Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 886 (2014) 1102–1127

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Dual conformal symmetry on the light cone

S.É. Derkachov [a], G.P. Korchemsky [b], A.N. Manashov [c,d]

[a] St. Petersburg Department of Steklov Mathematical Institute, Fontanka 27, 191023 St. Petersburg, Russia
[b] Institut de Physique Théorique [1], CEA Saclay, 91191 Gif-sur-Yvette Cedex, France
[c] Department of Theoretical Physics, St.-Petersburg State University, 199034 St.-Petersburg, Russia
[d] Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany

Received 24 February 2014; accepted 12 July 2014

Available online 16 July 2014

Editor: Stephan Stieberger

Abstract

We study the properties of conformal operators in the $SL(2)$ sector of planar $\mathcal{N}=4$ SYM and its supersymmetric $SL(2|2)$ extension. The correlation functions of these operators and their form factors with respect to asymptotic on-shell states are determined by two different polynomials which can be identified as eigenstates of the dilatation operator in the coordinate and momentum representations, respectively. We argue that, in virtue of integrability of the dilatation operator, the two polynomials satisfy a duality relation – they are proportional to each other upon an appropriate identification of momenta and coordinates. Combined with the conventional $\mathcal{N}=4$ superconformal symmetry, this leads to the dual superconformal symmetry of the dilatation operator. We demonstrate that this symmetry is powerful enough to fix the eigenspectrum of the dilatation operator to the lowest order in the coupling. We use the relation between the one-loop dilatation operator and Heisenberg spin chain to show that, to lowest order in the coupling, the dual symmetry is generated by the Baxter Q-operator in the limit of large spectral parameter.

1. Introduction

In this paper, we discuss the relation between dual superconformal symmetry in planar $\mathcal{N}=4$ SYM [1] and integrability of dilatation operator in the same theory (for a review, see [2]).

[1] Unité de Recherche Associée au CNRS URA 2306.

http://dx.doi.org/10.1016/j.nuclphysb.2014.07.014

At present, the dual symmetry is best understood for scattering amplitudes through their duality to light-like polygon (super) Wilson loops [3–8] and to correlation functions in the light-cone limit [9–11]. In a generic Yang–Mills theory, these objects depend on two different sets of variables (on-shell momenta of scattering particles versus coordinates of local operators in Minkowski space–time) and are not related to each other in a simple way. The very fact that such a relation exists in planar $\mathcal{N}=4$ SYM immediately leads to an enhancement of the symmetry – the conventional $\mathcal{N}=4$ superconformal symmetry of Wilson loops and correlation functions combined with the duality relation imply the dual superconformal symmetry of the scattering amplitudes.

The manifestation of the dual conformal symmetry can be also found in gauge theories with less supersymmetry including QCD. In particular, the dual symmetry has first emerged as the property of a particular class of scalar four-dimensional Feynman integrals [12,13]. It was also identified as the hidden symmetry of the BFKL equation [14] and its generalisations [15,16] and of the evolution equations governing the scale dependence of distribution amplitudes in QCD [17].

The dual superconformal symmetry acts naturally on the space of dual (super)coordinates (x_i, θ_i). For the scattering amplitudes, they are related to the (super)momenta of on-shell states (p_i, η_i) as [1]

$$p_i^{\alpha\dot\alpha} = (x_i - x_{i+1})^{\alpha\dot\alpha}, \qquad \eta_i^A \lambda_i^\alpha = (\theta_i - \theta_{i+1})^{A\,\alpha}, \tag{1.1}$$

where $p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}$ are light-like momenta of particles in the spinor-helicity notation (with $\alpha, \dot\alpha = 1, 2$) and Grassmann variables η_i^A (with $A = 1, \dots, 4$) serve to combine all asymptotic states into a single $\mathcal{N}=4$ on-shell superstate [18]. The dual symmetry is the exact symmetry of the scattering amplitudes in planar $\mathcal{N}=4$ SYM at tree level only. At loop level, it is believed that the scattering amplitudes in planar $\mathcal{N}=4$ SYM also respect the dual symmetry for arbitrary coupling, albeit in its anomalous form [19–22]. The tree-level amplitudes in $\mathcal{N}=4$ SYM are integrable in the sense that they are fixed by the dual and ordinary superconformal symmetry. The corresponding symmetry algebras do not commute and their (infinite-dimensional) closure has a Yangian structure [23]. This opens up the possibility to apply the quantum inverse scattering method to computing the tree-level amplitudes in $\mathcal{N}=4$ SYM [24].

In AdS/CFT description of the scattering amplitudes [3], the dual conformal symmetry arises at strong coupling from the symmetry of sigma-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background under the combined bosonic and fermionic T-duality [26–28]. Indeed, this sigma model is integrable and it possesses a lot of symmetries generated by the conserved charges [29,30]. The latter have been thoroughly studied in application to the energies of stringy excitations [2], or equivalently the eigenvalues of the dilatation operator in planar $\mathcal{N}=4$ SYM. The AdS/CFT correspondence suggests that, despite the fact that the scattering amplitudes and dilatation operator have different meaning in planar $\mathcal{N}=4$ SYM, they should have the same symmetries at strong coupling related to those of sigma-model on $\mathrm{AdS}_5 \times \mathrm{S}^5$. We can therefore ask what does integrability of the dilatation operator imply for the properties of the scattering amplitudes (and the S-matrix in general) and, vice versa, what is the manifestation of the dual conformal symmetry for the dilatation operator in planar $\mathcal{N}=4$ SYM?[2]

[2] Indeed, the relation between leading order corrections to the dilatation operator and tree-level scattering amplitudes in planar $\mathcal{N}=4$ SYM has been proposed in [25].

To address the question, in this paper we extend the dual symmetry to the so-called light-ray operators in $\mathcal{N}=4$ SYM

$$\mathbb{O}(z)=\mathrm{tr}\big[Z(nz_1)\dots Z(nz_L)\big]. \tag{1.2}$$

These are nonlocal single-trace operators built from L copies of a complex scalar field $Z(x)=Z^a(x)T^a$, with T^a being generators of the fundamental representation of the $SU(N_c)$ gauge group. All scalar fields in (1.2) are located along the same light ray defined by the light-like vector n^μ and variables $z=(z_1,\dots,z_L)$ denote the set of (real valued) light-cone coordinates. It is tacitly assumed that the gauge invariance of (1.2) is restored by inserting the path ordered exponentials $\mathrm{P}\exp(ig\int_{nz_i}^{nz_{i+1}}dx\cdot A(x))$ between the adjacent scalar fields on the right-hand side of (1.2). Such factors can be avoided by choosing the gauge $(n\cdot A(x))=0$.

The light-ray operators allow us to define two different functions that we shall denote as $\Phi_\alpha(z)$ and $\Psi_\alpha(\boldsymbol{p})$. The former depends on the light-cone coordinates of scalar fields and it is closely related to the operator product expansion of $\mathbb{O}(z)$. Namely, expansion of the light-ray operator (1.2) around $z_i=0$ produces an infinite set of local operators which mix under renormalisation and form a closed $SL(2)$ sector in $\mathcal{N}=4$ SYM. Diagonalising the corresponding mixing matrix, we can construct the conformal operators $\mathcal{O}_\alpha(0)$ having an autonomous scale dependence. Then, the expansion of the light-ray operators over the basis of conformal operators takes the form

$$\mathbb{O}(z)=\sum_\alpha \Phi_\alpha(z)\mathcal{O}_\alpha(0), \tag{1.3}$$

where the coefficient functions $\Phi_\alpha(z)$ are homogenous polynomials depending on light-cone coordinates of scalar fields $z=(z_1,\dots,z_L)$ as well as on the coupling constant. The explicit form of $\Phi_\alpha(z)$ can be found by diagonalising the dilatation operator in the $SL(2)$ sector.

The second function, $\Psi_\alpha(\boldsymbol{p})$, is the form factor defined as the matrix element of the conformal operator $\mathcal{O}_\alpha(0)$

$$\langle 0|\mathcal{O}_\alpha(0)|P_1,\dots,P_L\rangle\sim\overline{\Psi}_\alpha(\boldsymbol{p}),\qquad P_i^\mu=p_i\bar{n}^\mu, \tag{1.4}$$

where complex conjugation $\overline{\Psi}_\alpha(\boldsymbol{p})\equiv(\Psi_\alpha(\boldsymbol{p}))^*$ is introduced for the later convenience. Here the asymptotic state $|P_1,\dots,P_L\rangle$ consists of L massless particles carrying the momenta aligned along the same light-cone direction $\bar{n}^\mu$ (with $\bar{n}^2=n^2=0$ and $(n\bar{n})\neq 0$) and $\boldsymbol{p}=(p_1,\dots,p_L)$ being the corresponding light-cone components. The reason for such choice of particle momenta is motivated by the previous studies of analogous matrix elements in QCD. The matrix elements of the form (1.4) naturally appear in QCD description of hadrons as bound states of partons (quarks and gluons). In virtue of asymptotic freedom, the interaction between partons becomes weak at high energy. Therefore, when boosted into an infinite momentum frame, the hadron behaves as a collection of noninteracting partons moving along the same light-cone direction with the momenta $P_i^\mu=p_i\,\bar{n}^\mu$. Then, the function $\Psi_\alpha(\boldsymbol{p})$ defines the projection of the composite state $\mathcal{O}_\alpha(0)|0\rangle$ onto one of its Fock components $|P_1,\dots,P_L\rangle$ and has the meaning of light-cone distribution amplitude (for a review, see [31–33]).

As follows from their definition (1.3) and (1.4), the functions $\Phi_\alpha(z)$ and $\Psi_\alpha(\boldsymbol{p})$ have a different interpretation and, therefore, should be independent on each other. Nevertheless, previous studies of three-particle (baryon) distribution amplitudes in QCD revealed [17] that, upon identification of light-cone momenta and coordinates as $p_i=z_i-z_{i+1}$, the two functions are proportional to each other, to one-loop order at least. In the present paper, we generalise this relation to the states of arbitrary length L in planar $\mathcal{N}=4$ SYM,

$$\Psi_\alpha(\boldsymbol{p}) = \xi_\alpha \Phi_\alpha(\boldsymbol{z}), \qquad p_i = z_i - z_{i+1}, \tag{1.5}$$

with $z_{L+1} = z_1$. Notice that the proportionality factor ξ_α only depends on the quantum numbers of the conformal primary operator $\mathcal{O}_\alpha(0)$ and on the coupling constant, but not on the dual coordinates. The relation $p_i = z_i - z_{i+1}$ is very similar to the first relation in (1.1). In fact, the two relations are equivalent once we restrict the dual coordinates x_i^μ to be aligned along the same light-cone direction

$$P_i^\mu = \bar{n}^\mu p_i, \qquad x_i^\mu = \bar{n}^\mu z_i. \tag{1.6}$$

The relation (1.5) is similar to the duality relation between scattering amplitudes and light-like Wilson loops mentioned above but this time it establishes the correspondence between the coefficient functions and form factors of conformal primary operators.

As we explain below, to lowest order in the coupling, the duality relation (1.5) follows from integrability of the $SL(2)$ dilatation operator. More precisely, to one-loop order the functions $\Phi_\alpha(\boldsymbol{z})$ and $\Psi_\alpha(\boldsymbol{p})$ coincide with eigenstates of the $SL(2)$ Heisenberg spin chain in the coordinate and momentum representations, respectively, and their symmetry properties can be studied with a help of the Baxter Q-operator. This operator was first introduced by Baxter in solving the 8-vertex model [34] and has proven to be a very powerful tool in solving a variety of integrable models [35–37]. In the case of the $SL(2)$ Heisenberg spin chain, the Baxter operator $\mathbb{Q}(u)$ depends on an arbitrary complex parameter u and satisfies the defining relations summarised below in Section 4.1. Its explicit construction was carried out in Ref. [38].

The Baxter Q-operator is the generating function of integrals of motions of the $SL(2)$ spin chain. In particular, the one-loop dilatation operator in $\mathcal{N}=4$ SYM can be obtained from expansion of $\mathbb{Q}(u)$ around $u = \pm i/2$. As a consequence, to one-loop order, the functions $\Phi_\alpha(\boldsymbol{z})$ have to diagonalise the operator $\mathbb{Q}(u)$ for any u. The dual symmetry arises when we examine the action of the Baxter Q-operator on $\Phi_\alpha(\boldsymbol{z})$ for large values of the spectral parameter, $u \to \infty$ [38],

$$\mathbb{Q}(u)\Phi_\alpha(\boldsymbol{z}) \sim \Psi_\alpha(\boldsymbol{p}) + O(1/u), \qquad p_i = z_i - z_{i+1}. \tag{1.7}$$

Since $\Phi_\alpha(\boldsymbol{z})$ diagonalises the Baxter Q-operator, it follows from this relation that $\Phi_\alpha(\boldsymbol{z})$ has to be proportional to $\Psi_\alpha(\boldsymbol{p})$ thus leading to (1.5). In this manner, the duality relation (1.5) is generated, to the lowest order in the coupling, by the leading term in the asymptotic expansion of Baxter Q-operator at infinity.

According to (1.7), the Baxter operator automatically generates the transition to the dual coordinates, $p_i = z_i - z_{i+1}$, thus equating to zero the total momentum of $\Psi_\alpha(\boldsymbol{p})$. This means that, in distinction with the conventional conformal symmetry, the dual conformal symmetry is only present for the vanishing total momentum, $\sum_i p_i = 0$. The same property has been previously observed in the analysis of scattering amplitudes and form factors in planar $\mathcal{N}=4$ SYM. For the scattering amplitudes, the condition $\sum_i P_i^\mu = 0$ is automatically satisfied. For the form factors, $F_O = \int d^4x\, \mathrm{e}^{ix\cdot q} \langle 0|O(x)|P_1, \ldots, P_n\rangle$, the total momenta equals the momentum transferred, $\sum_i P_i^\mu = q^\mu$. At weak coupling, the explicit calculation of form factors showed [39] that the dual conformal symmetry is only present for the vanishing momentum transferred, $q^\mu = 0$. At strong coupling, the same result follows from a dual description of the form factor [40] in terms of minimal area attached to infinitely periodic zig-zag light-like contour located at the boundary of the AdS_5 and built from light-like momenta P_i^μ. The dual conformal symmetry of the form factor is broken for $q^\mu \neq 0$ because the above mentioned kinematical configuration is not stable under conformal transformations.

The duality relation (1.5) can be extended to a larger class of supersymmetric light-ray operators. These operators are obtained from (1.2) by replacing the scalar field $Z(z_i n)$ with gaugino and gauge strength fields in $\mathcal{N}=4$ SYM. To deal with such operators it is convenient to employ the light-cone superspace formalism [41,42]. It allows us to combine various components of fields into a single light-cone superfield $\mathcal{Z}(z_i n, \theta_i)$ (with $i=1,\dots,L$) and use it to construct the corresponding supersymmetric light-ray operator [43,44]. Going through the same steps as before, we can define supersymmetric extension of the coefficient functions $\Phi_\alpha(\boldsymbol{Z})$ and form factors $\Psi_\alpha(\boldsymbol{P})$ depending, respectively, on the set of L light-cone supercoordinates $\boldsymbol{Z}=\{z_i,\theta_i\}$ and conjugated supermomenta $\boldsymbol{P}=\{p_i,\vartheta_i\}$. The functions entering the duality relation (1.5) are the lowest components in the expansion of $\Phi_\alpha(\boldsymbol{Z})$ and $\Psi_\alpha(\boldsymbol{P})$ in powers of Grassmann variables. We show in this paper that the duality relation also holds for the remaining components

$$\Psi_\alpha(\boldsymbol{P})=\xi_\alpha\Phi_\alpha(\boldsymbol{Z}),\qquad p_i=z_i-z_{i+1},\qquad \vartheta_i^A=\theta_i^A-\theta_{i+1}^A. \tag{1.8}$$

As compared with the general form of duality transformation (1.1), the last two relations in (1.8) correspond to the collinear limit $x_i^{\alpha\dot\alpha}=z_i\,\bar n^{\alpha\dot\alpha}$ and $\theta_i^{A\,\alpha}=\theta_i^A\lambda^\alpha$ (with $\bar n^{\alpha\dot\alpha}=\lambda^\alpha\tilde\lambda^{\dot\alpha}$). As before, to the lowest order in the coupling, the duality relation (1.8) is generated by the Baxter Q-operator for supersymmetric generalisation of the $SL(2)$ Heisenberg spin chain [45,46].

The paper is organised as follows. In Section 2 we describe the properties of light-ray operators in $\mathcal{N}=4$ SYM and formulate the duality relation (1.5). In Section 3 we verify this relation at one loop by diagonalising the dilatation operator in the $SL(2)$ sector. In Section 4 we explain the origin of the dual conformal symmetry of the one-loop $SL(2)$ dilatation operator and demonstrate that it is generated by the leading term in the asymptotic expansion of the Baxter Q-operator for large spectral parameter. We also argue that the dual symmetry is powerful enough to uniquely fix the eigenstates of the one-loop dilatation operator. In Section 5 we discuss supersymmetric extension of the duality relation for a larger class of light-ray operators involving various components of gaugino and gauge fields in $\mathcal{N}=4$ SYM. Concluding remarks are presented in Section 6.

2. Light-ray operators

According to definition (1.2), the light-ray operator $\mathbb{O}(z)$ is given by the product of scalar fields located on the same light ray. Its expansion in powers of $\boldsymbol{z}=(z_1,\dots,z_L)$ produces an infinite set of local single-trace operators

$$\begin{aligned}
\mathbb{O}(\boldsymbol{z})&=\sum_{\boldsymbol{k}} z_1^{k_1}\dots z_L^{k_L}\,O_{\boldsymbol{k}}(0),\\
O_{\boldsymbol{k}}(0)&=\operatorname{tr}\left[\frac{D_+^{k_1}}{k_1!}Z(0)\dots\frac{D_+^{k_L}}{k_L!}Z(0)\right],
\end{aligned}\tag{2.1}$$

where the sum runs over nonnegative integers $\boldsymbol{k}=(k_1,\dots,k_L)$ and $D_+=(nD)$ stands for the light-cone component of the covariant derivative.

It is tacitly assumed that the operators $O_{\boldsymbol{k}}(0)$ are renormalised in a particular scheme (say minimal subtraction scheme) and depend on the renormalisation scale. The operators $O_{\boldsymbol{k}}(0)$ mix with each other under the change of this scale but we can diagonalised their mixing matrix and define the operators $\mathcal{O}_\alpha(0)$ having an autonomous scale dependence. They are given by a linear combination of the basis operators $O_{\boldsymbol{k}}(0)$

$$\mathcal{O}_\alpha(0) = \sum_{\boldsymbol{k}} \bar{c}_{\boldsymbol{k},\alpha}\big(g^2\big) O_{\boldsymbol{k}}(0), \tag{2.2}$$

with the expansion coefficients depending on 't Hooft coupling constant $g^2 = g^2_{\rm YM}N/(8\pi^2)$. The coefficients $\bar{c}_{\boldsymbol{k},\alpha} = (c_{\boldsymbol{k},\alpha})^*$ coincide with the eigenstates of the all-loop mixing matrix in planar $\mathcal{N} = 4$ SYM and are labelled by the index α (to be specified below).

2.1. Operator product expansion

Let us introduce the following polynomial

$$\overline{\Psi}_\alpha(\boldsymbol{p}) = \sum_{\boldsymbol{k}} \bar{c}_{\boldsymbol{k},\alpha}\big(g^2\big) \frac{p_1^{k_1}}{k_1!} \cdots \frac{p_L^{k_L}}{k_L!}. \tag{2.3}$$

It involves the same expansion coefficients as (2.2) and depends on the set of auxiliary variables $\boldsymbol{p} = (p_1, \dots, p_L)$. The polynomial (2.3) defines the symbol of the differential operator $\overline{\Psi}_\alpha(\boldsymbol{\partial}_z)$ which projects the light-ray operator $\mathbb{O}(\boldsymbol{z})$ onto local operator $\mathcal{O}_\alpha(0)$. Namely, the operator (2.2) is obtained from the light-ray operator (1.2) by substituting $p_i \to \partial_{z_i}$ on the right-hand side of (2.3) and applying the resulting differential operator $\overline{\Psi}_\alpha(\boldsymbol{\partial}_z)$ to both sides of (1.2)

$$\mathcal{O}_\alpha(0) = \overline{\Psi}_\alpha(\boldsymbol{\partial}_z)\mathbb{O}(\boldsymbol{z})\big|_{z=0}. \tag{2.4}$$

The p_i-variables in (2.3) are conjugated to light-cone coordinates z_i of scalar fields and have the meaning of the light-cone components of the momenta carried by scalar particles.

Inverting (2.2), we can expand $O_{\boldsymbol{k}}(0)$ over the basis of conformal operators $\mathcal{O}_\alpha(0)$ and rewrite the first relation in (2.1) as

$$\mathbb{O}(\boldsymbol{z}) = \sum_\alpha \Phi_\alpha(\boldsymbol{z})\mathcal{O}_\alpha(0), \tag{2.5}$$

where $\Phi_\alpha(\boldsymbol{z})$ are (homogenous) polynomials depending on the light-cone coordinates of scalar fields. The main advantage of (2.5) as compared with the first relation in (2.1) is that each term on the right-hand side of (2.5) has a definite scaling dimension.

The two polynomials entering the right-hand side of (2.4) and (2.5) carry a different information: $\Psi_\alpha(\boldsymbol{p})$ fixes the form of the local operator (2.4), whereas $\Phi_\alpha(\boldsymbol{z})$ determines its contribution to the operator expansion (2.5). Since the light-ray operator (1.2) is invariant under the cyclic shift of scalar fields inside the trace, the polynomials should be cyclically invariant functions of their arguments.[3] Together with (2.3) this implies that $c_{\boldsymbol{k},\alpha}(g^2)$ should be invariant under the cyclic shift of indices, $k_i \to k_{i+1}$.

The polynomials $\Psi_\alpha(\boldsymbol{p})$ and $\Phi_\alpha(\boldsymbol{z})$ are not independent on each other. Substituting (2.5) into the right-hand side of (2.4) and comparing the coefficients in front of $\mathcal{O}_\alpha(0)$, we find that the polynomials have to satisfy the orthogonality condition

$$\overline{\Psi}_\alpha(\boldsymbol{\partial}_z)\Phi_\beta(\boldsymbol{z})\big|_{z=0} = \delta_{\alpha\beta}. \tag{2.6}$$

In the similar manner, substitution of (2.4) into (2.5) yields the completeness condition

[3] This property is ultimately related to the fact that the light-ray operator (1.2) is built from the same complex field. If the operator (1.2) involved different fields, as it happens in QCD, the above condition should be relaxed.

$$\sum_\alpha \Phi_\alpha(z)\overline{\Psi}_\alpha(\boldsymbol{p}) = \frac{1}{L}\sum_{i=1}^{L} \exp(p_1 z_i + \cdots + p_L z_{i+L-1}), \tag{2.7}$$

where $z_{i+L} \equiv z_i$ and expressions on both sides of (2.7) are invariant under the cyclic shifts of $\boldsymbol{z}$ and $\boldsymbol{p}$. Note that relations (2.6) and (2.7) should hold for arbitrary coupling constant, independently on the choice of the renormalisation scheme.

2.2. Conformal symmetry

Let us specify the quantum numbers of conformal operators $\mathcal{O}_\alpha(0)$. According to (2.2), these operators are given by a linear combination of the basis operators $O_{\boldsymbol{k}}(0)$ built from L scalar fields and carrying the Lorentz spin $S = \sum_i k_i$ equal to the total number of covariant derivatives. In addition, the operators $\mathcal{O}_\alpha(0)$ have a definite scaling dimension $\Delta_{S,\alpha}$

$$\Delta_{S,\alpha} = L + S + \gamma_{S,\alpha}\big(g^2\big), \tag{2.8}$$

which receives an anomalous contribution $\gamma_{S,\alpha}(g^2)$. We use index α here to indicate that there exist few operators carrying the same Lorentz spin S.

In virtue of conformal symmetry, the operators $\mathcal{O}_\alpha(0)$ can be classified according to representation of the $SO(2,4)$ conformal group (for a review, see e.g. [33]). For the light-ray operators (1.2), the conformal symmetry reduces to its collinear $SL(2)$ subgroup. This subgroup leaves the light-ray $x^\mu = zn^\mu$ invariant and acts on the light-cone coordinates z as

$$z \to \frac{az+b}{cz+d}, \qquad ad - bc = 1. \tag{2.9}$$

The corresponding transformation properties of the operator $\mathcal{O}_\alpha \equiv \mathcal{O}_{S,\alpha}$ are

$$\mathcal{O}_{S,\alpha}(zn) \to (cz+d)^{-2j_{S,\alpha}}\mathcal{O}_{S,\alpha}\bigg(\frac{az+b}{cz+d}n\bigg), \tag{2.10}$$

where the conformal spin $j_{S,\alpha}$ is related to the Lorentz spin of the operator and its scaling dimension as

$$j_{S,\alpha} = \frac{1}{2}(S + \Delta_{S,\alpha}) = S + \frac{1}{2}L + \frac{1}{2}\gamma_{S,\alpha}\big(g^2\big). \tag{2.11}$$

The generators of the $SL(2)$ transformations (2.10) take the form of linear differential operators acting on the light-cone coordinates of the operators

$$\begin{aligned}
&L_-\mathcal{O}_{S,\alpha}(zn) = -\partial_z\, \mathcal{O}_{S,\alpha}(zn),\\
&L_0\mathcal{O}_{S,\alpha}(zn) = \big(z\partial_z + j_{S,\alpha}\big(g^2\big)\big)\mathcal{O}_{S,\alpha}(zn),\\
&L_+\mathcal{O}_{S,\alpha}(zn) = \big(z^2\partial_z + 2zj_{S,\alpha}\big(g^2\big)\big)\mathcal{O}_{S,\alpha}(zn).
\end{aligned} \tag{2.12}$$

It is straightforward to verify that the $SL(2)$ generators defined in this way satisfy the standard commutation relations $[L_0, L_\pm] = \pm L_\pm$ and $[L_+, L_-] = 2L_0$. The dependence of the last two relations in (2.12) on the coupling constant reflects the fact that the conformal generators L_0 and L_+ are modified by perturbative corrections. The generator L_- is related to the light-cone component of the total momentum operator and is protected from loop corrections.

The operator $\mathcal{O}_{S,\alpha}(zn)$ belongs to the $SL(2)$ representation labelled by the conformal spin $j_{S,\alpha}$. As follows from (2.12), the operator $\mathcal{O}_{S,\alpha}(0)$ defines the lowest weight of this representation and its descendants are given by total derivatives $(L_-)^\ell\mathcal{O}_{S,\alpha}(0) = (-\partial_z)^\ell\mathcal{O}_{S,\alpha}(zn)|_{z=0}$.

The conformal symmetry allows us to organise the sum on the right-hand side of (2.5) as the sum over different $SL(2)$ moduli

$$\mathbb{O}(z) = \sum_{S,\alpha} \big[\Phi_{S,\alpha}(z)\mathcal{O}_{S,\alpha}(0) + \text{descendants}\big], \tag{2.13}$$

where 'descendants' denote the contribution of the operators $(L_-)^\ell \mathcal{O}_{S,\alpha}(0)$ and the index α enumerates the conformal primary operators $\mathcal{O}_{S,\alpha}$ with the same Lorentz spin S. The conformal symmetry also fixes (up to an overall normalisation) the two-point correlation function of these operators

$$\big\langle \mathcal{O}_{S,\alpha}(x)\mathcal{O}_{S',\alpha'}(0)\big\rangle \sim \delta_{SS'}\delta_{\alpha\alpha'} \frac{(xn)^{2S}}{(x^2)^{S+\Delta_{S,\alpha}}}, \tag{2.14}$$

with the scaling dimension $\Delta_{S,\alpha}$ given by (2.8).

Let us now consider the correlation function $\langle \mathbb{O}(z)\mathcal{O}_{S,\alpha}(x)\rangle$. Replacing the light-ray operator with its expansion (2.13) and making use of (2.14), we find that the correlation function receives a nonzero contribution from the operators $(L_-)^\ell \mathcal{O}_{S,\alpha}(0)$ (with $\ell = 0, 1, \ldots$) in (2.13) belonging to the same $SL(2)$ moduli as $\mathcal{O}_{S,\alpha}(x)$. Moreover, taking the limit $x \to \infty$ we find that the leading contribution only comes from the conformal primary operator ($\ell = 0$) whereas the contribution of descendants is suppressed by factor of $1/|x|^\ell$ leading to

$$\big\langle \mathbb{O}(z)\mathcal{O}_{S,\alpha}(x)\big\rangle \overset{x\to\infty}{\sim} \Phi_{S,\alpha}(z) \frac{(xn)^{2S}}{(x^2)^{S+\Delta_{S,\alpha}}}. \tag{2.15}$$

Thus the polynomial $\Phi_{S,\alpha}(z)$ defines the leading asymptotic behaviour of the correlation function $\langle \mathbb{O}(z)\mathcal{O}_{S,\alpha}(x)\rangle$ at large distance $x \to \infty$.

2.3. *Form factors*

Let us examine matrix elements of the conformal primary operators with respect to asymptotic (on-shell) states $\langle 0|\mathcal{O}_{S,\alpha}(0)|P\rangle$.[4] Here the asymptotic state $|P\rangle$ consists of a fixed number of massless particles (scalars, gauginos and gluons) each carrying the on-shell momentum P_i^μ, certain helicity charge and the colour $SU(N)$ charge T^{a_i}. The total colour charge of the state is zero, $\sum_i T^{a_i} = 0$, and the total momentum equals $P^\mu = \sum_i P_i^\mu$.

Since the operators $\mathcal{O}_{S,\alpha}(0)$ arise from the expansion of the light-ray operator (2.13) it is natural to introduce the following quantity

$$F(z, P) = \langle 0|\mathbb{O}(z)|P\rangle = \sum_{S,\alpha} \Phi_{S,\alpha}(z)\langle 0|\mathcal{O}_{S,\alpha}(0)|P\rangle + O(P\cdot n), \tag{2.16}$$

which can be thought of as a generating function of the form factors $\langle 0|\mathcal{O}_{S,\alpha}(0)|P\rangle$. Making use of the orthogonality condition (2.6) we find from (2.16)

$$\langle 0|\mathcal{O}_{S,\alpha}(0)|P\rangle = \overline{\Psi}_{S,\alpha}(\partial_z)F(z, P)\big|_{z=0}. \tag{2.17}$$

The last term on the right-hand side of (2.16) describes the contribution of the $SL(2)$ descendant operators. Such operators involve total derivatives of $\mathcal{O}_{S,\alpha}(0)$ and their matrix elements

[4] More general matrix elements of the form $\langle P_1|\mathcal{O}_{S,\alpha}(0)|P_2\rangle$ can be obtained from $\langle 0|\mathcal{O}_{S,\alpha}(0)|P\rangle$ by allowing some particles inside the state $|P\rangle$ to carry negative energy.

are proportional to the light-cone component of the total momentum, $\langle 0|(L_-)^\ell \mathcal{O}_{S,\alpha}(0)|P\rangle \sim (Pn)^\ell \langle 0|\mathcal{O}_{S,\alpha}(0)|P\rangle$. Therefore we can eliminate the contribution of conformal descendants to (2.16) by choosing $(Pn) = 0$. We shall make use of this fact later in the paper.

Let us consider (2.16) in the special case when the state $|P\rangle$ consists of L scalars each carrying the light-like momentum P_i^μ aligned along the same light-like direction $\bar{n}_\mu$ (with $\bar{n}^2 = 0$)

$$P_i^\mu = p_i \bar{n}_\mu, \qquad P^\mu = (p_1 + \cdots + p_L)\bar{n}_\mu, \tag{2.18}$$

with $-\infty < p_i < \infty$. In what follows we shall denote such state as $|\boldsymbol{p}\rangle$. Then, evaluating the matrix element $\langle 0|\mathbb{O}(\boldsymbol{z})|P\rangle$ to the lowest order in the coupling, we can be replace scalar fields inside $\mathbb{O}(\boldsymbol{z})$ by plane waves $\langle 0|Z(nz)|P_i\rangle = \mathrm{e}^{i(P_i n)z} T^{a_i}$ to get

$$F(\boldsymbol{z}, \boldsymbol{p}) = \mathrm{tr}\big(T^{a_1} \dots T^{a_L}\big) \sum_{i=1}^{L} \mathrm{e}^{ip_1 z_i + \cdots + ip_L z_{i+L-1}} + \text{perm}, \tag{2.19}$$

where we put $(n\bar{n}) = 1$ for simplicity. Here the sum ensures the symmetry of $F(\boldsymbol{z}, \boldsymbol{p})$ under the cyclic shift of z's (with $z_{i+L} \equiv z_i$) and 'perm' denote terms with permutations of momenta and colour indices of particles. They are needed to restore the Bose symmetry of $F(\boldsymbol{z}, \boldsymbol{p})$.

Combining together (2.17) and (2.19), we obtain the following expression for the form factor of the conformal primary operator in the kinematical configuration (2.18), to the leading order in the coupling

$$\begin{aligned}\langle 0|\mathcal{O}_{S,\alpha}(0)|\boldsymbol{p}\rangle &= \overline{\Psi}_{S,\alpha}(\boldsymbol{\partial}_z) F(\boldsymbol{z}, \boldsymbol{p})\big|_{z=0} \\ &= \big[i^S L \overline{\Psi}_{S,\alpha}(\boldsymbol{p})\, \mathrm{tr}\big(T^{a_1} \dots T^{a_L}\big) + (\text{perm})\big].\end{aligned} \tag{2.20}$$

Here in the second relation we took into account that $\Psi_{S,\alpha}(\boldsymbol{p})$ is a cyclically invariant homogenous polynomial in $\boldsymbol{p} = (p_1, \dots, p_L)$ of degree S. We conclude from (2.20) that the polynomial $\Psi_{S,\alpha}(\boldsymbol{p})$ defines the form factor $\langle 0|\mathcal{O}_{S,\alpha}(0)|P\rangle$ in the multi-collinear kinematical configuration (2.18).

2.4. Duality

The polynomials $\Phi_{S,\alpha}(\boldsymbol{z})$ and $\Psi_{S,\alpha}(\boldsymbol{p})$ depend on two different sets of variables: the former depends on the light-cone coordinates of scalar fields, whereas the latter is a function of the conjugated light-cone momenta. They define the wave function of the same L-particle state in the coordinate and momentum representations, respectively, and satisfy the orthogonality condition (2.6).[5]

We recall that the explicit form of the polynomial $\Psi_{S,\alpha}(\boldsymbol{p})$, Eq. (2.3), is determined by the eigenstates of the mixing matrix in the $SL(2)$ sector. The polynomials $\Phi_{S,\alpha}(\boldsymbol{z})$ can then be obtained from the orthogonality condition (2.6). Since the functions $\Psi_{S,\alpha}(\boldsymbol{p})$ and $\Phi_{S,\alpha}(\boldsymbol{z})$ are defined in the two different representations, we do not expect them to be related to each other in an obvious way. The main goal of the present paper is to show that, due to integrability of the dilatation operator in planar $\mathcal{N} = 4$ SYM, there exists the following relation between the two polynomials to the leading order in the coupling

$$\Psi_{S,\alpha}(\boldsymbol{p}) = \xi_{S,\alpha} \Phi_{S,\alpha}(\boldsymbol{z}), \qquad p_i = z_i - z_{i+1}, \tag{2.21}$$

[5] The orthogonality condition (2.6) can be casted into the well-known quantum mechanical form. To see this we notice that $\int dx\, \bar{\phi}_\alpha(x)\phi_\beta(x) = \int dx\, \bar{\phi}_\alpha(x)\mathrm{e}^{x\partial_z}\phi_\beta(z)|_{z=0} = \overline{\psi}_\alpha(i\partial_z)\phi_\beta(z)|_{z=0}$, with $\psi_\alpha(k)$ being the Fourier transform of $\phi_\alpha(x)$.

where the proportionality factor $\xi_{S,\alpha}$ depends on the quantum numbers of the state and periodicity condition $z_{i+L} = z_i$ is implied. Extension of (2.21) beyond the leading order will be discussed in the forthcoming paper [47].

Notice that the duality relation (2.21) is formulated for the conformal primary operators $\mathcal{O}_{S,\alpha}(0)$ but not for their descendants. The reason for this is that (2.21) becomes trivial for the descendant operator $(\partial_+)^\ell \mathcal{O}_{S,\alpha}(0)$ (with $\ell \geq 1$) involving a power of the total light-cone derivative. According to definition (2.4), the Ψ-polynomial for such operator is given by $(\sum_i p_i)^\ell \Psi_{S,\alpha}(\boldsymbol{p})$ and it vanishes upon substitution into the left-hand side of (2.21). Then, the duality relation implies that the corresponding ξ-factor on the right-hand side of (2.21) vanishes as well.

The duality relation (2.21) leads to another interesting property of the polynomial $\Phi_{S,\alpha}(z)$. Let us consider the completeness condition (2.7) and substitute $p_i = w_i - w_{i+1}$ (with $w_{L+1} = w_1$). Since $\sum_i p_i = 0$, the conformal descendants produce a vanishing contribution to the left-hand side of (2.7). Then, we apply the duality relation (2.21) to get from (2.7)

$$\sum_{S,\alpha} \xi_{S,\alpha}\, \overline{\Phi}_{S,\alpha}(z)\Phi_{S,\alpha}(\boldsymbol{w}) = \frac{1}{L}\sum_{i=1}^{L} \exp\left(w_{12}z_i + \cdots + w_{L1}z_{i+L-1}\right), \tag{2.22}$$

where $w_{i,i+1} \equiv w_i - w_{i+1}$ and the expression on the right-hand side is invariant under translations and cyclic shifts of $\boldsymbol{z}$ and $\boldsymbol{w}$. Since $\Phi_{S,\alpha}(z)$ is a homogenous polynomial of degree S, we can further simplify (2.22) as

$$\sum_{\alpha} \xi_{S,\alpha}\, \overline{\Phi}_{S,\alpha}(z)\Phi_{S,\alpha}(\boldsymbol{w}) = \frac{1}{L}\sum_{i=1}^{L} \frac{1}{S!}\left(w_{12}z_i + \cdots + w_{L1}z_{i+L-1}\right)^S. \tag{2.23}$$

Here the sum on the right-hand side runs over the conformal primary operators carrying the same spin S.

We have demonstrated in the previous subsections that $\Psi_{S,\alpha}(\boldsymbol{p})$ and $\Phi_{S,\alpha}(z)$ have a simple interpretation, Eqs. (2.15) and (2.20), respectively. Then, the duality relation (2.21) establishes the correspondence between the correlation function $\langle \mathbb{O}(z)\mathcal{O}_{S,\alpha}(x)\rangle$ at large distances, $x \to \infty$, and the form factor $\langle 0|\mathcal{O}_{S,\alpha}(0)|\boldsymbol{p}\rangle$ evaluated for the special configuration of the light-cone momentum $p_i = z_i - z_{i+1}$.

In the rest of the paper, we demonstrate the validity of the duality relation (2.21) to the lowest order in the coupling and explain its relation to integrability of the dilatation operator.

3. Duality at the leading order

The polynomials $\Psi_{S,\alpha}(\boldsymbol{p})$ and $\Phi_{S,\alpha}(z)$ admit a perturbative expansion, e.g.

$$\Psi_{S,\alpha}(\boldsymbol{p}) = \Psi^{(0)}_{S,\alpha}(\boldsymbol{p}) + g^2\,\Psi^{(1)}_{S,\alpha}(\boldsymbol{p}) + O\left(g^4\right). \tag{3.1}$$

In what follows we shall restrict our consideration to the leading term $\Psi^{(0)}_{S,\alpha}(\boldsymbol{p})$. To simplify notations, we will not display the superscript '(0)'. The corresponding operator (2.4) is conformal primary at one loop – it does not mix with other conformal operators at one loop and diagonalises the dilatation operator at order $O(g^2)$.

3.1. Conformal Ward identity

Let us start with reviewing the constraints imposed by the conformal symmetry on the polynomials $\Psi_{S,\alpha}(\boldsymbol{p})$ and $\Phi_{S,\alpha}(z)$.

At the leading order in the coupling, the light-ray operator (1.2) is given by the product of free scalar fields $Z(nz_i)$. Each of them transforms under the conformal $SL(2)$ transformations according to (2.10) with the conformal weight $j_Z = 1/2$. Then, we use the relation (2.15) and require the correlation function $\langle \mathbb{O}(z)\mathcal{O}_{S,\alpha}(x)\rangle$ to be invariant under the conformal transformations generated by L_- and L_0, Eq. (2.12), to find that the polynomial $\Phi_{S,\alpha}(z)$ has to satisfy the conformal Ward identity

$$\sum_i \partial_{z_i}\Phi_{S,\alpha}(z) = \Big(\sum_i z_i\partial_{z_i} - S\Big)\Phi_{S,\alpha}(z) = 0, \tag{3.2}$$

so that $\Phi_{S,\alpha}(z)$ should be translationally invariant homogenous polynomial in z's of degree S. The Ward identity for the L_+ generator leads to the relation between $\Phi_{S,\alpha}(z)$ and the polynomials corresponding to the descendant operators.

To obtain analogous relations for $\Psi_{S,\alpha}(\boldsymbol{p})$, we examine transformation properties of both sides of (2.4) under the $SL(2)$ transformations, $L_a\mathcal{O}_{S,\alpha}(0) = \Psi_{S,\alpha}(\boldsymbol{\partial}_z)L_a\mathbb{O}(z)|_{z=0}$. Here the $SL(2)$ generators L_a act additively on each scalar field inside $\mathbb{O}(z)$ and are given by (2.12) with $j_{S,\alpha}$ replaced by the conformal spin of a free scalar field $j_Z = 1/2$. Then, we impose the conditions $(L_0 - j_{S,\alpha})\mathcal{O}_{S,\alpha}(0) = L_+\mathcal{O}_{S,\alpha}(0) = 0$ that follow from (2.12) to get

$$\Big(\sum_i p_i\partial_{p_i} - S\Big)\Psi_{S,\alpha}(\boldsymbol{p}) = \sum_i\big(p_i\partial^2_{p_i} + \partial_{p_i}\big)\Psi_{S,\alpha}(\boldsymbol{p}) = 0. \tag{3.3}$$

We would like to emphasise that the relations (3.2) and (3.3) were obtained to the lowest order in the coupling. To higher orders, only the second relation in (3.3) is modified by perturbative corrections.

We recall that the polynomials have to satisfy the orthogonality condition (2.6),

$$\bar{\Psi}_{S,\alpha}(\boldsymbol{\partial}_z)\Phi_{S',\alpha'}(z)\big|_{z=0} = \delta_{SS'}\delta_{\alpha\alpha'}. \tag{3.4}$$

As was already mentioned, this relation suggests that the two functions should be related to each other by a Fourier like transformation. Its explicit form has been worked out in Ref. [48][6]

$$\Phi_{S,\alpha}(z) = \int_0^\infty \prod_{i=1}^L dp_i \mathrm{e}^{-p_i}\Psi_{S,\alpha}(p_1z_1,\dots,p_Lz_L) = \Psi_{S,\alpha}(\partial_{\boldsymbol{w}})\prod_{i=1}^L(1-w_iz_i)^{-1}\Big|_{\boldsymbol{w}=0}. \tag{3.5}$$

An unusual feature of this transformation is that it maps one polynomial satisfying (3.2) into another polynomial verifying (3.3). Replacing $\Psi_{S,\alpha}(\boldsymbol{p})$ in (3.5) with its general expression (2.3), we obtain the following result for the polynomial $\Phi_{S,\alpha}(z)$

$$\Phi_{S,\alpha}(z) = \sum_{\boldsymbol{k}} c_{\boldsymbol{k},\alpha}\, z_1^{k_1}\dots z_L^{k_L}. \tag{3.6}$$

We recall that $\Phi_{S,\alpha}(z)$ should be translationally invariant and, therefore, the coefficients $c_{\boldsymbol{k},\alpha}$ are not independent. In addition, substituting (2.3) and (3.6) into (3.4) we find (for $S = S'$) that they

[6] The inverse relation takes the form of the Fourier transform $\Psi_{S,\alpha}(\boldsymbol{p}) = \int[\mathcal{D}z]\mathrm{e}^{\bar{z}\cdot\boldsymbol{p}}\Phi_{S,\alpha}(z)$, where $z\cdot\boldsymbol{p} \equiv \sum_i z_i p_i$ and integration goes over the unit disk, $z_i\bar{z}_i \le 1$, in the complex z-plane with the $SU(1,1)$ invariant measure of spin $j = 1/2$.

have to satisfy the orthogonality condition

$$\sum_{\boldsymbol{k}} (c_{\boldsymbol{k},\alpha})^* c_{\boldsymbol{k},\alpha'} = \delta_{\alpha\alpha'}, \tag{3.7}$$

where the sum runs over L nonnegative integers $\boldsymbol{k} = (k_1, \dots, k_L)$ such that $\sum_i k_i = S$.

As an example, let us consider length 2 operators. In this case, for $L = 2$, the conformal Ward identity (3.2) fixes $\Phi_S(\boldsymbol{z})$ up to an overall normalisation

$$\Phi_S(\boldsymbol{z}) = c\,(z_1 - z_2)^S. \tag{3.8}$$

Comparison with (3.6) shows that the expansion coefficients are given by binomial coefficients, $c_{\boldsymbol{k}} = c(-1)^k \binom{S}{k}$. Their substitution into (2.3) yields the polynomial $\Psi_S(\boldsymbol{p})$ as [49,50]

$$\Psi_S(\boldsymbol{p}) = c \sum_{k=0}^{S} \frac{(-1)^{S-k} S!}{(k!(S-k)!)^2} p_1^k p_2^{S-k} = \frac{c}{S!}(p_1 + p_2)^S \mathrm{C}_S^{1/2}\left(\frac{p_1 - p_2}{p_1 + p_2}\right), \tag{3.9}$$

where $\mathrm{C}_S^{1/2}(x)$ is the Gegenbauer polynomial and the normalisation factor $c = [(S!)^2/(2S)!]^{1/2}$ is fixed by (3.7). We can now test the duality relation (2.21) for $L = 2$. Replacing $p_1 = -p_2 = z_{12}$ on the right-hand side of (3.9) we get

$$\Psi_S(\boldsymbol{p})\big|_{p_i = z_{i,i+1}} = c \sum_{k=0}^{S} \frac{S!(z_1 - z_2)^S}{(k!(S-k)!)^2} = \frac{(2S)!}{(S!)^3} \Phi_S(\boldsymbol{z}), \tag{3.10}$$

in a perfect agreement with (2.21).

For $L \geq 3$ the conformal symmetry (3.2) and (3.3) is not sufficient to fix the polynomials $\Phi_{S,\alpha}(\boldsymbol{z})$ and $\Psi_{S,\alpha}(\boldsymbol{p})$. To find them, we have to use integrability of the dilatation operator in planar $\mathcal{N} = 4$ SYM.

3.2. Dilation operator at one loop

The explicit form of the conformal operators $\mathcal{O}_{S,\alpha}(0)$ and their anomalous dimensions $\gamma_{S,\alpha}$ can be obtained by diagonalising the dilatation operator in the $SL(2)$ sector of $\mathcal{N} = 4$ SYM. Making use of (2.4) and (2.5), the corresponding spectral problem can be reduced to solving a Schrödinger like equation for the polynomials $\Phi_{S,\alpha}(\boldsymbol{z})$ and $\Psi_{S,\alpha}(\boldsymbol{p})$, e.g.

$$\mathbb{H}\,\Phi_{S,\alpha}(\boldsymbol{z}) = \gamma_{S,\alpha}\big(g^2\big)\Phi_{S,\alpha}(\boldsymbol{z}). \tag{3.11}$$

To the lowest order in the coupling, the dilatation operator $\mathbb{H}$ can be mapped into a Hamiltonian of the $SL(2)$ Heisenberg spin $j = 1/2$ chain of length L

$$\mathbb{H} = g^2[H_{12} + \cdots + H_{L1}] + O\big(g^4\big). \tag{3.12}$$

Here the two-particle kernel $H_{i,i+1}$ acts locally on the light-cone coordinates z_i and z_{i+1} of two neighbouring particles and admits the following representation [51–53,17]

$$\begin{aligned} &H_{i,i+1}\phi(z_i, z_{i+1}) \\ &\quad = \int_0^1 \frac{d\tau}{\tau}\big[2\phi(z_i, z_{i+1}) - \phi\big((1-\tau)z_i + \tau z_{i+1}, z_{i+1}\big) - \phi\big(z_i, (1-\tau)z_{i+1} + \tau z_i\big)\big], \end{aligned} \tag{3.13}$$

with $\phi(z_i, z_{i+1})$ being a test function. This operator has a clear physical interpretation – it displays two particles with the coordinates z_i and z_{i+1} in the direction of each other.

The Hamiltonian $\mathbb{H}$ defined in (3.12) and (3.13) maps a homogenous polynomial in z of degree S into another homogenous polynomial of the same degree. Then, replacing $\Phi_{S,\alpha}(z)$ in (3.11) with its general expression (3.6) and comparing the coefficients in front of different powers of z's on the both sides of (3.11), we can obtain a system of linear homogenous equations for the expansion coefficients $c_{\boldsymbol{k},\alpha}$. The corresponding characteristic equation yields a polynomial equation for $\gamma_{S,\alpha}$. Solving the system we should also take into account that $\Phi_{S,\alpha}(z)$ has to be both cyclically and translationally invariant homogenous polynomial in z of degree S. This leads to the additional selection rule for the possible solutions.

In this way, it is straightforward to solve (3.11) for lowest values of the total spin S. For instance, for $S=0$, the Schrödinger equation (3.11) has a trivial solution $\Phi_{S=0}(z)=1$ and $\gamma_{S=0}(g^2)=0$. The corresponding conformal operator $\mathcal{O}_{S=0}=\mathrm{tr}[Z^L]$ is a half-BPS state and its anomalous dimension vanishes to all loops. For $S=2$ and $L=2,3,4$ we find the following expressions

$$\begin{aligned}
\Phi_2^{(L=2)} &= c_2\, z_{12}^2,\\
\Phi_2^{(L=3)} &= c_3\left[z_{12}^2+z_{23}^2+z_{31}^2\right],\\
\Phi_{2,\pm}^{(L=4)} &= c_{4,\pm}\left[z_{12}^2+z_{23}^2+z_{34}^2+z_{41}^2+(1\pm\sqrt{5})(z_{12}z_{34}+z_{23}z_{41})\right],
\end{aligned}\tag{3.14}$$

where $z_{ij}\equiv z_i-z_j$ and the normalisation factors are fixed by the condition (3.7) to be $c_2^2=1/6$, $c_3^2=1/24$ and $c_{4,\pm}^2=(3\mp\sqrt{5})/160$. The corresponding one-loop anomalous dimensions are

$$\gamma_2^{(L=2)}=6g^2,\qquad \gamma_2^{(L=3)}=4g^2,\qquad \gamma_{2,\pm}^{(L=4)}=(5\pm\sqrt{5})g^2.\tag{3.15}$$

Finally, we apply the relations (2.3) and (3.6) to obtain the corresponding expressions for the Ψ-polynomial

$$\begin{aligned}
\Psi_2^{(L=2)} &= c_2\left(\frac{1}{2}p_1^2-2p_1p_2+\frac{1}{2}p_2^2\right),\\
\Psi_2^{(L=3)} &= c_3\left[p_1^2+p_2^2+p_3^2-2(p_1p_2+p_2p_3+p_3p_1)\right],\\
\Psi_{2,\pm}^{(L=4)} &= c_4\big[p_1^2+p_2^2+p_3^2+p_4^2+2(1\pm\sqrt{5})(p_1p_3+p_2p_4)\\
&\qquad -(3\pm\sqrt{5})(p_1p_2+p_2p_3+p_3p_4+p_4p_1)\big].
\end{aligned}\tag{3.16}$$

By the construction, the polynomials (3.14) and (3.16) define the eigenstates of the one-loop dilatation operator in $\mathcal{N}=4$ SYM in the two representations.

Let us verify that the obtained expressions satisfy the duality relation (2.21). To this end, we substitute $p_i=z_i-z_{i+1}$ (with $z_{i+L}\equiv z_i$) into the right-hand side of (3.16) and compare the resulting expressions with (3.14). We find that, in a perfect agreement with (2.21), the polynomials are indeed proportional to each other,

$$\Psi_2^{(L)}(p_i=z_{i,i+1})=\xi_2^{(L)}\Phi_2^{(L)}(z_i),\tag{3.17}$$

with the proportionality factor given (to the lowest order in the coupling) by

$$\xi_2^{(L=2)}=3,\qquad \xi_2^{(L=3)}=2,\qquad \xi_{2,\pm}^{(L=4)}=\frac{5\pm\sqrt{5}}{2}.\tag{3.18}$$

For higher values of S and L, it is more efficient to construct the solution to (3.11) with a help of Algebraic Bethe Ansatz [54–57]. In this method, the Φ- and Ψ-polynomials coincide with the Bethe states in the z- and p-representations, respectively. Examining their explicit expressions one can verify that the duality relation (2.21) holds for arbitrary S and L.

4. Dual symmetry from the Baxter Q-operator

We have demonstrated in the previous section that the one-loop dilatation operator possesses the dual symmetry (2.21). To understand the origin of this symmetry we use the above mentioned relation between the one-loop dilatation operator and the $SL(2;\mathbb{R})$ Heisenberg spin chain. The latter model is exactly solvable and its eigenspectrum can be obtained using different technique [54–57]. For our purposes it is convenient to employ the method based on the Baxter Q-operator.

4.1. Baxter Q-operator

The method relies on the existence of the operator $\mathbb{Q}(u)$ which encodes an information about all conserved charges of the $SL(2;\mathbb{R})$ Heisenberg spin chain. It acts on the space of polynomials $\Phi(z)$, depends on an arbitrary complex parameter u and satisfies the following defining relations

$$\begin{aligned}&\big[\mathbb{Q}(u),\mathbb{Q}(v)\big]=\big[\mathbb{Q}(u),T(v)\big]=0,\\ &\mathbb{Q}(u+i)(u+i/2)^L+\mathbb{Q}(u-i)(u-i/2)^L=T(u)\mathbb{Q}(u),\end{aligned} \tag{4.1}$$

where $T(u)=2u^L+q_2u^{L-2}+\cdots+q_L$ is the so-called auxiliary transfer matrix and q_k are commuting conserved charges. Then, to the lowest order in the coupling, the Schrödinger equation (3.11) can be replaced with an analogous equation for the Baxter Q-operator

$$\mathbb{Q}(u)\Phi_{S,\alpha}(z)=Q_{S,\alpha}(u)\Phi_{S,\alpha}(z). \tag{4.2}$$

The solutions to (4.2) are characterised by the complete set of the conserved charges $q_2,\dots,q_L$. This allows us to identify index α with the set of their eigenvalues $\alpha=(q_2,\dots,q_L)$.

For the $SL(2;\mathbb{R})$ magnet of an arbitrary length L and (positive half-integer) spin j, the Baxter operator has been constructed in Ref. [38]. For our purposes we need its expression for $j=1/2$

$$\begin{aligned}&\mathbb{Q}(u)\Phi(z_1,\dots,z_L)\\ &\quad=c_Q\bigg[\Gamma\Big(iu+\frac{1}{2}\Big)\Gamma\Big(-iu+\frac{1}{2}\Big)\bigg]^{-L}\\ &\qquad\times\int_0^1\prod_{i=1}^L d\tau_i\,\tau_i^{-iu-1/2}(1-\tau_i)^{iu-1/2}\,\Phi\big(\tau_1z_1+(1-\tau_1)z_2,\dots,\tau_Lz_L+(1-\tau_L)z_1\big),\end{aligned} \tag{4.3}$$

where c_Q is a normalisation factor and $\Phi(z_1,\dots,z_L)$ is a test function. This operator satisfies the following relations

$$\mathbb{Q}(i/2)=c_Q,\qquad \mathbb{Q}(-i/2)=c_Q\,\mathbb{P}, \tag{4.4}$$

where $\mathbb{P}$ is the operator of cyclic shift, $z_i\mapsto z_{i+1}$. The Hamiltonian $\mathbb{H}$, or equivalently the one-loop dilatation operator (3.12), is related to the expansion of $\mathbb{Q}(u)$ around $u=\pm i/2$

$$\mathbb{H}=g^2\big[i\big(\ln\mathbb{Q}(i/2)\big)'-i\big(\ln\mathbb{Q}(-i/2)\big)'\big]+O\big(g^4\big), \tag{4.5}$$

where prime denotes a derivative with respect to the spectral parameter u.

The Schrödinger equation (3.11) is equivalent to the spectral problem (4.2) for the operator (4.3). As follows from the second relation in (4.1), the eigenvalues of the operator $Q_{S,\alpha}(u)$ satisfy the second-order finite-difference equation, the so-called TQ-relation. Having solved this equation, we can evaluate the one-loop anomalous dimension from (4.5) as

$$\gamma_{S,\alpha} = g^2\big[i\big(\ln Q_{S,\alpha}(i/2)\big)' - i\big(\ln Q_{S,\alpha}(-i/2)\big)'\big] + O\big(g^4\big). \tag{4.6}$$

In addition, the condition for $\Phi_{S,\alpha}(z)$ to be cyclic invariant function of z_i leads to the selection rule for $Q_{S,\alpha}(u)$

$$Q_{S,\alpha}(-i/2) = Q_{S,\alpha}(i/2), \tag{4.7}$$

which follows from $(\mathbb{P}-1)\Phi_{S,\alpha}(z) = 0$ combined with (4.4).

It is straightforward to verify that the eigenstates $\Phi_{S,\alpha}(z)$ found in the previous section, indeed diagonalised the Baxter Q-operator (4.3). For $S=0$, we substitute $\Phi_{S=0}(z)=1$ into (4.2) and (4.3) to find $Q_{S=0}(u) = c_Q$. For $S=2$, we use the relation (3.14) to get the corresponding eigenvalues

$$\begin{aligned}
&Q_2^{(L=2)}(u) = -\frac{1}{12} + u^2, \quad \left(q_2 = -\frac{13}{2}\right),\\
&Q_2^{(L=3)}(u) = -\frac{1}{4} + u^2, \quad \left(q_2 = -\frac{19}{2},\ q_3 = 0\right),\\
&Q_{2,\pm}^{(L=4)}(u) = -\frac{1}{4} \pm \frac{\sqrt{5}}{10} + u^2, \quad \left(q_2 = -13,\ q_3 = 0,\ q_4 = \frac{21}{8} \pm \sqrt{5}\right).
\end{aligned} \tag{4.8}$$

Here we also indicated the corresponding values of the conserved charges $q_2, \ldots, q_L$. They are uniquely fixed by the TQ-relation (4.1).[7] Also, the expressions (4.8) satisfy the normalisation condition $Q(u) \sim u^S$ at large u which fixes the constant c_Q in (4.3). The substitution of (4.8) into (4.6) yields the correct result for the one-loop anomalous dimensions (3.15).

For arbitrary total spin S and length L, the eigenvalues of the Baxter operator (4.3) are given by polynomials in u of degree S

$$Q_{S,\alpha}(u) = \sum_{k=0}^{S} c_k u^k = \prod_{k=1}^{S}(u - u_k), \tag{4.9}$$

with the expansion coefficients c_k and roots u_k depending on the conserved charges. The relation (4.9) corresponds to the particular choice of the normalisation factor c_Q in (4.3). We find from the first relation in (4.4) that it is given by

$$c_Q = Q_{S,\alpha}(i/2) = \prod_{k=1}^{S}\left(\frac{i}{2} - u_k\right). \tag{4.10}$$

Requiring $Q_{S,\alpha}(u)$ to satisfy the TQ-relation (4.1), we find (by putting $u = u_k$ on both sides of (4.1)) that the parameters u_k verify the Bethe equations for the $SL(2)$ Heisenberg magnet of spin 1/2

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{n\neq k} \frac{u_k - u_n - i}{u_k - u_n + i}. \tag{4.11}$$

For the $SL(2;\mathbb{R})$ magnet the roots u_k take real values only.

[7] It follows from the TQ-relation at large u that $q_2 = -(S+L/2)(S+L/2-1) - L/4$.

4.2. Large u expansion

In this subsection, we show following Ref. [38] that the dual symmetry can be derived from the asymptotic expansion of both sides of (4.2) at large u. In this limit, the eigenvalues of the Baxter Q-operator (4.9) scale as

$$Q_{S,\alpha}(u) = u^S\left[1 - \frac{\sum_k u_k}{u} + O\big(1/u^2\big)\right]. \tag{4.12}$$

This suggests that the operator $\mathbb{Q}(u)$ should admit similar expansion

$$\mathbb{Q}(u)\Phi_{S,\alpha}(z) = u^S\left[\mathbb{Q}^{(0)} + \frac{1}{u}\mathbb{Q}^{(1)} + O\big(1/u^2\big)\right]\Phi_{S,\alpha}(z), \tag{4.13}$$

with $\mathbb{Q}^{(0)}, \mathbb{Q}^{(1)}, \dots$ being mutually commuting operators. Substituting (4.12) and (4.13) into (4.2), we compare the coefficients in front of powers of u to find

$$\begin{aligned}
&\mathbb{Q}^{(0)}\Phi_{S,\alpha}(z) = \Phi_{S,\alpha}(z),\\
&\mathbb{Q}^{(1)}\Phi_{S,\alpha}(z) = -\Big(\sum_k u_k\Big)\Phi_{S,\alpha}(z), \quad \dots
\end{aligned} \tag{4.14}$$

To get explicit expression for the operators $\mathbb{Q}^{(0)}, \mathbb{Q}^{(1)}, \dots$, we apply (3.5) and replace a test polynomial $\Phi(z)$ on the right-hand side of (4.3) with its expression in terms of polynomial $\Psi(\boldsymbol{p})$,

$$\Phi(z) = \Psi(\partial_{\boldsymbol{w}})\prod_{i=1}^{L}(1 - w_i z_i)^{-1}\Big|_{\boldsymbol{w}=0}. \tag{4.15}$$

Then, integration over α-parameters in (4.3) yields another (equivalent) representation for the Q-operator

$$\mathbb{Q}(u)\Phi(z) = c_Q\Psi(\partial_{\boldsymbol{w}})\prod_{i=1}^{L}(1 - w_i z_i)^{iu-1/2}(1 - w_i z_{i+1})^{-iu-1/2}\Big|_{\boldsymbol{w}=0}. \tag{4.16}$$

Finally, we rescale the auxiliary parameters $w_i \to w_i/u$ and expand the product on the right-hand side in powers of $1/u$ to get after some algebra

$$\begin{aligned}
&\mathbb{Q}(u)\Phi(z)\\
&\quad = c_Q\left[1 + \frac{i}{2u}\sum_{i=1}^{L}(z_i + z_{i+1})\big(p_i\partial_{p_i}^2 + \partial_{p_i}\big) + O\big(u^{-2}\big)\right]\Psi(-iu\boldsymbol{p})\Big|_{p_i=z_{i,i+1}},
\end{aligned} \tag{4.17}$$

where we used a shorthand notation for $c\boldsymbol{p} \equiv (cp_1, \dots, cp_L)$. To match (4.13), we replace test functions in (4.17) by the eigenfunctions $\Phi_{S,\alpha}$ and $\Psi_{S,\alpha}$ to get

$$\begin{aligned}
&\mathbb{Q}^{(0)}\Phi_{S,\alpha}(z) = (-i)^S c_Q\,\Psi_{S,\alpha}(\boldsymbol{p})|_{p_i=z_{i,i+1}},\\
&\mathbb{Q}^{(1)}\Phi_{S,\alpha}(z) = \frac{1}{2}(-i)^{S-1}c_Q\sum_i(z_i + z_{i+1})\big(p_i\partial_{p_i}^2 + \partial_{p_i}\big)\Psi_{S,\alpha}(\boldsymbol{p})\Big|_{p_i=z_{i,i+1}}, \quad \dots
\end{aligned} \tag{4.18}$$

As follows from the first relation, the operator $\mathbb{Q}^{(0)}$ generates the duality transformation – it transforms the polynomial $\Phi_{S,\alpha}(z)$ from the coordinate to momentum representation and assigns the light-cone momenta as $p_i = z_i - z_{i+1}$. Since $\Phi_{S,\alpha}(z)$ diagonalises the operator $\mathbb{Q}^{(0)}$, Eq. (4.14), we conclude from (4.18) that it has to satisfy the duality relation (2.21) with

$$\xi_{S,\alpha} = i^S/c_Q = i^S/Q_{S,\alpha}(i/2) = \left[\prod_{k=1}^{S}\left(\frac{1}{2} + iu_k\right)\right]^{-1}, \tag{4.19}$$

where the second relation follows from (4.10). We verified that for $S = 2$ this relation is in an agreement with (3.18) and (4.8).

In the similar manner, we can equate the expressions for $\mathbb{Q}^{(1)}\Phi_{S,\alpha}(z)$ (as well as for the remaining subleading terms in the large u expansion of the Baxter operator) on both sides of (4.14) and (4.18) to obtain the additional relations between $\Psi_{S,\alpha}(\boldsymbol{p})$ and $\Phi_{S,\alpha}(z)$. As we show in the next subsection, for the polynomials verifying (2.21), these relations are automatically fulfilled.

4.3. Dual symmetry at work

Let us understand to which extend the dual symmetry (2.21) fixes the polynomials $\Psi_{S,\alpha}(\boldsymbol{p})$ and $\Phi_{S,\alpha}(z)$. Since the relation (2.21) involves $\Psi_{S,\alpha}(\boldsymbol{p})$ evaluated for $p_i = z_i - z_{i+1}$, we might expect that it will allow us to obtain $\Psi_{S,\alpha}(\boldsymbol{p})$ for the vanishing total momentum only, $\sum_i p_i = 0$. As we will see in a moment, this is not the case – the dual symmetry determines the polynomial $\Psi_{S,\alpha}(\boldsymbol{p})$ for arbitrary total momentum.

To begin with, we apply (3.5) and rewrite the duality relation (2.21) as

$$\Psi_{S,\alpha}(p_1, \dots, p_L)\big|_{p_i = z_i - z_{i+1}} = \xi_{S,\alpha} \int_0^\infty \prod_{i=1}^{L} dt_i\, \mathrm{e}^{-t_i}\, \Psi_{S,\alpha}(t_1 z_1, \dots, t_L z_L). \tag{4.20}$$

In general, $\Psi_{S,\alpha}(\boldsymbol{p})$ is a homogenous polynomial in $\boldsymbol{p} = (p_1, \dots, p_L)$ of degree S, Eq. (2.3). It is uniquely specified by the set of the expansion coefficients $c_{\boldsymbol{k},\alpha}$ invariant under the cyclic shift of integers $\boldsymbol{k} = (k_1, \dots, k_L)$. Substituting (2.3) into both sides of (4.20) we arrive at

$$\sum c_{\boldsymbol{k},\alpha} \frac{(z_1 - z_2)^{k_1}}{k_1!} \cdots \frac{(z_L - z_1)^{k_L}}{k_L!} = \xi_{S,\alpha} \sum c_{\boldsymbol{k},\alpha}\, z_1^{k_1} \dots z_L^{k_L}, \tag{4.21}$$

where the sum on both sides runs over nonnegative integers satisfying $k_1 + \ldots + k_L = S$.

Analysis of Eq. (4.21) goes along the same lines as in the beginning of Section 3.2. Namely, we compare coefficients in front of powers of z's on both sides of (4.21) and obtain the system of linear homogenous equations for the coefficients $c_{\boldsymbol{k},\alpha}$. The characteristic equation for this system yields a polynomial equation for $\xi_{S,\alpha}$. Then, each solution $\xi_{S,\alpha}$ leads to a definite expression for $c_{\boldsymbol{k},\alpha}$ (modulo an overall normalisation) which we can use to determine the polynomial $\Psi_{S,\alpha}(\boldsymbol{p})$, Eq. (2.3), for arbitrary total momentum $\sum_i p_i$. The overall normalisation of $c_{\boldsymbol{k},\alpha}$ is fixed by (3.7). It is straightforward to verify that for $S = 2$ and $L = 2, 3, 4$ the solutions to (4.21) obtained in this way coincide with those in (3.16).

Thus the duality relation (4.20) is powerful enough to determine the eigenstates of the one-loop dilatation operator (3.11). According to (4.2), the same polynomials diagonalised the Baxter operator $\mathbb{Q}(u)$ for arbitrary u. Then, it follows from (4.13) that at large u solutions to (4.20) diagonalised the operators $\mathbb{Q}^{(0)}, \mathbb{Q}^{(1)}, \dots$ defined in (4.18).

5. Supersymmetric extension of duality

In this section, we extend the duality symmetry to supersymmetric light-ray operators built from scalars and various components of gaugino and gauge strength fields in $\mathcal{N}=4$ SYM.

5.1. Light-cone superfields and superstates

Supersymmetric light-ray operators have the same form as (1.2) with complex scalar field $Z(x)$ replaced by the (chiral) superfield (see Eq. (5.8) below)

$$\mathcal{Z}(x,\theta)=\mathrm{e}^{\theta_a Q_n^a}Z(x)=Z(x)+\theta_a\psi_n^a(x)+\frac{1}{2}\theta_a\theta_b\epsilon^{ab}F_n(x). \tag{5.1}$$

Here θ_a are odd (Grassmann) coordinates and the generators of supersymmetric transformations Q_n^a (with $a=1,2$) are given by two linear combinations of $\mathcal{N}=4$ supercharges

$$Q_n^a=\lambda^\alpha(n)\,Q_\alpha^a,\qquad n_{\alpha\dot\alpha}=\lambda_\alpha(n)\tilde\lambda_{\dot\alpha}(n), \tag{5.2}$$

with $n_{\alpha\dot\alpha}=n_\mu(\sigma^\mu)_{\alpha\dot\alpha}$ (with $\alpha,\dot\alpha=1,2$) being the light-like vector entering the definition of the light-ray operator (2.4). The operators $\psi_n^a(x)$ and $F_n(x)$ are given by linear combinations of gaugino field, $\psi_\alpha^a(x)$, and self-dual part of the strength tensor, $F_{\alpha\beta}(x)$,

$$\psi_n^a(x)=\lambda^\alpha(n)\psi_\alpha^a(x),\qquad F_n(x)=\lambda^\alpha(n)\lambda^\beta(n)F_{\alpha\beta}(x). \tag{5.3}$$

Notice that the right-hand side of (5.1) does not involve all field components in $\mathcal{N}=4$ SYM. The reason for this is that constructing the superfield (5.1) we only used two (out of four) supercharges. In fact, as we show in Appendix A, the operator $\mathcal{Z}(x,\theta)$ defines the $\mathcal{N}=2$ part of the full $\mathcal{N}=4$ light-cone superfield. The latter superfield contains spurious field components (see Eq. (A.2)) whose contribution should be carefully separated [43]. To avoid this complication we prefer to deal with the $\mathcal{N}=2$ superfield (5.1).

To define form factors of supersymmetric light-cone operators, we also need a supersymmetric extension of the on-shell asymptotic states. The operators $(Z(x),\psi_n^a(x),F_n(x))$ create out of vacuum particles with helicity $h=(0,+1/2,+1)$, respectively, e.g.

$$\langle 0|\psi_\alpha^a(x)|p,\tfrac{1}{2},b\rangle=\lambda_\alpha(p)\,\delta^{ab}\mathrm{e}^{ipx},\qquad \langle 0|F_{\alpha\beta}(x)|p,1\rangle=\lambda_\alpha(p)\lambda_\beta(p)\,\mathrm{e}^{ipx}, \tag{5.4}$$

where $\lambda_\alpha(p)$ is defined by light-like momentum of particles $p_{\alpha\dot\alpha}=\lambda_\alpha(p)\tilde\lambda_{\dot\alpha}(p)$. To simplify formulae we do not display here colour indices. In a close analogy with (5.1), we can combine the single-particle states $|p,h\rangle$ into a single $\mathcal{N}=2$ superstate by introducing odd variables η^a (with $a=1,2$)

$$\begin{aligned}|p,\eta\rangle&=\left[a_0^\dagger(p)+\eta^a a_{1/2,a}^\dagger(p)+\frac{1}{2}\epsilon_{ab}\eta^a\eta^b\,a_1^\dagger(p)\right]|0\rangle\\&\equiv|p,0\rangle+\eta^a|p,\tfrac{1}{2},a\rangle+\frac{1}{2}\epsilon_{ab}\eta^a\eta^b|p,1\rangle.\end{aligned} \tag{5.5}$$

Here $a_h^\dagger(p)$ is the creation operator of a massless particle with light-like momentum p^μ and helicity h. To equate the helicity charge of three terms on the right-hand side of (5.5), we assign helicity $(-1/2)$ to odd variables η^a. As before, the superstate $|p,\eta\rangle$ describes $\mathcal{N}=2$ part of the full $\mathcal{N}=4$ supermultiplet of on-shell states.

We combine together relations (5.1), (5.4) and (5.5) to find

$$\langle 0|\mathcal{Z}(x,\theta)|p,\eta\rangle = \left[1 + \langle np\rangle(\theta\eta) + \frac{1}{2}\langle np\rangle^2(\theta\eta)^2\right] e^{ipx} = e^{ipx+\langle np\rangle(\theta\eta)}, \tag{5.6}$$

where $\langle np\rangle = \lambda^\alpha(n)\lambda_\alpha(p)$ and $(\theta\eta) = \theta_a\eta^a$. To define the supermomentum carried by the superfield, we perform Fourier transformation of (5.6) with respect to x^μ and θ_a

$$\int d^4x\, e^{-ikx} \int d^2\theta\, e^{-(\theta\vartheta)} \langle 0|\mathcal{Z}(x,\theta)|p,\eta\rangle = (2\pi)^4\delta^{(4)}(k-p)\delta^{(2)}\big(\vartheta - \langle np\rangle\eta\big). \tag{5.7}$$

We conclude from this relation that the odd momenta carried by the superfield and the superstate (5.5) are related to each other as $\vartheta^a = \langle np\rangle\eta^a$.

5.2. Light-ray operators in superspace

Supersymmetric generalisation of the light-ray operator (2.4) looks as

$$\mathbb{O}(\boldsymbol{Z}) = \mathrm{tr}\big[\mathcal{Z}(z_1 n,\theta_1)\dots Z(z_L n,\theta_L)\big], \tag{5.8}$$

where $\boldsymbol{Z} = (z_1,\theta_1),\dots,(z_L,\theta_L)$ denotes the coordinates of L supefields in the light-cone superspace. As before, the gauge invariance can be restored by inserting the light-like Wilson lines between the adjacent superfields. Similar to (2.1), the expansion of $\mathbb{O}(\boldsymbol{Z})$ in powers of z's produces local operators of length L containing an arbitrary number of covariant derivatives. Further expansion in powers of θ's yields operators $\mathrm{tr}[D_+^{k_1}X_1\dots D_+^{k_L}X_L]$ of different partonic content $X = (Z, \psi_n^a, F_n)$.

Different components of the superfield (5.1) are transformed under the conformal $SL(2)$ transformations according to (2.10) and carry the conformal spin, $j_Z = 1/2$, $j_\psi = 1$ and $j_F = 3/2$. When combined together, they define the so-called atypical, or chiral, representation of the superconformal $SL(2|2)$ group [43]. Namely, the light-cone superfield $\mathcal{Z}(zn,\theta)$ transforms linearly under the $SL(2|2)$ transformations

$$\delta_G\mathcal{Z}(zn,\theta) = G\mathcal{Z}(zn,\theta), \tag{5.9}$$

with the generators $G = \{L^\pm, L^0, W^{a,\pm}, V_a^\pm, T_b^a\}$ given by the differential operators acting on its coordinates

$$\begin{aligned}
&L^- = -\partial_z, \qquad L^+ = z + z^2\partial_z + z(\theta\cdot\partial_\theta), \qquad L^0 = \frac{1}{2} + z\partial_z + \frac{1}{2}(\theta\cdot\partial_\theta),\\
&W^{a,-} = \theta^a\,\partial_z, \qquad W^{a,+} = \theta^a\big[1 + z\partial_z + (\theta\cdot\partial_\theta)\big], \qquad T_b{}^a = \theta^a\partial_{\theta^b} - \frac{1}{2}\delta_b^a(\theta\cdot\partial_\theta),\\
&V_a^- = \partial_{\theta^a}, \qquad V_a^+ = z\partial_{\theta^a},
\end{aligned} \tag{5.10}$$

where $\partial_z \equiv \partial/\partial z$ and $\theta\cdot\partial_\theta \equiv \sum_{a=1,2}\theta^a\partial/\partial\theta^a$. A global form of the transformations (5.9) can be found in Ref. [43].

The super light-ray operator (5.8) belongs to the tensor product of L copies of the $SL(2|2)$ representations. Similarly to (2.13), we can classify all possible local operators entering the OPE expansion of $\mathbb{O}(\boldsymbol{Z})$ according to irreducible components that appear in this tensor product

$$\mathbb{O}(\boldsymbol{Z}) = \sum_{S,\alpha}\big[\Phi_{S,\alpha}(\boldsymbol{Z})\mathcal{O}_{S,\alpha}(0) + \text{descendants}\big], \tag{5.11}$$

where the contribution of each component involves the superconformal primary operator $\mathcal{O}_{S,\alpha}(0)$ and its $SL(2|2)$ descendants. The polynomial $\Phi_{S,\alpha}(\boldsymbol{Z})$ satisfies the lowest weight condition

$$L^- \Phi_{S,\alpha}(\boldsymbol{Z}) = W^{a,-} \Phi_{S,\alpha}(\boldsymbol{Z}) = V_a^- \Phi_{S,\alpha}(\boldsymbol{Z}) = 0, \tag{5.12}$$

and diagonalises the operators L^0 and $T_b^a T_a^b$. Here, the $SL(2|2)$ generators are given by the sum of differential operators (5.10) acting on the coordinates of L particles. In general, $\Phi_{S,\alpha}(\boldsymbol{Z})$ is given by a homogenous polynomial in z's and θ's of the total degree S. For $L = 2$ its explicit expression can be found in Ref. [44]. For $L \geq 3$ the relations (5.12) do not fix the polynomials $\Phi_{S,\alpha}(\boldsymbol{Z})$ completely. The additional condition for $\Phi_{S,\alpha}(\boldsymbol{Z})$ comes from integrability of the dilatation operator for the light-ray operators (5.8) which can be mapped, to the lowest order in the coupling, into Hamiltonian of the $SL(2|2)$ Heisenberg spin chain.

The operator $\mathcal{O}_{S,\alpha}(0)$ can be obtained from (5.11) as

$$\mathcal{O}_{S,\alpha}(0) = \overline{\Psi}_{S,\alpha}(\partial_{\boldsymbol{Z}})\, \mathbb{O}(\boldsymbol{Z})\big|_{\boldsymbol{Z}=0}, \tag{5.13}$$

where $\partial_{\boldsymbol{Z}} = (\partial_{z_1}, \partial_{\theta_1}), \dots, (\partial_{z_L}, \partial_{\theta_L})$ and $\Psi_{S,\alpha}(\boldsymbol{P})$ is a polynomial in $\boldsymbol{P} = (p_1, \vartheta_1), \dots, (p_L, \vartheta_L)$. The two (super)polynomials, $\Phi_{S,\alpha}(\boldsymbol{Z})$ and $\Psi_{S,\alpha}(\boldsymbol{P})$, uniquely determine the form of the superconformal operators and their contribution to the OPE (5.11). They satisfy the relations

$$\begin{aligned} &\overline{\Psi}_{S,\alpha}(\partial_{\boldsymbol{Z}})\Phi_{S',\alpha'}(\boldsymbol{Z})\big|_{\boldsymbol{Z}=0} = \delta_{SS'}\delta_{\alpha\alpha'}, \\ &\sum_{S,\alpha} \overline{\Psi}_{S,\alpha}(\boldsymbol{P})\Phi_{S,\alpha}(\boldsymbol{Z}) = \frac{1}{L}\sum_{i=1}^{L}\prod_{k=i}^{i+L-1} \exp(p_1 z_k + \vartheta_1\theta_k) \end{aligned} \tag{5.14}$$

(with $z_{k+L} = z_k$ and $\theta_{k+L} = \theta_k$) which generalise similar relations in the $SL(2)$ sector, Eqs. (2.6) and (2.7). The solution to these relations reads [44]

$$\begin{aligned} \Phi_{S,\alpha}(Z_1, \dots, Z_L) &= \int_0^\infty \prod_{k=1}^{L} dt_k\, \mathrm{e}^{-t_k} \Psi_{S,\alpha}(t_1 Z_1, \dots, t_L Z_L) \\ &= \Psi_{S,\alpha}(\partial_{W_1}, \dots, \partial_{W_L}) \prod_{k=1}^{L} (1 - w_k z_k - \zeta_k \cdot \theta_k)^{-1}\Big|_{w_k=\zeta_k=0}, \end{aligned} \tag{5.15}$$

where $Z_i \equiv (z_i, \theta_i^a)$ and $\partial_{W_k} \equiv (\partial_{w_k}, \partial_{\zeta_k})$.

The relations (5.15) are remarkably similar to those in the $SL(2)$ sector, Eqs. (3.5) and (4.15), and they are coincide for $\theta_i^a = 0$. Indeed, the super light-ray operator (5.8) reduces for $\theta_i^a = 0$ to its lowest component given by (2.4). This allows us to interpret the polynomial $\Phi_{S,\alpha}(\boldsymbol{Z})$ as being obtained from the analogous polynomial in the $SL(2)$ sector through the lift from the light-cone to the superspace $z_i \mapsto (z_i, \theta_i)$ [43].

5.3. Baxter operator for the $SL(2|2)$ spin chain

To lowest order in the coupling, the scale dependence of the super light-ray operators (5.8) in planar $\mathcal{N} = 4$ SYM can be determined from the eigenspectrum of the $SL(2|2)$ Heisenberg spin chain. Namely, the polynomials $\Phi_{S,\alpha}(\boldsymbol{Z})$ coincide with the eigenstates of the spin chain whereas the corresponding energies determine the one-loop correction to the anomalous dimension of the superconformal operators in (5.11).

As in the case of the $SL(2)$ spin chain, we replace the Schrödinger equation for the $SL(2|2)$ spin chain with the spectral problem for the Baxter Q-operator

$$\mathbb{Q}_i(u)\Phi_{S,\alpha}(\boldsymbol{Z}) = Q_{S,\alpha,i}(u)\Phi_{S,\alpha}(\boldsymbol{Z}). \tag{5.16}$$

Here we introduced the subscript $i = 1, \dots, 4$ to indicate that there are few Q-operators in this case. The operators $\mathbb{Q}_i(u)$ act on the tensor product of L copies of chiral $SL(2|2)$ representation and satisfy the same defining relations as in the previous case. Namely, the operators $\mathbb{Q}_i(u)$ commute with the $SL(2|2)$ transfer matrices and among themselves but the corresponding TQ relations are move involved in this case. A general approach to constructing such operators has been developed in Refs. [45,46].

For our purposes, we will only need an explicit expression for one of the Q-operators which plays a special role in finding the eigenspectrum of the model – it determines the Hamiltonian of the $SL(2|2)$ magnet through the same relation as for the $SL(2)$ spin chain, Eq. (4.5). Denoting this operator by $\mathbb{Q}(u)$, we find that it acts on a test polynomial $\Phi(Z_1, \dots, Z_L)$ depending on $Z_i = (z_i, \theta_i^a)$ as

$$\begin{aligned}
&\mathbb{Q}(u)\Phi(Z_1, \dots, Z_L) \\
&\quad = c_Q\left[\Gamma\left(iu + \frac{1}{2}\right)\Gamma\left(-iu + \frac{1}{2}\right)\right]^{-L} \int_0^1 \prod_{i=1}^L d\tau_i\, \tau_i^{-iu-1/2}(1-\tau_i)^{iu-1/2} \\
&\qquad \times \Phi\big(\tau_1 Z_1 + (1-\tau_1)Z_2, \dots, \tau_L Z_L + (1-\tau_L)Z_1\big),
\end{aligned} \tag{5.17}$$

where c_Q is a normalisation factor and we used the shorthand notation for $\alpha Z_1 + \beta Z_2 = (\alpha z_1 + \beta z_2, \alpha\theta_1 + \beta\theta_2)$. This operator satisfies the relations

$$\mathbb{Q}(i/2) = c_Q, \qquad \mathbb{Q}(-i/2) = c_Q\mathbb{P}, \tag{5.18}$$

where $\mathbb{P}$ is the operator of cyclic shifts $(z_i, \theta_i) \mapsto (z_{i+1}, \theta_{i+1})$. Applying the second relation in (5.15), we can obtain another, equivalent representation for the operator (5.17)

$$\begin{aligned}
&\mathbb{Q}(u)\Phi(\boldsymbol{Z}) \\
&\quad = c_Q\, \Psi(\partial_{\boldsymbol{W}}) \prod_{i=1}^L (1 - w_i z_i - \zeta_i\cdot\theta_i)^{iu-1/2}(1 - w_i z_{i+1} - \zeta_i\cdot\theta_{i+1})^{-iu-1/2}\bigg|_{\boldsymbol{W}=0},
\end{aligned} \tag{5.19}$$

where $\partial_{W_i} \equiv (\partial_{w_i}, \partial_{\zeta_i})$. As before, we observe a striking similarity of (5.17) with the analogous expression (4.3) for the $SL(2)$ magnet. This is not accidental of course since for $\theta_i^a = 0$ (with $i = 1, \dots, L$) the eigenstate $\Phi_{S,\alpha}(\boldsymbol{Z})$ reduces to its lowest component which satisfies the $SL(2)$ Baxter equation (4.3).

Let us examine the asymptotic expansion of the operator $\mathbb{Q}(u)$ at large u. We rescale $w_i \to w_i/u$ and $\zeta_i \to \zeta_i/u$ on the right-hand side of (5.19) and go along the same lines as in (4.17) to get

$$\mathbb{Q}(u)\Phi(\boldsymbol{Z}) = c_Q\big[1 + O\big(u^{-1}\big)\big]\Psi(-iu\boldsymbol{P})\big|_{P_i = Z_i - Z_{i+1}}, \tag{5.20}$$

where the Ψ-polynomial is evaluated for $P_i = (z_{i,i+1}, \theta_{i,i+1}^a)$. Finally, we replace a test polynomial in (5.20) with the eigenstate of the Q-operator, use the fact that $\Psi_{S,\alpha}(\boldsymbol{P})$ is a homogenous polynomial in $\boldsymbol{P}$ of degree S to find from (5.16)

$$\Psi_{S,\alpha}(\boldsymbol{P}) = \xi_{S,\alpha}\,\Phi_{S,\alpha}(\boldsymbol{Z}), \qquad P_i = Z_i - Z_{i+1}, \tag{5.21}$$

with $\xi_{S,\alpha} = i^S/c_Q = i^S/Q_{S,\alpha}(i/2)$. This relation extends the $SL(2)$ duality symmetry (2.21) to a larger $SL(2|2)$ sector in planar $\mathcal{N}=4$ SYM. As before, the duality relation (5.21) involves the polynomial $\Psi_{S,\alpha}(\boldsymbol{P})$ evaluated for the total supermomentum equal to zero, $\sum_i P_i = 0$. Nevertheless, similar to the situation in the $SL(2)$ sector, the duality relation (5.21) combined with (5.15) allows us to determine $\Psi_{S,\alpha}(\boldsymbol{P})$ for an arbitrary total momentum.

5.4. Dual superconformal symmetry

Let us show that the relation between the two sets of variables $P_i = Z_i - Z_{i+1}$ in (5.21) corresponds to the collinear limit of general dual superconformal transformations (1.1). To this end, we consider supersymmetric extension of form factors introduced in Section 2.3

$$F_{S,\alpha}(\boldsymbol{p},\boldsymbol{\eta}) = \langle 0|\mathcal{O}_{S,\alpha}(0)|\boldsymbol{p},\boldsymbol{\eta}\rangle = \overline{\Psi}_{S,\alpha}(\partial_{\boldsymbol{Z}})\langle 0|\mathbb{O}(\boldsymbol{Z})|\boldsymbol{p},\boldsymbol{\eta}\rangle\big|_{\boldsymbol{Z}=0}, \tag{5.22}$$

where in the second relation we applied (5.13). Here $|\boldsymbol{p},\boldsymbol{\eta}\rangle$ denotes the on-shell state of L superparticles (5.5) carrying the light-like momenta $p_i\bar{n}^\mu$ and the odd coordinates η_i.

The super form factor (5.22) has a well-defined expansion in powers of η's. The lowest component of the expansion, $F_{S,\alpha}(\boldsymbol{p},0)$ coincides with the scalar form factor considered in Section 2.3 whereas the remaining terms describe form factors evaluated over the states involving scalars, helicity $(+1/2)$ gaugino and helicity $(+1)$ gluons. To the leading order in the coupling, we replace each superfield in the light-ray operator (5.8) with the plane wave (5.6) to get

$$F_{S,\alpha}(\boldsymbol{p},\boldsymbol{\eta}) = L\,\mathrm{tr}\big(T^{a_1}\dots T^{a_L}\big)\overline{\Psi}_{S,\alpha}(\partial_{\boldsymbol{Z}})\prod_{i=1}^{L} \mathrm{e}^{ip_iz_i+\langle np_i\rangle(\theta_i\eta_i)}\bigg|_{\boldsymbol{Z}=0} + (\mathrm{perm}), \tag{5.23}$$

where 'perm' denote terms with permutations of (p_i, a_i, η_i) and we took into account the cyclic symmetry of the polynomial $\Psi_{S,\alpha}$. In this way, we obtain

$$F_{S,\alpha}(x;\boldsymbol{p},\boldsymbol{\eta}) = L\,\mathrm{tr}\big(T^{a_1}\dots T^{a_L}\big)\Psi_{S,\alpha}(\boldsymbol{P}) + (\mathrm{perm}), \tag{5.24}$$

where the polynomial on the right-hand side is evaluated for supermomentum $P_k = (ip_k, \langle np_k\rangle\eta_k^a)$.

According to (5.21), the form factor (5.24) is related to the coefficient function $\Phi_{S,\alpha}(\boldsymbol{Z})$ when expressed in term of the dual variables

$$p_i = z_i - z_{i+1}, \qquad \langle np_i\rangle\eta_i^a = \theta_i^a - \theta_{i+1}^a. \tag{5.25}$$

Then, the superconformal $SL(2|2)$ symmetry of $\Phi_{S,\alpha}(\boldsymbol{Z})$ is translated through the duality relation (5.21) into the dual superconformal symmetry of the form factor. Comparing (5.25) with the general expression for the dual variables (1.1) we observe that, in the collinear limit (2.18), the two sets of variables are related to each other as

$$x_i^\mu = z_i\bar{n}^\mu, \qquad \theta_i^{\alpha\,a} = \theta_i^a\frac{\lambda^\alpha(\bar{n})}{\langle n\bar{n}\rangle}, \tag{5.26}$$

where $\bar{n}^{\alpha\dot{\alpha}} = \lambda^\alpha(\bar{n})\tilde{\lambda}^{\dot{\alpha}}(\bar{n})$. Thus, the dual relation (5.21) can be interpreted as yet another manifestation of the dual superconformal symmetry in planar $\mathcal{N}=4$ SYM.

6. Conclusions and outlook

In this paper, we studied the properties of conformal operators in the $SL(2)$ sector and its supersymmetric extension. The correlation functions of these operators and their form factors with respect to asymptotic on-shell states are determined by two different polynomials which can be identified as eigenstates of the dilatation operator in the momentum and coordinate representations. As such, these polynomials respect the conventional $\mathcal{N}=4$ superconformal symmetry and are related to each other through an integral transformation which is analogous to the Fourier transformation for the discrete series representation of the $SL(2)$ group. We argued that, in virtue of integrability of the dilatation operator in planar $\mathcal{N}=4$ SYM, the two polynomials satisfy a duality relation – they are proportional to each other upon an appropriate identification of momenta and coordinates.[8]

The duality relations (1.5) and (1.8) imply that eigenstates of the $SL(2)$ dilatation operator possess the $\mathcal{N}=4$ dual superconformal symmetry. The dilation operator is believed to be integrable in planar $\mathcal{N}=4$ SYM and its eigenvalues can be found for any coupling [2]. The dual conformal symmetry allows us to extend integrability to the corresponding eigenstates. Indeed we have shown in this paper that the spectrum of one-loop eigenstates is integrable in the sense that it is uniquely fixed by the dual symmetry. What happens beyond the leading order? To verify the dual symmetry at higher loops, we can use the explicit expression for the dilatation operator in the $SL(2|2)$ sector constructed in Ref. [58]. Going through diagonalisation of this operator, it can be checked that the dual conformal symmetry survives to two loops [47]. All-loop proof of the dual symmetry remains an open problem.

We used the relation between the one-loop dilatation operator and the $SL(2)$ Heisenberg spin chain to show that the dual symmetry is generated by the Baxter operator $\mathbb{Q}(u)$, more precisely, by the leading term of its asymptotic expansion for large values of the spectral parameter u. Assuming that the dual conformal symmetry is present at higher loops, what could be the operator that generalises the Baxter Q-operator and generates the dual symmetry beyond one loop?

We employed the light-cone superfield formalism to obtain supersymmetric extension of the duality relation. In this formalism various field components in $\mathcal{N}=4$ SYM can be combined into a single light-cone superfield in such a way that the relations obtained in the $SL(2)$ sector can be easily extended to the $SL(2|2)$ sector. Trying to extend the duality relation to larger $SL(2|4)$ sector we encounter the following difficulty. The $\mathcal{N}=4$ light-cone superfield has spurious field components (see Appendix A) whose contribution to the duality relation should be carefully separated while preserving the superconformal symmetry.

Finally, it would be interesting to generalise the duality relation from the $SL(2|4)$ sector to the full $PSU(2,2|4)$ superconformal group. For this purpose, it is suggestive to relax the condition for all scalar fields in (1.2) to be located along the same light-ray and consider a more general operator like a supersymmetric light-like Wilson loop modified by the additional insertions of $\mathcal{N}=4$ superfields at the cusp points. Expansion of such operator in powers of like-like distances produces the most general local Wilson operators in $\mathcal{N}=4$ SYM. We expect that the duality relation analogous to (1.8) should hold in this case with the corresponding super-coordinates related through (1.1). This question deserves further investigation.

[8] The situation here is similar to the well-known property of harmonic oscillator Hamiltonian $H=p^2/2+x^2/2$ whose eigenstates in the coordinate and momentum representations are related to each other through Fourier transform and, at the same time, they coincide (up to an overall normalisation factor) upon identification $x \sim p$.

Acknowledgements

We are grateful to A. Belitsky, V. Braun, A. Gorsky and R. Pasechnik for a collaboration on various topics related to the present work. We would also like to thank V. Bazhanov, E. Sokatchev and A. Tseytlin for useful discussions. The work was supported in part by the French National Agency for Research (ANR) under contract StrongInt (BLANC-SIMI-4-2011), by CNRS/RFFI (PICS 6076) and CNRS-DNRF grants and by the DFG grant BR2021/5-2.

Appendix A. Light-cone superfield and superstates in $\mathcal{N}=4$ SYM

In this appendix, we review how various on-shell states and the corresponding quantum fields are described in the light-cone superspace approach in $\mathcal{N}=4$ SYM.

Asymptotic states in $\mathcal{N}=4$ SYM include six real scalars, gluons with helicity (± 1) and four gaugino with helicity $(\pm\frac{1}{2})$. These states can be combined into a single $\mathcal{N}=4$ superstate by introducing Grassman variables η^A (with $A=1,\dots,4$) [18]

$$|p,\eta\rangle_{\mathcal{N}=4}=\bigg[a^\dagger_{-1}(p)+\eta^A a^\dagger_{-1/2,A}(p)+\frac{1}{2}\eta^A\eta^B a^\dagger_{0,AB}(p) \\ +\frac{1}{3!}\epsilon_{ABCD}\eta^A\eta^B\eta^C a^{\dagger D}_{1/2}(p)+\frac{1}{4!}\epsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D a^\dagger_1(p)\bigg]|0\rangle, \tag{A.1}$$

where $a^\dagger_h(p)$ is the creation operator of a particle with helicity h and light-like momentum p^μ. The scalar and gaugino states carry the R-symmetry charge and their creation operators have the additional $SU(4)$ indices. The odd variables η^A have the helicity $(-1/2)$, so that each term in the expansion of the superstate has the same helicity charge (-1).

Each term in the expansion of the superstate (A.1) is associated with the corresponding quantum field. Similar to (A.1) we can combine various field components into a single light-cone superfield by introducing odd coordinates θ_A (with $A=1,\dots,4$) [41–43]

$$\Phi_{\mathcal{N}=4}(x,\theta_A)=(in\partial)^{-2}\bar F_n(x)+\theta_A(in\partial)^{-1}\bar\psi^A_n(x)+\frac{1}{2}\theta_A\theta_B\phi^{AB} \\ +\frac{1}{3!}\epsilon^{ABCD}\theta_A\theta_B\theta_C\psi_{nD}+\frac{1}{4!}\epsilon^{ABCD}\theta_A\theta_B\theta_C\theta_D F_n(x). \tag{A.2}$$

Here the coefficients in front of powers of θ_A involve special components of field operators which describe independent propagating degrees of freedom in $\mathcal{N}=4$ SYM theory quantised on the light-cone in the gauge $(n\cdot A(x))=0$ with $n^2=0$. They are given by gaugino and strength tensor fields projected onto (anti)holomorphic spinors defining the light-like vector $n^{\alpha\dot\alpha}=\lambda^\alpha(n)\tilde\lambda^{\dot\alpha}(n)$

$$\psi^A_n(x)=\lambda^\alpha(n)\,\psi^A_\alpha(x), \qquad \bar\psi_{nA}(x)=\tilde\lambda_{\dot\alpha}(n)\bar\psi^{\dot\alpha}_A(x), \\ F_n(x)=\lambda^\alpha(n)\lambda^\beta(n)F_{\alpha\beta}(x), \qquad \bar F_n(x)=\tilde\lambda^{\dot\alpha}(n)\tilde\lambda^{\dot\beta}(n)\bar F_{\dot\alpha\dot\beta}(x), \tag{A.3}$$

with $F_{\alpha\beta}$ and $\bar F_{\dot\alpha\dot\beta}$ being (anti) self-dual part of the strength tensor, $F_{\alpha\dot\alpha,\beta\dot\beta}=F_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}+\bar F_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}$. The remaining field components can be expressed in terms of those in (A.3) through the equations of motion.

The operator $\Phi(x,\theta_A)$ creates out of vacuum the $\mathcal{N}=4$ multiplet of single particle on-shell states (scalars, gaugino and gluons). Evaluating the matrix element $\langle 0|\Phi(x,\theta_A)|p,\eta\rangle$ with respect to the superstate (A.1) we find that each term of the expansion (A.2) produces a plane wave with the quantum numbers of the corresponding state

$$\langle 0|\Phi(x,\theta)|p,\eta\rangle = \mathrm{e}^{ipx}\left(\frac{1}{\langle np\rangle^2}+\frac{(\theta\cdot\eta)}{\langle np\rangle}+\frac{1}{2}(\theta\cdot\eta)^2+\frac{1}{3!}\langle np\rangle(\theta\cdot\eta)^3+\frac{1}{4!}\langle np\rangle^2(\theta\cdot\eta)^4\right), \tag{A.4}$$

where $(\theta\cdot\eta)=\theta_A\eta^A$ and $\langle np\rangle=\lambda^\alpha(n)\lambda_\alpha(p)$. The same relation can be rewritten in a compact form as

$$\langle 0|\Phi(x,\theta)|p,\eta\rangle = \langle np\rangle^{-2}\mathrm{e}^{ipx+\langle np\rangle(\theta\cdot\eta)}. \tag{A.5}$$

It allows us to identify the supermomentum conjugated to the odd coordinate θ_A to be $\langle np\rangle\eta^A$. Notice that the additional factor of $\langle np\rangle^{-2}$ on the right-hand side of (A.5) is needed to match helicity (-1) of the superstate (A.1).

The light-cone superfield (A.2) belongs to the representation of the $SL(2|4)$ superconformal group of spin $(-1/2)$ (see [43]). This representation is reducible due to the following unusual feature of (A.2). Notice that the first two terms on the right-hand side of (A.2) contain inverse derivatives and, therefore, the corresponding field operators are nonlocal. Expanding the superfield (A.2) around $x=0$ we can identify the nonlocal spurious components to be $\partial_+^{-2}\bar{F}_n(0)$, $\partial_+^{-1}\bar{F}_n(0)$ and $\partial_+^{-1}\bar{\psi}_n^A(0)$ (with $\partial_+\equiv(n\cdot\partial)$) and verify that they define an irreducible $SL(2|4)$ component of the $\mathcal{N}=4$ superfield. The presence of nonlocal spurious components inside $\Phi(x,\theta)$ complicates the construction of the dilatation operator in the $SL(2|4)$ sector [43].

We can avoid the problem with reducibility of $\mathcal{N}=4$ superfield by using the formulation of the same gauge theory in terms of $\mathcal{N}=2$ light-cone superfield. The latter can be identified as the coefficient in front of $\theta_3\theta_4$ in the expansion (A.2)

$$\Phi_{\mathcal{N}=2}(x,\theta_a)=\partial_{\theta_4}\partial_{\theta_3}\Phi_{\mathcal{N}=4}(x,\theta_A). \tag{A.6}$$

This superfield depends on two odd coordinates θ_a (with $a=1,2$) and its explicit expression is given by (5.1) with $Z=\phi^{34}$. It is straightforward to verify that the spurious components do not appear in $\Phi_{\mathcal{N}=2}(x,\theta_a)$ and, as a consequence, it belongs to the irreducible representation of the $SL(2|2)$ group of spin $1/2$. In the similar manner, we can project the $\mathcal{N}=4$ on-shell superstate (A.1) on its $\mathcal{N}=4$ component by retaining terms proportional to $\eta^3\eta^4$

$$\left|p,\eta^a\right\rangle_{\mathcal{N}=2}=\partial_{\eta^4}\partial_{\eta^3}\left|p,\eta^A\right\rangle_{\mathcal{N}=4} \tag{A.7}$$

leading to (5.5).

References

[1] J.M. Drummond, J. Henn, G.P. Korchemsky, E. Sokatchev, Nucl. Phys. B 828 (2010) 317, arXiv:0807.1095 [hep-th].
[2] N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond, L. Freyhult, N. Gromov, R.A. Janik, et al., Lett. Math. Phys. 99 (2012) 3, arXiv:1012.3982 [hep-th].
[3] L.F. Alday, J.M. Maldacena, JHEP 0706 (2007) 064, arXiv:0705.0303 [hep-th].
[4] J.M. Drummond, G.P. Korchemsky, E. Sokatchev, Nucl. Phys. B 795 (2008) 385, arXiv:0707.0243 [hep-th].
[5] A. Brandhuber, P. Heslop, G. Travaglini, Nucl. Phys. B 794 (2008) 231, arXiv:0707.1153 [hep-th].
[6] L.J. Mason, D. Skinner, JHEP 1012 (2010) 018, arXiv:1009.2225 [hep-th].
[7] S. Caron-Huot, JHEP 1107 (2011) 058, arXiv:1010.1167 [hep-th].
[8] A.V. Belitsky, G.P. Korchemsky, E. Sokatchev, Nucl. Phys. B 855 (2012) 333, arXiv:1103.3008 [hep-th].
[9] B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev, Nucl. Phys. B 869 (2013) 329, arXiv:1103.3714 [hep-th].
[10] B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev, Nucl. Phys. B 869 (2013) 378, arXiv:1103.4353 [hep-th].
[11] T. Adamo, M. Bullimore, L. Mason, D. Skinner, JHEP 1108 (2011) 076, arXiv:1103.4119 [hep-th].
[12] D.J. Broadhurst, Phys. Lett. B 307 (1993) 132.
[13] J.M. Drummond, J. Henn, V.A. Smirnov, E. Sokatchev, JHEP 0701 (2007) 064, arXiv:hep-th/0607160.

[14] L.N. Lipatov, Nucl. Phys. B 548 (1999) 328, arXiv:hep-ph/9812336.
[15] C. Gomez, J. Gunnesson, A.S. Vera, Phys. Lett. B 690 (2010) 78, arXiv:0908.2568 [hep-th].
[16] A. Prygarin, Phys. Rev. C 83 (2011) 055206, arXiv:0911.5279 [hep-ph].
[17] V.M. Braun, S.E. Derkachov, G.P. Korchemsky, A.N. Manashov, Nucl. Phys. B 553 (1999) 355, arXiv:hep-ph/9902375.
[18] V.P. Nair, Phys. Lett. B 214 (1988) 215.
[19] J.M. Drummond, J. Henn, G.P. Korchemsky, E. Sokatchev, Nucl. Phys. B 826 (2010) 337, arXiv:0712.1223 [hep-th].
[20] G.P. Korchemsky, E. Sokatchev, Nucl. Phys. B 832 (2010) 1, arXiv:0906.1737 [hep-th].
[21] S. Caron-Huot, S. He, JHEP 1207 (2012) 174, arXiv:1112.1060 [hep-th].
[22] M. Bullimore, D. Skinner, arXiv:1112.1056 [hep-th].
[23] J.M. Drummond, J.M. Henn, J. Plefka, JHEP 0905 (2009) 046, arXiv:0902.2987 [hep-th].
[24] D. Chicherin, S. Derkachov, R. Kirschner, Nucl. Phys. B 881 (2014) 467, arXiv:1309.5748 [hep-th].
[25] B.I. Zwiebel, J. Phys. A 45 (2012) 115401, arXiv:1111.0083 [hep-th].
[26] R. Kallosh, A.A. Tseytlin, JHEP 9810 (1998) 016, arXiv:hep-th/9808088.
[27] N. Berkovits, J. Maldacena, JHEP 0809 (2008) 062, arXiv:0807.3196 [hep-th].
[28] N. Beisert, R. Ricci, A.A. Tseytlin, M. Wolf, Phys. Rev. D 78 (2008) 126004, arXiv:0807.3228 [hep-th].
[29] G. Mandal, N.V. Suryanarayana, S.R. Wadia, Phys. Lett. B 543 (2002) 81, arXiv:hep-th/0206103.
[30] I. Bena, J. Polchinski, R. Roiban, Phys. Rev. D 69 (2004) 046002, arXiv:hep-th/0305116.
[31] G.P. Lepage, S.J. Brodsky, Phys. Rev. D 22 (1980) 2157.
[32] V.L. Chernyak, A.R. Zhitnitsky, Phys. Rep. 112 (1984) 173.
[33] V.M. Braun, G.P. Korchemsky, D. Mueller, Prog. Part. Nucl. Phys. 51 (2003) 311, arXiv:hep-ph/0306057.
[34] R.J. Baxter, Ann. Phys. 70 (1972) 193;
R.J. Baxter, Ann. Phys. 281 (2000) 187.
[35] V.V. Bazhanov, Y.G. Stroganov, J. Stat. Phys. 59 (1990) 799.
[36] M. Gaudin, V. Pasquier, J. Phys. A 25 (1992) 5243.
[37] V.V. Bazhanov, S.L. Lukyanov, A.B. Zamolodchikov, Commun. Math. Phys. 190 (1997) 247, arXiv:hep-th/9604044.
[38] S.E. Derkachov, J. Phys. A 32 (1999) 5299, arXiv:solv-int/9902015.
[39] A. Brandhuber, B. Spence, G. Travaglini, G. Yang, JHEP 1101 (2011) 134, arXiv:1011.1899 [hep-th].
[40] L.F. Alday, J. Maldacena, JHEP 0711 (2007) 068, arXiv:0710.1060 [hep-th].
[41] S. Mandelstam, Nucl. Phys. B 213 (1983) 149.
[42] L. Brink, O. Lindgren, B.E.W. Nilsson, Nucl. Phys. B 212 (1983) 401.
[43] A.V. Belitsky, S.E. Derkachov, G.P. Korchemsky, A.N. Manashov, Nucl. Phys. B 708 (2005) 115, arXiv:hep-th/0409120.
[44] A.V. Belitsky, S.E. Derkachov, G.P. Korchemsky, A.N. Manashov, Nucl. Phys. B 722 (2005) 191, arXiv:hep-th/0503137.
[45] A.V. Belitsky, S.E. Derkachov, G.P. Korchemsky, A.N. Manashov, J. Stat. Mech. 0701 (2007) P01005, arXiv:hep-th/0610332.
[46] V.V. Bazhanov, Z. Tsuboi, Nucl. Phys. B 805 (2008) 451, arXiv:0805.4274 [hep-th].
[47] G.P. Korchemsky, unpublished.
[48] S.E. Derkachov, S.K. Kehrein, A.N. Manashov, Nucl. Phys. B 493 (1997) 660.
[49] Y.M. Makeenko, Sov. J. Nucl. Phys. 33 (1981) 440, Yad. Fiz. 33 (1981) 842.
[50] T. Ohrndorf, Nucl. Phys. B 198 (1982) 26.
[51] I.I. Balitsky, V.M. Braun, Nucl. Phys. B 311 (1989) 541.
[52] V.M. Braun, S.E. Derkachov, A.N. Manashov, Phys. Rev. Lett. 81 (1998) 2020, arXiv:hep-ph/9805225.
[53] A.V. Belitsky, Phys. Lett. B 453 (1999) 59, arXiv:hep-ph/9902361.
[54] L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan, Theor. Math. Phys. 40 (1980) 688, Teor. Mat. Fiz. 40 (1979) 194.
[55] P.P. Kulish, E.K. Sklyanin, Lect. Notes Phys. 151 (1982) 61.
[56] E.K. Sklyanin, Quantum inverse scattering method selected topics, in: Mo-Lin Ge (Ed.), Quantum Group and Quantum Integrable Systems, in: Nankai Lectures in Mathematical Physics, World Scientific, Singapore, 1992, pp. 63–97, arXiv:hep-th/9211111.
[57] L.D. Faddeev, How Algebraic Bethe Anstz works for integrable model, in: A. Connes, K. Kawedzki, J. Zinn-Justin (Eds.), Quantum Symmetries/Symetries Qantiques, in: Proc. Les-Houches Summer School, vol. LXIV, North-Holland, 1998, pp. 149–211, arXiv:hep-th/9605187.
[58] B.I. Zwiebel, JHEP 0602 (2006) 055, arXiv:hep-th/0511109.