta_{\partial U}(f) = \Delta_{\overline{U}}(f) = \sum_{i=1}^m c_i \delta_{x_i}.$$

4.4.1. Clearly, both problems are uniquely solvable for the domain $U = \mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$ where
$x \in \mathbb{P}^1_{\mathrm{Berk}}$. But these problems are not solvable for any domain $U \subset \mathbb{P}^1_{\mathrm{Bek}}$ with $\partial U = \{x_1, \dots, x_m\}$. As an example, consider the domain $U = \mathbb{P}^1_{\mathrm{Bek}} \backslash \{a_1, \dots, a_m\}$ where $\{a_1, \dots, a_m\} \subset \mathbb{P}^1(K)$ and $m \geq 2$. Assume that there is a continuous function $f\colon \overline{U} \to \mathbb{R}$ which is harmonic on U. Since f is bounded by the Maximum Principle, f is constant on U by Corollary 4.3.9, and so constant on $\overline{U}$. Thus, there is no solution for the Dirichlet problem if $A_1, \dots, A_m$ are different and no solution for the Neumann problem if not all $c_1, \dots, c_m$ are equal to zero. If $\partial U = \{x_1, \dots, x_m\} \subset \mathbb{H}_{\mathrm{Berk}}$, we will see that both problems are solvable and the solution is given by the analogue of the Poisson formula.

Definition 4.4.2. We call a domain U a *finite-dendrite domain*, if U

i) is either a connected component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x\}$ for some $x \in \mathbb{H}_{\mathrm{Berk}}$, or
ii) is of the form $U = r_\Gamma^{-1}(\Gamma^0)$ for some finite subgraph $\Gamma \subset \mathbb{H}_{\mathrm{Berk}}$, where $\Gamma^0 := \Gamma \backslash \partial \Gamma$.

Remark. i) In the first case, the unique boundary point of U is x, and hence the main dendrite of U is empty. In the second case the main dendrite coincides with Γ^0. If

$$\Gamma = \bigcup_{i,j \in \{1,\dots,m\}} [x_i, x_j]$$

for a finite set of points $\{x_1, \dots, x_m\} \subset \mathbb{H}_{\mathrm{Berk}}$ and $U := r_\Gamma^{-1}(\Gamma^0)$. We have $\partial U = \{x_1, \dots, x_m\}$.

ii) On the other side, let U be a domain with $\partial U = \{x_1, \dots, x_m\} \subset \mathbb{H}_{\mathrm{Berk}}$. If $m = 1$, U is a connected component of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x_1\}$. If $m \geq 2$ and

$$\Gamma := \bigcup_{i,j \in \{1,\dots,m\}} [x_i, x_j],$$

then Γ is the finite subgraph such that $U = r_\Gamma^{-1}(\Gamma^0)$, where $\Gamma^0 := \Gamma\backslash\{x_1, \ldots, x_m\}$ is the main dendrite of U.

iii) By [BR, Lemma 2.28] a domain U is a *simple domain* if and only if U is an open Berkovich disc or $U = r_\Gamma^{-1}(\Gamma^0)$ for a non-trivial subgraph $\Gamma \subset \mathbb{H}_{\mathrm{Berk}}$ with endpoints of type II or III. Thus, the class of finite-dendrite domains contains the class of simple domains, which are regarded as the basic open neighborhoods in $\mathbb{P}^1_{\mathrm{Berk}}$.

In the following we consider a finite-dendrite domain V with $\partial V = \{x_1, \ldots, x_m\} \subset \mathbb{H}_{\mathrm{Berk}}$. Before we state the Poisson formula, it is shown that a harmonic function on V can be written as a piecewise linear function on a finite subgraph composed with the corresponding retraction map if $|\partial V| \geq 2$. This description is used in Chapter 5 to verify that Thuillier's definition of harmonic functions extends the one made in this chapter.

Proposition 4.4.3. *Let V be a finite-dendrite domain in $\mathbb{P}^1_{\mathrm{Berk}}$ with boundary points $x_1, \ldots, x_m \in \mathbb{H}_{\mathrm{Berk}}$. Then each harmonic function f on V belongs to* $\mathrm{BDV}(V)$ *and has a continuous extension $f\colon \overline{V} \to \mathbb{R}$.*

If $|\partial V| \geq 2$, then

$$f = \widetilde{f} \circ r_\Gamma$$

for a function $\widetilde{f} \in \mathrm{CPA}(\Gamma)$ and a finite subgraph Γ of $\mathbb{P}^1_{\mathrm{Berk}}$.

Proof. We consider at first the case $m = 1$. If V has only one boundary point, the main dendrite is empty and every harmonic function on V is constant by Proposition 4.2.6. We have seen in Example 4.1.3, that $f \in \mathrm{BDV}(V)$. Clearly, we can extend f to a continuous real-valued function on $\overline{V}$.

If $m \geq 2$, the main dendrite of V is the interior of the finite subgraph $\Gamma := \bigcup[x_i, x_j]$ which we denote by Γ^0. We know by Proposition 4.2.6 that f is constant on each branch off Γ^0. Hence, it suffices to show that the restriction of f to each edge of Γ^0 is affine. If $\widetilde{\Gamma}$ is a finite subgraph of V contained in Γ^0, we can find a subdomain U of V such that U is a finite-dendrite domain with $\overline{U} \subset V$ and $\widetilde{\Gamma}$ is contained in U. Lemma 4.1.7 says that f is strongly harmonic on U, and so f is piecewise affine on $\widetilde{\Gamma}$ by Lemma 4.1.9. Moreover, Lemma 4.1.9 implies

$$-\sum_{\vec{v} \in T_p(\Gamma)} d_{\vec{v}} f(p) = 0 \tag{4.11}$$

for all $p \in \Gamma^0$. Let S be a vertex set of Γ, i.e. S contains all endpoints $x_1, \ldots, x_m$ and all branch points of Γ. Let e be an edge of $\Gamma\backslash S$. If e is an edge between two branch points of Γ, we have seen above that f is piecewise affine on e. Since $|T_p(\Gamma)| = 2$ for each $p \in \Gamma\backslash S$, f has to be affine on e by Equation (4.11). We can choose points $y_1, \ldots, y_m$ closer and closer to the endpoints $x_1, \ldots, x_m$, and so the continuous function f has to be also affine on each edge $e \subset \Gamma^0$ between an endpoint and a branchpoint by the same reasons as above. Thus, the restriction of f to each edge of Γ^0 is affine, and so we can extend f continuously to $\overline{V}$. In particular, there is a $\widetilde{f} \in \mathrm{CPA}(\Gamma)$ such that

$$f = \widetilde{f} \circ r_\Gamma .$$

Due to $\mathrm{CPA}(\Gamma) \subset \mathrm{BDV}(\Gamma)$, we have seen in Example 3.2.2 that $f \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$, and so $f \in \mathrm{BDV}(V)$ by Lemma 3.1.29. □

Corollary 4.4.4. *Every harmonic function on a finite-dendrite domain V is strongly harmonic on V.*

Proof. If f is harmonic on V, then f belongs to $\mathrm{BDV}(V)$ by the previous proposition. Corollary 4.2.5 implies that f is strongly harmonic on V. □

Definition 4.4.5. Let V be a finite-dendrite domain with $\partial V = \{x_1, \dots, x_m\}$. For any $z \in \mathbb{P}^1_{\mathrm{Berk}}$, we define the real $(m+1) \times (m+1)$ matrix $M(z)$ as

$$M(z) := \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & -\log_v(\delta(x_1, x_1)_z) & \cdots & -\log_v(\delta(x_1, x_m)_z) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -\log_v(\delta(x_m, x_1)_z) & \cdots & -\log_v(\delta(x_m, x_m)_z) \end{pmatrix}.$$

We call that matrix $M(z)$ *Cantor matrix relative to z.*

Lemma 4.4.6. *For every $z \in \mathbb{P}^1_{\mathrm{Berk}}$, the matrix $M(z)$ is non-singular.*

Proof. We will show that the matrix $M := M(z)$ has a trivial kernel. Consider a vector $\vec{c} = (c_0, \dots, c_m)^T \in \mathbb{R}^{m+1}$ with $M\vec{c} = 0$. Then

$$\sum_{i=1}^{m} c_i = 0 \tag{4.12}$$

and

$$c_0 + \sum_{i=1}^{m} c_i(-\log_v(\delta(x_j, x_i)_z)) = 0$$

for each $j \in \{1, \dots, m\}$. The latter equation is equivivalent to the fact, that the function $f \colon \mathbb{P}^1_{\mathrm{Berk}} \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$f(x) := c_0 + \sum_{i=1}^{m} c_i(-\log_v(\delta(x, x_i)_z))$$

satisfies $f|_{\partial V} \equiv 0$. By Example 3.2.9, $f \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and

$$\begin{aligned}\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) &= \Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(c_0) + \sum_{i=1}^m c_i \cdot \Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(-\log_v(\delta(x,x_i)_z)) \\ &= \sum_{i=1}^m c_i(\delta_{x_i} - \delta_z) = \sum_{i=1}^m c_i\delta_{x_i},\end{aligned}$$

where the last equation is true by (4.12). Due to $\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) = \sum_{i=1}^m c_i\delta_{x_i}$ and $r_{\mathbb{P}^1_{\mathrm{Berk}},\overline{V}}(x_i) = x_i$ for each $i = 1, \dots m$, we have the identity

$$(r_{\mathbb{P}^1_{\mathrm{Berk}},\overline{V}})_*(\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f)) = \Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f).$$

Proposition 3.1.29 states that $f \in \mathrm{BDV}(V)$ and implies

$$\Delta_{\overline{V}}(f) = (r_{\mathbb{P}^1_{\mathrm{Berk}},\overline{V}})_*(\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f)) = \Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) = \sum_{i=1}^m c_i\delta_{x_i}.$$

Since $\partial V = \{x_1, \dots, x_m\}$, we have $\Delta_V(f) = 0$. The function f is continuous on V as the sum of continuous functions, i.e. f is strongly harmonic on V. We have already seen that $f \equiv 0$ on ∂V, so Theorem 4.3.1 ii), the Maximum Principle, says that $f \equiv 0$ on $\overline{V}$. Thus, f is constant on V. Hence, $\sum_{i=1}^m c_i\delta_{x_i} = \Delta_{\overline{V}}(f) = 0$ is true by [BR, Lemma 5.24], and so $c_i = \Delta_{\overline{V}}(f)(x_i) = 0$ for each $i \in \{1, \dots, m\}$. Further, we get

$$c_0 = f(x) - \sum_{i=1}^m c_i(-\log_v(\delta(x,x_i)_z)) = 0 - 0 = 0$$

for any $x \in V$ by the definition of f, i.e. $\vec{c} = 0$. Thus $\ker(M(z)) = \{0\}$. □

Theorem 4.4.7 (Poisson Formula, Version I)**.** *Let V be a finite-dendrite domain in $\mathbb{P}^1_{\mathrm{Berk}}$ with boundary points $x_1, \dots, x_m \in \mathbb{H}_{\mathrm{Berk}}$. For every $A_1, \dots, A_m \in \mathbb{R}^m$, there is a unique solution of the Dirichlet problem which is given as follows:*

Fix $z \in \mathbb{P}^1_{\mathrm{Berk}}$, and let $\vec{c} := (c_0, \dots, c_m)^T \in \mathbb{R}^{m+1}$ be the unique solution of the linear equation $M(z)\vec{c} = (0, A_1, \dots, A_m)^T$ (which is possible by the lemma above). Then

$$f(x) = c_0 + \sum_{i=1}^m c_i(-\log_v(\delta(x,x_i)_z))$$

for every $x \in \overline{V}$. (This should be understood as a limit, if z is of type I and $x = z \in V$.)

Moreover,

$$\Delta_{\overline{V}}(f) = \sum_{i=1}^m c_i\delta_{x_i}.$$

Proof. First, we show the uniqueness of a solution. Suppose there are two such functions f_1, f_2, then $f_1 - f_2$ is harmonic on V and $f_1 - f_2 \equiv 0$ on ∂V. By the Maximum Principle (Theorem 4.3.1 ii)), $f_1 - f_2 \equiv 0$ on V, and so $f_1 \equiv f_2$.

Now it remains to show that the given formula satisfies all required properties. By construction, $f(x_i) = A_i$ for all $i = 1, \dots, m$. Proposition 3.2.8 states that the generalized Hsia kernel $\delta(x, y)_z$ is continuous in every $x \in \mathbb{P}^1_{\mathrm{Berk}}$, and so f is continuous on $\overline{V}$. The function f belongs to the vector space $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ by Example 3.2.9, and hence to $\mathrm{BDV}(V)$ by Proposition 3.1.29. Furthermore, we know by Example 3.2.9 and $\sum_{i=1}^m c_i = {e_1}^T \cdot (M(z) \cdot \vec{c}) = 0$, that

$$\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) = \sum_{i=1}^m c_i(\delta_{x_i} - \delta_z) = \sum_{i=1}^m c_i \delta_{x_i}.$$

Due to $\partial V = \{x_1, \dots, x_m\}$, Proposition 3.1.29 implies

$$\Delta_V(f) = (\sum_{i=1}^m c_i \delta_{x_i})|_V = 0,$$

i.e. f is strongly harmonic on V, so particularly harmonic. As in Lemma 4.4.6, we have

$$\Delta_{\overline{V}}(f) = (r_{\mathbb{P}^1_{\mathrm{Berk}}, \overline{V}})_*(\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f)) = \Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f) = \sum_{i=1}^m c_i \delta_{x_i},$$

because $\Delta_{\mathbb{P}^1_{\mathrm{Berk}}}(f)$ is supported on ∂V. □

Remark. We also have a similar statement in the general case: If X is a smooth strictly k-analytic curve and Y an k-affinoid domain in X, then the restriction map defines an isomorphism from the space of harmonic functions on Y to $\mathrm{Hom}(\partial Y, \mathbb{R})$ (cf. [Th, Proposition 2.1.12] and [Th, Corollary 3.1.21]).

Remark 4.4.8. With the help of Cramer's rule, we can give an explicit formula for the coefficients c_i for $i = 0, \dots, m$,

$$c_i = \det(M_i(z, \vec{A})) / \det(M(z)),$$

where $M_i(z, \vec{A})$ denotes the matrix which we obtain by replacing the i^{th} column of $M(z)$ by $\vec{A} := (0, A_1, \dots, A_m)^T$. By the explicit formula for f in Theorem 4.4.7, we have the identity

$$f(z) = c_0 = \det(M_0(z, \vec{A})) / \det(M(z)).$$

Recall that a strict simple domain is a finite-dendrite domain whose boundary points are all of type II. The Poisson formula has the following corollaries:

Corollary 4.4.9. *If V is a strict simple domain with $\partial V = \{x_1, \dots, x_m\}$ and f a harmonic function on V, then there exist $c_0, \dots c_m \in \mathbb{R}$ and $a_1, \dots, a_m \in \mathbb{P}(K)$ not* *contained in $\overline{V}$ such that*

$$f(x) = c_0 - \sum_{i=1}^{m} c_i \log_v([T - a_i]_x)$$

for all $x \in \overline{V}$.

Proof. By a change of coordinates, we are allowed to assume that ∞ is not contained in V. Setting $z := \infty$, the Poisson formula Theorem 4.4.7 states the existence of $c_0, \ldots, c_m \in \mathbb{R}$ with $\sum_{i=1}^{m} c_i = 0$ such that

$$f(x) = c_0 - \sum_{i=1}^{m} c_i \log_v(\delta(x, x_i)_\infty)$$

for all $x \in \overline{V}$. We will show that we can find for every $x_i \in \partial V$ a point $a_i \notin \overline{V}$ of type I such that the path $[a_i, \infty]$ passes through x_i and $x \vee_\infty a_i = x \vee_\infty x_i$ for all $x \in \overline{V}$. Since V is connected and x_i is of type II, there is a connected component V_i of $\mathbb{P}^1_{\mathrm{Berk}} \backslash \{x_i\}$ such that $V \cap V_i = \emptyset$ and $\infty \notin V_i$. The connected component V_i is open, and so it has to contain a point a_i of the dense subset $\mathbb{P}^1(K)$ of $\mathbb{P}^1_{\mathrm{Berk}}$. This type I point a_i satisfies the required properties. Hence,

$$\delta(x, x_i)_\infty = \mathrm{diam}_\infty(x \vee_\infty x_i) = \mathrm{diam}_\infty(x \vee_\infty a_i) = \delta(x, a_i)_\infty$$

for all $x \in \overline{V}$. By [BR, Corollary 4.2], we have the identity $\delta(x, a_i)_\infty = [T - a_i]_x$ on $\overline{V}$. Thus,

$$f(x) = c_0 - \sum_{i=1}^{m} c_i \log_v([T - a_i]_x)$$

for all $x \in \overline{V}$. □

Corollary 4.4.10. *The Neumann problem for V is solvable. The solution is unique up to addition of a constant.*

Proof. Proposition 4.4.3 and Theorem 4.4.7 state that every $f \in \mathcal{H}(V)$ belongs to $\mathrm{BDV}(V)$, has a continuous extension $f \colon \overline{V} \to \mathbb{R}$, and $\Delta_{\overline{V}}(f) = \sum_{i=1}^{m} d_i \delta_{x_i}$ for suitable $d_i \in \mathbb{R}$. By [BR, Proposition 5.25], we have

$$0 = \Delta_{\overline{V}}(f)(\overline{V}) = \sum_{i=1}^{m} d_i. \tag{4.13}$$

$\vec{\partial}(f) := (d_1, \ldots, d_m) = 0$ if and only if $\Delta_{\overline{V}}(f) = \sum_{i=1}^{m} d_i \delta_{x_i} = 0$, what is equivalent to the fact that f is constant on $V \cap \mathbb{H}_{\mathrm{Berk}}$ by [BR, Lemma 5.24]. Since f is continuous on $\overline{V}$, f is constant on $V \cap \mathbb{H}_{\mathrm{Berk}}$ if and only if f is constant on $\overline{V}$. Consequently, $\vec{\partial}(f) = 0$ is equivalent to the fact that f is constant on $\overline{V}$.

If we have a given vector $\vec{A} := (A_1, \ldots, A_m)^T \in \mathbb{R}^m$, we denote the unique solution of the Dirichlet problem for $A_1, \ldots, A_m$ by $f_{\vec{A}}$. Then the following map

$$\begin{aligned} L : \mathbb{R}^m &\to \mathbb{R}^m \\ \vec{A} &\mapsto \vec{\partial}(f_{\vec{A}}) \end{aligned}$$

is $\mathbb{R}-$linear by the uniqueness of the Poisson formula. Furthermore, one can show that

$$\text{im}(L) = \{d \in \mathbb{R}^m | \sum_{i=1}^{m} d_i = 0\} =: H.$$

To see this, we will determine the dimension of the kernel of L. Suppose that $\vec{A} \in \mathbb{R}^m$ with $\vec{\partial}(f_{\vec{A}}) = 0$. We have seen above that this is equivalent to fact that $f_{\vec{A}}$ is constant on $\overline{V}$. Hence, $A_1 = \ldots = A_m$. But on the other hand, if $\vec{A} \in \mathbb{R}^m$ with $A_1 = \ldots = A_m$, then $f_{\vec{A}}$ is constant on $\overline{V}$ by the Maximum Principle (Theorem 4.3.1 ii)). Hence, $\vec{\partial}(f_{\vec{A}}) = 0$, i.e. $\vec{A} \in \ker(L)$. Thus, we have the identity $\ker(L) = \text{Diag}(\mathbb{R}^m)$. Therefore,

$$\dim(\text{im}(L)) = m - \dim(\ker(L)) = m - 1.$$

By Equation (4.13), $\text{im}(L) \subset H$, thus the image of L has to coincide with the hyperplane H. Therefore, for every $\vec{c} = (c_1, \ldots, c_m)^T \in \mathbb{R}^m$ with $\sum_{i=1}^m c_i = 0$ there exists a $\vec{A} = (A_1, \ldots, A_m)^T \in \mathbb{R}^m$ such that $\vec{\partial}(f_{\vec{A}}) = \vec{c}$, where $f_{\vec{A}}$ is the unique solution of the Dirichlet problem. This means that $f := f_{\vec{A}}$ is harmonic on V and continuous on $\overline{V}$ with

$$\Delta_{\partial V} = \Delta_{\overline{V}}(f) = \sum_{i=1}^{m} c_i \delta_{x_i}.$$

If f_1, f_2 are two harmonic functions on V which are continuous on $\overline{V}$ and

$$\Delta_{\overline{V}}(f_1) = \sum_{i=1}^{m} c_i \delta_{x_i} = \Delta_{\overline{V}}(f_2),$$

then $f_1 - f_2 \in \text{BDV}(V)$, $f_1 - f_2$ is continuous on $\overline{V}$ and $\Delta_{\overline{V}}(f_1 - f_2) = 0$. Hence, $f_1 - f_2$ is constant on $V \cap \mathbb{H}_{\text{Berk}}$ by Lemma 5.24 [BR] and so on $\overline{V}$. Consequently, the required function is unique up to addition of a constant. $\square$

4.5 Poisson Formula and the Equilibrium and the Poisson-Jensen Measure

Applying the Poisson formula for each boundary point separately will give us a further description of the solution of the Dirichlet problem. This version of the Poisson formula leads to an easier proof of the uniqueness of the Equilibrium measure in special cases. Moreover, we will define the Poisson-Jensen measure for a finite-dendrite domain V, and show that this measure coincides with the Equilibrium measure respectively to any point of V for the compacts set ∂V. Moreover, the second version of the Poisson formula enables us to characterize harmonic functions in a further way.

Definition 4.5.1. i) Let V be a finite-dendrite domain with $\partial V = \{x_1, \ldots, x_m\}$. We will call the unique harmonic function on V with a continuous extension on $\overline{V}$ and

$$h_i(x_j) = \delta_{ij},$$

which is given by Theorem 4.4.7 the *harmonic measure* for the boundary component x_i of V.

ii) If V is a finite-dendrite domain with $\partial V = \{x_1, \ldots, x_m\}$ and $z \in \mathbb{P}^1_{\mathrm{Berk}}$, we define the *Poisson-Jensen measure* $\mu_{z,V}$ on $\overline{V}$ relative to the point z as

$$\mu_{z,V} = \sum_{i=1}^{m} h_i(z)\delta_{x_i}.$$

4.5.2. By part ii) of the Maximum Principle, we know that $0 \leq h_i(x) \leq 1$ for all $x \in V$. Since part i) of the Maximum Principle says that h_i does not achieve an extremum on V, the inequality has to be strict, i.e.

$$0 < h_i < 1$$

on V. Furthermore, $h(x) := \sum_{i=1}^{m} h_i(x)$ is harmonic on V and continuous on $\overline{V}$ with $h(x) = \sum_{i=1}^{m} h_i(x) = 1$ for all $x \in \partial V$. Hence, the Maximum Principle part ii) implies that

$$\sum_{i=1}^{m} h_i = 1$$

on $\overline{V}$.

Proposition 4.5.3 (Poisson Formula, Version II)**.** *Let V be a finite-dendrite domain in $\mathbb{P}^1_{\mathrm{Berk}}$ with $\partial V = \{x_1, \ldots, x_m\}$ and $A_1, \ldots, A_m \in \mathbb{R}$. Then the solution of the Dirichlet problem f with $f(x_i) = A_i$ for each $i = 1, \ldots, m$ is given by*

$$f(z) = \sum_{i=1}^{m} A_i \cdot h_i(z)$$

for all $z \in \overline{V}$, where h_i is the harmonic measure for $x_i \in \partial V$.

Proof. Since the functions h_i are harmonic on V and continuous on $\overline{V}$ by construction, the same is true for the function $g(z) := \sum_{i=1}^{m} A_i \cdot h_i(z)$. Moreover,

$$g(x_j) = \sum_{i=1}^{m} A_i \cdot h_i(x_j) = A_j$$

for all $j \in \{1, \ldots, m\}$. Thus, the second version of the Poisson formula is a direct consequence of the uniqueness in the first version of the Poisson formula (cf. Theorem 4.4.7). □

Remark. By Remark 4.4.8, we have

$$h_i(z) = \det(M_0(z, \hat{e_i}))/\det(M(z))$$

for each $i = 1, \ldots, m$, where $\hat{e_i} \in \mathbb{R}^{m+1}$ is the vector which is 1 in the $(i{+}1)^{\text{st}}$ component and 0 elsewhere.

By the second version of the Poisson formula, we can characterize harmonic functions defined on $\overline{V}$. Afterwards, we extend this characterization to harmonic functions on general open sets.

Corollary 4.5.4. *If V is a finite-dendrite domain with $\partial V = \{x_1, \ldots, x_m\}$, then a continuous function $f : \overline{V} \to \mathbb{R}$ is harmonic on V if and only if*

$$f(z) = \int_{\partial V} f \; d\mu_{z,V}$$

for all $z \in V$.

Proof. Since

$$\int_{\partial V} f \; d\mu_{z,V} = \sum_{i=1}^{m} f(x_i) h_i(z),$$

the corollary follows directly from Proposition 4.5.3. □

Let U be an open subset in $\mathbb{P}^1_{\text{Berk}}$. Recall that every simple domain is a finite-dendrite domain. We can characterize harmonic functions on an open set U in the following way:

Corollary 4.5.5. *If U is an open subset of $\mathbb{P}^1_{\text{Berk}}$ and $f : U \to \mathbb{R}$ is a continuous function, then f harmonic on U if and only if for every simple subdomain V of U satisfying $\overline{V} \subset U$ we have*

$$f(z) = \int_{\partial V} f \; d\mu_{z,V}$$

for all $z \in V$.

Proof. The closures of simple domains form a fundamental system of compact neighborhoods for the topology on $\mathbb{P}^1_{\text{Berk}}$. The function f therefore is harmonic on U if and only if its restriction to every simple subdomain of U satisfying $\overline{V} \subset U$ is harmonic by Corollary 4.1.8. Hence, Corollary 4.5.4 implies the claim. □

Now, we will see that the Poisson-Jensen measure $\mu_{\zeta,V}$ coincides with the Equilibrium measure μ_ζ for ∂V relative to ζ. On the way to that, we need the following lemma, which gives also a simpler proof of the uniqueness of the Equilibrium measure for ∂V relative to ζ.

Lemma 4.5.6. *Let V be a finite-dendrite domain in $\mathbb{P}^1_{\text{Berk}}$, $\zeta \in V$ and $\nu \in \mathbb{P}(e)$ for $e := \partial V$.*

i) Then the following is equivalent:

a) $I_\zeta(\nu) = V_\zeta(e)$.

b) The potential function

$$u_\nu(x) := u_\nu(x,\zeta) = \int -\log_v \delta(x,y)_\zeta d\nu(y)$$

is constant on ∂V.

ii) The Equilibriums measure μ_ζ for ∂V relative to ζ is the unique probability measure satisfying these equivalent conditions.

Proof. Since e is finite and contained in $\mathbb{H}_{\text{Berk}}$, the set e is compact and has positive capacity by Lemma 4.3.5. Thus, the Equilibrium measure $\mu_\zeta \in \mathbb{P}(e)$ exists by [BR, Proposition 6.6] and is supported on $e = \partial V$ by [BR, Proposition 6.8]. Let $\partial V = \{x_1, \dots, x_m\}$, then any probability measure $\nu \in \mathbb{P}(e)$ is supported on $e = \{x_1, \dots, x_m\}$, i.e. $\nu = \sum_{i=1}^m \nu_i \delta_{x_i}$ for $\nu_i = \nu(x_i) \in \mathbb{R}$. Hence, $\sum_{i=1}^m \nu_i = \nu(\mathbb{P}^1_{\text{Berk}}) = 1$ and $\nu_i = \nu(x_i) \geq 0$ for all $i = 1, \dots, m$.

At first, we will show that a) implies b). Next, we will use this direction to show that there is a unique $\nu \in \mathbb{P}(e)$ satisfying b). Afterwards, we use this uniqueness to verify the other direction. Then part i) and ii) are true.

If $\nu \in \mathbb{P}(e)$ satisfies a), then ν is an Equilibrium measure for e relative to ζ. Since $e \subset \mathbb{H}_{\text{Berk}}$, every non-empty subset of e has positive capacity. Hence, $u_\nu(x) = V_\zeta(e)$ for every $x \in e$ by the Frostman's theorem [BR, Proposition 6.18]. Thus, u_ν is constant on $e = \partial V$.

Let $\nu \in \mathbb{P}(e)$ such that statement b) is satisfied. The potential function u_ν is constant on ∂V if and only if

$$M(\zeta)(\nu_0, \dots, \nu_m)^T = (\sum_{i=1}^m \nu_i, \nu_0 + u_\nu(x_1), \dots, \nu_0 + u_\nu(x_m))^T = (1, 0 \dots, 0)^T$$

for some $\nu_0 \in \mathbb{R}$. By Lemma 4.4.6, $M(\zeta)$ is non-singular, so there is a unique $\vec{\nu} \in \mathbb{R}^{m+1}$ such that b) is satisfied. Hence, the probability measure ν is unique.

To prove the other direction, we will use the just shown uniqueness which we have just showed. Let $\nu \in \mathbb{P}(e)$ such that b) is true. At the beginning, we have seen that there exists a Equilibriums measure μ_ζ for e relative to ζ, which is contained in $\mathbb{P}(e)$ and satisfies a) by definition. Hence, b) is true for μ_ζ by the first direction. Due to the uniqueness, μ_ζ has to coincide with ν, and so ν satisfies a). □

With this lemma we can show that the Poisson measure and the Equilibrium measure coincide:

Proposition 4.5.7. *Let V be a finite-dendrite domain with $\partial V = \{x_1, \ldots, x_m\}$, and let $\mu = \mu_{z,V}$ be the Poisson-Jensen measure for V relative to a point $z \in V$. Then μ is the Equilibrium measure for ∂V relative to z.*

We will verify that μ satisfies condition b) in Lemma 4.5.6:

Proof. For the Poisson-Jensen measure $\mu = \sum_{i=1}^m h_i(z)\delta_{x_i}$, the potential function

$$u_\mu(x) = \int -\log_v(\delta(x,y)_z)\ d\mu(y) = \sum_{i=1}^m -\log_v(\delta(x,x_i)_z) \cdot h_i(z)$$

is continuous on $\overline{V}$ since the generalized Hsia kernel $\delta(x,y)_z$ is continuous in x by Proposition 3.2.8. We have seen in Example 3.2.11 that $u_\mu(z) \in \mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk}})$ and $\Delta(u_\mu) = \mu - \delta_z$. By Lemma 3.1.29, u_μ belongs to $\mathrm{BDV}(V)$, and so $u_\mu \in \mathcal{C}(\overline{V}) \cap \mathrm{BDV}(V)$. Further,

$$\Delta_{\overline{V}}(u_\mu) = r_{\overline{V}*}(\mu - \delta_z) = \mu - \delta_z. \tag{4.14}$$

due to $x_1, \ldots, x_m, z \in \overline{V}$. Set $\nu_i = \delta_{x_i} - \delta_{x_1}$ for $i = 1, \ldots, m$, then ν_i is a finite signed Borel measure on $\overline{V}$ such that $\nu_i(\overline{V}) = 0$. By [BR, Proposition 5.28], there is a one-to-one correspondence between finite signed Borel measures of total mass zero on $\overline{V}$ and functions $h \in \mathrm{BDV}(V)$ modulu constant functions. Thus, there are $f_i \in \mathrm{BDV}(V)$, which are unique up to additive constants, such that

$$\Delta_{\overline{V}}(f_i) = \nu_i \tag{4.15}$$

for $i = 1, \ldots, m$. Since ν_i is supported on ∂V for all $i = 1, \ldots, m$, the functions f_i are all strongly harmonic on V. We have seen in Proposition 4.4.3 that any strongly harmonic function h on the finite-dendrite domain V can be extended to a continuous function on $\overline{V}$. So we can extend the function f_i such that $f_i \in \mathcal{C}(\overline{V}) \cap \mathrm{BDV}(V)$ for each $i = 1, \ldots, m$. Above we have showed that u_μ also belongs to $\mathcal{C}(\overline{V}) \cap \mathrm{BDV}(V)$. Therefore, the following integrals exist and coincide by [BR, Corollary 5.39]

$$\int_{\overline{V}} f_i \Delta_{\overline{V}}(u_\mu) = \int_{\overline{V}} u_\mu \Delta_{\overline{V}}(f_i). \tag{4.16}$$

Let $i = 1, \ldots, m$, then

$$u_\mu(x_i) - u_\mu(x_1) = \int_{\overline{V}} u_\mu \; d\nu_i = \int_{\overline{V}} u_\mu \Delta_{\overline{V}}(f_i)$$

by the definition of $\nu_i := \delta_{x_i} - \delta_{x_1}$ and Equation (4.15). Applying (4.16) and then (4.14), leads to the identity

$$u_\mu(x_i) - u_\mu(x_1) = \int_{\overline{V}} f_i \Delta_{\overline{V}}(u_\mu) = \left(\int_{\overline{V}} f_i \; d\mu \right) - f_i(z).$$

Finally, we have $u_\mu(x_i) - u_\mu(x_1) = 0$ for each $i = 1, \ldots, m$ by Corollary 4.5.4. Thus, u_μ is constant on ∂V. □

Remark. i) One can generalize the last Proposition for an arbitrary domain U if you require that ∂U has positive capacity. The proof of this generalization uses Green functions, and can be found in [BR, Proposition 7.43].

ii) By Proposition 4.5.7 and the proof of Lemma 4.5.6, $\mu_{z,V}$ is the unique measure μ supported on ∂V such that

$$M(z)(\mu_0, \mu(x_1), \ldots, \mu(x_m))^T = (1, 0, \ldots, 0)^T \in \mathbb{R}^{m+1}, \tag{4.17}$$

for some $\mu_0 \in \mathbb{R}$.

The last Remark and Cramer's rule provide a further explicit formula for the harmonic measure h_i:

Corollary 4.5.8. *Let $M_i(z)$ denote the matrix obtained by replacing the i^{th} column of $M(z)$ by $(1, 0, \ldots, 0)^T \in \mathbb{R}^{m+1}$. Then the harmonic measure $h_i(z)$ for $x_i \in \partial V$ is given by*

$$h_i(z) = \det(M_i(z)) / \det(M(z))$$

for each $i = 1, \ldots, m$.

Proof. Let μ denote the Poisson-Jensen measure which is given by $\mu = \sum_{i=1}^m h_i(z)\delta_{x_i}$. So we have $h_i(z) = \mu(x_i)$, and Equation (4.17) implies the formula. □

4.6 Uniform Convergence

In the complex potential theory, it follows immediately from Poisson formula that the limit of a sequence of harmonic functions on a domain which are converging locally uniformly is a harmonic function on the domain (cf. [Ra, Corollary 1.2.8]). In this section, we will see that this is also true in the potential theory on $\mathbb{P}^1_{\text{Berk}}$, even under a much weaker condition than is required classically. This fact is, as in the classical theory, a direct consequence of the Poisson formula. In Section 4.4, we have seen that every harmonic function on a strict simple domain can be described by functions of the form $\log_v([T - a_i]_x)$ for $a_i \in K$. At the end of this section, we extend this description to a harmonic function on an arbitrary domain using uniform convergence.

Proposition 4.6.1. *Let U be an open subset of $\mathbb{P}^1_{\mathrm{Berk}}$ and $f_1, f_2, \dots$ harmonic functions on U converging pointwise to a function $f\colon U \to \mathbb{R}$. Then f is harmonic on U, and the f_i converge uniformly to f on compact subsets of U.*

Proof. Consider a $x \in U$, then we can choose a simple domain U_x containing x such that $\overline{U_x} \subset U$. The functions f_k are harmonic on $U_x \subset U$ by Lemma 4.1.7 and continuous on $\overline{U_x}$ because they are continuous on U by definition. Note that every simple domain is a finite-dendrite domain. Let $\partial U_x = \{x_1, \dots, x_m\}$. The uniqueness in the second version of the Poisson formula, Proposition 4.5.3, implies that for each $k \geq 1$ the function f_k is given in the following way

$$f_k(z) = \sum_{i=1}^{m} f_k(x_i) h_i(z)$$

for all $z \in \overline{U_x}$. We have required that the sequence f_k converges pointwise to a function f on U, so $f_k(x_i)$ converges to $f(x_i)$ for each $i = 1, \dots, m$. Hence, $f_k(z) = \sum_{i=1}^{m} f_k(x_i)h_i(z)$ converges uniformly to $f(z) = \sum_{i=1}^{m} f(x_i)h_i(z)$ on $\overline{U_x}$. The first version of the Poisson formula, Theorem 4.4.7, states that the harmonic measures h_i are strongly harmonic on U_x, and so f is strongly harmonic on U_x as well. Thus, f is harmonic on U.

Every compact set $E \subset U$ can be covered by finitely many domains U_x. Therefore, the sequence $f_1, f_2, \dots$ converges uniformly to f on E. □

Corollary 4.6.2. *If U is a finite-dendrite domain, a sequence of harmonic functions $f_1, f_2, \dots$ converges pointwise to a function $f\colon U \to \mathbb{R}$ if and only if the sequence f_i converges uniformly to f.*

Proof. As we have seen in the proof above, $f_k(z) = \sum_{i=1}^{m} f_k(x_i)h_i(z)$ on $\overline{U}$, where $\partial U = \{x_1, \dots, x_m\}$. Since f_k converges pointwise to f, the sequence f_k converges uniformly to $f(x) = \sum_{i=1}^{m} f(x_i)h_i(z)$ as well. □

With the help of Corollary 4.4.9 we can describe a harmonic function on a domain in the following way:

Proposition 4.6.3. *If U is a domain and f is harmonic on U, there are rational functions $g_1(T), g_2(T), \dots \in K(T)$ and rational numbers $R_1, R_2, \dots \in \mathbb{Q}$ such that*

$$f(x) = \lim_{k \to \infty} R_k \cdot \log_v([g_k]_x)$$

uniformly on compact subsets of U.

Proof. If the main dendrite of U is empty, the harmonic function f on U is constant by Proposition 4.2.6. Let $c \in \mathbb{R}$ such that $f \equiv c$ on U. Since $\mathbb{Q} \subset \mathbb{R}$ is dense, there is a sequence $(R_k)_{k\in\mathbb{N}} \subset \mathbb{Q}$ such that

$$f(x) = \lim_{k\to\infty} R_k$$

for all $x \in U$. Let α be a constant in K such that $|\alpha| = q_v$, then the claim is true with $g_k \equiv \alpha$ for all $k \in \mathbb{N}$.

Now we assume that the main dendrite is non-empty. Therefore, we can change coordinates if $\infty \in U$, and so we are allowed to assume that ∞ is not contained in the domain U. By Corollary 4.2.4, we can consider an exhaustion $(U_k)_{k\geq 1}$ of U, where U_k are strict simple domains and $\overline{U_k} \subset U$ for $k \geq 1$, and each f_k is harmonic on U_k by Corollary 4.1.8. Let $\partial U_k = \{x_{k,1}, \dots, x_{k,m_k}\}$, then by Corollary 4.4.9 for each $k \geq 1$ there are $c_{k,0}, \dots, c_{k,m_k} \in \mathbb{R}$ with $\sum_{i=1}^{m_k} c_{k,i} = 0$ and points $a_{k,1}, \dots, a_{k,m_k} \notin \overline{U_k}$ of type I such that

$$f(x) = c_{k,0} - \sum_{i=1}^{m_k} c_{k,i} \log_v([T - a_{k,i}]_x)$$

for all $x \in \overline{U_k}$.

At next, we will construct a sequence $(f_k)_{k\geq 1}$ of functions on U converging uniformly to f on compact subsets of U. Afterwards, we will verify that these functions coincide with the functions in the claim. First, we show that the function $h_{k,i}(x) := \log_v(\delta(x, a_{k,i})_\infty) = \log_v([T - a_{k,i}]_x)$ is bounded on $\overline{U_k}$. The last identity is true by [BR, Corollary 4.2]. The function is continuous by [BR, Proposition 4.1]. Since $a_{k,i} \notin \overline{U_k}$ and $\infty \notin \overline{U_k}$, $x \vee_\infty a_{k,i}$ cannot be a point of type I. Hence, $\delta(x, a_{k,i})_\infty = \operatorname{diam}_\infty(x \vee_\infty a_{k,i}) \in \mathbb{R}_{>0}$, i.e. $h_{k,i}$ is real valued on $\overline{U_k}$ for each $i = 1, \dots, m_k$. Thus, $\log_v(\delta(x, a_{k,i})_\infty)$ is bounded on the compact set $\overline{U_k}$ by constants $\lambda_{k,i}$. Set $\lambda_k := \max_{i=1,\dots,m_k} \lambda_{k,i}$, and choose rational numbers $d_{k,i}$ such that $\sum_{i=1}^{m_k} d_{k,i} = 0$, and $|d_{k,i} - c_{k,i}| < \frac{1}{\lambda_k \cdot m_k \cdot 2k}$ for $i = 1, \dots, m_k$, and $|d_{k,0} - c_{k,0}| < \frac{1}{2k}$. Define

$$f_k(x) := d_{k,0} - \sum_{i=1}^{m_k} d_{k,i} h_{k,i}(x),$$

then

$$\begin{aligned} |f_k(x) - f(x)| &= |d_{k,0} - \sum_{i=1}^{m_k} d_{k,i} h_{k,i}(x) - c_{k,0} + \sum_{i=1}^{m_k} c_{k,i} h_{k,i}(x)| \\ &\leq |d_{k,0} - c_{k,0}| + \sum_{i=1}^{m_k} |d_{k,i} - c_{k,i}| \cdot |h_{k,i}(x)| \\ &< \frac{1}{2k} + m_k \cdot \lambda_k \cdot \frac{1}{\lambda_k \cdot m_k \cdot 2k} = \frac{1}{k} \end{aligned}$$

for each $x \in \overline{U_k}$. Further, $|f_n(x) - f(x)| < 1/n \leq 1/k$ for all $n \geq k$ since $\overline{U_k} \subset U_n$. Therefore, the sequence $(f_{k+l})_{l\in\mathbb{N}}$ converges uniformly to f on U_k for all $k \geq 1$. Thus, (f_k) converges uniformly to f on compact sets of U. It remains to show that the sequence (f_k) has the form from the claim. Let N_k be the common denominator for the $d_{k,i}$ and put $n_{k,i} = N_k \cdot d_{k,i} \in \mathbb{Z}$. Then we can find a constant $b_k \in K$ with $|b_k| = {q_v}^{-n_{k,0}}$. Setting

$$g_k(T) := b_k \cdot \prod_{i=1}^{m_k} (T - a_{k,i})^{n_{k,i}},$$

we get the following identity on $\overline{U_k}$

$$\begin{aligned} f_k(x) &= d_{k,0} - \sum_{i=1}^{m_k} d_{k,i} \log_v([T - a_{k,i}]_x) \\ &= -\frac{1}{N_k} \cdot (-n_{k,0} + \sum_{i=1}^{m_k} \log_v([T - a_{k,i}]_x^{n_{k,i}})) \\ &= -\frac{1}{N_k} \cdot (\log_v(\prod_{i=1}^{m_k} {q_v}^{-n_{k,0}} \cdot [T - a_{k,i}]_x^{n_{k,i}})) \\ &= -\frac{1}{N_k} \cdot (\log_v(\prod_{i=1}^{m_k} [b_k]_x \cdot [T - a_{k,i}]_x^{n_{k,i}})) \\ &= -\frac{1}{N_k} \cdot (\log_v([g_k]_x). \end{aligned}$$

□

4.7 Harnack's Principle

In the classical potential theory we have Harnacks's principle which describes the behavior of an ordered sequence of harmonic functions on a domain in $\mathbb{C}_\infty$, where $\mathbb{C}_\infty$ is the Riemann sphere. The principle says that either the sequence converges locally uniformly to ∞, or it converges locally uniformly to a harmonic function on the domain (cf. [Ra, Theorem 1.3.9]). In this section, an analogue of Harnack's principle is given. Note that we do not require that the sequence has to be non-negative as in [BR]. To prove the principle we will first give an analogue of Harnack's inequality, which is needed in the classical theory as well.

Lemma 4.7.1 (Harnack's Inequality). *Let $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a domain. Then for each $x_0 \in U$ and each compact set $X \subset U$, there is a constant $C = C(x_0, X)$ such that for any harmonic function h which is non-negative on U*

$$(1/C) \cdot h(x_0) \leq h(x) \leq C \cdot h(x_0) \tag{4.18}$$

is satisfied for all $x \in X$.

Proof. If the main dendrite D of U is empty, $h \equiv h(x_0)$ on U. Thus, Harnack's inequality (4.18) is true for all $C \geq 1$. So we may assume that $D \neq \emptyset$. If $h(x_0) = 0$, our harmonic function h is achieving a minimum on U since we have required that h is non-negative. Hence, the harmonic function h has to be constant with $h \equiv 0$ on U by the Maximum Principle. Again, the Inequality (4.18) is true for all $C \geq 1$. Therefore, it remains to consider the case where $D \neq \emptyset$ and $h(x_0) > 0$. We have seen in Proposition 4.2.6, that there is a point $\omega \in D$ such that $h(\omega) = h(x_0)$, so we may assume that x_0 is contained in the main dendrite D.

We start with the upper bound in (4.18). Let $\rho(x, y)$ be the logarithmic path distance on $\mathbb{P}^1_{\text{Berk}}$. By Proposition 4.2.3, the main dendrite D is finitely branched at every point $p \in D$, i.e. there is an $\varepsilon > 0$ such that the closed neighborhood of p in D defined by $\Gamma(p, \varepsilon) = \{x \in D |\ \rho(x, p) \leq \varepsilon\}$ is a star. This means that $\Gamma(p, \varepsilon)$ is the union of n closed segments of length ε emanating from p for some $n \geq 2$,

$$\Gamma(p, \varepsilon) = \bigcup_{i=1}^{n} [p, q_i],$$

where q_i are the endpoints which can be written as $q_i = p + \varepsilon \vec{v_i}$ for $i = 1, \dots, n$. We take ε as large as possible such that $\varepsilon \leq 1$. As in the proof of Proposition 4.4.3, the harmonic function h is linear on each of the segments $[p, q_i]$ and $\Delta_{\Gamma(p,\varepsilon)}(h)(p) = -\sum_{i=1}^{n} d_{\vec{v_i}} h(p) = 0$. Consider a point $x = p + t \cdot \vec{v_i} \in [p, q_i]$. Since the restriction of h to each segment is linear, the one-sided derivatives can be written as

$$d_{\vec{v_i}}(h) = \frac{h(q_i) - h(p)}{\varepsilon} = \frac{h(x) - h(p)}{t}.$$

Hence,

$$0 = \sum_{j=1}^{n} d_{\vec{v_j}}(h) = \frac{h(x) - h(p)}{t} + \sum_{j \neq i} \frac{h(q_j) - h(p)}{\varepsilon}.$$

This equality and the fact that $h(q_j) \geq 0$ for each $j \in \{1, \dots, n\}$ imply the following inequality

$$\begin{aligned} h(x) &= -\left(\sum_{j \neq i} \frac{h(q_i) - h(p)}{\varepsilon} \right) \cdot t + h(p) \\ &\leq \sum_{j \neq i} h(p) \cdot \frac{t}{\varepsilon} + h(p) \\ &\leq (n-1) h(p) + h(p) \\ &= h(p) \cdot n. \end{aligned}$$

So $h(x) \leq C_p \cdot h(p)$ for each $x \in \Gamma(p, \varepsilon)$ where $C_p := n$. Now we will use the compactness of X to get the upper bound for all $x \in X$. Since X is compact, there is a finite subgraph Γ of D such that the retraction of X to D is contained in the interior of Γ. This means there exists a finite subgraph $\Gamma \subset D$ such that $r_{\mathbb{P}^1_{\mathrm{Berk}},\overline{D}}(X) \subset \Gamma^0$, where Γ^0 denotes the interior of Γ.

If x_0 is not in contained in Γ, we can consider the union of the segment $[x_0, r_\Gamma(x_0)]$ and Γ instead of Γ. Since Γ is compact, there is a finite number of stars which cover Γ, i.e. $\Gamma \subset \bigcup_{i=1}^m \Gamma(p_i, \varepsilon_i)$. Starting at the point $p = x_0 \in \Gamma$ and proceeding stepwise, we get

$$h(x_0) \leq C \cdot h(x)$$

for all $x \in \Gamma$, where $C := \prod_{i=1}^m C_{p_i}$. Since $h(x) = h(r_{\mathbb{P}^1_{\mathrm{Berk}},\overline{D}}(x))$ for each $x \in X$ by Proposition 4.2.6, the upper bound holds for all $x \in X$.

For the lower bound, let $\{x_1, \ldots, x_m\}$ be the set of endpoints of Γ. Then $U_\Gamma^0 := r_\Gamma^{-1}(\Gamma^0)$ defines a subdomain of U with $\partial U_\Gamma^0 = \{x_1, \ldots, x_m\}$. Let $C_{\Gamma,i}$ be the constant which we have constructed above satisfying $h(x) \leq C_{\Gamma,i} \cdot h(x_i)$ on Γ and so on X, for each $i = 1, \ldots, m$. Taking $C'_\Gamma := \max_{i=1,\ldots,m} C_{\Gamma,i}$, then

$$h(x_0) \leq C'_\Gamma \cdot h(x_i)$$

for each $i = 1, \ldots, m$. Since h is harmonic on U and $\overline{U_\Gamma^0} \subset U$, h is harmonic on U_Γ^0 and continuous on $\overline{U_\Gamma^0}$. Thus,

$$\min(h(x_1), \ldots, h(x_m)) \leq h(x)$$

by the Maximum Principle for each $x \in \overline{U_\Gamma^0}$. As $r_{\mathbb{P}^1_{\mathrm{Berk}},\overline{D}}(X) \subset \Gamma^0$,

$$r_{\mathbb{P}^1_{\mathrm{Berk}},\Gamma}(X) = r_{\overline{D},\Gamma}(r_{\mathbb{P}^1_{\mathrm{Berk}},\overline{D}}(X)) \subset \Gamma^0,$$

and so $X \subset \overline{U_\Gamma^0}$. Altogether,

$$h(x_0) \leq C'_\Gamma \cdot \min_{i=1,\ldots,m} h(x_i) \leq C'_\Gamma \cdot h(x)$$

for all $x \in X$. Putting $C := \max(C_\Gamma, C'_\Gamma)$, we have

$$(1/C) \cdot h(x_0) \leq h(x) \leq C \cdot h(x_0)$$

for all $x \in X$. □

Theorem 4.7.2 (Harnack's Principle)**.** *Let $U \subset \mathbb{P}^1_{\mathrm{Berk}}$ be a domain and $f_1, f_2, \ldots$ harmonic functions on U with $f_1 \leq f_2 \leq \ldots$. Then either*

i) $\lim_{i\to\infty} f_i(x) = \infty$ *for each $x \in U$, or*

ii) $f(x) = \lim_{i\to\infty} f_i(x)$ *is finite for all $x \in U$, the f_i converge uniformly to f on compact subsets of U, and f is harmonic on U.*

Proof. First, we consider the case of a non-negative sequence $0 \leq f_1 \leq f_2 \leq \ldots$. Suppose that i) is not true, i.e. there is some $x_0 \in U$ such that $\lim_{i\to\infty} f_i(x_0)$ is finite. Then for any $x \in U$, we can apply Lemma 4.7.1 to the compact set $X := \{x\} \subset U$. Thus, there is a constant C such that

$$(1/C) \cdot f_i(x_0) \leq f_i(x) \leq C \cdot f_i(x_0)$$

for all $i = 1, 2, \ldots$. Since the sequence f_i is bounded in x_0, it is also bounded in our arbitrary x in U. Hence, the increasing sequence (f_i) converge pointwise to f with $f(x) = \lim_{i\to\infty} f_i(x) < \infty$ for each $x \in U$. We have seen in Proposition 4.6.1, that f is harmonic on U and the f_i converge uniformly to f on compact subsets of U.

Now, let a $f_1 \leq f_2 \leq \ldots$ be a sequence of harmonic functions on U which is not forced to be non-negative. Then we can apply the first case to the sequence

$$0 \leq f_2 - f_1 \leq f_3 - f_1 \leq \ldots$$

of harmonic functions on U. Since $f_1(x) \in \mathbb{R}$ for each $x \in U$, we either have $\lim_{i\to\infty} f_i(x) = \infty$ for all $x \in U$, or $f(x) = \lim_{i\to\infty} f_i(x)$ is finite for all $x \in U$ as well. For the rest of the proof, there was no need to be non-negative. Thus the claim is also true in the arbitrary case. □

Remark 4.7.3. If $\lim_{i\to\infty} f_i(x) = \infty$ for each $x \in U$, then the f_i converge uniformly to ∞ on compact subsets of U as well. This is a direct consequence of Harnack's inequality.

Remark. In the general case, Harnack's principle can be found in [Th, Proposition 3.1.2].

5 The link to smooth functions on analytic curves

In this chapter, we consider smooth functions and try to link it with harmonic functions. Antoine Chambert-Loir and Antoine Ducros introduced smooth functions on Berkovich analytic spaces and defined differential operators d' and d'' for them in [CD]. In particular, we study smooth functions and the operators d' and d'' on the analytification of an algebraic variety X over K following Walter Gubler in his paper [Gu]. This raises the question of whether there is a link between harmonic functions and smooth functions which belong to the kernel of $d'd''$. Thuillier introduced in [Th] harmonic functions on an arbitrary strictly analytic smooth curve X and stated in [Th, Théorème 2.3.21] two explicit conditions in which all harmonic functions are locally given by functions of the form $\log|f|$ where $f \in \mathcal{O}_X^\times$. This result has led to establish a connection between smoothness and functions of the form $\log|f|$ where $f \in \mathcal{O}_X^\times$ in Chapter 5.2. We will see that a function is smooth and belongs to the kernel of $d'd''$ if and only if it can be locally written as a linear combination of the just mentioned functions. If X is the projective line $\mathbb{P}^1_K$, the same can be shown for harmonic functions using some results from Chapter 4. Hence, the harmonic functions on $\mathbb{P}^1_{\mathrm{Berk}}$ coincide with the smooth functions contained in $\ker d'd''$. To find an answer in the general case, i.e. the analytification of a smooth algebraic variety X, we introduce Thuillier's definition of harmonic functions in Chapter 5.3, show that his definition is an extension to the one made in Chapter 4 and give the proof of [Th, Théorème 2.3.21]. At the end, we construct a smooth algebraic curve X over K such that one can find an open subset W of X^{an} and a harmonic function on W which is not smooth.

5.1 Differential forms and smooth functions on X^{an}

In this section, we consider an algebraically closed field K endowed with a non-trivial complete non-archimedean absolute value $|\ |$. Let X be an algebraic variety over K, i.e. X is an irreducible separated reduced scheme of finite type. To define smooth functions on X^{an} we introduce differential forms on the algebraic variety X. First, we recall (p,q)-superforms on open subsets of $\mathbb{R}^r$ which were introduced originally by Lagerberg in [La, §2]. This theory of superforms leads to superforms on polyhedral complexes developed in [CD]. With the help of Bieri-Groves one can give a definition of differential forms on algebraic varieties (cf. [Gu13]). We will see that a differential form of bidegree $(0,0)$ defines indeed a continuous function $f\colon X^{\mathrm{an}} \to \mathbb{R}$, and so we can define smooth functions as differential forms of bidegree $(0,0)$.

Definition 5.1.1. i) Let U be an open subset of $\mathbb{R}^r$, then a *superform of bidegree* (p,q) on U is an element of

$$A^{p,q}(U) := C^\infty(U) \otimes_{\mathbb{R}} \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*}.$$

If we choose a basis $x_1, \dots, x_r$ of $\mathbb{R}^r$, a superform α of bidegree (p,q) can be written as

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} d' x_I \wedge d'' x_J$$

where I (resp. J) consists of $i_1 < \cdots < i_p$ (resp. $j_1 < \cdots < j_q$) with $i_1, \dots, i_p, j_1, \dots, j_q \in \{1, \dots, r\}$, $\alpha_{IJ} \in C^\infty(U)$ and

$$d' x_I \wedge d'' x_J := (dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \otimes (dx_{j_1} \wedge \cdots \wedge dx_{j_q}).$$

There is a natural alternating wedge product $A^{p,q}(U) \times A^{p',q'}(U) \to A^{p+p',q+q'}(U)$ with $(\alpha, \beta) \mapsto \alpha \wedge \beta$.

ii) There are differential operators $d' \colon A^{p,q}(U) \to A^{p+1,q}(U)$ given by

$$d'\alpha := \sum_{|I|=p, |J|=q} \sum_{i=1}^r \frac{\partial \alpha_{IJ}}{\partial x_i} d' x_i \wedge d' x_I \wedge d'' x_J,$$

and $d'' \colon A^{p,q}(U) \to A^{p,q+1}(U)$ given by

$$d''\alpha := \sum_{|I|=p, |J|=q} \sum_{j=1}^r \frac{\partial \alpha_{IJ}}{\partial x_j} d'' x_j \wedge d' x_I \wedge d'' x_J.$$

Remark 5.1.2. Within the context of this thesis we are interested in the composition $d'd'' \colon A^{0,0}(U) \to A^{1,1}(U)$. Hence, we will just work with $A^{0,0}(U) = C^\infty(U)$, $A^{1,0}(U) = C^\infty(U) \otimes_{\mathbb{R}} \mathbb{R}^{r*}$, $A^{0,1}(U) = C^\infty(U) \otimes_{\mathbb{R}} \mathbb{R}^{r*}$, and $A^{1,1}(U) = C^\infty(U) \otimes_{\mathbb{R}} \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathbb{R}^{r*}$. In particular, the differential operators are given for any $f \in A^{0,0}(U) = C^\infty(U)$ as follows

$$d' f = \sum_{i=1}^r \frac{\partial f}{\partial x_i} d' x_i, \quad d'' f = \sum_{j=1}^r \frac{\partial f}{\partial x_j} d'' x_j,$$

and hence

$$d'd'' f = \sum_{i,j \in \{1,\dots,r\}} \frac{\partial^2 f}{\partial x_j \partial x_i} d' x_i \otimes d'' x_j. \tag{5.1}$$

Recall, that the linear map $dx_i \colon \mathbb{R}^r \to \mathbb{R}$ sends $v = \sum_{k=1}^r \lambda_k x_k$ to the coefficient λ_i respective to the basis $x_1, \dots, x_r$.

Lemma 5.1.3. *Let $U \subset \mathbb{R}^r$ be an open set and $f \in C^\infty(U)$. Then f is affine on U if and only if $d'd''f = 0$.*

Proof. This is a direct consequence of Equation (5.1). □

Next, we will define superforms on polyhedral complexes following [Gu13, §3] which are used later for the definition of differential forms on algebraic varieties.

Definition 5.1.4. i) A *polyhedron* in $\mathbb{R}^r$ is the intersection of finitely many half-spaces $H_i := \{w \in \mathbb{R}^r | \langle u_i, w\rangle \leq c_i\}$ with $u_i \in \mathbb{R}^{r*}$.

ii) A *polyhedral complex* $\mathscr{C}$ in $\mathbb{R}^r$ is a finite set of polyhedra in $\mathbb{R}^r$ satisfying the following two properties:

a) If τ is a face of a polyhedra $\sigma \in \mathscr{C}$, then $\tau \in \mathscr{C}$.

b) If $\sigma, \tau \in \mathscr{C}$, then $\sigma \cap \tau$ is a closed face of both.

Definition 5.1.5. Let $\mathscr{C}$ be a polyhedral complex in $\mathbb{R}^r$.

i) We say that $\mathscr{C}$ is of *dimension n* if the maximal dimension of its polyhedra is n. $\mathscr{C}$ is called *pure dimensional of dimension n* if every maximal polyhedron in $\mathscr{C}$ has dimension n.

ii) The *support* $|\mathscr{C}|$ of $\mathscr{C}$ is the union of all polyhedra in $\mathscr{C}$.

iii) Let $\sigma \in \mathscr{C}$, then $\mathbb{A}_\sigma$ denotes the affine space which is spanned by σ and $\mathbb{L}_\sigma$ denotes the corresponding linear subspace of $\mathbb{R}^r$.

iv) An open subset Ω of $|\mathscr{C}|$ is called *polyhedrally star shaped* with center z if there is a polyhedral complex $\mathscr{D}$ such that Ω is an open subset of $\mathscr{D}$ and for all maximal $\sigma \in \mathscr{D}$ the set $\sigma \cap \Omega$ is star shaped with center z in the sense that for all $x \in \sigma \cap \Omega$ and for all $t \in [0,1]$ the point $z + t(x - z)$ is contained in $\sigma \cap \Omega$ (cf. [Je, Definition 2.13]).

Definition 5.1.6. Let $\mathscr{C}$ be a polyhedral complex and Ω an open subset of $|\mathscr{C}|$. Then a *superform* $\alpha \in A^{p,q}(\Omega)$ *of bidegree* (p,q) on Ω is given by a superform $\alpha' \in A^{p,q}(V)$ where V is an open subset of $\mathbb{R}^r$ with $V \cap |\mathscr{C}| = \Omega$. Two forms $\alpha' \in A^{p,q}(V)$ and $\alpha'' \in A^{p,q}(W)$ with $V \cap |\mathscr{C}| = W \cap |\mathscr{C}| = \Omega$ define the same form in $A^{p,q}(\Omega)$ if we have for each $\sigma \in \mathscr{C}$

$$\langle \alpha'(x); v_1, \ldots, v_p, w_1, \ldots, w_q\rangle = \langle \alpha''(x); v_1, \ldots, v_p, w_1, \ldots, w_q\rangle$$

for all $x \in \sigma \cap \Omega$, $v_1, \ldots, v_p, w_1, \ldots, w_q \in \mathbb{L}_\sigma$. If this is true, we say that the restrictions $\alpha'|_\sigma$ and $\alpha''|_\sigma$ agree. If $\alpha \in A^{p,q}(\Omega)$ is given by $\alpha' \in A^{p,q}(V)$ we write

$$\alpha'|_\Omega = \alpha.$$

5.1.7. Let $F : \mathbb{R}^{r'} \to \mathbb{R}^r$ be an affine map. If $\mathscr{C}'$ is a polyhedral complex of $\mathbb{R}^{r'}$ and $\mathscr{C}$ a polyhedral complex of $\mathbb{R}^r$ with $F(|\mathscr{C}'|) \subset |\mathscr{C}|$, then the pullback $F^* : A^{p,q}(|\mathscr{C}|) \to$

$A^{p,q}(|\mathscr{C}'|)$ is well-defined and compatible with the differential operators d' and d''. Hence, we have also differential operators d' and d'' on $A^{p,q}(|\mathscr{C}|)$ given by the restriction of the corresponding operators on $A^{p,q}(\mathbb{R}^r)$.

To introduce (p,q)-forms on X^{an}, we first recall the analytification of X and define tropical charts of X^{an}.

Definition 5.1.8 (Analytification of X)**.** Let $U = \mathrm{Spec}(A)$ be an open affine subset of X, then let U^{an} be the set of all multiplicative seminorms on A extending $|\ |$ on K, endowed with the topology generated by the functions $U^{\mathrm{an}} \to \mathbb{R};\ p \mapsto p(a)$ with a ranging over A. By glueing, we get a topological space X^{an} which is connected locally compact and Hausdorff. We can endow it with a sheaf of analytic functions leading to a Berkovich analytic space over K which we call the *analytification* of X. We refer to [Be] for a more detailed definition and the fundamental properties of X^{an}.

5.1.9. If $\varphi : Y \to X$ is a morphism of algebraic varieties over K, we get an analytic morphism

$$\varphi^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$$

induced by composing the multiplicative seminorms with $\varphi^\sharp$ on suitable affine open subsets.

Definition 5.1.10. Let $T := \mathbb{G}_m^r$ be the split multiplicative torus of rank r with coordinates $z_1, \dots, z_r$.

i) We define the *tropicalization map* by

$$\mathrm{trop}\colon T^{\mathrm{an}} \to \mathbb{R}^r,\ p \mapsto (-\log p(z_1), \dots, -\log p(z_r)).$$

ii) Let Y be a closed subvariety of T. The *tropical variety associated to* Y is defined by

$$\mathrm{Trop}(Y) := \mathrm{trop}(Y^{\mathrm{an}}).$$

Remark 5.1.11. The tropicalization map is continuous.

Definition 5.1.12. Let U be an open subset of the algebraic variety X.

i) A *moment map* is a morphism $\varphi\colon U \to \mathbb{G}_m^r$.

ii) The *tropicalization* of φ is defined by

$$\varphi_{\mathrm{trop}} := \mathrm{trop} \circ \varphi^{\mathrm{an}}\colon U^{\mathrm{an}} \to \mathbb{R}^r.$$

iii) Let $U' \subset U$ be another open subset of X and $\varphi'\colon U' \to \mathbb{G}_m^{r'}$ a moment map. We say that φ' *refines* φ if there is an affine homomorphism of tori (i.e. a group homomorphism composed with a multiplicative translation) $\psi\colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$ such that $\varphi = \psi \circ \varphi'$ on U'.

Remark 5.1.13. If a moment map $\varphi' \colon U' \to \mathbb{G}_m^{r'}$ refines a moment map $\varphi \colon U \to \mathbb{G}_m^r$, the map $\psi \colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$ from above induces an affine map $\mathrm{Trop}(\psi) \colon \mathbb{R}^{r'} \to \mathbb{R}^r$ with $\varphi_{\mathrm{trop}} = \mathrm{Trop}(\psi) \circ \varphi'_{\mathrm{trop}}$ on $(U')^{\mathrm{an}}$.

Definition 5.1.14. If U is an open affine subset of X, one can construct a *canonical moment map* φ_U which is canonical up to multiplicative translation by an element of $T_U(K)$ and coordinate change: The abelian group $M_U := \mathcal{O}(U)^\times / K^\times$ is free of finite rank by [Sa, Lemme 1] and we choose representatives $\varphi_1, \dots, \varphi_r$ in $\mathcal{O}(U)^\times$ of a basis. Due to

$$\begin{aligned}\mathrm{Hom}_{K-\mathrm{Sch}}(U, \mathbb{G}_m^r) &= \mathrm{Hom}_{K-\mathrm{Alg.}}(\Gamma(\mathbb{G}_m^r, \mathcal{O}_{\mathbb{G}_m^r}), \Gamma(U, \mathcal{O}_U)) \\ &= \mathrm{Hom}_{K-\mathrm{Alg.}}(K[z_1^{\pm 1}, \dots, z_r^{\pm 1}], \Gamma(U, \mathcal{O}_U)) \\ &= (\Gamma(U, \mathcal{O}_U)^\times)^r,\end{aligned}$$

this leads to a moment map $\varphi_U \colon U \to \mathbb{G}_m^r$. We will write T_U for the canonical tori $\mathbb{G}_m^r$. By construction, φ_U refines all moment maps of U.

Definition 5.1.15. An open subset U of X is called *very affine* if U has a closed embedding into a multiplicative torus.

Remark 5.1.16. The very affine open subsets of X form a basis for the Zariski topology. If U is a very affine open subset of X, the canonical moment map φ_U from 5.1.14 is a closed embedding. These properties are stated in [Gu13, 4.13].

Definition 5.1.17. i) For a very affine open subset U of X we define

$$\mathrm{trop}_U := (\varphi_U)_{\mathrm{trop}},$$

and

$$\mathrm{Trop}(U) := \mathrm{trop}_U(U^{\mathrm{an}}).$$

ii) A *tropical chart* (V, φ_U) on X^{an} consists of an open subset V of X^{an} contained in U^{an} for a very affine open subset U of X with

$$V = \mathrm{trop}_U^{-1}(\Omega)$$

for some open subset Ω of $\mathrm{Trop}(U)$.

iii) We say that the tropical chart $(V', \varphi_{U'})$ is a *tropical subchart* of (V, φ_U) if $V' \subset V$ and $U' \subset U$.

Remark. i) If (V, φ_U) is a tropical chart on X^{an} as in the definition above, $\mathrm{trop}_U(V) = \Omega$ is open in $\mathrm{Trop}(U)$.

ii) The tropical charts form a basis of X^{an}, i.e. for every open subset W of X^{an} and for every element x in W there is a tropical chart (V, φ_U) such that $x \in V \subset W$ (cf. [Gu13, Proposition 4.16 a)]).

With the help of Bieri-Groves, we can introduce differential forms on algebraic varieties:

Proposition 5.1.18. *If X is an algebraic variety of dimension n over K and U is a very affine open subset of X, then* $\mathrm{Trop}(U)$ *is the support of an $\mathbb{R}$-affine polyhedral complex of pure dimension n.*

Proof. A reference and further explanations are given in [Gu12, Theorem 3.3]. □

5.1.19. The last proposition allows us to consider a superform $\alpha \in A^{p,q}(\mathrm{trop}_U(V))$ for a tropical chart (V, φ_U) of X^{an}. Let $(V', \varphi_{U'})$ be another tropical chart of X^{an}, then $(V \cap V', \varphi_{U\cap U'})$ is a tropical subchart of both by [Gu13, Proposition 4.16]. We get a canonical homomorphism $\psi_{U,U\cap U'} \colon \mathbb{G}_m^s \to \mathbb{G}_m^r$ of the underlying tori with

$$\varphi_U = \psi_{U,U\cap U'} \circ \varphi_{U\cap U'}$$

on $U \cap U'$ and an associated affine map $\mathrm{Trop}(\psi_{U,U\cap U'}) \colon \mathbb{R}^s \to \mathbb{R}^r$ such that

$$\mathrm{trop}_U = \mathrm{Trop}(\psi_{U,U\cap U'}) \circ \mathrm{trop}_{U\cap U'}$$

and the tropical variety $\mathrm{Trop}(U \cap U')$ is mapped onto $\mathrm{Trop}(U)$ (cf. [Gu13, 5.1]). We define the *restriction* of α to $\mathrm{trop}_{U\cap U'}(V \cap V')$ as

$$\mathrm{Trop}(\psi_{U,U\cap U'})^* \alpha \in A^{p,q}(\mathrm{trop}_{U\cap U'}(V \cap V'))$$

and write $\alpha|_{V\cap V'}$.

Definition 5.1.20. i) A *differential form* α of bidegree (p, q) on an open subset W of X^{an} is given by a covering $(V_i)_{i\in I}$ of W by tropical charts (V_i, φ_{U_i}) of X^{an} and superforms $\alpha_i \in A^{p,q}(\mathrm{trop}_{U_i}(V_i))$ such that

$$\alpha_i|_{V_i\cap V_j} = \alpha_j|_{V_i\cap V_j}$$

for every $i, j \in I$.

If α' is another differential form of bidegree (p, q) on W given by $\alpha'_j \in A^{p,q}(\mathrm{trop}_{U'_i}(V'_i))$ with respect to the tropical charts $(V'_i, \varphi_{U'_i})$ covering W, then we consider α and α' as the same differential forms if and only if

$$\alpha_i|_{V_i\cap V'_j} = \alpha'_j|_{V_i\cap V'_j}$$

for every $i \in I$ and $j \in J$.

ii) We denote the space of (p, q)-differential forms on an open subset W of X^{an} by $A^{p,q}(W)$.

iii) If $\alpha \in A^{p,q}(W)$ is given by a covering of tropical charts (V_i, φ_{U_i}) and superforms $\alpha_i \in A^{p,q}(\mathrm{trop}_{U_i}(V_i))$, then we define $d'\alpha$ resp. $d''\alpha$ to be given by (V_i, φ_{U_i}) and the superforms $d'\alpha_i \in A^{p+1,q}(\mathrm{trop}_{U_i}(V_i))$ resp. $d''\alpha_i \in A^{p,q+1}(\mathrm{trop}_{U_i}(V_i))$.

Remark 5.1.21. Let f be a differential form of bidegree $(0,0)$ on an open subset W of X^{an}. Then $f\colon W \to \mathbb{R}$ is a well-defined continuous map.

Proof. If f is given by a covering $(V_i, \varphi_{U_i})_{i\in I}$ of W and $f_i \in A^{0,0}(\mathrm{trop}_{U_i}(V_i))$, then

$$f = f_i \circ \mathrm{trop}_{U_i}$$

on V_i for every $i \in I$. Consider an arbitrary $x \in W$ which is contained in charts V_i and V_j. We have seen in 5.1.19 that

$$\mathrm{trop}_{U_i} = \mathrm{Trop}(\psi_{U_i, U_i \cap U_j}) \circ \mathrm{trop}_{U_i \cap U_j}$$

and

$$\mathrm{trop}_{U_j} = \mathrm{Trop}(\psi_{U_j, U_i \cap U_j}) \circ \mathrm{trop}_{U_i \cap U_j}.$$

We have required in the definition of differential forms that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$, i.e.

$$f_i \circ \mathrm{Trop}(\psi_{U_i, U_i \cap U_j}) = f_j \circ \mathrm{Trop}(\psi_{U_j, U_i \cap U_j}).$$

Hence, $f_i(\mathrm{trop}_{U_i}(x)) = f_j(\mathrm{trop}_{U_j}(x))$. Thus, $f(x)$ is independent of $i \in I$, and so f is a well-defined function on W. Further, f is continuous in every $x \in W$ as a composition of continuous functions. □

Definition 5.1.22. Let W be an open subset of X^{an}. We denote the space of smooth functions on W by $C^\infty(W) := A^{0,0}(W)$.

5.2 The link between the presheaf $\log|\mathcal{O}_X^\times|$ and smooth functions

Again, we consider an algebraically closed field K endowed with a non-trivial complete non-archimedean absolute value $|\ |$. Let X be an algebraic variety over K of dimension n. The goal of this section is to give a connection between smooth functions defined in Chapter 4.1 and functions of the form $\log|f|\colon X^{\mathrm{an}} \to \mathbb{R}$ for a $f \in \mathcal{O}_X^\times$. We will see that smooth functions in the kernel of $d'd''$ can be written locally as a linear combination of functions in $\log|\mathcal{O}_X^\times|$. Further, we show that $\log|f|$ is smooth and contained in the kernel of $d'd''$ for each $f \in \mathcal{O}_{X^{\mathrm{an}}}^\times$.

Lemma 5.2.1. *Let W be an open subset of X^{an} and $f \in C^\infty(W)$, then $f \in \ker d'd''$ if and only if for every $x \in W$ there is a tropical chart (V, φ_U) with $x \in V \subset W$ such that*

$$f = g \circ \mathrm{trop}_U$$

on V for an affine map $g\colon \mathbb{R}^r \to \mathbb{R}$ where $T_U = \mathbb{G}_m^r$.

Proof. We assume that f belongs to the kernel of $d'd''$ and consider an arbitrary $x \in W$. Due to $f \in C^\infty(W)$, we can find a tropical chart (V, φ_U) of X^{an} such that $x \in V \subset W$ and $f = g \circ \mathrm{trop}_U$ on V for a superform $g \in C^\infty(\mathrm{trop}_U(V))$. The neighborhood V of x has the form $\mathrm{trop}_U^{-1}(\Omega)$ for an open subset Ω of $\mathrm{Trop}(U) \subset \mathbb{R}^r$ which is the support of a polyhedral complex of pure dimension n. In particular, we have $\mathrm{trop}_U(V) = \Omega$. We can choose the tropical chart in the way that $\mathrm{trop}_U(V)$ is polyhedrally star shaped. We may assume (by translation) that the centre is the origin. Since we have required that f belongs to the kernel of $d'd''$, [Gu13, Proposition 5.6] implies that $d'd''g = 0$ in $A^{1,1}(\mathrm{trop}_U(V))$. Hence, g is affine on each polyhedron in $\mathrm{trop}_U(V)$. Let σ be such a polyhedron in $\mathrm{trop}_U(V)$ and ν a vector in $\mathbb{L}_\sigma$. The function g comes from a smooth function on an open set of $\mathbb{R}^r$, so the linear map $Dg\colon \mathbb{R}^r \to \mathbb{R}$ satisfies $Dg(0)(\nu) = \partial g / \partial \nu$. We have seen above that g is affine on σ, so g is given by $g(\nu) = \partial g/\partial \nu + g(0)$ on σ. Thus, g coincides with the affine map $Dg + g(0)$ on $\mathrm{trop}_U(V)$.

The other direction is a direct consequence of the definitions and Lemma 5.1.3. □

Remark 5.2.2. Let U be a very affine open subset of X and $f \in \mathcal{O}_X(U)^\times$, then the morphism $\varphi\colon U \to \mathbb{G}_m^1$ obtained by the map $K[z^{\pm 1}] \to \mathcal{O}_X(U);\ z \mapsto f$ is refined by the canonical moment map φ_U. By Remark 5.1.13, there is an affine map $\Psi\colon \mathbb{R}^r \to \mathbb{R}$ such that $\log|f| = \mathrm{trop} \circ \varphi^{\mathrm{an}} = -\Psi \circ \mathrm{trop}_U$ on U^{an}. Hence, $\log|f|\colon U^{\mathrm{an}} \to \mathbb{R}$ is smooth and belongs to $\ker(d'd'')$.

For the analytification X^{an} of the algebraic variety X one obtains a morphism of locally ringed spaces from X^{an} to X. If U is an open affine subset of X this morphism leads to an injective map $\mathcal{O}_X(U) \to \mathcal{O}_{X^{\mathrm{an}}}(U^{\mathrm{an}})$ (a description of the structure sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ can be found in [Th, Remarque 2.1.11]). In the following we therefore give a generalization of the previous Remark.

Proposition 5.2.3. *Let W be an open subset of X^{an} and $f \in \mathcal{O}_{X^{\mathrm{an}}}(W)^\times$, then the function $\log|f|\colon W \to \mathbb{R}$ is smooth and belongs to $\ker d'd''$.*

Proof. Let $f \in \mathcal{O}_{X^{\mathrm{an}}}(W)^\times$ and set $T := \mathbb{G}_m^1$. Then f defines the analytic morphism $\varphi\colon W \to T^{\mathrm{an}}$ which is locally given by $x \mapsto (F \mapsto |F(f)|_x)$, and satisfies $(\mathrm{trop} \circ \varphi)(x) = -\log|f(x)|$. [Gu13, Proposition 7.2] states that for every $x \in W$ there is a very affine open subset U of X with a moment map $\varphi'\colon U \to T$ and an open neighborhood V of x in $U^{\mathrm{an}} \cap W$ such that $\mathrm{trop} \circ \varphi = \varphi'_{\mathrm{trop}}$ on V. The canonical moment map φ_U refines $\varphi'\colon U \to T$, i.e. $\varphi'_{\mathrm{trop}} = \mathrm{Trop}(\psi) \circ \mathrm{trop}_U$ on U^{an} for an affine map $\mathrm{Trop}(\psi)$. Thus,

$$-\log|f| = \mathrm{Trop}(\psi) \circ \mathrm{trop}_U$$

is satisfied on an open neighborhood of x. Therefore, we can find a covering of W by tropical charts (V_i, φ_{U_i}) such that $\log|f| = f_i \circ \mathrm{trop}_{U_i}$ on V_i for $f_i \in A^{0,0}(\mathrm{trop}_{U_i}(V_i))$, and so the function $\log|f|$ is smooth on W. Moreover, Lemma 5.2.1 tells us that $-\log|f|$ belongs to $\ker(d'd'')$. □

Theorem 5.2.4. *Let W be an open subset of X^{an}. A function $f\colon W\to\mathbb{R}$ belongs to the kernel of $d'd''\colon C^\infty(W)\to A^{1,1}(W)$ if and only if for every $x\in W$ there is an open neighborhood V of x in W and an open subset U of X with $V\subset U^{\mathrm{an}}$ such that*

$$f=\sum_{i=1}^{r}\lambda_i\log|f_i|$$

on V where $f_1,\dots,f_r\in\mathcal{O}_X(U)^\times$ and $\lambda_1,\dots,\lambda_r\in\mathbb{R}$.

Proof. If $f\in\ker d'd''\subset C^\infty(W)$, then for every $x\in W$ there is a tropical chart (V,φ_U) such that

$$f=g\circ\mathrm{trop}_U$$

on V for an affine map $g\colon\mathbb{R}^r\to\mathbb{R}$ by Lemma 5.2.1. Due to the definition of the canonical moment map, there are $f_1,\dots,f_r\in\mathcal{O}_X(U)^\times$ such that

$$f=g\circ(-\log|f_1|,\dots,-\log|f_r|)$$

on V. Since g is affine, f is of the form $\sum_{i=1}^{r}\lambda_i\log|f_i|+C$ for $\lambda_i\in\mathbb{R}$ and a constant $C\in\mathbb{R}$. The absolute value $|\ |$ is non-trivial, so we can find $\lambda_{r+1}\in\mathbb{R}$ and $f_{r+1}\in K$ such that $C=\lambda_{r+1}\log|f_{r+1}|$.

If f has the described form, Remark 5.2.2 implies the other direction.

□

5.3 The link between the presheaf $\log|\mathcal{O}_X^\times|$ and harmonic functions

In Section 4, we have already defined harmonic functions on $\mathbb{P}^1_{\mathrm{Berk}}$. At the beginning of this section, we verify that a function on an open subset of $\mathbb{P}^1_{\mathrm{Berk}}$ is harmonic if and only if it can be written locally as a linear combination of $\log|f|$ where $f\in\mathcal{O}_X^\times$ for $X=\mathbb{P}^1_K$. Using Theorem 5.2.4, we can link the terms harmonic and smooth if $X=\mathbb{P}^1_K$. Afterwards, we will define harmonic functions on a smooth strictly analytic curve X generally (cf. [Th, §2.3]) and show that this definition is indeed an extension of the one made in Section 4. By [Th, Théorème 2.3.21], we get two explicit conditions in which the sheaf of harmonic functions coincides with the associated sheaf to the presheaf $\log|\mathcal{O}_X^\times|$. Thuillier considers in [Th] smooth strictly k-analytic curves over a field k which is complete with respect to a non-trivial non-archimedean absolute value $|\ |$. He does not require that k has to be algebraically closed different to Baker and Rumely in [BR] or Gubler in [Gu13]. Since we do not want to limit Thuillier's definition in [Th], we use the notion K if we require that the field has to be algebraically closed and k if not. For the link to smoothness, we consider again a smooth algebraic curve X over K. The analytification X^{an} is a smooth strictly K-analytic curve, and so we can apply Thuillier's theorem to X^{an}. The theorem and the characterization of $\ker d'd''$ (cf. Theorem 5.2.4) give us two explicit condition in which the harmonic functions coincides with the smooth functions in $\ker d'd''$. Further, one can construct a smooth algebraic curve X such that there is a harmonic function which is not smooth.

Theorem 5.3.1. *Let W be an open subset of $\mathbb{P}^1_{\mathrm{Berk}}$, then f is harmonic on W if and only if for every $x \in W$ there is an open neighborhood V of x in W and an open subset U of $\mathbb{P}^1_K$ with $V \subset U^{\mathrm{an}}$ such that*

$$f = \sum_{i=1}^{r} \lambda_i \log |f_i|$$

on V where $f_1, \ldots, f_r \in \mathcal{O}_{\mathbb{P}^1_K}(U)^\times$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$.

Proof. If f is harmonic on W, for every $x \in W$ there is a strict simple domain $V \subset W$ containing x such that f is harmonic on V. By Corollary 4.4.9, there are $c_0, \ldots, c_m \in \mathbb{R}$ and $a_1, \ldots, a_m \in \mathbb{P}(K)\backslash\overline{V}$ such that

$$f(x) = c_0 - \sum_{i=1}^{m} c_i \log([T - a_i]_x)$$

on $\overline{V}$ where $\partial V = \{x_1, \ldots, x_m\} \subset \mathbb{H}_{\mathrm{Berk}}$. The tropical charts form a basis of the Berkovich topology on $\mathbb{P}^1_{\mathrm{Berk}}$, so we can find a tropical chart $(\widetilde{V}, \varphi_{\widetilde{U}})$ with $x \in \widetilde{V} \subset V$ and $a_i \notin \widetilde{U}$ for $i = 1, \ldots, m$. Hence, $f_i := T - a_i \in \mathcal{O}_X(\widetilde{U})^\times$ for every $i = 1, \ldots, m$. Furthermore, we can find a $\lambda_{r+1} \in \mathbb{R}$ and an element $f_{r+1} \in \mathbb{P}(K)$ such that $\lambda_{r+1} \log |f_{r+1}| = c_0$. Since $[f_i]_x$ and $|f_i(x)|$ are just different notations, the claim is true.

Assume that for every $x \in W$ the function f has the described form on an open neighborhood V of x in W. Then we can find a domain $\widetilde{V}$ contained in V such that $x \in \widetilde{V}$. The functions $\log |f_i|$ are strongly harmonic on $\widetilde{V}$ by Example 4.1.5, and so f is strongly harmonic on $\widetilde{V}$. Thus, f is harmonic on W. □

Corollary 5.3.2. *A function f is harmonic on an open subset W of $\mathbb{P}^1_{\mathrm{Berk}}$ if and only if f is smooth on W and $d'd''f = 0$.*

Proof. This is a direct consequence of Theorem 5.2.4 and Theorem 5.3.1. □

Up to now, K was an algebraically closed field which is complete with respect to a non-trivial non-archimedean absolute value $| \ |$. From now on, we work over a field k and we do not require that k is algebraically closed. We set $k^\circ := \{a \in k | |a| \leq 1\}$ and $k^{\circ\circ} = \{a \in k | |a| < 1\}$. The residue field $k^\circ/k^{\circ\circ}$ is denoted by $\tilde{k}$. For a k°-algebra $\mathcal{A}$ we set $\mathrm{Spf}(\mathcal{A}) := \{\mathfrak{p} \in \mathcal{A} | \ k^{\circ\circ} \subset \mathfrak{p}\}$. Further, we use the notation $S := \mathrm{Spf}(k^\circ)$.

Definition 5.3.3. i) A Berkovich k-analytic space Y is called *strictly k-affinoid* if every $y \in Y$ admits a fundamental system of neighborhoods consisting of compact strictly k-affinoid domains.

ii) A *strictly k-analytic domain* of Y is a subset $V \subset Y$ which has a locally finite covering by strictly k-affinoid domains.

iii) If Y is a strictly k-affinoid space, we define the *rigid site* of Y as the category whose objects are the strictly k-analytic domains of Y, the morphisms are the inclusions and with the induced Grothendieck topology (see [Th, §2.1.1 p.20]). We will write Y_R for the rigid site of Y.

iv) A *strictly k-analytic curve X* is given by a paracompact topological space $|X|$ and a sheaf of k-algebras $\mathcal{O}_X$ on $|X|$ such that the ringed space $(|X|, \mathcal{O}_X)$ is locally isomorphic to $(Y \backslash \partial Y, \mathcal{O}_Y)$ where Y is a strictly k-affinoid space of pure dimension 1.

5.3.4. By [Th, Remarque 2.1.11], the analytification of a 1-dimensional algebraic variety over k is a strictly k-analytic curve, e.g. $\mathbb{P}^1_{\mathrm{Berk}}$. We have seen in Chapter 2 that one can classify the points of $\mathbb{P}^1_{\mathrm{Berk}}$ in four different types. This classification can be extended to an arbitrary strictly k-analytic curve X (cf. [Th, §2.1 p.27: Classification des points]).

Definition 5.3.5. i) A *S-curve $\mathcal{X}$* is a formal S-scheme which is locally of finite type, flat, separated and of pure dimension 1.

ii) We call a S-curve $\mathcal{X}$ *strictly semi-stable* if $\mathcal{X}$ has an open covering $(U_i)_{i \in I}$ such that there are $a_i \in k^\circ \backslash \{0\}$ and étale morphisms $\varphi_i \colon U_i \to \mathfrak{S}(a_i)$ where

$$\mathfrak{S}(a_i) := \mathrm{Spf}(k^\circ\{T_0, T_1\}/(T_0 T_1 - a_i)).$$

5.3.6. If $\mathcal{X}$ is a strictly semi-stable S-curve, then the generic fibre $\mathcal{X}_\eta$ is a strictly k-affinoid space which is rig-smooth (cf. [Th, Remarque 2.2.9] and for the definition of rig-smooth we refer to [Te, Definition 4.2.22]). For each strictly semi-stable S-curve $\mathcal{X}$ there is a unique pair $(S(\mathcal{X}), \tau_{\mathcal{X}})$ of a polyhedral complex $S(\mathcal{X})$ in $\mathcal{X}_\eta$ of dimension 1 and a retraction map $\tau_{\mathcal{X}} \colon \mathcal{X}_\eta \to S(\mathcal{X})$ satisfying certain compatibility conditions (cf. [Th, Théorèm 2.2.10]). Note that the subset $S(\mathcal{X})$ of X just contains points of type II and III (cf. [Th, Définition 2.2.13]).

Definition 5.3.7. If $\mathcal{X}$ is a strictly semi-stable S-curve, we call $S(\mathcal{X})$ the *skeleton* of $\mathcal{X}$.

In the following we will restrict to quasi-compact strictly semi-stable S-curves $\mathcal{X}$, which is equivalent to the fact that the topological space $|\mathcal{X}_\eta|$ is compact.

Definition 5.3.8. i) For a polyhedral complex S of dimension 1 and a locally finite subset Γ of S the space $H(S, \Gamma)$ is defined as the space of piecewise affine functions f on S satisfying $\sum_{\vec{v} \in T_p S} d_{\vec{v}} f(p) = 0$ for each $p \in S \backslash \Gamma$.

ii) Let $\mathcal{X}$ by a strictly semi-stable S-curve and $\partial\mathcal{X}_\eta$ the Berkovich boundary of $\mathcal{X}_\eta$ (see [Th, §2.1.2]). Then we define

$$H(\mathcal{X}) := \tau_{\mathcal{X}}^* H(S(\mathcal{X}), \partial\mathcal{X}_\eta)$$

which is a subspace of $C^0(|\mathcal{X}_\eta|, \mathbb{R})$.

5.3.9. Let Y be a strictly k-affinoid space of pure dimension 1 which is rig-smooth. By [Th, Théorèm 2.3.8] there exists a finite Galois extension k' of k, a strictly semi-stable curve $\mathcal{Y}$ and an isomorphism $\varphi\colon Y \otimes_k k' \to \mathcal{Y}_\eta$. Then the real subspace $(\varphi^* H(\mathcal{Y}))^{\mathrm{Gal}(k'/k)}$ of $C^0(|Y|, \mathbb{R})$ is independent of k', $\mathcal{Y}$ and φ by [Th, Proposition 2.3.3] and [Th, Proposition 2.3.7].

Definition 5.3.10. If Y is a strictly k-affinoid space of pure dimension 1 which is rig-smooth, we set $H(Y) := (\varphi^* H(\mathcal{Y}))^{\mathrm{Gal}(k'/k)}$ for a strictly semi-stable curve $\mathcal{Y}$ and an isomorphism $\varphi\colon Y \otimes_k k' \to \mathcal{Y}_\eta$.

5.3.11. Let X be a strictly k-analytic smooth curve. Let C be the category whose objects are the strictly k-affinoid spaces of pure dimension 1 which are rig-smooth. The morphisms in C are the affinoid immersions. By [Th, Proposition 2.3.3], we have a functor $H\colon C^{\mathrm{op}} \to \mathrm{Vect}_{\mathbb{R}}$. If V is an object in the category X_R, the strictly k-affinoid domains contained in V form a inductive filtered system $I(V)$ where the morphisms are the inclusions. Hence,

$$H_X(V) := \varprojlim_{V' \in I(V)} H(V')$$

defines a presheaf H_X on X_R. For each strictly k-affinoid domain Y in X the canonical homomorphism from $H_X(Y)$ to $H(Y)$ is an isomorphism. Note that H_X is no sheaf ([Th, Remarque 2.3.11]). Every open subset Ω of X has a local finite cover of strictly k-affinoid domains. Thus, Ω is an object in X_R and every open cover of Ω is a cover in X_R. We denote the site of the topological space $|X|$ by $|X|$ and we have a canonical morphism of sites $\iota\colon X_R \to |X|$. [Th, Corollaire 2.3.15] says that $\iota_* H_X$ is a sheaf on $|X|$.

Definition 5.3.12. We denote the sheaf $\iota_* H_X$ by $\mathcal{H}_X$ and call it the sheaf of *harmonic functions.*

Remark 5.3.13. The presheaf C_X of the germes of real continuous functions on $|X|$ is actually a sheaf on X_R (cf. [Th, Remarque 2.1.7]) and H_X is a subpresheaf of it. The elements of $\mathcal{H}_X(X)$ are the real continuous functions on $|X|$ whose restrictions belongs to $H(Y) \subset C^0(|Y|, \mathbb{R})$ for every strictly k-affinoid domain Y of X.

Before we start to state and verify the announced link, we will try to understand that this definition of harmonic functions on open subsets of $\mathbb{P}^1_{\mathrm{Berk}}$ coincides with the old one.

Proposition 5.3.14. *Let W be an open subset of $\mathbb{P}^1_{\mathrm{Berk}}$, then the vector space $\mathcal{H}(W)$ of harmonic functions on W introduced in [BR] coincides with the vector space $\mathcal{H}_X(W)$.*

Proof. Consider a function f in $\mathcal{H}(W)$, i.e. f is harmonic on W in the sense of Chapter 4. The last Remark tells us that it suffices to consider a strictly k-affinoid domain $Y \subset W$. We may assume that the interior of Y is connected, and so the interior of Y is a strict simple domain by [BR, Lemma 2.28], i.e. Y has a finite boundary and all boundary points are of type II. In particular, the interior of Y coincides with $r_\Gamma^{-1}(\Gamma^0)$ for a finite subgraph contained in Y whose endpoints are the boundary points of Y. In the setting of Chapter 4 the field K is algebraically closed, and so there is a strictly semi-stable S-curve $\mathcal{Y}$ such that Y is the generic fibre of $\mathcal{Y}$ by [Th, Théorèm 2.3.8]. The induced skeleton $S(\mathcal{Y})$ is connected (cf. [BPR, Proposition 3.9] and [BPR, Proposition 4.10]), and so a finite subgraph of $\mathbb{P}^1_{\mathrm{Berk}}$. Moreover, we may assume by [Th, Théorèm 2.2.22] that the boundary points of Y are vertices of the skeleton $S(\mathcal{Y}) \subset Y$ and Γ is contained in $S(\mathcal{Y})$. Our function f is harmonic on $r_\Gamma^{-1}(\Gamma^0)$ by Corollary 4.1.8 and we have the description $f = \widetilde{f} \circ r_\Gamma$ on Y for a function $\widetilde{f} \in \mathrm{CPA}(\Gamma)$ by Proposition 4.4.3. By Corollary 4.4.4, f is especially strongly harmonic on $r_\Gamma^{-1}(\Gamma^0)$. We know from Lemma 4.1.9 that the sum of outgoing slopes of $\widetilde{f}$ at any point in $\Gamma \backslash \partial Y$ is zero. Further, Proposition 4.2.6 states that f is constant on every path leading away from Γ. Hence, we can extend $\widetilde{f}$ properly to $S(\mathcal{Y})$ such that $f \in H(Y)$.

Now assume that $f \in \mathcal{H}_X(W)$. It suffices to show that f is harmonic on each strict simple subdomain V of W satisfying $\overline{V} \subset W$. By [BR, Lemma 2.27], the closure $Y := \overline{V}$ of V is a strictly $k-$affinoid domain contained in W. By assumption $f \in H(Y)$, and $H(Y) = \tau_{\mathcal{Y}}^*(S(\mathcal{Y}), \partial Y)$ for a strictly semi-stable S-curve $\mathcal{Y}$ with $Y = \mathcal{Y}_\eta$ by [Th, Théorèm 2.3.8]. Therefore, $f = \widetilde{f} \circ \tau_{\mathcal{Y}}$ on Y for a piecewise affine function $\widetilde{f}$ on $S(\mathcal{Y})$ with

$$\sum_{\vec{v} \in T_p} d_{\vec{v}} \widetilde{f}(p) = 0$$

for all $p \in S(\mathcal{Y}) \backslash \partial Y$. Note that $\Gamma := S(\mathcal{Y})$ is a finite subgraph of $\mathbb{P}^1_{\mathrm{Berk}}$ and $\tau_{\mathcal{Y}} = r_\Gamma$. Therefore, Example 3.2.2 implies that f is harmonic on V. □

Let X be a strictly k-analytic smooth curve. In this subsection we will link harmonic functions on X with the presheaf $\log|\mathcal{O}_X^\times|$.

Proposition 5.3.15. *For each section $f \in \Gamma(X, \mathcal{O}_X^\times)$ the function $\log|f|$ is harmonic on X.*

Proof. It suffices to show that the restriction $\log|f|$ to every strictly k-affinoid domain $Y \subset X$ belongs to the subspace $H(Y) \subset C^0(|Y|, \mathbb{R})$. By [Th, Théorèm 2.3.8] and [Th, Lemme 2.3.5], we may assume that Y is the generic fibre of a simple semi-stable S-curve $\mathcal{Y}$. [Th, Proposition 2.2.24] states that $\log|f| = \log|f| \circ \tau_{\mathcal{Y}}$ on Y and that the restriction of $\log|f|$ to $S(\mathcal{Y})$ is harmonic on $S(\mathcal{Y}) \backslash \partial Y$, i.e. $\log|f| \in H(\mathcal{Y})$. □

Definition 5.3.16. Let $\mathcal{F}_X$ denote the associated sheaf to the presheaf on $|X|$ which maps an open set U to the subspace of $C^0(|U|, \mathbb{R})$ generated by the functions $\log|f|$ where $f \in \Gamma(U, \mathcal{O}_X^\times)$.

Theorem 5.3.17. *Let X be a strictly k-analytic smooth curve over k. Then $\mathcal{F}_X$ is a subsheaf of $\mathcal{H}_X$ and $\mathcal{H}_X/\mathcal{F}_X$ is supported on a discrete set of points of type II. Moreover, $\mathcal{H}_X/\mathcal{F}_X$ is zero if one of the following conditions is satisfied:*

i) The residue field $\tilde{k}$ is algebraic over a finite field.

ii) The curve $X\widehat{\otimes}_k\widehat{k^a}$ is locally isomorphic to $\mathbb{P}^1_{\mathrm{Berk}}$ over $\widehat{k^a}$ where $\widehat{k^a}$ is the completion of the algebraic closure of k.

By Proposition 5.3.15 and [Ha, Exercise 1.4], $\mathcal{F}_X$ is a subsheaf of $\mathcal{H}_X$. To prove the rest of the theorem above, we need an analogue statement to a fact we have proved in Chapter 4 and further lemmata. Theorem 4.4.7 states that the Dirichlet problem is uniquely solvable on finite-dendrite domains. Similarly, we have the following lemma in the case of affinoid domains:

Lemma 5.3.18. *Let Y be a k-affinoid domain in X, then*

$$H(Y) \to \mathrm{Hom}(\partial Y, \mathbb{R});\ \ h \mapsto h|_{\partial Y}$$

is an isomorphism.

Proof. Note that the Shilov boundary of a k-affinoid domain in X coincides with its Berkovich boundary (cf. [Th, Proposition 2.1.12]) . With this fact, the lemma is proved in [Th, Corollaire 3.1.21]. □

Lemma 5.3.19. *Let x be a point of type II, k' a finite Galois extension of k and x' a point in $X' := X \otimes_k k'$ contained in the preimage of x under the canonical morphism $p\colon X \otimes_k k' \to X$. Then $\mathcal{F}_{X',x'} = \mathcal{H}_{X',x'}$ implies $\mathcal{F}_{X,x} = \mathcal{H}_{X,x}$.*

Proof. Assume that $\mathcal{F}_{X',x'} = \mathcal{H}_{X',x'}$. By Proposition 5.3.15, it remains to verify the inclusion $\mathcal{H}_{X,x} \subset \mathcal{F}_{X,x}$. Let V be a strictly k-affinoid neighborhood of x in X and V' the connected component containing x' in $V \otimes_k k'$. Consider a function $h \in H(V)$, then p^*h is harmonic on V' by [Th, Lemme 2.3.5]. Hence, $(p^*h)_{|V'}$ is a linear combination of functions $\log|f'|$ where $f' \in A_{V'}^\times$. The norm $N(f')$ is defined as the determinant of the A_V-algebra automorphism $A_{V'} \to A_{V'}$ given by the multiplication with f', where $A_{V'}$ is a free A_V-algebra of rank $[k' : k]$. By [Th, Proposition 2.1.8], we get that $f := N(f') \in A_V^\times$ and $p^*|f|_{|V'} = |f'|^{[k':k]}$. Using the norm, an easy calculation shows that h can be written as a linear combination of functions $\log|f|$ with $f \in A_V^\times$. Thus, $\mathcal{F}_{X',x'} = \mathcal{H}_{X',x'}$ implies $\mathcal{F}_{X,x} = \mathcal{H}_{X,x}$. □

Lemma 5.3.20. *If X is isomorphic to the generic fibre of a strictly semi-stable S-curve $\mathcal{X}$ and x is a point of type II corresponding to a proper, smooth and geometrically connected irreducible component C_x of the special fibre $\mathcal{X}_s$, then $\mathcal{H}_{X,x}/\mathcal{F}_{X,x}$ is canonically isomorphic to the vector space* $\mathrm{Pic}^0(C_x) \otimes_{\mathbb{Z}} \mathbb{R}$.

Proof. Let V be a strictly k-affinoid neighborhood of x. By [BL, Lemma 4.4], there is an admissible blow up $q\colon \mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is a strictly semi-stable $\mathrm{Spf}(k^\circ)$-curve and there is a formal scheme $\mathcal{U}'$ open in $\mathcal{X}'$ such that V is isomorphic to $\mathcal{U}'_\eta$. The morphism q induces an isomorphism from the irreducible component of $\mathcal{X}'$ associated to x to C_x.

For every $h \in H(V)$ we can define an $\mathbb{R}$-divisor $\mathrm{div}(h)$ on the $\widetilde{k}$-curve C_x. We can see this $\widetilde{k}$-curve as an irreducible component of $\mathcal{X}'_s$. Then the tangent space $T_xS(\mathcal{X}')$ can be canonically identified with the set of singular points of $\mathcal{X}'_s$ contained in C_x, and we denote the point corresponding to $\vec{v} \in T_xS(\mathcal{X}')$ by $\widetilde{x_{\vec{v}}}$. Since h is harmonic on the neighborhood V of x, we can set

$$\mathrm{div}(h) = \sum_{\vec{v}\in T_xS(\mathcal{X}')} d_{\vec{v}}h(x)[\widetilde{x_{\vec{v}}}],$$

and

$$\deg(\mathrm{div}(h)) = \sum_{t\in T_xS(\mathcal{X}')} d_{\vec{v}}h(x) = 0.$$

This leads to a linear map

$$\mathrm{div}\colon \mathcal{H}_{X,x} \to \mathrm{Div}(C_x) \otimes_{\mathbb{Z}} \mathbb{R},$$

and the following sequence

$$0 \to \mathbb{R} \to \mathcal{H}_{X,x} \xrightarrow{\mathrm{div}} \mathrm{Div}(C_x) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\deg} \mathbb{R} \to 0,$$

which can be verified to be exact. Obviously, the map $\mathbb{R} \to \mathcal{H}_{X,x}$, which maps a real number to a constant function, is injective. Moreover, $\mathrm{div}(h) = 0$ if and only if the harmonic function h is locally constant in x, and the map $\deg\colon \mathrm{Div}(C_x) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ is surjective. Further, we have seen above that $\mathrm{im}(\mathrm{div}) \subset \ker(\deg)$. So it remains to consider an $\mathbb{R}$-divisor

$$D = \sum_{\widetilde{x}\in|D|} n(\widetilde{x})[\widetilde{x}]$$

of degree 0, and find a function $h \in \mathcal{H}_{X,x}$ such that $\mathrm{div}(h) = D$. One can construct a strictly semi-stable S-curve $\mathcal{Y}$ such that the generic fibre $\mathcal{Y}_\eta$ is a neighborhood of x contained in $X = \mathcal{X}_\eta$ and

$$|D| \subset T_xS(\mathcal{Y}),$$

where $T_xS(\mathcal{Y})$ can be identified canonically with a finite set of closed points in C_x.

Consider the subscheme $Z := |D| \cap (\mathcal{X}_s \backslash \mathrm{Sing}(\mathcal{X}_s))$ in $\mathcal{X}_s$ and let $q\colon \mathcal{X}' \to \mathcal{X}$ be a blow up of Z. $\mathcal{X}'$ being strictly-semistable is equivalent to say that $\mathcal{X}'_\eta$ is smooth and $\mathcal{X}'_s$ is a $\widetilde{k}$-curve (locally algebraic) which has smooth irreducible components, and ordinary double points as singularities ([Th, Remarque 2.2.9]). Since the points in Z are no singularities, the blow up $\mathcal{X}'$ of Z is a strictly semi-stable S-curve as well. Let $E := q^{-1}(Z)$ be the exceptional divisor on $\mathcal{X}'$. By removing from each irreducible component of E a closed point disjoint of the strict transform of $\mathcal{X}$ in $\mathcal{X}'$, i.e. the closure of $\mathcal{X}\backslash Z$ in $\mathcal{X}'$, we obtain $\mathcal{Y}$. Then $\mathcal{Y}$ which is open in $\mathcal{X}'$ is a strictly semi-stable S-curve and we have $|D| \subset T_x S(\mathcal{Y})$.

Let H be a piecewise affine function on $S(\mathcal{Y})$ which satisfies

$$d_{\vec{v}} H(x) = \begin{cases} n(\widetilde{x_{\vec{v}}}); & \widetilde{x_{\vec{v}}} \in |D|, \\ 0; & \widetilde{x_{\vec{v}}} \notin |D|. \end{cases}$$

Due to $\sum_{t \in T_x S(\mathcal{Y})} d_{\vec{v}} H(x) = \deg(D) = 0$, the function H is harmonic in x. Hence, we can find a neighborhood of x in $S(\mathcal{Y})$ where the piecewise affine function H is harmonic on. Hence, the function $h := \tau_{\mathcal{Y}}^* H$ is harmonic on a neighborhood of x in X. By construction, we have $\mathrm{div}(h) = D$. Thus, $\ker(\deg) \subset \mathrm{im}(\mathrm{div})$, and so the sequence is exact.

To get the claim, we consider another short sequence

$$0 \to \mathbb{R} \to \mathcal{F}_{X,x} \xrightarrow{\mathrm{div}} \mathrm{Pr}(C_x) \to 0$$

where $\mathrm{Pr}(C_x)$ denotes the $\mathbb{R}$-vector space generated by the principal divisors on C_x. First, we show that $\mathrm{div}\colon \mathcal{F}_{X,x} \to \mathrm{Pr}(C_x)$ is well-defined. It suffices to consider a function of the form $\log|f|$ for $f \in \mathcal{O}_{X,x}^\times$ in the $\mathbb{R}$-vector space $\mathcal{F}_{X,x}$. Since x is of type II, we can find a $N \in \mathbb{N}_{\geq 1}$ such that $|f(x)|^N \in |k^\times|$ (cf. [Th, §2.1 p.27: Classification des points]). Let $\alpha \in k$ be an element such that $|f(x)|^N = |\alpha|$. If $\widetilde{x_{\vec{v}}}$ is the singular point in $\mathcal{X}'_s$ corresponding to $\vec{v} \in T_x S(\mathcal{X}')$, we can find a small enough neighborhood of $\widetilde{x_{\vec{v}}}$ containing only $\widetilde{x_{\vec{v}}}$ as a singularity, i.e. x is the endpoint of the corresponding skeleton, and we may apply [Th, Lemme 2.2.25]. This lemma says that there is a meromorphic function $\widetilde{f}$ on C_x induced by f^N/α such that

$$d_{\vec{v}}(N \log|f|)(x) = -\mathrm{ord}_{\widetilde{x_{\vec{v}}}}(\widetilde{f}).$$

Thus, $\mathrm{div}(h)$ belongs to the $\mathbb{R}$-vector space $\mathrm{Pr}(C_x)$ for any $h \in \mathcal{F}_{X,x}$. As above, the map $\mathbb{R} \to \mathcal{F}_{X,x}$ is injective and the image coincides with the kernel of the map div. Let f be a nonzero meromorphic function on C_x and $\mathrm{Div}(f)$ its principal divisor. Again, we can consider an admissible blow up such that every point in $|\mathrm{Div}(f)|$ is a singularity, and we may apply [Th, Lemme 2.2.25]. We therefore can find a $\lambda \in \mathbb{R}$ such that $\mathrm{div}(\lambda \cdot \log|f|) = \mathrm{Div}(f)$. Hence, $\mathrm{div}\colon \mathcal{F}_{X,x} \to \mathrm{Pr}(C_x)$ is surjective, and so the second sequence is exact as well.

Let $\mathrm{Div}^0(C_x)$ denote the $\mathbb{R}$-vector space generated by the divisors on C_x of degree zero. Then we have the following commutative diagramm of exact sequences

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{F}_{X,x} & \xrightarrow{\mathrm{div}} & \mathrm{Pr}(C_x) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{H}_{X,x} & \xrightarrow{\mathrm{div}} & \mathrm{Div}^0(C_x) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \mathcal{H}_{X,x}/\mathcal{F}_{X,x} & & \mathrm{Pic}^0(C_x)\otimes_{\mathbb{Z}}\mathbb{R} & &
\end{array}
$$

.

The snake lemma implies the isomorphism

$$\mathcal{H}_{X,x}/\mathcal{F}_{X,x} \xrightarrow{\sim} \mathrm{Pic}^0(C_x)\otimes_{\mathbb{Z}}\mathbb{R}.$$

□

Proof of Theorem 5.3.17. One can show that the germes of $\mathcal{H}_X$ and $\mathcal{F}_X$ coincide for each point x in X of type I, III or IV. If x is of type I or IV, there is a fundamental system of neighborhoods of x consisting of k-affinoid domains which have a unique boundary point. Lemma 5.3.18 states that the sections of $\mathcal{H}_X$, and so of $\mathcal{F}_X$ as well, are constant on these neighborhoods of x. Thus, the germes coincide. If x is of type III, there is a fundamental system of neighborhoods of x consisting of k-affinoid domains having exactly two boundary points. Then the $\mathbb{R}$-vector space $\mathcal{H}_{X,x}$ has dimension 2 by Lemma 5.3.18. Hence, $\mathcal{H}_{X,x}$ is generated by the germes of the constant function 1 and the function $\log|f|$, where f is a global section of $\mathcal{O}_X$ such that $|f|$ is not locally constant on the neighborhoods of x. This means that $\mathcal{H}_{X,x}$ coincides with its subspace $\mathcal{F}_{X,x}$.

Now we assume that x is a point of type II. By [Th, Théorèm 2.3.8], there is a finite seperable field extension k' of k such that $X\otimes_k k'$ is isomorphic to the generic fibre of a strictly semi-stable S-curve $\mathcal{X}$ and that x corresponds to a proper and geometrically connected irreducible component C_x of the special fibre $\mathcal{X}_s$. By Lemma 5.3.19 we may assume that for X.

To show that $\mathrm{supp}(\mathcal{H}_X/\mathcal{F}_X)$ is discrete, we verify that $\mathrm{supp}(\mathcal{H}_X/\mathcal{F}_X)$ is discrete at our arbitrary $x\in X$ of type II. For every point $y\in X$ of type II Lemma 5.3.20 states $\mathcal{H}_{X,y}/\mathcal{F}_{X,y}\cong \mathrm{Pic}^0(C_y)\otimes_{\mathbb{Z}}\mathbb{R}$ for the corresponding proper and smooth irreducible component C_y of $\mathcal{X}_s$, which is uniquely determined by its function field $\widetilde{\kappa(y)}$ (cf. [Ha, Corollary 4.5]). Let T be the set of all points $y \in X$ which are of type II and satisfy $C_y \not\cong \mathbb{P}^1_{\widetilde{k}}$. Further, let $S_0(\mathcal{X})$ denote the set consisting of the points in X which correspond to vertices of the skeleton $S(\mathcal{X})$. If $y \in X \backslash S(\mathcal{X})$, there is an affinoid neighborhood of y which is isomorphic to a closed ball by [Th, Définition 2.2.13], and if y is contained $S(\mathcal{X})$ but not in $S_0(\mathcal{X})$ there is an affinoid neighborhood of y which is isomorphic to an annulus. Hence, we can take y as a point of type II in $\mathbb{A}^1_k$, and so $\widetilde{\kappa(y)} \cong \widetilde{k}(T)$. Since C_y is uniquely determined by $\widetilde{\kappa(y)}$, $C_y = \mathbb{P}^1_{\widetilde{k}}$ for all $y \in X \backslash S_0(\mathcal{X})$, i.e. $T \subset S_0(\mathcal{X})$. The set of vertices of a skeleton is locally finite, and so $\operatorname{supp}(\mathcal{H}_X/\mathcal{F}_X)$ is discrete at x.

Now we come to the second part of the theorem. Above we have passed over to a finite field extension of k, so we show that for every $x \in X\widehat{\otimes}_k\widehat{k^a}$ of type II the equality $\operatorname{Pic}^0(C_x) \otimes_{\mathbb{Z}} \mathbb{R} = 0$ for the irreducible, proper and smooth $\widetilde{k^a}$-curve C_x is satisfied. This implies $\mathcal{H}_X/\mathcal{F}_X = 0$.

First, we assume that $\widetilde{k}$ is algebraic over a finite field. We know that $\operatorname{Pic}^0(C_x)$ is isomorphic to the group of $\widetilde{k^a}$-points of the Jacobian variety, i.e. $\operatorname{Pic}^0(C_x) \cong \mathrm{J}(C_x)(\widetilde{k^a})$. Since $\widetilde{k^a}$ is the algebraic closure of a finite field, we have

$$\operatorname{Pic}^0(C_x) \cong \bigcup_{k' \subset \widetilde{k^a}, |k'| < \infty} \mathrm{J}(C_x)(k').$$

The Jacobian variety is a $\widetilde{k^a}$-scheme of finite type, and so $\mathrm{J}(C_x)(k')$ is finite for every finite field k' contained in $\widetilde{k^a}$, i.e. in particular torsion. Hence, $\operatorname{Pic}^0(C_x) \otimes_{\mathbb{Z}} \mathbb{R} = 0$ for every $x \in X\widehat{\otimes}_k\widehat{k^a}$ of type II.

Now, we assume that the second condition is satisfied. We consider a point $x \in X\widehat{\otimes}_k\widehat{k^a}$ of type II and assume that $X\widehat{\otimes}_k\widehat{k^a}$ is locally isomorphic to the analytification of $\mathbb{P}^1_{\widehat{k^a}}$. As mentioned above it suffices to determine the function field $\widetilde{\kappa(x)}$ to get C_x. Hence, we may identify $X\widehat{\otimes}_k\widehat{k^a}$ with $\mathbb{P}^1_{\mathrm{Berk}}$ over the field $\widehat{k^a}$, and so $\widetilde{\kappa(x)} = \widetilde{k^a}(t)$ for the type II point x (cf. [BR, Proposition 2.3]). Therefore, the $\widetilde{k^a}$-curve C_x has to be isomorphic to $\mathbb{P}^1_{\widetilde{k^a}}$, and so $\operatorname{Pic}^0(C_x) = 0$. □

To link harmonic functions to smooth functions, we consider again an algebraically closed field K endowed with a non-trivial non-archimedean complete absolute value $|\ |$. If X is a smooth algebraic curve over K, the analytification X^{an} is a strictly K-analytic smooth curve. Hence, we may apply Theorem 5.3.17 to X^{an}. Using Theorem 5.2.4, we get the following corollary.

Corollary 5.3.21. *Let X be a smooth algebraic curve over K and assume that one of the following holds:*

i) $\widetilde{K}$ is algebraic over a finite field.

ii) X^{an} is locally isomorphic to $\mathbb{P}^1_{\mathrm{Berk}}$.

Then a function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} is harmonic if and only if it is smooth and $d'd''f = 0$.

Proof. If W is an open subset of X^{an}, the vector space $\mathcal{F}_{X^{\mathrm{an}}}(W)$ coincides with $\ker d'd'' \subset C^\infty(W)$ by Theorem 5.2.4 and Proposition 5.2.3. Hence, Theorem 5.3.17 implies the claim. □

In particular, one can see that Thuillier's theorem leads to the same result as Theorem 5.3.1 if $X = \mathbb{P}^1_K$. To give finally an answer to the question if every harmonic function on an open subset W of X^{an} is smooth, we state a further theorem:

Theorem 5.3.22. *Let X be a smooth algebraic curve over K. If a smooth function $f\colon W \to \mathbb{R}$ is harmonic, we have $d'd''f = 0$.*

Proof. Replacing X by its canonical smooth compactification, we may assume that X is proper. Consider a $x \in W$ and let V be a strictly K-affinoid neighborhood of x in W. We have required that K is algebraically closed, so we can find a strictly semi-stable $\mathrm{Spf}(K^\circ)$-curve $\mathcal{X}$ and a formal open subscheme $\mathcal{U}$ of $\mathcal{X}$ such that V is isomorphic to $\mathcal{U}_\eta$ by [Th, Théorèm 2.3.8] and [BL, Lemma 4.4]. Since f is harmonic on W, the function f belongs to $H(V)$ and is consequently given by $f = \varphi \circ \tau_{\mathcal{U}}$ on V for a piecewise affine map $\varphi\colon S(\mathcal{U}) \to \mathbb{R}$. [Th, Théorèm 2.2.10] implies that we can extend φ to a piecewise affine function on the skeleton $S(\mathcal{X})$ satisfying $f|_V = \varphi \circ \tau_{\mathcal{X}}$ on V. By [GH, Proposition B.7], there is a unique line bundle $\mathcal{L}$ on $\mathcal{X}$ with $\mathcal{L}|_X = \mathcal{O}_X$ such that $\varphi = -\log\|s\|_{\mathcal{L}}$ for the canonical invertible global section $s = 1$ of $\mathcal{O}_X$. Hence, $\|\ \|_{\mathcal{L}}$ coincides with the metric $\|\ \|_\varphi$ on $\mathcal{O}_X$ which is given by $\|1\|_\varphi := e^{-\varphi}$. Note that the metric $\|\ \|_{\mathcal{L}}$ is called a formal metric. For neat definitions of these metrics we refer to [GH, §1.2]. The metric $\|\ \|_\varphi = \|\ \|_{\mathcal{L}}$ leads to the following discrete measure

$$c_1(\mathcal{O}_X, |\ \|_\varphi) := \sum_Y \deg_{\mathcal{L}}(Y) \cdot \delta_{\zeta_Y}$$

where Y runs over all irreducible components of the special fibre $\mathcal{X}_s$ and ζ_Y is the unique point in X^{an} such that $\mathrm{red}(\zeta_y)$ is the generic point of Y (cf. [Be, Proposition 2.4.4]). This measure was introduced by Chambert-Loir and Ducros (in higher dimension) in [CD].

If W' is an open subset and $\alpha \in A^{p,q}(W')$, the *support* of α is the complement in W' of the set of points x of W' which have an open neighborhood V_x such that $\alpha|_{V_x} = 0$. Let $A_c^{p,q}(W')$ denote the space of (p,q)-differential forms with compact support in W'. Every $(1,1)$-differential form ω on X^{an} induces a unique signed Radon measure μ_ω on X^{an} such that $\int_{X^{\mathrm{an}}} g\, d\mu = \int_{X^{\mathrm{an}}} g \wedge \omega$ for every $g \in C_c^\infty(X^{\mathrm{an}})$ and we may identify ω with μ_ω (cf. [Gu13, Example 6.7] and [Gu13, Example 6.8]). For the definition of the integral of a differential form over W we refer to [Gu13, 5.14]. Next to differential forms, one can also define currents on X^{an}. A definition of them can be found in [Gu13, 6.2]. Setting $\widetilde{f} := \varphi \circ \tau_{\mathcal{X}}$, the function $\widetilde{f}$ on X^{an} is continuous and coincides with f on V. We can define the following current

$$[\widetilde{f}]\colon \mathcal{A}_c^{1,1}(X^{\mathrm{an}}) \to \mathbb{R};\ \omega \mapsto \int_{X^{\mathrm{an}}} \widetilde{f} d\mu_\omega$$

and consider $d'd''[\widetilde{f}]$ for the differential operators d' and d''. In this setting, [GK, Theorem 10.5] implies that

$$d'd''[\widetilde{f}] = c_1(\mathcal{O}_X, \|\ \|_\varphi),$$

where we understand $d'd''[\widetilde{f}]$ as a measure.

On the other hand, Thuillier defined in [Th, §3.2.4] a measure-valued Laplacian operator $\mathrm{dd^c}$ on a class of functions which contains $\widetilde{f}$ (cf. [Th, Théorème 3.2.10]). In particular, the kernel of $\mathrm{dd^c}$ is the sheaf of harmonic functions $\mathcal{H}_X$ (cf. [Th, Corollaire 3.2.11]). By [KRZB, Theorem 2.6], we have the identity

$$\mathrm{dd^c}\widetilde{f} = c_1(\mathcal{O}_X, \|\ \|_\varphi),$$

and so the measures $d'd''[\widetilde{f}]$ and $\mathrm{dd^c}\widetilde{f}$ coincide.

Since $f = \widetilde{f}|_V$ is harmonic on V, we have

$$d'd''[f]|_{V'} = d'd''[\widetilde{f}]|_{V'} = \mathrm{dd^c}\widetilde{f}|_{V'} = \mathrm{dd^c}f|_{V'} = 0$$

on an open neighborhood V' of x contained in V. Further, we have required that f is smooth on W, and so in particular on V'. [Gu13, Theorem 5.17] implies that $d'd''[f] = [d'd''f]$ on V' where $[d'd''f](g) := \int_{V'} d'd''f \wedge g$ for every $g \in C_c^\infty(V')$. Together, we get that $\int_{V'} d'd''f \wedge g = 0$ for every $g \in C_c^\infty(V')$ and so $d'd''f$ has to be zero on the open neighborhood V' of our arbitrary x in W. □

Altogether, we have the following conclusion:

Corollary 5.3.23. *Harmonic functions are not smooth in general, i.e. there is a smooth curve X over K and a harmonic function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} which is not smooth.*

Proof. Using Lemma 5.3.20 one can construct a smooth algebraic curve X over an algebraically closed field K such that $\mathcal{H}_{X^{\mathrm{an}}}/\mathcal{F}_{X^{\mathrm{an}}}$ is nonzero. Consider $\mathbb{C}(T)$ with the absolute value corresponding to the vanishing order at zero and set $K = \widehat{\mathbb{C}(T)^a}$. Let E be a curve over $\widetilde{K} = \mathbb{C}$ such that $\mathrm{Pic}^0(E) \otimes_{\mathbb{Z}} \mathbb{R}$ is nonzero, e.g. an elliptic curve of positive rank, and let ζ be the generic point of E. Consider the smooth algebraic curve $X := E \otimes_{\mathbb{C}} K$ over K, then we obtain a reduction map $\mathrm{red}\colon X^{\mathrm{an}} \to E$ such that there is a unique point $x \in X^{\mathrm{an}}$ satisfying $\widetilde{K(x)} = \widetilde{K}(\zeta)$ (cf. [Be, Proposition 2.4.4]). Since the corresponding irreducible curve C_x over $\widetilde{K}$ is uniquely determined by its function field, we have $C_x = E$. Hence, $\mathcal{H}_{X^{\mathrm{an}},x}/\mathcal{F}_{X^{\mathrm{an}},x} = \mathrm{Pic}^0(C_x) \otimes_{\mathbb{Z}} \mathbb{R}$ is nonzero by Lemma 5.3.20. We therefore can find an open subset W of X^{an} and a harmonic function $f\colon W \to \mathbb{R}$ which is not contained in $\mathcal{F}_{X^{\mathrm{an}}}(W)$. By Theorem 5.2.4 and Proposition 5.2.3, the vector space $\mathcal{F}_{X^{\mathrm{an}}}(W)$ coincides with the kernel of the linear operator $d'd''\colon C^\infty(W) \to A^{1,1}(W)$. Finally, Theorem 5.3.22 implies that f cannot be smooth. $\square$

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Selbständigkeitserklärung

Ich, Veronika Wanner, erkläre hiermit, dass ich die vorgelegte Masterarbeit mit dem Thema „Harmonic Functions on the Berkovich Projective Line“ selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Ort, Datum

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