Conformal symmetry breaking and evolution equations in Quantum Chromodynamics

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Abstract: In this thesis we present the evolution equation for non-singlet leading-twist operators in QCD to next-to-next-to-leading order in an expansion in the strong coupling constant. The method we use is based on conformal symmetry arguments. Using Ward identities we derive the conformal symmetry breaking in QCD in integer dimensions – also known as conformal anomaly – to two-loop accuracy. This result allows one to define a conformal invariant theory in $d = 4 - 2\epsilon$ dimension by tuning the strong coupling to a certain (critical) value. The symmetry revealed in that way allows us to solve for the evolution kernel to the three-loop accuracy. This result is given both in the formulation of non-local light-ray operators as well as local operators. In the latter case we present an explicit analytic solution of the NNLO evolution equation on the example of the pion distribution amplitude.

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Chapter 1

Introduction

Half a century has passed since the foundation of *Quantum Chromodynamics (QCD)* – the theory which describes the interaction of quarks via exchange of gluons. The concept of quarks was initially introduced by Gell-Mann and Zweig [1, 2] to avoid limitations due to "fermi's principle" by additional degrees of freedom. At that time it was not yet clear whether quarks are actual particles or just some "theoretical construction". The first DIS (deep inelastic scattering) experiments at the Stanford Linear Accelerator (SLAC) [3, 4, 5] encouraged the theoretical picture and gave profound reason to believe that nucleons are indeed bound states composed of almost free point-like constituents. Shortly later Gell-Mann and Fritzsch [6] proposed the theory of strong interactions – *QCD* – which turned out to give a precise description of the nature. Ever since this time there was an ongoing rally of both experimental and theoretical activities to gain a better understanding of the strong interaction. Despite all the successes we are still far from a complete, and in some concerns also detailed, picture. One of the major problems is that the strong coupling constant α_s acquires a peculiar dependence on the energy scale: at high energies (small distances) it tends to zero, quarks move as free particles and the theory gets trivial. This phenomenon is known as "asymptotic freedom" [7, 8]. Close to this regime one can construct a series in the small coupling and truncate this series at a sufficiently high order. However for small energies (large distances) the coupling grows and perturbation theory cannot be applied. Experiment shows that quarks and gluons cannot appear as free particles but only as bound states, a phenomenon called *confinement*. Among all non-perturbative formulations of the low-energy regime of the theory, Lattice QCD is the only one based on first principles and certainly the most successful one. Here the space-time is discretized in a finite volume and the theory is solved numerically. Just as a perturbative approach fails for large distances, the lattice formulation becomes obviously unreliable for small distances. The description of most processes requires an understanding of both regimes. The way out is given through so-called factorization theorems, which allow one to factorize some processes into a hard- (high energy-) and soft- (low energy-) part. Typically the hard part, often referred to as coefficient functions, depends on the process under consideration (but not the target particles). The soft part, given in terms of PDFs (parton distribution functions), TMDs (transverse momentum distributions), GPDs (generalized parton distributions) or DAs (distributions amplitudes) are process-independent universal functions (but depend on the particles). Due to their universality they can also be determined by a phenomenological fit to the data for a certain process and afterwards used for all kind of other processes. The factorization however introduces an additional unphysical parameter – the factorization scale. In addition the regularization of both soft- and hard-parts requires the introduction of renormalization scales. The dependence on these scales is governed by the RGEs (renormalization group equations) or evolution equations. Assuming both hard- and soft-part are exactly known, this ambiguity drops out. However, in reality both are just approximated to some accuracy and therefore some uncertainty in the physical quantities arises. In order to achieve a good precision one needs to keep this uncertainty small.

Studies of hard exclusive reactions contribute significantly to the research program at all major existing and planned accelerator facilities. The relevant non-perturbative input in such processes involves operator matrix elements between states with different momenta, dubbed GPDs, or vacuum-to-hadron matrix elements related to light-front hadron wave functions at small transverse separations, the DAs. The aim of this thesis is to make a step towards the NNLO QCD description of these reactions, which is the calculation of the scale dependence of the relevant parton distributions – GPDs and DAs – to the three-loop accuracy. This task is more complicated compared to the calculation of the scale dependence of usual parton distributions (DGLAP equations) because the distributions in question involve operator matrix elements sandwiched between states with different momentum. Thus, mixing with the operators containing total derivatives must be taken into account. From the technical point of view, the problem reduces to the calculation of the divergent parts of the relevant three-point functions involving two different external momenta. This is a much harder task as compared to the kinematics of forward scattering where only one external momentum is involved. Whereas in the case of forward kinematics NNLO [9, 10] results are available for quite some time and partial NNNLO results have been published recently [11, 12, 13, 14, 15, 16, 17], in the off-forward case NLO is the current accuracy [18, 19, 20, 21, 22, 23, 24, 25]. Due to the conformal symmetry of QCD at the classical level, the leading order exclusive evolution kernel can be deduced from the inclusive one and the non-trivial off-diagonal part appears at the two-loop level for the first time. It can be shown that this quantity can actually be determined by the conformal symmetry breaking at one-loop. In general it is known [26] that conformal symmetry (breaking) of the QCD Lagrangian allows one to restore full evolution kernels at a given order of perturbation theory from the spectrum of anomalous dimensions – alias the forward kernels – at the same order, and the calculation of the special conformal anomaly at one order less. This result was used to calculate the complete two-loop mixing matrix for twist-two operators in QCD [27, 23, 28], and derive the two-loop evolution kernels in momentum space for the GPDs [29, 24, 25]. In Ref. [30] an alternative technique has been suggested, the difference being that instead of studying conformal symmetry breaking in the physical theory [27, 23, 28] one uses exact conformal symmetry of a modified theory — QCD in $d = 4 - 2\epsilon$ dimensions at critical coupling. Exact conformal symmetry considerably simplifies the analysis and also suggests the optimal representation for the results in terms of light-ray operators. It is expected that these features will become increasingly advantageous in higher orders. This modified approach was illustrated in [30] on several examples to the two- and three-loop accuracy for scalar theories and in [31] to the two-loop accuracy in QCD, reproducing the known results [27, 23, 28].

The outline of the thesis is as follows: The first two chapters 2 and 3 are introductory, we establish the notion of a conformal field theory (CFT), give a reminder of the QCD Lagrangian and find a connection between QCD and CFT – conformal QCD at the critical point. We explain the concepts of the running coupling and evolution equations. Moreover we give a brief overview of the method. In chapter 4 we study the breakdown of conformal symmetry in integer-dimensional QCD and explicitly restore conformal symmetry in the modified conformal QCD. As the result we obtain three symmetry generators $S_\alpha(\alpha_s)$ which satisfy the conformal algebra to two-loop accuracy. In chapter 5 we use these findings to determine the evolution equations to the three-loop accuracy. We give explicit results in two different formulations – in terms of non-local light-ray operators and in terms of local operators. The latter are summarized in chapter 6, where we first need to establish a convenient conversion from one formulation to the other. In chapter 7 we apply our results to the evolution of the pion distribution amplitude, that is needed for the description of hard exclusive reactions involving production of energetic pions:

- the pion transition $\gamma + \gamma^* \to \pi^0$ and electromagnetic $\pi\pi\gamma^*$ form factors
- semi-leptonic and hadronic B-decays $B \to \pi\ell\nu_\ell$, $B \to \pi\pi$, etc.
- pion electroproduction $\gamma^* N \to \pi N$ and many others.

In chapter 8 we will conclude and give an outline of future progress in this direction. The main text will be followed by several appendices, which contain technical details and some relevant intermediate results.

Chapter 2

Conformal field theory

It is certainly beyond the scope of this thesis to present a comprehensive introduction to conformal field theory. Instead we want to refer to three books, lecture notes and reviews that have been important and inspiring for studies on that area: Firstly, there is the book by Di Francesco, Mathieu and Senechal [32], that can be seen as the standard book for conformal field theory. Secondly the lecture notes by Ginsparg [33] give an excellent introduction to the field. Finally, the review by Braun, Korchemsky and Müller [34] yields an perfect introduction and overview of the application of conformal symmetry to QCD. In the following we will restrict ourselves to a very brief resume of the main ideas from these references with focus on the areas that will be important for this thesis.

Since the dawn of science the concept of symmetries has been appealing to all kind of philosophers and researchers. While in the twentieth-century the idea of symmetries was often used for abstract constructions like gauge-symmetries, the more tangible notion of space-time symmetries is much older and has a long tradition. Even the breakdown of symmetries can be useful, as seen in the theory of phase transitions, critical phenomena [32] and electro-weak interactions. Modern particle physics is based on the concept of (relativistic) quantum field theory, where Poincaré-invariance is the fundamental space-time symmetry. A possible extension is given by scale invariance, which is the symmetry under global dilatations. In 1970 Polyakov [35] argued that for physical systems with local interactions it is reasonable to extend scale invariance by dilatations with a local scaling factor, which defines conformal transformations. Moreover, looking for transformations that leave the light-cone invariant, the conformal symmetry group turns out to be the maximal extension of the Poincaré-group. More precisely: In a conformal field theory (CFT) the usual Poincaré-symmetry of the classical theory is extended by scale transformations $x_\mu \mapsto \lambda x_\mu (\lambda \in \mathbb{R})$ and inversion $x_\mu \mapsto \frac{x_\mu}{x^2}$. The Poincaré group with these new added transformations forms the so-called conformal group. The Lie algebra of the Poincaré group is defined by the commutation relations

$$\begin{aligned}
&i[\mathbf{P}_\mu, \mathbf{P}_\nu] = 0, \qquad i[\mathbf{M}_{\alpha,\beta}, \mathbf{P}_\mu] = g_{\alpha\mu}\mathbf{P}_\beta - g_{\beta\mu}\mathbf{P}_\alpha, \\
&i[\mathbf{M}_{\alpha,\beta}, \mathbf{M}_{\mu\nu}] = g_{\alpha\mu}\mathbf{M}_{\beta\nu} - g_{\alpha\nu}\mathbf{M}_{\beta\mu} - g_{\beta\mu}\mathbf{M}_{\alpha\nu} + g_{\beta\nu}\mathbf{M}_{\alpha\mu},
\end{aligned} \tag{2.1}$$

and is extended by the following relations that generate the conformal algebra

$$\begin{aligned}
&i[\mathbf{D}, \mathbf{P}_\mu] = \mathbf{P}_\mu, \qquad i[\mathbf{D}, \mathbf{K}_\mu] = -\mathbf{K}_\mu, \\
&i[\mathbf{M}_{\alpha,\beta}, \mathbf{K}_\mu] = g_{\alpha\mu}\mathbf{K}_\beta - g_{\beta\mu}\mathbf{K}_\alpha, \qquad i[\mathbf{P}_\mu, \mathbf{K}_\nu] = -2g_{\mu\nu}\mathbf{D} + 2\mathbf{M}_{\mu\nu}, \\
&i[\mathbf{D}, \mathbf{M}_{\alpha,\beta}] = 0, \qquad i[\mathbf{K}_\mu, \mathbf{K}_\nu] = 0.
\end{aligned} \tag{2.2}$$

	#	finite action	generator of inf. action
Translation	4	$x_\mu \mapsto x_\mu + a_\mu$	$\mathbf{P}_\mu = -i\partial_\mu$
Rotation	6	$x_\mu \mapsto \omega_{\mu\nu}x^\nu$	$\mathbf{M}_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu - \Sigma_{\mu\nu})$
Dilatation	1	$x_\mu \mapsto \lambda x_\mu$	$\mathbf{D} = -i(x\cdot\partial + \Delta_\varphi)$
SCT	4	$x_\mu \mapsto \frac{x_\mu - a_\mu x^2}{1-2a\cdot x + a^2x^2}$	$\mathbf{K}_\mu = -i(2x_\mu x\cdot\partial - x^2\partial_\mu + 2\Delta_\varphi x_\mu - 2ix^\nu\Sigma_{\mu\nu})$

Table 2.1: The fifteen conformal transformations and their generators, see e.g. Ref. [34]. Δ_φ is the (canonical) scaling dimension of the field φ, for QCD explicit expressions will be given in Eq. (4.7)

For completeness, we collected the transformations and the corresponding generators in table 2.1. There we introduced the generator of spin rotations $\Sigma_{\mu\nu}$, which takes the following form for scalar, spinor (quark) and vector fields (gluon) fields

$$\Sigma_{\mu\nu}\phi = 0, \qquad \Sigma_{\mu\nu}q = \frac{i}{2}\sigma_{\mu\nu}q, \qquad \Sigma_{\mu\nu}A_\alpha = g_{\nu\alpha}A_\mu - g_{\mu\alpha}A_\nu, \tag{2.3}$$

respectively, where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ is the commutator of two Dirac matrices. As an inversion cannot be generated by an infinitesimal transformation, one usually considers the special conformal transformation (SCT) – the combination of inversion, translation and inversion.

During the past three decades CFT raised a lot of interest in both mathematics and physics and has built an ideal ground for the interplay of these two disciplines in the frame of mathematical physics. The main attention was put on CFT in two dimensions. In this special case the conformal algebra becomes infinite dimensional and thus imposes enough constraints to allow for an exact solution. In particular for string theory great achievements have been made by considering the string surface as a two-dimensional CFT. Despite all these appealing facts one should keep in mind that in dimensions greater than two the conformal algebra reduces to a finite one. Moreover most physical theories involve some natural dimensionful scales, provided by the masses of particles or the renormalization scale, and therefore one cannot expect these theories to feature scale or conformal invariance. Nevertheless it is known [36, 37, 38], that at the fixed point of the renormalization group flow the natural scale of the theory, given by the inverse correlation length, tends to zero and conformal symmetry emerges. This last point makes CFT also appealing for applications to statistical mechanics and condensed matter as it allows for the description of such systems close to the phase transition [39]. In that context the main topic is universality, meaning that several systems share common properties near the critical point and can be grouped into so-called universality classes.

In high-energy physics one most often considers light-cone dominated processes, i.e. ultra-relativistic particles moving close to the light-cone. To describe such kinematics it is appropriate to use light-cone variables

$$x^\mu = x_- n^\mu + x_+\bar{n}^\mu + x_\perp^\mu, \qquad x_+ = n\cdot x, \qquad x_- = \bar{n}\cdot x, \tag{2.4}$$

where $n^2 = 0$ and $\bar{n}^2 = 0$ define the two light-like directions, for convenience we can assume $n\cdot\bar{n} = 1$. In this picture a hadron is described by partons propagating along a collinear light-like direction

$$\varphi(x) \mapsto \varphi(zn) \equiv \varphi(z), \tag{2.5}$$

with some real number z. Whenever these kinematics apply for a process, the full conformal group reduces to a set of three symmetry generators acting non-trivial on the light-cone. These three generators correspond to the so–called collinear subgroup, $SL(2,\mathbb{R})$, of the conformal group, which consists of Moebius transformations,

$$z \mapsto z' = \frac{az+b}{cz+d}, \qquad \text{where } a,b,c,d,z \in \mathbb{R}, \qquad ad-bc=1. \tag{2.6}$$

Field transformations are given by:

$$\varphi(z) \mapsto T^j\varphi(z) = \frac{1}{(cz+d)^{2j}}\varphi(\frac{az+b}{cz+d}), \tag{2.7}$$

where the conformal spin $j = \frac{1}{2}(\Delta + s)$ of the field φ is given by half of the sum of the spin s and the scaling dimension. The generators for these three transformations are

$$\mathbf{L}_+ = -i\mathbf{P}_+, \qquad \mathbf{L}_- = \frac{i}{2}\mathbf{K}_-, \qquad \mathbf{L}_0 = \frac{i}{2}(\mathbf{D} - \mathbf{M}_{-+}). \tag{2.8}$$

They obey the $SL(2,\mathbb{R})$ - algebra

$$[\mathbf{L}_0, \mathbf{L}_\pm] = \pm\mathbf{L}_\pm, \qquad [\mathbf{L}_+, \mathbf{L}_-] = 2\mathbf{L}_0. \tag{2.9}$$

While the action of the generators on the quantum fields can be easily obtained from the definitions in table 2.1, it turns out to be more convenient to trade them for differential operators acting on the auxiliary variable z

$$\begin{aligned}
[\mathbf{L}_+, \varphi(z)] &= -\partial_z\varphi(z) \equiv S_-\varphi(z), \\
[\mathbf{L}_-, \varphi(z)] &= (z^2\partial_z + 2jz)\varphi(z) \equiv S_+\varphi(z), \\
[\mathbf{L}_0, \varphi(z)] &= (z\partial_z + j)\varphi(z) \equiv S_0\varphi(z),
\end{aligned} \tag{2.10}$$

which obey the same algebra

$$[S_0, S_\pm] = \pm S_\pm, \qquad [S_+, S_-] = 2S_0. \tag{2.11}$$

The n–particle generators are given by the sum of one–particle generators

$$\begin{aligned}
S_+ &= z_1^2\partial_{z_1} + 2j_1z_1 + \ldots + z_n^2\partial_{z_n} + 2j_nz_n, \\
S_0 &= z_1\partial_{z_1} + j_1 + \ldots + z_n\partial_{z_n} + j_n, \\
S_- &= -(\partial_{z_1} + \ldots + \partial_{z_n}).
\end{aligned} \tag{2.12}$$

The quadratic Casimir operator reads

$$\mathbb{C} = S_0^2 - S_0 + S_+S_-, \tag{2.13}$$

and commutes with all three generators

$$[\mathbb{C}, S_\alpha] = 0. \tag{2.14}$$

Its spectrum, as function of the conformal spin j, is given by $\mathbb{C} = j(j-1)$. The collinear subgroup will play the central role in our analysis.

Chapter 3

Theory and method

3.1 Conformal QCD

We consider massless QCD in the $d = 4 - 2\epsilon$ dimensional Euclidean space. The action reads

$$S = \int d^d x \Big\{ \bar{q} \not{D} q + \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} - \bar{c}^a \partial_\mu (D^\mu c)^a + \frac{1}{2\xi} (\partial_\mu A^{a,\mu})^2 \Big\}, \tag{3.1}$$

where $D_\mu = \partial_\mu - i g_B A^a_\mu T^a$ is the covariant derivative with T^a being the $SU(N)$ generators in the fundamental (adjoint) representation for quarks (ghosts). The field strength tensor is defined as usual

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_B f^{abc} A^b_\mu A^c_\nu. \tag{3.2}$$

Using dimensional regularization the dimension-shift ϵ serves as the regulator. For our purpose it will be useful in another context as we will explain later on. The bare coupling constant is $g_B = g\mu^\epsilon$ where μ is the scale parameter, introduced by the requirement to leave the action dimensionless. The most natural renormalization scheme in dimensional regularization is the *(modified) minimal subtraction* ($\overline{\text{MS}}$)-scheme [40]. The renormalized action is obtained from (3.1) by the replacements

$$q \to Z_q q, \qquad A \to Z_A A, \qquad c \to Z_c c, \qquad g \to Z_g g, \qquad \xi \to Z_\xi \xi, \tag{3.3}$$

where $Z_\xi = Z_A^2$ and the renormalization factors take the form (in ($\overline{\text{MS}}$)-scheme)

$$Z = 1 + \sum_{j=1}^{\infty} \epsilon^{-j} \sum_{k=j}^{\infty} z_{jk} \left(\frac{\alpha_s}{4\pi} \right)^k, \qquad \alpha_s = \frac{g^2}{4\pi}. \tag{3.4}$$

where z_{jk} are ϵ-independent constants. The anomalous dimensions of the fundamental fields

$$\gamma_\varphi = \mu \partial_\mu Z_\varphi, \tag{3.5}$$

are collected in appendix A to the three-loop accuracy. Note that we do not send $\epsilon \to 0$ in the action and the renormalized correlation functions so that they explicitly depend on ϵ.

Formally the theory has two charges — g and ξ. The corresponding beta-functions are

$$\beta(a) = \mu\frac{da}{d\mu} = -2a\big(\epsilon + \bar{\beta}(a)\big) \qquad\qquad \beta_\xi(\xi, g) = \mu\frac{d\xi}{d\mu} = -2\xi\gamma_A, \tag{3.6}$$

where $\bar{\beta} = \gamma_g$ is the anomalous dimension of the coupling constant [8, 7, 41, 42, 43]

$$\begin{aligned}\bar{\beta}(a) =& \mu\partial_\mu \ln Z_g = \beta_0 a + \beta_1 a^2 + \beta_2 a^3 + \mathcal{O}(a^4),\\ \beta_0 =& \frac{11}{3}N_c - \frac{2}{3}n_f, \qquad\qquad \beta_1 = \frac{2}{3}\left[17C_A^2 - 5C_A n_f - 3C_F n_f\right],\\ \beta_2 =& \frac{2857}{54}C_A^3 - \frac{1415}{54}C_A^2 n_f + \frac{79}{54}C_A n_f^2 + \frac{11}{9}C_F n_f^2 - \frac{205}{18}C_F C_A n_f + C_F^2 n_f.\end{aligned} \tag{3.7}$$

Here and in what follows we use the notation

$$a = \frac{\alpha_s}{4\pi}. \tag{3.8}$$

At the classical level one can set $d = 4$ and drop the scale parameter μ in (3.1), which becomes then invariant under conformal and scale transformations. In the interacting theory the introduction of the scale μ, however, explicitly breaks conformal symmetry of the theory. Even in the limit $d \to 4$, which can be taken after a proper renormalization prescription is done, the crucial μ-dependence remains. The same holds true for all known regularization schemes, e.g. a UV-cutoff or a lattice (where not even Poincar*é*-symmetry is preserved). To recover a conformal field theory and utilize all the powerful tools associated with it we will use the notion of renormalization-group-flow fixed points, which we will introduce in what follows.

It is known [37, 38] that there exists a critical value of the coupling (fixed point), $a = a_*(\epsilon)$ such that $\beta(a_*) = 0$. For a sufficiently large number of flavors n_f, the leading order beta-function changes its sign $\beta_0 < 0$. Therefore, in non-integer dimensions $d = 4 - 2\epsilon$, we find the critical point

$$\beta(a_*) = 0, \qquad\qquad a_* = -\frac{\epsilon}{\beta_0} - \left(\frac{\epsilon}{\beta_0}\right)^2\frac{\beta_1}{\beta_0} - \left(\frac{\epsilon}{\beta_0}\right)^3\frac{2\beta_1^2 - \beta_2\beta_0}{\beta_0^2} + \mathcal{O}(\epsilon^4). \tag{3.9}$$

From Eq. (3.6) we see that beta-function associated with the gauge parameter ξ vanishes identically in Landau gauge $\xi = 0$. As a consequence, Green's functions of quark and gluon fields in Landau gauge at critical coupling enjoy scale invariance. This can be seen as just a technical trick to consider QCD in $d = 4 - 2\epsilon$ as a scale invariant theory. We clearly need to stress that QCD in integer dimensions is not a scale invariant field theory.

So far we just discussed scale invariance. Although Bogoliubov's famous quote [44] – "There is no mathematical difference [between scale and conformal symmetry], but when some young people want to use a fancy word they call it conformal symmetry" – was certainly incorrect in the mathematical sense, in a physical context it seems to be true that scale invariance usually comes hand in hand with conformal invariance of the theory: It is believed that "physically reasonable" scale invariant theories are also conformally invariant, see Ref. [45] for a discussion. In non-gauge theories conformal invariance for the Green's functions of basic fields can be checked in perturbative expansions [46]. For local composite operators a proof of conformal invariance is based on analysis of pair counterterms for the product of the trace of the energy-momentum tensor and local operators [47]. In gauge theories, including QCD, conformal invariance does not hold for the correlators of basic fields and can be expected only for the Green's functions of gauge-invariant operators. In addition there are extra complications due to mixing of gauge-invariant operators with BRST variations and equation-of-motion (EOM) operators. We will discuss these issues briefly in what follows.

Renormalization ensures finiteness of the correlation functions of the basic fields that are encoded in the QCD partition function. Correlation functions with an insertion of a composite operator $\mathcal{O}_k$ – built of n fundamental fields φ – possess additional divergences that are removed by the operator renormalization,

$$[\mathcal{O}_k] = \sum_j \mathbb{Z}_{kj}\mathcal{O}_j, \tag{3.10}$$

where the sum goes over all operators with the same quantum numbers that get mixed; $\mathbb{Z}_{kj}$ are the renormalization factors that have a similar expansion in inverse powers of ϵ as in Eq. (3.4)

$$\mathbb{Z} = \sum_{i=1}^{\infty} \frac{\mathbb{Z}_i(a)}{\epsilon^i}, \qquad \mathbb{Z}_i(a) = \sum_{\ell=i}^{\infty} a^\ell Z_i^{(\ell)}. \tag{3.11}$$

Here and below we use square brackets to denote renormalized composite operators (in a minimal subtraction scheme).

Renormalized operators satisfy a RG equation with the anomalous dimension matrix (or evolution kernel, in a different representation)

$$\mathbb{H} = -(\mu\partial_\mu \mathbb{Z})\mathbb{Z}^{-1} = 2\sum_\ell \ell a^\ell \mathbb{Z}_1^{(\ell)}. \tag{3.12}$$

(up to field renormalization) which has a perturbative expansion with coefficients that in a minimal subtraction scheme do not depend on ϵ by construction. As a consequence, the anomalous dimension matrices are exactly the same for QCD in d dimensions that we consider at the intermediate step, and physical QCD in integer dimensions that is our final goal. To be specific, if in d-dimensional QCD at the critical point

$$\Big(\mu\partial_\mu + \mathbb{H}(a_*)\Big)[\mathcal{O}] = 0, \qquad \mathbb{H}(a_*) = a_*\mathbb{H}^{(1)} + a_*^2\mathbb{H}^{(2)} + \dots \tag{3.13}$$

then in $d = 4$ dimension and for arbitrary coupling

$$\Big(\mu\partial_\mu + \beta(a)\partial_a + \mathbb{H}(a)\Big)[\mathcal{O}] = 0, \qquad \mathbb{H}(a) = a\mathbb{H}^{(1)} + a^2\mathbb{H}^{(2)} + \dots \tag{3.14}$$

with *the same* matrices $\mathbb{H}^{(k)}$. All what one has to do in going over to the four-dimensional theory is to re-express consistently all occurrences of $\epsilon = (4-d)/2$ in terms of the critical coupling $\epsilon = \beta_0 a_* + \dots$ and replace $a_* \mapsto a$ in the resulting expressions. In other words: there is a hidden symmetry of *RG equations* in physical QCD in minimal subtraction schemes to all orders in perturbation theory. The requirement of large n_f for existence of the critical point is not principal since, staying within perturbation theory, the dependence on n_f is polynomial. In this sense the statement above holds for arbitrary number of flavors.

Let us briefly sketch the idea:

- As a first step we go over to a theory in non-integer dimensions, which enjoys exact scale and conformal invariance at the critical point.

- Using conformal Ward identities we will construct the explicit expressions for the symmetry generators of translations, dilatations and special conformal transformations.
- For QCD in integer dimensions this means we have found three operators that commute with the evolution kernel $\mathbb{H}$ and form an $SL(2)$ algebra.
- As we will see below, perturbative expansion of these commutation relations produces a nested set of equations that allow one to determine the non-diagonal parts of the anomalous dimension matrices with a relatively small effort.

Some parts of the construction become simpler and more transparent going over from local operators to the corresponding generating functions that are usually referred to as light-ray operators. This representation is introduced in the next section.

3.2 Conformal symmetry of evolution equations

In this chapter we explain this approach on a more technical level.

In order to make use of the (approximate) conformal symmetry of QCD it is natural to use a coordinate-space representation in which the symmetry transformations have a simple form, see [34] a review. The suitable objects are light-ray operators (see e.g. [48]) which can be understood as generating functions for the renormalized leading-twist local operators. For example

$$[\mathcal{O}^{(n)}](x;z_1,z_2) \equiv [\bar{q}(x+z_1 n)\not{n}q(x+z_2 n)] \equiv \sum_{m,k} \frac{z_1^m z_2^k}{m!k!}[(D_+^m\bar{q})(x)\not{n}(D_+^k q)(x)]. \tag{3.15}$$

Here the Wilson line is assumed between the quark fields on the light-cone, $D_+ = n_\mu D^\mu$ is a covariant derivative, n_μ is an auxiliary light-like vector, $n^2 = 0$, that ensures symmetrization and subtraction of traces of local operators. The square brackets $[\ldots]$ stand for the renormalization using dimensional regularization and $\overline{\text{MS}}$ subtraction. We tacitly assume the quarks to be of different flavor. Unless we denote it differently, the light-ray operator will be aligned in the "plus"-direction n.

The local composite operators defined by the OPE (3.15) do not have simple properties under the transformations defined in table 2.1. Therefore we aim to find operators that can be classified according to the irreducible representations of the SL(2)-algebra. This basis set of operators defines the so-called conformal tower of operators. The highest weight vector of the representation is the lowest operator in the conformal tower and defines the so-called *conformal operator* $\mathcal{Q}_N(x)$. It is an eigenfunction of the evolution equation (3.14). There are several equivalent ways to define a conformal operator, upon which the most common reads: The conformal operator must have the same transformation properties as the fundamental field, see. table 2.1. For operators located at the origin $x = 0$ this just means that it gets annihilated by a special conformal transformation

$$[\mathbf{L}_-, \mathcal{Q}_N(x=0)] = \frac{i}{2}\mathbf{K}_-\mathcal{Q}_n(x=0) = 0 \tag{3.16}$$

The higher operators in the conformal tower are obtained by adding total derivatives

$$\mathcal{Q}_{Nk} = (n\partial)^k \mathcal{Q}_N = [\mathbf{L}_+, [\mathbf{L}_+, \ldots, [\mathbf{L}_+, \mathcal{Q}_N]]]\,, \qquad k = 0,1,2,\ldots \tag{3.17}$$

From this definition and the SL(2)-algebra (2.9) one derives the transformation properties [1]

$$\begin{aligned}
[\mathbf{L}_+, \mathcal{Q}_{Nk}] &= \mathcal{Q}_{Nk+1}, \\
[\mathbf{L}_0, \mathcal{Q}_{Nk}] &= (j_N^* + k)\mathcal{Q}_{Nk}, \\
[\mathbf{L}_-, \mathcal{Q}_{Nk}] &= -k(2j_N^* + k - 1)\mathcal{Q}_{Nk-1},
\end{aligned} \tag{3.18}$$

where $j_N^* = N + 2 + \bar{\beta}(a_*) + \frac{1}{2}\gamma_N(a_*)$ is the conformal spin of the operator $\mathcal{Q}_N$. Obviously all operators $\mathcal{Q}_{Nk}$ in the tower have the same anomalous dimensions γ_N. The generators $\mathbf{L}_\pm$ act as raising and lowering operators on this representation. The OPE of the light-ray operator (3.15) in terms of this set of basis operators reads

$$[\mathcal{O}](z_1, z_2) = \sum_{Nk} \Psi_{Nk}(z_1, z_2)\mathcal{Q}_{Nk}, \tag{3.19}$$

where the coefficient functions $\Psi_{Nk}(z_1, z_2)$ are homogeneous polynomials of degree $N + k$. To fix the form of these coefficient functions we will trade the generators $\mathbf{L}_\alpha$ for the differential operators S_α a la Eq. (2.10)

$$[\mathbf{L}_{\pm,0}, O(z_1, z_2)] = \sum_{Nk} \Psi_{Nk}(z_1, z_2)\,[\mathbf{L}_{\pm,0}, \mathcal{Q}_{Nk}] = \sum_{Nk} S_{\mp,0}\Psi_{Nk}(z_1, z_2)\mathcal{Q}_{Nk}. \tag{3.20}$$

A simple algebra reveals that the generators $S_\pm$ act as raising and lowering operators on the basis of coefficient functions

$$\begin{aligned}
S_-\Psi_{Nk}(z_1, z_2) &= -\Psi_{Nk-1}(z_1, z_2), \\
S_0\Psi_{Nk}(z_1, z_2) &= (j_N^* + k)\Psi_{Nk}(z_1, z_2), \\
S_+\Psi_{Nk}(z_1, z_2) &= (k + 1)(2j_N^* + k)\Psi_{Nk+1}(z_1, z_2).
\end{aligned} \tag{3.21}$$

Thus the lowest weight coefficient function is annihilated by translations

$$S_-\Psi_{Nk=0}(z_1, z_2) \equiv S_-\Psi_N(z_1, z_2) = 0, \tag{3.22}$$

and therefore must be a shift-invariant function $\Psi_N(z_1, z_2) \simeq (z_1 - z_2)^N$. The higher weight functions are then obtained by successive action of the conformal generator

$$\Psi_{Nk}(z_1, z_2) \simeq (S_+)^k\Psi_N(z_1, z_2) \simeq (S_+)^k(z_1 - z_2)^N. \tag{3.23}$$

Explicit expressions for the proportionality factors are irrelevant for the moment but will be given later in the text.

Light-ray operators satisfy a renormalization-group equation

$$(\mu\partial_\mu + \beta(\alpha)\partial_\alpha + \mathbb{H})\,[\mathcal{O}(z_1, z_2)] = 0, \tag{3.24}$$

where $\mathbb{H}$ is an integral operator acting on the light-cone coordinates of the fields. It can be written as

$$\mathbb{H}[\mathcal{O}](z_1, z_2) = \int_0^1 d\alpha \int_0^1 d\beta h(\alpha, \beta)[\mathcal{O}](z_{12}^\alpha, z_{21}^\beta) + 2\gamma_q[\mathcal{O}](z_1, z_2), \tag{3.25}$$

[1] Here we still assume $x = 0$. The action of the symmetry generators on operators at arbitrary space-time points x takes more involved form and is derived in appendix B

where γ_q is the quark anomalous dimension,

$$z_{12}^{\alpha} = \bar{\alpha} z_1 + \alpha z_2, \qquad \bar{\alpha} = 1 - \alpha \tag{3.26}$$

and $h(\alpha, \beta)$ is a certain weight function, often also associated with the term "evolution kernel". The spectrum of the evolution kernel

$$\mathbb{H}(z_1 - z_2)^N = (\gamma_N + 2\gamma_q)(z_1 - z_2)^N, \tag{3.27}$$

is given by the well-known forward anomalous dimensions [9, 11, 12, 13, 14, 15, 16, 17]

$$\gamma_N = \int d\alpha d\beta\, h(\alpha, \beta)(1 - \alpha - \beta)^N. \tag{3.28}$$

In general the function $h(\alpha, \beta)$ is a function of two variables and therefore the knowledge of the eigenvalues – that are the anomalous dimensions γ_N – is not sufficient to fix it. However, to leading order the theory is conformally invariant and $\mathbb{H}$ must commute with the generators of the $SL(2)$ transformations $[\mathbb{H}, S_{\alpha}^{(0)}] = 0$, where to the leading order, cf. Eqs. (2.12),

$$\begin{aligned}
S_+ &= z_1^2 \partial_{z_1} + z_2^2 \partial_{z_2} + 2(z_1 + z_2) + \mathcal{O}(a_*) \equiv S_+^{(0)} + \mathcal{O}(a_*), \\
S_0 &= z_1 \partial_{z_1} + z_2 \partial_{z_2} + 2 + \mathcal{O}(a_*) \equiv S_0^{(0)} + \mathcal{O}(a_*), \\
S_- &= -\partial_{z_1} - \partial_{z_2} \equiv S_-^{(0)}.
\end{aligned} \tag{3.29}$$

In this case it can be shown that the function $h(\alpha, \beta)$ takes the form [49]

$$h(\alpha, \beta) = \bar{h}(\tau), \qquad \tau = \frac{\alpha\beta}{\bar{\alpha}\bar{\beta}} \tag{3.30}$$

and is effectively a function of one variable τ called the conformal ratio. This function can easily be reconstructed from its spectrum, alias from the anomalous dimensions.

Conformal symmetry of QCD is broken at the level of quantum corrections which implies that the symmetry of the evolution equations is lost at the two-loop level. In other words, writing the evolution kernel as an expansion in the coupling constant

$$\mathbb{H} = a\mathbb{H}^{(1)} + a^2\mathbb{H}^{(2)} + \dots \quad \leftrightarrow \quad h(\alpha, \beta) = a h^{(1)}(\alpha, \beta) + a^2 h^{(2)}(\alpha, \beta) + \dots \tag{3.31}$$

one expects that $h^{(1)}(\alpha, \beta)$ only depends on the conformal ratio whereas higher-order contributions remain to be nontrivial functions of two variables α and β. This prediction can be confirmed by an explicit calculation which reveals that the first-order kernel in QCD has a remarkably simple form

$$h^{(1)}(\alpha, \beta) = C_F \delta_+(\tau) - \theta(1 - \tau). \tag{3.32}$$

Taking appropriate matrix elements and a Fourier transformation to the momentum space one can check that the expression in Eq. (3.32) reproduces all classical LO QCD evolution equations: DGLAP, ERBL and GPD evolution equation [50, 51, 52]. The idea of Refs. [30, 31] is to consider a modified theory, QCD in non-integer $d = 4 - 2\epsilon$ dimensions. This theory enjoys exact scale and conformal invariance [37, 38] at the so–called critical point $a_* \sim \epsilon$, where $\beta(a_*) = 0$.

As a consequence, the renormalization group equations are exactly conformally invariant: the evolution kernels commute with the generators of the conformal group:

$$\boxed{[\mathbb{H}, S_{\alpha}] = 0.} \tag{3.33}$$

The generators are, however, modified by quantum corrections as compared to their canonical expressions (3.29):

$$S_\alpha = S_\alpha^{(0)} + a_* \Delta S_\alpha^{(1)} + a_*^2 \Delta S_\alpha^{(2)} + \dots \tag{3.34}$$

One can show that [31]

$$\begin{aligned} S_- &= S_-^{(0)}, \\ S_0 &= S_0^{(0)} - \epsilon + \frac{1}{2}\mathbb{H}(a_*), \\ S_+ &= S_+^{(0)} + (z_1 + z_2)\left(-\epsilon + a_* \mathbb{H}^{(1)}\right) + a_*(z_1 - z_2)\Delta^{(1)} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{3.35}$$

where

$$\Delta^{(1)}[O](z_1, z_2) = -C_F \int_0^1 \left(\frac{\bar\alpha}{\alpha} + \ln(\alpha)\right) \Big[[O](z_{12}^\alpha, z_2) - [O](z_1, z_{21}^\alpha)\Big] \tag{3.36}$$

i.e. the generator S_- (translation) is not deformed at all, the deformation of S_0 (dilatation) can be calculated exactly in terms of the evolution kernel (to all orders in perturbation theory), whereas the deformation of S_+ (special conformal transformation) is nontrivial and has to be calculated explicitly to the required accuracy. It can always be arranged in the form (3.35):

$$S_+ = S_+^{(0)} + (z_1 + z_2)\left(-\epsilon + \frac{1}{2}\mathbb{H}(\alpha_s)\right) + (z_1 - z_2)\Delta(\alpha_s). \tag{3.37}$$

and the main task will be to calculate the perturbative expansion for

$$\Delta(a) = a\Delta^{(1)} + a^2\Delta^{(2)} + \mathcal{O}(a^3). \tag{3.38}$$

From the pure technical point of view, this calculation replaces the evaluation of the conformal anomaly in the theory with broken symmetry in integer dimensions in the approach by D. Müller [27, 23, 28].

In order to derive Δ one needs to consider Conformal Ward identities (CWI) for the Green's function of two light-ray operators, one contracted with the auxiliary light-like vector n_μ, and the other one with $\bar n_\mu$ [2]:

$$\begin{aligned} G(x, z, w) &= \langle [\mathcal{O}^{(n)}](0, z)[\mathcal{O}^{(\bar n)}](x, w)\rangle \\ &= \mathcal{N}^{-1} \int D\varphi\, e^{-S_R(\varphi)} [\mathcal{O}^{(n)}](0, z)[\mathcal{O}^{(\bar n)}](x, w), \end{aligned} \tag{3.39}$$

where $\varphi \equiv \{q, \bar q, A, c, \bar c\}$ is the set of all fundamental fields and $z = \{z_1, z_2\},\ w = \{w_1, w_2\}$.

The statement is the following: conformal symmetry (in the modified theory at the critical point) imposes a constraint on this Green's function, called CWI. This Ward identity follows from the invariance of the Green's function under a change of variables $\varphi \mapsto \varphi + \delta\varphi$ in the functional integral,

$$\begin{aligned} &\langle \delta[\mathcal{O}^{(n)}](0, z)[\mathcal{O}^{(\bar n)}](x, w)\rangle + \langle [\mathcal{O}^{(n)}](0, z)\delta[\mathcal{O}^{(\bar n)}](x, w)\rangle \\ &\qquad - \langle \delta S_R [\mathcal{O}^{(n)}](0, z)[\mathcal{O}^{(\bar n)}](x, w)\rangle \stackrel{!}{=} 0, \end{aligned} \tag{3.40}$$

[2] both chosen orthogonal to x, i.e. $(nx) = (\bar n x) = 0$ and normalized as $(\bar n n) = 1$

where δ corresponds to either a dilatation $\delta = \delta_{\mathbf{D}}$ or a special conformal transformation $\delta = \delta_{\mathbf{K}}$. This equation is called Conformal Ward identity (CWI) and will be analyzed in great detail later on.

Conformal symmetry of the modified QCD at the critical coupling implies that the generators satisfy the usual $SL(2)$ commutation relations

$$[S_0, S_\pm] = \pm S_\pm, \tag{3.41a}$$

$$[S_+, S_-] = 2S_0. \tag{3.41b}$$

Due to the general form (3.35) only the first commutator, i.e. Eq. (3.41a) with a "+" sign, contains non-trivial information. Performing an expansion in powers of the coupling a_* one obtains a nested set of commutator relations [3]

$$\boxed{\begin{aligned} &[S_+^{(0)}, \mathbb{H}^{(1)}] = 0, \\ &[S_+^{(0)}, \mathbb{H}^{(2)}] = [\mathbb{H}^{(1)}, \Delta S_+^{(1)}], \\ &[S_+^{(0)}, \mathbb{H}^{(3)}] = [\mathbb{H}^{(1)}, \Delta S_+^{(2)}] + [\mathbb{H}^{(2)}, \Delta S_+^{(1)}], \\ &\qquad\qquad \vdots \\ &[S_+^{(0)}, \mathbb{H}^{(\ell)}] = \sum_{k=1}^{\ell-1} [\mathbb{H}^{(k)}, \Delta S_+^{(\ell-k)}]. \end{aligned}} \tag{3.42}$$

Note that the commutator of the canonical generator $S_+^{(0)}$ with the evolution kernel at order ℓ on the l.h.s. is given in terms of the evolution kernels $\mathbb{H}^{(k)}$ and the corrections to the generators $\Delta S_+^{(m)}$ at one order less, $k, m \leq \ell - 1$. The commutation relations Eq. (3.42) can be viewed as, essentially, inhomogeneous first-order differential equations on the evolution kernels. Their solution determines $\mathbb{H}^{(\ell)}$ up to an $SL(2)$-invariant term (solution of a homogeneous equation $[\mathbb{H}^{(\ell)}_{\text{inv}}, S_\alpha^{(0)}] = 0$), which can, again, be restored from the spectrum of the anomalous dimensions. Last but not least, in MS-like schemes the evolution kernels (anomalous dimensions) do not depend on the space-time dimension by construction. Thus the expressions derived in the d-dimensional (conformal) theory for the critical coupling allow one to restore the results for the theory in integer dimensions for arbitrary coupling; this procedure is straightforward and exact to all orders. This approach has been checked in calculations in scalar field theories to the three-loop accuracy [30] and in QCD to two loops [53, 31]. In both cases this technique proves to be very effective and the results can be presented in a compact analytic form.

[3] This is completely equivalent to the restriction $[\mathbb{H}, S_+] = 0$

Chapter 4

Conformal anomaly

In this chapter we will calculate the CWI to two-loop accuracy, which determines the corrections to the conformal generators.

4.1 Ward identities

Ward identities (WI) follow from invariance of path integrals under a change of variables, that corresponds to a symmetry transformation. The standard choice is the correlation function of the composite operator in question with a set of fundamental fields. In gauge theories and in particular in QCD it is more convenient to consider the correlation functions of light-ray operators, which are gauge-invariant.

As mentioned above, the operator S_+ in the light-ray operator representation is defined as the generator of special conformal transformations in the "minus" direction $\bar{n}$ acting on the light-ray operator aligned in the "plus" direction n and centered at the origin, $x = 0$:

$$\big[\mathbf{L}_-, [\mathcal{O}^{(n)}](0, z_1, z_2)\big] = \frac{i}{2}\big[\bar{n}\mathbf{K}, [\mathcal{O}^{(n)}](0, z_1, z_2)\big] = S_+\big[\mathcal{O}^{(n)}\big](0, z_1, z_2). \tag{4.1}$$

(Here we display explicitly the dependence on the auxiliary vector n in the definition of the light-ray operator). On the other hand, taking the transformation and operator along the "minus" direction $\bar{n}$ and choosing x such that $(x \cdot \bar{n}) = 0$ one gets, see Eq. (B.11)

$$\big[\mathbf{L}_-, [\mathcal{O}^{(\bar{n})}](x, z_1, z_2)\big] = \frac{i}{2}\big[\bar{n}\mathbf{K}, [\mathcal{O}^{(\bar{n})}](x, z_1, z_2)\big] = -\frac{1}{2}x^2(\bar{n}\partial_x)\big[\mathcal{O}^{(\bar{n})}\big](x, z_1, z_2). \tag{4.2}$$

Let us consider the Green's function of these two light-ray operators $[\mathcal{O}]^{(n)}(0, z)$ and $[\mathcal{O}]^{(\bar{n})}(x, w)$ that are separated by a transverse distance $(x \cdot n) = (x \cdot \bar{n}) = 0$:

$$\mathcal{G}(x; z, w) = \Big\langle [\mathcal{O}^{(n)}](0, z)\, [\mathcal{O}^{(\bar{n})}](x, w) \Big\rangle\,. \tag{4.3}$$

Here we use the shorthand notation $z = \{z_1, z_2\}$, $w = \{w_1, w_2\}$. The path-integral representation reads

$$\mathcal{G}(x; z, w) = \mathcal{N} \int D\varphi\, e^{-S_R(\varphi)} [\mathcal{O}^{(n)}](0, z)\, [\mathcal{O}^{(\bar{n})}](x, w). \tag{4.4}$$

Here $\mathcal{N}$ is a normalization factor, $S_R(\varphi)$ is the renormalized QCD action, $\varphi = \{A, q, \bar{q}, c, \bar{c}\}$ and the functional integration goes over all fields.

Let us perform the infinitesimal field transformations from table 2.1 in the path-integral

$$\varphi \mapsto \varphi + \lambda\delta_{\mathbf{D}}\varphi, \qquad \delta_{\mathbf{D}}\varphi = \big(x\partial_x + \Delta_\varphi\big)\varphi(x), \tag{4.5}$$

$$\varphi \mapsto \varphi + a_\mu\delta^\mu_{\mathbf{K}}\varphi, \qquad \delta^\mu_{\mathbf{K}}\varphi = \Big(2x_\mu(x\partial) - x^2\partial_\mu + 2\Delta_\varphi x_\mu - 2\Sigma_{\mu\nu}x^\nu\Big)\varphi(x), \tag{4.6}$$

corresponding to the dilatation and special conformal transformations, respectively. Ghost fields are scalars, thus they transform under spin rotations as follows, see Eq. (2.3),

$$\Sigma_{\mu\nu}c = \Sigma_{\mu\nu}\bar{c} = 0.$$

The definition for the scaling dimensions Δ_φ of the fundamental fields is a matter of taste, a convenient choice is [24]:

$$\Delta_q = \frac{3}{2} - \epsilon, \qquad \Delta_A = 1, \qquad \Delta_c = 0, \qquad \Delta_{\bar{c}} = 2 - 2\epsilon. \tag{4.7}$$

The choice $\Delta_A = 1$ ensures that the gluonic field strength tensor transforms covariantly under conformal transformations

$$\delta^\mu_{\mathbf{K}}F_{\alpha\beta} = \Big[2x_\mu(x\partial) - x^2\partial_\mu + 4x_\mu - 2\Sigma_{\mu\nu}x^\nu\Big]F_{\alpha\beta}, \tag{4.8}$$

and the reason for $\Delta_c = 0$ is that for this choice a covariant derivative of the ghost field $D_\rho c(x)$ transforms as a vector field of dimension one, i.e. in the same way as the gluon field A_ρ.

Demanding invariance of the Green's function $\mathcal{G}(x; z, w)$ in the path-integral representation under conformal variations implies the Ward identities

$$\boxed{\begin{aligned}&\Big\langle \delta[\mathcal{O}^{(n)}](0,z)\,[\mathcal{O}^{(\bar{n})}](x,w)\Big\rangle + \Big\langle [\mathcal{O}^{(n)}](0,z)\,\delta[\mathcal{O}^{(\bar{n})}](x,w)\Big\rangle \\ &\qquad - \Big\langle \delta S_R\,[\mathcal{O}^{(n)}](0,z)\,[\mathcal{O}^{(\bar{n})}](x,w)\Big\rangle = 0,\end{aligned}} \tag{4.9}$$

where $\delta = \delta_{\mathbf{D}}$ or $\delta = \delta_{\mathbf{K}} = \bar{n}_\mu\delta^\mu_K$ for either scale or conformal transformations, (4.5) and (4.6). The variation of the QCD action δS_R takes in both cases the following form

$$\begin{aligned}\delta_{\mathbf{D}}S_R &= \int d^dx\,\epsilon\mathcal{N}(x),\\ \delta^\mu_{\mathbf{K}}S_R &= \int d^dx\,2x^\mu\Big(\epsilon\mathcal{N}(x) - (d-2)\partial^\rho\mathcal{B}_\rho(x)\Big),\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}\mathcal{N}(x) &= 2\,\mathcal{L}_R^{YM+gf} = 2\left(\frac{1}{4}Z_A^2F^2 + \frac{1}{2\xi}(\partial A)^2\right),\\ \mathcal{B}_\rho(x) &= Z_c^2\bar{c}D^\rho c - \frac{1}{\xi}A^\rho(\partial A).\end{aligned} \tag{4.11}$$

It is known [46] that for QFTs with certain properties[1] one finds

$$2x^\mu \delta_{\mathbf{D}} \mathcal{L}_R = \delta^\mu_{\mathbf{K}} \mathcal{L}_R, \tag{4.12}$$

and accordingly scale invariance implies conformal invariance of the theory. For QCD, however, these properties are spoiled by an extra term $\sim \partial^\rho \mathcal{B}_\rho(x)$ in the conformal variation that does not vanish in the limit $\epsilon \to 0$. Hence the QCD action is not invariant under conformal transformations even for integer $d = 4$ dimensions. Luckily, since we only consider the correlator of gauge-invariant operators one can show that this operator $\mathcal{B}_\mu(x)$ does not give any contributions, as it can be written as a BRST variation of $\bar{c}^a A^a_\mu$ [24], see Appendix C.

On the other hand, conformal invariance of QCD at the critical point implies the constraint

$$\begin{aligned}\Big\langle [\mathbf{L}_-, [\mathcal{O}^{(n)}](0,z)]\, [\mathcal{O}^{(\bar{n})}](x,w) + [\mathcal{O}^{(n)}](0,z)\, [\mathbf{L}_-, [\mathcal{O}^{(\bar{n})}](x,w)] \Big\rangle \;&=\\ =\; \left[S_+^{(z)} - \frac{1}{2} x^2 (\bar{n}\partial_x)\right] \mathcal{G}(x; z, w) \;&=\; 0,\end{aligned} \tag{4.13}$$

where the superscript $S_+^{(z)}$ reminds that it is a differential operator acting on the z_1, z_2 coordinates. Equation (4.13) can be seen as a definition for what is meant by conformal symmetry at the critical point. By comparing the equations (4.9) and (4.13) one finds an expression for the exact conformal generator $S_+^{(z)}$ in the conformal theory.

In what follows we analyze the structure of the Ward identities (4.9) in detail.

4.1.1 Scale Ward identity

We start our analysis with the scale WI (SWI). First, the variation of the renormalized light-ray operator is given by

$$\delta_{\mathbf{D}} [\mathcal{O}^{(n)}](x,z) = Z \delta_{\mathbf{D}} \mathcal{O}^{(n)}(x,z) = \Big(x\partial_x + \sum_{i=1,2} z_i \partial_{z_i} + 3 - 2\epsilon\Big) [\mathcal{O}^{(n)}](x,z). \tag{4.14}$$

Here we used $Z \delta_{\mathbf{D}} Z^{-1} = \delta_{\mathbf{D}}$, which is true since the term $\Big(x\partial_x + \sum_{i=1,2} z_i \partial_{z_i} + 3 - 2\epsilon\Big)$ counts the canonical dimension of the operators and renormalization mixes only operators with the same canonical dimension. Thus we derive for Eq. (4.9)

$$\Big(x\partial_x + \sum_{i=1,2} (z_i \partial_{z_i} + w_i \partial_{w_i}) + 6 - 4\epsilon\Big) \mathcal{G}(x; z, w) = \Big\langle \delta_{\mathbf{D}} S_R\, [\mathcal{O}^{(n)}](0,z)\, [\mathcal{O}^{(\bar{n})}](x,w) \Big\rangle . \tag{4.15}$$

The l.h.s. is nothing else than the logarithmic derivative w.r.t. scale parameter $\sum_i (\Delta_i + x_i \cdot \partial_{x_i}) = \mu \partial_\mu$, and making use of the evolution equation (3.14) we arrive at the final form for the SWI

$$\Big\langle \delta_{\mathbf{D}} S_R\, [\mathcal{O}^{(n)}](0,z)\, [\mathcal{O}^{(\bar{n})}](x,w) \Big\rangle = - \Big(\beta(a)\partial_a + \mathbb{H}^{(z)}(a) + \mathbb{H}^{(w)}(a)\Big) \mathcal{G}(x; z, w). \tag{4.16}$$

[1] i. Interaction contains only fields without derivatives

ii. Translation- and rotation-invariance

iii. Kinetic terms in action are scale and conformally invariant

This results reflects the common wisdom that the SWI is nothing else than a renormalization group equation. Another way to understand this equation is to compare it with the standard way to derive the RGE. In the usual approach one calculates the Green's function and takes the $1/\epsilon$ pole part, while in the SWI one considers an insertion $\delta_{\mathbf{D}} S_R \sim \epsilon \int d^dx \mathcal{L}^{YM+gf}$ and takes the finite part. The two methods differ by additional insertions in gluon propagators, three- and four-gluon vertices, and gauge-fixing terms. Hence both Green's functions must be equal up to combinatorial factors. This factor is given by the number of gluon-lines and vertices of each diagram and is just equal to the number of loops of each diagram minus one , i.e. the eigenvalue of the operator $a\partial_a - 1$.

4.1.2 Conformal Ward identity

The two terms on the l.h.s. of the conformal Ward identity (CWI), Eq. (4.9), correspond to the variation of the light-ray operators. The first one can be expressed in terms of S_+,

$$\delta_{\mathbf{K}}[\mathcal{O}^{(n)}](0,z) = Z\delta_{\mathbf{K}}\mathcal{O}^{(n)}(0,z) = 2ZS_+^{(\epsilon)}\mathcal{O}^{(n)}(0,z) = 2ZS_+^{(\epsilon)}Z^{-1}[\mathcal{O}^{(n)}(0,z)], \tag{4.17}$$

where $S_+^{(\epsilon)} = S_+^{(0)} - \epsilon(z_1 + z_2)$, the term $-\epsilon(z_1 + z_2)$ is due to the modification of the quark scaling dimension $\Delta_q = \frac{3}{2} - \epsilon$, cf. (4.7).

The conformal variation of the second light-ray operator retains its leading order form (for our choice $(x \cdot \bar{n}) = 0$)

$$\delta_{\mathbf{K}}[\mathcal{O}^{(\bar{n})}](x,w) = -x^2(\bar{n}\cdot\partial_x)[\mathcal{O}^{(\bar{n})}](x,w). \tag{4.18}$$

Thus the CWI takes the form

$$\begin{aligned}\Big(2ZS_+^{(\epsilon)}Z^{-1} - x^2(\bar{n}\cdot\partial_x)\Big)\mathcal{G}(x;z,w) &= \Big\langle \delta_{\mathbf{K}}S_R\,[\mathcal{O}^{(n)}](0,z)\,[\mathcal{O}^{(\bar{n})}](x,w)\Big\rangle \\ &= 2\epsilon\int d^dy\,(\bar{n}\cdot y)\,\Big\langle \mathcal{N}(y)[\mathcal{O}^{(n)}](0,z)\,[\mathcal{O}^{(\bar{n})}](x,w)\Big\rangle\,,\end{aligned} \tag{4.19}$$

where in the second line we have discarded the term due to the BRST operator $\partial^\rho\mathcal{B}_\rho$ (4.10) as it does not contribute to gauge-invariant correlation functions. Comparison with eq. (4.13) yields the following constraint for the full conformal generator

$$\boxed{S_+\mathcal{G}(x;z,w) = Z\,S_+^{(\epsilon)}\,Z^{-1}\mathcal{G}(x;z,w) + \epsilon\int d^dy\,(\bar{n}\cdot y)\,\Big\langle \mathcal{N}(y)[\mathcal{O}^{(n)}](0,z)\,[\mathcal{O}^{(\bar{n})}](x,w)\Big\rangle} \tag{4.20}$$

The product $Z\,S_+^{(\epsilon)}\,Z^{-1}$ can be expanded (see Ref. [30]) as

$$\begin{aligned}Z\,S_+^{(\epsilon)}\,Z^{-1} &= S_+^{(\epsilon)} - \frac{1}{2}\int_0^a \frac{du}{u}\Big[\mathbb{H}(u), z_1+z_2\Big] + \ldots \\ &= S_+^{(\epsilon)} - \frac{1}{2}a\,[\mathbb{H}^{(1)}, z_1+z_2] - \frac{1}{4}a^2\,[\mathbb{H}^{(2)}, z_1+z_2] + \mathcal{O}(a^3) + \ldots\end{aligned} \tag{4.21}$$

where the ellipses denote singular $1/\epsilon$ terms. An explicit expression for the singular contributions is not needed since they must cancel in the sum of both terms on the right hand side of eq. (4.20). The remaining task is to determine the contribution due to the variation of the action. This will be done in a perturbative expansion by evaluation of the corresponding Feynman diagrams.

First we want to point out that correlation functions of elementary fields with the operator insertion $\mathcal{N}(x)$ must be finite. Hence we can express the operator insertion in terms of finite operators. The result reads [54, 25, 24, 34, 55, 47]

$$\mathcal{N}(y) = -\frac{\beta(a)}{a}\left[\mathcal{L}^{YM+gf}\right] - (\gamma_A + \gamma_g)\mathcal{E}_A - \sum_{\varphi\neq A}\gamma_\varphi\mathcal{E}_\varphi \\ + \frac{\gamma_A}{\xi}[(\partial A)^2] + r_{\bar{c}-c}\partial^\mu\mathcal{E}_\mu + r_{\mathcal{B}^\mu}\partial_\mu[\mathcal{B}^\mu], \tag{4.22}$$

where $\mathcal{E}_\varphi = [\mathcal{E}_\varphi]$ are EOM operators, $\mathcal{E}_\varphi = \varphi(y)\Big(\delta S_R/\delta\varphi(y)\Big)$ and $\partial^\mu\mathcal{E}_\mu = \mathcal{E}_{\bar{c}} - \mathcal{E}_c = \partial^\mu[\bar{c}D_\mu c - \partial_\mu\bar{c}\,c]$. The derivation of this expression can be found in Appendix D. The method explained there, however, does not work to constrain the constants $r_{\bar{c}-c}(a,\xi)$ and $r_{\mathcal{B}^\mu}(a,\xi)$. As we will see this issue will not cause any trouble, since the corresponding operators do not contribute to the CWI (4.20).

Let us analyze the individual terms in Eq. (4.22) separately.

i. The EOM contributions are simple, as they give rise to contact terms that can be evaluated using integration by parts in the path-integral

$$\Big\langle\mathcal{E}_\varphi(y)O_1\,O_2\Big\rangle = \Big\langle\varphi(y)\frac{\delta O_1}{\delta\varphi(y)}O_2\Big\rangle + \Big\langle O_1\varphi(y)\frac{\delta O_2}{\delta\varphi(y)}\Big\rangle. \tag{4.23}$$

The contribution due to the quark EOM reads

$$\Big\langle\delta_{\mathbf{K}}S_R\,[\mathcal{O}^{(n)}]\,[\mathcal{O}^{(\bar{n})}]\Big\rangle_{\mathcal{E}_q} = -2\gamma_q(z_1+z_2)\mathcal{G}(x;z,w) \;+\text{ singular terms.} \tag{4.24}$$

As it was mentioned we will drop singular terms since they must cancel in the end. The ghost EOM operator do not give any contribution to the correlator. In fact this is trivial since $\frac{\delta}{\delta c}\left([\mathcal{O}^{(n)}]\,[\mathcal{O}^{(\bar{n})}]\right) = 0$. In the case of gauge fields the situation is more peculiar due to the gauge links in the light-ray operators that produce an infinite amount of gluons. Later on we will find that this term will play a special role.

ii. The last three terms in Eq. (4.22) drop out in the CWI (4.20), as they are ghost EOM and BRST operators, see Eq. (C.6).

iii. The next contribution is due to

$$\Big\langle\delta_{\mathbf{K}}S_R\,[\mathcal{O}^{(n)}]\,[\mathcal{O}^{(\bar{n})}]\Big\rangle_{\mathcal{L}} = -\frac{\beta(a)}{a}\int d^dy\,2(\bar{n}\cdot y)\Big\langle[\mathcal{L}^{YM+gf}(y)]\,[\mathcal{O}^{(n)}]\,[\mathcal{O}^{(\bar{n})}]\Big\rangle. \tag{4.25}$$

We want to stress here again that the correlator is accompanied by an factor $-\frac{\beta(a)}{a} = 2(\epsilon+\bar{\beta})$. At the critical point $\beta(a^*) = 0$ we only need to find the divergent part of the correlation function [46, 47]. Since all three operators are renormalized, the divergent part must arise due to pair counterterms from the integration regions $y\to x$ and $y\to 0$. Thus we split

$$\Big\langle[\mathcal{L}]\,[\mathcal{O}^{(n)}]\,[\mathcal{O}^{(\bar{n})}]\Big\rangle = \Big\langle\Big[\mathcal{L}\mathcal{O}^{(n)}\mathcal{O}^{(\bar{n})}\Big]\Big\rangle - \Big\langle\mathrm{PCt}\Big(\mathcal{L}\mathcal{O}^{(n)}\Big)\,[\mathcal{O}^{(\bar{n})}]\Big\rangle \\ - \Big\langle[\mathcal{O}^{(n)}]\;\mathrm{PCt}\Big(\mathcal{L}\mathcal{O}^{(\bar{n})}\Big)\Big\rangle. \tag{4.26}$$

The first term on the r.h.s. of Eq. (4.26) can be dropped as it is finite by definition. One can show that pair counterterms for the contraction of two arbitrary operators $A(x)$ and $B(y)$, here denoted by $\mathrm{PCt}(A(x)B(y))$, have the following general structure [47]

$$\mathrm{PCt}\left(A(x)B(y)\right) = \delta(y-x)\widetilde{Z}_1\widetilde{O}^{(1)}(x) + (\partial_y^\mu \delta(y-x))\widetilde{Z}_2\widetilde{O}_\mu^{(2)}(x) + \ldots, \tag{4.27}$$

where $\widetilde{O}^{(1)}(x)$, $\widetilde{O}_\mu^{(2)}(x)$ are local operators and $\widetilde{Z}_i$ are singular coefficients. The ellipses stand for the contributions with more than one derivative acting on the δ-function. Due to the additional factor $(\bar{n}\cdot y)$ we need to keep the term with one derivative acting on the δ-function, which requires an explicit calculation. One cannot employ similar simplifications as for the SWI, where one only needs to keep the first term, which can be written in terms of the evolution kernel $\mathbb{H}$.

In the present case a representation like Eq. (4.27) is not suited for practical calculations. However, it reveals that one of the pair counterterms vanishes,

$$\int d^d y(\bar{n}y)\,\mathrm{PCt}\left(\mathcal{L}\mathcal{O}^{(\bar{n})}\right) \sim \bar{n}^2 = 0.$$

To calculate the contribution

$$-\frac{\beta(a)}{a}\int d^d y 2(\bar{n}y)\left\langle \mathrm{PCt}\left(\mathcal{L}\mathcal{O}^{(n)}\right)[\mathcal{O}^{(\bar{n})}]\right\rangle,$$

we can consider the Green's function

$$\left\langle \mathcal{O}^{(n)}q(p)\bar{q}(p')\right\rangle,$$

and insert an additional vertex $-\frac{\beta(a)}{a}\int d^d y 2(\bar{n}y)\mathcal{L}^{YM+gf}$. We will organize the contribution of a specific diagram $\mathscr{D}$ to the Green's function $\mathcal{G}(x;z,w) = \mathcal{G}_0(x;z,w) + \mathcal{O}(a)$ in the form

$$\mathrm{PCt}(\mathscr{D}) = -\mathrm{KR}'(\mathscr{D}) = Z_{\mathscr{D}}\mathcal{G}_0(x;z,w),$$

where $\mathrm{KR}'(\mathscr{D})$ denotes the pole part of a given diagram $\mathscr{D}$ with subtraction of divergent subdiagrams. The renormalization factor $Z_{\mathscr{D}}(a)$ is an integral operator acting on $z=(z_1,z_2)$ with the generic structure (3.4) $Z_{\mathscr{D}} = \sum_k \frac{Z_{\mathscr{D}}^{(1)}}{\epsilon^i}$. Once again we need to remind that there is a prefactor $\beta(a)/a = -2\big(\epsilon+\bar{\beta}(a)\big)$ and we conclude that the contribution to the conformal generator due to this term

$$-\frac{\beta(a)}{a}\int d^d y 2(\bar{n}y)\left\langle \mathrm{PCt}\left(\mathcal{L}^{YM+gf}\mathcal{O}^{(n)}\right)[\mathcal{O}^{(\bar{n})}]\right\rangle = -2\delta S_+(a)\,\mathcal{G}_0(x;z,w) + \ldots \tag{4.28}$$

is given by the simple pole of the renormalization factor

$$\boxed{\delta S_+(a) = \sum_{\mathscr{D}} Z_{\mathscr{D}}^{(1)}(a).} \tag{4.29}$$

There is one subtlety we have concealed so far. In the end one expects that $\mathcal{G}_0$ will be replaced by the full correlator

$$\mathcal{G}_0 \mapsto \mathcal{G}(a) = \mathcal{G}_0 + a\mathcal{G}_1 + \ldots$$

and one recovers the same operator $\delta S_+(a)$ in front of each coefficient. This property, however, is not true for the pair-counterterm contribution (4.28) itself, where different operators appear

$$\delta S_+(a)\,\mathcal{G}_0(x;z,w) + \delta\tilde{S}_+(a)\,\mathcal{G}_1(x;z,w) + \dots. \tag{4.30}$$

The expected form will be restored by adding the contributions due to gluon EOM operators $(\gamma_A+\gamma_g)\mathcal{E}_A$.

Finally, collecting the results from Eqs. (4.21), (4.24) and (4.28) we find an expression for the exact conformal generator at the critical point

$$\begin{aligned}
&\boxed{S_+(a_*) = S_+^{(\epsilon)} - \frac{1}{2}\int_0^{a_*}\frac{du}{u}\big[\mathbb{H}(u), z_1+z_2\big] + \gamma_q^*(z_1+z_2) + \delta S_+(a_*)} \\
&\qquad = S_+^{(0)} + \big(\gamma_q^* - \epsilon\big)(z_1+z_2) - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}a_*^k\big[\mathbb{H}^{(k)}, z_1+z_2\big] + \delta S_+(a_*) \\
&\qquad \equiv S_+^{(0)} + \Delta S_+(a_*).
\end{aligned} \tag{4.31}$$

Here we need to replace the parameter $\epsilon = (4-d)/2 \to \epsilon(a^*) = -\bar{\beta}(a_*) = -\beta_0 a_* - \beta_1 a_*^2 - \dots$ by its critical value and also the quark anomalous dimension is to be taken at this value $\gamma_q^* = \gamma_q(a_*)$. As already explained in the previous chapter we can expect that the correction term $\delta S_+(a_*)$ can be written in the form (3.37)

$$\delta S_+(a_*) = \frac{1}{2}\big[\mathbb{H}(a_*) - 2\gamma_q^*\big](z_1+z_2) + (z_1-z_2)\Delta_+(a_*), \tag{4.32}$$

where the operator Δ_+ commutes with S_- and anti-commutes with the permutation operator of quark coordinates $\mathbb{P}_{12}f(z_1,z_2) = f(z_2,z_1)$,

$$\mathbb{P}_{12}\Delta_+ = -\Delta_+\mathbb{P}_{12}. \tag{4.33}$$

The term $-\gamma_q^*(z_1+z_2)$ cancels the corresponding term in (4.31) such that the (gauge-dependent) quark anomalous dimension drops out of the final result. An explicit calculation will reveal that the structure (4.32) arises in a natural way.

Finally, at a fixed order ℓ in perturbation theory, we obtain the following expression for the correction to the generator of special conformal transformations:

$$\Delta S_+^{(\ell)} = \left(\beta_{\ell-1} + \frac{1}{2}\mathbb{H}^{(\ell)}\right)(z_1+z_2) - \frac{1}{2\ell}\big[\mathbb{H}^{(\ell)}, z_1+z_2\big] + z_{12}\Delta_+^{(\ell)}. \tag{4.34}$$

Here and throughout the rest of of this work we will use the abbreviation $z_{12} \equiv z_1 - z_2$. Comparing with the expression (3.37) we can identify

$$z_{12}\Delta^{(\ell)} = \frac{\ell-1}{2\ell}\big[\mathbb{H}^{(\ell)}, z_1+z_2\big] + z_{12}\Delta_+^{(\ell)} \tag{4.35}$$

In the next section we will explicitly derive an expression for the operator ΔS_+ up to two-loop accuracy. We will present an effective way to fix the result in the form (4.32) and determine the operator Δ_+ by an explicit calculation.

4.2 Perturbative calculation of conformal anomaly

In this section calculate the two-loop conformal generators at the critical point. As already mentioned we need to consider the diagrams associated with the correlation function

$$\int d^d y 2(\bar{n}\cdot y)\left\langle q(p_1)q(p_2)\mathcal{L}^{YM+gf}(y)\mathcal{O}^{(n)}(0;z)\right\rangle, \tag{4.36}$$

and we remind that we need only simple poles defined by the KR′-prescription.

4.2.1 Modified Feynman rules

Due to the insertion of the conformal variation δS_R into the correlation functions there appear a number of new Feynman rules in addition to the standard rules in QCD. Let us first examine in detail these new rules. First we rewrite

$$\begin{aligned}\mathcal{L}^{YM+gf} &= -\frac{1}{2}A^a_\mu K^{\mu\nu}A^a_\nu + \mathcal{L}^{YM+gf}_{\text{int}} + \frac{1+\xi}{2\xi}\partial_\mu\left(A^\mu(\partial\cdot A)\right)\\ &= -\frac{1}{2}A^a_\mu K^{\mu\nu}A^a_\nu + \mathcal{L}^{YM+gf}_{\text{int}} + \frac{1+\xi}{2}\partial_\mu\left(\bar{c}D^\mu c\right) + \text{ BRST variation },\end{aligned} \tag{4.37}$$

where $K^{\mu\nu}$ is the inverse gluon propagator

$$K^{\mu\nu} = g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu(1-\frac{1}{\xi}),$$

and we used $A^\mu(\partial\cdot A) = \xi\bar{c}D^\mu c + \text{BRST variation}$. The operator $\mathcal{L}^{YM+gf}_{\text{int}}$ contains three- and four-gluon vertices. To two-loop accuracy we can restrict ourselves to three-gluon vertices, as the four-gluon vertex generates three- and higher-loop diagrams.

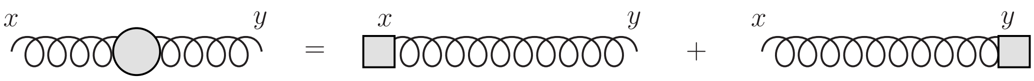
An insertion of the conformal variation $\int d^d y(\bar{n}\cdot y)\mathcal{L}^{YM+gf}(y)$ into the two-gluon vertex yields the modified gluon propagator

$$D^{\text{mod}}_{\mu\nu}(x-y) = n\cdot(x+y)D_{\mu\nu}(x-y),$$

where

$$D_{\mu\nu}(x-y) = g_{\mu\nu}\int\frac{d^d p}{(2\pi)^d}\frac{e^{ip(x-y)}}{p^2} \tag{4.38}$$

is the usual QCD gluon propagator [2]. In contrast to the standard propagator the modified one is not invariant under translations. In terms of Feynman diagrams we will present the modified propagator as follows



Here the gray boxes denote multiplication by $\bar{n}\cdot x$ and $\bar{n}\cdot y$. These factors are rather inconvenient unless they are attached to either external legs or the quark coordinates within the light-ray operator. Using

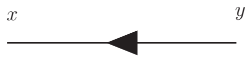
$$\bar{n}\cdot x = \bar{n}\cdot(x-y) + \bar{n}\cdot y, \tag{4.39}$$

[2] We use Feynman gauge $\xi = 1$ and Euclidean metric

we can shift these insertions in the diagrams until they are placed in a convenient position. The (shift-invariant) remnants $\bar{n} \cdot (x-y)$ can be replaced by simple derivatives $-i\bar{n} \cdot \partial_p$ in the momentum representation. For a generic propagator in the diagram we can represent this step diagrammatically by

In particular we find for the shift along gluon-, quark-propagators and gauge-links the following Feynman rules

$$= 2ig_{\mu\nu} \int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot(x-y)} \frac{(\bar{n}\cdot k)}{k^4},$$

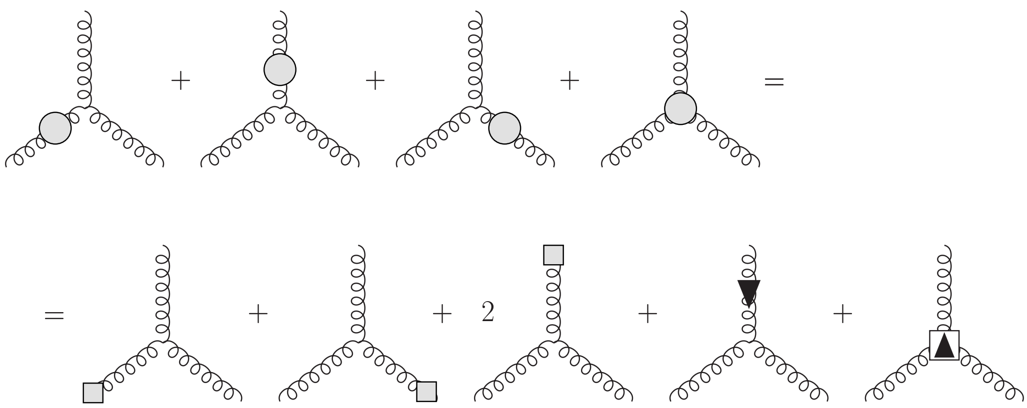


$$= -\int \frac{d^dk}{(2\pi)^d} e^{-ik\cdot(x-y)} \frac{\not{k}\not{\bar{n}}\not{k}}{k^4},$$

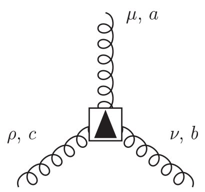
$$= z_{12}\,[z_1 n, z_2 n], \tag{4.40}$$

respectively.

In a similar manner one can derive an additional rule for the insertion of the modified three-gluon vertex. It is most convenient to consider this insertion accompanied by an insertion of the modified gluon propagator in all three attached lines. For this combination one establish the following rule:



where we have in addition to the previous rules an effective three-gluon vertex



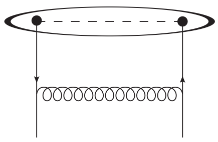
$$= g f^{abc}\Big(g^{\mu\rho}\bar{n}^{\nu} - g^{\mu\nu}\bar{n}^{\rho}\Big). \tag{4.41}$$

This vertex is defined with a distinct direction (here (μ, a)), denoted by the arrow.

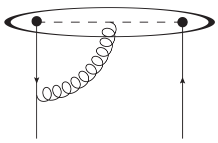
4.2.2 One-loop anomaly

To demonstrate the efficiency of our method it is most instructive to start with the one-loop calculation of the anomaly. We will see that the proposed structure (3.37) arises in a very natural way. Moreover, at leading order the result can be assembled easily from the calculation of the one-loop evolution kernel.

We start to consider the leading order diagrams for the evolution kernel. The results for the relevant diagrams [3] are known for quite some time [48] and read



$$\simeq\ \mathcal{H}^{(+)}\mathcal{O}(z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, \mathcal{O}(z_{12}^{\alpha}, z_{21}^{\beta}),$$



$$\simeq\ \hat{\mathcal{H}}_1\mathcal{O}(z_1, z_2) = \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha}\big[\mathcal{O}(z_1, z_2) - \mathcal{O}(z_{12}^{\alpha}, z_2)\big]. \tag{4.42}$$

Some details on the calculation of this Feynman diagrams can be found in Appendix E. This yields the known expression for the one-loop evolution kernel

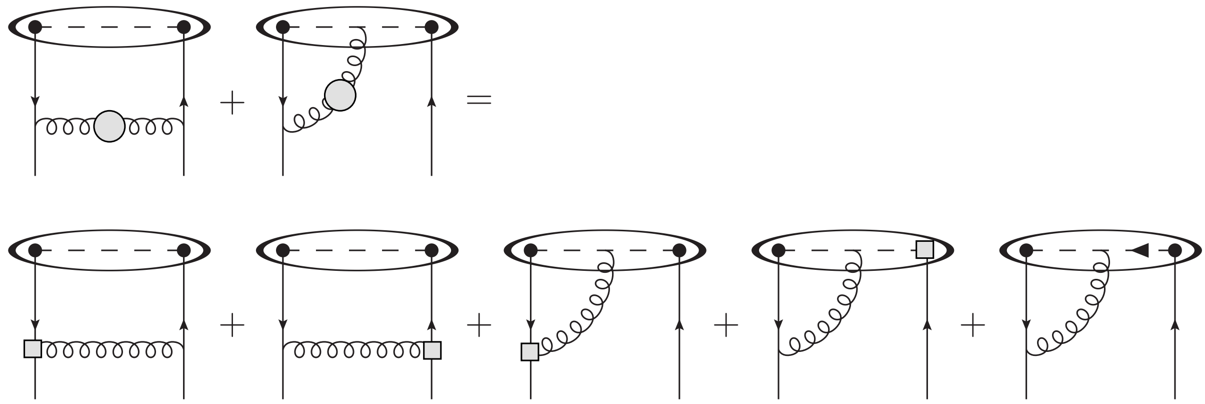
$$\mathbb{H}^{(1)} - 2\gamma_q = 4C_F\Big\{\hat{\mathcal{H}} - \mathcal{H}^{(+)}\Big\}, \tag{4.43}$$

where for the "single-particle operator"

$$\hat{\mathcal{H}}\mathcal{O}(z_1, z_2) = (\hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2)\mathcal{O}(z_1, z_2) = \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha}\big[2\mathcal{O}(z_1, z_2) - \mathcal{O}(z_{12}^{\alpha}, z_2) - \mathcal{O}(z_1, z_{21}^{\alpha})\big] \tag{4.44}$$

one needs to take into account also the mirror diagram with the gluon attached to the incoming quark. To leading accuracy the correction to the conformal generators δS_+ is given by the same two diagrams with insertions of the modified gluon propagator

[3] In all Feynman diagrams the light-ray operator is surrounded by an ellipse, the left point is z_1 and the right point z_2.



Assuming a diagram $\mathscr{D}$ (with ℓ loops) without any arrows or gray boxes gives a simple pole $\mathbb{Z}^{(\mathscr{D})} = \frac{\mathbb{H}^{(\mathscr{D})}}{2\ell}$, the recipe to receive the contribution to the anomaly is simple:

- diagrams with a gray box at the point z_i give a contribution

$$\delta S_+ = \frac{z_i \mathbb{H}^{(\mathscr{D})}}{2\ell} \tag{4.45}$$

- diagrams with a gray box attached to the external quark line give a contribution

$$\delta S_+ = \frac{\mathbb{H}^{(\mathscr{D})} z_i}{2\ell} \tag{4.46}$$

- all diagrams with arrows require an explicit calculation. (The case of an arrow on the gauge link is trivial as will be shown soon.)

The diagrams one to four in the second line obviously give a contribution

$$\delta S_+ \simeq -\mathcal{H}^{(+)}(z_1+z_2) + \hat{\mathcal{H}}_1 z_1 + z_2 \hat{\mathcal{H}}_1 = (-\mathcal{H}^{(+)} + \hat{\mathcal{H}}_1)(z_1+z_2), \tag{4.47}$$

where we used that $\hat{\mathcal{H}}_1$ does not act on the light-cone point z_2, i.e. $[\hat{\mathcal{H}}_1, z_2] = 0$. Together with the mirror diagram the correction reads

$$\delta S_+ = \mathbb{H}(z_1+z_2) + z_{12}\Delta_+, \tag{4.48}$$

where the operator Δ_+ is determined by the fifth diagram and its mirror counterpart. Let us examine this diagram in detail. To this end let us remind the definition of the gauge link

$$[z_1 n, z_2 n] = \mathrm{Pexp}\Big\{ igz_{12} \int_0^1 du\, n\cdot A(z_{21}^u) \Big\}. \tag{4.49}$$

An explicit derivation of the "single-particle kernel" $\hat{\mathcal{H}}$, see appendix E, shows that the gauge-link integration $\int du$ enters with an total derivative as integrand

$$\hat{\mathcal{H}}_1 \mathcal{O}(z_1,z_2) = \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha} \int_0^1 du\, \partial_u \mathcal{O}(z_{12}^{\alpha\bar{u}}, z_2), \tag{4.50}$$

which after integration gives rise to the typical structure of the δ_+-distribution $\sim \mathcal{O}(z_1, z_2) - \mathcal{O}(z_{12}^\alpha, z_2)$. According the rules from the previous section we get an additional line

$$(z_{21}^u - z_2)[z_{21}^u n, z_2 n] = u z_{12}[z_{21}^u n, z_2 n], \tag{4.51}$$

and thus the fifth diagram plus the symmetric one give rise to the contribution

$$\begin{aligned}
\delta S_+ \mathcal{O}(z_1, z_2) \simeq & z_{12} \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha} \int_0^1 du \big[u \partial_u \mathcal{O}(z_{12}^{\alpha\bar{u}}, z_2) + \bar{u}\partial_u \mathcal{O}(z_1, z_{21}^{\alpha u})\big] \\
= & z_{12} \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha} \int_0^1 du u \partial_u \big[\mathcal{O}(z_{12}^{\alpha\bar{u}}, z_2) - \mathcal{O}(z_1, z_{21}^{\alpha\bar{u}})\big] \\
= & - z_{12} \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha} \int_0^1 du \big[\mathcal{O}(z_{12}^{\alpha\bar{u}}, z_2) - \mathcal{O}(z_1, z_{21}^{\alpha\bar{u}})\big].
\end{aligned} \tag{4.52}$$

The boundary terms from partial integration cancel each other. This result can be easily generalized to any order in perturbation theory: Consider a ℓ-loop diagram $\mathscr{D}$ with only soft interaction (no interaction between the quark lines). Its contribution takes the form

$$\mathbb{H}^{(\mathscr{D})} \mathcal{O}(z_1, z_2) = \int_0^1 d\alpha h^{(\mathscr{D})}(\alpha) \big[2\mathcal{O}(z_1, z_2) - \mathcal{O}(z_{12}^\alpha, z_2) - \mathcal{O}(z_1, z_{21}^\alpha)\big]. \tag{4.53}$$

The relevant contributions to the conformal anomaly due to the corresponding diagrams with arrows on the gauge links read then

$$\delta S_+^{(\mathscr{D})} \mathcal{O}(z_1, z_2) = -\frac{z_{12}}{2\ell} \int_0^1 d\alpha h^{(\mathscr{D})}(\alpha) \int_0^1 du \big[\mathcal{O}(z_{12}^{\alpha\bar{u}}, z_2) - \mathcal{O}(z_1, z_{21}^{\alpha\bar{u}})\big]. \tag{4.54}$$

Re-scaling the arguments one of the integrations can be separated (if necessary)

$$\begin{gathered}
\int_0^1 d\alpha h^{(\mathscr{D})}(\alpha) \int_0^1 du \big[\mathcal{O}(z_{12}^{\alpha\bar{u}}, z_2) - \mathcal{O}(z_1, z_{21}^{\alpha\bar{u}})\big] = \\
\int_0^1 d\alpha \left(\int_\alpha^1 \frac{du}{u} h^{(\mathscr{D})}(u)\right) \big[\mathcal{O}(z_{12}^\alpha, z_2) - \mathcal{O}(z_1, z_{21}^\alpha)\big].
\end{gathered}$$

Finally we collect all contributions and recover indeed the result in the expected form (4.32)

$$\delta S_+^{(1)} = \frac{1}{2}(\mathbb{H}^{(1)} - 2\gamma_q)(z_1 + z_2) + z_{12}\Delta_+^{(1)}, \tag{4.55}$$

with the conformal anomaly

$$\begin{aligned}
\Delta_+^{(1)} \mathcal{O}(z_1, z_2) = & - 2C_F \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha} \int_0^1 du \big[\mathcal{O}(z_{12}^{\alpha\bar{u}}, z_2) - \mathcal{O}(z_1, z_{21}^{\alpha\bar{u}})\big] \\
= & - 2C_F \int_0^1 d\alpha \left(\frac{\bar{\alpha}}{\alpha} + \ln(\alpha)\right) \big[\mathcal{O}(z_{12}^\alpha, z_2) - \mathcal{O}(z_1, z_{21}^\alpha)\big].
\end{aligned} \tag{4.56}$$

The exact conformal generator (at the critical point) to one-loop accuracy reads then

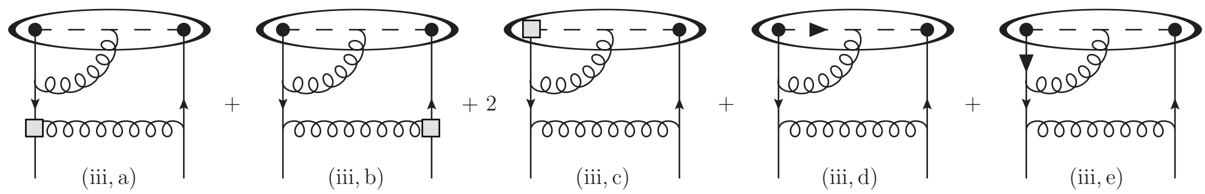
$$S_+ = S_+^{(0)} + a_* \Big\{(z_1 + z_2)\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) + z_{12}\Delta_+^{(1)}\Big\} + \mathcal{O}(a_*^2). \tag{4.57}$$

4.2.3 Two-loop anomaly

The calculation of the corrections to the conformal generators at NNLO goes along the same lines.

- First we consider all diagrams which contribute to the NNLO evolution kernel $\mathbb{H}^{(2)}$. A list of all diagrams entering the two-loop evolution kernel is shown in Fig. 4.1. The individual answers are collected in appendix F.
- From each diagram in Fig. 4.1 we generate several new ones, by replacing one after another each gluon line, ghost line and three-gluon vertex by the modified Feynman rules from section 4.2.1.
- We move all gray boxes to the external points. This choice is motivated by the expected form (4.32).
- We calculate all diagrams with arrows on the propagators explicitly.

This straightforward procedure will be illustrated once again on an example. Let us consider the diagram (iii) in Fig. 4.1 . Replacing the gluon lines and performing some rearrangements results in the sum of the following five diagrams



Taking into account that there is also a symmetric contribution we find for the contribution of diagrams (iii,a-c)

$$\delta S_+^{(\mathrm{iii,a-c})} = \frac{1}{4}\left\{\mathbb{H}^{(\mathrm{iii})}, z_1+z_2\right\} = \frac{1}{2}\mathbb{H}^{(\mathrm{iii})}(z_1+z_2) + z_{12}\Delta_+^{(\mathrm{iii,a-c})}, \tag{4.58}$$

where $\mathbb{H}^{(iii)}$ is the contribution to the evolution kernel from this diagram. Moving the factor z_1+z_2 to the right of the evolution kernel (in diagrammatic language, moving the gray boxes from the operator to the external legs) causes an additional term due to the commutator

$$z_{12}\Delta_+^{(\mathrm{iii,a-c})} = -\frac{1}{4}[\mathbb{H}^{(\mathrm{iii})}, z_1+z_2], , \tag{4.59}$$

There are two additional contributions to the operator $\Delta_+^{(\mathrm{iii})}$ from the last two diagrams. They both contain lines with an arrow, i.e. with the modified Feynman rules (4.40), and therefore require a separate calculation. The answer reads

$$\begin{aligned}
\Delta_+^{(\mathrm{iii,c/d})}\mathcal{O}(z_1,z_2) =& 2C_F^2\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\left(w^{(\mathrm{iii,d/e})}(\alpha) - w^{(\mathrm{iii,d/e})}(\beta)\right)\mathcal{O}(z_{12}^\alpha, z_{21}^\beta),\\
w^{(\mathrm{iii,d})}(\alpha) =& 2\ln\bar\alpha + (\frac{1}{\bar\alpha}-4)\ln\alpha - \frac{1}{2\bar\alpha}\ln^2\alpha - \frac{2}{\bar\alpha}\left(\mathrm{Li}_2(\alpha)-\mathrm{Li}_2(1)\right)\\
w^{(\mathrm{iii,e})}(\alpha) =& \frac{1}{2}\ln^2\bar\alpha - (1+\alpha)\ln\bar\alpha + \alpha\ln\alpha + \alpha,
\end{aligned}\tag{4.60}$$

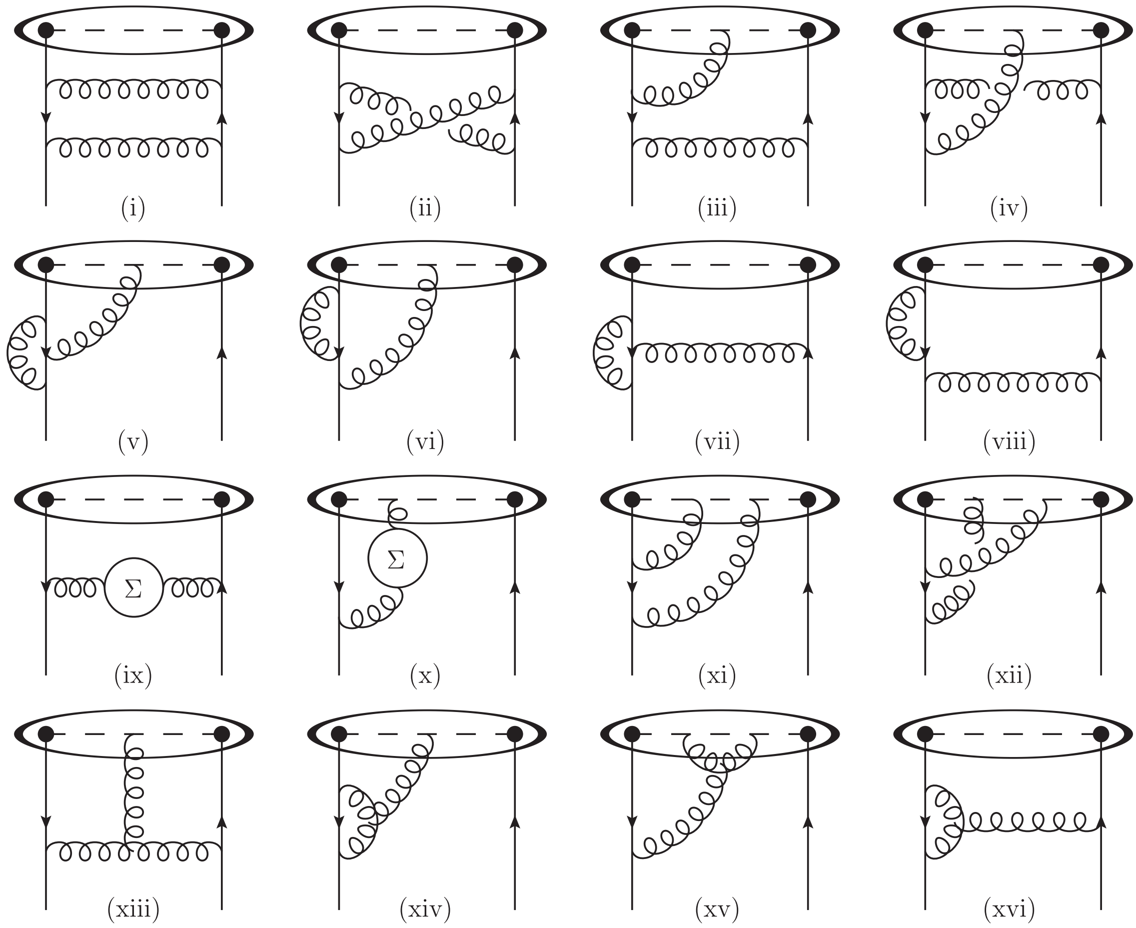
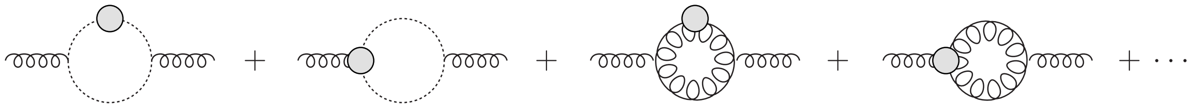


Figure 4.1: All Feynman diagrams contribution to the two-loop evolution kernel $\mathbb{H}^{(2)}$. The Σ-circle denotes the sum of quark-, gluon- and ghost-loops.

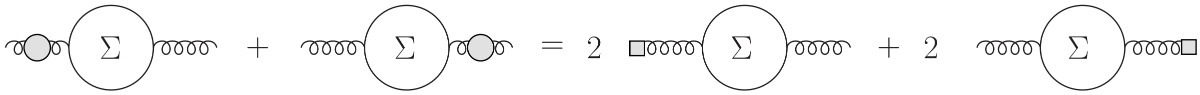
Collecting all findings we obtain as complete contribution from this diagram to the operator Δ_+

$$\begin{aligned}
\Delta_+^{(\mathrm{iii})}\mathcal{O}(z_1,z_2) &= \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \left[w^{(\mathrm{iii})}(\alpha,\beta) - w^{(\mathrm{iii})}(\alpha,\beta)\right] O(z_{12}^{\alpha}, z_{21}^{\beta}) \\
w^{(\mathrm{iii})}(\alpha,\beta) &= 2C_F^2\left(w^{(\mathrm{iii,d})}(\alpha) + w^{(\mathrm{iii,e})}(\alpha)\right) + \frac{1}{4}(\alpha-\beta)h^{(\mathrm{iii})}(\alpha) \\
&= 2C_F^2\bigg\{(\beta-\alpha)\bigg[4 + 3\ln\alpha - 2\ln\bar\alpha + \frac{1}{2}\ln^2\alpha \\
&\qquad + \frac{1}{2}\ln^2\bar\alpha + 2\big(\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(1)\big)\bigg] - \frac{2}{\bar\alpha}\Big[\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(1)\Big] \\
&\qquad - \frac{1}{2\bar\alpha}\ln^2\alpha + \frac{1}{2}\ln^2\bar\alpha + \left[\frac{\alpha}{\bar\alpha} - 3 + \alpha\right]\ln\alpha + \bar\alpha\ln\bar\alpha + \alpha\bigg\}, \qquad (4.61)
\end{aligned}$$

The answers for all other diagrams can be worked out in the same manner. There is one simplification that is worth to be mentioned: The third term in Eq. (4.37) gives rise to corrections to ghost-lines and ghost-gluon vertices. To two-loop accuracy ghosts appear only as self-energy insertions in the gluon propagator. There are four (plus four symmetric) possibilities to modify this vacuum energy insertion



An explicit calculation shows that in the sum these corrections cancel each other, thus insertions into self-energy diagrams can be ignored. For the insertion into one of the gluon lines one finds



where the Σ-circle is the sum of quark-, gluon- and ghost-loops.

Collecting the results for all topologies in figure 4.1 we find the following expression for the two-loop conformal anomaly:

$$\begin{aligned}\Delta_+^{(2)}\mathcal{O}(z_1,z_2) &= \int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\Big[w(\alpha,\beta)+w^{\mathbb{P}}(\alpha,\beta)\mathbb{P}_{12}\Big]\Big[\mathcal{O}(z_{12}^\alpha,z_{21}^\beta)-\mathcal{O}(z_{12}^\beta,z_{21}^\alpha)\Big]\\ &\quad+\int_0^1 du\int_0^1 dt\, v(t)\,\Big[\mathcal{O}(z_{12}^{ut},z_2)-\mathcal{O}(z_1,z_{21}^{ut})\Big].\end{aligned}\tag{4.62}$$

We split the kernel functions $v(t)$, $w(\alpha,\beta)$, $w^{\mathbb{P}}(\alpha,\beta)$ into contributions of three different color structures

$$\begin{aligned}v(t) &= C_F^2\, v_{ff}(t)+C_FC_A\, v_{fA}(t)+C_F\beta_0 v_{bF}(t),\\ w(\alpha,\beta) &= C_F^2\, w_{ff}(\alpha,\beta)+C_FC_A\, w_{fA}(\alpha,\beta),\\ w^{\mathbb{P}}(\alpha,\beta) &= C_F^2\, w_{ff}^{\mathbb{P}}(\alpha,\beta)+C_FC_A\, w_{fA}^{\mathbb{P}}(\alpha,\beta).\end{aligned}\tag{4.63}$$

This splitting is ambiguous, another common choice is to separate the contributions of planar diagrams and the non-planar $1/N_c$ suppressed corrections

$$\begin{aligned}v(t) &= C_F^2\, v_{\mathrm{P}}(t)+\frac{C_F}{N_C}\, v_{fA}(t)+C_F\beta_0 v_{bF}(t),\\ w(\alpha,\beta) &= C_F^2\, w_{\mathrm{P}}(\alpha,\beta)+\frac{C_F}{N_C} w_{fA}(\alpha,\beta),\\ w^{\mathbb{P}}(\alpha,\beta) &= \frac{C_F}{N_C} w_{fA}^{\mathbb{P}}(\alpha,\beta),\end{aligned}\tag{4.64}$$

with the connection $v_{\mathrm{P}}=v_{ff}+2v_{fA}$ etc.. Note that the terms involving quark permutations on the light cone do not receive planar contributions, $w_{fA}^{\mathbb{P}}=-\frac{1}{2}w_{ff}^{\mathbb{P}}$. The term $\sim\beta_0$ in $v(t)$ arises by rewriting the contribution proportional to the number of quark flavors n_f in terms of β_0. In contrast, the terms $(z_1+z_2)\beta_{\ell-1}$ in the expression for $\Delta S_+^{(\ell)}$ involve the "genuine" QCD β-function.

Explicit expressions for the "two-particle" kernels $w, w^{\mathbb{P}}$ are:

$$
\begin{aligned}
w_{ff}(\alpha,\beta) &= 4\Bigg[\left(\alpha-\frac{1}{\alpha}\right)\left[\mathrm{Li}_2\left(\frac{\beta}{\bar\alpha}\right)-\mathrm{Li}_2(\beta)-\mathrm{Li}_2(\alpha)-\frac{1}{4}\ln^2\bar\alpha\right]-\alpha\Big[\mathrm{Li}_2(\alpha)-\mathrm{Li}_2(1)\Big]\\
&\quad-\frac{\alpha+\beta}{2}\ln\alpha\ln\bar\alpha+\frac{1}{4}\left(\beta\ln^2\bar\alpha-\alpha\ln^2\alpha\right)-\frac{\alpha}{\tau}\Big(\tau\ln\tau+\bar\tau\ln\bar\tau\Big)\\
&\quad+\frac{1}{4}\left(\beta-2\bar\alpha+\frac{2\beta}{\alpha}\right)\ln\bar\alpha+\frac{1}{2}\left(\bar\alpha-\frac{\alpha}{\bar\alpha}-3\beta\right)\ln\alpha-\frac{15}{4}\alpha\Bigg],\\
w_{fA}(\alpha,\beta) &= 2\Bigg[\left(\frac{1}{\alpha}-\alpha\right)\left[\mathrm{Li}_2\left(\frac{\beta}{\bar\alpha}\right)-\mathrm{Li}_2(\beta)-2\,\mathrm{Li}_2(\alpha)-\ln\alpha\ln\bar\alpha\right]\\
&\quad+\frac{\alpha}{\tau}\Big(\tau\ln\tau+\bar\tau\ln\bar\tau\Big)-\bar\beta\ln\alpha-\frac{\bar\alpha}{\alpha}\ln\bar\alpha\Bigg],\\
w^{\mathbb{P}}_{fA}(\alpha,\beta) &= 2\Bigg[\left(\bar\alpha-\frac{1}{\bar\alpha}\right)\left[\mathrm{Li}_2\left(\frac{\alpha}{\bar\beta}\right)-\mathrm{Li}_2(\alpha)-\ln\bar\alpha\ln\bar\beta\right]+\alpha\bar\tau\ln\bar\tau+\frac{\beta^2}{\bar\beta}\ln\bar\alpha\Bigg],
\end{aligned}
\tag{4.65}
$$

and for the planar combination $w_{\mathrm{P}} = w_{ff} + 2w_{fA}$

$$
\begin{aligned}
w_{\mathrm{P}}(\alpha,\beta) &= \frac{4}{\alpha}\Big[\mathrm{Li}_2(\bar\alpha)-\mathrm{Li}_2(1)\Big]+\frac{1}{\alpha}\ln^2\bar\alpha-(\alpha-\beta)\ln^2\left(\frac{\alpha}{\bar\alpha}\right)-\beta\ln^2\alpha\\
&\quad+2\alpha\left(\frac{\pi^2}{3}-\frac{15}{2}\right)-2\left(\alpha+\beta+\frac{1}{\bar\alpha}\right)\ln\alpha+\Big(\beta-2\bar\alpha\Big)\left(1+\frac{2}{\alpha}\right)\ln\bar\alpha.
\end{aligned}
\tag{4.66}
$$

For the "one-particle" kernels $v(t)$ we obtain

$$
\begin{aligned}
v_{bF}(t) &= -2\frac{\bar t}{t}\Big(\ln\bar t+\frac{5}{3}\Big),\\
v_{fA}(t) &= \frac{2\bar t}{t}\left\{(2+t)\Big[\mathrm{Li}_2(\bar t)-\mathrm{Li}_2(t)\Big]-(2-t)\Big(\frac{t}{\bar t}\ln t+\ln\bar t\Big)-\frac{\pi^2}{6}t-\frac{4}{3}-\frac{t}{2}\left(1-\frac{t}{\bar t}\right)\right\},\\
v_{\mathrm{P}}(t) &= 4\bar t\Big[\mathrm{Li}_2(\bar t)-\mathrm{Li}_2(1)\Big]+4\left(\frac{t^2}{\bar t}-\frac{2\bar t}{t}\right)\Big[\mathrm{Li}_2(t)-\mathrm{Li}_2(1)\Big]-2t\ln t\ln\bar t-\frac{\bar t}{t}(2-t)\ln^2\bar t\\
&\quad+\frac{t^2}{\bar t}\ln^2 t-2\left(1+\frac{1}{t}\right)\ln\bar t-2\left(1+\frac{1}{\bar t}\right)\ln t-\frac{16}{3}\frac{\bar t}{t}-1-5t.
\end{aligned}
\tag{4.67}
$$

The last expression can also be rewritten as

$$
\begin{aligned}
v_{\mathrm{P}}(t) &= -4\bar t\,\mathrm{Li}_2(1)+4\left(\frac{1}{\bar t}-\frac{2}{t}\right)\Big[\mathrm{Li}_2(t)-\mathrm{Li}_2(1)\Big]-2(2-t)\ln t\ln\bar t-\frac{\bar t}{t}(2-t)\ln^2\bar t\\
&\quad+\frac{t^2}{\bar t}\ln^2 t-2\left(1+\frac{1}{t}\right)\ln\bar t-2\left(1+\frac{1}{\bar t}\right)\ln t-\frac{16}{3}\frac{\bar t}{t}-1-5t.
\end{aligned}
\tag{4.68}
$$

For sake of brevity we do not give a separate expression for $v_{ff}(t) = v_{\mathrm{P}}(t) - 2v_{fA}(t)$.

Note that the "single-particle" contribution to Δ_+ (second line in (4.62)) can be rewritten as a single integration (by a simple re-scaling $ut \to t$) with the kernel function $\nu(t) = \int_t^1 \frac{du}{u} v(u)$. However, the resulting function $\nu(t)$ is somewhat more complicated, of course.

This completes our results for the conformal generator to two-loop accuracy, to wit we have found:

$$
\boxed{S_+(a_*) = S_+^{(0)} + a_*\Delta S_+^{(1)} + a_*^2\Delta S_+^{(2)} + \mathcal{O}(a_*^3),}
\tag{4.69}
$$

with the anomaly term

$$\boxed{\begin{aligned}\Delta S_+^{(1)} =&(z_1+z_2)(\frac{1}{2}\beta_0+\mathbb{H}^{(1)})+z_{12}\Delta_+^{(1)},\\ \Delta S_+^{(2)} =&(z_1+z_2)(\frac{1}{2}\beta_1+\mathbb{H}^{(2)})+\frac{1}{4}[\mathbb{H}^{(2)},z_1+z_2]+z_{12}\Delta_+^{(2)}.\end{aligned}} \tag{4.70}$$

The explicit expressions for the operators $\Delta_+^{(1,2)}$ are given in Eqs. (4.56) and (4.62). Expressions for the evolution kernels $\mathbb{H}^{(i)}$ at one-loop ($i=1$) and two-loop ($i=2$) will be derived in the next chapter. In order to match the form proposed in Eq. (3.37) we can identify

$$z_{12}\Delta^{(1)} = z_{12}\Delta_+^{(1)}, \qquad\qquad z_{12}\Delta^{(2)} = z_{12}\Delta_+^{(2)} + \frac{1}{4}[\mathbb{H}^{(2)}, z_1+z_2]. \tag{4.71}$$

Chapter 5

Evolution equations to NNLO

5.1 A short digression on the history of evolution equations

The idea to study the scale dependence of physical quantities in QFT was anticipated already in the 50's by Stückelberg and Petermann [56]. In the very beginning of the 70‘s it was Wilson [57] who inspired the notion of renormalization group. At the same time Callan [58] and Symanzik [59] proposed a very powerful formalism to investigate the scaling behavior – the framework of *renormalization group equation (RGE)*, often also referred to as "Callan-Symanzik-equation". It is most natural to start to study the evolution, and connected to it the anomalous dimensions, of the basic ingredients of the Lagrangian: the canonical fields, their masses and the couplings of the theory, which give rise to the famous β-functions. Correspondingly these quantities are nowadays determined to a very high accuracy, e.g. the β-function in QCD is known up to five-loops [60, 61, 62].

However, understanding the basic ingredients of a theory is not enough. Once one combines the canonical fields to so-called "composite operators", additional divergences arise which are not covered by the renormalization of the basic constituents. Thus in the full theory a sheer amount of objects needs to be investigated.

The leading-twist two-particle operator introduced in Eq. (3.15) is an example for such a composite object. The first formulation of its scaling behavior was due to Dokshitzer [63], Gribov, Lipatov [64], Altarelli and Parisi [65] (DGLAP) in the 70‘s. They considered the evolution for parton distributions, that are defined by the forward matrix elements of the operators. A few years later Yndurain et. al. [66, 67] extended this study to the NLO. Around the turn of the century Larin, van Ritbergen [10], Moch, Vermaseren and Vogt [9] investigated the forward-kinematics to the NNLO. Since then a number of authors, including Velizhanin [11, 12, 13], Baikov, Chetyrkin and Kühn [14] and Moch, Ruijl, Ueda, Vermaseren, Vogt and Davies [15, 16, 17], extended this study to obtain (partial) non-singlet results to the four-loop accuracy.

The first formulation for non-forward kinematics, describing the evolution of exclusive processes was given by the ERBL-equation (Efremov, Radyushkin [68, 69], Brodsky, Lepage [70]). The NLO calculation of the off-diagonal elements of the evolution equation of local operators was performed in the middle of the 80‘s by a number of authors, all of them limiting themselves to results in the non-singlet sector [18, 19, 20, 21, 22, 50]. During the 90‘s Dieter Müller revived this topic using a more sophisticated method based on conformal symmetry [23]. In collaboration with Belitsky he managed to describe the full mixing between gluonic and fermionic operators in both the local representation as well as in momentum representation [25, 24]. The coordinate space representation of the LO evolution equation was formulated by Balitsky and Braun [48]. Recently this was extended to NLO accuracy [31], using an approach equivalent to the one by D. Müller. In this work we will continue this project to the three-loop accuracy. On the way we will demonstrate the equivalence to the method by Müller. Based on that we will also present NNLO results for the evolution of local operators.

5.2 Details of the method

Once more it is worth to explain the approach in detail. Using CWI we have found three operators

$$\begin{aligned} S_-(a) =& S_-^{(0)}, \\ S_0(a) =& S_0^{(0)} + \left(\bar{\beta}(a) + \frac{1}{2}\mathbb{H}(a)\right), \\ S_+(a) =& S_+^{(0)} + a\Delta S_+^{(1)} + a^2\Delta S_+^{(2)} + \mathcal{O}(a^3), \end{aligned} \tag{5.1}$$

which are supposed to commute with the evolution kernel at the critical point

$$[S_\alpha(a_*), \mathbb{H}(a_*)] = 0, \tag{5.2}$$

up to four-loop corrections. The operators $S_-(a)$ and $S_0(a)$, which are fixed to any order in perturbation theory, do not yield any non-trivial information. It can be easily seen by examination of the commutator (5.2) with the evolution kernel in the generic form (3.25).

Expanding (5.2) for $S_\alpha = S_+$ in a powers of the strong coupling and comparing each order gives the set of equations (3.42)

$$[S_+^{(0)}, \mathbb{H}^{(1)}] = 0, \tag{5.3a}$$

$$[S_+^{(0)}, \mathbb{H}^{(2)}] = [\mathbb{H}^{(1)}, \Delta S_+^{(1)}], \tag{5.3b}$$

$$[S_+^{(0)}, \mathbb{H}^{(3)}] = [\mathbb{H}^{(2)}, \Delta S_+^{(1)}] + [\mathbb{H}^{(1)}, \Delta S_+^{(2)}]. \tag{5.3c}$$

This system of equations can be solved iteratively, i.e. the ℓ-loop evolution kernel $\mathbb{H}^{(\ell)}$ on the l.h.s. is determined by lower-order operator $\mathbb{H}^{(k)}, \Delta S_+^{(k)},\ k \le \ell - 1$. We want to emphasize that these equations constrain the evolution kernel only up to so-called canonically invariant contributions, i.e. solutions of the homogeneous equation $[S_+^{(0)}, \mathbb{H}_{\text{inv}}] = 0$. Therefore it is most sensible to split the full evolution kernel into a canonically invariant and a non-invariant part

$$\mathbb{H} = \mathbb{H}_{\text{inv}} + \mathbb{H}_{\text{ninv}}. \tag{5.4}$$

We can replace the evolution kernel on the l.h.s. of Eqs. (5.3) by the non-invariant part

$$[S_+^{(0)}, \mathbb{H}^{(\ell)}] \equiv [S_+^{(0)}, \mathbb{H}^{(\ell)}_{\text{ninv}}]. \tag{5.5}$$

The requirement that the spectrum of the evolution kernel equals the forward anomalous dimensions of the local leading-twist operators

$$\mathbb{H}z_{12}^N = (\mathbb{H}_{\text{inv}} + \mathbb{H}_{\text{ninv}})z_{12}^N = \big(\gamma_{\text{inv}}(N) + \gamma_{\text{ninv}}(N)\big)z_{12}^N = \gamma(N)z_{12}^N, \tag{5.6}$$

turns out to be sufficient to fix the SL(2)-invariant part of the kernel. The anomalous dimensions are known up to three-loop accuracy [9], some fixed moments even to four-loops [11, 12, 13, 14, 15, 16, 17].

The evolution kernel can be written the generic form (3.25)

$$\begin{aligned}\mathbb{H}\mathcal{O}(z_1,z_2) =& H_{\text{const}}\mathcal{O}(z_1,z_2) + \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta h(\alpha,\beta)\mathcal{O}(z_{12}^\alpha, z_{21}^\beta)\\ &+ \int_0^1 d\alpha \frac{\bar\alpha}{\alpha} h^\delta(\alpha)\Big(2\mathcal{O}(z_1,z_2) - \mathcal{O}(z_{12}^\alpha,z_2) - \mathcal{O}(z_1,z_{21}^\beta)\Big)\,. \end{aligned} \tag{5.7}$$

The commutator on the l.h.s. of Eq. (5.3) can be easily transformed into a differential equation on the kernel functions $h(\alpha,\beta), h^\delta(\alpha)$. We find

$$\begin{aligned}[S_+^{(0)},\mathbb{H}] =& z_{12}\int_0^1 d\alpha \int_0^1 d\beta \left(\alpha\bar\alpha\partial_\alpha - \beta\bar\beta\partial_\beta\right) h(\alpha,\beta)\mathcal{O}(z_{12}^\alpha, z_{21}^\beta)\\ &+ z_{12}\int_0^1 d\alpha\bar\alpha^2\partial_\alpha h^\delta(\alpha)\Big(\mathcal{O}(z_{12}^\alpha,z_2) - \mathcal{O}(z_1,z_{21}^\beta)\Big)\,. \end{aligned} \tag{5.8}$$

The straightforward way to deal with the Eqs. (5.3) is to write the r.h.s. as

$$[\mathbb{H}^{(i)},\Delta S_+^{(j)}] = z_{12}\int_0^1 d\alpha \int_0^1 d\beta \Big[f^{(ij)}(\alpha,\beta) - f^{(ij)}(\beta,\alpha)\Big]\mathcal{O}(z_{12}^\alpha, z_{21}^\beta) \tag{5.9}$$

$$+z_{12}\int_0^1 d\alpha g^{(ij)}(\alpha)\Big(\mathcal{O}(z_{12}^\alpha,z_2) - \mathcal{O}(z_1,z_{21}^\beta)\Big)\,. \tag{5.10}$$

and solve the inhomogeneous differential equations

$$\left(\alpha\bar\alpha\partial_\alpha - \beta\bar\beta\partial_\beta\right) h_{\text{ninv}}^{(\ell)}(\alpha,\beta) = \sum_{i+j=\ell} f^{(ij)}(\alpha,\beta) - f^{(ij)}(\beta,\alpha), \tag{5.11a}$$

$$\bar\alpha^2\partial_\alpha h_{\text{ninv}}^{\delta,(\ell)}(\alpha) = \sum_{i+j=\ell} g^{(ij)}(\alpha), \tag{5.11b}$$

to find the non-invariant kernels.

The invariant kernels are the solutions of the homogeneous equation

$$\begin{aligned}\left(\alpha\bar\alpha\partial_\alpha - \beta\bar\beta\partial_\beta\right) h_{\text{inv}}(\alpha,\beta) = 0, \quad &\Longleftrightarrow \quad h_{\text{inv}}(\alpha,\beta) = f_{\text{inv}}\left(\frac{\alpha\beta}{\bar\alpha\bar\beta}\right) = f_{\text{inv}}(\tau),\\ \bar\alpha^2\partial_\alpha h_{\text{inv}}^{\delta,(\ell)}(\alpha) = 0, \quad &\Longleftrightarrow \quad h_{\text{inv}}^{\delta,(\ell)}(\alpha) = \text{ const }. \end{aligned} \tag{5.12}$$

As already mentioned these functions needs to be adjusted in such a way that the spectrum of evolution kernel reproduces the known anomalous dimensions

$$\int d\alpha d\beta h_{\text{inv}}(\tau)(1-\alpha-\beta)^N = \gamma(N) - \gamma_{\text{ninv}}(N) = \gamma_{\text{inv}}(N). \tag{5.13}$$

This can be done by means of the inversion formula

$$h_{\text{inv}}(\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dN\gamma_{\text{inv}}(N)(2N+1)P_N\left(\frac{1+\tau}{1-\tau}\right). \tag{5.14}$$

where $P_N(x)$ is the Legendre function. The integration contour goes along the imaginary axis and all poles of $\gamma_{\rm inv}(N)$ lie to the left of it.

Up to NLO this method works out pretty well, see Ref. [31]. At NNLO we will face two major problems: the algebra required to obtain Eq. (5.10) and consequently solve Eq. (5.11) turns out to be rather cumbersome. Moreover, an analytic solution of Eq. (5.14) is (at present) not feasible at all.

The approach that we are going to explain in the upcoming two subsections allows for a solution of both problems. The (non-invariant part of the) evolution kernel will be represented as product of simpler, i.e. lower-order operators $\mathbb{H}^{(\ell)}_{\rm ninv} \simeq \mathbb{A}^{(\ell-i)}\mathbb{B}^{(i)}$. This ansatz appears promising as the r.h.s. of Eqs. (5.3) are given as products, too. In addition the invariant part of the kernel, that is defined through this solution, turns out to be a special one: it is a function of the canonical Casimir operator (2.13).

5.2.1 Similarity transformation

As it was shown in Sec. 4.2.3 the correction to the generator S_+ has the form (3.37)

$$\Delta S_+(a) = (z_1+z_2)\left(\beta(a) + \frac{1}{2}\mathbb{H}(a)\right) + z_{12}\Delta(a). \tag{5.15}$$

The first term on the r.h.s. is fixed to any order in terms of the evolution kernel, the anomaly operator $\Delta(a)$ needs a separate calculation. It is worthwhile to separate these two contributions. To this end we construct a transformation

$$\mathbb{U} : [\mathcal{O}(z_1,z_2)] \mapsto [\mathcal{O}(z_1,z_2)]^{\mathbb{U}} = \mathbb{U}[\mathcal{O}(z_1,z_2)], \tag{5.16}$$

that brings the operators $S_\alpha(a)$ into the form

$$\begin{aligned}
\mathbf{S}_-(a) =& \mathbb{U}S_-(a)\mathbb{U}^{-1} = S_-(a),\\
\mathbf{S}_0(a) =& \mathbb{U}S_0(a)\mathbb{U}^{-1} = S_0^{(0)} + \bar\beta(a) + \frac{1}{2}\mathbb{U}\mathbb{H}(a)\mathbb{U}^{-1},\\
\mathbf{S}_+(a) =& \mathbb{U}^{-1}S_+(a)\mathbb{U} = S_+^{(0)} + (z_1+z_2)\left(\bar\beta(a) + \frac{1}{2}\mathbb{U}\mathbb{H}(a)\mathbb{U}^{-1}\right).
\end{aligned} \tag{5.17}$$

Note that the anomaly term $\Delta(a)$ is absent in $\mathbf{S}_+(a)$. We will therefore refer to the boldface operator $\mathbf{S}_+$ as "canonical" generator. Parametrizing

$$\mathbb{U}(a) = e^{\mathbb{X}(a)} = e^{a\mathbb{X}^{(1)} + a^2\mathbb{X}^{(2)} + \mathcal{O}(a^2)}, \tag{5.18}$$

and comparing Eq. (5.17) with (3.37) we find the following constraints for the operator $\mathbb{X}$

$$\begin{aligned}
[S_+^{(0)}, \mathbb{X}^{(1)}] =& z_{12}\Delta^{(1)},\\
[S_+^{(0)}, \mathbb{X}^{(2)}] =& z_{12}\Delta^{(2)} + [\mathbb{X}^{(1)}, z_1+z_2](\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}) + \frac{1}{2}[\mathbb{X}^{(1)}, z_{12}\Delta^{(1)}].
\end{aligned} \tag{5.19}$$

These equations fix the operator $\mathbb{X}(a)$ up to canonically invariant pieces. This rotation can be seen as a finite renormalization scheme transformation, namely the operators $[\mathcal{O}]^{\mathbb{U}}$ satisfy the following evolution equation

$$(\mu\partial_\mu + \beta(a)\partial_a + \mathbb{U}\mathbb{H}(a)\mathbb{U}^{-1} - \beta(a)(\partial_a\mathbb{U})\mathbb{U}^{-1})[\mathcal{O}(z_1,z_2)]^{\mathbb{U}} = 0. \tag{5.20}$$

Let us consider the operator

$$\mathbf{H}(a) = \mathbb{U}\mathbb{H}(a)\mathbb{U}^{-1}, \tag{5.21}$$

that can be seen as evolution kernel in the rotated scheme (at the critical point). It has the same spectrum as the evolution kernel in $\overline{\text{MS}}$-scheme and must satisfy the constraint

$$[\mathbf{S}_\alpha(a), \mathbf{H}(a)] = 0. \tag{5.22}$$

Since the anomaly term is absent in the "canonical" generators $\mathbf{S}_+$ this equation turns out to have a much simpler structure than the initial one in $\overline{\text{MS}}$-scheme (5.2):

$$[S_+^{(0)}, \mathbf{H}^{(1)}] = 0, \tag{5.23a}$$

$$[S_+^{(0)}, \mathbf{H}^{(2)}] = \big[\mathbf{H}^{(1)}, z_1 + z_2\big]\left(\beta_0 + \frac{1}{2}\mathbf{H}^{(1)}\right), \tag{5.23b}$$

$$[S_+^{(0)}, \mathbf{H}^{(3)}] = \big[\mathbf{H}^{(1)}, z_1 + z_2\big]\left(\beta_1 + \frac{1}{2}\mathbf{H}^{(2)}\right) + \big[\mathbf{H}^{(2)}, z_1 + z_2\big]\left(\beta_0 + \frac{1}{2}\mathbf{H}^{(1)}\right). \tag{5.23c}$$

The solutions to these equations have a rather simple and transparent form

$$\begin{aligned}
\mathbf{H}^{(1)} &= \mathbf{H}^{(1)}_{\text{inv}}, \\
\mathbf{H}^{(2)} &= \mathbf{H}^{(2)}_{\text{inv}} + \mathbb{T}^{(1)}\left(\beta_0 + \frac{1}{2}\mathbf{H}^{(1)}_{\text{inv}}\right), \\
\mathbf{H}^{(3)} &= \mathbf{H}^{(3)}_{\text{inv}} + \mathbb{T}^{(1)}\left(\beta_1 + \frac{1}{2}\mathbf{H}^{(2)}_{\text{inv}}\right) + \mathbb{T}^{(1)}_1\left(\beta_0 + \frac{1}{2}\mathbf{H}^{(1)}_{\text{inv}}\right)^2 \\
&\qquad + \left(\mathbb{T}^{(2)} + \frac{1}{2}\big(\mathbb{T}^{(1)}\big)^2\right)\left(\beta_0 + \frac{1}{2}\mathbf{H}^{(1)}_{\text{inv}}\right),
\end{aligned} \tag{5.24}$$

where $\mathbf{H}^{(1)}_{\text{inv}}$ are (canonically) $SL(2)$-invariant operators. The operators $\mathbb{T}^{(i)}$ commute with $S_-^{(0)}$ and $S_0^{(0)}$ and obey the following equations:

$$\begin{aligned}
&[S_+^{(0)}, \mathbb{T}^{(1)}] = [\mathbf{H}^{(1)}_{\text{inv}}, z_1 + z_2], \\
&[S_+^{(0)}, \mathbb{T}^{(2)}] = [\mathbf{H}^{(2)}_{\text{inv}}, z_1 + z_2], \qquad [S_+^{(0)}, \mathbb{T}^{(1)}_1] = [\mathbb{T}^{(1)}, z_1 + z_2].
\end{aligned} \tag{5.25}$$

These equations define the operators $\mathbb{T}$ up to $SL(2)$ (canonically) invariant terms.

Note that the expressions for the perturbative expansion of the evolution kernel in Eq. (5.24) can be assembled in the following single expression:

$$\bar\beta(a) + \frac{1}{2}\mathbf{H}(a) = \left\{\mathbb{1} - \frac{1}{2}\left(a\mathbb{T}^{(1)} + a^2\left(\mathbb{T}^{(2)} + \mathbb{T}^{(1)}_1\mathbf{H}_{\text{inv}}(a)\right) + \mathcal{O}(a^3)\right)\right\}^{-1}\left(\bar\beta(a) + \frac{1}{2}\mathbf{H}_{\text{inv}}(a)\right). \tag{5.26}$$

Finally, performing the rotation back to $\overline{\text{MS}}$-scheme, $\mathbb{H}(a) = \mathbb{U}^{-1}\mathbf{H}(a)\mathbb{U}$, we obtain the following results for the first three orders of the non-invariant part of the evolution kernel in the $\overline{\text{MS}}$ scheme[1]:

$$
\begin{aligned}
\mathbb{H}^{(1)}_{\text{ninv}} &= 0,\\
\mathbb{H}^{(2)}_{\text{ninv}} &= \mathbf{H}^{(2)}_{\text{ninv}} + [\mathbf{H}^{(1)}, \mathbb{X}^{(1)}] = \mathbb{T}^{(1)}\Big(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\Big) + [\mathbb{H}^{(1)}, \mathbb{X}^{(1)}],\\
\mathbb{H}^{(3)}_{\text{ninv}} &= \mathbf{H}^{(3)}_{\text{ninv}} + [\mathbf{H}^{(2)}, \mathbb{X}^{(1)}] + [\mathbf{H}^{(1)}, \mathbb{X}^{(2)}] + \frac{1}{2}\big[[\mathbf{H}^{(1)}, \mathbb{X}^{(1)}], \mathbb{X}^{(1)}\big]\\
&= \mathbb{T}^{(1)}(\beta_1 + \frac{1}{2}\mathbb{H}^{(2)}_{\text{inv}}) + \left(\mathbb{T}^{(2)} + \frac{1}{2}\big(\mathbb{T}^{(1)}\big)^2\right)\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) + \mathbb{T}^{(1)}_1\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right)^2\\
&\quad + \big[\mathbb{H}^{(2)}, \mathbb{X}^{(1)}\big] + [\mathbb{H}^{(1)}, \mathbb{X}^{(2)}] - \frac{1}{2}\big[[\mathbb{H}^{(1)}, \mathbb{X}^{(1)}], \mathbb{X}^{(1)}\big].
\end{aligned}
\tag{5.27}
$$

The explicit determination of the relevant operators will be considered later on. Note that we have chosen $\mathbb{H}^{(i)}_{\text{inv}} = \mathbf{H}^{(i)}_{\text{inv}}$. It means that we will put any invariant piece that arises by the rotation into the non-invariant part of the evolution kernel.

5.2.2 Large-spin expansion and reciprocity

To complete our results for the evolution kernel we need to find a suitable way to restore the invariant part of the kernels. In the present case "suitable" means that it should come in a very natural way and it should be given in a simple form. Our solution will be motivated by the study of the large-spin expansion of the anomalous dimensions.

It is well-known that conformal symmetry of the theory implies that the evolution kernel needs to commute with the Casimir operator $\mathbb{C}$ (2.13) of the conformal symmetry group. It means the evolution kernel is a function of the Casimir $\mathbb{H} = h(\mathbb{C})$. Hence the spectrum of the evolution kernel can be expressed in terms of the spectrum of the Casimir $\gamma(N) = h\big(C(N)\big)$, which reads to leading accuracy $C^{(0)}(N) = (N+1)(N+2)$. Beyond LO the Casimir operator receives corrections just as the spin generators do. One can show that these modifications result in the following spectrum

$$
C(N) = \big(N + 1 + \bar{\beta} + \frac{1}{2}\gamma(N)\big)\big(N + 2 + \bar{\beta} + \frac{1}{2}\gamma(N)\big). \tag{5.28}
$$

Therefore a natural representation for the anomalous dimensions is

$$
\gamma(N) = f\big(N + 2 + \bar{\beta} + \frac{1}{2}\gamma(N)\big) = f(j_N), \tag{5.29}
$$

where we introduced the conformal spin $j_N = N + 2 + \bar{\beta} + \frac{1}{2}\gamma(N)$.

Comparing both sides of Eq. (5.29) in a perturbative series we find that the values of $\gamma^{(k)}(N)$ and

[1] note that compared to our publication in Ref. [71] a typo in the three-loop expression has been fixed

$f^{(k)}(j_N)$ at the same order of perturbation theory are equal up to addition of certain lower-order terms

$$\gamma^{(1)}(N) = f^{(1)}(j_N^{(0)}), \tag{5.30a}$$
$$\begin{aligned}\gamma^{(2)}(N) &= f^{(2)}(j_N^{(0)}) + \frac{d}{dN}\left(\beta_0 + \frac{1}{2}\gamma^{(1)}(N)\right)^2 \\ &= \gamma^{(2)}(N) + \left(\beta_0 + \frac{1}{2}\gamma^{(1)}(N)\right)\frac{d}{dN} f^{(1)}(j_N^{(0)}),\end{aligned} \tag{5.30b}$$
$$\begin{aligned}\gamma^{(3)}(N) &= f^{(3)}(j_N^{(0)}) + \left(\beta_1 + \frac{1}{2}\gamma^{(2)}(N)\right)\frac{d}{dN} f^{(1)}(j_N^{(0)}) \\ &\quad + \frac{1}{2}\left(\beta_0 + \frac{1}{2}\gamma^{(1)}(N)\right)^2 \frac{d^2}{dN^2} f^{(1)}(j_N^{(0)}) + \left(\beta_0 + \frac{1}{2}\gamma^{(1)}(N)\right)\frac{d}{dN} f^{(2)}(j_N^{(0)}),\end{aligned} \tag{5.30c}$$

etc., where $j_N^{(0)} = N+2$ is the LO conformal spin.

Now let us come back to equation (5.24). It is possible to fix the operators $\mathbb{T}$ such that their spectrum have the spectrum $\mathbb{T}^{(i)} z_{12}^N = T^{(i)}(N) z_{12}^N$ reads

$$\begin{aligned}T^{(1)} &= \frac{\mathrm{d}}{\mathrm{d}N}\gamma_{\rm inv}^{(1)}(N), \\ T^{(2)} &= \frac{\mathrm{d}}{\mathrm{d}N}\gamma_{\rm inv}^{(2)}(N), \\ T_1^{(1)} &= \frac{\mathrm{d}^2}{\mathrm{d}N^2}\gamma_{\rm inv}(N).\end{aligned} \tag{5.31}$$

Then we find from Eq. (5.27) the following spectrum of the evolution kernel

$$\begin{aligned}\gamma^{(1)}(N) &= \gamma_{\rm inv}^{(1)}(N)\,, \\ \gamma^{(2)}(N) &= \gamma_{\rm inv}^{(2)}(N) + \left(\beta_0 + \frac{1}{2}\gamma^{(1)}(N)\right)\frac{d}{dN}\gamma_{\rm inv}^{(1)}(N)\,, \\ \gamma^{(3)}(N) &= \gamma_{\rm inv}^{(3)}(N) + \left(\beta_1 + \frac{1}{2}\gamma_{\rm inv}^{(2)}(N)\right)\frac{d}{dN}\gamma_{\rm inv}^{(1)}(N) + \frac{1}{2}\left(\beta_0 + \frac{1}{2}\gamma_{\rm inv}^{(1)}(N)\right)^2\frac{d^2}{dN^2}\gamma_{\rm inv}^{(1)}(N) \\ &\quad + \left(\beta_0 + \frac{1}{2}\gamma_{\rm inv}^{(1)}(N)\right)\left[\frac{d}{dN}\gamma_{\rm inv}^{(2)}(N) + \frac{1}{2}\left(\frac{d}{dN}\gamma_{\rm inv}^{(1)}(N)\right)^2\right].\end{aligned} \tag{5.32}$$

Thus we can conclude that

$$\boxed{f(j_N^{(0)}) \equiv \gamma_{\rm inv}(N).} \tag{5.33}$$

We have found a representation which is sufficiently simple and moreover we will see that the required normalization of the $\mathbb{T}$-operators comes in a very natural way.

Recently it has been shown [72, 73, 74] that the asymptotic expansion of the function $f(j_N)$ at large j_N is invariant under the reflection $j_N \to 1 - j_N$. More precisely, the expansion takes the form [72]

$$f(j) = 2\Gamma_{\rm cusp}\ln\left(Je^{\gamma_E}\right) + f_0 + \sum_{n=1}^{\infty}\frac{f_n\Big(\ln\left(Je^{\gamma_E}\right)\Big)}{\left(J^2\right)^n}, \qquad J^2 = j(1-j), \tag{5.34}$$

and thus it is indeed a function of the Casimir. In all known cases the function $f(j)$ turns out to have a simpler structure than the anomalous dimensions. By comparing the expansion (5.34) with the generic form for the invariant kernel in Eq. (5.7), we find that the first two terms in the large-spin expansion stem from the constant and "one-particle" operator contributions in invariant evolution kernel. This observation allows us to fix the corresponding coefficients

$$\begin{aligned}\mathbb{H}_{\text{inv}}\mathcal{O}(z_1,z_2) =&(f_0+2\Gamma_{\text{cusp}})\mathcal{O}(z_1,z_2)+\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta h_{\text{inv}}(\tau)\mathcal{O}(z_{12}^\alpha,z_{21}^\beta)\\ &+\Gamma_{\text{cusp}}\int_0^1 d\alpha\frac{\bar\alpha}{\alpha}\Big(2\mathcal{O}(z_1,z_2)-\mathcal{O}(z_{12}^\alpha,z_2)-\mathcal{O}(z_1,z_{21}^\beta)\Big)\,. \end{aligned}\tag{5.35}$$

The leading-asymptotic coefficients f_0 and Γ_{cusp} are known to three-loop accuracy [9, 72]. The kernel $h_{\text{inv}}(\tau)$ of the "two-particle" operator remains to be fixed. It is determined by the structure of the $\sim 1/N$ suppressed contributions in the asymptotic expansion (5.34).

5.3 One-loop evolution kernel

At leading accuracy (canonical) conformal symmetry is unbroken, thus the evolution kernel is invariant, $\mathbb{H}^{(1)}=\mathbb{H}^{(1)}_{\text{inv}}$, and the spectrum is given by

$$\gamma^{(1)}(N)=\gamma^{(1)}_{\text{inv}}(N)=f^{(1)}(N+2)=2C_F\left(4S_1(N+1)-\frac{2}{(N+1)(N+2)}-3\right).\tag{5.36}$$

Here $S_1(n)=\sum_{k=1}^n\frac{1}{k}$ is the harmonic sum. By comparison with the large-spin expansion (5.34) we find

$$\Gamma^{(1)}_{\text{cusp}}=4C_F,\qquad f_0^{(1)}=-6C_F,\qquad f_1^{(1)}=-4C_F.\tag{5.37}$$

The invariant function $h^{(1)}_{\text{inv}}(\tau)$ is fixed by the equation

$$\int_0^1 d\alpha\int_0^{\bar\alpha}d\beta h^{(1)}_{\text{inv}}(\tau)(1-\alpha-\beta)^N=\frac{-4C_F}{(N+1)(N+2)},\tag{5.38}$$

that results in $h^{(1)}_{\text{inv}}(\tau)=-4C_F$. Thus the one-loop evolution kernel takes the form

$$\boxed{\begin{aligned}\mathbb{H}^{(1)}\mathcal{O}(z_1,z_2) =&2C_F\mathcal{O}(z_1,z_2)-4C_F\int_0^1 d\alpha\int_0^{\bar\alpha}d\beta\mathcal{O}(z_{12}^\alpha,z_{21}^\beta)\\ &+4C_F\int_0^1 d\alpha\frac{\bar\alpha}{\alpha}\Big(2\mathcal{O}(z_1,z_2)-\mathcal{O}(z_{12}^\alpha,z_2)-\mathcal{O}(z_1,z_{21}^\beta)\Big)\,,\end{aligned}}\tag{5.39}$$

This can be easily verified by a direct calculation as shown in subsection 4.2.2.

5.4 Two-loop evolution kernel

The two-loop result was obtained in Ref. [31] with the method described on page 47. Here we reproduce the results from Ref. [31] using the new method presented in subsections 5.2.1 and 5.2.2.

To fix the non-invariant part at NLO we just need to determine the operators $\mathbb{X}^{(1)}$ and $\mathbb{T}^{(1)}$, defined by Eqs. (5.19) and (5.25). We easily find the solutions

$$\mathbb{X}^{(1)}\mathcal{O}(z_1,z_2) = 2C_F\int_0^1 d\alpha\frac{\ln\alpha}{\alpha}\Big[2\mathcal{O}(z_1,z_2)-\mathcal{O}(z_{12}^\alpha,z_2)-\mathcal{O}(z_1,z_{21}^\alpha)\Big]+\Delta\mathbb{X}^{(1)}_{\rm inv}, \tag{5.40a}$$

$$\begin{aligned}\mathbb{T}^{(1)}\mathcal{O}(z_1,z_2) = {}&-\Gamma^{(1)}_{\rm cusp}\int_0^1 d\alpha\frac{\bar\alpha\ln\bar\alpha}{\alpha}\Big(\mathcal{O}(z_{12}^\alpha,z_2)+\mathcal{O}(z_1,z_{21}^\alpha)\Big)\\ &+\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\ln(1-\alpha-\beta)h^{(1)}_{\rm inv}(\tau)\mathcal{O}(z_{12}^\alpha,z_{21}^\beta)+\Delta\mathbb{T}^{(1)}_{\rm inv}.\end{aligned} \tag{5.40b}$$

The choice of the invariant operators $\Delta\mathbb{X}^{(1)}_{\rm inv}$ and $\Delta\mathbb{T}^{(1)}_{\rm inv}$ is ad hoc arbitrary. For the operator $\mathbb{T}^{(1)}$ we set $\Delta\mathbb{T}^{(1)}_{\rm inv}=0$, in order to fulfill the requirement (5.31). Indeed, one easily verifies

$$\begin{aligned}\mathbb{T}^{(1)}z_{12}^N ={}&z_{12}^N\Big[-2\Gamma^{(1)}_{\rm cusp}\int_0^1 d\alpha\frac{\bar\alpha\ln\bar\alpha}{\alpha}\bar\alpha^N\\ &+\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\ln(1-\alpha-\beta)h^{(1)}_{\rm inv}(\tau)(1-\alpha-\beta)^N\Big]\\ ={}&z_{12}^N\partial_N\Big[-2\Gamma^{(1)}_{\rm cusp}\int_0^1 d\alpha\frac{\bar\alpha}{\alpha}(\bar\alpha^N-1)+\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta h^{(1)}_{\rm inv}(\tau)(1-\alpha-\beta)^N\Big]\\ ={}&z_{12}^N\partial_N\gamma^{(1)}(N).\end{aligned} \tag{5.41}$$

For the operator $\mathbb{X}$ we also we set $\Delta\mathbb{X}^{(1)}_{\rm inv}=0$. Another more natural way to fix the invariant part is to require $\mathbb{X}^{(1)}z_{12}^N=0$. That choice, however, cannot be realized in an easy way.

Note that one can convolute the operator product in Eq. (5.27) to find an expression for the non-invariant part as a single integral operator

$$\mathbb{H}^{(2)}_{\rm ninv}\mathcal{O}(z_1,z_2) = \Big(\mathbb{T}^{(1)}\Big(\beta_0+\frac{1}{2}\mathbb{H}^{(1)}_{\rm inv}\Big)+[\mathbb{H}^{(1)}_{\rm inv},\mathbb{X}^{(1)}]\Big)\mathcal{O}(z_1,z_2) \tag{5.42a}$$

$$\begin{aligned}={}&C_F^2c^{(2)}_{\rm ninv}\mathcal{O}(z_1,z_2)+C_F^2\int_0^1 d\alpha\int_0^{\bar\alpha}d\beta h^{(2)}_{\rm ninv}(\alpha,\beta)\mathcal{O}(z_{12}^\alpha,z_{21}^\alpha)\\ &+C_F^2\int_0^1 d\alpha\frac{\bar\alpha}{\alpha}\hat h^{(2)}_{\rm ninv}(\alpha)\Big(2\mathcal{O}(z_1,z_2)-\mathcal{O}(z_{12}^\alpha,z_2)-\mathcal{O}(z_1,z_{21}^\alpha)\Big).\end{aligned} \tag{5.42b}$$

After some algebra one gets the following expressions for the functions $c^{(2)}_{\rm ninv}$, $\hat h^{(2)}_{\rm ninv}(\alpha)$ and $h^{(2)}_{\rm ninv}(\alpha,\beta)$:

$$\begin{aligned}c^{(2)}_{\rm ninv} &= \frac{16}{3}\pi^2-48\zeta_3-11,\\ \hat h^{(2)}_{\rm ninv}(\alpha) &= 8\ln(\bar\alpha)\Big(\ln(\bar\alpha)-\frac{(2-\alpha)\ln(\alpha)}{\bar\alpha}-\frac{3}{2}\Big)+16(1-\frac{\pi^2}{6}),\\ h^{(2)}_{\rm ninv}(\alpha,\beta) &= 16(\frac{\pi^2}{6}-1)+4\ln(\bar\tau\tau)-4\ln^2(1-\alpha-\beta)\\ &\quad-8\Big(\frac{1}{2}\ln^2(\bar\alpha)-\ln(\alpha)\ln(1-\alpha)+\frac{\bar\alpha\ln(\bar\alpha)}{\alpha}-\frac{\ln(\alpha)}{2}+\alpha\leftrightarrow\beta\Big).\end{aligned} \tag{5.43}$$

However, the factorized representation in Eq. (5.42a) seems to be more convenient for practical applications.

Next we need to find an expression for the invariant part. The function $f(N) = f_+(N) + (-1)^N f_-(N)$ reads to NLO accuracy

$$\begin{aligned}
f_+^{(2)}(N) =& 2C_AC_F\left(8S_3 + (\frac{4\pi^2}{3} + \frac{268}{9})S_1 + \frac{2\pi^2}{3\mathbb{J}^2} - \frac{302}{9\mathbb{J}^2} - \frac{8}{(\mathbb{J}^2)^2} + \frac{4}{(\mathbb{J}^2)^3} + 12\zeta(3) - \frac{22\pi^2}{9} - \frac{17}{6}\right) \\
&- 2C_Fn_f\left(16S_3 + \frac{40}{9}S_1 - \frac{44}{9\mathbb{J}^2} - \frac{4\pi^2}{9} - \frac{1}{3}\right) + 2C_F^2\left(\frac{14}{(\mathbb{J}^2)^2} + \frac{8}{(\mathbb{J}^2)^3} - 24\zeta(3) + 2\pi^2 - \frac{3}{2}\right), \\
f_-^{(2)}(N) =& 8C_F(2C_F - C_A)\left(\frac{1}{(\mathbb{J}^2)^3} + \frac{2}{(\mathbb{J}^2)^2} + (-1)^N\left[2S_{-3} - 4S_{1,-2} - \frac{2S_{-2}}{\mathbb{J}^2} - \frac{\pi^2}{6\mathbb{J}^2} - \frac{\pi^2}{3}S_1 + \zeta_3\right]\right).
\end{aligned} \tag{5.44}$$

Here $\mathbb{J}^2 \equiv (N+1)(N+2)$ is the canonical Casimir and $S_{\vec{a}} \equiv S_{\vec{a}}(N+1)$ are so-called nested harmonic sums of weight $w = \sum_i |a_i|$. They are defined recursively by [75]

$$S_{\pm a}(n) = \sum_{i=1}^{n} \frac{(\pm)^i}{i^a}, \qquad\qquad S_{\pm a,\vec{b}}(n) = \sum_{i=1}^{n} \frac{(\pm)^i}{i^a} S_{\vec{b}}(i). \tag{5.45}$$

The leading asymptotic term is given by the coefficient of the first harmonic sum

$$S_1(N+1) \simeq \ln(N+1). \tag{5.46}$$

Thus we easily read off the leading asymptotic terms which enter Eq. (5.35)

$$\begin{aligned}
f_0^{(2)} =& 8C_F\left[C_A(\frac{17}{24} + \frac{11}{18}\pi^2 - 3\zeta_3) - C_F(\frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3) - n_f(\frac{1}{12} + \frac{\pi^2}{9})\right] \\
\Gamma_{\text{cusp}}^{(2)} =& 16C_F\left[C_A\left(\frac{67}{36} - \frac{\pi^2}{12}\right) - \frac{5}{18}n_f\right],
\end{aligned} \tag{5.47}$$

We need to find the "two-particle"-kernel $h_{\text{inv}}^{(2)}(\tau)$ which yields the sub-leading structure

$$\begin{aligned}
&\int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta h_{\text{inv}}^{(2)}(\tau)(1-\alpha-\beta)^N \\
&= f^{(2)}(N) - f_0 - 2\Gamma_{\text{cusp}}^{(2)} \ln(N+1) = \sum_{n=1}^{\infty} \frac{f_n^{(2)}(\ln\mathbb{J})}{(\mathbb{J}^2)^n}.
\end{aligned} \tag{5.48}$$

In principle, using the "one"- and "two"-particle operators defined in Eqs. (4.42) and (4.44) we can write the solution formally as

$$\mathbb{H}_{\text{inv}}^{(2)} \simeq \sum_{n=1}^{\infty} f_n^{(2)}(\hat{\mathcal{H}}) \left(\mathcal{H}^{(+)}\right)^n. \tag{5.49}$$

Unfortunately the sub-leading structure is even to NLO accuracy pretty complicated. Unlike the one-loop case the asymptotic series (5.34) does not end up in a finite series but an infinite amount of expansion coefficients $f_n^{(2)}$ is required. In order to check the reciprocity relation (5.34), let us formulate the problem more concise, by examining the asymptotic expansion of e.g. the first term in Eq. (5.44)

$$S_3(N+1) = \zeta_3 - \frac{1}{2}\frac{1}{\mathbb{J}^2} + \frac{1}{4}\frac{1}{(\mathbb{J}^2)^2} - \frac{1}{6}\frac{1}{(\mathbb{J}^2)^3} + \frac{1}{6}\frac{1}{(\mathbb{J}^2)^4} - \frac{4}{15}\frac{1}{(\mathbb{J}^2)^5} + \frac{2}{3}\frac{1}{(\mathbb{J}^2)^6} + \ldots \tag{5.50}$$

which respects reciprocity but cannot be expressed in a finite nor closed form. Even worse is the situation for most of the other harmonic sums, as they cannot even be expressed in a series of the inverse Casimir and therefore they explicitly break reciprocity. Only certain linear combinations of them do fulfill this property, thus one needs to rearrange the answer in terms of so-called "r(eciprocity) r(especting)" harmonic sums $\Omega_{\vec{a}}$ [76], which are linear combinations of the nested ones. Using the algorithm proposed in [77] we find

$$S_1 = \Omega_1, \qquad S_3 = \Omega_3, \qquad S_{-2} = \Omega_{-2}, \qquad S_{1,-2} - \frac{1}{2}S_{-3} = \Omega_{1,-2}, \tag{5.51}$$

Thus the NLO f-functions can be written in terms of "rr"-harmonic sums $\Omega_{\vec{a}}$ and inverse powers of the Casimir

$$\begin{aligned}
f_+^{(2)}(N) =& 2C_AC_F\Bigg(8\Omega_3 + (\frac{4\pi^2}{3} + \frac{268}{9})\Omega_1 + \frac{2\pi^2}{3\mathbb{J}^2} - \frac{302}{9\mathbb{J}^2} - \frac{8}{(\mathbb{J}^2)^2} + \frac{4}{(\mathbb{J}^2)^3} + 12\zeta_3 \\
& - \frac{22\pi^2}{9} - \frac{17}{6}\Bigg) - 2C_Fn_f\left(16\Omega_3 + \frac{40}{9}\Omega_1 - \frac{44}{9\mathbb{J}^2} - \frac{4\pi^2}{9} - \frac{1}{3}\right) \\
& + 2C_F^2\left(\frac{14}{(\mathbb{J}^2)^2} + \frac{8}{(\mathbb{J}^2)^3} - 24\zeta(3) + 2\pi^2 - \frac{3}{2}\right), \\
f_-^{(2)}(N) =& 8C_F(2C_F - C_A)\Bigg(\frac{1}{(\mathbb{J}^2)^3} + \frac{2}{(\mathbb{J}^2)^2} \\
& + (-1)^N\left[-4\Omega_{1,-2} - \frac{2\Omega_{-2}}{\mathbb{J}^2} - \frac{\pi^2}{6\mathbb{J}^2} - \frac{\pi^2}{3}\Omega_1 + \zeta_3\right]\Bigg).
\end{aligned} \tag{5.52}$$

The corresponding invariant kernel is accordingly split into $h_{\text{inv}}^{(2)}(\tau) = \chi_{\text{inv}}^{(2+)}(\tau) + \chi_{\text{inv}}^{(2-)}(\tau)\mathbb{P}_{12}$ with the permutation operator $\mathbb{P}_{12}f(z_1,z_2) = f(z_2,z_2)$. We find

$$\begin{aligned}
\chi_{\text{inv}}^{(2+)}(\tau) &= 4C_F\Bigg\{-\frac{11}{3}\beta_0 + C_F\left[\ln\bar\tau - \frac{20}{3} + \frac{2\pi^2}{3}\right] \\
&\quad - \frac{2}{N_c}\left(\text{Li}_2(\tau) + \frac{1}{2}\ln^2\bar\tau - \frac{1}{\tau}\ln\bar\tau - \frac{\pi^2}{6} + \frac{5}{3}\right)\Bigg\}, \\
\chi_{\text{inv}}^{(2-)}(\tau) &= -\frac{4C_F}{N_c}\left(\ln^2\bar\tau - 2\tau\ln\bar\tau\right).
\end{aligned} \tag{5.53}$$

The functions $\chi_{\text{inv}}^{(2+)}(\tau)$ and $\chi_{\text{inv}}^{(2-)}(\tau)$ correspond to $f_+^{(2)}(N)$ and $f_-^{(2)}(N)$, respectively. In practice, to two-loop accuracy we have made great benefit from the knowledge of the kernel function from the explicit calculation, see appendix F.

5.5 Three-loop evolution kernel

Whereas at the first two orders of perturbation theory the problem was easy enough to be solved in a compact and analytic way, at NNLO accuracy we have to content ourselves with a solution that is semi-analytic and not quite compact.

We start to reexamine the non-invariant part already proposed in equation (5.27). The evolution kernels $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$, and so the operators $\mathbb{X}^{(1)}$ and $\mathbb{T}^{(1)}$ have already been determined in the previous two sections. We are left with the operators $\mathbb{T}_1^{(1)}, \mathbb{T}^{(2)}$ and $\mathbb{X}^{(2)}$. As mentioned before, the solution for the $\mathbb{T}$-type operators is rather simple. One can easily show that to any order in perturbation theory the constraint

$$[S_+^{(0)}, \mathbb{T}^{(\ell)}] = [\mathbb{H}_{\rm inv}^{(\ell)}, z_1 + z_2], \tag{5.54}$$

is solved by

$$\begin{aligned}\mathbb{T}^{(\ell)}\mathcal{O}(z_1,z_2) = &-\Gamma_{\rm cusp}^{(\ell)}\int_0^1 d\alpha \frac{\bar\alpha \ln\bar\alpha}{\alpha}\Big(\mathcal{O}(z_{12}^\alpha, z_2) + \mathcal{O}(z_1, z_{21}^\alpha)\Big) \\ &+ \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \ln(1-\alpha-\beta) h_{\rm inv}^{(\ell)}(\tau)\mathcal{O}(z_{12}^\alpha, z_{21}^\beta).\end{aligned} \tag{5.55}$$

where $\Gamma_{\rm cusp}$ and $h_{\rm inv}(\tau)$ is the input defined by the generic form (5.35). The relevant functions are given in equations (5.37), (5.47) and (5.53). For the operator $\mathbb{T}_1^{(1)}$ we find the solution [2]

$$\begin{aligned}\mathbb{T}_1^{(1)}\mathcal{O}(z_1,z_2) = &-\frac{1}{2}\Gamma_{\rm cusp}^{(1)}\int_0^1 d\alpha \frac{\bar\alpha \ln^2\bar\alpha}{\alpha}\Big(\mathcal{O}(z_{12}^\alpha, z_2) + \mathcal{O}(z_1, z_{21}^\alpha)\Big) \\ &+ \frac{1}{2}\int_0^1 d\alpha \int_0^{\bar\alpha} d\beta h_{\rm inv}^{(1)}(\tau) \ln^2(1-\alpha-\beta)\mathcal{O}(z_{12}^\alpha, z_{21}^\beta).\end{aligned} \tag{5.56}$$

Again, both solutions can be modified by the addition of invariant kernels. The solutions (5.55) and (5.56) are chosen such that the moments comply with (5.31)

$$\mathbb{T}^{(\ell)} z_{12}^N = \partial_N \gamma^{(\ell)} z_{12}^N, \qquad\qquad \mathbb{T}_1^{(\ell)} z_{12}^N = \frac{1}{2}\partial_N^2 \gamma^{(\ell)} z_{12}^N, \tag{5.57}$$

as can be checked along the same lines as for $\mathbb{T}^{(1)}$ in Eq. (5.41)

The derivation of the $\mathbb{X}^{(2)}$ kernel is much more intricate. Some details and the result can be found in appendix G.

Due to the complexity of the involved operators the representation (5.27) as product of operators seems to be best suited for an analytic representation of the results. Performing the convolutions and rewriting the evolution kernel as one operator a la Eq. (3.25) is not practical is this situation.

The invariant part takes the form (5.35)

$$\begin{aligned}\mathbb{H}_{\rm inv}^{(3)}\mathcal{O}(z_1,z_2) =&(f_0^{(3)} + 2\Gamma_{\rm cusp}^{(3)})\mathcal{O}(z_1,z_2) + \Gamma_{\rm cusp}^{(3)}\int_0^1 d\alpha\frac{\bar\alpha}{\alpha}\Big(2\mathcal{O}(z_1,z_2) - \mathcal{O}(z_{12}^\alpha, z_2) - \mathcal{O}(z_1, z_{21}^\beta)\Big) \\ &+ \int_0^1 d\alpha\int_0^{\bar\alpha} d\beta \left(\chi_{\rm inv}^{(3+)}(\tau) + \chi_{\rm inv}^{(3-)}(\tau)\mathbb{P}_{12}\right)\mathcal{O}(z_{12}^\alpha, z_{21}^\beta).\end{aligned} \tag{5.58}$$

The leading-asymptotic coefficients $f_0^{(3)}$, $\Gamma_{\rm cusp}^{(3)}$ can be found in [9]. The cusp anomalous dimension reads

$$\begin{aligned}\Gamma_{\rm cusp}^{(3)} = 64\bigg[&C_A^2 C_F\left(\frac{245}{96} - \frac{67\pi^2}{216} + \frac{11\pi^4}{720} + \frac{11}{24}\zeta_3\right) + C_A C_F n_f \frac{1}{2}\left(-\frac{209}{216} + \frac{5\pi^2}{54} - \frac{7}{6}\zeta_3\right) \\ &+ C_f^2 n_f \frac{1}{2}\left(\zeta_3 - \frac{55}{48}\right) - \frac{1}{108}C_F n_f^2\bigg].\end{aligned} \tag{5.59}$$

[2]Please note that the typo in Eq. (C.4) of Ref. [71] is fixed here

and for the constant part we find $f_0^{(3)} + 2\Gamma_{\rm cusp}^{(3)} \equiv \chi_0^{(3)}$

$$\begin{aligned}
\chi_0^{(3)} &= C_F^3\Big[\frac{3176}{9}\zeta_3 - 320\zeta_5 + \frac{1672\pi^4}{135} - \frac{23954\pi^2}{81} + \frac{13454}{9}\Big] \\
&+ C_F^2 n_f\Big[-\frac{752}{9}\zeta_3 - \frac{128\pi^4}{135} + \frac{3452\pi^2}{81} - \frac{6242}{27}\Big] + C_F n_f^2\Big[\frac{32}{9}\zeta_3 - \frac{80\pi^2}{81} + \frac{70}{27}\Big] \\
&+ \frac{C_f^2}{N_c}\Big[-\frac{16}{3}\pi^2\zeta_3 + \frac{9464}{9}\zeta_3 - 560\zeta_5 + \frac{322\pi^4}{27} - \frac{27158\pi^2}{81} + \frac{28789}{18}\Big] \\
&+ \frac{C_F n_f}{N_c}\Big[-\frac{1072}{9}\zeta_3 - \frac{2\pi^4}{45} + \frac{1816\pi^2}{81} - \frac{2752}{27}\Big] \\
&+ \frac{C_F}{N_c^2}\Big[\frac{3632}{9}\zeta_3 - 80\zeta_5 + \frac{31\pi^4}{15} - \frac{7712\pi^2}{81} + \frac{7537}{18}\Big].
\end{aligned} \tag{5.60}$$

All sub-leading corrections

$$\Delta\gamma_{\rm inv}^{(3)}(N) = f^{(3)}(N+2) - 2\Gamma_{\rm cusp} S_1(N+1) - f_0^{(3)} = \sum_n \frac{f_n^{(3)}(\ln\mathbb{J})}{(\mathbb{J}^2)^n}. \tag{5.61}$$

enter in the one-variable functions $\chi_{\rm inv}^{(3\pm)}(\tau)$. One usually separates the contributions

$$\Delta\gamma_{\rm inv}^{(3)}(N) = \Delta\gamma_{\rm inv}^{(3)+}(N) + (-1)^N \Delta\gamma_{\rm inv}^{(3)-}(N) \tag{5.62}$$

to distinguish between the moments for even and odd N. Using the knowledge of the spectrum

$$\begin{aligned}
\int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \chi_{\rm inv}^{(3+)}(\tau)(1-\alpha-\beta)^N &= \Delta\gamma_{\rm inv}^{(3)+}(N), \\
\int_0^1 d\alpha \int_0^{\bar\alpha} d\beta \chi_{\rm inv}^{(3-)}(\tau)(1-\alpha-\beta)^N &= \Delta\gamma_{\rm inv}^{(3)-}(N),
\end{aligned} \tag{5.63}$$

one can, in principle, invert for the desired kernels

$$\begin{aligned}
\chi_{\rm inv}^{(3+)}(\tau) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dN(2N+3)\Delta\gamma_{\rm inv}^{(3+)}(N) P_{N+1}\left(\frac{1+\tau}{1-\tau}\right), \\
\chi_{\rm inv}^{(3-)}(\tau) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dN\,(2N+3)\,\Delta\gamma_{\rm inv}^{(3-)}(N)\, P_{N+1}\left(\frac{1+\tau}{1-\tau}\right).
\end{aligned} \tag{5.64}$$

In practice, the algebraic structure of the NNLO f-function is too complicated to allow for an analytical solution. With the help of computer algebra tools we can fix analytically the sub-leading structure $\sim c_{mn}\frac{S_1(N+1)^m}{(\mathbb{J}^2)^n}$, to an arbitrary, but finite order. More precisely, we will present analytic results for

- the contribution $\sim \frac{f_1^{(3)}(\ln\mathbb{J})}{\mathbb{J}^2}$ that dominates at large spin j_N
- the leading singularities $\sim \frac{1}{\mathbb{J}^{2n}}$ (pole at $N=-1$) in the complex plane up to $n \leq 5$.

For the remainder, which is numerically rather small, we provide a simple (in the sense of the number of parameters) fit.

5.5.1 Splitting functions

For later convenience we introduce the splitting function $H(x)$, defined by

$$-\int_0^1 dxH(x)x^N = \gamma(N). \tag{5.65}$$

The splitting function is known to the same accuracy as the anomalous dimension [9] and given in terms of harmonic poly-logarithms (HPL) [78]. There exists a one-to-one correspondence between harmonic sums with weight w and HPLs of weight $w-1$. The HPLs are defined recursively by

$$\begin{aligned} f_0(x) &= \frac{1}{x}, & f_\pm(x) &= \frac{1}{1\mp x}, \\ H_{0,\dots,0}(x) &= \frac{1}{n!}\ln^n(x), & H_{a_1,a_2,\dots,a_n}(x) &= \int_0^x dy f_{a_1}(y)H_{a_2,\dots,a_n}(x). \end{aligned} \tag{5.66}$$

There are two major advantages compared to kernel functions $h_{\text{inv}}(\tau)$:

- The reciprocity condition can be formulated in a clear way:

$$H_{\text{inv}}(x) \stackrel{!}{=} -xH_{\text{inv}}\left(\frac{1}{x}\right). \tag{5.67}$$

  Thus we can easily implement the reciprocity condition in a fit ansatz.

- There exist various computer algebra tools dedicated to HPLs, see e.g. Refs. [78, 79]. They allow for an easy and fast extraction of the asymptotics, which in the language of splitting functions is given by the end-point behavior at $x=1$ (large N) and $x=0$ (leading pole).

In the following we will briefly explain how to extract the asymptotic limits from the splitting functions.

Let us start to examine the behavior at $x\to 0$. To this end we need to extract logarithmic singularities $\sim \ln^i(x)$. It is known, that to a fixed perturbative order ℓ the splitting functions involve singularities up to $\ln^{2(\ell-1)}(x)$. In terms of HPLs $H_{\vec{w}}(x)$ these singularities are separated by looking for "trailing zeros" $H_{\vec{w},0}(x)$ and decomposing

$$H_{w_1\dots w_n,0}(x) = H_{w_1\dots w_n}(x)\ln(x) - H_{0,w_1\dots w_n}(x) - H_{w_1,0\dots w_n}(x) - \ldots - H_{w_1\dots 0,w_n}(x). \tag{5.68}$$

Is the penultimate index w_n also zero, this step has to be repeated until no more "trailing zeros" appear. A HPL without a "trailing zero" is finite in the limit $x\to 0$. At NNLO the maximal number of iterations is $2(3-1)=4$. Extracting only logarithms would spoil reciprocity $\ln(x)\neq -x\ln(1/x)$. To this end we define a set of functions $\phi_i(x)\sim \ln^i(x)$ such that

$$\int_0^1 dxx^N\phi_i(x) = \frac{1}{[(N+1)(N+2)]^{i+1}} \tag{5.69}$$

They are defined recursively by

$$\phi_0(x) = \bar{x}, \qquad \phi_i(x) = \int_x^1 \frac{dz}{z}\phi_{i-1}(z)\phi_0(x/z). \tag{5.70}$$

In detail, we have found [3]

$$\begin{aligned}
\phi_1(x) &= -2\bar{x} - (1+x)\ln(x), \\
\phi_2(x) &= 6\bar{x} + 3(x+1)\ln(x) + \frac{1}{2}\bar{x}\ln^2(x), \\
\phi_3(x) &= -20\bar{x} - 10(x+1)\ln(x) - 2\bar{x}\ln^2(x) - \frac{1}{6}(x+1)\ln^3(x), \\
\phi_4(x) &= 70\bar{x} + +35(x+1)\ln(x) + \frac{15}{2}\bar{x}\ln^2(x) + \frac{5}{6}(x+1)\ln^3(x) + \frac{1}{24}\bar{x}\ln^4(x).
\end{aligned} \tag{5.71}$$

In the limit $x \to 1$ the leading singularity of the splitting function is given by [4] $\sim \frac{\Gamma_{\text{cusp}}}{(1-x)_+}$ and already isolated. The remainder vanishes at $x \to 1$. Thus we need to identify contributions $\sim (1-x)\ln(1-x)^i$. Bearing in mind the reciprocity condition (5.67) we define $\omega(x) = \bar{x}\ln\left(\frac{x}{\bar{x}^2}\right)$. We suggest the following expression for the splitting function

$$\begin{aligned}
\Delta H_{\text{inv}}^{(3+)}(x) &= \sum_{k=0}^{4} B_k^{(3+)}\phi_k(x) + C_1^{(3+)}\omega(x) + \delta H_{\text{inv}}^{(3+)}(x), \\
H_{\text{inv}}^{(3-)}(x) &= \sum_{k=0}^{4} B_k^{(3-)}\phi_k(x) + \delta H_{\text{inv}}^{(3-)}(x).
\end{aligned} \tag{5.72}$$

The analytic expressions for all coefficients $B_i^{(3\pm)}$ and $C_1^{(3+)}$ can be found in table 5.2, together with the numbers for the leading coefficients $\chi_0^{(3)}$ and $\Gamma_{\text{cusp}}^{(3)}$. In all cases we show the coefficients for the following color decomposition:

$$F = C_F^3 F_{\langle 1\rangle} + C_F^2 n_f F_{\langle 2\rangle} + C_F n_f^2 F_{\langle 3\rangle} + \frac{C_F^2}{N_c}F_{\langle 4\rangle} + \frac{C_F n_f}{N_c}F_{\langle 5\rangle} + \frac{C_F}{N_c^2}F_{\langle 6\rangle} \tag{5.73}$$

where $F = C_1^{(3+)}, B_k^{(3\pm)}, \chi_0^{(3)}, \Gamma_{\text{cusp}}^{(3)}$.

The addenda, $\delta H_{\text{inv}}^{(3+)}(x)$ and $\delta H_{\text{inv}}^{(3-)}(x)$ are finite in the limit $x \to 0$ and vanish at [5] $x \to 1$

$$\delta H_{\text{inv}}^{(3\pm)}(x) \underset{x\to 1}{=} \mathcal{O}(\bar{x}^3), \qquad \delta H_{\text{inv}}^{(3\pm)}(x) \underset{x\to 0}{=} H_0 + \mathcal{O}(x). \tag{5.74}$$

These functions are numerically rather small. For illustration we plot the ratio $\delta H_{\text{inv}}^{(3+)}(x)/H_{\text{inv}}^{(3+)}(x)$ for $N_c = 3$ and $n_f = 4$ in Fig. 5.1 (dashed curve on the left panel). One can see that $\delta H_{\text{inv}}^{(3+)}(x)$ contributes at most $\sim 8\%$ to the full splitting function in the whole range $0 < x < 1$ so that for all practical purposes it can be approximated by a simple expression with a few parameters.

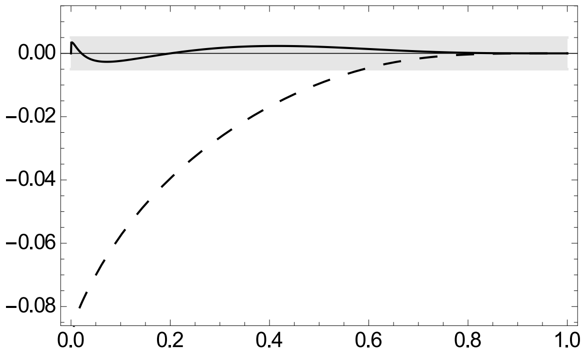
We parametrize the functions $\delta H_{\text{inv}}^{(3\pm)}(x)$ in a form that respects the reciprocity condition (5.67)

$$\delta H_{\text{inv}}^{(3\pm)}(x) = \bar{x}\, h_\pm(x/\bar{x}^2). \tag{5.75}$$

Motivated by the constraints (5.74) we employ the following ansatz

$$h_\pm(t) = H_0^\pm \frac{a_\pm}{t + a_\pm}\left(1 + \frac{b_\pm\, t}{t + a_\pm}\right), \tag{5.76}$$

[3] Please note the typo in Eq. (5.15) of Ref. [71]
[4] the subscript $()_+$ denotes regularization in the sense of the δ_+ distribution.
[5] explicit inspection shows that no $\mathcal{O}(\bar{x}^2)$ terms appear



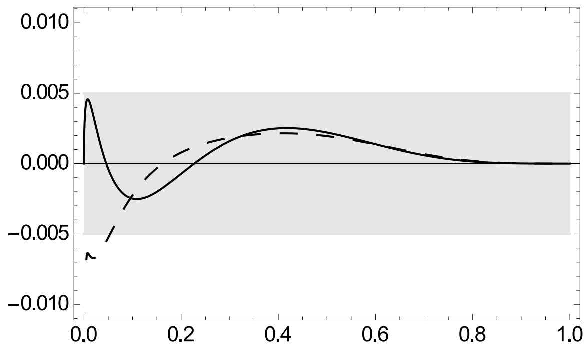


Figure 5.1: Left panel shows the ratio $\delta H_{\rm inv}^{(3+)}(x)/H_{\rm inv}^{(3+)}(x)$ (dashed curve) for exact splitting functions and the error in using the approximation (5.76), $(\delta H_{\rm inv}^{(3+)}|_{\rm fit} - \delta H_{\rm inv}^{(3+)}|_{\rm exact})/H_{\rm inv}^{(3+)}$ (solid curve) for $n_f = 4$. The shaded area indicates an error band of 0.5%. The similarly defined approximation error for the combinations $H^{(+)} + H^{(-)}$ (dashes) and $H^{(+)} - H^{(-)}$ (solid) which give rise to moments with odd and even N, respectively, is shown on the right panel.

where $a_\pm$ and $b_\pm$ are fit parameters and the normalization constants $H_0^\pm$ are determined analytically from the condition (5.74). This constants together with the fitted values of the parameters $a_\pm$ and $b_\pm$ for the different color structures can be found in table 5.1. Using this simple parametrization we reduce the deviation from the exact splitting functions to approximately 0.5%, see Fig. 5.1.

	H_0^+	a_+	b_+	H_0^-	a_-	b_-
C_F^3	$\frac{272}{3} - \frac{28\pi^2}{9} - \frac{64\pi^4}{45}$	0.2263	0	0	0	0
$C_F^2 n_f$	$\frac{32}{3} - \frac{16\pi^2}{9}$	0.5340	0	0	0	0
$\frac{C_F^2}{N_c}$	$-\frac{368\zeta_3}{3} - \frac{992}{9} + \frac{176\pi^2}{9} + \frac{4\pi^4}{9}$	0.05174	4.116	$128\zeta_3 - 24 - \frac{2200\pi^2}{27} + \frac{28\pi^4}{9}$	0.4040	−0.7986
$\frac{C_F n_f}{N_c}$	$-\frac{32\zeta_3}{3} + \frac{256}{9} - \frac{8\pi^2}{9}$	0.09626	−1.526	$\frac{176\pi^2}{27} - 16\zeta_3$	0.1252	0
$\frac{C_F}{N_c^2}$	$-\frac{328\zeta_3}{3} - \frac{736}{9} + \frac{140\pi^2}{9} + \frac{8\pi^4}{5}$	0.06595	0	$64\zeta_3 - 24 - \frac{1208\pi^2}{27} + \frac{7\pi^4}{9}$	0.2206	−1.077

Table 5.1: Values of all parameters in the ansatz for $\delta H_{\rm inv}^{(3\pm)}(x)$ (5.75), (5.76).

	$\chi_0^{(3)}$	$\Gamma_{\rm cusp}^{(3)}$
C_F^3	$\frac{3176}{9}\zeta_3 - 320\zeta_5 + \frac{1672\pi^4}{135} - \frac{23954\pi^2}{81} + \frac{13454}{9}$	$\frac{352}{3}\zeta_3 + \frac{1960}{3} - \frac{2144\pi^2}{27} + \frac{176\pi^4}{45}$
$C_F^2 n_f$	$-\frac{752}{9}\zeta_3 - \frac{128\pi^4}{135} + \frac{3452\pi^2}{81} - \frac{6242}{27}$	$-\frac{128}{3}\zeta_3 - \frac{2662}{27} + \frac{160\pi^2}{27}$
$C_F n_f^2$	$\frac{32}{9}\zeta_3 - \frac{80\pi^2}{81} + \frac{70}{27}$	$-\frac{16}{27}$
$\frac{C_F^2}{N_c}$	$-\frac{16}{3}\pi^2\zeta_3 + \frac{9464}{9}\zeta_3 - 560\zeta_5 + \frac{322\pi^4}{27} - \frac{27158\pi^2}{81} + \frac{28789}{18}$	$\frac{352}{3}\zeta_3 + \frac{1960}{3} - \frac{2144\pi^2}{27} + \frac{176\pi^4}{45}$
$\frac{C_F n_f}{N_c}$	$-\frac{1072}{9}\zeta_3 - \frac{2\pi^4}{45} + \frac{1816\pi^2}{81} - \frac{2752}{27}$	$-\frac{112}{3}\zeta_3 - \frac{836}{27} + \frac{80\pi^2}{27}$
$\frac{C_F}{N_c^2}$	$\frac{3632}{9}\zeta_3 - 80\zeta_5 + \frac{31\pi^4}{15} - \frac{7712\pi^2}{81} + \frac{7537}{18}$	$\frac{88}{3}\zeta_3 + \frac{490}{3} - \frac{536\pi^2}{27} + \frac{44\pi^4}{45}$

	$B_0^{(3+)}$	$B_1^{(3+)}$	$B_2^{(3+)}$	$B_3^{(3+)}$	$B_4^{(3+)}$	$C_1^{(3+)}$
C_F^3	$\frac{352}{3}\zeta_3 + \frac{70768}{27} - \frac{3488\pi^2}{27} + \frac{176\pi^4}{45}$	$-\frac{746}{9} - \frac{40\pi^2}{9}$	$20 - \frac{16\pi^2}{3}$	0	0	$-\frac{184}{3}$
$C_F^2 n_f$	$-\frac{128}{3}\zeta_3 - \frac{11966}{27} + \frac{256\pi^2}{27}$	$\frac{28}{9} - \frac{16\pi^2}{9}$	0	0	0	$-\frac{16}{3}$
$C_F n_f^2$	$\frac{64}{9}$	0	0	0	0	0
$\frac{C_F^2}{N_c}$	$\frac{352}{3}\zeta_3 + \frac{74428}{27} - \frac{3488\pi^2}{27} + \frac{176\pi^4}{45}$	$-176\zeta_3 + \frac{1886}{3} - \frac{52\pi^2}{9}$	$\frac{3632}{9} - \frac{16\pi^2}{3}$	$-\frac{520}{3}$	-64	$\frac{16\pi^2}{3} - \frac{376}{3}$
$\frac{C_F n_f}{N_c}$	$-\frac{112}{3}\zeta_3 - \frac{5932}{27} + \frac{128\pi^2}{27}$	$-\frac{512}{9} - \frac{8\pi^2}{9}$	$-\frac{400}{9}$	$-\frac{16}{3}$	0	$-\frac{8}{3}$
$\frac{C_F}{N_c^2}$	$\frac{88}{3}\zeta_3 + \frac{17902}{27} - \frac{836\pi^2}{27} + \frac{44\pi^4}{45}$	$-112\zeta_3 + \frac{3272}{9} - \frac{76\pi^2}{9}$	$\frac{1816}{9} - \frac{16\pi^2}{3}$	$-\frac{176}{3}$	-24	$\frac{8\pi^2}{3} - \frac{196}{3}$

	$B_0^{(3-)}$	$B_1^{(3-)}$	$B_2^{(3-)}$	$B_3^{(3-)}$	$B_4^{(3-)}$
$\frac{C_F^2}{N_c}$	$\frac{88}{3}$	$-128\zeta_3 + \frac{7288}{9} + \frac{52\pi^2}{9}$	$\frac{3632}{9} + \frac{32\pi^2}{3}$	$-\frac{520}{3}$	-64
$\frac{C_F n_f}{N_c}$	$-\frac{8}{3}$	$\frac{16\pi^2}{9} - \frac{488}{9}$	$-\frac{400}{9}$	$-\frac{16}{3}$	0
$\frac{C_F}{N_c^2}$	$\frac{44}{3}$	$-112\zeta_3 + \frac{3860}{9} + \frac{164\pi^2}{9}$	$\frac{1816}{9} + \frac{44\pi^2}{3}$	$-\frac{176}{3}$	-24

Table 5.2: Cusp anomalous dimension $\Gamma_{\rm cusp}^{(3)}$ (5.59), constant term $\chi_0^{(3)}$ (5.60) and the coefficients $B_k^{(3\pm)}$, $C_1^{(3+)}$ (5.72) in the splitting function representation of the invariant kernel.

5.5.2 From "x"- to "τ"-space via Mellin transformation

To match our proposed representation (5.58) we need to translate the results for $H^{(3\pm)}_{\mathrm{inv}}(x)$ into the functions $\chi^{(3\pm)}_{\mathrm{inv}}(\tau)$. The straightforward way – plugging the Mellin representation of the anomalous dimensions (5.65) into the inversion formula (5.64) – is again not the simplest strategy. To this end it turns out to be very useful to define a Mellin transform for the functions $\chi^{(3\pm)}_{\mathrm{inv}}(\tau)$ in analogy to (5.65) by

$$\chi(\tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\rho\, \widetilde{\chi}(\rho)\, (\bar{\tau}/\tau)^{-\rho}\,, \tag{5.77a}$$

$$\widetilde{\chi}(\rho) = \int_0^1 \frac{d\tau}{\tau\bar{\tau}}\, (\bar{\tau}/\tau)^{\rho}\, \chi(\tau). \tag{5.77b}$$

The integration contour in the first integral, (5.77a), must be chosen in the analyticity strip of the second integral, (5.77b), (the strip where integral converges). The anomalous dimensions (Mellin transform of the splitting function) are related by

$$\Delta\gamma^{(3+)}_{\mathrm{inv}}(N) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\rho\, \widetilde{\chi}^{(3+)}_{\mathrm{inv}}(\rho)\Gamma^2(1+\rho)\frac{\Gamma(j_N-1-\rho)}{\Gamma(j_N+1+\rho)}, \tag{5.78a}$$

$$\gamma^{(3-)}_{\mathrm{inv}}(N) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\rho\, \widetilde{\chi}^{(3-)}_{\mathrm{inv}}(\rho)\Gamma^2(1+\rho)\frac{\Gamma(j_N-1-\rho)}{\Gamma(j_N+1+\rho)}. \tag{5.78b}$$

It turns out that this Mellin space kernels take a very simple form compared to to anomalous dimensions. The Mellin transform of the one- and two-loop kernels read:

$$\widetilde{\chi}^{(1)}_{\mathrm{inv}}(\rho) = -4C_F\Big[2\pi i\delta(\rho)\Big], \tag{5.79a}$$

$$\widetilde{\chi}^{(2+)}_{\mathrm{inv}}(\rho) = 4C_F\left\{r_0(2\pi i)\delta(-i\rho) - \frac{\pi}{\rho\sin\pi\rho}\left(C_F + \frac{2}{N_c}\frac{2\rho+1}{\rho(\rho+1)}\right)\right\},$$

$$\widetilde{\chi}^{(2-)}_{\mathrm{inv}}(\rho) = -\frac{8C_F}{N_c}\frac{\pi}{\rho\sin\pi\rho}(\rho-1)S_1(\rho-1), \tag{5.79b}$$

where

$$r_0 = -\frac{11}{3}\beta_0 + C_F\left[\frac{2\pi^2}{3} - \frac{20}{3}\right] - \frac{2}{N_c}\left(\frac{8}{3} - \frac{\pi^2}{6}\right). \tag{5.80}$$

If the anomalous dimensions are written in terms of the splitting functions, Eq.(5.65),

$$\gamma^{(k\pm)}_{\mathrm{inv}}(N) = -\int_0^1 dx\, x^N H^{(k\pm)}_{\mathrm{inv}}(x), \tag{5.81}$$

the corresponding Mellin-transformed invariant kernels can be calculated as

$$\widetilde{\chi}^{(k\pm)}_{\mathrm{inv}}(\rho) = -\frac{\Gamma(2\rho+2)}{\Gamma^2(1+\rho)}\int_0^1 dx\, H^{(k\pm)}_{\mathrm{inv}}(x)\,\frac{1}{x\bar{x}}\left(\frac{1+x}{1-x}\right)\left(\frac{x}{\bar{x}^2}\right)^{\rho}. \tag{5.82}$$

For the remainder function $\delta H^{(3\pm)}_{\rm inv}(x)$ in the form (5.75) this simplifies to

$$\delta\widetilde{\chi}^{(3\pm)}_{\rm inv}(\rho) = -\frac{\Gamma(2\rho+2)}{\Gamma^2(1+\rho)}\int_0^\infty dt\, h_\pm(t)\, t^{\rho-1}. \tag{5.83}$$

Using the simple ansatz in Eq. (5.76) we find

$$\delta\widetilde{\chi}^{(3\pm)}_{\rm inv}(\rho) = -\frac{\Gamma(2\rho+2)}{\Gamma^2(1+\rho)}\frac{\pi}{\sin(\pi\rho)} H_0^\pm(1+b_\pm\rho)a_\pm^\rho. \tag{5.84}$$

The kernels in τ space can finally be obtained by the inverse Mellin transformation (5.77a). We find the following contribution to the invariant kernel

$$\delta\chi^{(3\pm)}_{\rm inv}(\tau) = \frac{H_0^\pm}{(1+4a_\pm\tau/\bar\tau)^{5/2}}\Big[1+a_\pm\frac{\tau}{\bar\tau}(4-6\,b_\pm)\Big] - H_0^\pm. \tag{5.85}$$

The expressions for the functions ϕ_k (5.69) in ρ- and τ-space are defined by

$$\widetilde{\phi}_k(\rho) \equiv -\frac{\Gamma(2\rho+2)}{\Gamma^2(1+\rho)}\int_0^1 dx\,\phi_k(x)\,\frac{1}{x\bar x}\left(\frac{1+x}{1-x}\right)\left(\frac{x}{\bar x^2}\right)^\rho \equiv \int_0^1 \frac{d\tau}{\tau\bar\tau}\,(\bar\tau/\tau)^\rho\,\varphi_k(\tau). \tag{5.86}$$

and take the following form

$$\begin{aligned}
\widetilde{\phi}_0(\rho) &= -2\pi i\delta(\rho),\\
\widetilde{\phi}_1(\rho) &= -\pi/(\rho\sin\pi\rho),\\
\widetilde{\phi}_2(\rho) &= \widetilde{\phi}_1(\rho)\,\frac{\bar\rho}{\rho},\\
\widetilde{\phi}_3(\rho) &= \widetilde{\phi}_1(\rho)\Big(-2\,\frac{\bar\rho}{\rho}+\psi'(\rho)-\frac{\pi^2}{6}\Big),\\
\widetilde{\phi}_4(\rho) &= \widetilde{\phi}_1(\rho)\Big(5\,\frac{\bar\rho}{\rho}+\frac{(1-3\rho)}{\rho}\Big[\psi'(\rho)-\frac{\pi^2}{6}\Big]\Big),
\end{aligned} \tag{5.87}$$

and according to the transformation (5.77a)

$$\begin{aligned}
\varphi_0(\tau) &= -1,\\
\varphi_1(\tau) &= \ln\bar\tau = H_1\left(-\frac{\tau}{\bar\tau}\right),\\
\varphi_2(\tau) &= -\varphi_1(\tau) + H_{01}\left(-\frac{\tau}{\bar\tau}\right) = -\ln\bar\tau + \mathrm{Li}_2\left(-\tau/\bar\tau\right),\\
\varphi_3(\tau) &= -2\varphi_2(x) - H_{101}\left(-\frac{\tau}{\bar\tau}\right) = 2H_1\left(-\frac{\tau}{\bar\tau}\right) - 2H_{01}\left(-\frac{\tau}{\bar\tau}\right) - H_{101}\left(-\frac{\tau}{\bar\tau}\right),\\
&= 2\ln\bar\tau - 2\,\mathrm{Li}_2\left(-\tau/\bar\tau\right) - 2\Big(\mathrm{Li}_3(\bar\tau)-\mathrm{Li}_3(1)\Big) + \ln\bar\tau\left(\mathrm{Li}_2(\bar\tau)+\frac{\pi^2}{6}\right) + \frac{1}{6}\ln^3\bar\tau,\\
\varphi_4(\tau) &= -3\varphi_3(\tau) - \varphi_2(\tau) - H_{0101}\left(-\frac{\tau}{\bar\tau}\right),\\
&= -5H_1\left(-\frac{\tau}{\bar\tau}\right) + 5H_{01}\left(-\frac{\tau}{\bar\tau}\right) + 3H_{101}\left(-\frac{\tau}{\bar\tau}\right) - H_{0101}\left(-\frac{\tau}{\bar\tau}\right).
\end{aligned} \tag{5.88}$$

In the same manner one can derive the transformation rule for the function $\omega(x)$

$$\omega(x) = \bar x\ln(x/\bar x^2) \qquad \mapsto \qquad \tilde\omega(\tau) = 2 + \ln(\tau/\bar\tau). \tag{5.89}$$

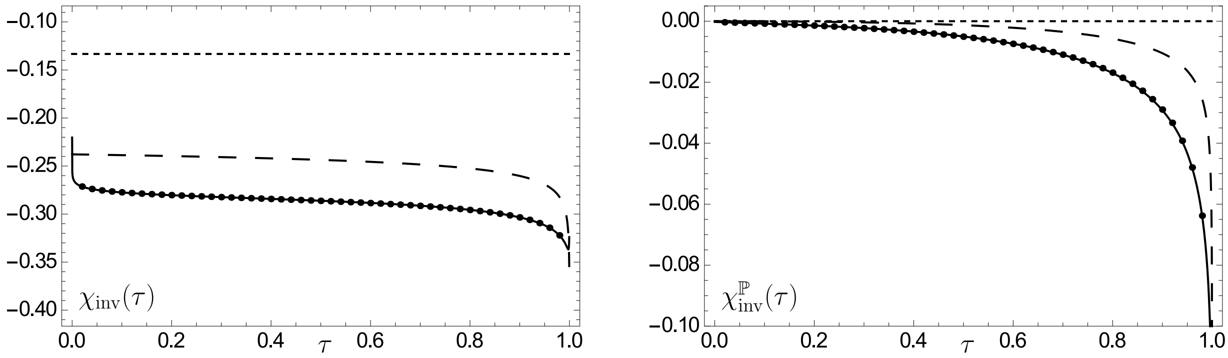


Figure 5.2: Invariant functions $\chi_{\rm inv}(\tau)$ (left panel) and $\chi^{\mathbb{P}}_{\rm inv}(\tau)$ (right panel) for $\alpha_s/\pi = 0.1$. The LO result (short dashes) is shown together with the NLO (long dashes) and NNLO (solid curves). The NNLO results using exact $\mathcal{O}(a^3)$ functions obtained by the numerical integration of Eq. (5.64) are shown by black dots for comparison.

Our final for the invariant kernel (5.58) reads

$$\begin{aligned}
\chi^{(3+)}_{\rm inv}(\tau) &= \sum_{k=0}^{4} B^{(3+)}_k \varphi_k(\tau) + C^{(3+)}_1 \tilde{\omega}(\tau) + \delta\chi^{(3+)}_{\rm inv}(\tau),\\
\chi^{(3-)}_{\rm inv}(\tau) &= \sum_{k=0}^{4} B^{(3-)}_k \varphi_k(\tau) + \delta\chi^{(3-)}_{\rm inv}(\tau),
\end{aligned} \tag{5.90}$$

where the functions $\varphi_k(\tau)$ are defined in Eq. (5.88), the coefficients $B^{(3\pm)}_k$, $C^{(3\pm)}_1$, $\chi^{(3)}_0$, $\Gamma^{(3)}_{\rm cusp}$ can be found in table 5.2 and the parameters for $\delta\chi^{(3\pm)}_{\rm inv}(\tau)$ (5.85) are collected in table 5.1.

To illustrate the numerical impact of our findings we compare the full NNLO invariant functions $\chi(a) = a\chi^{(1)} + a^2\chi^{(2)} + a^3\chi^{(3)}$ with the NLO, $\mathcal{O}(a^2)$, and the LO, $\mathcal{O}(a)$, results for a typical value of the coupling $\alpha_s/\pi = 0.1$ and four different quarks $n_f = 4$, see Fig. 5.2. Furthermore we took the exact expressions for the $f^{(3)}_\pm$-functions and performed the inversion Eq. (5.64) numerically. The result can be seen as a kind of exact result and is shown by dots. We conclude that the accuracy of our parametrization is pretty good. The remaining entries in the invariant kernel are, for the same values $n_f = 4$ and $\alpha_s/\pi = 0.1$,

$$\begin{aligned}
\Gamma_{\rm cusp} &= a\Gamma^{(1)}(1 + 8.019a + 80.53a^2 + \ldots) = a\Gamma^{(1)}(1 + 0.2005 + 0.0503 + \ldots),\\
\chi_0 &= a\chi^{(1)}_0(1 - 0.7935a - 141.3a^2 + \ldots) = a\chi^{(1)}_0(1 - 0.0198 - 0.0883 + \ldots).
\end{aligned} \tag{5.91}$$

Chapter 6

Evolution equation for local operators

6.1 Formulation in terms of local operators

So far we have studied the evolution of light-ray operators. This formulation can be viewed as very general, since the light-ray operator is nothing else but a generating function for renormalized (flavor-nonsinglet) leading-twist local operators

$$[\mathcal{O}](z_1,z_2) = \sum_{m,k} \widetilde{\mathcal{P}}_{mk}(z_1,z_2)[\mathcal{Q}_{mk}], \qquad\qquad \mathcal{Q}_{mk} = \bar{q}(x)\mathcal{P}_{mk}(\overleftarrow{D}_+,\overrightarrow{D}_+)q(x), \tag{6.1}$$

with $D_\mu = \partial_\mu - igA_\mu$ being the covariant derivative and $\mathcal{P}_{mk}(x,y)$ and $\widetilde{\mathcal{P}}_{mk}(x,y)$ being some polynomials of degree $m+k$, c.f. Eq. (3.15) for an example. Note that the polynomials $\mathcal{P}_{mk}(x,y)$ relate the local operators to the light-ray operator

$$[\mathcal{Q}_{mk}] = \mathcal{P}_{mk}(\partial_{z_1},\partial_{z_2})[\mathcal{O}](z_1,z_2)\Big|_{z_1=z_2=0}. \tag{6.2}$$

From this observation (6.2) and the expansion (6.1) it follows that the polynomials $\mathcal{P}_{mk}$ and $\widetilde{\mathcal{P}}_{mk}$ are orthogonal to each other

$$\mathcal{P}_{mk}(\partial_x,\partial_y)\widetilde{\mathcal{P}}_{mk}(x,y)\Big|_{x=y=0} = \delta_{mm'}\delta_{kk'}. \tag{6.3}$$

This property can be seen as a definition for a scalar product.

To rewrite the evolution equation for the light-ray operator

$$\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + \mathbb{H}\right)[\mathcal{O}(z_1,z_2)] = 0, \tag{6.4}$$

in terms of the local operators

$$\sum_{mk}\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\delta^{kk'}_{mm'} + \gamma^{kk'}_{mm'}\right)[\mathcal{Q}_{m'k'}] = 0, \tag{6.5}$$

one needs to act with the projection polynomials $\mathcal{P}_{mk}(x,y)$ on equation (6.4). In this way one derives the following relation between the evolution kernel and the mixing matrix

$$\boxed{\gamma^{kk'}_{mm'} \equiv \mathcal{P}_{mk}(\partial_{z_1},\partial_{z_2})\mathbb{H}\widetilde{\mathcal{P}}_{m'k'}(z_1,z_2)\Big|_{z_1=z_2=0},} \tag{6.6}$$

i.e. the mixing matrix is obtained as matrix elements of the evolution kernel w.r.t. the scalar product defined in equation (6.3).

While the non-local light-ray formulation is much more general, the local formulation turns out to be more useful and applicable for practical calculations, e.g. in the descriptions of distribution amplitudes (DA) and generalized parton distributions (GPD). The moments of these DAs and GPDs are matrix elements of local operators and their calculation using lattice QCD is an active field, with current precision that requires to be matched with NNLO accuracy on the perturbative side.

The procedure we are going to introduce is very general and works for any set of local operators. To get explicit results we need to stick to a particular example – we choose the so-called tower of local conformal operators, already introduced in chapter 3[1]. The matrix elements of these operators are commonly used in the description of DAs and GPDs and therefore lattice- and sum-rule-estimates at low scales are well established. At leading accuracy these operators diagonalize the evolution equations. This choice is not only convenient from the technical point of view, but also allows us to compare our findings with known results from literature [26, 23, 28, 27, 25, 24] where NLO expression are presented in just this basis. On the technical side we will benefit from two advantages: Firstly, solving the conformal constraint $[S_+,\mathbb{H}]=0$ is sufficiently easier in this language, however, we have also obtained exact analytic results for the non-invariant kernels in the light-cone formulation. Secondly, the crucial step of reconstructing the invariant kernels from the eigenvalues can be completely avoided here. We will see that due to Poincar*é*-invariance the forward anomalous dimensions enter the matrices as diagonal elements.

6.1.1 Gegenbauer basis operators

The desired set of operators – so-called conformal tower of operators – is defined by

$$\mathcal{Q}_{nk} \equiv \rho_n \langle \Big(S_+^{(0)}\Big)^{k-n} z_{12}^n|\mathcal{O}(z_1,z_2)\rangle_{(1,1)} = \partial_+^k C_n^{3/2}\left(\frac{\partial_-}{\partial_+}\right)\mathcal{O}(z_1,z_2)\Big|_{z_1=z_2=0}, \qquad k>n$$

where we use $\partial_\pm = \partial_{z_1}\pm\partial_{z_2}$ and $\rho_n = \frac{(n+1)(n+2)!}{2}$. The lowest-weight (conformal invariant) operator is given by $\mathcal{Q}_n \equiv \mathcal{Q}_{nn}$. Here and in what follows $C_n^{3/2}(x)$ are Gegenbauer polynomials. The $s\ell(2)$-invariant scalar product $\langle\cdot|\cdot\rangle_{(j_1,j_2)}$ is defined by

$$\langle\varphi(z_1,z_2)|\psi(z_1,z_2)\rangle_{(j_1,j_2)} = \left(\prod_{i=1}^{2}\frac{2j_i-1}{\pi}\int_{|z_i|<1} d^2z_i(1-|z_i|^2)^{2j_i-2}\right)\varphi(z_1^*,z_2^*)\phi(z_1,z_2), \tag{6.7}$$

where the integration goes over the unit disc in the complex plane and z_i^* denotes the complex conjugated variable. The representation in terms of this scalar product turns out to be more useful for our context,

[1]Just as the symmetry generators get perturbative corrections, the conformal operators do as well, of course. Sticking to the conventions used in literature, we will consider the conformal operators just to leading order accuracy. For the exact operators to NNLO, including radiative corrections, see appendix H.

while the representation in terms of Gegenbauer polynomials is the one used throughout most of the literature.

The operators $\mathcal{Q}_{nk}$ are labeled according to their Poincaré representation (index k, i.e. number of total derivatives) and conformal representation (index n). To leading order in the strong coupling both symmetries are preserved, and the operators diagonalize the RGE (6.5). Beyond LO conformal symmetry breaks down and operators with different conformal spin start to mix under evolution, while the number of total derivatives remains to be conserved under evolution to all orders. We will see that this fact simplifies the mixing pattern significantly.

The expansion of the light-ray operator in terms of these operators reads [80]

$$\mathcal{O}(x; z_1, z_2) = \sum_{n=0,k=n}^{\infty} \Phi_{nk}(z_1, z_2)\mathcal{Q}_{nk}(x), \tag{6.8}$$

where we define the "wave functions"

$$\Phi_{nk}(z_1, z_2) = \tilde{\omega}_{nk}(S_+^{(0)})^{k-n} z_{12}^n, \qquad \tilde{\omega}_{nk} = 2\frac{2n+3}{(k-n)!}\frac{\Gamma(n+2)}{\Gamma(n+k+4)}. \tag{6.9}$$

It is important to note that we use the canonical expressions for both the spin generator S_+ as well as the conformal operator $\mathcal{Q}_n(x)$.

The "wave functions" form an orthogonal and complete set of functions w.r.t. to the canonical $s\ell(2)$-scalar product

$$\langle \Phi_{nk} | \Phi_{n'k'} \rangle = \delta_{kk'}\delta_{nn'} ||\Phi_{nk}||^2. \tag{6.10}$$

The numerical factor $||\Phi_{nk}||^2 = \tilde{\omega}_{nk}\rho_n^{-1}$ is the norm of the "wave function". To get the connection to the scalar product (6.3) we notice that the Gegenbauer polynomials are related via a Fourier transformation

$$||\Phi_{nk}||^{-2} \langle e^{z_1 u_1 + z_2 u_2} | \Phi_{nk} \rangle = (u_1 + u_2)^k C_n^{3/2}\left(\frac{u_1 - u_2}{u_1 + u_2}\right). \tag{6.11}$$

The spin generators $S_\pm$ act as rising and lowering operators on the set of these coefficient functions, while S_0 acts diagonal

$$\begin{aligned} S_0^{(0)}\Phi_{nk}(z_1, z_2) &= (k+2)\Phi_{nk}(z_1, z_2), \\ S_+^{(0)}\Phi_{nk}(z_1, z_2) &= (k-n+1)(n+k+4)\Phi_{nk+1}(z_1, z_2), \\ S_-\Phi_{nk}(z_1, z_2) &= -\Phi_{nk-1}(z_1, z_2). \end{aligned}$$

Let us now consider a general quantity $\mathcal{A}$ acting on the quantum fields. In terms of local operators the action is realized by an operator expansion, with "matrix elements" serving as expansion coefficients

$$[\mathcal{A}, \mathcal{Q}_{nk}] = \sum_{n'k'} A_{nn'}^{kk'} \mathcal{Q}_{n'k'}. \tag{6.12}$$

In the case of non-local operators we can trade the action of $\mathcal{A}$ for an integro-differential operator $\mathbf{A}$ acting on the light-ray operator $\mathcal{O}(z_1, z_2)$, and by means of the expansion (6.8) on the wave functions,

$$[\mathcal{A}, \mathcal{O}](z_1, z_2) = [\mathbf{A}\mathcal{O}](z_1, z_2) = \sum_{nk} [\mathbf{A}\Phi_{nk}](z_1, z_2)\mathcal{Q}_{nk}. \tag{6.13}$$

However, from Eq. (6.12) it is clear that we can expand the resulting coefficient functions as

$$[\mathbf{A}\Phi_{nk}](z_1,z_2) = \sum_{n'k'} A^{k'k}_{n'n}\Phi_{n'k'}(z_1,z_2), \tag{6.14}$$

with the expansion coefficients given by the transposed matrix elements. Using the orthogonality relation (6.10) one can easily solve for the expansion coefficients

$$A^{kk'}_{nn'} = ||\Phi_{nk}||^{-2}\langle\Phi_{nk}(z_1,z_2)|[\mathbf{A}\Phi_{n'k'}](z_1,z_2)\rangle_{(1,1)} \equiv \langle nk|\mathbf{A}|n'k'\rangle, \tag{6.15}$$

The "matrix elements" $A^{kk'}_{nn'}$ depend in general on four indices, the upper (lower) ones label the Poincare (conformal) representation. Therefore it is clear that the upper indices need to fulfill $k' = k + d(\mathbf{A})$, where $d(\mathbf{A})$ denotes the canonical dimension of the quantity $\mathbf{A}$, i.e. $[S^{(0)}_0, \mathbf{A}] = d(\mathbf{A})\mathbf{A}$. This reduces the number of independent indices by one and allows one to write $A^{kk'}_{nn'} \equiv A_{nn'}(k)$. If, in addition, the observable $\mathbf{A}$ is translation invariant, $[S_-, \mathbf{A}] = 0$ there is no mixing w.r.t. the upper index $A^{kk'}_{nn'} \equiv A_{nn'}$ at all.

Using these findings the evolution equation Eq. (6.5) for the local operators simplifies to

$$\boxed{\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a}\right)[\mathcal{Q}_{nk}] = -\sum_{n'=0}^{n}\gamma_{nn'}\,[\mathcal{Q}_{n'k}],} \tag{6.16}$$

with a lower triangular mixing matrix $\gamma_{nn'}$. Its diagonal elements are equal to the anomalous dimensions

$$\gamma_{nn'} = 0 \quad \text{if} \quad n' > n, \qquad\qquad \gamma_{nn} = \gamma_n. \tag{6.17}$$

Since $\gamma_{nn'}$ does not depend on k, the second subscript k for the operators is essentially redundant. In what follows we will use a "hat" for the anomalous dimensions and other quantities in matrix notation

$$\widehat{\boldsymbol{\gamma}} \equiv \gamma_{nn'}. \tag{6.18}$$

The constraint on the operator mixing in the light-ray operator representation that follows from conformal algebra $[S_+, \mathbb{H}] = 0$ takes the form (3.42)

$$\left[S^{(0)}_+, \mathbb{H}(a)\right] = \left[\mathbb{H}(a), z_1+z_2\right]\left(\bar\beta(a) + \frac{1}{2}\mathbb{H}(a)\right) + \left[\mathbb{H}(a), z_{12}\Delta(a)\right]. \tag{6.19}$$

To translate this equation into the local operator representation, we define the matrices

$$\begin{aligned}
\mathbf{a}_{mn}(k) &= \langle m,k|S^{(0)}_+|n,k-1\rangle,\\
\mathbf{b}_{mn}(k) &= \langle m,k|z_1+z_2|n,k-1\rangle,\\
\boldsymbol{\gamma}_{mn} &= \langle m,k|\mathbb{H}|n,k\rangle,\\
\mathbf{w}_{mn} &= \langle m,k|z_{12}\Delta|n,k-1\rangle.
\end{aligned} \tag{6.20}$$

The latter two are nontrivial and need to be calculated in a perturbative expansion

$$\begin{aligned}
\widehat{\boldsymbol{\gamma}}(a) &= a\widehat{\boldsymbol{\gamma}}^{(1)} + a^2\widehat{\boldsymbol{\gamma}}^{(2)} + a^3\widehat{\boldsymbol{\gamma}}^{(3)} + \dots, \qquad a = \frac{\alpha_s}{4\pi},\\
\widehat{\mathbf{w}}(a) &= a\widehat{\mathbf{w}}^{(1)} + a^2\widehat{\mathbf{w}}^{(2)} + \dots.
\end{aligned} \tag{6.21}$$

The first two matrix elements are easily computed,

$$\begin{aligned}\mathbf{a}_{mn}(k) &= -(m-k)(m+k+3)\delta_{mn} \equiv -\mathbf{a}(m,k)\delta_{mn},\\ \mathbf{b}_{mn}(k) &= 2(k-n)\delta_{mn} - 2(2n+3)\vartheta_{mn},\end{aligned} \tag{6.22}$$

where we introduced a discrete step function

$$\vartheta_{mn} = \begin{cases} 1 \text{ if } m-n>0 \text{ and even} \\ 0 \text{ else.} \end{cases}$$

The conformal constraint (6.19) reads in matrix notation

$$\boxed{[\widehat{\mathbf{a}}, \widehat{\boldsymbol{\gamma}}(a)] = [\widehat{\gamma}(a), \widehat{\mathbf{b}}]\left(\bar{\beta}(a) + \frac{1}{2}\widehat{\gamma}(a)\right) + [\widehat{\gamma}, \widehat{\mathbf{w}}(a)].} \tag{6.23}$$

Note that the matrices $\widehat{\mathbf{a}}(k)$ and $\widehat{\mathbf{b}}(k)$ (6.22) depend in principle on the total number of derivatives k. However, due to the fact that only diagonal elements depend on this parameter, the dependence on k drops out in the commutator. Hence we can safely omit it.

In complete analogy to the light-ray operator formulation, this equation fixes the non-diagonal (i.e. canonically non-invariant) part of the anomalous dimension matrix. Indeed, the commutator on the l.h.s. of Eq. (6.23) takes the form

$$[\widehat{\mathbf{a}}, \widehat{\boldsymbol{\gamma}}(a)]_{mn} = (-\mathbf{a}(m,k) + \mathbf{a}(n,k))\gamma_{mn} = -\mathbf{a}(m,n)\gamma_{mn}, \tag{6.24}$$

that vanishes for $n = m$. Therefore we decompose

$$\widehat{\gamma}(a) = \widehat{\boldsymbol{\gamma}}^{\mathrm{D}}(a) + \widehat{\boldsymbol{\gamma}}^{\mathrm{ND}}(a), \tag{6.25}$$

so that the diagonal elements are given by the forward anomalous dimensions

$$\widehat{\boldsymbol{\gamma}}^{\mathrm{D}}(a) = \mathrm{diag}\{\gamma_0(a), \gamma_1(a), \gamma_2(a), \ldots\}, \tag{6.26}$$

that are known to four-loop accuracy [11, 12, 13, 14, 15, 16, 17] and the non-diagonal elements of the mixing matrix are given by [23]

$$\boxed{\widehat{\boldsymbol{\gamma}}^{\mathrm{ND}}(a) = \mathcal{G}\left\{[\widehat{\gamma}(a), \widehat{\mathbf{b}}]\left(\frac{1}{2}\widehat{\gamma}(a) + \bar{\beta}(a)\right) + [\widehat{\gamma}(a), \widehat{\mathbf{w}}(a)]\right\},} \tag{6.27}$$

where

$$\mathcal{G}\{\widehat{M}\}_{mn} = -\frac{M_{mn}}{\mathbf{a}(m,n)}. \tag{6.28}$$

This solution (6.27) is valid to any order. Again, in a perturbative expansion the non-diagonal elements of the anomalous dimension on the l.h.s. are fixed by products of lower-order operators on the r.h.s. of Eq. (6.27).

6.1.2 Two-loop anomalous dimension matrix

The expansion of $\widehat{\boldsymbol{\gamma}}^{\mathrm{ND}} = \sum\limits_{\ell=2}^{\infty} a^\ell \widehat{\boldsymbol{\gamma}}^{(\ell),\mathrm{ND}}$ starts at the order $\mathcal{O}(a^2)$ and is uniquely fixed by the lower-order anomalous dimension, β-function and conformal anomaly. In particular to the leading accuracy we find

$$\widehat{\boldsymbol{\gamma}}^{(2),\mathrm{ND}} = \mathcal{G}\left\{[\widehat{\boldsymbol{\gamma}}^{(1)}, \widehat{\mathbf{b}}]\left(\frac{1}{2}\widehat{\boldsymbol{\gamma}}^{(1)} + \beta_0\right) + [\widehat{\boldsymbol{\gamma}}^{(1)}, \widehat{\mathbf{w}}^{(1)}]\right\}. \tag{6.29}$$

Here $\widehat{\boldsymbol{\gamma}}^{(1)}$ is the well-known (diagonal) matrix of one-loop anomalous dimensions

$$\gamma^{(1)}_{mn} = \gamma^{(1)}_n \delta_{mn}, \qquad \gamma^{(1)}_n = 2C_F\big(4S_1(n+1) - \frac{2}{(n+1)(n+2)} - 3\big), \tag{6.30}$$

and $\widehat{\mathbf{w}}^{(1)}$ is the one-loop conformal anomaly

$$\mathbf{w}^{(1)}_{mn} = 4C_F(2n+3)\,\mathbf{a}(m,n)\left(\frac{A_{mn} - S_1(m+1)}{(n+1)(n+2)} + \frac{2A_{mn}}{\mathbf{a}(m,n)}\right)\vartheta_{mn}, \tag{6.31}$$

where

$$A_{mn} = S_1\left(\frac{m+n+2}{2}\right) - S_1\left(\frac{m-n-2}{2}\right) + 2S_1(m-n-1) - S_1(m+1). \tag{6.32}$$

Collecting everything one obtains the two-loop anomalous dimension matrix:

$$\gamma^{(2)}_{mn} = \delta_{mn}\gamma^{(2)}_n - \frac{\gamma^{(1)}_m - \gamma^{(1)}_n}{\mathbf{a}(m,n)}\left\{-2(2n+3)\left(\beta_0 + \frac{1}{2}\gamma^{(1)}_n\right)\vartheta_{mn} + \mathbf{w}^{(1)}_{mn}\right\}. \tag{6.33}$$

The first few elements $(0 \le n \le 7,\ 0 \le m \le 7)$ for $N_c = 3$ are

$$\gamma^{(2)}_{mn} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{23488}{243} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{260}{9} & 0 & \frac{34450}{243} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{8668}{243} & 0 & \frac{5241914}{30375} & 0 & 0 & 0 & 0 \\
\frac{52}{9} & 0 & \frac{8512}{243} & 0 & \frac{662846}{3375} & 0 & 0 & 0 \\
0 & \frac{120692}{8505} & 0 & \frac{261232}{7875} & 0 & \frac{83363254}{385875} & 0 & 0 \\
-\frac{2054}{14175} & 0 & \frac{34243}{2025} & 0 & \frac{2208998}{70875} & 0 & \frac{718751707}{3087000} & 0 \\
0 & \frac{226526}{35721} & 0 & \frac{982399}{55125} & 0 & \frac{7320742}{250047} & 0 & \frac{557098751203}{2250423000}
\end{pmatrix}$$

$$- n_f \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{512}{81} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{40}{9} & 0 & \frac{830}{81} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{88}{27} & 0 & \frac{26542}{2025} & 0 & 0 & 0 & 0 \\
\frac{104}{45} & 0 & \frac{1064}{405} & 0 & \frac{31132}{2025} & 0 & 0 & 0 \\
0 & \frac{1144}{567} & 0 & \frac{232}{105} & 0 & \frac{1712476}{99225} & 0 & 0 \\
\frac{4108}{2835} & 0 & \frac{242}{135} & 0 & \frac{1804}{945} & 0 & \frac{3745727}{198450} & 0 \\
0 & \frac{2372}{1701} & 0 & \frac{506}{315} & 0 & \frac{2860}{1701} & 0 & \frac{36241943}{1786050}
\end{pmatrix}. \tag{6.34}$$

We have checked that our expressions for the one-loop conformal anomaly and the two-loop anomalous dimension matrix coincide identically with the results in [24].[2]

[2] The explicit relation to the notations in [24] is as follows: $\widehat{\mathbf{a}}(k)|_{[24]} = -\frac{1}{2}\widehat{\mathbf{a}}(k)$, $\widehat{\mathbf{b}}(k)|_{[24]} = \widehat{\mathbf{b}}(k)$, $\widehat{\mathbf{w}}^{(1)}|_{[24]} = -\widehat{\mathbf{w}}^{(1)}$, and $\widehat{\gamma}^{(i)}_{[24]} = 2^i\widehat{\gamma}^{(i-1)}$. A perturbative expansion in [24] is done in powers of $\alpha_s/(2\pi)$, e.g., $\widehat{\boldsymbol{\gamma}}(a) = \sum_i (2a)^i \widehat{\boldsymbol{\gamma}}^{(i-1)}$.

6.1.3 Three-loop anomalous dimension matrix

Expanding Eq. (6.27) to the third order, we obtain the three-loop nondiagonal anomalous dimension matrix in the form

$$\widehat{\gamma}^{(3),\mathrm{ND}} = \mathcal{G}\Big\{[\widehat{\gamma}^{(2)}, \widehat{\mathbf{b}}](\frac{1}{2}\widehat{\gamma}^{(1)} + \beta_0) + [\widehat{\gamma}^{(2)}, \widehat{\mathbf{w}}^{(1)}] + [\widehat{\gamma}^{(1)}, \widehat{\mathbf{b}}](\frac{1}{2}\widehat{\gamma}^{(2)} + \beta_1) + [\widehat{\gamma}^{(1)}, \widehat{\mathbf{w}}^{(2)}]\Big\}. \tag{6.35}$$

In addition to the already known quantities, this expression involves the matrix element of the two-loop conformal anomaly (4.62)

$$\mathbf{w}^{(2)}_{mn} = \langle m, k|z_{12}\Delta^{(2)}|n, k-1\rangle = \frac{1}{4}[\widehat{\gamma}^{(2)}, \widehat{\mathbf{b}}] + \Delta\mathbf{w}^{(2)}_{mn}, \tag{6.36}$$

where

$$\Delta\mathbf{w}^{(2)}_{mn} = \langle m, k|z_{12}\Delta^{(2)}_{+}|n, k-1\rangle. \tag{6.37}$$

The explicit expression for the operator $\Delta^{(2)}_{+}$ can be found in Eq. (4.62). The determination of $\Delta\mathbf{w}^{(2)}_{mn}$ for given values m, n is straightforward[3]. Splitting the result into different color structures

$$\Delta\widehat{\mathbf{w}}^{(2)} = C_F^2\Delta\widehat{\mathbf{w}}^{P} + \frac{C_F}{N_c}\Delta\widehat{\mathbf{w}}^{FA} + \beta_0 C_F\Delta\widehat{\mathbf{w}}^{bF},$$

[3]The calculation of analytic results for arbitrary n, m is currently not feasible. On a standard desktop-PC fixed-m, n values can be easily evaluated to the order $m, n \simeq 100$. If an application requires a certain axis of the anomaly matrix, a possible strategy is to fit the asymptotic behavior along this axis. In particular we have found the following asymptotic behavior along the

- vertical axis (n is fixed): $\mathbf{w}_{mn} \sim \ln m$.

- diagonal axis ($m - n$ is fixed): $\mathbf{w}_{m,m-(m-n)} \sim m$.

For the RGE-mixing the horizontal axis is required, which is bounded due to the triangularity of $\hat{\mathbf{w}}$: $\mathbf{w}_{mn} = 0$ for $m \leq n$.

we get for the first few elements ($0 \le n \le 5,\ 1 \le m \le 7$)

$$
\Delta\widehat{\mathbf{w}}^{FA} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{75}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{5075}{108} & 0 & 0 & 0 & 0 \\
-\frac{679}{15} & 0 & -\frac{58723}{720} & 0 & 0 & 0 \\
0 & -\frac{7399}{90} & 0 & -\frac{724339}{6000} & 0 & 0 \\
-\frac{1070777}{16800} & 0 & -\frac{12001}{96} & 0 & -\frac{123357091}{756000} & 0 \\
0 & -\frac{22974677}{211680} & 0 & -\frac{101507627}{588000} & 0 & -\frac{308384869}{1481760}
\end{pmatrix},
$$

$$
\Delta\widehat{\mathbf{w}}^{P} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2965}{144} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1176553}{10800} & 0 & 0 & 0 & 0 \\
-\frac{140959}{9000} & 0 & -\frac{7387709}{36000} & 0 & 0 & 0 \\
0 & -\frac{75208391}{617400} & 0 & -\frac{2111899581}{6860000} & 0 & 0 \\
-\frac{68372343}{5488000} & 0 & -\frac{5045910661}{21168000} & 0 & -\frac{307457793929}{740880000} & 0 \\
0 & -\frac{99911324293}{800150400} & 0 & -\frac{808931234579}{2222640000} & 0 & -\frac{2942615103467}{5601052800}
\end{pmatrix},
$$

$$
\Delta\widehat{\mathbf{w}}^{bF} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
35 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1855}{27} & 0 & 0 & 0 & 0 \\
\frac{105}{2} & 0 & \frac{2555}{24} & 0 & 0 & 0 \\
0 & \frac{2891}{30} & 0 & \frac{146839}{1000} & 0 & 0 \\
\frac{6459}{100} & 0 & \frac{390313}{2700} & 0 & \frac{2552407}{13500} & 0 \\
0 & \frac{202829}{1764} & 0 & \frac{4798313}{24500} & 0 & \frac{14365013}{61740}
\end{pmatrix}. \tag{6.38}
$$

Using these expressions and the diagonal matrix elements from [9] we obtain the full three-loop anomalous dimension matrix

$$
\widehat{\gamma}^{(3)} = \mathrm{diag}\{\gamma_0^{(3)}, \gamma_1^{(3)}, \ldots\} + \widehat{\gamma}^{(3)}_{\langle 1\rangle} + n_f\, \widehat{\gamma}^{(3)}_{\langle n_f\rangle} + n_f^2\, \widehat{\gamma}^{(3)}_{\langle n_f^2\rangle}, \tag{6.39}
$$

where the off-diagonal matrices for $N_c = 3$ and different powers of n_f in the range $0 \leq n \leq 7,\ 0 \leq m \leq 7$ are given by the following expressions:

$$\widehat{\gamma}^{(3)}_{\langle 1\rangle} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{49024}{81} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{36623912}{54675} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3911}{27} & 0 & \frac{23599891}{36450} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8049304723}{31255875} & 0 & \frac{320657981731}{520931250} & 0 & 0 & 0 & 0 \\ \frac{281851388261}{7501410000} & 0 & \frac{208052194247}{714420000} & 0 & \frac{21898269506047}{37507050000} & 0 & 0 & 0 \\ 0 & \frac{7192640196053}{56710659600} & 0 & \frac{159898280729473}{525098700000} & 0 & \frac{220023775251709}{396974617200} & 0 & 0 \end{pmatrix},$$

$$\widehat{\gamma}^{(3)}_{\langle n_f\rangle} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{28700}{243} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5762188}{54675} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1279108}{30375} & 0 & -\frac{26434828}{273375} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{849255644}{18753525} & 0 & -\frac{516077668}{5788125} & 0 & 0 & 0 & 0 \\ -\frac{54942827}{2500470} & 0 & -\frac{636248861}{13395375} & 0 & -\frac{77507831071}{937676250} & 0 & 0 & 0 \\ 0 & -\frac{1660976917}{67512690} & 0 & -\frac{7496172461}{156279375} & 0 & -\frac{36406093529}{472588830} & 0 & 0 \end{pmatrix},$$

$$\widehat{\gamma}^{(3)}_{\langle n_f^2\rangle} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{236}{81} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3172}{1215} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1316}{2025} & 0 & \frac{41356}{18225} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{187412}{178605} & 0 & \frac{22012}{11025} & 0 & 0 & 0 & 0 \\ \frac{55169}{893025} & 0 & \frac{10793}{9450} & 0 & \frac{176539}{99225} & 0 & 0 & 0 \\ 0 & \frac{726193}{1607445} & 0 & \frac{681503}{595350} & 0 & \frac{515359}{321489} & 0 & 0 \end{pmatrix}. \tag{6.40}$$

For completeness we also list the first few anomalous dimensions [9]:

$$\begin{aligned}
\gamma_0^{(3)} =& 0, \\
\gamma_1^{(3)} =& \frac{2560}{81}\zeta_3 + \frac{11028416}{6561} - \left(\frac{2560}{27}\zeta_3 + \frac{334400}{2187}\right) n_f - \frac{1792}{729} n_f^2, \\
\gamma_2^{(3)} =& \frac{2200}{81}\zeta_3 + \frac{64486199}{26244} - \left(\frac{4000}{27}\zeta_3 + \frac{967495}{4374}\right) n_f - \frac{2569}{729} n_f^2, \\
\gamma_3^{(3)} =& \frac{11512}{405}\zeta_3 + \frac{245787905651}{82012500} - \left(\frac{5024}{27}\zeta_3 + \frac{726591271}{2733750}\right) n_f - \frac{384277}{91125} n_f^2, \\
\gamma_4^{(3)} =& \frac{11312}{405}\zeta_3 + \frac{559048023977}{164025000} - \left(\frac{5824}{27}\zeta_3 + \frac{90842989}{303750}\right) n_f - \frac{431242}{91125} n_f^2, \\
\gamma_5^{(3)} =& \frac{558896}{19845}\zeta_3 + \frac{10337334685136687}{2756768175000} - \left(\frac{45376}{189}\zeta_3 + \frac{713810332943}{2187911250}\right) n_f - \frac{160695142}{31255875} n_f^2,
\end{aligned}$$

$$
\begin{aligned}
\gamma_6^{(3)} &= \frac{185482}{6615}\zeta_3 + \frac{59388575317957639}{14702763600000} - \left(\frac{16432}{63}\zeta_3 + \frac{12225186887503}{35006580000}\right) n_f - \frac{1369936511}{250047000} n_f^2, \\
\gamma_7^{(3)} &= \frac{5020814}{178605}\zeta_3 + \frac{46028648192099544431}{10718314664400000} - \left(\frac{158128}{567}\zeta_3 + \frac{349136571992501}{945177660000}\right) n_f - \frac{38920977797}{6751269000} n_f^2.
\end{aligned}
\tag{6.41}
$$

Finally we want to illustrate the size of the three-loop correction. In order to do so we consider the full NNLO anomalous dimension matrix for $n_f = 4$ in the range $(0 \le n \le 4,\ 0 \le m \le 4)$:

$$
\widehat{\gamma} = a\widehat{\gamma}^{(1)}\left[\mathbb{1} + a\Delta\widehat{\gamma}\right],
$$

with

$$
\Delta\widehat{\gamma} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 10 + 86a & 0 & 0 & 0 \\
1 + 16a & 0 & 9.1 + 78a & 0 & 0 \\
0 & 1.6 + 21a & 0 & 8.6 + 72a & 0 \\
-0.21 - 0.82a & 0 & 1.5 + 18a & 0 & 8.3 + 71a
\end{pmatrix}.
$$

One notices a strong suppression of the non-diagonal elements compared to the diagonal ones. Moreover, for realistic values of the strong coupling $a = \alpha_s/(4\pi) \sim 1/40$ the three-loop contribution is on the average about 30% of the two-loop result.

Chapter 7

Sample application: Pion distribution amplitude

In this chapter we guide towards a possible strategy how to use our results for phenomenological applications. Our results are aimed for the description of exclusive reactions. On the theoretical level such processes are much more challenging compared to inclusive ones. The theoretical description is, whenever factorization is applicable, given in terms of a perturbative "hard scattering part" and non-perturbative "soft functions". The latter is usually formulated by DAs (vacuum-to-hadron matrix element) or GPDs (hadron-to-hadron matrix element). In the following we will focus on the description of DAs. In contrast to the usual parton distribution functions, which represent the overall probability to find a parton with a certain momentum fraction in the respective hadron, DAs denote the probability amplitude of finding the valence Fock state ($|q\bar{q}\rangle$ for a meson) with each parton carrying a certain momentum fraction. In general all higher states, i.e. states with additional quark-antiquark-pairs and gluons as $|q\bar{q}G\rangle, |q\bar{q}q\bar{q}\rangle \ldots$ etc. also contribute to physical processes. However, for reactions with large momentum transfer Q, the higher states are suppressed by additional powers of $1/Q^2$ compared to the leading Fock state.

There is a manifold field of applications of distribution amplitudes. In the mesonic sector, where the results from this work can be applied, the classical example is the pion DA involved in the theoretical description of the process

$$\gamma\gamma^* \to \pi^0, \tag{7.1}$$

that is still arousing interest in B-factories like BaBar [81] and Belle [82]. Further applications of the pion DA include the electromagnetic pion form factor, and the semi-leptonic [83, 84, 85] and hadronic B-meson decays [86, 87]

$$B \to \bar{\nu}_\ell \, \ell \, \pi, \qquad\qquad B \to \pi \, \pi, \tag{7.2}$$

respectively.

7.1 Definition of the leading-twist pion distribution amplitude

The pion wave function is defined as the vacuum-to-pion matrix element of the non-local light-ray operator

$$\mathcal{O}_A^{(n)}(z_1, z_2) = \bar{q}(z_1 n)\not{n}\gamma_5 q(z_2 n). \tag{7.3}$$

As before, n is a light-like direction, $n^2 = 0$. The subscript $_A$ denotes the additional γ_5 matrix structure compared to the definition (3.15). For non-singlet states the evolution of this axial-vector operator is governed by the same evolution equation as for the vector operator in Eq. (3.15). Therefore we will omit this subscript in what follows.

In term of the pion wave-function the distribution amplitude is defined by

$$\boxed{\langle 0|[\mathcal{O}^{(n)}](z_1, z_2)|\pi(p)\rangle = if_\pi (p\cdot n)\int_0^1 du e^{iz_{12}^u (pn)}\phi(u).} \tag{7.4}$$

Here $f_\pi = 93$ MeV is the pion decay constant and square brackets denote renormalization in $\overline{\text{MS}}$-scheme. The function $\phi(u)$ is called pion distribution amplitude (DA) and contains all non-perturbative input. It is a function of the momentum fraction u of each of the two valence quarks and, although not denoted explicitly, it depends on the scale μ.

The normalization is fixed by the requirement that the local operator matrix element is equal to the pion decay constant

$$\langle 0|\bar{q}(0)\gamma_\mu\gamma_5 q(0)|\pi(p)\rangle = if_\pi p_\mu, \qquad \boxed{\int_0^1 du\phi(u) = 1.} \tag{7.5}$$

The scale dependence of the DA is governed by the evolution equation for the light-ray operator

$$\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + \mathbb{H}\right)\otimes[\mathcal{O}^{(n)}](z_1, z_2) = 0, \tag{7.6}$$

where $\mathbb{H}$ is the evolution kernel presented in chapter 5 to three-loop accuracy. For practical applications the coordinate representation of the evolution equation is rather inconvenient. The original formulation of the ERBL equation [69, 70] is given in momentum space. In this formalism the evolution kernel is given as an integral operator acting on the momentum fraction

$$\left(\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + \mathbb{V}\right)\otimes\phi(u) = 0, \qquad \mathbb{V}\otimes\phi(u) = \int_0^1 dw V(u, w)\phi(w), \tag{7.7}$$

The kernel function $V(u, w) = aV^{(1)}(u, w) + a^2 V^{(2)}(u, w) + \mathcal{O}(a^3)$ is known to NLO accuracy [27]. While it is an easy exercise to verify the equivalence between the coordinate kernel $\mathbb{H}$ and the momentum kernel $\mathbb{V}$ by mapping both to a local matrix representation, see chapter 6, the explicit restoration of $\mathbb{V}$ to three-loop accuracy from the coordinate representation becomes a tedious task.

Alternatively one can employ the OPE of the light-ray operator to expand the pion DA in terms of (matrix elements of) local operators. We expand the non-local operator as in Eq. (6.8)

$$\mathcal{O}(z_1, z_2) = \sum_{n=0,k=n} \Phi_{nk}(z_1, z_2)\mathcal{Q}_{nk}, \tag{7.8}$$

with the same coefficient functions $\Phi_{nk}(z_1, z_2)$, see Eq. (6.9), and the operators

$$\mathcal{Q}_{nk} \equiv (\partial_{z_1} + \partial_{z_2})^k C_n^{3/2}\left(\frac{\partial_{z_1} - \partial_{z_2}}{\partial_{z_1} + \partial_{z_2}}\right) \mathcal{O}(z_1, z_2)\bigg|_{z_1=z_2=0}. \tag{7.9}$$

We want to remind that we use again the same notation despite the additional γ_5 structure. We can derive an explicit expression for the pion DA by acting with the differential operator $\partial_+^n C_n^{3/2}\left(\frac{\partial_-}{\partial_+}\right)$ on both side of the definition (7.4) of the pion wave function: The l.h.s. reproduces the (matrix elements of) local operators (7.9) while the r.h.s. is given by the integral of the pion DA with a Gegenbauer polynomial

$$\langle 0|\mathcal{Q}_{nn}|\pi(p)\rangle = f_\pi (ip\cdot n)^{n+1} \int_0^1 du\phi(u) C_n^{3/2}(2u-1)\,. \tag{7.10}$$

Using orthogonality of the Gegenbauer polynomials

$$\int_0^1 du(u\bar{u}) C_m^{3/2}(2u-1) C_n^{3/2}(2u-1) = \mathcal{N}_n \delta_{nm}, \tag{7.11}$$

where $\mathcal{N}_n = \frac{(n+1)(n+2)}{4(2n+3)}$, we can establish the *conformal wave expansion* for the pion DA

$$\boxed{\phi(u) = u\bar{u} \sum_n \mathcal{N}_n^{-1} C_n^{3/2}(2u-1)\langle\!\langle \mathcal{Q}_n \rangle\!\rangle,} \tag{7.12}$$

with the so-called *moments* of the DA

$$\boxed{\langle\!\langle \mathcal{Q}_n \rangle\!\rangle \equiv \frac{\langle 0|\mathcal{Q}_{nn}|\pi(p)\rangle}{f_\pi (ip\cdot n)^{n+1}} = \int_0^1 du\phi(u) C_n^{3/2}(2u-1)\,.} \tag{7.13}$$

In the literature one usually parametrizes the moments by $\langle\!\langle \mathcal{Q}_n \rangle\!\rangle = 6\mathcal{N}_n a_n$. Using the transformation properties of the pion wave function (7.4) under G-parity one derives the symmetry property

$$\phi(u) = \phi(\bar{u}), \tag{7.14}$$

imposing that the conformal wave expansion (7.12) covers only even moments in the sum.

There are three major advantages of the representation (7.12):

i. The whole renormalization scale dependence is encoded in the moments $\langle\!\langle \mathcal{Q}_n \rangle\!\rangle = \langle\!\langle \mathcal{Q}_n(\mu) \rangle\!\rangle$, all other parts do not depend on the scale.

ii. Due to the definition (7.13) the scale dependence of the moments is governed by the evolution equation of the local operators (6.16), i.e.

$$\boxed{(\mu\partial_\mu + \beta(a)\partial_a)\langle\!\langle \mathcal{Q}_n \rangle\!\rangle = -\sum_{m=0}^{n} \gamma_{nm} \langle\!\langle \mathcal{Q}_m \rangle\!\rangle,} \tag{7.15}$$

that is known to the three-loop accuracy, see chapter 6.

iii. the expectation values for these couplings $\langle\!\langle \mathcal{Q}_n \rangle\!\rangle$ at low scales have been studied extensively in non-perturbative calculations such as sum rules [88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 84] and lattice simulations [100, 101, 102].

n	0	2	4	6	8	10	12	...
a_n^{latt}	1	0.136	n.a.	n.a.	n.a.	n.a.	n.a.	...
a_n^{AdS}	1	0.146	0.0573	0.0305	0.0189	0.0129	0.00935	...

Table 7.1: Conformal moments of pion DA from the lattice [102] and the AdS/CFT-model [103, 104] at the scale $\mu = 2$ GeV.

The third point, however, exposes also the limitations of the formalism: only a limited number of moments can be accessed via non-perturbative methods and therefore in practice one needs to truncate the conformal wave expansion (7.12) after the first few terms. At large scale the higher moments are sufficiently suppressed due to the renormalization group flow, that drives all moments except the leading one to zero

$$\lim_{\mu\to\infty} \langle\!\langle \mathcal{Q}_n \rangle\!\rangle = \begin{cases} 1 \text{ if } n = 0, \\ 0 \text{ else.} \end{cases} \tag{7.16}$$

The zeroth moment is constant as it corresponds to a conserved current and its value is fixed to $\langle\!\langle \mathcal{Q}_0 \rangle\!\rangle = 1$ due to the normalization of the pion DA, see Eq. (7.5). Therefore in the asymptotic limit the pion DA reduces to

$$\lim_{\mu\to\infty} \phi(u) \equiv \phi^{\text{as}}(u) = 6u\bar{u}. \tag{7.17}$$

We want to introduce and compare two different models for the pion DA (at reference scale $\mu = 2\,\text{GeV}$):

- the one inspired by the conformal wave expansion with truncation after the first two terms $\phi^{\text{latt}}(u) = 6u\bar{u}(1 + C_2^{3/2}(2u-1)a_2^{\text{latt}})$. For the second moment we choose the value $a_2^{\text{latt}}(2\,\text{GeV}) = 0.136$ from Ref. [102], that is in very good agreement with recent lattice and sum rule calculations and in accordance with the second model.
- the model inspired by AdS/CFT duality $\phi^{\text{AdS}}(u) = \frac{8}{\pi}\sqrt{u\bar{u}}$ [103]. It coincides with the experimental data quite well. In fact, it was already suggested 30 years ago by Mikhailov and Radyushkin [104]. Its first few moments in a conformal wave expansion can be found in table 7.1.

In table 7.1 we list the first few moments for both models. We see that the second moment a_2 is comparable in both models. While on the lattice the calculation of higher moments is currently not feasible, the AdS-moments a_n^{AdS} show a decent decrease with increasing n. Both models converge to the asymptotic DA $\phi^{\text{as}}(u)$ in the large scale limit.

To close this section some comments are in order: In the derivation of the DA only leading twist contributions were taken into account. In principle all higher twists give rise to separate contributions to distribution amplitudes. Usually these contributions are significant at low scales. At higher scales they become negligible as they are suppressed by powers $1/Q$. A consideration of these effects goes beyond the scope of this work. So far also perturbative corrections to the leading-twist DA have been ignored, they will be investigated in detail in the next section.

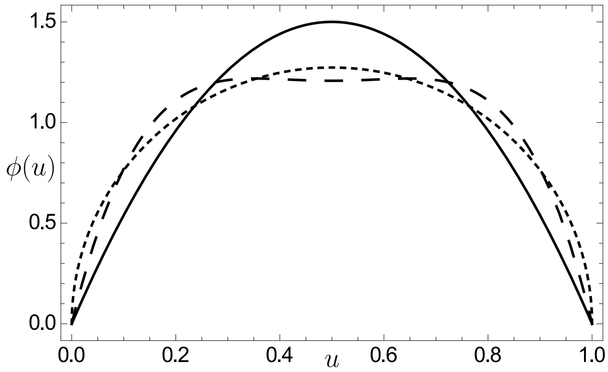


Figure 7.1: Plot of pion DA models (at $\mu = 2\,\mathrm{GeV}$) in the range $u = 0 \dots 1$: $\phi^{\mathrm{as}}(u)$ (solid), $\phi^{\mathrm{latt}}(u)$ (long dashes) and $\phi^{\mathrm{AdS}}(u)$ (short dashes).

7.2 Perturbative evolution of the pion DA

To incorporate the radiative corrections to the pion DA in the form (7.12) it is sufficient to restore the scale dependence of the moments $\langle\!\langle \mathcal{Q}_n(\mu) \rangle\!\rangle$. Technically speaking, we need to solve the evolution equation (7.15) – a set of first order partial differential equations supplied with initial conditions provided by the low-scale values for the moments. To tackle this problem it is instructive to start with a simpler exercise, that is the evolution of eigenstates $\langle\!\langle \mathcal{Q}_n^{(\mathrm{co})} \rangle\!\rangle$ of the equation (7.15). In that case the equations for the moments $\langle\!\langle \mathcal{Q}_n^{(\mathrm{co})} \rangle\!\rangle$ decouple from each other:

$$\big(\mu\partial_\mu + \beta(a)\partial_a + \gamma_n(a)\big)\langle\!\langle \mathcal{Q}_n^{(\mathrm{co})} \rangle\!\rangle = 0, \tag{7.18}$$

where $\gamma_n(a)$ are the forward anomalous dimensions. The general solution is given by

$$\langle\!\langle \mathcal{Q}_n^{(\mathrm{co})}(\mu) \rangle\!\rangle = \exp\left\{ -\int_{a(\mu_0)}^{a(\mu)} da' \frac{\gamma_n(a')}{\beta(a')} \right\} \langle\!\langle \mathcal{Q}_n^{(\mathrm{co})}(\mu_0) \rangle\!\rangle, \tag{7.19}$$

where $\langle\!\langle \mathcal{Q}_n^{(\mathrm{co})}(\mu_0) \rangle\!\rangle$ are some initial values. To leading order the operators in Eq. (7.9) indeed diagonalize the evolution equation. To that accuracy Eq. (7.19), provided with the initial conditions from the table 7.1 is the desired solution to the RGE (7.15).

To solve the evolution equation (7.15) beyond leading order one can factorize the solution into a purely diagonal and non-diagonal contribution. In that sense we make the Ansatz

$$\langle\!\langle \vec{\mathcal{Q}}(\mu) \rangle\!\rangle = \hat{\mathcal{B}} \otimes \hat{\mathcal{A}} \otimes \langle\!\langle \vec{\mathcal{Q}}(\mu_0) \rangle\!\rangle, \tag{7.20}$$

where the matrices $\hat{\mathcal{B}} \equiv \mathcal{B}_{nk}(\mu, \mu_0)$ and $\hat{\mathcal{A}} \equiv \delta_{nk}\mathcal{A}_n(\mu, \mu_0)$ encode the non-diagonal and diagonal evolution, respectively. Both matrices need to fulfill the boundary conditions

$$\mathcal{B}(\mu_0, \mu_0) = \mathbb{1}, \qquad\qquad \mathcal{A}(\mu_0, \mu_0) = \mathbb{1}. \tag{7.21}$$

Obviously the combination $\langle\!\langle \vec{\mathcal{Q}}^{(\mathrm{co})}(\mu)\rangle\!\rangle \equiv \hat{\mathcal{A}} \otimes \langle\!\langle \vec{\mathcal{Q}}(\mu_0)\rangle\!\rangle$ defines a set of moments that are eigenfunctions of the evolution equation with initial values $\langle\!\langle \vec{\mathcal{Q}}(\mu_0)\rangle\!\rangle$. Henceforth we can view the matrix $\hat{\mathcal{B}}$ as a transformation from the diagonal to the non-diagonal basis

$$\langle\!\langle \vec{\mathcal{Q}}(\mu)\rangle\!\rangle = \hat{\mathcal{B}} \otimes \langle\!\langle \vec{\mathcal{Q}}^{(\mathrm{co})}(\mu)\rangle\!\rangle. \tag{7.22}$$

The diagonal matrix $\hat{\mathcal{A}}$ is given by the expression from Eq. (7.19)

$$\mathcal{A}_k(a, a_0) = \exp\left\{-\int_{a_0}^{a} da' \frac{\gamma_k(a')}{\beta(a')}\right\}, \tag{7.23}$$

where $a \equiv a(\mu), a_0 = a(\mu_0)$ implicitly depend on the renormalization and reference scale. To the first three orders we find

$$\begin{aligned}
\mathcal{A}_k(a, a_0) =& L^{-\frac{\gamma_k^{(1)}}{2\beta_0}} \left[1 + aA^{(1)}(L) + a^2 A^{(2)}(L) + \mathcal{O}(a^3)\right], \\
\mathcal{A}_k^{(1)}(L) =& (1 - L)\left[\frac{\gamma_k^{(2)}}{2\beta_0} - \frac{\beta_1}{\beta_0}\frac{\gamma_k^{(1)}}{2\beta_0}\right], \\
\mathcal{A}_k^{(2)}(L) =& \frac{1}{2}\left[A_k^{(1)}(L)\right]^2 + \frac{1 - L^2}{4\beta_0}\left[\gamma_k^{(3)} - \frac{\beta_1}{\beta_0}\gamma_k^{(2)} - \frac{\beta_1^2 - \beta_2\beta_0}{\beta_0}\gamma_k^{(1)}\right].
\end{aligned} \tag{7.24}$$

Here and in what follows we use the scaling factor $L = \frac{a_0}{a}$.

To find a solution for $\hat{\mathcal{B}}$ we plug the ansatz (7.20) into the RGE (7.15) and use fact that $\mathcal{A}_n \langle\!\langle \mathcal{Q}_n(\mu_0)\rangle\!\rangle$ satisfies the RGE (7.18). In this way we derive a differential equation for the matrix $\hat{\mathcal{B}}(a, a_0)$:

$$\boxed{\beta(a)\partial_a \hat{\mathcal{B}} + [\hat{\boldsymbol{\gamma}}^{\mathrm{D}}(a), \hat{\mathcal{B}}] + \hat{\boldsymbol{\gamma}}^{\mathrm{ND}}(a)\hat{\mathcal{B}} = 0, \qquad \hat{\mathcal{B}}(a_0, a_0) = \mathbb{1},} \tag{7.25}$$

where we employed again the splitting into diagonal and off-diagonal anomalous dimensions as in Eq. (6.25).

The general solution for the differential equation (7.25) with an initial condition $\hat{\mathcal{B}}(a_0, a_0) = \mathbb{1}$ can be obtained iteratively and reads

$$\hat{\mathcal{B}}(a, a_0) = \frac{1}{1 - \mathcal{L}\hat{\boldsymbol{\gamma}}^{\mathrm{ND}}} = \mathbb{1} - \int_{a_0}^{a} da' \frac{\hat{\boldsymbol{\gamma}}^{\mathrm{ND}}(a')}{\beta(a')} + \ldots \tag{7.26}$$

where the all order solution is defined by

$$\mathcal{L}\gamma_{nk}^{\mathrm{ND}} = -\int_{a_0}^{a} da' \frac{\gamma_{nk}^{\mathrm{ND}}(a')}{\beta(a')} \exp\left\{-\int_{a'}^{a} da'' \frac{\gamma_n(a'') - \gamma_k(a'')}{\beta(a'')}\right\} \tag{7.27}$$

We find to NNLO accuracy $\mathcal{B}_{nk}(a, a_0) = \delta_{nk} + a\mathcal{B}^{(1)}_{nk}(a, a_0) + a^2\mathcal{B}^{(2)}_{nk}(a, a_0) + \mathcal{O}(a^3)$ with

$$\begin{aligned}
\mathcal{B}^{(1)}_{nk}(a, a_0) =& R_{nk}(a, a_0|1)\frac{\gamma^{\mathrm{ND}(2)}_{nk}}{2\beta_0},\\
\mathcal{B}^{(2)}_{nk}(a, a_0) =& [R_{nk}(a, a_0|1) - R_{nk}(a, a_0|2)]\left[\frac{\gamma^{(2)}_n - \gamma^{(2)}_k}{2\beta_0} - \frac{\beta_1}{\beta_0}\frac{\gamma^{(1)}_n - \gamma^{(1)}_k}{2\beta_0}\right]\frac{\gamma^{\mathrm{ND}(2)}_{nk}}{2\beta_0}\\
&+ R_{nk}(a, a_0|2)\left[\frac{\gamma^{\mathrm{ND}(3)}_{nk}}{2\beta_0} - \frac{\beta_1}{\beta_0}\frac{\gamma^{\mathrm{ND}(2)}_{nk}}{2\beta_0}\right]\\
&+ \sum_m \frac{\gamma^{\mathrm{ND}(2)}_{nm}}{2\beta_0}\frac{R_{mk}(a, a_0|1) - R_{nk}(a, a_0|2)}{1 - \frac{\gamma^{(1)}_n}{2\beta_0} + \frac{\gamma^{(1)}_m}{2\beta_0}}\frac{\gamma^{\mathrm{ND}(2)}_{mk}}{2\beta_0},
\end{aligned} \tag{7.28}$$

where

$$R_{nk}(a, a_0|n) = \frac{1}{n - \frac{\gamma^{(1)}_n}{2\beta_0} + \frac{\gamma^{(1)}_k}{2\beta_0}}\left[1 - L^{n - \frac{\gamma^{(1)}_n}{2\beta_0} + \frac{\gamma^{(1)}_k}{2\beta_0}}\right]. \tag{7.29}$$

To summarize, the NNLO solution of the evolution equation (7.15) reads

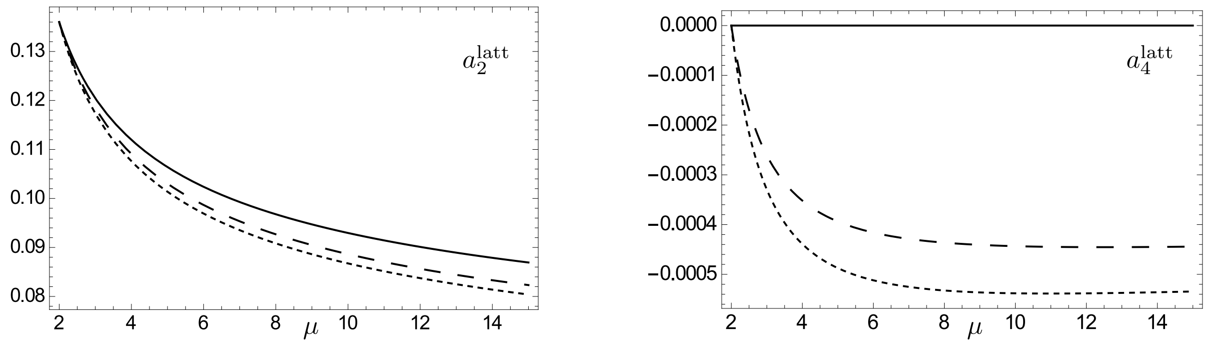
$$\boxed{\begin{aligned}
\langle\!\langle \mathcal{Q}_n(a)\rangle\!\rangle =& \sum_{k=0}^{n} L^{-\frac{\gamma^{(1)}_k}{2\beta_0}}\Big\{\delta_{nk} + a\left[\delta_{nk}\mathcal{A}^{(1)}_k + \mathcal{B}^{(1)}_{nk}\right](a, a_0)\\
&+ a^2\left[\delta_{nk}\mathcal{A}^{(2)}_k + \mathcal{B}^{(2)}_{nk} + \mathcal{B}^{(1)}_{nk}\mathcal{A}^{(1)}_k\right](a, a_0)\Big\}\langle\!\langle \mathcal{Q}_k(a_0)\rangle\!\rangle.
\end{aligned}} \tag{7.30}$$

To conclude this chapter we investigate the numerical size of the perturbative corrections. Let us consider the second and fourth moments for the two models introduced above. To restore the strong coupling as a function of the scale we choose a brute force integration of the RGE (3.6) with the same truncation order as in Eq. (3.7) and the initial condition $a(1\,\mathrm{GeV}) = \alpha_s(1\,\mathrm{GeV})/(4\pi) = 1/(8\pi)$. Setting the number of flavors to $n_f = 4$, we evolve the moments from the reference scale to, e.g., $\mu = 10\,\mathrm{GeV}$ and obtain

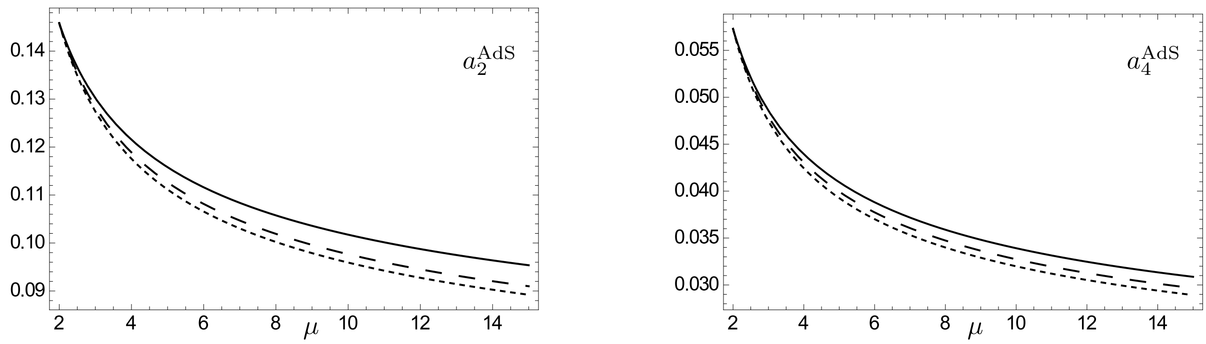
$$\begin{array}{llll}
 & \text{LO} & \text{NLO} & \text{NNLO}\\
a_2^{\mathrm{AdS}} = 0.10\Big(1 & & -0.04 & -0.02\Big),\\
a_4^{\mathrm{AdS}} = 0.03\Big(1 & & -0.03 & -0.02\Big),
\end{array}
\qquad
\begin{array}{llll}
 & \text{LO} & \text{NLO} & \text{NNLO}\\
a_2^{\mathrm{Latt}} = & 0.09\Big(1 & -0.04 & -0.02\Big),\\
a_4^{\mathrm{Latt}} = 4\cdot 10^{-4} & \Big(0 & -1 & -0.2\Big).
\end{array}$$

Supporting these findings, we visualize our results for the scale-dependence of the moments in Fig. 7.2. We notice that under evolution to asymptotic scale the couplings tend towards zero and the perturbative corrections enhance this behavior. The evolution of the coupling a_4^{latt} deserves some explanation. In the lattice model we have set its value (and all higher moments) to zero at reference scale, due to lack of any information about it. While to leading accuracy it stays zero as it scales multiplicativly, NLO and NNLO evolution give rise to a non-vanishing contribution. The reason for that is mixing with the lower moments a_0^{latt}, and a_2^{latt}. We observe a maximum of $a_4^{\mathrm{latt}}(\mu)$ around $\mu = 12\,\mathrm{GeV}$, then the slope changes and it tends to zero for high scales. Numerically this effect is quite small.

We conclude with the remark that the numerical impact of the perturbative corrections is rather small. However, aiming for a high-precision theoretical prediction it is indispensable to take into account both NLO and NNLO corrections.



(a) The second and fourth moments a_2^{latt} (left panel) and a_4^{latt} (right panel), respectively, for $n_f = 4$.



(b) The second and fourth moments a_2^{AdS} (left panel) and a_4^{AdS} (right panel), respectively, for $n_f = 4$.

Figure 7.2: Plots of the second and fourth moments in both models in the range $\mu = 2 \ldots 15$ GeV. We compare the LO expression (solid) against NLO (long dashes) and NNLO (short dashes).

Chapter 8

Conclusions

8.1 Summary

In this work we present the evolution equation for flavor non-singlet leading-twist operators up to three-loop accuracy. The results apply to both local and non-local operators. We use a novel technique that is based on the use of conformal symmetry arguments in a slightly modified theory. In chapter 2 we introduce conformal symmetry with all aspects necessary for our analysis. In chapter 3 we explain the theoretical set-up. We define a modified theory in $4-2\epsilon$ dimensions, in which conformal symmetry is restored by tuning the strong coupling to its critical value. We explain the ideas and technical issues of the underlying method. In chapter 4 we employ (conformal) Ward identities to fix perturbative corrections to the conformal spin generators. We perform the perturbative calculations that determine the spin operators to two-loop accuracy. In doing so we reveal a hidden symmetry of the QCD evolution equations. In chapter 5 we use this symmetry and the method introduced in chapter 3 to solve for the evolution kernel in coordinate representation to three-loop accuracy. On this step no additional perturbative calculation are required, only the knowledge of the spin generators and the forward anomalous dimensions is utilized. Moreover in chapter 6 we show how to convert this result to the language of local operators and present the corresponding mixing matrices to the three-loop accuracy. Finally in chapter 7 we introduce the pion distribution amplitude to demonstrate the applicability and numerical impact of our results. We present an explicit solution for the evolution equation of the local operators to three-loops, that is required to calculate the scale dependence of the pion distribution amplitude to next-to-next-to-leading order accuracy.

8.2 Main results

The NNLO evolution kernel for light-ray operators in Eq. $\boxed{(5.27)}$ is intended to be the main result of this work. In addition we also present the NNLO anomalous dimension matrix for local operators in $\boxed{(6.39)}$. Whereas the light-cone representation Eq. (5.27) can be seen as more general compared to the local representation (6.39), the latter turns out to be more useful in practice, as it allows one to solve the evolution equation in an economic way, see equation $\boxed{(7.30)}$. On the way to derive the three-loop evolution kernel we pass another major outcome that is the conformal anomaly in Eq. $\boxed{(4.34)}$. It is worth to be mentioned as a stand-alone result as it has also different applications than fixing the evolution kernel. One can utilize it to derive a mapping between the $\overline{\text{MS}}$-scheme and a so-called conformal scheme [26] – i.e. one uses the conformal anomaly to construct operators that transform covariantly under conformal transformation and thus are eigenstates of the evolution equation. Some details of this mapping can be found in appendix H. In that scheme the construction of off-diagonal anomalous dimensions can be completely avoided. While being a very elegant construction, the conformal scheme lacks in general utility compared to the $\overline{\text{MS}}$-scheme. In that sense we favor transparency over sophistication, bearing in mind that our results are aimed to be embedded in phenomenological calculations – most often done in $\overline{\text{MS}}$-scheme.

8.3 Outlook

The outcome of our work serves as motivation for further studies in this direction. The most important goal is to extend the analysis to singlet operators, taking into account mixing with the gluonic light-ray operator $\mathcal{O}^F(z_1, z_2) = F^a_{n\mu}(z_1 n) F^{a,\mu n}(z_2 n)$ and to consider also axial-vector operators. This extension would allow one to apply the results to a by far broader spectrum of processes, including DVCS and deeply-virtual meson production. To carry out this project, beside some conceptual problems like the definition of the γ_5-matrix in dimensional regularization and subtleties on the operator level, a large amount of diagrams needs to be calculated. It is likely that one needs to rely on computer algebra, which anyway has become the standard tool for many-loop calculations in the last decade. The NLO results provided by Müller and Belitsky [27] can serve as comparison and guideline.

Another appealing direction is to investigate other implications of conformal symmetry. For example, in a CFT the form of two- and three-point functions is constrained up to normalization. In this way, in general, one can fix physical quantities $\mathcal{F}$ up to terms proportional to the β-function

$$\mathcal{F}(a) = \mathcal{F}^{\text{CFT}}(a) + \beta(a)\Delta\mathcal{F}(a), \tag{8.1}$$

which require a separate but typically simpler calculation. This idea already found applications for the OPE of two electromagnetic currents needed to evaluate the scattering amplitude in DVCS [27, 105] or the pion electromagnetic form factor [28]. Moreover there is a novel idea [106] to access the hadron distribution amplitudes by comparing suitably chosen Euclidean correlation functions, calculated on the lattice, to the perturbative prediction (in terms of the DA). Recently a preliminary study has been provided by the Regensburg Lattice-QCD collaboration [107]. As it was demonstrated in [106], conformal symmetry can be used to simplify the perturbative analysis.

Appendix A

Collection of renormalization factors

For completeness we list the quark, gluon and ghost anomalous dimensions in $\overline{\text{MS}}$-scheme (Feynman gauge $\xi = 1$) to the NNLO accuracy. From Refs. [108, 109] we find

$$
\begin{aligned}
\gamma_c =& \frac{3}{2}a + \left(\frac{147}{8} - \frac{5}{4}n_f\right)a^2 \\
&+ \frac{1}{144}\left(-140n_f^2 - (2376\zeta_3 + 3255)n_f + 2916\zeta_3 + 32976\right)a^3 + \mathcal{O}\left(a^4\right), \\
\gamma_A =& \left(5 - \frac{2}{3}n_f\right)a + \left(\frac{207}{4} - \frac{61}{6}n_f\right)a^2 \\
&+ \frac{1}{432}\left(3440n_f^2 + (14256\zeta_3 - 93972)n_f - 17496\zeta_3 + 328131\right)a^3 + \mathcal{O}\left(a^4\right), \\
\gamma_q =& -\frac{4}{3}a + \frac{2}{3}(2n_f - 47)a^2 + \left(\frac{1}{54}(3759 - 40n_f)n_f + 26\zeta_3 - \frac{24941}{36}\right)a^3 + \mathcal{O}(a^4),
\end{aligned}
\tag{A.1}
$$

where for brevity we have set $N_c = 3$.

Appendix B

Transformation properties of (non-)local operators

In order to generalize the identities (3.18) from $x = 0$ to arbitrary x we need to employ the translation operator $e^{ix\mathbf{P}}$

$$\mathcal{Q}_{Nk}(x) = e^{ix\mathbf{P}}\mathcal{Q}_{Nk}(0)e^{-ix\mathbf{P}}. \tag{B.1}$$

Let us investigate how the generators transform, which can be obtained by straightforward algebra from the commutation relations (2.1) and (2.2):

- For translations, one obviously obtains

$$[\mathbf{L}_{+}\mathcal{Q}_{Nk}(x)] = \mathcal{Q}_{Nk+1}. \tag{B.2}$$

- For dilatations we find

$$e^{ix\mathbf{P}}\mathbf{D}e^{-ix\mathbf{P}} = \mathbf{D} - (x\mathbf{P}), \tag{B.3}$$

  and therefore the action on the operator $\mathcal{Q}_{Nk}(x)$ is given by

$$i[\mathbf{D}, \mathcal{Q}_{Nk}(x)] = \Big(x\partial_x + \Delta^*_{Nk}\Big)\mathcal{Q}_{Nk}(x). \tag{B.4}$$

  where $\Delta^*_{Nk} = N + 3 + 2\bar{\beta}(a_*) + \gamma_N(a_*) + k$.
  For the Lorentz rotations we have

$$e^{i\mathbf{P}x}i\mathbf{M}_{\alpha\beta}e^{-i\mathbf{P}x} = \mathbf{D} - i(x\mathbf{P}) = i\Big(\mathbf{M}_{\alpha\beta} - x_\alpha\mathbf{P}_\beta + x_\beta\mathbf{P}_\alpha\Big), \tag{B.5}$$

  that results in

$$i[\mathbf{M}_{\mu\nu}, \mathcal{Q}_{Nk}(x)] = \left(x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu} + n_\mu\frac{\partial}{\partial n^\nu} - n_\nu\frac{\partial}{\partial n^\mu}\right)\mathcal{Q}_{Nk}(x). \tag{B.6}$$

  Then one receives for the action of the generator $\mathbf{L}_0$

$$[\mathbf{L}_0, \mathcal{Q}_{Nk}(x)] = \frac{1}{2}\left((n\bar{n})\Big[x\partial_x + n\partial_n + \Delta^*_{Nk}\Big] - n^\mu\bar{n}^\nu\left(x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}\right)\right)\mathcal{Q}_{Nk}(x). \tag{B.7}$$

- For the conformal generator we get

$$e^{i\mathbf{P}x}i\mathbf{K}_\mu e^{-i\mathbf{P}x} = \mathbf{D} - i(x\mathbf{P}) = i\Big(\mathbf{K}_\mu + 2x_\mu(x\mathbf{P}) - x^2\mathbf{P}_\mu - 2x_\mu\mathbf{D} + 2x^\mu\mathbf{M}_{\nu\mu}\Big), \tag{B.8}$$

so that

$$\begin{aligned}[\mathbf{L}_+, \mathcal{Q}_{Nk}(x)] &= \frac{1}{2}\Big(2(x\bar{n})(x\partial) - x^2(\bar{n}\partial) + 2(x\bar{n})\Delta^*_{Nk} - 2(xn)(\bar{n}\partial_n) + 2(n\bar{n})(x\partial_n)\Big)\mathcal{Q}_{Nk}(x)\\ &\quad + (n\bar{n})k(2j^*_N + k - 1)\mathcal{Q}_{Nk-1}(x).\end{aligned} \tag{B.9}$$

Note that we have restored here the factor $(n\bar{n})$ which was set to one before.

Moreover we need the variation of the light-ray operator. We have two possibilities to specify the light-like direction of the constituent fields:

- the operator is aligned in the "plus" direction n: Then

$$\begin{aligned}\Big[\mathbf{L}_+, \mathcal{O}^{(n)}(x,z_1,z_2)\Big] &= -(n\partial)\mathcal{O}^{(n)}(x,z_1,z_2) = -(\partial_{z_1} + \partial_{z_2})\mathcal{O}^{(n)}(x,z_1,z_2)\\ \Big[\mathbf{L}_0, \mathcal{O}^{(n)}(x,z_1,z_2)\Big] &= \Big[(n\bar{n})S_0 + \frac{1}{2}\Big((n\bar{n})x\partial + (\bar{n}x)(n\partial) - (n\bar{x})(\bar{n}\partial)\Big)\Big]\,\mathcal{O}^{(n)}(x,z_1,z_2)\\ \Big[\mathbf{L}_-, \mathcal{O}^{(n)}(x,z_1,z_2)\Big] &= \Big[(n\bar{n})S_+ + 2(x\bar{n})S_0 + (x\bar{n})(x\partial) - \frac{1}{2}x^2(\bar{n}\partial)\\ &\quad + (n\bar{n})(x\partial_n) - (xn)(\bar{n}\partial_n) - (x\bar{n})(n\partial_n)\Big]\mathcal{O}^{(n)}(x,z_1,z_2)\end{aligned} \tag{B.10}$$

- The operator is aligned in the "minus" direction $\bar{n}$. Then one gets

$$\begin{aligned}\Big[\mathbf{L}_+, \mathcal{O}^{(\bar{n})}(x,z_1,z_2)\Big] &= -(n\partial)\mathcal{O}^{(\bar{n})}(x,z_1,z_2),\\ \Big[\mathbf{L}_0, \mathcal{O}^{(\bar{n})}(x,z_1,z_2)\Big] &= \Big[(n\bar{n})\left(1 - \epsilon + \frac{1}{2}\mathbb{H}(a_*)\right) +\\ &\quad \frac{1}{2}\Big((n\bar{n})x\partial + (\bar{n}x)(n\partial) - (n\bar{x})(\bar{n}\partial)\Big)\Big]\mathcal{O}^{(\bar{n})}(x,z_1,z_2),\\ \Big[\mathbf{L}_-, \mathcal{O}^{(\bar{n})}(x,z_1,z_2)\Big] &= \Big[(x\bar{n})(x\partial) - \frac{1}{2}x^2(\bar{n}\partial) + (x\bar{n})\big(2 - 2\epsilon + \mathbb{H}(a_*)\big)\Big]\mathcal{O}^{(\bar{n})}(x,z_1,z_2).\end{aligned} \tag{B.11}$$

Appendix C

BRST transformations

The quantization of QFTs with a gauge symmetry is a delicate procedure. In particular for non-abelian theories like QCD the introduction of ghost particles by Faddeev and Popov [110] seems more heuristic than profound. A much more rigorous approach is given due to the work of Becchi, Rouet, Stora [111] and Tyutin [112]. They introduce a new symmetry of the QCD action (3.1), given by the transformations $\varphi \mapsto \varphi + \delta_B \varphi$, with the variations

$$\delta_B q = igt^a c^a q \delta\lambda, \qquad \delta_B A_\mu = \left(\partial_\mu c^a + g f^{abc} A^b_\mu c^c\right) \delta\lambda,$$
$$\delta_B c^a = \frac{1}{2} g f^{abc} c^b c^c \delta\lambda, \qquad \delta_B \bar{c}^a = -\frac{1}{\xi}(\partial A^a)\delta\lambda, \tag{C.1}$$

where λ is the BRST parameter – an anti-commuting number. The key message that we are going to use from their formalism is the observation that every gauge-invariant operator $O_{\text{gauge inv}}$ is automatically BRST-invariant

$$\delta_B O_{\text{gauge inv}} = 0. \tag{C.2}$$

As a consequence, correlation functions between gauge invariant operators and any BRST variation $\delta_B \tilde{O}$ vanish

$$\langle O_{\text{gauge inv}} \delta_B \tilde{O} \rangle = 0, \tag{C.3}$$

as can be seen by partial integration. In our analysis two operators appear, which can be expressed as BRST variations. One of them is $\mathcal{B}_\mu$, see Eq. (4.11), which appears to be a BRST variation of $\bar{c}^a A^a_\mu$ [24]

$$\mathcal{B}_\mu(x) = Z_c^2 \bar{c} D^\mu c - \frac{1}{\xi} A^\mu (\partial A) = \frac{\delta}{\delta \lambda_R} \bar{c}^a A^a_\mu. \tag{C.4}$$

The other one is

$$\mathcal{B} = -\frac{1}{\xi}(\partial A)^2 + Z_c^2 \bar{c} \partial D c = \frac{\delta}{\delta \lambda_R} \bar{c}^a (\partial A^a). \tag{C.5}$$

Due to this the gauge fixing term in the action can be represented as a sum of the EOM and BRST exact operators

$$\frac{1}{\xi}(\partial A)^2 = -\mathcal{B} + \mathcal{E}_{\bar{c}}, \qquad \mathcal{E}_{\bar{c}} = Z_c^2 \bar{c} \partial D c = \bar{c}(x) \frac{\delta S_R}{\delta \bar{c}(x)}. \tag{C.6}$$

One can show that $\mathcal{B}_\mu$ is a finite operator, i.e. $[\mathcal{B}_\mu] = \mathcal{B}_\mu$, while $\mathcal{B}$ is not, $[\mathcal{B}] = Z_B \mathcal{B} + Z_{B^\mu} \partial^\mu \mathcal{B}_\mu +$ EOM.

Appendix D

Renormalization of gauge invariant operators

A naïve assumption, that gauge invariant operators mix under renormalization only with gauge invariant operators turns out to be wrong [113]. In fact, using finiteness of the correlation function of a renormalized gauge invariant operator $[O_{\text{gauge inv}}] = O^{\text{bare}}_{\text{gauge inv}} + O^{\text{count}}_{\text{gauge inv}}$ with elementary fields, one can show that the counterterms $O^{\text{count}}_{\text{gauge inv}}$ must be BRST invariant operators [55]

$$\delta_B O^{\text{count}}_{\text{gauge inv}} = 0. \tag{D.1}$$

We conclude that suitable candidates must be gauge invariant operators or, as BRST transformations are nilpotent $\delta_B \delta_B = 0$, BRST variations $\delta_B \tilde{O}$. However, trying to verify that theorem by explicit calculations, one finds [113] in addition the appearance of operators that vanish by means of the equations of motion. Another result [55] that we are going to use is that the mixing between gauge invariant, BRST-exact and EOM operators takes the triangular form

$$\begin{pmatrix} [O_{\text{gauge inv}}] \\ [O_{\text{BRST}}] \\ [O_{\text{EOM}}] \end{pmatrix} = \begin{pmatrix} Z^{\text{gi}}_{\text{gi}} & Z^{\text{gi}}_{\text{brst}} & Z^{\text{gi}}_{\text{eom}} \\ 0 & Z^{\text{brst}}_{\text{brst}} & Z^{\text{brst}}_{\text{eom}} \\ 0 & 0 & Z^{\text{eom}}_{\text{eom}} \end{pmatrix} \begin{pmatrix} O_{\text{gauge inv}} \\ O_{\text{BRST}} \\ O_{\text{EOM}} \end{pmatrix}. \tag{D.2}$$

For a correlation function of two gauge-invariant operators we find

$$\langle [O_{\text{gauge inv}}][\tilde{O}_{\text{gauge inv}}] \rangle = Z^{\text{gi}}_{\text{gi}} \tilde{Z}^{\text{gi}}_{\text{gi}} \langle O_{\text{gauge inv}} \tilde{O}_{\text{gauge inv}} \rangle, \tag{D.3}$$

since correlators of gauge invariant operators with BRST variations and EOM operators vanish (up to contact terms).

Now we want to apply these findings to the correlator for the conformal Ward identities (4.9). Our goal is to express the conformal variation (4.11)

$$\begin{aligned} \mathcal{N}(x) &= \frac{1}{2} Z_A^2 F^2 + \frac{1}{\xi} (\partial A)^2 \\ &= \frac{1}{2} Z_A^2 F^2 - \mathcal{B} + \mathcal{E}_{\bar{c}}, \end{aligned} \tag{D.4}$$

in terms of renormalized operators. Note that in the correlator of $\mathcal{N}$ with any gauge invariant operator the last two terms will disappear. Bearing in mind the triangular mixing pattern mentioned above we schematically find

$$\mathcal{N} = \frac{1}{2} Z'_F [F^2] - Z'_B [\mathcal{B}] + Z'_{\mathcal{B}_\mu} \partial^\mu [\mathcal{B}_\mu] + \sum_\varphi Z'_\varphi [\mathcal{E}_\varphi], \tag{D.5}$$

where all $Z's$ are some (singular) coefficients

$$Z'_F, Z'_{\mathcal{B}} \sim 1 + \mathcal{O}(\frac{1}{\epsilon}) \qquad\qquad Z'_{\mathcal{B}_\mu}, Z'_\varphi \sim \mathcal{O}(\frac{1}{\epsilon}). \tag{D.6}$$

For the conformal variation δS, see Eq. (4.10), we need to take into account an additional factor of ϵ and find

$$\epsilon \mathcal{N} = \frac{1}{2} r_F [F^2] - r_{\mathcal{B}} [\mathcal{B}] + r_{\mathcal{B}_\mu} \partial^\mu [\mathcal{B}_\mu] + \sum_\varphi r_\varphi [\mathcal{E}_\varphi] + \mathcal{O}(\epsilon), \tag{D.7}$$

where all $r_i \equiv r_i(a, \xi)$ are residues of the $Z's$ and depend only on the strong coupling and the gauge parameter but not on ϵ.

In order to fix these coefficients, let us have a look at the method of differential vertex operator insertions. This study allows one to determine the coefficients for operators that do not involve total derivatives. The combinations $\mathcal{E}_{\bar{q}} - \mathcal{E}_q = \partial^\mu \bar{q} \gamma_\mu q$ and $\mathcal{E}_{\bar{c}} - \mathcal{E}_c = \partial^\mu [\bar{c} D_\mu c - \partial_\mu \bar{c} c]$, however, cannot be determined in that way, as they correspond to total derivatives. For the quark EOM we can luckily invoke charge symmetry to find $r_{\bar{q}} = r_q$, while for the ghost EOM we can only access the combination $r_{\bar{c}} + r_c$. The combination $r_{\bar{c}-c} = r_{\bar{c}} - r_c$, as well as the coefficient $r_{\mathcal{B}_\mu}$, cannot be fixed in this way.

Let us consider the Green's function of n fundamental fields. Its derivatives w.r.t. the couplings g and ξ generate insertions of the so-called differential vertex operator

$$\begin{aligned} g\partial_g \langle \varphi(x_1) \dots \varphi(x_n) \rangle &= \langle g\partial_g S_R \varphi(x_1) \dots \varphi(x_n) \rangle, \\ \xi\partial_\xi \langle \varphi(x_1) \dots \varphi(x_n) \rangle &= \langle \xi\partial_\xi S_R \varphi(x_1) \dots \varphi(x_n) \rangle \end{aligned} \tag{D.8}$$

As the l.h.s. is just a derivative of a finite number, one concludes that the r.h.s. must be finite, too. In consequence, the operators $g\partial_g \mathcal{L}_R$ and $\xi\partial_\xi \mathcal{L}_R$ can be expressed in terms of renormalized operators, up to total derivatives. To investigate the explicit form, we define for convenience $A_0 \mapsto G_0 = g_0 A_0$ such that

$$\mathcal{L}_R = \mathcal{L}(\varphi_0, g_0, \xi_0) = \frac{1}{g_0^2} \left(\frac{1}{4} F_0^2 + \frac{1}{2\xi_0} (\partial G_0)^2 \right) + \bar{q}_0 i \not{D} q_0 + \bar{c} (\partial \cdot D) c. \tag{D.9}$$

Then

$$\partial_g \mathcal{L}_R = \frac{\delta \mathcal{L}}{\delta \varphi_0} \frac{\partial \varphi_0}{\partial g} + \frac{\partial \mathcal{L}}{\partial g_0} \frac{\partial g_0}{\partial g} + \frac{\partial \mathcal{L}}{\partial \xi_0} \frac{\partial \xi_0}{\partial g}, \tag{D.10}$$

and similar for the ξ-derivative. Working out all derivatives yields

$$\begin{aligned} g\partial_g \mathcal{L}_R &= \frac{\epsilon a}{\beta(a)} (\mathcal{L}_R^{\mathrm{YM}} + \mathcal{L}_R^{\mathrm{gf}}) + \mathcal{E}_A g\partial_g \ln(g Z_g Z_A) + \sum_{\varphi \neq A} \mathcal{E}_\varphi g\partial_g \ln Z_\varphi - \frac{1}{\xi} (\partial A)^2 g\partial_g \ln Z_A, \\ \xi\partial_\xi \mathcal{L}_R &= -\frac{1}{2\xi} (\partial A)^2 (1 + 2\xi\partial_\xi \ln Z_A) + \sum_\varphi \mathcal{E}_\varphi \, \xi\partial_\xi \ln Z_\varphi, \end{aligned} \tag{D.11}$$

where $\mathcal{L}_R^{\mathrm{YM(gf)}}$ is the Yang-Mills (gauge-fixing) term of the renormalized Lagrangian. We can separate the r.h.s. of both equations in (D.11) as follows:

$$g\partial_g S_R = -2\left(\mathcal{L}^{YM} + \mathcal{L}^{gf} - \frac{1}{2}A\frac{\delta\mathcal{L}}{\delta A}\right) + \mathcal{O}(1/\epsilon), \qquad \xi\partial_\xi\mathcal{L}_R = -\frac{1}{2\xi}(\partial A)^2 + \mathcal{O}(1/\epsilon). \tag{D.12}$$

As the l.h.s. is a renormalized operator in $\overline{\mathrm{MS}}$-scheme, the r.h.s. must be as well, meaning that the singular $\mathcal{O}(1/\epsilon)$ terms just provide subtraction of divergences, and thus by definition

$$g\partial_g\mathcal{L}_R = -2\left[\mathcal{L}^{YM} + \mathcal{L}^{gf} - \frac{1}{2}\mathcal{E}_A\right], \qquad \xi\partial_\xi\mathcal{L}_R = -\frac{1}{2\xi}[(\partial A)^2]. \tag{D.13}$$

It follows from Eqs. (D.4) and (D.11)

$$\begin{aligned}\epsilon\mathcal{N}(x) =& -\frac{\beta(a)}{a}\left[\mathcal{L}^{YM} + \mathcal{L}^{gf}\right] - \mathcal{E}_A\mathcal{D}_g\ln(Z_g Z_A) - \sum_{\varphi\neq A}\mathcal{E}_\varphi\,\mathcal{D}_g\ln Z_\varphi \\ &+ \frac{1}{\xi}(\partial A)^2\mathcal{D}_g\ln Z_A\,, \end{aligned}\tag{D.14a}$$

$$\frac{1}{2\xi}[(\partial A)^2] = \frac{1}{2\xi}(\partial A)^2(1 + 2\xi\partial_\xi\ln Z_A) - \sum_\varphi\mathcal{E}_\varphi\xi\partial_\xi\ln Z_\varphi, \tag{D.14b}$$

where $\mathcal{D}_g = \beta(a)\partial_g$ and we want to remind that these equations are valid upon integration in $\int d^4x$. Using $\gamma_\varphi = \mu\partial_\mu\ln Z_\varphi = (\beta(a)\partial_g + \beta_\xi\partial_\xi)\ln Z_\varphi$ and $\beta_\xi = -2\xi\gamma_A$ one obtains $\mathcal{D}_g\ln Z_A = \gamma_A(1 + 2\xi\partial_\xi\ln Z_A)$. It follows then from the Eqs. (D.14a)

$$\epsilon\mathcal{N}(x) = -\frac{\beta(a)}{a}\left[\mathcal{L}^{YM} + \mathcal{L}^{gf}\right] - (\gamma_A + \gamma_g)\mathcal{E}_A - \sum_{\varphi\neq A}\gamma_\varphi\mathcal{E}_\varphi + \frac{\gamma_A}{\xi}[(\partial A)^2] + \ldots, \tag{D.15}$$

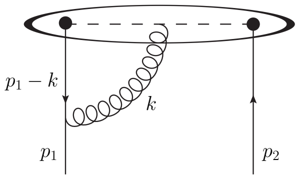
where the dots stand for total derivative operators. Comparing Eqs. (D.15) and (D.7) we can solve for the coefficients r_i in Eq. (D.7)

$$r_F = \gamma_g, \qquad r_B = r_A = \gamma_g + \gamma_A, \qquad r_q = r_{\bar q} = \gamma_q, \qquad r_c + r_{\bar c} = 2\gamma_c + \gamma_g + \gamma_A. \tag{D.16}$$

Appendix E

Sample Feynman diagram calculation

To illustrate some basic concepts of Feynman diagram calculus and technical details specific for the calculation of light-ray evolution kernels, we explicitly derive here the solution for the following diagram contributing to the one-loop evolution kernel:



Its contribution is given by the following Feynman integral

$$-g^2 \int_0^1 du\, iz_{12} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \not{n} \frac{\not{p}_1 - \not{k}}{(p_1-k)^2} \not{n} e^{i(p_1-k)\cdot nz_1 + ik\cdot nz_{21}^u + ip_2 z_2}, \tag{E.1}$$

which is understood to be sandwiched between the external lines $\frac{\not{p}_1}{p_1^2} \ldots \frac{\not{p}_2}{p_2^2}$. As a first step one needs to combine the propagators by introducing so-called Feynman (Schwinger) - parameters

$$\frac{1}{A_1^{a_1} \ldots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_i \Gamma(a_i)} \int_0^1 dx_1 \ldots dx_n \delta(\sum_i x_i - 1) \frac{\prod_i x_i^{a_i - 1}}{(\sum_i x_i A_i)^a}, \tag{E.2}$$

where $a = \sum_i a_i$. We find for the integral (E.1)

$$-g^2 \int_0^1 du\, iz_{12} \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{\not{n}(\not{p}_1 - \not{k})\not{n}}{(\alpha(p_1-k)^2 + \bar{\alpha}k^2)^2} e^{i(p_1-k)\cdot nz_1 + ik\cdot nz_{21}^u + ip_2 z_2}. \tag{E.3}$$

Here we already performed one of the α-integrations by means of the δ-distribution. We shift the momentum integration variable $k \to l = k - \alpha p$, such that the integral (E.3) becomes

$$-g^2 \int_0^1 du\, iz_{12} \int_0^1 d\alpha \int \frac{d^dl}{(2\pi)^d} \frac{\not{n}(\bar{\alpha}\not{p}_1 - \not{l})\not{n}}{(l^2 + \alpha\bar{\alpha}p_1^2)^2} e^{ip_1\cdot nz_{12}^{\alpha\bar{u}} + il\cdot n(z_2 - z_1)u}. \tag{E.4}$$

Next we need to perform the momentum integration. To do so we expand the exponential $e^{il\cdot n(z_2-z_1)u} = 1 + il \cdot n(z_2 - z_1)u + \ldots$ and use the formulas

$$\begin{aligned} \int \frac{d^dl}{(2\pi)^d} \frac{1}{(l^2+\Delta)^a} &= \frac{\Delta^{d/2-a}}{(4\pi)^{d/2}} \frac{\Gamma(a-d/2)}{\Gamma(a)}, \\ \int \frac{d^dl}{(2\pi)^d} \frac{l^{\mu_1} \ldots l^{\mu_n}}{(l^2+\Delta)^a} &= \frac{\sum_{\mathcal{P}} \prod_{(i,j)\in\mathcal{P}} g^{\mu_i\mu_j}}{n} \frac{\Delta^{d/2-a+n/2}}{(4\pi)^{d/2}} \frac{\Gamma(a-d/2-n/2)}{\Gamma(a)}, \end{aligned} \tag{E.5}$$

where the sum goes over all sets $\mathcal{P}$ of different index pairs (μ_i, μ_j) with $i \neq j \in (1, \dots, n)$. Eq. (E.5) is valid if n is even, otherwise the integral is zero. One will notice that in this example only the scalar integral survives, all tensor integrals yield factors $n^2 = 0$. We receive

$$-\frac{g^2}{(4\pi)^d}\Gamma(\epsilon)\int_0^1 du\, iz_{12}\int_0^1 d\alpha\bar{\alpha}\not{n}\not{p}_1\not{n}(\alpha\bar{\alpha}p_1^2)^{-\epsilon}e^{ip_1\cdot nz_{12}^{\alpha\bar{u}}+ip_2 z_2}. \tag{E.6}$$

At this point we expand in ϵ and get

$$-2\frac{\alpha_s}{\epsilon}\int_0^1 du\int_0^1 d\alpha\bar{\alpha}\, iz_{12}(n\cdot p_1)\not{n}e^{ip_1\cdot nz_{12}^{\alpha\bar{u}}+ip_2\cdot nz_2} + \text{ finite.} \tag{E.7}$$

We can rewrite $iz_{12}(n\cdot p_1) \mapsto \partial_u$ and integrate by parts, which yields the desired result for the pole contribution

$$KR'(D) = -2\frac{\alpha_s}{\epsilon}\frac{\not{p}_1}{p_1^2}\not{n}\frac{\not{p}_2}{p_2^2}\int_0^1 d\alpha\frac{\bar{\alpha}}{\alpha}\left(e^{ip_1\cdot nz_1+ip_2\cdot nz_2} - e^{ip_1\cdot nz_{12}^{\alpha}+ip_2\cdot nz_2}\right). \tag{E.8}$$

In terms of the non-interacting diagram

$$\mathrm{LO} = \frac{\not{p}_1}{p_1^2}\not{n}\frac{\not{p}_2}{p_2^2}\left(e^{ip_1\cdot nz_1+ip_2\cdot nz_2}\right), \tag{E.9}$$

one receives

$$KR'(D) = -2\frac{\alpha_s}{\epsilon}[\hat{\mathcal{H}}_1\otimes \mathrm{LO}], \tag{E.10}$$

with $\hat{\mathcal{H}}_1$ given by the integral operator in Eq. (4.42).

Appendix F

Results for individual diagrams

Here we want to collect the answers for all individual Feynman diagrams that are relevant for the computation of the two-loop evolution kernel and conformal anomaly, see Fig. 4.1. All results are given in Feynman gauge, i.e. $\xi = 1$.

F.1 Evolution kernel

The contributions to the evolution kernel from the diagrams in Fig. 4.1(i)–(xvi) (including symmetric diagrams with the interchange of the quark and the antiquark) can be written in the following form:

$$\begin{aligned}
[\mathbb{H}\mathcal{O}](z_1 z_2) = &- 4\int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \Big[h(\alpha,\beta) + h^{\mathbb{P}}(\alpha,\beta)\mathbb{P}_{12}\Big]\Big[\mathcal{O}(z_{12}^{\alpha}, z_{21}^{\beta}) + \mathcal{O}(z_{12}^{\beta}, z_{21}^{\alpha})\Big] \\
&- 4\int_0^1 du\, \hat{h}(u)\Big[2\mathcal{O}(z_1,z_2) - \mathcal{O}(z_{12}^{u}, z_2) - \mathcal{O}(z_1, z_{21}^{u})\Big]\,,
\end{aligned} \tag{F.1}$$

where $\mathbb{P}_{12}$ is the permutation operator

$$\mathbb{P}_{12}\mathcal{O}(z_1,z_2) = \mathcal{O}(z_2,z_1)\,. \tag{F.2}$$

One obtains:

$$
\begin{aligned}
\hat{h}_{(\mathrm{iv})}(u) &= 2\Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\frac{\bar{u}}{u}\Big[-2\,\mathrm{Li}_2(u) + \frac{u}{\bar{u}}\ln u \ln\bar{u} - \frac{1}{2}\ln^2\bar{u} - \frac{u}{\bar{u}}\ln u\Big],\\
\hat{h}_{(\mathrm{v})}(u) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\frac{\bar{u}}{u}\Big[\ln^2\bar{u} - 3\frac{u}{\bar{u}}\ln u + 3\ln\bar{u} - \ln u - 1\Big],\\
\hat{h}_{(\mathrm{vi})}(u) &= C_F^2\frac{\bar{u}}{u}\left[\ln u + 1\right],\\
\hat{h}_{(\mathrm{vii})}(u) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\ \ln u\,,\\
\hat{h}_{(\mathrm{x})}(u) &= C_F\frac{\bar{u}}{u}\Big[(2C_A - \beta_0)\ln\bar{u} + \frac{8}{3}C_A - \frac{5}{3}\beta_0\Big],\\
\hat{h}_{(\mathrm{xi+xii})}(u) &= 2\,C_F^2\frac{\bar{u}}{u}\left[2\big(\mathrm{Li}_2(1) - \mathrm{Li}_2(\bar{u})\big) - \ln^2\bar{u} + 2\frac{u}{\bar{u}}\ln u\right]\\
&\quad + C_F C_A\frac{\bar{u}}{u}\Big[2\big(\mathrm{Li}_2(\bar{u}) - \mathrm{Li}_2(u)\big) + \frac{1}{2}\ln^2\bar{u} - \frac{1}{2}\ln^2 u - \frac{1+u}{\bar{u}}\ln u - 2\Big],\\
\hat{h}_{(\mathrm{xiii})}(u) &= C_F C_A\frac{\bar{u}}{u}\Big[\mathrm{Li}_2(u) + \frac{1}{\bar{u}}\ln u\ln\bar{u} - \frac{1}{4}\ln^2\bar{u} - \frac{u}{4\bar{u}}\ln^2 u - \frac{u}{\bar{u}}\ln u\Big],\\
\hat{h}_{(\mathrm{xiv})}(u) &= \frac{1}{2}C_F C_A\frac{\bar{u}}{u}\left[\frac{1}{2}\left(1 - \frac{u}{\bar{u}}\right)\ln^2 u + \ln\bar{u} - 3\right],\\
\hat{h}_{(\mathrm{xv})}(u) &= -C_F C_A\frac{\bar{u}}{u}\left[\mathrm{Li}_2(\bar{u}) - \mathrm{Li}_2(1) + 1 + \frac{1}{4}\ln^2\bar{u} + \ln\bar{u} - \frac{1+u}{2\bar{u}}\ln u\left(\frac{1}{2}\ln u + 1\right)\right],
\end{aligned}
\tag{F.3}
$$

and

$$
\begin{aligned}
h_{(\mathrm{i})}(\alpha,\beta) &= C_F^2\left[2\ln\bar{\tau} + 1 + \frac{1}{2}\ln^2(1-\alpha-\beta) - \ln^2\bar{\alpha}\right],\\
h_{(\mathrm{ii})}(\alpha,\beta) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\Big[\ln^2\bar{\alpha} + 4\ln\bar{\alpha}\Big],\\
h_{(\mathrm{iii})}(\alpha,\beta) &= C_F^2\Big[2\ln\tau + 8 + 4\big(\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(1)\big) + \ln^2\alpha + \ln^2\bar{\alpha} + 2\ln\alpha\Big],\\
h_{(\mathrm{iv})}(\alpha,\beta) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\Big[2\left(2 + \frac{1}{\tau}\right)\ln\bar{\tau} - 3\ln\tau - \ln^2\bar{\tau} - 2\,\mathrm{Li}_2(\tau) + \ln^2(1-\alpha-\beta)\\
&\quad - \ln^2(\alpha\bar{\alpha}) - 4\Big[\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(1)\Big] + \frac{2}{\alpha}\ln\bar{\alpha} - 2\big[2 + \mathrm{Li}_2(1) - 3\zeta(3)\big]\delta(\alpha)\delta(\beta)\Big],\\
h_{(\mathrm{vii})}(\alpha,\beta) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\Big[\ln^2\bar{\alpha} - 8\ln\bar{\alpha} - \ln^2(1-\alpha-\beta) - 7\ln\bar{\tau} - \frac{1}{2}\ln\tau - 6 + \delta(\alpha)\delta(\beta)\Big],\\
h_{(\mathrm{viii})}(\alpha,\beta) &= -C_F^2\Big[\ln\alpha + 3\Big],\\
h_{(\mathrm{ix})}(\alpha,\beta) &= C_F\left[\frac{1}{6}(C_A - \beta_0)\delta(\alpha)\delta(\beta) - \left(C_A - \frac{1}{2}\beta_0\right)\ln(1-\alpha-\beta) - \frac{10}{3}C_A + \frac{11}{6}\beta_0\right],\\
h_{(\mathrm{xiii})}(\alpha,\beta) &= C_F C_A\left[-\frac{3}{4}\ln\tau + \mathrm{Li}_2(\bar{\alpha}) - \mathrm{Li}_2(\alpha) + \frac{1}{\alpha}\ln\bar{\alpha} + \big[\mathrm{Li}_2(1) - 2\big]\delta(\alpha)\delta(\beta)\right],\\
h_{(\mathrm{xvi})}(\alpha,\beta) &= -\frac{1}{2}C_F C_A\Big[\ln(1-\tau) + 2\ln\tau - 4 + \ln^2\alpha - \ln^2\bar{\alpha}\Big].
\end{aligned}
\tag{F.4}
$$

The non-vanishing contributions to $h^{\mathbb{P}}(\alpha,\beta)$ originate from two diagrams only:

$$\begin{aligned} h^{\mathbb{P}}_{(\mathrm{ii})}(\alpha,\beta) &= -\Big[C_F^2-\frac{1}{2}C_FC_A\Big]\left[4\ln\bar\tau-2\ln\bar\alpha^2+\ln^2(1-\alpha-\beta)\right], \\ h^{\mathbb{P}}_{(\mathrm{iv})}(\alpha,\beta) &= \Big[C_F^2-\frac{1}{2}C_FC_A\Big]\Big[6\ln\bar\tau-\ln^2\bar\tau-2\bar\tau\ln\bar\tau-2\ln^2\bar\alpha+\ln^2(1-\alpha-\beta)\Big]. \end{aligned} \tag{F.5}$$

In all expressions here and below

$$\tau=\frac{\alpha\beta}{\bar\alpha\bar\beta}\,, \qquad\qquad \beta_0=\frac{11}{3}C_A-\frac{2}{3}n_f\,. \tag{F.6}$$

Note that we do not display vanishing contributions.

F.2 Conformal anomaly

Here we consider the diagrams in Fig. 4.1 with an insertion of the conformal variation of the action. As shown in the chapter 4 the correction to the conformal generator can be written in the form

$$\Delta S_+=\frac{1}{2}\mathbb{H}(z_1+z_2)+z_{12}\Delta_+, \tag{F.7}$$

where $\mathbb{H}$ is the corresponding contribution to the evolution kernel. The contributions to Δ_+ from the diagrams in Fig. 4.1 (including symmetric diagrams with the interchange of the quark and the antiquark) can be brought to the following form:

$$\begin{aligned} [\Delta_+\mathcal{O}](z_1,z_2) &= \int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\Big[w(\alpha,\beta)+w^{\mathbb{P}}(\alpha,\beta)\mathbb{P}_{12}\Big]\Big[\mathcal{O}(z_{12}^\alpha,z_{21}^\beta)-\mathcal{O}(z_{12}^\beta,z_{21}^\alpha)\Big] \\ &\quad+\int_0^1 du\int_0^1 dt\,v(t)\Big[\mathcal{O}(z_{12}^{ut},z_2)-\mathcal{O}(z_1,z_{21}^{ut})\Big]. \end{aligned} \tag{F.8}$$

We obtain (again we omit vanishing contributions)

$$
\begin{aligned}
v_{(\mathrm{iv})}(t) &= \Big[C_F^2-\frac{1}{2}C_FC_A\Big]\Big\{\frac{4}{\bar t}\Big[\operatorname{Li}_2(t)-\operatorname{Li}_2(1)\Big]-4t\Big[\operatorname{Li}_2(t)-\operatorname{Li}_2(1)\Big]+4\bar t\operatorname{Li}_2(1)\\
&\quad -2t\ln t\ln\bar t+\frac{t}{\bar t}\ln^2 t+\bar t\ln^2\bar t-4t\ln\bar t+\frac{2t}{\bar t}(2-3t)\ln t+2\Big\},\\
v_{(\mathrm{v})}(t) &= \Big[C_F^2-\frac{1}{2}C_FC_A\Big]\Big[t\ln^2 t+\frac{2\bar t}{t}\ln^2\bar t+\frac{6\bar t}{t}\ln\bar t-\frac{\bar t}{t}(3t+2)\ln t-9t+8-\frac{1}{t}\Big],\\
v_{(\mathrm{vi})}(t) &= C_F^2\left[\frac{1}{t}+\frac{1+\bar t}{t}\ln t\right],\\
v_{(\mathrm{vii})}(t) &= \Big[C_F^2-\frac{1}{2}C_FC_A\Big]\Big[-t\ln t-1\Big],\\
v_{(\mathrm{x})}(t) &= -2C_F\frac{\bar t}{t}\Big[(\beta_0-2C_A)\ln\bar t-\frac{8}{3}C_A+\frac{5}{3}\beta_0\Big],\\
v_{(\mathrm{xi+xii})}(t) &= -4C_F^2\Big\{t\Big[\operatorname{Li}_2(t)-\operatorname{Li}_2(1)\Big]+2\frac{\bar t}{t}\Big[\operatorname{Li}_2(\bar t)-\operatorname{Li}_2(1)\Big]+\frac{\bar t}{t}\ln^2\bar t+\frac{1}{2}t\ln^2 t+2\bar t\ln\bar t\\
&\quad -\frac{3}{2}(1-2t)\ln t+2\Big\}+C_FC_A\frac{\bar t}{t}\Big\{4\Big[\operatorname{Li}_2(\bar t)-\operatorname{Li}_2(t)\Big]+\frac{1}{2}(2+t)\ln^2\bar t\\
&\quad -\left(1-\frac{t^2}{2\bar t}\right)\ln^2 t-2(1-2t)\ln\bar t-\left(5t+\frac{1}{\bar t}\right)\ln t-3+2t\Big\},\\
v_{(\mathrm{xiii})}(t) &= C_FC_A\Big\{\frac{2t}{\bar t}\Big[\operatorname{Li}_2(t)-\operatorname{Li}_2(1)\Big]+\bar t\Big[\operatorname{Li}_2(\bar t)-\operatorname{Li}_2(1)\Big]-t\ln t\ln\bar t\\
&\quad +\frac{1}{4}\bar t\ln^2\bar t+\frac{1}{4}\frac{t(3-t)}{\bar t}\ln^2 t-\frac{t^2}{\bar t}\ln t+\frac{1}{2}\ln t-\frac{1+t}{t}\ln\bar t+1\Big\},\\
v_{(\mathrm{xiv})}(t) &= C_FC_A\Big\{\frac{\bar t}{t}\left[\frac{1-2t}{2\bar t}\ln^2 t+\ln\bar t-3\right]\\
&\quad +\frac{1}{2}\left[\frac{1}{2}\ln^2 t-\bar t\ln^2\bar t+\frac{t^2-\bar t}{t}\ln t-2\bar t\ln\bar t-1-\bar t\right]\Big\},\\
v_{(\mathrm{xv})}(t) &= C_FC_A\frac{\bar t}{t}\Big\{t\Big[\operatorname{Li}_2(\bar t)-\operatorname{Li}_2(1)\Big]+\frac{1}{4}t\ln^2\bar t+\frac{1}{4}(2+t)\ln^2 t-(3-t)\ln\bar t\\
&\quad +\frac{1}{2}\left(1-\frac{t^2}{\bar t}\right)\ln t-\bar t-\frac{3}{2}\Big\}.
\end{aligned}
\tag{F.9}
$$

The function $w^{\mathbb{P}}(\alpha,\beta)$ originates from two diagrams only:

$$
\begin{aligned}
w^{\mathbb{P}}_{(\mathrm{ii})}(\alpha,\beta) &= -C_F\left(C_F-\frac{1}{2}C_A\right)2\beta\Big(\ln^2\bar\alpha+4\ln\bar\alpha\Big),\\
w^{\mathbb{P}}_{(\mathrm{iv})}(\alpha,\beta) &= -4C_F\left(C_F-\frac{1}{2}C_A\right)\left\{\left(\bar\alpha-\frac{1}{\bar\alpha}\right)\left[\operatorname{Li}_2\left(\frac{\alpha}{\bar\beta}\right)-\operatorname{Li}_2(\alpha)-\ln\bar\alpha\ln\bar\beta\right]+\alpha\bar\tau\ln\bar\tau\right.\\
&\quad \left.+\frac{\beta^2}{\bar\beta}\ln\bar\alpha-\frac{1}{2}\beta\Big(\ln^2\bar\alpha+4\ln\bar\alpha\Big)\right\}.
\end{aligned}
\tag{F.10}
$$

The non-vanishing contributions to $w(\alpha,\beta)$ are

$$
\begin{aligned}
w_{(\mathrm{i})}(\alpha,\beta) &= -C_F^2\,\beta\Big[\ln^2\bar\alpha + 4\ln\bar\alpha - 2\Big],\\
w_{(\mathrm{ii})}(\alpha,\beta) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\Big[\bar\alpha\ln\bar\alpha\big(\ln\bar\alpha + 2\big) + \beta\big(\ln^2\bar\alpha + 4\ln\bar\alpha - 2\big)\Big],\\
w_{(\mathrm{iii})}(\alpha,\beta) &= 2C_F^2\Big\{(\beta-\alpha)\Big[4 + 3\ln\alpha - 2\ln\bar\alpha + \frac{1}{2}\ln^2\alpha + \frac{1}{2}\ln^2\bar\alpha + 2\big(\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(1)\big)\Big]\\
&\quad - \frac{2}{\bar\alpha}\Big[\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(1)\Big] - \frac{1}{2\bar\alpha}\ln^2\alpha + \frac{1}{2}\ln^2\bar\alpha + \Big[\frac{\alpha}{\bar\alpha} - 3 + \alpha\Big]\ln\alpha + \bar\alpha\ln\bar\alpha + \alpha\Big\},\\
w_{(\mathrm{iv})}(\alpha,\beta) &= -4\Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\Big\{\Big(\beta - \frac{1}{\beta}\Big)\Big[\mathrm{Li}_2(\alpha/\bar\beta) - \mathrm{Li}_2(\alpha) - \mathrm{Li}_2(\beta)\Big]\\
&\quad + \Big(\beta - \frac{1}{\bar\alpha}\Big)\Big[\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(1)\Big] + \frac{1}{2}(\alpha+\beta)\ln\alpha\ln\bar\alpha + \frac{1}{2}\beta\ln^2\bar\alpha\\
&\quad - \frac{1}{4}\frac{\bar\alpha}{\alpha}\ln^2\bar\alpha - \frac{1}{4}\frac{\alpha}{\bar\alpha}\ln^2\alpha + \alpha\frac{\bar\tau}{\tau}\ln\bar\tau + 3\beta\ln\bar\alpha + \frac{\alpha}{\bar\alpha}\ln\alpha - \frac{1}{2}\frac{\bar\alpha}{\alpha}\ln\bar\alpha\\
&\quad - (1-\alpha-\beta)\Big[\frac{1}{4}\ln^2\alpha - \frac{1}{4}\ln^2\bar\alpha + \frac{3}{2}\ln\alpha - \frac{3}{2}\ln\bar\alpha - \frac{1}{2\alpha}\ln\bar\alpha\Big]\Big\},\\
w_{(\mathrm{vii})}(\alpha,\beta) &= \Big[C_F^2 - \frac{1}{2}C_F C_A\Big]\Big[\alpha\ln^2\alpha + \beta\ln^2\bar\alpha + 7\beta\ln\bar\alpha + \bar\beta\ln\alpha + 4\alpha\ln\alpha - 4\alpha\Big],\\
w_{(\mathrm{viii})}(\alpha,\beta) &= C_F^2\,\bar\beta\Big[\ln\alpha + 3\Big],\\
w_{(\mathrm{xiii})}(\alpha,\beta) &= C_F C_A\Big\{\frac{2}{\bar\alpha}\Big[\mathrm{Li}_2(\alpha) - \mathrm{Li}_2(1)\Big] + \frac{2}{\alpha}\Big[\mathrm{Li}_2(\bar\alpha) - \mathrm{Li}_2(1)\Big] + \frac{1}{2}\frac{\bar\alpha}{\alpha}\ln^2\bar\alpha\\
&\quad + \frac{1}{2}\frac{\alpha}{\bar\alpha}\ln^2\alpha - \frac{2}{\bar\alpha}\ln\alpha - \frac{2}{\alpha}\ln\bar\alpha + \bar\alpha\ln\alpha + \alpha\ln\bar\alpha + \frac{3}{2}\ln\bar\alpha + \frac{1}{2}\ln\alpha\\
&\quad + \frac{3}{2}\alpha\ln\tau - (\alpha-\beta)\Big[\frac{1}{\alpha}\ln\bar\alpha + \mathrm{Li}_2(\bar\alpha) - \mathrm{Li}_2(\alpha)\Big]\Big\},\\
w_{(\mathrm{xvi})}(\alpha,\beta) &= C_F C_A\Big[\frac{1}{2}\beta\ln^2\bar\alpha + \frac{1}{2}\bar\beta\ln^2\alpha + 3\beta\ln\bar\alpha + 2\bar\beta\ln\alpha + \frac{1}{2}\alpha\ln\alpha - \frac{1}{2}\bar\alpha\ln\bar\alpha - \alpha\Big].
\end{aligned}
\tag{F.11}
$$

Note that the only contribution of the diagram in Fig. 4.1(ix) is through the corresponding term $\sim h_{(\mathrm{ix})}(\alpha,\beta)$ in the evolution kernel.

Appendix G

$\mathbb{X}$ kernels

In this appendix we sample explicit expressions for the kernels $\mathbb{X}^{(k)}$ appearing in the similarity transformation (5.18).

As a reminder, the one-loop kernel $\mathbb{X}^{(1)}$ is defined as a solution to the differential equation

$$\big[S_+^{(0)}, \mathbb{X}^{(1)}\big] = z_{12}\Delta^{(1)}, \tag{G.1}$$

where $\Delta^{(1)}$ is the $\mathcal{O}(a)$ conformal anomaly

$$\Delta^{(1)} f(z_1, z_2) = -2C_F \int_0^1 d\alpha \Big(\frac{\bar\alpha}{\alpha} + \ln\alpha\Big)\Big[f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\Big]. \tag{G.2}$$

As already shown in the chapter 5 the result reads

$$\mathbb{X}^{(1)} f(z_1, z_2) = 2C_F\left(\int_0^1 d\alpha \frac{\ln\alpha}{\alpha}\Big[2f(z_1,z_2) - f(z_{12}^\alpha, z_2) - f(z_1, z_{21}^\alpha)\Big] + \Delta\mathbb{X}_{\rm inv}^{(1)}\right), \tag{G.3}$$

where $\Delta\mathbb{X}_{inv}^{(1)}$ can be any invariant kernel. For convenience we put it to zero.

The two-loop kernel $\mathbb{X}^{(2)}$ is defined as a solution to equation

$$\big[S_+^{(0)}, \mathbb{X}^{(2)}\big] = z_{12}\Delta^{(2)} + \Big[\mathbb{X}^{(1)}, z_1 + z_2\Big]\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) + \frac{1}{2}\Big[\mathbb{X}^{(1)}, z_{12}\Delta^{(1)}\Big], \tag{G.4}$$

with the anomaly $\Delta^{(2)}$ which can be found in Eq. (4.71).

Since this commutator gives rise to a linear differential equation we can split the solution as a sum of three terms corresponding to the three contributions on the r.h.s. of Eq. (G.4)

$$\mathbb{X}^{(2)} = \mathbb{X}_{\rm I}^{(2)} + \mathbb{X}^{(2,1)}\left(\beta_0 + \frac{1}{2}\mathbb{H}^{(1)}\right) - \frac{1}{2}\mathbb{X}^{(2,2)} + \Delta\mathbb{X}_{\rm inv}^{(2)}, \tag{G.5}$$

where $\Delta\mathbb{X}_{\rm inv}^{(2)}$ is an arbitrary invariant part (we will discard it) and the operators must satisfy

$$[S_+^{(0)}, \mathbb{X}_{\rm I}^{(2)}] = z_{12}\Delta^{(2)}, \qquad [S_+^{(0)}, \mathbb{X}^{(2,1)}] = [\mathbb{X}^{(1)}, z_1 + z_2], \qquad [S_+^{(0)}, \mathbb{X}^{(2,2)}] = [z_{12}\Delta^{(1)}, \mathbb{X}^{(1)}]. \tag{G.6}$$

The latter two equations take rather simple solutions

$$\mathbb{X}^{(2,1)}\mathcal{O}(z_1,z_2) = 2C_F\Big\{-\int_0^1 d\alpha\left(\frac{\bar\alpha}{\alpha}\ln\bar\alpha+\ln\alpha\right)\Big[2\mathcal{O}(z_1,z_2)-\mathcal{O}(z_{12}^\alpha,z_2)-\mathcal{O}(z_1,z_{21}^\alpha)\Big], \tag{G.7}$$

and

$$\begin{aligned}
&\mathbb{X}^{(2,2)}\mathcal{O}(z_1,z_2) = \\
&= 4C_F^2\Big\{\int_0^1 d\alpha\int_0^1 du\left[\frac{\ln\bar\alpha}{\alpha}\left(\frac{1}{2}\ln\bar\alpha+2\right)+\frac{\bar u}{u}\frac{\vartheta(\alpha)}{\bar\alpha}\right]\Big[2\mathcal{O}(z_1,z_2)-\mathcal{O}(z_{12}^{\alpha u},z_2)-\mathcal{O}(z_1,z_{21}^{\alpha u})\Big]\\
&\quad+\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\Big[\frac{1}{\tau}\Big(\vartheta_+(\alpha)+\vartheta_+(\beta)\Big)\Big[\mathcal{O}(z_{12}^\alpha,z_{21}^\beta)-\mathcal{O}(z_1,z_{21}^\beta)-\mathcal{O}(z_{12}^\alpha,z_2)+\mathcal{O}(z_1,z_2)\Big]\\
&\qquad+\Big(\vartheta_0(\alpha)+\vartheta_0(\beta)\Big)\mathcal{O}(z_{12}^\alpha,z_{21}^\beta)\Big]\Big\},
\end{aligned} \tag{G.8}$$

where

$$\begin{aligned}
\vartheta_+(\alpha) &= -\frac{1}{\bar\alpha}\Big(\ln\alpha\ln\bar\alpha+2\alpha\ln\alpha+2\bar\alpha\ln\bar\alpha\Big),\\
\vartheta_0(\alpha) &= 2\Big(\mathrm{Li}_3(\bar\alpha)-\mathrm{Li}_3(\alpha)-\ln\bar\alpha\,\mathrm{Li}_2(\bar\alpha)+\ln\alpha\,\mathrm{Li}_2(\alpha)\Big)+\frac{1}{\alpha}\ln\alpha\ln\bar\alpha+\frac{2}{\alpha}\ln\bar\alpha,\\
\vartheta(\alpha) &= \frac{\alpha}{\bar\alpha}\Big(\mathrm{Li}_2(\bar\alpha)-\ln^2\alpha\Big)-\frac{1}{2}\frac{\bar\alpha}{\alpha}\ln^2\bar\alpha+\left(\alpha-\frac{2}{\alpha}\right)\ln\alpha\ln\bar\alpha-\left(3+\frac{1}{\bar\alpha}\right)\ln\alpha-(\alpha-\bar\alpha)\frac{\bar\alpha}{\alpha}\ln\bar\alpha-2.
\end{aligned} \tag{G.9}$$

The operator $\mathbb{X}_{\rm I}^{(2)}$ arises due to the two-loop anomaly and requires a more involved calculation. First we can split

$$\mathbb{X}_{\rm I}^{(2)} = \frac{1}{4}\left(\mathbb{T}^{(2)}+\beta_0\mathbb{T}_1^{(1)}\right)+C_F\left(1-\frac{\pi^2}{6}\right)\mathbb{T}^{(1)}+\mathbb{X}_{\rm IA}^{(2)}+\mathbb{X}_{\rm IB}^{(2)}. \tag{G.10}$$

The operators $\mathbb{T}^{(1)},\mathbb{T}^{(2)}$ and $\mathbb{T}_1^{(1)}$ are given in Eqs. (5.55) and (5.56). For the operators $\mathbb{X}_{\rm IA}^{(2)}$, $\mathbb{X}_{\rm IB}^{(2)}$ we obtain

$$\begin{aligned}
\mathbb{X}_{\rm IA}^{(2)}\mathcal{O}(z_1,z_2) &= \int_0^1 du\,\frac{\bar u}{u}\int_0^1\frac{d\alpha}{\bar\alpha}\,\big[v(\alpha)-v(1)\big]\Big[2\mathcal{O}(z_1,z_2)-\mathcal{O}(z_{12}^{\alpha u},z_2)-\mathcal{O}(z_1,z_{21}^{\alpha u})\Big]\\
&\quad+\int_0^1 d\alpha\,\xi_{\rm IA}(\alpha)\Big[2\mathcal{O}(z_1,z_2)-\mathcal{O}(z_{12}^\alpha,z_2)-\mathcal{O}(z_1,z_{21}^\alpha)\Big],
\end{aligned} \tag{G.11}$$

where $v(\alpha)$ is one of the functions entering the two-loop conformal anomaly (4.62) and

$$\begin{aligned}
\xi_{\rm IA}(\alpha) &= 2C_F^2\frac{\bar\alpha}{\alpha}\Big[-\mathrm{Li}_3(\bar\alpha)+\ln\bar\alpha\,\mathrm{Li}_2(\bar\alpha)+\frac{1}{3}\ln^3\bar\alpha+\mathrm{Li}_2(\alpha)+\frac{1}{\bar\alpha}\ln\alpha\ln\bar\alpha-\frac{1}{4}\ln^2\bar\alpha\\
&\quad-\frac{3\alpha}{\bar\alpha}\ln\alpha-3\ln\bar\alpha\Big]+\frac{C_F}{N_c}\left(\ln\alpha+\frac{\bar\alpha}{\alpha}\ln\bar\alpha\right).
\end{aligned} \tag{G.12}$$

Finally

$$\mathbb{X}_{\rm IB}^{(2)}\mathcal{O}(z_1,z_2) = C_F\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\Big[C_F\,\xi_{\rm IB}^f(\alpha,\beta)+\frac{1}{N_c}\Big(\xi_{\rm IB}^A(\alpha,\beta)+\xi_{\rm IB}^{A\mathbb{P}}(\alpha,\beta)\mathbb{P}_{12}\Big)\Big]\mathcal{O}(z_{12}^\alpha,z_{21}^\beta), \tag{G.13}$$

where

$$\xi_{\mathrm{IB}}^{A\mathbb{P}}(\alpha,\beta) = -2\bigg[\mathrm{Li}_3(\bar\alpha) - \mathrm{Li}_3\left(1-\frac{\alpha}{\bar\beta}\right) + \frac{1}{\bar\alpha}\left[\mathrm{Li}_2(\alpha) - \mathrm{Li}_2\left(\frac{\alpha}{\bar\beta}\right)\right] + \ln\bar\tau\ \mathrm{Li}_2\left(1-\frac{\alpha}{\bar\beta}\right)$$
$$+ \frac{1+\alpha}{2\bar\alpha}\ln\bar\alpha\ln\bar\beta + (\alpha\leftrightarrow\beta)\bigg],$$
$$\xi_{\mathrm{IB}}^{A}(\alpha,\beta) = 2\bigg[\mathrm{Li}_3(\bar\beta) - 2\,\mathrm{Li}_3(\beta) - \mathrm{Li}_3\left(1-\frac{\beta}{\bar\alpha}\right) - \mathrm{Li}_3\left(\frac{\beta}{\bar\alpha}\right) + \ln\bar\tau\ \mathrm{Li}_2\left(1-\frac{\beta}{\bar\alpha}\right) + \ln\left(\frac{\beta}{\bar\alpha}\right)\mathrm{Li}_2(\beta)$$
$$+ \frac{1}{\alpha}\left(\mathrm{Li}_2(\beta) - \mathrm{Li}_2\left(\frac{\beta}{\bar\alpha}\right)\right) + 2\frac{\bar\beta}{\beta}\,\mathrm{Li}_2(\beta) + \frac{1}{2}\ln\bar\alpha\ln\bar\beta + \frac{\bar\beta}{\beta}\ln\beta\ln\bar\beta + (\alpha\leftrightarrow\beta)\bigg], \tag{G.14}$$

and

$$\xi_{\mathrm{IB}}^{f}(\alpha,\beta) = \ln(1-\alpha-\beta)\ln(\tau\,\bar\tau) - \frac{1}{3}\ln^3(1-\alpha-\beta) + 3\ln\bar\alpha\ln\bar\beta - \ln\alpha\ln\beta$$
$$+ \bigg[-6\,\mathrm{Li}_3(\bar\alpha) - 10\,\mathrm{Li}_3(\alpha) + 2\ln\bar\alpha\,\mathrm{Li}_2(\bar\alpha) + 6\ln\alpha\,\mathrm{Li}_2(\alpha) + \ln\alpha\ln\bar\alpha\big(\ln\alpha + \ln\bar\alpha - 2\big)$$
$$- 4\frac{1+\alpha}{\alpha}\Big(\mathrm{Li}_2(\bar\alpha) - \mathrm{Li}_2(1)\Big) - \frac{1}{3}\ln^3\bar\alpha - \frac{\bar\alpha}{\alpha}\ln^2\bar\alpha + \frac{1}{2}\ln^2\alpha - \frac{2}{\bar\alpha}\ln\alpha + \frac{4}{\alpha}\ln\bar\alpha + 15\ln\bar\alpha$$
$$+ (\alpha\leftrightarrow\beta)\bigg]. \tag{G.15}$$

In all expressions $\tau = \frac{\alpha\beta}{\bar\alpha\bar\beta}$.

Appendix H

Transformation to conformal scheme

In this appendix we will comment on the solution (7.30) of the evolution equation (6.16) for the local operators. As mentioned in chapter 7 the matrix $\hat{\mathcal{B}}(a, a_0)$ can be seen as a transformation from the tree-level conformal operators, which mix under renormalization starting from $\mathcal{O}(a^2)$, to a diagonal basis of operators that evolves autonomously (up to $\mathcal{O}(a^4)$ corrections). This set of operators

$$\mathcal{Q}_n^{(\mathrm{co})}(a, a_0) = \sum_{m=0}^{n} \mathcal{B}_{nm}^{-1}(a, a_0)\mathcal{Q}_m(a), \tag{H.1}$$

can be identified as *conformal operators* to NNLO. The solution (7.28) is valid for arbitrary coupling a and reference scale a_0, which is defined by the condition

$$\mathcal{Q}_n^{(\mathrm{co})}(a_0, a_0) = \mathcal{Q}_n(a_0). \tag{H.2}$$

In the following we will discuss a simplified version of the transformation, namely $\hat{\mathcal{B}}_* \equiv \hat{\mathcal{B}}(a_*, a_0 = 0)$. It is valid only at the conformal fixed point a_* of the theory and takes a perturbative expansion $\hat{\mathcal{B}}_* = \mathbb{1} + a_*\hat{\mathcal{B}}_*^{(1)} + \dots$ with expansion coefficients $\hat{\mathcal{B}}_*^{(i)}$ that do not depend on the scale. We can easily convince ourselves that at the critical point a_* the differential equation (7.25) for the transformation $\mathcal{B}$ reduces to

$$\hat{\mathcal{B}}^{-1}(a_*)\hat{\gamma}(a_*)\hat{\mathcal{B}}(a_*) = \hat{\gamma}^{\mathrm{D}}(a_*). \tag{H.3}$$

Eq. (H.3) relates the anomalous dimension matrix $\hat{\gamma}$ at ℓ-loop accuracy to the transformation matrix $\hat{\mathcal{B}}_*$ at $(\ell - 1)$-loops. The matrix $\hat{\mathcal{B}}_*$ can be derived by the following simple considerations: Looking for a transformation, that maps the full conformal generator to its canonical form

$$\hat{\mathcal{B}}_*^{-1}\left(\hat{\boldsymbol{a}} + \hat{\gamma}(\bar{\beta} + \frac{1}{2}\hat{\gamma}) + \hat{\mathbf{w}}\right)\hat{\mathcal{B}}_* = \hat{\boldsymbol{a}}, \tag{H.4}$$

one obtains the solution [28]

$$\hat{\mathcal{B}}_* = \frac{1}{1 + \mathcal{G}\big\{\hat{\boldsymbol{b}}(\bar{\beta} + \frac{1}{2}\hat{\gamma}) + \hat{\mathbf{w}}\big\}}. \tag{H.5}$$

In fact, one easily verifies that this matrix (H.5) also fulfills Eq. (H.3). To this end it is not surprising that this matrix can be written in terms of the similarity transformation $\mathbb{U}$ derived Sec. 5.2.1. Let us transform the operator $\mathbb{U}$, see Eq. (5.18), to local matrix representation according to the rules from chapter 6, see Eq. (6.15)

$$\boldsymbol{U}_{nm} = \langle nk|\mathbb{U}|mk\rangle, \tag{H.6}$$

where we discard perturbative corrections to the diagonal elements $\boldsymbol{U}_{nn} \overset{!}{=} 1$[1]. In practice, this can be achieved by the proper choice of the invariant kernel $\Delta\mathbb{X}$ in Eqs. (G.3) and (G.5). In terms of the similarity transformation the matrix $\hat{\mathcal{B}}_*$ factorizes to

$$\hat{\mathcal{B}}_* = \hat{\boldsymbol{U}}\,\hat{\boldsymbol{S}}, \tag{H.7}$$

where $\hat{\mathcal{S}}$ is the transformation that removes the anomaly part $\sim \hat{\boldsymbol{b}}(\bar{\beta}(a_*) + \frac{1}{2}\hat{\gamma}(a_*))$. It is defined through the index shift of Gegenbauer polynomials

$$C_n^\alpha(x) = \sum_{m=0}^{n} \mathbf{s}_{nm}^{(\alpha\to\beta)} C_m^\beta(x), \qquad \mathbf{s}_{nm}^{(\alpha,\beta)} = \begin{cases} \frac{(\alpha)_{\frac{n+m}{2}}(\alpha-\beta)_{\frac{n-m}{2}}}{(\beta)_{\frac{n+m}{2}}} \frac{\beta+m}{\beta+\frac{n+m}{2}} \frac{1}{(\frac{n-m}{2})!}, & \text{if } n-m \text{ even} \\ 0, & \text{else} \end{cases} \tag{H.8}$$

as follows

$$\boldsymbol{S}_{nm} = \frac{(\lambda_m)_m}{(3/2)_m} \mathbf{s}_{nm}^{(3/2\to\lambda_m)}, \qquad \lambda_m = 3/2 + \bar{\beta}(a_*) + \frac{1}{2}\gamma_m(a_*). \tag{H.9}$$

Here $(x)_n$ is the Pochhammer symbol. In Ref. [28] the transformation matrix $\hat{\mathcal{B}}_*$ has been derived to NLO which agrees with the result from our work.

[1]This choice allows us also to omit the dependence on the total number of derivatives k in Eq. (H.6) as this number can enter the diagonal elements.

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