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Abstract

We consider the shape optimization of an object in Navier–Stokes flow by employing a combined phase field and porous medium approach, along with additional perimeter regularization. By considering integral control and state constraints, we extend the results of earlier works concerning the existence of optimal shapes and the derivation of first order optimality conditions. The control variable is a phase field function that prescribes the shape and topology of the object, while the state variables are the velocity and the pressure of the fluid. In our analysis, we cover a multitude of constraints which include constraints on the center of mass, the volume of the fluid region, and the drag of the object. Finally, we present numerical results of the optimization problem that is solved using the variable metric projection type (VMPT) method proposed by Blank and Rupprecht, where we consider one example of topology optimization without constraints and one example of maximizing the lift of the object with a state constraint, as well as a comparison with earlier results for the drag minimization.

Key words. Topology optimization, shape optimization, phase field approach, Navier–Stokes flow, integral state constraints, Lagrange multipliers.

AMS subject classification. 35Q35, 35Q56, 35R35, 49Q10, 49Q12, 65M22, 65M60, 76S05

1 Introduction

Fundamental to the design of aircraft and cars, as well as any technologies that would involve an object traveling within a fluid, such as wind turbines and drug delivery in biomedical applications, is the consideration of hydrodynamic forces acting on the object, for example the drag and lift forces. The desire to construct an object with minimal drag or with maximal lift-to-drag ratio naturally leads to the notion of shape optimization in fluids, in which the problem can often be formulated in terms of an optimal control problem with PDE constraints.

Let us assume that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with Lipschitz boundary, and contains a non-permeable object B. We will denote the boundary of B by $\Gamma := \partial B \cap \Omega$

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with the outer unit normal $\boldsymbol{\nu}$. A fluid is present in the complement region $E := \Omega \setminus B$, and we assume that the velocity $\boldsymbol{u}$ and the pressure p of the fluid in the region E obey the stationary Navier–Stokes equations with no-slip conditions on Γ, namely,

$$-\mu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } E, \tag{1.1a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \text{ in } E, \tag{1.1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma, \tag{1.1c}$$

$$\boldsymbol{u} = \boldsymbol{g} \text{ on } \partial E \cap \partial\Omega. \tag{1.1d}$$

Here $\boldsymbol{f}$ denotes the external body force, μ denotes the viscosity, and $\boldsymbol{g}$ models the inflow and outflow on the boundary $\partial\Omega$ such that $\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{\nu}_{\partial\Omega}\,\mathrm{d}\mathcal{H}^{d-1} = 0$, where ν_Ω denotes the outer unit normal on $\partial\Omega$.

We now introduce a design function $\varphi : \Omega \to \{\pm 1\}$, where $\{\varphi = 1\} = E$ describes the fluid region and $\{\varphi = -1\} = B$ is its complement. The natural function space for the design functions is the space of bounded variations that take values ± 1, i.e., $\varphi \in BV(\Omega, \{\pm 1\})$, which implies that the fluid region E has finite perimeter $P_\Omega(E)$. If φ is a function of bounded variation, its distributional derivative $\mathrm{D}\varphi$ is a finite Radon measure which can be decomposed into a positive measure $|\mathrm{D}\varphi|$ and a S^{d-1}-valued function $\boldsymbol{\nu}_\varphi \in L^1(\Omega, |\mathrm{D}\varphi|)^d$. The total variation for $\varphi \in BV(\Omega,\{\pm 1\})$, denoted by $|\mathrm{D}\varphi|(\Omega)$ satisfies

$$|\mathrm{D}\varphi|(\Omega) = 2P_\Omega(\{\varphi = 1\}),$$

and thus we can express the Hausdorff measure on Γ as $\frac{1}{2}\,\mathrm{d}\,|\mathrm{D}\varphi|(\Omega)$. Furthermore, the S^{d-1}-valued function $\boldsymbol{\nu}_\varphi$ can be considered as a generalized normal on $\partial\{\varphi = 1\}$. For a more detailed introduction to the theory of sets of finite perimeter and functions of bounded variation we refer to [1,10,17]. For functions $b : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, we consider the following shape optimization problem with perimeter regularization:

$$\begin{aligned}\min_{(\varphi,\boldsymbol{u},p)} \mathcal{J}_0(\varphi,\boldsymbol{u},p) &:= \int_\Omega b(x,\boldsymbol{u},\nabla\boldsymbol{u},p,\varphi)\,\mathrm{dx} \\ &\quad + \int_\Omega \frac{1}{2} h(x,\nabla\boldsymbol{u},p,\boldsymbol{\nu}_\varphi)\,\mathrm{d}\,|\mathrm{D}\varphi|(\Omega) + \frac{\gamma}{2}|\mathrm{D}\varphi|(\Omega),\end{aligned} \tag{1.2}$$

subject to $\varphi \in BV(\Omega,\{\pm 1\})$ and $(\boldsymbol{u},p) \in \boldsymbol{H}^1(E) \times L^2(E)$ fulfilling

$$-\mu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } E = \{\varphi = 1\}, \tag{1.3a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \text{ in } E, \tag{1.3b}$$

$$\boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega \cap \partial E, \tag{1.3c}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma = \Omega \cap \partial E. \tag{1.3d}$$

In addition, for fixed $m_1, m_2 \in \mathbb{N}\cup\{0\}$, we impose the m_1 integral equality constraints and m_2 integral inequality constraints:

$$H_i(\varphi,\boldsymbol{u},p) = 0 \text{ for } 1 \le i \le m_1, \quad H_i(\varphi,\boldsymbol{u},p) \ge 0 \text{ for } m_1 + 1 \le i \le m_1 + m_2, \tag{1.4}$$

where $\{H_i\}_{i=1}^{m_1+m_2}$ all take the form

$$H_i(\varphi,\boldsymbol{u},p) := \int_\Omega L_0(x,\boldsymbol{u},\nabla\boldsymbol{u},p,\varphi)\,\mathrm{dx} + \int_\Omega \boldsymbol{L}_1(x,\nabla\boldsymbol{u},p)\cdot\boldsymbol{\nu}_\varphi\,\mathrm{d}\,|\mathrm{D}\varphi|(\Omega), \tag{1.5}$$

for some functions $L_0 : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $z : \mathbb{R} \to \mathbb{R}$ and $\boldsymbol{L}_1 : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}^d$. The parameter $\gamma > 0$ in (1.2) is the weighting factor for the perimeter regularization, b and

L_0 are volume functions defined on the whole of Ω, while h and $\boldsymbol{L}_1$ are surface functions defined on Γ. Let us point out that one of the main difficulties in the mathematical treatment of shape optimization problems is that it is unknown if a minimizer/optimal shape to the problem exists (see [23, 26, 34]). There are positive results in this direction if perimeter penalization is additionally imposed, see for instance the recent results of [33]. This is the motivation of the perimeter regularization term $\frac{\gamma}{2}|\mathrm{D}\varphi|(\Omega)$ that appears in (1.2).

Let us now give some examples of the volume cost b that we would like to include in our present approach:

- the total potential power

$$\frac{1+\varphi}{2}\left(\frac{\mu}{2}|\nabla \boldsymbol{u}|^2 - \boldsymbol{f}\cdot\boldsymbol{u}\right), \tag{1.6}$$

- the construction cost of the object $\frac{1-\varphi}{2}w(x)$, where w denotes a cost function per unit volume,

- the least square approximation

$$\frac{1+\varphi}{2}\chi_Q(x)(\delta_1|p-p_{\mathrm{tar}}|^2 + \delta_2|\boldsymbol{u}-\boldsymbol{u}_{\mathrm{tar}}|^2)$$

  to a target velocity profile $\boldsymbol{u}_{\mathrm{tar}}$ and a target pressure profile p_{tar} in an observation region $Q \subset E$. Here δ_1 and δ_2 denote nonnegative constants.

- the square difference of the pressure

$$\frac{1+\varphi}{2}|\chi_{A_1}(x)p - \chi_{A_2}(x)p|^2$$

  for two disjoint regions $A_1, A_2 \subset E$.

An example for the surface cost h which we will focus on is the hydrodynamic force component in the direction of the unit vector $\boldsymbol{a}$, which is given as

$$\boldsymbol{a}\cdot\left(\mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\top}\right) - p\mathbf{I}\right)\boldsymbol{\nu}_{\varphi}, \tag{1.7}$$

where $\mathbf{I}$ denotes the identity tensor. The drag of the object is given when $\boldsymbol{a}$ is parallel to the flow direction $\boldsymbol{U}$, while the lift of the object is given when $\boldsymbol{a} = \boldsymbol{U}^{\perp}$, the unit vector perpendicular to the flow direction.

Examples of integral constraints we will study include

- volume constraints on the amount of fluid $L_0 = \beta_2 - \varphi$ and $L_0 = \varphi - \beta_1$ for fixed constants $-1 < \beta_1 \le \beta_2 < 1$, leading to

$$\frac{\beta_1+1}{2}|\Omega| \le \int_\Omega \frac{1+\varphi}{2}\,\mathrm{dx} = |E| \le \frac{\beta_2+1}{2}|\Omega| \Leftrightarrow \beta_1|\Omega| \le \int_\Omega \varphi\,\mathrm{dx} \le \beta_2|\Omega|, \tag{1.8}$$

- the prescribed mass of the object $L_0 = M|\Omega|^{-1} - \frac{1-\varphi}{2}\rho(x)$, where $\rho(x)$ is a mass density and $M > 0$ is a target/maximal mass, leading to

$$\int_\Omega \tfrac{1}{2}\rho(x)(1-\varphi)\,\mathrm{dx} \le M, \tag{1.9}$$

- the prescribed center of mass of the object (with uniform mass density) at a point y in the interior of Ω, i.e., $y \notin \partial\Omega$, $L_0 = \frac{1-\varphi}{2}(x_i - y_i)$ for $1 \le i \le d$, leading to

$$\int_\Omega \tfrac{1}{2}(1-\varphi)(x_i - y_i)\,\mathrm{dx} = 0 \text{ for } i = 1, 2, \ldots, d, \tag{1.10}$$

- the prescribed drag of the object, $\boldsymbol{L}_1 = -\mu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top)\boldsymbol{a} + p\boldsymbol{a}$, $L_0 = D\,|\,\Omega\,|^{-1}$, where $\boldsymbol{a}$ is the unit vector parallel to the flow direction $\boldsymbol{U}$ and $D > 0$ is a maximal drag value, leading to

$$\boldsymbol{a} \cdot \int_\Omega \big(\mu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top)\boldsymbol{\nu}_\varphi - p\boldsymbol{\nu}_\varphi\big)\,\frac{1}{2}\,\mathrm{d}\,|\,\mathrm{D}\varphi\,|\,(\Omega) \le D.$$

In the examples of the cost functional described above, the problem involving minimizing the drag of the object has received much attention and is well-studied in the literature. Let us point out that the hydrodynamic force component (using the notation $\mathcal{H}^{d-1}$ to denote the $(d-1)$-dimensional Hausdorff measure and $\boldsymbol{\nu}$ to denote the unit normal of Γ pointing from E to B)

$$\int_\Gamma \boldsymbol{a} \cdot \big(\mu\big(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\big) - p\,\mathbf{I}\,\big)\,\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1} \tag{1.11}$$

can be expressed in terms of a volume integral over $E = \{\varphi = 1\}$, and this reformulation has been used extensively in numerical simulations, see [9, §5.1], [16, §2.2], and [22, §9]: Given the unit vector $\boldsymbol{a}$, let $\boldsymbol{\eta}$ be a smooth vector field such that

$$\boldsymbol{\eta} = \boldsymbol{a} \text{ on } \Gamma \text{ and } \boldsymbol{\eta} = \mathbf{0} \text{ on } \partial\Omega. \tag{1.12}$$

Then, by taking the scalar product of (1.3a) with $\boldsymbol{\eta}$, we obtain

$$0 = \int_E -\operatorname{div}\,(\mu\nabla\boldsymbol{u})\cdot\boldsymbol{\eta} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\eta} + \nabla p\cdot\boldsymbol{\eta} - \boldsymbol{f}\cdot\boldsymbol{\eta}\,\mathrm{dx}\,.$$

Integrating by parts and noting that the boundary integrals over $\partial\Omega$ vanish due to $\boldsymbol{\eta} = \mathbf{0}$ on $\partial\Omega$, we then obtain

$$\begin{aligned} -\int_E \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{\eta} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\eta} - p\operatorname{div}\boldsymbol{\eta} - \boldsymbol{f}\cdot\boldsymbol{\eta}\,\mathrm{dx} &= \int_\Gamma \boldsymbol{a}\cdot(\mu\nabla\boldsymbol{u} - p\,\mathbf{I}\,)\,\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1} \\ = \int_\Gamma \boldsymbol{a}\cdot\big(\mu\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\big) - p\,\mathbf{I}\,\big)\,\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1}\,. \end{aligned} \tag{1.13}$$

Here we have also used that $\boldsymbol{u}$ has no tangential component on Γ due to the no-slip condition $\boldsymbol{u} = \mathbf{0}$ on Γ. Together with the divergence-free condition, we obtain that $(\nabla\boldsymbol{u})^\top\boldsymbol{\nu} = \mathbf{0}$ on Γ (see [15, §2] for more details). This implies that we can also consider the volume form of the drag as a possible cost functional

$$b = -\frac{1+\varphi}{2}\,(\mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{\eta} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\eta} - p\operatorname{div}\boldsymbol{\eta} - \boldsymbol{f}\cdot\boldsymbol{\eta}) \tag{1.14}$$

or as a possible constraint

$$L_0 = D\,|\,\Omega\,|^{-1} + \frac{1+\varphi}{2}\,(\mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{\eta} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\eta} - p\operatorname{div}\boldsymbol{\eta} - \boldsymbol{f}\cdot\boldsymbol{\eta})\,. \tag{1.15}$$

In addition, using integration by parts and the boundary conditions $\boldsymbol{\eta} = \mathbf{0}$ on $\partial\Omega$ and $\boldsymbol{u} = \mathbf{0}$ on Γ, we see that

$$\int_E (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\eta}\,\mathrm{dx} = -\int_E (\boldsymbol{u}\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{u}\,\mathrm{dx}\,,$$

and so we may also use instead of (1.14) and (1.15) the following

$$b = -\frac{1+\varphi}{2}\left(\mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{\eta} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{u} - p\operatorname{div}\boldsymbol{\eta} - \boldsymbol{f}\cdot\boldsymbol{\eta}\right), \tag{1.16}$$

$$L_0 = D|\Omega|^{-1} + \frac{1+\varphi}{2}\left(\mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{\eta} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{\eta}\cdot\boldsymbol{u} - p\operatorname{div}\boldsymbol{\eta} - \boldsymbol{f}\cdot\boldsymbol{\eta}\right), \tag{1.17}$$

In particular, we will use the formation (1.16) instead of (1.14) in our numerical investigation below.

While the problem of shape optimization in fluid flow has been investigated intensively by several authors, see for example [3, 6, 27, 28, 31] and the references therein, and also [30] for the derivation of shape derivatives for general volume and boundary objective functionals in Navier–Stokes flow, to the authors best knowledge, the shape optimization problem with integral state constraint has not received much attention. Our present contribution arises from a previous numerical study of the shape optimization problem of maximizing the lift-to-drag ratio in [15]. In two dimensions, the lift-to-drag ratio is defined as

$$\frac{L}{D} := \frac{\int_\Gamma \boldsymbol{u}_\infty^\perp\cdot\left(\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\top\right)-p\mathbf{I}\right)\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1}}{\int_\Gamma \boldsymbol{u}_\infty\cdot\left(\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\top\right)-p\mathbf{I}\right)\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^{d-1}}, \tag{1.18}$$

where $\boldsymbol{u}_\infty$ is the flow direction and $\boldsymbol{u}_\infty^\perp$ is the vector perpendicular to $\boldsymbol{u}_\infty$. In [15] the authors obtain an optimal shape similar to an airfoil. However, an obstacle to a rigorous mathematical treatment of the problem is that it is unknown if the lift-to-drag ratio (1.18) is bounded from below. Hence, our present contribution attempts to study a related problem involving maximizing the lift while the drag is constrained to be below a certain threshold.

The rest of the paper is organized as follows: In Section 2, we present the phase field approximation of (1.2)-(1.5) that utilizes the porous-medium approach of Borrvall and Petersson [7], and state several preliminary results on the state equations. In Section 3, we state the assumptions on b, h, L_0 and $\boldsymbol{L}_1$, and then establish the existence of minimizers to the phase field shape optimization problem. In Section 4 we outline the assumptions on the differentiability of b, h, L_0 and $\boldsymbol{L}_1$, for the existence of Lagrange multipliers, the solvability of the adjoint system, and the derivation of the necessary optimality conditions. In Section 5 we verify the conditions of the existence of Lagrange multipliers for consider the special case where the integral constraints are acting only on φ and not on the state variables $(\boldsymbol{u}, p)$. These will be constraints on the mass, center of mass and volume. In Section 6 we briefly outline our numerical approach to solving the optimality conditions, and present several numerical simulations in Section 7.

2 Phase field formulation

In the phase field formulation, we relax the condition that φ takes only values in $\{\pm 1\}$ and now allow φ to be a function with values in $\mathbb{R}$ and inherits $H^1(\Omega)$ regularity. This leads to the development of interfacial layers $\{-1 < \varphi < 1\}$ in between the fluid region $E = \{\varphi = 1\}$ and the object region $B = \{\varphi = -1\}$. This interfacial layer replaces the boundary Γ of B and a parameter $\varepsilon > 0$ is associated to the thickness of the interfacial layer. The idea is to use the Ginzburg–Landau energy functional

$$\mathcal{E}_\varepsilon(\varphi) = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\Psi(\varphi),$$

where Ψ is a potential with equal minima at $\varphi = \pm 1$, to approximate the perimeter functional P_Ω. It has been shown in [25] that $\mathcal{E}_\varepsilon$ approximates $\varphi \mapsto c_0 |\mathrm{D}\varphi|(\Omega) = 2c_0 P_\Omega(\{\varphi = 1\})$ in the sense of Γ-convergence, where c_0 is a positive constant depending only on Ψ:

$$c_0 := \frac{1}{2}\int_{-1}^{1} \sqrt{2\Psi(s)}\,\mathrm{d}s\,. \tag{2.1}$$

By introducing an interfacial region between the fluid and the object, we have relaxed the non-permeability assumption of the object in the vicinity of its boundary. Therefore, we use the so-called porous medium approach and replace the object B with a porous medium of small permeability $(\overline{\alpha_\varepsilon})^{-1} \ll 1$. A function $\alpha_\varepsilon(\varphi)$ is introduced to interpolate between the inverse permeabilities of the fluid region $\alpha_\varepsilon(1) = 0$ and the porous medium $\alpha_\varepsilon(-1) = \overline{\alpha_\varepsilon}$, which satisfies

$$\overline{\alpha_\varepsilon} \to \infty \text{ as } \varepsilon \to 0.$$

With this, we extend the state equations from E to the whole domain Ω by the addition of the term $\alpha_\varepsilon(\varphi)\boldsymbol{u}$:

$$\alpha_\varepsilon(\varphi)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega, \tag{2.2a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \tag{2.2b}$$

$$\boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega. \tag{2.2c}$$

We note that this additional term vanishes in the fluid region, and in the limit $\varepsilon \to 0$, one expects that the velocity $\boldsymbol{u}$ in the object region to vanish. Later we will add $\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{dx}$ to the cost functional to ensure that in the limit $\varepsilon \to 0$, the velocity $\boldsymbol{u}$ vanishes outside the fluid region. This combination of the porous medium approach of Borrvall and Petersson [7] and a Ginzburg–Landau regularization as in the work of Bourdin and Chambolle [8] has been used in [13] for Stokes' flow and in [12, 14, 15, 24] for Navier–Stokes flow. In [15] a phase field approach to handle boundary object functionals is introduced. The idea is to approximate the generalized normal $\boldsymbol{\nu}_\varphi$ by $\frac{\nabla\varphi}{|\nabla\varphi|}$ and use the so-called equipartition of the Ginzburg–Landau energy to approximate $c_0|\mathrm{D}\varphi|(\Omega)$ by $\frac{2}{\varepsilon}\Psi(\varphi)$. In the case where h is one-homogeneous with respect to its last variable, which is true for the applications we have in mind, the phase field approximation can be further simplified, where one uses the vector-valued measure with density $\frac{1}{2}\nabla\varphi$ as an approximation to $\boldsymbol{\nu}_\varphi\,\mathrm{d}\,|\mathrm{D}\varphi|(\Omega)$, see [15, §3.2] for more details. This in turn gives us

$$\int_\Omega \frac{1}{2}h(x, \nabla\boldsymbol{u}, p, \nabla\varphi)\,\mathrm{dx}$$

as a phase field approximation to the surface cost functional h in (2.4). Then, we may also approximate the hydrodynamic force component (1.11) by

$$\int_\Omega \frac{1}{2}\nabla\varphi\cdot\left(\mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\right) - p\,\mathbf{I}\right)\boldsymbol{a}\,\mathrm{dx}\,. \tag{2.3}$$

For the rest of this work, we will assume that h is one-homogeneous in its last variable. The phase field approximation to (1.2)-(1.5) is given by the following optimal control problem:

$$\begin{aligned}\min_{(\varphi,\boldsymbol{u},p)} \mathcal{J}_\varepsilon(\varphi,\boldsymbol{u},p) := &\int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2\,\mathrm{dx} + \int_\Omega b(x,\boldsymbol{u},\nabla\boldsymbol{u},p,\varphi)\,\mathrm{dx}\\ &+ \int_\Omega \frac{1}{2}h(x,\nabla\boldsymbol{u},p,\nabla\varphi)\,\mathrm{dx} + \frac{\gamma}{2c_0}\int_\Omega \frac{1}{\varepsilon}\Psi(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2\,\mathrm{dx}\,,\end{aligned} \tag{2.4}$$

subject to

$$
\begin{aligned}
&\varphi \in H^1(\Omega),\\
&\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) := \left\{\boldsymbol{h} \in \boldsymbol{H}^1(\Omega) \,|\, \operatorname{div} \boldsymbol{h} = 0 \text{ in } \Omega \text{ and } \boldsymbol{h} = \boldsymbol{g} \text{ on } \partial\Omega\right\},\\
&p \in L^2_0(\Omega) := \left\{h \in L^2(\Omega) \,\Big|\, \int_\Omega h \,\mathrm{dx} = 0\right\}
\end{aligned}
$$

satisfying

$$
\int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} - p \operatorname{div}\boldsymbol{v}\,\mathrm{dx} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \tag{2.5}
$$

and equality/inequality integral constraints of the form

$$
G(\boldsymbol{u}, p, \varphi) := \int_\Omega L_0(x, \boldsymbol{u}, \nabla\boldsymbol{u}, p, \varphi) + \frac{1}{2}\nabla\varphi\cdot\boldsymbol{L}_1(x, \nabla\boldsymbol{u}, p)\,\mathrm{dx}.
$$

We point out that (2.5) is the weak formulation of the porous medium Navier–Stokes equations (2.2), and that the phase field approximations for the cost functions and the constraints from Section 1 have the same functional form.

2.1 Preliminaries on the state equations

The strong form for (2.5) is

$$
\alpha_\varepsilon(\varphi)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = f \text{ in } \Omega, \tag{2.6a}
$$

$$
\operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \tag{2.6b}
$$

$$
\boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega. \tag{2.6c}
$$

In this section, we state the assumptions on the function α_ε and the results regarding the above state equations.

Assumption 2.1. *We assume that $\alpha_\varepsilon \in C^{1,1}(\mathbb{R})$ is non-negative, with $\alpha_\varepsilon(1) = 0$, $\alpha_\varepsilon(-1) = \overline{\alpha}_\varepsilon > 0$, and that there exists s_a, $s_b \in \mathbb{R}$ with $s_a \le -1$ and $s_b \ge 1$ such that*

$$
\begin{aligned}
\alpha_\varepsilon(s) = \alpha_\varepsilon(s_a)\ \forall s \le s_a,\\
\alpha_\varepsilon(s) = \alpha_\varepsilon(s_b)\ \forall s \ge s_b.
\end{aligned}
$$

Furthermore, the inverse permeability $\overline{\alpha}_\varepsilon$ tends to infinity as $\varepsilon \to 0$.

Note that, for any $\varphi \in L^1(\Omega)$, its truncation $\tilde{\varphi} := \max(s_a, \min(s_b, \varphi))$ satisfies $\alpha_\varepsilon(\varphi) = \alpha_\varepsilon(\tilde{\varphi})$. Hence, for the rest of this paper, without loss of generality, we assume that $\varphi \in H^1(\Omega)$ with $s_a \le \varphi \le s_b$ a.e. in Ω.

Lemma 2.1. *[15, Lemma 4.3] Under Assumption 2.1, for every $\varphi \in L^1(\Omega)$ there exists at least one pair $(\boldsymbol{u}, p) \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ such that (2.5) is satisfied. Furthermore, there exists a positive constant $C = C(\mu, \alpha_\varepsilon, \boldsymbol{f}, \boldsymbol{g}, \Omega)$ independent of φ such that*

$$
\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|p\|_{L^2(\Omega)} \le C. \tag{2.7}
$$

By the above existence result, we can define a set-valued solution operator

$$
\boldsymbol{S}_\varepsilon(\varphi) := \left\{(\boldsymbol{u}, p) \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega) \,|\, (\boldsymbol{u}, p) \text{ satisfies (2.5)}\right\} \tag{2.8}
$$

for any $\varphi \in L^1(\Omega)$. In general, we do not have a unique solution to (2.5), but under an additional assumption, there is a conditional uniqueness result.

Lemma 2.2. *([12, Lemma 5], [19, Lemma 12.2]) If there exists* $\boldsymbol{u} \in \boldsymbol{S}_\varepsilon(\varphi)$ *with*

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} < \frac{\mu}{K_\Omega}, \tag{2.9}$$

where

$$K_\Omega := \begin{cases} \frac{1}{2}|\Omega|^{\frac{1}{2}} & \text{for } d = 2, \\ \frac{2\sqrt{2}}{3}|\Omega|^{\frac{1}{6}} & \text{for } d = 3. \end{cases} \tag{2.10}$$

Then, $\boldsymbol{S}_\varepsilon(\varphi) = \{(\boldsymbol{u}, p)\}$*. That is, there is exactly one solution of* (2.5) *corresponding to* $\varphi \in L^1(\Omega)$*.*

Remark 2.1. *Let us point out that from the derivation of [15, Equation (4.10)] one obtains*

$$\begin{aligned} \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \leq \frac{4}{\mu}\Big((\max_{s\in[s_a,s_b]} \alpha_\varepsilon(s) + 2\mu)\|\boldsymbol{G}\|^2_{\boldsymbol{H}^1(\Omega)} + 2C_p^2\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}\Big) \\ + \frac{4}{\mu}\Big(2(C_p+1)^2\|\boldsymbol{G}\cdot\nabla\boldsymbol{G}\|^2_{\boldsymbol{H}^{-1}(\Omega)}\Big) + 2\|\nabla\boldsymbol{G}\|^2_{\boldsymbol{L}^2(\Omega)}, \end{aligned} \tag{2.11}$$

where C_p *denotes the positive constant from the Poincaré inequality and* $\boldsymbol{G} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$ *is a vector field satisfying* $\boldsymbol{G} = \boldsymbol{g}$ *on* $\partial\Omega$ *(see [11, Lemma IX.4.2] or [12, Lemma 3]). Thus, the condition* (2.9) *can be achieved for small data* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *or with high viscosity* μ*. However, there are also settings in which* (2.9) *can be justified a posteriorly, and for this we refer the reader to [12] for more details.*

Remark 2.2. *The subsequent analysis is valid in a neighborhood of an isolated local solution to* (2.6) *if* (2.9) *is neglected.*

Next, we state a continuity property of the solution operator.

Lemma 2.3. *[15, Lemma 4.4 and 4.5] Under Assumption 2.1, let* $(\varphi_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ *be a sequence with corresponding solution* $(\boldsymbol{u}_k, p_k) \in \boldsymbol{S}_\varepsilon(\varphi_k) \subset \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ *for each* $k \in \mathbb{N}$*. Suppose there exists* $\varphi \in L^1(\Omega)$ *such that*

$$\|\varphi_k - \varphi\|_{L^1(\Omega)} \to 0 \text{ as } k \to \infty.$$

Then, there exists a subsequence, denoted by the same index, and functions $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$*,* $p \in L^2(\Omega)$ *such that*

$$\|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \to 0, \quad \|p_k - p\|_{L^2(\Omega)} \to 0 \text{ as } k \to \infty,$$

with the property that $(\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)$*. Furthermore, it holds that*

$$\lim_{k\to\infty} \int_\Omega \alpha_\varepsilon(\varphi_k)|\boldsymbol{u}_k|^2 \,\mathrm{dx} = \int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2 \,\mathrm{dx}.$$

We now state the Fréchet differentiability of the solution operator $\boldsymbol{S}_\varepsilon$, which will require the uniqueness of solutions to the state equations (2.6).

Lemma 2.4. *[15, Lemma 4.8] Under Assumption 2.1, let* $\varphi_\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega)$ *be given such that* $\boldsymbol{S}_\varepsilon(\varphi_\varepsilon) = \{(\boldsymbol{u}_\varepsilon, p_\varepsilon)\}$*. Then, there exists a neighborhood* N *of* φ_ε *in* $H^1(\Omega) \cap L^\infty(\Omega)$ *such that for every* $\psi \in N$*, the solution operator consists of exactly one pair, and we may write* $\boldsymbol{S}_\varepsilon : N \to \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$*. This mapping is differentiable at* φ_ε *with*

$$\mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)(\psi) =: (\boldsymbol{u}, p) \in \boldsymbol{H}^1_0(\Omega) \times L^2_0(\Omega),$$

where $(\boldsymbol{u}, p)$ *is the unique solution to the linearized state system*

$$\alpha_\varepsilon'(\varphi_\varepsilon)\psi \boldsymbol{u}_\varepsilon + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_\varepsilon + (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{0} \ \textit{in } \Omega, \tag{2.12a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \ \textit{in } \Omega, \tag{2.12b}$$

$$\boldsymbol{u} = \boldsymbol{0} \ \textit{on } \partial\Omega. \tag{2.12c}$$

In particular, the hypothesis of Lemma 2.4 can be satisfied with the condition (2.9), which holds for small data $\boldsymbol{f}$ and $\boldsymbol{g}$ or large viscosity μ. Alternatively, one can work in the neighborhood of an isolated local solution to (2.6).

3 Existence of a minimizer

We make the following assumptions for the potential Ψ and the functions b, h, L_0, and $\boldsymbol{L}_1$.

Assumption 3.1. *Let* $\Psi \in C^{1,1}(\mathbb{R})$ *be a non-negative function such that* $\Psi(s) = 0$ *if and only if* $s = \pm 1$, *and that there exist positive constants* c_1, c_2, t_0 *such that*

$$c_1 t^k \le \Psi(t) \le c_2 t^k \quad \forall\, |t| \ge t_0, k \ge 2.$$

Assumption 3.2. *Let* $b : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $h : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *be Carathéodory functions fulfilling*

- $b(\cdot, \boldsymbol{w}, \boldsymbol{A}, s, t) : \Omega \to \mathbb{R}$ *and* $h(\cdot, \boldsymbol{A}, s, \boldsymbol{w}) : \Omega \to \mathbb{R}$ *are measurable for each* $s, t \in \mathbb{R}$, $\boldsymbol{w} \in \mathbb{R}^d$ *and* $\boldsymbol{A} \in \mathbb{R}^{d\times d}$,
- $b(x, \cdot, \cdot, \cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $h(x, \cdot, \cdot, \cdot) : \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *are continuous for almost all* $x \in \Omega$.

Moreover, we suppose that b *is of the form*

$$b(x, \boldsymbol{w}, \boldsymbol{A}, s, t) := B(x, \boldsymbol{w}, \boldsymbol{A}, s)\, z(x, t), \tag{3.1}$$

for some Carathéodry functions $B : \Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\times\mathbb{R} \to \mathbb{R}$, $z : \Omega\times\mathbb{R} \to \mathbb{R}$, *and there exist non-negative functions* $y_b, y_h \in L^1(\Omega)$, $\{y_{b,i}\}_{i=1}^4, \{y_{h,k}\}_{k=1}^3 \subset L^\infty(\Omega)$ *such that for almost every* $x \in \Omega$, *it holds for any* $r \ge 0$, $p \ge 2$ *in two-dimensions and* $2 \le p \le 6$ *in three-dimensions*

$$\begin{aligned} |B(x, \boldsymbol{w}, \boldsymbol{A}, s)| &\le y_b(x) + y_{b,1}(x)|\boldsymbol{w}|^p + y_{b,2}(x)|\boldsymbol{A}|^2 + y_{b,3}(x)|s|^2, \\ |z(x,t)| &\le y_{b,4}(x)|t|^r, \\ |h(x, \boldsymbol{A}, s, \boldsymbol{w})| &\le y_h(x) + y_{h,1}(x)|\boldsymbol{A}|^2 + y_{h,2}(x)|s|^2 + y_{h,3}(x)|\boldsymbol{w}|^2, \end{aligned}$$

for all $s, t \in \mathbb{R}$, $\boldsymbol{w} \in \mathbb{R}^d$ *and* $\boldsymbol{A} \in \mathbb{R}^{d\times d}$. *Furthermore, the functionals* $\mathcal{B} : H^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ *and* $\mathcal{H} : H^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ *defined as*

$$\mathcal{B}(\varphi, \boldsymbol{u}, p) := \int_\Omega B(x, \boldsymbol{u}, \nabla\boldsymbol{u}, p)\, z(x, \varphi)\,\mathrm{dx}, \quad \mathcal{H}(\varphi, \boldsymbol{u}, p) := \int_\Omega \frac{1}{2} h(x, \nabla\boldsymbol{u}, p, \nabla\varphi)\,\mathrm{dx}$$

satisfy the following properties

- $\mathcal{B}|_{\mathbb{K}_{ad}\times\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)}$ *and* $\mathcal{H}|_{\mathbb{K}_{ad}\times\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)}$ *are bounded from below, and*
- $\mathcal{B}$ *is weakly lower semicontinuous, and for all* $\varphi_n \rightharpoonup \varphi$ *in* $H^1(\Omega)$, $\boldsymbol{u}_n \to \boldsymbol{u}$ *in* $\boldsymbol{H}^1(\Omega)$, $p_n \to p$ *in* $L^2(\Omega)$, *it holds that*

$$\mathcal{H}(\varphi, \boldsymbol{u}, p) \le \liminf_{n\to\infty} \mathcal{H}(\varphi_n, \boldsymbol{u}_n, p_n).$$

Assumption 3.3. *Let* $L_0 : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\boldsymbol{L}_1 : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}^d$ *be Carathédory functions fulfilling*

- $L_0(\cdot, \boldsymbol{w}, \boldsymbol{A}, s, t) : \Omega \to \mathbb{R}$ *and* $\boldsymbol{L}_1(\cdot, \boldsymbol{A}, s) : \Omega \to \mathbb{R}^d$ *are measurable for each* $s, t \in \mathbb{R}$, $\boldsymbol{w} \in \mathbb{R}^d$ *and* $\boldsymbol{A} \in \mathbb{R}^{d\times d}$,
- $L_0(x, \cdot, \cdot, \cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $\boldsymbol{L}_1(x, \cdot, \cdot) : \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}^d$ *are continuous for almost all* $x \in \Omega$.

Moreover, we suppose that L_0 *is of the form*

$$L_0(x, \boldsymbol{w}, \boldsymbol{A}, s, t) := L(x, \boldsymbol{w}, \boldsymbol{A}, s)\, y(x, t) + k(x), \tag{3.2}$$

for some $k \in L^1(\Omega)$ *and Carathédory functions* $L : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}$, $y : \Omega \times \mathbb{R} \to \mathbb{R}$, *and there exists non-negative functions* $z_1, z_6 \in L^1(\Omega)$, $z_2, z_3, z_4, z_5, z_7, z_8 \in L^\infty(\Omega)$ *such that for almost every* $x \in \Omega$, *it holds for any* $r \geq 0$, $p \in [2, \infty)$ *in two-dimensions and* $p \in [2, 6)$ *in three-dimensions*

$$\begin{aligned} |L(x, \boldsymbol{w}, \boldsymbol{A}, s)| &\leq z_1(x) + z_2(x)|\boldsymbol{w}|^p + z_3(x)|\boldsymbol{A}|^2 + z_4(x)|s|^2, \\ |y(x,t)| &\leq z_5(x)|t|^r, \\ |\boldsymbol{L}_1(x, \boldsymbol{A}, s)| &\leq z_6(x) + z_7(x)|\boldsymbol{A}| + z_8(x)|s|. \end{aligned}$$

Remark 3.1. *The particular forms of* b *and* L_0 *are motivated from the discussions in Section 1, where* z *and* y *would typically be functions of the form* $\frac{1+\varphi}{2}$, *and the function* k *would be of the form* $D|\Omega|^{-1}$.

We make the following definition.

Definition 3.1. *For fixed* $\varepsilon \in (0, 1]$ *and for any* $1 \leq i \leq m_1 + m_2$, *we define the functional* $\mathcal{G}_i : H^1(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$ *as*

$$\mathcal{G}_i(\varphi) := \int_\Omega L(x, \boldsymbol{u}, \nabla \boldsymbol{u}, p)\, y(x, \varphi) + \frac{1}{2} \boldsymbol{L}_1(x, \nabla \boldsymbol{u}, p) \cdot \nabla \varphi \,\mathrm{dx} \;\; \text{for } (\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi).$$

We say that $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ *is an admissible design function if and only if* $\mathcal{G}_i(\varphi) = 0$ *for* $1 \leq i \leq m_1$ *and* $\mathcal{G}_i(\varphi) \geq 0$ *for* $m_1 + 1 \leq i \leq m_1 + m_2$. *We denote by* $\mathbb{K}_{ad} := \{\varphi \in H^1(\Omega) : s_a \leq \varphi \leq s_b$ *a.e. in* Ω *for admissible* $\varphi\}$ *as the space of admissible design functions.*

Lemma 3.1. *Let* $\{\varphi_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{K}_{ad}$ *such that* $\varphi_n \rightharpoonup \varphi \in H^1(\Omega)$ *for some* $\varphi \in H^1(\Omega)$, *then* $\varphi \in \mathbb{K}_{ad}$.

Proof. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{K}_{ad}$ be a sequence with $\{(\boldsymbol{u}_n, p_n)\}_{n\in\mathbb{N}} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ such that $(\boldsymbol{u}_n, p_n) \in \boldsymbol{S}_\varepsilon(\varphi_n)$ and there exists some element $\varphi \in H^1(\Omega)$ such that $\varphi_n \rightharpoonup \varphi \in H^1(\Omega)$ and $s_a \leq \varphi \leq s_b$ a.e. in Ω. Then, by the compact embedding $H^1(\Omega) \subset\subset L^p(\Omega)$ for $p \in [1, \infty)$ in two-dimensions and $p \in [1, 6)$ in three-dimensions, and the assertions of Lemma 2.3, we find a subsequence, denoted by the same index, such that $\varphi_n \to \varphi$ in $L^p(\Omega)$, $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ and $p_n \to p$ in $L^2(\Omega)$ where $(\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)$.

By the continuity of $\boldsymbol{L}_1$ with respect to its second and third variables, it holds that $\boldsymbol{L}_1(x, \nabla \boldsymbol{u}_n, p_n) \to \boldsymbol{L}_1(x, \nabla \boldsymbol{u}, p)$ a.e. in Ω. Using the growth conditions in Assumption 3.3 for $\boldsymbol{L}_1$, the strong convergences for $\{\boldsymbol{u}_n, p_n\}_{n\in\mathbb{N}}$ and the generalized Lebesgue dominated convergence theorem leads to

$$\boldsymbol{L}_1(x, \nabla \boldsymbol{u}_n, p_n) \to \boldsymbol{L}_1(x, \nabla \boldsymbol{u}, p) \text{ strongly in } \boldsymbol{L}^2(\Omega) \text{ as } n \to \infty. \tag{3.3}$$

Together with the weak convergence $\nabla\varphi_n$ to $\nabla\varphi$ in $\boldsymbol{L}^2(\Omega)$, we have

$$\lim_{n\to\infty}\int_\Omega \frac{1}{2}\nabla\varphi_n\cdot \boldsymbol{L}_1(x,\nabla\boldsymbol{u}_n,p_n)\,\mathrm{dx} = \int_\Omega \frac{1}{2}\nabla\varphi\cdot \boldsymbol{L}_1(x,\nabla\boldsymbol{u},p)\,\mathrm{dx}\,.$$

Note that $s_a \le \varphi_n, \varphi \le s_b$ a.e. in Ω for all $n\in\mathbb{N}$, and thus there exists a constant $M>0$ such that $\sup_{x\in\Omega}(|y(x,\varphi_n)|,|y(x,\varphi)|)\le M$ for all $n\in\mathbb{N}$. Using the splitting

$$\begin{aligned}
&\left|\int_\Omega L(x,\boldsymbol{u}_n,\nabla\boldsymbol{u}_n,p_n)\,y(x,\varphi_n) - L(x,\boldsymbol{u},\nabla\boldsymbol{u},p)\,y(x,\varphi)\,\mathrm{dx}\right|\\
&\quad\le\left|\int_\Omega (L(x,\boldsymbol{u}_n,\nabla\boldsymbol{u}_n,p_n) - L(x,\boldsymbol{u},\nabla\boldsymbol{u},p))\;y(x,\varphi_n)\,\mathrm{dx}\right|\\
&\quad\quad+\left|\int_\Omega L(x,\boldsymbol{u},\nabla\boldsymbol{u},p)\;(y(x,\varphi_n)-y(x,\varphi))\,\mathrm{dx}\right| =: I_1 + I_2,
\end{aligned}$$

we can show that $\lim_{n\to\infty}\mathcal{G}_i(\varphi_n) = \mathcal{G}_i(\varphi)$ once we demonstrate that $I_1, I_2\to 0$ as $n\to\infty$. This would then imply that $\varphi\in\mathbb{K}_{ad}$. Using the growth conditions in Assumption 3.3 for L, the strong convergences for $\{\boldsymbol{u}_n,p_n\}_{n\in\mathbb{N}}$ and the generalized Lebesgue dominated convergence theorem yields that

$$L(x,\boldsymbol{u}_n,\nabla\boldsymbol{u}_n,p_n)\to L(x,\boldsymbol{u},\nabla\boldsymbol{u},p)\text{ strongly in }L^1(\Omega)\text{ as }n\to\infty.$$

Then, the assertion that $I_1\to 0$ as $n\to\infty$ follows from the above strong convergence in $L^1(\Omega)$ and the boundedness of $y(x,\varphi_n)$ in $L^\infty(\Omega)$. Meanwhile, dominating the sequence $\{L(x,\boldsymbol{u},\nabla\boldsymbol{u},p)\,y(x,\varphi_n)\}_{n\in\mathbb{N}}$ by the function $\|z_5\|_{L^\infty(\Omega)}M\,|L(x,\boldsymbol{u},\nabla\boldsymbol{u},p)|\in L^1(\Omega)$, and the application of the usual Lebesgue dominating convergence theorem yields

$$\lim_{n\to\infty}\int_\Omega L(x,\boldsymbol{u},\nabla\boldsymbol{u},p)\,y(x,\varphi_n)\,\mathrm{dx} = \int_\Omega L(x,\boldsymbol{u},\nabla\boldsymbol{u},p)\,y(x,\varphi)\,\mathrm{dx}\,,$$

and hence $I_2\to 0$ as $n\to\infty$. ☐

We state the existence result for a minimizer of the problem (2.4).

Theorem 3.2. *Suppose $\mathbb{K}_{ad}$ is non-empty, then under Assumptions 2.1, 3.1, 3.3 and 3.2 there exists at least one minimizer to the problem* (2.4).

Proof. By Assumption 3.2, $(\mathcal{B}+\mathcal{H})|_{\mathbb{K}_{ad}\times\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)}$ is bounded from below by a constant $C_0\in\mathbb{R}$. Then, by the non-negativity of α_ε and Ψ, we find that there exists a constant $C_1\in\mathbb{R}$ such that $\mathcal{J}_\varepsilon:\mathbb{K}_{ad}\times\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)\to\mathbb{R}$ is bounded from below by C_1. Thus, we can choose a minimizing sequence $(\varphi_n,\boldsymbol{u}_n,p_n)_{n\in\mathbb{N}}\subset\mathbb{K}_{ad}\times\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)$ such that $(\boldsymbol{u}_n,p_n)\in\boldsymbol{S}_\varepsilon(\varphi_n)$ for all $n\in\mathbb{N}$ and

$$\lim_{n\to\infty}\mathcal{J}_\varepsilon(\varphi_n,\boldsymbol{u}_n,p_n) = \inf_{\varphi\in\mathbb{K}_{ad},(\boldsymbol{u},p)\in\boldsymbol{S}_\varepsilon(\varphi)}\mathcal{J}_\varepsilon(\varphi,\boldsymbol{u},p)\ge C_1 > -\infty.$$

Then, for arbitrary $\eta>0$, there exists $N\in\mathbb{N}$ such that for $n>N$,

$$C_0 + \frac{\gamma\varepsilon}{4c_0}\|\nabla\varphi_n\|^2_{\boldsymbol{L}^2(\Omega)}\le\mathcal{J}_\varepsilon(\varphi_n,\boldsymbol{u}_n,p_n)\le\inf_{\varphi\in\mathbb{K}_{ad},(\boldsymbol{u},p)\in\boldsymbol{S}_\varepsilon(\varphi)}\mathcal{J}_\varepsilon(\varphi,\boldsymbol{u},p)+\eta.$$

The above estimate implies that $\{\varphi_n\}_{n\in\mathbb{N}}\subset\mathbb{K}_{ad}$ is bounded uniformly in $H^1(\Omega)\cap L^\infty(\Omega)$. Thus, we may choose a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ such that $\varphi_{n_k}\to\varphi$ strongly in $L^p(\Omega)$ and almost everywhere in Ω for $2\le p<\infty$ in two-dimensions and $2\le p<6$ in three-dimensions.

Furthermore, by Lemma 3.1 we also have that $\varphi \in \mathbb{K}_{ad}$. By Lemma 2.3, there is a subsequence $(\boldsymbol{u}_{n_k}, p_{n_k})_{n\in\mathbb{N}} \subset \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ such that

$$\lim_{k\to\infty} \|\boldsymbol{u}_{n_k} - \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = 0, \quad \lim_{k\to\infty} \|p_{n_k} - p\|_{L^2(\Omega)} = 0,$$

for some $(\boldsymbol{u}, p) \in \boldsymbol{S}_\varepsilon(\varphi)$, and

$$\lim_{k\to\infty} \int_\Omega \alpha_\varepsilon(\varphi_{n_k}) |\boldsymbol{u}_{n_k}|^2 \,\mathrm{dx} = \int_\Omega \alpha_\varepsilon(\varphi) |\boldsymbol{u}|^2 \,\mathrm{dx}.$$

The continuity of Ψ and the fact that $(\varphi_{n_k})_{k\in\mathbb{N}} \subset L^\infty(\Omega)$ implies that $(\Psi(\varphi_{n_k}))_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\Omega)$. The application of the dominated convergence theorem yields that $\Psi(\varphi_{n_k})$ converges strongly to $\Psi(\varphi)$ in $L^1(\Omega)$ as $k \to \infty$. Furthermore, by the weak lower semicontinuity assumptions of $\mathcal{B}$ and $\mathcal{H}$, and the weak lower semicontinuity of the mapping $\varphi \mapsto \|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}$, we find that

$$\mathcal{J}_\varepsilon(\varphi, \boldsymbol{u}, p) \le \liminf_{k\to\infty} \mathcal{J}_\varepsilon(\varphi_{n_k}, \boldsymbol{u}_{n_k}, p_{n_k}) = \inf_{\phi\in\mathbb{K}_{ad}, (\boldsymbol{v},q)\in\boldsymbol{S}_\varepsilon(\phi)} \mathcal{J}_\varepsilon(\phi, \boldsymbol{v}, q),$$

and so $(\varphi, \boldsymbol{u}, p) \in \mathbb{K}_{ad} \times \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ is a minimizer of (2.4). □

4 Optimality conditions

For this section, we assume that the state equations are uniquely solvable. That is, we may appeal to condition (2.9) to obtain a unique pair $(\boldsymbol{u}, p) \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega) \times L^2_0(\Omega)$ of solutions to the state equation (2.5) for every $\varphi \in L^1(\Omega)$. To derive first order optimality conditions, we will need to establish the Fréchet differentiability of the objective functional and the integral constraints, and in addition show the existence of Lagrange multipliers for each of these $m_1 + m_2$ integral constraints. We make the following assumptions on the differentiability of B, z, h, L, y, and $\boldsymbol{L}_1$.

Assumption 4.1. *In addition to Assumption 3.2, assume further that* $x \mapsto k(x)$, $x \mapsto B(x, \boldsymbol{w}, \boldsymbol{A}, s)$, $x \mapsto z(x,t)$ *and* $x \mapsto h(x, \boldsymbol{A}, s, \boldsymbol{w})$ *belong to* $W^{1,1}(\Omega)$ *for all* $\boldsymbol{w} \in \mathbb{R}^d$, $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, $s, t \in \mathbb{R}$, *and the partial derivatives*

$$\begin{aligned}&\mathrm{D}_2 B(x, \cdot, \boldsymbol{A}, s),\ \mathrm{D}_3 B(x, \boldsymbol{w}, \cdot, s),\ \mathrm{D}_4 B(x, \boldsymbol{w}, \boldsymbol{A}, \cdot),\\ &\mathrm{D}_2 z(x, \cdot), \quad \mathrm{D}_2 h(x, \cdot, s, \boldsymbol{w}),\ \mathrm{D}_3 h(x, \boldsymbol{A}, \cdot, \boldsymbol{w}),\ \mathrm{D}_4 h(x, \boldsymbol{A}, s, \cdot)\end{aligned}$$

exist for all $\boldsymbol{w} \in \mathbb{R}^d$, $s \in \mathbb{R}$, $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, *and almost all* $x \in \Omega$ *as Carathéodory functions with*

$$|\mathrm{D}_2 B(x, \boldsymbol{w}, \boldsymbol{A}, s)| \le \tilde{c}(x) + \tilde{b}_1(x) |\boldsymbol{w}|^{p-1} + \tilde{b}_2(x) |\boldsymbol{A}| + \tilde{b}_3(x) |s|, \tag{4.1a}$$

$$\left|\mathrm{D}_{(3,4)} B(x, \boldsymbol{w}, \boldsymbol{A}, s)\right| \le \tilde{a}(x) + \tilde{b}_1(x) |\boldsymbol{w}|^{p/2} + \tilde{b}_2(x) |\boldsymbol{A}| + \tilde{b}_3(x) |s|, \tag{4.1b}$$

$$\left|\mathrm{D}_{(2,3,4,5)} h(x, \boldsymbol{A}, s, \boldsymbol{w})\right| \le \tilde{a}(x) + \tilde{b}_1(x) |\boldsymbol{A}| + \tilde{b}_2(x) |s| + \tilde{b}_3(x) |\boldsymbol{w}|, \tag{4.1c}$$

$$|\mathrm{D}_2 z(x,t)| \le \tilde{b}_1(x), \tag{4.1d}$$

for some non-negative functions $\tilde{a} \in L^2(\Omega)$, $\tilde{c} \in L^{\frac{p}{p-1}}(\Omega)$, $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3 \in L^\infty(\Omega)$, *where* $p \ge 2$ *in two-dimensions and* $p \in [2, 6]$ *in three-dimension.*

Here, we use the notation $\mathrm{D}_{(3,4)} f$ to mean that the assumption holds for the partial derivatives $\mathrm{D}_3 f$ and $\mathrm{D}_4 f$ individually. From Assumption 4.1 we see that

$$\begin{aligned}(L^2(\Omega))^{d\times d} \ni \boldsymbol{A} &\mapsto \mathrm{D}_2 h(\cdot, \boldsymbol{A}, s, \boldsymbol{w}) \in L^2(\Omega) \quad \forall s \in L^2(\Omega), \boldsymbol{w} \in (L^2(\Omega))^d,\\ L^2(\Omega) \ni s &\mapsto \mathrm{D}_3 h(\cdot, \boldsymbol{A}, s, \boldsymbol{w}) \in L^2(\Omega) \quad \forall \boldsymbol{A} \in (L^2(\Omega))^{d\times d}, \boldsymbol{w} \in (L^2(\Omega))^d,\\ (L^2(\Omega))^d \ni \boldsymbol{w} &\mapsto \mathrm{D}_4 h(\cdot, \boldsymbol{A}, s, \boldsymbol{w}) \in L^2(\Omega) \quad \forall \boldsymbol{A} \in (L^2(\Omega))^{d\times d}, s \in L^2(\Omega),\end{aligned}$$

are well-defined Nemytskii operators if and only if (4.1c) is fulfilled (see [35, §4.3.3] or [18, Theorems 1 and 3]). Moreover, the operator

$$(L^2(\Omega))^{d\times d} \times L^2(\Omega) \times (L^2(\Omega))^d \ni (\boldsymbol{A}, s, \boldsymbol{w}) \mapsto h(\cdot, \boldsymbol{A}, s, \boldsymbol{w}) \in L^1(\Omega)$$

is continuously Fréchet differentiable (see [18, Theorem 7] or [35, §4.3.3] with $p = r = 2$ and $q = 1$). Hence, we find that

$$\mathcal{H} : \big(H^1(\Omega) \cap L^\infty(\Omega)\big) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \ni (\varphi, \boldsymbol{u}, p) \mapsto \int_\Omega \frac{1}{2} h(x, \nabla \boldsymbol{u}, p, \nabla\varphi)\,\mathrm{dx}$$

is continuously Fréchet differentiable and its distributional derivative is given as

$$\mathrm{D}\mathcal{H}(\varphi, \boldsymbol{u}, p)(\eta, \boldsymbol{v}, s) = \int_\Omega \frac{1}{2} (\mathrm{D}_2 h, \mathrm{D}_3 h, \mathrm{D}_4 h)\,|_{(x,\nabla\boldsymbol{u},p,\nabla\varphi)} \cdot (\nabla\boldsymbol{v}, s, \nabla\eta)\,\mathrm{dx}. \tag{4.2}$$

Here we use the notation

$$(\mathrm{D}_2 h, \mathrm{D}_3 h, \mathrm{D}_4 h)\,|_{(x,\nabla\boldsymbol{u},p,\nabla\varphi)} \cdot (\nabla\boldsymbol{v}, s, \nabla\eta) := (\mathrm{D}_2 h) : \nabla\boldsymbol{v} + (\mathrm{D}_3 h)s + (\mathrm{D}_4 h)\cdot\nabla\varphi,$$

where the partial derivatives are evaluated at $(x, \nabla\boldsymbol{u}, p, \nabla\varphi)$. With a similar argument, the growth conditions on the partial derivatives of B and z yields that

$$\begin{aligned}
(L^p(\Omega))^d \ni \boldsymbol{w} &\mapsto \mathrm{D}_2 B(\cdot, \boldsymbol{w}, \boldsymbol{A}, s) \in L^{\frac{p}{p-1}}(\Omega),\\
(L^2(\Omega))^{d\times d} \ni \boldsymbol{A} &\mapsto \mathrm{D}_3 B(\cdot, \boldsymbol{w}, \boldsymbol{A}, s) \in L^2(\Omega),\\
L^2(\Omega) \ni s &\mapsto \mathrm{D}_4 B(\cdot, \boldsymbol{w}, \boldsymbol{A}, s) \in L^2(\Omega),\\
L^\infty(\Omega) \ni t &\mapsto \mathrm{D}_2 z(\cdot, t) \in L^\infty(\Omega),
\end{aligned}$$

are well-defined Nemytskii operators for all $\boldsymbol{w} \in (L^p(\Omega))^d$, $\boldsymbol{A} \in (L^2(\Omega))^{d\times d}$, $s \in L^2(\Omega)$ and $t \in L^\infty(\Omega)$. Thus, the operator

$$(L^p(\Omega))^d \times (L^2(\Omega))^{d\times d} \times L^2(\Omega) \times L^\infty(\Omega) \ni (\boldsymbol{w}, \boldsymbol{A}, s, t) \mapsto b(\cdot, \boldsymbol{w}, \boldsymbol{A}, s, t) \in L^1(\Omega)$$

is continuously Fréchet differentiable, and hence

$$\mathcal{B} : (H^1(\Omega) \cap L^\infty(\Omega)) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \ni (\varphi, \boldsymbol{u}, p) \mapsto \int_\Omega b(x, \boldsymbol{u}, \nabla\boldsymbol{u}, p, \varphi)\,\mathrm{dx}$$

is continuously Fréchet differentiable with distributional derivative

$$\begin{aligned}
\mathrm{D}\mathcal{B}(\varphi, \boldsymbol{u}, p)(\eta, \boldsymbol{v}, s) &= \int_\Omega z(x, \varphi)\ (\mathrm{D}_2 B, \mathrm{D}_3 B, \mathrm{D}_4 B)|_{(x,\boldsymbol{u},\nabla\boldsymbol{u},p)} \cdot (\boldsymbol{v}, \nabla\boldsymbol{v}, s)\,\mathrm{dx}\\
&\quad + \int_\Omega B(x, \boldsymbol{u}, \nabla\boldsymbol{u}, p)\ \mathrm{D}_2 z(x, \varphi)\eta\,\mathrm{dx}.
\end{aligned} \tag{4.3}$$

With a similar set of assumptions for the differentiability of L_0, and $\boldsymbol{L}_1$, we deduce the continuous Fréchet differentiability of the function $\mathcal{G}$.

Assumption 4.2. *In addition to Assumption 3.3, assume further that the mappings $x \mapsto L(x, \boldsymbol{w}, \boldsymbol{A}, s)$, $x \mapsto y(x, t)$, and $x \mapsto \boldsymbol{L}_1(x, \boldsymbol{A}, s)$ belong to $W^{1,1}(\Omega)$ for all $\boldsymbol{w} \in \mathbb{R}^d$, $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, $s, t \in \mathbb{R}$ and the partial derivatives*

$$\begin{aligned}
&\mathrm{D}_2 L(x, \cdot, \boldsymbol{A}, s),\ \mathrm{D}_3 L(x, \boldsymbol{w}, \cdot, s),\ \mathrm{D}_4 L(x, \boldsymbol{w}, \boldsymbol{A}, \cdot),\\
&\mathrm{D}_2 y(x, \cdot),\quad \mathrm{D}_2 \boldsymbol{L}_1(x, \cdot, s),\ \mathrm{D}_3 \boldsymbol{L}_1(x, \boldsymbol{A}, \cdot)
\end{aligned}$$

exist for all $\boldsymbol{w} \in \mathbb{R}^d$*,* $s \in \mathbb{R}$*,* $\boldsymbol{A} \in \mathbb{R}^{d\times d}$*, and almost all* $x \in \Omega$ *as Carathéodory functions. Moreover, we assume that*

$$
\begin{aligned}
|\mathrm{D}_2 L(x,\boldsymbol{w},\boldsymbol{A},s)| &\le \tilde{c}(x) + \tilde{b}_1(x)|\boldsymbol{w}|^{p-1} + \tilde{b}_2(x)|\boldsymbol{A}| + \tilde{b}_3(x)|s|,\\
\left|\mathrm{D}_{(3,4)} L(x,\boldsymbol{w},\boldsymbol{A},s)\right| &\le \tilde{a}(x) + \tilde{b}_1(x)|\boldsymbol{w}|^{p/2} + \tilde{b}_2(x)|\boldsymbol{A}| + \tilde{b}_3(x)|s|,\\
\left|\mathrm{D}_{(2,3)} \boldsymbol{L}_1(x,\boldsymbol{A},s)\right| &\le \tilde{b}_1(x),\\
|\mathrm{D}_2 y(x,t)| &\le \tilde{b}_1(x),
\end{aligned}
$$

for some non-negative functions $\tilde{a} \in L^2(\Omega)$*,* $\tilde{c} \in L^{\frac{p}{p-1}}(\Omega)$ *and* $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3 \in L^\infty(\Omega)$*, where* $p \ge 2$ *in two-dimensions and* $p \in [2,6]$ *in three-dimensions.*

Applying similar arguments as above, we obtain that the operator

$$
(L^p(\Omega))^d \times (L^2(\Omega))^{d\times d} \times L^2(\Omega) \times L^\infty(\Omega) \ni (\boldsymbol{w},\boldsymbol{A},s,t) \mapsto L_0(\cdot,\boldsymbol{w},\boldsymbol{A},s,t) \in L^1(\Omega),
$$

is continuously Fréchet differentiable. Let $P : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ denote the operator

$$
P(x,\boldsymbol{A},s,\boldsymbol{w}) := \boldsymbol{L}_1(x,\boldsymbol{A},s)\cdot\boldsymbol{w}.
$$

Then, a short computation shows that

$$
\mathrm{D}_{(2,3)} P(x,\boldsymbol{A},s,\boldsymbol{w}) = \mathrm{D}_{(2,3)}\boldsymbol{L}_1(x,\boldsymbol{A},s)\cdot\boldsymbol{w}, \quad \mathrm{D}_4 P(x,\boldsymbol{A},s,\boldsymbol{w}) = \boldsymbol{L}_1(x,\boldsymbol{A},s),
$$

where the product operator in $\mathrm{D}_{(2,3)}\boldsymbol{L}_1(x,\boldsymbol{A},s)\cdot\boldsymbol{w}$ can be a tensor product or a scalar product depending the form of $\boldsymbol{L}_1$. By Assumptions 3.3 and 4.2, we see that

$$
\begin{aligned}
(L^2(\Omega))^{d\times d} \ni \boldsymbol{A} &\mapsto \mathrm{D}_2 P(\cdot\boldsymbol{A},s,\boldsymbol{w}) \in L^2(\Omega),\\
L^2(\Omega) \ni s &\mapsto \mathrm{D}_3 P(\cdot\boldsymbol{A},s,\boldsymbol{w}) \in L^2(\Omega),\\
(L^2(\Omega))^d \ni \boldsymbol{w} &\mapsto \mathrm{D}_4 P(\cdot\boldsymbol{A},s,\boldsymbol{w}) \in (L^2(\Omega))^d,
\end{aligned}
$$

are well-defined Nemytskii operators for all $\boldsymbol{w} \in (L^2(\Omega))^d$, $\boldsymbol{A} \in (L^2(\Omega))^{d\times d}$ and $s \in L^2(\Omega)$. This yields that the operator

$$
(L^2(\Omega))^{d\times d} \times L^2(\Omega) \times (L^2(\Omega))^d \ni (\boldsymbol{A},s,\boldsymbol{w}) \mapsto \boldsymbol{L}_1(\cdot,\boldsymbol{A},s)\cdot\boldsymbol{w} \in L^1(\Omega)
$$

is continuously Fréchet differentiable. Hence, for $\varphi \in H^1(\Omega)\cap L^\infty(\Omega)$ and $(\boldsymbol{u},p) = \boldsymbol{S}_\varepsilon(\varphi)$, we obtain that

$$
\mathcal{G} : (H^1(\Omega)\cap L^\infty(\Omega)) \ni \varphi \mapsto \int_\Omega L_0(x,\boldsymbol{u},\nabla\boldsymbol{u},p,\varphi) + \frac{1}{2}\boldsymbol{L}_1(x,\nabla\boldsymbol{u},p)\cdot\nabla\varphi\,\mathrm{dx}
$$

is continuously Fréchet differentiable with distributional derivatives

$$
\begin{aligned}
\mathrm{D}\mathcal{G}(\varphi)(\phi) = &\int_\Omega y(x,\varphi)\ (\mathrm{D}_2 L, \mathrm{D}_3 L, \mathrm{D}_4 L)|_{(x,\boldsymbol{u},\nabla\boldsymbol{u},p)}\cdot(\boldsymbol{v},\nabla\boldsymbol{v},w)\,\mathrm{dx}\\
&+ \int_\Omega L(x,\boldsymbol{u},\nabla\boldsymbol{u},p)\ \mathrm{D}_2 z(x,\varphi)\phi\,\mathrm{dx}\\
&+ \frac{1}{2}\int_\Omega \nabla\phi\cdot\boldsymbol{L}_1(x,\nabla\boldsymbol{u},p) + \nabla\varphi\cdot\left((\mathrm{D}_2\boldsymbol{L}_1, \mathrm{D}_3\boldsymbol{L}_1)|_{(x,\nabla\boldsymbol{u},p)}\cdot(\nabla\boldsymbol{v},w)\right)\mathrm{dx},
\end{aligned}
\tag{4.4}
$$

where the pair of functions $(\boldsymbol{v},w) = \mathrm{D}\boldsymbol{S}_\varepsilon(\varphi)(\phi)$ is the solution to the linearized state system given in Lemma 2.4 for ϕ in the neighbourhood $N \subset H^1(\Omega)\cap L^\infty(\Omega)$ of φ.

Remark 4.1. *We point out that the boundedness assumption of* $\mathrm{D}_{(2,3)}\boldsymbol{L}_1$ *is a consequence of the fact that every differentiable Nemytskii operator from* $L^q(\Omega)$ *to* $L^q(\Omega)$ *is affine, see [18, §3.1] or [2, §3.9]. However, for the choice of* $\boldsymbol{L}_1$ *we are interested in (see Section 1), i.e., for the hydrodynamic force component in direction* $\boldsymbol{a}$*, we have*

$$\boldsymbol{L}_1(x, \boldsymbol{A}, s) = (\mu\,(\boldsymbol{A} + \boldsymbol{A}^\top) - s\,\mathbf{I}\,)\boldsymbol{a},$$

and a short computation shows that

$$\mathrm{D}_2\boldsymbol{L}_1(x, \boldsymbol{A}, s)\cdot \boldsymbol{w} = \mu(\boldsymbol{w}\otimes\boldsymbol{a} + \boldsymbol{a}\otimes\boldsymbol{w}), \quad \mathrm{D}_3\boldsymbol{L}_1(x, \boldsymbol{A}, s)\cdot\boldsymbol{w} = -\boldsymbol{a}\cdot\boldsymbol{w}$$

and thus the assumption is satisfied with $\tilde{b}_1(x) = \max(\mu, 1)$*. Furthermore, a simple computation shows that the volume formulations* (1.15) *and* (1.17) *of the drag, and the total potential power* (1.6) *satisfy the differentiability conditions in Assumption 4.2.*

4.1 Fréchet differentiability of the objective functional

Due to the well-posedness of the state equations, we may now write the problem (2.4) as a minimizing problem for a reduced objective functional defined on an open set in $H^1(\Omega)\cap L^\infty(\Omega)$ with the help of Lemma 2.4. Let $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon) \in \mathbb{K}_{ad}\times \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)$ denote a minimizer of (2.4), whose existence is guaranteed by Theorem 3.2. By Lemma 2.4, there exists a neighborhood $N\subset H^1(\Omega)\cap L^\infty(\Omega)$ of φ_ε such that for every $\psi\in N$, the state equations (2.6) are uniquely solvable. We define the reduced functional $j_\varepsilon : N\to\mathbb{R}$ by

$$j_\varepsilon(\psi) \coloneqq \mathcal{J}_\varepsilon(\psi, \boldsymbol{S}_\varepsilon(\psi)) \qquad \text{for all } \psi\in N.$$

We now show that, as a mapping from $H^1(\Omega)\cap L^\infty(\Omega)\to\mathbb{R}$, j_ε is Fréchet differentiable at φ_ε. As Lemma 2.4 guarantees the Fréchet differentiability of the solution operator $\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ as a mapping from $H^1(\Omega)\cap L^\infty(\Omega)$ to $\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$, we focus on the dependence of $\mathcal{J}_\varepsilon$ on the first variable.

By Assumption 2.1, α_ε and α'_ε are uniformly bounded and so

$$L^6(\Omega)\ni h\mapsto \alpha'_\varepsilon(\varphi)h\in L^6(\Omega)$$

is a well-defined mapping from $H^1(\Omega)\subset L^6(\Omega)$ to $L^6(\Omega)$ for all $\varphi\in H^1(\Omega)$. Then, by [35, §4.3.3], we see that α_ε defines a Fréchet differentiable Nemytskii operator as a mapping from $L^6(\Omega)$ to $L^3(\Omega)$. Meanwhile, the local Lipschitz continuity of the potential Ψ and [35, Lemma 4.13] imply that $\Psi(\varphi)$ is continuously Fréchet differentiable Nemytskii operator as a mapping from $L^\infty(\Omega)$ to $L^\infty(\Omega)$. Combined with the Fréchet differentiability of the mapping $H^1(\Omega)\ni\varphi\mapsto\int_\Omega|\nabla\varphi|^2\,\mathrm{dx}$, $\mathcal{B}$ and $\mathcal{H}$, we obtain that $j_\varepsilon : N\to\mathbb{R}$ is Fréchet differentiable.

4.2 Existence of Lagrange multipliers

To show the existence of Lagrange multipliers for the constraints, we make use of the Zowe–Kurcyusz constraint qualification (ZKCQ), see [36] and [35, §6.1.2] for more details. For this purpose, we introduce the notation

$$\begin{aligned} K &\coloneqq \{\boldsymbol{y}\in\mathbb{R}^{m_1+m_2}\,|\, y_i = 0, y_j\geq 0 \text{ for } 1\leq i\leq m_1, m_1+1\leq j\leq m_1+m_2\}\subset Y\coloneqq\mathbb{R}^{m_1+m_2},\\ g(\varphi) &\coloneqq (\mathcal{G}_1(\varphi),\dots,\mathcal{G}_{m_1}(\varphi),\mathcal{G}_{m_1+1}(\varphi),\dots,\mathcal{G}_{m_1+m_2}(\varphi)),\\ C &\coloneqq \left\{f\in H^1(\Omega)\,|\, s_a\leq f(x)\leq s_b \text{ for a.e. } x\in\Omega\right\}. \end{aligned}$$

Then, C is a closed convex subset of $H^1(\Omega)$ and K is a closed convex cone in Y with vertex at the origin, i.e., $\delta_1 K + \delta_2 K \subset K$ for $\delta_1, \delta_2 > 0$. In the notation of [36], we introduce the sets

$$C(\varphi_\varepsilon) = \{\lambda(\varphi - \varphi_\varepsilon) \,|\, \varphi \in C, \lambda \geq 0\},$$
$$K(g(\varphi_\varepsilon)) = \{\boldsymbol{k} - \lambda g(\varphi_\varepsilon) \,|\, \boldsymbol{k} \in K, \lambda \geq 0\}.$$

Furthermore, the continuous Fréchet differentiability of $\mathcal{G}$ allow us to deduce that

$$g'(\varphi_\varepsilon)\varphi = (\mathrm{D}\mathcal{G}_1(\varphi_\varepsilon)(\varphi), \dots, \mathrm{D}\mathcal{G}_{m_1}(\varphi_\varepsilon)(\varphi), \mathrm{D}\mathcal{G}_{m_1+1}(\varphi_\varepsilon)(\varphi), \dots, \mathrm{D}\mathcal{G}_{m_1+m_2}(\varphi_\varepsilon)(\varphi))$$

where the derivative $\mathrm{D}\mathcal{G}$ is computed in (4.4). We require that φ_ε is a regular point in the sense of [36], which means that the so-called Zowe–Kurcyusz constraint qualification

$$Y = g'(\varphi_\varepsilon)C(\varphi_\varepsilon) - K(g(\varphi_\varepsilon)) \tag{4.5}$$

has to hold. In our setting this translates to the following.

Assumption 4.3. *In addition to Assumption 4.2, assume further that for any $\boldsymbol{z} \in Y = \mathbb{R}^{m_1+m_2}$, there exists $\varphi \in C$, $\boldsymbol{\tau} \in \mathbb{R}^{m_1+m_2}$, $\boldsymbol{\xi}, \boldsymbol{k} \in \mathbb{R}^{m_2}$ such that $\tau_i, \xi_j, k_j \geq 0$ for $1 \leq i \leq m_1 + m_2$, $1 \leq j \leq m_2$ and*

$$z_i = \tau_i \mathrm{D}\mathcal{G}_i(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon), \qquad \text{for } 1 \leq i \leq m_1$$
$$z_{m_1+j} = \tau_{m_1+j}\mathrm{D}\mathcal{G}_{m_1+j}(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) - k_j + \xi_j \mathcal{G}_{m_1+j}(\varphi_\varepsilon), \text{ for } 1 \leq j \leq m_2,$$

where φ_ε is the minimizer of (2.4).

Then, by [36, Theorem 3.1 and Theorem 4.1], the set of Lagrange multipliers associated to φ_ε is non-empty and bounded. In particular, we obtain the existence of Lagrange multipliers $\boldsymbol{\lambda} := (\lambda_1, \dots, \lambda_{m_1+m_2}) \in K^+ := \{\boldsymbol{y} \in \mathbb{R}^{m_1+m_2} \,|\, \boldsymbol{y} \cdot \boldsymbol{k} \geq 0 \ \forall \boldsymbol{k} \in K\}$ satisfying

$$\boldsymbol{\lambda} \cdot g(\varphi_\varepsilon) = 0, \text{ and } \langle \mathrm{D}j_\varepsilon(\varphi_\varepsilon) - \boldsymbol{\lambda} \cdot g'(\varphi_\varepsilon), \varphi - \varphi_\varepsilon \rangle \geq 0 \quad \forall \varphi \in C,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^1(\Omega)$ and its dual. This means that there exist $\lambda_1, \dots, \lambda_{m_1} \in \mathbb{R}$, $\lambda_{m_1+1}, \dots, \lambda_{m_1+m_2} \in \mathbb{R}_{\geq 0}$ such that

$$\mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) + \sum_{i=1}^{m_1} \lambda_i \mathrm{D}\mathcal{G}_i(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) + \sum_{j=m_1+1}^{m_1+m_2} \lambda_j \mathrm{D}\mathcal{G}_j(\varphi_\varepsilon)(\varphi - \varphi_\varepsilon) \geq 0 \tag{4.6}$$

for all $\varphi \in C$, and the following complementary slackness conditions for the inequality constraints

$$\lambda_j \mathcal{G}_j(\varphi_\varepsilon) = 0 \text{ for } m_1 + 1 \leq j \leq m_1 + m_2. \tag{4.7}$$

Remark 4.2. *We mention that* (4.5) *is equivalent (see [36, §3] and [21, Theorem 1.56]) to the following interior point/linearized Slater condition (which is also commonly known as the Robinson regularity condition [29]):*

$$0 \in \operatorname{int}\big(g(\varphi_\varepsilon) + g'(\varphi_\varepsilon)(C - \varphi_\varepsilon) - K\big). \tag{4.8}$$

Remark 4.3. *In Section 5 below we will verify Assumption 4.3 and the non-emptiness of $\mathbb{K}_{ad}$ when all integral constraints are for the design variable φ, for example volume constraints* (1.8)*, mass constraint* (1.9) *and prescribed center of mass* (1.10) *on the object. However, in cases where at least one of constraints involve the state variables, we will simply assume that $\mathbb{K}_{ad}$ is non-empty and that Assumption 4.3 is satisfied, which in turn guarantees the existence of bounded Lagrange multipliers.*

4.3 Adjoint system

We now introduce the Lagrangian $\mathbb{L} : (H^1(\Omega)\cap L^\infty(\Omega))\times \boldsymbol{H}^1(\Omega)\times L^2(\Omega)\times \boldsymbol{H}^1_0(\Omega)\times L^2(\Omega) \to \mathbb{R}$ as

$$\begin{aligned}\mathbb{L}(\varphi,\boldsymbol{u},p,\boldsymbol{q},\pi) &:= \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2 + \frac{\gamma}{2c_0}\Big(\frac{1}{\varepsilon}\Psi(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2\Big)\,\mathrm{dx} \\ &+ \int_\Omega b(x,\boldsymbol{u},\nabla\boldsymbol{u},p,\varphi) + \frac{1}{2}h(x,\nabla\boldsymbol{u},p,\nabla\varphi)\,\mathrm{dx} \\ &- \int_\Omega \alpha_\varepsilon(\varphi)\boldsymbol{u}\cdot\boldsymbol{q} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{q} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{q} - p\,\mathrm{div}\,\boldsymbol{q} - \boldsymbol{f}\cdot\boldsymbol{q} - \pi\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{dx} \\ &+ \int_\Omega \sum_{i=1}^{m_1+m_2}\lambda_i\Big(L_{0,i}(x,\boldsymbol{u},\nabla\boldsymbol{u},p,\varphi) + \frac{1}{2}\nabla\varphi\cdot\boldsymbol{L}_{1,i}(x,\nabla\boldsymbol{u},p)\Big) + \theta p\,\mathrm{dx}\end{aligned}$$

where λ_i is the Lagrange multiplier for the integral constraint $\mathcal{G}_i$ and θ is a Lagrange multiplier for the constraint $\int_\Omega p\,\mathrm{dx} = 0$ for the pressure. A formal computation of $\mathrm{D}_{\boldsymbol{u}}\mathbb{L}$ and $\mathrm{D}_p\mathbb{L}$ yields the following adjoint system,

$$\begin{aligned}&\alpha_\varepsilon(\varphi)\boldsymbol{q}_\varepsilon - \mu\,\mathrm{div}\,(\nabla\boldsymbol{q}_\varepsilon + (\nabla\boldsymbol{q}_\varepsilon)^\top) + (\nabla\boldsymbol{u}_\varepsilon)^\top\boldsymbol{q}_\varepsilon - (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{q}_\varepsilon + \nabla\pi_\varepsilon \\ &\quad = \alpha_\varepsilon(\varphi)\boldsymbol{u}_\varepsilon + \mathrm{D}_2 b - \mathrm{div}\,(\mathrm{D}_3 b + \tfrac{1}{2}\mathrm{D}_2 h) \\ &\quad + \sum_{i=1}^{m_1+m_2}\Big(\lambda_i\mathrm{D}_2 L_{0,i} - \mathrm{div}\,\Big(\lambda_i\big(\mathrm{D}_3 L_{0,i} + \tfrac{1}{2}\nabla\varphi_\varepsilon\cdot\mathrm{D}_2\boldsymbol{L}_{1,i}\big)\Big)\Big) \quad \text{in } \Omega, && (4.9a)\\ &\mathrm{div}\,\boldsymbol{q}_\varepsilon = -\mathrm{D}_4 b - \tfrac{1}{2}\mathrm{D}_3 h - \theta - \sum_{i=1}^{m_1+m_2}\big(\lambda_i\mathrm{D}_4 L_{0,i} + \tfrac{1}{2}\lambda_i\nabla\varphi_\varepsilon\cdot\mathrm{D}_3\boldsymbol{L}_{1,i}\big) \quad \text{in } \Omega, && (4.9b)\\ &\boldsymbol{q}_\varepsilon = \boldsymbol{0} \quad \text{on } \partial\Omega, && (4.9c)\end{aligned}$$

where $\mathrm{D}_{(2,3,4)}b$ are evaluated at $(x,\boldsymbol{u}_\varepsilon,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\varphi_\varepsilon)$, $\mathrm{D}_{(2,3)}h$ are evaluated at $(x,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\nabla\varphi_\varepsilon)$, $\mathrm{D}_{(2,3,4)}L_0$ are evaluated at $(x,\boldsymbol{u}_\varepsilon,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\varphi_\varepsilon)$, and $\mathrm{D}_{(2,3)}\boldsymbol{L}_1$ are evaluated at $(x,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon)$, and upon integrating the divergence equation for $\boldsymbol{q}_\varepsilon$, we obtain

$$\theta = \frac{1}{|\Omega|}\int_\Omega -\mathrm{D}_4 b - \tfrac{1}{2}\mathrm{D}_3 h - \sum_{i=1}^{m_1+m_2}\big(\lambda_i\mathrm{D}_4 L_{0,i} + \tfrac{1}{2}\lambda_i\nabla\varphi_\varepsilon\cdot\mathrm{D}_3\boldsymbol{L}_{1,i}\big)\,\mathrm{dx}. \tag{4.10}$$

Let us also recall from (3.1) and (3.2) that

$$\begin{aligned}\mathrm{D}_{(2,3,4)}b(x,\boldsymbol{u},\nabla\boldsymbol{u},p,\varphi) &= z(x,\varphi)\,\mathrm{D}_{(2,3,4)}B(x,\boldsymbol{u},\nabla\boldsymbol{u},p),\\ \mathrm{D}_{(2,3,4)}L_{0,i}(x,\boldsymbol{u},\nabla\boldsymbol{u},p,\varphi) &= y(x,\varphi)\,\mathrm{D}_{(2,3,4)}L_i(x,\boldsymbol{u},\nabla\boldsymbol{u},p).\end{aligned}$$

We now show that the adjoint system is well-posed.

Lemma 4.1. *Let $\varphi_\varepsilon \in H^1(\Omega)\cap L^\infty(\Omega)$ and $\boldsymbol{u}_\varepsilon \in \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ be given such that* (2.9) *is satisfied. Then, under Assumptions 4.1, 4.2 and 4.3, there exists a unique weak solution pair $(\boldsymbol{q}_\varepsilon,\pi_\varepsilon)\in\boldsymbol{H}^1_0(\Omega)\times L^2(\Omega)$ to the adjoint system* (4.9).

Proof. By Assumptions 4.1 and 4.2, the boundedness of the Lagrange multipliers and (4.10), we see that

$$g := -\mathrm{D}_4 b - \tfrac{1}{2}\mathrm{D}_3 h - \theta - \sum_{i=1}^{m_1+m_2}\lambda_i\big(\mathrm{D}_4 L_{0,i} + \tfrac{1}{2}\nabla\varphi_\varepsilon\cdot\mathrm{D}_3\boldsymbol{L}_{1,i}\big) \in L^2_0(\Omega), \tag{4.11}$$

Applying [32, Lemma II.2.1.1], we find a $\boldsymbol{G}\in\boldsymbol{H}^1_0(\Omega)$ such that

$$\mathrm{div}\,\boldsymbol{G} = g \text{ in } \Omega, \text{ and } \|\nabla\boldsymbol{G}\|_{\boldsymbol{L}^2(\Omega)} \le C\|g\|_{L^2(\Omega)}$$

for some constant $C > 0$ depending only on Ω. We define the bilinear form $a : \boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{H}^1_{0,\sigma}(\Omega) \to (\boldsymbol{H}^1_{0,\sigma}(\Omega))'$ by

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{v} + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\nabla\boldsymbol{u}_\varepsilon)^\top\boldsymbol{u}\cdot\boldsymbol{v} - (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{dx}\,. \tag{4.12}$$

Using (2.9), Poincaré's inequality, Hölder's inequality, the boundedness of α_ε and properties of the trilinear form $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{dx}$ (see [15, Lemma 4.1]), it is shown in [15, Proof of Lemma 4.9] that $a(\cdot,\cdot)$ is a bounded and coercive bilinear form. Furthermore,

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{v}) := &\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v} + (\mathrm{D}_3 b + \tfrac{1}{2}\mathrm{D}_2 h)\cdot\nabla\boldsymbol{v} + \mathrm{D}_2 b\cdot\boldsymbol{v}\,\mathrm{dx}\\ &+ \int_\Omega \sum_{i=1}^{m_1+m_2}\lambda_i\left(\mathrm{D}_2 L_{0,i}\cdot\boldsymbol{v} + \left(\tfrac{1}{2}\nabla\varphi_\varepsilon\cdot\mathrm{D}_2\boldsymbol{L}_{1,i} + \mathrm{D}_3 L_{0,i}\right)\cdot\nabla\boldsymbol{v}\right)\mathrm{dx}\end{aligned} \tag{4.13}$$

is a bounded linear form on $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ due to Assumptions 4.1 and 4.2 and the boundedness of the Lagrange multipliers from [36, Theorem 3.1 and Theorem 4.1]. Thus, by the Lax–Milgram theorem, we obtain a unique $\hat{\boldsymbol{q}} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ such that

$$a(\hat{\boldsymbol{q}},\boldsymbol{v}) = \boldsymbol{F}(\boldsymbol{v}) - a(\boldsymbol{G},\boldsymbol{v}) \quad \forall\boldsymbol{v}\in\boldsymbol{H}^1_{0,\sigma}(\Omega).$$

Using the density of $\boldsymbol{C}^\infty_{0,\sigma}(\Omega) := \{\boldsymbol{v}\in(C^\infty_0(\Omega))^d \mid \operatorname{div}\boldsymbol{v} = 0\}$ in $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ (see [32, Lemma II.2.2.3]), one finds that for any $\boldsymbol{y}\in\boldsymbol{H}^1_0(\Omega)$, $\boldsymbol{v}\in\boldsymbol{H}^1_{0,\sigma}(\Omega)$ that

$$\int_\Omega \nabla\boldsymbol{y}\cdot(\nabla\boldsymbol{v})^\top\,\mathrm{dx} = 0.$$

This implies that the solution $\boldsymbol{q}_\varepsilon := \hat{\boldsymbol{q}} + \boldsymbol{G} \in \boldsymbol{H}^1_0(\Omega)$ satisfies

$$\begin{aligned}a(\boldsymbol{q}_\varepsilon,\boldsymbol{v}) &= \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{q}_\varepsilon\cdot\boldsymbol{v} + \mu(\nabla\boldsymbol{q}_\varepsilon + (\nabla\boldsymbol{q}_\varepsilon)^\top)\cdot\nabla\boldsymbol{v} + (\nabla\boldsymbol{u}_\varepsilon)^\top\boldsymbol{q}_\varepsilon\cdot\boldsymbol{v} - (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{q}_\varepsilon\cdot\boldsymbol{v}\,\mathrm{dx}\\ &= \boldsymbol{F}(\boldsymbol{v}) \quad \forall\boldsymbol{v}\in\boldsymbol{H}^1_{0,\sigma}(\Omega)\end{aligned}$$

and

$$\operatorname{div}\boldsymbol{q}_\varepsilon = \operatorname{div}\boldsymbol{G} = g.$$

The existence of the adjoint pressure $\pi_\varepsilon \in L^2(\Omega)$ follows from standard results, see for instance [32, Lemma II.2.2.1]. Thus $(\boldsymbol{q}_\varepsilon,\pi_\varepsilon)$ is the unique weak solution to the adjoint system (4.9). □

4.4 Necessary optimality conditions

Now we can formulate the first order necessary optimality conditions for our optimal control problem.

Theorem 4.2. *Let $(\varphi_\varepsilon,\boldsymbol{u}_\varepsilon,p_\varepsilon)\in\mathbb{K}_{ad}\times\boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)\times L^2_0(\Omega)$ be a minimizer of* (2.4) *satisfying* (2.9). *Then, under Assumptions 4.1, 4.2 and 4.3, the following optimality system is fulfilled,*

$$\begin{aligned}0 \le &\left\langle \alpha'_\varepsilon(\varphi_\varepsilon)\left(\frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\right) + \frac{\gamma}{2c_0\varepsilon}\Psi'(\varphi_\varepsilon) + \mathrm{D}_5 b + \sum_{i=1}^{m_1+m_2}\lambda_i\mathrm{D}_5 L_{0,i}, \varphi - \varphi_\varepsilon\right\rangle_{L^2(\Omega)}\\ &+ \left\langle \frac{\gamma\varepsilon}{2c_0}\nabla\varphi_\varepsilon + \frac{1}{2}\mathrm{D}_4 h + \frac{1}{2}\sum_{i=1}^{m_1+m_2}\lambda_i\boldsymbol{L}_{1,i}, \nabla(\varphi-\varphi_\varepsilon)\right\rangle_{\boldsymbol{L}^2(\Omega)}\end{aligned} \tag{4.14}$$

for all $\varphi \in C := \{f \in H^1(\Omega) \,|\, s_a \le f(x) \le s_b$ *for a.e.* x *in* $\Omega\}$*, where* $\mathrm{D}_4 h$ *is evaluated at* $(x, \nabla \boldsymbol{u}_\varepsilon, p_\varepsilon, \nabla\varphi_\varepsilon)$*,* $\boldsymbol{L}_1$ *is evaluated at* $(x, \nabla\boldsymbol{u}_\varepsilon, p_\varepsilon)$*, and* $\mathrm{D}_5 b = B(x, \boldsymbol{u}_\varepsilon, \nabla\boldsymbol{u}_\varepsilon, p_\varepsilon)\,\mathrm{D}_2 z(x,\varphi_\varepsilon)$*,* $\mathrm{D}_5 L_{0,i} = L_i(x, \boldsymbol{u}_\varepsilon, \nabla\boldsymbol{u}_\varepsilon, p_\varepsilon)\,\mathrm{D}_2 y_i(x,\varphi_\varepsilon)$*.*

Proof. In Section 4.1 we have that the reduced functional $j_\varepsilon(\varphi_\varepsilon) := \mathcal{J}_\varepsilon(\varphi_\varepsilon, \boldsymbol{S}_\varepsilon(\varphi_\varepsilon))$ is Fréchet differentiable with respect to φ_ε, and in Section 4.2 we derived the gradient equation (4.6). We now want to rewrite (4.6) into a more convenient form using the adjoint system. We find for every $\zeta \in H^1(\Omega) \cap L^\infty(\Omega)$,

$$\begin{aligned} \mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\zeta) = &\int_\Omega \frac{1}{2}\alpha_\varepsilon'(\varphi_\varepsilon)\zeta\,|\boldsymbol{u}_\varepsilon|^2 + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u} + \frac{\gamma}{2c_0}\left(\frac{1}{\varepsilon}\Psi'(\varphi_\varepsilon)\zeta + \varepsilon\nabla\varphi_\varepsilon\cdot\nabla\zeta\right)\mathrm{dx} \\ &+ \int_\Omega (\mathrm{D}_2 b, \mathrm{D}_3 b, \mathrm{D}_4 b, \mathrm{D}_5 b)|_{(x,\boldsymbol{u}_\varepsilon,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\varphi_\varepsilon)}\cdot(\boldsymbol{u},\nabla\boldsymbol{u},p,\zeta)\,\mathrm{dx} \\ &+ \int_\Omega \frac{1}{2}(\mathrm{D}_2 h, \mathrm{D}_3 h, \mathrm{D}_4 h)|_{(x,\nabla\boldsymbol{u}_\varepsilon,p_\varepsilon,\nabla\varphi_\varepsilon)}\cdot(\nabla\boldsymbol{u},p,\nabla\zeta)\,\mathrm{dx}\,, \end{aligned} \tag{4.15}$$

where $(\boldsymbol{u}_\varepsilon, p_\varepsilon) = \boldsymbol{S}_\varepsilon(\varphi_\varepsilon)$ and $(\boldsymbol{u}, p) := \mathrm{D}\boldsymbol{S}_\varepsilon(\varphi_\varepsilon)(\zeta)$ is the solution of the linearized state equations (2.12). Using the adjoint state $\boldsymbol{q}_\varepsilon$ as a test function in (2.12) leads to

$$\begin{aligned} 0 = &\int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\zeta\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon + \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}\cdot\boldsymbol{q}_\varepsilon + \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{q}_\varepsilon\,\mathrm{dx} \\ &+ \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon + (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{q}_\varepsilon - pg\,\mathrm{dx}\,, \end{aligned} \tag{4.16}$$

where g is defined in (4.11). Using the linearized state $\boldsymbol{u}$ as a test function in the adjoint system (4.9) leads to

$$\boldsymbol{F}(\boldsymbol{u}) = \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{q}_\varepsilon\cdot\boldsymbol{u} + \mu\nabla\boldsymbol{q}_\varepsilon\cdot\nabla\boldsymbol{u} + (\nabla\boldsymbol{u}_\varepsilon)^\top\boldsymbol{q}_\varepsilon\cdot\boldsymbol{u} - (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{q}_\varepsilon\cdot\boldsymbol{u}\,\mathrm{dx}\,, \tag{4.17}$$

where we used that $\operatorname{div}\boldsymbol{u} = 0$ and $\boldsymbol{F}$ is defined in (4.13). Upon comparing terms in (4.16) and (4.17) we find that

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{u}) + \int_\Omega (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{q}_\varepsilon\cdot\boldsymbol{u}\,\mathrm{dx} &= \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{q}_\varepsilon\cdot\boldsymbol{u} + \mu\nabla\boldsymbol{q}_\varepsilon\cdot\nabla\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\,\mathrm{dx} \\ &= \int_\Omega pg - \alpha_\varepsilon'(\varphi_\varepsilon)\zeta\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon - (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{q}_\varepsilon\,\mathrm{dx}\,. \end{aligned} \tag{4.18}$$

Using that $p \in L^2_0(\Omega)$, $\operatorname{div}\boldsymbol{u}_\varepsilon = 0$ in Ω, $\boldsymbol{q}_\varepsilon = \boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$, and thus

$$\int_\Omega p\theta\,\mathrm{dx} = \theta\int_\Omega p\,\mathrm{dx} = 0, \quad \int_\Omega (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{q}_\varepsilon\cdot\boldsymbol{u} + (\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{q}_\varepsilon\,\mathrm{dx} = \int_\Omega \boldsymbol{u}_\varepsilon\cdot\nabla(\boldsymbol{q}_\varepsilon\cdot\boldsymbol{u})\,\mathrm{dx} = 0,$$

we can simplify (4.18) into

$$\begin{aligned} &\int_\Omega p\left(-\mathrm{D}_4 b - \tfrac{1}{2}\mathrm{D}_3 h - \sum_{i=1}^{m_1+m_2}\lambda_i\left(\mathrm{D}_4 L_{0,i} + \tfrac{1}{2}\nabla\varphi_\varepsilon\cdot\mathrm{D}_3\boldsymbol{L}_{1,i}\right)\right) - \alpha_\varepsilon'(\varphi_\varepsilon)\zeta\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\,\mathrm{dx} \\ &= \int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u} + \left(\mathrm{D}_3 b + \tfrac{1}{2}\mathrm{D}_2 h\right)\cdot\nabla\boldsymbol{u} + \mathrm{D}_2 b\cdot\boldsymbol{u}\,\mathrm{dx} \\ &+ \int_\Omega \sum_{i=1}^{m_1+m_2}\lambda_i\left(\mathrm{D}_2 L_{0,i}\cdot\boldsymbol{u} + \left(\tfrac{1}{2}\nabla\varphi_\varepsilon\cdot\mathrm{D}_2\boldsymbol{L}_{1,i} + \mathrm{D}_3 L_{0,i}\right)\cdot\nabla\boldsymbol{u}\right)\mathrm{dx}\,, \end{aligned}$$

and upon rearranging we obtain

$$\begin{aligned} &\int_\Omega \alpha_\varepsilon(\varphi_\varepsilon)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u} + (\mathrm{D}_2 b, \mathrm{D}_3 b, \mathrm{D}_4 b)\cdot(\boldsymbol{u},\nabla\boldsymbol{u},p)\,\mathrm{dx} + \frac{1}{2}(\mathrm{D}_2 h, \mathrm{D}_3 h)\cdot(\nabla\boldsymbol{u},p)\,\mathrm{dx} \\ &= \int_\Omega -\alpha_\varepsilon'(\varphi_\varepsilon)\zeta\boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon - \sum_{i=1}^{m_1+m_2}\lambda_i(\mathrm{D}_2 L_{0,i}, \mathrm{D}_3 L_{0,i}, \mathrm{D}_4 L_{0,i})\cdot(\boldsymbol{u},\nabla\boldsymbol{u},p)\,\mathrm{dx} \\ &- \int_\Omega \sum_{i=1}^{m_1+m_2}\lambda_i\frac{1}{2}\nabla\varphi_\varepsilon\cdot(\mathrm{D}_2\boldsymbol{L}_{1,i}, \mathrm{D}_3\boldsymbol{L}_{1,i})\cdot(\nabla\boldsymbol{u},p)\,\mathrm{dx}\,. \end{aligned} \tag{4.19}$$

Substituting (4.19) into (4.15), we obtain

$$\begin{aligned} \mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\zeta) &= \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\left(\frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\right)\zeta + \frac{\gamma}{2c_0}\left(\frac{1}{\varepsilon}\Psi'(\varphi_\varepsilon)\zeta + \varepsilon\nabla\varphi_\varepsilon\cdot\nabla\zeta\right)\mathrm{dx} \\ &+ \int_\Omega \mathrm{D}_5 b\zeta + \frac{1}{2}\mathrm{D}_4 h\cdot\nabla\zeta\,\mathrm{dx} - \int_\Omega \sum_{i=1}^{m_1+m_2}\lambda_i(\mathrm{D}_2L_{0,i},\mathrm{D}_3L_{0,i},\mathrm{D}_4L_0)\cdot(\boldsymbol{u},\nabla\boldsymbol{u},p)\,\mathrm{dx} \\ &- \int_\Omega \sum_{i=1}^{m_1+m_2}\lambda_i\frac{1}{2}\nabla\varphi_\varepsilon\cdot(\mathrm{D}_2\boldsymbol{L}_{1,i},\mathrm{D}_3\boldsymbol{L}_{1,i})\cdot(\nabla\boldsymbol{u},p)\,\mathrm{dx}. \end{aligned} \tag{4.20}$$

Then, using (4.4) and substituting $\zeta = \varphi - \varphi_\varepsilon$, we see that

$$\begin{aligned} 0 &\le \mathrm{D}j_\varepsilon(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon) + \sum_{i=1}^{m_1+m_2}\lambda_i\mathrm{D}\mathcal{G}_i(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon) \\ &= \int_\Omega \alpha_\varepsilon'(\varphi_\varepsilon)\left(\frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon\cdot\boldsymbol{q}_\varepsilon\right)(\varphi-\varphi_\varepsilon) + \frac{\gamma}{2c_0}\left(\frac{1}{\varepsilon}\Psi'(\varphi_\varepsilon)(\varphi-\varphi_\varepsilon) + \varepsilon\nabla\varphi_\varepsilon\cdot\nabla(\varphi-\varphi_\varepsilon)\right)\mathrm{dx} \\ &+ \int_\Omega \mathrm{D}_5 b(\varphi-\varphi_\varepsilon) + \frac{1}{2}\mathrm{D}_4 h\cdot\nabla(\varphi-\varphi_\varepsilon) + \sum_{i=1}^{m_1+m_2}\lambda_i\left(\mathrm{D}_5L_{0,i}(\varphi-\varphi_\varepsilon) + \frac{1}{2}\nabla(\varphi-\varphi_\varepsilon)\cdot\boldsymbol{L}_{1,i}\right)\mathrm{dx} \end{aligned}$$

which is (4.14). □

Remark 4.4. *In the case where there is only a volume constraint, i.e.,* $m_1 + m_2 = 1$ *with* $\mathcal{G}(\varphi) := \int_\Omega \varphi - \beta\,\mathrm{dx}$ *for a fixed constant* $\beta \in (-1,1)$*, the existence of Lagrange multipliers using the Zowe–Kurcyusz constraint qualification has been shown in [19, Proof of Theorem 7.1] (for the case of inequality constraint), see also [13, Proof of Theorem 3] for another argument using geometric variations. For the case of equality constraint, we refer to [15, Proof of Theorem 4.10] which is based on a different argument.*

5 Existence of Lagrange multipliers for constraints on mass, center of mass and volume

In this section, we verify Assumptions 3.3 and 4.2 for the a specific set of design constraints, namely volume constraints (1.8), mass constraint (1.9) and prescribed center of mass (1.10) on the object. Let us recall the set

$$C := \{f \in H^1(\Omega)\,|\,s_a \le f(x) \le s_b \text{ for a.e. } x \in \Omega\},$$

and we denote a minimizer obtained from Theorem 3.2 as $(\varphi_\varepsilon, \boldsymbol{u}_\varepsilon, p_\varepsilon)$.

Let the desired centre of mass $\boldsymbol{y} \in \Omega \subset \mathbb{R}^2$ be given, and without loss of generality, suppose $\boldsymbol{y}$ is the origin in $\mathbb{R}^2$, which can be achieved by translating the domain Ω. Let $M > 0$, $-1 < \beta < 1$ be fixed constants, and a bounded mass density function $\rho : \Omega \to \mathbb{R}_{>0}$ be given. We impose the integral constraints

$$\int_\Omega \tfrac{1}{2}(1-\varphi)x_i\,\mathrm{dx} = 0 \text{ for } i = 1,2, \quad \int_\Omega \tfrac{1}{2}\rho(x)(1-\varphi)\,\mathrm{dx} \le M, \quad \beta \le \frac{1}{|\Omega|}\int_\Omega \varphi\,\mathrm{dx} \tag{5.1}$$

to the optimisation problem (2.4). The last constraint implies that the object can only occupy a maximal volume of $\frac{1-\beta}{2}|\Omega|$. We define

$$\begin{aligned} &\mathcal{G}_i(\varphi) := \int_\Omega \tfrac{1}{2}(1-\varphi)x_i\,\mathrm{dx} \text{ for } i = 1,2, \quad \mathcal{G}_3(\varphi) := \int_\Omega M|\Omega|^{-1} - \tfrac{1}{2}\rho(x)(1-\varphi)\,\mathrm{dx}, \\ &\mathcal{G}_4(\varphi) := \int_\Omega \varphi - \beta\,\mathrm{dx} \end{aligned}$$

so that (5.1) can be expressed as

$$\mathcal{G}_1(\varphi) = \mathcal{G}_2(\varphi) = 0, \quad \mathcal{G}_3(\varphi) \geq 0, \quad \mathcal{G}_4(\varphi) \geq 0. \tag{5.2}$$

As Ω is a bounded domain, we have that $x_i \in L^\infty(\Omega)$ for $i = 1,2$, and as $\rho \in L^\infty(\Omega)$ we see that Assumption 3.3 is satisfied for $\mathcal{G}_1, \dots, \mathcal{G}_4$. Furthermore, we compute that

$$\begin{aligned} &\boldsymbol{L}_{1,i} = \mathbf{0}, \quad \mathrm{D}_2 L_{0,i} = \mathrm{D}_3 L_{0,i} = \mathbf{0}, \quad \mathrm{D}_4 L_{0,i} = 0 \text{ for } i = 1,2,3,4, \\ &\mathrm{D}_5 L_{0,1} = -\tfrac{1}{2} x_1, \quad \mathrm{D}_5 L_{0,2} = -\tfrac{1}{2} x_2, \quad \mathrm{D}_5 L_{0,3} = -\tfrac{1}{2}\rho, \quad \mathrm{D}_5 L_{0,4} = 1, \end{aligned}$$

and so Assumption 4.2 is also fulfilled. To verify Assumption 4.3, we have to show that for an arbitrary $\boldsymbol{z} = (z_1, z_2, z_3, z_4) \in Y := \mathbb{R}^4$ there exists a $\varphi \in C$, along with non-negative constants $\tau_1, \dots, \tau_4, \xi_1, \xi_2, k_1, k_2$ such that

$$2z_1 = \tau_1 \int_\Omega (\varphi_\varepsilon - \varphi) x_1 \,\mathrm{dx}, \quad 2z_2 = \tau_2 \int_\Omega (\varphi_\varepsilon - \varphi) x_2 \,\mathrm{dx}, \tag{5.3a}$$

$$z_3 = \tau_3 \int_\Omega \tfrac{1}{2}\rho(x)(\varphi - \varphi_\varepsilon)\,\mathrm{dx} - k_1 + \xi_1 \Big(M - \int_\Omega \tfrac{1}{2}\rho(x)(1 - \varphi_\varepsilon)\,\mathrm{dx} \Big), \tag{5.3b}$$

$$z_4 = \tau_4 \int_\Omega \varphi - \varphi_\varepsilon \,\mathrm{dx} - k_2 + \xi_2 \Big(\int_\Omega \varphi_\varepsilon - \beta \,\mathrm{dx} \Big). \tag{5.3c}$$

Using the fact that $\mathcal{G}_1(\varphi_\varepsilon) = \mathcal{G}_2(\varphi_\varepsilon) = 0$ we can simplify the first two conditions to

$$2z_i = \tau_i \int_\Omega (1 - \varphi) x_i \,\mathrm{dx} \ \text{ for } i = 1,2.$$

We first argue for the equality constraints. Since $\boldsymbol{y}$ is the origin and $\boldsymbol{y} \notin \partial\Omega$, this implies that the domain Ω has non-empty intersection with the four quadrants of $\mathbb{R}^2$. Let $Q_1, \dots, Q_4$ denote the quadrant where $(x_1, x_2 > 0)$, $(x_1 < 0, x_2 > 0)$, $(x_1, x_2 < 0)$ and $(x_1 > 0, x_2 < 0)$, respectively. Without loss of generality, suppose $z_1, z_2 \neq 0$, since if $z_i = 0$ we choose $\tau_i = 0$. For arbitrary non-zero z_1, z_2, we choose a $\varphi \in C$ with $\beta < \varphi \leq 1$ a.e. in Ω such that

$$A := \operatorname{supp}(1 - \varphi) \subset\subset Q_i \cap \Omega \text{ if } (z_1, z_2) \in Q_i \text{ for } i = 1,2,3,4, \quad |A| < \frac{2M}{(1-\beta)\|\rho\|_{L^\infty(\Omega)}},$$

and

$$\tau_i := \frac{2z_i}{\int_\Omega (1 - \varphi) x_i \,\mathrm{dx}}.$$

We point out that as $\varphi \leq 1$ a.e. in Ω, this implies that $1 - \varphi$ is non-negative and only positive in $A = \operatorname{supp}(1 - \varphi)$. Furthermore, the location of A implies that the integrand $(1 - \varphi)x_i$ has the same sign as z_i for $i = 1,2$, and thus $\tau_i > 0$ for $i = 1,2$. This shows the conditions (5.3a) for the equality constraints. The constraint on the Lebesgue measure of A will be used below for the mass constraint.

For the inequality constraints, we have to show that the same φ chosen above simultaneously satisfies (5.3b) and (5.3c). We now perform the analysis for $\mathcal{G}_3$, which can be divided into two cases: suppose the inequality constraint $\mathcal{G}_3$ is inactive, i.e., φ_ε satisfies $\int_\Omega \frac{1}{2}\rho(x)(1 - \varphi_\varepsilon)\,\mathrm{dx} < M$, then we can choose $\tau_3 = 0$ and it holds that

$$\Big\{ -k_1 + \xi_1 \Big(M - \int_\Omega \tfrac{1}{2}\rho(x)(1 - \varphi_\varepsilon)\,\mathrm{dx} \Big) \,|\, k_1, \xi_1 \geq 0 \Big\} = \mathbb{R},$$

which implies that (5.3b) is fulfilled without making use of the function φ. Now suppose the inequality constraint $\mathcal{G}_3$ is active, i.e., $\int_\Omega \frac{1}{2}\rho(x)(1-\varphi_\varepsilon)\,\mathrm{dx} = M$, then (5.3b) simplifies to

$$z_3 = \tau_3\left(M + \int_\Omega \tfrac{1}{2}\rho(x)(\varphi - 1)\,\mathrm{dx}\right) - k_1.$$

If the quantity in the bracket is positive, then we have that

$$\left\{\tau_3\left(M + \int_\Omega \tfrac{1}{2}\rho(x)(\varphi - 1)\,\mathrm{dx}\right) - k_1 \,|\, k_1, \tau_1 \geq 0\right\} = \mathbb{R},$$

which implies that (5.3b) is also fulfilled. This follows from the fact that the function $\varphi \in C$ chosen in the analysis of (5.3a) satisfies $\beta < \varphi \leq 1$ a.e. in Ω and so

$$M - \int_\Omega \tfrac{1}{2}\rho(x)(1-\varphi)\,\mathrm{dx} = M - \int_A \tfrac{1}{2}\rho(x)(1-\varphi)\,\mathrm{dx} \geq M - \tfrac{1}{2}\|\rho\|_{L^\infty(\Omega)}(1-\beta)\,|A| > 0.$$

For (5.3c), a similar case analysis applies. If $\mathcal{G}_4$ is inactive, then $\int_\Omega \varphi_\varepsilon - \beta\,\mathrm{dx} > 0$ and thus

$$\left\{-k_2 + \xi_2\left(\int_\Omega \varphi_\varepsilon - \beta\,\mathrm{dx}\right) \,|\, k_2, \xi_2 \geq 0\right\} = \mathbb{R}.$$

If $\mathcal{G}_4$ is active, i.e., $\int_\Omega \varphi_\varepsilon\,\mathrm{dx} = \beta\,|\Omega|$, then it suffices to show that $\int_\Omega \varphi - \varphi_\varepsilon\,\mathrm{dx} = \int_\Omega \varphi - \beta\,\mathrm{dx} > 0$. By construction, $\varphi > \beta$ a.e. in Ω, and so (5.3c) is fulfilled. This shows that Assumption 4.3 is fulfilled and the existence of Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_3, \lambda_4 \in \mathbb{R}_{\geq 0}$ for the constraints (5.2) are guaranteed.

It remains to show that the space of admissible design function $\mathbb{K}_{ad}$ is non-empty so that Theorem 3.2 applies. We can always choose a $\phi \in C$ such that

$$\beta < \frac{1}{|\Omega|}\int_\Omega \phi\,\mathrm{dx}, \quad \int_\Omega \tfrac{1}{2}(1-\phi)x_i\,\mathrm{dx} = 0 \text{ for } i = 1, 2,$$

which amounts to choosing an object with its centre of mass at the origin such that its volume is bounded above by $\frac{1-\beta}{2}|\Omega|$. Furthermore, the mapping $\phi \mapsto \int_\Omega \frac{1}{2}\rho(x)(1-\phi)\,\mathrm{dx}$ is continuous from C to $\mathbb{R}$, and thus we can always decrease the volume of the object region $\{\phi = -1\}$ to ensure that the mass is bounded above by M. Hence the set $\mathbb{K}_{ad}$ is non-empty and Theorem 3.2 guarantees the existence of a minimiser to the optimization problem (2.4) with the integral state constraints (5.1). Furthermore, from Theorem 4.2 the first order optimality condition (4.6) becomes

$$\begin{aligned}
0 \leq &\left\langle \frac{\gamma\varepsilon}{2c_0}\nabla\varphi_\varepsilon + \frac{1}{2}\mathrm{D}_4 h, \nabla(\varphi - \varphi_\varepsilon)\right\rangle_{\boldsymbol{L}^2(\Omega)} \\
&+ \left\langle \alpha_\varepsilon'(\varphi_\varepsilon)\left(\frac{1}{2}|\boldsymbol{u}_\varepsilon|^2 - \boldsymbol{u}_\varepsilon \cdot \boldsymbol{q}_\varepsilon\right) + \frac{\gamma}{2c_0\varepsilon}\Psi'(\varphi_\varepsilon) + \mathrm{D}_5 b, \varphi - \varphi_\varepsilon\right\rangle_{L^2(\Omega)} \\
&+ \left\langle -\tfrac{1}{2}\lambda_1 x_1 - \tfrac{1}{2}\lambda_2 x_2 - \tfrac{1}{2}\lambda_3\rho(x) + \lambda_4, \varphi - \varphi_\varepsilon\right\rangle_{L^2(\Omega)} \quad \forall \varphi \in C,
\end{aligned} \tag{5.4}$$

together with the complementary slackness conditions

$$\lambda_3\left(M - \int_\Omega \tfrac{1}{2}\rho(x)(1-\varphi_\varepsilon)\,\mathrm{dx}\right) = 0, \quad \lambda_4\left(\int_\Omega \varphi_\varepsilon - \beta\,\mathrm{dx}\right) = 0.$$

6 Numerical implementation

Let us now describe how we can use the above results to compute optimal topologies in given flow settings. Since our optimization variable is a phase field, and thus has the natural regularity $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, we use the variable metric projection type (VMPT) method proposed in [5] to solve the resulting minimization problems. Note that $H^1(\Omega) \cap L^\infty(\Omega)$ is not a Hilbert space and thus a standard projected gradient method can not be used for the constraint minimization problem.

We use the double-obstacle free energy for Ψ, namely

$$\Psi(\varphi_\varepsilon) = \begin{cases} \frac{1}{2}(1 - \varphi_\varepsilon^2) & \text{if } |\varphi_\varepsilon| \le 1, \\ \infty & \text{else.} \end{cases} \tag{6.1}$$

From this we obtain the constraint $|\varphi_\varepsilon| \le 1$, and $c_0 = \frac{\pi}{2}$, where c_0 is the constant defined in (2.1). Although the double-obstacle potential (6.1) does not satisfy Assumption 3.1, the analysis is not affected once we choose $s_a = -1$ and $s_b = 1$, so that $|\varphi_\varepsilon| \le 1$ and the potential becomes $\Psi(\varphi_\varepsilon) = \frac{1}{2}(1 - \varphi_\varepsilon^2)$. We refer the reader to [12, 14] which also uses the double-obstacle potential (6.1). For $\alpha_\varepsilon(\varphi_\varepsilon)$ we choose

$$\alpha_\varepsilon(\varphi_\varepsilon) = \frac{\overline{\alpha}}{2\varepsilon}(1 - \varphi_\varepsilon),$$

with a fixed $\overline{\alpha}$, and Assumption 2.1 is fulfilled with $s_a = -1$ and $s_b = 1$.

6.1 Spatial discretization

We use finite elements for the numerical discretization of the minimization problem. We use piecewise linear and globally continuous finite elements for the representation of φ_ε, p_ε, and π_ε and piecewise quadratic and globally continuous finite elements for $\boldsymbol{u}_\varepsilon$ and $\boldsymbol{q}_\varepsilon$ on a conforming triangulation of Ω.

As φ_ε undergoes rapid variations across the interface an adaptive concept for the spatial resolution is indispensable. This especially is true as we use the PDAS approach as presented in [4] to solve certain constraint optimization problems that are stated at the heart of the VMPT. Thus we always need a certain amount of inactive degrees of freedom, i.e., with $|\varphi_\varepsilon| < 1$ to be able to solve these problems. On the other hand, it is reasonable to base the spatial discretization not only on the phase field alone, but to include further variables and especially the velocity field into the adaptive concept.

We use the dual weighted residuals (DWR) approach and derive this approach along the lines of [20]. The DWR approach is only applicable if for a given triangulation an optimal solution is already found and uses this information to calculate error indicators. On the other hand, the PDAS strategy requires a certain amount of inactive degrees of freedom. For this reason, whenever the number of inactive degrees is smaller than 0.02 times the number of all degrees of freedom of the phase field, we use mesh adaptation that is based on φ_ε only, namely we use the jumps of the normal derivatives of φ_ε across edges as proposed in [14] to be able to proceed with the PDAS.

We stop the adaptation loop as soon as a given maximum number of degrees of freedom is reached.

7 Numerical examples

In this section we show some numerical examples to illustrate our present approach.

7.1 A tube through heavy ground

Using a phase field for the representation of the desired topology allows us to deal with situations where no a priori information is available. Here we consider the situation of searching a tube that connects the inflow at the bottom to the outflow at the top through the domain $\Omega = (0,1)^2$ with some impermeable rocks inside, see Figure 1. Constructing a tube through the rocks is expensive and therefore a tube that avoids these regions is desired. So this is a setting where we want to minimize the cost of an object. The inflow and the outflow regions as well as the location of the rocks are a priori known. We define the inflow and outflow conditions as

$$g_{in}(x) = \begin{pmatrix} 0 \\ \max\left(2\left(1-\left(\frac{x_1-0.5}{1/6}\right)^2\right), 0.0\right)\end{pmatrix}, \quad g^i_{out}(x) \quad = \begin{pmatrix} \max\left((-1)^i\left(1-\left(\frac{x_2-0.8}{1/12}\right)^2\right), 0.0\right) \\ 0 \end{pmatrix}$$

for $i = 1, 2$. For the objective functional we define a 'rock' centered at $\boldsymbol{m}$ with radius σ and cost c as

$$R[\boldsymbol{m},\sigma,c](x) := (c-1)\left(\frac{\phi_0(-\frac{1}{\varepsilon}(\|\frac{x-\boldsymbol{m}}{\sigma}\|-1))+1}{2}\right)+1, \quad \phi_0(z) = \begin{cases}\sin(z) & \text{if } |z| \le \frac{\pi}{2}, \\ \operatorname{sign}(z) & \text{else.}\end{cases}$$

We consider the volume function

$$b(x,\boldsymbol{u},\nabla\boldsymbol{u},p,\varphi) := \left(\frac{1+\varphi}{2}\right)\prod_{i=1}^{4} R[\boldsymbol{m}_i,\sigma,c](x),$$
$$\text{i.e., } z(x,\varphi) := \frac{1+\varphi}{2}, \quad B(x,\boldsymbol{u},\nabla\boldsymbol{u},p) = \prod_{i=1}^{4} R[\boldsymbol{m}_i,\sigma,c](x),$$

where

$$\boldsymbol{m}_1 = (0.5, 0.3)^\top, \quad \boldsymbol{m}_2 = (0.15, 0.45)^\top, \quad \boldsymbol{m}_3 = (0.85, 0.45)^\top, \quad \boldsymbol{m}_4 = (0.5, 0.75)^\top.$$

The optimization problem (2.4) then becomes

$$\min_{(\varphi,\boldsymbol{u},p)} \mathcal{J}_\varepsilon(\varphi,\boldsymbol{u},p) = \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2 + \frac{1+\varphi}{2}\prod_{i=1}^{4} R[\boldsymbol{m}_i,\sigma,c] + \frac{\gamma}{\pi}\left(\frac{1}{\varepsilon}\Psi(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2\right)\,\mathrm{dx}\,,$$

subject to $\varphi \in H^1(\Omega)$, $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$, $p \in L^2_0(\Omega)$ satisfying (2.5). For this example we do not apply any integral constraints as this serves to demonstrate the strength of topology optimization with the phase field approach.

We start the optimization procedure with no information, i.e., $\varphi^0_\varepsilon \equiv 0$, on a homogeneous coarse grid with mesh size $h = 1/20$ yielding 685 degrees of unknowns for φ_ε. We stop the solution and adaptation procedures as soon as an optimal solution with more than 100000 degrees of freedom is found. It turns out that it is not necessary to use any integral constraints. The numerical parameters are: $\sigma = 0.15$, $c = 50$, $\varepsilon = 0.01$, $\overline{\alpha} = 5$, $\mu = 0.02$ and $\gamma = 0.001$.

To stress the benefits of our approach in Figure 2 we show φ_ε at certain steps of the optimization procedure.

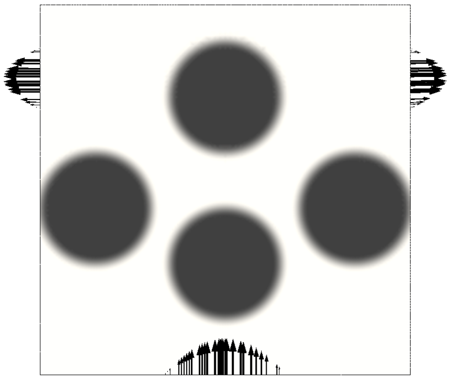

Figure 1: The inflow and outflow conditions for the Navier–Stokes equations together with the location of the rocks.

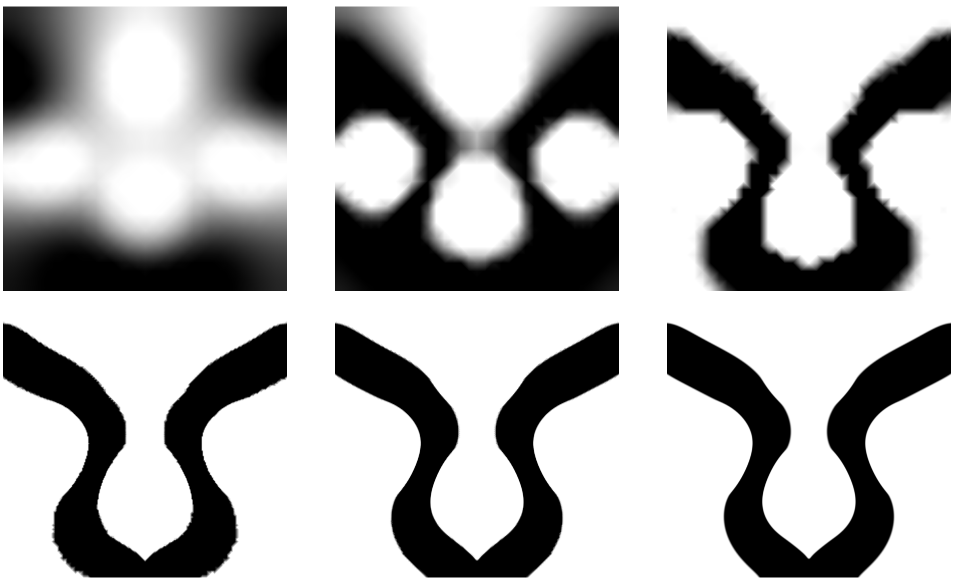

Figure 2: The iterations 20,30,40,80,120,190 of the VMPT to minimize the cost of a tube through heavy ground. We see that after 40 steps already the correct structure is found and that in subsequent steps mostly the resolution of the structure is improved. Let us also note that at iteration 20 we have only 923 degrees of freedom for φ_ε and in iteration 40 still only 1398 degrees of freedom. The final iteration has 125069 degrees of freedom.

7.2 Reproduction of results on drag minimization from earlier works

We now reproduce the numerical results for the surface formulation of drag minimization presented by the authors in [15]. That is, we consider the optimization problem

$$\min_{(\varphi,\boldsymbol{u},p)} \mathcal{J}_\varepsilon(\varphi,\boldsymbol{u},p) = \int_\Omega \frac{1}{2}\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2 + \frac{1}{2}\boldsymbol{a}\cdot\left(\mu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\top\right) - p\mathbf{I}\right)\nabla\varphi\,\mathrm{dx}$$
$$+ \int_\Omega \frac{\gamma}{\pi}\left(\frac{1}{\varepsilon}\Psi(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2\right)\mathrm{dx}$$

subject to $\varphi \in H^1(\Omega)$, $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{g},\sigma}(\Omega)$, $p \in L^2_0(\Omega)$ satisfying (2.5) and the volume constraint (see (1.8))

$$\int_\Omega \varphi\,\mathrm{dx} \le \beta_2|\Omega| \text{ for } \beta_2 \in (-1,1).$$

We use the parameters from [15], namely $\Omega = (0,1.7)\times(0,0.4)$, $\varepsilon = 0.00025$, $\overline{\alpha} = 0.03$, $\mu = 0.001$ and $\gamma = 0.01$. The boundary velocity is set to $\boldsymbol{g} = (1,0)^\top$ to stay close to the analysis and we initialize the optimization with $\varphi^0_\varepsilon(x) := -R[(0.5,0.2)^\top,0.25,-1](x)$, i.e., a ball around $m = (0.5,0.2)^\top$ with radius $r = 0.25$. For the volume constraint, we choose $\beta_2 = \beta = 0.975$, i.e., $\int_\Omega \varphi\,\mathrm{dx} \le 0.663$.

To be able to use only a small number of unknown as long as possible, we start the optimization with $\varepsilon = 0.008$ and a maximum number of allowed degrees of freedom of 10000. We halve the value of ε as soon as an optimal solution is found with current maximum allowed number of degrees of freedom and increase this value by 20%, resulting in 45000 unknowns for the final result. In Figure 3 we show the optimal shape for different values of ε, namely $\varepsilon \in \{0.008, 0.004, 0.002, 0.001, 0.0005, 0.00025\}$. In Table 1 we show the diffuse interface drag F^D_ε using formulation (2.3) and the sharp interface drag F^D by evaluation of (1.11) with $\boldsymbol{a} = (1.0,0.0)^\top$ over $\Gamma = \{\varphi_\varepsilon = 0\}$.

We reproduce the results found in [15] where a gradient flow approach is applied that is based on an artificial time evolution. We stress, that using a gradient flow approach, the interface has to be resolved in each time step of the temporal evolution, which leads to a large numerical effort. To be precise, while the results shown here are found in few hours using the VMPT, the results in [15] required several days of calculation using the gradient flow.

7.3 Comparision of volume and surface formulation

In (1.16) we introduced a volume formulation of the drag functional. We now compare results using the volume formulation (1.16) and the surface formulation (2.3). We again consider the setup from Section 7.2 and for the volume formulation we set $\boldsymbol{\eta} \equiv \boldsymbol{a}$ on $(0.15,1.0)\times(0.13,0.27)$.

Using the surface formulation we observe that for larger values of $\overline{\alpha}$ an interfacial region $\{|\varphi_\varepsilon| < 1\}$ that is neither fluid nor object appearing in front of the object. A similar behaviour was observed in the work on [15] with another minimization algorithm. In any case, a sufficiently impermeable object can be obtained by using smaller values of ε. We stress, that in Section 7.2 for $\varepsilon = 0.00025$ the the velocity $|\boldsymbol{u}_\varepsilon|$ inside the object is five orders of magnitude smaller than outside the object (see [15, Figure 1]).

On the other hand, using the volume formulation (1.16) we have to define the extension of the unit vector field $\boldsymbol{a}$, named the vector field $\boldsymbol{\eta}$ which has to vanish at $\partial\Omega$. We define $\boldsymbol{\eta}$ as the solution of a Poisson problem on Ω with $\boldsymbol{\eta} = \boldsymbol{a}$ on a square around the object and

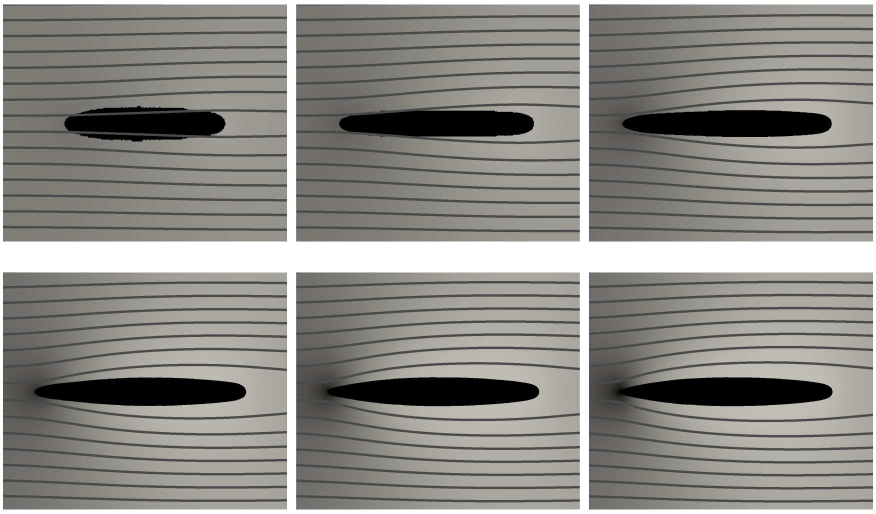

Figure 3: The optimal shapes for the minimization of drag in the surface formulation with the parameters from [15] for $\varepsilon = 0.008, 0.004, 0.002, 0.001, 0.0005, 0.00025$ (left upper to right lower). The shape is shown in black. The pressure is shown in gray, where darker gray means larger pressure, and some streamlines of the velocity are shown in black.

ε	0.008	0.004	0.002
F_ε^D	1.0570×10^{-2}	1.9806×10^{-2}	2.8370×10^{-2}
F^D	1.1103×10^{-2}	2.0519×10^{-2}	2.9025×10^{-2}
ε	0.001	0.0005	0.00025
F_ε^D	3.4255×10^{-2}	3.8184×10^{-2}	4.0739×10^{-2}
F^D	3.4777×10^{-2}	3.8572×10^{-2}	4.1012×10^{-2}

Table 1: The diffuse (F_ε^D) and sharp (F^D) drag for the parameters from [15] and different values of ε. Note that $\alpha_\varepsilon(-1) \to \infty$ for $\varepsilon \to 0$, i.e., the object becomes less permeable and thus the drag increases with $\varepsilon \to 0$. In [15] for $\varepsilon = 0.00025$ we observed $F_\varepsilon^D = 3.9117 \times 10^{-2}$ and $F^D = 3.9499 \times 10^{-2}$.

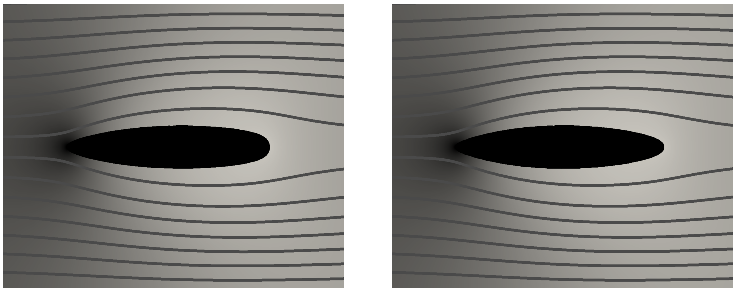

Figure 4: The optimized shapes of the object using the surface formulation ((2.3), left) and the volume formulation ((1.16), right) of the drag with $\mu = 0.01$ and $\overline{\alpha} = 0.03$. We observe that the rear of the object is slightly more pronounced when the volume formulation is used, while the drag measured on the zero level-line in both cases is nearly identical.

$\boldsymbol{\eta} = 0$ on $\partial\Omega$. That is, let S denote a square such that $\{\varphi_\varepsilon = -1\} \subset S$ and $\partial S \cap \partial\Omega = \varnothing$, then we solve

$$-\Delta\boldsymbol{\eta} = \mathbf{0} \text{ in } \Omega \setminus S, \quad \boldsymbol{\eta} = \mathbf{0} \text{ on } \partial\Omega, \quad \boldsymbol{\eta} = \boldsymbol{a} \text{ in } \overline{S}. \tag{7.1}$$

Then, for small values of $\overline{\alpha}$, we observe that the object splits and the solid is collected close to the inflow outflow boundaries. We believe this behavior is due to the following: On the one hand, due to the boundary condition $\boldsymbol{\eta} = \mathbf{0}$ on $\partial\Omega$, the magnitude $|\boldsymbol{\eta}|$ is small close to the inflow and outflow boundaries, which results in small drag forces. On the other hand, for $\overline{\alpha}$ small, the porous-medium penalization term $\int_\Omega \alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2 \,\mathrm{dx}$ is small, and thus the value of the objective functional can be reduced by placing material in regions where $|\boldsymbol{\eta}|$ is small. Therefore, in contrast to the surface formulation, large values of $\overline{\alpha}$ are needed for the volume formulation to obtain reasonable optimal shapes, which additionally allows us to construct sufficiently impermeable objects when we use larger values of ε.

We use the set up from Section 7.2 with only one modification, that we set $\mu = 0.01$. In Figure 4 the optimal shapes of the objects using the surface and the volume formulation of the drag are shown. We observe that the front of the object with both formulations is rather similar, while the surface formulation leads to a less pronounced rear. The corresponding drag values in sharp interface evaluation as defined in Section 7.2 are $F^D = 0.106467052$ (volume formulation) and $F^D = 0.106470276$ (surface formulation).

As described above, using the volume formulation we can use larger values for $\overline{\alpha}$ to model objects with smaller permeability. To show the influence of $\overline{\alpha}$ in Figure 5 we show the optimal shape for the above parameters, but using a larger value $\overline{\alpha} = 1$ and $\mu = 0.01$ (left) and $\mu = 0.001$ (right). For $\mu = 0.01$ we observe, that we get a sharper rear of the object, while the magnitude of the velocity inside the object is of order 10^{-4}, which is two orders of magnitudes smaller than in the case $\overline{\alpha} = 0.03$. We also mention that the shapes obtain here bear similarities to the optimized shape for the minimization of the dissipative energy, as presented in [14, Figures 4 and 5]. For $\overline{\alpha} = 1$, and $\mu = 0.001$ we observe a symmetric airfoil shape.

7.4 Maximizing the lift with constraints on the total potential power

We give an example of dealing with a state constraint, namely we consider the maximization of the lift of an object under the constraint that the total potential power is bounded by some given value. This is a non-linear constraint on the state variables of the constraint optimization problem, namely the velocity field. To treat the highly non-linear dissipative power constraint we use Moreau–Yosida relaxation.

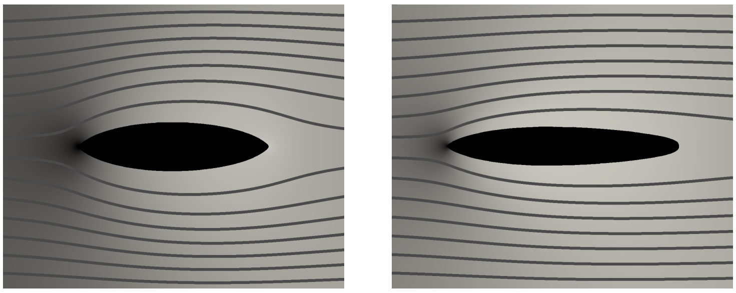

Figure 5: The optimized shape of the object using the volume formulation and $\overline{\alpha} = 1$ with $\mu = 0.01$ (left) and $\mu = 0.001$ (right). Compared to Figure 4 we observe a sharper rear and for $\mu = 0.001$ a symmetric airfoil shape emerges. For $\mu = 0.01$ the drag is $F^D = 0.205542595$ and the velocity inside the object is of order 10^{-4}, which is two orders of magnitude smaller than in the case $\overline{\alpha} = 0.03$. For $\mu = 0.001$ the drag is $F^D = 0.041090517$ and the velocity inside the object is of order 10^{-6}.

The optimization problem (2.4) becomes

$$\min_{(\varphi,\boldsymbol{u},p)} \mathcal{J}_\varepsilon^s(\varphi) := \int_\Omega \frac{\alpha_\varepsilon(\varphi)|\boldsymbol{u}|^2}{2} + \frac{\boldsymbol{a}}{2}\cdot\left(\mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top - p\mathbf{I}\right)\nabla\varphi\right) + \frac{\gamma}{\pi}\left(\frac{1}{\varepsilon}\Psi(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2\right)\mathrm{dx}$$
$$+\frac{s}{2}\max\left(0.0, \int_\Omega \frac{1+\varphi}{2}\left(\frac{\mu}{2}|\nabla\boldsymbol{u}|^2 - \boldsymbol{u}\cdot\boldsymbol{f}\right) - D|\Omega|^{-1}\,\mathrm{dx}\right)^2$$

where $s > 0$ is a parameter that penalizes violation of the constraint that the total dissipative power of the fluid region must be less than or equal to a prescribed value D:

$$\int_\Omega \frac{1+\varphi}{2}\left(\frac{\mu}{2}|\nabla\boldsymbol{u}|^2 - \boldsymbol{u}\cdot\boldsymbol{f}\right)\mathrm{dx} \le D.$$

In terms of (3.2) we set

$$y(x,\varphi) = -\frac{1+\varphi}{2}, \quad L(x,\boldsymbol{u},\nabla\boldsymbol{u},p) = \frac{\mu}{2}|\nabla\boldsymbol{u}|^2 - \boldsymbol{u}\cdot\boldsymbol{f}, \quad k(x) = D|\Omega|^{-1}.$$

To realize the maximization of lift, we set $\boldsymbol{a} = (-1,0)^\top$ as the negative unit vector perpendicular to the flow direction. The set up is similar to Section 7.2. We set $\Omega = (0.0,1.7)\times(0.0,0.4)$, $\boldsymbol{g} = (1,0)^\top$, $\varphi_\varepsilon^0(x) := -R[(0.5,0.2)^\top, 0.25, -1](x)$, i.e., a circle around $\boldsymbol{m} = (0.5,0.2)^\top$ with radius $r = 0.25$. For the penalization parameter, we set $s = 100$. For additional control constraints we consider restricting the volume of the fluid domain so that $0.663 \le \int_\Omega \varphi_\varepsilon\,\mathrm{dx} \le 0.665$ and fixing the center of mass at $(0.5,0.2)^\top$. Further numerical parameters are $\varepsilon = 0.02$, $\overline{\alpha} = 2$, $\mu = 0.01$, $\gamma = 0.001$, and $D = 0.06$.

In Figure 6 we show the resulting optimal shape of the object. As expected we observe an inclined structure in order to maximize lift, but due to the constraint on the dissipative power, the angle of attack is restricted.

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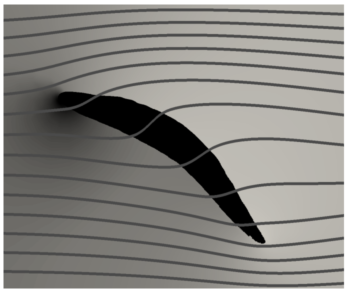

Figure 6: The optimal shape for the maximization of the lift of an object, under a constraint on the dissipative power. We observe an inclined shape.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems (Oxford Mathematical Monographs).* Oxford University Press, USA, 2000.

[2] J. Appell and P.P. Zabrejko. *Nonlinear Superposition Operators.* Cambridge University Press, Cambridge, 1990.

[3] J. A. Bello, E. Fernández-Cara, J. Lemoine, and J. Simon. The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier–Stokes flow. *SIAM J. Control Optim.*, 35(2):626–640, 1997.

[4] L. Blank, M. Butz, and H. Garcke. Solving the Cahn–Hilliard variational inequality with a semi-smooth Newton method. *ESAIM Control Optim. Calc. Var.*, 17(4):931–954, 2011.

[5] L. Blank and C. Rupprecht. An extension of the projected gradient method to a Banach space setting with application in structural topology optimization. Preprint arXiv:1503.03783v2, 2015.

[6] S. Boisgérault and J.P. Zolésio. Shape derivative of sharp functionals governed by Navier–Stokes flow. In W. Jäger, J. Nečas, O. John, K. Najzar, and J. Stará, editors, *Partial Differential Equations: Theory and Numerical Solution*, pages 49–63. Chapman and Hall/CRC, 1993.

[7] T. Borrvall and J. Petersson. Topology optimization of fluids in Stokes flow. *Internat. J. Numer. Methods Fluids*, 41(1):77–107, 2003.

[8] B. Bourdin and A. Chambolle. Design-dependent loads in topology optimization. *ESAIM Control Optim. Calc. Var.*, 9:19–48, 2003.

[9] C. Brandenburg, F. Lindemann, M. Ulbrich, and S. Ulbrich. A Continuous Adjoint Approach to Shape Optimization for Navier Stokes Flow. In K. Kunisch, J. Sprekels, G. Leugering, and F. Tröltzsch, editors, *Optimal Control of Coupled Systems of Partial Differential Equations*, volume 158 of *Internat. Ser. Numer. Math.*, pages 35–56. Birkhäuser, 2009.

[10] L.C. Evans and R.F. Gariepy. *Measure Theory and Fine Properties of Functions.* Studies in advanced mathematics. CRC Press, 1992.

[11] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier–Stokes Equations.* Springer Monographs in Mathematics. Springer, New York, NY, 2011.

[12] H. Garcke and C. Hecht. Applying a phase field approach for shape optimization of a stationary Navier-Stokes flow. *ESAIM: Control Optim. Calc. Var.*, 2015.

[13] H. Garcke and C. Hecht. Shape and topology optimization in Stokes flow with a phase field approach. *Appl. Math. Optim.*, pages 1–48, 2015.

[14] H. Garcke, C. Hecht, M. Hinze, and C. Kahle. Numerical approximation of phase field based shape and topology optimization for fluids. *SIAM J. Sci. Comput.*, 37(4):A1846–A1871, 2015.

[15] H. Garcke, C. Hecht, M. Hinze, C. Kahle, and K.F. Lam. Shape optimization for surface functionals in Navier–Stokes flow using a phase field approach. *Interfaces Free Bound.*, 18(2):219–261, 2016.

[16] M. Giles, M. Larson, M. Levenstam, and E. Süli. Adaptive error control for finite element approximations of the lift and drag coefficients in viscous flow. *Technical Report NA-79/06, Oxford University Computing Laboratory*, 1997.

[17] E. Giusti. *Minimal surfaces and functions of bounded variation.* Notes on pure mathematics. Dept. of Pure Mathematics, 1977.

[18] H. Goldberg, W. Kampowsky, and F. Tröltzsch. On Nemytskij operators in L_p-spaces of abstract functions. *Math. Nachr.*, 155:127–140, 1992.

[19] C. Hecht. *Shape and topology optimization in fluids using a phase field approach and an application in structural optimization.* PhD thesis, University of Regensburg, 2014.

[20] M. Hintermüller, M. Hinze, C. Kahle, and T. Keil. A goal-oriented dual-weighted adaptive finite elements approach for the optimal control of a Cahn–Hilliard–Navier–Stokes system. Preprint Hamburger Beiträge zur Angewandten Mathematik 2016-25, 2016.

[21] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE Constraints.* Mathematical Modelling: Theory and Applications. Springer Netherlands, 2009.

[22] J. Hoffman and C. Johnson. Adaptive Finite Element Methods for Incompressible Fluid Flow. In T.J. Barth and H. Deconinck, editors, *Error Estimation and Adaptive Discretization Methods in Computational Fluid Dynamics*, volume 25, pages 95–157. Springer Berlin Heidelberg, 2003.

[23] B. Kawohl, O. Pironneay, L. Tartar, and J.-P. Zolesio. *Optimal Shape Design: Lectures Given at the Joint C.I.M./C.I.M.E. Summer School Held in Troia (Portugal), June 1-6, 1998.* Lecture Notes in Mathematics / C.I.M.E. Foundation Subseries. Springer-Verlag Berlin Heidelberg, 2000.

[24] T. Kondoh, T. Matsumori, and A. Kawamoto. Drag minimization and lift maximization in laminar flows via topology optimization employing simple objective function expressions based on body force integration. *Struct. Multidiscip. Optim.*, 45(5):693–701, 2012.

[25] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Ration. Mech. Anal.*, 98(2):123–142, 1987.

[26] F. Murat. Contre-exemples pour divers problèmes où le contrôle intervient dans les coefficients. *Ann. Mat. Pura Appl., Serie 4*, 112(1):49–68, 1977.

[27] O. Pironneau. On optimum design in fluid mechanics. *J. Fluid Mech.*, 64:97–110, 5 1974.

[28] P.I. Plotnikov and J. Sokolowski. Shape derivative of drag functional. *SIAM J. Control Optim.*, 48(7):4680–4706, 2010.

[29] S.M. Robinson. Stability theorems for systems of inequalities, Part II: Differentiable nonlinear systems. *SIAM J. Numer. Anal.*, 13(4):497–513, 1976.

[30] S. Schmidt and V. Schulz. Shape Derivatives for General Objective Functions and the Incompressible Navier–Stokes Equations. *Control Cybernet.*, 39(3):677–713, 2010.

[31] J. Simon. Domain variation for drag in Stokes flow. In *Control Theory of Distributed Parameter Systems and Applications*, volume 159 of *Lecture Notes in Control and Information Sciences*. Springer, Berlin, Heidelberg, 1991.

[32] H. Sohr. *The Navier-Stokes Equations: An Elementary Functional Analytic Approach.* Birkhäuser Advanced Texts. Springer Verlag, 2001.

[33] K. Sturm, M. Hintermüller, and D. Hömberg. Distortion compensation as a shape optimization problem for a sharp interface model. *Comput. Optim. Appl.*, 64:557–588, 2016.

[34] L. Tartar. Problemes de Controle des Coefficients Dans des Equations aux Derivees Partielles. In A. Bensoussan and J.L. Lions, editors, *Control Theory, Numerical Methods and Computer Systems Modelling*, volume 107 of *Lecture Notes in Economics and Mathematical Systems*, pages 420–426. Springer Berlin Heidelberg, 1975.

[35] F. Tröltzsch. *Optimal Control of Partial Differential Equations: Theory, Methods, and Applications.* Graduate studies in mathematics. AMS, Providence, RI, 2010.

[36] J. Zowe and S. Kurcyusz. Regularity and Stability for the Mathematical Programming Problem in Banach Spaces. *Appl. Math. Optim.*, 5:49–62, 1979.