

Unitarity violation in noninteger dimensional Gross-Neveu-Yukawa model

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We construct an explicit example of unitarity violation in fermionic quantum field theories in noninteger dimensions. We study the two-point correlation function of four-fermion operators. We compute the one-loop anomalous dimensions of these operators in the Gross-Neveu-Yukawa model. We find that at one-loop order, the four-fermion operators split into three classes with one class having negative norms. This implies that the theory violates unitarity, following the definition in Ref. [1].

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I. INTRODUCTION

Conformal field theories (CFTs) have always been an area of active research due to their rich mathematical structure and physical applications. In unitary theories conformal symmetry imposes severe constraints on the spectrum of operator dimensions. It is believed that these dimensions can be determined with the help of the conformal bootstrap technique [2,3]. This technique has proved to be extremely useful for solving two-dimensional CFTs. The effective numerical algorithms for solving the bootstrap equations for higher-dimensional CFTs have been proposed in Ref. [4] (see also Refs. [5–7], [8–13], and [14–18] for more details and recent developments in $d = 3$, $d = 4$, and $d = 5$ dimensions, respectively). One of the advantages of this approach is that it allows one to obtain operator dimensions directly in various integer dimensions.

The standard technique for the calculation of the operator dimensions, the so-called ϵ -expansion [19,20], is based on calculation of the scaling dimensions in $d = 4 - 2\epsilon$ dimensional theory and interpolation of the relevant critical indices to the physical dimension. The critical indices for many CFTs are known with high precision. One of the recent achievements is the calculation of the six-loop β function in the φ^4 theory [21]. In order to get a better understanding of the new conformal bootstrap technique it was quite natural to apply it to theories in noninteger dimensions, $d = 4 - 2\epsilon$ (see Refs. [22,23]). At the same time, one of the assumptions which most of the conformal

bootstrap relies on is the unitarity of the theory. One can hardly expect that this assumption—unitarity—will be true for theories in noninteger dimensions. This question was raised in Refs. [1,24], where unitarity violation in φ^4 theory was demonstrated by constructing states (operators) with negative norm. The first “negative norm” operator in φ^4 theory has a rather high scaling dimension ($\Delta = 23$), and it is expected that unitarity breaking effects will appear only in high orders of ϵ expansion. Negative norm operators necessarily have to be evanescent operators, i.e., operators that are vanishing in integer dimensions. In scalar theories the building blocks for the operators are fields, and their derivatives and therefore evanescent operators are must have a high dimension. The situation is quite different in theories with fermions where there are evanescent (scalar) operators of canonical dimension six [25].

The aim of this article is to demonstrate the existence of the negative norm states in the $d = 4 - 2\epsilon$ dimensional Gross—Neveu—Yukawa (GNY) model [26]. It was argued in [1] that unitarity implies the positiveness of the coefficient C in the correlator

$$\langle \mathcal{O}^\dagger(x)\mathcal{O}(0) \rangle = C/x^{2\Delta}, \quad (1)$$

where \mathcal{O} is a conformal operator with scaling dimension Δ . In an integer dimensional CFT, violation of this condition indicates the presence of negative norm states in the theory [1]. We consider the renormalization of an infinite set of scalar four-fermion operators in $d = 4 - 2\epsilon$ dimensions and show that the positiveness condition is broken for infinitely many operators. Since the canonical dimension of these operators is not large, $\Delta^{\text{can}} = 6$, one can wonder about the effect of negative norm operators to the conformal bootstrap technique.

The article is organized as follows: In Sec. II we discuss the two-point correlation function of scalar four-fermion operators in free theory. We find that the theory contains

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evanescent operators which could generate negative norm states.

In order to continue our discussion, we then compute in Sec. III the anomalous dimension of the physical and evanescent operators at one-loop order in the GNY model. It turns out that all the evanescent operators split into two classes of definite anomalous dimension. We show that the negative norm states are generated by one of these two classes, depending on the number of fermion flavors of the theory.

II. FOUR-FERMION CORRELATION FUNCTION IN NONINTEGER DIMENSIONS

The GNY model describes an interacting fermion-boson system with the Lagrangian given by the following expression [26,27]:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \bar{\Psi}_i\partial\Psi_i + g_1\sigma\bar{\Psi}_i\Psi_i + \frac{1}{24}g_2\sigma^4, \quad (2)$$

where the index $i = 1, \dots, n_f$ enumerates different fermion flavors and σ is a scalar field.

The model has an infrared stable fixed point in $d = 4 - 2\epsilon$ dimensions [28]. At one loop the critical couplings take the form

$$u_* = \frac{(g_1^*)^2}{(4\pi)^2} = \frac{\epsilon}{N_f + 6},$$

$$v_* = \frac{(g_2^*)^2}{(4\pi)^2} = \frac{6 - N_f + \sqrt{N_f^2 + 132N_f + 36}}{6(N_f + 6)}\epsilon, \quad (3)$$

where $N_f \equiv n_f \text{tr}(\mathbb{1}_d)$. The basic critical indices are now known with four-loop accuracy and can be found in Ref. [29].

Let us consider an infinite system of four-fermion local operators in $d = 4 - 2\epsilon$ dimensions

$$\mathcal{O}^{(m)} = \frac{1}{m!}(\bar{\Psi}\Gamma_\mu^{(m)}\Psi)(\bar{\Psi}\Gamma_\mu^{(m)}\Psi). \quad (4)$$

A summation over flavor index inside each bracket is tacitly assumed. The notation $\Gamma_\mu^{(m)}$ stands for an antisymmetric product of m γ -matrices

$$\Gamma_\mu^{(m)} = \Gamma_{\mu_1 \dots \mu_m} \equiv \frac{1}{m!} \sum_{s \in S_m} (-1)^P \gamma_{\mu_{s_1}} \dots \gamma_{\mu_{s_m}}. \quad (5)$$

The sum goes over all permutations and P is the parity of a permutation.

Before taking a closer look at correlators of the operators (4), let us state a few things about the $\Gamma_{(m)}$ matrices. The Dirac γ -matrices satisfy the basic anticommutation relation in d -dimensional space

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_d, \quad g^{\mu\nu}g_{\mu\nu} = d, \quad (6)$$

where $g^{\mu\nu}$ is the metric tensor. In integer dimensions there are only d distinct gamma matrices $\gamma^0, \dots, \gamma^{d-1}$. This restricts the maximum number of different antisymmetrized matrices $\Gamma_{(m)}$.¹ Namely, $0 \leq m \leq d(\leq d-1)$ for even (odd) dimensional spaces.

In noninteger dimensions, however, the situation is different. There exists an infinite number of γ -matrices, and therefore it is possible to construct infinitely many nonvanishing and distinct $\Gamma^{(m)}$. As a result, the parameter m in Eq. (4) takes any positive integer values. However, in $d = 4 - 2\epsilon$ dimensional space the operators (4) with $m \geq 5$ have to vanish in the limit $\epsilon \rightarrow 0$, and therefore they are called evanescent operators.

The renormalized operators $[\mathcal{O}_m]$ satisfy the renormalization group equation

$$(M\partial_M + \beta_u\partial_u + \beta_v\partial_v)[\mathcal{O}_m] = -\gamma_{\mathcal{O}}^{m,n}(u,v)[\mathcal{O}_n], \quad (7)$$

where M is the renormalization scale, $\beta_{u,v}$ are the corresponding β -functions, $\beta_u = \frac{du}{d\ln M}$, $\beta_v = \frac{dv}{d\ln M}$, and $\gamma_{\mathcal{O}}^{m,n}$ is the anomalous dimension matrix. The structure of the operator mixing of the four-fermion operators was considered in great detail in [25,30,31].

At the critical point $\beta_u(u_*, v_*) = \beta_v(u_*, v_*) = 0$, the problem of constructing operators with autonomous scale dependence is equivalent to the eigenproblem for the matrix $\gamma_{\mathcal{O}}^{m,n}$. This means that if c_γ^m is the left eigenvector of the anomalous dimension matrix

$$c_\gamma^m \gamma_{\mathcal{O}}^{m,n}(u_*, v_*) = \gamma c_\gamma^n, \quad (8)$$

then the operator $\mathcal{O}_\gamma = \sum_m c_\gamma^m [\mathcal{O}_m]$ has an autonomous scale dependence

$$(M\partial_M + \gamma)\mathcal{O}_\gamma = 0. \quad (9)$$

The operator \mathcal{O}_γ transforms in a proper way under conformal transformations, and according to a general theory the correlators of operators with different scaling dimensions ($\Delta_\gamma = 6 - 4\epsilon + \gamma$) vanish, i.e.,

$$\langle \mathcal{O}_\gamma^\dagger(x) \mathcal{O}_\gamma(0) \rangle = \delta_{\gamma\gamma'} C_\gamma / x^{2\Delta_\gamma}. \quad (10)$$

In an unitary theory the coefficients C_γ have to be positive [1]. We calculate the one-loop anomalous dimension matrix $\gamma_{\mathcal{O}}^{m,n}$ in the next section, while in the rest of this section we study the correlator (10) in more detail.

¹Note that in even dimensions, $\Gamma_{(m>d)}$ vanishes because of the antisymmetrization of gamma matrices. In odd dimension d , $\Gamma_{(d)}$ is removed from the independent basis since $\Gamma_{(d)} \propto \Gamma_{(0)}$.

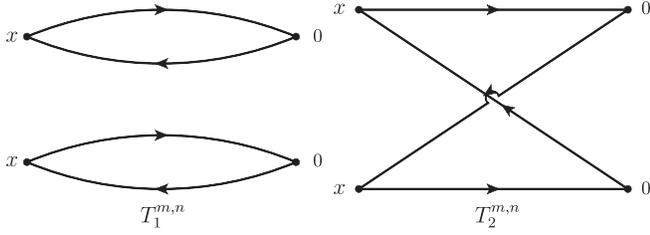


FIG. 1. Feynman diagrams of leading order.

Let us write the correlator (10) in the form

$$\langle \mathcal{O}_\gamma^\dagger(x) \mathcal{O}_\gamma(0) \rangle = \sum_{m,n} (c_\gamma^m)^\dagger C^{m,n}(x) c_\gamma^n, \quad (11)$$

where $C^{m,n}$ is the correlator of the basic operators defined in Eq. (4) (note that $(\mathcal{O}_n)^\dagger = \mathcal{O}_n$ and $d = 4 - 2\epsilon$)

$$\begin{aligned} C^{m,n}(x) &= \langle \mathcal{O}^{(m)}(x) \mathcal{O}^{(n)}(0) \rangle \\ &= \frac{\mathcal{C}^{m,n}(d)}{|x^2|^{2d-2}} (1 + \mathcal{O}(u_*, v_*)). \end{aligned} \quad (12)$$

In $d = 4 - 2\epsilon$ dimensions, it is expected that for the physical operators ($m, n \leq 4$), $C^{m,n}(d) \sim \mathcal{O}(1)$ and for one of the indices $m, n \geq 5$, $C^{m,n}(d) \sim \mathcal{O}(\epsilon)$. Thus one gets the following expression for the constant C_γ at the leading order

$$C_\gamma = \sum_{n,m} (c_\gamma^m)^\dagger C^{m,n}(x) c_\gamma^n \equiv (c_\gamma, C c_\gamma). \quad (13)$$

At leading order only the two Feynman diagrams shown in Fig. 1 contribute to $C^{m,n}(d)$. Using the expression for the fermion propagator in Euclidean space

$$\langle \Psi(x) \bar{\Psi}(0) \rangle = A \frac{\not{x}}{|x^2|^{d/2}}, \quad A = \frac{\Gamma(d/2)}{2\pi^{d/2}}, \quad (14)$$

we find

$$C^{m,n}(x) = \Delta^{m,n} A^4 N_f |x^2|^{2-2d}, \quad (15)$$

where

$$\begin{aligned} \Delta^{m,n} &= N_f T_1^{m,n} + T_2^{m,n}, \\ T_1^{m,n} &= \frac{\delta_{m,n}}{x^4 \text{tr}^2(\mathbb{1}_d) (m!)^2} [\text{tr}(\Gamma_\mu^{(m)} \not{x} \Gamma_\nu^{(m)} \not{x})]^2, \\ T_2^{m,n} &= \frac{-1}{x^4 \text{tr}(\mathbb{1}_d) m! n!} \text{tr}(\not{x} \Gamma_\mu^{(m)} \not{x} \Gamma_\nu^{(n)} \not{x} \Gamma_\mu^{(m)} \not{x} \Gamma_\nu^{(n)}). \end{aligned} \quad (16)$$

The summation between upper and lower indices is here implied. The calculation of the traces in (16) is discussed in

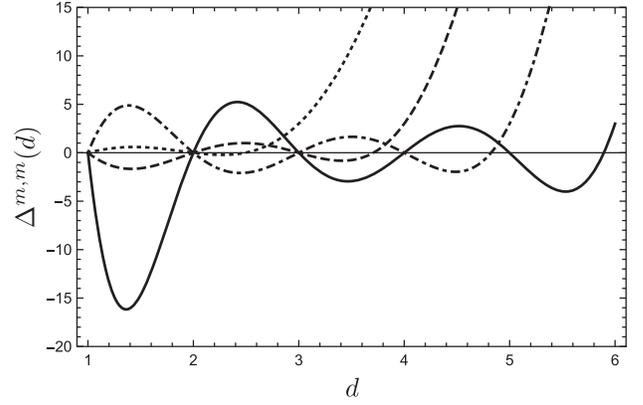


FIG. 2. $\Delta^{m,m}(d)$, with $N_f = 4$ as function of space dimension d for $m = 3, 4, 5, 6$ presented by dotted, dashed, dot-dashed, and solid curves, respectively.

Appendices B and C (see Ref. [32] for a general treatment of contracting infinitely many antisymmetrized gamma matrices); here we present the final result²

$$T_1^{m,n} = \frac{\Gamma(d+1)}{m! \Gamma(d-m+1)} \delta^{m,n}, \quad (17)$$

$$T_2^{m,n} = -\frac{1}{2} i^{m(m+1)+n(n+1)} a^{m,n}. \quad (18)$$

Note that $T_1^{m,n}$ and $T_2^{m,n}$ are x -independent. The coefficients $a^{m,n}$ are encoded by the generating function

$$\begin{aligned} F(x, y) &= \sum_{m,n=0}^{\infty} a^{m,n} x^m y^n \\ &= (1 - x + y + xy)^d + (1 + x - y + xy)^d \\ &\quad - (1 + x + y - xy)^d + (1 - x - y - xy)^d. \end{aligned} \quad (19)$$

We point out that $T_1^{m,n}$ and $T_2^{m,n}$ are symmetric regarding the exchange of $m \leftrightarrow n$ and in contrast to the first diagram, which is proportional to $\delta^{m,n}$, $a^{m,n}$ contributes to cases of both $m = n$ and $m \neq n$. Both diagrams are polynomials in the spacetime dimension d and can become negative valued in noninteger dimensions. Therefore the coefficient $\Delta^{m,n}$ is negative valued in some regions (see Fig. 2). A detailed analysis of $a^{m,n}$ shows that $|T_2^{m,m}| \gg |T_1^{m,m}|$ for $m \gg 1$ and therefore gives the main contribution at large m for $\Delta^{m,m}$. The fact that $\Delta^{m,m}$ ($\sim C^{m,m}$) can become negative valued suggests the possibility of having conformal operators with negative norms. For this reason we compute the one-loop anomalous dimension of the operators $\mathcal{O}^{(m)}$ in the next section in order to classify them by their one-loop anomalous dimensions.

²A nontrivial relation in integer dimensions $\text{tr}^2(\mathbb{1}) [T_1 T_2^{-1}]^2 = \mathbb{1}$ serves as an additional check for our results of $T_1^{m,n}$ and $T_2^{m,n}$. This relation is obtained from considering the Fierz identities.

III. ANOMALOUS DIMENSIONS AND UNITARITY IN THE GNY MODEL

So far our calculations are rather general and can be applied to any fermionic theory in noninteger dimensions. In order to continue our study of norm states in a conformal theory, according to Eq. (10), it is necessary to find eigenstates with definite anomalous dimensions and study correlation functions between them. It is therefore more instructive to consider an explicit example, the GNY model, and compute the one-loop anomalous dimensions of the operators $\mathcal{O}^{(m)}$ defined in Eq. (4) in this model. The Feynman diagrams needed for this calculation are given in Figs. 3 and 4. Note that diagrams in Fig. 4 contribute only to the anomalous dimension of physical operators.

Then it is straightforward to compute these one-loop diagrams and obtain the anomalous dimension matrix $\gamma_{\mathcal{O}}^{m,n}$. Interestingly, we find that the anomalous dimension matrix has a simple block diagonal form (the calculation details can be found in Appendix D),

$$\gamma_{\mathcal{O}}^{m,n} = 2u_* \text{diag}(\gamma_0, \gamma_1, \gamma_2, \dots)^{m,n}, \quad (20)$$

where γ_0 is a 5×5 anomalous dimension matrix

$$\gamma_0 = \begin{pmatrix} N_f + 2 & 0 & 0 & 0 & 0 \\ -4 & 0 & -2 & 0 & 0 \\ 6 & -3 & 1 & 0 & 0 \\ 4 & 0 & 0 & 2 & -4 \\ -1 & 0 & 0 & -1 & -1 \end{pmatrix} \quad (21)$$

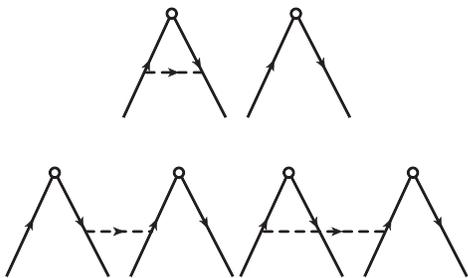


FIG. 3. Feynman diagrams for calculating anomalous dimensions of evanescent operators.

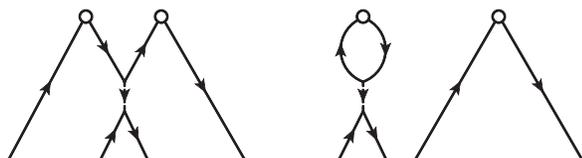


FIG. 4. Additional Feynman diagrams contributing to the anomalous dimensions of physical operators.

involving only physical operators, while $\gamma_{k \geq 1}$ are 2×2 matrices

$$\gamma_k = \begin{pmatrix} 2k + 2 & -2k - 4 \\ 2k - 1 & -2k - 1 \end{pmatrix} \quad (22)$$

describing the mixing between evanescent operators $\mathcal{O}^{(2k+3)}$ and $\mathcal{O}^{(2k+4)}$ at one-loop order.

It is clear from the explicit expression of the anomalous dimension matrix that the physical and evanescent operators decouple at one-loop order. We can therefore study them separately and find the conformal basis in each case.

Let us write the physical operators in conformal basis as $\tilde{\mathcal{O}}$. Then

$$\begin{pmatrix} \tilde{\mathcal{O}}_0^{(0)} \\ \tilde{\mathcal{O}}_+^{(1)} \\ \tilde{\mathcal{O}}_-^{(1)} \\ \tilde{\mathcal{O}}_+^{(2)} \\ \tilde{\mathcal{O}}_-^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{10}{1-N_f} & -1 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ \frac{5}{N_f-1} & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1/4 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}^{(0)} \\ \mathcal{O}^{(1)} \\ \mathcal{O}^{(2)} \\ \mathcal{O}^{(3)} \\ \mathcal{O}^{(4)} \end{pmatrix} \quad (23)$$

where the operators $\tilde{\mathcal{O}}_0^{(0)}$, $\tilde{\mathcal{O}}_+^{(k)}$, and $\tilde{\mathcal{O}}_-^{(k)}$ have anomalous dimension $\gamma_0 = 2(N_f + 2)u_*$, $\gamma_+ = 6u_*$, and $\gamma_- = -4u_*$ at one-loop order, respectively. Note that we use the bold font letters for anomalous dimension matrices and common ones for the eigenvalues.

The conformal basis for evanescent operators, denoted as $\tilde{\mathcal{O}}_{\pm}^{(k)}$, is

$$\begin{pmatrix} \tilde{\mathcal{O}}_+^{(k)} \\ \tilde{\mathcal{O}}_-^{(k)} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ \frac{1-2k}{2(k+2)} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}^{(2k+3)} \\ \mathcal{O}^{(2k+4)} \end{pmatrix}, \quad (24)$$

with $k \geq 1$ and $\tilde{\mathcal{O}}_{\pm}^{(k)}$ having anomalous dimension $6u_*$ and $-4u_*$, respectively. These results allow us to classify the operators by their one-loop anomalous dimension. More explicitly, the evanescent operators form two and the physical operators form three classes (two for $N_f = 1$). At this point one should mention that the two-loop anomalous dimensions of the operators $\mathcal{O}^{(m)}$ probably allow us to make further classifications. The anomalous dimensions of the different operators are collected in Table I.

In order to find the negative norm states of the theory, we have to consider correlation functions between operators of the same anomalous dimension. According to Eq. (10), this corresponds to the study of the coefficient C_γ . We point out

TABLE I. Anomalous dimensions (AD) of the different physical and evanescent operators.

	$\tilde{\mathcal{O}}_0^{(0)}$	$\tilde{\mathcal{O}}_+^{(k)}$	$\tilde{\mathcal{O}}_-^{(k)}$	$\tilde{\mathcal{O}}_+^{(k)}$	$\tilde{\mathcal{O}}_-^{(k)}$
AD	γ_0	γ_+	γ_-	γ_+	γ_-

that at one-loop accuracy, the orthogonality condition in Eq. (10) is realized by the expressions

$$\begin{aligned} \langle (\tilde{\mathcal{O}}_{0,\pm}^{(k_1)}(x))^\dagger \tilde{\mathcal{O}}_{\mp}^{(k_2)}(0) \rangle &= \mathcal{O}(\epsilon), \\ \langle (\tilde{\mathcal{O}}_{0,\pm}^{(k_1)}(x))^\dagger \tilde{\mathcal{O}}_{\mp}^{(k_2)}(0) \rangle &= \mathcal{O}(\epsilon), \\ \langle (\tilde{\mathcal{O}}_{\pm}^{(k_1)}(x))^\dagger \tilde{\mathcal{O}}_{\mp}^{(k_2)}(0) \rangle &= \mathcal{O}(\epsilon^2), \end{aligned} \quad (25)$$

which is exactly what we find from Eqs. (15), (17), and (18).

With the orthogonality condition checked at the one-loop order, let us now focus on the evanescent operators in the conformal basis. We write the correlator as

$$\begin{aligned} \langle (\tilde{\mathcal{O}}_{\pm}^{(k_1)}(x))^\dagger \tilde{\mathcal{O}}_{\pm}^{(k_2)}(0) \rangle &= A^4 \frac{N_f}{|x^2|^{2d-2}} \bar{\Delta}_{\pm}^{k_1, k_2} + \mathcal{O}(\epsilon^2), \\ \bar{\Delta}_{\pm}^{k_1, k_2} &= N_f T_{1\pm}^{k_1, k_2} + T_{2\pm}^{k_1, k_2}. \end{aligned} \quad (26)$$

Here both $T_{1\pm}$ and $T_{2\pm}$ are proportional to ϵ and correspond to the first and second diagram in Fig. 1, respectively. A is defined in Eq. (14).

The matrices $T_{1\pm}$ are diagonal matrices. It is easy to see that all matrix elements of $T_{1-}(T_{1+})$ are positive (negative) numbers. This implies that $(f, T_{1-}f) > 0$ and $(f, T_{1+}f) < 0$ for arbitrary nonzero vectors f , i.e., $T_{1-}(T_{1+})$ is a positive (negative) definite matrix. The situation with the matrices $T_{2\pm}$ is a bit more complicated since they are not diagonal. But we checked numerically³ to confirm that all truncated matrices $T_{2\pm}^N = (T_{2\pm})^{n,m}$ with $n, m \leq N$ are positive definite (T_{2+}) and negative definite (T_{2-}) matrices for $N \leq 80$. The definiteness of $T_{1,2\pm}$ implies

- (1) In the large N_f limit, the matrices $\bar{\Delta}_{\pm} \sim T_{1\pm}(1 + \mathcal{O}(1/N_f))$ and therefore $\bar{\Delta}_+$ is negative definite, and $\bar{\Delta}_-$ is positive definite.
- (2) On the contrary, for small values of N_f ($N_f \lesssim 5$), $|T_{2\pm}|$ dominates over $|T_{1\pm}|$ and $\bar{\Delta}_+$ is positive definite, and $\bar{\Delta}_-$ is negative definite.

As we have seen, the one-loop corrections are not enough to resolve the operator mixing since infinitely many operators have the same anomalous dimension at one loop. Nevertheless, it allows one to argue that a general conformal operator with anomalous dimension $\gamma_+ = 6u_* + \mathcal{O}(\epsilon^2)$ has the form⁴

³It is possible to show that all the leading principal minors of T_{2+} are positive, while the k -th order leading minor of T_{2-} is negative (positive) for odd (even) k .

⁴For the sake of clarity one should mention that the mixing with physical operators is here neglected. But it can be shown that the effect of the physical operators is at order $\mathcal{O}(\epsilon^2)$ by considering the orthogonality condition of the conformal operators. More specifically, one finds that the coefficients in front of the physical operators are at $\mathcal{O}(\epsilon)$ order.

$$\mathcal{O}_{\gamma_+} = \sum c_i^+ \tilde{\mathcal{O}}_+^{(i)} + \sum c_j^- \tilde{\mathcal{O}}_-^{(j)}, \quad (27)$$

where the coefficients $c_i^+ \sim \mathcal{O}(1)$, while $c_k^- \sim \mathcal{O}(\epsilon)$. One can easily see that the coefficient C_{γ_+} , corresponding to the correlator of such operators, is given by

$$C_{\gamma_+} = A^4 \frac{N_f}{|x^2|^{2d-2}} (c^+, \bar{\Delta}_+ c^+) + \mathcal{O}(\epsilon^2). \quad (28)$$

Similarly, for a conformal operator with anomalous dimension $\gamma_- = -4u_* + \mathcal{O}(\epsilon^2)$, one gets

$$C_{\gamma_-} = A^4 \frac{N_f}{|x^2|^{2d-2}} (c^-, \bar{\Delta}_- c^-) + \mathcal{O}(\epsilon^2). \quad (29)$$

As we have shown, the coefficients C_{γ_+} and C_{γ_-} have opposite signs at order $\mathcal{O}(\epsilon)^5$ for either small N_f or $N_f \rightarrow \infty$. Therefore, one class of the operators inevitably generates the negative norm states of the theory, according to the criteria given in Ref. [1]. In particular, at the lower bound of $N_f = 1$, all negative norm states are generated by $\tilde{\mathcal{O}}_-^{(i)}$. In this case, the fermion field has only one degree of freedom and the GNY model may become supersymmetric as suggested in Ref. [33].

We then conclude that the negative norm states are an integral part of the GNY model in $d = 4 - 2\epsilon$ dimensions. At one-loop order, all negative norm states are generated by operators with anomalous dimension γ_- for $N_f \lesssim \mathcal{O}(1)$. We believe the negative norm states exist in other theories with fermionic degrees of freedom in noninteger dimensions as well because Eqs. (15), (17), and (18) are valid for any fermionic theory with any number of flavors.

IV. CONCLUSION

We have demonstrated the existence of negative norm states in the Gross-Neveu-Yukawa model in $d = 4 - 2\epsilon$ dimensions through the study of the two-point correlation functions of four-fermion operators and their one-loop anomalous dimension matrix. The negative norm states we found are unavoidable, as the two-point correlation functions are an integral part of the theory. They are generated by evanescent operators with anomalous dimension $-4u_*$ at one-loop order when the fermion flavor number is small. We argue that the negative norm states are a general feature of fermionic theories in noninteger dimensions.

It is now clear that unitarity violation occurs in both the scalar and fermionic case. In addition, a recent study also reveals that unitarity is violated in noninteger dimensional nonrelativistic conformal field theory [34], where unitarity is defined by the notion of reflection positivity. Therefore, it seems that unitarity violation is a general property of CFTs in noninteger dimensions.

⁵This is because $\bar{\Delta}_{\pm}$ are order $\mathcal{O}(\epsilon)$.

We can't see any way to consistently remove these negative norm states from the fermionic field theory in noninteger dimensions. They have no effect, however, on theories in integer dimensions where all the negative norm states vanish.

One should mention, however, that although the loss of unitarity prohibits imposing extra constraints while applying the bootstrap technique, the “non-unitary bootstrap” technique, which has no reliance on unitarity, still works [35–37].

It would be a natural extension of our current study to compute the two-loop anomalous dimension matrix and investigate how the operators in the conformal basis at the two-loop order further classify the negative and positive norm states. It would be interesting to investigate the appearance of negative norm states in other fermion/scalar conformal field theories as well.

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APPENDIX A: GENERAL FORMULAE

We provide some key steps in our calculations. The Feynman diagrams in Fig. 1 are translated into $T_1^{m,n}$ and $T_2^{m,n}$ in Eq. (16) as

$$\begin{aligned} T_1^{m,n} &= \frac{\delta_{m,n}}{x^4 \text{tr}^2(\mathbb{1}_d)(m!)^2} [\text{tr}(\Gamma_\mu^{(m)} \not{x} \Gamma_\nu^{(m)} \not{x})]^2, \\ T_2^{m,n} &= \frac{-1}{x^4 \text{tr}(\mathbb{1}_d)m!n!} \text{tr}(\not{x} \Gamma_\mu^{(m)} \not{x} \Gamma_\nu^{(n)} \not{x} \Gamma_{(m)}^\mu \not{x} \Gamma_{(n)}^\nu), \end{aligned} \quad (\text{A1})$$

where $\Gamma_\mu^{(m)}$ and alike (e.g., $\Gamma_\nu^{(n)}$ and etc.,) are given in Eq. (5) with arbitrary integer m, n and indices μ, ν . Two identities which proved to be the most useful in our study are

$$\gamma^\nu \Gamma^{\mu_1 \dots \mu_n} = \Gamma^{\nu \mu_1 \dots \mu_n} + \sum_{i=1}^n (-1)^{i+1} g^{\nu \mu_i} \Gamma^{\mu_1 \dots \hat{\mu}_i \dots \mu_n}, \quad (\text{A2})$$

$$(-1)^n \Gamma^{\mu_1 \dots \mu_n} \gamma^\nu = \Gamma^{\nu \mu_1 \dots \mu_n} + \sum_{i=1}^n (-1)^i g^{\nu \mu_i} \Gamma^{\mu_1 \dots \hat{\mu}_i \dots \mu_n}. \quad (\text{A3})$$

Here $\hat{\mu}_i$ denotes that the index μ_i is omitted. These equations are a consequence of the basic anticommutation relation between gamma matrices and the antisymmetric structure of $\Gamma_{(n)}^\mu$. By combining the latter two equations one finds

$$2\Gamma^{\nu \mu_1 \dots \mu_n} = \gamma^\nu \Gamma^{\mu_1 \dots \mu_n} + (-1)^n \Gamma^{\mu_1 \dots \mu_n} \gamma^\nu. \quad (\text{A4})$$

The general formula for contracting the antisymmetrized products of gamma matrices reads

$$\Gamma^{\nu_1 \dots \nu_m \mu_n \dots \mu_1} \Gamma_{\mu_1 \dots \mu_n} = \prod_{i=0}^{n-1} (d - m - i) \Gamma^{\nu_1 \dots \nu_m}. \quad (\text{A5})$$

Finally, we have

$$x_a x_b \Gamma^{\mu_1 \dots a \dots b \dots \nu_m} = 0, \quad (\text{A6})$$

as a direct consequence of the definition of $\Gamma^{\mu_1 \dots a \dots b \dots \nu_m}$.

We then calculate $T_1^{m,n}$ and $T_2^{m,n}$ separately in the following two sections.

APPENDIX B: $T_1^{m,n}$

The cyclic property of the trace together with the anticommutation relation between gamma matrices allows us to first conclude that $T_1^{m,n} = 0$ for $m \neq n$. One thus writes,

$$\begin{aligned} T_1^{m,n} &= \frac{\delta_{m,n}}{x^4 \text{tr}^2(\mathbb{1}_d)(m!)^2} [\text{tr}(\Gamma_\mu^{(m)} \not{x} \Gamma_\nu^{(m)} \not{x})]^2, \\ &= B_m \frac{\delta_{m,n}}{(m!)^2}. \end{aligned} \quad (\text{B1})$$

Then we note

$$B_m = \frac{1}{\text{tr}^2(\mathbb{1}_d)} \text{tr}(\Gamma_\mu^{(m)} \Gamma_\nu^{(m)}) \text{tr}(\Gamma_{(m)}^\mu \Gamma_{(m)}^\nu), \quad (\text{B2})$$

which can be proven by the cyclic property of the trace together with Eqs. (A2) and (A6).

By setting the default ordering of $\Gamma_\mu^{(m)}$ and $\Gamma_\nu^{(m)}$ to be μ_1, \dots, μ_m and ν_1, \dots, ν_m , while noting $\mu_i \neq \mu_j$ and $\nu_i \neq \nu_j$ for $i \neq j$, B_m can then be rewritten as

$$\begin{aligned} B_m &= \frac{1}{\text{tr}^2(\mathbb{1}_d)} \text{tr}(\gamma_{\mu_1 \dots \mu_m} \gamma_{\nu_1 \dots \nu_m}) \text{tr}(\gamma^{\mu_1 \dots \mu_m} \gamma^{\nu_1 \dots \nu_m}) \\ &= R_{\mu_1 \dots \mu_m} R^{\mu_1 \dots \mu_m}. \end{aligned} \quad (\text{B3})$$

By moving γ_{μ_1} to the right of the product in Eq. (B3) and using the cyclic property of the trace, one finds

$$R_{\mu_1 \dots \mu_m} = \frac{(-1)^{m-1}}{\text{tr}(\mathbb{1}_d)} \sum_{i=1}^m (-1)^{i+1} g_{\mu_1 \nu_i} \times \text{tr}(\gamma_{\mu_2 \dots \mu_m} \gamma_{\nu_1 \dots \hat{\nu}_i \dots \nu_m}), \quad (\text{B4})$$

where again $\hat{\nu}_i$ denotes the omitted index.

Here we have reduced a trace with $2m$ indices to a sum of traces with $2(m-1)$ indices. By repeatedly applying Eq. (B4) one can reduce the trace with $2m$ indices to $\text{tr}(\mathbb{1}_d)$ with an appropriate combination of coefficients. It is clear that each further reduction step produces one extra summation and one extra set of $g^{\mu_i \nu_j}$ with the appropriate sign in front. To this end, we write,

$$R_{\mu_1 \dots \nu_m} = (-1)^{\frac{m(m-1)}{2}} \text{tr}(\mathbb{1}_d) \left\{ \sum_{i_1=1}^m \dots \sum_{i_m=1}^m g_{\mu_1 \nu_{i_1}} \dots \times g_{\mu_m \nu_{i_m}} \Omega(i_1, \dots, i_m) \right\}, \quad (\text{B5})$$

where the overall factor $(-1)^{\frac{m(m-1)}{2}}$ is accumulated from the repeated use of Eq. (B4) and $\Omega(i_1, \dots, i_m) \in \{0, \pm 1\}$. Since each index ν_k appears only once in the trace, one straightforwardly concludes that $\Omega(i_1, \dots, i_k, \dots, i_k, \dots, i_m) = 0$. A more detailed analysis also reveals that $\Omega(i_1, \dots, i_k, i_{k+1}, \dots, i_m) = -\Omega(i_1, \dots, i_{k+1}, i_k, \dots, i_m)$, which is a property inherited from the antisymmetric nature of $\Gamma^{(m)}$. Finally by noting that $\Omega(1, \dots, m) = 1$, which corresponds to eliminate $\gamma_{\nu_1}, \gamma_{\nu_2}, \dots, \gamma_{\nu_m}$ in order, one identifies

$$\Omega(i_1, \dots, i_m) = \epsilon_{i_1 \dots i_m}.$$

Therefore, one finds

$$B_m = \sum_{\substack{i_1 \dots i_m=1 \\ j_1 \dots j_m=1}}^m m g_{\mu_{j_1}}^{\mu_{i_1}} \dots g_{\mu_{j_m}}^{\mu_{i_m}} \epsilon_{i_1 \dots i_m} \epsilon^{j_1 \dots j_m}, \quad (\text{B6})$$

with

$$\sum_{\substack{i_1 \dots i_m=1 \\ j_1 \dots j_m=1}}^m \equiv \sum_{i_1=1}^m \dots \sum_{i_m=1}^m \sum_{j_1=1}^m \dots \sum_{j_m=1}^m. \quad (\text{B7})$$

This summation can be worked out by dividing the general case into two scenarios:

$$i_k = j_k = m, \quad k = 1, \dots, m,$$

$$\text{or } i_k = j_l = m, k \neq l, \quad m, l = 1, \dots, m. \quad (\text{B8})$$

The summation is then easily carried out and leads to a recurrence relation for B_m ,

$$B_m = m(d - m + 1)B_{m-1}, \quad (\text{B9})$$

for $m > 1$. Combined with the initial condition of the sequence $B_0 = 1$, we obtain the final expression for $T_1^{m,n}$,

$$T_1^{m,n} = \frac{\Gamma(d+1)}{m! \Gamma(d-m+1)} \delta_{m,n}. \quad (\text{B10})$$

It is clear that $T_1^{m,n}$ vanishes for theories in even d dimensions if $m > d$. This observation is in accordance with the fact that there are d numbers of gamma matrices in even d dimensions, and consequently, the antisymmetrized product of m gamma matrices $\Gamma_\mu^{(m)}$ vanishes if $m > d$. In odd dimension d , $\Gamma^{(d)}$ is no longer an independent matrix since $\Gamma^{(d)} \propto \Gamma^{(0)}$, and therefore this redundancy must be removed ‘‘by hand.’’ If the dimension d is no longer an integer, however, then $T_1^{m,n}$ never vanishes and can take negative values.

APPENDIX C: $T_2^{m,n}$

We proceed to calculate $T_2^{m,n}$ next:

$$T_2^{m,n} = \frac{-1}{x^4 \text{tr}(\mathbb{1}_d) m! n!} \text{tr}(\not{x} \Gamma_\mu^{(m)} \not{x} \Gamma_\nu^{(n)} \not{x} \Gamma_\mu^{(m)} \not{x} \Gamma_\nu^{(n)}), \quad (\text{C1})$$

which can be simplified using Eqs. (A2)–(A6) as

$$T_2^{m,n} = \frac{-1}{\text{tr}(\mathbb{1}_d) m! n!} \text{tr}(\Gamma_\mu^{(m)} \Gamma_\nu^{(n)} \Gamma_\mu^{(m)} \Gamma_\nu^{(n)}). \quad (\text{C2})$$

Then again with Eqs. (A4) and (A5), one finds

$$\begin{aligned} T_2^{m,n} &= \frac{-1}{2 \text{tr}(\mathbb{1}_d) m! n!} \text{tr}[\Gamma^{\nu_2 \dots \nu_n} \Gamma^{(m)} \gamma_{(n)} \Gamma^{(m)} \gamma^{\nu_1} \\ &\quad + (-1)^{m+n-1} (d-2m) \Gamma^{(n-1)} \Gamma^{(m)} \gamma_{(n-1)} \Gamma^{(m)}] \\ &= \frac{-1}{2 \text{tr}(\mathbb{1}_d) m! n!} (s_1 + s_2), \end{aligned} \quad (\text{C3})$$

where $\gamma_{(n-k)} = \gamma_{\nu_1 \dots \nu_{n-k}} = \gamma_{\nu_1} \dots \gamma_{\nu_{n-k}}$ is a product of gamma matrices of standard ordering. Then,

$$\begin{aligned} s_1 &= \text{tr} \left\{ 2 \sum_{i=1}^{n-1} (d-n+2) \Gamma^{(n-2)} \Gamma^{(m)} \gamma_{(n-2)} \Gamma^{(m)} \right. \\ &\quad \left. + (-1)^{m+n-1} (d-2m) \Gamma^{(n-1)} \Gamma^{(m)} \gamma_{(n-1)} \Gamma^{(m)} \right\}. \end{aligned} \quad (\text{C4})$$

One then obtains a recursion relation for $T_2^{m,n}$,

$$\begin{aligned} T_2^{m,n} &= (-1)^{m+n-1} (d-2m) T_2^{m,n-1} \\ &\quad + (n-1)(d-n+2) T_2^{m,n-2}, \end{aligned} \quad (\text{C5})$$

with boundary conditions

$$\begin{aligned} T_2^{m,0} &= -\frac{(-1)^{\frac{m(m-1)}{2}}}{m!} \frac{\Gamma(d+1)}{\Gamma(d-m+1)}, \\ T_2^{m,1} &= -\frac{(-1)^{\frac{m(m+1)}{2}}}{m!} \frac{(d-2m)\Gamma(d+1)}{\Gamma(d-m+1)}. \end{aligned} \quad (\text{C6})$$

The recurrence relation can be expressed as

$$T_2^{m,n} = -\frac{1}{2} i^{m(m+1)+n(n+1)} a_{m,n}, \quad (\text{C7})$$

where $a_{m,n}$ are the coefficients of the generating function

$$F(x, y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n. \quad (\text{C8})$$

Then the recursive behavior in Eq. (C5) is inherited by $a_{m,n}$ and combined with the boundary conditions Eq. (C6). The generating function is found to be

$$\begin{aligned} F(x, y) &= (1-x+y+xy)^d + (1+x-y+xy)^d \\ &\quad - (1+x+y-xy)^d + (1-x-y-xy)^d. \end{aligned} \quad (\text{C9})$$

APPENDIX D: ANOMALOUS DIMENSIONS

The Feynman diagrams in the first row of Fig. 3 have a divergent part which reads

$$I_1^{(m)} = (g_1^*)^2 \frac{m-2}{16\pi^2\epsilon} (-1)^m \mathcal{O}^{(m)}. \quad (\text{D1})$$

The Feynman diagrams in the last row of Fig. 3 together yield

$$I_2^{(m)} \stackrel{\text{div}}{=} \frac{(g_1^*)^2}{64\pi^2 m! \epsilon} \bar{\Psi}(x) [\Gamma^{(m)}, \gamma_\mu] \Psi(x) \\ \times \bar{\Psi}(x) [\Gamma_{(m)}, \gamma^\mu] \Psi(x), \quad (\text{D2})$$

which requires some algebra. For odd m , we get

$$I_2^{(m=\text{odd})} = \frac{(g_1^*)^2 (m+1)}{16\pi^2 \epsilon} \mathcal{O}^{(m+1)}, \quad (\text{D3})$$

while for the even case, we have

$$I_2^{(m=\text{even})} = \frac{(g_1^*)^2 (5-m)}{16\pi^2 \epsilon} \mathcal{O}^{(m-1)}. \quad (\text{D4})$$

The Feynman diagrams in Fig. 4 yield a divergent piece which reads

$$I_3^{(m)} = \frac{(g_1^*)^2 \mathcal{O}^{(0)}}{16\pi^2 \epsilon} \left[\frac{(-1)^{m(m-1)/2} m^{-1}}{m!} \prod_{i=0}^{m-1} (4-i) - N_f \delta_{m,0} \right]. \quad (\text{D5})$$

Consequently, the operator renormalization matrix is found to be

$$Z_{\mathcal{O}}^{m,n} = I_1 + I_2 + I_3 \\ = \frac{(g_1^*)^2}{16\pi^2 \epsilon} \left[(n-2)(-1)^n \delta_{m,n} + n \delta_{m,n-1} \text{mod}(m,2) \right. \\ \left. + (4-n) \delta^{m,n+1} \text{mod}(n,2) - N_f \delta_{m,0} \delta_{n,0} \right. \\ \left. + \frac{4!(-1)^{m(m-1)/2}}{m!(4-m)!} \delta_{n,0} \right]. \quad (\text{D6})$$

The one-loop renormalization of the fermion self energy in the GNY model reads

$$Z_{\Psi}^{m,n} = 1 + \frac{(g_1^*)^2}{32\pi^2 \epsilon} \delta_{m,n}, \quad (\text{D7})$$

and therefore one obtains the one-loop anomalous dimensions matrix

$$\gamma_{\mathcal{O}}^{m,n} = \frac{d\alpha_1}{d \ln \mu} \frac{\partial}{\partial \alpha_1} \ln(Z_{\mathcal{O}} Z_{\Psi}^{-2}) \\ = 2u_* \left[(1 - (-1)^n (n-2)) \delta_{m,n} - n \delta_{m,n-1} \text{mod}(m,2) \right. \\ \left. + (n-4) \delta_{m,n+1} \text{mod}(n,2) + N_f \delta_{m,0} \delta_{n,0} \right. \\ \left. - \frac{4!(-1)^{m(m-1)/2}}{m!(4-m)!} \delta_{n,0} \right]. \quad (\text{D8})$$

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