Differential Forms on Tropical Spaces

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Abstract

We show some basic cohomological properties of the double complex of differential forms on tropical spaces and the associated derived dual complexes. We then use these results to show that the tropical projective space satisfies an analogue of the dd^c-lemma for complex manifolds.

Contents

Introduction

Tropical geometry

Tropical algebraic geometry is the study of certain finite rational polyhedral complexes equipped with some additional structure. Some of the most important applications come from algebraic geometry where one can associate tropical varieties to algebraic varieties through a so-called tropicalization process. One then hopes to get a dictionary between properties in the tropical world and properties in the algebraic-geometric world. Results in this vein can be very powerful, mainly because the purely combinatorical nature of tropical varieties makes them much more accessible to computations and more direct constructions. Suitably, some of the most prominent applications of tropical geometry lie in enumerative algebraic geometry, e.g. Mikhalkin's Correspondence theorem [Mik05, Thm. 1].

A more recent development has been the introduction of tropical homology and cohomology groups in [MZ13] (or [IKMZ16]). Again, these can be given in a combinatorial manner and many direct applications to tropical and algebraic geometry have already been found. Apart from the original papers [MZ13] and [IKMZ16] we refer here to Shaw's study of the intersection product on tropical surfaces in [Sha15] which makes extensive use of tropical homology groups.

But as it turns out tropical geometry also is a very useful language for the study of non-archimedian analytic spaces (in the sense of Berkovich). Not only can the topology of the Berkovich analytification of an algebraic variety be described through its tropicalizations ([Pay09, Thm. 1.1]) but tropical methods also allow one to define bigraded sheaves of differential forms on Berkovich spaces. Building upon Lagerberg's superforms [Lag12], the latter were first introduced in [CD12] where Chambert-Loir and Ducros use them to define Monge-Ampère measures and first Chern classes in a 'classical' manner and prove several of their properties. This proceeds to be a very active field of study, with recent advances for instance in [Liu17].

Main Results

In the present paper, we will concern ourselves only with the tropical side of this construction: with the double complex of sheaves of differential forms $\mathcal{A}_X^{\bullet,\bullet}$ on a tropical space X. The connection between bigraded differential forms on $\mathbb{R}^N$ and tropical geometry was first discussed in [Lag12]. Lagerbergs results on positive closed currents are also central to the theory developed in [CD12]. We will deviate from this, focussing purely on cohomological properties of $\mathcal{A}_X^{\bullet,\bullet}$. The first major result in this direction has been Jell's Poincaré lemma in [Jel16a, 2.18], where he shows that the complexes $\mathcal{A}_X^{p,\bullet}$ are fine resolutions of the respective kernels

$$\mathcal{L}_X^p = \ker(\mathcal{A}_X^{p,0} \to \mathcal{A}_X^{p,1}).$$

Together with Philipp Jell and Kristin Shaw we were able to derive from this that the tropical cohomology groups of X from [MZ13] are canonically isomorphic to the sheaf cohomology groups of $\mathcal{L}_X^p$, [JSS15, 3.15]:

Theorem 1. *Let X be a tropical space. Then the tropical cohomology groups of X with real coefficients are canonically isomorphic to the Dolbeault cohomology groups on X:*

$$\mathrm{H}^{p,q}_{\mathrm{trop}}(X) \cong \mathrm{H}^q(X, \mathcal{L}_X^p) \cong \mathrm{H}^q\left(\Gamma(X, \mathcal{A}_X^{p,\bullet})\right).$$

In particular, this gives an answer to the question raised in [CD12, p.12], establishing a mediate connection between tropical cohomology and the cohomology of superforms on

Berkovich spaces. In section 2 we will give a proof of this result (theorem 2.16), differing from [JSS15] in the computation of the sheaves $\mathcal{L}^p_X$.

Given a double complex like $\mathcal{A}^{\bullet;\bullet}_X$ one might also be interested in the properties of its total complex $\mathcal{A}^{\bullet}_X = \mathrm{tot}^{\bullet}(\mathcal{A}^{\bullet;\bullet}_X)$. In proposition 3.5 we first show that up to quasi-isomorphism, $\mathcal{A}^{\bullet}_X$ has a simple direct sum decomposition:

Proposition 2. *Let X be a tropical space. Then there exists a canonical quasi-isomorphism of complexes of sheaves on X,*

$$\bigoplus_{p\in\mathbb{Z}} \mathcal{L}^p_X[-p] \xrightarrow{\sim} \mathcal{A}^{\bullet}_X .$$

For tropical manifolds, this also allows us to transfer Poincaré duality – proved in [JSS15] for the vertical complexes $\mathcal{A}^{p;\bullet}_X$ – to the total complex $\mathcal{A}^{\bullet}_X$ in theorem 3.28. We will phrase this result in terms of the complex $\mathcal{D}^{\bullet}_X$ of linear currents which represents the derived dual of $\mathcal{A}^{\bullet}_X$ in the derived category of sheaves on X (c.f. example 3.3f):

Theorem 3. *Let X be a tropical manifold of pure dimension n. Then there exists a canonical quasi-isomorphism*

$$\mathcal{A}^{\bullet}_X[2n] \xrightarrow{\sim} \mathcal{D}^{\bullet}_X,$$

induced by the wedge product of forms and a natural integration map $\Gamma_c(X, \mathcal{A}^{2n}_X) \to \mathbb{R}$.

We will usually consider tropical spaces as topological spaces locally isomorphic to the support of polyhedral complexes in $\mathbb{T}^N$, where $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ is the tropical affine line, equipped with the topology of a half open interval. This forces us to pay special attention to the points where one or more coordinates are $\{-\infty\}$, leading us to define *sedentarities* or more specifically *good sedentarities* as closed subsets at infinity which have certain global properties in X (c.f. definition 1.29). For the complex $\mathcal{D}^{\bullet}_X$, we have a nice description of the cohomology with support in a good sedentarity in theorem 3.14:

Theorem 4. *Let X be a regular tropical space and let $\imath : Z \subset X$ be the closed embedding of a good sedentarity. Then there exists a canonical isomorphism in the derived category of sheaves on X:*

$$\mathrm{R}\,\imath_!\,\mathcal{D}^{\bullet}_X \xrightarrow{\sim} \mathrm{R}\,\Gamma_Z\,\mathcal{D}^{\bullet}_X .$$

Apart from the cohomology of the complexes $\mathcal{A}^{p;\bullet}_X$, $\mathcal{A}^{\bullet;q}_X$ and of its total complex $\mathcal{A}^{\bullet}_X$, the double complex $(\mathcal{A}^{\bullet;\bullet}_X, d', d'')$ of forms on X also gives rise to *Bott-Chern* and *Aeppli cohomology* groups on X,

$$\mathrm{H}^{p,q}_{BC}(X) = \frac{\ker(d') \cap \ker(d'') \cap \mathcal{A}^{p;q}_X(X)}{\mathrm{im}(d'd'')},$$
$$\mathrm{H}^{p,q}_{A}(X) = \frac{\ker(d'd'') \cap \mathcal{A}^{p;q}_X(X)}{\mathrm{im}(d') + \mathrm{im}(d'')}.$$

It is an interesting question to ask if these groups are canonically isomorphic: For instance, the corresponding statement for compact symplectic manifolds is equivalent to the Hard Lefschetz property (c.f. [AT15, 5.2]). Here, we only give a first result in this direction, using a construction of Schweitzer to show that $\mathbb{P}^N$ satisfies this property (theorem 4.21):

Theorem 5. *The tropical projective space $\mathbb{P}^N$ of dimension N satisfies the $d'd''$-lemma, i.e. for every $p, q \in \mathbb{Z}$ the canonical map*

$$\mathrm{H}^{p,q}_{BC}\left(\mathbb{P}^N\right) \xrightarrow{\sim} \mathrm{H}^{p,q}_{A}\left(\mathbb{P}^N\right)$$

is an isomorphism.

Note that in the main text we will work with differentials d_1 and d_2 – which differ from d' and d'' only by sign – in order to end up with double complexes with commuting squares.

Lastly, we give a possible construction for a locally convex topology on the $\mathbb{R}$-vector spaces $\mathcal{A}_X^{p,q}(X)$ in section 5.1. This allows us to define the subcomplex $\tilde{\mathcal{D}}_X^\bullet \subset \mathcal{D}_X^\bullet$ of continuous currents on a tropical space X. The integration morphism $\mathcal{A}_X^\bullet[2n] \to \mathcal{D}_X^\bullet$ factors through the embedding $\tilde{\mathcal{D}}_X^\bullet \to \mathcal{D}_X^\bullet$ and from theorem 3 one can derive a smoothing-of-cohomology type statement (theorem 5.17), similar to the classical case:

Theorem 6. *Let X be a smooth tropical space of pure dimension n. Then the canonical morphism of complexes of sheaves*

$$\mathcal{A}_X^\bullet[2n] \xrightarrow{\sim} \tilde{\mathcal{D}}_X^\bullet$$

is a quasi-isomorphism. In particular: Up to an exact continuous current, every closed continuous current is given by a closed smooth form on X.

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1 Tropical spaces and tropical homology

1.1 Polyhedral complexes in tropical affine space

We recall the definitions and notations from [IKMZ16, Sect. 2]. Throughout, for a natural number $N \in \mathbb{N}\setminus\{0\}$ we will use the shorthand $[N] := \{1, \ldots, N\}$.

Definition 1.1. The *tropical affine space* of dimension N is the topological space $\mathbb{T}^N := [-\infty, \infty)^n$, stratified by the family $\{\mathbb{R}^N_I\}_{I\subset[N]}$, where

$$\mathbb{R}^N_I := (\mathbb{T}^N_I)^\circ := \{(x_i)_{i\in[N]} \in \mathbb{T}^n; x_i = -\infty \text{ if and only if } i \in I\}.$$

We denote the topological closure of $\mathbb{R}^N_I$ by

$$\mathbb{T}^n_I := \{(x_i)_{i\in[N]} \in \mathbb{T}^n; x_i = -\infty \text{ if } i \in I\}.$$

For $I \subset J \subset [N]$ we write π^J_I for the obvious projection maps $\mathbb{T}^N_J \to \mathbb{T}^N_I$ as well as $\mathbb{R}^N_J \to \mathbb{R}^N_I$. Via these maps we can identify $\mathbb{R}^N_I$ with $\mathbb{R}^N / \mathbb{R}^{|I|}$ and we fix the integral structure $\mathbb{Z}^N_I = \mathbb{Z}^N / \mathbb{Z}^{|I|}$ on each stratum $\mathbb{R}^N_I$.

Definition 1.2. For any subset $X \subset \mathbb{T}^N$ and $I \subset [N]$, we fix the following notation:

$$X_I := X \cap \mathbb{T}^N_I, \qquad X^\circ_I := X \cap \mathbb{R}^N_I.$$

For $I = \emptyset$ we will generally omit the subscript I, i.e. we have $X^\circ = X \cap \mathbb{R}^N$ etc. We will call X° the *finitary part* of X and say that X is *finitary* if $X = X^\circ$.

Definition 1.3. 1. A *convex (rational) polyhedral domain* or simply *(rational) polyhedron* σ in $\mathbb{R}^N$ is the intersection of a finite number of half-spaces $H \subset \mathbb{R}^N$ of the form

$$H = \{x \in \mathbb{T}^N; m \cdot x \leq a\},$$

with $m \in \mathbb{R}^N$ ($m \in \mathbb{Z}^N$) and $a \in \mathbb{R}$.

2. The *dimension* of a polyhedron σ is its dimension as a topological space.

3. A *face* of a polyhedron σ in $\mathbb{R}^N$ is the intersection of σ with some boundaries

$$\partial H := \{x \in \mathbb{R}^N; m \cdot x = 0\}$$

of the halfspaces H defining σ.

4. We write $\gamma \prec \sigma$ if γ is a face of σ.

5. The *relative interior* $\operatorname{relint}(\sigma)$ of a polyhedron σ in $\mathbb{R}^N$ is the complement in σ of all of its proper faces.

6. The *linear space* $\mathbb{L}(\sigma) := \mathbb{L}_\mathbb{R}(\sigma)$ and – in the rational case – the *lattice* $\mathbb{L}_\mathbb{Z}(\sigma)$ *associated to* σ are defined by

$$\mathbb{L}_A(\sigma) := \operatorname{span}_\mathbb{R}(x - y; x, y \in \operatorname{relint}(\sigma)) \cap A^N,\ \ A \in \{\mathbb{Z}, \mathbb{R}\}.$$

Definition 1.4. A *(rational) polyhedral complex* in $\mathbb{R}^N$ is a finite set Σ of (rational) polyhedra in $\mathbb{R}^N$ satisfying:

1. For each $\sigma \in \Sigma$, Σ contains all faces of σ.

2. For each two $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face of σ.

The face relation makes Σ into a poset. The *dimension* of Σ is the maximal dimension among polyhedra in Σ; if each maximal polyhedron in Σ has dimension n, then Σ is called *purely n-dimensional.*

The *support* of Σ is the closed subset $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma \subset \mathbb{R}^N$.

We write

$$\Sigma_k := \{\sigma \in \Sigma; \dim(\sigma) = k\}$$

for $k \in \mathbb{N}$.

The following lemma (c.f. [IKMZ16, 4]) describes the behavior of polyhedral complexes in $\mathbb{R}^N$ when taking their closure in $\mathbb{T}^N$.

Lemma 1.5. *Let Σ° be an n-dimensional polyhedral complex in $\mathbb{R}^N$ with support X and let $\overline{X}$ be the closure of X in $\mathbb{T}^N$. Then the intersection $\overline{X}_I^\circ = \overline{X} \cap \mathbb{R}_I^N$ is the support of a polyhedral complex in $\mathbb{R}_I^N$ of dimension $\leq (n-1)$.*

In particular, the proof of this lemma shows that if σ° is an n-dimensional polyhedron in $\mathbb{R}^N$ and σ its closure in $\mathbb{T}^N$, then the intersection $\sigma_I^\circ := \sigma \cap \mathbb{R}_I^N$ is a polyhedron in $\mathbb{R}_I^N$ of dimension $\leq (n-1)$. We take this as motivation for the following definition:

Definition 1.6.

1. A *(rational) polyhedron in* $\mathbb{T}^N$ is the closure σ in $\mathbb{T}^N$ of a (rational) polyhedron in $\mathbb{R}_I^N$ for some $I \subset [N]$.
2. The *dimension* of a polyhedron σ in $\mathbb{T}^N$ is its dimension as a topological space. Its *sedentarity* $\text{sed}(\sigma)$ is the unique subset $I \subset [N]$ such that σ is the closure in $\mathbb{T}^N$ of a polyhedron in $\mathbb{R}_I^N$.
3. A *mobile face* of a polyhedron σ of sedentarity I in $\mathbb{T}^N$ is a polyhedron $\gamma \subset \sigma$ of sedentarity I in $\mathbb{T}^N$ such that γ_I° is a face of σ_I° in $\mathbb{R}_I^N$. A *sedentary face* of σ is the intersection $\gamma_J := \gamma \cap \mathbb{T}_J^N$ for some mobile face γ of σ and a subset $I \subsetneq J \subset [N]$. A *face* of σ is either a mobile or a sedentary face; we write $\gamma \prec \sigma$ if γ is a face of σ.
4. The *relative interior* $\text{relint}(\sigma)$ of a polyhedron σ of sedentarity $\text{sed}(\sigma) = I$ in $\mathbb{T}^N$ is the relative interior of the polyhedron $\sigma_I^\circ := \sigma \cap \mathbb{R}_I^N$ in $\mathbb{R}_I^N$. It is equal to the complement in σ of the union of proper faces of σ.
5. The *linear space* $\mathbb{L}(\sigma) := \mathbb{L}_\mathbb{R}(\sigma)$ and – in the rational case – the *lattice* $\mathbb{L}_\mathbb{Z}(\sigma)$ *associated to* a polyhedron σ of sedentarity I in $\mathbb{T}^N$ are defined by

$$\mathbb{L}_R(\sigma) := \text{span}_\mathbb{R}(x - y; x, y \in \text{relint}(\sigma)) \cap R_I^N \subset \mathbb{R}_I^N, \ R \in \{\mathbb{Z}, \mathbb{R}\}.$$

   If $\gamma \prec \sigma$ is a mobile face of σ, then there exists a natural *inclusion map* $\mathbb{L}(\gamma) \to \mathbb{L}(\sigma)$. For the sedentary face $\sigma_J \prec \sigma$, we get a natural *projection map* $\mathbb{L}(\sigma) \to \mathbb{L}(\sigma_J)$ induced by the projection $\pi_{IJ} : \mathbb{R}_I^N \to \mathbb{R}_J^N$.

Definition 1.7. A *(rational) polyhedral complex* Σ *in* $\mathbb{T}^N$ is a finite family of (rational) polyhedra $\sigma \subset \mathbb{T}^N$ with $I \subset [N]$ satisfying the following conditions:

1. For $\sigma \in \Sigma$ and every face $\gamma \prec \sigma$, we have $\gamma \in \Sigma$.
2. For each two polyhedra σ, $\sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face of σ.

We will always assume that Σ is *of sedentarity* $\emptyset$, i.e. all maximal polyhedra σ of Σ are of sedentarity $\text{sed}(\sigma) = \emptyset$.

We write $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma \subset \mathbb{T}^N$ for the *support* of Σ and $\Sigma_I := \{\sigma \cap \mathbb{T}_I^N; \sigma \in \Sigma\}$ for the *induced polyhedral complex in sedentarity* $I \subset [N]$. Both are equipped with the induced topology.

For every polyhedron $\sigma \in \Sigma$, we define the *open star* of σ to be $U_\sigma := \bigcup_{\sigma \prec \tau} \operatorname{relint}(\tau)$ (this is in fact an open subset of $|\Sigma|$).

If every maximal face $\sigma \in \Sigma$ has dimension n, Σ is called *purely n-dimensional.*

We also write

$$\Sigma_k := \{\sigma \in \Sigma; \dim(\sigma) = k\}$$

for $k \in \mathbb{N}$.

Definition 1.8.

1. A polyhedron σ in $\mathbb{T}^N$ of sedentarity $\emptyset$ is called *regular* (or *regular at infinity*) if the underlying polyhedron σ° in $\mathbb{R}^N$ can be given as a finite intersection of halfspaces
$$H = \{x \in \mathbb{R}^N; m \cdot x \leq a\}$$
with $m \in \mathbb{R}^N$, $a \in \mathbb{R}$, with the additional requirement that $m_i \geq 0$ whenever $\sigma_{\{i\}} = \sigma \cap \mathbb{T}^N_{\{i\}}$ is non-empty.

2. A polyhedral complex Σ in $\mathbb{T}^N$ is *regular* if all of its maximal polyhedra (which have empty sedentarity by assumption) are regular.

The most important properties of regular polyhedral complexes for us are encapsuled in the following lemma from [IKMZ16, 9]:

Lemma 1.9. *Let X be the support of a regular rational complex Σ in $\mathbb{T}^N$ and let $X_I := X \cap \mathbb{T}^N_I$ be non-empty. Then, Σ_I is a regular rational polyhedral complex in $\mathbb{T}^N_I$ with support X_I; in particular, all maximal polyhedra of Σ_I have sedentarity I. Moreover, for sufficiently small $\epsilon > 0$, the neighborhood*

$$X_I^\epsilon := \{x \in X; x_i < \log(\epsilon), i \in I\}$$

of X_I splits as the product

$$X_I^\epsilon = X_I \times \mathbb{T}^I_\epsilon$$

where $\mathbb{T}^I_\epsilon := \{(x_i)_{i \in I} \in \mathbb{T}^I; x_i < \log(\epsilon)\}$.

As remarked in [MZ13, 1.4], parent faces are uniquely determined in regular polyhedral complexes:

Lemma 1.10. *Let Σ be a regular polyhedral complex in $\mathbb{T}^N$ (of empty sedentarity) and $\sigma_I \neq \emptyset$ a polyhedron in Σ_I. Then for every $J \subset I$, there exists a unique polyhedron σ_I^J in Σ_J with $\sigma_I = \sigma_I^J \cap \mathbb{T}^N_I$, i.e. the parent face of sedentarity J of σ_I is uniquely determined.*

Remark 1.11. Occasionally, we will consider several different polyhedral complexes at once. In this case we will distinguish the corresponding associated linear spaces by an index; for example, if X is the support of a completed polyhedral complex Σ in $\mathbb{T}^N$ and $\sigma \in \Sigma$, then we set

$$\mathbb{L}_X(\sigma) := \mathbb{L}(\sigma) := \mathbb{L}_\mathbb{R}(\sigma).$$

1.2 Extended affine Maps

Definition 1.12. Let $\tilde{U} \subset \mathbb{T}^N$, $\tilde{U}' \subset \mathbb{T}^{N'}$ be open subsets.

1. An *extended affine map* $F : \tilde{U} \to \tilde{U}'$ is a continuous map $F : \tilde{U} \to \tilde{U}'$ such that for every $I \subset [N]$ there exists $I' \subset [N']$ such that
$$F|_{\tilde{U}_I^\circ} : \tilde{U}_I^\circ \to (\tilde{U}')^\circ_{I'}$$
is well defined and the restriction of an affine map
$$\mathbb{R}^N_I \to \mathbb{R}^{N'}_{I'}.$$

2. Let $U \subset \mathbb{T}^N$, $U' \subset \mathbb{T}^{N'}$ be locally closed subsets. An *extended affine map* $F : U \to U'$ is an extended affine map $\tilde{F} : \tilde{U} \to \tilde{U}'$, where $U \subset \tilde{U}$ and $U' \subset \tilde{U}'$ are open neighbourhoods. We identify two extended affine maps $F : U \to U'$ and $G : U \to U'$ if they agree on U. An extended affine map F is *rational* if all the maps

$$\mathbb{R}^N_I \to \mathbb{R}^{N'}_{I'}$$

in the definition above are rational, i.e. their linear part is $\mathbb{Z}$-linear.

1.3 Weigted complexes and the balancing condition

Definition 1.13. Let Σ be a purely n-dimensional regular rational polyhedral complex in $\mathbb{T}^N$. A *weight* on Σ is a map

$$w : \Sigma_n \to \mathbb{Z},$$

and (Σ, w) is called a *weighted polyhedral complex* in $\mathbb{T}^N$. It is said to be *balanced* or to satisfy the *balancing condition* if for every $\sigma \in \Sigma_{n-1}$ we have

$$\sum_{\sigma \prec \tau \in \Sigma_n} w(\tau) v_\sigma^\tau \in \mathbb{L}(\sigma),$$

where v_σ^τ is a representant of the primitive outward-pointing generator of $\mathbb{L}_\mathbb{Z}(\tau)/\mathbb{L}_\mathbb{Z}(\sigma) \cong \mathbb{Z}$. Let (Σ, w) and (Σ', w') be weighted polyhedral complexes of pure dimension n in $\mathbb{T}^N$. Then Σ' is a *refinement of* Σ if $|\Sigma'| = |\Sigma|$ and for every $\sigma' \in \Sigma'$ there exists $\sigma \in \Sigma$ with $\sigma' \subset \sigma$. If for every $\sigma' \in \Sigma'_n$ we also have $w'(\sigma') = w(\sigma)$, then (Σ', w') is called a *refinement of* (Σ, w). Two weighted polyhedral complexes (Σ, w) and (Σ', w') in $\mathbb{T}^N$ are *equivalent* if they have a common refinement.

Remark 1.14. For a balanced polyhedral complex Σ in $\mathbb{T}^N$ – a tropical cycle – and an extended affine map $F : |\Sigma| \to \mathbb{T}^M$ one can define the pushforward of $[\Sigma, w]$ similar to [Gub13, 3.9ff] or [AR10, ch.7]. This faciliates an intersection product of tropical cycles. As in [Gub13], this pushforward is well-behaved with respect to the integration pairing. It would be interesting to see how much of the intersection theory of tropical cycles can equivalently be formulated in terms of the differential forms on tropical spaces discussed below. We will not pursue this question further here.

1.4 Tropical spaces

We can now consider spaces equipped with an atlas of charts to polyhedral subspaces in $\mathbb{T}^N$. As in [JSS15, 2.22] we first define general polyhedral spaces and then specialize to tropical spaces.

Definition 1.15. Let X be a topological space. A *polyhedral atlas on* X is a collection of maps

$$\mathfrak{A} = \{\varphi_i : U_i \to V_i \subset X_i\}_{i \in I}$$

such that:

1. The U_i are open subsets of X and the V_i are open subsets of the supports X_i of polyhedral complexes in some $\mathbb{T}^{N_i}$.

2. The maps

$$\varphi_i : U_i \to V_i$$

are homeomorphisms for every $i \in I$.

3. For all $i, j \in I$ the transition map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is an extended affine map.

A polyhedral atlas as above is a *tropical atlas* if it satisfies the following additional conditions:

1. The X_i are the supports of balanced weighted rational polyhedral complexes in $\mathbb{T}^{N_i}$ with positive weights.
2. The transition maps

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are integral extended affine maps and they are weight preserving.

Two (tropical) polyhedral atlases on X are *(tropically) equivalent* if their union is a (tropical) atlas on X.

Definition 1.16. 1. A *polyhedral space* X is a paracompact, second countable Hausdorff topological space together with an equivalence class of polyhedral atlases on X. A *morphism* of polyhedral spaces $X \to Y$ is a map

$$f : X \to Y$$

such that for some choice of atlases for X and Y, f restricts to extended affine maps on all charts. We denote by Poly the category of polyhedral spaces.

2. A polyhedral space X is *regular* or *regular at infinity* if it has an atlas as above such that each X_i is a regular polyhedral complex in $\mathbb{T}^{N_i}$.
3. If all the V_i are subsets of $\mathbb{R}^{N_i}$, then X is a *finitary* polyhedral space.

Definition 1.17. 1. A *tropical space* is a paracompact, second countable Hausdorff topological space together with a tropical equivalence class of tropical atlases. A *morphism* of tropical spaces $X \to Y$ is a map

$$f : X \to Y$$

such that for some choice of atlases for X and Y, f restricts to integral extended affine maps on all charts. We denote by Trop the category of polyhedral spaces.

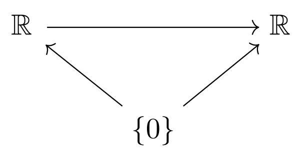
2. If all the X_i can be chosen to be smooth, then X is called a *tropical manifold.*
3. *Regular* and *finitary* tropical spaces are defined analogously to regular and finitary polyhedral spaces.

Remark 1.18. The canonical functor

$$\Phi : \mathrm{Trop} \to \mathrm{Poly}$$

is faithful but neither full nor essentially surjective:

The unit interval $[0, 1] \subset \mathbb{R}^1$ is a polyhedral space which does not lie in the essential image of Φ, so Φ is not essentially surjective. On the other hand, the polyhedral spaces $\{0\}$ and $\mathbb{R}$ lie in the essential image of Φ. While the number of commuting diagrams



(0 maps to 0 in $\mathbb{R}$) is countable in Trop, it is uncountable in Poly. This precludes Φ from being full. It is clear that Φ is faithful.

1.5 Starshaped open subsets

Often, when examining local properties of tropical spaces, we are in need of a suitable basis of topology which faciliates the computation of various cohomology groups. In those cases, we will make use of polyhedrally starshaped open subsets:

Definition 1.19. Let X be a (tropical) polyhedral space.

1. A (tropical) polyhedral chart $\phi : U \to V \subset \mathbb{T}^N$ is *polyhedrally starshaped* (*with center* $x \in U$) if there is a polyhedral complex Σ in $\mathbb{T}^N$ such that V is the open star of $\sigma \in \Sigma$ with $\phi(x) \in \operatorname{relint}(\sigma)$.

2. An open subset $U \subset X$ is *polyhedrally starshaped* (*with center* $x \in U$) if there exists a (tropical) polyhedral chart $\phi' : U' \to V' \subset \mathbb{T}^N$ for X with $U \subset U'$ such that the restricted chart
$$\phi'|_U : U \to \phi'(U) \subset \mathbb{T}^N$$
is polyhedrally starshaped (with center x).

Whenever ambiguity is ruled out, we will simply speak of starshaped charts and starshaped open subsets.

Remark 1.20.
1. Every polyhedral or tropical space X has an atlas consisting of starshaped charts. Similarly, every $x \in X$ has a neighbourhood system consisting of starshaped open subsets with center x.

2. Also, if X is the support of a polyhedral complex Σ in $\mathbb{R}^N$ and $U \subset X$ is polyhedrally starshaped with center $x \in X$, then U also is polyhedrally starshaped in the sense of [Jel16b, Definition 2.2.11], i.e. for some polyhedral complex Σ' in $\mathbb{R}^N$ with support X and every maximal polyhedron $\tau \in \Sigma'$, the intersection $\tau \cap U$ is starshaped with center x in $\mathbb{R}^N$.

1.6 Bergman fans of matroids and linear tropical subspaces of $\mathbb{T}^N$

We will mainly work with smooth tropical spaces. These are modelled locally on Bergman fans of matroids which we will define here.

Definition 1.21.
1. A *matroid* is a finite set M together with a rank function $r : \mathcal{P}(M) \to \mathbb{N}$, defined on the power set $\mathcal{P}(M)$ of M, satisfying the following properties:
   - For $A, B \subset M$ we have
   $$r(A \cup B) + r(A \cap B) \leq r(A) + r(B).$$
   - Every $A \subset M$ satisfies $r(A) \leq |A|$.
   - For $A \subset B \subset M$ we have $r(A) \leq r(B)$.

2. Let M be a matroid and $A \subset M$ a subset. Then A is *independent* if $r(A) = |A|$ holds. Otherwise A is called *dependent*. An independent subset $B \subset M$ with $r(B) = r(M)$ is called a *basis* for M.

3. A *flat* of a matroid M is a subset $F \subset M$ which is maximal with rank $r(F)$; i.e. $F \subset G$ and $r(F) = r(G)$ implies $G = F$.

4. A *loop* of a matroid M is a subset $A \subset M$ with $r(A) = 0$. If $\emptyset$ is the only loop of M, M is called *loopless*.

5. A *coloop* of a matroid M is a subset $C \subset M$ with $C \subset B$ for every basis B for M.

Deletion and *restriction* are two constructions to obtain new matroids from a given one; they play a crucial role in Proposition [Sha13, 2.25] which is central to the proof of Poincaré duality for tropical manifolds in [JSS15, 4.21ff].

Definition 1.22. Let M be a matroid, $S \subset M$ a subset and $T = M \setminus S$ its complement. We define two different matroids on the base set $S = M \setminus T$:

1. The *restriction* of M to S, written $M|S$, is the matroid on the set S whose independent sets are the independent sets of M that are contained in S. Equivalently, its rank function is that of M restricted to subsets of S. We call $M \setminus T := M|S$ the *deletion* of T from M. If $T = \{i\}$ consists of a single element $i \in M$, we also write $M \setminus i = M \setminus \{i\}$.

2. If T is a subset of M, the *contraction* of M by T, written M/T, is the matroid $(M \setminus T, r')$ whose rank function is given by

$$r'(A) = r(A \cup T) - r(T).$$

   Once again, if $T = \{i\}$ consists of a single element, we write M/i for brevity.

Bergman fans of loopless matroids will form the basic building blocks for *smooth* tropical spaces. They are constructed as follows:

Definition 1.23. Let M be a loopless matroid with rank function r. For $m := |M|$ let $B = \{e_1, \dots, e_m\} \subset \mathbb{Z}^{m-1}$ be a set of integral vectors such that $\sum_{j \in M} e_j = 0$ holds and such that every proper subset of B is a basis of $\mathbb{Z}^{m-1}$.

1. For every flat $F \subset M$, we denote by e_F the integral vector

$$e_F := \sum_{j \in F} e_j \in \mathbb{Z}^{m-1}.$$

2. A *flag* of flats in M is a sequence

$$\mathfrak{F} : F_1 \subset \cdots \subset F_k$$

   with $F_i \neq F_{i+1}$, $1 \leq i \leq k-1$.

3. Let $\mathfrak{F}$ be a flag of flats in M. The *cone associated to* $\mathfrak{F}$ is the cone $\sigma_{\mathfrak{F}}$ generated by the vectors e_F, where F runs through the flats in $\mathfrak{F}$.

4. The *Bergman fan* of M (associated to B) is the $(r(M)-1)$-dimensional fan $\Sigma(M) := \Sigma_B(M)$ in $\mathbb{R}^{m-1}$ whose cones are precisely the cones associated to flags of flats in M.

Remark 1.24. The Bergman fan of a loopless matroid M is clearly a rational polyhedral complex in $\mathbb{R}^{m-1}$. When equipped with the constant weight function 1, it becomes a balanced weighted polyhedral complex.

We adopt the following naming convention from [Sha15, 2.5]:

Definition 1.25. 1. A k-dimensional *fan tropical linear space* $L \subset \mathbb{R}^N$ is a tropical space in $\mathbb{R}^N$ given by the Bergman fan $\Sigma_B(M)$ for some $\mathbb{Z}^N$-basis B and a matroid M of rank $k+1$, equipped with weight 1 on all of its maximal polyhedra.

2. A k-dimensional *fan tropical linear space* $L \subset \mathbb{T}^N$ is a tropical space in $\mathbb{T}^N$ given by the Bergman fan $\Sigma_B(M)$ for the $\mathbb{Z}^N$-basis $B = \{-e_1, \cdots, -e_N, \sum_{i=1}^N e_i\}$ and a matroid M of rank $k+1$, equipped with weight 1 on all of its maximal polyhedra (in particular, it is the closure in $\mathbb{T}^N$ of a k-dimensional fan tropical linear space in $\mathbb{R}^N$).

1.7 Smooth tropical varieties

Definition 1.26. Let Σ be a regular polyhedral complex of pure dimension n in $\mathbb{T}^N$ and $\sigma \in \Sigma$ a polyhedron of sedentarity $\emptyset$. Let $x \in \mathrm{relint}(\sigma)$ be a point in the relative interior of σ and consider the *tangent cone*

$$T_x X := \{v \in \mathbb{R}^N; x + \epsilon v \in X \quad 0 < \epsilon \ll 1\}.$$

We call $F_\sigma := T_x X / \mathbb{L}(\sigma)$ the *relative fan* of σ. It is a polyhedral fan of dimension $n - \dim(\sigma)$ in $\mathbb{R}^N / \mathbb{L}(\sigma)$.

Definition 1.27. Let $X \subset \mathbb{T}^N$ be the support of a regular polyhedral complex Σ. Then X is called *smooth at* a mobile face $\sigma \in \Sigma$ if the relative fan F_σ has the same support as the Bergman fan $\Sigma(M)$ for some loopless matroid M. If X is smooth at every mobile face of Σ then $(\Sigma, 1)$ is balanced and we call $(X, \Sigma, 1)$ a *smooth* affine tropical variety.

1.8 Sedentarities of tropical spaces

Both lemma 1.5 and lemma 1.9 do not generalize immediately to arbitrary polyhedral spaces X. We will usually restrict ourselves to cases where they do. First one needs an appropriate replacement for taking the intersection with some $\mathbb{T}^N_I$ in the affine case, which will be accomplished by the notion of a *sedentarity* S in X:

Definition 1.28. A *sedentarity* of a (tropical) polyhedral space X is the closure $S = \overline{S'}$ of a connected subset $S' \subset X$ such that, for some (tropical) atlas $\mathfrak{A}$ of X and for every chart $\phi_U : U \to V_U \subset \mathbb{T}^N$ in $\mathfrak{A}$, the intersection $\phi_U(S' \cap U)$ is either empty or equal to the intersection $V_U \cap \mathbb{R}^N_I$ for some $I \subset [N]$. Setting $S \prec T$ for two sedentarities S, T with $S \subset T$, we make the set of sedentarities of X into a poset.

We will frequently require sedentarities to fulfill the following splitting property:

Definition 1.29. Let X be a (tropical) polyhedral space.

1. A sedentarity $S \subset X$ is *good*, if there exists an open neighbourhood $S \subset U$ of S in X such that there is a commuting diagram of morphisms

$$\begin{array}{ccc} S & \xrightarrow{\mathrm{id}} & U \\ & \searrow^{i} & \downarrow j \\ & & S \times \mathbb{T}^d, \end{array}$$

   where j is an open embedding and $i : S \to S \times \mathbb{T}^d$ is the map $s \mapsto (s, -\infty, \ldots, -\infty)$.

2. If all sedentarities of X (of codimension d) are good, X is said to have *good sedentarities (in codimension d)*.

1.9 Examples of tropical spaces

Let us look at two instructive examples.

Example 1.30 (Tropical projective space)**.**

As a set, we define N-dimensional (tropical) projective space by

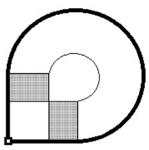
$$\mathbb{P}^N := \mathbb{P}^N_{\mathbb{T}} := \left(\mathbb{T}^{N+1} \setminus \{(-\infty, \ldots, -\infty)\}\right) / \sim,$$

where $(t_0, \dots, t_N)$ and $(s_0, \dots, s_N)$ are considered equivalent if there exists $a \in \mathbb{R}$ with $t_i = a + s_i$ for every $0 \leq i \leq N$. We write $[s_j]_j$ for the equivalence class of $(s_j)_j$.

For $0 \leq i \leq N$ fixed we define $U_i := \{[s_j]_j \in \mathbb{P}^N; s_i \neq -\infty\}$ and bijections

$$\varphi_i : U_i \to \mathbb{T}^N, \qquad [s_j]_j \mapsto (s_j - s_i)_{j \neq i}.$$

This makes $\mathbb{P}^N$ into a N-dimensional compact tropical manifold.



The complements Z_i of the charts $U_i \cong \mathbb{T}^N$ are isomorphic to $\mathbb{P}^{N-1}$ and they are precisely the closed $N-1$-dimensional closed sedentarities of $\mathbb{P}^N$. For every $0 \leq j \leq N$, the intersection $Z_i \cap U_j \subset U_j$ corresponds to $\mathbb{R}^N_i \subset \mathbb{T}^N$ via the isomorphism $U_i \to \mathbb{T}^N$. One can see from this that $\mathbb{P}^N$ has good sedentarities of dimension $N-1$. Inductively it follows that $\mathbb{P}^N$ has good sedentarities.

Example 1.31 (The tropical eye)**.**

The 'tropical eye' depiceted above has a bad sedentarity: Let X be given by charts $\phi_1 : U_1 \to V_1$, $\phi_2 : U_2 \to V_2$ with

$$\begin{aligned} V_1 &: = \{(x,y) \in \mathbb{T}^2; x < 0 \text{ and } y < -1\} \cup \{(x,y) \in \mathbb{T}^2; y < 0 \text{ and } x < -1\}, \\ V_2 &: = \{(x,y) \in \mathbb{T}^2; -1 < x < 1 \text{ and } y < -1\}; \end{aligned}$$

$$\begin{aligned} \phi_1(U_1 \cap U_2) &= \{-1 < x < 0, y < -1\} \sqcup \{0 < y < 1, x < -1\}, \\ \phi_2(U_1 \cap U_2) &= \{-1 < x < 0, y < -1\} \sqcup \{0 < x < 1, y < -1\}; \\ \phi_1 \circ \phi_2^{-1}(x,y) &= \begin{cases} (x,y), & x < 0, \\ (y,-x), & x > 0. \end{cases} \end{aligned}$$

Note that X has exactly three sedentarities $S_0 \prec S_1 \prec S_2$, where S_0 is a single point, S_1 is homeomorphic to $\mathbb{S}^1$ and S_2 is homeomorphic to an annulus in $\mathbb{R}^2$. The sedentarities S_0 and S_2 are good, while the sedentarity S_1 of dimension 1 is a bad sedentarity.

1.10 Constructible sheaves on tropical spaces

Let $X \subset \mathbb{T}^N$ be the support of a polyhedral complex Σ in $\mathbb{T}^N$. Topologically, after a suitable refinement of Σ (possibly allowing countably many pieces), we may think of Σ as a simplicial complex and $X = |\Sigma|$ its topological realization (c.f. [KS90, 8.1]). This way, we can transfer the definitions of constructible sheaves from [KS90, 8.1.3] to X, retaining their properties. Instead of giving the somewhat cumbersome proofs we will refer to the corresponding statements from [KS90] from which they can be deduced.

Once again, let R be either $\mathbb{Z}$ or $\mathbb{R}$. We denote by $\mathrm{Shv}(X,R)$ the category of sheaves of R-modules on X and by $\mathrm{D}^b(X,R)$ its bounded derived category (see appendix A.1).

Definition 1.32. Let $\mathcal{F}^\bullet$ in $\mathrm{D}^b(X,R)$, the derived category of sheaves of R-modules on X.

1. We call $\mathcal{F}^\bullet$ *weakly constructible* (with respect to Σ), if the cohomology sheaves $\mathrm{H}^k(\mathcal{F}^\bullet)|_{\mathrm{relint}(\sigma)}$ are constant for every $k \in \mathbb{Z}$ and $\sigma \in \Sigma$.
2. If $\mathcal{F}^\bullet$ is weakly constructible and moreover $\mathrm{H}^k(\mathcal{F}^\bullet_x)$ is finitely generated for every $x \in X$ and $k \in \mathbb{Z}$, then we call $\mathcal{F}^\bullet$ *constructible* (with respect to Σ).

A sheaf $\mathcal{F}$ on X is *(weakly) constructible* if it is so as an object in $\mathrm{D}^b(X, R)$.

Proposition 1.33. *Let $\mathcal{F}$ be a weakly constructible sheaf on X. Then for every $\sigma \in \Sigma$ and $x \in \operatorname{relint}(\sigma)$, we have isomorphisms*

1. $\mathrm{H}^0(U_\sigma, \mathcal{F}) \cong \mathrm{H}^0(\operatorname{relint}(\sigma), \mathcal{F}|_{\operatorname{relint}(\sigma)}) \cong \mathcal{F}_x$,
2. $\mathrm{H}^k(U_\sigma, \mathcal{F}) = \mathrm{H}^k(\operatorname{relint}(\sigma), \mathcal{F}|_{\operatorname{relint}(\sigma)}) = 0$ *for* $k \neq 0$.

Proof. This follows from [KS90, 8.1.4]. □

Remark 1.34. In particular, this applies to starshaped open subsets of polyhedral spaces: Say $\phi : U \to V \subset \mathbb{T}^N$ is a starshaped chart of a polyhedral space X with center $x \in U$, where V is the open star of $\sigma \in \Sigma$ for a polyhedral complex Σ in $\mathbb{T}^N$. Assume that $\mathcal{F}$ is a sheaf on X such that $\mathcal{F}|_U = \phi^*(\mathcal{F}')$ with a weakly Σ-constructible sheaf $\mathcal{F}'$ on $|\Sigma| \subset \mathbb{T}^N$. Then the natural maps

$$\Gamma(U, \mathcal{F}) \to \mathcal{F}_x, \qquad \Gamma(U, \mathcal{F}) \to \mathrm{R}\,\Gamma(U, \mathcal{F})$$

are isomorphisms in Mod_R and $\mathrm{D}(\mathrm{Mod}_R)$ respectively.

Proposition 1.35. *Let* $\mathrm{Shv}_\Sigma(X) = \mathrm{Shv}_\Sigma(X, R)$ *be the full (abelian) subcategory of* $\mathrm{Shv}(X, R)$ *consisting of constructible sheaves, and let* $\mathrm{D}^b_\Sigma(X) = \mathrm{D}^b_\Sigma(X, R)$ *be the full triangulated subcategory of* $\mathrm{D}^b(X)$ *consisting of constructible objects (both with respect to* Σ*).*

Then the natural functor

$$\mathrm{D}^b(\mathrm{Shv}_\Sigma(X)) \to \mathrm{D}^b_\Sigma(X)$$

is an equivalence of categories.

Proof. This is due to [KS90, 8.1.11]. □

Proposition 1.36. *Let X be the support of a polyhedral complex Σ in $\mathbb{T}^N$ and let $U \subset X$ be a relatively compact open subset. Let $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$ be constructible. Then $\mathrm{R}^k\,\Gamma(U, \mathcal{F}^\bullet)$ and $\mathrm{R}^k\,\Gamma_c(U, \mathcal{F}^\bullet)$ are finitely generated R-modules.*

Proof. This follows from [KS90, 8.4.11]. □

Proposition 1.37. *For $\sigma \in \Sigma$, let $\imath_\sigma : \operatorname{relint}(\sigma) \to X = |\Sigma|$ be the canonical embedding and let M be a finitely generated R-module. Then the sheaf $M_\sigma := (\imath_\sigma)_* M$ is constructible on X. Moreover, every sheaf $\mathcal{F}$ in* $\mathrm{Shv}_\Sigma(X, R)$ *can be embedded in a finite product of such sheaves. For $R = \mathbb{R}$, the sheaves $\mathbb{R}_\sigma$ are injective in* $\mathrm{Shv}_\Sigma(X, \mathbb{R})$.

Proof. It is clear that M_σ is constructible. If $\mathcal{F}$ in $\mathrm{Shv}_\Sigma(X, R)$ is constructible, every sheaf $\mathcal{F}|_{\operatorname{relint}(\sigma)}$ is finitely generated and constant, i.e. we find M in Mod_R finitely generated with

$$(\imath_\sigma)_* \imath_\sigma^{-1} \mathcal{F} = M_\sigma.$$

From the adjunction $(\imath_\sigma^{-1}, (\imath_\sigma)_*)$ we get canonical morphisms $\mathcal{F} \to (\imath_\sigma)_* \imath_\sigma^{-1} \mathcal{F}$. These are isomorphisms on stalks in $x \in \operatorname{relint}(\sigma)$. Taking the product over $\sigma \in \Sigma$ gives us a monomorphism

$$\mathcal{F} \hookrightarrow \prod_{\sigma \in \Sigma} M_\sigma,$$

as required.

By the adjunction $(\imath_\sigma^{-1}, (\imath_\sigma)_*)$ and the definition of constructible sheaves, we have a canonical isomorphisms

$$\mathrm{Hom}_X(\mathcal{F}, \mathbb{R}_\sigma) \cong \mathrm{Hom}_X(\imath_\sigma^{-1} \mathcal{F}, \mathbb{R}) = \mathrm{Hom}_R(\mathcal{F}_x, \mathbb{R})$$

for every $\mathcal{F} \in \mathrm{Shv}_\Sigma(X)$ and each $\sigma \in \Sigma$, $x \in \operatorname{relint}(\sigma)$. This is an exact functor, as required for the last statement. □

1.11 Tropical homology and cohomology

In [MZ13], tropical homology and cohomology groups on a tropical space X are introduced via singular (co)chain complexes with coefficients. In [MZ13, Sect. 2.4], they give an equivalent definition using the language of cosheaves and sheaves on X; this latter description – as detailed below – will be the most useful for us. The particular cosheaves $\mathcal{F}_p$ and sheaves $\mathcal{F}^p$ used by Mikhalkin and Zharkov are constructed using a 'canonical' stratification of X. We will – in order to keep the notation simple – work around this by using a starshaped open covering of X instead. Since we will not pursue cosheaves on topological spaces further after this section, we just refer to [Bre97] and [Bre68] as entry points to this particular theory.

But first, let us start with the combinatorical situation of a rational polyhedral complex Σ in $\mathbb{T}^N$. In this case, we can define the (co)sheaves $\mathcal{F}_p$ and $\mathcal{F}^p$ as (co)sheaves on the poset Σ (c.f. appendix B for the basic definitions on sheaves on cosheaves on posets).

Definition 1.38. Let Σ be a rational regular polyhedral complex in $\mathbb{T}^N$ and let X be its support.

1. We define the cosheaves $\mathcal{F}_p := \mathcal{F}_p^{\mathbb{R}} \in \operatorname{CoShv}(\Sigma, \mathbb{R})$ und $\mathcal{F}_p^{\mathbb{Z}} \in \operatorname{CoShv}(\Sigma, \mathbb{Z})$ on Σ by
$$\begin{aligned} \mathcal{F}_p^R : \Sigma^{op} &\to \operatorname{Mod}_R, \\ \sigma &\mapsto \sum_{\sigma \prec \tau \in \Sigma_{\operatorname{sed}(\sigma)}} \bigwedge^p \mathbb{L}_R(\tau), \end{aligned}$$
$(R \in \{\mathbb{Z}, \mathbb{R}\})$. If $\sigma \prec \tau$ is a pair of polyhedra of the same sedentarity, then
$$\mathcal{F}_p^R(\tau) \to \mathcal{F}_p^R(\sigma)$$
is the embedding map. If $\sigma = \tau \cap \mathbb{T}^N_{\operatorname{sed}(\sigma)}$, then
$$\mathcal{F}_p^R(\tau) \to \mathcal{F}_p^R(\sigma)$$
is given by the projection map $\mathbb{R}^N_{\operatorname{sed}(\tau)} \to \mathbb{R}^N_{\operatorname{sed}(\sigma)}$. All other corestriction maps are determined by functoriality.

2. Dually, we define the sheaves $\mathcal{F}^p := \mathcal{F}^p_{\mathbb{R}} \in \operatorname{Shv}(\Sigma, \mathbb{R})$ and $\mathcal{F}^p_{\mathbb{Z}} \in \operatorname{Shv}(\Sigma, \mathbb{Z})$ on Σ by
$$\begin{aligned} \mathcal{F}^p_R : \Sigma &\to \operatorname{Mod}_R, \\ \sigma &\mapsto \operatorname{Hom}_R(\mathcal{F}_p^R(\sigma), R), \end{aligned}$$
$(R \in \{\mathbb{Z}, \mathbb{R}\})$, with obvious restriction maps.

Remark 1.39. When we equip the poset Σ with its Alexandrov topology (see appendix B.1), the map $\Phi : X \to \Sigma$ determined by $x \in \operatorname{relint}(\Phi(x))$ is continuous. The (co)sheaves $\mathcal{F}_p^R$ and $\mathcal{F}^p_R$ on the poset Σ correspond uniquely to (co)sheaves on the topological space Σ. This allows us to consider the pullbacks to X via Φ of these (co)sheaves, which we will later again denote by $\mathcal{F}_p^R$ and $\mathcal{F}^p_R$. See also remark 1.42.

Proceeding to an arbitrary tropical space X, we now need a good grasp on the local description of X. Here the starshaped charts and starshaped open subsets from definition 1.19 come in handy. For the following recall the definition of constructible sheaves from section 1.10.

Construction 1.40. Let now X be a regular tropical space with an atlas $\mathfrak{A}$ consisting of starshaped tropical charts. Following the recipe of [MZ13, Sect. 2.4], we will define certain constructible sheaves $\mathcal{F}^p$ and cosheaves $\mathcal{F}_p$ on X, starting on charts in $\mathfrak{A}$:

Fix a starshaped chart $\phi : U \to V \subset \mathbb{T}^N$ together with a polyhedral complex Σ in $\mathbb{T}^N$ as in definition 1.19. We furthermore assume that Σ is maximal with respect to refinement. If $U' \subset U$ is another open subset, we can consider the poset $\Sigma_{U'}$ of connected components of $\phi^{-1}(\mathrm{relint}(\tau)) \cap U'$ with $\tau \in \Sigma$, ordered by adjacency. Let $\tilde{\tau} \in \Sigma_{U'}$ be a connected component of $\phi^{-1}(\mathrm{relint}(\tau)) \cap U'$, $\tau \in \Sigma$; we then can set

$$\mathcal{F}_p^R(\tilde{\tau}) := \mathcal{F}_p^R(\tau),$$
$$\mathcal{F}_R^p(\tilde{\tau}) := \mathcal{F}_R^p(\tau),$$

and, for $\tilde{\sigma} \prec \tilde{\tau}$ in $\Sigma_{U'}$, we get the obvious transition maps from the (co)sheaves $\mathcal{F}_p^R$ and $\mathcal{F}_R^p$ on Σ. As in [MZ13, Def. 2.6] we can then define

$$\mathcal{F}_p^R(U') := \mathrm{colim}_{\tilde{\tau} \in \Sigma_{U'}} \mathcal{F}_p^R(\tilde{\tau}),$$
$$\mathcal{F}_R^p(U') := \lim_{\tilde{\tau} \in \Sigma_{U'}} \mathcal{F}_p^R(\tilde{\tau}).$$

As in [MZ13] one shows that this defines (co)sheaves $\mathcal{F}_p^R$ and $\mathcal{F}_R^p$ on each such starshaped open subset $U \subset X$ – and that we can glue them to obtain cosheaves $\mathcal{F}_p^R$ and constructible sheaves $\mathcal{F}_R^p$ on X.

As noted before, the sheaves and cosheaves considered in [MZ13] arise from considering the stratification of U' obtained from the *canonical* stratification of X (which we do not define in this paper; c.f. [MZ13, Def. 1.12]). However, it is easy to see that on a starshaped open subset U the stratification induced by the canonical stratification agrees with the stratification considered here. In [MZ13, Prop. 2.7] Mikhalkin and Zharkov show that their construction does not depend on the atlas chosen for X. This shows that the cosheaves $\mathcal{F}_p^{\mathbb{Z}}$ and sheaves $\mathcal{F}_{\mathbb{Z}}^p$ constructed here agree with the cosheaves $\mathcal{F}_p$ and sheaves $\mathcal{F}^p$ constructed in [MZ13]. By [MZ13, Prop. 2.8], this allows us to define tropical (co)homology as follows:

Definition 1.41. *Tropical homology groups* and *tropical cohomology groups* (with integral coefficients) of a regular tropical space X are defined as cosheaf homology and sheaf cohomology groups

$$\mathrm{H}_{p,q}^{\mathrm{trop}}(X) := \mathrm{H}_q(X, \mathcal{F}_p^{\mathbb{Z}}), \qquad \mathrm{H}^{p,q}_{\mathrm{trop}}(X) := \mathrm{H}^q(X, \mathcal{F}_{\mathbb{Z}}^p).$$

Remark 1.42. Both in definition 1.38 and in construction 1.40, sheaves and cosheaves on a poset Σ (or $\Sigma_{U'}$) play a crucial role. In proposition B.5 we recall that the categories of (co)sheaves on the poset Σ are equivalent to the categories of (co)sheaves on the topological space $|\Sigma|$, equipped with the Alexandrov topology.

- In the notation of construction 1.40, we have a canonical continuous map $\Phi_U : U \to \Sigma_U$, defined by $x \in \Phi(x)$. One then can show, that the (co)sheaves $\mathcal{F}_p^A$ and $\mathcal{F}_A^p$ are in fact the pullbacks of the corresponding (co)sheaves on the poset Σ_U.
- If X is the support of a rational polyhedral complex Σ, we also get a continuous map $\Phi : X \to \Sigma$, defined by $x \in \mathrm{relint}\, \Phi(x)$ for $x \in X$. In this case one can show that the (co)sheaves $\mathcal{F}_p^A$ and $\mathcal{F}_A^p$ on X from construction 1.40 are canonically isomorphic to the pullbacks via Φ of the corresponding (co)sheaves on the poset Σ.
- One can extend these considerations to arbitrary tropical spaces by using the poset induced by the canonical stratification on X (as defined in [MZ13, Def. 1.12]).

This is useful because often, derived functors on the categories of (co)sheaves on the topological space X, constructible with respect to a certain stratification, can be computed using corresponding derived functors on the categories of (co)sheaves on the poset of strata of X – which are often much more easily understood.

1.12 Tropical modifications

We borrow the terms and definitions regarding tropical modifications from [Sha15, sect. 2.5]; c.f also [Sha13, sect. 2.3] for details.

Definition 1.43. Let U be a connected open subset of $\mathbb{T}^N$ and let $S = \mathrm{Sed}(U) = \bigcup_{x \in U} \mathrm{sed}(x)$. A *tropical regular function* $f : U \to \mathbb{T}$ is a tropical Laurent polynomial

$$f(x) = \max\{r_\alpha + \alpha \cdot x; \alpha \in \Delta\}$$

with $r_\alpha \in \mathbb{R}$ for $\alpha \in \Delta$, where $\emptyset \neq \Delta \subset \mathbb{Z}^N$ is a finite set such that $\alpha_i \geq 0$ for all $i \in S$ and $\alpha \in \Delta$.

Remark 1.44. It is clear that every tropical regular function is a piecewise affine convex function with integral slopes and that its graph is a finite polyhedral complex in $\mathbb{T}^{N+1}$. The represenation of a tropical regular function as a Laurent polynomial is *not* unique, as can be seen in the example

$$f(x) = \max\{0, x, 2x\} = \max\{0, 2x\}$$

on $U = \mathbb{T}$.

Construction 1.45. Let X be a purely n-dimensional affine tropical variety in $\mathbb{T}^N$ and consider a tropical regular function $f : \mathbb{T}^N \to \mathbb{T}$. Then its *graph*

$$\Gamma_f(X) := \{(x, f(x)); x \in X\} \subset \mathbb{T}^N \times \mathbb{T}$$

is the support of a rational polyhedral complex of dimension n and it inherits weights from the maximal polyhedra of some weighted polyhedral complex representing X. However, since f is only piecewise linear, $\Gamma_f(X)$ may not be balanced. To repair this, we attach on each $n-1$-dimensional polyhedron σ of $\Gamma_f(X)$ which fails the balancing condition, the n-dimensional polyhedron

$$\mu_\sigma := \{x - te_{n+1}; x \in \sigma, t \in [0, -\infty]\},$$

equipping it with the appropriate positive integral weight to enforce the balancing condition in σ.

Definition 1.46 (Tropical Modifications)**.** Consider X, f and $\Gamma_f(X)$ as in the construction above.

1. The *elementary tropical modification* of X with respect to f is the polyhedral subspace
$$\tilde{X} = \Gamma_f(X) \cup \bigcup_\sigma \mu_\sigma$$
of $\mathbb{T}^N \times \mathbb{T}$, together with the canonical projection map $\delta : \tilde{X} \to X$. When equipped with the weights described above, $\tilde{X}$ becomes an affine tropical subspace of $\mathbb{T}^N \times \mathbb{T}$ and $\delta : \tilde{X} \to X$ is a morphism of affine tropical spaces

2. We call the union $\mathfrak{U}_f(X) = \bigcup_\sigma \mu_\sigma$ of all such σ the *undergraph* of the elementary tropical modification δ.

3. The *divisor* of the elementary tropical modification δ is the subset
$$\mathrm{div}_X(f) = \delta(\mathfrak{U}_f(X)) \cup f^{-1}(-\infty)$$
of X. Assume for simplicity that $f^{-1}(-\infty) \cap X = \emptyset$; then, when equipped with the weights inherited from $\mathfrak{U}_f(X)$, the divisor is a $n-1$-dimensional tropical subspace of X (see [BIMS15, 5.27] for the general case).

4. Let $X' := \tilde{X} \cap \left(\mathbb{T}^N \times \mathbb{R}\right)$. Then the restriction

$$\delta|_{X'} : X' \to X$$

of δ to X' is called the *open elementary tropical modification* of X with respect to f.

Definition 1.47. We say that an elementary tropical modification $\delta : \tilde{X} \to X$ given by a tropical regular function f is *regular* if $f^{-1}(-\infty) \cap \operatorname{div}_X(f) = \emptyset$.

Definition 1.48. Suppose $L \subset \mathbb{T}^N$ is a fan tropical linear space (definition 1.25) and let f be a tropical rational function on $\mathbb{T}^N$ such that $\operatorname{div}_L(f)$ is also a fan tropical linear space in $\mathbb{T}^N$. Then the elementary tropical modification $\delta : \tilde{L} \to L$ along f is said to be a degree 1 modification of $L \subset \mathbb{T}^N$.

Definition 1.49. Let $\tilde{X}$ and X be a pair of tropical manifolds and let $\delta : \tilde{X} \to X$ be a morphism of tropical spaces.

1. The morphism δ is a *elementary tropical modification* if there exist atlases $\tilde{\mathfrak{A}}$ for $\tilde{X}$ and $\mathfrak{A}$ for X and for every $\tilde{x}$ in $\tilde{X}$ there are charts $\tilde{U}$ in $\tilde{\mathfrak{A}}$ around $\tilde{x}$ and U in $\mathfrak{A}$ around $\delta(\tilde{x})$ such that
$$\delta : \tilde{U} \to U$$
is an affine elementary tropical modification of degree 1.

2. The morphism δ is called a *tropical modification* if it is a finite composition of elementary tropical modifications.

Remark 1.50. The proof of Poincaré duality in [JSS15, ch.4] relies heavily on properties of tropical modifications – the main result being [Sha13, Prop. 2.25], which implies that n-dimensional Bergman fans (which form the basic building blocks for smooth tropical spaces; c.f. section 1.6 and section 1.7) can be contracted to $\mathbb{R}^n$ in a finite number of tropical modifications. Another important property is that tropical cohomology is invariant under tropical modifications; this will be discussed in section 1.13 below.

1.13 Tropical modifications and cohomology

Next we give a different version of the comparison result for tropical cohomology along tropical modifications as in [JSS15, 4.22]. The crucial lemma lemma 1.52 might have applications beyond the scope of this thesis (for instance, sheaves locally isomorphic to $\mathcal{F}^p_X$ do satisfy the preconditions of the lemma). However, in contrast to the approach chosen in [JSS15], here the compatibility with Poincaré duality does not become obvious.

Let us first recall some of the notation from section 1.12. We consider a regular tropical modification $\delta : \tilde{X} \to X$ of $X \subset \mathbb{T}^N$ with respect to some regular tropical function $f : x \mapsto \max\{\nu \cdot x + a_\nu; \nu \in \Delta\}$ on $\mathbb{T}^r$. Then $\tilde{X}$ is the disjoint union of the graph $\Gamma_f(X) \subset \mathbb{T}^{N+1}$, the (open) undergraph $\mathfrak{U}(f)^\circ := \mathfrak{U}(f) \cap \left(\mathbb{T}^N \times \mathbb{R}\right)$ and the divisor of the modification, $D \subset X$. We write $\mathfrak{U}(f)$ for the preimage of D under δ, the (closed) undergraph. If we embed X into $\mathbb{T}^{N+1}$ as a subset of $\mathbb{T}^{N+1}_{N+1} \cong \mathbb{T}^N$, we may assume that $\mathfrak{U}(f)^\circ$ is an open subset of the preimage of D under the canonical projection $\mathbb{T}^{N+1} \to \mathbb{T}^{N+1}_{N+1}$, i.e. of $D \times \mathbb{T}$. We then have the following diagram of topological spaces (*),

$$\begin{array}{ccccc}
\tilde{U} & \xrightarrow{\tilde{\jmath}} & \tilde{X} & \xleftarrow[\tilde{\imath}]{} & \mathfrak{U}(f) \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\
U & \xrightarrow{\jmath} & X & \xleftarrow[\imath]{} & D.
\end{array}$$

where $\tilde{U}$ (resp. U) is the open complement of $\mathfrak{U}(f) \subset \tilde{X}$ (resp. $D \subset X$). We will later use the fact that both squares are cartesian.

Let X be an open subset of the support of a weighted polyhedral complex Σ in $\mathbb{T}^N$ and $\tilde{X}$ by a weighted polyhedral complex $\tilde{\Sigma}$ in $\mathbb{T}^{N+1}$. We may assume that for a face $\sigma \subset D$, the preimage $\delta^{-1}(\mathrm{relint}(\sigma))$ consists of (the relative interiors of) exactly three non-empty faces $\sigma_0 = \sigma$, $\sigma_u = \delta^{-1}(\mathrm{relint}(\sigma)) \cup \mathfrak{U}(f)^\circ \subset \sigma \times \mathbb{R}\, e_{N+1}$ and $\sigma^f = \delta^{-1}(\mathrm{relint}(\sigma)) \cap \Gamma_f(X)$ in $\tilde{\Sigma}$.

The following proposition lets us compare cohomologies along tropical modifications:

Proposition 1.51. *Let $\delta : \tilde{X} \to X$ be a regular tropical modification of $X \subset \mathbb{T}^N$ as above and let $\mathcal{F}$ be a constructible sheaf on $\tilde{X}$ with respect to some completed polyhedral complex $\tilde{\Sigma}$ in $\mathbb{T}^{N+1}$ representing $\tilde{X}$ as above. Assume that $\mathcal{F}(\sigma^f) \to \mathcal{F}(\sigma_u)$ is an epimorphism for every face $\sigma \subset D$ in $\tilde{\Sigma}$. Then the canonical morphism*

$$\delta_* \mathcal{F} \to \mathrm{R}\, \delta_* \mathcal{F}$$

is a quasi-isomorphism. In particular,

$$\mathrm{R}^q\, \delta_*(\mathcal{F}) = 0$$

for $q > 0$.

This follows immediately from the following Lemma:

For simplicity, from now on we write $\mathcal{G}(\tau) := \mathcal{G}(U_\tau)$ for sheaves $\mathcal{G}$ constructible with respect to some polyhedral complex Σ representing X, and $\tau \in \mathcal{C}$ with open star $U_\tau \subset X$.

Lemma 1.52. *Let $\delta : \tilde{X} \to X$, $\tilde{\Sigma}$ and Σ be as above. Consider the class $\mathcal{I}$ of sheaves $\mathcal{F}$ constructible with respect to $\tilde{\Sigma}$ such that $\mathcal{F}(\sigma^f) \to \mathcal{F}(\sigma_u)$ is an epimorphism for every face $\sigma \subset D$ in Σ. Then for every short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ with $\mathcal{F}'$ in $\mathcal{I}$, the sequence*

$$0 \to \delta_* \mathcal{F}' \to \delta_* \mathcal{F} \to \delta_* \mathcal{F}'' \to 0$$

is exact. Also, the canonical morphism

$$\delta_* \mathcal{F} \to \mathrm{R}\, \delta_* \mathcal{F}$$

in $\mathrm{D}(X)$ *is an isomorphism.*

Proof. Note that δ is a proper morphism of Hausdorff spaces where every open subset is paracompact. Hence, we may apply the proper base change theorem and for every $q \geq 0$ and $x \in X$ we have a canonical isomorphism

$$(\mathrm{R}^q\, \delta_* \mathcal{F})_x \cong \mathrm{H}^q(\delta^{-1}(x), \mathcal{F}|_{\delta^{-1}(x)}).$$

We have two distinct cases: For $x \in U$, $\delta^{-1}(x)$ consists of a single point $\tilde{x} \in \Gamma_f(X)$, so for $q > 0$ we have $(\mathrm{R}^q\, \delta_* \mathcal{F})_x = 0$ immediately. This implies that the sequence is exact, when restricted to U. For $x \in D$, choose a $\sigma \in \Sigma$ with $x \in \mathrm{relint}(\sigma)$. It suffices to show that

$$0 \to \mathcal{F}'|_{\delta^{-1}(x)}(\delta^{-1}(x)) \to \mathcal{F}|_{\delta^{-1}(x)}(\delta^{-1}(x)) \to \mathcal{F}''|_{\delta^{-1}(x)}(\delta^{-1}(x)) \to 0$$

is exact. Consider the diagram of exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}'(\sigma^f) & \xrightarrow{\mu_1} & \mathcal{F}(\sigma^f) & \xrightarrow{\lambda_1} & \mathcal{F}''(\sigma^f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi''} & & \\
0 & \longrightarrow & \mathcal{F}'(\sigma_u) & \xrightarrow{\mu_u} & \mathcal{F}(\sigma_u) & \xrightarrow{\lambda_u} & \mathcal{F}''(\sigma_u) & \longrightarrow & 0 \\
 & & {\scriptstyle \rho'}\uparrow & & {\scriptstyle \rho}\uparrow & & {\scriptstyle \rho''}\uparrow & & \\
0 & \longrightarrow & \mathcal{F}'(\sigma_0) & \xrightarrow{\mu_0} & \mathcal{F}(\sigma_0) & \xrightarrow{\lambda_0} & \mathcal{F}''(\sigma_0) & \longrightarrow & 0.
\end{array}$$

Now let ζ be an element of $\mathcal{F}''(\delta^{-1}(x)) := \mathcal{F}''|_{\delta^{-1}(x)}(\delta^{-1}(x))$. We have cartesian diagrams

$$\begin{array}{ccc} \mathcal{F}''(\delta^{-1}(x)) & \longrightarrow & \mathcal{F}''(\sigma^f) \\ \downarrow & & \downarrow \pi'' \\ \mathcal{F}''(\sigma_0) & \xrightarrow{\rho''} & \mathcal{F}''(\sigma_u), \end{array}$$

and similarly for $\mathcal{F}'$ and $\mathcal{F}$. In particular, we can write

$$\zeta = (\zeta_0, \zeta_1) \in \mathcal{F}''(\sigma_0) \times \mathcal{F}''(\sigma^f)$$

with $\pi''(\zeta_1) = \rho''(\zeta_0)$. We may choose $\omega_0 \in \mathcal{F}(\sigma_0)$ and $\omega_1 \in \mathcal{F}(\sigma^f)$ with $\lambda_i \omega_i = \zeta_i$. By commutativity of above diagram we have $\pi(\omega_1) - \rho(\omega_0) \in \ker(\lambda_u)$ and hence we may choose $\eta_u \in \mathcal{F}'(\sigma_u)$ with $\mu_u(\eta_u) = \pi(\omega_1) - \rho(\omega_0)$. But η_u has a preimage $\eta_1 \in \mathcal{F}'(\sigma^f)$ by surjectivity of π' and we have

$$\pi(\omega_1 - \mu_1(\eta_1)) = \rho(\omega_0),$$

and hence $(\omega_0, \omega_1 - \mu_1(\eta_1)) \in \mathcal{F}(\delta^{-1}(x))$ is a preimage of $(\zeta_0, \zeta_1) = \zeta$. This shows, that the sequence is exact at every $x \in D$.

The category $\mathrm{Shv}_{\tilde{\Sigma}}(\tilde{X})$ of sheaves constructible with respect to the decomposition induced by $\tilde{\Sigma}$ contains a cogenerating system of injective sheaves which also belong to $\mathcal{I}$ (c.f. proposition 1.37). Then lemma 1.52 shows in particular that the class $\mathcal{I}$ is δ_*-acyclic in $\mathrm{Shv}_{\tilde{\Sigma}}(\tilde{X})$. Now proposition 1.35 allows us to compute $\mathrm{R}\,\delta_*(\mathcal{F})$ in the category $\mathrm{D}^b(\mathrm{Shv}_{\tilde{\Sigma}}(\tilde{X}))$ for a sheaf $\mathcal{F}$ constructible with respect to $\tilde{\Sigma}$ (note that the injective sheaves in $\mathrm{Shv}_{\tilde{\Sigma}}(\tilde{X})$ mentioned before are flabby and hence we may apply [KS90, 1.8.7] to the composition of functors $\mathrm{Shv}_{\tilde{\Sigma}}(\tilde{X}) \to \mathrm{Shv}(\tilde{X}) \to \mathrm{Shv}(X)$). Hence, if $\mathcal{F}$ belongs to $\mathcal{I}$, we get that the canonical morphism $\delta_* \mathcal{F} \to \mathrm{R}\,\delta_* \mathcal{F}$ in $\mathrm{D}(X)$ is an isomorphism. □

We now want to apply this to the sheaves $\mathcal{F}^p_X$ from section 1.11 which give rise to tropical cohomology groups. To compare sheaf cohomology, we first define an isomorphism of sheaves $\delta_* : \delta_* \mathcal{F}^p_{\tilde{X}} \to \mathcal{F}^p_X$ as explained in the following proof:

Lemma 1.53. *If $\delta : \tilde{X} \to X$ is a regular tropical modification, then δ induces an isomorphism*

$$\delta_* \mathcal{F}^p_{\tilde{X}} \xrightarrow{\sim} \mathcal{F}^p_X$$

of sheaves on X.

Proof. For every $x \in X$ we may choose an open neighbourhood $x \in U_x \subset \mathbb{T}^N$ with decomposition represented by a polyhedral complex Σ in $\mathbb{T}^N$ such that $\delta^{-1}(U_x)$ is a subset of $\mathbb{T}^N \times \mathbb{T}$ with polyhedral structure represented by $\tilde{\Sigma}$ in $\mathbb{T}^{N+1}$ and δ is given by the projection $\mathbb{R}^{N+1} \to \mathbb{R}^{N+1} / \mathbb{R}\, e_{N+1}$. We may assume that U_x is the open star with center x in Σ and that $\delta(\mathrm{relint}(\tilde{\sigma}))$ is the open interior of a polyhedron in Σ for every $\tilde{\sigma} \in \tilde{\Sigma}$ intersecting $\delta^{-1}(U_x)$.

First assume that $x \in X$ is finitary, i.e. $x \in \mathbb{R}^N$. We have two different cases: For $x \notin D$, $\delta : \delta^{-1}(U_x) \to U_x$ is a homeomorphism and the morphism $\delta_* \mathcal{F}^p_{\tilde{X}}(U_x) \to \mathcal{F}^p_X(U_x)$ induced by $\mathbb{R}^{N+1} \to \mathbb{R}^{N+1} / \mathbb{R}\, e_{N+1}$ is an isomorphism.

Now, let $x \in D$. Let $\tilde{x} \in \Gamma_f(X)$ and $x_u \in \mathfrak{U}(f)^\circ$ with $\delta(\tilde{x}) = \delta(x_u) = x$. We may assume that U_x satisfies

$$\delta^{-1}(U_x) = \tilde{U}(\tilde{x}) \cup \tilde{U}(x_u) \cup \tilde{U}(x),$$

where $\tilde{U}(\tilde{x})$, $\tilde{U}(x)$ and $\tilde{U}(x_u)$ are the open stars of $\tilde{x}$, x and of x_u in $\tilde{X}$ with respect to $\tilde{\Sigma}$. Using the explicit description in proposition 2.8, we then get a cartesian diagram

$$\begin{array}{ccc} \delta_* \mathcal{F}^p_{\tilde{X}}(U_x) & \longrightarrow & \mathcal{F}^p_{\tilde{X}}(\tilde{U}(\tilde{x})) \\ \downarrow & & \downarrow \iota^* \\ \mathcal{F}^p_{\tilde{X}}(\tilde{U}(x)) & \xrightarrow{\pi^*} & \mathcal{F}^p_{\tilde{X}}(\tilde{U}(x_u)), \end{array}$$

where ι^* is induced by the embedding

$$\iota : \sum_{x_u \in \tau \in \tilde{\Sigma}} \bigwedge^p \mathbb{L}(\tau) \to \sum_{\tilde{x} \in \tau} \bigwedge^p \mathbb{L}(\tau),$$

and π^* is induced by the projection

$$\pi : \sum_{x_u \in \tau \in \tilde{\Sigma}} \bigwedge^p \mathbb{L}(\tau) \to \sum_{x_u \in \tau} \bigwedge^p (\mathbb{L}(\tau) / \mathbb{R}\, e_{N+1}).$$

This way we get isomorphisms

$$\delta_* \mathcal{F}^p_{\tilde{X}}(U_x) \xrightarrow{\sim} \left(\sum_{\tilde{x} \in \tau} \bigwedge^p (\mathbb{L}(\tau) / \mathbb{R}\, e_{N+1}) \right)^*,$$

where the right hand side is canonically isomorphic to $\mathcal{F}^p_X(U_x)$.

Now, let $x \in U_x \subset \mathbb{T}^N$ as above, but assume $\emptyset \neq \operatorname{sed}(x) = I \subset [N]$. For $x \notin D$ we may assume $U_x \cap D = \emptyset$ and the restriction $\delta|_{\delta^{-1}(U_x)}$ is an ismorphism of weighted stratified spaces. Using proposition 2.8, one can show that similar to before the projection $\mathbb{R}^{N+1}_I \to \mathbb{R}^{N+1}_I / \mathbb{R}\, e_{N+1}$ induces an isomorphism $\delta_* \mathcal{F}^p_{\tilde{X}}(U_x) \to \mathcal{F}^p_X(U_x)$. If on the other hand $x \in D \cap \mathbb{T}^N_I$, then we may argue as before, replacing $\mathbb{L}_{\tilde{X}}(\tau)$ by $\pi_I(\mathbb{L}_{\tilde{X}}(\tau)) \subset \mathbb{R}^{N+1} / \sum_{i \in I} \mathbb{R}\, e_i$.

For varying $x \in X$ we get a covering by open subsets $U_x \subset X$ which are open stars of their respective decompositions and the isomorphisms $\delta_* \mathcal{F}^p_{\tilde{X}}(U_x) \to \mathcal{F}^p_X(U_x)$ glue to an isomorphism

$$\delta_* \mathcal{F}^p_{\tilde{X}} \xrightarrow{\sim} \mathcal{F}^p_X .$$

□

Corollary 1.54. *If $\delta : \tilde{X} \to X$ is a regular tropical modification with $X \subset \mathbb{T}^N$ and $\tilde{X} \subset \mathbb{T}^{N+1}$ then, for all p, q, δ induces isomorphisms:*

$$\mathrm{H}^q(\tilde{X}, \mathcal{F}^p_{\tilde{X}}) \cong \mathrm{H}^q(X, \mathcal{F}^p_X),$$
$$\mathrm{H}^q_c(\tilde{X}, \mathcal{F}^p_{\tilde{X}}) \cong \mathrm{H}^q_c(X, \mathcal{F}^p_X).$$

Proof. Note that the conditions of proposition 1.51 apply to the sheaves $\mathcal{F}^p_{\tilde{X}}$. Let $\Gamma_\Phi(X, \cdot)$ denote the functors $\Gamma(X, \cdot)$ resp. $\Gamma_c(X, \cdot)$. We then get the following quasi-isomorphisms:

$$\begin{aligned} \mathrm{R}\,\Gamma_\Phi(\tilde{X}, \mathcal{F}^p_{\tilde{X}}) &\cong \mathrm{R}\,\Gamma_\Phi(X, \mathrm{R}\,\delta_* \mathcal{F}^p_{\tilde{X}}) \\ &\cong \mathrm{R}\,\Gamma_\Phi(X, \delta_* \mathcal{F}^p_{\tilde{X}}) \\ &\cong \mathrm{R}\,\Gamma_\Phi(X, \mathcal{F}^p_X). \end{aligned}$$

Since δ is proper we have $\delta_! = \delta_*$ and hence both $\Gamma_c(\tilde{X}, \cdot) = \Gamma_c(X, \delta_!(\cdot))$ and $\Gamma(\tilde{X}, \cdot) = \Gamma(X, \delta_*(\cdot))$; this implies the first isomorphism. The second one is proposition 1.51 and the third one is lemma 1.52. □

Remark 1.55. The isomorphism $\mathrm{H}^q(\tilde{X}, \mathcal{F}^p_{\tilde{X}}) \cong \mathrm{H}^q(X, \mathcal{F}^p_X)$ has also be shown in [Sha15, 4.13] by a different method.

Corollary 1.56. *Assume $\delta : \tilde{X} \to X$ is a composition of finitely many elementary tropical modifications. Then for every (p, q) we have isomorphisms*

$$\mathrm{H}^q(\tilde{X}, \mathcal{F}^p_{\tilde{X}}) \cong \mathrm{H}^q(X, \mathcal{F}^p_X), \qquad \mathrm{H}^q_c(\tilde{X}, \mathcal{F}^p_{\tilde{X}}) \cong \mathrm{H}^q_c(X, \mathcal{F}^p_X)$$

induced by δ.

Definition 1.57. Let X be a tropical space. A *tropical modification* of X is a morphism $\delta : \tilde{X} \to X$ of tropical spaces, where X has an atlas $(\phi_U : X \to V_U \subset \mathbb{T}^{N_U})_{U \in \mathfrak{U}}$ such that there exists an atlas $(\psi_U : \delta^{-1}(U) \to \tilde{V}_U \subset \mathbb{T}^{M_U})_{U \in \mathfrak{U}}$ of $\tilde{X}$ satisfying that

$$\phi_U \circ \delta \circ \psi_U^{-1} : \tilde{V}_U \to V_U$$

is a composition of finitely many elementary tropical modifications for every $U \in \mathfrak{U}$.

Corollary 1.58. *Assume $\delta : \tilde{X} \to X$ is a tropical modification of a tropical space X. Then, δ induces isomorphisms*

$$\mathrm{H}^q_c(\tilde{X}, \mathcal{F}^p_{\tilde{X}}) \cong \mathrm{H}^q_c(X, \mathcal{F}^p_X), \qquad \mathrm{H}^q(\tilde{X}, \mathcal{F}^p_{\tilde{X}}) \cong \mathrm{H}^q(X, \mathcal{F}^p_X).$$

2 Differential forms and tropical cohomology

2.1 Differential forms on polyhedral spaces

Bigraded differential forms on a (tropical) polyhedral space are our main object of study. Locally, they are constructed starting with the 'classical' sheaves of differential forms on $\mathbb{R}^N$ and then extending this definition, first to $\mathbb{T}^N$ and then to polyhedral spaces contained in $\mathbb{T}^N$. By glueing one obtains a bigraded complex $\mathcal{A}_X^{\bullet,\bullet}$ of sheaves of differential forms on arbitrary polyhedral spaces. The idea to consider bigraded complexes of differential forms and currents and then to relate them to tropical geometry goes back to [Lag12] and it plays a major role in the theory of Monge-Ampère measures on Berkovich spaces initiated by [CD12]. By recent results of Liu ([Liu17]), there exists a strong link between the bigraded complex forms on a Berkovich space and its deRham cohomology. We will not work with Berkovich spaces in the present paper, focussing instead purely on the cohomological properties of $\mathcal{A}_X^{\bullet,\bullet}$ in the tropical world.

Construction 2.1. Let $V = \mathbb{R}^N$ be the N-dimensional real affine space and let $(\Omega_V^\bullet, d)$ be the complex of sheaves of smooth differential forms on V. We write $\mathcal{A}_V^{p,q} := \Omega_V^p \otimes_{\mathcal{C}_V^\infty} \Omega_V^q$ and we call $\mathcal{A}_V^{p,q}$ the *sheaf of* (p,q)*-forms* on V. In coordinates, every (p,q)-for α on V can be represented in the form

$$\alpha = \sum_{I,J} \alpha_{IJ} d'x_I \otimes d''x_J,$$

with $\alpha_{IJ} \in \mathcal{C}^\infty(V)$, where for $I = \{i_1, \dots, i_p\} \subset [N]$ with $i_k < i_{k+1}$ we set $d'x_I := d'x_{i_1} \wedge \dots \wedge d'x_{i_p}$, and similarly for $J \subset [N]$ and $d''x_J$.

The differentials $d : \Omega_V^p \to \Omega_V^{p+1}$ induce differentials

$$d' := d \otimes 1 : \mathcal{A}_V^{p,q} \to \mathcal{A}_V^{p+1,q}, \qquad d'' := (-1)^p \otimes d : \mathcal{A}_V^{p,q} \to \mathcal{A}_V^{p,q+1},$$

and the wedge product gives a morphism $\wedge : \mathcal{A}_V^{p',q'} \otimes \mathcal{A}_V^{p'',q''} \to \mathcal{A}_V^{p'+p'',q'+q''}$. In coordinates, these can be given as follows:

$$d'\left(\sum_{I,J} \alpha_{IJ} d'x_I \otimes d''x_J\right) := \sum_{I,J,i} \partial_i \alpha_{IJ} d'x_i \wedge d'x_I \otimes d''x_J,$$

$$d''\left(\sum_{I,J} \alpha_{IJ} d'x_I \otimes d''x_J\right) := \sum_{I,J,j} (-1)^{|I|} \partial_j \alpha_{IJ} d'x_I \otimes d''x_j \wedge d''x_J,$$

$$\left(\sum_{I,J} \alpha_{IJ} d'x_I \otimes d''x_J\right) \wedge \left(\sum_{K,L} \beta_{KL} d'x_K \otimes d''x_L\right) := \sum_{I,J,K,L} (-1)^{|J||K|} \alpha_{IJ}\beta_{KL} d'x_I \wedge d'x_K \otimes d''x_J \wedge d''x_L.$$

Remark 2.2.

1. We have

$$d'd'' = -d''d', \qquad d'd' = 0, \qquad d''d'' = 0,$$

   in every degree. There are two different ways to introduce double complexes: Either one demands that they have commuting squares or anticommuting squares. In section 3 we will start working with double complexes and the double complex of forms in particular. Since for us a double complex will have commuting squares, we will then have to adjust the signs of the differentials accordingly.

2. The following *Leibniz formulas* for $\alpha \in \mathcal{A}_V^{p',q'}$ are easily derived:

$$d'(\alpha \wedge \beta) = d'\alpha \wedge \beta + (-1)^{p'+q'} \alpha \wedge d'\beta$$
$$d''(\alpha \wedge \beta) = d''\alpha \wedge \beta + (-1)^{p'+q'} \alpha \wedge d''\beta,$$

Now, we can define the sheaves of (p,q)-forms and the corresponding differential maps on polyhedral spaces as follows:

Definition 2.3. First, let $X \subset \mathbb{R}^N$ be the support of a polyhedral complex and let $\imath : X^{reg} \subset \mathbb{R}^N$ be the smooth manifold of regular points of X.

1. The restriction of forms defines a morphism of sheaves $\rho : \mathcal{A}^{p,q}_{\mathbb{R}^N} \to \imath_* \mathcal{A}^{p,q}_{X^{reg}}$ and we define the *sheaf of forms vanishing on* X to be the sheaf $\mathcal{K}^{p,q}_X := \ker(\rho)$ on $\mathbb{R}^N$. The *sheaf of* (p,q)*-forms on* X is the sheaf
$$\mathcal{A}^{p,q}_X := \left(\mathcal{A}^{p,q}_{\mathbb{R}^N} \,/\, \mathcal{K}^{p,q}_X\right)\Big|_X .$$
2. It is easy to see that the morphisms d', d'' and $\wedge$ on (p,q)-forms on $\mathbb{R}^N$ induce morphisms of sheaves
$$d' : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q}_X, \qquad d'' : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p,q+1}_X, \qquad \wedge : \mathcal{A}^{p',q'}_X \otimes \mathcal{A}^{p'',q''}_X \to \mathcal{A}^{p'+p'',q'+q''}_X .$$

Definition 2.4. Let now X be the support of a polyhedral complex in $\mathbb{T}^N$ and let $U \subset X$ be an open subset. For each $I \subset [N]$ define $X_I^\circ := X \cap \mathbb{R}^N_I$ which is the support of a polyhedral complex in $\mathbb{R}^N_I$ and $U_I^\circ := U \cap \mathbb{R}^N_I$ as an open subset of X_I°. We write $\mathrm{Sed}(U)$ for the set of subsets $I \subset [N]$ with $U_I^\circ \neq \emptyset$. A (p,q)*-form* α on U is given by a collection of forms $(\alpha_I)_{I \in \mathrm{Sed}(U)}$ such that,

1. $\alpha_I \in \mathcal{A}^{p,q}_{X_I^\circ}(U_I^\circ)$ for all I with $U_I^\circ \neq \emptyset$,
2. for each point $x \in U \subset \mathbb{T}^N$ of sedentarity I, there exists a neighbourhood U_x of x contained in U such that for each $J \subset I$ with $U^\circ_{x,J} \neq \emptyset$ the projection satisfies $\pi_{IJ}(U^\circ_{x,J}) = U^\circ_{x,I}$, and
$$(\pi^*_{IJ}\alpha_I)|_{U^\circ_{x,J}} = \alpha_J|_{U^\circ_{x,J}},$$
where $\pi_{IJ} : \mathbb{R}^N_J \to \mathbb{R}^N_I$ is the natural projection.

Since the projections commute with d' and d'' we may define for $\alpha = (\alpha_I)_I \in \mathcal{A}^{p',q'}_X(U)$ and $\beta = (\beta_I)_I \in \mathcal{A}^{p'',q''}_X(U)$,
$$d'\alpha := (d'\alpha_I)_I, \qquad d''\alpha := (d'\alpha_I)_I, \qquad \alpha \wedge \beta := (\alpha_I \wedge \beta_I)_I,$$
obtaining morphisms of sheaves as expected. By construction, they retain the properties listed above, i.e. we have
$$d'd'' = -d''d', \qquad d'd' = 0, \qquad d''d'' = 0,$$
and
$$d'(\alpha \wedge \beta) = d'\alpha \wedge \beta + (-1)^{p'+q'} \alpha \wedge d'\beta, \qquad d''(\alpha \wedge \beta) = d''\alpha \wedge \beta + (-1)^{p'+q'} \alpha \wedge d''\beta,$$
for (p',q')-forms α and any form β.

Proposition 2.5. *Let $f : X \to Y$ be a morphism of polyhedral spaces. Then f induces a natural morphism of sheaves*
$$f^* : \mathcal{A}^{p,q}_Y \to f_* \mathcal{A}^{p,q}_X$$
for each $p,q \in \mathbb{Z}$, satisfying
$$f^*d'\alpha = d'f^*\alpha, \qquad f^*d''\alpha = d''f^*\alpha, \qquad f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta,$$
for every $\alpha \in \mathcal{A}^{p',q'}_Y(U)$, $\beta \in \mathcal{A}^{p'',q''}_Y(U)$, $U \subset Y$ open.

Proof. We give a short sketch of the proof: First assume that X and Y are the supports of polyhedral complexes in $\mathbb{T}^N$ and $\mathbb{T}^M$ respectively, and $f : X \to Y$ is the restriction of an extended affine map $F : \mathbb{T}^N \to \mathbb{T}^M$ to X (c.f. definition 1.12). For an open subset $V \subset Y$ and $\alpha \in \mathcal{A}_Y^{p,q}(V)$, there exists an open subset $\tilde{V} \subset \mathbb{T}^M$ with $V = \tilde{V} \cap Y$ and $f^{-1}(V) = F^{-1}(\tilde{V}) \cap X$. The pullback map

$$F^* : \mathcal{A}^{p,q}_{\mathbb{T}^M}(\tilde{V}) \to F_* \mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{V}), \qquad (\alpha_I)_{I \in \mathrm{Sed}(\tilde{V})} \mapsto (F_I^* \alpha_I)_{I \in \mathrm{Sed}(\tilde{V})}$$

is well defined and as in [Gub13, 3.2], F^* maps $\mathcal{K}_Y^{p,q}(\tilde{V})$ to $\mathcal{K}_X^{p,q}(F^{-1}(\tilde{V}))$. We obtain an induced map of the quotients, $f^* : \mathcal{A}_Y^{p,q}(V) \to \mathcal{A}_X^{p,q}(f^{-1}(V))$, independent of the choice of $\tilde{V}$. The construction is functorial and compatible with restrictions, differential maps and the wedge product, so by choosing atlases for X and Y as in definition 1.16 we obtain morphisms $f^* : \mathcal{A}_Y^{p,q} \to f_* \mathcal{A}_X^{p,q}$ of sheaves as required. □

2.2 Closed $(0, q)$-forms

In the following, we present a variation of the proof of [JSS15, 3.20]. Arguably, this approach is more cumbersome than the one presented in [JSS15] but it has merits of its own, in particular in highlighting the interplay of sheaf theory with classical analysis more clearly.

Definition 2.6. Let X be a polyhedral space. We define $\mathcal{L}_X^p := \ker\left(\mathcal{A}_X^{p,0} \to \mathcal{A}_X^{p,1}\right)$. Later, we will often consider the complex

$$\mathcal{L}^\bullet = \mathcal{L}_X^\bullet := \bigoplus_{p \in \mathbb{Z}} \mathcal{L}_X^p[-p]$$

of sheaves of $\mathbb{R}$-vector spaces on X. Note that $\mathcal{L}_X^p$ is canonically isomorphic to the kernel of $d' : \mathcal{A}_X^{0,p} \to \mathcal{A}_X^{1,p}$ by symmetry.

We start by describing the sheaves $\mathcal{L}_X^p$ on the support of a polyhedral complex in $\mathbb{R}^N$ and will then subsequently expand this to completed polyhedral complexes in $\mathbb{T}^N$ and finally general polyhedral spaces.

Fix a vectorspace $V = \mathbb{R}^N$ and the support $X = |\Sigma|$ of a finite polyhedral complex Σ of pure dimension n in V. In this section, we will use the canonical isomorphism

$$\mathcal{A}_V^{p,q}(\tilde{U}) = \mathcal{C}^\infty\left(\tilde{U}, \mathrm{Hom}\left(\bigwedge^q V, \bigwedge^p V^*\right)\right)$$

given by $\bigwedge^p V^* \otimes \bigwedge^q V^* \cong \mathrm{Hom}(\bigwedge^q V, \bigwedge^p V^*)$. As before, for a polyhedron $\sigma \in \Sigma$ we will denote by $\mathbb{L}(\sigma)$ the $\mathbb{R}$-linear space

$$\mathbb{L}(\sigma) := \mathrm{span}_{\mathbb{R}}(x - x'; x, x' \in \mathrm{relint}(\sigma))$$

of σ in V.

Lemma 2.7. *Let $\tilde{U}$ be an open subset of V and let $\phi : \tilde{U} \to \mathrm{Hom}(\bigwedge^q V, \bigwedge^p V^*)$ be an element of $\mathcal{A}_V^{p,q}(\tilde{U})$. Then ϕ lies in $\mathcal{K}_X^{p,q}(\tilde{U})$ if and only if*

$$\phi(x)(v) \in \left(\sum_{x \in \tau} \bigwedge^p \mathbb{L}(\tau)\right)^\perp$$

for every $x \in \tilde{U}$ and for every $v \in \bigcap_{x \in \tau \in \Sigma_n} \bigwedge^q \mathbb{L}(\tau)$.

Proof. If ϕ lies in $\mathcal{K}_X^{p,q}(\tilde{U})$, then $\phi(x)(v)(\alpha)$ vanishes for every regular point $x \in \operatorname{relint}(\sigma)$ with $\sigma \in \Sigma$ maximal and for every $v \in \bigwedge^q \mathbb{L}(\sigma)$ and every $\alpha \in \bigwedge^p \mathbb{L}(\sigma)^\perp$. By continuity of ϕ, this implies that $\phi(x)$ maps $\bigwedge^q \mathbb{L}(\sigma)$ to $(\bigwedge^p \mathbb{L}(\sigma))^\perp$ for every $x \in \sigma$. If now x lies in $\sigma \cap \sigma'$ for two maximal cells σ and σ', then $\phi(x)(v)$ has to vanish on $\bigwedge^p \mathbb{L}(\sigma) + \bigwedge^p \mathbb{L}(\sigma')$ for every $v \in \bigwedge^q \mathbb{L}(\sigma) \cap \bigwedge^q \mathbb{L}(\sigma')$. Iterating this argument implies the *only if* part.

For the *if* part, it suffices to consider $x \in X^{reg}$. If $x \in \operatorname{relint}(\sigma)$ for a maximal cell $\sigma \in \Sigma$ then the claim follows immediately. Else, there is a polyhedral complex Σ' with support X and a maximal cell $\sigma' \in \Sigma'$ with $x \in \operatorname{relint}(\sigma')$. After refining Σ, we may choose Σ' such that for every maximal $\tau \in \Sigma$ with $x \in \tau$, we have $\tau \subset \sigma'$. We then have

$$\bigcap_{x \in \tau \in \Sigma_n} \bigwedge^q \mathbb{L}(\tau) = \bigwedge^q \mathbb{L}(\sigma'),$$
$$\sum_{x \in \tau} \bigwedge^p \mathbb{L}(\tau) = \bigwedge^p \mathbb{L}(\sigma'),$$

which implies the claim. □

Proposition 2.8. *Let X be the support of a polyhedral complex Σ in $V = \mathbb{R}^N$ and consider the sheaves $\mathcal{A}_X^{p,q}$ of differential forms on X as defined above. Then, the kernel $\mathcal{L}_X^p$ of d' : $\mathcal{A}_X^{p,0} \to \mathcal{A}_X^{p,1}$ is given as the sheafification of the presheaf*

$$U \mapsto \left(\bigwedge^p V\right)^* / \bigcap_{\tau \cap U \neq \emptyset} \left(\bigwedge^p \mathbb{L}(\tau)\right)^\perp$$
$$= \left(\sum_{\tau \cap U \neq \emptyset} \bigwedge^p \mathbb{L}(\tau)\right)^*$$

with obvious restriction maps, where $\mathbb{L}(\tau) = \operatorname{span}_{\mathbb{R}}(v' - v; v, v' \in \tau)$ denotes the linear subspace of V associated to τ.

Proof. First, assume that $X = V$, $\Sigma = \{V\}$. An element ζ of $\mathcal{A}_V^{p,0}(U) = \mathcal{A}_X^{p,0}(U)$ is then given as a formal sum $\zeta = \sum_J f_J d' x_J$ where J runs through subsets $J \subset [N] = \{1, \ldots, N\}$ with p elements, $f_J \in \mathcal{C}^\infty(U)$ and $d' x_J = \bigwedge_{j \in J} d' x_j \in \bigwedge^p V^*$. The $d' x_J$ are linearly independent and hence $d'' \zeta = 0$ implies that all f_J are constant functions. Hence, the kernel of d'' is equal to the p-th exterior product

$$\ker d''_V = \bigwedge^p V^*.$$

For general $X = |\Sigma|$, we consider the following diagram of sheaves of abelian groups on V with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}^p & \longrightarrow & \mathcal{K}^{p,0} & \longrightarrow & \mathcal{G}^p & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigwedge^p V^* & \longrightarrow & \mathcal{A}^{p,0} & \longrightarrow & d'' \mathcal{A}^{p,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L}_X^p & \longrightarrow & \mathcal{A}_X^{p,0} & \longrightarrow & \mathcal{A}_X^{p,1} & & \\
 & & \downarrow & & \downarrow & & & & \\
\cdots & \longrightarrow & \mathcal{C} & \longrightarrow & 0 & & & &
\end{array}
$$

Where

$$
\begin{aligned}
\mathcal{L}_X^p &:= \ker(d'' : \mathcal{A}_X^{p,0} \to \mathcal{A}_X^{p,1}), \\
\mathcal{G}^p &:= \ker((d'' \mathcal{A}^{p,0}) \to \mathcal{A}_X^{p,1}), \\
\mathcal{K}^p &:= \ker \left(\bigwedge^p V^* \to \mathcal{L}_X^p \right), \\
\mathcal{C} &:= \operatorname{coker} \left(\bigwedge^p V^* \to \mathcal{L}_X^p \right).
\end{aligned}
$$

We first show $\mathcal{C} = 0$ by using the snake lemma. Hence, we have to show that $\mathcal{K}^{p,0} \to \mathcal{G}^p$ is an epimorphism of sheaves of abelian groups. Let $\xi \in \mathcal{G}^p(U) = \ker((d'' \mathcal{A}^{p,0})(\tilde{U}) \to \mathcal{A}_X^{p,1}(\tilde{U}))$. By shrinking $\tilde{U}$, we may assume that $\xi = d''\zeta$ with $\zeta \in \mathcal{A}^{p,0}(\tilde{U})$ and that $U = \tilde{U} \cap X$ is polyhedrally starshaped with center x_0; in particular $\tau \cap U \neq \emptyset$ implies $x_0 \in \tau$ and $\tau \cap U$ is simply connected for every τ.

Because $d''\zeta = 0$ in $\mathcal{A}_X^{p,1}(\tilde{U})$ we may conclude that

$$d''\zeta(y)(v) \in \left(\sum_{y \in \tau} \bigwedge^p \mathbb{L}(\tau) \right)^{\perp}$$

for every $y \in U$ and $v \in \bigcap_{y \in \tau \in \Sigma_n} \mathbb{L}(\tau)$ by the previous lemma. Moreover, by continuity we have

$$d''\zeta(x_0)(v) \in \left(\sum_{y \in \tau} \bigwedge^p \mathbb{L}(\tau) \right)^{\perp}$$

for every $v \in \bigcap_{y \in \tau \in \Sigma_n} \mathbb{L}(\tau)$ as well. This implies

$$(d''\zeta(y) - d''\zeta(x_0))(v) \in \left(\sum_{y \in \tau} \bigwedge^p \mathbb{L}(\tau) \right)^{\perp}$$

for every $v \in \bigcap_{y \in \tau \in \Sigma_n} \mathbb{L}(\tau)$ by linearity.

The path $\gamma : t \mapsto t(y - x_0) + x_0$ from x_0 to y is contained in U and using Stokes' theorem, we obtain

$$\begin{aligned}\zeta(y)-\zeta(x_0) &= \int_\gamma d\zeta \\ &= \int_0^1 d\zeta(\gamma(t))\gamma'(t)dt \\ &= \int_0^1 d\zeta(\gamma(t))(y-x_0)dt.\end{aligned}$$

Now if y lies in a cell σ, then for every $0 \leq t \leq 1$, $\gamma(t)$ lies in σ as well and $y - x_0$ lies in $\mathbb{L}(\sigma)$. Hence for every t, $d''\zeta(\gamma(t))(y-x_0)$ lies in $\left(\sum_{y\in\tau}\bigwedge^p \mathbb{L}(\tau)\right)^\perp$. This implies that $\zeta(y)-\zeta(x_0)$ vanishes on $\sum_{y\in\tau}\bigwedge^p \mathbb{L}(\tau)$ and we may conclude that $\zeta - \zeta(x_0)$ lies in $\mathcal{K}^{p,0}(\tilde{U})$.

Using the snake lemma once again, we see that the kernel $\mathcal{K}^p$ of $\bigwedge^p V^* \to \mathcal{L}^p_X$ is isomorphic to the kernel of $\mathcal{K}^{p,0} \to \mathcal{K}^{p,1}$. But the latter kernel is easily seen to be equal to the subsheaf $\mathfrak{L}_p^\perp$ of $\bigwedge^p V^*$, i.e. the sheaf associated to

$$U \mapsto \left(\sum_{\tau\cap U\neq\emptyset} \bigwedge^p \mathbb{L}(\tau)\right)^\perp,$$

with the obvious restriction maps.

Hence $\mathcal{L}^p_X = \operatorname{coker}(\mathfrak{L}_p^\perp \to \bigwedge^p V^*)$ is canonically isomorphic to the sheafification of the presheaf

$$U \mapsto \left(\sum_{\tau\cap U\neq\emptyset} \bigwedge^p \mathbb{L}(\tau)\right)^*.$$

□

Remark 2.9. It is obvious that the roles of d' and d'' can be switched.

We can extend this to polyhedral complexes in $\mathbb{T}^N$ (recall also the definition of constructible sheaves from section 1.10):

Proposition 2.10. *Let $X = |\Sigma|$ be the support of a polyhedral complex Σ in $\mathbb{T}^N$.*

1. *For $x \in \operatorname{relint}(\sigma)$ with $\sigma \in \Sigma$ and $\operatorname{sed}(\sigma) = I$ there exists a basis of open neighbourhoods U_x of x in X such that the natural maps*

$$\mathcal{L}^p_X(U_x) \cong \mathcal{L}^p_{X,x} \cong \mathcal{L}^p_{X_I^\circ,x} \cong \left(\sum_{x\in\tau\in\Sigma_I} \bigwedge^p \mathbb{L}(\tau)\right)^*$$

   are isomorphisms of $\mathbb{R}$-vecor spaces.

2. *For every p, the complex $\mathcal{A}^{p,\bullet}_X$ is exact in positive degrees and every sheaf $\mathcal{A}^{p,q}_X$ is a fine sheaf of $\mathbb{R}$-vector spaces.*

3. *The complex $\mathcal{A}^{p,\bullet}_X$ of sheaves is constructible on X (with respect to Σ).*

Proof. 1. Let U_x be any polyhedrally starshaped open subset of X with center $x \in \operatorname{relint}(\sigma)$, $\sigma \in \Sigma$ (c.f. definition 1.19). Let $(\alpha_J)_{J\subset I} \in \mathcal{A}^{p,0}_X(U_x)$ be a d''-closed $(p,0)$-form on U_x. Then each α_J is a d''-closed $(p,0)$-form on $(U_x)^\circ_J \subset \mathbb{R}^N_J$. Together with the condition of compatibility from definition 2.4(2), we obtain easily from proposition 2.8 that $(\alpha_J)_J$ is uniquely determined by α_I and that the above chain consists of isomorphisms.

2. This is due to Jell, c.f. [Jel16b, 2.1.59] and [Jel16b, 2.2.18].

3. Statement (1) shows that the stalk $\mathrm{H}^0(\mathcal{A}_X^{p,\bullet})_x = \mathcal{L}_{X,x}^p$ only depends on $\sigma \in \Sigma$ with $x \in \operatorname{relint}(\sigma)$; it is also clear that they are finite dimensional. With (2) we get that $\mathrm{H}^k(\mathcal{A}_X^{p,\bullet})_x = 0$ for $k \neq 0$.

□

Proposition 2.11. *Let X be a polyhedral space and let $\phi : U \to V \subset \mathbb{T}^N$ be a polyhedrally starshaped chart with center x for X (definition 1.19).*

1. *The canonical morphisms*
$$\Gamma(U, \mathcal{L}_X^p) \to \mathrm{R}\,\Gamma(U, \mathcal{L}_X^p)$$
are isomorphisms; i.e. we have
$$\mathrm{H}^q(U, \mathcal{L}_X^p) = 0$$
for $q > 0$.

2. *The canonical morphisms*
$$\mathrm{H}^0(U, \mathcal{L}_X^p) \to (\mathcal{L}_X^p)_x$$
are isomorphisms of finite dimensional $\mathbb{R}$-vector spaces.

Proof. This is just proposition 1.33. □

Corollary 2.12. *Let X be the support of a polyhedral complex Σ in $\mathbb{T}^N$. The cohomology groups of $\mathcal{L}_X^p$,*
$$\mathrm{H}^q(X, \mathcal{L}_X^p) = \mathrm{H}^q(\Gamma(X, \mathcal{A}_X^{p,\bullet})),$$
and the cohomology groups with compact support,
$$\mathrm{H}_c^q(X, \mathcal{L}_X^p) = \mathrm{H}^q\left(\Gamma_c(X, \mathcal{A}_X^{p,\bullet})\right),$$
are finite dimensional.

Remark 2.13. For $\mathrm{H}^q(X, \mathcal{L}_X^p)$ this this is due to Philipp Jell, c.f. [Jel16b, 2.2.34].

Proof of corollary 2.12. This follows from proposition 1.36 (note that Σ is a finite set by assumption and hence can be covered by finitely many open stars; Čech cohomology finishes the proof). □

Remark 2.14. Let X be a polyhedral space.

1. For an open subset $U \subset X$, the canonical morphisms $\mathcal{L}_X^p\,|_U \to \mathcal{L}_U^p$ are isomorphisms.

2. The cohomology sheaves $\mathrm{H}^q\left(\mathcal{A}_X^{p,\bullet}\right)$ are zero for $q \neq 0$. For $q = 0$ we have, in every chart $\phi_U : U \to V \subset \mathbb{T}^N$,
$$\mathcal{L}_X^p\,|_U = \mathrm{H}^0\left(\mathcal{A}_X^{p,\bullet}\right)|_U \cong \phi_U^*\,\mathcal{L}_V^p\,.$$

We will repeatedly make use of the following property of d''-closed $(p,0)$-forms:

Corollary 2.15. *Let X be a polyhedral space and $U \subset X$ an open subset. Then every d''-closed form $\alpha \in \mathcal{A}_X^{p,0}(U)$ is d'-closed. Similarly, every d'-closed form in $\mathcal{A}_X^{0,q}(U)$ is d''-closed.*

Proof. Let $\alpha \in \mathcal{A}_X^{p,0}(U)$ be d''-closed. By proposition 2.10(1) and the previous remark we may assume that X is the support of a polyhedral complex Σ in $\mathbb{R}^N$. Then by proposition 2.8 $\alpha \in \mathcal{L}_X^p(U)$ has a representative of the form
$$\alpha = \sum_I c_I d' x_I$$
with $c_I \in \mathbb{R}$ constant. This implies $d'\alpha = 0$. □

2.3 Comparison of tropical cohomology and cohomology of differential forms

Putting the results of the previous sections together, we obtain the comparison result for tropical cohomology (section 1.11) and cohomology of differential forms on a tropical space X. Its proof has first been published in [JSS15].

Theorem 2.16. *Let X be a tropical space. Then the tropical cohomology groups of X with real coefficients are canonically isomorphic to the Dolbeault cohomology groups on X:*

$$\mathrm{H}^{p,q}_{\mathrm{trop}}(X) \otimes \mathbb{R} \cong \mathrm{H}^q(X, \mathcal{F}^p) \cong \mathrm{H}^q(X, \mathcal{L}^p_X) \cong \mathrm{H}^q\left(\Gamma(X, \mathcal{A}^{p,\bullet}_X)\right).$$

Proof. It suffices that the sheaves $\mathcal{F}^p$ and $\mathcal{L}^p_X$ are isomorphic and this can be tested on any atlas for X. We choose an atlas of starshaped charts for X as in construction 1.40. For each chart $\phi : U \to V \subset \mathbb{T}^N$ we can choose a polyhedral complex Σ in $\mathbb{T}^N$, maximal with the 'starshaped' property from definition 1.19. Let σ be the minimal polyhedron in Σ with $\sigma \cap V \neq \emptyset$ and set $I = \mathrm{sed}(\sigma)$. Now we easily get isomorphisms

$$\mathcal{F}^p_X(U) \cong \left(\sum_{\sigma \leq \tau \in \Sigma_I} \bigwedge^p \mathbb{L}_{\mathbb{R}}(\tau) \right)^* \cong \mathcal{L}^p_X(U).$$

These are compatible with the respective restriction maps and hence glue to an isomorphism

$$\mathcal{F}^p_X \cong \mathcal{L}^p_X$$

of constructible sheaves of $\mathbb{R}$-vector spaces on X. This provides us with the middle isomorphisms in the statement of the theorem. The left isomorphism is clear by definition and the right isomorphism is clear because $\mathcal{A}^{p,\bullet}_X$ is a soft resolution of $\mathcal{L}^p_X$. □

3 Total complexes of forms and currents

3.1 Some notations

The cohomology groups $\mathrm{H}^q(X, \mathcal{L}_X^p)$ considered in the previous section can be understood as Dolbeault cohomology groups of the double complex $\mathcal{A}_X^{\bullet,\bullet}$. In this section, we will use them to derive properties of the total complex $\mathcal{A}_X^\bullet$ of $\mathcal{A}_X^{\bullet,\bullet}$. One might think of this as analogous to the de Rham cohomology of complex manifolds. Note however that this analogy is not perfect; for instance, the global section cohomology of $\mathcal{A}_X^\bullet$ does not compute the singular cohomology groups of X. Most constructions and arguments in this section come from homological algebra and we will first introduce some standard notations we will be using:

Definition 3.1. 1. A *double complex* $(A^{\bullet,\bullet}, d_1, d_2)$ (with values in an abelian category $\mathfrak{A}$) is a collection $(A^{k,l})_{k,l\in\mathbb{Z}}$ of objects of $\mathfrak{A}$ together with morphisms

$$d_1^{p,q} : A^{p,q} \to A^{p+1,q}, \qquad d_2^{p,q} : A^{p,q} \to A^{p,q+1},$$

satisfying

$$d_1 \circ d_1 = 0, \qquad d_2 \circ d_2 = 0, \qquad d_1 \circ d_2 = d_2 \circ d_1.$$

We say that $A^{\bullet,\bullet}$ is *bounded* if $A^{p,q} = 0$ for $|p|$ and $|q| \geq k$ for some $k \in \mathbb{N}$.

2. The *total complex* $(\mathrm{tot}^\bullet A^{\bullet,\bullet}, d)$ of a bounded double complex $A^{\bullet,\bullet}$ is defined by

$$\mathrm{tot}^k A^{\bullet,\bullet} = \bigoplus_{p+q=k} A^{p,q},$$
$$d|_{A^{p,q}} = d_1 + (-1)^p d_2.$$

If α is an element of $\mathrm{tot}^k A^{\bullet,\bullet}$, we write $\alpha^{p,q} \in A^{p,q}(X)$ for the image of α in $A^{p,q}(X)$ under the natural projection.

Definition 3.2. Let $(A^\bullet; d_A)$ and $(B^\bullet; d_B)$ be two bounded complexes of modules over a ring or a sheaf of rings R.

1. The *Hom-complex* $(\mathrm{Hom}_R^\bullet(A^\bullet, B^\bullet), \partial)$ is the total complex of the bounded double complex $(\mathrm{Hom}_R^{\bullet,\bullet}(A^\bullet, B^\bullet), (d_B)_*, (d_A)^*)$ with

$$\mathrm{Hom}_R^{k,l}(A^\bullet, B^\bullet) := \mathrm{Hom}_R(A^{-l}, B^k),$$
$$(d_B)_*^{k,l} = \mathrm{Hom}(A^{-l}, d_B^k),$$
$$((d_A)^*)^{k,l} = (-1)^l \,\mathrm{Hom}_R(d_A^{-(l+1)}, B^k).$$

2. We define the *(complex) tensor product* $(A^\bullet \otimes_R^\bullet B^\bullet, d_\otimes)$ to be the total complex of the double complex $((A^\bullet \otimes_R B^\bullet)^{\bullet,\bullet}; d_1, d_2)$ with

$$(A^\bullet \otimes_R B^\bullet)^{k',k''} := A^{k'} \otimes_R B^{k''},$$
$$d_1^{k',k''} = d_A^{k'} \otimes \mathrm{id}_{B^{k''}},$$
$$d_2^{k',k''} = \mathrm{id}_{A^{k'}} \otimes d_B^{k''}.$$

From here on, a double complex will always be a double complex with commuting squares as in definition 3.1. The triple $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$ introduced in section 2.1 does *not* form a double complex in this sense (rather it would be a 'double complex with anticommuting squares' as we have $d'd'' = -d''d'$). We will compensate for this by using the following sign convention:

Example 3.3. 1. A double complex of prime interest for us is the bounded double complex $(\mathcal{A}_X^{\bullet,\bullet}, d_1, d_2)$ of sheaves of (p,q)-forms on a tropical space X, where we set

$$d_1^{p,q} := (d')^{p,q}, \qquad d_2^{p,q} := (-1)^p (d'')^{p,q}.$$

We write $(\mathcal{A}_X^{\bullet}, d) := \operatorname{tot}^\bullet \mathcal{A}_X^{\bullet,\bullet}$ for its total complex. A *k-form* on X is an element $\alpha \in \mathcal{A}_X^k(U)$ for some open subset $U \subset X$. Note that we have

$$d|_{\mathcal{A}_X^{p,q}} = d_1 + (-1)^p d_2 = d' + d''.$$

2. The second main double complex we will be concerned with is the double complex $(\mathcal{D}_X^{\bullet,\bullet}, \partial_2, \partial_1)$ of presheaves of *linear currents*,

$$\begin{aligned} \mathcal{D}_X^{r,s}(U) &:= \operatorname{Hom}_{\mathbb{R}}(\Gamma_c(U, \mathcal{A}_X^{-s,-r}), \mathbb{R}), \\ \partial_2^{r,s} &:= (-1)^r \operatorname{Hom}(d_2^{-s,-r-1}, \mathbb{R}), \\ \partial_1^{r,s} &:= (-1)^s \operatorname{Hom}(d_1^{-s-1,-r}, \mathbb{R}), \end{aligned}$$

with restriction maps induced by the embeddings $\Gamma_c(U, \mathcal{A}_X^{-s,-r}) \to \Gamma_c(U', \mathcal{A}_X^{-s,-r})$ for open subsets $U \subset U' \subset X$. We write $(\mathcal{D}_X^\bullet, \partial)$ for the total complex $\operatorname{tot}^\bullet \mathcal{D}_X^{\bullet,\bullet}$. For some immediate properties of $\mathcal{D}_X^{\bullet,\bullet}$ see proposition 3.4 below. We will relate $\mathcal{D}_X^\bullet$ to a more classical notion of (continuous) currents in section 5.4.

3. Note that the double complex $(\mathcal{A}_X^{\bullet,\bullet}, d_1, d_2)$ is canonically isomorphic to the tensor product double complex

$$\mathcal{A}_X^{\bullet,\bullet} \simeq (\mathcal{A}_X^{\bullet,0} \otimes_{\mathcal{A}_X^{0,0}} \mathcal{A}_X^{\bullet,0})^{\bullet,\bullet} \simeq (\mathcal{A}_X^{\bullet,0} \otimes_{\mathcal{A}_X^{0,0}} \mathcal{A}_X^{0,\bullet})^{\bullet,\bullet},$$

when we consider the complexes $\mathcal{A}_X^{\bullet,0}$ resp. $\mathcal{A}_X^{0,\bullet}$ to be equipped with the differential maps d_1 and d_2 respectively.

Proposition 3.4. *Let X be a polyhedral space.*

1. *For each $(r,s) \in \mathbb{Z}^2$, the presheaf $\mathcal{D}_X^{r,s}$ is a flabby sheaf of $\mathbb{R}$-vector spaces.*
2. *For every open $U \subset X$ we have canonical isomorphisms of complexes of $\mathbb{R}$-vector spaces*

$$\mathcal{D}_X^\bullet(U) = \operatorname{tot}^\bullet \mathcal{D}_X^{\bullet,\bullet}(U) \cong \operatorname{Hom}_{\mathbb{R}}^\bullet(\operatorname{tot}^\bullet \Gamma_c(U, \mathcal{A}_X^{\bullet,\bullet}), \mathbb{R}) = \operatorname{Hom}_{\mathbb{R}}^\bullet(\Gamma_c(U, \mathcal{A}_X^\bullet), \mathbb{R})$$

3. *The derived duals (see A.20) of the complexes $\mathcal{A}_X^\bullet$, $\mathcal{A}_X^{p,\bullet}$ and $\mathcal{A}_X^{\bullet,q}$ can be represented by the complexes $(\mathcal{D}_X^\bullet, \partial)$, $(\mathcal{D}_X^{\bullet,-p}, \partial_2)$ and $(\mathcal{D}_X^{-q,\bullet}, \partial_1)$ of flabby sheaves on X:*

$$\mathcal{D}(\mathcal{A}_X^\bullet) = \mathcal{D}_X^\bullet, \qquad \mathcal{D}(\mathcal{A}_X^{p,\bullet}) = \mathcal{D}_X^{\bullet,-p}, \qquad \mathcal{D}(\mathcal{A}_X^{\bullet,q}) = \mathcal{D}_X^{-q,\bullet}.$$

Proof. (1) These are indeed flabby sheaves: The sheaves $\mathcal{A}_X^{-s,-r}$ are soft on X, so for two open subsets $U, U' \subset X$ we have short exact sequences

$$0 \to \Gamma_c(U \cap U', \mathcal{A}_X^{-s,-r}) \to \Gamma_c(U, \mathcal{A}_X^{-s,-r}) \oplus \Gamma_c(U', \mathcal{A}_X^{-s,-r}) \to \Gamma_c(U \cup U', \mathcal{A}_X^{-s,-r}) \to 0.$$

Because $\operatorname{Hom}_{\mathbb{R}}(\cdot, \mathbb{R})$ is left exact, this implies that $\mathcal{D}_X^{r,s}$ is a sheaf. Because $\Gamma_c(U, \mathcal{A}_X^{-s,-r}) \to \Gamma_c(U'', \mathcal{A}_X^{-s,-r})$ is an injection for open subsets $U \subset U''$ of X, right exactness of $\operatorname{Hom}_{\mathbb{R}}(\cdot, \mathbb{R})$ implies that $\mathcal{D}_X^{r,s}$ is flabby.

(2) A quick computation shows that $\mathcal{D}_X^\bullet(U)$ is canonically isomorphic to $\operatorname{Hom}_{\mathbb{R}}^\bullet(\Gamma_c(U, \mathcal{A}_X^\bullet), \mathbb{R})$: We only have to show that the differential maps of both complexes agree, which follows from

$$\begin{aligned} \partial \phi^{r,s} &= \partial_2 \phi^{r,s} + (-1)^r \partial_1 \phi^{r,s} \\ &= (-1)^r \phi^{r,s} \circ d_2^{-s,-r-1} + (-1)^{r+s} \phi^{r,s} \circ d_1^{-s-1,-r} \\ &= (-1)^{r+s} \phi^{r,s} \circ (d_1^{-s-1,-r} + (-1)^s d_2^{-s,-r-1}) \\ &= (-1)^{r+s} \phi^{r,s} \circ d^{-r-s-1}, \end{aligned}$$

for $\phi^{r,s} : \Gamma_c(U, \mathcal{A}_X^{-s,-r}) \to \mathbb{R}$ in $\mathcal{D}_X^{r,s}(U) \subset \mathcal{D}_X^{r+s}(U)$.

(3) This follows directly from the construction A.17 of the dualizing complex $\omega_X = \pi^! \mathbb{R}$, where $\pi : X \to \mathrm{pt}$ is the projection to a point (c.f. definition A.20, example A.18).

□

3.2 The total complex of forms

Proposition 3.5. *Let X be a tropical variety of pure dimension n and $\mathcal{L}_X^p := \ker(d_2 : \mathcal{A}_X^{p,0} \to \mathcal{A}_X^{p,1})$. We consider the complex $\mathcal{L}_X^\bullet := \bigoplus_{p\in\mathbb{Z}} \mathcal{L}_X^p[-p]$ as a double complex. Then the canonical morphism of double complexes $\mathcal{L}_X^\bullet \to \mathcal{A}_X^{\bullet,\bullet}$ given by the inclusions $\mathcal{L}_X^p \to \mathcal{A}_X^{p,0}$ induces a quasi-isomorphism of the respective total complexes*

$$\mathcal{L}_X^\bullet \xrightarrow{\sim} \mathcal{A}_X^\bullet .$$

Proof. By [KS90, 1.9.3] we have to show that $\mathrm{H}_1 \mathrm{H}_2(\mathcal{L}_X^\bullet) \to \mathrm{H}_1 \mathrm{H}_2(\mathcal{A}_X^{\bullet,\bullet})$ is an isomorphism of double complexes, where for a double complex (A, f_1, f_2), $(\mathrm{H}_1(A), h_1, h_2)$ and $(\mathrm{H}_2(A), g_1, g_2)$ are the double complexes

$$\mathrm{H}_1^{p,q}(A) := \ker(f_1^{p,q}) / \operatorname{im}(f_1^{p-1,q}), \qquad h_1 := 0, \qquad h_2 := f_2,$$

$$\mathrm{H}_2^{p,q}(A) := \ker(f_2^{p,q}) / \operatorname{im}(f_2^{p,q-1}), \qquad g_1 := f_1, \qquad g_2 := 0.$$

Now for $p \in \mathbb{Z}$ the map

$$\mathcal{L}_X^p = \mathrm{H}_2(\mathcal{L}_X^\bullet)^{p,0} \to \mathrm{H}_2(\mathcal{A}_X^{\bullet,\bullet})^{p,0} = \ker(d_2)^{p,0} = \ker(d'')^{p,0}$$

already is an isomorphism and both double complexes $\mathrm{H}_2(\mathcal{L}_X^\bullet)$ and $\mathrm{H}_2(\mathcal{A}_X^{\bullet,\bullet})$ are trivial otherwise, so the claim follows from the fact that $d_1|_{\mathcal{L}_X^p} = 0$ for every $p \in \mathbb{Z}$ (corollary 2.15). □

Remark 3.6. In particular this shows that we have canonical isomorphisms in the derived category of sheaves of $\mathbb{R}$-vector spaces on X,

$$\mathcal{D}(\mathcal{L}_X^\bullet) \xrightarrow{\sim} \mathcal{D}(\mathcal{A}_X^\bullet) \xrightarrow{=} \mathcal{D}_X^\bullet,$$

where $\mathcal{D}_X^\bullet$ is the complex of sheaves described in the previous section.

Corollary 3.7. *For every left exact functor F on $\mathrm{Shv}(X)$, there are isomorphisms*

$$\mathrm{R}^k F(\mathcal{A}_X^\bullet) \cong \bigoplus_{p+q=k} \mathrm{R}^q F(\mathcal{L}_X^p).$$

In particular, we have direct sum decompositions

$$\mathrm{R}^k \Gamma(X, \mathcal{A}_X^\bullet) \cong \bigoplus_{p+q=k} \mathrm{H}^q(X, \mathcal{L}_X^p),$$

$$\mathrm{R}^k \Gamma_c(X, \mathcal{A}_X^\bullet) \cong \bigoplus_{p+q=k} \mathrm{H}_c^q(X, \mathcal{L}_X^p).$$

Proof. This is purely formal: Derived functors commute with finite direct sums, so by 3.5 we have

$$\mathrm{R}^k F(\mathcal{A}_X^\bullet) = \mathrm{R}^k F(\bigoplus_{p\in\mathbb{Z}} \mathcal{L}_X^p[-p]) = \bigoplus_{p\in\mathbb{Z}} \mathrm{R}^k F(\mathcal{L}_X^p[-p]) = \bigoplus_{p\in\mathbb{Z}} \mathrm{R}^{k-p} F(\mathcal{L}_X^p).$$

□

Corollary 3.8. *Let X be a polyhedrally starshaped polyhedral space. Then the map*

$$\phi^k : \mathrm{H}^0(X, \mathcal{L}_X^k) \xrightarrow{\sim} \mathrm{R}^k\,\Gamma(X, \mathcal{A}_X^\bullet),$$

mapping α to the class of the $(k,0)$-form α, is an isomorphism.

Proof. Let us denote by

$$\mathrm{Z}_{d_2}^{k,0} := \{\alpha \in \mathcal{A}_X^{k,0}(X); d_2\alpha = 0\} = \mathcal{L}_X^k(X),$$

$$\mathrm{Z}_d^{p,0} := \{\alpha \in \mathcal{A}_X^{p,0}; d\alpha = 0\}, \qquad \mathrm{Z}_d^k := \{\alpha \in \mathcal{A}_X^k; d\alpha = 0\}$$

the three spaces of closed forms we will consider.

We first show that ϕ^k is well defined: Because for $\alpha \in \mathcal{A}_X^{k,0}(X)$, $d_2\alpha = 0$ implies $d_1\alpha = 0$ by corollary 2.15, we get a chain of morphisms

$$\begin{aligned} \mathrm{H}^0(X, \mathcal{L}_X^k) &= \mathrm{Z}_{d_2}^{k,0} \subset \mathrm{Z}_d^{k,0} \\ &\subset \mathrm{Z}_d^k \to \mathrm{H}^k\left(\Gamma(X, \mathcal{A}_X^\bullet)\right) \\ &= \mathrm{R}^k\,\Gamma(X, \mathcal{A}_X^\bullet), \end{aligned}$$

as required.

From the previous corollary we already know that there is an isomorphism

$$\mathrm{R}^k\,\Gamma(X, \mathcal{A}_X^\bullet) \cong \bigoplus_{p+q=k} \mathrm{H}^q(X, \mathcal{L}_X^p) = \mathrm{H}^0(X, \mathcal{L}_X^k),$$

and the latter $\mathbb{R}$-vector space is finite dimensional by proposition 2.11. Note that $\mathrm{H}^q(X, \mathcal{L}_X^p)$ vanishes for $q \neq 0$ by proposition 2.11 as well. Hence, if we can show that $\phi^k : \mathrm{H}^0(X, \mathcal{L}_X^k) \to \mathrm{R}^k\,\Gamma(X, \mathcal{A}_X^\bullet)$ is injective, it has to be an isomorphism.

Assume that $\alpha \in \mathrm{Z}_{d_2}^{k,0}$ maps to zero in $\mathrm{R}^k\,\Gamma(X, \mathcal{A}_X^\bullet)$, i.e.

$$\alpha = d\beta \text{ with } \beta = (\beta^{p,q})_{p+q=k-1} \in \mathcal{A}_X^{k-1}(X).$$

In particular, we have $\alpha = d_1\beta^{k-1,0}$ and $d_2\beta^{p,q-1} = (-1)^{p-1}d_1\beta^{p-1,q}$ for every $q \neq 0$. We show that this implies that α is already zero itself, i.e. ϕ^k is injective. This is obvious for $k = 0$. For $k = 1$, $\beta^{0,0}$ satisfies $d_2\beta^{0,0} = 0$ and hence we have $\alpha = d_1\beta^{0,0} = 0$ as well.

Let us now consider the case $k > 1$. Inductively, we construct a sequence

$$\gamma^{p,q} \in \mathcal{A}_X^{p,q}, p + q = k - 2,$$

for $p \leq k - 2$ such that

$$\beta^{p,q} + (-1)^p d_1\gamma^{p-1,q} = d_2\gamma^{p,q-1}$$

for $p + q = k - 1$, $p \leq k - 2$:

For $p < 0$ we set $\gamma^{p,q} = 0$. For $p = 0$ we have $d_2\beta^{0,k-1} = 0$. Because $\mathcal{A}_X^{0,\bullet}(X)$ is exact in positive degrees by proposition 2.11 and $k-1 > 0$ by assumption, we find $\gamma^{0,k-2} \in \mathcal{A}_X^{0,k-2}(X)$ with $d_2\gamma^{0,k-2} = \beta^{0,k-1} = \beta^{0,k-1} - d_1\gamma^{-1,k-1}$. For $0 < p \leq k - 2$, assume that $\gamma^{p-2,q+1}$ and $\gamma^{p-1,q}$ have already been constructed for $p + q = k - 1$. We then have

$$\begin{aligned} d_2 d_1\gamma^{p-1,q} &= d_1\left(\beta^{p-1,q+1} + (-1)^{p-1}d_1\gamma^{p-2,q+1}\right) \\ &= (-1)^{p-1}d_2\beta^{p,q}, \end{aligned}$$

and hence $d_2\left(\beta^{p,q} + (-1)^p d_1\gamma^{p-1,q}\right) = 0$. Once again, $\mathcal{A}_X^{p,\bullet}(X)$ is exact in positive degrees and $q = k - 1 - p > 0$ by assumption, so we find $\gamma^{p,q-1} \in \mathcal{A}_X^{p,q-1}(X)$ as required.

This allows us to show that $\alpha = 0$ in the case $k > 1$ as well: For $p = k-1$ we now get

$$\begin{aligned} d_1 d_2 \gamma^{k-2,0} &= d_1(\beta^{k-2,1} + (-1)^{k-2} d_1 \gamma^{k-3,1}) \\ &= (-1)^{k-2} d_2 \beta^{k-1,0} \end{aligned}$$

and hence $\beta^{k-1,0} + (-1)^{k-1} d_1 \gamma^{k-2,0}$ is a d_2-closed $(k-1,0)$-form. But with corollary 2.15 this implies that

$$\alpha = d_1 \beta^{k-1,0} = d_1 \left(\beta^{k-1} + (-1)^{k-1} d_1 \gamma^{k-2,0} \right) = 0.$$

This shows that ϕ^k is injective for every $k \in \mathbb{Z}$, finishing the proof. □

Remark 3.9. Corollary 3.8 could also be deduced from [Jel16b, 2.2.35], where surjectivity of the map in question is shown.

3.3 Wedge and cap products

In the following subsections we will translate known operations on bigraded forms and currents to the total complexes $(\mathcal{A}_X^\bullet, d)$ and $(\mathcal{D}_X^\bullet, \partial)$ introduced in the previous section.

Proposition 3.10. *Let X be a tropical space of dimension n.*

1. *The wedge product $\wedge : \mathcal{A}_X^{p',q'} \otimes \mathcal{A}_X^{p'',q''} \to \mathcal{A}_X^{p,q}$ from construction 2.1 induces a* wedge product *on the total complexes by*

$$\begin{aligned} \wedge : \mathcal{A}_X^\bullet \otimes^\bullet \mathcal{A}_X^\bullet &\to \mathcal{A}_X^\bullet, \\ \wedge : \Gamma(U, \mathcal{A}_X^\bullet) \otimes^\bullet \Gamma_c(U, \mathcal{A}_X^\bullet) &\to \Gamma_c(U, \mathcal{A}_X^\bullet), \\ \alpha^{p',q'} \otimes \beta^{p'',q''} &\mapsto \alpha^{p',q'} \wedge \beta^{p'',q''}. \end{aligned}$$

2. *The wedge product induces a* cap product *on the total complexes*

$$\begin{aligned} \cap : \mathcal{D}_X^\bullet \otimes^\bullet \mathcal{A}_X^\bullet &\to \mathcal{D}_X^\bullet, \\ (\delta^r \cap \alpha^k)(\eta) &:= (-1)^k \delta^r(\alpha^k \wedge \eta). \end{aligned}$$

In particular, for $\alpha \in \mathcal{A}_X^k(U)$, $\delta \in \mathcal{D}_X^r(U)$ for some open subset $U \subset X$, we get equations

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta, \qquad \partial(\delta \cap \alpha) = \partial\delta \cap \alpha + (-1)^r \delta \cap d\alpha.$$

Proof. 1. We have to show that for a (p,q)-form α with $p+q=k$ and an (r,s)-form β with $r+s=l$ we have $d \circ \wedge(\alpha \otimes \beta) = \wedge \circ d_\otimes(\alpha \otimes \beta)$:

$$\begin{aligned} d \circ \wedge(\alpha \otimes \beta) &= d(\alpha \wedge \beta) \\ &= d'(\alpha \wedge \beta) + d''(\alpha \wedge \beta) \\ &= d'\alpha \wedge \beta + (-1)^k \alpha \wedge d'\beta + d''\alpha \wedge \beta + (-1)^k \alpha \wedge d''\beta \\ &= (d_1 + (-1)^p d_2)\alpha \wedge \beta + (-1)^k \alpha \wedge (d_1 + (-1)^r d_2)\beta \\ &= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \\ &= \wedge \circ d_\otimes(\alpha \otimes \beta). \end{aligned}$$

This shows what we needed. For the following calculation we keep in mind that we have shown the identity

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

2. First, let us see if degrees match appropriately: For $\alpha \in \mathcal{A}^k_X(U)$ and $\delta \in \mathcal{D}^r_X(U)$, $\delta \cap \alpha$ is a map defined on all compactly supported forms η on U of degree $-r-k$. In other words, $\delta \cap \alpha$ is an element of $\mathcal{D}^{k+r}_X(U)$. This is what we needed.

   As before, we have to show for a k-form α, $\delta \in \mathcal{D}^r_X$ and a $-r-k-1$-form η that we have $\partial \circ \cap(\delta \otimes \alpha)(\eta) = \cap \circ d_\otimes(\delta \otimes \alpha)(\eta)$. This can be seen as follows:

$$\begin{aligned}
\partial(\delta \cap \alpha)(\eta) &= (-1)^{r+k} \delta \cap \alpha(d\eta) \\
&= (-1)^r \delta(\alpha \wedge d\eta) \\
&= (-1)^{r+k} \delta(d(\alpha \wedge \eta) - d\alpha \wedge \eta) \\
&= (-1)^k \partial\delta(\alpha \wedge \eta) + (-1)^r \delta \cap d\alpha(\eta) \\
&= (\partial\delta \cap \alpha)(\eta) + (-1)^r (\delta \cap d\alpha)(\eta) \\
&= \cap(d_\otimes(\delta \otimes \alpha)).
\end{aligned}$$

□

3.4 A projection formula

Recall, that a *morphism* of tropical spaces X, Y is a map $f : Y \to X$ that can be given by extended affine maps in suitable charts. Recall the definition of the pullback morphism from proposition 2.5:

Definition 3.11. Let $f : Y \to X$ be a morphism of tropical spaces.

1. Let U, V be charts of X, Y on which f is an extended affine map. Then, for every form $\alpha \in \mathcal{A}^{p,q}_X(U)$, the pullback $f^*\alpha \in \mathcal{A}^{p,q}_Y(V)$ is well defined and compatible with the differential maps d_1 and d_2 and the wedge product. This is compatible with restriction and transition maps, yielding a morphism of double complexes

$$f^* : \mathcal{A}^{\bullet,\bullet}_X \to f_* \mathcal{A}^{\bullet,\bullet}_Y .$$

2. Similarly, we denote by

$$f^* : \mathcal{A}^\bullet_X \to f_* \mathcal{A}^\bullet_Y$$

   the induced morphism on the total complexes. Using the quasi-isomorphism $\mathcal{L}^\bullet_X \to \mathcal{A}^\bullet_X$, we may also interprete this as a morphism $f^* : \mathcal{L}^\bullet_X \to \mathrm{R} f_* \mathcal{L}^\bullet_Y$ in the derived category of sheaves on X.

We will now work with the 'lower shriek' functor $f_!$ and its right derived functor $\mathrm{R} f_!$ as introduced in remark A.15.

Proposition 3.12. *Let $f : Y \to X$ be a morphism of tropical spaces of dimensions n and m respectively. Then f induces natural morphisms of bounded complexes of sheaves and in $\mathrm{D}^b(X)$,*

$$f_* : f_! \mathcal{D}^\bullet_Y \to \mathcal{D}^\bullet_X,$$
$$f_* : \mathrm{R} f_! \mathcal{D}(\mathcal{L}^\bullet_Y) \to \mathcal{D}(\mathcal{L}^\bullet_X).$$

Proof. The morphism in the derived category can be obtained in a purely formal way from the adjunction $(\mathrm{R} f_!, f^!)$, but we will construct the morphism $f_* : f_! \mathcal{D}^\bullet_Y \to \mathcal{D}^\bullet_X$ directly:

Let $\phi \in f_! \mathcal{D}^k_Y(U)$ for $U \subset X$ open and $\eta \in \Gamma_c(U, \mathcal{A}^{-k}_X)$. Then the intersection

$$K := \mathrm{supp}(\phi) \cap \mathrm{supp}(f^*\eta) \subset \mathrm{supp}(\phi) \cap f^{-1}(\mathrm{supp}(\eta)) \subset f^{-1}(U)$$

is compact. Hence, we find a relatively compact neighbourhood K' of K and a form $\eta' \in \mathcal{A}_Y^{-k}(U)$ with $\eta' = \phi^*\eta$ in a neighbourhood of K and $\operatorname{supp}(\eta') \subset K'$. Then

$$f_*\phi(\eta) := \phi(f^*\eta) := \phi(\eta')$$

is independent of the choice of η' and defines a linear map $\Gamma_c(U, \mathcal{A}_X^{-k}) \to \mathbb{R}$. It is easy to see that this definition is compatible with the differential and restriction maps.

This way, for every k we get a morphism of sheaves

$$f_* : f_! \, \mathcal{D}_Y^k \to \mathcal{D}_X^k$$

inducing a morphism

$$f_* : f_! \, \mathcal{D}_Y^\bullet \to \mathcal{D}_X^\bullet$$

as required (because $df^*\alpha = f^*d\alpha$ for $\alpha \in \mathcal{A}_X^{-k}(U)$).

Note that the canonical morphism $f_! \, \mathcal{D}_Y^\bullet \to \mathrm{R} \, f_! \, \mathcal{D}_Y^\bullet$ is an isomorphism in the derived category because $\mathcal{D}_Y^\bullet$ is a complex of flabby sheaves; with proposition 3.4 and proposition 3.5 we get the morphism in the derived category as needed. ☐

Proposition 3.13 (Projection formula). *Let $f : Y \to X$ be a morphism of smooth tropical spaces. We have a canonical commuting diagram of morphisms in $\mathrm{D}^b(X)$, natural in f,*

$$\begin{array}{ccccc}
\mathrm{R}\, f_! \, \mathcal{D}(\mathcal{L}_Y^\bullet) \otimes^L \mathcal{L}_X^\bullet & \xrightarrow{\mathrm{id} \otimes f^*} & \mathrm{R}\, f_! \, \mathcal{D}(\mathcal{L}_Y^\bullet) \otimes^L \mathrm{R}\, f_* \, \mathcal{L}_Y^\bullet & \longrightarrow & \mathrm{R}\, f_! \left(\mathcal{D}(\mathcal{L}_Y^\bullet) \otimes^L \mathcal{L}_Y^\bullet\right) \\
 & & & & \big\downarrow \cap \\
\big\downarrow f_* \otimes \mathrm{id} & & & & \mathrm{R}\, f_! \, \mathcal{D}(\mathcal{L}_Y^\bullet) \\
 & & & & \big\downarrow f_* \\
\mathcal{D}(\mathcal{L}_X^\bullet) \otimes^L \mathcal{L}_X^\bullet & & \xrightarrow{\quad\quad\cap\quad\quad} & & \mathcal{D}(\mathcal{L}_X^\bullet).
\end{array}$$

Here, $(\cdot) \otimes^L (\cdot)$ denotes the derived tensor product (c.f. example A.11).

Proof. Via the isomorphisms $\mathcal{L}_X^\bullet \to \mathcal{A}_X^\bullet$ and $\mathcal{D}(\mathcal{L}_X^\bullet) \to \mathcal{D}_X^\bullet$, we obtain this diagram from proposition 3.10. Commutativity of the diagram follows in a purely formal fashion: For $U \subset X$ open, $\phi^{r,s} \in \operatorname{Hom}_{\mathbb{R}}(\Gamma_c(f^{-1}(U), \mathcal{A}_Y^{-s,-r}), \mathbb{R})$ with support proper over X, $\alpha^{p,q} \in \mathcal{A}_X^{p,q}(U)$ and $\eta \in \Gamma_c(U, \mathcal{A}_X^{-s-p,-r-q})$ we have

$$\begin{aligned}
(f_*\phi^{r,s} \cap \alpha^{p,q})(\eta) &= (-1)^{p+q} f_*\phi^{r,s}(\alpha^{p,q} \wedge \eta) \\
&= (-1)^{p+q} \phi^{r,s}(f^*(\alpha^{p,q} \wedge \eta)) \\
&= (-1)^{p+q} \phi^{r,s}(f^*\alpha^{p,q} \wedge f^*\eta) \\
&= (\phi^{r,s} \cap f^*\alpha^{p,q})(f^*\eta) \\
&= f_*(\phi^{r,s} \cap f^*\alpha^{p,q})(\eta).
\end{aligned}$$

☐

3.5 Pushforward and sedentarity

In the following we will investigate how the complexes $\mathcal{D}_X^\bullet$ (this section) and $\mathcal{L}_X^\bullet$ (next section) behave when restricted to closed good sedentarities $S \subset X$ as defined in section 1.8. We will make use of the properties of the functor Γ_S of local sections with support in S on sheaves on X, as defined in appendix A.2. In the end, this will give us a (very slight) generalization of [JSS15, 4.23].

Theorem 3.14. *Let X be a regular tropical space and let $\imath : S \subset X$ be the closed embedding of a good sedentarity. Then the pushforward morphism $\imath_* : \mathrm{R}\,\imath_!\,\mathcal{D}(\mathcal{L}_S^\bullet) \to \mathcal{D}(\mathcal{L}_X^\bullet)$ induces a canonical isomorphism in the derived category of sheaves on X:*

$$\mathrm{R}\,\imath_!\,\mathcal{D}(\mathcal{L}_S^\bullet) \xrightarrow{\sim} \mathrm{R}\,\Gamma_S\,\mathcal{D}(\mathcal{L}_X^\bullet).$$

Proof. Because the sheaves $\mathcal{D}^k$ are flabby, we have

$$\mathrm{R}\,\imath_!\,\mathcal{D}_S^\bullet = \imath_!\,\mathcal{D}_S^\bullet,$$

$$\mathrm{R}\,\Gamma_S(\mathcal{D}_X^\bullet) = \Gamma_S(\mathcal{D}_X^\bullet).$$

Also, $\mathcal{D}_X(\mathcal{L}_X^\bullet)$ can be represented by the complex $\mathcal{D}_X^\bullet$, and similarly for S. The claim now follows from the following proposition: □

Proposition 3.15. *Let X be a regular tropical space of dimension n and $\imath : S \subset X$ a closed good sedentarity of X. Then there are canonical isomorphisms of complexes of sheaves on X,*

$$\imath_!\,\mathcal{D}_S^\bullet \to \Gamma_S(\mathcal{D}_X^\bullet),$$

induced by $\imath_*$.

Proof. For every open subset $U \subset X$ and $V = U \cap S$, the maps $\imath_* : \mathcal{D}_S^k(V) \to \mathcal{D}_X^k(U)$ are compatible with restrictions. Moreover, it is clear that $\imath_*\delta|_{U\setminus V} = 0$. We get a natural map

$$\imath_* : \mathcal{D}_S^k(V) \to \Gamma_V(U, \mathcal{D}_X^k),$$
$$\delta \mapsto \imath_*\delta.$$

Assume now that $\imath_*\delta$ is the zero map. Every $-k$-form η on S with compact support in V can be continued to a $-k$-form η' on X with compact support in U such that $\eta'|_V = \eta$. Hence δ has to be zero. This shows that $\imath_*$ is injective.

Now let $\delta' \in \Gamma_{S\cap U}(U, \mathcal{D}_X^k)$; we want to show that δ' lies in the image of $\imath_*$. First, we choose a locally finite covering of U by open subsets U_i and U'_j, $i \in I$ and $j \in J$, such that each U'_j is contained in $U \setminus V$ and each U_i is isomorphic to $(U_i \cap V) \times [-\infty, r_i)^c$ for some $r_i \in \mathbb{R}$, $c \in \mathbb{N}$, and contains $V_i := U_i \cap V$ as the set $V_i \times \{-\infty\}^c$ (via this isomorphism; this is possible due to definition 1.29). We choose a partition of unity $(\chi_i)_{i\in I\cup J}$ subordinate to this cover (in particular we have $\chi_i \in \mathcal{A}_X^{0,0}(U_i)$ for every $i \in I$) and we set $\chi := \sum_{i\in I} \chi_i$. Then χ is a smooth 0-form with $\chi|_V \equiv 1$.

We denoty by $\pi_i : U_i \cong V_i \times [-\infty, r_i)^c \to V_i$ the natural projection. If η is a $(-k)$-form on S with compact support in V, then $\chi_i \cdot \pi_i^*\eta$ is a $(-k)$-form on X with compact support in U_i. The sum $\sigma(\eta) := \sum_{i\in I} \chi_i \cdot \pi_i^*\eta$ is a well defined $(-k)$-form on X with compact support in U. The map

$$\delta : \eta \mapsto \delta'(\sigma(\eta))$$

is linear on $\Gamma_c(V, \mathcal{A}_S^{-k})$, i.e. $\delta \in \mathcal{D}_S^k(V)$. We claim that $\imath_*\delta = \delta'$: By definition 2.4 we have $\eta' - \sigma(\imath^*\eta') = 0$ for every $\eta' \in \Gamma_c(U, \mathcal{A}_X^{-k})$ in a neighbourhood U' of V in X, i.e. $\mathrm{supp}(\eta' - \sigma(\imath^*\eta')) \subset U \setminus V$. Hence,

$$(\delta' - \imath_*\delta)(\eta') = \delta'(\eta' - \sigma(\imath^*\eta')) = 0$$

because $\delta'|_{U\setminus V} = 0$ by assumption.

This shows $\imath_*\delta = \delta'$. We have shown that $\imath_* : \mathcal{D}_S^k(V) \to \Gamma_V(U, \mathcal{D}_X^k)$ is both surjective and injective. It is then obvious that $\imath_* : \mathcal{D}_S^\bullet \to \Gamma_V(\mathcal{D}_X^\bullet)$ is an isomorphism of complexes of sheaves. □

Applying the global section functor $\Gamma(X,\cdot)$ to the short exact sequence of complexes of sheaves on X,

$$0 \to \imath_! \mathcal{D}_S^\bullet \to \mathcal{D}_X^\bullet \to \jmath_* \jmath^{-1} \mathcal{D}_X^\bullet \to 0,$$

we get a long exact sequence of cohomology groups. Together with

$$\mathrm{R}^k \Gamma(X, \mathcal{D}_X^\bullet) = \mathrm{R}^{-k} \Gamma_c(X, \mathcal{L}_X^\bullet)^*,$$

we arrive at the announced generalization of [JSS15, 4.23]:

Corollary 3.16. *Let $S \subset X$ be a good sedentarity. Then the restriction to S of forms on X induces a long exact sequence of cohomology groups*

$$\cdots \to \mathrm{R}^k \Gamma_c(S, \mathcal{L}_S^\bullet)^* \to \mathrm{R}^k \Gamma_c(X, \mathcal{L}_X^\bullet)^* \to \mathrm{R}^k \Gamma_c(U, \mathcal{L}_X^\bullet)^* \to \mathrm{R}^{k-1} \Gamma_c(S, \mathcal{L}_S^\bullet)^* \to \cdots,$$

which decomposes into a direct sum of long exact sequences of the form

$$\cdots \to \mathrm{R}^q \Gamma_c(S, \mathcal{L}_S^p)^* \to \mathrm{R}^q \Gamma_c(X, \mathcal{L}_X^p)^* \to \mathrm{R}^q \Gamma_c(U, \mathcal{L}_X^p)^* \to \mathrm{R}^{q-1} \Gamma_c(S, \mathcal{L}_S^p)^* \to \cdots.$$

In fact, when going through the proof of proposition 3.15 one can see that we could have worked equally well with the bigraded parts $\mathcal{D}_X^{r,s}$ instead of $\mathcal{D}_X^k$. Then one arrives at the following, slightly stronger statement:

Corollary 3.17. *Let X be a regular tropical space of dimension n and $\imath : S \subset X$ a closed good sedentarity of X of codimension c, with complement $\jmath : U = X \setminus S \subset X$. Then the short sequence of double complexes,*

$$0 \to \imath_! \mathcal{D}_S^{\bullet,\bullet} \xrightarrow{\imath_*} \mathcal{D}_X^{\bullet,\bullet} \xrightarrow{\rho} \jmath_* \mathcal{D}_U^{\bullet,\bullet} \to 0$$

is exact (in every bidegree $(r,s) \in \mathbb{Z}^2$).

3.6 Closed forms at sedentarity

We now investigate the behaviour of $\mathcal{L}_X^\bullet$ at a good sedentarity, generalizing [JSS15, 4.28] slightly.

Construction 3.18. Let X be a tropical space and let $S \subset X$ be a closed good sedentarity of codimension 1. Then every open subset U of X has a locally finite covering $\{U_0, U_i; i \in I\}$ by open subsets such that $\overline{U_0} \cap S = \emptyset$ and U_i is isomorphic to $V_i \times [-\infty, r_i)$ for some $r_i \in \mathbb{R}$, $V_i := U_i \cap S$ and every $i \in I$. For every $i \in I$ and $(x,t) \in U_i \setminus V_i$ let $v_i(x,t)$ be the unique integral tangential vector at (x,t) pointing towards V_i, i.e. $(x,t) + tv_i(x,t)$ converges to $x = \pi_i(x,t)$ in U_i, where $\pi_i : U_i \to V_i$ is the natural projection. Then $v_i(x,t)$ is in fact independent from the chosen covering and also from $t \in [-\infty, r_i)$, and we get a well defined vector $v(x)$ for every $x \in \bigcup_{i \in I} V_i$.

For every form $\alpha \in \mathcal{L}_X^p(U_i \setminus V_i)$, we get a well defined form $\imath'_v(\alpha) \in \mathcal{L}_X^{p-1}(U_i \setminus V_i)$, where $\imath_v \alpha(x,t) := \imath_{v(x)} \alpha(x,t)$ denotes the interior product of α with the family of vectors v defined before. Note that because α has locally constant coefficients, $\alpha(x,t)$ only depends on x on $U_i \setminus V_i$ and hence, writing $U' := \bigcup_{i \in I} U_i$, we get a well defined form

$$\imath'_v(\alpha|_{U'}) \in \mathcal{L}_S^{p-1}(S \cap U)$$

for every $\alpha \in \mathcal{L}_X^p(U \setminus (S \cap U))$.

Proposition 3.19. *Let X be a tropical space and let $\imath : S \hookrightarrow X$ be the embedding of a good closed sedentarity of codimension* 1. *Denote by $\jmath : X \setminus S \to X$ the corresponding open embedding. Then there exists a short exact sequence of complexes*

$$0 \to \mathcal{L}_X^\bullet \xrightarrow{\eta} \jmath_* \jmath^{-1} \mathcal{L}_X^\bullet \xrightarrow{\imath'_v} \imath_* \mathcal{L}_S^\bullet[-1] \to 0,$$

where $\imath'_v$ is the map constructed above. In fact, up to the obvious shifts, this short exact sequence decomposes into a direct sum of short exact sequences of sheaves,

$$0 \to \mathcal{L}_X^p \xrightarrow{\eta} \jmath_*\jmath^{-1}\,\mathcal{L}_X^p \xrightarrow{\imath'_v} \imath_*\,\mathcal{L}_S^{p-1} \to 0.$$

Proof. We only show the second half of the statement, which induces the first half immediately by taking the direct sum. It is clear for $p = 0$ (setting $\mathcal{L}_S^{-1} = 0$). For $p \geq 1$ we can show this on stalks. First, let $x \in X \setminus S$. Then the canonical morphism $\eta_x : (\mathcal{L}_X^p)_x \to (\jmath_*\jmath^{-1}\,\mathcal{L}_X^p)_x$ is an isomorphism and $\left(\imath_*\,\mathcal{L}_S^{p-1}\right)_x = 0$, so the claim is trivial.

Let us now fix some notations for the case $p \geq 1$: Let $x \in S$ and let $\phi : U \subset |\Sigma_U| \subset \mathbb{T}^N$ be a tropical chart near x in X, where Σ_U is a polyhedral complex in $\mathbb{T}^N$ which is regular at infinity. We may assume that U is polyhedrally starshaped with center x and that U is of the form $U = V \times [-\infty, r)$ with $V \subset S$ open and $r \in \mathbb{R}$. In fact, we can assume that U is the open star of a face $\sigma \in \Sigma_U$ contained in V with $x \in \operatorname{relint}(\sigma)$ and that V is the open star of σ in the polyhedral complex $\Sigma_V := \Sigma_U|_V := \{\tau \cap V; \tau \in \Sigma_U\}$ in $\mathbb{T}_N^N$. Recall that we write $\mathbb{T}_N^N := \{t \in \mathbb{T}^N; t_N = -\infty\}$. Let $I := \operatorname{sed}(\sigma) \subset [N]$ be the sedentarity of σ, let σ^N be the unique parent face of σ in $\mathbb{T}^N_{I\setminus\{N\}}$ and let σ' be the unique parent face of σ of empty sedentarity (c.f. lemma 1.10).

Now we can show the claim by the following computation: The map $\eta_x : (\mathcal{L}_X^p)_x \to (\jmath_*\jmath^{-1}\,\mathcal{L}_X^p)_x$ is the map dual to the projection

$$p : \sum_{\sigma' \leq \tau'} \bigwedge^p \left(\mathbb{L}(\tau')/\,\mathbb{R}^N_{\operatorname{sed}(\sigma^N)}\right) \to \sum_{\sigma' \leq \tau'} \bigwedge^p \left(\mathbb{L}(\tau')/\,\mathbb{R}^N_{\operatorname{sed}(\sigma)}\right) \to 0.$$

Using the isomorphisms $\left(\mathbb{L}(\tau')/\,\mathbb{R}^N_{\operatorname{sed}(\sigma^N)}\right) \xrightarrow{\sim} \left(\mathbb{L}(\tau')/\,\mathbb{R}^N_{\operatorname{sed}(\sigma)}\right) \oplus \mathbb{R}\,e_N$, we can see that the map

$$0 \to \sum_{\sigma' \leq \tau'} \bigwedge^{p-1} \left(\mathbb{L}(\tau')/\,\mathbb{R}^N_{\operatorname{sed}(\sigma)}\right) \to \sum_{\sigma' \leq \tau'} \bigwedge^p \left(\mathbb{L}(\tau')/\,\mathbb{R}^N_{\operatorname{sed}(\sigma^N)}\right),$$

induced by $\omega \mapsto e_N \wedge \omega$ is then a kernel of p. Dualizing the resulting short exact sequence, we arrive at the short exact sequence

$$0 \to (\mathcal{L}_X^p)_x \xrightarrow{\eta_x} (\jmath_*\jmath^{-1}\,\mathcal{L}_X^p)_x \xrightarrow{(\imath'_v)_x} (\imath_*\,\mathcal{L}_S^{p-1})_x \to 0$$

as required. □

Corollary 3.20. *Let X, S, $\imath$ and $\jmath$ be as above. Then there exists a (non-canonical) isomorphism in the derived category of sheaves of $\mathbb{R}$-vector spaces on X:*

$$\mathrm{R}\,\imath_*\,\mathcal{L}_S^\bullet \simeq \mathrm{R}\,\Gamma_S(\mathcal{L}_X^\bullet)[2].$$

Proof. First note that the canonical morphism $\jmath_*\jmath^{-1}\,\mathcal{L}_X^\bullet \to \jmath_*\jmath^{-1}\,\mathcal{A}_X^\bullet = \mathrm{R}\,\jmath_*\jmath^{-1}\,\mathcal{L}_X^\bullet$ is an isomorphism in the derived category: This can be shown using that for every polyhedrally starshaped open set U, the open set $U \setminus (S \cap U)$ is starshaped as well and then applying proposition 2.11.

We now have a diagram in the derived category

$$\begin{array}{ccccccc}
\mathrm{R}\,\imath_*\,\mathcal{L}_S^\bullet[-2] & \longrightarrow & \mathcal{L}_X^\bullet & \longrightarrow & \mathrm{R}\,\jmath_*\jmath^{-1}\,\mathcal{L}_X^\bullet & \longrightarrow & \mathrm{R}\,\imath_*\,\mathcal{L}_S^\bullet[-1] \\
\vdots & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \vdots \\
\mathrm{R}\,\Gamma_S(\mathcal{L}_X^\bullet) & \longrightarrow & \mathcal{L}_X^\bullet & \longrightarrow & \mathrm{R}\,\jmath_*\jmath^{-1}\,\mathcal{L}_X^\bullet & \longrightarrow & \mathrm{R}\,\Gamma_S(\mathcal{L}_X^\bullet)[1],
\end{array}$$

where the top row is the distinguished triangle obtained from the proposition (use [KS90, 1.7.5]) and the bottom row is the canonical distinguished triangle associated to the functor $\mathrm{R}\Gamma_S$. It is obvious that the solid square commutes and hence we get the dotted arrows by one of the axioms of a triangulated category (c.f. definition A.1(5)). By [KS90, 1.5.5], the dotted arrows are isomorphisms. □

3.7 Integration of forms

In order to formulate Poincaré duality, we now give a short reminder on the integration map

$$\delta_X : \Gamma_c(X, \mathcal{A}_X^{n,n}) \to \mathbb{R}, \qquad \eta \mapsto \int_X \eta,$$

for a tropical space X. In the finitary case of $X \subset \mathbb{R}^N$, this has been introduced in [CD12, 1.5]. For the general case, we will rely on the exposition in [JSS15, 4.1ff]. Note that it is essential that X is tropical, both for the definition of δ_X (we need an integral structure) and for $\int_X$ to vanish on exact forms (this uses the balancing condition).

The following lemma allows us to treat compactly supported $2n$-forms α on X as if they were given by compactly supported forms on some $\mathbb{R}^N$:

Lemma 3.21. *Let X be a tropical space of dimension n and let $\alpha \in \mathcal{A}_X^{p,q}(X)$. Then for every sedentarity S of X of dimension $\dim(S) < \max(p,q)$, the support of α is contained in the complement of S,*

$$\operatorname{supp}(\alpha) \subset X \setminus S.$$

In particular, if $\max(p,q) = n$, the support of α is finitary.

Proof. This can be shown as in [JSS15, 4.1]. □

Definition 3.22. Let X be a tropical space of dimension n and consider a form $\alpha \in \Gamma_c(X, \mathcal{A}_X^{2n})$.

1. First assume that X is a tropical space in $\mathbb{T}^N$, represented by a weighted polyhedral complex (Σ, w). For every $\sigma \in \Sigma$ with $\dim(\sigma) = n$, choose a basis $(x_1^\sigma, \dots, x_n^\sigma)$ of the lattice $\mathbb{L}_\mathbb{Z}(\sigma) \subset \mathbb{L}(\sigma)$. Then $\alpha|_{\sigma^\circ}$ has the form

   $$\alpha|_{\sigma^\circ} = f_\sigma\,(d'x_1^\sigma \wedge \ldots \wedge d'x_n^\sigma) \otimes (d''x_1^\sigma \wedge \ldots \wedge d''x_n^\sigma)$$

   with a smooth function f_σ on the manifold with boundary $\sigma^\circ = \sigma \cap \mathbb{R}^N$ and we set

   $$\int_X \alpha := (-1)^{\frac{n(n-1)}{2}} \sum_{\sigma \in \Sigma_n} w_\sigma \int_{\sigma^\circ} f_\sigma d\lambda_\sigma,$$

   where λ_σ is the measure on $\sigma^\circ \subset \mathbb{R}^N$ given by the lattice basis $(x_i^\sigma)_{i=1}^n$.

2. In general, we choose an atlas $\mathfrak{A}$ for X of tropical charts $\phi_U : U \to V \subset \mathbb{T}^{N_U}$ and a smooth partition of unity $(\chi_U)_{U \in \mathfrak{A}}$ on X subordinate to this covering. Writing $\alpha_U := \chi_U \alpha \circ \phi_U^{-1} \in \Gamma_c(V, \mathcal{A}_V^{n,n})$ we may then define

   $$\int_X \alpha := \sum_{U \in \mathfrak{A}} \int_{\phi_U(U)} \alpha_U.$$

Remark 3.23. We refer to [Gub13, 2.4ff] or [Jel16b, 2.1.43ff] for a more thorough discussion of $\int_X$. The sign in the definition above comes from the equality

$$\left(\bigwedge_{i=1}^n d'x_i\right) \otimes \left(\bigwedge_{i=1}^n d''x_i\right) = (-1)^{\frac{n(n-1)}{2}} \bigwedge_{i=1}^n (d'x_i \otimes d''x_i).$$

Proposition 3.24. *Let X be a tropical space. Then the integration map $\int_X : \Gamma_c(X, \mathcal{A}_X^{n,n}) \to \mathbb{R}$ induces a morphism of complexes,*

$$\Gamma_c(X, \mathcal{A}_X^{\bullet})[2n] \to \mathbb{R},$$
$$\alpha \mapsto \int_X \alpha.$$

In particular, the integration map $\int_X : \Gamma_c(X, \mathcal{A}_X^{n,n}) \to \mathbb{R}$ is a ∂-closed linear current in $\mathcal{D}_X^{-2n}(X)$. Hence, it is also ∂_2- and ∂_1-closed in $\mathcal{D}_X^{-n,-n}(X)$.

Proof. This follows from [Gub13, 3.8]. □

Remark 3.25. Henceforth we will usually write δ_X for the closed current $\int_X(\cdot)$, especially when considering it as a cohomology class in $\mathrm{R}^{-2n}\,\Gamma(X, \mathcal{D}_X^{\bullet})$. For smooth tropical spaces X, we will see next that the cap product with δ_X gives rise to natural Poincaré duality type statements.

3.8 Poincaré duality for Dolbeault cohomology

In [JSS15, ch.4] it has been shown that for every $p, q \in \mathbb{Z}$ the morphism of complexes

$$\Gamma(X, \mathcal{A}_X^{p,\bullet}) \otimes \Gamma_c(X, \mathcal{A}_X^{n-p,\bullet}[n]) \to \mathbb{R},$$
$$\alpha \otimes \eta \mapsto \int_X \alpha \wedge \eta,$$

gives rise to the following Poincaré duality statement ([JSS15, 4.33]):

Proposition 3.26. *Let X be a tropical manifold. Then for every $p, q \in \mathbb{Z}$, the Poincaré map*

$$\mathrm{H}^q(X, \mathcal{L}_X^p) = \mathrm{R}^q\,\Gamma(X, \mathcal{A}_X^{p,\bullet}) \to \mathrm{Hom}_{\mathbb{R}}(\mathrm{R}^{n-q}\,\Gamma_c(X, \mathcal{A}_X^{n-p,\bullet}), \mathbb{R}) = \mathrm{H}_c^{n-q}(X, \mathcal{L}_X^{n-p})^*$$

induced by

$$\alpha \mapsto [\eta \mapsto \int_X \alpha \wedge \eta].$$

is an isomorphism.

For the sake of convenience we will emphasize the formulation in terms of the double complexes $\mathcal{A}_X^{\bullet,\bullet}$ and $\mathcal{D}_X^{\bullet,\bullet}$ (c.f. example 3.3), already present in the proof of [JSS15, 4.33]:

Corollary 3.27. *Let X be a tropical manifold of dimension n.*

1. *For every $q \in \mathbb{Z}$, the Poincaré map induces a quasi-isomorphism of complexes of sheaves on X,*

$$\delta_X \cap (\cdot) : (\mathcal{A}_X^{\bullet,q}, d_1)[n] \to (\mathcal{D}_X^{q-n,\bullet}, \partial_1)$$
$$\delta_X \cap \alpha^{p,q}(\eta) := (-1)^{p+q} \int_X \alpha \wedge \eta.$$

2. *For every $p \in \mathbb{Z}$, the Poincaré map induces a quasi-isomorphism of complexes of sheaves on X,*

$$\delta_X \cap (\cdot) : (\mathcal{A}_X^{p,\bullet}, d_2)[n] \to (\mathcal{D}_X^{\bullet,p-n}, \partial_2)$$
$$\delta_X \cap \alpha^{p,q}(\eta) := (-1)^{p+q} \int_X \alpha \wedge \eta.$$

3.9 Poincaré duality for the total complexes

We now want to prove the analogous version of corollary 3.27 in terms of the total complex $\mathcal{A}_X^\bullet$. It will follow from corollary 3.27 by algebraic means.

Theorem 3.28. *Let X be a tropical manifold of dimension n. Then the cap product $\mathcal{D}_X^\bullet \otimes \mathcal{A}_X^\bullet \to \mathcal{D}_X^\bullet$ induces a quasi-isomorphism of complexes of sheaves on X,*

$$\begin{aligned} \delta_X \cap (\cdot) : \mathcal{A}_X^\bullet[2n] &\xrightarrow{\sim} \mathcal{D}_X^\bullet, \\ \alpha &\mapsto \delta_X \cap \alpha, \end{aligned}$$

with $\delta_X : \eta \mapsto \int_X \eta$ in $\mathcal{D}_X^{-2n}(X)$.

Lemma 3.29. *Let X be a tropical manifold. Then there exists an isomorphism*

$$\mathcal{L}_X^\bullet[2n] \cong \mathcal{D}(\mathcal{L}_X^\bullet)$$

in the derived category of sheaves on X.

Proof. By corollary 3.27 we have isomorphisms $\mathcal{L}_X^p[n] \to \mathcal{D}(\mathcal{L}_X^{n-p})$ in the derived category of sheaves on X for every $p \in \mathbb{Z}$. This gives us

$$\begin{aligned} \mathcal{L}_X^\bullet[2n] &= \bigoplus_{p\in\mathbb{Z}} \mathcal{L}_X^p[-p+2n] \\ &\cong \bigoplus_{p\in\mathbb{Z}} \mathcal{D}(\mathcal{L}_X^{n-p})[-p+n] \\ &\cong \mathcal{D}(\bigoplus_{p\in\mathbb{Z}} \mathcal{L}_X^{n-p}[-n+p]) \\ &\cong \mathcal{D}(\mathcal{L}_X^\bullet). \end{aligned}$$

□

Lemma 3.30. *Let X be a tropical manifold of dimension n and let $U \subset X$ be a polyhedrally starshaped open subset. Then the morphism*

$$\Gamma(U, \mathcal{A}_X^\bullet[2n]) \to \Gamma(U, \mathcal{D}_X^\bullet)$$

induced by $\delta_X\cap$ is a quasi-isomorphism. In other words, $\delta_X\cap$ induces a canonical isomorphism

$$\mathrm{R}\,\Gamma(U, \mathcal{A}_X^\bullet[2n]) \to \mathrm{R}\,\Gamma(U, \mathcal{D}_X^\bullet).$$

Proof. Recall that $\mathcal{D}_X^\bullet$ is a complex of sheaves on X representing $\mathcal{D}(\mathcal{L}_X^\bullet)$. Using the previous lemma, we get the existence of isomorphisms

$$\begin{aligned} \mathrm{R}^k\,\Gamma(U, \mathcal{A}_X^\bullet[2n]) &\cong \mathrm{R}^k\,\Gamma(U, \mathcal{L}_X^\bullet[2n]) \\ &\cong \mathrm{R}^k\,\Gamma(U, \mathcal{D}(\mathcal{L}_X^\bullet)) \\ &\cong \mathrm{R}^k\,\Gamma(U, \mathcal{D}_X^\bullet). \end{aligned}$$

Because by proposition 2.11 we have an isomorphism $\mathrm{R}^k\,\Gamma(U, \mathcal{L}_X^\bullet[2n]) \cong \mathrm{H}^0(U, \mathcal{L}_X^{2n+k})$ of finite dimensional $\mathbb{R}$-vector spaces (corollary 2.12), which are furthermore isomorphic to $\mathrm{R}^k\,\Gamma(U, \mathcal{D}_X^\bullet)$ by lemma 3.29, it suffices to show that

$$\mathrm{H}^k(\delta_X\cap) : \mathrm{R}^k\,\Gamma(U, \mathcal{A}_X^\bullet[2n]) \to \mathrm{R}^k\,\Gamma(U, \mathcal{D}_X^\bullet)$$

is injective. Note that because $\mathcal{A}_X^\bullet$ and $\mathcal{D}_X^\bullet$ are soft and flabby sheaves respectively, it suffices to consider the cohomology groups of the complexes $\mathcal{A}_X^\bullet(U)[2n]$ and $\mathcal{D}_X^\bullet(U)$ respectively.

By corollary 3.8, every d-closed form in $\mathcal{A}_X^{2n+k}(U)$ can be represented up to a d-exact form by a d_2-closed form in $\mathcal{A}_X^{2n+k,0}(U) \subset \mathcal{A}_X^{2n+k}(U)$. Let $\alpha = \alpha^{2n+k,0} \in \mathcal{A}_X^{2n+k,0}(U)$ be such a d_2-closed form and assume that

$$[\alpha] := \delta_X \cap \alpha = \partial\phi \in \mathcal{D}_X^{-n,n+k}(U) \subset \mathcal{D}_X^k(U)$$

with $\phi \in \Gamma(U, \mathcal{D}_X^{k-1})$. We now have to show that this already implies $\alpha = 0$. Recall from 3.3(2) that the complex $\mathcal{D}_X^\bullet$ is the total complex of $(\mathcal{D}_X^{\bullet,\bullet}, \partial_2, \partial_1)$ with $\mathcal{D}_X^{r,s}(U) = \mathrm{Hom}_{\mathbb{R}}(\Gamma_c(U, \mathcal{A}_X^{-s,-r}), \mathbb{R})$ and

$$\partial_2^{r,s} = (-1)^r \mathrm{Hom}_{\mathbb{R}}(d_2, \mathbb{R}), \qquad \partial_1^{r,s} = (-1)^s \mathrm{Hom}_{\mathbb{R}}(d_1, \mathbb{R}).$$

We can write $\phi = (\phi^{r,s})_{r+s=k-1}$ with linear maps $\phi^{r,s} : \Gamma_c(U, \mathcal{A}_X^{-s,-r}) \to \mathbb{R}$. From $[\alpha] = \partial\phi$ we get

$$\begin{aligned} [\alpha]^{-n,k+n} &= \partial_2\phi^{-n-1,k+n} + (-1)^n \partial_1\phi^{-n,n+k-1} + \\ &= (-1)^n \partial_1\phi^{-n,n+k-1}, \\ 0 &= \partial_2\phi^{-q,-p} + (-1)^q \partial_1\phi^{-q+1,-p-1}, \end{aligned} \tag{1}$$

for $p + q = -k + 1$, $q \neq n + 1$. For $2n + k \leq 0$,

$$p + q = -k + 1 \geq 2n + 1$$

implies $p > n$ or $q > n$ and we get $\phi = 0$ immediately. Hence in this case $[\alpha] = 0$ and subsequently, with corollary 3.27, $\alpha = 0$. For $2n + k = 1$ we have

$$0 \overset{(1)}{=} \partial_2\phi^{-n,-n} + (-1)^{-n+1}\partial_1\phi^{-n+1,-n-1} = \partial_2\phi^{-n,-n}.$$

Again by corollary 3.27 we may represent $\phi^{-n,-n}$ by a d_2-closed $(0,0)$-form β. By corollary 2.15 this already implies $d_1\beta = 0$ and with equation (1) we get $[\alpha] = (-1)^n\partial_1[\beta] = 0$ as required.

Now let $2n + k > 1$. We then have

$$0 \overset{(1)}{=} \partial_2\phi^{n+k-1,-n} + (-1)^{n+k}\partial_1\phi^{n+k,-n-1} = \partial_2\phi^{n+k-1,-n},$$

with $n + k - 1 > -n$. Note that once again by corollary 3.27, the complexes $(\mathcal{D}_X^{\bullet,s}(U), \partial_2)$ are exact in degrees $r \neq -n$. We then find a linear map

$$\psi^{n+k-2,-n} : \Gamma_c(U, \mathcal{A}_X^{n,-n-k+2}) \to \mathbb{R} \qquad \text{with} \qquad \partial_2\psi^{n+k-2,-n} = \phi^{n+k-1,-n}.$$

Inductively we now can construct a sequence of maps $\psi^{-q-1,-p} : \Gamma_c(U, \mathcal{A}_X^{p,q+1}) \to \mathbb{R}$ for each $p + q = -k + 1$ with $q < n$ such that

$$\phi^{-q,-p} - \partial_2\psi^{-q-1,-p} = (-1)^q\partial_1\psi^{-q,-p-1} :$$

For $p = n$, $q = -n - k + 1$ we are already done when choosing $\psi^{n+k-1,-n-1} = 0$ and $\psi^{n+k-2,-n}$ as constructed before.

Assume then that the maps $\psi^{-q-1,-p}$ and $\psi^{-q,-p-1}$ have already been constructed for $p+q = -k + 1$ with $q < n - 1$. Then we have

$$0 = \partial_1(\phi^{-q,-p} - \partial_2\psi^{-q-1,-p}) \overset{(1)}{=} (-1)^{q+1}\partial_2\phi^{-q-1,-p+1} - \partial_2\partial_1\psi^{-q-1,-p},$$

and hence $(\phi^{-q-1,-p+1} + (-1)^q\partial_1\psi^{-q-1,-p})$ is ∂_2-closed. Again with $q + 1 < n$ we find $\psi^{-q-2,-p+1}$ with

$$\phi^{-q-1,-p+1} - \partial_2\psi^{-q-2,-p+1} = (-1)^{q+1}\partial_1\psi^{-q-1,-p}.$$

This is enough to finish the induction step.

From the final step $q = n-1$ we now get

$$\begin{aligned} 0 &= \partial_1 \left(\phi^{-n+1,n+k-2} - \partial_2 \psi^{-n,n+k-2}\right) \\ &\overset{(1)}{=} (-1)^n \partial_2 \phi^{-n,n+k-1} - \partial_2 \partial_1 \psi^{-n,n+k-2} \\ &= (-1)^{-n} \partial_2 \left(\phi^{-n,n+k-1} - (-1)^{-n} \partial_1 \psi^{-n,n+k-2}\right). \end{aligned}$$

By corollary 3.27 again, the ∂_2-closed linear map $\phi^{-n,n+k-1} - \partial_1 \psi^{-n,n+k-2}$ can be represented by a d_2-closed $(2n+k-1,0)$-form β, i.e. we have

$$\phi^{-n,n+k-1} = [\beta] + (-1)^{-n} \partial_1 \psi^{-n,n+k-2}.$$

Because $d_1\beta$ has to vanish as well by corollary 2.15,

$$[\alpha] = \partial_1 \phi^{-n,n+k-1} = \partial_1 [\beta] = 0$$

follows. This implies $\alpha = 0$ and finishes the proof. □

Proof of theorem 3.28. Because the homotopy category $\mathrm{K}^b(X)$ of bounded complexes of sheaves on X is a triangulated category, we find a distinguished triangle

$$\mathcal{A}_X^\bullet[2n] \xrightarrow{\delta_X \cap} \mathcal{D}_X^\bullet \to \mathcal{F}^\bullet \to \mathcal{A}_X^\bullet[2n+1]$$

in $\mathrm{K}^b(X)$ with definition A.1(4). Because talking stalks is an exact functor, this induces a distinguished triangle of $\mathbb{R}$-vector spaces for every $x \in X$,

$$\mathcal{A}_{X,x}^\bullet[2n] \xrightarrow{\delta_X \cap} \mathcal{D}_{X,x}^\bullet \to \mathcal{F}_x^\bullet \to \mathcal{A}_{X,x}^\bullet[2n+1].$$

For every $x \in X$ this triangle can be obtained by taking the colimit of the sequences

$$\mathcal{A}_X^\bullet(U)[2n] \xrightarrow{\delta_X \cap} \mathcal{D}_X^\bullet(U) \to \mathcal{F}^\bullet(U) \to \mathcal{A}_X^\bullet(U)[2n+1]$$

in $\mathrm{Mod}_{\mathbb{R}}$, running through polyhedrally starshaped open subsets U with center $x \in U$. Taking this colimit is an exact functor on $\mathrm{Mod}_{\mathbb{R}}$ and because every single $\mathcal{A}_X^\bullet(U)[2n] \to \mathcal{D}_X^\bullet(U)$ is a quasi-isomorphism by lemma 3.30, so is the colimit $\mathcal{A}_{X,x}^\bullet[2n] \to \mathcal{D}_{X,x}^\bullet$.

Taking the long exact sequence of the distinguished triangle of stalks we get that $\mathrm{H}^k(\mathcal{F}_x^\bullet) = 0$ for every $x \in X$ and every $k \in \mathbb{Z}$. This implies that $\mathcal{F}^\bullet$ is exact. Then the first distinguished triangle implies that $\delta_X \cap (\cdot)$ is an isomorphism in the derived category and hence a quasi-isomorphism of complexes. □

Corollary 3.31. *Let X be a tropical manifold of dimension n. Then the integration morphism $\int_X : \Gamma_c(X, \mathcal{A}_X^\bullet[2n]) \to \mathbb{R}$ given by $\alpha \mapsto \int_X \alpha$ for $\alpha \in \mathcal{A}_X^{2n}(X) = \mathcal{A}_X^{n,n}(X)$ induces a non-degenerate pairing in cohomology*

$$\int_X : \mathrm{R}^{n-k}\,\Gamma(X, \mathcal{A}_X^\bullet) \times \mathrm{R}^{n+k}\,\Gamma_c(X, \mathcal{A}_X^\bullet) \to \mathbb{R}$$

for every $k \in \mathbb{Z}$.

Proof. The map

$$\int_X : \mathrm{R}^{n-k}\,\Gamma(X, \mathcal{A}_X^\bullet) \times \mathrm{R}^{n+k}\,\Gamma_c(X, \mathcal{A}_X^\bullet) \to \mathbb{R}$$

is obviously bilinear. Now choose a closed form $\alpha \in \Gamma(X, \mathcal{A}_X^{n-k})$. To have

$$\int_X \alpha \wedge \eta = 0$$

for every closed form $\eta \in \Gamma_c(X, \mathcal{A}_X^{n+k})$ implies

$$\delta_X \cap \alpha = 0 \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{R}^{n+k}\,\Gamma_c(X, \mathcal{A}_X^\bullet), \mathbb{R}) = \mathrm{R}^{-n-k}\,\Gamma(X, \mathcal{D}_X^\bullet).$$

Using theorem 3.28 we get that α is exact.

On the other hand, let $\eta \in \Gamma_c(X, \mathcal{A}_X^{n+k})$ be a closed form with compact support such that $\int_X \alpha \wedge \eta = 0$ for every closed form $\alpha \in \Gamma(X, \mathcal{A}_X^{n-k})$. In order to show that η is exact, we first choose a basis $(\eta_j)_{j\in J}$ for $\mathrm{R}^{n+k}\,\Gamma_c(X, \mathcal{A}_X^\bullet) = \mathrm{H}^{n+k}\,(\Gamma_c(X, \mathcal{A}_X^\bullet))$ and consider the dual basis $(\sigma_j)_{j\in J}$ for $\mathrm{R}^{n+k}\,\Gamma_c(X, \mathcal{A}_X^\bullet)^* = \mathrm{R}^{-n-k}\,\Gamma(X, \mathcal{D}_X^\bullet)$. Using theorem 3.28 we find closed forms $\alpha_j \in \Gamma(X, \mathcal{A}_X^{n-k})$ with

$$\delta_X \cap \alpha_j = \sigma_j \in \mathrm{R}^{-n-k}\,\Gamma(X, \mathcal{D}_X^\bullet).$$

By assumption we then have $\sigma_j(\eta) = (-1)^{n+k} \int_X \alpha_j \wedge \eta = 0$ for each $j \in J$. By the choice of $(\sigma_j)_{j\in J}$, η has to be zero in $\mathrm{R}^{n+k}\,\Gamma_c(X, \mathcal{A}_X^\bullet)$. □

3.10 The conjugation morphism

Definition 3.32. Let X be a polyhedral space, $U \subset X$ an open subset.

1. Writing $\alpha \in \mathcal{A}_X^{p,q}(U)$ in coordinates as $\alpha = \sum_{I,J} \alpha_{IJ} d'x_I \otimes d''x_J$, we define the (q,p)-form $J\alpha$ by
$$J\alpha = (-1)^{pq} \sum_{IJ} \alpha_{IJ} d'x_J \otimes d''x_I,$$
in coordinates. We occasionally use the notation $\overline{\alpha}$ for the form $(-1)^{pq} J\alpha = \sum_{IJ} \alpha_{IJ} d'x_J \otimes d''x_I$.

2. We say that $\alpha \in \mathcal{A}_X^k(U)$ is *symmetric* if $J\alpha = \alpha$ and *antisymmetric* if $J\alpha = -\alpha$ and we call
$$\alpha^+ := \frac{\alpha + J\alpha}{2}, \qquad \alpha^- := \frac{\alpha - J\alpha}{2},$$
the *symmetric* resp. *antisyymetric* component of α. We also set
$$\mathcal{A}_X^{k,+}(U) := \{\alpha; \alpha \text{ is symmetric})\}, \qquad \mathcal{A}_X^{k,-}(U) := \{\alpha; \alpha \text{ is antisymmetric}\}.$$

Lemma 3.33. *Let $\alpha \in \mathcal{A}_X^{p',q'}(U) \subset \mathcal{A}_X^{k'}(U)$ and $\beta \in \mathcal{A}_X^{p'',q''}(U) \subset \mathcal{A}_X^{k''}(U)$ with $p' + q' = k'$ and $p'' + q'' = k''$. Then we have the following formulas:*

$$\begin{aligned}
d_1 J\alpha &= (-1)^{p'} J d_2 \alpha,\\
d_2 J\alpha &= (-1)^{q'} J d_1 \alpha,\\
dJ\alpha &= Jd\alpha,\\
J(\alpha \wedge \beta) &= J\alpha \wedge J\beta,\\
J(\alpha \wedge J\beta) &= (-1)^{k'k''} \beta \wedge J\alpha.
\end{aligned}$$

Proof. These are easy computations. We show the latter three equations, starting with $dJ = Jd$: For every $p + q = k + 1$ we have

$$\begin{aligned}
(dJ\alpha)^{q,p} &= d_1 (J\alpha)^{q-1,p} + (-1)^q d_2 (J\alpha)^{q,p-1}\\
&= (-1)^{qp-p} d_1 \overline{\alpha^{p,q-1}} + (-1)^{qp} d_2 \overline{\alpha^{p-1,q}}\\
&= (-1)^{qp} \overline{(d_1 \alpha^{p-1,q} + (-1)^p d_2 \alpha^{p,q-1})}\\
&= (Jd\alpha)^{q,p}.
\end{aligned}$$

Next, we write $\alpha = \sum \alpha_{IJ} d'x_I \otimes d''x_J$ and $\beta = \sum \beta_{KL} d'x_K \otimes d''x_L$ in coordinates. Recall (section 2.1) that the wedge product $\alpha \wedge \beta$ can be given in coordinates as

$$\sum (-1)^{|J||K|} \alpha_{IJ} \beta_{KL} d'x_I \wedge d'x_K \otimes d''x_J \wedge d''x_L,$$

where the sum run through all subsets $I, J, K, L \subset [N]$ with $(|I|, |J|, |K|, |L|) = (p', q', p'', q'')$ by assumption (for a suitable $N \in \mathbb{N}$). We then arrive at

$$J(\alpha \wedge \beta) = (-1)^{(p'+p'')(q'+q'')+q'p''} \sum \alpha_{IJ} \beta_{KL} d'x_{JL} \otimes d''x_{IK},$$
$$J\alpha \wedge J\beta = (-1)^{p'q'+p''q''+p'q''} \sum \alpha_{IJ} \beta_{KL} d'x_{JL} \otimes d''x_{IK}.$$

This shows $J(\alpha \wedge \beta) = J\alpha \wedge J\beta$.

Lastly, $J(\alpha \wedge J\beta) = (-1)^{k'k''} \beta \wedge J\alpha$ is a simple application of

$$(d'x_I \otimes d''x_J) \wedge (d'x_K \otimes d''x_L) = (-1)^{(|I|+|J|)(|K|+|L|)} (d'x_K \otimes d''x_L) \wedge (d'x_I \otimes d''x_J)$$

and the previous equation. □

As a direct application of lemma 3.33 we obtain:

Proposition 3.34. *Let X be a polyhedral space.*

1. *Both $\mathcal{A}_X^{k,+} : U \mapsto \mathcal{A}_X^{k,+}(U)$ and $\mathcal{A}_X^{k,-} : U \mapsto \mathcal{A}_X^{k,-}(U)$ define subsheaves of $\mathcal{A}_X^k$ and we have morphisms of sheaves $(\cdot)^+ : \mathcal{A}_X^k \to \mathcal{A}_X^{k,+}$ and $(\cdot)^- : \mathcal{A}_X^k \to \mathcal{A}_X^{k,-}$.*
2. *Together with the induced differentials on $\mathcal{A}_X^{k,+}$ and $\mathcal{A}_X^{k,-}$, this gives a direct sum decomposition of complexes*
$$\mathcal{A}_X^\bullet = \mathcal{A}_X^{\bullet,+} \oplus \mathcal{A}_X^{\bullet,-}.$$

In particular, we have direct sum decompositions

$$\mathrm{R}^k\,\Gamma(U, \mathcal{A}_X^\bullet) = \mathrm{R}^k\,\Gamma(U, \mathcal{A}_X^{\bullet,+}) \oplus \mathrm{R}^k\,\Gamma(U, \mathcal{A}_X^{\bullet,-}),$$
$$\mathrm{R}^k\,\Gamma_c(U, \mathcal{A}_X^\bullet) = \mathrm{R}^k\,\Gamma_c(U, \mathcal{A}_X^{\bullet,+}) \oplus \mathrm{R}^k\,\Gamma_c(U, \mathcal{A}_X^{\bullet,-}).$$

Proof.
1. This is clear.
2. With lemma 3.33, $d \circ (\cdot)^+ = (\cdot)^+ \circ d$ and $d \circ (\cdot)^- = (\cdot)^- \circ d$ follows immediately. This shows that $(\cdot)^+$ and $(\cdot)^-$ are in fact morphisms of complexes. It is clear that for $\alpha \in \mathcal{A}_X^k(U)$ symmetric we have $\alpha^+ = \alpha$, and similarly for α antisymmetric. Together with $\alpha = \alpha^+ + \alpha^-$, this shows the claim.

□

This allows us to introduce the following pairing on compact manifolds:

Proposition 3.35. *Let X be a compact n-dimensional tropical manifold.*

1. *The pairing*
$$e_X : \mathrm{R}^n\,\Gamma(X, \mathcal{A}_X^\bullet) \times \mathrm{R}^n\,\Gamma(X, \mathcal{A}_X^\bullet) \to \mathbb{R},$$
$$(\alpha, \beta) \mapsto \int_X \alpha \wedge J\beta,$$
is symmetric and non-degenerate.

2. *If n is even, then the subspaces $\mathrm{R}^n\,\Gamma(X, \mathcal{A}_X^{\bullet;+})$ and $\mathrm{R}^n\,\Gamma(X, \mathcal{A}_X^{\bullet;-})$ are orthogonal to each other with respect to this pairing. If n is odd, then $\mathrm{R}^n\,\Gamma(X, \mathcal{A}_X^{\bullet;+})$ and $\mathrm{R}^n\,\Gamma(X, \mathcal{A}_X^{\bullet;-})$ are othogonal to themselves.*

Proof. 1. Symmetry of e_X follows from lemma 3.33: For α and β in $\mathcal{A}_X^n(X)$ we have

$$\begin{aligned} e_X(\alpha, \beta) &= \int_X \alpha \wedge J\beta \\ &= \int_X \overline{\alpha \wedge J\beta} \\ &= (-1)^{n^2} \int_X J(\alpha \wedge J\beta) \\ &= \int_X \beta \wedge J\alpha \\ &= e_X(\beta, \alpha). \end{aligned}$$

From Poincaré Duality 3.31 it follows directly that e_X is non-degenerate.

2. Let n be even. For $\alpha = \alpha^+$ symmetric and $\beta = \beta^-$ antisymmetric, we can compute as follows:

$$\begin{aligned} e_X(\alpha^+, \beta^-) &= \int_X J(\alpha^+ \wedge J\beta^-) \\ &= \int_X J\alpha^+ \wedge J^2\beta^- \\ &= -e_X(\alpha^+, \beta^-) \end{aligned}$$

and hence we get $e_X(\alpha^+, \beta^-) = 0$ as alleged. The case n odd follows similarly with $e_X(\cdot, \cdot) = -\int_X J(\cdot \wedge J\cdot)$.

□

3.11 A Künneth formula

Proposition 3.36. *Let X and Y be tropical spaces and let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the canonical projections. Then we have a canonical isomorphism of complexes of sheaves on $X \times Y$,*

$$\mathcal{L}_{X\times Y}^{\bullet} \xrightarrow{\sim} (p_X^{-1}\,\mathcal{L}_X^{\bullet}) \otimes_{\mathbb{R}}^{\bullet} (p_Y^{-1}\,\mathcal{L}_Y^{\bullet}).$$

Proof. The tensor product of complexes $(p_X^{-1}\,\mathcal{L}_X^{\bullet}) \otimes_{\mathbb{R}}^{\bullet} (p_Y^{-1}\,\mathcal{L}_Y^{\bullet})$ (c.f. definition 3.2) has a direct sum decomposition

$$\begin{aligned} (p_X^{-1}\,\mathcal{L}_X^{\bullet}) \otimes_{\mathbb{R}}^{\bullet} (p_Y^{-1}\,\mathcal{L}_Y^{\bullet}) &= \left(\bigoplus_{p' \in \mathbb{Z}} p_X^{-1}\,\mathcal{L}_X^{p'}[-p'] \right) \otimes_{\mathbb{R}}^{\bullet} \left(\bigoplus_{p'' \in \mathbb{Z}} p_Y^{-1}\,\mathcal{L}_Y^{p''}[-p''] \right) \\ &= \bigoplus_{p', p'' \in \mathbb{Z}} \left(p_X^{-1}\,\mathcal{L}_X^{p'} \right) \otimes_{\mathbb{R}} \left(p_Y^{-1}\,\mathcal{L}_Y^{p''} \right) [-p' - p'']. \end{aligned}$$

Hence, it remains to show that $\mathcal{L}_{X\times Y}^p$ is isomorphic to the sheaf $\bigoplus_{p'+p''=p} \pi_X^*\,\mathcal{L}_X^{p'} \otimes \pi_Y^*\,\mathcal{L}_Y^{p''}$. First, we assume that X and Y are tropical spaces in $\mathbb{T}^N$ and $\mathbb{T}^M$ respectively. By proposition 2.10, we then see that, for $(x, y) \in X \times Y$ and a suitable open neighbourhood U of

(x, y), we have isomorphisms

$$\begin{aligned}
\mathcal{L}^p_{X\times Y}(U) &= \left(\sum_{(x,y)\in\tau'\times\tau''} \bigwedge^p \mathbb{L}_{X\times Y}(\tau'\times\tau'') \right)^* \\
&= \left(\sum_{x\in\tau', y\in\tau''} \bigwedge^p \mathbb{L}_X(\tau')\times\mathbb{L}_Y(\tau'') \right)^* \\
&\cong \left(\bigoplus_{p'+p''=p} \left(\sum_{x\in\tau', y\in\tau''} \bigwedge^{p'} \mathbb{L}_X(\tau') \otimes \bigwedge^{p''} \mathbb{L}_Y(\tau'') \right) \right)^* \\
&\cong \bigoplus_{p'+p''=p} \left(\sum_{x\in\tau'} \bigwedge^{p'} \mathbb{L}_X(\tau') \right)^* \otimes \left(\sum_{y\in\tau''} \bigwedge^{p''} \mathbb{L}_Y(\tau'') \right)^* \\
&\cong \bigoplus_{p'+p''=p} \left(\pi_X^{-1}\mathcal{L}^{p'}_X \otimes \pi_Y^{-1}\mathcal{L}^{p''}_Y \right)(U),
\end{aligned}$$

compatible with restrictions. Together, this describes an isomorphism of constructible sheaves

$$\mathcal{L}^p_{X\times Y} \cong \bigoplus_{p'+p''=p} \pi_X^{-1}\mathcal{L}^{p'}_X \otimes \pi_Y^{-1}\mathcal{L}^{p''}_Y$$

on $X \times Y$.

If X and Y are general tropical spaces, we can choose atlases $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ for X and Y. On each open subset $U_i \times V_j \subset X \times Y$ we get an isomorphism as above and one can show that these glue to an isomorphism $\mathcal{L}^p_{X\times Y} \cong \bigoplus_{p'+p''=p} \left(\pi_X^{-1}\mathcal{L}^{p'}_X \otimes \pi_Y^{-1}\mathcal{L}^{p''}_Y \right)$ as before. This finishes the proof. □

Corollary 3.37. *We get the following Kuenneth formula for cohomology groups with compact support in open subsets $U \subset X$, $V \subset Y$:*

$$\begin{aligned}
\mathrm{R}^k\,\Gamma_c(U\times V, \mathcal{L}^\bullet_{X\times Y}) &\cong \bigoplus_{k'+k''=k} \mathrm{R}^{k'}\,\Gamma_c(U, \mathcal{L}^\bullet_X) \otimes \mathrm{R}^{k''}\,\Gamma_c(V, \mathcal{L}^\bullet_Y), \\
\mathrm{H}^q_c(U\times V, \mathcal{L}^p_{X\times Y}) &\cong \bigoplus_{q'+q''=q}\ \bigoplus_{p'+p''=p} \mathrm{H}^{q'}_c(U, \mathcal{L}^{p'}_X) \otimes \mathrm{H}^{q''}_c(V, \mathcal{L}^{p''}_Y).
\end{aligned}$$

Proof. For this we just have to note that every sheaf of $\mathbb{R}$-vector spaces is flat and hence

$$(p_X^{-1}\mathcal{L}^\bullet_X) \otimes_{\mathbb{R}} (p_Y^{-1}\mathcal{L}^\bullet_Y) = (p_X^{-1}\mathcal{L}^\bullet_X) \otimes^L_{\mathbb{R}} (p_Y^{-1}\mathcal{L}^\bullet_Y),$$

with the derived tensor product from example A.11. Then we can apply [KS90, Ex. II.18]. □

3.12 Some examples of smooth tropical surfaces

We will now compute the Dolbeault cohomology groups $\mathrm{H}^q(X, \mathcal{L}^p_X)$ for several choice smooth tropical surfaces. For now we will restrict ourselves to surfaces locally isomorphic to $\mathbb{T}^2$.

In order to keep the exposition neat we will use the following shorthands:

Definition 3.38. Let X be a smooth tropical surface. Then we will denote by $h^{\bullet,\bullet}_{d_2}(X)$ the 3×3-matrix

$$h^{\bullet,\bullet}_{d_2}(X) = \begin{pmatrix} h^{0,0}_X & h^{0,1}_X & h^{0,2}_X \\ h^{1,0}_X & h^{1,1}_X & h^{1,2}_X \\ h^{2,0}_X & h^{2,1}_X & h^{2,2}_X \end{pmatrix}$$

with $h_X^{p,q} := \dim_{\mathbb{R}} \mathrm{H}^q(X, \mathcal{L}_X^p)$.

Let $X = U_1 \cup U_2$ with open subsets U_1, $U_2 \subset X$ and set $U := U_1 \cap U_2$. We then say that the *type of the long exact sequence in cohomology associated to* (U_1, U_2), or the *type of* (U_1, U_2) is the matrix

$$\left(\begin{array}{ccc|ccc|ccc} h_X^{0,0} & h_{U_1}^{0,0} + h_{U_2}^{0,0} & h_U^{0,0} & h_X^{0,1} & h_{U_1}^{0,1} + h_{U_2}^{0,1} & h_U^{0,1} & h_X^{0,2} & h_{U_1}^{0,2} + h_{U_2}^{0,2} & h_U^{0,2} \\ h_X^{1,0} & h_{U_1}^{1,0} + h_{U_2}^{1,0} & h_U^{1,0} & h_X^{1,1} & h_{U_1}^{1,1} + h_{U_2}^{1,1} & h_U^{1,1} & h_X^{1,2} & h_{U_1}^{1,2} + h_{U_2}^{1,2} & h_U^{1,2} \\ h_X^{2,0} & h_{U_1}^{2,0} + h_{U_2}^{2,0} & h_U^{2,0} & h_X^{2,1} & h_{U_1}^{2,1} + h_{U_2}^{2,1} & h_U^{2,1} & h_X^{2,2} & h_{U_1}^{2,2} + h_{U_2}^{2,2} & h_U^{2,2} \end{array} \right).$$

Note that the entries in the p-th row correspond to the dimensions of the spaces appearing in the Mayer-Vietoris sequence for $\mathcal{L}_X^p$,

$$\cdots \to \mathrm{H}^q(X, \mathcal{L}_X^p) \to \mathrm{H}^q(U_1, \mathcal{L}_X^p) \oplus \mathrm{H}^q(U_1, \mathcal{L}_X^p) \to \mathrm{H}^q(U_1 \cap U_2, \mathcal{L}_X^p) \to \mathrm{H}^{q+1}(X, \mathcal{L}_X^p) \to \cdots,$$

associated to the covering (U_1, U_2) (c.f. [KS90, 2.6.10]).

We start of with an easy example:

Example 3.39. Consider the tropical spaces $\mathbb{T}^N$ and $\mathbb{R}^N$. We then have

$$h_{d_2}^{0,0}(\mathbb{T}^N) = 1 \text{ and } h_{d_2}^{p,q}(\mathbb{T}^N) = 0 \text{ otherwise, and}$$

$$h_{d_2}^{p,0}(\mathbb{R}^N) = \begin{pmatrix} N \\ p \end{pmatrix} \text{ for } 0 \leq p \leq N, \text{ and } h_{d_2}^{p,q}(\mathbb{R}^N) = 0 \text{ otherwise.}$$

For later reference, the Dolbeault cohomology of $\mathbb{R} \times \mathbb{T}$ is given by

$$h_{d_2}^{\bullet,\bullet}(\mathbb{R} \times \mathbb{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This follows from computing $\mathrm{H}^0(X, \mathcal{L}_X^p)$ for $X \in \{\mathbb{T}^N, \mathbb{R}^N, \mathbb{R} \times \mathbb{T}\}$ directly and then noting that each of those spaces is polyhedrally starshaped, so we get $\mathrm{H}^q(X, \mathcal{L}_X^p) = 0$ for $q > 0$ from proposition 2.11.

Example 3.40. The Dolbeault cohomology of $\mathbb{R} \times \mathbb{P}^1$, $\mathbb{T} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$h_{d_2}^{\bullet,\bullet}(\mathbb{R} \times \mathbb{P}^1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, h_{d_2}^{\bullet,\bullet}(\mathbb{T} \times \mathbb{P}^1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_{d_2}^{\bullet,\bullet}(\mathbb{P}^1 \times \mathbb{P}^1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We prove this by covering each $X = \mathbb{R} \times \mathbb{P}^1, \mathbb{T} \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1$ by two open subsets $X = U_1 \cup U_2$ and applying the Mayer-Vietoris sequence.

For $X = \mathbb{R} \times \mathbb{P}^1$, we can choose $U_1 \cong U_2 \cong \mathbb{T} \times \mathbb{R}$ and $U_1 \cap U_2 \cong \mathbb{R}^2$. The type of the covering (U_1, U_2) is then

$$\left(\begin{array}{ccc|ccc|ccc} h_X^{0,0} & 2 & 1 & h_X^{0,1} & 0 & 0 & h_X^{0,2} & 0 & 0 \\ h_X^{1,0} & 2 & 2 & h_X^{1,1} & 0 & 0 & h_X^{1,2} & 0 & 0 \\ h_X^{2,0} & 0 & 1 & h_X^{2,1} & 0 & 0 & h_X^{2,2} & 0 & 0 \end{array} \right).$$

We have $h_X^{0,0} = \dim_{\mathbb{R}} \mathrm{H}^0(X, \mathbb{R}) = 1$ and $h_Y^{1,0} = 1$ is easy to see. Exactness of the Mayer-Vietoris sequences for $\mathcal{L}_X^0$, $\mathcal{L}_X^1$ and $\mathcal{L}_X^2$ then shows $h_X^{1,1} = 1$, $h_X^{2,1} = 1$ and $h_X^{p,q} = 0$ otherwise.

For $X = \mathbb{T} \times \mathbb{P}^1$, we can choose $U_1 \cong U_2 \cong \mathbb{T}^2$ and $U_1 \cap U_2 \cong \mathbb{R} \times \mathbb{T}$. The type of the covering (U_1, U_2) is then

$$\left(\begin{array}{ccc|ccc|ccc} h_X^{0,0} & 2 & 1 & h_X^{0,1} & 0 & 0 & h_X^{0,2} & 0 & 0 \\ h_X^{1,0} & 0 & 1 & h_X^{1,1} & 0 & 0 & h_X^{1,2} & 0 & 0 \\ h_X^{2,0} & 0 & 0 & h_X^{2,1} & 0 & 0 & h_X^{2,2} & 0 & 0 \end{array} \right).$$

We have $h_X^{0,0} = \dim_{\mathbb{R}} \mathrm{H}^0(X, \mathbb{R}) = 1$. Again by exactness we see $h_X^{1,1} = 1$ and $h_X^{p,q} = 0$ otherwise.

For $X = \mathbb{P}^1 \times \mathbb{P}^1$, we can choose $U_1 \cong U_2 \cong \mathbb{T} \times \mathbb{P}^1$ and $U_1 \cap U_2 \cong \mathbb{R} \times \mathbb{P}^1$. The type of the covering (U_1, U_2) is then

$$\left(\begin{array}{ccc|ccc|ccc} h_X^{0,0} & 2 & 1 & h_X^{0,1} & 0 & 0 & h_X^{0,2} & 0 & 0 \\ h_X^{1,0} & 0 & 1 & h_X^{1,1} & 2 & 1 & h_X^{1,2} & 0 & 0 \\ h_X^{2,0} & 0 & 0 & h_X^{2,1} & 0 & 1 & h_X^{2,2} & 0 & 0 \end{array} \right).$$

We obviously have $h_X^{0,0} = 1$, $h_X^{1,0} = 0$ and $h_X^{2,0} = 0$. We also see $h_X^{0,1} = 0$, $h_X^{2,1} = 0$, $h_X^{0,2} = 0$ and $h_X^{2,2} = 1$ immediately. Because X is compact and smooth, we may deduce $h_X^{1,2} = h_X^{1,0} = 0$ from Poincaré duality. This finally implies $h_X^{1,1} = 2$ as required.

Example 3.41. The Dolbeault cohomology of the tropical projective space $\mathbb{P}^n$ is given by

$$h_{d_2}^{\bullet,\bullet}(\mathbb{P}^n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The claim is trivial for $\mathbb{P}^0 = \{0\}$. For $n \geq 1$, we may partition $\mathbb{P}^n = \mathbb{T}^n \cup S$ with a good closed sedentarity $S \subset \mathbb{P}^n$ of codimension 1, $S \cong \mathbb{P}^{n-1}$. The long exact sequence in cohomology obtained from the (second) short exact sequence from proposition 3.19 reads as

$$\mathrm{H}^q(\mathbb{P}^n, \mathcal{L}^p) \longrightarrow \mathrm{H}^q(\mathbb{T}^n, \mathcal{L}^p) \longrightarrow \mathrm{H}^q(\mathbb{P}^{n-1}, \mathcal{L}^{p-1}) \to \mathrm{H}^{q+1}(\mathbb{P}^n, \mathcal{L}^p),$$

for $p, q \in \mathbb{Z}$. By induction we may assume $\mathrm{H}^q(\mathbb{P}^{n-1}, \mathcal{L}^p) = 1$ for $0 \leq p = q < n$ and $\mathrm{H}^q(\mathbb{P}^{n-1}, \mathcal{L}^p) = 0$ otherwise. For $p = q = 0$ we get $\mathrm{H}^0(\mathbb{P}^n, \mathcal{L}^0) = \mathrm{H}^0(\mathbb{T}^n, \mathcal{L}^0) = \mathbb{R}$ from $\mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{L}^{-1}) = 0$ (this is also obvious because $\mathbb{P}^n$ as well as $\mathbb{T}^n$ are simply connected).

For $q = 1$, $p \neq 1$ we have $\mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{L}^{p-1}) = \mathrm{H}^1(\mathbb{T}^n, \mathcal{L}^p) = 0$ and hence $\mathrm{H}^1(\mathbb{P}^n, \mathcal{L}^p) = 0$ as well. For $p = q = 1$, $\mathrm{H}^0(\mathbb{T}^n, \mathcal{L}^1) = \mathrm{H}^1(\mathbb{T}^n, \mathcal{L}^1) = 0$ implies $\mathrm{H}^1(\mathbb{P}^n, \mathcal{L}^1) = \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{L}^0) = \mathbb{R}$.

For $q > 1$, $\mathrm{H}^{q-1}(\mathbb{T}^n, \mathcal{L}^p) = \mathrm{H}^q(\mathbb{T}^n, \mathcal{L}^p) = 0$ implies $\mathrm{H}^q(\mathbb{P}^n, \mathcal{L}^p) = \mathrm{H}^{q-1}(\mathbb{P}^{n-1}, \mathcal{L}^{p-1})$, which is one dimensional for $p = q \leq n$ and vanishes otherwise, by assumption.

Example 3.42. Two copies U_1 and U_2 of $(-1, 1) \times \mathbb{P}^1$ can be glued by identifying $(-1, 0) \times \mathbb{P}^1 \subset U_i$ with $(0, 1) \times \mathbb{P}^1 \subset U_j$, $i \neq j$. If we take each transition map to be given by the map $-1 + \mathrm{id}_{\mathbb{R}} \times \mathrm{id}_{\mathbb{P}}$, we end up with a space (homeomorphic to) $\mathbb{S}^1 \times \mathbb{P}^1$. On the other hand, if we take one transition map to be $(-1 + \mathrm{id}_{\mathbb{R}}) \times \mathrm{id}_{\mathbb{P}}$ and the other one to be $(-1 + \mathrm{id}_{\mathbb{R}}) \times (-\mathrm{id}_{\mathbb{P}})$, then we end up with a space M homeomorphic to the Moebius strip (with boundary). Their Dolbeault cohomology is given by

$$h_{d_2}^{\bullet,\bullet}(\mathbb{S}^1 \times \mathbb{P}^1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, h_{d_2}^{\bullet,\bullet}(M) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

For both $X \in \{\mathbb{S}^1 \times \mathbb{P}^1, M\}$, the type of the respective covering (U_1, U_2) with $U_1 \cap U_2 \cong \mathbb{R} \times \mathbb{P}^1 \sqcup \mathbb{R} \times \mathbb{P}^1$ is given by

$$\left(\begin{array}{ccc|ccc|ccc} h_X^{0,0} & 2 & 2 & h_X^{0,1} & 0 & 0 & h_X^{0,2} & 0 & 0 \\ h_X^{1,0} & 2 & 2 & h_X^{1,1} & 2 & 2 & h_X^{1,2} & 0 & 0 \\ h_X^{2,0} & 0 & 0 & h_X^{2,1} & 2 & 2 & h_X^{2,2} & 0 & 0 \end{array} \right).$$

Successively we get $h_X^{0,0} = 1 = h_X^{0,1}$ and $h_X^{2,1} = 1 = h_X^{2,2}$ by using exactness of the long exact sequences, and Poincaré duality afterwards. The form $d'x_1$ generates the space of d''-closed

$(1,0)$-forms on $U_1 = (-1,1) \times \mathbb{P}^1$ and has a unique continuation to a closed $(1,0)$-form on both $X \in \{\mathbb{S} \times \mathbb{P}, M\}$. On the other hand, every closed $(1,0)$-form on X restricts to a closed $(1,0)$-form on U_1, in other words to a multiple of $d'x_1$. This shows that in either case $\mathrm{H}^0(X, \mathcal{L}^1_X)$ is one dimensional, generated by the aforementioned continuation. From Poincaré duality we obtain $h^{1,0}_X = 1 = h^{1,2}_X$ and exactness of the long exact sequences then implies $h^{1,1}_X = 2$ for both $X \in \{\mathbb{S} \times \mathbb{P}, M\}$.

4 Towards a d_1d_2-lemma for polyhedral spaces

4.1 The d_1d_2-Lemma

Let $(A^{\bullet,\bullet}, d_1, d_2)$ be a double complex of $\mathbb{R}$-vector spaces. We introduce the following 'cohomology' groups:

Definition 4.1. Let $(A^{\bullet,\bullet}, d_1, d_2)$ be a double complex of $\mathbb{R}$-vector spaces.

1. The *Dolbeault cohomology groups* of $A^{\bullet,\bullet}$ are
$$\mathrm{H}^{p,q}_{d_1}(A^{\bullet,\bullet}) := \ker(d_1^{p,q})/\operatorname{im}(d_1^{p-1,q}), \qquad \mathrm{H}^{p,q}_{d_2}(A^{\bullet,\bullet}) := \ker(d_2^{p,q})/\operatorname{im}(d_2^{p,q-1}).$$

2. The *total cohomology groups* of $A^{\bullet,\bullet}$ are the cohomology groups of the total complex $(\mathrm{tot}^\bullet(A^{\bullet,\bullet}), d)$:
$$\mathrm{H}^k_d(A^{\bullet,\bullet}) := \ker(d^k)/\operatorname{im}(d^{k-1}).$$

3. The *Bott-Chern* and *Aeppli cohomology groups* of $A^{\bullet,\bullet}$ are the groups
$$\begin{aligned}\mathrm{H}^{p,q}_{BC}(A^{\bullet,\bullet}) &:= \ker(d_1^{p,q}) \cap \ker(d_2^{p,q})/\operatorname{im}(d_1^{p-1,q}d_2^{p-1,q-1}),\\ \mathrm{H}^{p,q}_{A}(A^{\bullet,\bullet}) &:= \ker(d_1^{p,q+1}d_2^{p,q})/\left(\operatorname{im}(d_1^{p-1,q}) + \operatorname{im}(d_2^{p,q-1})\right).\end{aligned}$$

We have the following well-known result on the relation between those groups ([DGMS75, 5.15f]):

Proposition 4.2. *Let $A^{\bullet,\bullet}$ be a bounded double complex of $\mathbb{R}$-vector spaces. The identity induces a commuting diagram of $\mathbb{Z}^2$- and $\mathbb{Z}$-graded $\mathbb{R}$-vector spaces*

$$\begin{array}{ccccc} & & \mathrm{H}^{\bullet,\bullet}_{BC}(A^{\bullet,\bullet}) & & \\ & \swarrow & \downarrow & \searrow & \\ \mathrm{H}^{\bullet,\bullet}_{d_1}(A^{\bullet,\bullet}) & & \mathrm{H}^{\bullet}_{d}(A^{\bullet,\bullet}) & & \mathrm{H}^{\bullet,\bullet}_{d_2}(A^{\bullet,\bullet}) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{H}^{\bullet,\bullet}_{A}(A^{\bullet,\bullet}). & & \end{array}$$

The following conditions are equivalent for every $k \in \mathbb{Z}$:

1. *The map $\bigoplus_{p+q=k} \mathrm{H}^{p,q}_{BC}(A^{\bullet,\bullet}) \to \mathrm{H}^k_d(A^{\bullet,\bullet})$ is injective.*
2. *The maps $\mathrm{H}^{p,q}_{BC}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{d_i}(A^{\bullet,\bullet})$ are injective for all $p+q=k$, $i \in \{1,2\}$.*
3. *The maps $\mathrm{H}^{p,q}_{BC}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{A}(A^{\bullet,\bullet})$ are injective for all $p+q=k$.*
4. *The map $\mathrm{H}^{k-1}_d(A^{\bullet,\bullet}) \to \bigoplus_{p+q=k-1} \mathrm{H}^{p,q}_{A}(A^{\bullet,\bullet})$ is surjective.*
5. *The maps $\mathrm{H}^{p,q}_{d_i}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{A}(A^{\bullet,\bullet})$ are surjective for all $p+q=k-1$ and $i \in \{1,2\}$.*
6. *The maps $\mathrm{H}^{p,q}_{BC}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{A}(A^{\bullet,\bullet})$ are surjective for all $p+q=k-1$.*

If these equivalent conditions hold for every $k \in \mathbb{Z}$, then all the maps are isomorphisms.

Definition 4.3. We say that a double complex $(A^{\bullet,\bullet}, d_1, d_2)$ *satisfies the d_1d_2-Lemma* if the equivalent conditions of proposition 4.2 hold for every $k \in \mathbb{Z}$.

In section 4.6 we will collect some examples of compact 2-dimensional tropical manifolds satisfying the d_1d_2-Lemma (I do not know examples of compact tropical surfaces *not* satisfying the d_1d_2-lemma) by considering the following exact sequences connecting Dolbeault with Bott-Chern and Aeppli cohomology ([Var86, 3.1], see also the proof of [AT15, 3.4]):

Proposition 4.4 (J. Varuchas exact sequences)**.** *Let $(A^{\bullet,\bullet}, d_1, d_2)$ be a double complex of $\mathbb{R}$-vector spaces. Then there exist exact sequences, induced by the identity:*

1.

$$0 \to \frac{\mathrm{im}(d_1) \cap \ker(d_2)}{\mathrm{im}(d_1 d_2)} \to \mathrm{H}^{p,q}_{BC}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{d_1}(A^{\bullet,\bullet}) \to \frac{\ker(d_1 d_2)}{\mathrm{im}(d_1) + \ker(d_2)} \to \frac{\ker(d_1 d_2)}{\ker(d_1) + \ker(d_2)} \to 0,$$

2.

$$0 \to \frac{\ker(d_1) \cap \mathrm{im}(d_2)}{\mathrm{im}(d_1 d_2)} \to \mathrm{H}^{p,q}_{BC}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{d_2}(A^{\bullet,\bullet}) \to \frac{\ker(d_1 d_2)}{\ker(d_1) + \mathrm{im}(d_2)} \to \frac{\ker(d_1 d_2)}{\ker(d_1) + \ker(d_2)} \to 0,$$

3.

$$0 \to \frac{\mathrm{im}(d_1) \cap \mathrm{im}(d_2)}{\mathrm{im}(d_1 d_2)} \to \frac{\ker(d_1) + \mathrm{im}(d_2)}{\mathrm{im}(d_1 d_2)} \to \mathrm{H}^{p,q}_{d_1}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{A}(A^{\bullet,\bullet}) \to \frac{\ker(d_1 d_2)}{\ker(d_1) + \mathrm{im}(d_2)} \to 0,$$

4.

$$0 \to \frac{\mathrm{im}(d_1) \cap \mathrm{im}(d_2)}{\mathrm{im}(d_1 d_2)} \to \frac{\mathrm{im}(d_1) + \ker(d_2)}{\mathrm{im}(d_1 d_2)} \to \mathrm{H}^{p,q}_{d_2}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{A}(A^{\bullet,\bullet}) \to \frac{\ker(d_1 d_2)}{\mathrm{im}(d_1) + \ker(d_2)} \to 0.$$

4.2 The d_1d_2-lemma for tropical spaces

We now want to consider the double complex $\mathcal{A}^{\bullet;\bullet}_X(X)$ for some examples of smooth tropical spaces and apply the notions of the previous section. For a polyhedral space X, we will write $\mathrm{H}^{p,q}_{BC}(X)$ for the vector space $\mathrm{H}^{p,q}_{BC}(\mathcal{A}^{\bullet;\bullet}_X(X))$, and similarly for the other cohomology groups defined above.

Sadly, we are not able to prove the d_1d_2-lemma for tropical spaces in general. In section 4.6 we will give some simple examples of surfaces satisfying it though.

First, we have the following reformulation of corollary 2.15:

Corollary 4.5. *Let X be a polyhedral space. Then the canonical maps*

$$\mathrm{H}^{p,0}_{BC}(X) \to \mathrm{H}^{p,0}_{d_2}(X), \qquad \mathrm{H}^{0,q}_{BC}(X) \to \mathrm{H}^{0,q}_{d_1}(X)$$

are isomorphisms.

Example 4.6. Let $U = \mathbb{R}^n$ with $n > 0$. Then U does not satisfy the d_1d_2-lemma:

This follows directly from the previous corollary: We have $\mathrm{H}^{0,1}_{BC}(U) = \mathrm{H}^{0,1}_{d_1}(U) \cong \mathbb{R}^n$ and $\mathrm{H}^{0,1}_{d_2}(U) = 0$, so the canonical map $\mathrm{H}^{0,1}_{BC}(U) \to \mathrm{H}^{0,1}_{d_2}(U)$ is not injective.

Proposition 4.7. *Let X be a polyhedral space satisfying the d_1d_2-Lemma. Then we get canonical isomorphisms*

$$\mathrm{H}^{p,q}_{d_1}(X) \cong \mathrm{H}^{q,p}_{d_1}(X)$$

for every $p, q \in \mathbb{Z}$.

Proof. This is clear since the maps $J : \mathcal{A}_X^{p,q}(X) \to \mathcal{A}_X^{q,p}(X)$ induce isomorphisms

$$\mathrm{H}^{p,q}_{BC}(X) \xrightarrow{J} \mathrm{H}^{q,p}_{BC}(X).$$

The canonical maps $\mathrm{H}^{p,q}_{BC}(X) \to \mathrm{H}^{p,q}_{d_1}(X)$ are isomorphisms by assumption and the claim follows. □

Proposition 4.8. *Assume that X is a compact tropical manifold satisfying the d_1d_2-Lemma. Then the decomposition*

$$\mathrm{R}^n\,\Gamma(X,\mathcal{L}_X^\bullet) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}_{BC}(X) \cong \bigoplus_{p+q=n} \mathrm{H}^q(X,\mathcal{L}_X^p)$$

induced by proposition 4.2 is an orthogonal decomposition for the non-degenerate symmetric integration pairing from proposition 3.35

$$\mathrm{R}^n\,\Gamma(X,\mathcal{L}_X^\bullet)\times \mathrm{R}^n\,\Gamma(X,\mathcal{L}_X^\bullet) \to \mathbb{R},$$
$$(\alpha,\beta) \mapsto \int_X \alpha \wedge J\beta.$$

Proof. It is clear that $\int_X \alpha\wedge J\beta = 0$ for every two d-closed forms $\alpha \in \mathcal{A}_X^{p',q'}(X)$, $\beta \in \mathcal{A}_X^{p'',q''}(X)$ with $(p',q') \neq (p'',q'')$, $p'+q' = p''+q'' = n$. By assumption the d_1d_2-lemma holds and hence the canonical maps

$$\bigoplus_{p+q=n} \mathrm{H}^{p,q}_{BC}(X) \to \mathrm{H}^n_d(X) = \mathrm{R}^n\,\Gamma(X,\mathcal{L}_X^\bullet)$$

and

$$\bigoplus_{p+q=n} \mathrm{H}^{p,q}_{BC}(X) \to \bigoplus_{p+q=n} \mathrm{H}^{p,q}_{d_2}(X) = \bigoplus_{p+q=n} \mathrm{H}^q(X,\mathcal{L}_X^p)$$

are isomorphisms by proposition 4.2. This finishes the proof. □

For the $(0,0)$-Aeppli cohomology, we have the following maximum principle, due to [Jel16b, 2.1.66]:

Proposition 4.9. *Let X be a tropical space and $f \in \mathrm{H}^{0,0}_A(X)$, i.e. $f \in \ker(d_1d_2) \cap \mathcal{A}_X^{0,0}(X)$. Then, if $f : X \to \mathbb{R}$ has a local maximum at $x \in X$, f is locally constant at x.*

For compact tropical spaces, we immediately get the following corollary:

Corollary 4.10. *Let X be a connected compact tropical space.*

1. *The canonical map $\mathrm{H}^0_d(X) \to \mathrm{H}^{0,0}_A(X)$ is an isomorphism, i.e. $\mathrm{H}^{0,0}_A(X) = \mathbb{R}$.*
2. *The canonical map $\mathrm{H}^{1,0}_{BC}(X) \oplus \mathrm{H}^{0,1}_{BC}(X) \to \mathrm{H}^1_d(X)$ is injective.*
3. *The canonical map $\mathrm{H}^{0,1}_{BC}(X) \to \mathrm{H}^{0,1}_{d_2}(X) = \mathrm{H}^1(X,\mathbb{R})$ is injective.*

Note that this gives a topological upper bound for $\mathrm{H}^0(X,\mathcal{L}_X^1) \cong \mathrm{H}^{0,1}_{BC}(X) \cong \mathrm{H}^{1,0}_{BC}(X)$.

Proof. Statement (1) is [Jel16b, 2.1.67] while (2) and (3) follow directly from proposition 4.2. □

4.3 The local solvability lemma

Following the presentation of [Schw07, 4.1] we prove the following local solvability lemmata for tropical modifications of $\mathbb{T}^N$ (in terms of forms 4.11 and linear currents 4.12 respectively). This will subsequently allow us to prove that the tropical projective space $\mathbb{P}^N$ satisfies the $d_1 d_2$-lemma. Throughout this section, for a k-form α on $U \subset X$ we will write $\alpha^{p,q} \in \mathcal{A}_X^{p,q}(U)$ for its degree (p,q)-part ($p+q=k$).

Proposition 4.11. *Let X be a tropical manifold such that there exists a regular tropical modification $\delta : X \to \mathbb{T}^N$.*

1. *Let $\alpha \in \mathcal{A}_X^k(X)$ be a d-closed k-form on X, $k > 0$, such that each $\alpha^{p,q} \in \mathcal{A}_X^{p,q}(X)$ vanishes unless $p_1 \leq p \leq p_2$ with $p_1 < p_2$. Then there exists $\beta \in \mathcal{A}_X^{k-1}(X)$ with $d\beta = \alpha$ such that $\beta^{p,q}$ vanishes unless $p_1 \leq p \leq p_2 - 1$.*
2. *Let $\alpha = \alpha^{p,q} \in \mathcal{A}_X^{p,q}(X)$ be a d-closed (p,q)-form, i.e. $\alpha \in \ker(d_1) \cap \ker(d_2)$.*
   (a) *If $p \geq 1$ or $q \geq 1$, then $\alpha = d_1 d_2 \gamma^{p-1,q-1}$ with $\gamma \in \mathcal{A}_X^{p-1,q-1}(X)$.*
   (b) *If $q = 0$ and $p = 0$, then $\alpha \in \mathbb{R}$.*
3. *Let $\alpha = \alpha^{p,q} \in \mathcal{A}_X^{p,q}(U)$ be $d_1 d_2$-closed, i.e. $d_1 d_2 \alpha = 0$. Then α is the sum of a d_1-closed and a d_2-closed form. In other words:*
   (a) *If $p \geq 1$ or $q \geq 1$ then $\alpha = d_1 \beta^{p-1,q} + (-1)^p d_2 \beta^{p,q-1}$.*
   (b) *If $q = 0$ and $p = 0$, then $\alpha \in \mathbb{R}$.*
4. *Let $\alpha \in \mathcal{A}_X^k(X)$, $k > 0$, be a* nearly d-closed *k-form, i.e. there exist $p_1 < p_2$ with $\alpha^{p,q} = 0$ unless $p_1 \leq p \leq p_2$ and we have $d\alpha \in \mathcal{A}_X^{p_1,q_1+1}(X) \oplus \mathcal{A}_X^{p_2+1,q_2}(X)$. Then there exists $\beta \in \mathcal{A}_X^{k-1}(X)$ with $\beta^{p,q} = 0$ unless $p_1 \leq p \leq p_2 - 1$ such that*

$$\alpha = d\beta + \tilde{\alpha}^{p_1,q_1} + \tilde{\alpha}^{p_2,q_2},$$

   where $\tilde{\alpha}^{p_1,q_1}$ is a d_1-closed (p_1,q_1)-form and $\tilde{\alpha}^{p_2,q_2}$ is a d_2-closed (p_2,q_2)-form.

Proposition 4.12. *Let X be a tropical manifold such that there exists a regular tropical modification $\delta : X \to \mathbb{T}^N$.*

1. *Let $\phi \in \mathcal{D}_X^k(X)$ be a ∂-closed k-current on X with $k > -2n$ such that each $\phi^{r,s} \in \mathrm{Hom}_{\mathbb{R}}(\Gamma_c(X, \mathcal{A}_X^{-s,-r}), \mathbb{R}) = \mathcal{D}_X^{r,s}(X)$ vanishes unless $r_1 \leq r \leq r_2$ with $r_1 < r_2$. Then there exists $\psi \in \mathcal{D}_X^{k-1}(X)$ with $d\psi = \phi$ such that $\psi^{r,s}$ vanishes unless $r_1 \leq r \leq r_2 - 1$.*
2. *Let $\phi = \phi^{r,s} \in \mathcal{D}_X^{r,s}(X)$ be a ∂-closed (r,s)-current, i.e. $\phi \in \ker(\partial_1) \cap \ker(\partial_2)$.*
   (a) *If $r \geq -n+1$ or $s \geq -n+1$, then $\phi = \partial_1 \partial_2 \rho^{r-1,s-1}$ with $\rho \in \mathcal{D}_X^{r-1,s-1}(X)$.*
   (b) *If $r = -n$ and $s = -n$, then $\phi \in \mathbb{R} \cdot \delta_X$.*
3. *Let $\phi = \phi^{r,s} \in \mathcal{D}_X^{r,s}(X)$ be $\partial_1 \partial_2$-closed, i.e. $\partial_1 \partial_2 \phi = 0$. Then ϕ is the sum of a ∂_1-closed and a ∂_2-closed form. In other words:*
   (a) *If $r \geq -n+1$ or $s \geq -n+1$ then $\phi = \partial_2 \psi^{r-1,s} + (-1)^r \partial_1 \psi^{r,s-1}$.*
   (b) *If $r = -n$ and $s = -n$, then $\phi \in \mathbb{R} \cdot \delta_X$.*
4. *Let $\phi \in \mathcal{D}_X^k(X)$, $k > -2n$, be a* nearly ∂-closed *k-current, i.e. there exist $r_1 < r_2$ with $\phi^{r,s} = 0$ unless $r_1 \leq r \leq r_2$ and we have $\partial\phi \in \mathcal{D}_X^{r_1,s_1+1}(X) \oplus \mathcal{D}_X^{r_2+1,s_2}(X)$. Then there exists $\psi \in \mathcal{D}_X^{k-1}(X)$ with $\psi^{r,s} = 0$ unless $r_1 \leq r \leq r_2 - 1$ such that*

$$\phi = \partial\psi + \tilde{\psi}^{r_1,s_1} + \tilde{\psi}^{r_2,s_2},$$

   where $\tilde{\psi}^{r_1,s_1}$ is a ∂_2-closed (r_1,s_1)-current and $\tilde{\psi}^{r_2,s_2}$ is a ∂_1-closed (r_2,s_2)-current.

Proof. We will only prove the proposition on forms. It will turn out that it only depends on the following facts, following from the computation of

$$\mathrm{R}^k\,\Gamma(X,\mathcal{A}_X^\bullet) = \bigoplus_{p+q=k} \mathrm{H}^q(X,\mathcal{L}_X^p) = \bigoplus_{p+q=k} \mathrm{H}^q(X,\mathcal{F}_X^p) = \bigoplus_{p+q=k} \mathrm{H}^q(\mathbb{T}^N,\mathcal{F}_{\mathbb{T}^N}^p)$$

with corollary 3.7, theorem 2.16, corollary 1.54 and example 3.39:

- The total complex $\mathcal{A}_X^\bullet(X)$ of forms is exact in positive degrees.
- The vertical (horizontal) complexes $\mathcal{A}_X^{p,\bullet}(X)$ (resp. $\mathcal{A}_X^{\bullet,q}(X)$) are exact for $p>0$ $(q>0)$. For $p=0$ $(q=0)$ they are exact in positive degrees.

Using lemma 3.30 and corollary 3.27 one can transfer these crucial exactness results to $\mathcal{D}_X^{\bullet,\bullet}(X)$ and use essentially the same proof for the proposition on 'currents'.

1. For $p_1=0$ and $p_2=k$ this follows immediately from exactness of $\mathcal{A}_X^\bullet(X)$ in positive degrees, so we can assume $k\geq 2$. Let then $p_1>0$. Because $\mathcal{A}_X^\bullet(X)$ is exact in positive degrees, we can write $\alpha=d\beta$ with $\beta\in\mathcal{A}_X^{k-1}(X)$. This implies $d_2\beta^{0,k-1}=\alpha^{0,k}=0$ because we assumed $p_1>0$. Now $\mathcal{A}_X^{p,\bullet}(X)$ is exact in positive degrees as well and we can write $\beta^{0,k-1}=d_2\gamma^{0,k-2}$. Then $\beta':=\beta-d\gamma^{0,k-2}$ still maps to α and we have $(\beta')^{0,k-1}=0$. We may hence assume that we had $\beta^{0,k-1}=0$ to begin with. Repeating this process we may assume that $\beta^{p,q}=0$ for each $p<p_1$. Similarly, if $p_2<k$, we can use that the complexes $\mathcal{A}_X^{q,\bullet}(U)$ are exact in positive degrees to reduce to the case $\beta^{p,q}=0$ for $p>p_2-1$ by a similar inductive process.

2. For case (a) we assume $p\geq 1$. We may then apply (1) to the form α, with $p_1=p$ and $p_2=p+1$. This gives us a form $\beta=\beta^{p,q-1}$ with $d\beta=\alpha$. In particular, we have $d_2\beta^{p,q-1}=\alpha^{p,q}$ and $d_1\beta^{p,q-1}=0$. Because the complex $\mathcal{A}_X^{\bullet,q-1}(X)$ is exact in positive degrees, we find $\gamma^{p-1,q-1}$ with $d_1\gamma^{p-1,q-1}=\beta^{p,q-1}$. We then have $\alpha=d_1d_2\gamma^{p-1,q-1}$.

   Case (b) follows immediately from $\mathrm{H}^0(X,\mathcal{A}_X^\bullet)=\mathrm{H}^0(X,\mathcal{A}_X^{0,\bullet})=\mathbb{R}$.

3. In case (a) we first assume $p=0$. Since we have $\mathrm{H}^0(X,\mathcal{A}_X^{\bullet,q})=0$ for $q>0$, $d_1d_2\alpha=0$ implies $d_2\alpha=0$. Now $\mathcal{A}_X^{0,\bullet}(X)$ is exact in positive degrees and hence α is d_2-exact, as needed. So let us assume that $p>0$ and $q>0$. The $(p+1,q)$-form $\lambda=d_1\alpha$ is d_2-exact by assumption and we have $d\lambda=0$. With (2) we can write $\lambda=d_1d_2\beta^{p,q-1}$. Then the (p,q)-form $\alpha_1:=\alpha-d_2\beta^{p,q-1}$ is d_1-closed and we have $\alpha=\alpha_1+d_2\beta^{p,q-1}$. Since $\mathcal{A}_X^{\bullet,q}(U)$ is exact in positive degrees, we can write $\alpha_1=d_1\beta^{p-1,q}$ as required.

   For case (b), note that $d_2\alpha\in\mathcal{A}_X^{0,1}(X)$ is d_1-closed. This implies $d_2\alpha=0$ and hence $\alpha\in\mathbb{R}$.

4. We first show that both α^{p_1,q_1} and α^{p_2,q_2} are d_1d_2-closed: By assumption we have $(d\alpha)^{p_1+1,q_1}=d_1\alpha^{p_1,q_1}-(-1)^{p_1}d_2\alpha^{p_1+1,q_1-1}=0$. This implies $d_1d_2\alpha^{p_1,q_1}=0$. The other case follows similarly.

   We can now apply (3a) to α^{p_1,q_1} and α^{p_2,q_2}. This gives us

   $$\alpha^{p_1,q_1}=\tilde{\alpha}^{p_1,q_1}+(-1)^{p_1}d_2\beta^{p_1,q_1-1},$$

   with $d_1\tilde{\alpha}^{p_1,q_1}=0$. With

   $$d_2(\alpha^{p_1+1,q_1-1}-d_1\beta^{p_1,q_1-1})=d_1((-1)^{p_1}\alpha^{p_1,q_1}-d_2\beta^{p_1,q_1-1})=0$$

   we find $\beta^{p_1+1,q_1-2}\in\mathcal{A}_X^{p_1+1,q_1-2}(X)$ with

   $$\alpha^{p_1+1,q_1-1}=d_1\beta^{p_1,q_1-1}+(-1)^{p_1+1}d_2\beta^{p_1+1,q_1-2}.$$

This way we construct inductively $\beta = \beta^{p_1,q_1-1} + \cdots + \beta^{p_2-1,q_2} \in \mathcal{A}_X^{k-1}(X)$. With $\tilde{\alpha}^{p_2,q_2} := \alpha^{p_2,q_2} - d_1\beta^{p_2-1,q_2}$ we then have

$$d_2\tilde{\alpha}^{p_2,q_2} = d_2\alpha^{p_2,q_2} - d_1d_2\beta^{p_2-1,q_2} = (-1)^{p_2-1}d_1\alpha^{p_2-1,q_2+1} - d_1d_2\beta^{p_2-1,q_2}.$$

By construction of β^{p_2-1,q_2}, the form $\alpha^{p_2-1,q_2+1} + (-1)^{p_2}d_2\beta^{p_2-1,q_2}$ is d_1-exact and hence $d_2\tilde{\alpha}^{p_2,q_2} = 0$. We now have $\alpha = d\beta + \tilde{\alpha}^{p_1,q_1} + \tilde{\alpha}^{p_2,q_2}$ as required.

□

Definition 4.13. Let X be a connected tropical manifold of pure dimension n. Then X is said to *satisfy the $\partial_1\partial_2$-lemma*, if the double complex $(\mathcal{D}_X^{\bullet;\bullet}(X), \partial_2, \partial_1)$ satisfies the $\partial_1\partial_2$-lemma.

Corollary 4.14. *The tropical manifold $\mathbb{T}^N$ satisfies both the d_1d_2-lemma and the $\partial_1\partial_2$-lemma.*

Proof. From proposition 4.11(2) we get that $\mathrm{H}_{BC}^{p,q}(\mathbb{T}^N) = 0$ for $(p,q) \neq (0,0)$ and $\mathrm{H}_{BC}^{0,0}(\mathbb{T}^N) = \mathbb{R}$. This implies that the maps $\mathrm{H}_{d_1}^{p,q}(\mathbb{T}^N) \longleftarrow \mathrm{H}_{BC}^{p,q}(\mathbb{T}^N) \longrightarrow \mathrm{H}_{d_2}^{p,q}(\mathbb{T}^N)$ are injective, as required by proposition 4.2. Using proposition 4.12(2) we get the corresponding results for $\mathcal{D}_{\mathbb{T}^N}^{\bullet;\bullet}(\mathbb{T}^N)$. □

This prompts the following conjecture:

Conjecture 4.15. *Let X be a connected tropical manifold. Then X satisfies the d_1d_2-lemma if and only if X satisfies the $\partial_1\partial_2$-lemma.*

It is clear that this conjecture is true for X compact:

Proposition 4.16. *Let X be a compact connected tropical manifold. Then X satisfies the d_1d_2-lemma if and only if X satisfies the $\partial_1\partial_2$-lemma.*

Proof. Because X is compact we have $\mathcal{D}_X^{r,s}(X) = \mathrm{Hom}_{\mathbb{R}}(\mathcal{A}_X^{-s,-r}(X), \mathbb{R})$. The d_1d_2-lemma for X is equivalent to

$$\ker(d_1) \cap \mathrm{im}(d_2) \cap \mathcal{A}_X^{p,q}(X) = d_1d_2\,\mathcal{A}_X^{p-1,q-1}(X),$$
$$\mathrm{im}(d_1) \cap \ker(d_2) \cap \mathcal{A}_X^{p,q}(X) = d_1d_2\,\mathcal{A}_X^{p-1,q-1}(X),$$

for every $p, q \in \mathbb{Z}$ by proposition 4.4(1,2) and proposition 4.2. Similarly, the $\partial_1\partial_2$-lemma for X is equivalent to

$$\ker(\partial_1\partial_2) \cap \mathcal{D}_X^{r,s}(X) = (\mathrm{im}(\partial_1) + \ker(\partial_2)) \cap \mathcal{D}_X^{r,s}(X),$$
$$\ker(\partial_1\partial_2) \cap \mathcal{D}_X^{r,s}(X) = (\ker(\partial_1) + \mathrm{im}(\partial_2)) \cap \mathcal{D}_X^{r,s}(X),$$

for every $r, s \in \mathbb{Z}$ by proposition 4.4(3,4) and proposition 4.2.

Assume now that X satisfies the d_1d_2-lemma. Then for every $r, s \in \mathbb{Z}$ we have a chain of equalities of subsets of $\mathcal{D}_X^{r,s}(X)$:

$$\begin{aligned}\ker(\partial_1\partial_2) &= (\mathrm{im}(d_1d_2))^{\perp}\\ &= (\ker(d_1) \cap \mathrm{im}(d_2))^{\perp}\\ &= \ker(d_1)^{\perp} + \mathrm{im}(d_2)^{\perp}\\ &= \mathrm{im}(\partial_1) + \ker(\partial_2),\end{aligned}$$

and similarly $\ker(\partial_1\partial_2) = \ker(\partial_1) + \mathrm{im}(\partial_2)$ by reversing the roles of d_1 and d_2. Hence, $\mathcal{D}_X^{\bullet;\bullet}(X)$ satisfies the $\partial_1\partial_2$-lemma.

The other implication is shown with the same arguments. □

4.4 Bott-Chern and Aeppli cohomology as sheaf cohomology

Following [Schw07, ch. 4] we will introduce complexes $\mathcal{B}^\bullet_{p,q}$ of sheaves on a polyhedral space X such that the Bott-Chern cohomology groups and Aeppli cohomology groups of X appear as

$$\mathrm{H}^{p,q}_{BC}(X) = \mathrm{R}^{p+q}\,\Gamma(X, \mathcal{B}^\bullet_{p,q}[-1]), \qquad \mathrm{H}^{p,q}_{A}(X) = \mathrm{R}^{p+q}\,\Gamma(X, \mathcal{B}^\bullet_{p+1,q+1}).$$

We will also see that the canonical map $\mathrm{H}^{p,q}_{BC}(X) \to \mathrm{H}^{p,q}_{A}(X)$ is induced by a morphism of complexes of sheaves

$$\mu : \mathcal{B}^\bullet_{p,q} \to \mathcal{B}^\bullet_{p-1,q-1}[1].$$

This will later allow us to prove the $\partial_1\partial_2$-lemma for $\mathbb{P}^N$ by using a long exact sequence of cohomology groups.

Definition 4.17. Let X be a tropical manifold, $p \geq 0$, $q \geq 0$ and $r \geq -n$, $s \geq -n$.

1. The complex $\mathcal{B}^\bullet_{p,q}$ of sheaves of forms on X is given by

$$\mathcal{B}^k_{p,q} := \bigoplus_{a+b=k, a<p, b<q} \mathcal{A}^{a,b}_X \text{ for } k \leq p+q-2,$$
$$\mathcal{B}^k_{p,q} := \bigoplus_{a+b=k+1, a\geq p, b\geq q} \mathcal{A}^{a,b}_X \text{ for } k \geq p+q-1,$$

with differentials

$$\begin{aligned} \mathcal{B}^0_{p,q} \xrightarrow{\pi\circ d} \cdots \xrightarrow{\pi\circ d} \mathcal{B}^{k-1}_{p,q} \xrightarrow{\pi\circ d} \mathcal{B}^{k}_{p,q} \xrightarrow{\pi\circ d} \cdots \\ \cdots \xrightarrow{\pi\circ d} \mathcal{B}^{p+q-2}_{p,q} \xrightarrow{d_1 d_2} \mathcal{B}^{p+q-1}_{p,q} \xrightarrow{d} \cdots \\ \cdots \xrightarrow{d} \mathcal{B}^{l-1}_{p,q} \xrightarrow{d} \mathcal{B}^{l}_{p,q} \xrightarrow{d} \cdots \\ \cdots \xrightarrow{d} \mathcal{B}^{2n-1}_{p,q} \to 0, \end{aligned}$$

where $\pi : \mathcal{A}^k_X \to \mathcal{B}^k_{p,q}$ denotes the projection map.

2. We define a morphism $\mu : \mathcal{B}^\bullet_{p,q} \to \mathcal{B}^\bullet_{p+1,q+1}[1]$ of complexes of sheaves, where μ^k is given by

$$\mu^k : \mathcal{B}^k_{p,q} \to \mathcal{A}^{k-q+1,q-1}_X \xrightarrow{(-1)^k d_2} \mathcal{A}^{k-q+1,q}_X \to \mathcal{B}^{k+1}_{p+1,q+1}$$

for $k \leq p+q-2$,

$$\mu^k : \mathcal{B}^k_{p,q} = \mathcal{A}^{p,q}_X \xrightarrow{(-1)^k \mathrm{id}} \mathcal{A}^{p,q}_X = \mathcal{B}^{k+1}_{p+1,q+1}$$

for $k = p+q-1$ and

$$\mu^k : \mathcal{B}^k_{p,q} \to \mathcal{A}^{k-q,q}_X \xrightarrow{(-1)^k d_2} \mathcal{A}^{k-q,q+1}_X \to \mathcal{B}^{k+1}_{p+1,q+1}$$

for $k \geq p+q$.

3. Similarly, the complex $\mathcal{E}^\bullet_{r,s}$ of sheaves of currents on X is given by

$$\mathcal{E}^k_{r,s} := \bigoplus_{a+b=k, a<r, b<r} \mathcal{D}^{a,b}_X \text{ for } k \leq r+s-2,$$
$$\mathcal{E}^k_{r,s} := \bigoplus_{a+b=k+1, a\geq r, b\geq s} \mathcal{D}^{a,b}_X \text{ for } k \geq r+s-1,$$

with differentials

$$\begin{aligned}
&0 \to \mathcal{E}_{r,s}^{-2n} \xrightarrow{\pi\circ\partial} \mathcal{E}_{r,s}^{1-2n} \xrightarrow{\pi\circ\partial} \mathcal{E}_{r,s}^{2-2n} \xrightarrow{\pi\circ\partial} \cdots \\
&\cdots \xrightarrow{\pi\circ\partial} \mathcal{E}_{r,s}^{r+s-3} \xrightarrow{\pi\circ\partial} \mathcal{E}_{r,s}^{r+s-2} \xrightarrow{\partial_1\partial_2} \mathcal{E}_{r,s}^{r+s-1} \xrightarrow{\partial} \cdots \\
&\cdots \xrightarrow{\partial} \mathcal{E}_{r,s}^{l-1} \xrightarrow{\partial} \mathcal{E}_{r,s}^{l} \xrightarrow{\partial} \cdots \\
&\cdots \xrightarrow{\partial} \mathcal{E}_{r,s}^{-1} \to 0,
\end{aligned}$$

where $\pi : \mathcal{D}_X^k \to \mathcal{E}_{r,s}^k$ denotes the projection map.

4. We define a morphism $\nu : \mathcal{E}_{r,s}^\bullet \to \mathcal{E}_{r+1,s+1}^\bullet[1]$ of complexes of sheaves, where ν^k is given by

$$\nu^k : \mathcal{E}_{r,s}^k \to \mathcal{D}_X^{k-s+1,s-1} \xrightarrow{(-1)^k\partial_1} \mathcal{D}_X^{k-s+1,s} \to \mathcal{E}_{r+1,s+1}^{k+1}$$

for $k \le r+s-2$,

$$\nu^k : \mathcal{E}_{r,s}^k = \mathcal{D}_X^{r,s} \xrightarrow{(-1)^k\,\mathrm{id}} \mathcal{D}_X^{r,s} = \mathcal{E}_{r+1,s+1}^{k+1}$$

for $k = r+s-1$ and

$$\nu^k : \mathcal{E}_{r,s}^k \to \mathcal{D}_X^{k-s,s} \xrightarrow{(-1)^k\partial_1} \mathcal{D}_X^{k-s,s+1} \to \mathcal{E}_{r+1,s+1}^{k+1}$$

for $k \ge r+s$.

Remark 4.18. We give a proof that μ is indeed a morphism of complexes below, but since it is rather bulky let us first consider the following proposition:

Proposition 4.19. *We have natural isomorphisms for $p, q \ge 1$ and $r, s \ge -n$*

$$\begin{aligned}
\mathrm{H}_{BC}^{p,q}(X) := \mathrm{H}_{BC}^{p,q}(\mathcal{A}_X^{\bullet,\bullet}(X)) &= \mathrm{R}^{p+q-1}\,\Gamma(X, \mathcal{B}_{p,q}^\bullet), \\
\mathrm{H}_{A}^{p,q}(X) := \mathrm{H}_{A}^{p,q}(\mathcal{A}_X^{\bullet,\bullet}(X)) &= \mathrm{R}^{p+q}\,\Gamma(X, \mathcal{B}_{p+1,q+1}^\bullet), \\
\mathrm{H}_{BC}^{r,s}(\mathcal{D}_X^{\bullet,\bullet}(X)) &= \mathrm{R}^{r+s-1}\,\Gamma(X, \mathcal{E}_{r,s}^\bullet), \\
\mathrm{H}_{A}^{r,s}(\mathcal{D}_X^{\bullet,\bullet}(X)) &= \mathrm{R}^{r+s}\,\Gamma(X, \mathcal{E}_{r+1,s+1}^\bullet),
\end{aligned}$$

and the canonical morphisms $\mathrm{H}_{BC}^{p,q}(X) \to \mathrm{H}_{A}^{p,q}(X)$ and $\mathrm{H}_{BC}^{r,s}(\mathcal{D}_X^{\bullet,\bullet}(X)) \to \mathrm{H}_{A}^{r,s}(\mathcal{D}_X^{\bullet,\bullet}(X))$ are given by μ^{p+q-1} and ν^{r+s-1}.

Proof. Because the sheaves $\mathcal{A}_X^{a,b}$ and $\mathcal{D}_X^{a,b}$ are soft resp. flabby on X, so are the sheaves $\mathcal{B}_{p,q}^k$ and $\mathcal{E}_{r,s}^k$. The cohomology groups on the right hand side can then be computed by taking global sections. The claim is now a direct consequence of the definition of Bott-Chern and Aeppli cohomology. □

As promised, we now take a closer look at the map μ:

Lemma 4.20. *The maps*

$$\mu : \mathcal{B}_{p,q}^\bullet \to \mathcal{B}_{p+1,q+1}^\bullet[1], \qquad \nu : \mathcal{E}_{r,s}^\bullet \to \mathcal{E}_{r+1,s+1}^\bullet[1]$$

are morphisms of complexes.

Proof. We only prove the claim for μ since the proof for ν is similar. It is easy to check that each $\mu^k : \mathcal{B}_{p,q}^k \to \mathcal{B}_{p+1,q+1}^{k+1}$ is well defined and that it is a morphism of sheaves. In order to show that it commutes with the differential maps, we have to consider the following four cases of squares of sheaves on X:

$$
\begin{array}{ccc}
\mathcal{B}^k_{p,q} & \xrightarrow{\quad \pi\circ d \quad} & \mathcal{B}^{k+1}_{p,q} \\
\big\downarrow & & \big\downarrow \\
\mathcal{A}_X^{k-q+1,q-1} & \underline{k \leq p+q-3} & \mathcal{A}_X^{k-q+2,q-1} \\
\big\downarrow{\scriptstyle (-1)^k d_2} & & {\scriptstyle (-1)^{k+1} d_2}\big\downarrow \\
\mathcal{A}_X^{k-q+1,q} & & \mathcal{A}_X^{k-q+2,q} \\
\big\downarrow & & \big\downarrow \\
\mathcal{B}^{k+1}_{p+1,q+1} & \xrightarrow{\quad -\pi\circ d \quad} & \mathcal{B}^{k+2}_{p+1,q+1},
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{B}^k_{p,q} & \xrightarrow{\quad d_1 d_2 \quad} & \mathcal{B}^{k+1}_{p,q} \\
\big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle =} \\
\mathcal{A}_X^{p-1,q-1} & \underline{k = p+q-2} & \mathcal{A}_X^{p,q} \\
\big\downarrow{\scriptstyle (-1)^k d_2} & & {\scriptstyle (-1)^{k+1}\,\mathrm{id}}\big\downarrow \\
\mathcal{A}_X^{p-1,q} & & \mathcal{A}_X^{p,q} \\
\big\downarrow & & \big\downarrow{\scriptstyle =} \\
\mathcal{B}^{k+1}_{p+1,q+1} & \xrightarrow{\quad -\pi\circ d \quad} & \mathcal{B}^{k+2}_{p+1,q+1},
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{B}^k_{p,q} & \xrightarrow{\quad d \quad} & \mathcal{B}^{k+1}_{p,q} \\
\big\downarrow{\scriptstyle =} & & \big\downarrow \\
\mathcal{A}_X^{p,q} & \underline{k = p+q-1} & \mathcal{A}_X^{p+1,q} \\
\big\downarrow{\scriptstyle (-1)^k\,\mathrm{id}} & & {\scriptstyle (-1)^{k+1} d_2}\big\downarrow \\
\mathcal{A}_X^{p,q} & & \mathcal{A}_X^{p+1,q+1} \\
\big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle =} \\
\mathcal{B}^{k+1}_{p+1,q+1} & \xrightarrow{\quad -d_1 d_2 \quad} & \mathcal{B}^{k+2}_{p+1,q+1},
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{B}^k_{p,q} & \xrightarrow{\quad d \quad} & \mathcal{B}^{k+1}_{p,q} \\
\big\downarrow & & \big\downarrow \\
\mathcal{A}_X^{k-q+1,q-1} & \underline{k \geq p+q} & \mathcal{A}_X^{k-q+2,q-1} \\
\big\downarrow{\scriptstyle (-1)^k d_2} & & {\scriptstyle (-1)^{k+1} d_2}\big\downarrow \\
\mathcal{A}_X^{k-q+1,q} & & \mathcal{A}_X^{k-q+2,q} \\
\big\downarrow & & \big\downarrow \\
\mathcal{B}^{k+1}_{p+1,q+1} & \xrightarrow{\quad -d \quad} & \mathcal{B}^{k+2}_{p+1,q+1}.
\end{array}
$$

Note that we have to consider the differential maps on the bottom with sign -1 because they are the differential maps of $\mathcal{B}^\bullet_{p+1,q+1}$ shifted by [1]. Now for each square, a simple computation shows that both composite maps $\mu^{k+1} \circ d_{\mathcal{B}}$ and $d_{\mathcal{B}} \circ \mu^k : \mathcal{B}^k_{p,q} \to \mathcal{B}^{k+1}_{p+1,q+1}$ are induced by $(-1)^{k+1} d_1 d_2$ on a direct summand of $\mathcal{B}^k_{p,q}$. In particular, all four squares commute and hence μ is a morphism of complexes. $\square$

4.5 The $d_1 d_2$-lemma for $\mathbb{P}^N$

Theorem 4.21. *The tropical projective space $\mathbb{P}^N$ of dimension $N \geq 0$ satisfies the $d_1 d_2$-lemma.*

Proof. By proposition 4.16 it suffices to show that $X = \mathbb{P}^N$ satisfies the $\partial_1\partial_2$-lemma. By proposition 4.2 it is enough to show that the natural map

$$\mathrm{H}^{r,s}_{BC}(\mathcal{D}_X^{\bullet,\bullet}(X)) \to \mathrm{H}^{r,s}_{A}(\mathcal{D}_X^{\bullet,\bullet}(X))$$

is injective for every $r, s \in \mathbb{Z}$. This is obvious for $N = 0$, so let $N \geq 1$ and fix $r, s \in \mathbb{Z}$. Assume by induction that $\mathbb{P}^{N-1}$ satisfies the $\partial_1\partial_2$-lemma.

We can partition $\mathbb{P}^N$ as

$$\mathbb{P}^N = U \sqcup Z,$$

where $Z \subset \mathbb{P}^N$ is a good sedentarity of dimension $N-1$ with $Z \cong \mathbb{P}^{N-1}$ and $U \cong \mathbb{T}^N$. By corollary 3.17 we have a short exact sequence of sheaves

$$0 \to \imath_* \mathcal{D}_Z^{a,b} \to \mathcal{D}_X^{a,b} \to \jmath_* \mathcal{D}_U^{a,b} \to 0$$

for every $a, b \in \mathbb{Z}$, where $\imath : Z \to \mathbb{P}^N$ and $\jmath : U \to \mathbb{P}^N$ are the closed and open embeddings respectively. This gives rise to a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \imath_* \mathcal{E}_Z^\bullet & \longrightarrow & \mathcal{E}_X^\bullet & \longrightarrow & \jmath_* \mathcal{E}_U^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nu_Z} & & \downarrow{\scriptstyle \nu_X} & & \downarrow{\scriptstyle \nu_U} & & \\ 0 & \longrightarrow & \imath_* \tilde{\mathcal{E}}_Z^\bullet[1] & \longrightarrow & \tilde{\mathcal{E}}_X^\bullet[1] & \longrightarrow & \jmath_* \tilde{\mathcal{E}}_U^\bullet[1] & \longrightarrow & 0 \end{array}$$

where we write $\mathcal{E}_Z^\bullet$ for the complex $\mathcal{E}_{r,s}^\bullet$ and $\tilde{\mathcal{E}}_Z$ for the complex $\mathcal{E}_{r+1,s+1}^\bullet$ on Z and similarly for X and U.

By proposition 4.19, applying $\mathrm{R}^{r+s-1}\,\Gamma(X, \bullet)$ yields long exact sequences in cohomology

$$\begin{array}{ccccccc} \mathrm{R}^{r+s-2}\,\Gamma(U, \mathcal{E}_U^\bullet) & \longrightarrow & \mathrm{H}_{BC}^{r,s}(\mathcal{D}_Z^{\bullet,\bullet}(Z)) & \xrightarrow{\imath_*} & \mathrm{H}_{BC}^{r,s}(\mathcal{D}_X^{\bullet,\bullet}(X)) & \xrightarrow{\rho} & \mathrm{H}_{BC}^{r,s}(\mathcal{D}_U^{\bullet,\bullet}(U)) \\ \downarrow & & \downarrow{\scriptstyle \nu_Z} & & \downarrow{\scriptstyle \nu_X} & & \downarrow{\scriptstyle \nu_U} \\ \mathrm{R}^{r+s-2}\,\Gamma(U, \tilde{\mathcal{E}}_U^\bullet[1]) & \xrightarrow{\tilde{\delta}} & \mathrm{H}_A^{r,s}(\mathcal{D}_Z^{\bullet,\bullet}(Z)) & \xrightarrow{\imath_*} & \mathrm{H}_A^{r,s}(\mathcal{D}_X^{\bullet,\bullet}(X)) & \longrightarrow & \mathrm{H}_A^{r,s}(\mathcal{D}_U^{\bullet,\bullet}(U)). \end{array}$$

The map $\nu_Z : \mathrm{H}_{BC}^{r,s}(\mathcal{D}_Z^{\bullet,\bullet}(Z)) \to \mathrm{H}_A^{r,s}(\mathcal{D}_Z^{\bullet,\bullet}(Z))$ is an isomorphism by assumption and the map $\nu_U : \mathrm{H}_{BC}^{r,s}(\mathcal{D}_U^{\bullet,\bullet}(U)) \to \mathrm{H}_A^{r,s}(\mathcal{D}_U^{\bullet,\bullet}(U))$ is an isomorphism by corollary 4.14.

This immediately implies $\mathrm{H}_{BC}^{r,s}(\mathcal{D}_X^{\bullet,\bullet}(X)) = 0 = \mathrm{H}_A^{r,s}(\mathcal{D}_X^{\bullet,\bullet}(X))$ for $(r,s) = (-n, -n+1)$ or $(r,s) = (-n+1, -n)$: We have $\mathrm{H}_A^{r,s}(\mathcal{D}_Z^{\bullet,\bullet}(Z)) = 0$ by $\dim(Z) < n$ and we have $\mathrm{H}_A^{r,s}(\mathcal{D}_U^{\bullet,\bullet}(U)) = 0$ from proposition 4.12(3a).

For all other cases, we next show $\mathrm{R}^{r+s-2}\,\Gamma(U, \tilde{\mathcal{E}}_U^\bullet[1]) = 0$ as follows: Recall that by definition 4.17(3), the relevant part of $\tilde{\mathcal{E}}_U^\bullet = \mathcal{E}_{r+1,s+1}^\bullet$ on U is

$$\begin{aligned} \cdots \longrightarrow \mathcal{E}_{r+1,s+1}[1]^{r+s-3} &= \mathcal{D}_U^{r-2,s} \oplus \mathcal{D}_U^{r-1,s-1} \oplus \mathcal{D}_U^{r,s-2} \longrightarrow \\ \mathcal{E}_{r+1,s+1}[1]^{r+s-2} &= \mathcal{D}_U^{r-1,s} \oplus \mathcal{D}_U^{r,s-1} \longrightarrow \\ \mathcal{E}_{r+1,s+1}[1]^{r+s-1} &= \mathcal{D}_U^{r,s} \longrightarrow \cdots, \end{aligned}$$

with differentials given by projecting the image of the usual differential map $\partial : \mathcal{D}_U^k \to \mathcal{D}_U^{k+1}$. Keep in mind that the sheaves $\tilde{\mathcal{E}}_U^k$ are flabby so we have $\mathrm{R}^k\,\Gamma(U, \tilde{\mathcal{E}}_U^\bullet[1]) = \mathrm{H}^k(\tilde{\mathcal{E}}_U^\bullet(U)[1])$.

First assume $(r,s) = (-n,-n)$; then $\mathrm{R}^{r+s-2}\,\Gamma(U, \tilde{\mathcal{E}}_U^\bullet[1]) = 0$ is trivial by $\tilde{\mathcal{E}}_U[1]^{-2n-2} = 0$. For $r + s > -2n + 1$, we show $\mathrm{R}^{r+s-2}\,\Gamma(U, \tilde{\mathcal{E}}_U^\bullet[1]) = 0$ by using proposition 4.12(4): The kernel of

$$\Gamma(U, \tilde{\mathcal{E}}_U^{r+s-1}) = \mathcal{D}_U^{r,s-1}(U) \oplus \mathcal{D}_U^{r-1,s}(U) \xrightarrow{\pi \circ \partial} \mathcal{D}_U^{r,s}(U) = \Gamma(U, \tilde{\mathcal{E}}_U^{r+s})$$

consists precisely of the nearly ∂-closed currents (with $r_1 = r-1$ and $r_2 = r$). If ϕ is such a nearly ∂-closed current, proposition 4.12(4) provides us with $\tilde{\phi}^{r-1,s} \in \mathcal{D}_U^{r-1,s}(U)$ and $\tilde{\phi}^{r,s-1} \in \mathcal{D}_U^{r,s-1}(U)$ which are ∂_2-closed and ∂_1-closed respectively such that

$$\phi = \tilde{\phi}^{r-1,s} + \partial \tilde{\psi}^{r-1,s-1} + \tilde{\phi}^{r,s-1},$$

for some $\tilde{\psi}^{r-1,s-1} \in \mathcal{D}_U^{r-1,s-1}(U)$. The complexes $\mathcal{D}_U^{r-1,\bullet}(U)$ and $\mathcal{D}_U^{\bullet,s-1}(U)$ are exact in degrees $> -n$ for $r-1 = -n$ and $s-1 = -n$, and exact in every degree otherwise (by $U \cong \mathbb{T}^N$, Poincaré duality 3.27 and example 3.39). With $r+s > -2n+1$ this implies that we can write $\tilde{\phi}^{r-1,s} = \partial_2 \tilde{\psi}^{r-2,s}$ and $\tilde{\phi}^{r,s-1} = (-1)^r \partial_1 \tilde{\psi}^{r,s-2}$. Then

$$\psi := \tilde{\psi}^{r-2,s} + \tilde{\psi}^{r-1,s-1} + \tilde{\psi}^{r,s-2} \in \Gamma(U, \tilde{\mathcal{E}}_U^{r+s-2})$$

maps to ϕ via the differential map $\pi \circ \partial$ of $\tilde{\mathcal{E}}_U^\bullet$. This shows $\mathrm{R}^{r+s-2}\,\Gamma(U, \tilde{\mathcal{E}}_U^\bullet[1]) = 0$ as claimed.

Let now $\phi \in \mathrm{H}^{r,s}_{BC}(\mathcal{D}^{\bullet;\bullet}_X(X))$ with $\nu_X(\phi) = 0$. We then show $\phi = 0$ by a simple diagram chase: First, $\nu_U\rho(\phi) = 0$ implies $\rho(\phi) = 0$ because ν_U is an isomorphism, and hence we can write $\phi = \imath_*(\psi)$. Because $\imath_*\nu_Z(\psi) = 0$ by assumption, this implies $\nu_Z(\psi) = \tilde{\delta}(\zeta)$. But we have shown that $\mathrm{R}^{r+s-2}\,\Gamma(U, \tilde{\mathcal{E}}^{\bullet}_U[1]) = 0$ and hence $0 = \tilde{\delta}(\zeta) = \nu_Z(\psi)$. By induction, ν_Z is an isomorphism, so we have $\psi = 0$. This implies $\phi = \imath_*(\psi) = 0$ and shows that ν_X is injective, finishing the proof. □

4.6 The d_1d_2-lemma for certain simple examples

In this chapter, we collect some direct proofs of the d_1d_2-lemma for very simple tropical surfaces. But first, we give the following simple application of theorem 4.21:

Example 4.22. For the tropical projective space $X = \mathbb{P}^N$ the canonical morphism

$$(\cdot)^- : \mathrm{R}^2\,\Gamma(X, \mathcal{A}^{\bullet}_X) \to \mathrm{R}^2\,\Gamma(X, \mathcal{A}^{\bullet;-}_X)$$

from proposition 3.34 is an isomorphism.

Proof. Let $\alpha \in \Gamma(X, \mathcal{A}^2_X)$ be a d-closed form. By the d_1d_2-lemma and example 3.41, we may chose $\alpha = \alpha^{1,1} \in \mathcal{A}^{1,1}_X(X) \cap \ker(d)$. Now $\mathbb{P}^N$ has a covering by open subsets $U \cong \mathbb{T}^N$ and on each U we can write $\alpha|_U = d_1d_2 f_U$ with $f_U \in \mathcal{A}^{0,0}_X(U)$ by proposition 4.11(2). We now have

$$\begin{aligned} J\alpha|_U &= J(d_1d_2 f_U) \\ &= -(d_2d_1 J f_U) \\ &= -d_1d_2 f_U \\ &= -\alpha|_U. \end{aligned}$$

This implies $\alpha^- = \frac{\alpha - J\alpha}{2} = \alpha$ and $\alpha^+ = \frac{\alpha + J\alpha}{2} = 0$. By proposition 3.34, the maps $(\cdot)^+$ and $(\cdot)^-$ induce isomorphisms

$$\mathrm{R}^k\,\Gamma(X, \mathcal{A}^{\bullet}_X) \to \mathrm{R}^k\,\Gamma(X, \mathcal{A}^{\bullet;+}_X) \oplus \mathrm{R}^k\,\Gamma(X, \mathcal{A}^{\bullet;-}_X)$$

for every k. Since we have shown that $(\cdot)^+$ vanishes on $\mathrm{R}^2\,\Gamma(X, \mathcal{A}^{\bullet}_X)$, this finishes the proof. □

It is quite possible that this (and more) can be obtained directly, without using the d_1d_2-lemma, but I think this proof is nice too. Now let us start with the computations:

Example 4.23. The spaces $L := \mathbb{S}^1 \times \mathbb{P}^1$ and M, the tropical Moebius strip from example 3.42, both satisfy the d_1d_2-Lemma.

Proof. We will prove both statements in one go because they are very similar:

Recall that the Dolbeault cohomology of $X \in \{L, M\}$ is given by

$$h^{\bullet;\bullet}_{d_2}(X) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We now show that $\mathrm{H}^{p,q}_{BC}(X) \to \mathrm{H}^q(X, \mathcal{L}^p_X)$ and $\mathrm{H}^{p,q}_{BC}(X) \to \mathrm{H}^p(X, \mathcal{L}^p_X)$ are injective, using the exact sequences

$$0 \to \frac{\mathrm{im}(d_1) \cap \ker(d_2)}{\mathrm{im}(d_1d_2)} \to \mathrm{H}^{p,q}_{BC}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{d_1}(A^{\bullet,\bullet}),$$

$$0 \to \frac{\ker(d_1) \cap \operatorname{im}(d_2)}{\operatorname{im}(d_1 d_2)} \to \mathrm{H}^{p,q}_{BC}(A^{\bullet,\bullet}) \to \mathrm{H}^{p,q}_{d_2}(A^{\bullet,\bullet}),$$

from proposition 4.4.

Case $p + q \leq 1$: The case $p = q = 0$ is trivial and for $p+q = 1$ it follows from corollary 4.10(2) and proposition 4.2.

Cases $(p, q) = (0, 2)$ or $(p, q) = (2, 0)$: Since we have $h^{0,2}_{d_1}(X) = h^{0,2}_{d_2}(X) = 0$ for both L and M we also get $\mathrm{H}^{p,q}_{BC}(X) = 0$ and injectivity follows immediately.

Case $(p, q) = (1, 1)$: By symmetry we only have to consider the second exact sequence above for the map $\mathrm{H}^{1,1}_{BC}(X) \to \mathrm{H}^{1,1}_{d_2}(X)$. So for $\alpha \in \ker(d_1) \cap \operatorname{im}(d_2)$, we have to show that α lies in the image of $d_1 d_2$. We can write $\alpha = d_2\beta$ with $\beta \in \mathcal{A}^{1,0}_X(X)$ and $d_2 d_1 \beta = 0$. From $h^{2,0}_{d_2}(X) = 0$ we get that β is d_1-closed. For both $X \in \{L, M\}$, $\mathrm{H}^1(X, \mathcal{L}^1_X)$ is one dimensional. When considering the chart $U = (-1, 1) \times \mathbb{P}$ in either case, we can continue the closed $(1, 0)$-form $d'x_1$ to all of X. Moreover, $d'x_1$ is not exact: Else there would be a smooth function f on X with $d_1 f|_U = d'x_1$ which implies that up to a constant $f|_U(x_1, x_2)$ is linear with non-zero slope in the coordinate x_1, and constant in x_2. Such a function has no continuous continuation to either space. Hence $d'x_1$ generates $\mathrm{H}^1(X, \mathcal{L}^1_X)$ and we have $\beta = cd'x_1 + d_1 f$ with $f \in \mathcal{A}^{0,0}_X(X)$, $c \in \mathbb{R}$. Hence we have $\alpha = d_2(cd'x_1 + d_1 f) = d_1 d_2 f \in \operatorname{im}(d_1 d_2)$ and we are done.

Case $(p, q) = (1, 2)$ or $(p, q) = (2, 1)$: Once again, we only show the first case, the second one follows by exchanging the roles of d_1 and d_2. First, consider $\alpha \in \operatorname{im}(d_1) \cap \ker(d_2)$, i.e. $\alpha = d_1\beta$ with $\beta \in \mathcal{A}^{0,2}_X(X)$. Since we have $h^{0,2}_{d_2}(X) = 0$, we can write $\beta = d_2\gamma$ and $\alpha = d_1 d_2 \gamma \in \operatorname{im}(d_1 d_2)$. Hence the map $\mathrm{H}^{1,2}_{BC}(X) \to \mathrm{H}^{1,2}_{d_1}(X)$ is injective.

Next we show that $\mathrm{H}^{1,2}_{BC}(X) \to \mathrm{H}^{1,2}_{d_2}(X)$ is injective as well: Let then $\alpha \in \ker(d_1) \cap \operatorname{im}(d_2)$, $\alpha = d_2\beta$. We then have $d_2 d_1 \beta = 0$, i.e. $d_1\beta \in \ker(d_2) \subset \mathcal{A}^{2,1}_X(X)$.

The space $\mathrm{H}^{2,1}_{d_2}(X)$ is one dimensional and generated by a form ρ given by

$$\rho|_U = h(x_2) d'x_1 \wedge d'x_2 \otimes d''x_2$$

on the chart $U = (-1, 1) \times \mathbb{P} \subset X$, where $0 \neq h : \mathbb{R} \subset \mathbb{P} \to \mathbb{R}$ is a symmetric non-negative function with compact support: It is clear that $\rho|_U$ has a unique continuation $\rho \in \mathcal{A}^{2,1}_X(X)$ and that ρ is d_2-closed. If we assume by contradiction that $\rho = d_2\zeta$ is d_2-exact with $\zeta|_U = f(x_1, x_2) d'x_1 \wedge d'x_2$ with $f : U \to \mathbb{R}$ smooth, we get

$$\partial_1 f = 0, \qquad \partial_2 f(x_1, x_2) = h(x_2).$$

In particular, when fixing $x_1 \in (-1, 1)$, we get $f(x_1, -\infty) < f(x_1, \infty)$. In a neighbourhood of $S = (-1, 1) \times \{\pm\infty\}$ the form ζ has to be the pullback of a form on S. This implies that the support of f has empty intersection with S, so we end up with a contradiction and ρ indeed generates the space $\mathrm{H}^{2,1}_{d_2}(X)$.

We can now write $d_1\beta = c\rho + d_2\zeta$ with $\zeta \in \mathcal{A}^{2,0}_X(X)$. But once again we have $h^{2,0}_{d_1}(X) = 0$, so $\zeta = d_1\gamma$ with $\gamma \in \mathcal{A}^{1,0}_X(X)$. This implies that $c\rho = d_1(\beta - d_2\gamma)$ is d_1-exact. Writing

$$\beta - d_2\gamma = f_1 d'x_1 \otimes d''x_1 + g_1 d'x_1 \otimes d''x_2 + f_2 d'x_2 \otimes d''x_1 + g_2 d'x_2 \otimes d''x_2 \in \mathcal{A}^{1,1}_X(X),$$

we get

$$ch(x_2) = \partial_1 g_2(x_1, x_2) - \partial_2 g_1(x_1, x_2),$$

on $U = (-1, 1) \times \mathbb{P}$. Let us assume that $c \neq 1$ and derive a contradiction. Without restricting generality, we may assume $c = 1$. Again, since $\beta - d_2\gamma$ is the pullback of a form on $S = (-1,1) \times \{\pm\infty\}$ near S, both $g_1(x_1, \cdot)$ and $g_2(x_1, \cdot)$ have to have compact support in $\mathbb{R} \subset \mathbb{P}$ for every $x_1 \in (-1,1)$. This implies

$$0 < \int_{\mathbb{R}} h(x_2)dx_2 = \int_{\mathbb{R}} (\partial_1 g_2(x_1,x_2) - \partial_2 g_1(x_1,x_2))dx_2 = \int_{\mathbb{R}} \partial_1 g_2(x_1,x_2)dx_2.$$

for every $x_1 \in \mathbb{R}$ (we integrate with respect to the usual measure on $\mathbb{R}$).

In both cases $X \in \{L, M\}$, we may interprete $g_2(\cdot, x_2)$ as a continuous function on $I = [-1,1]$ with $g_2(-1,x_2) = g_2(1,x_2)$ for $X = L$ and $g_2(-1,x_2) = g_2(1,-x_2)$ for $X = M$. We can then integrate the above (in)equality again over I. This results in

$$0 < \int_{I\times\mathbb{R}} h(x_2)dx_2dx_1 = \int_{\mathbb{R}}\int_I \partial_1 g_2(x_1,x_2)dx_1dx_2 = 0.$$

This is a contradiction, and we get $c = 0$.

To recapitulate, we now have $\alpha = d_2\beta = d_2(\beta - d_2\gamma)$ with $d_1(\beta - d_2\gamma) = 0$, $\gamma \in \mathcal{A}_X^{1,0}(X)$. The space $\mathrm{H}_{d_1}^{1,1}(X)$ is two dimensional and one can easily see that it is generated by continuations of the forms

$$d'x_1 \otimes d''x_1, \qquad h(x_2)d'x_2 \otimes d''x_2$$

on $U = (-1,1) \times \mathbb{P}$ in both cases, where $h : \mathbb{R} \subset \mathbb{P} \to \mathbb{R}$ is a symmetric function with compact support as above: Note first that indeed both forms are d_1-closed and have unique continuations in $\mathcal{A}_X^{1,1}(X)$. Say they are linearly dependent in $\mathrm{H}_{d_1}^{1,1}(X)$. Then we find a form $\alpha \in \mathcal{A}_X^{0,1}(X)$, restricting to $fd''x_1 + gd''x_2 \in \mathcal{A}_X^{0,1}(U)$ and satisfying an equation of the following kind:

$$d_1(fd''x_1 + gd''x_2) = c_1 d'x_1 \otimes d''x_1 + c_2 h(x_2)d'x_2 \otimes d''x_2$$

on U. Similar to before we can see that this is only possible if f is constant and g vanishes and hence $c_1 = c_2 = 0$ follows. In particular, the forms generate $\mathrm{H}_{d_1}^{1,1}(X)$.

At last, we then can write

$$\begin{aligned}\alpha &= d_2(\beta - d_2\gamma)\\ &= d_2\left(c_1 d'x_1 \otimes d''x_1 + c_2 h d'x_2 \otimes d''x_2 + d_1\lambda\right)\\ &= d_1 d_2 \lambda \in \mathrm{im}(d_1 d_2),\end{aligned}$$

with $c_1, c_2 \in \mathbb{R}$. This finishes this part of the proof.

Case $(p,q) = (2,2)$: By symmetry, it suffices to show that $\mathrm{H}_{BC}^{2,2}(X) \to \mathrm{H}_{d_2}^{2,2}(X)$ is injective. Let now $\alpha \in \mathrm{im}(d_2)$, $\alpha = d_2\beta$ with $\beta \in \mathcal{A}_X^{2,1}(X)$. In the previous step, we have seen that the $(2,1)$-form $h(x_2)d'x_1 \wedge d'x_2 \otimes d''x_2$ is neither d_1- nor d_2-exact. Hence, it generates the one dimensional space $\mathrm{H}_{d_1}^{2,1}(X) = \mathcal{A}_X^{2,1}(X)/\operatorname{im}(d_1)$. We now can write

$$\alpha = d_2\beta = d_2\left(ch(x_2)d'x_1 \wedge d'x_2 \otimes d''x_2 + d_1\gamma\right) = d_1 d_2 \gamma,$$

with $\gamma \in \mathcal{A}_X^{1,1}(X)$, $c \in \mathbb{R}$. This finishes this last case and hence L and M both satisfy the $d_1 d_2$-lemma. □

5 Cohomology of currents

5.1 Topology of differential forms on tropical spaces in $\mathbb{T}^N$

In this section, for a tropical space X in $\mathbb{T}^N$ we will first equip the spaces $\mathcal{A}_X^{p,q}(U)$ with a structure of locally convex spaces. Note that they are not necessarily complete with respect to this topology. In the next section we will extend this to general tropical spaces. We will then use this construction to define complexes of sheaves of currents on X. If U is finitary (definition 1.17), this will give the same objects as considered in [Gub13]. In the subsequent sections we will then collect some properties of these complexes, including a 'smoothing of cohomology' statement in theorem 5.17.

First we need to define a family of seminorms on forms on $\mathbb{T}^N$:

Recall that for an open subet $\tilde{U} \subset \mathbb{T}^N$ we write $\mathrm{Sed}(\tilde{U})$ for the set of subsets $I \subset [N]$ with $\tilde{U}_I^\circ := \tilde{U} \cap \mathbb{R}_I^N \neq \emptyset$.

Definition 5.1. Let $\tilde{U}$ be an open subset of $\mathbb{T}^N$ and consider $f = (f_I)_{I \in \mathrm{Sed}(\tilde{U})} \in \mathcal{A}_{\mathbb{T}^N}^{0,0}(\tilde{U})$.

1. For $I \in \mathrm{Sed}(\tilde{U})$, $x \in \mathbb{R}_I^N$ and $\nu \in \mathbb{N}^N$ we define
$$|f(x)|^\nu := 0, \text{ if } \nu_i > 0 \text{ for some } i \in I \text{ and}$$
$$|f(x)|^\nu := \left| \frac{d^{\nu_1} \cdots d^{\nu_N}}{dx_1^{\nu_1} \cdots dx_N^{\nu_N}} f_I(x) \right|, \text{ otherwise.}$$

2. For $x \in \mathbb{T}^N$ and $k \in \mathbb{N}$ we define
$$|f(x)|^k := \max_{|\nu| \leq k} |f(x)|^\nu.$$

3. For $K \subset \tilde{U}$ compact, $k \in \mathbb{N}$ we define
$$|f|_K^k := \sup_{x \in K} |f(x)|^k.$$

Lemma 5.2. *Let $\tilde{U} \subset \mathbb{T}^N$ be an open subset. For every $f \in \mathcal{A}_{\mathbb{T}^N}^{0,0}(\tilde{U})$, the map*
$$|f|^k : X \to \mathbb{R}_0^+, \qquad x \mapsto |f(x)|^k,$$
is continuous. In particular, for $K \subset \tilde{U}$ compact we have $|f|_K^k < \infty$.

Proof. This follows from the fact that for $J \subset I \in \mathrm{Sed}(\tilde{U})$ and $x \in \tilde{U}_I^\circ$, f_J is equal to $\pi_{IJ}^* f_I$ in a neighbourhood of x (c.f. definition 2.4). □

Construction 5.3. For every open subset $\tilde{U} \subset \mathbb{T}^N$, the family of seminorms $(|\cdot|_K^k)_{K,k}$ induces a locally convex topology on $\mathcal{A}_{\mathbb{T}^N}^{p,q}(\tilde{U})$ such that for $\tilde{U}' \subset \tilde{U}$ the restriction map $\mathcal{A}_{\mathbb{T}^N}^{p,q}(\tilde{U}) \to \mathcal{A}_{\mathbb{T}^N}^{p,q}(\tilde{U}')$ as well as the differential maps
$$d' : \mathcal{A}_{\mathbb{T}^N}^{p,q}(\tilde{U}) \to \mathcal{A}_{\mathbb{T}^N}^{p+1,q}(\tilde{U}) \qquad \text{and} \qquad d'' : \mathcal{A}_{\mathbb{T}^N}^{p,q}(\tilde{U}) \to \mathcal{A}_{\mathbb{T}^N}^{p,q+1}(\tilde{U})$$
are continuous: We simply put, for $\alpha = (\alpha_I)_{I \in \mathrm{Sed}(\tilde{U})} \in \mathcal{A}_{\mathbb{T}^N}^{p,q}(\tilde{U})$, $\alpha_I = \sum_{\mu,\lambda \subset [N]} f_I^{\mu\lambda} d'x_\mu \otimes d''x_\lambda$, and for $x \in \tilde{U}_I^\circ$:
$$|\alpha(x)|^k := \max_{\mu,\lambda} |f_I^{\mu\lambda}(x)|^k$$
and then, for $K \subset \tilde{U}$ compact, we may take the maximum over all $x \in K$, defining seminorms $|\cdot|_K^k$ on $\mathcal{A}_{\mathbb{T}^N}^{p,q}(\tilde{U})$ as required.

Lemma 5.4. *Let $\tilde{U} \subset \mathbb{T}^N$ be an open subset.*

1. *Every $\mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U})$ is a metrizable locally convex space, i.e. its topology is generated by a countable set of seminorms.*

2. *If $\tilde{U}$ is finitary, i.e. $\tilde{U} \subset \mathbb{R}^N$, then the map $\mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U}) \to \mathcal{C}^\infty(\tilde{U}) \otimes \bigwedge^p(\mathbb{R}^N)^* \otimes \bigwedge^q(\mathbb{R}^N)^*$ is an isomorphism of locally convex spaces. In particular, $\mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U})$ is a Fréchet space.*

Proof. For (1) note that $\mathbb{T}^N$ is homeomorphic to $[0,1)^N$. Hence the directed set of compact subsets of $\tilde{U} \subset \mathbb{T}^N$ contains a countable cofinal family of compact subsets of $\tilde{U}$. It suffices to consider seminorms p_K^k with K in this family. For an open subset $\tilde{U}$ of $\mathbb{R}^N$ it is enough to note that the seminorms considered on $\mathcal{A}^{0,0}(\tilde{U})$ are the same as the seminorms used for the topology of $\mathcal{C}^\infty(\tilde{U})$. □

For a polyhedral complex Σ in $\mathbb{T}^N$ and $X = |\Sigma|$ recall the definition of the sheaves $\mathcal{K}^{p,q}_X$ of (p,q)-forms on $\mathbb{T}^N$ vanishing on X (definition 2.3).

Lemma 5.5. *Let U be an open subset of $X = |\Sigma|$ for a polyhedral complex Σ in $\mathbb{T}^N$. Then for every open subset $\tilde{U} \subset \mathbb{T}^N$ with $\tilde{U} \cap X = U$ the space $\mathcal{K}^{p,q}_X(\tilde{U})$ of forms vanishing on X is closed in $\mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U})$.*

Proof. First, let U be finitary so that we can find $\tilde{U} \subset \mathbb{R}^N$ with $\tilde{U} \cap X = U$. Then the embedding $\imath : U^{\mathrm{reg}} \to \tilde{U}$ is a morphism of smooth manifolds and $\mathcal{K}^{p,q}_X(\tilde{U})$ is the kernel of the continuous map of Fréchet spaces $\imath^* : \mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U}) \to \mathcal{A}^{p,q}_{U^{\mathrm{reg}}}(U^{\mathrm{reg}})$. In particular, it is a closed subspace of $\mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U})$.

For general $\tilde{U}$, let $(\alpha^j)_{j\in J}$ be a net in $\mathcal{K}^{p,q}_X(\tilde{U})$ with $\alpha^j = (\alpha^j_I)_{I\in\mathrm{Sed}(\tilde{U})}$, converging to $\alpha = (\alpha_I)_{I\in\mathrm{Sed}(\tilde{U})} \in \mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U})$. Then for every $I \in \mathrm{Sed}(\tilde{U})$, $(\alpha^j_I)_{j\in J}$ is a net in $\mathcal{K}^{p,q}_{X_I^\circ}(\tilde{U}_I^\circ)$ converging to $\alpha_I \in \mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U}_I^\circ)$. By the finitary case discussed before, each α_I lies in $\mathcal{K}^{p,q}_{X_I^\circ}(\tilde{U}_I^\circ)$. It follows that α is an element of $\mathcal{K}^{p,q}_X(\tilde{U})$. □

Construction 5.6. Let X be a polyhedral space in $\mathbb{T}^N$. Then, for every open subset $U \subset X$, the space $\mathcal{A}^{p,q}_X(U)$ is the quotient of $\mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U})$ by the closed subspace $\mathcal{K}^{p,q}_X(\tilde{U})$, where $\tilde{U}$ is an open subset of $\mathbb{T}^N$ with $\tilde{U} \cap X = U$. We equip $\mathcal{A}^{p,q}_X(U)$ with the (locally convex) quotient topology.

Proposition 5.7. *Let $U' \subset U \subset X$ be open subsets.*

1. *Assume that U is finitary, i.e. $U \subset \mathbb{R}^N$. Then $\mathcal{A}^{p,q}_X(U)$ is a Fréchet space for every $p, q \in \mathbb{Z}$.*

2. *The restriction maps $\mathcal{A}^{p,q}_X(U') \to \mathcal{A}^{p,q}_X(U)$ are continuous.*

3. *The maps $d_1 : \mathcal{A}^{p,q}_X(U) \to \mathcal{A}^{p+1,q}_X(U)$, $d_2 : \mathcal{A}^{p,q}_X(U) \to \mathcal{A}^{p,q+1}_X(U)$ and $\wedge : \mathcal{A}^{p',q'}_X(U) \times \mathcal{A}^{p'',q''}_X(U) \to \mathcal{A}^{p'+p'',q'+q''}_X(U)$ are continuous.*

Proof. (1) follows directly from proposition C.14: $\mathcal{A}^{p,q}_X(U)$ is the quotient of a Fréchet space by a closed subspace, equipped with the quotient topology. Hence it is Fréchet.

(2) and (3) follow directly from the definitions: The corresponding maps are continuous on $\mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U})$ for $\tilde{U} \cap X = U$ and they restrict to continuous maps on the sections of $\mathcal{K}^{p,q}_X(\tilde{U})$. Using the universal property of the cokernel $\mathcal{A}^{p,q}_X(U) = \mathrm{coker}(\mathcal{K}^{p,q}_X(\tilde{U}) \to \mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U}))$ in lcs, we get continuous maps as required. □

Remark 5.8. The space $\mathcal{A}^{0,0}_{\mathbb{T}}(\mathbb{T})$ is not complete: consider a sequence of smooth functions $f_k : \mathbb{T} \to \mathbb{R}$ such that $f_k(t) = 0$ for $t < -k$, $f_k(t) = 1$ for $t > 1 - k$ and $0 \leq f_k(t) \leq 1$ everywhere. Then, for every $n \in \mathbb{N}$, the function $g_n := \sum_{k=0}^{n} \frac{1}{n^2} f_n$ lies in $\mathcal{A}^{0,0}_{\mathbb{T}}(\mathbb{T})$ and $(g_n)_n$ is a Cauchy sequence in $\mathcal{A}^{0,0}_{\mathbb{T}}(\mathbb{T})$ that does not converge.

5.2 Topology of differential forms on general tropical spaces

Lemma 5.9. *Let U be an open subset of the support X of a polyhedral complex Σ in $\mathbb{T}^N$ and U' an open subset of the support X' of Σ' in $\mathbb{T}^{N'}$. Let $F : U \to U'$ be an extended affine map.*

1. *The induced pullback maps $F^* : \mathcal{A}^{p,q}_{X'}(U') \to \mathcal{A}^{p,q}_{X}(U)$ are continuous.*
2. *If F is an isomorphism, then F^* is an isomorphism of locally convex spaces.*

Proof. We first consider the case $(p, q) = (0, 0)$ and $X = \mathbb{T}^N$, $X' = \mathbb{T}^{N'}$. Let $\alpha^j = (\alpha^j_{I'})_{I \in \mathrm{Sed}(U')}$, $j \in J$, be a sequence in $\mathcal{A}^{0,0}_{X}(U')$, converging to $\alpha \in \mathcal{A}^{0,0}_{X}(U')$. For every $I \in \mathrm{Sed}(U)$, we find $I' \in \mathrm{Sed}(U')$ such that $F|_{U_I^\circ} : U_I^\circ \to (U')^\circ_{I'}$ is an affine map on the open subsets of $\mathbb{R}^N_I$ and $\mathbb{R}^{N'}_{I'}$ respectively. In particular, the induced map $(F|_{U_I^\circ})^* : \mathcal{A}^{0,0}((U')^\circ_{I'}) \to \mathcal{A}^{0,0}(U_I^\circ)$ is continuous. This implies that each net $(F|_{U_I^\circ})^* \alpha^j_{I'}$ converges to $(F|_{U_I^\circ})^* \alpha_{I'}$ and hence $F^*\alpha^j$ converges to $F^*\alpha$. The cases $(p, q) \neq (0, 0)$ follow analoguously.

If Σ and Σ' are arbitrary polyhedral complexes, we can first extend F to an extended affine map $\tilde{F} : \tilde{U} \to \tilde{U}'$, where $\tilde{U}$ and $\tilde{U}'$ are open subsets of $\mathbb{T}^N$ and $\mathbb{T}^{N'}$ respectively with $U = \tilde{U} \cap X$ and $U' = \tilde{U}' \cap X'$. We then get a commuting square of continuous maps of locally convex spaces

$$\begin{array}{ccc} \mathcal{K}^{p,q}_{X'}(\tilde{U}') & \xrightarrow{\tilde{F}} & \mathcal{K}^{p,q}_{X}(\tilde{U}) \\ \downarrow & & \downarrow \\ \mathcal{A}^{p,q}_{\mathbb{T}^{N'}}(\tilde{U}') & \xrightarrow{\tilde{F}} & \mathcal{A}^{p,q}_{\mathbb{T}^N}(\tilde{U}), \end{array}$$

where the closed subspaces $\mathcal{K}^{p,q}_{X'}(\tilde{U}')$ and $\mathcal{K}^{p,q}_{X}(U)$ are equipped with the induced topology. Taking the cokernels of the vertical maps gives us the continuous map $\tilde{F} : \mathcal{A}^{p,q}_{X'}(U') \to \mathcal{A}^{p,q}_{X}(U)$ as required for (1).

For (2), let F be an isomorphism. Then there exists an extended affine map $G : U' \to U$ such that GF and FG are the identity on U and U' respectively. On $\mathcal{A}^{p,q}_{X}(U)$ and $\mathcal{A}^{p,q}_{X'}(U')$ we have $F^*G^* = \mathrm{id}$ and $G^*F^* = \mathrm{id}$ and by part (1), G^* is continuous as well. Hence F^* is an isomorphism of locally convex spaces. □

This allows us to extend the locally convex topology to sections of $\mathcal{A}^{p,q}_{X}$ over more general tropical spaces:

Proposition 5.10. *Let X be a tropical space which has an atlas $\mathcal{U} = (U_k)_{j \in J}$ of tropical charts $\phi^j : U_j \to V_j \subset \mathbb{T}^{N^k}$. Then, for every open subset $U \subset X$, the subspace topology on $\mathcal{A}^{p,q}_{X}(U)$ with respect to the embedding*

$$0 \to \mathcal{A}^{p,q}_{X}(U) \to \prod_{j \in J} \mathcal{A}^{p,q}_{X}(\phi^j(U_j \cap U)),$$

is independent of the chosen covering. Here, the rightmost space is equipped with locally convex topology of the product ([Pro00, 2.1.3]).

Proof. Let $\mathcal{V} = (V_l)_{l \in L}$ be a second covering and consider the common refinement $\mathcal{U} \cap \mathcal{V} = (U_j \cap V_l)_{j \in J, l \in L}$. We then have a commuting diagram of $\mathbb{R}$-vector spaces:

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathcal{A}_X^{p,q}(U) & \longrightarrow & \prod_{j \in J} \mathcal{A}_X^{p,q}(\phi_j(U_j)) \\
 & & \big\downarrow = & & \big\downarrow \\
0 & \longrightarrow & \mathcal{A}_X^{p,q}(U) & \longrightarrow & \prod_{j \in J} \prod_{l \in L} \mathcal{A}_X^{p,q}(\phi_j(U_j \cap V_l)) \\
 & & \big\downarrow = & & \big\downarrow (\phi_j \circ \psi_l^{-1})^* \\
0 & \longrightarrow & \mathcal{A}_X^{p,q}(U) & \longrightarrow & \prod_{j \in J} \prod_{l \in L} \mathcal{A}_X^{p,q}(\psi_l(U_j \cap V_l)) \\
 & & = \big\uparrow & & \big\uparrow \\
0 & \longrightarrow & \mathcal{A}_X^{p,q}(U) & \longrightarrow & \prod_{l \in L} \mathcal{A}_X^{p,q}(\psi_l(V_l)).
\end{array}$$

The morphisms $(\phi_j \circ \psi_l^{-1})^*$ are isomorphisms of locally convex spaces for every $j \in J$, $l \in L$ by the previous lemma. Moreover, for every $j \in J$, the morphism

$$\mathcal{A}_X^{p,q}(\phi_j(U_j)) \to \prod_{l \in L} \mathcal{A}_X^{p,q}(\phi_j(U_j \cap V_l))$$

identifies the topology on $\mathcal{A}_X^{p,q}(\phi_j(U_j))$ with the subspace topology with respect to this embedding (this can be seen by using that the compact subsets of $\phi_j(U_j)$ and the compact subsets of $\phi_j(U_j \cap V_l)$ for varying l are cofinal). This implies that the top three horizontal maps all induce the same subspace topology on $\mathcal{A}_X^{p,q}(U)$. The same argument shows that the bottom three maps also induce the same topology. Hence, the topology is independent from the chosen covering. $\square$

5.3 Currents

Definition 5.11. Let X be a tropical space. For every open subset $U \subset X$ and any compact subset $K \subset U$, we consider the closed subspaces

$$\Gamma_K(U, \mathcal{A}_X^{p,q}) := \ker(\mathcal{A}_X^{p,q}(U) \to \mathcal{A}_X^{p,q}(U \setminus K))$$

of $\mathcal{A}^{p,q}(U)$, equipped with the induced topology.

The space of *compactly supported* (p, q)*-forms* on U is the set

$$\Gamma_c\,(U, \mathcal{A}_X^{p,q}) := \operatorname{colim}_K \Gamma_K\,(U, \mathcal{A}_X^{p,q})\,,$$

equipped with the inductive limit topology. Here, the limit runs through all compact subsets of U.

Remark 5.12. Note that these are just the usual definitions of $\Gamma_K(U, \mathcal{A}_X^{p,q})$ and $\Gamma_c\,(U, \mathcal{A}_X^{p,q})$, additionally equipped with a topology. Conversely, these topological spaces are in fact the kernel of $\mathcal{A}_X^{p,q}(U) \to \mathcal{A}_X^{p,q}(U \setminus K)$ and the colimit $\operatorname{colim}_K \Gamma_K\,(U, \mathcal{A}_X^{p,q})$ in the quasi-abelian category lcs of locally convex vector spaces.

Lemma 5.13. *The natural embeddings $\Gamma_c(U', \mathcal{A}_X^{p,q}) \to \Gamma_c(U, \mathcal{A}_X^{p,q})$ for $U' \subset U$ as well as the restrictions of the differentials, $d_1 : \Gamma_c(U, \mathcal{A}_X^{p,q}) \to \Gamma_c(U, \mathcal{A}_X^{p+1,q})$ and $d_2 : \Gamma_c(U, \mathcal{A}_X^{p,q}) \to \Gamma_c(U, \mathcal{A}_X^{p,q+1})$ are continuous maps. For every finitary open subset $U \subset X$, $\Gamma_c(U, \mathcal{A}_X^{p,q})$ is a LF-space.*

Proof. If $U \subset X$ is finitary and $K \subset U$ is compact, then $\mathcal{A}_X^{p,q}(U) \to \mathcal{A}_X^{p,q}(U \setminus K)$ is a continuous map of Fréchet spaces by proposition 5.7(1), proposition 5.10 and proposition C.14 (since the topology of X is second countable, we may use a countable covering in 5.10 and a countable direct product of Fréchet spaces is Fréchet). Hence by proposition C.14, $\Gamma_K(U, \mathcal{A}_X^{p,q})$ is a Fréchet space as well. The preordered set of all compact subsets of U has a directed cofinal countable subset and hence $\Gamma_c(U, \mathcal{A}_X^{p,q})$ is a countable direct limit of Fréchet spaces. Hence, it is an LF-space. Continuity of all maps follows directly from the construction. □

Definition 5.14. Let X be a tropical space. For every r, $s \in \mathbb{Z}$ and every open subset $U \subset X$ we define

$$\tilde{\mathcal{D}}_X^{r,s}(U) := \operatorname{Hom}_{\mathrm{lcs}}\left(\Gamma_c(U, \mathcal{A}_X^{-s,-r}), \mathbb{R}\right),$$

where lcs denotes the category of locally convex $\mathbb{R}$-vector spaces. Using the obvious restriction maps, we get a presheaf $\tilde{\mathcal{D}}_X^{p,q}$ on X. Because the differentials d_1 and d_2 are continuous on $\Gamma_c(U, \mathcal{A}^{-s,-r})$ for every open subset $U \subset X$, their dual maps

$$\partial_1 = (-1)^s \operatorname{Hom}_{\mathrm{lcs}}(d_1, \mathbb{R}) \text{ and } \partial_2 = (-1)^r \operatorname{Hom}_{\mathrm{lcs}}(d_2, \mathbb{R})$$

make $\left(\tilde{\mathcal{D}}_X^{\bullet,\bullet}, \partial_2, \partial_1\right)$ into a double complex of presheaves of $\mathbb{R}$-vector spaces.

We will write $\tilde{\mathcal{D}}_X^{\bullet} := \operatorname{tot}^{\bullet} \tilde{\mathcal{D}}_X^{\bullet,\bullet}$ for the total complex of $\tilde{\mathcal{D}}_X^{\bullet,\bullet}$.

Proposition 5.15. *Let X be a tropical space of dimension n. Then the presheaves $\tilde{\mathcal{D}}_X^{r,s}$ are flabby sheaves of $\mathbb{R}$-vector spaces on X. We have a canonical morphism of double complexes of sheaves on X,*

$$\imath : \tilde{\mathcal{D}}_X^{\bullet,\bullet} \to \mathcal{D}_X^{\bullet,\bullet}$$

induced by the embedding map $\operatorname{Hom}_{\mathrm{lcs}}(\Gamma_c(U, \mathcal{A}_X^{-s,-r}), \mathbb{R}) \to \operatorname{Hom}_{\mathbb{R}}(\Gamma_c(U, \mathcal{A}_X^{-s,-r}), \mathbb{R})$.

Proof. The sheaves $\mathcal{A}_X^{-s,-r}$ are soft and $\mathbb{R}$ is strongly injective in lcs, i.e. $\operatorname{Hom}_{\mathrm{lcs}}(\cdot, \mathbb{R})$ preserves arbitrary kernels and cokernels. This implies that the presheaves $\tilde{\mathcal{D}}_X^{r,s}$ are sheaves. Because the map $\Gamma_c(U', \mathcal{A}_X^{-s,-r}) \to \Gamma_c(U, \mathcal{A}_X^{-s,-r})$ is injective and continuous for $U' \subset U$ open in X it follows immediately that $\tilde{\mathcal{D}}_X^{r,s}$ is flabby. The last claim is obvious from the definitions. □

Proposition 5.16. *Let X be a tropical space of dimension n and $U \subset X$ an open subset. The integration map*

$$\delta_X : \Gamma_c(U, \mathcal{A}_X^{n,n}) \to \mathbb{R}, \qquad \eta \mapsto \int_X \eta$$

is continuous.

Proof. We may assume that X is the support of a polyhedral complex in $\mathbb{T}^N$. Let $\tilde{U} \subset \mathbb{T}^N$ be an open subset with $\tilde{U} \cap X = U$. From the definition of $\int_X(\cdot)$ in section 3.7 it follows that δ_X is continuous as a map $\mathcal{A}_{\mathbb{T}^N}^{n,n}(\tilde{U}) \to \mathbb{R}$ and it vanishes on $\mathcal{K}_X^{n,n}(\tilde{U})$. Hence it is a continuous map on the cokernel $\mathcal{A}_X^{n,n}(U) = \operatorname{coker}\left(\mathcal{K}_X^{n,n}(\tilde{U}) \to \mathcal{A}_{\mathbb{T}^N}^{n,n}(\tilde{U})\right)$. □

5.4 Smoothing of cohomology

Theorem 5.17 (Smoothing of cohomology)**.** *Let X be a tropical manifold. Then, the Poincaré map induces quasi-isomorphisms of complexes of sheaves of $\mathbb{R}$-vector spaces on X,*

$$\delta_X \cap : \mathcal{A}_X^{\bullet}[2n] \xrightarrow{\sim} \tilde{\mathcal{D}}_X^{\bullet},$$
$$\delta_X \cap : \mathcal{A}_X^{p,\bullet}[n] \xrightarrow{\sim} \tilde{\mathcal{D}}_X^{\bullet,n-p},$$
$$\delta_X \cap : \mathcal{A}_X^{\bullet,q}[n] \xrightarrow{\sim} \tilde{\mathcal{D}}_X^{n-q,\bullet}.$$

We only show the first quasi-isomorphism, using Poincaré duality (theorem 3.28) and the following lemma. The same proof works for the other two morphisms, when replacing 3.28 with 3.27.

Lemma 5.18. *The natural morphism $\imath : \tilde{\mathcal{D}}_X^\bullet \to \mathcal{D}_X^\bullet$ induces injective maps in cohomology,*

$$\imath : \mathrm{R}^k\,\Gamma(U, \tilde{\mathcal{D}}_X^\bullet) \to \mathrm{R}^k\,\Gamma(U, \mathcal{D}_X^\bullet)$$

for every $k \in \mathbb{Z}$ and every open subset $U \subset X$.

Proof. First, we need the following purely algebraic statement: If $(A^\bullet, d)$ is a complex in any abelian category, we get canonical isomorphisms

$$\mathrm{H}^q(A^\bullet) \cong \ker\left(\operatorname{coker}(d^{q-1}) \to \operatorname{coker}\ker(d^q)\right)$$

for every $q \in \mathbb{Z}$. This follows immediately from the Snake lemma, applied to the diagram

$$\begin{array}{ccccccc} & & \ker\operatorname{coker}(d^{q-1}) & \longrightarrow & \ker(d^q) & \longrightarrow & \mathrm{H}^q(A^\bullet) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^q & \xrightarrow{\mathrm{id}} & A^q & \longrightarrow & 0 & & \end{array} .$$

We can now apply this to our case: In particular, for the complex $(\tilde{\mathcal{D}}_X^\bullet, \partial)$ of currents on X we have canonical isomorphisms

$$\begin{aligned} \mathrm{R}^k\,\Gamma(U, \tilde{\mathcal{D}}_X^\bullet) &= \mathrm{H}^k(\tilde{\mathcal{D}}_X^\bullet(U)) \\ &= \ker(\operatorname{coker}(\partial) \to \operatorname{coker}\ker(\partial)) \\ &= \mathrm{Hom}_{\mathrm{lcs}}\left(\operatorname{coker}(\ker\operatorname{coker}(d) \to \ker(d)), \mathbb{R}\right), \end{aligned}$$

where d is the (continuous) restriction of the usual differential map to sections with compact support and $\ker(d)$, for instance, is the kernel of d in the quasi-abelian category lcs (c.f. proposition C.7). Note that in the last equation we use that the functor

$$\mathrm{Hom}_{\mathrm{lcs}}(\cdot, \mathbb{R}) : \mathrm{lcs}^{op} \to \mathrm{Mod}_{\mathbb{R}}, \qquad E \mapsto \mathrm{Hom}_{\mathrm{lcs}}(E, \mathbb{R})$$

of quasi-abelian categories is *strongly exact*, i.e. it preserves arbitrary kernels and cokernels, by the Hahn-Banach theorem for locally convex spaces (proposition C.9).

Now by proposition C.7, as an $\mathbb{R}$-vector space (forgetting the topology), the cokernel of $\ker\operatorname{coker}(d) \to \ker(d)$ in lcs is just $\ker(d)/\operatorname{im}(d) = \mathrm{R}^{-k}\,\Gamma_c(U, \mathcal{A}_X^\bullet)$, i.e. we may consider $\mathrm{R}^k\,\Gamma(U, \tilde{\mathcal{D}}_X^\bullet)$ as the linear subspace of continuous maps in $\mathrm{Hom}_{\mathbb{R}}(\mathrm{R}^{-k}\,\Gamma_c(U, \mathcal{A}_X^\bullet), \mathbb{R}) = \mathrm{R}^k\,\Gamma(U, \mathcal{D}_X^\bullet)$. This suffices to prove the lemma. ☐

Now the theorem is a purely formal consequence of Poincaré duality:

Proof of theorem 5.17. Because the wedge product and the integration map are continuous (5.7 and 5.16) we have a commuting triangle of complexes of sheaves of $\mathbb{R}$-vector spaces

$$\begin{array}{ccc} \mathcal{A}_X^\bullet[2n] & \xrightarrow{\delta_X \cap} & \tilde{\mathcal{D}}_X^\bullet \\ & \searrow^{\delta_X \cap} & \downarrow \imath \\ & & \mathcal{D}_X^\bullet, \end{array}$$

where $\imath$ is the natural injection from proposition 5.15 and $\delta_X \cap$ is the Poincare map. Now the morphism $\mathcal{A}_X^\bullet[2n] \to \mathcal{D}_X^\bullet$ is a quasi-isomorphism which implies that $\mathrm{H}^k(\imath)$ is an epimorphism for every $k \in \mathbb{Z}$ and – by the previous remark – $\mathrm{H}^k(\imath)$ also has to be a monomorphism. It follows that $\imath$ is quasi-isomorphism and hence $\delta_X \cap : \mathcal{A}_X^\bullet[2n] \to \tilde{\mathcal{D}}_X^\bullet$ is a quasi-isomorphism as well. ☐

A Sheaf cohomology

A.1 Derived categories and functors

The language of derived categories and functors is a useful framework for the study of cohomological functors, in particular for those appearing in the cohomology of sheaves. here, we will give a short overview on the most important notions and methods in this area. For an in-depth treatment of the subject matter, we refer to [KS90, ch.I].

First and foremost, every derived category (as defined below) is a *triangulated category*:

Definition A.1. A *triangulated category* is an additive category $\mathfrak{C}$ together with

1. an automorphism $T : \mathfrak{C} \to \mathfrak{C}$, the *shift functor* of $\mathfrak{C}$, and
2. a family of *triangles* in $\mathfrak{C}$, i.e. of chains

   $$X \to Y \to Z \to TX$$

   of morphisms in $\mathfrak{C}$. The triangles in this family are called *distinguished triangles*. A *morphism of triangles* is a commuting diagram

   $$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle T\phi} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & TX'. \end{array}$$

These data are subject to the following conditions:

1. A triangle isomorphic to a distinguished triangle is distinguished.
2. The triangle $X \overset{\mathrm{id}}{\to} X \to 0 \to TX$ is distinguished for every object X in $\mathfrak{C}$.
3. The triangle $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} TX$ is distinguished if and only if $Y \overset{g}{\to} Z \overset{h}{\to} TX \overset{-Tf}{\longrightarrow} TY$ is distinguished.
4. Every morphism $f : X \to Y$ is part of a distinguished triangle $X \to Y \to Z \to TX$.
5. Let $D : X \overset{f}{\to} Y \to Z \to TX$ and $D' : X' \overset{f'}{\to} Y' \to Z' \to TX'$ be distinguished triangles. Then every commuting square

   $$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ \downarrow & & \downarrow \\ X' & \overset{f'}{\longrightarrow} & Y' \end{array}$$

   can be completed to a morphism of triangles $D \to D'$.
6. For every three distinguished triangles

   $$X \overset{f}{\to} Y \to Z' \to TX,$$
   $$Y \overset{g}{\to} Z \to X' \to TY,$$
   $$X \overset{g\circ f}{\to} Z \to Y' \to TX,$$

there exists a distinguished triangle

$$Z' \to Y' \to X' \to TZ'$$

such that the following diagram commutes:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & TX \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow{gf} & Z & \longrightarrow & Y' & \longrightarrow & TX \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle Tf} \\
Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & TY \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & TZ'.
\end{array}$$

Definition A.2. Let D and D' be triangulated categories with shift functors T and T'. An additive functor $F : D \to D'$ is a *functor of triangulated categories* if

1. $T'F = FT$ holds and
2. for every distinguished triangle $X \to Y \to Z \to TX$ in D, $FX \to FY \to FZ \to T'FX$ is distinguished as well.

For an abelian category $\mathfrak{A}$, we now define the derived category $\mathrm{D}(\mathfrak{A})$ (as well as its close relatives $\mathrm{D}^+(\mathfrak{A})$, $\mathrm{D}^-(\mathfrak{A})$ and $\mathrm{D}^b(\mathfrak{A})$) in the following three steps. Once again, we refer to [KS90, 1.3-7] for details:

Construction A.3. Let $\mathfrak{A}$ be an abelian category.

1. We denote by $\mathrm{C}(\mathfrak{A})$ the category of *complexes* in $\mathfrak{A}$ and we write $\mathrm{C}^-(\mathfrak{A})$, $\mathrm{C}^+(\mathfrak{A})$ and $\mathrm{C}^b(\mathfrak{A})$ for the full subcategories of $\mathrm{C}(\mathfrak{A})$ consisting of bounded above, bounded below and bounded complexes respectively.
2. The *homotopy category of complexes* in $\mathfrak{A}$ is the category $\mathrm{K}(\mathfrak{A})$ with the same objects as $\mathrm{C}(\mathfrak{A})$ and with

   $$\mathrm{Mor}_{\mathrm{K}(\mathfrak{A})}(A^\bullet, B^\bullet) := \mathrm{Mor}_{\mathrm{C}(\mathfrak{A})}(A^\bullet, B^\bullet)/\,\mathrm{Hot}(A^\bullet, B^\bullet),$$

   where $\mathrm{Hot}(A^\bullet, B^\bullet)$ is the group of morphisms $A^\bullet \to B^\bullet$ homotopic to zero. The *shift functor* [1] given by

   $$\begin{aligned}
   [1] : \mathrm{K}(\mathfrak{A}) &\to \mathrm{K}(\mathfrak{A}), \\
   C^\bullet[1]^k &:= C^{k+1}, \\
   d^k_{C^\bullet[1]} &:= -d^{k+1}_{C^\bullet}, \\
   f^\bullet[1]^k &:= f^{k+1},
   \end{aligned}$$

   makes $\mathrm{K}(\mathfrak{A})$ into a triangulated category, when choosing the mapping cone triangles as family of distinguished triangles (c.f. [KS90, 1.4.1-3] for the definition of mapping cones and the associated triangles). Similarly, we obtain triangulated categories $\mathrm{K}^-(\mathfrak{A})$, $\mathrm{K}^+(\mathfrak{A})$ and $\mathrm{K}^b(\mathfrak{A})$.

3. Let $\mathfrak{N} \subset \mathrm{K}(\mathfrak{A})$ be the family of exact complexes and denote by $S(\mathfrak{N})$ the family of morphisms $f : X \to Y$ that belong to a distinguished triangle

$$X \to Y \to N \to X[1]$$

with N in $\mathfrak{N}$. The *derived category* $\mathrm{D}(\mathfrak{A})$ is the localization

$$\mathrm{D}(\mathfrak{A}) := \mathrm{K}(\mathfrak{A})/\,\mathfrak{N} := \mathrm{K}(\mathfrak{A})_{S(\mathfrak{N})}$$

of the triangulated category $\mathrm{K}(\mathfrak{A})$ by the multiplicative system $S(\mathfrak{N})$ of morphisms. The canonical morphism $\mathrm{K}(\mathfrak{A}) \to \mathrm{D}(\mathfrak{A})$ induces a structure of a triangulated category on $\mathrm{D}(\mathfrak{A})$. Once again, we define the triangulated categories $\mathrm{D}^-(\mathfrak{A})$, $\mathrm{D}^+(\mathfrak{A})$ and $\mathrm{D}^b(\mathfrak{A})$ analogously.

Remark A.4. By [KS90, 1.6.9], the localization functor $\mathrm{K}(\mathfrak{A}) \to \mathrm{D}(\mathfrak{A})$ satisfies the following property: every functor $F : \mathrm{K}(\mathfrak{A}) \to D$ with $F(N) = 0$ for every N in $\mathfrak{N}$ factors uniquely through the canonical functor $\mathrm{K}(\mathfrak{A}) \to \mathrm{D}(\mathfrak{A})$.

The functor $\mathrm{H}^k : \mathrm{C}(\mathfrak{A}) \to \mathfrak{A}$, $(A^\bullet, d^\bullet) \mapsto \ker(d^k)/\operatorname{im}(d^{k-1})$ induces a well-defined functor

$$\mathrm{H}^k : \mathrm{D}(\mathfrak{A}) \to \mathfrak{A}\,.$$

For every distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathrm{D}(\mathfrak{A})$, we get a long exact sequence

$$\mathrm{H}^k(X) \to \mathrm{H}^k(Y) \to \mathrm{H}^k(Z) \to \mathrm{H}^{k+1}(X)$$

in $\mathfrak{A}$ in a functorial manner.

Let now $F : \mathfrak{A} \to \mathfrak{B}$ be an additive functor of abelian categories. Then F induces a functor $\mathrm{K}^+(F) : \mathrm{K}^+(\mathfrak{A}) \to \mathrm{K}^+(\mathfrak{B})$ of the corresponding homotopy categories of complexes. However, to get an associated functor between the associated (bounded below) derived categories, one has to be more careful. The starting point is the following definition:

Definition A.5. Let $F : \mathfrak{A} \to \mathfrak{B}$ be an additive functor of abelian categories and denote by $Q_{\mathfrak{A}} : \mathrm{K}^+(\mathfrak{A}) \to \mathrm{D}^+(\mathfrak{A})$, $Q_{\mathfrak{B}} : \mathrm{K}^+(\mathfrak{B}) \to \mathrm{D}^+(\mathfrak{B})$ the canonical localization functors. A *right derived functor* of F is a functor $T : \mathrm{D}^+(\mathfrak{A}) \to \mathrm{D}^+(\mathfrak{B})$ of triangulated categories together with a morphism of functors

$$s : Q_{\mathfrak{B}} \circ \mathrm{K}^+(F) \to T \circ Q_{\mathfrak{A}}$$

such that, for any functor $G : \mathrm{D}^+(\mathfrak{A}) \to \mathrm{D}^+(\mathfrak{B})$ of triangulated categories, the morphism

$$\mathrm{Hom}(T, G) \to \mathrm{Hom}(Q_{\mathfrak{B}} \circ \mathrm{K}^+(F), G \circ Q_{\mathfrak{A}})$$

induced by s is an isomorphism.

If a right derived functor T for F exists we say that F is *right derivable*. The derived functor T is then uniquely determined up to isomorphism and we write $\mathrm{R}\,F := T$. The functor

$$\mathrm{R}^k\,F := \mathrm{H}^k \circ \mathrm{R}\,F : \mathrm{D}^+(\mathfrak{A}) \to \mathfrak{B}$$

is called the *k-th right derived functor* of F.

For simplicity, we will usually write $F : \mathrm{K}^+(\mathfrak{A}) \to \mathrm{K}^+(\mathfrak{B})$ for the functor of homotopy categories induced by F.

When F is left exact, in order to prove the existence of derived functors and also when making computations, one usually depends upon so-called F-injective subcategories of $\mathfrak{A}$:

Definition A.6. Let $F : \mathfrak{A} \to \mathfrak{B}$ be a left exact functor of abelian categories. A full additive subcategory $\mathfrak{I} \subset \mathfrak{A}$ of $\mathfrak{A}$ is *F-injective* if the following conditions are satisfied:

1. For every A in $\mathfrak{A}$ there exists a monomorphism $A \to I$ with I in $\mathfrak{I}$.
2. If $0 \to I' \to I \to X'' \to 0$ is a short exact sequence in $\mathfrak{A}$ with I' and I in $\mathfrak{I}$, then X'' is in $\mathfrak{I}$ as well.
3. If $0 \to I' \to I \to I'' \to 0$ is a short exact sequence in $\mathfrak{A}$ with I', I and I'' in $\mathfrak{I}$, then the sequence $0 \to F(I') \to F(I) \to F(I'') \to 0$ is exact as well.

Construction A.7. Let $F : \mathfrak{A} \to \mathfrak{B}$ be a left exact functor of abelian categories and assume that $\mathfrak{A}$ has an F-injective full subcategory $\mathfrak{I}$. Then, by [KS90, 1.8.3] F has a right derived functor $\mathrm{R}\,F : \mathrm{D}^+(\mathfrak{A}) \to \mathrm{D}^+(\mathfrak{B})$. This functor can be constructed as follows:

1. Let A be any object of $\mathrm{D}^+(\mathfrak{A})$. Then A can be represented by a complex $I^\bullet$ in $\mathrm{C}^+(\mathfrak{A})$ with I^k_A in $\mathfrak{I}$ for every $k \in \mathbb{Z}$ by [KS90, 1.7.7]. Fix one such representation for every A in $\mathrm{D}^+(\mathfrak{A})$.
2. The class of the complex $F(I^\bullet_A)$ in $\mathrm{D}^+(\mathfrak{B})$ only depends on A and every morphism $A \to B$ in $\mathrm{D}^+(\mathfrak{A})$ induces a morphism $F(I^\bullet_A) \to F(I^\bullet_B)$ in $\mathrm{D}^+(\mathfrak{B})$ in a functorial way by [KS90, p.51]. One can show that this indeed defines a derived functor for F, i.e. we have
$$\mathrm{R}\,F(A) = F(I^\bullet_A)$$
in $\mathrm{D}^+(\mathfrak{B})$, and similarly for morphisms.

We will now give some well-known examples:

Example A.8. Let $\mathfrak{A}$ be an abelian category and assume that $\mathfrak{A}$ *has enough injectives*, i.e. for every object A in $\mathfrak{A}$ there exists a monomorphism $A \to I$ with I injective. Then the full subcategory $\mathfrak{I}$ of $\mathfrak{A}$ consisting of injective objects is F-injective for *every* left exact functor F. In particular, every left exact functor on $\mathfrak{A}$ is right derivable and the right derived functor $\mathrm{R}\,F$ can be constructed by choosing injective resolutions.

Example A.9. Let X be a topological space and let $\mathfrak{A} := \mathrm{Shv}(X)$ be the category of sheaves of abelian groups on X.

1. The category $\mathrm{Shv}(X)$ is abelian and has enough injectives. We denote its derived category by $\mathrm{D}(X)$. Similarly, we write $\mathrm{D}^+(X)$, $\mathrm{D}^-(X)$ and $\mathrm{D}^b(X)$ for the derived categories of bounded below, bounded above and bounded complexes of sheaves of $\mathbb{Z}$-modules on X.
2. The functor
$$\Gamma(X, \cdot) : \mathrm{Shv}(X) \to \mathrm{Ab}, \qquad \mathcal{F} \mapsto \mathcal{F}(X),$$
of *global sections* is left exact. The category $\mathrm{Shv}(X)$ has enough injectives and hence $\Gamma(X, \cdot)$ has a right derived functor $\mathrm{R}\,\Gamma(X, \cdot)$. For a sheaf $\mathcal{F}$ on X we write
$$\mathrm{H}^k(X, \mathcal{F}) := \mathrm{R}^k\,\Gamma(X, \mathcal{F})$$
and call it the *k-th cohomology group* (of global sections) of $\mathcal{F}$.
3. Similarly, the functor
$$\Gamma_c(X, \bullet) : \mathrm{Shv}(X) \to \mathrm{Ab}, \qquad \mathcal{F} \mapsto \Gamma_c(X, \mathcal{F}),$$
of *global sections with compact support* is left exact, has a right derived functor $\mathrm{R}\,\Gamma_c(X, \cdot)$ and we write
$$\mathrm{H}^k_c(X, \mathcal{F}) := \mathrm{R}^k\,\Gamma_c(X, \mathcal{F})$$
for the *k-th cohomology group* (of global sections) *with compact support* of $\mathcal{F}$.

Example A.10. Most of the topological spaces of interest for us are locally compact. Under this condition on X, the following classes of sheaves on X are both $\Gamma(X, \cdot)$- and $\Gamma_c(X, \cdot)$-injective

- injective sheaves,
- flabby sheaves,
- c-soft sheaves (we will generally refer to them as soft sheaves),
- fine sheaves.

This is useful for us, because injective sheaves tend to be quite large and unwieldy for computations while simultaneously the theory of differential forms we consider in the main text presents us with complexes of c-soft (even fine) sheaves which are quite easy to manipulate. At least to some extent.

Example A.11. Similarly to the right derived functor $\mathrm{R}\,F$ and F-injective subcategories for a left exact functor F, one can define the *left derived functor* $\mathrm{L}\,G$ of a right exact functor using *G-projective* subcategories. The most important functor for us is the tensor product $(\cdot) \otimes (\cdot) : \mathrm{Shv}(X) \times \mathrm{Shv}(X) \to \mathrm{Shv}(X)$. For $\mathcal{F}$ in $\mathrm{Shv}(X)$, the functor $G = (\cdot) \otimes \mathcal{F}$ is right exact and the class of flat sheaves on X is G-projective in $\mathrm{Shv}(X)$. Hence one can define the left derived functor of G. As is customary, we write

$$\mathrm{L}\,G(\mathcal{G}) =: \mathcal{G} \otimes^L \mathcal{F}\,.$$

In the next sections, we will consider several other important functors between derived categories of sheaves.

A.2 Sections with support in a closed subset

Definition A.12. Let $\mathcal{F}$ be a sheaf on X and let $Z \subset X$ be a closed subset.

1. For every open subset $U \subset X$ and $V := U \cap Z \subset U$ we define the group of *sections with support in* V of $\mathcal{F}$ by

$$\Gamma_V(U, \mathcal{F}) := \ker(\Gamma(U, \mathcal{F}) \to \Gamma(U \setminus V, \mathcal{F}).$$

2. We define the *sheaf of sections* of $\mathcal{F}$ *with support in* Z by

$$\Gamma_Z(\mathcal{F})(U) := \Gamma_{U \cap Z}(U, \mathcal{F}).$$

Proposition A.13. *Let $Z \subset X$ be a closed subset and denote by $\jmath : X \setminus Z \to X$ the open embedding of the complement.*

1. *The functor $\Gamma_Z(X, \cdot) : \mathrm{Shv}(X) \to \mathrm{Ab}$ is left exact.*

2. *For every sheaf $\mathcal{F}$ on X, the presheaf $\Gamma_Z(\mathcal{F})$ is a sheaf and the functor*

$$\begin{aligned} \Gamma_Z : \mathrm{Shv}(X) &\to \mathrm{Shv}(X), \\ \mathcal{F} &\mapsto \Gamma_Z(\mathcal{F}) \end{aligned}$$

   is left exact.

3. *Let* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *be a short exact sequence of sheaves on* X *and assume that* $\mathcal{F}'$ *is flabby. Then both sequences*

$$0 \to \Gamma_Z(X, \mathcal{F}') \to \Gamma_Z(X, \mathcal{F}) \to \Gamma_Z(X, \mathcal{F}'') \to 0,$$

$$0 \to \Gamma_Z(\mathcal{F}') \to \Gamma_Z(\mathcal{F}) \to \Gamma_Z(\mathcal{F}'') \to 0$$

are exact. Moreover, $\Gamma_Z(\mathcal{F}')$ *is a flabby sheaf.*

4. *For every sheaf* $\mathcal{F}$ *on* X*, the sequence*

$$0 \to \Gamma_Z(\mathcal{F}) \to \mathcal{F} \xrightarrow{\eta} \jmath_* \jmath^{-1} \mathcal{F},$$

is exact, where $\eta : \mathcal{F} \to \jmath_* \jmath^{-1} \mathcal{F}$ *is the unit morphism of the adjunction* $(\jmath^{-1}, \jmath_*)$*.*

[KS90, 2.4.8], [KS90, 2.4.6], [KS90, 2.3.9]

Next, we collect some properties of the derived functors $\mathrm{R}\,\Gamma_Z$ and $\mathrm{R}\,\Gamma_Z(X, \cdot)$:

Proposition A.14. *Let* $Z, Z' \subset X$ *be a closed subsets and* $\jmath : X \setminus Z \to X$ *the embedding of the open complement of* Z*.*

1. *Let* $\mathcal{F}^\bullet$ *be in* $\mathrm{D}^+(X)$*. Then the canonical morphism*

$$\mathrm{R}\,\Gamma(X, \mathrm{R}\,\Gamma_Z(\mathcal{F}^\bullet)) \xrightarrow{\sim} \mathrm{R}\,\Gamma_Z(X, \mathcal{F}^\bullet)$$

is an isomorphism.

2. *For* $\mathcal{F}^\bullet$ *in* $\mathrm{D}^+(X)$ *we have canonical distinguished triangles*

$$\mathrm{R}\,\Gamma_Z(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet \to \mathrm{R}\,\jmath_* \jmath^{-1} \mathcal{F}^\bullet \xrightarrow{+1},$$

$$\mathrm{R}\,\Gamma_{Z \cap Z'}(\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma_Z(\mathcal{F}^\bullet) \oplus \mathrm{R}\,\Gamma_{Z'}(\mathcal{F}^\bullet) \to \mathrm{R}\,\Gamma_{Z \cup Z'}(\mathcal{F}^\bullet) \xrightarrow{+1}.$$

[KS90, p.111], [KS90, p.115]

A.3 Poincaré-Verdier duality

In this section we will recall some of the basic constructions and properties concerning Poincaré-Verdier duality, following the exposition in [KS90].

For a topological space X and a ring R we write $\mathrm{Shv}(X)$ resp. $\mathrm{Shv}_R(X)$ for the categories of sheaves of abelian groups and sheaves of R-modules on X respectively. A 'sheaf on X' will always be a sheaf of abelian groups on X. We write $\mathrm{D}(X)$ for the derived category $\mathrm{D}(\mathrm{Shv}_R(X))$ and $\mathrm{D}^b(X)$, $\mathrm{D}^+(X)$ resp. $\mathrm{D}^-(X)$ for the subcategories generated by bounded, bounded below resp. bounded above complexes.

Remark A.15. Let $f : X \to Y$ be a continuous map. We then consider the following functors associated to f:

1. The pushforward $f_* : \mathrm{Shv}_R(X) \to \mathrm{Shv}_R(Y)$ and pullback $f^{-1} : \mathrm{Shv}_R(Y) \to \mathrm{Shv}_R(X)$ functors,

2. for an open embedding $j : U \subset X$:

$$(\cdot)_U := \jmath_* \jmath^{-1} : \mathrm{Shv}_R(X) \to \mathrm{Shv}_R(X),$$

3. the pushforward with compact support $f_! : \mathrm{Shv}_R(X) \to \mathrm{Shv}_R(Y)$, where

$$\Gamma(U, f_! \mathcal{F}) = \{s \in \Gamma(f^{-1}U, \mathcal{F}); f : \mathrm{supp}(s) \to Y \text{ is proper}\}.$$

The functor

$$f_! : \mathrm{Shv}(X) \to \mathrm{Shv}(Y),$$

is left exact and has a right derived functor

$$\mathrm{R}\, f_! : \mathrm{D}(X) \to \mathrm{D}(Y).$$

We say that $f_!$ *has finite cohomological dimension* if there exists an $r \in \mathbb{Z}$ such that $\mathrm{R}^j\, f_! := \mathrm{H}^j\, \mathrm{R}\, f_! = 0$ for every $j > r$.

The following is the main result here:

Theorem A.16 (Poincaré-Verdier duality)**.** *Let $f : Y \to X$ be a continuous map of locally compact spaces such that $f_!$ has finite cohomological dimension. Then the functor of triangulated categories*

$$\mathrm{R}\, f_! : \mathrm{D}^+(\mathrm{Shv}_A(Y)) \to \mathrm{D}^+(\mathrm{Shv}_A(X))$$

has a right adjoint functor of triangulated categories

$$f^! : \mathrm{D}^+(\mathrm{Shv}_A(X)) \to \mathrm{D}^+(\mathrm{Shv}_A(Y)).$$

[KS90, 3.1.5]

The functor $f^!$ can be given as follows:

Construction A.17. Let K be a flat and f-soft sheaf on Y and let F be an injective sheaf of A-modules on X.

1. The presheaf

$$f^!_K : V \mapsto \mathrm{Hom}_{\mathrm{Shv}_A(X)}\left(f_!\left(A_Y \otimes_{\mathbb{Z}_Y} K_V\right), F\right),$$

   is a sheaf and it is injective as a sheaf of R-modules.

2. For every sheaf of R-modules G on Y we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Shv}_A(X)}\left(f_!\left(G \otimes_{\mathbb{Z}_Y} K\right), F\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Shv}_A(Y)}(G, f^!_K F),$$

   functorial in G.

3. Let $\mathcal{I}(X)$ denote the category of all injective sheaves on X. For a flat and f-soft resolution $K : 0 \to K^0 \to \cdots \to K^r \to 0$ of $\mathbb{Z}_Y$ and for $F \in \mathrm{K}^+(\mathcal{I}(X))$ let $f^!_K F$ be the total complex associated to the double complex

$$(f^!_{K^{-q}}(F^p))^{p,q}.$$

   Then $f^!_K$ is a functor of triangulated categories

$$f^!_K : \mathrm{K}^+(\mathcal{I}(X)) \to \mathrm{K}^+(\mathcal{I}(X))$$

   such that the canonical diagram

$$\begin{array}{ccc} \mathrm{K}^+(\mathcal{I}(X)) & \xrightarrow{f^!_K} & \mathrm{K}^+(\mathcal{I}(Y)) \\ \downarrow \approx & & \downarrow \approx \\ \mathrm{D}^+(\mathrm{Shv}_A(X)) & \xrightarrow{f^!} & \mathrm{D}^+(\mathrm{Shv}_A(Y)) \end{array}$$

   is commutative.

[KS90, 3.1.2-3.1.5]

Example A.18. Assume that we have a quasi-isomorphism $\phi : \mathcal{L}^\bullet \to \mathcal{A}^\bullet$ of bounded complexes of sheaves of $\mathbb{R}$-vector spaces on X such that $\mathcal{A}^k$ is soft for every $k \in \mathbb{Z}$. Consider the projection to a point $\pi : X \to \mathrm{pt}$. Because $\mathbb{R}$ is injective in the category of $\mathbb{R}$-vector spaces, we have a quasi-isomorphism

$$\mathcal{D}^\bullet \to \mathrm{R}\,\mathcal{H}om_X(\mathcal{L}^\bullet, \pi^!\,\mathbb{R}),$$

given by ϕ, where $\mathcal{D}^\bullet$ is the complex of sheaves given by

$$\mathcal{D}^\bullet(U) := \mathrm{Hom}^\bullet_{\mathbb{R}}\left(\Gamma_c(U, \mathcal{A}^\bullet), \mathbb{R}\right).$$

The functor $f^!$ has the following properties:

Proposition A.19. *Let $f_!$ have finite cohomological dimension.*

1. *Consider a cartesian square of topological spaces*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ X & \xrightarrow{f} & Y. \end{array}$$

   Then $g_!$ has finite cohomological dimension and we have a canonical isomorphism of functors
$$f^! \circ \mathrm{R}\,y_* = \mathrm{R}\,x_* \circ g^!.$$

2. *We have canonical isomorphisms of bifunctors on $(\mathrm{D}^b)^{op} \times \mathrm{D}^+$:*

$$\begin{aligned} \mathrm{R}\,\mathrm{Hom}_{\mathrm{Shv}(Y)}(\mathrm{R}\,f_!(\cdot), \cdot) &= \mathrm{R}\,\mathrm{Hom}_{\mathrm{Shv}(X)}(\cdot, f^!(\cdot)), \\ \mathrm{R}\,\mathcal{H}om_Y(\mathrm{R}\,f_!(\cdot), \cdot) &= \mathrm{R}\,f_*\,\mathrm{R}\,\mathcal{H}om_X(\cdot, f^!(\cdot)), \\ f^!\,\mathrm{R}\,\mathcal{H}om_X(\cdot, \cdot) &= \mathrm{R}\,\mathcal{H}om_Y(f^{-1}(\cdot), f^!(\cdot)). \end{aligned}$$

3. *Assume $f : X \to Y$ is a homeomorphism onto a locally closed subset $Z \subset Y$. Then*

$$f^! = f^{-1} \circ \mathrm{R}\,\Gamma_Z.$$

[KS90, 3.1.9-13]

A.4 Dualizing complex and derived dual

Definition A.20. Let X and Y be locally compact topological spaces.

1. Let $f : X \to Y$ be a continuous map and assume that $f_!$ has finite cohomological dimension. We then write
$$\omega_{X/Y} := f^!\,\mathbb{R};$$
   if $\pi : X \to \mathrm{pt}$ is the projection to a point then we write $\omega_X = \pi^!\,\mathbb{R}$ and call ω_X the *dualizing complex* on X.

2. Assume X has finite c-soft dimension and let $\mathcal{F} \in \mathrm{D}^b(\mathrm{Shv}_{\mathbb{R}}(X))$. We then set
$$\mathcal{D}(\mathcal{F}) = \mathrm{R}\,\mathcal{H}om_X(\mathcal{F}, \omega_X)$$
   and call $\mathcal{D}(\mathcal{F})$ the *derived dual* or *dual complex* to $\mathcal{F}$.

Remark A.21. Due to [KS90, 8.4.2] a complex $\mathcal{F}^\bullet$ of sheaves on a real analytic manifold is $\mathbb{R}$-*constructible*, if it has a locally finite covering $X = \bigcup_i X_i$ by subanalytic subsets, such that for every i, and every q, the sheaves $\mathrm{H}^q(\mathcal{F}^\bullet)|_{X_i}$ are locally constant. We will skip the definition of subanalytic subsets ([KS90, 8.2.1]), instead pointing out that the class of constructible sheaves introduced in section 1.10 satisfies this condition. Hence, if one desires to, the following proposition is applicable in our case.

Proposition A.22. *Let X be a real analytic manifold and let $\mathcal{F}$ be an $\mathbb{R}$-constructible complex of sheaves on X.*

1. *The dual complex $\mathcal{D}(\mathcal{F})$ is $\mathbb{R}$-constructible.*
2. *The canonical morphism $\mathcal{F} \to \mathcal{D}\,\mathcal{D}(\mathcal{F})$ is an isomorphism.*
3. *For any $x \in X$, we have isomorphisms*

$$\mathrm{R}\,\Gamma_{\{x\}}(X, \mathcal{D}(\mathcal{F})) \cong \mathrm{R}\,\mathrm{Hom}(\mathcal{F}_x, \mathbb{R}), \qquad \mathcal{D}(\mathcal{F})_x \cong \mathrm{R}\,\mathrm{Hom}(\mathrm{R}\,\Gamma_{\{x\}}(X, \mathcal{F}), \mathbb{R}).$$

Proof. This follows from the more general statement [KS90, 3.4.3], applied to the particular case via [KS90, 8.4.9]. □

B Sheaves and cosheaves on posets

B.1 Sheaves and cosheaves on posets

Definition B.1 (Posets)**.** A *poset* is a set Σ together with a relation $\leq$ on Σ such that for γ, σ and $\tau \in \Sigma$

1. $\sigma \leq \sigma$,
2. $\gamma \leq \sigma$ and $\sigma \leq \tau$ implies $\gamma \leq \tau$ and
3. $\sigma \leq \tau$ and $\tau \leq \sigma$ implies $\sigma = \tau$.

We may interpret a poset Σ as a small category, whose objects are the faces $\sigma \in \Sigma$ and, for σ, $\tau \in \Sigma$, the set of morphisms $\mathrm{Mor}_\Sigma(\sigma, \tau)$ consists of exactly one element if and only if $\sigma \leq \tau$ and is empty otherwise.

A *morphism of posets* $f : \Sigma \to \Sigma'$ is a map satisfying $f(\sigma) \leq f(\tau)$ for $\sigma \leq \tau$ in Σ.

We denote by $\hat{\Sigma}$ the poset $\Sigma \cup \{0_\Sigma, 1_\Sigma\}$ where $0_\Sigma \leq \sigma \leq 1_\Sigma$ for every $\sigma \in \Sigma$.

The poset Σ is *locally finite* if every closed interval is a finite set; it is *topologically finite* if all open stars are finite sets. A finite poset Σ is *graded* if all maximal chains in Σ have the same length. In this case, we have a well defined function $\dim : \Sigma \to \mathbb{N}$ mapping σ to the maximal length of chains ending at σ.

Definition B.2. Let Σ be a poset. The *Alexandrov topology* on Σ is the topology α whose open sets are those subsets $U \subset \Sigma$ which satisfy

$$\sigma \in U, \sigma \leq \tau \Rightarrow \tau \in U.$$

A basis of this topology is given by the *open stars* $U_\sigma := \{\tau \in \Sigma; \sigma \leq \tau\}$, $\sigma \in \Sigma$. We write Σ_α for the topological space Σ, equipped with the Alexandrov topology.

Remark B.3. A map $f : \Sigma \to \Sigma'$ is a morphism of posets if and only if it is continuous with respect to the respective Alexandrov topologies.

Definition B.4. Let Σ be a poset and $\mathcal{A}$ an abelian category.

1. The category of *sheaves on* Σ with values in $\mathcal{A}$ is the category of functors

$$\mathrm{Shv}_{\mathcal{A}}(\Sigma) := \mathrm{Fct}(\Sigma, \mathcal{A}).$$

If $\sigma \leq \tau$ in Σ and $\mathcal{F}$ is a sheaf on Σ, then we write $\rho^{\mathcal{F}}_{\sigma,\tau} : \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ for the induced *restriction* morphism in $\mathcal{A}$.

2. The category of *cosheaves on* Σ with values in $\mathcal{A}$ is the category of functors

$$\mathrm{CoShv}_{\mathcal{A}}(\Sigma) := \mathrm{Fct}(\Sigma^{\mathrm{op}}, \mathcal{A}),$$

where Σ^{op} denotes the opposite category to Σ. If $\sigma \leq \tau$ in Σ and $\mathfrak{A}$ is a sheaf on Σ, then we write $\lambda^{\sigma,\tau}_{\mathfrak{A}} : \mathfrak{A}(\tau) \to \mathfrak{A}(\sigma)$ for the induced *corestriction* morphism in $\mathcal{A}$.

If Σ is a poset and α the Alexandrov topology on Σ then, using the left resp. right Kan extension of a sheaf resp. cosheaf along the functor $\Sigma \to \alpha^{\mathrm{op}}$, $\sigma \mapsto U_\sigma$, one can show the following:

Proposition B.5. *Let $\mathcal{A}$ be an abelian category which is both complete and cocomplete. Then, we have canonical equivalences of abelian categories*

$$\mathrm{Shv}_{\mathcal{A}}(\Sigma) \cong \mathrm{Shv}_{\mathcal{A}}(\Sigma_\alpha), \qquad \mathrm{CoShv}_{\mathcal{A}}(\Sigma) \cong \mathrm{CoShv}_{\mathcal{A}}(\Sigma_\alpha).$$

A sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ in $\mathrm{Shv}_{\mathcal{A}}(\Sigma)$ is exact if and only if all induced sequences

$$\mathcal{F}'(\sigma) \to \mathcal{F}(\sigma) \to \mathcal{F}''(\sigma)$$

for $\sigma \in \Sigma$ are exact. A similar statement holds for sequences of cosheaves.

[Cur14, 4.2.10]

Definition B.6. Let $f : \Sigma \to \Sigma'$ be a morphism of posets and $\mathcal{A}$ a complete and cocomplete abelian category. Let $\mathcal{F}$ and $\mathfrak{A}$ be a sheaf and a cosheaf on Σ and $\mathcal{G}$ and $\mathfrak{B}$ a sheaf and a cosheaf on Σ', each with values in $\mathcal{A}$.

1. The *pullback sheaf* $f^*\mathcal{G}$ on Σ is given by

$$f^*\mathcal{G}(\sigma) = \mathcal{G}(f(\sigma)), \qquad \rho^{f^*\mathcal{G}}_{\sigma,\tau} = \rho^{\mathcal{G}}_{f(\sigma),f(\tau)}.$$

2. The *pullback cosheaf* $f^*\mathfrak{B}$ on Σ is given by

$$f^*\mathfrak{B}(\sigma) = \mathfrak{B}(f(\sigma)), \qquad \lambda^{f^*\mathfrak{B}}_{\sigma,\tau} = \lambda^{\mathfrak{B}}_{f(\sigma),f(\tau)}.$$

3. The *pushforward sheaf* $f_*\mathcal{F}$ on Σ' is given by

$$f_*\mathcal{F}(\sigma') = \lim_{\sigma' \leq f(\tau)} \mathcal{F}(\tau),$$

with restriction maps given by the respective universal properties.

4. The *pushforward cosheaf* $f_*\mathfrak{A}$ on Σ' is given by

$$f_*\mathfrak{A}(\sigma') = \mathrm{colim}_{\sigma' \leq f(\tau)} \mathfrak{A}(\tau),$$

with restriction maps given by the respective universal properties.

Since these constructions are functorial, we get *pushforward* and *pullback functors*

$$f^* : \mathrm{Shv}_{\mathcal{A}}(\Sigma') \to \mathrm{Shv}_{\mathcal{A}}(\Sigma), \qquad f_* : \mathrm{Shv}_{\mathcal{A}}(\Sigma) \to \mathrm{Shv}_{\mathcal{A}}(\Sigma'),$$
$$f^* : \mathrm{CoShv}_{\mathcal{A}}(\Sigma') \to \mathrm{CoShv}_{\mathcal{A}}(\Sigma), \qquad f_* : \mathrm{CoShv}_{\mathcal{A}}(\Sigma) \to \mathrm{CoShv}_{\mathcal{A}}(\Sigma').$$

Proposition B.7. *Let $f : \Sigma \to \Sigma'$ be a morphism of posets and let $\mathcal{A}$ be a complete and cocomplete abelian category.*

1. *The pullback functor $f^* : \mathrm{Shv}_{\mathcal{A}}(\Sigma') \to \mathrm{Shv}_{\mathcal{A}}(\Sigma)$ is exact and left adjoint to the pushforward functor $f_* : \mathrm{Shv}_{\mathcal{A}}(\Sigma) \to \mathrm{Shv}_{\mathcal{A}}(\Sigma')$. Moreover, f^* has a left adjoint, $f_\dagger$.*
2. *The pullback functor $f^* : \mathrm{CoShv}_{\mathcal{A}}(\Sigma') \to \mathrm{CoShv}_{\mathcal{A}}(\Sigma)$ is exact and right adjoint to the pushforward functor $f_* : \mathrm{CoShv}_{\mathcal{A}}(\Sigma) \to \mathrm{CoShv}_{\mathcal{A}}(\Sigma')$. Moreover, f^* has a right adjoint $f_\dagger$.*

[Cur14, 5.3.1]

B.2 Cohomology of sheaves and cosheaves on posets

In [Cur14] it has been illustrated how the categories of sheaves on posets are well posed to construct examples of constructible sheaves on cell complexes and, if given such a constructible sheaf, to compute its cohomology. In fact, this can be done very easily in terms of derived functors on $\mathrm{Shv}(\Sigma)$. Since we do not make much use of this in the main text, we will however just give the very first definitions below.

The following result (proven in [Cur14, 7.1.5ff] in the context of vector spaces), often allows to compute the values of derived functors on sheaves on posets quickly and explicitly:

Proposition B.8. *Let $\mathcal{A}$ be a complete and cocomplete abelian category and Σ a poset.*

1. *If $\mathcal{A}$ has enough injectives, then $\mathrm{Shv}_{\mathcal{A}}(\Sigma)$ has enough injectives. If $\mathcal{I}$ is a cogenerating set of injectives for $\mathcal{A}$, then a cogenerating set of injective sheaves for $\mathrm{Shv}_{\mathcal{A}}(\Sigma)$ is given by the sheaves*
$$(\imath_\sigma)_* I, \text{ for } \sigma \in \Sigma \text{ and } I \in \mathcal{I},$$
where $\imath_\sigma : \{\sigma\} \to \Sigma$ is the inclusion map.
2. *If $\mathcal{A}$ has enough projectives, then $\mathrm{CoShv}_{\mathcal{A}}(\Sigma)$ has enough projectives. If $\mathcal{P}$ is a generating set of projectives for $\mathcal{A}$, then a generating set of projective sheaves for $\mathrm{Shv}_{\mathcal{A}}(\Sigma)$ is given by the sheaves*
$$(\imath_\sigma)_* P, \text{ for } \sigma \in \Sigma \text{ and } P \in \mathcal{P},$$
where $\imath_\sigma : \{\sigma\} \to \Sigma$ is the inclusion map.

Proof. We prove (1) as an example: First, let I be injective in $\mathcal{A}$ and let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence in $\mathrm{Shv}_{\mathcal{A}}(\Sigma)$. In particular, for every $\sigma \in \Sigma$, the induced sequence
$$0 \to \mathcal{F}'(\sigma) \to \mathcal{F}(\sigma) \to \mathcal{F}''(\sigma) \to 0$$
is exact. We now have natural isomorphisms
$$\mathrm{Hom}_\Sigma(\mathcal{F}, (\imath_\sigma)_* I) \cong \mathrm{Hom}_{\mathcal{A}}(\imath_\sigma^* \mathcal{F}, I) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}(\sigma), I),$$
which shows that $\mathrm{Hom}_\Sigma(\cdot, (\imath_\sigma)_* I)$ is an exact functor and hence $(\imath_\sigma)_* I$ is injective.

For every σ, the canonical map $\mathcal{F}(\sigma) \to (\imath_\sigma)_* \imath_\sigma^* \mathcal{F}(\sigma)$ is injective, hence $\mathcal{F} \to \bigoplus_{\sigma \in \Sigma} (\imath_\sigma)_* \imath_\sigma^* \mathcal{F}$ is a monomorphism. For every σ, we also find a monomorphism $\mathcal{F}(\sigma) = \imath_\sigma^* \mathcal{F}(\sigma) \to I_\sigma$, with I_σ in $\mathcal{I}$, since $\mathcal{I}$ is cogenerating in $\mathcal{A}$. The combined morphism $\mathcal{F} \to \bigoplus_{\sigma \in \Sigma} (\imath_\sigma)_* I_\sigma$ is a monomorphism of sheaves and hence the set in question is cogenerating. □

Definition B.9. Let Σ be a poset and $\pi : \Sigma \to \mathrm{pt}$ the projection to a point. Let $\mathcal{A}$ be a complete and cocomplete abelian category with enough injectives resp. projectives. Then we define, for every sheaf $\mathcal{F}$ in $\mathrm{Shv}_{\mathcal{A}}(\Sigma)$ resp. cosheaf $\mathfrak{A}$ in $\mathrm{CoShv}_{\mathcal{A}}(\Sigma)$ the *global section cohomology groups* resp. *homology groups*

$$\mathrm{H}^q(\Sigma, \mathcal{F}) := \mathrm{R}^q\, \pi_*\, \mathcal{F}, \qquad \mathrm{H}_q(\Sigma, \mathfrak{A}) := \mathrm{L}_q\, \pi_*\, \mathfrak{A}\,.$$

Remark B.10. As explained in [Cur14, 7.3], if $\mathcal{F}$ comes from a constructible sheaf $\tilde{\mathcal{F}}$ on a cell complex X, these combinatorical cohomology groups compute usual sheaf cohomology; i.e. we have canonical isomorphisms

$$\mathrm{H}^q(\Sigma, \mathcal{F}) \cong \mathrm{H}^q(X, \tilde{\mathcal{F}})$$

for every $q \in \mathbb{Z}$.

C Locally convex vector spaces

C.1 Quasi-abelian categories

In section 5 we will equip our sheaves of forms with locally convex topologies on their sections and use this to define sheaves of currents. In order to get cohomological properties of the so–defined complexes, we need a good understanding of the categories we are working in. The category of locally convex spaces is quasi-abelian and in [Schn99], Schneiders introduced the necessary tools to do homological algebra in quasi-abelian categories. In this appendix, we will introduce some of the basic notions developed there.

Definition C.1 (Quasi-abelian category)**.** Let $\mathcal{E}$ be an additive category with kernels and cokernels. We write $\mathrm{im}(f) := \ker \operatorname{coker}(f)$ and $\mathrm{coim}(f) := \operatorname{coker} \ker(f)$ for a morphism f in $\mathcal{E}$.

1. A morphism f in $\mathcal{E}$ is called *strict*, if the canonical morphism $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.
2. The category $\mathcal{E}$ is *quasi-abelian*, if for every cartesian square

$$\begin{array}{ccc} E' & \xrightarrow{f'} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & F \end{array}$$

and every strict epimorphism f, f' is a strict epimorphism as well, and for every cocartesian square

$$\begin{array}{ccc} E' & \xrightarrow{f'} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & F \end{array}$$

and every strict monomorphism f', f is a strict monomorphism as well.

Lemma C.2. *Kernels and cokernels are strict.*

Proof. [Schn99, 1.1.2] ◻

Lemma C.3. *Let $\mathcal{E}$ be a quasi-abelian category and*

$$\begin{array}{ccc} E & \xrightarrow{u} & F \\ & \searrow^{f} & \downarrow{\scriptstyle v} \\ & & G \end{array}$$

a commutative diagram in $\mathcal{E}$. Then,

1. *if f is a strict monomorphism, u is a strict monomorphism,*

2. *if f is a strict epimorphism, v is a strict epimorphism.*

Proof. [Schn99, 1.1.8] □

Remark C.4. It is clear that every abelian category is quasi-abelian. Moreover, if $\mathcal{E}$ is a quasi-abelian category, then the dual category $\mathcal{E}^{op}$ is quasi-abelian as well (this follows directly from the definitions).

Schneider proceeds to introduce several notions (of varying strength) of exact sequences in a quasi-abelian category $\mathcal{E}$, which allows him to define the derived category $\mathrm{D}(\mathcal{E})$. Furthermore, in [Schn99, sect. 1.2.1] he shows that this derived category allows two – possibly different – canonical T-structures with hearts $\mathcal{LH}(\mathcal{E})$ (the *left heart* of $\mathcal{E}$) and $\mathcal{RH}(\mathcal{E})$ (the *right heart* of $\mathcal{E}$). This gives a good foundation to develop a theory of derived functors of functors between quasi-abelian categories, largely similar to the classical theory for abelian categories. We will not use this in the main text, but we refer to proposition C.12 which might prove useful in our context.

Here, we restrict ourselves to introduce a very special kind of exact functor:

Definition C.5. A functor $F : \mathcal{E} \to \mathcal{E}'$ of quasi-abelian categories is *strongly exact* if it preserves arbitrary kernels and cokernels.

C.2 Locally convex spaces

The two main quasi-abelian categories we are working with are the category lcs of locally convex spaces and its full subcategory fre of Fréchet spaces. In this subsection we will concern ourselves with the basic properties of lcs and the completion functor $\mathrm{Cpl} : \mathrm{lcs} \to \mathrm{lcs}$ while in the next section we will take a closer look at Fréchet spaces.

Definition C.6. A (real) *locally convex space* is a real vector space E together with a family $(p_i)_{i \in I}$ of seminorms on E. We will in this case equip E with the finest topology making each map $p_i : E \to \mathbb{R}_0^+$ continuous. We write lcs for the category of locally convex spaces with continuous linear maps as morphisms.

Proposition C.7. *The category* lcs *is complete and cocomplete quasi-abelian. Let $f : E' \to E$ be a morphism in* lcs.

1. *The kernel of f is the embedding $f^{-1}(0) \to E'$, where $f^{-1}(0)$ is equipped with the induced topology as a subspace of E'.*

2. *The cokernel of f is the projection $E \to E/f(E')$, where $E/f(E')$ is equipped with the quotient topology.*

3. *The image of f is the embedding $f(E') \to E$, where $f(E')$ is equipped with the induced topology.*

4. *The coimage of f is the projection $E' \to E'/f^{-1}(0)$, where $E'/f^{-1}(0)$ is equipped with the quotient topology.*

Proof. [Pro00, 2.1.8], [Pro00, 2.1.11], [Pro00, 2.2.1] □

Corollary C.8. *Let $f : E \to F$ be a morphism in* lcs. *Then the following conditions are equivalent:*

1. *f is strict;*
2. *f is relatively open;*
3. *for any semi-norm p of E, there is a semi-norm q of F and $C > 0$ such that for every $x \in E$*

$$\inf_{f(e)=0} p(x+e) \leq Cq(f(x)).$$

Proof. [Pro00, 2.1.9] □

The most important fact on lcs for us is the following Hahn-Banach theorem:

Proposition C.9. *The vector space $\mathbb{R}$ (together with the usual topology) is a strongly injective object of* lcs, *i.e. the functor*

$$\mathrm{Hom}_{\mathrm{lcs}}(\cdot, \mathbb{R}) : \mathrm{lcs}^{op} \to \mathrm{Mod}_{\mathbb{R}}$$

preserves arbitrary kernels and cokernels.

Proof. The representable functors are always strongly left exact, so here we have natural isomorphisms

$$\ker\left(\mathrm{Hom}_{\mathrm{lcs}}(E, \mathbb{R}) \to \mathrm{Hom}_{\mathrm{lcs}}(E', \mathbb{R})\right) = \mathrm{Hom}_{\mathrm{lcs}}(E/fE', \mathbb{R})$$

for every continuous map $f : E' \to E$ in lcs. The crucial part is exactness on the right and for this consider a continuous map $g : E \to E''$ of locally convex vector spaces. The kernel of g in lcs is the vector space $g^{-1}(0) \subset E$, equipped with the subspace topology. Now one version of the Hahn-Banach theorem for locally convex spaces tells us that every continuous linear form on $g^{-1}(0)$ can be lifted to a continuous linear form on E. Hence the map

$$\mathrm{Hom}_{\mathrm{lcs}}(E, \mathbb{R}) \to \mathrm{Hom}_{\mathrm{lcs}}(\ker(g), \mathbb{R})$$

is surjective, which finishes the proof. □

Since the locally convex spaces $\mathcal{A}_X^{p,q}(X)$ of differential forms on a tropical space X considered in section 5 are not necessarily complete, their completion might in fact be the more interesting space to consider. However, it is not clear if the exactness properties of $\mathcal{A}_X^{\bullet,\bullet}(U)$ are preserved during completion. To answer this question, the following together with proposition C.15 might prove useful:

Definition C.10. Let E be a locally convex space with defining family $(p_i)_{i \in I}$ of seminorms.

1. A net $(e_j)_{j \in J}$ in E is a *Cauchy net* if for every $i \in I$ and every $\epsilon > 0$ there is a $j_0 \in J$ such that

$$p_i(e_j - e_{j'}) > \epsilon$$

for every $j, j' \geq j_0$.

2. The locally convex space E is *complete* if it is separated – i.e. $\{0\}$ is closed in E – and every Cauchy net in E converges.

Construction C.11. As usual, we can define the *completion* $\mathrm{Cpl}(E)$ of a locally convex space E as a set of equivalence classes of Cauchy sequences. A defining family $(p_i)_{i \in I}$ of seminorms for E induces a family $(\hat{p}_i)_{i \in I}$ of seminorms on $\mathrm{Cpl}(E)$.

Proposition C.12. *The completion functor* $\mathrm{Cpl} : \mathrm{lcs} \to \mathrm{lcs}$ *has the following properties:*

1. *The functor* Cpl *is left exact and has a right derived functor* $\mathrm{R}\,\mathrm{Cpl} : \mathrm{D}^+(\mathrm{lcs}) \to \mathrm{D}^+(\mathrm{lcs})$.

2. *For every* E *in* lcs *we have* $\mathrm{R}\,\mathrm{Cpl}(E) = \mathrm{R}\,\mathrm{Cpl}(\mathrm{Cpl}(E))$.

Proof. [Pro00, 4.2.2], [Pro00, 4.3.14] □

C.3 Fréchet spaces and (LF)-spaces

Fréchet spaces and (LF)-spaces are two of the 'basic' classes of locally convex spaces used in analytic applications. Because they appear prominently in section 5, we recall their definition and first properties here.

Definition C.13. A locally convex vector space E is a *Fréchet space* if it is complete and its topology can be given by a countable family $(p_n)_{n \in \mathbb{N}}$ of seminorms. We write fre for the full subcategory of lcs consisting of Fréchet spaces.

Proposition C.14. *The category* fre *is quasi-abelian and has enough injectives. The embedding functor* fre $\to$ lcs *preserves strict morphisms, arbitrary kernels and cokernels of strict morphisms.*

Proof. [Pro00, 4.4.2-6] □

Proposition C.15. *Every Fréchet space* F *is* cohomologically complete*, i.e. the canonical morphism*

$$F \to \mathrm{R}\,\mathrm{Cpl}(F)$$

is an isomorphism in $\mathrm{D}(\mathrm{lcs})$.

Proof. [Pro00, 4.4.7] □

Definition C.16. A locally convex space E is an *(LF)-space* if there is a system

$$E_1 \hookrightarrow E_2 \hookrightarrow E_3 \hookrightarrow \cdots$$

of Fréchet spaces and continuous embeddings $\phi_i : E_i \to E_{i+1}$ and an isomorphism of locally convex spaces $E = \mathrm{colim}_i\, E_i$. The system (E_i, ϕ_i) is called a *defining spectrum* for E.

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