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Perturbation theory for critical points of causal variational principles

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PERTURBATION THEORY FOR CRITICAL POINTS OF CAUSAL VARIATIONAL PRINCIPLES

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Abstract. The perturbation theory for critical points of causal variational principles is developed. We first analyze the class of perturbations obtained by multiplying the universal measure by a weight function and taking the push-forward under a diffeomorphism. Then the constructions are extended to convex combinations of such measures, leading to perturbation expansions for the mean and the fluctuation of the measure, both being coupled in higher order perturbation theory. It is explained how our methods and results apply to the causal action principle for causal fermion systems. It is shown how the perturbation expansion in the continuum limit and the effect of microscopic mixing are recovered in specific limiting cases.

Contents

1. Introduction

The theory of causal fermion systems is an approach to describe fundamental physics. Giving quantum mechanics, general relativity and quantum field theory as limiting cases, it is a candidate for a unified physical theory (see [7] or the survey article [9]). So far, the connection to perturbative quantum field theory has been established by first taking the continuum limit [7] and then including the mechanism of microscopic mixing (see [6]). Although this procedure gives the correct limiting case with an interaction described by a unitary time evolution on Fock spaces (see [6, Section 8]), the derivation is not quite convincing conceptually because it is based on the perturbation expansion for solutions of the Dirac equation coupled to classical bosonic fields as obtained in the continuum limit (see [7, §3.8.4] and [6, Section 2]). In order to clarify the mathematical structure of the theory, it is desirable to perform the perturbation expansion directly for the universal measure of the causal fermion system, without referring to specific limiting cases. This also opens up the research program to explore how the perturbation theory for causal fermion systems differs from perturbative quantum field theory, with the goal of making experimental predictions.

In this paper the general perturbation theory for causal fermion systems is developed. We thus succeed in extending the methods of perturbative quantum field theory to non-smooth situations where space-time has a non-trivial, possibly discrete microscopic structure and the physical equations are no longer obtained by quantizing partial differential equations. We work in the jet formalism introduced in [10] in the more general and at the same time more convenient framework of causal variational principles in the non-compact setting. Our perturbation expansion has the nice feature that the bosonic and fermionic perturbations are described on the same footing in terms of jet spaces containing bosonic and fermionic subspaces.

The paper is organized as follows. Section 2 provides the necessary background on causal variational principles and the jet formalism. In Section 3 the perturbation theory is developed for the class of perturbations obtained by multiplying the universal measure by a weight function and taking the push-forward under a diffeomorphism (see (3.1)). After bringing the combinatorics into a convenient form (Section 3.1), we invert the linearized equations with Green's operators (see Definition 3.3 in Section 3.2). The resulting perturbation expansion is summarized in Section 3.3. In Section 3.4 it is explained how, starting from a linearized solution, one can construct a one-parameter family of nonlinear solutions of the field equations. In Sections 3.5 it is shown how, perturbing the vacuum by an inhomogeneity, one can construct a corresponding nonlinear solution of the field equations.

In Section 4 the perturbation theory is extended to measures of the more general form (4.1), allowing for the possibility that the measure is "decomposed" into several components and the support of the measure is "enlarged" (see Figure 3 on page 12). The key for developing this so-called *perturbation theory with fragmentation* is to decompose suitable jets describing the perturbation into their mean and the fluctuations (see (4.4) in Section 4.1). Intuitively speaking, the fluctuations take into account the effects of a *second quantization* of the bosonic fields (as is made precise by the limiting case analyzed in [6]). A technical complication is that the fluctuations of the scalar component drop out of the field equations to first and second order in perturbation theory, making it necessary to determine them from the third and higher order

contributions (Section 4.2). The resulting perturbation expansion is summarized in Section 4.3.

In Section 5 we explain how our methods and results apply to the setting of causal fermion systems. After the necessary preliminaries (Section 5.1), the perturbation expansion for the wave evaluation operator is derived (Section 5.2). After identifying jets with perturbations of the wave evaluation operator, the general perturbation expansion applies in a straightforward way (Section 5.3).

In Section 6, it is shown that by a suitable choice of the jet spaces one recovers the analysis in the continuum limit as carried out in [7]. Finally, in Section 7 we describe how to incorporate the effect of microscopic mixing as analyzed in [6].

2. Preliminaries

2.1. **Causal Variational Principles in the Non-Compact Setting.** We consider causal variational principles in the non-compact setting as introduced in [10, Section 3]. Thus let $\mathcal{F}$ be a (possibly non-compact) smooth manifold of dimension $m \geq 1$. Moreover, we are given a non-negative function $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$ (the *Lagrangian*) with the following properties:

(i) $\mathcal{L}$ is lower semi-continuous, i.e. for all sequences $x_n \to x$ and $y_{n'} \to y$,

$$\mathcal{L}(x,y) \leq \lim_{n,n' \to \infty} \mathcal{L}(x_n, y_{n'}) \,.$$

(ii) $\mathcal{L}$ is symmetric: $\mathcal{L}(x,y) = \mathcal{L}(y,x)$ for all $x, y \in \mathcal{F}$.

Next, we let ρ be a (positive) Borel measure on $\mathcal{F}$ (the *universal measure*). The *causal variational principle* is to minimize the action

$$\mathcal{S} = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y)\, \mathcal{L}(x,y)$$

under variations of the measure ρ, keeping the total volume $\rho(\mathcal{F})$ fixed. If the total volume is infinite, one can make mathematical sense of variations of $\mathcal{S}$ by considering variations of ρ of finite total variation and zero volume (for details see [10, Section 3]). Here we do not enter the details of the minimization procedure and of the properties of the minimizing measure. Instead, we restrict attention to the resulting Euler-Lagrange (EL) equations as derived in [10, Lemma 3.3]:

Definition 2.1. *A Borel measure ρ on $\mathcal{F}$ is a* **critical point** *of the causal variational principle if it has the following properties:*

- *The measure ρ is locally finite.*
- *The function $\mathcal{L}(x,.)$ is ρ-integrable for all $x \in \mathcal{F}$.*
- *For a suitable value of the parameter $\nu > 0$, the function ℓ defined by*

$$\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) - \frac{\nu}{2} \tag{2.1}$$

is minimal and vanishes on the support of ρ,

$$\ell|_{\text{supp}\,\rho} \equiv \inf_{\mathcal{F}} \ell = 0 \,. \tag{2.2}$$

We remark that the value of the parameter ν can be changed arbitrarily by rescaling the measure according to

$$\rho \to \lambda \rho \qquad \text{with} \qquad \lambda > 0 \,. \tag{2.3}$$

With this in mind, we shall always keep ν fixed when varying or perturbing the measure.

2.2. **The Weak Euler-Lagrange Equations.** Let ρ be a critical point of the causal variational principle. We introduce *space-time* M as the support of this measure,

$$M := \operatorname{supp} \rho \subset \mathcal{F} .$$

The idea behind the formulation of the weak EL equations is to use only part of the information contained in the EL equations (2.2). Namely, we evaluate them only on M, taking into account first derivatives. Moreover, we restrict attention to directions where ℓ is differentiable. This leads to the following construction: We let $\Gamma = C^\infty(M, T\mathcal{F})$ be the smooth vector fields on $\mathcal{F}$ restricted to M (thus every $u \in \Gamma$ has a smooth extension to $\mathcal{F}$). Moreover, we let Γ^{diff} be those vector fields for which the directional derivative of the function ℓ exists,

$$\Gamma^{\text{diff}} = \big\{ u \in C^\infty(M, T\mathcal{F}) \;\big|\; D_u \ell(x) \text{ exists for all } x \in M \big\} .$$

Next, we introduce the space of one-jets

$$\mathfrak{J}^{\text{diff}} := C^\infty(M, \mathbb{R}) \oplus \Gamma^{\text{diff}} \;\subset\; C^\infty(M, \mathbb{R}) \oplus C^\infty(M, T\mathcal{F}) .$$

For a jet $\mathfrak{u} = (a, u) \in \mathfrak{J}^{\text{diff}}$ we define $\nabla_\mathfrak{u}$ as the linear combination of scalar multiplication and directional derivative, i.e.

$$\nabla_\mathfrak{u} \ell(x) := a(x)\, \ell(x) + \big(D_u \ell\big)(x) .$$

We choose a linear subspace $\mathfrak{J}^{\text{test}} \subset \mathfrak{J}^{\text{diff}}$ with the property that its scalar and vector components are both vector spaces,

$$\mathfrak{J}^{\text{test}} = C^{\text{test}}(M, \mathbb{R}) \oplus \Gamma^{\text{test}} \;\subseteq\; \mathfrak{J}^{\text{diff}} .$$

Then the *weak EL equations* read (for details cf. [10, (eq. (4.10))])

$$\nabla_\mathfrak{u} \ell|_M = 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J}^{\text{test}} . \tag{2.4}$$

2.3. **Solutions of the Linearized Field Equations.** Linearized solutions are linear perturbations of ρ which preserve the weak EL equations (2.4). We now give the precise definition (for more details cf. [10, Section 4.2]).

Definition 2.2. *A jet $\mathfrak{v} \in \mathfrak{J} := C^\infty(\mathcal{F}, \mathbb{R}) \oplus C^\infty(\mathcal{F}, T\mathcal{F})$ is referred to as a* **solution of the linearized field equations** *if it has the following properties:*

(l1) *For all $y \in M$ and all x in an open neighborhood of M, the following combination of directional derivatives exists,*

$$\big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathcal{L}(x, y) . \tag{2.5}$$

(l2) *Integrating the expression (2.5) over y with respect to the measure ρ, the resulting function (defined on an open neighborhood of M) is differentiable in the direction of every jet $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$ and*

$$\nabla_{\mathfrak{u}(x)} \int_M \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathcal{L}(x, y)\, d\rho(y) = \nabla_\mathfrak{u} \nabla_\mathfrak{v}\, \frac{\nu}{2} \qquad \text{for all } x \in M .$$

The vector space of all linearized solutions is denoted by $\mathfrak{J}^{\text{lin}} \subset \mathfrak{J}$.

Here ∇_1 and ∇_2 denote the partial derivatives of $\mathcal{L}(.,.)$ with respect to the first and second argument, respectively. The combination of directional derivatives in (2.5) is defined by

$$\big(D_{1,v} + D_{2,v}\big)\mathcal{L}(x,y) := \frac{d}{d\tau}\mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\big|_{\tau=0}\,,$$

where F_τ is the flow of the vector field v.

We also remark that it would suffice to define the jets in $\mathfrak{J}$ on an open neighborhood of M. But, keeping in mind that such mappings can be extended smoothly to all of $\mathcal{F}$, there is no loss in generality to define the jets on all of $\mathcal{F}$. Similarly, we will also introduce other functions later on as functions on $\mathcal{F}$, even if they are needed only in an open neighborhood of M.

3. The Abstract Perturbation Expansion

3.1. **Perturbation Expansion for the Universal Measure.** Let ρ be a measure (not necessarily a critical point of the causal variational principle). We want to construct a measure $\tilde{\rho}$ which satisfies the weak EL equations. To this end, we make the ansatz

$$\tilde{\rho} = F_*\big(f\,\rho\big)\,, \tag{3.1}$$

where f and F are smooth,

$$f \in C^\infty\big(\mathcal{F}, \mathbb{R}^+\big) \qquad \text{and} \qquad F \in C^\infty\big(\mathcal{F}, \mathcal{F}\big)\,. \tag{3.2}$$

Moreover, we assume that the total derivative $DF|_x : T_x\mathcal{F} \to T_{F(x)}\mathcal{F}$ has maximal rank at every $x \in M$. Intuitively speaking, the ansatz (3.1) implies that the general structure of the measure ρ (like being discrete or continuous, being supported on a smooth submanifold, etc.) is preserved by the perturbation. This ansatz is motivated mainly by its simplicity. More general perturbations of the universal measure will be studied in Section 4.

Exactly as in [10, Section 5.2], it is most convenient to evaluate the weak EL equations (2.4) on the unperturbed space-time $M := \operatorname{supp}\rho$. Then the weak EL equations can be written as (see [10, Lemma 4.2])

$$\nabla_{\mathfrak{u}(x)}\bigg(\int_M f(x)\,\mathcal{L}\big(F(x),F(y)\big)\,f(y)\,d\rho(y) - \frac{\nu}{2}\,f(x)\bigg) = 0\,, \tag{3.3}$$

to be satisfied for all $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$ and $x \in M$. We point out that in this equation, the derivative $\nabla_{\mathfrak{u}(x)}$ acts on both $f(x)$ and the first argument of $\mathcal{L}((F(x),F(y))$, where we must apply the product and chain rules, i.e.

$$\nabla_{\mathfrak{u}(x)}\mathcal{L}\big(F(x),F(y)\big) = a(x)\,\mathcal{L}\big(F(x),F(y)\big) + D_1\mathcal{L}\big(F(x),F(y)\big)\cdot DF|_x\cdot u(x)\,,$$

where $\mathfrak{u} = (a,u)$ and $DF|_x : T_x\mathcal{F} \to T_{F(x)}\mathcal{F}$ again denotes the total derivative.

We now assume that ρ is close to a critical point in the sense that

$$\nabla_{\mathfrak{u}}\bigg(\int_M \mathcal{L}(x,y)\,d\rho(y) - \frac{\nu}{2}\bigg) = \lambda\,\nabla_{\mathfrak{u}}E^{(1)} \tag{3.4}$$

with an error term $E^{(1)}$, where $\lambda \in \mathbb{R}$ is a small parameter. We expand both f and F in a power series in λ. For the function f, we make the perturbation ansatz

$$f(x) = \sum_{p=0}^\infty \lambda^p\, f^{(p)}(x) \qquad \text{with} \qquad f^{(0)}(x) = 1\,, \tag{3.5}$$

where the choice of $f^{(0)}$ will ensure that the measure $\tilde{\rho}$ goes over to the unperturbed measure ρ in the limit $\lambda \to 0$. For the expansion of F, we choose a chart around x and write $F(x)$ in components as $(F(x)^\alpha)_{\alpha=1,\ldots,m}$. Then we can expand F componentwise,

$$F(x)^\alpha = \sum_{p=0}^\infty \lambda^p\, F^{(p)}(x)^\alpha \qquad \text{with} \qquad F^{(0)}(x)^\alpha = x^\alpha\,. \tag{3.6}$$

For ease in notation, we shall omit the index α from now on. But one should keep in mind that the expansion of $F(x)$ always involves the choice of a chart around x.

Next, we want to evaluate (3.3) to any order $p = 1, 2, \ldots$ in λ. A Taylor expansion using (3.5) yields

$$\begin{aligned} 0 &= \frac{1}{p!}\frac{d^p}{d\lambda^p} \nabla_{\mathrm{u}(x)} \bigg(\int_M f(x)\, \mathcal{L}\big(F(x), F(y)\big)\, f(y)\, d\rho(y) - \frac{\nu}{2}\, f(x) \bigg) \bigg|_{\lambda=0} \\ &= \nabla_{\mathrm{u}(x)} \int_M \sum_{\substack{l,r,q \geq 0 \\ \text{with } l+r+q=p}} f^{(l)}(x)\, f^{(r)}(y)\, \frac{1}{q!} \frac{d^q}{d\lambda^q} \mathcal{L}\big(F(x), F(y)\big)\Big|_{\lambda=0}\, d\rho(y) \\ &\quad - \frac{\nu}{2}\, \nabla_{\mathrm{u}(x)} f^{(p)}(x)\,. \end{aligned} \tag{3.7}$$

In order to compute the λ-derivatives of the Lagrangian, we first expand it in a Taylor series in both arguments,

$$\begin{aligned} \mathcal{L}\big(F(x), F(y)\big) &= \sum_{a=0}^\infty \frac{1}{a!} \sum_{b=0}^\infty \frac{1}{b!}\, \big(D_{1,F(x)-x}\big)^a \big(D_{2,F(y)-y}\big)^b \mathcal{L}(x,y) \\ &= \sum_{k=0}^\infty \frac{1}{k!} \Big(D_{1,F(x)-x} + D_{2,F(y)-y}\Big)^k \mathcal{L}(x,y)\,. \end{aligned}$$

Here D_1 and D_2 denote the partial derivatives of $\mathcal{L}(.,.)$ with respect to the first and second argument, respectively. It is important to observe that these partial derivatives do not act on the arguments $F(x) - x$ or $F(y) - y$. Taking the q^{th} derivative with respect to λ and evaluating at $\lambda = 0$, we obtain

$$\begin{aligned} \frac{1}{q!}\frac{d^q}{d\lambda^q}\, \mathcal{L}\big(F(x), F(y)\big)\Big|_{\lambda=0} &= \frac{1}{q!} \sum_{k=0}^q \frac{1}{k!} \sum_{\substack{q_1,\ldots,q_k \geq 1 \\ \text{with } q_1+\cdots+q_k=q}} \binom{q}{q_1\, \cdots\, q_k} \\ &\quad \times\, q_1!\, \big(D_{1,F^{(q_1)}} + D_{2,F^{(q_1)}}\big) \cdots q_k!\, \big(D_{1,F^{(q_k)}} + D_{2,F^{(q_k)}}\big) \mathcal{L}(x,y) \\ &= \sum_{k=0}^q \frac{1}{k!} \sum_{\substack{q_1,\ldots,q_k \geq 1 \\ \text{with } q_1+\cdots+q_k=q}} \big(D_{1,F^{(q_1)}} + D_{2,F^{(q_1)}}\big) \cdots \big(D_{1,F^{(q_k)}} + D_{2,F^{(q_k)}}\big) \mathcal{L}(x,y)\,. \end{aligned} \tag{3.8}$$

Using this formula in (3.7), we obtain the equations

$$\begin{aligned} &\nabla_{\mathrm{u}(x)} \int_M \sum_{k=0}^p \frac{1}{k!} \sum_{\substack{l,r \geq 0 \text{ and } q_1,\ldots,q_k \geq 1 \\ \text{with } l+r+q_1+\cdots+q_k=p}} f^{(l)}(x)\, f^{(r)}(y) \\ &\qquad \times\, \big(D_{1,F^{(q_1)}} + D_{2,F^{(q_1)}}\big) \cdots \big(D_{1,F^{(q_k)}} + D_{2,F^{(q_k)}}\big) \mathcal{L}(x,y)\, d\rho(y) \\ &= \frac{\nu}{2}\, \nabla_{\mathrm{u}} f^{(p)}(x)\,, \end{aligned} \tag{3.9}$$

which must be satisfied for all $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$, $x \in M$ and $p = 1, 2, \ldots$. We again point out that the partial derivatives D_1 and D_2 act only on $\mathcal{L}$, but not on the operators F^{q_k}. However, the derivative $\nabla_{\mathfrak{u}(x)}$ acts on all the x-dependencies inside the integral.

In the next lemma, we combine the multiplications and derivatives in the integrand of (3.9) using the operation ∇ and simplify the combinatorics. To this end, we set

$$b(x) = \log f(x) \tag{3.10}$$

and expand $b(x)$ similar to (3.5) as

$$b(x) = \sum_{p=0}^{\infty} \lambda^p \, b^{(p)}(x) \qquad \text{with} \qquad b^{(0)}(x) = 0\,. \tag{3.11}$$

Moreover, we combine the $b^{(p)}$ and $F^{(p)}$ to jets $\mathfrak{v}^{(p)}$, i.e.

$$\mathfrak{v}^{(p)} := \big(b^{(p)}, F^{(p)}\big)\,. \tag{3.12}$$

Lemma 3.1. *To every order $p = 1, 2, \ldots$, the weak EL equations* (3.3) *can be written as*

$$\begin{aligned} 0 = \nabla_{\mathfrak{u}(x)} \sum_{\ell=1}^{p} \frac{1}{\ell!} & \sum_{\substack{q_1, \ldots, q_\ell \geq 1 \\ \text{with } q_1 + \cdots + q_\ell = p}} \\ & \times \bigg\{ \int_M \big(\nabla_{1,\mathfrak{v}^{(q_1)}} + \nabla_{2,\mathfrak{v}^{(q_1)}}\big) \cdots \big(\nabla_{1,\mathfrak{v}^{(q_\ell)}} + \nabla_{2,\mathfrak{v}^{(q_\ell)}}\big) \mathcal{L}(x,y)\, d\rho(y) \\ & \quad - \frac{\nu}{2}\, b^{(q_1)}(x) \cdots b^{(q_\ell)}(x) \bigg\}\,. \end{aligned} \tag{3.13}$$

Proof. We first compute the factor $f^{(p)}$ on the right side of (3.9). Using (3.5) as well as (3.10) and (3.11), we know that

$$\begin{aligned} f^{(p)} = \frac{1}{p!}\, \frac{d^p}{d\lambda^p} e^b \Big|_{\lambda=0} &= \frac{1}{p!} \sum_{\ell=1}^{p} \frac{1}{\ell!} \sum_{\substack{q_1, \ldots, q_\ell \geq 1 \\ \text{with } q_1 + \cdots + q_\ell = p}} \binom{p}{q_1 \;\cdots\; q_a} \, q_1!\, b^{(q_1)} \,\cdots\, q_a!\, b^{(q_a)} \\ &= \sum_{\ell=1}^{p} \frac{1}{\ell!} \sum_{\substack{q_1, \ldots, q_\ell \geq 1 \\ \text{with } q_1 + \cdots + q_\ell = p}} b^{(q_1)} \,\cdots\, b^{(q_a)}\,. \end{aligned}$$

This gives the term involving ν in (3.13).

In order to treat the factors $f^{(l)}$ and $f^{(r)}$ on the left side of (3.9), we first compute their Cauchy product with the binomial theorem,

$$\sum_{\substack{l, r \geq 0 \\ \text{with } l+r=L}} f^{(l)}(x)\, f^{(r)}(y) = \sum_{\substack{l, r \geq 0 \\ \text{with } l+r=L}} \frac{1}{l!}\, \frac{d^l}{d\lambda^l} e^{b(x)}\, \frac{1}{r!}\, \frac{d^r}{d\lambda^r} e^{b(y)} \Big|_{\lambda=0} = \frac{1}{L!} \frac{d^L}{d\lambda^L} e^{b(x)+b(y)} \Big|_{\lambda=0}\,.$$

Proceeding similar as in (3.8), we obtain

$$\begin{aligned}\sum_{\substack{l,r\geq 0\\ \text{with } l+r=L}} f^{(l)}(x)\, f^{(r)}(y) &= \frac{1}{L!}\sum_{a=0}^{L} \frac{1}{a!} \sum_{\substack{l_1,\ldots,l_a\geq 1\\ \text{with } l_1+\cdots+l_a=L}} \binom{L}{l_1\ \cdots\ l_a} \\ &\quad \times l_1!\left(b^{(l_1)}(x)+b^{(l_1)}(y)\right)\ \cdots\ l_a!\left(b^{(l_a)}(x)+b^{(l_a)}(y)\right) \\ &= \sum_{a=0}^{L} \frac{1}{a!} \sum_{\substack{l_1,\ldots,l_a\geq 1\\ \text{with } l_1+\cdots+l_a=L}} \left(b^{(l_1)}(x)+b^{(l_1)}(y)\right)\ \cdots\ \left(b^{(l_a)}(x)+b^{(l_a)}(y)\right).\end{aligned}$$

Using this formula, the integrand in (3.9) can be written as

$$\begin{aligned}&\sum_{\ell=1}^{p} \sum_{\substack{a,k\geq 0\\ \text{with } a+k=\ell}} \frac{1}{a!\,k!} \sum_{\substack{l_1,\ldots,l_a,q_1,\ldots,q_k\geq 1\\ \text{with } l_1+\cdots+l_a+q_1+\cdots+q_k=p}} \left(b^{(l_1)}(x)+b^{(l_1)}(y)\right)\ \cdots\ \left(b^{(l_a)}(x)+b^{(l_a)}(y)\right) \\ &\qquad \times \left(D_{1,F^{(q_1)}}+D_{2,F^{(q_1)}}\right)\cdots\left(D_{1,F^{(q_k)}}+D_{2,F^{(q_k)}}\right)\mathcal{L}(x,y)\,.\end{aligned}$$

Again applying the binomial formula, we obtain the result. □

3.2. **Green's Operators.** In Lemma 3.1 we rewrote the weak EL equations as the system of equations (3.13), to be satisfied for every $p = 1, 2, \ldots$. In order to solve this system of equations, we bring the contribution involving $\mathfrak{v}^{(p)}$ to the left. We thus obtain the equation

$$\nabla_{\mathfrak{u}(x)}\left(\int_M \left(\nabla_{1,\mathfrak{v}^{(p)}}+\nabla_{2,\mathfrak{v}^{(p)}}\right)\mathcal{L}(x,y)\,d\rho(y) - \frac{\nu}{2}\,b^{(p)}(x)\right) = -\nabla_{\mathfrak{u}} E^{(p)}(x)\,, \tag{3.14}$$

where $E^{(1)}$ is given by (3.4), whereas for $p > 1$ we have

$$\begin{aligned}E^{(p)} = \sum_{\ell=2}^{p} \frac{1}{\ell!} \sum_{\substack{q_1,\ldots,q_\ell\geq 1\\ \text{with } q_1+\cdots+q_\ell=p}} &\left\{ -\frac{\nu}{2}\, b^{(q_1)}(x)\cdots b^{(q_\ell)}(x)\right. \\ &\left. + \int_M \left(\nabla_{1,\mathfrak{v}^{(q_1)}}+\nabla_{2,\mathfrak{v}^{(q_1)}}\right)\cdots\left(\nabla_{1,\mathfrak{v}^{(q_\ell)}}+\nabla_{2,\mathfrak{v}^{(q_\ell)}}\right)\mathcal{L}(x,y)\,d\rho(y)\right\}.\end{aligned} \tag{3.15}$$

Before solving for $\mathfrak{v}^{(p)}$, we need to specify the jet space used for varying the measure: We denote the smooth global one-jets of the cotangent bundle restricted to M by

$$\mathfrak{J}^* := C^\infty(M,\mathbb{R}) \oplus C^\infty(M, T^*\mathcal{F})\,.$$

We let $(\mathfrak{J}^{\text{test}})^*$ be the quotient space

$$\begin{aligned}(\mathfrak{J}^{\text{test}})^* := \mathfrak{J}^* \Big/ \big\{(g,\varphi)\in\mathfrak{J}^* \,\big|\, &g(x)\,a(x) + \langle \phi(x), u(x)\rangle = 0 \\ &\text{for all } \mathfrak{u}=(a,u)\in\mathfrak{J}^{\text{test}} \text{ and } x\in M\big\}\,,\end{aligned}$$

where $\langle .,. \rangle$ denotes the dual pairing of $T_x^*\mathcal{F}$ and $T_x\mathcal{F}$.

Definition 3.2. *The jet space $\mathfrak{J}^{\text{vary}} \subset C^\infty(\mathcal{F}) \oplus \Gamma(T\mathcal{F})$ is defined as the vector space of jets with the following properties:*

(v1) *For all $y \in M$ and all x in an open neighborhood of M, in suitable charts around x and y the directional derivatives*

$$\big(\nabla_{1,\mathfrak{v}_1} + \nabla_{2,\mathfrak{v}_1}\big) \cdots \big(\nabla_{1,\mathfrak{v}_\ell} + \nabla_{2,\mathfrak{v}_\ell}\big) \mathcal{L}(x,y) \tag{3.16}$$

exist for all $\ell \in \mathbb{N}$ and all $\mathfrak{v}_1, \ldots, \mathfrak{v}_\ell \in \mathfrak{J}^{\text{vary}}$.

(v2) *Integrating the expression (3.16) in y over M with respect to the measure ρ, the resulting function (defined on an open neighborhood of M) is differentiable in the direction of every jet $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$. This gives rise to a mapping*

$$\Delta_\ell : \underbrace{\mathfrak{J}^{\text{vary}} \times \cdots \times \mathfrak{J}^{\text{vary}}}_{\ell \text{ factors}} \to (\mathfrak{J}^{\text{test}})^* , \tag{3.17}$$

$$\big\langle \mathfrak{u}, \Delta_\ell\big[\mathfrak{v}_1, \ldots, \mathfrak{v}_\ell\big] \big\rangle(x) = \frac{1}{\ell!}\, \nabla_{\mathfrak{u}(x)} \bigg(\int_M \big(\nabla_{1,\mathfrak{v}_1} + \nabla_{2,\mathfrak{v}_1}\big) \cdots \big(\nabla_{1,\mathfrak{v}_\ell} + \nabla_{2,\mathfrak{v}_\ell}\big) \mathcal{L}(x,y)\, d\rho(y) - \frac{\nu}{2}\, b_1(x) \cdots b_\ell(x) \bigg) ,$$

valid for any $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$.

We remark for clarity that the mapping Δ_ℓ is symmetric in its ℓ arguments. Choosing $\ell = 1$, we obtain the mapping $\Delta \equiv \Delta_1 : \mathfrak{J}^{\text{vary}} \to (\mathfrak{J}^{\text{test}})^*$ given by

$$\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x) = \nabla_{\mathfrak{u}} \bigg(\int_M \Big(\nabla_{1,\mathfrak{v}} \mathcal{L}(x,y) + \nabla_{2,\mathfrak{v}} \mathcal{L}\big(x,y\big) \Big)\, d\rho(y) - \nabla_{\mathfrak{v}}\, \frac{\nu}{2} \bigg) . \tag{3.18}$$

Definition 3.3. *A linear mapping $S : (\mathfrak{J}^{\text{test}})^* \to \mathfrak{J}^{\text{vary}}$ is referred to as a* **Green's operator** *if*

$$\Delta\, S\, \mathfrak{v} = -\mathfrak{v} \qquad \text{for all } v \in (\mathfrak{J}^{\text{test}})^* . \tag{3.19}$$

Clearly, a Green's operator exists if and only if the mapping Δ is surjective. In analogy to the situation for hyperbolic PDEs, the Green's operators need not be unique. Indeed, just as in classical field theory, the difference of two Green's operators is a solution of the linearized field equations (see Definition 2.2).

With the above notions, we can write (3.14) as

$$\Delta \mathfrak{v}^{(p)} = -E^{(p)} \in (\mathfrak{J}^{\text{test}})^* .$$

Having a Green's operator to our disposal, we can solve this equation for $\mathfrak{v}^{(p)}$,

$$\mathfrak{v}^{(p)} = S\, E^{(p)} . \tag{3.20}$$

Combining this equation with (3.4) and (3.15), we have obtained an iterative procedure for constructing measures which satisfy the weak EL equations (3.3). We again point out that the Green's operator S is not unique. Indeed, there is the freedom to choose a different Green's operator to every order in perturbation theory. Exactly as in the analogous situation for hyperbolic PDEs, taking this freedom into account gives rise to the *general* solution to the weak EL equations. In order to make this non-uniqueness manifest, we prefer to write (3.20) as

$$\mathfrak{v}^{(p)} = S^{(p)}\, E^{(p)} , \tag{3.21}$$

where $S^{(1)}, S^{(2)}, \ldots$ are arbitrary Green's operators.

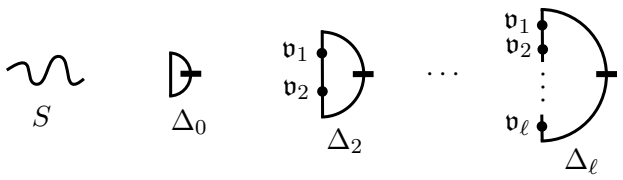

FIGURE 1. Building blocks of Feynman diagrams.

3.3. **Diagrams and Feynman Rules.** We now summarize the above construction and formulate it in a diagrammatic language. For simplicity, we leave out the parameter λ, which was used merely as a book-keeping device in order to keep track of the different orders in perturbation theory. We introduce the operators Δ_ℓ by (see (3.17))

$$\Delta_0(x) = \int_M \mathcal{L}(x,y)\, d\rho(y) - \frac{\nu}{2} \tag{3.22}$$

$$\Delta_\ell\big[\mathfrak{v}_1, \ldots, \mathfrak{v}_\ell\big](x) = \frac{1}{\ell!}\bigg(\int_M \big(\nabla_{1,\mathfrak{v}_1} + \nabla_{2,\mathfrak{v}_1}\big) \cdots \big(\nabla_{1,\mathfrak{v}_\ell} + \nabla_{2,\mathfrak{v}_\ell}\big) \mathcal{L}(x,y)\, d\rho(y) - \frac{\nu}{2}\, b_1(x) \cdots b_\ell(x) \bigg) \qquad (\text{for } \ell \geq 1) \tag{3.23}$$

and choose Green's operators $S^{(p)}$ with $p = 1, 2, \ldots$ as minus the inverse of $\Delta \equiv \Delta_1$ (see Definition 3.3),

$$\Delta\, S^{(p)}\, \mathfrak{v} = -\mathfrak{v} \qquad \text{for all } v \in (\mathfrak{J}^{\text{test}})^* \,. \tag{3.24}$$

Then the jets $v^{(p)}$ are defined iteratively by (see (3.21))

$$\mathfrak{v}^{(p)} = S^{(p)}\, E^{(p)} \,, \tag{3.25}$$

where $E^{(p)}$ depends on the previous jets $v^{(1)}, \ldots v^{(p-1)}$ by (see (3.4) and (3.15))

$$E^{(1)}(x) = \Delta_0(x) \tag{3.26}$$

$$E^{(p)}(x) = \sum_{\ell=2}^{p} E_\ell^{(p)}(x) \qquad (\text{for } p \geq 2) \tag{3.27}$$

$$E_\ell^{(p)}(x) = \sum_{\substack{q_1, \ldots, q_\ell \geq 1 \\ \text{with } q_1 + \cdots + q_\ell = p}} \Delta_\ell\big[\mathfrak{v}^{(q_1)}, \ldots, \mathfrak{v}^{(q_\ell)}\big](x) \,. \tag{3.28}$$

The universal measure $\tilde{\rho}$ is obtained as follows: In the chosen chart on $\mathcal{F}$, we set

$$\mathfrak{v}^{(0)}(x) = (0, x) \qquad \text{and} \qquad \tilde{\mathfrak{v}}(x) = \sum_{p=0}^{\infty} \mathfrak{v}^{(p)}(x) \,. \tag{3.29}$$

Then the universal measure is given by (see (3.1), (3.5), (3.6), (3.10) and (3.12))

$$\tilde{\rho} = F_*\big(e^a\, \rho\big) \qquad \text{where} \qquad \tilde{\mathfrak{v}} =: (a, F) \,. \tag{3.30}$$

For the graphical representation, we denote the Green's operator by a wiggled line and the operators Δ_ℓ by semicircles (see Figure 1). Then the contributions to the perturbation expansion can be depicted by Feynman diagrams as illustrated in Figure 2. The combinatorial factors are to be chosen as in (3.27), (3.28) and (3.29). We point out that our perturbation expansion only involves tree diagrams.

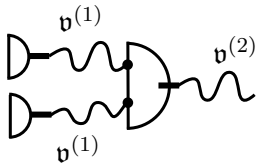

FIGURE 2. A simple Feynman diagram.

Remark 3.4. (alternative form of the perturbation expansion) For completeness, we now give an alternative form of the perturbation expansion which might be useful for future applications. Namely, the weak EL equations (3.3) can also be written alternatively as

$$\nabla_{\check{\mathfrak{u}}(x)} \left(\mathcal{L}\big(F(x), F(y)\big)\, f(y)\, d\rho(y) - \frac{\nu}{2} \right) = 0\,, \tag{3.31}$$

to be evaluated on the transformed jet space

$$\check{\mathfrak{J}}^{\text{test}} := \Big\{ \big(a + D_u \log f\big), u\big) \text{ with } \mathfrak{u} = (a,u) \in \mathfrak{J}^{\text{test}} \Big\}\,. \tag{3.32}$$

Expanding the equations in this form, one obtains the same perturbation expansion as above, except that the operator Δ_ℓ in (3.23) is to be modified to

$$\check{\Delta}_\ell\big[\mathfrak{v}_1, \ldots, \mathfrak{v}_\ell\big](x) = \frac{1}{\ell!} \int_M \big(D_{1,v_1} + \nabla_{2,\mathfrak{v}_1}\big) \cdots \big(D_{1,v_\ell} + \nabla_{2,\mathfrak{v}_\ell}\big) \mathcal{L}(x,y)\, d\rho(y)\,. \tag{3.33}$$

This formulation has the advantage that the Lagrange multiplier ν drops out. Moreover, it becomes clearer that the scalar component of the jets only enters at the point y (as is obvious in (3.31) where only $f(y)$ appears). The disadvantage is that (3.33) is less symmetric in the variables x and y (in particular, the form (3.23) is of advantage for the derivation of conservation laws for surface layer integrals in [10, 8]). We also remark that instead of first choosing $\mathfrak{J}^{\text{test}}$ and then defining $\check{\mathfrak{J}}^{\text{test}}$ by (3.32), one could proceed in the opposite way by first choosing $\check{\mathfrak{J}}^{\text{test}}$ and then defining $\mathfrak{J}^{\text{test}}$ using the inverse of the transformation in (3.32). Then the conceptual difficulties related to the fact that $\check{\mathfrak{J}}^{\text{test}}$ in (3.32) depends on the a-priori unknown function f would disappear. ♢

3.4. **Constructing Non-Linear Solutions of the Field Equations.** We now explain how the general construction of Section 3.1 can be adapted in order to construct non-linear solutions of the field equations. We consider the setting that ρ is a minimizing measure, and we again assume that we are given a Green's operator S (see Definition 3.3). Moreover, we are given a jet $\mathfrak{v}^{(1)} \in \mathfrak{J}^{\text{vary}}$ being a solution of the linearized field equations (see Definition 2.2). Our goal is to construct a family of solutions $(\tilde{\rho}_\tau)_{\tau \in \mathbb{R}}$ of the weak EL equations of the form (3.1) whose first variation coincides with $\mathfrak{v}^{(1)}$, i.e.

$$\tilde{\rho}_\tau|_{\tau=0} = \rho \qquad \text{and} \qquad \big(\partial_\tau f, \partial_\tau F\big)\big|_{\tau=0} = \mathfrak{v}^{(1)}\,.$$

To this end, we construct the jets $\mathfrak{v}^{(2)}, \mathfrak{v}^{(3)}, \ldots$ iteratively again by (3.25), (3.27) and (3.28) with Δ_ℓ according to (3.23) (note that now $\Delta_0 = \Delta_1[v^{(1)}] = 0$). The desired family of measures $(\tilde{\rho}_\tau)$ is then defined similar to (3.29) and (3.30) by inserting

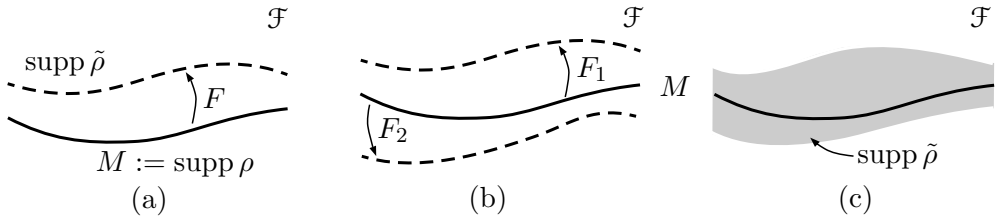

FIGURE 3. Fragmentation of the measure ρ.

powers of τ, i.e.

$$\tilde{\mathfrak{v}}_\tau(x) = \sum_{p=0}^{\infty} \tau^p\, \mathfrak{v}^{(p)}(x) \qquad \text{and} \qquad \tilde{\rho}_\tau = (F_\tau)_*\big(e^{a_\tau}\, \rho\big)\,,$$

where we again set $\mathfrak{v}^{(0)}(x) := (x, 1)$ and $\tilde{\mathfrak{v}}_\tau =: (a_\tau, F_\tau)$.

3.5. **Perturbing a Vacuum Measure.** In the applications, one often knows a critical measure which typically describes the vacuum of the system. Then the system is perturbed for example by introducing particles and/or fields. The task is to construct a solution of the weak EL equations. We now adapt the construction of Section 3.1 to this setting. To this end, we assume that ρ is measure which satisfies the EL equations (2.2). Moreover, we assume that we are given a Green's operator S (see Definition 3.3). The perturbed system is described by a measure $\hat{\rho}$, which in analogy to (3.1) we assume to be of the form

$$\hat{\rho} = H_*\big(h\, \rho\big)\,, \tag{3.34}$$

where h and H are smooth,

$$h \in C^\infty\big(\mathcal{F}, \mathbb{R}^+\big) \qquad \text{and} \qquad H \in C^\infty\big(\mathcal{F}, \mathcal{F}\big)\,.$$

Clearly, the measure $\hat{\rho}$ is no longer a solution of the weak EL equations. Similar to (3.10), (3.12) and (3.29), we combine h and H into a jet $\tilde{\mathfrak{w}}$ and expand,

$$\big(\log h, H\big) =: \tilde{\mathfrak{w}} = \sum_{p=0}^{\infty} \mathfrak{w}^{(p)} \qquad \text{with} \qquad \mathfrak{w}^{(p)} \in \mathfrak{J}^{\text{vary}}\,. \tag{3.35}$$

Here we need to assume that the resulting jets $\mathfrak{w}^{(p)}$ are in $\mathfrak{J}^{\text{vary}}$.

In order to construct a corresponding solution of the EL equations, we again make the ansatz (3.1) and describe f and F by jets $\mathfrak{v}^{(p)}$ (see (3.6), (3.10), (3.11) and (3.12)). Now we perform the perturbation expansion similar to (3.27)–(3.30), taking into account the inhomogeneity $\mathfrak{w}^{(p)}$ to every order in perturbation theory. More precisely, (3.25) is to be replaced by

$$\mathfrak{v}^{(p)} = \mathfrak{w}^{(p)} + S^{(p)}\left(E^{(p)} + \Delta\mathfrak{w}^{(p)}\right). \tag{3.36}$$

4. Perturbation Theory with Fragmentation

The perturbation expansion of the previous section was based on the ansatz that the perturbed measure $\tilde{\rho}$ should be of the form (3.1) with f and F according to (3.2). Intuitively speaking, this ansatz means that the support of the measure is changed smoothly as a whole (see Figure 3 (a)), but it is impossible to model a situation

where the measure ρ "disintegrates" into several "components" which are perturbed differently (see Figure 3 (b)). We now extend the constructions Section 3 such as to allow for such so-called *fragmentations* of the universal measure.

In order to introduce the setting, we let ρ be a universal measure which satisfies the EL equations (2.2) (typically describing the vacuum). We choose a parameter $L \in \mathbb{N}$ and consider mappings

$$f_\mathfrak{a} \in C^\infty\big(\mathcal{F}, \mathbb{R}^+\big)\,, \quad F_\mathfrak{a} \in C^\infty\big(\mathcal{F}, \mathcal{F}\big) \qquad \text{with } \mathfrak{a} = 1, \ldots, L\,.$$

For the *universal measure with fragmentation* we make the ansatz

$$\tilde{\rho} = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} (F_\mathfrak{a})_* \big(f_\mathfrak{a}\, \rho\big)\,. \tag{4.1}$$

We refer to L as the *number of subsystems*, and $\mathfrak{a}$ as the *subsystem index*. Clearly, for one subsystem, (4.1) reduces to our earlier ansatz (3.1). The larger L is chosen, the more freedom we have in perturbing the measure ρ. We point out that we may choose L arbitrarily large. In the limit $L \to \infty$, one can even describe situations where the support of the measure ρ is "enlarged" by the perturbation as shown in Figure 3 (c). We also note that a universal measure of the form (4.1) is closely related to the mechanism of microscopic mixing as introduced in [6]; this will be explained further in Section 7.

Before going on, we remark that the ansatz (4.1) could be generalized considerably. For instance, as in [6] one could replace the normalized sum $L^{-1} \sum_{\mathfrak{a}=1}^{L} \cdots$ by $\sum_{\mathfrak{a}=1}^{L} c_\mathfrak{a} \cdots$ with weight factors $c_\mathfrak{a} \geq 0$ normalized by $c_1 + \cdots + c_L = 1$. Even more, one could replace $c_\mathfrak{a} \rho$ by a more general measure on $M \times \{1, \ldots, L\}$. However, all these generalizations do not give anything essentially new because by increasing L and forming groups of subsystems with identical $f_\mathfrak{a}$ and $F_\mathfrak{a}$, one can approximate any other measure of the above form. With this in mind, it seems preferable to keep the setting as simple as possible by taking (4.1) as the starting point.

4.1. **Linearized Field Equations for Fluctuations.** Adapted to the measure (4.1), the weak EL equations (3.3) read

$$\nabla_{\mathfrak{u}_\mathfrak{a}(x)} \bigg(\frac{1}{L} \sum_{\mathfrak{b}=1}^{L} \int_M f_\mathfrak{a}(x)\, \mathcal{L}\big(F_\mathfrak{a}(x), F_\mathfrak{b}(y)\big)\, f_\mathfrak{b}(y)\, d\rho(y) - f_\mathfrak{a}(x)\, \frac{\nu}{2} \bigg) = 0\,,$$

to be satisfied for all jets $\mathfrak{u} \in (\mathfrak{J}^{\text{test}})^L$ as well as for all $x \in M$ and $a \in \{1, \ldots, L\}$. Since in finite dimension, pointwise evaluation is equivalent to weak evaluation, we can write this equation equivalently as

$$\frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \nabla_{\mathfrak{u}_\mathfrak{a}(x)} \bigg(\frac{1}{L} \sum_{\mathfrak{b}=1}^{L} \int_M f_\mathfrak{a}(x)\, \mathcal{L}\big(F_\mathfrak{a}(x), F_\mathfrak{b}(y)\big)\, f_\mathfrak{b}(y)\, d\rho(y) - f_\mathfrak{a}(x)\, \frac{\nu}{2} \bigg) = 0\,, \tag{4.2}$$

which must hold for all $\mathfrak{u} \in (\mathfrak{J}^{\text{test}})^L$ and all $x \in M$.

In order to get an idea for how to set up the perturbation expansion, we expand (4.2). Using a similar notation as in (3.18), we obtain to first order

$$\begin{aligned}\big\langle \mathfrak{u}, \Delta\big[\mathfrak{v}^{(1)}\big]\big\rangle(x) &:= \frac{1}{L^2} \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \\ &\times \nabla_{\mathfrak{u}_\mathfrak{a}(x)} \int_M \Big(\big(D_{1,v_\mathfrak{a}^{(1)}} + D_{2,v_\mathfrak{b}^{(1)}}\big)\mathcal{L}(x,y) + \mathcal{L}(x,y)\, f_\mathfrak{b}^{(1)}(y) \Big)\, d\rho(y)\end{aligned} \tag{4.3}$$

with $v_\mathfrak{a}^{(1)} = F_\mathfrak{a}^{(1)}$ (here we used that ρ satisfies the EL equations (2.2) and thus $\nabla_{\mathfrak{u}_\mathfrak{a}(x)} f_\mathfrak{a}^{(1)}(x)\big(\ell(x) - \nu/2\big) = 0$). The vector component of the jet $\mathfrak{v}_\mathfrak{a}^{(1)}$ shifts the support of the universal measure in each subsystem independently (as shown in Figure 3 (b)).

At this point, it is helpful to decompose the jets into components independent of the subsystem index and components whose mean vanishes, i.e.

$$\mathfrak{u} = \bar{\mathfrak{u}} + \mathfrak{u}_\mathrm{F} \qquad \text{with} \qquad \bar{\mathfrak{u}}_\mathfrak{a}(x) := \frac{1}{L} \sum_{\mathfrak{b}=1}^{L} \mathfrak{u}_\mathfrak{b}(x)\,. \tag{4.4}$$

Here the subscript "F" can be thought of as referring to the "fragmentation" of the universal measure or as describing the "fluctuations" of the jets in the subsystems. For a convenient notation, we usually omit the subsystem index of $\bar{u}$. The above splitting gives rise to a direct sum decomposition of the jet spaces, which we write as

$$\mathfrak{J}^L = \bar{\mathfrak{J}} \oplus \mathfrak{J}_\mathrm{F}$$

and similarly for the jet spaces $\mathfrak{J}^\mathrm{test}$ and $\mathfrak{J}^\mathrm{vary}$.

Using these notions, we can immediately carry out the $\mathfrak{b}$-sum in (4.3) to obtain

$$\big\langle \mathfrak{u}, \Delta\big[\mathfrak{v}^{(1)}\big]\big\rangle(x) = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \nabla_{\mathfrak{u}_\mathfrak{a}(x)} \int_M \Big(\big(D_{1,v_\mathfrak{a}^{(1)}} + D_{2,\bar{v}^{(1)}}\big)\mathcal{L}(x,y) + \mathcal{L}(x,y)\, \bar{f}^{(1)}(y) \Big)\, d\rho(y)\,.$$

The fluctuations drop out completely when testing in $\bar{\mathfrak{J}}^\mathrm{test}$,

$$\big\langle \bar{\mathfrak{u}}, \Delta\big[\mathfrak{v}^{(1)}\big]\big\rangle(x) = \nabla_{\bar{\mathfrak{u}}(x)} \int_M \Big(\big(D_{1,\bar{v}^{(1)}} + D_{2,\bar{v}^{(1)}}\big)\mathcal{L}(x,y) + \mathcal{L}(x,y)\, \bar{f}^{(1)}(y) \Big)\, d\rho(y)\,. \tag{4.5}$$

Since this expression does not involve the fluctuations, we can define a Green's operator as in Section 3:

Definition 4.1. *We define the operator* $\bar{\Delta} : \bar{\mathfrak{J}}^\mathrm{vary} \to (\bar{\mathfrak{J}}^\mathrm{test})^*$ *by*

$$\langle \bar{\mathfrak{u}}, \bar{\Delta}\bar{\mathfrak{v}}\rangle(x) = \nabla_{\bar{\mathfrak{u}}} \int_M \Big(\big(D_{1,\bar{v}} + D_{2,\bar{v}}\big)\mathcal{L}(x,y) + \mathcal{L}(x,y)\, \bar{b}(y) \Big)\, d\rho(y)$$

(where the scalar and vector components of $\bar{\mathfrak{v}}$ *are again denoted by* $\bar{b}$ *and* $\bar{v}$*, respectively) or equivalently by*

$$\langle \bar{\mathfrak{u}}, \bar{\Delta}\bar{\mathfrak{v}}\rangle(x) = \nabla_{\bar{\mathfrak{u}}} \bigg(\int_M \big(\nabla_{1,\bar{\mathfrak{v}}} + \nabla_{2,\bar{\mathfrak{v}}}\big)\mathcal{L}(x,y)\, d\rho(y) - \nabla_{\bar{\mathfrak{v}}} \frac{\nu}{2} \bigg)\,.$$

A **linearized solution of the averaged field equations** *is a jet satisfying the equation* $\bar{\Delta}\bar{\mathfrak{v}} = 0$*. A linear mapping* $\bar{S} : (\bar{\mathfrak{J}}^\mathrm{test})^* \to \bar{\mathfrak{J}}^\mathrm{vary}$ *is referred to as an* **averaged Green's operator** *if*

$$\bar{\Delta}\, \bar{S}\, \bar{\mathfrak{v}} = -\bar{\mathfrak{v}} \quad \textit{for all } \bar{\mathfrak{v}} \in (\bar{\mathfrak{J}}^\mathrm{test})^*\,.$$

But clearly, the fluctuations are visible when testing in $\mathfrak{J}_{\mathrm{F}}^{\mathrm{test}}$,

$$\big\langle \mathfrak{u}_{\mathrm{F}}, \Delta\big[\mathfrak{v}^{(1)}\big] \big\rangle(x) = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \nabla_{\mathfrak{u}_{\mathrm{F},\mathfrak{a}}(x)} \int_M D_{1, v_{\mathrm{F},\mathfrak{a}}^{(1)}} \mathcal{L}(x,y)\, d\rho(y)\,. \tag{4.6}$$

It is a useful observation that the fluctuations $f_{\mathrm{F},\bullet}$ of the scalar component do not enter. Moreover, the vector component $v_{\mathrm{F},\bullet}$ enters only via a directional derivative acting on the variable x. In addition, the resulting jet in $E^{(1)}$ has no scalar component, because

$$\int_M D_{1, v_{\mathrm{F},\mathfrak{a}}^{(1)}} \mathcal{L}(x,y)\, d\rho(y) = D_{v_{\mathrm{F},\mathfrak{a}}^{(1)}} \ell(x,\mathfrak{a}) \overset{(2.2)}{=} 0 \qquad \text{for all } x \in M \text{ and } \mathfrak{a} \in \{1,\ldots,L\}\,.$$

For this reason, it is impossible to invert the operator $(\mathcal{F}^{(1)}, f^{(1)}) \mapsto E^{(1)}$, implying that no Green's operator exists. But we can hope that there is a Green's operator on the vector component $(\Gamma_{\mathrm{F}}^{\mathrm{test}})^*$. This leads us to the following definition.

Definition 4.2. *We define the operator* $\Delta_{\mathrm{F}} : \Gamma_{\mathrm{F}}^{\mathrm{vary}} \to (\Gamma_{\mathrm{F}}^{\mathrm{test}})^* \subset (\mathfrak{J}_{\mathrm{F}}^{\mathrm{test}})^*$ *by*

$$\langle \mathfrak{u}_{\mathrm{F}}, \Delta_{\mathrm{F}} v_{\mathrm{F}} \rangle(x) = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \nabla_{\mathfrak{u}_{\mathrm{F},\mathfrak{a}}(x)} \int_M D_{1, v_{\mathrm{F},\mathfrak{a}}} \mathcal{L}(x,y)\, d\rho(y)\,. \tag{4.7}$$

A **linearized solution of the field equations for fluctuations** *is a jet satisfying the equation* $\Delta_{\mathrm{F}} v_{\mathrm{F}} = 0$.

A linear mapping $S_{\mathrm{F}} : (\Gamma_{\mathrm{F}}^{\mathrm{test}})^* \to \Gamma_{\mathrm{F}}^{\mathrm{vary}}$ *is referred to as a* **vector Green's operator for fluctuations** *if*

$$\Delta_{\mathrm{F}}\, S_{\mathrm{F}}\, v_{\mathrm{F}} = -v_{\mathrm{F}} \quad \textit{for all } v_{\mathrm{F}} \in (\Gamma_{\mathrm{F}}^{\mathrm{test}})^*\,.$$

We finally explain how, given an inhomogeneity, one can construct a linearized solution of the field equations. Similar as in Section 3.5, we perturb the vacuum measure ρ by an inhomogeneity, which we describe by a jet $\tilde{\mathfrak{w}}$ having the expansion (3.35). Then the first order the perturbation is described by a jet $\mathfrak{w}_{\mathfrak{a}}^{(1)} = (c_{\mathfrak{a}}^{(1)}, w_{\mathfrak{a}}^{(1)}) \in (\mathfrak{J}^{\mathrm{vary}})^L$. Similar to (4.3), we obtain a linear contribution to the weak EL equations (4.2) of the form

$$\Delta\big[\mathfrak{w}^{(1)}\big](x,\mathfrak{a}) := \frac{1}{L} \sum_{\mathfrak{a}}^{L} \int_M \Big(\big(D_{1,w_{\mathfrak{a}}} + D_{2,w_{\mathfrak{b}}}\big) \mathcal{L}(x,y) + \mathcal{L}(x,y)\, c_{\mathfrak{b}}(y) \Big)\, d\rho(y)\,.$$

Similar to (4.5) and (4.6), one can decompose this contribution into the average and the fluctuations (the latter being purely vectorial), i.e.

$$\Delta\mathfrak{w}^{(1)} = \bar{\Delta}\bar{\mathfrak{w}}^{(1)} + \Delta_{\mathrm{F}}\mathfrak{w}_{\mathrm{F}}^{(1)} \qquad \text{with} \qquad \bar{\Delta}\bar{\mathfrak{w}}^{(1)} \in (\bar{\mathfrak{J}}^{\mathrm{test}})^*\,,\;\; \Delta_{\mathrm{F}}\big[\mathfrak{w}_{\mathrm{F}}^{(1)}\big] \in (\Gamma_{\mathrm{F}}^{\mathrm{test}})^*\,.$$

Then, using a block matrix notation in the scalar and vector components, the jet $\mathfrak{v}^{(1)}$ defined by

$$\mathfrak{v}^{(1)} := \mathfrak{w}^{(1)} + \bar{S}\bar{\Delta}\bar{\mathfrak{w}}^{(1)} + \begin{pmatrix} 0 & 0 \\ 0 & S_{\mathrm{F}} \end{pmatrix} \Delta_{\mathrm{F}}\big[\mathfrak{w}_{\mathrm{F}}^{(1)}\big] + \begin{pmatrix} b_{\mathrm{F}}^{(1)} \\ 0 \end{pmatrix} \tag{4.8}$$

is a solution of the linearized field equations $\Delta\mathfrak{v}^{(1)} = 0$. Here $b_{\mathrm{F}}^{(1)}$ is a fluctuation of the scalar component. It can be chosen arbitrarily, because it does not enter the linearized field equations.

4.2. **Treating the Scalar Fluctuations.** In analogy to the procedure in Section 3.4, we want to construct a family $(f_{\mathfrak{a},\tau}, F_{\mathfrak{a},\tau})$ of solutions of the weak EL equations. We again make a perturbation ansatz

$$f_{\mathfrak{a},\tau} = \sum_{p=0}^{\infty} \tau^p\, f_{\mathfrak{a}}^{(p)}\,, \qquad F_{\mathfrak{a},\tau} = \sum_{p=0}^{\infty} \tau^p\, v_{\mathfrak{a}}^{(p)} \tag{4.9}$$

with

$$\big(f_{\mathfrak{a}}^{(0)}, F_{\mathfrak{a}}^{(0)}\big) = \big(1_{\mathcal{F}}, \mathbb{1}_F\big) \qquad \text{and} \qquad \big(f_{\mathfrak{a}}^{(1)}, v_{\mathfrak{a}}^{(1)}\big) = \mathfrak{v}^{(1)}\,, \tag{4.10}$$

where $\mathfrak{v}^{(1)}$ is the linearized solution (4.8). We assume that the perturbations to every order lie in the jet space $\mathfrak{J}^{\text{vary}}$, i.e.

$$\mathfrak{v}^{(p)} = \big(b^{(p)}, v^{(p)}\big) := \big(f^{(p)}, F^{(p)}\big) \;\in\; \mathfrak{J}^{\text{vary}}\,. \tag{4.11}$$

Note that the fluctuation of the scalar component $b_{\text{F}}^{(1)}$ is not yet determined and could be modified arbitrarily. Likewise, the linearized field equations do not involve a contribution in direction of the fluctuation of the scalar component.

The key for making our constructions work is that the fluctuation of the scalar component vanishes even quadratically in τ.

Lemma 4.3. *Taking the ansatz* (4.9) *and* (4.10)*, for any jet* $\mathfrak{u}$ *which is fluctuating and scalar, i.e.*

$$\mathfrak{u} = \mathfrak{u}_{\text{F}} = (a_{\text{F}}, 0) \in \mathfrak{J}_{\text{F}}^{\text{test}}\,,$$

the weak EL equations (4.2) *are satisfied up to order* $\mathcal{O}(\tau^3)$*.*

Proof. Our task is to show that the contribution $\sim \tau^2$ vanishes. A straightforward computation gives for this contribution

$$\frac{1}{L^2} \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} a_{\text{F},\mathfrak{a}}(x) \int_M \Big\{ \Big(\big(D_{1,v_{\mathfrak{a}}^{(2)}} + D_{2,v_{\mathfrak{b}}^{(2)}}\big)\mathcal{L}(x,y) + \mathcal{L}(x,y)\, f_{\mathfrak{b}}^{(2)}(y) \tag{4.12}$$

$$+ \big(D_{1,v_{\mathfrak{a}}^{(1)}} + D_{2,v_{\mathfrak{b}}^{(1)}}\big)^2 \mathcal{L}(x,y) + 2\,\big(D_{1,v_{\mathfrak{a}}^{(1)}} + D_{2,v_{\mathfrak{b}}}^{(1)}\big)\mathcal{L}(x,y)\, f_{\mathfrak{b}}^{(1)}(y) \tag{4.13}$$

$$+ f_{\mathfrak{a}}^{(1)}(x) \Big(\big(D_{1,v_{\mathfrak{a}}^{(1)}} + D_{2,v_{\mathfrak{b}}^{(1)}}\big)\mathcal{L}(x,y) + \mathcal{L}(x,y)\, f_{\mathfrak{b}}^{(1)}(y) \Big) \Big\}\, d\rho(y)\,. \tag{4.14}$$

The term (4.14) vanishes because $\mathfrak{v}^{(1)}$ satisfies the linearized field equations. Using that the sums over $\mathfrak{a}$ and $\mathfrak{b}$ vanish if one fluctuating factor appears in one of the sums, only the following terms remain:

$$\frac{1}{L^2} \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} a_{\text{F},\mathfrak{a}}(x) \int_M \Big(D_{1,v_{\mathfrak{a}}^{(2)}} \mathcal{L}(x,y) + \big(D_{1,v_{\text{F},\mathfrak{a}}^{(1)}}\big)^2 \mathcal{L}(x,y) \Big)\, d\rho(y)\,.$$

The first summand in the integrand vanishes after integration because ρ satisfies the EL equations (2.2). Likewise, the second summand vanishes because $v_{\text{F}}^{(1)}$ satisfies the linearized field equations for fluctuations (see Definition 4.2). This concludes the proof. □

Using this result, we can solve the EL equations of order τ^2 by choosing $\mathfrak{v}^{(2)}$ purely vectorial. Then the fluctuating scalar contribution $\sim \tau^3$ can be computed as in

Lemma 4.3. Indeed, similar as in (4.14), the terms involving $f^{(1)}_{\mathfrak{a}}(x)$ and $f^{(2)}_{\mathfrak{a}}(x)$ drop out as a consequence of the EL equations. The only remaining term is

$$\frac{1}{L^2} \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} a_{\mathrm{F},\mathfrak{a}}(x) \int_M D_{1,v^{(1)}_{\mathrm{F},\mathfrak{a}}} D_{2,v^{(1)}_{\mathrm{F},\mathfrak{b}}} \mathcal{L}(x,y)\, b^{(1)}_{\mathrm{F},\mathfrak{b}}(y)\, d\rho(y)\,.$$

Our strategy is to solve for $b^{(1)}_{\mathrm{F}}$ by inverting this operator. This leads us to the following notions.

Definition 4.4. *The vector component of the linearized solution $v^{(1)}_{\mathrm{F}}$ gives rise to a* **well-posed fragmentation** *if the linear operator M_{F} defined by*

$$M_{\mathrm{F}} \,:\, C^{\mathrm{vary}}_{\mathrm{F}}(\mathscr{M},\mathbb{R}) \to \big(C^{\mathrm{test}}_{\mathrm{F}}(\mathscr{M},\mathbb{R})\big)^*\,,$$

$$\big(M_{\mathrm{F}}\, b_{\mathrm{F}}\big)(x,a) = \frac{1}{L} \sum_{\mathfrak{b}=1}^{L} \int_M D_{1,v^{(1)}_{\mathrm{F},\mathfrak{a}}} D_{2,v^{(1)}_{\mathrm{F},\mathfrak{b}}} \mathcal{L}(x,y)\, b_{\mathrm{F},\mathfrak{b}}(y)\, d\rho(y)$$

is surjective. An operator $T_{\mathrm{F}} : \big(C^{\mathrm{test}}_{\mathrm{F}}(\mathscr{M},\mathbb{R})\big)^* \to C^{\mathrm{vary}}_{\mathrm{F}}(\mathscr{M},\mathbb{R})$ with the property

$$M_{\mathrm{F}}\, T_{\mathrm{F}}\, \mathfrak{v} = -\mathfrak{v} \qquad \text{for all } v \in \big(C^{\mathrm{test}}_{\mathrm{F}}(\mathscr{M},\mathbb{R})\big)^*$$

is called **scalar Green's operator for fluctuations**.

4.3. **Constructing Non-Linear Solutions of the Field Equations.** After the above preparations, we are in the position to perform the general perturbation expansion with fragmentation. We have two applications in mind: The first application is to construct non-linear solutions of the field equations. Then, similar as in Section 3.4, the starting point is a solution $\mathfrak{v}^{(1)}$ of the linearized field equations $\Delta\mathfrak{v}^{(1)} = 0$. In contrast to the setting in Section 3.4, we must not prescribe $\mathfrak{v}^{(1)}$ completely, but we need to leave the freedom to modify the scalar fluctuations. Hence we are given

$$\mathfrak{v}^{(1)} \quad \text{up to a contribution } b^{(1)}_{\mathrm{F}} \in C^{\mathrm{vary}}_{\mathrm{F}}(\mathscr{M},\mathbb{R})\,.$$

The second application is that, as in Section 3.5, we perturb the vacuum in such a way that fragmentation occurs. Then, describing the perturbation linearly again by a jet $\mathfrak{w}^{(1)}$, we construct a corresponding linearized solution $\mathfrak{v}^{(1)}$ according to (4.8). This linearized solution is again defined modulo scalar fluctuations. The only difference to the first application is that we now also have higher order inhomogeneities $\mathfrak{w}^{(2)}, \mathfrak{w}^{(3)}, \ldots$, which need to be taken into account in each order in perturbation theory.

Assume that $v^{(1)}$ defines a well-posed fragmentation (see Definition 4.4). Moreover, we assume that we are given Green's functions $\bar{S}^{(p)}$, $S^{(p)}_{\mathrm{F}}$ and $T^{(p)}_{\mathrm{F}}$. Then in our first application of constructing non-linear solutions of the field equations, the perturbation expansion is obtained iteratively by

$$b^{(1)}_{\mathrm{F}} = \begin{pmatrix} T^{(1)}_{\mathrm{F}} & 0 \end{pmatrix} E^{(3)}_{\mathrm{F}} \tag{4.15}$$

$$\mathfrak{v}^{(p)} = \bar{S}^{(p)} \bar{E}^{(p)} + \begin{pmatrix} 0 & 0 \\ 0 & S^{(p)}_{\mathrm{F}} \end{pmatrix} E^{(p)}_{\mathrm{F}} + \begin{pmatrix} T^{(p)}_{\mathrm{F}} & 0 \\ 0 & 0 \end{pmatrix} E^{(p+2)}_{\mathrm{F}} \qquad (p \geq 2)\,. \tag{4.16}$$

Here the jets are again the Taylor coefficients of f and F; see (4.9) and (4.11). In our second application of treating perturbations of a vacuum measure, the inhomogeneities $\mathfrak{w}^{(2)}, \mathfrak{w}^{(3)}, \ldots$ are taken into account by modifying (4.16) in analogy of (3.36)

to

$$b_{\mathrm{F}}^{(1)} = \begin{pmatrix} T_{\mathrm{F}}^{(1)} & 0 \end{pmatrix} E_{\mathrm{F}}^{(3)} + \begin{pmatrix} \mathbb{1} + T_{\mathrm{F}}^{(1)} M_{\mathrm{F}} & 0 \end{pmatrix} \mathfrak{w}_{\mathrm{F}}^{(1)}$$

$$\mathfrak{v}^{(p)} = \bar{S}^{(p)} \bar{E}^{(p)} + \begin{pmatrix} 0 & 0 \\ 0 & S_{\mathrm{F}}^{(p)} \end{pmatrix} E_{\mathrm{F}}^{(p)} + \begin{pmatrix} T_{\mathrm{F}}^{(p)} & 0 \\ 0 & 0 \end{pmatrix} E_{\mathrm{F}}^{(p+2)}$$
$$+ \mathfrak{w}^{(p)} + \bar{S}^{(p)}\, \bar{\Delta} \bar{\mathfrak{w}}^{(p)} + \begin{pmatrix} T_{\mathrm{F}}^{(p)} M_{\mathrm{F}} & 0 \\ 0 & S_{\mathrm{F}}^{(p)} \Delta_{\mathrm{F}} \end{pmatrix} \mathfrak{w}_{\mathrm{F}}^{(p)} .$$

We close with a few remarks. In order to simplify the combinatorics of the perturbation expansion, it is again helpful to again work instead of $\bar{f}$ with the function $\bar{b} := \log \bar{f}$ (just as explained in Lemma 3.1). Also, it might again be useful to work with the alternative formulation introduced in Remark 3.4. However, we point out that the decomposition into mean and fluctuations must be performed for f, i.e.

$$f = \bar{f} + f_{\mathrm{F}} \qquad \text{with} \qquad \bar{f}_{\mathfrak{a}}(x) := \frac{1}{L} \sum_{\mathfrak{b}=1}^{L} f_{\mathfrak{b}}(x) ,$$

because decomposing $b = \log f$ instead would give rise to additional quadratic contributions in Lemma 4.3 and to the operator M_{F} in Definition 4.4. Generally speaking, it does not seem beneficial to describe the scalar fluctuations by functions $b_{\mathrm{F}} = \log f_{\mathrm{F}}$.

5. Perturbation Expansion for Causal Fermion Systems

5.1. **Preliminaries.** We briefly recall how the causal action principle for causal fermion systems fits into the framework of causal variational principles in the non-compact setting (see also [10, Section 2.3]). Compared to the setting in [7, Section 1.1], we here incorporate the trace constraint by restricting attention to operators of fixed trace. Moreover, we treat the boundedness constraint with a Lagrange multiplier κ. Finally, by assuming the unperturbed measure has the property that all space-time points are regular (see [7, Definition 1.1.5]), we may assume that all operators have exactly n positive and n negative eigenvalues. This leads to the following setting:

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be a complex Hilbert space. Moreover, we are given parameters $n \in \mathbb{N}$ (the spin dimension), $c > 0$ (the constraint for the local trace) and $\kappa > 0$ (the Lagrange multiplier of the boundedness constraint). We let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all self-adjoint operators on $\mathcal{H}$ with the following properties:

- F is self-adjoint, has finite rank and (counting multiplicities) has at n positive and n negative eigenvalues.
- The local trace is constant, i.e.

$$\mathrm{tr}(F) = c \,. \tag{5.1}$$

On $\mathcal{F}$ we consider the topology induced by the sup-norm on $\mathrm{L}(\mathcal{H})$. For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. We denote its non-trivial eigenvalues counting algebraic multiplicities by $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$. We introduce the *spectral weight* $|\,.\,|$ of an operator as the sum of the absolute values of its eigenvalues. In particular, the spectral weights of the operator products xy and $(xy)^2$ are defined by

$$|xy| = \sum_{i=1}^{2n} \left|\lambda_i^{xy}\right| \qquad \text{and} \qquad \left|(xy)^2\right| = \sum_{i=1}^{2n} \left|\lambda_i^{xy}\right|^2 . \tag{5.2}$$

We introduce the Lagrangian by

$$\mathcal{L}(x,y) = \big|(xy)^2\big| - \frac{1}{2n}\,|xy|^2 + \kappa\,|xy|^2\,. \tag{5.3}$$

Clearly, this Lagrangian is non-negative and continuous on $\mathcal{F} \times \mathcal{F}$. Thus we are back in the setting of Section 2.1. The EL equations in Definition 2.1 agree with the EL equations as derived for the causal action principle with constraints in [1] (see [1, Theorem 1.1]).

Before going on, we make a few remarks. Since in the present setting, the Lagrange multiplier term $\kappa\,|xy|^2$ in (5.3) is always present, we can simplify the notation in [7] by omitting the subscript κ. We also point out that we shall always keep the constants c and κ in (5.1) and (5.3) fixed when varying or perturbing the measure ρ. This is justified as follows. The constant c can be changed arbitrarily by rescaling the measure according to

$$\rho(\Omega) \to \rho\Big(\big\{\alpha x \,\big|\, x \in \Omega\big\}\Big) \qquad \text{with} \qquad \alpha \in \mathbb{R}\,.$$

Combining this transformation with our previous transformation (2.3), the freedom in rescaling the universal measure is exhausted. Therefore, the parameter κ must be regarded as a physical parameter of the system. The reason for keeping it fixed is that we want to describe *localized physical systems*, meaning that the perturbations of ρ are spatially compact or that the resulting space-time is asymptotically flat. In such situations, the parameter κ is determined by the asymptotic form of the universal measure at infinity, which is kept fixed in our variations and perturbations. More generally, κ can be kept fixed if we assume that there is a macroscopic region in space-time where no interaction takes place.

We now recall the definition of a few other basic objects needed for the analysis of causal fermion systems (for more details see [7, Section 1.1]). For every $x \in \mathcal{F}$ we define the *spin space* S_x by $S_x = x(\mathcal{H})$; it is a subspace of $\mathcal{H}$ of dimension $2n$. On the spin space S_x, the *spin scalar product* $\prec .|. \succ_x$ is defined by

$$\prec u|v\succ_x = -\langle u|xu\rangle_{\mathcal{H}} \qquad (\text{for all } u, v \in S_x)\,.$$

We let π_x be the orthogonal projection on $S_x \subset \mathcal{H}$. Then, for any $x, y \in M$ we define the *kernel of the fermionic projector* $P(x,y)$ by

$$P(x,y) = \pi_x\, y|_{S_y} \;:\; S_y \to S_x\,.$$

The kernel of the fermionic projector is very useful because, forming the closed chain A_{xy} by

$$A_{xy} := P(x,y)\,P(y,x) = \pi_x\, y\, x|_{S_x} \;:\; S_x \to S_x\,,$$

the eigenvalues of A_x coincide with the eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ in (5.2). In this way, the Lagrangian can be expressed in terms of the kernel of the fermionic projector.

A *wave function* ψ is defined as a function which to every $x \in M$ associates a vector of the corresponding spin space,

$$\psi \;:\; M \to \mathcal{H} \qquad \text{with} \qquad \psi(x) \in S_x M \quad \text{for all } x \in M\,.$$

A wave function is said to be *continuous* at x if for every $x \in M$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$\big\|\sqrt{|y|}\,\psi(y) - \sqrt{|x|}\,\psi(x)\big\|_{\mathcal{H}} < \varepsilon \qquad \text{for all } y \in M \text{ with } \|y - x\| \leq \delta\,.$$

The vector space of continuous wave functions is denoted by $C^0(M, SM)$. For every $u \in \mathcal{H}$, the corresponding *physical wave function* ψ^u is the wave function obtained by projecting to the spin spaces, i.e.

$$\psi^u(x) := \pi_x u \in S_x M \,.$$

The physical wave functions are all continuous. The *wave evaluation operator* Ψ is the linear operator which to every Hilbert space vector associates the corresponding physical wave function,

$$\Psi \,:\, \mathcal{H} \to C^0(M, SM)\,, \qquad u \mapsto \psi^u \,. \tag{5.4}$$

Evaluating at a fixed space-time point gives the mapping

$$\Psi(x) \,:\, \mathcal{H} \to S_x M\,, \qquad u \mapsto \psi^u(x)\,.$$

The operator x as well as the kernel of the fermionic projector can be expressed in terms of the wave evaluation operator by (see [7, Lemma 1.1.3])

$$x = -\Psi(x)^*\, \Psi(x) \qquad \text{and} \qquad P(x,y) = -\Psi(x)\, \Psi(y)^* \,. \tag{5.5}$$

5.2. **Perturbation Expansion for the Wave Evaluation Operator.** The perturbation expansion in Section 3 was performed in a chart on $\mathcal{F}$. We now explain how to construct such a chart. Working in this chart will also immediately give a perturbation expansion for the wave evaluation operator. Given $x \in M$, we consider the mapping

$$R \,:\, \mathrm{L}(\mathcal{H}, S_x) \to \mathcal{F}\,, \qquad \psi \mapsto \frac{c}{\mathrm{tr}(\psi^* \psi)}\, \psi^* \psi \,. \tag{5.6}$$

Clearly, this mapping is surjective, because every operator in $\mathcal{F}$ can be represented in the required form (note that the rescaling by the prefactor $c/\mathrm{tr}(\psi^*\psi)$ is needed in order to satisfy the trace condition (5.1)). But the mapping R has a kernel for two reasons: First, due to the rescaling, multiplying ψ by a real number leaves $R(\psi)$ unchanged. Second, a local unitary transformation

$$\psi \to U\, \psi \qquad \text{with} \qquad U \in \mathrm{U}(S_x)$$

preserves the combination $\psi^* \psi$ and thus leaves $R(\psi)$ unchanged. We choose a subspace $E \subset \mathrm{L}(\mathcal{H}, S_x)$ which contains $\Psi(x)$ such that

$$R|_E \,:\, E \to \mathcal{F}$$

is a local diffeomorphism around $\psi = \Psi(x)$. Then its local inverse

$$X := (R|_E)^{-1} \,:\, U \subset \mathcal{F} \to E$$

defines a chart (X, U) around x. Denoting a basis of E by $(e_1, \ldots, e_m)$, we thus write $F(x)$ in components $F(x)^\alpha$ with

$$X\big(F(x)\big) = \sum_{\alpha=1}^m F(x)^\alpha\, e_\alpha \,.$$

Choosing for every $x \in M$ a chart of this form and choosing a suitable jet space $\mathfrak{J}^{\text{test}}$, we are back in the setting of Sections 3.1. After determining the $F^{(p)}$, the corresponding perturbation of the wave evaluation operator is given simply by the component in our chart, i.e.

$$\Psi^{(p)}(x) = F^{(p)}(x)^\alpha\, e_\alpha \;\in\; E \subset \mathrm{L}(\mathcal{H}, S_x)\,.$$

5.3. **Perturbing the Vacuum.** We now explain how the construction in Section 3.5 applies to causal fermion systems. Let ρ be a universal measure describing the vacuum (for example, a regularized Dirac sea configuration as constructed in [7, Section 1.2]). Introducing particles and/or anti-particles (as described in [7, Section §2.1.7]) amounts to modifying the wave evaluation operator Ψ to

$$\hat{\Psi} := \Psi + \Delta\Psi \;:\; \mathcal{H} \to C^0(M, SM)\,. \tag{5.7}$$

At this point, the complication arises that the local correlation operators defined in analogy to (5.5) by $\hat{F}(x) = -\hat{\Psi}(x)^* \hat{\Psi}(x)$ (see [7, eq. (1.4.12)]) will in general violate our trace condition (5.1). In order to resolve this problem, we rescale the local correlation operators similar as in (5.6) by setting

$$\hat{H}(x) := \frac{c}{\mathrm{tr}\big(\hat{\Psi}(x)^* \hat{\Psi}(x)\big)}\, \hat{\Psi}(x)^* \hat{\Psi}(x)\,. \tag{5.8}$$

We now introduce the corresponding universal measure $\hat{\rho}$ as the push-forward of $\hat{H}$,

$$\hat{\rho} := \hat{H}_* \rho\,. \tag{5.9}$$

Now we are back in the setting of Section 3.5. We remark that the rescaling (5.8) seems unproblematic because in physical applications it affects only the higher orders in ε relative to the length scale of macroscopic physics (for details on this point see [7, Section 2.5]).

6. Example: Perturbation Expansion in the Continuum Limit

6.1. **Preliminaries.** We now recall a few constructions of the continuum analysis in [7] which will be of relevance here. In [7, §1.4.1] the EL equations are written in a form which is particular convenient for a detailed analysis. These EL equations are obtained by considering a particular class of variations of the wave evaluation operator Ψ:

Definition 6.1. *A* **variation of the physical wave functions** $(\Psi_\tau)_{\tau \in (-\delta, \delta)}$ *with* $\Psi_0 = \Psi$ *is* **smooth** *and* **compact** *if the family of operators has the following properties:*

(a) *The variation is trivial on the orthogonal complement of a finite-dimensional subspace $I \subset \mathcal{H}$, i.e.*

$$\Psi_\tau|_{I^\perp} = \Psi \qquad \text{for all } \tau \in (-\delta, \delta)\,.$$

(b) *There is a compact subset $K \subset M$ outside which the variation is trivial, i.e.*

$$\big(\Psi_\tau(u)\big)\big|_{M \setminus K} = \big(\Psi(u)\big)\big|_{M \setminus K} \qquad \text{for all } \tau \in (-\delta, \delta) \text{ and } u \in \mathcal{H}\,.$$

(c) *The Lagrangian is continuously differentiable in the sense that the derivative*

$$\frac{d}{d\tau} \mathcal{L}\big(x, F_\tau(y)\big)\big|_{\tau=0}$$

exists and is continuous on $M \times M$.

According to (5.5), the first variation $\delta\Psi = \partial_\tau \Psi|_{\tau=0}$ defines a corresponding variation of the kernel of the fermionic projector given by

$$\delta P(x,y) = -\delta\Psi(x)\, \Psi(y)^* - \Psi(x)\, \delta\Psi(y)^*\,. \tag{6.1}$$

The resulting first variation of the Lagrangian can be written as (see [4, Section 5.2] and [7, eq. (1.4.16)])

$$\delta\mathcal{L}(x,y) = \mathrm{Tr}_{S_y}\big(Q(y,x)\, \delta P(x,y)\big) + \mathrm{Tr}_{S_x}\big(Q(x,y)\, \delta P(x,y)^*\big) \tag{6.2}$$

with a kernel $Q(x,y) : S_y \to S_x$ which is symmetric in the sense that

$$Q(x,y)^* = Q(y,x)\,.$$

Then the EL equations corresponding to the above variations can be written as (see [7, Proposition 1.4.3])

$$\int_M Q(x,y)\,\psi^u(y)\,d\rho(y) = \frac{\lambda}{2}\,\psi^u(x) \qquad \text{for all } u \in \mathcal{H} \text{ and } x \in M\,. \tag{6.3}$$

The connection to the weak EL equations (2.4) is not obvious and will be explained in Section 6.2 below.

In the *continuum limit* (for details see [7, §3.5.2]), the EL equations (6.3) are evaluated for a physical wave function ψ^u having the form of an *ultrarelativistic wave packet* of negative energy, meaning that the wave packet has frequency of the order $|\Omega|$ and is spatially localized on the scale δ (as measured in a chosen reference frame). Moreover, we assume that the spatial distance of the ultrarelativistic wave packet from the space-time point x is on the scale ℓ with (see [7, eq. (3.5.28) and Figure 3.1])

$$\varepsilon \ll |\Omega|^{-1} \ll \delta \ll \ell, \ell_{\text{macro}}, m^{-1} \tag{6.4}$$

(where m^{-1} is the Compton scale and ℓ_{macro} denotes the length scales of atomic or high energy physics). Moreover, the equations (6.3) are evaluated weakly with a test function ϕ which is supported in a δ-neighborhood of the point x (with Euclidean distances measured again in a chosen reference frame). Then the supports of ϕ and ψ^u are disjoint, so that the right side of (6.3) vanishes (see [7, eqs (3.5.24) and (3.5.29)])

$$\int_M d\rho(x) \int_M d\rho(y)\, \prec\!\phi(x)\,|\,Q(x,y)\,\psi^u(y)\!\succ_x = 0\,. \tag{6.5}$$

Written in this form, the main contribution to the EL equations comes from the behavior of $Q(x,y)$ on the light cone, making it possible to analyze the equations in detail in the formalism of the continuum limit (for details see [7, Section 2.4 and Chapters 3-5]).

In the resulting continuum description, the kernel of the fermionic projector is a solution of the Dirac equation in the presence of a classical gauge field. In order to keep the setting a simple as possible, we here restrict attention to one generation of elementary particles and a U(1) gauge field (the generalizations to several generations and more general gauge fields are carried out in detail in [7, Chapters 3–5]). Then the Dirac equation reads

$$\left(i\partial\!\!\!/ + A\!\!\!/ - m\right) P(x,y) = 0\,, \tag{6.6}$$

where A can be thought of as the electromagnetic potential. In order to construct the kernel of the fermionic projector in the presence of the electromagnetic potential, one expands the Dirac equation (6.6) in powers of the potential and solves the equations iteratively with the help of Dirac Green's operators s defined by

$$(i\partial\!\!\!/ - m)\, s(x,y) = \delta^4(x-y)\,. \tag{6.7}$$

The resulting *causal perturbation expansion* becomes unique by making use of the underlying causal structure (for details see [7, Section 2.1]).

6.2. **Choosing the Jet Spaces and the Green's Operator.** In this section we explain how the weak EL equations (2.4) and the perturbation expansion of Section 3 are related to the analysis in the continuum limit. Our first task is to introduce the jet spaces. It is useful that, similar as explained in (5.7) and (5.8) for finite variations, tangent vectors to $\mathcal{F}$ on M can be described by infinitesimal variations of the wave evaluation operator. Thus we describe a tangent vector $u \in T_x F$ at a space-time point $x \in M$ as

$$u = \delta\hat{F}(x) = -\delta\Psi(x)^*\,\Psi(x) - \Psi(x)^*\,\delta\Psi(x) + \frac{x}{c}\,\mathrm{tr}\left(\delta\Psi(x)^*\,\Psi(x) + \Psi(x)^*\,\delta\Psi(x)\right) \tag{6.8}$$

(where we used that $\mathrm{tr}\, x = -\,\mathrm{tr}\,\Psi(x)^*\Psi(x) = c$) with

$$\delta\Psi \;:\; \mathcal{H} \to C^\infty(M, SM)\,. \tag{6.9}$$

Our next goal is to introduce the space of test jets $\mathfrak{J}^{\text{test}}$ in such a way that the weak EL equations (2.4) agree with the EL equations in the continuum limit (6.5) for ψ^u an ultrarelativistic wave packet (6.4). We say that a physical wave function ψ^u is *macroscopic* if its energy and momentum is much smaller than the Planck energy. We choose u such that ψ^u is macroscopic and is an *ultrarelativistic wave packet* as defined before (6.4). Next, we choose $\delta\psi^u$ as a wave function with compact support such that its spatial distance to the ultrarelativistic wave packet scales like

$$\varepsilon \ll \mathrm{dist}\big(\,\mathrm{supp}\,\delta\psi^u, \mathrm{supp}\,\psi^u\big) \ll \ell_{\text{macro}}\,. \tag{6.10}$$

We define the corresponding variation of the wave evaluation operator $\delta\Psi$ as the unique linear mapping with the properties that

$$\delta\Psi \;:\; v \mapsto \begin{cases} \delta\psi^u & \text{if } v = u \\ 0 & \text{if } v \perp u\,. \end{cases}$$

Since ψ^u and $\delta\psi^u$ have disjoint supports by definition, the trace in (6.8) vanishes. Therefore, the vector field described by $\delta\Psi$ is given by

$$u = \delta\hat{F}(x) = -\delta\Psi(x)^*\,\Psi(x) - \Psi(x)^*\,\delta\Psi(x)\,.$$

We choose Γ^{test} as the span of all the vector fields u for $\delta\Psi$ as specified above. Since in the weak evaluation on the light cone, only variations of the wave functions are considered, we choose the scalar component of $\mathfrak{J}^{\text{test}}$ trivially,

$$\mathfrak{J}^{\text{test}} = \{0\} \oplus \Gamma^{\text{test}} \;\subset\; C^\infty(M, \mathbb{R}) \oplus C^\infty(M, T\mathcal{F})\,. \tag{6.11}$$

We remark that there is no point in making (6.10) mathematically more precise, because in the formalism of the continuum limit one also works merely with the scaling behavior.

The next lemma gives the connection between the weak EL equations (2.4) and their continuum limit (6.5).

Lemma 6.2. *For any $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$ and all $x \in M$,*

$$\nabla_{\mathfrak{u}}\ell(x) = -2\,\mathrm{Re}\int_M \mathrm{tr}\left(\delta\Psi(x)^*\,Q(x,y)\,\Psi(y)\right)\,d\rho(y)\,.$$

Proof. Since $\mathfrak{u}$ has no scalar component, the term involving ν in (2.1) drops out. Using (6.1) together with the fact that we only vary x but leave y unchanged,

$$\nabla_{\mathfrak{u}}(x)P(x,y) = -\delta\Psi(x)\,\Psi(y)^*\,.$$

Using this formula in (6.2), we obtain

$$\begin{aligned}\nabla_{\mathrm{u}}(x)\mathcal{L}(x,y) &= -\operatorname{Tr}_{S_y}\big(Q(y,x)\,\delta\Psi(x)\,\Psi(y)^*\big) - \operatorname{Tr}_{S_x}\big(Q(x,y)\,\Psi(y)\,\delta\Psi(x)^*\big) \\ &= -2\,\mathrm{Re}\,\mathrm{tr}\big(\delta\Psi(x)^*\,Q(x,y)\,\Psi(y)\big)\,,\end{aligned}$$

where in the last step we cyclically commuted the factors inside the trace. Integrating over y gives the result. □

We next turn attention to the jets used for perturbing the measure. The abstract Definition 3.2 is intended to make $\mathfrak{J}^{\text{vary}}$ as large as possible, giving the largest possible freedom for the perturbations. But not all of the degrees of freedom of $\mathfrak{J}^{\text{vary}}$ are needed in the applications. Therefore, we must specify those subspaces of $\mathfrak{J}^{\text{vary}}$ which are of relevance here. We first consider jets which are needed to describe particle and anti-particle states.

Definition 6.3. *A vector field u of the form* (6.8) *where the variation $\delta\Psi$ is a mapping of finite rank with the property that for every $u \in \mathcal{H}$, either Ψu or $\delta\Psi u$ is macroscopic, is called* **fermionic vector field**. *The vector space of fermionic vector fields is referred to as Γ^{f}. The* **fermionic jets** *are defined by*

$$\mathfrak{J}^{\mathrm{f}} = \{0\} \oplus \Gamma^{\mathrm{f}}\,.$$

In the next definition we introduce the jets describing the bosons, for simplicity for an electromagnetic potential.

Definition 6.4. *Let $A \in C^\infty(M, T^*M)$ be a smooth one-form. A vector field u of the form* (6.8) *with*

$$\big(\delta\Psi\big)(x) = -\int_M s(x,y)\,\not{A}(y)\,\Psi(y)\,d\rho(y) \tag{6.12}$$

is called **bosonic vector field** *(here $s(x,y)$ is a Dirac Green's function* (6.7)*). The vector space of bosonic vector fields is referred to as Γ^{b}. The* **bosonic jets** *are defined by*

$$\mathfrak{J}^{\mathrm{b}} = \{0\} \oplus \Gamma^{\mathrm{b}}\,.$$

Clearly, the fermionic and bosonic jets are subspaces of $\mathfrak{J}^{\text{vary}}$,

$$\mathfrak{J}^{\mathrm{f}}, \mathfrak{J}^{\mathrm{b}} \subset \mathfrak{J}^{\text{vary}}\,.$$

We now explain how the perturbative description in the continuum limit is described in our setting. In the formalism of Section 3.5, the particles and anti-particles as introduced in [7, §3.4.3] correspond to a perturbation H of the vacuum measure in (3.34). The corresponding jets in (3.35) are fermionic,

$$\mathfrak{w}^{(p)} \in \mathfrak{J}^{\mathrm{f}}\,.$$

The resulting contributions to the weak EL equations are compensated by bosonic fields. Consequently, we here introduce the Green's operator S (see Definition 3.3) as a mapping to the bosonic jets,

$$S\,:\,(\mathfrak{J}^{\text{test}})^* \to \mathfrak{J}^{\mathrm{b}} \subset \mathfrak{J}^{\text{vary}}\,. \tag{6.13}$$

The condition (3.19) means that the potential $\mathcal{B}$ in (6.12) satisfies the inhomogeneous classical field equations. In the example of an electromagnetic potential (6.6) a Maxwell field, these equations become

$$\partial_{jk}(S\mathfrak{v})^k - \Box(S\mathfrak{v})_j = -c\,v_j \tag{6.14}$$

(or equivalently with differential forms $\delta d\, S\, \mathfrak{v} = -c\, \mathfrak{v}$, where the constant c depends on the detailed form of the regularization parameters in [7, Chapter 3]). This is the usual equation for the Maxwell propagator. It involves the freedom in choosing a gauge. For example, in the Lorenz gauge, one may choose S as the multiplication operator in momentum space $S(k) = c/k^2$. But S can also be given in any other gauge. Generally speaking, the choice of the Green's operator (6.13) involves a choice of gauge.

6.3. **Discussion and Remarks.** We now clarify the previous constructions by a few remarks. We first note that, in order to simplify the computations, it is often convenient to assume that the rescaling term in (6.8) vanishes, i.e.

$$\text{tr}\big(\Psi(x)^*\, \delta\Psi(x)\big) = 0 \qquad \text{for all } x \in M\,. \tag{6.15}$$

This can be arranged for example by the transformation

$$\delta\Psi \to \delta\Psi + \text{tr}\big(\Psi(x)^*\, \delta\Psi(x)\big)\, \frac{\Psi}{c}\,.$$

Thinking in terms of the charts constructed in Section 5.2, with the condition (6.15) one restricts attention to a special class of charts around x.

We next point out that, as explained in [7, Section 2.5], the rescaling terms in (6.8) give rise to terms of higher order in $\varepsilon/l_{\text{macro}}$. With this in mind, in many applications it is admissible to simply leave out the rescaling and to ignore the condition (6.15).

We also remark that all the above jet spaces have a natural *complex structure*. In order to understand how this comes about, we recall that according to (6.8) the vector fields on M were described by variations of the wave evaluation operator (6.9). Since the spin spaces are complex vector spaces, pointwise multiplication by complex scalars gives a natural complex structure on $\delta\Psi$. Using the notation (6.8), we thus obtain a corresponding almost complex structure J on $T_x\mathcal{F}$ given by

$$\begin{aligned} &J\, \delta\hat{F}[\delta\Psi](x) = \delta\hat{F}[i\delta\Psi](x) \\ &= i\delta\Psi(x)^*\, \Psi(x) - i\Psi(x)^*\, \delta\Psi(x) + \frac{x}{c}\, \text{tr}\big(-i\delta\Psi(x)^*\, \Psi(x) + i\Psi(x)^*\, \delta\Psi(x)\big)\,. \end{aligned}$$

This also gives rise to a complex structure on the vector spaces of vectorial jets on M like $\mathfrak{J}^{\text{f}}$ and $\mathfrak{J}^{\text{b}}$. This complex structure is of no relevance to the constructions in [7] but might be of importance for future developments.

We finally point out that the choice of test jets in (6.11) is very restrictive. In other words, the analysis of the continuum limit only uses very little of the information contained in the EL equations. On the other hand, this information seems to capture precisely what is needed in order to describe the effective macroscopic interaction. One shortcoming of the analysis in the continuum limit is that the test jets do not intersect the bosonic jets,

$$\mathfrak{J}^{\text{test}} \cap \mathfrak{J}^{\text{b}} = \varnothing\,.$$

This implies that the symplectic form as introduced in [10] is undefined for the bosonic jets. Moreover, since the intersection of the test jets with the fermionic jets only contains the very restrictive class of jets formed of ultrarelativistic wave packets, also the conserved surface layer integrals in [8] cannot be evaluated for interesting fermionic jets. This last shortcoming is closely related to the fact that the Green's operator (6.13) is purely bosonic, whereas the fermionic dynamics is encoded in the Dirac equation (6.6). Taking into account that in [7, Section 3.10] the validity of the Dirac equation is justified from the causal action principle by ruling out nonlocal potentials, this procedure is conceptually convincing as a first step. But eventually, one would like to have more

general test jets, giving rise to a unified description of the interaction in terms of Green's operators composed of a fermionic and a bosonic component. Such a description will be developed in the future papers [2, 11].

7. Example: Perturbation Expansion with Microscopic Mixing

7.1. **Preliminaries.** The method of microscopic mixing of wave functions was introduced in [6] (based on preliminary considerations in [5]). Using our present notation, the basic construction is summarized as follows. One first decomposes space-time into disjoint subsystems $M_1, \ldots, M_L$,

$$M = M_1 \cup \cdots \cup M_L \qquad \text{and} \qquad M_\mathfrak{a} \cap M_\mathfrak{b} = \varnothing \quad \text{if } a \neq b\,.$$

For each subsystem, one introduces a unitary operator $V_\mathfrak{a}$ with the property that $\mathbb{1} - V_\mathfrak{a}$ is an operator of finite rank which maps particle and anti-particle states to sea states and vice versa (for details see [6, Section 2.2]). Then the kernel of the fermionic projector with microscopic mixing is introduced by

$$P^\varepsilon(x,y) = \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \chi_{M_\mathfrak{a}}(x)\, P^{\mathfrak{a},\mathfrak{b}}(x,y)\, \chi_{M_\mathfrak{b}}(y) \tag{7.1}$$

$$P^{\mathfrak{a},\mathfrak{b}}(x,y) = -\Psi(x)\, V_\mathfrak{a}\, V_\mathfrak{b}^*\, \Psi(y)^* \tag{7.2}$$

(where $\chi_{M_\mathfrak{a}}$ is the characteristic function). In [6], this kernel of the fermionic projector is used as the starting point for a perturbative treatment based on the methods of the analysis in the continuum limit. It is shown that in a suitable limiting case, one obtains an effective interaction in terms of bosonic and fermionic field operators acting on Fock spaces.

7.2. **A Synchronization Mechanism.** In preparation for getting a connection to the setting of Section 4, we recast microscopic mixing in terms of the universal measure (for a similar construction see [7, §1.5.3]). To this end, for a unitary operator $V \in \mathrm{U}(\mathcal{H})$ we define the measure $V(\rho)$ by

$$(V\rho)(\Omega) = \rho\big(V \Omega V^{-1}\big)\,. \tag{7.3}$$

We introduce the measure $\hat{\rho}$ as the convex combination

$$\hat{\rho} = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \rho_\mathfrak{a} \qquad \text{with} \quad \rho_\mathfrak{a} = V_\mathfrak{a} \rho\,.$$

Then the resulting space-time $\hat{M} := \operatorname{supp} \hat{\rho}$ is given by

$$\hat{M} = \bigcup_{\mathfrak{a}=1}^{N} M_\mathfrak{a} \qquad \text{with} \qquad M_\mathfrak{a} := V_\mathfrak{a}\, M\, V_\mathfrak{a}^{-1}\,.$$

Comparing the unitary transformation $x \to V x V^{-1}$ in (7.3) with the first equation in (5.5), one sees that the wave evaluation operator (5.4) is transformed to

$$\hat{\Psi} \,:\, \mathcal{H} \to C^0\big(\hat{M}, S\hat{M}\big)\,, \qquad \hat{\Psi}(x_\mathfrak{a}) = \Psi(x)\, V_\mathfrak{a}\,.$$

Applying this relation in the second equation in (5.5), one recovers (7.2).

Next, we rewrite the wave evaluation operator of the $\mathfrak{a}^{\text{th}}$ subsystem as

$$\hat{\Psi}_\mathfrak{a} = \Psi + \Delta\Psi_\mathfrak{a} \qquad \text{with} \qquad \Delta\Psi_\mathfrak{a} = \Psi(x)\,\big(V_\mathfrak{a} - \mathbb{1}\big)\,.$$

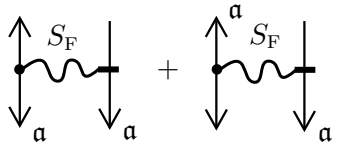

FIGURE 4. Synchronization of fluctuations.

Exactly as explained in Section 5.3, the resulting transformation of the universal measure can be written as (cf. (5.7) and (5.9))

$$\rho_{\mathfrak{a}} = (H_{\mathfrak{a}})_* \rho \,.$$

Expanding $H_{\mathfrak{a}}$ in a given chart on $\mathcal{F}$ similar to (3.35), one obtains inhomogeneities $\mathfrak{w}_{\mathfrak{a}}^{(p)}$ in the EL equations which depend on the subsystem. Following the constructions in Section 4.1 for the linearized inhomogeneity $\mathfrak{w} = \mathfrak{w}^{(1)}$, one gets a corresponding linearized solution of the field equations $\mathfrak{v}^{(1)}$ which involves fluctuations (4.8). The higher orders in perturbation theory are obtained just as in Sections 4.2 and 4.3. The only condition for the construction to work is that the resulting fragmentation must be well-posed (see Definition 4.4).

Choosing the jet spaces as in the continuum limit in Section 6.2, the above construction simplifies because the jet spaces do not have a scalar component. As a consequence, one may disregard the scalar fluctuations and ignore the condition of well-posedness in Definition 4.4. In this limiting case, one recovers the perturbation expansion in [6] with one important exception: the perturbation expansion with fragmentation gives rise to an additional *synchronization mechanism*. Namely, leaving out the scalar component, the perturbation expansion (4.16) simplifies to

$$\mathfrak{v}^{(p)} = \bar{S}^{(p)} \bar{E}^{(p)} + S_{\mathrm{F}}^{(p)} E_{\mathrm{F}}^{(p)} \,.$$

The Green's operators $\bar{S}^{(p)}$ can be chosen exactly as in Section 6.2 as the usual bosonic Green's operator S in (6.14). According to Definition 4.2, the fluctuating Green's function acts on each subsystem separately and can again be formed of the Green's operator without fragmentation,

$$\big(S_{\mathrm{F}}^{(p)}\big)_{\mathfrak{b}}^{\mathfrak{a}} = \delta_{\mathfrak{b}}^{\mathfrak{a}}\, S \,.$$

However, from (4.7) one sees that it couples only to the current generated by Dirac wave functions in the subsystem $\mathfrak{a}$ (see the Feynman diagrams in Figure 4). This seems to make it unnecessary to consider the stochastic background field in [6, Section 4] for synchronization. Also, the recombination of subsystems in [6, Section 7] needs to be reconsidered. The consequences of this synchronization mechanism will be analyzed in detail in a separate publication [12].

7.3. **Gauge Potentials are Subsystem-Diagonal.** The previous constructions yield an interaction which can be described by a Dirac equation with an electromagnetic potential for each subsystem, i.e.

$$(i\partial\!\!\!/ + A\!\!\!/_{\mathfrak{a}} - mY)\, \psi^{\mathfrak{a}}(x) = 0 \qquad \text{for all } \mathfrak{a} = 1, \ldots, L \,.$$

More generally, one could consider a matrix potential which mixes the subsystems, i.e.

$$\sum_{\mathfrak{b}=1}^{L} (i\partial\!\!\!/ + A\!\!\!/_{\mathfrak{b}}^{\mathfrak{a}} - mY)\, \psi^{\mathfrak{b}} = 0 \qquad \text{for all } \mathfrak{a} = 1, \ldots, L\,. \tag{7.4}$$

We now give an independent general argument which conveys a good intuitive understanding for why such subsystem-mixing potentials must not occur.

The matrix potential in (7.4) can be regarded as a $\mathrm{U}(L)$ gauge potential. To leading degree on the light cone, this gauge potentials affects the kernel of the fermionic projector via generalized phase transformations (for details see [3] or [7, §3.6.2 and §4.3.2]). Considering for simplicity a gauge transformation, the Dirac wave functions transforms according to

$$\psi_{\mathfrak{a}}(x) \to \sum_{\mathfrak{b}=1}^{L} U_{\mathfrak{b}}^{\mathfrak{a}}(x)\, \psi_{\mathfrak{b}}(x)\,.$$

Using this transformation law in (7.2) in the special case with trivial mixing matrices $V_1 = \cdots = V_N = \mathbb{1}$, one finds that the kernel of the fermionic projector transforms according to

$$P^{\mathfrak{a},\mathfrak{b}}(x,y) \to (U(x)\, v)^{\mathfrak{a}}\, P^{\mathfrak{a},\mathfrak{b}}(x,y)\, (U(y)\, v)^{\mathfrak{b}}\,,$$

where

$$v = (1, \ldots, 1) \in \mathbb{C}^L\,.$$

Since the Lagrangian is homogeneous of degree four in $P(x,y)$, it transforms like

$$\mathcal{L}(x,y) \to \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \left|(U(x)\, v)^{\mathfrak{a}}\right|^4 \left|(U(y)\, v)^{\mathfrak{b}}\right|^4 \mathcal{L}(x,y)\,.$$

Thus, seeking for minimizers of the causal action, one must

$$\text{minimize} \qquad \sum_{\mathfrak{a}=1}^{L} \left|(Uv)^{\mathfrak{a}}\right|^4. \tag{7.5}$$

We would like to show that the minimizers of this functional are precisely the subsystem-diagonal potentials. However, the situation is not quite so simple, as the following counter example shows:

Example 7.1. Choose $L = 2$ and consider the one-parameter group of unitary matrices $(U_t)_{t \in \mathbb{R}}$

$$U_t = \exp\left(\frac{it}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right).$$

Using that the matrix in the exponent is twice a projection operator, a short computation yields

$$U_t = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{e^{it}}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\,.$$

Thus

$$U_t\, v = e^{it} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence the relations

$$\left|(U_t\, v)^{\mathfrak{a}}\right| = 1 \qquad \text{for all } \mathfrak{a} = 1, 2$$

hold, although the unitary operators U_t are not diagonal. This shows that the diagonal unitary matrices cannot be singled out by minimizing (7.5). ◊

We now enter the general analysis. Given a compact connected Lie subgroup $\mathcal{G} \subset \mathrm{U}(L)$, we set

$$\mathcal{G}v := \{Uv \,|\, U \in \mathcal{G}\} \subset \mathbb{C}^L .$$

Moreover, we introduce the *diagonal* and *orthogonal subgroups* by

$$\mathcal{G}^{\mathrm{d}} = \left\{ U \in \mathcal{G} \,\middle|\, U = \left(e^{i\varphi_1}, \ldots, e^{i\varphi_L}\right) \text{ with } \varphi_{\mathfrak{a}} \in \mathbb{R} \right\} \tag{7.6}$$

$$\mathcal{G}^{\perp} = \left\{ U \in \mathcal{G} \,\middle|\, U|_{\mathcal{G}v} = \mathbb{1}_{\mathcal{G}v} \right\} . \tag{7.7}$$

The vector v is called *cyclic* if $\mathcal{G}v = \mathbb{C}^L$. Clearly, if v is cyclic, then $\mathcal{G}^{\perp}$ is trivial.

Proposition 7.2. *The infimum of the functional in* (7.5) *is given by*

$$\inf_{U \in \mathcal{G}} \sum_{\mathfrak{a}=1}^{L} \left|(Uv)^{\mathfrak{a}}\right|^4 = \inf_{U \in \mathrm{U}(L)} \sum_{\mathfrak{a}=1}^{L} \left|(Uv)^{\mathfrak{a}}\right|^4 = L . \tag{7.8}$$

Moreover, if this functional is minimal on all of $\mathcal{G}$, i.e.

$$\sum_{\mathfrak{a}=1^L} \left|(Uv)^{\mathfrak{a}}\right|^4 = L \qquad \text{for all } U \in \mathcal{G} ,$$

then every $U \in \mathcal{G}$ has a unique decomposition into a diagonal and an orthogonal element,

$$U = U^{\mathrm{d}}\, U^{\perp} \qquad \text{with} \qquad U^{\mathrm{d}} \in \mathcal{G}^{\mathrm{d}} \text{ and } U^{\perp} \in \mathcal{G}^{\perp} . \tag{7.9}$$

Before giving the proof, we explain what this result means. Generally speaking, this proposition gives strong constraints for the form of the subsystem-mixing gauge potentials. They may only be non-trivial if the vector v is not cyclic. But the vector v will be cyclic whenever each subsystem has its own dynamics. Namely, in this case, the subsystem-diagonal gauge potentials will be different in each subsystems, giving rise to different U(1)-phases in each subsystem. As a consequence, the group $\mathcal{G}$ will contain the abelian subgroup of all diagonal unitary matrices, implying that v is cyclic.

Proof of Proposition 7.2. We first prove (7.8). Since the rows of a unitary matrix are unit vectors, we know that

$$\sum_{\mathfrak{b}=1}^{n} |U^{\mathfrak{a}}_{\mathfrak{b}}|^2 = 1 .$$

As a consequence, using the Schwarz inequality,

$$L = \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \left|U^{\mathfrak{a}}_{\mathfrak{b}}\right|^2 = \sum_{\mathfrak{a}=1}^{L} \left|(Uv)^{\mathfrak{a}}\right|^2 \leq \sqrt{L} \left(\sum_{\mathfrak{a}=1}^{L} \left|(Uv)^{\mathfrak{a}}\right|^4 \right)^{\frac{1}{2}} , \tag{7.10}$$

implying that

$$\sum_{\mathfrak{a}=1}^{L} \left|(Uv)^{\mathfrak{a}}\right|^4 \geq L .$$

Equality is attained in the case $U = \mathbb{1}$, proving (7.8). More generally, equality holds if and only if all the summands in (7.10) coincide, i.e.

$$\left|(Uv)^{\mathfrak{a}}\right| = 1 \qquad \text{for all } \mathfrak{a} = 1, \ldots, L . \tag{7.11}$$

Next, we prove uniqueness of the decomposition (7.9). Suppose that a unitary operator U has the representation (7.9). Then, using (7.7), we know that $Uv = U^{\mathrm{d}} v$.

This relation uniquely determines all the phases $\varphi_1, \ldots, \varphi_L$ in (7.6). Hence U^{d} is unique, which also determines $U^\perp$ uniquely by $U^\perp = (U^{\mathrm{d}})^{-1} U$.

It remains to construct the decomposition (7.9). Let $A \in \mathfrak{g} \subset \mathfrak{u}(L)$ be a vector of the Lie algebra of $\mathcal{G}$. Then (7.11) implies that for any vector $w \in \mathcal{G}v$, the equation

$$\big|(e^{itA} w)^{\mathfrak{a}}\big| = 1 \qquad \text{holds for all } t \in \mathbb{R} \text{ and all } \mathfrak{a} = 1, \ldots, L\,.$$

Employing a spectral decomposition of the Hermitian matrix A,

$$A = \sum_{k=1}^K \lambda_k\, E_k\,, \qquad e^{itA} = \sum_{k=1}^K e^{i\lambda_k t}\, E_k\,,$$

we obtain

$$1 = \big|(e^{itA} w)^{\mathfrak{a}}\big|^2 = \sum_{k,k'=1}^K e^{i(\lambda_k - \lambda_{k'})t}\, \overline{\big(E_{k'} w\big)^{\mathfrak{a}}} \big(E_k w\big)^{\mathfrak{a}}\,. \tag{7.12}$$

We want to conclude that at most one summand is non-zero, i.e.

$$\big(E_k w\big)^{\mathfrak{a}} = 0 \qquad \text{for all } k \neq \ell$$

and a suitable $\ell = \ell(\mathfrak{a}, w)$. To this end, assume conversely that $(E_k w)^{\mathfrak{a}}$ and $(E_{k'} w)^{\mathfrak{a}}$ are both non-zero for $k \neq k'$. We choose k and k' such that $\lambda_k - \lambda_{k'}$ is maximal. Then the right side of (7.12) involves non-zero Fourier terms $\sim e^{\pm i(\lambda_k - \lambda_k')t}$, a contradiction.

Let us show that ℓ can be chosen independent of w. We proceed indirectly and assume that $k := \ell(\mathfrak{a}, w_1) \neq \ell(\mathfrak{a}, w_2) =: k'$. Then, evaluating (7.12) for $w = w_1 + w_2$, one gets a non-zero contribution

$$\mathrm{Re}\left(e^{i(\lambda_k - \lambda_{k'})t}\, \overline{\big(E_{k'} w_2\big)^{\mathfrak{a}}} \big(E_k w_1\big)^{\mathfrak{a}} \right).$$

Varying the phase of w_2, one again gets a contradiction. We conclude that

$$\big(E_k w\big)^{\mathfrak{a}} = 0 \qquad \text{for all } k \neq \ell(\mathfrak{a}) \text{ and all } w \in \mathcal{G}v\,. \tag{7.13}$$

Using the completeness of the spectral projectors, we obtain

$$w^{\mathfrak{a}} = \sum_{j=1}^L \big(E_j w\big)^{\mathfrak{a}} = \sum_{j=\ell(\mathfrak{a})} \big(E_j w\big)^{\mathfrak{a}} = \big(E_{\ell(\mathfrak{a})} w\big)^{\mathfrak{a}}\,.$$

Combining this relation with (7.13), it follows that

$$(E_k w)^{\mathfrak{a}} = \delta_{k, \ell(\mathfrak{a})}\, w^{\mathfrak{a}}\,.$$

Since $w \in \mathcal{G}v$ is arbitrary, we can also write this relation as

$$E_k\big|_{\mathcal{G}v} = E_k^{\mathrm{d}}\big|_{\mathcal{G}v} \qquad \text{with} \qquad \big(E_k^{\mathrm{d}}\big)^{\mathfrak{a}}_{\mathfrak{b}} = \delta^{\mathfrak{a}}_{\mathfrak{b}}\, \delta_{k, \ell(\mathfrak{a})}\,.$$

As a consequence, the matrix

$$U^{\mathrm{d}} := \sum_{k=1}^K e^{i\lambda_k t}\, E_k^{\mathrm{d}}$$

is diagonal. Moreover, the matrix $U^\perp := (U^{\mathrm{d}})^{-1} U$ is trivial on $\mathcal{G}v$, giving the desired decomposition (7.9). □

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