Chow groups of zero- and one-cycles on schemes over local fields and henselian discrete valuation rings

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Zusammenfassung

In dieser Dissertation studieren wir drei Vermutungen von Kerz, Esnault und Wittenberg zu Restriktionsabbildungen für relative Nullzykel auf glatten projektiven Schemata über henselschen diskreten Bewertungsringen. Das Thema der ersten beiden Vermutungen ist eine Basiswechseleigenschaft mit endlichen Koeffizienten, die prim zur Restklassencharakteristik der Basis sind, für sogenannte höhere Nullzykel und Nullzykel mit Koeffizienten in Milnor K-Theorie. Die dritte Vermutung behandelt Chow-Gruppen relativer Nullzykel mit Koeffizienten, die nicht prim zur Restklassencharakteristik sind. Dies führt unter anderem zum Studium von Deformationen von Nullzykeln. Die drei Vermutungen hängen eng mit Vermutungen von Colliot-Thélène zur Struktur von Chow-Gruppen von Nullzykeln glatter projektiver Varietäten über p-adischen lokalen Körpern zusammen.

Wir beweisen einen Spezialfall der ersten Vermutung, beweisen die Basiswechseleigenschaft für Nullzykel mit Koeffizienten in Milnor K-Theorie und geben zwei verschiedene Beweise einer Aussage zur Algebraisierung von Nullzykeln.

Abstract

In this thesis, we study three conjectures by Kerz, Esnault and Wittenberg on restriction maps for relative zero-cycles on smooth projective schemes over henselian discrete valuation rings. The first two concern base change properties with finite coefficients prime to the residue characteristic of the base for higher zero-cycles and zero-cycles with coefficients in Milnor K-theory. The last one concerns the p-part, i.e. the part not prime to the residue characteristic of the base, and involves studying the deformations of zero-cycles. These conjectures are closely related to conjectures of Colliot-Thélène on the structure of the Chow group of zero-cycles of smooth projective schemes over p-adic local fields.

We give evidence for the first conjecture proving a special case, prove the base change property for zero-cycles with coefficients in Milnor K-theory and give two very different proofs of an algebraization theorem for zero-cycles.

Für meinen Freund François

Contents

Introduction

Let k be a field and X_k a k-variety. The Chow group of zero-cycles of X_k is defined to be the free abelian group $Z_0(X_k)$ generated by the closed points of X_k modulo the group $Z_0(X_k)_{\mathrm{rat}}$ generated by divisors of rational functions on integral curves $C \subset X_k$, i.e.

$$\mathrm{CH}_0(X_k) := Z_0(X_k)/Z_0(X_k)_{\mathrm{rat}}.$$

An element of $Z_0(X_k)$ is called a zero-cycle. The degree of a zero-cycle is defined by $\deg(\sum_x n_x x) = \sum_x n_x [k(x) : k]$. If X_k is proper over k, then the degree of a zero-cycle is invariant modulo rational equivalence, i.e. there is a factorisation

$$\begin{array}{ccc} Z_0(X_k) & \xrightarrow{\deg} & \mathbb{Z}. \\ \downarrow & \nearrow & \\ \mathrm{CH}_0(X_k) & & \end{array}$$

We denote the kernel of deg by $A_0(X_k)$.

If $k = \mathbb{F}$ is finite, then the study of $\mathrm{CH}_0(X_{\mathbb{F}})$ is the study of class field theory, the Kato conjectures and the étale fundamental group. One of the main results is that $A_0(X_{\mathbb{F}})$ is finite. In this thesis we are mainly interested in the structure of the group $\mathrm{CH}_0(X_K)$ for a p-adic field K, i.e. a finite extension of $\mathbb{Q}_p$ for a prime p.

Let K be a p-adic field, $\mathcal{O}_K$ the ring of integers in K and X_K a smooth projective geometrically integral K-variety. In [10], Colliot-Thélène poses, among others, the following questions which have been guiding the study of zero cycles over p-adic fields:

(a) Is $A_0(X_K)/n$ finite for any $n \in \mathbb{N}_{>0}$? Furthermore, is $A_0(X_K)/n$ zero for almost all n?

(b) Is $A_0(X_K)[n]$ finite for any $n \in \mathbb{N}_{>0}$?

(c) Is the group $A_0(X_K)_{\mathrm{tors}}$ finite?

(d) Is the maximal divisible subgroup $D(X_K)$ of $A_0(X_K)$ uniquely divisible?

(e) Is the group $A_0(X_K)/D(X_K)$, up to a finite group, isomorphic to a group of the form $\mathbb{Z}_p^m$ for some $m \in \mathbb{N}$?

One notes that (a) is equivalent to

(f) $A_0(X_K)$ is the direct sum of a finite group and a p'-divisible group, i.e. a group divisible for any prime $p' \neq p$.

if we require $(n, p) = 1$ in (a) and that (e) is implied by

(g) Is there a (non-canonical) isomorphism

$$\mathrm{CH}_0(X_K) \cong \mathbb{Z} \oplus \mathbb{Z}_p^m \oplus \text{(finite group)} \oplus \text{(divisible group)}?$$

In particular, a positive answer to (g) would imply the finiteness of $A_0(X_K)/p$.

In the case of curves, i.e. $\dim X_K = 1$, all of these questions have an affirmative answer since in that case the Albanese map $A_0(X_K) \to J(K)$ is injective with finite cokernel and the group of rational points of the Jacobian of X_K, $J(K)$, is known to be the direct sum of a finite group and $\mathcal{O}_K^m$ for some $m \in \mathbb{N}$ by a result of Mattuck ([55]).

Let us turn to the case of higher dimension. For surfaces a few results on these questions were shown in the 1980s using ideas of Bloch involving algebraic K-theory and the theorem of Merkurjev-Suslin. Most notable is the finiteness of the n-torsion which was shown by Colliot-Thélène, Sansuc and Soulé (see [14]). The first breakthrough in arbitrary dimension was achieved by Saito and Sato in 2006 in [66]. In [66], Saito and Sato propose to study not zero-cycles on X_K but relative zero-cycles on a regular model X of X_K over $\mathcal{O}_K$ in order to approach Question (a). More generally, let from now on $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with residue field k of characteristic p and quotient field K. Let X be regular, flat, quasi-semistable and projective over $\mathcal{O}_K$ with generic fiber X_K and special fiber X_0. Let $n \in \mathbb{N}$ with $(n, p) = 1$. Let d be the fiber dimension of X over $\mathcal{O}_K$. The main result of [66] is that the cycle class map

$$cl_X : \mathrm{CH}_1(X)/n \to H^{2d}_{\text{ét}}(X, \mu_n^{\otimes d})$$

is an isomorphism if k is finite or separably closed. From this result they deduce that Question (f) has a positive answer. Furthermore they deduce, using class field theory, that the restriction map

$$res^{\mathrm{CH}} : \mathrm{CH}_1(X)/n \to \mathrm{CH}_0(X_0)/n$$

induced by restricting an integral one-cycle Z on X in good position, i.e. Z is flat over $\mathcal{O}_K$ and does not meet the singular locus of X_k, to $[Z \cap X_k]$, is an isomorphism if X is smooth over $\mathcal{O}_K$. The proof in [66] involves homology theories, a variant of the Kato complexes, Lefschetz theorems for étale cohomology and an extension of a base change theorem of Rapoport and Zink ([63, Satz 2.19]).

More recently, Bloch gave a new and purely geometric proof of the theorem of Saito and Sato that cl_X is an isomorphism if k is separably closed (see [17, App.]). In order to show that $\ker(cl_X)$ is n-divisible, Bloch's idea is to normalise cycles in the kernel of cl_X, which in this case is just the degree map, and to embed them into $\mathbb{P}^N_X$ for some $N \in \mathbb{N}$. Since the embedding dimension of these nonsingular cycles is ≤ 1, one may use Bertini theorems in $\mathbb{P}^N_X$ to reduce the n-divisibility to relative dimension 1 in which case it is known.

Building on this idea Kerz, Esnault and Wittenberg show in [46] that the restriction map $res^{\mathrm{CH}} : \mathrm{CH}_1(X)/n \to \mathrm{CH}_0(X_0)/n$ is an isomorphism if X is smooth over $\mathcal{O}_K$ for arbitrary perfect residue fields k. In fact they show a more general statement assuming that X_0 is a simple normal crossing divisor (see the introduction of Chapter 1). For simplicity, and since we make this assumption throughout this thesis, we assume from now on that X is smooth over $\mathcal{O}_K$ or that X_K has a smooth model over O_K. Let $\mathrm{CH}^s(X,t)_\Lambda := \mathrm{CH}^s(X,t,\mathbb{Z}/nZ)$ denote Bloch's higher Chow groups with coefficients in $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Generalising this restriction isomorphism, Kerz, Esnault and Wittenberg state the following conjectures:

(h) The restriction map

$$res^{\mathrm{CH}} : \mathrm{CH}^d(X,j)_\Lambda \to \mathrm{CH}^d(X_0,j)_\Lambda$$

is an isomorphism for all $j \geq 0$.

(i) The restriction map

$$res^{\mathrm{CH}} : \mathrm{CH}^{d+j}(X,j)_\Lambda \to \mathrm{CH}^{d+j}(X_0,j)_\Lambda$$

is an isomorphism for all $j \geq 0$.

(j) If $\mathrm{ch}(K) = 0$ and if k is perfect of characteristic $p > 0$, then the restriction map

$$res : \mathrm{CH}^d(X) \otimes \mathbb{Z}/p^r\mathbb{Z} \to \text{"}\lim_n\text{"}\, H^d(X_1, \mathcal{K}^M_{d,X_n}/p^r)$$

is an isomorphism in the category of pro-abelian groups pro-Ab considering $\mathrm{CH}^d(X) \otimes \mathbb{Z}/p^r\mathbb{Z}$ as a constant pro-system.

For the definition of the restriction maps in (h) and (i) see Chapter 1 and for the definition of the restriction map in (j), assuming the Gersten conjecture for the Milnor K-theory sheaf $\mathcal{K}^M_X$, see Chapter 3. We make a few remarks on the above questions and conjectures and their relations:

1. Conjecture (h) is related to the following conjecture by Saito and Sato stated in [66]:

   (k) If k is finite or separably closed, then the Kato complex $KC(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ defined in [66, Sec. 2] is exact except in degree 1.

2. Conjecture (h) is related to Question (b) (see Remark 1.2.3 and Proposition 2.5.4).

3. Conjecture (j) would imply a positive answer to Question (a) for $n = p$ and is related to Question (g) (see Remark 4.4.7).

4. In [2], Asakura and Saito show the existence of a smooth projective surface over a p-adic field with infinite torsion in the Chow group of zero-cycles giving a negative answer to Question (c). In particular, if (g) is true, then (d) is false.

Let us list our main results chapter by chapter:

In Chapter 1 we show the following theorem which makes progress on Conjecture (h):

Theorem A. *Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with quotient field K and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$ and assume that $1/n \in k^\times$. Let X be a smooth and projective scheme over* $\mathrm{Spec}\mathcal{O}_K$ *of fiber dimension d. Let X_0 denote the reduced special fiber. Then the restriction map*

$$res^{\mathrm{CH}} : \mathrm{CH}^d(X,1)_\Lambda \twoheadrightarrow \mathrm{CH}^d(X_0,1)_\Lambda$$

is surjective.

If $\dim X = 2$ we also show that res^{CH} is injective. We thereby obtain a new proof of the finiteness of $\mathrm{CH}^d(X_K)[n]$ for a smooth projective surface X_K with good reduction.

In Chapter 2 we study zero-cycles with coefficients in Milnor K-theory. Our main result is the following:

Theorem B. *Let the notation be as in Theorem A. Then Conjecture (i) holds.*

The proofs of Theorem A and B are purely geometric and make use of the above mentioned idea of Bloch. In an excursus we also study the n-torsion groups $\mathrm{CH}^{d+j}(X,j)[n]$ for a smooth (projective) scheme X of dimension d over a finite or local field for $j \geq 1$.

In Chapter 3 we study Conjecture (j). The topic of this and the following chapter is different from that of the first two chapters in the sense that here we study the deformation of zero-cycles on smooth projective schemes over henselian discrete valuation rings and are interested in the p-part of $\mathrm{CH}_0(X_K)$. Our main theorem is the following:

Theorem C. *Let A be a henselian discrete valuation ring with uniformising parameter π and residue field k. Let X be a smooth projective scheme over* $\mathrm{Spec}(A)$ *of relative dimension d. Let $X_n := X \times_A A/(\pi^n)$. Assume that the Gersten conjecture holds for the Milnor K-theory sheaf $\mathcal{K}^M_X$. Then the restriction map $res : H^d(X, \mathcal{K}^M_{d+j,X}) \to H^d(X_1, \mathcal{K}^M_{d+j,X_n})$ is surjective. In particular the map of pro-systems*

$$res : H^d(X, \mathcal{K}^M_{d+j,X}) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d+j,X_n})$$

is an epimorphism in the category of pro-abelian groups pro-Ab for all $j \geq 0$. Here we consider $H^d(X, \mathcal{K}^M_{d+j,X})$ as a constant pro-system.

The proof uses an idelic argument which shows that $H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}) = 0$ for all $j \geq 0$. We also show Conjecture (j) for $d = 1$ and $j = 0$ and give a sufficient condition for it to be true in general. Taking into account the known results on the Gersten conjecture for Milnor K-theory, we get the following corollary:

Corollary D. *(i) If A is equi-characteristic, then the map*

$$res : \mathrm{CH}^{d+j}(X,j) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d+j,X_n})$$

is is an epimorphism in pro-Ab for all $j \geq 0$.

(ii) If A is of mixed characteristic $(0,p)$ with $p > d+j-1$, then the map

$$res : \mathrm{CH}^{d+j}(X, j, \mathbb{Z}/p^r) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d+j,X_n}/p^r)$$

is is an epimorphism in pro-Ab for all $j \geq 0$.

In the last section of Chapter 3 we relate the restriction map *res* to the p-adic cycle class map $\varrho^{d,0}_{p^r} : \mathrm{CH}^d(X)/p^r \to H^{2d}_{\text{ét}}(X, \mathcal{T}_r(d))$ which was shown to be surjective by Saito and Sato in [67].

In Chapter 4 we give a completely different proof of Theorem C by studying the exact sequence

$$H^d(X_1, \Omega^{d-1}_{X_1}) \to H^d(X_1, \mathcal{K}^M_{d,X_n}) \to H^d(X_1, \mathcal{K}^M_{d,X_{n-1}})$$

induced by the short exact sequence

$$\Omega^{d-1}_{X_1} \to \mathcal{K}^M_{d,X_n} \to \mathcal{K}^M_{d,X_{n-1}} \to 0.$$

We first prove Theorem C in the case $\dim X_1 = 2$ using a moving lemma for $H^2(X_1, \Omega^1_{X_1})$. Then the general case is reduced to this one via a Lefschetz theorem.

In the Appendix we review p-adic étale Tate twists and homology theories and show why homology theories, which are used in the approach of [66], can't be used to attack Conjecture (j).

The content of chapter 1, 2 and 3 has appeared in essentially the same form in the following (pre-)publications:

Chapter 1: Morten Lüders, *On a base change conjecture for higher zero cycles*, Homology, Homotopy and Applications, Volume 20 (2018) Number 1, p. 59-68.

Chapter 2: Morten Lüders, *A restriction isomorphism for zero-cycles with coefficients in Milnor K-theory*, preprint 2017.

Chapter 3: Morten Lüders, *Algebraization for zero-cycles and the p-adic cycle class map*, preprint 2017.

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Chapter 1

Base change for higher zero-cycles

Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with quotient field K and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$ and always assume that $1/n \in k^\times$. Let X be a regular scheme, flat and projective over $\mathrm{Spec}\mathcal{O}_K$ of fiber dimension d. Let X_K denote the generic fiber and X_0 the reduced special fiber. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$.

In [66, Cor. 9.5] and [17, App.] it is shown that for $X \to \mathrm{Spec}\mathcal{O}_K$ smooth and projective and k finite or algebraically closed, the restriction map

$$\mathrm{CH}_1(X)_\Lambda \xrightarrow{\cong} \mathrm{CH}_0(X_0)_\Lambda$$

is an isomorphism of Chow groups with coefficients in Λ. This result is reproven in [46] for more general residue fields and generalised to the case that X_0 is a simple normal crossings divisor. In that case one needs to replace $\mathrm{CH}_0(X_0)$ by $H^{2d}_{\mathrm{cdh}}(X_0, \mathbb{Z}/n\mathbb{Z}(d))$, i.e. the hypercohomology of the motivic complex $\mathbb{Z}/n\mathbb{Z}(d)$ in the cdh-topology, which is isomorphic to $\mathrm{CH}_0(X_0)$ for X_0/k smooth. The result then says that if k is finite, or algebraically closed, or $(d-1)!$ prime to m, or A is of equal characteristic, or $X/\mathcal{O}_K$ is smooth and always assuming that k is perfect, then there is an isomorphism

$$\mathrm{CH}_1(X)_\Lambda \xrightarrow{\cong} H^{2d}_{\mathrm{cdh}}(X_0, \mathbb{Z}/n\mathbb{Z}(d))$$

which is induced by restricting a one-cycle in general position to a zero-cycle on X_0^{sm}, where X_0^{sm} is the smooth locus of X_0. Generalising this result, the following conjecture is stated in Section 10 of [46]:

Conjecture 1.0.1. *The restriction homomorphism*

$$res : H^{i,d}(X, \mathbb{Z}/n\mathbb{Z}) \to H^{i,d}_{\mathrm{cdh}}(X_0, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for all $i \geq 0$.

Here $H^{i,d}(X, \mathbb{Z}/m\mathbb{Z}) = H^i(X, \mathbb{Z}/m\mathbb{Z}(d))$ are the motivic cohomology groups for schemes over Dedekind rings defined in [70]. In this chapter we consider the corresponding restriction map on higher Chow groups of zero-cycles with coefficients in Λ

$$res^{\mathrm{CH}} : \mathrm{CH}^d(X, 2d-i)_\Lambda \to \mathrm{CH}^d(X_0, 2d-i)_\Lambda$$

for $X/\mathcal{O}_K$ smooth which we define to be induced by the following composition:

$$res^{\mathrm{CH}} : \mathrm{CH}^n(X, m) \to \mathrm{CH}^n(X_K, m) \xrightarrow{\cdot(-\pi)} \mathrm{CH}^{n+1}(X_K, m+1) \xrightarrow{\partial} \mathrm{CH}^n(X_0, m).$$

Here $\cdot(-\pi)$ is the product with $-\pi \in \mathrm{CH}^1(K, 1) = K^\times$ defined in [4, Sec. 5], π is a local parameter for the discrete valuation on K and ∂ is the boundary map coming from the localization sequence for higher Chow groups (see [50]). We call the composition

$$sp_\pi^{\mathrm{CH}} : \mathrm{CH}^n(X_K, m) \xrightarrow{\cdot(-\pi)} \mathrm{CH}^{n+1}(X_K, m+1) \xrightarrow{\partial} \mathrm{CH}^n(X_0, m)$$

a specialisation map for higher Chow groups. We note that res^{CH} does not depend on the choice of π whereas sp_π^{CH} does. For a detailed discussion of the specialisation map see also [16, Sec. 3].

Our main theorem is the following:

Theorem 1.0.2. *Let $X/\mathcal{O}_K$ be smooth. Then the restriction map*

$$res^{\mathrm{CH}} : \mathrm{CH}^d(X, 1)_\Lambda \twoheadrightarrow \mathrm{CH}^d(X_0, 1)_\Lambda$$

is surjective. This implies in particular the surjectivity part of Conjecture 1.0.1 for the pair $(2d-1, d)$.

This implies the following corollary:

Corollary 1.0.3. *Let $X/\mathcal{O}_K$ be smooth. Then the specialisation map*

$$sp_\pi^{\mathrm{CH}} : \mathrm{CH}^d(X_K, 1)_\Lambda \twoheadrightarrow \mathrm{CH}^d(X_0, 1)_\Lambda$$

is surjective.

The restriction map in the special degree of Theorem 1.0.2 is of particular interest since, as noted in the introduction, it is related to Conjecture (b) on the finiteness of $\mathrm{CH}^d(X_K)[n]$ for K a p-adic field. In Section 1.2 we show the injectivity for $d = 2$. Furthermore, Theorem 1.0.2 together with the main result of [46] may be considered as a generalisation to perfect residue fields of the vanishing of the Kato homology group $KH_3(X, \mathbb{Q}_\ell/Z_\ell)$ defined in [66] where it is proven for k finite or separably closed.

1.1 Main theorem

Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with quotient field K and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$ and always assume that $1/n \in k^\times$. From now on let X be a smooth and projective scheme over $\mathrm{Spec}\mathcal{O}_K$ of fiber dimension d in which case we also say that X is of relative dimension d over $\mathcal{O}_K$. Let X_K denote the generic fiber and X_0 the reduced special fiber. By $X_{(p)}$ we denote the set of points $x \in X$ such that $\dim(\overline{\{x\}}) = p$, where $\overline{\{x\}}$ denotes the closure of x in X.

We are going to use the following notation for Rost's Chow groups with coefficients in Milnor K-theory (see [64, Sec. 5]):

$$C_p(X,m) = \bigoplus_{x\in X_{(p)}} (K^M_{m+p}k(x)) \otimes \mathbb{Z}/n\mathbb{Z},$$

$$Z_p(X,m) = \ker[\partial : C_p(X,m) \to C_{p-1}(X,m)],$$
$$A_p(X,m) = H_p(C_*(X,m)).$$

We write $Z_k(X)$ for the group of k-cycles on X, i.e. the free abelian group generated by k-dimensional closed subschemes of X.

Let π be some fixed a local parameter of $\mathcal{O}_K$. We define the restriction map

$$res_\pi : C_p(X,m) \to C_{p-1}(X_0, m+1)$$

to be the composition

$$res_\pi : C_p(X,m) \to C_{p-1}(X_K, m+1) \xrightarrow{\cdot\{-\pi\}} C_{p-1}(X_K, m+2) \xrightarrow{\partial} C_{p-1}(X_0, m+1).$$

In the above composition the map $C_p(X,m) \to C_{p-1}(X_K, m+1)$ is defined to be the identity on all elements supported on $X_{(p)} \setminus X_{0(p)}$ and zero on $X_{0(p)}$. The map ∂ is defined to be the boundary map induced by the tame symbol on Milnor K-theory for discrete valuation rings. More precisely, ∂ is defined as follows: Let $\overline{\{x\}}$ be the subscheme corresponding to $x \in X_{(p)}$. Let us assume for simplicity that $\overline{\{x\}}$ is normal. Otherwise we take the normalisation and use the norm map. Now if $y \in \overline{\{x\}}_{(p-1)}$, then y defines a discrete valuation on $k(x)$. Let π' be a local parameter of $k(x)$. Let $\partial^x_y : K^M_{n+1}k(x) \to K^M_n k(y)$ be the tame symbol defined by sending $\{\pi', u_1, ..., u_n\}$ to $\{\bar{u_1}, ..., \bar{u_n}\}$, where the u_i are units in the discrete valuation ring of $k(x)$ and the $\bar{u}_i$ their images in $k(y)$. ∂ is defined to be the sum of all ∂^x_y taken over all $x \in X_{(p)}$ and all $y \in \overline{\{x\}}_{(p-1)}$. Note that the restriction map res_π has to be distinguished from the specialisation map

$$sp^x_{y,\pi'} = \partial^x_y \circ \{-\pi'\} : K^M_n k(x) \to K^M_n k(y).$$

$sp^x_{y,\pi'}$ sends $\{\pi'^{i_1}u_1, ..., \pi'^{i_n}u_n\}$ to $\{\bar{u_1}, ..., \bar{u_n}\}$, where again the u_i are units in the discrete valuation ring of $k(x)$ and the $\bar{u}_i$ their images in $k(y)$.

The map res_π depends on the choice of π but the induced map on homology

$$res : A_p(X,m) \to A_{p-1}(X_0, m+1)$$

is independent of the choice. This can be seen as follows: Let $u \in \mathcal{O}_K^\times$ and $\alpha \in C_p(X,m)$. Then $res_{u\pi}(\alpha) = \partial(\{-\pi u\}\cdot\alpha) = \partial(\{-\pi\}\cdot\alpha) + \partial(\{u\}\cdot\alpha) = res_\pi(\alpha) + \partial(\{u\}\cdot\alpha)$. Now if $\alpha \in A_p(X,m)$, then $\partial(\{u\}\cdot\alpha) = 0$ and $res_{u\pi}(\alpha) = res_\pi(\alpha)$. In the following we will write res for res_π, fixing a local parameter $\pi \in O_K$.

We now turn to our principle interest of study, the restriction map

$$res : C_2(X,-1) \to C_1(X_0, 0).$$

We start with the following lemma:

Lemma 1.1.1. *The map* $res : C_2(X,-1) \to C_1(X_0,0)$*, after having fixed* π*, is surjective.*

Proof. Let $\bar{u} \in K_1^M k(x)$ for some $x \in X_0^{(d-1)}$. As in the proof of [66, Lem. 7.2] we can find a relative surface $Z \subset X$ containing x and being regular at x and such that $Z \cap X_0$ contains $\overline{\{x\}}$ with multiplicity 1. Let $Z_0 = \cup_{i \in I} Z_0^{(i)} \cup \overline{\{x\}}$ be the union of the pairwise different irreducible components of the special fiber of Z with those irreducible components different from $\overline{\{x\}}$ indexed by I. Since all maximal ideals, $\mathfrak{m}_i$ corresponding to $Z_0^{(i)}$ and $\mathfrak{m}_x$ corresponding to $\overline{\{x\}}$, in the semi-local ring $\mathcal{O}_{Z,Z_0}$ are coprime, the map $\mathcal{O}_{Z,Z_0} \to \prod_{i \in I} \mathcal{O}_{Z,Z_0}/\mathfrak{m}_i \times \mathcal{O}_{Z,Z_0}/\mathfrak{m}_x$ is surjective. Therefore we can find a lift $u \in K_1^M k(z)$, z being the generic point of Z, of $\bar{u}$ which specialises to $\bar{u}$ in $K(\overline{\{x\}})^\times$ and to 1 in $K(Z_0^{(i)})^\times$ for all $i \in I$. □

The main result we are going to prove is the following:

Proposition 1.1.2. *The restriction map* $res : A_2(X,-1) \to A_1(X_0,0)$ *is surjective.*

It will be implied by the following key lemma:

Key lemma 1.1.3. *Let* $\xi \in \ker[Z_1(X)/n \overset{res}{\to} Z_0(X_0)/n]$*, then there is a* $\xi' \in \ker[C_2(X,-1) \overset{res}{\to} C_1(X_0,0)]$ *with* $\partial(\xi') = \xi$*.*

Proof of Proposition 1.1.2. Let $\xi_0 \in \ker[C_1(X_0,0) \overset{\partial}{\to} C_0(X_0,0)]$. By Lemma 1.1.1 there is a $\xi \in C_2(X,-1)$ with $res(\xi) = \xi_0$. As $res(\partial(\xi)) = \partial(res(\xi)) = 0$, key Lemma 1.1.3 tells us that there is a $\xi' \in \ker(C_2(X,-1) \to C_1(X_0,0))$ with $\partial\xi' = \partial\xi$. As res is a homomorphism, it follows that $\xi_0 = res(\xi - \xi')$ and $\partial(\xi - \xi') = 0$. Hence $res : Z_2(X,-1) \to Z_1(X_0,0)$ is surjective and the commutativity of ∂ and res implies that $res : A_2(X,-1) \to A_1(X_0,0)$ is surjective. □

Proof of Key lemma 1.1.3. We start with the case of relative dimension $d = 1$, i.e. X is a smooth fibered surface over $\mathcal{O}_K$, and consider the following diagram:

$$\begin{array}{ccc} C_2(X,-1) = K(X)^* \otimes \mathbb{Z}/n\mathbb{Z} & \xrightarrow{res} & C_1(X_0,0) = K(X_0)^* \otimes \mathbb{Z}/n\mathbb{Z} \\ \downarrow \partial & & \downarrow \partial \\ Z_1(X)/n & \xrightarrow{res} & Z_0(X_0)/n \end{array}$$

where we write $Z_i(X)/n$ for $C_i(X,-i)$ which are just the cycles of dimension i modulo n. The restriction map in the lowest degree $res : Z_1(X)/n \to Z_0(X_0)/n$ agrees with the specialisation map on cycles defined by Fulton in [19, Rem. 2.3] since X_0 is a principle Cartier divisor and $\partial_y^x(\{-\pi\}) = \mathrm{ord}_{\mathcal{O}_{\overline{\{x\}},y}}(\pi)$. Modifying $\xi \in \ker[Z_1(X)/n \overset{res}{\to} Z_0(X_0)/n]$ by elements equivalent to zero in $Z_1(X)/n$, we may represent it by an element $x \in \ker[Z_1(X) \to Z_0(X_0)]$.

We consider the following short exact sequence of sheaves:

$$0 \to \mathcal{O}^*_{X;X_0} \to \mathcal{M}^*_{X;X_0} \to \mathcal{D}iv(X,X_0) \to 0, \tag{1.1.1}$$

where $\mathcal{M}^*_{X;X_0}$(resp. $\mathcal{O}^*_{X;X_0}$) denotes the sheaf of invertible meromorphic functions (resp. invertible regular functions) relative to $\mathrm{Spec}\mathcal{O}_K$ and congruent to 1 in the generic point of X_0, i.e. in $\mathcal{O}_{X,\mu}$, where μ is the generic point of X_0, and $\mathcal{D}iv(X, X_0)$ is the sheaf associated to $\mathcal{M}^*_{X;X_0}/\mathcal{O}^*_{X;X_0}$. In other words, $\mathcal{D}iv(X, X_0)(U)$ is the set of relative Cartier divisors on $U \subset X$ which specialise to zero in X_0. For the concept of relative meromorphic functions and divisors see [29, Sec. 20, 21.15].

We want to show that $(\mathcal{D}iv(X, X_0)(X)/\mathcal{M}^*_{X;X_0}(X))/n = 0$.

Claim 1.1.4. $\mathrm{Pic}(X, X_0) \cong \mathcal{D}iv(X, X_0)(X)/\mathcal{M}^*_{X;X_0}(X)$.

Short exact sequence (1.1.1) induces the following exact sequence:

$$\mathcal{O}^*_{X;X_0}(X) \to \mathcal{M}^*_{X;X_0}(X) \to \mathcal{D}iv(X, X_0)(X) \to \mathrm{Pic}(X, X_0) \to H^1(X, \mathcal{M}^*_{X;X_0})$$

Now $\mathrm{Pic}(X, X_0) = H^1(X, \mathcal{O}^*_{X;X_0})$ can also be described as the group of isomorphism classes of pairs $(\mathcal{L}, \psi)$ of an invertible sheaf $\mathcal{L}$ with a trivialisation $\psi : \mathcal{L}|_{X_0} \cong \mathcal{O}_{X_0}$ (see e.g. [71, Lem. 2.1]).

The following argument shows that the map $\mathcal{D}iv(X, X_0)(X) \to \mathrm{Pic}(X, X_0)$ is surjective: Let $(\mathcal{L}, \psi) \in \mathrm{Pic}(X, X_0)$. The trivialisation ψ gives an isomorphism $\psi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0} \xrightarrow{\cong} \mathcal{O}_{X_0}$ and by localising an isomorphism $\psi_\mu : \mathcal{L}_\mu \otimes_{\mathcal{O}_{X,\mu}} \mathcal{O}_{X_0,\mu} \xrightarrow{\cong} \mathcal{O}_{X_0,\mu}$, where μ again denotes the generic point of X_0. Let s denote a lift of $\psi_\mu^{-1}(1)$ under the surjective map $\mathcal{L}_\mu \twoheadrightarrow \mathcal{L}_\mu \otimes_{\mathcal{O}_{X,\mu}} \mathcal{O}_{X_0,\mu}$. Then s is a meromorphic section of $\mathcal{L}$ and the divisor $\mathrm{div}(s) \in \mathcal{D}iv(X, X_0)(X)$ maps to $(\mathcal{L}, \psi)$.

It follows that $\mathrm{Pic}(X, X_0) \cong \mathcal{D}iv(X, X_0)(X)/\mathcal{M}^*_{X;X_0}(X)$. □

Claim 1.1.5. $\mathrm{Pic}(X, X_0)$ *is uniquely* n*-divisible.*

Since

$$\mathrm{Pic}(X, X_0) \cong \varprojlim_m \mathrm{Pic}(X_m, X_0) \cong \varprojlim_m H^1(X_0, 1 + \pi\mathcal{O}_{X_m}),$$

where the first isomorphism follows from [28, Thm. 5.1.4], it suffices to show that $H^1(X_0, 1+\pi\mathcal{O}_{X_m})$ is uniquely n-divisible. This can be seen as follows:

$$1 + \pi\mathcal{O}_{X_m} \supset 1 + \pi^2\mathcal{O}_{X_m} \supset ... \supset 1$$

defines a finite filtration on the sheaf $1 + \pi\mathcal{O}_{X_m}$ with graded pieces $gr^n = (\pi)^n/(\pi)^{n+1} \cong \mathcal{O}_{X_0} \otimes (\pi)^n$. We use this filtration to define a filtration on $H^1(X_0, 1 + \pi\mathcal{O}_{X_m})$ by

$$F^n := \mathrm{Im}(H^1(X_0, 1 + \pi^n\mathcal{O}_{X_m}) \to H^1(X_0, 1 + \pi\mathcal{O}_{X_m})).$$

The unique divisibility of $H^1(X_0, 1+\pi\mathcal{O}_{X_m})$ follows now by descending induction from the exact sequence

$$0 \to 1 + \pi^{n+1}\mathcal{O}_{X_m} \to 1 + \pi^n\mathcal{O}_{X_m} \to gr^n \to 0,$$

the unique divisibility of $H^i(X_0, \mathcal{O}_{X_0} \otimes \pi^n)$ as a finitely generated k-module and the five-lemma. □

It follows that $\mathrm{Pic}(X, X_0)/n \cong (\mathcal{D}iv(X, X_0)(X)/\mathcal{M}^*_{X;X_0}(X))/n = 0$ and therefore that the class of x in $Z_1(X)/n$, i.e. ξ, is in the image of $\ker[C_2(X, -1) \stackrel{res}{\to} C_1(X_0, 0)]$ under ∂.

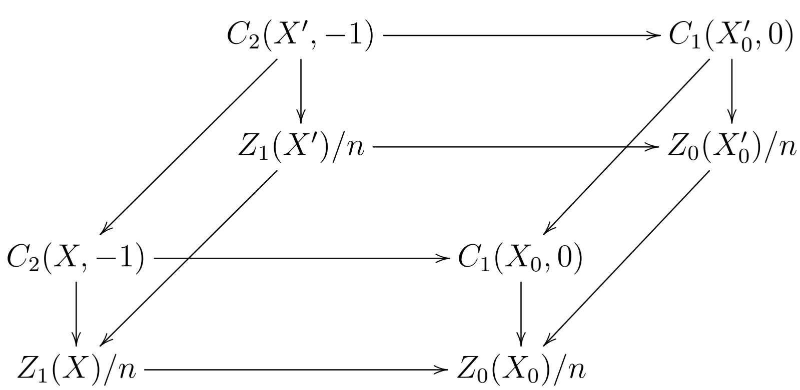
Let now $d > 1$. We start with some reduction steps.

As above we may represent ξ by an element of $\ker[Z_1(X) \to Z_0(X_0)]$ and as in the proof of [46, Prop. 4.1] we may assume that ξ is represented by a cycle of the form $[x] - r[y] \in \ker[Z_1(X) \to Z_0(X_0)]$ with x and y integral and such that y is regular and has intersection number 1 with X_0. Let us recall the argument: First note that one can lift a reduced closed point of X_0 to an integral horizontal one-cycle having intersection number 1 with X_0. Now if $\xi = \sum_{i=1}^{s} n_i[x_i] \in \ker[Z_1(X) \to Z_0(X_0)]$, then we lift $(x_i \cap X_0)_{\mathrm{red}}$ to a one-cycle y_i of the aforementioned type. Furthermore, we choose the same y_i for all the x_i intersecting X_0 in the same closed point. Let r_i be the intersection multiplicity of x_i with X_0. Then also $\sum_{i=1}^{s} n_i r_i [y_i] \in \ker[Z_1(X) \to Z_0(X_0)]$ and it suffices to show the statement for each $x_i - r_i y_i$ separately, i.e. the claim follows.

By an idea of Bloch put forward in [17, App.] we may assume furthermore that x is regular: Let $\tilde{x}$ be the normalisation of x. Since $\mathcal{O}_K$ is excellent, $\tilde{x}$ is finite over x. This implies that there is an imbedding $\tilde{x} \hookrightarrow X' := X \times_{\mathrm{Spec}\mathcal{O}_K} \mathbb{P}^N$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{x} & \longrightarrow & X' = X \times_{\mathrm{Spec}\mathcal{O}_K} \mathbb{P}^N \\ \downarrow & & \downarrow {\scriptstyle pr_X} \\ x & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\mathcal{O}_K & \xrightarrow{=} & \mathrm{Spec}\mathcal{O}_K \end{array}$$

Let $[\tilde{x} \cap X'_0] = r'[\bar{z}]$ for $\bar{z}$ an integral zero-dimensional subscheme of X'_0. We take a regular lift z of $\bar{z}$ in $y \times \mathbb{P}^N \subset X'$ which has intersection number 1 with X'_0 and get that $[\tilde{x}] - r'[z] \in \ker[Z_1(X') \to Z_0(X'_0)]$ and $pr_{X*}([\tilde{x}] - r'[z]) = [x] - r[y] = \xi$. The commutativity of the following diagram implies that proving the key lemma for a cycle $[x] - r[y] \in \ker[Z_1(X) \to Z_0(X_0)]$ with x regular and y intersecting the special fiber transversally suffices to prove it in general.



The commutativity of the diagram follows from [64, Sec. 4] since all the maps in question are defined in terms of the 'four basic maps' which are compatible.

We now prove the key lemma in the situation that $\xi = [x] - r[y] \in \ker[Z_1(X) \to Z_0(X_0)]$ with x regular and y intersecting the special fiber transversally by an induction on the relative dimension of X over $\mathrm{Spec}\mathcal{O}_K$, assuming that the key lemma holds in this situation for relative dimension $d-1$. By a standard norm argument we may from now on assume that k is infinite. We need the following Bertini theorem deduced from Bertini theorems by Altman and Kleiman:

Lemma 1.1.6. *There exist smooth closed subschemes $Z, Z' \subset X$ with the following properties:*

1. *Z has fiber dimension one, Z' has fiber dimension $d-1$.*
2. *Z contains $\tilde{x}$, Z' contains z.*
3. *The intersection $Z \cap Z' \cap X_0$ consist of reduced points.*

Proof. First note that for a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ we have the following short exact sequence:

$$0 \to \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-[X_0])(M) \to \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_X(M) \to \mathcal{J} \otimes_{\mathcal{O}_X} i_*\mathcal{O}_{X_0}(M) \to 0$$

for $i : X_0 \hookrightarrow X$ and $M \in \mathbb{Z}$. For $M \gg 0$ Serre vanishing implies that $H^1(X, \mathcal{F}(M)) = 0$ for $\mathcal{F}$ coherent and therefore that the map

$$\Gamma(\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_X(M)) \twoheadrightarrow \Gamma(\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}(M))$$

is surjective. This allows us to lift the sections on the right defining subvarieties of X_0 to sections of a twisted sheaf of ideals on X'.

Let $\mathcal{J}_{\tilde{x}}$ be the sheaf of ideals defining $\tilde{x}$ and $\mathcal{J}_z$ be the sheaf of ideals defining z. Let $p \in \tilde{x} \cap X_0$ ($q \in z \cap X_0$). Then $\dim_{X_0}(p) = d \geq 2$ and since $\tilde{x}$ (resp. z) is regular, we have that $e_{\tilde{x}\cap X_0}(p) \leq e_{\tilde{x}}(p) = \dim_{k(p)}(\Omega^1_{\tilde{x}}(p)) = 1 < 2$, where $e_{\tilde{x}}(p)$ is the embedding dimension of $\tilde{x}$ at p and analogously for q. Therefore by [49, Thm. 7] we can find sections in $\bar{\sigma}_1, ..., \bar{\sigma}_{d-1} \in \mathcal{J}_{\tilde{x}}|_{X_0}(M)$ (resp. $\bar{\sigma}' \in \mathcal{J}_{\tilde{x}}|_{X_0}(M)$) defining smooth subschemes containing p (resp. q) that intersect transversally. Let $\sigma_1, ..., \sigma_{d-1}$ (resp. σ') be liftings under the surjections $\Gamma(\mathcal{J}_{\tilde{x}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(M)) \twoheadrightarrow \Gamma(\mathcal{J}_{\tilde{x}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}(M))$ and $\Gamma(\mathcal{J}_z \otimes_{\mathcal{O}_X} \mathcal{O}_X(M)) \twoheadrightarrow \Gamma(\mathcal{J}_z \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}(M))$. Then the complete intersections $Z := V(\sigma_1, ..., \sigma_{d-1})$ and $Z' := V(\sigma')$ have the desired properties. □

Using these subschemes, we can now do the induction step. Since $Z \cap Z' \cap X_0$ consists of reduced points, the component z' of $Z \cap Z'$ that contains $z \cap X_0$ has intersection number 1 with X_0 and is a regular curve as it is regular over the closed point of $\mathrm{Spec}\mathcal{O}_K$. Now since Z' is of relative dimension $d-1$ and z and z' both lie in Z' and satisfy $res([z'] - [z]) = 0$, we get by the induction assumption that there is a ξ with support on Z' restricting to 1 and with $\partial(\xi) = [z'] - [z]$.

By the relative dimension one case proved in the beginning we get that for $\tilde{x}, z' \subset Z$ and $[\tilde{x}] - r'[z']$, which also restricts to 0, there is a ξ' with support on Z such that $res(\xi') = 0$ and $\partial(\xi') = [\tilde{x}] - r'[z']$. It follows that $res(\xi' + r\xi) = 1$ and $\partial(\xi' + r\xi) = [\tilde{x}] - r'[z]$. □

Corollary 1.1.7. *The restriction map*

$$res^{\mathrm{CH}} : \mathrm{CH}^d(X,1)_\Lambda \to \mathrm{CH}^d(X_0,1)_\Lambda$$

defined in the introduction is surjective.

Proof. We first show that the homology of the sequence

$$\bigoplus_{x\in X_0^{(d-2)}} K_2^M k(x) \to \bigoplus_{x\in X_0^{(d-1)}} K_1^M k(x) \to \bigoplus_{x\in X_0^{(d)}} K_0^M k(x)$$

is isomorphic to $\mathrm{CH}^d(X_0,1)$ which implies that $A_1(X_0,0)\cong \mathrm{CH}^d(X_0,1)_\Lambda$. This follows from the spectral sequence

$$E_1^{p,q} = \bigoplus_{x\in X_0^{(p)}} \mathrm{CH}^{r-p}(\mathrm{Spec}k(x),-p-q) \Rightarrow \mathrm{CH}^r(X_0,-p-q) \tag{1.1.2}$$

(see [4, Sec. 10]) for $r=d=\dim X_0$, the fact that $\mathrm{CH}^r(k(x),r)\cong K_r^M(k(x))$ and the vanishing of $\mathrm{CH}^r(\mathrm{Spec}k(x),j)$ for $r>j$ as well as the vanishing of $\mathrm{CH}^0(k(x),1)$.

Using a limit argument and the localization sequence for schemes over a regular noetherian base B of dimension one constructed in [50], we also get the existence of spectral sequence (1.1.2) for $X/\mathcal{O}_K$. Now for the same reasons as above this spectral sequence implies that the homology of

$$\bigoplus_{x\in X^{(d-2)}} K_2^M k(x) \to \bigoplus_{x\in X^{(d-1)}} K_1^M k(x) \to \bigoplus_{x\in X^{(d)}} K_0^M k(x)$$

is isomorphic to $\mathrm{CH}^d(X,1)$ which implies that $A_2(X,-1)\cong \mathrm{CH}^d(X,1)_\Lambda$.

The claim now follows from Proposition 1.1.2 and the compatibility of res and res^{CH}. □

Remark 1.1.8. *The isomorphism $A_1(X_0,0)\cong \mathrm{CH}^d(X_0,1)_\Lambda$ also follows from the isomorphism $\mathrm{CH}^p(X_0,1)\cong H^{p-1}(X_0,\mathcal{K}_p)$ for $p\geq 0$ and $\mathcal{K}_p$ the K-theory sheaf (see e.g. [58, Cor. 5.3]).*

1.2 Remarks on the injectivity of res

In this section we prove the injectivity of the restriction map for $d=2$ in our case and remark on implications of the conjectured injectivity.

Conjecture 1.2.1. *The map $res : A_2(X,-1)\to A_1(X_0,0)$ is injective.*

Proposition 1.2.2. *Conjecture 1.2.1 holds for $X/\mathcal{O}_K$ of relative dimension 2.*

Proof. Let $\Lambda := \mathbb{Z}/n$ and $\Lambda(q) := \mu_n^{\otimes q}$. We use the coniveau spectral sequence

$$E_1^{p,q}(X,\Lambda(c)) = \bigoplus_{x\in X^p} H_x^{p+q}(X,\Lambda(c)) \Rightarrow H_{\text{ét}}^{p+q}(X,\Lambda(c)),$$

where H_x^* is étale cohomology with support in x.

Cohomological purity (respectively absolute purity) gives isomorphisms $H_x^{p+q}(X,\Lambda(c)) \cong H^{q-p}(k(x),\Lambda(c-p))$ which lets us write the above spectral sequence in the following form:

$$E_1^{p,q}(X,\Lambda(c)) = \bigoplus_{x\in X^p} H^{q-p}(k(x),\Lambda(c-p)) \Rightarrow H_{\text{ét}}^{p+q}(X,\Lambda(c)).$$

For more details see for example [12]. Writing out this spectral sequence for X and X_0 respectively and using the norm residue isomorphism $K_n^M(k)/m \cong H^n(k,\mu_m^{\otimes n})$ for $n \le 2$ (see [56]), we get injective edge morphisms $A_2(X,-1) \hookrightarrow H_{\text{ét}}^3(X,\Lambda(2))$ and $A_1(X_0,-1) \hookrightarrow H_{\text{ét}}^3(X_0,\Lambda(2))$ for dimensional reasons. The restriction map induces a map between these spectral sequences and therefore a commutative diagram

$$\begin{array}{ccc} A_2(X,-1) & \longrightarrow & A_1(X_0,0) \\ \downarrow & & \downarrow \\ H_{\text{ét}}^3(X,\Lambda(2)) & \xrightarrow{\cong} & H_{\text{ét}}^3(X_0,\Lambda(2)) \end{array}$$

whose lower horizontal morphism is an isomorphism by proper base change. It follows that $A_2(X,-1) \to A_1(X_0,0)$ is injective. □

Remark 1.2.3. *The injectivity of res would have implications for a finiteness conjecture on the n-torsion of $\mathrm{CH}_0(X_K)$ for X_K a smooth scheme over a p-adic field with finite residue field and good reduction (see for example [10]). More precisely, using the coniveau spectral sequence, we can see that the group $A_1(X_K,0)$ is isomorphic to $H_{\text{Zar}}^{2d-1}(X_K,\mathbb{Z}/n(d))$ and therefore surjects onto $\mathrm{CH}_0(X_K)[n]$. Furthermore it fits into the exact sequence (see [64, Sec. 5])*

$$A_2(X,-1) \to A_1(X_K,0) \to A_1(X_0,-1) \cong \mathrm{CH}_1(X_0)/n.$$

Now Conjecture 1.2.1 implies that there is a sequence of injections

$$A_2(X,-1) \hookrightarrow A_1(X_0,0) \hookrightarrow H_{\text{ét}}^{2d-1}(X_0,\mathbb{Z}/n(d))$$

into the finite group $H_{\text{ét}}^{2d-1}(X_0,\mathbb{Z}/n(d))$. Note that the second injection follows from the Kato conjectures. More precisely, there is an exact sequence

$$KH_3(X_0,\mathbb{Z}/n\mathbb{Z}) \to A_1(X_0,0) \cong \mathrm{CH}^d(X_0,1)_\Lambda \to H_{\text{ét}}^{2d-1}(X_0,\mathbb{Z}/n(d))$$

(see [36, Lem. 6.2]) and the Kato homology group $KH_3(X_0,\mathbb{Z}/n\mathbb{Z})$ is zero due to the Kato conjectures (see [47]). Therefore the finiteness of $\mathrm{CH}_0(X_K)[n]$ would depend on the finiteness of $\mathrm{CH}_1(X_0)/n$.

In the case of relative dimension 2 the finiteness of $\mathrm{CH}_1(X_0)/n \cong \mathrm{Pic}(X_0)/n$ *can be shown using the injection* $\mathrm{Pic}(X_0)/n \hookrightarrow H^2_{\text{ét}}(X_0, \mu_n)$ *and the finiteness of* $H^2_{\text{ét}}(X_0, \mu_n)$ *(see f.e. [57, VI.2.8]). Therefore Proposition 1.2.2 implies the finiteness of* $\mathrm{CH}_0(X_K)[n]$ *for* X_K *a smooth surface over a p-adic field with finite residue field and good reduction which is a well-known result by Colliot-Thélène, Sansuc and Soulé. For more details on torsion questions see Section 2.5.*

Remark 1.2.4. *In the light of Remark 1.2.3 and the base change conjecture for higher zero-cycles stated in the introduction one might ask if*

$$\mathrm{CH}^d(X_K, i)[n]$$

is finite for all $i \geq 0$ *for smooth schemes over p-adic fields.*

Chapter 2

Zero-cycles with coefficients in Milnor K-theory

Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with quotient field K and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$ and always assume that $1/n \in k^\times$. Let X be a smooth and projective scheme over $\mathrm{Spec}\mathcal{O}_K$ of fiber dimension d. Let X_K denote the generic fiber and X_0 the reduced special fiber. By $X_{(p)}$ we denote the set of points $x \in X$ such that $\dim(\overline{\{x\}}) = p$, where $\overline{\{x\}}$ denotes the closure of x in X.

We call the groups

$$\mathrm{coker}(\bigoplus_{x\in X_0^{(d-1)}} K^M_{j-d+1}k(x) \to \bigoplus_{x\in X_0^{(d)}} K^M_{j-d}k(x))$$

and

$$H(\bigoplus_{x\in X^{(d-1)}} K^M_{j-d+1}k(x) \to \bigoplus_{x\in X^{(d)}} K^M_{j-d}k(x) \to \bigoplus_{x\in X^{(d+1)}} K^M_{j-d-1}k(x))$$

as well as the higher Chow groups

$$\mathrm{CH}^j(X_0, j-d)$$

and

$$\mathrm{CH}^j(X, j-d)$$

Chow groups of zero-cycles with coefficients in Milnor K-theory. Here $H(A \to B \to C) := \ker(B \to C)/\mathrm{im}(A \to B)$ for abelian groups A, B, C. We will see in Section 2.1 that the groups $\mathrm{coker}(\oplus_{x\in X_0^{(d-1)}} K^M_{j-d+1}k(x) \to \oplus_{x\in X_0^{(d)}} K^M_{j-d}k(x))$ and $\mathrm{CH}^j(X_0, j-d)$ are isomorphic. The identification of $H(\oplus_{x\in X^{(d-1)}} K^M_{j-d+1}k(x) \to \oplus_{x\in X^{(d)}} K^M_{j-d}k(x) \to \oplus_{x\in X^{(d+1)}} K^M_{j-d-1}k(x))$ and $\mathrm{CH}^j(X, j-d)$ depends on the Gersten conjecture for a henselian DVR for higher Chow groups if we work integrally. However, which is sufficient for our purposes, they are isomorphic if considered with finite coefficients (see Section 2.1).

In this chapter we study the restriction homomorphism on higher Chow groups

$$res^{\mathrm{CH}} : \mathrm{CH}^j(X, 2j-i) \to \mathrm{CH}^j(X_0, 2j-i)$$

for $i-j=d$. We recall the definition of the restriction homomorphism res^{CH} from Chapter 1. It is defined to be the composition

$$\mathrm{CH}^j(X,2j-i)\to \mathrm{CH}^j(X_K,2j-i)\xrightarrow{\cdot(-\pi)}\mathrm{CH}^{j+1}(X_K,2j-i+1)\xrightarrow{\partial}\mathrm{CH}^j(X_0,2j-i).$$

Here $\cdot(-\pi)$ is the product with a local parameter $-\pi\in\mathrm{CH}^1(K,1)=K^\times$ defined in [4, Sec. 5] and ∂ is the boundary map coming from the localization sequence for higher Chow groups (see [50]). We call the composition

$$\mathrm{CH}^j(X_K,2j-i)\xrightarrow{\cdot(-\pi)}\mathrm{CH}^{j+1}(X_K,2j-i+1)\xrightarrow{\partial}\mathrm{CH}^j(X_0,2j-i)$$

a specialisation map and denote it by sp_π^{CH}. One notes that res^{CH} is independent of the choice of π whereas sp_π^{CH} depends on it. We denote higher Chow groups with coefficients in a ring Λ by $\mathrm{CH}^j(X,j-d)_\Lambda$. From now on let $\Lambda=\mathbb{Z}/n\mathbb{Z}$. Our main result in this chapter is the following:

Theorem 2.0.5. *The restriction map*

$$res^{\mathrm{CH}}:\mathrm{CH}^j(X,j-d)_\Lambda\to\mathrm{CH}^j(X_0,j-d)_\Lambda$$

is an isomorphism for all j.

As mentioned in the introduction, this was conjectured in [46, Conj. 10.3] by Kerz, Esnault and Wittenberg. More precisely, they conjecture that the corresponding restriction homomorphism for motivic cohomology $res: H^{i,j}(X,\mathbb{Z}/n\mathbb{Z})\to H^{i,j}_{\mathrm{cdh}}(X_0,\mathbb{Z}/n\mathbb{Z})$ is an isomorphism for $i-j=d$. The case $j=d$ was first proved in [66] assuming that k is finite or separably closed and then generalised to arbitrary perfect residue fields in [46] using an idea of Bloch put forward in [17].

For $j=d+1$ and k finite, Theorem 2.0.5 also follows from the Kato conjectures. In fact, Jannsen and Saito observe that for $j=d+1$ and k finite, the étale cycle class map

$$\rho_X^{j,j-d}:\mathrm{CH}^j(X,j-d)_\Lambda\to H^{d+j}_{\text{ét}}(X,\Lambda(j))$$

fits into the exact sequence

$$\begin{aligned}...\to KH^0_{2+a}(X,\mathbb{Z}/n\mathbb{Z})\to\mathrm{CH}^{d+1}(X,a)_\Lambda\to H^{2d+2-a}_{\text{ét}}(X,\Lambda(d+1))\\ \to KH^0_{1+a}(X,\mathbb{Z}/n\mathbb{Z})\to\mathrm{CH}^{d+1}(X,a-1)_\Lambda\to...,\end{aligned}$$

where $KH^0_a(X,\mathbb{Z}/n\mathbb{Z})$ denotes the homology of certain complexes $C^0_n(X)$ in degree a defined by Kato. For more details see Section 2.2.

Theorem 2.0.5 implies the following two well-known corollaries:

Corollary 2.0.6. *Let X_K be a smooth projective scheme of dimension d with good reduction over a local field K with finite residue field k of characteristic p. Then the groups*

1. $\mathrm{CH}^j(X_K,j-d)_\Lambda$ *are finite for all $j\geq 0$.*

2. $\mathrm{CH}^j(X_K, j-d)_\Lambda = 0$ *for* $j \geq d+3$.

Corollary 2.0.7. *Let* X *be a smooth projective scheme over an excellent henselian discrete valuation ring* $\mathcal{O}_K$ *with finite or algebraically closed residue field. Then*

$$\rho_X^{j,j-d} : \mathrm{CH}^j(X, j-d)_\Lambda \to H^{d+j}_{\text{ét}}(X, \Lambda(j))$$

is an isomorphism for all j.

In the last two sections of this chapter, we turn to torsion questions for Chow groups of zero-cycles with coefficients in Milnor K-theory. We show the following two propositions:

Proposition 2.0.8. *Let* X_K *be a smooth and proper scheme over a local field* K *with ring of integers* $\mathcal{O}_K$ *and finite residue field* k *of characteristic* p*. Assume that* X_K *has good reduction over* $\mathcal{O}_K$ *and let* $n > 0$ *be a natural number prime to* p*. Then for all* $j \geq 1$ *the groups*

$$\mathrm{CH}^{d+j}(X_K, j)[n]$$

are finite.

Proposition 2.0.9. *Let* X_k *be a smooth projective scheme of dimension* d *over a finite field* k *of characteristic* $p > 0$*. Then the group*

$$\mathrm{CH}^{d+1}(X_k, 1)$$

is finite.

2.1 Identification of zero-cycles with coefficients in Milnor K-theory

We start by recalling some basic facts about Milnor K-theory.

Definition 2.1.1. *Let* k *be a field. We define the* n*-th Milnor K-group* $K_n^M(k)$ *to be the quotient of* $(k^\times)^{\otimes n}$ *by the Steinberg group, i.e. the subgroup of* $(k^\times)^{\otimes n}$ *generated by elements of the form* $a_1 \otimes ... \otimes a_n$ *satisfying* $a_i + a_j = 1$ *for some* $1 \leq i < j \leq n$*. Elements of* $K_n^M(k)$ *are called symbols and the image of* $a_1 \otimes ... \otimes a_n$ *in* $K_n^M(k)$ *is denoted by* $\{a_1, ..., a_n\}$*.*

One can easily see that in $K_2^M(k)$ the following relations hold:

$$\{x, -x\} = 0 \text{ and } \{x, x\} = \{x, -1\}$$

This implies the following lemma:

Lemma 2.1.2. *Let* K *be a discrete valuation ring with ring of integers* A*, local parameter* π *and residue field* k*. Then* $K_n^M(k)$ *is generated by symbols of the form*

$$\{\pi, u_2, ..., u_n\} \text{ and } \{u_1, u_2, ..., u_n\}$$

with $u_i \in A^\times$ *for* $1 \leq i \leq n$*.*

Keeping the notation of Lemma 2.1.2 and denoting the image of u_i in $K_n^M(k)$ by $\bar{u}_i$, one can show that there exists a unique homomorphism

$$\partial : K_n^M(K) \to K_{n-1}^M(k)$$

satisfying

$$\partial(\{\pi, u_2, ..., u_n\}) = \{\bar{u}_2, ..., \bar{u}_n\},$$

called the tame symbol, and a unique homomorphism

$$sp_\pi : K_n^M(K) \to K_n^M(k)$$

satisfying

$$sp_\pi(\{\pi^{i_1}u_1, \pi^{i_2}u_2, ..., \pi^{i_n}u_n\}) = \{\bar{u}_1, ..., \bar{u}_n\},$$

called the specialisation map. Note that sp_π, unlike ∂, depends on the choice of a local parameter in K and that $\partial(\{u_1, u_2, ..., u_n\}) = 0$, if $u_i \in A^\times$ for all $1 \leq i \leq n$.

We now return to the situation of the introduction: Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with quotient field K and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$ and always assume that $1/n \in k^\times$. Let X be a smooth and projective scheme over $\mathrm{Spec}\mathcal{O}_K$ of fiber dimension d. Let X_K denote the generic fiber and X_0 the reduced special fiber. By $X_{(p)}$ we denote the set of points $x \in X$ such that $\dim(\overline{\{x\}}) = p$, where $\overline{\{x\}}$ denotes the closure of x in X.

We use the following notation for Rost's Chow groups with coefficients in Milnor K-theory (see [64, Sec. 5]):

$$C_p(X, m) = \bigoplus_{x \in X_{(p)}} (K_{m+p}^M k(x)) \otimes \mathbb{Z}/n\mathbb{Z},$$

$$Z_p(X, m) = \ker[\partial : C_p(X, m) \to C_{p-1}(X, m)],$$

$$A_p(X, m) = H_p(C_*(X, m))$$

and similarly for X_0 (resp. X_K) replacing X by X_0 (resp. X_K). Furthermore, let

$$C_p^g(X, m) = \bigoplus_{x \in X_{(p)}^g} (K_{m+p}^M k(x)) \otimes \mathbb{Z}/n\mathbb{Z}$$

and

$$Z_p^g(X, m) = \ker[\partial : C_p^g(X, m) \to C_{p-1}(X, m)]$$

be the corresponding groups supported on cycles in good position, i.e. the sum is taken over all $x \in X_{(p)}$ such that $\overline{\{x\}}$ is flat over $\mathcal{O}_K$. Note that $C_k(X, -k) = Z_k(X) \otimes \mathbb{Z}/n\mathbb{Z}$, the group of k-cycles on X, i.e. the free abelian group generated by k-dimensional closed subschemes of X, tensored with $\mathbb{Z}/n\mathbb{Z}$.

Let now π be a local parameter of $\mathcal{O}_K$. We define the restriction map

$$res_\pi : C_p(X, m) \to C_{p-1}(X_0, m + 1)$$

as in Chapter 1 to be the composition

$$res_\pi : C_p(X,m) \to C_{p-1}(X_K, m+1) \xrightarrow{\cdot\{-\pi\}} C_{p-1}(X_K, m+2) \xrightarrow{\partial} C_{p-1}(X_0, m+1),$$

where $C_p(X,m) \to C_{p-1}(X_K, m+1)$ is defined to be the identity on all elements supported on $X_{(p)} \setminus X_{0(p)}$ and zero on $X_{0(p)}$ and ∂ is the boundary map induced by the tame symbol. For the fact that this composition is compatible with the corresponding cycle complexes see [64, Sec. 4]. For more details see Chapter 1. We just recall that the restriction map res_π depends on choice of π but the induced map $res : A_p(X,m) \to A_{p-1}(X_0, m+1)$ is independent of such a choice. From now on, we will fix a $\pi \in \mathcal{O}_K$ and write res instead of res_π.

We now prove the identifications of the two versions of Chow groups of zero-cycles with coefficients in Milnor K-theory stated in the introduction.

Proposition 2.1.3. *For all $j \geq 0$, there are the following isomorphisms:*

1. $\mathrm{CH}^j(X_0, j-d) \cong \mathrm{coker}(\bigoplus_{x \in X_0^{(d-1)}} K^M_{j-d+1}k(x) \to \bigoplus_{x \in X_0^{(d)}} K^M_{j-d}k(x))$.
2. $\mathrm{CH}^j(X, j-d)_\Lambda \cong A_1(X, j-d-1)$.

Proof. In [4, Sec. 10], Bloch proves the existence of the spectral sequence

$$^{\mathrm{CH}}E_1^{p,q} = \bigoplus_{x \in X_0^{(p)}} \mathrm{CH}^{r-p}(\mathrm{Spec} k(x), -p-q) \Rightarrow \mathrm{CH}^r(X_0, -p-q). \tag{2.1.1}$$

Using the localization sequence for higher Chow groups for schemes over a regular noetherian base (see [50]) and a limit argument, one also gets the existence of the spectral sequence

$$^{\mathrm{CH}}E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \mathrm{CH}^{r-p}(\mathrm{Spec} k(x), -p-q) \Rightarrow \mathrm{CH}^r(X, -p-q). \tag{2.1.2}$$

Setting $j = r$, and noting that $\mathrm{CH}^r(k(x), r) \cong K^M_r(k(x))$, it follows from spectral sequence (2.1.1) that $\mathrm{CH}^j(X_0, j-d)$ is isomorphic to the cokernel of

$$\bigoplus_{x \in X_0^{(d-1)}} K^M_{j-d+1}k(x) \to \bigoplus_{x \in X_0^{(d)}} K^M_{j-d}k(x)$$

since $E_1^{\bullet, \geq -j+1} = 0$ as $\mathrm{CH}^s(k(x), t) = 0$ for $s > t$.

Similarly, we get from spectral sequence (2.1.2) that $\mathrm{CH}^j(X, j-d)_\Lambda$ is isomorphic to $A_1(X, j-d-1)$, noting that the map

$$\bigoplus_{x \in X^{(d)}} \mathrm{CH}^{j-d}(k(x), j-d+1)_\Lambda \to \bigoplus_{x \in X^{(d+1)}} \mathrm{CH}^{j-d-1}(k(x), j-d)_\Lambda$$

is surjective for all j (see Figure 2.1). This can be seen as follows: Let $x \in X^{(d+1)}$ and let y be the generic point of a regular lift Z of x to X which is flat over $\mathcal{O}_K$. Now by the Beilinson-Lichtenbaum conjecture (see Theorem 2.2.3), $\mathrm{CH}^{j-d}(k(x), j-d+1)_\Lambda$ is isomorphic to

$H^{j-d-1}(k(y),\Lambda(j-d))$ and $\mathrm{CH}^{j-d-1}(k(x),j-d)_\Lambda$ is isomorphic to $H^{j-d-2}(k(x),\Lambda(j-d-1))$. The assertion now follows from the surjectivity of the map

$$\partial : H^{j-d-1}(k(y),\Lambda(j-d)) \to H^{j-d-2}(k(x),\Lambda(j-d-1)).$$

Since $\mathcal{O}_{Z,x}$ is henselian, $H^{j-d-2}(k(x),\Lambda(j-d-1)) \cong H^{j-d-2}(\mathcal{O}_{Z,x},\Lambda(j-d-1))$ by rigidity for étale cohomology. An element $\alpha \in H^{j-d-2}(k(x),\Lambda(j-d-1))$ corresponding to an element $\alpha' \in H^{j-d-2}(\mathcal{O}_{Z,x},\Lambda(j-d-1))$ lifts to an element $\alpha' \cup s \in H^{j-d-1}(k(y),\Lambda(j-d))$, s being a generator of the maximal ideal of $\mathcal{O}_{Z,x}$, with $\partial(\alpha' \cup s) = \alpha$ (see also [38, Lem. 1.4 (2)]). □

Remark 2.1.4. 1. *For similar identifications for X_0 with motivic cohomology see also [65].*

2. *In order to show the isomorphism*

$$\mathrm{CH}^j(X,j-d) \cong H(\bigoplus_{x\in X^{(d-1)}} K^M_{j-d+1}k(x) \to \bigoplus_{x\in X^{(d)}} K^M_{j-d}k(x) \to \bigoplus_{x\in X^{(d+1)}} K^M_{j-d-1}k(x))$$

integrally, one would need to show the Gersten conjecture for a henselian DVR for higher Chow groups, i.e. the exactness of the sequence

$$0 \to \mathrm{CH}^r(\mathrm{Spec}A,q) \to \mathrm{CH}^r(\mathrm{Spec}K,q) \to \mathrm{CH}^{r-1}(\mathrm{Spec}k,q-1) \to 0$$

for a henselian discrete valuation ring A with field of fractions K and residue field k.

3. *Let A be as in* (2)*. If k is of characteristic $p>0$, then the sequence*

$$0 \to \mathrm{CH}^r(A,\mathbb{Z}/p^r\mathbb{Z},q) \to \mathrm{CH}^r(K,\mathbb{Z}/p^r\mathbb{Z},q) \to \mathrm{CH}^{r-1}(k,\mathbb{Z}/p^r\mathbb{Z},q-1) \to 0$$

is exact. This follows from the fact that in this case $\mathrm{CH}^{r-1}(\mathrm{Spec}k,\mathbb{Z}/p^r\mathbb{Z},q-1)=0$ for $r \neq q$ by [22, Thm. 1.1] and that

$$\mathrm{CH}^r(K,\mathbb{Z}/p^r\mathbb{Z},r) \cong K^M_r(K)/p^r \to K^M_{r-1}(k)/p^r \cong \mathrm{CH}^{r-1}(k,\mathbb{Z}/p^r\mathbb{Z},r-1)$$

is surjective which implies that the long exact localization sequence

$$\ldots \to \mathrm{CH}^r(A,\mathbb{Z}/p^r\mathbb{Z},q) \to \mathrm{CH}^r(K,\mathbb{Z}/p^r\mathbb{Z},q) \to \mathrm{CH}^{r-1}(k,\mathbb{Z}/p^r\mathbb{Z},q-1) \to \ldots$$

splits (see also [21, Cor. 4.3]).

4. *If k is finite, then $\mathrm{CH}^j(X_0,j-d)=0$ for $j>d+1$ since $K^M_2(k)=0$ in that case. If K is a local field, then $\mathrm{CH}^j(X_K,j-d)_\Lambda=0$ for $j>d+2$ since $K^M_n(K)$ is uniquely divisible for $n\geq 3$ (see [78, VI. 7.1]).*

		d		$d+1$
$-j+1$	0	0		0
$-j$	...	$\bigoplus_{x\in X^{(d)}} \mathrm{CH}^{j-d}(k(x), j-d)$	$\longrightarrow$	$\bigoplus_{x\in X^{(d+1)}} \mathrm{CH}^{j-d-1}(k(x), j-d-1)$
$-j-1$	...	$\bigoplus_{x\in X^{(d)}} \mathrm{CH}^{j-d}(k(x), j-d+1)$	$\longrightarrow$	$\bigoplus_{x\in X^{(d+1)}} \mathrm{CH}^{j-d-1}(k(x), j-d)$
	...	...		...

Figure 2.1: Table of ${}^{\mathrm{CH}}E_1^{p,q}$ for $X/\mathcal{O}_K$.

2.2 Relation with Kato complexes

In this section we recall some facts about the Kato conjectures which we will need in the following sections.

Let X be an excellent scheme. In [38], Kato defines the following complexes:

$$C_n^i(X) : \ldots \to \bigoplus_{x\in X_a} H^{i+a+1}(k(x), \mathbb{Z}/n(i+a)) \to \ldots \to \bigoplus_{x\in X_1} H^{i+2}(k(x), \mathbb{Z}/n(i+1)) \to \bigoplus_{x\in X_0} H^{i+1}(k(x), \mathbb{Z}/n(i))$$

Here the term $\oplus_{x\in X_a} H^{i+a+1}(k(x), \mathbb{Z}/n(i+a))$ is placed in degree a. We denote the homology of $C_n^i(X)$ in degree a by $KH_a^i(X, \mathbb{Z}/n\mathbb{Z})$. The groups $H^{i+a+1}(k(x), \mathbb{Z}/n(i+a))$ are the étale cohomology groups of $\mathrm{Spec} k(x)$ with coefficients in $\mathbb{Z}/n(i+a) := \mu_n^{\otimes i+a}$ if n is invertible on X and $\mathbb{Z}/n(i+a) := W_r\Omega^{i+a}_{X_1,\log}[-(i+a)] \oplus \mathbb{Z}/m(i)$ if $n = mp^r$ is not invertible on X and X is smooth over a field of characteristic p.

The complex $C_n^0(X)$ for a proper smooth scheme X over a finite field or the ring of integers in a (1-)local field is the subject of the study of the Kato conjectures. The Kato conjectures say the following and have been fully proved in case the coefficient characteristic is invertible on X by Jannsen, Kerz and Saito (see [47]):

Conjecture 2.2.1. *Let X be a proper smooth scheme over a finite field. Then*

$$KH_a^0(X, \mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for } a > 0.$$

Conjecture 2.2.2. *Let X be a regular scheme proper and flat over $\mathrm{Spec}(\mathcal{O}_k)$, where $\mathcal{O}_k$ is the ring of integers in a local field. Then*

$$KH_a^0(X, \mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for } a \geq 0.$$

In [36, Lem. 6.2], Jannsen and Saito relate the complex $C_n^0(X)$ for a smooth scheme X over a finite field to the étale cycle class map

$$\rho_X^{r,2r-s} : \mathrm{CH}^r(X, 2r-s)_\Lambda \to H^s_{\text{ét}}(X, \Lambda(r))$$

for $r = d$. More precisely, they show that there is an exact sequence

$$\begin{aligned} \ldots \to KH^0_{q+2}(X,\mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^d(X,q)_\Lambda \to H^{2d-q}_{\text{ét}}(X,\Lambda(d)) \\ \to KH^0_{q+1}(X,\mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^d(X,q-1)_\Lambda \to \ldots \end{aligned} \tag{2.2.1}$$

This sequence is a tool to deduce finiteness results for Chow groups of higher zero cycles with finite coefficients from the Kato conjectures. The proof of the exactness of (2.2.1) uses the coniveau spectral sequence for the domain and target of $\rho_X^{r,q}$ and the following theorem of Voevodsky:

Theorem 2.2.3. *(Beilinson-Lichtenbaum conjecture, see [77]) Let X be a smooth scheme over a field. Then the étale cycle map*

$$\rho_X^{r,2r-s} : \mathrm{CH}^r(X,2r-s)_\Lambda \to H^s_{\text{ét}}(X,\Lambda(r))$$

is an isomorphism for $s \leq r$.

We recall the following Proposition from [46, Prop. 9.1]:

Proposition 2.2.4. *Let X be a proper smooth scheme over a finite or algebraically closed field. Then the étale cycle map*

$$\rho_X^{j,j-d+a} : \mathrm{CH}^j(X,j-d+a)_\Lambda \to H^{j+d-a}_{\text{ét}}(X,\Lambda(j))$$

is an isomorphism for all $j \geq d$ and all a except possibly if k is finite, $j = d$ and $a = -1$. In particular the groups $\mathrm{CH}^j(X,j-d+a)_\Lambda$ are finite if $j \geq d, a \geq 0$.

Proof. We consider the spectral sequences

$${}^{\mathrm{CH}}E_1^{p,q}(X) = \oplus_{x\in X^{(p)}}\mathrm{CH}^{j-p}(\mathrm{Spec}k(x),-p-q)_\Lambda \Rightarrow \mathrm{CH}^j(X,-p-q)_\Lambda$$

and

$${}^{\text{ét}}E_1^{p,q}(X,\Lambda(j)) = \oplus_{x\in X^p}H^{q-p}(k(x),\Lambda(j-p)) \Rightarrow H^{p+q}_{\text{ét}}(X,\Lambda(j)). \tag{2.2.2}$$

The étale cycle class map $\rho_X^{r,q}$ induces a map of spectral sequences

$$\rho_X^{r,q} : {}^{\mathrm{CH}}E_1^{p,q} \to {}^{\text{ét}}E_1^{p,q+2j}$$

which by Theorem 2.2.3 is an isomorphism for $q \leq -j$. By cohomological dimension, the difference between the two spectral sequences is given by

$${}^{\text{ét}}E_1^{\bullet,j+1} = C_n^{j-d}(X)$$

which is equal to the zero-complex if $j > d$ or if k is an algebraically closed field. If $j = d$, then the complex ${}^{\text{ét}}E_1^{\bullet,d+1} = C_n^0(X)$ is exact except for possibly the last term on the right due to Conjecture 2.2.1. □

We now turn to the arithmetic case (see also [47, Sec. 9]).

Proposition 2.2.5. *Let X be a proper smooth scheme over a henselian discrete valuation ring $\mathcal{O}_K$ with finite residue field k. Let d be the relative dimension of X over $\mathcal{O}_K$. Then for $j = d+1$, there is an exact sequence*

$$\begin{aligned} \dots \to KH^0_{2+a}(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+1}(X,a)_\Lambda \to H^{2d+2-a}_{\text{ét}}(X, \Lambda(d+1)) \\ \to KH^0_{1+a}(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+1}(X, a-1)_\Lambda \to \dots \end{aligned} \tag{2.2.3}$$

and for $j > d+1$, there are isomorphisms

$$\mathrm{CH}^j(X, j-d+a)_\Lambda \to H^{j+d-a}_{\text{ét}}(X, \Lambda(j)).$$

Proof. We keep the notation of the proof of Proposition 2.2.4. Like there, we get a map of spectral sequences

$$\rho^{r,q}_X : {}^{\mathrm{CH}}E^{p,q}_1 \to {}^{\text{ét}}E^{p,q+2j}_1$$

which by Theorem 2.2.3 is an isomorphism for $q \leq -j$. The difference between the two spectral sequences is given by

$${}^{\text{ét}}E^{\bullet,j+1}_1 = C^{j-d-1}_n(X)$$

if $j \geq d+1$ since all other rows vanish by cohomological dimension and $C^{j-d-1}_n(X) = 0$ for $j \geq d+2$ again by cohomological dimension. This implies the proposition. □

Remark 2.2.6. *For X be a scheme over an excellent henselian discrete valuation ring with finite residue field and $j = d$ we are in the situation of [66] which is more complex since there are two rows (${}^{\text{ét}}E^{\bullet,j+1}_1 = C^{-1}_n(X)$ and ${}^{\text{ét}}E^{\bullet,j+2}_1$) which might not be quasi-isomorphic to zero. In [66], Saito and Sato show that $KH^{-1}_a(X, \mathbb{Q}_n/\mathbb{Z}_n) = 0$ for $a = 2, 3$.*

We keep the notation of Proposition 2.2.5. It follows from Conjecture 2.2.2 that

$$res^{\mathrm{CH}} : \mathrm{CH}^{d+1}(X,a)_\Lambda \to \mathrm{CH}^{d+1}(X_0,a)_\Lambda$$

is an isomorphism for all a. In the next section, we generalise this result for $a = 1$ to arbitrary residue fields. It remains an open problem if $res^{\mathrm{CH}} : \mathrm{CH}^{d+1}(X,a)_\Lambda \to \mathrm{CH}^{d+1}(X_0,a)_\Lambda$ is an isomorphism for arbitrary residue fields for all a.

2.3 Main theorem

We keep the notation of the introduction of this chapter. In this section we prove Theorem 2.0.5. By the identifications of Proposition 2.1.3, this comes down to studying the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{x \in X^{(d-1)}} K^M_{j-d+1} k(x) & \xrightarrow{sp} & \bigoplus_{x \in X_0^{(d-1)}} K^M_{j-d+1} k(x) \\ \downarrow \partial & & \downarrow \partial \\ \bigoplus_{x \in X^{(d)}} K^M_{j-d} k(x) & \xrightarrow{sp} & \bigoplus_{x \in X_0^{(d)}} K^M_{j-d} k(x) \\ \downarrow \partial & & \\ \bigoplus_{x \in X^{(d+1)}} K^M_{j-d-1} k(x) & & \end{array}$$

j	...	$\bigoplus_{x\in X^{(1)}} H^{j-1}(k(x),\Lambda(j-1))$	$\to$	$\bigoplus_{x\in X^{(2)}} H^{j-2}(k(x),\Lambda(j-2))$
$j-1$	...	$\bigoplus_{x\in X^{(1)}} H^{j-2}(k(x),\Lambda(j-1))$	$\to$	$\bigoplus_{x\in X^{(2)}} H^{j-3}(k(x),\Lambda(j-2))$
	...	...		...
2	...	$\bigoplus_{x\in X^{(1)}} H^{1}(k(x),\Lambda(j-1))$	$\longrightarrow$	$\bigoplus_{x\in X^{(2)}} H^{0}(k(x),\Lambda(j-2))$
1	...	$\bigoplus_{x\in X^{(1)}} H^{0}(k(x),\Lambda(j-1))$		0
0	...	0		0
	0	1		2

Figure 2.2: Table of $E^1_{p,q}(X,\Lambda)$ for $X/\mathcal{O}_K$ and $d=1$.

We first note the surjectivity of res^{CH} resp. res.

Proposition 2.3.1. *The specialisation map*

$$res : A_1(X, j-d-1) \to A_0(X_0, j-d)$$

is surjective for all j.

Proof. Let $\{\bar{u}_1, ..., \bar{u}_{j-d}\} \in K^M_{j-d}k(x)$ for some $x \in X_0^{(d)}$. Let $y \in X^{(d)}$ be the generic point of a lift Z of x which intersects X_0 transversally in x. Let A be the stalk of Z at y and denote by $u_i \in A^\times$ a lift of $\bar{u}_i$ to the units of A. Then $res(\{u_1, ..., u_{j-d}\}) = \{\bar{u}_1, ..., \bar{u}_{j-d}\}$ and $\partial(\{u_1, ..., u_{j-d}\}) = 0$. □

Key lemma 2.3.2. *Let*

$$\alpha \in \ker(Z^g_1(X, j-d-1) \to Z_0(X_0, j-d)).$$

Then $\alpha \equiv 0 \in A_0(X, j-d-1)$. In particular there is a well-defined map

$$\phi : Z_0(X_0, j-d) \to A_1(X, j-d-1).$$

Proof. We start with the case of relative dimension $d=1$. Let $\Lambda := \mathbb{Z}/n$ and $\Lambda(q) := \mu_n^{\otimes q}$. We consider the coniveau spectral sequence

$$E_1^{p,q}(X, \Lambda(j)) = \oplus_{x\in X^p} H^{q-p}(k(x), \Lambda(j-p)) \Rightarrow H^{p+q}_{\text{ét}}(X, \Lambda(j))$$

for X and X_0 respectively (see Figure 2.2 for $E^1_{p,q}(X,\Lambda)$ for $X/\mathcal{O}_K$ and $d=1$) and the norm residue isomorphism $K^M_n(k)/m \cong H^n(k, \mu_m^{\otimes n})$ to show that there are injective edge morphisms

$$A_1(X, j-d-1) = E_2^{d,j}(X) \hookrightarrow H^{d+j}_{\text{ét}}(X, \Lambda(j))$$

and
$$A_0(X_0, j-d) = E_2^{d,j}(X_0) \hookrightarrow H^{d+j}_{\text{ét}}(X_0, \Lambda(j)).$$
The injectivity in the second case, i.e. for X_0, is trivial since we just have two non-trivial columns for dimensional reasons. In the first case, we have three columns but $E_2^{2,j}(X, \Lambda(c))$ is equal to zero for all j since the map

$$\oplus_{x\in X^{(1)}} H^{j-2}(k(x), \Lambda(j-1)) \to \oplus_{x\in X^{(2)}} H^{j-3}(k(x), \Lambda(j-2))$$

is surjective by the same arguments as in the proof of 2.1.3. The restriction map induces a map between the respective spectral sequences for X and X_0 and therefore a commutative diagram

$$\begin{array}{ccc} A_1(X, j-d-1) & \longrightarrow & A_0(X_0, j-d) \\ \downarrow & & \downarrow \\ H^{d+j}_{\text{ét}}(X, \Lambda(j)) & \xrightarrow{\cong} & H^{d+j}_{\text{ét}}(X_0, \Lambda(j)) \end{array}$$

whose lower horizontal morphism is an isomorphism by proper base change. It follows that $A_1(X, j-d-1) \to A_0(X_0, j-d)$ is injective.

Let now $d > 1$. We start with some reduction steps. Let

$$\alpha \in \ker(Z_1^g(X, j-d-1) \to Z_0(X_0, j-d)).$$

By definition,

$$\alpha = \sum_{x\in X^g_{(1)}} \alpha_x \in \ker(res : \oplus_{x\in X^g_{(1)}} K^M_{j-d}k(x) \to \oplus_{x\in X_0^{(d)}} K^M_{j-d}k(x))$$

and $\alpha \in \ker(\partial : \oplus_{x\in X^{(d)}} K^M_{j-d}k(x) \to \oplus_{x\in X^{(d+1)}} K^M_{j-d-1}k(x))$ with $\alpha_x \in K^M_{j-d}k(x)$. We may assume that
$$\alpha \in \ker(res : \oplus_{x\in X^g_{(1)}} K^M_{j-d}k(x) \to K^M_{j-d}k(x_0))$$
for some $x_0 \in X_0^{(d)}$.

Let $y \in X^g_{(1)}$ be the generic point of a lift Z_1 of x_0 which intersects X_0 transversally in x_0. We may now assume that

$$\alpha = (\alpha_y, \alpha_z) \in \ker(res : K^M_{j-d}k(y) \oplus K^M_{j-d}k(z) \to K^M_{j-d}k(x_0))$$

for some $z \in X^g_{(1)}$. This follows from the fact that for every $z \in X^g_{(1)}$ which intersects X_0 in x_0, we can lift $res(\alpha_z)$ to an element $\alpha_y \in K^M_{j-d}k(y)$ such that $\partial(\alpha_y) = \partial(\alpha_z)$. This can be seen as follows: Let $\alpha'_y \in K^M_{j-d}k(y)$ be a lift of $res(\alpha_z)$. If

$$\partial(\alpha'_y) - \partial(\alpha_z) = \sum_{s\in S} \{\bar{u}_1^{(s)}, ..., \bar{u}^{(s)}_{j-d-1}\} \neq 0$$

for some $\bar{u}_i^{(s)} \in k(x_0)^\times, 1 \leq i \leq j-d-1$ and some finite index set S, then choosing $u_i^{(s)}$ to be a lift of $\bar{u}_i^{(s)}$ to a unit in the discrete valuation ring of $k(y)$, we can set

$$\alpha_y := \alpha'_y - \sum_{s\in S}\{\pi, u_1^{(s)}, ..., u_{j-d}^{(s)}\}.$$

This has the required properties since $res(\{\pi, u_1^{(s)}, ..., u_{j-d}^{(s)}\}) = 0$ and $\partial(\{\pi, u_1^{(s)}, ..., u_{j-d}^{(s)}\}) = \{\bar{u}_1^{(s)}, ..., \bar{u}_{j-d-1}^{(s)}\}$.

We now apply an idea of Bloch to reduce our situation to the case that $Z_1 = \overline{\{y\}}$ intersects X_0 transversally and that $Z_2 = \overline{\{z\}}$ is regular (see [17, App.]). Let $\tilde{Z}_2$ be the normalisation of Z_2. Since $\mathcal{O}_K$ is excellent, $\tilde{Z}_2 \to Z_2$ is finite and projective. This implies that there is an imbedding $\tilde{Z}_2 \hookrightarrow X' := X \times_{\mathrm{Spec}\mathcal{O}_K} \mathbb{P}^N$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{Z}_2 & \longrightarrow & X' = X \times_{\mathrm{Spec}\mathcal{O}_K} \mathbb{P}^N \\ \downarrow & & \downarrow{\scriptstyle pr_X} \\ Z_2 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\mathcal{O}_K & \xrightarrow{=} & \mathrm{Spec}\mathcal{O}_K \end{array}$$

Let $(\tilde{Z}_2 \cap X'_0)_{\mathrm{red}} = x'_0$ for x'_0 an integral zero-dimensional subscheme of X'_0. Let $\tilde{Z}_1$ be a regular lift of x' in $Z_1 \times \mathbb{P}^N \subset X'$ which has intersection number 1 with X'_0. We denote the generic points of $\tilde{Z}_1$ and $\tilde{Z}_2$ by y' and z' respectively. Note first now that $\alpha_z \in K^M_{j-d}k(z')$. Then, taking into account ramification, we can lift $res(\alpha_z) \in K^M_{j-d}k(x'_0)$ to an element $\alpha'_y \in K^M_{j-d}k(y')$ such that (α'_y, α_z) lies in the kernel of $res : \oplus_{x\in X'^{(d+N)}} K^M_{j-d+1}k(x) \to \oplus_{x\in X_0'^{(d+N)}} K^M_{j-d}k(x)$ as well as the kernel of $\partial : \oplus_{x\in X'^{(d+N)}} K^M_{j-d}k(x) \to \oplus_{x\in X'^{(d+N-1)}} K^M_{j-d}k(x)$ and such that, furthermore, we have that $pr_X((\alpha'_y, \alpha_z)) = (\alpha_y, \alpha_z)$. It therefore remains to show that (α'_y, α_z) is in the image of the boundary map

$$\partial : \oplus_{x\in X'^{(d+N+1)}} K^M_{j-d+1}k(x) \to \oplus_{x\in X'^{(d+N)}} K^M_{j-d}k(x).$$

We show that the key lemma holds for

$$\alpha = (\alpha_y, \alpha_z) \in \ker(res : K^M_{j-d}k(y) \oplus K^M_{j-d}k(z) \to K^M_{j-d}k(x_0))$$

as above assuming that $Z_1 = \overline{\{y\}}$ intersects X_0 transversally and that $Z_2 = \overline{\{z\}}$ is regular by an induction on the relative dimension d of X over $\mathrm{Spec}\mathcal{O}_K$.

Using a standard norm argument, we may assume that the residue field of $\mathrm{Spec}\mathcal{O}_K$ is infinite. By standard Bertini arguments (cf. [46, Sec. 4] or Lemma 1.1.6), we can find smooth closed subschemes $S_1, S_2 \subset X$ with the following properties:

1. S_1 has fiber dimension one, S_2 has fiber dimension $d-1$.

2. S_1 contains Z_1, S_2 contains Z_2.

3. The intersection $S_1 \cap S_2 \cap X_0$ consist of reduced points.

Let Z_3 denote the component of $S_1 \cap S_2$ containing x_0 and let t denote its generic point. Then again there is an $\alpha_t \in K^M_{j-d}k(t)$ such that $res(\alpha_t) = res(\alpha_y) = -res(\alpha_z)$ and $\partial(\alpha_t) = \partial(\alpha_y) = -\partial(\alpha_z)$. Now by our induction assumption, both (α_y, α_t) and (α_x, α_t) map to zero in $A_1(X, j-d-1)$ and therefore so does (α_y, α_z). □

Proposition 2.3.3. *The restriction map*

$$res^{\mathrm{CH}} : \mathrm{CH}^j(X, j-d)_\Lambda \to \mathrm{CH}^j(X_0, j-d)_\Lambda$$

is an isomorphism.

Proof. By the identification of Proposition 2.1.3, it suffices to show that $res : A_1(X, j-d-1) \to A_0(X_0, j-d)$ is an isomorphism.

We need to show that the map $\phi : Z_0(X_0, j-d) \to A_1(X, j-d-1)$ factorises through the group $A_0(X_0, j-d)$. In other words, we need to show that there is a $\overline{\phi} : A_0(X_0, j-d) \to A_1(X, j-d-1)$ such that the following diagram commutes:

$$\begin{array}{ccc} A_0(X_0, j-d) & & \\ \uparrow & \overset{\overline{\phi}}{\searrow} & \\ Z_0(X_0, j-d) & \xrightarrow{\phi} & A_1(X, j-d-1) \end{array}$$

Then $res \circ \overline{\phi} = id$ and since $\overline{\phi}$ is surjective the result follows.

Let $\alpha_0 = (\alpha_0^1, ..., \alpha_0^{j-d+1}) \in C_1(X_0, j-d) = \oplus_{x \in X_0^{(d-1)}} K^M_{j-d+1}k(x)$ be supported on some $x \in X_0^{(d-1)}$. As in the proof of [66, Lem. 7.2], we can find a relative surface $Z \subset X$ containing x which is regular at x and such that $Z \cap X_0$ contains $\overline{\{x\}}$ with multiplicity 1. Let Z_0 be the special fiber of Z and denote by $\cup_{i \in I} Z_0^{(i)} \cup \overline{\{x\}}$ the union of the pairwise different irreducible components of Z_0. Here the irreducible components different from $\overline{\{x\}}$ are indexed by I. Let z be the generic point of Z. Now as in the proof of Lemma 1.1.1, we can for all $1 \le t \le j-d+1$ find a lift $\alpha^t \in k(z)^\times$ of α_0^t which specialises to α_0^t in $k(x)^\times$ and to 1 in $K(Z_0^{(i)})^\times$ for all $i \in I$. Let $\alpha = (\alpha^1, ..., \alpha^{j-d+1})$. Then $\phi(\partial(\alpha_0)) = \partial(\alpha) = 0$ in $A_1(X, j-d-1)$ which implies the above factorisation.

The surjectivity of $\overline{\phi}$ follows from the surjectivity of ϕ which follows from key Lemma 2.3.2: Let $\alpha \in A_1(X, j-d-1)$. By arguments as in the last paragraph, one sees that $Z_1^g(X, j-d-1)$ generates $A_1(X, j-d-1)$. We may therefore assume that $\alpha \in Z_1^g(X, j-d-1)$. Let α_0 be the restriction of α to $Z_0(X-0, j-d)$ and α' be a lift of α_0 to $Z_1^g(X, j-d-1)$. Then, by key Lemma 2.3.2, we have that $\alpha \equiv \alpha' \in A_1(X, j-d-1)$. □

2.4 Applications and open problems

We list some applications of Proposition 2.3.3:

Corollary 2.4.1. *Let X be a smooth projective scheme of relative dimension d over an excellent henselian discrete valuation ring $\mathcal{O}_K$ with finite or algebraically closed residue field. Then*

$$\rho_X^{j,j-d} : \mathrm{CH}^j(X, j-d)_\Lambda \to H^{d+j}_{\text{ét}}(X, \Lambda(j))$$

is an isomorphism for all j.

Proof. Consider the diagram

$$\begin{array}{ccc} \mathrm{CH}^j(X, j-d)_\Lambda & \xrightarrow{\rho_X^{j,j-d}} & H^{d+j}_{\text{ét}}(X, \Lambda(j)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{CH}^j(X_0, j-d)_\Lambda & \xrightarrow{\cong} & H^{d+j}_{\text{ét}}(X_0, \Lambda(j)). \end{array}$$

The left vertical isomorphism follows from Proposition 2.3.3, the lower horizontal isomorphism from Proposition 2.2.4 and the right vertical isomorphism from proper base change. This implies that $\rho_X^{j,j-d}$ is an isomorphism for all j. □

Corollary 2.4.2. *Let X_K be a smooth projective scheme of dimension d with good reduction over a local field K with finite residue field k of characteristic p. Then the groups*

1. $\mathrm{CH}^j(X_K, j-d)_\Lambda$ *are finite for all* $j \geq 0$.
2. $\mathrm{CH}^j(X_K, j-d)_\Lambda = 0$ *for* $j \geq d+3$.

Proof. Consider the localisation sequence

$$\mathrm{CH}^{j-1}(X_0, j-d)_\Lambda \xrightarrow{i_*} \mathrm{CH}^j(X, j-d)_\Lambda \xrightarrow{j^*} \mathrm{CH}^j(X_K, j-d)_\Lambda \to \mathrm{CH}^{j-1}(X_0, j-d-1)_\Lambda.$$

By Proposition 2.3.3 and Proposition 2.2.4, the groups $\mathrm{CH}^j(X, j-d)_\Lambda$ are finite for all j and vanish for $j \geq d+2$. By Proposition 2.2.4, the groups $\mathrm{CH}^{j-1}(X_0, j-d-1)_\Lambda$ are finite for all j and vanish for $j \geq d+3$. This implies that $\mathrm{CH}^j(X_K, j-d)_\Lambda$ is finite for all j, vanishes for $j \geq d+3$. □

We now state a conjecture generalising the moving Lemma of [20] to zero-cycles with coefficients in Milnor K-theory. This would make it possible to generalise Theorem 2.0.5 to the non-smooth case.

Conjecture 2.4.3. *Let X be a regular, flat and quasi-projective scheme over a discrete valuation ring S. Let Z_1^g the set of one-cycles in good position, i.e. the set of 1-cycles Z such that $Z \cap X_0^{\text{sing}} = \emptyset$ and let $X_{(1)}^g$ be the set of all $x \in X_{(1)}$ such that $\overline{\{x\}} \in Z_1^g(X)$. Let $C_1^g(X, j-d-1) = \oplus_{x \in X_{(1)}^g} K_{j-d}^M k(x)$ and $Z_1^g(X, j-d-1) = \ker[\partial : C_p^g(X, m) \to C_{p-1}(X, m)]$. Then the map*

$$Z_1^g(X, j-d-1) \to A_1(X, j-d-1)$$

is surjective.

We also expect the following conjecture, motivated by the remarks at the end of Section 2.2, to hold:

Conjecture 2.4.4. *Let X be as in the introduction of this chapter. Then the restriction map*

$$res^{\mathrm{CH}} : \mathrm{CH}^i(X, j)_\Lambda \to \mathrm{CH}^i(X_0, j)_\Lambda$$

is an isomorphism for all $i \geq d+1$ and $j \geq 0$.

2.5 Torsion

In this section we prove finiteness theorems for torsion subgroups of some higher Chow groups of zero-cycles with coefficients in Milnor K-theory for smooth (projective) schemes over p-adic local fields. These theorems generalise results of [14] (see also [15]) and [73]. The proofs are very similar.

Notation 2.5.1. *For an abelian group A we denote by $A[n]$ the kernel of the multiplication by n. For a prime l we denote by $A\{l\}$ the l-primary torsion subgroups of A and by A_{tors} the entire torsion subgroup of A.*

For X a scheme we denote by $\mathcal{H}^q(\mu_n^{\otimes m})$ the Zariski sheaf associated to the presheaf $U \mapsto H^q_{\text{ét}}(U, \mu_n^{\otimes m})$.

Let X be a smooth variety over a field. Recall that by [8] the Leray spectral sequence associated to the canonical morphism of sites $X_{\text{ét}} \to X_{\mathrm{Zar}}$

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mu_n^{\otimes m})) \Rightarrow H^{p+q}_{\text{ét}}(X, \mu_n^{\otimes m}) \tag{2.5.1}$$

and the coniveau spectral sequence

$$E_1^{p,q} = \oplus_{x \in X^p} H^{q-p}(k(x), \mu_n^{\otimes m-p}) \Rightarrow H^{p+q}_{\text{ét}}(X, \mu_n^{\otimes m}) \tag{2.5.2}$$

agree from E_2 onwards and that therefore in particular

$$H^p(X, \mathcal{H}^q(\mu_n^{\otimes m})) = 0 \text{ for } p > q.$$

Proposition 2.5.2. *Let S be a smooth surface over a field k. Let $n > 0$ be a natural number prime to the characteristic of k. Then for all $j \geq 0$ there is a surjection*

$$\mathrm{CH}^{2+j}(S, j+1, \mathbb{Z}/n\mathbb{Z}) \twoheadrightarrow \mathrm{CH}^{2+j}(S, j)[n]$$

and $\mathrm{CH}^{2+j}(S, j+1, \mathbb{Z}/n\mathbb{Z})$ is an extension of $H^1(S, \mathcal{K}^M_{2+j}/n)$ by a finite group. Furthermore, we have the following diagram:

$$\begin{array}{ccccc}
H^1(S, \mathcal{K}^M_{2+j}/n) & & & & \\
\downarrow \simeq & & & & \\
H^1(S, \mathcal{H}^{2+j}(\mu_n^{\otimes 2+j})) & \xrightarrow{\simeq} & E_\infty^{1,2+j} & & \\
 & & \uparrow\!\!\uparrow & & \\
 & & F^1 H^{3+j} & \hookrightarrow & H^{3+j}_{\text{ét}}(S, \mu_n^{\otimes 2+j})
\end{array}$$

In particular, if k is either separably closed, local of dimension 1 or finite, then the groups

$$\mathrm{CH}^{2+j}(S,j)[n]$$

are finite.

Proof. The surjectivity of $\mathrm{CH}^{2+j}(S,j+1,\mathbb{Z}/n\mathbb{Z}) \twoheadrightarrow \mathrm{CH}^{2+j}(S,j)[n]$ is clear. For the second statement consider the spectral sequence

$${}^{\mathrm{CH}}E_1^{p,q}(S) = \oplus_{x\in S^{(p)}}\mathrm{CH}^{2+j-p}(\mathrm{Spec}k(x), -p-q)_\Lambda \Rightarrow \mathrm{CH}^{2+j}(S,-p-q)_\Lambda$$

and use Theorem 2.2.3 to show that $E_\infty^{1,-(2-j)}(S) \cong H^1(S,\mathcal{K}^M_{2+j}/n)$ and that $E_\infty^{2,-(3-j)}(S)$ is finite.

We now turn to the sequence of arrows in the diagram of the proposition. By the Bloch-Kato conjecture (see [77]) and the results recalled from [8] at the beginning of this section, we have an isomorphism

$$H^1(S,\mathcal{K}^M_{2+j}/n) \cong H^1(S,\mathcal{H}^{2+j}(\mu_n^{\otimes 2+j}))$$

in the Zariski topology. Since $\dim(S) = 2$, we have that $H^1(S,\mathcal{H}^{2+j}(\mu_n^{\otimes 2+j})) \cong E_2^{1,2+j} \cong E_\infty^{1,2+j}$ which is a quotient of $F^1H^{3+j} \subset H^{3+j}(S,\mu_n^{\otimes 2+j})$. If k is separably closed, then $H^{3+j}_{\text{ét}}(S,\mu_n^{\otimes 2+j})$ is finite by [30, Ch. XVI, Thm. 5.2]. This implies the finiteness for k finite of local of dimension 1 by the Hochschild-Serre spectral sequence. □

Remark 2.5.3. *1. The case $j = 0$ of Proposition 2.5.2 was first shown in [14] and [15]. The case $j = 1$ is shown in [73] assuming that S is proper.*

2. Let X be a smooth projective scheme of dimension d over a p-adic field K. In [2, Sec. 1, Sec. 5], Asakura and Saito show that neither the group $\mathrm{CH}^d(X)_{\mathrm{tors}}$ nor the group $\mathrm{CH}^{d+1}(X,1)_{\mathrm{tors}}$ may be expected to be finite.

3. It would be interesting to have a conjecture on the expected structure of $\mathrm{CH}^j(X,j-d)^{\wedge p}$, the p-completion of $\mathrm{CH}^j(X,j-d)$, for X a variety over a p-adic field for $j > 0$. For $j = 0$ see [10, Sec. 1].

For proper smooth schemes one can generalise the above proposition for $j \geq 1$ to arbitrary dimension:

Proposition 2.5.4. *Let X_K be a smooth and proper scheme over a local field K with ring of integers $\mathcal{O}_K$ and finite residue field k of characteristic p. Assume that X_K has good reduction over $\mathcal{O}_K$ and let $n > 0$ be a natural number prime to p. Then for all $j \geq 1$ the groups*

$$\mathrm{CH}^{d+j}(X_K,j)[n]$$

are finite.

Proof. For higher Chow groups we have the following exact sequence:

$$0 \to \mathrm{CH}^s(X_K, t)/n \to \mathrm{CH}^s(X_K, t, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^s(X_K, t-1)[n] \to 0$$

In order to study $\mathrm{CH}^{d+j}(X_K, j)[n]$, one can therefore study the group $\mathrm{CH}^{d+j}(X_K, j+1, \mathbb{Z}/n\mathbb{Z})$. By Levine's localization sequence for higher Chow groups, $\mathrm{CH}^{d+j}(X_K, j+1, \mathbb{Z}/n\mathbb{Z})$ fits into the exact sequence

$$\mathrm{CH}^{d+j}(X, j+1, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+j}(X_K, j+1, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^{d+j-1}(X_k, j, \mathbb{Z}/n\mathbb{Z}).$$

By Proposition 2.2.4, Proposition 2.2.5 and the Kato conjectures, the groups $\mathrm{CH}^{d+j}(X, j+1, \mathbb{Z}/n\mathbb{Z})$ and $\mathrm{CH}^{d+j-1}(X_k, j, \mathbb{Z}/n\mathbb{Z})$ are finite if $j \geq 1$. This implies that $\mathrm{CH}^{d+j}(X_K, j)[n]$ is finite if $j \geq 1$. □

2.6 A finiteness theorem

In [42], Kato and Saito prove the following theorem:

Theorem 2.6.1. *([42, Thm. 7.1]) Let F be a number field, $\mathcal{O}_F$ the ring of integers of F and C an open subset of $\mathcal{O}_F$. Let X be projective and integral over C and K be the function field of X. Let $d = \dim X$. Then the following statements hold:*

(1) If $n > d+1$, then the group $H^d_\Sigma(X_{\mathrm{Nis}}, \mathcal{K}^M_{n+1}(\mathcal{O}_K, \mathcal{I}))$ vanishes for any non-zero ideal $\mathcal{I}$ of $\mathcal{O}_K$.

(2) For a sufficiently small non-zero ideal $\mathcal{I}$ of $\mathcal{O}_K$, there exists a canonical isomorphism

$$s_X : H^d_\Sigma(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1}(\mathcal{O}_K, \mathcal{I})) \cong \mu(K).$$

Here $\mathcal{K}^M_{d+1}(\mathcal{O}_K, \mathcal{I}) := \ker(\mathcal{K}^M_{d+1}(\mathcal{O}_K) \to \mathcal{K}^M_{d+1}(\mathcal{O}_K/\mathcal{I}))$ and $\mu(K)$ is the group of all roots of 1 in K. For the exact definition of $H^d_\Sigma(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1}(\mathcal{O}_K, \mathcal{I}))$ see [42, (1.4.1)]. We just note that $H^d_\Sigma(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1}(\mathcal{O}_K, \mathcal{I})) = H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d+1}(\mathcal{O}_K, \mathcal{I}))$ if $\mathrm{ch}(K) \neq 0$. Theorem 2.6.1 implies in particular the finiteness of $\mathrm{CH}^{d+1}(X, 1)$ for a smooth projective scheme X of dimension d over a finite field. In this section we give a different proof of this finiteness using ideas of [14] and [15]. Note that $\mathrm{CH}^d(X)$ is studied in unramified class field theory and treated for example in [15], [41] and [74, Sec. 8].

Let X be a smooth scheme over a field k. Let $\mathbb{Z}(i)$ denote the motivic complex on the Zariski site of X defined by Suslin and Voevodsky (see [72]). For a ring Λ we denote $\mathbb{Z}(i) \otimes \Lambda$ by $\Lambda(i)$. In the following let l be a prime number. We recall that $\mathbb{Z}(i)$ has the following properties:

1. There is an isomorphism $H^n(X, \mathbb{Z}(i)) \cong CH^i(X, 2i-n)$.

2. Let $\pi : X_{\text{ét}} \to X_{\text{Zar}}$ be the canonical map of sites. Let

$$\mathbb{Z}/l^n\mathbb{Z}(i) := \begin{cases} \mu_{l^n}^{\otimes i} & l \neq p \\ \upsilon_n(i)[-i] & l = p \end{cases}$$

in the derived category of étale sheaves on X. There are quasi-isomorphisms

$$\pi^*\mathbb{Z}(i) \otimes \mathbb{Z}/l^n\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/l^n\mathbb{Z}(i),$$

and

$$\mathbb{Z}(i) \otimes \mathbb{Z}/l^n\mathbb{Z} \xrightarrow{\cong} \tau_{\leq i} R\pi_*\mathbb{Z}/l^n\mathbb{Z}(i),$$

in the derived category of étale and Zariski sheaves on X respectively. For $l \neq p$ these isomorphisms are shown in [72] and [71] respectively if k is of characteristic 0 and without assumption on k in [22] and [23]. For $l = p$ they are shown in [22, Thm. 8.5].

We now note that for all $j \geq 0$ we have the following commutative diagram:

$$\begin{array}{ccc} H^{2d+j-1}(X, \mathbb{Z}/l^n\mathbb{Z}(d+j)) & \twoheadrightarrow & \mathrm{CH}^{d+j}(X, j)[l^n] \\ \downarrow & & \downarrow \\ H^{2d+j-1}_{\text{ét}}(X, \mathbb{Z}/l^n\mathbb{Z}(d+j)) & \longrightarrow & H^{2d+j}_{\text{ét}}(X, \mathbb{Z}/l^m\mathbb{Z}(d+j)) \end{array}$$

Let us recall the construction (cf. [73, Sec. 3], where the commutativity of the above diagram is shown in detail, and [74, Sec. 8]):

1. The upper horizontal map comes from the distinguished triangle

$$\mathbb{Z}(i) \xrightarrow{l^n} \mathbb{Z}(i) \to \mathbb{Z}/l^n\mathbb{Z}(i) \to \mathbb{Z}(i)[1].$$

2. The lower horizontal map comes from the exact sequence

$$0 \to \mu_{l^m}^{\otimes d+j} \to \mu_{l^{mn}}^{\otimes d+j} \to \mu_{l^n}^{\otimes d+j} \to 0$$

on $X_{\text{ét}}$ inducing the distinguished triangle

$$R\pi_*\mu_{l^m}^{\otimes d+j} \to R\pi_*\mu_{l^{mn}}^{\otimes d+j} \to R\pi_*\mu_{l^n}^{\otimes d+j} \to R\pi_*\mu_{l^m}^{\otimes d+j}[1].$$

in the $\mathrm{D}(X_{\text{Zar}})$.

3. The two vertical maps are induced by the change of sites map $\pi : X_{\text{ét}} \to X_{\text{Zar}}$, i.e.

$$H^{2d+j-1}(X, \mathbb{Z}/l^n\mathbb{Z}(d+j)) \xrightarrow{\simeq} H^{2d+j-1}(X, \tau_{\leq d+j} R\pi_*\mathbb{Z}/l^n\mathbb{Z}(d+j))$$
$$\to H^{2d+j-1}(X, R\pi_*\mathbb{Z}/l^n\mathbb{Z}(d+j)) \xrightarrow{\simeq} H^{2d+j-1}_{\text{ét}}(X, \mathbb{Z}/l^n\mathbb{Z}(d+j))$$

and

$$H^{2d+j}(X, \mathbb{Z}(d+j)) \to H^{2d+j}(X, \tau_{\leq d+j} R\pi_*\mathbb{Z}/l^n\mathbb{Z}(d+j)) \to H^{2d+j}_{\text{ét}}(X, \mathbb{Z}/l^n\mathbb{Z}(d+j)).$$

Taking the colimit over n and the limit over m in the above commutative diagram, we get the following commutative diagram:

$$\begin{array}{ccc} H^{2d+j-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+j)) & \twoheadrightarrow & \mathrm{CH}^{d+j}(X, j)\{l\} \\ \downarrow & & \downarrow \\ H^{2d+j-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+j)) & \longrightarrow & H^{2d+j}_{\text{ét}}(X, \mathbb{Z}_\ell(d+j)) \end{array} \tag{2.6.1}$$

Proposition 2.6.2. *Let X be a smooth projective scheme of dimension d over a finite field k of characteristic $p > 0$. Then the group*

$$\mathrm{CH}^{d+1}(X, 1)$$

is finite and the cycle map

$$\rho : \mathrm{CH}^{d+1}(X, 1)\{l\} \to H^{2d+1}_{\text{ét}}(X, \mathbb{Z}_\ell(d+1))$$

is an isomorphism for l a prime different from p.

Proof. We first note that $\mathrm{CH}^{d+1}(X, 1) = \mathrm{CH}^{d+1}(X, 1)_{\text{tors}}$. It therefore suffices to show the finiteness of $\mathrm{CH}^{d+1}(X, 1)_{\text{tors}}$.

Let us first show that $\mathrm{CH}^{d+1}(X, 1)$ does not contain any p-torsion. Since $\mathcal{K}^M_{X,*}$ is p-torsion free by [35], we get an exact sequence

$$0 \to \mathcal{K}^M_{X,*} \xrightarrow{p^n} \mathcal{K}^M_{X,*} \to \mathcal{K}^M_{X,*}/p^n \to 0.$$

This induces a surjection

$$H^{d-1}(X, \mathcal{K}^M_{X,d+1}/p^n) \twoheadrightarrow H^d(X, \mathcal{K}^M_{X,d+1})[p^n] \cong \mathrm{CH}^{d+1}(X, 1)[p^n].$$

Since $H^{d-1}(X, \mathcal{K}^M_{X,d+1}/p^n) \cong H^{d-1}(X, v_n(d+1)) = 0$ by the Bloch-Kato-Gabber theorem, the statement follows.

Let l be a prime different from p. For $j = 1$, diagram (2.6.1) takes the form

$$\begin{array}{ccc} H^{2d}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) & \twoheadrightarrow & \mathrm{CH}^{d+1}(X, 1)\{l\} \\ \downarrow \cong & & \downarrow \rho \\ H^{2d}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) & \xrightarrow{\cong} & H^{2d+1}_{\text{ét}}(X, \mathbb{Z}_\ell(d+1)) \end{array}$$

The left vertical morphism is an isomorphism by Proposition 2.2.4 ($j = d+1$ and $a = 1$). That the lower horizontal map is an isomorphism follows from the vanishing of $H^{2d}_{\text{ét}}(X, \mathbb{Q}_\ell(d+1))$ and $H^{2d+1}_{\text{ét}}(X, \mathbb{Q}_\ell(d+1))$ which follows from the fact that the groups

$$H^i_{\text{ét}}(X, \mathbb{Z}_\ell(r))$$

are torsion for $i \neq 2r, 2r+1$. This follows from the Weil conjectures (see [15, Sec. 2]). In particular ρ is an isomorphism. The finiteness of $\mathrm{CH}^{d+1}(X,1)_{\mathrm{tors}}$ now follows from the above diagram and the fact that for a smooth projective scheme X over a finite field k, the groups

$$H^i_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))$$

are finite for $i \neq 2r, 2r+1$ and zero for almost all ℓ (see [14, Thm. 2] and [15, Sec. 2]) and the fact that $\mathrm{CH}^{d+1}(X,1)$ does not contain any p-torsion which we showed in the beginning. □

Remark 2.6.3. *Two remarks are in order:*

1. *The injectivity of ρ, implying the finiteness of $\mathrm{CH}^{d+1}(X,1)$, may be reduced to curves as follows: Let C be a smooth curve over k and l be a prime number prime to p. In this case we have the following commutative diagram by the the same arguments as in the proof of Proposition 2.6.2:*

$$\begin{array}{ccc}
H^2(C, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) & \twoheadrightarrow & \mathrm{CH}^2(C,1)\{l\} \\
\downarrow \cong & & \downarrow \\
H^2_{\text{ét}}(C, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) & \xrightarrow{\cong} & H^3_{\text{ét}}(C, \mathbb{Z}_\ell(2))
\end{array}$$

In particular, the map $\mathrm{CH}^2(C,1)\{l\} \to H^3_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ is an isomorphism.

We now show that the map $\mathrm{CH}^{d+1}(X,1)\{l\} \to H^{2d+1}_{\text{ét}}(X, \mathbb{Z}_\ell(d+1))$ is injective. Let $\alpha \in \ker(\mathrm{CH}^{d+1}(X,1)\{l\} \to H^{2d+1}_{\text{ét}}(X, \mathbb{Z}_\ell(d+1)))$. By the Bertini theorem of Poonen (see [62]) and noting that

$$\mathrm{CH}^{d+1}(X,1) \cong \mathrm{coker}\Big(\bigoplus_{x \in X^{(d-1)}} K^M_{d+2}k(x) \to \bigoplus_{x \in X^{(d)}} K^M_{d+1}k(x)\Big),$$

we can find a smooth curve C containing the support of α. Since $\mathrm{CH}^2(C,1)$ is torsion, $\alpha \in \mathrm{CH}^2(C,1)\{lm\}$ for m prime to l. Furthermore m is prime to p since $\mathrm{CH}^2(C,1)$ is p-torsion free. The injectivity now follows from the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{CH}^2(C,1)\{lm\} & \hookrightarrow & H^3_{\text{ét}}(C, \mathbb{Z}_{\ell m}(2)) \\
\downarrow & & \downarrow \simeq \\
\mathrm{CH}^{d+1}(X,1)\{lm\} & \longrightarrow & H^{2d+1}_{\text{ét}}(X, \mathbb{Z}_{\ell m}(d+1)) \\
\uparrow & & \uparrow \\
\mathrm{CH}^{d+1}(X,1)\{l\} & \longrightarrow & H^{2d+1}_{\text{ét}}(X, \mathbb{Z}_\ell(d+1))
\end{array}$$

Note that the upper horizontal map is injective since $H^i_{ét}(X, \mathbb{Z}_{\ell m}(r)) \cong H^i_{ét}(X, \mathbb{Z}_\ell(r)) \times H^i_{ét}(X, \mathbb{Z}_m(r))$. The isomorphism on the right follows from the weak Lefschetz theorem (see [57, Thm. 7.1]) and the Hochschild-Serre spectral sequence (cf. [15, p. 793]).

2. *Let X be a smooth projective scheme of dimension d over a finite field k of characteristic $p > 0$. For $j = 0$, diagram (2.6.1) takes the form*

$$\begin{array}{ccc} H^{2d-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \twoheadrightarrow & \mathrm{CH}^d(X)\{l\} \\ \downarrow \cong & & \downarrow \\ H^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \hookrightarrow & H^{2d}_{\text{ét}}(X, \mathbb{Z}_\ell(d)) \end{array}$$

The left vertical arrow is an isomorphism by Proposition 2.2.4 and the injectivity of the lower horizontal map follows from the vanishing of $H^{2d-1}_{\text{ét}}(X, \mathbb{Q}_\ell(d))$. In particular the map $\mathrm{CH}^d(X)\{l\} \to H^{2d}_{\text{ét}}(X, \mathbb{Z}_\ell(d))$ is injective (see [15, Thm. 22(iii)] for the surface case).

Now again the fact that the groups

$$H^i_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))$$

are finite for $i \neq 2r, 2r+1$ and zero for almost all ℓ (see [14, Thm. 2] and [15, Sec. 2]) implies that $\mathrm{CH}^d(X)\{l\}$ is finite and zero for almost all l. A similar argument works for the p-primary torsion part. In fact there is a surjection $\mathrm{CH}^d(X, 1; \mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{CH}^d(X)[p^n]$ and $\mathrm{CH}^d(X, q; \mathbb{Z}/p^r\mathbb{Z}) \to H^{2d-q}_{\text{ét}}(X, \mathbb{Z}/p^r\mathbb{Z}(d))$ is an isomorphism since the Kato homology groups $KH^0_i(X, \mathbb{Z}/p^r\mathbb{Z})$ vanishes for $1 \leq i \leq 4$ by [36, Thm. 0.3] (see also the proof of Lemma 3.6.7). Noting that $A_0(X)$ is torsion (see [15, Prop. 4]), this implies that

$$A_0(X)$$

is finite. We therefore see that a reduction to surfaces, which was used in [13] (see also [9, Sec. 5]), is not necessary if one uses the Kato conjectures. This was already remarked upon in [74, p. 25].

Remark 2.6.4. *That $\mathrm{CH}^{d+1}(X, 1)$ is torsion is a special case of Parshin's conjecture saying that $K_i(X) \otimes \mathbb{Q} = 0$ for a smooth projective scheme over a finite field if $i > 0$. Proposition 2.6.2 says more: $\mathrm{CH}^{d+1}(X, 1)$ is not just torsion but finite. This proves a special case of Bass's finiteness conjecture saying that $\mathrm{CH}^r(X, q)$ should be finitely generated for all $r, q \in \mathbb{N}$ if X is of finite type over $\mathbb{Z}$ or a finite field.*

We finish this chapter with the following conjecture:

Conjecture 2.6.5. *Let X be proper and of finite type over $\mathbb{Z}$. Then the group*

$$\mathrm{CH}^{d+1}(X, 1)$$

is finite.

Chapter 3

An idelic approach to the algebraization of zero-cycles

Let A be an excellent henselian discrete valuation ring with uniformising parameter π and residue field k. Let X be a smooth projective scheme over $\mathrm{Spec}(A)$ of relative dimension d. Let $X_n := X \times_A A/(\pi^n)$, i.e. X_1 is the special fiber and the X_n are the respective thickenings of X_1.

For n invertible in k and $\Lambda = \mathbb{Z}/n\mathbb{Z}$ the following commutative diagram has been studied extensively:

$$\begin{array}{ccc} \mathrm{CH}_1(X)/n & \xrightarrow{\rho} & \mathrm{CH}_0(X_1)/n \\ \downarrow{\scriptstyle cl_X} & & \downarrow{\scriptstyle cl_{X_s}} \\ H^{2d}_{\text{ét}}(X, \Lambda(d)) & \xrightarrow{\sim} & H^{2d}_{\text{ét}}(X_1, \Lambda(d)) \end{array} \tag{3.0.1}$$

The lower horizontal map is an isomorphism by proper base change. The map cl_{X_s} is an isomorphism assuming that k is finite or separably closed by unramified class field theory (see [15, Thm. 5, Rem. 3] and [41]). In [66], Saito and Sato show that cl_X is an isomorphism if k is finite or separably closed which implies that ρ is an isomorphism under these conditions. That ρ is in fact an isomorphism for arbitrary perfect residue fields is shown in [46] without using étale realizations by making use of a method introduced by Bloch in [17, App.]. In Chapter 2 we generalised these results to zero-cycles with coefficients in Milnor K-theory.

Let $\mathcal{K}^M_{X,d}$ be the improved Milnor K-sheaf defined in [44] and $\mathcal{K}^M_{d,X_n}$ its restriction to X_n. In this chapter we study the restriction map

$$res_{X_n} : \mathrm{CH}^{d+j}(X, j) \xrightarrow{\cong} H^d(X, \mathcal{K}^M_{d+j,X}) \xrightarrow{res_{X_n}} H^d(X_1, \mathcal{K}^M_{d+j,X_n}).$$

If $j = 0$, we assume the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{n,X}$ (see Definition 3.2.1) for the isomorphism on the left. By [43] and [44] it holds if X is equi-characteristic. If $j > 0$, we additionally assume the Gersten conjecture for the sheaf $\mathcal{CH}^r(-q)$ associated to the presheaf $U \mapsto \mathrm{CH}^r(U, -q)$ for the isomorphism on the left (see Chapter 2). This holds with finite coefficients or again if X is equi-characteristic (see e.g. Chapter 2). For our applications we will need the following additional result which is well-known to the expert and easily follows from what is known about the Gersten conjecture for Quillen K-theory with finite coefficients (see Section 3.2):

Proposition 3.0.6. *Let* $\mathrm{ch}(k) = p > (n-1)$*. Then the Gersten conjecture holds for the sheaf* $\mathcal{K}^M_{n,X}/p^r$ *for all* $r \geq 1$*.*

The main theorem of this chapter is the following:

Theorem 3.0.7. *The restriction map* $res : H^d(X, \mathcal{K}^M_{d+j,X}) \to H^d(X_1, \mathcal{K}^M_{d+j,X_n})$ *is surjective. In particular the map of pro-systems*

$$res : H^d(X, \mathcal{K}^M_{d+j,X}) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d+j,X_n})$$

is is an epimorphism in the category of pro-abelian groups pro-Ab for all $j \geq 0$*.*

Theorem 3.0.7 is a partial response to conjecture (j) of the introduction by Kerz, Esnault and Wittenberg saying that assuming the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{n,X}$ the restriction map $res : \mathrm{CH}^d(X) \otimes \mathbb{Z}/p^r\mathbb{Z} \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d,X_n}/p^r)$ is an isomorphism if $\mathrm{ch}(\mathrm{Quot}(A)) = 0$ and if k is perfect of characteristic $p > 0$ (see [46, Sec. 10]).

Let again $\mathrm{ch}(\mathrm{Quot}(A)) = 0$ and k be perfect of characteristic $p > 0$. In the final section of this chapter we relate the restriction map res to the p-adic cycle class map $\varrho^{d+j,j}_{p^r} : \mathrm{CH}^{d+j}(X, j)/p^r \to H^{2d+j}_{\text{ét}}(X, \mathcal{T}_r(j))$. The $\mathcal{T}_r(n)$ are the complexes defined in [69] and called p-adic étale Tate twists. $\mathcal{T}_r(n)$ is an object in the derived category $D^b(X, \mathbb{Z}/p^r\mathbb{Z})$ of bounded complexes of étale $\mathbb{Z}/p^r\mathbb{Z}$-sheaves on X. $\mathcal{T}_r(n)$ is expected to agree with $\mathbb{Z}(n)^{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z}$, where $\mathbb{Z}(n)^{\text{ét}}$ denotes the conjectural étale motivic complex of Beilinson-Lichtenbaum (see [67, Sec. 1.3]). If $p > n+1$, then $i^*\mathcal{T}_r(n) \cong \mathcal{S}_r(n)$, where i is the inclusion $X_1 \to X$ and $\mathcal{S}_r(n)$ is the syntomic complex defined in [40] (see [69, Sec. 1.4]). In [67], Saito and Sato show the following result on the p-adic cycle class map:

Theorem 3.0.8. *([67, Thm. 1.3.1]) Let X be a regular scheme which is proper flat of finite type over the ring of integers A of a p-adic local field K. Assume that X has good or semistable reduction over A and let d be the fiber dimension of X over A. Then the cycle class map*

$$\varrho^{d,0}_{p^r} : \mathrm{CH}^d(X)/p^r \to H^{2d}_{\text{ét}}(X, \mathcal{T}_r(d))$$

defined in [69, Cor. 6.1.4] is surjective.

We will show the following proposition:

Proposition 3.0.9. *Let $W(k)$ be the Witt ring of a finite field k of characteristic p and $p > d$. Let X be smooth and projective over $W(k)$. Then for all $j \geq d$ the map*

$$\text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{j,X_n}/p^r) \to H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j))$$

is an isomorphism of pro-abelian groups. Here we consider $H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j))$ as a constant pro-system.

In sum we establish, making use of the above result on the Gersten conjecture, the following commutative diagram analogous to diagram (3.0.1) for X smooth over $A = W(k)$ for a finite field k of characteristic p, $j \geq 0$ and $p > d + j + 1$ (see Proposition/Definition 3.6.10):

$$\begin{array}{ccccc} \mathrm{CH}^{d+j}(X, j, \mathbb{Z}/p^r\mathbb{Z}) & \xrightarrow{\cong} & H^d(X, \mathcal{K}^M_{d+j,X}/p^r) & \xrightarrow{res} & \text{"}\lim_n\text{"}\, H^d(X_1, \mathcal{K}^M_{d+j,X_n}/p^r) \\ & & \downarrow & & \cong \downarrow \\ & & H^{2d+j}_{\text{ét}}(X, \mathcal{T}_r(d+j)) & \xrightarrow{\cong} & H^{2d+j}_{\text{ét}}(X_1, \mathcal{S}_r(d+j)) \end{array} \tag{3.0.2}$$

Notation. *Unless otherwise specified, all cohomology groups are taken over the Zariski topology.*

3.1 Parshin chains

Let X be an excellent scheme.

Definition 3.1.1. *1. A chain on X is a sequence of points $P = (p_0, ..., p_s)$ on X such that*

$$\overline{\{p_0\}} \subset \overline{\{p_1\}} \subset ... \subset \overline{\{p_s\}}.$$

2. *A Parshin chain on X is a chain $P = (p_0, ..., p_s)$ such that $\dim \overline{\{p_i\}} = i$ for all $0 \leq i \leq s$.*

3. *A Q-chain on X is a chain $Q = (p_0, ..., p_{s-2}, p_s)$ such that $\dim \overline{\{p_i\}} = i$ for $i \in \{0, 1, ..., s-2, s\}$. We denote by $B(Q)$ the set of all $x \in X$ such that $Q(x) = (p_0, ..., p_{s-2}, x, p_s)$ is a Parshin chain.*

4. *Let Z be a closed subscheme of X and $U = X - Z$. A Parshin chain (resp. $\mathcal{Q}$-chain) on (X, Z) is a Parshin chain $P = (p_0, ..., p_s)$ (resp. $\mathcal{Q}$-chain $Q = (p_0, ..., p_{s-2}, p_s)$) such that $p_i \in Z$ for $i \leq s-1$ and $p_s \in U$ (resp. $p_i \in Z$ for $i \leq s-2$ and $p_s \in U$). A Parshin chain (resp. $\mathcal{Q}$-chain) on (X, X) is a Parshin chain in the sense of (2) (resp. $\mathcal{Q}$-chain in the sense of (3)).*

5. *We say that a Parshin chain $P = (p_0, ..., p_s)$ on X is supported on a closed subscheme Z of X if $p_i \in Z$ for all $0 \leq i \leq s$.*

6. *The dimension $d(P)$ of a chain $P = (p_0, ..., p_s)$ is defined to be $\dim \overline{\{p_s\}}$.*

Definition 3.1.2. *Let $P = (p_0, ..., p_s)$ be a chain on X.*

1. *We define $\mathcal{O}_{X,P} = \mathcal{O}_{X,p_s}$ and $k(P) = k(p_s)$.*

2. *We define the finite product of henselian local rings $\mathcal{O}^h_{X,P}$ inductively as follows: If $s=0$, then $\mathcal{O}^h_{X,P} = \mathcal{O}^h_{X,p_0}$. If $s>0$, then assume that the ring $\mathcal{O}^h_{X,P'}$ over $\mathcal{O}_{X,p_0}$ has already been defined for $P' = (p_0, ..., p_{s-1})$. Denote $\mathcal{O}^h_{X,P'}$ by R. Let T be the finite set of prime ideals of $\mathcal{O}^h_{X,P'}$ lying over p_s and*

$$\mathcal{O}^h_{X,P} = \prod_{\mathfrak{p}\in T} R^h_{\mathfrak{p}}.$$

Let $k^h(P)$ denote the finite product of residue fields of $\mathcal{O}^h_{X,P}$.

We note that T in Definition 3.1.2(2) is finite by [29, Thm. 18.6.9 (ii)] since X is excellent and therefore in particular noetherian. For a Parshin chain P on X we denote Spec$\mathcal{O}_{X,P}$ by X_P and Spec$\mathcal{O}^h_{X,P}$ by X^h_P. For more details on Parshin chains see [42, Sec. 1.6].

We will need the following facts (see e.g. [31, Ch. IV.]): Let X be a locally noetherian scheme, $\mathcal{F}$ be a sheaf of abelian groups on X and $\tau \in \{\text{Zar,Nis}\}$. Then there are coniveau spectral sequences

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} H_x^{p+q}(X_\tau, \mathcal{F}) \Rightarrow H^{p+q}(X_\tau, \mathcal{F})$$

and isomorphisms

$$H^q_x(X_{\text{Zar}}, \mathcal{F}) \cong H^q_x(\mathcal{O}_{X,x}, \mathcal{F}) \text{ and } H^q_x(X_{\text{Nis}}, \mathcal{F}) \cong H^q_x(\mathcal{O}^h_{X,x}, \mathcal{F})$$

for every $x \in X$ and $q \geq 0$. From the coniveau spectral sequence we get complexes

$$... \to \bigoplus_{x\in X^{(p-1)}} H_x^{p+q-1}(X_\tau, \mathcal{F}) \to \bigoplus_{x\in X^{(p)}} H_x^{p+q}(X_\tau, \mathcal{F}) \to \bigoplus_{x\in X^{(p+1)}} H_x^{p+q+1}(X_\tau, \mathcal{F}) \to ..$$

We denote a morphism $H_y^{p+q}(X_\tau, \mathcal{F}) \to H_x^{p+q+1}(X_\tau, \mathcal{F})$ arising this way by ∂_{yx}. We explain this notation as follows: If $\mathcal{F} = \mathcal{K}^M_{n,X}$, $y \in X^{(p+q)}$, $x \in X^{(p+q+1)}$, and if the Gesten conjecture holds for $\mathcal{K}^M_{n,X}$ (see Section 3.2.2), then the diagram

$$\begin{array}{ccc} H_y^{p+q}(X_\tau, \mathcal{K}^M_{n,X}) & \longrightarrow & H_x^{p+q+1}(X_\tau, \mathcal{K}^M_{n,X}) \\ \downarrow \cong & & \downarrow \cong \\ K^M_{n-p-q}(k(y)) & \longrightarrow & K^M_{n-p-q-1}(k(x)) \end{array}$$

commutes, where the lower horizontal map is the tame symbol defined by passing to the normalisation and using the norm map for Milnor K-theory (see f.e. [24, 8.1.1]).

Finally recall that the cohomological dimension of X_{Zar} and X_{Nis} is at most equal to $\dim(X)$.

3.2 The Gersten conjecture for Milnor K-theory mod p

Let X be an excellent scheme and let $\mathcal{K}^M_{n,X}$ be the improved Milnor K-sheaf defined in [44].

Definition 3.2.1. *We say that the Gersten conjecture holds for the (Milnor K-)sheaf $\mathcal{K}^M_{n,X}$ if the sequence of sheaves*

$$0 \to \mathcal{K}^M_{n,X} \to \bigoplus_{x\in X^{(0)}} i_{x,*}K^M_n(k(x)) \to \bigoplus_{x\in X^{(1)}} i_{x,*}K^M_{n-1}(k(x)) \to ...$$

is exact.

This conjecture is known to hold for $\mathcal{K}^M_{n,X}$ if all local rings of X are regular and equi-characteristic (see [43] and [44, Prop. 10(8)]). If X is smooth over a henselian local discrete valuation ring of mixed characteristic $(0,p)$, then the Gersten conjecture is not known to hold for the sheaf $\mathcal{K}^M_{n,X}$. However, if $p > n-1$, then we have the following much weaker result which we will use in Section 3.6.

Proposition 3.2.2. *Let A be a discrete valuation ring with uniformising parameter π and residue field k of characteristic $p > 0$. Let B be a local ring, essentially smooth over A with field of fractions F and let $p > (n-1)$. Then the sequence*

$$0 \to K^M_n(B)/p^r \xrightarrow{i_n} K^M_n(F)/p^r \to \bigoplus_{x\in X^{(1)}} K^M_{n-1}(x)/p^r \to ...$$

is exact for all $r \geq 1$.

Proof. First note that the Gersten conjecture for Quillen K-theory with finite coefficients holds for B by [22, Thm. 8.2].

We consider the following commutative diagram:

$$\begin{array}{ccccc}
K^M_n(B)/p^r & \xrightarrow{i_n} & K^M_n(F)/p^r & \longrightarrow & \bigoplus_{x\in X^{(1)}} K^M_{n-1}(x)/p^r \\
\downarrow & & \downarrow & & \downarrow \\
K^Q_n(B,\mathbb{Z}/p^r) & \xrightarrow{i^Q_n} & K^Q_n(F,\mathbb{Z}/p^r) & \longrightarrow & \bigoplus_{x\in X^{(1)}} K^Q_{n-1}(x,\mathbb{Z}/p^r) \\
\downarrow & & \downarrow & & \\
K^M_n(B)/p^r & \longrightarrow & K^M_n(F)/p^r & \longrightarrow & \bigoplus_{x\in X^{(1)}} K^M_{n-1}(x)/p^r
\end{array}$$

The composition $\mu : K^M_n(B) \to K^Q_n(B) \to K^M_n(B)$ is multiplication by $(n-1)!$ by [59, Sec. 4] and [44, Prop. 10(6)]. Let us first show the injectivity of i_n: Let $\alpha \in K^M_n(B)/p^r$ and suppose that $i_n(\alpha) = 0$. Then $(n-1)! \cdot \alpha = 0$ since i^Q_n is injective. For $p > (n-1)$ we have that $(p,(n-1)!) = 1$. This implies that $\alpha = 0$. The exactness at $K^M_n(F)/p^r$ can be seen as follows: Let $\alpha \in \ker[K^M_n(F)/p^r \to \bigoplus_{x\in X^{(1)}} K^M_{n-1}(x)/p^r]$. Then $(n-1)!\alpha \in \mathrm{im}(i_n)$ since the square on the upper right commutes (see [78, p. 449f.]) and the middle row is exact at $K^Q_n(F,\mathbb{Z}/p^r)$. Again since $(p,(n-1)!) = 1$ it follows that $\alpha \in \mathrm{im}(i_n)$.

Exactness at the other places follows for example from [21, Cor. 4.3]. □

Remark 3.2.3. *See [59, Cor. 4.4] for a similar result.*

We will repeatedly use the following purity statement which follows from the Gersten conjecture:

Lemma 3.2.4. *Let $x \in X$ be a point which is not be contained in D and assume the Gersten conjecture for the sheaf $\mathcal{K}_{n,X}^M$. Then for $t = \operatorname{codim}_X(x)$ there is a canonical isomorphism*

$$H_x^t(X, \mathcal{K}_{n,X|D}^M) \cong K_{n-t}^M(k(x)).$$

(for the definition of $\mathcal{K}_{n,X|D}^M$ see Definition 3.3.1(2)).

3.3 Some topology on Milnor K-groups

In this section we define a topology on Milnor K-groups and state two lemmas which we will need in the proof of our main theorem.

Recall that the naive Milnor K-sheaf $\mathcal{K}_n^{M,\text{naive}}$ is defined to be the sheafification of the functor

$$R \mapsto (R^\times)^{\otimes n} / < a_1 \otimes ... \otimes a_n | a_i + a_j = 1 \text{ for some } i \neq j >$$

from the category of commutative rings to abelian groups and that there is a natural homomorphism of sheaves

$$\mathcal{K}_n^{M,\text{naive}} \to \mathcal{K}_n^M$$

to the improved Milnor K-sheaf which is surjective (see [44]). In particular there is the following commutative diagram for a commutative local ring R, an ideal $I \subset R$ and $K = \operatorname{Frac}(R)$:

$$\begin{array}{ccccc}
\mathcal{K}_n^M(R) & & \longrightarrow & & \mathcal{K}_n^M(R/I) \\
\uparrow & \searrow & & & \uparrow \\
 & & K_n^M(K) & & \\
\uparrow & \nearrow & & & \uparrow \\
\mathcal{K}_n^{M,\text{naive}}(R) & & \longrightarrow & & \mathcal{K}_n^{M,\text{naive}}(R/I)
\end{array}$$

This implies that that when defining a topology on $K_n^M(K)$ as in the following Definition 3.3.1(4) we may work with both $\mathcal{K}_n^{M,\text{naive}}$ or $\mathcal{K}_n^M$. We will use the improved Milnor K-sheaf and at some points implicitly use its generation by symbols.

Definition 3.3.1. 1. *For a commutative ring R and an ideal $I \subset R$ we define $K_n^M(R, I)$ to be $\ker[\mathcal{K}_n^M(R) \to \mathcal{K}_n^M(R/I)]$ and similarly for $K_n^{M,\text{naive}}(R, I)$.*

2. *Let D be an effective Cartier divisor on X. We define $\mathcal{K}_{n,X|D}^M$ to be the kernel of the restriction map $\mathcal{K}_{n,X}^M \to i_* \mathcal{K}_{n,D}^M$ for $i : D \to X$ the inclusion. Again similarly for $K_{n,X|D}^{M,\text{naive}}$.*

3. *Let R be an excellent semi-local integral domain of dimension 1 with field of fractions K. We endow R with the J_R-adic topology, where J_R is the Jacobsen radical of R. We endow $K_n^M(K)$ with the structure of a topological group by taking the subgroups generated by $\{U_1, ..., U_n\}$, where U_i ranges over all open subgroups of $R^\times$, as a fundamental system of neighbourhoods of 0 in $K_n^M(K)$.*

4. *For a Parshin chain $P = (p_0, ..., p_{s-1}, p_s)$, and $P' = (p_0, ..., p_{s-1})$, on an excellent scheme X and $Y = \overline{\{p_s\}}$ we define a topology on $K_n^M(k(P))$ (resp. $K_n^M(k^h(P))$) by taking the images of $K_n^M(\mathcal{O}_{Y,P'}, I)$ (resp. $K_n^M(\mathcal{O}^h_{Y,P'}, I)$) as a fundamental system of neighbourhoods of 0, where I ranges over all open ideals, with respect to the topology defined in (3), of the one dimensional local ring $\mathcal{O}_{Y,P'}$ (resp. semi-local ring $\mathcal{O}^h_{Y,P'}$).*

Remark 3.3.2. *If we set $R := \mathcal{O}_{Y,P'}$ (resp. $\mathcal{O}^h_{Y,P'}$), then the topologies on $K_n^M(K), K = Frac(R)$, defined in (3) and (4) coincide.*

Example 3.3.3. *Let $m \geq 0$ be an integer. If R in (3) is a discrete valuation ring with quotient field K, maximal ideal $\mathfrak{p} \subset R$ and generic point η, then the subgroups generated by $K_n^M(K, m) := \{1 + \mathfrak{p}^m, R^\times, ..., R^\times\}$ of $K_n^M(K)$ generate the topology on $K_n^M(K)$ with respect to the Parshin chain $(\mathfrak{p}, \eta)$.*

Lemma 3.3.4. *(Cf. [39, Prop. 2]) Let R be an excellent semi-local integral domain of dimension 1 with field of fractions K. Let $\tilde{R}$ be the integral closure of R in K. Then the topology of $K_n^M(K)$ defined by $\tilde{R}$ coincides with that defined by R.*

Proof. Since R is excellent, the normalisation is finite and there is some $f \in J_R \setminus \{0\}$ such that $f\tilde{R} \subset R$. Therefore for every $i \geq 1$

$$1 + f^{i+1}\tilde{A} \subset 1 + f^i A.$$

□

Lemma 3.3.5. *(cf. [42, Prop. 2.7], [45, Lem. 6.2]) Let X be an excellent integral scheme. Let U be a regular open subscheme of X and D an effective Weil divisor with support $X - U$. Let $y \in U$ and x be of codimension 1 on $\overline{\{y\}}$. Let $dim\mathcal{O}_{X,y} = t$ and assume the Gersten conjecture for the sheaf $\mathcal{K}^M_{n,U}$. Then the map*

$$\partial_{yx} : K^M_{n-t}(k(y)) \cong H^t_y(X, \mathcal{K}^M_{n,X|D}) \to H^{t+1}_x(X, \mathcal{K}^M_{n,X|D})$$

annihilates the image of $K^M_{n-t}(\mathcal{O}_{Y,x}, J_x)$ for some non-zero ideal $J_x \subset \mathcal{O}_{Y,x}$. In particular the kernel of ∂_{yx} is open with respect to the topology defined in Definition 3.3.1(4) and the Parshin chain (x, y).

Proof. We proceed by induction on t. The case $t = 0$ is clear since in that case $Y = X$ and $H^1_x(X, \mathcal{K}^M_{n,X|D}) \cong K_n^M(k(y))/K_n^M(\mathcal{O}_{Y,x}, J)$ for $x \in D^{(0)}$ and J corresponding to D.

If $t \geq 1$, then we take some point $z \in X^{t-1}$ such that y lies in the regular locus of $\overline{\{z\}}$. Consider the complex

$$H^{t-1}_z(X, \mathcal{K}^M_{n,X|D}) \to \bigoplus_{y' \in \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x} \setminus D} H^t_{y'}(X, \mathcal{K}^M_{n,X|D}) \oplus \bigoplus_{y'' \in \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x} \cap D} H^t_{y''}(X, \mathcal{K}^M_{n,X|D}) \to H^{t+1}_x(X, \mathcal{K}^M_{n,X|D}) \quad (3.3.1)$$

coming from the coniveau spectral sequence in Section 3.1. Applying the induction assumption to $H^{t-1}_z(X, \mathcal{K}^M_{n,X|D}) \to H^t_{y''}(X, \mathcal{K}^M_{n,X|D})$ for all $y'' \in \mathrm{Spec}\mathcal{O}^{(1)}_{Z_x} \cap D$, we see that it suffices to show that the map

$$K^M_{n-t+1}(k(z)) \xrightarrow{(\partial,\mathrm{Id})} \bigoplus_{y' \in \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x} \setminus D} K^M_{n-t}(k(y')) \oplus \bigoplus_{y'' \in \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x} \cap D} K^M_{n-t+1}(K(z))/K^M_{n-t+1}(\mathcal{O}_{Z,y''}, J_{y''})$$

annihilates the image of $K^M_{n-t}(\mathcal{O}_{Y,x}, J_x)$ in $K^M_{n-t}(k(y))$ for some non-zero ideal $J_x \subset \mathcal{O}_{Y,x}$ given some non-zero ideals $J_{y''} \subset \mathcal{O}_{Z,y''}$. Indeed, in that case if $\alpha \in \mathrm{Im}(K^M_{n-t}(\mathcal{O}_{Y,x}, J_x) \to K^M_{n-t}(k(y)))$, then there is some $\beta \in K^M_{n-t+1}(k(z))$ such that $\partial_{zy}(\beta) = \alpha$ and such that $(\bigoplus_{y' \neq y \in \mathrm{Spec}\mathcal{O}^{(1)}_{Z_x}} \partial_{zy''}, \mathrm{Id})(\beta) = 0$. Since (3.3.1) is a complex, this implies that $\partial_{yx}(\alpha) = 0$.

Now let $A := \mathcal{O}_{Z,x}$. By Lemma 3.3.4 we may assume that the $\mathcal{O}_{Z,y''}$ are normal (semi-local) rings. By the definition of ∂ we may work with the normalisation $\tilde{A}$ of A. Let $\{y''_1, ..., y''_r\} = \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x} \cap D$ and let $J^{(y''_i)}$ be ideals in $\tilde{A}$ such that $J^{(y''_i)}\mathcal{O}_{Z,y''_i} = J_{y''_i}$. Let $\mathfrak{q}$ be the prime ideal corresponding to y. Let $\pi \in A$ such that $v_{\mathfrak{q}}(\pi) = 1$. Let $\{\mathfrak{p}_{1+r}, ..., \mathfrak{p}_t\}$ be the finite set of prime ideals in $\tilde{A}$ such that $v_{\mathfrak{p}_i}(\pi) > 0, 1 + r \leq i \leq t$. By a standard approximation lemma (see e.g. [51, Lem. 9.1.9(b)]) we can choose an element π_i for all i with $r + 1 \leq i \leq t$ satisfying $v_{\mathfrak{p}_i}(\pi_i) = 1$ and $v_{\mathfrak{q}}(\pi_i) = 0$. Now we can choose a non-zero ideal

$$J^{(x)} \subset J^{(y''_1)}...J^{(y''_r)}(\pi_{r+1})...(\pi_t)(\tilde{A}/\mathfrak{q}) \twoheadleftarrow J^{(y''_1)}...J^{(y''_r)}(\pi_{r+1})...(\pi_t)\tilde{A}. \quad (3.3.2)$$

Let $J_x := J^{(x)}\mathcal{O}_{Y,x}$. Now given a symbol $\alpha := \{\bar{a}_1, ..., \bar{a}_{n-t}\} \in K^M_{n-t}(\mathcal{O}_{Y,x}, J_x)$ with $\bar{a}_1 \in 1 + J_x$, lift α to $\beta := \{\pi, a_1, ..., a_{n-t}\} \in K^M_{n-t+1}(k(z))$ lifting $\bar{a}_1$ via the surjection in (3.3.2) to a_1 and lifting the other $\bar{a}_i$ arbitrarily. Then β satisfies the required properties since

1. $\partial_{zy}(\beta) = \alpha$.

2. If $\tilde{y}' \notin \mathrm{div}(\pi)$, then $\partial_{z\tilde{y}'} = 0$ since $\pi, a_1, ..., a_{n-t} \in \mathcal{O}^\times_{\tilde{A},\tilde{y}'}$.

3. If $\tilde{y}' \in \mathrm{div}(\pi)$, i.e. $\tilde{y}' \sim \mathfrak{p}_i$, then $\partial_{z\tilde{y}'} = 0$ since $a_1 = 1 \bmod (\pi_i)$.

4. $a_1 \in K^M_{n-t+1}(\mathcal{O}_{Z,y''}, J_{y''})$ for all $y'' \in \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x} \cap D$.

□

Remark 3.3.6. *In [42, Prop. 2.7] the above lemma was proved in the Nisnevich topology. The proof in the Zariski topology follows the argument in loc. cit. We recall the proof for the convenience of the reader and to convince them of this claim. In [45, Lem. 6.2] the last step of the proof is given under the assumption that A is a two-dimensional excellent henselian local ring.*

Lemma 3.3.7. *Given a family of inequivalent discrete valuations $v_1, ..., v_s$ on a valued field F, the diagonal map*

$$K_n^M(F) \to \bigoplus_{v_i} K_n^M(F_{v_i}),$$

has dense image, were we write F_{v_i} instead of F in order to indicate which valuation defines the topology on F.

Proof. This follows from standard approximation theorems for F. See e.g. [60, II.3.4]. □

3.4 Main theorem

In this section we prove Theorem 3.0.7.

We return to the situation of the introduction. Let A be an excellent henselian discrete valuation ring with uniformising parameter π and residue field k and let X be a smooth projective scheme over Spec(A) of relative dimension d. Let $X_n := X \times_A A/(\pi^n)$, i.e. X_1 is the special fiber and the X_n are the respective thickenings of X_1.

Proposition 3.4.1. *For all $j \geq 0$ the group*

$$H^{d+1}_{\mathrm{Zar}}(X, \mathcal{K}^M_{j+d,X|X_n}) = 0.$$

Proof. By the coniveau spectral sequence and cohomological vanishing we have to show that the map

$$\bigoplus_{y \in X^{(d)}} H^d_y(X, \mathcal{K}^M_{d+j,X|X_n}) \to \bigoplus_{x \in X^{(d+1)}} H^{d+1}_x(X, \mathcal{K}^M_{d+j,X|X_n})$$

is surjective. In order to show this, we show that the map

$$\bigoplus_{y \in (\mathrm{Spec}\mathcal{O}_{X,x}[\frac{1}{\pi}])^d} H^d_y(X, \mathcal{K}^M_{d+j,X|X_n}) \to H^{d+1}_x(X, \mathcal{K}^M_{d+j,X|X_n})$$

is surjective for any $x \in X^{(d+1)}$. This suffices since $(\mathrm{Spec}\mathcal{O}_{X,x}[\frac{1}{\pi}])^d \subset X^{(d)}$ and since, as A is henselian, any $y \in (\mathrm{Spec}\mathcal{O}_{X,x}[\frac{1}{\pi}])^d$ restricts to just one closed point $x \in X^{(d+1)}$. Let us start with the case $d = 0$: Let $X'_x := X_x - x$. Then

$$H^1_x(X, \mathcal{K}^M_{j,X|X_n}) \cong H^0(X'_x, \mathcal{K}^M_{j,X|X_n})/H^0(X_x, \mathcal{K}^M_{j,X|X_n}),$$

and $H^0_\mu(X, \mathcal{K}^M_{j,X|X_n})$, μ being the generic point of X, surjects onto $H^1_x(X, \mathcal{K}^M_{j,X|X_n})$ since $H^0_\mu(X, \mathcal{K}^M_{j,X|X_n})$ is isomorphic to $H^0(X'_x, \mathcal{K}^M_{j,X|X_n})$.

Let $d \geq 1$ and $x \in X^{(d+1)}$. We have that

$$H_x^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}) \cong H^d(X'_x, \mathcal{K}^M_{d+j,X|X_n})$$

and again it follows from the coniveau spectral sequence and cohomological vanishing that $H^d(X'_x, \mathcal{K}^M_{d+j,X|X_n})$ is isomorphic to

$$\mathrm{coker}(\bigoplus_{z \in (X_x)^{d-1}} H_z^{d-1}(X_x, \mathcal{K}^M_{d+j,X|X_n}) \to \bigoplus_{y \in (X_x)^d} H_y^d(X_x, \mathcal{K}^M_{d+j,X|X_n})).$$

By Lemma 3.2.4 we have that

$$H_y^d(X_x, \mathcal{K}^M_{d+j,X|X_n}) \cong \mathcal{K}^M_{j,X}(k(x,y))$$

for $y \in (X_x[\frac{1}{\pi}])^d$. It therefore suffices to move elements of $H_y^d(X_x, \mathcal{K}^M_{d+j,X|X_n})$ for $y \in X_x^{d-r} \setminus (X_x[\frac{1}{\pi}])^d$ to the horizontal components, i.e. with $y \in (X_x[\frac{1}{\pi}])^d$, using the 'Q-chains' $H_z^{d-1}(X_x, \mathcal{K}^M_{d+j,X|X_n})$.

We write P_r for a Parshin chain $(x, ...)$ of dimension r and let x_{P_r} denote the closed point of X_{P_r} and X'_{P_r} the open subscheme $X_{P_r} \setminus \{x_{P_r}\}$. We proceed by descending in induction in $r \geq 0$, starting with $r = d$, to show that the map

$$\bigoplus_{y \in (X_{P_r}[\frac{1}{\pi}])^{d-r}} H_y^{d-r}(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n}) \to H_{x_{P_r}}^{d-r+1}(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n})$$

is surjective for all Parshin chains P_r supported on X_1.

The group $H_{x_{P_r}}^{d-r+1}(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n})$ is isomorphic to

$$\begin{aligned}\mathrm{coker}(\bigoplus_{z \in (X_{P_r})^{d-1-r}} H_z^{d-1-r}(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n}) \to \bigoplus_{y \in (X_{P_r})^{d-r}} H_y^{d-r}(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n})) \\ \cong H^{d-r}(X'_{P_r}, \mathcal{K}^M_{d+j,X|X_n})\end{aligned}$$

for $r < d$ and to

$$H^0(X'_{P_d}, \mathcal{K}^M_{d+j,X|X_n})/H^0(X_{P_d}, \mathcal{K}^M_{d+j,X|X_n})$$

for $r = d$. If $r = d$, then $H_y^0(X_{P_d}, \mathcal{K}^M_{d+j,X|X_n})$ is isomorphic to $H^0(X'_{P_d}, \mathcal{K}^M_{d+j,X|X_n})$ which implies the induction beginning.

We now do the induction step. Let $\alpha \in H_y^{d-r}(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n})$ for $y \in X_{P_r}^{d-r} \setminus (X_{P_r}[\frac{1}{\pi}])^{d-r}$. Then

$$\bigoplus_{z \in (X_{(P_r,y)})^{d-r-1}} H_z^{d-r-1}(X_{(P_r,y)}, \mathcal{K}^M_{d+j,X|X_n}) \twoheadrightarrow H_y^{d-r}(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n})$$

and by assumption we have that

$$\mathrm{coker}(\bigoplus_{t \in (X_{(P_r,y)})^{d-r-2}} H_t^{d-r-2}(X_{(P_r,y)}, \mathcal{K}^M_{d+j,X|X_n}) \to \bigoplus_{z \in (X_{(P_r,y)})^{d-r-1}} H_z^{d-r-1}(X_{(P_r,y)}, \mathcal{K}^M_{d+j,X|X_n}))$$

is generated by $K^M_{r+j+1}(k(P))$ for all Parshin chains $P=(P_r,y,z)$ of dimension $r+2$ on (X,X_1). We may assume that α is in the image of $K^M_{r+j+1}(k(P))$ for some such P. Then by Lemma 3.3.5 the kernel of the map

$$\partial_{zy}-\alpha : K^M_{r+j+1}(k(P)) \to H^{d-r}_y(X_{P_r},\mathcal{K}^M_{d+j,X|X_n})$$

is open in $K^M_{r+j+1}(k(P))$ and the kernel of the map

$$\partial_{zy'} : K^M_{r+j+1}(k(P)) \to H^{d-r}_{y'}(X_{P_r},\mathcal{K}^M_{d+j,X|X_n})$$

is open in $K^M_{r+j+1}(k(P_r,y',z))$ for all $y'\neq y \in X^{d-r}_{P_r}\cap X_1$ with $\overline{\{x_{P_r}\}}\subset\overline{\{y'\}}\subset\overline{\{z\}}$. By Lemma 3.3.7 the diagonal image of

$$K^M_{r+j+1}(k(Q)) \cong H^{d-r-1}_z(X_{P_r},\mathcal{K}^M_{d+j,X|X_n})$$

for a Q-chain $Q=(P_r,z)$ is dense in the direct sum (with finitely many summands) $K^M_{r+j+1}(k(P)) \oplus_{y'\neq y\in X^{d-r}_{P_r}\cap\overline{\{z\}}_1} K^M_{r+j+1}(k(P_r,y',z))$ which implies that α is in the image of some $\beta\in H^{d-r-1}_z(X_x,\mathcal{K}^M_{d+j,X|X_n})$ with β mapping to zero in $H^{d-r}_{y'}(X_x,\mathcal{K}^M_{d+j,X|X_n})$ for all $y'\neq y\in((X_x)_1)^d$. □

Remark 3.4.2. *The proof of Proposition 3.4.1 is inspired by the proof of Theorem 2.5 in [42] and the proof of Theorem 8.2 in [45]. In both of these articles the authors work in the Nisnevich topology. We note that the proof of Proposition 3.4.1 also works in the Nisnevich topology if for every Parshin chain $P=(p_0,...,p_s)$ we replace $\mathcal{O}_{X,P}=\mathcal{O}_{X,p_s}$ by $\mathcal{O}^h_{X,P}$ according to Definition 3.1.2. We therefore get that*

$$H^{d+1}_{\text{Nis}}(X,\mathcal{K}^M_{j+d,X|X_n})=0$$

for all $j\geq 0$.

Remark 3.4.3. *If $\mathrm{ch}(k)=0$ and $A=k[[t]]$ or if A is the Witt ring $W(k)$ of a perfect field k of $ch(k)>2$, then there are exact sequences of sheaves*

$$0\to\Omega^{r-1}_{X_1}\to\mathcal{K}^M_{r,X_n}\to\mathcal{K}^M_{r,X_{n-1}}\to 0$$

and

$$0\to\Omega^{r-1}_{X_1}/B_{n-2}\Omega^{r-1}_{X_1}\to\mathcal{K}^M_{r,X_n}\to\mathcal{K}^M_{r,X_{n-1}}\to 0$$

respectively, by [5, Sec. 2] and [6, Sec. 12]. Under the above assumptions this implies that the canonical map

$$H^d_{\text{Zar}}(X_1,\mathcal{K}^M_{d+j,X_n})\to H^d_{\text{Nis}}(X_1,\mathcal{K}^M_{d+j,X_n})$$

is an isomorphism for all $n\in\mathbb{N}_{>0}$. Indeed, it follows from the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{*,X_1}$, that the maps $H^i_{\text{Zar}}(X_1,\mathcal{K}^M_{d+j,X_1})\to H^i_{\text{Nis}}(X_1,\mathcal{K}^M_{d+j,X_1})$ are isomorphisms for all i and the sheaves $\Omega^{r-1}_{X_1}$ and $\Omega^{r-1}_{X_1}/B_{n-2}\Omega^{r-1}_{X_1}$ are coherent. The claim now follows by induction on n.

Corollary 3.4.4. *1. The restriction map res* : $H^d(X, \mathcal{K}^M_{d+j,X}) \to H^d(X_1, \mathcal{K}^M_{d+j,X_n})$ *is surjective. In particular the map of pro-systems*

$$res : H^d(X, \mathcal{K}^M_{d+j,X}) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d+j,X_n})$$

is an epimorphism in pro-Ab for all $j \geq 0$.

2. The restriction map res : $H^d(X, \mathcal{K}^M_{d+j,X}/p^r) \to H^d(X_1, \mathcal{K}^M_{d+j,X_n}/p^r)$ *is surjective. In particular the map of pro-systems*

$$res : H^d(X, \mathcal{K}^M_{d+j,X}/p^r) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d+j,X_n}/p^r)$$

is an epimorphism in pro-Ab for all $j \geq 0$.

Here and in the following we always consider $H^d(X, \mathcal{K}^M_{d+j,X})$ *(resp.* $H^d(X, \mathcal{K}^M_{d+j,X}/p^r)$*) as a constant pro-system in in pro-Ab.*

Proof. For (1) consider the short exact sequence

$$0 \to \mathcal{K}^M_{d+j,X|X_n} \to \mathcal{K}^M_{d+j,X} \to \mathcal{K}^M_{d+j,X_n} \to 0$$

and the induced long exact sequence

$$... \to H^d(X, \mathcal{K}^M_{d+j,X|X_n}) \xrightarrow{i} H^d(X, \mathcal{K}^M_{d+j,X}) \xrightarrow{res} H^d(X_1, \mathcal{K}^M_{d+j,X_n}) \to H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}) \to ...$$

The statement now follows from Proposition 3.4.1 and the fact that "$\lim_n$" is exact when considered as a functor $\mathrm{Hom}(I^{\mathrm{op}}, \mathrm{Ab}) \to$ pro-Ab, where I is a small filtering category (see [1, App. Prop. 4.1]).

(2) can be seen as follows: Since $\otimes\mathbb{Z}/p^r\mathbb{Z}$ is right exact, there is a short exact sequence

$$0 \to \mathcal{K}^M_{d+j,X|X_n}/p^r/\mathcal{I} \to \mathcal{K}^M_{d+j,X}/p^r \to \mathcal{K}^M_{d+j,X_n}/p^r \to 0$$

for some sheaf of abelian groups $\mathcal{I}$. This induces an exact sequence

$$H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r) \to H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r/\mathcal{I}) \to H^{d+2}(X, \mathcal{I}).$$

By [27, Thm. 3.6.5] the group $H^{d+2}(X, \mathcal{I})$ vanishes for dimensional reasons. The group $H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r)$ vanishes by the same arguments as in the integral case or in fact from the surjectivity of the map $H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}) \to H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r)$ which also holds for dimensional reasons. Together this implies the vanishing of $H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r/\mathcal{I})$. This implies the statement by the same argument as in the proof of (1). □

Corollary 3.4.5. *If A is equi-characteristic, then the map*

$$res : \mathrm{CH}^{d+j}(X, j) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d+j,X_n})$$

is an epimorphism in pro-Ab for all $j \geq 0$. *If A is of mixed characteristic $(0, p)$ with $p > d + j - 1$, then the map*

$$res : \mathrm{CH}^{d+j}(X, j, \mathbb{Z}/p^r) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d+j,X_n}/p^r)$$

is an epimorphism in pro-Ab for all $j \geq 0$.

Proof. For $j = 0$, Corollary 3.4.4 implies the first assertion since the Gersten conjecture for the sheaf $\mathcal{K}^M_{n,X}$ holds for regular schemes of equal characteristic and the second assertion since the Gersten conjecture holds for $\mathcal{K}^M_{n,X}/p^r$ if $p > n-1$ by Proposition 3.2.2.

If $j > 0$, then the identification of $\mathrm{CH}^{d+j}(X, j)$ with $H^d(X, \mathcal{K}^M_{d+j,X})$ additionally requires the Gersten conjecture for higher Chow groups (see Chapter 2). This holds if A is equi-characteristic by [4, Sec. 10] and the method developed by Panin in [61] to extend the Gersten conjecture to the equi-dimensional setting. In mixed characteristic the Gersten conjecture for higher Chow groups with $\mathbb{Z}/p^r\mathbb{Z}$-coefficients holds by [21, Cor. 4.3]. □

3.5 Remarks on the injectivity of *res*

In this section we make a few remarks on the conjectured injectivity of *res*. Whether it is true or not is at this point unclear.

Let the notation be as in Section 3.4 and consider again the short exact sequence

$$0 \to \mathcal{K}^M_{d,X|X_n} \to \mathcal{K}^M_{d,X} \to \mathcal{K}^M_{d,X_n} \to 0$$

and the induced long exact sequence

$$\ldots \to H^d(X, \mathcal{K}^M_{d,X|X_n}) \xrightarrow{i} H^d(X, \mathcal{K}^M_{d,X}) \xrightarrow{res} H^d(X_1, \mathcal{K}^M_{d,X_n}) \xrightarrow{0} H^{d+1}(X, \mathcal{K}^M_{d,X|X_n}) \to \ldots$$

We denote the image of $H^d(X, \mathcal{K}^M_{d,X|X_n})$ under i by F^X_n.

As mentioned in the introduction, Kerz, Esnault and Wittenberg conjecture in [46, Sec. 10] that if $\mathrm{ch}(\mathrm{Quot}(A)) = 0$ and k is perfect of characteristic $p > 0$ and if we assume that the Gersten conjecture for $\mathcal{K}^M_X$ holds, then the map

$$res_X : \mathrm{CH}_1(X) \otimes \mathbb{Z}/p^r \to \text{"}\lim_n\text{"} H^d(X_n, \mathcal{K}^M_{X_n,d}/p^r)$$

is an isomorphism in the category of pro-systems of abelian groups.

We note that this conjecture would be implied by the following conjecture:

Conjecture 3.5.1.

$$F^X_n = < g_* F^Y_n | g : Y \to X \ \textit{projective}, Y/A \ \textit{smooth projective relative curve} > .$$

This can be seen as follows: $F^Y_n \subset H^1(Y, \mathcal{K}^M_{1,Y})$ is the image of $H^1(Y, \mathcal{K}^M_{1,Y|Y_n}) = H^1(Y, \mathcal{O}^\times_{Y|Y_n})$. By the p-adic logarithm isomorphism, assuming that p is large enough, $H^1(Y, \mathcal{O}^\times_{Y|Y_n}) \cong H^1(Y, p^n\mathcal{O}_Y)$ and therefore the composition

$$H^1(Y, \mathcal{O}^\times_{Y|Y_{n+1}}) \to H^1(Y, \mathcal{O}^\times_{Y|Y_n})$$

is multiplication by p. This implies that $\text{"}\lim_n\text{"} F^X_n \otimes \mathbb{Z}/p^r = 0$.

The following conjecture and the following Bertini theorem reduce the injectivity of *res* to the case of two horizontal one-cycles intersecting the special fiber transversally.

Conjecture 3.5.2. *Let Y be a smooth projective subscheme of X of relative dimension $d-1$. Let $x \in Y_1$. Then the map*

$$H^d_x(Y, \mathcal{K}^M_{d-1,Y|Y_{n+1}}) \to H^{d+1}_x(X, \mathcal{K}^M_{d,X|X_{n+1}})$$

is injective for all $n \geq 1$.

Proposition 3.5.3. *Let A be a henselian discrete valuation ring with uniformising parameter π and infinite residue field k. Let X be a smooth projective scheme over* $\mathrm{Spec}(A)$ *of relative dimension d. Let X_1 denote the special fiber. Let $Z_1, Z_2 \subset X$ be regular horizontal subschemes of dimension* 1 *such that $Z_i \pitchfork X_1$ for $i = 1, 2$. Then there is a smooth subscheme $H \subset X$ of relative dimension* 1 *over A containing Z_1 and Z_2.*

Proof. We may assume that $d = 2$. The case $Z_1 \cap Z_2 = \emptyset$ is clear. Let us therefore assume that Z_1 and Z_2 intersect in the closed point x. We start with a local analysis and then use this to find a global hypersurface section containing Z_1 and Z_2. Let $S_1, S_2 \subset X$ (resp. $T_1, T_2 \subset X$) be smooth relative curves containing Z_1 (resp. Z_2) and such that $S_1 \cap S_2 \cap X_1$ (resp. $T_1 \cap T_2 \cap X_1$) is reduced. Let

$$(x_1, x_2, \pi)$$

and

$$(y_1, y_2, \pi)$$

be the respective local parameter systems at x coming from the S_i and T_i. Z_1 is defined by the ideal (x_1, x_2) and Z_2 by the ideal (y_1, y_2). We may express the x_i in terms of (y_1, y_2, π):

$$x_1 = \alpha_1 y_1 + \beta_1 y_2 + \gamma_1 \pi$$

and

$$x_2 = \alpha_2 y_1 + \beta_2 y_2 + \gamma_2 \pi$$

with $\alpha_i, \beta_i, \gamma_i \in \mathcal{O}_{X,x}, i \in \{1, 2\}$. Now $(x_1, x_2) = (x_1 + bx_2, x_2)$ with $b \in \mathcal{O}_{X,x}$ and

$$x_1 + bx_2 = (\alpha_1 + b\alpha_2)y_1 + (\beta_1 + b\beta_2)y_2 + (\gamma_1 + b\gamma_2)\pi.$$

Therefore if $\gamma_1 + b\gamma_2 = 0$, then $x_1 + bx_2 \in (x_1, x_2)$ and $x_1 + bx_2 \in (y_1, y_2)$. We now distinguish the following two cases:

1. Case: γ_2 is invertible (if γ_1 is invertible, then we start by replacing x_2 by $x_2 + ax_1$). Then we set $b := -\gamma_1/\gamma_2$.

2. Case: Neither γ_1 nor γ_2 is invertible. We may assume that γ_1 is of the form $\pi^i \cdot u_1$ and γ_2 is of the form $\pi^j \cdot u_2$ with $i > j$ and $u_1, u_2 \in \mathcal{O}_{X,x}^\times$. Then we set $b := -\pi^{i-j}u_1/u_2$.

We now turn to the global situation. Since $V(x_1)$ and $V(x_2)$ are smooth at x and transversal, $V(x_1 + bx_2)$ is also smooth at x and contains Z_1 and Z_2. This implies that the embedding dimension

$$e_{(Z_1 \cup Z_2|_{X_1})}(x) \leq 1.$$

Let $\mathcal{I}$ denote the sheaf of ideals defining $Z_1 \cup Z_2$. Then for n large enough the natural homomorphism

$$H^0(X, \mathcal{I}(n)) \to H^0(X_1, \mathcal{I}(n)|_{X_1})$$

is surjective by Serre vanishing. By [49, Thm. 7] and possibly enlarging n, there is a section $s_1 \in H^0(X_1, \mathcal{I}(n)|_{X_1})$ defining a smooth hypersurface section of X_1. Any lift $s \in H^0(X, \mathcal{I}(n))$ of s_1 satisfies the properties of the proposition. □

Remark 3.5.4. *We may in general not find a smooth hypersurface section containing more than two, let's say Z_1, Z_2, Z_3, horizontal one-cycles intersecting X_1 transversally since the embedding dimension $e_{(Z_1 \cup Z_2 \cup Z_3|_{X_1})}(x)$ might be bigger than 1. This is not the case though if they are in a certain sense collinear.*

3.6 Relation with the p-adic cycle class map

In this section we prove Proposition 3.0.9.

Let k be a finite field of $\mathrm{ch}(k) = p > 0$, $A = W(k)$ and X be a smooth projective scheme over A of fiber dimension d. We let X_1/k denote the reduced special fiber. Let $\tau \in \{\text{Nis, ét}\}$ and $X_{1,\tau}$ be the respective small site. Let $\epsilon : X_{1,\text{ét}} \to X_{1,\text{Nis}}$ be the canonical map of sites.

Definition 3.6.1. *([6, Def. A.3])*

(a) By $\mathrm{Sh}(X_{1,\tau})$ we denote the category of sheaves of abelian groups on $X_{1,\tau}$. By $\mathrm{C}(X_{1,\tau})$ we denote the category of unbounded complexes in $\mathrm{Sh}(X_{1,\tau})$.

(b) By $\mathrm{Sh}_{\mathrm{pro}}(X_{1,\tau})$ we denote the category of pro-sytems in $\mathrm{Sh}(X_{1,\tau})$.

(c) By $\mathrm{C}_{\mathrm{pro}}(X_{1,\tau})$ we denote the category of pro-systems in $\mathrm{C}(X_{1,\tau})$.

(d) By $\mathrm{D}_{\mathrm{pro}}(X_{1,\tau})$ we denote the Verdier localization of the homotopy category of $\mathrm{C}_{\mathrm{pro}}(X_{1,\tau})$, where we kill objects which are represented by systems of complexes which have level-wise vanishing cohomology sheaves.

Definition 3.6.2. *We define*

$$W.\Omega^\bullet_{X_1} \in \mathrm{C}_{\mathrm{pro}}(X_1)_\tau$$

to be the pro-system of de Rham-Witt complexes in the étale or Nisnevich topology (see [33]). We define

$$W.\Omega^r_{X_1,\log} \in \mathrm{Sh}_{\mathrm{pro}}(X_1)_\tau$$

to be the pro-system of étale or Nisnevich subsheaves in $W_r\Omega^j_{X_1}$ which are locally generated by symbols

$$d\log\{[a_1]\} \cdot ... \cdot d\log\{[a_j]\}$$

with $a_1, ..., a_j \in \mathcal{O}^\times_{X_1}$ local sections and where $[-]$ is the Teichmüller lift (see [33, p. 505, (1.1.7)]).

Definition 3.6.3. *Assuming $j < p$, we define $\mathcal{S}_r(j)_{\text{ét}}$ to be the syntomic complex defined in [40, Def. 1.6]. We denote the corresponding object in $\mathrm{D}_{\text{pro}}(X_1)_{\text{ét}}$ by $\mathcal{S}_{X_.}(j)_{\text{ét}}$.*

Definition 3.6.4. *([6, Sec. 4]) We define $\mathcal{S}_r(j)_{\text{Nis}} := \tau_{\leq j} R\epsilon_* \mathcal{S}_r(j)_{\text{ét}}$ and $\mathcal{S}_{X_.}(j)_{\text{Nis}} := \tau_{\leq j} R\epsilon_* \mathcal{S}_{X_.}(j)_{\text{ét}}$, where $\tau_{\leq j}$ is the good truncation.*

Let $j < p$. In [6, Sec. 7], Bloch, Esnault and Kerz define a motivic pro-complex

$$\mathbb{Z}_{X_.}(j) := \text{cone}(\mathcal{S}_{X_.}(j) \oplus \mathbb{Z}_{X_1}(j) \to W_.\Omega^j_{X_1,\log}[-j])[-1]$$

in the Nisnevich topology. $\mathbb{Z}_{X_.}(j)$ is an object in $\mathrm{D}_{\text{pro}}(X_{1,\text{Nis}})$ with the following properties:

Proposition 3.6.5. *([6, Prop. 7.2])*

(0) $\mathbb{Z}_{X_.}(0) = \mathbb{Z}$, *the constant sheaf in degree* 0.

(1) $\mathbb{Z}_{X_.}(j) = \mathbb{G}_{m,X_.}[-1]$.

(2) $\mathbb{Z}_{X_.}(j)$ *is supported in degrees* $\leq j$ *and in* $[1, j]$ *if* $j \geq 1$ *and if the Beilinson-Soulé conjecture holds.*

(3) $\mathbb{Z}_{X_.}(j) \otimes^L_{\mathbb{Z}} \mathbb{Z}/p^\cdot = \mathcal{S}_{X_.}(j)_{\text{Nis}}$ *in* $\mathrm{D}_{\text{pro}}(X_{1,\text{Nis}})$.

(4) $\mathcal{H}^j(\mathbb{Z}_{X_.}(j)) = \mathcal{K}^M_{j,X_.}$ *in* $\mathrm{Sh}_{\text{pro}}(X_{1,\text{Nis}})$.

(5) There is a canonical product structure $\mathbb{Z}_{X_.}(j) \otimes^L_{\mathbb{Z}} \mathbb{Z}_{X_.}(j') \to \mathbb{Z}_{X_.}(j + j')$.

We now start the proof of Proposition 3.0.9 proving the following lemmas:

Lemma 3.6.6. *Let $j < p$. Then the map*

$$H^d_{\text{Nis}}(X_1, \mathcal{K}^M_{j,X_.} \otimes \mathbb{Z}/p^\cdot) \to H^{d+j}_{\text{Nis}}(X_1, \mathcal{S}_.(j))$$

is an isomorphism in the category of pro-abelian groups.

Proof. By properties (2)-(4) of Proposition 3.6.5 we have that $\mathcal{K}^M_{j,X_.} \otimes \mathbb{Z}/p^\cdot \cong \mathcal{H}^j(\mathcal{S}_.(j)_{\text{Nis}})$. Let us be more precise:

$$\mathcal{K}^M_{j,X_.} \otimes \mathbb{Z}/p^\cdot \overset{(4)}{\cong} \mathcal{H}^j(\mathbb{Z}_{X_.}(j)) \otimes \mathbb{Z}/p^\cdot \overset{(2)}{\cong} \mathcal{H}^j(\mathcal{S}_.(j)_{\text{Nis}} \otimes \mathbb{Z}/p^\cdot) \overset{(3)}{\cong} \mathcal{H}^j(\mathcal{S}_.(j)_{\text{Nis}}).$$

Note that we do not work with the derived tensor product here and that $\mathbb{Z}_{X_\cdot}(j) \otimes_{\mathbb{Z}} \mathbb{Z}/p^\cdot = \mathcal{S}_{X_\cdot}(j)_{\mathrm{Nis}}$ follows from the same arguments as in the proof of [6, Prop. 7.2(3)]. This implies that

$$H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^\cdot) \cong H^d_{\mathrm{Nis}}(X_1, \mathcal{H}^j(\mathcal{S}_\cdot(j))).$$

The hypercohomology spectral sequence

$$E_2^{pq} = H^p_{\mathrm{Nis}}(X_1, \mathcal{H}^q(\mathcal{S}_r(j))) \Rightarrow \mathbb{H}^{p+q}_{\mathrm{Nis}}(X_1, \mathcal{S}_r(j))$$

together with the Nisnevich cohomological dimension of X_1 and the concentration of $\mathcal{S}_r(j)_{\mathrm{Nis}}$ in degrees $\leq j$ implies that $H^d_{\mathrm{Nis}}(X_1, \mathcal{H}^j(\mathcal{S}_r(j))) \cong H^{d+j}_{\mathrm{Nis}}(X_1, \mathcal{S}_r(j))$ and therefore that $H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^\cdot) \to H^{d+j}_{\mathrm{Nis}}(X_1, \mathcal{S}_\cdot(j))$. □

Lemma 3.6.7. *The natural map*

$$H^{2d-q}_{\mathrm{Nis}}(X_1, W_r\Omega^d_{X_1,\log}[-d]) \to H^{2d-q}_{\text{ét}}(X_1, W_r\Omega^d_{X_1,\log}[-d])$$

is an isomorphism for $q \leq 2$.

Proof. Let $KH^0_a(X_1, \mathbb{Z}/p^r\mathbb{Z})$ denote the so called Kato homology groups, i.e. the homology in degree a of the complex $C^0_{p^r}$ defined in [38]. By [36, Lem. 6.2] (see also [47, Sec. 9]) there is a long exact sequence

$$\begin{aligned} ... \to KH^0_{q+2}(X_1, \mathbb{Z}/p^r\mathbb{Z}) \to \mathrm{CH}^d(X_1, q; \mathbb{Z}/p^r\mathbb{Z}) \to H^{2d-q}_{\text{ét}}(X_1, \mathbb{Z}/p^r\mathbb{Z}(d)) \to \\ KH^0_{q+1}(X_1, \mathbb{Z}/p^r\mathbb{Z}) \to \mathrm{CH}^d(X_1, q-1; \mathbb{Z}/p^r\mathbb{Z}) \to H^{2d-q+1}_{\text{ét}}(X_1, \mathbb{Z}/p^r\mathbb{Z}(d)) \to ... \end{aligned}$$

where $\mathbb{Z}/p^r\mathbb{Z}(d) = W_r\Omega^d_{X_1,\log}[-d]$. We first identify the group $\mathrm{CH}^d(X_1, q; \mathbb{Z}/p^r\mathbb{Z})$ with $H^{2d-q}_{\mathrm{Nis}}(X_1, W_r\Omega^d_{X_1,\log}[-d])$ for $q = 0, 1$. Consider the spectral sequence

$${}^{\mathrm{CH}}E_1^{p,q}(X_1) = \oplus_{x \in X_1^{(p)}} \mathrm{CH}^{d-p}(\mathrm{Spec}k(x), -p-q, \mathbb{Z}/p^r\mathbb{Z}) \Rightarrow \mathrm{CH}^d(X_1, -p-q, \mathbb{Z}/p^r\mathbb{Z})$$

from [4, Sec. 10] and note that

$$\mathrm{CH}^a(\mathrm{Spec}k(x), a, \mathbb{Z}/p^r\mathbb{Z}) \cong K^M_a(k(x))/p^r \cong W_r\Omega^a_{k(x),\log}$$

for all $a \geq 0$. The first isomorphism follows from [59, Thm. 4.9] (see also [76]) and the fact that $\mathrm{CH}^a(\mathrm{Spec}k(x), a, \mathbb{Z}/p^r\mathbb{Z}) \cong \mathrm{CH}^a(\mathrm{Spec}k(x), a) \otimes \mathbb{Z}/p^r\mathbb{Z}$. The second isomorphism follows from the Bloch-Gabber-Kato theorem (see [7]). This implies the identification since $\mathrm{CH}^0(k(x), 1) = 0$ and since

$$\bigoplus_{x \in X^0} i_{x*}W_r\Omega^d_{k(x),\log} \to \bigoplus_{x \in X^1} i_{x*}W_r\Omega^{d-1}_{k(x),\log} \to ... \to \bigoplus_{x \in X^d} i_{x*}W_r\Omega^0_{k(x),\log}$$

is a (Gersten-)resolution for the sheaf $W_r\Omega^d_{X_1,\log}$ considered in the Zariski topology (see [26]) and therefore also in the Nisnevich topology. In particular $H^i_{\mathrm{Zar}}(X_1, W_r\Omega^d_{X_1,\log}) \cong H^i_{\mathrm{Nis}}(X_1, W_r\Omega^d_{X_1,\log})$ for all $i \geq 0$.

Now the Kato homology groups $KH^0_i(X_1, \mathbb{Z}/p^r\mathbb{Z})$ vanishes for $1 \leq i \leq 4$ by [36, Thm. 0.3] (see also [47, Thm. 8.1]) which implies the lemma. □

Lemma 3.6.8. *Let $j < p$. Then*

$$H^{j+d}_{\mathrm{Nis}}(X_1, \mathcal{S}_{\cdot}(j)) \to H^{j+d}_{\text{ét}}(X_1, \mathcal{S}_{\cdot}(j))$$

is an isomorphism for all $j \geq d$.

Proof. By [6, Thm 5.4] we have an exact triangle

$$p(j)\Omega^{\leq j}_{X_\cdot}[-1] \to S_{X_\cdot}(j)_{\mathrm{Nis}} \to W_{\cdot}\Omega^j_{X_1,\log}[-j] \xrightarrow{[1]} ..$$

in $\mathrm{D}_{\mathrm{pro}}(X_1)_{\mathrm{Nis}}$ which comes from the exact triangle

$$p(j)\Omega^{\leq j}_{X_\cdot}[-1] \to S_{X_\cdot}(j)_{\text{ét}} \to W_{\cdot}\Omega^j_{X_1,\log}[-j] \xrightarrow{[1]} ..$$

in $\mathrm{D}_{\mathrm{pro}}(X_1)_{\text{ét}}$ by applying the functor $\tau_{\leq j} \circ R\epsilon_*$. This induces the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
H^{d+j-1}_{\mathrm{Nis}}(X_1, W_{\cdot}\Omega^j_{X_1,\log}[-j]) & \longrightarrow & H^{d+j}_{\mathrm{Nis}}(X_1, p(j)\Omega^{\leq j}_{X_\cdot}[-1]) & \longrightarrow & H^{d+j}_{\mathrm{Nis}}(X_1, S_{X_\cdot}(j)_{\mathrm{Nis}}) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow \\
H^{d+j-1}_{\text{ét}}(X_1, W_{\cdot}\Omega^j_{X_1,\log}[-j]) & \longrightarrow & H^{d+j}_{\text{ét}}(X_1, p(j)\Omega^{\leq j}_{X_\cdot}[-1]) & \longrightarrow & H^{d+j}_{\text{ét}}(X_1, S_{X_\cdot}(j)_{\text{ét}})
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^{d+j}_{\mathrm{Nis}}(X_1, W_{\cdot}\Omega^j_{X_1,\log}[-j]) & \longrightarrow & H^{d+j+1}_{\mathrm{Nis}}(X_1, p(j)\Omega^{\leq j}_{X_\cdot}[-1]) \\
& \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\
\longrightarrow & H^{d+j}_{\text{ét}}(X_1, W_{\cdot}\Omega^j_{X_1,\log}[-j]) & \longrightarrow & H^{d+j+1}_{\text{ét}}(X_1, p(j)\Omega^{\leq j}_{X_\cdot}[-1])
\end{array}$$

Now α and γ are isomorphisms by Lemma 3.6.7 and the fact that $W_{\cdot}\Omega^j_{X_1,\log}[-j] = 0$ for $j > d$. The maps β and δ are isomorphisms since $p(j)\Omega^{\leq j}_{X_\cdot}[-1]$ is a complex of coherent sheaves. The result follows by the five-lemma. □

Proposition 3.6.9. *Let $p > j$ and $j \geq d$. Then the map*

$$\text{"}\lim_n\text{"} H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_n}/p^r) \to H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j))$$

is an isomorphism of pro-abelian groups.

Proof. It follows from Lemma 3.6.6 and Lemma 3.6.8 that

$$H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^{\cdot}) \to H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_{\cdot}(j)).$$

Tensoring with $\mathbb{Z}/p^r$ gives the desired result since it is right exact and both cohomology groups are taken in the top degree. Note furthermore that $\mathcal{S}_{\cdot}(j)_{\text{ét}} \otimes \mathbb{Z}/p^r \cong \mathcal{S}_r(j)_{\text{ét}}$. This follows for example from [21, Thm. 1.3]. □

Proposition/Definition 3.6.10. *Diagram (3.0.2) of the introduction commutes.*

Proof. Let X_K be the generic fiber of X. Let $i : X_1 \hookrightarrow X$ and $j : X_K \hookrightarrow X$ be the canonical inclusions.

The exact Kummer sequence

$$0 \to \mu_{p^n} \to \mathcal{O}_{X_K}^\times \xrightarrow{p^n} \mathcal{O}_{X_K}^\times \to 0$$

on $X_{K,\text{ét}}$ induces an exact sequence

$$j_*\mathcal{O}_{X_K}^\times \xrightarrow{p^n} j_*\mathcal{O}_{X_K}^\times \to R^1 j_*\mu_{p^n} \to 0$$

on $X_{\text{ét}}$ which induces a Galois symbol map

$$j_*\mathcal{K}_{q,X_K}^M \to R^q j_*(\mathbb{Z}/p^r\mathbb{Z}(q))$$

(see [7, (1.2)]). This map induces a map

$$\mathcal{K}_{q,X}^M \to \ker(\sigma_{X,r}^q : R^q j_*(\mathbb{Z}/p^r\mathbb{Z}(q)) \to W_r\Omega_{X_1,\log}^{q-1}) \cong \mathcal{H}^q(\mathcal{T}_r(q))$$

in the étale topology. For the definition of $\sigma_{X,r}^q$ see [69, Sec. 3.2] and for the isomorphism on the right see [69, Def. 4.2.4]. Taking cohomology groups we get the following commutative diagram:

$$\begin{array}{ccc}
H^d_{\text{ét}}(X, \mathcal{K}^M_{q,X}/p^r) & \longrightarrow & H^d_{\text{ét}}(X_1, \mathcal{K}^M_{q,X_n}/p^r) \\
\downarrow & & \downarrow (*) \\
H^d_{\text{ét}}(X, \mathcal{H}^q(\mathcal{T}_r(q))) & \xrightarrow{\cong} & H^d_{\text{ét}}(X_1, \mathcal{H}^q(\mathcal{S}_r(q))) \\
\downarrow \cong & & \downarrow \cong \\
H^{d+q}_{\text{ét}}(X, \mathcal{T}_r(q)) & \xrightarrow{\cong} & H^{d+q}_{\text{ét}}(X_1, \mathcal{S}_r(q))
\end{array}$$

Here $(*)$ is induced by Kato's syntomic regulator map ([40, Sec. 3]) and the commutativity of the upper sqare follows from [40, Lem. 4.2]. The horizontal isomorphisms follow from proper base change and the fact that $i^*\mathcal{T}_r(n) \cong \mathcal{S}_r(n)$ if $p > n+1$. For the isomorphism on the right see the proof of Proposition 3.6.6. The change of sites $\epsilon : X_{\text{ét}} \to X_{\text{Zar}}$ now gives the result. □

As a corollary we get the following result:

Corollary 3.6.11. *Let $j+1 < p$. Then the cycle class map*

$$\varrho_{p^r}^{j,j-d} : \mathrm{CH}^j(X, j-d, \mathbb{Z}/p^r\mathbb{Z}) \to H^{d+j}_{\text{ét}}(X, \mathcal{T}_r(j))$$

is surjective for all $j \geq d$.

Proof. By Corollary 3.4.5 and Remark 3.4.3 the map

$$res : \mathrm{CH}^j(X, j-d, \mathbb{Z}/p^r\mathbb{Z}) \to \text{"}\lim_n\text{"} H^d_{\mathrm{Zar}}(X_1, \mathcal{K}^M_{j,X_n}/p^r) \cong H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_n}/p^r)$$

is surjective for $j-1<p$. By Proposition 3.6.9 we have that "$\lim_n$"$H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_n}/p^r) \cong H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j))$ for $j<p$. Furthermore, for $j+1<p$ we have that $H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j)) \cong H^{d+j}_{\text{ét}}(X, \mathcal{T}_r(j))$ (see [69, Sec. 1.4]). The result now follows from the commutativity of (3.0.2). □

Remark 3.6.12. *As we noted in the introduction, Saito and Sato show in [67] that the the cycle class map*

$$\varrho^{d,0}_{p^r} : \mathrm{CH}^d(X)/p^r \to H^{2d}_{\text{ét}}(X, \mathcal{T}_r(d))$$

defined in [69, Cor. 6.1.4] is surjectiv for X a regular scheme which is proper, flat, of finite type and which has semistable reduction over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a p-adic local field K, by [67]. We expect that this map coincides with the map defined in Proposition 3.6.10.

Finally we note the following injectivity result for curves:

Proposition 3.6.13. *Let X be smooth projective of relative dimension 1 over a p-adic local ring A. Then the cycle class map*

$$\varrho^{1,0}_{p^r} : \mathrm{CH}_1(X)/p^r \to H^2_{\text{ét}}(X, \mathcal{T}_r(1))$$

is injective.

Proof. This follows immediately from the spectral sequence

$$E_1^{u,v} = \oplus_{x \in X^u} H^{v+u}_{\text{ét},x}(X, \mathcal{T}_r(d)) \Rightarrow H^{v+u}_{\text{ét}}(X, \mathcal{T}_r(d))$$

since by absolute cohomological purity and the purity property of p-adic étale Tate twists $E_2^{1,1} \cong \mathrm{CH}_1(X)/p^r$ (see [68]). □

Keeping the assumptions of Proposition 3.6.13 we get a sequence of isomorphisms

$$H^2_{\text{ét}}(X, \mathcal{T}_r(1)) \xleftarrow{\cong} \mathrm{CH}_1(X)/p^r \xrightarrow{\cong} \text{"}\lim_n\text{"} H^1(X_1, \mathcal{K}^M_{X_n,1}/p^r).$$

The isomorphism on the right follows from Section 3.5. It would be interesting to have a similar result for $\mathrm{CH}^2(X, 1, \mathbb{Z}/p^r\mathbb{Z})$.

Chapter 4

Deformation theory of zero-cycles

Let A be a henselian discrete valuation ring with uniformising parameter π and residue field k. Let X be a smooth projective scheme over $\mathrm{Spec}(A)$ of relative dimension d. Let $X_n := X \times_A A/(\pi^n)$, i.e. X_1 is the special fiber and the X_n are the respective thickenings of X_1. Let $\mathcal{K}^M_{*,X}$ be the improved Milnor K-sheaf defined in [43] and $\mathcal{K}^M_{*,X_n}$ its restriction to X_n.

In this chapter we assume that either (1) A is the Witt ring $W(k)$ of a perfect field k of $\mathrm{ch}(k) > 2$ or that (2) k is of characteristic 0 and $A = k[[t]]$. In each of these two cases there exists a well-defined exponential map

$$\exp : \Omega^{d-1}_{X_1} \to \mathcal{K}^M_{d,X_n}$$

defined by

$$x d\log(y_1) \wedge ... \wedge d\log(y_{d-1}) \mapsto \{1 + x\pi^{n-1}, y_1, ..., y_{d-1}\}$$

(see [6, Sec. 12] for (1) and [5, Sec. 2] for (2)). In these two cases we therefore get an exact sequence

$$\Omega^{d-1}_{X_1} \to \mathcal{K}^M_{d,X_n} \to \mathcal{K}^M_{d,X_{n-1}} \to 0$$

which we use to study the restriction map

$$res_{X_n} : \mathrm{CH}^d(X) \xrightarrow{\cong} H^d(X, \mathcal{K}^M_{d,X}) \xrightarrow{res_{X_n}} H^d(X_1, \mathcal{K}^M_{d,X_n})$$

assuming the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{*,X}$ (see Definition 3.2.1) for the isomorphism on the left. One may consider $H^d(X_1, \mathcal{K}^M_{d,X_n})$ to be an ad hoc cohomological definition of the Chow group $\mathrm{CH}_0(X_n)$ of zero-cycles on X_n. This definition was also used in [5, Sec. 4].

By proving a moving lemma for $H^2(X_1, \Omega^1_{X_1})$ if $d = 2$ and a Lefschetz theorem, we give a different proof of the following theorem which we already saw in Chapter 3.

Theorem 4.0.14. *With the above notation, and assuming the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{*,X}$, the map*

$$res : \mathrm{CH}_1(X) \to (H^d(X_1, \mathcal{K}^M_{d,X_n}))_n$$

is an epimorphism in pro-Ab.

Furthermore, we give a detailed proof of Conjecture (j) for $d = 1$. Recall that Conjecture (j) says the following:

Conjecture 4.0.15. *(see [46, Sec. 10]) The map*

$$res : \mathrm{CH}^d(X) \otimes \mathbb{Z}/p^r\mathbb{Z} \to \text{"}\lim_n\text{"}\, H^d(X_1, \mathcal{K}^M_{d,X_n}/p^r)$$

is an isomorphism in pro-Ab if $\mathrm{ch}(\mathrm{Quot}(A)) = 0$ *and if k is perfect of characteristic $p > 0$* .

4.1 Local cohomology and some calculations

In this section we recall some definitions and calculations in local cohomology which we will need later on. A standard reference for the following is [31, Ch. IV].

Let X be a locally noetherian scheme. To any sheaf of abelian groups $\mathcal{F}$ on X we can associate a coniveau complex of sheaves

$$\mathcal{C}(\mathcal{F}) := \bigoplus_{x \in X^{(0)}} i_{x,*} H^0_x(X, \mathcal{F}) \to \bigoplus_{x \in X^{(1)}} i_{x,*} H^1_x(X, \mathcal{F})) \to ...$$

where $i_x : x \to X$ is the natural inclusion. This complex is also called the Cousin complex of $\mathcal{F}$.

Definition 4.1.1. *A sheaf $\mathcal{F}$ on X is called Cohen-Macaulay, or simply* CM*, if for every $x \in X$ it holds that $H^i_x(X, \mathcal{F}) = 0$ for $i \neq \mathrm{codim}(x)$.*

Via the coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H^{p+q}_x(X, \mathcal{F}) \Rightarrow H^n(X, \mathcal{F})$$

one can easily deduce that the property of being CM for $\mathcal{F}$ is equivalent to $\mathcal{C}(\mathcal{F})$ being an acyclic resolution of $\mathcal{F}$ (see [31, Ch. IV, prop. 2.6]). In that case one can use $\mathcal{C}(\mathcal{F})$ to calculate the cohomology of $\mathcal{F}$, i.e. $H^*(X, \mathcal{F}) \cong H^*(X, \mathcal{C}(\mathcal{F}))$.

Locally free sheaves are CM (see [31, p.239]) so in particular the sheaf of differential forms Ω^1_X and its exterior powers Ω^a_X are CM if X is a smooth variety over a field. These sheaves are our main tool in this chapter to study the deformation of zero-cycles on a model X/A as above. We can make the following calculations:

Lemma 4.1.2. *Let k be a field and X_1 be a scheme of dimension* 1 *over* $\mathrm{Spec}(k)$*. Let $x \in X_1$ be a regular closed point and f a local parameter at x. Then*

$$\mathcal{O}_{X_1,x}[\frac{1}{f}]/\mathcal{O}_{X_1,x} \cong H^1_x(X_1, \mathcal{O}_{X_1}).$$

Proof. We calculate $H^1_x(X_1, \mathcal{O}_{X_1})$ locally as follows: Let $X_{1,x} := \mathrm{Spec}\mathcal{O}_{X_1,x}$. Applying Motif B of [31, p.217] to the triple $(x, X_1, X_1 - x)$, we get a short exact sequence

$$H^0(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}) \to H^0(X_{1,x}-x, \mathcal{O}_{X_1}|_{X_{1,x}-x}) \to H^1_x(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}) \to H^1(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}).$$

Since $H^1(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}) = 0$, this gives an isomorphism

$$\mathcal{O}_{X_1,x}[\frac{1}{f}]/\mathcal{O}_{X_1,x} \cong H^1_x(X_{1,x}, \mathcal{O}_{X_1}|_{X_{1,x}}).$$

□

We now turn to the higher dimensional case. Similar calculations can be found in [3, Sec. 5].

Lemma 4.1.3. *Let k be a field and X_1 be a scheme of dimension $d > 1$ over* $\mathrm{Spec}(k)$*. Let $x \in X_1$ be a regular closed point and $f_1, ..., f_d \in \mathfrak{m}_x$ a local parameter system at x. Then $H^d_x(X, \Omega^{d-1}_{X_1})$ is generated by elements of the form*

$$\frac{df_1 \wedge ... \wedge \hat{df_i} \wedge ... \wedge df_d}{f_1^{n_1}...f_d^{n_d}}$$

modulo $\frac{df_1\wedge...\wedge\hat{df_i}\wedge...\wedge df_d}{f_1^{n_1}...\hat{f_j}...f_d^{n_d}}$ over $\mathcal{O}_{X_1,x}$.

Proof. Let U be an affine neighbourhood of $x \in X_1$. Let $\mathcal{V} = \{V_i := U - V(f_i)\}$ be a covering of $U - x$. Then the Čech complex

$$0 \to \prod \Omega^{d-1}_{X_1}(V_i) \to \prod_{i \neq j} \Omega^{d-1}_{X_1}(V_i \cap V_j) \to ... \to \Omega^{d-1}_{X_1}(V_1 \cap ... \cap V_d)$$

gives an isomorphism of $\mathrm{coker}(\prod \Omega^{d-1}(V_1 \cap ... \cap \hat{V_i} \cap ... \cap V_d)) \to \Omega^{d-1}(V_1 \cap ... \cap V_d))$ with $\Gamma(U, R^{d-1}j_*(\Omega^{d-1}|_{U-x}))$, where j is the inclusion $X_1 - x \hookrightarrow X_1$. Now $R^{d-1}j_*(\Omega^{d-1}|_{X_1-x})$ is isomorphic to the sheaf $\mathcal{H}^d_x(X_1, \Omega^{d-1}_{X_1})$ (again Motif B of [31, p.217] and that $d \geq 2$). In other words, $\Gamma(U, \mathcal{H}^d_x(X, \Omega^{d-1}_{X_1}))$ is generated by elements of the form $\frac{df_1\wedge...\wedge\hat{df_i}\wedge...\wedge df_d}{f_1^{n_1}...f_d^{n_d}}$ modulo $\frac{df_1\wedge...\wedge\hat{df_i}\wedge...\wedge df_d}{f_1^{n_1}...\hat{f_j}...f_d^{n_d}}$ over $\mathcal{O}(U)$. Passing to the limit, we get the desired result. □

Proposition 4.1.4. *Let k be a perfect field with* $\mathrm{ch}(k) = p > 2$ *and let X be a smooth scheme over $A := W(k)$. Then there is an exact sequence*

$$0 \to \Omega^{r-1}_{X_1}/B_{n-1}\Omega^{r-1}_{X_1} \to K^M_{r,X_{n+1}} \to K^M_{r,X_n} \to 0. \qquad (4.1.1)$$

Proof. Let R_n be an essentially smooth local ring over A/π^n. We define a filtration $U^iK^M_r(R_n) \subset K^M_r(R_n)$ by

$$U^iK^M_r(R_n) :=< \{1 + \pi^i x, x_2, ..., x_r \mid x \in R_n, x_2, ..., x_r \in R^*_n\} > .$$

The U^i fit into the following exact sequences:

$$0 \to U^n K_r^M(R_{n+1}) \to K_r^M(R_{n+1}) \to K_r^M(R_n) \to 0.$$

By [6, 12.3] there is an isomorphism

$$\Omega_{R_1}^{r-1}/B_{i-1}\Omega_{R_1}^{r-1} \cong gr^i K_r^M(R_n) \cong U^i K_r^M(R_n)/U^{i+1}K_r^M(R_n)$$

and since $U^{n+1}(K_r^M(R_{n+1})) = 0$, this implies that $U^n(K_r^M(R_{n+1})) \cong \Omega_{R_1}^{r-1}/B_{n-1}\Omega_{R_1}^{r-1}$ and therefore the exact sequence

$$0 \to \Omega_{R_1}^{r-1}/B_{n-1}\Omega_{R_1}^{r-1} \to K_r^M(R_{n+1}) \to K_r^M(R_n) \to 0.$$

□

Remark 4.1.5. *For k a field of characteristic 0, $S_n = \mathrm{Spec} k[t]/(t^n)$, $S = \mathrm{Spec} k[[t]]$ and X smooth, separated and of finite type over S, there exists a short exact sequence*

$$0 \to \Omega_{X_1}^{r-1} \to \mathcal{K}_{r,X_n}^M \to \mathcal{K}_{r,X_{n-1}}^M \to 0$$

by [5, Prop. 2.3].

Corollary 4.1.6. *Let X be as in Proposition 4.1.4. Then the sheaf $\mathcal{K}_{r,X_n}^M$ is* CM.

Proof. Applying the derived functor $H_x^i(X_1, -)$ to (4.1.1), we get the exact sequence

$$H_x^i(X_1, \Omega_{R_1}^{r-1}/B_{n-1}\Omega_{R_1}^{r-1}) \to H_x^i(X_1, \mathcal{K}_{r,X_2}^M) \to H_x^i(X_1, \mathcal{K}_{r,X_1}^M).$$

By [33, Cor. 3.9, p. 572], the sheaf $\Omega_{X_1}^{r-1}/B_{n-1}\Omega_{X_1}^{r-1}$ is locally free and therefore CM. The sheaf $\mathcal{K}_{r,X_1}^M$ is CM by [43] and [44]. The result follows inductively. □

Remark 4.1.7. *Let X be as in Proposition 4.1.4. Then if $\mathcal{K}_{r,X}^M$ satisfies the Gersten conjecture, then the relative Milnor K-sheaf $\mathcal{K}_{r,X|X_n}^M$ of Definition 3.3.1 is* CM*: The short exact sequence*

$$0 \to \mathcal{K}_{r,X|nX_1}^M \to \mathcal{K}_{r,X}^M \to \mathcal{K}_{r,X_n}^M \to 0$$

induces the exact sequence

$$... \to H_x^{c-1}(X_1, \mathcal{K}_{r,X_n}^M) \to H_x^c(X, \mathcal{K}_{r,X|X_n}^M) \to H_x^c(X, \mathcal{K}_{r,X}^M) \to ...$$

The statement follows from Lemma 4.1.6 and the assumption that $\mathcal{K}_{r,X}^M$ satisfies the Gersten conjecture.

4.2 $\mathrm{CH}^1(X)/p^i \cong$ "lim"$\mathrm{Pic}(X_n)/p^i$ for relative dimension 1

Before we state the main proposition of this section, we quickly review the theory of pro-objects. Standard references are [1] and [30].

Let $\mathcal{C}$ be a category. The category of pro-objects pro-$\mathcal{C}$ in $\mathcal{C}$ is defined as follows: A pro-object is a contravariant functor

$$X : I^\circ \to \mathcal{C},$$

from a filtered index category I to $\mathcal{C}$, i.e. an inverse system of objects X_i in $\mathcal{C}$. We denote X also by "lim"X_i or $(X_i)_i$. The morphisms between two objects $X =$ "lim"X_i and $Y =$ "lim"$Y_i \in$ in pro-$\mathcal{C}$ are given by

$$\mathrm{Hom}(X, Y) = \varprojlim_j (\varinjlim_i \mathrm{Hom}(X_i, Y_j)).$$

There is a natural fully faithful embedding of $\mathcal{C}$ into pro-$\mathcal{C}$ which associates to an object $C \in \mathcal{C}$ the constant diagram C. This functor has a right adjoint pro-$\mathcal{C} \to \mathcal{C}$, "lim"$X_i \mapsto \varprojlim_i X_i$. If $\mathcal{C}$ has finite direct (inverse) limits, then the functor

$$\mathrm{Hom}(I^\circ, \mathcal{C}) \to \text{pro-}\mathcal{C}$$

commutes with finite direct (inverse) limits. In particular if $\mathcal{C}$ has finite direct and inverse limits, then the above functor is exact (see [1, p.163]).

A criterion for when a map of pro-systems is an isomorphism is given by the following proposition (see [34, Lem. 2.3]):

Proposition 4.2.1. *A level map $A \to B$ in* pro-$\mathcal{C}$*, i.e. a map between pro-systems with the same index category and maps $A_s \to B_s$ for all $s \in I$, is an isomorphism if and only if for all s there exists a $t \geq s$ and a commutative diagram*

$$\begin{array}{ccc} A_t & \longrightarrow & B_t \\ \downarrow & \swarrow & \downarrow \\ A_s & \longrightarrow & B_s. \end{array}$$

We now give a proof of Conjecture 4.0.15 for $d = 1$.

Theorem 4.2.2. *Let k be a finite field of characteristic $p > 2$ and $A = W(k)$ the Witt ring of k. Let X be a smooth projective scheme of relative dimension 1 over A. Then the map*

$$res : \mathrm{CH}^1(X) \otimes \mathbb{Z}/p^i\mathbb{Z} \to \text{"lim"} H^1(X_1, \mathcal{K}^M_{1,X_n}/p^i)$$

is an isomorphism in the category of pro-systems of abelian groups.

Proof. We first note that $\mathrm{CH}^1(X) \cong \mathrm{Pic}(X)$ and that $\mathrm{Pic}(X) \cong \varprojlim \mathrm{Pic}(X_n)$ by [28, Thm. 5.1.4]. Furthermore, $H^1(X_1, \mathcal{K}^M_{1,X_n}) = H^1(X_1, \mathcal{O}^\times_{X_n}) \cong \mathrm{Pic}(X_n)$. It therefore suffices to show that

$$\varprojlim \mathrm{Pic}(X_n) \otimes \mathbb{Z}/p^i\mathbb{Z} \to \text{"lim"} \mathrm{Pic}(X_n) \otimes \mathbb{Z}/p^i\mathbb{Z}$$

is an isomorphism.

Using the p-adic logarithm isomorphism $1 + p\mathcal{O}_{X_n} \xrightarrow{\cong} p\mathcal{O}_{X_n}$, the short exact sequence

$$1 \to (1 + p^j\mathcal{O}_{X_n}) \to \mathcal{O}^\times_{X_n} \to \mathcal{O}^\times_{X_j} \to 1$$

induces a short exact sequence

$$0 \to H^1(X_1, p^j\mathcal{O}_{X_n}) \to H^1(X_1, \mathcal{O}^*_{X_n}) \to H^1(X_1, \mathcal{O}^*_{X_j}) \to H^2(X_1, p^j\mathcal{O}_{X_n}) = 0$$

(the last equality following for dimension reasons). Applying the Functor $\varprojlim_n$, we get an exact sequence

$$\varprojlim_n H^1(X_1, p^j\mathcal{O}_{X_n}) \to \varprojlim_n \mathrm{Pic}(X_n) \to \mathrm{Pic}(X_j) \to \varprojlim_n{}^1 H^1(X_1, p^j\mathcal{O}_{X_n}).$$

Now $\varprojlim^1_n H^1(X_1, p^j\mathcal{O}_{X_n}) = 0$ since the inverse system $(H^1(X_1, p^j\mathcal{O}_{X_n}))_n$ satisfies Mittag-Leffler being an inverse system of finite dimensional vector spaces. Tensoring with $\mathbb{Z}/p^i\mathbb{Z}$ gives the exact sequence

$$\varprojlim_n H^1(X_1, p^j\mathcal{O}_{X_n}) \otimes \mathbb{Z}/p^i\mathbb{Z} \to \varprojlim_n \mathrm{Pic}(X_n) \otimes \mathbb{Z}/p^i\mathbb{Z} \to \mathrm{Pic}(X_j) \otimes \mathbb{Z}/p^i\mathbb{Z} \to 0.$$

We now apply the exact functor "$\varprojlim_j$" to this sequence. By the theorem on formal functions, there is an isomorphism

$$\text{"}\varprojlim_j\text{"} \varprojlim_n H^1(X_1, p^j\mathcal{O}_{X_n}) \otimes \mathbb{Z}/p^i\mathbb{Z} \cong \text{"}\varprojlim_j\text{"} \varprojlim_n H^1(X, p^j\mathcal{O}_X) \otimes_A A/\pi^n A \otimes \mathbb{Z}/p^i\mathbb{Z}.$$

Since the image of the inclusion $p^{i+j}\mathcal{O}_X \hookrightarrow p^j\mathcal{O}_X$ vanishes modulo p^i, the same holds for the image of the morphism $H^1(X, p^{i+j}\mathcal{O}_X) \to H^1(X, p^j\mathcal{O}_X)$. By Proposition 4.2.1 this implies that

$$\text{"}\varprojlim_j\text{"} \varprojlim_n H^1(X_1, p^j\mathcal{O}_{X_n}) \otimes \mathbb{Z}/p^i\mathbb{Z}$$

is pro-isomorphic to zero and therefore that the theorem holds. □

4.3 A Lefschetz theorem

In this section we prove a Kodaira vanishing theorem which implies a Lefschetz theorem allowing us later in Section 4.4 to reduce our main theorem to relative dimension 2.

In this section let X_1, unless otherwise stated, be a smooth projective scheme over a field k. Let $H \subset X_1$ be a hyperplane section and $\mathcal{L}(d) = |dH|, d > 0$, be the linear system of hypersurface sections of degree d. We say that a hypersurface section $Y_1 \subset X_1$ is of high or sufficiently high degree if $Y_1 \in \mathcal{L}(d)$ with d sufficiently large such that certain higher cohomology groups vanish by Serre vanishing.

Proposition 4.3.1. *Let Y_1 be a smooth hypersurface of X_1 and $d = \dim X_1$. If Y_1 is of sufficiently high degree, then*

$$H^a(X_1, \Omega^b_{Y_1} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}(Y_1)) = H^a(Y_1, \Omega^b_{Y_1} \otimes_{\mathcal{O}_{Y_1}} \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}}) = 0$$

for $a + b > d - 1$.

Proof. Note that the first equality in the statement follows from the projection formula since $i_*\Omega^b_{Y_1} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}(Y_1) = i_*(\Omega^b_{Y_1} \otimes_{\mathcal{O}_{Y_1}} i^*\mathcal{O}_{X_1}(Y_1))$ for i the inclusion $Y_1 \hookrightarrow X_1$.

We first show that for $\omega_{X_1} = \Omega^d_{X_1}$ and $\omega_{Y_1} = \Omega^{d-1}_{Y_1}$, we have that

$$H^{a>0}(X_1, \omega_{Y_1} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}(Y_1)) = 0$$

if Y_1 is of high degree. By [32, Ch. II, Prop. 8.20] we know that

$$\omega_{Y_1} \cong \omega_{X_1} \otimes \mathcal{O}_{Y_1} \otimes \mathcal{O}_{X_1}(Y_1).$$

This implies that $\omega_{X_1}|_{Y_1} = \omega_{Y_1}(-Y_1)$ and therefore that the sequences

$$0 \to \omega_{X_1}(Y_1) \to \omega_{X_1}(2Y_1) \to \omega_{Y_1}(Y_1) \to 0$$

and

$$H^a(X_1, \omega_{X_1}(2Y_1)) \to H^a(X_1, \omega_{Y_1}(Y_1)) \to H^{a+1}(X_1, \omega_{X_1}(Y_1))$$

are exact. Since by Serre vanishing $H^a(X_1, \omega_{X_1}(2Y_1)) = H^a(X_1, \omega_{X_1}(Y_1)) = 0$ for $a > 0$ and Y_1 of sufficiently high degree, this implies that if Y_1 is of sufficiently high degree we also have that $H^{a>0}(X_1, \omega_{Y_1}(Y_1)) = H^{a>0}(Y_1, \omega_{Y_1} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}}) = 0$.

We now consider the exact sequence

$$0 \to \Omega^{p-1}_{Y_1}(-Y_1) \to \Omega^p_{X_1}|_{Y_1} \to \Omega^p_{Y_1} \to 0$$

coming from the conormal exact sequence $0 \to \mathcal{O}_{Y_1}(-Y_1) \to \Omega^1_{X_1}|_{Y_1} \to \Omega^1_{Y_1} \to 0$. Tensoring with $\mathcal{O}_{Y_1}(2Y_1)$ gives an exact sequence

$$0 \to \Omega^{p-1}_{Y_1}(Y_1) \to \Omega^p_{X_1}(2Y_1)|_{Y_1} \to \Omega^p_{Y_1}(2Y_1) \to 0.$$

This implies that the sequence

$$H^a(X_1, \Omega^b_{Y_1}(2Y_1)) \to H^{a+1}(X_1, \Omega^{b-1}_{Y_1}(Y_1)) \to H^{a+1}(X_1, \Omega^b_{X_1}(2Y_1))$$

is exact. The proposition follows inductively. □

Proposition 4.3.2. *Let Y_1 be a smooth hypersurface section of X_1 and $d = \dim X_1$. Let i denote the inclusion $Y_1 \hookrightarrow X_1$. Then the map*

$$i^* : H^q(X_1, \Omega^p_{X_1}) \to H^q(Y_1, \Omega^p_{Y_1})$$

is an isomorphism for $p + q < d - 1$ and injective for $p + q = d - 1$ if Y_1 is of high degree.

Proof. We factorise the map $i^* : \Omega^p_{X_1} \to i_* \Omega^p_{Y_1}$ by

$$\Omega^p_{X_1} \to i_*(\Omega^p_{X_1}|_Y)$$

followed by

$$i_*(\Omega^p_{X_1}|_{Y_1}) \to i_* \Omega^p_{Y_1}$$

and show that each of these maps induce isomorphisms, resp. injections, on cohomology in the stated range.

We first consider the exact sequence

$$0 \to \Omega^p_{X_1}(-Y_1) \to \Omega^p_{X_1} \to \Omega^p_{X_1}|_{Y_1} \to 0.$$

This induces the exact sequence

$$H^q(X_1, \Omega^p_{X_1}(-Y_1)) \to H^q(X_1, \Omega^p_{X_1}) \to H^q(Y_1, \Omega^p_{X_1}|_{Y_1}) \to H^{q+1}(X_1, \Omega^p_{X_1}(-Y_1)).$$

By Serre duality $H^q(X, \Omega^p_{X_1}(-Y_1)) \cong H^{d-q}(X, \Omega^{d-p}_{X_1}(Y_1))$. This implies that $H^q(X_1, \Omega^p_{X_1}) \to H^q(Y_1, \Omega^p_{X_1}|_{Y_1})$ is an isomorphism for $p+q < d-1$ and injective for $p+q = d-1$ if Y_1 is of sufficiently high degree by Serre vanishing.

We now consider the exact sequence

$$0 \to \Omega^{p-1}_{Y_1}(-Y_1) \to \Omega^p_{X_1}|_{Y_1} \to \Omega^p_{Y_1} \to 0$$

on Y_1. This induces the exact sequence

$$H^q(Y_1, \Omega^{p-1}_{Y_1}(-Y_1)) \to H^q(Y_1, \Omega^p_{X_1}|_{Y_1}) \to H^q(Y_1, \Omega^p_{Y_1}) \to H^{q+1}(Y_1, \Omega^{p-1}_{Y_1}(-Y_1))$$

which by Serre duality and Proposition 4.3.1 implies that $H^q(Y_1, \Omega^p_{X_1}|_{Y_1}) \to H^q(Y, \Omega^p_{Y_1})$ is an isomorphism for $p+q < d+1$ and injective for $p+q = d+2$ if Y_1. □

Proposition 4.3.3. *Let Y_1 be a hypersurface section of X_1 and $d = \dim X_1$. Then there is a map*

$$\phi : H^{d-1}(Y_1, \Omega^{d-2}_{Y_1}) \to H^d(X_1, \Omega^{d-1}_{X_1})$$

which is an isomorphism for $d \geq 4$ and surjective for $d = 3$ if Y_1 is of high degree.

Proof. Let i denote the inclusion $Y_1 \hookrightarrow X_1$. We define ϕ to be the composition

$$H^{d-1}(Y_1, \Omega^{d-2}_{Y_1}) \to H^{d-1}(Y_1, R^1 i^! \Omega^{d-1}_{X_1}) \cong H^d_{Y_1}(X_1, \Omega^{d-1}_{X_1}) \to H^d(X_1, \Omega^{d-1}_{X_1})$$

where the first map is induced by the Gysin map

$$g : \Omega^{d-2}_{Y_1} \to R^1 i^! \Omega^{d-1}_{X_1}, \omega \mapsto \omega \wedge \frac{df_d}{f_d}$$

(see [25, Ch. II, (3.2.13)]) with f_d is the regular parameter defining Y_1. Since $H^d(X_1 - Y_1, \Omega^{d-1}_{X_1 - Y_1}) = 0$ for $d \geq 1$, we have that

$$H^d_{Y_1}(X_1, \Omega^{d-1}_{X_1}) \cong H^d(X_1, \Omega^{d-1}_{X_1})$$

for $d \geq 2$. We are therefore reduced to showing that g induces an isomorphism on H^{d-1} for $d-1 \geq 3$ and a surjection for $d-1=2$. We define a filtration

$$g(\Omega_{Y_1}^{d-2}) = \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \cup_{i \geq 0} \mathcal{F} = R^1 i^! \Omega_{X_1}^{d-1},$$

letting $\mathcal{F}_i$ be the subsheaf of $R^1 i^! \Omega_{X_1}^{d-1}$ locally defined by

$$< \omega \wedge \frac{df_d}{f_d^{n_d}} | n_d \geq i > .$$

This is independent of the choice of parameters.

Let $gr_i R^1 i^! \Omega_{X_1}^{d-1} := \mathcal{F}_{i+1}/\mathcal{F}_i$. Then $gr_i R^1 i^! \Omega_{X_1}^{d-1} \cong \Omega_{Y_1}^{d-2} \otimes_{\mathcal{O}_{Y_1}} \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}}$ and the short exact sequence

$$0 \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to gr_i R^1 i^! \Omega_{X_1}^{d-1} \to 0$$

induces the following exact sequence on cohomology groups:

$$\begin{aligned} H^{d-2}(Y_1, \Omega_{Y_1}^{d-2} \otimes_{\mathcal{O}_{Y_1}} \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}}) \to H^{d-1}(Y_1, \mathcal{F}_i) \to H^{d-1}(Y_1, \mathcal{F}_{i+1}) \\ \to H^{d-1}(Y_1, \Omega_{Y_1}^{d-2} \otimes_{\mathcal{O}_{Y_1}} \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}}) \end{aligned} \tag{4.3.1}$$

By Proposition 4.3.1 we have that if Y_1 is of high degree, then $H^{d-1}(Y_1, \Omega_{Y_1}^{d-2} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}})$ vanishes for $d > 2$ and $H^{d-2}(Y_1, \Omega_{Y_1}^{d-2} \otimes_{\mathcal{O}_{X_1}} \mathcal{O}_{X_1}(Y_1)|_{\mathcal{O}_{Y_1}})$ for $d > 3$.

This implies that the maps $H^a(Y_1, \mathcal{F}_i) \to H^a(Y_1, \mathcal{F}_{i+1})$ are isomorphisms for $d \geq 4$ and surjective for $d = 3$ if Y_1 is of sufficiently high degree. Since $H^a(Y_1, \varinjlim \mathcal{F}_i) \cong \varinjlim H^a(Y_1, \mathcal{F}_i)$ (see [32, Ch. III, Prop. 2.9]), the same holds for the maps $H^a(Y_1, \mathcal{F}_1 = \Omega_{Y_1}^{d-2}) \to H^a(Y_1, R^1 i^! \Omega_{X_1}^{d-1})$. In particular, for $d = \dim X_1 = 3$ we get that $H^{d-1}(Y_1, \Omega_{Y_1}^{d-2}) \to H^d(X_1, \Omega_{X_1}^{d-1})$ is surjective and for $d = \dim X_1 \geq 4$ that $H^{d-1}(Y_1, \Omega_{Y_1}^{d-2}) \to H^d(X_1, \Omega_{X_1}^{d-1})$ is an isomorphism. □

Corollary 4.3.4. *Let X be as in the introduction of this chapter and $d \geq 3$. Let Y_1 a smooth hypersurface section of X_1. Let $\alpha \in H^d(X_1, \mathcal{K}_{d,X_n}^M)$. If Y_1 is of sufficiently high degree and contains the image of α in $\mathrm{CH}_0(X_1)$ under the restriction map $H^d(X_1, \mathcal{K}_{d,X_n}^M) \to H^d(X_1, \mathcal{K}_{d,X_1}^M) \cong \mathrm{CH}_0(X_1)$, then α is in the image of $H^{d-1}(Y_1, \mathcal{K}_{d-1,Y_n}^M) \to H^d(X_1, \mathcal{K}_{d,X_n}^M)$.*

Proof. We do the $n = 2$ case. The general case follows inductively. Consider the commutative diagram

$$\begin{array}{ccccccc} H^d(X_1, \Omega_{X_1}^{d-1}) & \longrightarrow & H^d(X_1, \mathcal{K}_{d,X_2}^M) & \longrightarrow & H^d(X_1, \mathcal{K}_{d,X_1}^M) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ H^{d-1}(Y_1, \Omega_{Y_1}^{d-2}) & \longrightarrow & H^{d-1}(Y_1, \mathcal{K}_{d-1,Y_2}^M) & \longrightarrow & H^{d-1}(Y_1, \mathcal{K}_{d-1,Y_1}^M) & \longrightarrow & 0 \end{array}$$

induced by the (right-)exact sequence of sheaves

$$\Omega^{d-1} \to \mathcal{K}_2^M \to \mathcal{K}_2^M \to 0$$

on X_1 and Y_1. The statement follows from Proposition 4.3.3 and a simple diagram chase. □

4.4 Main theorem

We return to the situation of the introduction of this chapter. Let A be a henselian discrete valuation ring with uniformising parameter π and residue field k. Let X be a smooth projective scheme over $\mathrm{Spec}(A)$ of relative dimension d. Let $X_n := X \times_A A/(\pi^n)$, i.e. X_1 is the special fiber and the X_n are the respective thickenings of X_1. We assume furthermore that either (1) A is the Witt ring $W(k)$ of a perfect field k of $\mathrm{ch}(k) > 2$ or that (2) k is of characteristic 0 and $A = k[[t]]$.

Let us first recall how one can lift a regular closed point $x \in X_1$ to a 1-cycle on X: Let $\{f_1, ..., f_d\} \subset \mathcal{O}_{X_1,x}$ be a generating set of local parameters and let $\{\tilde{f}_1, ..., \tilde{f}_d\}$ be lifts of these generators to $\mathcal{O}_{X,x}$. The ideal $\tilde{f}_1\mathcal{O}_{X,x} + ... + \tilde{f}_d\mathcal{O}_{X,x}$ defines a subscheme of $\mathrm{Spec}\mathcal{O}_{X,x}$ and its closure in X defines a subscheme Z of X. The unique irreducible component of Z containing x is a prime-cycle $C \in Z_1(X)$ which is flat and finite over A. Such liftings are of course not unique.

We also introduce the following notation: Let X be a scheme and Z an effective Cartier divisor on X. Let C be a curve in X, i.e. an effective 1-cycle on X. Let

$$(Z, C)_x := \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/I_Z + I_C)$$

be the intersection multiplicity of Z and C at x. We say that Z and C intersect transversally at x if $(Z, C)_x = 1$ and if Z and C are regular at x. If Z and C intersect transversally everywhere, we denote this by $Z \pitchfork C$.

In this section we show the following proposition:

Proposition 4.4.1. *Let X be of relative dimension 2 over A. Then, assuming the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{*,X}$, the map*

$$res : \mathrm{CH}_1(X) \to \text{"}\lim_n\text{"} H^2(X_1, \mathcal{K}^M_{2,X_n})$$

is an epimorphism in pro-Ab.

We need some preparation for the proof. Consider the (right-)exact sequence

$$\Omega^1_{X_1} \to \mathcal{K}^M_{2,X_2} \to \mathcal{K}^M_{2,X_1} \to 0.$$

We will lift elements which lie in the kernel of $res : H^2(X_1, \mathcal{K}^M_{2,X_2}) \to H^2(X_1, \mathcal{K}^M_{2,X_1})$ in a compatible way to $\mathrm{CH}_1(X)$. The kernel of res is in the image of $H^2(X_1, \Omega^1_{X_1})$. Now since $\Omega^1_{X_1}$ is CM, $H^2(X_1, \Omega^1_{X_1})$ is isomorphic to

$$\mathrm{coker}(\oplus_{x \in X_1^{(1)}} H^1_x(X_1, \Omega^1_{X_1}) \to \oplus_{x \in X_1^{(2)}} H^2_x(X_1, \Omega^1_{X_1})).$$

In order to proceed, we need to study this cokernel and the occurring local cohomology groups a bit further. By Lemma 4.1.3 we have that $H^2_x(X_1, \Omega^1_{X_1})$ is generated by differential forms of the form

$$\frac{df_1}{f_1^{n_1} f_2^{n_2}}\mathcal{O}_{X_1,x} \oplus \frac{df_2}{f_1^{n'_1} f_2^{n'_2}}\mathcal{O}_{X_1,x} \quad \mathrm{mod} \quad \frac{df_i}{f_j^{n_j}}\mathcal{O}_{X_1,x}$$

for $\{f_1, f_2\}$ a system of local parameters in $\mathcal{O}_{X_1,x}$ and $i, j \in \{1, 2\}$.

We define subgroups

$$F_r := < \frac{df_1}{f_1^{n_1} f_2^{n_2}} + \frac{df_2}{f_1^{n_1'} f_2^{n_2'}} | n_1 + n_2 - 1 \leq r, n_1' + n_2' - 1 \leq r >$$

of $H_x^2(X_1, \Omega_{X_1}^1)$ with respect to a system of local parameters. Sometimes we therefore write $H_x^2(X_1, \Omega_{X_1}^1)_{(f_1,f_2)}$ instead of $H_x^2(X_1, \Omega_{X_1}^1)$ to indicate with respect to which system of local parameters we are working. Then

$$0 \subset F_1 \subset F_2 \subset ... \subset H_x^2(X_1, \Omega_{X_1}^1)_{(f_1,f_2)}$$

defines an ascending filtration on $H_x^2(X_1, \Omega_{X_1}^1)$. We will call elements of F_1 forms with simple poles. The following lemma shows that this definition is in fact independent of the chosen parameter system, meaning that there is a natural isomorhism between $H_x^2(X_1, \Omega_{X_1}^1)_{(f_1,f_2)}$ and $H_x^2(X_1, \Omega_{X_1}^1)_{(f_1',f_2')}$ for two local parameter systems $\{f_1, f_2\}$ and $\{f_1', f_2'\}$ inducing isomorphisms on the respective filtrations. This isomorphism is given by considering a differential form with respect to the respective defining parameter systems.

Lemma 4.4.2. *(1) Let $x \in X_1$ be a closed point. Then subgroups $F_r \subset H_x^2(X_1, \Omega_{X_1}^1)$ are independent of the local parameter system we consider them in.*

(2) In particular, the subgroup

$$F_1 = < \frac{df_1}{f_1 f_2}\mathcal{O}_{X_1,x} \oplus \frac{df_2}{f_1 f_2}\mathcal{O}_{X_1,x} > \subset H_x^2(X_1, \Omega_{X_1}^1)$$

is independent of the chosen local parameter system of $\mathcal{O}_{X_1,x}$. We therefore denote it by Λ_x.

(3) The graded pieces

$$F_{r+1}/F_r$$

are independent of the chosen local parameter system of $\mathcal{O}_{X_1,x}$.

Proof. It suffices to show the proposition for two parameter systems (f_1, f_2) and (f_1, f_2') and $f_2 = f_2' + \beta f_1$. We saw in Section 4.1 that $H_x^2(X_1, \Omega_{X_1}^1)$ can be calculated locally as $\hat{H}^1(\mathrm{Spec}\mathcal{O}_{X_1,x} \setminus \{x\}, \Omega_{X_1}^1)$ with respect to coverings of of $\mathrm{Spec}\mathcal{O}_{X_1,x} \setminus \{x\}$. Now considering $\frac{df_2}{f_1^{n_1} f_2^{n_2}} \in \hat{H}^1(\mathrm{Spec}\mathcal{O}_{X_1,x} \setminus \{x\}, \Omega_{X_1}^1)$ for the covering $D(f_1) \cup D(f_2)$ of $\mathrm{Spec}\mathcal{O}_{X_1,x} \setminus \{x\}$, we can pass to the smaller covering $D(f_1) \cup D(f_2 f_2')$ of $\mathrm{Spec}\mathcal{O}_{X_1,x} \setminus \{x\}$. With respect to this covering

$$\frac{df_2}{f_1^{n_1} f_2^{n_2}} = \frac{df_2'}{f_1^{n_1} f_2'^{n_2}(1 + \beta\frac{f_1}{f_2'})^{n_2}} + \frac{\beta df_1}{f_1^{n_1} f_2'^{n_2}(1 + \beta\frac{f_1}{f_2'})^{n_2}}$$

is equivalent to

$$\frac{(\sum_{n=0}^{\infty}(-\beta\frac{f_1}{f_2'})^n)^{n_2}df_2'}{f_1^{n_1}f_2'^{n_2}}+\frac{(\sum_{n=0}^{\infty}(-\beta\frac{f_1}{f_2'})^n)^{n_2}\beta df_1}{f_1^{n_1}f_2'^{n_2}}$$

since $\frac{1}{f_1^{n_1}f_2'^{n_2}(1+\beta\frac{f_1}{f_2'})^{n_2}}$ converges to $\frac{(\sum_{n=0}^{\infty}(-\beta\frac{f_1}{f_2'})^n)^{n_2}}{f_1^{n_1}f_2'^{n_2}}$ in $\mathcal{O}_{X_1,x}[\frac{1}{f_1f_2f_2'}]/\mathcal{O}_{X_1,x}[\frac{1}{f_2f_2'}]$, which lies again in $F_{n_1+n_2-1}$ considering it as an element of $\hat{H}^1(\mathrm{Spec}\mathcal{O}_{X_1,x}\setminus\{x\},\Omega^1_{X_1})$ with respect to the covering $D(f_1)\cup D(f_2')$. This proves (1). (2) and (3) follow immediately. □

In order to prove Proposition 4.4.1, we need to prove key Lemma 4.4.4. Its proof is inspired by the techniques of [48] from which we cite the following lemma:

Lemma 4.4.3. *([48, Lemma 10.2]) Let X be a noetherian scheme, E an effective Cartier divisor on X, and A be an effective Cartier divisor on E. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that*

$$H^1(X,\mathcal{F}\otimes\mathcal{O}_X(-E))=H^1(E,\mathcal{F}|_E\otimes\mathcal{O}_E(-A))=0.$$

Then the restriction $res_A: H^0(X,\mathcal{F})\to H^0(A,\mathcal{F}|_A)$ is surjective.

By Lemma 4.4.2 we can talk about the pole order of elements of $H^2_x(X_1,\Omega^1_{X_1})$ independent of the parameter system chosen. In particular, the following lemma makes sense:

Key lemma 4.4.4. *Every element $\gamma\in H^2_x(X_1,\Omega^1_{X_1})$ is equivalent to a sum of forms with simple poles in $H^2(X_1,\Omega^1_{X_1})$.*

Proof. Without loss of generality we work with $\gamma=\frac{\alpha df_1}{f_1^{n_1}f_2^{n_2}}\in H^2_x(X_1,\Omega^1_{X_1})$. Let D_1 be a regular curve containing x and f_1' a local parameter of D_1 at x. We consider $\frac{\alpha df_1}{f_1^{n_1}f_2^{n_2}}$ in $H^2_x(X_1,\Omega^1_{X_1})_{(f_1',f_2)}$. By Lemma 4.4.2, γ is still in F_{r+1} for $r+1=n_1+n_2$. We may assume that it is of the form $\frac{\alpha df_1'}{f_1'^{n_1}f_2^{n_2}}$.

Let $H\subset X_1$ be a hyperplane section and for an integer $d>0$ let $\mathcal{L}(d)=|dH|$ be the linear system of hypersurface sections of degree d.

Now for $d\gg 0$ there exists an $F^1\in\mathcal{L}(d)$ such that

1. $x\in F^1$,
2. $F^1 \pitchfork D_1$ at any $y\in F^1\cap D_1$.

For d' sufficiently large relative to d, there exists an $F^2\in\mathcal{L}(d')$ such that

1. $x\in F^2$,
2. $F^2 \pitchfork D_1$ at any $y\in F^2\cap D_1$,
3. $F^1\cap F^2\cap D_1-x=\emptyset$.

We choose $F^3, ..., F^{n_2}$ analogously. Furthermore, we choose F^{n_2} to be of sufficiently high degree so that

$$H^1(X_1, \Omega^1_{X_1}((n_1-1)D_1+F^1+...+F^{n_2})) = H^1(D, \Omega^1_{X_1}(n_1D_1+F^1+...+F^{n_2})|_D \otimes \mathcal{O}_D(-x)) = 0$$

holds by Serre vanishing. By Lemma 4.4.3, the last condition implies that the restriction map

$$H^0(X_1, \Omega^1_{X_1}(n_1D_1 + F^1 + ... + F^{n_2})) \xrightarrow{res} H^0(x, \Omega^1_{X_1}(n_1D_1 + F^1 + ... + F^{n_2}) \otimes k(x))$$

is surjective. Let y be the generic point of D_1. By construction, the diagram

$$\begin{array}{ccc} H^0(X_1, \Omega^1_{X_1}(n_1D_1 + F^1 + ... + F^{n_2})) & \xrightarrow{res} & \Omega^1_{X_1}(n_1D_1 + F^1 + ... + F^{n_2}) \otimes k(x) \\ \downarrow & & \downarrow \\ H^1_y(X_1, \Omega^1_{X_1}) & \xrightarrow{d_x} & F_{r+1}/F_rH^2_x(X_1, \Omega^1_{X_1}) \end{array}$$

is commutative and $\frac{\alpha df_1}{f_1^{n_1}f_2^{n_2}}$ lies in $\Omega^1_{X_1}(n_1D_1 + F^1 + ... + F^{n_2}) \otimes k(x)$. Notice that the map on the right is well-defined. This implies that there is a $\gamma \in H^1_y(X_1, \Omega^1_{X_1})$ such that $d_x(\gamma) = \frac{\alpha df_1}{f_1^{n_1}f_2^{n_2}}$. Furthermore for any $x' \in |D_1| - x$, the form $d_{x'}(\gamma)$ has at most simple poles in f_2 at x', i.e. $d_{x'}(\gamma) \in F_{n_2+1}$. Now we apply the same construction to the form $d_{x'}(\gamma)$ which completes the proof. □

Proof of Proposition 4.4.1. Let $x \in X$ be a closed point and $X_{1,x}$ be the spectrum of the stalk of $\mathcal{O}_{X_1}$ in x. The Čech to derived functor spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \mathcal{H}^q(X, \mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

induces an edge map

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \to H^i(X, \mathcal{F}).$$

Since this edge map is functorial in $\mathcal{F}$, we get a commutative diagram

$$\begin{array}{ccc} \check{H}^1(X_{1,x} - x, \Omega^1_{X_1}) & \longrightarrow & \check{H}^1(X_{1,x} - x, \mathcal{K}^M_{2,X_n}) \\ \cong\downarrow & & \downarrow \\ H^1(X_{1,x} - x, \Omega^1_{X_1}) \cong H^2_x(X_1, \Omega^1_{X_1}) & \longrightarrow & H^1(X_{1,x} - x, \mathcal{K}^M_{2,X_n}) \cong H^2_x(X_1, \mathcal{K}^M_{2,X_n}) \end{array}$$

for a closed point $x \in X_1$. We saw in Lemma 4.1.3 that $\check{H}^1(X_{1,x} - x, \Omega^1_{X_1})$ is generated by elements of the form $\frac{\alpha_1}{f_1^{n_1}}\frac{df_2}{f_2^{n_2}} + \frac{\alpha_2}{f_2^{n_2}}\frac{df_1}{f_1^{n_1}}$ for a local parameter system $(f_1, f_2) \in \mathcal{O}_{X_1,x}$ and by key Lemma 4.4.4 we may assume that it has simple poles. The point of the proof is that for forms with simple poles we can write down explicit lifts to $\mathrm{CH}_1(X)$.

Without loss of generality we consider the form $\alpha df_1/(f_1f_2)$. The upper horizontal map in the above diagram maps $\alpha df_1/(f_1f_2)$ to $\{f_1, 1 + \pi^{n-1}\alpha/f_2\}$. Now the map

$$\mathrm{CH}_1(X) \to H^2(X, \mathcal{K}^M_{2,X}),$$

induced by the assumption of the Gersten conjecture, sends the cycle $\overline{V(\tilde{f}_1, \tilde{f}_2 + \pi^{n-1}\alpha)} - \overline{V(\tilde{f}_1, \tilde{f}_2)}$ to $(\{\tilde{f}_1, \tilde{f}_2 + \pi^{n-1}\alpha\}, -\{\tilde{f}_1, \tilde{f}_2\})$ in

$$\check{H}^1(X_{\overline{V(\tilde{f}_1, \tilde{f}_2+\pi^{n-1}\alpha)}} - \overline{V(\tilde{f}_1, \tilde{f}_2 + \pi^{n-1}\alpha)}, \mathcal{K}^M_{2,X}) \oplus \check{H}^1(X_{\overline{V(\tilde{f}_1,\tilde{f}_2)}} - \overline{V(\tilde{f}_1, \tilde{f}_2)}, \mathcal{K}^M_{2,X}).$$

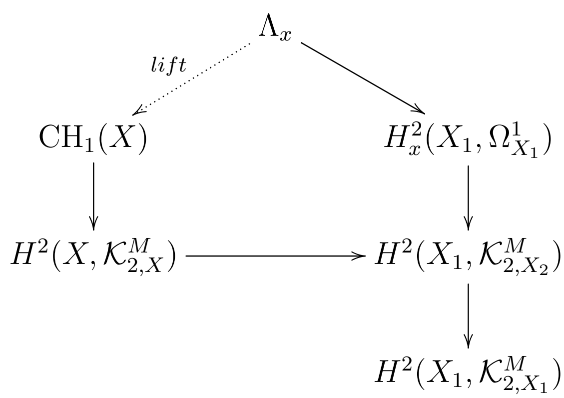
The restriction map

$$H^2(X, \mathcal{K}^M_{2,X}) \to H^2(X, \mathcal{K}^M_{2,X_n})$$

sends the tuple of Čech-cycles $(\{\tilde{f}_1, \tilde{f}_2+\pi^{n-1}\alpha\}, -\{\tilde{f}_1, \tilde{f}_2\})$ to $\{\tilde{f}_1, 1+\pi^{n-1}\alpha/\tilde{f}_2\} \in \check{H}^1(X_1 - x, \mathcal{K}^M_{2,X_n})$. This shows in particular that $\ker(H^2(X_1, \mathcal{K}^M_{2,X_n}) \to H^2(X_1, \mathcal{K}^M_{2,X_{n-1}}))$ is in the image of $\mathrm{CH}_1(X)$.

The surjectivity of $\mathrm{CH}_1(X) \to H^2(X_n, \mathcal{K}^M_{2,X_n})$ now follows by induction: For $n = 1$ this is just the surjectivity of $\mathrm{CH}_1(X) \to \mathrm{CH}_0(X_1)$. For $n > 1$, let $\alpha \in H^2(X_1, \mathcal{K}^M_{2,X_n})$. Let α_{n-1} be the image of α in $H^2(X_1, \mathcal{K}^M_{2,X_{n-1}})$. By assumption there is a cycle $Z \in \mathrm{CH}_1(X)$ mapping to α_{n-1}. Denote the image of Z in $H^2(X_n, \mathcal{K}^M_{2,X_n})$ by α_Z. Now $\alpha - \alpha_Z$ is in the kernel of $H^2(X_1, \mathcal{K}^M_{2,X_n}) \to H^2(X_1, \mathcal{K}^M_{2,X_{n-1}})$ and by the above construction lifts to an element $Z' \in \mathrm{CH}_1(X)$. Now $Z' - Z$ maps to α. □

The proof of Proposition 4.4.1 can be summed up in the following diagram:



The question remains if there is a well-defined map $\Lambda_x \to \mathrm{CH}_1(X)/F_n$ for some filtration

$$.. \subset F_2 \subset F_1 \subset \mathrm{CH}_1(X)$$

making the above diagram commutative and vanishing in the pro-sense mod p over suitable bases.

Remark 4.4.5. *Let X be of relative dimension 1 over A. Let F_n be the subgroup of $\mathrm{CH}_1(X)$ generated by all cycles vanishing on X_n. By Lemma 4.1.2 we know that $H^1_x(X_1, \mathcal{O}_{X_1}) \cong O_{X_1,x}[\frac{1}{f}]/\mathcal{O}_{X_1,x}$ for a local parameter $f \in \mathcal{O}_{X_1,x}$. In this case we can define a map*

$$\gamma_x : O_{X_1,x}[\frac{1}{f}]/\mathcal{O}_{X_1,x} \to \mathrm{CH}_1(X)/F_n$$

by

$$\alpha = \frac{\alpha_0}{f^m} \mapsto V(\tilde{f}^m + \alpha_0 \pi^{n-1}) - V(\tilde{f}^m).$$

That γ_x is well-defined can be seen as follows: Let $\tilde{f}_1, \tilde{f}_2 \in \mathcal{O}_{X,x}$ be liftings of f. Then $(1 + \frac{\alpha_0}{\tilde{f}_1^m}\pi^{n-1})/(1 + \frac{\alpha_0}{\tilde{f}_2^m}\pi^{n-1}) \equiv (1 + (\frac{\alpha_0}{\tilde{f}_1^m} - \frac{\alpha_0}{\tilde{f}_2^m})\pi^{n-1}) = 1 \mod F_n$. An indication of how to define a good candidate for F_n is also given in Section 3.5.

Corollary 4.4.6. *Let X be as in Proposition 4.4.1 but of arbitrary relative dimension d over A. Then, assuming the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{*,X}$, the map*

$$res : \mathrm{CH}_1(X) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d,X_n})$$

is an epimorphism in pro-Ab.

Proof. This follows immediately from Corollary 4.3.4, Proposition 4.4.1 and standard Bertini arguments. □

Remark 4.4.7. *In [46, Sec. 10], a relation between Conjecture 4.0.15 and Question (g) is postulated. Let us sketch this relationship. Let A be the ring of integers in a p-adic local field K. Let X be asmooth and projective of relative dimension d over A. For a smooth projective (over A) subscheme of codimension one $Y \subset X$ we expect that the map*

$$\text{"}\lim_n\text{"} H^{d-1}(Y_1, \mathcal{K}^M_{d-1,Y_n}/p^r) \to \text{"}\lim_n\text{"} H^d(X_1, \mathcal{K}^M_{d,X_n}/p^r)$$

is an isomorphism for $d \geq 3$ and surjective for $d = 2$ if we choose Y to be of high degree. For $n = 1$, this follows from class field theory and standard Lefschetz theorems for the étale fundamental group. In order to prove this for arbitrary n, the Lefschetz theorem of Section 4.3, Proposition 4.3.3, needs to be improved by one degree. This and the injectivity of res for arbitrary dimension would imply that the map

$$\mathrm{CH}_1(Y)/p^r \to \mathrm{CH}_1(X)/p^r$$

is bijective for $d \geq 3$ and surjective for $d = 2$ and since $\mathrm{CH}_1(Y) \to \mathrm{CH}_0(Y_K)$ and $\mathrm{CH}_1(X) \to \mathrm{CH}_0(X_K)$ are surjective, the same statement for

$$\mathrm{CH}_0(Y_K)/p^r \to \mathrm{CH}_0(X_K)/p^r.$$

For a curve C_K we know, as mentioned in the introduction, that

$$A_0(C_K) \cong A^m \oplus (\text{finite group})$$

for some $m \in \mathbb{N}$. This implies, under the above assumptions, the same result for the p-completion of $\mathrm{CH}_1(X_K)$ and therefore give a positive answer to Question (g).

The corresponding weak Lefschetz theorem for l prime to p saying that the map

$$\mathrm{CH}_1(Y)/l^r \to \mathrm{CH}_1(X)/l^r$$

is bijective for $d \geq 3$ and surjective for $d = 2$ is proved in [66, Cor. 9.6].

Appendices

Appendix A

p-adic étale Tate twists and homology theories

Let B be the spectrum of a discrete valuation ring with residue field k. In [66], Saito and Sato define a homology theory

$$H_q(X,\Lambda) := H^{2-q}(X, Rf^!\Lambda)$$

for X a quasi-projective B-scheme with structural morphism $f: X \to B$ and coefficients $\Lambda = \Lambda_n = \mathbb{Z}/l^n\mathbb{Z}$ with l a prime different from $\mathrm{ch}(k)$. This homology theory is used to construct a cycle class map

$$\rho_X : \mathrm{CH}_1(X)/n \to H^{2d}_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z})$$

for X of relative dimension d over B via the niveau spectral sequence associated to a homology theory. The localization property of the constructed homology theory is then used to show that ρ_X is an isomorphism if X is regular flat and projective over B and if B is an excellent henselian discrete valuation ring with finite or separably closed residue field. For a nice survey of the work of Saito and Sato and see [11].

In this appendix we give the definition of p-adic étale Tate twists and explain why they can't be used to construct a homology theory playing the role of $H_q(X,\Lambda)$ in the p-adic case. This is probably well-known to experts since this approach has not been taken to study the cycle class map

$$cl^d_{X,r} : \mathrm{CH}^d(X)/p^r \to H^{2d}_{\text{ét}}(X, \mathcal{T}_r(d))$$

(see f.e. [67]).

In this chapter all cohomology groups are taken over the étale cohomology.

A.0.1 Homology theories

Definition A.0.8. *Let S be the spectrum of a field or of an excellent Dedekind domain. We denote by $\mathcal{C}_S$ the category of S-schemes X, which are separated and of finite type over*

S. We denote by $\mathcal{C}_{S*}$ the category whose objects are the same as those of $\mathcal{C}_S$ and whose morphisms are proper maps in $\mathcal{C}_S$.

A homology theory is defined as follows:

Definition A.0.9. *A homology theory $H = \{H_a\}_{a\in\mathbb{Z}}$ on $\mathcal{C}_S$ is a sequence of covariant functors*

$$H_a(-) : \mathcal{C}_{S*} \to Ab$$

satisfying the following conditions:

(i) For each open immersion $j : V \to X$ in $\mathcal{C}_S$, there is a map $j^* : H_a(X) \to H_a(V)$, associated to j in a functorial way.

(ii) If $i : Y \to X$ is a closed immersion in X, with open complement $j : V \to X$, there is a long exact functorial localization sequence

$$\ldots \xrightarrow{\partial} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(V) \xrightarrow{\partial} H_{a-1}(Y) \to \ldots$$

A morphism between homology theories H and H' is a morphism $\psi : H \to H'$ of functors on $\mathcal{C}_{S*}$, which is compatible with the long exact sequences in (ii).

Given a homology theory H on $\mathcal{C}_{S*}$, one can associate to it for every $X \in \mathcal{C}_S$ the niveau spectral sequence

$$E^1_{a,b}(X) = \oplus_{x\in X_{(a)}} H_{a+b}(x) \Rightarrow H_{a+b}(X), \tag{A.0.1}$$

where $H_a(x) = \varinjlim_{V\subseteq\overline{\{x\}}} H_a(V)$ with the limit being taken over all non-empty open subsets V of $\overline{\{x\}}$. For more details see [8].

Example A.0.10. *Let X be a scheme which is separated and of finite type over a noetherian scheme S with structural morphism $f : X \to B$. Let $\Lambda \in D^b(S_{\text{ét}})$ be a bounded complex of étale sheaves. Let $Rf^!$ be the right adjoint of $Rf_!$ defined in [30, XVIII, 3.1.4]. Then*

$$H_a(X, \Lambda) := H^{-a}_{\text{ét}}(X, Rf^!\Lambda)$$

defines a homology theory. For more details see [37, Ex. 2.2].

A.0.2 p-adic étale Tate twist

Let us recall some basic facts about p-adic étale Tate twist. Let B be the spectrum of a discrete valuation ring of mixed characteristic $(0, p)$ and X a regular noetherian scheme, flat of finite type over B and with semistable reduction. Let j be the open immersion $X[1/p] \hookrightarrow X$ and Y be the special fiber. In [69], Sato proves the following theorem:

Theorem A.0.11. *([69, Thm. 1.1.1.]) For each $n \geq 0$ and $r \geq 1$, there exists an object $\mathcal{T}_r(n)_X \in D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$, which we call a p-adic étale Tate twist, satisfying the following properties:*

1. *There is an isomorphism* $t : j^*\mathcal{T}_r(n)_X \simeq \mu_{p^r}^{\otimes n}$.

2. $\mathcal{T}_r(n)_X$ *is concentrated in* $[0, n]$, *i.e. the* q*-th cohomology sheaf is zero unless* $0 \le q \le n$.

3. *For a locally closed regular subscheme* $i : Z \hookrightarrow X$ *of characteristic* p *and of codimension* $c\,(\ge 1)$, *there is a Gysin isomorphism*

$$W_r\Omega^{n-c}_{Z,\log}[-n-c] \xrightarrow{\cong} \tau_{\le n+c} Ri^! \mathcal{T}_r(n)_X \text{ in } D^b(Z_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}).$$

4. *Let* $i_y : y \hookrightarrow X$ *and* $i_x : x \hookrightarrow X$ *be points on* X *with* $\mathrm{ch}(x) = p$, $x \in \overline{\{y\}}$ *and* $\mathrm{codim}_X(x) = \mathrm{codim}_X(y) + 1$. *Put* $c := \mathrm{codim}_X(x)$. *Then the connecting homomorphism*

$$R^{n+c-1} i_{y*}(Ri_y^! \mathcal{T}_r(n)_X) \longrightarrow R^{n+c} i_{x*}(Ri_x^! \mathcal{T}_r(n)_X)$$

in localization theory agrees with the (sheafified) boundary map of Galois cohomology groups due to Kato

$$\begin{Bmatrix} R^{n-c+1} i_{y*}\mu_{p^r}^{\otimes n-c+1} & (\mathrm{ch}(y) = 0) \\ i_{y*} W_r\Omega^{n-c}_{Z,\log}[-n-c] & (\mathrm{ch}(y) = p) \end{Bmatrix} \longrightarrow i_{x*} W_r\Omega^{n-c}_{Z,\log}[-n-c]$$

up to a sign depending only on $(\mathrm{ch}(y), c)$ *via Gysin isomorphisms.*

5. *There is a unique morphism*

$$\mathcal{T}_r(m)_X \otimes^{\mathbb{L}} \mathcal{T}_r(n)_X \longrightarrow \mathcal{T}_r(m+n)_X \quad \text{in } D^-(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

that extends the natural isomorphism $\mu_{p^r}^{\otimes m} \otimes \mu_{p^r}^{\otimes n} \simeq \mu_{p^r}^{\otimes m+n}$ *on* $X[1/p]$.

An important property of $\mathcal{T}_r(n)_X$ is that it fits into the following exact triangle:

$$i_* \nu^{n-1}_{Y,r}[n-1] \to \mathcal{T}_r(n)_X \to \tau_{\le n} Rj_* \mu_{p^r}^{\otimes n} \to i_* \nu^{n-1}_{Y,r}[n] \tag{A.0.2}$$

Sato shows furthermore that the pair $(\mathcal{T}_r(n)_X, t)$ is unique up to isomorphism in $D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ ([69, Thm. 1.3.5]). From now on we will fix such a pair for all $n \ge 0, r > 0$ and denote it by $\mathcal{T}_r(n)_X$.

A.0.3 Cycle class map

For the following see also [68, Sec. 4]. There is a localization spectral sequence

$$E_1^{u,v} = \oplus_{x \in X^u} H_x^{v+u}(X, \mathcal{T}_r(n)_X) \Rightarrow H_{\text{ét}}^{v+u}(X, \mathcal{T}_r(n)_X). \tag{A.0.3}$$

Let us introduce the following notation:

$$\mathbb{Z}/p^r\mathbb{Z}(n)_x := \begin{cases} \mu_{p^r}^{\otimes n} & \text{if } \mathrm{ch}(x) \neq p. \\ W_r\Omega^n_{x,\log}[-n] & \text{if } \mathrm{ch}(x) = p. \end{cases}$$

One notes that

$$E_1^{u,v} \cong \oplus_{x\in X^u} H^{v-u}(x, \mathbb{Z}/p^r\mathbb{Z}(n-u)_x), \text{ if } v \leq n$$

since

1. if $\mathrm{ch}(x) \neq p$, then

$$H_x^{v+u}(X, \mathcal{T}_r(n)_X) = H_x^{v+u}(\mathcal{O}_{X,x}, \mathcal{T}_r(n)_X) \cong H^{v-u}(x, \mu_{p^r}^{\otimes n})$$

   by the absolute purity isomorphism

$$\mathrm{Gys}_\psi^n : \mu_{p^r}^{\otimes n-c}[-2c] \xrightarrow{\cong} R\psi^! \mu_{p^r}^{\otimes n}$$

   in $D^b(U_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$, where $U := \overline{\{x\}}[1/p]$ and $\psi : U \to X_K$ (see [18], [75]).

2. if $\mathrm{ch}(x) = p$, then

$$H_x^{v+u}(X, \mathcal{T}_r(n)_X) \cong H^{v-u}(x, W_r\Omega^n_{x,\log}[-n])$$

   if $v \leq n$ by Theorem A.0.11(3).

In particular, $E_2^{n,n} \cong \mathrm{CH}^n(X)/p^r$ and there is a cycle class map given by the edge map

$$cl^n_{X,r} : \mathrm{CH}^n(X)/p^r \to H^{2n}_{\text{ét}}(X, \mathcal{T}_r(n))$$

Proposition A.0.12. *The map*

$$cl^d_{X,r} : \mathrm{CH}^d(X)/p^r \to H^{2d}_{\text{ét}}(X, \mathcal{T}_r(d))$$

is injective for $d = 1$ and the map (again induced by (A.0.3))

$$\mathrm{CH}^d(X, 1, \mathbb{Z}/p^r\mathbb{Z}) \to H^{2d-1}_{\text{ét}}(X, \mathcal{T}_r(d))$$

is injective for $d \leq 2$.

Proof. This follows immediately from spectral sequence (A.0.3). □

We would like to pose the following question:

Question A.0.13. *Let $d = n$. Is the complex*

$$E_1^{\bullet, d+1}$$

exact?

Remark A.0.14. *The main difficulty or obstruction seems to be the lack of full purity for logarithmic de Rham-Witt sheaves.*

A.0.4 p-adic homology theories

Ideally one would like the cycle class map

$$cl^d_{X,r} : \mathrm{CH}^d(X)/p^r \to H^{2d}_{\text{ét}}(X, \mathcal{T}_r(d))$$

to arise from the niveau spectral sequence associated to a homology theory. This would allow us to use the localization sequence of Definition A.0.9(ii) to study it. We will now explain the problems with this approach.

Denote $\mathcal{T}_r(n)_B$ by $\Lambda(n)$. Let $f_X : X \to B$ be as in Example A.0.10 and assume that X is quasi-projective over B. Then

$$H_q(X, \Lambda(n)) := H^{2-q}_{\text{ét}}(X, Rf^!_X \Lambda(n))$$

defines a homology theory and there is an associated homologically graded spectral sequence

$$E^1_{a,b}(X, \Lambda(n)) = \oplus_{x \in X_a} H_{a+b}(x, \Lambda(n)) \Rightarrow H_{a+b}(X, \Lambda(n)) \tag{A.0.4}$$

Lemma A.0.15. *There is a trace isomorphism*

$$\mathcal{T}_r(d+1)_X[2d] \to Rf^!_X \mathcal{T}_r(1)_B$$

Proof. This follows from the exact triangle A.0.2 and the trace isomorphisms for $i_* v^d_{Y,r}[d]$ and $\tau_{\leq d+1} Rj_* \mu_{p^r}^{\otimes d+1}$. □

Proposition A.0.16. *Let Y be a subscheme of X and d be the relative dimension of X over B. Let i denote the inclusion $Y \hookrightarrow X$. Then there is an isomorphism*

$$H^{2d+2-q}_Y(X, \mathcal{T}_r(d+1)) \to H_q(Y, \Lambda(1)).$$

Proof. By Lemma A.0.15 we have that

$$\begin{aligned} H^{2d+2-q}_Y(X, \mathcal{T}_r(d+1)) &= H^{2d+2-q}(Y, Ri^! \mathcal{T}_r(d+1)) = H^{2-q}(Y, Ri^! \mathcal{T}_r(d+1)[2d]) \\ &\cong H^{2-q}(Y, Ri^! Rf^!_X \mathcal{T}_r(1)_B) \cong H^{2-q}(Y, Rf^!_Y \mathcal{T}_r(1)_B) = H_q(Y, \Lambda(1)). \end{aligned}$$

□

It follows from Proposition A.0.16 that for $n = 1$, (A.0.4) is isomorphic to

$$E^1_{a,b}(X, \Lambda(1)) = \oplus_{x \in X_a} H^{a-b}(x, \Lambda(a)) \Rightarrow H_{a+b}(X, \Lambda(1)) \tag{A.0.5}$$

More precisely, for $a = \dim \overline{\{x\}}$

$$H_{a+b}(x, \Lambda(1)) = \varinjlim_{V \subseteq \overline{\{x\}}} H_{a+b}(V, \Lambda(1)) \cong \varinjlim_{V \subseteq \overline{\{x\}}} H^{a-b}(V, \Lambda(d+1)) = H^{a-b}(x, \Lambda(d+1)).$$

We finally note now that the spectral sequence (A.0.5) induces an edge (or cycle) map

$$E^2_{0,0} \cong \mathrm{CH}_0(X)/p^r \to H_0(X, \Lambda(1)) \cong H^{2d+2}_{\text{ét}}(X, \mathcal{T}_r(d+1)).$$

In order to study $\mathrm{CH}_1(X)/p^r$, one would need a purity isomorphism $H^{2d+2-q}_Y(X, \mathcal{T}_r(d)) \to H_q(Y, \Lambda(0))$ as in Proposition A.0.16 which fails due to the lack of full purity for logarithmic de Rham Witt sheaves.

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