Eisenstein series via the Poincaré bundle and applications

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Introduction and Overview

One of the famous open problems in the 17th and 18th century was the question of the value of the convergent series
$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$
This question was first asked by Mengoli in 1644. Many great mathematicians of that time tried to solve this problem in vain. It took almost 100 years until Euler solved it in 1734 and proved
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
using state of the art techniques in analysis. From a nowadays point of view this formula can be seen as the first contribution to the study of special values of L-functions. On the other hand, at the time of Euler no one would have considered this formula as a statement of number-theoretical interest. The bare appearance of numbers in a mathematical statement does not make it a number-theoretical statement. So, it took another 100 years until L-functions became objects of number-theoretical interest. In 1837, Dirichlet introduced L-functions associated with Dirichlet characters and used them to prove the existence of an infinite number of primes in arithmetic progressions. Only shortly afterwards, Dirichlet gave the first instance of a class number formula for quadratic fields. The analytic class number formula

$$\lim_{s\to 0} s^{-(r_1+r_2-1)}\zeta_K(s) = -\frac{h_K r_K}{w_K}$$

for a general number field K goes back to Dedekind. Here, h_K is the order of the class group, r_K is the regulator, w_K is the number of the roots of unity contained in K and $r_1 + r_2 - 1$ is the rank of the group of units of $\mathcal{O}_K$. This beautiful formula assembles all the basic invariants of a number field in a single equation. From the mid 19th century on, L-functions have been a central and important object of study in number theory. Furthermore, they provide a source of deep and beautiful conjectures. The analytic class number formula can be seen as the prototype of the very general Tamagawa number conjecture (TNC) of Bloch and Kato on special values of L-functions. The Tamagawa number conjecture and its p-adic analogue, the Perrin-Riou conjecture, express special L-values in terms of various realizations of motivic cohomology classes. While the TNC in its general form is far out of scope of today's methods, there has been some progress in proving particular cases. Let us concentrate here on some results concerning progress in the case of Hecke characters of number fields. For the Riemann zeta function the proof of the Tamagawa number conjecture (TNC) goes back to Bloch and Kato and was completed by Huber and Wildeshaus in [HW98]. For Dirichlet characters it was proven by Huber and Kings [HK03]. For non-critical L-values of CM-elliptic curves the TNC was settled by Kings in [Kin01]. For critical L-values of Hecke characters of imaginary quadratic fields the most general result has been obtained by Tsuji [Tsu04] as an application of a very general explicit reciprocity law.

In order to tackle particular cases of the Tamagawa number conjecture first we need a way to construct enough motivic cohomology classes, then we have to understand their realizations and relate those cohomology classes to special values of L-functions. An important source of such motivic cohomology classes is provided by the *polylogarithm*. The bridge between the specializations of those polylogarithm classes and L-values is often given via *Eisenstein series* and their cohomology classes. Let us take a closer look at Eisenstein series and the elliptic polylogarithm and thereby explain the main results of this thesis.

Eisenstein series

Eisenstein series provide a key tool in studying special values of L-functions. For example, consider real-analytic Eisenstein series for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Their values at CM-points play an important role in studying special values of L-functions associated with Hecke characters of imaginary quadratic fields. For

$$(\omega_1,\omega_2)\in \mathrm{GL}^+ := \left\{(\omega_1,\omega_2)\in\mathbb{C}^2 : \mathrm{Im}\left((\omega_1)^{-1}\omega_2\right)>0\right\},$$

let us consider the lattice Γ in $\mathbb{C}$ spanned by ω_1,ω_2. For $b>a+2$ and $x_0\in\frac{1}{N}\Gamma$ the series

$$\frac{(-1)^{b+1}b!}{(\mathrm{Im}\,\bar{\omega}_1\omega_2)^a}\sum_{\gamma\in\Gamma\setminus\{-x_0\}}\frac{(\bar{x}_0+\bar{\gamma})^a}{(x_0+\gamma)^b}$$

converges absolutely. It defines a real-analytic modular form of level N. For $a=0$ the resulting modular form is algebraic. The non-holomorphic Eisenstein series

$$\frac{(-1)^{b+1}b!}{(\mathrm{Im}\,\bar{\omega}_1\omega_2)^a}\sum_{\gamma\in\Gamma\setminus\{-x_0\}}\frac{(\bar{x}_0+\bar{\gamma})^a}{(x_0+\gamma)^b}$$

can be obtained by applying a times the *Weil operator*

$$W=-\frac{1}{\mathrm{Im}\,\bar{\omega}_1\omega_2}\left(\bar{\omega}_1\frac{\partial}{\partial\omega_1}+\bar{\omega}_2\frac{\partial}{\partial\omega_2}\right)$$

to the algebraic modular form

$$(-1)^{b-a+1}(b-a)!\sum_{\gamma\in\Gamma\setminus\{-x_0\}}\frac{1}{(x_0+\gamma)^{b-a}}\quad b-a>2.$$

At first sight, there seems to be little hope to give an algebraic interpretation of the non-holomorphic Eisenstein series for $a > 0$. However, by an ingenious insight of Katz the Weil operator admits a cohomological interpretation. This cohomological description allows a purely algebraic interpretation of real-analytic Eisenstein series as de Rham cohomology classes. Let us recall the approach of Katz [Kat76], cf. the exposition in [Tsu04, II. §2]. To each point $(\omega_1, \omega_2) \in \mathrm{GL}^+$ we can associate a complex elliptic curve

$$E = \mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}), \quad (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} E[N](\mathbb{C}), (x,y) \mapsto x\frac{\omega_1}{N} + y\frac{\omega_2}{N}$$

with $\Gamma(N)$ level structure. For $N > 3$ this gives us a map

$$\mathrm{GL}^+ \to M_N(\mathbb{C})$$

to the $\mathbb{C}$-valued points of the modular curve of level $\Gamma(N)$. This map allows us to view real-analytic modular forms as real-analytic sections of the line bundle $\underline{\omega}(\mathcal{C}^\infty)^{a+b}$ with $\omega := e^*\Omega^1_{E^{\mathrm{univ}}/M_N}$. Let us denote by $\underline{H}^1_{\mathrm{dR}}$ the first relative de Rham cohomology of the universal elliptic curve. Following Katz, let us use the Gauss–Manin connection on $\underline{H}^1_{\mathrm{dR}}$ and the Kodaira–Spencer isomorphism to define the algebraic differential operator:

$$\theta\colon \underline{\omega}^k \hookrightarrow \underline{\mathrm{Sym}}^k\, \underline{H}^1_{\mathrm{dR}} \xrightarrow{\nabla} \underline{\mathrm{Sym}}^k\, \underline{H}^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{M_N}} \Omega^1_{M_N} \xleftarrow{\sim} \underline{\mathrm{Sym}}^k\, \underline{H}^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{M_N}} \underline{\omega}^2 \hookrightarrow \underline{\mathrm{Sym}}^{k+2}\, \underline{H}^1_{\mathrm{dR}}$$

Composing this with the projection obtained from the Hodge decomposition, we obtain the differential operator

$$\theta(\mathcal{C}^\infty)\colon \underline{\omega}(\mathcal{C}^\infty)^k \xrightarrow{\theta} \underline{\mathrm{Sym}}^{k+2}\, \underline{H}^1_{\mathrm{dR}}(\mathcal{C}^\infty) \twoheadrightarrow \underline{\omega}(\mathcal{C}^\infty)^{k+2}.$$

Now, Katz showed that under the above identification the Weil operator coincides with the differential operator $\theta(\mathcal{C}^\infty)$. The benefit of this interpretation is that the definition of θ is purely algebraic. This gives us a purely algebraic interpretation of real-analytic Eisenstein series as sections of

$$\underline{\mathrm{Sym}}^{a+b}\, \underline{H}^1_{\mathrm{dR}}.$$

Katz' cohomological construction of real-analytic Eisenstein series via the Gauss–Manin connection allows us to study algebraic and p-adic properties of real-analytic Eisenstein series. On the other hand, the usage of the differential operator θ has an obvious disadvantage. It is far from being functorial. For the definition of θ it is essential that the Kodaira–Spencer map is an isomorphism. As an example, let us recall that the value of real-analytic Eisenstein series on CM elliptic curves is one of the main tools for studying L-values of Hecke characters for imaginary quadratic fields. The non-functoriality of the above construction forces us to study the universal situation although we are just interested in the value of real-analytic Eisenstein classes for a single CM elliptic curve.

In [BK10b] Bannai and Kobayashi have observed that the *Kronecker theta function* and certain translates of it are generating series for an important class of real-analytic Eisenstein series, the *Eisenstein–Kronecker series*. These are real-analytic modular forms defined for $b > a + 2 \geq 0$ by

$$e^*_{a,b}(z_0, w_0; \Gamma) := \sum_{\gamma \in \Gamma \setminus \{-z_0\}} \frac{(\bar{z}_0 + \bar{\gamma})^a}{(z_0 + \gamma)^b} \langle \gamma, w_0 \rangle_\Gamma$$

with $\langle z,w\rangle_\Gamma := \exp\left(\frac{z\bar{w}-w\bar{z}}{A(\Gamma)}\right)$ and $A(\Gamma) := \frac{\Im \omega_2\bar{\omega}_1}{\pi}$. For general $b>0, a\geq 0$ they can be defined by analytic continuation, cf. [BK10b, §1.1]. For CM elliptic curves Bannai and Kobayashi applied Mumford's theory of algebraic theta functions to study algebraicity properties of the Kronecker theta function. At least for CM elliptic curves this gives a new approach for studying real-analytic Eisenstein series and their algebraic properties, which avoids considering the universal situation. Unfortunately, the approach via the Kronecker theta function does not directly generalize to more general elliptic curves.

The first main result of this thesis gives a new and purely algebraic construction of cohomology classes of real-analytic Eisenstein series via the Poincaré bundle. This construction is compatible with base change and works for arbitrary families of elliptic curves E over a general base scheme S. We are building on the work of Bannai and Kobayashi. Instead of working with theta functions, we will work with the underlying section of the Poincaré bundle. This algebraic section will be defined in the first chapter and will be called the *canonical section*

$$s_{\mathrm{can}} \in \Gamma\left(E\times_S E^\vee, \mathcal{P}\otimes\Omega^1_{E\times_S E^\vee/E^\vee}\left([E\times e]+[e\times E^\vee]\right)\right)$$

of the Poincaré bundle $\mathcal{P}$. After passing to the universal vectorial extensions of both E and its dual, we get canonical connections on the Poincaré bundle. Applying them iteratively to certain translates of the canonical section, allows us to give a functorial construction of elements

$$E^{a,b+1}_{s,t} \in \Gamma\left(S, \underline{\mathrm{Sym}}^a\,\underline{H}^1_{\mathrm{dR}}\left(E^\vee/S\right)\otimes\underline{\mathrm{Sym}}^{b+1}\,\underline{H}^1_{\mathrm{dR}}\left(E/S\right)\right)$$

for torsion sections $s\in E^\vee[D](S)$ and $t\in E[N](S)$. The first main result of this thesis is the following:

Theorem (cf. Theorem[1] 1.7.2)**.** *The Hodge decomposition on the universal elliptic curve of level $\Gamma(ND)$ identifies $E^{a,b+1}_{s,t}$ with the real-analytic Eisenstein–Kronecker series*

$$(-1)^{a+b}a!b!\frac{e^*_{a,b+1}(D\tilde{s},N\tilde{t})}{A^a a!}dz^{\otimes(a+b+1)}.$$

Here, $\tilde{s}$ and $\tilde{t}$ are certain analytic lifts of the torsion sections s and t.

As an application of our purely algebraic construction of real-analytic Eisenstein series via the Poincaré bundle, we will give a new construction of the two-variable p-adic measure of Katz interpolating real-analytic Eisenstein series on the ordinary locus of the modular curve. Like Katz we will work with the universal trivialized elliptic curve $E^{\mathrm{triv}}/M^{\mathrm{triv}}$. Norman's theory of p-adic theta functions allows us to associate p-adic theta functions

$${}_D\vartheta_s(T_1,T_2)\in V\left(\mathbb{Z}_p,\Gamma(N)\right)[\![T_1,T_2]\!]$$

[1] for a more detailed version of the theorem, see the main body of the text

with coefficients in Katz' ring $V(\mathbb{Z}_p, \Gamma(N))$ of generalized p-adic modular forms to certain translates of the canonical section. Via Amice' isomorphism between functions on $\widehat{\mathbb{G}}_m$ and p-adic measures we obtain a two-variable measure

$$\mu_{D,s}^{\mathrm{Eis}} \in \mathrm{Meas}\Big(\mathbb{Z}_p \times \mathbb{Z}_p, V(\mathbb{Z}_p, \Gamma(N))\Big)$$

with values in p-adic modular forms. We call this measure the *p-adic Eisenstein–Kronecker measure*. Our second main result gives a bridge between p-adic theta functions and generalized p-adic modular forms:

Theorem (cf. Theorem[2] 4.5.3)**.** *The measure $\mu_{D,s}^{\mathrm{Eis}}$ interpolates p-adic variants ${}_D\mathcal{E}_s^{k,r+1}$ of the Eisenstein–Kronecker series p-adically, i.e.:*

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} x^k y^r d\mu_{D,s}^{\mathrm{Eis}}(x,y) = {}_D\mathcal{E}_s^{k,r+1}.$$

This result is motivated by the construction of two-variable p-adic measures for CM-elliptic curves at ordinary primes by Bannai–Kobayashi [BK10b]. Further, we compare our measure to that of Katz.

The remaining results of this thesis concern the algebraic de Rham realization and the syntomic realization of the elliptic polylogarithm:

The elliptic polylogarithm

The cohomological polylogarithm is an important tool for constructing cohomology classes of motivic origin as needed for studying particular cases of conjectures on special values of L-functions. To give an example, polylogarithmic classes play a key role in the above mentioned proofs of the TNC for Dirichlet characters by Huber-Kings [HK03] and CM elliptic curves by Kings [Kin01].

But let us start with the classical polylogarithmic functions. The functions

$$\mathrm{ln}_k\, x := \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad |x| < 1, k \geq 1$$

have already been studied by Euler in 1768. The relation to the classical logarithm is given by $\mathrm{ln}_1\, x = -\log(1-x)$. On the open unit disc these functions satisfy the integral relation

$$\mathrm{ln}_k\, x = \int_0^x \mathrm{ln}_{k-1}\, z \frac{dz}{z}, \quad |x| < 1, k \geq 2.$$

Using this identity, the classical polylogarithmic functions can be analytically continued to multivalued holomorphic functions on $\mathbb{C} \setminus \{0,1\}$. The starting point towards a modern treatment of the classical polylogarithmic functions can be seen in the monodromy computation of Ramakrishnan of the polylogarithmic functions [Ram82], see also the

[2] for a more detailed version of the theorem, see the main body of the text

exposition by Hain [Hai94]. The above integral equation can be reformulated as a linear differential equation. Let us define $\omega_0 = \frac{dz}{z}$ and $\omega_1 := \frac{dz}{1-z}$ and

$$W := \begin{pmatrix} 0 & \omega_1 & 0 & \cdots & 0 \\ & \ddots & \omega_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & \omega_0 \\ 0 & & \cdots & & 0 \end{pmatrix} \in \mathrm{H}^0(\mathbb{P}^1(\mathbb{C}), \Omega^1_{\mathbb{P}^1}(\log\{0,1,\infty\})) \otimes \mathrm{Gl}_{n+1}(\mathbb{C}),$$

then the multivalued polylogarithmic functions can be encoded as solutions of the linear differential equation

$$d\underline{\lambda} = \underline{\lambda} W, \quad \underline{\lambda} : \mathbb{C} \setminus \{0,1\} \to \mathbb{C}^{n+1}.$$

Again, we can reformulate this. Let us consider the connection

$$\nabla(f) = df - fW$$

on the $n+1$-dimensional trivial vector bundle $\mathbb{C}^{n+1} \times \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$ and let us denote the local system of horizontal sections by $\mathcal{P}ol$. This local system captures the solutions of the above differential equation. Then, one can show that $\mathcal{P}ol$ sits in a non-trivial extension

$$0 \to \mathcal{L}og^n \to \mathcal{P}ol \to \underline{\mathbb{C}} \to 0$$

of local systems. Here $\underline{\mathbb{C}}$ is the constant local system and $\mathcal{L}og^n$ is the local system on $\mathbb{C} \setminus \{0\}$ associated with the representation of $\pi_1 := \pi_1(\mathbb{C} \setminus \{0\})$

$$\mathbb{C}[\pi_1]/J^{n+1}, \quad J := \ker\left(\mathbb{C}[\pi_1] \twoheadrightarrow \mathbb{C}\right).$$

Let us observe that every unipotent representation of π_1 of length n is a module under $\mathbb{C}[\pi_1]/J^{n+1}$. Thus, the sheaf $\mathcal{L}og^n$ has a distinguished role under all unipotent local systems of length n and is called n-th *logarithm* sheaf. Let us summarize the above by saying that the classical polylogarithmic functions are encoded in the above non-trivial extension. It is this sheaf-theoretical interpretation of the classical polylogarithm functions which can be fruitfully generalized to other more general settings.

It was observed by Deligne that the above extension of local systems underlies a variation of mixed Hodge structures. Beilinson extended the definition of the polylogarithm for $\mathbb{P} \setminus \{0,1,\infty\}$ to the theory of mixed sheaves in the l-adic setting . The motivic origin of the polylogarithm for $\mathbb{P} \setminus \{0,1,\infty\}$ was worked out by Huber and Wildeshaus [HW98]. In the seminal work “The elliptic polylogarithm” Beilinson and Levin have extended the definition of the polylogarithm to elliptic curves. They have defined the polylogarithm for any mixed sheaf theory and have proven its motivic origin. Besides, they have shown that the period functions of the polylogarithm in the $\mathbb{R}$-Hodge realization are given by certain Eisenstein–Kronecker series. From here on many generalizations and applications of polylogarithmic cohomology classes have been given. Let us concentrate here on the syntomic realization of the elliptic polylogarithm. The importance of syntomic cohomology comes from its interpretation as absolute p-adic Hodge cohomology, see Bannai [Ban02] and Deglise–Nizioł [DN15]. Thus, syntomic cohomology can be seen as a p-adic version of Deligne–Beilinson cohomology. This explains the importance of syntomic cohomology for the formulation of the p-adic Beilinson conjecture and the conjecture of Perrin-Riou on special values of L-functions.

The rigid syntomic realization of the elliptic polylogarithm for CM elliptic curves has been studied by Bannai, Kobayashi and Tsuji [BKT10]. They first give an explicit description of the de Rham realization of the elliptic polylogarithm. Building on this they relate the rigid syntomic realization of the elliptic polylogarithm to overconvergent functions obtained as certain moment functions of the p-adic distribution interpolating the Eisenstein–Kronecker numbers. On the other hand, the syntomic Eisenstein classes obtained by specializing the polylogarithm class on the ordinary locus of the modular curve have been described by Bannai and Kings [BK10a]. These Eisenstein classes are related to moments of Katz' two variable p-adic Eisenstein measure interpolating real-analytic Eisenstein series. Again, the de Rham Eisenstein classes are an important intermediate step for understanding the syntomic Eisenstein classes.

The remaining main results of this thesis are concerning the de Rham realization and the syntomic realization of the elliptic polylogarithm. For the de Rham realization of the elliptic polylogarithm we are building on previous results in the PhD thesis of Scheider [Sch14]. Like Scheider, we consider arbitrary families of elliptic curves over a smooth base scheme over a field of characteristic zero. Scheider shows that the (relative) de Rham logarithm sheaves for families of elliptic curves can be constructed by restricting the Poincaré bundle $\mathcal{P}^\dagger$ on $E \times_S E^\dagger$ to infinitesimal thickenings of $E \times e$:

$$\mathcal{L}_n^\dagger := (\mathrm{pr}_E)_*\Big(\mathcal{P}^\dagger\Big|_{E\times \mathrm{Inf}_e^n E^\dagger} \Big).$$

Restricting the *canonical section* to such infinitesimal neighbourhoods allows us to construct sections

$$L_n^D \in \Gamma\left(E, \Omega^1_E(E[D]) \otimes \mathcal{L}_n^\dagger\right).$$

Indeed, these logarithmic 1-forms with values in the logarithm sheaves represent the polylogarithm in de Rham cohomology:

Theorem (cf. Theorem[3] 5.2.10)**.** *The de Rham realization of the D-variant of the elliptic polylogarithm is explicitly given as follows:*

$$\left([L_n^D]\right)_{n\geq 0} = \mathrm{pol}_{D,\mathrm{dR}} \in \varprojlim_n H^1_{\mathrm{dR}}\left(U_D, \mathrm{Log}^n_{\mathrm{dR}}\right)$$

This can be seen as an algebraic version of previous results of Scheider [Sch14, §3]. Together with Scheider's result on the logarithm sheaves this gives an algebraic description of the de Rham realization of the elliptic polylogarithm purely out of the Poincaré bundle. Building on this result allows us to give an explicit description of the rigid syntomic

[3] for a more detailed version of the theorem, see the main body of the text

realization of the elliptic polylogarithm on the ordinary locus of the modular curve. This generalizes the results of Bannai–Kings and Bannai–Kobayashi–Tsuji:

Theorem (cf. Theorem[4] 6.5.3)**.** *There is a compatible system of overconvergent sections in the syntomic logarithm sheaves $\rho_n \in \Gamma\left(\bar{\mathscr{E}}_K, j_D^\dagger(\mathrm{Log}^n_{\mathrm{syn}})\right)$ describing the D-variant of the syntomic polylogarithm on the ordinary locus of the modular curve*

$$\mathcal{pol}_{D,\mathrm{syn}} = ([\rho_n])_{n\geq 0} \in \varprojlim_n H^1_{\mathrm{syn}}\left(\mathscr{U}_D, \mathcal{Log}^n_{\mathrm{syn}}(1)\right).$$

In tubular neighbourhoods $]t[$ of torsion sections there is a canonical decomposition of these overconvergent sections

$$\rho_n|_{]t[} = \sum_{k+l\leq n} \hat{\mathrm{e}}_{t,(k,l)}\hat{\omega}^{[k,l]}$$

in terms of certain generators $(\hat{\omega}^{[k,l]})_{k+l\leq n}$ of the logarithm sheaves. The rigid-analytic functions $\hat{\mathrm{e}}_{t,(k,l)}$ appearing in this decomposition are moment functions of the p-adic Eisenstein–Kronecker measure

$$\hat{\mathrm{e}}_{t,(k,l)}(s) = (-1)^l l! \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} y^k x^{-(l+1)}(1+s)^x d\mu^{\mathrm{Eis}}_{D,t}(x,y)$$

with values in p-adic modular forms.

Both, the description of the de Rham and the syntomic realization, are developed out of the Poincaré bundle. We hope that the approach via the Poincaré bundle might give also new insights for studying polylogarithms for higher dimensional Abelian varieties.

Overview

Let us give a more detailed overview over the single chapters of this thesis:

Chapter 1: Eisenstein series via the Poincaré bundle

In the first chapter we will present a new construction of real-analytic Eisenstein series via the Poincaré bundle. Let $\pi : E \to S$ be an elliptic curve and $(\mathcal{P}, r_0, s_0)$ be the bi-rigidified Poincaré bundle on $E \times_S E^\vee$. After choosing the autoduality isomorphism

$$E \xrightarrow{\sim} E^\vee, \quad P \mapsto [\mathcal{O}_E([-P] - [e])],$$

we get the more explicit description

$$\begin{aligned}(\mathcal{P}, r_0, s_0) &= \left(\mathcal{O}_{E\times E}(-[e\times E] - [E\times e] + \Delta) \otimes_{\mathcal{O}_{E\times E}} \pi^*_{E\times E}\underline{\omega}^{\otimes -1}_{E/S}, r_0, s_0\right) = \\ &= \left(\mathrm{pr}_1^*\mathcal{O}_E([e])^{\otimes -1} \otimes \mathrm{pr}_2^*\mathcal{O}_E([e])^{\otimes -1} \otimes \mu^*\mathcal{O}_E([e]) \otimes \pi^*_{E\times E}\underline{\omega}^{\otimes -1}_{E/S}, r_0, s_0\right)\end{aligned}$$

[4] for a more detailed version of the theorem, see the main body of the text

where Δ is the anti-diagonal and the rigidifications $r_0 : (e \times \mathrm{id})^*\mathcal{P} \xrightarrow{\sim} \mathcal{O}_E$ and $s_0 : (\mathrm{id} \times e)^*\mathcal{P} \xrightarrow{\sim} \mathcal{O}_E$ are induced by the canonical isomorphism

$$e^*\mathcal{O}_E(-[e]) \xrightarrow{\sim} \underline{\omega}_{E/S}.$$

We can view the line bundle $\mathcal{O}_{E\times E}(-[e\times E]-[E\times e]+\Delta)$ in a canonical way as a Cartier divisor. Now, it comes with a canonical meromorphic section. This section induces a section

$$s_{\mathrm{can}} \in \Gamma\left(E \times_S E^\vee, \mathcal{P} \otimes \Omega^1_{E\times_S E^\vee/E^\vee}\left([E \times e] + [e \times E^\vee]\right)\right).$$

which will be called the *canonical section*. Further, we will define translation operators

$$\mathcal{U}^{N,D}_{s,t} : (T_s \times T_t)^*([D] \times [N])^*\mathcal{P} \longrightarrow ([D] \times [N])^*\mathcal{P}.$$

for $N, D \geq 1$, and torsion sections $s \in E[N](S)$, $t \in E^\vee[D](S)$. For later applications we will further define a *D-variant of the canonical section* s^D_{can}. Passing to the universal vectorial extensions of both E and $E^\vee$

$$E^\sharp \times_S E^\dagger \twoheadrightarrow E \times_S E^\vee$$

and denoting by $\mathcal{P}^{\sharp,\dagger}$ the pullback of the Poincaré bundle gives canonical integrable connections

$$\mathcal{P}^{\sharp,\dagger} \xrightarrow{\nabla_\sharp} \mathcal{P}^{\sharp,\dagger} \otimes_{\mathcal{O}_{E^\sharp\times E^\dagger}} \Omega^1_{E^\sharp\times E^\dagger/E^\sharp} \cong \mathcal{P}^{\sharp,\dagger} \otimes_{\mathcal{O}_S} \underline{H}^1_{\mathrm{dR}}\left(E^\vee/S\right)$$

$$\mathcal{P}^{\sharp,\dagger} \xrightarrow{\nabla_\dagger} \mathcal{P}^{\sharp,\dagger} \otimes_{\mathcal{O}_{E^\sharp\times E^\dagger}} \Omega^1_{E^\sharp\times E^\dagger/E^\dagger} \cong \mathcal{P}^{\sharp,\dagger} \otimes_{\mathcal{O}_S} \underline{H}^1_{\mathrm{dR}}\left(E/S\right).$$

Applying these connections iteratively to translates of the canonical section and evaluating at the zero section gives a functorial construction of *algebraic Eisenstein–Kronecker series*

$$E^{k,r+1}_{s,t} \in \Gamma\left(S, \underline{\mathrm{Sym}}^k\, \underline{H}^1_{\mathrm{dR}}\left(E^\vee/S\right) \otimes \underline{\mathrm{Sym}}^{r+1}_{\mathcal{O}_S}\, \underline{H}^1_{\mathrm{dR}}\left(E/S\right)\right).$$

Our main result of the first chapter relates the *algebraic Eisenstein–Kronecker series* to real-analytic Eisenstein series via the Hodge decomposition on the analytification of the universal elliptic curve. Furthermore, we prove a distribution relation for translates of the canonical sections generalizing the distribution relation of Bannai–Kobayashi [BK10b]. While the definition of the canonical section a priori involves the choice of an autoduality isomorphism, we give an intrinsic characterization of the canonical section. This intrinsic characterization of the canonical section is motivated by the definition of the Kato–Siegel functions of Kato [Kat04, Prop. 1.3]. The symmetry of our constructions via the Poncaré bundle is reflected by the functional equation of the Eisenstein–Kronecker series.

Chapter 2: The geometric de Rham logarithm sheaves

In the second chapter we recall results of the PhD thesis of Scheider [Sch14]. Let us denote the pullback of the Poincaré bundle to $E \times_S E^\dagger$ by $\mathcal{P}^\dagger$. Here, $E^\dagger$ is the universal vectorial extension of $E^\vee$. By restricting the Poincaré bundle $\mathcal{P}^\dagger$ along $E \times_S \operatorname{Inf}_e^n E^\dagger$ we obtain the *geometric logarithm sheaves*

$$\mathcal{L}_n^\dagger := (\mathrm{pr}_E)_* \Big(\mathcal{P}^\dagger\big|_{E\times \operatorname{Inf}_e^n E^\dagger} \Big).$$

The canonical connection on $\mathcal{P}^\dagger$ induces an integrable S-connection $\nabla_{\mathcal{L}_n^\dagger}$ on $\mathcal{L}_n^\dagger$. One of the main results of Scheider says that $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger})$ satisfies the universal property of the n-th relative de Rham logarithm sheaf. In the second chapter we recall the most important properties of the geometric logarithm sheaves from [Sch14]. Along the same lines we study the properties of

$$\mathcal{L}_n := (\mathrm{pr}_E)_* \Big(\mathcal{P}|_{E\times \operatorname{Inf}_e^n E^\vee} \Big).$$

It turns out that there are canonical inclusions $\mathcal{L}_n \hookrightarrow \mathcal{L}_n^\dagger$. A good way to think about $\mathcal{L}_n$ is as the first non-trivial filtration step in the Hodge filtration of the (geometric) logarithm sheaves $\mathcal{L}_n^\dagger$. Restricting the D-variant of the canonical section s_{can}^D to $E \times_S \operatorname{Inf}_e^n E^\vee$ gives us a family of sections

$$l_n^D \in \Gamma(E, \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D])).$$

These sections will be called *canonical sections of the geometric logarithm sheaves*. These sections will be important for describing the de Rham realization of the elliptic polylogarithm.

Chapter 3: The Katz splitting

As outlined above, the geometric logarithm sheaves serve as relative versions of the de Rham logarithm sheaves. For studying the algebraic de Rham realization of the polylogarithm, we need to extend the relative connections $\nabla_{\mathcal{L}_n^\dagger}$ to absolute connections. In chapter 3 we develop the necessary tools for overcoming these technical difficulties. First, we recall a construction of Katz of a functorial cross-section to the canonical projection

$$E^\sharp \twoheadrightarrow E$$

of the universal vectorial extension over the open subset $U := E \setminus \{e\}$. This gives us a U-valued point of the universal vectorial extension. A result of Mazur and Messing [MM74, (2.6.7)] allows us to view this U-valued point as an infinitesimal rigidification on the line bundle $\mathcal{P}|_{U\times E^\vee}$. Restricting this infinitesimal rigidification to $U \times \operatorname{Inf}_e^1 E^\vee$, gives a functorial way to split the canonical extension of the first geometric logarithm sheaf

$$0 \longrightarrow \pi^*\underline{\omega}_{E^\vee/S} \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{O}_E \longrightarrow 0 \tag{L}$$

over the open subset $U = E \setminus \{e\} \subseteq E$. For an elliptic curve E/S over a smooth T-scheme this allows us at the same time to split the restriction to U of the short exact sequence

$$0 \longrightarrow \pi^*\Omega^1_{S/T} \longrightarrow \Omega^1_{E/T} \longrightarrow \Omega^1_{E/S} \longrightarrow 0. \tag{Ω}$$

Indeed, we prove that tensorizing the pushout of (L) along the Kodaira–Spencer map

$$\pi^*\underline{\omega}_{E^\vee/S} \to \pi^*\Omega^1_{S/T} \otimes_{\mathcal{O}_E} \pi^*\underline{\omega}^{\otimes -1}_{E^\vee/S}$$

with $\pi^*\underline{\omega}_{E^\vee/S} = \Omega^1_{E/S}$ gives the exact sequence (Ω). This allows us to transfer the canonical splitting of the short exact sequence $(\mathrm{L})|_U$ to a canonical splitting of $(\Omega)|_U$. These splittings allow us to explicitly extend the relative connection on $\mathcal{L}_n^\dagger$ to an absolute one.

Chapter 4: P-adic interpolation of real analytic Eisenstein series via p-adic Theta functions

In the fourth chapter we will construct Katz' two-variable p-adic Eisenstein measure via the Poincaré bundle. Following Norman, we will associate p-adic theta functions to sections of the Poincaré bundle for ordinary elliptic over p-adic rings. Applying this construction to the D-variant of the canonical section on the universal trivialized elliptic curve $E^{\mathrm{triv}}/M^{\mathrm{triv}}$ allows us to construct p-adic theta functions

$${}_D\vartheta_s(T_1, T_2) \in V\left(\mathbb{Z}_p, \Gamma(N)\right)[\![T_1, T_2]\!]$$

for non-trivial torsion sections $e \neq s \in E^{\mathrm{triv}}[N](M^{\mathrm{triv}})$. Here, $V\left(\mathbb{Z}_p, \Gamma(N)\right)$ is the ring of (generalized) p-adic modular forms of level N. The Amice transform between p-adic measures and functions on $\widehat{\mathbb{G}}_m$ allows us to associate a two-variable p-adic measure $\mu^{\mathrm{Eis}}_{D,s}$ with values in generalized p-adic modular forms to ${}_D\vartheta_s(T_1, T_2)$. This measure will be called *D-variant of the p-adic Eisenstein–Kronecker measure.* Finally, we will compare this measure to Katz' two-variable measure interpolating real-analytic Eisenstein series p-adically.

Chapter 5: The algebraic de Rham realization of the elliptic polylogarithm

In this chapter we construct the algebraic de Rham realization of the elliptic polylogarithm completely out of the Poincaré bundle. Let K be a field of characteristic zero and E/S an elliptic curve over a smooth K-scheme. Following Scheider, we will recall the definition of the D-variant of the elliptic polylogarithm in de Rham cohomology. Using analytic methods, Scheider has already given an analytic description of the de Rham realization via the Jacobi theta function on the universal elliptic curve [Sch14, §3]. Our description of the algebraic de Rham realization can be seen as an algebraic version of his results. The Katz splitting allows us to lift the relative 1-forms

$$l_n^D \in \Gamma(E, \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D]))$$

to absolute ones

$$L_n^D \in \Gamma(E, \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E}(E[D])).$$

The main theorem of this chapter is that the (D-variant of the) de Rham polylogarithm is represented by the pro-system L_n^D:

$$([L_n^D])_{n\geq 0} = \mathrm{pol}_{D,\mathrm{dR}} \in \varprojlim_n H^1_{\mathrm{dR}}\left(U_D, \mathrm{Log}^n_{\mathrm{dR}}\right).$$

Chapter 6: The syntomic realization of the elliptic polylogarithm for ordinary elliptic curves

In the last chapter we will give an explicit description of the rigid syntomic realization of the elliptic polylogarithm over the ordinary locus of the modular curve. This can be seen as a generalization of the description of the syntomic Eisenstein classes in [BK10a] and the description of the syntomic polylogarithm for CM elliptic curves in [BKT10].

Let us consider a morphism of syntomic data $\mathscr{E} \to \mathscr{S}$ - in the sense of Bannai [Ban00] - underlying an elliptic curve. As in Bannai–Kings and Bannai–Kobayashi–Tsuji, the de Rham realization $([L_n^D])_{n\geq 0}$ determines the syntomic realization uniquely. More precisely, the differential equation

$$\nabla_{\mathrm{Log}^n_{\mathrm{dR}}}(\rho_n) = (1-\phi)(L_n^D)$$

characterizes a unique system ρ_n of overconvergent sections of the syntomic logarithm sheaves. This system describes the syntomic polylogarithm class. In the above differential equation, ϕ is obtained via the Frobenius structure of the syntomic logarithm. If we restrict the universal elliptic curve to the ordinary locus of the modular curve, we can describe the overconvergent sections ρ_n more explicitly. In tubular neighbourhoods of torsion sections the overconvergent sections ρ_n are given by moment functions of the p-adic Eisenstein–Kronecker measure constructed in chapter 4. The proof heavily exploits the fact that both the de Rham realization L_n^D and the p-adic Eisenstein–Kronecker measure are constructed via the same object: namely, the canonical section of the Poincaré bundle.

Notation and Conventions

All schemes are assumed to be separated and locally Noetherian. For a group scheme G over a basis S we will usually write $\pi : G \to S$ for the structure map and $e : S \to G$ for the unit section. If G is Abelian, the multiplication by N morphism will be denoted by $[N]$. Whenever we are working over a fixed base scheme S, morphisms are supposed to be S-morphisms. If we are working over a fixed base scheme, products are taken in the category of S-schemes.

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1 Eisenstein series via the Poincaré bundle

In [Kat76] Katz constructed algebraic sections in symmetric powers of the first de Rham cohomology of the universal elliptic curve representing real-analytic Eisenstein series. His construction involves the Gauss–Manin connection and the Kodaira–Spencer isomorphism on the universal elliptic curve. In [BK10b] Bannai and Kobayashi observed that the *Kronecker theta function* and certain translates of it are generating series for real-analytic Eisenstein series. Motivated by this work, we will give an algebraic construction of real-analytic Eisenstein series for arbitrary families of elliptic curves via the Poincaré bundle in this chapter. To be more precise, we will construct in a functorial way a *canonical section* of the Poincaré bundle for elliptic curves over arbitrary base schemes. Passing to the universal vectorial extensions of both, the elliptic curve and its dual, gives us two integrable connections on the Poincaré bundle. By applying them iteratively and evaluating at torsion points allows us to construct the desired sections representing real-analytic Eisenstein series. On the way we will prove a distribution relation for the canonical section. Furthermore, the symmetry of the geometric situation immediately yields the symmetry of the real-analytic Eisenstein series predicted by the functional equation. As far as possible we try to make all constructions independent of unnecessary choices like the choice of an autoduality isomorphism between E and its dual.

1.1 The Poincaré bundle

Let E/S be an elliptic curve. Let us recall the definition of the Poincaré bundle and thereby fix some notation. A *rigidification* on a line bundle $\mathcal{L}$ on E is an isomorphism

$$r : e^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_S.$$

A morphism of rigidified line bundles is a morphism of line bundles respecting the rigidification. The dual elliptic curve $E^\vee$ represents the connected component of the functor

$$T \mapsto \mathrm{Pic}(E_T/T) := \{\text{iso. classes of rigidified line bundles } (\mathcal{L}, r) \text{ on } E_T/T\}$$

on the category of S-schemes. The dual elliptic curve is again an elliptic curve. Since a rigidified line bundle has no non-trivial automorphisms, an isomorphism class of a rigidified line bundle determines the line bundle up to unique isomorphism. This implies the existence of a universal rigidified line bundle $(\mathcal{P}, r_0)$ on $E \times_S E^\vee$ with the following universal property: For any rigidified line bundle of degree zero $(\mathcal{L}, r)$ on E_T/T there is a unique morphism

$$f : T \to E^\vee$$

such that $(\mathrm{id}_E \times f)^*(\mathcal{P}, r_0) \xrightarrow{\sim} (\mathcal{L}, r)$. In particular, we obtain for any isogeny

$$\varphi : E \to E'$$

the *dual isogeny* as the morphism $\varphi^\vee : (E')^\vee \to E^\vee$ classifying the rigidified line bundle $(\varphi \times \mathrm{id})^*\mathcal{P}'$ obtained as the pullback of the Poincaré bundle $\mathcal{P}'$ on $E' \times_S (E')^\vee$. By the universal property of the Poincaré bundle, we get a unique isomorphism of rigidified line bundles

$$\gamma_{\mathrm{id},\varphi^\vee} : (\mathrm{id}_E \times \varphi^\vee)^*\mathcal{P} \xrightarrow{\sim} (\varphi \times \mathrm{id}_{(E')^\vee})^*\mathcal{P}'.$$

Of particular interest for us is the case $\varphi = [N]$. In this case the dual $[N]^\vee$ is just the N-multiplication $[N]$ on $E^\vee$. Let us simplify notation and write $\gamma_{1,N}$ instead of $\gamma_{\mathrm{id},[N]}$. The inverse of $\gamma_{1,N}$ will be denoted by

$$\gamma_{N,1} : ([N] \times \mathrm{id})^*\mathcal{P} \xrightarrow{\sim} (\mathrm{id} \times [N])^*\mathcal{P}.$$

For $N, D \geq 1$ define

$$\gamma_{N,D} : ([N] \times [D])^*\mathcal{P} \xrightarrow{\sim} ([D] \times [N])^*\mathcal{P}$$

as the composition in the following commutative diagram

$$\begin{array}{ccc} ([N] \times [D])^*\mathcal{P} & \xrightarrow{([N]\times\mathrm{id})^*\gamma_{1,D}} & ([ND] \times \mathrm{id})^*\mathcal{P} \\ {\scriptstyle (\mathrm{id}\times[D])^*\gamma_{N,1}}\big\downarrow & & \big\downarrow{\scriptstyle ([D]\times\mathrm{id})^*\gamma_{N,1}} \\ (id \times [DN])^*\mathcal{P} & \xrightarrow{(\mathrm{id}\times[N])^*\gamma_{1,D}} & ([D] \times [N])^*\mathcal{P}. \end{array} \tag{1.1}$$

Indeed, this diagram is commutative since all maps are isomorphisms of rigidified line bundles. Furthermore, rigidified line bundles do not have any non-trivial automorphisms, i. e. there can be at most one isomorphism between rigidified line bundles. By the same argument we obtain the following identities.

Lemma 1.1.1. *Let $N, N', D, D' \geq 1$ then*

(a) $([D] \times \mathrm{id})^*\gamma_{1,D'} \circ (\mathrm{id} \times [D'])^*\gamma_{1,D} = \gamma_{1,DD'}$

(b) $(\mathrm{id} \times [N])^*\gamma_{N',1} \circ ([N'] \times \mathrm{id})^*\gamma_{N,1} = \gamma_{NN',1}$

(c) $([D] \times [N])^*\gamma_{N',D'} \circ ([N'] \times [D'])^*\gamma_{N,D} = \gamma_{NN',DD'}$

For a section $s \in E(S)$ let us write $T_s \colon E \to E$ for the translation morphism. We can now define the following translation operators for the Poincaré bundle.

Definition 1.1.2. For $N, D \geq 1$, $s \in E[N](S)$, $t \in E^\vee[D](S)$ we define an isomorphism

$$\mathcal{U}_{s,t}^{N,D} : (T_s \times T_t)^*([D] \times [N])^*\mathcal{P} \longrightarrow ([D] \times [N])^*\mathcal{P}.$$

of $\mathcal{O}_{E\times_S E^\vee}$-modules via

$$\mathcal{U}_{s,t}^{N,D} := \gamma_{N,D} \circ (T_s \times T_t)^* \gamma_{D,N}.$$

In the most important case $N = 1$ we will simply write $\mathcal{U}_t^D := \mathcal{U}_{e,t}^{1,D}$.

For a given torsion point $t \in E[D](T)$ for some S-scheme T let us write Nt instead of $[N](t)$. We have the following behaviour under composition.

Corollary 1.1.3. *Let $D, D', N, N' \geq 1$ be integers. For $s \in E[N](S)$, $s' \in E[N'](S)$ and $t \in E^\vee[D](S)$, $t' \in E^\vee[D'](S)$ we get:*

$$\left(([D]\times[N])^*\mathcal{U}_{Ds',Nt'}^{N',D'}\right) \circ (T_{s'} \times T_{t'})^*([D']\times[N'])^*\mathcal{U}_{D's,N't}^{N,D} = \mathcal{U}_{s+s',t+t'}^{NN',DD'}$$

Proof. The general case decomposes into the following special cases

$$\left(([D]\times \mathrm{id})^*\mathcal{U}_{e,t'}^{1,D'}\right) \circ (\mathrm{id}\times T_{t'})^*([D']\times \mathrm{id})^*\mathcal{U}_{e,t}^{1,D} = \mathcal{U}_{e,t+t'}^{1,DD'} \tag{1.2}$$

$$\left((\mathrm{id}\times[N])^*\mathcal{U}_{s',e}^{N',1}\right) \circ (T_{s'}\times \mathrm{id})^*(\mathrm{id}\times[N'])^*\mathcal{U}_{s,e}^{N,1} = \mathcal{U}_{s+s',e}^{NN',1} \tag{1.3}$$

and

$$\left(([D]\times \mathrm{id})^*\mathcal{U}_{Ds',e}^{N',1}\right) \circ (T_{s'}\times \mathrm{id})^*(\mathrm{id}\times[N'])^*\mathcal{U}_{e,N't}^{1,D} = \mathcal{U}_{s',t}^{N',D} \tag{1.4}$$

$$\left((\mathrm{id}\times[N])^*\mathcal{U}_{e,Nt'}^{1,D'}\right) \circ (\mathrm{id}\times T_{t'})^*([D']\times \mathrm{id})^*\mathcal{U}_{D's,e}^{N,1} = \mathcal{U}_{s,t'}^{N,D'}. \tag{1.5}$$

Using Lemma 1.1.1(a), we will prove (1.2) and (1.4). The other cases are completely analogous. We have

$$\begin{aligned}
&\left(([D]\times \mathrm{id})^*\mathcal{U}_{e,t'}^{1,D'}\right) \circ (\mathrm{id}\times T_{t'})^*([D']\times \mathrm{id})^*\mathcal{U}_{e,t}^{1,D} \overset{\text{Def.}}{=} \\
=&([D]\times \mathrm{id})^*\gamma_{1,D'} \circ ([D]\times T_{t'})^*\gamma_{D',1} \circ ([D']\times T_{t'})^*\gamma_{1,D} \circ ([D']\times T_{t'})^*(\mathrm{id}\times T_t)^*\gamma_{D,1} \overset{(1.1)}{=} \\
=&([D]\times \mathrm{id})^*\gamma_{1,D'} \circ (\mathrm{id}\times T_{t'})^* \left((\mathrm{id}\times[D'])^*\gamma_{1,D} \circ (\mathrm{id}\times[D])^*\gamma_{D,1}\right) \circ ([D']\times T_{t+t'})^*\gamma_{D,1} = \\
=&([D]\times \mathrm{id})^*\gamma_{1,D'} \circ (\mathrm{id}\times[D'])^*\gamma_{1,D} \circ (\mathrm{id}\times T_{t'})^*(\mathrm{id}\times[D])^*\gamma_{D,1} \circ ([D']\times T_{t+t'})^*\gamma_{D,1} \overset{\text{Lem.}}{=} \\
=&\gamma_{1,DD'} \circ (\mathrm{id}\times T_{t+t'})^*\gamma_{DD',1} = \mathcal{U}_{e,t+t'}^{1,DD'}
\end{aligned}$$

and

$$\begin{aligned}
&\left(([D]\times \mathrm{id})^*\mathcal{U}_{Ds',e}^{N',1}\right) \circ \left((T_{s'}\times[N'])^*\mathcal{U}_{e,N't}^{1,D}\right) = \\
=&([D]\times \mathrm{id})^* \left(\gamma_{N',1} \circ (T_{Ds'}\times \mathrm{id})^*\gamma_{1,N'}\right) \circ (T_{s'}\times[N'])^* \left(\gamma_{1,D} \circ (\mathrm{id}\times T_{N't})^*\gamma_{D,1}\right) \overset{\text{Lem.}}{=} \\
=&([D]\times \mathrm{id})^*\gamma_{N',1} \circ ([N']\times \mathrm{id})^*\gamma_{1,D} \circ (T_{s'}\times[D])^*\gamma_{1,N'} \circ (T_{s'}\times T_t)^*(\mathrm{id}\times[N'])^*\gamma_{D,1} = \\
=&\gamma_{N',D} \circ (T_{s'}\times T_t)^*\gamma_{D,N'} = \\
=&\mathcal{U}_{s',t}^{N',D}.
\end{aligned}$$

$\square$

1.2 The canonical section of the Poincaré bundle

In this section we will state the characterizing property of the canonical section. The characterizing property will not depend on a chosen autoduality isomorphism. The proof of existence of the canonical section will be the content of the next section. There we will choose an autoduality isomorphism and construct the canonical section quite explicitly. The advantage of this approach is that we have both an intrinsic characterization independent of any choices and an explicit description useful for computations.

Let E/S be an elliptic curve over a separated locally Noetherian base scheme S. Let us fix once and for all a rigidified line bundle $(\mathcal{P}, r_0)$ satisfying the universal property of the Poincaré bundle. The line bundle $(\mathrm{id} \times e)^*(\mathcal{P}, r_0)$ is a trivial line bundle on E together with a rigidification. By rigidity there is a unique isomorphism

$$s_0 : (\mathrm{id} \times e)^*(\mathcal{P}, r_0) \xrightarrow{\sim} (\mathcal{O}_E, \mathrm{can})$$

of rigidified line bundles. The triple $(\mathcal{P}, r_0, s_0)$ will be called the birigidified Poincaré bundle. Let

$$\Omega^1_{E\times_S E^\vee/E^\vee}\left([E\times e]+[e\times E^\vee]\right) := \mathcal{O}_{E\times E^\vee}\left([E\times e]+[e\times E^\vee]\right)\otimes_{\mathcal{O}_{E\times_S E^\vee}} \Omega^1_{E\times_S E^\vee/E^\vee}$$

be the sheaf of relative Kähler differentials on $E \times_S E^\vee$ tensorized with the line bundle $\mathcal{O}_{E\times E^\vee}\left([E\times e]+[e\times E^\vee]\right)$. In this section we will define the *canonical section* of the Poincaré bundle

$$s_{\mathrm{can}} \in \Gamma\left(E\times_S E^\vee, \mathcal{P}\otimes\Omega^1_{E\times_S E^\vee/E^\vee}\left([E\times e]+[e\times E^\vee]\right)\right).$$

1.2.1 The canonical 1-forms

Let us first recall the residue map for Kähler differentials. Let E/S be an elliptic curve and $D > 1$ invertible on S. Then, the group scheme $E[D]$ is a finite étale group scheme over S, i.e. there is a finite étale morphism $f : T \to S$ with $E_T[D] \cong (\mathbb{Z}/D\mathbb{Z})^2_T$. We have a Cartesian diagram

$$\begin{array}{ccc} E_T & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}$$

and the canonical map $\tilde{f}^*\Omega^1_{E/S} \to \Omega^1_{E_T/T}$ is an isomorphism. Let us write

$$i_{E[D]} : E[D] \hookrightarrow E$$

for the closed immersion. The residue map

$$\mathrm{Res} : \Omega^1_{E/S}(\log E[D]) \to (i_{E[D]})_*\mathcal{O}_{E[D]}$$

can be described explicitly after the finite étale base change to T as follows. Using the decomposition

$$(i_{E_T[D]})_*\mathcal{O}_{E_T[D]} \cong \bigoplus_{t\in E_T[D](T)} t_*\mathcal{O}_S, \tag{1.6}$$

let us write Res_t, $t \in E_T[D](T)$ for the composition of Res with the projection to the t component in (1.6). If $t \in E_T[D](T)$ is locally cut out by the equation $X_t = 0$, then $\mathrm{Res}_t(\frac{dX_t}{X_t}) = 1$. Let us finally remark that the universal property of log-differentials gives a canonical map

$$\Omega^1_{E/S}(\log E[D]) \to \Omega^1_{E/S}(E[D]) := \Omega^1_{E/S} \otimes_{\mathcal{O}_E} \mathcal{O}_E(E[D]) \tag{1.7}$$

which is easily checked to be an isomorphism. Indeed, (1.7) restricted to $\Omega^1_{E/S}$ is the identity and if $t \in E[D]$ is locally cut out by $X_t = 0$ then (1.7) maps $\frac{dX_t}{X_t}$ to $dX_t \otimes 1/X_t$. Since $\Omega^1_{E/S}(\log E[D])$ can be étale locally described as the smallest subsheaf of $(j_{U_D})_*\Omega^1_{U_D/T}$ generated by $\Omega^1_{E_T/T}$ and $\frac{dX_t}{X_t}$ for $t \in E_T[D](T)$, while $\mathcal{O}_E(E[D])$ is locally near $t \in E_T[D](S)$ generated as fractional ideal by $1/X_t$, we conclude that (1.7) is an isomorphism. For every $N > 1$ define the Zariski open subset $S[\frac{1}{N}] := S \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[\frac{1}{N}]$ of S and $E_{S[\frac{1}{N}]}$ as the base change of E along $S[\frac{1}{N}] \to S$. Before we state the characterizing property of the canonical section, we will need the following result. It is motivated by the characterization of the logarithmic derivative of the Kato–Siegel functions. Indeed, we will see later that we obtain the logarithmic derivatives of the Kato–Siegel functions as a special case.

Proposition 1.2.1. *Let E/S be an elliptic curve, $D > 1$ invertible on S. For every $e \neq t \in E^\vee[D](S)$ there exists a unique element*

$$\omega^D_t \in \Gamma(E, \Omega^1_{E/S}(E[D]))$$

satisfying the following properties:

(a) *For all finite étale $f : T \to S$ with $|E_T[D](T)| = D^2$ and all $\tilde{t} \in E_T[D](T)$ we have*

$$\mathrm{Res}_{\tilde{t}}(\omega^D_t) = \langle \tilde{t}, t\rangle_D.$$

Here, $\langle\cdot,\cdot\rangle_D$ denotes the canonical pairing

$$E[D] \times_S E^\vee[D] \to \mu_{D,S}$$

as in [Oda69, Section 1].

(b) *For all $N > 1$ coprime to D we have the following trace compatibility*

$$\mathrm{Tr}_{[N]}\left(\omega^D_{Nt}\Big|_{E_{S[\frac{1}{N}]}}\right) = \omega^D_t\Big|_{E_{S[\frac{1}{N}]}}$$

where $\mathrm{Tr}_{[N]} : \Omega^1_{E/S} \to \Omega^1_{E/S}$ is the trace map induced by $[N] : E \to E$.

We call ω_t^D the canonical 1-form associated with the torsion section t. If we want to fix the dependence of ω_t^D on the elliptic curve E in the notation, we will write $\omega_{t,E}^D$.

Proof. We only prove uniqueness here. The explicit construction of ω_T^D is contained in the next section. For the proof of uniqueness let ω_t^D and $\tilde{\omega}_t^D$ be two sections satisfying the conditions of the statement. By (a) the difference $\omega_0 := \omega_t^D - \tilde{\omega}_t^D$ is contained in $\Gamma(E, \ker(\mathrm{Res})) = \Gamma(E, \Omega^1_{E/S})$. Choose $a \geq 1$ with $N^a \equiv 1 \mod D$. Then, by (b) we have

$$\mathrm{Tr}_{[N^a]}\left(\omega_t^D\Big|_{E_{S[\frac{1}{N}]}}\right) = \omega_t^D\Big|_{E_{S[\frac{1}{N}]}}$$

and similarly for $\tilde{\omega}_t^D$. But the trace $\mathrm{Tr}_{[N]}$ acts by multiplication by N on the global sections $\Gamma(E_{S[\frac{1}{N}]}, \Omega^1_{E_{S[\frac{1}{N}]}/S[\frac{1}{N}]})$. This gives

$$N^a \cdot \left(\omega_0|_{E_{S[\frac{1}{N}]}}\right) = \mathrm{Tr}_{[N]}^{\circ a}(\omega_0|_{E_{S[\frac{1}{N}]}}) = \mathrm{Tr}_{[N]}^{\circ a}(\omega_t^D\Big|_{E_{S[\frac{1}{N}]}} - \tilde{\omega}_t^D\Big|_{E_{S[\frac{1}{N}]}}) = \omega_0|_{E_{S[\frac{1}{N}]}}$$

and we conclude $\omega_0|_{E_{S[\frac{1}{N\cdot(N^a-1)}]}} = 0$. Now, the fact that $\left(E_{S[\frac{1}{N\cdot(N^a-1)}]}\right)_{N,a\geq 1, N^a\equiv 1 \mod D}$ is a Zariski covering of E implies $\omega_0 = 0$. □

We would like to emphasize that the definition of ω_t^D is intrinsic in the sense that it does not make use of any autoduality. Furthermore, note that the characterizing property of ω_t shows that ω_t is compatible with base change along arbitrary maps $T \to S$.

1.2.2 Density of torsion sections

Let us briefly recall the following definition.

Definition 1.2.2. Let X be a scheme and $(f_\lambda : Z_\lambda \to X)_{\lambda\in\Lambda}$ be a family of morphisms of schemes.

(a) The family $(f_\lambda)_{\lambda\in\Lambda}$ is called schematically dominant if for any $U \subseteq X$ and any $s \in \Gamma(U, \mathcal{O}_X)$
$$f_\lambda^*(s) = 0, \quad \forall\, \lambda \in \Lambda$$
implies $s = 0$. Here, $f_\lambda^* : \Gamma(U, \mathcal{O}_X) \to \Gamma(f_\lambda^{-1}(U), \mathcal{O}_{Z_\lambda})$ denotes the pullback map on sections.

(b) For an S-scheme X the family $(f_\lambda)_{\lambda\in\Lambda}$ is called universally schematically dominant relative S if the family $(f'_\lambda)_\lambda$
$$f'_\lambda : Z_\lambda \times_S S' \to X \times_S S'$$
obtained by base change is schematically dense for all S-schemes S'.

Let us recall that we have assumed our base scheme to be separated and locally Noetherian.

Proposition 1.2.3. *(density of torsion sections) Let $N > 1$ and E/S be an elliptic curve with N invertible on S. The family of N^n-torsion points*

$$\{t \in E[N^n](T) : n \geq 0,\ T \to S \text{ finite étale}\}$$

in the category of S-schemes is universally schematically dominant for E relative S.

Proof. By [Gro66, Thm. 11.10.9] we are reduced to prove the result in the case $S = \operatorname{Spec} k$ for a field k. In this case the result is well-known, cf. [EGM12, (5.30) Thm, and the remark (2) afterwards]. □

Remark 1.2.4. The restriction to étale morphisms $T \to S$ in the above family is just for making the indexing family a set.

Often we will apply the following reformulation.

Corollary 1.2.5. *Let $N > 1$ and E/S be an elliptic curve with N invertible on S. For $\mathcal{F}$ a locally free $\mathcal{O}_E$-module of finite rank, $U \subseteq E$ open and $s \in \Gamma(U, \mathcal{F})$ we have: The section s is zero, if and only if*

$$t^* s = 0$$

for all $T \to S$ finite étale, $n \geq 0$ and $t \in E[N^n](T)$.

Proof. By the sheaf property we may prove this locally and reduce to the case $\mathcal{F} = \mathcal{O}_E^r$, $r \geq 0$. In this case the corollary follows immediately from the above proposition. □

Remark 1.2.6. If we take all torsion points different from zero, we still get a universally schematically dense family. Indeed, a priori the family is then only universally schematically dense in the open subscheme $U = E \setminus \{e(S)\}$, but the inclusion $U \hookrightarrow E$ is also universally schematically dense, since it is the complement of a divisor [GW10, cf. the remark after Lemma 11.33].

1.2.3 Characterizing property of the canonical section

In order to define the canonical section let us come back to the translation operators $\mathcal{U}_{s,t}^{N,D}$ for $s \in E[N](S), t \in E[D](T)$ defined in the previous section.

Definition 1.2.7. For $U \subseteq E \times_S E^\vee$ and $f \in \Gamma\left(U, \mathcal{P} \otimes \Omega^1_{E \times_S E^\vee / E^\vee}([E \times e] + [e \times E^\vee])\right)$ define

$$U_{s,t}^{N,D}(f) := \left(\mathcal{U}_{s,t}^{N,D} \otimes \mathrm{id}_{\Omega^1}\right)((T_s \times T_t)^*([D] \times [N])^* f)\,.$$

Then, $U_{s,t}^{N,D}(f)$ is a section over the open set $(T_s\times T_t)^{-1}([D]\times[N])^{-1}(U)$ in the sheaf

$$\begin{aligned}&([D]\times[N])^*\left(\mathcal{P}\otimes(T_{Ds}\times T_{Nt})^*\Omega^1_{E\times_S E^\vee/E^\vee}\left([E\times e]+[e\times E^\vee]\right)\right)\cong\\ &\cong([D]\times[N])^*\left(\mathcal{P}\otimes\Omega^1_{E\times_S E^\vee/E^\vee}\left([E\times(-Nt)]+[(-Ds)\times E^\vee]\right)\right)\end{aligned}$$

As above we write $U_t^D(f):=U_{e,t}^{1,D}(f)$ in the case $N=1$.

For E/S, $e\neq t\in E^\vee[D](S)$ and $f\in\Gamma(E\times_S E^\vee,\mathcal{P}\otimes\Omega^1_{E\times_S E^\vee/E^\vee}([E\times e]+[e\times E^\vee]))$ we have

$$U_t^D(f)\in\Gamma\left(E\times_S E^\vee,([D]\times\mathrm{id})^*\left(\mathcal{P}\otimes\Omega^1_{E\times_S E^\vee/E^\vee}\left([E\times(-t)]+[e\times E^\vee]\right)\right)\right).$$

Now, we compute

$$\begin{aligned}&(\mathrm{id}\times e)^*([D]\times\mathrm{id})^*\left(\mathcal{P}\otimes_{\mathcal{O}_{E\times_S E^\vee}}\Omega^1_{E\times_S E^\vee/E^\vee}\left([E\times(-t)]+[e\times E^\vee]\right)\right)=\\ =&[D]^*\left((\mathrm{id}\times e)^*\mathcal{P}\otimes_{\mathcal{O}_E}(\mathrm{id}\times e)^*\Omega^1_{E\times_S E^\vee/E^\vee}\left([E\times(-t)]+[e\times E^\vee]\right)\right)\overset{*}{\cong}\\ \cong&[D]^*\left(\Omega^1_{E/S}\left([e]\right)\right)\cong\\ \cong&\Omega^1_{E/S}(E[D])\end{aligned}\tag{1.8}$$

where we have used the rigidification of the Poincaré bundle as well as $e\neq t$ in $(*)$. The above identification allows us to view $(\mathrm{id}\times e)^*\left(U_t^D(f)\right)$ as a global section of $\Omega^1_{E/S}(E[D])$. We will implicitly use this identification in the following. The canonical section of the Poincaré bundle is characterized by the canonical 1-forms in the following precise sense.

Theorem 1.2.8. *Let E/S be an elliptic curve. There exists a section*

$$s_{\mathrm{can}}\in\Gamma(E\times_S E^\vee,\mathcal{P}\otimes_{\mathcal{O}_{E\times_S E^\vee}}\Omega^1_{E\times_S E^\vee/E^\vee}([E\times e]+[e\times E^\vee])$$

which is compatible with pullback and uniquely characterized by the following property: For every $D>1$ the section s_{can} satisfies the following condition

$(*)_D$ *For every pair (T,t) consisting of a finite étale S-scheme T with $D\in\mathcal{O}_T^\times$ and $t\in E^\vee[D](T)$ we have the following equality in $\Gamma\left(E_T,\Omega^1_{E_T/T}(E_T[D])\right)$*

$$\omega^D_{t,E_T}=(\mathrm{id}_{E_T}\times e)^*U_t^D(s_{can,E_T}).\tag{1.9}$$

Here, we write s_{can,E_T} for the pullback of s_{can} to E_T/T.

The section s_{can} will be called canonical section *of the Poincaré bundle.*

Proof. Again, we just prove uniqueness and give an explicit construction in the next section. Let us first assume that $N>1$ is invertible on S, i.e. $S[\frac{1}{N}]=S$. We claim that already the condition $(*)_{N^n}$ for all $n\geq 1$ uniquely determines s_{can}. This follows by a density of torsion sections argument as follows. Assume we have two candidates $s,\tilde{s}$

both satisfying condition $(*)_{N^n}$ for all $n \geq 1$. I. e. for all $n \geq 1$ and all pairs (T, t) as in $(*)_{N^n}$ the equation

$$(\mathrm{id}_{E_T} \times e)^* U_t^{N^n}(s) = (\mathrm{id}_{E_T} \times e)^* U_t^{N^n}(\tilde{s})$$

holds. Using the definition of $U_t^{N^n}$, we can restate this as

$$\begin{aligned}&(\mathrm{id}_{E_T} \times e)^*(\gamma_{1,N^n} \otimes \mathrm{id}_{\Omega^1})\left((\mathrm{id} \times T_t)^*\left((\gamma_{N^n,1} \otimes \mathrm{id}_{\Omega^1})\left(([N^n] \times \mathrm{id})^*(\tilde{s})\right)\right)\right) = \\ =&(\mathrm{id}_{E_T} \times e)^*(\gamma_{1,N^n} \otimes \mathrm{id}_{\Omega^1})\left((\mathrm{id} \times T_t)^*\left((\gamma_{N^n,1} \otimes \mathrm{id}_{\Omega^1})\left(([N^n] \times \mathrm{id})^*(s)\right)\right)\right)\end{aligned}$$

and using $(\mathrm{id}_{E_T} \times e)^* \gamma_{1,N^n} = \mathrm{id}_{\mathcal{O}_E}$ we get

$$\begin{aligned}&(\mathrm{id} \times t)^*\left((\gamma_{N^n,1} \otimes \mathrm{id}_{\Omega^1})\left(([N^n] \times \mathrm{id})^*(\tilde{s})\right)\right) = \\ =&(\mathrm{id} \times t)^*\left((\gamma_{N^n,1} \otimes \mathrm{id}_{\Omega^1})\left(([N^n] \times \mathrm{id})^*(s)\right)\right).\end{aligned}$$

Using that $\gamma_{N^n,1}$ is an isomorphism and pullback of sections along $[N^n]$ is injective, we get for all $n \geq 0$ and all pairs (T, t) as in $(*)_{N^n}$ the equality

$$(\mathrm{id} \times t)^* \tilde{s} = (\mathrm{id} \times t)^* s.$$

Now, we conclude $s = \tilde{s}$ by density of torsion sections for $E \times_S E^\vee$ relative E.

For general base scheme S assume that $s, \tilde{s}$ satisfy the condition $(*)_{D,E}$ for all $D > 1$. By the above argument we conclude

$$s|_{E_{S[\frac{1}{N}]}} = \tilde{s}|_{E_{S[\frac{1}{N}]}}.$$

Since $(E_{S[\frac{1}{N}]})_{N \geq 1}$ is a Zariski covering, we conclude $s = \tilde{s}$. □

1.3 Explicit construction of the canonical section of the Poincaré bundle

In this section we will construct the canonical section explicitly and thereby prove its existence. Let us choose the following autoduality isomorphism:

$$\begin{aligned}\lambda : E &\longrightarrow \underline{\mathrm{Pic}}^0_{\mathrm{E/S}} =: E^\vee \\ P &\longmapsto (\mathcal{O}_E([-P] - [e]) \otimes_{\mathcal{O}_E} \pi^* e^* \mathcal{O}_E([-P] - [e])^{-1}, \mathrm{can})\end{aligned} \tag{1.10}$$

Here, can is the canonical rigidification given by the canonical isomorphism

$$e^* \mathcal{O}_E([-P] - [e]) \otimes_{\mathcal{O}_S} e^* \mathcal{O}_E([-P] - [e])^{-1} \xrightarrow{\sim} \mathcal{O}_S.$$

With this choice we can describe the pullback of the Poincaré bundle $\mathcal{P}_\lambda := (\mathrm{id} \times \lambda)^* \mathcal{P}$ as follows

$$\begin{aligned}(\mathcal{P}_\lambda, r_0, s_0) :&= \left(\mathcal{O}_{E \times E}(-[e \times E] - [E \times e] + \Delta) \otimes_{\mathcal{O}_{E \times E}} \pi^*_{E \times E} \underline{\omega}^{\otimes -1}_{E/S}, r_0, s_0\right) = \\ &= \left(\mathrm{pr}_1^* \mathcal{O}_E([e])^{\otimes -1} \otimes \mathrm{pr}_2^* \mathcal{O}_E([e])^{\otimes -1} \otimes \mu^* \mathcal{O}_E([e]) \otimes \pi^*_{E \times E} \underline{\omega}^{\otimes -1}_{E/S}, r_0, s_0\right).\end{aligned}$$

Here, $\Delta = \ker(\mu : E \times E \to E)$ is the anti-diagonal and r_0, s_0 are the ridifications induced by the canonical isomorphism

$$e^*\mathcal{O}_E(-[e]) \xrightarrow{\sim} \underline{\omega}_{E/S}.$$

This description of the Poincaré bundle gives the following isomorphisms of locally free $\mathcal{O}_{E\times E}$-modules, i. e. all tensor products over $\mathcal{O}_{E\times E}$:

$$\begin{aligned}
\mathcal{P}_\lambda \otimes \mathcal{P}_\lambda^{\otimes -1} &= \mathcal{P}_\lambda \otimes \left(\mathcal{O}_{E\times E}(-[e\times E] - [E\times e] + \Delta) \otimes \pi_{E\times E}^*\omega_{E/S}^{\otimes -1}\right)^{\otimes -1} \\
&\cong \mathcal{P}_\lambda \otimes \mathcal{O}_{E\times E}([e\times E] + [E\times e]) \otimes \left(\pi_{E\times E}^*\omega_{E/S}\right) \otimes \mathcal{O}_{E\times E}(-\Delta) \\
&\cong \mathcal{P}_\lambda \otimes \mathcal{O}_{E\times E}([e\times E] + [E\times e]) \otimes \left(\mathrm{pr}_1^*\Omega^1_{E/S}\right) \otimes \mathcal{O}_{E\times E}(-\Delta) \\
&\cong \mathcal{P}_\lambda \otimes \mathcal{O}_{E\times E}([e\times E] + [E\times e]) \otimes \Omega^1_{E\times E/E} \otimes \mathcal{O}_{E\times E}(-\Delta) \\
&\cong \mathcal{P}_\lambda \otimes \Omega^1_{E\times E/E}([e\times E] + [E\times e]) \otimes \mathcal{O}_{E\times E}(-\Delta)
\end{aligned} \tag{1.11}$$

The line bundle $\mathcal{O}_{E\times E}(-\Delta)$ can be identified with the ideal sheaf $\mathcal{J}_\Delta$ of the anti-diagonal Δ in $E\times_S E$ in a canonical way. If we combine the inclusion

$$\mathcal{O}_{E\times E}(-\Delta) \cong \mathcal{J}_\Delta \hookrightarrow \mathcal{O}_{E\times E}$$

with (1.11), we get a morphism of $\mathcal{O}_{E\times E}$-modules

$$\mathcal{P}_\lambda \otimes \mathcal{P}_\lambda^{\otimes -1} \hookrightarrow \mathcal{P}_\lambda \otimes \Omega^1_{E\times E/E}([e\times E] + [E\times e]). \tag{1.12}$$

Definition 1.3.1. Let

$$s_{\mathrm{can}}^\lambda \in \Gamma\left(E\times_S E, \mathcal{P}_\lambda \otimes_{\mathcal{O}_{E\times E}} \Omega^1_{E\times E/E}([e\times E] + [E\times e])\right)$$

be the image of the identity element $\mathrm{id}_{\mathcal{P}_\lambda}$ under (1.12).

In the rest of this section we will prove that s_{can}^λ satisfies the characterizing properties of the canonical section under the identification

$$E\times E \xrightarrow{\sim} E\times E^\vee.$$

Remark 1.3.2. One could define $s_{\mathrm{can}} := (\mathrm{id}\times\lambda^{-1})^* s_{\mathrm{can}}^\lambda$. But then it is not immediately clear that this does not depend on the chosen autoduality. Thus, we have preferred to first give an intrinsic characterization and then a non-intrinsic construction.

For given $D > 1$ and $t \in E[D](S)$ let us, by slight abuse of notation, write U_t^D for the pullback of $U_{\lambda(t)}^D$ along the autoduality isomorphism. In particular, we get

$$(\mathrm{id}\times e)^* U_t^D(s_{\mathrm{can}}^\lambda) \in \Gamma\left(E, \Omega^1_{E/S}(E[D])\right)$$

as in (1.8). The following result is a first step towards the existence of ω_t^D. We will construct a section $\omega_t^{D,\lambda}$. Later, we will prove that it satisfies the characterizing property of ω_t^D under the autoduality isomorphism.

Proposition 1.3.3. *Let E/S be an elliptic curve with $D \in \mathcal{O}_S^\times$ and let $e \neq t \in E[D](S)$. The section*

$$\omega_t^{D,\lambda} := (\mathrm{id} \times e)^* U_t^D(s_{\mathrm{can}}^\lambda)$$

satisfies the following properties:

(a) *For each finite étale S-scheme T with $|E[D](T)| = D^2$ we have*

$$\mathrm{Res}_{\tilde{t}}\, \omega_t^{D,\lambda} = \langle \tilde{t}, \lambda(t) \rangle$$

for all $\tilde{t} \in E[D](T)$.

(b) *The section $\omega_t^{D,\lambda} \in \Gamma\left(E, \Omega^1_{E/S}(E[D])\right)$ is contained in the $\mathcal{O}_E$-submodule*

$$\Omega^1_{E/S}\left([D]^*([e] - [t])\right)$$

of $\Omega^1_{E/S}(E[D])$.

Further, $\omega_t^{D,\lambda}$ is the unique section of $\Omega^1_{E/S}(E[D])$ satisfying (a) and (b).

Proof. For uniqueness let $\tilde{\omega}_1$ and $\tilde{\omega}_2$ both satisfy (a) and (b). By (a) the difference satisfies:

$$\tilde{\omega}_1 - \tilde{\omega}_2 \in \Gamma(E, \ker(\mathrm{Res})) = \Gamma(E, \Omega^1_{E/S})$$

On the other hand, (b) shows that $\tilde{\omega}_1 - \tilde{\omega}_2$ vanishes along the divisor $[D]^*[t]$ and we conclude $\tilde{\omega}_1 - \tilde{\omega}_2 = 0$.

Let us now prove that $\omega_t^{D,\lambda}$ satisfies (b). By its definition s_{can}^λ is contained in the submodule

$$\mathcal{P}_\lambda \otimes \Omega^1_{E\times E/E}([e \times E] + [E \times e]) \otimes \mathcal{O}_{E\times E}(-\Delta)$$

of $\mathcal{P}_\lambda \otimes \Omega^1_{E\times E/E}([e \times E] + [E \times e])$. By the definition of the translation operator the global section $(\mathrm{id} \times e)^* U_t^D(s_{\mathrm{can}}^\lambda)$ of $\Omega^1_{E/S}(E[D])$ is a global section of the $\mathcal{O}_E$-submodule

$$\Omega^1_{E/S}\left([D]^*([e] - [t])\right).$$

This proves (b).

The residue map is compatible with base change. Combining this with the isomorphism

$$f^*\Omega^1_{E/S}(E[D]) \xrightarrow{\sim} \Omega^1_{E_T/T}(E_T[D])$$

for $f : T \to S$ finite étale, allow us to check (a) after finite étale base change. Thus, we may assume that $|E[D](S)| = D^2$. Before we do the residue computation, let us recall the definition of Oda's pairing

$$\langle \cdot, \cdot \rangle : \ker \varphi \times_S \ker \varphi^\vee \to \mathbb{G}_{m,S}$$

for an isogeny $\varphi : E \to E'$. Let $t \in (\ker \varphi)(S)$ and $[\mathcal{L}] \in (\ker \varphi^\vee)(S)$. Since we have assumed $[\mathcal{L}] \in (\ker \varphi^\vee)(S)$, the line bundle $\varphi^*\mathcal{L}$ is trivial and we can choose an isomorphism

$$\alpha : \varphi^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_E.$$

The chosen isomorphism α gives rise to a chain of isomorphisms

$$\mathcal{O}_E \xrightarrow[\sim]{\alpha^{-1}} \varphi^*\mathcal{L} = T_t^*\varphi^*\mathcal{L} \xrightarrow[\sim]{T_t^*\alpha} T_t^*\mathcal{O}_E = \mathcal{O}_E$$

and $\langle t, [\mathcal{L}]\rangle_\varphi$ is defined as the image of 1 under this isomorphism. It is not hard to check that this pairing is independent of the chosen α. In the case of $\varphi = [D]$ we get the pairing

$$\langle\cdot,\cdot\rangle_D : E[D] \times_S E^\vee[D] \to \mu_{D,S}.$$

As remarked above, we may assume $|E[D](S)| = D^2$. Our first aim is to prove

$$T_{t'}^*\omega_t^{D,\lambda} = \langle t', \lambda(t)\rangle_D \cdot \omega_t^{D,\lambda}$$

for every $t' \in E[D](S)$. By our choice of autoduality, $\lambda(t)$ is represented by the isomorphism class $[\mathcal{O}_E([-t] - [e])] = [(\mathrm{id} \times t)^*\mathcal{P}_\lambda]$. We apply Oda's pairing to t' and $[\mathcal{L}]$ with $\mathcal{L} := (\mathrm{id} \times t)^*\mathcal{P}_\lambda$. We have the following canonical choice for α:

$$\alpha : [D]^*\mathcal{L} = ([D] \times t)^*\mathcal{P}_\lambda \xrightarrow{(\mathrm{id}\times t)^*\gamma_{D,1}^\lambda} (\mathrm{id} \times t)^*(\mathrm{id} \times [D])^*\mathcal{P} = (\mathrm{id} \times e)^*\mathcal{P} \cong \mathcal{O}_E.$$

Here, we have written $\gamma_{D,1}^\lambda$ for the isomorphism $([D] \times \mathrm{id})^*\mathcal{P}_\lambda \to (\mathrm{id} \times [D])^*\mathcal{P}_\lambda$ induced from $\gamma_{D,1}$ via autoduality. Note that

$$\begin{aligned}
\omega_t^{D,\lambda} &:= (\mathrm{id} \times e)^* U_t^D(s_{\mathrm{can}}^\lambda) \overset{\mathrm{def}}{=} \\
&= (\mathrm{id} \times e)^* \left(\left[\left(\gamma_{1,D}^\lambda \circ (\mathrm{id} \times T_t)^*\gamma_{D,1}^\lambda\right) \otimes \mathrm{id}_{\Omega^1}\right]((D \times T_t)^*(s_{\mathrm{can}}))\right) = \\
&= \left((\mathrm{id} \times t)^*\gamma_{D,1}^\lambda \otimes \mathrm{id}_{\Omega^1}\right)(([D] \times t)^* s_{\mathrm{can}}^\lambda) = (\alpha \otimes \mathrm{id}_{\Omega^1})(([D] \times t)^* s_{\mathrm{can}}^\lambda).
\end{aligned} \tag{1.13}$$

After tensorizing

$$\begin{tikzcd}
\mathcal{O}_E \arrow[r, "\alpha^{-1}"] \arrow[rr, bend right, "\cdot\langle t',\mathcal{L}\rangle"'] & \varphi^*\mathcal{L} = T_{t'}^*\varphi^*\mathcal{L} \arrow[r, "T_{t'}^*\alpha"] & \mathcal{O}_E
\end{tikzcd}$$

with $\otimes_{\mathcal{O}_E}\Omega^1_{E/S}(E[D])$, we obtain

$$\begin{tikzcd}
\Omega^1_{E/S}(E[D]) \arrow[rr, bend right, "\cdot\langle t',\mathcal{L}\rangle"'] & \varphi^*\mathcal{L} \otimes \Omega^1_{E/S}(E[D]) = T_{t'}^*\varphi^*\mathcal{L} \otimes \Omega^1_{E/S}(E[D]) \arrow[l, "\alpha\otimes\mathrm{id}_\Omega"'] \arrow[r, "T_{t'}^*\alpha\otimes\mathrm{id}_\Omega"] & \Omega^1_{E/S}(E[D])
\end{tikzcd}$$

This diagram together with (1.13) proves

$$\begin{aligned}
T_{t'}^*\omega_t^{D,\lambda} &= T_{t'}^*\Big[(\alpha \otimes \mathrm{id}_{\Omega^1})(([D] \times t)^* s_{\mathrm{can}}^\lambda)\Big] = (T_{t'}^*\alpha \otimes \mathrm{id}_{\Omega^1})\left(([D] \times t)^* s_{\mathrm{can}}^\lambda\right) = \\
&= \langle t',\mathcal{L}\rangle \cdot (\alpha \otimes \mathrm{id}_{\Omega^1})(([D] \times t)^* s_{\mathrm{can}}^\lambda) = \langle t',\mathcal{L}\rangle \cdot \omega_t^{D,\lambda} = \langle t',\lambda(t)\rangle \cdot \omega_t^{D,\lambda}
\end{aligned}$$

as desired.

The equation

$$T_{t'}^*\omega_t^{D,\lambda} = \langle t', \lambda(t)\rangle_D \cdot \omega_t^{D,\lambda}$$

reduces the proof of (a) to the claim

$$\mathrm{Res}_e\, \omega_t^{D,\lambda} = 1$$

which can be checked by an explicit computation in a neighbourhood of the zero section as follows. If we view $\mathcal{O}_E(-[e])$ as fractional ideal, it identifies with the ideal sheaf of the zero section. Thus, we can choose a covering $(U_i)_{i\in I}$ and local generators $f_i \in \Gamma(U_i, \mathcal{O}_E)$ s.t.

$$\mathcal{O}_E(-[e])|_{U_i} = f_i\mathcal{O}_{U_i}.$$

The residue map and the construction of $\omega_t^{D,\lambda}$ are compatible with base change, thus we may check the equality

$$\mathrm{Res}_e\, \omega_t^{D,\lambda} = 1$$

locally on the base. In particular, we may assume that $t(S) \subseteq U_i$ for some i and $e(S) \subseteq U_j$ for some j. Let us rename these open sets as $U_t := U_i$ and similarly $U_e := U_j$. The corresponding local generators of $\mathcal{O}_E(-[e])$ will be called $f_t := f_i$ resp. $f_e := f_j$. Through the canonical isomorphism

$$e^*\mathcal{O}_E(-[e]) \cong \omega_{E/S}$$

the local generator f_e of the ideal sheaf of the zero section gives rise to some generator $\omega_0 := e^*(df_e) \in \Gamma(S, \underline{\omega}_{E/S}) = \Gamma(S, e^*\Omega^1_{E/S}) = \Gamma(E, \Omega^1_{E/S})$. Since df_e generates $\Omega^1_{E/S}$ in some neighbourhood V_0 of $e(S)$, we can write $\omega_0 = g df_e$ for some $g \in \Gamma(V_0, \mathcal{O}_E)$. We have

$$\omega_0 = (e^*g)\cdot(e^*df_e) = e^*g\omega_0, \quad \Longrightarrow\ e^*g = 1 \tag{1.14}$$

by the definition of ω_0. If we denote by $\omega_0^\vee \in \Gamma(S, \omega_{E/S}^\vee) = \Gamma(S, \omega_{E/S}^{\otimes -1})$ the dual basis, we can trivialize the Poincaré bundle

$$\mathcal{P}_\lambda = \mu^*\mathcal{O}_E([e]) \otimes \mathrm{pr}_1^*\mathcal{O}_E(-[e]) \otimes \mathrm{pr}_2^*\mathcal{O}_E(-[e]) \otimes (\pi_{E\times E})^*\underline{\omega}_{E/S}^\vee$$

on $U_{ijk} := pr_1^{-1}U_i \cap pr_2^{-1}U_j \cap \mu^{-1}U_k$ as follows:

$$\mathcal{P}_\lambda|_{U_{ijk}} = (\mathrm{pr}_1^* f_i)\cdot(\mathrm{pr}_2^* f_j)\cdot(\mu^* f_k)^{-1}\cdot\mathcal{O}_{U_{ijk}} \otimes (\pi_{E\times E})^*\omega_0^\vee$$

For simplicity write $f_{ijk} := (\mathrm{pr}_1^* f_i)\cdot(\mathrm{pr}_2^* f_j)\cdot(\mu^* f_k)^{-1} \otimes (\pi_{E\times E})^*\omega_0^\vee$ for the induced local generator of $\mathcal{P}_\lambda$ over U_{ijk}. Following the above notation, let us write

$$U_{eet} := \mathrm{pr}_1^{-1}U_e \cap \mathrm{pr}_2^{-1}U_e \cap \mu^{-1}U_t.$$

The intersection

$$U := ([D]\times\mathrm{id})^{-1}U_{ett} \cap (\mathrm{id}\times[D])^{-1}U_{eee} \subseteq E\times E$$

is non-empty, since $(e \times t)$ factors through U. The element ω_0 in the definition of f_{e**} is chosen in such a way that $(e \times \mathrm{id})^* f_{e**}$ maps to $1 \in \Gamma((e \times \mathrm{id})^{-1} U_{e**}, \mathcal{O}_S)$ under the canonical rigidification

$$(e \times \mathrm{id})^* \mathcal{P}_\lambda = \pi^* e^* \mathcal{O}_E(-[e]) \otimes \pi^* \omega_{E/S}^\vee \xrightarrow{\sim} \mathcal{O}_E.$$

We deduce that the isomorphism

$$\gamma_{D,1}^\lambda : ([D] \times \mathrm{id})^* \mathcal{P}_\lambda \xrightarrow{\sim} (\mathrm{id} \times [D])^* \mathcal{P}_\lambda$$

identifies $([D] \times \mathrm{id})^* f_{ett}$ with $(\mathrm{id} \times [D])^* f_{eee}$ on U. Indeed, this follows from the definition of f_{e**} since $\gamma_{D,1}^\lambda$ is compatible with the rigidifications. The identity section of

$$\mathcal{P}_\lambda \otimes \mathcal{P}_\lambda^\vee$$

was used in order to define s_{can}^λ. Thus, s_{can}^λ is locally on U_{ett} given by

$$\Big((\mathrm{pr}_1^* f_e)^{-1} \cdot (\mathrm{pr}_2^* f_t)^{-1} \cdot (\mu^* f_t) \otimes (\pi_{E \times E})^* \omega_0 \Big) \otimes f_{ett}.$$

Using the explicit local description of $\gamma_{D,1}$ and the canonical splitting, we can compute

$$\omega_t^{D,\lambda}\Big|_V \in \Gamma(V, \Omega^1_{E/S}(E[D]))$$

on $V = (\mathrm{id} \times t)^{-1} U \subseteq E$:

$$\begin{aligned} \omega_t^{D,\lambda}\Big|_V &= ([D] \times t)^* \Big((\mathrm{pr}_1^* f_e)^{-1} \cdot (\mathrm{pr}_2^* f_t)^{-1} \cdot (\mu^* f_t) \otimes (\pi_{E \times E})^* \omega_0 \Big)\Big|_V && (1.15) \\ &= ([D]^* T_t^* f_t) \cdot (t^* f_t)^{-1} \cdot ([D]^* f_e)^{-1} \otimes [D]^* \omega_0 && (1.16) \\ &= ([D]^* T_t^* f_t) \cdot (t^* f_t)^{-1} \cdot ([D]^* f_e)^{-1} \otimes [D]^* (g df_e) \\ &= ([D]^* T_t^* f_t) \cdot (t^* f_t)^{-1} \cdot ([D]^* g) \otimes \frac{d([D]^* f_e)}{[D]^* f_e} \end{aligned}$$

Since U contains $(e \times t)$, the open subset $V = (\mathrm{id} \times t)^{-1} U$ of E is a non-empty neighborhood of e. Since $[D]$ is étale, $[D]^* f_e$ is a local parameter in some neighbourhood of e. This allows us to compute the residue of $\omega_t^{D,\lambda}$ as:

$$\mathrm{Res}_e\, \omega_t^{D,\lambda} = e^* \Big(([D]^* T_t^* f_t) \cdot (t^* f_t)^{-1} \cdot ([D]^* g) \Big) = (t^* f_t) \cdot (t^* f_t)^{-1} \cdot e^* g \overset{(1.14)}{=} 1$$

This finishes the proof of the proposition. □

The following result was obtained during the proof of the above proposition.

Corollary 1.3.4. *Let E/S be an elliptic curve with $|E[D](S)| = D^2$ and D invertible on S. Then we have the following equality for all $t, \tilde{t} \in E[D](S)$ with $t \neq e$:*

$$T_{\tilde{t}}^* \omega_t^{D,\lambda} = \langle \tilde{t}, \lambda(t) \rangle_D \cdot \omega_t^{D,\lambda}$$

The above proposition does not yet prove that $(\mathrm{id} \times \lambda)^* \omega_t^D = \omega_t^{D,\lambda}$. The problem is that the characterizing properties of $\omega_t^{D,\lambda}$ and ω_t^D do not coincide. Our next aim is to prove the trace compatibility for $\omega_t^{D,\lambda}$.

Lemma 1.3.5. *Let E/S be an elliptic curve, $N, D \geq 1$ coprime integers and assume that N and D are invertible on S.*

(a) Assume that $|E[N](S)| = N^2$. Then, for all $s \in E[N](S)$ and $e \neq t \in E[D](S)$:
$$\omega_{s+t}^{N\cdot D,\lambda} = \sum_{s' \in E[N](S)} \langle s', Ds \rangle_N \cdot (T_{-s'})^* \omega_{Nt}^{D,\lambda}$$

(b) For $e \neq s \in E[N](S)$:
$$\omega_s^{N\cdot D,\lambda} = [D]^* \omega_s^{N,\lambda}$$

(c) Assume $|E[N](S)| = N^2$. For $e \neq t \in E[D](S)$:
$$\sum_{s' \in E[N](S)} (T_{s'})^* \omega_{Nt}^{D,\lambda} = [N]^* \omega_t^{D,\lambda}$$

Proof. Since N, D are invertible, there is a finite étale map $f : T \to S$ s.t. $|E_T[D](T)| = D^2$ and $|E_T[N](T)| = N^2$. The canonical map
$$f^* \Omega^1_{E/S}(E[D]) \to \Omega^1_{E_T/T}(E_T[D])$$
is an isomorphism and since the construction of $\omega_t^{D,\lambda}$ is compatible with base change, we may assume during the proof that $|E[D](S)| = D^2$ and $|E[N](S)| = N^2$.

(a): Both sides of the equation in (a) are elements of $\Gamma(E, \Omega^1_{E/S}(E[ND]))$. In a first step we show that the difference of both sides has no residue, i. e.
$$\omega_0 := \left(\omega_{s+t}^{N\cdot D,\lambda} - \sum_{s' \in E[N](S)} \langle s', Ds \rangle_N \cdot (T_{-s'})^* \omega_{Nt}^{D,\lambda} \right) \in \Gamma(E, \ker \mathrm{Res})).$$
Using the characterization of $\omega_*^{*,\lambda}$, allows us to compare the residues of both sides. Immediately from Proposition 1.3.3 (a) we deduce
$$\mathrm{Res}_{\tilde{s}}\, \omega_{s+t}^{N\cdot D,\lambda} = \langle \tilde{s}, s+t \rangle_{N\cdot D} \quad \forall \tilde{s} \in E[ND](S)$$
and again using Proposition 1.3.3:
$$\begin{aligned}
\mathrm{Res}_{\tilde{s}} \left(\sum_{s' \in E[N](S)} \langle s', Ds \rangle_N \cdot (T_{-s'})^* \omega_{Nt}^{D,\lambda} \right) &= \sum_{s' \in E[N](S)} \langle Ds', s \rangle_N \cdot \mathrm{Res}_{\tilde{s}} \left((T_{-s'})^* \omega_{Nt}^{D,\lambda} \right) = \\
&= \sum_{s' \in E[N](S)} \begin{cases} \langle Ds', s \rangle_N \cdot \langle \tilde{s} - s', Nt \rangle_D & \tilde{s} - s' \in E[D](S) \\ 0 & \tilde{s} - s' \notin E[D](S) \end{cases} \\
&= \sum_{s' \in E[N](S)} \begin{cases} \langle Ds', s \rangle_N \cdot \langle N\tilde{s}, t \rangle_D & D\tilde{s} = Ds' \\ 0 & D\tilde{s} \neq Ds' \end{cases} \\
&= \langle D\tilde{s}, s \rangle_N \cdot \langle N\tilde{s}, t \rangle_D = \langle \tilde{s}, s+t \rangle_{N\cdot D}
\end{aligned}$$

This shows $\omega_0 \in \Gamma(E, \Omega^1_{E/S})$. In particular, ω_0 is translation-invariant, i. e.

$$D^2\omega_0 = \sum_{\tilde{t}\in E[D](S)} T^*_{\tilde{t}}\omega_0.$$

On the other hand, we can use the behaviour of $\omega_t^{D,\lambda}$ under translation (cf. Corollary 1.3.4) to compute:

$$\begin{aligned}
D^2\omega_0 &= \sum_{\tilde{t}\in E[D](S)} T^*_{\tilde{t}}\omega_0 = \sum_{\tilde{t}\in E[D](S)} T^*_{\tilde{t}}\left(\omega^{N\cdot D,\lambda}_{s+t} - \sum_{s'\in E[N](S)} \langle s', Ds\rangle_N \cdot (T_{-s'})^*\omega^{D,\lambda}_{Nt}\right) = \\
&= \sum_{\tilde{t}\in E[D](S)} \left(T^*_{\tilde{t}}\omega^{N\cdot D,\lambda}_{s+t} - \sum_{s'\in E[N](S)} \langle s', Ds\rangle_N \cdot (T_{-s'})^* T^*_{\tilde{t}}\omega^{D,\lambda}_{Nt}\right) \overset{\text{Cor.1.3.4}}{=} \\
&= \sum_{\tilde{t}\in E[D](S)} \left(\langle \tilde{t}, t+s\rangle_{ND}\cdot\omega^{N\cdot D,\lambda}_{s+t} - \sum_{s'\in E[N](S)} \langle s', Ds\rangle_N \cdot \langle \tilde{t}, Nt\rangle_D \cdot (T_{-s'})^*\omega^{D,\lambda}_{Nt}\right) = \\
&= \sum_{\tilde{t}\in E[D](S)} \left(\langle \tilde{t}, Nt\rangle_{D}\cdot\omega^{N\cdot D,\lambda}_{s+t} - \sum_{s'\in E[N](S)} \langle s', Ds\rangle_N \cdot \langle \tilde{t}, Nt\rangle_D \cdot (T_{-s'})^*\omega^{D,\lambda}_{Nt}\right) = \\
&= \underbrace{\left(\sum_{\tilde{t}\in E[D](S)} \langle \tilde{t}, Nt\rangle_D\right)}_{=0} \cdot \left(\omega^{N\cdot D,\lambda}_{s+t} - \sum_{s'\in E[N](S)} \langle s', Ds\rangle_N \cdot (T_{-s'})^*\omega^{D,\lambda}_{Nt}\right) = 0
\end{aligned}$$

Since D is invertible on S, we conclude $\omega_0 = 0$ as desired.

(b): We use the same strategy as in (a). For $\tilde{s} \in E[ND](S)$:

$$\operatorname{Res}_{\tilde{s}}\omega^{N\cdot D,\lambda}_s = \langle \tilde{s}, s\rangle_{ND} = \operatorname{Res}_{\tilde{s}}[D]^*\omega^{N,\lambda}_s$$

Thus,

$$\omega_0 := \omega^{N\cdot D,\lambda}_s - [D]^*\omega^{N,\lambda}_s \in \Gamma(E, \Omega^1_{E/S})$$

is a global section of $\Omega^1_{E/S}$. From

$$\begin{aligned}
N^2\omega_0 &= \sum_{\tilde{s}\in E[N](S)} T^*_{\tilde{s}}\omega_0 = \sum_{\tilde{s}\in E[N](S)} T^*_{\tilde{s}}\left(\omega^{N\cdot D,\lambda}_s - [D]^*\omega^{N,\lambda}_s\right) = \\
&= \sum_{\tilde{s}\in E[N](S)} \left(\langle \tilde{s}, s\rangle_{ND}\omega^{N\cdot D,\lambda}_s - [D]^* T^*_{D\tilde{s}}\omega^{N,\lambda}_s\right) = \\
&= \sum_{\tilde{s}\in E[N](S)} \left(\langle \tilde{s}, s\rangle_{ND}\omega^{N\cdot D,\lambda}_s - \langle D\tilde{s}, s\rangle_N [D]^*\omega^{N,\lambda}_s\right) = \\
&= \underbrace{\left(\sum_{\tilde{s}\in E[N](S)} \langle \tilde{s}, s\rangle_{ND}\right)}_{=0} \cdot \left(\omega^{N\cdot D,\lambda}_s - [D]^*\omega^{N,\lambda}_s\right) = 0
\end{aligned}$$

we conclude $\omega_0 = 0$ since N is invertible on S.

(c): Setting $s = 0$ in (a) and using (b) results in:

$$[N]^*\omega_t^{D,\lambda} = \omega_t^{ND,\lambda} = \sum_{s' \in E[N](S)} T_{s'}^* \omega_{Nt}^{D,\lambda}$$

□

Corollary 1.3.6. *Let E/S be an elliptic curve, $N, D \geq 1$ coprime integers with $D, N \in \mathcal{O}_S^\times$ and $e \neq t \in E[D](S)$. Then:*

$$\operatorname{Tr}_{[N]}\left(\omega_{Nt}^{D,\lambda}\right) = \omega_t^{D,\lambda}$$

Proof. Working étale locally, we may assume $|E[N](S)| = N^2$. In this case the statement of the corollary is equivalent to Lemma 1.3.5 (c). □

This above corollary was the missing step to relate $\omega_t^{D,\lambda}$ and ω_t^D.

Corollary 1.3.7. *Let E/S be an elliptic curve and $D > 1$ invertible on S. For every $e \neq t \in E[D](S)$ the uniquely characterized section $\omega_t^D \in \Gamma(E, \Omega^1_{E/S}(E[D]))$ defined in Theorem 1.2.8 exists and is explicitly given by*

$$\omega_{\lambda(t)}^D = \omega_t^{D,\lambda}.$$

Proof. The residue condition characterizing $\omega_{\lambda(t)}^D$ coincides with the residue condition characterizing $\omega_t^{D,\lambda}$. That $\omega_t^{D,\lambda}$ also satisfies the second characterizing property of $\omega_{\lambda(t)}^D$, i. e. the trace-compatibility, was proven in Corollary 1.3.6. □

Finally, we get the desired relation between s_{can} and s_{can}^λ.

Corollary 1.3.8. *Let E/S be an elliptic curve then*

$$s_{\mathrm{can}} = \left(\mathrm{id} \times \lambda^{-1}\right)^* (s_{\mathrm{can}}^\lambda).$$

Proof. By Proposition 1.3.3 and Corollary 1.3.7 the section $s = \left(\mathrm{id} \times \lambda^{-1}\right)^* (s_{\mathrm{can}}^\lambda)$ satisfies for all $D > 1$ and all pairs (T, t)

$$(\mathrm{id} \times t)^* U_t^D(s) = \omega_{\lambda^{-1}(t)}^{D,\lambda} = \omega_t^D.$$

This property characterizes s_{can} uniquely, i. e.

$$s_{\mathrm{can}} = \left(\mathrm{id} \times \lambda^{-1}\right)^* (s_{\mathrm{can}}^\lambda).$$

□

We close this section with a symmetry property of the canonical section. For every choice of autoduality
$$\tilde{\lambda} : E \xrightarrow{\sim} E^\vee$$
the bi-rigidified line bundle $\mathcal{P}_{E^\vee} := (\tilde{\lambda}^{-1} \times \tilde{\lambda})^* \mathcal{P}$ on $E^\vee \times_S E = E^\vee \times (E^\vee)^\vee$ satisfies the universal property of the Poincaré bundle for the elliptic curve $E^\vee$. Let us take $\mathcal{P}_{E^\vee}$ as our fixed Poincaré bundle for $E^\vee$ and let

$$s_{\mathrm{can},E^\vee} \in \Gamma\left(E^\vee \times_S E, \mathcal{P}_{E^\vee} \otimes \Omega^1_{E^\vee \times E/E}\left([E^\vee \times e] + [e \times E]\right)\right)$$

be the uniquely characterized canonical section associated with the elliptic curve $E^\vee/S$.

Corollary 1.3.9. *For every choice of autoduality isomorphism $\tilde{\lambda} : E \xrightarrow{\sim} E^\vee$ we have:*

$$(\tilde{\lambda}^{-1} \times \tilde{\lambda})^* (s_{\mathrm{can},E}) = s_{\mathrm{can},E^\vee}$$

Proof. Two autoduality isomorphisms λ and $\tilde{\lambda}$ differ by an automorphism $\alpha := \tilde{\lambda}^{-1} \circ \lambda$. Using our explicit description of $s_{\mathrm{can}} = \left(\mathrm{id} \times \lambda^{-1}\right)^* (s^\lambda_{\mathrm{can}})$ for λ as above, the claim boils down to the symmetry of the line bundle

$$\mathcal{O}_{E\times E}(-[E \times e] - [e \times E] + \Delta)$$

i. e. for any automorphism $\alpha : E \xrightarrow{\sim} E$:

$$(\alpha^{-1} \times \alpha)^* \mathcal{O}_{E\times E}(-[E \times e] - [e \times E] + \Delta) = \mathcal{O}_{E\times E}(-[E \times e] - [e \times E] + \Delta).$$

□

1.4 The distribution relation

In this section we state a distribution relation for the canonical section. As a technical tool we need a few relations satisfied by the canonical 1-forms.

Lemma 1.4.1. *Let E/S be an elliptic curve, $N, D \geq 1$ coprime integers. Assume that N and D are invertible on S and $|E^\vee[N](S)| = N^2$.*

(a) For all $s \in E^\vee[N](S)$ and $e \neq t \in E^\vee[D](S)$:

$$\omega^{N\cdot D}_{s+t} = \sum_{s' \in E[N](S)} \langle s', Ds\rangle_N \cdot (T_{-s'})^* \omega^D_{Nt}$$

(b) For $e \neq s \in E^\vee[N](S)$:

$$\omega^{N\cdot D}_s = [D]^* \omega^N_s$$

(c) For $e \neq t \in E^\vee[D](S)$:

$$\sum_{s' \in E[N](S)} (T_{s'})^* \omega^D_{Nt} = [N]^* \omega^D_t$$

(d) *For $e \neq t \in E^\vee[D](S)$:*

$$\sum_{s \in E^\vee[N](S)} \omega_{t+s}^{ND} = N^2 \omega_{Nt}^D$$

Proof. (a), (b), (c) follow immediately from the corresponding properties for $\omega_*^{*,\lambda}$ as stated in Lemma 1.3.5. The remaining statement (d) follows by summing (a) over all $s \in E^\vee[N](S)$:

$$\sum_{s \in E^\vee[N](S)} \omega_{s+t}^{N \cdot D} = \sum_{s' \in E[N](S)} \underbrace{\sum_{s \in E^\vee[N](S)} \langle s', Ds \rangle_N}_{=0 \text{ for } s' \neq e} \cdot (T_{-s'})^* \omega_{Nt}^D = N^2 \omega_{Nt}^D$$

□

For the proof of the distribution relation we need the following observations.

Lemma 1.4.2. *Let N, D be positive integers.*

(a) *For $s \in E[N](S)$ and $e \neq t \in E^\vee[D](S)$ we have*

$$(\mathrm{id} \times e)^* U_{s,t}^{N,D}(s_{\mathrm{can}}) = T_s^* \omega_{Nt}^D .$$

(b) *For $\tilde{D} \geq 1$ and $\tilde{t} \in E^\vee[\tilde{D}](S)$ we have:*

$$([\tilde{D}] \times \mathrm{id})^* \gamma_{N,D} \circ ([N] \times [D])^* \mathcal{U}_{D\tilde{t}}^{\tilde{D}} = ([D] \times [N])^* \mathcal{U}_{N\tilde{t}}^{\tilde{D}} \circ (\tilde{D} \times T_{\tilde{t}})^* \gamma_{N,D}$$

Proof. (a): By definition we have

$$U_{s,t}^{N,D}(s_{\mathrm{can}}) = \left(\mathcal{U}_{s,t}^{N,D} \otimes \mathrm{id}_{\Omega^1}\right) \left((T_s \times T_t)^* ([D] \times [N])^* s_{\mathrm{can}}\right)$$

and Corollary 1.1.3 gives

$$\mathcal{U}_{s,t}^{N,D} = \left(([D] \times \mathrm{id})^* \mathcal{U}_{Ds,e}^{N,1}\right) \circ \left((T_s \times [N])^* \mathcal{U}_{e,Nt}^{1,D}\right) .$$

Since $\gamma_{N,1}$ is compatible with the rigidifications, we see that $(\mathrm{id} \times e)^* \gamma_{N,1}$ is the canonical isomorphism

$$[N]^* \mathcal{O}_E \xrightarrow{\sim} \mathcal{O}_E .$$

Applying this to the definition

$$\mathcal{U}_{s,e}^{N,1} := \gamma_{N,1} \circ (T_s \times \mathrm{id})^* \gamma_{1,N}$$

of $\mathcal{U}_{s,e}^{N,1}$ we conclude that $(\mathrm{id} \times e)^* \mathcal{U}_{s,e}^{N,1}$ is the canonical isomorphism

$$T_s^* \mathcal{O}_E \xrightarrow{\sim} \mathcal{O}_E$$

induced by T_s. Thus, we compute

$$(\mathrm{id}\times e)^*U^{N,D}_{s,t}(s_{\mathrm{can}}) = T_s^*\Big((\mathrm{id}\times e)^*U^D_{Nt}(s_{\mathrm{can}})\Big) = T_s^*\omega^D_{Nt}.$$

(b): We have the following commutative diagrams. The first diagram

$$\begin{tikzcd}
(\mathrm{id}\times T_{\tilde t})^*([N\tilde D]\times[D])^*\mathcal{P} \arrow[r, "(\mathrm{id}\times T_{\tilde t})^*([N]\times[D])^*\gamma_{\tilde D,1}"] \arrow[d, equal] & (\mathrm{id}\times T_{\tilde t})^*([N]\times[D\tilde D])^*\mathcal{P} \arrow[d, "(\mathrm{id}\times T_{\tilde t})^*\gamma_{N,D\tilde D}"] \\
(\mathrm{id}\times T_{\tilde t})^*([N\tilde D]\times[D])^*\mathcal{P} \arrow[r, "(\mathrm{id}\times T_{\tilde t})^*([\tilde D]\times\mathrm{id})^*\gamma_{N,D}"] & (\mathrm{id}\times T_{\tilde t})^*([\tilde D D]\times[N])^*\mathcal{P}
\end{tikzcd}$$

follows from $\gamma_{N,D\tilde D} = ([\tilde D]\times\mathrm{id})^*\gamma_{N,D}\circ([N]\times[D])^*\gamma_{1,\tilde D}$ cf. Lemma 1.1.1. The second diagram

$$\begin{tikzcd}
([N]\times[D\tilde D])^*\mathcal{P} \arrow[r, "([N]\times[D])^*\gamma_{1,\tilde D}"] \arrow[d, "(\mathrm{id}\times T_{\tilde t})^*\gamma_{N,D\tilde D}"] & ([N\tilde D]\times[D])^*\mathcal{P} \arrow[d, "\gamma_{N\tilde D,D}"] \\
(\mathrm{id}\times T_{\tilde t})^*([\tilde D D]\times[N])^*\mathcal{P} \arrow[r, "(\mathrm{id}\times T_{\tilde t})^*([D]\times[N])^*\gamma_{\tilde D,1}"] & ([D]\times[N\tilde D])^*\mathcal{P}
\end{tikzcd}$$

encodes the identity

$$\begin{aligned}
(\mathrm{id}\times T_t)^*\Big(([D]\times[N])^*\gamma_{\tilde D,1}\circ\gamma_{N,D\tilde D}\Big) &\overset{\textit{Lemma } 1.1.1}{=} (\mathrm{id}\times T_t)^*\Big((\mathrm{id}\times[\tilde D])^*\gamma_{N,D}\Big) = \\
&= (\mathrm{id}\times[\tilde D])^*\gamma_{N,D} \overset{\textit{Lemma } 1.1.1}{=} \\
&= \gamma_{N\tilde D,D}\circ([N]\times[D])^*\gamma_{1,\tilde D}.
\end{aligned}$$

And the last diagram

$$\begin{tikzcd}
([N\tilde D]\times[D])^*\mathcal{P} \arrow[r, "([\tilde D]\times\mathrm{id})^*\gamma_{N,D}"] \arrow[d, "\gamma_{N\tilde D,D}"] & ([D\tilde D]\times[N])^*\mathcal{P} \arrow[d, equal] \\
([D]\times[N\tilde D])^*\mathcal{P} \arrow[r, "([D]\times[N])^*\gamma_{1,\tilde D}"] & ([D\tilde D]\times[N])^*\mathcal{P}
\end{tikzcd}$$

is equivalent to $\gamma_{N\tilde D,D} = ([D]\times[N])^*\gamma_{\tilde D,1}\circ([\tilde D]\times\mathrm{id})^*\gamma_{N,D}$, see Lemma 1.1.1. The composition of the upper horizontal arrows in the three diagrams is

$$([\tilde D]\times\mathrm{id})^*\gamma_{N,D}\circ([N]\times[D])^*\gamma_{1,\tilde D}\circ(\mathrm{id}\times T_{\tilde t})^*([N]\times[D])^*\gamma_{\tilde D,1}$$

while the composition of the lower horizontal arrows is:

$$([D]\times[N])^*\gamma_{1,\tilde D}\circ(\mathrm{id}\times T_{\tilde t})^*([D]\times[N])^*\gamma_{\tilde D,1}\circ([\tilde D]\times T_{\tilde t})^*\gamma_{N,D}$$

The commutativity shows that both compositions are equal, i. e. it gives the middle equality in

$$\begin{aligned}
&([\tilde D]\times\mathrm{id})^*\gamma_{N,D}\circ([N]\times[D])^*\mathcal{U}^{\tilde D}_{D\tilde t} = \\
=&([\tilde D]\times\mathrm{id})^*\gamma_{N,D}\circ([N]\times[D])^*\gamma_{1,\tilde D}\circ(\mathrm{id}\times T_{\tilde t})^*([N]\times[D])^*\gamma_{\tilde D,1} = \\
=&([D]\times[N])^*\gamma_{1,\tilde D}\circ(\mathrm{id}\times T_{\tilde t})^*([D]\times[N])^*\gamma_{\tilde D,1}\circ([\tilde D]\times T_{\tilde t})^*\gamma_{N,D} = \\
=&([D]\times[N])^*\mathcal{U}^{\tilde D}_{N\tilde t}\circ(\tilde D\times T_{\tilde t})^*\gamma_{N,D}
\end{aligned}$$

while the other equalities follow from the definition of the translation operators. □

Now, we can state the distribution relation which is motivated from the theta function distribution relation given in [BK10b, Proposition 1.16].

Theorem 1.4.3. *Let E/S be an elliptic curve. Assume $N, N', D, D' \geq 1$ are pairwise coprime and invertible on S. Furthermore, assume $|E^\vee[D'](S)| = (D')^2$ and $|E[N'](S)| = (N')^2$. Then, for $t \in E^\vee[D](S)$, $s \in E[N](S)$:*

$$\sum_{\substack{\alpha\in E[N'](S),\\ \beta\in E^\vee[D'](S)}} U^{NN',DD'}_{s+\alpha,t+\beta}(s_{\mathrm{can}}) = (D')^2 \cdot \left(([D]\times[N])^*\gamma_{N',D'}\right)\left(([N']\times[D'])^* U^{N,D}_{N's,D't}(s_{\mathrm{can}})\right)$$

Proof. As in the proof of Theorem 1.2.8 using the Zariski covering $(S[\frac{1}{\tilde{D}}])_{\tilde{D}>1,(\tilde{D},NN'DD')=1}$ together with a density of torsion section argument we reduce the proof of the equality to the following claim. For all $\tilde{D} > 1$ coprime to $NN'DD'$ we have:

$(*)_{\tilde{D}}$ For all pairs $(T, \tilde{t})$ with T an S-scheme, $\tilde{D}$ invertible on T and $e \neq \tilde{t} \in E^\vee[\tilde{D}](T)$ we have

$$\begin{aligned}(\mathrm{id}\times\tilde{t})^* &\sum_{\substack{\alpha\in E[N'](S),\\ \beta\in E^\vee[D'](S)}} U^{NN',DD'}_{s+\alpha,t+\beta}(s_{\mathrm{can}}) = \\ =&(\mathrm{id}\times\tilde{t})^*(D')^2\cdot\left(([D]\times[N])^*\gamma_{N',D'}\right)\left(([N']\times[D'])^*U^{N,D}_{N's,D't}(s_{\mathrm{can}})\right)\end{aligned}$$

Since the pullback of sections along faithfully flat maps is injective and $(\mathrm{id}\times e)^*\mathcal{U}^{\tilde{D}}_{NN'\tilde{t}}$ is an isomorphism, condition $(*)_{\tilde{D}}$ is equivalent to

$(*)_{\tilde{D}}$ For all pairs $(T, \tilde{t})$ with T an S-scheme, $\tilde{D}$ invertible on T and $e \neq \tilde{t} \in E^\vee[\tilde{D}](T)$ we have

$$\begin{aligned}&\left(([DD']\times e)^*\mathcal{U}^{\tilde{D}}_{NN'\tilde{t}}\otimes\mathrm{id}_\Omega\right)\left[([\tilde{D}]\times\tilde{t})^*\sum_{\substack{\alpha\in E[N'](S),\\ \beta\in E^\vee[D'](S)}} U^{NN',DD'}_{s+\alpha,t+\beta}(s_{\mathrm{can}})\right] = \\ &=(D')^2\cdot\left(([DD']\times e)^*\mathcal{U}^{\tilde{D}}_{NN'\tilde{t}}\otimes\mathrm{id}_\Omega\right)\circ \\ &\circ([\tilde{D}]\times\tilde{t})^*\left(([D]\times[N])^*\gamma_{N',D'}\right)\left(([N']\times[D'])^*U^{N,D}_{N's,D't}(s_{\mathrm{can}})\right)\end{aligned} \tag{1.17}$$

We compute the left hand side for arbitrary $(T,\tilde{t})$:

$$\Big(([DD']\times e)^*\mathcal{U}^{\tilde{D}}_{NN'\tilde{t}}\otimes\mathrm{id}\Big)\left(([\tilde{D}]\times\tilde{t})^*\sum_{\substack{\alpha\in E[N'](S),\\ \beta\in E^\vee[D'](S)}} U^{NN',DD'}_{s+\alpha,t+\beta}(s_{\mathrm{can}})\right)\overset{\text{Cor.1.1.3}}{=} \tag{1.18}$$

$$=(\mathrm{id}\times e)^*\left(\sum_{\substack{\alpha\in E[N'](S),\\ \beta\in E^\vee[D'](S)}} U^{NN',DD'\tilde{D}}_{(\tilde{D})^{-1}(s+\alpha),\tilde{t}+t+\beta}(s_{\mathrm{can}})\right)\overset{\text{Lem.1.4.2}}{=}$$

$$=\sum_{\substack{\alpha\in E[N'](S),\\ \beta\in E^\vee[D'](S)}}(T_{(\tilde{D})^{-1}(s+\alpha)})^*\omega^{DD'\tilde{D}}_{NN'\tilde{t}+NN't+NN'\beta}=$$

$$=\sum_{\substack{\alpha\in E[N'](S),\\ \beta\in E^\vee[D'](S)}}(T_{(\tilde{D})^{-1}s+\alpha})^*\omega^{DD'\tilde{D}}_{NN'\tilde{t}+NN't+\beta}\overset{\text{Lem.1.4.1}(d)}{=}$$

$$=(D')^2\cdot(T_{(\tilde{D})^{-1}s})^*\sum_{\alpha\in E[N'](S)}(T_\alpha)^*\omega^{D\tilde{D}}_{D'(NN'\tilde{t}+NN't)}\overset{\text{Lem.1.4.1}(c)}{=}$$

$$=(D')^2\cdot(T_{(\tilde{D})^{-1}s})^*[N']^*\omega^{D\tilde{D}}_{D'N\tilde{t}+D'Nt}$$

Before we simplify the right hand side of the above equation, we use Lemma 1.4.2 to simplify the following expression:

$$\begin{aligned}
&(D')^2\cdot(\mathrm{id}\times e)^*\Big[([DD']\times[NN'])^*\mathcal{U}^{\tilde{D}}_{NN'\tilde{t}}\circ([\tilde{D}]\times T_{\tilde{t}})^*([D]\times[N])^*\gamma_{N',D'}\circ\\
&\circ([\tilde{D}]\times T_{\tilde{t}})^*([N']\times[D'])^*\mathcal{U}^{N,D}_{N's,D't}\Big]=\\
=&(D')^2\cdot(\mathrm{id}\times e)^*\Big[([D]\times[N])^*\Big(([D']\times[N'])^*\mathcal{U}^{\tilde{D}}_{NN'\tilde{t}}\circ([\tilde{D}]\times T_{N\tilde{t}})^*\gamma_{N',D'}\Big)\circ\\
&\circ([\tilde{D}]\times T_{\tilde{t}})^*([N']\times[D'])^*\mathcal{U}^{N,D}_{N's,D't}\Big]\overset{\text{Lem.1.4.2}(b)}{=}\\
=&(D')^2\cdot(\mathrm{id}\times e)^*\Big[([D]\times[N])^*\Big(([\tilde{D}]\times\mathrm{id})^*\gamma_{N',D'}\circ([N']\times[D'])^*\mathcal{U}^{\tilde{D}}_{ND'\tilde{t}}\Big)\circ\\
&\circ([\tilde{D}]\times T_{\tilde{t}})^*([N']\times[D'])^*\mathcal{U}^{N,D}_{N's,D't}\Big]=\\
=&(D')^2\cdot(\mathrm{id}\times e)^*\Big[([D\tilde{D}]\times[N])^*\gamma_{N',D'}\circ\\
&\circ([N']\times[D'])^*\Big(([D]\times[N])^*\mathcal{U}^{\tilde{D}}_{ND'\tilde{t}}\circ([\tilde{D}]\times T_{\tilde{D}'t})^*\mathcal{U}^{N,D}_{N's,D't}\Big)\Big]\overset{\text{Cor.1.1.3}}{=}\\
=&(D')^2\cdot(\mathrm{id}\times e)^*\Big[([D\tilde{D}]\times[N])^*\gamma_{N',D'}\circ([N']\times[D'])^*\Big(\mathcal{U}^{N,D\tilde{D}}_{N'\tilde{D}^{-1}s,D'(t+\tilde{t})}\Big)\Big]=\\
=&(D')^2\cdot(\mathrm{id}\times e)^*\Big[([N']\times[D'])^*\Big(\mathcal{U}^{N,D\tilde{D}}_{N'\tilde{D}^{-1}s,D'(t+\tilde{t})}\Big)\Big]
\end{aligned}$$

Here, we have used the compatibility of $\gamma_{N',D'}$ with rigidifications in the last step. Using this and again Lemma 1.4.2, the right hand side of(1.17) is:

$$(D')^2 \cdot (\mathrm{id} \times e)^* \left(([N'] \times [D'])^* U^{N,D\tilde{D}}_{N'\tilde{D}^{-1}s, D'(t+\tilde{t})}(s_{\mathrm{can}}) \right) =$$
$$=(D')^2 \cdot [N']^* T_{N'\tilde{D}^{-1}s} \omega^{D\tilde{D}}_{ND'(t+\tilde{t})} = (D')^2 \cdot (T_{\tilde{D}^{-1}s})^* [N']^* \omega^{D\tilde{D}}_{ND'(t+\tilde{t})}$$

Comparing this last equation to (1.18) shows that the equation in $(*)_{\tilde{D}}$ holds for all pairs $(T, \tilde{t})$. By density of torsion sections this finishes the proof of the theorem. □

Remark 1.4.4. One can slightly generalize the above distribution relation by summing over all torsion points in the kernel of more general isogenies $\varphi : E \to E'$.

In its simplest form the distribution relation specializes to the following equality:

Corollary 1.4.5. *For E/S with D invertible on S and $|E[D](S)| = D^2$:*

$$\sum_{e \neq t \in E^\vee[D](S)} U^D_t(s_{\mathrm{can}}) = D^2 \cdot \gamma_{1,D}\left((\mathrm{id} \times [D])^*(s_{\mathrm{can}})\right) - ([D] \times \mathrm{id})^*(s_{\mathrm{can}})$$

Proof. Substituting N', D, N by 1 in Theorem 1.4.3 gives this corollary. □

Definition 1.4.6. For E/S, $D > 1$ invertible on S define

$$s^D_{\mathrm{can}} := D^2 \cdot \gamma_{1,D}\left((\mathrm{id} \times [D])^*(s_{\mathrm{can}})\right) - ([D] \times \mathrm{id})^*(s_{\mathrm{can}}).$$

This is a priori an element in

$$\Gamma\left(E \times_S E^\vee, ([D] \times \mathrm{id})^* \left[\mathcal{P} \otimes \Omega^1_{E \times E^\vee / E^\vee}\left([E \times E^\vee[D]] + [E[D] \times E]\right)\right]\right).$$

We call s^D_{can} the *D-variant of the canonical section* of the Poincaré bundle.

An immediate consequence of the above corollary is the following result which roughly says that the above construction of the D-variant removes a pole along $E \times e$.

Corollary 1.4.7. *For E/S and D invertible on S the section s^D_{can} is contained in*

$$\Gamma\left(E \times_S E^\vee, ([D] \times \mathrm{id})^* \left[\mathcal{P} \otimes \Omega^1_{E \times E^\vee / E^\vee}\left([E \times (E^\vee[D] \setminus \{e\})] + [E[D] \times E]\right)\right]\right).$$

In particular, $(\mathrm{id} \times e)^* s^D_{\mathrm{can}} \in \Gamma(E, \Omega^1_{E/S}(E[D]))$.

Proof. We can check the claim after some finite étale base change and may thus assume $|E[D](S)| = D^2$. Now, the claim follows from the distribution relation in Corollary 1.4.5 and the observation that $U^D_t(s_{\mathrm{can}})$ is an element of

$$\Gamma\left(E \times_S E^\vee, ([D] \times \mathrm{id})^* \left[\mathcal{P} \otimes \Omega^1_{E \times E^\vee / E^\vee}\left([E \times (-t)] + [E[D] \times E]\right)\right]\right).$$

□

Remark 1.4.8. Later, we will give an explicit description of s_{can} on the analytification of the universal elliptic curve with $\Gamma(N)$-level structure via theta functions. In this case, the distribution relation gives back the distribution relation of [BK10b, Prop. 1.16] for the Kronecker theta function. Thus, we can see the above distribution relation as an algebraic version of the distribution relation in [BK10b].

1.5 The canonical section, Kato–Siegel units and Eisenstein series

The aim of this section is to relate the canonical section to the logarithmic derivative of the Kato–Siegel function and to Kato's Eisenstein series. Later, we will show that far more general real-analytic Eisenstein series can be constructed via the Poincaré bundle.

Definition 1.5.1. Let E/S be an elliptic curve with D invertible and define

$$\omega^D := (\mathrm{id} \times e)^* s^D_{\mathrm{can}} \in \Gamma(E, \Omega^1_{E/S}(E[D])).$$

Remark 1.5.2. Let E/S be an elliptic curve with D invertible. For $T \to S$ finite étale with $|E[D](S)| = D^2$ we have

$$\omega^D = \sum_{e \neq t \in E^\vee_T[D](T)} \omega^D_t \in \Gamma(E_T, \Omega_{E_T/T}(E_T[D]))$$

by the distribution relation.

Proposition 1.5.3. *Let $D > 1$ be an integer coprime to 6 and invertible on S. The section $\omega^D \in \Gamma(E, \Omega_{E/S}(E[D]))$ coincides with the logarithmic derivative of the Kato–Siegel function ${}_D\theta$:*

$$\omega^D = \mathrm{dlog}\, {}_D\theta$$

Proof. The logarithmic derivative $\mathrm{dlog}\, {}_D\theta \in \Gamma(E, \Omega^1_{E/S}(E[D]))$ is uniquely determined by the following two properties:

(a) Its residue is

$$\mathrm{Res}(\mathrm{dlog}\, {}_D\theta) = D^2 \mathbb{1}_e - \mathbb{1}_{E[D]}$$

where

$$\mathrm{Res} : \Omega^1_{E/S}(\log E[D]) \to (i_{E[D]})_* \mathcal{O}_{E[D]}$$

is the residue map and $\mathbb{1}_e$ resp. $\mathbb{1}_{E[D]}$ are the functions in $(i_{E[D]})_* \mathcal{O}_{E[D]}$ which have the constant value one along e resp. $E[D]$.

(b) It is trace invariant, i. e. $\mathrm{Tr}_{[N]}\, \mathrm{dlog}\, {}_D\theta = \mathrm{dlog}\, {}_D\theta$.

But both properties are satisfied by ω^D. It is trace compatible since it is a finite sum of trace compatible elements. And étale locally we can compute the residue of ω^D by:

$$\mathrm{Res}_{\tilde{t}}\, \omega^D = \sum_{e \neq t \in E^\vee[D](T)} \mathrm{Res}_{\tilde{t}}\, \omega^D_t = \sum_{e \neq t \in E^\vee[D](T)} \langle \tilde{t}, t \rangle_D = \begin{cases} D^2 - 1 & \tilde{t} = e \\ -1 & \tilde{t} \neq e \end{cases}$$

□

Remark 1.5.4. The restriction $(D, 6) = 1$ is necessary for the construction of the Kato–Siegel function ${}_D\theta$. It is remarkable that we do not have to assume $(D, 6) = 1$ for the construction of its logarithmic derivative ω^D.

By abuse of notation let us write d for the map

$$\left(\Omega^1_{E/S}\right)^{\otimes k} = \mathcal{O}_E \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1}\omega^{\otimes k}_{E/S} \xrightarrow{d\otimes \mathrm{id}} \Omega^1_{E/S} \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1}\omega^{\otimes k}_{E/S} = \left(\Omega^1_{E/S}\right)^{\otimes k+1}$$

and let $d^{\circ k}\omega^D$ for $\omega^D \in \Gamma(E, \Omega^1_{E/S}(E[D]))$ be understood as $d^{\circ k}\left(\omega^D\Big|_{E\setminus E[D]}\right)$ for $\left(\omega^D\Big|_{E\setminus E[D]}\right) \in \Gamma(E \setminus E[D], \Omega^1_{E/S})$. The definition of Kato's geometric Eisenstein series ${}_D\mathrm{E}^{(k)}_s$ via logarithmic derivatives of the Kato–Siegel function gives immediately the following corollary:

Corollary 1.5.5. *Let $D > 1$ coprime to 6 and invertible on S. For $N > 1$ coprime to D and $s \in E[N](S)$ we have*

$$s^*\left(d^{\circ(k-1)}\omega^D\right) = {}_D\mathrm{E}^{(k)}_s$$

where ${}_D\mathrm{E}^{(k)}_s$ are the geometric Eisenstein series defined by Kato in [Kat04, §3].

Kato's Eisenstein series ${}_D\mathrm{F}^{(k)}_s$ for $s \in E[N](S)$ can also be constructed in a very natural way via the Poincaré bundle. We will give this construction later.

1.6 Analytification of the Poincaré bundle

1.6.1 Analytification of the Poincaré bundle

Let $N > 3$ and $\Gamma(N) := \ker(\mathrm{Sl}_2(\mathbb{Z}) \to \mathrm{Sl}_2(\mathbb{Z}/N\mathbb{Z}))$. In this section we recall an explicit model for the analytification of the universal elliptic curve E_N/M_N with $\Gamma(N)$-level structure. We follow Scheider [Sch14, §3.4] and refer there for details. As an application we can relate the canonical section to theta functions of the Poincaré bundle. Furthermore, the explicit description of the analytification of the Poincaré bundle on the universal elliptic curve is the foundation for the proof of the main result of this chapter.

The complex-analytic space associated with the $\mathbb{C}$-valued points of E_N/M_N can be described as follows. Let $\Gamma(N)$ act on $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ via fractional linear transformations and let $\left(\begin{pmatrix} m \\ n \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \mathbb{Z}^2 \rtimes \Gamma(N)$ act on $(z, \tau, j) \in \mathbb{C}\times\mathbb{H}\times(\mathbb{Z}/N\mathbb{Z})^\times$ via

$$\left(\begin{pmatrix} m \\ n \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).(z,\tau,j) := \left(\frac{z + m\tau + n}{c\tau + d}, \frac{a\tau + b}{c\tau + d}, j\right).$$

The obtained complex analytic family

$$E_N^{an} := \left(\mathbb{Z}^2 \rtimes \Gamma(N) \setminus \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times\right) \xrightarrow{\pi_E^{an}} M_N^{an} := \left(\Gamma(N) \setminus \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times\right)$$

of complex elliptic curves gives an explicit model for the analytification of E_N/M_N. The universal covering spaces of E_N^{an} and M_N^{an} are:

$$\widetilde{E}_N := \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\tilde{p}_E} E_N^{an} = \mathbb{Z}^2 \rtimes \Gamma(N) \setminus \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times$$

and

$$\widetilde{M}_N := \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\tilde{p}_M} M_N^{an} = \Gamma(N) \setminus \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times .$$

As a lift of $e : M_N \to E_N$ we choose $(0) : \widetilde{M}_N \to \widetilde{E}_N, (\tau, j) \mapsto (0, \tau, j)$. We summarize the introduced notation in the following diagram

$$\begin{tikzcd}
\widetilde{E}_N \arrow[r, "\tilde{p}_E"] \arrow[d, "\tilde{\pi}_E"] & E_N^{an} \arrow[d, "\pi_E^{an}"] \\
\widetilde{M}_N \arrow[r, "\tilde{p}_M"] \arrow[u, bend left, "(0)"] & M_N^{an}. \arrow[u, bend left, "e"]
\end{tikzcd}$$

Using the autoduality isomorphism $\lambda : E \xrightarrow{\sim} E^\vee$ from 1.10, we get an explicit description of the analytification of the dual elliptic curve $(E_N^\vee)^{an}$ and its universal covering space

$$\widetilde{E}_N^\vee := \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\widetilde{p}_{E^\vee}} (E_N^\vee)^{an} = \mathbb{Z}^2 \rtimes \Gamma(N) \setminus \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times .$$

Let us write (w, τ, j) for the coordinates on $\widetilde{E}_N^\vee := \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times$. The classical theta function

$$\vartheta(z, \tau) := \exp\left(\frac{z^2}{2}\eta(1, \tau)\right) \cdot \sigma(z, \tau)$$

gives us a trivializing section for the line bundle $\mathcal{O}_{E_N^{an}}^{an}(-[e])$:

$$\begin{aligned}
\mathcal{O}_{\widetilde{E}_N}^{an} = \mathcal{O}_{\mathbb{C}\times\mathbb{H}\times(\mathbb{Z}/N\mathbb{Z})^\times}^{an} &\longrightarrow \tilde{p}_E^*\mathcal{O}_{E_N^{an}}^{an}(-[e]) \\
1 &\longmapsto \vartheta(z, \tau).
\end{aligned}$$

Here, $\eta(1, \tau) := \zeta(z, \tau) - \zeta(z + 1, \tau)$ is the period of the Weierstrass zeta function. We should remark that the chosen trivializing section $\vartheta(z, \tau)$ of $\tilde{p}_E^*\mathcal{O}_{E_N^{an}}^{an}(-[e])$ induces via the canonical isomorphism

$$\tilde{p}_M^*\underline{\omega}_{E_N^{an}/M_N^{an}} \xrightarrow{\sim} \tilde{p}_M^* e^*\mathcal{O}_{E_N^{an}}^{an}(-[e]) \xrightarrow{\sim} (0)^*\tilde{p}_E^*\mathcal{O}_{E_N^{an}}^{an}(-[e])$$

a basis of $\tilde{p}_M^*\underline{\omega}_{E_N^{an}/M_N^{an}}$ which coincides with dz. Using our fixed autoduality isomorphism, allows us to describe the analytification of the Poincaré bundle as

$$\mathcal{P}^{an} = \mu^*\mathcal{O}_{E_N}^{an}([e]) \otimes_{\mathcal{O}_{E\times E}^{an}} \mathrm{pr}_1^*\mathcal{O}_{E_N}^{an}(-[e]) \otimes_{\mathcal{O}_{E\times E}^{an}} \mathrm{pr}_2^*\mathcal{O}_{E_N}^{an}(-[e]) \otimes_{\mathcal{O}_{E\times E}^{an}} \pi_{E\times E}^*\underline{\omega}_{E_N/M_N}^\vee .$$

The trivializations of $\mathcal{O}_{E_N}^{an}(-[e])$ and $\underline{\omega}_{E_N^{an}/M_N^{an}}$ induce a trivializing section

$$\tilde{\mathfrak{t}} := \frac{1}{J(z, w, \tau)} \otimes (dz)^\vee$$

of $\tilde{\mathcal{P}} := \tilde{p}_E^*\mathcal{P}^{an}$. Here,

$$J(z, w, \tau) := \frac{\vartheta(z + w, \tau)}{\vartheta(z, \tau)\vartheta(w, \tau)} \tag{1.19}$$

is the *Jacobi theta function*. By the above discussion the canonical trivializations of $\tilde{\mathcal{P}}$ are explicitly given by

$$(\mathrm{id}\times(0))^*\tilde{\mathcal{P}}\xrightarrow{\sim}\mathcal{O}^{an}_{\widetilde{E}_N},\quad (\mathrm{id}\times(0))^*\tilde{\mathfrak{t}}\mapsto 1$$

and

$$((0)\times\mathrm{id})^*\tilde{\mathcal{P}}\xrightarrow{\sim}\mathcal{O}^{an}_{\widetilde{E}_N},\quad ((0)\times\mathrm{id})^*\tilde{\mathfrak{t}}\mapsto 1.$$

By abuse of notation let us again write $[D]:\widetilde{E}_N\to\widetilde{E}_N,(z,\tau,j)\mapsto(D\cdot z,\tau,j)$ for the lift of $[D]$ to the universal covering.

Lemma 1.6.1 ([Sch14, Lemma 3.5.10])**.** *The canonical isomorphism*

$$\tilde{\gamma}_{1,D}:(\mathrm{id}\times[D])^*\tilde{\mathcal{P}}\xrightarrow{\sim}([D]\times\mathrm{id})^*\tilde{\mathcal{P}}$$

is the unique $\mathcal{O}^{an}_{\widetilde{E}_N\times\widetilde{E}_N}$-linear map given by

$$(\mathrm{id}\times[D])^*\tilde{\mathfrak{t}}\mapsto([D]\times\mathrm{id})^*\tilde{\mathfrak{t}}.$$

Proof. For details we refer to [Sch14, Lemma 3.5.10]. As in the algebraic case, there is only one isomorphism

$$(\mathrm{id}\times[D])^*\mathcal{P}^{an}\xrightarrow{\sim}([D]\times\mathrm{id})^*\mathcal{P}^{an}$$

which is compatible with the rigidifications. The map given in the lemma descends to a map

$$(\mathrm{id}\times[D])^*\mathcal{P}^{an}\xrightarrow{\sim}([D]\times\mathrm{id})^*\mathcal{P}^{an}.$$

Indeed, this boils down to the fact that the function $\frac{J(z,Dw,\tau)}{J(Dz,w,\tau)}$ is invariant under the action of $\mathbb{Z}^2\rtimes\Gamma(N)$, which is easily checked. Further, the given map is compatible with the rigidifications by the above explicit formulas and the claim follows by uniqueness. □

Lemma 1.6.2. *The pullback of the analytification of the canonical section along the universal covering*

$$\tilde{p}_E\times\tilde{p}_E:\widetilde{E}_N\times_{\widetilde{M}_N}\widetilde{E}_N\to E_N^{an}\times_{M_N^{an}}E_N^{an}$$

is given by

$$\tilde{s}_{\mathrm{can}}:=(\tilde{p}_E\times\tilde{p}_E)^*(s_{\mathrm{can}})^{an}=J(z,w,\tau)\left(\tilde{\mathfrak{t}}\otimes dz\right)$$

Proof. By the explicit description of the canonical section in Corollary 1.3.8, we can describe $\tilde{s}_{\mathrm{can}}$ as the image of the identity $\mathrm{id}_{\mathcal{P}}$ under

$$\begin{aligned}&\tilde{\mathcal{P}}\otimes\tilde{\mathcal{P}}^{\otimes-1}=\\&=\tilde{\mathcal{P}}\otimes(\tilde{p}_E\times\tilde{p}_E)^*\Big(\mathcal{O}^{an}_{E_N^{an}\times E_N^{an}}(-[e\times E_N^{an}]-[E_N^{an}\times e]+\Delta)\otimes(\pi^{an}_{E\times E})^*\omega^{-1}_{E_N^{an}/M_N^{an}}\Big)^{\otimes-1}\cong\\&\cong\tilde{\mathcal{P}}\otimes\tilde{p}_E^*\left(\Omega^1_{E_N^{an}\times E_N^{an}/E_N^{an}}([e\times E_N^{an}]+[E_N^{an}\times e]-\Delta)\right).\end{aligned}$$

The identity is given by $\mathrm{id}_{\tilde{\mathcal{P}}}=\tilde{\mathfrak{t}}\otimes\tilde{\mathfrak{t}}^{-1}$. Since $\tilde{\mathfrak{t}}^{-1}=J(z,w,\tau)\otimes dz$, the element $\mathrm{id}_{\tilde{\mathcal{P}}}$ maps to

$$J(z,w,\tau)\tilde{\mathfrak{t}}\otimes dz$$

under the above isomorphism. □

1.6.2 Analytification of the universal vectorial extension

The analytification of the universal vectorial extension $E_N^\dagger$ of $E_N^\vee$ can be described as follows. By [MM74, Ch I, 4.4] $E_N^{\dagger,an}$ sits in an exact sequence

$$0 \longrightarrow R^1(\pi_E^{an})_*(2\pi i\mathbb{Z}) \longrightarrow \underline{H}^1_{\mathrm{dR}}\left(E_N^{an}/M_N^{an}\right) \longrightarrow E_N^{\dagger,an} \longrightarrow 0.$$

In particular, the geometric vector bundle associated with $\tilde{p}_M^*\underline{H}^1_{\mathrm{dR}}\left(E_N^{an}/M_N^{an}\right)$ serves as a universal covering space of $E_N^{\dagger,an}$. Choosing coordinates on this universal covering is equivalent to choosing a basis on the cotangent space $\left(\tilde{p}_M^*\underline{H}^1_{\mathrm{dR}}\left(E_N^{an}/M_N^{an}\right)\right)^\vee$. If we choose $[\omega]=[dz]$ and $[\eta]=[\wp(z,\tau)dz]$ as basis of $\tilde{p}_M^*\underline{H}^1_{\mathrm{dR}}\left(E_N^{an}/M_N^{an}\right)$, we obtain coordinates (w',u) associated with the dual basis $([\eta]^\vee,[\omega]^\vee)$. I. e. via the above identification of the cotangent space with $\left(\tilde{p}_M^*\underline{H}^1_{\mathrm{dR}}\left(E_N^{an}/M_N^{an}\right)\right)^\vee$, we have

$$dw' = [\eta]^\vee, \quad du = [\omega]^\vee.$$

With these coordinates we obtain the following universal covering of $E_N^{\dagger,an}$

$$\widetilde{E}_N^\dagger = \mathbb{C}^2 \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \to E_N^{\dagger,an}.$$

This universal covering fits into the following commutative diagram, cf. [Sch14, (3.4.10)]:

$$\begin{tikzcd}
\widetilde{E}_N^\dagger = \mathbb{C}^2 \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "-\mathrm{pr}_1"] \arrow[d] & \widetilde{E}_N^\vee = \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \arrow[d] \\
E_N^{\dagger,an} \arrow[r] & E_N^{\vee,an}.
\end{tikzcd} \tag{1.20}$$

The reason for the minus sign in the above commutative diagram is the following. The canonical projection of the universal vectorial extension in (1.20) induces the map

$$\underline{\omega}_{E_N^\vee/M_N} \hookrightarrow \underline{H}^1_{\mathrm{dR}}\left(E_N^\vee/M_N\right) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}\left(E_N/M_N\right)^\vee. \tag{1.21}$$

on the cotangent spaces. Here, the isomorphism

$$\underline{H}^1_{\mathrm{dR}}\left(E_N^\vee/M_N\right) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}\left(E_N/M_N\right)^\vee.$$

is the canonical isomorphism coming from Deligne's pairing (cf. [Del74], [Ber09]). The coordinate on the analytification of $E_N^\vee$ was induced from the autoduality isomorphism. Now, the minus sign appears since, by our choice of autoduality, (1.21) maps $\lambda^*(dz)$ to $-[\eta]^\vee$.

Let us denote the pullback of $\mathcal{P}$ along the canonical projection

$$E_N \times_{M_N} E_N^\dagger \to E_N \times_{M_N} E_N^\vee$$

by $\mathcal{P}^\dagger$. The rigidifications of $\mathcal{P}$ induce rigidifications on $\mathcal{P}^\dagger$. We will denote the pullback of $\tilde{\mathcal{P}}$ to $\widetilde{E}_N^\dagger$ by $\tilde{\mathcal{P}}^\dagger$. Let us write

$$q^\dagger : E_N^\dagger \to E_N^\vee$$

for the canonical projection and $\tilde{q}^\dagger$ for the corresponding map on covering spaces. The commutative diagram (1.20) and our chosen trivialization of $\tilde{\mathcal{P}}$ induce a trivializing section for the line bundle $\tilde{\mathcal{P}}^\dagger$

$$\tilde{\mathfrak{t}}^\dagger := (\tilde{q}^\dagger)^*\tilde{\mathfrak{t}} = \frac{1}{J(z,-w',\tau)} \otimes (dz)^\vee .$$

The birigidified Poincaré bundle $\mathcal{P}^\dagger$ on $E_N \times_{M_N} E_N^\dagger$ is equipped with a unique integrable $E_N^\dagger$ connection making it universal among line bundles on E_N with integrable connection $\nabla_{\mathcal{P}^\dagger}$. For details we refer to the exposition in [Sch14, §0.1.1].

Lemma 1.6.3. *The induced connection on $\tilde{\mathcal{P}}^\dagger$ is the unique $\widetilde{E}_N^\dagger$-connection with*

$$\nabla_{\tilde{\mathcal{P}}^\dagger}(\tilde{\mathfrak{t}}^\dagger) = (\eta(1,\tau)w' + u)\tilde{\mathfrak{t}}^\dagger \otimes dz$$

Proof. [Sch14, eq. (3.4.16)]. □

In the same way we can describe the universal vectorial extension $E_N^\sharp$ of E_N. Again, $E_N^{\sharp,an}$ sits in an exact sequence

$$0 \longrightarrow R^1(\pi_{E^\vee}^{an})_*(2\pi i\mathbb{Z}) \longrightarrow \underline{H}^1_{\mathrm{dR}}\left((E_N^\vee)^{an}/M_N^{an}\right) \longrightarrow E_N^{\sharp,an} \longrightarrow 0.$$

In particular, the basis $([\omega],[\eta])$ of

$$\tilde{p}_M^*\underline{H}^1_{\mathrm{dR}}\left(E_N^{an}/M_N^{an}\right) \xrightarrow{\sim} \tilde{p}_M^*\underline{H}^1_{\mathrm{dR}}\left((E_N^\vee)^{an}/M_N^{an}\right)^\vee$$

induces coordinates (z,v) on the universal covering

$$\widetilde{E}_N^\sharp = \mathbb{C}^2 \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \to E_N^{\sharp,an}.$$

We have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{E}_N^\sharp = \mathbb{C}^2 \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times & \xrightarrow{\mathrm{pr}_1} & \widetilde{E}_N = \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \\ \downarrow & & \downarrow \\ E_N^{\sharp,an} & \longrightarrow & E_N^{an}. \end{array} \tag{1.22}$$

Let us denote by $\mathcal{P}^\sharp$ the birigidified Poincaré bundle on $E_N^\sharp \times_{M_N} E_N^\vee$ obtained by pullback of $\mathcal{P}$ and by $\tilde{\mathcal{P}}^\sharp$ its pullback to the universal covering. Let us write

$$q^\sharp : E_N^\sharp \to E_N$$

for the canonical projection and $\tilde{q}^\sharp$ for the corresponding map on covering spaces. The commutative diagram (1.22) and our chosen trivialization of $\tilde{\mathcal{P}}$ induce a trivializing section for $\tilde{\mathcal{P}}^\dagger$

$$\tilde{\mathfrak{t}}^\sharp := (\tilde{q}^\sharp)^* \tilde{\mathfrak{t}} = \frac{1}{J(z,w,\tau)} \otimes (dz)^\vee .$$

Again, the Poincaré bundle $\mathcal{P}^\sharp$ on $E_N^\sharp \times_{M_N} E_N^\vee$ is equipped with a unique integrable $E_N^\sharp$ connection $\nabla_{\mathcal{P}^\sharp}$ making it universal among line bundles on $E_N^\vee$ with integrable connection.

Lemma 1.6.4. *The induced connection on $\tilde{\mathcal{P}}^\sharp$ is the unique $\widetilde{E}_N^\sharp$-connection with*

$$\nabla_{\tilde{\mathcal{P}}^\sharp}(\tilde{\mathfrak{t}}^\sharp) = (-\eta(1,\tau)z + v)\tilde{\mathfrak{t}}^\sharp \otimes dz$$

Proof. Our autoduality induces an isomorphism

$$\tilde{p}_M^* \underline{H}^1_{\mathrm{dR}}\left(E_N^{an}/M_N^{an}\right) \xrightarrow{\sim} \tilde{p}_M^* \underline{H}^1_{\mathrm{dR}}\left((E_N^\vee)^{an}/M_N^{an}\right).$$

It identifies (z,v) with $(-w,u)$ and the result follows from Lemma 1.6.3. □

Finally, let us consider the following diagram:

$$\begin{tikzcd}
 & E_N^\sharp \times_{M_N} E_N^\dagger \arrow[dl, "\mathrm{id}\times q^\dagger"'] \arrow[dr, "q^\sharp\times\mathrm{id}"] & \\
E_N^\sharp \times_{M_N} (E_N^\vee)^{an} \arrow[dr, "q^\sharp\times\mathrm{id}"'] & & E_N \times_{M_N} E_N^\dagger \arrow[dl, "\mathrm{id}\times q^\dagger"] \\
 & E_N \times_{M_N} E_N^\vee &
\end{tikzcd}$$

We have obtained the Poincaré bundles with connections $\mathcal{P}^\dagger$ resp. $\mathcal{P}^\sharp$ by pullback of $\mathcal{P}$ along the lower maps in this diagram. Let us denote the pullback of $\mathcal{P}$ along $q^\sharp \times q^\dagger$ by $\mathcal{P}^{\sharp,\dagger}$. Since it is a pullback of $\mathcal{P}^\dagger$, it is equipped with an integrable $E_N^\dagger$-connection

$$\nabla_\dagger : \mathcal{P}^{\sharp,\dagger} \to \mathcal{P}^{\sharp,\dagger} \otimes_{\mathcal{O}_{E_N^\sharp \times E_N^\dagger}} \Omega^1_{E_N^\sharp \times_{M_N} E_N^\dagger / E_N^\dagger}.$$

At the same time it is a pullback of $\mathcal{P}^\sharp$ and thus equipped with an integrable $E_N^\sharp$-connection

$$\nabla_\sharp : \mathcal{P}^{\sharp,\dagger} \to \mathcal{P}^{\sharp,\dagger} \otimes_{\mathcal{O}_{E_N^\sharp \times E_N^\dagger}} \Omega^1_{E_N^\sharp \times_{M_N} E_N^\dagger / E_N^\sharp}.$$

Let us denote the pullback of $\mathcal{P}^{\sharp,\dagger}$ to the universal covering $\widetilde{E}_N^\sharp \times_{\widetilde{M}_N} \widetilde{E}_N^\dagger$ by $\tilde{\mathcal{P}}^{\sharp,\dagger}$. The trivializing section $\tilde{\mathfrak{t}}$ of $\tilde{\mathcal{P}}$ induces a trivializing section $\tilde{\mathfrak{t}}^{\sharp,\dagger} := (q^\sharp \times q^\dagger)^* \tilde{\mathfrak{t}}$. Let us write

$\tilde{\nabla}_\dagger$ and $\tilde{\nabla}_\sharp$ for the corresponding connections on the universal covering. By the above formulas for $\nabla_{\tilde{\mathcal{P}}^\sharp}$ and $\nabla_{\tilde{\mathcal{P}}^\dagger}$ we get the following explicit formulas for $\tilde{\nabla}_\dagger$ and $\tilde{\nabla}_\sharp$

$$\tilde{\nabla}_\dagger(f(z,v,w',u,\tau,j)\tilde{\mathfrak{t}}^{\sharp,\dagger}) = (\partial_z f)\tilde{\mathfrak{t}}^{\sharp,\dagger}\otimes dz + (\partial_v f)\tilde{\mathfrak{t}}^{\sharp,\dagger}\otimes dv + f\cdot(\eta(1,\tau)w'+u)\,\tilde{\mathfrak{t}}^{\sharp,\dagger}\otimes dz$$

and

$$\tilde{\nabla}_\sharp(f(z,v,w',u,\tau,j)\tilde{\mathfrak{t}}^{\sharp,\dagger}) = (\partial_w f)\tilde{\mathfrak{t}}^{\sharp,\dagger}\otimes dw' + (\partial_u f)\tilde{\mathfrak{t}}^{\sharp,\dagger}\otimes du + f\cdot(-\eta(1,\tau)z+v)\,\tilde{\mathfrak{t}}^{\sharp,\dagger}\otimes dw'.$$

Here, $f(z,v,w',u,\tau,j)$ is an analytic function on the universal covering of $E_N^\sharp \times E_N^\dagger$ in the coordinates introduced above.

1.6.3 The Jacobi and the Kronecker theta function

We recall some properties of the Jacobi theta function which was defined in Eq. (1.19). Then we give an explicit description of the canonical section via the Jacobi theta function. Furthermore, we discuss the relation to the Kronecker theta function. The Jacobi theta function has the following behaviour under the $\mathbb{Z}^2\times\mathbb{Z}^2$-action:

Lemma 1.6.5 ([Sch14, Corollary 3.3.14]). *For $m,n,k,l\in\mathbb{Z}$ we have:*

$$J(z+m\tau+n, w+k\tau+l,\tau) = J(z,w)\exp\left(-2\pi i\left(k\cdot z+m\cdot w\right)\right)$$

Let us briefly discuss the relation between the Jacobi theta function and the *Kronecker theta function* as considered in [BK10b]. The Kronecker theta function and certain translates of it have the advantage of being a generating function for the Eisenstein–Kronecker series $e^*_{a,b}(z_0,w_0,\tau)$. The *Eisenstein–Kronecker series* are defined for $b > a+2\geq 0$ and $\tau\in\mathbb{H}$ by

$$e^*_{a,b}(z_0,w_0;\tau) := \sum_{\gamma\in\Gamma_\tau\setminus\{-z_0\}} \frac{(\bar{z}_0+\bar{\gamma})^a}{(z_0+\gamma)^b}\langle\gamma,w_0\rangle_\tau$$

with $\Gamma_\tau=\mathbb{Z}+\tau\mathbb{Z}$ and $\langle z,w\rangle_\tau := \exp\left(\frac{2\pi i(z\bar{w}-w\bar{z})}{\tau-\bar{\tau}}\right)$. For general $b>a\geq 0$ the Eisenstein–Kronecker series can be defined via analytic continuation, cf. [BK10b, §1.1]. The *Kronecker theta function* is defined as

$$\Theta(z,w,\tau) := \frac{\theta(z+w,\tau)}{\theta(z,\tau)\theta(w,\tau)}$$

with

$$\theta(z,\tau) := \exp\left(-\frac{e_2^*(\tau)}{2}z^2\right)\sigma(z,\tau),\quad e_2^*(\tau) := e^*_{0,2}(0,0;\tau).$$

Note that the Kronecker theta function varies non-holomorphically in τ. The definitions of $J(z,w,\tau)$ and $\Theta(z,w,\tau)$ immediately yield the following equation:

$$J(z,w,\tau) = \exp(-\frac{zw}{A(\tau)})\Theta(z,w,\tau) \tag{1.23}$$

with $A(\tau) := \frac{\tau-\bar{\tau}}{2\pi i}$ varying non-holomorphically in τ. Bannai and Kobayashi define the following translates of $\Theta(z, w, \tau)$

$$\Theta_{z_0,w_0}(z, w, \tau) := \exp\left(-\frac{z_0\bar{w}_0 + z\bar{w}_0 + w\bar{z}_0}{A(\tau)}\right)\Theta(z + z_0, w + w_0, \tau). \tag{1.24}$$

They prove the following.

Theorem 1.6.6 ([BK10b, Thm. 1.17.]). *For fixed τ the Laurent expansion of translates of the Kronecker theta function $\Theta_{z_0,w_0}(z, w, \tau)$ is given by*

$$\begin{aligned}\Theta_{z_0,w_0}(z, w, \tau) =& \langle w_0, z_0\rangle\frac{\delta(z_0)}{z} + \frac{\delta(w_0)}{w} + \\ &+ \sum_{a\geq 0, b\geq 0}(-1)^{a+b}\frac{e^*_{a,b+1}(z_0, w_0)}{a!A^a}z^b w^a\end{aligned} \tag{1.25}$$

where $\delta(x) = 0$ for $x \notin \mathbb{Z} + \tau\mathbb{Z}$ and $\delta(x) = 1$ else.

In analogy with $\Theta_{z_0,w_0}(z, w, \tau)$ let us define

$$\begin{aligned}J_{z_0,w_0}(z, w, \tau) :&= \exp\left(-\frac{zw}{A(\tau)}\right)\Theta_{z_0,w_0}(z, w, \tau) = \\ &= \exp\left(-\frac{zw}{A(\tau)}\right)\exp\left(-\frac{z_0\bar{w}_0 + z\bar{w}_0 + w\bar{z}_0}{A(\tau)}\right)\Theta(z + z_0, w + w_0, \tau) = \\ &= \exp\left(z\frac{w_0 - \bar{w}_0}{A(\tau)}\right)\exp\left(w\frac{z_0 - \bar{z}_0}{A(\tau)}\right)\exp\left(z_0\frac{w_0 - \bar{w}_0}{A(\tau)}\right)J(z + z_0, w + w_0, \tau).\end{aligned} \tag{1.26}$$

Above we have described the canonical section of the Poincaré bundle on the universal covering $\tilde{\mathcal{P}}$ as $J(z, w, \tau)\tilde{\mathfrak{t}} \otimes dz$. A similar result holds if we apply the translation operator $U^{N,D}_{s,t}$ to the canonical section. For the rest of the chapter consider coprime integers $N, D \geq 1$ and let E_{ND} be the universal elliptic curve with $\Gamma(ND)$-level structure. For given $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $(c, d) \in (\mathbb{Z}/D\mathbb{Z})^2$ we get torsion sections $s = s_{a,b} \in E_{ND}[N](M_{ND})$ and $t = t_{c,d} \in E_{ND}[D](M_{ND})$. After passing to the analytification, we can express

$$s, t : M^{an}_{ND} \to E^{an}_{ND}$$

explicitly as

$$\begin{aligned}(\tau, j) &\mapsto s(\tau, j) := (\frac{a}{N}j\tau + \frac{b}{N}, \tau, j)\\ (\tau, j) &\mapsto t(\tau, j) := (\frac{c}{D}j\tau + \frac{d}{D}, \tau, j).\end{aligned}$$

These formulas describe at the same time distinguished lifts

$$\tilde{s}, \tilde{t} : \tilde{M}_{ND} \to \tilde{E}_{ND}.$$

to the universal coverings.

Proposition 1.6.7. *The pullback of $U^{N,D}_{s,t}(s_{\mathrm{can}})$ to the universal covering is given by the explicit formula*

$$([D]\times[N])^*\left(J_{D\tilde{s},N\tilde{t}}(z,w,\tau)\tilde{\mathfrak{t}}\otimes dz\right)$$

where we write $J_{D\tilde{s},N\tilde{t}}(z,w,\tau)$ for $J_{D\tilde{s}(\tau,j),N\tilde{t}(\tau,j)}(z,w,\tau)$ with $\tilde{s}(\tau,j) := \frac{a}{N}j\tau+\frac{b}{N}$ and $\tilde{t}(\tau,j) := \frac{c}{D}j\tau+\frac{d}{D}$.

Proof. Let us denote by $\tilde{s}_{\mathrm{can}}$ the pullback of the analytification of s_{can} to the universal covering. Similarly, let us write $\tilde{U}^{N,D}_{s,t}$ for the pullback of the analytification of the translation operator. Before we give an explicit description of $\tilde{U}^{N,D}_{s,t}(\tilde{s}_{\mathrm{can}})$, let us do the following computation:

$$\begin{aligned}(T_{\tilde{s}}\times T_{\tilde{t}})^*\left([N]\times[D]\right)^*(\tilde{\mathfrak{t}}) &= (T_{\tilde{s}}\times T_{\tilde{t}})^*([N]\times[D])^*\left(\frac{1}{J(z,w,\tau)}\otimes\tilde{\omega}\right) = \\ &= \frac{1}{J(Nz+aj\tau+b,Dw+cj\tau+d,\tau)}\otimes([N]\times[D])^*\tilde{\omega} \overset{\text{L.1.6.5}}{=} \\ &= \exp\left(2\pi i(Nzcj+Dwaj+acj^2\tau)\right)([N]\times[D])^*(\tilde{\mathfrak{t}})\end{aligned} \tag{1.27}$$

By Eq. (1.26) we have

$$\begin{aligned}&J_{D\tilde{s},N\tilde{t}}(Dz,Nw,\tau) := \\ &= \exp\left(2\pi i\cdot(Nzcj+Dwaj+acj^2\tau)\right)J(Dz+D\tilde{s},Nw+N\tilde{t},\tau).\end{aligned}$$

The definition

$$U^{N,D}_{s,t}(s_{\mathrm{can}}) := (\gamma_{N,D}\otimes\mathrm{id}_\Omega)\left((T_s\times T_t)^*\left[(\gamma_{D,N}\otimes\mathrm{id}_\Omega)\left(([D]\times[N])^*s_{\mathrm{can}}\right)\right]\right)$$

gives us the following explicit description of $\tilde{U}^{N,D}_{s,t}(\tilde{s}_{\mathrm{can}})$:

$$\begin{aligned}\tilde{U}^{N,D}_{s,t}(\tilde{s}_{\mathrm{can}}) &= (\tilde{\gamma}_{N,D}\otimes\mathrm{id}_\Omega)\left((T_{\tilde{s}}\times T_{\tilde{t}})^*\left[(\tilde{\gamma}_{D,N}\otimes\mathrm{id}_\Omega)\left(([D]\times[N])^*\tilde{s}_{\mathrm{can}}\right)\right]\right) \overset{\text{Lem.1.6.1+Lem.1.6.2}}{=} \\ &= (\tilde{\gamma}_{N,D}\otimes\mathrm{id}_\Omega)\left((T_{\tilde{s}}\times T_{\tilde{t}})^*\left[J(Dz,Nw,\tau)([N]\times[D])^*(\tilde{\mathfrak{t}}\otimes dz)\right]\right) = \\ &= (\tilde{\gamma}_{N,D}\otimes\mathrm{id}_\Omega)\big(J(Dz+D\tilde{s},Nw+N\tilde{t},\tau)(T_{\tilde{s}}\times T_{\tilde{t}})^*([N]\times[D])^*(\tilde{\mathfrak{t}}\otimes dz)\big) \overset{(1.27)}{=} \\ &= (\tilde{\gamma}_{N,D}\otimes\mathrm{id}_\Omega)\left(J_{D\tilde{s},N\tilde{t}}(Dz,Nw)([N]\times[D])^*(\tilde{\mathfrak{t}}\otimes dz)\right) = \\ &= J_{D\tilde{s},N\tilde{t}}(Dz,Nw)([D]\times[N])^*(\tilde{\mathfrak{t}}\otimes dz) = \\ &= ([D]\times[N])^*\left(J_{D\tilde{s},N\tilde{t}}(z,w)\tilde{\mathfrak{t}}\otimes dz\right)\end{aligned}$$

□

Finally, we remark that the distribution relation combined with the above explicit formula for the translation operators gives the following corollary. Alternatively, this can be obtained from the distribution relation for the Kronecker theta function proved by Bannai and Kobayashi.

Corollary 1.6.8. *For D, D', N, N' coprime and $(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$, $(c,d) \in (\mathbb{Z}/D\mathbb{Z})^2$ define $s = s_{a,b}$ and $t = t_{c,d}$ and the lifts $\tilde{s}, \tilde{t}$ as above. Then,*

$$\sum_{\substack{\alpha \in E^{an}_{N'}[N'](M^{an}_{N'}),\\ \beta \in E^{an}_{D'}[D'](M^{an}_{D'})}} J_{D'\tilde{s}+\tilde{\alpha}, N'\tilde{t}+\tilde{\beta}}(D'z, N'w, \tau) = D'N' J_{N'\tilde{s}, D'\tilde{t}}(N'z, D'w, \tau) \tag{1.28}$$

where we denote by $\tilde{\alpha}$ resp. $\tilde{\beta}$ are arbitrary lifts of N' resp. D'-torsion sections to the universal covering.

Proof. This follows by a straight forward computation from the distribution relation Theorem 1.4.3 and the explicit description of $\widetilde{U}^{*;*}_{*,*}(\tilde{s}_{\mathrm{can}})$. □

Remark 1.6.9. As remarked above there is no reason to choose the Jacobi theta function to trivialize the Poincaré bundle. Every other theta function with the same divisor gives a trivialization of the Poincaré bundle and would work equally well. We would like to emphasize that for our purposes it would equally well be possible to view $\widetilde{E}_N$ as a real manifold and work with a non-holomorphic theta function. The point is that we only need the analytification as a tool for comparing purely algebraic constructions. Thus, its no problem to do this comparison after the injection $\mathcal{O}^{an}_{\widetilde{E}_N} \hookrightarrow \mathcal{C}^\infty(\widetilde{E}_N)$. In particular, we could trivialize the Poincaré bundle using the *Kronecker theta function*. Then, the algebraic distribution relation immediately specializes to the distribution relation in [BK10b].

1.7 Real-analytic Eisenstein series via the Poincaré bundle

In this section we give a functorial construction of real-analytic Eisenstein series as classes in

$$\underline{\mathrm{Sym}}^k_{\mathcal{O}_S} \underline{H}^1_{\mathrm{dR}}(E/S).$$

The Hodge decomposition on the universal elliptic curve

$$\underline{\mathrm{Sym}}^k \underline{H}^1_{\mathrm{dR}}(E_N/M_N) \otimes \mathcal{C}^\infty(M_N^{an}) \twoheadrightarrow \underline{\omega}^{\otimes k}_{E_N/M_N} \otimes \mathcal{C}^\infty(M_N^{an})$$

allows us to get the link back to classical $\mathcal{C}^\infty$-modular forms.

Let E/S be an elliptic curve over some scheme S. We denote by

$$E^\sharp \xrightarrow{q^\sharp} E \qquad \text{and} \qquad E^\dagger \xrightarrow{q^\dagger} E^\vee$$

the universal vectorial extension of E and $E^\vee$. Let us write $\mathcal{P}^\sharp$ resp. $\mathcal{P}^\dagger$ for the pullbacks of $\mathcal{P}$ to $E^\sharp \times_S E^\vee$ resp. $E \times_S E^\dagger$. Then, $\mathcal{P}^\sharp$ resp. $\mathcal{P}^\dagger$ are equipped with canonical integrable

$E^\sharp$- resp. $E^\dagger$-connections. Let us write $\mathcal{P}^{\sharp,\dagger}$ for the pullback of $\mathcal{P}$ along $q^\sharp\times q^\dagger$

$$\begin{tikzcd}[column sep=tiny] & E_N^\sharp \times_{M_N} E_N^\dagger \arrow[dl, "\mathrm{id}\times q^\dagger"'] \arrow[dr, "q^\sharp\times\mathrm{id}"] & \\ E_N^\sharp \times_{M_N} (E_N^\vee)^{an} \arrow[dr, "q^\sharp\times\mathrm{id}"'] & & E_N \times_{M_N} E_N^\dagger \arrow[dl, "\mathrm{id}\times q^\dagger"] \\ & E_N \times_{M_N} E_N^\vee & \end{tikzcd}$$

Then, $\mathcal{P}^{\sharp,\dagger}$ is in a natural way equipped with both an integrable $E^\sharp$- and an integrable $E^\dagger$-connection

$$\mathcal{P}^{\sharp,\dagger} \xrightarrow{\nabla_\sharp} \mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_{E^\sharp\times E^\dagger}} \Omega^1_{E^\sharp\times E^\dagger/E^\sharp}$$

$$\mathcal{P}^{\sharp,\dagger} \xrightarrow{\nabla_\dagger} \mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_{E^\sharp\times E^\dagger}} \Omega^1_{E^\sharp\times E^\dagger/E^\dagger}.$$

We have

$$\begin{aligned}\Omega^1_{E^\sharp\times E^\dagger/E^\sharp} &\xrightarrow{\sim} \mathrm{pr}^*_{E^\dagger}\Omega^1_{E^\dagger/S} \xrightarrow{\sim} (\pi_{E^\sharp\times E^\dagger})^* e^*\Omega^1_{E^\dagger/S} \xrightarrow{\sim} (\pi_{E^\sharp\times E^\dagger})^*\underline{H}^1_{\mathrm{dR}}(E^\vee/S)\\ \Omega^1_{E^\sharp\times E^\dagger/E^\dagger} &\xrightarrow{\sim} \mathrm{pr}^*_{E^\sharp}\Omega^1_{E^\sharp/S} \xrightarrow{\sim} (\pi_{E^\sharp\times E^\dagger})^* e^*\Omega^1_{E^\sharp/S} \xrightarrow{\sim} (\pi_{E^\sharp\times E^\dagger})^*\underline{H}^1_{\mathrm{dR}}(E/S).\end{aligned} \tag{1.29}$$

Let us abbreviate for the moment

$$\mathcal{H}^{i,j} := \underline{\mathrm{Sym}}^i_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}(E^\vee/S)\otimes_{\mathcal{O}_S}\underline{\mathrm{Sym}}^j_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}(E/S).$$

Since both $\nabla_\sharp$ and $\nabla_\dagger$ are $(\pi_{E^\sharp\times E^\dagger})^{-1}\mathcal{O}_S$-linear, we can define the following differential operators:

$$\nabla_\sharp\colon \mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\mathcal{H}^{i,j} \xrightarrow{\nabla_\sharp\otimes\mathrm{id}} \mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\mathcal{H}^{i+1,j}$$

$$\nabla_\dagger\colon \mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\mathcal{H}^{i,j} \xrightarrow{\nabla_\dagger\otimes\mathrm{id}} \mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\mathcal{H}^{i,j+1}$$

Applying $\nabla_\sharp$ and $\nabla_\dagger$ iteratively leads to

$$\nabla^{k,r}_{\sharp,\dagger}\colon \mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\mathcal{H}^{i,j} \xrightarrow{\nabla_\sharp^{\circ k}\circ\nabla_\dagger^{\circ r}} \mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\mathcal{H}^{i+k,j+r}.$$

Let us remark that $\nabla_\sharp$ and $\nabla_\dagger$ do in general not commute. Similarly, we can define for $N,D\geq 1$

$$([D]\times[N])^*\nabla^{k,r}_{\sharp,\dagger}\colon ([D]\times[N])^*\mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\mathcal{H}^{i,j}\longrightarrow ([D]\times[N])^*\mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\mathcal{H}^{i+k,j+r}$$

by iteratively applying the pullback connections $([D]\times[N])^*\nabla_\sharp$ and $([D]\times[N])^*\nabla_\dagger$.

For coprime integers $N, D \geq 1$ and $e \neq s \in E[N](S), e \neq t \in E^\vee[D](S)$ define

$$\sigma_{s,t}^{N,D} := U_{s,t}^{N,D}(s_{\mathrm{can}}) \in \Gamma\left(U, ([N]\times[D])^*\mathcal{P} \otimes \Omega^1_{E\times_S E^\vee/E^\vee}\right)$$

where $U := ([D]\times[N])^{-1}(T_{Ds}\times T_{Nt})^{-1}\left(E\times E^\vee \setminus \{E\times e \amalg e\times E^\vee\}\right)$. Here, we have implicitly used the canonical isomorphism

$$([D]\times[N])^*(T_{Ds}\times T_{Nt})^*\Omega^1_{E\times_S E^\vee/E^\vee}\left([E\times e]+[e\times E^\vee]\right)\Big|_U \cong \Omega^1_{E\times_S E^\vee/E^\vee}\Big|_U .$$

Since we have assumed $e \neq s, e \neq t$ and N, D coprime, the morphism

$$(e\times e): S = S\times_S S \to E\times_S E^\vee$$

factors through the open subset U. Via the isomorphism in (1.29) we obtain a canonical inclusion

$$(q^\sharp\times q^\dagger)^*\left[([N]\times[D])^*\mathcal{P}\otimes_{\mathcal{O}_{E\times_S E^\vee}}\Omega^1_{E\times_S E^\vee/E^\vee}\right] \hookrightarrow ([N]\times[D])^*\mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}(E/S)$$

which allows us to view

$$(q^\sharp\times q^\dagger)^*\sigma_{s,t}^{N,D}\in\Gamma\left((q^\sharp\times q^\dagger)^{-1}U, ([N]\times[D])^*\mathcal{P}^{\sharp,\dagger}\otimes_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}(E/S)\right).$$

Definition 1.7.1. For coprime integers $N, D \geq 1$, torsion sections $e \neq s \in E[N](S)$, $e \neq t \in E^\vee[D](S)$ and $k, r \geq 0$ define

$$E_{s,t}^{k,r+1}\in\Gamma\left(S, \underline{\mathrm{Sym}}^k_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}(E^\vee/S)\otimes_{\mathcal{O}_S}\underline{\mathrm{Sym}}^{r+1}_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}(E/S)\right)$$

via

$$E_{s,t}^{k,r+1} := (e\times e)^*\left[\left(([D]\times[N])^*\nabla^{k,r}_{\sharp,\dagger}\right)\left((q^\sharp\times q^\dagger)^*\sigma_{s,t}^{N,D}\right)\right].$$

We call $E_{s,t}^{k,r+1}$ *algebraic Eisenstein–Kronecker series.*

We will prove that the algebraic Eisenstein–Kronecker series give rise to the real-analytic Eisenstein–Kronecker series

$$\frac{e_{k,r+1}(s,t)}{A^k k!}$$

via the Hodge decomposition on the analytification of the universal elliptic curve with $\Gamma(ND)$-level structure. Again, our definition of $E_{s,t}^{k,r+1}$ is intrinsic, i. e. does not refer to a chosen autoduality isomorphism. In order to relate

$$E_{s,t}^{k,r+1}\in\Gamma\left(S, \underline{\mathrm{Sym}}^k_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}(E^\vee/S)\otimes_{\mathcal{O}_S}\underline{\mathrm{Sym}}^{r+1}_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}(E/S)\right)$$

to $\mathcal{C}^\infty$-modular forms, it will nevertheless be convenient to fix an autoduality isomorphism. Until the end of this section let us identify E with $E^\vee$ via our chosen autoduality from Section 1.3

$$\lambda : E \xrightarrow{\sim} E^\vee.$$

Let $N, D > 1$ be coprime as above and let $E_{ND} \to M_{ND}$ be the universal elliptic curve with $\Gamma(ND)$-level structure. Note that $N, D > 1$ coprime implies $ND > 3$ and thus the moduli problem is representable. Let E_{ND} be the universal elliptic curve with $\Gamma(ND)$-level structure. We take the explicit description of the analytification of E_{ND} as introduced in Section 1.6.

Theorem 1.7.2. *Let $N, D > 1$ be coprime and let us write $M = M_{ND}$ resp. $E = E_{MN}$ for the modular curve resp. the universal elliptic curve of level $\Gamma(ND)$. For chosen $(0,0) \neq (a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $(0,0) \neq (c,d) \in (\mathbb{Z}/D\mathbb{Z})^2$ let $s = s_{a,b} \in E[N](M)$ and $t = t_{c,d} \in E^\vee[D](M) \simeq E[D](M)$ be the associated torsion sections. The algebraic Eisenstein–Kronecker series $E_{s,t}^{k,r+1}$ on the universal elliptic curve with $\Gamma(ND)$-level structure is uniquely determined by its associated $\mathcal{C}^\infty$-modular form obtained by the Hodge decomposition*

$$\left(\underline{\mathrm{Sym}}^{k+r+1}_{\mathcal{O}_M} \underline{H}^1_{\mathrm{dR}}(E/M)\right)^{an} \otimes_{\mathcal{O}_M^{an}} \mathcal{C}^\infty(M^{an}) \twoheadrightarrow \left(\underline{\omega}_{E^{an}/M^{an}}(\mathcal{C}^\infty)\right)^{\otimes(k+r+1)}.$$

The $\mathcal{C}^\infty$-modular form associated with $E_{s,t}^{k,r+1}$ is the real-analytic Eisenstein series

$$(-1)^{r+k} k! r! \frac{e^*_{k,r+1}(D\tilde{s}, N\tilde{t})}{A^k k!} dz^{\otimes(k+r+1)},$$

where we write $\tilde{s} = \tilde{s}(\tau, j) = \frac{a}{N} j\tau + \frac{b}{N}$ and $t = t(\tau, j) = \frac{c}{D} j\tau + \frac{d}{D}$.

Proof. The construction of $E_{s,t}^{r,k+1}$ is compatible with base change and isomorphisms of elliptic curves. Thus, $E_{s,t}^{r,k+1}$ is uniquely determined by its value on the universal elliptic curve $(E/M, \alpha_{ND}^{univ})$ with $\Gamma(ND)$-level structure. Further, M is flat over $\operatorname{Spec} \mathbb{Z}[\frac{1}{ND}]$ and $\underline{\mathrm{Sym}}^{k+r+1} \underline{H}^1_{\mathrm{dR}}(E/M)$ is locally free of finite rank. Since M is affine,

$$\Gamma\left(M, \underline{\mathrm{Sym}}^{k+r+1} \underline{H}^1_{\mathrm{dR}}(E/M)\right)$$

is a flat $\mathbb{Z}[\frac{1}{ND}]$-module and the inclusion

$$\begin{aligned}\Gamma\left(M, \underline{\mathrm{Sym}}^{k+r+1} \underline{H}^1_{\mathrm{dR}}(E/M)\right) \hookrightarrow &\Gamma\left(M, \underline{\mathrm{Sym}}^{k+r+1} \underline{H}^1_{\mathrm{dR}}(E/M)\right) \otimes_{\mathbb{Z}[\frac{1}{N}]} \mathbb{Q} = \\ &= \Gamma\left(M_{\mathbb{Q}}, \underline{\mathrm{Sym}}^{k+r+1} \underline{H}^1_{\mathrm{dR}}(E_{QQ}/M_{QQ})\right)\end{aligned}$$

shows that $E_{s,t}^{N,D}$ is uniquely determined by its value on the universal elliptic curve $E_{\mathbb{Q}}/M_{\mathbb{Q}}$ with $\Gamma(ND)$-level structure over $\mathbb{Q}$. Further, the map given by analytification and Hodge decomposition

$$\Gamma\left(M_{\mathbb{C}}, \underline{\mathrm{Sym}}^{k+r+1} \underline{H}^1_{\mathrm{dR}}(E_{\mathbb{C}}/M_{\mathbb{C}})\right) \hookrightarrow \Gamma\left(M_{\mathbb{C}}^{an}, \left(\underline{\omega}_{E^{an}/M^{an}}(\mathcal{C}^\infty)\right)^{\otimes(k+r+1)}\right)$$

is easily seen to be injective, cf. [Urb14, §2]. We conclude that $E_{s,t}^{N,D}$ is indeed uniquely determined by the image under the Hodge decomposition on the universal elliptic curve.

Thus, it remains to compute the value of $E^{k,r+1}_{s,t}$ under Hodge decomposition on the universal elliptic curve. Let us write $\left(E^{k,r+1}_{s,t}\right)^{an} \in \Gamma\left(M^{an}, \left(\underline{\mathrm{Sym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}(E/M)\right)^{an}\right)$ for the analytification of $E^{k,r+1}_{s,t}$ evaluated on $(E/M,\alpha)$. In the following we use the explicit models for the analytifications of E, $E^\vee$, $E^\dagger$, $E^\sharp$ and the corresponding universal covering spaces $\tilde{E}$, $\tilde{E}^\vee$, $\tilde{E}^\dagger$, $\tilde{E}^\sharp$. Let us summarize the notation from the last section in the following diagrams:

$$\begin{tikzcd}
\tilde{E} \arrow[r, "\tilde{p}_E"] \arrow[d, "\tilde{\pi}_E"] & E^{an} \arrow[d, "\pi_E^{an}"] \\
\tilde{M} \arrow[r, "\tilde{p}_M"] \arrow[u, bend left, "(0)"] & M^{an} \arrow[u, bend left, "e"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\tilde{E}^\dagger \arrow[r, "\tilde{q}^\dagger"] \arrow[d, "\tilde{\pi}_{E^\dagger}"'] & \tilde{E}^\vee \arrow[dl, "\pi^{an}_{E^\vee}"] \\
\tilde{M} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{E}^\sharp \arrow[r, "\tilde{q}^\sharp"] \arrow[d, "\tilde{\pi}_{E^\sharp}"'] & \tilde{E} \arrow[dl, "\pi^{an}_{E}"] \\
\tilde{M} &
\end{tikzcd}$$

Let us define

$$\tilde{\mathcal{H}}^{i,j} := \tilde{p}_M^*\left(\mathcal{H}^{i,j}\right)^{an}$$

and $\tilde{\nabla}_\sharp$, $\tilde{\nabla}_\dagger$ and $\tilde{\nabla}^{k,r}_{\sharp,\dagger}$ for the induced differential operators on $\tilde{\mathcal{P}}^{\sharp,\dagger}$. If we combine the canonical derivations with the isomorphisms from (1.29), we obtain:

$$d_{\tilde{E}^\dagger}: \mathcal{O}^{an}_{\tilde{E}^\sharp\times_{\tilde{M}}\tilde{E}^\dagger} \xrightarrow{d} \Omega^1_{\tilde{E}^\sharp\times_{\tilde{M}}\tilde{E}^\dagger/\tilde{E}^\dagger} \xrightarrow{\sim} \mathcal{O}^{an}_{\tilde{E}^\sharp\times_{\tilde{M}}\tilde{E}^\dagger}\otimes_{\mathcal{O}^{an}_{\tilde{M}}} (\tilde{p}_M)^*\left(\underline{H}^1_{\mathrm{dR}}(E/M)\right)^{an}$$

$$d_{\tilde{E}^\sharp}: \mathcal{O}^{an}_{\tilde{E}^\sharp\times_{\tilde{M}}\tilde{E}^\dagger} \xrightarrow{d} \Omega^1_{\tilde{E}^\sharp\times_{\tilde{M}}\tilde{E}^\dagger/\tilde{E}^\sharp} \xrightarrow{\sim} \mathcal{O}^{an}_{\tilde{E}^\sharp\times_{\tilde{M}}\tilde{E}^\dagger}\otimes_{\mathcal{O}^{an}_{\tilde{M}}} (\tilde{p}_M)^*\left(\underline{H}^1_{\mathrm{dR}}(E^\vee/M)\right)^{an}$$

In particular, we get

$$d_{E^\dagger}: \tilde{\mathcal{H}}^{i,j} \to \tilde{\mathcal{H}}^{i,j+1}, \quad d_{E^\sharp}: \tilde{\mathcal{H}}^{i,j} \to \tilde{\mathcal{H}}^{i+1,j}.$$

Using the Leibniz rule and the explicit formulas for $\tilde{\nabla}_\sharp$ and $\tilde{\nabla}_\sharp$ from Section 1.6.2, we obtain

$$\tilde{\nabla}_\sharp^{\circ k}\left(f(z,v,w',u,\tau,j)\tilde{\mathfrak{t}}\right) = \sum_{j=0}^{k}\binom{k}{j} d^{\circ(k-j)}_{\tilde{E}^\sharp}(f)\tilde{\nabla}_\sharp^{\circ j}(\tilde{\mathfrak{t}}) = \tag{1.30}$$

$$= \sum_{j=0}^{k}\binom{k}{j}\left(-\eta(1,\tau)z+v\right)^j d^{\circ(k-j)}_{\tilde{E}^\sharp}(f)\tilde{\mathfrak{t}}\otimes([\eta^\vee])^j \tag{1.31}$$

and

$$\tilde{\nabla}_\dagger^{\circ r}\left(f(z,v,w',u,\tau,j)\tilde{\mathfrak{t}}\right) = \sum_{i=0}^{r}\binom{r}{i} d^{\circ(r-i)}_{\tilde{E}^\dagger}(f)\tilde{\nabla}_\dagger^{\circ i}(\tilde{\mathfrak{t}}) = \tag{1.32}$$

$$= \sum_{i=0}^{r}\binom{r}{i}\left(\eta(1,\tau)w'+u\right)^i d^{\circ(r-i)}_{\tilde{E}^\dagger}(f)\tilde{\mathfrak{t}}\otimes([\omega])^i. \tag{1.33}$$

The pullback $\tilde{\sigma}^{N,D}_{s,t}$ of $(\sigma^{N,D}_{s,t})^{an} := \left(U^{N,D}_{s,t}(s_{\mathrm{can}})\right)^{an}$ to $\tilde{E} \times_{\tilde{M}} \tilde{E}$ is given according to Proposition 1.6.7 explicitly by the formula

$$\tilde{\sigma}^{N,D}_{s,t} = ([D] \times [N])^* \left(J_{D\tilde{s},N\tilde{t}}(z,w,\tau)\tilde{\mathfrak{t}} \otimes dz\right).$$

This gives the following explicit description

$$(\tilde{q}^\sharp \times \tilde{q}^\dagger)^* \tilde{\sigma}^{N,D}_{s,t} = ([D] \times [N])^* \left(J_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\tilde{\mathfrak{t}}^{\sharp,\dagger} \otimes [\omega]\right).$$

Now, we have everything at hand to compute $(\tilde{q}^\sharp \times \tilde{q}^\dagger)^* \left(E^{k,r+1}_{s,t}\right)^{an}$ explicitly:

$$\begin{aligned}
(\tilde{p}_M)^* \left(E^{k,r+1}_{s,t}\right)^{an} &= (0\times 0)^* \left[\left(([D]\times[N])^* \tilde{\nabla}^{k,r}_{\sharp,\dagger}\right)\left((q^\sharp \times q^\dagger)^* \tilde{\sigma}^{N,D}_{s,t}\right)\right] = \\
=&(0\times 0)^* \left[([D]\times[N])^* \left(\tilde{\nabla}^{k,r}_{\sharp,\dagger}\left(J_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\tilde{\mathfrak{t}}\right)\right)\right] \otimes [\omega] = \\
=&(0\times 0)^* \left[\tilde{\nabla}^{k,r}_{\sharp,\dagger}\left(J_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\tilde{\mathfrak{t}}\right)\right] \otimes [\omega] \overset{(1.30)}{=} \\
=&(0\times 0)^* \left[\tilde{\nabla}^{\circ k}_{\sharp}\left(\sum_{i=0}^{r}\binom{r}{i}(\eta(1,\tau)w'+u)^i \partial_z^{\circ(r-i)} J_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\tilde{\mathfrak{t}}\right)\right] \otimes [\omega]^{\otimes(r+1)} \overset{(1.32)}{=} \\
=& \sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{r}\binom{r}{i} d^{\circ(k-j)}_{\tilde{E}^\sharp}\left[(\eta(1,\tau)w'+u)^i\partial_z^{\circ(r-i)}J_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\right]\Big|_{\substack{z=v=0\\ w'=u=0}} \cdot \\
&\cdot \underbrace{(-\eta(1,\tau)z+v)^j}_{=0 \text{ for } j\neq 0, z=v=0}\Big|_{\substack{z=v=0\\ w'=u=0}} \otimes [\omega]^{\otimes(r+1)} = \\
=& \sum_{i=0}^{r}\binom{r}{i} d^{\circ k}_{\tilde{E}^\dagger}\left((\eta(1,\tau)w'+u)^i\partial_z^{\circ(r-i)}J_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\right)\Big|_{\substack{z=v=0\\ w'=u=0}} \otimes [\omega]^{\otimes(r+1)} = \\
=& d^{\circ k}_{\tilde{E}^\dagger}\left(\sum_{i=0}^{r}\binom{r}{i}(\eta(1,\tau)w'+u)^i\, \partial_z^{\circ(r-i)}J_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\Big|_{z=v=0}\right)\Big|_{w'=u=0} \otimes [\omega]^{\otimes(r+1)} = \\
=& d^{\circ k}_{\tilde{E}^\dagger}\left[\partial_z^{\circ r}\left(\exp\left[z(w'\eta(1,\tau)+u)\right] J_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\right)\Big|_{z=v=0}\right]\Big|_{w'=u=0} \otimes [\omega]^{\otimes(r+1)} = \\
=& d^{\circ k}_{\tilde{E}^\dagger}\left[\partial_z^{\circ r}\left(\exp\left[z(w'\eta(1,\tau)+u)\right] J_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\right)\right]\Big|_{\substack{z=v=0\\ w'=u=0}} \otimes [\omega]^{\otimes(r+1)} \overset{(1.23)}{=} \\
=& d^{\circ k}_{\tilde{E}^\dagger}\left[\partial_z^{\circ r}\left(\exp\left[z(w'\eta(1,\tau)+u)\right]\exp\left[\frac{zw'}{A(\tau)}\right]\Theta_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\right)\right]\Big|_{\substack{z=v=0\\ w'=u=0}} \otimes [\omega]^{\otimes(r+1)} \overset{\text{Leibniz}}{=} \\
=& d^{\circ k}_{\tilde{E}^\dagger}\left[\sum_{i=0}^{r}\binom{r}{i}\left(w'\eta(1,\tau)+u+\frac{w'}{A(\tau)}\right)^i \partial_z^{\circ(r-i)}\Theta_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\right]\Big|_{\substack{z=v=0\\ w'=u=0}} \otimes [\omega]^{\otimes(r+1)} =
\end{aligned}$$

$$= \sum_{i=0}^{r} \binom{r}{i} d^{\circ k}_{\tilde{E}^\dagger} \left[\left(w'\eta(1,\tau) + u + \frac{w'}{A(\tau)} \right)^i \partial_z^{\circ(r-i)} \Theta_{D\tilde{s},N\tilde{t}}(z,-w',\tau) \right] \Bigg|_{\substack{z=v=0\\ w'=u=0}} \otimes [\omega]^{\otimes(r+1)} =$$

$$= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{k} \binom{k}{j} \underbrace{d^{\circ j}_{\tilde{E}^\dagger} \left[\left(w'\eta(1,\tau) + u + \frac{w'}{A(\tau)} \right)^i \right]}_{=0 \text{ for } i\neq j} \Bigg|_{w'=u=0} \cdot$$

$$\cdot\, \partial_{w'}^{\circ(k-j)} \partial_z^{\circ(r-i)} \Theta_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\Big|_{z=w'=0} \otimes [\omega]^{\otimes(r+1)} =$$

$$= \sum_{i=0}^{\min(k,r)} \binom{r}{i} \binom{k}{i} \partial_{w'}^{\circ(k-i)} \partial_z^{\circ(r-i)} \Theta_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\Big|_{z=w'=0} \otimes$$

$$\otimes \left(\eta(1,\tau)[\eta^\vee] + [\omega^\vee] + \frac{[\eta^\vee]}{A(\tau)} \right)^{\otimes i} \otimes [\eta^\vee]^{\otimes(k-i)} \otimes [\omega]^{\otimes(r+1)}$$

Our choice of autoduality induces an isomorphism

$$\tilde{p}_M^* \left(\left(\underline{H}^1_{\mathrm{dR}}(E/M) \right)^\vee \right)^{an} \xrightarrow{\sim} \tilde{p}_M^* \left(\underline{H}^1_{\mathrm{dR}}(E/M) \right)^{an}$$

which identifies $[\eta]^\vee \mapsto -[\omega]$ and $[\omega]^\vee \mapsto [\eta]$. Using this, we can summarize the above computation as the following equality in $\tilde{p}_M^* \left(\underline{\mathrm{Sym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}(E/M) \right)^{an} \otimes \mathcal{C}^\infty$:

$$(\tilde{p}_M)^* \left(E^{k,r+1}_{s,t} \right)^{an} = \sum_{i=0}^{\min(k,r)} \binom{r}{i} \binom{k}{i} (-1)^{k-i}\, \partial_w^{\circ(k-i)} \partial_z^{\circ(r-i)} \Theta_{D\tilde{s},N\tilde{t}}(z,-w,\tau)\Big|_{z=w=0} \otimes$$

$$\otimes \left(-\eta(1,\tau)[\omega] + [\eta] - \frac{[\omega]}{A(\tau)} \right)^{\otimes i} \otimes [\omega]^{\otimes(k+r+1-i)}$$

To conclude the theorem it suffices now to remark that we have the following equality in $\tilde{p}_M^* \underline{H}^1_{\mathrm{dR}}(E/M)^{an} \otimes \mathcal{C}^\infty(\tilde{M})$:

$$\frac{[d\bar{z}]}{A(\tau)} = \eta(1,\tau)[\omega] - [\eta] + \frac{[\omega]}{A(\tau)}$$

Here, $[\bar{dz}]$ is the class of $d\bar{z}$ in $\tilde{p}_M^* \underline{H}^1_{\mathrm{dR}}(E/M)^{an} \otimes \mathcal{C}^\infty(\tilde{M})$ and $\eta(1,\tau) = \zeta(z,\tau) - \zeta(z+1,\tau)$ is the period of the Weierstrass zeta function. The above formula can be deduced from [Kat76, p. 1.3.4]. Using this formula, the image of $(\tilde{p}_M)^* \left(E^{k,r+1}_{s,t} \right)^{an}$ under the Hodge decomposition is:

$$\mathit{Hodge}\left[(\tilde{p}_M)^* \left(E^{k,r+1}_{s,t} \right)^{an} \right] = (-1)^k\, \partial_{w'}^{\circ k} \partial_z^{\circ r} \Theta_{D\tilde{s},N\tilde{t}}(z,-w',\tau)\Big|_{z=w=0} \otimes \omega^{\otimes(k+r+1)} =$$

$$= (-1)^{r+k} k! r! \frac{e^*_{k,r+1}(D\tilde{s}, N\tilde{t})}{A^k k!}$$

This concludes the proof of the theorem. ∎

Remark 1.7.3. The construction of the algebraic Eisenstein–Kronecker series is compatible with base change and isomorphisms of elliptic curves. Thus, we can view the construction

$$(E/S, s, t) \mapsto E^{k,r+1}_{s,t} \in \underline{\mathrm{Sym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}(E/S)$$

as a geometric nearly holomorphic modular form as discussed by Urban in [Urb14, §2]. We refer to [Urb14] for more on nearly holomorphic modular forms.

1.8 Symmetry and the functional equation

In the last section we have defined for $N, D > 1$ coprime integers and $e \neq s \in E[N](S)$, $e \neq t \in E^\vee[D](S)$ algebraic Eisenstein–Kronecker series

$$E^{k,r+1}_{s,t} \in \Gamma\left(S, \underline{\mathrm{Sym}}^k_{\mathcal{O}_S}\, \underline{H}^1_{\mathrm{dR}}(E^\vee/S) \otimes_{\mathcal{O}_S} \underline{\mathrm{Sym}}^{r+1}_{\mathcal{O}_S}\, \underline{H}^1_{\mathrm{dR}}(E/S)\right).$$

Further, we have proved that they correspond to Eisenstein–Kronecker series

$$(-1)^{r+k} k! r! \frac{e^*_{k,r+1}(D\tilde{s}, N\tilde{t})}{A^k k!}$$

with $\tilde{s} = \tilde{s}(\tau, j) = \frac{a}{N} j\tau + \frac{b}{N}$ and $\tilde{t} = \tilde{t}(\tau, j) = \frac{c}{D} j\tau + \frac{d}{D}$ under the Hodge decomposition. It is a well-known consequence (cf. [BK10b]) of the functional equation of the Eisenstein–Kronecker–Lerch series, which for $\operatorname{Re} s > a/2 + 1$ is defined by

$$K^*_a(z_0, w_0, s; \Gamma_\tau) := \sum_{-z_0 \neq \gamma \in \Gamma_\tau} \frac{(\bar{z}_0 + \bar{\gamma})^a}{|z + \gamma|^{2s}} \langle \gamma, w_0 \rangle_\tau$$

and for general s by analytic continuation that we have

$$r! \frac{e^*_{k,r+1}(z_0, w_0)}{A^k} = k! \frac{e^*_{r+1,k}(w_0, z_0)}{A^{r+1}} \langle w_0, z_0 \rangle_\tau. \tag{F.E.}$$

In our construction this aspect of the functional equation can be seen as symmetry of the Poincaré bundle. For every choice of autoduality $\tilde{\lambda} : E \xrightarrow{\sim} E^\vee$ the maps

$$\lambda^* : \underline{H}^1_{\mathrm{dR}}(E^\vee/S) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}(E/S), \quad (\lambda^{-1})^* : \underline{H}^1_{\mathrm{dR}}(E/S) \to \underline{H}^1_{\mathrm{dR}}(E^\vee/S)$$

induce isomorphisms

$$(\tilde{\lambda}^{-1} \otimes \tilde{\lambda})^* : \mathcal{H}^{k,r+1} \xrightarrow{\sim} \mathcal{H}^{r+1,k}.$$

If we apply the construction of the algebraic Eisenstein–Kronecker series to the elliptic curve $E^\vee/S$ for given $\sigma \in E^\vee[N](S)$, $\tau \in (E^\vee)^\vee[D](S) = E[D](S)$, we obtain

$$E^{k,r+1}_{\sigma,\tau,(E^\vee/S)} \in \Gamma\left(S, \underline{\mathrm{Sym}}^k\, \underline{H}^1_{\mathrm{dR}}(E/S) \otimes \underline{\mathrm{Sym}}^{r+1}\, \underline{H}^1_{\mathrm{dR}}(E^\vee/S)\right).$$

The symmetry of the Poincaré bundle gives us immediately the following geometric functional equation.

Proposition 1.8.1 (Geometric functional equation)**.** *Let $N, D > 1$ coprime and consider non-zero torsion sections $e \neq s \in E[N](S)$ and $e \neq t \in E^\vee[D](S)$. For every choice of autoduality $\tilde{\lambda} : E \xrightarrow{\sim} E^\vee$ we have the following geometric functional equation:*

$$(\tilde{\lambda}^{-1} \otimes \tilde{\lambda})^* E^{k,r+1}_{s,t} = E^{k,r+1}_{\tilde{\lambda}^{-1}(t),\tilde{\lambda}(s),(E^\vee/S)}$$

Proof. We have already seen that $(\tilde{\lambda} \times \tilde{\lambda}^{-1})^*(s_{\mathrm{can,E}}) = s_{\mathrm{can,E^\vee}}$ under the identification

$$(\tilde{\lambda} \times \tilde{\lambda}^{-1})^* \mathcal{P}_E = \mathcal{P}_{E^\vee}.$$

The isomorphism

$$(\tilde{\lambda} \times \tilde{\lambda}^{-1}) : E \times_S E^\vee \xrightarrow{\sim} E^\vee \times_S E$$

induces an isomorphism

$$(\tilde{\lambda} \times \tilde{\lambda}^{-1}) : E^\sharp \times_S E^\dagger \xrightarrow{\sim} E^\dagger \times_S E^\sharp.$$

and we get $(\tilde{\lambda} \times \tilde{\lambda}^{-1})^* \mathcal{P}^{\sharp,\dagger}_E = \mathcal{P}^{\sharp,\dagger}_{E^\vee}$. One easily checks that $(\tilde{\lambda} \times \tilde{\lambda}^{-1})^* \nabla_\sharp = \nabla_\dagger$ and $(\tilde{\lambda} \times \tilde{\lambda}^{-1})^* \nabla_\dagger = \nabla_\sharp$ under this identification. Now, the claim follows from the definition of $E^{k,r+1}_{s,t}$. □

Viewing $E^{k,r+1}_{s,t}$ as nearly holomorphic modular forms this means that the involution on test objects

$$(E, s, t) \mapsto (E^\vee, \lambda^{-1}(t), \lambda(s))$$

corresponds to pullback along $(\lambda^{-1} \otimes \lambda)$.

Remark 1.8.2. Indeed, this reflects the functional equation. If we choose $\lambda : E \xrightarrow{\sim} E^\vee$, the geometric functional equation gives the following identity after applying the Hodge decomposition:

$$(-1)^{k+r} r! \frac{e^*_{k,r+1}(D\tilde{s}, N\tilde{t})}{A^k} = (-1)^{k+r} k! \frac{e^*_{r+1,k}(N\tilde{t}, D\tilde{s})}{A^{r+1}}$$

this is almost the above analytic functional equation (F.E.) up to the missing factor $\langle D\tilde{s}, Nt\rangle_\tau$. But

$$\langle D\tilde{s}, N\tilde{t}\rangle_\tau = \langle N\tilde{s}, D\tilde{t}\rangle_\tau = 1.$$

2 The geometric de Rham logarithm sheaves

One of the main ingredients of the proof of the Tamagawa number conjecture for CM elliptic curves by Kings in [Kin01] is an explicit 1-motivic description of the polylogarithm class in étale cohomology. At least for the logarithm sheaves a similar 1-motivic description for the algebraic de Rham realization was worked out by Scheider in his PhD thesis [Sch14]. He showed that the restriction of the Poincaré bundle with integrable connection on $E \times E^\dagger$ to infinitesimal thickenings of $E \times e \hookrightarrow E \times E^\dagger$ satisfies the defining property of the relative de Rham logarithm sheaves. For the precise statement we refer to [Sch14, §2] or our later chapter on the de Rham realization of the elliptic polylogarithm. For the moment we do not even give the defining property of the abstract de Rham logarithm sheaves. Instead, we study the basic properties of the restrictions of the Poincaré bundle to infinitesimal thickenings of $e \times E$. Nevertheless, it is good to keep the result of Scheider in mind as motivation. This chapter does not contain any new results. We are just recalling and slightly generalizing results from [Sch14] or restating results from the first chapter in terms of infinitesimal restrictions of the Poincaré bundle.

2.1 The geometric logarithm sheaves

Most material contained in this section is scattered all over the PhD thesis of Scheider [Sch14]. Since Scheider restricts himself to the case of smooth varieties over $\mathbb{Q}$ while we want to work more generally over separated locally Noetherian schemes, sometimes minor modifications are needed.

As always, let $\pi : E \to S$ be an elliptic curve over a separated locally Noetherian base scheme S. Let us fix once for all a bi-rigidified Poincaré bundle $(\mathcal{P}, r_0, s_0)$. By its universal property, this choice is unique up to unique isomorphism. Let $q^\dagger : E^\dagger \to E^\vee$ be the universal vectorial extension of $E^\vee$. Let us denote by $\mathcal{P}^\dagger$ the birigidified Poincaré bundle on $E \times E^\dagger$ obtained by pullback. It is equipped with a canonical integrable $E^\dagger$-connection

$$\nabla_{\mathcal{P}^\dagger} : \mathcal{P}^\dagger \to \mathcal{P}^\dagger \otimes \Omega^1_{E \times_S E^\dagger / E^\dagger}.$$

2.1.1 Definition and basic properties

Let

$$\iota_n^\dagger : E_n^\dagger := \mathrm{Inf}_e^n E^\dagger \hookrightarrow E^\dagger$$
$$\iota_n : E_n^\vee := \mathrm{Inf}_e^n E^\vee \hookrightarrow E^\vee$$

denote the inclusions of the infinitesimal thickenings of e in $E^\dagger$ resp. $E^\vee$.

Definition 2.1.1. For $n \geq 0$ define

$$\mathcal{L}_n^\dagger : = (\mathrm{pr}_E)_*(\mathrm{id}_E \times \iota_n^\dagger)^*\mathcal{P}^\dagger$$
$$\mathcal{L}_n : = (\mathrm{pr}_E)_*(\mathrm{id}_E \times \iota_n)^*\mathcal{P}.$$

Both $\mathcal{L}_n$ and $\mathcal{L}_n^\dagger$ are locally free $\mathcal{O}_E$-modules of finite rank equipped with canonical isomorphisms

$$\mathrm{triv}_e : e^*\mathcal{L}_n \xrightarrow{\sim} \mathcal{O}_{E_n^\vee}$$
$$\mathrm{triv}_e : e^*\mathcal{L}_n^\dagger \xrightarrow{\sim} \mathcal{O}_{E_n^\dagger}$$

induced by the rigidifications of the Poincaré bundle. Furthermore, $\nabla_{\mathcal{P}^\dagger}$ induces an integrable S-connection on $\mathcal{L}_n^\dagger$. We call $\mathcal{L}_n$ resp. $\mathcal{L}_n^\dagger$ the *n-th geometric logarithm sheaf* (resp. with connection). If it is not clear from the context to which elliptic curve we are referring, we will write $\mathcal{L}_{n,E}$.

In the following, we will write $\mathcal{L}_n^{(\dagger)}$ resp. $\mathcal{P}^{(\dagger)}$ if a statement holds for both $\mathcal{L}_n$ and $\mathcal{L}_n^\dagger$ resp. $\mathcal{P}$ and $\mathcal{P}^\dagger$. The compatibility of $\mathcal{P}^{(\dagger)}$ with base change along $f : T \to S$ shows immediately that the geometric logarithm sheaves are compatible with base change, i. e.

$$\mathrm{pr}_E^*\mathcal{L}_{n,E/S}^{(\dagger)} \xrightarrow{\sim} \mathcal{L}_{n,E_T/T}^{(\dagger)}$$

where $\mathrm{pr}_E : E_T = E \times_S T \to E$ is the projection. The commutative diagrams

$$\begin{tikzcd} E_n^\dagger \arrow[d, hook] \arrow[dr, hook, "\iota_n^\dagger"] & \\ E_{n+1}^\dagger \arrow[r, hook, "\iota_{n+1}^\dagger"'] & E^\dagger \end{tikzcd} \qquad \begin{tikzcd} E_n^\vee \arrow[d, hook] \arrow[dr, hook, "\iota_n"] & \\ E_{n+1}^\vee \arrow[r, hook, "\iota_{n+1}"'] & E^\vee \end{tikzcd}$$

induce transition maps

$$\mathcal{L}_{n+1}^{(\dagger)} \twoheadrightarrow \mathcal{L}_n^{(\dagger)}.$$

For $\mathcal{L}_n^\dagger$ the transition maps are horizontal. The rigidification $(\mathrm{id} \times e)^*\mathcal{P}^{(\dagger)} \cong \mathcal{O}_E$ induces an isomorphism $\mathcal{L}_0^{(\dagger)} \cong \mathcal{O}_E$. Further, we have canonical injections

$$\mathcal{L}_n \hookrightarrow \mathcal{L}_n^\dagger.$$

Indeed, the commutative diagram

$$\begin{tikzcd}
\mathrm{Inf}_e^n E^\dagger \arrow[r, "\iota_n^\dagger"] \arrow[d, two heads, "q_n^\dagger := q_n^\dagger|_{\mathrm{Inf}_e^n E^\dagger}"'] & (E')^\vee \arrow[d, two heads, "q^\dagger"] \\
\mathrm{Inf}_e^n E^\vee \arrow[r, "\iota_n^\vee"] & E^\vee
\end{tikzcd}$$

combined with the adjunction $(\mathrm{id} \times \iota_n)^*\mathcal{P} \hookrightarrow (\mathrm{id} \times q_n^\dagger)_*(\mathrm{id} \times q_n^\dagger)^*(\mathrm{id} \times \iota_n)^*\mathcal{P}$ gives

$$\mathcal{L}_n \hookrightarrow (\mathrm{pr}_E)_*(\mathrm{id} \times q_n^\dagger)_*(\mathrm{id} \times q_n^\dagger)^*(\mathrm{id} \times \iota_n)^*\mathcal{P} = (\mathrm{pr}_E)_*(\mathrm{id} \times \iota_n^\dagger)^*(\mathrm{id} \times q^\dagger)^*\mathcal{P} = \mathcal{L}_n^\dagger.$$

These inclusions are compatible with transition maps and base change. Let us introduce the notation

$$\mathcal{H} := \underline{H}^1_{\mathrm{dR}}\left(E^\vee/S\right) \overset{(*)}{\cong} \underline{H}^1_{\mathrm{dR}}\left(E/S\right)^\vee$$

where we have used the canonical isomorphism induced by Deligne's pairing in $(*)$. Furthermore, we will write $\mathcal{H}_E := \pi^*\mathcal{H}$.

Lemma 2.1.2. *The transition maps $\mathcal{L}_1^{(\dagger)} \to \mathcal{L}_0^{(\dagger)} = \mathcal{O}_E$ fit into the following diagram of short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\underline{\omega}_{E^\vee/S} \arrow[r] \arrow[d, hook] & \mathcal{L}_1 \arrow[r] \arrow[d, hook] & \mathcal{O}_E \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathcal{H}_E \arrow[r] & \mathcal{L}_1^\dagger \arrow[r] & \mathcal{O}_E \arrow[r] & 0
\end{tikzcd}$$

where $\underline{\omega}_{E^\vee/S} \hookrightarrow \mathcal{H}$ is the natural inclusion. Further, the lower exact sequence is horizontal if we equip $\mathcal{O}_E$ and $\mathcal{H}_E$ with the canonical S-connection obtained via pullback of the trivial S-connections on S.

Proof. $\mathcal{O}_{\mathrm{Inf}_e^1 E^\vee}$ and $\mathcal{O}_{\mathrm{Inf}_e^1 E^\dagger}$ sit in the following short exact sequence of $\mathcal{O}_S$-modules

$$\begin{tikzcd}
0 \arrow[r] & \underline{\omega}_{E^\vee/S} \arrow[r] \arrow[d] & \mathcal{O}_{\mathrm{Inf}_e^1 E^\vee} \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{O}_{\mathrm{Inf}_e^1 E^\dagger} \arrow[r] & \mathcal{O}_S \arrow[r] & 0.
\end{tikzcd}$$

After applying $(\pi_E)_*\left[(\mathcal{O}_E \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1}(\cdot)) \otimes_{\mathcal{O}_{E\times E^\vee}} \mathcal{P}\right]$ (which is exact), we obtain

$$\begin{tikzcd}
0 \arrow[r] & (\pi_E)_*\left[\left(\mathcal{O}_E \otimes_{\mathcal{O}_S} \underline{\omega}_{E^\vee/S}\right) \otimes_{\mathcal{O}_{E\times E^\vee}} \mathcal{P}\right] \arrow[r] \arrow[d] & \mathcal{L}_1 \arrow[r] \arrow[d] & \mathcal{O}_E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\pi_E)_*\left[(\mathcal{O}_E \otimes_{\mathcal{O}_S} \mathcal{H}) \otimes_{\mathcal{O}_{E\times E^\vee}} \mathcal{P}\right] \arrow[r] & \mathcal{L}_1^\dagger \arrow[r] & \mathcal{O}_E \arrow[r] & 0.
\end{tikzcd}$$

Now, the result follows from

$$(\pi_E)_*\left[(\mathcal{O}_E\otimes_{\mathcal{O}_S}\mathcal{H})\otimes_{\mathcal{O}_{E\times E^\vee}}\mathcal{P}\right]=\mathcal{H}\otimes_{\pi^{-1}\mathcal{O}_S}(\mathrm{id}_E\times e)^*\mathcal{P}= \\ =\mathcal{H}_E$$

and

$$(\pi_E)_*\left[\left(\mathcal{O}_E\otimes_{\mathcal{O}_S}\underline{\omega}_{E^\vee/S}\right)\otimes_{\mathcal{O}_{E\times E^\vee}}\mathcal{P}\right]=\underline{\omega}_{E^\vee/S}\otimes_{\pi^{-1}\mathcal{O}_S}(\mathrm{id}_E\times e)^*\mathcal{P}= \\ =\pi^*\underline{\omega}_{E^\vee/S}$$

The connection on the Poincaré bundle $\mathcal{P}^\dagger$ induces an integrable connection on

$$0\longrightarrow\underbrace{\mathcal{H}\otimes_{\pi^{-1}\mathcal{O}_S}(\mathrm{id}_E\times e)^*\mathcal{P}^\dagger}_{=(\pi_E)^*\mathcal{H}}\longrightarrow\mathcal{L}_1^\dagger\longrightarrow\mathcal{L}_0^\dagger=\mathcal{O}_E\longrightarrow 0$$

making this sequence horizontal. □

2.1.2 Extension classes of the first geometric logarithm sheaves

Above we have seen that $\mathcal{L}_1$ and $\mathcal{L}_1^\dagger$ sit in short exact sequences

$$0\longrightarrow\pi^*\underline{\omega}_{E^\vee/S}\longrightarrow\mathcal{L}_1\longrightarrow\mathcal{O}_E\longrightarrow 0 \tag{2.1}$$

and

$$0\longrightarrow\mathcal{H}_E\longrightarrow\mathcal{L}_1^\dagger\longrightarrow\mathcal{O}_E\longrightarrow 0. \tag{2.2}$$

In the following, we will have a closer look at the corresponding extension classes. The short exact sequence (2.1) gives rise to an extension class

$$[\mathcal{L}_1]\in\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E,\pi^*\underline{\omega}_{E^\vee/S})=\mathrm{H}^1(E,\pi^*\underline{\omega}_{E^\vee/S}).$$

The Leray spectral sequence gives a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(S,\underline{\omega}_{E^\vee/S}) \arrow[r, "\pi^*"] & \mathrm{H}^1(E,\pi^*\underline{\omega}_{E^\vee/S}) \arrow[l, bend right, "e^*"'] \arrow[r, "\delta"] & \underbrace{\Gamma\left(S,R^1\pi_*\left(\pi^*\underline{\omega}_{E^\vee/S}\right)\right)}_{=\Gamma(S,\underline{\omega}_{E^\vee/S}\otimes\underline{\omega}^\vee_{E^\vee/S})} \arrow[r] & 0. \end{tikzcd}$$

Since $e^*\mathcal{L}_1$ is equipped with a canonical splitting, the class $[\mathcal{L}_1]$ maps to 0 under e^*. Before we give an explicit description of the extension class of $\mathcal{L}_1$ under δ, let us remark that $\delta([\mathcal{E}])$ for an extension

$$0\longrightarrow\pi^*\underline{\omega}_{E^\vee/S}\longrightarrow\mathcal{E}\longrightarrow\mathcal{O}_E\longrightarrow 0 \tag{2.3}$$

coincides with the image of $1 \in \Gamma(S, \pi_* \mathcal{O}_E) = \Gamma(S, \mathcal{O}_S)$ under the connecting homomorphism

$$\Gamma(S, \pi_* \mathcal{O}_E) \to \Gamma\left(S, R^1\pi_*(\pi^* \underline{\omega}_{E^\vee/S})\right) = \Gamma(S, \underline{\omega}_{E^\vee/S} \otimes \underline{\omega}_{E^\vee/S}^\vee)$$

obtained from (2.3) by applying $R\pi_*$.

Proposition 2.1.3. *We have*

$$\delta([\mathcal{L}_1]) = \mathrm{id}_{\underline{\omega}_{E^\vee/S}} \in \Gamma(S, \underline{\omega}_{E^\vee/S}^\vee \otimes_{\mathcal{O}_S} \underline{\omega}_{E^\vee/S}).$$

Proof. In [FC90, S. 81 f.] it is stated that the extension class of the restriction of the Poincaré bundle to the first-order infinitesimal neighbourhood of $E \times e$ coincides with the extension class of the universal vectorial extension. Using this, the assertion of this proposition follows immediately. Since no proof is given in loc. cit., let us reduce the proof to a statement which will be proven in Chapter 5. The case of a smooth separated scheme over $\mathbb{Q}$ will be treated in Corollary 5.1.18. We reduce the general case to this case as follows: the claim is compatible with base change, i. e. if the claim holds for E/S, then it also holds for E_T/T. Furthermore,

$$T \mapsto \underline{\omega}_{E_T^\vee/T}^\vee \otimes_{\mathcal{O}_T} \underline{\omega}_{E_T^\vee/T}$$

defines a sheaf on the small étale site of S. By the compatibility with base change and by the sheaf condition, it is enough to show the equality

$$\delta([\mathcal{L}_1]) = \mathrm{id}_{\underline{\omega}_{E^\vee/S}}$$

étale locally on the base. Using the Zariski covering $\left(S[\frac{1}{N}]\right)_{N>3}$ and the fact that a $\Gamma(N)$-level structure exists étale locally, we are reduced to the case $(E/S, \alpha)$ where $\alpha : (\mathbb{Z}/N\mathbb{Z})_S^2 \xrightarrow{\sim} E[N]$ is a $\Gamma(N)$-level structure. By compatibility with base change it is enough to prove the claim in the universal situation, i. e. for E_N/M_N the universal elliptic curve with $\Gamma(N)$-level structure. Let us write E for E_N and M for M_N. Since M is flat over $\operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$ and $\underline{\omega}_{E^\vee/M}^\vee \otimes_{\mathcal{O}_M} \underline{\omega}_{E^\vee/M}$ is locally free of rank 1, we obtain an injection

$$\underline{\omega}_{E^\vee/M}^\vee \otimes_{\mathcal{O}_M} \underline{\omega}_{E^\vee/M} \hookrightarrow \underline{\omega}_{E_\mathbb{Q}/M_\mathbb{Q}}^\vee \otimes \underline{\omega}_{E_\mathbb{Q}/M_\mathbb{Q}}.$$

This injection and again the compatibility with base change reduces the claim to the case $E_\mathbb{Q}/M_\mathbb{Q}$. Since $M_{N,\mathbb{Q}}$ is a smooth separated $\mathbb{Q}$-scheme, we have reduced the claim to the case which will be proven in Corollary 5.1.18. □

Corollary 2.1.4. *Let E/S be an elliptic curve, M be a locally free $\mathcal{O}_S$-module of finite rank and let $(\mathcal{F}, \sigma)$ be a pair consisting of an extension*

$$\mathcal{F}: \quad 0 \longrightarrow \pi^* M \longrightarrow F \longrightarrow \mathcal{O}_E \longrightarrow 0$$

together with a splitting of $e^*\mathcal{F}$. I. e. σ is an isomorphism $e^* F \xrightarrow{\sim} \mathcal{O}_S \oplus M$ which is compatible with the extension structure. Then, there is a unique morphism

$$\varphi : \underline{\omega}_{E^\vee/S} \to M$$

such that the pair $(\mathcal{F},\sigma)$ *is the pushout of the pair* $(\mathcal{L}_1, \mathrm{triv}_e : e^*\mathcal{L}_1 \xrightarrow{\sim} \mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S})$ *along* φ*.*

Proof. The group

$$\ker\left(\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E, \pi^*M) \xrightarrow{e^*} \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, M)\right)$$

classifies pairs $(\mathcal{F},\sigma)$ as above up to (unique) isomorphism. The Leray spectral sequence gives an isomorphism

$$\ker\left(\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E, \pi^*M) \xrightarrow{e^*} \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, M)\right) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_{E^\vee/S}, M). \tag{2.4}$$

Thus, a map $f : M \to N$ induces

$$f_* : \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_{E^\vee/S}, M) \to \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_{E^\vee/S}, N)$$

and we obtain via (2.4) a map

$$\ker\left(\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E, \pi^*M) \xrightarrow{e^*} \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, M)\right) \xrightarrow{f_*} \ker\left(\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E, \pi^*N) \xrightarrow{e^*} \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, N)\right).$$

The map f_* maps $[(\mathcal{F},\sigma)]$ to the extension class which is obtained from $(\mathcal{F},\sigma)$ by pushout along f. Since there are no non-trivial isomorphisms of an extension $(\mathcal{F},\sigma)$ compatible with the fixed splitting, a pair $(\mathcal{F},\sigma)$ is uniquely, up to unique isomorphism, determined by its extension class. The corollary follows now from $\delta([\mathcal{L}_1, can]) = \mathrm{id}_{\underline{\omega}_{E^\vee/S}}$. □

Similarly, the horizontal exact sequence for $\mathcal{L}_1^\dagger$ induces a long exact sequence in relative de Rham cohomology $\underline{H}^\bullet_{\mathrm{dR}}(E/S, \cdot)$. Let us denote by

$$\delta : \mathcal{O}_S = \underline{H}^0_{\mathrm{dR}}(E/S) \to \underline{H}^1_{\mathrm{dR}}(E/S, \mathcal{H}_E) = \underline{H}^1_{\mathrm{dR}}(E/S) \otimes_{\mathcal{O}_S} \mathcal{H} \cong \mathcal{H}^\vee \otimes_{\mathcal{O}_S} \mathcal{H}$$

the connecting homomorphism.

Proposition 2.1.5 ([Sch14, Thm. 2.3.1])**.** *We have*

$$\delta(1) = \mathrm{id}_{\mathcal{H}}.$$

Proof. If S is a smooth separated scheme over $\mathbb{Q}$, this was proven by Scheider [Sch14, Thm. 2.3.1], see also Corollary 5.1.13. The general case can be reduced to this case in exactly the same way as in the above proof for $\mathcal{L}_1$. □

Remark 2.1.6. Let us make a remark about de Rham cohomology over arbitrary schemes. For any smooth map $\pi : X \to S$ and any $\mathcal{O}_X$-module with integrable S-connection one can define the relative de Rham cohomology as the derived direct image of the relative de Rham complex. In general, i. e. for schemes which are not smooth over a field of characteristic zero, this definition is not well behaved. On the other hand, for an elliptic curve $E \to S$ this definition is reasonable with all good properties over *any* base scheme. In this case $\underline{H}^1_{\mathrm{dR}}(E/S)$ is always locally free of rank 2 and compatible with base change, i. e. for $f : T \to S$ we have canonical isomorphisms $f^*\underline{H}^1_{\mathrm{dR}}(E/S) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}(E_T/T)$. For the above result this is all we need.

Corollary 2.1.7. *Let E/S be an elliptic curve. The short exact sequence* (2.2) *associated with $\mathcal{L}_1^\dagger$ is the pushout of the short exact sequence* (2.1) *associated with $\mathcal{L}_1$ along*

$$\underline{\omega}_{E^\vee/S} \hookrightarrow \mathcal{H}.$$

Proof. This can be directly seen using the fact that $\mathcal{P}^\dagger$ is the pullback of $\mathcal{P}$. Alternatively it can be deduced from the above proposition. Indeed, the (Zariski) cohomology class of the extension (2.2) corresponds to the canonical inclusion $\underline{\omega}_{E^\vee/S} \hookrightarrow \mathcal{H}$. □

2.1.3 Behavior under isogenies

For a given isogeny $\varphi : E \to E'$ we consider the following diagrams

$$\begin{tikzcd} \mathrm{Inf}_e^n(E')^\vee \arrow[r, "\iota_n'"] \arrow[d, "\varphi_n^\vee := \varphi^\vee|_{\mathrm{Inf}_e^n}"'] & (E')^\vee \arrow[d, "\varphi^\vee"] \\ \mathrm{Inf}_e^n E^\vee \arrow[r, "\iota_n"] & E^\vee \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Inf}_e^n(E')^\dagger \arrow[r, "\iota_n'^\dagger"] \arrow[d, "\varphi_n^\dagger := \varphi^\dagger|_{\mathrm{Inf}_e^n}"'] & (E')^\dagger \arrow[d, "\varphi^\dagger"] \\ \mathrm{Inf}_e^n E^\dagger \arrow[r, "\iota_n^\dagger"] & E^\dagger \end{tikzcd}$$

where $\varphi^\vee : (E')^\vee \to E^\vee$ and $\varphi^\dagger : (E')^\dagger \to E^\dagger$ are the induced maps on $E^\vee$ and its universal vectorial extension. The adjunction maps

$$(\mathrm{id} \times \iota_n)^*\mathcal{P} \to (\mathrm{id} \times \varphi_n^\vee)_*(\mathrm{id} \times \varphi_n^\vee)^*(\mathrm{id} \times \iota_n)^*\mathcal{P}$$

induce canonical maps

$$(\mathrm{pr}_E)_*(\mathrm{id}\times\iota_n)^*\mathcal{P} \to (\mathrm{pr}_E)_*(\mathrm{id}\times\varphi_n^\vee)_*(\mathrm{id}\times\varphi_n^\vee)^*(\mathrm{id}\times\iota_n)^*\mathcal{P} = (\mathrm{pr}_E)_*(\mathrm{id}\times\iota_n)^*(\mathrm{id}\times\varphi^\vee)^*\mathcal{P}.$$

Let us denote the resulting map by

$$Ad_\varphi : \mathcal{L}_n \to (\mathrm{pr}_E)_*(\mathrm{id} \times \iota_n)^*(\mathrm{id} \times \varphi^\vee)^*\mathcal{P}.$$

And similarly the adjunction for $(\mathrm{id} \times \varphi_n^\dagger)$ induces

$$Ad_\varphi^\dagger : \mathcal{L}_n^\dagger \to (\mathrm{pr}_E)_*(\mathrm{id} \times \iota_n^\dagger)^*(\mathrm{id} \times \varphi^\dagger)^*\mathcal{P}^\dagger.$$

Definition 2.1.8. Let $\varphi : E \to E'$ be an isogeny. The composition of the adjunction maps with the isomorphism

$$\gamma_{1,\varphi} : (\mathrm{id} \times \varphi^{(\dagger)})^*\mathcal{P}_E^{(\dagger)} \xrightarrow{\sim} (\varphi \times \mathrm{id})^*\mathcal{P}_{E'}^{(\dagger)}$$

gives

$$\Phi_\varphi^{(\dagger)} : \mathcal{L}_{n,E}^{(\dagger)} \xrightarrow{Ad_\varphi^{(\dagger)}} (\mathrm{pr}_E)_*(\mathrm{id} \times \iota_n^{(\dagger)})^*(\mathrm{id} \times \varphi^{(\dagger)})^*\mathcal{P}_E^{(\dagger)} \xrightarrow{\gamma_{1,\varphi}} \varphi^*\mathcal{L}_{n,E'}^{(\dagger)}.$$

We summarize some further properties of $\mathcal{L}_n$ and $\mathcal{L}_n^\dagger$.

Lemma 2.1.9. *Let $\varphi : E \to E'$ be an isogeny.*

(a) *The maps*
$$\mathcal{L}^{(\dagger)}_{n,E} \xrightarrow{\Phi^{(\dagger)}_{\varphi}} \varphi^*\mathcal{L}^{(\dagger)}_{n,E'}.$$
are compatible with the inclusions $\mathcal{L}_n \hookrightarrow \mathcal{L}^\dagger_n$, transition maps and base change. Furthermore, $\Phi^\dagger_\varphi$ is horizontal.

(b) *The maps $e^*\Phi_\varphi$ and $e^*\Phi^\dagger_\varphi$ coincide with the maps of sheaves underlying the morphism φ_n resp. $\varphi^\dagger_n$, i. e.*
$$e^*\Phi_\varphi = (\varphi^\vee_n)^\# : \mathcal{O}_{\mathrm{Inf}^n_e E^\vee} \to \mathcal{O}_{\mathrm{Inf}^n_e (E')^\vee}$$
$$e^*\Phi^\dagger_\varphi = (\varphi^\dagger_n)^\# : \mathcal{O}_{\mathrm{Inf}^n_e E^\dagger} \to \mathcal{O}_{\mathrm{Inf}^n_e (E')^\dagger}.$$

(c) *If $\varphi^\vee$ is étale, the map*
$$\mathcal{L}_{n,E} \xrightarrow{\Phi_{\varphi}} \varphi^*\mathcal{L}_{n,E'}.$$
is an isomorphism.

(d) *If both φ and $\varphi^\vee$ are étale, the map*
$$\mathcal{L}^{\dagger}_{n,E} \xrightarrow{\Phi^{\dagger}_{\varphi}} \varphi^*\mathcal{L}^{\dagger}_{n,E'}.$$
is an isomorphism.

(e) *For isogenies*
$$E \xrightarrow{\varphi} E' \xrightarrow{\psi} E''$$
we have
$$\Phi^{(\dagger)}_{(\psi\circ\varphi)} = \varphi^*\Phi^{(\dagger)}_{\psi} \circ \Phi^{(\dagger)}_{\varphi} : \quad \mathcal{L}^{(\dagger)}_{n,E} \to (\psi\circ\varphi)^*\mathcal{L}^{(\dagger)}_{n,E''}.$$

Proof. (a) and (b) follow immediately from the definitions. For (c) and (d) it is enough to show that the adjunction maps Ad_φ and $Ad^\dagger_\varphi$ are isomorphisms under the given assumptions. We will show this in the case (d). The other case is analogous. Consider the maps
$$(\varphi^\dagger_n)^\# : \mathcal{O}_{\mathrm{Inf}^n_e E^\dagger} \to (\varphi^\dagger_n)_*\mathcal{O}_{\mathrm{Inf}^n_e (E')^\dagger}.$$
We want to show that, under the assumptions that φ and $\varphi^\vee$ are étale, these maps are isomorphisms for all $n \geq 0$. The claim follows then by the definition of $Ad^\dagger_\varphi$. The diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathrm{Sym}}^n_{\mathcal{O}_S}\mathcal{H} \arrow[r] \arrow[d, "\underline{\mathrm{Sym}}^n(\varphi^\vee)^*"] & \mathcal{O}_{\mathrm{Inf}^n_e E^\dagger} \arrow[r] \arrow[d, "(\varphi^\dagger_n)^\#"] & \mathcal{O}_{\mathrm{Inf}^{n-1}_e E^\dagger} \arrow[r] \arrow[d, "(\varphi^\dagger_{n-1})^\#"] & 0 \\
0 \arrow[r] & \underline{\mathrm{Sym}}^n_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}((E')^\vee/S) \arrow[r] & \mathcal{O}_{\mathrm{Inf}^n_e E^\dagger} \arrow[r] & \mathcal{O}_{\mathrm{Inf}^{n-1}_e (E')^\dagger} \arrow[r] & 0
\end{tikzcd}$$

shows that

$$(\varphi_n^\dagger)^\# : \mathcal{O}_{\mathrm{Inf}_e^n E^\dagger} \to (\varphi_n^\dagger)_* \mathcal{O}_{\mathrm{Inf}_e^n (E')^\dagger}$$

is an isomorphism for all $n \geq 0$ if and only if

$$(\varphi^\vee)^* : \mathcal{H} = \underline{H}^1_{\mathrm{dR}}\left(E^\vee/S\right) \to \underline{H}^1_{\mathrm{dR}}\left((E')^\vee/S\right)$$

is an isomorphism. But the compatibility of $(\varphi^\vee)^*$ with the Hodge filtration shows that $(\varphi^\vee)^*$ is an isomorphism if and only if

$$(\varphi^\vee)^* : \underline{\omega}_{E^\vee/S} \to \underline{\omega}_{(E')^\vee/S}$$

and

$$(\varphi^*)^\vee : \underline{\omega}^\vee_{E'/S} \to \underline{\omega}^\vee_{E/S}$$

are isomorphisms. This is equivalent to φ and $\varphi^\vee$ being étale.

(e) follows immediately from the equality

$$(\varphi \times \mathrm{id})^* \gamma_{1,\psi^\vee} \circ \gamma_{1,\varphi^\vee} = \gamma_{1,(\psi\circ\varphi)^\vee}. \tag{2.5}$$

□

The most important case will be $\varphi = [N]$ for $N \geq 1$. In this case we will often write $\Phi_N^{(\dagger)}$ for $\Phi_{[N]}^{(\dagger)}$.

Lemma/Definition 2.1.10. Let $N \geq 1$ be an integer and assume that N is invertible on S.

(a) For $t \in E[N](S)$ let us define the following translation operator

$$\mathrm{trans}_t^{(\dagger)} : T_t^* \mathcal{L}_n^{(\dagger)} \xrightarrow[\sim]{T_t^*\Phi_N^{(\dagger)}} T_t^*[N]^* \mathcal{L}_n^{(\dagger)} = [N]^* \mathcal{L}_n^{(\dagger)} \xleftarrow[\sim]{\Phi_N^{(\dagger)}} \mathcal{L}_n^{(\dagger)}$$

which is independent of the chosen N.

(b) For $t \in E[N](S)$ we obtain trivializations of $t^*\mathcal{L}_n^{(\dagger)}$ as

$$\mathrm{triv}_t^\dagger : t^*\mathcal{L}_n^\dagger \xrightarrow[\sim]{e^*\mathrm{trans}_t^\dagger} e^*\mathcal{L}_n^\dagger \xrightarrow[\sim]{\mathrm{triv}_e^\dagger} \mathcal{O}_{\mathrm{Inf}_e^n E^\dagger}$$

resp.

$$\mathrm{triv}_t : t^*\mathcal{L}_n \xrightarrow[\sim]{e^*\mathrm{trans}_t} e^*\mathcal{L}_n \xrightarrow[\sim]{\mathrm{triv}_e} \mathcal{O}_{\mathrm{Inf}_e^n E^\vee}.$$

These are independent of the chosen N.

Proof. For the statement about independence on N we may assume N minimal with $t \in E[N](S)$ and compare for arbitrary $M \geq 1$

$$T_t^* \mathcal{L}_n^{(\dagger)} \xrightarrow[\sim]{T_t^*\Phi_N^{(\dagger)}} T_t^*[N]^* \mathcal{L}_n^{(\dagger)} = [N]^* \mathcal{L}_n^{(\dagger)} \xleftarrow[\sim]{\Phi_N^{(\dagger)}} \mathcal{L}_n^{(\dagger)}$$

to

$$T_t^*\mathcal{L}_n^{(\dagger)} \xrightarrow[\sim]{T_t^*\Phi_{NM}^{(\dagger)}} T_t^*[NM]^*\mathcal{L}_n^{(\dagger)} = [NM]^*\mathcal{L}_n^{(\dagger)} \xleftarrow[\sim]{\Phi_{NM}^{(\dagger)}} \mathcal{L}_n^{(\dagger)}.$$

Using

$$T_t^*\Phi_{NM}^{(\dagger)} = T_t^*\Phi_{[N]\circ[M]}^{(\dagger)} = T_t^*[N]^*\Phi_{[M]}^{(\dagger)} \circ T_t^*\Phi_{[N]}^{(\dagger)} = [N]^*\Phi_{[M]}^{(\dagger)} \circ T_t^*\Phi_{[N]}^{(\dagger)},$$

it follows immediately that the above maps coincide. □

Remark 2.1.11. For $t \in E[N](S)$ one could also use

$$t^*\Phi_N : t^*\mathcal{L}_n \to t^*[N]^*\mathcal{L}_n = e^*\mathcal{L}_n \cong \mathcal{O}_{\mathrm{Inf}_e^n E^\vee}$$

to trivialize $t^*\mathcal{L}_n$. But this trivialization is not independent of the chosen N. Indeed, by Lemma 2.1.9 (b) they differ by the morphism

$$([N]|_{\mathrm{Inf}_e^n E^\vee})^\# : \mathcal{O}_{\mathrm{Inf}_e^n E^\vee} \to \mathcal{O}_{\mathrm{Inf}_e^n E^\vee}.$$

The same remark applies to $\mathrm{triv}_t^\dagger$.

2.1.4 The comultiplication maps and symmetric tensors

Let us recall the construction of certain natural comultiplication maps on $\mathcal{L}_n^{(\dagger)}$. For the construction of the comultiplication maps we follow [Sch14, §2.4.2] and refer to loc. cit. for more details.

We keep the notation $\mathcal{L}_n^{(\dagger)}$ resp. $\mathcal{P}^{(\dagger)}$ for denoting either of $\mathcal{L}_n$ or $\mathcal{L}_n^\dagger$ and $\mathcal{P}$ or $\mathcal{P}^\dagger$. Further, it will be convenient to introduce a similar notation for $E^\vee$ and $E^\dagger$. So, let us write $E^{(\dagger)}$ if some statement holds for both $E^\vee$ and $E^\dagger$. As before,

$$\iota_n^{(\dagger)} : \mathrm{Inf}_e^n E^{(\dagger)} \hookrightarrow E^{(\dagger)}$$

denotes the inclusion of the infinitesimal neighbourhood. For the time being we will work over a fixed S-scheme and use the convention to denote by $\times$ the product in the category of S-schemes.

Recall that the Poincaré bundle $\mathcal{P}^{(\dagger)}$ is equipped with a natural $\mathbb{G}_{m,S}$-biextension structure, i. e. isomorphisms

$$\begin{aligned} (\mu_E \times \mathrm{id}_{E^{(\dagger)}})^*\mathcal{P}^{(\dagger)} &\xrightarrow{\sim} \mathrm{pr}_{1,3}^*\mathcal{P} \otimes \mathrm{pr}_{2,3}^*\mathcal{P}^{(\dagger)} \quad \text{on} \quad E \times E \times E^{(\dagger)} \\ (\mathrm{id}_E \times \mu_{E^{(\dagger)}})^*\mathcal{P}^{(\dagger)} &\xrightarrow{\sim} \mathrm{pr}_{1,2}^*\mathcal{P} \otimes \mathrm{pr}_{1,3}^*\mathcal{P}^{(\dagger)} \quad \text{on} \quad E \times E^{(\dagger)} \times E^{(\dagger)} \end{aligned} \tag{2.6}$$

satisfying certain compatibilities, cf. [SGA7, exp. VII]. Here, μ denotes the multiplication and $\mathrm{pr}_{i,j}$ the projection on the i-th and j-th component of the product. Now, fix some integers $n, m \geq 1$ and define $\mathcal{P}_n^{(\dagger)} := (\mathrm{id} \times \iota_n^{(\dagger)})^*\mathcal{P}^{(\dagger)}$. Restricting

$$\mu_{E^{(\dagger)}} : E^{(\dagger)} \times E^{(\dagger)} \to E^{(\dagger)}$$

to $E_n^{(\dagger)} \times E_m^{(\dagger)}$ gives

$$\mu_{n,m} : E_n^{(\dagger)} \times E_m^{(\dagger)} \to E_{n+m}^{(\dagger)}.$$

Restricting (2.6) along

$$E \times E_n^{(\dagger)} \times E_m^{(\dagger)} \hookrightarrow E \times E^{(\dagger)} \times E^{(\dagger)}$$

results in

$$\mathcal{P}_{n+m}^{(\dagger)} \to (\mathrm{pr}_{12})^* \mathcal{P}_n^{(\dagger)} \otimes_{\mathcal{O}_{E \times E_n^{(\dagger)} \times E_m^{(\dagger)}}} (\mathrm{pr}_{13})^* \mathcal{P}_m^{(\dagger)}.$$

Using the unit of the adjunction between $(\mathrm{id} \times \mu_{n,m})_*$ and $(\mathrm{id} \times \mu_{n,m})^*$, we obtain

$$\mathcal{P}_{n+m}^{(\dagger)} \to (\mathrm{id} \times \mu_{n,m})_* \left[(\mathrm{pr}_{12})^* \mathcal{P}_n^{(\dagger)} \otimes (\mathrm{pr}_{13})^* \mathcal{P}_m^{(\dagger)} \right]$$

and taking the direct image along pr_E gives:

$$\xi_{n,m} : \mathcal{L}_{n+m}^{(\dagger)} \to \mathcal{L}_n^{(\dagger)} \otimes_{\mathcal{O}_E} \mathcal{L}_m^{(\dagger)}$$

For details we refer to [Sch14, §2.4.2]. For $\mathcal{P}^\dagger$ the $\mathbb{G}_{m,S}$-biextension structure is compatible with the connection. This implies that $\xi_{n,m}$ is horizontal. Using the compatibilities of the $\mathbb{G}_{m,S}$-biextension structure, one deduces the following commutative diagrams:

$$\begin{tikzcd} \mathcal{L}_{n+m}^{(\dagger)} \arrow[r, "\xi_{n,m}"] \arrow[dr, "\xi_{m,n}"'] & \mathcal{L}_n^{(\dagger)} \otimes_{\mathcal{O}_E} \mathcal{L}_m^{(\dagger)} \arrow[d, "can"] \\ & \mathcal{L}_m^{(\dagger)} \otimes_{\mathcal{O}_E} \mathcal{L}_n^{(\dagger)} \end{tikzcd} \tag{2.7}$$

and

$$\begin{tikzcd} \mathcal{L}_{n+m+l}^{(\dagger)} \arrow[r, "\xi_{n+m,l}"] \arrow[d, "\xi_{n,m+l}"'] & \mathcal{L}_{n+m}^{(\dagger)} \otimes_{\mathcal{O}_E} \mathcal{L}_l^{(\dagger)} \arrow[d, "\xi_{n,m}\otimes\mathrm{id}"] \\ \mathcal{L}_n^{(\dagger)} \otimes_{\mathcal{O}_E} \mathcal{L}_{m+l}^{(\dagger)} \arrow[r, "\mathrm{id}\otimes\xi_{m,l}"] & \mathcal{L}_n^{(\dagger)} \otimes_{\mathcal{O}_E} \mathcal{L}_m^{(\dagger)} \otimes_{\mathcal{O}_E} \mathcal{L}_l^{(\dagger)}. \end{tikzcd} \tag{2.8}$$

Thus, we obtain well-defined maps

$$\mathcal{L}_n^{(\dagger)} \to \underbrace{\mathcal{L}_1^{(\dagger)} \otimes_{\mathcal{O}_E} \ldots \otimes_{\mathcal{O}_E} \mathcal{L}_1^{(\dagger)}}_{n \text{ times}}.$$

The diagram (2.7) shows that this map is invariant under transposing any of the n factors on the right hand side. Thus, letting the symmetric group S_n act on

$$\mathcal{L}_1^{(\dagger)} \otimes_{\mathcal{O}_E} \ldots \otimes_{\mathcal{O}_E} \mathcal{L}_1^{(\dagger)}$$

by permuting the factors we see that $\mathcal{L}_n^{(\dagger)} \to \left(\mathcal{L}_1^{(\dagger)}\right)^{\otimes n}$ factors through the invariants of the S_n action. We denote the resulting map by

$$\mathcal{L}_n^{(\dagger)} \to \underline{\mathrm{TSym}}^n_{\mathcal{O}_E} \mathcal{L}_1^{(\dagger)} := \left[\left(\mathcal{L}_1^{(\dagger)}\right)^{\otimes n} \right]^{S_n}. \tag{2.9}$$

The transition maps make $(\mathcal{L}_n^{(\dagger)})_{n\geq 0}$ a pro-system of sheaves. The maps

$$\underbrace{\mathcal{L}_1^{(\dagger)} \otimes_{\mathcal{O}_E} \dots \otimes_{\mathcal{O}_E} \mathcal{L}_1^{(\dagger)}}_{n+1 \text{ times}} \xrightarrow{\mathrm{id}^{\otimes n}\otimes \mathrm{pr}} \underbrace{\mathcal{L}_1^{(\dagger)} \otimes_{\mathcal{O}_E} \dots \otimes_{\mathcal{O}_E} \mathcal{L}_1^{(\dagger)}}_{n \text{ times}} \otimes_{\mathcal{O}_E} \mathcal{L}_0^{(\dagger)} \xrightarrow{\simeq} \underbrace{\mathcal{L}_1^{(\dagger)} \otimes_{\mathcal{O}_E} \dots \otimes_{\mathcal{O}_E} \mathcal{L}_1^{(\dagger)}}_{n \text{ times}}$$

induce projections

$$\underline{\mathrm{TSym}}^{n+1}_{\mathcal{O}_E} \mathcal{L}_1^{(\dagger)} \to \underline{\mathrm{TSym}}^{n}_{\mathcal{O}_E} \mathcal{L}_1^{(\dagger)}$$

and the commutative diagram (2.8) shows that these maps are compatible with the transition maps $\mathcal{L}_{n+1}^{(\dagger)} \to \mathcal{L}_n^{(\dagger)}$. In other words, (2.9) induces a morphism of pro-systems:

$$\mathbb{D}^{(\dagger)} : (\mathcal{L}_n^{(\dagger)})_{n\geq 0} \to \left(\underline{\mathrm{TSym}}^{n+1}_{\mathcal{O}_E} \mathcal{L}_1^{(\dagger)}\right)_{n\geq 0}.$$

For $(\mathcal{L}_n^{\dagger})$ this morphism is horizontal if we equip $\underline{\mathrm{TSym}}^n \mathcal{L}_1^{\dagger}$ with the connection induced by $\nabla_{\mathcal{L}_1^\dagger}$ on the tensor product. Let us summarize some properties of symmetric tensors in the following remark:

Remark 2.1.12. In this remark let X be a scheme and $\mathcal{F}$ a locally free $\mathcal{O}_X$-module of finite rank. Let S_n act on $\mathcal{F}^{\otimes n} := \mathcal{F} \otimes_{\mathcal{O}_X} \dots \otimes_{\mathcal{O}_X} \mathcal{F}$ via permutations and define

$$\underline{\mathrm{TSym}}^{n}_{\mathcal{O}_X} \mathcal{F} := \left(\mathcal{F}^{\otimes n}\right)^{S_n}.$$

For $k, l \geq 0$ define the following subgroup of S_{k+l}

$$S_{k,l} := \Big\{\sigma \in S_{k+l} : \sigma(1) < \sigma(2) < \dots < \sigma(k), \quad \sigma(k+1) < \dots < \sigma(k+l)\Big\}.$$

The shuffle product of $z \in \underline{\mathrm{TSym}}^k \mathcal{F}$ and $z' \in \underline{\mathrm{TSym}}^l \mathcal{F}$ is defined by

$$z \otimes z' \mapsto \sum_{\sigma \in S_{k,l}} \sigma(z \otimes z').$$

This defines a commutative ring structure on $\underline{\mathrm{TSym}}^{\bullet}_{\mathcal{O}_X} \mathcal{F} := \bigoplus_{n\geq 0} \underline{\mathrm{TSym}}^{n}_{\mathcal{O}_X}$. Furthermore,

$$[\cdot]^n : \underline{\mathrm{TSym}}^{\geq 1}_{\mathcal{O}_X} \to \underline{\mathrm{TSym}}^{\geq 1}_{\mathcal{O}_X}, \quad x \mapsto [x]^n := \underbrace{x \otimes \dots \otimes x}_{n \text{ times}}$$

defines a divided power structure on the ideal $\underline{\mathrm{TSym}}^{\geq 1}_{\mathcal{O}_X}$. Thus, one obtains a canonical morphism

$$\underline{\Gamma}_{\mathcal{O}_X}(\mathcal{F}) \to \underline{\mathrm{TSym}}^{\bullet}_{\mathcal{O}_X} \mathcal{F}$$

from the universal P.D.-algebra $\underline{\Gamma}_{\mathcal{O}_X}(\mathcal{F})$ to $\underline{\mathrm{TSym}}^{\bullet}_{\mathcal{O}_X} \mathcal{F}$ which is easily checked to be an isomorphism. More generally this holds for flat $\mathcal{O}_X$-modules [SGA4, p. 5.5.2.5.]. In particular, $\underline{\mathrm{TSym}}^{\bullet}_{\mathcal{O}_X} \mathcal{F}$ inherits all good properties of $\underline{\Gamma}_{\mathcal{O}_X}(\mathcal{F})$, e.g. compatibility with base change. In general, i. e. for non-flat modules, $\underline{\mathrm{TSym}}^{\bullet}$ is not necessary compatible with base change. Finally, let us note that the map

$$\underline{\mathrm{Sym}}^{1}_{\mathcal{O}_X} \mathcal{F} = \mathcal{F} = \underline{\mathrm{TSym}}^{1}_{\mathcal{O}_X} \mathcal{F} \hookrightarrow \underline{\mathrm{TSym}}^{\bullet}_{\mathcal{O}_X} \mathcal{F}$$

induces a canonical homomorphism of sheaves of $\mathcal{O}_X$-algebras

$$\underline{\mathrm{Sym}}^\bullet_{\mathcal{O}_X}\mathcal{F} \to \underline{\mathrm{TSym}}^\bullet_{\mathcal{O}_X}\mathcal{F}.$$

If X is a scheme over $\mathbb{Q}$, this map is an isomorphism.

Lemma 2.1.13. *If S is a scheme over a field of characteristic zero, then*

$$\mathcal{L}_n^{(\dagger)} \to \underline{\mathrm{TSym}}^n_{\mathcal{O}_E}\mathcal{L}_1^{(\dagger)}$$

is an isomorphism for all $n \geq 0$.

Proof. This can be shown as in [Sch14, §2.4.2]. More precisely, it is proven in loc. cit. around equation (2.4.7). $\square$

Next let us study the pullback of $\mathbb{D}^{(\dagger)}$ along e. If we combine $e^*\mathbb{D}^{(\dagger)}$ with the isomorphism

$$\begin{aligned} e^*\mathcal{L}_1 &\xrightarrow{\sim} \mathcal{O}_{\mathrm{Inf}^1_e E^\vee} = \mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S} \\ e^*\mathcal{L}_1^\dagger &\xrightarrow{\sim} \mathcal{O}_{\mathrm{Inf}^1_e E^\dagger} = \mathcal{O}_S \oplus \mathcal{H}, \end{aligned}$$

we get

$$\begin{aligned} e^*\mathcal{L}_n &\xrightarrow{e^*\mathbb{D}} \underline{\mathrm{TSym}}^n_{\mathcal{O}_S}\left(\mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S}\right) \xrightarrow{\sim} \bigoplus_{i=0}^n \underline{\mathrm{TSym}}^i_{\mathcal{O}_S}\underline{\omega}_{E^\vee/S} \\ e^*\mathcal{L}_n^\dagger &\xrightarrow{e^*\mathbb{D}^\dagger} \underline{\mathrm{TSym}}^n_{\mathcal{O}_S}\left(\mathcal{O}_S \oplus \mathcal{H}\right) \xrightarrow{\sim} \bigoplus_{i=0}^n \underline{\mathrm{TSym}}^i_{\mathcal{O}_S}\mathcal{H}. \end{aligned} \tag{2.10}$$

Here, we have used the canonical isomorphisms for locally free modules of finite rank

$$\underline{\mathrm{TSym}}^n_{\mathcal{O}_S}(\mathcal{F}_1 \oplus \mathcal{F}_2) \cong \bigoplus_{i=0}^n \left(\underline{\mathrm{TSym}}^{n-i}_{\mathcal{O}_S}\mathcal{F}_1 \otimes \underline{\mathrm{TSym}}^i_{\mathcal{O}_S}\mathcal{F}_2\right)$$

as well as the isomorphism $\underline{\mathrm{TSym}}^k_{\mathcal{O}_S}\mathcal{O}_S \xrightarrow{\sim} \mathcal{O}_S, \quad [1]^k \mapsto 1$. If we define on the right hand side of (2.10) transition maps by the projection maps

$$\mathrm{pr} : \bigoplus_{i=0}^{n+1} \underline{\mathrm{TSym}}^i_{\mathcal{O}_S}(\cdot) \twoheadrightarrow \bigoplus_{i=0}^{n} \underline{\mathrm{TSym}}^i_{\mathcal{O}_S}(\cdot),$$

then the maps in (2.10) are compatible with the transition maps on both sides. If we write $\hat{E}^\vee$ and $\hat{E}^\dagger$ for the formal completions, we get canonical isomorphisms

$$\begin{aligned} \varprojlim_n e^*\mathcal{L}_n &\xrightarrow{\sim} \varprojlim_n \mathcal{O}_{\mathrm{Inf}^n_e E^\vee} \xrightarrow{\sim} \mathcal{O}_{\hat{E}^\vee} \\ \varprojlim_n e^*\mathcal{L}_n^\dagger &\xrightarrow{\sim} \varprojlim_n \mathcal{O}_{\mathrm{Inf}^n_e E^\dagger} \xrightarrow{\sim} \mathcal{O}_{\hat{E}^\dagger}. \end{aligned}$$

Using this identification, we may rewrite the above maps as

$$\mathbb{D}_e : \mathcal{O}_{\hat{E}^\vee} \xrightarrow{\sim} \varprojlim_n e^*\mathcal{L}_n \longrightarrow \widehat{\underline{\mathrm{TSym}}}^{\bullet}_{\mathcal{O}_S}\underline{\omega}_{E^\vee/S}$$

$$\mathbb{D}_e^\dagger : \mathcal{O}_{\hat{E}^\dagger} \xrightarrow{\sim} \varprojlim_n e^*\mathcal{L}_n^\dagger \longrightarrow \widehat{\underline{\mathrm{TSym}}}^{\bullet}_{\mathcal{O}_S}\mathcal{H}$$

where completion $\widehat{\underline{\mathrm{TSym}}}^{\bullet}$ is with respect to the respective augmentation ideals $\underline{\mathrm{TSym}}^{\geq 1}$. The next result compares $\mathbb{D}_e^{(\dagger)}$ to the map obtained by applying iteratively the universal invariant derivation of the formal groups $\hat{E}^\vee$ resp. $\hat{E}^\dagger$. The most convenient way to formulate the result is via the bigebra of invariant differential operators of the formal groups, cf. [BKL14, Remark 1.1.8]:

Lemma 2.1.14. *Let $\mathbb{D}_e$ and $\mathbb{D}_e^\dagger$ be the maps defined above.*

(a) Let us write $\mathbb{H}_{\hat{E}^\dagger}$ for the sheaves of bigebras of invariant differential operators on the formal groups $\hat{E}^\dagger$. We have a canonical isomorphism $\mathcal{O}_{\hat{E}^\dagger} \cong \mathrm{Hom}_{\mathcal{O}_S}(\mathbb{H}_{\hat{E}^\dagger}, \mathcal{O}_S)$, and the inclusion of the Lie algebra $\mathcal{H}^\vee$ of $\hat{E}^\dagger$ into $\mathbb{H}_{\hat{E}^\dagger}$ induces a canonical map

$$\underline{\mathrm{Sym}}^{\bullet}_{\mathcal{O}_S}\mathcal{H}^\vee \to \mathbb{H}_{\hat{E}^\dagger}.$$

Then, the composition

$$\mathcal{O}_{\hat{E}^\dagger} \cong \mathrm{Hom}_{\mathcal{O}_S}(\mathbb{H}_{\hat{E}^\dagger}, \mathcal{O}_S) \to \mathrm{Hom}_{\mathcal{O}_S}\left(\underline{\mathrm{Sym}}^{\bullet}_{\mathcal{O}_S}\mathcal{H}^\vee, \mathcal{O}_S\right) \cong \widehat{\underline{\mathrm{TSym}}}^{\bullet}_{\mathcal{O}_S}\mathcal{H}$$

coincides with $\mathbb{D}_e^\dagger$.

(b) Similarly, in the case of $\hat{E}^\vee$ the inclusion of the Lie algebra $(\underline{\omega}_{E^\vee/S})^\vee$ into the sheaf of invariant differential operators $\mathbb{H}_{\hat{E}^\vee}$ induces a map

$$\mathcal{O}_{\hat{E}^\vee} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(\mathbb{H}_{\hat{E}^\vee}, \mathcal{O}_S) \to \mathrm{Hom}_{\mathcal{O}_S}\left(\underline{\mathrm{Sym}}^{\bullet}_{\mathcal{O}_S}\underline{\omega}^\vee_{E^\vee/S}, \mathcal{O}_S\right) \cong \widehat{\underline{\mathrm{TSym}}}^{\bullet}_{\mathcal{O}_S}\underline{\omega}_{E^\vee/S}$$

which coincides with $\mathbb{D}_e$.

Proof. We just consider the case of $\hat{E}^\dagger$ the case of $\hat{E}^\vee$ is completely analog. Let

$$\mathcal{O}_{\mathrm{Inf}_e^n E^\dagger} \xrightarrow{\sim} e^*\mathcal{L}_n^\dagger \to \bigoplus_{i=0}^{n} \underline{\mathrm{TSym}}^i_{\mathcal{O}_S}\mathcal{H} \tag{2.11}$$

be the map induced by the $\mathbb{G}_{m,S}$-biextension structure on $\mathcal{P}^\dagger$. In [Sch14, S. 124] it is shown that its dual

$$\bigoplus_{i=0}^{n} \underline{\mathrm{Sym}}^i_{\mathcal{O}_E}\mathcal{H}^\vee \to \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_{\mathrm{Inf}_e^n E^\dagger}, \mathcal{O}_S) = \mathbb{H}^n_{\hat{E}^\dagger} \tag{2.12}$$

is a homomorphism of $\mathcal{O}_S$-algebras. Thus, in order to identify this map with the morphism induced by the inclusion of the Lie algebra $\mathcal{H}^\vee$ into the sheaf of differential operators $\mathbb{H}^n_{\hat{E}^\dagger}$ of order $\leq n$ it is, by the universal property of $\underline{\mathrm{Sym}}^\bullet$, enough to show that (2.12) restricted to $\mathcal{H}^\vee$ is the canonical inclusion. The compatibility of (2.11), which is dual to (2.12), with transition maps allows us to reduce to the case $n = 1$. But in the case $n = 1$ the claim is obvious. Thus, (2.12) coincides with the map induced by the canonical inclusion of the Lie algebra into the sheaf of differential operators of order $\leq n$. Taking duals and passing to the limit proves the claim. □

We can reformulate this as:

Corollary 2.1.15. *Let*

$$d : \mathcal{O}_{\hat{E}^\dagger} \to \mathcal{O}_{\hat{E}^\dagger} \otimes_{\mathcal{O}_S} \mathcal{H}$$

be the map induced by the invariant derivation. Applying d iteratively and evaluating at zero gives a map

$$\mathcal{O}_{\hat{E}^\dagger} \to \underline{\mathrm{TSym}}^n_{\mathcal{O}_S} \mathcal{H}, f \mapsto e^*\left(d^{\circ n} f\right).$$

Then,

$$\mathbb{D}^\dagger_e : \mathcal{O}_{\hat{E}^\dagger} \to \widehat{\underline{\mathrm{TSym}}}^\bullet_{\mathcal{O}_S} \mathcal{H}$$

is given by $f \mapsto \left(e^*\left(d^{\circ n} f\right)\right)_{n \geq 0}$. And the analog statement holds for $E^\vee$.

Remark 2.1.16. We have chosen the notation $\mathbb{D}^{(\dagger)}$ to emphasize that this map specializes after pullback along e to the map taking all iterated invariant derivatives at zero.

2.2 The canonical section and the geometric logarithm sheaves

The purpose of this section is to construct sections of the geometric logarithm sheaves via the canonical section.

Definition 2.2.1. Let E/S be an elliptic curve with D invertible on S. The canonical isomorphism

$$[D]^*\Omega^1_{E/S}([e]) \xrightarrow{\sim} \Omega^1_{E/S}(E[D])$$

tensorized with $(\Phi_D)^{-1}$ gives

$$(\Phi_D)^{-1} \otimes \mathrm{can} : [D]^*\mathcal{L}_n \otimes_{\mathcal{O}_E} [D]^*\Omega^1_{E/S}([e]) \xrightarrow{\sim} \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D]). \tag{2.13}$$

Here, we have used that D is invertible on S for Φ_D being an isomorphism.

(a) Define

$$l^D_n \in \Gamma\left(E, \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D])\right)$$

as the image of

$$(\mathrm{pr}_E)_*(\mathrm{id} \times \iota_n)^*(s^D_{\mathrm{can}}) \in \Gamma\left(E, [D]^*\left[\mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}([e])\right]\right)$$

under (2.13). Here, s^D_{can} is the D-variant of the canonical section of the Poincaré bundle defined in Chapter 1.

(b) For $e \neq t \in E^\vee[D](S)$ define

$$l^D_{t,n} \in \Gamma\left(E, \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}\left(E[D]\right)\right)$$

as the image of

$$(\mathrm{pr}_E)_*(\mathrm{id} \times \iota_n)^*(U^D_t(s_{\mathrm{can}})) \in \Gamma\left(E, [D]^*\left[\mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}\left([e]\right)\right]\right)$$

under (2.13).

Sometimes we will use the canonical inclusions

$$\mathcal{L}_n \hookrightarrow \mathcal{L}^\dagger_n$$

and view l^D_n or $l^D_{t,n}$ as global sections of $\mathcal{L}^\dagger_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}\left(E[D]\right)$.

Remark 2.2.2. The canonical section s_{can} is a global section of

$$\mathcal{P} \otimes \Omega^1_{E\times E^\vee/E^\vee}\left([E \times e] + [e \times E^\vee]\right).$$

Restricting it to $\mathcal{L}_n$ and using $e^*\mathcal{O}_{E^\vee}([e]) = \underline{\omega}^\vee_{E^\vee/S}$ gives a global section

$$l_n := (\mathrm{pr}_E)_*(\mathrm{id} \times \iota_n)^*(s_{\mathrm{can}}) \in \Gamma\left(E, \mathcal{L}_n \otimes_{\mathcal{O}_E} (\pi_E)^*\left[\underline{\omega}^\vee_{E^\vee/S}\right] \otimes_{\mathcal{O}_E} \Omega^1_{E/S}([e])\right).$$

Later, we will use the sections l^D_n to describe the D-variant of the elliptic polylogarithm in de Rham cohomology. If one prefers to describe the classical elliptic polylogarithm in de Rham cohomology, one can do this along the same lines but starting with l_n instead of l^D_n.

2.3 Analytification of the logarithm sheaves

In [Sch14, §3.5] Scheider studied the analytification of the geometric logarithm sheaves $\mathcal{L}^\dagger_n$. Using the Jacobi theta function, he was able to describe the analytification of the de Rham realization explicitly in terms of theta functions. In this section we show that the algebraically defined sections $(l^D_n)_{n\geq 0}$ correspond to the (relative version of the) analytic sections used by Scheider to describe the de Rham realization of the elliptic polylogarithm on the universal elliptic curve.

In order to compare the algebraically defined sections l^D_n with the analytic description given by Scheider, we follow the analytification given in [Sch14, §3.5] closely. We keep the notation introduced in Section 1.6 for describing the analytification of the universal elliptic curve $\pi_E : E_N \to M_N$ with $\Gamma(N)$-level structure for $N > 3$. We summarize the introduced notation in the following diagrams

$$\begin{tikzcd} \widetilde{E}_N := \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "\tilde{p}_E"] \arrow[d, "\tilde{\pi}_E"] & E_N^{an} = \mathbb{Z}^2 \rtimes \Gamma(N) \setminus \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \arrow[d, "\pi_E^{an}"] \\ \widetilde{M}_N := \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "\tilde{p}_M"] \arrow[u, bend left, "(0)"] & M_N^{an} = \Gamma(N) \setminus \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times . \arrow[u, bend left, "e"] \end{tikzcd}$$

with coordinates (z, τ, j) on $\widetilde{E}_N$. The analytification of $E_N^\vee$ and its universal vectorial extension is summarized in the following diagram

$$\begin{tikzcd} \widetilde{E}_N^\dagger = \mathbb{C}^2 \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "-\mathrm{pr}_1"] \arrow[d] & \widetilde{E}_N^\vee = \mathbb{C} \times \mathbb{H} \times (\mathbb{Z}/N\mathbb{Z})^\times \arrow[d] \\ E_N^{\dagger,an} \arrow[r] & E_N^{\vee,an}. \end{tikzcd} \tag{2.14}$$

As before, write (w', v, τ, j) for the coordinates on $\widetilde{E}_N^\dagger$. Furthermore, let us again write $\tilde{\mathcal{P}}$ for the pullback of the Poincaré bundle to $\widetilde{E}_N \times_{\widetilde{M}_N} \widetilde{E}_N^\vee$ and $\tilde{\mathcal{P}}^\dagger$ for the pullback of the Poincaré bundle on $E \times_S E^\dagger$ to the universal covering $\widetilde{E}_N \times_{\widetilde{M}_N} \widetilde{E}_N^\dagger$. Using the Jacobi theta function, we have obtained trivializations:

$$\tilde{\mathcal{P}} \xrightarrow{\sim} \mathcal{O}^{an}_{\widetilde{E}_N \times_{\widetilde{M}_N} \widetilde{E}_N^\vee}, \quad \tilde{\mathfrak{t}} \mapsto 1 \tag{2.15}$$

$$\tilde{\mathcal{P}}^\dagger \xrightarrow{\sim} \mathcal{O}^{an}_{\widetilde{E}_N \times_{\widetilde{M}_N} \widetilde{E}_N^\dagger}, \quad \tilde{\mathfrak{t}}^\dagger \mapsto 1. \tag{2.16}$$

Restricting the isomorphisms (2.15) resp. (2.16) to $\widetilde{E}_N \times \mathrm{Inf}^1_e \widetilde{E}_N^\vee$ resp. $\widetilde{E}_N \times \mathrm{Inf}^1_e \widetilde{E}_N^\dagger$ gives analytic trivializations

$$\tilde{\mathcal{L}}_1 := \tilde{p}_E^* \mathcal{L}_1 = (\mathrm{pr}_{\widetilde{E}_N})_* (\mathrm{id} \times \tilde{\iota}_1)^* \tilde{\mathcal{P}} \xrightarrow{\sim} (\mathrm{pr}_{\widetilde{E}_N})_* \mathcal{O}^{an}_{\widetilde{E}_N \times_{\widetilde{M}_N} \mathrm{Inf}^1_e \widetilde{E}_N^\vee} \xrightarrow{\sim} \mathcal{O}^{an}_{\widetilde{E}_N} \oplus \underline{\tilde{\omega}}^{an}_{E^\vee}$$

and

$$\tilde{\mathcal{L}}_1^\dagger := \tilde{p}_E^* \mathcal{L}_1^\dagger = (\mathrm{pr}_{\widetilde{E}_N})_* (\mathrm{id} \times \tilde{\iota}_1^\dagger)^* \tilde{\mathcal{P}}^\dagger \xrightarrow{\sim} \mathcal{O}^{an}_{\widetilde{E}_N} \oplus \tilde{\mathcal{H}}^{an}_E.$$

Here, we have used the notation $\tilde{\mathcal{H}}^{an}_E$ and $\underline{\tilde{\omega}}^{an}_{E^\vee}$ for the pullback of $\mathcal{H}_E$ and $\underline{\omega}_{E^\vee/S}$ to the universal covering of E_N. Combining the analytic trivializations with the maps $\mathbb{D}^\dagger$ gives isomorphisms

$$\widetilde{\mathrm{split}} : \tilde{\mathcal{L}}_n \xrightarrow{\sim} \underline{\mathrm{TSym}}^n \, \tilde{\mathcal{L}}_1 \xrightarrow{\sim} \bigoplus_{i=0}^n \underline{\mathrm{TSym}}^n \, \underline{\tilde{\omega}}^{an}_{E^\vee}$$

and

$$\widetilde{\mathrm{split}}^\dagger : \tilde{\mathcal{L}}_n^\dagger \xrightarrow{\sim} \underline{\mathrm{TSym}}^n \, \tilde{\mathcal{L}}_1^\dagger \xrightarrow{\sim} \bigoplus_{i=0}^n \underline{\mathrm{TSym}}^n \, \tilde{\mathcal{H}}^{an}_E.$$

We call these *the analytic splitting* maps. Note that these splittings depend on the theta function which is chosen in the trivialization of the Poincaré bundle. We can use the chosen generators $\omega = dw$ of $\underline{\tilde{\omega}}^{an}_{E^\vee}$ and $[\eta]^\vee, [\omega]^\vee$ of $\tilde{\mathcal{H}}^{an}_{E}$ to define a basis of the free $\mathcal{O}^{an}_{\widetilde{E}_N}$-modules $\tilde{\mathcal{L}}^\dagger_n$ and $\tilde{\mathcal{L}}_n$:

$$\begin{aligned}
\tilde{\omega}^{[k]} &: = \widetilde{\mathrm{split}}^{-1}(\omega^{[i]}), \quad \forall\, 0 \leq i \leq n \\
\tilde{\omega}^{[k,l]} &: = (\widetilde{\mathrm{split}}^{\dagger})^{-1}\left(([\eta]^\vee)^{[k]} \cdot ([\omega]^\vee)^{[l]}\right), \quad \forall\, 0 \leq k+l \leq n
\end{aligned}$$

These form a basis for $\tilde{\mathcal{L}}_n$ resp. $\tilde{\mathcal{L}}^\dagger_n$:

$$\tilde{\mathcal{L}}_n = \bigoplus_{i=0}^{n} \tilde{\omega}^{[i]} \mathcal{O}^{an}_{\widetilde{E}_N} \tag{2.17}$$

$$\tilde{\mathcal{L}}^\dagger_n = \bigoplus_{i+j \leq n} \tilde{\omega}^{[i,j]} \mathcal{O}^{an}_{\widetilde{E}_N} \tag{2.18}$$

Remark 2.3.1. Scheider uses $\underline{\mathrm{Sym}}$ instead of $\underline{\mathrm{TSym}}$, i. e. he uses the isomorphism

$$\tilde{\mathcal{L}}^\dagger_n \xrightarrow{\sim} \underline{\mathrm{Sym}}^n\, \tilde{\mathcal{L}}^\dagger_1 \xrightarrow{\sim} \bigoplus_{i=0}^{n} \underline{\mathrm{Sym}}^i\, \tilde{\mathcal{H}}^{an}_E \xrightarrow{\frac{1}{n!}} \bigoplus_{i=0}^{n} \underline{\mathrm{Sym}}^i\, \tilde{\mathcal{H}}^{an}_E.$$

He chooses

$$\frac{f^i g^j}{(n-i-j)!} := \frac{([\eta]^\vee)^i([\omega]^\vee)^j}{(n-i-j)!} \in \underline{\mathrm{Sym}}^{i+j}\, \tilde{\mathcal{H}}^{an}_E$$

as a basis to trivialize $\tilde{\mathcal{L}}^\dagger_n$. It is straightforward to check that our basis is related to his basis via:

$$\frac{e^i f^j}{(n-i-j)!} \quad \leftrightsquigarrow \quad i!j!\tilde{\omega}^{[i,j]}.$$

Lemma 2.3.2. *The inclusion*

$$\tilde{\mathcal{L}}_n \hookrightarrow \tilde{\mathcal{L}}^\dagger_n$$

identifies $\tilde{\omega}^{[i]}$ *with* $(-1)^i\tilde{\omega}^{[i,0]}$.

Proof. Using the definitions of $\tilde{\omega}^{[i]}$ and $\tilde{\omega}^{[i,0]}$, this boils down to the fact that

$$\underline{\tilde{\omega}}^{an}_{E^\vee} \xrightarrow{\sim} \tilde{p}^*_M \underline{\omega}^{an}_{E_N/M_N} \hookrightarrow \tilde{p}^*_M \underline{H}^1_{\mathrm{dR}}\,(E_N/M_N)^{an} \xrightarrow{\sim} \tilde{\mathcal{H}}^{an}_E$$

identifies $\omega = dw$ with $-[\eta]^\vee$. $\square$

Lemma 2.3.3. *The connection on* $\mathcal{L}^\dagger_n$ *induces a connection on* $\tilde{\mathcal{L}}^\dagger_n$*:*

$$\nabla_{\tilde{\mathcal{L}}^\dagger_n} : \tilde{\mathcal{L}}^\dagger_n \to \tilde{\mathcal{L}}^\dagger_n \otimes_{\mathcal{O}^{an}_{\widetilde{E}_N}} \Omega^1_{\widetilde{E}_N/\widetilde{M}_N}$$

This connection is explicitly given by the formula

$$\tilde{\omega}^{[i,j]} \mapsto (i+1)\eta(1,\tau)\tilde{\omega}^{[i+1,j]} \otimes dz + (j+1)\tilde{\omega}^{[i,j+1]} \otimes dz.$$

Proof. The map
$$\tilde{\mathcal{L}}_n^\dagger \xrightarrow{\sim} \underline{\mathrm{TSym}}^n\, \tilde{\mathcal{L}}_1^\dagger$$
is horizontal if we equip $\underline{\mathrm{TSym}}^n\, \tilde{\mathcal{L}}_1^\dagger$ with the connection induced by $\nabla_{\tilde{\mathcal{L}}_1^\dagger}$ on the tensor product. The explicit description of the connection $\nabla_{\tilde{\mathcal{P}}^\dagger}$ gives us the following explicit description of $\nabla_{\tilde{\mathcal{L}}_1^\dagger}$: It is explicitly given as $\nabla_{\tilde{\mathcal{L}}_1^\dagger}(\tilde{\omega}^{[0,0]}) = \eta(1,\tau)\tilde{\omega}^{[1,0]} + \tilde{\omega}^{[0,1]}$ and $\nabla_{\tilde{\mathcal{L}}_1^\dagger}(\tilde{\omega}^{[1,0]}) = \nabla_{\tilde{\mathcal{L}}_1^\dagger}(\tilde{\omega}^{[0,1]}) = 0$. The isomorphism
$$\underline{\mathrm{TSym}}^n\, \tilde{\mathcal{L}}_1^\dagger \xrightarrow{\sim} \bigoplus_{i=0}^n \underline{\mathrm{TSym}}^i\, \tilde{\mathcal{H}}_E^{an}$$
identifies $\left(\tilde{\omega}^{[0,0]}\right)^{[n-i-j]}\left(\tilde{\omega}^{[1,0]}\right)^{[i]}\left(\tilde{\omega}^{[0,1]}\right)^{[j]}$ with $\tilde{\omega}^{[i,j]}$. Using this isomorphism, the claim follows from:
$$\begin{aligned}
&\nabla_{\underline{\mathrm{TSym}}^n\, \mathcal{L}_1^\dagger}\left[(\left(\tilde{\omega}^{[0,0]}\right)^{[n-i-j]}\left(\tilde{\omega}^{[1,0]}\right)^{[i]}\left(\tilde{\omega}^{[0,1]}\right)^{[j]})\right] = \\
=&\nabla_{\tilde{\mathcal{L}}_1^\dagger}(\tilde{\omega}^{[0,0]})\cdot\left(\tilde{\omega}^{[0,0]}\right)^{[n-i-j-1]}\cdot\left(\tilde{\omega}^{[1,0]}\right)^{[i]}\cdot\left(\tilde{\omega}^{[0,1]}\right)^{[j]} = \\
=&(i+1)\eta(1,\tau)\left(\tilde{\omega}^{[0,0]}\right)^{[n-i-j-1]}\cdot\left(\tilde{\omega}^{[1,0]}\right)^{[i+1]}\cdot\left(\tilde{\omega}^{[0,1]}\right)^{[j]}\otimes dz+ \\
&+(j+1)\left(\tilde{\omega}^{[0,0]}\right)^{[n-i-j-1]}\cdot\left(\tilde{\omega}^{[1,0]}\right)^{[i]}\cdot\left(\tilde{\omega}^{[0,1]}\right)^{[j+1]}\otimes dz
\end{aligned}$$
□

Let us denote the induced translation operator on the universal covering by
$$\tilde{\mathrm{trans}}_t : T_t^*\tilde{\mathcal{L}}_n^{(\dagger)} \to \tilde{\mathcal{L}}_n^{(\dagger)}$$
and the invariance under isogenies map by
$$\tilde{\Phi}_D : \tilde{\mathcal{L}}_n^{(\dagger)} \xrightarrow{\sim} [D]^*\tilde{\mathcal{L}}_n^{(\dagger)}.$$

Lemma 2.3.4 ([Sch14, Prop. 3.5.9])**.** *We have the following explicit formulas for the translation operators and the behaviour under isogenies.*

(a) *For $D \geq 1$ we have*
$$\begin{aligned}
\tilde{\Phi}_D : \tilde{\mathcal{L}}_n^\dagger &\xrightarrow{\quad\sim\quad} [D]^*\tilde{\mathcal{L}}_n^\dagger \\
\textstyle\sum_{i+j\leq n} f_{i,j}\tilde{\omega}^{[i,j]} &\longmapsto \textstyle\sum_{i+j\leq n} D^{i+j} f_{i,j}[D]^*\left(\tilde{\omega}^{[i,j]}\right)
\end{aligned}$$

(b) *Let $(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$ and consider the lift $\tilde{s} = j\frac{a}{N}\tau + \frac{b}{N}$ of the torsion section $s = s_{a,b} \in E_N^{an}[N](M_N^{an})$ corresponding to (a,b). Then, the pullback of* trans_s *to the universal covering is given by:*
$$\begin{aligned}
\mathrm{trans}_{\tilde{s}} : T_{\tilde{s}}^*\tilde{\mathcal{L}}_n^\dagger &\xrightarrow{\quad\sim\quad} \tilde{\mathcal{L}}_n^\dagger \\
\textstyle\sum_{i+j\leq n} f_{i,j}T_{\tilde{s}}^*\tilde{\omega}^{[i,j]} &\longmapsto \left(\textstyle\sum_{i+j\leq n} f_{i,j}\tilde{\omega}^{[i,j]}\right)\cdot\exp\left[-\tilde{\omega}^{[1,0]}\frac{\tilde{s}-\bar{\tilde{s}}}{A(\tau,j)}\right]
\end{aligned}$$

where the product is taken in $\underline{\mathrm{TSym}}^\bullet$ *and* $\exp\left[\tilde{\omega}^{[1,0]}\varphi\right] = \sum_{i=0}^n \varphi^i\tilde{\omega}^{[i,0]}$ *for any* $\varphi \in \mathcal{O}_{E_N}^{an}$.

Proof. Let us first observe that in the case $n = 1$ the basis (e, f, g) of $\mathcal{L}_1^\dagger$ used by Scheider coincides with our basis $(\tilde{\omega}^{[0,0]}, \tilde{\omega}^{[1,0]}, \tilde{\omega}^{[0,1]})$.
(a): The case $n = 1$ is exactly the equation $(**)$ in the proof of [Sch14, Prop. 3.5.9]. Further, Φ_D is compatible with the isomorphism

$$\mathcal{L}_n^\dagger \xrightarrow{\sim} \underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger$$

and we can deduce the general case from $n = 1$ by applying $\underline{\mathrm{TSym}}^n$.
(b): The case $n = 1$ is treated in [Sch14, Prop. 3.5.9] and the general case follows by applying $\underline{\mathrm{TSym}}^n$. $\square$

Lemma 2.3.5. *The analytifications of the sections* l_n^D *and* $l_{t,n}^D$ *can be described on the universal covering as follows:*

(a) *For* $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^2$ *and* $\tilde{s} = j\frac{a}{N}\tau + \frac{b}{N}$ *lifting* $s = s_{a,b} \in E_N[N]$ *the section* $\tilde{l}_{s,n}^N \in \tilde{\mathcal{L}}_n^\dagger \otimes \tilde{p}_E^*\Omega^1_{E_N/M_N}(E_N[N])$ *is given as*

$$\tilde{l}_{s,n}^N = \sum_{i=0}^n i!\frac{s_{\tilde{s},i}^N}{N^i}(z,\tau)\tilde{\omega}^{[i,0]} \otimes dz$$

where $s_{\tilde{s},i}^N$ *is defined via the expansion*

$$NJ_{o,\tilde{s}}(Nz, -w', \tau) = \sum_{i\geq 0} s_{\tilde{s},i}^N (w')^i$$

(b) *For* $D > 1$ *the section* $\tilde{l}_n^D \in \tilde{\mathcal{L}}_n^\dagger \otimes \tilde{p}_E^*\Omega^1_{E_N/M_N}(E_N[D])$ *is given as*

$$\sum_{i=0}^n i!\frac{s_i^D(z,\tau)}{D^i}\tilde{\omega}^{[i,0]} \otimes dz$$

where $s_i^D(z, \tau, j)$ *is defined via the expansion*

$$D^2J(z, -Dw', \tau) - DJ(Dz, -w', \tau) = \sum_{i\geq 0} s_i^D(z,\tau)(w')^i$$

Proof. (a): From Proposition 1.6.7 we know that the pullback of $U_{1,s}^{1,N}(s_{\mathrm{can}})$ to the universal covering is given by

$$([N] \times \mathrm{id})^*\left[J_{0,\tilde{s}}(z, -w', \tau)\tilde{\mathfrak{t}}^\dagger \otimes dz\right].$$

applying $(\mathrm{pr}_{\widetilde{E}_N})(\mathrm{id} \times \tilde{\iota})^*$ gives

$$\sum_{i=0}^n i! s_{\tilde{s},i}^N [N]^*\left(\tilde{\omega}^{[i,0]} \otimes dz\right).$$

By its definition, $\tilde{l}^N_{s,n}$ is the image of this element under

$$\Phi_N^{-1} \otimes \mathrm{can} : [N]^* \left[\tilde{\mathcal{L}}^\dagger_n \otimes \tilde{p}_E^* \Omega^1_{E_N/M_N}([e])\right] \xrightarrow{\sim} \tilde{\mathcal{L}}^\dagger_n \otimes \tilde{p}_E^* \Omega^1_{E_N/M_N}(E_N[N]) .$$

Now, the result follows using Lemma 2.3.4 (a) and the fact that $[N]^* dz$ maps to $N dz$ under the canonical isomorphism $\mathrm{can} : [N]^* \Omega^1_{\widetilde{E}_N/\widetilde{M}_N}([e]) \xrightarrow{\sim} \Omega^1_{\widetilde{E}_N/\widetilde{M}_N}(E_N[N])$. (b) follows similarly. □

2.4 Real analytic Eisenstein series via the geometric logarithm sheaves

We give an 'infinitesimal' version of our construction of Eisenstein–Kronecker series via the Poincaré bundle (cf. Section 1.7). In order to avoid choosing level structures on both E and $E^\vee$, let us fix $E \xrightarrow{\sim} E^\vee$. For $s \in E[N](S)$ let us define

$$\mathbb{D}^\dagger_s : s^* \mathcal{L}^\dagger_n \xrightarrow{\sim} \bigoplus_{i=0}^n \underline{\mathrm{TSym}}^i_{\mathcal{O}_S} \mathcal{H}$$

as the composition

$$s^* \mathcal{L}^\dagger_n \xrightarrow{\sim} e^* \mathcal{L}^\dagger_n \xrightarrow{\sim} \bigoplus_{i=0}^n \underline{\mathrm{TSym}}^i_{\mathcal{O}_S} \mathcal{H}.$$

Furthermore, the connection $\nabla_{\mathcal{L}^\dagger_n}$ induces maps

$$\nabla_{\mathcal{L}^\dagger_n} : \mathcal{L}^\dagger_n \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1} \underline{\omega}^{\otimes k}_{E/S} \xrightarrow{\nabla_{\mathcal{L}^\dagger_n} \otimes \mathrm{id}} \mathcal{L}^\dagger_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S} \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1} \underline{\omega}^{\otimes k}_{E/S} \xrightarrow{\sim} \mathcal{L}^\dagger_n \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1} \underline{\omega}^{\otimes k+1}_{E/S}.$$

Let us write $U_D := E \setminus E[D]$. Via the identification

$$\left(\mathcal{L}^\dagger_n \otimes \Omega^1_{E/S}(E[D])\right)|_{U_D} = \left(\mathcal{L}^\dagger_n \otimes \Omega^1_{E/S}\right)|_{U_D}$$

we will view l^D_n as section of $\Gamma(U_D, \mathcal{L}^\dagger_n \otimes \Omega^1_{E/S})$.

Definition 2.4.1.

(a) Let N, D be coprime integers. Let E/S be an elliptic curve with $\Gamma(N)$-level structure α. For $(0,0) \neq (a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$ let $s := s_{(a,b)} \in E[N](S)$ be the corresponding N-torsion section of E. We define

$$_D TE^{k,r+1}_{(a,b)} \in \Gamma\left(S, \underline{\mathrm{TSym}}^k_{\mathcal{O}_S} \mathcal{H} \otimes_{\mathcal{O}_S} \underline{\omega}^{\otimes(r+1)}_{E/S}\right)$$

via

$$\left({}_D TE^{k,r+1}_{(a,b)}\right)^n_{k=0} := (\mathbb{D}_s \otimes \mathrm{id}_{\omega^{r+1}})(s^* \left[\nabla^{(\circ r)}_{\mathcal{L}^\dagger_n}(l^D_n)\right]).$$

(b) Let N, D be coprime integers. Let E/S be an elliptic curve with $\Gamma(ND)$-level structure α. For $(0,0) \neq (a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $(0,0) \neq (c,d) \in (\mathbb{Z}/D\mathbb{Z})^2$ consider the torsion sections $s = s_{(a,b)} \in E[N](S)$ and $t = t_{(c,d)} \in E^\vee[D](S) \cong E[D](S)$. We define
$$TE^{k,r+1}_{(a,b),(c,d)} \in \Gamma\left(S, \underline{\mathrm{TSym}}^k_{\mathcal{O}_S} \mathcal{H} \otimes_{\mathcal{O}_S} \underline{\omega}^{\otimes(r+1)}_{E/S}\right)$$
via
$$\left(TE^{k,r+1}_{(a,b),(c,d)}\right)_{k=0}^n := (\mathbb{D}_s \otimes \mathrm{id}_{\omega^{r+1}})(s^* \left[\nabla^{(\circ r)}_{\mathcal{L}_n^\dagger}(l^D_{t,n})\right]).$$

Note that ${}_DTE^{k,r+1}_{(a,b)}$ does not depend on the chosen $n \geq k$ by compatibility with the transition maps.

Remark 2.4.2. The T in the notation of $TE^{k,r+1}_{(a,b),(c,d)}$ and ${}_DTE^{k,r+1}_{(a,b)}$ is to express the usage of $\underline{\mathrm{TSym}}$ instead of $\underline{\mathrm{Sym}}$.

The construction
$$(E, \alpha) \mapsto {}_DTE^{k,r+1}_{(a,b)}, (\text{ resp. } TE^{k,r+1}_{(a,b),(c,d)})$$
is easily checked to be compatible with base change and isomorphisms of elliptic curves with level structure. The Hodge decomposition on the universal elliptic curve induces a projection
$$\underline{\mathrm{TSym}}^k \mathcal{H}^{an} \otimes \mathcal{C}^\infty \twoheadrightarrow \underline{\mathrm{TSym}}^k \underline{\omega}_{E/M}(\mathcal{C}^\infty) \xrightarrow{\sim} \underline{\omega}^{\otimes k}_{E/M}(\mathcal{C}^\infty)$$
where the last map is given by $(dz)^{[k]} \mapsto (dz)^{\otimes k}$.

Theorem 2.4.3. *Let $N, D > 1$ coprime.*

(a) *On the universal elliptic curve $E_{ND} \to M_{ND}$ with $\Gamma(ND)$-level structure the Hodge decomposition*
$$\left(\underline{\mathrm{TSym}}^k_{\mathcal{O}_M} \mathcal{H}^{an} \otimes \underline{\omega}^{\otimes(r+1)}_{E/M}\right) \otimes \mathcal{C}^\infty(M_{ND}) \twoheadrightarrow \underline{\omega}^{\otimes(k+r+1)}_{E/M}(\mathcal{C}^\infty)$$
maps
$$TE^{k,r+1}_{(a,b),(c,d)}$$
to
$$\left((-1)^{k+r} k! r! D^{r+1-k} \frac{e^*_{k,r+1}(Ds, \tilde{t})}{A(\tau)^k k!} dz^{\otimes(k+r+1)}\right)_{k=0}^n.$$

(b) *On the universal elliptic curve $E_N \to M_N$ with $\Gamma(N)$-level structure the Hodge decomposition*
$$\left(\underline{\mathrm{TSym}}^k_{\mathcal{O}_M} \mathcal{H}^{an} \otimes \underline{\omega}^{\otimes(r+1)}_{E/M}\right) \otimes \mathcal{C}^\infty(M_{ND}) \twoheadrightarrow \underline{\omega}^{\otimes(k+r+1)}_{E/M}(\mathcal{C}^\infty)$$
maps
$${}_DTE^{k,r+1}_{(a,b)}$$
to
$$\left((-1)^{k+r} k! r! \left[D^2 \frac{e^*_{k,r+1}(s, \tilde{t})}{A(\tau)^k k!} - D^{r+1-k} \frac{e^*_{k,r+1}(D\tilde{s}, \tilde{t})}{A(\tau)^k k!}\right] dz^{\otimes(k+r+1)}\right)_{k=0}^n.$$

Proof. It is not hard to deduce this from Theorem 1.7.2. Let us nevertheless give a direct proof using the analytification of the logarithm sheaves.

(a): We compute the image of

$$(\mathbb{D}_s \otimes \mathrm{id}_{\omega^{r+1}})(s^* \left[\nabla^{(\circ r)}_{\mathcal{L}^\dagger_n}(l^D_{t,n})\right])$$

under the Hodge decomposition on the universal elliptic curve with $\Gamma(ND)$-level structure. In this proof let us drop the subscript ND, i.e. we write E/M, $\tilde{E}/\tilde{M}$ for E_{ND}/M_{ND}, $\tilde{E}_{ND}/\tilde{M}_{ND}$ and so on. Let

$$\tilde{\mathbb{D}}^\dagger_s : \tilde{s}^*\tilde{\mathcal{L}}^\dagger_n \xrightarrow{\sim} \bigoplus_{i=0}^{n} \underline{\mathrm{TSym}}^i_{\mathcal{O}_{\tilde{M}}} \tilde{p}^*_M \underline{H}^1_{\mathrm{dR}}\left(E^\vee/M\right)^{an}$$

be the lift of $\mathbb{D}^\dagger_s$ to the universal covering.

Claim: The image of

$$(\tilde{\mathbb{D}}_s \otimes \mathrm{id}_{\omega^{r+1}})(\tilde{s}^* \left[\nabla^{(\circ r)}_{\tilde{\mathcal{L}}^\dagger_n}(\tilde{l}^D_{t,n})\right])$$

in

$$\bigoplus_{k=0}^{n}\left(\underline{\mathrm{TSym}}^k_{\mathcal{O}^{an}_{\tilde{M}}} \tilde{p}^*_M \underline{H}^1_{\mathrm{dR}}\left(E/M\right)^{an} \otimes \tilde{p}^*_M \underline{\omega}^{\otimes(r+1)}_{E^{an}/M^{an}}\right) \otimes \mathcal{C}^\infty(M^{an})$$

is given by

$$r!\sum_{k=0}^{n}\sum_{i=0}^{\min(r,n-k)} D^{1+r-i-k}(-1)^{r+k}\frac{e^*_{k,r+1-i}(D\tilde{s},\tilde{t})}{A(\tau)^k}[dz]^{[k]}([d\bar{z}]^{[i]}) \otimes dz^{\oplus(r+1)}.$$

Before we prove the claim, let us remark that the explicit description in the theorem follows by applying the Hodge decomposition to the above equation.

Pf. of the claim: By Lemma 2.3.5 we have

$$\tilde{l}^D_{t,n} = \sum_{k=0}^{n} k!\frac{1}{D^k}s^D_{\tilde{t},k}(z,\tau)\tilde{\omega}^{[k,0]} \otimes dz$$

where $s^N_{\tilde{t},k}$ is defined via the expansion

$$NJ_{0,\tilde{t}}(Dz,-w',\tau) = \sum_{k\geq 0} s^D_{\tilde{t},k}(w')^k.$$

Further, by Lemma 2.3.3 and the Leibniz rule we have

$$\begin{aligned}
\nabla^{\circ r}_{\tilde{\mathcal{L}}^\dagger_n}(\tilde{l}^D_{t,n}) &= \sum_{k=0}^{n} k!\sum_{i=0}^{r}\binom{r}{i}\frac{1}{D^k}\partial_z^{\circ(r-i)}s^D_{\tilde{t},k}(z,\tau)\nabla^{\circ i}_{\tilde{\mathcal{L}}^\dagger_n}(\tilde{\omega}^{[k,0]}) \otimes dz^{r-i+1} = \\
&= \sum_{k=0}^{n} k!\sum_{i=0}^{\min(r,n-k)}\binom{r}{i}\frac{1}{D^k}\partial_z^{\circ(r-i)}s^D_{\tilde{t},k}(z,\tau)\tilde{\omega}^{[k,0]}\left(\eta(1,\tau)\tilde{\omega}^{[1,0]}+\tilde{\omega}^{[0,1]}\right)^i \otimes dz^{r+1} = \\
&= \sum_{k=0}^{n} k!\sum_{i=0}^{\min(r,n-k)} i!\binom{r}{i}\frac{1}{D^k}\partial_z^{\circ(r-i)}s^D_{\tilde{t},k}(z,\tau)\tilde{\omega}^{[k,0]}\left(\eta(1,\tau)\tilde{\omega}^{[1,0]}+\tilde{\omega}^{[0,1]}\right)^{[i]} \otimes dz^{r+1}
\end{aligned}$$

From Lemma 2.3.4 we get:

$$(\tilde{\mathbb{D}}_s \otimes \mathrm{id}_{\omega^{r+1}})(\tilde{s}^*\left[\nabla^{(\circ r)}_{\tilde{\mathcal{L}}_n^\dagger}(\tilde{l}^D_{t,n})\right]) = \sum_{k=0}^{n}\sum_{i=0}^{\min(r,n-k)} \frac{k!i!}{D^k}\binom{r}{i}\left(\partial_z^{\circ(r-i)} s^D_{\tilde{t},k}\right)\Big|_{z=\tilde{s}} \cdot \tag{2.19}$$
$$\cdot \exp\left(-\frac{\tilde{s}-\bar{\tilde{s}}}{A(\tau)}\tilde{\omega}^{[1,0]}\right)\tilde{\omega}^{[k,0]}\left(\eta(1,\tau)\tilde{\omega}^{[1,0]}+\tilde{\omega}^{[0,1]}\right)^{[i]} \otimes dz^{r+1}$$

We consider the formal power series ring $\mathcal{O}_M^{an}[\![z,w',u]\!]$ and compute the formal derivative

$$D\cdot\partial_z^{\circ r}\left[\exp(zu)\exp\left(-\frac{\tilde{s}-\bar{\tilde{s}}}{A}w'\right)J_{0,t}(D(z+\tilde{s}),-w')\right]\Big|_{z=0}$$

in two different ways:

$$D\cdot\partial_z^{\circ r}\left[\exp(zu)\exp\left(-\frac{\tilde{s}-\bar{\tilde{s}}}{A}w'\right)J_{0,t}(D(z+\tilde{s}),-w')\right]\Big|_{z=0} =$$
$$=\left[\sum_{i=0}^{r}\binom{r}{i}\partial_z^{\circ i}\exp(zu)\cdot\exp\left(-\frac{\tilde{s}-\bar{\tilde{s}}}{A}w'\right)D\cdot\partial_z^{\circ(r-i)}J_{0,t}(D(z+\tilde{s}),-w')\right]\Big|_{z=0} =$$
$$=\sum_{k\geq 0}\sum_{i=0}^{r}k!i!\binom{r}{i}\cdot\left(\partial_z^{\circ(r-i)}s^D_{\tilde{t},k}\right)\Big|_{z=\tilde{s}}\exp\left(-\frac{\tilde{s}-\bar{\tilde{s}}}{A(\tau)}w'\right)\frac{w'^k}{k!}\frac{u^i}{i!} \tag{2.20}$$

On the other hand,

$$\partial_z^{\circ r}\left[\exp(zu)\exp\left(-\frac{\tilde{s}-\bar{\tilde{s}}}{A}w'\right)D\cdot J_{0,\tilde{t}}(D(z+\tilde{s}),-w')\right]\Big|_{z=0} \overset{(1.26)}{=}$$
$$= D\cdot\partial_z^{\circ r}\left[\exp(zu)\exp\left(-\frac{\tilde{s}-\bar{\tilde{s}}}{A}w'\right)\exp\left(\frac{D(z+\tilde{s})w'}{A}\right)\cdot\Theta_{0,\tilde{t}}(D(z+\tilde{s}),-w')\right]\Big|_{z=0} \overset{(1.24)}{=}$$
$$= D\cdot\partial_z^{\circ r}\left[\exp\left[z\left(\frac{Dw'}{A}+u\right)\right]\cdot\Theta_{D\tilde{s},\tilde{t}}(Dz,-w')\right]\Big|_{z=0} =$$
$$= D\cdot\left[\sum_{i=0}^{r}\binom{r}{i}\left(u+\frac{Dw'}{A(\tau)}\right)^i\cdot\partial_z^{\circ(r-i)}\Theta_{D\tilde{s},\tilde{t}}(Dz,-w')\Big|_{z=0}\right] \overset{(1.25)}{=}$$
$$= D\cdot\left[\sum_{i=0}^{r}\binom{r}{i}i!\frac{\left(u+\frac{Dw'}{A(\tau)}\right)^i}{i!}\cdot(r-i)!D^{r-i}\sum_{k\geq 0}(-1)^{r-i+k}\frac{e^*_{k,r-i+1}(D\tilde{s},\tilde{t})}{A(\tau)^k k!}(-w')^k\right] =$$
$$= r!\sum_{i=0}^{r}\sum_{k\geq 0}\cdot D^{1+r-i}(-1)^{r+k}\frac{e^*_{k,r-i+1}(D\tilde{s},\tilde{t})}{A(\tau)^k}\frac{(-w')^k}{k!}\frac{\left(-u-\frac{Dw'}{A(\tau)}\right)^i}{i!} \tag{2.21}$$

Comparing (2.20) and (2.20) we obtain the equation

$$\sum_{k\geq 0}\sum_{i=0}^{r} k!i!\binom{r}{i}\cdot\left(\partial_z^{\circ(r-i)} s_{\tilde{t},k}^D\right)\Big|_{z=\tilde{s}} \exp\left(-\frac{\tilde{s}-\bar{\tilde{s}}}{A(\tau)}w'\right)\frac{w'^k}{k!}\frac{u^i}{i!}= \tag{2.22}$$

$$=r!\sum_{k\geq 0}\sum_{i=0}^{r}\cdot D^{1+r-i}(-1)^{r+k}\frac{e^*_{k,r-i+1}(D\tilde{s},\tilde{t})}{A(\tau)^k}\frac{(-w')^k}{k!}\frac{\left(-u-\frac{Dw'}{A(\tau)}\right)^i}{i!} \tag{2.23}$$

in the formal power series ring $\mathcal{O}_{\tilde{M}}^{an}[\![w',u]\!]$. We specialize this identity using the ring homomorphism

$$\mathcal{O}_{\tilde{M}}^{an}[\![w',u]\!]\twoheadrightarrow\bigoplus_{i=0}^{n}\underline{\mathrm{TSym}}^i\,\tilde{\mathcal{H}}_E, w'\mapsto\frac{\tilde{\omega}^{[1,0]}}{D}, u\mapsto\left(\eta(1,\tau)\tilde{\omega}^{[1,0]}+\tilde{\omega}^{[0,1]}\right)$$

and obtain

$$\sum_{k=0}^{n}\sum_{i=0}^{\min(r,n-k)}\frac{k!i!}{D^k}\binom{r}{i}\left(\partial_z^{\circ(r-i)} s_{\tilde{t},k}^D\right)\Big|_{z=\tilde{s}}\exp\left(-\frac{\tilde{s}-\bar{\tilde{s}}}{A(\tau)}\tilde{\omega}^{[1,0]}\right)\tilde{\omega}^{[k,0]}\left(\eta(1,\tau)\tilde{\omega}^{[1,0]}+\tilde{\omega}^{[0,1]}\right)^{[i]}=$$

$$r!\sum_{k=0}^{n}\sum_{i=0}^{\min(r,n-k)}\cdot D^{1+r-k-i}(-1)^{r+k}\frac{e^*_{k,r-i+1}(D\tilde{s},\tilde{t})}{A(\tau)^k}\frac{\left(-\tilde{\omega}^{[1,0]}\right)^k}{k!}\frac{\left(-\eta(1,\tau)\tilde{\omega}^{[1,0]}-\tilde{\omega}^{[0,1]}-\frac{\tilde{\omega}^{[1,0]}}{A(\tau)}\right)^i}{i!}.$$

Now, combining this equation with (2.19) and using the identification

$$\tilde{p}_M^*\,\underline{\mathrm{TSym}}^k\,\underline{H}^1_{\mathrm{dR}}\left(E/M\right)^\vee\xrightarrow{\sim}\tilde{p}_M^*\underline{H}^1_{\mathrm{dR}}\left(E^\vee/M\right)\xrightarrow{\sim}\tilde{p}_M^*\,\underline{\mathrm{TSym}}^k\,\underline{H}^1_{\mathrm{dR}}\left(E/M\right)$$

which maps $\tilde{\omega}^{[i,j]}$ to $(-1)^i[\omega]^{[i]}[\eta]^{[j]}$ proves the claim:

$$(\tilde{\mathbb{D}}_s\otimes\mathrm{id}_{\omega^{r+1}})(\tilde{s}^*\left[\nabla^{(\circ r)}_{\tilde{\mathcal{L}}_n^\dagger}(\tilde{l}^D_{t,n})\right])=$$

$$=r!\sum_{k=0}^{n}\sum_{i=0}^{\min(r,n-k)}\cdot D^{1+r-k-i}(-1)^{r+k}\frac{e^*_{k,r-i+1}(D\tilde{s},\tilde{t})}{A(\tau)^k}[\omega]^{[k]}\left(\eta(1,\tau)[\omega]-[\eta]+\frac{[\omega]}{A(\tau)}\right)^{[i]}\otimes dz^{r+1}=$$

$$=r!\sum_{k=0}^{n}\sum_{i=0}^{\min(r,n-k)}\cdot D^{1+r-k-i}(-1)^{r+k}\frac{e^*_{k,r-i+1}(D\tilde{s},\tilde{t})}{A(\tau)^{k+i}}[dz]^{[k]}[d\bar{z}]^{[i]}\otimes dz^{r+1}$$

Here, we have used the equality $\frac{[d\bar{z}]}{A(\tau)}=\eta(1,\tau)[\omega]-[\eta]+\frac{[\omega]}{A(\tau)}$ in $\tilde{p}_M^*\underline{H}^1_{\mathrm{dR}}\left(E/M\right)^{an}\otimes\mathcal{C}^\infty(\tilde{M})$ where $\eta(1,\tau)=\zeta(z,\tau)-\zeta(z+1,\tau)$ is the period of the zeta function. For example, the formula for $[d\bar{z}]$ can be deduced from [Kat76, p. 1.3.4].

(b): Follows either by summing over all $e\neq t\in E^\vee[D]$ or by a similar computation. □

Corollary 2.4.4. *Let E/S be an elliptic curve with $\Gamma(N)$-level structure. Let $(0,0)\neq(a,b)\in(\mathbb{Z}/N\mathbb{Z})^2$ and $s_{a,b}$ be the associated N-torsion section. The image of $s^*l_n^D$ under*

$$\mathbb{D}_s\otimes\mathrm{id}_\omega: s^*\mathcal{L}_n\otimes_{\mathcal{O}_E}\pi^*\underline{\omega}_{E/S}\xrightarrow{\sim}\bigoplus_{k=0}^{n}\left(\underline{\mathrm{TSym}}^k\,\underline{\omega}_{E^\vee/S}\right)\otimes_{\mathcal{O}_S}\underline{\omega}_{E/S}\xrightarrow{\sim}\underline{\omega}_{E/S}^{k+1}$$

gives Kato's Eisenstein series $\left({}_D\mathrm{F}^{(k+1)}_{(a,b)}\right)_{k=0}^{n}$ cf. [Kat04, §4.2].

Proof. We can reduce the proof to the universal case. Then, this follows from the particular case $r = 0$ in the above theorem together with the following computation:

$$\begin{aligned}\frac{e^*_{k,1}(s,0)}{A^k k!}(dz)^{\otimes(k+1)} &\overset{(1)}{=} \frac{K^*_{k+1}(s,0,1;\Gamma_\tau)}{A^k k!}(dz)^{\otimes(k+1)} \overset{(2)}{=} \\ &= K^*_{k+1}(0,s,k+1;\Gamma_\tau)(dz)^{\otimes(k+1)} \overset{(3)}{=} \\ &= \frac{(-1)^{k+1}}{k!}\mathrm{F}^{(k+1)}_{(a,b)}(\tau)(2\pi i dz)^{\otimes(k+1)}\end{aligned}$$

Here, $K^*_{k+1}(s,0,1;\Gamma_\tau)$ is the Eisenstein–Kronecker–Lerch series defined in [BK10b, Def. 1.1]. (1) and (2) follow from [BK10b, Def. 1.5 and Prop. 1.3]. For (3) we refer to [Sch14, Theorem 3.3.16]. □

Remark 2.4.5. Using similar methods as in Section 1.5, it is possible to prove the above corollary algebraically.

3 The Katz splitting

In [Kat77, Appendix C] Katz gave a geometric construction of Eisenstein series of weight one. His main tool was a construction of a distinguished cross-section to the canonical projection of the universal vectorial extension over the open subscheme $E \setminus \{e\}$. In this chapter we will recall his construction. Then, we will use the relation between the universal vectorial extension and various other geometric objects like the geometric logarithm sheaves or absolute Kähler differentials to obtain splittings of those objects over $E \setminus \{e\}$. These splittings will be important technical tools for studying the algebraic de Rham realization of the elliptic polylogarithm. The splitting of the short exact sequence of Kähler differentials will enable us to extend the relative integrable connection on $\mathcal{L}_1^\dagger$ to an absolute one.

Let E/S be an elliptic curve. As always S is assumed to be separated and locally Noetherian. Since we are following the construction in [Kat77, Appendix C], we will assume during the whole chapter that 6 is invertible on S. We keep the introduced notation. In particular, $\pi : E \to S$ is the structure morphism and

$$q^\dagger : E^\dagger \twoheadrightarrow E^\vee$$

denotes the universal vectorial extension.

3.1 The Katz section of the universal vectorial extension

Let E/S be an elliptic curve over S and assume that 6 is invertible on S. First, let us recall Katz' construction of a distinguished section

$$E^\vee \setminus \{e\} \to E^\dagger$$

to the projection $q^\dagger$. The universal vectorial extension of $E^\vee$ sits in the following short exact sequence of S-group schemes

$$0 \longrightarrow V_S(\underline{\omega}_{E/S}) \longrightarrow E^\dagger \longrightarrow E^\vee \longrightarrow 0$$

where $V_S(\underline{\omega}_{E/S})$ is the vector group with T-valued points $V_S(\underline{\omega}_{E/S})(T) = \underline{\omega}_{E_T/T}$. For the moment let us assume that S is affine and $\underline{\omega}_{E/S}$ is a free $\mathcal{O}_S$-module of rank 1. After fixing a generator $\omega_0 \in \Gamma(S, \underline{\omega}_{E/S})$, there is a unique Weierstrass equation for E/S

$$y^2 = 4x^3 - g_2 x - g_3 \quad g_2, g_3 \in \Gamma(S, \mathcal{O}_S).$$

such that $\frac{dx}{y} = \omega_0$. Now, Katz proves the following:

Lemma 3.1.1 ([Kat77, p. C.2.1])**.** *Let* (E, ω_0) *be as above and* $U := E \setminus \{e\}$*. Then, for* $P \in U(S)$ *a point with Weierstrass coordinates* $x = a$*,* $y = b$ *the* 1*-form*

$$\omega_P := \frac{(y+b)}{2(x-a)} \frac{dx}{y} \in \Gamma(E, \Omega^1_{E/S}([e] + [P]))$$

has residues

$$\mathrm{Res}_P\, \omega_P = 1, \quad \mathrm{Res}_e\, \omega_P = -1.$$

The construction of ω_P *is independent of the auxiliary choice of* ω_0*.*

Let us come back to the general case. Let E/S be an elliptic curve and $e \neq P \in E(S)$. Since $\underline{\omega}_{E/S}$ is always locally free, we may construct ω_P locally on the base S. The fact that ω_P does not depend on the auxiliary ω_0 allows us to glue the construction and we obtain a well-defined section

$$\omega_P \in \Gamma(E, \Omega^1_{E/S}([e] + [P]))$$

with residues 1 at P and -1 at e. Using this, Katz defines an integrable S-connection on $\mathcal{O}_E([P] - [e])$ as

$$\nabla_P : \mathcal{O}_E([P] - [e]) \to \mathcal{O}_E([P] - [e]) \otimes_{\mathcal{O}_E} \Omega^1_{E/S}, \quad f \mapsto df + f\omega_P.$$

Now, Katz defines $U \to E^\dagger$ through its T-valued points $P \in U(T)$ by

$$P \mapsto [(\mathcal{O}_E([P] - [e]), \nabla_P)] \in E^\dagger(T)$$

where $[(\mathcal{O}_E([P] - [e]), \nabla_P)]$ denotes the isomorphism class of the line bundle $\mathcal{O}_E([P]-[e])$ with its integrable S-connection ∇_P. But unfortunately this definition makes use of the chosen autoduality

$$E(T) \ni P \mapsto [\mathcal{O}_E([P] - [e])] \in E^\vee.$$

However, this can easily be fixed:

Definition 3.1.2. Set $V := E^\vee \setminus \{e\}$ and define

$$\kappa_{E^\dagger} : V \to (q^\dagger)^{-1}(V) \subseteq E^\dagger$$

as the unique morphism of S-schemes which is given on T-valued points as

$$\kappa_{E^\dagger} : V(T) \ni [\mathcal{L}] \mapsto [\mathcal{O}_E([P] - [e]), \nabla_P]$$

with $P \in V(T)$ the unique point with $[\mathcal{L}] = [\mathcal{O}_E([P] - [e])]$. We call $\kappa_{E^\dagger}$ the *Katz section* of the universal vectorial extension of $E^\vee$.

Remark 3.1.3.

(a) The above definition does no more depend on the choice of an autoduality isomorphism.

(b) Let us write $E^\sharp$ for the universal vectorial extension of E. The canonical isomorphism $E \xrightarrow{\sim} (E^\vee)^\vee$ gives us a section

$$\kappa_{E^\sharp} : U = E \setminus \{e\} \to E^\sharp$$

to the projection $q^\sharp$.

(c) The Katz section is compatible with base change.

Let us recall Katz' construction of the Eisenstein series of weight 1 [Kat77]. Let $N > 1$ and assume that $6N$ is invertible on S. For $e \neq t \in E[N](S)$ let $\tilde{t}$ be the unique N-torsion section of $E^\sharp$ lifting t. Define $A_1(E,t) \in \Gamma(S, \underline{\omega}_{E^\vee/S})$ via

$$A_1(E,t) := \kappa_{E^\sharp}(t) - \tilde{t} \in \ker(q^\sharp)(S) = V_S(\underline{\omega}_{E^\vee/S})(S) = \Gamma(S, \underline{\omega}_{E^\vee/S}).$$

Remark 3.1.4.

(a) If one fixes an autoduality isomorphism $E \xrightarrow{\sim} E^\vee$, one obtains a geometric modular form

$$(E,t) \mapsto A_1(E,t) \in \Gamma(S, \underline{\omega}_{E^\vee/S}) \cong \Gamma(S, \underline{\omega}_{E/S})$$

of level $\Gamma_{00}(N)$ and weight 1. This is Katz' construction of a geometric Eisenstein series of weight one.

(b) For a moment let us identify E with its dual using the principal polarization λ associated with the ample line bundle $\mathcal{O}_E([e])$. Katz gives the following transcendental description for $A_1(E,t)$ evaluated on the complex elliptic curve $\mathbb{C}/\Lambda_\tau, t = \frac{a}{N}\tau + \frac{b}{N}$, $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ for $\tau \in \mathbb{H}$:

$$A_1(E,t) = \left((\zeta(\frac{a}{N}\tau + \frac{b}{N}) + \eta\left(\frac{a}{N}\tau + \frac{b}{N}, \tau\right)\right) dz$$

(c) For an elliptic curve E/S with $\Gamma(N)$-level structure we can associate to a given pair $(0,0) \neq (a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$ a torsion section $s_{a,b} \in E[N](S)$. The resulting geometric modular form

$$(E,\alpha) \mapsto A_1(E, s_{a,b})$$

coincides with Kato's $E^1_{(a,b)}$ as is easily seen from the above transcendental description.

We want to give a slightly different description of $A_1(E,t)$. Let E/S be an elliptic curve and let us keep the assumption that $6N$ is invertible on S. For the moment let us write $[N]^*E^\sharp := E^\sharp \times_{E,[N]} E$ for the base change of $E^\sharp$ along $[N] : E \to E$. The

N-multiplication $[N] : E^\sharp \to E^\sharp$ factors through the pullback $[N]^*E^\sharp$ as follows:

$$\begin{tikzcd}
0 \arrow[r] & V_S(\underline{\omega}_{E^\vee/S}) \arrow[r] \arrow[d, "\cdot N"] & E^\sharp \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V_S(\underline{\omega}_{E^\vee/S}) \arrow[r] \arrow[d, equal] & {[N]^*E^\sharp} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[r] \arrow[d, "{[N]}"] & 0 \\
0 \arrow[r] & V_S(\underline{\omega}_{E^\vee/S}) \arrow[r] & E^\sharp \arrow[r] & E \arrow[r] & 0
\end{tikzcd} \tag{3.1}$$

In other words, the short exact sequence of f.p.p.f-groups over S

$$0 \longrightarrow V_S(\underline{\omega}_{E^\vee/S}) \longrightarrow [N]^*E^\sharp \longrightarrow E \longrightarrow 0$$

is (uniquely) isomorphic to the pushout of

$$0 \longrightarrow V_S(\underline{\omega}_{E^\vee/S}) \longrightarrow E^\sharp \longrightarrow E \longrightarrow 0$$

along $(\cdot N) : V_S(\underline{\omega}_{E^\vee/S}) \to V_S(\underline{\omega}_{E^\vee/S})$. This map is an isomorphism since N is invertible on S by assumptions. Thus, we obtain a canonical isomorphism of E-schemes

$$E^\sharp \xrightarrow{\sim} [N]^*E^\sharp. \tag{3.2}$$

By similar notation let us write $T_t^*E^\dagger := E^\sharp \times_{E,T_t} E$ for the base change of $E^\dagger$ along the translation $T_t : E \to E$ for $t \in E(S)$. The isomorphism $E^\sharp \xrightarrow{\sim} [N]^*E^\sharp$ induces an isomorphism

$$\mathrm{trans}_{E^\sharp,t} : T_t^*E^\sharp \xrightarrow{T_t^*(3.2)} T_t^*[N]^*E^\sharp = [N]^*E^\sharp \xleftarrow{\ \sim\ } E^\sharp$$

of E-schemes, here $T_t^*E^\sharp = E^\sharp \times_{E,T_t} E$ is equipped with the structure morphism to E given by the projection. The base change of $\mathrm{trans}_{E^\sharp,t}$ along $e : S \to E$ gives us a canonical way to trivialize the fiber over $t \in E[N](S)$:

$$\mathrm{triv}_{E^\sharp,t} : t^*E^\sharp := E^\sharp \times_{E,t} S \xrightarrow{\sim} V_S(\underline{\omega}_{E^\vee/S}).$$

The reader has certainly noticed the similarities between the above constructions and the corresponding constructions for the geometric logarithm sheaves given in Section 2.1.3. This is no accident. Later, we will give a construction of $\mathcal{L}_1$ via the universal vectorial extension $E^\sharp$. The Katz section $\kappa_{E^\sharp}$ will induce a canonical splitting of $\mathcal{L}_1|_U$. Then, in order to restate Katz' construction of the Eisenstein series of weight one in terms of a natural splitting on the geometric logarithm sheaves, it will be convenient to have a reformulation of Katz' construction of Eisenstein series of weight one, which also works on the geometric logarithm sheaves.

Remark 3.1.5. The pullback of $[N]^*E^\sharp \xrightarrow{\sim} E^\sharp$ along $t : S \to E$ gives another trivialization

$$t^*E^\sharp \xrightarrow{\sim} V_S(\underline{\omega}_{E^\vee/S}).$$

This coincides with $N \cdot \mathrm{triv}_{E^\sharp,t}$ as can be seen from (3.1). One checks easily that $\mathrm{triv}_{E^\sharp,t}$ is independent of the chosen N.

Now, consider the following diagram

$$\begin{tikzcd}
V_S(\underline{\omega}_{E^\vee/S}) & E^\dagger \times_{E,t} S \arrow[l, "\mathrm{triv}_{E^\dagger,t}"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E^\dagger \arrow[d] \\
& S \arrow[ul, "\mathrm{triv}_{E^\dagger,t}\circ t^*\kappa_{E^\sharp}"] \arrow[u, bend left, "t^*\kappa_{E^\sharp}"] \arrow[r, "t"] & U.
\end{tikzcd}$$

We can restate Katz' construction as follows:

Lemma 3.1.6. *We have the following equalities in $V_S(\underline{\omega}_{E^\vee/S})(S) = \Gamma(S, \underline{\omega}_{E^\vee/S})$:*

$$A_1(E,t) = \frac{1}{N}[N](\kappa_{E^\sharp} \circ t) = \mathrm{triv}_{E^\sharp,t} \circ t^*\kappa_{E^\sharp}. \tag{3.3}$$

Proof. The S-valued point $[N](\kappa_{E^\sharp} \circ t)$

$$S \xrightarrow{t} U \xrightarrow{\kappa} E^\sharp \xrightarrow{[N]} E^\sharp$$

of $E^\sharp$ is contained in $\ker(q^\sharp)(S) = V_S(\underline{\omega}_{E^\vee/S})(S)$. The first equality

$$A_1(E,t) = \frac{1}{N}[N](\kappa \circ t)$$

follows from the commutative diagram

$$\begin{tikzcd}
& V_S(\underline{\omega}_{E^\vee/S}) \arrow[r, equal] \arrow[d] & V_S(\underline{\omega}_{E^\vee/S}) \arrow[r, "\cdot N"] \arrow[d] & V_S(\underline{\omega}_{E^\vee/S}) \arrow[d] \\
S \arrow[ur, "\kappa\circ t - \tilde{t}"] \arrow[r, "\kappa\circ t-\tilde{t}"] & E^\sharp \arrow[r] \arrow[rr, bend right, "{[N]}"'] & {[N]^*E^\sharp} \arrow[r] & E^\sharp.
\end{tikzcd}$$

Indeed, the composition of the lower horizontal map is $[N](\kappa \circ t)$ while $\kappa \circ t - \tilde{t}$ followed by the upper horizontal map gives $N \cdot (\kappa \circ t - \tilde{t}) = N \cdot A_1(E,t)$. For the second equality $\frac{1}{N}[N](\kappa \circ t) = \mathrm{triv}_{E^\sharp,t} \circ t^*\kappa$ consider first

$$\begin{tikzcd}
V_S(\underline{\omega}_{E^\vee/S}) = t^*[N]^*E^\sharp \arrow[r, hook] & E^\sharp \times_{E,[N]} E \\
t^*E^\sharp \arrow[u, "t^*(3.2)"] \arrow[r, hook] & E^\sharp. \arrow[u, "(3.2)"']
\end{tikzcd}$$

Note that $N \cdot \mathrm{triv}_{E^\sharp,t}$ coincides with the map $t^*(3.2)$ in this diagram, cf. Remark 3.1.3. If we compose this diagram with the lower left corner of the diagram (3.1), we obtain

$$\begin{tikzcd} V_S(\underline{\omega}_{E^\vee/S}) \arrow[r, hook] & E^\sharp \\ t^*E^\sharp \arrow[u, "N\cdot\mathrm{triv}_{E^\sharp,t}"] \arrow[r, hook] & E^\sharp. \arrow[u, "{[N]}"'] \end{tikzcd}$$

Finally, the composition of this diagram with the commutative diagram

$$\begin{tikzcd} t^*E^\sharp \arrow[r, hook] & E^\sharp \\ S \arrow[u, "t^*\kappa"] \arrow[r, hook, "t"] & E \arrow[u, "\kappa"'] \end{tikzcd}$$

gives the diagram

$$\begin{tikzcd} V_S(\underline{\omega}_{E^\vee/S}) \arrow[r, hook] & E^\sharp \\ t^*E^\sharp \arrow[u, "N\cdot\mathrm{triv}_{E^\sharp,t}"] \arrow[r, hook] & E^\sharp \arrow[u, "{[N]}"'] \\ S \arrow[u, "t^*\kappa"] \arrow[r, hook, "t"] & E. \arrow[u, "\kappa"'] \end{tikzcd}$$

This diagram shows $\frac{1}{N}[N](\kappa \circ t) = \mathrm{triv}_{E^\sharp,t} \circ t^*\kappa$. □

3.2 Splitting the first geometric logarithm sheaf

The aim of this section is to combine the Katz section $\kappa_{E^\sharp} : U \to E^\sharp$ with a result of Mazur–Messing about (infinitesimally) rigidified $\mathbb{G}_m$-extensions in order to split $\mathcal{L}_1|_U$. Afterwards, we will discuss various properties of this splitting.

3.2.1 Rigidified extensions and the Katz splitting

Let us start with recalling the definition of (infinitesimally) rigidified $\mathbb{G}_m$-extensions and the relation to the universal vectorial extension.

Definition 3.2.1. An *infinitesimal rigidified (or inf-rigidified) line bundle* on E/S is a pair $(\mathcal{L}, r_{\mathrm{inf}})$ consisting of:

(a) a line bundle $\mathcal{L}$ on E

(b) an isomorphism $r_{\mathrm{inf}} : \mathcal{L}|_{\mathrm{Inf}^1_e E} \xrightarrow{\sim} \mathcal{O}_{\mathrm{Inf}^1_e E}$, which will be called *infinitesimal rigidification* or *inf-rigidification*.

A morphism of inf-rigidified line bundles is a morphism of line bundles respecting the infinitesimal rigidifications r_{inf}. Let $\underline{\mathrm{Picrig}}^0_{\mathrm{E/S}}$ be the f.p.p.f sheaf on S which is locally given by

$$T \mapsto \underline{\mathrm{Picrig}^0}\,(E_T/T) := \{\text{Iso. classes } (\mathcal{L}, r_{\mathrm{inf}}) : \mathcal{L} \text{ alg. eq. to zero }\}.$$

Remark 3.2.2. In [MM74] a rigidified $\mathbb{G}_m$-extension of E over S is defined as an extension of f.p.p.f sheaves over S

$$0 \longrightarrow \mathbb{G}_{m,S} \longrightarrow \mathcal{E} \longrightarrow E \longrightarrow 0.$$

together with an infinitesimal section $\mathrm{Inf}^1_e\, E \to \mathcal{E}$ to the projection. $\underline{\mathrm{Extrig}}^0(E, \mathbb{G}_m)$ is defined to be the f.p.p.f sheaf of isomorphism classes of rigidified $\mathbb{G}_m$-extensions. By Barsotti–Rosenlicht–Weil (cf. [Sch14, Theorem 0.1.26]) the map

$$\underline{\mathrm{Pic}}^0_{\mathrm{E/S}}\,(E_T/T) \to \underline{\mathrm{Ext}}^1_{fppf}\,(E_T, \mathbb{G}_{m,T})$$

given by associating to a line bundle the underlying $\mathbb{G}_m$-torsor is an isomorphism. The additional datum of an inf-rigidification r_{inf} induces a rigidification in the sense of Mazur–Messing on the $\mathbb{G}_m$-extension. This gives the isomorphism

$$\underline{\mathrm{Picrig}^0}\,(E_T/T) \xrightarrow{\sim} \underline{\mathrm{Extrig}}^0(E, \mathbb{G}_m).$$

Remark 3.2.3. One might wonder why we call r_{inf} an infinitesimal rigidification and not just rigidification as in Mazur–Messing. The reason is that we have already used the notion *rigidification* for line bundles with a fixed isomorphism

$$e^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_S.$$

In cases where confusion might arise we will write zero-rigidification for what we have called just rigidification before. An inf-rigidified line bundle is in particular rigidified by restricting the infinitesimal rigidification to the zero section. This also shows that there are no non-trivial isomorphisms of inf-rigidified line bundles and so the isomorphism class $[\mathcal{L}, r_{\mathrm{inf}}]$ of $(\mathcal{L}, r_{\mathrm{inf}})$ determines $(\mathcal{L}, r_{\mathrm{inf}})$ up to unique isomorphism. In particular, we obtain a map

$$\underline{\mathrm{Picrig}}^0_{\mathrm{E/S}} \to \underline{\mathrm{Pic}}^0_{\mathrm{E/S}}$$

by restricting the infinitesimal rigidification.

Recall that an S-connection

$$\nabla : \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_E} \Omega^1_{E/S}$$

on a line bundle $\mathcal{L}$ on E can be equivalently be expressed as an $\mathcal{O}_{\mathrm{Inf}^1_\Delta(E\times_S E)}$-linear isomorphism

$$\nabla : p_1^*\mathcal{L} \xrightarrow{\sim} p_2^*\mathcal{L}$$

where $p_i : \mathrm{Inf}^1_\Delta(E \times_S E) \hookrightarrow E \times_S E \xrightarrow{\mathrm{pr}_i} E$ is the projection from the first infinitesimal neighbourhood of the diagonal to the components [MM74, §3.1]. Now, consider an S-scheme T and a zero-rigidified line bundle $(\mathcal{L}, r_0)$ on E_T/T with integrable T-connection $\nabla_{\mathcal{L}}$. Such a datum gives rise to a T-valued point

$$[\mathcal{L}, r_0, \nabla_{\mathcal{L}}] \in E^\dagger(T).$$

Following Mazur–Messing, we explain how one can associate to $(\mathcal{L}, r_0, \nabla_{\mathcal{L}})$ an inf-rigidified line bundle $(\mathcal{L}, r_{\mathrm{inf}})$: Let

$$\tau : \mathrm{Inf}^1_e E \to \mathrm{Inf}^1_\Delta(E \times_S E)$$

be the unique morphism fitting into the diagram

$$\begin{tikzcd} & E \\ \mathrm{Inf}^1_e E \arrow[ur, "\iota_1"] \arrow[r, "\tau"] \arrow[dr, "e \circ \pi_{\mathrm{Inf}^1_e E}"'] & \mathrm{Inf}^1_\Delta(E \times_S E) \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & E. \end{tikzcd}$$

Here, $\iota_1 : \mathrm{Inf}^1_e E \to E$ is the inclusion and $\pi_{\mathrm{Inf}^1_E E}$ is the structure map to S. If we view the connection $\nabla_{\mathcal{L}}$ as an isomorphism

$$\nabla_{\mathcal{L}} : p_1^*\mathcal{L} \xrightarrow{\sim} p_2^*\mathcal{L},$$

then $\tau^*\nabla_{\mathcal{L}}$ induces an infinitesimal rigidification on $\mathcal{L}$

$$r_{\mathrm{inf}} : \mathcal{O}_{\mathrm{Inf}^1_e E} \xrightarrow[\sim]{(\pi_{\mathrm{Inf}^1_e})^* r_0} \pi^*_{\mathrm{Inf}^1_E E} e^*\mathcal{L} \xrightarrow[\sim]{\tau^*\nabla_{\mathcal{L}}} (\iota_1)^*\mathcal{L}.$$

This induces a map

$$E^\dagger \to \underline{\mathrm{Picrig}}^0_{\mathrm{E/S}}.$$

Theorem 3.2.4 ([MM74, (2.6.7)]). *The map*

$$E^\dagger \to \underline{\mathrm{Picrig}}^0_{\mathrm{E/S}}$$

is an isomorphism of f.p.p.f.-sheaves on S.

Proof. In [MM74, (2.6.7) Proposition] this is proven for $\underline{\mathrm{Extrig}}(E, \mathbb{G}_m)$ and the statement for $\underline{\mathrm{Picrig}}^0_{\mathrm{E/S}}$ follows from the isomorphism $\underline{\mathrm{Picrig}}^0_{\mathrm{E/S}} \xrightarrow{\sim} \underline{\mathrm{Extrig}}(E, \mathbb{G}_m)$ cf. Remark 3.2.2. $\square$

This theorem allows us to construct a functorial splitting of $\mathcal{L}_1$ using the Katz section. Since we make use of the Katz section $\kappa_{E^\sharp}$, we assume from now on that 6 is invertible on S. The Katz section $U \to E^\sharp$ gives an U-valued point of $E^\sharp$ which corresponds to a triple

$$(\mathcal{P}_U, r_0, \nabla_{\mathcal{P}_U})$$

consisting of

(a) a line bundle $\mathcal{P}_U$ on $E^\vee_U := E^\vee \times_S U$.

(b) a zero-rigidification $r_0 : e^*\mathcal{P}_U \xrightarrow{\sim} \mathcal{O}_U$

(c) an integrable U-connection on $\mathcal{P}_U$.

The commutativity of

$$\begin{tikzcd} E^\sharp \arrow[r, two heads] & (E^\vee)^\vee \arrow[d, equal] \\ U \arrow[u, "\kappa_{E^\sharp}"] \arrow[r, hook] & E \end{tikzcd}$$

shows that the bundle $\mathcal{P}_U$ on $E^\vee \times_S U$ is the restriction of the Poincaré bundle $\mathcal{P}$ on $E^\vee \times_S (E^\vee)^\vee$. By the above theorem we have a bijection

$$E^\sharp(U) \xrightarrow{\sim} \underline{Picrig}^0_{E^\vee/S}(U).$$

By the above construction we have an inf-rigidification on $\mathcal{P}_U = \mathcal{P}|_{E^\vee \times_S U}$ which means an isomorphism

$$r_{\mathrm{inf}} : \mathcal{O}_{(\mathrm{Inf}^1_e E^\vee)\times_S U} \xrightarrow{\sim} (\iota_1 \times U)^*\mathcal{P}.$$

The pushforward of this isomorphism along $\mathrm{pr}_U : \mathrm{Inf}^1_e E^\vee \times_S U \twoheadrightarrow U$ gives

$$\kappa_{\mathcal{L}_1} : \mathcal{O}_U \oplus \pi_U^*\underline{\omega}_{E^\vee/S} = (\mathrm{pr}_U)_*\mathcal{O}_{(\mathrm{Inf}^1_e E^\vee)\times U} \xrightarrow[\sim]{(\mathrm{pr}_U)_* r_{\mathrm{inf}}} (\mathrm{pr}_U)_*(\iota_1 \times \mathrm{id}_U)^*\mathcal{P} = \mathcal{L}_1|_U .$$

Since the map $\mathcal{L}_1 \to \mathcal{L}_0 = \mathcal{O}_E$ is induced by the inclusion $\mathrm{Inf}^0_e E^\vee \hookrightarrow \mathrm{Inf}^1_e E^\vee$, it follows immediately that

$$\mathcal{O}_U \hookrightarrow \mathcal{O}_U \oplus \pi_U^*\underline{\omega}_{E^\vee/S} \xrightarrow{\sim} \mathcal{L}_1|_U$$

is a section to the canonical epimorphism $\mathcal{L}_1|_U \twoheadrightarrow \mathcal{O}_U$.

Definition 3.2.5.

(a) The map

$$\kappa_{\mathcal{L}_1} : \mathcal{O}_U \oplus \pi_U^*\underline{\omega}_{E^\vee/S} \xrightarrow{\sim} \mathcal{L}_1|_U$$

corresponding to the Katz section $\kappa_{E^\sharp} \in E^\sharp(U)$ via the isomorphism of Mazur–Messing will be called the *Katz splitting* of the first geometric logarithm sheaf and

$$\mathcal{O}_U \hookrightarrow \mathcal{O}_U \oplus \pi_U^*\underline{\omega}_{E^\vee/S} \xrightarrow{\sim} \mathcal{L}_1|_U$$

will be called the *Katz section* of the first geometric logarithm sheaf.

(b) Since $\mathcal{L}_1^\dagger$ is the pushout of

$$0 \longrightarrow \pi_E^*\underline{\omega}_{E^\vee/S} \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{O}_E \longrightarrow 0$$

along the canonical inclusion

$$\pi_E^*\underline{\omega}_{E^\vee/S} \hookrightarrow \mathcal{H}_E,$$

we also obtain a canonical splitting

$$\kappa_{\mathcal{L}_1^\dagger} : \mathcal{O}_U \oplus \mathcal{H}_U \xrightarrow{\sim} \mathcal{L}_1^\dagger\Big|_U$$

of $\mathcal{L}_1^\dagger$ which will also be called *Katz splitting*.

3.2.2 A characterization of the Katz splitting in terms of Eisenstein series

The geometric construction of the Eisenstein series $A_1(E,t)$ of Katz can now be translated to the geometric logarithm sheaf $\mathcal{L}_1$. For T an S-scheme and a non-zero N-torsion section $e \neq t \in E[N](T) = E_T[N](T)$ we consider the image of $t^*\kappa_{\mathcal{L}_1}(1,0)$ under:

$$t^*\mathcal{L}_{1,E} \xrightarrow{\sim} t^*\mathcal{L}_{1,E_T} \xrightarrow[\sim]{\mathrm{triv}_t} \mathcal{O}_{\mathrm{Inf}_e^1 E_T^\vee} = \mathcal{O}_T \oplus \underline{\omega}_{E_T^\vee/T}$$

By Lemma 3.1.6 it is not surprising that this gives rise to Katz' Eisenstein series of weight one.

Proposition 3.2.6. *Let E/S be an elliptic curve with 6 invertible on S.*

(a) *For $N>1$ invertible on S and $e \neq t \in E[N](S)$ consider $(1,0) \in \mathcal{O}_U \oplus \pi_U^*\underline{\omega}_{E^\vee/S}$. We have*

$$\mathrm{triv}_t(t^*\kappa_{\mathcal{L}_1}(1,0)) = (1, A_1(E,t)) \in \Gamma(S, \mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S})$$
$$\textit{resp. } \mathrm{triv}_t^\dagger(t^*\kappa_{\mathcal{L}_1^\dagger}(1,0)) = (1, \imath A_1(E,t)) \in \Gamma(S, \mathcal{O}_S \oplus \mathcal{H})$$

where $\imath : \underline{\omega}_{E^\vee/S} \hookrightarrow \mathcal{H}$ is the canonical inclusion.

(b) *Further, $\kappa_{\mathcal{L}_1}$ is the unique functorial splitting of the extension*

$$0 \longrightarrow \pi_U^*\underline{\omega}_{E^\vee/S} \longrightarrow \mathcal{L}_1|_U \longrightarrow \mathcal{O}_U \longrightarrow 0$$

satisfying the following property for every $N>1$:

$(*)_N$ *For every T-scheme S with N invertible on T, every $n \geq 1$ and every $e \neq t \in E[N](T) = E_T[N](T)$ we have*

$$\mathrm{triv}_{t,E_T}(t^*\kappa_{\mathcal{L}_1}(1,0)) = (1, A_1(E,t))$$

in $\Gamma(T, \mathcal{O}_T \oplus \underline{\omega}_{E_T^\vee/T})$.

A similar property characterizes $\kappa_{\mathcal{L}_1^\dagger}$.

Proof. Once the first part of the statement is proven, the second part follows by the compatibility of the Katz splitting with base change and density of torsion sections. Thus, it is enough to prove

$$\mathrm{triv}_t(t^*\kappa_{\mathcal{L}_1}(1,0)) = (1, A_1(E,t)) \in \Gamma(S, \mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S}).$$

The idea is to translate the statement via the isomorphism of Mazur–Messing to the corresponding statement for the Katz splitting of $E^\sharp$. This reduces the proof to Lemma 3.1.6. First, recall that an inf-rigidified line bundle is in particular zero-rigidified, i. e. we have

$$\underline{\mathrm{Picrig}}^0_{E^\vee/S} \twoheadrightarrow \underline{\mathrm{Pic}}^0_{E^\vee/S}, \quad (\mathcal{L}, r_{\mathrm{inf}}) \mapsto (\mathcal{L}, r_0).$$

By rigidity, i. e. the fact that there are no non-trivial isomorphisms, we deduce that for every

$$(\mathcal{L}, r_{\mathrm{inf}}) \in \ker\left(\underline{\mathrm{Picrig}}^0_{E^\vee/S} \twoheadrightarrow \underline{\mathrm{Pic}}^0_{E^\vee/S}\right)(S)$$

there is a unique isomorphism

$$\varphi : \mathcal{L} \xrightarrow{\sim} \mathcal{O}_{E^\vee}$$

of zero rigidified line bundles, where the right hand side is equipped with the canonical zero-rigidification. Thus, we can write every element in

$$\ker\left(\underline{\mathrm{Picrig}}^0_{E^\vee/S} \twoheadrightarrow \underline{\mathrm{Pic}}^0_{E^\vee/S}\right)(S)$$

as $(\mathcal{O}_{E^\vee}, r_{\mathrm{inf}})$ with r_{inf} a inf-rigidification compatible with the canonical zero-rigidification. Using this, we can define

$$\ker\left(\underline{\mathrm{Picrig}}^0_{E^\vee/S} \twoheadrightarrow \underline{\mathrm{Pic}}^0_{E^\vee/S}\right) \xrightarrow{\sim} V_S(\underline{\omega}_{E^\vee/S}) \tag{3.4}$$

which is given on T-valued points as:

$$(\mathcal{O}_{E_T}, r_{\mathrm{inf}}) \mapsto (\mathrm{pr}_{\underline{\omega}} \circ r_{\mathrm{inf}})(1) \in \Gamma(T, \underline{\omega}_{E_T^\vee/T})$$

where $\mathrm{pr}_{\underline{\omega}} \circ r_{\mathrm{inf}}$ is $\mathcal{O}_{\mathrm{Inf}_e^1 E_T^\vee} \xrightarrow{\sim} \mathcal{O}_T \oplus \underline{\omega}_{E_T^\vee/T} \to \underline{\omega}_{E_T^\vee/T}$. Using the definition of the map (3.4) and the explicit description of

$$E^\sharp \xrightarrow{\sim} \underline{\mathrm{Picrig}}^0_{E^\vee/S},$$

one checks the commutativity of the diagram

$$\begin{tikzcd}
\ker\left(E^\sharp \to E\right) \arrow[r, "\sim", "M.M."'] \arrow[d, "\cong"] & \ker\left(\underline{\mathrm{Picrig}}^0_{E^\vee/S} \twoheadrightarrow \underline{\mathrm{Pic}}^0_{E^\vee/S}\right) \arrow[d, "(3.4)"] \\
V_S(\underline{\omega}_{E^\vee/S}) \arrow[r, equal] & V_S(\underline{\omega}_{E^\vee/S}).
\end{tikzcd}$$

where $M.M.$ is the map induced by Mazur–Messing. In Lemma 3.1.6 we have reformulated Katz' construction of $A_1(E,t)$ in the following way

$$N \cdot A_1(E,t) = [N] \circ \kappa_{E^\sharp} \circ t.$$

We have the following commutative diagram:

$$\begin{array}{ccc} E^\sharp(U) & \xrightarrow{M.M.} & \underline{\mathrm{Picrig}}^0_{E^\vee/S}(U) \\ \big\downarrow{\scriptstyle [N]} & & \big\downarrow{\scriptstyle [N]^*} \\ E^\sharp(U) & \xrightarrow{M.M.} & \underline{\mathrm{Picrig}}^0_{E^\vee/S}(U) \\ \big\downarrow{\scriptstyle t^*} & & \big\downarrow{\scriptstyle t^*} \\ E^\sharp(S) & \xrightarrow{M.M.} & \underline{\mathrm{Picrig}}^0_{E^\vee/S}(S) \end{array}$$

The U-valued section $\kappa_{E^\sharp}$ maps under the left lower composition to $[N] \circ \kappa_{E^\sharp} \circ t \in V_S(\underline{\omega}_{E^\vee/S})$ which is $N \cdot A_1(E,t)$. Under the upper right composition it maps to the inf-rigidified line bundle

$$\Big((t \times [N])^*\mathcal{P}, (t \times [N])^* r_{\inf}\Big) \in \ker\Big(\underline{\mathrm{Picrig}}^0_{E^\vee/S} \twoheadrightarrow \underline{\mathrm{Pic}}^0_{E^\vee/S}\Big)(S).$$

Thus, we know that

$$\Big((t \times [N])^*\mathcal{P}, (t \times [N])^* r_{\inf}\Big) \tag{3.5}$$

corresponds under

$$\ker\Big(\underline{\mathrm{Picrig}}^0_{E^\vee/S} \twoheadrightarrow \underline{\mathrm{Pic}}^0_{E^\vee/S}\Big)(S) \xrightarrow{\sim} V_S(\underline{\omega}_{E^\vee/S})(S) = \Gamma(S, \underline{\omega}_{E^\vee/S}) \tag{3.6}$$

to $N \cdot A_1(E,t)$. It remains to relate $N \cdot \mathrm{pr}_{\underline{\omega}}\left(\mathrm{triv}_t(t^*\kappa_{\mathcal{L}_1}(1,0))\right)$ to the image of (3.5) under (3.6). We claim that the image of (3.5) under (3.6) is:

$$\mathrm{pr}_{\underline{\omega}}\left((\pi_{\mathrm{Inf}^1_e E})_* \left[(t \times \iota_1)^*\gamma_{1,N} \circ (t \times [N]|_{\mathrm{Inf}^1_e E^\vee})^* r_{\inf}(1)\right]\right) \tag{3.7}$$

Indeed,

$$(t \times \iota_1)^*\gamma_{1,N} : (t \times [N])^*\mathcal{P} \xrightarrow{\sim} (t \times \mathrm{id})^*([N] \times \mathrm{id}^*)\mathcal{P} = \mathcal{O}_{E^\vee}$$

is the unique isomorphism of zero-rigidified line bundles, here we equip $\mathcal{O}_{E^\vee}$ with its canonical rigidification and $(t \times [N])^*\mathcal{P}$ with $(t \times [N])^* r_0$. Thus, the claim follows from the explicit description of (3.6) given above.

It remains to relate (3.7) to $N \cdot \mathrm{pr}_{\underline{\omega}}\left(\mathrm{triv}_t(t^*\kappa_{\mathcal{L}_1}(1,0))\right)$ but this is straight forward:

$$\begin{aligned} & N \cdot \mathrm{pr}_{\underline{\omega}}\left(\mathrm{triv}_t(t^*\kappa_{\mathcal{L}_1}(1,0))\right) \overset{(A)}{=} \mathrm{pr}_{\underline{\omega}}\left(t^*\Phi_N \circ t^*\kappa_{\mathcal{L}_1}(1,0)\right) = \\ & = \mathrm{pr}_{\underline{\omega}}\left(t^*\left[\Phi_N \circ \kappa_{\mathcal{L}_1}(1,0)\right]\right) \overset{(B)}{=} \mathrm{pr}_{\underline{\omega}}\left(t^*\left[\Phi_N \circ (\mathrm{pr}_E)_* r_{\inf}(1,0)\right]\right) \overset{(C)}{=} \\ & = \mathrm{pr}_{\underline{\omega}}\left(t^*(\mathrm{pr}_E)_*\left[(\mathrm{id}_E \times \iota_1)^*\gamma_{1,N} \circ (\mathrm{id} \times [N]|_{\mathrm{Inf}^1_e E^\vee})^* r_{\inf}(1,0)\right]\right) = \\ & = \mathrm{pr}_{\underline{\omega}}\left((\pi_{\mathrm{Inf}^1_e E^\vee})_*\left[(t \times \iota_1)^*\gamma_{1,N} \circ (t \times [N]|_{\mathrm{Inf}^1_e E^\vee})^* r_{\inf}(1,0)\right]\right) \end{aligned}$$

Here, we have used:

(A) The fact that $\mathrm{pr}_{\underline{\omega}} \circ t^*\Phi_N : t^*\mathcal{L} \to t^*[N]^*\mathcal{L} = \mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S} \to \underline{\omega}_{E^\vee/S}$ differs from $\mathrm{pr}_{\underline{\omega}} \circ \mathrm{triv}_t : t^*\mathcal{L} \to \underline{\omega}_{E^\vee/S}$ by multiplication with N, cf. Remark 2.1.11.

(B) we have used the definition of $\kappa_{\mathcal{L}_1}$

(C) follows from the definition of $\Phi_N : \mathcal{L}_1 \xrightarrow{\sim} [N]^*\mathcal{L}_1$ cf. Section 2.1.3.

In total, we have proven

$$N \cdot \mathrm{pr}_{\underline{\omega}} (\mathrm{triv}_t(t^*\kappa_{\mathcal{L}_1}(1,0))) = N \cdot A_1(E,t).$$

Since N is invertible, we conclude the desired equality. □

3.2.3 The Katz splitting and the logarithmic derivative of the Kato–Siegel function

Using the Katz splitting of the geometric logarithm sheaves, many natural constructions for $\mathcal{L}_1$ and $\mathcal{L}_1^\dagger$ can be made more explicit. In this section, we will show that the isomorphism $\mathcal{L}_1 \xrightarrow{\sim} [N]^*\mathcal{L}_1$ is closely related to the logarithmic derivative $d\log_D\theta$ of the Kato–Siegel functions. First, observe that the inclusion $\underline{\omega}_{E^\vee/S} \hookrightarrow \mathcal{L}_1$ fits into the following commutative diagram

$$\begin{tikzcd} \pi^*\underline{\omega}_{E^\vee/S} \arrow[r, hook] \arrow[d, "\cdot N"] & \mathcal{L}_1 \arrow[d, "\Phi_N"] \\ \underbrace{\pi^*\underline{\omega}_{E^\vee/S}}_{=[N]^*\pi^*\underline{\omega}_{E^\vee/S}} \arrow[r, hook] & [N]^*\mathcal{L}_1. \end{tikzcd}$$

Indeed, the restriction of Φ_N to $\pi^*\underline{\omega}_{E^\vee/S}$ comes from the morphism $\underline{\omega}_{E^\vee/S} \to \underline{\omega}_{E^\vee/S}$ induced by $[N]^\vee = [N] : E^\vee \to E^\vee$.

Proposition 3.2.7. *Let E/S be an elliptic curve and $D > 1$ an integer with $6D$ invertible on S. Define the open subschemes $U_D := E \setminus E[D]$ and $U = E \setminus \{e\}$. Let us denote by $(\lambda \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})^* (d\log_D\theta)$ the image of $d\log_D\theta$ under the map*

$$\Gamma(U_D, \Omega^1_{E/S}) = \Gamma\left(U_D, \underline{\omega}_{E/S} \otimes_{\pi^{-1}\mathcal{O}_S} \mathcal{O}_E\right) \xrightarrow{(\lambda^*\otimes_{\mathcal{O}_S}\mathrm{id}_{\mathcal{O}_E})} \Gamma\Big(U_D, \underbrace{\underline{\omega}_{E^\vee/S} \otimes_{\pi^{-1}\mathcal{O}_S} \mathcal{O}_E}_{=\pi^*\underline{\omega}_{E^\vee/S}}\Big)$$

induced by the autoduality isomorphism (1.10)*. The restriction of Φ_D to U_D*

$$\mathcal{L}_1|_{U_D} \xrightarrow{\Phi_D|_{U_D}} ([D]^*\mathcal{L}_1)|_{U_D} = [D]^*\left(\mathcal{L}_1|_U\right)$$

can be expressed via the Katz splitting as follows:

$$\begin{tikzcd}
\mathcal{L}_1|_{U_D} \arrow[r, "\Phi_D|_{U_D}"] & {[D]^*\left(\mathcal{L}_1|_U\right)} \\
\mathcal{O}_{U_D} \oplus \pi^*_{U_D}\underline{\omega}_{E^\vee/S} \arrow[u, "\kappa_{\mathcal{L}_1}"] \arrow[r] & \mathcal{O}_{U_D} \oplus \pi^*_{U_D}\underline{\omega}_{E^\vee/S} \arrow[u, "{[D]^*\kappa_{\mathcal{L}_1}}"'] \\
(1,0) \arrow[r, mapsto] & \left(1, \frac{1}{D}(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})\left(d\log{}_D\theta\right)\right)
\end{tikzcd}$$

Proof. Since Φ_D respects the extension structure, we already know that Φ_D is given in terms of the Katz splitting by a map

$$\begin{aligned}
\mathcal{O}_{U_D} \oplus \pi^*_{U_D}\underline{\omega}_{E^\vee/S} &\longrightarrow \mathcal{O}_{U_D} \oplus \pi^*_{U_D}\underline{\omega}_{E^\vee/S} \\
(1,0) &\longmapsto (1,\vartheta)
\end{aligned}$$

for some $\vartheta \in \Gamma(U_D, \pi^*\underline{\omega}_{E^\vee/S})$. Further, the restriction of Φ_D to $\pi^*\underline{\omega}_{E^\vee/S}$ is multiplication by D. Thus, ϑ determines Φ_D on U_D uniquely. Our aim is to show the equality $\vartheta = \frac{1}{D}(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})\left(d\log{}_D\theta\right)$. The idea is to show

$$s^*\vartheta = DA_1(E_T, s) - A_1(E_T, Ds)$$

for all N^n-torsion points for an auxiliary integer N and the desired equality will follow by density of torsion sections. Let N be some integer prime to D and assume we have some N^n torsion section $e \neq s \in E_T[N^n](T) = E[N^n](T)$ with $n \geq 1$ and T some S-scheme with N invertible on T. The commutative diagram

$$\begin{tikzcd}
{[N^n]^*\mathcal{L}_{1,E_T}} \arrow[r, "{[N^n]^*\Phi_D}"] & {[D\cdot N^n]^*\mathcal{L}_{1,E_T}} \\
\mathcal{L}_{1,E_T} \arrow[u, "\Phi_{N^n}"] \arrow[r, "\Phi_D"] & {[D]^*\mathcal{L}_{1,E_T}} \arrow[u, "{[D]^*\Phi_{N^n}}"']
\end{tikzcd}$$

proven in Lemma 2.1.9 (e) shows the commutativity of

$$\begin{tikzcd}
e^*\mathcal{L}_{1,E_T} \arrow[r, "e^*\Phi_D"] & e^*\mathcal{L}_{1,E_T} \\
s^*\mathcal{L}_{1,E_T} \arrow[u, "\mathrm{triv}_s"] \arrow[r, "s^*\Phi_D"] & (Ds)^*\mathcal{L}_{1,E_T}. \arrow[u, "\mathrm{triv}_{Ds}"']
\end{tikzcd}$$

This fits into the big commutative diagram

$$\begin{tikzcd}
\mathcal{O}_T \oplus \underline{\omega}_{E_T^\vee/T} \arrow[r, "\mathrm{id}\oplus(\cdot D)"] \arrow[d, equal] & \mathcal{O}_T \oplus \underline{\omega}_{E_T^\vee/T} \arrow[d, equal] \\
e^*\mathcal{L}_{1,E_T} \arrow[r, "e^*\Phi_D"] & e^*\mathcal{L}_{1,E_T} \\
s^*\mathcal{L}_{1,E_T} \arrow[r, "s^*\Phi_D"] \arrow[u, "\mathrm{triv}_s"] & (Ds)^*\mathcal{L}_{1,E_T} \arrow[u, "\mathrm{triv}_{Ds}"'] \\
\mathcal{O}_T \oplus \underline{\omega}_{E_T^\vee/T} \arrow[r] \arrow[u, "s^*\kappa_{\mathcal{L}_1}"] & \mathcal{O}_T \oplus \underline{\omega}_{E_T^\vee/T} \arrow[u, "(Ds)^*\kappa_{\mathcal{L}_1}"']
\end{tikzcd}$$

$$(1,0) \longmapsto (1, s^*\vartheta)$$

The composition of the left vertical and the upper horizontal map applied to $(1,0)$ gives

$$e^*\Phi_D\Big(\big(\mathrm{triv}_s \circ s^*\kappa_{\mathcal{L}_1}\big)(1,0)\Big),$$

while the composition of the lower horizontal with the right vertical map gives

$$\Big(\big(\mathrm{triv}_{Ds} \circ (Ds)^*\kappa_{\mathcal{L}_1}\big)(1,0)\Big) + (0, s^*\vartheta).$$

Comparing both sides and observing Proposition 3.2.6 gives:

$$D \cdot A_1(E_T, s) = A_1(E, Ds) + s^*\vartheta$$

i. e. $s^*\vartheta = D \cdot A_1(E_T, s) - A_1(E, Ds)$. Observing Remark 3.1.4, $d\log_D\theta$ is by density of torsion sections uniquely determined by the following property: For all N prime to D, all S-schemes T with N invertible on T and all $e \neq s \in E_T[N^n](T)$ for some $n \geq 1$ we have

$$s^* d\log_D\theta = (\lambda^{-1})^* \Big[D^2 \cdot A_1(E_T, s) - D \cdot A_1(E_T, Ds)\Big].$$

And we can conclude $\vartheta = \frac{1}{D}(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})\,(d\log_D\theta)$ as desired. □

Remark 3.2.8. We are trying to avoid choosing autoduality isomorphisms. Since the Katz splitting and Φ_D are defined intrinsically, i. e. without choosing any autoduality, the appearance of $(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})\,(d\log_D\theta)$ in the above statement seams to be strange and there should be an intrinsic way to define $(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})\,(d\log_D\theta)$. Indeed, this is possible. It is not hard to see, using the symmetry of the Poincaré bundle and the equality $(\mathrm{id}_E \times e)^* s^D_{\mathrm{can}} = d\log_D\theta$ that

$$(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})\,(d\log_D\theta) = (\mathrm{id}_E \times e)^*(s^D_{\mathrm{can},E^\vee}) \in \Gamma(E, \pi^*\underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_E} \mathcal{O}(E[D]))$$

where $s^D_{\mathrm{can},E^\vee}$ is the D-variant of the canonical section associated with the elliptic curve $E^\vee$ instead of E. It is just because of readability that we have preferred to use the established Kato–Siegel functions instead of $(\mathrm{id}_E \times e)^*(s^D_{\mathrm{can},E^\vee})$.

Since $\mathcal{L}_1^\dagger$ and $\kappa_{\mathcal{L}_1^\dagger}$ are obtained by pushout from $\mathcal{L}_1$ and $\kappa_{\mathcal{L}_1}$, we deduce the following:

Corollary 3.2.9. *Let E/S be an elliptic curve $D > 1$ and $6D$ invertible on S. The restriction of $\Phi_D^\dagger$ to $U_D := E \setminus E[D]$ can be expressed via the Katz splitting as follows:*

$$\begin{tikzcd}
\mathcal{L}_1^\dagger\big|_{U_D} \arrow[r, "\Phi_D^\dagger|_{U_D}"] & {[D]^*\left(\mathcal{L}_1^\dagger\big|_U\right)} \\
\mathcal{O}_{U_D} \oplus \mathcal{H}_{U_D} \arrow[r] \arrow[u, "\kappa_{\mathcal{L}_1}"] & \mathcal{O}_{U_D} \oplus \mathcal{H}_{U_D} \arrow[u, "{[D]^*\kappa_{\mathcal{L}_1}}"']
\end{tikzcd}$$

$$(1,0) \longmapsto \left(1, i\left[\tfrac{1}{D}(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})\,(d\log{}_D\theta)\right]\right)$$

where we consider $\frac{1}{D}(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})\,(d\log{}_D\theta) \in \Gamma(U_D, \pi^*\underline{\omega}_{E^\vee/S})$ as element of $\Gamma(U_D, \mathcal{H}_E)$ via the inclusion $i : \underline{\omega}_{E^\vee/S} \hookrightarrow \mathcal{H}$.

3.2.4 The Katz splitting and the connection on the geometric logarithm sheaves

The geometric logarithm sheaf $\mathcal{L}_1^\dagger$ is equipped with a canonical connection $\nabla_{\mathcal{L}_1^\dagger}$. The Katz splitting is not horizontal if we equip $\mathcal{O}_U \oplus \mathcal{H}_U$ with the canonical S-connection obtained by pullback of the trivial connection on each summands. The aim of this section is to describe the corresponding connection on $\mathcal{O}_U \oplus \mathcal{H}_U$ explicitly.

Let us first recall that the canonical short exact sequence

$$0 \longrightarrow \mathcal{H}_E \longrightarrow \mathcal{L}_1^\dagger \longrightarrow \mathcal{O}_E \longrightarrow 0$$

is horizontal if we equip $\mathcal{H}_E$ with the canonical S-connection obtained from pullback of the trivial S-connection on $\mathcal{H}$. The Katz splitting can be seen as an isomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}_U \arrow[r] & \mathcal{L}_1^\dagger\big|_U \arrow[r] & \mathcal{O}_U \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{H}_U \arrow[r] \arrow[u, equal] & \mathcal{O}_U \oplus \mathcal{H}_U \arrow[r] \arrow[u, "\kappa_{\mathcal{L}_1^\dagger}"] & \mathcal{O}_U \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Thus, the S-connection

$$\mathcal{O}_U \oplus \mathcal{H}_U \to \left(\mathcal{O}_U \oplus \mathcal{H}_U\right) \otimes_{\mathcal{O}_E} \Omega^1_{E/S}$$

on $\mathcal{O}_U \oplus \mathcal{H}_U$ making $\kappa_{\mathcal{L}_1^\dagger}$ horizontal is the unique S-connection ∇_κ with

(a) $\nabla_\kappa|_{\mathcal{H}_U}$ is the pullback connection on the subspace $\mathcal{H}_U$ and

(b) $\nabla_\kappa(1,0) = \nabla_{\mathcal{L}_1^\dagger}\left(\kappa_{\mathcal{L}_1^\dagger}(1,0)\right) \in \Gamma(U, \mathcal{H}_U \otimes \Omega^1_{U/S})$.

We will describe $\nabla_\kappa(1,0)$ explicitly. By restricting the counit of the adjunction

$$(\pi_E)^*(\pi_E)_*\Omega^1_{E/S}(2[e]) \to \Omega^1_{E/S}(2[e]) \tag{3.8}$$

to $j : U \hookrightarrow E$ we obtain

$$(\pi_U)^*(\pi_E)_*\Omega^1_{E/S}(2[e]) \to j^*\Omega^1_{E/S}(2[e]) = \Omega^1_{U/S}.$$

On the other hand, we have the morphism

$$(\pi_E)_*\Omega^1_{E/S}(2[e]) \to (\pi_E)_*j_*\Omega^1_{U/S} = (\pi_U)_*\Omega^1_{U/S} \twoheadrightarrow \underline{H}^1_{\mathrm{dR}}(U/S) \xleftarrow{\sim} \underline{H}^1_{\mathrm{dR}}(E/S),$$

which is in fact an isomorphism [Kat73, A 1.2.3]. This morphism induces after pullback to U the isomorphism

$$(\pi_U)^*(\pi_E)_*\Omega^1_{E/S}(2[e]) \xrightarrow{\sim} \pi_U^*\underline{H}^1_{\mathrm{dR}}(E/S) = \mathcal{H}_U^\vee. \tag{3.9}$$

Now, $(3.9)^\vee \otimes (3.8)$ gives the map

$$\left((\pi_U)^*(\pi_E)_*\Omega^1_{E/S}(2[e])\right)^\vee \otimes_{\mathcal{O}_U} \left((\pi_U)^*(\pi_E)_*\Omega^1_{E/S}(2[e])\right) \to \mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^1_{U/S}.$$

The image of the identity

$$\mathrm{id}_{(\pi_U)^*(\pi_E)_*\Omega^1_{E/S}(2[e])} \in \Gamma\left(U, \left((\pi_U)^*(\pi_E)_*\Omega^1_{E/S}(2[e])\right)^\vee \otimes_{\mathcal{O}_U} \left((\pi_U)^*(\pi_E)_*\Omega^1_{E/S}(2[e])\right)\right)$$

under this morphism gives rise to some element

$$\mathit{Id}_{\mathcal{H}} \in \Gamma\left(U, \mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^1_{U/S}\right).$$

Remark 3.2.10. A more explicit description of

$$\mathit{Id}_{\mathcal{H}} \in \Gamma\left(\mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^1_{U/S}\right)$$

can be given as follows. Let us first assume $\underline{\omega}_{E/S}$ is free and generated by $\omega \in \Gamma(S, \underline{\omega}_{E/S})$ since we are still assuming that 6 is invertible on S we get a Weierstrass equation with $\omega = \frac{dx}{y}$. The elements ω and $\eta = x\frac{dx}{y}$ form a basis of the free $\mathcal{O}_S$-module $(\pi_E)_*\Omega^1_{E/S}(2[e])$. Via the isomorphism

$$(\pi_E)_*\Omega^1_{E/S}(2[e]) \to (\pi_E)_*j_*\Omega^1_{U/S} = (\pi_U)_*\Omega^1_{U/S} \twoheadrightarrow \underline{H}^1_{\mathrm{dR}}(U/S) \xleftarrow{\sim} \underline{H}^1_{\mathrm{dR}}(E/S)$$

we obtain a basis of $([\omega], [\eta])$ of the free $\mathcal{O}_S$-module $\underline{H}^1_{\mathrm{dR}}(E/S)$. Let $([\omega]^\vee, [\eta]^\vee)$ be the dual basis of $\underline{H}^1_{\mathrm{dR}}(E/S)^\vee \cong \underline{H}^1_{\mathrm{dR}}(E^\vee/S)$. The element $\mathit{Id}_{\mathcal{H}}$ is then explicitly given by

$$\mathit{Id}_{\mathcal{H}} = [\omega]^\vee \otimes \omega + [\eta]^\vee \otimes \eta \in \Gamma(U, \pi_U^*\underline{H}^1_{\mathrm{dR}}(E/S)^\vee \otimes \Omega^1_{U/S}).$$

It is straightforward to check that this explicit definition of $\mathit{Id}_{\mathcal{H}}$ does not depend on the chosen generator ω of $\Gamma(S, \underline{\omega}_{E/S})$. Since $\underline{\omega}_{E/S}$ is locally free of rank one, we can glue this local definition. This gives an alternative way to define $\mathit{Id}_{\mathcal{H}}$.

Proposition 3.2.11. *Let ∇_κ be the unique integrable S-connection on $\mathcal{O}_U \oplus \mathcal{H}_U$ making*

$$\kappa_{\mathcal{L}_1^\dagger} : (\mathcal{O}_U \oplus \mathcal{H}_U, \nabla_\kappa) \xrightarrow{\sim} (\mathcal{L}_1^\dagger, \nabla_{\mathcal{L}_1^\dagger})$$

horizontal. Then, ∇_κ is the unique S-connection which is the pullback connection on $\mathcal{H}_U$ and satisfies

$$\nabla_\kappa(1,0) = \mathit{Id}_{\mathcal{H}} \in \Gamma(U, \mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^1_{U/S}).$$

Proof. By the above discussion the connection is uniquely determined by its restriction to $\mathcal{H}_U$ and by $\nabla_\kappa(1,0)$. Thus, it is enough to prove the equality

$$\nabla_\kappa(1,0) = \mathit{Id}_{\mathcal{H}}.$$

Since all constructions are compatible with base change, this equality can be checked locally on the base. Thus, we can assume that S is affine, that there exists some $D > 1$ which is invertible on S and that $\underline{\omega}_{E/S}$ is freely generated by $\omega \in \Gamma(S, \underline{\omega}_{E/S})$. In particular, the associated Weierstrass equation gives us generators $[\omega], [\eta]$ of $\underline{H}^1_{\mathrm{dR}}(U/S) \cong \underline{H}^1_{\mathrm{dR}}(E/S)$ and we can write

$$\mathit{Id}_{\mathcal{H}} = [\omega]^\vee \otimes \omega + [\eta]^\vee \otimes \eta$$

The short exact sequence

$$0 \longrightarrow \mathcal{H}_U \longrightarrow \mathcal{O}_U \oplus \mathcal{H}_U \longrightarrow \mathcal{O}_U \longrightarrow 0$$

is horizontal if we equip $\mathcal{O}_U \oplus \mathcal{H}_U$ with ∇_κ. Thus, the connecting homomorphism

$$\delta : \mathcal{O}_S = \underline{H}^0_{\mathrm{dR}}(U/S) \to \underline{H}^1_{\mathrm{dR}}(U/S, \mathcal{H}_U) \cong \underline{H}^1_{\mathrm{dR}}(U/S)^\vee \otimes_{\mathcal{O}_S} \underline{H}^1_{\mathrm{dR}}(U/S)$$

in relative de Rham cohomology can be computed as

$$\delta(1) = [\nabla_\kappa(1,0)].$$

Here, $[\cdot]$ is the canonical map

$$[\cdot] : \Gamma\left(U, \mathcal{H}_U \otimes \Omega^1_{U/S}\right) \to \Gamma\left(S, \underline{H}^1_{\mathrm{dR}}(U/S)^\vee \otimes_{\mathcal{O}_S} \underline{H}^1_{\mathrm{dR}}(U/S)\right).$$

Now, Proposition 2.1.5 tells us that

$$[\nabla_\kappa(1,0)] = \delta(1) = [\mathit{Id}_{\mathcal{H}}] \tag{3.10}$$

and the rest of the proof will consist in showing that the equation $[\nabla_\kappa(1,0)] = [\mathit{Id}_{\mathcal{H}}]$ lifts along $[\cdot]$ to an equality in

$$\Gamma\left(U, \mathcal{H}_U \otimes \Omega^1_{U/S}\right).$$

By equation (3.10) we know that

$$\nabla_\kappa(1,0) = [\omega]^\vee \otimes \omega + [\eta]^\vee \otimes \eta + [\omega]^\vee \otimes \omega_1 + [\eta]^\vee \otimes \omega_2 \tag{3.11}$$

with
$$\omega_1, \omega_2 \in \Gamma\left(U, \operatorname{Im}\left(d : \mathcal{O}_U \to \Omega^1_{U/S}\right)\right).$$
Recall that the map $\Phi_D^\dagger : \mathcal{L}_1^\dagger \to [D]^*\mathcal{L}_1^\dagger$ is horizontal. Now, Corollary 3.2.9 shows that this map is given via the Katz splitting by:
$$\begin{aligned} \mathcal{O}_{U_D} \oplus \mathcal{H}_{U_D} &\longrightarrow \mathcal{O}_{U_D} \oplus \mathcal{H}_{U_D} = [D]^*\left(\mathcal{O}_U \oplus \mathcal{H}_U\right) \\ (1,0) &\longmapsto \left(1, i\left(\tfrac{1}{D}(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id}_{\mathcal{O}_E})\left(d\log{}_D\theta\right)\right)\right) \end{aligned} \tag{3.12}$$
This map is horizontal if we equip the left hand side with the connection ∇_κ and the right hand side with the connection $[D]^*\nabla_\kappa$. Here, $i : \pi_{U_D}^* \underline{\omega}_{E^\vee/S} \to \mathcal{H}_{U_D}$ is the natural inclusion. Let us write for a moment Φ' for the map (3.12). In particular, the horizontality of (3.12) implies the equality
$$([D]^*\nabla_\kappa)(\Phi'(1,0)) = (\Phi' \otimes \mathrm{id}_{\Omega^1_{U/S}})\left(\nabla_\kappa(1,0)\right) \tag{3.13}$$
Before we make this equality explicitly, let us consider the image of the logarithmic derivative of the Kato-Siegel function $(\lambda^* \otimes_{\mathcal{O}_S} \mathrm{id})(d\log{}_D\theta) \in \Gamma(U_D, \pi_{U_D}^*\underline{\omega}_{E^\vee/S})$ under
$$d : \pi_{U_D}^*\underline{\omega}_{E^\vee/S} = \pi_{U_D}^{-1}\underline{\omega}_{E^\vee/S} \otimes_{\pi_{U_D}^{-1}\mathcal{O}_S} \mathcal{O}_{U_D} \xrightarrow{\mathrm{id}\otimes d} \pi_{U_D}^*\underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/S}.$$
It is given by the formula
$$-D^2(\lambda^*\omega) \otimes \eta + D(\lambda^*\omega) \otimes [D]^*\eta. \tag{3.14}$$
This can be checked by a direct transcendental computation in the universal situation which boils down to
$$\partial_z\left[D^2\zeta(z,\tau) - D\zeta(Dz,\tau)dz\right] = -D^2\wp(z,\tau)dz + D[D]^*(\wp(z,\tau)dz)$$
or see [Kat04, (3.5.1)] but observe the missing minus sign in loc. cit. The left hand side of (3.13) can be computed as follows:
$$\begin{aligned} &([D]^*\nabla_\kappa)(\Phi'(1,0)) = ([D]^*\nabla_\kappa)\left(1, i\left(\frac{1}{D}(\lambda^* \otimes \mathrm{id})(d\log{}_D\theta)\right)\right) = \\ =&([D]^*\nabla_\kappa)(1,0) + (i \otimes \mathrm{id}_{\Omega^1_{U_D/S}})\left(d\left[\frac{1}{D}(\lambda^* \otimes \mathrm{id})(d\log{}_D\theta)\right]\right) \overset{(3.11),(3.14)}{=} \\ =&\left([\omega]^\vee \otimes [D]^*(\omega+\omega_1) + [\eta]^\vee \otimes [D]^*(\eta+\omega_2)\right) - D\cdot i(\lambda^*\omega)\otimes\eta + i(\lambda^*\omega)\otimes[D]^*\eta = \\ =&\left([\omega]^\vee \otimes [D]^*(\omega+\omega_1) + [\eta]^\vee \otimes [D]^*(\eta+\omega_2)\right) + D\cdot[\eta]^\vee\otimes\eta - [\eta]^\vee\otimes[D]^*\eta = \\ =&D[\omega]^\vee \otimes \omega + [\omega]^\vee \otimes [D]^*\omega_1 + [\eta]^\vee \otimes [D]^*\omega_2 + D\cdot[\eta]^\vee \otimes \eta \end{aligned}$$
While the right hand side of (3.13) gives:
$$\begin{aligned} (\Phi' \otimes \mathrm{id}_{\Omega^1_{U/S}})\left(\nabla_\kappa(1,0)\right) &= (\Phi' \otimes \mathrm{id}_{\Omega^1_{U/S}})\left([\omega]^\vee \otimes (\omega+\omega_1) + [\eta]^\vee \otimes (\eta+\omega_2)\right) = \\ &= D[\omega]^\vee \otimes (\omega+\omega_1) + D[\eta]^\vee \otimes (\eta+\omega_2) \end{aligned}$$

Comparing both sides of (3.13) gives the equations

$$[D]^*\omega_1 = D\omega_1, \quad [D]^*\omega_2 = D\omega_2.$$

But this implies $\omega_1, \omega_2 \in \Gamma(U, \pi^{-1}\underline{\omega}_{E/S})$. From $\underline{\omega}_{E/S} \hookrightarrow \underline{H}^1_{\mathrm{dR}}(U/S)$ we deduce

$$\Gamma(U, \mathrm{Im}(d) \cap \pi^{-1}\underline{\omega}_{E/S}) = 0$$

and we conclude $\omega_1 = \omega_2 = 0$. □

Remark 3.2.12. For a family of complex elliptic curves we have explained in Section 2.3 how to construct an analytic splitting starting from a Theta function, which trivializes the Poincaré bundle. It is straightforward to show that the Katz splitting coincides with the analytic splitting obtained via the Theta function $\Xi(z, w)$ appearing in [BKT10, §1]. Further, in [BKT10] an explicit splitting of the first logarithm sheaf for en elliptic curve over $\mathbb{C}$ is constructed from an explicit Čech cocycle. Also, this splitting coincides with the Katz splitting. This explains why the explicit formula for the connection on the first logarithm sheaf in loc. cit. coincides with the above formula.

3.3 The Katz splitting for relative Kähler differentials

In this section we consider the following setup. Let $f : S \to T$ be a smooth morphism, let $\pi : E \to S$ be an elliptic curve and assume that 6 is invertible on S. We have the following fundamental short exact sequences of Kähler differentials:

$$0 \longrightarrow \pi^*\Omega^1_{S/T} \longrightarrow \Omega^1_{E/T} \longrightarrow \Omega^1_{E/S} \longrightarrow 0 \tag{3.15}$$

and

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow e^*\Omega^1_{E/T} \longrightarrow \Omega^1_{S/T} \longrightarrow 0$$

with $\mathcal{I}$ the ideal sheaf defining the zero section $e : S \to E$. Using $\mathcal{I}/\mathcal{I}^2 = \underline{\omega}_{E/S} = e^*\Omega^1_{E/S}$, allows us to rewrite the second short exact sequence as:

$$0 \longrightarrow e^*\Omega^1_{E/S} \longrightarrow e^*\Omega^1_{E/T} \longrightarrow e^*\pi^*\Omega^1_{S/T} \longrightarrow 0 \tag{3.16}$$

This exact sequence provides us with a splitting of the pullback of (3.15) along e. For later reference let us fix this observation in a lemma:

Lemma 3.3.1. *Let $E/S/T$ as above then the short exact sequence* (3.16) *provides a canonical splitting to the pullback of the short exact sequence*

$$0 \longrightarrow \pi^*\Omega^1_{S/T} \longrightarrow \Omega^1_{E/T} \longrightarrow \Omega^1_{E/S} \longrightarrow 0$$

along $e : S \to E$. Let us denote the resulting splitting

$$\sigma : e^*\Omega^1_{E/S} \oplus \Omega^1_{S/T} \xrightarrow{\sim} e^*\Omega^1_{E/T}$$

by σ.

Let us denote by $\mathrm{KS} : \underline{\omega}_{E^\vee/S} \to \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \underline{\omega}^\vee_{E/S}$ the Kodaira–Spencer map. By pullback along π we obtain a map $\mathrm{KS}_E : \pi^*\underline{\omega}_{E^\vee/S} \to \pi^*\Omega^1_{S/T} \otimes_{\mathcal{O}_E} \left(\Omega^1_{E/S}\right)^\vee$. If we tensorize the pushout of the first geometric logarithm sheaf

$$0 \longrightarrow \pi^*\underline{\omega}_{E^\vee/S} \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{O}_E \longrightarrow 0$$

along the Kodaira–Spencer map with $\otimes_{\mathcal{O}_E}\Omega^1_{E/S}$, we obtain a short exact sequence

$$0 \to \pi^*\Omega^1_{S/T} \to \left(\mathcal{L}_1 \coprod_{(\pi^*\underline{\omega}_{E^\vee/S})} \left(\pi^*\Omega^1_{S/T} \otimes \left(\Omega^1_{E/S}\right)^\vee\right)\right) \otimes \Omega^1_{E/S} \to \Omega^1_{E/S} \to 0. \tag{3.17}$$

Here, we write $\coprod$ for the coproduct in the category of $\mathcal{O}_E$-modules. The canonical splitting of the geometric logarithm sheaf along e induces a canonical splitting

$$\mathrm{can}_{\mathrm{KS}} : e^*\Omega^1_{E/S} \oplus \Omega^1_{S/T} \xrightarrow{\sim} e^* \left[\left(\mathcal{L}_1 \coprod_{(\pi^*\underline{\omega}_{E^\vee/S})} \left(\pi^*\Omega^1_{S/T} \otimes \left(\Omega^1_{E/S}\right)^\vee\right)\right) \otimes \Omega^1_{E/S}\right]$$

of the short exact sequence (3.17).

Proposition 3.3.2. *There is a unique isomorphism between the short exact sequence* (3.17) *and the short exact sequence*

$$0 \longrightarrow \pi^*\Omega^1_{S/T} \longrightarrow \Omega^1_{E/T} \longrightarrow \Omega^1_{E/S} \longrightarrow 0$$

which is compatible with the canonical splittings σ and $\mathrm{can}_{\mathrm{KS}}$ *along e.*

Proof. There are many equivalent ways to define the Kodaira–Spencer map. We recall the following definition from [FC90, p. 80] which fits best for our purposes. Let

$$0 \longrightarrow \pi^*\Omega^1_{S/T} \otimes_{\mathcal{O}_E} \left(\Omega^1_{E/S}\right)^\vee \longrightarrow \Omega^1_{E/T} \otimes_{\mathcal{O}_E} \left(\Omega^1_{E/S}\right)^\vee \longrightarrow \mathcal{O}_E \longrightarrow 0$$

be the short exact sequence obtained by tensorizing the short exact sequence of Kähler differentials with $\otimes_{\mathcal{O}_E} \left(\Omega^1_{E/S}\right)^\vee$. The connecting homomorphism obtained by applying $R\pi_*$ to this sequence gives a map

$$\delta : \mathcal{O}_S = \pi_*\mathcal{O}_E \to R^1\pi_* \left[\pi^*\Omega^1_{S/T} \otimes_{\mathcal{O}_E} \left(\Omega^1_{E/S}\right)^\vee\right] = R^1\pi_*\mathcal{O}_E \otimes_{\mathcal{O}_S} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \underline{\omega}_{E/S}.$$

Using $R\pi_*\mathcal{O}_E = \underline{\omega}^\vee_{E^\vee/S}$, the Kodaira–Spencer map is the image of $1 \in \Gamma(S, \mathcal{O}_S)$ under δ. Using this definition of the Kodaira–Spencer map the claim follows immediately from Corollary 2.1.4. $\square$

The Katz splitting of $\mathcal{L}_1|_U$ induces a splitting on the pushout of $\mathcal{L}_1|_U$ along KS_E. If we tensorize with $\Omega^1_{E/S}$, we obtain a canonical splitting of the short exact sequence (3.17) restricted to U. The above proposition allows us to define a splitting of the short exact sequence of Kähler differentials restricted to U:

Definition 3.3.3. The splitting

$$\kappa_\Omega : \Omega^1_{U/S} \oplus \pi_U^* \Omega^1_{S/T} \xrightarrow{\sim} \Omega^1_{U/T}$$

induced by Proposition 3.3.2 and the Katz splitting on $\mathcal{L}_1$ will be called *Katz splitting of* $\Omega^1_{U/T}$.

Let us note that the Katz splitting is compatible with base change.

Remark 3.3.4. If the Kodaira–Spencer map is an isomorphism, e.g. in the case of the universal elliptic curve with some level structure, the short exact sequence of Kähler differentials is essentially equivalent to the first geometric logarithm sheaf $\mathcal{L}_1$.

3.3.1 Characterization of the Katz splitting for Kähler differentials

For $N > 1$ and $t \in E[S]$ we obtain a canonical isomorphism

$$T_t^* \Omega^1_{E/T} \xrightarrow{\sim} \Omega^1_{E/T},$$

which induces

$$\mathrm{triv}_{t,\Omega} : t^* \Omega^1_{E/T} \to e^* \Omega^1_{E/T} \cong \underline{\omega}_{E/S} \oplus \Omega^1_{S/T}.$$

The following characterization is an immediate consequence of the corresponding result for $\mathcal{L}_1$.

Proposition 3.3.5. *The Katz splitting*

$$\kappa_\Omega : \Omega^1_{U/S} \oplus \pi_U^* \Omega^1_{S/T} \xrightarrow{\sim} \Omega^1_{U/T} \xrightarrow{\sim} \Omega^1_{U/T}$$

is the unique splitting of the short exact sequence of relative Kähler differentials with the following property: For every $N > 1$ and every S-scheme X with N invertible on X and every $e \neq t \in E_X[N](X)$ we have

$$\mathrm{triv}_{t,\Omega}(t^*(\kappa_\Omega(\tilde{\omega}, 0))) = (t^*\tilde{\omega}, \mathrm{KS}_{E_X}(t^*\tilde{\omega} \otimes A_1(E_X, t)) , \quad \forall\ \tilde{\omega} \in \Gamma(U, \Omega^1_{U/S})$$

where

$$\mathrm{KS}_E : \underline{\omega}_{E/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{E^\vee/S} \to \Omega^1_{S/T}$$

is the Kodaira–Spencer map.

Proof. This follows immediately from Proposition 3.2.6 and the fact that κ_Ω was defined by pushout of $\kappa_{\mathcal{L}_1}$ along the Kodaira–Spencer map. $\square$

3.3.2 The Katz splitting of Kähler differentials on the Weierstrass curve

The definition of the Katz splitting for Kähler differentials was rather indirect. Indeed, one can give a very explicit description of the Katz splitting in terms of Weierstrass equations. We do the computation in the universal case. Let $E^{\mathrm{Weier}}/M^{\mathrm{Weier}}$ be the universal elliptic curve with a fixed invariant differential, i. e. $M^{\mathrm{Weier}} = \operatorname{Spec} \mathbb{Z}[g_1, g_3]\left[\frac{1}{6\Delta}\right]$ where $\Delta = g_2^3 - 27g_3^2$ and E^{Weier} is given explicitly by the Weierstrass equation:

$$y^2 = 4x^3 - g_2 x - g_3$$

We have a fixed invariant differential $\omega = \frac{dx}{y}$ on $E^{\mathrm{Weier}}/M^{\mathrm{Weier}}$.

Proposition 3.3.6. *Let $E^{\mathrm{Weier}}/M^{\mathrm{Weier}}$ be as above and $U := E^{\mathrm{Weier}} \setminus \{e\}$. Then, the Katz splitting is the unique $\mathcal{O}_U$-linear map*

$$\Omega^1_{U/M^{\mathrm{Weier}}} \oplus \pi_U^* \Omega^1_{M^{\mathrm{Weier}}/\mathbb{Z}} \to \Omega^1_{E^{\mathrm{Weier}}/\mathbb{Z}}$$

which induces on $\pi_U^* \Omega^1_{M^{\mathrm{Weier}}/\mathbb{Z}}$ the canonical inclusion

$$\pi_U^* \Omega^1_{M^{\mathrm{Weier}}/\mathbb{Z}} \hookrightarrow \Omega^1_{E^{\mathrm{Weier}}/\mathbb{Z}}$$

and on $\Omega^1_{U/M^{\mathrm{Weier}}}$ the map $\Omega^1_{U/M^{\mathrm{Weier}}} \to \Omega^1_{E^{\mathrm{Weier}}/\mathbb{Z}}$ given by

$$\frac{dx}{y} \mapsto \left(-18\frac{g_2}{\Delta}xy + 27\frac{g_3}{\Delta}y\right)dx + \left(12\frac{g_2}{\Delta}x^2 - 18\frac{g_3}{\Delta}x - 2\frac{g_2^2}{\Delta}\right)dy + \frac{3}{2}\frac{g_2}{\Delta}y\,dg_2.$$

Proof. Let us write

$$\tilde{\omega} := \left(-18\frac{g_2}{\Delta}xy + 27\frac{g_3}{\Delta}y\right)dx + \left(12\frac{g_2}{\Delta}x^2 - 18\frac{g_3}{\Delta}x - 2\frac{g_2^2}{\Delta}\right)dy + \frac{3}{2}\frac{g_2}{\Delta}y\,dg_2.$$

From the definition of $\tilde{\omega}$ it is not even clear that $\tilde{\omega}$ maps to ω under the restriction of Kähler differentials

$$\Omega^1_U \twoheadrightarrow \Omega^1_{U/M^{\mathrm{Weier}}}$$

which is a necessary condition for $\omega \mapsto \tilde{\omega}$ defining a section of $\Omega^1_U \twoheadrightarrow \Omega^1_{U/M^{\mathrm{Weier}}}$. Indeed, using $y^2 = 4x^3 - g_2x - g_3$ and the corresponding equation on Kähler differentials $2ydy =$

$(12x^2 - g_2)dx - xdg_2 - dg_3$ allows us to reformulate:

$$\begin{aligned}
\tilde{\omega} :=& \left(-18\frac{g_2}{\Delta}xy + 27\frac{g_3}{\Delta}y\right)dx + \left(12\frac{g_2}{\Delta}x^2 - 18\frac{g_3}{\Delta}x - 2\frac{g_2^2}{\Delta}\right)dy + \frac{3}{2}\frac{g_2}{\Delta}ydg_2 = \\
=& \left(-18\frac{g_2}{\Delta}x + 27\frac{g_3}{\Delta}\right)\left(4x^3 - g_2x - g_3\right)\frac{dx}{y} + \\
&+ \left(12\frac{g_2}{\Delta}x^2 - 18\frac{g_3}{\Delta}x - 2\frac{g_2^2}{\Delta}\right)\frac{((12x^2 - g_2)dx - xdg_2 - dg_3)}{2y} + \frac{3g_2}{2\Delta}y^2\frac{dg_2}{y} = \\
=& \left(-18\frac{g_2}{\Delta}x + 27\frac{g_3}{\Delta}\right)\left(4x^3 - g_2x - g_3\right)\frac{dx}{y} + \\
&+ \left(12\frac{g_2}{\Delta}x^2 - 18\frac{g_3}{\Delta}x - 2\frac{g_2^2}{\Delta}\right)\frac{((12x^2 - g_2)dx - xdg_2 - dg_3)}{2y} + \\
&+ \frac{3g_2}{2\Delta}\left(4x^3 - g_2x - g_3\right)\frac{dg_2}{y} = \\
=& \left[\left(-18\frac{g_2}{\Delta}x + 27\frac{g_3}{\Delta}\right)\left(4x^3 - g_2x - g_3\right) + \left(6\frac{g_2}{\Delta}x^2 - 9\frac{g_3}{\Delta}x - \frac{g_2^2}{\Delta}\right)\left(12x^2 - g_2\right)\right]\frac{dx}{y} + \\
&+ \left[\frac{3g_2}{2\Delta}\left(4x^3 - g_2x - g_3\right) - x\left(6\frac{g_2}{\Delta}x^2 - 9\frac{g_3}{\Delta}x - \frac{g_2^2}{\Delta}\right)\right]\frac{dg_2}{y} - \\
&- \left(6\frac{g_2}{\Delta}x^2 - 9\frac{g_3}{\Delta}x - \frac{g_2^2}{\Delta}\right)\frac{dg_3}{y} \\
=& \frac{dx}{y} + \left(9\frac{g_3}{\Delta}x^2 - \frac{g_2^2}{2\Delta}x - \frac{3g_2g_3}{2\Delta}\right)\frac{dg_2}{y} - \left(6\frac{g_2}{\Delta}x^2 - 9\frac{g_3}{\Delta}x - \frac{g_2^2}{\Delta}\right)\frac{dg_3}{y}
\end{aligned} \tag{3.18}$$

From the last expression it is obvious that $\tilde{\omega}$ lifts $\frac{dx}{y}$. It remains to prove $\kappa_\Omega(\frac{dx}{y}, 0) = \tilde{\omega}$. This equality can be checked after extensions of scalars to $\mathbb{C}$ and analytification. We have the following explicit description of the $\mathbb{C}$-valued points of M^{Weier}:

$$\begin{array}{ccc}
(\mathbb{C} \times \mathbb{H}/\mathbb{Z}^2 \rtimes \mathrm{Sl}_2(\mathbb{Z})) \times \mathbb{C}^\times & \xrightarrow{\sim} & E^{\mathrm{Weier}}(\mathbb{C}) \\
\downarrow & & \downarrow \\
(\mathbb{H}/\mathrm{Sl}_2(\mathbb{Z})) \times \mathbb{C}^\times & \xrightarrow{\sim} & M^{\mathrm{Weier}}(\mathbb{C}) = \{(g_2, g_3) \in \mathbb{C}^2 : g_2^3 - 27g_3^2 \neq 0\}
\end{array}$$

Here, the lower horizontal map is given by

$$(\tau, \lambda) \mapsto (g_2(\lambda\Lambda_\tau), g_3(\lambda\Lambda_\tau))$$

with $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ and g_2 and g_3 the classical invariants associated with the lattice Λ_τ. The upper horizontal map in the diagram is the map

$$(z, \tau, \lambda) \mapsto (x, y) = (\lambda^{-2}\wp(z, \tau), \lambda^{-3}\partial_z\wp(z, \tau))$$

given by complex uniformization. The fixed invariant differential $\frac{dx}{y}$ corresponds via this isomorphism to

$$\frac{\lambda^{-2}d\wp(z, \tau)}{\lambda^{-3}\partial_z\wp(z, \tau)} = \lambda dz.$$

Using $g_2(\lambda\tau) = \lambda^{-4} g_2(\tau)$, $g_3(\lambda\tau) = \lambda^{-6} g_3(\tau)$ and $\Delta(\lambda\tau) = \lambda^{-12}\Delta(\tau)$, the analytification of $\tilde{\omega}$ expresses via the above isomorphism as:

$$\begin{aligned}\tilde{\omega} =& \lambda^3 \frac{d\left(\lambda^{-2}\wp(z,\tau)\right)}{\partial_z \wp(z,\tau)} + \\ &+ \lambda^5 \left(9\frac{g_3(\tau)}{\Delta(\tau)}\wp(z,\tau)^2 - \frac{g_2(\tau)^2}{2\Delta(\tau)}\wp(z,\tau) - \frac{3g_2(\tau)g_3(\tau)}{2\Delta(\tau)}\right) \frac{d\left(\lambda^{-4} g_2(\tau)\right)}{\partial_z \wp(z,\tau)} - \\ &- \lambda^7 \left(6\frac{g_2(\tau)}{\Delta(\tau)}\wp(z,\tau)^2 - 9\frac{g_3(\tau)}{\Delta(\tau)}\wp(z,\tau) - \frac{g_2(\tau)^2}{\Delta(\tau)}\right) \frac{d\left(\lambda^{-6} g_3(\tau)\right)}{\partial_z \wp(z,\tau)}\end{aligned}$$

For the following calculations let us agree on dropping the variable, i. e. we will write $\wp$, g_2, g_3 for $\wp(z,\tau)$, $g_2(\tau)$, $g_3(\tau)$ etc. Furthermore, let us write $(\cdot)' := \partial_\tau(\cdot)$ for partial derivation with respect to τ if a function only depends on τ:

$$\begin{aligned}\tilde{\omega} =& \lambda^3 \frac{d\left(\lambda^{-2}\wp\right)}{\partial_z \wp} + \\ &+ \lambda^5 \left(9\frac{g_3}{\Delta}\wp^2 - \frac{g_2^2}{2\Delta}\wp - \frac{3g_2 g_3}{2\Delta}\right) \frac{d\left(\lambda^{-4} g_2\right)}{\partial_z \wp} - \\ &- \lambda^7 \left(6\frac{g_2}{\Delta}\wp^2 - 9\frac{g_3}{\Delta}\wp - \frac{g_2^2}{\Delta}\right) \frac{d\left(\lambda^{-6} g_3\right)}{\partial_z \wp} = \\ =& \lambda dz + \\ &+ \lambda \left[\partial_\tau \wp + \left(9\frac{g_3}{\Delta}\wp^2 - \frac{g_2^2}{2\Delta}\wp - \frac{3g_2 g_3}{2\Delta}\right) g_2' - \left(6\frac{g_2}{\Delta}\wp^2 - 9\frac{g_3}{\Delta}\wp - \frac{g_2^2}{\Delta}\right) g_3'\right] \frac{d\tau}{\partial_z \wp} \\ &+ \left[-2\wp - 4\left(9\frac{g_3}{\Delta}\wp^2 - \frac{g_2^2}{2\Delta}\wp - \frac{3g_2 g_3}{2\Delta}\right) g_2 + 6\left(6\frac{g_2}{\Delta}\wp^2 - 9\frac{g_3}{\Delta}\wp - \frac{g_2^2}{\Delta}\right) g_3\right] \frac{d\lambda}{\partial_z \wp} =\end{aligned}$$

$$\begin{aligned}=& \lambda dz + \\ &+ \lambda \left[\partial_\tau \wp + \left(9\frac{g_3}{\Delta}\wp^2 - \frac{g_2^2}{2\Delta}\wp - \frac{3g_2 g_3}{2\Delta}\right) g_2' - \left(6\frac{g_2}{\Delta}\wp^2 - 9\frac{g_3}{\Delta}\wp - \frac{g_2^2}{\Delta}\right) g_3'\right] \frac{d\tau}{\partial_z \wp} + \\ &+ \left[-2\wp - 2\frac{g_2^3}{\Delta}\wp - 54\frac{g_3^2}{\Delta}\wp\right] \frac{d\lambda}{\partial_z \wp} \overset{\text{Def. } \Delta}{=} \\ =& \lambda dz + \\ &+ \lambda \left[\partial_\tau \wp + \left(9\frac{g_3}{\Delta}\wp^2 - \frac{g_2^2}{2\Delta}\wp - \frac{3g_2 g_3}{2\Delta}\right) g_2' - \left(6\frac{g_2}{\Delta}\wp^2 - 9\frac{g_3}{\Delta}\wp - \frac{g_2^2}{\Delta}\right) g_3'\right] \frac{d\tau}{\partial_z \wp}\end{aligned}$$

This can be simplified further using Ramanujan's identities. Let E_{2k} be the classical Eisenstein series with normalized constant term in the q-expansion, i. e.

$$E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n, \quad E_4 = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n, \quad E_6 = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n$$

Then, we have the following identities due to Ramanujan, cf. [Ram00, eq. (30)] or [SS12, (3.1)-(3.4)]:

$$\frac{1}{2\pi i}E_4' = \frac{1}{3}\left(E_2E_4 - E_6\right)$$
$$\frac{1}{2\pi i}E_6' = \frac{1}{2}\left(E_2E_6 - E_4^2\right)$$
$$E_2 = \frac{1}{2\pi i}\frac{\Delta'}{\Delta}$$

The Eisenstein series E_4 and E_6 are related to the Eisenstein series g_2, g_3 defined above via $g_2 = \frac{4}{3}\pi^4 E_4$ and $g_3 = \frac{8}{27}\pi^6 E_6$. Furthermore, E_2 is related to the period function $\eta(1,\tau) = \zeta(z+1,\tau) - \zeta(z,\tau)$ of the Weierstrass zeta function via $\eta(1,\tau) = \frac{(2\pi i)^2}{12}E_2$. We can restate Ramanujan's identities as:

$$g_2' = \frac{1}{3}\frac{\Delta'}{\Delta}g_2 + \frac{6}{2\pi i}g_3 \tag{R1}$$
$$g_3' = \frac{1}{2}\frac{\Delta'}{\Delta}g_3 + \frac{1}{3}\frac{g_2^2}{2\pi i} \tag{R2}$$
$$\eta(1,\tau) = 2\pi i\frac{1}{12}\frac{\Delta'}{\Delta} \tag{R3}$$

Using Ramanujan's identities, we can further simplify the analytic expression for $\tilde{\omega}$ by a straightforward computation:

$$\begin{aligned}\tilde{\omega} =&\lambda dz+\\ &+\lambda\left[\partial_\tau\wp + \left(9\frac{g_3}{\Delta}\wp^2 - \frac{g_2^2}{2\Delta}\wp - \frac{3g_2g_3}{2\Delta}\right)g_2' - \left(6\frac{g_2}{\Delta}\wp^2 - 9\frac{g_3}{\Delta}\wp - \frac{g_2^2}{\Delta}\right)g_3'\right]\frac{d\tau}{\partial_z\wp} \overset{(R1,R2)}{=}\\ =&\lambda dz+\\ &+\lambda\Big[\partial_\tau\wp + \left(9\frac{g_3}{\Delta}\wp^2 - \frac{g_2^2}{2\Delta}\wp - \frac{3g_2g_3}{2\Delta}\right)\left(\frac{1}{3}\frac{\Delta'}{\Delta}g_2 + \frac{6}{2\pi i}g_3\right) -\\ &-\left(6\frac{g_2}{\Delta}\wp^2 - 9\frac{g_3}{\Delta}\wp - \frac{g_2^2}{\Delta}\right)\left(\frac{1}{2}\frac{\Delta'}{\Delta}g_3 + \frac{1}{3}\frac{g_2^2}{2\pi i}\right)\Big]\frac{d\tau}{\partial_z\wp} =\\ =&\lambda dz + \lambda\left[\partial_\tau\wp - 2\frac{1}{2\pi i}\wp^2 - \frac{1}{6}\frac{\Delta'}{\Delta}\wp + \frac{1}{3}\frac{g_2}{2\pi i}\right]\frac{d\tau}{\partial_z\wp}\end{aligned}$$

N.B.: This is a good point to remark that the above computations show that the analytification of the map

$$\Omega^1_{U/M^{\text{Weier}}} \to \Omega^1_U, \quad \frac{dx}{y} \mapsto \tilde{\omega}$$

does not depend on λ. Indeed, the above computations show that this map is analytically given by

$$\lambda dz \mapsto \lambda\cdot dz + \lambda\cdot\left[\partial_\tau\wp - 2\frac{1}{2\pi i}\frac{\Delta'}{\Delta}\wp^2 - \frac{1}{6}\frac{\Delta'}{\Delta}\wp + \frac{1}{3}\frac{g_2}{2\pi i}\right]\frac{d\tau}{\partial_z\wp}$$

or equivalently by

$$dz \mapsto dz + \left[\partial_\tau \wp - 2\frac{1}{2\pi i}\frac{\Delta'}{\Delta}\wp^2 - \frac{1}{6}\frac{\Delta'}{\Delta}\wp + \frac{1}{3}\frac{g_2}{2\pi i}\right]\frac{d\tau}{\partial_z \wp}. \tag{3.19}$$

The independence of λ is a necessary condition in order to give $\frac{dx}{y} \mapsto \tilde{\omega}$ a chance to coincide with κ_Ω. The Katz splitting does not depend on a chosen invariant differential. The corresponding property that $\frac{dx}{y} \mapsto \tilde{\omega}$ is independent of the choice of the invariant differential is exactly reflected by the fact that this map is independence of λ.

In order to compare the above map with the Katz splitting we should describe the Katz splitting on $E^{\mathrm{Weier}}(\mathbb{C})/M^{\mathrm{Weier}}(\mathbb{C})$ analytically. We claim that the map

$$\Omega_{U/M^{\mathrm{Weier}}} \hookrightarrow \Omega^1_{U/M^{\mathrm{Weier}}} \oplus \pi^*\Omega^1_{M^{\mathrm{Weier}}} \xrightarrow{\sim} \Omega^1_U$$

is analytically given by

$$dz \mapsto dz + \frac{1}{2\pi i}\left(\zeta(z,\tau) + \eta(1,\tau)\cdot z\right)d\tau \tag{3.20}$$

Indeed, it is easily checked that the 1-form on the right hand side is invariant under the action of $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ and thus defines a holomorphic 1-form on $U(\mathbb{C})$. If we view

$$dz + \frac{1}{2\pi i}\left(\zeta(z,\tau) + \eta(1,\tau)\cdot z\right)d\tau.$$

as holomorphic 1-form on the complex elliptic curve $\mathbb{C}\times\mathbb{H}/\mathbb{Z}^2$ where $(m,n)\in\mathbb{Z}^2$ acts as $(z,\tau)\mapsto(z+m\tau+n)$, we see that $\frac{1}{2\pi i}\left(\zeta(z,\tau)+\eta(1,\tau)\cdot z\right)d\tau$ specializes along the map $\tau\mapsto(\frac{a}{N}\tau+\frac{b}{N})$ to

$$\frac{1}{2\pi i}\left(\zeta(\frac{a}{N}\tau+\frac{b}{N},\tau)+\eta\left(\frac{a}{N}\tau+\frac{b}{N},\tau\right)\right)d\tau$$

but by the explicit formula for $A_1(E,t)$ this coincides with $\mathrm{KS}(dz\otimes A_1(E,t))$. Thus, the map (3.20) coincides with the analytification of κ_Ω on a dense subset and the claim follows.

Thus, by the above equations (3.20) and (3.19) it remains to prove the following equality of meromorphic functions on $\mathbb{C}\times\mathbb{H}$

$$\frac{1}{2\pi i}\left(\zeta(z,\tau)+\eta(1,\tau)\cdot z\right) = \frac{1}{\partial_z\wp}\left[\partial_\tau\wp - 2\frac{1}{2\pi i}\frac{\Delta'}{\Delta}\wp^2 - \frac{1}{6}\frac{\Delta'}{\Delta}\wp + \frac{1}{3}\frac{g_2}{2\pi i}\right]. \tag{3.21}$$

We first show that applying ∂_z to both sides gives equality and then care about the constant term in the Laurent series expansion in z. Using again one of the Ramanujan identities, the left hand side gives

$$\partial_z\frac{1}{2\pi i}\left(\zeta(z,\tau)+\eta(1,\tau)\cdot z\right) = -\frac{1}{2\pi i}\wp + \frac{\eta(1,\tau)}{2\pi i} \overset{R3}{=} -\frac{1}{2\pi i}\wp + \frac{1}{12}\frac{\Delta'}{\Delta}.$$

Before we apply ∂_z to the right hand side, let us note that we obtain the following equations by differentiating the Weierstrass equation $(\partial_z\wp)^2 = 4\wp^3 - g_3\wp - g_2$:

$$\partial_z\partial_z\wp = 6\wp^2 - \frac{1}{2}g_2 \tag{I}$$

$$\partial_z\wp\cdot\partial_\tau\partial_z\wp = 6\wp^2\partial_\tau\wp - \frac{1}{2}\wp g_2' - \frac{1}{2}g_2\partial_\tau\wp - \frac{1}{2}g_3' \tag{II}$$

Using these equations, we can simplify ∂_z applied to the r.h.s:

$$\begin{aligned}
&\partial_z\left(\frac{1}{\partial_z\wp}\left[\partial_\tau\wp - 2\frac{1}{2\pi i}\frac{\Delta'}{\Delta}\wp^2 - \frac{1}{6}\frac{\Delta'}{\Delta}\wp + \frac{1}{3}\frac{g_2}{2\pi i}\right]\right) = \\
=&\frac{1}{\partial_z\wp}\left[-4\frac{1}{2\pi i}\wp\partial_z\wp - \frac{1}{6}\frac{\Delta'}{\Delta}\partial_z\wp\right] + \\
&+\frac{\partial_z\partial_\tau\wp}{\partial_z\wp} - \frac{\partial_z\partial_z\wp}{(\partial_z\wp)^2}\cdot\left[\partial_\tau\wp - 2\frac{1}{2\pi i}\wp^2 - \frac{1}{6}\frac{\Delta'}{\Delta}\wp + \frac{1}{3}\frac{g_2}{2\pi i}\right] = \\
=&-4\frac{1}{2\pi i}\wp - \frac{1}{6}\frac{\Delta'}{\Delta} + \\
&+\frac{1}{(\partial_z\wp)^2}\left(\partial_z\wp\cdot\partial_z\partial_\tau\wp - \partial_z\partial_z\wp\cdot\left[\partial_\tau\wp - 2\frac{1}{2\pi i}\wp^2 - \frac{1}{6}\frac{\Delta'}{\Delta}\wp + \frac{1}{3}\frac{g_2}{2\pi i}\right]\right) \overset{(I)}{=}
\end{aligned}$$

$$\begin{aligned}
\overset{(I)}{=}&-4\frac{1}{2\pi i}\wp - \frac{1}{6}\frac{\Delta'}{\Delta} + \\
&+\frac{1}{(\partial_z\wp)^2}\left(\partial_z\wp\cdot\partial_z\partial_\tau\wp - \partial_z\partial_z\wp\cdot\partial_\tau\wp + \left(6\wp^2 - \frac{1}{2}g_2\right)\cdot\left[2\frac{1}{2\pi i}\wp^2 + \frac{1}{6}\frac{\Delta'}{\Delta}\wp - \frac{1}{3}\frac{g_2}{2\pi i}\right]\right) \overset{(I,II)}{=} \\
=&-4\frac{1}{2\pi i}\wp - \frac{1}{6}\frac{\Delta'}{\Delta} + \\
&+\frac{1}{(\partial_z\wp)^2}\left(-\frac{1}{2}\wp g_2' - \frac{1}{2}g_3' + \left(12\frac{1}{2\pi i}\wp^4 + \frac{\Delta'}{\Delta}\wp^3 - 3\frac{g_2}{2\pi i}\wp^2 - \frac{1}{12}\frac{\Delta'}{\Delta}g_2\wp + \frac{1}{6}\frac{g_2^2}{2\pi i}\right)\right) = \\
=&-4\frac{1}{2\pi i}\wp - \frac{1}{6}\frac{\Delta'}{\Delta} + \frac{12\frac{1}{2\pi i}\wp^4 + \frac{\Delta'}{\Delta}\wp^3 - 3\frac{g_2}{2\pi i}\wp^2 - \left(\frac{1}{12}\frac{\Delta'}{\Delta}g_2 + \frac{1}{2}g_2'\right)\wp + \frac{1}{6}\frac{g_2^2}{2\pi i}}{(\partial_z\wp)^2} \overset{(I,II)}{=} \\
=&-4\frac{1}{2\pi i}\wp - \frac{1}{6}\frac{\Delta'}{\Delta} + \frac{12\frac{1}{2\pi i}\wp^4 + \frac{\Delta'}{\Delta}\wp^3 - 3\frac{g_2}{2\pi i}\wp^2 - \left(\frac{1}{4}\frac{\Delta'}{\Delta}g_2 + \frac{3}{2\pi i}g_3\right)\wp + \frac{1}{6}\frac{g_2^2}{2\pi i}}{4\wp^3 - g_3\wp - g_2} = \\
=&-4\frac{1}{2\pi i}\wp - \frac{1}{6}\frac{\Delta'}{\Delta} + \frac{3}{2\pi i}\wp + \frac{1}{4}\frac{\Delta'}{\Delta} = -\frac{1}{2\pi i}\wp + \frac{1}{12}\frac{\Delta'}{\Delta}
\end{aligned}$$

Comparing the above formulas we get

$$\frac{1}{2\pi i}\partial_z\left(\zeta(z,\tau) + \eta(1,\tau)\cdot z\right) = \partial_z\left(\frac{1}{\partial_z\wp}\left[\partial_\tau\wp - 2\frac{1}{2\pi i}\frac{\Delta'}{\Delta}\wp^2 - \frac{1}{6}\frac{\Delta'}{\Delta}\wp + \frac{1}{3}\frac{g_2}{2\pi i}\right]\right).$$

In order to prove the theorem it remains to compare the constant terms in the Laurent expansion of both sides. The Laurent series of the left hand side of (3.21) starts as follows:

$$\frac{1}{2\pi i}\left(\zeta(z,\tau)+\eta(1,\tau)\cdot z\right)=\frac{1}{2\pi i}z^{-1}+O(z)$$

Thus, there is no constant term in the Laurent expansion in z. Recall that $\wp(z,\tau)=z^{-2}+O(z^2)$ which implies $\frac{1}{\partial_z\wp}=-\frac{1}{2}z^3+O(z^5)$. Using this, we have

$$\frac{1}{\partial_z\wp}\left[\partial_\tau\wp-\frac{1}{6}\frac{\Delta'}{\Delta}\wp+\frac{1}{3}\frac{g_2}{2\pi i}\right]=O(z)$$

and using $\wp^2(z)=z^{-4}+O(1)$ gives

$$\frac{1}{\partial_z\wp}\left[-2\frac{1}{2\pi i}\frac{\Delta'}{\Delta}\wp^2\right]=\frac{1}{2\pi i}z^{-1}+O(z).$$

Combining the above gives

$$\frac{1}{\partial_z\wp}\left[\partial_\tau\wp-2\frac{1}{2\pi i}\frac{\Delta'}{\Delta}\wp^2-\frac{1}{6}\frac{\Delta'}{\Delta}\wp+\frac{1}{3}\frac{g_2}{2\pi i}\right]=\frac{1}{2\pi i}z^{-1}+O(z)\,,$$

which concludes the proof. □

Remark 3.3.7. Since $\underline{\omega}_{E/S}$ is always locally free of rank 1, this gives us, at least locally on the base, a quite explicit and purely algebraic description of the Katz splitting for Kähler differentials.

3.3.3 Lifting the connection via the Katz splitting

The geometric logarithm sheaf $\mathcal{L}_n^\dagger$ comes with an integrable S-connection. Our aim is to extend this connection to an integrable T-connection whenever S is a smooth T-scheme. In a first step we want to relate the Katz splitting of Kähler differentials to the Gauss–Manin connection. Let again E/S be an elliptic curve with 6 invertible on S and let S be smooth over T. The relative de Rham cohomology is equipped with a canonical connection

$$\nabla_{\mathrm{GM}}:\underline{H}^1_{\mathrm{dR}}(E/S)\to\Omega^1_{S/T}\otimes_{\mathcal{O}_S}\underline{H}^1_{\mathrm{dR}}(E/S)$$

coming from the spectral sequence associated with the filtration induced by the short exact sequence of relative Kähler differentials for $E/S/T$ cf. [KO68] resp. [Kat70]. An explicit way to compute the Gauss–Manin connection is already given in [KO68]. We follow the exposition in [Ked08, §3.3, §3.4]. The canonical surjection

$$(\pi_U)_*\Omega^1_{U/S}\twoheadrightarrow\underline{H}^1_{\mathrm{dR}}(U/S)\cong\underline{H}^1_{\mathrm{dR}}(E/S) \tag{3.22}$$

induces an isomorphism (cf. [Kat73, A1.2,p. 163]):

$$\pi_*\Omega^1_{E/S}(2[e])\xrightarrow{\sim}\underline{H}^1_{\mathrm{dR}}(E/S) \tag{3.23}$$

Observe that we can use this isomorphism to split the surjection (3.22) via

$$\underline{H}^1_{\mathrm{dR}}(E/S)\cong\pi_*\Omega^1_{E/S}(2[e])\hookrightarrow\pi_*j_*\Omega^1_{U/S}=(\pi_U)_*\Omega^1_{U/S} \tag{3.24}$$

where $j : U \hookrightarrow E$ is the inclusion. The short exact sequence

$$0 \longrightarrow (\pi_U)^*\Omega^1_{S/T} \longrightarrow \Omega^1_{U/T} \longrightarrow \Omega^1_{U/S} \longrightarrow 0$$

induces a canonical surjection

$$\Omega^2_{U/T} = \Lambda^2\Omega^1_{U/T} \twoheadrightarrow (\pi_U)^*\Omega^1_{S/T} \otimes_{\mathcal{O}_U} \Omega^1_{U/S}.$$

Applying $(\pi_U)_*$, which is exact since it is affine, and using the projection formula we obtain:

$$(\pi_U)_*\Omega^2_{U/T} \twoheadrightarrow (\pi_U)_*\left((\pi_U)^*\Omega^1_{S/T} \otimes_{\mathcal{O}_U} \Omega^1_{U/S}\right) = \Omega^1_{S/T} \otimes_{\mathcal{O}_S} (\pi_U)_*\Omega^1_{U/S}. \tag{3.25}$$

The explicit method from [Ked08, §3.3, §3.4] can be summarized in the commutative diagram:

$$\begin{tikzcd}
(\pi_U)_*\Omega^1_{U/S} \arrow[r, dotted] \arrow[dd, two heads, "(3.22)"] & (\pi_U)_*\Omega^1_{U/T} \arrow[r, "d_{U/T}"] & (\pi_U)_*\Omega^2_{U/T} \arrow[d, two heads, "(3.25)"] \\
 & & \Omega^1_{S/T} \otimes_{\mathcal{O}_S} (\pi_U)_*\Omega^1_{U/S} \arrow[d, two heads, "\mathrm{id}\otimes(3.22)"] \\
\underline{H}^1_{\mathrm{dR}}(E/S) \arrow[uu, bend left, "(3.24)"] \arrow[rr, "\nabla_{\mathrm{GM}}"] & & \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \underline{H}^1_{\mathrm{dR}}(E/S)
\end{tikzcd} \tag{3.26}$$

Here, the dotted arrow in the upper left corner is an arbitrary section to the canonical projection. One possible and natural choice for the dotted arrow is the map induced by the Katz splitting for relative Kähler differentials, but every other choice works equally well. The choice of the Katz splitting for the dotted arrow makes the above diagram compatible with the canonical splittings on both sides.

Lemma 3.3.8. *For $S \to T$ smooth with 6 invertible on S and E/S an elliptic curve the following diagram commutes:*

$$\begin{tikzcd}
 & (\pi_U)_*\Omega^1_{U/S} \arrow[r, "\kappa_\Omega"] & (\pi_U)_*\Omega^1_{U/T} \arrow[r, "d_{U/T}"] & (\pi_U)_*\Omega^2_{U/T} \\
\underline{\omega}_{E/S} \arrow[ur, hook] \arrow[dr, hook] & & & \Omega^1_{S/T} \otimes_{\mathcal{O}_S} (\pi_U)_*\Omega^1_{U/S} \arrow[u, "\mathrm{id}\wedge\kappa"] \\
 & \underline{H}^1_{\mathrm{dR}}(E/S) \arrow[rr, "\nabla_{\mathrm{GM}}"] & & \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \underline{H}^1_{\mathrm{dR}}(E/S) \arrow[u, "\mathrm{id}\otimes(3.24)"]
\end{tikzcd} \tag{3.27}$$

Proof. The Gauss–Manin connection as well as the Katz splitting are compatible with base change. Thus, the claim can be checked locally on the base and we may assume that $\underline{\omega}_{E/S}$ is freely generated by $\omega \in \Gamma(S, \underline{\omega}_{E/S})$. Again, by compatibility with base change we can prove the claim in the universal situation, i. e. we can reduce to the case of the Weierstrass curve $E = E^{\text{Weier}}$, $S = M^{\text{Weier}}$ and $T = \operatorname{Spec} \mathbb{Z}[1/6]$ with invariant differential $\omega = \frac{dx}{y}$. Since both compositions in the above diagram are $\mathcal{O}_T$-linear derivations on $\underline{\omega}_{E/S}$, it suffices to check the commutativity on the generator ω. Using the explicit description of the Katz splitting in terms of the Weierstrass equation, a straightforward but lengthy computation with Kähler differentials gives:

$$
\begin{aligned}
&d_U\left(\kappa_\Omega(\omega)\right) = \\
=&d_U\left[\left(-18\frac{g_2}{\Delta}xy + 27\frac{g_3}{\Delta}y\right)dx + \left(12\frac{g_2}{\Delta}x^2 - 18\frac{g_3}{\Delta}x - 2\frac{g_2^2}{\Delta}\right)dy + \frac{3}{2}\frac{g_2}{\Delta}ydg_2\right] = \ldots \\
=&\left(3\frac{g_2}{\Delta}dg_3 - \frac{9}{2}\frac{g_3}{\Delta}dg_2\right)\wedge x\frac{dx}{y} - \frac{1}{12}\frac{d\Delta}{\Delta}\wedge\frac{dx}{y} + \frac{1}{y}\left(\frac{3}{\Delta}x^2 - \frac{1}{4}\frac{g_2}{\Delta}\right)dg_2\wedge dg_3
\end{aligned}
$$

In particular, we deduce from this formula and the commutative diagram (3.26) the following formula for $\nabla_{\text{GM}}(\omega)$:

$$
\nabla_{\text{GM}}(\omega) = \left(3\frac{g_2}{\Delta}dg_3 - \frac{9}{2}\frac{g_3}{\Delta}dg_2\right)\otimes[\eta] - \frac{1}{12}\frac{d\Delta}{\Delta}\otimes[\omega]
$$

where $[\omega]$ and $[\eta]$ are the de Rham cohomology classes associated with $\frac{dx}{y}$ resp. $x\frac{dx}{y}$. It remains to compute the image of $\nabla_{\text{GM}}(\omega)$ under

$$
\Omega^1_M \otimes_{\mathcal{O}_M} \underline{H}^1_{\text{dR}}(E/M) \xrightarrow{\text{id}\otimes(3.24)} \Omega^1_M \otimes_{\mathcal{O}_M} (\pi_U)_*\Omega^1_{U/M} \xrightarrow{\text{id}\wedge\kappa} (\pi_U)_*\Omega^2_U.
$$

By the above formula for $\nabla_{\text{GM}}(\omega)$ this is given by:

$$
\begin{aligned}
&\left((\text{id}\wedge\kappa)\circ(\text{id}\otimes(3.24))\right)\left(\nabla_{\text{GM}}(\omega)\right) = \\
=&\left(3\frac{g_2}{\Delta}dg_3 - \frac{9}{2}\frac{g_3}{\Delta}dg_2\right)\wedge\kappa\left(x\frac{dx}{y}\right) - \frac{1}{12}\frac{d\Delta}{\Delta}\wedge\kappa\left(\frac{dx}{y}\right) \overset{(3.18)}{=} \\
=&\left(3\frac{g_2}{\Delta}dg_3 - \frac{9}{2}\frac{g_3}{\Delta}dg_2\right)\wedge x\frac{dx}{y} + \\
&+\left(3\frac{g_2}{\Delta}dg_3 - \frac{9}{2}\frac{g_3}{\Delta}dg_2\right)\wedge\left[\left(9\frac{g_3}{\Delta}x^2 - \frac{g_2^2}{2\Delta}x - \frac{3g_2g_3}{2\Delta}\right)\frac{dg_2}{y} - \left(6\frac{g_2}{\Delta}x^2 - 9\frac{g_3}{\Delta}x - \frac{g_2^2}{\Delta}\right)\frac{dg_3}{y}\right] \\
&-\frac{1}{12}\frac{d\Delta}{\Delta}\wedge\frac{dx}{y} - \\
&-\frac{1}{12}\frac{d\Delta}{\Delta}\wedge\left[\left(9\frac{g_3}{\Delta}x^2 - \frac{g_2^2}{2\Delta}x - \frac{3g_2g_3}{2\Delta}\right)\frac{dg_2}{y} - \left(6\frac{g_2}{\Delta}x^2 - 9\frac{g_3}{\Delta}x - \frac{g_2^2}{\Delta}\right)\frac{dg_3}{y}\right] = \\
=&\left(3\frac{g_2}{\Delta}dg_3 - \frac{9}{2}\frac{g_3}{\Delta}dg_2\right)\wedge x\frac{dx}{y} - \frac{1}{12}\frac{d\Delta}{\Delta}\wedge\frac{dx}{y} + \left[\frac{3}{\Delta}x^2 - \frac{1}{4}\frac{g_2}{\Delta}\right]\frac{dg_2\wedge dg_3}{y}
\end{aligned}
$$

Now, a comparison with the above equation for $d_U\left(\kappa_\Omega(\omega)\right)$ proves the result. □

Remark 3.3.9. In the proof we have deduced, as a by product, an explicit formula for the Gauss–Manin connection on the Weierstrass family:

$$\nabla_{\mathrm{GM}}(\omega) = [\eta] \otimes \left(3\frac{g_2}{\Delta} dg_3 - \frac{9}{2}\frac{g_3}{\Delta} dg_2 \right) - \frac{1}{12}[\omega] \otimes \frac{d\Delta}{\Delta}$$

As a consistency check one can compare our formula for $\nabla_{\mathrm{GM}}(\omega)$ with the transcendental description of the Gauss–Manin connection on $\mathbb{C} \times \mathbb{H}/\mathbb{Z}^2$ given by Katz, cf. [Kat73, §A1.3]. Our above formula gives the transcendental formula

$$\nabla_{\mathrm{GM}}(\omega) = [\eta] \otimes \left(3\frac{g_2 g_3'}{\Delta} - \frac{9}{2}\frac{g_3 g_2'}{\Delta} \right) d\tau - \frac{1}{12}[\omega] \otimes \frac{d\Delta}{\Delta}.$$

Using Ramanujan's identities (cf. proof of Proposition 3.3.6)

$$g_2' = \frac{1}{3}\frac{\Delta'}{\Delta} g_2 + \frac{6}{2\pi i} g_3 \tag{R1}$$

$$g_3' = \frac{1}{2}\frac{\Delta'}{\Delta} g_3 + \frac{1}{3}\frac{g_2^2}{2\pi i} \tag{R2}$$

$$\eta(1,\tau) = 2\pi i \frac{1}{12}\frac{\Delta'}{\Delta}, \tag{R3}$$

we can rewrite this as

$$\nabla_{\mathrm{GM}}(\omega) = -\frac{1}{2\pi i}[\eta] \otimes d\tau - \frac{1}{2\pi i}\eta(1,\tau)[\omega] \otimes d\tau$$

which is exactly the formula in [Kat73, A1.3].

Corollary 3.3.10. *Let $E/S/T$ be an elliptic curve over S with 6 invertible and let S be smooth over T. Let us write $\mathcal{H}_U := \pi_U^* \underline{H}^1_{\mathrm{dR}}(E/S)^\vee \cong \pi_U^* \underline{H}^1_{\mathrm{dR}}(E^\vee/S)$. Then, there is a unique integrable T-connection*

$$\nabla_U : \mathcal{L}_1^\dagger|_U \to \mathcal{L}_1^\dagger|_U \otimes_{\mathcal{O}_U} \Omega^1_{U/T}$$

on

$$0 \longrightarrow \mathcal{H}_U \longrightarrow \mathcal{L}_1^\dagger|_U \longrightarrow \mathcal{O}_U \longrightarrow 0$$

satisfying the following conditions:

(a) *If we restrict the T-connection ∇_U to an S-connection, then we obtain the canonical connection $\nabla_{\mathcal{L}_1^\dagger}|_U$.*

(b) *The above short exact sequence is T-horizontal if we equip $\mathcal{L}_1^\dagger|_U$ with ∇_U, $\mathcal{O}_U$ with the canonical T-derivation and $\mathcal{H}_U$ with the pullback connection of the Gauss–Manin connection.*

(c) *We have the explicit lifting formula*

$$\nabla_U(\kappa(1,0)) = \left(\mathrm{id}_{\mathcal{L}_1^\dagger} \otimes \kappa_\Omega\right)(\mathit{Id}_{\mathcal{H}}).$$

Proof. Indeed, we have $\mathcal{L}_1^\dagger \cong \mathcal{O}_U \oplus \mathcal{H}_U$ under the Katz splitting. Thus, it is straightforward to construct a unique connection satisfying (b) and (c). (a) is an immediate consequence of (b) and (c) using $\nabla_{\mathcal{L}_1^\dagger}(\kappa_{\mathcal{L}}(1,0)) = \mathcal{Id}_{\mathcal{H}}$. The crucial point is to prove that the defined T-connection is integrable.

The pullback of the Gauss–Manin connection induces a canonical T-connection on $\mathcal{H}_U$. Let

$$d^{(1)}_{\mathcal{H}_U} : \mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^1_{U/T} \to \mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^2_{U/T}$$

be the second differential in the de Rham complex $\Omega^\bullet_{U/T}(\mathcal{H}_U)$. The dual $\mathcal{H}_U^\vee$ is equipped with a canonical T-connection. Now, let

$$d^{(0)}_{\mathcal{H}_U\otimes\mathcal{H}^\vee} : \mathcal{H}_U \otimes \mathcal{H}^\vee \to \mathcal{H}_U \otimes \mathcal{H}^\vee \otimes_{\mathcal{O}_U} \Omega^1_{U/T}$$

be the first differential in the de Rham complex of $\mathcal{H}_U \otimes_{\mathcal{O}_U} \mathcal{H}_U^\vee$ with the tensor product connection induced by $\mathcal{H}_U$ and $\mathcal{H}_U^\vee$. Then, we claim that

$$\begin{tikzcd}
\mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^1_{U/S} \arrow[r, "\mathrm{id}\otimes\kappa"] & \mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^1_{U/T} \arrow[r, "d^{(1)}_{\mathcal{H}_U}"] & \mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^2_{U/T} \\
\mathcal{H}_U \otimes_{\mathcal{O}_U} \pi_U^*\underline{\omega}_{E/S} \arrow[u, "\cong"] \arrow[d, hook] & & \mathcal{H}_U \otimes_{\mathcal{O}_U} \Omega^1_{U/S} \otimes_{\mathcal{O}_U} \Omega^1_{U/T} \arrow[u, "\mathrm{id}\otimes(\kappa\wedge\mathrm{id})"'] \\
\mathcal{H}_U \otimes_{\mathcal{O}_U} \mathcal{H}_U^\vee \arrow[rr, "d^{(0)}_{\mathcal{H}_U\otimes\mathcal{H}^\vee}"] & & \mathcal{H}_U \otimes_{\mathcal{O}_U} \mathcal{H}_U^\vee \otimes_{\mathcal{O}_U} \otimes_{\mathcal{O}_U} \Omega^1_{U/T} \arrow[u, "\mathrm{id}\otimes\pi_U^*((3.24))"']
\end{tikzcd}$$

commutes. Both compositions are $\mathcal{O}_T$-linear derivations on $\mathcal{H}_U \otimes \Omega^1_{U/S}$ with values in $\mathcal{H}_U \otimes \Omega^2_{U/S}$. We may check the commutativity locally. By working locally on the base we may assume that $\underline{\omega}_{E/S}$ is freely generated by $\omega \in \Gamma(S, \underline{\omega}_{E/S})$. Then, $\mathcal{H}_U \otimes \Omega^1_{U/S}$ is a free $\mathcal{O}_U$-module of rank 2 with generators $[\eta]^\vee \otimes \omega$ and $[\omega]^\vee \otimes \omega$. By the Leibniz rule it is enough to show that both compositions in the diagram coincide on those generators. But this follows immediately from Lemma 3.3.8.

Using the commutativity of the above diagram, allows us to show the integrability of ∇_U. The pullback of the Gauss–Manin connection is integrable. Thus, the claimed integrability of ∇_U boils down to the equation:

$$d^{(1)}_{\mathcal{H}_U}\left(\nabla_U(\kappa(1,0))\right) = 0.$$

By (c) this is equivalent to

$$d^{(1)}_{\mathcal{H}_U}\left((\mathrm{id}\otimes\kappa)(\mathcal{Id}_{\mathcal{H}})\right) = 0.$$

Using the above commutative diagram, this is equivalent to $\mathcal{Id}_{\mathcal{H}}$ mapping to 0 under the other composition of the diagram. But $\mathcal{Id}_{\mathcal{H}}$ maps by definition of $\mathcal{Id}_{\mathcal{H}}$ to $\mathrm{id}_{\mathcal{H}_U} \in \Gamma(U, \mathcal{H}_U \otimes \mathcal{H}_U^\vee)$ under the left vertical composition. Since the identity $\mathrm{id}_{\mathcal{H}_U} : \mathcal{H}_U \to \mathcal{H}_U$ is horizontal for the canonical connection on $\mathcal{H}_U$, it maps to zero under the lower horizontal map. This finishes the proof. □

4 P-adic interpolation of Eisenstein–Kronecker series via p-adic theta functions

In the paper [Kat76] Katz constructs p-adic variants of real-analytic Eisenstein series and proves the existence of a two-variable p-adic measure interpolating them. The construction of the p-adic real-analytic Eisenstein series is completely analogous to his construction via the Gauss–Manin connection in the $\mathcal{C}^\infty$ case. Let us briefly recall Katz' construction. By applying the Gauss–Manin connection on the universal elliptic curve iteratively to classical algebraic Eisenstein series one obtains global sections of the sheaf of symmetric powers of the first relative de Rham cohomology. In order to construct real-analytic Eisenstein series from those classes one can pass to the analytification and apply the Hodge-decomposition. Instead of passing to the analytification one can restrict to the ordinary locus and pass to the moduli space of trivialized elliptic curves. Here, one can use the unit-root decomposition instead of the Hodge-decomposition to construct p-adic modular forms out of those classes in symmetric powers of the first de Rham cohomology.

In Chapter 1 we have provided a different construction of real-analytic Eisenstein series via the Poincaré bundle. The aim of this section is to provide a new construction of Katz' two variable p-adic measure via our approach. We will proceed as follows: for every elliptic curve we have trivializations

$$e^*\mathcal{L}_n \overset{\sim}{\to} \mathcal{O}_{\mathrm{Inf}_e^n E^\vee}.$$

of the geometric logarithm sheaves along e. For elliptic curves over p-adic rings with ordinary reduction we will extend this splitting to infinitesimal neighbourhoods of the zero section. This gives us an infinitesimal trivialization

$$\varprojlim_n \mathcal{L}_n|_{\hat{E}} \overset{\sim}{\to} \mathcal{O}_{\widehat{E\times_S E^\vee}}.$$

If one further assumes the existence of a rigidification on E, i. e. an isomorphism $\widehat{\mathbb{G}}_{m,S} \overset{\sim}{\to} \hat{E}$, one can use this isomorphism to construct p-adic theta functions associated with translates of the canonical section. In the main theorem of this chapter we will show that the Amice transform of this p-adic theta function gives rise to a variant of Katz' two variable p-adic measure interpolating real-analytic Eisenstein series p-adically.

This approach is motivated by the construction of a two-variable p-adic L-function for CM-elliptic curves by Bannai and Kobayashi [BK10b]. They are using Mumford's theory of algebraic theta functions to deduce algebraicity results for the coefficients. This allows them to construct a p-adic measure via the Kronecker theta function. Our result can be seen as a generalization of this to arbitrary families of ordinary elliptic curves. The infinitesimal trivialization of the Poincaré bundle which is used in our construction is motivated by the work [Nor86] of Norman. While his theory does not verbatim apply to our situation the essential idea goes back to his construction of p-adic theta functions in the ordinary case.

As always we assume our base scheme to be separated and locally Noetherian.

4.1 Trivializing the geometric logarithm sheaf along finite subgroups

Let E/S be an elliptic curve and $i_C : C \hookrightarrow E$ a finite subgroup scheme over S. Let us consider the isogeny

$$\varphi : E \to E/C =: E'$$

and assume that the dual isogeny $\varphi^\vee : (E')^\vee \to E^\vee$ is étale. The aim of this section is to construct an isomorphism of $\mathcal{O}_C$-modules

$$\mathcal{L}_n|_C \xrightarrow{\sim} (\mathrm{pr}_C)_* \mathcal{O}_{C \times_S \mathrm{Inf}^n_e E^\vee}.$$

Let $E_C := E \times_S C$. The diagonal $\Delta_C : C \to E_C$ induces a canonical C-valued point of $\ker\left(\varphi_C : E_C \stackrel{\varphi \times \mathrm{id}_C}{\longrightarrow} E'_C\right)$:

$$\begin{tikzcd} E_C \arrow[r] \arrow[d] & E \arrow[d] \\ C \arrow[u, bend left, "\Delta_C"] \arrow[ur, "i_C"] \arrow[r] & S \end{tikzcd}$$

Let us write $\iota_n : \mathrm{Inf}^n_e E^\vee \hookrightarrow E^\vee$. The composition

$$C \times_S \mathrm{Inf}^n_e E^\vee \xrightarrow{\Delta_C \times \iota_n} C \times_S E \times_S E^\vee \xrightarrow{\mathrm{pr}_{E \times E^\vee}} E \times_S E^\vee$$

coincides with $i_C \times \iota_n$. Thus, we have an isomorphism of $\mathcal{O}_S$-modules

$$\begin{aligned} \mathcal{L}_n|_C := i_C^* \mathcal{L}_n = (\mathrm{pr}_C)_*(i_C \times \iota_n)^* \mathcal{P}_E = (\mathrm{pr}_C)_*(\Delta_C \times \iota_n)^* \mathrm{pr}^*_{E \times E^\vee} \mathcal{P}_E \stackrel{(*)}{\cong} \\ \cong (\mathrm{pr}_C)_*(\Delta_C \times \iota_n)^* \mathcal{P}_{E_C} = \Delta_C^* \mathcal{L}_{n,E_C} \end{aligned}$$

where we have used the compatibility of the Poincaré bundle with base change in (*). Composing this morphism with the morphism

$$\mathrm{triv}_{\Delta_C} : \Delta_C^* \mathcal{L}_{n,E_C} \xrightarrow[\sim]{\Delta_C^* \Phi_\varphi} \Delta_C^* \varphi_C^* \mathcal{L}_{n,E'_C} = e^* \mathcal{L}_{n,E'_C} \xleftarrow[\sim]{e^* \Phi_\varphi} e^* \mathcal{L}_{n,E_C} \cong (\mathrm{pr}_C)_* \mathcal{O}_{C \times_S \mathrm{Inf}^n_e E^\vee}$$

(cf. Section 2.1.3) gives our trivialization along C

$$\mathrm{triv}_C : \mathcal{L}_n|_C \cong \Delta_C^* \mathcal{L}_{n,E_C} \stackrel{\mathrm{triv}_{\Delta_C}}{\longrightarrow} (\mathrm{pr}_C)_* \mathcal{O}_{C \times_S \mathrm{Inf}^n_e E^\vee}.$$

Here, Φ_φ is an isomorphism since we have assumed $\varphi^\vee$ being étale, cf. Lemma 2.1.9.

4.2 The infinitesimal splitting and p-adic theta functions

Let p be a fixed prime. In this section we apply the construction of the last section to the subgroup given by the connected component C_n of the subgroup scheme $E[p^n]$ for an elliptic curve $E/\operatorname{Spec} R$ with ordinary reduction over a p-adic ring R. Finally, this will give trivializations of $\mathcal{L}_n$ along the formal completion of E along torsion sections.

Let R be a p-adic ring, i. e. R is complete and separated in its p-adic topology. An elliptic curve E/S with $S = \operatorname{Spec} R$ will be said to have ordinary reduction if $E \times_S \operatorname{Spec} R/pR$ is fiber-wise an ordinary elliptic curve. Let C_n be the connected component of $E[p^n]$. We define

$$\varphi_n : E \twoheadrightarrow E/C_n$$

and note that its dual is étale since we assumed E/S to have ordinary reduction. Thus, we are in a situation where we can apply the construction of the last section and obtain isomorphisms

$$\mathrm{triv}_{C_m} : \mathcal{L}_n|_{C_m} \xrightarrow{\sim} (\mathrm{pr}_{C_m})_* \mathcal{O}_{C_m \times_S \mathrm{Inf}_e^n E^\vee}.$$

These trivializations along C_m are compatible with the canonical morphisms obtained by restriction along

$$\begin{tikzcd} C_m \arrow[d, hook] \arrow[dr, hook, "i_{C_m}"] & \\ C_{m+1} \arrow[r, hook, "i_{C_{m+1}}"'] & E \end{tikzcd}$$

as well as with the transition maps of the geometric logarithm sheaves. Let $\hat{E}$ be the formal completion of the elliptic curve with respect to the unit section e and

$$\iota_{\hat{E}} : \hat{E} \hookrightarrow E$$

the inclusion of the formal scheme $\hat{E}$. Let us write $\mathcal{L}_n|_{\hat{E}} := (\iota_{\hat{E}})^* \mathcal{L}_n$. The inclusions $C_m \hookrightarrow \hat{E}$ induce an isomorphism of formal schemes $\varinjlim_m C_m \xrightarrow{\sim} \hat{E}$. We define the following isomorphism of $\mathcal{O}_{\hat{E}}$-modules

$$\mathrm{triv}_{\hat{E}} : \mathcal{L}_n|_{\hat{E}} = \varprojlim_m i^*_{C_m} \mathcal{L}_n \xrightarrow[\sim]{\varprojlim \mathrm{triv}_{C_m}} \varprojlim_m (\mathrm{pr}_{C_m})_* \mathcal{O}_{C_m \times_S \mathrm{Inf}_e^n E^\vee} = \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}_e^n E^\vee}.$$

Since $\mathrm{triv}_{\hat{E}}$ is compatible with transition maps, we obtain

$$\mathrm{triv}_{\hat{E}} : \varprojlim_n \mathcal{L}_n|_{\hat{E}} \xrightarrow[\sim]{\varprojlim_n \mathrm{triv}_{\hat{E}}} \mathcal{O}_{\hat{E}} \hat{\otimes}_{\mathcal{O}_S} \mathcal{O}_{\hat{E}^\vee}.$$

For $s \in E[N](S)$ and $(N, p) = 1$ we have isomorphisms

$$T_s^* \mathcal{L}_n \xrightarrow{\sim} \mathcal{L}_n$$

which are compatible with transition maps. Restricting this isomorphism to $\hat{E}$ and applying $\mathrm{triv}_{\hat{E}}$ gives:

$$(T_s^*\mathcal{L}_n)|_{\hat{E}} \xrightarrow{\sim} \mathcal{L}_n|_{\hat{E}} \xrightarrow{\sim} \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}_e^n E^\vee} \tag{4.1}$$

and passing to the limit

$$\varprojlim_n (T_s^*\mathcal{L}_n)|_{\hat{E}} \xrightarrow{\sim} \mathcal{O}_{\hat{E}} \hat{\otimes}_{\mathcal{O}_S} \mathcal{O}_{\hat{E}^\vee}.$$

If we denote by $\hat{E}_s$ the completion of E along the torsion section s and observe that translation by s induces a commutative diagram

$$\begin{array}{ccc} \hat{E}_s & \xhookrightarrow{\iota_{\hat{E}_s}} & E \\ \cong\uparrow & & \uparrow T_s \\ \hat{E} & \xhookrightarrow{\iota_{\hat{E}}} & E, \end{array}$$

we can rewrite the above isomorphism (4.1) as

$$\mathrm{triv}_{\hat{E}_s} : \mathcal{L}_n|_{\hat{E}_s} \xrightarrow{\sim} \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}_e^n E^\vee}.$$

Definition 4.2.1. Let p be fixed and E/S an ordinary elliptic curve over a p-adic ring. Let $N, D > 1$ such that N, D, p are pairwise coprime. For $e \neq s \in E[N](S)$ define

$${}_D\vartheta_s \in \Gamma\left(\hat{E} \times \hat{E}^\vee, \mathcal{O}_{\hat{E}\times_S \hat{E}^\vee} \otimes_{\mathcal{O}_S} \underline{\omega}_{E/S}\right)$$

as the image of

$$\left(l_n^D|_{\hat{E}_s}\right)_{n\geq 0} \in \Gamma\left(\hat{E}_s, \varprojlim_n \mathcal{L}_n|_{\hat{E}_s} \otimes_S \underline{\omega}_{E/S}\right)$$

under

$$\mathrm{triv}_{\hat{E}_s} \otimes \mathrm{id}_{\underline{\omega}} : \varprojlim_n \mathcal{L}_n|_{\hat{E}_s} \otimes_S \underline{\omega}_{E/S} \xrightarrow{\sim} \mathcal{O}_{\hat{E}\times_S \hat{E}^\vee} \otimes_{\mathcal{O}_S} \underline{\omega}_{E/S}$$

We call ${}_D\vartheta_s$ the p-adic theta function associated with $U_s^N(s_{\mathrm{can}}^D)$.

Remark 4.2.2. In the first chapter we constructed real-analytic Eisenstein series via the Poincaré bundle. Later, we reformulated the construction 'infinitesimally' along $E \times e \hookrightarrow E \times E^\vee$ in terms of the geometric logarithm sheaves. In this chapter we again had the choice of formulating results either 'globally' or 'infinitesimally'. We decided to present only the 'infinitesimal' formulation of all constructions since this formulation has the advantage of fitting better to our later application on the rigid syntomic realization of the elliptic polylogarithm. The global formulation has the advantage of being more symmetric. Thus, let us at least mention that what we did above is actually a construction of an infinitesimal trivialization of the Poincaré bundle for ordinary elliptic curves

$$\mathcal{P}|_{\widehat{E\times E^\vee}} \xrightarrow{\sim} \mathcal{O}_{\widehat{E\times E^\vee}}.$$

This trivialization allows us to associate a function on the formal completion of $E \times E^\vee$ to sections of the Poincaré bundle. From this point of view, the name p-adic theta function is justified for the above construction. In this sense ${}_D\vartheta_t$ can be seen as the theta function associated with the section $U_s^N(s_{\mathrm{can}}^D)$.

Let us close this section with some results about the compatibility of the infinitesimal trivialization of $\mathcal{L}_n$ with Φ_φ where φ is the quotient isogeny associated with the canonical subgroup.

Lemma 4.2.3. *Let $\varphi : E \to E/C_1 =: E'$ the quotient map by the canonical subgroup. Let $\hat{\varphi}$ be the induced map on the formal groups and let us denote by $\varphi_n^\vee : \mathrm{Inf}_e^n(E')^\vee \to \mathrm{Inf}_e^n E^\vee$ the restriction of $\varphi^\vee$ to the n-th infinitesimal neighbourhood. Then, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{L}_n|_{\hat{E}} \arrow[r, "\Phi_\varphi|_{\hat{E}}"] \arrow[d, "\mathrm{triv}_{\hat{E}}"] & (\varphi^*\mathcal{L}_{n,E'})|_{\hat{E}} \arrow[d, "\hat{\varphi}^*\mathrm{triv}_{\hat{E}'}"] \\
\mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}_e^n E^\vee} \arrow[r, "\mathrm{can}\otimes(\varphi_n^\vee)^\#"] & \hat{\varphi}^*\mathcal{O}_{\hat{E}'} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}_e^n(E')^\vee}
\end{tikzcd}$$

Proof. Let C'_{m-1} be the connected component of $E'[p^{m-1}]$ and $\varphi_{C'_{m-1}} : E' \to E'/C'_{m-1}$. Since the image of C_m under φ is C'_{m-1}, we can identify the composition

$$E \xrightarrow{\varphi} E/C_1 = E' \xrightarrow{\varphi_{C'_{m-1}}} E'/C'_{m-1} = E/C_m$$

with the isogeny corresponding to the quotient $\varphi_{C_m} : E \to E/C_m$. By Lemma 2.1.9 we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{L}_{n,E} \arrow[r, "\Phi_\varphi"] \arrow[d, "\Phi_{\varphi_{C_m}}"] & \varphi^*\mathcal{L}_{n,E'} \arrow[d, "\varphi^*\Phi_{\varphi_{C'_{m-1}}}"] \\
\varphi_{C_m}^*\mathcal{L}_{n,(E/C_m)} \arrow[r, equal] & \varphi^*\varphi_{C'_{m-1}}^*\mathcal{L}_{n,(E/C_m)}.
\end{tikzcd}$$

This proves the commutativity of the left upper two squares in the following diagram:

$$\begin{tikzcd}
\Delta_{C_m}^*\mathcal{L}_{n,E_{C_m}} \arrow[r, "\Delta_{C_m}^*\Phi_\varphi"] \arrow[d, "\Delta_{C_m}^*\Phi_{\varphi_{C_m}}"] & \Delta_{C_m}^*\varphi^*\mathcal{L}_{n,E'_{C_m}} \arrow[r, "\sim"] \arrow[d, "\Delta_{C_m}^*\varphi^*\Phi_{\varphi_{C'_{m-1}}}"] & (\varphi|_{C_m})^*\Delta_{C'_{m-1}}^*\mathcal{L}_{n,(E')_{C'_{m-1}}} \arrow[d, "(\varphi|_{C_m})^*\Delta_{C'_{m-1}}^*\Phi_{\varphi_{C'_{m-1}}}"] \\
e^*\mathcal{L}_{n,(E/C_m)_{C_m}} \arrow[r, equal] & e^*\mathcal{L}_{n,(E/C_m)_{C_m}} \arrow[r, "\sim"] & (\varphi|_{C_m})^*e^*\mathcal{L}_{n,(E/C_m)_{C'_{m-1}}} \\
e^*\mathcal{L}_{n,E_{C_m}} \arrow[r, "e^*\Phi_\varphi"] \arrow[u, "e^*\Phi_{\varphi_{C_m}}"] & e^*\mathcal{L}_{n,E'_{C_m}} \arrow[r, "\sim"] \arrow[u, "e^*\varphi^*\Phi_{\varphi_{C'_{m-1}}}"] & (\varphi|_{C_m})^*e^*\mathcal{L}_{n,(E')_{C'_{m-1}}} \arrow[u, "(\varphi|_{C_m})^*e^*\Phi_{\varphi_{C'_{m-1}}}"] \\
\mathcal{O}_{C_m} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}_e^n E^\vee} \arrow[r, "\mathrm{id}\otimes(\varphi_n^\vee)^\#"] \arrow[u, "\cong"] & \mathcal{O}_{C_m} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}_e^n(E')^\vee} \arrow[r, "\mathrm{can}\otimes\mathrm{id}"] \arrow[u, "\cong"] & (\varphi|_{C_m})^*\mathcal{O}_{C'_{m-1}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}_e^n(E')^\vee} \arrow[u, "\cong"']
\end{tikzcd}$$

The commutativity of the lower left square is Lemma 2.1.9 (b). The squares in the right column of the above diagram commute by the compatibility of the geometric logarithm sheaves and the maps Φ_φ with base change. Indeed, let us for example consider the upper right square in the above diagram. The Cartesian diagram

$$\begin{tikzcd} E'_{C_m} \arrow[r, "\mathrm{id}\times(\varphi|_{C_m})"] \arrow[d] & E'_{C'_{m-1}} \arrow[d] \\ C'_m \arrow[r, "(\varphi|_{C_m})"] & C'_{m-1} \end{tikzcd}$$

induces a canonical isomorphism

$$\mathcal{L}_{n,E'_{C_m}} \xrightarrow{\sim} (\mathrm{id}\times(\varphi|_{C_m}))^*\mathcal{L}_{n,E'_{C'_{m-1}}}.$$

The compatibility of $\Phi_{\varphi_{C'_{m-1}}}$ with base change can be expressed as the commutative diagram

$$\begin{tikzcd} \mathcal{L}_{n,E'_{C_m}} \arrow[r, "\sim"] \arrow[d, "\Phi_{\varphi_{C'_{m-1}}}"] & (\mathrm{id}\times(\varphi|_{C_m}))^*\mathcal{L}_{n,E'_{C'_{m-1}}} \arrow[d, "(\mathrm{id}\times(\varphi|_{C_m}))^*\Phi_{\varphi_{C'_{m-1}}}"] \\ \varphi^*_{C'_{m-1}}\mathcal{L}_{n,(E/C_m)_{C_m}} \arrow[r, "\sim"] & (\mathrm{id}\times(\varphi|_{C_m}))^*\varphi^*_{C'_{m-1}}\mathcal{L}_{n,(E/C_m)_{C'_{m-1}}}. \end{tikzcd}$$

If we pullback this diagram along $(\Delta_{C_m})^*\varphi^*$ and use the commutativity of

$$\begin{tikzcd} C_m \arrow[r, "\Delta_{C_m}"] \arrow[dd, "\varphi|_{C_m}"] & E_{C_m} \arrow[d, "\varphi"] \\ & E'\times C_m = E'_{C_m} \arrow[d, "\mathrm{id}\times\varphi|_{C_m}"] \\ C'_{m-1} \arrow[r, "\Delta_{C'_{m-1}}"] & E'\times C'_{m-1} = E'_{C'_{m-1}}, \end{tikzcd}$$

we obtain the upper right square of the above big diagram. The commutativity of the remaining squares follows in a similar way.

The left vertical composition in the above big diagram yields the isomorphism

$$\mathrm{triv}_{\Delta_{C_m}}:\Delta^*_{C_m}\mathcal{L}_{n,E_{C_m}} \xrightarrow{\sim} \mathcal{O}_{C_m}\otimes_{\mathcal{O}_S}\mathcal{O}_{\mathrm{Inf}^n_e E^\vee},$$

while the vertical right composition gives $(\varphi|_{C_m})^*\mathrm{triv}_{\Delta_{C'_{m-1}}}$. Thus, we can rewrite the above big diagram as

$$\begin{tikzcd} \mathcal{L}_{n,E}|_{C_m} \arrow[r, "(\Phi_\varphi)|_{C_m}"] \arrow[d, "\mathrm{triv}_{C_m}"] & \left(\varphi^*\mathcal{L}_{n,E'}\right)|_{C_m} \arrow[r, "\sim"] & (\varphi|_{C_n})^*\left(\mathcal{L}_{n,E'}|_{C'_{m-1}}\right) \arrow[d, "(\varphi|_{C_n})^*\mathrm{triv}_{C'_{m-1}}"] \\ \mathcal{O}_{C_m}\otimes_{\mathcal{O}_S}\mathcal{O}_{\mathrm{Inf}^n_e E^\vee} \arrow[rr, "\mathrm{can}\otimes(\varphi_n^\vee)^\#"] & & (\varphi|_{C_m})^*\mathcal{O}_{C_m}\otimes_{\mathcal{O}_S}\mathcal{O}_{\mathrm{Inf}^n_e E^\vee}. \end{tikzcd}$$

One easily checks compatibility with restriction $C_m \hookrightarrow C_{m+1}$ and passing to the limit gives the desired diagram

$$\begin{tikzcd}
\mathcal{L}_{n,E}|_{\hat{E}} \arrow[r, "(\Phi_\varphi)|_{\hat{E}}"] \arrow[d, "\mathrm{triv}_{\hat{E}}"] & (\varphi^*\mathcal{L}_{n,E'})\,|_{\hat{E}} \arrow[r, "\sim"] & (\varphi|_{\hat{E}})^*\left(\mathcal{L}_{n,E'}|_{\hat{E}'}\right) \arrow[d, "(\varphi|_{\hat{E}})^*\mathrm{triv}_{\hat{E}'}"] \\
\mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}^n_e E^\vee} \arrow[rr, "\mathrm{can}\otimes(\varphi_n^\vee)^\#"] & & (\varphi|_{\hat{E}})^*\mathcal{O}_{\hat{E}'} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}^n_e E^\vee}.
\end{tikzcd}$$

□

Finally, let us remark that we cannot directly trivialize $\mathcal{L}_n^\dagger$ via the same construction. The reason is that $\Phi_\varphi^\dagger$ is not an isomorphism for $\varphi : E \to E/C_n$ since $\varphi : E \to E/C_n$ is not étale for $n \geq 1$. But we already know that $\mathcal{L}_1^\dagger$ is the pushout of $\mathcal{L}_1$. This allows us at least to construct an infinitesimal trivialization of the first geometric logarithm sheaf

$$\mathrm{triv}^\dagger_{\hat{E}} : \mathcal{L}_1^\dagger\Big|_{\hat{E}} \xrightarrow{\sim} \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}^1_e E^\dagger}$$

via pushout. This, combined with the canonical map

$$\mathbb{D}^\dagger : \mathcal{L}_n^\dagger \to \underline{\mathrm{TSym}}^n\, \mathcal{L}_1^\dagger\,,$$

is often enough for our purposes. As immediate corollary of the above lemma we obtain:

Corollary 4.2.4. *Let $\varphi : E \to E/C_1 =: E'$ be the quotient map by the canonical subgroup. The map φ induces morphisms $\hat{\varphi} : \hat{E} \to \hat{E}'$ and $\varphi^\dagger : (E')^\dagger \to E^\dagger$. Let us denote by $\varphi_1^\dagger : \mathrm{Inf}^1_e(E')^\dagger \to \mathrm{Inf}^1_e E^\dagger$ the restriction of $\varphi^\dagger$ to the first infinitesimal neighbourhood. Then, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{L}_1^\dagger|_{\hat{E}} \arrow[r, "\Phi_\varphi^\dagger|_{\hat{E}}"] \arrow[d, "\mathrm{triv}^\dagger_{\hat{E}}"] & \left(\varphi^*\mathcal{L}^\dagger_{1,E'}\right)\Big|_{\hat{E}} \arrow[d, "\hat{\varphi}^*\mathrm{triv}_{\hat{E}'}"] \\
\mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}^1_e E^\dagger} \arrow[r, "\mathrm{can}\otimes(\varphi_1^\vee)^\#"] & \hat{\varphi}^*\mathcal{O}_{\hat{E}'} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}^1_e(E')^\dagger}
\end{tikzcd}$$

Proof. This follows immediately from Lemma 4.2.3. □

We define the isomorphism

$$\hat{\mathrm{split}} : (\underline{\mathrm{TSym}}^n_{\mathcal{O}_E}\, \mathcal{L}_1)|_{\hat{E}} \to \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \left(\bigoplus_{k=0}^{n} \underline{\mathrm{TSym}}^k_{\mathcal{O}_S}\, \underline{\omega}_{E^\vee/S}\right)$$

as the composition of

$$(\underline{\mathrm{TSym}}^n_{\mathcal{O}_E}\, \mathcal{L}_1)\Big|_{\hat{E}} \xrightarrow{\mathrm{triv}_{\hat{E}}} \underline{\mathrm{TSym}}^n_{\mathcal{O}_{\hat{E}}} \left(\mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \mathcal{O}_{\mathrm{Inf}^1_e\, \underline{\omega}_{E^\vee/S}}\right)$$

with the canonical isomorphisms:

$$\begin{aligned}\mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \underline{\mathrm{TSym}}^n_{\mathcal{O}_S} \left(\mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S}\right) = \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \underline{\mathrm{TSym}}^n_{\mathcal{O}_S} \left(\mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S}\right) \cong \\ \cong \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \left(\bigoplus_{k=0}^n \underline{\mathrm{TSym}}^k_{\mathcal{O}_S} \underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_S} \underline{\mathrm{TSym}}^{n-k}\, \mathcal{O}_S\right) \cong \\ \cong \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \left(\bigoplus_{k=0}^n \underline{\mathrm{TSym}}^k_{\mathcal{O}_S} \underline{\omega}_{E^\vee/S}\right).\end{aligned}$$

We can define in the same way the isomorphism

$$\hat{\mathrm{split}}^\dagger : (\underline{\mathrm{TSym}}^n_{\mathcal{O}_E} \mathcal{L}_1^\dagger)|_{\hat{E}} \to \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_S} \left(\bigoplus_{k=0}^n \underline{\mathrm{TSym}}^k_{\mathcal{O}_S} \mathcal{H}\right). \tag{4.2}$$

4.3 The infinitesimal splitting for the universal trivialized elliptic curve

Let $E/S = \operatorname{Spec} R$ be an elliptic curve over a p-adic ring R. A trivialization of E is an isomorphism

$$\beta : \hat{E} \xrightarrow{\sim} \widehat{\mathbb{G}}_{m,S}$$

of formal groups over R. For $N \geq 1$ a natural number coprime to p, a trivialized elliptic curve with $\Gamma(N)$-level structure is a triple (E, β, α_N) consisting of an elliptic curve E/S a rigidification β and a level structure $\alpha_N : (\mathbb{Z}/N\mathbb{Z})^2_S \xrightarrow{\sim} E[N]$. Let $(E^{\mathrm{triv}}, \beta, \alpha_N)$ be the universal trivialized elliptic curve with $\Gamma(N)$-level structure over $M^{\mathrm{triv}} = \operatorname{Spec} V(\mathbb{Z}_p, \Gamma(N))$. For more details we refer to [Kat76, Ch. V]. The ring $V(\mathbb{Z}_p, \Gamma(N))$ will be called ring of generalized p-adic modular forms.

4.3.1 The unit root decomposition

Let us recall the definition of the unit root decomposition. Dividing E^{triv} by its canonical subgroup C, again gives a trivialized elliptic curve

$$(E' = E^{\mathrm{triv}}/C, \beta', \alpha'_N)$$

with $\Gamma(N)$-level structure over $\operatorname{Spec} V(\mathbb{Z}_p, \Gamma(N))$. The corresponding morphism

$$\mathrm{Frob} : V(\mathbb{Z}_p, \Gamma(N)) \to V(\mathbb{Z}_p, \Gamma(N))$$

classifying this quotient will be called *Frobenius morphism* of $V(\mathbb{Z}_p, \Gamma(N))$. In particular, the quotient map $E^{\mathrm{triv}} \to E' = E^{\mathrm{triv}}/C$ induces a Frob-linear map

$$F : \mathrm{Frob}^* \underline{H}^1_{\mathrm{dR}}\left(E^{\mathrm{triv}}/M^{\mathrm{triv}}\right) = \underline{H}^1_{\mathrm{dR}}\left(E'/M^{\mathrm{triv}}\right) \to \underline{H}^1_{\mathrm{dR}}\left(E^{\mathrm{triv}}/M^{\mathrm{triv}}\right)$$

which is easily seen to respect the Hodge filtration

$$0 \longrightarrow \underline{\omega}_{E^{\mathrm{triv}}/M^{\mathrm{triv}}} \longrightarrow \underline{H}^1_{\mathrm{dR}}\left(E^{\mathrm{triv}}/M^{\mathrm{triv}}\right) \longrightarrow \underline{\omega}^\vee_{(E^{\mathrm{triv}})^\vee/M^{\mathrm{triv}}} \longrightarrow 0.$$

Further, the induced Frob-linear endomorphism of $\underline{\omega}^\vee_{(E^{\mathrm{triv}})^\vee/M^{\mathrm{triv}}}$ is bijective while the induced Frob-linear map on $\underline{\omega}_{E^{\mathrm{triv}}/M^{\mathrm{triv}}}$ is divisible by p. This induces a decomposition

$$\underline{H}^1_{\mathrm{dR}}\left(E^{\mathrm{triv}}/M^{\mathrm{triv}}\right) = \underline{\omega}_{E^{\mathrm{triv}}/M^{\mathrm{triv}}} \oplus U \tag{4.3}$$

where $U \subseteq \underline{H}^1_{\mathrm{dR}}\left(E^{\mathrm{triv}}/M^{\mathrm{triv}}\right)$ is the unique F-invariant $\mathcal{O}_{M^{\mathrm{triv}}}$-submodule on which F is invertible. U is called the unit root space and (4.3) is called unit root decomposition. The unit root decomposition induces via the Deligne pairing

$$\underline{H}^1_{\mathrm{dR}}\left((E^{\mathrm{triv}})^\vee/M^{\mathrm{triv}}\right) \times \underline{H}^1_{\mathrm{dR}}\left(E^{\mathrm{triv}}/M^{\mathrm{triv}}\right) \to \mathcal{O}_{M^{\mathrm{triv}}}$$

a decomposition

$$\mathcal{H} = \underline{H}^1_{\mathrm{dR}}\left((E^{\mathrm{triv}})^\vee/M^{\mathrm{triv}}\right) \xrightarrow{\sim} \underline{\omega}_{(E^{\mathrm{triv}})^\vee/M^{\mathrm{triv}}} \oplus V$$

where $\underline{\omega}_{(E^{\mathrm{triv}})^\vee/M^{\mathrm{triv}}} \xrightarrow{\sim} U^\vee$ and $V \xrightarrow{\sim} \underline{\omega}^\vee_{E^{\mathrm{triv}}/M^{\mathrm{triv}}}$. Note that the dual of the Frobenius

$$F^\vee : \mathcal{H} \to \mathrm{Frob}^*\mathcal{H}$$

respects the above decomposition. It is invertible on $\underline{\omega}_{(E^{\mathrm{triv}})^\vee/M^{\mathrm{triv}}}$ and divisible by p on the subspace V.

4.3.2 The Frobenius lift

Let $\varphi : E^{\mathrm{triv}} \to E' = E^{\mathrm{triv}}/C$ be the quotient of the universal trivialized elliptic curve by its canonical subgroup. We have the following commutative diagram

$$\begin{tikzcd} E^{\mathrm{triv}} \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & E' \arrow[r, "\widetilde{\mathrm{Frob}}"] \arrow[d, "\pi_{E'}"] \arrow[dr, phantom, "\lrcorner"] & E^{\mathrm{triv}} \arrow[d, "\pi"] \\ & M^{\mathrm{triv}} \arrow[r, "\mathrm{Frob}"] & M^{\mathrm{triv}} \end{tikzcd} \tag{4.4}$$

Let us define

$$\phi := \widetilde{\mathrm{Frob}} \circ \varphi : E^{\mathrm{triv}} \to E^{\mathrm{triv}}.$$

Then, ϕ gives us a canonical lift of the absolute Frobenius morphism. We can define a canonical $\mathcal{O}_{E^{\mathrm{triv}}}$-linear map

$$\Psi^\dagger : \mathcal{L}^\dagger_n \to \phi^*\mathcal{L}^\dagger_n$$

as follows. We have a canonical isomorphism $\widetilde{\mathrm{Frob}}^*\mathcal{L}^\dagger_{n,E^{\mathrm{triv}}} \cong \mathcal{L}^\dagger_{n,E'}$ by the Cartesian diagram in (4.4) and the compatibility of the geometric logarithm sheaves with base change. Combining this with the morphism

$$\Phi_\varphi : \mathcal{L}^\dagger_n \to \varphi^*\mathcal{L}^\dagger_{n,E'}$$

gives

$$\Psi^\dagger : \mathcal{L}_n^\dagger \to \varphi^*\mathcal{L}_{n,E'}^\dagger \cong \varphi^*\widetilde{\mathrm{Frob}}^*\mathcal{L}_n^\dagger = \phi^*\mathcal{L}_n^\dagger.$$

Since both maps $\widetilde{\mathrm{Frob}}^*\mathcal{L}_{n,E^{\mathrm{triv}}}^\dagger \cong \mathcal{L}_{n,E'}^\dagger$ and Φ_φ are horizontal, we deduce that $\Psi^\dagger$ is horizontal for the canonical M^{triv}-connections on both sides. In the same way we obtain

$$\Psi : \mathcal{L}_n \to \phi^*\mathcal{L}_n.$$

The map Ψ is invertible since Φ_φ is an isomorphism. Let us define

$$\Phi : \phi^*\mathcal{L}_n \to \mathcal{L}_n \tag{4.5}$$

as the inverse of Ψ. The map $\Psi^\dagger : \mathcal{L}_n^\dagger \to \phi^*\mathcal{L}_n^\dagger$ will be important for the p-adic realization since it induces a canonical Frobenius structure on $\mathcal{L}_n^\dagger$.

4.3.3 Tensor symmetric powers of the geometric logarithm sheaves

We have already observed that we cannot trivialize $\mathcal{L}_n^\dagger$ along $\hat{E}$ in the same way as $\mathcal{L}_n$ since $\varphi : E^{\mathrm{triv}} \to E'$ is not étale. But the fact that $\mathcal{L}_1^\dagger$ is the pushout of $\mathcal{L}_1$ allows us to define a trivialization of $\mathcal{L}_1^\dagger$. Since $V(\mathbb{Z}_p, \Gamma(N))$ is flat over $\mathbb{Z}_p$, the canonical map

$$\mathcal{L}_n^\dagger \to \underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger$$

is injective. And we can split $\underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger$ via the isomorphism

$$\hat{\mathrm{split}}^\dagger : (\underline{\mathrm{TSym}}^n_{\mathcal{O}_E}\,\mathcal{L}_1^\dagger)|_{\hat{E}} \to \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_{M^{\mathrm{triv}}}} \left(\bigoplus_{k=0}^n \underline{\mathrm{TSym}}^k_{\mathcal{O}_{M^{\mathrm{triv}}}}\,\mathcal{H}\right).$$

defined in (4.2). The above injection combined with this splitting is sufficient to describe many constructions of the geometric logarithm sheaves on the universal trivialized elliptic curve explicitly. The trivialization β gives us a canonical generator $\omega := \beta^*\left(\frac{dT}{(1+T)}\right)$ of $\underline{\omega}_{E^{\mathrm{triv}}/M^{\mathrm{triv}}}$. Let us define a generator $u \in U$ by the normalization $\langle \omega, u\rangle = 1$. The dual basis $u^\vee, \omega^\vee$ gives us generators of $\mathcal{H} = \underline{H}^1_{\mathrm{dR}}\left(E^{\mathrm{triv}}/M^{\mathrm{triv}}\right)^\vee$. Let us observe that $[u]^\vee$ generates the submodule $\underline{\omega}_{(E^{\mathrm{triv}})^\vee/M^{\mathrm{triv}}}$, while $[\omega]^\vee$ generates the submodule V of $\mathcal{H}$. Using the basis $[u]^\vee, [\omega]^\vee$ of $\mathcal{H}$, we define

$$\hat{\omega}^{[k,l]} := (\hat{\mathrm{split}}^\dagger)^{-1}\left[([u]^\vee)^{[k]} \cdot ([\omega]^\vee)^{[l]}\right] \quad k + l \le n.$$

Then, $(\hat{\omega}^{[k,l]})_{k+l\le n}$ gives us a basis for $\left(\underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger\right)|_{\hat{E}}$. We have the following explicit description of the Frobenius structure on $\mathcal{L}_n^\dagger$ under the canonical inclusion:

Lemma 4.3.1. *The $\mathcal{O}_{E^{\mathrm{triv}}}$-linear map*

$$\left(\underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger\right)|_{\hat{E}} \to \phi^*\left(\underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger\right)|_{\hat{E}}$$

induced by $\Psi : \mathcal{L}_1^\dagger \to \phi^*\mathcal{L}_1^\dagger$ is explicitly given by

$$\hat{\omega}^{[k,l]} \mapsto p^l \cdot \phi^*\left(\hat{\omega}^{[k,l]}\right).$$

Proof. Since the map in the statement is induced from the case $n=1$ by applying $\underline{\mathrm{TSym}}$, it is enough to prove the statement in the case $n=1$. The generators $\hat{\omega}^{[0,0]}$,$\hat{\omega}^{[1,0]}=[u]^\vee$ and $\hat{\omega}^{[0,1]}=[\omega]^\vee$ are induced via the infinitesimal trivialization

$$\underline{\mathrm{TSym}}^1\,\mathcal{L}_1^\dagger|_{\hat{E}}=\mathcal{L}_1^\dagger|_{\hat{E}}\xrightarrow{\sim}\mathcal{O}_{\hat{E}}\oplus\pi_{\hat{E}}^*\mathcal{H}.$$

It follows from Corollary 4.2.4 that Ψ maps $\hat{\omega}^{[0,0]}$ to $\phi^*(\hat{\omega}^{[0,0]})$ and coincides on $\pi_{\hat{E}}^*\mathcal{H}$ with the dual of the Frobenius:

$$\pi_{\hat{E}}^*(F^\vee):\pi_{\hat{E}}^*\mathcal{H}\to\pi_{\hat{E}}^*\mathrm{Frob}^*\mathcal{H}=\phi^*\mathcal{H}.$$

This map sends $[u]^\vee$ to $\phi^*([u]^\vee)$ and $[\omega]^\vee$ to $p\cdot\phi^*([\omega]^\vee)$, which can be deduced as in [BK10a, §4.3, p.22]. □

Let us now give an explicit description of the connection on $\underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger|_{\hat{E}}$ via the infinitesimal splitting:

Lemma 4.3.2. *Let $\nabla_{\underline{\mathrm{TSym}}}$ be the M^{triv}-connection on $\underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger$ induced from the M^{triv}-connection on $\mathcal{L}_1^\dagger$. Then,*

$$\nabla_{\underline{\mathrm{TSym}}}\left(\hat{\omega}^{[k,l]}\right)=(l+1)\hat{\omega}^{[k,l+1]}\otimes\omega\in\Gamma(\hat{E},\underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger|_{\hat{E}},\otimes_{\mathcal{O}_{\hat{E}}}\hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}).$$

Proof. For simplicity, let us write E for E^{triv} during the proof. Let us first prove the following:
Claim: Let $s:\mathcal{L}_1|_{\hat{E}}\xrightarrow{\sim}\mathcal{O}_{\hat{E}}\oplus\pi_{\hat{E}}^*\underline{\omega}_{E^\vee/M^{\mathrm{triv}}}$ be any splitting of the short exact sequence

$$0\longrightarrow\pi_{\hat{E}}^*\underline{\omega}_{E^\vee/M^{\mathrm{triv}}}\longrightarrow\mathcal{L}_1|_{\hat{E}}\longrightarrow\mathcal{O}_{\hat{E}}\longrightarrow 0$$

and define $s^\dagger:\mathcal{L}_1^\dagger|_{\hat{E}}\xrightarrow{\sim}\mathcal{O}_{\hat{E}}\oplus\pi_{\hat{E}}^*\mathcal{H}$ via pushout. Then,

$$\begin{aligned}\mathcal{O}_{\hat{E}}\oplus\pi_{\hat{E}}^*\mathcal{H}\xrightarrow{\sim}\mathcal{L}_1^\dagger|_{\hat{E}}\xrightarrow{\nabla|_{\hat{E}}}\mathcal{L}_1^\dagger|_{\hat{E}}\otimes\hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}\xrightarrow{\sim}\left(\mathcal{O}_{\hat{E}}\oplus\pi_{\hat{E}}^*\mathcal{H}\right)\otimes\hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}\overset{\mathrm{pr}}{\twoheadrightarrow}\\ \overset{\mathrm{pr}}{\twoheadrightarrow}\left(\mathcal{O}_{\hat{E}}\oplus\pi_{\hat{E}}^*\underline{\omega}^\vee_{E/M^{\mathrm{triv}}}\right)\otimes\hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}\xrightarrow{\sim}\hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}\oplus\left(\hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}\right)^\vee\otimes\hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}\end{aligned}$$

does not depend on the chosen splitting. Furthermore, this map is explicitly given by

$$(1,0)\mapsto\left(0,\mathrm{id}_{\hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}}\right).$$

Let us first prove this claim. For independence of s, let s_1 and s_2 be two such splittings. Since $s_1^\dagger$ and $s_2^\dagger$ are defined via pushout and since we have a canonical decomposition

$\mathcal{H} \cong \underline{\omega}_{E^\vee/M^{\mathrm{triv}}} \oplus V$, we get a commutative diagram

$$\begin{tikzcd}[column sep=small] & \mathcal{O}_{\hat{E}} \oplus \pi_{\hat{E}}^* \underline{\omega}_{E^\vee/M^{\mathrm{triv}}} \oplus \pi_{\hat{E}}^* V \arrow[dd, "\cdot \begin{pmatrix} \mathrm{id} & 0 & 0 \\ f & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}"] \\ \mathcal{L}_1^\dagger|_{\hat{E}} = \mathcal{L}_1|_{\hat{E}} \oplus \pi_{\hat{E}}^* V \arrow[ur, "s_1^\dagger"] \arrow[dr, "s_2^\dagger"'] & \\ & \mathcal{O}_{\hat{E}} \oplus \pi_{\hat{E}}^* \underline{\omega}_{E^\vee/M^{\mathrm{triv}}} \oplus \pi_{\hat{E}}^* V \end{tikzcd}$$

for some $f \in \underline{\mathrm{Hom}}_{\mathcal{O}_{\hat{E}}}(\mathcal{O}_{\hat{E}}, \pi_{\hat{E}}^* \underline{\omega}_{E^\vee/M^{\mathrm{triv}}})$. From this we obtain

$$\begin{aligned} &\nabla_{\mathcal{L}_1^\dagger}\left((s_1^\dagger)^{-1}(1,0,0)\right) - \nabla_{\mathcal{L}_1^\dagger}\left((s_2^\dagger)^{-1}(1,0,0)\right) = \\ &\nabla_{\mathcal{L}_1^\dagger}\left((s_2^\dagger)^{-1}(1,f(1),0)\right) - \nabla_{\mathcal{L}_1^\dagger}\left((s_2^\dagger)^{-1}(1,0,0)\right) = \\ =&\nabla_{\mathcal{L}_1^\dagger}\left(i(f(1))\right) \in \Gamma(\hat{E}, \pi_{\hat{E}}^* \underline{\omega}_{E^\vee/M^{\mathrm{triv}}} \otimes_{\mathcal{O}_{\hat{E}}} \hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}) \end{aligned}$$

where $i : \pi_{\hat{E}}^* \underline{\omega}_{E^\vee/M^{\mathrm{triv}}} \hookrightarrow \pi_{\hat{E}}^* \mathcal{H}$. But this is contained in the kernel of the quotient map

$$\left(\mathcal{O}_{\hat{E}} \oplus \pi_{\hat{E}}^* \mathcal{H}\right) \otimes \hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}} \overset{\mathrm{pr}}{\twoheadrightarrow} \left(\mathcal{O}_{\hat{E}} \oplus \pi_{\hat{E}}^* \left(\underline{\omega}_{E/M^{\mathrm{triv}}}\right)^\vee\right) \otimes \hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}$$

which is induced by the Hodge filtration. This shows the independence of the chosen infinitesimal splitting. It remains to show that the map in the above claim maps $(1,0) \in \mathcal{O}_{\hat{E}} \oplus \pi_{\hat{E}}^* \mathcal{H}$ to $\left(0, \mathrm{id}_{\hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}}\right)$. We can show this after passing to some finite étale cover $T \to M^{\mathrm{triv}}$ of the base. Thus, we may assume, at least after some finite étale base change, that there exists some $t \in U(M^{\mathrm{triv}})$ where $U = E \setminus \{e(M^{\mathrm{triv}})\}$, e.g. take $e \neq t \in E[D](T)$ for D prime to p and T finite étale over M^{triv}. Translation by t induces an isomorphism

$$\hat{E} \xrightarrow{\sim} \hat{E}_t$$

between the completion at e and the completion at t. Using this isomorphism and the Katz splitting[1] for $\mathcal{L}_1$, induces another infinitesimal splitting

$$s_\kappa : \mathcal{L}_1|_{\hat{E}} \xrightarrow{\sim} \mathcal{L}_1|_{\hat{E}_t} = (\mathcal{L}_1|_U)|_{\hat{E}_t} \overset{\kappa|_{\hat{E}_t}}{\xrightarrow{\sim}} \mathcal{O}_{\hat{E}_t} \oplus \pi_{\hat{E}_t}^* \underline{\omega}_{E^\vee/M^{\mathrm{triv}}} \xrightarrow{\sim} \mathcal{O}_{\hat{E}} \oplus \pi_{\hat{E}}^* \underline{\omega}_{E^\vee/M^{\mathrm{triv}}}.$$

For this particular splitting it follows from

$$\nabla_{\mathcal{L}_1^\dagger}(\kappa_{\mathcal{L}}(1,0)) = \mathit{Id}_{\mathcal{H}} = \omega \otimes [\omega]^\vee + \eta \otimes [\eta]^\vee$$

[1] For the construction of the Katz splitting we had to assume that 6 is invertible on the base. So for $p = 2, 3$ a minor modification in the proof is necessary. $V(\mathbb{Z}_p, \Gamma(N))$ is flat over $\mathbb{Z}_p$ thus we can prove the explicit formula in the claim after tensorizing with $\mathbb{Q}_p$ and proceed as above.

that

$$(1,0) \mapsto (0, \omega \otimes \omega^\vee) = (0, \mathrm{id}_{\Omega^1_{\hat{E}/M}})$$

under the map in the claim.

Let us now come to the proof of the lemma. The M^{triv}-connection on $\underline{\mathrm{TSym}}^n \mathcal{L}_1^\dagger$ is induced by the M^{triv}-connection on $\mathcal{L}_1^\dagger$. Thus, we can reduce the proof of the lemma to the case $n = 1$ and we have to prove $\nabla_{\mathcal{L}_1^\dagger}(\hat{\omega}^{[0,0]}) = \hat{\omega}^{[1,0]} \otimes \omega$ and

$$\nabla_{\mathcal{L}_1^\dagger}(\hat{\omega}^{[0,1]}) = 0 = \nabla_{\mathcal{L}_1^\dagger}(\hat{\omega}^{[1,0]}).$$

The last two equalities hold since $\nabla_{\mathcal{L}_1^\dagger}$ coincides with the pullback of the trivial M^{triv}-connection on $\mathcal{H}$. Let us show $\nabla_{\mathcal{L}_1^\dagger}(\hat{\omega}^{[0,0]}) = \hat{\omega}^{[1,0]} \otimes \omega$. We already know that

$$\nabla_{\mathcal{L}_1^\dagger}(\hat{\omega}^{[0,0]}) \in \Gamma(\hat{E}, \pi_{\hat{E}}^* \mathcal{H} \otimes_{\mathcal{O}_{\hat{E}}} \hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}})$$

since the connection and the splitting are both compatible with the extension structure of $\mathcal{L}_1^\dagger$. Thus, we make the following ansatz:

$$\nabla_{\mathcal{L}_1^\dagger}(\hat{\omega}^{[0,0]}) = f_1[u]^\vee \otimes \omega + f_2[\omega]^\vee \otimes \omega$$

for $f_1, f_2 \in \Gamma(\hat{E}, \mathcal{O}_{\hat{E}})$. By the above claim we already know $f_2 = 1$. It remains to show $f_1 = 0$ but this is easily deduced from the fact that $\nabla_{\mathcal{L}_1^\dagger}$ is compatible with the Frobenius structure. The horizontality of

$$\Psi : \mathcal{L}_1^\dagger \to \phi^* \mathcal{L}_1^\dagger$$

expresses, using Corollary 4.2.4, as

$$f_1 \phi^*(\hat{\omega}^{[1,0]}) \otimes \omega + p \cdot \phi^*(\hat{\omega}^{[0,1]}) \otimes \omega = \phi^*(\hat{\omega}^{[1,0]}) \otimes \phi^*(f_1 \omega) + \phi^*(\hat{\omega}^{[0,1]}) \otimes \phi^*(\omega).$$

Using $\phi^*(\omega) = p\omega$, this equation reduces to

$$f_1 \omega = \phi^*(f_1 \omega).$$

If $f_1 \neq 0$, there exists a minimal $r \geq 0$ with $f_1 \in p^r \mathcal{O}_{\hat{E}}$. But on the other hand we have $\phi^*(f_1\omega) \in p^{r+1} \hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}}$. This is in contradiction to the equation $f_1 \omega = \phi^*(f_1 \omega)$ and the minimality of r. Thus, we conclude $f_1 = 0$ and obtain

$$\nabla_{\mathcal{L}_1^\dagger}(\hat{\omega}^{[0,0]}) = \hat{\omega}^{[0,1]} \otimes \omega$$

as desired. □

4.4 Real analytic Eisenstein series as p-adic modular forms

As explained in the introduction to this chapter, Katz defines p-adic variants of real analytic Eisenstein series via the unit root decomposition on the universal trivialized elliptic curve. For simplicity let us write E for E^{triv} and M for M^{triv} in this section. In Section 2.4, we have defined

$$_{D}TE^{k,r+1}_{(a,b)} \in \Gamma\left(M, \underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\,\mathcal{H}\otimes_{\mathcal{O}_M}\underline{\omega}^{\otimes(r+1)}_{E/M}\right).$$

We use these classes and the unit root decomposition in order to construct generalized p-adic modular forms. Let us use our chosen autoduality isomorphism

$$\underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\,\mathcal{H}\otimes_{\mathcal{O}_M}\underline{\omega}^{\otimes(r+1)}_{E/M} \xrightarrow{\sim} \underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\,\underline{H}^1_{\mathrm{dR}}(E/M)\otimes_{\mathcal{O}_M}\underline{\omega}^{\otimes(r+1)}_{E/M}$$

in this section. The trivialization β gives a canonical generator $\omega := \beta^*\left(\frac{dT}{1+T}\right)$ of $\underline{\omega}_{E/M}$.

Definition 4.4.1. Let E/M be the universal trivialized elliptic curve with $\Gamma(N)$-level structure. For $(0,0)\neq(a,b)\in(\mathbb{Z}/N\mathbb{Z})^2$ define

$$_{D}\mathcal{E}^{k,r+1}_{(a,b)} \in V(\mathbb{Z}_p,\Gamma(N)) = \Gamma(M,\mathcal{O}_M)$$

as the image of

$$_{D}TE^{k,r+1}_{(a,b)} \in \Gamma\left(M, \underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\,\underline{H}^1_{\mathrm{dR}}(E/M)\otimes_{\mathcal{O}_M}\underline{\omega}^{\otimes(r+1)}_{E/M}\right)$$

under

$$\underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\,\underline{H}^1_{\mathrm{dR}}(E/M)\otimes_{\mathcal{O}_M}\underline{\omega}^{\otimes(r+1)}_{E/M} \to \underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\,\underline{\omega}_{E/M}\otimes_{\mathcal{O}_M}\underline{\omega}^{\otimes(r+1)}_{E/M}\xrightarrow{\sim}\mathcal{O}_M$$

where the last isomorphism is given by

$$\underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\,\underline{\omega}_{E/M}\otimes_{\mathcal{O}_M}\underline{\omega}^{\otimes(r+1)}_{E/M}\xrightarrow{\sim}\mathcal{O}_M,\quad \omega^{[k]}\otimes\omega^{\otimes(r+1)}\mapsto 1.$$

Katz defines generalized p-adic modular forms $2\Phi_{k,r,f}\in V(\mathbb{Z}_p,\Gamma(N))$ for $k,r\geq 1$ and $f:(\mathbb{Z}/N\mathbb{Z})^2\to\mathbb{Z}_p$. For the precise definition we refer to [Kat76, §5.11]. Essentially, he applies the differential operator

$$\Theta : \underline{\mathrm{Sym}}^k\,\underline{H}^1_{\mathrm{dR}}(E/M)\to\underline{\mathrm{Sym}}^k\,\underline{H}^1_{\mathrm{dR}}(E/M)\otimes_{\mathcal{O}_M}\Omega^1_{M/\mathbb{Z}_p}\hookrightarrow\underline{\mathrm{Sym}}^{k+2}\,\underline{H}^1_{\mathrm{dR}}(E/M)$$

obtained by Gauss–Manin connection and Kodaira–Spencer isomorphism to classical Eisenstein series and finally uses the unit root decomposition in order to obtain p-adic modular forms. We have the following comparison result.

Proposition 4.4.2. *We have the following equality of p-adic modular forms:*

$$_{D}\mathcal{E}^{k,r+1}_{(a,b)} = 2N^{-k}\left[D^2\Phi_{r,k,\delta_{(a,b)}} - D^{r+1-k}\Phi_{r,k,\delta_{(Da,Db)}}\right]$$

where $\delta_{(a,b)}$ is the function on $(\mathbb{Z}/N\mathbb{Z})^2$ with $\delta_{(a,b)}(a,b)=1$ and zero else.

Proof. Let us start with recalling the definition of Katz [Kat76, Lemma 5.11.4]: Let f be a $\mathbb{Z}_p$-valued function on $(\mathbb{Z}/N\mathbb{Z})^2$. Let $2G_{k,0,f}$ be the algebraic Eisenstein series of level $\Gamma(N)$ defined in [Kat76, Theorem 3.6.9]. Let us for a moment assume the following setup: Let E/S be an elliptic curve with level N-structure and $S \to T$ be a smooth morphism. Let us further assume that the Kodaira–Spencer map

$$\underline{\omega}_{E/S}^{\otimes 2} \to \Omega^1_{S/T}$$

is an isomorphism. For $k \geq r \geq 1$ define

$$2\phi_{k,r,f} \in \Gamma\left(S, \underline{\mathrm{Sym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}\,(E/S)\right)$$

as the image of $2G_{k+1-r,0,f}$ under

$$(N \cdot \Theta)^{\circ r} : \underline{\omega}_{E/S}^{\otimes(k+1-r)} \to \underline{\mathrm{Sym}}^{k-r}\, \underline{H}^1_{\mathrm{dR}}\,(E/S) \to \underline{\mathrm{Sym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}\,(E/S)\,.$$

For $r \geq k \geq 1$ define $2\phi_{k,r,f}$ as the image of $2G_{r+1-k,0,\hat{f}}$ under $N^k \cdot \Theta^{\circ k}$ where $\hat{f}$ is the symplectic Fourier transformation as in [Kat76, p. 3.0.1][2]. If we apply this construction to the universal trivialized elliptic curve with level N-structure, we can describe the p-adic modular forms $2\Phi_{k,r,f}$ of Katz as the image of $2\phi_{k,r,f}$ under the unit root decomposition:

$$\underline{\mathrm{Sym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}\,(E/M) \twoheadrightarrow \underline{\omega}_{E/M}^{\otimes(k+1+r)} \xrightarrow{\sim} V\left(\mathbb{Z}_p, \Gamma(N)\right).$$

In order to compare $\phi_{k,r,f}$ to ${}_DTE^{k,r+1}_{(a,b)}$ we have to embed both $\underline{\mathrm{Sym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}\,(E/M)$ and $\underline{\mathrm{TSym}}^k \otimes \underline{\omega}_{E/M}^{\otimes(r+1)}$ into $\underline{\mathrm{TSym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}\,(E/M)$. For the moment, let us write $\underline{H}^1_{\mathrm{dR}}$ for $\underline{H}^1_{\mathrm{dR}}\,(E/M)$. Since $V\left(\mathbb{Z}_p, \Gamma(N)\right)$ is flat over $\mathbb{Z}_p$, the canonical map

$$\underline{\mathrm{Sym}}^{\bullet}\, \underline{H}^1_{\mathrm{dR}}\,(E/M) \hookrightarrow \underline{\mathrm{TSym}}^{\bullet}\, \underline{H}^1_{\mathrm{dR}}\,(E/M) \tag{4.6}$$

is a monomorphism. Consider the following commutative diagram:

$$\begin{tikzcd}
\underline{\mathrm{Sym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}} \arrow[r, two heads] \arrow[d, hook] & \underline{\mathrm{Sym}}^{k+r+1}\, \underline{\omega}_{E/M} \arrow[r, "\sim"] \arrow[d, hook] & V\left(\mathbb{Z}_p, \Gamma(N)\right) \arrow[d, "\cdot(k+r+1)"] \\
\underline{\mathrm{TSym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}} \arrow[r, two heads] & \underline{\mathrm{TSym}}^{k+r+1}\, \underline{\omega}_{E/M} \arrow[r, "\sim"] & V\left(\mathbb{Z}_p, \Gamma(N)\right) \\
\underline{\mathrm{TSym}}^{k}\, \underline{H}^1_{\mathrm{dR}} \otimes \underline{\mathrm{TSym}}^{r+1}\, \underline{\omega}_{E/M} \arrow[r, two heads] \arrow[u, "k!r!\mathrm{mult}"] & \underline{\mathrm{TSym}}^{k}\, \underline{\omega}_{E/M} \otimes \underline{\mathrm{TSym}}^{r+1}\, \underline{\omega}_{E/M} \arrow[r, "\sim"] \arrow[u, "k!r!"] & V\left(\mathbb{Z}_p, \Gamma(N)\right) \arrow[u, "\cdot(k+r+1)!"'] \\
\underline{\mathrm{TSym}}^{k}\, \underline{H}^1_{\mathrm{dR}} \otimes \underline{\omega}_{E/M}^{\otimes(r+1)} \arrow[r, two heads] \arrow[u, "\cong"] & \underline{\mathrm{TSym}}^{k}\, \underline{\omega}_{E/M} \otimes \underline{\omega}_{E/M}^{\otimes(r+1)} \arrow[r, "\sim"] \arrow[u, "\cong"] & V\left(\mathbb{Z}_p, \Gamma(N)\right) \arrow[u, equal]
\end{tikzcd}$$

[2] Katz works with a $\Gamma(N)^{\mathrm{arith}}$-level structure which is probably the better choice. But in order to be consistent with the previous chapters we keep to our choice of a $\Gamma(N)$-level structure. Anyway, for our applications N is always invertible on the base and in this case there is not much difference between a $\Gamma(N)$ and a $\Gamma(N)^{\mathrm{arith}}$-structure. This leads to minor modifications, e.g. the appearance of symplectic Fourier transforms instead of transposition cf. the discussion in [Kat76, §2.0, §3.6,] and [Kat76, Lemma3.2.4].

Here, all isomorphisms are induced by identifying the canonical generators, i. e. $\omega^{\otimes k}$, $\omega^{[k]}$ resp. 1, of the free $V(\mathbb{Z}_p, \Gamma(N))$-modules $\underline{\omega}_{E/M}^{\otimes k}$, $\underline{\mathrm{TSym}}^k\, \underline{\omega}_{E/M}$ resp. $V(\mathbb{Z}_p, \Gamma(N))$. All epimorphisms are induced by the unit root decomposition and all monomorphisms are induced by (4.6). To check the commutativity of the diagram is straightforward. The image of

$$2N^{-k}\left[D^2\phi_{k,r,\delta_{(b,a)}} - D^{r+1-k}\phi_{k,r,\delta_{(Db,Da)}}\right] \tag{4.7}$$

under the upper horizontal map gives the p-adic modular form in the right hand side of the statement. The image of ${}_D TE^{k,r+1}_{(a,b)}$ under the lower horizontal map gives the p-adic modular form in the left hand side of the statement. Thus, by the commutativity of the diagram it is enough to show the equality of (4.7) and ${}_D TE^{k,r+1}_{(a,b)}$ in $\underline{\mathrm{TSym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}$. Let us denote by

$$i_1 : \underline{\mathrm{Sym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}} \hookrightarrow \underline{\mathrm{TSym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}$$

and

$$i_2 : \underline{\mathrm{TSym}}^{k}\, \underline{H}^1_{\mathrm{dR}} \otimes \underline{\omega}_{E/M}^{\otimes(r+1)} \hookrightarrow \underline{\mathrm{TSym}}^{k+r+1}\, \underline{H}^1_{\mathrm{dR}}$$

the inclusions in the above diagram. The construction of ${}_D TE^{k,r+1}_{(a,b)}$ is compatible with arbitrary base change. As we have indicated above, the construction of Katz via the Gauss–Manin connection works whenever the Kodaira–Spencer map is an isomorphism. Since Gauss–Manin connection and Kodaira–Spencer map are compatible with base change, we are reduced to check the equality

$$i_1\left(2N^{-k}\left[D^2\phi_{r,k,\delta_{(a,b)}} - D^{r+1-k}\phi_{r,k,\delta_{(Da,Db)}}\right]\right) = i_2({}_D TE^{k,r+1}_{(a,b)})$$

on the universal elliptic curve with $\Gamma(N)$-level structure. By the usual argument we are reduced to show this equality after analytification. Thus, it is enough to show that the corresponding $\mathcal{C}^\infty$-modular forms obtained via the Hodge decomposition coincide. The $\mathcal{C}^\infty$-modular form associated with

$$2\phi_{k,r,f}$$

is according to Katz [Kat76, pp. 3.6.5, 3.0.5] given by

$$(2\phi_{k,r,f})^{an} = (2G_{k+r+1,-r,f})^{an} = (-1)^{k+r+1}k!\left(\left(\frac{N}{A(\tau)}\right)^r \zeta_{k+r+1}\left(\frac{k-r+1}{2}, 1, \tau, f\right)\right)$$

where ζ_{k+r+1} is the Epstein zeta function obtained by analytic continuation of

$$\zeta_k(s,1,\tau,f) = N^{2s}\sum_{(0,0)\neq(n,m)} \frac{f(n,m)}{(m\tau+n)^k|m\tau+n|^{2s-k}}, \quad \mathrm{Re}(s) > 1. \tag{4.8}$$

The Eisenstein–Kronecker series $e^*_{k,r+1}(\frac{a}{N}\tau + \frac{b}{N}, 0)$ appearing in the description of the $\mathcal{C}^\infty$-modular form ${}_D TE^{k,r+1}_{(a,b)}$ can be defined as

$$\begin{aligned}\frac{e^*_{k,r+1}(\frac{a}{N}\tau + \frac{b}{N}, 0)}{A(\tau)^k k!} &= \frac{K^*_{k+r+1}(\frac{a}{N}\tau + \frac{b}{N}, 0, r+1, \tau)}{A(\tau)^k k!} \overset{F.E.}{=} \\ &= \frac{K^*_{k+r+1}(0, \frac{a}{N}\tau + \frac{b}{N}, k+1, \tau)}{A(\tau)^r r!}\end{aligned}$$

with the Eisenstein–Kronecker–Lerch $K_k^*(0, \frac{a}{N}\tau + \frac{b}{N}, s, \tau)$ series which is given by analytic continuation of

$$K_k^*\left(0, \frac{a}{N}\tau + \frac{b}{N}, s, \tau\right) := \sum_{(0,0)\neq(m,n)} \frac{(m\bar{\tau}+n)^k}{|m\tau+n|^{2s}} \exp\left(2\pi i \frac{ma-nb}{N}\right). \tag{4.9}$$

Comparing (4.8) and (4.9) shows

$$K_k^*\left(0, \frac{a}{N}\tau + \frac{b}{N}, s + \frac{k}{2}, \tau\right) = N^{1-2s}\zeta_k(s, 1, \tau, \hat{\delta}_{(a,b)}). \tag{4.10}$$

Using this, we compute

$$\begin{aligned} N^{-k}(2\phi_{k,r,\hat{\delta}_{a,b}})^{an} &= N^{-k}(-1)^{k+r+1}k!\left(\frac{N}{A(\tau)}\right)^r \zeta_{k+r+1}(\frac{k-r+1}{2}, 1, \tau, \hat{\delta}_{a,b}) = \\ &= N^{-k}(-1)^{k+r+1}k!\left(\frac{N}{A(\tau)}\right)^r N^{k-r}K^*_{k+r+1}(0, s, k+1; \tau) = \\ &= (-1)^{k+r+1}k!r!\frac{e^*_{k,r+1}}{A(\tau)^k k!} \end{aligned}$$

Finally, let us recall from [Kat76] the identity $\phi_{k,r,f} = \phi_{r,k,\hat{f}}$. Now, the analytic identity

$$\begin{aligned} N^{-k}&\left[D^2\left(2\phi_{r,k,\delta_{a,b}}\right)^{an} - D^{r+1-k}\left(2\phi_{r,k,\delta_{Da,Db}}\right)^{an}\right] = \\ =&(-1)^{k+r+1}k!r!\left[D^2\frac{e^*_{k,r+1}(s,0)}{A(\tau)^k k!} - D^{r+1-k}\frac{e^*_{k,r+1}(Ds,0)}{A(\tau)^k k!}\right] \end{aligned}$$

proves the desired algebraic identity on the universal elliptic curve and thereby the proposition. □

4.5 p-adic interpolation of Eisenstein–Kronecker series via p-adic theta functions

Let $(E^{\mathrm{triv}}/M^{\mathrm{triv}}, \beta, \alpha_N)$ be the universal trivialized elliptic curve with $\Gamma(N)$-level structure. Katz' construction of the p-adic measure interpolating the p-adic modular forms $\Phi_{k,r,f}$ consists essentially in checking the predicted congruences among the $\Phi_{k,r,f}$ on the Tate curve by using the q-expansion principle for p-adic modular forms. On the other hand, the very definition of the moduli problem for trivialized elliptic curves includes an isomorphism

$$\beta : \widehat{\mathbb{G}}_{m,V(\mathbb{Z}_p,\Gamma(N))} \xrightarrow{\sim} \hat{E}^{\mathrm{triv}}.$$

The Amice transform

$$\Gamma\left(R, \mathcal{O}_{\widehat{\mathbb{G}}_{m,R}} \hat{\otimes} \mathcal{O}_{\widehat{\mathbb{G}}_{m,R}}\right) \xrightarrow{\sim} \mathrm{Meas}(\mathbb{Z}_p^2, R)$$

between functions on the formal group $\widehat{\mathbb{G}}_{m,R} \times \widehat{\mathbb{G}}_{m,R}$ and R-valued measures for a p-adic ring R suggests a construction of Katz' Eisenstein measure via functions on the formal completion of $E^{\mathrm{triv}} \times E^{\mathrm{triv}}$. The aim of this section will be to show that the p-adic theta function ${}_D\vartheta_{s_{a,b}}$ corresponds to a two-variable p-adic measure with values in generalized p-adic modular forms interpolating the p-adic modular forms $({}_D\mathcal{E}^{k,r+1}_{(a,b)})_{k,r}$.

Let R be a p-adic ring. A p-adic measure on a pro-finite Abelian group G is an R-linear continuous map $C(G,R) \to R$, where $C(G,R)$ denotes the R-module of R-valued continuous functions on G. Let us write $\mathrm{Meas}(\mathbb{Z}_p^2, R)$ for the set of all R-valued measures on $\mathbb{Z}_p^2$.

Proposition 4.5.1. *Let R be a p-adic ring. There is an isomorphism*

$$\Gamma\left(R, \mathcal{O}_{\widehat{\mathbb{G}}_{m,R}} \hat{\otimes} \mathcal{O}_{\widehat{\mathbb{G}}_{m,R}}\right) \xrightarrow{\sim} \mathrm{Meas}(\mathbb{Z}_p^2, R), \quad f \mapsto \mu_f$$

which is uniquely characterized by

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} x^a y^b d\mu_f(x,y) = \partial_1^{\circ a} \partial_2^{\circ b} f\Big|_{T_1 = T_2 = 0}$$

where T_1, T_2 are the coordinates of $\widehat{\mathbb{G}}_{m,R} \times \widehat{\mathbb{G}}_{m,R}$ and $\partial_1 = (1+T_1)\frac{\partial}{\partial T_1}$, $\partial_2 = (1+T_2)\frac{\partial}{\partial T_2}$ are the invariant derivatives associated with the coordinates. The inverse is given by

$$\mu \mapsto A_\mu(T_1, T_2) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p} (1+T_1)^x (1+T_2)^y d\mu(x,y)$$

and will be called Amice transform *of the measure μ.*

Proof. For a proof see e.g. [Hid93, §3.7., Theorem 1] □

Let us use our autoduality λ and the trivialization β to obtain an isomorphism

$$\hat{E}^{\mathrm{triv}} \times (\hat{E}^{\mathrm{triv}})^\vee \xrightarrow{\sim} \widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}} \times \widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}$$

of formal groups. Here, the products are taken in the category of formal schemes over M^{triv}. Let us denote the coordinates on $\widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}} \times_{M^{\mathrm{triv}}} \widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}$ by T_1 and T_2 and let us use the canonical generator $\omega = \beta^* \frac{dT}{1+T}$ of $\underline{\omega}_{E^{\mathrm{triv}}/M^{\mathrm{triv}}}$ to obtain the isomorphism $\underline{\omega}_{E^{\mathrm{triv}}/M^{\mathrm{triv}}} \xrightarrow{\sim} \mathcal{O}_{M^{\mathrm{triv}}}$.

Definition 4.5.2. For $e \neq s \in E^{\mathrm{triv}}[N](M^{\mathrm{triv}})$ let us write

$${}_D\vartheta_s(T_1, T_2) \in V(\mathbb{Z}_p, \Gamma(N))[\![T_1, T_2]\!] = \Gamma\left(M^{\mathrm{triv}}, \mathcal{O}_{\widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}} \hat{\otimes}_{\mathcal{O}_{M^{\mathrm{triv}}}} \mathcal{O}_{\widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}}\right)$$

for the image of ${}_D\vartheta_s$ under the isomorphism

$$\mathcal{O}_{\hat{E}^{\mathrm{triv}} \times (\hat{E}^{\mathrm{triv}})^\vee} \otimes_{\mathcal{O}_{M^{\mathrm{triv}}}} \underline{\omega}_{E^{\mathrm{triv}}/M^{\mathrm{triv}}} \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}} \hat{\otimes}_{\mathcal{O}_{M^{\mathrm{triv}}}} \mathcal{O}_{\widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}}$$

induced by the trivialization β and the generator ω. Let

$$\mu_{D,s}^{\mathrm{Eis}} \in \mathrm{Meas}\Big(\mathbb{Z}_p \times \mathbb{Z}_p, V\left(\mathbb{Z}_p, \Gamma(N)\right)\Big)$$

be the p-adic measure having the p-adic theta function ${}_D\vartheta_s$ as Amice transform. The measure $\mu_{D,s}^{\mathrm{Eis}}$ will be called *D-variant of the p-adic Eisenstein–Kronecker measure.*

Indeed, the measure $\mu_{D,s}^{\mathrm{Eis}}$ has the p-adic Eisenstein series ${}_D\mathcal{E}_{(a,b)}^{k,r+1}$ as moments:

Theorem 4.5.3. *For $e \neq s \in E^{\mathrm{triv}}[N](M^{\mathrm{triv}})$ corresponding to $(0,0) \neq (a,b)$ via α we have*

$$\partial_1^{\circ r}\partial_2^{\circ k}\ {}_D\vartheta_s(T_1,T_2)\Big|_{T_1=T_2=0} = {}_D\mathcal{E}_{(a,b)}^{k,r+1}$$

where T_1, T_2 are the coordinates on $\widehat{\mathbb{G}}_{m,R} \times \widehat{\mathbb{G}}_{m,R}$ and $\partial_1 = (1+T_1)\frac{\partial}{\partial T_1}$, $\partial_2 = (1+T_2)\frac{\partial}{\partial T_2}$ are the invariant derivatives associated with the coordinates. Stated differently,

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} x^k y^r \, d\mu_{D,s}^{\mathrm{Eis}}(x,y) = {}_D\mathcal{E}_{(a,b)}^{k,r+1}.$$

Proof. Let us simply write E for E^{triv} and M for M^{triv} in the following proof. Denote by $d_{\hat{E}}$ and $d_{\hat{E}^\vee}$ the universal continuous derivations on the formal groups $\hat{E}$ and $\hat{E}^\vee$, i. e.

$$d_{\hat{E}} : \mathcal{O}_{\hat{E}} \to \hat{\Omega}^1_{\hat{E}/M^{\mathrm{triv}}} \cong \underline{\omega}_{E/M^{\mathrm{triv}}} \otimes_{\mathcal{O}_{M^{\mathrm{triv}}}} \mathcal{O}_{\hat{E}}$$

and

$$d_{\hat{E}^\vee} : \mathcal{O}_{\hat{E}^\vee} \to \hat{\Omega}^1_{\hat{E}^\vee/M^{\mathrm{triv}}} \cong \underline{\omega}_{E^\vee/M} \otimes_{\mathcal{O}_M} \mathcal{O}_{\hat{E}^\vee}.$$

Since $d_{\hat{E}}$ and $d_{\hat{E}^\vee}$ are $\mathcal{O}_M$-linear, we get differential operators

$$d_{\hat{E}} : \mathcal{O}_{\hat{E}\times\hat{E}^\vee} \otimes_{\mathcal{O}_M} \underline{\omega}_{E/M}^{\otimes i} \otimes_{\mathcal{O}_M} \underline{\omega}_{E^\vee/M}^{\otimes j} \xrightarrow{d_{\hat{E}}\otimes\mathrm{id}} \mathcal{O}_{\hat{E}\times\hat{E}^\vee} \otimes_{\mathcal{O}_M} \underline{\omega}_{E/M}^{\otimes(i+1)} \otimes_{\mathcal{O}_M} \underline{\omega}_{E^\vee/M}^{\otimes j}$$

and similarly for $d_{\hat{E}^\vee}$. We can reformulate the statement of the theorem in the following more intrinsic way

$$(e_{\hat{E}} \times e_{\hat{E}^\vee})^* \left[d_{\hat{E}}^{\circ r} d_{\hat{E}^\vee}^{\circ k}\ {}_D\vartheta\right] = (u \times \mathrm{id}_{\underline{\omega}^{r+1}})\left({}_DTE_{(a,b)}^{k,r+1}\right) \in \Gamma(M, \underline{\omega}_{E^\vee/M}^{\otimes k} \otimes \underline{\omega}_{E/M}^{\otimes(r+1)}) \quad (4.11)$$

where u is the map

$$u : \underline{\mathrm{TSym}}^k\,\mathcal{H} \to \underline{\mathrm{TSym}}^k\,\underline{\omega}_{E^\vee/M} \xrightarrow{\sim} \underline{\omega}_{E^\vee/M}^{\otimes k}$$

induced by the unit root decomposition. Recall that we have constructed

$$\widehat{\mathrm{split}} : (\underline{\mathrm{TSym}}^n_{\mathcal{O}_E}\,\mathcal{L}_1)|_{\hat{E}} \to \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \left(\bigoplus_{k=0}^n \underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\,\underline{\omega}_{E^\vee/M}\right)$$

and

$$\hat{\mathrm{split}}^{\dagger} : (\underline{\mathrm{TSym}}^n_{\mathcal{O}_E}\, \mathcal{L}_1^{\dagger})|_{\hat{E}} \to \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \left(\bigoplus_{k=0}^{n} \underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\, \mathcal{H} \right).$$

Let us write for the moment $\mathcal{H}^{[k]} := \underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\, \mathcal{H}$ and $\underline{\omega}^{[k]}_{E^\vee/M} := \underline{\mathrm{TSym}}^k_{\mathcal{O}_M}\, \underline{\omega}_{E^\vee/M}$. We claim that the equality (4.11) follows once we have shown the commutativity of the following diagram:

$$\begin{tikzcd}
(T_s^*\mathcal{L}_n)|_{\hat{E}} \arrow[r, hook] \arrow[d, "\mathrm{trans}_s"] & \left(T_s^*\mathcal{L}_n^{\dagger}\right)\Big|_{\hat{E}} \arrow[r, "T_s^*\nabla^{\circ r}_{\mathcal{L}^\dagger}|_{\hat{E}}"] \arrow[d, "\mathrm{trans}_s^{\dagger}|_{\hat{E}}"] & \left(T_s^*\mathcal{L}_n^{\dagger} \otimes_{\mathcal{O}_E} (\Omega^1_{E/M})^{\otimes r}\right)\Big|_{\hat{E}} \arrow[d, "(\mathrm{trans}_s^{\dagger}\otimes\mathrm{id})|_{\hat{E}}"] \\
\mathcal{L}_n|_{\hat{E}} \arrow[r, hook] \arrow[d, "\mathbb{D}|_{\hat{E}}"] & \mathcal{L}_n^{\dagger}\Big|_{\hat{E}} \arrow[r, "\nabla^{\circ r}_{\mathcal{L}^\dagger}|_{\hat{E}}"] \arrow[d, "\mathbb{D}^{\dagger}|_{\hat{E}}"] & \left(\mathcal{L}_n^{\dagger} \otimes_{\mathcal{O}_E} (\Omega^1_{E/M})^{\otimes r}\right)\Big|_{\hat{E}} \arrow[d, "(\mathbb{D}\otimes\mathrm{id})|_{\hat{E}}"] \\
\left(\underline{\mathrm{TSym}}^n_{\mathcal{O}_E}\, \mathcal{L}_1\right)|_{\hat{E}} \arrow[r, hook] \arrow[d, "\hat{\mathrm{split}}"] & \left(\underline{\mathrm{TSym}}^n_{\mathcal{O}_E}\, \mathcal{L}_1^{\dagger}\right)|_{\hat{E}} \arrow[r, "\nabla_{\mathrm{TSym}}"] \arrow[d, "\hat{\mathrm{split}}^{\dagger}"] \arrow[dr, phantom, "\bigstar"] & \left[\left(\underline{\mathrm{TSym}}^n_{\mathcal{O}_E}\, \mathcal{L}_1^{\dagger}\right) \otimes_{\mathcal{O}_E} \Omega^1_{E/M}\right]|_{\hat{E}} \arrow[d, "\hat{\mathrm{split}}^{\dagger}\otimes\mathrm{id}"] \\
\mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \left(\bigoplus_{k=0}^{n} \underline{\omega}^{[k]}_{E^\vee/M}\right) \arrow[r, hook] \arrow[d, equal] & \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \left(\bigoplus_{k=0}^{n} \mathcal{H}^{[k]}\right) \arrow[d, "\mathrm{id}\otimes u"] & \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \left(\bigoplus_{k=0}^{n} \mathcal{H}^{[k]} \otimes \underline{\omega}^{\otimes r}_{E/M}\right) \arrow[d, "\mathrm{id}\otimes u\otimes\mathrm{id}"] \\
\mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \left(\bigoplus_{k=0}^{n} \underline{\omega}^{[k]}_{E^\vee/M}\right) \arrow[r, equal] & \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \left(\bigoplus_{k=0}^{n} \underline{\omega}^{[k]}_{E^\vee/M}\right) \arrow[r, "d^{\circ r}_{\hat{E}}"] & \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \left(\bigoplus_{k=0}^{n} \underline{\omega}^{[k]}_{E^\vee/M} \otimes \underline{\omega}^{\otimes r}_{E/M}\right)
\end{tikzcd}$$

All squares in the above diagram are easily seen to be commutative except the one denoted by $\bigstar$. Before we prove the commutativity of $\bigstar$, let us explain how it proves the theorem. Consider the composition of the upper horizontal maps followed by the right vertical maps and tensor the resulting map with $\otimes_{\mathcal{O}_M}\underline{\omega}_{E/M}$. The image of $s^* l_n^D$ under the pullback of this map along e gives $(u \times \mathrm{id}_{\underline{\omega}^{r+1}})\left({}_D TE^{k,r+1}_{(a,b)}\right)$. Since the lower horizontal map of the diagram is $d_{\hat{E}}$, it suffices in view of (4.11) to show that the left horizontal map tensorized with $\underline{\omega}_{E/M}$ sends $T_s^* l_n^D$ to $\left((\mathrm{id}_{\hat{E}} \times e)^* \left[d^{\circ k}_{\hat{E}^\vee}\, {}_D\vartheta_s\right]\right)$. This again follows from the following commutative diagram

$$\begin{tikzcd}
s^*l_n^D \arrow[dd, mapsto] & \in & T_s^*\mathcal{L}_n|_{\hat{E}} \otimes_{\mathcal{O}_M} \underline{\omega}_{E/M} \arrow[d, "\mathrm{trans}_s|_{\hat{E}}\otimes\mathrm{id}"] & \\
 & & \mathcal{L}_n|_{\hat{E}} \otimes \underline{\omega}_{E/M} \arrow[r, "\mathbb{D}|_{\hat{E}}\otimes\mathrm{id}"] \arrow[d, "\mathrm{triv}_{\hat{E}}"] & \left(\underline{\mathrm{TSym}}^n\,\mathcal{L}_1\right)|_{\hat{E}} \otimes \underline{\omega}_{E/M} \arrow[d, "\mathrm{TSym}(\mathrm{triv}_{\hat{E}})"] \arrow[dd, bend left=60, "\widehat{\mathrm{split}}\otimes\mathrm{id}"] \\
{}_D\vartheta_s \arrow[drr, mapsto] & \in & \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \mathcal{O}_{\mathrm{Inf}_e^n E^\vee} \otimes \underline{\omega}_{E/M} \arrow[r, "\mathrm{id}\otimes\mathbb{D}_e\otimes\mathrm{id}"] & \mathcal{O}_{\hat{E}} \otimes_{\mathcal{O}_M} \underline{\mathrm{TSym}}^n_{\mathcal{O}_M}(\mathcal{O}_{\mathrm{Inf}_e^1 E^\vee}) \otimes \underline{\omega}_{E/M} \arrow[d, "\cong"] \\
 & & \left((\mathrm{id}_{\hat{E}}\times e)^*\left[d^{\circ k}_{\hat{E}^\vee}\,{}_D\vartheta_s\right]\right)_{k=0}^n \arrow[r, phantom, "\in"] & \mathcal{O}_{\hat{E}} \otimes \bigoplus_{k=0}^n \underline{\omega}^{[k]}_{E^\vee/M} \otimes \underline{\omega}_{E/M}
\end{tikzcd}$$

where we have used Corollary 2.1.15 for

$${}_D\vartheta_s \mapsto \left((\mathrm{id}_{\hat{E}}\times e)^*\left[d^{\circ k}_{\hat{E}^\vee}\,{}_D\vartheta_s\right]\right)_{k=0}^n .$$

It remains to show the commutativity of ★. It is enough to consider the case $r = 1$. The general case follows by composing r times the following diagram. For $r = 1$ the diagram ★ is:

$$\begin{tikzcd}
\left(\underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger\right)|_{\hat{E}} \arrow[r, "\nabla_{\underline{\mathrm{TSym}}}"] \arrow[d, "\widehat{\mathrm{split}}"] & \left[\left(\underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger\right) \otimes_{\mathcal{O}_E} \Omega^1_{E/M}\right]|_{\hat{E}} \arrow[d, "\widehat{\mathrm{split}}\otimes\mathrm{id}"] \\
\left(\bigoplus_{k=0}^n \pi_{\hat{E}}^*\mathcal{H}^{[k]}\right) \arrow[d, "u"] & \left(\bigoplus_{k=0}^n \pi_{\hat{E}}^*\mathcal{H}^{[k]}\right) \otimes_{\mathcal{O}_{\hat{E}}} \hat{\Omega}^1_{\hat{E}/M} \arrow[d, "u"] \\
\left(\bigoplus_{k=0}^n \pi_{\hat{E}}^*\underline{\omega}^{[k]}\right) \arrow[r, "d_{\hat{E}}"] & \left(\bigoplus_{k=0}^n \pi_{\hat{E}}^*\underline{\omega}^{[k]}\right) \otimes_{\mathcal{O}_{\hat{E}}} \hat{\Omega}^1_{\hat{E}/M}
\end{tikzcd}$$

The commutativity of this diagram follows from the explicit formulas for $\nabla_{\underline{\mathrm{TSym}}}$ given in Lemma 4.3.2. □

4.6 Restriction of the measure and the Frobenius morphism

As always when one has a p-adic measure μ on $\mathbb{Z}_p$ having some sequence of interest as moments, it is only possible to define the moment function

$$\mathbb{Z}_p \ni s \mapsto \int_{\mathbb{Z}_p^\times} \langle x\rangle^s d\mu(x)$$

with $\langle\cdot\rangle\colon \mathbb{Z}_p^\times = (1+p\mathbb{Z}_p)\times\mu_{p-1} \twoheadrightarrow (1+p\mathbb{Z}_p)$ after restriction to $\mathbb{Z}_p^\times$. Let us consider again the universal trivialized elliptic curve $(E^{\mathrm{triv}}/M^{\mathrm{triv}}, \beta, \alpha_N)$ with $\Gamma(N)$-level structure. For

$e \neq s \in E^{\mathrm{triv}}[N](M^{\mathrm{triv}})$ corresponding to (a,b) via the level structure we have defined the p-adic measure $\mu^{\mathrm{Eis}}_{D,s}$. From Katz [Kat76, §6.3] and Proposition 4.4.2 we can easily deduce

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} f(x,y) d\mu^{\mathrm{Eis}}_{D,s} = \int_{\mathbb{Z}_p \times \mathbb{Z}_p} f(x,y) d\mu^{\mathrm{Eis}}_{D,s} - \mathrm{Frob} \int_{\mathbb{Z}_p \times \mathbb{Z}_p} f(p \cdot x, y) d\mu^{\mathrm{Eis}}_{D,s}.$$

Katz deduces this result by comparing the q-expansions of the moments of both sides. In this section we sketch an alternative proof of this result. In some sense it can be seen as a shadow of the distribution relation for the canonical section.

Let us define

$$\mu^{\mathrm{Eis,(p)}}_{D,s} := \mu^{\mathrm{Eis}}_{D,s}|_{\mathbb{Z}_p^\times \times \mathbb{Z}_p}.$$

Sometimes it will be convenient to view $\mu^{\mathrm{Eis,(p)}}_{D,s}$ as a measure on $\mathbb{Z}_p \times \mathbb{Z}_p$ which is supported on $\mathbb{Z}_p^\times \times \mathbb{Z}_p$. In particular, we use Proposition 4.5.1 to define ${}_D\vartheta_s^{(p)}$ as the Amice transform of $\mu^{\mathrm{Eis,(p)}}_{D,s}$. It is well-known how the Amice transform of a p-adic measure behaves under the operation of restricting to $\mathbb{Z}_p^\times$. In the particular case of the p-adic ring $V(\mathbb{Z}_p, \Gamma(N))$ this reads as:

Lemma 4.6.1. *As before, let T_1 and T_2 be the coordinates of $\widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}} \times \widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}$. Define $R := V(\mathbb{Z}_p, \Gamma(N)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mu_p]$ by adjoining the p-th roots of unity. Then,*

$$\sum_{\zeta \in \widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}[p](R)} {}_D\vartheta_s(\zeta +_{\hat{\mathbb{G}}_m} T_1, T_2) \in p\mathcal{O}_{\hat{\mathbb{G}}_{m,R} \times \hat{\mathbb{G}}_{m,R}}$$

and we have

$${}_D\vartheta_s^{(p)}(T_1,T_2) = {}_D\vartheta_s(T_1,T_2) - \frac{1}{p} \sum_{\zeta \in \widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}[p](R)} {}_D\vartheta_s(\zeta +_{\hat{\mathbb{G}}_m} T_1, T_2).$$

Proof. [Sha87, §3.3, (7')] or [Col04, §1.5.2]. □

The following result can be deduced from a slightly generalized version of the distribution relation, see Remark 1.4.4.

Lemma 4.6.2. *Let $\varphi : E^{\mathrm{triv}} \to E' := E^{\mathrm{triv}}/C$ be the quotient of E^{triv} by its canonical subgroup. Write $R := V(\mathbb{Z}_p, \Gamma(N)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mu_p]$ and let E_R^{triv} and E'_R be the base changes of E^{triv} and E' to* Spec R. *We have the following equation in $\mathcal{L}_{n,E_R^{\mathrm{triv}}} \otimes \Omega^1_{E^{\mathrm{triv}}/R}$:*

$$\sum_{\tau \in C(R)} \mathrm{trans}_\tau \left(T_\tau^* l^D_{n,E^{\mathrm{triv}}} \right) = p\Phi_\varphi^{-1} \left(\varphi^*(l^D_{n,E'}) \right).$$

Proof. Let us give a direct proof, since we did not formulate the distribution relation for more general isogenies than multiplication by N. Since $V(\mathbb{Z}_p, \Gamma(N)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\mu_p]$ is flat over $\mathbb{Z}_p$ and $\varphi^*\mathcal{L}_{n,E'_R} \otimes \Omega^1_{E^{\mathrm{triv}}/R}$ is locally free, we may check the equality after base change to $\mathbb{Q}_p$. Now, the claim can be deduced by some density of torsion sections argument: the pullback of the left hand side along some l^n-torsion points s gives according to Corollary 2.4.4

$$\left(\sum_{\tau\in\ker\varphi} {}_D\mathrm{F}^{(k)}_{s+\tau}\right)_{k=1}^{n+1}$$

while observing Lemma 2.1.9 (b), the pullback of the right hand side gives

$$\left(p\cdot {}_D\mathrm{F}^{(k+1)}_{\varphi(s)}\right)_{k=0}^{n}.$$

Now, the claimed equality of the lemma follows from

$$\sum_{\tau\in\ker\varphi} {}_D\mathrm{F}^{(k)}_{s+\tau} = p\cdot {}_D\mathrm{F}^{(k+1)}_{\varphi(s)} \tag{4.12}$$

and density of torsion sections. The distribution property (4.12) can be deduced from the explicit q-expansion formulas given in [Kat76, Lemma 5.11.0.] observing the equality

$${}_D\mathrm{F}^{(k)}_{s_{a,b}} = D^2 G_{k,0,\delta_{a,b}} - D^{2-k} G_{k,0,\delta_{Da,Db}}.$$

□

In (4.5) we have defined a canonical lift of the absolute Frobenius $\varphi: E^{\mathrm{triv}}\to E^{\mathrm{triv}}$ and defined

$$\Phi:\phi^*\mathcal{L}_n\to\mathcal{L}_n.$$

Corollary 4.6.3. *Under the map*

$$\left(\varprojlim_n \mathcal{L}_n\otimes\Omega^1_{E^{\mathrm{triv}}/M^{\mathrm{triv}}}\right)\Big|_{\hat{E}_s} \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}}\hat{\otimes}_{\mathcal{O}_{M^{\mathrm{triv}}}}\mathcal{O}_{\widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}} \tag{4.13}$$

the compatible system

$$\left[\left(l_n^D-\Phi(\phi^* l_n^D)\right)_{n\geq 0}\right]|_{\hat{E}_s}$$

corresponds to ${}_D\vartheta_s^{(p)}(T_1,T_2)$.

Proof. From the definition of $\Phi:\phi^*\mathcal{L}_n\to\mathcal{L}_n$ we deduce

$$\Phi(\phi^* l^D_{n,E^{\mathrm{triv}}}) = \Phi_\varphi^{-1}\left(\varphi^* l^D_{n,E'}\right).$$

Combined with the above lemma, we obtain:

$$l^D_{n,E^{\mathrm{triv}}}-\Phi(\phi^* l^D_{n,E^{\mathrm{triv}}}) = l^D_{n,E^{\mathrm{triv}}}-\frac{1}{p}\sum_{\tau\in C(R)}\mathrm{trans}_\tau\left(T_\tau^* l^D_{n,E^{\mathrm{triv}}}\right)$$

The right hand side restricted to $\hat{E}_s$ maps under (4.13) to

$$ {}_D\vartheta_s(T_1,T_2) - \frac{1}{p} \sum_{\zeta \in \widehat{\mathbb{G}}_{m,M^{\mathrm{triv}}}[p](R)} {}_D\vartheta_s(\zeta +_{\hat{\mathbb{G}}_m} T_1, T_2) $$

and the corollary follows from Lemma 4.6.1. □

Now, we can easily deduce Katz' result:

Corollary 4.6.4. *The moments of the restricted measure are given as follows:*

$$ \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} f(x,y) d\mu_{D,s}^{\mathrm{Eis}} = \int_{\mathbb{Z}_p \times \mathbb{Z}_p} f(x,y) d\mu_{D,s}^{\mathrm{Eis}} - \mathrm{Frob} \int_{\mathbb{Z}_p \times \mathbb{Z}_p} f(p \cdot x, y) d\mu_{D,s}^{\mathrm{Eis}}. $$

Proof. From the above corollary together with the commutativity of

$$ \begin{array}{ccc} \hat{E}^{\mathrm{triv}} & \xrightarrow{\phi|_{\hat{E}^{\mathrm{triv}}}} & \hat{E}^{\mathrm{triv}} \\ \downarrow \beta & & \downarrow \beta \\ \widehat{\mathbb{G}}_{m,\mathbb{Z}_p} \times_{\mathbb{Z}_p} M^{\mathrm{triv}} & \xrightarrow{[p]\times \mathrm{Frob}} & \widehat{\mathbb{G}}_{m,\mathbb{Z}_p} \times_{\mathbb{Z}_p} M^{\mathrm{triv}} \end{array} $$

we deduce

$$ {}_D\vartheta_s^{(p)}(T_1,T_2) = {}_D\vartheta_s(T_1,T_2) - \mathrm{Frob}\Big[{}_D\vartheta_s\left([p](T_1), T_2\right) \Big]. $$

The statement of the corollary follows by passing to the corresponding measure under the Amice transform. □

5 The algebraic de Rham realization of the elliptic polylogarithm

The aim of this section is to give an explicit description of the algebraic de Rham realization of the elliptic polylogarithm in terms of the Poincaré bundle. We are building on previous results of the PhD-thesis of Scheider [Sch14]. Scheider has given a purely algebraic description of the de Rham logarithm sheaves in terms of infinitesimal restrictions of the Poincaré bundle. In our terminology this means that the geometric logarithm sheaves satisfy the universal property of the relative de Rham logarithm sheaves. Building on this, he gave an explicit analytic description of the differential forms representing the elliptic polylogarithm in de Rham cohomology on the $\mathbb{C}$-valued points of the universal elliptic curve using the Jacobi theta function. The canonical section of the Poincaré bundle allows us to make his description algebraic, i.e we give a purely algebraic description of the de Rham realization of the elliptic polylogarithm for arbitrary families of elliptic curves over smooth schemes over a field of characteristic zero. One of the technical difficulties we have to overcome is that the Poincaré bundle a priori gives only relative connections. But as already explained in Chapter 3 the Katz splitting gives us an explicit way to extend the connections.

For the whole chapter, let us fix a field K of characteristic zero. Our base scheme S will always be assumed to be a smooth and separated K-scheme.

5.1 The de Rham logarithm sheaves

Let us recall the basic definitions and properties of the de Rham logarithm sheaves. Most material of this section can be found in a more detailed way in the PhD-thesis of [Sch14].

5.1.1 The universal property of the de Rham logarithm sheaves

For a smooth morphism $\pi : S \to T$ between smooth separated schemes of finite type over K let us denote by $\mathrm{VIC}\,(S/T)$ the category of vector bundles on S with integrable T-connection and horizontal maps as morphisms. Since every coherent $\mathcal{O}_S$-module with integrable K-connection is a vector bundle, the category $\mathrm{VIC}\,(S/K)$ is Abelian (cf. [BO78, §2, Note 2.17]). The pullback along a smooth map $\pi : S \to T$ of smooth K-schemes induces an exact functor

$$\pi^* : \mathrm{VIC}\,(T/K) \to \mathrm{VIC}\,(S/K)\,.$$

By restricting the connection we get a forgetful map

$$\mathrm{VIC}\,(S/K) \to \mathrm{VIC}\,(S/T)\,.$$

For details we refer to [Sch14, §0.2].

Definition 5.1.1. Let $\pi : S \to T$ be a smooth morphism of smooth separated K-schemes and E/S an elliptic curve.

(a) An object $\mathcal{U} \in ob\mathrm{VIC}\,(E/T)$ is called unipotent of length n for $E/S/T$ if there exists a descending filtration in the category $\mathrm{VIC}\,(E/T)$

$$\mathcal{U} = A^0\mathcal{U} \subseteq A^1\mathcal{U} \subseteq \ldots \subseteq A^{n+1}\mathcal{U} = 0$$

such that for all $0 \le i \le n$, $\mathrm{gr}^i_A\mathcal{U} = A^i\mathcal{U}/A^{i+1}\mathcal{U} = \pi^*Y_i$ for some $Y_i \in \mathrm{VIC}\,(S/T)$.

(b) Let $U^\dagger_n(E/S/K)$ be the full subcategory of objects of $\mathrm{VIC}\,(E/K)$ which are unipotent of length n for $E/S/K$.

(c) Let $U^\dagger_n(E/S)$ be the full subcategory of objects of $\mathrm{VIC}\,(E/S)$ which are unipotent of length n for $E/S/S$.

Restricting the forgetful functor $\mathrm{VIC}\,(E/K) \to \mathrm{VIC}\,(E/S)$ to the full subcategories of unipotent objects gives $U^\dagger_n(E/S/K) \to U^\dagger_n(E/S)$. Further, the category $\mathrm{VIC}\,(S/K)$ is equipped in a natural way with a tensor product

$$\otimes : \mathrm{VIC}\,(S/K) \times \mathrm{VIC}\,(S/K) \to \mathrm{VIC}\,(S/K)$$

and an internal Hom functor

$$\underline{\mathcal{H}om} : \mathrm{VIC}\,(S/K)^{op} \times \mathrm{VIC}\,(S/K) \to \mathrm{VIC}\,(S/K)$$

making $\mathrm{VIC}\,(S/K)$ a closed symmetric monoidal category. For $F, G \in \mathrm{VIC}\,(S/K)$ let us write $\underline{\mathrm{Hom}}_{\mathrm{VIC}(S/K)}(F,G)$ for the sheaf of horizontal morphisms. We have

$$\pi_*\underline{\mathrm{Hom}}_{\mathrm{VIC}(S/K)}(F,G) = \underline{H}^0_{\mathrm{dR}}\left(S/T, \underline{\mathcal{H}om}(F,G)\right).$$

In particular, the Gauss–Manin connection gives a K-connection on $\pi_*\underline{\mathrm{Hom}}_{\mathrm{VIC}(S/K)}(F,G)$.

Lemma/Definition 5.1.2. There exists a pair $(\mathrm{Log}^n_{\mathrm{dR}}, \mathbb{1}^n)$, consisting of a unipotent vector bundle with K-connection

$$\mathrm{Log}^n_{\mathrm{dR}} = (\mathrm{Log}^n_{\mathrm{dR}}, \nabla_{\mathrm{Log}^n_{\mathrm{dR}}}) \in \mathrm{U}^\dagger_n\,(E/S/K)$$

and a horizontal section $\mathbb{1}^n \in \Gamma(S, e^*\mathrm{Log}^n_{\mathrm{dR}})$, such that the following universal property holds: The pair $(\mathrm{Log}^n_{\mathrm{dR}}, \mathbb{1}^n)$ is the unique pair such that for all $\mathcal{U} \in \mathrm{U}^\dagger_n\,(E/S/K)$ the map

$$\pi_*\underline{\mathrm{Hom}}_{\mathrm{VIC}(E/K)}(\mathrm{Log}^n_{\mathrm{dR}}, \mathcal{U}) \to e^*\mathcal{U}, \quad f \mapsto (e^*f)(\mathbb{1}^n)$$

is an isomorphism in $\mathrm{VIC}\,(S/K)$. The pair $(\mathrm{Log}^n_{\mathrm{dR}}, \mathbb{1}^n)$ is called the n-th *(absolute de Rham) logarithm sheaf.*

Proof. The uniqueness is clear by the universal property. We refer to [Sch14, Theorem 1.3.6] for a proof of existence. Indeed, $\mathrm{Log}^n_{\mathrm{dR}}$ is defined differently in [Sch14, §1.1] and the universal property is shown as one of its properties. $\square$

Remark 5.1.3. Another way to formulate the universal property is as follows. Consider the category consisting of pairs $(\mathcal{U}, s)$ with $\mathcal{U} \in \mathrm{U}_n^{\dagger}(E/S/K)$ and a fixed horizontal section $s \in \Gamma(S, e^*\mathcal{U})$. Morphisms are supposed to be horizontal and respect the fixed section after pullback along e. Then, the universal property reformulates as the fact that this category has an initial object. This initial object is $(\mathrm{Log}^n_{\mathrm{dR}}, \mathbb{1}^n)$.

By restricting the K-connection on the logarithm sheaf to an S-connection we obtain a pair $(\mathrm{Log}^n_{\mathrm{dR,rel}\,S}, \mathbb{1}^n)$ with $\mathrm{Log}^n_{\mathrm{dR,rel}\,S} \in \mathrm{U}_n^{\dagger}(E/S)$ and $\mathbb{1} \in \Gamma(S, e^*\mathrm{Log}^n_{\mathrm{dR,rel}\,S})$. The universal property for $\mathrm{Log}^n_{\mathrm{dR}}$ implies the following universal property for $\mathrm{Log}^n_{\mathrm{dR,rel}\,S}$:

Corollary 5.1.4. *The pair* $(\mathrm{Log}^n_{\mathrm{dR,rel}\,S}, \mathbb{1}^n)$ *is uniquely characterized by the following property: for all* $\mathcal{U} \in \mathrm{U}_n^{\dagger}(E/S)$ *the map*

$$\pi_*\underline{\mathrm{Hom}}_{\mathrm{VIC}(E/S)}(\mathrm{Log}^n_{\mathrm{dR,rel}\,S}, \mathcal{U}) \to e^*\mathcal{U}, \quad f \mapsto (e^*f)(\mathbb{1})$$

is an isomorphism. The pair $(\mathrm{Log}^n_{\mathrm{dR,rel}\,S}, \mathbb{1}^n)$ *is called the* n*-th* relative (de Rham) logarithm sheaf*.*

5.1.2 Basic properties of the de Rham logarithm sheaves

Let us list the basic properties of $\mathrm{Log}^n_{\mathrm{dR}}$. Let us write $\mathcal{H} := \underline{H}^1_{\mathrm{dR}}(E/S)^{\vee}$ and $\mathcal{H}_E := \pi_E^*\mathcal{H}$. The pullback of the Gauss–Manin connection induces a canonical K-connection on $\mathcal{H}_E$.

Proposition 5.1.5. *Let* $n \geq 1$.

(a) *(Transition maps) There are horizontal transition maps* $\mathrm{Log}^{n+1}_{\mathrm{dR}} \to \mathrm{Log}^n_{\mathrm{dR}}$ *identifying* $\mathbb{1}^{n+1}$ *with* $\mathbb{1}^n$ *after pullback along* e*.*

(b) *(short exact sequence) The transition map* $\mathrm{Log}^1_{\mathrm{dR}} \to \mathrm{Log}^0_{\mathrm{dR}}$ *fits into a short exact sequence of vector bundles with integrable* K*-connections*

$$0 \longrightarrow \mathcal{H}_E \longrightarrow \mathrm{Log}^1_{\mathrm{dR}} \longrightarrow \mathcal{O}_E \longrightarrow 0. \tag{5.1}$$

Here, $\mathcal{H}_E$ *is equipped with the pullback of the Gauss–Manin connection.*

(c) *(invariance) There is a horizontal isomorphism* $\mathrm{Log}^n_{\mathrm{dR}} \xrightarrow{\sim} [N]^*\mathrm{Log}^n_{\mathrm{dR}}$ *identifying* $\mathbb{1}^n$ *with* $\mathbb{1}^n$ *after pullback along* e*.*

(d) *(symmetric powers) There are horizontal isomorphisms* $\mathrm{Log}^n_{\mathrm{dR}} \xrightarrow{\sim} \underline{\mathrm{Sym}}^n\,\mathrm{Log}^1_{\mathrm{dR}}$ *mapping* $\mathbb{1}^n$ *to* $\frac{(\mathbb{1}^1)^{\otimes n}}{n!}$ *after pullback along* e*.*

(e) *(Extension class) The connecting homomorphism of the short exact sequence* (5.1) *in relative de Rham cohomology gives a connecting homomorphism*

$$\delta : \mathcal{O}_S = \underline{H}^0_{\mathrm{dR}}(E/S) \to \underline{H}^1_{\mathrm{dR}}(E/S, \mathcal{H}_E) = \mathcal{H} \otimes_{\mathcal{O}_S} \mathcal{H}^{\vee}$$

mapping 1 *to* $\mathrm{id}_{\mathcal{H}}$*.*

(f) *(Compatibility with base change) For a Cartesian diagram*

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "\pi"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

the pullback of $(\mathrm{Log}^n_{\mathrm{dR}}, \mathbb{1})$ *along the Cartesian diagram satisfies the universal property of the n-th logarithm sheaf for E'/S'.*

Proof. These are more or less trivial consequences of the universal property. We refer to [Sch14, Chapter 1.1, and Chapter 1.3] where all properties are discussed. □

Remark 5.1.6. We have used $\underline{\mathrm{Sym}}^n$ in Proposition 5.1.5 (d) to be consistent with [Sch14]. The appearance of $\frac{(\mathbb{1}^1)^{\otimes n}}{n!}$ already indicates that the better choice is using $\underline{\mathrm{TSym}}^n$ as we did for the geometric logarithm sheaves. Since we are working over a field of characteristic zero, there is not much difference between either of both.

The short exact sequence (5.1) gives an extension class $[\mathrm{Log}^1_{\mathrm{dR}}] \in \mathrm{Ext}^1_{\mathrm{VIC}(E/K)}(\mathcal{O}_E, \mathcal{H}_E)$.

Corollary 5.1.7. *The Leray spectral sequence gives us a split short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{H}) \underset{e^*}{\overset{\pi^*}{\rightleftarrows}} \mathrm{Ext}^1_{\mathrm{VIC}(E/K)}(\mathcal{O}_E, \mathcal{H}_E) \xrightarrow{\delta} \mathrm{Hom}_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{H} \otimes \mathcal{H}^\vee) \longrightarrow 0.$$

The class $[\mathrm{Log}^1_{\mathrm{dR}}]$ *is the unique class with* $e^*[\mathrm{Log}^1_{\mathrm{dR}}] = 0$ *and* $\delta([\mathrm{Log}^1_{\mathrm{dR}}]) = \mathrm{id}_{\mathcal{H}}$.

Proof. This is the way [Sch14, §1] defines the first de Rham logarithm sheaf, see the remark below. By the universal property [Sch14, Theorem 1.3.6] both definitions are equivalent. □

Remark 5.1.8. An extension

$$0 \longrightarrow \mathcal{H}_E \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

representing $[\mathrm{Log}^1_{\mathrm{dR}}]$ has in general non-trivial automorphisms. If one additionally fixes a splitting

$$\sigma : e^*\mathcal{E} \xrightarrow{\sim} \mathcal{O}_S \oplus \mathcal{H},$$

then the identity is the only automorphism of the extension class respecting the fixed splitting. This gives an alternative definition of $\mathrm{Log}^1_{\mathrm{dR}}$ as an extension together with a fixed splitting. Then, $\mathrm{Log}^n_{\mathrm{dR}}$ can be defined by taking symmetric powers. This is the definition of Scheider cf. [Sch14, Chapter 1.1].

5.1.3 The geometric logarithm sheaves

The aim of this section is to show that the geometric logarithm sheaves $\mathcal{L}_n^\dagger$ give us a concrete geometric realization of the abstractly defined relative de Rham logarithm sheaves. This is one of the main results of Scheider [Sch14, Theorem 2.3.1]. By taking the universal property seriously we can give a much simpler proof then the original one.

As in Scheider's proof, we need an interpretation of $\mathcal{L}_n^\dagger$ as a Fourier–Mukai transform. Thus, we start recalling some definitions and results appearing in the work of Scheider. As before, let $E^\dagger$ be the universal vectorial extension of $E^\vee$.

Definition 5.1.9. Let $\mathcal{J}$ be the ideal sheaf of $\mathcal{O}_{E^\dagger}$ defined by the unit section. Let $\mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right)$ be the full subcategory of the category of quasi-coherent $\mathcal{O}_{E^\dagger}$-modules $\mathcal{F}$, s.t. $\mathcal{J}^{n+1}\mathcal{F}=0$ and $\mathcal{J}^i\mathcal{F}/\mathcal{J}^{i+1}\mathcal{F}$ is a locally free $\mathcal{O}_S=\mathcal{O}_{E^\dagger}/\mathcal{J}$-module of finite rank for $i=0,...,n$.

In particular, $\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}=\mathcal{O}_{\mathrm{Inf}_e^n E^\dagger}$ is an object in $\mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right)$. For an object $\mathcal{F}\in\mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right)$ let us define

$$\hat{\mathcal{F}}^\dagger := (\mathrm{pr}_E)_*\left((\mathrm{pr}_{E^\dagger})^*\mathcal{F}\otimes_{\mathcal{O}_{E\times_S E^\dagger}}\mathcal{P}^\dagger\right).$$

The integrable $E^\dagger$-connection $\nabla_{\mathcal{P}^\dagger}$ on $\mathcal{P}^\dagger$ induces a canonical S-connection $\nabla_{\hat{\mathcal{F}}^\dagger}$ on $\hat{\mathcal{F}}^\dagger$. We call $\hat{\mathcal{F}}^\dagger$ the Fourier–Mukai transform of $\mathcal{F}$. The following result can be seen as a particular case of the general Fourier–Mukai equivalence due to Laumon [Lau96] between the derived category of $\mathcal{O}_{E^\dagger}$-modules and the derived category of $\mathcal{D}_{E/S}$-modules. The reason for the following non-derived version is that the derived Fourier–Mukai transform of $\mathcal{F}\in\mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right)$ is cohomologically concentrated in one degree.

Proposition 5.1.10 ([Sch14, Thm. 2.2.12, Prop. 2.2.16])**.**

(a) *The Fourier–Mukai transform induces an equivalence of categories:*

$$\mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right)\xrightarrow{\sim}\mathrm{U}_n^\dagger(E/S),\quad \mathcal{F}\mapsto(\hat{\mathcal{F}}^\dagger,\nabla_{\hat{\mathcal{F}}^\dagger})$$

(b) *For* $\mathcal{F}\in\mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right)$ *there is a canonical isomorphism*

$$e^*\hat{\mathcal{F}}^\dagger\xrightarrow{\sim}(\pi_n)_*\mathcal{F}.$$

Here, $\pi_n:\mathrm{Inf}_e^n E^\dagger\to S$ *is the structure morphism of* $\mathrm{Inf}_e^n E^\dagger$*.*

The geometric logarithm sheaves can be expressed as follows:

$$\mathcal{L}_n^\dagger\overset{\text{Def.}}{=}(\mathrm{pr}_E)_*(\mathrm{id}\times\iota_n^\dagger)^*\mathcal{P}^\dagger=(\mathrm{pr}_E)_*\left((\mathrm{pr}_{E^\dagger})^*\mathcal{O}_{\mathrm{Inf}_e^n E^\dagger}\otimes_{\mathcal{O}_{E\times E^\dagger}}\mathcal{P}^\dagger\right)=\left(\widehat{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right)^\dagger$$

Furthermore, the canonical trivialization of $\mathcal{L}_n^\dagger$

$$e^*\mathcal{L}_n^\dagger\xrightarrow{\sim}\mathcal{O}_{\mathrm{Inf}_e^n E^\dagger}=\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}$$

gives us a canonical section $1\in\Gamma(S,\mathcal{O}_{\mathrm{Inf}_e^n E^\dagger})\cong\Gamma(S,e^*\mathcal{L}_n^\dagger)$ of $e^*\mathcal{L}_n^\dagger$.

Theorem 5.1.11 ([Sch14, §2.3, cf. Thm 2.3.1])**.** *The pair $(\mathcal{L}_n^\dagger, 1)$ satisfies the universal property of the n-th relative de Rham logarithm sheaf.*

Proof. Let us sketch a simpler proof than the original one in [Sch14]. Let $\mathcal{G} \in \mathrm{U}_n^\dagger(E/S)$. By the equivalence

$$\widehat{(\cdot)}^\dagger : \mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right) \xrightarrow{\sim} \mathrm{U}_n^\dagger(E/S)$$

we may assume $\mathcal{G} = \hat{\mathcal{F}}^\dagger$ for some $\mathcal{F} \in \mathrm{U}_n^\dagger(E/S)$. Then, we have the following chain of isomorphisms

$$\pi_* \underline{\mathrm{Hom}}_{\mathrm{U}_n^\dagger(E/S)}\left(\widehat{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}^\dagger, \mathcal{G}\right) \overset{(A)}{\cong} (\pi_n)_* \underline{\mathrm{Hom}}_{\mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right)}\left(\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}, \mathcal{F}\right) =$$

$$= (\pi_n)_* \mathcal{F} \overset{(B)}{\cong} e^* \hat{\mathcal{F}}^\dagger = e^* \mathcal{G}$$

where (A) is induced by the Fourier–Mukai type equivalence of categories and (B) is Proposition 5.1.10 (b). It is straightforward to check that this chain of isomorphisms sends f to $(e^* f)(1)$, which proves the universal property of $\left(\widehat{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}^\dagger, 1\right)$. □

Remark 5.1.12. We have already remarked that $(\mathrm{Log}^n_{\mathrm{dR,rel}\,S}, \mathbb{1}^n)$ can be seen as initial object in the category of pairs $(\mathcal{U}, s)$ with $\mathcal{U} \in \mathrm{U}_n^\dagger(E/S)$ and $s \in \Gamma(S, e^*\mathcal{U})$. Similarly, one can define the category of pairs $(\mathcal{F}, s)$ with $\mathcal{F} \in \mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}\right)$ and $s \in \Gamma(S, (\pi_n)_*\mathcal{F})$ whose morphisms are assumed to identify the chosen sections. This category has an obvious initial object $(\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}, 1)$. Then, the above result is nothing else than the observation that $\hat{(\cdot)}^\dagger$ induces an equivalence of categories between both categories of pairs. In particular, it identifies initial objects.

Corollary 5.1.13. *The connecting homomorphism*

$$\delta : \mathcal{O}_S = \underline{H}^0_{\mathrm{dR}}(E/S) \to \underline{H}^1_{\mathrm{dR}}(E/S, \mathcal{H}_E) = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{H}, \mathcal{H})$$

associated with the short exact sequence of $\mathcal{L}_1^\dagger$ in relative de Rham cohomology satisfies $\delta(1) = \mathrm{id}_{\mathcal{H}}$.

Proof. We have $[\mathcal{L}_n^\dagger] = [\mathrm{Log}^n_{\mathrm{dR,rel}\,S}]$. Now, use Proposition 5.1.5 (e). □

Up to now we have considered just the geometric logarithm sheaves $\mathcal{L}_n^\dagger$ in this section. In Chapter 2 we also defined and studied $\mathcal{L}_n$. Indeed, in complete analogy we can define and prove a corresponding universal property for the pair $(\mathcal{L}_n, 1)$ with $1 \in \Gamma(S, e^*\mathcal{L}_n)$:

Definition 5.1.14.

(a) Write $VB(E)$ for the category of vector bundles on E. A vector bundle $\mathcal{U}$ on E is called unipotent of length n for E/S if there is a descending filtration $A^\bullet\mathcal{U}$

$$\mathcal{U} = A^0\mathcal{U} \subseteq A^1\mathcal{U} \subseteq ... \subseteq A^{n+1}\mathcal{U} = 0$$

s.t. $\mathrm{gr}^i_A = A^i\mathcal{U}/A^{i+1}\mathcal{U} = \pi^*\mathcal{G}$ for some vector bundle $\mathcal{G}$ on S.

(b) Let $U_n(E/S)$ be the full subcategory of $VB(E)$ consisting of unipotent vector bundles of length n for E/S.

(c) Let $\mathcal{I} \subseteq \mathcal{O}_{E^\vee}$ be the ideal sheaf of the zero section. Let $\mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\vee}/\mathcal{I}^{n+1}}\right)$ be the full subcategory of the category of quasi-coherent $\mathcal{O}_{E^\vee}$-modules consisting of $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_{E^\vee}/\mathcal{I}^{n+1}}$ s.t. $\mathcal{I}^i\mathcal{F}/\mathcal{I}^{i+1}\mathcal{F}$ is a locally free $\mathcal{O}_S$-module of finite rank.

For $\mathcal{F} \in \mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\vee}/\mathcal{I}^{n+1}}\right)$ we define the Fourier–Mukai transform

$$\hat{\mathcal{F}} := (\mathrm{pr}_E)_*\left((\mathrm{pr}_{E^\vee})^*\mathcal{F} \otimes_{\mathcal{O}_{E\times E^\vee}} \mathcal{P}\right).$$

In complete analogy to Proposition 5.1.10 we have:

Proposition 5.1.15.

(a) *The Fourier–Mukai transform induces an equivalence of categories:*

$$\mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\vee}/\mathcal{I}^{n+1}}\right) \xrightarrow{\sim} \mathrm{U}_n(E/S), \quad \mathcal{F} \mapsto \hat{\mathcal{F}}$$

(b) *For $\mathcal{F} \in \mathrm{U}_n\left(\mathrm{Mod}_{\mathcal{O}_{E^\vee}/\mathcal{I}^{n+1}}\right)$ there is a canonical isomorphism:*

$$e^*\hat{\mathcal{F}} \xrightarrow{\sim} (\pi_n)_*\mathcal{F}.$$

Here, $\pi_n : \mathrm{Inf}_e^n E^\vee \to S$ is the structure morphism of $\mathrm{Inf}_e^n E^\vee$.

Proof. The proof of the corresponding result in [Sch14, Thm. 2.2.12, Prop. 2.2.16] transfers verbatim to the situation without connection by replacing the Fourier–Mukai–Laumon equivalence by the classical Fourier–Mukai equivalence. □

We have

$$\mathcal{L}_n \overset{\text{Def.}}{=} (\mathrm{pr}_E)_*(\mathrm{id}\times\iota_n)^*\mathcal{P} = (\mathrm{pr}_E)_*\left((\mathrm{pr}_{E^\vee})^*\mathcal{O}_{\mathrm{Inf}_e^n E^\vee} \otimes_{\mathcal{O}_{E\times E^\vee}} \mathcal{P}\right) = \widehat{\mathcal{O}_{E^\vee}/\mathcal{I}^{n+1}}$$

Theorem 5.1.16. *The pair $(\mathcal{L}_n, 1)$ is the unique pair, up to unique isomorphism, consisting of a unipotent vector bundle of length n for E/S and a section*

$$1 \in \Gamma(S, e^*\mathcal{L}_n) = \Gamma(S, \mathcal{O}_{\mathrm{Inf}_e^n E^\vee})$$

such that the following universal property holds: For all $\mathcal{U} \in \mathrm{U}_n(E/S)$ the map

$$\pi_*\underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{L}_n, \mathcal{G}) \to e^*\mathcal{G}, \quad f \mapsto (e^*f)(1)$$

is an isomorphism of $\mathcal{O}_S$-modules.

Proof. "Un-daggering" the proof of Theorem 5.1.11. □

Corollary 5.1.17. *By forgetting the connection of $\mathcal{L}_n^\dagger$ the universal property of $\mathcal{L}_n$ gives a map $\mathcal{L}_n \to \mathcal{L}_n^\dagger$. This map coincides with the canonical inclusion*

$$\mathcal{L}_n \hookrightarrow \mathcal{L}_n^\dagger.$$

Proof. The pullback of the canonical inclusion along e maps 1 to 1. $\square$

Corollary 5.1.18. *Let*

$$\delta : \ker\left(\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E, \pi^*\underline{\omega}_{E/S}) \xrightarrow{e^*} \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, \underline{\omega}_{E/S})\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_{E/S}, \underline{\omega}_{E/S})$$

be the isomorphism obtained from the Leray spectral sequence in Zariski cohomology. Then

$$\delta([\mathcal{L}_1]) = \mathrm{id} \in \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_{E/S}, \underline{\omega}_{E/S}).$$

Proof. Let us recall from Section 2.1.2 that the group

$$\ker\left(\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E, \pi^*\underline{\omega}_{E/S}) \xrightarrow{e^*} \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, \underline{\omega}_{E/S})\right)$$

classifies pairs $(\mathcal{F}, \sigma)$ consisting of an extension $\mathcal{F}$ of $\mathcal{O}_E$ by $\pi^*\underline{\omega}_{E/S}$ together with a fixed splitting $\sigma : \mathcal{O}_S \oplus \underline{\omega}_{E/S} \xrightarrow{\sim} e^*\mathcal{F}$. Such a pair has no non-trivial automorphisms. In particular, there is a unique class

$$[(\mathit{L}, \sigma)] \in \ker\left(\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{O}_E, \pi^*\underline{\omega}_{E/S}) \xrightarrow{e^*} \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_S, \underline{\omega}_{E/S})\right)$$

corresponding to the identity $\mathrm{id}_{\underline{\omega}_{E/S}}$. This class is unique up to unique isomorphism. Further, any other extension with a fixed splitting is obtained in a unique way as a pushout of (L, σ) (cf. proof of Corollary 2.1.4). But this is just a reformulation of the fact that the pair

$$(\mathit{L}, \sigma(1, 0))$$

satisfies the universal property of Theorem 5.1.16. Thus, there is a unique isomorphism between $\mathcal{L}_1$ and L compatible with the splittings along e. In particular, the extension classes coincide and the corollary follows. $\square$

5.1.4 Extending the connection of the logarithm sheaves

Owing to Theorem 5.1.11 we have a geometric interpretation of the relative de Rham logarithm sheaves. For the description of the de Rham realization of the elliptic polylogarithm we will need the absolute versions. Let

$$\mathrm{res}_{/S} : \mathrm{VIC}\,(E/K) \to \mathrm{VIC}\,(E/S)\,, \quad (\mathcal{F}, \nabla_{\mathcal{F}}) \mapsto (\mathcal{F}, \mathrm{res}_{/S}(\nabla_{\mathcal{F}}))$$

be the functor restricting an absolute connection to an S-connection. As already observed by Scheider one can immediately deduce the following result from the universal property of the de Rham logarithm sheaves:

Lemma/Definition 5.1.19. There exists a unique K-connection $\nabla_{\mathcal{L}_{n,abs}^\dagger}$ on the geometric logarithm sheaf $\mathcal{L}_n^\dagger$ extending the connection $\nabla_{\mathcal{L}_n^\dagger}$, i. e. $\mathrm{res}_{/S}(\nabla_{\mathcal{L}_{n,abs}^\dagger}) = \nabla_{\mathcal{L}_n^\dagger}$, such that $\left(\left(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_{n,abs}^\dagger}\right), \mathbb{1}^n\right)$ satisfies the universal property of the absolute n-th de Rham logarithm sheaf.

Proof. [Sch14, Prop. 2.1.4] Let us sketch the proof. Let $\left((\mathrm{Log}_{\mathrm{dR}}^n, \nabla_{\mathrm{Log}_{\mathrm{dR}}^n}), \mathbb{1}\right)$ satisfy the universal property of the absolute de Rham logarithm sheaf. Then, the pair $\left((\mathrm{Log}_{\mathrm{dR}}^n, \mathrm{res}_{/S}(\nabla_{\mathrm{Log}_{\mathrm{dR}}^n})), \mathbb{1}\right)$ satisfies the universal property of the n-th relative de Rham logarithm sheaf. Thus, there is a unique S-horizontal isomorphism

$$\alpha : \mathcal{L}_n^\dagger \xrightarrow{\sim} \mathrm{Log}_{\mathrm{dR}}^n.$$

By transport of structure along the isomorphism α we obtain an integrable K-connection $\alpha^*\nabla_{\mathrm{Log}_{\mathrm{dR}}^n}$ on $\mathcal{L}_n^\dagger$ which restricts to $\nabla_{\mathcal{L}_n^\dagger}$ and satisfies the universal property of the absolute de Rham logarithm sheaves. This proves existence. Uniqueness follows after applying $\mathrm{res}_{/S}$ and using the universal property of the relative logarithm sheaf. □

Let us define $\mathcal{H}_U := \pi_U^*\mathcal{H}$ for π_U the structure map of $U = E \setminus e(S)$. The aim of this section is to give a more explicit description of the absolute connection on the geometric de Rham logarithm sheaves via the Katz splitting. More precisely, in Corollary 3.3.10 we have already constructed an integrable K-connection ∇_U prolonging $\nabla_{\mathcal{L}_1^\dagger}$ and making

$$0 \longrightarrow \mathcal{H}_U \longrightarrow \mathcal{L}_1^\dagger|_U \longrightarrow \mathcal{O}_U \longrightarrow 0.$$

horizontal. We want to prove that ∇_U coincides with the unique prolongation defined in Lemma/Definition 5.1.19. While $(\mathrm{Log}_{\mathrm{dR}}^n, \mathbb{1}^n)$ satisfies a universal property, there is no characterizing property for $\mathrm{Log}_{\mathrm{dR}}^n|_U$. Indeed, $\mathrm{Log}_{\mathrm{dR}}^n|_U$ has in general non-trivial automorphism, which is an obvious obstruction against a universal property. This makes it more difficult to characterize the unique K-connection on $\mathrm{Log}_{\mathrm{dR}}^n|_U$. Let us have a slightly closer look on the automorphisms of the extension

$$0 \longrightarrow \mathcal{H}_E \longrightarrow \mathrm{Log}_{\mathrm{dR}}^1 \longrightarrow \mathcal{O}_E \longrightarrow 0. \tag{5.2}$$

It is easily seen that the automorphism group of (5.2) is given by

$$\mathrm{Hom}_{\mathrm{VIC}(E/K)}\left(\mathcal{O}_E, \mathcal{H}_E\right) \cong \underline{H}_{\mathrm{dR}}^0\left(E/K, \mathcal{H}_E\right) \cong \underline{H}_{\mathrm{dR}}^0\left(S/K, \mathcal{H}\right).$$

Similarly, the automorphism group of

$$0 \longrightarrow \mathcal{H}_U \longrightarrow \mathrm{Log}_{\mathrm{dR}}^1|_U \longrightarrow \mathcal{O}_U \longrightarrow 0.$$

is

$$\mathrm{Hom}_{\mathrm{VIC}(U/K)}\left(\mathcal{O}_U, \mathcal{H}_U\right) \cong \underline{H}_{\mathrm{dR}}^0\left(U/K, \mathcal{H}_U\right) \cong \underline{H}_{\mathrm{dR}}^0\left(S/K, \mathcal{H}\right).$$

In this context the following result is useful:

Lemma 5.1.20. *Let $N > 3$ and E_N/M_N be the universal elliptic curve with $\Gamma(N)$-level structure over K. Then*

$$H^0_{\mathrm{dR}}\left(M_N, \mathrm{Sym}^k\mathcal{H}\right) = 0$$

for all $k \geq 1$.

Proof. A purely algebraic proof is implicitly given in [Sch98]. Alternatively, one can show the vanishing of $H^0_{\mathrm{dR}}\left(M_N, \underline{\mathrm{Sym}}^k\,\mathcal{H}\right)$ after analytification. Then, the statement boils down, using the Riemann–Hilbert correspondence, to the obvious vanishing result

$$H^0(\Gamma(N), \underline{\mathrm{Sym}}^k\,\mathbb{Z}^2) = 0, \quad k \geq 1$$

in group cohomology. Here, $\mathbb{Z}^2$ is the regular representation of $\Gamma(N) \subseteq \mathrm{Sl}_2(\mathbb{Z})$ on $\mathbb{Z}^2$. $\square$

We get immediately the following corollaries:

Corollary 5.1.21. *Let E_N/M_N be the universal elliptic curve with $\Gamma(N)$-level structure over K and let $\mathrm{Log}^1_{\mathrm{dR}}$ be the first absolute de Rham logarithm sheaf for E_N/M_N. Then, the short exact sequences in the Abelian category $\mathrm{VIC}\,(E/K)$ resp. $\mathrm{VIC}\,(U/K)$*

$$0 \longrightarrow \mathcal{H}_E \longrightarrow \mathrm{Log}^1_{\mathrm{dR}} \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

and

$$0 \longrightarrow \mathcal{H}_U \longrightarrow \mathrm{Log}^1_{\mathrm{dR}}|_U \longrightarrow \mathcal{O}_U \longrightarrow 0.$$

have no non-trivial automorphisms.

This result will be helpful since it makes the above extensions of $\mathrm{Log}^1_{\mathrm{dR}}$ resp. $\mathrm{Log}^1_{\mathrm{dR}}|_U$ in the universal situation rigid without fixing some additional structure like a splitting. Let us return to the general situation of this chapter i. e. an elliptic curve $E/S/K$. Restricting an extension on E to U gives an isomorphism of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{H}) \arrow[r, "\pi^*"] \arrow[d, equal] & \mathrm{Ext}^1_{\mathrm{VIC}(E/K)}(\mathcal{O}_E, \mathcal{H}_E) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathrm{VIC}(S/K)}(\mathcal{H}, \mathcal{H}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{H}) \arrow[r, "\pi^*"] & \mathrm{Ext}^1_{\mathrm{VIC}(U/K)}(\mathcal{O}_U, \mathcal{H}_U) \arrow[r] & \mathrm{Hom}_{\mathrm{VIC}(S/K)}(\mathcal{H}, \mathcal{H}) \arrow[r] & 0
\end{tikzcd}$$

Using the above rigidity result allows us to characterize the absolute connection $\nabla_{\mathcal{L}^\dagger_1, abs}$ on U as follows:

Corollary 5.1.22. *Let E_N/M_N be the universal elliptic curve with $\Gamma(N)$-level structure over K and let $(\mathcal{L}^\dagger_1, \nabla_{\mathcal{L}^\dagger_1})$ be the first geometric logarithm sheaf for E_N/M_N. The K-connection $\nabla_{\mathcal{L}^\dagger_1, abs}|_U$ is the unique integrable K-connection on $\mathcal{L}^\dagger_1|_U$ which extends the S-connection $\nabla_{\mathcal{L}^\dagger_1}|_U$, makes*

$$0 \longrightarrow \mathcal{H}_U \longrightarrow \mathcal{L}^\dagger_1|_U \longrightarrow \mathcal{O}_U \longrightarrow 0$$

a short exact sequence in $\mathrm{VIC}\,(U/K)$ and satisfies $[\mathcal{L}^\dagger_1, \nabla_{\mathcal{L}^\dagger_1, abs}] = [\mathrm{Log}^1_{\mathrm{dR}}|_U]$.

Proof. Assume we have a second integrable K-connection $\tilde{\nabla}_U$ on $\mathcal{L}_1^\dagger$ extending the S-connection $\nabla_{\mathcal{L}_1^\dagger}$ and giving rise to the extension class of the absolute logarithm sheaf. By the above rigidity property (cf. Corollary 5.1.21) and the above isomorphism

$$\mathrm{Ext}^1_{\mathrm{VIC}(E/K)}(\mathcal{O}_E, \mathcal{H}_E) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{VIC}(U/K)}(\mathcal{O}_U, \mathcal{H}_U)$$

both connections $\tilde{\nabla}_U$ and $\nabla_{\mathcal{L}_n^\dagger}|_U$ extend uniquely to integrable K-connections on $\mathcal{L}^\dagger$. Let us denote the extended K-connection of $\tilde{\nabla}_U$ by $\tilde{\nabla}$. The unique extension of $\nabla_{\mathcal{L}_1^\dagger,abs}|_U$ is $\nabla_{\mathcal{L}_1^\dagger,abs}$. By the universal property of $\left((\mathcal{L}_1^\dagger, \nabla_{\mathcal{L}_n^\dagger,abs}), 1\right)$ there is a unique K-horizontal isomorphism

$$\varphi : (\mathcal{L}_1^\dagger, \nabla_{\mathcal{L}_n^\dagger,abs}) \xrightarrow{\sim} (\mathcal{L}_1^\dagger, \tilde{\nabla})$$

which identifies $1 \in \Gamma(S, e^*\mathcal{L}_1^\dagger)$ with 1. Applying the restriction functor $\mathrm{res}_{/S}$ to this morphism gives an S-horizontal isomorphism

$$(\mathcal{L}_1^\dagger, \nabla_{\mathcal{L}_n^\dagger}) \xrightarrow{\sim} (\mathcal{L}_1^\dagger, \mathrm{res}_{/S}(\tilde{\nabla})).$$

Since $\nabla_{\mathcal{L}_1^\dagger}$ and $\mathrm{res}_{/S}(\tilde{\nabla})$ coincide on the schematically dense open subset U, we deduce $\nabla_{\mathcal{L}_n^\dagger} = \mathrm{res}_{/S}(\tilde{\nabla})$. Thus, φ is an automorphism of $(\mathcal{L}_1^\dagger, \nabla_{\mathcal{L}_n^\dagger})$ mapping 1 to 1 after applying e^*. By its universal property we conclude $\varphi = \mathrm{id}$ and thereby $\nabla_{\mathcal{L}_n^\dagger,abs} = \tilde{\nabla}$. $\square$

Now, we can prove the following result.

Proposition 5.1.23. *Let $E/S/K$ be an elliptic curve over a smooth separated K-scheme. Let ∇_U be the K-connection defined in Corollary 3.3.10. Then, $\nabla_{\mathcal{L}_1^\dagger,abs} = \nabla_U$, i. e. the absolute connection $\nabla_{\mathcal{L}_1^\dagger,abs}|_U$ is the unique K-connection making*

$$0 \longrightarrow \mathcal{H}_U \longrightarrow \mathcal{L}_1^\dagger|_U \longrightarrow \mathcal{O}_U \longrightarrow 0. \tag{5.3}$$

horizontal and satisfying $\nabla_{\mathcal{L}_1^\dagger,abs}(\kappa_{\mathcal{L}}(1,0)) = (\mathrm{id}_{\mathcal{H}_U} \otimes \kappa_\Omega)(\mathcal{I}d_{\mathcal{H}})$.

Proof. Let us first show that $(\mathcal{L}_1^\dagger|_U, \nabla_U)$ defines the same extension class as $\mathrm{Log}^1_{\mathrm{dR}}|_U$. The commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathrm{VIC}(U/K)}(\mathcal{O}_U, \mathcal{H}_U) & \xrightarrow{\delta_{U/K}} & \mathrm{Hom}_{\mathrm{VIC}(S/K)}(\mathcal{H}, \mathcal{H}) \\ \downarrow{\scriptstyle \mathrm{res}_{/S}} & & \downarrow \\ \mathrm{Ext}^1_{\mathrm{VIC}(U/S)}(\mathcal{O}_U, \mathcal{H}_U) & \xrightarrow{\delta_{U/S}} & \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{H}, \mathcal{H}) \end{array}$$

and the definition of ∇_U show that $[\mathcal{L}_1^\dagger, \nabla_U]$ maps to $\mathrm{id}_{\mathcal{H}}$ under $\delta_{U/K}$. By the characterization of the class $[\mathrm{Log}^1_{\mathrm{dR}}]$ in Corollary 5.1.7 and the isomorphism

$$\mathrm{Ext}^1_{E/K}(\mathcal{O}_U, \mathcal{H}_U) \xrightarrow{\sim} \mathrm{Ext}^1_{U/K}(\mathcal{O}_U, \mathcal{H}_U)$$

it remains to prove that $[\mathcal{L}_1^\dagger, \nabla_U]$ maps to zero under

$$\begin{aligned}\mathrm{Ext}^1_{U/K}(\mathcal{O}_U, \mathcal{H}_U) \xleftarrow{\sim} \mathrm{Ext}^1_{E/K}(\mathcal{O}_E, \mathcal{H}_E) \xrightarrow{e^*} \mathrm{Ext}^1_{S/K}(\mathcal{O}_S, \mathcal{H}) = \\ = \underline{H}^1_{\mathrm{dR}}\,(S/K, \mathcal{H}) \overset{\lambda^*}{\cong} \\ \cong H^1_{\mathrm{dR}}\left(S/K, \underline{H}^1_{\mathrm{dR}}\,(E/S)\right). \qquad (5.4)\end{aligned}$$

Here, we have used our fixed autoduality λ, as in Eq. (1.10), to identify $\mathcal{H}$ with $\underline{H}^1_{\mathrm{dR}}\,(E/S)$. Let us write $\xi_{E/S} \in H^1_{\mathrm{dR}}\left(S/K, \underline{H}^1_{\mathrm{dR}}\,(E/S)\right)$ for the image of $[\mathcal{L}_1^\dagger, \nabla_U]$ under (5.4). The assignment

$$E/S \mapsto \xi_{E/S} \in H^1_{\mathrm{dR}}\left(S/K, \underline{H}^1_{\mathrm{dR}}\,(E/S)\right)$$

gives a well-defined cohomology class for any elliptic curve over a smooth separated K-scheme. Furthermore, the assignment is compatible with base change. Thus, $\xi_{E/S}$ defines (the cohomology class of) a modular form of weight 3 and full level $\mathrm{Sl}_2(\mathbb{Z})$. Since there is no such non-zero modular form, we conclude $\xi_{E/S} = 0$ for any E/S. This proves that $[\mathcal{L}_1^\dagger, \nabla_U] = [\mathrm{Log}^1_{\mathrm{dR}}]$. It remains to show $\nabla_U = \nabla_{\mathcal{L}_1^\dagger, abs}|_U$. So far, we only know that they are isomorphic. But we can prove the asserted equality after a finite étale base change. In particular, if we fix $N > 3$ we may assume that E/S is equipped with a level N-structure. Since both ∇_U and $\nabla_{\mathcal{L}_1^\dagger, abs}|_U$ are compatible with base change, it is enough to prove the equality for the universal elliptic curve E_N with level N-structure. But in this case the equality of both connections follows from the 'rigidity' in the universal situation, cf. Corollary 5.1.22. □

The above proposition gives us an explicit way to describe the absolute logarithm sheaf via the Poincaré bundle and the Katz splitting.

5.2 The de Rham realization of the elliptic polylogarithm

The aim of this section is to give an explicit description of the de Rham realization of the elliptic polylogarithm in terms of the canonical section of the Poincaré bundle for arbitrary families of elliptic curves E/S over a smooth separated K-scheme S. This can be seen as an algebraic version of the work of Scheider. Indeed, on the analytification of the universal elliptic curve with level $\Gamma(N)$-structure we can choose the Jacobi theta function as trivializing section of the Poincaré bundle. Then, it is straight-forward to check that the analytification of our description specializes to the explicit analytic description in [Sch14, §3].

As before, we assume S to be smooth separated over $\mathrm{Spec}\, K$.

5.2.1 Definition of the polylogarithm class

Let us recall the definition of the de Rham cohomology class of the polylogarithm. We refer to [Sch14, Chapter 1.5] for more details. The kernel $\ker(\mathrm{Log}^n_{\mathrm{dR}} \twoheadrightarrow \mathrm{Log}^i_{\mathrm{dR}})$ for $0 \leq i \leq n$ defines the *unipotent* filtration on $\mathrm{Log}^n_{\mathrm{dR}}$. The spectral sequence associated with this filtration allows to compute the relative de Rham cohomology of the logarithm sheaves:

Proposition 5.2.1 ([Sch14, Thm 1.2.1])**.**

(a) *For $i = 0, 1$ the transition maps $\mathrm{Log}^{n+1}_{\mathrm{dR}} \to \mathrm{Log}^n_{\mathrm{dR}}$ induce the zero map*

$$\underline{H}^i_{\mathrm{dR}}\left(E/S, \mathrm{Log}^{n+1}_{\mathrm{dR}}\right) \xrightarrow{0} \underline{H}^i_{\mathrm{dR}}\left(E/S, \mathrm{Log}^n_{\mathrm{dR}}\right)$$

in de Rham cohomology. In particular, the pro-system $\left(\underline{H}^i_{\mathrm{dR}}\left(E/S, \mathrm{Log}^n_{\mathrm{dR}}\right)\right)_{n\geq 0}$ is Mittag-Leffler zero for $i = 0, 1$.

(b) *For $i = 2$ the transition maps induce the following chain of isomorphisms*

$$\underline{H}^2_{\mathrm{dR}}\left(E/S, \mathrm{Log}^n_{\mathrm{dR}}\right) \xrightarrow{\sim} \underline{H}^2_{\mathrm{dR}}\left(E/S, \mathrm{Log}^{n-1}_{\mathrm{dR}}\right) \xrightarrow{\sim} \ldots \underline{H}^2_{\mathrm{dR}}\left(E/S, \mathcal{O}_E\right) \xrightarrow{\sim} \mathcal{O}_S$$

where the last isomorphism is the trace isomorphism $\underline{H}^2_{\mathrm{dR}}\left(E/S, \mathcal{O}_E\right) \xrightarrow{\sim} \mathcal{O}_S$.

Let us fix some $D > 1$. Let us define the sections $1_e, 1_{E[D]} \in \Gamma(E[D], \mathcal{O}_{E[D]})$ as follows: Let $1_{E[D]}$ correspond to $1 \in \mathcal{O}_{E[D]}$ and 1_e correspond to the image of $1 \in e_*\mathcal{O}_S$ under the canonical map

$$e_*\mathcal{O}_S \to \mathcal{O}_{E[D]}.$$

Combining the above result about the cohomology of $\mathrm{Log}^n_{\mathrm{dR}}$ with the localization sequence for

$$\begin{tikzcd} U_D := E \setminus E[D] \arrow[r, hook, "j_D"] \arrow[dr, "\pi_{U_D}"'] & E \arrow[d, "\pi"] & E[D] \arrow[l, hook', "i_D"'] \arrow[dl, "\pi_{E[D]}"] \\ & S & \end{tikzcd} \tag{5.5}$$

in de Rham cohomology gives the following.

Lemma 5.2.2 ([Sch14, §1.5.2, Lemma 1.5.4])**.** *Let us write $\mathcal{H}_{E[D]} := \pi^*_{E[D]}\mathcal{H}$ and $\mathcal{H}_{U_D} := \pi^*_{U_D}\mathcal{H}$. The localization sequence in de Rham cohomology for* (5.5) *induces an exact sequence:*

$$0 \longrightarrow \varprojlim_n H^1_{\mathrm{dR}}\left(U_D/K, \mathrm{Log}^n_{\mathrm{dR}}\right) \xrightarrow{\mathrm{Res}} \prod_{k=0}^\infty H^0_{\mathrm{dR}}\left(E[D]/K, \underline{\mathrm{Sym}}^k\, \mathcal{H}_{E[D]}\right) \xrightarrow{\sigma} K.$$

If we view the horizontal section $D^2 \cdot 1_e - 1_{E[D]} \in \Gamma(E[D], \mathcal{O}_{E[D]})$ as sitting in degree zero of

$$\prod_{k=0}^{\infty} \underline{H}^0_{\mathrm{dR}}\left(E[D]/K, \underline{\mathrm{Sym}}^k\, \mathcal{H}_{E[D]}\right),$$

it is contained in the kernel of the augmentation map σ.

Proof. For details see [Sch14, §1.5.2]. For the convenience of the reader let us recall how to deduce the above sequence from the localization sequence. The localization sequence and the vanishing of $\varprojlim_n H^1_{\mathrm{dR}}(E, \mathrm{Log}^n_{\mathrm{dR}}) = 0$ gives

$$0 \longrightarrow \varprojlim_n H^1_{\mathrm{dR}}(U_D, \mathrm{Log}^n_{\mathrm{dR}}) \xrightarrow{\mathrm{Res}} \varprojlim_n H^0_{\mathrm{dR}}(E[D], i_D^*\mathrm{Log}^n_{\mathrm{dR}}) \longrightarrow \varprojlim_n H^2_{\mathrm{dR}}(E, \mathrm{Log}^n_{\mathrm{dR}}).$$

Now, the exact sequence in the claim follows by Proposition 5.2.1 and the isomorphism

$$i_D^*\mathrm{Log}^n_{\mathrm{dR}} \xrightarrow{\sim} i_D^*[D]^*\mathrm{Log}^n_{\mathrm{dR}} = \pi_{E[D]}^* e^*\mathrm{Log}^n_{\mathrm{dR}} \xrightarrow{\sim} \bigoplus_{k=0}^{n} \underline{\mathrm{Sym}}^k\, \mathcal{H}_{E[D]}.$$

□

Definition 5.2.3. Let $\mathrm{pol}_{D,\mathrm{dR}} = (\mathrm{pol}^n_{D,\mathrm{dR}})_{n\geq 0} \in \varprojlim_n \underline{H}^1_{\mathrm{dR}}(U_D/K, \mathrm{Log}^n_{\mathrm{dR}})$ be the unique pro-system of cohomology classes which maps to $D^2 1_e - 1_{E[D]}$ under the residue map. We call $\mathrm{pol}_{D,\mathrm{dR}}$ *the (D-variant) of the elliptic polylogarithm.*

Remark 5.2.4. The *classical polylogarithm* in de Rham cohomology

$$(\mathrm{pol}^n_{\mathrm{dR}})_{n\geq 0} \in \varprojlim_n \underline{H}^1_{\mathrm{dR}}(U/K, \mathcal{H}_U^\vee \otimes_{\mathcal{O}_U} \mathrm{Log}^n_{\mathrm{dR}})$$

is defined as the unique element mapping to $\mathrm{id}_{\mathcal{H}}$ under the isomorphism

$$\varprojlim_n \underline{H}^1_{\mathrm{dR}}(U/K, \mathcal{H}_U^\vee \otimes_{\mathcal{O}_U} \mathrm{Log}^n_{\mathrm{dR}}) \xrightarrow{\sim} \prod_{k=1}^{\infty} \underline{H}^0_{\mathrm{dR}}\left(S/K, \mathcal{H}^\vee \otimes_{\mathcal{O}_S} \underline{\mathrm{Sym}}^k\, \mathcal{H}\right).$$

This isomorphism comes from the localization sequence for $U := E \setminus \{e\} \hookrightarrow E$. For details we refer to [Sch14, §1.5.1]. Indeed, there is not much difference between the classical polylogarithm and its D-variant. For a comparison of both we refer to [Sch14, §1.5.3].

5.2.2 Lifting the canonical sections of the geometric logarithm sheaves

In the previous section we saw that the canonical S-connection $\nabla_{\mathcal{L}_n^\dagger}$ on $\mathcal{L}_n^\dagger$ extends uniquely to an integrable K-connection $\nabla_{\mathcal{L}_n^\dagger,abs}$ such that $\mathcal{L}_n^\dagger$ satisfies the universal property of the n-th absolute de Rham logarithm sheaf. We would like to relate the canonical sections of the geometric logarithm sheaves

$$l_n^D \in \Gamma\left(E, \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D])\right) \subseteq \Gamma\left(E, \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D])\right)$$

to the de Rham realization of the elliptic polylogarithm. In a first step we have to lift them to absolute 1-forms with values in the logarithm sheaves which are in the kernel for the differential in the absolute de Rham complex of $\mathcal{L}_1^\dagger$. As always it is the Katz splitting which allows us to do this. The map induced by the Katz splitting

$$\mathrm{split}_\kappa : \mathcal{L}_n|_U \xrightarrow{\sim} \bigoplus_{k=0}^{n} \underline{\mathrm{TSym}}^k\, \pi_U^*(\underline{\omega}_{E^\vee/S})$$

is an isomorphism since we are working over a field of characteristic zero. Further, recall that this splitting is compatible with the transition maps of $\mathcal{L}_n$, i. e., by passing to the limit we obtain

$$\varprojlim_n \mathcal{L}_n|_U \xrightarrow{\sim} \prod_{k=0}^{\infty} \underline{\mathrm{TSym}}^k\, \pi_U^*(\underline{\omega}_{E^\vee/S}). \tag{5.6}$$

Define $(\lambda_k^D)_{k\geq 0}$, $\lambda_k^D \in \Gamma\left(U_D, \underline{\mathrm{TSym}}^k\, \pi_{U_D}^*(\underline{\omega}_{E^\vee/S})\right)$ as the image of l_n^D under (5.6). For $i \geq 0$ let

$$\mathrm{KS}_{i+1} : \left(\underline{\mathrm{TSym}}^{i+1}\, \pi_{U_D}^*(\underline{\omega}_{E^\vee/S})\right) \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/S} \to \left(\underline{\mathrm{TSym}}^{i}\, \pi_{U_D}^*(\underline{\omega}_{E^\vee/S})\right) \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/K}$$

be the composition of

$$\left(\underline{\mathrm{TSym}}^{i+1}\, \pi_{U_D}^*(\underline{\omega}_{E^\vee/S})\right) \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/S} \xleftarrow[\sim]{\frac{\mathrm{mult}}{i+1}\otimes\mathrm{id}} \pi_{U_D}^* \left(\underline{\mathrm{TSym}}^{i}\, \underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{E/S}\right)$$

with the Kodaira–Spencer map

$$\pi_{U_D}^* \left(\underline{\mathrm{TSym}}^{i}\, \underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_S} \underline{\omega}_{E/S}\right) \xrightarrow{\pi^*(\mathrm{id}\otimes\mathrm{KS})} \pi_{U_D}^* \left(\underline{\mathrm{TSym}}^{i}\, \underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_S} \Omega^1_{S/K}\right)$$

and the inclusion

$$\pi_{U_D}^* \left(\underline{\mathrm{TSym}}^{i}\, \underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_S} \Omega^1_{S/K}\right) \hookrightarrow \underline{\mathrm{TSym}}^{i}\, \pi_{U_D}^* \underline{\omega}_{E^\vee/S} \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/K}$$

Definition 5.2.5. Define

$$\Lambda_k^D := (\mathrm{id} \otimes \kappa_\Omega)(\lambda_k^D) - \mathrm{KS}_{k+1}(\lambda_{k+1}^D) \in \Gamma\left(U_D, \underline{\mathrm{TSym}}^k\, \pi_{U_D}^* \underline{\omega}_{E^\vee/S} \otimes \Omega^1_{U_D/K}\right)$$

and let

$$L_n^D \in \Gamma\left(U_D, \mathcal{L}_n \otimes \Omega^1_{U_D/K}\right)$$

be the image of $(\Lambda_k^D)_{k=0}^n$ under

$$\mathrm{split}_\kappa^{-1} \otimes \mathrm{id} : \bigoplus_{k=0}^{n} \underline{\mathrm{TSym}}^k\, \pi_U^*(\underline{\omega}_{E^\vee/S}) \otimes \Omega^1_{U_D/K} \xrightarrow{\sim} \mathcal{L}_n|_U \otimes \Omega^1_{U_D/K}.$$

Let us call Λ_k^D the *k-th absolute connection form* and λ_k^D the *k-th relative connection form.*

Remark 5.2.6. The name *connection form* is chosen in analogy to the connection functions defined by Bannai, Kobayashi and Tsuji in [BKT10, Def. 1.4.]. Indeed, if we consider a CM-elliptic curve $E/\operatorname{Spec} K$ together with a fixed invariant differential ω over $\mathcal{O}_K$, one can use the differential to trivialize $\underline{\mathrm{TSym}}^k\, \pi_U^*(\underline{\omega}_{E^\vee/S}) \otimes \Omega^1_{U_D/K} \cong \mathcal{O}_{U_D}$. It is not hard to prove that λ_k^D are D-variants of the connection functions defined in [BKT10]. The reason is that the analytic splitting of the logarithm sheaves obtained by choosing the theta function $\Xi(z,w,\tau)$ coincides with the Katz splitting. Thus, the relative and absolute connection forms defined above can be seen as generalizations of the connection functions appearing in [BKT10].

A first step in proving that the absolute connection forms represent the polylogarithm is the following:

Proposition 5.2.7. *Let us view L_n^D as section of $\mathcal{L}_n^\dagger|_{U_D} \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/K}$ via the inclusion $\mathcal{L}_n \hookrightarrow \mathcal{L}_n^\dagger$. Then*

$$L_n^D \in \Gamma\left(U_D, \ker\left(\mathcal{L}_n^\dagger \otimes_{\mathcal{O}_E} \Omega^1_{E/K} \xrightarrow{d^{(1)}} \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_E} \Omega^2_{E/K}\right)\right)$$

where $d^{(1)}$ is the second differential in the absolute de Rham complex of $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger,abs})$.

Proof. We want to show the vanishing of

$$\left(\nabla_{\mathcal{L}_n^\dagger,abs} \wedge \mathrm{id} + \mathrm{id} \otimes d\right)(L_n^D) \in \Gamma\left(U_D, \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_E} \Omega^2_{E/K}\right). \tag{5.7}$$

Since $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger,abs})$ and the construction of L_n^D are compatible with base change, we may check the vanishing étale locally on the base. Thus, we may assume the existence of a $\Gamma(N)$-level structure for some $N > 3$ prime to D and reduce, again by compatibility with base change, to the situation of the universal elliptic curve $E_{N,K}/M_{N,K}$. Furthermore, we can check the vanishing of (5.7) after analytification. Passing to the universal covering and using the explicit model for M_N^{an} and its universal covering $\tilde{M}$ from Section 1.6, allows us to compute the analytification of the left hand side of Eq. (5.7) explicitly. Indeed, we will show that the system $(L_n^D)_{n\geq 0}$ corresponds to the analytic pro-system of 1-forms used in [Sch14] to describe the analytification of the de Rham realization of the elliptic polylogarithm on the universal elliptic curve.

Let us write $\underline{\tilde{\omega}}^{an}_{E^\vee/M}$ for the pullback of $\underline{\omega}_{E^\vee/M}$ to the universal covering $\tilde{E}$ of E_N^{an}. Recall that the choice of the Jacobi theta function as trivializing section of the Poincaré bundle and the basis $dw = \lambda^* dz$ of $\underline{\tilde{\omega}}$ induce a basis

$$\tilde{\mathcal{L}}_n = \bigoplus_{i+j\leq n} \tilde{\omega}^{[i]} \mathcal{O}^{an}_{\tilde{E}_N}$$

on $\tilde{\mathcal{L}}_n^\dagger$, cf. Section 2.3. Let $\tilde{U}_D$ be the preimage of U_D^{an} on the universal covering. The Katz splitting gives another isomorphism

$$\widetilde{\mathrm{split}}_\kappa : \tilde{\mathcal{L}}_n|_{\tilde{U}_D} \xrightarrow{\sim} \bigoplus_{k=0}^{n} \underline{\mathrm{TSym}}^k_{\mathcal{O}^{an}_{\tilde{U}_D}} \underline{\tilde{\omega}}_{E^\vee/M}.$$

Via

$$\omega_\kappa^{[i]} := \tilde{\mathrm{split}}_\kappa^{-1}(dw)^{[i]}$$

we obtain a different basis $(\omega_\kappa^{[i]})_{i\leq n}$ of $\tilde{\mathcal{L}}_n|_{\tilde{U}_D}$. Let us first compare both bases. Therefore, let us write $E_1(z,\tau) := \zeta(z,\tau) + \eta(1,\tau)\cdot z$.

Claim 1: We have

$$\omega_\kappa^{[i]} = \tilde{\omega}^{[i]} \cdot \exp\left(\tilde{\omega}^{[1]} \cdot (E_1(z,\tau))\right)$$
$$\tilde{\omega}^{[i]} = \omega_\kappa^{[i]} \cdot \exp\left(-\omega_\kappa^{[1]} \cdot (E_1(z,\tau))\right)$$

where exp in the ring $\underline{\mathrm{TSym}}^\bullet$ is understood as

$$\exp\left(f\tilde{\omega}^{[1]}\right) := \sum_{i=0}^{n} f^i (\tilde{\omega}^{[1]})^{[i]} = \sum_{i=0}^{n} f^i (\tilde{\omega}^{[i]})$$

and the product is taken in $\underline{\mathrm{TSym}}^\bullet\,\mathcal{H}$, i.e using the shuffle product.

Pf. of claim 1: The splittings are compatible with the isomorphism

$$\mathcal{L}_n \xrightarrow{\sim} \underline{\mathrm{TSym}}^n\,\mathcal{L}_1.$$

This allows us to reduce to the case $n=1$. In the case $n=1$ it only remains to show

$$\omega_\kappa^{[0]} = \tilde{\omega}^{[0]} + (E_1(z,\tau))\cdot\tilde{\omega}^{[1]}$$

and

$$\tilde{\omega}^{[0]} = \omega_\kappa^{[0]} - (E_1(z,\tau))\cdot\omega_\kappa^{[1]}$$

Indeed, both equations are equivalent since we have $\tilde{\omega}^{[1]} = dw = \omega_\kappa^{[1]}$ in $\tilde{\mathcal{L}}_1$. Since both bases are compatible with the extension structure of $\tilde{\mathcal{L}}_1^\dagger$, we already know that

$$\omega_\kappa^{[0]} = \tilde{\omega}^{[0]} + f(z,\tau)\cdot\tilde{\omega}^{[1]}.$$

for some holomorphic function f on $\tilde{U}_D$. It remains to show

$$f(z,\tau) = (E_1(z,\tau)).$$

This can be done using the characterizing property of the Katz splitting. For N' with $N|N'$ we have a finite étale map

$$M_{N',K} \to M_{N,K}.$$

Let $(a,b)\in(\mathbb{Z}/N'\mathbb{Z})^2$ and $t_{(a,b)} : M_{N',K} \to E_{N,K}$ be the corresponding N'-torsion point. By Lemma 2.3.4 (b) and Lemma 2.3.2 we get the formula

$$\mathrm{triv}_{\tilde{t}}(\tilde{\omega}^{[0]} + f\tilde{\omega}^{[1]}) = \left(1, \left[f(\tilde{t},\tau) + \frac{\tilde{t}-\bar{\tilde{t}}}{A(\tau)}\right] dw\right) \in \Gamma(\tilde{M}, \mathcal{O}_{\tilde{M}}^{an} \oplus \underline{\tilde{\omega}}_{E^\vee/M}^{an}).$$

where $\tilde{t}$ is a lift of the torsion section t to the universal covering. On the other hand the characterizing property of the Katz splitting in terms of the Eisenstein series $A_1(E_N,t)$ gives

$$\begin{aligned}\mathrm{triv}_{\tilde{t}}(\tilde{t}^*(\tilde{\omega}^{[0]}+f\tilde{\omega}^{[1]})) = \mathrm{triv}_{\tilde{t}}(\tilde{t}^*(\omega_\kappa^{[0]})) = \mathrm{triv}_{\tilde{t}}(\tilde{t}^*\kappa_{\mathcal{L}_1}(1,0)) = (1, A_1(E_N,t)^{an}) = \\ = (1, \left[\zeta(\tilde{t},\tau)+\eta(\tilde{t},\tau)\right] dw).\end{aligned}$$

Comparing both formulas and using the Legendre relation $\eta(\tau,\tau)=2\pi i+\eta(1,\tau)\cdot\tau$ gives

$$f(\tilde{t},\tau)=\zeta(\tilde{t},\tau)+\eta(\tilde{t},\tau)-\frac{\tilde{t}-\bar{\tilde{t}}}{A(\tau)}=\zeta(\tilde{t},\tau)+\eta(1,\tau)\cdot\tilde{t}=E_1(\tilde{t},\tau). \tag{5.8}$$

The image of all lifts $\tilde{t}$ of all N'-torsion sections $e\neq t$ for all N' with $N|N'$ is a dense subset of the universal covering $\tilde{U}$. Thus, the holomorphic functions $f(z,\tau)$ and $E_1(z,\tau)$ coincide, since they coincide by (5.8) on this dense subset.

By essentially the same argument we obtain the following explicit form of the analytification of the Katz splitting of absolute Kähler differentials

$$\kappa_\Omega(dz)=dz+\frac{1}{2\pi i}\cdot E_1(z,\tau)d\tau,$$

compare also Eq. (3.20).

Claim 2: The composition

$$\begin{tikzcd}
\tilde{\mathcal{L}}_{n+1}|_{U_D}\otimes\Omega^{1,an}_{\tilde{U}/\tilde{M}} \arrow[d, "\cong"', "\tilde{\mathrm{split}}_\kappa\otimes\mathrm{id}"] & & \tilde{\mathcal{L}}_{n}|_{U_D}\otimes\Omega^{1,an}_{\tilde{U}/\mathbb{C}} \\
\bigoplus_{k=0}^{n+1}\underline{\mathrm{TSym}^k\,\tilde{\omega}}\otimes\Omega^{1,an}_{\tilde{U}/\tilde{M}} \arrow[rr, "\oplus_k \mathrm{KS}_{k+1}"] & & \bigoplus_{k=0}^{n}\underline{\mathrm{TSym}^k\,\tilde{\omega}}\otimes\Omega^{1,an}_{\tilde{U}/\mathbb{C}} \arrow[u, "\cong", "(\tilde{\mathrm{split}}_\kappa)^{-1}\otimes\mathrm{id}"']
\end{tikzcd}$$

maps $\sum_{k=0}^{n+1}a_k\tilde{\omega}^{[k]}\otimes dz$ to

$$-\frac{1}{2\pi i}\left(-\left(E_1(z,\tau)\right)\cdot\left(\sum_{k=0}^{n}a_k\tilde{\omega}^{[k]}\right)+\sum_{k=0}^{n}a_{k+1}\tilde{\omega}^{[k]}\right)\otimes d\tau.$$

Pf. of claim 2: We have defined $\underline{\tilde{\omega}}$ as the pullback of $\underline{\omega}_{E^\vee/M}$ to the universal covering $\tilde{U}$ and we have chosen dw as basis of $\underline{\tilde{\omega}}$. Define

$$\partial:\bigoplus_{k=0}^{n+1}\underline{\mathrm{TSym}^k\,\tilde{\omega}}\to\bigoplus_{k=0}^{n}\underline{\mathrm{TSym}^k\,\tilde{\omega}},\quad \sum_{k=0}^{n+1}a_k dw^{[k]}\mapsto\sum_{k=0}^{n}a_{k+1}dw^{[k]}.$$

This corresponds under the isomorphism of $\mathcal{O}^{an}_{\tilde{U}}$-algebras

$$\mathcal{O}^{an}_{\tilde{U}}[t]/t^{n+1}\to\bigoplus_{k=0}^{n}\underline{\mathrm{TSym}^k\,\tilde{\omega}},\quad t\mapsto dw$$

to the usual formal derivation with respect to t. In particular, ∂ satisfies all the properties of the formal derivation, e.g. the Leibniz rule. By the definition of KS_{k+1} we have

$$\begin{aligned}\mathrm{KS}_{k+1}(dw^{[k+1]}\otimes dz) = \mathrm{KS}_{k+1}(\frac{dw^{[k]}\cdot dw\otimes dz}{k+1}) = dw^{[k]}\mathrm{KS}(dw\otimes dz) = \\ = -\frac{1}{2\pi i}dw^{[k]}\otimes d\tau = -\frac{1}{2\pi i}\partial(dw^{[k+1]})\otimes d\tau.\end{aligned}$$

By linearity we deduce

$$(\oplus_k \mathrm{KS}_{k+1})\,(f\otimes dz) = -\frac{1}{2\pi i}\partial f\otimes d\tau. \tag{5.9}$$

for $f\in\bigoplus_{k=0}^{n+1}\underline{\mathrm{TSym}}^k\,\underline{\tilde{\omega}}$. From the first claim we deduce

$$(\tilde{\mathrm{split}}_\kappa\otimes\mathrm{id})\left(\sum_{k=0}^{n+1}a_k\tilde{\omega}^{[k]}\otimes dz\right) = \exp\left(-\left(E_1(z,\tau)\right)dw^{[1]}\right)\cdot\left(\sum_{k=0}^{n+1}a_k dw^{[k]}\right)\otimes dz \tag{5.10}$$

where exp and multiplication are understood in $\underline{\mathrm{TSym}}$ as above. We compute:

$$\begin{aligned}&(\oplus_k\mathrm{KS}_{k+1})\circ(\tilde{\mathrm{split}}_\kappa\otimes\mathrm{id})(\sum_{k=0}^{n+1}a_k\tilde{\omega}^{[k]}\otimes dz) \overset{(5.10)}{=}\\ =&(\oplus_k\mathrm{KS}_{k+1})\left(\exp\left(-E_1\cdot(dw)^{[1]}\right)\cdot\left(\sum_{k=0}^{n+1}a_k(dw)^{[k]}\right)\otimes dz\right)\overset{(5.9)}{=}\\ =&-\frac{1}{2\pi i}\partial\left[\exp\left(-E_1\cdot(dw)^{[1]}\right)\cdot\left(\sum_{k=0}^{n+1}a_k(dw)^{[k]}\right)\right]\otimes d\tau =\\ =&-\frac{1}{2\pi i}\left(\partial\exp\left(-E_1\cdot(dw)^{[1]}\right)\right)\cdot\left(\sum_{k=0}^{n}a_k(dw)^{[k]}\right)\otimes d\tau -\\ &-\frac{1}{2\pi i}\exp\left(-E_1\cdot(dw)^{[1]}\right)\cdot\partial\left(\sum_{k=0}^{n+1}a_k(dw)^{[k]}\right)\otimes d\tau =\\ =&-\frac{1}{2\pi i}\exp\left(-E_1\cdot(dw)^{[1]}\right)\cdot\left[-E_1\cdot\left(\sum_{k=0}^{n}a_k(dw)^{[k]}\right)+\sum_{k=0}^{n}a_{k+1}(dw)^{[k]}\right]\otimes d\tau\end{aligned}$$

This maps to

$$-\frac{1}{2\pi i}\left[-\left(E_1(z,\tau)\right)\cdot\left(\sum_{k=0}^{n}a_k\tilde{\omega}^{[k]}\right)+\sum_{k=0}^{n}a_{k+1}\tilde{\omega}^{[k]}\right]\otimes d\tau$$

under $\tilde{\mathrm{split}}_\kappa^{-1}\otimes\mathrm{id}$, cf. (5.10). This finishes the proof of claim 2.

Let us recall from Lemma 2.3.5 that the section $\tilde{l}_n^D$ is given in terms of the analytic basis as

$$\sum_{k=0}^{n}(-1)^k k!\, s_k^D\tilde{\omega}^{[k]}\otimes dz$$

where s_i^D is defined via the expansion

$$D^2 J(z, -w, \tau) - DJ(Dz, -\frac{w}{D}, \tau) = \sum_{i\geq 0} s_i^D(z,\tau) w^i.$$

Let us write $\tilde{L}_n^D$ for the analytification of L_n^D.
Claim 3: We have

$$\tilde{L}_n^D = \sum_{k=0}^{n} (-1)^k k! s_k^D \tilde{\omega}^{[k]} \otimes dz - \frac{1}{2\pi i}(-1)^{k+1}(k+1)! s_{k+1}^D \tilde{\omega}^{[k]} \otimes d\tau.$$

Pf. of claim 3: It follows from the definition of L_n^D that $\tilde{L}_n^D$ is the image of $\tilde{l}_n^D$ under the composition

$$\begin{array}{ccc} \tilde{\mathcal{L}}_{n+1}|_{U_D} \otimes \Omega^{1,an}_{\tilde{U}/\tilde{M}} & & \tilde{\mathcal{L}}_n|_{U_D} \otimes \Omega^{1,an}_{\tilde{U}/\tilde{M}} \\ \cong\Big\downarrow \mathrm{sp\tilde{l}it}_\kappa \otimes \mathrm{id} & & \cong\Big\uparrow \mathrm{sp\tilde{l}it}_\kappa{}^{-1} \otimes \mathrm{id} \\ \bigoplus_{k=0}^{n+1} \underline{\mathrm{TSym}}^k\, \underline{\tilde{\omega}} \otimes \Omega^{1,an}_{\tilde{U}/\tilde{M}} & \xrightarrow{\mathrm{id}\otimes\kappa_\Omega - \oplus_k \mathrm{KS}_{k+1}} & \bigoplus_{k=0}^{n} \underline{\mathrm{TSym}}^k\, \underline{\tilde{\omega}} \otimes \Omega^{1,an}_{\tilde{U}/\tilde{M}} \end{array}$$

By claim 2 and the explicit formula for κ_Ω, the image of $l_n^D = \sum_{k=0}^n (-1)^k k! s_k^D \tilde{\omega}^{[k]} \otimes dz$ under this composition is

$$\left(\sum_{k=0}^{n} (-1)^k k! s_k^D \tilde{\omega}^{[k]}\right) \otimes \left(dz + \frac{E_1}{2\pi i} d\tau\right) + \\ + \frac{1}{2\pi i}\left[-E_1 \sum_{k=0}^{n} (-1)^k k! s_k^D \tilde{\omega}^{[k]} + \sum_{k=0}^{n} (-1)^{k+1}(k+1)! s_{k+1}^D \tilde{\omega}^{[k]}\right] \otimes d\tau$$

which simplifies to

$$\sum_{k=0}^{n} (-1)^k k! s_k^D \tilde{\omega}^{[k]} \otimes dz - \frac{1}{2\pi i}(-1)^{k+1}(k+1)! s_{k+1}^D \tilde{\omega}^{[k]} \otimes d\tau.$$

This proves the third claim.

Now, we observe that the injection

$$\tilde{\mathcal{L}}_n \hookrightarrow \tilde{\mathcal{L}}_n^\dagger$$

maps $\tilde{\omega}^{[i]}$ to $(-1)^i \tilde{\omega}^{[i,0]}$ and we obtain the formula

$$\tilde{L}_n^D = \sum_{k=0}^{n} k! s_k^D \tilde{\omega}^{[k,0]} \otimes dz - \frac{1}{2\pi i}(k+1)! s_{k+1}^D \tilde{\omega}^{[k,0]} \otimes d\tau.$$

Finally, using Remark 2.3.1 this expresses as

$$\tilde{L}_n^D = \sum_{k=0}^{n} s_k^D \frac{e^{n-k} f^k}{(n-k)!} \otimes dz - \frac{1}{2\pi i}(k+1) s_{k+1}^D \frac{e^{n-k} f^k}{(n-k)!} \otimes d\tau.$$

in the basis $\left(\frac{e^{n-k-j}f^k g^j}{(n-k-j)!}\right)_{j+k\leq n}$ chosen by Scheider in [Sch14]. From here on we can follow the proof of [Sch14, Theorem 3.6.2]. The essential point is that one can use the explicit description of $\tilde{L}_n^D$ and translate the vanishing of

$$\left(\nabla_{\mathcal{L}_n^\dagger,abs} \wedge \mathrm{id} + \mathrm{id} \otimes d\right)(\tilde{L}_n^D)$$

into the system of differential equations

$$\partial_\tau s_k^D = -\frac{k+1}{2\pi i}\partial_z s_{k+1}^D.$$

This is satisfied by the *mixed heat equation* for the Jacobi theta function:

$$2\pi i \partial_\tau J(z,w,\tau) = \partial_z \partial_w J(z,w,\tau)$$

We refer to [Sch14, Theorem 3.6.2] and [Sch14, Proposition 3.5.22] for details. □

Remark 5.2.8. The above result shows that $[(L_n^D)]_{n\geq 0}$ defines a cohomology class in

$$\varprojlim_n \underline{H}^1_{\mathrm{dR}}\left(U_D/K, \mathrm{Log}^n_{\mathrm{dR}}\right).$$

Using the Leray spectral sequence combined with the vanishing results of Proposition 5.2.1, we deduce an isomorphism:

$$\varprojlim_n H^1_{\mathrm{dR}}\left(U_D/K, \mathrm{Log}^n_{\mathrm{dR}}\right) \cong \varprojlim_n H^0_{\mathrm{dR}}\left(S/K, \underline{H}^1_{\mathrm{dR}}\left(U_D/S, \mathrm{Log}^n_{\mathrm{dR}}\right)\right). \tag{5.11}$$

The relative 1-forms $(l_n^D)_{n\geq 0}$ define a compatible system

$$[l_n^D] \in \varprojlim_n \Gamma\left(S, \underline{H}^1_{\mathrm{dR}}\left(U_D/S, \mathrm{Log}^n_{\mathrm{dR}}\right)\right)$$

and the above result shows that this is a horizontal section of $\underline{H}^1_{\mathrm{dR}}\left(U_D/S, \mathrm{Log}^n_{\mathrm{dR}}\right)$ with respect to the Gauss–Manin connection. From this point of view, the Picard–Fuchs equation expressing the horizontality of $[l_n^D]$ in a suitable trivialization of $\mathrm{Log}^n_{\mathrm{dR}}$ turns out to be the mixed heat equation of the Jacobi theta function.

Remark 5.2.9. The pullback of the absolute logarithm sheaves $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger,abs})$ along some torsion section t is isomorphic to

$$\bigoplus_{k=0}^n \underline{\mathrm{TSym}^k}\,\mathcal{H}$$

with each summand equipped with the connection induced by the Gauss–Manin connection. With this observation we can use the vanishing of

$$d^{(1)}(L_n^D) = 0$$

to relate our construction of real-analytic Eisenstein series via $\nabla_{\mathcal{L}_n^\dagger}$ to the construction of Katz via the Gauss–Manin connection on the universal elliptic curve. In other words, the mixed heat equation can be seen as a bridge between both constructions of real-analytic Eisenstein series: Katz' approach via the Gauss–Manin connection and our approach via the connection of the Poincaré bundle.

5.2.3 The polylogarithm class via the Poincaré bundle

As before, let $E/S/K$ be an elliptic curve with S a smooth separated K-scheme over some field K of characteristic zero. Let us fix $\left((\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger, abs}), 1\right)$ as absolute de Rham logarithm sheaf. The Hodge filtration on $\mathcal{H}$ induces a descending filtration of $\mathcal{O}_E$-modules on $\mathcal{L}_1^\dagger$ such that all morphisms in

$$0 \to \mathcal{H}_E \to \mathcal{L}_1^\dagger \to \mathcal{O}_E \to 0$$

are strictly compatible with the filtration. Here, $\mathcal{O}_E$ is considered to be concentrated in filtration step 0. Explicitly this filtration is given as

$$F^{-1}\mathcal{L}_1^\dagger = \mathcal{L}_1^\dagger \supseteq F^0\mathcal{L}_1^\dagger = \mathcal{L}_1 \supseteq F^1\mathcal{L}_1^\dagger = 0$$

The edge morphism $E_2^{1,0} \to E^1$ in the Hodge-to-de-Rham spectral sequence

$$E_1^{p,q} = H^q(U_D, \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_{U_D}} \Omega^p_{U_D/K}) \Rightarrow E^{p+q} = H^{p+q}_{\mathrm{dR}}\left(U_D/K, \mathcal{L}_n^\dagger\right)$$

induces a morphism

$$\Gamma\left(U_D, \ker\left(\mathcal{L}_n^\dagger \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/K} \xrightarrow{d^{(1)}} \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_{U_D}} \Omega^2_{U_D/K}\right)\right) \xrightarrow{[\cdot]} H^1_{\mathrm{dR}}\left(U_D/K, \mathcal{L}_n^\dagger\right). \quad (5.12)$$

We show that $\mathrm{pol}_{D,\mathrm{dR}}$ is represented by the compatible system $(L_n^D)_{n\geq 0}$ under (5.12).

Theorem 5.2.10. *The D-variant of the elliptic polylogarithm in de Rham cohomology is explicitly given by*

$$\mathrm{pol}_{D,\mathrm{dR}} = ([L_n^D])_{n\geq 0}$$

where $[L_n^D]$ is the de Rham cohomology class associated with L_n^D via (5.12).

Proof. First let us recall that

$$L_n^D \in \Gamma\left(U_D, \ker\left(\mathcal{L}_n^\dagger \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/K} \xrightarrow{d^{(1)}} \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_{U_D}} \Omega^2_{U_D/K}\right)\right)$$

by Proposition 5.2.7. Thus, $[L_n^D]$ is well-defined. Further, the question is étale locally on the base. Indeed, for a Cartesian diagram

$$\begin{array}{ccc} E_T & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}$$

with f finite étale we have an isomorphism

$$\tilde{f}^*\left(\mathcal{L}_n^\dagger \otimes \Omega^1_{E/K}\right) \xrightarrow{\sim} \mathcal{L}^\dagger_{n,E_T} \otimes \Omega^1_{E_T/K}$$

which identifies $(\tilde{f}^* L_n^D)_{n\geq 0}$ with $(L_n^D)_{n\geq 0}$. Furthermore, the canonical map

$$H^1_{\mathrm{dR}}\left(U_D, \mathcal{L}_n^\dagger\right) \xrightarrow{\sim} H^1_{\mathrm{dR}}\left(U_D \times_S T, \mathcal{L}^\dagger_{n,E_T}\right)$$

is an isomorphism and identifies the polylogarithm classes. Thus, we may prove the claim after a finite étale base change. Now, choose an arbitrary $N > 3$. Since we are working over a scheme of characteristic zero, the integer N is invertible and there exists étale locally a $\Gamma(N)$-level structure. Again, by compatibility with base change it is enough to prove the claim for the universal elliptic curve $E_{N,K}$ with $\Gamma(N)$-level structure over $M_{N,K}$. From now on let $E := E_{N,K}$ and $M := M_{N,K}$. By the defining property of the polylogarithm we have to show

$$\mathrm{Res}\left(([L_n^D])_{n\geq 0}\right) = D^2 1_e - 1_{E[D]}.$$

We split this into two parts:

(A) $\mathrm{Res}\, L_0^D = D^2 1_e - 1_{E[D]}$

(B) The image of $\mathrm{Res}\left(([L_n^D])_{n\geq 0}\right)$ under

$$\prod_{n=0}^{\infty} H^0_{\mathrm{dR}}\left(E[D], \underline{\mathrm{TSym}}^k\, \mathcal{H}_{E[D]}\right) \twoheadrightarrow \prod_{n=1}^{\infty} H^0_{\mathrm{dR}}\left(E[D], \underline{\mathrm{TSym}}^k\, \mathcal{H}_{E[D]}\right)$$

is zero.

(A): Since M is affine, the Leray spectral sequence for de Rham cohomology shows that we obtain the localization sequence for $n = 1$ by applying $H^0_{\mathrm{dR}}(M, \cdot)$ to

$$0 \longrightarrow \underline{H}^1_{\mathrm{dR}}(E/S) \longrightarrow \underline{H}^1_{\mathrm{dR}}(U_D/S) \xrightarrow{\mathrm{Res}} \underline{H}^0_{\mathrm{dR}}(E/S).$$

This exact sequence can be obtained by applying $R\pi_*$ to the short exact sequence

$$0 \longrightarrow \Omega^\bullet_{E/S} \longrightarrow \Omega^\bullet_{E/S}(E[D]) \xrightarrow{\mathrm{res}} (i_{E[D]})_* \mathcal{O}_{E[D]}[-1] \longrightarrow 0$$

of complexes. Thus, it is enough to show $\mathrm{res}(l_0^D) = D^2 1_e - 1_{E[D]}$. But we have already shown this residue property in the proof of Proposition 1.5.3. This proves (A).

(B): By the vanishing of

$$H^0_{\mathrm{dR}}\left(E[D], \underline{\mathrm{TSym}}^k\, \mathcal{H}_{E[D]}\right) = 0, \quad \text{for } k > 0,$$

which is Lemma 5.1.20, there is nothing to show. □

Remark 5.2.11. Let us note that it is possible to construct the classical elliptic polylogarithm along the same lines. Using the canonical section s_{can} instead of its D-variant, it is possible to construct a section

$$l_n \in \mathcal{H}_E^\vee \otimes \mathcal{L}_n^\dagger \otimes \Omega^1_{E/S}([e]),$$

cf. Remark 2.2.2. Using the Katz splitting, it is then possible to lift it to a section

$$L_n \in \mathcal{H}_U^\vee \otimes \mathcal{L}_n^\dagger \otimes \Omega^1_{U/K}([e])$$

representing the classical elliptic polylogarithm.

5.2.4 Uniqueness of the absolute connection forms

In the previous section we constructed lifts of the l_n^D to absolute 1-forms L_n^D with values in the logarithm sheaves. Further, we showed that they represent the pro-system of polylogarithm classes in de Rham cohomology. Still one might expect that this compatible system is only one of many ways to represent the polylogarithm class explicitly. In this section we will show that this is not the case. The constructed system is essentially unique. The edge morphism in the Hodge-to-de-Rham spectral sequence gives the following map:

$$[\cdot]:\Gamma\left(U_D,\ker\left(\mathcal{L}_n^\dagger\otimes_{\mathcal{O}_E}\Omega^1_{E/K}\xrightarrow{d^{(1)}}\mathcal{L}_n^\dagger\otimes_{\mathcal{O}_E}\Omega^2_{E/K}\right)\right)\to H^1_{\mathrm{dR}}\left(U_D/K,\mathcal{L}_n^\dagger\right).$$

Proposition 5.2.12. *The compatible system $(L_n^D)_{n\geq 0}$ is the unique compatible system of sections in the projective system*

$$\Gamma\left(U_D,\ker\left(\mathcal{L}_n^\dagger\otimes_{\mathcal{O}_E}\Omega^1_{E/K}\xrightarrow{d^{(1)}}\mathcal{L}_n^\dagger\otimes_{\mathcal{O}_E}\Omega^2_{E/K}\right)\cap\left(\Omega^1_{E/K}\otimes\mathcal{L}_n\right)\right)_{n\geq 0}$$

representing the polylogarithm class in de Rham cohomology.

Proof. We already know that the pro-system $(L_n^D)_{n\geq 0}$ satisfies the above properties. By compatibility with base change we may check the uniqueness locally on the base. We may thus assume $S=\operatorname{Spec}R$ is affine with $\underline{\omega}_{E/S}$ being freely generated by some $\omega\in\Gamma(S,\underline{\omega}_{E/S})$. There exists a unique Weierstrass equation with $\frac{dx}{y}=\omega$. Let us define $\eta:=x\frac{dx}{y}$. Let

$$\partial:\mathcal{O}_{U_D}\to\mathcal{O}_{U_D}$$

be the derivation dual to ω. We have

$$\bigcap_{k\geq 0}\Gamma(U_D,\operatorname{Im}\partial^{\circ k})=\{0\}.$$

One possible way to see this is by a direct computation on the affine locus using the Weierstrass equation. In a first step we show that there is no compatible system of relative coboundaries contained in $\mathcal{L}_n\otimes\Omega^1_{U_D/S}$:

Claim: If

$$(a_n)_{n\geq 0}\in\Gamma\left(U_D,(\operatorname{Im}\nabla_{\mathcal{L}_n^\dagger})\cap\left(\mathcal{L}_n\otimes_{\mathcal{O}_E}\Omega^1_{E/S}\right)\right)_{n\geq 0}$$

is compatible with the transition maps, then $a_n=0$ for all $n\geq 0$.

Pf. of the claim: Since U_D is affine, the sequence

$$0\longrightarrow\Gamma(U_D,\ker\nabla_{\mathcal{L}_n^\dagger})\longrightarrow\Gamma(U_D,\mathcal{L}_n^\dagger)\longrightarrow\Gamma(U_D,\operatorname{Im}\nabla_{\mathcal{L}_n^\dagger})\longrightarrow 0$$

is exact. The localization sequence for $U_D\hookrightarrow E$ induces an isomorphism

$$\underline{H}^0_{\mathrm{dR}}\left(E/S,\mathcal{L}_n^\dagger\right)\xrightarrow{\sim}\underline{H}^0_{\mathrm{dR}}\left(U_D/S,\mathcal{L}_n^\dagger\right)$$

which is compatible with transition maps. Thus, we deduce from Proposition 5.2.1 that

$$\underline{H}^0_{\mathrm{dR}}\left(U_D/S, \mathcal{L}_n^\dagger\right) = \ker \nabla_{\mathcal{L}_n^\dagger}$$

is Mittag-Leffler zero. In particular, the above short exact sequence induces an isomorphism:

$$\varprojlim_n \Gamma(U_D, \mathcal{L}_n^\dagger) \xrightarrow{\sim} \varprojlim_n \Gamma(U_D, \operatorname{Im} \nabla_{\mathcal{L}_n^\dagger})$$

The transition maps on the left hand side are surjective. In particular, there is a unique compatible sequence $(b_n)_{n\geq 0}$ with $b_n \in \Gamma(U_D, \mathcal{L}_n^\dagger)$ and $\nabla_{\mathcal{L}_n^\dagger}(b_n) = a_n$ for all $n \geq 0$. Since we are working over a field of characteristic zero, the Katz splitting induces isomorphisms

$$\mathrm{split}_\kappa^\dagger : \mathcal{L}_n^\dagger\Big|_{U_D} \xrightarrow{\sim} \bigoplus_{k=0}^{n} \underline{\mathrm{TSym}}^k\, \mathcal{H}_{U_D}.$$

We obtain an explicit basis

$$\omega_\kappa^{[i,j]} := (\mathrm{split}_\kappa^\dagger)^{-1}\left(([\eta]^\vee)^{[i]} \cdot ([\omega]^\vee)^{[j]}\right)$$

of $\mathcal{L}_n^\dagger$. Using this basis, let us write

$$b_n = \sum_{k+l\leq n} \beta_{k,l}\omega_\kappa^{[k,l]}, \quad \beta_{k,l} \in \Gamma(U_D, \mathcal{O}_{U_D}).$$

Observe that $\beta_{k,l}$ does not depend on the chosen $n \geq k+l$ by compatibility with transition maps. Similarly, let us write

$$a_n = \sum_{k=0}^{n} \alpha_k \omega_\kappa^{[k,0]} \otimes \omega, \quad \alpha_k \in \Gamma(U_D, \mathcal{O}_{U_D}).$$

The explicit formula for $\nabla_{\mathcal{L}_n^\dagger}$ in Proposition 3.2.11 implies

$$\nabla_{\mathcal{L}_n^\dagger}(\omega_\kappa^{[k,l]}) = (k+1)\omega_\kappa^{[k+1,l]} \otimes \eta + (l+1)\omega_\kappa^{[k,l+1]} \otimes \omega.$$

With this formula we can rewrite the differential equation $\nabla(b_n) = a_n$ as the formal sum:

$$\sum_{k,l\geq 0}\left(\partial\beta_{k,l}\omega_\kappa^{[k,l]} \otimes \omega + (k+1)\beta_{k,l}\omega_\kappa^{[k+1,l]} \otimes \eta + (l+1)\beta_{k,l}\omega_\kappa^{[k,l+1]} \otimes \omega\right) = \sum_{k\geq 0} \alpha_k \omega_\kappa^{[k,0]} \otimes \omega.$$

We claim that this implies $\beta_{k,l} = 0$ for all $k, l \geq 0$. Let us prove this by induction on k. For $k = 0$ comparing coefficients of the form $\omega_\kappa^{[0,l]}$ gives the following system of differential equations for $(\beta_{0,l})_{l\geq 0}$:

$$\begin{aligned} \partial\beta_{0,0} &= \alpha_0 \\ \partial\beta_{0,l} &= -l \cdot \beta_{0,l-1} \quad \text{for } l > 0 \end{aligned}$$

In particular, this implies for any $l \geq 0$

$$\partial\beta_{0,l} \in \bigcap_{k\geq 0} \Gamma(U_D, \operatorname{Im}\partial^{\circ k}) = \{0\}.$$

We conclude $(\beta_{0,l})_{l\geq 0} = (0)_{l\geq 0}$. For the induction step, let us assume $(\beta_{j,l})_{l\geq 0} = (0)_{l\geq 0}$ for all $j < k$. Comparing coefficients of $\omega_k^{[k,*]}$ under the induction hypothesis gives

$$\begin{aligned} \partial\beta_{k,0} &= \alpha_k \\ \partial\beta_{k,l} &= -l \cdot \beta_{k,l-1} \qquad \text{for } l > 0 \end{aligned}$$

and as above we conclude $(\beta_{k,l})_{l\geq 0} = (0)_{l\geq 0}$. This proves $b_k = 0$ for all $k \geq 0$ and thereby $a_k = \nabla_{\mathcal{L}_n^\dagger}(b_k) = 0$ for all $k \geq 0$ as desired. The claim follows.

For the uniqueness in the statement of the proposition, we consider the difference between two compatible systems. This difference gives a compatible system of coboundaries

$$(A_n)_{n\geq 0} \in \Gamma(U_D, \operatorname{Im}(\nabla_{\mathcal{L}_n^\dagger, abs}) \cap (\mathcal{L}_n \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/K}))_{n\geq 0}$$

thus the proposition follows from the following:
Claim 2: Every sequence

$$(A_n)_{n\geq 0} \in \Gamma(U_D, \operatorname{Im}(\nabla_{\mathcal{L}_n^\dagger, abs}) \cap (\mathcal{L}_n \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/K}))_{n\geq 0}$$

which is compatible with the transition maps is the zero sequence, i.e $A_n = 0$ for all $n \geq 0$.

Pf. of Claim 2: As in the previous claim one proves that there is a unique compatible system $(B_n)_{n\geq 0}$ with $\nabla_{\mathcal{L}_n^\dagger, abs}(B_n) = A_n$. Restricting $A_n \in \Gamma(U_D, \mathcal{L}_n \otimes \Omega^1_{U_D/K})$ to a 1-form relative S gives a compatible system

$$(a_n)_{n\geq 0} \in \Gamma(U_D, \mathcal{L}_n \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/S})_{n\geq 0}.$$

Since

$$\begin{tikzcd} & \Gamma\left(U_D, \mathcal{L}_n^\dagger \otimes \Omega^1_{U_D/K}\right) \arrow[dd, two heads] \\ \Gamma\left(U_D, \mathcal{L}_n^\dagger\right) \arrow[ur, "\nabla_{\mathcal{L}_n^\dagger, abs}"] \arrow[dr, "\nabla_{\mathcal{L}_n^\dagger}"'] & \\ & \Gamma\left(U_D, \mathcal{L}_n^\dagger \otimes \Omega^1_{U_D/S}\right) \end{tikzcd}$$

commutes, we have $\nabla_{\mathcal{L}_n^\dagger}(B_n) = a_n$ for all $n \geq 0$. The proof of Claim 2 shows then $B_n = 0$ for all $n \geq 0$. In particular, we have $A_n = \nabla_{\mathcal{L}_n^\dagger, abs}(B_n) = 0$ for all $n \geq 0$. This proves the second claim and thereby the proposition. □

Remark 5.2.13. The de Rham realization of the elliptic polylogarithm can be seen as an invariant in de Rham cohomology of an elliptic curve. From this point of view we have the following interpretation of the above result. The connection forms $(\Lambda_n^D)_{n\geq 0}$ or equivalently the compatible system $(L_n^D)_{n\geq 0}$ forms a distinguished system of cocycles representing the cohomology class of the polylogarithm. Thus, we can see the connection forms as a distinguished and functorial refinement of the invariant given by the elliptic polylogarithm. On the other hand, the theory of the canonical section developed in the first and the second chapter shows that this distinguished refinement has also a nice geometric interpretation via the Poincaré bundle. Even better: It is not only a refinement in the sense that we have found a canonical representative of the underlying cohomology class, it even allows constructions which have not been possible so far. While a construction of real-analytical Eisenstein series or their p-adic analogue seems to be impossible via the cohomology class of the elliptic polylogarithm, the results of Section 2.4 and Section 4.5 show that the refined invariant $(l_n^D)_{n\geq 0}$ allows such a construction. From this point of view this thesis fits into the frame of the collaborative research center[1] 'Higher Invariants' whose scope is studying structural and hierarchical refinements of classical invariants.

[1] SFB 1085, 'Higher Invariants - Interactions between Arithmetic Geometry and Global Analysis', Funded by the DFG

6 The syntomic realization of the elliptic polylogarithm for ordinary elliptic curves

The aim of this chapter is to describe the rigid syntomic realization of the elliptic polylogarithm for families of ordinary elliptic curves. The case of a single CM elliptic curve has been treated by Bannai, Kobayashi and Tsuji in [BKT10]. On the other hand the syntomic Eisenstein classes obtained by specializing the polylogarithm class on the ordinary locus of the modular curve have been described by Bannai and Kings in [BK10a]. Thus, loosely speaking our result can be seen as the least common generalization of [BKT10] (for ordinary primes) and [BK10a]. So, it is not surprising that the used methods are very much inspired by both [BKT10] and [BK10a].

For this chapter it might be helpful to recall two steps used in [BKT10] to describe the polylogarithm for a single CM-elliptic curve. Once the de Rham realization is settled Bannai, Kobayashi and Tsuji proceed in two steps to describe the rigid syntomic realization for CM elliptic curves. In a first step they build a system of differential equations for overconvergent functions starting with the connection functions appearing in the de Rham realization. The solution of this differential equation describes the Frobenius structure on the polylogarithm sheaf. In a second step this system of differential equations is solved on tubular neighbourhoods of torsion sections. This is possible since the connection functions on such tubular neighbourhoods are closely related to moment functions of p-adic distributions. In this step it is exploited that the p-adic Fourier transform of Schneider–Teitelbaum translates this differential equation into a differential equation which is more or less obviously satisfied by the moment functions.

6.1 Rigid syntomic cohomology

Syntomic cohomology can be seen as the p-adic analogue of Deligne–Beilinson cohomology. Indeed, in the case of good reduction Bannai has proven that syntomic cohomology can be seen as absolute p-adic Hodge cohomology [Ban02]. The recent work of Deglise and Nizioł generalizes this to arbitrary smooth proper schemes over a discretely valued field of mixed characteristic[DN15]. The approach of Deglise–Nizioł allows further the construction of a ring spectrum in the motivic homotopy category of Morel–Voevodsky representing syntomic cohomology. In their approach coefficients for syntomic cohomology can be defined abstractly as modules over this ring spectrum. Nevertheless, we will use rigid syntomic cohomology as developed by Bannai for describing the syntomic realization of the elliptic polylogarithm. Indeed, since we want an explicit description of the polylogarithm class, we need explicit complexes computing syntomic cohomology.

In this section we briefly recall the definition and basic properties of rigid syntomic cohomology. We follow closely the appendix of [BK10a]. In particular, we use their modification of the definition of smooth pair allowing overconvergent Frobenii which are not globally defined. Let $K/\mathbb{Q}_p$ be a finite unramified extension with ring of integers $\mathcal{O}_K$, residue field k and Frobenius morphism $\sigma : K \to K$.

Definition 6.1.1.

(a) A smooth pair is a tuple $\mathscr{X} = (X, \bar{X})$ consisting of a smooth scheme X of finite type over $\mathcal{O}_K$ together with a smooth compactification $\bar{X}$ of X with complement $D := \bar{X} \setminus X$ a simple normal crossing divisor relative $\operatorname{Spec} \mathcal{O}_K$. We denote the formal completion of X with respect to $X_k := X \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} k$ by $\mathcal{X}$ and the formal completion of $\bar{X}$ with respect to $\bar{X}_k$ by $\bar{\mathcal{X}}$. The rigid analytic spaces associated with $\mathcal{X}$ resp. $\bar{\mathcal{X}}$ will be denoted by $\mathcal{X}_K$ resp. $\bar{\mathcal{X}}_K$.

(b) An overconvergent Frobenius $\phi_X = (\phi, \phi_V)$ on a smooth pair $\mathscr{X} = (X, \bar{X})$ consists of: A morphism of $\mathcal{O}_K$-formal schemes

$$\phi : \mathcal{X} \to \mathcal{X}$$

lifting the absolute Frobenius on X_k and an extension of ϕ to a morphism of rigid analytic spaces

$$\phi_V : V \to \bar{\mathcal{X}}_K$$

to some strict neighbourhood V of $\mathcal{X}_K$ in $\bar{\mathcal{X}}_K$.

(c) A smooth pair together with an overconvergent Frobenius $\mathscr{X} = (X, \bar{X}, \phi, \phi_V)$ will be called *syntomic datum*.

For a smooth pair $\mathscr{X} = (X, \bar{X})$ let us write X_K and $\bar{X}_K$ for the generic fibers and X_K^{an} resp. $\bar{X}^{an}$ for the associated rigid analytic spaces. Then, X_K^{an} is a strict neighbourhood of $j : \mathcal{X}_K \hookrightarrow \bar{\mathcal{X}}_K$. A coherent module M on $\bar{X}_K$ with integrable connection

$$\nabla : M \to M \otimes \Omega^1_{\bar{X}_K}(\log D)$$

and logarithmic poles along D induces an overconvergent connection (M^{rig}, ∇^{rig}) on $M^{rig} := j^\dagger(M|_{X_K^{an}})$. The category of filtered overconvergent F-isocrystals on $\mathscr{X}$ serves as coefficients for rigid syntomic cohomology and may be realized as follows.

Definition 6.1.2. Let the category $S(\mathscr{X})$ of filtered overconvergent F-isocrystals on $\mathscr{X} = (X, \bar{X})$ be the category consisting of 4-tuples

$$\mathcal{M} = (M, \nabla, F^\bullet, \Phi_M)$$

with: M a coherent $\mathcal{O}_{\bar{X}_K}$-module with integrable connection

$$\nabla : M \to M \otimes \Omega^1_{\bar{X}_K}(\log D)$$

with logarithmic poles along $D = \bar{X}_K \setminus X_K$. $F^\bullet$ a descending exhaustive and separating filtration on M satisfying Griffith transversality:

$$\nabla(F^\bullet M) \subseteq F^{\bullet-1}(M) \otimes \Omega^1_{\bar{X}_K}(\log D)$$

And a horizontal isomorphism

$$\Phi_M : F_\sigma^* M^{rig} \to M^{rig}.$$

where F_σ is the Frobenius endofunctor on the category of overconvergent isocrystals defined in [Ber97]. Φ_M will be called a Frobenius structure. Morphisms in this category are morphisms of $\mathcal{O}_{\bar{X}_K}$-modules respecting the additional structures.

If one has a fixed overconvergent Frobenius on the smooth pair $\mathscr{X} = (X, \bar{X})$, one can realize a Frobenius structure more concretely as a horizontal morphism

$$\phi_V^* M^{rig} \to M^{rig}.$$

A morphism of pairs $\mathscr{X} = (X, \bar{X}) \to \mathscr{Y} = (Y, \bar{Y})$ is a morphism $f : \bar{X} \to \bar{Y}$ such that $f(X) \subseteq Y$. A morphism of pairs is called smooth, proper, ... etc, if $f|_X$ is smooth, proper, etc. For smooth morphisms of smooth pairs we define the higher direct image as follows. Let $D' := \bar{Y} \setminus Y$. The sheaf of relative logarithmic differentials is defined as the cokernel in the following short exact sequence:

$$0 \longrightarrow f^*\Omega^1_{\bar{Y}}(\log D') \longrightarrow \Omega^1_{\bar{X}}(\log D) \longrightarrow \Omega^1_{\bar{X}/\bar{Y},\log} \longrightarrow 0$$

and $\Omega^p_{\bar{X}/\bar{Y},\log} := \Lambda^p \Omega^1_{\bar{X}/\bar{Y},\log}$. For $\mathcal{M} = (M, \nabla, F^\bullet, \Phi_M) \in S(\mathscr{X})$ we can define the following algebraic and rigid relative de Rham complexes

$$\mathrm{DR}^\bullet_{X/Y}(M) := M \otimes_{\mathcal{O}_{\bar{X}}} \Omega^1_{\bar{X}/\bar{Y},\log}$$

and

$$\mathrm{DR}^\bullet_{X/Y}(M^{rig}) := M^{rig} \otimes_{j^\dagger \mathcal{O}_{\bar{\mathcal{X}}_K}} j^\dagger \Omega^1_{\bar{\mathcal{X}}_K/\bar{\mathcal{Y}}_K}$$

and their higher direct images

$$R^p f_* \mathrm{DR}^\bullet_{X/Y}(M), \quad R^p f_{rig,*} \mathrm{DR}^\bullet_{X/Y}(M^{rig}).$$

In the special case $\mathscr{X} \xrightarrow{f} \mathscr{V} := (\mathcal{O}_K, \mathcal{O}_K)$ both

$$H^p_{\mathrm{dR}}(X, M) := R^p f_* \mathrm{DR}^\bullet_{X/\mathcal{O}_K}(M), \quad H^p_{\mathrm{rig}}\left(X_k, M^{rig}\right) := R^p f_{rig,*} \mathrm{DR}^\bullet_{X/\mathcal{O}_K}(M^{rig})$$

are K-vector spaces.

While $R^p f_* \mathrm{DR}^\bullet_{X/Y}(M)$ is equipped with the Hodge-Filtration $F^\bullet$ and the Gauss–Manin connection ∇_{GM}, the rigid cohomology $R f_{rig,*} \mathrm{DR}^\bullet_{X/Y}(M^{rig})$ is equipped with a Frobenius structure Φ. If we write $j_Y : \mathcal{Y}_K \hookrightarrow \bar{\mathcal{Y}}_K$ for the inclusion, we have a comparison map

$$\Theta_{\mathscr{X}/\mathscr{Y}} : j_Y^\dagger \left(R^p f_* \mathrm{DR}^\bullet_{X/Y}(M)|_{Y_K^{an}}\right) \to R^p f_{rig,*} \mathrm{DR}^\bullet_{X/Y}(M^{rig}).$$

Whenever $\Theta_{\mathscr{X}/\mathscr{Y}}$ is an isomorphism, we obtain a structure of a filtered overconvergent F-isocrystal over $\mathscr{Y}$:

$$\underline{H}^p_{\mathrm{syn}}(\mathscr{X}/\mathscr{Y},\mathcal{M}) := \left(Rf_*\mathrm{DR}^\bullet_{X/Y}(M), \nabla_{\mathrm{GM}}, F^\bullet, \Phi\right) \in S(\mathscr{Y}).$$

It is known that for proper maps $\pi : \mathscr{X} \to \mathscr{Y}$ the comparison map $\Theta_{\mathscr{X}/\mathscr{Y}}$ is always an isomorphism [BK10a, Prop. A.7.].

Definition 6.1.3. A filtered overconvergent F-isocrystal $\mathcal{M} = (M, \nabla, F^\bullet, \Phi_M) \in S(\mathscr{X})$ is called *admissible* if:

(a) The Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^p(\bar{X}_K, \mathrm{gr}^p_F \mathrm{DR}^\bullet_{X/K}(M)) \implies H^{p+q}_{\mathrm{dR}}(X_K, M)$$

degenerates at E_1.

(b) $\Theta_{\mathscr{X}/(\mathcal{O}_K,\mathcal{O}_K)} : H^p_{\mathrm{dR}}(X_K, M) \to H^p_{\mathrm{rig}}(X_k, M^{rig})$ is an isomorphism.

(c) The K-vector space $H^p_{\mathrm{dR}}(X_K, M) \xrightarrow{\sim} H^p_{\mathrm{rig}}(X_k, M^{rig})$ with Hodge filtration coming from H^p_{dR} and Frobenius structure coming from H^p_{rig} is weakly admissible in the sense of Fontaine.

Let us write $S(\mathscr{X})^{adm}$ for the full subcategory of admissible objects.

We will also need the following relative version of 'admissible' from [Sol08, Def. 5.8.12]:

Definition 6.1.4. Let $\pi : \mathscr{X} \to \mathscr{Y}$ be a smooth morphism of smooth pairs. A filtered overconvergent F-isocrystal $\mathcal{M} = (M, \nabla, F^\bullet, \Phi_M) \in S(\mathscr{X})$ is called π-*admissible* if:

(a) $\Theta_{\mathcal{X}/\mathcal{Y}}$ is an isomorphism.

(b) The obtained filtered overconvergent F-isocrystals over $\mathscr{Y}$

$$\underline{H}^p_{\mathrm{syn}}(\mathscr{X}/\mathscr{Y},\mathcal{M}) := \left(R^pf_*\mathrm{DR}^\bullet_{X/Y}(M), \nabla_{\mathrm{GM}}, F^\bullet, \Phi\right) \in S(\mathscr{Y}).$$

are admissible.

Let us write $S(\mathscr{X})^{\pi-adm}$ for the full subcategory of π-admissible objects.

For $\pi : \mathscr{X} \to \mathscr{Y}$ a smooth morphism of smooth pairs we obtain functors

$$\underline{H}^p_{\mathrm{syn}}(\mathscr{X}/\mathscr{Y},\cdot) : S(\mathscr{X})^{\pi-adm} \to S(\mathscr{Y})^{adm}.$$

Let us briefly recall the definition of rigid-syntomic cohomology as given by Bannai. We follow the exposition in [BK10a]: Let $\mathscr{X} = (X, \bar{X}, \phi, \phi_V)$ be a syntomic datum and $\mathcal{M} = (M, \nabla, F^\bullet, \Phi_M)$ be a filtered overconvergent F-isocrystal. For a finite Zariski covering $\mathfrak{U} = (\bar{U}_i)_{i\in I}$ of $\bar{X}$ set $\bar{U}_{i_0,\ldots,i_n,K} := \bigcap_{0\leq j\leq n} \bar{U}_{i_j,K}$. $\mathfrak{U}$ induces a covering $(\mathcal{U}_{i,K})_{i\in I}$ of $\mathcal{X}_K$ obtained via the completion of $U_i \cap X$ along its special fiber. Let us write

$$j_{i_0,\ldots,i_n} : \mathcal{U}_{i_0,\ldots,i_n,K} := \bigcap_{0\leq j\leq n} \mathcal{U}_{i_j,K} \hookrightarrow \bar{\mathcal{X}}_K$$

for the inclusion. The total complex associated with the Čech complex

$$\prod_i \Gamma\left(\bar{U}_{i,K}, \mathrm{DR}^\bullet_{\mathrm{dR}}(M)\right) \to \prod_{i_0,i_1} \Gamma\left(\bar{U}_{i_0,i_1,K}, \mathrm{DR}^\bullet_{\mathrm{dR}}(M)\right) \to ...$$

will be denoted by $R^\bullet_{\mathrm{dR}}(\mathfrak{U}, \mathcal{M})$. Similarly, let us define $R^\bullet_{\mathrm{rig}}(\mathfrak{U}, \mathcal{M})$ as the total complex associated with:

$$\prod_i \Gamma\left(\bar{\mathcal{X}}_K, j_i^\dagger \mathrm{DR}^\bullet_{rig}(M^{rig})\right) \to \prod_{i_0,i_1} \Gamma\left(\bar{\mathcal{X}}_K, j_{i_0,i_1}^\dagger \mathrm{DR}^\bullet_{rig}(M^{rig})\right) \to ...$$

The Frobenius structure Φ_M together with the overconvergent Frobenius $\phi_X = (\phi, \phi_V)$ induce

$$\phi_{\mathfrak{U}} : K \otimes_{\sigma,K} R^\bullet_{\mathrm{rig}}(\mathfrak{U}, \mathcal{M}) \to R^\bullet_{\mathrm{rig}}(\mathfrak{U}, \mathcal{M})$$

and the comparison map $\Theta_{X/K}$ induces

$$\Theta_{\mathfrak{U}} : R^\bullet_{\mathrm{dR}}(\mathfrak{U}, \mathcal{M}) \to R^\bullet_{\mathrm{rig}}(\mathfrak{U}, \mathcal{M}).$$

Let

$$R^\bullet_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M}) := \mathrm{Cone}\left(F^0 R^\bullet_{\mathrm{dR}}(\mathfrak{U}, \mathcal{M}) \xrightarrow{(1-\phi_{\mathfrak{U}})\circ\Theta_{\mathfrak{U}}} R^\bullet_{\mathrm{rig}}(\mathfrak{U}, \mathcal{M})\right)[1]$$

where $F^\bullet$ is the filtration induced by the Hodge filtration.

Definition 6.1.5. The rigid syntomic cohomology of $\mathscr{X}$ with coefficients in $\mathcal{M}$ is defined as

$$H^n_{\mathrm{syn}}(\mathscr{X}, \mathcal{M}) := \varinjlim_{\mathfrak{U}} H^n\left(R^\bullet_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M})\right)$$

where the limit is taken over all coverings with respect to refinements.

By its very definition we have a long exact sequence

$$.. \longrightarrow F^0 H^m_{\mathrm{dR}}(X_K, M) \longrightarrow H^m_{\mathrm{rig}}(X_k, M^{rig}) \longrightarrow H^{m+1}_{\mathrm{syn}}(\mathscr{X}, \mathcal{M}) \longrightarrow ...$$

Above we have defined functors

$$\underline{H}^p_{\mathrm{syn}}(\mathscr{X}/\mathscr{Y}, \cdot) : S(\mathscr{X})^{\pi-adm} \to S(\mathscr{Y})^{adm}.$$

The reason for the chosen notation is the following spectral sequence [Sol08, Theorem 5.9.1]. For $\mathcal{M} = (M, \nabla, F^\bullet, \Phi_M) \in S(\mathscr{X})^{\pi-adm}$ and $\pi : \mathscr{X} \to \mathscr{Y}$ a smooth morphism of smooth pairs there is a Leray spectral sequence:

$$E_2^{p,q} = H^p_{\mathrm{syn}}\left(\mathscr{Y}, \underline{H}^p_{\mathrm{syn}}(\mathscr{X}/\mathscr{Y}, \mathcal{M})\right) \implies E^{p+q} = H^{p+q}_{\mathrm{syn}}(\mathscr{X}, \mathcal{M}).$$

Either by this spectral sequence or directly by the above long exact sequence, we deduce the following:

Corollary 6.1.6. *For $\mathscr{V} := (\mathcal{O}_K, \mathcal{O}_K)$ we have the short exact sequence:*

$$0 \to H^1_{\mathrm{syn}}\left(\mathscr{V}, \underline{H}^m_{\mathrm{syn}}(\mathscr{X}, \mathcal{M})\right) \to H^{m+1}_{\mathrm{syn}}(\mathscr{X}, \mathcal{M}) \to H^0_{\mathrm{syn}}\left(\mathscr{V}, \underline{H}^{m+1}_{\mathrm{syn}}(\mathscr{X}, \mathcal{M})\right) \to 0$$

Definition 6.1.7. The *boundary map*

$$\delta : H^m_{\mathrm{syn}}(\mathscr{X}, \mathcal{M}) \to H^m_{\mathrm{dR}}(X_K, M)$$

is defined as the composition

$$H^m_{\mathrm{syn}}(\mathscr{X}, \mathcal{M}) \to H^0_{\mathrm{syn}}\left(\mathscr{V}, \underline{H}^m_{\mathrm{syn}}(\mathscr{X}, \mathcal{M})\right)$$

with the inclusion

$$H^0_{\mathrm{syn}}\left(\mathscr{V}, \underline{H}^m_{\mathrm{syn}}(\mathscr{X}, \mathcal{M})\right) = \ker\left(F^0 H^m_{\mathrm{dR}}(X_K, M) \xrightarrow{1-\phi} H^m_{\mathrm{rig}}\left(X_k, M^{rig}\right)\right) \subseteq H^m_{\mathrm{dR}}(X_K, M)$$

In general, the category $S(\mathscr{X})$ is not Abelian. As in [Ban00, Rem 1.15] we will regard the category $S(\mathscr{X})$ as an exact category with exact sequences given by sequences

$$0 \to M' \to M \to M'' \to 0$$

such that the underlying sequence of $\mathcal{O}_{\bar{X}_K}$-modules is exact and the morphisms in the sequence are strictly compatible with the filtrations. The Tate objects $K(n) \in S(\mathscr{X})$ are defined as

$$K(n) = (\mathcal{O}_{\bar{X}_K}, d, F^\bullet, \Phi)$$

with $F^{-j}\mathcal{O}_{\bar{X}_K} = \mathcal{O}_{\bar{X}_K} \subseteq F^{-j+1}\mathcal{O}_{\bar{X}_K} = 0$ and $\Phi(1) = p^{-j}$.

Proposition 6.1.8 ([Ban00, Proposition 4.4])**.** *For $i = 0, 1$ there is a canonical isomorphism*

$$\mathrm{Ext}^i_{S(\mathscr{X})}(K(0), \mathcal{M}) \xrightarrow{\sim} H^i_{\mathrm{syn}}(\mathscr{X}, \mathcal{M})$$

fitting into the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^i_{S(\mathscr{X})}(K(0), \mathcal{M}) \arrow[r, "\mathrm{For}"] \arrow[d, "\cong"] & \mathrm{Ext}^i_{\mathrm{VIC}(X_K/K)}(K(0), M) \arrow[d, "\cong"] \\
H^i_{\mathrm{syn}}(\mathscr{X}, \mathcal{M}) \arrow[r, "\delta"] & H^i_{\mathrm{dR}}(X_K, M)
\end{tikzcd}$$

where For *is the map forgetting the Hodge filtration and the Frobenius structure.*

Last but not least, let us recall the following useful description of classes in $H^1_{\mathrm{syn}}(\mathscr{X}, \mathcal{M})$ if $F^0 M = 0$:

Proposition 6.1.9 ([BK10a, Proposition A.16]). *Let $\mathcal{M} = (M, \nabla, F, \Phi)$ be an admissible filtered overconvergent F-isocrystal with $F^0 M = 0$. A cohomology class*

$$[\alpha] \in H^1_{\mathrm{syn}}(\mathscr{X}, \mathcal{M})$$

is given uniquely by a pair (α, ξ) with

$$\alpha \in \Gamma(\bar{\mathcal{X}}_K, M^{rig}), \quad \xi \in \Gamma(\bar{X}_K, F^{-1}M \otimes \Omega^1_{\bar{X}_K}(\log D))$$

satisfying the conditions:

$$\nabla(\alpha) = (1 - \Phi)(\xi), \quad \nabla(\xi) = 0$$

In particular, this result will apply to the polylogarithm class. Indeed, we will see that the differential equation of overconvergent functions describing the rigid syntomic polylogarithm class is just a restatement of the abstract differential equation

$$\nabla(\alpha) = (1 - \Phi)(\xi)$$

in terms of the basis obtained by the Katz splitting.

Corollary 6.1.10 ([BK10a, Corollary A.17]). *Suppose $(\alpha, \xi) = [\alpha] \in H^1_{\mathrm{syn}}(\mathscr{X}, \mathcal{M})$ is as in the previous proposition. Then, the image of $[\alpha]$ under*

$$H^1_{\mathrm{syn}}(\mathscr{X}, \mathcal{M}) \to H^1_{\mathrm{dR}}(\mathscr{X}, M)$$

is given by $[\xi]$.

Remark 6.1.11. In particular, this corollary shows that forgetting the Frobenius structure of the syntomic polylogarithm class gives a distinguished system of cocycles representing the de Rham cohomology class. We have shown in Proposition 5.2.12 that this conclusion holds even without referring to any compatibility with the Frobenius structure.

6.2 Definition of the rigid syntomic logarithm sheaves

We recall the definitions of the rigid syntomic logarithm sheaves. As before, let $K/\mathbb{Q}_p$ be a finite unramified extension. Let $\pi : \mathscr{E} = (E, \bar{E}, \phi_E) \to \mathscr{S} = (S, \bar{S}, \phi_S)$ be a morphism of syntomic data with $\pi : E \to S$ an elliptic curve over some affine scheme S.

Since π is proper, $\mathcal{H} := \underline{H}^1_{\mathrm{syn}}(\mathscr{E}/\mathscr{S}, K(1)) \in S(\mathscr{S})^{ad}$ and $\mathcal{H}^\vee \in S(\mathscr{S})^{ad}$ are well defined admissible filtered overconvergent F-isocrystals. Applying the Leray spectral sequence for syntomic cohomology to $\mathcal{H}_E := \pi^*\mathcal{H}$ and using Proposition 6.1.8 gives a split short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{S(\mathscr{S})}(K(0), \mathcal{H}) \arrow[r, "\pi^*"] & \mathrm{Ext}^1_{S(\mathscr{E})}(K(0), \mathcal{H}_E) \arrow[l, bend right, "e^*"'] \arrow[r, "\tilde{\delta}"] & \mathrm{Hom}_{S(\mathscr{S})}(\mathcal{H}, \mathcal{H}) \arrow[r] & 0 \end{tikzcd}$$

Lemma/Definition 6.2.1. Let $[\mathcal{L}og^1_{\mathrm{syn}}] \in \mathrm{Ext}^1_{S(\mathscr{E})}(K(0), \mathcal{H}_E)$ be the extension class which is uniquely determined by $e^*[\mathcal{L}og^1_{\mathrm{syn}}] = 0$ and $\tilde{\delta}[\mathcal{L}og^1_{\mathrm{syn}}] = \mathrm{id}_{\mathcal{H}}$ in the above split short exact sequence. For every sequence

$$0 \longrightarrow \mathcal{H}_E \longrightarrow \mathcal{L}og^1_{\mathrm{syn}} \longrightarrow K(0) \longrightarrow 0 \tag{6.1}$$

representing $[\mathcal{L}og^1_{\mathrm{syn}}]$ there is exactly one splitting

$$e^*\mathcal{L}og^1_{\mathrm{syn}} \xrightarrow{\sim} K(0) \oplus \mathcal{H}$$

in the category $S(\mathscr{S})$. In particular, there are no non-trivial automorphisms of (6.1). We call

$$\mathcal{L}og^1_{\mathrm{syn}} = (\mathrm{Log}^1_{\mathrm{syn}}|_{E_K}, \nabla_{\mathcal{L}og_{\mathrm{syn}}}, F^\bullet_{\mathcal{L}og_{\mathrm{syn}}}, \Phi_{\mathcal{L}og_{\mathrm{syn}}}) =$$

the first syntomic logarithm sheaf. We define $\mathcal{L}og^n_{\mathrm{syn}} := \underline{\mathrm{TSym}}^n\, \mathcal{L}og^1_{\mathrm{syn}}$.

Proof. The only assertion to check that there is only one splitting

$$e^*\mathcal{L}og^1_{\mathrm{syn}} \xrightarrow{\sim} K(0) \oplus \mathcal{H}$$

compatible with filtration, connection and Frobenius structure. Two splittings differ by a map $f \in \mathrm{Hom}_{S(\mathscr{S})}(K(0), \mathcal{H})$. Compatibility with filtration shows that f factors through $F^0\mathcal{H}$. By compatibility with the connection we deduce $f = 0$. □

The canonical map $\mathcal{L}og^1_{\mathrm{syn}} \twoheadrightarrow K(0)$ induces transition maps $\mathcal{L}og^{n+1}_{\mathrm{syn}} \twoheadrightarrow \mathcal{L}og^n_{\mathrm{syn}}$. The canonical isomorphism

$$e^*\mathcal{L}og^1_{\mathrm{syn}} \xrightarrow{\sim} K(0) \oplus \mathcal{H}$$

induces isomorphisms

$$e^*\mathcal{L}og^n_{\mathrm{syn}} \xrightarrow{\sim} \bigoplus_{k=0}^{n} \underline{\mathrm{TSym}}^k\, \mathcal{H}.$$

In particular, $1 \in K(0)$ gives us a canonical horizontal section $\mathbb{1}^n$ in $e^*\mathcal{L}og^n_{\mathrm{syn}}$.

Remark 6.2.2.

(a) The Hodge filtrations on $\mathcal{H}_E$ and $K(0)$ determine the Hodge filtration on $\mathcal{L}og^1_{\mathrm{syn}}$, since we have assumed morphisms in exact sequences to be strictly compatible with the filtrations.

(b) Assume we have fixed an absolute de Rham logarithm sheaf $(\mathrm{Log}^n_{\mathrm{dR}}, \nabla_{\mathrm{Log}^n_{\mathrm{dR}}}, \mathbb{1}^n)$ for E/S. Then, the universal property of the de Rham logarithm sheaves gives us a unique horizontal isomorphism

$$(\mathrm{Log}^n_{\mathrm{dR}}, \nabla_{\mathrm{Log}^n_{\mathrm{dR}}}) \xrightarrow{\sim} (\mathrm{Log}^n_{\mathrm{syn}}, \nabla_{\mathcal{L}og_{\mathrm{syn}}})|_{E_K}$$

identifying $\mathbb{1}^n$ with $\mathbb{1}^n$. In particular, the Frobenius structure on $\mathcal{L}og^n_{\mathrm{syn}}$ induces a unique Frobenius structure on $(\mathrm{Log}^n_{\mathrm{dR}})^{rig}$. Applying this to the geometric logarithm sheaves with its absolute connection $(\mathcal{L}^\dagger_n, \nabla_{\mathcal{L}^\dagger_n, abs}, 1)$ gives us a unique Frobenius structure $\Phi_{\mathcal{L}}$ on $(\mathcal{L}^\dagger_n)^{rig}$.

By the above two remarks we may assume

$$(\mathrm{Log}^1_{\mathrm{syn}}|_{E_K}, \nabla_{\mathcal{L}og_{\mathrm{syn}}}, F^{\bullet}_{\mathcal{L}og_{\mathrm{syn}}}, \Phi_{\mathcal{L}og_{\mathrm{syn}}}) = (\mathcal{L}_1^{\dagger}, \nabla_{\mathcal{L}_1^{\dagger},abs}, F^{\bullet}, \Phi_{\mathcal{L}}).$$

Here, $(\mathcal{L}_1^{\dagger}, \nabla_{\mathcal{L}_1^{\dagger},abs})$ is the absolute connection on the first geometric logarithm sheaf, $F^{\bullet}$ is the Hodge filtration

$$\mathcal{L}_1^{\dagger} = F^{-1}\mathcal{L}_1^{\dagger} \supseteq F^0\mathcal{L}_1^{\dagger} = \mathcal{L}_1 \supseteq F^1\mathcal{L}_1^{\dagger} = 0$$

and $\Phi_{\mathcal{L}}$ is the unique Frobenius structure described above. Later, we will give a more explicit description of the Frobenius structure in some cases.

Since π is proper, the map $\Theta_{\mathscr{E}/\mathscr{S}}$ is an isomorphism. In particular,

$$\underline{H}^i_{\mathrm{syn}}\left(\mathscr{E}/\mathscr{S}, \mathcal{L}og^n_{\mathrm{syn}}\right) \in S(\mathscr{S})$$

is well defined. Along the same lines as in the de Rham realization [Sch14, §1.2], one can prove the following result:

Proposition 6.2.3 ([Sol08, Lemma 6.3.3.])**.** *Let $\pi : \mathscr{E} \to \mathscr{S}$ be as before.*

(a)

$$\underline{H}^i_{\mathrm{syn}}\left(\mathscr{E}/\mathscr{S}, \mathcal{L}og^n_{\mathrm{syn}}\right) \cong \begin{cases} \underline{\mathrm{TSym}}^k\,\mathcal{H} & i=0 \\ \underline{\mathrm{TSym}}^{k+1}\,\mathcal{H}(-1) & i=1 \\ K(-1) & i=2 \end{cases}$$

(b) The transition maps

$$\underline{H}^i_{\mathrm{syn}}\left(\mathscr{E}/\mathscr{S}, \mathcal{L}og^{n+1}_{\mathrm{syn}}\right) \to \underline{H}^i_{\mathrm{syn}}\left(\mathscr{E}/\mathscr{S}, \mathcal{L}og^n_{\mathrm{syn}}\right)$$

are zero for $i = 0,1$ and are isomorphisms for $i = 2$. In particular, the trace isomorphism for $i = 2$ gives canonical isomorphisms

$$\underline{H}^2_{\mathrm{syn}}\left(\mathscr{E}/\mathscr{S}, \mathcal{L}og^n_{\mathrm{syn}}\right) \xrightarrow{\sim} \dots \xrightarrow{\sim} \underline{H}^2_{\mathrm{syn}}\left(\mathscr{E}/\mathscr{S}, \mathcal{L}og^0_{\mathrm{syn}}\right) \xrightarrow{\sim} K(0)$$

Using Liebermann's trick, i. e. the isomorphism [KLL15, Theorem 3.2.3.]

$$H^i_{\mathcal{T}}\left(\mathscr{S}, \underline{\mathrm{TSym}}^k\,\mathcal{H}(j)\right) \cong H^{i+k}_{\mathcal{T}}\left(\mathscr{E}^k, K(j+k)\right)(\epsilon_k)$$

for $\mathcal{T} \in \{\mathrm{dR}, \mathrm{rig}, \mathrm{syn}\}$, one can deduce that $\underline{\mathrm{TSym}}^k\,\mathcal{H}(j)$ is admissible. This combined with the above computation of the relative cohomology of $\mathcal{L}og^n_{\mathrm{syn}}$ shows:

Corollary 6.2.4. *The overconvergent filtered F-isocrystals $\mathcal{L}og^n_{\mathrm{syn}}$ are π-admissible.*

6.3 Definition of the rigid syntomic polylogarithm class

For the definition of the (D-variant) of the elliptic polylogarithm in rigid syntomic cohomology consider the following diagram of smooth pairs: For $D > 1$ define $U_D := E \backslash E[D]$.

$$\begin{tikzcd} \mathscr{U}_D := (U_D, \bar{E}) \arrow[r, hook, "j_D"] \arrow[dr, "\pi_{\mathscr{U}_D}"'] & \mathscr{E} := (E, \bar{E}) \arrow[d, "\pi_{\mathscr{E}}"] & \mathscr{E}[D] := (E[D], \overline{E[D]}) \arrow[l, hook', "i_D"'] \arrow[dl, "\pi_{\mathscr{E}[D]}"] \\ & \mathscr{S} = (S, \bar{S}) & \end{tikzcd}$$

Lemma 6.3.1. *The localization sequence induces an exact sequence:*

$$0 \longrightarrow \varprojlim_n H^1_{\mathrm{syn}}\left(\mathscr{U}_D, \mathcal{L}og^n_{\mathrm{syn}}(1)\right) \longrightarrow \varprojlim_n \underline{H}^0_{\mathrm{syn}}\left(\mathscr{E}[D], \mathcal{L}og^n_{\mathrm{syn}}(1)|_{\mathscr{E}[D]}\right) \xrightarrow{\mathrm{aug}} K$$

Proof. Let us first note that the localization sequences for de Rham and rigid cohomology induce the following exact sequence in the category $S(\mathscr{V})$ for $\mathscr{V} = (\mathcal{O}_K, \mathcal{O}_K)$:

$$0 \to \underline{H}^1_{\mathrm{syn}}\left(\mathscr{E}/\mathscr{V}, \mathcal{L}og^n_{\mathrm{syn}}(1)\right) \to \underline{H}^1_{\mathrm{syn}}\left(\mathscr{U}_D/\mathscr{V}, \mathcal{L}og^n_{\mathrm{syn}}(1)\right) \to \underline{H}^0_{\mathrm{syn}}\left(\mathscr{E}[D]/\mathscr{V}, \mathcal{L}og^n_{\mathrm{syn}}(1)|_{\mathscr{E}[D]}\right) \to K(0)$$

The strict compatibility of filtrations can be deduced from Hodge theory since we are considering de Rham cohomology over a field of characteristic 0. The compatibility of the localization sequence in rigid cohomology with Frobenii is shown in [Tsu99]. The term $K(0)$ follows from Proposition 6.2.3.

The exact sequence in the statement follows now by applying $H^0_{\mathrm{syn}}(\mathscr{V}, \cdot)$ and observing Corollary 6.1.6 as well as the vanishing results

$$\varprojlim_n \underline{H}^1_{\mathrm{syn}}\left(\mathscr{E}/\mathscr{V}, \mathcal{L}og^n_{\mathrm{syn}}(1)\right) = 0, \quad \varprojlim_n \underline{H}^0_{\mathrm{syn}}\left(\mathscr{U}_D/\mathscr{V}, \mathcal{L}og^n_{\mathrm{syn}}(1)\right) = 0$$

which are deduced from Proposition 6.2.3. □

Let $D^2 1_e - 1_{E[D]}$ be defined as in Section 5.2.1. The exact sequence

$$0 \longrightarrow \varprojlim_n H^0_{\mathrm{syn}}\left(\mathscr{E}[D], \mathcal{L}og^n_{\mathrm{syn}}\right) \longrightarrow \varprojlim_n F^0 H^0_{\mathrm{dR}}\left(E[D]_K, \mathrm{Log}^n_{\mathrm{dR}}\right) \xrightarrow{1-\phi}$$
$$\xrightarrow{1-\phi} \varprojlim_n H^0_{\mathrm{rig}}\left(E[D]_k, (\mathrm{Log}^n_{\mathrm{dR}})^{rig}\right)$$

allows us to view $D^2 1_e - 1_{E[D]} \in \ker(1-\phi)$ as element of $\varprojlim_n H^0_{\mathrm{syn}}\left(\mathscr{E}[D], \mathcal{L}og^n_{\mathrm{syn}}\right)$.

Definition 6.3.2. The D-variant of the syntomic polylogarithm $\mathcal{pol}_{D,\mathrm{syn}}$ is the unique pro-system in

$$\varprojlim_n H^1_{\mathrm{syn}}\left(\mathscr{U}_D, \mathcal{L}og^n_{\mathrm{syn}}(1)\right)$$

mapping to $D^2 1_e - 1_{E[D]}$ under the residue map Res in the localization sequence.

6.4 The differential equation associated with the Katz splitting

In this section we want to set up a system of overconvergent differential equations describing the Frobenius structure of the polylogarithm class. This works for any syntomic datum underlying an elliptic curve.

Let $\pi : \mathscr{E} = (E, \bar{E}, \phi_E) \to \mathscr{S} = (S, \bar{S}, \phi_S)$ be a morphism of syntomic data with $\pi : E \to S$ an elliptic curve over some affine scheme S. For simplicity let us further assume that $\underline{\omega}_{E/S}$ is freely generated by some $\omega \in \Gamma(S, \underline{\omega}_{E/S})$. By Remark 6.2.2 we may assume:

$$(\mathrm{Log}^n_{\mathrm{syn}}|_{E_K}, \nabla_{\mathcal{L}og_{\mathrm{syn}}}, F^\bullet_{\mathcal{L}og_{\mathrm{syn}}}, \Phi_{\mathcal{L}og_{\mathrm{syn}}}) = (\mathcal{L}^\dagger_n, \nabla_{\mathcal{L}^\dagger_n, abs}, F^\bullet, \Phi_{\mathcal{L}}) \tag{6.2}$$

Further, observe $F^1\mathcal{L}^\dagger_n = 0$ and $F^0\mathcal{L}^\dagger_n = \mathcal{L}_n$. Let us recall the definition of the map ϕ used in the definition of rigid syntomic cohomology:

$$\phi : \Gamma(\bar{\mathcal{E}}_K, (\mathrm{Log}_{\mathrm{syn}})^{rig}) \xrightarrow{\phi^*_E} \Gamma(\bar{\mathcal{E}}_K, \phi^*_E(\mathrm{Log}_{\mathrm{syn}})^{rig}) \xrightarrow{\Phi_{\mathcal{L}og_{\mathrm{syn}}}} \Gamma(\bar{\mathcal{E}}_K, (\mathrm{Log}_{\mathrm{syn}})^{rig})$$

Combining [BK10a, Proposition A.16] with Theorem 5.2.10 we obtain:

Corollary 6.4.1. *There exists a unique compatible sequence $(\rho_n)_{n\geq 0}$ of overconvergent sections $\rho_n \in \Gamma\left(\bar{\mathscr{E}}_K, j^\dagger_D(\mathrm{Log}^n_{\mathrm{syn}})\right)$ satisfying*

$$\nabla_{\mathcal{L}^\dagger, abs}(\rho_n) = (1-\phi)(L^D_n) \tag{6.3}$$

where we refer to Definition 5.2.5 for the definition of $L^D_n \in \Gamma\left((U_D)_K, F^{-1}\mathcal{L}^\dagger_n \otimes \Omega^1_{(U_D)_K}\right)$. The pair $\left(\rho_n, L^D_n\right)_{n\geq 0}$ is the unique pair representing $\mathit{pol}^n_{D,\mathrm{syn}} \in \underline{H}^1_{\mathrm{syn}}\left(\mathscr{U}_D, \mathcal{L}og^n_{\mathrm{syn}}(1)\right)$.

Proof. By [BK10a, Proposition A.16] there exists a unique pair (ρ_n, ξ_n) representing $\mathit{pol}^n_{D,\mathrm{syn}}$ with $\rho_n \in \Gamma(\bar{\mathscr{E}}_K, j^\dagger_D(\mathrm{Log}^n_{\mathrm{syn}}))$ and $\xi_n \in \Gamma(\bar{E}_K, F^{-1}\mathrm{Log}^1_{\mathrm{syn}} \otimes \Omega^1_{\bar{E}_K}(\log D))$ satisfying

$$\nabla_{\mathcal{L}og_{\mathrm{syn}}}(\rho_n) = (1-\phi)(\xi_n), \quad \nabla^{(1)}_{\mathcal{L}og_{\mathrm{syn}}}(\xi_n) = 0.$$

The restriction to U_D and the identification (6.2) give an injection:

$$\Gamma\left(\bar{E}_K, \mathrm{Log}^n_{\mathrm{syn}} \otimes_{\mathcal{O}_{\bar{E}_K}} \Omega^1_{\bar{E}_K/K}(\log D)\right) \subseteq \Gamma\left((U_D)_K, \mathcal{L}^\dagger_n \otimes_{\mathcal{O}_{U_{D,K}}} \Omega^1_{U_{D,K}/K}\right), s \mapsto s|_{U_{D,K}}.$$

By Corollary 6.1.10, ξ_n represents the corresponding de Rham class under the boundary map δ, i.e $[\xi_n] = \mathrm{pol}_{D,\mathrm{dR}}$. In Proposition 5.2.12 we have characterized

$$(L^D_n)_{n\geq 0} \in \Gamma\left((U_D)_K, \mathcal{L}^\dagger_n \otimes \Omega^1_{U_D/K}\right)_{n\geq 0} \tag{6.4}$$

as the unique compatible system representing $\mathrm{pol}_{D,\mathrm{dR}}$ which is contained in the filtration step $F^0\mathcal{L}^\dagger_n \otimes \Omega^1_{U_D/K} = \mathcal{L}_n \otimes \Omega^1_{U_D/K}$. Since $\xi_n|_{U_{D,K}}$ satisfies these properties, we obtain $\xi_n|_{U_{D,K}} = L^D_n$ and the corollary follows. Let us note that, by slightly abusing notation, we have identified L^D_n with the preimage of L^D_n under the inclusion (6.4). $\square$

This corollary gives a differential equation $\nabla_{\mathcal{L}^\dagger,abs}(\rho_n) = (1-\phi)(L_n^D)$ for ρ_n. In particular, the functions ρ_n satisfy the relative version of this differential equation

$$\nabla_{\mathcal{L}^\dagger}(\rho_n) = (1-\phi)(l_n^D)$$

since L_n^D was defined as a lift of l_n^D with respect to the forgetful map

$$\mathcal{L}_n^\dagger \otimes \Omega^1_{U_D/K} \to \mathcal{L}_n^\dagger \otimes \Omega^1_{U_D/S}.$$

We will use the Katz splitting in order to reformulate it as a differential equation for overconvergent functions. The Katz splitting gives isomorphisms

$$\mathrm{split}_\kappa : \mathcal{L}_n^\dagger\Big| U_D \xrightarrow{\sim} \underline{\mathrm{TSym}}^n \, \mathcal{L}_1^\dagger\Big|_{U_D} \xrightarrow{\sim} \bigoplus_{k=0}^{n} \underline{\mathrm{TSym}}^k \, \mathcal{H}_{U_D}.$$

Recall that we have fixed a generator ω of $\underline{\omega}_{E/S}$. The inclusion $\underline{\omega}_{E/S} \subseteq \underline{H}^1_{\mathrm{dR}}(E/S)$ gives us a global section $[\omega]$ of $\underline{H}^1_{\mathrm{dR}}(E/S)$. Let us extend $[\omega]$ to some basis $[\omega'], [\omega]$ of $\underline{H}^1_{\mathrm{dR}}(E/S)$ with $\omega' \in \Gamma(E, \Omega^1_{E/S}(2[e]))$. The dual basis $[\omega']^\vee, [\omega]^\vee$ generates $\mathcal{H}_{U_D}$ as $\mathcal{O}_{U_D}$-module. Let us write

$$\omega^{[k,l]} := \mathrm{split}_\kappa^{-1}\left(([\omega']^\vee)^{[k]}([\omega]^\vee)^{[l]}\right).$$

Remark 6.4.2. Usually we will make the choice $\omega' = \eta := x\frac{dx}{y}$. In some situations there might be other bases for $\underline{H}^1_{\mathrm{dR}}(E/S)$ which might be better suited. For example for a CM elliptic curve the basis ω, ω^* with $\omega^* = -\eta - e_2^*\omega$ used in [BKT10] is compatible with the Frobenius structure on $\underline{H}^1_{\mathrm{dR}}(E/S)$.

Thus, we obtain

$$\mathcal{L}_n^\dagger|_{U_D} = \bigoplus_{k+l\le n} \omega^{[k,l]}\mathcal{O}_{U_D} \tag{6.5}$$

and via $(\mathcal{L}_n^\dagger)^{rig} := j_D^\dagger \mathcal{L}_n^\dagger$

$$(\mathcal{L}_n^\dagger)^{rig} = \bigoplus_{k+l\le n} \omega^{[k,l]} j_D^\dagger \mathcal{O}_{\bar{\mathcal{E}}_K}. \tag{6.6}$$

These decompositions are compatible with the transition maps. Using this isomorphism, let us decompose

$$(1-\phi)(l_n^D) = \sum_{k\le n} \lambda_k^{(p)} \omega^{[k,0]} \otimes \omega$$

with overconvergent functions $\lambda_k^{(p)} \in \Gamma(\bar{\mathcal{E}}_K, j_D^\dagger \mathcal{O}_{\bar{\mathcal{E}}_K})$. Since (6.6) is compatible with transition maps, the overconvergent functions $\lambda_k^{(p)}$ do not depend on the chosen $n \ge k$.

Proposition 6.4.3. *There is a unique family of overconvergent functions $(D_{m,n}^{(p)})_{m,n\ge 0}$ satisfying the following system of differential equations:*

$$\begin{aligned} dD_{k,0}^{(p)} &= -kD_{k-1,0}^{(p)}\omega' + \lambda_k^{(p)}\omega, \quad k \ge 0 \\ dD_{k,l}^{(p)} &= -kD_{k-1,l}\omega' - lD_{k,l-1}\omega, \quad k \ge 0, l > 0. \end{aligned}$$

Here, we use the convention that $D_{m,n} := 0$ whenever $m < 0$ or $n < 0$. Furthermore, the unique system $(\rho_n)_{n\geq 0}$ describing the syntomic polylogarithm class is related to $(D^{(p)}_{m,n})$ as follows:

$$\rho_n = \sum_{k+l\leq n} D^{(p)}_{k,l}\omega^{[k,l]}.$$

Proof. Let us denote by

$$\partial : j^\dagger_D \mathcal{O}_{\bar{\mathcal{E}}_K} \to j^\dagger_D \mathcal{O}_{\bar{\mathcal{E}}_K}$$

the derivative dual to ω, i. e. $ds = (\partial s)\omega$ for $s \in \Gamma(\bar{\mathcal{E}}_K, j^\dagger_D \mathcal{O}_{\bar{\mathcal{E}}_K})$.
Claim: $\Gamma(\bar{\mathcal{E}}_K, \ker\partial \cap \operatorname{Im}\partial) = 0$.
Pf. of Claim: Let us first observe the following properties of the relative rigid cohomology of U_D over S:

$$j^\dagger_S \mathcal{O}_{\bar{\mathcal{S}}_K} \otimes_{\mathcal{O}_{S_K}} \underline{\omega}_{E_K/S_K} \hookrightarrow R^1\pi_{rig,*}\mathrm{DR}^\bullet_{U_D/S}(j^\dagger_D\mathcal{O}_{\bar{\mathcal{E}}_K}), \tag{A}$$

$$j^\dagger_S \mathcal{O}_{\bar{\mathcal{S}}_K} \xrightarrow{\sim} R^0\pi_{rig,*}\mathrm{DR}^\bullet_{U_D/S}(j^\dagger_D\mathcal{O}_{\bar{\mathcal{E}}_K}) \tag{B}$$

where $j_S : \mathcal{S}_K \to \bar{\mathcal{S}}_K$ is the inclusion of the rigid analytic spaces associated with the completions of S and $\bar{S}$ with respect to their special fibers. These properties are easily shown directly or deduced from the comparison isomorphism [Ger07, (2.2)]

$$R^i\pi_{rig,*}\mathrm{DR}^\bullet_{U_D/S}(j^\dagger_D\mathcal{O}_{\bar{\mathcal{E}}_K}) \xrightarrow{\sim} j^\dagger_S\mathcal{O}_{\bar{\mathcal{S}}_K} \otimes_{\mathcal{O}_{S_K}} \underline{H}^1_{\mathrm{dR}}\left(U_{D,K}/S_K\right)$$

and the corresponding statements for de Rham cohomology. The isomorphism (B) implies

$$\Gamma(\bar{\mathcal{E}}_K, \ker\partial) = \Gamma(\bar{\mathcal{E}}_K, \pi^{-1}j^\dagger_S\mathcal{O}_{\bar{\mathcal{S}}_K}) \tag{6.7}$$

while the injection (A) gives

$$\Gamma\left(\bar{\mathcal{E}}_K, \operatorname{Im} d \cap \pi^{-1}\left(j^\dagger_S\mathcal{O}_{\bar{\mathcal{S}}_K} \otimes_{\mathcal{O}_{S_K}} \underline{\omega}_{E_K/S_K}\right)\right) = \{0\}.$$

This can be reformulated as

$$\Gamma(\bar{\mathcal{E}}_K, \operatorname{Im}\partial \cap \pi^{-1}j^\dagger_S\mathcal{O}_{\bar{\mathcal{S}}_K}) = \{0\}. \tag{6.8}$$

Combining (6.7) and (6.8) proves the claim.

Let us now prove uniqueness of the solution in the statement. By considering the difference of two solutions it is enough to prove that $(\tilde{D}_{k,l})_{k,l} = (0)_{k,l\geq 0}$ is the only sequence satisfying:

$$d\tilde{D}_{k,0} = -k\tilde{D}_{k-1,0}\omega', \quad \forall k \geq 0$$
$$d\tilde{D}_{k,l} = -k\tilde{D}_{k-1,l}\omega' - l\tilde{D}_{k,l-1}\omega, \quad \forall k \geq 0, l > 0.$$

Here, again we use the convention that $\tilde{D}_{k,0} := 0$ for $k < 0$. Assume there were a non-zero sequence $(\tilde{D}_{k,l})_{k,l}$ satisfying this differential equation. Then

$$N := \min\{k : \exists l \geq 0 \text{ s.t. } \tilde{D}_{k,l} \neq 0\}$$

exists. The sequence $(\tilde{D}_{N,l})_{l\geq 0}$ satisfies then:

$$\begin{aligned}\partial \tilde{D}_{N,0} &= 0\\ \partial \tilde{D}_{N,l} &= -l\tilde{D}_{N,l-1}, \quad l>0\end{aligned}$$

Using *Claim 1*, we prove by induction $\tilde{D}_{N,l} = 0$ for all $l \geq 0$. This contradicts the minimality of N and we conclude the uniqueness.

Existence of the solution in the statement is a direct translation of the differential equation

$$\nabla_{\mathcal{L}^\dagger}(\rho_n) = (1-\phi)(l_n^D) \tag{6.9}$$

in terms of the basis $\omega^{[k,l]}$ obtained via the Katz splitting. Let us write

$$\rho_n = \sum_{k,l\leq n} D_{k,l}^{(p)}\omega^{[k,l]}$$

with overconvergent functions $D_{k,l}^{(p)} \in \Gamma(\bar{\mathcal{E}}_K, j_D^\dagger \mathcal{O}_{\bar{\mathcal{E}}_K})$. The explicit description of the relative connection $\nabla_{\mathcal{L}_1^\dagger}$ gives:

$$\begin{aligned}\nabla_{\mathcal{L}_1^\dagger}(\omega^{[k,l]}) &= [\omega']^\vee \cdot \omega^{[k,l]} \otimes \omega' + [\omega]^\vee \cdot \omega^{[k,l]} \otimes \omega =\\ &= (k+1)\omega^{[k+1,l]} \otimes \omega' + (l+1)\omega^{[k,l+1]} \otimes \omega\end{aligned}$$

This allows us to rewrite (6.9) as:

$$\begin{aligned}&\sum_{k+l\leq n}\left(d(D_{k,l}^{(p)})\omega^{[k,l]} + (k+1)D_{k,l}^{(p)}\omega^{[k+1,l]}\otimes\omega' + (l+1)D_{k,l}^{(p)}\omega^{[k,l+1]}\otimes\omega\right) = \qquad (6.10)\\ &= \sum_{k\leq n}\lambda_k^{(p)}\omega^{[k,0]}\otimes\omega\end{aligned}$$

for all $n \geq 0$. With the convention that $D_{k,l}^{(p)} = 0$ for $k<0$ or $l<0$ the system (6.10) is equivalent to the following system of differential equations:

$$\begin{aligned}d(D_{k,0}^{(p)}) &= -kD_{k-1,0}^{(p)}\omega' + \lambda_k^{(p)}\omega, \quad k\geq 0\\ d(D_{k,l}^{(p)}) &= -kD_{k-1,l}^{(p)}\omega' - lD_{k,l-1}^{(p)}\omega, \quad k\geq 0, l>0\end{aligned}$$

Thus, $D_{k,l}^{(p)}$ is the unique solution of the system of differential equations in the statement. By construction we have:

$$\rho_n = \sum_{k,l\leq n} D_{k,l}^{(p)}\omega^{[k,l]}.$$

□

Remark 6.4.4.

(a) The map ϕ and thus also the definition of $\lambda_n^{(p)}$ depend on the overconvergent Frobenius chosen in the fixed syntomic datum. The functions $D_{k,l}^{(p)}$ further depend on the chosen basis ω, ω'.

(b) In the case of a CM-elliptic curve we have a canonical Frobenius lift. If we choose $\omega' = \omega^*$ as in [BKT10], the above differential equation is a D-variant of the differential equation considered in [BKT10, Theorem 3.3.].

(c) For ordinary elliptic curves we have a canonical overconvergent Frobenius lift obtained by dividing by the canonical subgroup. From this point one might try to proceed as in [BKT10] and relate this differential equation to moment functions of p-adic measures in tubular neighbourhoods of torsion sections. But we have already remarked that the above explicit differential equation is just a shadow of the abstract differential equation

$$\nabla_{\mathcal{L}_n^\dagger}(\rho_n) = l_n^D$$

under the Katz splitting. For ordinary elliptic curves we have another splitting, the infinitesimal splitting, in tubular neighbourhoods around torsion sections. Since the infinitesimal splitting gave rise to the p-adic Eisenstein–Kronecker measure, it will be much more natural to work with the infinitesimal splitting instead of the Katz splitting when we want to relate the p-adic realization of the polylogarithm to moment functions of the Eisenstein measure.

6.5 The rigid syntomic polylogarithm for ordinary elliptic curves

Let p be a prime and $N > 3$ be an integer prime to p. Let $K = \mathbb{Q}_p$ and denote by $\mathscr{V}$ the smooth pair $\mathscr{V} = (\operatorname{Spec} \mathbb{Z}_p, \operatorname{Spec} \mathbb{Z}_p)$ over $\mathbb{Z}_p$. For the modular curve $M = M_{N,\mathbb{Z}_p}$ with $\Gamma(N)$-level structure over $\mathbb{Z}_p$ choose a smooth compactification $\bar{M}$ and let $(E = E_N, \alpha_N)$ be the universal elliptic curve with level N-structure over M. Let $\bar{E}$ be the Neron model of E over $\bar{M}$. Then

$$(E, \bar{E}) \xrightarrow{\pi} (M, \bar{M})$$

is a smooth proper morphism of smooth pairs. If we restrict to the ordinary locus $M^{\mathrm{ord}} \subseteq M$ defined as the complement of the vanishing locus of the Eisenstein series $E_{(p-1)} \in \Gamma(M, \underline{\omega}_{E/M}^{\otimes(p-1)})$ and define $E^{\mathrm{ord}} := E \times_M M^{\mathrm{ord}}$, we obtain a smooth proper morphism of smooth pairs:

$$(E^{\mathrm{ord}}, \bar{E}) \to (M^{\mathrm{ord}}, \bar{M})$$

6.5.1 Canonical Frobenius structures

Let $\mathcal{E}^{\mathrm{ord}}, \bar{\mathcal{E}}$ resp. $\mathcal{M}^{\mathrm{ord}}, \bar{\mathcal{M}}$ be the formal completions of $E^{\mathrm{ord}}, \bar{E}$ resp. $M^{\mathrm{ord}}, \bar{M}$ with respect to their special fibers. Then, $\mathcal{M}^{\mathrm{ord}}$ classifies ordinary elliptic curves with level

N-structure over p-adic rings. If we divide an ordinary elliptic curve with level N-structure (E, α) by its canonical subgroup, we obtain another ordinary elliptic curve $(E' = E/C, \alpha')$ with level N-structure. In particular, the map $(E, \alpha) \mapsto (E/C, \alpha')$ induces a map

$$\mathrm{Frob} : \mathcal{M}^{\mathrm{ord}} \to \mathcal{M}^{\mathrm{ord}}$$

lifting the Frobenius morphism on the special fiber. By [Kat73, Chapter 3] the induced Frobenius $\mathcal{M}^{\mathrm{ord}}_{\mathbb{Q}_p} \to \mathcal{M}^{\mathrm{ord}}_{\mathbb{Q}_p}$ on the associated rigid analytic space $\mathcal{M}^{\mathrm{ord}}_{\mathbb{Q}_p}$ is overconvergent. In particular, we have a canonical overconvergent Frobenius ϕ_M on the smooth pair $(M^{\mathrm{ord}}, \bar{M})$. The associated syntomic datum will be denoted by

$$\mathscr{M}^{\mathrm{ord}} := (M^{\mathrm{ord}}, \bar{M}, \phi_M).$$

For the moment let us write $E^{\mathrm{ord}}|_{\mathcal{M}^{\mathrm{ord}}}$ for the pullback of the universal elliptic curve to the formal completion. Similarly, as in Eq. (4.4) the commutative diagram

$$\begin{tikzcd}
E^{\mathrm{ord}}|_{\mathcal{M}^{\mathrm{ord}}} \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & E'|_{\mathcal{M}^{\mathrm{ord}}} := (E^{\mathrm{ord}}/C)|_{\mathcal{M}^{\mathrm{ord}}} \arrow[r, "\widetilde{\mathrm{Frob}}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E^{\mathrm{ord}}|_{\mathcal{M}^{\mathrm{ord}}} \arrow[d, "\pi"] \\
& \mathcal{M}^{\mathrm{ord}} \arrow[r, "\mathrm{Frob}"] & \mathcal{M}^{\mathrm{ord}}
\end{tikzcd}$$

induces a Frobenius lift $E^{\mathrm{ord}}|_{\mathcal{M}^{\mathrm{ord}}} \to E^{\mathrm{ord}}|_{\mathcal{M}^{\mathrm{ord}}}$ on $E^{\mathrm{ord}}|_{\mathcal{M}^{\mathrm{ord}}}$ which gives us a canonical overconvergent Frobenius ϕ_E on the smooth pair $(E^{\mathrm{ord}}, \bar{E})$. The associated syntomic datum is

$$\mathscr{E}^{\mathrm{ord}} := (E^{\mathrm{ord}}, \bar{E}, \phi_E)$$

and $\pi : \mathscr{E}^{\mathrm{ord}} \to \mathscr{M}^{\mathrm{ord}}$ is a morphism of syntomic data. As remarked above, if we fix the de Rham part

$$(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger, abs}, F^\bullet) \tag{6.11}$$

there is a unique Frobenius structure

$$\Phi_{\mathcal{L}_n^\dagger} : \phi_E^* \left(\mathcal{L}_n^\dagger\right)^{rig} \to \left(\mathcal{L}_n^\dagger\right)^{rig}$$

which is compatible with the structure in the given de Rham datum (5.8). We can construct such a morphism along the same lines as in Eq. (4.5). The isogeny

$$\varphi : (E^{\mathrm{ord}})|_{\mathcal{M}^{\mathrm{ord}}} \twoheadrightarrow E'|_{\mathcal{M}^{\mathrm{ord}}} := (E^{\mathrm{ord}}/C)|_{\mathcal{M}^{\mathrm{ord}}}$$

induces the morphism $\Phi_\varphi^\dagger : \mathcal{L}_n^\dagger \to \varphi^* \mathcal{L}_{n,E'}^\dagger$. Combining this with the compatibility of $\mathcal{L}_n^\dagger$ with base change gives:

$$\Psi^\dagger : \mathcal{L}_n^\dagger \xrightarrow{\Phi_\varphi^\dagger} \varphi^* \mathcal{L}_{n,E'}^\dagger \cong \varphi^* \widetilde{\mathrm{Frob}}^* \mathcal{L}_n^\dagger = \phi_E^* \mathcal{L}_n^\dagger.$$

Further, this map is horizontal and compatible with the Hodge filtration. While $\Phi_\varphi^\dagger$ is not an isomorphism since φ is not étale, the induced map on the generic fiber is an isomorphism. Thus, on the associated rigid analytic space the map $\Psi_K^\dagger$ induces an isomorphism and its inverse gives us *the* unique Frobenius structure

$$\Phi_{\mathcal{L}_n^\dagger} : \phi_E^* \left(\mathcal{L}_n^\dagger\right)^{rig} \to \left(\mathcal{L}_n^\dagger\right)^{rig}$$

compatible with the connection and the Hodge filtration.

Remark 6.5.1. Thus, for ordinary elliptic curves we can construct the syntomic logarithm sheaves in a canonical way out of the Poincaré bundle:

$$(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger, abs}, F^\bullet, \Phi_{\mathcal{L}})$$

The geometric logarithm sheaf with connection $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger, abs})$ was constructed in Chapter 2 by restricting $\mathcal{P}^\dagger$ to $E \times \mathrm{Inf}^n E^\dagger$. The Hodge filtration is induced by the inclusion $\mathcal{L}_1 \hookrightarrow \mathcal{L}_1^\dagger$ and the Frobenius structure arises naturally by dividing by the canonical subgroup.

Our next aim is to describe the syntomic polylogarithm class for $\mathscr{E}^{\mathrm{ord}}$ more explicitly along tubular neighborhoods of torsion sections. Let $(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$ and let $t = t_{(a,b)}$ be the associated torsion section of $E^{\mathrm{ord}}/M^{\mathrm{ord}}$. Let $]t[$ be the tubular neighbourhood in $\mathcal{E}_K^{\mathrm{ord}}$ of the reduction of t. As in the previous section choose $D > 0$ prime to p and let $\mathscr{U}_D$ be the syntomic datum associated to the complement of $E[D]$ in E. Let $(\rho_n)_{n \geq 0}$ be the unique system describing the pro-system

$$([\rho_n])_{n\geq 0} = (\mathcal{pol}^n_{D,\mathrm{syn}})_{n \geq 0} \in \varprojlim_n \underline{H}^1_{\mathrm{syn}} \left(\mathscr{U}_D, \mathcal{Log}^n_{\mathrm{syn}}(1)\right).$$

Our aim will be to relate $\rho_n|_{]t[} \in \Gamma(]t[, (\mathcal{L}_n^\dagger)^{rig})$ to moment functions of the two-variable Eisenstein–Kronecker measure constructed in Chapter 4. In order to do this it will be convenient to pass to the moduli space of trivialized elliptic curves.

6.5.2 Passing to the moduli space of trivialized elliptic curves

Let $E^{\mathrm{triv}} \to \mathcal{M}^{\mathrm{triv}} = \mathrm{Spf}\, V(\mathbb{Z}_p, \Gamma(N))$ be the formal moduli space classifying elliptic curves with level N-structure and a given rigidification

$$\beta : \hat{E} \xrightarrow{\sim} \widehat{\mathbb{G}}_{m,R}.$$

The formal moduli space $\mathcal{M}^{\mathrm{triv}} = \mathrm{Spf}\, V(\mathbb{Z}_p, \Gamma(N))$ is the formal completion of the moduli space $M^{\mathrm{triv}} = \mathrm{Spec}\, V(\mathbb{Z}_p, \Gamma(N))$, considered in Section 4.3, along its special fiber. The existence of a trivialization on an elliptic curve already implies that the curve is ordinary. Thus, the forgetful map

$$(E, \alpha, \beta) \mapsto (E, \alpha)$$

induces a map $\mathcal{M}^{\mathrm{triv}} \to \mathcal{M}^{\mathrm{ord}}$. The induced map on rigid analytic spaces sits in the following Cartesian diagram

$$\begin{tikzcd} \mathcal{E}^{\mathrm{triv}}_{\mathbb{Q}_p} \arrow[r, "\tilde{p}"] \arrow[d] & \mathcal{E}^{\mathrm{ord}}_{\mathbb{Q}_p} \arrow[d] \\ \mathcal{M}^{\mathrm{triv}}_{\mathbb{Q}_p} \arrow[r] & \mathcal{M}^{\mathrm{ord}}_{\mathbb{Q}_p}. \end{tikzcd}$$

Let $\tilde{t} = \tilde{t}_{a,b}$ be the associated torsion section on E^{triv} and $]\tilde{t}[$ be the tubular neighbourhood of the reduction of $\tilde{t}$ in $\mathcal{E}^{\mathrm{triv}}_{\mathbb{Q}_p}$. Pullback along the covering map $\tilde{p}$ induces an injection

$$\Gamma(]t[, (\mathcal{L}^{\dagger}_{n,E^{\mathrm{ord}}})^{rig}) \hookrightarrow \Gamma(]\tilde{t}[, (\mathcal{L}^{\dagger}_{n,E^{\mathrm{triv}}})^{rig}).$$

The advantage of describing $\tilde{p}^*(\rho_n)$ instead of ρ_n is that the infinitesimal splitting gives a canonical basis $\hat{\omega}^{[k,l]}$ of $(\mathcal{L}^{\dagger}_{n,E^{\mathrm{triv}}})^{rig}$:

$$(\mathcal{L}^{\dagger}_{n,E^{\mathrm{triv}}})^{rig}|_{]\tilde{t}[} \xrightarrow{\sim} \bigoplus_{k+l\leq n} \hat{\omega}^{[k,l]}\mathcal{O}^{an}_{B^-(0,1)\times\mathcal{M}^{\mathrm{triv}}_{\mathbb{Q}_p}}$$

Indeed, in Section 4.3.3 we have constructed a basis

$$(\underline{\mathrm{TSym}}^n\, \mathcal{L}^{\dagger}_1)|_{\hat{E}^{\mathrm{triv}}} = \bigoplus_{k+l\leq n} \hat{\omega}^{[k,l]}\mathcal{O}_{\hat{E}^{\mathrm{triv}}}.$$

Combining this with

$$\mathcal{L}^{\dagger}_n|_{\hat{E}^{\mathrm{triv}}} \to (\underline{\mathrm{TSym}}^n\, \mathcal{L}^{\dagger}_1)|_{\hat{E}^{\mathrm{triv}}_{\tilde{t}}} \xrightarrow{\sim} (\underline{\mathrm{TSym}}^n\, \mathcal{L}^{\dagger}_1)|_{\hat{E}^{\mathrm{triv}}}$$

and the canonical isomorphism

$$\mathcal{O}_{\hat{E}^{\mathrm{triv}}} \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}\times\mathcal{M}^{\mathrm{triv}}}$$

gives after analytification a canonical isomorphism

$$(\mathcal{L}^{\dagger}_{n,E^{\mathrm{triv}}})^{rig}|_{]\tilde{t}[} \xrightarrow{\sim} \bigoplus_{k+l\leq n} \hat{\omega}^{[k,l]}\mathcal{O}^{an}_{B^-(0,1)\times\mathcal{M}^{\mathrm{triv}}_{\mathbb{Q}_p}}.$$

where $B^-(0,1)$ is the open unit disc obtained as the rigid analytic space associated with $\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}$. This allows us to write

$$\tilde{\rho}_n|_{]\tilde{t}[} = \tilde{p}^*\rho_n|_{]\tilde{t}[} = \sum_{k+l\leq n} \hat{\mathrm{e}}_{t,(k,l)}\hat{\omega}^{[k,l]}$$

with $\hat{\mathrm{e}}_{t,(k,l)} \in \Gamma\left(B^-(0,1)\times\mathcal{M}^{\mathrm{triv}}, \mathcal{O}^{an}_{B^-(0,1)\times\mathcal{M}^{\mathrm{triv}}_{\mathbb{Q}_p}}\right)$. It will be convenient to view $\hat{\mathrm{e}}_{t,(k,l)}$ as analytic functions on the open unit disc with values in the ring of generalized p-adic modular forms $V(\mathbb{Z}_p, \Gamma(N)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \Gamma(\mathcal{M}^{\mathrm{triv}}_{\mathbb{Q}_p}, \mathcal{O}_{\mathcal{M}^{\mathrm{triv}}_{\mathbb{Q}_p}})$:

$$B^-(0,1) \to V(\mathbb{Z}_p, \Gamma(N)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \quad x \mapsto \hat{\mathrm{e}}_{t,(k,l)}(x)$$

We want to describe $\hat{\mathrm{e}}_{t,(k,l)}$ explicitly. The idea is the same as in Section 6.4. We will use the infinitesimal splitting and reformulate the differential equation

$$\nabla_{\mathcal{L}}(\tilde{\rho}_n) = (1-\phi)(l_n^D)$$

to characterize the functions $\hat{\mathrm{e}}_{t,(k,l)}$. But as in the CM-case [BKT10], it turns out that the corresponding differential equation on the open unit ball does not have a unique solution. In [BKT10, Lem. 3.9] this problem is solved by imposing a trace-zero condition making the solution unique. We follow this strategy and prove in a first step that $x \mapsto \hat{\mathrm{e}}_{t,(k,l)}(x)$ satisfies a trace-zero condition:

Lemma 6.5.2. *The functions $s \mapsto \hat{\mathrm{e}}_{t,(k,l)}(s)$ satisfy:*

$$\sum_{\zeta \in \widehat{\mathbb{G}}_m[p](\mathbb{C}_p)} \hat{\mathrm{e}}_{t,(k,l)}\left(s +_{\widehat{\mathbb{G}}_m} \zeta\right) = 0, \quad \forall s \in B^-(0,1)(\mathbb{C}_p)$$

Proof. Let us recall the algebraic translation operators

$$\mathrm{trans}_\tau : T_\tau^* \mathcal{L}_n \xrightarrow{T_\tau^* \Phi_\varphi} \varphi^* \mathcal{L}_n \xrightarrow{\Phi_\varphi^{-1}} \mathcal{L}_n$$

for some torsion section $\tau \in \ker \varphi$.
Claim:

$$\sum_{\tau \in \hat{E}^{\mathrm{triv}}[p]} \mathrm{trans}_\tau(T_\tau^* \left[(1-\phi)(l_n^D)\right]) = 0.$$

Pf. of Claim: Indeed, in Lemma 4.6.2 we have proven the equation

$$\sum_{\tau \in \hat{E}^{\mathrm{triv}}[p]} \mathrm{trans}_\tau(T_\tau^* l_n^D) = p\Phi_\varphi^{-1}\left(\varphi^* l^D_{n,E^{\mathrm{triv}}/C}\right). \tag{6.12}$$

The claim follows by summing this equation over all torsion sections in the canonical subgroup $\tau \in \hat{E}^{\mathrm{triv}}[p]$: The definition of the map ϕ gives us the equality

$$\phi(l_n^D) = \Phi_{\mathcal{L}}(\phi_E(l_n^D)) = \Phi_\varphi^{-1}(\varphi^* l^D_{n,E^{\mathrm{triv}}/C}).$$

Using this, we compute:

$$\begin{aligned}
\mathrm{trans}_\tau(T_\tau^* l_n^D) &= (\Phi_\varphi^{-1} \circ T_\tau^* \Phi_\varphi)\left(T_\tau^* \left[(1-\phi)(l_n^D)\right]\right) = \\
&= \mathrm{trans}_\tau\left(T_\tau^* l_n^D\right) - (\Phi_\varphi^{-1} \circ T_\tau^* \Phi_\varphi)\left(T_\tau^* \Phi_\varphi^{-1}\left[T_\tau^* \varphi^* l^D_{n,E^{\mathrm{triv}}/C}\right]\right) = \\
&= \mathrm{trans}_\tau\left(T_\tau^* l_n^D\right) - (\Phi_\varphi^{-1})\left(\varphi^* l^D_{n,E^{\mathrm{triv}}/C}\right)
\end{aligned}$$

Summing this over all torsion sections τ in the canonical subgroup and using (6.12) proves the claim.

Passing to the associated rigid analytic space gives translation operators

$$\mathrm{trans}_\tau^\dagger : T_\tau^*(\mathcal{L}_n^\dagger)^{rig} \to (\mathcal{L}_n^\dagger)^{rig}$$

which are horizontal with respect to the canonical connections on both sides. Further, they are compatible with trans_τ via the inclusion $(\mathcal{L}_n)^{rig} \hookrightarrow (\mathcal{L}_n^\dagger)^{rig}$. In particular, we get the differential equation:

$$\nabla_{\mathcal{L}_n^\dagger}\left(\sum_{\tau\in\hat{E}^{\mathrm{triv}}[p]} \mathrm{trans}_\tau^\dagger(T_\tau^*\tilde{\rho}_n)\right) = \sum_{\tau\in\hat{E}^{\mathrm{triv}}[p]} \mathrm{trans}_\tau^\dagger(T_\tau^*\left[(1-\phi)l_n^D\right]).$$

Using the above claim, this can be written as:

$$\nabla_{\mathcal{L}_n^\dagger}\left(\sum_{\tau\in\hat{E}^{\mathrm{triv}}[p]} \mathrm{trans}_\tau(T_\tau^*\tilde{\rho}_n)\right) = 0 \tag{6.13}$$

In the proof of Proposition 6.4.3 we have shown implicitly that $\tilde{D}_n = (0)_{n\geq 0}$ is the only solution of the differential equation

$$\nabla_{\mathcal{L}_n^\dagger}(\tilde{D}_n) = 0.$$

Applying this to the system (6.13) gives

$$\sum_{\tau\in\hat{E}^{\mathrm{triv}}[p]} \mathrm{trans}_\tau(T_\tau^*\tilde{\rho}_n) = 0, \quad \forall n \geq 0.$$

Restricting this equality to the tubular neighbourhood $]\tilde{t}[$ proves the equality

$$\sum_{\zeta\in\widehat{\mathbb{G}}_m[p](\mathbb{C}_p)} \hat{\mathrm{e}}_{t,(k,l)}(s +_{\widehat{\mathbb{G}}_m} \zeta) = 0, \quad \forall x \in B^-(0,1)(\mathbb{C}_p)$$

cf. Corollary 4.6.3 for a similar argument. □

6.5.3 The syntomic realization on the ordinary locus of the modular curve

Finally, we give an explicit description of the syntomic realization of the elliptic polylogarithm on the ordinary locus of the modular curve in tubular neighbourhoods of torsion sections. We keep the notation from the beginning of this section. As above, let $\mathscr{E}^{\mathrm{ord}} \to \mathscr{M}^{\mathrm{ord}}$ be the syntomic datum associated with the ordinary locus of the modular curve with $\Gamma(N)$-level structure. Further, let $\mathscr{U}_D = (E^{\mathrm{ord}} \setminus E[D], \bar{E}^{\mathrm{ord}}, \phi_E)$ be the syntomic datum associated with the complement of D-torsion points. The syntomic polylogarithm class for $\mathscr{E}^{\mathrm{ord}}/\mathscr{M}^{\mathrm{ord}}$ is uniquely given by the compatible system

$$(\rho_n)_{n\geq 0}.$$

Let us write as above $\tilde{\rho}_n|_{]\tilde{t}[} = \tilde{p}^*\left(\rho_n|_{]t[}\right)$ for the image of $\rho_n|_{]t[}$ under the inclusion

$$\Gamma(]t[, (\mathcal{L}_{n,E^{\mathrm{ord}}}^\dagger)^{rig}) \hookrightarrow \Gamma(]\tilde{t}[, (\mathcal{L}_{n,E^{\mathrm{triv}}}^\dagger)^{rig}).$$

Theorem 6.5.3. *For $(a, b) \neq (0, 0)$ let $t = t_{a,b}$ be the associated N-torsion section on the universal elliptic curve E^{ord} with $\Gamma(N)$-level structure. The decomposition*

$$\tilde{\rho}_n\big|_{]\tilde{t}[} = \sum_{k+l \leq n} \hat{\mathrm{e}}_{t,(k,l)} \hat{\omega}^{[k,l]}$$

gives us rigid analytic functions $(s \mapsto \hat{\mathrm{e}}_{t,(k,l)}(s))_{k,l \geq 0}$ on the open unit disc with values in the ring of generalized p-adic modular forms which are explicitly given by:

$$\hat{\mathrm{e}}_{t,(k,l)}(s) = (-1)^l l! \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} y^k x^{-(l+1)} (1+s)^x d\mu_{D,t}^{\mathrm{Eis},(\mathrm{p})}(x, y)$$

Proof. In the following let us consider the elliptic curve E^{triv}, i. e. when we write l_n^D we mean $l_{n,E^{\mathrm{triv}}}^D$ and so on. Recall from Corollary 4.6.3 that $\left((1-\phi)(l_n^D)\right)_{n \geq 0}$ is mapped to

$$\sum_{k \geq 0} \partial_2^{\circ k} {}_D\vartheta_t^{(p)}(T_1, T_2)|_{T_2=0} \hat{\omega}^{[k,0]} \otimes \omega$$

under

$$\left(\varprojlim_n \mathcal{L}_n \otimes \Omega^1_{E^{\mathrm{triv}}/S}\right)\Bigg|_{\hat{E}_t} \overset{\mathrm{triv}_{\hat{E}_s}}{\rightarrow} \bigoplus_{k \geq 0} \mathcal{O}_{\widehat{\mathbb{G}}_{m,\mathcal{M}^{\mathrm{triv}}}} \hat{\omega}^{[k,0]} \otimes \omega.$$

From now on let us write $B^-(0,1)$ for the rigid analytic space associated with $\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}$. Thus

$${}_D\vartheta_t^{(p)}(T_1, T_2) \in V\left(\mathbb{Z}_p, \Gamma(N)\right) [\![T_1, T_2]\!]$$

induces an analytic function

$$B^-(0,1) \times B^-(0,1) \to V\left(\mathbb{Z}_p, \Gamma(N)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

and we will write (s, s') for the coordinates on $B^-(0,1) \times B^-(0,1)$ induced by T_1, T_2. Thus, the differential equation

$$\nabla_{\mathcal{L}_n^\dagger}(\tilde{\rho}_n) = (1-\phi)(l_n^D)$$

can be rewritten using the infinitesimal splitting as:

$$\nabla_{\mathcal{L}_n^\dagger}\left(\sum_{k+l \leq n} \hat{\mathrm{e}}_{t,(k,l)}(s) \hat{\omega}^{[k,l]}\right) = \sum_{k=0}^{n} \left((1+s')\frac{\partial}{\partial s'}\right)^{\circ k} {}_D\vartheta_t^{(p)}(s, s')\Bigg|_{s'=0} \hat{\omega}^{[k,0]} \otimes \omega, \quad \forall n \geq 0$$

Recall from Lemma 4.3.2 that the connection $\nabla_{\mathcal{L}_n^\dagger}$ expresses via the infinitesimal splitting on $\bigoplus_{k \geq 0} \mathcal{O}_{\widehat{\mathbb{G}}_{m,\mathcal{M}^{\mathrm{triv}}}} \omega^{[k,0]}$ as

$$\nabla_{\mathcal{L}_n^\dagger}(\hat{\omega}^{[k,l]}) = (l+1)\hat{\omega}^{[k,l+1]} \otimes \omega.$$

Further, let us recall that ${}_D\vartheta_t^{(p)}(s,s')$ is the Amice transform of the p-adic measure $\mu_{D,t}^{\mathrm{Eis,(p)}}$. Thus, we obtain the following explicit system of differential equations satisfied by $\hat{\mathrm{e}}_{t,(k,l)}(s)$:

$$(1+s)\frac{\partial}{\partial s}\hat{\mathrm{e}}_{t,(k,0)}(s) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} y^k(1+s)^x d\mu_{D,t}^{\mathrm{Eis,(p)}}(x,y), \quad k \geq 0$$
$$(1+s)\frac{\partial}{\partial s}\hat{\mathrm{e}}_{t,(k,l)}(s) = -l\hat{\mathrm{e}}_{t,(k,l-1)}(s), \quad l>0, k\geq 0.$$

Here, we have used the fact that ${}_D\vartheta_t^{(p)}$ is the Amice transform of the measure $\mu_{D,t}^{\mathrm{Eis,(p)}}$. Further, by Lemma 6.5.2 the functions $\hat{\mathrm{e}}_{t,(k,l)}(s)$ satisfy the following trace-zero condition:

$$\sum_{\zeta\in\widehat{\mathbb{G}}_m[p](\mathbb{C}_p)} \hat{\mathrm{e}}_{t,(k,l)}(s +_{\widehat{\mathbb{G}}_m} \zeta) = 0, \quad \forall s \in B^-(0,1)(\mathbb{C}_p)$$

Claim: The system $\left(\hat{\mathrm{e}}_{t,(k,l)}(s)\right)_{k,l\geq 0}$ is the only system of analytic functions on $B^-(0,1)$ with values in $V(\mathbb{Z}_p,\Gamma(N))\otimes\mathbb{Q}_p$ satisfying:

(a) $(1+s)\frac{\partial}{\partial s}\hat{\mathrm{e}}_{t,(k,0)}(s) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} y^k(1+s)^x d\mu_{D,t}^{\mathrm{Eis,(p)}}(x,y), \quad k \geq 0$

(b) $(1+s)\frac{\partial}{\partial s}\hat{\mathrm{e}}_{t,(k,l)}(s) = -l\hat{\mathrm{e}}_{t,(k,l-1)}(s), \quad l>0, k\geq 0$

(c) $\sum_{\zeta\in\widehat{\mathbb{G}}_m[p](\mathbb{C}_p)} \hat{\mathrm{e}}_{t,(k,l)}(s +_{\widehat{\mathbb{G}}_m} \zeta) = 0, \quad \forall s \in B^-(0,1)(\mathbb{C}_p).$

Pf. of the claim: The functions $\hat{\mathrm{e}}_{t,(k,l)}(s)$ satisfy the above conditions. For uniqueness let $k,l\geq 0$. By induction it is enough to show that any analytic function F on $B^-(0,1)$ with values in $V(\mathbb{Z}_p,\Gamma(N))\otimes\mathbb{Q}_p$ satisfying

(A) $(1+s)\frac{\partial}{\partial s}F = \begin{cases} \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} y^k(1+s)^x d\mu_{D,t}^{\mathrm{Eis,(p)}}(x,y) & \text{if } l=0 \\ -l\hat{\mathrm{e}}_{t,(k,l-1)}(s) & \text{if } l>0 \end{cases}$

(B) $\sum_{\zeta\in\widehat{\mathbb{G}}_m[p](\mathbb{C}_p)} F(s +_{\widehat{\mathbb{G}}_m} \zeta) = 0, \quad \forall s \in B^-(0,1)(\mathbb{C}_p)$

satisfies $F=\hat{\mathrm{e}}_{t,(k,l)}$. Indeed, since any analytic function is given by a power series, one deduces from (A) that the difference of two solutions is a constant $c\in V(\mathbb{Z}_p,\Gamma(N))\otimes\mathbb{Q}_p$. By (B) we conclude $p\cdot c = \sum_{\zeta\in\widehat{\mathbb{G}}_m[p](\mathbb{C}_p)} c = 0$ which implies $c=0$ and proves the claim.

Now, the theorem follows from the following observation: The sequence $(e'_{k,l})_{k,l\geq 0}$ defined by

$$e'_{k,l}(s) := (-1)^l l! \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} y^k x^{-(l+1)}(1+s)^x d\mu_{D,t}^{\mathrm{Eis,(p)}}(x,y)$$

satisfies:

(a) $(1+s)\frac{\partial}{\partial s}e'_{k,0}(s) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} y^k(1+s)^x d\mu_{D,t}^{\mathrm{Eis,(p)}}(x,y), \quad k \geq 0$

(b) $(1+s)\frac{\partial}{\partial s}e'_{k,l}(s) = -l\cdot e'_{k,l-1}(s), \quad k\geq 0, l>0$

(c) $\sum_{\zeta\in\widehat{\mathbb{G}}_m[p](\mathbb{C}_p)} e'_{k,l}(s +_{\widehat{\mathbb{G}}_m} \zeta) = 0, \quad \forall s \in B^-(0,1)(\mathbb{C}_p), k,l\geq 0.$

Indeed, (a) and (b) are obvious and (c) follows since $e'_{k,l}$ is the Amice transform of a p-adic measure which is supported on $\mathbb{Z}_p^\times$. From the above claim we deduce $e'_{k,l}(s) = \hat{\mathrm{e}}_{t,(k,l)}(s)$ which proves the theorem. $\square$

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