Homogenization of Layered Materials with Stiff Components

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Zusammenfassung

Im Rahmen der Homogenisierung elastoplastischer Materialien untersuchen wir das makroskopische Verhalten von Schichtmaterialien mit steifen Komponenten. Für die Modellierung des Materials wählen wir einen variationellen Zugang, der mit den Annahmen finiter Elastizität verträglich ist.

Im Hinblick auf das makroskopische Materialverhalten steht die leichte Verformbarkeit einzelner dünner Schichten der Steifheit des Materials gegenüber, was die Frage des optimalen Skalierungsverhältnisses zwischen Schichtdicke und Steifheit aufwirft. Die Antwort unterscheidet zwei Skalierungsregime. Für ausreichende Materialsteifheit zeigen wir ein neuartiges asymptotisches Rigiditätsresultat, das die relativ eingeschränkten Möglichkeiten der Materialverformung vollständig charakterisiert. Insbesondere zeigt sich, dass unter der Annahme lokaler Volumenerhaltung in zwei Dimensionen lediglich globale Scherdeformationen möglich sind. Dagegen belegen wir anhand konkreter Beispiele, dass bei kleinen elastischen Konstanten Biege- und Wrinkling-Effekte weit mehr Materialverformungen erlauben.

Mit Hilfe dieser Charakterisierung können Homogenisierungsformeln für eine große Zahl variationeller Modelle elastoplastischer Materialien durch Γ-Konvergenz bestimmt werden. In diesem Werk wird dies für zwei Materialmodelle, eines für rein elastische Schichtmaterialien und eines im Kontext der Kristallplastizität, gezeigt.

Für Materialmodelle, in denen die weichen Schichten durch eine elastische Energiedichte mit übereinstimmender poly- und quasikonvexer Einhüllenden beschrieben werden, geben wir eine explizite Homogenisierungsformel an und erläutern deren Beziehung zu klassischen Zell- und Multizellformeln.

Des Weiteren betrachten wir ein steifes Schichtmaterial in zwei Dimensionen, mit einem aktiven Gleitsystem in den weichen Schichten. In diesem Fall wird die Homogenisierungsformel stark von der Orientierung des Gleitsystems beeinflusst. Insbesondere bestätigen die Ergebnisse die Erwartung, dass ein senkrecht zur den steifen Schichten orientiertes Gleitsystem von diesen blockiert wird, während eine Scherung entlang der steifen Schichten unbeeinflusst bleibt.

Nach den beiden periodischen Homogenisierungsresultaten betrachten wir abschließend noch ein Material mit zufälliger Schichtdicke, dessen steife Komponente vollkommen rigide ist, sich in jeder zweiten Schicht jedoch entlang eines aktiven Gleitsystems plastisch verformen lässt.

Abstract

In the context of homogenization of elastoplastic materials, we study the effects of a stiff component on the macroscopic behavior of a material of fine layered structure. To model these materials a variational approach is chosen in accordance to the assumptions of finite elasticity.

In view of the macroscopic material response, the elasticity of the individual thin layer stands in contrast to the stiffness of the components, leading to the question of optimal scaling relations between layer thickness and stiffness. We answer this question by identifying two regimes. For sufficiently strong stiffness, we provide a new type of asymptotic rigidity theorem, which enables us to give a full characterization of the rather restricted class of macroscopic material responses. In particular, we show that in two dimensions, if volume is preserved locally, this class comprises only globally rotated shear deformations. In contrast, we illustrate with explicit examples that for small elastic constants bending and wrinkling of layers leads to much broader possibilities for deformations.

This characterization result allows to determine homogenization formulas for a manifold of variational models for finite elastoplastic materials via Γ-convergence. In this work, we provide two homogenization results, one for elastic materials and one in the context of crystal plasticity.

Firstly, assuming that the elastic softer layers are described by an energy density whose polyconvex envelope coincides with its quasiconvex envelope, we establish an explicit homogenization formula and discuss its relations to cell and multicell formulas.

Secondly, we consider a stiff material in two dimensions with one active slip system in every other layer. Here, the homogenization formula strongly depends on the orientation of the slip system. In particular, the intuition that a slip system orthogonal to the stiff layers should be blocked, while for a parallel orientation it should be unhindered, is rigorously confirmed. Due to the distinct differential inclusion constraints imposed on different layers on admissible deformations, the proof requires tailor-made recovery sequences for which we give explicit constructions.

While the above homogenization results both concern periodic layered materials, we also give a homogenization result for randomly layered material featuring totally rigid layers and one active slip system in every other layer.

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1

Elastoplastic Materials of Layered Structure

1.1 Introduction

The behavior of an elastoplastic body under external forces is determined by a large number of different material properties, ranging from its dimensions and shape over the polycrystalline texture, the plastic deformation processes such as twinning and movement of dislocations to the specific structure of inter-atomic bonds [80, 128]. The physical processes determining these properties take place on different length scales, with the actual macroscopic material response resulting from the configuration of and the interaction between these underlying systems, see also Figure 1.1. To describe these effects a broad range of ad-hoc models has been in use, each expressing the behavior with respect to the individual length scale, but in recent years interest grew in the transition between different length scales [107].

With this approach the asymptotic behavior of material models is determined computationally or, as in this work, analytically, and evaluated if the limit is a viable physical model on the next length scale. Accordingly, parameters specific to the original length scale fade into irrelevance by averaging effects in the homogenized model. Progress has been made by various authors on length scale transition between e.g. discrete atomistic [23] or dislocation models [121] to continuous material descriptions, to name two individual contributions.

In this work, we focus on a specific geometric layout of the material by assuming that it is of layered structure. Materials of this type are constructed for various applications such as reinforcement, but are also of natural origin, e.g. in nacre. Furthermore, we assume that one material component is stiff in the sense that deformations of this component are always close to rigid body motions. From a physical point of view this corresponds to a large elastic constant.

Devoted to the study of the macroscopic material behavior of these layered materials with stiff components, our interest lies particularly in the influence of the stiff layers. More precisely, we aim to answer the following questions:

- How can the macroscopic response of a periodically bilayered material with stiff layers be characterized? What is the optimal scaling relation between the stiffness and the layer thickness for this characterization to hold?

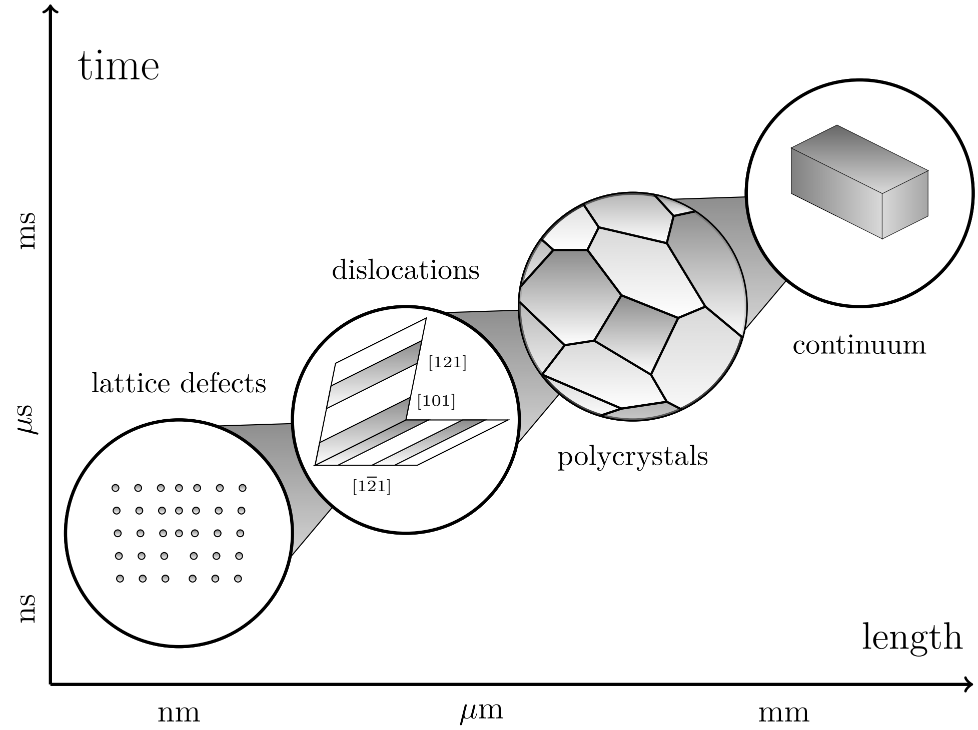

Figure 1.1: The macroscopic behavior of a material under external forces results from physical phenomena on different length scales, ranging from the movement of lattice defects on an atomistic scale, over the interaction of dislocations and specific characteristics of the polycrystalline texture to actual macroscopic features like size or shape of the body.

- Is there a general homogenization formula for periodically bilayered elastic materials with a stiff material component?
- How can we characterize the macroscopic behavior of a crystalline material that is elastically stiff as a whole, yet every other layer can be plastically deformed along one active slip system?
- Assuming that the layer thickness varies randomly, do the above characterizations still hold under suitable assumptions?

Geometry of layered materials. In this work, the elastoplastic body is represented by an n-dimensional, $n \in \mathbb{N}$, $n \geq 2$ bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. To describe the periodic structure of fine layers we introduce a length parameter $\epsilon > 0$ and set $P := \mathbb{R}^{n-1} \times [0, 1)$ for the periodic cell, see also Figure 1.2.

The periodic cell P itself consists of two components, the stiff and the soft one, with a ratio of λ, which we denote by

$$P_{\text{soft}} := \mathbb{R}^{n-1} \times (0, \lambda] \quad \text{and} \quad P_{\text{stiff}} := \mathbb{R}^{n-1} \times (\lambda, 1]. \tag{1.1}$$

Thus, we refer to the whole soft component in Ω by $\epsilon P_{\text{soft}} \cap \Omega$.

Variational approach to finite elastoplasticity. We choose a variational approach to study the properties of elastoplastic bodies. Accordingly, the physical properties of the materials are derived from given stored energy functionals $E : W^{1,p}(\Omega; \mathbb{R}^n) \to \mathbb{R}$, where

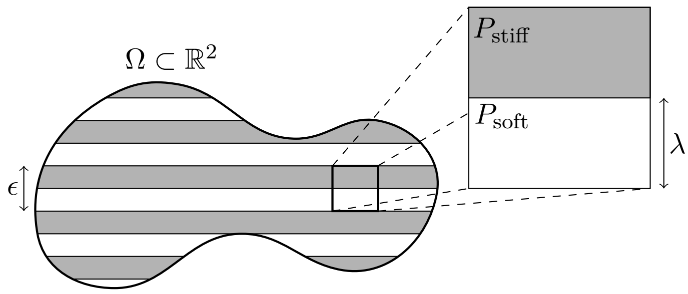

Figure 1.2: The first chapters of this work concern periodically layered materials with respect to the periodic cell $P = \mathbb{R}^{n-1} \times (0, 1]$, which we split according to a fixed ratio parameter $\lambda > 0$ in a soft part, denoted by $P_{\text{soft}} = \mathbb{R}^{n-1} \times (0, \lambda]$, and a stiff part, denoted by $P_{\text{stiff}} = \mathbb{R}^{n-1} \times (\lambda, 1]$. To describe the layer thickness we introduce a length parameter $\epsilon > 0$. Thus, $\epsilon P_{\text{stiff}} \cap \Omega$ refers to the whole stiff component throughout Ω. In certain sections of this work, we also consider more generally ϵ-dependent ratios λ_ϵ or random ratios.

$1 < p < \infty$. We distinguish two fundamentally different types of deformation of physical bodies. One is the elastic deformation of a body, the other is the plastic deformation.

For the elastic energy $E_{\text{el}} : W^{1,p}(\Omega; \mathbb{R}^n) \to [0, \infty)$, we consider the prototypical energy density given by distance to $SO(n)$, i.e.

$$E_{\text{el}}(u) = \int_\Omega \operatorname{dist}^p (\nabla u, SO(n))\, \mathrm{d}x, \quad u \in W^{1,p}(\Omega; \mathbb{R}^n),$$

which is continuous, frame-indifferent and assigns zero elastic energy to rigid body motions. To model the stiffness of the material we introduce a penalization related to the length parameter ϵ via a scaling parameter $\alpha > 0$. Moreover, incorporating the layered structure in the sense that we only apply the penalized elastic energy to the stiff layers, we set $E_\epsilon^\alpha : L_0^p(\Omega; \mathbb{R}^n) \to [0, \infty]$ to be given for $u \in W^{1,p}(\Omega; \mathbb{R}^n) \cap L_0^p(\Omega; \mathbb{R}^n)$ by

$$E_\epsilon^\alpha(u) = \int_\Omega \frac{1}{\epsilon^\alpha} \operatorname{dist}^p (\nabla u, SO(n))\, \mathbb{1}_{\epsilon P_{\text{stiff}}} + W(\nabla u)\, \mathbb{1}_{\epsilon P_{\text{soft}}}\, \mathrm{d}x, \tag{1.2}$$

where $W : \mathbb{R}^{n \times n} \to [0, \infty)$ is a general continuous energy density satisfying

(i) (p-growth) For all $F \in \mathbb{R}^{n \times n}$ and constants $C, c, d > 0$

$$c|F|^p - d \le W(F) \le C(1 + |F|^p);$$

(ii) (Lipschitz-condition) For all $F, G \in \mathbb{R}^{n \times n}$ and a constant $L > 0$

$$|W(F) - W(G)| \le L(1 + |F|^{p-1} + |G|^{p-1})|F - G|.$$

For technical reasons, we extend E_ϵ^α to $L_0^p(\Omega; \mathbb{R}^n)$ by ∞, where we denote by $L_0^p(\Omega; \mathbb{R}^n)$ the set of all mean value free $L^p(\Omega; \mathbb{R}^n)$-functions.

Regarding plastic deformations we consider a two-dimensional model for crystalline materials featuring one active slip system determined by a slip direction $s \in \mathbb{S}^1$ and a slip plane normal $m \in \mathbb{S}^1$. Mathematically, this amounts to an energy functional that only takes finite value for shear deformations in the plane determined by s and m. Introducing a parameter $\gamma \in \mathbb{R}$ measuring the amount of shear, the corresponding energy $E_{\text{slip}} : W^{1,2}(\Omega;\mathbb{R}^2) \to [0,\infty]$ is given by

$$E_{\text{slip}}(u) = \begin{cases} \int_\Omega \gamma^2 \,\mathrm{d}x = \int_\Omega |\nabla u m|^2 - 1 \,\mathrm{d}x & \text{if } \nabla u = \mathbb{I} + \gamma s \otimes m, \\ \infty & \text{otherwise.} \end{cases}$$

In the presence of elastic as well as plastic material response, we assume in accordance to finite strain theory a multiplicative decomposition of the deformation gradient in an elastic and a plastic part. As this decomposition is not unique, it is appropriate to consider a condensed energy $E_{\text{con}} : W^{1,1}(\Omega;\mathbb{R}^2) \to \bar{\mathbb{R}}$, which minimizes the sum of both energy contributions over all possible decomposition, i.e.

$$E_{\text{con}}(u) = \int_\Omega \inf_{\gamma\in\mathbb{R}} \left(\operatorname{dist}^2\left(\nabla u(\mathbb{I} - \gamma s \otimes m), SO(2)\right) + \gamma^2 \right) \mathrm{d}x.$$

Imposing a penalization on the elastic energy yet in contrast to the previous model throughout the material via a scaling parameter $\beta > 0$ and incorporating the layered structure for the plastic energy we set $E_\epsilon^\beta : L_0^1(\Omega;\mathbb{R}^2) \to [0,\infty]$ to be given for $u \in W^{1,1}(\Omega;\mathbb{R}^2) \cap L_0^1(\Omega;\mathbb{R}^2)$ by

$$E_\epsilon^\beta(u) = \int_{\epsilon P_{\text{stiff}}\cap\Omega} \epsilon^{-\beta} \operatorname{dist}^2(\nabla u, SO(2)) \,\mathrm{d}x \\ + \int_{\epsilon P_{\text{soft}}\cap\Omega} \inf_{\gamma\in\mathbb{R}} \left(\epsilon^{-\beta} \operatorname{dist}^2\left(\nabla u(\mathbb{I} - \gamma s \otimes m), SO(2)\right) + \gamma^2\right) \mathrm{d}x.$$

Notice that in contrast to E_ϵ^α, the penalization of the elastic energy in E_ϵ^β also effects the soft layers, motivating the change of notation. Again, for technical reasons, E_ϵ^β is defined on $L_0^1(\Omega;\mathbb{R}^2)$. To mitigate the complexities arising from the condensed energy, often the rigid-plastic idealization is studied, which allows only rigid body motions as elastic deformations. The corresponding energy $E_\epsilon^\infty : W^{1,2}(\Omega;\mathbb{R}^2) \to [0,\infty]$ is then only of finite value if

$$\nabla u(x) \in \begin{cases} SO(2) & \text{if } x \in \epsilon P_{\text{stiff}} \cap \Omega, \\ \mathcal{M} = \{R(\mathbb{I} + \gamma s \otimes m) \mid R \in SO(2), \gamma \in \mathbb{R}\} & \text{if } x \in \epsilon P_{\text{soft}} \cap \Omega, \end{cases}$$

in which case

$$E_\epsilon^\infty(u) = \int_{\epsilon P_{\text{soft}}\cap\Omega} \gamma^2 \,\mathrm{d}x.$$

Results. In the following, we give an overview on the results proven in the later chapters and discuss how they provide answers to the key questions asked above. Though these theorems hold on a large class of bounded Lipschitz domains, certain conditions on the geometry of the domain are required, which will be discussed to some extent in Chapter 3 and fully in Chapter 4. To avoid these rather technical considerations here, we assume that Ω is of cylindrical shape, by which we mean that $\Omega = S \times (a, a+h)$, where $S \subset \mathbb{R}^{n-1}$, $n \in \mathbb{N}$, $n \geq 2$ is a bounded Lipschitz domain of $n-1$ dimensions, $a \in \mathbb{R}$ and $h > 0$.

The first theorem gives a full characterization of the effective material responses.

Theorem 1.1.1 (Asymptotic characterization of fine bilayered functions with stiff components)**.** *For $n \geq 2$ and $1 < p < \infty$ let $(u_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n)$ satisfy for $\alpha > p$, a constant $C > 0$ and all $\epsilon > 0$*

$$\int_{\epsilon P_{\text{stiff}}\cap\Omega} \operatorname{dist}^p\left(\nabla u_\epsilon, SO(n)\right) \mathrm{d}x < C\epsilon^\alpha. \tag{1.3}$$

If $u_\epsilon \rightharpoonup u$ *in* $W^{1,p}(\Omega;\mathbb{R}^n)$ *as* $\epsilon \to 0$ *for some* $u \in W^{1,p}(\Omega;\mathbb{R}^n)$*, then there are functions* $R \in W^{1,p}(\Omega; SO(n))$ *and* $b \in W^{1,p}(\Omega;\mathbb{R}^n)$ *with* $\partial_i R = 0$ *and* $\partial_i b = 0$ *for* $i = 1, \dots, n-1$ *such that for a.e.* $x \in \Omega$

$$u(x) = R(x)x + b(x). \tag{1.4}$$

Moreover, if $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ *satisfies* (1.4)*, then there is a sequence* $(u_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n)$ *such that* (1.3) *holds for* $\alpha > p$*.*

Remark 1.1.2. a) Since R and b depend locally only on x_n, R and b are continuous and thus so is also u.

b) For general bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$, the statement merely holds true for $R \in W^{1,p}_{\text{loc}}(\Omega; SO(n))$. Yet, for Lipschitz domains of cylindrical shape with respect to the e_n-direction, we have full integrability on Ω, i.e. $R \in W^{1,p}(\Omega; SO(n))$.

c) In Chapter 3 we show a version of the necessary statement of Theorem 1.1.1 that is more general in several regards addressing also the following relevant cases:

1) If $(u_\epsilon)_\epsilon \subset W^{1,1}(\Omega;\mathbb{R}^n)$ converges weakly to a limit function $u \in W^{1,1}(\Omega;\mathbb{R}^n)$ and satisfies for $1 < q < \infty$, an $\alpha > q$, a constant $C > 0$ and for all $\epsilon > 0$ the condition

$$\int_{\epsilon P_{\text{stiff}} \cap \Omega} \operatorname{dist}^q \left(\nabla u_\epsilon, SO(n)\right) \mathrm{d}x < C\epsilon^\alpha,$$

then, there is an $R \in BV_{\text{loc}}(\Omega; SO(n))$ as well as a function $b \in BV_{\text{loc}}(\Omega;\mathbb{R}^n)$ with $\partial_i R = 0$ and $\partial_i b = 0$ for $i = 1, \dots, n-1$ such that

$$u(x) = R(x)x + b(x).$$

2) We may replace the constant ratio λ between the material components by an ϵ-dependent sequence $(\lambda_\epsilon)_\epsilon$. In this case, the results still holds if $\alpha > 0$ and λ_ϵ are such that

$$1 - \lambda_\epsilon \gg \epsilon^{\frac{\alpha}{p} - 1}.$$

In the presence of a local volume preservation condition, the above result implies an even stronger restriction of limit functions.

Corollary 1.1.3 (Asymptotic rigidity)**.** *Assume additionally that* $u \in W^{1,r}(\Omega;\mathbb{R}^n)$ *for* $r \geq n$ *and that* u *locally preserves volume, i.e.* $\det \nabla u = 1$ *a.e. Then, there are a constant rotation* $Q \in SO(n)$*, and functions* $R \in L^\infty(\Omega; SO(n-1))$ *and* $a \in L^p(\Omega;\mathbb{R}^n)$ *with* $\partial_i R = 0$ *for all* $i \in \{1, \dots, n-1\}$ *and* $e_n \cdot a = 0$ *such that*

$$\nabla u = Q \operatorname{diag}(R, 1)(\mathbb{I} + a \otimes e_n),$$

where $\operatorname{diag}(R, 1)$ *denotes the block matrix with* R *and* 1 *on the diagonal.*

Remark 1.1.4. a) In the two-dimensional case $n = 2$, we have $R = \pm 1$. Hence, for a constant rotation $Q \in SO(2)$ it holds that

$$\nabla u = Q(\mathbb{I} + a \otimes e_n).$$

This result in the setting of totally rigid layers has been established by the author together with his adviser Carolin Kreisbeck in [42].

b) These results are optimal in the sense that there are explicit examples of different macroscopic behavior in the regime $0 < \alpha < p$.

The characterization theorems are key to understand the effects of the stiff layers on the macroscopic material behavior. Based on these results we give two homogenization results. The first concerns $(E^\alpha_\epsilon)_\epsilon$, which describes a bilayered material with stiff layers and a general energy density W on the soft layers.

Theorem 1.1.5 (Homogenization of periodically layered materials with stiff components). *If $\alpha > p > n \geq 2$ and the quasiconvex hull W^{qc} of W is polyconvex, then the family of energy functionals $E^\alpha_\epsilon : L^p_0(\Omega;\mathbb{R}^n) \to [0,\infty]$ defined by (1.2), converges in the sense of Γ-convergence with respect to the strong L^p-topology to the limit functional $E : L^p_0(\Omega;\mathbb{R}^n) \to [0,\infty]$ given for $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ with $u(x) = R(x)x + b(x)$ for a.e. $x \in \Omega$, $R \in W^{1,p}(\Omega; SO(n))$, $b \in L^p(\Omega;\mathbb{R}^n)$ with $\partial_i R = 0$ and $\partial_i b = 0$, $i = 1, \dots, n-1$ by*

$$E(u) = \int_\Omega \lambda W^{\mathrm{qc}}(\lambda^{-1}(\nabla u - (1-\lambda)R))\,\mathrm{d}x,$$

and $E(u) = \infty$ otherwise in $L^p_0(\Omega;\mathbb{R}^n)$.

Furthermore, sequences $(u_\epsilon)_\epsilon \subset L^p_0(\Omega;\mathbb{R}^n)$ that are of bounded energy with respect to $(E^\alpha_\epsilon)_\epsilon$, i.e. for a constant $C > 0$ it holds that $E^\alpha_\epsilon(u_\epsilon) \leq C$ for all $\epsilon > 0$ are relatively compact in $L^p_0(\Omega;\mathbb{R}^n)$.

The second homogenization result determines the Γ-limit of $(E^\beta_\epsilon)_\epsilon$, which models a stiff material with one active slip system in every other layer.

Theorem 1.1.6 (Homogenization of bilayered materials with one active slip system). *For $n = 2$ and $\beta > 2$ the sequence $E^\beta_\epsilon : L^1_0(\Omega;\mathbb{R}^2) \to [0,\infty]$ converges in the sense of Γ-convergence with respect to the strong L^1-topology to a functional $E : L^1_0(\Omega;\mathbb{R}^2) \to [0,\infty]$. Using the notation*

$$K_{s,\lambda} = \begin{cases} \{0\} & \text{if } s = e_2, \quad [-2\frac{s_1}{s_2}\lambda, 0] \quad \text{if } s_1 s_2 > 0, \\ \mathbb{R} & \text{if } s = e_1, \quad [0, -2\frac{s_1}{s_2}\lambda] \quad \text{if } s_1 s_2 < 0, \end{cases}$$

the limit functional E is given for $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with $R \in SO(2)$, $\gamma \in L^2(\Omega)$, $\gamma \in K_{s,\lambda}$ a.e. in Ω by

$$E(u) = \lambda \int_\Omega \left|(\nabla u m - (1-\lambda)R)m\right|^2 - 1\,\mathrm{d}x,$$

and $E(u) = \infty$ otherwise in $L^1_0(\Omega;\mathbb{R}^2)$.

Furthermore, sequences $(u_\epsilon)_\epsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ that are of bounded energy with respect to $(E^\beta_\epsilon)_\epsilon$, i.e. for a constant $C > 0$ it holds that $E^\beta_\epsilon(u_\epsilon) \leq C$ for all $\epsilon > 0$, are relatively compact in $L^1_0(\Omega;\mathbb{R}^2)$.

Lastly, we consider a model with random component ratio. To keep the focus on the stochastic challenges, we only consider the rigid plastic idealization $\beta = \infty$ with E^∞_ϵ associated to the slip system $s = e_1, m = e_2$. The ratio between each neighboring rigid and soft layer is determined by a valued stochastic process $(\lambda^i)_{i\in\mathbb{Z}}$ taking values in $(0,1)$ and defined on a probability space $(\Xi, \mathcal{A}, \mathbb{P})$. We adapt the notion of P_{stiff} and P_{soft} accordingly, setting for the i-th layer $P^i_{\text{soft}} = (0, \lambda_i]$.

Theorem 1.1.7 (Homogenization of randomly layered materials with rigid layers). *For $\lambda \in (0,1)$ let $(\lambda^i)_{i\in\mathbb{Z}}$ be a stationary and ergodic process with $\lambda^i > \lambda$ for all $i \in \mathbb{Z}$. Then, the family of energy functionals $E^\infty_\epsilon : L^2_0(\Omega;\mathbb{R}^n) \times \Xi \to [0,\infty]$ converges almost surely in the sense of Γ-convergence with respect to the strong L^2-topology to a functional $E : L^2_0(\Omega;\mathbb{R}^2) \to [0,\infty]$ given for $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ a.e. in Ω, $R \in SO(2)$, $\gamma \in \mathbb{R}$ by*

$$E(u) = \frac{1}{\mathbb{E}(\lambda^i)} \int_\Omega \gamma^2\,\mathrm{d}x,$$

and $E(u) = \infty$ otherwise in $L^2_0(\Omega;\mathbb{R}^2)$.

The lower bound on λ^i is required for the deterministic characterization of the limit function of Theorem 1.1.1 to be applicable.

Relations to active fields of research. The results presented are related to several active fields of research. In the following, we want to give an overview on associated works, focusing on the relations to recent developments in different communities rather than completeness.

As the necessary statement of Theorem 1.1.1 allows to conclude on the global features of the limit u in the directions $x_1, \dots, x_{n-1}$ by local properties, it can be seen as a rigidity result. This interpretation is even more pronounced by Corollary 1.1.3 and Remark 1.1.4. The classic geometric rigidity result in this context is the Liouville theorem [87, Section 2.3], which was generalized to lower regularity by Rešetnjak, see [122]. In [72], Friesecke, James and Müller provided a quantitative estimate of this result, which was extended to p-growth conditions for $1 \leq p < \infty$ by Conti and Schweizer [55]. Newer developments include rigidity results for incompatible fields [115], mixed growth conditions [54], SBV-functions [38] and rigidity results for two- and multiwell problems, such as [55, 101, 40, 90].

The difference to classic geometric rigidity results lies in the fact that the stiffness condition is only satisfied for non-connected subdomains and thus rigidity can only be expected to hold in the limit of the layer thickness tending towards zero.

Theorem 1.1.5 falls in the context of homogenization of periodic integral functionals in terms of Γ-convergence. While homogenization can also be studied on the level of partial differential equations, with an entry point to the large body of literature dedicated to this topic given by [44, 126, 129], introductory works to the homogenization of integral functionals are given by [91, 30] and with a focus on Γ-convergence by [29].

Our result features an explicit homogenization formula, for which we will show that it corresponds to the classic cell formula. The first publication providing an explicit homogenization formula for convex energy densities was given by Marcellini [105], and Müller [112] showed that for general non-convex energy densities this formula does not necessarily hold, and gave a multicell formula for non-convex energy densities featuring the exact growth and Lipschitz conditions we assume for the energy density on the soft layers. A similar result for non-convex energy densities obtained by different techniques is due to Braides [27].

In recent years, the knowledge on homogenization of integral functionals has been expanded by results assuming less restrictive growth conditions such as (quasi-)convex growth [10, 7], as well as results concerning Young measures [9] and singular integrals [6, 8].

Furthermore, the relation between cell and multicell formula has been the focus of further research, with additional counterexamples given [17] and the equivalence of both formulas shown near $SO(n)$ in the context of the Cauchy-Born-rule [50].

Notice that the characteristic feature of our problem is the approximate differential inclusion in $SO(n)$. On the topic of homogenization of integral functions under constraints a large body of literature has emerged, such as restrictions to manifolds [14], partial differential equation constraints [31, 71, 106, 62] and pointwise gradient constraints [35, 36, 34, 43].

Note that the explicit homogenization formulas of Theorem 1.1.5 and Theorem 1.1.6 both feature a strong relation to progress made in the relaxation of integral functionals particularly in the context of plasticity models. For the former note that the condition of matching quasi- and polyconvex envelopes is also found in relaxation results under determinant constraints by Conti and Dolzmann [48]. The latter theorem builds on works on relaxation of plastic single slip models [56, 47], in the context of which recent progress has been made on two [52] and three [49] slip systems of certain geometry. An essential building block of our result is the compactness result obtained in [45].

For the literature on stochastic homogenization an extensive review was given by Gloria

[76]. On the level of equations, a first result was given by [93], while integral functions were considered in [60, 61], corresponding to first results on convex density functions in periodic homogenization, e.g. [105]. The progress by Müller [112] on non-convex density functions is reflected in the stochastic context by [108]. Yet note, that the recent achievements in this field by the schools of Gloria and Otto [77, 75] and Armstrong and Scott [12] concerning convergence rates have no direct relation to the result given in this work.

Moreover, we want to point out similar physical models considered in the literature for example in the context of linear elasticity. In view of the geometry, a relation can be seen in two dimensions to models for fiber reinforced materials. A general introduction to this topic is given in [104], while there is a large body of literature studying these materials analytically [26, 85, 24, 88, 21] and computationally [109].

Another characteristic of our models is the penalization of the elastic deformations on the stiff layers with decreasing layer thickness. This feature can also be found in models for high-contrast materials, which have been subject of ongoing mathematical research for the last decades, studying the influence of stiff inclusions [32, 39] as well as materials of layered structure, also called stratified materials [19, 20]. But while the intentions of the latter are similar to the ones in this work, the difference lies in the fact that these works consider comparatively explicit models in the context of linear elasticity, while our approach is more general and uses assumptions more compatible with finite elasticity.

Overview of thesis. Besides this introduction and the outlook at the end, this thesis is comprised of five main chapters.

The first one following the introduction gives an overview on main results of the modern calculus of variations that are relevant to this work. Starting from the direct method to show existence of solutions for minimization problems, we focus on the importance of lower semicontinuity, which for integral functionals is related to convexity properties of the density. In particular, the cases of scalar and vectorial problems are distinguished to motivate the introduction of generalized notions of convexity, such as quasiconvexity. We then proceed with a discussion of relaxation of minimization problems and constraint minimization problems. Next, we focus on Γ-convergence, a form of variational convergence for energy functionals, which is defined and its main properties stated. This chapter concludes with an overview on the homogenization of material models, reviewing cell and multicell formulas for convex and non-convex problems.

The third chapter concerns the necessary statement of Theorem 1.1.1. After a first example an interlayer estimate crucial to the proof is established, followed by a short review of geometric rigidity results such as the celebrated result by Friesecke, James and Müller [72]. A self-contained proof of this result along the lines of the original is given in the appendix to this chapter. We then proceed with the proof of a slight generalization of the necessary statement of Theorem 1.1.1. To establish optimality of the scaling parameter α in the theorem, we also provide explicit bending constructions showing asymptotic behavior that is deviating from the characterization result for less stiffness than required in the theorem.

The beginning of the fourth chapter gives a quick introduction to non-linear elasticity. Afterwards, we prove the sufficiency statement of Theorem 1.1.1 by an explicit construction for approximating sequences, which will also provide the basis for the subsequent construction of sequences with optimal energy. We then proceed with the proof of Theorem 1.1.5 establishing the result for affine limit functions first, followed by suitable localization arguments. A crucial ingredient for the proof is Theorem 1.1.1 of the previous chapter. We conclude this chapter pointing out the relations to cell and multicell formulas.

In the fifth chapter we give another application of Theorem 1.1.1 in the context of crystal plasticity. We will start with a short introduction to the physical background and proceed with the proof of Theorem 1.1.6. First, a discussion of admissible deformations is needed, followed by a short review of results on convex integration and the discussion of compactness results in the context of these elastoplastic models. Building on these results, we then give an explicit construction for the recovery sequences and a lower bound estimate to prove Theorem 1.1.6. The results presented in this chapter constitute a full reproduction of the results published together with the adviser Carolin Kreisbeck in the context of totally rigid layers in [42] incorporating the more general setting of stiff layers and minor technical variations.

In contrast to the previous two chapters which concerned the homogenization of periodically layered materials, we consider in the sixth chapter layered materials of random layer thickness. To study the stochastic effects, we again consider a model of crystal plasticity, but in contrast to chapter 5 we restrict ourselves to the less involved rigid-plastic idealization. At first, we establish the result for a Bernoulli model describing the layer thickness, using Kolmogorov's law of large numbers to obtain a homogenization result. To generalize the class of admissible random variables modeling the layer thickness, we then give an overview on results of ergodic theory. At the end of the chapter, we apply these ergodic theorems to generalize the homogenization results obtained using Kolmogorov's law of large numbers.

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1.2 Notation

The set of all natural numbers is $\mathbb{N} := \{1, 2, \dots\}$, while $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a function $f : M \to N$ between two sets M and N and a pointwise condition P we use the shorthand $\{f \text{ satisfies } P\} := \{x \in M \mid f(x) \text{ satisfies } P\}$.

For a set X and a subset $A \subset X$, we use the notation $\mathbb{1}_A$ for the indicator function corresponding to A and χ_A for the characteristic function corresponding to A, i.e. for $x \in X$

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \chi_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{otherwise.} \end{cases}$$

The space dimension is denoted by $n \in \mathbb{N}$. We call a subset $\Omega \subset \mathbb{R}^n$ domain, if it is open and (path-)connected. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \leq p \leq \infty$. For $f \in L^1(\Omega; \mathbb{R}^n)$, we denote the partial derivative in e_i-direction in the sense of distributions by $\partial_i f$. By the mean value of f on Ω, we mean $\fint_\Omega f \, \mathrm{d}x$ and we denote the subspace of $L^p(\Omega; \mathbb{R}^n)$ of all functions with vanishing mean value by $L^p_0(\Omega; \mathbb{R}^n)$. The space of all Sobolov functions that are p-integrable and k-times weakly differentiable by $W^{k,p}(\Omega; \mathbb{R}^n)$.

We view a function $f \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $\partial_i f = 0$ for $i = 1, \ldots, n-1$ as locally one-dimensional and consequently denote the derivative in e_n-direction by $f' = \partial_n f$, see also the Appendix of Chapter 4. We call a function $s \in L^\infty(\Omega; \mathbb{R}^n)$ simple if there are finitely many disjoint $\Omega_i \subset \Omega$ and $s_i \in \mathbb{R}^n$, $i = 1, \ldots, N$, $N \in \mathbb{N}$, such that

$$s = \sum_{i=1}^{N} s_i \, \mathbb{1}_{\Omega_i} \, .$$

Continuous parameters are written as subindices while discrete parameters or flags are written as superindices. The set of all real matrices with m rows and n columns is denoted by $\mathbb{R}^{m \times n}$, $m, n \in \mathbb{N}$, elements of which are always indicated by capital letters. For $F \in \mathbb{R}^{m \times n}$ we denote by F_{ij} the component in the i-th row and the j-th column, where $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$. For $1 \leq q \leq \infty$ we define on $\mathbb{R}^{m \times n}$ the q-norm $|\cdot|_q : \mathbb{R}^{m \times n} \to \mathbb{R}$ given for $F \in \mathbb{R}^{m \times n}$ by

$$|F|_q = \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} |F_{ij}|^q \Big)^{\frac{1}{q}} \quad \text{if } 1 \leq q < \infty \quad \text{and} \quad |F|_\infty = \max_{i=1,\ldots,m; j=1,\ldots,n} |F_{ij}|.$$

In the euclidean case $q = 2$ we suppress the index $|\cdot| = |\cdot|_2$. We use the same notations for the norms of vectors in $\mathbb{R}^n$ identifying $\mathbb{R}^n \cong \mathbb{R}^{1 \times n}$. For the m-dimensional identity matrix we use the notation $\mathbb{I}_m$, suppressing the index if it coincides with the space dimension, i.e. $\mathbb{I} = \mathbb{I}_n$.

For $n \in \mathbb{N}$ we denote by $GL(n) = \{F \in \mathbb{R}^{n \times n} \mid F \text{ is invertible}\}$ the general linear group, the orthogonal group by $O(n) = \{Q \in GL(n) \mid Q^T Q = Q Q^T = \mathbb{I}\}$, and the special orthogonal group by $SO(n) = \{Q \in O(n) \mid \det(Q) = 1\}$. We set $\operatorname{dist}(F, SO(n)) = \min_{Q \in SO(n)} |F - Q|$.

All geometric arguments in $\mathbb{R}^n$ throughout this work are to be read with respect to the euclidean scalar product denoted by $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ inducing the euclidean norm $|\cdot|$ as the metric. Consequently, geometric objects defined with respect to a metric such as balls are defined with respect to the euclidean norm. For example, the open n-dimensional ball around the point $x \in \mathbb{R}^n$ with radius $\ell > 0$ is denoted by

$$B(0, \ell) = B_n(0, \ell) = \{x \in \mathbb{R}^n \mid |x| < \ell\},$$

suppressing the index if the dimension of the ball is the same as the space considered. A subset $Q \subset \mathbb{R}^n$ is called a cuboid, if there is an $a \in \mathbb{R}^n$ and $\ell_i \in [0, \infty)$, $i \in \{1, \ldots, n\}$, such that

$$Q = a + (0, \ell_1) \times (0, \ell_2) \times \ldots, \times (0, \ell_n).$$

If $\ell_i = \ell \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$ we call Q a cube. In particular, we will always assume these objects to have sides parallel to the coordinate axes.

Throughout this work we use generic constants.

2

Preliminary Results from the Calculus of Variations

In this section we want to give a short introduction to modern calculus of variations and some key concepts that we will use in the following chapters. At the beginning, we discuss the existence of solutions to minimization problems, in particular the direct method, explaining the necessity of lower semicontinuity and its relation to notions of convexity. We will proceed with the concept of relaxation, with a focus on integral functionals. Also, differential inclusion constraints will be addressed. Afterwards, we define with Γ-convergence a key notion to formulate the results of later chapters and review some of its properties. One application of this type of variational convergence is homogenization, which we discuss next, citing in particular results on explicit cell and multicell formulas for the homogenized energy.

2.1 Existence of solutions for minimization problems

Let X be a set and $f : X \to \mathbb{R}$ a function. One approach to study the behavior of f is to determine its extrema and the points at which they are obtained. This corresponds to the goal of optimizing parameters in the various fields that utilize mathematical modeling. Thus, by the fact that minimizers and maximizers interchange by considering $-f$ instead of f, the basic object of our deliberations is the set of minimizers M, i.e.

$$M \subset X \quad \text{with} \quad f(x) = \min_{x \in X} f(x) \quad \text{for all } x \in M.$$

The first question to answer is if M is non-empty, i.e. if minimizers do exist at all. Simply considering the identity on an open interval of $\mathbb{R}$ shows this is not a given and surely there is no general answer for functions $f : X \to \mathbb{R}$. Yet, even under seemingly reasonable assumptions on X and f existence of minimizers may fail as the history of the Dirichlet-principle and the well-known example of Weierstraß shows [11, Section 8.2].

While at the time some authors viewed the existence of minimizers for integral functionals among all continuous functions as self-evident, Weierstraß gave a counterexample, considering for $a, b \in \mathbb{R}$ with $a \neq b$ the function space [11, Section 8.2.2],

$$X = \{\varphi \in C^1([0,1]) \mid \varphi(-1) = a, \varphi(1) = b\},$$

together with the integral functional $E : X \to \mathbb{R}$ given by

$$E(\varphi) = \int_{-1}^{1} t^2 \left(\varphi'(t)\right)^2 \mathrm{d}t.$$

On the one hand as the integrand is non-negative, so is E. On the other hand, let for $\epsilon > 0$ the functions $\varphi : [-1, 1] \to \mathbb{R}$ be given by

$$\varphi_\epsilon(t) = \frac{a+b}{2} + \frac{b-a}{2} \frac{\arctan \frac{x}{\epsilon}}{\arctan \frac{1}{\epsilon}}, \quad t \in [-1, 1].$$

Notice that $\varphi_\epsilon \in X$ for each $\epsilon > 0$. Furthermore, since the derivative reads

$$\varphi_\epsilon'(t) = \frac{b-a}{2 \arctan \frac{1}{\epsilon}} \cdot \frac{\epsilon}{x^2 + \epsilon^2}, \quad t \in [-1, 1]$$

we obtain the estimate

$$\begin{aligned} E(\varphi_\epsilon) = \int_{-1}^{1} t^2 \left(\varphi'(t)\right)^2 \mathrm{d}t &\leq \int_{-1}^{1} (t^2 + \epsilon^2) \left(\varphi'(t)\right)^2 \mathrm{d}t \\ &= \epsilon \frac{(b-a)^2}{(2 \arctan \frac{1}{\epsilon})^2} \int_{-1}^{1} \frac{\epsilon}{x^2 + \epsilon^2} \,\mathrm{d}t \\ &= \frac{\epsilon}{2} \frac{(b-a)^2}{\arctan \frac{1}{\epsilon}} \to 0 \quad \text{as } \epsilon \to 0. \end{aligned}$$

This implies that the minimal value of E cannot be positive. Yet, if $\bar{\varphi} \in X$ satisfies $E(\bar{\varphi}) = 0$, then $\mathrm{id}_{(-1,1)} \bar{\varphi}' = 0$ on $[-1, 1]$. Hence, $\bar{\varphi}' = 0$ a.e. in $[-1, 1]$ and thus the continuity of $\bar{\varphi}'$ implies that $\bar{\varphi}$ is constant, which contradicts the boundary conditions. Overall, we see that no minimizer can exist.

A framework to show existence of minimizers is known in the calculus of variations as the "direct method", while the general outline of the argument can be found nameless in fields of mathematics ranging from differential geometry to numerics.

Proposition 2.1.1 (Abstract existence result, c.f. [57, Section 3.1])**.** *Let X be a metric space and let $K \subset X$ be a non-empty compact subset. Furthermore, let $f : X \to [0, \infty]$ be bounded from below and sequentially lower semicontinuous, i.e. for all $(x_k)_{k \in \mathbb{N}} \subset X$ with $x_k \to x$ for some $x \in X$ it holds that*

$$\liminf_{k \to \infty} f(x_k) \geq f(x).$$

Then, there is a minimizer $x_{\min} \in K \subset X$ of f, i.e.

$$f(x_{\min}) = \min_{x \in K} f(x).$$

Proof. Since $f(K)$ is bounded from below, the infimum of this set exists and we find a sequence $(x_k)_{k \in \mathbb{N}}$ such that

$$\lim_{k \to \infty} f(x_k) = \inf_{x \in K} f(x).$$

By compactness of K there is a convergent subsequence $(x_{k_\ell})_{\ell \in \mathbb{N}}$ with limit $x \in K$. Yet now we have

$$f(x) \leq \liminf_{\ell \to \infty} f(x_{k_\ell}) = \lim_{k \to \infty} f(x_k) = \inf_{x \in K} f(x).$$

Hence, x is a minimizer of f. □

We see that besides compactness, lower semicontinuity with respect to the topology considered plays a central role. Note that, depending on the topology, it may also be necessary to distinguish between lower semicontinuity and sequential lower semicontinuity.

2.1.1 Necessity of convexity for existence of minimizers for scalar problems

In this subsection we discuss how the lower semicontinuity of integral functions is related to the convexity of the energy density. In particular, for scalar valued problems the energy density has to satisfy the classic notion of convexity, while for vectorial valued problems generalized notions of convexity are required. This overview is mainly composed from the work of Dacorogna [57]. Alternatively, this topic is discussed e.g. in [28, Chapter 2 and 12].

In the following, let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a bounded open set, and $W : \Omega \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$, $m \in \mathbb{N}$ an energy density corresponding to the energy functional $E : W^{1,p}(\Omega; \mathbb{R}^m) \to [0, \infty]$, $p \in (1, \infty]$ given by

$$E(u) = \int_\Omega W(x, u(x), \nabla u(x))\,\mathrm{d}x, \quad u \in W^{1,p}(\Omega; \mathbb{R}^m).$$

The next theorem gives a precise formulation of the statement that under suitable assumptions if u is scalar, i.e. $m = 1$ or $n = 1$, weak semicontinuity of E implies that W is convex.

Theorem 2.1.2 ([57, Chapter 3.3, Theorem 3.1])**.** *Let $W : \Omega\times\mathbb{R}^m\times\mathbb{R}^{n\times m} \to \mathbb{R}$ be continuous and assume that*

$$|W(x, z, F)| \le a(x, |z|, |F|) \quad \text{for all } (x, z, F) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{n\times m},$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is increasing with respect to $|z|$ and $|F|$ and locally integrable in x. If E is weak-$*$ lower semicontinuous in $W^{1,\infty}(\Omega; \mathbb{R}^m)$, then $F \mapsto f(x, z, F)$ is convex.

Note that in the scalar case, convexity of the energy density is not only a necessary but also a sufficient condition of lower semicontinuity of the associated energy functional, see [57, Chapter 3.3, Theorem 3.4]. The proof of Theorem 2.1.2 is based on the following lemma, which is of interest on its own, as it also holds in the vectorial case, i.e. for $n \ge 2$ and $m \ge 2$.

Lemma 2.1.3 ([57, Chapter 3.3, Lemma 3.3])**.** *Let $W : \Omega \times \mathbb{R}^m \times \mathbb{R}^{n\times m} \to \mathbb{R}$ be continuous and assume that*

$$|W(x, z, F)| \le a(x, |z|, |F|) \quad \text{for all } (x, z, F) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{n\times m},$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is increasing with respect to $|z|$ and $|F|$ and locally integrable in x. If E is weak-$*$ lower semicontinuous in $W^{1,\infty}(\Omega; \mathbb{R}^m)$, then for all cubes $D \subset \Omega$, for all $(x_0, z_0, F_0) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{n\times m}$ and all $\varphi \in W_0^{1,\infty}(D, \mathbb{R}^m)$ we have

$$\frac{1}{|D|}\int_D W(x_0, z_0, F_0 + \nabla\varphi(y))\,\mathrm{d}y \ge W(x_0, z_0, F_0). \tag{2.1}$$

In the vectorial case, it does not hold in general that the integral has to be convex for the energy functional to be lower semicontinuous. Yet, with (2.1) still valid, this motivates the introduction of general notions of convexity, for which we follow [58, Chapter 5].

The notion of convexity which is necessary and sufficient for the lower semicontinuity of vectorial energy functional under suitable assumptions is quasiconvexity, see [58, Chapter 8]. It was first introduced by Morrey in [111], while we will use the terminology by Ball introduced in [16], see [58, Remark 5.2].

In one dimension this notation of course coincides with convexity.

Definition 2.1.4 (Quasi-convexity [58, Definition 5.1, (ii)])**.** A measurable and locally bounded function $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ is called *quasiconvex* if for every bounded open set $D \subset \mathbb{R}^n$, every $\xi \in \mathbb{R}^{n\times n}$ and every $\varphi \in W_0^{1,\infty}(D, \mathbb{R}^n)$ it holds that

$$f(\xi) \leq \frac{1}{|D|} \int_D f(\xi + \nabla\varphi(x))\,\mathrm{d}x. \tag{2.2}$$

Remark 2.1.5. a) Observe that (2.2) can be seen as Jensen's inequality for gradients.

b) The test functions $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^n)$ can as well be chosen as $\varphi \in C_0^\infty(D; \mathbb{R}^n)$, see [58, Remark 5.2].

c) If (2.2) holds for one bounded open set $D \subset \mathbb{R}^n$, then it holds for all such sets [58, Proposition 5.11]. Hence, the definition of quasiconvexity does not depend on the choice of D.

d) Note that we only defined quasiconvexity for real valued functions, not for functions taking values in $[0, \infty]$. As discussed in [58, Remark 5.2] the reason lies in the fact that there is no proof that quasiconvexity of the energy density fully characterizes the lower semicontinuity of the associated energy functional. While necessity has been shown, the sufficiency of quasiconvexity is still an open problem.

The notion of quasiconvexity is rather abstract and since is not a pointwise condition it is quite involved to verify. Therefore, it is convenient to define two additional notions of convexity, which are easier to verify - one weaker, known as *rank one convexity* and one stronger, referred to as *polyconvexity.*

Definition 2.1.6 (Rank one convexity [58, Definition 5.1, (i)])**.** A function $f : \mathbb{R}^{n\times n} \to [0, \infty]$ is called *rank one convex* if for all $\lambda \in [0, 1]$ and all $\xi, \eta \in \mathbb{R}^{n\times n}$ with $\mathrm{rk}(\xi - \eta) = 1$ it holds that

$$f(\lambda\xi + (1 - \lambda)\eta) \leq \lambda f(\xi) + (1 - \lambda)f(\eta).$$

To define polyconvexity, we first need the definition of a minor (determinant).

Definition 2.1.7 (Minor (determinant)[124, Definition 2.34], [25, Section 7.3])**.** For $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$ let

$$I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\} \quad \text{and} \quad J = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$$

with $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$. The *minor (determinant)* μ_{IJ} *of order* k associated to I and J is given by the determinant of the submatrix with lines and rows specified by I and J, meaning that by the Leibniz formula we have for the symmetric group S_k on k letters

$$\mu_{IJ}(A) = \det\left((A_{ij})_{i\in I, j\in J}\right) = \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma) \prod_{\ell=1}^{k} A_{i_{\sigma(\ell)}, j_\ell} \quad \text{for } A \in \mathbb{R}^{n\times n}.$$

The vector of all minors of all orders is denoted by $\mathcal{M} \in \mathbb{R}^{\tau_n}$, where $\tau_n = \binom{n}{1}^2 \times \dots \times \binom{n}{n}^2$.

Definition 2.1.8 (Polyconvexity [58, Definition 5.1, (iii)])**.** A function $f : \mathbb{R}^{n\times n} \to [0, \infty]$ is called *polyconvex* if there is a convex function $g : \mathbb{R}^{\tau_n} \to [0, \infty]$ such that

$$f(\xi) = g(\mathcal{M}(\xi)).$$

The relations between the different generalized notions of convexity have been studied in detail.

Theorem 2.1.9 (Relations between generalized notions of convexity [58, Theorem 5.3])**.** *If* $f : \mathbb{R}^{n\times n} \to \mathbb{R}$*, then*

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank one convex.}$$

If $f : \mathbb{R}^{n\times n} \to [0,\infty]$*, then*

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ rank one convex.}$$

Furthermore, if $n = 1$*, then all these notions of convexity are equivalent.*

2.1.2 Relaxation of minimization problems

In the previous subsections, we have seen that lower semicontinuity is needed to show existence of minimizers by the direct method, while for integral functionals this relates in the scalar case to the convexity of the energy density.

But what can be said, if the energy density is non-convex and therefore the energy functional is not lower semicontinuous? In this section, we discuss how to apply the techniques of the previous section to obtain some information on the infima of the energy functional, following [58, Section 9.1].

For a metric space X let $E : X \to [0,\infty]$ be an energy functional on X. Suppose that E is not sequentially lower semicontinuous, so the direct method cannot be applied to show existence of minimizers. The crucial idea to obtain at least partial information on elements $u \in X$ for which E takes a small value is to replace the original minimization problem by a *relaxed* problem, meaning that instead of considering the original functional E we consider a *relaxed* energy functional E_{rel}, which satisfies

$$\inf_{u\in X} E(u) = \inf_{u\in X} E_{\text{rel}}(u)$$

and, if suitable coercivity conditions are satisfied, attains its infima. Note that in view of coercivity conditions and compactness it might also be prudent to widen the scope of admissible functions, considering a Banach space X' with $X \subset X'$ on which E_{rel} is defined.

We will see that under suitable assumptions, E_{rel} is again an integral functional with energy density W_{rel}. Notice that this is not self-evident as non-local behavior may occur. However, as the energy function ought to be lower semicontinuous, the above results yield that in the scalar case W_{rel} has to be convex. The fact that this is in general not a necessary condition in the vectorial case motivates the definition of envelopes corresponding to the generalized notions of convexity introduced above. We start this section with these definitions before formulating explicit relaxation results.

Definition 2.1.10 (Envelopes for generalized notions of convexity [58, Section 6.1])**.** Let $f : \mathbb{R}^{n\times n} \to \mathbb{R}$, then the *rank one convex, quasiconvex, polyconvex and convex envelope* is given for $\xi \in \mathbb{R}^{n\times n}$

$$\begin{aligned}
f^{\text{rk}}(\xi) &= \sup\{g(\xi) \mid g \leq f \text{ and } g \text{ rank one convex}\},\\
f^{\text{qc}}(\xi) &= \sup\{g(\xi) \mid g \leq f \text{ and } g \text{ quasiconvex}\},\\
f^{\text{pc}}(\xi) &= \sup\{g(\xi) \mid g \leq f \text{ and } g \text{ polyconvex}\},\\
f^{\text{c}}(\xi) &= \sup\{g(\xi) \mid g \leq f \text{ and } g \text{ convex}\},
\end{aligned}$$

respectively.

An example of a non-(quasi)-convex function arising in the theory of non-linear elasticity is the density of the Saint Venant-Kirchhoff energy, which is in dimension $n \in \mathbb{N}$ given for a parameter $\nu \in (0, 1/2)$ up to rescaling by [58, Section 6.6.6]

$$W_{\mathrm{SK}}(F) = |F^T F - \mathbb{I}|^2 + \frac{\nu}{1-2\nu}(|F|^2 - n)^2.$$

The relaxation of this energy for $n = 3$ has been given by Le Dret and Raoult [97]. We limit ourselves to the result for $n = 2$.

Proposition 2.1.11 (Envelopes for the Saint Venant-Kirchhoff energy density [58, Theorem 6.29])**.** *For a matrix $F \in \mathbb{R}^{2\times 2}$ we denote its singular values by $0 \le \lambda_1(F) \le \lambda_2(F)$. For $n = 2$ the Saint Venant-Kirchhoff energy density $W_{\mathrm{SK}} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is given for $F \in \mathbb{R}^{2\times 2}$ by*

$$\begin{aligned} W_{\mathrm{SK}}(F) &= |F^T F - \mathbb{I}|^2 + \frac{\nu}{1-2\nu}(|F|^2 - 2)^2 \\ &= \left(\lambda_1^2(F) - 1\right)^2 + \left(\lambda_2^2(F) - 1\right)^2 + \frac{\nu}{1-2\nu}\left(\lambda_1^2(F) + \lambda_2^2(F) - 2\right)^2. \end{aligned}$$

Distinguishing the sets

$$\begin{aligned} D_1 &= \{F \in \mathbb{R}^{2\times 2} \mid (1-\nu)\lambda_1^2(F) + \nu\lambda_2^2(F) < 1 \text{ and } \lambda_2(F) < 1\} \\ &= \{F \in \mathbb{R}^{2\times 2} \mid \lambda_1(F) \le \lambda_2(F) < 1\}, \\ D_2 &= \{F \in \mathbb{R}^{2\times 2} \mid (1-\nu)\lambda_1^2(F) + \nu\lambda_2^2(F) < 1 \text{ and } \lambda_2(F) \ge 1\}, \end{aligned}$$

we define the function $g : \mathbb{R}^{2\times 2} \to \mathbb{R}$ for $F \in \mathbb{R}^{2\times 2}$ by

$$g(F) = \begin{cases} W_{\mathrm{SK}}(F) & \text{if } F \in D_1 \cup D_2, \\ \frac{1}{1-\nu}(\lambda_2^2(F) - 1)^2 & \text{if } F \in D_2, \\ 0 & \text{if } F \in D_1. \end{cases}$$

Then, for $F \in \mathbb{R}^{2\times 2}$

$$W_{\mathrm{SK}}^{\mathrm{c}}(F) = W_{\mathrm{SK}}^{\mathrm{pc}}(F) = W_{\mathrm{SK}}^{\mathrm{qc}}(F) = W_{\mathrm{SK}}^{\mathrm{rc}}(F) = g(F).$$

In particular, all envelopes of generalized convexity coincide with the convex envelopes.

We see that for the special case of the Saint Venant-Kirchhoff energy all the envelopes coincide in two dimensions. This is also the case for $n = 3$, see [97].

The relation between relaxation and the quasiconvex envelope is established by relaxation theorems such as the next, which only concerns integral functions where the integrand depends only on the gradient. Note that far more general relaxation results establishing the relation to the quasiconvex envelope of the energy density are known, see e.g. [58, Section 9.2.2].

Theorem 2.1.12 (Relaxation theorem [58, Theorem 9.1])**.** *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a bounded open set and $1 \le p < \infty$. Let $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ be a measurable function that satisfies for a quasiconvex function $g : \mathbb{R}^{n\times n} \to \mathbb{R}$ and $\alpha > 0$ for every $F \in \mathbb{R}^{n\times n}$*

$$g(F) \le W(F) \quad \text{and} \quad |g(F)| + |W(F)| \le \alpha(1 + |F|^p).$$

We consider the energy functional $E : W^{1,p}(\Omega; \mathbb{R}^n) \to \mathbb{R}$ given for $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ by

$$E(u) = \int_\Omega W(\nabla u)\,\mathrm{d}x.$$

Then, the energy function $E_{\mathrm{rel}} : W^{1,p}(\Omega;\mathbb{R}^n) \to \mathbb{R}$ *given for* $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ *by*

$$E_{\mathrm{rel}}(u) = \int_\Omega W^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x$$

satisfies for $u_0 \in W^{1,p}(\Omega;\mathbb{R}^n)$

$$\inf_{u \in u_0 + W^{1,p}(\Omega;\mathbb{R}^n)} E(u) = \inf_{u \in u_0 + W^{1,p}(\Omega;\mathbb{R}^n)} E_{\mathrm{rel}}(u).$$

More precisely, for every $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ *there is a sequence* $(u_k)_{k\in\mathbb{N}} \subset u + W_0^{1,p}(\Omega;\mathbb{R}^n)$ *such that*

$$u_k \to u \text{ in } L^p(\Omega;\mathbb{R}^n) \quad \text{and} \quad E(u_k) \to E_{\mathrm{rel}}(u) \quad \text{as } k \to \infty.$$

2.1.3 Minimization problems under differential inclusion constraints

As in the previous sections, let X be a metric space and $E : X \to [0,\infty]$ an energy functional. Notice that up to this point, all considerations focused on the energy functional rather than the space X, and all results were formulated for rather broad classes of functions. Yet for applications, the class of functions suitable for the given problem, also referred to as the admissible functions for the problem, are more specific, as they should satisfy additional conditions. Typical examples range from classic boundary conditions in the field of partial differential equations, over additional partial differential conditions [31, 71, 106, 62], which arise for example in electrodynamics, to pointwise gradient constraints [35, 36, 34, 43]. The mathematical interest lies in the fact that these constraints may hinder direct application of standard results of the calculus of variations.

Example 2.1.13 (Local volume preservation)**.** For $n \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $u \in W^{1,2}(\Omega;\mathbb{R}^n)$. A typical example for a pointwise constraint on the gradient ∇u of u is the requirement that u preserves volume locally, which corresponds mathematically to the condition $\det \nabla u = 1$ a.e.

The category of pointwise constraints also comprises *differential inclusion constraints*, which are pointwise restrictions on the gradient of a function. More precisely, let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $1 \le p \le \infty$ and $u \in W^{1,p}(\Omega;\mathbb{R}^n)$. Furthermore, let $K \subset \mathbb{R}^{n\times n}$, then we say that u satisfies the *exact differential inclusion constraint*, if $\nabla u \in K$. If merely a constraint on the distance $\mathrm{dist}(\nabla u, K)$ of ∇u to the set K is imposed, we will refer to this condition as an *approximate differential inclusion constraint.*

Similar to the way relaxation is related to the generalized notions of convexity for functions, differential inclusion constraints motivate generalized notion for convex sets. In analogy to the different convex envelopes for a function, we introduce different notions of convex hulls for the set K. We cite the definitions given by Müller. Alternatively, these hulls are discussed in [58, Section 7.1].

Definition 2.1.14 (Generalized notions of convex sets [114, Section 4.4])**.** Let $K \subset \mathbb{R}^{n\times n}$, then the rank one convex, quasiconvex, polyconvex and convex hull of K is given by

$$\begin{aligned}
K^{\mathrm{rc}} &= \big\{F \in \mathbb{R}^{n\times n} \mid f(F) \le \inf_K f \quad \text{for all } f : \mathbb{R}^{n\times n} \to \mathbb{R} \text{ rank one convex}\big\},\\
K^{\mathrm{qc}} &= \big\{F \in \mathbb{R}^{n\times n} \mid f(F) \le \inf_K f \quad \text{for all } f : \mathbb{R}^{n\times n} \to \mathbb{R} \text{ quasiconvex}\big\},\\
K^{\mathrm{pc}} &= \big\{F \in \mathbb{R}^{n\times n} \mid f(F) \le \inf_K f \quad \text{for all } f : \mathbb{R}^{n\times n} \to \mathbb{R} \text{ polyconvex}\big\},\\
K^{\mathrm{c}} &= \big\{F \in \mathbb{R}^{n\times n} \mid f(F) \le \inf_K f \quad \text{for all } f : \mathbb{R}^{n\times n} \to \mathbb{R} \text{ convex}\big\},
\end{aligned}$$

respectively.

Simple laminates. In the context of differential inclusion constraints, a particular problem is to construct functions $u \in W^{1,p}(\Omega;\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ is a bounded Lipschitz domain, satisfying for a given set $K \subset \mathbb{R}^{n\times n}$ an approximate or exact differential inclusion constraint. Due to their gradients being piecewise constant, particular useful constructions are simple laminates.

Let $A, B \in \mathbb{R}^{n\times n}$. Our goal is to construct - if possible - a function $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ of layered structure with the property that $\nabla u \in \{A, B\}$. More precisely, we are aiming for a function whose gradient is oscillating layerwise between A and B in the direction determined by a vector $\nu \in \mathbb{S}^n$, and is constant in all directions normal to ν, see also Figure 2.1.

The following result by Ball and James shows that for such a inclusion condition to hold, the matrices A and B have to be rank one connected, i.e. $\operatorname{rank}(B - A) = 1$.

Proposition 2.1.15 (Simple laminates [15, Proposition 1])**.** *For $n \in \mathbb{N}$ and $A, B \in \mathbb{R}^{n\times n}$ let $\Omega \subset \mathbb{R}^n$ be a domain that decomposes in two disjoint measurable sets of positive measure Ω_A, Ω_B, i.e. $\Omega = \Omega_A \cup \Omega_B$.*

Let $u \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ satisfy

$$\nabla u = \begin{cases} A & \text{on } \Omega_A, \\ B & \text{on } \Omega_B. \end{cases}$$

Then, there are vectors $c, \nu \in \mathbb{R}^n$, $|\nu| = 1$ such that

$$A - B = c \otimes \nu, \tag{2.3}$$

a point $x_0 \in \mathbb{R}^n$, $x_0 \cdot c = 0$ and a function $\theta \in W^{1,\infty}(\Omega)$ satisfying $\nabla\theta = \nu\, \mathbb{1}_{\Omega_A}$ a.e. such that

$$u(x) = x_0 + Bx + \theta(x)c, \quad x \in \Omega. \tag{2.4}$$

Remark 2.1.16 (Simple laminate associated to layered structure)**.** Assume that $A, B \in \mathbb{R}^{n\times n}$ are rank one connected, i.e. suppose that (2.3) holds. Then, we can define for $\epsilon > 0$ functions $u_\epsilon \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ with layerwise oscillating gradient via (2.4) by setting $\theta_\epsilon \in W^{1,\infty}(\Omega)$ such that $\nabla\theta_\epsilon = \nu\, \mathbb{1}_{\epsilon P_{\text{soft}} \cap \Omega}$.

2.2 The notion of Γ-convergence

As we aim to determine asymptotic descriptions of material models, a suitable notion of limit and therefore of convergence is needed. Since in accordance to a variational approach the materials are modeled using energy functionals $(E_\epsilon)_\epsilon$ in dependence of a certain parameter $\epsilon > 0$ whose limiting behavior is to be studied, we require a type of convergence for energy functionals that retains the variational character in the sense the minimizers for E_ϵ converge to minimizers of the limiting functional. Such a notion is given by Γ-convergence, which was introduced by De Giorgi and Franzoni [65, 63]. Introductions to this topic are available in several textbooks [64, 59, 28].

For our purposes it is more suitable to formulate Γ-convergence in terms of sequences. Therefore, we state the basic definition of Γ-convergence also in sequential form.

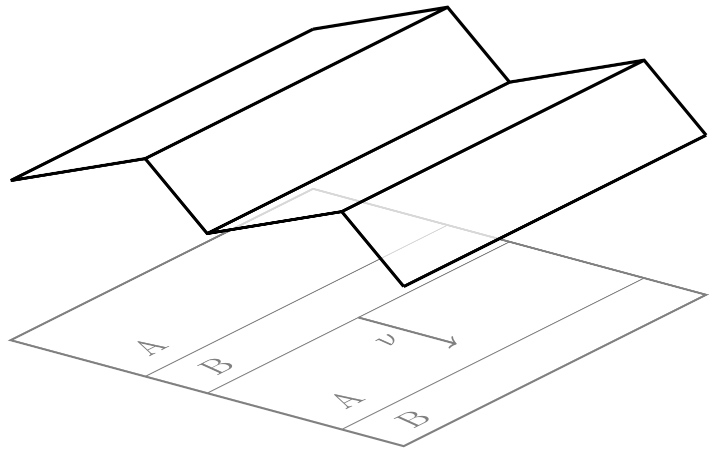

Figure 2.1: Sketch of the first component $e_1 \cdot u$ for a simple laminate $u : \mathbb{R}^2 \to \mathbb{R}^2$ with $\nabla u \in \{A, B\}$ for rank one connected matrices $A, B \in \mathbb{R}^{2\times 2}$ orientated normal to $\nu \in \mathbb{S}^1$.

2.2.1 Definition of Γ-convergence

Definition 2.2.1 (Γ-convergence, [28, Definition 1.5])**.** Let X be a metric space and for each $k \in \mathbb{N}$ let $f_k : X \to [0, \infty]$ be a function. The sequence $(f_k)_{k\in\mathbb{N}}$ *Γ-converges with respect to the topology induced by the metric of* X to a function $f_\infty : X \to [0, \infty]$ if for all $x \in X$ it holds that

(i) (lim inf-inequality) for every sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \to x$

$$f_\infty(x) \leq \liminf_{k\to\infty} f_k(x_k),$$

(ii) (lim sup-inequality) there exists a sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \to x$ such that

$$f_\infty(x) \geq \limsup_{k\to\infty} f_k(x_k).$$

In this case, we call the unique function f_∞ the Γ-*limit of* $(f_k)_{k\in\mathbb{N}}$, which is denoted by $f_\infty = \Gamma\text{-}\lim_{k\to\infty} f_k$.

Remark 2.2.2 (Recovery sequences)**.** Since the sequence $(x_k)_{k\in\mathbb{N}}$ satisfying the lim sup-inequality also has to satisfy the lim inf-inequality, we have

$$f_\infty(x) \leq \liminf_{k\to\infty} f_k(x_k) \leq \limsup_{k\to\infty} f_k(x_k) \leq f_\infty(x),$$

which implies that the limit $\lim_{k\to\infty} f_k(x_k) = f_\infty(x)$ exists. For that reason, a sequence satisfying this condition is called *recovery sequence* and one may replace the lim sup-inequality by requiring the existence of recovery sequences [28, Section 1.2, (ii)'].

Before discussing the properties of Γ-convergence, we consider a first example.

Example 2.2.3 (A first Γ-limit [28, Section 1, Example 1.11])**.** Let $f_1 : \mathbb{R} \to \mathbb{R}$ be given by

$$f_1(t) = \begin{cases} 1 & \text{if } t = 1, \\ -1 & \text{if } t = -1, \\ 0 & \text{otherwise.} \end{cases} \qquad t \in \mathbb{R}$$

For $k \in \mathbb{N}$ consider the sequence $(f_k)_{k\in\mathbb{N}}$ of real functions $f_k : \mathbb{R} \to \mathbb{R}$ given by $f_k(t) = f_1(kt)$ for $t \in \mathbb{R}$.

We are going to show that $(f_k)_{k\in\mathbb{N}}$ converges in the sense of Γ-convergence with respect to the euclidean metric to the function $f_\infty : \mathbb{R} \to \mathbb{R}$ given by

$$f_\infty(t) = \begin{cases} 0 & \text{if } t \neq 0, \\ -1 & \text{if } t = 0. \end{cases}$$

First, we consider the lim inf-inequality. Let $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ with $x_k \to x$ as $k \to \infty$. We have to show that

$$\liminf_{k\to\infty} f_k(x_k) \geq f_\infty(x).$$

If $x \neq 0$, we see that for $K \in \mathbb{N}$ with $K > 1/|x|$ there is a ball $B \subset \mathbb{R}$ around x (with radius smaller than $|x| - 1/K$) such that $f_k|_B = 0$ for all $k \in \mathbb{N}$ with $k > K$. Hence, for k large enough all x_k lie in B and thus for $x \neq 0$ we have $\lim_{k\to\infty} f_k(x_k) = 0 = f_\infty(x)$. If $x = 0$, then $f_\infty(x) = f_\infty(0) = -1$ and since $f_k \geq -1$ for all $k \in \mathbb{N}$ the desired estimate holds.

Secondly, we have to construct for each $x \in \mathbb{R}$ a sequence $(y_k)_k \in \mathbb{R}$ such that $y_k \to x$ and $\limsup_{k\to\infty} f_k(y_k) \leq f(x)$. If $x \neq 0$, then by arguing as for the lim inf-inequality, any sequences $(y_k)_{k\in\mathbb{N}}$ with $y_k \to x$ satisfies $\limsup_{k\to\infty} f_k(y_k) = 0 = f(x)$, and thus we may choose in particular the constant sequence given by $y_k = x$. If $x = 0$, then we consider the sequence $(y_k)_{k\in\mathbb{N}}$ given by $y_k = -1/k$, which satisfies $f_k(y_k) = -1 = f_\infty(0) = f_\infty(x)$.

Overall, we have established that $\Gamma\text{-}\lim_{k\to\infty} f_k = f_\infty$. Observe that the Γ-limit of $(f_k)_{k\in\mathbb{N}}$ does not coincide with the pointwise limit of the sequences, which would be given by 0.

It is also possible to define Γ-convergence in purely topological terms. This has the advantage that this formulation is valid in general topological spaces.

Definition 2.2.4 (Γ-convergence in terms of topology, [59, Chap 4., Definition 4.1])**.** Let X be a topological space and denote by $N(x)$ the set of all open neighborhoods of x. For each $k \in \mathbb{N}$ let $f_k : X \to [0, \infty]$ be a function. Then, the Γ-*lower limit* and the Γ-*upper limit* of the sequence $(f_k)_{k\in\mathbb{N}}$ are for all $x \in X$ given by

$$\Gamma\text{-}\liminf_{k\to\infty} f_k(x) = \sup_{U\in N(x)} \liminf_{k\to\infty} \inf_{y\in U} f_k(y),$$
$$\Gamma\text{-}\limsup_{k\to\infty} f_k(x) = \sup_{U\in N(x)} \limsup_{k\to\infty} \inf_{y\in U} f_k(y).$$

If the two coincide, i.e. if there is a function $f_\infty : X \to [0, \infty]$ such that

$$f_\infty = \Gamma\text{-}\liminf_{k\to\infty} f_k = \Gamma\text{-}\limsup_{k\to\infty} f_k,$$

then we write $f_\infty = \Gamma\text{-}\lim_{k\to\infty} f_k$ and we say that the sequence $(f_k)_{k\in\mathbb{N}}$ Γ-converges to f_∞ in X or that f_∞ is the Γ-limit of $(f_k)_{k\in\mathbb{N}}$ in X.

As pointed out in the work of Dal Maso, there is a close relation of Γ-convergence of a sequence of functions and set convergence in the sense of Kuratowski of their epi-graphs. Hence, some authors tend to call the former epi-convergence, which is for example the case in some literature on stochastic homogenization cited later in this thesis. Therefore, we want to give a short overview on these results.

Definition 2.2.5 (Set convergence in the sense of Kuratowski [59, Chap 4., Definition 4.10])**.** Let X be a topological space and $(E_k)_{k\in\mathbb{N}}$ be a sequence of subsets $E_k \subset X$.

The *K-lower limit of* $(E_k)_{k\in\mathbb{N}}$, which we denote by $K\text{-}\liminf_{k\to\infty} E_k$, is the set of all points $x \in X$ such that for every $U \in N(x)$ there is a $k \in \mathbb{N}$ such that for every $h \geq k$ it holds that $U \cap E_k \neq \emptyset$.

The *K-upper limit of* $(E_k)_{k\in\mathbb{N}}$, which we denote by $K\text{-}\limsup_{k\to\infty} E_k$, is the set of all points $x \in X$ such that for every $U \in N(x)$ and every $k \in \mathbb{N}$ there is an $h \geq k$ such that it holds that $U \cap E_k \neq \emptyset$.

If both coincide, i.e. if there is an $E \subset X$ with $E = K\text{-}\liminf_{k\to\infty} E_k = K\text{-}\limsup_{k\to\infty} E_k$, then $E = \lim_{k\to\infty} E_k$ is said to be the *limit of E_k in the sense of Kuratowski.*

The following theorem determines the relation between Γ-convergence and K-convergence of epi-graphs. Recall that the epi-graph of a function $f : X \to [0,\infty]$ is given by

$$\operatorname{epi}(f) = \{(x,t) \in X \times \mathbb{R} \mid f(x) \leq t\}.$$

Theorem 2.2.6 (Γ-convergence and K-convergence of the epi-graphs, [59, Chap 4., Thm 4.16])**.** *Let $f_\infty, f_k : X \to [0,\infty]$ for all $k \in \mathbb{N}$. Then, the sequence $(f_k)_{k\in\mathbb{N}}$ Γ-converges to f_∞ in X if and only if the sequence of sets $(\operatorname{epi}(f_k))_{k\in\mathbb{N}}$ K-converges to $\operatorname{epi}(f)$ in $X \times \mathbb{R}$ with respect to the product topology of $X \times \mathbb{R}$.*

2.2.2 Properties of Γ-convergence

Γ-convergence satisfies three key properties by design, which we cite from [28], but can be found for example in [59], as well. In the following, let X be a metric space and for each $\epsilon > 0$ let $f_\epsilon : X \to [0,\infty]$ be a function.

Proposition 2.2.7 (Lower semicontinuity of the Γ-limit [28, Proposition 1.28])**.** *The Γ-upper and Γ-lower limit of a sequence $(f_\epsilon)_\epsilon$ are lower semicontinuous functions.*

Proposition 2.2.8 (Stability under continuous perturbations [28, Remark 1.7])**.** *Assume that $g : X \to [0,\infty]$ is a continuous function. If f_ϵ converges to $f : X \to [0,\infty]$ in the sense of Γ-convergence, then $f_\epsilon + g$ also Γ-converges to $f + g$.*

Proposition 2.2.9 (Convergence of global minimizers [28, Theorem 1.21])**.** *Let $(f_\epsilon)_\epsilon$ be equi-mildly coercive, by which we mean that there exists a non-empty compact set $K \subset X$ such that $\inf_X f_\epsilon = \inf_K f_\epsilon$ for all $\epsilon > 0$. Furthermore, assume that $(f_\epsilon)_\epsilon$ converges in the sense of Γ-convergence to $f : X \to [0,\infty]$. Then, the minimum of f_ϵ on X exists and*

$$\min_X f = \lim_{\epsilon\to 0} \inf_X f_\epsilon.$$

Furthermore, if $(x_\epsilon)_\epsilon \subset X$ is a precompact sequence such that $\lim_{\epsilon\to 0} f_\epsilon(x_\epsilon) = \lim_{\epsilon\to 0} \inf_X f_\epsilon$, then every limit of a subsequence of $(x_\epsilon)_\epsilon$ is a minimizer for f.

2.3 Homogenization of material models

To find a mathematical model describing real world systems or processes it is crucial to identify the relevant quantities. A good model comprises all factors that influence the system or process significantly but also neglects marginal effects to keep the model manageable with respect to data collection and computation. Thus, while the underlying parameters and processes for the small scale system are well understood, it may seem prudent to not consider

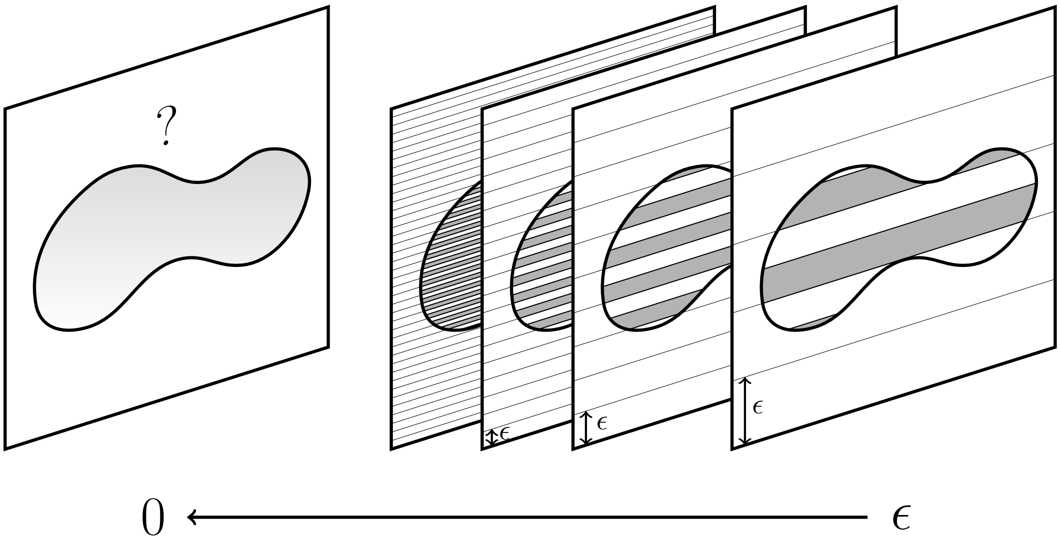

Figure 2.2: One goal of this work is to determine a suitable macroscopic model for materials with fine layered structure. To that end, we consider a sequence of variational models with decreasing layer thickness ϵ and aim to determine an asymptotic model in terms of Γ-convergence as $\epsilon \to 0$.

them for a model describing a large scale situation. Yet, conceiving an ad-hoc model for each scale does not utilize the knowledge on the underlying relations.

In such situations, homogenization can be used to obtain rigorous limit models that make precise the notion of parameters becoming negligible as they relate to averaging quantities.

While examples for applications of these techniques can be found throughout many fields from physics to computational science, applications in material science are of particular interest, as elastoplastic bodies feature multiple length scales with different underlying physical processes. As mentioned in the introduction, see also Figure 1.1, the different length scales range from atomistic models to dislocation models to continuum descriptions of the body.

In this work, we are in particular interested in a macroscopic description of materials that feature a fine layered structure, see Figure 2.2. The main model parameter is the layer thickness $\epsilon > 0$. Since we follow a variational approach, with the material described by energy functionals, the right language to formulate the forthcoming homogenization results in is Γ-convergence as introduced in the last section.

An essential tool for the homogenization of periodic structures is the classic lemma on weak convergence of highly oscillating functions we are citing as formulated by Dacorogna, alternatively see e.g. [28, Example 2.4].

Lemma 2.3.1 (Weak convergence of highly oscillating functions [57, Chapter 2, Theorem 1.5])**.** *Let $\Omega = \prod_{i=1}^{n}(a_i, b_i)$ and let $u \in L^p(\Omega)$, $1 \leq p \leq \infty$. Extend u periodically to $\mathbb{R}^n$ with respect to Ω and set $u_\epsilon(x) = u(\epsilon^{-1}x)$, then if $1 \leq p < \infty$*

$$u_\epsilon \rightharpoonup \bar{u} \text{ in } L^p(\Omega) \text{ as } \epsilon \to 0, \quad \text{where} \quad \bar{u}(x) = \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x_n,$$

and if $p = \infty$ we have $u_\epsilon \overset{*}{\rightharpoonup} \bar{u}$ in $L^\infty(\Omega)$, as $\epsilon \to 0$.

In later chapters, we will prove two generalizations of this result concerning functions that are oscillating only in a one-dimensional subspace of $\mathbb{R}^n$ in Lemma 3.4.1 and weakly convergent sequences of functions in Lemma 5.3.1

Lemma 2.3.1 allows us in particular to determine the weak limit of simple laminates, see also Remark 2.1.16, which will be essential in later chapters. Here, we want to give a first example for the asymptotic behavior of a sequence of simple laminates that can be interpreted as shear deformation of a layered material.

Example 2.3.2 (Weak limit of oscillating shear deformation)**.** Let P_{soft} and P_{stiff} as in (1.1). For $n \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $u_1 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ be determined by

$$\nabla u_1 = \mathbb{I} + \gamma e_1 \otimes e_2 \, \mathbb{1}_{P_{\text{soft}}} .$$

Observe that by choosing the mean value of u_1 to vanish, the specification of the gradient indeed determines the potential uniquely. Furthermore, for $\epsilon > 0$ we set $u_\epsilon = u_1(\epsilon^{-1}\cdot)$.

Since we can identify the function $\mathbb{1}_{\epsilon P_{\text{soft}}}$ with the ϵ-periodic extension of the one-dimensional function $\mathbb{1}_{\epsilon(0,\lambda]} : [0,\epsilon] \to \mathbb{R}$ to $\mathbb{R}$ via $\mathbb{1}_{\epsilon P_{\text{soft}}}(x) = \mathbb{1}_{\epsilon(0,\lambda]}(x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, Lemma 2.3.1 yields for the gradients

$$\nabla u_\epsilon = \mathbb{I} + \gamma e_1 \otimes e_2 \, \mathbb{1}_{P_{\text{soft}}}(\epsilon^{-1}\cdot) \rightharpoonup \mathbb{I} + \lambda\gamma e_1 \otimes e_2 \quad \text{in } L^2(\Omega;\mathbb{R}^2) \quad \text{as } \epsilon \to 0.$$

Again, since for each $\epsilon > 0$ the function u_ϵ has mean value zero, we obtain by the Poincaré inequality for $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ determined by $\nabla u = \mathbb{I} + \lambda\gamma e_1 \otimes e_2$ that

$$u_\epsilon \rightharpoonup u \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^2) \quad \text{as } \epsilon \to 0.$$

In the context of integral functionals, homogenization aims to determine explicit formulas for the homogenized energy density, similar to the desire for explicit relaxation formulas. In certain cases, these can be expressed in the form of what is known as cell and multicell formulas.

Definition 2.3.3 (Cell and multicell formulas [112])**.** Let $W : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to [0,\infty]$ be an energy density periodic on $(0,1)^n$ with respect to the first variable. Then, the *cell formula* $W^{\text{cell}} : \mathbb{R}^{n\times n} \to \mathbb{R}$ *for the homogenized energy density corresponding to* W is given for $F \in \mathbb{R}^{n\times n}$ by

$$W^{\text{cell}}(F) = \inf_{\psi \in W^{1,p}_{\text{per}}((0,1)^n;\mathbb{R}^n)} \int_{(0,1)^n} W(x, F + \nabla\psi(x)) \, \mathrm{d}x,$$

and the *multicell formula* $W^{\text{mult}} : \mathbb{R}^{n\times n} \to \mathbb{R}$ *for the homogenized energy density corresponding to* W is given for $F \in \mathbb{R}^{n\times n}$ by

$$W^{\text{mult}}(F) = \inf_{k\in\mathbb{N}} \inf_{\psi \in W^{1,p}_{\text{per}}((0,k)^n;\mathbb{R}^n)} \int_{(0,k)^n} W(x, F + \nabla\psi) \, \mathrm{d}x.$$

Both these formulas play a central role in the key homogenization results we present next.

The first result concerns the case of a convex energy density. Marcellini has first shown for scalar u and under polynomial growth conditions that the homogenized functional is an integral functional whose density can be expressed via the cell formula [105, Theorem 4.4]. To remain in the vector-valued framework, we cite a version of this result formulated by Müller. Note that Müller himself generalized this result in the same work to more general growth conditions.

Theorem 2.3.4 (Convex homogenization [112, Theorem 3.3]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $W : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ be periodic on $(0,1)^n$ with respect to the first variable and such that*

(i) $W(x,F)$ is convex in F for all $x \in \Omega$;

(ii) $a|F|^p \le W(x,F) \le b(1+|F|^p)$ for constants $a,b>0$ and $p>1$.

Furthermore, let $\epsilon > 0$ be the energy functional $E_\epsilon : W^{1,p}(\Omega;\mathbb{R}^n) \to \mathbb{R}$ that is given for $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ by

$$E_\epsilon(u) = \int_\Omega W\big(\tfrac{x}{\epsilon}, \nabla u(x)\big)\,\mathrm{d}x.$$

Then, the family $(E_\epsilon)_\epsilon$ converges in the sense of Γ-convergence with respect to the strong L^p-topology to the energy functional $E_{\mathrm{hom}} : W^{1,p}(\Omega;\mathbb{R}^n) \to \mathbb{R}$ given for $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ by

$$E_{\mathrm{hom}}(u) = \int_\Omega W^{\mathrm{cell}}(\nabla u)\,\mathrm{d}x.$$

In the same work, Müller also shows a homogenization result for non-convex energy densities leading to a homogenized functional that is an integral functional whose energy density is determined via the multicell formula. Such a result was also obtained by Braides building on different techniques [27].

Theorem 2.3.5 (Non-convex homogenization [112, Theorem 1.3]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $W : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ be periodic on $(0,1)^n$ with respect to the first variable and such that for constants $a,b,L>0$ and $1<p<\infty$*

(i) $a|F|^p \le W(x,F) \le b(1+|F|^p)$ for $F \in \mathbb{R}^{n\times n}$;

(ii) $|W(x,F) - W(x,G)| \le L(1+|F|^{p-1}+|G|^{p-1})|F-G|$ for $F,G \in \mathbb{R}^{n\times n}$.

Furthermore, let $\epsilon > 0$ be the energy functional $E_\epsilon : W^{1,p}(\Omega;\mathbb{R}^n) \to \mathbb{R}$ that is given for $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ by

$$E_\epsilon(u) = \int_\Omega W\big(\tfrac{x}{\epsilon}, \nabla u(x)\big)\,\mathrm{d}x.$$

Then, the family $(E_\epsilon)_\epsilon$ converges in the sense of Γ-convergence with respect to the strong L^p-topology to the energy functional $E_{\mathrm{hom}} : W^{1,p}(\Omega;\mathbb{R}^n) \to \mathbb{R}$ given for $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ by

$$E_{\mathrm{hom}}(u) = \int_\Omega W^{\mathrm{mult}}(\nabla u)\,\mathrm{d}x.$$

Moreover, if the energy density in Theorem 2.3.5 is in fact convex, then the multicell formula reduces to the cell formula [112, Lemma 4.1].

To show that the cell and multicell formula determine in general different energy densities, Müller also gives in [112] an explicit example of an energy density and deformations. We want to give a small recap of these results, as the construction involves a bilayered structure.

Lemma 2.3.6 (Müller's polyconvex energy density [112, Lemma 4.2]). *Let $W_{\mathrm{M}} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be given for $0 < a < \frac{1}{2}$ by*

$$W_{\mathrm{M}}(F) = |F|^4 + h(\det(F)) \quad \textit{where} \quad h(\delta) = \begin{cases} \frac{8(1+a)^2}{\delta+a} - 8(1+a) - 4 & \textit{if } \delta > 0, \\ \frac{8(1+a)^2}{a} - 8(1+a) - 4 - \frac{8(1+a)^2}{a^2}\delta & \textit{if } \delta \le 0. \end{cases}$$

Then, it holds for W_{M} that *(i)* $W_{\mathrm{M}} \geq 0$ *and* $W_{\mathrm{M}}(F) = 0$ *if and only if* $F^T F = \mathbb{I}$ *and* $\det F > 0$*;*

(ii) *for* $\det F > 0$ *we have* $W_{\mathrm{M}}(F) = O(|F^T F - \mathbb{I}|^2)$ *as* $|F^T F - \mathbb{I}| \to 0$*;*

(iii) W_{M} *is polyconvex, i.e. there is a convex function* $g : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ *such that for* $F \in \mathbb{R}^{2\times 2}$ *we have* $W_{\mathrm{M}}(F) = g(F, \det F)$*;*

(iv) *the growth condition* $|F|^4 - C \leq W_{\mathrm{M}}(F) \leq b(1 + |F|^4)$*,* $F \in \mathbb{R}^{2\times 2}$*;*

(v) *the local Lipschitz condition* $|W_{\mathrm{M}}(F) - W_{\mathrm{M}}(G)| \leq C(1 + |F|^3 + |G|^3)|F - G|$*,* $F, G \in \mathbb{R}^{2\times 2}$*.*

In the next lemma, the energy density W_{M} is used on both the stiff and the soft layers, only differently weighted by a factor of $\mu > 0$. For the proof, the idea is to use the fact that by (ii) in Lemma 2.3.6 the value of W_{M} is small for deformations close to a rigid body motion, by constructing a sequences whose gradient is close to a rotation on the heavily weighted stiff material component. We will construct similar deformations in Section 3.4 and revisit parts of Müller's arguments in Section 4.4.

Proposition 2.3.7 (Cell vs. multicell formula [112, Theorem 4.3])**.** *Let* $Y = [0,1]^2 \subset \mathbb{R}^2$ *denote the unit square. For a weight parameter* $\mu > 0$ *we consider the layerwise defined energy density* $W_\mu : \Omega \times \mathbb{R}^{2\times 2} \to [0, \infty]$ *given by*

$$W_\mu(x, F) = \big(\mathbb{1}_{P_{\mathrm{stiff}}}(x) + \mu\, \mathbb{1}_{P_{\mathrm{soft}}}(x)\big) W_{\mathrm{M}}(F), \tag{2.5}$$

where W_{M} *as in 2.3.6.*

Then, there is an $F \in \mathbb{R}^{2\times 2}$ *and a* $\mu_0 > 0$ *such that for all* $0 < \mu < \mu_0$ *we have*

$$W_\mu^{\mathrm{mult}}(F) < W_\mu^{\mathrm{cell}}(F).$$

3

Rigidity for Periodically Layered Materials

We begin the study of the macroscopic behavior of materials with stiff layers by analyzing the asymptotic behavior of sequences of functions whose gradients are close to rigid body motions on every other layer. In particular, we determine the optimal scaling relation between the layer thickness and the stiffness parameter, identifying two regimes. The result in the stiff regime is comprised in Theorem 1.1.1, for which we prove a slightly more general version of the necessity statement in this chapter, while the sufficiency statement is proven in the next. After an introductory example we will discuss known geometric rigidity results, before proceeding to the proof of our new asymptotic characterization result. At the end of this chapter the optimality of the scaling relation is established by examples of limit functions deviating from the above characterization in the regime corresponding to insufficient stiffness.

3.1 Introduction

Firstly, let us make precise the notion of a periodic bilayered structure and the related notation used throughout this chapter.

Definition 3.1.1 (Periodic bilayered structure)**.** For $n \in \mathbb{N}$, $n \geq 2$ we define $P^0 = \mathbb{R}^{n-1} \times (0, 1]$ and set $P^i = ie_n + P^0$ for $i \in \mathbb{Z}$. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ we denote the smallest and largest index of the layers intersecting Ω by

$$i_\epsilon^{\Omega,\min} = \min\{k \in \mathbb{Z} \mid \epsilon P^k \cap \Omega \neq \emptyset\} \quad \text{and} \quad i_\epsilon^{\Omega,\max} = \max\{k \in \mathbb{Z} \mid \epsilon P^k \cap \Omega \neq \emptyset\}$$

and gather all indices of the layers in between in the index set

$$I_\epsilon^\Omega = \{k \in \mathbb{Z} \mid i_\epsilon^{\Omega,\max} < k < i_\epsilon^{\Omega,\max}\}.$$

We suppress the superscript Ω if the reference is apparent from context.

For $(\lambda_\epsilon)_\epsilon \subset (0, 1)$ the ratio between the stiff and soft component, the *periodic bilayered structure corresponding to* $(\lambda_\epsilon)_\epsilon$ is the sequence of sets $(P_{\text{stiff}})_\epsilon$ and its complements $(P_{\text{soft}})_\epsilon$ given by

$$P_{\text{soft}} = \mathbb{R}^{n-1} \times (0, \lambda_\epsilon] \quad \text{and} \quad P_{\text{stiff}} = \mathbb{R}^{n-1} \times (\lambda_\epsilon, 1].$$

We identify these sets with their P^0-periodic extensions in e_n-direction to $\mathbb{R}^n$.

Lastly, the projection of $x \in \Omega$ on the closest layer interface below (in e_n-direction) is denoted by

$$\lfloor x \rfloor_\epsilon = (x_1, \ldots, x_{n-1}, \lfloor x_n \rfloor_\epsilon) \quad \text{where for } t \in \mathbb{R} \text{ we set} \quad \lfloor t \rfloor_\epsilon = \epsilon \lfloor \tfrac{t}{\epsilon} \rfloor.$$

To gain familiarity with our model for materials of layered structure and the rigidity effects that may occur in this context, we consider a first example of deformations in two dimensions. As mentioned in the introduction, we regard a component as stiff if its elastic deformation apart from a mere rigid body motion requires large amounts of energy. The deformation considered in this example actually satisfies the limiting case that the stiff layers are in fact totally rigid, i.e., their only possible deformations are rigid body motions. In other words, for $\epsilon > 0$ the deformation u_ϵ of the body satisfies the exact differential inclusion constraint $\nabla u_\epsilon \in SO(2)$ on $\epsilon P_{\text{stiff}}$.

Example 3.1.2. Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded Lipschitz domain representing an elastic body which features for $\epsilon > 0$ a layered structure in the sense of Definition 3.1.1. We consider the deformations $(u_\epsilon)_\epsilon \subset W^{1,\infty}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ that do not deform the rigid component at all, i.e. $\nabla u_\epsilon|_{\epsilon P_{\text{stiff}}} = \mathbb{I}$, but shear each soft layer along $(\cos {}^{\pi}\!/6, \sin {}^{\pi}\!/6)^T$ and rotate it so that it is compatible to the non-deformed rigid layers. More precisely, let $s = (\cos {}^{\pi}\!/6, \sin {}^{\pi}\!/6)^T = (\sqrt{3}/2, 1/2)^T$ denote the shear direction and $m = s^\perp = (-1/2, \sqrt{3}/2)^T$ its normal. For a given amount of shear $\gamma \in \mathbb{R}$ and a rotation $R \in SO(2)$, let the deformation gradient on the soft layer be given by

$$R^{-1} \nabla u_\epsilon|_{\epsilon P_{\text{soft}}} = \mathbb{I} + \gamma s \otimes m = \mathbb{I} + \gamma \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} = \mathbb{I} + \frac{\gamma}{4} \begin{pmatrix} -\sqrt{3} & 3 \\ -1 & \sqrt{3} \end{pmatrix}.$$

In view of Proposition 2.1.15, the gradients have to be rank one connected to ensure compatibility. Hence, let $\gamma = 2\frac{s_1}{s_2} = 2\sqrt{3}$ and

$$R = \begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

so that

$$R^{-1} - \mathbb{I} - \gamma s \otimes m = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{2\sqrt{3}}{4} \begin{pmatrix} \sqrt{3} & -3 \\ 1 & -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -\sqrt{3} \\ -3 \end{pmatrix} \otimes e_2.$$

Thus, $\nabla u_\epsilon|_{\epsilon P_{\text{stiff}}} = \mathbb{I}$ and $\nabla u_\epsilon|_{\epsilon P_{\text{soft}}} = R(\mathbb{I} + \gamma s \otimes m)$ are indeed rank one connected. Hence, Proposition 2.1.15 entails that $(u_\epsilon)_\epsilon \subset W^{1,\infty}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ is well-defined and given by a simple laminate.

Next, let us determine the asymptotic behavior of $(u_\epsilon)_\epsilon$. By the classic Lemma 2.3.1 on weak convergence of highly oscillating functions we obtain

$$\begin{aligned} R^{-1} \nabla u_\epsilon &= R^{-1} \mathbb{1}_{\epsilon P_{\text{stiff}}} + \mathbb{I} \, \mathbb{1}_{\epsilon P_{\text{soft}}} + \gamma s \otimes m \, \mathbb{1}_{\epsilon P_{\text{soft}}} \\ &\rightharpoonup (1-\lambda) R^{-1} + \lambda \mathbb{I} + \lambda \gamma s \otimes m \quad \text{in } L^2(\Omega; \mathbb{R}^2) \quad \text{as } \epsilon \to 0. \end{aligned}$$

The Poincaré inequality then yields for the function $u \in W^{1,2}(\Omega) \cap L^2_0(\Omega; \mathbb{R}^2)$, determined by

$$R^{-1} \nabla u = (1-\lambda) R^{-1} + \lambda \mathbb{I} + \lambda \gamma s \otimes m,$$

that $u_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$. For a simplified expression for ∇u, observe that since

$$R \cdot \frac{1}{2}\begin{pmatrix} -1 & (1+2\lambda)\sqrt{3} \\ -\sqrt{3} & -1+6\lambda \end{pmatrix} = \begin{pmatrix} 1 & -2\lambda\sqrt{3} \\ 0 & 1 \end{pmatrix} = \mathbb{I} + \lambda 2\sqrt{3} e_1 \otimes e_2,$$

it holds that

$$\begin{aligned} R^{-1}\nabla u &= \frac{1-\lambda}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \frac{\lambda 2\sqrt{3}}{4}\begin{pmatrix} -\sqrt{3} & 3 \\ -1 & \sqrt{3} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -1 & (1+2\lambda)\sqrt{3} \\ -\sqrt{3} & -1+6\lambda \end{pmatrix} = R^{-1}(\mathbb{I} + \lambda 2\sqrt{3} e_1 \otimes e_2). \end{aligned}$$

Hence, $\nabla u = \mathbb{I} + \lambda 2\sqrt{3} e_1 \otimes e_2$ and we see that the resulting macroscopic deformation is a shear along e_2, i.e. along the layer direction. Also, the rotation R of the soft layers does not enter the formula for ∇u explicitly.

This example features two major simplifications. Firstly, the deformation satisfies the differential inclusion constraint exactly, while we intend to study functions satisfying the approximate inclusion constraint. This issue will be addressed in Section 3.2. Secondly, the deformation gradients are identical on every other layer. For a general deformation of a material with a stiff component, this will not hold. It is therefore crucial to obtain a good estimate between the rigid body motions which the deformation is close to on different stiff layers.

To that end, the next lemma considers a cuboid that consists of two neighboring rigid smaller cuboids and a softer component in between, see Figure 3.1. An estimate on the gradients of the rigid body motions is then derived using the gradient structure of the deformation.

Lemma 3.1.3 (Estimate on rigid body motions on different rigid layers)**.** *For $n \in \mathbb{N}$, $n \geq 2$ and $\ell, \ell_j \in (0,\infty)$, $j \in \{1,\dots,n\}$ with $\ell \leq \ell_j$ let $P := (0,\ell_1) \times (0,\ell_2) \times \cdots \times (0,\ell_n)$ be a cuboid. For $a \in \mathbb{R}^n$ and $\xi \geq 0$ set $P_1 = a + P$, $P_2 = a + \xi e_n + P$. For $1 \leq p \leq \infty$ and the cuboid*

$$Q = \bigcup_{t \in (0,\xi)} (a + te_n) + P,$$

let $u \in W^{1,p}(Q;\mathbb{R}^n)$ be a deformation which coincides with a rigid body motion on P_1 and P_2, i.e. for $i \in \{1,2\}$ there are $R_i \in SO(n)$ and $c_i \in \mathbb{R}$ such that $u(x) = R_i x + c_i$ for $x \in P_i$.

Then,

$$\|R_2 - R_1\|_{L^p(P;\mathbb{R}^{n\times n})} \leq C\ell^{-1}\xi \|\nabla u\|_{L^p(Q;\mathbb{R}^{n\times n})}. \tag{3.1}$$

Proof. We may assume that $\xi > \ell_n$, since otherwise $R_1 = R_2$ in which case there is nothing to show. To avoid the need for more involved arguments close to the boundary, we introduce $\tilde{P} = (0,\ell_1) \times (0,\ell_2) \times \cdots \times (0, \ell_n/2)$ and set

$$\tilde{P}_1 = a + \tfrac{\ell_n}{2} e_n + \tilde{P} \quad \text{and} \quad \tilde{P}_2 = a + \xi e_n + \tilde{P}.$$

By the estimate on difference quotients by weak derivatives from Proposition 3.5.10 we obtain for $d = c_2 - c_1 + (\xi - \ell_n/2) R_2 e_n$

$$\|(R_2 - R_1)x + d\|_{L^p(\tilde{P}_1;\mathbb{R}^n)} \leq \left\|u(x + (\xi - \tfrac{\ell_n}{2})e_n) - u(x)\right\|_{L^p(\tilde{P};\mathbb{R}^n)} \leq \xi \|\nabla u\|_{L^p(Q;\mathbb{R}^{n\times n})}.$$

This, together with the estimate of the subsequent Lemma 3.1.4 applied to the left hand side concludes the proof. ☐

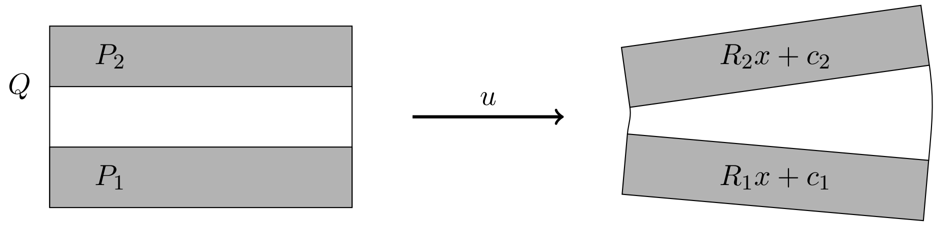

Figure 3.1: A key ingredient in the proof of the asymptotic characterization result is a good estimate between rigid body motions on different stiff layers. For a two-dimensional domain Q, two neighboring layers, P_1 and P_2 are sketched here, on which the deformation u coincides with a rigid body motion, determined by rotations $R_1, R_2 \in SO(2)$ and a translation vectors $c_1, c_2 \in \mathbb{R}^2$.

The next lemma establishes an inverse Poincaré type inequality for differences of rigid body motions on thin cuboids. Notice that the constant is invariant under scaling of the thin direction, which is not the case for usual estimates for derivatives of harmonic maps and will be essential later on.

Lemma 3.1.4 (Estimate on differences of rigid body motions on thin cuboids)**.** *For $n \in \mathbb{N}$ with $n \geq 2$ and $\ell, \ell_j \in (0, \infty)$ with $\ell \leq \ell_j$, $j \in \{1, \dots, n-1\}$ let $O = (-\ell_1, \ell_1) \times \dots \times (-\ell_{n-1}, \ell_{n-1})$. We denote by $a \in \mathbb{R}^n$ and $h \in (0, \infty)$ the cuboid $P_a = a + O \times (-h, h)$. Furthermore, let $R_1, R_2 \in SO(n)$ be two rotations and $d \in \mathbb{R}^n$. Then, for $1 \leq p < \infty$ there is a constant $C > 0$ only depending on the space dimension n and p such that*

$$\int_{P_a} |R_2 - R_1|^p \, \mathrm{d}x \leq C\ell^{-p} \int_{P_a} |(R_2 - R_1)x + d|^p \, \mathrm{d}x. \tag{3.2}$$

Proof. Step 1: Optimizing in d. To shorten the notation set $A = R_2 - R_1$. Since the L^p-norm is convex and $t \mapsto t^p$ is monotone on $[0, \infty)$, the linearity of $x \mapsto Ax$ implies that the value of

$$\int_{P_a} |Ax + d|^p \, \mathrm{d}x = \int_{P_0} |Ax + (d - Aa)|^p \, \mathrm{d}x$$

is minimal for $d - Aa = 0$, see Lemma 3.5.7 for the explicit calculations.

Hence, denoting the $(n-1)$-dimensional ball around zero with radius ℓ by $B_{n-1}(0, \ell)$ and setting $\tilde{P}_0 = B_{n-1}(0, \ell) \times (-h, h) \subset O \times (-h, h)$ we obtain

$$\begin{aligned} \int_{P_a} |Ax + d|^p \, \mathrm{d}x &\geq \int_O \int_{-h}^{h} |Ax|^p \, \mathrm{d}x_n \, \mathrm{d}x' \\ &= \int_O \int_{-h}^{h} \left(\langle A^T A x, x\rangle\right)^{\frac{p}{2}} \, \mathrm{d}x_n \, \mathrm{d}x', \\ &\geq \int_{\tilde{P}_0} \left(\langle A^T A x, x\rangle\right)^{\frac{p}{2}} \, \mathrm{d}x. \end{aligned} \tag{3.3}$$

Step 2: Utilizing specific structure of $A = R_2 - R_1$. To derive an estimate with a constant $C > 0$ independent of h, we use the specific structure of A as a difference of two rotations, leading to

$$A^T A = (R_2 - R_1)^T (R_2 - R_1) = 2\mathbb{I} - R_2^T R_1 - R_1^T R_2 = 2\mathbb{I} - R_2^T R_1 - (R_2^T R_1)^T.$$

Observe that $R_2^T R_1 \in SO(n)$. Furthermore, note that for even n, each rotation $R \in SO(n)$ is similar to a matrix featuring only planar rotations on the diagonal with the transformation matrix an element of $SO(n)$, see e.g. [92, Satz 8.3.10], i.e. there are a coordinate transformation $U \in SO(n)$ and planar rotations $\Theta_1, \dots, \Theta_{n/2} \in SO(2)$ such that

$$R = U \operatorname{diag}(\Theta_1, \dots, \Theta_{n/2}) U^T.$$

For odd n, there is an additional 1 in the last entry of the diagonal.

Let $\bar{U} \in SO(n)$ and $\bar{\Theta}_1, \dots, \bar{\Theta}_{n/2} \in SO(2)$ be the matrices of the representation corresponding to $R_2^T R_1$, then

$$R_2^T R_1 = \bar{U} \operatorname{diag}(\bar{\Theta}_1, \dots, \bar{\Theta}_{n/2}) \bar{U}^T \quad \text{and} \quad (R_2^T R_1)^T = \bar{U} \operatorname{diag}(\bar{\Theta}_1^T, \dots, \bar{\Theta}_{n/2}^T) \bar{U}^T$$

for even n and similar with an additional 1 in the last entry of the diagonal in the odd case.

Hence, with $\bar{\theta}_{j,1}$ denoting the upper left component of $\bar{\Theta}_j$, $j \in \{1, \dots, n/2\}$ we obtain for $A^T A$ for even n

$$U^T A^T A U = 2 \operatorname{diag}(1 - \bar{\theta}_{1,1}, 1 - \bar{\theta}_{1,1}, \dots, 1 - \bar{\theta}_{n/2,1}, 1 - \bar{\theta}_{n/2,1}) =: D.$$

Observe that each eigenvalue appears at least twice and is non-negative. In the case of odd n, we obtain analogously a representation with an additional 0 in the last entry of diagonal, which may be the only simple eigenvalue. Thus, for $j \in \{1, \dots, \lfloor n/2 \rfloor\}$ we have

$$\begin{aligned} \int_{\tilde{P}_0} (\langle A^T A x, x \rangle)^{\frac{p}{2}} \,\mathrm{d}x &= \int_{U\tilde{P}_0} (\langle Dx, x \rangle)^{\frac{p}{2}} \,\mathrm{d}x \\ &\geq (1 - \bar{\theta}_{j,1})^{\frac{p}{2}} \int_{U\tilde{P}_0} (x_{2j}^2 + x_{2j+1}^2)^{\frac{p}{2}} \,\mathrm{d}x. \end{aligned} \tag{3.4}$$

Step 3: Estimate on the eigenvalues of $A^T A$. Let $i \in \{1, \dots, \lfloor n/2 \rfloor\}$ be arbitrary but fixed. Notice that the integrand depends on the projection onto the plane spanned by $\{e_{2j}, e_{2j+1}\}$, denoted by $\operatorname{span}\{e_{2j}, e_{2j+1}\}$. The intersection of $\operatorname{span}\{e_{2j}, e_{2j+1}\}$ with the $(n-1)$-dimensional subspace orthogonal to Ue_n, denoted by $\operatorname{span}\{Ue_n\}^\perp$ that encompasses UO is at least of dimension one, in formulas

$$\dim \big(\operatorname{span}\{e_{2j}, e_{2j+1}\} \cap \operatorname{span}\{Ue_n\}^\perp \big) \geq 1.$$

Since $x \mapsto (x_{2j}^2 + x_{2j+1}^2)^{\frac{p}{2}}$ is invariant under rotation in the plane $\operatorname{span}\{e_{2j}, e_{2j+1}\}$ we may assume that

$$\operatorname{span}\{e_j, e_{j+1}\} \cap \operatorname{span}\{Ue_n\}^\perp = \operatorname{span}\{e_j\}.$$

Hence, using the notation $x = (x_1, \dots, x_n) = (x_j, \tilde{x}) \in \mathbb{R}^n$ we have

$$\int_{U\tilde{P}_0} (x_j^2 + x_{j+1}^2)^{\frac{p}{2}} \,\mathrm{d}x \geq 2 \int_{(-\ell/2, \ell/2)^{n-2} \times (-h,h)} \int_0^{\ell/2} x_j^p \,\mathrm{d}x_j \,\mathrm{d}\tilde{x} = \frac{1}{2^{p-1}(p+1)} h \ell^{n+p-1}. \tag{3.5}$$

Step 4: Deriving an estimate for $\|A\|_{L^p(P_a;\mathbb{R}^{n\times n})}$. The inequality (3.5), together with (3.3) and (3.4), provides an estimate on the eigenvalue of $A^T A$, which is for a constant $C > 0$ of the form

$$(1 - \bar{\theta}_{j,1}) \leq C \ell^{-2} \Big(\fint_{P_a} |Ax + d|^p \,\mathrm{d}x \Big)^{\frac{2}{p}}.$$

Taking the sum over all eigenvalues of $A^T A$ and using the fact that all norms on finite dimensional vector spaces are equivalent, we obtain

$$|P_a|^{\frac{1}{p}} \operatorname{tr}(A^T A) \leq C\ell^{-1}\Big(\int_{P_a} |Ax+d|^p \,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

Finally, note that the trace of $A^T A$ provides a direct estimate for the euclidean spectral norm, which is related to the Frobenius norm by a constant depending only on the space dimension n. Overall, for an absolute constant $C > 0$ only depending on n and p we have

$$\|A\|^p_{L^p(P_a;\mathbb{R}^{n\times n})} \leq C|P_a|\big(\operatorname{tr}(A^T A)\big)^p \leq C\ell^{-p}\int_{P_a} |Ax+d|^p \,\mathrm{d}x$$

as desired. □

Remark 3.1.5 (Estimate for layerwise rigid body motions)**.** Since the constant on the right hand side of (3.2) is independent of h, the lemma by itself generalizes to functions that coincide with (in general different) rigid body motions on every layer. In particular, we do not require any gradient structure for this estimate to hold. Indeed, let $Q \subset \mathbb{R}^{n-1}$ be a cuboid with side lengths larger than $\ell \in (0,\infty)$ and for $h_1, h_2 > 0$ and $a_n \in \mathbb{R}$ let $P_1 := Q \times (a_n, a_n + h_1)$ and $P_2 := Q \times (a_n, a_n - h_2)$. Suppose that $R_1, R_2 \in L^\infty(P_1 \cup P_2; SO(n))$ with $R_k|_{P_i} = R_k^i \in SO(n)$, $i, k = 1, 2$ and $d \in L^\infty(P_1 \cup P_2; \mathbb{R}^n)$ with $d|_{P_i} = d_i \in \mathbb{R}^n$.

Then, we have for $i = 1, 2$ the estimates

$$\|R_1^i - R_2^i\|^p_{L^p(P_i,\mathbb{R}^{n\times n})} \leq C\ell^{-p}\int_{P_i} \big|(R_1^i - R_2^i)x + d_i\big|^p \,\mathrm{d}x.$$

Adding both estimates we obtain

$$\|R_1 - R_2\|^p_{L^p(P_1\cup P_2,\mathbb{R}^{n\times n})} \leq C\ell^{-p}\int_{P_1\cup P_2} \big|(R_1 - R_2)x + d\big|^p \,\mathrm{d}x.$$

3.2 Qualitative and quantitative geometric rigidity

In the last subsection we discussed only functions that coincided on parts of the material with rigid body motions. In general we require the deformations on the stiff layers to satisfy the approximate differential inclusion constraint that the deformation gradient is close to $SO(n)$. Therefore a quantitative rigidity result is necessary, that establishes that if the approximate differential inclusion constraint is satisfied, then the deformation is close to a rigid body motion. Such a result has been proven by Friesecke, James and Müller in [72, Section 3].

The prototype of rigidity results regarding the differential inclusion constraint in $SO(n)$ is the classic Liouville theorem for smooth functions $u : \Omega \to \mathbb{R}^n$ defined on a bounded Lipschitz domain Ω satisfying the exact inclusion constraint $\nabla u \in SO(n)$.

Theorem 3.2.1 (Liouville Theorem [72, Section 3])**.** *For $n \in \mathbb{N}$ with $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $u \in C^\infty(\Omega;\mathbb{R}^n)$ be such that $\nabla u \in SO(n)$. Then, u is a rigid body motion.*

Proof. Since for a general matrix $A \in \mathbb{R}^{n\times n}$ the cofactor matrix $\operatorname{cof}(A)$ satisfies [25, Section 4.4, Satz 3]

$$\operatorname{cof}(A)A^T = \det(A)\mathbb{I},$$

all rotations $R \in SO(n)$ satisfy $\operatorname{cof}(R) = R$. Consequently, as for all $u \in C^\infty(\Omega;\mathbb{R}^n)$ it holds that $\operatorname{div}(\operatorname{cof}\nabla u) = 0$, see [68, Section 8.1.4], the assumption $\nabla u \in SO(n)$ implies

$$\Delta u = \operatorname{div}(\nabla u) = \operatorname{div}(\operatorname{cof}\nabla u) = 0,$$

which means that u is harmonic. Furthermore, we see that $|\nabla u|^2 - n = |\mathbb{I}| - n = 0$, so taking the Laplacian yields by the rules for differentiation of inner products and the fact that the gradient of a harmonic function is also harmonic

$$0 = \tfrac{1}{2}\Delta\big(|\nabla u|^2 - n\big) = \nabla u : \Delta\nabla u + |\nabla^2 u|^2 = |\nabla^2 u|^2. \tag{3.6}$$

Hence, u is affine and since $\nabla u \in SO(n)$ we obtain that $\nabla u = R$ for a fixed rotation $R \in SO(n)$. Thus, u is indeed a rigid body motion. □

Remark 3.2.2. This result is closely related to the Liouville theorem classifying conformal maps between $\mathbb{R}^n$ for $n > 2$, see [87, Section 1.3]. Historically, a challenging question was the right amount of regularity necessary for this statement to hold. A key contribution was made by Rešetnjak, generalizing the Liouville theorem for conformal maps to a Sobolev setting without additional assumptions [122].

Correspondingly, the following generalization of Theorem 3.2.1 is restated in full for later reference. Notice that the proof above also holds if the derivatives are interpreted in this weaker notion of differentiability.

Theorem 3.2.3 (Rešetnjak Theorem [72, Section 3])**.** *For $n \in \mathbb{N}$ with $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ such that $\nabla u \in SO(n)$. Then, u is a rigid body motion.*

In the context of their new approach to plate theory [73], Friesecke, James and Müller established a quantified version of Rešetnjaks theorem.

Theorem 3.2.4 (Quantitative Geometric Rigidity [72, Theorem 3.1])**.** *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$ be a bounded Lipschitz domain. Then, there exists a constant $C > 0$, depending on Ω such that for each $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^2(\Omega)} \leq C(\Omega)\|\operatorname{dist}(\nabla u, SO(n))\|_{L^2(\Omega)}. \tag{3.7}$$

Remark 3.2.5 (Scaling behavior)**.** a) While the fact that this result also holds in terms of $L^p(\Omega)$ for $1 < p < \infty$ in the sense that instead of (3.7), it holds for all $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ that

$$\|\nabla u - R\|_{L^p(\Omega)} \leq C(\Omega, p)\|\operatorname{dist}(\nabla u, SO(n))\|_{L^p(\Omega)} \tag{3.8}$$

was already announced in the original paper, the proof is not contained therein. As established by Conti, the original proof can be generalized when supplemented by suitable singular integral estimates for the Laplace operator [55, 46].

b) The constant $C(\Omega, p)$ in (3.8) is invariant under uniform scaling and translation of the domain Ω [72, Section 3, Remark]. To see this, let $Y = (0,1)^n$ be the unit cube and $u \in W^{1,p}(Y;\mathbb{R}^n)$. By a), there is a $C > 0$ and a rotation $R \in SO(n)$ such that

$$\|\nabla u - R\|_{L^p(Y;\mathbb{R}^{n\times n})} \leq C(Y, p)\|\operatorname{dist}(\nabla u, SO(n))\|_{L^p(Y)}.$$

For $a \in \mathbb{R}^n$ and $h > 0$ we define $v_h \in W^{1,p}(a + hY;\mathbb{R}^n)$ by

$$v_h(x) := hu(h^{-1}(x - a)), \quad x \in a + hY.$$

Since $\nabla v_h(x) = \nabla u(h^{-1}(x-a))$, $x \in a+hY$ the change of variables formula for $x = a+hy$ applied twice yields

$$\int_{a+hY} |\nabla v_h - R|^p \,\mathrm{d}x = \int_Y h^n |\nabla u - R|^p \,\mathrm{d}y$$
$$\leq C \int_Y h^n \operatorname{dist}^p(\nabla u, SO(n)) \,\mathrm{d}y = C \int_{a+hY} \operatorname{dist}^p(\nabla v_h, SO(n)) \,\mathrm{d}x. \quad (3.9)$$

Hence, we see that the estimate (3.8) is invariant under uniform scaling and translation.

c) In [73, Section 4], Friesecke, James and Müller show for the case $p = 2$ and $n = 3$, that for a thin domain $P_\epsilon = (0,1)^2 \times (0,\epsilon)$, the constant scales with a factor of ϵ^{-1}.

For our intended applications, we require a quantitative rigidity result for $1 < p < \infty$ and for general dimension $n \geq 2$ as well as the explicit scaling behavior on thin domains. The arguments by Friesecke, James, Müller, Conti and Schweizer are also valid in this more general setting. For later reference, we end this section with a restatement of the theorem with explicit scaling of the constant on thin domains. A self-contained proof of this result is given in the Appendix for completion following the original arguments.

Theorem 3.2.6 (Quantitative rigidity on thin domains)**.** *For $n \in \mathbb{N}$, $n \geq 2$ let $O \subset \mathbb{R}^{n-1}$ be a bounded Lipschitz domain, $a_n \in \mathbb{R}$ and $\delta \in (0,1)$. Set $P_\delta = O \times (a_n, a_n + \delta)$ and let $1 < p < \infty$. Then, there exists a constant $C > 0$, depending on n, p, O but not ϵ such that for each $u \in W^{1,p}(P_\delta; \mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^p(\Omega)} \leq C\delta^{-1} \| \operatorname{dist}(\nabla u, SO(n))\|_{L^p(\Omega)}. \quad (3.10)$$

3.3 Asymptotic characterization for fine bilayered functions with stiff components

All rigidity results presented in the last subsection address functions u that satisfy the differential inclusion constraint $\nabla u \in SO(n)$ in the exact or approximate sense throughout a domain Ω. In our context of layered materials with stiff components these rigidity theorems are therefore merely applicable to each individual stiff layer.

The next theorem, which is the main theorem of this chapter, utilizes quantitative rigidity in the form of Theorem 3.2.6, and builds on the ideas of Lemma 3.1.3 to establish an asymptotic rigidity result.

Theorem 3.3.1 (Asymptotic characterization)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ with $n \geq 2$ be bounded Lipschitz domain, $1 \leq p < \infty$ and $1 < q < \infty$ with $p \leq q$. For $(\lambda_\epsilon)_\epsilon \subset (0,1)$ let $(P_{\text{stiff}})_\epsilon$ be a periodic bilayered structure. Furthermore, let $(u_\epsilon)_\epsilon \subset W^{1,p}(\Omega; \mathbb{R}^n)$ satisfy for $\alpha > 0$, a constant $C > 0$ and all $\epsilon > 0$*

$$\int_{\epsilon P_{\text{stiff}} \cap \Omega} \operatorname{dist}^q(\nabla u_\epsilon, SO(n)) \,\mathrm{d}x < C\epsilon^\alpha$$

and such that for a $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ we have $u_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$ as $\epsilon \to 0$.

If $p > 1$ and

$$r_\epsilon := \epsilon^{\frac{\alpha}{q}-1}(1-\lambda_\epsilon)^{-1+\frac{1}{p}-\frac{1}{q}} \to 0 \quad \text{as } \epsilon \to 0, \quad (3.11)$$

then there is an $R \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathrm{SO}(n))$ *as well as a function* $b \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ *with* $\partial_j R = 0$ *and* $\partial_j b = 0$ *for all* $j \in \{1, \dots, n-1\}$ *such that*

$$u(x) = R(x)x + b(x) \quad \text{for all } x \in \Omega. \tag{3.12}$$

Furthermore, if $p = 1$ *and* (3.11) *holds for constant* $\lambda_\epsilon = \lambda \in (0, 1)$*, then there exists an* $R \in BV_{\mathrm{loc}}(\Omega; \mathrm{SO}(n))$ *and a function* $b \in BV_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ *with* $\partial_j R = 0$ *and* $\partial_j b = 0$ *for all* $j \in \{1, \dots, n-1\}$ *such that* (3.12) *holds.*

Remark 3.3.2 (Integrability of R' close to the boundary)**.** With the proof based on the arguments of Lemma 3.1.3, the best we may expect in the case of a general Lipschitz domain Ω for the integrability of R is $R \in BV_{\mathrm{loc}}(\Omega; \mathrm{SO}(n))$ and $R \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathrm{SO}(n))$, respectively. This becomes apparent in the fact that the constant in (3.1) is proportional to ℓ^{-1}, which implies that rigid body motions on layers of small diameter cannot be uniformly controlled. This will be reflected by the dependence of the constants in Lemma 3.3.12 and Proposition 3.3.15 on ℓ. Such layers of small diameter occur for example for cones pointing in the e_n-direction. Also, observe that arguments based on the transformation of boundary pieces to half spaces by a Lipschitz map are not expedient as they in general perturb the layered structure.

However, under certain assumptions on the geometry of Ω, integrability of R' for the whole domain Ω can be concluded, in the sense that $R \in W^{1,p}(\Omega; \mathrm{SO}(n))$, see Corollary 3.3.8.

If we assume that the limit gradient preserves volume locally, the class of possible limit functions is even more restricted.

Corollary 3.3.3 (Asymptotic rigidity)**.** *Assume additionally that* $u \in W^{1,r}(\Omega; \mathbb{R}^n)$ *with* $r \geq n$ *and* $\det \nabla u = 1$*. Then, there are* $S \in SO(n)$*,* $R \in L^\infty(\Omega; SO(n-1))$ *and* $d \in L^p(\Omega; \mathbb{R}^n)$ *with* $\partial_j R = 0$ *for all* $j \in \{1, \dots, n-1\}$ *and* $e_n \cdot d = 0$ *such that*

$$\nabla u = S \operatorname{diag}(R, 1)(\mathbb{I} + d \otimes e_n).$$

Example 3.3.4. Notice that the general formulation of Theorem 3.3.1 covers several interesting cases. In particular for constant $\lambda_\epsilon = \lambda \in (0, 1)$ and $p = q = 2$ the condition (3.11) is satisfied for $\alpha > 2$. Furthermore, if we consider $n = 2$ and assume a volume preserving condition for the limit gradient then $S \operatorname{diag}(R, 1)$ is constant, which means that the limits are given by globally rotated shear deformations in the direction of the layers. Under these assumptions, also the matter of lesser integrability of $R \in BV_{\mathrm{loc}}(\Omega; SO(2))$ is mute.

Before proving the results of this section, we introduce notation convenient for layered materials and used throughout this work. Here, in particular the layered structure and the compactness arguments motivate a more elaborate partition of cuboids.

Definition 3.3.5 (Notation for nested cuboids)**.** For $n \in \mathbb{N}$ with $n \geq 2$ let $O \subset \mathbb{R}^{n-1}$ be a cube and $J = [a, b] \subset \mathbb{R}$ an interval and set $Q := O \times J \subset \mathbb{R}^n$. We define $J' := 2J - \frac{a+b}{2}$ and correspondingly $Q' := O \times J'$. For the layer index sets, we use the abbreviation $I'_\epsilon = I^{Q'}_\epsilon$ if the set Q in reference is apparent from context. Also, we will apply this notation iteratively, in the sense that $Q'' := (Q')'$, so that $Q'' = O \times J''$ for $J'' = (J')'$, see Figure 3.2.

Proof of Theorem 3.3.1. The proof is based on two preliminary propositions, the first being the existence of a sequence $(w_\epsilon)_\epsilon$ of layerwise rigid body motions approximating $(u_\epsilon)_\epsilon$, while the second establishes the compactness for $(w_\epsilon)_\epsilon$, which allows us to obtain information on the limit function u. Both propositions will be separately proven in the next subsections.

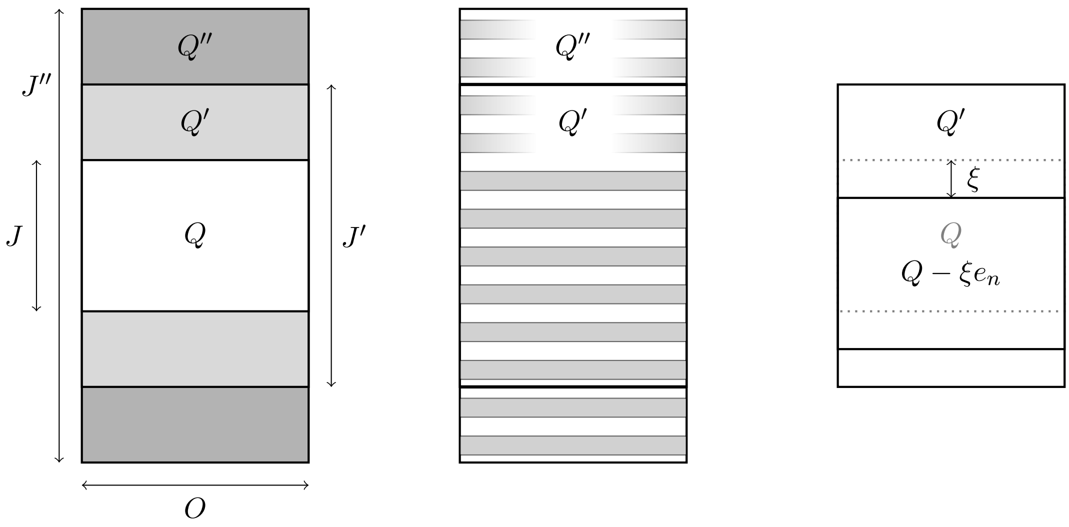

Figure 3.2: The partition of the cuboid Q'' in Q' and Q, of half and a third the size of Q''. While Q' is motivated by the layered structure, see Proposition 3.3.10, Q is necessary for the application of compactness theorems, see Proposition 3.3.15.

To avoid problems arising from varying length of layers, we establish the result for cuboids $Q \subset \Omega$ with $Q'' \subset \Omega$ using the notation of Definition 3.3.5. For a general Lipschitz domain Ω, we argue by covering Ω with cuboids Q (thin enough to satisfy $Q'' \subset \Omega$) that overlap only finitely many times, obtaining on each that the restriction of limit u is of the form (3.12). However, since $\nabla u e_i = Re_i$ for $i = 1, \dots, n-1$, this representation is unique, and thus holds on the whole domain Ω.

Subsequently, let $Q \subset \Omega$ such that $Q'' \subset\subset \Omega$ be fixed. Firstly, we approximate each u_ϵ by a layerwise rigid body motion $w_\epsilon \in L^q(Q''; \mathbb{R}^n)$ with respect to the strong L^p-norm, using on each individual stiff layer the quantitative rigidity result on thin domains from Theorem 3.2.6. This is done in Proposition 3.3.10, which provides the existence of a sequence $(\Sigma_\epsilon)_\epsilon \subset L^\infty(Q''; SO(n))$ with $\Sigma_\epsilon|_{\epsilon P^i} = R^i_\epsilon \in SO(n)$ constant for each $i \in I''_\epsilon$ and a sequence $(b_\epsilon)_\epsilon \subset L^q(Q''; \mathbb{R}^n)$ with $\partial_j b_\epsilon = 0$, $j \in \{1, \dots, n-1\}$ and $\|b_\epsilon\|_{L^q(Q''; \mathbb{R}^n)} < C(1-\lambda_\epsilon)^{-1}$ such that there are functions $w_\epsilon \in L^q(Q''; \mathbb{R}^n)$ with $w_\epsilon(x) = \Sigma_\epsilon(x)x + b_\epsilon(x)$ for $x \in Q'$ that satisfy

$$\begin{aligned} \|u_\epsilon - w_\epsilon\|_{L^p(Q'; \mathbb{R}^n)} &\leq C\epsilon(1-\lambda_\epsilon)^{-\frac{1}{p}} + C\epsilon^{-1}(1-\lambda_\epsilon)^{-1+\frac{1}{p}-\frac{1}{q}} \|\operatorname{dist}(\nabla u, SO(n))\|_{L^q(Q'')} \\ &\leq C\epsilon(1-\lambda_\epsilon)^{-\frac{1}{p}} + C\epsilon^{\frac{\alpha}{q}-1}(1-\lambda_\epsilon)^{-1+\frac{1}{p}-\frac{1}{q}} < Cr_\epsilon. \end{aligned} \tag{3.13}$$

To establish compactness for $(w_\epsilon)_\epsilon$, an additional estimate on the different rigid body motions that constitute each w_ϵ is needed. Building on the ideas of Lemma 3.1.4, Lemma 3.3.12 establishes that for every $\xi \in \mathbb{R}$ with $Q + \xi e_n \subset Q'$ it holds that

$$\|\Sigma_\epsilon(\,\cdot\, + \xi e_n) - \Sigma_\epsilon\|^p_{L^p(Q; \mathbb{R}^n)} \leq C(\xi^p + r_\epsilon) \quad \text{and} \quad \|b_\epsilon\|^p_{L^p(Q; \mathbb{R}^n)} \leq C(1 + r_\epsilon). \tag{3.14}$$

To show compactness for $(w_\epsilon)_\epsilon$ we have to distinguish two cases. In the case $p > 1$, the estimate (3.14), implies in particular that $(b_\epsilon)_\epsilon$ is bounded in $L^p(Q; \mathbb{R}^n)$. This, together with the estimate on $(\Sigma_\epsilon)_\epsilon$ allows us to conclude by Proposition 3.3.15, which is based on the Fréchet-Kolmogorov-Riesz compactness arguments, that there is a $\Sigma_0 \in W^{1,p}(Q; SO(n))$ and a function $b_0 \in L^q(Q; \mathbb{R}^n)$ with $\partial_j \Sigma_0 = 0$ and $\partial_j b_0 = 0$ for $j \in \{1, \dots, n-1\}$ such that up to a subsequence

$$\Sigma_\epsilon \to \Sigma_0 \quad \text{in } L^p(Q; \mathbb{R}^n) \quad \text{and} \quad w_\epsilon \rightharpoonup w_0 = \Sigma_0 x + b_0 \quad \text{in } L^p(Q; \mathbb{R}^n) \quad \text{as } \epsilon \to 0. \tag{3.15}$$

If $p = 1$, the assumption that $\lambda_\epsilon = \lambda$ is constant implies that Proposition 3.3.10 provides a uniform L^q-bound on $(b_\epsilon)_\epsilon$. Hence, again by Proposition 3.3.15, we obtain that there is a $\Sigma_0 \in BV(Q; SO(n))$ and a function $b_0 \in L^q(Q; \mathbb{R}^n)$ with $\partial_j \Sigma_0 = 0$ and $\partial_j b_0 = 0$ for $j \in \{1, \dots, n-1\}$ such that up to a subsequence (3.15) holds.

Finally, (3.13) and the uniqueness of the weak limit entails

$$u_\epsilon \rightharpoonup w_0 \quad W^{1,p}(\Omega; \mathbb{R}^n) \quad \text{as } \epsilon \to 0 \quad \text{with} \quad w_0(x) = \Sigma_0(x)x + b_0(x), \quad x \in \Omega.$$

From (3.12) it then follows for $p > 1$ that $b_\epsilon \in W^{1,p}(\Omega; \mathbb{R}^n)$ and in the case $p = 1$ that $b_\epsilon \in BV_{\text{loc}}(\Omega; \mathbb{R}^n)$, respectively.

□

Proof of Corollary 3.3.3. By the product rule for Sobolev or BV-functions [5, Example 3.97], respectively, we have for a.e. $x \in \Omega$ and $d(x) = R'(x)x + b'(x)$

$$\nabla u(x) = R(x) + R'(x)x \otimes e_n + b'(x) \otimes e_n = R(x) + d(x) \otimes e_n$$

The local volume preserving condition $\det \nabla u = 1$ implies for a.e. $x \in \Omega$

$$\det(\nabla u)(x) = \det(R(x) + d(x) \otimes e_n) = \det(\mathbb{I} + R^T(x)d(x) \otimes e_n).$$

So by the Laplace expansion of the determinant we obtain for a.e. $x \in \Omega$

$$1 = \det(\mathbb{I} + R^T(x)d(x) \otimes e_n) = 1 + R(x)e_n \cdot d(x). \tag{3.16}$$

Thus, $Re_n \cdot d = 0$ a.e. Since $d(x) = R'(x)x + b'(x)$ for a.e. $x \in \Omega$ with $\partial_j R = 0$ and $\partial_j b = 0$ for $j \in \{1, \dots, n-1\}$, differentiating (3.16) yields

$$Re_n \cdot R'e_i = 0 \quad \text{for all } i \in \{1, \dots, n-1\}.$$

Furthermore, the product rule for Sobolev or BV-functions entails on the one hand for $i \in \{1, \dots, n-1\}$

$$0 = \partial_n(Re_i \cdot Re_n) = R'e_i \cdot Re_n + Re_i \cdot R'e_n,$$

so that we obtain $Re_i \cdot R'e_n = 0$, and on the other hand

$$0 = \partial_n |Re_n|^2 = \partial_n(Re_n \cdot Re_n) = 2Re_n \cdot R'e_n.$$

Thus, $R'e_n = 0$ as $\{Re_1, \dots, Re_n\}$ forms a basis of $\mathbb{R}^n$. This implies that Re_n is constant and therefore there is a constant rotation $S \in SO(n)$ and $\tilde{R} \in L^\infty(\Omega; SO(n-1))$ such that $R = S\tilde{R}$.

□

Lastly, we want to address the issue of integrability close to the boundary. As mentioned before in Remark 3.3.2, certain geometries have to be excluded for full integrability to hold, which we do with the following definitions.

Definition 3.3.6 ($e_n^\perp$-connected domain)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set and for $t \in \mathbb{R}$ denote by H_t the hyperplane $H_t = \{x \in \mathbb{R}^n \mid x_n = t\}$. We say that Ω is *e_n-orthogonally connected*, or *$e_n^\perp$-connected* for short, if for every $t \in \mathbb{R}$ the intersection $\Omega_t = H_t \cap \Omega$ is connected.

Definition 3.3.7 (e_n-flatness)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and for $t \in \mathbb{R}$ denote by $\Omega_t = H_t \cap \bar{\Omega}$ the intersection of the hyperplane $H_t = \{x \in \mathbb{R}^n \mid x_n = t\}$ with the closure of Ω. We say that Ω is *e_n-flat* if for every $t \in \mathbb{R}$ it holds that either Ω_t is empty, or it has non-empty relative interior in the sense that for every $x \in \Omega_t$ there is a $\delta > 0$ such that

$$B(x, \delta) \cap H_t \subset \Omega_t.$$

Corollary 3.3.8 (Full integrability up to the boundary)**.** *Assume additionally that $\Omega \subset \mathbb{R}^n$ is $e_n^\perp$-connected and e_n-flat. Then, if $p > 1$ we have $R \in W^{1,p}(\Omega; \mathrm{SO}(n))$.*

Remark 3.3.9. Note that the restriction to $e_n^\perp$-connected domains directly generalizes to Lipschitz domains that can be decomposed in finitely many $e_n^\perp$-connected domains. Such decompositions will be studied in Chapter 4, see Proposition 4.2.8.

Proof of Corollary 3.3.8. Let

$$a := \inf \{x_n \in \mathbb{R} \mid \exists x' \in \mathbb{R}^{n-1} : (x', x_n) \in \Omega\},$$
$$b := \sup \{x_n \in \mathbb{R} \mid \exists x' \in \mathbb{R}^{n-1} : (x', x_n) \in \Omega\}.$$

Since Ω is $e_n^\perp$-connected, there is by Lemma 4.5.6 a function $\tilde{R} \in W^{1,p}_{\mathrm{loc}}(a, b; SO(n))$ such that $R(x', x_n) = \tilde{R}(x_n)$ for a.e. $x = (x', x_n) \in \Omega$. By the e_n-flatness of Ω, we know that there is a $y \in \Omega_b = H_b \cap \Omega$, where H_b is the hyperplane given by $H_b = \{x_n = b\}$, such that for some $\delta_0 > 0$ we have $B(y, \delta_0) \cap H_b \subset \Omega_t$. Notice that all elements of Ω_b are boundary points, so that the Lipschitz property of the domain Ω, see [58, Definition 12.10], implies that there is a ball $B(x, \delta_1)$ with radius $0 < \delta_1 < \delta_0$ such that $B(x, \delta_1) \cap \{y_n < x_n\} \subset \Omega$.

Hence, we know that for $d := b - \delta_1/\sqrt{n}$

$$Q := [-\delta_1/\sqrt{n}, \delta_1/\sqrt{n}]^{n-1} \times (-(b-d), 0) + y \subset \Omega.$$

Therefore, if we exhaust Q up to a null set by a disjoint family of cuboids $(Q_i)_{i\in\mathbb{N}}$ of the same cross section as Q with respect to the e_n-direction but of decreasing height so that the distance to the boundary suffices to apply Proposition 3.3.10, Lemma 3.3.12 and Proposition 3.3.15 on each Q_i, these results yield for each $i \in \mathbb{N}$ an estimate on $\|R\|_{W^{1,p}(Q_i;SO(n))}$ by $\|u\|_{W^{1,p}(Q_i;\mathbb{R}^n)}$ with uniform constant. Hence, by the identification of R with $\tilde{R}$, we obtain $\tilde{R} \in W^{1,p}(d, b; \mathrm{SO}(n))$. Arguing similarly at a, we obtain $\tilde{R} \in W^{1,p}(a, b; \mathrm{SO}(n))$ as desired. □

The following subsections contain the propositions and lemmata crucial for the proof of Theorem 3.3.1.

3.3.1 Approximation by layerwise rigid body motions

A key observation to characterize the limit function of a sequence $(u_\epsilon)_\epsilon$ that is close to rigid body motions on the stiff material component is that it can be approximated by a sequence $(w_\epsilon)_\epsilon$ of layerwise rigid body motions.

Proposition 3.3.10 (Layerwise affine approximation)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$ be a bounded Lipschitz domain, $Q = O \times J \subset \Omega$, $O \subset \mathbb{R}^{n-1}$, $J \subset \mathbb{R}$ be a cuboid such that $Q' \subset\subset \Omega$ with Q' as in Definition 3.3.5 and let $0 < \epsilon < \frac{1}{2}|J|$. For $(\lambda_\epsilon)_\epsilon \subset (0, 1)$ let $(P_{\mathrm{stiff}})_\epsilon$ be a periodic bilayered structure in the sense of Definition 3.1.1 and for $1 \leq p < \infty$ and $1 < q < \infty$ with $p \leq q$ let $(u_\epsilon)_\epsilon \subset W^{1,p}(\Omega; \mathbb{R}^n)$ be uniformly bounded in $W^{1,p}(\Omega; \mathbb{R}^n)$.*

Then, there is a sequence $(\Sigma_\epsilon)_\epsilon \subset L^\infty(Q'; SO(n))$ *with* $\Sigma_\epsilon|_{\epsilon P^i} = R^i_\epsilon \in SO(n)$ *constant for each* $i \in I'_\epsilon$ *and a sequence* $(b_\epsilon)_\epsilon \subset L^q(Q'; \mathbb{R}^n)$ *with* $\partial_j b_\epsilon = 0$, $j \in \{1, \dots, n-1\}$ *and* $\|b_\epsilon\|_{L^q(Q';\mathbb{R}^n)} < C(1-\lambda_\epsilon)^{-1}$ *such that for* $C > 0$ *not depending on* ϵ *and* λ_ϵ

$$\|\nabla u_\epsilon - \Sigma_\epsilon\|_{L^q(\epsilon P_{\text{stiff}} \cap Q; \mathbb{R}^n)} < C\big(\epsilon(1-\lambda_\epsilon)\big)^{-1} \| \operatorname{dist}(\nabla u_\epsilon, SO(n))\|_{L^q(\epsilon P_{\text{stiff}} \cap Q')}. \tag{3.17}$$

Furthermore, there are functions $w_\epsilon \in L^q(Q'; \mathbb{R}^n)$ *with* $w_\epsilon(x) = \Sigma_\epsilon(x)x + b_\epsilon(x)$ *for* $x \in Q$ *satisfying for a constant* $C > 0$ *not depending on* ϵ *and* λ_ϵ

$$\|u_\epsilon - w_\epsilon\|_{L^p(Q;\mathbb{R}^n)} \leq C\epsilon + C\epsilon^{-1}(1-\lambda_\epsilon)^{-1+\frac{1}{p}-\frac{1}{q}} \| \operatorname{dist}(\nabla u_\epsilon, SO(n)\|_{L^q(Q')}. \tag{3.18}$$

Example 3.3.11. An important case to keep in mind is the one of constant $\lambda_\epsilon = \lambda \in (0,1)$ and $p = q = 2$. Here, we have the simpler estimates

$$\|\nabla u_\epsilon - \Sigma_\epsilon\|_{L^2(\epsilon P^i_{\text{stiff}} \cap Q; \mathbb{R}^n)} < C\epsilon^{-1} \| \operatorname{dist}(\nabla u_\epsilon, SO(n))\|_{L^2(\epsilon P_{\text{stiff}} \cap Q')},$$

and

$$\|u_\epsilon - w_\epsilon\|_{L^2(Q;\mathbb{R}^n)} \leq C\big(\epsilon + \epsilon^{-1} \| \operatorname{dist}(\nabla u_\epsilon, SO(n))\|_{L^2(\epsilon P_{\text{stiff}} \cap Q')}\big).$$

Hence, we see that in this case it suffices to require that for some $\alpha > 2$ it holds that

$$\| \operatorname{dist}(\nabla u_\epsilon, SO(n))\|^2_{L^2(\epsilon P_{\text{stiff}} \cap Q)} < C\epsilon^\alpha,$$

to obtain with $(w_\epsilon)_\epsilon$ a sequence of layerwise rigid body motions such that

$$\|u_\epsilon - w_\epsilon\|_{L^2(\Omega;\mathbb{R}^n)} \to 0 \quad \text{in } L^2(\Omega; \mathbb{R}^n) \quad \text{as } \epsilon \to 0.$$

Proof of Proposition 3.3.10. In the following we will denote the union of all layers which lie completely in Q' by $Q_\epsilon := \bigcup_{i \in I'_\epsilon} \epsilon P^i \cap Q'$. Notice that since $0 < \epsilon < \frac{1}{2}|J|$, we have $Q \subset Q_\epsilon$.

Step 1: Construction of $(\Sigma_\epsilon)_\epsilon$. For each $i \in I'_\epsilon$ we apply the quantitative rigidity result on thin domains from Theorem 3.2.6 to the sets $\epsilon P^i_{\text{stiff}} \cap \Omega$, which yields the existence of a constant $C > 0$ depending neither on i nor on ϵ, and sequences of rotations $(R^i_\epsilon)_\epsilon \subset SO(n)$ with the property that for every $\epsilon > 0$, and $i \in I'_\epsilon$ the estimate

$$\|\nabla u_\epsilon - R^i_\epsilon\|_{L^q(\epsilon P^i_{\text{stiff}} \cap Q'; \mathbb{R}^{n\times n})} \leq C(1-\lambda_\epsilon)^{-1}\epsilon^{-1} \| \operatorname{dist}(\nabla u_\epsilon, SO(n))\|_{L^q(\epsilon P^i_{\text{stiff}} \cap Q')} \tag{3.19}$$

is satisfied. We define $(\Sigma_\epsilon)_\epsilon \subset L^\infty(Q'; SO(n))$ to be given by the rotations R^i_ϵ on each layer $i \in I'_\epsilon$ and extended by $\mathbb{I}$ on Q', in formulas

$$\Sigma_\epsilon = \sum_{i \in I'_\epsilon} R^i_\epsilon \, \mathbb{1}_{\epsilon P^i \cap Q'} + \mathbb{I} \, \mathbb{1}_{Q' \setminus Q_\epsilon}.$$

From the estimates on the individual layers in (3.19), we obtain

$$\begin{aligned} \|\nabla u_\epsilon - \Sigma_\epsilon\|^q_{L^q(\epsilon P_{\text{stiff}} \cap Q_\epsilon; \mathbb{R}^{n\times n})} &= \sum_{i \in I'_\epsilon} \|\nabla u_\epsilon - \Sigma_\epsilon\|^q_{L^q(\epsilon P^i_{\text{stiff}} \cap Q'; \mathbb{R}^{n\times n})} \\ &\leq \sum_{i \in I'_\epsilon} C\big(\epsilon(1-\lambda_\epsilon)\big)^{-q} \| \operatorname{dist}(\nabla u_\epsilon, SO(n))\|^q_{L^q(\epsilon P^i_{\text{stiff}} \cap Q')} \\ &\leq C(1-\lambda_\epsilon)^{-q}\epsilon^{-q} \| \operatorname{dist}(\nabla u_\epsilon, SO(n))\|^q_{L^q(\epsilon P_{\text{stiff}} \cap Q')}. \end{aligned} \tag{3.20}$$

Step 2: Construction of $(w_\epsilon)_\epsilon$. For the construction of the sequence $(w_\epsilon)_\epsilon$ of layerwise rigid body motions we specify its layerwise gradients to be given by $(\Sigma_\epsilon)_\epsilon$. Accordingly, we introduce $(\sigma_\epsilon)_\epsilon \subset L^\infty(Q';\mathbb{R}^n)$ and $(b_\epsilon)_\epsilon \subset L^\infty(Q';\mathbb{R}^n)$ by setting

$$\sigma_\epsilon(x) = \Sigma_\epsilon(x)x = \sum_{i\in I'_\epsilon} (R^i_\epsilon x)\, \mathbb{1}_{\epsilon P^i}(x) \quad x \in Q'$$

and

$$b_\epsilon(x) = \sum_{i\in I'_\epsilon} b^i_\epsilon\, \mathbb{1}_{\epsilon P^i}(x), \quad x \in Q' \quad \text{with} \quad b^i_\epsilon = \frac{1}{|\epsilon P^i_{\text{stiff}} \cap Q'|} \int_{\epsilon P^i_{\text{stiff}}\cap Q'} u_\epsilon - R^i_\epsilon x \,\mathrm{d}x.$$

By the fact that $R^i_\epsilon \in SO(n)$ and $(u_\epsilon)_\epsilon \subset L^q(\epsilon P^i_{\text{stiff}} \cap Q';\mathbb{R}^n)$ is uniformly bounded, Jensen's inequality applied for the convex function $x \mapsto |x|^q$ yields for $i \in I'_\epsilon$

$$|b^i_\epsilon|^q \le \frac{C}{|\epsilon P^i_{\text{stiff}} \cap Q'|}\Big(\|u_\epsilon\|^q_{L^q(\epsilon P^i_{\text{stiff}}\cap Q';\mathbb{R}^n)} + \|R^i_\epsilon x\|^q_{L^q(\epsilon P^i_{\text{stiff}}\cap Q';\mathbb{R}^n)}\Big).$$

Hence, integrating over $\epsilon P^i \cap Q'$ and summing over all layers $i \in I'_\epsilon$ entails

$$\|b_\epsilon\|_{L^q(Q';\mathbb{R}^n)} < C(1-\lambda_\epsilon)^{-1}.$$

Finally, we define for all $\epsilon > 0$ the function $w_\epsilon = \sigma_\epsilon + b_\epsilon$. Again, the fact that $R^i_\epsilon \in SO(n)$ yields $\|\sigma_\epsilon\|_{L^\infty(Q')} \le n\|x\|_{L^\infty(Q')} \le C$ for all $\epsilon > 0$. Hence, by the choice of b_ϵ we obtain $u_\epsilon - w_\epsilon \in L^q_0(\epsilon P^i_{\text{stiff}} \cap Q';\mathbb{R}^n)$ for each $i \in I'_\epsilon$, which enables the application of the Poincaré inequality on each stiff layer. In particular, this implies for each layer $i \in I'_\epsilon$

$$\|u_\epsilon - w_\epsilon\|^q_{L^q(\epsilon P^i_{\text{stiff}}\cap Q';\mathbb{R}^n)} \le C\|\nabla u_\epsilon - \Sigma_\epsilon\|^q_{L^q(\epsilon P^i_{\text{stiff}}\cap Q';\mathbb{R}^{n\times n})}.$$

Summing over $i \in I'_\epsilon$ together with (3.20) entails (3.17) and

$$\begin{aligned}\|u_\epsilon - w_\epsilon\|_{L^q(\epsilon P_{\text{stiff}}\cap Q;\mathbb{R}^n)} &\le C\|\nabla u_\epsilon - \Sigma_\epsilon\|_{L^q(\epsilon P_{\text{stiff}}\cap Q_\epsilon;\mathbb{R}^{n\times n})}\\ &< C(1-\lambda_\epsilon)^{-1}\epsilon^{-1}\|\operatorname{dist}(\nabla u_\epsilon, SO(n))\|_{L^q(\epsilon P_{\text{stiff}}\cap Q')}.\end{aligned} \tag{3.21}$$

Step 3: Proof of (3.18). Lastly, it remains to estimate the difference of $(w_\epsilon)_\epsilon$ from $(u_\epsilon)_\epsilon$ with respect to the strong L^p-topology on the whole set Q. To that end, notice that on the stiff layers we may directly utilize (3.17) proven in Step 1. For the soft layers, we need to combine (3.17) with a one-dimensional version of the Poincaré inequality. Let $\eta^i : [i, i+1] \to \mathbb{R}$ be a smooth cut-off function with

$$\eta^i = 1 \text{ on } [i, i+\lambda_\epsilon], \qquad \eta^i(i+1) = 0, \qquad |\eta'| < 2,$$

and set $\eta^i_\epsilon := \eta^i(\epsilon^{-1}(1-\lambda_\epsilon)^{-1}\,\cdot\,)$. For fixed $0 < \epsilon < \frac{1}{2}|J|$ and $i \in I'_\epsilon$, a one-dimensional version of the Poincaré inequality yields on the soft layers

$$\begin{aligned}&\int_{\epsilon P^i_{\text{soft}}\cap Q'} |u_\epsilon - w_\epsilon|^p \,\mathrm{d}x\\ &\quad\le \int_{\epsilon P^i\cap Q'} \big|(\eta^i_\epsilon(x_n)u_\epsilon(x) + (1-\eta^i_\epsilon(x_n))w_\epsilon(x)) - w_\epsilon(x)\big|^p \,\mathrm{d}x\\ &\quad= \int_O \int_{\epsilon i}^{\epsilon(i+1)} \big|\eta^i_\epsilon(x_n)\cdot(u_\epsilon - w_\epsilon)(x)\big|^p \,\mathrm{d}x_n\,\mathrm{d}x'\\ &\quad\le \int_O C\epsilon^p \int_{\epsilon i}^{\epsilon(i+1)} \big|\partial_n\big(\eta^i_\epsilon(x_n)\cdot(u_\epsilon - w_\epsilon)(x)\big)\big|^p \,\mathrm{d}x_n\,\mathrm{d}x'\\ &\quad\le 2C\epsilon^p \int_{\epsilon P^i\cap Q'} |(\nabla u_\epsilon - R^i_\epsilon)e_n|^p \,\mathrm{d}x + C\epsilon^p \int_{\epsilon P^i_{\text{stiff}}\cap Q'} |(\eta^i_\epsilon)'(x_n)|^p|u_\epsilon - w_\epsilon|^p(x)\,\mathrm{d}x\\ &\quad\le C\epsilon^p\|\nabla u_\epsilon - R^i_\epsilon\|^p_{L^p(\epsilon P^i\cap Q';\mathbb{R}^{n\times n})} + \frac{C}{(1-\lambda_\epsilon)^p}\|u_\epsilon - w_\epsilon\|^p_{L^p(\epsilon P^i_{\text{stiff}}\cap Q';\mathbb{R}^n)}.\end{aligned}$$

Summing over all $i \in I_\epsilon$ we obtain the estimate

$$\|u_\epsilon - w_\epsilon\|_{L^p(Q;\mathbb{R}^n)} \leq C\epsilon\|\nabla u_\epsilon - R^i_\epsilon\|_{L^p(Q';\mathbb{R}^{n\times n})} + \frac{C}{(1-\lambda_\epsilon)}\|u_\epsilon - w_\epsilon\|_{L^p(\epsilon P_{\text{stiff}}\cap Q_\epsilon;\mathbb{R}^n)}.$$

For the first term on the right hand side, we argue that by the uniform bound on $(\nabla u_\epsilon)_\epsilon$ it holds that

$$\begin{aligned}\|\nabla u_\epsilon - R^i_\epsilon\|_{L^p(Q';\mathbb{R}^{n\times n})} &\leq \|\nabla u_\epsilon\|_{L^p(Q';\mathbb{R}^{n\times n})} + \|R^i_\epsilon\|_{L^p(Q';\mathbb{R}^{n\times n})}\\ &\leq \|\nabla u_\epsilon\|_{L^p(Q';\mathbb{R}^{n\times n})} + n|Q'| < C,\end{aligned}$$

while for the second term on the right hand side, and Hölder's inequality for the exponent q/p and corresponding conjugated exponent $q/q-p$ yields together with (3.21)

$$\begin{aligned}\int_{\epsilon P_{\text{stiff}}\cap Q_\epsilon} |u_\epsilon - w_\epsilon|^p \,\mathrm{d}x &\leq |\epsilon P_{\text{stiff}}\cap Q_\epsilon|^{1-\frac{p}{q}}\Big(\int_{\epsilon P_{\text{stiff}}\cap Q_\epsilon}|u_\epsilon - w_\epsilon|^q\,\mathrm{d}x\Big)^{\frac{p}{q}}\\ &\leq |Q'|\epsilon^{-p}(1-\lambda_\epsilon)^{1-p-\frac{p}{q}}\|\operatorname{dist}(\nabla u_\epsilon, SO(n))\|^p_{L^q(\epsilon P_{\text{stiff}}\cap Q')}.\end{aligned}$$

Overall, we obtain (3.18). □

In the next lemma, we establish for an approximating sequence $(w_\epsilon)_\epsilon$ of layerwise rigid body motions as in Proposition 3.3.10, an estimate in the spirit of Lemma 3.1.3. This estimate will be key to show compactness for $(w_\epsilon)_\epsilon$.

Lemma 3.3.12 (Interlayer estimate for $(w_\epsilon)_\epsilon$)**.** *For $n \in \mathbb{N}$, $n \geq 2$ let $Q = O \times J$, $J \subset \mathbb{R}$ be a cuboid with side length of $O \subset \mathbb{R}^{n-1}$ larger than $\ell > 0$. For $0 < \epsilon < \frac{1}{2}|J|$ and $1 \leq p < \infty$ let $(u_\epsilon)_\epsilon \subset W^{1,p}(Q';\mathbb{R}^n)$ be uniformly bounded. Moreover, let $(\Sigma_\epsilon)_\epsilon \subset L^\infty(Q'; SO(n))$ and $(b_\epsilon)_\epsilon \subset L^p(Q';\mathbb{R}^n)$ with $\Sigma_\epsilon|_{\epsilon P^i}$ and $b_\epsilon|_{\epsilon P^i}$ constant for each $P^i = \mathbb{R}^{n-1}\times(i, i+1]$, $i \in I'_\epsilon$ such that $w_\epsilon \in L^p(Q';\mathbb{R}^n)$ defined by $w_\epsilon(x) = \Sigma_\epsilon(x)x + b_\epsilon(x)$ for $x \in Q$ satisfies for a sequence $(r_\epsilon)_\epsilon$*

$$\|u_\epsilon - w_\epsilon\|_{L^p(Q;\mathbb{R}^n)} \leq Cr_\epsilon. \tag{3.22}$$

Then, for every $\xi \in \mathbb{R}$ with $Q + \xi e_n \subset Q'$ it holds that

$$\|\Sigma_\epsilon(\,\cdot\,+\xi e_n) - \Sigma_\epsilon\|^p_{L^p(Q;\mathbb{R}^{n\times n})} \leq C\ell^{-p}\big(|\xi|^p + r^p_\epsilon\big) \quad \textit{and} \quad \|b_\epsilon\|^p_{L^p(Q;\mathbb{R}^n)} \leq C\big(1 + r^p_\epsilon\big).$$

Proof. The proof of this result follows the general ideas developed in Lemma 3.1.3. The difference lies in the fact that here, we have not one Sobolev function that coincides with a rigid body motion on every other layer, but for each ϵ, we have two related functions - on the one hand, u_ϵ that is weakly differentiable on Q and on the other hand an approximation w_ϵ that coincides with a rigid body motion on each layers. Also, we do not compare the values of w_ϵ on different layers, but compare shifts of w_ϵ to the function itself.

Firstly, the estimate on difference quotients by weak derivatives from Proposition 3.5.10 yields

$$\|u_\epsilon(\,\cdot\,+\xi e_n) - u_\epsilon\|_{L^p(Q;\mathbb{R}^n)} \leq |\xi|C\|\nabla u_\epsilon e_n\|_{L^p(Q';\mathbb{R}^n)}. \tag{3.23}$$

By the triangle inequality we obtain for each $\xi \in \mathbb{R}$ such that $Q + \xi e_n \subset Q'$

$$\begin{aligned}\|w_\epsilon(\,\cdot\,+\xi e_n) - w_\epsilon\|_{L^p(Q;\mathbb{R}^n)} &\leq \|u_\epsilon(\,\cdot\,+\xi e_n) - u_\epsilon\|_{L^p(Q;\mathbb{R}^n)} + \|(w_\epsilon - u_\epsilon)(\,\cdot\,+\xi e_n)\|_{L^p(Q;\mathbb{R}^n)}\\ &\qquad + \|w_\epsilon - u_\epsilon\|_{L^p(Q;\mathbb{R}^n)}\\ &\leq \|u_\epsilon(\,\cdot\,+\xi e_n) - u_\epsilon\|_{L^p(Q;\mathbb{R}^n)} + 2\|w_\epsilon - u_\epsilon\|_{L^p(Q';\mathbb{R}^n)}.\end{aligned}$$

To the first term on the right hand side we apply (3.23), while for the second an estimate is provided by the assumption of (3.22). Thus, we have

$$\begin{aligned}\|w_\epsilon(\,\cdot\,+\xi e_n) - w_\epsilon\|^p_{L^p(Q;\mathbb{R}^n)} &\leq |\xi|^p C\|\nabla u_\epsilon e_n\|^p_{L^p(Q';\mathbb{R}^n)} + C\|w_\epsilon - u_\epsilon\|^p_{L^p(Q';\mathbb{R}^n)} \\ &\leq C(|\xi|^p + r_\epsilon). \end{aligned} \tag{3.24}$$

Furthermore, since it holds for $d_\epsilon(x) := \Sigma_\epsilon(x+\xi e_n)(\xi e_n) + b_\epsilon(x+\xi e_n) - b_\epsilon(x)$, $x \in Q$ that

$$\|w_\epsilon(\,\cdot\,+\xi e_n) - w_\epsilon\|^p_{L^p(Q;\mathbb{R}^n)} = \int_Q |(\Sigma_\epsilon(x+\xi e_n) - \Sigma_\epsilon(x))x + d_\epsilon(x)|^p \,\mathrm{d}x,$$

Lemma 3.1.4 and Remark 3.1.5 yield

$$\|\Sigma_\epsilon(\,\cdot\,+\xi e_n) - \Sigma_\epsilon\|^p_{L^p(Q;\mathbb{R}^{n\times n})} \leq C\ell^{-p}\|w_\epsilon(\,\cdot\,+\xi e_n) - w_\epsilon\|^p_{L^p(Q;\mathbb{R}^n)},$$

which together with (3.24) implies the claimed estimate on $(\Sigma_\epsilon)_\epsilon$. For the estimate on $(b_\epsilon)_\epsilon$ we argue

$$\begin{aligned}\|b_\epsilon\|_{L^p(Q;\mathbb{R}^n)} &= \|w_\epsilon(x) - \Sigma_\epsilon(x)x\|_{L^p(Q;\mathbb{R}^n)} \\ &\leq \|u_\epsilon\|_{L^p(Q;\mathbb{R}^n)} + \|\Sigma_\epsilon\|_{L^\infty(Q;\mathbb{R}^{n\times n})}\|\mathrm{id}_{\mathbb{R}^n}\|_{L^p(Q;\mathbb{R}^n)} + \|u_\epsilon - w_\epsilon\|_{L^p(Q;\mathbb{R}^n)} \\ &\leq C(1 + |Q| + r_\epsilon),\end{aligned}$$

completing the proof. □

3.3.2 Compactness for layerwise approximating sequences

Theorem 3.3.13 (Fréchet-Kolmogorov-Riesz [3, U2.21])**.** *Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a measurable bounded set. Then, a subset $\mathcal{F} \subset L^p(\Omega;\mathbb{R}^n)$ is relatively compact if the following conditions are satisfied:*

(i) $\mathcal{F}$ is bounded, i.e., there exists a constant $C > 0$ such that for all $f \in \mathcal{F}$ it holds that $\|f\|_{L^p(\Omega;\mathbb{R}^n)} \leq C$;

(ii) For all $h \in \mathbb{R}^n$ it holds that

$$\sup_{f\in\mathcal{F}} \int_{\{x\in\mathbb{R}^n \,|\, x, x+h\in\Omega\}} |f(x+h) - f(x)|^p \,\mathrm{d}x \to 0 \quad \text{as } |h| \to 0;$$

(iii) The family $\mathcal{F}$ does not concentrate on the boundary in the sense that we have for $\Omega_\epsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \epsilon\}$

$$\sup_{f\in\mathcal{F}} \int_{\Omega\cap\Omega_\epsilon^c} |f(x)|^p \,\mathrm{d}x \to 0 \quad \text{as } \epsilon \to 0.$$

Remark 3.3.14. For a cuboid $Q = O \times J \subset \mathbb{R}^n$ let $\mathcal{F} \subset W^{1,p}(Q';\mathbb{R}^n)$ with $\partial_j f = 0$ for all $j \in \{1, \dots, n-1\}$. Then, if for $0 < \xi < \frac{1}{2}|J|$ and all $f \in \mathcal{F}$ it holds that

$$\lim_{\xi\to 0} \sup_{f\in\mathcal{F}} \|f(\,\cdot\,+\xi e_n) - f\|^p_{L^p(Q;\mathbb{R}^n)} = 0,$$

then $\mathcal{F}$ also satisfies condition 2. of Theorem 3.3.13 for $\Omega = Q$. Indeed, since Q is convex, all f are constant along lines parallel to $e_1, \dots, e_{n-1}$ and due to the fact that $Q \subset\subset Q'$ the integration domain is independent of h.

Proposition 3.3.15 (Compactness for $(w_\epsilon)_\epsilon$). *For $n \in \mathbb{N}$, $n \geq 2$ let $Q = O \times J$, $J \subset \mathbb{R}$ be a cuboid with side length of $O \subset \mathbb{R}^{n-1}$ larger than $\ell > 0$. For $0 < \epsilon < \frac{1}{2}|J|$ and $1 \leq p < \infty$ let $(\Sigma_\epsilon)_\epsilon \subset L^\infty(Q'; SO(n))$ such that for a sequence $(r_\epsilon)_\epsilon$ with $r_\epsilon \to 0$ as $\epsilon \to 0$ and every $\xi \in \mathbb{R}$ with $Q + \xi e_n \subset Q'$ it holds that*

$$\|\Sigma_\epsilon(\,\cdot\, + \xi e_n) - \Sigma_\epsilon\|^p_{L^p(Q;\mathbb{R}^n)} \leq C\ell^p(\xi^p + r^p_\epsilon), \tag{3.25}$$

and for $1 < q < \infty$ with $p \leq q$ let $(b_\epsilon)_\epsilon \subset L^q(Q'; \mathbb{R}^n)$ be uniformly bounded. Lastly, let $w_\epsilon \in L^p(Q'; \mathbb{R}^n)$ satisfy $w_\epsilon(x) = \Sigma_\epsilon(x)x + b_\epsilon(x)$ for $x \in Q$.

Then, if $p > 1$ there is a $\Sigma_0 \in W^{1,p}(Q; SO(n))$ with $\|\Sigma_0\|_{W^{1,p}(Q;\mathbb{R}^{n\times n})} \leq C(1 + \ell^{-1})$ and a function $b_0 \in L^q(Q; \mathbb{R}^n)$ with $\partial_j \Sigma_0 = 0$ and $\partial_j b_0 = 0$ for $j \in \{1, \dots, n-1\}$ such that up to a subsequence

$$\Sigma_\epsilon \to \Sigma_0 \quad \text{in } L^p(Q;\mathbb{R}^n) \quad \text{and} \quad w_\epsilon \rightharpoonup w_0 = \Sigma_0 x + b_0 \quad \text{in } L^p(Q;\mathbb{R}^n) \quad \text{as } \epsilon \to 0. \tag{3.26}$$

If $p = 1$, then there is a $\Sigma_0 \in BV(Q; SO(n))$ with $\|\Sigma_0\|_{BV(Q;\mathbb{R}^{n\times n})} \leq C(1 + \ell^{-1})$ and a function $b_0 \in L^q(Q; \mathbb{R}^n)$ with $\partial_j \Sigma_0 = 0$ and $\partial_j b_0 = 0$ for $j \in \{1, \dots, n-1\}$ such that up to a subsequence (3.26) *holds.*

Proof. We define $\sigma_\epsilon \in L^\infty(Q; \mathbb{R}^n)$ by $\sigma_\epsilon(x) = \Sigma_\epsilon(x)x$ for all $x \in Q$. In the following we establish compactness for $(\sigma_\epsilon)_\epsilon$ in the strong L^p-topology and $(b_\epsilon)_\epsilon$ in the weak L^q-topology. For the latter, the uniform bound on $(b_\epsilon)_\epsilon$ in the reflexive space $L^q(\Omega; \mathbb{R}^n)$ yields existence of a weak limit $b_0 \in L^q(Q; \mathbb{R}^n)$ such that for a subsequence (not relabeled) $b_\epsilon \rightharpoonup b_0$ in $L^q(Q; \mathbb{R}^n)$. Hence, it remains to establish the existence of a subsequence $(\epsilon_j)_{j\in\mathbb{N}}$ and a $\Sigma_0 \in W^{1,p}(Q; SO(n))$ such that $\sigma_\epsilon \to \sigma_0$ in $L^p(Q; \mathbb{R}^n)$ as $\epsilon \to 0$ where $\sigma_0(x) = \Sigma_0(x)x$ since this shows the existence of a subsequence $(w_{\epsilon_j})_{j\in\mathbb{N}}$ (not relabeled) such that $w_\epsilon \rightharpoonup w_0$ in $L^p(Q; SO(n))$, where $w_0 \in L^p(Q; \mathbb{R}^n)$ is given by

$$w_0(x) = \Sigma_0(x)x + b_0 \quad \text{for } x \in Q.$$

The existence of a weak limit σ_0 of $(\sigma_\epsilon)_\epsilon$ may be concluded in analogy to $(b_\epsilon)_\epsilon$, but we claim that σ_0 bears more structure and in fact is a strong limit of (σ_ϵ). To prove this, the Fréchet-Kolmogorov compactness Theorem 3.3.13 is key, using (3.25) in an argument inspired by Friesecke, James and Müller [72, Proof of Theorem 4.1].

Let $(\epsilon_j)_{j\in\mathbb{N}}$ with $0 < \epsilon_j < \frac{1}{2}|J|$ and $\epsilon_j \to 0$ as $j \to \infty$. Since $(\Sigma_{\epsilon_j})_{\epsilon_j}$ is equibounded in $L^\infty(Q; \mathbb{R}^{n\times n})$ it suffices in accordance to Remark 3.3.14 to establish

$$\lim_{|\xi|\to 0} \sup_{j\in\mathbb{N}} \|\Sigma_{\epsilon_j}(\,\cdot\, + \xi e_n) - \Sigma_{\epsilon_j}\|^p_{L^p(Q;\mathbb{R}^{n\times n})} = 0. \tag{3.27}$$

Let $\delta > 0$ be arbitrary but fixed. Then, we split $\mathbb{N}$ in the finite set $N_\delta := \{j \in \mathbb{N} \mid r_{\epsilon_j} \geq \delta\}$ and the complement $\mathbb{N} \setminus N_\delta$. Observe that taking the supremum over $\mathbb{N}$ yields the same value as taking the supremum of the supremum over N_δ and the supremum over $\mathbb{N} \setminus N_\delta$. For fixed ϵ_j it holds for the L^p-function Σ_{ϵ_j}, see [3, Satz 2.14, ⟨1⟩]

$$\lim_{|\xi|\to 0} \|\Sigma_{\epsilon_j}(\,\cdot\, + \xi e_n) - \Sigma_{\epsilon_j}\|^p_{L^p(Q;\mathbb{R}^{n\times n})}.$$

Hence, since N_δ is finite, we obtain

$$\lim_{|\xi|\to 0} \sup_{j\in N_\delta} \|\Sigma_{\epsilon_j}(\,\cdot\, + \xi e_n) - \Sigma_{\epsilon_j}\|^p_{L^p(Q;\mathbb{R}^{n\times n})} = 0.$$

To address the supremum over $\mathbb{N} \setminus N_\delta$ we use (3.25), which yields

$$\lim_{|\xi|\to 0} \sup_{j\in\mathbb{N}\setminus N_\delta} \|\Sigma_{\epsilon_j}(\,\cdot\, + \xi e_n) - \Sigma_{\epsilon_j}\|^p_{L^p(Q;\mathbb{R}^{n\times n})} \leq \lim_{|\xi|\to 0} \sup_{j\in\mathbb{N}\setminus N_\delta} C(\xi^p + r_\epsilon) = C\delta.$$

As $\delta > 0$ was arbitrary, this yields (3.27).

Therefore, by the Fréchet-Kolmogorov-Riesz compactness Theorem 3.3.13 and with regard to Remark 3.3.14 there is a subsequence $\epsilon_j \to 0$ and a $\Sigma_0 \in L^p(Q;\mathbb{R}^{n\times n})$ such that $\Sigma_{\epsilon_j} \to \Sigma_0$ in $L^p(Q;\mathbb{R}^{n\times n})$ as $j \to \infty$. Notice that since for each $\epsilon > 0$ we have $\partial_j\Sigma_\epsilon = 0$ for all $j \in \{1,\dots,n-1\}$ the same holds true for the limit, i.e. $\partial_j\Sigma_0 = 0$ for all $j \in \{1,\dots,n-1\}$.

Next, we turn to the question of the regularity of Σ_ϵ. By (3.25) we can also derive an estimate on difference quotients

$$\int_Q \Big|\frac{\Sigma_\epsilon(\,\cdot\, + \xi e_n) - \Sigma_\epsilon}{\xi}\Big|^p \,\mathrm{d}x \leq C\ell^p\Big(1 + \frac{r_\epsilon}{\xi^p}\Big).$$

Therefore, we obtain by taking the limit $\epsilon \to 0$ an estimate on the difference quotient of Σ_0, namely

$$\int_Q \Big|\frac{\Sigma_0(\,\cdot\, + \xi e_n) - \Sigma_0}{\xi}\Big|^p \,\mathrm{d}x \leq C\ell^p. \tag{3.28}$$

This estimate, together with the fact that $\partial_j\Sigma_0 = 0$ for all $j \in \{1,\dots,n-1\}$, which on the convex set Q implies that Σ_0 only depends on x_n, yields in the case $p > 1$ that $\Sigma_0 \in W^{1,p}(Q;\mathbb{R}^{n\times n})$, see e.g. [68, Section 5.8, Theorem 3], while it entails for $p = 1$ that $\Sigma_0 \in BV(Q;\mathbb{R}^{n\times n})$, see e.g. [99, Theorem 13.48]. Furthermore, since strong L^p-convergence implies pointwise convergence of a subsequence we may assume that Σ_ϵ converges pointwise to Σ_0. By continuity of the absolute value and the determinant, it follows that $\Sigma_0(x) \in SO(n)$ for a.e. $x \in Q$. This, together with (3.28) yields in particular

$$\|\Sigma_0\|_{W^{1,p}(Q;\mathbb{R}^{n\times n})} \leq C(1+\ell^{-1}), \quad \text{or} \quad \|\Sigma_0\|_{BV(Q;\mathbb{R}^{n\times n})} \leq C(1+\ell^{-1}), \quad \text{respectively.}$$

Lastly, let us consider σ_ϵ. By the estimate

$$\|\sigma_\epsilon(x) - \Sigma_0(x)x\|_{L^p(Q;\mathbb{R}^n)} \leq \int_Q |\Sigma_\epsilon(x) - \Sigma_0(x)|^p|x|^p \,\mathrm{d}x \leq C\,\mathrm{diam}(Q)\|\Sigma_\epsilon - \Sigma_0\|_{L^p(Q;\mathbb{R}^{n\times n})},$$

we obtain strong convergence of $(\sigma_\epsilon)_\epsilon$ in $L^p(Q;\mathbb{R}^n)$ to $\sigma_0 \in L^p(Q;\mathbb{R}^n)$ given by $\sigma_0(x) = \Sigma_0(x)x$ for a.e. $x \in Q$. □

Discussion of compactness arguments.

In [42] a version of Corollary 3.3.3 was shown, namely Proposition 2.1 in that work, for the specific case of totally rigid layers. In that case instead of the Fréchet-Kolmogorov-Riesz compactness theorem, Helly's selection theorem was applied with the same intent. For the rest of this subsection, we want to argue the advantages in using the Fréchet-Kolmogorov-Riesz compactness theorem. First, let us recall Helly's selection theorem.

Theorem 3.3.16 (Helly's selection theorem [84, Theorem 12])**.** *Let $[a,b] \subset \mathbb{R}$ be a bounded interval, and $(f_\epsilon)_\epsilon \subset BV([a,b];\mathbb{R}^n)$, $n \in \mathbb{N}$. If there is a constant C such that*

$$\|f_\epsilon\|_{L^\infty([a,b];\mathbb{R}^n)} < C \quad \text{and} \quad V(f_\epsilon,[a,b]) < C \quad \text{for all } \epsilon > 0,$$

where $V(f_\epsilon, [a,b])$ denotes the variation of f_ϵ on $[a,b]$ defined by

$$V(f_\epsilon, [a,b]) = \sup\Big\{ \sum_{k=1}^{\ell-1} |f(t_{k+1}) - f(t_k)| \;\Big|\; a = t_1 < t_2 < \cdots < t_\ell = b \Big\},$$

then, there is a subsequence $(f_\epsilon)_\epsilon$ which converges pointwise everywhere and with respect to the $L^1([a,b])$-norm to a function $f \in BV([a,b];\mathbb{R}^n)$.

Remark 3.3.17 (Fréchet-Kolmogorov vs. Helly)**.** While Hanche-Olsen and Holden have shown in [84] that the Fréchet-Kolmogorov theorem and the Arzela-Ascoli theorem can be derived from the same abstract compactness result, the relation between the Fréchet-Kolmogorov theorem and Helly's selection theorem is more direct, as the latter can be shown as a corollary of the former. This is due to fact that for bounded interval $J \subset \mathbb{R}$ any function $f : J \to \mathbb{R}^n$, $n \in \mathbb{N}$ satisfies for all $\tilde{J} \subset\subset J$ the estimate

$$\int_{\tilde{J}} |f(t+\xi) - f(t)| \,\mathrm{d}x \leq |\xi| V(f, J) \quad \text{for all } \xi \in \mathbb{R} \text{ such that } \tilde{J} + \xi \subset J.$$

The next example establishes that small oscillations may prohibit to conclude compactness from Helly's selection theorem, while it can be derived from the Fréchet-Kolmogorov theorem.

Example 3.3.18 (Small scale oscillations leading to unbound variation)**.** Let $J \subset \mathbb{R}$ be a bounded interval and $g : \mathbb{R} \to \mathbb{R}$ be 1-periodic with $g|_{[0,1)} = \mathbb{1}_{[0,\frac{1}{2}]}$. For $\beta \in (0,1)$ and $\epsilon > 0$, we define $(f_\epsilon) \subset L^\infty(J)$ by

$$f_\epsilon(t) = \epsilon^\beta g(\epsilon^{-1} t), \quad t \in J.$$

Then, the variation of f_ϵ can be estimated by

$$V(f_\epsilon, I) \geq 2\epsilon^\beta \lfloor \epsilon^{-1} |J| \rfloor \to \infty \quad \text{as } \epsilon \to 0,$$

and is thus not uniformly bounded in ϵ. In contrast, we have for every $J' \subset\subset J$ and all $\xi \in \mathbb{R}$ with $J' + \xi \subset J$ that

$$\|f_\epsilon(\,\cdot\, + \xi) - f_\epsilon\|_{L^1(J')} \leq 2\epsilon^\beta |J|.$$

Hence, the Fréchet-Kolmogorov theorem yields that $(f_\epsilon)_\epsilon$ is compact which can be confirmed directly as the limit is obviously given by the zero function.

For our application this becomes relevant in the context of the compactness of the sequences $(\Sigma_\epsilon)_\epsilon$. As the next example shows, certain rather artificial choices in the construction of $(\Sigma_\epsilon)_\epsilon$ in Proposition 3.3.10 may lead to sequences of functions of unbounded variation, that are nevertheless admissible to Proposition 3.3.15. The underlying question, if the construction can be improved in such a way that $(\Sigma_\epsilon)_\epsilon$ features a bounded variation is still open, see also the outlook in Chapter 7.

Example 3.3.19 (Variation of $(\Sigma_\epsilon)_\epsilon$)**.** For this example we restrict ourselves to the case of $\Omega = (0,1)^2$, $p = q = 2$ and $\lambda_\epsilon = \lambda \in (0,1)$ constant for all $\epsilon > 0$. For $C > 0$ and $\alpha > 2$, let $(u_\epsilon)_\epsilon \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ be the sequence of functions determined by

$$\nabla u_\epsilon = \mathbb{I} + \epsilon^{\frac{\alpha}{2}} e_1 \otimes e_1.$$

This sequence satisfies the estimate

$$\|\nabla u_\epsilon - \mathbb{I}\|_{L^2(\epsilon P_{\text{stiff}}\cap\Omega;\mathbb{R}^{2\times 2})} = \epsilon^{\frac{\alpha}{2}}|\epsilon P_{\text{stiff}}\cap\Omega| \le C\epsilon^{\frac{\alpha}{2}}.$$

Besides, for $Q_\epsilon \in SO(2)$ satisfying $|Q_\epsilon - \mathbb{I}|^2 = \epsilon^{\alpha-2}$, we obtain

$$\begin{aligned}\|\nabla u_\epsilon - Q_\epsilon\|^2_{L^2(\epsilon P^i_{\text{stiff}}\cap\Omega;\mathbb{R}^{2\times 2})} &\le \|\nabla u_\epsilon - \mathbb{I}\|_{L^2(\epsilon P_{\text{stiff}}\cap\Omega;\mathbb{R}^{2\times 2})} + \|\mathbb{I} - Q_\epsilon\|_{L^2(\epsilon P_{\text{stiff}}\cap\Omega;\mathbb{R}^{2\times 2})}\\ &\le |\epsilon P_{\text{stiff}}\cap\Omega|(\epsilon^{\frac{\alpha}{2}} + \epsilon^{\frac{\alpha}{2}-1}) \le 2|\epsilon P_{\text{stiff}}\cap\Omega|\epsilon^{\frac{\alpha}{2}-1} \le C\epsilon^{\frac{\alpha}{2}-1}.\end{aligned}$$

Hence, $(\Sigma_\epsilon)_\epsilon \subset L^\infty(\Omega; SO(2))$ given by

$$\Sigma_\epsilon = \sum_{i\in I_\epsilon, i \text{ even}} \mathbb{I}\,\mathbb{1}_{\epsilon P^i\cap\Omega} + \sum_{i\in I_\epsilon, i \text{ odd}} Q_\epsilon\,\mathbb{1}_{\epsilon P^i\cap\Omega}$$

satisfies

$$\|\nabla u_\epsilon - \Sigma_\epsilon\|^2_{L^2(\epsilon P_{\text{stiff}}\cap\Omega;\mathbb{R}^{2\times 2})} \le C\epsilon^{\alpha-2},$$

since, both Q_ϵ and $\mathbb{I}$ satisfy the requirements imposed for Σ_ϵ in Proposition 3.3.10, that are derived from the quantitative geometric rigidity Theorem 3.2.4.

But, for the total variation of Σ_ϵ (seen as a one-dimensional function), we obtain

$$V(\Sigma_\epsilon, (0,1)) \ge \lfloor \tfrac{1}{\epsilon} \rfloor |Q_\epsilon - \mathbb{I}| = \lfloor \tfrac{1}{\epsilon} \rfloor \epsilon^{\frac{\alpha-2}{2}} \ge C\epsilon^{\frac{\alpha-4}{2}}.$$

Hence, for $2 < \alpha < 4$ the variation of $(\Sigma_\epsilon)_\epsilon$ is not uniformly bounded in ϵ.

3.4 Examples of softer asymptotic behavior for small stiffness

In this section we want to illustrate with explicit examples that if the stiffness is insufficiently large the asymptotic behavior deviates from the characterization by Theorem 3.3.1. These examples will be based on a common Lemma 3.4.3 containing the calculations of the limits and the energies involved.

We start by establishing that the well-known Lemma 2.3.1 on weak convergence on highly-oscillating functions generalizes directly to functions only oscillating in one component by adapting the proof of Braides for the classic result [28, Example 2.4], alternatively see [57, Theorem 1.5]. Convergence of functions oscillating in proper subspaces can also be studied via an unfolding operator approach to two-scale convergence, see for example [116, Section 6] for an application of these techniques in the context of in-plane oscillations in thin films.

Recall that we use for $t \in \mathbb{R}$ and $x = (x', x_n) \in \mathbb{R}^n$ the short hand

$$\lfloor t \rfloor_\epsilon = \epsilon \lfloor \tfrac{t}{\epsilon} \rfloor \quad \text{and} \quad \lfloor x \rfloor_\epsilon = (x', \lfloor x_n \rfloor_\epsilon)$$

in order to refer to the next lower layer boundary.

Lemma 3.4.1 (Weak convergence of functions oscillating highly in one component)**.** *For $n \in \mathbb{N}$, $n \ge 2$, $1 \le p \le \infty$ and $\ell \in (0,\infty)$ let $u \in L^p_{\text{loc}}(\mathbb{R}^n)$ with $u(x + \ell e_n) = u(x)$ for a.e. $x \in \mathbb{R}^n$. Define $u_\epsilon \in L^p_{\text{loc}}(\mathbb{R}^n)$ by $u_\epsilon(x', x_n) = u(x', x_n/\epsilon)$ for all $x = (x', x_n) \in \mathbb{R}^n$.*

Then, for all domains $\Omega \subset \mathbb{R}^n$ we have $u_\epsilon \rightharpoonup \bar{u}$ in $L^p(\Omega)$ as $\epsilon \to 0$, where

$$\bar{u}(x', x_n) = \frac{1}{\ell}\int_0^\ell u(x', x_n)\,\mathrm{d}x_n \quad \textit{for a.e. } x = (x', x_n) \in \Omega,$$

if $1 \le p < \infty$. The same statement holds for $p = \infty$ with weak convergence replaced by weak-$*$-convergence.

Proof. We adapt the proof by Braides [28, Example 2.4] which utilizes Lemma 3.4.2. We start proving the special case $p = \infty$ and $\ell = 1$ first. Under these assumptions, it holds for a constant $C > 0$ that $\|u_\epsilon\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\Omega)} < C$. Also, for each $A = (0,a)^n + b$, where $a \in (0,\infty)$, $b = (b', b_n) \in \mathbb{R}^n$, we obtain

$$\begin{aligned}\int_A u_\epsilon \,\mathrm{d}x &= \int_{(0,a)^{n-1}+b'} \int_{b_n}^{b_n+a} u_\epsilon \,\mathrm{d}x_n \,\mathrm{d}x' \\ &= \int_{(0,a)^{n-1}+b'} \Big(\int_{b_n}^{\lfloor b_n \rfloor_\epsilon + \epsilon} u_\epsilon \,\mathrm{d}x_n + \int_{\lfloor b_n \rfloor_\epsilon+\epsilon}^{\lfloor b_n+a \rfloor_\epsilon} u_\epsilon \,\mathrm{d}x_n + \int_{\lfloor b_n+a \rfloor_\epsilon}^{b_n+a} u_\epsilon \,\mathrm{d}x_n \Big) \,\mathrm{d}x'. \end{aligned} \tag{3.29}$$

For the first and analogously for the third integral, we argue

$$\begin{aligned}\Big| \int_{(0,a)^{n-1}+b'} \int_{b_n}^{\lfloor b_n \rfloor_\epsilon+\epsilon} u_\epsilon \,\mathrm{d}x_n \,\mathrm{d}x' \Big| &\le a^{n-1} \|u\|_{L^\infty(\mathbb{R}^n)} \big| \lfloor b_n \rfloor_\epsilon + \epsilon - b_n \big| \\ &\to 0 \quad \text{as} \quad \epsilon \to 0. \end{aligned}$$

For the remaining second integral in (3.29), we obtain

$$\begin{aligned}\int_{(0,a)^{n-1}+b'} \int_{\lfloor b_n \rfloor_\epsilon+\epsilon}^{\lfloor b_n+a \rfloor_\epsilon} u_\epsilon \,\mathrm{d}x_n \,\mathrm{d}x' &= \int_{(0,a)^{n-1}+b'} \frac{\lfloor b_n + a \rfloor_\epsilon - \lfloor b_n \rfloor_\epsilon - \epsilon}{\epsilon} \int_0^\epsilon u_\epsilon \,\mathrm{d}x_n \,\mathrm{d}x' \\ &= \int_{(0,a)^{n-1}+b'} \big(\lfloor b_n + a \rfloor_\epsilon - \lfloor b_n \rfloor_\epsilon - \epsilon \big) \int_0^1 u \,\mathrm{d}x_n \,\mathrm{d}x' \\ &\to \int_{(0,a)^n+b'} \bar{u} \,\mathrm{d}x \quad \text{as } \epsilon \to 0. \end{aligned} \tag{3.30}$$

Therefore, it holds that

$$\int_A u_\epsilon \,\mathrm{d}x \to \int_{(0,a)^n+b'} \bar{u} \,\mathrm{d}x \quad \text{as } \epsilon \to 0,$$

and thus the next Lemma 3.4.2 yields the claim.

Now, let $u \in L^p(\mathbb{R}^n)$ for $1 \le p < \infty$. We define for $c > 0$ the approximation

$$u^c = u\, \mathbb{1}_{\{|u| \le c\}} + c\, \mathbb{1}_{\{u > c\}} + (-c)\, \mathbb{1}_{\{u < -c\}}$$

that is bounded in $L^\infty(\mathbb{R}^n)$ by c and inherits the 1-periodicity in x_n from u. Hence, by the above arguments $u^c_\epsilon = u^c({}^t\!/\!_\epsilon)$ converges to $\overline{u^c}$.

Now, let $\Omega \subset \mathbb{R}^n$ be a domain. To check the weak convergence of u_ϵ let $\varphi \in L^{p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and consider

$$\int_\Omega (u_\epsilon - \bar{u})\varphi \,\mathrm{d}x = \int_\Omega (u_\epsilon - u^c_\epsilon)\varphi \,\mathrm{d}x + \int_\Omega (u^c_\epsilon - \overline{u^c})\varphi \,\mathrm{d}x + \int_\Omega (\overline{u^c} - \bar{u})\varphi \,\mathrm{d}x.$$

The second term turns 0 as $\epsilon \to 0$ by weak convergence of u^c_ϵ. For the first term we have by Hölder's inequality and the same transformation used above in the case $p = \infty$

$$\lim_{\epsilon \to 0} \int_\Omega (u^c_\epsilon - u_\epsilon)\varphi \,\mathrm{d}y \le C\|\varphi\|_{L^{p'}(\Omega)} \int_\Omega \Big(\int_0^1 |u^c - u|^p \,\mathrm{d}x_n \Big)^{\frac{1}{p}} \,\mathrm{d}y,$$

while for the third term we have by applying Hölder's inequality twice

$$\int_\Omega (\overline{u^c} - \bar{u})\varphi \,\mathrm{d}y \le C\|\varphi\|_{L^{p'}(\Omega)} \int_\Omega \Big(\int_0^1 |u^c - u|^p \,\mathrm{d}x_n \Big)^{\frac{1}{p}} \,\mathrm{d}y.$$

Therefore, both terms tend towards zero as $c \to \infty$ since by dominated convergence

$$\int_\Omega \Big(\int_0^1 |u^c - u|^p \,\mathrm{d}x_n \Big)^{\frac{1}{p}} \mathrm{d}y \le \int_\Omega \Big(\int_0^1 |u|^p \mathbb{1}_{\{|u|>c\}} \,\mathrm{d}x_n \Big)^{\frac{1}{p}} \mathrm{d}y \to 0 \quad \text{as } c \to \infty.$$

For $\ell \neq 1$ the transformation $y = \ell x$ provides a 1-periodic function and resubstitution via the transformation formula leads to the claimed representation. □

Lemma 3.4.2 (Characterization of weak convergence [57, Lemma 1.4])**.** *Let Ω be a bounded open set and let $(f_k)_{k\in\mathbb{N}} \subset L^p(\Omega)$ be a bounded sequence. If $1 < p < \infty$, then $f_k \rightharpoonup f$ in $L^p(\Omega)$ if and only if $\|f_k\|_{L^p(\Omega)} \le C$ for $C > 0$ and*

$$\int_A f_k \,\mathrm{d}x \to \int_A f \,\mathrm{d}x \quad \textit{for all } A = (0,a)^n + b \subset \Omega, a \in \mathbb{R}, b \in \mathbb{R}^n.$$

The same statement holds for $p = \infty$ with weak convergence replaced by weak-$*$ convergence and for $p = 1$ if we require equi-integrability of $(f_k)_{k\in\mathbb{N}}$ in view of the Dunford-Pettis theorem.

The goal of this section is to give illustrative examples of macroscopic material responses showing the differences between the softer regime and the more restricted beyond the critical threshold, as well as the implications of the volume condition. The common idea for these examples is to deform the stiff layers according to a vector field f_ϵ that specifies the deformation of the middle fiber of each stiff layer, while using linear interpolation on the soft layers. The following lemma provides a joint framework and contains the necessary calculations regarding energy and convergence.

The constructions will be formulated on the two-dimensional unit cube, i.e. $\Omega = (0,1)^2 \subset \mathbb{R}^2$, but can be simply extended to higher dimensional unit cubes $Y^n = (0,1)^n \subset \mathbb{R}^n$ by defining for the respective two-dimensional deformations $u_\epsilon = (u_\epsilon^1, u_\epsilon^2)$ the n-dimensional deformation $u_\epsilon^n : Y^n \to \mathbb{R}^n$, given by

$$u_\epsilon^n(x) = (u_\epsilon^1(x_1, x_n), x_2, \dots, x_{n-1}, u_\epsilon^2(x_1, x_n)) \quad \text{for} \quad x = (x_1, \dots, x_n) \in Y^n.$$

For the projection to the mid-fiber of the stiff layer, we use in the following for $t \in \mathbb{R}$ and $x = (x', x_n) \in \mathbb{R}^n$ the notation

$$[\![t]\!]_\epsilon = \epsilon\lceil \tfrac{t}{\epsilon} \rceil - \epsilon + \tfrac{1+\lambda}{2}\epsilon \quad \text{and} \quad [\![x]\!]_\epsilon = (x', [\![x_n]\!]_\epsilon).$$

Lemma 3.4.3 (Deformations specified by mid-fibers of stiff layers)**.** *For $Q = [0,1] \times [0,2]$ and $\epsilon \in (0,1)$ let $f_\epsilon \in C^2(Q;\mathbb{R}^2)$ with $|\partial_1 f_\epsilon| = 1$. For $\lambda \in (0,1)$ let $(P_{\text{stiff}})_\epsilon$ be a periodic bilayered structure. Let $\Omega = (0,1)^2$ and for $1 < p < \infty$ let $u_\epsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ on the stiff layers be given by*

$$u_\epsilon(x) := f_\epsilon([\![x]\!]_\epsilon) + (x_2 - [\![x_2]\!]_\epsilon)(\partial_1 f_\epsilon)^\perp([\![x]\!]_\epsilon) \quad \textit{for } x \in \epsilon P_{\text{stiff}} \cap \Omega,$$

and linear interpolation in e_2-direction on the soft layers $\epsilon P_{\text{soft}} \cap Q$.

Then, we have for $1 < q < \infty$ and all $\epsilon \in (0,1)$

$$\int_{\epsilon P_{\text{stiff}} \cap \Omega} \operatorname{dist}^q(\nabla u_\epsilon, SO(2)) \,\mathrm{d}x \le C\epsilon^q \|\partial_{11} f_\epsilon\|^q_{L^\infty(Q;\mathbb{R}^2)}.$$

Furthermore, if

(i) $\epsilon\|\nabla^2 f_\epsilon\|_{L^\infty(Q;\mathbb{R}^{2\times2\times2})} \to 0$ as $\epsilon \to 0$;

(ii) $\nabla f_\epsilon \rightharpoonup F$ *in* $L^p(Q;\mathbb{R}^{2\times 2})$ *as* $\epsilon \to 0$ *for some* $F \in L^p(Q;\mathbb{R}^{2\times 2})$*;*

(iii) $\partial_2(\nabla f_\epsilon) = 0$ *for all* $\epsilon \in (0,1)$ *or* $\nabla f_\epsilon \to F$ *in* $L^p(Q;\mathbb{R}^2)$ *as* $\epsilon \to 0$*;*

then $\nabla u_\epsilon \rightharpoonup F$ *in* $L^p(Q;\mathbb{R}^{2\times 2})$ *as* $\epsilon \to 0$.

Proof. Recall throughout these calculations that by the definition of the layered geometry, in particular the allocation of boundaries, it holds for each $i \in \mathbb{Z}$ and $x \in \epsilon P^i$ that $\lfloor x \rfloor_\epsilon \in P^{i-1}$, recalling the notation

$$\lfloor x \rfloor_\epsilon = (x_1, \dots, x_{n-1}, \lfloor x_n \rfloor_\epsilon) \quad \text{where for } t \in \mathbb{R} \text{ we set} \quad \lfloor t \rfloor_\epsilon = \epsilon \lfloor \tfrac{t}{\epsilon} \rfloor.$$

We first observe that in the interior of both the stiff and the soft layer, u_ϵ is differentiable in the classic sense, while it is only weakly differentiable on the whole domain. On the stiff layers, i.e. for $x \in \epsilon P_{\text{stiff}} \cap Q$ we have $\partial_1 [\![x]\!]_\epsilon = e_1$ and $\partial_2 [\![x]\!]_\epsilon = 0$ and thus

$$\nabla u_\epsilon(x) = \partial_1 f_\epsilon([\![x]\!]_\epsilon) \otimes e_1 + (x_2 - [\![x_2]\!]_\epsilon)(\partial_{11} f_\epsilon)^\perp([\![x]\!]_\epsilon) \otimes e_1 + (\partial_1 f_\epsilon)^\perp([\![x]\!]_\epsilon) \otimes e_2.$$

Concerning the estimate on the distance of ∇u_ϵ from $SO(2)$, the main observation is that since $|\partial_1 f_\epsilon| = 1$ it follows that $F_\epsilon := \partial_1 f_\epsilon \otimes e_1 + (\partial_1 f_\epsilon)^\perp \otimes e_2 \in SO(2)$. Hence, as for $x_2 \in \mathbb{R}$

$$\big|x_2 - [\![x_2]\!]_\epsilon\big| = \epsilon \big|\tfrac{x_2}{\epsilon} - \lceil \tfrac{x_2}{\epsilon} \rceil + 1 - \tfrac{1+\lambda}{2}\big| \le 2\epsilon,$$

we obtain for each $x \in \Omega$

$$\begin{aligned} \operatorname{dist}^q\left(\nabla u_\epsilon(x), SO(2)\right) &\le |\nabla u_\epsilon(x) - F_\epsilon|^q \\ &\le |x_2 - [\![x_2]\!]_\epsilon|^q |(\partial_{11} f_\epsilon)^\perp([\![x]\!]_\epsilon)|^q \\ &\le (2\epsilon)^q \|\partial_{11} f_\epsilon\|^q_{L^\infty(Q;\mathbb{R}^2)}, \end{aligned} \tag{3.31}$$

which implies

$$\int_{\epsilon P_{\text{stiff}} \cap \Omega} \operatorname{dist}^q\left(\nabla u_\epsilon, SO(2)\right) \mathrm{d}x \le C\epsilon^q |\Omega| \cdot \|\partial_{11} f_\epsilon\|^q_{L^\infty(Q;\mathbb{R}^2)}.$$

On the soft layer, i.e. $x \in \epsilon P_{\text{soft}} \cap Q$ we need to explicitly calculate the linear interpolation first. Note that u_ϵ is on the interfaces between the layers given by

$$u_\epsilon(\lfloor x \rfloor_\epsilon) = f_\epsilon([\![x]\!]_\epsilon - \epsilon e_2) + \tfrac{1-\lambda}{2} \epsilon (\partial_1 f_\epsilon)^\perp([\![x]\!]_\epsilon - \epsilon e_2)$$

and

$$u_\epsilon(\lfloor x \rfloor_\epsilon + \lambda \epsilon e_2) = f_\epsilon([\![x]\!]_\epsilon) - \tfrac{1-\lambda}{2} \epsilon (\partial_1 f_\epsilon)^\perp([\![x]\!]_\epsilon).$$

Therefore, for $x \in \epsilon P_{\text{soft}} \cap Q$ we have

$$\begin{aligned} u_\epsilon(x) &= \left(1 - \tfrac{1}{\epsilon\lambda}(x_2 - \lfloor x_2 \rfloor_\epsilon)\right) u_\epsilon(\lfloor x \rfloor_\epsilon) + \tfrac{1}{\epsilon\lambda}(x_2 - \lfloor x_2 \rfloor_\epsilon) u_\epsilon(\lfloor x \rfloor_\epsilon + \lambda\epsilon e_2) \\ &= \left(1 - \tfrac{1}{\epsilon\lambda}(x_2 - \lfloor x_2 \rfloor_\epsilon)\right)\left(f_\epsilon([\![x]\!]_\epsilon - \epsilon e_2) + \tfrac{1-\lambda}{2}\epsilon(\partial_1 f_\epsilon)^\perp([\![x]\!]_\epsilon - \epsilon e_2)\right) \\ &\qquad + \tfrac{1}{\epsilon\lambda}(x_2 - \lfloor x_2 \rfloor_\epsilon)\left(f_\epsilon([\![x]\!]_\epsilon) - \tfrac{1-\lambda}{2}\epsilon(\partial_1 f_\epsilon)^\perp([\![x]\!]_\epsilon)\right) \\ &= f_\epsilon([\![x]\!]_\epsilon - \epsilon e_2) + \tfrac{1-\lambda}{2}\epsilon(\partial_1 f_\epsilon)^\perp([\![x]\!]_\epsilon - \epsilon e_2) \\ &\qquad\qquad + \tfrac{1}{\epsilon\lambda}(x_2 - \lfloor x_2 \rfloor_\epsilon)\left(f_\epsilon([\![x]\!]_\epsilon) - f_\epsilon([\![x]\!]_\epsilon - \epsilon e_2)\right) \\ &\qquad\qquad - \tfrac{1-\lambda}{2\lambda}(x_2 - \lfloor x_2 \rfloor_\epsilon)\left((\partial_1 f_\epsilon)^\perp([\![x]\!]_\epsilon - \epsilon e_2) + (\partial_1 f_\epsilon)^\perp([\![x]\!]_\epsilon)\right). \end{aligned}$$

Thus, the gradient reads for $x \in \epsilon P_{\text{soft}} \cap Q$

$$\begin{aligned}\nabla u_\epsilon(x) = \big(&\partial_1 f_\epsilon(\llbracket x \rrbracket_\epsilon - \epsilon e_2) + \tfrac{1-\lambda}{2}\epsilon(\partial_{11} f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon - \epsilon e_2)\big) \otimes e_1\\ &+ \tfrac{1}{\epsilon\lambda}(x_2 - \lfloor x_2 \rfloor_\epsilon)\big(\partial_1 f_\epsilon(\llbracket x \rrbracket_\epsilon) - \partial_1 f_\epsilon(\llbracket x \rrbracket_\epsilon - \epsilon e_2)\big) \otimes e_1\\ &- \tfrac{1-\lambda}{2\lambda}(x_2 - \lfloor x_2 \rfloor_\epsilon)\big((\partial_{11} f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon - \epsilon e_2) + (\partial_{11} f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon)\big) \otimes e_1\\ &+ \tfrac{1}{\epsilon\lambda}\big(f_\epsilon(\llbracket x \rrbracket_\epsilon) - f_\epsilon(\llbracket x \rrbracket_\epsilon - \epsilon e_2)\big) \otimes e_2\\ &- \tfrac{1-\lambda}{2\lambda}\big((\partial_1 f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon - \epsilon e_2) + (\partial_1 f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon)\big) \otimes e_2.\end{aligned}$$

Next, we determine the weak limit of $(\nabla u_\epsilon)_\epsilon$ in $L^p(Q;\mathbb{R}^{n\times n})$ as $\epsilon \to 0$. Since ∇u_ϵ contains both terms converging in the strong and in the weak sense we introduce an auxiliary sequence $V_\epsilon \in L^\infty(Q;\mathbb{R}^{2\times 2})$ that only contains the weakly convergent terms. Precisely, we set

$$V_\epsilon = \big(\partial_1 f_\epsilon \otimes e_1 + (\partial_1 f_\epsilon)^\perp \otimes e_2\big)\mathbb{1}_{\epsilon P_{\text{stiff}}} + \big(\partial_1 f_\epsilon \otimes e_1 + \big(\tfrac{1}{\lambda}\partial_2 f_\epsilon - \tfrac{1-\lambda}{\lambda}(\partial_1 f_\epsilon)^\perp\big) \otimes e_2\big)\mathbb{1}_{\epsilon P_{\text{soft}}},$$

and show that for all $x \in \Omega$ it holds that $|\nabla u_\epsilon - V_\epsilon|(x) \to 0$ as $\epsilon \to 0$. By the fact that $f_\epsilon \in C^2(Q;\mathbb{R}^2)$ we may apply the mean value theorem in x_2-direction to obtain on the stiff layers, i.e. for $x \in \epsilon P_{\text{stiff}} \cap Q$

$$\begin{aligned}|\nabla u_\epsilon - V_\epsilon|(x) &\le \big|\partial_1 f_\epsilon(\llbracket x \rrbracket_\epsilon) - \partial_1 f_\epsilon(x)\big| + \big|(x_2 - \llbracket x_2 \rrbracket_\epsilon)(\partial_{11} f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon)\big|\\ &\qquad + \big|(\partial_1 f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon) - (\partial_1 f_\epsilon)^\perp(x)\big|\\ &\le |x_2 - \lfloor x_2 \rfloor_\epsilon|\big(\|\partial_{12} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)} + \|\partial_{11} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)} + \|(\partial_{12} f_\epsilon)^\perp\|_{L^\infty(Q;\mathbb{R}^2)}\big),\end{aligned}$$

while on the soft layers, i.e. for $x \in \epsilon P_{\text{soft}} \cap Q$,

$$\begin{aligned}|\nabla u_\epsilon e_1 - V_\epsilon e_1|(x) &\le \big|\partial_1 f_\epsilon(\llbracket x \rrbracket_\epsilon - \epsilon e_2) - \partial_1 f_\epsilon(x)\big| + \tfrac{1-\lambda}{2}\epsilon\big|(\partial_{11} f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon - \epsilon e_2)\big|\\ &\quad + \tfrac{1}{\epsilon\lambda}|x_2 - \lfloor x_2 \rfloor_\epsilon + \epsilon e_2|\big|\partial_1 f_\epsilon(\llbracket x \rrbracket_\epsilon) - \partial_1 f_\epsilon(\llbracket x \rrbracket_\epsilon - \epsilon e_2)\big|\\ &\quad + \tfrac{1-\lambda}{2\lambda}|x_2 - \lfloor x_2 \rfloor_\epsilon|\big|(\partial_{11} f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon - \epsilon e_2) + (\partial_{11} f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon)\big|\\ &\le |x_2 - \llbracket x_2 \rrbracket_\epsilon|\|\partial_{12} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)} + \tfrac{1-\lambda}{2}\epsilon\|\partial_{11} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)}\\ &\quad + \tfrac{1}{\lambda}|x_2 - \lfloor x \rfloor_\epsilon|\|\partial_{12} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)} + \tfrac{1-\lambda}{\lambda}\epsilon\|\partial_{11} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)}\\ &\le \epsilon C\big(\|\partial_{11} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)} + \|\partial_{12} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)}\big)\end{aligned}$$

and for some $\xi \in \big(\llbracket x_2 \rrbracket_\epsilon - \epsilon, \llbracket x_2 \rrbracket_\epsilon\big)$, implying $|x_2 - \xi| \le |x_2 - \llbracket x_2 \rrbracket_\epsilon| + |\xi - \llbracket x_2 \rrbracket_\epsilon| \le 2\epsilon$,

$$\begin{aligned}|\nabla u_\epsilon e_2 - V_\epsilon e_2|(x) &\le \tfrac{1}{\lambda}\big|\tfrac{1}{\epsilon}\big(f_\epsilon(\llbracket x \rrbracket_\epsilon) - f_\epsilon(\llbracket x \rrbracket_\epsilon - \epsilon e_2)\big) - \partial_2 f_\epsilon(x)\big|\\ &\quad + \tfrac{1-\lambda}{2\lambda}\big|(\partial_1 f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon - \epsilon e_2) + (\partial_1 f_\epsilon)^\perp(\llbracket x \rrbracket_\epsilon) - 2(\partial_1 f_\epsilon^\perp)(x)\big|\\ &\le \tfrac{1}{\lambda}\big|\partial_2 f_\epsilon(x_1,\xi) - \partial_2 f_\epsilon(x)\big| + \tfrac{1-\lambda}{\lambda}\epsilon\|\partial_{12} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)}\\ &\le \tfrac{1}{\lambda}|x_2 - \xi|\|\partial_{22} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)} + \tfrac{1-\lambda}{\lambda}\epsilon\|\partial_{12} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)}\\ &\le \epsilon C\big(\|\partial_{22} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)} + \|\partial_{12} f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)}\big).\end{aligned}$$

Hence, $|\nabla u_\epsilon - V_\epsilon| < \epsilon C\|\nabla^2 f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)}$ and since we have $\epsilon\|\nabla^2 f_\epsilon\|_{L^\infty(Q;\mathbb{R}^2)} \to 0$ as $\epsilon \to 0$ by assumption (i), it follows that $\|\nabla u_\epsilon - V_\epsilon\|_{L^p(\Omega;\mathbb{R}^2)} \to 0$ as $\epsilon \to 0$.

It remains to discuss the convergence of V_ϵ. Here, the necessity of the different cases in assumption (iii) becomes apparent in view of the products of ϵ-dependent functions like e.g. $\partial_1 f_\epsilon \otimes e_1 \mathbb{1}_{\epsilon P_{\text{stiff}} \cap Q}$. If $(f_\epsilon)_\epsilon$ converges strongly together with the fact that $\mathbb{1}_{\epsilon P_{\text{stiff}} \cap Q} \overset{*}{\rightharpoonup} 1 - \lambda$ in $L^\infty(Q)$ as $\epsilon \to 0$ by the classic Lemma 2.3.1 on highly oscillating functions we obtain

$$\partial_1 f_\epsilon \otimes e_1 \mathbb{1}_{\epsilon P_{\text{stiff}} \cap Q} \rightharpoonup (1-\lambda)F e_1 \quad \text{in } L^p(Q;\mathbb{R}^{2\times 2}) \quad \text{as } \epsilon \to 0,$$

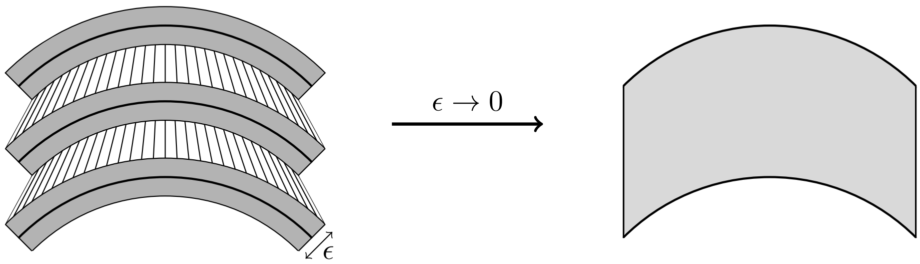

Figure 3.3: The framework of Lemma 3.4.3 comprises deformations of the whole body that are determined by deformations of the stiff layers with respect to the mid-fibers and linear interpolation on the soft layers. Hence, a first example is given by uniform bending of the layers. Depicted here is uniform bending with respect to a circular arc and the corresponding limit as $\epsilon \to 0$ in two dimensions.

as products of weakly and strongly convergent sequences converge weakly to the product of the respective limits.

For the alternative in assumption (iii) holds, i.e. if $\partial_2(\nabla f_\epsilon) = 0$ for all $\epsilon \in (0,1)$, we argue that the convergence takes place in different variables and by Lemma 3.4.1 applied to the product we obtain the same weak limit. Arguing similarly for all other terms in V_ϵ yields

$$\begin{aligned} V_\epsilon &\rightharpoonup Fe_1 \otimes e_1 + (1-\lambda)(Fe_1)^\perp \otimes e_2 + Fe_2 \otimes e_2 - (1-\lambda)(Fe_1)^\perp \otimes e_2 \\ &= F \quad \text{in } L^p(Q;\mathbb{R}^{2\times 2}) \quad \text{as } \epsilon \to 0. \end{aligned}$$

This completes the proof. □

In the following we discuss four examples building on the lemma just proven. Hence, we will use its specific notation for the rest of the section.

Example 3.4.4 (Uniform bending).
Construction. In this example we bend all stiff layers in the same way, determined by the deformation of the mid-fiber which is specified by a C^2-curve $g : [0,1] \to \mathbb{R}^2$ that is parametrized by arc length, i.e. g satisfies for all $x \in [0,1]$ the condition $|g'(x)| = 1$, see Figure 3.3. This corresponds to setting

$$f_\epsilon(x) = f(x) = g(x_1) + x_2 e_2, \quad x \in Q.$$

By shifting u_ϵ by a constant if necessary, we may assume $u_\epsilon \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L^p_0(\Omega;\mathbb{R}^2)$.

For a particular illustrative example we may choose for $r > 4/\pi$

$$g(t) = r \sin\left(r^{-1}(t - \tfrac{1}{2})\right)e_1 + r \cos\left(r^{-1}(t - \tfrac{1}{2})\right)e_2, \quad t \in [0,1],$$

which describes a circular arc whose radius is controlled by the parameter r chosen larger than $4/\pi$ to avoid self-intersection and fits the common associations with bending.

Energies. Looking at the elastic energy of u_ϵ on the stiff layers, Lemma 3.4.3 provides for $1 < q < \infty$ and all $\epsilon > 0$ the estimate

$$\int_{\epsilon P_{\text{stiff}} \cap Q} \text{dist}^q(\nabla u_\epsilon, SO(2))\,\mathrm{d}x < C\epsilon^q \|g''\|^q_{L^\infty(0,1;\mathbb{R}^2)}. \tag{3.32}$$

Thus, for $\alpha = q$ and g such that $\|g''\|_{L^\infty(0,1;\mathbb{R}^2)} < C$, which is for example satisfied in the case of the circular arc, we have for all $\epsilon > 0$

$$\int_{\epsilon P_{\text{stiff}} \cap Q} \operatorname{dist}^q(\nabla u_\epsilon, SO(2))\, \mathrm{d}x < C.$$

Macroscopic material behavior. Since $f_\epsilon = f$ we can apply Lemma 3.4.3 and obtain $\nabla u_\epsilon \rightharpoonup \nabla f$ in $L^p(\Omega;\mathbb{R}^{2\times 2})$ as $\epsilon \to 0$ and since the mean values of u_ϵ vanish on Ω we have $u_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$ for $u \in W^{1,p}(\Omega;\mathbb{R}^2) \cap L^p_0(\Omega;\mathbb{R}^2)$ characterized by $\nabla u = \nabla f$.

Hence, for R defined for $x \in \Omega$ by $R(x) = g'(x_1) \otimes e_1 + g'(x_1)^\perp \otimes e_2$ we have for $x \in \Omega$

$$\nabla u = \nabla f(x) = g'(x_1) \otimes e_1 + e_2 \otimes e_2 = R(x)\big(\mathbb{I} + (g_2'(x_1)e_1 + g_1'(x_1)e_2) \otimes e_2\big).$$

Thus, for example in the case of the circular arc, we have $\partial_1 R \neq 0$, which establishes that the limit deformation differs from the characterization given in Theorem 3.3.1, which would hold for deformations satisfying $\alpha > q$. Also, notice that for non-trivial g, the limit deformation u does not locally preserve the volume.

For later purposes we also study another representation of the limit u. More precisely, we show that $u(x) = g(x_1)e_1 + x_2 e_2 = Fx + \psi(x_1)$ for all $x \in \Omega$, where $F \in \mathbb{R}^{2\times 2}$ and $\psi \in W^{1,p}_{\text{per}}(\Omega;\mathbb{R}^2)$ with $\Omega = (0,1)^2$. Since g and ψ only depend on x_1, the matrix F is fully determined by Fe_1, as $Fe_2 = e_2$. By the zero boundary values of ψ, we have

$$\begin{aligned} Fe_1 &= \int_0^1 F + \nabla\psi(x_1)\, \mathrm{d}x \\ &= \int_0^1 g'(t)\, \mathrm{d}t = \big[r \sin\big(r^{-1}(t - \tfrac{1}{2})\big)e_1 + r\cos\big(r^{-1}(t-\tfrac{1}{2})\big)e_2\big]_0^1 = \big(2r\sin\big((2r)^{-1}\big), 0\big)^T. \end{aligned}$$

Thus, we obtain

$$u = \operatorname{diag}\big(2r\sin((2r)^{-1}), 1\big) \cdot \big(x_1 - \tfrac{1}{2}, x_2\big)^T + \big(g(x_1) - (x_1 - \tfrac{1}{2})2r\sin\big((2r)^{-1}\big)\big)e_1.$$

Depending on r, the term $2r\sin\big((2r)^{-1}\big)$ can take values greater and smaller than 1. For example for $r = 8/\pi$ we have $16/\pi \sin(\pi/16) \approx 0.0175 < 1$.

In the next example, we adapt the bending of the stiff layers in such a way, that the limit deformation locally preserves volume.

Example 3.4.5 (Macroscopically volume-preserving bending).
Construction. For this example the intuitive picture is a stack of paper, bent as a whole. While the deformation of each rigid layer is of similar general form, there is a certain adjustment from layer to layer in order to locally preserve the volume. Considering again the specific case that the deformation of the mid fiber is given by a circular arc, this can be achieved by monotonically increasing the radius of the arc and thus the curvature from layer to layer.

More precisely, we consider for a C^2-curve $g : [0,1] \to \mathbb{R}^2$ parametrized by arc length the functions $f_\epsilon = f$ given by

$$f(x) = (x_2 + 1)g\Big(\frac{x_1}{x_2+1}\Big), \quad x \in \Omega.$$

Energies. As in the previous example, u_ϵ again by Lemma 3.4.3 satisfies (3.32).

Macroscopic material behavior. From Lemma 3.4.3 we also obtain $\nabla u_\epsilon \rightharpoonup \nabla u = \nabla f$ in $L^p(\Omega;\mathbb{R}^{2\times 2})$ as $\epsilon \to 0$ with ∇u given for $x \in \Omega$ by

$$\nabla u(x) = \nabla f(x) = g'\Big(\frac{x_1}{x_2+1}\Big) \otimes e_1 - \frac{x_1}{x_2+1} g'\Big(\frac{x_1}{x_2+1}\Big) \otimes e_2 + g\Big(\frac{x_1}{x_2+1}\Big) \otimes e_2.$$

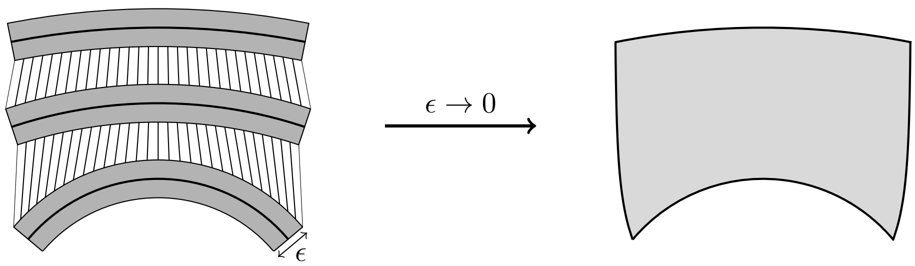

Figure 3.4: We can vary the uniform bending construction with respect to a circular arc by adapting the curvature of the arcs in such a way that the limit is a locally volume preserving deformation. The shape of the deformed body in the limit is reminiscent of the one obtained by bending a stack of paper, but does not resemble a globally rotated shear deformation as the asymptotic rigidity result of Corollary 3.3.3 would suggest in two dimensions in the regime $\alpha > q$.

Denoting for $x \in \Omega$ by $R(x) \in SO(2)$ the rotation determined by $R(x)e_1 = g'(\frac{x_1}{x_2+1})$, yields the representation

$$\nabla u(x) = R(x)\Big(\mathbb{I} + \Big(- e_2 - \frac{x_1}{x_2+1} e_1 + g'(x_1) \cdot g(x_1) e_1 - g'(x_1) \cdot g^{\perp}(x_1) e_2\Big) \otimes e_2\Big), \quad x \in \Omega.$$

This shows that any g satisfying $g' \cdot g^{\perp} = -1$ leads to a locally volume preserving limit deformation u, i.e. $\det \nabla u = \det \nabla f = 1$ in Ω. One such example is given by the circular arc, i.e. by $g(t) = \sin(t - \frac{1}{2})e_1 + \cos(t - \frac{1}{2})e_2$, $t \in [0, 1]$.

In the study of plates a well known effect is the wrinkling of plates [72, Section 5], which can also be observed here, where it manifests in an asymptotic shortening of the stiff layers.

Example 3.4.6 (Wrinkling).
Construction. The idea of this example is to construct deformations that feature on the rigid layers fine periodic oscillations which in the limit lead to a reduction of the length of the material, see Figure 3.5. Note that the macroscopic deformation may still be volume preserving, compensating the reduction in length by an increase in height.

More precisely, let $\beta \in \mathbb{R}$ be the parameter that describes the increase in height. For the fine oscillation, let $g : \mathbb{R} \to \mathbb{R}^2$ be an 1-periodic C^2-function parametrized by arc length and for the length scale of the oscillations we introduce $\gamma \in (0, 1)$ and define for $\epsilon \in (0, 1)$ the family of functions $g_\epsilon : [0, 1] \to \mathbb{R}^2$ by $g_\epsilon(t) = \epsilon^\gamma g(\epsilon^{-\gamma} t)$, $t \in [0, 1]$.

The reduction in length will be due to the weak convergence of g_ϵ as the weak convergence of highly oscillating functions implies

$$g'_\epsilon \rightharpoonup \bar{g}' = \int_0^1 g'(t)\,\mathrm{d}t = g(1) - g(0) \quad \text{in } L^1(0, 1; \mathbb{R}^2) \quad \text{as } \epsilon \to 0.$$

Hence for non-constant g' we have $|\bar{g}'| < 1$.

Correspondingly, we define f_ϵ by

$$f_\epsilon(x) = g_\epsilon(x_1) + \beta x_2 e_2 \quad x \in Q.$$

Observe that $\partial_2(\nabla f_\epsilon) = 0$ and $\nabla f_\epsilon \rightharpoonup F$ in $L^1(Q;\mathbb{R}^2)$ as $\epsilon \to 0$, with

$$F = \bar{g}' \otimes e_1 + \beta e_2 \otimes e_2.$$

An illustrative example is given by an explicit construction by Friesecke, James and Müller for the wrinkling of plates [72, Section 5], which we recall in the following.

Let $R : \mathbb{R} \to SO(2)$ be given by $R(t) = \cos(t)e_1 \otimes e_1 - \sin(t)e_1 \otimes e_2 + \sin(t)e_2 \otimes e_1 + \cos(t)e_2 \otimes e_2$ and $\theta_\epsilon : \mathbb{R} \to \mathbb{R}$ be a parametrization which we specify as follows: For $1 > \gamma > \beta > 0$ and two values $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_2 > \theta_1$ we set θ_ϵ to be a smooth periodic function of period ϵ^β satisfying

$$\theta_\epsilon(x_1) = \begin{cases} \theta_1^\epsilon & \text{if} \quad x_1 \in (0, \frac{1}{4}\epsilon^\beta - \frac{1}{2}\epsilon^\gamma] \text{ or } x_1 \in (\frac{3}{4}\epsilon^\beta + \frac{1}{2}\epsilon^\gamma, \epsilon^\beta], \\ \theta_2^\epsilon & \text{if} \quad x_1 \in (\frac{1}{4}\epsilon^\beta + \frac{1}{2}\epsilon^\gamma, \frac{3}{4}\epsilon^\beta - \frac{1}{2}\epsilon^\gamma], \end{cases}$$

and $|\frac{\mathrm{d}\theta_\epsilon}{\mathrm{d}x_1}| < 2(\theta^2 - \theta^1)\epsilon^\gamma$.

Now, consider the function $g_\epsilon : [0,1] \to \mathbb{R}^2$

$$g_\epsilon(t) = \int_0^t R(\theta_\epsilon(s))e_1 \,\mathrm{d}s.$$

For this choice of g_ϵ we have $g_\epsilon'(t) = R(\theta_\epsilon(t))e_1$ and

$$|g_\epsilon''(t)| = |R'(\theta_\epsilon)\theta_\epsilon' e_1| < 2(\theta^2 - \theta^1)\epsilon^\gamma < C\epsilon^\gamma.$$

Therefore, this example satisfies all the assumptions on g_ϵ imposed above and by the classic Lemma 2.3.1 on highly oscillating functions, we obtain that $g_\epsilon' \rightharpoonup g'$ in $L^1(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$ with g' the mean value of g_ϵ on the periodic cell of length ϵ^β and in particular constant. Therefore the limit is of the claimed form

$$\nabla u = g' \otimes e_1 + \beta e_2 \otimes e_2,$$

and since g' is a constant we can choose β such that $\det(\nabla u) = 1$.

Next, let us consider the elastic energy of the deformations u_ϵ. The gradient on the rigid layers reads

$$\nabla u_\epsilon = R(\theta_\epsilon)\Big(\mathbb{I} - (x_2 - [\![x_2]\!]_\epsilon)\frac{\mathrm{d}\theta_\epsilon}{\mathrm{d}x_1} e_1 \otimes e_1\Big).$$

Energies. For simplicity, we restrict ourselves to the case $p = q = 2$. Lemma 3.4.3 provides the energy estimate

$$\int_{\epsilon P_{\mathrm{stiff}} \cap \Omega} \operatorname{dist}^2(\nabla u_\epsilon, SO(2))\,\mathrm{d}x \le 4\epsilon^2 \|g_\epsilon''\|^2_{L^\infty(Q;\mathbb{R}^2)} \le 4\epsilon^{2-2\gamma}\|g''\|^2_{L^\infty(0,1;\mathbb{R}^2)} < C\epsilon^{2-2\gamma}.$$

For the particular construction by Friesecke, James and Müller, we have

$$\begin{aligned} \frac{1}{\epsilon^\alpha}\|\operatorname{dist}(\nabla u_\epsilon, SO(2))\|^2_{L^2(\Omega)} &\le C\epsilon^{2-\alpha}\Big\|\frac{\mathrm{d}\theta_\epsilon}{\mathrm{d}x_1}\Big\|^2_{L^2(Q)} \\ &= C\lambda^2\epsilon^{2-\alpha}\int_0^1 \Big|\frac{\mathrm{d}\theta_\epsilon}{\mathrm{d}x_1}\Big|^2 \mathrm{d}x_1 \\ &\le C(\theta_2 - \theta_1)^2\epsilon^{2-\alpha-\gamma}. \end{aligned} \tag{3.33}$$

Therefore, we see that for every $0 \le \alpha < 2$ we find a γ small enough such that the elastic energy of the corresponding sequence of deformations u_ϵ converges towards zero.

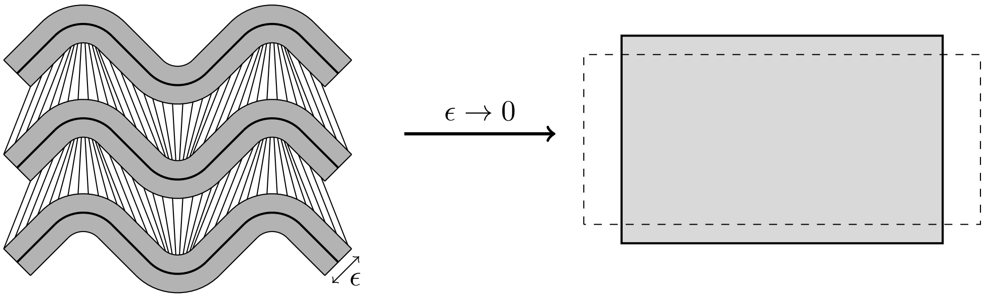

Figure 3.5: A typical effect occurring in the context of bending insufficient stiff plates or rods is wrinkling: Small oscillations leading to an asymptotic decrease of length. Accordingly, small oscillations of the stiff layers indeed lead to a loss of volume in the limit.

Macroscopic material behavior. Furthermore, Lemma 3.4.3 yields $\nabla u_\epsilon \rightharpoonup \nabla u = F$ in $L^1(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$. Since $\det \nabla u = \det F = \beta \bar{g}_1'$ we see that the volume is in general not preserved for $\beta = 1$. Yet, the localvolume constraint can be met for suitable values of β for constructions featuring constant $\bar{g}_1'$. One is given by the adaption of the explicit construction by Friesecke, James and Müller for the wrinkling of plates given above.

Lastly, we give an example which realizes in the regime $\alpha > q$ only the weaker restrictions on limit gradients yet does not satisfy the restrictions related to the volume preserving condition.

Example 3.4.7 (Deformation with non-constant rotation).
Construction. The goal of this example is to construct a macroscopic deformation u with a gradient of the form $\nabla u = R(\mathbb{I} + a \otimes e_2)$, where $R \in L^\infty(\Omega; SO(2))$ and $a \in L^2(\Omega;\mathbb{R}^2)$ featuring in particular a non-constant rotation, i.e. $\partial_2 R \neq 0$. This can be achieved by rotating each rigid layer by a subsequently increasing rotation angle, see Figure 3.6. Accordingly, we specify $f_\epsilon = f$, $\epsilon \in (0,1)$ to be given by

$$f(x) = (x_1 - \tfrac{1}{2})R(x_2)e_1 + \tfrac{1}{2}e_1 + x_2 e_2 \quad x \in Q.$$

Energies. Then, as $\partial_{11} f = 0$, the corresponding u_ϵ satisfies for all $1 \leq q < \infty$ and $\epsilon \in (0,1)$

$$\int_{\epsilon P_{\text{stiff}} \cap \Omega} \operatorname{dist}^q(\nabla u_\epsilon, SO(2))\, \mathrm{d}x = 0.$$

Macroscopic material behavior. By Lemma 3.4.1, we obtain $\nabla u_\epsilon \rightharpoonup \nabla u = F$ in $L^1(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$ with

$$\begin{aligned} \nabla u = F &= R(x_2)e_1 \otimes e_1 + (x_1 - \tfrac{1}{2})R'(x_2)e_1 \otimes e_2 + e_2 \otimes e_2 \\ &= R(x_2)(\mathbb{I} + (-e_2 + (x_1 - \tfrac{1}{2})R^T(x_2)R'(x_2)e_1 + R^T(x_2)e_2) \otimes e_2) \quad x \in Q. \end{aligned}$$

Thus, indeed, ∇u is of the form suggested by the asymptotic characterization Theorem 3.3.1 for $\alpha > q$, yet with the volume preserving condition not satisfied by the limit deformation u, Corollary 3.3.3 does not apply.

At the end of this subsection, we turn to the case of ϵ-dependent layer ratio λ_ϵ. The next example illustrates that for the asymptotic characterization of Theorem 3.3.1 to hold in the context of decreasing volume of the stiff layers, the stiffness of the layers has to increase. We restrict ourselves to the case $p = q = 2$.

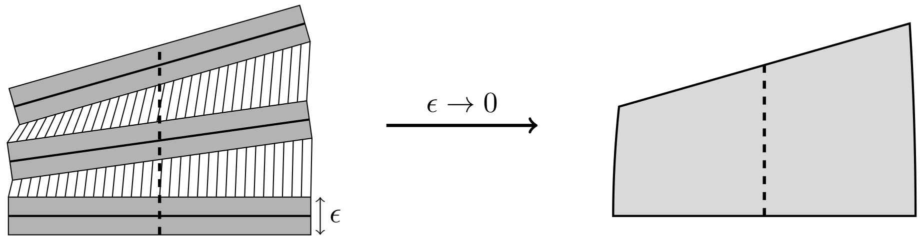

Figure 3.6: We have seen with the asymptotic rigidity result of Corollary 3.3.3 that if we assume in two dimensions that the limit locally preserves volume the only possible macroscopic deformations are globally rotated shear deformations. Here, want to illustrate the distinction between the characterizations of Corollary 3.3.3 and Theorem 3.3.1 by giving an example of a sequence of deformations satisfying the exact differential inclusion constraint, whose limit is not a globally rotated shear deformation. This is obtained by rotating the stiff layers by a subsequently increasing rotation angle as depicted.

Example 3.4.8 (Decreasing volume of stiff component)**.** Consider an affine deformation $u_\epsilon = u \in W^{1,2}(\Omega, \mathbb{R}^n)$ determined by $\nabla u = F$ for a matrix $F \in \mathbb{R}^{n\times n}$. Then, if $(1-\lambda_\epsilon) \leq \epsilon^\beta$ for $\beta > 0$ we have

$$\begin{aligned}\int_{\epsilon P_{\text{stiff}}\cap\Omega} \operatorname{dist}^2(\nabla u_\epsilon, SO(n))\,\mathrm{d}x &= |\epsilon P_{\text{stiff}} \cap \Omega|\,|\operatorname{dist}^2(F, SO(n))| \\ &\leq C(1-\lambda_\epsilon)|\operatorname{dist}^2(F, SO(n))| \leq C\epsilon^\beta,\end{aligned}$$

and obviously, $(u_\epsilon)_\epsilon$ converges to the affine limit, which for general $F \in \mathbb{R}^{n\times n}$ does not comply with the characterization by Theorem 3.3.1. In particular, we see that although the elastic energy on the stiff layers does not decrease relative to the volume, it does in absolute terms.

This illustrates, that for Theorem 3.3.1 to hold in the context of decreasing volume of the stiff layers, their stiffness has to increase in relation to the volume decrease. This can be seen from r_ϵ in (3.11) requiring here that $\alpha > 0$ satisfies

$$r_\epsilon = \epsilon^{\frac{\alpha-2}{2}}(1-\lambda_\epsilon)^{-1} = \epsilon^{\frac{\alpha-2-2\beta}{2}} \to 0,$$

which would be satisfied for $\alpha > 2\beta + 2$.

3.5 Appendix

3.5.1 Quantitative rigidity in L^p and its scaling behavior on thin domains

Definition 3.5.1 ((ℓ, L)-Lipschitz equivalent domains)**.** Let $n \in \mathbb{N}$. For $\ell, L \in (0, \infty)$ two bounded Lipschitz domains $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^n$ are called (ℓ, L)-Lipschitz equivalent if there is a Bilipschitz map $\varphi : \Omega_1 \to \Omega_2$, i.e. a bijective Lipschitz map φ whose operator norm $|||\nabla\varphi||| = \|\nabla\varphi\|_{\mathcal{L}(\Omega_1;\Omega_2)}$ satisfies

$$|||\nabla\varphi||| \leq L \quad \text{and} \quad |||\nabla\varphi^{-1}||| \leq \ell.$$

As our interest lies in applications on thin domains representing the stiff layers we focus on the scaling behavior of the constant in (3.10). To that end, the next Theorem is a version of

Theorem 3.2.4 in L^p for $p \geq 2$ concerning (ℓ, L)-Lipschitz equivalent domains. For completion, we give the proof of the theorem along the original arguments from [72] and [73], incorporating the adjustments by Conti for the L^p-case as presented in [46].

Theorem 3.5.2 (Quantitative rigidity estimate for (ℓ, L)-Lipschitz equivalent domains)**.** *Let $n \in \mathbb{N}$, $n \geq 2$, $1 < p < \infty$ and $\ell, L \in (0, \infty)$. For an index set I let $(\Omega_k)_{k \in I}$ be a family of bounded Lipschitz domains $\Omega_k \subset \mathbb{R}^n$ that are (ℓ, L)-Lipschitz equivalent. Then, there exists a constant $C > 0$ depending on p, n, ℓ, L and $(\Omega_k)_{k \in I}$ but not each specific $k \in I$, such that for all $k \in I$ there is for each $u \in W^{1,p}(\Omega_0; \mathbb{R}^n)$ an associated rotation $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^p(\Omega_k; \mathbb{R}^{n \times n})} \leq C \int_{\Omega_k} \operatorname{dist}^p(\nabla u, SO(n)) \, \mathrm{d}x. \tag{3.34}$$

We first show a preliminary result on a cube in the case $p = 2$.

Proposition 3.5.3 ([72, Proposition 3.4])**.** *Let $Q'' = (-1, 1)^n$, $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ and $1 < p < \infty$. Then, there is a constant $C > 0$ only depending on n such that for each $u \in W^{1,2}(Q''; \mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^2(Q; \mathbb{R}^{n \times n})} \leq C \|\operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q'')}.$$

Furthermore, this estimate is invariant under uniform scaling and translation.

Proof. Step 1: Approximation. In this first step we show that we may assume for a constant $M > 0$ that $\|\nabla u\|_{L^\infty(Q'')} \leq M$. Firstly, observe that for matrices of large norm, we can estimate the norm by its distance to $SO(n)$. More precisely, it holds for each matrix $A \in \mathbb{R}^{n \times n}$ with $|A| \geq 2\sqrt{n}$ that

$$\tfrac{1}{2}|A| \leq |A| - \sqrt{n} \leq \inf_{R \in SO(n)} \big||A| - |R|\big| \leq \operatorname{dist}(A, SO(n)). \tag{3.35}$$

By Lemma 3.5.9 for $\mu = 4\sqrt{n}$ there is a function $v \in W^{1,\infty}(Q''; \mathbb{R}^n)$ satisfying for a constant $C > 0$

$$\|\nabla v\|_{L^\infty(Q''; \mathbb{R}^{n \times n})} \leq 4\sqrt{C} =: M$$

and

$$\begin{aligned}\|\nabla u - \nabla v\|^2_{L^2(Q''; \mathbb{R}^{n \times n})} &\leq C \int_{\{|\nabla u| > 2\sqrt{n}\}} |\nabla u|^2 \, \mathrm{d}x \\ &\leq 4C \int_{Q''} \operatorname{dist}^2(\nabla u, SO(n)) \, \mathrm{d}x.\end{aligned}$$

Thus, if the theorem holds for v, we obtain for u by the triangle inequality applied twice

$$\begin{aligned}\|\nabla u - R\|_{L^2(Q; \mathbb{R}^{n \times n})} &\leq \|\nabla u - \nabla v\|_{L^2(Q''; \mathbb{R}^{n \times n})} + \|\nabla v - R\|_{L^2(Q''; \mathbb{R}^{n \times n})} \\ &\leq C \|\operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q''; \mathbb{R}^{n \times n})} + C \|\operatorname{dist}(\nabla v, SO(n))\|_{L^2(Q''; \mathbb{R}^{n \times n})} \\ &\leq C \|\operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q''; \mathbb{R}^{n \times n})}.\end{aligned}$$

Step 2: Decomposition. The goal is now to decompose u in a harmonic function w and a controllable rest $z \in W^{1,p}(Q''; \mathbb{R}^n)$ that vanishes on the boundary of Q''. By the fact that $\operatorname{div} \operatorname{cof} \nabla u = 0$, [68, Section 8.1.4], it holds that

$$-\Delta u = \operatorname{div}(\operatorname{cof} \nabla u - \nabla u). \tag{3.36}$$

The function $A \mapsto |A - \operatorname{cof} A|^2$ is smooth, non-negative and vanishes on $SO(n)$, cf. also Proof of Theorem 3.2.1. Thus, its derivative is bounded on the compact set $\{A \in \mathbb{R}^{n\times n} \mid |A| \le M\}$, which yields

$$|A - \operatorname{cof} A|^2 \le C \operatorname{dist}^2(A, SO(n)) \quad \text{for all } A \in \mathbb{R}^{n\times n} \text{ with } |A| \le M.$$

This motivates to choose z as the unique solution of

$$-\Delta z = \operatorname{div}(\operatorname{cof} \nabla u - \nabla u) \quad \text{and} \quad z = 0 \text{ on } \partial Q''.$$

Testing this equation with z and applying an absorption argument, we obtain the estimate

$$\|\nabla z\|_{L^2(Q'';\mathbb{R}^{n\times n})} \le C\| \operatorname{cof} \nabla u - \nabla u\|_{L^2(Q'';\mathbb{R}^{n\times n})} \le C\| \operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q'')}. \tag{3.37}$$

Consequently, we set $w := u - z$, which implies that $\Delta w = 0$ on Q'', so w is harmonic as desired.

Step 3: L^∞-estimate for the harmonic part. In this step we set $Q' = (-\frac{3}{4}, \frac{3}{4})^n$, such that $Q \subset Q' \subset Q''$, and show the existence of a constant $C > 0$ such that there is a rotation $R \in SO(n)$ associated to w satisfying

$$\|\nabla w - R\|_{L^2(Q;\mathbb{R}^{n\times n})} \le C\| \operatorname{dist}(\nabla w, SO(n))\|_{L^2(Q'';\mathbb{R}^{n\times n})}.$$

For a cutoff function $\eta \in C_0^\infty(Q'')$ with $\eta \ge 0$ and $\eta = 1$ on Q' we derive from (3.6) and the fact that $\|\nabla u\|_{L^\infty(Q'';\mathbb{R}^{n\times n})} \le M$ the estimate

$$\begin{aligned}\int_{Q''} |\nabla^2 w|^2 \eta \,\mathrm{d}x &\le \sup_{Q''}(\Delta\eta) \int_{Q''} ||\nabla w|^2 - n| \,\mathrm{d}x \\ &\le C\Big(\int_{Q''} ||\nabla u|^2 - n| \,\mathrm{d}x + 2\int_{Q''} |\nabla u||\nabla z| \,\mathrm{d}x + \int_{Q''} |\nabla z|^2 \,\mathrm{d}x\Big) \\ &\le C\| \operatorname{dist}(\nabla u, SO(n))\|_{L^1(Q'')} + C\|\nabla z\|_{L^1(Q'';\mathbb{R}^{n\times n})} + C\|\nabla z\|^2_{L^2(Q'';\mathbb{R}^{n\times n})}.\end{aligned}$$

Now, for the first and the second term on the right hand side, we apply Hölder's inequality to obtain an estimate in terms of L^2 and then use in the case of the second term (3.37). For the third term we utilize the square of (3.37) and then deduce from the L^∞-bound on ∇u for the estimate

$$\| \operatorname{dist}(\nabla u, SO(n))\|^2_{L^2(Q'')} \le C\| \operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q'')}. \tag{3.38}$$

Overall, it holds that

$$\int_{Q'} |\nabla^2 w|^2 \,\mathrm{d}x \le C\| \operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q'')}.$$

Now, w and therefore $\nabla^2 w$ is harmonic in Q''. Thus, the mean value theorem yields for $r = \operatorname{dist}(Q', \partial Q)$

$$\sup_{x\in Q} |\nabla^2 w|^2 \le \sup_{x\in Q} \Big| \frac{1}{|B(x,r)|} \int_{B(x,r)} \nabla^2 w(x) \,\mathrm{d}x\Big|^2 \le C\| \operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q'')},$$

and the Poincaré inequality entails the existence of an $R \in \mathbb{R}^{n\times n}$ satisfying

$$\|\nabla w - R\|_{L^\infty(Q;\mathbb{R}^{n\times n})} \le C\| \operatorname{dist}(\nabla u, SO(n))\|^{\frac{1}{2}}_{L^2(Q'')}. \tag{3.39}$$

Furthermore, by (3.37), we obtain by the triangle inequality

$$\int_{Q''} \operatorname{dist}^2(\nabla w, SO(n))\,\mathrm{d}x \le 2\int_{Q''} \operatorname{dist}^2(\nabla v, SO(n)) + |\nabla z|^2\,\mathrm{d}x \\ \le C\|\operatorname{dist}(\nabla u, SO(n))\|^2_{L^2(Q'';\mathbb{R}^{n\times n})},$$

and analogously,

$$|Q''|\operatorname{dist}^2(R, SO(n)) = \int_{Q''} \operatorname{dist}^2(R, SO(n))\,\mathrm{d}x \\ \le 2\int_{Q''} \operatorname{dist}^2(\nabla w, SO(n)) + |\nabla w - R|^2\,\mathrm{d}x \\ \le C\|\operatorname{dist}(\nabla u, SO(n))\|^2_{L^2(Q'';\mathbb{R}^{n\times n})}.$$

Thus, again by (3.38), it holds that

$$\operatorname{dist}(R, SO(n)) \le C\|\operatorname{dist}(\nabla u, SO(n))\|^{\frac{1}{2}}_{L^2(Q'';\mathbb{R}^{n\times n})}, \tag{3.40}$$

and we may assume in (3.39) that $R \in SO(n)$.

Step 4: Linearizing $\operatorname{dist}(\,\cdot\,, SO(n))$. We may assume $R = \mathbb{I}$, arguing for $R^T v$ and $R^T w$ instead of v and w, otherwise. To linearize $\operatorname{dist}(\,\cdot\,, SO(n))$ near the identity, we derive by the Taylor expansion [81, Chapter 9, Section 29], [66, Section 6.1.9]

$$\operatorname{dist}(A, SO(n)) = |\tfrac{1}{2}(A + A^T) - \mathbb{I}| + \mathcal{O}(|A - \mathbb{I}|^2), \quad A \in \mathbb{R}^{n\times n}.$$

Applying this estimate to ∇w, together with (3.39), yields

$$\|\tfrac{1}{2}(\nabla w + (\nabla w)^T)\|_{L^2(Q;\mathbb{R}^{n\times n})} \le \|\operatorname{dist}(\nabla w, SO(n))\|_{L^2(Q)} + \|\nabla w - \mathbb{I}\|_{L^2(Q;\mathbb{R}^{n\times n})} \\ \le \|\operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q'')}.$$

Now we use Korn's inequality, which states that for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and $1 < p < \infty$ that there is a constant $C > 0$ such that for each $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ there exists an associated skew-symmetric matrix $S \in \mathbb{R}^{n\times n}$, $S = -S^T$ such that

$$\|\nabla u - S\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \le C\|\nabla u + (\nabla u)^T\|_{L^p(\Omega;\mathbb{R}^{n\times n})}.$$

This mathematically well-studied estimate has mostly been considered for $p = 2$, in which case also the dependence of the constant on the geometry of the domain is known, see [100], and the references therein. For the general case $1 < p < \infty$, see for example [74].

We apply Korn's inequality to the displacement $v(x) = w(x) - x$, $x \in Q''$, which shows that there is an antisymmetric matrix $S \in \mathbb{R}^{n\times n}$ such that

$$\|\nabla w - \mathbb{I} - S\|_{L^2(Q;\mathbb{R}^{n\times n})} \le C\|\tfrac{1}{2}(\nabla w + (\nabla w)^T) - \mathbb{I}\|_{L^2(Q;\mathbb{R}^{n\times n})} \\ \le C\|\operatorname{dist}(\nabla w, SO(n))\|_{L^2(Q)} + c\|\nabla w - \mathbb{I}\|_{L^2(Q;\mathbb{R}^{n\times n})} \\ \le C\|\operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q'')}.$$

Arguing as in (3.40), we may choose $\mathbb{I} - S \in SO(n)$.

This, together with (3.37) yields the estimate for $Q'' = (-1,1)^n$. The fact that the constant is invariant under uniform scaling and translation follows by the same arguments as in Remark 3.2.5 b). □

Proof of Theorem 3.5.2. As in the Proof of Proposition 3.5.3, we start by approximating $u \in W^{1,p}(\Omega_k;\mathbb{R}^n)$, $k \in I$ by a Lipschitz function $v \in W^{1,\infty}(\Omega_k;\mathbb{R}^n)$. More precisely, we apply Lemma 3.5.9 for $\mu = 4\sqrt{n}$ which yields the existence of a function $v \in W^{1,\infty}(\Omega_k;\mathbb{R}^n)$ and a constant $M > 0$ such that $\|\nabla v\|_{L^\infty(\Omega_k;\mathbb{R}^{n\times n})} \leq M$. Notice that since all Ω_k are (ℓ, L)-Lipschitz equivalent, we may choose M independent of the specific $k \in I$.

Similar to Step 2 of the Proof of 3.5.3 we aim to decompose u additively in a harmonic part w and a rest z. Yet, for the particular definition of the decomposition we differ from the one given there, since, as Conti and Schweizer point out in [55], for general Lipschitz domains L^p-estimates on solutions of (3.36) may not be available, see [89]. Yet, for the proof of this theorem, the specific boundary values are not important, so we may follow Conti and Schweizer in setting $z = \operatorname{div}\psi$, where $\psi : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfies the equation

$$-\Delta\psi = \begin{cases} \operatorname{cof}\nabla u - \nabla u & \text{on } Q'', \\ 0 & \text{on } \mathbb{R}^n \setminus Q'', \end{cases}$$

componentwise, with zero boundary data at infinity [55]. Notice that by the L^∞-bound on ∇u, the right hand side of this equation is in L^∞. Thus, by well-known singular integral estimates on $\mathbb{R}^n$, see [1, Section 4], namely L^p-estimates on the Riesz-operator [1, Example 4.17] via the Mikhlin Multiplier Theorem [1, Theorems 4.23, 4.28] we have for a constant $C > 0$ and all $1 < q < \infty$

$$\begin{aligned}\|\nabla z\|_{L^q(Q'';\mathbb{R}^{n\times n})} \leq \|\nabla^2\psi\|_{L^q(Q'';\mathbb{R}^{n\times n})} &\leq C\|\operatorname{cof}\nabla u - \nabla u\|_{L^q(Q'';\mathbb{R}^{n\times n})} \\ &\leq \|\operatorname{dist}(\nabla u, SO(n))\|_{L^q(Q'';\mathbb{R}^{n\times n})}, \end{aligned} \tag{3.41}$$

in particular for $q = p$ and $q = 2$, where we have used in the last estimate similar arguments as in (3.37).

To estimate the harmonic part w, we argue by exhaustion of P_δ by cubes of the form $Q(a,r) = a + r\big(-\frac{1}{2},\frac{1}{2}\big)^n$ with side length $r > 0$ centered at $a \in \mathbb{R}^n$. More precisely, we choose a covering $(Q_i)_{i\in\mathbb{N}}$, $Q_i = Q(a_i, r_i)$ with

$$4r_i \leq \operatorname{dist}(a_i, \partial\Omega_k) \leq Cr_i, \tag{3.42}$$

such that for a fixed natural number $N \in \mathbb{N}$ each point $x \in P_\delta$ lies in at most N cubes Q_i.

Now, since w is harmonic, we obtain by the mean value property estimate on the derivatives, [68, Chapter 2, Theorem 7]

$$r_i^{n+2}\|\nabla^2 w\|^2_{L^\infty(Q(a_i,r_i))} \leq C(n)\int_{Q(a_i,2r_i)} |\nabla w - R_i|^2 \,\mathrm{d}x.$$

Furthermore, since $w \in W^{1,2}(\Omega_k;\mathbb{R}^n)$, Proposition 3.5.3 together with (3.41) for $q = 2$ entails

$$\begin{aligned} r_i^{n+2}\|\nabla^2 w\|^2_{L^\infty(Q(a_i,r_i))} &\leq C\int_{Q(a_i,2r_i)} |\nabla w - R_i|^2 \,\mathrm{d}x \\ &\leq C\|\operatorname{dist}(\nabla w, SO(n))\|^2_{L^2(Q(a_i,4r_i))} \\ &\leq C\|\operatorname{dist}(\nabla u, SO(n))\|_{L^2(Q(a_i,4r_i))}, \end{aligned} \tag{3.43}$$

where we have again used the L^∞-bound on ∇u.

Now, we distinguish two cases. First, let us assume that $2 \le p < \infty$. Then, taking (3.43) to the $p/2$-th power and applying Jensen's inequality, we obtain

$$\begin{aligned} r_i^{p+\frac{pn}{2}} \|\nabla^2 w\|^p_{L^\infty(Q(a_i,r_i))} &\le C\| \operatorname{dist}(\nabla u, SO(n))\|^p_{L^2(Q(a_i,4r_i))} \\ &\le C r_i^{\frac{pn}{2}-n} \| \operatorname{dist}(\nabla u, SO(n))\|^p_{L^p(Q(a_i,4r_i))}. \end{aligned}$$

Hence, we obtain

$$|Q(a_i,r_i)| r_i^p \|\nabla^2 w\|^p_{L^\infty(Q(a_i,r_i))} \le C\| \operatorname{dist}(\nabla u, SO(n))\|^p_{L^p(Q(a_i,4r_i))}.$$

Using the relation between r_i and the distance of points in a cube to the boundary $\partial\Omega_k$ from (3.42) we have

$$\int_{Q(a_i,r_i)} \operatorname{dist}^p(x,\partial\Omega_k)|\nabla^2 w|^p \,\mathrm{d}x \le C\| \operatorname{dist}(\nabla u, SO(n))\|^p_{L^p(Q(a_i,4r_i))}.$$

By summation over i, keeping in mind that at each point at most N cubes of the exhaustion overlap, we arrive at

$$\int_{\Omega_k} \operatorname{dist}^p(x,\partial\Omega_k)|\nabla^2 w|^p \,\mathrm{d}x \le C\| \operatorname{dist}(\nabla u, SO(n))\|^p_{L^p(\Omega_k)}. \tag{3.44}$$

In the case that $1 < p < 2$, note that we may apply Hölder's inequality and the L^∞-bound on ∇u, to derive the estime directly from the just proven estimate (3.44) in the case $p = 2$. Indeed,

$$\begin{aligned} \int_{\Omega_k} \operatorname{dist}^p(x,\partial\Omega_k)|\nabla^2 w|^p \,\mathrm{d}x &\le \int_{\Omega_k} \operatorname{dist}^2(x,\partial\Omega_k)|\nabla^2 w|^2 \,\mathrm{d}x \\ &\le C\| \operatorname{dist}(\nabla u, SO(n))\|^2_{L^2(\Omega_k)} \le C\| \operatorname{dist}(\nabla u, SO(n))\|^p_{L^p(\Omega_k)}. \end{aligned}$$

Hence, (3.44) holds for all $1 < p < \infty$.

Lastly, we apply the following weighted Poincaré inequality

$$\min_{G\in\mathbb{R}^{n\times n}} \int_{\Omega_k} |f-G|^q \,\mathrm{d}x \le C \int_{\Omega_k} \operatorname{dist}^q(x,\partial\Omega_k)|\nabla f|^q \,\mathrm{d}x,$$

which has been established in [72, Equation (3.29)] by combining an embedding theorem from the theory of weighted Sobolev spaces [96] and a standard Poincaré inequality. We have restated the result as Lemma 3.5.4 in which we determined the particular scaling behavior of the Poincaré constant.

Utilizing this estimate for $q = p$ and $f = \nabla w$, we obtain for a matrix $G \in \mathbb{R}^{n\times n}$

$$\|\nabla w - G\|_{L^p(\Omega_k)} \le C\Big(\int_{\Omega_k} \operatorname{dist}^p(x,\Omega_k)|\nabla^2 w|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \le C\| \operatorname{dist}(\nabla v, SO(n))\|_{L^p(\Omega_k)}.$$

□

Lemma 3.5.4. *Let $1 < p < \infty$ and $\ell, L \in (0,\infty)$. Let Ω and Ω_0 be bounded Lipschitz domains such that Ω is (ℓ, L)-Lipschitz equivalent to Ω_0. Suppose that there is a constant $C > 0$ only depending on n, p and Ω_0, such that for all $f \in W^{1,p}(\Omega_0,\mathbb{R}^n)$ there is an associated $G \in \mathbb{R}^n$ such that*

$$\int_{\Omega_0} |f-G|^p \,\mathrm{d}x \le C \int_{\Omega_0} \operatorname{dist}^p(x,\partial\Omega_0)|\nabla f|^p \,\mathrm{d}x. \tag{3.45}$$

Then, it holds for all $f \in W^{1,p}(\Omega,\mathbb{R}^n)$ that

$$\int_{\Omega} |f-G|^p \,\mathrm{d}x \le C n^{\frac{p}{2}} \ell^{2p} L^{2p} \int_{\Omega} \operatorname{dist}^p(x,\partial\Omega)|\nabla f|^p \,\mathrm{d}x.$$

Proof. The key lies in understanding the transformation behavior of the distance function. Denote by $\varphi : \Omega_0 \to \Omega$ the bijective Bilipschitz map whose gradient satisfies

$$|||\nabla\varphi^{-1}||| \leq \ell \quad \text{and} \quad |||\nabla\varphi||| \leq L,$$

that exists by the definition of (ℓ, L)-Lipschitz equivalence.

Then, it holds for $x \in \Omega$ that

$$L^{-1}\operatorname{dist}(x, \partial\Omega) \leq \operatorname{dist}(\varphi^{-1}(x), \partial\Omega_0) \leq \ell \operatorname{dist}(x, \partial\Omega). \tag{3.46}$$

since for $x, y \in \Omega_0$ we have $|\varphi(y) - \varphi(x)| \leq L|y - x|$ and $|y - x| \leq \ell|\varphi(y) - \varphi(x)|$. Also notice that, if the Lipschitz bounds are sharp for points on the boundary, i.e. for $x \in \partial\Omega_0$ or $y \in \partial\Omega_0$, then the estimates in (3.46) are sharp.

Hence, by (3.45) there is a $G \in \mathbb{R}^n$ associated to $f \circ \varphi$, for which the change of variables formula, (3.46) and the estimate $|\nabla\varphi| \leq \sqrt{n}|||\nabla\varphi|||$ between the operator norm and the Frobenius norm that also provides an estimate on $\det(\nabla\varphi)$ yields

$$\begin{aligned}
\int_\Omega |f - G|^p \,\mathrm{d}x &= \int_{\Omega_0} |f \circ \varphi - G|^p \det(\nabla\varphi) \,\mathrm{d}x \\
&\leq CL^p \int_{\Omega_0} |f \circ \varphi - G|^p \,\mathrm{d}x \\
&\leq CL^p \int_{\Omega_0} \operatorname{dist}^p(x, \partial\Omega_0)|(\nabla f) \circ \varphi|^p |\nabla\varphi|^p \,\mathrm{d}x \\
&\leq n^{\frac{p}{2}} L^{2p} \ell^p C \int_{\varphi(\Omega_0)} \operatorname{dist}^p(\varphi^{-1}(x), \partial\Omega_0)|\nabla f|^p \,\mathrm{d}x \\
&\leq n^{\frac{p}{2}} \ell^{2p} L^{2p} C \int_\Omega \operatorname{dist}^p(x, \partial\Omega)|\nabla f|^p \,\mathrm{d}x.
\end{aligned}$$

This completes the proof. □

Remark 3.5.5 (Scaling behavior of the harmonic part w)**.** Notice that the Proof of Theorem 3.5.2 provides an estimate of the form (3.34) for the harmonic part w of u. But while the behavior of the constant in this estimate under anisotropic scaling can be determined by Lemma 3.5.4, this does not determine the overall scaling behavior of the constant in (3.34) for the whole function u. The reason lies in the first step of the proof, where u function is approximated by a Lipschitz function whose Lipschitz constant is not invariant under anisotropic scaling. As we do not know at this point in the proof, that in fact, ∇u is not only close to a Lipschitz function, but on cubes close to constant rotations, the scaling behavior of the estimate can be improved using this a forterior information.

Corollary 3.5.6 (Approximation by rotations on thin domains [73, Theorem 6])**.** *For a Lipschitz domain $O \subset \mathbb{R}^{n-1}$, $h \in \mathbb{R}$ and $\delta \in (0, 1)$ let $P_\delta = O \times (h, h + \delta)$ and $1 < p < \infty$. Then, there exists a constant $C > 0$, depending on n, p, O but not on δ such that for all $u \in W^{1,p}(P_\delta; \mathbb{R}^n)$, $1 < p < \infty$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^p(P_\delta; \mathbb{R}^{n\times n})} \leq C\delta^{-p} \int_{P_\delta} \operatorname{dist}^p\left(\nabla u, SO(n)\right) \mathrm{d}x. \tag{3.47}$$

Proof. The proof follows closely the arguments by Friesecke, James and Müller given in [73, Theorem 6], only adapted for general dimension $n \in \mathbb{N}$, $n \geq 2$ and $1 < p < \infty$.

The strategy is to use the knowledge for uniform scaling of domains obtained in Remark 3.2.5 b) to construct a function of higher regularity close to ∇u whose gradient can be estimated with the right scaling, and then to apply the Poincaré inequality to estimate the distance of ∇u from a fixed rotation.

Firstly, since O is a Lipschitz domain, there is a covering $(U_k)_{k\in\{1,\dots,N\}}$, $N \in \mathbb{N}$ with the property, that after rotating the coordinates if necessary, there are bounded open (generalized) intervals $J_k \subset \mathbb{R}^{n-1}$ and Lipschitz maps $L_k : J_k \to \mathbb{R}$ such that

$$\begin{aligned} U_k \cap O &= \{x = (x', x_n) \in U_k \mid x' \in J_k, x_n > L(x')\}, \\ U_k \cap \partial O &= \{x = (x', x_n) \in U_k \mid x' \in J_k, x_n = L(x')\}. \end{aligned} \tag{3.48}$$

In the next two steps, we argue for fixed $k \in \{1, \dots, N\}$ returning to the global perspective in Step 3. To shorten the notation, we drop the index in the next two steps and write U instead of U_k.

Step 1: Estimates far from the boundary. Let $K \subset U$ be a compact subset and assume $0 < \delta < \frac{1}{3} \inf_{x\in\partial U, y\in K} |y - x|_\infty$. We denote by $\overline{\nabla u}$ the vertical average given by

$$\overline{\nabla u}(x') = \fint_{(h,h+\delta)} \nabla u(x', x_n)\,\mathrm{d}x_n.$$

For $x' \in \mathbb{R}^{n-1}$ we denote by $Q_{x',\delta}$ the cube with side length δ and lower left corner x', i.e.

$$Q_{x',\delta} = x' + (0, \delta)^{n-1}.$$

Using a non-negative standard mollifier $\psi \in C_0^\infty((0,1)^{n-1})$, by which we mean a function of compact support in the unit cube, with $\psi \geq 0$ and $\int_{(0,1)^{n-1}} \psi\,\mathrm{d}x = 1$ and the notation $\psi_\delta = \delta^{-n+1}\psi(\delta^{-1}\,\cdot)$ we introduce the smoothed vertical average $R : K \to \mathbb{R}^n$ by setting

$$R(x') = (\psi_h * \overline{\nabla u})(x') = \int_{Q_{x',\delta}\times(h,h+\delta)} \delta^{-n+1}\psi\Big(\frac{x'-z'}{\delta}\Big)\nabla u(z)\,\mathrm{d}z'\,\mathrm{d}z_n.$$

Now, let us study the properties of R. By Theorem 3.5.2 there is a constant $C > 0$ and rotations $R_{x',\delta} \in SO(n)$ such that on each cube $Q_{x',\delta}$ we have

$$\int_{Q_{x',\delta}\times(h,h+\delta)} |\nabla u(z) - R_{x',\delta}|^p\,\mathrm{d}z \leq C \int_{Q_{x',\delta}\times(h,h+\delta)} \operatorname{dist}^p(\nabla u, SO(n))\,\mathrm{d}z.$$

Since $\int_{(0,\delta)^{n-1}} \psi_\delta = 1$ we can apply Jensen's inequality to the convex function $x \mapsto |x|^p$ with respect to the probability measure $\psi_\delta \cdot \lambda^{n-1}$, where λ^k for $k \in \mathbb{N}$ denotes the k-dimensional Lebesgue measure which together with the L^∞-bound on ψ yields

$$\begin{aligned} |R(x') - R_{x',h}|^p &\leq \frac{C}{\delta^{n-1}} \int_{Q_{x',\delta}\times(h,h+\delta)} |\nabla u(z) - R_{x',\delta}|^p\,\mathrm{d}z \\ &\leq \frac{C}{\delta^{n-1}} \int_{Q_{x',\delta}\times(h,h+\delta)} \operatorname{dist}^p(\nabla u, SO(n))\,\mathrm{d}z. \end{aligned}$$

For the gradient of R, we argue that since $\int_{(0,\delta)^{n-1}} \nabla\psi_\delta\,\mathrm{d}x = 0$ due to the compact support of ψ, it holds for $\xi' \in Q_{x',\delta}$ that

$$\begin{aligned} |\nabla R(\xi')|^p &\leq \Big| \int_{Q_{\xi',\delta}\times(h,h+\delta)} \delta^{-n}\nabla\psi\Big(\frac{\xi'-z'}{h}\Big) \cdot (1,\dots,1)^T \nabla u(z)\,\mathrm{d}z'\,\mathrm{d}z_n\Big|^p \\ &= \Big|\delta^{-n+1} \int_{Q_{\xi',\delta}\times(h,h+\delta)} \delta^{-1}\nabla\psi\Big(\frac{\xi'-z'}{h}\Big) \cdot (1,\dots,1)^T (\nabla u(z) - R_{x',\delta})\,\mathrm{d}z'\,\mathrm{d}z_n\Big|^p \\ &\leq \delta^{-n+1} \int_{Q_{\xi',\delta}\times(h,h+\delta)} \delta^{-p}\Big|\nabla\psi\Big(\frac{\xi'-z'}{h}\Big) \cdot (1,\dots,1)^T\Big|^p |\nabla u(z) - R_{x',\delta}|^p\,\mathrm{d}z'\,\mathrm{d}z_n \\ &\leq \delta^{-np+1} C \int_{Q_{\xi',\delta}\times(h,h+\delta)} |\nabla u(z) - R_{x',\delta}|^p\,\mathrm{d}z. \end{aligned}$$

Hence,

$$\begin{aligned}|\nabla R(\xi')|^p &\leq \delta^{-np+1} C \int_{Q_{\xi',\delta}\times(h,h+\delta)} |\nabla u(z) - R_{x',\delta}|^p \,\mathrm{d}z \\ &\leq \delta^{-np+1} C \int_{Q_{x',2\delta}\times(h,h+\delta)} \operatorname{dist}^p(\nabla u, SO(n)) \,\mathrm{d}z,\end{aligned}$$

and thus by the mean value theorem, there is for $\xi' \in Q_{x',h}$ a point $y \in (\xi', x')$ such that

$$|R(\xi') - R(x')|^p \leq \delta^p |\nabla u(y) \cdot (\xi' - x')|^p \leq \frac{C}{\delta^{n-1}} \int_{Q_{x',2\delta}} \operatorname{dist}^p(\nabla u, SO(n)) \,\mathrm{d}z.$$

This implies

$$\int_{Q_{x',2\delta}} |\nabla u(z) - R(z')|^p \,\mathrm{d}z \leq C \int_{Q_{x',2\delta}\times(h,h+\delta)} \operatorname{dist}^p(\nabla u, SO(n)) \,\mathrm{d}z.$$

Lastly, we estimate the distance of R to $SO(n)$ itself. We set $g(\zeta) = \operatorname{dist}\left(R(x' + \delta\zeta), SO(n)\right)$ for $\zeta \in \mathbb{R}^{n-1}$. Then from the above estimates we obtain

$$\int_{(0,1)^{n-1}} |g|^p \,\mathrm{d}\zeta + \sup_{\zeta\in(0,1)^{n-1}} |\nabla g|^p(\zeta) \leq \frac{1}{\delta^{n-1}} \int_{Q_{x',2\delta}\times(h,h+\delta)} \operatorname{dist}^p(\nabla u, SO(n)) \,\mathrm{d}z.$$

Hence, denoting the average of g on $(0,1)^{n-1}$ by $(g)_{(0,1)^{n-1}}$ we see by the Poincaré inequality that

$$\begin{aligned}\sup_{\zeta\in(0,1)^{n-1}} |g(\zeta) - (g)_{(0,1)^{n-1}}| &\leq \sup_{\zeta\in(0,1)^{n-1}} |\nabla g(\zeta)| \\ &\leq \frac{1}{\delta^{n-1}} \int_{Q_{x',2\delta}\times(h,h+\delta)} \operatorname{dist}^p(\nabla u, SO(n)) \,\mathrm{d}z.\end{aligned}$$

Finally, we cover U by a lattice of mesh size δ, sum over all cubes that intersect K and obtain the estimate

$$\int_{K\times(h,h+\delta)} |R - \nabla u|^p + \delta^p |\nabla R|^p \,\mathrm{d}z \leq \int_{U\times(h,h+\delta)} \operatorname{dist}^p(\nabla u, SO(n)) \,\mathrm{d}z. \tag{3.49}$$

Step 2: Estimates close to the boundary. We consider the case of a flat piece of boundary first. Let U be such that $U \cap O = U \cap \mathbb{R}^{n-1}_+$, $U \cap \partial O = U \cap \partial\mathbb{R}^{n-1}_+$, using the notation $\mathbb{R}^{n-1}_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \mid x_n > 0\}$ for the upper half space. Let $K \subset U$ be a compact subset. For $x' \in K \cap \mathbb{R}^{n-1}_+$ we define as in Step 1 the smoothed vertical average $R(x') = (\psi_\delta * \overline{\nabla u})(x')$. As the support of ψ lies in the upper half space, R is well defined and we may argue as in Step 1 to obtain a version of (3.49) with $K \cap O$ and $U \cap O$ replacing K and U respectively.

Now, returning to the general case we have for U, where $U \cap O$ and $U \cap \partial O$ can be represented by a Lipschitz function L in the sense of (3.48). We introduce the map $\varphi : U \cap \overline{O} \to \overline{\mathbb{R}^{n-1}_+}$ given by $\varphi(x', x_n) = (x', x_n - L(x'))$, which flattens the boundary and is Bilipschitz as well as area preserving. Analogously to the case of flat boundary, we now define

$$(R \circ \varphi^{-1}) = \psi_\delta * (\overline{\nabla u} \circ \varphi^{-1}).$$

Again, for $\xi' \in \varphi(K)$ the value $(R\circ\varphi^{-1})(\xi')$ is well defined for sufficiently small δ. By Theorem 3.5.2, there is a constant $C > 0$ and rotations $R_{\xi',\delta} \in SO(n)$ such that

$$\begin{aligned}\int_{Q_{\xi',\delta}\times(h,h+\delta)} |\nabla u(\varphi^{-1}(\zeta'), z_n) - R_{\xi',\delta}|^p \,\mathrm{d}\zeta'\,\mathrm{d}z_n &= \int_{\varphi^{-1}(Q_{\xi',\delta})\times I} |\nabla u(z', z_n) - R_{\xi',\delta}|^p \,\mathrm{d}z'\,\mathrm{d}z_n \\ &\leq C\int_{\varphi^{-1}(Q_{\xi',\delta})\times I} \operatorname{dist}^p(\nabla u(z', z_n); SO(n))\,\mathrm{d}z'\,\mathrm{d}z_n \\ &= C\int_{\varphi^{-1}(Q_{\xi',\delta})\times I} \operatorname{dist}^p(\nabla u, SO(n))\circ\varphi^{-1}(\zeta', z_n)\,\mathrm{d}\zeta'\,\mathrm{d}z_n.\end{aligned}$$

This enables us to argue as in Step 1 to obtain firstly

$$\int_{Q_{\xi',\delta}} |R\circ\varphi - R_{\xi',\delta}|^p \,\mathrm{d}\zeta' \leq C\int_{Q_{\xi',\delta}\times(h,h+\delta)} \operatorname{dist}^p(\nabla u, SO(n))\circ\varphi^{-1}(\zeta', z_n)\,\mathrm{d}\zeta'\,\mathrm{d}z_n,$$

as well as all other estimates now in terms of $R\circ\varphi^{-1}$ and $\nabla u\circ\varphi^{-1}$ instead of R and ∇u. In particular, we obtain

$$\int_{(K\cap O)\times(h,h+\delta)} |R-\nabla u|^p + \delta^p|\nabla R|^p \,\mathrm{d}z \leq \int_{(U\cap S)\times(h,h+\delta)} \operatorname{dist}^p(\nabla u, SO(n))\,\mathrm{d}z. \tag{3.50}$$

Step 3: Merging the estimates by partition of unity. We return to the global view and reestablish the indices on the covering open sets U_k, $k \in \{1,\dots,N\}$. Accordingly, let $\tilde{R}_k$, $k \in \{1,\dots,N\}$ denote the functions constructed in Step 2 and let $(\eta_k)_{k\in\{1,\dots,N\}}$ be a partition of unity subordinate to the cover $(U_k)_{k\in\{1,\dots,N\}}$, which means that

$$\eta_k \in C_0^\infty(U_k), \quad \eta_k \geq 0, \quad \sum_{k=1}^N \eta_i = 1 \ \text{ on } O.$$

We introduce $\tilde{R} := \sum_{k=1}^N \eta_k\tilde{R}_k$, for which we have

$$\tilde{R} - \nabla u = \sum_{k=1}^N \eta_k(\tilde{R}_k - \nabla u)$$

and

$$\nabla\tilde{R} = \sum_{k=1}^N \eta_k\nabla\tilde{R}_k + \sum_{k=1}^N \nabla\eta_k(\tilde{R}_k - \nabla u).$$

By (3.49) and (3.50) for $K_k = \operatorname{supp}\eta_k$ we have

$$\int_{O\times(h,h+\delta)} |\tilde{R}-\nabla u|^p + \delta^p|\nabla\tilde{R}|^p \leq C\int_\Omega \operatorname{dist}^p(\nabla u, SO(n))\,\mathrm{d}x. \tag{3.51}$$

Finally, the Poincaré inequality together with (3.51) yield the existence of a matrix $R \in \mathbb{R}^{n\times n}$ such that

$$\|\tilde{R} - R\|_{L^p(P_\delta;\mathbb{R}^{n\times n})} \leq C\delta^{-p}\int_\Omega \operatorname{dist}^p(\nabla u, SO(n))\,\mathrm{d}x.$$

Arguing as for (3.40), we may assume that $R \in SO(n)$ and again by (3.51) we obtain overall

$$\|\nabla u - R\|_{L^p(P_\delta;\mathbb{R}^{n\times n})} \leq C\delta^{-p}\int_\Omega \operatorname{dist}^p(\nabla u, SO(n))\,\mathrm{d}x,$$

as desired. □

3.5.2 Technical preliminaries

Lemma 3.5.7. *For $n \in \mathbb{N}$ let $P \subset \mathbb{R}^n$ be the cuboid that is given for $\ell_1, \dots, \ell_n > 0$ by $P = (-\ell_1, \ell_1) \times \cdots \times (-\ell_n, \ell_n)$ and $1 \le p < \infty$. Then, for each $A \in \mathbb{R}^{n\times n}$*

$$\min_{d\in\mathbb{R}^n} \|Ax - d\|_{L^p(P;\mathbb{R}^n)} = \min_{d\in\mathbb{R}^n} \|Ax\|_{L^p(d+P;\mathbb{R}^n)} = \|Ax\|_{L^p(P;\mathbb{R}^n)}.$$

Graphically speaking, the volume between the paraboloid described by Ax and a cuboid $d + P$ for $d \in \mathbb{R}^n$ is minimized for $d = 0$, i.e. if the paraboloid is centered in the cuboid.

Proof. Since the L^p-norm is convex, we know that each critical point of $d \mapsto \|Ax - d\|_{L^p(P;\mathbb{R}^n)}$ is a minimium. Together with the fact that $t \mapsto t^p$ is monotone on $[0, \infty)$, it thus suffices to show that $d = 0$ is a critical point of $d \mapsto \|Ax - d\|^p_{L^p(P;\mathbb{R}^n)}$. Since the integrand $|Ax - d| \le \operatorname{diam}(P)|A| + |d|$, is bounded for all $x \in P$, we obtain for the derivative with respect to the variable d

$$\begin{aligned} D_d \int_P |Ax - d|^p \,\mathrm{d}x = \int_P D_d \langle Ax - d\rangle^{\frac{p}{2}} \,\mathrm{d}x &= \int_P \frac{p}{2} |Ax - d|^{p-2} 2(Ax - d) \,\mathrm{d}x \\ &= p \int_P |Ax - d|^{p-2}(Ax - d) \,\mathrm{d}x. \end{aligned}$$

Hence, for $d = 0$, for $O := (-\ell_1, \ell_1) \times \cdots \times (-\ell_{n-1}, \ell_{n-1}) \subset \mathbb{R}^{n-1}$ the symmetry of the integrand yields

$$\begin{aligned} \Big(D_d \int_P |Ax - d|^p \,\mathrm{d}x\Big)(0) &= p \int_P |Ax|^{p-2} Ax \,\mathrm{d}x = p \int_O \int_{-\ell_n}^{\ell_n} |Ax|^{p-2} Ax \,\mathrm{d}x' \,\mathrm{d}x_n \\ &= p \int_O \int_{-\ell_n}^{0} |Ax|^{p-2} Ax \,\mathrm{d}x' \,\mathrm{d}x_n + p \int_O \int_0^{\ell_n} |Ax|^{p-2} Ax \,\mathrm{d}x' \,\mathrm{d}x_n \\ &= -p \int_O \int_0^{\ell_n} |Ax|^{p-2} Ax \,\mathrm{d}x' \,\mathrm{d}x_n + p \int_O \int_0^{\ell_n} |Ax|^{p-2} Ax \,\mathrm{d}x' \,\mathrm{d}x_n = 0, \end{aligned}$$

which sufficed to show the claim. □

Lemma 3.5.8. *Let $A \in \mathbb{R}^{2\times 2}$. If $Ae_1 - (Ae_2)^\perp \neq 0$, then*

$$\min_{R\in SO(2)} |A - R|^2 = |A|^2 + 2 - 2\sqrt{|A|^2 + 2\det A}$$

and a minimizer is given by

$$R_A = \begin{pmatrix} \frac{Ae_1 - (Ae_2)^\perp}{|Ae_1 - (Ae_2)^\perp|} & \frac{(Ae_1)^\perp + Ae_2}{|Ae_1 - (Ae_2)^\perp|} \end{pmatrix}.$$

Otherwise, every $R \in SO(2)$ minimizes the problem and

$$\min_{R\in SO(2)} |A - R|^2 = |A|^2 + 2.$$

Proof. Note that

$$\begin{aligned}
\min_{R\in SO(2)} |A-R|^2 &= \min_{R\in SO(2)} \left(|Ae_1 - Re_1|^2 + |e_2 - Re_2|^2\right) \\
&= \min_{|v|=1} \left(|Ae_1 - v|^2 + |Ae_2 - v^\perp|^2\right) \\
&= \min_{|v|=1} \left(|Ae_1|^2 + |Ae_2|^2 + |v|^2 + |v^\perp|^2 - 2Ae_1\cdot v - 2Ae_2\cdot v^\perp\right) \\
&= \min_{|v|=1} \left(|A|^2 + 2 + 2v\cdot(-Ae_1 + (Ae_2)^\perp)\right) \\
&= |A|^2 + 2 - 2\frac{|Ae_1 - (Ae_2)^\perp|^2}{|Ae_1 - (Ae_2)^\perp|} \\
&= |A|^2 + 2 - 2|Ae_1 - (Ae_2)^\perp| \\
&= |A|^2 + 2 - 2\sqrt{|A|^2 - 2Ae_1\cdot(Ae_2)^\perp} \\
&= |A|^2 + 2 - 2\sqrt{|A|^2 + 2\det A},
\end{aligned}$$

where we have used that the problem has the minimizer

$$v = \frac{Ae_1 - (Ae_2)^\perp}{|Ae_1 - (Ae_2)^\perp|},$$

if this expression is nonzero, otherwise we may chose v arbitrarily in $\mathbb{S}^1$. Consequently, we have

$$R_A = (v \,|\, v^\perp) = \begin{pmatrix} \frac{Ae_1-(Ae_2)^\perp}{|Ae_1-(Ae_2)^\perp|} & \frac{(Ae_1)^\perp+Ae_2}{|Ae_1-(Ae_2)^\perp|}\end{pmatrix}.$$

□

The following lemma is a version of a truncation argument by Friesecke, James and Müller to approximate $W^{1,p}(U;\mathbb{R}^n)$-functions, $1 \le p < \infty$ on a bounded Lipschitz domain U by $W^{1,\infty}(U;\mathbb{R}^n)$-functions. The arguments themselves are similar to approximation arguments in [69, Section 6.6.2, 6.6.3]. For the proof we restrict ourselves to only showing the claimed scaling behavior of the constants, referring for the rest to the original paper.

Lemma 3.5.9 (Approximation of $W^{1,p}$- by $W^{1,\infty}$-functions on thin domains [72, Proposition A.1])**.** *Let $n \in \mathbb{N}$, $n \ge 2$, $1 < p < \infty$ and $\ell, L \in (0,\infty)$. Let $\Omega, \Omega_0 \subset \mathbb{R}^n$ be bounded Lipschitz domains such that Ω is (ℓ, L)-Lipschitz equivalent to Ω_0. Then, there exists a constant $C > 0$, depending on n, p and Ω_0 but not on Ω such that for each $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ and each $\mu > 0$ there exists a function $v \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ such that*

(i) $\|\nabla v\|_{L^\infty(\Omega;\mathbb{R}^{n\times n})} \le C\ell\mu;$

(ii) $\|\nabla u - \nabla v\|^p_{L^p(\Omega;\mathbb{R}^{n\times n})} \le L^{2p}\ell^p n^{\frac{p}{2}} C \int_{\left\{|\nabla u| > L^{-1} n^{-\frac{p}{2}}\mu\right\}} |\nabla u|^p \,\mathrm{d}x.$

Proof. We will prove this statement under the assumption that it holds for $\Omega = \Omega_0$, while for the proof of that statement we refer to [72, Proposition A.1]. Denote by $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be the bijective Bilipschitz map whose gradient satisfies

$$|||\nabla\varphi^{-1}||| \le \ell \quad\text{and}\quad |||\nabla\varphi||| \le L$$

that exists by the definition of (ℓ, L)-Lipschitz equivalence.

Now, let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ and $\mu > 0$, then $u \circ \varphi \in W^{1,p}(\Omega_0, \mathbb{R}^n)$, hence by [72, Proposition A.1] there is a function $v \in W^{1,\infty}(\Omega_0; \mathbb{R}^n)$ such that

$$\|\nabla v\|_{L^\infty(\Omega_0;\mathbb{R}^{n\times n})} \leq C\mu \quad \text{and} \quad \big|\{u \neq v\}\big| \leq \frac{C}{\mu^p} \int_{\{|\nabla u|>\mu\}} |\nabla u|^p \,\mathrm{d}x.$$

We consider the function $v \circ \varphi^{-1} \in W^{1,\infty}(\Omega; \mathbb{R}^n)$ since by the fact that the operator norm relates to the Frobenius norm by $|\nabla\varphi| \leq \sqrt{n}|||\nabla\varphi|||$

$$\|\nabla(v \circ \varphi^{-1})\|_{L^\infty(\Omega;\mathbb{R}^{n\times n})} \leq \|\nabla v\|_{L^\infty(\Omega_0;\mathbb{R}^{n\times n})}\|\nabla\varphi^{-1}\|_{L^\infty(\Omega;\mathbb{R}^{n\times n})} \leq \sqrt{n}\ell C\mu.$$

Thus, (i) holds. For (ii), we obtain using the change of variables formula and estimates on the determinant $\det(\nabla u)$ by the bounds of the operator norm of ∇u

$$\begin{aligned}\int_\Omega |\nabla u - \nabla v \circ \varphi^{-1}|^p \,\mathrm{d}x &= \int_{\Omega_0} |\nabla u \circ \varphi - v|^p| \det(\nabla\varphi)| \,\mathrm{d}x \\ &\leq CL^p \int_{\{|\nabla(u\circ\varphi)|>\mu\}} |\nabla(u \circ \varphi)|^p \,\mathrm{d}x \\ &\leq L^{2p}\ell^p n^{\frac{p}{2}} C \int_{\{|\nabla u|\circ\varphi>L^{-1}n^{-\frac{p}{2}}\mu\}} (|\nabla u|^p \circ \varphi_\delta)| \det(\nabla\varphi)| \,\mathrm{d}x \\ &\leq L^{2p}\ell^p n^{\frac{p}{2}} C \int_{\{|\nabla u|>L^{-1}n^{-\frac{p}{2}}\mu\}} |\nabla u|^p \,\mathrm{d}x.\end{aligned}$$

□

Proposition 3.5.10 (Difference quotients [68, Section 5.8, Theorem 3])**.** *For $n \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $V \subset\subset \Omega$. Furthermore, let $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega; \mathbb{R}^n)$. Then, there is a constant $C > 0$ such that for all $h \in \mathbb{R}$ with $1 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial\Omega)$ and $j \in \{1, \ldots, n\}$ we have*

$$\|u(x + he_j) - u(x)\|_{L^p(V;\mathbb{R}^n)} \leq Ch\|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}.$$

4

Homogenization of Elastic Materials of Periodically Layered Structure

The goal of this chapter is to prove a homogenization result for layered materials with one stiff component in terms of Γ-convergence with the energy on the soft layers specified by a general class of energy densities. After a short introduction to non-linear elasticity to motivate the models considered, we show for a broader class of domains Ω the sufficiency statement of Theorem 1.1.1, giving an explicit construction for the microscopic approximation of macroscopic deformations. This sequence will form the basis for the construction of sequences with optimal energy, with which we proceed afterwards. The Γ-convergence result is then completed by a suitable lower bound estimate. We end this chapter by showing that the homogenized energy density obtained explicitly in the theorem coincides with a cell formula for a model with totally rigid layers.

4.1 Introduction to non-linear elasticity

In this section we give a brief overview on the essentials of non-linear elasticity in the language of continuum mechanics, along the lines of the book of Gurtin on this subject [81]. Let Ω represent the undeformed body, which we choose as reference configuration. We denote the material points by $x \in \Omega$. The deformation u maps each material point x to a point y in the deformed body. For the deformation gradient $F(x) = \nabla u(x)$ we assume $\det F > 0$. Adding a dependence of the deformation on time, y specifies the motion of the body.

Crucial quantities to understand the deformation of a body are the forces on the surface and inside the body. By Cauchy's theorem the balance of momentum implies the existence of a unique symmetric tensor field $T(y, t)$, known as (Cauchy) stress, to describe the surface force s and the body force b by the relations

$$s(n) = Tn \quad \text{and} \quad b = \rho\dot{v} - \operatorname{div} T,$$

where n is a unit vector normal to the surface, v denotes the velocity corresponding to y and ρ is the density [81, Sections 14, 15]. Therefore, to model a body's behavior it suffices to specify T according to particular constitutive assumptions. As the deformation gradient $F(x, t)$, which depends on the material point x and the time t, measures local distance changes, it is natural for the stress tensor to depend only on the deformation gradient F and the material point x [81, Section 25], i.e.

$$T(y,t) = \hat{T}(F(x,t),x).$$

Furthermore, the deformation should not depend on the orientation of the body, which means that the response of the elastic body is independent of the observer. It can be shown that this condition is satisfied if and only if it holds that

$$Q\hat{T}(F)Q^T = \hat{T}(QF) \quad \text{for all } Q \in SO(n) \text{ and } F \in \mathbb{R}^{n\times n} \text{ with } \det(F) > 0.$$

Notice that the stress T depends on y and thus was formulated in terms of the deformed configuration. To obtain a formulation with respect to a reference configuration, we introduce the *Piola-Kirchhoff stress* [81, Section 27]

$$S = (\det F)T(y(x,t),t)F^{-T},$$

which satisfies for a part $\mathcal{P}$ of the body Ω and its configuration $\mathcal{P}_t$ at time t, the relation

$$\int_{\partial\mathcal{P}_t} Tm \,\mathrm{d}s = \int_{\partial P} (\det F)T(y(x,t),t)F^{-T}n \,\mathrm{d}s = \int_{\partial P} Sn \,\mathrm{d}s,$$

where m and n are outward unit normals to $\partial\mathcal{P}_t$ and $\partial\mathcal{P}$, respectively. Under the constitutive assumptions for an elastic body and assuming independence of the observer we obtain that $S = \hat{S}(F)$ and the relation

$$\hat{S}(QF) = Q\hat{S}(F) \quad \text{for all } Q \in SO(n) \text{ and } F \in \mathbb{R}^{n\times n} \text{ with } \det(F) > 0. \tag{4.1}$$

Up to this point, we have not taken into account restrictions imposed by the thermodynamic principles such as the requirement of non-negative work in closed processes. For an elastic deformation it can be shown that this requirement reads

$$\int_{t_0}^{t_1} \int_{\mathcal{P}} S \cdot \dot{F} \,\mathrm{d}x \,\mathrm{d}t \geq 0$$

for a closed process with starting time t_0 and end time t_1. However, this inequality yields the existence of a scalar function $W(F,x)$, called *strain-energy density* such that [81, Section 28]

$$\hat{S}(F,x) = \frac{\partial}{\partial F} W(F,x).$$

An elastic body with a Piola-Kirchhoff stress satisfying this relation is called *hyperelastic*. The *strain energy* of a part $\mathcal{P}$ of the body is then given by

$$\int_{\mathcal{P}} W \,\mathrm{d}x.$$

From (4.1) it can be deduced that

$$W(QF) = W(F) \quad \text{for all } Q \in SO(n) \text{ and } F \in \mathbb{R}^{n\times n} \text{ with } \det(F) > 0.$$

A well-known example of such an energy density is given by the Saint Venant-Kirchhoff-material [66, Chapter 5], where

$$W(F) = \tfrac{\lambda}{2}\big(\operatorname{tr}(G)\big)^2 + \mu \operatorname{tr}(G^2)\big) \quad \text{with} \quad G = \tfrac{1}{2}(F^TF - \mathbb{I}), \quad F \in \mathbb{R}^{n\times n},$$

and λ and μ are the Lamé constants.

In the following mathematical treatment, we study macroscopic deformations of elastic materials, not their evolution. Consequently, we can disregard the time dependence. Furthermore, we will express all quantities in terms of the reference configuration and the deformation, denoting the body by Ω, the material points by x and the deformation by u, with the deformation gradient given by ∇u. In accordance with the deliberations above, we consider strain energies $E_{\text{el}} : W^{1,p}(\Omega;\mathbb{R}^n) \to \mathbb{R}$ given by a strain energy density $W_{\text{el}} : \mathbb{R}^{n\times n} \to \mathbb{R}$ via

$$E(u) = \int_\Omega W_{\text{el}}(\nabla u)\,\mathrm{d}x,$$

where W_{el} satisfies at least the following common assumptions of geometrically non-linear elasticity theory, see [72, Section 2]

(i) W_{el} is continuous;

(ii) W_{el} is frame indifferent, i.e. for all $F \in \mathbb{R}^{n\times n}$ and $R \in SO(n)$ we have $W(F) = W(RF)$;

(iii) $W_{\text{el}}(F) \geq C \operatorname{dist}^p(F, SO(n))$ and $W_{\text{el}}(F) = 0$ if $F \in SO(n)$.

While the second assumption is motivated from the above considerations, the first and the third are mathematical assumptions, normalizing the elastic energy of rigid body motions to zero and assuming continuity and quadratic growth of the energy density. These conditions are satisfied for example by

$$W_{\text{el}}(F) = \operatorname{dist}^p(F, SO(n)),$$

which will be the prototypical elastic energy used in the following. Notice that condition (iii) relates the elastic energy to the differential inclusion constraint discussed in Chapter 3.

4.2 Sequences of finite energy approximating possible macroscopic deformations

The goal of this section is to establish the sufficiency statement of Theorem 1.1.1 for bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ with $n \geq 2$ that are e_n-flat in the sense of Definition 3.3.7.

Throughout this section, we assume that $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ with $n \geq 2$ is a bounded Lipschitz domain, with further requirements to be made for the specific result. Furthermore, we consider only materials of fixed ratio between the stiff and soft component, i.e. we consider the periodic bilayered structure $(P_{\text{stiff}})_\epsilon$ associated with $\lambda_\epsilon = \lambda \in (0,1)$.

The claim is that if for $1 < p < \infty$ we have $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that for functions $R \in W^{1,p}(\Omega; SO(n))$ and $b \in W^{1,p}(\Omega;\mathbb{R}^n)$ with $\partial_i R = 0$ and $\partial_i b = 0$ for $i \in \{1,\dots,n-1\}$ it holds that

$$u(x) = R(x)x + b(x), \quad x \in \Omega, \tag{4.2}$$

then, there is a sequence $(v_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n)$ with $v_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ which satisfies the exact differential inclusion constraint, i.e. $\nabla v_\epsilon \in SO(n)$ on $\epsilon P_{\text{stiff}} \cap \Omega$.

In the following, we give an explicit construction for the sequence $(v_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n)$. Since (4.2) yields for the gradient

$$\nabla u(x) = R(x) + R'(x)x \otimes e_n + b'(x) \otimes e_n = R(x)(\mathbb{I} + a(x) \otimes e_n), \quad x \in \Omega, \tag{4.3}$$

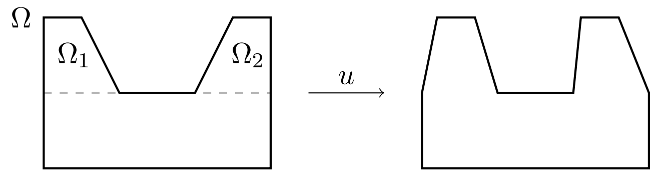

Figure 4.1: Due to the non-connectedness of cross sections of the domain with respect to the e_n-direction, the two upper branches Ω_1, Ω_2 may be deformed differently, with the deformation u still satisfying $\partial_i u = 0$ for $i \in \{1, \dots, n-1\}$. An example of such a deformation is given by $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ determined by $\nabla u|_{\Omega_1} = \mathbb{I} + 2e_1 \otimes e_2$, $\nabla u|_{\Omega_2} = \mathbb{I} - 3e_1 \otimes e_2$ and $\nabla u = \mathbb{I}$ otherwise in Ω.

where $a(x) = R^T(x)(R'(x)x \otimes e_n + b'(x) \otimes e_n)$, we introduce for given $U := \nabla u$ the notation

$$U_\lambda(x) = R(x) + \frac{1}{\lambda} R'(x)x \otimes e_n + \frac{1}{\lambda} b'(x) \otimes e_n, \quad x \in \Omega. \tag{4.4}$$

In the case of affine limits a suitable approximation of the limit can be constructed by simple laminates.

Lemma 4.2.1 (Approximation of affine admissible limits). *Let $u \in W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ be an affine function, with $\nabla u = R + d \otimes e_n =: F$, where $R \in SO(n)$ and $d \in \mathbb{R}^n$. Furthermore, let $v^F_\epsilon \in W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ be the simple laminate given by*

$$\nabla v^F_\epsilon := \begin{cases} R & \text{if } x \in \epsilon P_{\text{stiff}} \cap \Omega, \\ F_\lambda := R + \frac{1}{\lambda} d \otimes e_n & \text{if } x \in \epsilon P_{\text{soft}} \cap \Omega. \end{cases}$$

Then, $v^F_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$ and for each $\epsilon > 0$ the function v^F_ϵ satisfies the exact differential inclusion constraint $\nabla v^F_\epsilon = R \in SO(n)$ on the stiff layers $\epsilon P_{\text{stiff}} \cap \Omega$.

Proof. Since $v^F_\epsilon(x) = v^F_1(\frac{x}{\epsilon})$ for all $x \in \Omega$, the weak convergence is a direct consequence of the classic Lemma 2.3.1 on weak convergence of highly oscillating functions. Indeed, the fact that

$$\mathbb{1}_{\epsilon P_{\text{stiff}}} \overset{*}{\rightharpoonup} 1 - \lambda \quad \text{and} \quad \mathbb{1}_{\epsilon P_{\text{soft}}} \overset{*}{\rightharpoonup} \lambda \quad \text{both in } L^\infty(\Omega;\mathbb{R}^n) \quad \text{as } \epsilon \to 0,$$

implies

$$\nabla v^F_\epsilon = R\, \mathbb{1}_{\epsilon P_{\text{stiff}}} + (R + \tfrac{1}{\lambda} d \otimes e_n)\, \mathbb{1}_{\epsilon P_{\text{soft}}} \overset{*}{\rightharpoonup} R + d \otimes e_n \quad \text{in } L^\infty(\Omega;\mathbb{R}^n) \quad \text{as } \epsilon \to 0.$$

Notice that since $\nabla v^F_\epsilon\, \mathbb{1}_{\epsilon P_{\text{soft}}} = F_\lambda = (\nabla u)_\lambda$ the notation is consistent with (4.4). □

Beyond affine limit functions, the geometry of Ω takes a more decisive role. The reason is that while for a function $f \in W^{1,p}(\Omega;\mathbb{R}^n)$, $n \in \mathbb{N}$, $n \geq 2$, $1 < p < \infty$, the condition $\partial_i f = 0$ for $i \in \{1, \dots, n-1\}$ implies for domains Ω that are $e_n^\perp$-connected in the sense of Definition 3.3.6, that f depends only on x_n, see Lemma 4.5.6, on general bounded Lipschitz domains Ω, f can still depend on x' for $x = (x', x_n) \in \Omega$, see Figure 4.1.

Therefore, we will construct an approximation for $e_n^\perp$-connected domains first. Afterwards we provide a general strategy to decompose Lipschitz domains in $e_n^\perp$-connected domains and generalize the result to Lipschitz domains that feature partitions in finitely many $e_n^\perp$-connected domains.

Proposition 4.2.2 (Approximation of general admissible limits on $e_n^\perp$-connected domains)**.** *Let Ω be a simply connected bounded Lipschitz domain, that is $e_n^\perp$-connected. Furthermore, let $1 < p < \infty$ and assume $u \in W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ with $u(x) = R(x)x + b(x)$ for all $x \in \Omega$, where $R \in W^{1,p}(\Omega; SO(n))$ and $b \in L^p(\Omega;\mathbb{R}^n)$ with $\partial_i R = 0$ and $\partial_i b = 0$ for $i = 1,\dots,n-1$. Then, there is a sequence $(v_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ such that $v_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$ and the property, that there are sequences $(R_\epsilon)_\epsilon \subset W^{1,p}(\Omega; SO(n))$ such that*

$$\nabla v_\epsilon = R_\epsilon \quad \textit{in } \epsilon P_{\text{stiff}} \cap \Omega \tag{4.5}$$

and

$$R_\epsilon \rightharpoonup R \textit{ in } W^{1,p}(\Omega;\mathbb{R}^{n\times n}) \quad \textit{and} \quad \|\nabla v_\epsilon - U_\lambda\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^{n\times n})} \to 0, \textit{ both as } \epsilon \to 0. \tag{4.6}$$

Proof. Firstly, by Lemma 4.2.3 there is a sequence $(R_\epsilon)_\epsilon \subset W^{1,p}(\Omega; SO(n))$ such that $R_\epsilon \rightharpoonup R$ in $W^{1,p}(\Omega;\mathbb{R}^{n\times n})$ as $\epsilon \to 0$ satisfying

$$R'_\epsilon = 0 \text{ on } \epsilon P_{\text{stiff}} \cap \Omega \quad \text{and} \quad \|R'_\epsilon - \lambda^{-1} R'\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega,\mathbb{R}^{n\times n})} \to 0 \text{ as } \epsilon \to 0. \tag{4.7}$$

We set for $x \in \Omega$

$$U_{\lambda,\epsilon}(x) = R_\epsilon(x) + R'_\epsilon(x)x \otimes e_n + \lambda^{-1} b'(x) \otimes e_n$$

and

$$U_\epsilon = R_\epsilon \mathbb{1}_{\epsilon P_{\text{stiff}}\cap\Omega} + U_{\lambda,\epsilon} \mathbb{1}_{\epsilon P_{\text{soft}}\cap\Omega}$$

Then, by (4.7) we obtain

$$\begin{aligned} U_\epsilon &= R_\epsilon + R'_\epsilon x \otimes e_n \mathbb{1}_{\epsilon P_{\text{soft}}} + \lambda^{-1} b' \otimes e_n \mathbb{1}_{\epsilon P_{\text{soft}}} \\ &\rightharpoonup R + \lambda \cdot \lambda^{-1} R' x \otimes e_n + \lambda \cdot \lambda^{-1} b' \otimes e_n = \nabla u, \quad \text{in } L^p(\Omega;\mathbb{R}^{n\times n}) \quad \text{as } \epsilon \to 0. \end{aligned}$$

and

$$\begin{aligned} \|U_\epsilon - U_\lambda\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^{n\times n})} &\le \operatorname{diam}(\Omega)\big(\|R_\epsilon - R\|_{L^p(\Omega;\mathbb{R}^{n\times n})} + \|R'_\epsilon - \lambda^{-1} R'\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^{n\times n})}\big) \\ &\to 0 \quad \text{as } \epsilon \to 0. \end{aligned}$$

Therefore, if we establish for $\epsilon > 0$ that U_ϵ has a vanishing curl in the sense of distributions, then the mean value free potential v_ϵ of U_ϵ, existing on any simply connected domain Ω satisfies exactly the desired properties. Since $R_\epsilon \in W^{1,p}(\Omega;\mathbb{R}^n)$ with $\partial_n R = 0$ on $\epsilon P_{\text{stiff}} \cap \Omega$ and $\partial_i R_\epsilon = 0$, $\partial_i b = 0$ for $i \in \{1,\dots,n-1\}$ on Ω we obtain for $k, \ell \in \{1,\dots,n\}$, $k < \ell$ and $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$ in the case $\ell < n$

$$\int_\Omega U_\epsilon e_k \cdot \partial_\ell \varphi \,\mathrm{d}x - \int_\Omega U_\epsilon e_\ell \cdot \partial_k \varphi \,\mathrm{d}x = \int_\Omega R_\epsilon e_k \cdot \partial_\ell \varphi - R_\epsilon e_\ell \cdot \partial_k \varphi \,\mathrm{d}x = 0,$$

and in the case $\ell = n$

$$\begin{aligned} \int_\Omega U_\epsilon e_k \cdot \partial_n \varphi \,\mathrm{d}x - \int_\Omega U_\epsilon e_n \cdot \partial_k \varphi \,\mathrm{d}x &= \int_\Omega R_\epsilon e_k \cdot \partial_n \varphi - (R_\epsilon e_n + (R'_\epsilon x + \lambda^{-1} b')\mathbb{1}_{\epsilon P_{\text{soft}}}) \cdot \partial_k \varphi \,\mathrm{d}x \\ &= \int_\Omega R'_\epsilon e_k \mathbb{1}_{\epsilon P_{\text{soft}}} \cdot \varphi - R'_\epsilon e_k \mathbb{1}_{\epsilon P_{\text{soft}}} \cdot \varphi \,\mathrm{d}x = 0. \end{aligned}$$

Hence, $\operatorname{curl} U_\epsilon = 0$ as claimed. □

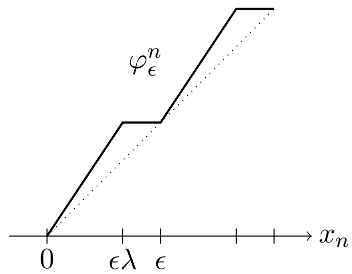

Figure 4.2: The main idea of the construction of an approximating sequence $(R_\epsilon)_\epsilon$ of R in accordance to the layered structure is to approximate its parametrization rather than the function itself. To that end we introduce the sequence φ_ϵ which on most of Ω coincides with the identity for its first $n-1$ components, while in its n-th component φ_ϵ^n is constant on the stiff layers and is given by linear interpolation on the soft layers as sketched. Since this means that φ_ϵ anticipates the values taking by the identity, the last pair of a soft and a stiff layer has to be adapted to ensure $\varphi_\epsilon(\Omega) \subset \Omega$.

The restriction to $e_n^\perp$-connected domains in Proposition 4.2.2 merely reflects the geometric assumptions under which the approximation of R constructed in the next lemma holds.

Lemma 4.2.3 (Approximation of R on $e_n^\perp$-connected domains)**.** *Let Ω be an $e_n^\perp$-connected bounded Lipschitz domain. For $1 < p < \infty$ let $R \in W^{1,p}(\Omega; SO(n))$ with $\partial_i R = 0$ for $i \in \{1, \dots, n-1\}$. Then, there exists a sequence $(R_\epsilon)_\epsilon \subset W^{1,p}(\Omega; SO(n))$ with $R_\epsilon \rightharpoonup R$ in $W^{1,p}(\Omega; \mathbb{R}^{n\times n})$ as $\epsilon \to 0$ such that*

$$R'_\epsilon = 0 \text{ a.e. in } \epsilon P_{\text{stiff}} \cap \Omega \quad \text{and} \quad \|R'_\epsilon - \lambda^{-1} R'\|_{L^p(\epsilon P_{\text{soft}} \cap \Omega; \mathbb{R}^{n\times n})} \to 0 \quad \text{as } \epsilon \to 0.$$

Proof. The idea of the construction is to adapt the parametrization of R in such a way that the resulting function satisfies the differential inclusion constraint on the stiff layers exactly, which means that, graphically speaking, we stop R on these sets, while accelerating the parametrization of R on the soft layers.

Accordingly, we set for $\epsilon > 0$

$$\tilde{\Omega}_\epsilon = \bigcup_{i \in I_\epsilon \cup \{i_\epsilon^{\min}\}} \epsilon P^i \cap \Omega = \Omega_\epsilon \cup \epsilon P^{i_\epsilon^{\min}},$$

and introduce the piecewise affine function $\varphi_\epsilon : \Omega \to \mathbb{R}^n$ defined for $(x', x_n) = x \in \tilde{\Omega}_\epsilon$ by

$$\varphi_\epsilon(x', x_n) = \begin{cases} (x', \lfloor x_n \rfloor_\epsilon + \epsilon) & \text{if } x \in \epsilon P_{\text{stiff}}, \text{ i.e. } \lfloor x_n \rfloor_\epsilon + \epsilon\lambda < x_n \le \lfloor x_n \rfloor_\epsilon + \epsilon, \\ (x', \frac{1}{\lambda}(x_n - \lfloor x_n \rfloor_\epsilon) + \lfloor x_n \rfloor_\epsilon) & \text{if } x \in \epsilon P_{\text{soft}}, \text{ i.e. } \lfloor x_n \rfloor_\epsilon < x_n \le \lfloor x_n \rfloor_\epsilon + \epsilon\lambda, \end{cases}$$

see also Figure 4.2. To ensure $\varphi_\epsilon(\Omega) \subset \Omega$ we set on the last layer, i.e. for $x \in \epsilon P^{i_\epsilon^{\max}}$

$$\varphi_\epsilon = \mathrm{id}_{\mathbb{R}^n} \text{ on } \epsilon P_{\text{soft}}, \quad \text{and} \quad \varphi_\epsilon(x', x_n) = (x', \lfloor x_n \rfloor_\epsilon + \lambda\epsilon) \text{ if } x \in \epsilon P_{\text{stiff}}.$$

Observe that $\varphi_\epsilon(\Omega) \subset \Omega$ and the sequence $(\varphi_\epsilon)_\epsilon$ converges uniformly to the identity $\mathrm{id}_{\mathbb{R}^n}$.

Hence, $(R_\epsilon)_\epsilon$ given by $R_\epsilon = R \circ \varphi_\epsilon$ is a well-defined Sobolev function and converges pointwise to R as R is continuous by Lemma 4.5.4. Also, since $R(x) \in SO(n)$ a.e., we obtain by dominated convergence that $R_\epsilon \to R$ in $L^p(\Omega; \mathbb{R}^{n\times n})$.

Furthermore, since R is absolutely continuous, see Remark 4.5.5, and φ_ϵ is a monotone Lipschitz function, the composition R_ϵ is absolutely continuous. Hence, the fact that R_ϵ is constant on the stiff layers, the chain rule [99, Theorem 3.44] and the change of variables formula yield

$$\begin{aligned}\|R'_\epsilon\|^p_{L^p(\Omega;\mathbb{R}^{n\times n})} &= \int_{\epsilon P_{\text{soft}}\cap\Omega} |R'(\varphi_\epsilon)\varphi'_\epsilon|^p \,\mathrm{d}x \leq \frac{1}{\lambda^p}\int_{\epsilon P_{\text{soft}}\cap\Omega} |R'(\varphi_\epsilon)|^p \,\mathrm{d}x \\ &= \frac{1}{\lambda^p}\int_\Omega |R'|^p|\varphi'_\epsilon|^{-1}\,\mathrm{d}x \leq \frac{1}{\lambda^{p-1}}\|R'\|^p_{L^p(\Omega;\mathbb{R}^{n\times n})}.\end{aligned}$$

Therefore, R'_ϵ is uniformly bounded in the reflexive space $L^p(\Omega;\mathbb{R}^{n\times n})$ and thus we have by a Urysohn argument, cf. Lemma 4.5.15, that $(R_\epsilon)_\epsilon$ converges weakly to R in $W^{1,p}(\Omega;\mathbb{R}^n)$.

Lastly, note that since $R'_\epsilon = \frac{1}{\lambda}R'\circ\varphi_\epsilon$ on $\epsilon P_{\text{soft}}\cap\tilde{\Omega}_\epsilon$ and $\varphi_\epsilon = \mathrm{id}_{\mathbb{R}^n}$ on $\epsilon P_{\text{soft}}^{i_\epsilon^{\max}}$, Lemma 4.2.4 yields

$$\|R'_\epsilon - \lambda^{-1}R'\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^{n\times n})} \to 0 \quad \text{as } \epsilon\to 0.$$

This finishes the proof. □

The next lemma establishes the strong convergence of R'_ϵ on the softer layers. The arguments are similar to the proof of the continuity of the shift operator, see e.g. [70, Proposition 8.5]

Lemma 4.2.4. *For $n\in\mathbb{N}$, $n\geq 2$ let $\Omega\subset\mathbb{R}^n$ be a bounded domain. Let $f\in L^p(\Omega;\mathbb{R}^n)$ and for $\epsilon>0$ let $\varphi_\epsilon:\mathbb{R}^n\to\mathbb{R}^n$ be the piecewise affine function defined for $(x',x_n)=x\in\Omega$ by*

$$\varphi_\epsilon(x',x_n) = \begin{cases}(x', \lfloor x_n\rfloor_\epsilon + \epsilon) & \text{if } x\in\epsilon P_{\text{stiff}},\ \text{i.e. } \lfloor x_n\rfloor_\epsilon + \epsilon\lambda < x_n \leq \lfloor x_n\rfloor_\epsilon + \epsilon,\\ (x', \frac{1}{\lambda}(x_n - \lfloor x_n\rfloor_\epsilon) + \lfloor x_n\rfloor_\epsilon) & \text{if } x\in\epsilon P_{\text{soft}},\ \text{i.e. } \lfloor x_n\rfloor_\epsilon < x_n \leq \lfloor x_n\rfloor_\epsilon + \epsilon\lambda.\end{cases}$$

Then,

$$\|f\circ\varphi_\epsilon - f\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^n)} \to 0 \quad \text{as } \epsilon\to 0.$$

Proof. We prove this result by approximation. Firstly, let us consider the special case of $g\in C_c(\Omega,\mathbb{R}^n)$ instead of f. Then, by the continuity of g we see on the one hand that the pointwise convergence of $(\varphi_\epsilon)_\epsilon$ to the identity yields pointwise convergence of $(g\circ\varphi_\epsilon)_\epsilon$ to g. On the other hand, since g is compactly supported in the bounded set Ω, g is bounded and therefore we obtain by dominated convergence that

$$\|g\circ\varphi_\epsilon - g\|_{L^p(\Omega;\mathbb{R}^n)} \to 0 \quad \text{as } \epsilon\to 0. \tag{4.8}$$

Now, let $\delta>0$ be given. Then, as $C_c(\Omega,\mathbb{R}^n)$ is dense in $L^p(\Omega;\mathbb{R}^n)$, see e.g. [70, Proposition 7.9], there is a $g\in C_c(\Omega,\mathbb{R}^n)$ such that $\|f-g\|_{L^p(\Omega;\mathbb{R}^n)} < \delta/3$. Furthermore, by (4.8), it holds for ϵ small enough, that $\|g\circ\varphi_\epsilon - g\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^n)} < \delta/3$. Lastly, observe that by the change of variable formula we have

$$\begin{aligned}\|f\circ\varphi_\epsilon - g\circ\varphi_\epsilon\|^p_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^n)} &= \int_{\epsilon P_{\text{soft}}\cap\Omega} |f-g|^p(x', \tfrac{1}{\lambda}(x_n - \lfloor x_n\rfloor_\epsilon) + \lfloor x_n\rfloor_\epsilon)\,\mathrm{d}x \\ &= \lambda\int_\Omega |f-g|^p\,\mathrm{d}x \leq \|f-g\|^p_{L^p(\Omega;\mathbb{R}^n)} < \frac{\delta}{3}.\end{aligned}$$

Overall, we obtain

$$\begin{aligned}\|f\circ\varphi_\epsilon - f\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^n)} &= \|f\circ\varphi_\epsilon - g\circ\varphi_\epsilon\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^n)} + \|g\circ\varphi_\epsilon - g\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^n)} \\ &\quad + \|f-g\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^n)} \\ &\leq \frac{\delta}{3}+\frac{\delta}{3}+\frac{\delta}{3} = \delta.\end{aligned}$$

Since $\delta>0$ was chosen arbitrary, the claim follows. □

For the rest of this subsection, we want to address non-$e_n^\perp$-connected domains and in particular generalize Lemma 4.2.3. As mentioned above, Lemma 4.2.2 then also generalizes to this larger class of Lipschitz domains without changes.

To assess this task, consider the non-$e_n^\perp$-connected domain sketched in Figure 4.1. First of all, notice that we cannot apply the construction of φ_ϵ from the Proof of Lemma 4.2.3 directly. The underlying problem is that φ_ϵ anticipates the values of the identity function it approximates and thus if R takes different values on Ω_1 and Ω_2 the map $R_\epsilon = R \circ \varphi$ may already take on Ω close to the boundary to Ω_1 and Ω_2 the values that R takes on these sets, causing incompatibilities in the layer intersected by the boundary. In contrast, to each of the domains Ω_1, Ω_2 and $\Omega \setminus (\Omega_1 \cup \Omega_2)$, which are $e_n^\perp$-connected, we may apply the construction of φ_ϵ, yet incompatibilities at the boundary may arise. However, these can be resolved adapting the construction. In this explicit case, one may reduce the speed of the parametrization on the last layer of $\Omega \setminus (\Omega_1 \cup \Omega_2)$, and accelerate the parametrization once Ω_2 and Ω_1 are reached.

Therefore, our first task is to find a general way to obtain a decomposition in $e_n^\perp$-connected domains and then, assuming that the set consists only of finitely many of these, adapt the construction of φ as suggested for the specific Ω of Figure 4.1.

To construct the decomposition in $e_n^\perp$ connected domains, we first introduce the notion of *monotonically connected* points.

Definition 4.2.5 (Monotonically connected points with respect to the e_n-direction)**.** For $n \in \mathbb{N}$, $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We call a continuous path $\gamma : [0,1] \to \Omega$ *monotonically decreasing in e_n-direction* or *e_n-monotonically decreasing* for short, if the n-th component function γ_n is monotonically decreasing.

We say $x, y \in \Omega$ are *e_n-monotonically connected*, if there is a continuous path $\gamma : [0,1] \to \Omega$ that is e_n-monotonically decreasing with $\gamma(0) = x$ and $\gamma(1) = y$ or $\gamma(0) = y$ and $\gamma(1) = x$. For $x \in \Omega$, we denote the set of all $y \in \Omega$ that are e_n-monotonically connected to x by

$$M_x = \{y \in \Omega \mid x \text{ and } y \text{ are } e_n\text{-monotonically connected}\}.$$

We say that a set $M \subset \Omega$ is *e_n-monotonically connected* if all points in M are e_n-monotonically connected.

Remark 4.2.6. a) Notice that the same goals could be achieved using e_n-monotonically increasing maps instead of decreasing maps. For clarity, we fix one direction.

b) Note that in general, e_n-monotonically connectedness is not transitive, i.e. if $M \subset \Omega$ is a set and $x, y, z \in M$, such that x, y and y, z are e_n-monotonically connected, then this does in general not imply that x, z are e_n-monotonically connected. This is due to the directed character of the definition via paths that are monotonically decreasing in the e_n-direction. This implies that for $x \in M$ the set M_x of points monotonically connected to x is in general not monotonically connected. However, if M is e_n-monotonically connected, then e_n-monotonically connectedness is trivially transitive as all elements of M are e_n-monotonically connected.

Lemma 4.2.7 (Equivalence between $e_n^\perp$-connectedness and e_n-monotonically connectedness)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain. Then, Ω is $e_n^\perp$-connected if and only if Ω is e_n-monotonically connected.*

Proof. Step 1: Necessity. We show that if $\Omega \subset \mathbb{R}^n$ is e_n-monotonically connected, then Ω is $e_n^\perp$-connected. Indeed, assume for $t \in \mathbb{R}$ that $x, y \in \Omega \cap H_t$, where H_t is the hyperplane given by $H_t = \{x_n = t\} \subset \mathbb{R}^n$. Since Ω is an e_n-monotonically connected set, we may assume without loss of generality that there is an e_n-monotonically decreasing continuous path $\gamma : [0,1] \to \Omega$ from x to y. But since $x_n = y_n = t$ we have $\gamma_n(s) = t$ for all $s \in [0,1]$, hence $\gamma([0,1]) \subset \Omega \cap H_t$. Thus, $\Omega \cap H_t$ is (path-)connected for all $t \in \mathbb{R}$ which is the definition of $e_n^\perp$-connectedness.

Step 2: Sufficiency. We show that if $\Omega \subset \mathbb{R}^n$ is $e_n^\perp$-connected, then Ω is e_n-monotonically connected. Accordingly, we establish for $x, y \in \Omega$ that there is an e_n-monotonically decreasing path connecting x and y.

Since Ω is path-connected there is a continuous path $\tilde{\gamma} : [0,1] \to \Omega$ with $\tilde{\gamma}(0) = x$ and $\tilde{\gamma}(1) = y$. By the fact that Ω is open and $\tilde{\gamma}([0,1]) \subset \Omega$ is compact, there is a finite cover $(B_i)_{i\in\{1,\dots,N\}}$, $N \in \mathbb{N}$ of $\gamma([0,1])$ by open balls $B_i \subset \Omega$, $i \in \{1,\dots,N\}$, i.e.

$$\tilde{\gamma}([0,1]) \subset \bigcup_{i=1}^{N} B_i \subset \Omega.$$

Now, we construct a new path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$ using only the balls $(B_i)_{i\in\{1,\dots,N\}}$ and the $e_n^\perp$-connectedness of the domain Ω, yet not $\tilde{\gamma}$ directly. Without loss of generality we may assume $x_n > y_n$. Denoting the projection to the n-th component by $\mathrm{proj}_n : \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto x_n$, we have

$$[y_n, x_n] \subset \mathrm{proj}_n\big(\gamma[0,1]\big) \subset \bigcup_{i=1}^{N} \mathrm{proj}_n(B_i),$$

which implies that there is a finite partition $t_0 = y_n < t_1 < \dots t_k = x_n$, $k \in \mathbb{N}$ of $[y_n, x_n]$ such that for each $i \in \{0, \dots, k-1\}$ there is an $a_i \in \mathbb{R}^{n-1}$ such that $\{a_i\} \times [t_i, t_{i+1}] \subset B_j$ for some $j \in \{1, \dots, n\}$. Now, let $\gamma_i : [i, i + 1/2] \to \Omega$, $i \in \{0, \dots, k-1\}$ be the paths given by

$$\gamma_i(s) = \big(a_i, 2(s-i)(t_{i+1} - t_i) + t_i\big), \quad s \in [i, i + 1/2],$$

and let $\bar{\gamma}_i : [i + 1/2, i + 1] \to \Omega$, $i \in \{0, \dots, k-1\}$ be the paths connecting $\{a_i\} \times \{t_{i+1}\}$ and $\{a_{i+1}\} \times \{t_{i+1}\}$ in the hyperplane $H_{t_{i+1}} = \{z_n = t_{i+1}\}$ that exists since Ω is $e_n^\perp$-connected. Then, $\gamma : [0,1] \to \Omega$ is given by connecting γ_1, $\bar{\gamma}_1$, γ_2, $\dots$, γ_k in this order and reparametrize by $s = \frac{2}{2k-1} t$ to $[0,1]$. □

The next proposition constructs a decomposition of a domain in e_n-monotonically connected domains. Starting from the set M_x the idea is to take the intersection with all M_y with $y \in M_x$ to ensure the transitivity of e_n-monotonically connectedness on these components. Furthermore, to obtain a disjoint partition, we also remove points e_n-monotonically connected to elements of Ω that are not in M_x.

Proposition 4.2.8 (e_n-monotonically connected components)**.** *For $n \in \mathbb{N}$, $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a domain. We set*

$$C_x := M_x \cap \bigcap_{y \in M_x} M_y \cap \bigcup_{y \in \Omega \setminus M_x} M_y^c, \qquad x \in \Omega,$$

where we denote for a set $A \subset \Omega$ the complement of A in Ω by A^c. Then, for $x \in \Omega$, the set C_x is e_n-monotonically connected, for $x, y \in \Omega$, either $y \in C_x$ in which case $C_x = C_y$ or C_x and C_y are disjoint, and $\Omega = \bigcup_{x\in\Omega} C_x$. We call $(C_x)_{x\in\Omega}$ the e_n-monotonically connected components of Ω.

Furthermore, the shared boundaries between the e_n-monotonically connected components are subsets of hyperplanes $H_t = \{x_n = t\}$, $t \in \mathbb{R}$.

Proof. We start with the statement, that C_x is e_n-monotonically connected. Suppose that for $x \in \Omega$ there are $y, z \in C_x$ that are not e_n-monotonically connected. Then, $y, z \in C_x$, but $z \notin M_y$. However, since $y \in C_x \subset M_x$, we have by definition of C_x

$$z \notin C_x \cap M_y = \Big(M_x \cap \bigcap_{y \in M_x} M_y \cap \bigcup_{y \in \Omega \setminus M_x} M_y^c\Big) \cap M_y = C_x, \tag{4.9}$$

which is a contradiction. Thus, we obtain in particular that for each $x \in \Omega$ we have on the set C_x that e_n-monotonically connectedness is transitive.

Hence, if $x, y \in \Omega$ with $y \in C_x$, then $C_x = C_y$ by transitivity of e_n-monotonically connectedness. Now, assume that for $y \in \Omega \setminus C_x$ we have $z \in C_x \cap C_y \neq \emptyset$. Then, $z \in M_y$. Thus, if $y \notin M_x$, then by the same argument as for (4.9), we have the contradiction $z \notin C_x \cap M_y^c = C_x$.

If $y \in M_x$, then there is an $a \in M_x$ with $y \notin M_a$, but $z \in M_a$, or there is a $b \in \Omega \setminus M_x$ with $y \in M_b$, but $z \notin M_b$. But in both cases, it follows that $y \notin C_y$, which is again a contradiction.

By definition, boundaries of the sets M_x are always subsets of hyperplanes of the form $H_t = \{x_n = t\}$ for $t \in \mathbb{R}$. Indeed, if for $x \in \Omega$ one point in a connected component of $\Omega \cap H_t$ is e_n-monotonically connected, all points of this connected component are. The same is true for C_x by construction. □

Remark 4.2.9. 1) Note that it does not suffice to set for $x \in \Omega$

$$\tilde{C}_x := M_x \cap \bigcap_{y \in M_x} M_y$$

to obtain a disjoint decomposition. Indeed, consider the set Ω of Figure 4.1. Then, for $x \in \Omega_1$ we have $\tilde{C}_x = \Omega \setminus \Omega_2 = M_x$, while $C_x = \Omega_1$ as desired. This motivates to interpret the definition of C_x as starting from M_x and eliminating all $y \in M_x$ for which there is a $z \in \Omega$ that is connected to either x or y but not the other.

2) Note that while the construction of the decomposition of Ω in e_n-monotonically connected components in Proposition 4.2.8 does not depend on choices, it is not the only partition of Ω in e_n-monotonically connected components. In general, it can be made finer by partitioning components further in e_n-direction, or coarse, as can be seen for example for the set Ω in Figure 4.1, where $\Omega_1, \Omega_2, \Omega \setminus (\Omega_1 \cup \Omega_2)$ would be the decomposition constructed in Proposition 4.2.8, while $\Omega_2, \Omega \setminus \Omega_2$ would be another partition in e_n-monotonically connected components.

3) Since cubes are e_n-monotonically connected and each domain $\Omega \subset \mathbb{R}^n$ can be exhausted by at most countably many cubes, the union of all e_n-monotonically connected components $\bigcup_{x \in \Omega} C_x = \Omega$ is a union of at most countably many sets.

4) Consider again Ω as in Figure 4.1. In light of the partition into e_n-monotonically connected components, observe that for $x \in \Omega_1$ the boundary between Ω_1 and $\Omega \setminus (\Omega_1 \cup \Omega_2)$ belongs to C_x. This can be seen by the fact that for $y \in \Omega_2$ it does not belong to M_y. In particular, we see that e_n-connected components may be neither closed nor open.

In the following, we will assume that for Ω the decomposition given by Proposition 4.2.8 is a partition of Ω in finitely many sets. Though it is surely possible to show under suitable assumption that a large class of bounded Lipschitz domains satisfies this condition, this is rather a geometric topic which lies beyond the intentions of this section. Hence, we restrict ourselves to observe, that Figure 4.1 shows an example of such a set, and proceed with the necessary adaption to Lemma 4.2.3.

Lemma 4.2.10 (Approximation on domains with finitely many e_n-monotonically connected components)**.** *Assume that for a bounded Lipschitz domain Ω the decomposition given by Proposition 4.2.8 is a partition of Ω in finitely many sets. For $1 < p < \infty$ let $R \in W^{1,p}(\Omega; SO(n))$*

with $\partial_i R = 0$ for $i \in \{1, \dots, n-1\}$. Then, there exists a sequence $(R_\epsilon)_\epsilon \subset W^{1,p}(\Omega; SO(n))$ with $R_\epsilon \rightharpoonup R$ in $W^{1,p}(\Omega; \mathbb{R}^{n\times n})$ as $\epsilon \to 0$ such that

$$R'_\epsilon = 0 \text{ a.e. in } \epsilon P_{\text{stiff}} \cap \Omega \quad \text{and} \quad \|R'_\epsilon - \lambda^{-1} R'\|_{L^p(\epsilon P_{\text{soft}} \cap \Omega; \mathbb{R}^{n\times n})} \to 0 \quad \text{as } \epsilon \to 0.$$

Proof. Since for each $x \in \Omega$ the e_n-monotonically connected component C_x is $e_n^\perp$-connected, we merely have to adapt the construction of φ_ϵ in the Proof of Lemma 4.2.3 to be well-defined if C_x borders more than one other e_n-monotonically connected component.

More precisely, consider for fixed $x \in \Omega$ the e_n-monotonically connected component C_x. To shorten the notation, set $C = C_x$. As usual, we index the layers in C by $I_\epsilon = I_\epsilon^C$, see Definition 3.1.1.

Let $\epsilon > 0$ be fixed. If $\# I_\epsilon \geq 1$, we define the function $\psi_\epsilon : C \to \mathbb{R}^n$ layerwise, distinguishing three cases. Firstly, if not specified otherwise below, we set for $i \in I_\epsilon \cup \{i_\epsilon^{\min}, i_\epsilon^{\max}\}$

$$\psi_\epsilon^i(x', x_n) = \begin{cases} (x', \lfloor x_n \rfloor_\epsilon + \epsilon) & \text{if } x \in \epsilon P^i_{\text{stiff}}, \\ (x', \frac{1}{\lambda}(x_n - \lfloor x_n \rfloor_\epsilon) + \lfloor x_n \rfloor_\epsilon) & \text{if } x \in \epsilon P^i_{\text{soft}}. \end{cases}$$

Now, if C is bordering more than one other e_n-monotonically connected component on the upper edge, we set for $i_\epsilon^{\max}$

$$\psi_\epsilon^{i_\epsilon^{\max}}(x', x_n) = \begin{cases} (x', \lfloor x_n \rfloor_\epsilon + \lambda\epsilon) & \text{if } x \in \epsilon P^i_{\text{stiff}}, \\ (x', x_n) & \text{if } x \in \epsilon P^i_{\text{soft}}. \end{cases}$$

If C is bordering exactly one component on the lower edge, then the value for $i_\epsilon^{\min}$ is already defined by that component and we set for $i_\epsilon^{\min} + 1$

$$\psi_\epsilon^{i_\epsilon^{\min}+1}(x', x_n) = \begin{cases} (x', \lfloor x_n \rfloor_\epsilon + \epsilon) & \text{if } x \in \epsilon P^i_{\text{stiff}}, \\ (x', \frac{2-\lambda}{\lambda}(x_n - \lfloor x_n \rfloor_\epsilon) + \lfloor x_n \rfloor_\epsilon - (1-\lambda)\epsilon) & \text{if } x \in \epsilon P^i_{\text{soft}}. \end{cases}$$

As there are only finitely many e_n-monotonically connected components the condition $\# I_\epsilon \geq 1$ is satisfied for ϵ small enough. This construction ensures compatibility between the different e_n-monotonically connected components and the claim follows arguing as in Lemma 4.2.3. □

4.3 Homogenization of layered materials with stiff components

In this section we will apply the characterization result of Theorem 3.3.1 and the approximation constructed in Proposition 4.2.2 to establish a homogenization result for bilayered material models with one stiff component and the softer modeled by an energy density belonging to a rather broad class of functions.

More precisely, let $W : \mathbb{R}^{n\times n} \to [0, \infty)$ be a continuous energy density satisfying the following conditions:

(i) (Quasiconvex envelope is polyconvex) It holds that $W^{\text{qc}} = W^{\text{pc}}$;

(ii) (p-growth) For all $F \in \mathbb{R}^{n\times n}$ and constants $C, c > 0$ and $d > 0$

$$c|F|^p - d \leq W(F) \leq C(1 + |F|^p);$$

(iii) (Lipschitz-condition) For all $F, G \in \mathbb{R}^{n\times n}$ and a constant $L > 0$

$$|W(F) - W(G)| \leq L(1 + |F|^{p-1} + |G|^{p-1})|F - G|.$$

For $\epsilon > 0$ and $\alpha > 0$ let $W_\epsilon^\alpha : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to [0, \infty)$ be given for $(x, F) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$ by

$$W_\epsilon^\alpha(x, F) = \begin{cases} \epsilon^{-\alpha} \operatorname{dist}^p(F, SO(n)) & \text{if } x \in \epsilon P_{\text{stiff}}, \\ W(F) & \text{if } x \in \epsilon P_{\text{soft}}. \end{cases}$$

Accordingly, we define the energy $E_\epsilon^\alpha : L_0^p(\Omega; \mathbb{R}^n) \to [0, \infty]$ to be given for $u \in W^{1,p}(\Omega; \mathbb{R}^n) \cap L_0^p(\Omega; \mathbb{R}^n)$ by

$$E_\epsilon^\alpha(u) = \int_\Omega W_\epsilon^\alpha(x, \nabla u)\, \mathrm{d}x, \tag{4.10}$$

and extend E_ϵ^α by ∞ to $L_0^p(\Omega; \mathbb{R}^n)$.

The next theorem determines the Γ-limit of the energies $(E_\epsilon^\alpha)_\epsilon$ as ϵ tends towards zero for sufficient large stiffness parameter $\alpha > 0$.

Theorem 4.3.1 (Homogenization of bilayered materials with one stiff component)**.** *For $n \in \mathbb{N}$, $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a simply connected bounded Lipschitz domain that is $e_n^\perp$-connected and e_n-flat in the sense of Definition 3.3.6 and Definition 3.3.7, respectively, and let $n < p < \infty$.*

If $\alpha > p$, then, the family $(E_\epsilon^\alpha)_\epsilon$ of energy functionals $E_\epsilon^\alpha : L_0^p(\Omega; \mathbb{R}^n) \to [0, \infty]$ given by (4.10), converges in the sense of Γ-convergence with respect to the strong L^p-topology to the limit functional $E : L_0^p(\Omega; \mathbb{R}^n) \to [0, \infty]$ given for $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $u(x) = R(x)x + b(x)$ for a.e. $x \in \Omega$, $R \in W^{1,p}(\Omega; SO(n))$, $b \in L^p(\Omega; \mathbb{R}^n)$ with $\partial_i R = 0$ and $\partial_i b = 0$, $i = 1, \dots, n-1$ by

$$E(u) = \int_\Omega \lambda W^{\mathrm{qc}}(\lambda^{-1}(\nabla u - (1-\lambda)R))\, \mathrm{d}x, \tag{4.11}$$

and $E(u) = \infty$ otherwise in $L_0^p(\Omega; \mathbb{R}^n)$.

Furthermore, sequences $(u_\epsilon)_\epsilon \subset L_0^p(\Omega; \mathbb{R}^n)$ that are of bounded energy with respect to $(E_\epsilon^\alpha)_\epsilon$, i.e. for a constant $C > 0$ it holds that $E_\epsilon^\alpha(u_\epsilon) \leq C$ for all $\epsilon > 0$ are relatively compact in $L_0^p(\Omega; \mathbb{R}^n)$.

In the following, we will refer to the homogenized energy density by

$$W_{\text{hom}}(F) = \lambda W^{\mathrm{qc}}(\lambda^{-1}(F - (1-\lambda)R)), F \in \mathbb{R}^{n\times n}, F = R + d \otimes e_n \text{ for } R \in SO(n), d \in \mathbb{R}^n.$$

Before proving this result, we want to give an explicit application.

Example 4.3.2 (Saint Venant-Kirchhoff-energy)**.** Recall that the quasiconvex and polyconvex envelope for the Saint Venant-Kirchhoff-Energy in two and three dimensions are explicitly known, see Proposition 2.1.11. In particular the quasiconvex and polyconvex envelope coincide, and the growth and Lipschitz conditions are satisfied. Hence, Theorem 4.3.1 in combination with Proposition 2.1.11 provides an explicit homogenization formula.

Remark 4.3.3 (Energy densities W taking values in $\mathbb{R}$)**.** The statement of Theorem 4.3.1 generalizes directly to the case of energy densities $W : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfying conditions (i)-(iii), which by the growth condition (ii) from below amounts to energy densities taking values in $[-d, \infty)$. This follows from applying Theorem 4.3.1 to the non-negative energy density $W + d$ and then using the fact that $(W + d)^{\mathrm{qc}} = W^{\mathrm{qc}} + d$.

4.3.1 The case of affine limit functions

In the first lemma, we give an energy estimate for affine limits, using the fact that on the stiff layers, u_ϵ is close to a rigid body motion. For a similar result in the context of convex energy densities, see Lemma 5.2.23.

Lemma 4.3.4 (Energy estimate for affine limits)**.** *For $n \in \mathbb{N}$, $n \geq 2$ let $Q \subset \mathbb{R}^n$ be a cuboid and let $2 \leq n < p < \infty$. For $R \in SO(n)$ and $d \in \mathbb{R}^n$ let $F = R + d \otimes e_n$. Furthermore, let $u \in W^{1,p}(Q;\mathbb{R}^n)$ with $\nabla u = F$ and let $(u_\epsilon)_\epsilon \subset W^{1,p}(Q;\mathbb{R}^n)$ such that $u_\epsilon \rightharpoonup u$ in $W^{1,p}(Q;\mathbb{R}^n)$ as $\epsilon \to 0$. Furthermore, let there be a sequence $(\Xi_\epsilon)_\epsilon \in L^\infty(Q;\mathbb{R}^{n\times n})$ with $\Xi_\epsilon \to R$ in $L^p(Q;\mathbb{R}^{n\times n})$ as $\epsilon \to 0$ such that*

$$\|(\nabla u_\epsilon - \Xi_\epsilon)\|_{L^p(\epsilon P_{\mathrm{stiff}}\cap Q;\mathbb{R}^{n\times n})} \to 0 \quad \text{as } \epsilon \to 0. \tag{4.12}$$

Then,

$$\liminf_{\epsilon\to 0} \int_{\epsilon P_{\mathrm{soft}}\cap Q} W(x,\nabla u_\epsilon)\,\mathrm{d}x \geq \lambda \int_Q W^{\mathrm{pc}}\big(\tfrac{1}{\lambda}(F-(1-\lambda)R)\big)\,\mathrm{d}x.$$

Proof. By the strong convergence of $(\Xi_\epsilon)_\epsilon$ and (4.12) we obtain by the classic Lemma 2.3.1 on weak convergence of highly oscillating functions applied to the vector of minors

$$\begin{aligned}\mathcal{M}(\nabla u_\epsilon)\,\mathbb{1}_{\epsilon P_{\mathrm{stiff}}} &= \big(\mathcal{M}(\nabla u_\epsilon) - \mathcal{M}(\Xi_\epsilon)\big)\,\mathbb{1}_{\epsilon P_{\mathrm{stiff}}} + \mathcal{M}(\Xi_\epsilon)\,\mathbb{1}_{\epsilon P_{\mathrm{stiff}}}\\ &\rightharpoonup (1-\lambda)\mathcal{M}(R) \quad \text{in } L^1(Q;\mathbb{R}^{\tau_n}) \quad \text{as } \epsilon\to 0,\end{aligned}$$

where $\tau_n = \binom{n}{1}^2 \times \cdots \times \binom{n}{n}^2$, see Definition 2.1.7 for the notation on minors. Since for $p > n$ one has $\mathcal{M}(\nabla u_\epsilon) \rightharpoonup \mathcal{M}(\nabla u)$ in $L^1(Q;\mathbb{R}^{\tau_n})$ as $\epsilon \to 0$ [58, Theorem 8.20, Part 4], this implies

$$\begin{aligned}\mathcal{M}(\nabla u_\epsilon)\,\mathbb{1}_{\epsilon P_{\mathrm{soft}}} &= \mathcal{M}(\nabla u_\epsilon) - \mathcal{M}(\nabla u_\epsilon)\,\mathbb{1}_{\epsilon P_{\mathrm{stiff}}}\\ &\rightharpoonup \mathcal{M}(\nabla u) - (1-\lambda)\mathcal{M}(R) \quad \text{in } L^1(Q;\mathbb{R}^{\tau_n}) \quad \text{as } \epsilon\to 0.\end{aligned}$$

By definition of the polyconvex envelope, there is a convex function $g : \mathbb{R}^{\binom{n}{1}^2\times\cdots\times\binom{n}{n}^2} \to \mathbb{R}$ such that $W^{\mathrm{pc}}(F) = g(\mathcal{M}(F))$ for all $F \in \mathbb{R}^{n\times n}$, so that

$$\int_{\epsilon P_{\mathrm{soft}}\cap Q} W(x,\nabla u_\epsilon)\,\mathrm{d}x \geq \int_{\epsilon P_{\mathrm{soft}}\cap Q} W^{\mathrm{pc}}(\nabla u_\epsilon)\,\mathrm{d}x \geq \sum_{i\in I_\epsilon} \int_{\epsilon P^i_{\mathrm{soft}}\cap Q} g\big(\mathcal{M}(\nabla u_\epsilon)\big)\,\mathrm{d}x.$$

By application of Jensen's inequality first in the continuous then in the discrete case we obtain for $i_0 \in I_\epsilon$

$$\begin{aligned}\sum_{i\in I_\epsilon} \int_{\epsilon P^i_{\mathrm{soft}}\cap Q} g\big(\mathcal{M}(\nabla u_\epsilon)\big)\,\mathrm{d}x &\geq \sum_{i\in I_\epsilon} |\epsilon P^{i_0}_{\mathrm{soft}}\cap Q|\, g\Big(\fint_{\epsilon P^i_{\mathrm{soft}}\cap Q} \mathcal{M}(\nabla u_\epsilon)\,\mathrm{d}x\Big)\\ &\geq |\epsilon P^{i_0}_{\mathrm{soft}}\cap Q|\#I_\epsilon \cdot g\Big(\frac{1}{\#I_\epsilon}\sum_{i\in I_\epsilon} \fint_{\epsilon P^i\cap Q} \mathcal{M}(\nabla u_\epsilon)\,\mathbb{1}_{\epsilon P_{\mathrm{soft}}}\,\mathrm{d}x\Big).\end{aligned}$$

Finally, using the notation $Q_\epsilon := \bigcup_{i\in I_\epsilon} \epsilon P^i \cap Q$, the fact that $|Q_\epsilon| \to |Q|$ as $\epsilon \to 0$ together with the uniform bound on the gradients ∇u_ϵ, the continuity of the convex function g and the upcoming Lemma 4.3.6 yields

$$\begin{aligned}\liminf_{\epsilon\to 0} \int_{\epsilon P_{\mathrm{soft}}\cap Q} W(x,\nabla u_\epsilon)\,\mathrm{d}x &\geq \liminf_{\epsilon\to 0} \lambda|Q_\epsilon| \cdot g\Big(\frac{1}{\lambda|Q_\epsilon|}\int_{Q_\epsilon} \mathcal{M}(\nabla u_\epsilon)\,\mathbb{1}_{\epsilon P_{\mathrm{soft}}}\,\mathrm{d}x\Big)\\ &= \lambda|Q| \cdot g\big(\tfrac{1}{\lambda}(\mathcal{M}(\nabla u) - (1-\lambda)\mathcal{M}(R))\big)\\ &= \lambda \int_Q g\big(\mathcal{M}(R + \tfrac{1}{\lambda} d\otimes e_n)\big).\end{aligned}$$

□

Remark 4.3.5. The restriction that $p > n$ is due to the fact that we need enough integrability for ∇u such that its minors are also integrable. A result proven by similar strategy but for $W^{1,1}(\Omega;\mathbb{R}^2)$-functions with additional properties that guarantee convergence of the minors in the context of crystal plasticity is given in Lemma 5.2.23.

Next, we prove the identity for minors used in the proof of Lemma 4.3.4.

Lemma 4.3.6 (An identity for minors)**.** *Let $A \in \mathbb{R}^{n\times n}$, $d \in \mathbb{R}^n$ and $\lambda \in (0,1)$. Then, for every minor μ it holds that*

$$\tfrac{1}{\lambda}\big(\mu(A + d\otimes e_n) - (1-\lambda)\mu(A)\big) = \mu\big(A + \tfrac{1}{\lambda} d\otimes e_n\big).$$

Proof. Though by rearranging the terms, this identity can alternatively be deduced from the affinity of minors along rank one lines, we give a proof by explicit calculation of the minors.

Let μ be of order k and associated to the index sets

$$I = \{i_1,\dots,i_k\} \subset \{1,\dots,n\} \quad \text{and} \quad J = \{j_1,\dots,j_k\} \subset \{1,\dots,n\}$$

with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. We distinguish two cases. If $j_k \neq n$, the n-th row of a matrix does not affect the value of μ, which means

$$\tfrac{1}{\lambda}\big(\mu(A + d\otimes e_n) - (1-\lambda)\mu(A)\big) = \tfrac{1}{\lambda}\big(\mu(A) - (1-\lambda)\mu(A)\big) = \mu(A) = \mu\big(A + \tfrac{1}{\lambda} d\otimes e_n\big).$$

However, if $j_k = n$ we obtain

$$\begin{aligned}
&\tfrac{1}{\lambda}\big(\mu(R + d\otimes e_n) - (1-\lambda)\mu(R)\big)\\
&\quad= \frac{1}{\lambda}\Big(\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)(A_{i_{\sigma(k)},n} + d_{i_{\sigma(k)}})\cdot\prod_{\ell=1}^{k-1}A_{i_{\sigma(\ell)},j_\ell}\Big) - \frac{1-\lambda}{\lambda}\Big(\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\prod_{\ell=1}^{k}A_{i_{\sigma(\ell)},j_\ell}\Big)\\
&\quad= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\Big(\frac{1}{\lambda}A_{i_{\sigma(k)},n} - \frac{1-\lambda}{\lambda}A_{i_{\sigma(k)},n} + \frac{1}{\lambda}d_{i_{\sigma(k)}}\Big)\cdot\prod_{\ell=1}^{k-1}A_{i_{\sigma(\ell)},j_\ell}\\
&\quad= \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\Big(A_{i_{\sigma(k)},n} + \frac{1}{\lambda}d_{i_{\sigma(k)}}\Big)\cdot\prod_{\ell=1}^{k-1}A_{i_{\sigma(\ell)},j_\ell}\\
&\quad= \mu\big(A + \tfrac{1}{\lambda}d\otimes e_n\big).
\end{aligned}$$

This finishes the proof. □

The lower bound in the case of affine limits follows directly from Lemma 4.3.4, if we mind the fact that $(\Sigma_\epsilon)_\epsilon$ from Proposition 3.3.10 that takes the role of $(\Xi_\epsilon)_\epsilon$ is only defined on cubes contained in Ω, not on the whole set.

Corollary 4.3.7 (Affine lower bound)**.** *For $2 \le n < p < \infty$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $(u_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ with $E^\alpha_\epsilon(u_\epsilon) < C$ and $u_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$ for $u \in W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ such that $\nabla u = F$ for some $F \in \mathbb{R}^{n\times n}$. Then,*

$$\liminf_{\epsilon\to 0} E^\alpha_\epsilon(u_\epsilon) \ge E(u).$$

Proof. Since the sequence is of bounded energy, by the asymptotic characterization established in Theorem 3.3.1 there are $R \in SO(n)$ and $d \in \mathbb{R}^n$ such that $\nabla u = F = R + d\otimes e_n$. Furthermore, let $(Q_i)_{i\in\mathbb{N}}$ be an exhaustion of Ω with disjoint open cuboids up to a null set. By splitting the cuboids if necessary we may assume for each $x \in Q_i$ that $\operatorname{dist}(x,\partial\Omega) > 3\operatorname{diam}(Q_i)$. Then, by Proposition 3.3.10, there is for each Q_i a sequence $(\Sigma_\epsilon^i)_\epsilon \subset L^\infty(Q_i; SO(n))$ such that $\|\nabla u_\epsilon - \Sigma_\epsilon\|_{L^p(\epsilon P_{\text{stiff}} \cap Q_i; \mathbb{R}^n)} \to 0$ as $\epsilon \to 0$ and by Lemma 3.3.12 and Proposition 3.3.15 it also satisfies $\Sigma_\epsilon \to R$ in $L^p(Q_i; \mathbb{R}^{n\times n})$ as $\epsilon \to 0$. Hence, Lemma 4.3.4 implies that

$$\liminf_{\epsilon\to 0} \int_{\epsilon P_{\text{soft}} \cap Q_i} W(\nabla u_\epsilon) \geq \lambda \int_{Q_i} W^{\text{pc}}\Big(R + \frac{1}{\lambda} d \otimes e_n\Big).$$

Summing over all $i \in \mathbb{N}$ we obtain

$$\liminf_{\epsilon\to 0} E_\epsilon^\alpha(u_\epsilon) \geq \liminf_{\epsilon\to 0} \int_{\epsilon P_{\text{soft}} \cap \Omega} W(\nabla u_\epsilon) \geq \lambda \int_{\Omega} W^{\text{pc}}\Big(R + \frac{1}{\lambda} d \otimes e_n\Big) = E(u). \tag{4.13}$$

□

Next, we turn to the matter of a recovery sequence in the affine case, based on the laminate construction introduced in Lemma 4.2.1.

Lemma 4.3.8 (Recovery sequence for affine limits)**.** *For $2 \leq n < p < \infty$ let Ω be a simply connected bounded Lipschitz domain. Let $u \in W^{1,p}(\Omega; \mathbb{R}^n) \cap L_0^p(\Omega; \mathbb{R}^n)$ with $\nabla u = F$, $F \in \mathbb{R}^{n\times n}$ such that $E(u) < \infty$. Then, there is a sequence $(u_\epsilon)_\epsilon \subset W^{1,p}(\Omega; \mathbb{R}^n) \cap L_0^p(\Omega; \mathbb{R}^n)$ such that $u_\epsilon \to u$ in $L^p(\Omega; \mathbb{R}^n)$ as $\epsilon \to 0$,*

$$\limsup_{\epsilon\to 0} E_\epsilon^\alpha(u_\epsilon) = E(u) \quad \textit{and} \quad u_\epsilon - v_\epsilon^F \in W_0^{1,p}(\Omega; \mathbb{R}^n),$$

where v_ϵ^F is the simple laminate constructed in Lemma 4.2.1.

Proof. The general idea for this construction is inspired by Müller [112, Lemma 2.1 (a)]. Since $E(u) < \infty$ we have $\nabla u = F = R + d \otimes e_n$ with $R \in SO(n)$ and $d \in \mathbb{R}^n$. In the following, we use the notation $F_\lambda = R + \frac{1}{\lambda} d \otimes e_n$ as introduced in Lemma 4.2.1. Accordingly, W_{hom} reads for $\nabla u = F$

$$W_{\text{hom}}(F) = \lambda W^{\text{qc}}(\lambda^{-1}(F - (1-\lambda)R)) = \lambda W^{\text{qc}}(F_\lambda).$$

By the fact that the quasiconvex envelope is independent of the domain, cf. Remark 2.1.5, it holds that

$$W_{\text{hom}}(F) = \lambda \inf\Big\{ \fint_{(0,1)^{n-1}\times(0,\lambda)} W(F_\lambda + \nabla\psi)\,\mathrm{d}x \;\Big|\; \psi \in W_0^{1,p}((0,1)^{n-1}\times(0,\lambda); \mathbb{R}^n) \Big\}.$$

For $\delta > 0$ let the function $\psi_\delta \in W_0^{1,p}((0,1)^{n-1}\times(0,\lambda); \mathbb{R}^n)$ be an almost minimizer in the sense that

$$W_{\text{hom}}(F) \leq \int_{(0,1)^{n-1}\times(0,\lambda)} W(F_\lambda + \nabla\psi_\delta)\,\mathrm{d}x \leq W_{\text{hom}}(F) + \delta.$$

We extend ψ_δ first by zero to the unit cube and then periodically to $\mathbb{R}^n$ with respect to the unit cube. Furthermore, for

$$N_\epsilon = \{k \in \mathbb{Z}^n \mid \epsilon(k + (0,1)^n) \subset \Omega\} \quad \text{and} \quad \Omega_\epsilon = \bigcup_{i\in N_\epsilon} \epsilon(k + (0,1)^n),$$

and $(v_\epsilon^F)_\epsilon \subset W^{1,p}(\Omega; \mathbb{R}^n)$ as in Lemma 4.2.1 we define $u_{\epsilon,\delta} \in W^{1,p}(\Omega; \mathbb{R}^n) \cap L_0^p(\Omega; \mathbb{R}^n)$ by

$$u_{\epsilon,\delta}(x) = \begin{cases} v_\epsilon^F(x) + \epsilon\psi_\delta(\frac{x}{\epsilon}) + c & \text{if } x \in \Omega_\epsilon, \\ v_\epsilon^F(x) + c & \text{if } x \in \Omega \setminus \Omega_\epsilon, \end{cases}$$

where $c \in \mathbb{R}^n$ is chosen such that the mean value of $u_{\epsilon,\delta}$ vanishes. Graphically speaking, we add the oscillations that almost obtain the optimal energy on every ϵ-cube contained in Ω to the laminate v_ϵ^F that fits the layered structure, while we only retain the laminate on the boundary area that is of decreasing volume. Thus, we obtain the desired boundary values for the recovery sequence by construction.

Furthermore, a change of variables $y = \epsilon^{-1}x$ and the 1-periodicity of W and ψ_δ yield for each $k \in N_\epsilon$

$$\begin{aligned}\int_{k+(0,\epsilon)^n} W_\epsilon^\alpha(\tfrac{x}{\epsilon}, \nabla u_{\epsilon,\delta})\,\mathrm{d}x &= \epsilon^n \int_{(0,1)^n} W_1^\alpha(y, \nabla v_1^F + \nabla\psi_\delta)\,\mathrm{d}y \\ &= \epsilon^n \int_{(0,1)^{n-1}\times(0,\lambda)} W(F_\lambda + \nabla\psi_\delta)\,\mathrm{d}y.\end{aligned}$$

Since ψ_δ is an almost minimizer we obtain from taking the sum over all $k \in N_\epsilon$ the estimate

$$|\Omega_\epsilon| W_{\text{hom}}(F) \le \int_{\Omega_\epsilon} W_\epsilon^\alpha(\tfrac{x}{\epsilon}, \nabla u_{\epsilon,\delta})\,\mathrm{d}x \le |\Omega_\epsilon|\big(W_{\text{hom}}(F) + \delta\big).$$

Thus, we have

$$|\Omega_\epsilon| W_{\text{hom}}(F) \le \int_{\Omega} W_\epsilon^\alpha(\tfrac{x}{\epsilon}, \nabla u_{\epsilon,\delta})\,\mathrm{d}x \le |\Omega_\epsilon|\big(W_{\text{hom}}(F) + \delta\big) + \int_{\Omega} W_\epsilon^\alpha(\tfrac{x}{\epsilon}, \nabla v_\epsilon^F)\,\mathbb{1}_{\Omega\setminus\Omega_\epsilon}\,\mathrm{d}x.$$

Since $W_\epsilon^\alpha(\frac{x}{\epsilon}, \nabla v_\epsilon^F) = W_\epsilon^\alpha(\frac{x}{\epsilon}, R) = 0$ for $x \in \epsilon P_{\text{stiff}} \cap \Omega$ and the energy density is bounded due to the Lipschitz condition on W, we see that $\nabla v_\epsilon^F\,\mathbb{1}_{\Omega\setminus\Omega_\epsilon}$ is bounded and converges pointwise to zero as $\epsilon \to 0$. Thus, as $|\Omega_\epsilon| \to |\Omega|$, dominated convergence implies

$$E(F) \le \limsup_{\epsilon\to 0} E_\epsilon^\alpha(u_{\epsilon,\delta}) \le E(F) + \delta|\Omega|. \tag{4.14}$$

Now, consider

$$a_{\epsilon,\delta} := |E_\epsilon^\alpha(u_{\epsilon,\delta}) - E(u)| + \|u_{\epsilon,\delta} - u\|_{L^p(\Omega;\mathbb{R}^n)}.$$

Since $u_{\epsilon,\delta} = v_\epsilon^F$ on $\Omega \setminus \Omega_\epsilon$ and $\epsilon\psi_\delta(\epsilon^{-1}\,\cdot\,) \to 0$ strongly in $L^p(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$, the Poincaré inequality together with Lemma 4.2.1 yields $u_{\epsilon,\delta} \to u$ in $L^p(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$. Thus, by (4.14) we have overall

$$\limsup_{\delta\to 0}\limsup_{\epsilon\to 0} a_{\epsilon,\delta} = 0.$$

By the Attouch diagonalization Lemma 4.5.14 there is a function $\delta(\epsilon)$ with $\lim_{\epsilon\to 0} a_{\epsilon,\delta(\epsilon)} = 0$, where existence of the limit is due to the fact that a is non-negative.

We set $u_\epsilon = u_{\epsilon,\delta(\epsilon)}$. This sequence satisfies by construction

$$E_\epsilon^\alpha(u_\epsilon) \to E(u) \quad \text{and} \quad u_\epsilon \to u \text{ in } L^p(\Omega;\mathbb{R}^n) \quad \text{both as } \epsilon \to 0.$$

Furthermore, the growth condition on W yields that the sequence ∇u_ϵ is uniformly bounded. Therefore, u_ϵ is uniformly bounded in the reflexive space $W^{1,p}(\Omega;\mathbb{R}^n)$ implying that every subsequence of u_ϵ contains a convergent subsequence with L^p-limit u. Hence, by the Urysohn lemma, cf. Lemma 4.5.15, we have the whole sequence $u_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$. □

The next proposition also builds on ideas by Müller [112] to establish that the homogenized energy density satiesfies the similar growth and Lipschitz conditions as W.

Proposition 4.3.9 (Growth and Lipschitz condition for the homogenized energy, cf. [112, Lemma 2.1])**.** *For the homogenized energy density $W_{\hom}$ hold similar growth and Lipschitz conditions as for W, i.e. there are constants $c, C > 0$ and $d > 0$ such that for all $F \in \mathbb{R}^{n\times n}$ with $F = R + d \otimes e_n$, $R \in SO(n)$ and $d \in \mathbb{R}^n$*

$$c|F|^p - d \leq W_{\hom}(F) \leq C(1 + |F|^p),$$

and a constant $L > 0$ such that for all $F, G \in \mathbb{R}^{n\times n}$ with $F = R_F + d_F \otimes e_n$, $G = R_G + d_G \otimes e_n$, where $R_F, R_G \in SO(n)$, $d_F, d_G \in \mathbb{R}^n$

$$|W_{\hom}(G) - W_{\hom}(F)| \leq L\big(1 + |F|^{p-1} + |G|^{p-1}\big)\big(|G - F| + |R_G - R_F|\big).$$

Proof. Growth condition. Firstly, notice that for all $F \in \mathbb{R}^{n\times n}$ with $F = R + d \otimes e_n$, $R \in SO(n)$ and $d \in \mathbb{R}^n$, it follows from

$$\tfrac{1}{\lambda}|F| = |\tfrac{1}{\lambda}R + \tfrac{1}{\lambda}d \otimes e_n| \leq (\tfrac{1}{\lambda} - 1)|R| + |F_\lambda|$$

and

$$|F_\lambda| = |R + \tfrac{1}{\lambda}d \otimes e_n| \leq \tfrac{1}{\lambda}|F| + |1 - \tfrac{1}{\lambda}||R|$$

that for $d' = |1 - \frac{1}{\lambda}|$ and $c' = \frac{1}{\lambda}$ the estimate

$$c'|F| - d' \leq |F_\lambda| \leq c'|F| + d' \tag{4.15}$$

holds.

By the admissible choice $\varphi = 0$ in the definition of $W_{\hom}$, we obtain for all $F \in \mathbb{R}^{n\times n}$ with $F = R + d \otimes e_n$, $R \in SO(n)$ and $d \in \mathbb{R}^n$ that $W_{\hom}(F_\lambda) \leq W(F_\lambda)$, which together with (4.15) shows the upper bound. For the lower bound, consider for $Y = (0,1)^n$ and given $F \in \mathbb{R}^{n\times n}$ with $F = R + d \otimes e_n$ a δ-minimizer $\varphi_\delta \in W_0^{1,p}(Y;\mathbb{R}^n)$ in the definition of $W_{\hom}$. It holds by the growth condition on W that

$$\delta + W_{\hom}(F) \geq \int_Y W(F_\lambda + \nabla\varphi_\delta)\,\mathrm{d}x \geq \int_Y c|F_\lambda + \nabla\varphi_\delta|^p\,\mathrm{d}x - d \geq c|F_\lambda|^p - d,$$

by convexity of $F \mapsto |F|^p - d$. Taking $\delta \to 0$ together with (4.15) yields the desired estimate.

Lipschitz condition. To obtain this result we apply the upper and lower bound estimates established in the case of affine limits for $\Omega = Y = (0,1)^n$. Interchanging F and G otherwise, we may assume that $W_{\hom}(G) > W_{\hom}(F)$. Let $f, g \in W^{1,p}(Y;\mathbb{R}^n) \cap L_0^p(Y;\mathbb{R}^n)$ be defined by $\nabla f = F$ and $\nabla g = G$ and let $(v_\epsilon)_\epsilon \subset W^{1,p}(Y;\mathbb{R}^n) \cap L_0^p(Y;\mathbb{R}^n)$ be a recovery sequence for f as constructed in Lemma 4.3.8, and let $(v_\epsilon^F)_\epsilon, (v_\epsilon^G)_\epsilon \subset W^{1,p}(Y;\mathbb{R}^n) \cap L_0^p(Y;\mathbb{R}^n)$ approximating sequences constructed in Lemma 4.2.1 such that

$$v_\epsilon^G \rightharpoonup g \quad \text{and} \quad v_\epsilon^F \rightharpoonup f \quad \text{both in } W^{1,p}(\Omega;\mathbb{R}^n) \quad \text{as } \epsilon \to 0.$$

Furthermore, set $(w_\epsilon)_\epsilon \subset W^{1,p}(Y;\mathbb{R}^n) \cap L_0^p(Y;\mathbb{R}^n)$ to be given by $w_\epsilon = v_\epsilon^G - v_\epsilon^F + v_\epsilon$. Then, by construction of $v_\epsilon^F, v_\epsilon^G$ and the recovery sequence v_ϵ

$$w_\epsilon \rightharpoonup g \text{ in } W^{1,p}(Y;\mathbb{R}^n) \text{ as } \epsilon \to 0 \quad \text{and} \quad \nabla w_\epsilon = R_G \in SO(n) \text{ on } \epsilon P_{\text{stiff}} \cap Y.$$

Hence, by the lower bound estimate of (4.13) we have

$$W_{\hom}(G) \leq \liminf_{\epsilon\to 0} \int_{\epsilon P_{\text{soft}} \cap Y} W(\nabla w_\epsilon)\,\mathrm{d}x,$$

and consequently

$$W_{\text{hom}}(G) - W_{\text{hom}}(F) \leq \liminf_{\epsilon\to 0} \int_{\epsilon P_{\text{soft}}\cap Y} W(\nabla w_\epsilon) - W(v_\epsilon)\,\mathrm{d}x.$$

Since the Lipschitz condition on W implies

$$\begin{aligned}\int_{\epsilon P_{\text{soft}}\cap Y} |W(\nabla w_\epsilon) - W(\nabla v_\epsilon)|\,\mathrm{d}x &\leq C\big(1 + \|\nabla w_\epsilon\|^p_{L^p(Y;\mathbb{R}^n)} + \|\nabla v_\epsilon\|^p_{L^p(Y;\mathbb{R}^n)}\big)^{\frac{p-1}{p}} \lambda|G_\lambda - F_\lambda|\\ &\leq C\big(1 + \|\nabla w_\epsilon\|^p_{L^p(Y;\mathbb{R}^n)} + \|\nabla v_\epsilon\|^p_{L^p(Y;\mathbb{R}^n)}\big)^{\frac{p-1}{p}} \big(|G-F| + (1-\lambda)|R_G - R_F|\big),\end{aligned}$$

it remains only to establish that $\|\nabla w_\epsilon\|^p_{L^p(Y;\mathbb{R}^n)}$ and $\|\nabla v_\epsilon\|^p_{L^p(Y;\mathbb{R}^n)}$ can be estimated from above in terms of $|F|$ and $|G|$ in the limit $\epsilon \to 0$. On the one hand, by the lower growth estimate on W we obtain $|\nabla v_\epsilon|^p(x) \leq \frac{1}{c}W(\nabla v_\epsilon)(x)$ for all $x \in Y$, from which it follows by the upper growth estimate on W_{hom}

$$\limsup_{\epsilon\to 0} \|\nabla v_\epsilon\|^p_{L^p(Y;\mathbb{R}^n)} \leq \limsup_{\epsilon\to 0} c^{-1} E^\alpha_\epsilon(v_\epsilon) \leq c^{-1} W_{\text{hom}}(F) \leq Cc^{-1}\big(|F|^p + 1\big).$$

On the other hand $\nabla w_\epsilon = G - F + \nabla v_\epsilon$, so that

$$\limsup_{\epsilon\to 0} \int_{\epsilon P_{\text{soft}}\cap Y} |W(\nabla w_\epsilon) - W(\nabla v_\epsilon)|\,\mathrm{d}x \leq C\big(1 + |F|^p + |G|^p\big)^{\frac{p-1}{p}} \big(|G-F| + |R_G - R_F|\big).$$

Overall, we obtain

$$W_{\text{hom}}(G) - W_{\text{hom}}(F) \leq C\big(1 + |F|^p + |G|^p\big)^{\frac{p-1}{p}} \big(|G-F| + |R_G - R_F|\big),$$

as desired. □

4.3.2 The case of general limit functions

Note that by the construction of an approximating sequence satisfying the exact differential inclusion constraint, a large step towards a recovery sequence is already taken. To illustrate this point, we show that in the case of a convex energy density, the approximating sequence indeed constitutes a recovery sequence.

Proposition 4.3.10 (Recovery sequence for convex energies)**.** *Suppose that $W = W_c$ is convex. Let Ω be a simply connected bounded Lipschitz domain that is $e_n^\perp$-connected. Furthermore, let $u \in W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ such that $u(x) = R(x)x + b(x)$ for a.e. $x \in \Omega$, where $R \in W^{1,p}(\Omega; SO(n))$ and $b \in L^p(\Omega;\mathbb{R}^n)$ with $\partial_i R = 0$ and $\partial_i b = 0$ for $i = 1, \dots, n-1$.*

Then, there is a sequence $(v_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ with $v_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$ satisfying

$$\limsup_{\epsilon\to 0} E^\alpha_\epsilon(v_\epsilon) \leq E(u).$$

Proof. By Proposition 4.2.2 we obtain a sequence $(v_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n)$ with $\nabla v_\epsilon \in SO(n)$ on $\epsilon P_{\text{stiff}}$ and (4.6), in particular

$$\|\nabla v_\epsilon - U_\lambda\|_{L^p(\epsilon P_{\text{soft}}\cap\Omega;\mathbb{R}^{n\times n})} \to 0. \tag{4.16}$$

Hence, the fact that $\|\operatorname{dist}(\nabla v_\epsilon, SO(n))\|_{L^p(\epsilon P_{\text{stiff}}\cap\Omega)} = 0$ and the Lipschitz continuity of the convex energy density yield

$$E_\epsilon^\alpha(v_\epsilon) = \int_\Omega W_\epsilon^\alpha(x, \nabla v_\epsilon)\,\mathrm{d}x = \int_{\epsilon P_{\mathrm{soft}}\cap\Omega} W_c(\nabla v_\epsilon)\,\mathrm{d}x$$
$$\leq \int_\Omega W_c(U_\lambda)\,\mathbb{1}_{\epsilon P_{\mathrm{soft}}}\,\mathrm{d}x + \|\nabla v_\epsilon - U_\lambda\|_{L^p(\Omega;\mathbb{R}^n)}.$$

Taking the superior limit $\epsilon \to 0$, we obtain by (4.16) and the weak convergence of $\mathbb{1}_{\epsilon P_{\mathrm{soft}}} \overset{*}{\rightharpoonup} \lambda$ in $L^\infty(\Omega)$ as $\epsilon \to 0$ that

$$\limsup_{\epsilon\to 0} E_\epsilon^\alpha(v_\epsilon) \leq \lambda \int_\Omega W_c(U_\lambda)\,\mathrm{d}x = \lambda \int_\Omega W_c(R(x) + \lambda^{-1}R'(x)x \otimes e_n + \lambda^{-1}d(x)\otimes e_n)$$
$$= \lambda \int_\Omega W_c(\lambda^{-1}(\nabla u - (1-\lambda)R)) = E(u).$$

This shows the claim. □

Now, for non-convex W we have to incorporate additional microstructure in the recovery sequence as done in the affine case in Lemma 4.3.8. One approach, featured for example in the above mentioned work by Müller [112] is to tie the general case to the already established affine case by approximating the limit function u by piecewise affine functions. Here, this is not practical, as the class of functions u whose gradients are piecewise constant and of the form $\nabla u = R + d \otimes e_n$, $R \in SO(n)$, $d \in \mathbb{R}^n$ is rather restricted. For more details in two dimensions, see also Lemma 5.2.3.

Hence, the idea is not to use a piecewise affine function as a starting point, but rather the approximating sequence $(v_\epsilon)_\epsilon$ constructed in Lemma 4.2.2, and glue the microstructure that is locally optimal with respect the specific values of u on v_ϵ. Note that at this point we have to restrict ourselves to simply connected bounded Lipschitz domains that are $e_n^\perp$-connected.

Lemma 4.3.11 (Recovery sequence for general limits)**.** *Let Ω be a simply connected bounded Lipschitz domain that is $e_n^\perp$-connected and let $u \in L_0^p(\Omega;\mathbb{R}^n)$. Then, there is a sequence $(u_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n) \cap L_0^p(\Omega;\mathbb{R}^n)$ with $u_\epsilon \to u$ in $L^p(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$ satisfying*

$$\limsup_{\epsilon\to 0} E_\epsilon^\alpha(u_\epsilon) \leq E(u).$$

Proof. Firstly, note that we may assume that $E(u) < \infty$. Otherwise the constant sequence $u_\epsilon = \mathrm{id}_{\mathbb{R}^n}$ for all $\epsilon > 0$ satisfies all requirements. If $E(u) < \infty$, we have by definition of E that $u \in W^{1,p}(\Omega;\mathbb{R}^n) \cap L_0^p(\Omega;\mathbb{R}^n)$ with $u(x) = R(x)x + b(x)$, for a.e. $x \in \Omega$, where $R \in W^{1,p}(\Omega; SO(n))$ and $b \in L^p(\Omega;\mathbb{R}^n)$ such that $\partial_i R = 0$ and $\partial_i b = 0$ for $i = 1, \dots, n-1$.

Step 1: Piecewise constant approximation of ∇u. By differentiating u in the representation $u(x) = R(x)x + b(x)$, a.e. $x \in \Omega$ as in (4.3), we obtain for $d \in L^p(\Omega;\mathbb{R}^n)$ given for $x \in \Omega$ by $d(x) := R'(x)x + b'$,

$$\nabla u(x) = R(x) + R'(x)x \otimes e_n + b'(x) \otimes e_n = R(x) + d(x) \otimes e_n, \quad \text{for a.e. } x \in \Omega.$$

Based on this representation we approximate ∇u by approximating each of the functions R, R', b' and the identity map $\mathrm{id}_{\mathbb{R}^n}$ by simple functions on a common partition of Ω. More precisely, there is for every $\delta > 0$ a sufficiently fine approximation Ω_δ of Ω by finitely many cuboids $Q_\delta^j \subset \Omega$, which we index by J_δ such that $\Omega_\delta = \bigcup_{j\in J_\delta} Q_\delta^j$ satisfies $|\Omega \setminus \Omega_\delta| \to 0$ as $\delta \to 0$, as well as simple functions

$$R_\delta = \sum_{j\in J_\delta} R_\delta^j\,\mathbb{1}_{Q_\delta^j}, \quad d_\delta = \sum_{j\in J_\delta} d_\delta^j\,\mathbb{1}_{Q_\delta^j}, \quad G_\delta = \sum_{j\in J_\delta} G_\delta^j\,\mathbb{1}_{Q_\delta^j}, \quad \xi_\delta = \sum_{j\in J_\delta} \xi_\delta^j\,\mathbb{1}_{Q_\delta^j},$$

where $d_\delta^j, \xi_\delta^j \in \mathbb{R}^n$, $G_\delta^j \in \mathbb{R}^{n\times n}$ and $R_\delta^j \in SO(n)$ such that

$$\begin{aligned}\|R_\delta - R\|_{L^p(\Omega;\mathbb{R}^{n\times n})} &+ \|G_\delta - R'\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \\ &+ \|d_\delta - b'\|_{L^p(\Omega;\mathbb{R}^n)} + \|\xi_\delta - \mathrm{id}_{\mathbb{R}^n}\|_{L^\infty(\Omega;\mathbb{R}^n)} < \delta. \end{aligned} \tag{4.17}$$

For approximation by simple functions under manifold constraints, see Lemma 4.5.12. Overall, we define the simple function $U_\delta : \Omega \to \mathbb{R}$ by

$$U_\delta = R_\delta + G_\delta \xi_\delta \otimes e_n + d_\delta \otimes e_n = \sum_{j\in J_\delta} U_\delta^j\, \mathbb{1}_{Q_\delta^j},$$

where

$$U_\delta^j = R_\delta^j + G_\delta^j \xi_\delta^j \otimes e_n + d_\delta^j \otimes e_n, \quad j \in J_\delta.$$

By (4.17) we have the estimate

$$\begin{aligned}\|U_\delta - \nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})} &\le \|R_\delta - R\|_{L^p(\Omega;\mathbb{R}^{n\times n})} + \mathrm{diam}(\Omega)\|G_\delta - R'\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \\ &\quad + \|d_\delta - b'\|_{L^p(\Omega;\mathbb{R}^n)} + \|R'\|_{L^p(\Omega;\mathbb{R}^{n\times n})}\|\xi_\delta - \mathrm{id}_{\mathbb{R}^n}\|_{L^\infty(\Omega;\mathbb{R}^n)} \\ &< C\delta. \end{aligned} \tag{4.18}$$

Step 2: Determining locally optimal microstructure. Applying Lemma 4.3.8 to each Q_δ^j, $j \in J_\delta$, we obtain sequences $(z_{\delta,\epsilon}^j)_\epsilon \subset W^{1,p}(Q_\delta^j;\mathbb{R}^n)$ such that

$$\nabla z_{\delta,\epsilon}^j \rightharpoonup U_\delta \quad \text{in } L^p(Q_\delta^j;\mathbb{R}^{n\times n}) \quad \text{as } \epsilon \to 0$$

and

$$\lim_{\epsilon\to 0} \int_{Q_\delta^j} W_\epsilon^\alpha(x, \nabla z_\delta^j)\, \mathrm{d}x = \int_{Q_{\delta,\epsilon}^j} W_{\mathrm{hom}}(U_\delta)\, \mathrm{d}x.$$

Also, the boundary values of $z_{\delta,\epsilon}^j$ coincide with the laminate $w_{\delta,\epsilon}^j := v_\epsilon^{U_\delta^j} \in W^{1,\infty}(Q_\delta^j;\mathbb{R}^n)$ by construction, so that $\varphi_{\delta,\epsilon}^j : Q_\delta^j \to \mathbb{R}^n$ given by $\varphi_{\delta,\epsilon}^j = z_{\delta,\epsilon}^j - w_{\delta,\epsilon}^j$ satisfies $\varphi_{\delta,\epsilon}^j \in W_0^{1,p}(Q_\delta^j;\mathbb{R}^n)$. We gather this local function in $\varphi_{\delta,\epsilon} \in W^{1,p}(\Omega;\mathbb{R}^n)$ defined by $\varphi_{\delta,\epsilon} = \sum_{j\in J_\delta} \varphi_{\delta,\epsilon}^j\, \mathbb{1}_{Q_\delta^j}$.

The local Lipschitz condition satisfied by the homogenized energy density W_{hom}, see Lemma 4.3.9, together with (4.17) and (4.18) yields for $\Omega_\delta = \bigcup_{j\in J_\delta} Q_\delta^j$

$$\begin{aligned}\int_{\Omega_\delta} W_{\mathrm{hom}}(U_\delta)\, \mathrm{d}x &\le \int_\Omega W_{\mathrm{hom}}(\nabla u)\, \mathrm{d}x + C\|U_\delta - \nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})} + \|R_\delta - R\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \\ &\le \int_\Omega W_{\mathrm{hom}}(\nabla u)\, \mathrm{d}x + C\delta. \end{aligned}$$

Thus, taking the sum over all $j \in J_\delta$ and the limit $\epsilon \to 0$ entails

$$\limsup_{\epsilon\to 0} \int_{\Omega_\delta} \sum_{j\in J_\delta} W_\epsilon^\alpha(x, \nabla w_{\delta,\epsilon}^j)\, \mathbb{1}_{Q_\delta^j}\, \mathrm{d}x \le \int_\Omega W_{\mathrm{hom}}(\nabla u)\, \mathrm{d}x + C\delta. \tag{4.19}$$

Step 3: Adding locally optimal microstructure to approximating sequence. As mentioned before, the problem of the above approximation is that while it is suitable to determine the locally optimal microstructure, the aggregate of laminates $w_{\epsilon,\delta} := \sum_{j\in J_\delta} w_{\delta,\epsilon}^j\, \mathbb{1}_{Q_\delta^j}$ does not feature gradient structure on the whole of Ω. Therefore, we add the local microstructure $\varphi_{\delta,\epsilon}$

to v_ϵ constructed in Proposition 4.2.2 that has shown its potential as a recovery sequence in the case of convex energy densities in Proposition 4.3.10, where no additional microstructure is needed.

Accordingly, let $u_{\delta,\epsilon} \in W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ be given by

$$u_{\delta,\epsilon} := v_\epsilon + \varphi_{\delta,\epsilon} + c_{\delta,\epsilon},$$

where $c_{\delta,\epsilon} \in \mathbb{R}$ is chosen such that the mean value vanishes.

It remains to estimate the difference in energy caused by switching from the laminates $w^j_{\delta,\epsilon}$ to v_ϵ. Note that on the stiff layers there is no change in energy as the approximating functions v_ϵ as well as the laminates $w^j_{\delta,\epsilon}$ satisfy the exact differential inclusion constraint. For the soft layers observe that the L^p-norms of ∇v_ϵ, $\nabla w_{\delta,\epsilon}$ and $\nabla\varphi_{\delta,\epsilon}$ on Ω are uniformly bounded with respect to ϵ and δ. Indeed, the uniform bound of ∇v_ϵ on soft layers follows from Proposition 4.2.2, while the uniform bound on $\nabla w_{\delta,\epsilon}$ and $\nabla\varphi_{\delta,\epsilon}$ is due to the construction in the Proof of Lemma 4.3.8, with $w^j_{\delta,\epsilon}$ being a simple laminate, $\varphi_{\delta,\epsilon}$ derived from an almost minimizer of W and W controlling the gradient by the growth condition from below. In other words, there is a constant $\tilde{C} > 0$ independent of δ such that

$$\int_\Omega 1 + |\nabla v_\epsilon|^p + \sum_{j\in J_\delta} |\nabla w^j_{\delta,\epsilon}|^p \,\mathbb{1}_{Q^j_\delta} + |\nabla\varphi_{\delta,\epsilon}|^p \,\mathrm{d}x < \tilde{C}.$$

By the Lipschitz condition satisfied by W, we obtain summing over all Q^j_δ, $j \in J_\delta$, using the notation $(\nabla u)_\lambda$ as in (4.4) and Hölder's inequality for integrals and sums

$$\begin{aligned}
&\Big|\int_{\epsilon P_{\mathrm{soft}}\cap\Omega_\delta} W(\nabla u_{\delta,\epsilon}) - \sum_{j\in J_\delta} W(\nabla w^j_{\delta,\epsilon})\,\mathbb{1}_{Q^j_\delta}\,\mathrm{d}x\Big| \\
&\le \sum_{j\in J_\delta} \int_{\epsilon P_{\mathrm{soft}}\cap Q^j_\delta} \big|W(\nabla u_{\delta,\epsilon}) - W(\nabla w^j_{\delta,\epsilon})\big|\,\mathrm{d}x \\
&\le \sum_{j\in J_\delta} C \int_{\epsilon P_{\mathrm{soft}}\cap Q^j_\delta} (1 + |\nabla v_\epsilon|^{p-1} + |\nabla w^j_{\delta,\epsilon}|^{p-1} + |\nabla\varphi_{\delta,\epsilon}|^{p-1})\,|\nabla v_\epsilon - \nabla w^j_{\delta,\epsilon}|\,\mathrm{d}x \\
&\le C\cdot\tilde{C}^{\frac{p-1}{p}}\Big(\sum_{j\in J_\delta} \int_{\epsilon P_{\mathrm{soft}}\cap Q^j_\delta} |\nabla v_\epsilon - \nabla w^j_{\delta,\epsilon}|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \\
&\le C\|\nabla v_\epsilon - (\nabla u)_\lambda\|_{L^p(\epsilon P_{\mathrm{soft}}\cap\Omega;\mathbb{R}^{n\times n})} + C\|(\nabla u)_\lambda - (U_\delta)_\lambda\|_{L^p(\epsilon P_{\mathrm{soft}}\cap\Omega_\delta;\mathbb{R}^{n\times n})} \\
&\le C\|\nabla v_\epsilon - (\nabla u)_\lambda\|_{L^p(\epsilon P_{\mathrm{soft}}\cap\Omega;\mathbb{R}^{n\times n})} + C\delta,
\end{aligned} \tag{4.20}$$

where in the last estimate we have used the fact that (4.18) together with (4.17) yields for a constant $C > 0$ independent of δ

$$\begin{aligned}
&\|(\nabla u)_\lambda - (U_\delta)_\lambda\|_{L^p(\epsilon P_{\mathrm{soft}}\cap\Omega_\delta;\mathbb{R}^{n\times n})} \\
&\qquad \le \big(\tfrac{1}{\lambda} - 1\big)\|R - R_\delta\|_{L^p(\epsilon P_{\mathrm{soft}}\cap\Omega;\mathbb{R}^{n\times n})} + \tfrac{1}{\lambda}\|U - U_\delta\|_{L^p(\Omega;\mathbb{R}^{n\times n})} < C\delta.
\end{aligned} \tag{4.21}$$

Note that by the same reasoning

$$\begin{aligned}
&\int_\Omega \big|W(\nabla u_{\delta,\epsilon}) - W(\nabla v_\epsilon)\big|\,\mathrm{d}x \\
&\quad \le \int_\Omega \Big|W(\nabla u_{\delta,\epsilon}) - \sum_{j\in J_\delta} W(\nabla w^j_{\delta,\epsilon})\,\mathbb{1}_{Q^j_\delta}\Big| + \Big|\sum_{j\in J_\delta} W(\nabla w^j_{\delta,\epsilon})\,\mathbb{1}_{Q^j_\delta} - W(\nabla v_\epsilon)\Big|\,\mathrm{d}x \\
&\quad \le C\delta.
\end{aligned} \tag{4.22}$$

Step 4: Diagonalization. Since $\nabla u_{\delta,\epsilon}$ satisfies the differential inclusion constraint on the stiff layers exactly, we obtain by (4.20), (4.19) and the estimates of Proposition 4.2.2 that

$$\begin{aligned}\limsup_{\epsilon\to 0} E_\epsilon^\alpha(u_{\delta,\epsilon}) &= \limsup_{\epsilon\to 0}\int_{\Omega_\delta} W(\nabla u_{\delta,\epsilon})\,\mathrm{d}x + \limsup_{\epsilon\to 0}\int_{\Omega\setminus\Omega_\delta} W(\nabla u_{\delta,\epsilon})\,\mathrm{d}x \\ &\leq \int_\Omega W_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x + C\delta + \limsup_{\epsilon\to 0}\int_{\Omega\setminus\Omega_\delta} W(\nabla u_{\delta,\epsilon})\,\mathrm{d}x.\end{aligned}$$

For the last term, (4.22), the growth condition on W from above and the fact that $(\nabla v_\epsilon)_\epsilon$ is strongly convergent in $L^p(\Omega;\mathbb{R}^{n\times n})$ on the soft layers in the sense of (4.6), entail for $(\nabla u)_\lambda$

$$\begin{aligned}\limsup_{\epsilon\to 0}\int_{\Omega\setminus\Omega_\delta} W(\nabla u_{\delta,\epsilon})\,\mathrm{d}x &\leq \limsup_{\epsilon\to 0}\int_{\epsilon P_{\mathrm{soft}}\cap\Omega} W(\nabla v_\epsilon)\,\mathbb{1}_{\Omega\setminus\Omega_\delta}\,\mathrm{d}x + C\delta \\ &\leq \limsup_{\epsilon\to 0}\int_{\epsilon P_{\mathrm{soft}}\cap\Omega} |\nabla v_\epsilon|^p\,\mathbb{1}_{\Omega\setminus\Omega_\delta}\,\mathrm{d}x + C\delta \\ &\leq \int_\Omega |(\nabla u)_\lambda|^p\,\mathbb{1}_{\Omega\setminus\Omega_\delta}\,\mathrm{d}x + C\delta \to 0 \quad \text{as } \delta\to 0,\end{aligned}$$

where the convergence in the last line follows by dominated convergence.

Furthermore, by construction of $u_{\delta,\epsilon}$, we see that

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} u_{\delta,\epsilon} = \lim_{\delta\to 0}\lim_{\epsilon\to 0}(v_\epsilon + \varphi_{\delta,\epsilon} + c_{\delta,\epsilon}) = u.$$

Hence, by the Attouch diagonalization Lemma 4.5.14 there is a sequence $\delta(\epsilon)$ such that $u_\epsilon := u_{\epsilon,\delta(\epsilon)} \in W^{1,p}(\Omega;\mathbb{R}^n)\cap L_0^p(\Omega;\mathbb{R}^n)$ satisfies $u_\epsilon \to u$ in $L^p(\Omega;\mathbb{R}^n)$ and

$$\limsup_{\epsilon\to 0} E_\epsilon^\alpha(u_\epsilon) \leq \int_\Omega W_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x = E(u).$$

Finally, since the sequence $(u_\epsilon)_\epsilon$ is uniformly bounded with respect to the $W^{1,p}(\Omega;\mathbb{R}^n)$-norm we obtain by a Urysohn argument, cf. Lemma 4.5.15, that $u_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon\to 0$. This shows that u_ϵ has the desired properties of a recovery sequence. ☐

Lemma 4.3.12 (Compactness). *Let Ω be a simply connected bounded Lipschitz domain that is $e_n^\perp$-connected and e_n-flat. Furthermore, let $(u_\epsilon)_\epsilon \subset L_0^p(\Omega;\mathbb{R}^n)$ be of bounded energy with respect to $(E_\epsilon^\alpha)_\epsilon$, i.e. for a constant $C>0$ it holds that $E_\epsilon^\alpha(u_\epsilon)\leq C$ for all $\epsilon>0$. Then, there is a function $u\in W^{1,p}(\Omega;\mathbb{R}^n)\cap L_0^p(\Omega;\mathbb{R}^n)$ of the form (4.3) such that for a subsequence $(u_{\epsilon_i})_{i\in\mathbb{N}}$*

$$u_{\epsilon_i} \rightharpoonup u \quad \text{in } W^{1,p}(\Omega;\mathbb{R}^n) \quad \text{as } \epsilon\to 0.$$

Proof. Observe that the growth condition on W and the fact that the norm of can be estimated by the distance to $SO(n)$, see also (3.35), yield that $(\nabla u_\epsilon)_\epsilon$ is uniformly bounded in $L^p(\Omega;\mathbb{R}^n)$. Since u_ϵ is mean value free, we obtain by the Poincaré inequality that $(\nabla u_\epsilon)_\epsilon$ is uniformly bounded in $W^{1,p}(\Omega;\mathbb{R}^n)$. Hence, by the reflexivity of L^p for $1<p<\infty$ there is a subsequence (u_{ϵ_i}) and a $u\in W^{1,p}(\Omega;\mathbb{R}^n)\cap L_0^p(\Omega;\mathbb{R}^n)$ such that $u_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon\to 0$.

Regarding (4.3), the additional properties of u follow directly by Theorem 3.3.1 and in particular Corollary 3.3.8. ☐

Proposition 4.3.13 (Lower bound). *Let Ω be a simply connected bounded Lipschitz domain that is $e_n^\perp$-connected and e_n-flat. Let $(u_\epsilon)_\epsilon \subset L_0^p(\Omega;\mathbb{R}^n)$ with $E_\epsilon^\alpha(u_\epsilon)<C$ such that $u_\epsilon\to u$ in $L^p(\Omega;\mathbb{R}^n)$ as $\epsilon\to 0$ for $u\in L_0^p(\Omega;\mathbb{R}^n)$.*

Then,

$$\liminf_{\epsilon\to 0} E_\epsilon^\alpha(u_\epsilon) \geq E(u).$$

Proof. As we may assume that $(u_\epsilon)_\epsilon$ is of bounded energy and $(u_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$, Lemma 4.3.12 yields that u is of the form (4.3), which implies for the gradient that

$$\nabla u(x) = R(x) + R'(x)x \otimes e_n + b'(x) \otimes e_n \quad \text{for a.e. } x \in \Omega, \tag{4.23}$$

where $R \in W^{1,p}(\Omega; SO(n))$ and $b \in W^{1,p}(\Omega;\mathbb{R}^n)$ with $\partial_i R = 0$ and $\partial_i b = 0$ for $i \in \{1,\dots,n-1\}$. By the non-negativity of the energy density it suffices for the validity of the lim inf-inequality to show that

$$\liminf_{\epsilon\to 0} \int_{\epsilon P_{\mathrm{soft}} \cap \Omega} W(\nabla u_\epsilon)\,\mathrm{d}x \geq \lambda \int_\Omega W^{\mathrm{qc}}((\nabla u)_\lambda)\,\mathrm{d}x = \lambda \int_\Omega W^{\mathrm{qc}}(\lambda^{-1}(\nabla u - (1-\lambda)R))\,\mathrm{d}x.$$

To establish this estimate we follow a classic strategy, previously applied e.g. in [112, Proof of Theorem 1.3], to reduce the general case to the one of affine limits using a suitable piecewise constant approximation of the limit gradient. Notice that for this argument a gradient structure of the approximation is only required locally on each piece, so that the results for the affine case can be applied to each piece separately.

Step 1: Construction of an approximating sequence. Based on the representation (4.23) we approximate ∇u as in Step 1 of the Proof of Lemma 4.3.11 using an approximation for each of the functions R, R', b' and the identity map $\mathrm{id}_{\mathbb{R}^n}$ by simple functions on a common lattice. To avoid repetition, we adopt the same notation as above and retain in particular the estimate (4.17).

For $\delta > 0$ there exists by Lemma 4.2.1 a sequence $(w_{\delta,\epsilon}) \subset L^p_0(\Omega_\delta;\mathbb{R}^n)$ defined separately on each Q^j_δ satisfying $w^j_{\delta,\epsilon} = w_{\delta,\epsilon}|_{Q^j_\delta} \in W^{1,p}(Q^j_\delta;\mathbb{R}^n) \cap L^p_0(Q^j_\delta;\mathbb{R}^n)$ and

$$w_{\delta,\epsilon} \rightharpoonup U^j_\delta x + c^j_\delta \quad \text{in } W^{1,p}(Q^j_\delta;\mathbb{R}^n) \quad \text{as } \epsilon \to 0,$$

where c^j_δ is such that the mean value of $x \mapsto U^j_\delta x + c^j_\delta$ vanishes on Q^j_δ.

Besides, let $(v_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ the sequence constructed in Proposition 4.2.2, satisfying (4.5) and (4.6).

Lastly, we introduce the sequence $(z_{\delta,\epsilon})_\epsilon \subset L^p_0(\Omega_\delta;\mathbb{R}^n)$, given by $z_{\delta,\epsilon} := u_\epsilon - v_\epsilon + w_{\delta,\epsilon}$. Notice that $z^j_{\delta,\epsilon} = z_{\delta,\epsilon}|_{Q^j_\delta} \in W^{1,p}(Q^j_\delta;\mathbb{R}^n)$, $j \in J_\delta$, and

$$\nabla z_{\delta,\epsilon} = \nabla u_\epsilon - \nabla v_\epsilon + \nabla w_{\delta,\epsilon} \rightharpoonup U^j_\delta \quad \text{in } L^p(Q^j_\delta;\mathbb{R}^n) \quad \text{as } \epsilon \to 0 \quad \text{for all } j \in J_\delta.$$

Additionally, we may assume, that in the notation of Definition 3.3.5 we have $(Q^j_\delta)'' \subset \Omega$, splitting the cuboids Q^j_δ if necessary. For fixed $\delta > 0$ let $(\Sigma^j_\epsilon)_\epsilon \subset L^\infty(Q^j_\delta; SO(n))$, $j \in J_\delta$ with

$$\|\nabla u_\epsilon - \Sigma^j_\epsilon\|_{L^p(\epsilon P_{\mathrm{stiff}} \cap Q^j_\delta;\mathbb{R}^{n\times n})} < C\epsilon^{\frac{\alpha}{p}-1},$$

from Proposition 3.3.10, for which we know by Lemma 3.3.12 and Proposition 3.3.15 that $\Sigma^j_\epsilon \to R$ in $L^p(Q^j_\delta;\mathbb{R}^{n\times n})$ as $\epsilon \to 0$. Then, we obtain by (4.6)

$$\begin{aligned}
\|\nabla z_{\delta,\epsilon} - R_\epsilon\|^p_{L^p(\epsilon P_{\mathrm{stiff}} \cap Q^j_\delta;\mathbb{R}^{n\times n})} &= \|\nabla u_\epsilon - R_\epsilon\|^p_{L^p(\epsilon P_{\mathrm{stiff}} \cap Q^j_\delta;\mathbb{R}^{n\times n})} \\
&= \|\nabla u_\epsilon - \Sigma^j_\epsilon\|^p_{L^p(\epsilon P_{\mathrm{stiff}} \cap Q^j_\delta;\mathbb{R}^{n\times n})} + \|\Sigma^j_\epsilon - R\|^p_{L^p(\epsilon P_{\mathrm{stiff}} \cap Q^j_\delta;\mathbb{R}^{n\times n})} \\
&\qquad + \|R - R_\epsilon\|^p_{L^p(\epsilon P_{\mathrm{stiff}} \cap Q^j_\delta;\mathbb{R}^{n\times n})} \\
&\to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

Thus, we can apply Lemma 4.3.4 to each Q_δ^j, $j \in J_\delta$ separately and taking the sum over all $j \in J_\delta$ yields

$$\begin{aligned}\liminf_{\epsilon\to 0} \int_{\epsilon P_{\mathrm{soft}}\cap\Omega_\delta} \sum_{i\in J_\delta} W(x, \nabla z_{\delta,\epsilon})\, \mathbb{1}_{Q_\delta^j}\, \mathrm{d}x &\geq \lambda \int_{\Omega_\delta} W^{\mathrm{qc}}(\lambda^{-1}(\nabla u - (1-\lambda)R))\, \mathrm{d}x \\ &= \int_{\Omega_\delta} W_{\mathrm{hom}}(\nabla u)\, \mathrm{d}x. \end{aligned} \tag{4.24}$$

Step 2: Energy estimates for the approximating sequence.

In this step, we show that both the left and the right hand side in (4.24) are close to the respective sides in the desired lim inf-inequality for $(u_\epsilon)_\epsilon$, in the sense that taking the limit $\delta \to 0$ yields the claim.

The Lipschitz condition established for W_{hom} in Proposition 4.3.9 together with (4.17) and (4.18) yields for the right hand side of (4.24)

$$\|W_{\mathrm{hom}}(U_\delta) - W_{\mathrm{hom}}(\nabla u)\|_{L^1(\Omega_\delta)} \leq C\|U_\delta - \nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})} + \|R_\delta - R\|_{L^p(\Omega;\mathbb{R}^{n\times n})} < C\delta.$$

On the left hand side of (4.24), we obtain by the assumed Lipschitz condition for W arguing similarly to (4.20),

$$\begin{aligned}&\int_{\epsilon P_{\mathrm{soft}}\cap\Omega} \sum_{j\in J_\delta} |W(x, \nabla z_{\delta,\epsilon}^j) - W(x,\nabla u)|\, \mathbb{1}_{Q_\delta^j}\, \mathrm{d}x \\ &\qquad = \int_{\epsilon P_{\mathrm{soft}}\cap\Omega} \sum_{i\in J_\delta} |W(\nabla u_\epsilon - \nabla v_\epsilon + \nabla w_{\delta,\epsilon}^j) - W(\nabla u_\epsilon)|\, \mathbb{1}_{Q_\delta^j}\, \mathrm{d}x \\ &\qquad \leq C\Big\|\nabla v_\epsilon - \sum_{j\in J_\delta} (\nabla w_{\delta,\epsilon}^j)\, \mathbb{1}_{Q_\delta^j}\Big\|_{L^p(\epsilon P_{\mathrm{soft}}\cap\Omega_\delta;\mathbb{R}^{n\times n})}.\end{aligned}$$

Furthermore, we have in light of (4.21) and (4.17)

$$\Big\|\nabla v_\epsilon - \sum_{j\in J_\delta} (\nabla w_{\delta,\epsilon}^j)\, \mathbb{1}_{Q_\delta^j}\Big\|_{L^p(\epsilon P_{\mathrm{soft}}\cap\Omega_\delta;\mathbb{R}^{n\times n})} < \|\nabla v_\epsilon - (\nabla u)_\lambda\|_{L^p(\epsilon P_{\mathrm{soft}}\cap\Omega_\delta;\mathbb{R}^{n\times n})} + C\delta,$$

where by (4.6)

$$\|\nabla v_\epsilon - (\nabla u)_\lambda\|_{L^p(\epsilon P_{\mathrm{soft}}\cap\Omega_\delta;\mathbb{R}^{n\times n})} \to 0 \quad \text{as } \epsilon \to 0.$$

Hence, taking the limit $\delta \to 0$ yields the claim. □

4.4 Relation to (multi-)cell formulas

In this section we discuss the relation of the homogenized energy (4.11) to cell and multicell formulas, see also Section 2.3. As we will show in the first proposition, the formula obtained coincides with the cell formula for an energy density modeling totally rigid layers.

Proposition 4.4.1 (Relation to cell formula)**.** *Let $W_\epsilon^\infty : \Omega \times \mathbb{R}^{n\times n} \to [0,\infty]$ be given by*

$$W_\epsilon^\infty(x, F) = \chi_{SO(n)}(F)\, \mathbb{1}_{P_{\mathrm{stiff}}}(x) + W(F)\, \mathbb{1}_{P_{\mathrm{soft}}}(x) \quad x \in \Omega, F \in \mathbb{R}^{n\times n}.$$

Then, we have

$$W_{\mathrm{rig}}^{\mathrm{cell}}(F) = \lambda W^{\mathrm{qc}}(\lambda^{-1}(F - (1-\lambda)R)) = W_{\mathrm{hom}}(F)$$

for all $F \in \mathbb{R}^{n\times n}$ of the form $F = R + d \otimes e_n$, $R \in SO(n)$, $d \in \mathbb{R}^n$ and both sides taking the value ∞, otherwise.

Proof. Let $Y = (0,1)^n$ denote the unit cube. By definition of W_{rig} and the cell formula, we see that if for $F \in \mathbb{R}^{n\times n}$

$$W_{\mathrm{rig}}^{\mathrm{cell}}(F) = \inf_{\psi \in W^{1,p}_{\mathrm{per}}(Y,\mathbb{R}^n)} \int_Y W_{\mathrm{rig}}(x, F + \nabla\psi)\,\mathrm{d}x < \infty,$$

then $F + \nabla\psi \in SO(n)$ a.e. on $P_{\mathrm{stiff}} \cap Y$ and thus by Rešetnjak's Rigidity Theorem 3.2.3 there is a rotation $R \in SO(n)$ such that

$$F + \nabla\psi = R \text{ on } P_{\mathrm{stiff}} \cap Y. \tag{4.25}$$

Hence, by the periodicity of ψ in the e_i-direction for $i = 1,\dots,n-1$ we have

$$Fe_i = Fe_i + \int_Y \partial_i\psi\,\mathrm{d}x = \int_Y Re_i\,\mathrm{d}x = Re_i. \tag{4.26}$$

This shows that for some $d \in \mathbb{R}^n$, we have

$$F = R + d \otimes e_n. \tag{4.27}$$

From (4.25) we conclude that $\nabla\psi = -d \otimes e_n$ on P_{stiff}, meaning that ψ has the representation $\psi = \varphi_F + \varphi$, where $\varphi_F \in W^{1,\infty}_{\mathrm{per}}(Y;\mathbb{R}^n) \cap L^p_0(\Omega;\mathbb{R}^n)$ is the piecewise affine function determined by

$$\nabla\varphi_F = (-\mathbb{1}_{P_{\mathrm{stiff}}} + \tfrac{1-\lambda}{\lambda}\mathbb{1}_{P_{\mathrm{soft}}})d \otimes e_n$$

and $\varphi \in W^{1,p}_{\mathrm{per}}(Y;\mathbb{R}^n)$ satisfies $\nabla\varphi = 0$ on $P_{\mathrm{stiff}} \cap Y$. Therefore,

$$\begin{aligned}
&\inf_{\psi \in W^{1,p}_{\mathrm{per}}(Y;\mathbb{R}^n)} \int_Y W_{\mathrm{rig}}(x, F + \nabla\psi)\,\mathrm{d}x \\
&\quad = \inf\Big\{ \int_{P_{\mathrm{soft}}\cap Y} W(F + \nabla\psi)\,\mathrm{d}x \;\Big|\; \psi \in W^{1,p}_{\mathrm{per}}(Y;\mathbb{R}^n), \nabla\psi = -d \otimes e_n \text{ on } P_{\mathrm{stiff}} \Big\} \\
&\quad = \inf\Big\{ \int_{P_{\mathrm{soft}}\cap Y} W(F + \tfrac{1-\lambda}{\lambda} d \otimes e_n + \nabla\varphi)\,\mathrm{d}x \;\Big|\; \varphi \in W^{1,p}_{\mathrm{per}}(Y;\mathbb{R}^n), \nabla\varphi = 0 \text{ on } P_{\mathrm{stiff}} \Big\} \\
&\quad = \inf\Big\{ \lambda \fint_{P_{\mathrm{soft}}\cap Y} W(F - (1-\lambda)R + \nabla\varphi)\,\mathrm{d}x \;\Big|\; \varphi \in W^{1,p}_{\mathrm{per}}(Y;\mathbb{R}^n), \nabla\varphi = 0 \text{ on } P_{\mathrm{stiff}} \Big\} \\
&\quad = \lambda \inf_{\varphi \in W^{1,p}_0(P_{\mathrm{soft}}\cap Y;\mathbb{R}^n)} \fint_{P_{\mathrm{soft}}\cap Y} W(F - (1-\lambda)R + \nabla\varphi)\,\mathrm{d}x \\
&\quad = \lambda W^{\mathrm{qc}}(F - (1-\lambda)R),
\end{aligned}$$

where we have used that in the formula for the quasiconvex envelope zero boundary values can be replaced by periodic boundary values, see [58, Proposition 5.13]. This establishes the claim. □

Notice in particular, that the structure of the limit gradient F as determined by the asymptotic characterization of Theorem 3.3.1 is also recovered from the cell formula in (4.27). Hence, the representation of the homogenized energy via the cell formula cannot hold in the regime $0 < \alpha < p$ as we have seen the existence of sequences in this regime in Section 3.4 satisfying the approximate differential inclusion constraint, yet not complying with the limit characterization suggested by Theorem 3.3.1 for the regime $\alpha > p$.

At the end of this section, we want to illustrate this fact with an explicit example in two dimensions, which is inspired by arguments of Müller, made in the context of the discussion of cell and multicell formulas [112, Theorem 4.2], see also Proposition 2.3.7.

Example 4.4.2 (Counterexample for insufficient stiffness)**.** Let $Y = [0,1]^2 \subset \mathbb{R}^2$ denote the two-dimensional unit square, $p > 2$. In the following, we show that there is a sequence $(u_\epsilon)_\epsilon \subset W^{1,p}(Y;\mathbb{R}^2)$ with $u_\epsilon \rightharpoonup u$ in $W^{1,p}(Y;\mathbb{R}^2)$ for some $u \in W^{1,p}(Y;\mathbb{R}^2)$ of the form $u = \operatorname{diag}(d,1)x + \psi$ with $0 < d < 1$ and $\psi \in W^{1,p}_{\mathrm{per}}(Y;\mathbb{R}^2)$ and such that for any $0 < \alpha < p$

$$\liminf_{\epsilon \to 0} \int_Y W^\alpha_\epsilon(x, \nabla u_\epsilon)\,\mathrm{d}x < C\lambda, \tag{4.28}$$

yet,

$$\int_Y W_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x = \int_Y W^{\mathrm{cell}}_{\mathrm{rig}}(\nabla u)\,\mathrm{d}x = \infty, \tag{4.29}$$

where $W^{\mathrm{cell}}_{\mathrm{rig}}$ denotes cell formula associated to W_{rig} as defined in Definition 2.3.3.

The following arguments are inspired by Müller's discussion of cell and multicell formulas [112, Theorem 4.2], see also Proposition 2.3.7.

Recall from Example 3.4.4 that there is a sequence of deformations $(u_\epsilon)_\epsilon \subset W^{1,\infty}(Y;\mathbb{R}^2)$ such that $u_\epsilon \rightharpoonup \operatorname{diag}(d,1)x + \psi$ in $W^{1,p}(Y;\mathbb{R}^2)$ as $\epsilon \to 0$, where $0 < d < 1$ and $\psi \in W^{1,p}_{\mathrm{per}}(Y;\mathbb{R}^2)$ and satisfying by (3.32)

$$\int_{\epsilon P_{\mathrm{stiff}} \cap \Omega} \operatorname{dist}^p\left(\nabla u_\epsilon, SO(2)\right)\mathrm{d}x \le C\epsilon^p.$$

Furthermore, $(\nabla u_\epsilon)_\epsilon$ is uniformly bounded in $L^p(Y;\mathbb{R}^2)$, i.e. $\|\nabla u_\epsilon\|_{L^p(Y;\mathbb{R}^{2\times 2})} < C$ for a $C > 0$, which together with the growth condition on W allows us to estimate the energy on the soft layers by $C\lambda$.

Overall, we obtain

$$\int_Y W^\alpha_\epsilon(x, \nabla u_\epsilon)\,\mathrm{d}x \le C\epsilon^{p-\alpha} + C\lambda,$$

and taking the inferior limit $\epsilon \to 0$ yields (4.28).

Regarding (4.29), note that this follows directly from the arguments leading to (4.27). In particular, (4.26) cannot hold for $F = \operatorname{diag}(d,1)$ with $0 < d < 1$.

Observe that since the the construction in Example 3.4.4 can be extended to arbitrary dimension $n \in \mathbb{N}$, $n \ge 2$, all arguments in this example generalize to n dimensions.

4.5 Appendix

4.5.1 Properties of locally one-dimensional functions

In the following we study the properties of functions that are locally only depending on the n-th component x_n for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $n \ge 2$.

Definition 4.5.1 (Locally one-dimensional function)**.** For $n \in \mathbb{N}$, $n \ge 2$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \le p \le \infty$. We say a function $f \in L^p(\Omega;\mathbb{R}^n)$ is a *locally one-dimensional function with respect to* x_n, if $\partial_i f = 0$ for all $i = 1, \dots, n-1$ in the sense of distributions.

We restrict our considerations to Sobolev functions, but many arguments may also hold for BV-functions.

Lemma 4.5.2 (Local approximation). *For $n \in \mathbb{N}$, $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $1 \leq p < \infty$ let $f \in W^{1,p}(\Omega;\mathbb{R}^n)$ be locally one-dimensional. Then, for every compact subset $Q \subset \Omega$ there exist locally one-dimensional functions $f_\epsilon \in C^\infty(\bar{Q};\mathbb{R}^n)$ such that*

$$f_\epsilon \to f \quad \text{in } W^{1,p}(Q;\mathbb{R}^n) \quad \text{as } \epsilon \to 0.$$

Proof. The approximation argument constructing a sequence $f_\epsilon = \eta_\epsilon * f$ using convolution with a mollifier η_ϵ is standard, see [68, Section 5.3, Theorem 2 and Appendix C.4, Theorem 6]. It remains to check that the approximation is locally one-dimensional, which follows directly from the derivation rule for convolution. More precisely, for $i = 1, \dots, n-1$, we have

$$\partial_i(\eta_\epsilon * f) = \eta_\epsilon * \partial_i f = 0.$$

□

Lemma 4.5.3 (Local identification with one-dimensional function). *For $n \in \mathbb{N}$, $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $1 \leq p < \infty$ let $f \in W^{1,p}(\Omega;\mathbb{R}^n)$ be locally one-dimensional. Then, for every cube $Q = [0,\ell]^n + a$ for $\ell > 0$ and $a \in \mathbb{R}$ such that $Q \subset \Omega$, there is a function $\tilde{f} \in W^{1,p}(a_n, a_n + \ell;\mathbb{R}^n)$ such that*

$$\tilde{f}(x_n) = f(x', x_n) \quad \text{for a.e. } x = (x', x_n) \in Q.$$

Proof. Notice that the task is essentially to construct a suitable trace operator on Q for locally one-dimensional functions. In accordance to the usual approach to trace operators we approximate $f|_Q$ in view of Lemma 4.5.2 by locally one-dimensional functions $f_\epsilon \in C^\infty(\bar{Q};\mathbb{R}^n)$, i.e. $f_\epsilon \to f$ in $W^{1,p}(Q;\mathbb{R}^n)$ as $\epsilon \to 0$. Since each f_ϵ is smooth, $\partial_i f_\epsilon = 0$, $i = 1, \dots, n-1$ implies that $f_\epsilon(x) = f_\epsilon(y)$ for all $x, y \in Q$ with $x_n = y_n$. Hence, we obtain for $\epsilon_k, \epsilon_j > 0$

$$\begin{aligned} \ell^{n-1} \|f_{\epsilon_k} - f_{\epsilon_j}\|^p_{W^{1,p}(a_n,a_n+\ell;\mathbb{R}^n)} &= \ell^{n-1} \int_{a_n}^{a_n+\ell} |f_{\epsilon_k} - f_{\epsilon_j}|^p + |\nabla f_{\epsilon_k} - \nabla f_{\epsilon_j}|^p \,\mathrm{d}x_n \\ &= \int_Q |f_{\epsilon_k} - f_{\epsilon_j}|^p + |\nabla f_{\epsilon_k} - \nabla f_{\epsilon_j}|^p \,\mathrm{d}x \\ &= \|f_{\epsilon_k} - f_{\epsilon_j}\|^p_{W^{1,p}(Q;\mathbb{R}^n)}. \end{aligned} \tag{4.30}$$

Thus, f_ϵ constitutes a Cauchy sequence in the Banach space $W^{1,p}(a_n, a_n + \ell;\mathbb{R}^n)$, and we denote its limit by $\tilde{f} \in W^{1,p}(a_n, a_n + \ell;\mathbb{R}^n)$. By extending $\tilde{f}$ constantly to Q, we obtain

$$\begin{aligned} \|f_\epsilon - \tilde{f}\|^p_{W^{1,p}(Q;\mathbb{R}^n)} &= \int_Q |f_\epsilon - \tilde{f}|^p + |\nabla f_\epsilon - \nabla \tilde{f}|^p \,\mathrm{d}x \\ &= \ell^{n-1} \int_{a_n}^{a_n+\ell} |f_\epsilon - \tilde{f}|^p + |\nabla f_\epsilon - \nabla \tilde{f}|^p \,\mathrm{d}x_n \\ &= \ell^{n-1} \|f_\epsilon - \tilde{f}\|^p_{W^{1,p}(a_n,a_n+\ell;\mathbb{R}^n)}. \end{aligned} \tag{4.31}$$

Thus, we see that f_ϵ converges to f as well as the extension of $\tilde{f}$ in $W^{1,p}(Q;\mathbb{R}^n)$ as $\epsilon \to 0$. Hence by the uniqueness of the limit both coincide almost everywhere. □

Lemma 4.5.4 (Continuity). *For $n \in \mathbb{N}$, $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 < p < \infty$. If $f \in W^{1,p}(\Omega;\mathbb{R}^n)$ is locally one-dimensional, then $f \in C^0(\Omega;\mathbb{R}^n)$.*

Proof. For each $x \in \Omega$ there is a cube $Q = [0, \ell]^n + a$ for $\ell > 0$ and $a \in \mathbb{R}$ such that $x \in Q$. By Lemma 4.5.3, we may identify f a.e. with a one-dimensional function, i.e. we may assume $f|_Q$ depends only on x_n. Thus, Morrey's inequality yields

$$\|f\|_{C^{0,1-1/p}(Q;\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(Q;\mathbb{R}^n)}. \tag{4.32}$$

In particular, $f \in C^0(Q; \mathbb{R}^n)$. Arguing this way for each $x \in \Omega$, we obtain by the uniqueness of a continuous representative that $f \in C^0(\Omega; \mathbb{R}^n)$. □

Remark 4.5.5 (Absolute continuity)**.** Notice that for a locally one-dimensional function $f \in W^{1,p}(\Omega; \mathbb{R}^n)$ actually stronger notions of continuity hold. Firstly, we disregarded that (4.32) provides Hölder continuity. Furthermore, note by absolute continuity of Sobolev functions on lines, see [99, Theorem 10.35], we obtain that f is actually absolutely continuous.

Lemma 4.5.6 (Global identification with functions of one variable)**.** *For $n \in \mathbb{N}$, $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain that is $e_n^\perp$-connected and e_n-flat and let*

$$a := \inf \{x_n \in \mathbb{R} \mid \exists x' \in \mathbb{R}^{n-1} : (x', x_n) \in \Omega\},$$
$$b := \sup \{x_n \in \mathbb{R} \mid \exists x' \in \mathbb{R}^{n-1} : (x', x_n) \in \Omega\}.$$

Further, let $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ for $1 \leq p < \infty$ be locally one-dimensional. Then, there is a function $\tilde{f} \in W^{1,p}(a, b; \mathbb{R}^n)$ such that $f(x', x_n) = \tilde{f}(x_n)$ for a.e. $x = (x', x_n) \in \Omega$.

Proof. Since f is continuous by Lemma 4.5.4, the condition $\partial_i f = 0$ for $i = 1, \dots, n-1$ implies that f is constant on all connected components of $\Omega_t = H_t \cap \Omega$, where H_t is the hyperplane $H_t = \{x_n = t\}$. As Ω is $e_n^\perp$-connected, we see that in fact, f is constant on each Ω_t.

Arguing as in the Proof of Corollary 3.3.8 via the Lipschitz property of Ω, the e_n-flatness of Ω yields that there are for $\ell_1, \ell_2 > 0$, $b_1, a_2 \in (a, b)$, $d_1, d_2 \in \mathbb{R}^{n-1}$ cuboids

$$Q_1 = (0, \ell_1)^{n-1} \times (a, b_1) + (d_1, 0)^T \subset \Omega \quad \text{and} \quad Q_2 = (0, \ell_2)^{n-1} \times (a_2, b) + (d_2, 0)^T \subset \Omega.$$

Since $[a_2, b_1]$ is compact, there is a finite partition of (a_2, b_1) up to a λ^1-null set in disjoint subintervals $(a_i, b_i)_{i \in N}$, indexed by the finite index set $N \subset \{3, 4, \dots\}$, with the property, that there is an $\ell_i > 0$ and a $d_i \in \mathbb{R}^{n-1}$ such that

$$Q_i := (0, \ell_i)^{n-1} \times (a_i, b_i) + (d_i, 0)^T \subset \Omega.$$

Lastly, we set $\bar{N} = N \cup \{1, 2\}$ and $\ell := \min_{i \in \bar{N}} \ell_i$.

By virtue of Lemma 4.5.3 we may identify f on each Q_i with a one-dimensional function $\tilde{f}_i \in W^{1,p}(a_i, b_i; \mathbb{R}^n)$. Then, $\tilde{f} := \sum_{i \in \bar{N}} \tilde{f}_i \mathbb{1}_{(a_i, b_i)}$ is a $W^{1,p}(a, b; \mathbb{R}^n)$-function and since f is constant on each Ω_t we have

$$\begin{aligned}
\|f - \tilde{f}\|^p_{W^{1,p}(\Omega;\mathbb{R}^n)} &= \int_\Omega |f - \tilde{f}|^p + |\nabla f - \nabla \tilde{f}|^p \,\mathrm{d}x \\
&\leq \operatorname{diam}(\Omega)^{n-1} \sum_{i \in \bar{N}} \int_{a_i}^{b_i} |f - \tilde{f}_i|^p + |\nabla f - \nabla \tilde{f}_i|^p \,\mathrm{d}x_n \\
&\leq \ell^{-n+1} \operatorname{diam}(\Omega)^{n-1} \sum_{i \in \bar{N}} \int_{Q_i} |f - \tilde{f}_i|^p + |\nabla f - \nabla \tilde{f}_i|^p \,\mathrm{d}x = 0.
\end{aligned}$$

□

Remark 4.5.7. Since all arguments are made locally, notice that for general $e_n^\perp$-connected Lipschitz domains Ω Lemma 4.5.6 holds for $f \in W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^n)$ with $\tilde{f} \in W^{1,p}_{\text{loc}}(a,b;\mathbb{R}^n)$.

Lemma 4.5.8 (Global approximation)**.** *For $n \in \mathbb{N}$, $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be an $e_n^\perp$-connected bounded Lipschitz domain. For $1 < p < \infty$ let $f \in W^{1,p}(\Omega;\mathbb{R}^n)$ be a locally one-dimensional function. Then, there exist locally one-dimensional functions $f_\epsilon \in C^\infty(\bar{\Omega};\mathbb{R}^n)$ such that*

$$f_\epsilon \to f \quad \textit{in } W^{1,p}(\Omega;\mathbb{R}^n) \quad \textit{as } \epsilon \to 0.$$

Proof. Let

$$a := \inf\{x_n \in \mathbb{R} \mid \exists x' \in \mathbb{R}^{n-1} : (x', x_n) \in \Omega\},$$
$$b := \sup\{x_n \in \mathbb{R} \mid \exists x' \in \mathbb{R}^{n-1} : (x', x_n) \in \Omega\}.$$

By Lemma 4.5.6 we can identify f with a function $f \in W^{1,p}(a,b;\mathbb{R}^n)$. By standard approximation theorems [68, Theorem 3, Section 5.3.3] there is a sequence $f_\epsilon \in C^\infty([a,b];\mathbb{R}^n)$ with $f_\epsilon \to f$ in $W^{1,p}(a,b;\mathbb{R}^n)$, and by arguing for the constant extensions as in (4.31) also in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$. □

Lemma 4.5.9 (Extension)**.** *For $n \in \mathbb{N}$, $n \geq 2$ let $\Omega \subset \mathbb{R}^n$ be an $e_n^\perp$-connected bounded Lipschitz domain and for $1 < p < \infty$ let $f \in W^{1,p}(\Omega;\mathbb{R}^n)$ be locally one-dimensional. Furthermore, let $Q \subset \mathbb{R}^n$ be the cuboid of minimal height containing Ω. Then, there is an extension $\tilde{f} \in W^{1,p}(Q';\mathbb{R}^n)$, with Q' as in the Proof of Theorem 3.3.1, such that $\tilde{f}|_\Omega = f$ and $\tilde{f}$ is also locally one-dimensional.*

Proof. By Lemma 4.5.6, we may identify f with a one-dimensional function and then extend it constantly to a function $f \in W^{1,p}(Q;\mathbb{R}^n)$. Then, using the approximation result of Lemma 4.5.8 we may apply the usual higher-order reflection of f on the top and bottom of Q with respect to the e_n direction, which yields a function $\tilde{f} \in W^{1,p}(Q';\mathbb{R}^n)$ with

$$\|\tilde{f}\|_{W^{1,p}(Q';\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(Q;\mathbb{R}^n)},$$

and inherits from f by reflection the property that $\tilde{f}$ only depends on x_n, see [99, Exercise 10.37] or [68, Theorem 1, Section 5.4]. □

The next lemma addresses the question of approximation under manifold constraints, which in our case would be the restriction to $SO(n)$. The main ingredient is the tubular neighborhood theorem.

Theorem 4.5.10 (Tubular neighborhoods [33, Theorem 11.4])**.** *Let N be a compact smooth submanifold of $\mathbb{R}^k$ and let $\Xi(N,\epsilon) = \{(x,v) \in N \times \mathbb{R}^k \mid v \perp T_xN, \|v\| < \epsilon\}$ be the ϵ-neighborhood of 0 in the normal bundle. Then there is an $\epsilon > 0$ such that $\theta : \Xi(N,\epsilon) \to \mathbb{R}^k$ given by $\theta(x,v) = x + v$ is a diffeomorphism onto the neighborhood $\{y \in \mathbb{R}^k \mid \operatorname{dist}(N,y) < \epsilon\}$ of N in $\mathbb{R}^k$.*

The next lemma combines classic approximation results for Sobolev functions on manifolds [82, Theorem 2.1] with the properties of locally one-dimensional functions.

Lemma 4.5.11 (Smooth approximation under manifold constraints)**.** *Let N be a compact manifold isometrically embedded in $\mathbb{R}^k$ for $k \in \mathbb{N}$. If $1 < p < \infty$ and $f \in W^{1,p}(\Omega;N)$ is locally one-dimensional, then there is a locally one-dimensional sequence $(f_\epsilon)_\epsilon \subset C^\infty(\overline{\Omega};N)$ with $f_\epsilon \to f$ in $W^{1,p}(\Omega;\mathbb{R}^k)$ as $\epsilon \to 0$.*

Proof. We follow the general ideas outlined in [82, Theorem 2.1] applied to locally one-dimensional functions. By Lemma 4.5.8 there is a locally one-dimensional approximating sequence $(f_\epsilon)_\epsilon \subset C^\infty(\overline{\Omega}, \mathbb{R}^k)$ with $f_\epsilon \to f$ in $W^{1,p}(\Omega; \mathbb{R}^k)$ as $\epsilon \to 0$.

By Morrey's inequality, see (4.32), locally one-dimensional $W^{1,p}(\Omega; \mathbb{R}^k)$-functions embed in a Hölder space, so the convergence of $(f_\epsilon)_\epsilon$ is uniform. Thus, for ϵ small enough, all f_ϵ lie in a tubular neighborhood $U \subset \mathbb{R}^k$ of the embedded manifold N, whose existence follows from Theorem 4.5.10. Therefore, the nearest point projection $\pi : U \to N$ is well-defined and smooth. Thus, $\pi \circ f_\epsilon$ is smooth and $\pi \circ f_\epsilon \to \pi \circ f = f$ in $W^{1,p}(\Omega, \mathbb{R}^k)$ as $\epsilon \to 0$. ☐

Lemma 4.5.12 (Approximation by simple functions)**.** *Let Ω be an $e_n^\perp$-connected domain and let N be a compact manifold isometrically embedded in $\mathbb{R}^k$ for $k \in \mathbb{N}$. If $1 < p < \infty$ and $f \in W^{1,p}(\Omega; N)$ is locally one-dimensional, then, there is a sequence of locally one-dimensional simple functions $(s_\epsilon)_\epsilon \subset L^\infty(\Omega; N)$, such that $s_\epsilon \to f$ in $L^p(\Omega; \mathbb{R}^k)$ as $\epsilon \to 0$.*

Proof. By Lemma 4.5.11 there is a sequence $(f_\delta)_\delta \subset C^\infty(\overline{\Omega}; N)$ of locally one-dimensonal functions with $f_\delta \to f$ in $W^{1,p}(\Omega; \mathbb{R}^k)$ as $\delta \to 0$, in particular $f_\delta \to f$ as in $L^p(\Omega; \mathbb{R}^k)$ as $\delta \to 0$. Since f_δ is locally one-dimensional and Ω is $e_n^\perp$-connected, we may identify it for

$$a := \inf \{x_n \in \mathbb{R} \mid \exists x' \in \mathbb{R}^{n-1} : (x', x_n) \in \Omega\},$$
$$b := \sup \{x_n \in \mathbb{R} \mid \exists x' \in \mathbb{R}^{n-1} : (x', x_n) \in \Omega\},$$

with a function $f_\delta \in C^\infty([a, b]; N)$. By standard approximation theorems, see e.g. [4, Theorem VI.1.2], there is for each $\delta > 0$ a sequence of simple functions $(z_{\delta,\epsilon})_\epsilon \subset L^\infty([a, b]; \mathbb{R}^k)$ such that $z_{\delta,\epsilon} \to f_\delta$ uniformly on the compact set $[a, b]$ as $\epsilon \to 0$. As in the proof of Lemma 4.5.11, we have for ϵ small enough, that all $z_{\delta,\epsilon}$ lie in a tubular neighborhood $U \subset \mathbb{R}^k$ of the embedded manifold N, see Theorem 4.5.10. Hence, applying again the smooth nearest point projection $\pi : U \to N$ we obtain that $\pi \circ z_{\delta,\epsilon}$ satisfies $\pi \circ z_{\delta,\epsilon} \to \pi \circ f_\delta = f_\delta$ in $L^p(\Omega; \mathbb{R}^k)$ as $\epsilon \to 0$. Hence, we have

$$\lim_{\delta \to 0} \lim_{\epsilon \to 0} \|\pi \circ z_{\delta,\epsilon} - f\|_{L^p(\Omega;\mathbb{R}^k)} = \lim_{\delta \to 0} \|f_\delta - f\|_{L^p(\Omega;\mathbb{R}^k)} = 0.$$

Thus, by Attouch's diagonalization argument of Lemma 4.5.14, there is a subsequence $(s_\epsilon)_\epsilon \subset L^\infty(\Omega; N)$ of $(\pi \circ z_{\delta,\epsilon})_{\delta,\epsilon}$ such that

$$s_\epsilon \to f \quad \text{in } L^p(\Omega; \mathbb{R}^k) \quad \text{as } \epsilon \to 0,$$

as desired. ☐

Remark 4.5.13. Observe that Lemma 4.5.12 generalizes directly to bounded Lipschitz domains Ω for which the decomposition given by Proposition 4.2.8 is a partition of Ω in finitely many sets. Indeed, it suffice to apply Lemma 4.5.12 to each e_n-monotonically connected component yields a simple function on Ω that is locally one-dimensional.

4.5.2 Technical tools

Lemma 4.5.14 (Attouch diagonalization argument, [13, Lemma 1.15, Corollary 16])**.** *Let $(a_{i,j})_{i,j \in \mathbb{N}} \subset [0, \infty]$ be a doubly indexed family. Then,*

(i) there exist maps $i \mapsto j(i)$ increasing to ∞, such

$$\liminf_{i \to \infty} a_{i,j(i)} \geq \liminf_{j \to \infty} \big(\liminf_{i \to \infty} a_{i,j}\big);$$

(ii) there exist maps $i \mapsto j(i)$ increasing to ∞, such

$$\limsup_{i\to\infty} a_{i,j(i)} \leq \limsup_{j\to\infty} \big(\limsup_{i\to\infty} a_{i,j} \big).$$

Lemma 4.5.15 (A Urysohn argument)**.** *For $n \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $1 < p < \infty$. Furthermore, let $(u_\epsilon)_\epsilon \subset W^{1,p}(\Omega;\mathbb{R}^n)$ with $\|u_\epsilon\|_{W^{1,p}(\Omega;\mathbb{R}^n)} < C$ for a constant $C > 0$ and all $\epsilon > 0$ and $u_\epsilon \to u$ in $L^p(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$ for a $u \in W^{1,p}(\Omega;\mathbb{R}^n)$. Then, $u_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$.*

Proof. Since $(u_\epsilon)_\epsilon$ is uniformly bounded in the reflective space $W^{1,p}(\Omega;\mathbb{R}^n)$, every subsequence of $(u_\epsilon)_\epsilon$ has a convergent subsequence $(u_{\epsilon_j})_{j\in\mathbb{N}}$ and since $W^{1,p}(\Omega;\mathbb{R}^n)$ embeds continuously in $L^p(\Omega;\mathbb{R}^n)$, uniqueness of the limit yields $u_{\epsilon_j} \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $j \to \infty$. As this holds for any subsequence of $(u_\epsilon)_\epsilon$, we obtain by the Urysohn property, that $u_\epsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^n)$ as $\epsilon \to 0$. □

5

Homogenization of Bilayered Structures in Crystalline Plasticity

In this chapter we obtain an explicit homogenization formula for a single slip crystal plasticity model. The results proven generalize and reproduce the results published together with the adviser Carolin Kreisbeck in the context of totally rigid layers in [42], incorporating the more general setting of stiff layers besides minor technical changes. We will begin with a short overview on the physical background of the crystal plasticity model analyzed in this chapter. After a discussion of admissible micro- and macroscopic deformations for these material models and an overview on the technique of convex integration we proceed with the proof of the main Theorem 5.2.1 formulated in terms of Γ-convergence. As typical for a Γ-convergence result, we will establish compactness, construct recovery sequences and give the lower bound estimate in that order.

5.1 Introduction to crystalline plasticity

In this first section, we give a short introduction to the underlying physical principles, for which we follow the work of Lubarda [102].

Besides the reversible elastic behavior discussed in Chapter 4 many materials exhibit an irreversible material response to outside forces, which we refer to as the plastic deformation of the material. In the following we restrict ourselves to models for materials that are known as rate-independent models. Considering the evolution of a deformation in time, the current configuration of a rate-independent material depends only on the history of the evolution, yet not on its rate. In rate-independent models the notion of a yield locus is introduced, which is bounded by the yield surface [102, Chapter 8]. Stress variations contained in the yield locus entail a purely elastic response, while an instantaneous plastic material response occurs if they exceed the yield surface.

The properties of the yield surface reflect the underlying microscopic processes on different length scales that govern the macroscopic plastic deformation such as twinning and movement of dislocations or formation of microstructures within the material, which can be observed in experiments. Furthermore, due to effects like hardening, i.e. the increasing resistance of the material to plastic deformation over prolonged deformation, which itself is influenced for example by the interaction of dislocations, the macroscopic deformation depends on the history of the evolution. From a modeling point of view, the plentiful microscopic parameters involved motivate what is known as an internal variable approach [102, Section 4.4], introducing additional variables whose evolution is prescribed by flow rules.

We will specify the yield surface according to common assumptions of crystal plasticity. Considering the atomic lattice structure of a metal, the experimental observation that the energy required to move whole planes of atoms against each other exceeds by far the real values for plastic deformation of the lattice, lead to the finding that the underlying process is the movement of discrete lattice defects, called dislocations. Our point of view from a larger length scale does not resolve this dislocation structure but merely comprises its effects in the assumption of one active slip system describing the possibility of a shear deformation by a continuous parameter [102, Chapter 12].

5.1.1 A single-slip model for finite crystal plasticity

In the following we define a specific model for crystal plasticity, following the work by Carstensen, Hackl and Mielke [37]. Alternatively, see [118]. While the plastic variables are governed by a flow rule corresponding to one active slip system, the time dependence of the process is addressed by a discretization process. As we choose a variational approach, for each time step a minimization is introduced that is associated to the equations of the incremental problem. Overall, the resulting variational model will describe the plastic deformation of a crystalline material with one active slip system in the first time step of the discretization.

Note that a different approach is given by the framework of energetic solutions developed by Mielke and several coworkers, recasting the evolution laws in a global stability condition and an energy inequality. For an introduction to this framework, see [110].

In accordance to the work of Kröner [95] and Lee [98] we assume multiplicative decomposition of the deformation gradient $F = F_{\mathrm{e}} \cdot F_{\mathrm{p}}$ in an elastic part F_{e} and a plastic part F_{p} [102, Chapter 11]. Note that this decomposition is non-unique and corresponds to the idea of a stress free intermediate state that is not observed experimentally. For a recent mathematical approach to justify this assumption see [119, 121, 120]. Due to the problems arising by the non-uniqueness of the decomposition, a common simplification is the *rigid-plastic idealization*, which disregards elastic deformations by only allowing plastic deformation and rotation, corresponding to the restriction $F_{\mathrm{e}} \in SO(n)$.

We introduce the internal plastic variables $p \in \mathbb{R}^m$ and assume that the Helmholtz free energy is described by an integral functional with a density of the form

$$W(F, F_{\mathrm{p}}, p) = \bar{W}(F_{\mathrm{e}}, p),$$

see also [118, Section 2.1]. The Piola-Kirchhoff stress tensor T, the plastic stress tensor Q and the internal forces q are then given by

$$T = \frac{\partial W}{\partial F}, \qquad Q = -\frac{\partial W}{\partial F_{\mathrm{p}}^{-1}}, \qquad q = -\frac{\partial W}{\partial p}.$$

To simplify the notation, let $P = F_{\mathrm{p}}^{-1}$. The evolution of the plastic variables (P, p) is now determined by the yield function φ describing the yield surface. More precisely, assuming the principle of maximum plastic dissipation [86, Chapter 3], [125, Section 1.4] holds, a flow rule for (P, p) can be derived from φ.

Modeling the plasticity of a crystalline material with one active slip system determined by a slip direction $s \in \mathbb{S}^{n-1}$ and a slip plain normal $m \in \mathbb{S}^{n-1}$, $s \cdot m = 0$, we consider the

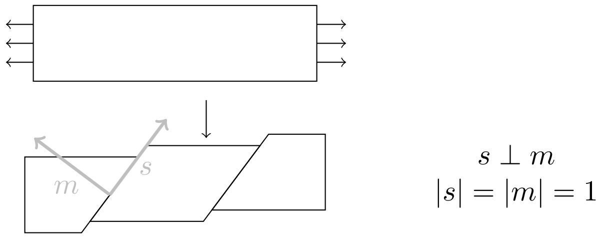

Figure 5.1: Typical ductility experiments apply forces to the bases of cylindrical mono-crystals. The material responds by deforming along slip planes, determined by the slip direction s and the slip plane normal m, which allows to conclude on the underlying movements of dislocations.

particular yield function given by [37, Section 6.]

$$\varphi(P^T Q, q) = |s^T P^T Q m| - r - q,$$

see also Figure 5.1. Physically, $s^T P^T Q m$ represents the resolved shear stress on the plane normal to m, while $r \geq 0$ specifies the initial yield stress [83, Section 3.6]. Notice that $P^T Q = -F_{\mathrm{e}}^T \partial_{F_{\mathrm{e}}} \bar{W}(F_{\mathrm{e}}, p)$ does not depend on the plastic deformation F_{p}.

The associated flow rule to φ is then for a slip rate parameter $\dot{\lambda} \geq 0$ given by

$$(P^{-1}\dot{P}, \dot{p}) = \dot{\lambda}(\operatorname{sign}(s^T P^T Q m) s \otimes m, -1),$$

see also [118, Section 3.2], a result going back to the work of Rice [123].

Assuming the body to be initially not deformed, i.e. setting the initial condition $P_0 = \mathbb{I}$, it can be deduced that for $\gamma = \dot{\lambda} \operatorname{sign}(s^T P^T Q m)$

$$P = \mathbb{I} + \gamma s \otimes m.$$

The plastic variables (P, p) are thus determined by the parameter γ.

To resolve the time dependence of the variables we partition the time interval $[0, T]$ in $\ell \in \mathbb{N}$ time steps $0 = t_0 < t_1 < \cdots < t_\ell = T$ and since each time step is of the same general structure, we merely consider the first one with initial data (P_0, p_0) given and $(P, p) = (P_1, p_1)$ to be determined in the time period $\tau = t_1 - t_0$. Accordingly, the flow rule needs to be discretized. Notice that in particular $P^{-1}\dot{P}$ allows the application of different discretization schemes, with the discretized P^{-1} to be chosen e.g. as a certain value or a convex combination of values taken in the specific time step.

To determine the internal plastic parameters (P_1, p_1), we assume that in each time step the energy of the system is minimized by these quantities. Hence, a time independent minimization problem is introduced, incorporating the discrete flow rule and additional constitutive relations in the sense that these are satisfied by stationary points of the energy functional, see [37, Section 4,5]. The resulting energy functional is of the form

$$E(u, \gamma, P, p) = \int_\Omega \bar{W}(\nabla u P, p) + r|\gamma - \gamma_0| - f \cdot u \, \mathrm{d}x.$$

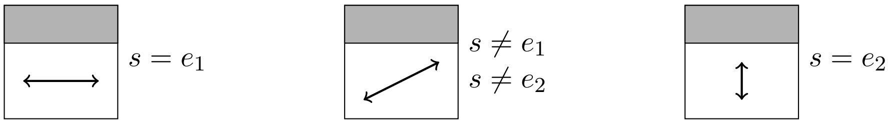

Figure 5.2: In the context of layered materials with stiff components, the orientation of the slip direction of an active slip system in the softer layer plays a decisive role. Graphically speaking, if $s = e_1$ the slip system is unblocked, while for $s = e_2$, the slip is orthogonal and thus intuitively impeded by the stiff layers. This will be reflected by the macroscopic behavior of the material determined by Theorem 5.2.1.

Splitting the energy $\bar{W}$ additively in an elastic part W_{e} and a plastic part that is associated with isotropic, linear hardening with hardening modulus $a > 0$ we obtain

$$\bar{W}(F_{\mathrm{e}}, p) = W_{\mathrm{e}}(F_{\mathrm{e}}) + \frac{a}{2}p^2.$$

This leads under the assumption that $r = 0$ and $a = 2$ to

$$E(u, P, p) = \int_\Omega W_{\mathrm{e}}(\nabla u P) + p^2 - f \cdot u \,\mathrm{d}x.$$

Lastly, choosing p_0, γ_0 such that $p = -|\gamma|$ and using the fact that $P = \mathbb{I} + \gamma s \otimes m$ results in the condensed energy

$$E(u) = \min_{\gamma \in \mathbb{R}} \int_\Omega W_{\mathrm{e}}(\nabla u(\mathbb{I} + \gamma s \otimes m)) + |\gamma|^2 - f \cdot u \,\mathrm{d}x. \tag{5.1}$$

In the following sections we want to study homogenization of models for layered material involving this type of plastic deformation energy.

5.1.2 A model for layered materials with stiff components in finite crystal plasticity

In the following we study two-dimensional models for bilayered materials that feature a stiff component. Yet, in contrast to the previous chapter, we assume that the whole material is stiff in the sense that the elastic constants are large. However, we also assume that every other layers can be plastically deformed along one active slip system. On these layers the differential inclusion constraint is therefore imposed on $\nabla u P$ rather than ∇u, where $P = \mathbb{I} + \gamma s \otimes m$ with $\gamma \in \mathbb{R}$.

In view of the exact differential inclusion constraint, which requires $\nabla u P \in SO(2)$ we introduce the sets

$$\begin{aligned}\mathcal{M}_s &= \{F \in \mathbb{R}^{2\times 2} \mid F = R(\mathbb{I} + \gamma s \otimes m), R \in SO(2), \gamma \in \mathbb{R}\} \\ &= \{F \in \mathbb{R}^{2\times 2} \mid \det F = 1, |Fs| = 1\},\end{aligned}$$

and

$$\mathcal{N}_s = \{F \in \mathbb{R}^{2\times 2} \mid \det F = 1, |Fs| \leq 1\}.$$

The relation between these two sets will be discussed later, see Remark 5.2.7.

Furthermore, we introduce energies motivated by (5.1). However, we disregard the loading term, specify our choice for the elastic energy, which comprises again a penalization factor and tailor the notation to the particular case of two dimensions.

Accordingly, we define the energy density $W_{\text{slip}} : \mathbb{R}^{2\times 2} \to [0,\infty]$ given for $F \in \mathbb{R}^{2\times 2}$ by

$$W_{\text{slip}}(F) = \begin{cases} \gamma^2 = |Fm|^2 - 1 & \text{if } F = R(\mathbb{I} + \gamma s \otimes m) \in \mathcal{M}_s, \\ \infty & \text{otherwise,} \end{cases} \tag{5.2}$$

which describes the deformation along one active slip system. To model the elastic energy we introduce for $\beta > 0$ the penalized energy density $W_{\text{e}}^{\beta} : \mathbb{R}^{2\times 2} \to [0,\infty)$ familiar from previous chapters that is given by

$$W_{\text{e}}^{\beta}(F) = \epsilon^{-\beta} \operatorname{dist}^2\left(F, SO(2)\right), \quad F \in \mathbb{R}^{2\times 2}.$$

In the context of the rigid-plastic idealization we consider $W_{\text{e}}^{\infty} : \mathbb{R}^{2\times 2} \to [0,\infty]$ that is for $F \in \mathbb{R}^{2\times 2}$ defined by

$$W_{\text{e}}^{\infty}(F) = \begin{cases} 0 & \text{if } F \in SO(2), \\ \infty & \text{otherwise.} \end{cases} \tag{5.3}$$

We define the layered structure by specifying the energy density for $x \in \Omega$ and $F \in \mathbb{R}^{2\times 2}$ by

$$W^{\beta}(x, F) = \min\left\{W_{\text{e}}^{\beta}(F_e) + W_{\text{slip}}(F_p)\,\mathbb{1}_{P_{\text{soft}}} \;\middle|\; F = F_{\text{e}}F_{\text{p}}\right\},$$

see also Figure 5.2, using the notation W^{∞} for the energy density with W_{e}^{∞} in place of W_{e}^{β}. Lastly, for $\epsilon > 0$ we introduce the energy functional $E_{\epsilon}^{\beta} : L_0^1(\Omega;\mathbb{R}^2) \to [0,\infty]$ that is defined for $u \in W^{1,1}(\Omega;\mathbb{R}^2)$ by

$$E_{\epsilon}^{\beta}(u) = \int_{\Omega} W^{\beta}\left(\tfrac{x}{\epsilon}, \nabla u(x)\right) \mathrm{d}x$$

and $E_{\epsilon}^{\beta}(u) = \infty$ otherwise in $L_0^1(\Omega;\mathbb{R}^2)$. Note that the relaxation of the energy W^{β}, that models the soft layers here has been studied in [45].

Before proceeding with the analysis of the model, we want to give a first example of a simple macroscopic shear deformation to illustrate the setting.

Example 5.1.1 (Macroscopic shear deformation). Suppose we have one active slip system in the soft layers that allows to shear this component in the direction along the layers, i.e. $s = e_1$. The question is, if a macroscopic shear deformation can be obtained, i.e. if there is a convergent sequence of deformations $(u_{\epsilon}) \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ whose weak limit $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ is a simple shear deformation of the form

$$\nabla u = \mathbb{I} + \gamma e_1 \otimes e_2,$$

where $\gamma \in \mathbb{R}$ describes the amount of shear. While Theorem 5.2.1 will answer this question positively, we want to give an explicit construction for $(u_{\epsilon})_{\epsilon}$ at this point.

As on the microscopic level, the energy functionals $(E_{\epsilon}^{\beta})_{\epsilon}$ impose a penalization on the deformation of the stiff layers, the idea of the construction is not to shear these layers at all, but compensate on the soft layers by shearing more. Accordingly we consider the deformations $(u_{\epsilon}) \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ given by

$$\nabla u_{\epsilon} = \mathbb{I} + \frac{1}{\lambda}\gamma\,\mathbb{1}_{\epsilon P_{\text{soft}}}\, e_1 \otimes e_2.$$

Then, by the classic Lemma 2.3.1 on weak convergence of highly oscillating functions

$$\nabla u_\epsilon = \mathbb{I} + \frac{1}{\lambda}\gamma\, \mathbb{1}_{\epsilon P_{\text{soft}}}\, e_1 \otimes e_2 \rightharpoonup \mathbb{I} + \gamma e_1 \otimes e_2 \quad \text{in } L^2(\Omega; \mathbb{R}^{2\times 2}) \quad \text{as } \epsilon \to 0.$$

Thus by the vanishing mean values and the Poincaré inequality $u_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$. Since we do not shear the stiff layers, $\|\nabla u_\epsilon\|^2_{L^2(\Omega;\mathbb{R}^2)} \leq (2+\gamma^2/\lambda^2)|\Omega| < C$ for a constant $C > 0$. For the energies, we obtain by the same arguments

$$E_\epsilon^\beta(u_\epsilon) = \int_\Omega \frac{1}{\lambda^2}\gamma^2\, \mathbb{1}_{\epsilon P_{\text{soft}}}\, \mathrm{d}x \to \frac{1}{\lambda}\int_\Omega \gamma^2\, \mathrm{d}x = E(u) \quad \text{as } \epsilon \to 0,$$

with E as in Theorem 5.2.1.

5.2 Homogenization of layered materials with stiff components in crystal plasticity

In this section we utilize the asymptotic characterization result of Theorem 3.3.1 and build on the results by Conti, Dolzmann and Kreisbeck [45], in particular for the question of compactness to show the following homogenization result.

Theorem 5.2.1 (Homogenization of layered stiff material with one active slip system)**.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded Lipschitz domain that is $e_n^\perp$-connected in the sense of Definition 3.3.6 and $(E_\epsilon^\beta)_\epsilon$ as specified above. If $\beta > 2$, then $(E_\epsilon^\beta)_\epsilon$ converges to a functional $E : L_0^2(\Omega;\mathbb{R}^2) \to [0,\infty]$ in the sense of Γ-convergence with respect to the strong $L^2(\Omega;\mathbb{R}^2)$-topology. Using the notation*

$$K_{s,\lambda} = \begin{cases} \{0\} & \text{if } s = e_2, \quad \big[-2\lambda\frac{s_1}{s_2}, 0\big] \quad \text{if } s_1 s_2 > 0, \\ \mathbb{R} & \text{if } s = e_1, \quad \big[0, -2\lambda\frac{s_1}{s_2}\big] \quad \text{if } s_1 s_2 < 0, \end{cases}$$

the limit functional E is given for each $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$, where $R \in SO(2)$, $\gamma \in L^2(\Omega)$, $\gamma \in K_{s,\lambda}$ a.e. in Ω, by

$$E(u) = \lambda \int_\Omega \big|\tfrac{1}{\lambda}(\nabla u - (1-\lambda)R)m\big|^2 - 1\, \mathrm{d}x,$$

and $E(u) = \infty$ otherwise in $L^2(\Omega;\mathbb{R}^2)$. Moreover, sequences of bounded energy with respect to $(E_\epsilon^\beta)_\epsilon$, i.e. sequences $(u_\epsilon)_\epsilon \subset W^{1,2}(\Omega;\mathbb{R}^2)$ with $E_\epsilon^\beta(u_\epsilon) \leq C$ for all $\epsilon > 0$, are relatively compact in $L^2(\Omega;\mathbb{R}^2)$.

Remark 5.2.2. a) An alternative representation for the homogenized energy E is given for $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $E(u) < \infty$ by

$$E(u) = \lambda \int_\Omega \Big|\frac{1}{\lambda}(\nabla u - (1-\lambda)R)m\Big|^2 - 1\, \mathrm{d}x = \frac{s_1^2}{\lambda}\int_\Omega \gamma^2\, \mathrm{d}x - 2 s_1 s_2 \int_\Omega \gamma\, \mathrm{d}x.$$

b) The set $K_{s,\lambda}$ can be written as

$$K_{s,\lambda} = \{\gamma \in \mathbb{R} \mid |s + \frac{\gamma}{\lambda} s_2 e_1| \leq 1\}.$$

Indeed, we have

$$1 \geq \Big|s + \frac{\gamma}{\lambda} s_2 e_1\Big|^2 = \Big(s_1 + \frac{\gamma}{\lambda} s_2\Big)^2 + s_2^2 = 1 + 2\frac{\gamma}{\lambda} s_1 s_2 + \frac{\gamma^2}{\lambda^2} s_2^2,$$

which is equivalent to

$$0 \geq s_2\Big(s_1\gamma + \frac{s_2}{2\lambda}\gamma^2\Big) = s_1 s_2 \gamma\Big(1 + \frac{1}{2\lambda}\frac{s_2}{s_1}\gamma\Big).$$

This inequality holds for $\gamma \in \mathbb{R}$ if and only if $\gamma \in K_{s,\lambda}$.

c) The theorem generalizes directly to bounded Lipschitz domains Ω for which the decomposition given by Proposition 4.2.8 is a partition of Ω in finitely many sets.

d) A special case of the this theorem was proven in [42, Theorem 1.1] assuming that the layers are rigid in the sense of a rigid-plastic idealization.

The following subsections concern the proof of Theorem 5.2.1. To avoid repetition, we assume throughout the rest of this section that Ω is a simply connected bounded Lipschitz domain that is $e_n^\perp$-connected

5.2.1 Admissible micro- and macroscopic deformations

In preparation for the proof of Theorem 5.2.1, we consider sufficient and necessary conditions for microscopic deformations u_ϵ and macroscopic deformations u to be admissible. For the former, this corresponds by our choice of a variational approach to the study of deformations u_ϵ of finite energy with respect to E_ϵ^β, i.e. functions satisfying $E_\epsilon^\beta(u_\epsilon) < \infty$. For the latter, from the point of view of Γ-convergence, in particular the lim inf-inequality, the subjects of interest are the limits u of weakly convergent sequences of microscopic deformations u_ϵ of finite energy, i.e. $u_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$ with $E_\epsilon^\beta(u_\epsilon) < C$ for a constant $C > 0$. The goal of this section is in particular to establish a relation between admissible micro- and macroscopic deformations, providing for all possible macroscopic deformations u explicit microscopic deformations u_ϵ that converge to u.

We first tend to microscopic deformations. Notice that we have seen a first example of a laminate construction of finite energy with respect to E_ϵ^β for $s = (\sqrt{3}/2, 1/2)^T$ in Example 3.1.2. Our first task is to determine all possible laminate constructions satisfying the constraints imposed by E_ϵ^β in dependence of $s \in \mathbb{S}^1$, which means determining all rank one connections in $\mathcal{M}_s$, see also [42, Lemma 3.1] and [41].

Lemma 5.2.3 (Rank one connections in $\mathcal{M}_s$)**.** *For $R, Q \in SO(2)$, $\gamma, \zeta \in \mathbb{R}$ let $F, G \in \mathcal{M}_s$ be given by $F = R(\mathbb{I} + \gamma s \otimes m)$ and $G = Q(\mathbb{I} + \zeta s \otimes m)$. Then, F and G are rank one connected, i.e.* $\operatorname{rank}(F - G) = 1$*, if and only if one of the following conditions is satisfied:*

(i) $R = Q$ and $\gamma \neq \zeta$, in which case $F - G = (\gamma - \zeta)Rs \otimes m$;

(ii) $R \neq Q$ and $\gamma - \zeta = 2\tan(\frac{\theta}{2})$, where $\theta \in (-\pi, \pi)$ denotes the rotation angle corresponding to $Q^T R$ given by $Q^T R e_1 = \cos(\theta)e_1 + \sin(\theta)e_2$, in which case

$$F - G = \frac{\gamma - \zeta}{4 + (\gamma - \zeta)^2} Q\big((\zeta - \gamma)s + 2m\big) \otimes \big(2s + (\gamma + \zeta)m\big).$$

Proof. Since multiplying $F - G$ by the rotation $S = (s, m) \in SO(2)$ does not affect its rank and $S^T s \otimes m S = e_1 \otimes e_2$ implies $FS, GS \in \mathcal{M}_{e_1}$, it suffices to consider $s = e_1$ and $Q = \mathbb{I}$. In that case, for $F - G$ to be of rank smaller than 2 it is necessary that

$$\begin{aligned} 0 = \det(F - G) &= -(F - G)e_1 \cdot \big((F - G)e_2\big)^\perp = (\mathbb{I} - R)e_1 \cdot \big((R - \mathbb{I})e_2 + (\gamma R - \zeta\mathbb{I})e_1\big)^\perp \\ &= (\mathbb{I} - R)e_1 \cdot \big((\mathbb{I} - R)e_1 - (\zeta\mathbb{I} - \gamma R)e_2\big) = 2 - Re_1 \cdot (2e_1 - (\zeta - \gamma)e_2). \end{aligned}$$

Thus $Re_1 = (r_1, r_2)^T$ has to be a solution of the problem

$$2r_1 + (\gamma - \zeta) r_2 = 2 \quad \text{subject to } r_1^2 + r_2^2 = 1.$$

Geometrically, the equation describes a line in $\mathbb{R}^2$ through $(1,0)^T$, while the constraint describes a circle around 0 with radius 1, and consequently for $\gamma \neq \zeta$ this problem has two solutions. Algebraically, substituting r_1 as in the equation into the constraint leads to a quadratic equation in r_2 reading

$$r_2\Big((\gamma - \zeta) - \frac{4 + (\gamma - \zeta)^2}{4} r_2\Big) = 0,$$

with the solutions $r_2 = 0$ and

$$r_2 = \frac{4(\gamma - \zeta)}{4 + (\gamma - \zeta)^2},$$

corresponding to $r_1 = 1$ and

$$r_1 = 1 - \frac{\gamma - \zeta}{2} \frac{4(\gamma - \zeta)}{4 + (\gamma - \zeta)^2} = \frac{4 - (\gamma - \zeta)^2}{4 + (\gamma - \zeta)^2},$$

respectively. Furthermore, as by definition $r_1 = \cos(\theta)$ and $r_2 = \sin(\theta)$ the trigonometric identities

$$\cos(2x) = \frac{1 - \tan^2(x)}{1 + \tan^2(x)} \quad \text{and} \quad \sin(2x) = \frac{2\tan(x)}{1 + \tan^2(x)}$$

yield $\gamma - \zeta = 2\tan(\frac{\theta}{2})$.

Consequently, we obtain for Re_1 the representation $Re_1 = e_1$ or

$$Re_1 = \frac{4 - (\gamma - \zeta)^2}{4 + (\gamma - \zeta)^2} e_1 + \frac{4(\gamma - \zeta)}{4 + (\gamma - \zeta)^2} e_2.$$

In the former case, we see immediately that $F - G = (\gamma - \zeta) Re_1 \otimes e_2$. In the latter case, the calculation is more extensive. By the above representation we have

$$(F - G)e_1 = Re_1 - e_1 = \frac{-2(\gamma - \zeta)^2}{4 + (\gamma - \zeta)^2} e_1 + \frac{4(\gamma - \zeta)}{4 + (\gamma - \zeta)^2} e_2.$$

Also, since $Re_2 = (Re_1)^\perp$ it holds that

$$\begin{aligned}
(F - G)e_2 &= Re_2 - e_2 + \gamma Re_1 - \zeta e_1 = (Re_1 - e_1)^\perp + \gamma Re_1 - \zeta e_1 \\
&= \frac{-2(\gamma - \zeta)^2}{4 + (\gamma - \zeta)^2} e_2 - \frac{4(\gamma - \zeta)}{4 + (\gamma - \zeta)^2} e_1 + \gamma \frac{4 - (\gamma - \zeta)^2}{4 + (\gamma - \zeta)^2} e_1 + \gamma \frac{4(\gamma - \zeta)}{4 + (\gamma - \zeta)^2} e_2 - \zeta e_1 \\
&= \frac{-4(\gamma - \zeta) + 4\gamma - \gamma(\gamma - \zeta)^2 - 4\zeta - \zeta(\gamma - \zeta)^2}{4 + (\gamma - \zeta)^2} e_1 + \frac{-2(\gamma - \zeta)^2 + 4\gamma(\gamma - \zeta)}{4 + (\gamma - \zeta)^2} e_2 \\
&= -\frac{(\gamma + \zeta)(\gamma - \zeta)^2}{4 + (\gamma - \zeta)^2} e_1 + \frac{2(\gamma - \zeta)(\gamma + \zeta)}{4 + (\gamma - \zeta)^2} e_2.
\end{aligned}$$

Overall we obtain

$$F - G = \frac{\gamma - \zeta}{4 + (\gamma - \zeta)^2} \big((\zeta - \gamma)e_1 + 2e_2\big) \otimes \big(2e_1 + (\gamma + \zeta)e_2\big).$$

Lastly, since we only consider two dimensions, matrices of rank smaller than 2 can only have rank 1 or 0. To exclude the latter, i.e. $F = G$, the requirement that $R \neq Q$ or $\gamma \neq \zeta$ suffices. This concludes the proof. □

Since we aim to construct laminates tailored to the layered structure, only particular rank one connections are of interest to us, see also [42, Remark 3.2].

Corollary 5.2.4 (Rank one connections compatible to the layered structure)**.** *For rotations $R, Q \in SO(2)$ and $\gamma, \zeta \in \mathbb{R}$ let $F, G \in \mathcal{M}_s$ be given by $F = R(\mathbb{I}+\gamma s\otimes m)$ and $G = Q(\mathbb{I}+\zeta s\otimes m)$ such that $F - G = a \otimes e_2$ for some $a \in \mathbb{R}^2 \setminus \{0\}$. Then, either*

(i) $s = e_1$, in which case it must hold that $R = Q$ and $\gamma \neq \zeta$; or

(ii) $s \neq e_1$, in which case it must hold that $\gamma + \zeta = 2\frac{s_1}{s_2}$. This implies that a given γ determines ζ and therefore $Q^T R$ is determined via θ.

Now that we have determined sufficient conditions for a simple laminate construction to be admissible for E_ϵ^β we intend to characterize all possible macroscopic deformations. The arguments will feature prominently the rigidity results of Chapter 3. Note that the additional assumption on the strong convergence of $\nabla u_\epsilon s$ can also be derived from the uniform bound on the energy of $(u_\epsilon)_\epsilon$, which we will establish in the compactness arguments later, in particular in (5.6). The proof is similar to the arguments given in the beginning of [42, Section 3.1].

Lemma 5.2.5 (Necessary condition for admissible macroscopic deformations)**.** *For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ let $(u_\epsilon)_\epsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ with $E_\epsilon(u_\epsilon) < C$ for all $\epsilon > 0$ such that $u_\epsilon \rightharpoonup u$ in $W^{1,1}(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$ for some $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $\det \nabla u = 1$ and $\|\nabla u_\epsilon s\|_{L^2(\Omega;\mathbb{R}^2)} \to |\Omega|$ as $\epsilon \to 0$.*

Then, there is a constant rotation $R \in SO(2)$ and a $\gamma \in L^2(\Omega)$ such that

$$\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2) \quad \text{and} \quad \gamma \in K_{s,\lambda} \text{ a.e. in } \Omega.$$

Proof. The asymptotic rigidity result of Corollary 3.3.3 for $r = n = 2$ yields the existence of $R \in SO(2)$ and a $\gamma \in L^2(\Omega)$ such that $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$, or, in other words, $\nabla u \in \mathcal{M}_{e_1}$.

Furthermore, Conti, Dolzmann and Kreisbeck established in [45, Theorem 1.1], that for a.e. $x \in \Omega$

$$\nabla u(x) \in \mathcal{N}_s = \{F \in \mathbb{R}^{n\times n} \mid \det F = 1, |Fs| \leq 1\}.$$

For more details on the set $\mathcal{N}_s$, see also Remark 5.2.7. Thus, we know that for a.e. $x \in \Omega$

$$\nabla u(x) \in \mathcal{M}_{e_1} \cap \mathcal{N}_s.$$

From this it follows directly that for a.e. $x \in \Omega$

$$|s + \gamma(x)s_2 e_1|^2 = |(\mathbb{I} + \gamma(x)e_1 \otimes e_2)s|^2 = |R(\mathbb{I} + \gamma(x)e_1 \otimes e_2)s|^2 = |\nabla u(x)s|^2 \leq 1,$$

which in view of Remark 5.2.2 is the case if and only if $\gamma \in K_{s,1}$.

At this point, a second consequence of the stiff layers, which are of asymptotic volume fraction of $|P_{\text{stiff}}| = 1 - \lambda$, enters by an argument similar to Lemma 4.3.4 and Corollary 4.3.7. Using the notation of Definition 3.3.5, let for $x \in \Omega$ be $Q \subset \Omega$ such that $x \in Q$ and $Q'' \subset\subset \Omega$, then, by Proposition 3.3.10, Lemma 3.3.12 and Proposition 3.3.15, there is a sequence $(\Sigma_\epsilon)_\epsilon \subset L^\infty(Q'; SO(2))$ such that $\|\nabla u_\epsilon - \Sigma_\epsilon\|_{L^1(\epsilon P_{\text{stiff}} \cap Q';\mathbb{R}^{2\times 2})} \to 0$ and $\Sigma_\epsilon \to R$ in $L^1(Q;\mathbb{R}^{2\times 2})$ as $\epsilon \to 0$. Thus, on the stiff layers

$$\nabla u_\epsilon \, \mathbb{1}_{\epsilon P_{\text{stiff}}} = (\nabla u_\epsilon - \Sigma_\epsilon)\, \mathbb{1}_{\epsilon P_{\text{stiff}}} + \Sigma_\epsilon \, \mathbb{1}_{\epsilon P_{\text{stiff}}} \rightharpoonup (1-\lambda)R \quad \text{in} \quad L^1(Q;\mathbb{R}^2) \quad \text{as } \epsilon \to 0,$$

while on the soft layers

$$(\nabla u_\epsilon s)\,\mathbb{1}_{\epsilon P_{\text{soft}}} = \nabla u_\epsilon s - (\nabla u_\epsilon s)\,\mathbb{1}_{\epsilon P_{\text{stiff}}} \rightharpoonup R(\lambda\mathbb{I} + \gamma e_1 \otimes e_2)s \quad \text{in } L^1(Q;\mathbb{R}^2) \quad \text{as } \epsilon \to 0.$$

Also, the assumption $\|\nabla u_\epsilon s\|_{L^2(\Omega;\mathbb{R}^2)} \to |\Omega|$ implies $|(\nabla u_\epsilon s)\,\mathbb{1}_{\epsilon P_{\text{soft}}}| = |\nabla u_\epsilon s|\,\mathbb{1}_{\epsilon P_{\text{soft}}} \rightharpoonup \lambda$ in $L^2(\Omega)$ both as $\epsilon \to 0$. Hence, for any open ball $B \subset Q$ with $x \in B$ the weak lower semicontinuity of the L^1-norm implies

$$\int_B |R(\lambda\mathbb{I} + \gamma e_1 \otimes e_2)s|\,\mathrm{d}x \leq \lim_{\epsilon\to 0} \int_B |(\nabla u_\epsilon s)\,\mathbb{1}_{\epsilon P_{\text{soft}}}|\,\mathrm{d}x = |B|\lambda.$$

Dividing the inequality by its right hand side we obtain

$$⨍_B |s + \lambda^{-1}\gamma s_2 e_1|\,\mathrm{d}x \leq 1.$$

Therefore, by the Lebesgue point theorem and by Remark 5.2.2 we have $\gamma \in K_{s,\lambda}$ a.e. in Ω. □

The next lemma shows that elements of $\mathcal{N}_s$ can be written as convex combinations of elements of $\mathcal{M}_s$. This problem was first studied in the context of relaxation of energy functionals describing slip systems by Conti and Theil [56] and for this specific energy with linear hardening it was obtained by Conti [47]. Here, we give a variation of the argument also featured in [42, Lemma 3.3].

Lemma 5.2.6 (Convex decomposition of elements in $\mathcal{N}_s \setminus \mathcal{M}_s$)**.** *Let $N \in \mathcal{N}_s \setminus \mathcal{M}_s$. Then, there are matrices $F, G \in \mathcal{M}_s$ with $|Fs| = |Gs| = 1$ and* $\operatorname{rank}(F-G) = 1$ *and a $\mu \in (0,1)$ such that*

$$\textit{(i)}\ N = \mu F + (1-\mu)G \qquad \textit{and} \qquad \textit{(ii)}\ |Nm| = |Fm| = |Gm|.$$

Proof. As $F, G \in \mathcal{M}_s$ we have to find rotations $R, Q \in SO(2)$ and $\gamma, \zeta \in \mathbb{R}$ such that the statement holds for $F = R(\mathbb{I} + \gamma s \otimes m)$, $G = Q(\mathbb{I} + \zeta s \otimes m)$ and a suitable $\mu \in (0,1)$. For (ii) to be satisfied it is therefore necessary that

$$|\gamma|^2 + 1 = |m + \gamma s|^2 = |Fm|^2 = |Nm|^2 \quad \text{and} \quad |\zeta|^2 + 1 = |m + \zeta s|^2 = |Gm|^2 = |Nm|^2.$$

These conditions are satisfied for $\bar\gamma = \sqrt{|Nm|^2 - 1}$ and $\bar\zeta = -\bar\gamma$, which are well-defined since $1 = \det N \leq |Ns||Nm|$ and $|Ns| < 1$ imply that $|Nm| > 1$. It remains to determine μ and the rotations R, Q. Since F and G are supposed to be rank one connected, and $R = Q$ would entail $N \in \mathcal{M}_s$, Lemma 5.2.3 yields

$$\begin{aligned} Ns &= Gs + \mu(F-G)s \\ &= Q(\mathbb{I} + \zeta s \otimes m)s + \mu\frac{\gamma - \zeta}{4 + (\gamma-\zeta)^2} Q\big((\zeta - \gamma)s + 2m\big) \otimes \big(2s + (\gamma+\zeta)m\big)s \\ &= Qs + 2\mu\frac{\gamma - \zeta}{4 + (\gamma-\zeta)^2} Q\big((\zeta-\gamma)s + 2m\big). \end{aligned}$$

Hence, for our particular choices of $\bar\gamma$ and $\bar\zeta$

$$Ns = Q\Big(s + \frac{2\bar\gamma\mu}{1 + \bar\gamma^2}(m - \bar\gamma s)\Big). \tag{5.4}$$

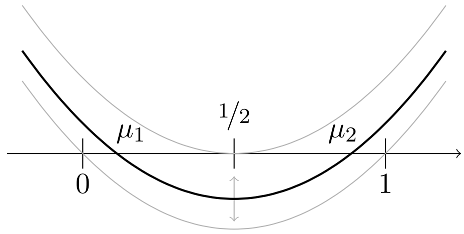

Figure 5.3: Since the function $x \mapsto x(x-1)+t$, $t \in \mathbb{R}$ takes its minimum at $1/2$ with the value $1/4 + t$ it has for $t \in (0, 1/4)$ two zeros at $\mu_1, \mu_2 \in (0,1)$ with $\mu_1 + \mu_2 = 1$ by symmetry.

To determine candidates for μ we take the squared norms on both sides of the equation and obtain

$$\begin{aligned}
|Ns|^2 &= \Big| e_1 + \frac{2\bar\gamma\mu}{1+\bar\gamma^2}(e_2 - \bar\gamma e_1)\Big|^2 = \Big(1 - \frac{2\bar\gamma^2\mu}{1+\bar\gamma^2}\Big)^2 + \Big(\frac{2\bar\gamma\mu}{1+\bar\gamma^2}\Big)^2 \\
&= 1 - \frac{4\bar\gamma^2\mu}{1+\bar\gamma^2} + (\bar\gamma^2+1)\Big(\frac{2\bar\gamma\mu}{1+\bar\gamma^2}\Big)^2 \\
&= \frac{4\bar\gamma^2}{1+\bar\gamma^2}\mu^2 - \frac{4\bar\gamma^2}{1+\bar\gamma^2}\mu + 1 \\
&= \frac{4\bar\gamma^2}{1+\bar\gamma^2}\mu(\mu-1) + 1.
\end{aligned}$$

Hence, for $\bar\gamma$ we have

$$0 = \mu(\mu-1) + \frac{(1+\bar\gamma^2)(1-|Ns|^2)}{4\bar\gamma^2} = \mu(\mu-1) + \frac{|Nm|^2}{4}\cdot\frac{1-|Ns|^2}{|Nm|^2-1}.$$

Since $\frac{|Nm|^2(1-|Ns|^2)}{|Nm|^2-1} < 1$ is equivalent to $1 < |Ns||Nm|$, which is always satisfied, the equation has always two solutions $\mu_1 \in (0, \frac{1}{2})$ and $\mu_2 \in (\frac{1}{2}, 1)$ with $\mu_1 + \mu_2 = 1$, see Figure 5.3. Notice that if the equation holds for the norms, we always find a $Q_\mu \in SO(2)$ in dependence of the choice of μ, such that (5.4) holds. Furthermore, by Lemma 5.2.3 we know that R_μ is determined via the relation $\bar\gamma - 2\tan(\frac{\theta}{2})$, where θ is the rotation angle of $Q_\mu^T R_\mu$.

Therefore, it remains to choose from μ_1 and μ_2 the one satisfying

$$Nm = \mu Fm + (1-\mu)Gm.$$

To that end, observe that for a G satisfying (5.4) it holds that

$$(Ns)^\perp \cdot Gm = \Big(m + \frac{2\gamma\mu}{1+\gamma^2}(s - \gamma m)\Big)\cdot(m - \gamma s) = 1.$$

Thus, in any case, independent of the choice of μ, Gm is an element of the set

$$\{a \in \mathbb{R}^2 \mid (Ns)^\perp \cdot a = 1\} \cap \{a \in \mathbb{R}^2 \mid |a| = |Nm|\}.$$

Notice that geometrically the former set describes a line in $\mathbb{R}^2$ while the latter describes a circle, and since Ns lies on the line and inside the circle as $|Ns| < |Nm|$, the intersection has exactly two elements, one of them being Nm. Hence, we may choose the $\mu \in \{\mu_1, \mu_2\}$ corresponding to $Gm = Nm$. From that choice it is immediate that $Nm = \mu Fm + (1-\mu)Gm$. This equality also shows that interchanging γ and ζ in the beginning amounts to switching F and G. □

Remark 5.2.7 (Quasi-convex hull of $\mathcal{M}_s$)**.** As established by Conti and Theil, the quasiconvex hull of $\mathcal{M}_s$ is given by the set [56, Theorem 1]

$$\mathcal{N}_s = \{F \in \mathbb{R}^{2\times 2} \mid \det F = 1, |Fs| \leq 1\}.$$

Furthermore, it coincides with the rank one and polyconvex hull of $\mathcal{M}_s$. The upper bound, i.e. that $\mathcal{N}_s$ is the rank one hull of $\mathcal{M}_s$ can be seen from Lemma 5.2.6, while the lower bound, that $\mathcal{N}_s$ is the polyconvex hull of $\mathcal{M}_s$ follows from the fact that the defining conditions of $\mathcal{N}_s$ are polyconvex.

In [45], Conti, Dolzmann and Kreisbeck established in the compactness argument of [45, Theorem 1.1] that weak limits of sequences of deformations that satisfy an approximate differential inclusion in $\mathcal{M}_s$ are pointwise contained in $\mathcal{N}_s$ as well. Note that we will recall parts of their argument in Proposition 5.2.13.

Lemma 5.2.8 ([42, Lemma 3.4])**.** *Let $\lambda \in (0,1)$ and $s \in \mathbb{S}^1 \setminus \{e_1\}$. Then,*

(i) For given $\gamma \in K_{s,\lambda}$ and $R \in SO(2)$ there is an $N \in \mathcal{M}_{e_1} \cap \mathcal{N}_s$, namely $N = R(\mathbb{I} + \frac{\gamma}{\lambda} e_1 \otimes e_2)$ satisfying

$$Ne_1 = Re_1 \quad \text{and} \quad \lambda N + (1-\lambda)R = R(\mathbb{I} + \gamma e_1 \otimes e_2);$$

(ii) For given $N \in \mathcal{N}_s$ and $R \in SO(2)$ with $Re_1 = Ne_1$ there is a $\gamma \in K_{s,\lambda}$ such that

$$\lambda N + (1-\lambda)R = R(\mathbb{I} + \gamma e_1 \otimes e_2).$$

Proof. (i) Consider $N = R(\mathbb{I} + \frac{\gamma}{\lambda} e_1 \otimes e_2) \in \mathcal{M}_{e_1}$. Then, $Ne_1 = Re_1$ and $|Ns| = |s + \frac{\gamma}{\lambda} s_2 e_1| \leq 1$ by Remark 5.2.2 as $\gamma \in K_{s,\lambda}$. Hence, $N \in \mathcal{N}_s$. Lastly,

$$\lambda N + (1-\lambda)R = \lambda R + \gamma R e_1 \otimes e_2 + (1-\lambda)R = R(\mathbb{I} + \gamma e_1 \otimes e_2),$$

as desired.

(ii) Note that since $Ne_1 = Re_1$ and $1 = \det N = (Ne_1)^\perp \cdot Ne_2$ we have

$$Re_2 = (Re_1)^\perp = \big((Ne_1)^\perp \cdot Ne_2\big)(Ne_1)^\perp.$$

Also,

$$Ne_2 = (Ne_1 \cdot Ne_2)Ne_1 + \big((Ne_1)^\perp \cdot Ne_2\big)(Ne_1)^\perp = (Ne_1 \cdot Ne_2)Ne_1 + Re_2.$$

Hence, since $Ne_1 = Re_1$, it suffices to observe for Ne_2 that

$$\lambda Ne_2 + (1-\lambda)Re_2 = Re_2 + \lambda(Ne_1 \cdot Ne_2)Ne_1 = R(\mathbb{I} + \lambda(Ne_1 \cdot Ne_2)e_1 \otimes e_2)e_2,$$

so that by $s = s_1 e_1 + s_2 e_2$ we have

$$|s + (Ne_1 \cdot Ne_2)s_2 e_1| = \lambda^{-1}|(R(\mathbb{I} + \lambda(Ne_1 \cdot Ne_2)e_1 \otimes e_2)s - (1-\lambda)Rs| = |Ns| \leq 1,$$

which implies $\lambda Ne_1 \cdot Ne_2 \in K_{s,\lambda}$ by Remark 5.2.2. Thus, $\gamma = \lambda Ne_1 \cdot Ne_2$ satisfies all claimed properties. □

5.2.2 The technique of convex integration

As the previous section suggests, the key to satisfy the differential inclusion constraint is the interplay between $\mathcal{M}_s$ and its quasiconvex hull $\mathcal{N}_s$, in the sense that functions with gradients in $\mathcal{N}_s$ can be approximated by functions with gradients in $\mathcal{M}_s$. Problems of these structure arise in many branches of mathematics, and the associated theory is known as convex integration [78, 79]. In the context of elastoplasticity, this theory has been advanced by the work of Müller and Šverák [113] that will also find application here. For our purposes it is furthermore necessary to know very explicitly the structure of the gradients of the approximating sequences. For $\mathcal{M}_s \subset \mathcal{N}_s$ such constructions have been developed by Conti and Theil [56] whose results we summarize in our notation in Corollary 5.2.9.

The goal of this section is to condense the aspects of the work of Müller, Šverák and Conti, Theil relevant to us in the following corollary, see also [42, Corollary 3.7].

Corollary 5.2.9 (Convex integration of $\mathcal{N}_s$)**.** *Let $N \in \mathcal{N}_s$. If $N \in \mathcal{N}_s \setminus \mathcal{M}_s$ let $F, G \in \mathcal{M}_s$ and $\mu \in (0,1)$ as in Lemma 5.2.6, otherwise let $F = G = N \in \mathcal{M}_s$ and $\mu \in (0,1)$.*

Then, for every $\delta > 0$ there exists $\psi_\delta \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ and $\Omega_\delta \subset \Omega$ with $|\Omega \setminus \Omega_\delta| < \delta$ such that ψ_δ coincides with a simple laminate between F and G with weights μ and $1-\mu$ and period $h_\delta < \delta$ in Ω_δ,

$$\nabla\psi_\delta \in \mathcal{M}_s \text{ a.e. in } \Omega, \quad \psi_\delta = Nx \text{ on } \partial\Omega, \quad |\nabla\psi_\delta m| < |Nm| + \delta \text{ a.e. in } \Omega. \tag{5.5}$$

In particular, $|\nabla\psi_\delta m| = |Nm|$ a.e. in Ω_δ, and $\nabla\psi_\delta \overset{*}{\rightharpoonup} N$ in $L^\infty(\Omega;\mathbb{R}^{2\times 2})$ as $\delta \to 0$.

This corollary results from two major theorems. The first one is a construction by Conti and Theil that approximates for $N \in \mathcal{N}_s \setminus \mathcal{M}_s$ the function $u(x) = Nx$, $x \in \Omega$ by a finitely piecewise affine function that coincides with a laminate with gradients in $\mathcal{M}_s$ on a large part of Ω and features a gradient in $\mathcal{N}_s$ on the rest.

Theorem 5.2.10 ([56, Theorem 4])**.** *Let $\mu \in (0,1)$ and suppose that $F, G \in \mathcal{M}_s$ are rank one connected with $Fs \neq Gs$ and $N = \mu F + (1-\mu)G \in \mathcal{N}_s$.*

Then, for every $\delta > 0$ there are $h_\delta^0 > 0$ and $\Omega_\delta \subset \Omega$ with $|\Omega \setminus \Omega_\delta| < \delta$ such that the restriction to Ω_δ of any simple laminate between the gradients F and G with weights μ and $1-\mu$ and period $h < h_\delta^0$ can be extended to a finitely piecewise affine function $\psi_\delta : \Omega \to \mathbb{R}^2$ with $\nabla\psi_\delta \in \mathcal{N}_s$ a.e. in Ω, $\psi_\delta(x) = Nx$ for $x \in \partial\Omega$, and $\operatorname{dist}(\nabla\psi_\delta, [F,G]) < \delta$ a.e. in Ω, where $[F,G] = \{tF + (1-t)G \mid t \in [0,1]\}$.

To obtain a function whose gradient is almost everywhere in $\mathcal{M}_s$, Conti and Theil [56, Section 3] suggest to apply the convex integration results by Müller and Šverák.

Theorem 5.2.11 ([113, Theorem 1.3])**.** *Let $\mathcal{M} \subset \{F \in \mathbb{R}^{2\times 2} \mid \det F = 1\}$. Suppose that $(U_i)_{i\in\mathbb{N}}$ is an in-approximation of $\mathcal{M}$, i.e., the sets U_i are open in $\{F \in \mathbb{R}^{2\times 2} \mid \det F = 1\}$ and uniformly bounded, U_i is contained in the rank one convex hull of U_{i+1}, i.e. $U_i \subset U_{i+1}^{rc}$ for every $i \in \mathbb{N}$ and $(U_i)_i$ converges to $\mathcal{M}$ in the sense that if $F_i \in U_i$ for $i \in \mathbb{N}$ and $|F_i - F| \to 0$ as $i \to \infty$, then $F \in \mathcal{M}$.*

Then, for any $F \in U_1$ and any open domain $\Omega \subset \mathbb{R}^2$, there exists $\psi \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ such that $\nabla\psi \in \mathcal{M}$ a.e. in Ω and $\psi(x) = Fx$ for $x \in \partial\Omega$.

Proof of Corollary 5.2.9. For $N \in \mathcal{M}_s$ there is nothing to do. If $N \in \mathcal{N}_s \setminus \mathcal{M}_s$, Theorem 5.2.10 yields for $\delta > 0$ that there is a set Ω_δ and a function $\psi_\delta : \Omega \to \mathbb{R}^2$ that is piecewise affine, coincides on Ω_δ with a simple laminate with the gradient oscillating between F and G with a period of $h_\delta < \min\{\delta, h_\delta^0\}$, and features $\nabla\psi_\delta \in \mathcal{N}_s$ a.e. in Ω and $\psi_\delta(x) = Nx$ for

$x \in \partial\Omega$. As F and G are chosen according to Lemma 5.2.6, we have $|\nabla\psi_\delta m| = |Nm|$ a.e. on Ω_δ and on the rest of Ω, we have for a.e. $x \in \Omega$ the estimate

$$\big||\nabla\psi_\delta|(x) - |Nm|\big| \leq \min_{t\in[0,1]} \big|\nabla\psi_\delta(x)m - \big(tFm + (1-t)Gm\big)\big| \leq \operatorname{dist}\big(\nabla\psi_\delta(x), [F,G]\big) < \delta.$$

Now, we apply Theorem 5.2.11 to each of the finitely many domains, where $\nabla\psi_\delta \notin \mathcal{M}_s$ using the fact that the sets $(U_i^\delta)_{i\in\mathbb{N}}$ defined for each $i \in \mathbb{N}$ by

$$U_i^\delta := \big\{F \in \mathbb{R}^{2\times 2} \;\big|\; \det F = 1, 1 - 2^{-(i-1)} < |Fs| < 1, |Fm| < |Nm| + \delta\big\}$$

are an in-approximation of $\mathcal{M}_s \cap \{F \in \mathbb{R}^{n\times n} \mid |Fm| < |Nm| + \delta\}$. □

5.2.3 Compactness for sequences of bounded energy

A mathematical challenge in this model of crystal plasticity is the issue of compactness. We recall the common example of the decomposition in elastic and plastic part of the curve $F : \mathbb{R} \to \mathbb{R}^{n\times n}$ in the space of all deformation gradients given by [51, Remark 1.2, (iv)]

$$F(t) = \begin{pmatrix} t & t^2 \\ 0 & t \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = F_{\mathrm{e}}(t)F_{\mathrm{p}}(t), \quad t \in \mathbb{R}.$$

By concatenating F with a function $t \in L^2(\Omega;\mathbb{R}^2)$ we observe that due to the component F_{12}, we only obtain $F \circ t \in L^1(\Omega;\mathbb{R}^{2\times 2})$. This also implies that if a sequence of deformations $(u_\epsilon)_\epsilon \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ is of bounded energy with respect to E_ϵ^β, i.e. for $C > 0$ it holds that $E_\epsilon^\beta(u_\epsilon) < C$, it merely follows for a constant $C > 0$ that $\|\nabla u_\epsilon\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} < C$ for all $\epsilon > 0$ which is not enough to conclude that $(u_\epsilon)_\epsilon$ is relatively compact with respect to the weak topology of $W^{1,1}(\Omega;\mathbb{R}^{2\times 2})$.

Hence, a more involved analysis is needed, a problem that was comprehensively solved by Conti, Dolzmann and Kreisbeck in [45]. A main issue to overcome is to establish that $\det\nabla u = 1$ whose solution is based on a generalization of the classic div-curl lemma due to Murat and Tartar in their study of compensated compactness by Conti, Dolzmann and Müller.

Theorem 5.2.12 (Generalized div-curl lemma [45, Theorem 2.2][53, Corollary])**.** *Assume $(u_\epsilon)_\epsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ and let $\nabla u_\epsilon = A_\epsilon + B_\epsilon$ for sequences $(A_\epsilon)_\epsilon \subset L^2(\Omega;\mathbb{R}^{2\times 2})$ such that $\det(A_k)$ is equi-integrable and $A_k \rightharpoonup A$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$ for some $A \in L^2(\Omega;\mathbb{R}^{2\times 2})$ and $(B_\epsilon)_\epsilon \subset L^1(\Omega;\mathbb{R}^{2\times 2})$ such that $B_\epsilon \to 0$ in $L^1(\Omega;\mathbb{R}^{2\times 2})$ both as $\epsilon \to 0$. Then,*

$$\det A_k \rightharpoonup \det A \quad \textit{in } L^1(\Omega;\mathbb{R}^{2\times 2}) \quad \textit{as } \epsilon \to 0.$$

Proposition 5.2.13 (Compactness)**.** *Let $(u_\epsilon)_\epsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ be a sequence of bounded energy with respect to $(E_\epsilon^\beta)_\epsilon$, i.e. $E_\epsilon^\beta(u_\epsilon) < C$ for a constant $C > 0$. Then, there is a function $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ and a subsequence $(u_\epsilon)_\epsilon$ (not relabeled) such that $u_\epsilon \rightharpoonup u$ in $W^{1,1}(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$. Furthermore, there is a rotation $R \in SO(2)$ and a function $\gamma \in L^2(\Omega)$ with $\partial_1\gamma = 0$ and $\gamma \in K_{s,\lambda}$ a.e. in Ω such that $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$.*

Proof. It suffices to show the existence of a weak limit $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ with $\det(\nabla u) = 1$. The additional properties of the statement then follow by Lemma 5.2.5.

To establish the existence of u, we follow the original arguments by Conti, Dolzmann and Kreisbeck [45, Section 3]. We start with a closer study of the bounds on ∇u_ϵ. Note that we can represent each $Q \in SO(2)$ by a vector $a \in \mathbb{S}^1$ via $Q = a \otimes s + a^\perp \otimes m$. Furthermore, observe that since the function $\gamma \mapsto \gamma^2$ is convex and of quadratic growth and $\mathbb{S}^1$ is a compact set, the minimization problem imposed by the condensed energy has for each $F \in \mathbb{R}^{2\times 2}$ a solution $(a_\epsilon^F, \gamma_\epsilon^F) \in \mathbb{S}^1 \times \mathbb{R}$, which means that

$$
\begin{aligned}
W(F) &= \inf_{\gamma\in\mathbb{R}, a\in\mathbb{S}^1} \Big(\epsilon^{-\beta}\big|F(\mathbb{I}-\gamma s\otimes m) - a\otimes s - a^\perp \otimes m\big|^2 + |\gamma|^2\Big) \\
&= \min_{\gamma\in\mathbb{R}, a\in\mathbb{S}^1} \Big(\epsilon^{-\beta}\big|F(s\otimes s + m\otimes m-\gamma s\otimes m) - a\otimes s - a^\perp \otimes m\big|^2 + |\gamma|^2\Big) \\
&= \min_{\gamma\in\mathbb{R}, a\in\mathbb{S}^1} \Big(\epsilon^{-\beta}\big(|Fs-a|^2 + |Fm-\gamma Fs - a^\perp|^2\big) + |\gamma|^2\Big) \\
&= \epsilon^{-\beta}\big(|Fs-a_\epsilon^F|^2 + |Fm - \gamma_\epsilon^F Fs - (a_\epsilon^F)^\perp|^2\big) + |\gamma_\epsilon^F|^2.
\end{aligned} \tag{5.6}
$$

Now, rewriting ∇u_ϵ with respect to the basis $s, m \in \mathbb{R}^2$, and rearranging terms to mirror the terms appearing in (5.6) we obtain

$$\nabla u_\epsilon = A_\epsilon + B_\epsilon,$$

where for $a_\epsilon(x) := a_\epsilon^{(\nabla u)(x)}$ and $\gamma_\epsilon(x) := \gamma_\epsilon^{(\nabla u)(x)}$

$$
\begin{aligned}
A_\epsilon &= a_\epsilon \otimes s + (\gamma_\epsilon a_\epsilon + a_\epsilon^\perp)\otimes m \\
B_\epsilon &= \big((\nabla u_\epsilon)s - a_\epsilon\big)\otimes s + \big((\nabla u_\epsilon)m - \gamma_\epsilon(\nabla u_\epsilon)s - a_\epsilon^\perp\big)\otimes m + \gamma_\epsilon\big((\nabla u_\epsilon)s - a_\epsilon\big)\otimes m.
\end{aligned}
$$

Furthermore, it follows from (5.6) that $\|\gamma_\epsilon\|_{L^2(\Omega)} < C$ and

$$\|(\nabla u_\epsilon)s - a_\epsilon\|^2_{L^2(\Omega;\mathbb{R}^2)} + \|(\nabla u_\epsilon)m - \gamma_\epsilon(\nabla u_\epsilon)s - a_\epsilon^\perp\|^2_{L^2(\Omega;\mathbb{R}^2)} \le C\epsilon^\beta \to 0 \quad \text{as } \epsilon \to 0.$$

Also, by Hölder's inequality we have

$$\big\|\gamma_\epsilon\big((\nabla u_\epsilon)s - a_\epsilon\big)\big\|_{L^1(\Omega;\mathbb{R}^2)} \to 0 \quad \text{as } \epsilon \to 0.$$

These estimates imply the existence of a subsequence (not relabeled) such that

$$
\begin{aligned}
&a_\epsilon \rightharpoonup a \quad \text{in } L^\infty(\Omega;\mathbb{R}^2), && A_\epsilon \rightharpoonup A \quad \text{in } L^2(\Omega;\mathbb{R}^{2\times 2}), \\
&B_\epsilon \to 0 \quad \text{in } L^1(\Omega;\mathbb{R}^{2\times 2}), && u_\epsilon \to u \quad \text{in } L^1(\Omega;\mathbb{R}^2), \qquad \text{as } \epsilon \to 0.
\end{aligned}
$$

Thus, $\int_\Omega u \,\mathrm{d}x = 0$ and $\nabla u_\epsilon \rightharpoonup A$ in $L^1(\Omega;\mathbb{R}^{2\times 2})$ as $\epsilon \to 0$, which yields $\nabla u = A \in L^2(\Omega;\mathbb{R}^{2\times 2})$. This shows the existence of the limit u and it remains to establish that $\det \nabla u = 1$. Though ∇u is merely in $L^1(\Omega;\mathbb{R}^{2\times 2})$ we have $A_\epsilon \in L^2(\Omega;\mathbb{R}^{2\times 2})$ and $A_\epsilon \rightharpoonup \nabla u$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Hence, we can calculate

$$\det A_\epsilon = \det\big(a_\epsilon \otimes s + (\gamma_\epsilon a_\epsilon + a_\epsilon^\perp)\otimes m\big) = a_\epsilon^\perp \cdot (\gamma_\epsilon a_\epsilon + a_\epsilon^\perp) = 1.$$

In particular, $\det A_\epsilon$ is equi-integrable and thus, by the generalized div-curl lemma of Theorem 5.2.12, we obtain

$$\det \nabla u = \lim_{\epsilon\to 0} \det A_\epsilon = 1,$$

as desired. This finishes the proof. □

Remark 5.2.14. Note that the arguments establishing the existence of the limit u and $\det(\nabla u) = 1$ do not depend on the power of β, while of course the rest of the statements only holds for $\beta > 2$ as they are derived from the asymptotic rigidity result of Corollary 3.3.3.

5.2.4 Construction of recovery sequences

The construction necessary in the case $s \neq e_1$ is overly complicated if $s = e_1$ and a simpler approach actually allows to prove a slightly more general result. Therefore, the first part of this subsection will concern the case $s = e_1$ only, followed by the case $s \neq e_1$ afterwards.

We first introduce the more general energy density modeling mixed hardening. Let for $\tau \geq 0$ the energy density $W^\tau_{\mathrm{slip}} : \mathbb{R}^{2\times 2} \to [0,\infty]$ for $F \in \mathbb{R}^{2\times 2}$ be given by

$$W^\tau_{\mathrm{slip}} = \begin{cases} \gamma^2 + \tau|\gamma| & \text{if } F = R(\mathbb{I} + \gamma e_1 \otimes e_2) \text{ with } R \in SO(2), \gamma \in \mathbb{R}, \\ \infty & \text{otherwise,} \end{cases}$$

and for $\epsilon > 0$ let $E^\tau_\epsilon : L^2_0(\Omega;\mathbb{R}^2) \to [0,\infty]$ be defined analogously to E^β_ϵ.

Furthermore, we define the functional $E^{\tau,\beta} : L^2_0(\Omega;\mathbb{R}^2) \to [0,\infty]$ by

$$E^\tau(u) = \begin{cases} \displaystyle\int_\Omega \frac{1}{\lambda}\gamma^2 + \tau|\gamma| \,\mathrm{d}x & \text{if } u \in W^{1,2}(\Omega;\mathbb{R}^2), \nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2) \\ & \quad \text{with } R \in SO(2),\ \gamma \in L^2(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

We start with the upper bound estimate for $s = e_1$, see also [42, Section 4]

Proposition 5.2.15 (Upper bound for $s = e_1$)**.** *Let $u \in L^2_0(\Omega;\mathbb{R}^2)$, then there is a sequence $(v_\epsilon)_\epsilon \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ such that $v_\epsilon \to u$ in $L^2(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$ and*

$$\lim_{\epsilon\to 0} E^{\tau,\beta}_\epsilon(v_\epsilon) \leq E^\tau(u).$$

Proof. Firstly notice, that if $E^\tau(u) = \infty$, we may just choose the constant sequence $(u)_\epsilon$, which is why we may assume the case $E^\tau(u) < \infty$. Therefore, there are $R \in SO(2)$ and $\gamma \in L^2(\Omega)$ such that $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$. The idea to avoid penalization of the gradient by the constraints imposed by $E^{\tau,\beta}_\epsilon$ is not to shear the stiff layers and compensate by additional shearing of the soft layers, see also Figure 5.4

Accordingly, we define for $\epsilon > 0$ the oscillating function $\gamma_\epsilon \in L^2(\Omega)$ by

$$\gamma_\epsilon = \frac{\gamma}{\lambda}\,\mathbb{1}_{\epsilon P_{\mathrm{soft}} \cap \Omega}\,.$$

The classic Lemma 2.3.1 on the weak convergence of highly oscillating functions yields $\mathbb{1}_{\epsilon P_{\mathrm{soft}}\cap\Omega} \overset{*}{\rightharpoonup} \lambda$ in $L^\infty(\Omega)$ and thus, $\gamma_\epsilon \rightharpoonup \gamma$ in $L^2(\Omega)$, both as $\epsilon \to 0$. As Ω is simply connected and $\mathrm{curl}(R(\mathbb{I} + \gamma_\epsilon e_1 \otimes e_2)) = 0$ in the sense of distributions, there are functions $v_\epsilon \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ such that

$$\begin{aligned} \nabla v_\epsilon = R(\mathbb{I} + \gamma_\epsilon e_1 \otimes e_2) &= R\big(\mathbb{I} + \tfrac{1}{\lambda}\gamma\,\mathbb{1}_{\epsilon P_{\mathrm{soft}}\cap\Omega}\, e_1 \otimes e_2\big) \\ &\rightharpoonup R(\mathbb{I} + \gamma e_1 \otimes e_2) = \nabla u \qquad \text{in } L^2(\Omega;\mathbb{R}^{2\times 2}) \quad \text{as } \epsilon \to 0. \end{aligned}$$

As we assumed that the mean of v_ϵ vanishes on Ω the Poincaré inequality yields $v_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$ and thus $v_\epsilon \to u$ in $L^2(\Omega;\mathbb{R}^2)$. Finally, we obtain for the energies

$$\begin{aligned} E^{\tau,\beta}_\epsilon(v_\epsilon) &= \int_{\epsilon P_{\mathrm{soft}}\cap\Omega} \gamma_\epsilon^2 + \tau|\gamma_\epsilon| \,\mathrm{d}x = \int_\Omega \Big(\frac{\gamma^2}{\lambda^2} + \tau\frac{|\gamma|}{\lambda}\Big)\,\mathbb{1}_{\epsilon P_{\mathrm{soft}}\cap\Omega}\,\mathrm{d}x \\ &\to \int_\Omega \frac{1}{\lambda}\gamma^2 + \tau|\gamma| \,\mathrm{d}x = E^\tau(u), \qquad \text{in } L^2(\Omega) \quad \text{as } \epsilon \to 0, \end{aligned}$$

as desired. □

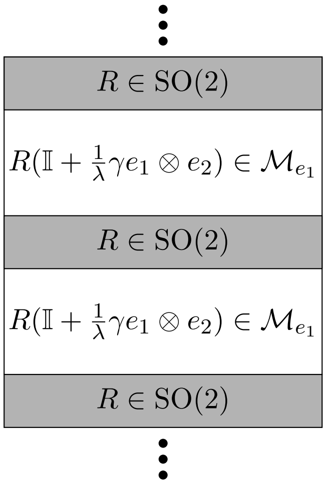

Figure 5.4: For the recovery sequence, the task is to construct macroscopically a globally rotated shear deformation. On the microscopic level, the shear is impeded by the stiff layers. The idea is to forgo shearing the stiff layers at all, yet compensate on the softer layers by shearing more, namely by a factor of $1/\lambda$. Since $s = e_1$, we obtain for any amount of shear a compatible construction.

Remark 5.2.16. Since Ω is assumed to be $e_2^\perp$-connected in the sense of Definition 3.3.6, the function v_ϵ constructed in the proof has a very simple representation given by

$$v_\epsilon(x) = R\Big(x + \int_0^{x_2} \gamma_\epsilon(t) e_1 \,\mathrm{d}t\Big) + c, \quad x \in \Omega,$$

where $c \in \mathbb{R}^2$ is chosen such that the mean value of u_ϵ on Ω vanishes.

Next, we will construct a recovery sequence for general $s \in \mathbb{S}^1$ which features in contrast to the one constructed for $s = e_1$ the formation of microstructure, see also [42, Section 5]. Since the relaxation for a mixed energy density like W^τ_{soft} is not known [47, 3. Discussion], we can only handle energy densities modeling linear hardening.

Proposition 5.2.17 (Upper bound for $s \neq e_1$)**.** *Let $u \in L^2_0(\Omega;\mathbb{R}^2)$, then there is a sequence $(u_\epsilon)_\epsilon \subset L^2_0(\Omega;\mathbb{R}^2)$ such that $u_\epsilon \to u$ in $L^2(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$ and*

$$\lim_{\epsilon \to 0} E^\beta_\epsilon(u_\epsilon) = E(u).$$

Proof. Firstly, note that for $E(u) = \infty$, we may choose any sequence $(u_\epsilon)_\epsilon \subset L^2_0(\Omega;\mathbb{R}^2)$ with $u_\epsilon \to u$ in $L^2(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$, in particular the constant sequence $u_\epsilon = u$ for all $\epsilon > 0$. Hence, we may assume that $E(u) < \infty$, in which case $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$, where $R \in SO(2)$ and $\gamma \in L^2(\Omega)$ with $\gamma \in K_{s,\lambda}$. In this case Proposition 5.2.21 provides the desired sequence. The idea of the construction is that we may approximate the general limit function with a piecewise affine function. This is done in the Proof of Proposition 5.2.21. For each piece, one could use a laminate with gradients in $\mathcal{N}_s$ to approximate the affine limit, which is shown in Lemma 5.2.18. But such a laminate construction is in general not admissible for E^β_ϵ. Hence, we replace in Lemma 5.2.19 large parts of the soft layers by a finer laminate converging to a function with gradient in $\mathcal{N}_s$ and use convex integration to fit the boundary data, see also Figure 5.5. To ensure compatibility between the different affine pieces, the construction has to be adapted on the boundaries, which is done in Lemma 5.2.20. ☐

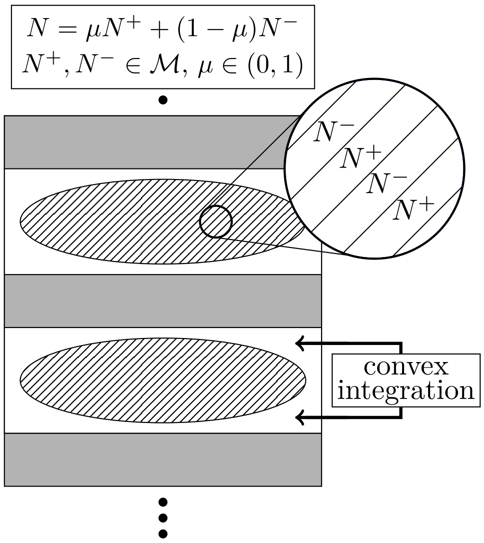

Figure 5.5: Compared to the case $s = e_1$, the construction of a recovery sequence in the case $s \neq e_1$ is more involved, in particular to ensure compatibility of the different parts of the material. Firstly, we construct a laminate which takes on the softer layers values $N \in \mathcal{N}_s$. Afterwards, we replace N by a finer laminate with gradients $N^+, N^- \in \mathcal{M}_s$ approximating N on most of the softer layer and apply convex integration on the rest to obtain compatible boundary values.

In the following we proof the individual lemmata needed in the proof of the upper bound for $s \neq e_1$.

Lemma 5.2.18 (Laminate for constant γ with gradients in $\mathcal{N}_s$)**.** *For each affine function $u \in W^{1,2}(\Omega;\mathbb{R}^2)\cap L^2_0(\Omega;\mathbb{R}^2)$ with $\nabla u = R(\mathbb{I}+\gamma e_1\otimes e_2)$ and $\gamma \in K_{s,\lambda}$, there is an $N \in \mathcal{M}_{e_1}\cap\mathcal{N}_s$, namely $N = R(\mathbb{I}+\frac{\gamma}{\lambda}e_1 \otimes e_2)$, such that for the simple laminate $v_1 \in W^{1,\infty}(\mathbb{R}^2,\mathbb{R}^2)$ given by*

$$\nabla v_1 = R\,\mathbb{1}_{P_{\text{stiff}}} + N\,\mathbb{1}_{P_{\text{soft}}},$$

the sequence $(v_\epsilon)_\epsilon \subset W^{1,\infty}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ defined by $\nabla v_\epsilon = \nabla v_1(\epsilon^{-1}\,\cdot\,)$ satisfies $v_\epsilon \rightharpoonup u$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$ as $\epsilon \to 0$ and

$$\int_\Omega |\nabla v_\epsilon m|^2 - 1\,\mathrm{d}x \to \lambda \int_\Omega |Nm|^2 - 1\,\mathrm{d}x = E(u) \quad \text{as } \epsilon \to 0.$$

Proof. Lemma 5.2.8 yields the existence of an $N \in \mathcal{N}_s$ such that

$$\lambda N + (1-\lambda)R = R(\mathbb{I} + \gamma e_1 \otimes e_2) \quad \text{and} \quad Ne_1 = Re_1.$$

Hence, R and N are rank one connected and with $N - R = \frac{\gamma}{\lambda} Re_1 \otimes e_2$ and therefore v_ϵ is well-defined. Furthermore, we obtain by the weak convergence of highly oscillating functions

$$\nabla v_\epsilon \rightharpoonup \lambda N + (1-\lambda)R = \nabla u \quad \text{in } L^2(\Omega;\mathbb{R}^{2\times 2}) \quad \text{as } \epsilon \to 0,$$

while for the corresponding energies we have

$$\begin{aligned}\int_\Omega |\nabla v_\epsilon m|^2 - 1\,\mathrm{d}x &= \int_\Omega (|Nm|^2 - 1)\,\mathbb{1}_{\epsilon P_{\text{soft}}\cap\Omega}\,\mathrm{d}x \\ &\to \lambda \int_\Omega |Nm|^2 - 1\,\mathrm{d}x = E(u) \quad \text{in } L^2(\Omega;\mathbb{R}^2) \quad \text{as } \epsilon \to 0.\end{aligned}$$

□

Lemma 5.2.19 (Admissible laminate for constant γ)**.** *For $R \in SO(2)$ and $N \in \mathcal{N}_s$ let $v_1 \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ satisfy*

$$\nabla v_1 = R\,\mathbb{1}_{P_{\text{stiff}}} + N\,\mathbb{1}_{P_{\text{soft}}},$$

and let $(v_\epsilon)_\epsilon \subset W^{1,\infty}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ given by $\nabla v_\epsilon = \nabla v_1(\epsilon^{-1}\,\cdot\,)$ converge weakly to a function $v \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$, i.e. $v_\epsilon \rightharpoonup v$ in $W^{1,2}(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$.

Then, there is a sequence $(w_\epsilon)_\epsilon \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ such that

$$w_\epsilon \rightharpoonup v \text{ in } W^{1,2}(\Omega;\mathbb{R}^2) \text{ as } \epsilon \to 0 \quad \text{and} \quad \limsup_{\epsilon\to 0} E^\beta_\epsilon(w_\epsilon) \leq E(v).$$

Proof. Let $C = a + (0,\ell)^2$, $a \in \mathbb{R}^2$, $\ell > 0$ be a cube containing Ω with $\operatorname{dist}(x, \partial C) > 1$ for all $x \in \Omega$. Choosing C large enough we may assume $|P^0 \cap C| = 1 \cdot \ell$, i.e. P^0 is not intersected by the upper or lower edge of C. By Corollary 5.2.9 there is for each $\delta = \epsilon \in (0,1)$ a function $\psi_\epsilon \in W^{1,\infty}(P^0_{\text{soft}} \cap C;\mathbb{R}^2)$ such that (5.5) is satisfied for $\delta = \epsilon$ and $\nabla\psi_\epsilon \overset{*}{\rightharpoonup} N$ in $L^\infty(P^0_{\text{soft}} \cap C;\mathbb{R}^{2\times 2})$ as $\epsilon \to 0$. For $x \in P^0 \cap C$ we set

$$\varphi_\epsilon(x) = \Big(\psi_\epsilon(x) - Nx\Big)\,\mathbb{1}_{P^0 \cap C},$$

and extend this function periodically with respect to P^0 in e_2-direction to $\varphi_\epsilon \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$. Furthermore, we set $z_\epsilon = v_1 + \varphi_\epsilon$ and obtain by the fact that $\nabla\varphi_\epsilon \overset{*}{\rightharpoonup} 0$ in $L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})$ as $\epsilon \to 0$, that

$$\nabla z_\epsilon \overset{*}{\rightharpoonup} \nabla v_1 \quad \text{in } L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2}) \quad \text{as } \epsilon \to 0. \tag{5.7}$$

Lastly, we introduce $w_\epsilon \in W^{1,2}(\mathbb{R}^2;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ given by

$$\nabla w_\epsilon = \nabla z_\epsilon(\epsilon^{-1}\,\cdot\,) = \nabla v_\epsilon + \nabla\varphi_\epsilon(\epsilon^{-1}\,\cdot\,).$$

Notice that by construction, $\nabla w_\epsilon \in \mathcal{M}_s$ a.e. in Ω and in particular $\nabla w_\epsilon = R \in SO(2)$ on $\epsilon P_{\text{stiff}} \cap \Omega$. Furthermore, due to (5.7) Lemma 5.3.1 yields

$$\nabla w_\epsilon \rightharpoonup \int_0^1 \nabla v_1 \,\mathrm{d}x_2 = \lambda N + (1-\lambda)R = \nabla u \quad \text{in } L^2(\Omega;\mathbb{R}^{2\times 2}) \quad \text{as } \epsilon \to 0.$$

Finally, we consider the energies. Since $|\nabla w_\epsilon m| = |\psi_\epsilon| < |Nm| + \epsilon$ a.e. in $\epsilon P_{\text{soft}} \cap C$ and $|Rm| = 1$ and $1 \leq |Nm|$ it follows that

$$\begin{aligned}\limsup_{\epsilon\to 0} E^\beta_\epsilon(w_\epsilon) &= \limsup_{\epsilon\to 0} \int_\Omega (|\nabla w_\epsilon m|^2 - 1)\,\mathbb{1}_{\epsilon P_{\text{soft}} \cap \Omega}\,\mathrm{d}x \\ &\leq \lim_{\epsilon\to 0} \int_\Omega (|Nm|^2 - 1)\,\mathbb{1}_{\epsilon P_{\text{soft}} \cap \Omega}\,\mathrm{d}x = E(u).\end{aligned}$$

□

Lemma 5.2.20 (Recovery sequence for piecewise affine limits)**.** *For each (finitely) piecewise affine function $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ for $R \in SO(2)$ and $\gamma \in L^2(\Omega)$ piecewise constant satisfying $\partial_1\gamma = 0$ and $\gamma \in K_{s,\lambda}$, there is a sequence $(w_\epsilon)_\epsilon \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ such that*

$$w_\epsilon \rightharpoonup u \text{ in } W^{1,2}(\Omega;\mathbb{R}^2) \text{ as } \epsilon \to 0 \quad \text{and} \quad \limsup_{\epsilon\to 0} E^\beta_\epsilon(w_\epsilon) \leq E(u).$$

Proof. We denote by Ω^i, indexed by a set I_γ the domains, where ∇u and thus γ takes the constant value $\gamma^i \in \mathbb{R}$. In accordance to with Lemma 5.2.18 we set $N^i = R(\mathbb{I} + \frac{\gamma^i}{\lambda} e_1 \otimes e_2)$. Since Ω is assumed to have Lipschitz boundary and $\partial_1 \gamma = 0$ in the sense of distributions it follows that the interfaces $\Gamma^{ij} = \overline{\Omega^i} \cap \overline{\Omega^j}$, $i, j \in I_\gamma$ are either empty or there are $t_k \in \mathbb{R}$, indexed by a set I_Γ such that $\Gamma^{ij} \subset \mathbb{R} \times \{t_k\}$. Furthermore, this implies that Ω^i is itself a bounded Lipschitz domain and thus Lemma 5.2.19 yields that there is a recovery sequence $(w^i_\epsilon)_\epsilon \subset W^{1,2}(\Omega^i; \mathbb{R}^2)$ for constant γ^i such that $\nabla w^i_\epsilon = R$ on $\epsilon P_{\text{stiff}} \cap \Omega$. Overall, we define $(w^i_\epsilon)_\epsilon \subset W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ by

$$\nabla w_\epsilon = R + \sum_{i \in I_\gamma} (\nabla w^i_\epsilon \, \mathbb{1}_{\Omega^i} - R) \, \mathbb{1}_{\Omega \setminus \bigcup_{k \in I_\Gamma} \mathbb{R} \times (\lfloor t_k \rfloor_\epsilon + \epsilon, \lfloor t_{k+1} \rfloor_\epsilon)}.$$

Since ∇w^i_ϵ and R are compatible along the layer interfaces $(\mathbb{R} \times \epsilon\mathbb{Z}) \cap \Omega$, it follows that u_ϵ has gradient structure and by choice of w^i_ϵ and the fact that

$$\Big|\Omega \setminus \bigcup_{k \in I_\Gamma} \mathbb{R} \times (\lfloor t_k \rfloor_\epsilon + \epsilon, \lfloor t_{k+1} \rfloor_\epsilon)\Big| \leq 2 \operatorname{diam}(\Omega) |I_\Gamma| \quad \text{as } \epsilon \to 0$$

we have

$$\nabla w_\epsilon \rightharpoonup \sum_{i \in I_\gamma} (\lambda N^i + (1 - \lambda) R) \, \mathbb{1}_{\Omega^i} = \nabla u \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}) \quad \text{as } \epsilon \to 0,$$

while we obtain for the energies

$$\begin{aligned}\int_\Omega |\nabla w_\epsilon m|^2 - 1 \, \mathrm{d}x &= \int_{\epsilon P_{\text{soft}} \cap \Omega} \sum_{i \in I_\gamma} (|N^i m|^2 - 1) \, \mathbb{1}_{|\Omega \setminus \bigcup_{k \in I_\Gamma} \mathbb{R} \times (\lfloor t_k \rfloor_\epsilon + \epsilon, \lfloor t_{k+1} \rfloor_\epsilon)} \, \mathrm{d}x \\ &\to \lambda \int_\Omega \sum_{i \in I_\gamma} (|N^i m|^2 - 1) \, \mathbb{1}_{\Omega^i} \, \mathrm{d}x = E(u) \quad \text{as } \epsilon \to 0,\end{aligned}$$

as desired. □

Proposition 5.2.21 (Recovery sequence for general limits)**.** *Let $u \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ for $R \in SO(2)$ and $\gamma \in L^2(\Omega)$ with $\partial_1 \gamma = 0$ and $\gamma \in K_{s,\lambda}$. Then, there exists a sequence $(u_\epsilon)_\epsilon \subset W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ such that*

$$u_\epsilon \rightharpoonup u \text{ in } W^{1,2}(\Omega; \mathbb{R}^2) \text{ as } \epsilon \to 0 \quad \text{and} \quad \limsup_{\epsilon \to 0} E^\beta_\epsilon(u_\epsilon) \leq E(u).$$

Proof. Based on Lemma 5.2.20 we argue by approximation and diagonalization. Accordingly, let $(\zeta^k)_{k \in \mathbb{N}} \subset L^2(\Omega, K_{s,\lambda})$ be a sequence of locally one-dimensional simple functions approximating the locally one-dimensonal function γ in the sense that $\zeta^k \to \gamma$ in $L^2(\Omega)$, see Lemma 4.5.12. Correspondingly, for $k \in \mathbb{N}$ we introduce $(w^k)_{k \in \mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ given by $\nabla w^k = R(\mathbb{I} + \zeta^k e_1 \otimes e_2)$. Note that by the Poincaré inequality there is a constant $C > 0$ such that for all $k \in \mathbb{N}$

$$\begin{aligned}\|u - w^k\|_{W^{1,2}(\Omega; \mathbb{R}^2)} &\leq C \|\nabla u - \nabla w^k\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \\ &= C \big\| R(\mathbb{I} + \gamma e_1 \otimes e_2) - R(\mathbb{I} + \zeta^k e_1 \otimes e_2) \big\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} = C \|\gamma - \zeta\|_{L^2(\Omega)}\end{aligned}$$

and, since γ and ζ_k are bounded in $L^\infty(\Omega)$, it holds that

$$|E(u) - E(w^k)| \leq \int_\Omega |\gamma^2 - (\zeta^k)^2| \, \mathrm{d}x \leq C \|\gamma - \zeta^k\|_{L^2(\Omega)}.$$

Now, let $(w_\epsilon^k)_\epsilon \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ be a recovery sequence for w_k given by Lemma 5.2.20, so that for all $k \in \mathbb{N}$

$$\|w_\epsilon^k - w^k\|_{L^2(\Omega;\mathbb{R}^2)} \to 0 \quad \text{and} \quad E_\epsilon^\beta(w_\epsilon^k) \to E(w^k) \quad \text{both as } \epsilon \to 0.$$

Thus,

$$\begin{aligned}\lim_{k\to\infty}\lim_{\epsilon\to 0}\left(\|w_\epsilon^k - u\|_{L^2(\Omega;\mathbb{R}^2)} + |E_\epsilon^\beta(w_\epsilon^k) - E(u)|\right) &\leq \lim_{k\to\infty}\left(\|w^k - u\|_{L^2(\Omega;\mathbb{R}^2)} + |E(w^k) - E(u)|\right)\\ &= 0.\end{aligned}$$

Hence, the diagonalization Lemma 4.5.14 by Attouch yields the existence of a sequence $u_\epsilon = w_\epsilon^{k(\epsilon)}$ satisfying

$$u_\epsilon \to u \quad \text{in } L^2(\Omega;\mathbb{R}^2) \quad \text{and} \quad E_\epsilon^\beta(u_\epsilon) \to E(u) \quad \text{both as } \epsilon \to 0.$$

Furthermore, since u_ϵ is uniformly bounded in $W^{1,2}(\Omega;\mathbb{R}^2)$ we obtain by the Urysohn argument, cf. Lemma 4.5.15, that $u_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$. □

Remark 5.2.22. Observe that the $e_2^\perp$-connectedness of Ω is only needed in Proposition 5.2.21 for the approximation argument by simple functions. Hence by Remark 4.5.13, Proposition 5.2.21 generalizes directly to bounded Lipschitz domains Ω for which the decomposition given by Proposition 4.2.8 is a partition of Ω in finitely many sets.

5.2.5 The lower bound estimate

For the lower bound, we follow the general strategy applied for the elastic energy in Chapter 4, in particular of Lemma 4.3.4 and Proposition 4.3.13. Yet note, that these results cannot be directly applied. In particular Lemma 4.3.4 only holds for sequences in $W^{1,p}(\Omega;\mathbb{R}^n)$ with $p > n \geq 2$ to guarantee the weak continuity of the minors. For the model of crystal plasticity discussed here, the deformations are merely functions in $W^{1,1}(\Omega;\mathbb{R}^n)$ so we have to adapt the arguments using the techniques developed by Conti, Dolzmann and Kreisbeck in the context of compactness, see Subsection 5.2.3.

Though our application is solely restricted to $n = p = 2$, the next lemma holds in all dimensions for a convex energy density W, in which case in contrast to Lemma 4.3.4 the requirement of a gradient structure and the restriction to $p > n$ can be dropped.

Lemma 5.2.23 (Energy estimate for affine limits and convex energies)**.** *For $n \in \mathbb{N}$ and $1 \leq p < \infty$ let $W : \mathbb{R}^{n\times n} \to [0,\infty)$ be a convex energy density. Let $U_\epsilon \rightharpoonup F$ in $L^p(\Omega;\mathbb{R}^{n\times n})$ as $\epsilon \to 0$ with $F = R + d \otimes e_n$ and let there be a sequence $(\Xi_\epsilon)_\epsilon \subset L^\infty(\Omega;\mathbb{R}^{n\times n})$ with $\Xi_\epsilon \to R$ in $L^p(\Omega;\mathbb{R}^{n\times n})$ and $\|U_\epsilon - \Xi_\epsilon\|_{L^p(\Omega\cap\epsilon P_{\text{stiff}};\mathbb{R}^{n\times n})} \to 0$ both as $\epsilon \to 0$.*

Then,

$$\liminf_{\epsilon\to 0} \int_\Omega W_\epsilon(x, U_\epsilon)\,\mathrm{d}x \geq \lambda \int_\Omega W(R + \lambda^{-1} d \otimes e_n)\,\mathrm{d}x.$$

Proof. Firstly, we have to determine the asymptotic behavior of the sequence $U_\epsilon \mathbb{1}_{\epsilon P_{\text{stiff}}}$. Since $\|U_\epsilon - \Xi_\epsilon\|_{L^p(\Omega\cap\epsilon P_{\text{stiff}};\mathbb{R}^{n\times n})} \to 0$, $\Xi_\epsilon \to R$ in $L^p(\Omega;\mathbb{R}^{n\times n})$ and by the classic Lemma 2.3.1 on weak convergence of highly oscillating functions $\mathbb{1}_{\epsilon P_{\text{stiff}}} \overset{*}{\rightharpoonup} (1-\lambda)$ as $\epsilon \to 0$ it holds that

$$U_\epsilon \mathbb{1}_{\epsilon P_{\text{stiff}}} = (U_\epsilon - \Xi_\epsilon)\mathbb{1}_{\epsilon P_{\text{stiff}}} + \Xi_\epsilon \mathbb{1}_{\epsilon P_{\text{stiff}}} \rightharpoonup (1-\lambda)R \quad \text{in } L^p(\Omega;\mathbb{R}^{n\times n}) \quad \text{as } \epsilon \to 0.$$

Together with the fact that $U_\epsilon \rightharpoonup F$ in $L^p(\Omega;\mathbb{R}^{n\times n})$ this implies

$$U_\epsilon\, \mathbb{1}_{\epsilon P_{\text{soft}}} = U_\epsilon - U_\epsilon\, \mathbb{1}_{\epsilon P_{\text{stiff}}} \rightharpoonup F - (1-\lambda)R \quad \text{in } L^p(\Omega;\mathbb{R}^{n\times n}) \quad \text{as } \epsilon \to 0.$$

Hence, by Jensens inequality and the continuity of convex functions yields

$$\begin{aligned}
\liminf_{\epsilon\to 0} \int_\Omega W_\epsilon(x, U_\epsilon)\,\mathrm{d}x &\geq \liminf_{\epsilon\to 0} \int_{\epsilon P_{\text{soft}}\cap\Omega} W(U_\epsilon)\,\mathrm{d}x \\
&\geq \lambda|\Omega| \liminf_{\epsilon\to 0} W\Big(\frac{1}{|\epsilon P_{\text{soft}}\cap\Omega|}\int_{\Omega\cap\epsilon P_{\text{soft}}} U_\epsilon\,\mathrm{d}x\Big) \\
&= \lambda|\Omega| \liminf_{\epsilon\to 0} W\Big(\frac{1}{|\epsilon P_{\text{soft}}\cap\Omega|}\int_{\Omega} U_\epsilon\, \mathbb{1}_{\epsilon P_{\text{soft}}}\,\mathrm{d}x\Big) \\
&= \lambda|\Omega| W(\lambda^{-1}(F - R(1-\lambda))) \\
&= \lambda \int_\Omega W(R + \lambda^{-1} d\otimes e_n)\,\mathrm{d}x, \qquad (5.8)
\end{aligned}$$

which is the estimate desired. □

As for the results concerning elastic models in Chapter 4, we can use this lemma to establish a lower bound in the case of affine limits, while the general case builds directly on the lemma, rather than the result for the affine case. We therefore proceed directly to the general lower bound, see also [42, Section 5].

Proposition 5.2.24 (Lower bound for general limits)**.** *For $u \in L^2_0(\Omega;\mathbb{R}^2)$ suppose that $(u_\epsilon)_\epsilon \subset L^2_0(\Omega;\mathbb{R}^2)$ is a sequence with $u_\epsilon \to u$ in $L^2(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$. Then,*

$$\liminf_{\epsilon\to 0} E^\beta_\epsilon(u_\epsilon) \geq E(u).$$

Proof. If $\liminf_{\epsilon\to 0} E^\beta_\epsilon(u_\epsilon) = \infty$ there is nothing to show. We therefore may assume that $E^\beta_\epsilon(u_\epsilon) < C$ for a constant $C > 0$, arguing for a subsequence of $(u_\epsilon)_\epsilon$ if necessary. By Lemma 5.2.13 and a Urysohn argument, cf. Lemma 4.5.15, we see that there is a rotation $R \in SO(2)$ and a function $\gamma \in L^2(\Omega)$ with $\partial_1\gamma = 0$ and $\gamma \in K_{s,\lambda}$ a.e. in Ω such that $\nabla u = R(\mathbb{I} + \gamma e_1\otimes e_2)$.

Now, we approximate the γ by essentially one-dimensional simple functions, using Lemma 4.5.12 and the same notation as in Lemma 5.2.20. Let $(\zeta_k)_{k\in\mathbb{N}} \subset L^2(\Omega)$ be a sequence of simple functions with $\zeta_k \to \gamma$ in $L^2(\Omega)$ as $k \to \infty$ of the form $\zeta_k = \sum_{i=1}^{n_k} \zeta^i_k\, \mathbb{1}_{\Omega^i_k}$ such that the partitions $\Omega_k = (\Omega^i_k)_{i\in\{1,\dots,n_k\}}$ are nested in the sense that each Ω^i_k can be written as a finite union of sets Ω^j_{k+1}. For $s \neq e_1$ we may furthermore assume that the sequence is monotone , i.e. $\zeta_k \leq \zeta_{k+1}$. Since Ω is assumed to be $e_n^\perp$-connected and $\partial_1\gamma = 0$, the interfaces $\Gamma^{ij}_k = \overline{\Omega^i_k} \cap \overline{\Omega^j_k}$, $i,j \in \{1,\dots,n_k\}$ are either empty or there are $t_\ell \in \mathbb{R}$ indexed by a set I^Γ_k such that $\Gamma^{ij}_k \subset \mathbb{R}\times\{t_\ell\}$. Furthermore, we introduce $w_k \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ determined by $\nabla w_k = R(\mathbb{I} + \zeta_k e_1\otimes e_2)$ for $k \in \mathbb{N}$, which will serve as an approximation of u.

Regarding $(u_\epsilon)_\epsilon$ the discussion of compactness in Proposition 5.2.13 suggests to reuse the decomposition of $\nabla u_\epsilon = A_\epsilon + B_\epsilon$ from the proof of this proposition. In particular, the fact that

$$\liminf_{\epsilon\to 0} \int_\Omega |A_\epsilon|^2 - 1\,\mathrm{d}x \leq \liminf_{\epsilon\to 0} \int_\Omega W(\nabla u_\epsilon)\,\mathrm{d}x \qquad (5.9)$$

allows us to argue for the convex energy density $A \mapsto |A|^2 - 1$ rather than the more involved condensed energy [45, Section 3]. Naturally, this applies only to the soft layers, which suggests to consider instead of ∇u_ϵ the functions $(U_\epsilon)_\epsilon \subset L^2(\Omega;\mathbb{R}^{2\times 2})$ given by

$$U_\epsilon = (\nabla u_\epsilon)\, \mathbb{1}_{\epsilon P_{\text{stiff}}} + A_\epsilon\, \mathbb{1}_{\epsilon P_{\text{soft}}}.$$

Next, our goal is to obtain an approximation $(Z_\epsilon^k)_\epsilon \subset L^2(\Omega;\mathbb{R}^2)$ of ∇w_k on the level of finite $\epsilon > 0$ that is close to U_ϵ. More precisely, we define $Z_\epsilon^k := U_\epsilon - \nabla v_\epsilon + \nabla v_\epsilon^k$ for sequences $(v_\epsilon)_\epsilon$ and $(v_\epsilon^k)_\epsilon \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ that satisfy for all $k \in \mathbb{N}$

$$v_\epsilon \rightharpoonup u \quad \text{and} \quad v_\epsilon^k \rightharpoonup w_k \quad \text{both in } W^{1,2}(\Omega;\mathbb{R}^2) \text{ as } \epsilon \to 0. \tag{5.10}$$

Furthermore, to avoid interference with the differential inclusion constraints on the stiff layers the constructed sequences should satisfy for all $k \in \mathbb{N}$ and $\epsilon > 0$

$$\nabla v_\epsilon^k = \nabla v_\epsilon \quad \text{a.e. in } \epsilon P_{\text{stiff}} \cap \Omega, \tag{5.11}$$

and, to control the distance to U_ϵ in terms of k, we seek an estimate of the form

$$\|\nabla v_\epsilon^k - \nabla v_\epsilon\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq C\|\zeta_k - \gamma\|_{L^2(\Omega)}. \tag{5.12}$$

Choosing for $(v_\epsilon)_\epsilon$ and $(v_\epsilon^k)_\epsilon$ recovery sequences for u and w_k, respectively, would provide sequences that satisfy (5.10) and (5.11), yet (5.12) is obscured by the here unnecessary microstructure attached to recovery sequences that ensure the deformations are admissible. Therefore, we choose sequences constructed along the same lines as the recovery sequences in Lemma 5.2.18 , Lemma 5.2.20 and Proposition 5.2.21, but skipping the contributions of Lemma 5.2.19.

In accordance to Lemma 5.2.18 and Lemma 5.2.20 we define for $N_k^i = R(\mathbb{I} + \frac{\gamma_k^i}{\lambda} e_1 \otimes e_2)$ the functions $v_\epsilon^k \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ by

$$\nabla v_\epsilon^k = R + \sum_{i=1}^{n_k} (N_k^i - R)\, \mathbb{1}_{\Omega_k^i}\, \mathbb{1}_{\epsilon P_{\text{soft}}}\, \mathbb{1}_{\Omega \setminus \bigcup_{\ell \in I_k^\Gamma} \mathbb{R} \times (\lfloor t_\ell \rfloor_\epsilon + \epsilon, \lfloor t_{\ell+1} \rfloor_\epsilon)}.$$

For each $k \in \mathbb{N}$, the weak convergence of highly oscillating functions applied to each Ω_k^i together with the fact that

$$\Big|\Omega \setminus \bigcup_{k \in I_k^\Gamma} \mathbb{R} \times \left(\lfloor t_k \rfloor_\epsilon + \epsilon, \lfloor t_{k+1} \rfloor_\epsilon\right)\Big| \leq 2\,\text{diam}(\Omega)\epsilon \cdot \# I_k^\Gamma \to 0 \quad \text{as } \epsilon \to 0$$

yields $v_\epsilon^k \rightharpoonup w_k$ in $W^{1,2}(\Omega;\mathbb{R}^2)$.

In the case $s \neq e_1$, we obtain from $(v_\epsilon^k)_\epsilon$ the sequence $(v_\epsilon)_\epsilon$ by applying the Attouch diagonalization Lemma 4.5.14 as in Proposition 5.2.21 a subsequence $v_\epsilon = v_\epsilon^{k(\epsilon)} \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$. Hence, we see that (5.10) and (5.11) hold. To establish (5.12) observe that for $k, K \in \mathbb{N}$ and $\epsilon > 0$ the definition of both N_k^i and Ω_k^i yields

$$\begin{aligned}
\|\nabla v_\epsilon^k - \nabla v_\epsilon^K\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} &\leq \Big\| \sum_{i=1}^{n_k} \sum_{j=1}^{n_K} (N_i^k - N_j^K)\, \mathbb{1}_{\Omega_k^i}\, \mathbb{1}_{\Omega_K^j} \Big\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \\
&= \frac{1}{\lambda} \Big\| \sum_{i=1}^{n_k} \sum_{j=1}^{n_K} (\zeta_i^k - \zeta_j^K)\, \mathbb{1}_{\Omega_k^i}\, \mathbb{1}_{\Omega_K^j} \Big\|_{L^2(\Omega)} \\
&= \frac{1}{\lambda} \|\zeta_k - \zeta_K\|_{L^2(\Omega)}.
\end{aligned}$$

Thus, since the sequence $(\zeta_k)_{k\in\mathbb{N}}$ is for $s \neq e_1$ assumed to be monotone, we have

$$\|\nabla v_\epsilon^k - \nabla v_\epsilon\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq \sup_{K \geq k} \|\nabla v_\epsilon^k - \nabla v_\epsilon^K\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq \frac{1}{\lambda} \|\zeta_k - \gamma\|_{L^2(\Omega)}.$$

For $s = e_1$ we may choose for v_ϵ the much simpler recovery sequence constructed in Proposition 5.2.15, in which case (5.12) follows immediately.

Now, let us return to $(Z_\epsilon^k)_\epsilon \subset L^2(\Omega;\mathbb{R}^{2\times 2})$ given by

$$Z_\epsilon^k = U_\epsilon - \nabla v_\epsilon + \nabla v_\epsilon^k$$

and consider its properties. By (5.11) we see that $\nabla v_\epsilon^k \,\mathbb{1}_{\epsilon P_{\text{stiff}}} = \nabla v_\epsilon \,\mathbb{1}_{\epsilon P_{\text{stiff}}} = R$ and thus we have for each $x \in \epsilon P_{\text{stiff}} \cap \Omega$

$$\operatorname{dist}(Z_\epsilon^k, SO(2)) = \operatorname{dist}(\nabla u_\epsilon, SO(2)).$$

Thus, arguing by exhaustion as in Corollary 4.3.7 we may assume by Proposition 3.3.10, that there is a sequence $(\Sigma_\epsilon^i)_\epsilon \subset L^\infty(\Omega; SO(n))$ such that $\|Z_\epsilon^k - \Sigma_\epsilon\|_{L^2(\Omega;\mathbb{R}^2)} \to 0$ as $\epsilon \to 0$ and by Lemma 3.3.12 and Proposition 3.3.15 it also satisfies $\Sigma_\epsilon \to R$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$ as $\epsilon \to 0$. Furthermore, on each Ω_k^i we have that $\nabla w_\epsilon = R(\mathbb{I} + \frac{1}{\lambda}\zeta_i^k e_1 \otimes e_2)$ and $\nabla z_\epsilon^k \rightharpoonup \nabla w_k$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$ as $\epsilon \to 0$. Hence, Lemma 5.2.23 applied to each Ω_k^i and summing over $i \in \{1,\dots,n_k\}$ yields

$$\liminf_{\epsilon\to 0} \int_\Omega |Z_\epsilon^k|^2 - 1 \,\mathrm{d}x \geq \lambda \int_\Omega W(R + \lambda^{-1}\zeta_k e_1 \otimes e_2)\,\mathrm{d}x = E(w_k). \tag{5.13}$$

Finally, it remains to show that taking the limit $k \to \infty$ in (5.13) establishes the claim. We observe for the right hand side that by Hölders inequality the fact that $(\nabla w_k)_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega;\mathbb{R}^{2\times 2})$

$$\begin{aligned} |E(w_k) - E(u)| &\leq \int_\Omega \big||\nabla w_k|^2 - |\nabla u|^2\big|\,\mathrm{d}x \\ &\leq C\Big(\int_\Omega |\nabla w_k - \nabla u|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \leq C\|\zeta_k - \gamma\|_{L^2(\Omega)} \to 0 \quad \text{as } k \to 0. \end{aligned}$$

For the left hand side of (5.13) we argue analogously using (5.11) and (5.12)

$$\begin{aligned} \Big|\int_\Omega |Z_\epsilon^k|^2 - 1\,\mathrm{d}x - \int_\Omega |A_\epsilon|^2 - 1\,\mathrm{d}x\Big| &\leq C\int_{\epsilon P_{\text{soft}}\cap\Omega} |\nabla v_\epsilon^k - \nabla v_\epsilon|^2\,\mathrm{d}x \\ &\leq \frac{C}{\lambda}\|\zeta_k - \gamma\|_{L^2(\Omega)} \to 0 \quad \text{as } \epsilon \to 0. \end{aligned}$$

This, together with (5.9) yields the claim. □

Remark 5.2.25. Notice that the $e_2^\perp$-connectedness of Ω is only needed for the approximation argument by simple functions. By arguing locally on cubes $C \subset\subset \Omega$ and exhausting Ω with such cubes, the result can be extended to general bounded Lipschitz domains. However, with Proposition 5.2.21 requiring this restriction on the geometry of Ω, we may also assume it here.

Remark 5.2.26 (Addition of a term linear γ)**.** For $E_\epsilon^{\tau,\beta}$ note that by the lower semicontinuity of the L^1-norm, a lower bound estimate can be easily derived from the lower bound for E_ϵ^β. Indeed, the uniform bound on $(E_\epsilon^\beta(u_\epsilon))_\epsilon$ yields weak convergence of $\gamma_\epsilon \to \gamma$ in $L^2(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$. Hence, by lower semicontinuity of the L^1-norm we have the estimate

$$\liminf_{\epsilon\to 0}\int_\Omega |\gamma_\epsilon|\,\mathrm{d}x \geq \int_\Omega |\gamma|\,\mathrm{d}x.$$

Overall, we obtain

$$\liminf_{\epsilon\to 0} E_\epsilon^{\tau,\beta}(u_\epsilon) = \liminf_{\epsilon\to 0} E_\epsilon^{\tau,\beta}(u_\epsilon) + \liminf_{\epsilon\to 0} \tau\int_\Omega |\gamma_\epsilon|\,\mathrm{d}x \geq E(u) + \tau\int_\Omega |\gamma|\,\mathrm{d}x = E^\tau(u).$$

Notice that this estimate holds independent of the slip direction, but we have shown a matching upper bound only in the case $s = e_1$.

5.3 Appendix

The following result is a generalized version of Lemma 3.4.1 on the weak convergence of periodic functions in one component. Here, we restrict ourselves to the case of $p = \infty$ as this requires only a slight adaption of the original proof.

For weakly convergent functions periodic to the unit cell $(0,1)^n \subset \mathbb{R}^n$ a self-contained proof was given by Lukkassen and Wall [103]. As mentioned before, a more comprehensive study of this type of results motivates the introduction of the notion of two-scale convergence and techniques such as the unfolding operator.

Lemma 5.3.1 (Weak convergence of highly oscillating weakly convergent functions)**.** *For $n \in \mathbb{N}$ and $\ell \in (0,\infty)$ let $(u_\epsilon)_\epsilon \subset L^\infty(\mathbb{R}^n)$ be uniformly bounded with $u_\epsilon(x + \ell e_n) = u_\epsilon(x)$ for all $x \in \mathbb{R}^n$ such that for some $u \in L^\infty(\mathbb{R}^n)$ we have $u_\epsilon \overset{*}{\rightharpoonup} u$ in $L^\infty(\mathbb{R}^n)$ as $\epsilon \to 0$. Define $(v_\epsilon)_\epsilon \subset L^\infty(\mathbb{R}^n)$ by $v_\epsilon(x', x_n) = u_\epsilon(x', {x_n}/{\epsilon})$ for all $x = (x', x_n) \in \mathbb{R}^n$. Then, for all domains $\Omega \subset \mathbb{R}^n$ we have*

$$v_\epsilon \overset{*}{\rightharpoonup} \bar{u} \text{ in } L^\infty(\Omega) \text{ as } \epsilon \to 0, \quad \text{where} \quad v(x) = \frac{1}{\ell}\int_0^\ell u \,\mathrm{d}x_n.$$

Proof. We merely adapt the proof of Lemma 3.4.1. Arguing as before, we may assume that $\ell = 1$ and obtain by the uniform bound on $(u_\epsilon)_\epsilon$ for each $A = (0,a)^n + b$, where $a \in \mathbb{R}$, and $b = (b', b_n) \in \mathbb{R}^n$ that

$$\lim_{\epsilon\to 0}\int_A v_\epsilon \,\mathrm{d}x = \lim_{\epsilon\to 0}\int_{\lfloor b_n\rfloor_\epsilon}^{\lfloor b_n + a\rfloor_\epsilon} v_\epsilon \,\mathrm{d}x_n \,\mathrm{d}x'.$$

Thus, the key is to substitute (3.30) by

$$\begin{aligned}\int_{(0,a)^{n-1}+b'}\int_{\lfloor b_n\rfloor_\epsilon+\epsilon}^{\lfloor b_n + a\rfloor_\epsilon} f_\epsilon \,\mathrm{d}x_n \,\mathrm{d}x' &= \int_{(0,a)^{n-1}+b'} \frac{\lfloor b_n + a\rfloor_\epsilon - \lfloor b_n\rfloor_\epsilon - \epsilon}{\epsilon}\int_0^\epsilon u_\epsilon(x', \tfrac{x_n}{\epsilon}) \,\mathrm{d}x_n \,\mathrm{d}x' \\ &= \big(\lfloor b_n + a\rfloor_\epsilon - \lfloor b_n\rfloor_\epsilon - \epsilon\big)\int_{(0,a)^{n-1}+b'}\int_0^1 u_\epsilon(x', x_n) \,\mathrm{d}x_n \,\mathrm{d}x' \\ &\to \int_{(0,a)^n+b'} \bar{u} \,\mathrm{d}x \quad \text{as } \epsilon \to 0.\end{aligned}$$

Hence, Lemma 3.4.2 entails the claim.

□

6

Homogenization of Materials with Randomly Layered Structure

Up to this point, we have only considered strictly periodically layered materials. In this chapter we want to broaden our view towards layered materials of random layer thickness. To keep the focus on stochastics, we will restrict our considerations to the rigid-plastic idealization of the elastoplastic model considered in Chapter 5 with the slip system $s = e_1, m = e_2$ active in every other layer.

To illustrate the necessary changes, we consider in the first section a simple Bernoulli model. In the following sections, we then give a short overview of ergodic theory to extend the result to more general stochastic processes.

6.1 Introduction to stochastic homogenization

6.1.1 Transition from periodic to random layer thickness

Before considering explicit material models, we want to address the key question arising in the transition from periodic to stochastic homogenization: What suitable property of a stochastic process can substitute periodicity and what technical arguments take the place of convergence results such as the classic Lemma 2.3.1 on the weak convergence of periodically oscillating functions?

Let us first review the periodic structures. For a constant ratio $\lambda \in (0,1)$ between stiff and soft layers we previously defined

$$P_{\text{soft}} = \mathbb{R} \times (0, \lambda] \quad \text{and} \quad P_{\text{stiff}} = \mathbb{R} \times (\lambda, 1],$$

and extended this structure periodically to $\mathbb{R}^2$ with respect to the periodic cell $\mathbb{R} \times (0, 1]$. We may also define periodic functions tailored to this layered structure, for example, the function $f \in L^\infty(\mathbb{R}^2, \mathbb{R})$ given by

$$f(x) = \mathbb{1}_{P_{\text{soft}}}(x) = \mathbb{1}_{(0,\lambda]}(x_2) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

Notice that f is periodic with respect to $\mathbb{R} \times (0, 1]$ in the sense that $f(x + e_2) = f(x)$ for all $x \in \mathbb{R}^2$.

Now, assume that the ratio between a pair of stiff and soft layers can take two values $\lambda^1, \lambda^2 \in (0,1)$ at random. More precisely, we introduce for $\rho \in (0,1)$ a family of independent random variables $X = (X_i)_{i\in\mathbb{Z}}$ that are Bernoulli distributed with parameter ρ and are each defined on the probability space $(\Xi_i, \mathcal{A}_i, \mathbb{P}_i)$ with their product space denoted by $(\Xi, \mathcal{A}, \mathbb{P})$. In the following we will identify X_i with the respective projections on the product space $(\Xi, \mathcal{A}, \mathbb{P})$. For the two possible component ratios $\lambda^1, \lambda^2 \in (0,1)$ we define for each $i \in \mathbb{Z}$ the random variable $\lambda_i = \lambda^1 + (\lambda^2 - \lambda^1)X_i$ and set correspondingly for $i \in \mathbb{Z}$

$$P^i_{\text{soft}} = \mathbb{R} \times (i, i+\lambda_i) \quad \text{and} \quad P^i_{\text{stiff}} = \mathbb{R} \times (i+\lambda_i, i+1), \tag{6.1}$$

see also Figure 6.1. Coming back to the key question asked in the beginning, consider for $P^0 = \mathbb{R} \times (0,1]$ the function $\mathbb{1}_{(0,\lambda_0]} : P^0 \times \Xi \to \mathbb{R}$, where we set $\mathbb{1}_{(0,\lambda_0]}(x,\omega) = \mathbb{1}_{(0,\lambda_0(\omega)]}(x)$ of $(x,\omega) \in P^0 \times \Xi$. In contrast to the periodicity condition, under which the function would coincide with its shift by ie_2, $i \in \mathbb{Z}$, the value of the function should be related to the layer thickness determined by the random variable X_i. Accordingly, we would expect the extension of $\mathbb{1}_{(0,\lambda_0]}$ to $\mathbb{R}^2$ to satisfy the relation

$$\mathbb{1}_{(0,\lambda_0]}(x+ie_2, \,\cdot\,) = \mathbb{1}_{(0,\lambda_i]}(x, \,\cdot\,), \quad (x,\omega) \in \mathbb{R}^2 \times \Xi.$$

Note that for given x this is a relation between two random variables, as $\mathbb{1}_{(0,\lambda_i]}$ is depending on a spatial as well as a stochastic variable. In general, a function f would for a given point x still depend on the stochastic variable, so a meaningful analogon to a periodic function f is given by a function $f \in L^\infty(\mathbb{R}^2; L^1(\Xi, \mathcal{A}, \mathbb{P}; \mathbb{R}))$, where $L^\infty(\mathbb{R}^2; L^1(\Xi, \mathcal{A}, \mathbb{P}; \mathbb{R}))$ denotes the usual Banach space valued L^p-space, such that there is a function $g \in L^\infty(P^0 \times \{0,1\}; \mathbb{R})$ with

$$f|_{P^0}(x, \,\cdot\,) = g(x, X_0) \quad \text{and} \quad f(x+ie_2, \cdot) = g(x, X_i) \quad i \in \mathbb{Z} \quad \text{for } x \in P^0.$$

Notice that $g(\,\cdot\,, X_i)$ is $\mathcal{B}^2 \otimes \mathcal{A}_i$-measurable, as concatenation of the $\mathcal{B}^2 \otimes P(\{0,1\})$-measurable function g with the $\mathcal{B}^2 \otimes \mathcal{A}_i$-$\mathcal{B}^2 \otimes \mathcal{B}^1$-measurable function $(\mathrm{id}_{\mathbb{R}^2}, X_i)$, where $\mathcal{B}^2$ denotes the two-dimensional Borel sets.

This definition of f implies for the integrals of f over P_i that for $i \in \mathbb{Z}$

$$\int_{P^i} f(x)\,\mathrm{d}x = \int_{P^0} g(x, X_i)\,\mathrm{d}x \in L^1(\Xi_i, \mathcal{A}_i, \mathbb{P}_i; \mathbb{R}).$$

In particular, for each $i \in Z$ we have that $\int_{P^i} f(x)\,\mathrm{d}x$ is $\mathcal{A}_i$-measurable by the theory usually developed in the context Fubini's theorem, see e.g. [67, V.2, Satz 2.1]. Furthermore, the family of random variables $(\int_{P^i} f(x)\,\mathrm{d}x)_{i\in\mathbb{Z}}$ is independent.

In the next sections, more precisely, in Subsection 6.3.1, we will see how to incorporate the features obtained via the stochastic process $(X_i)_{i\in\mathbb{Z}}$ directly into the probability space.

For our applications we need to substitute Lemma 2.3.1 on weak convergence of highly oscillating functions by a suitable result on the asymptotic behavior of means of random variables. The simplest theorem of this type is given by the law of large numbers, whose descendants branch into the fields of ergodic theory, renewal theory and others. Later on, we will utilize ergodic theory to derive a more general substitute for Lemma 2.3.1.

Theorem 6.1.1 (Kolmogorov's strong law of large numbers [18, Korollar 12.2])**.** *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of independent, identically distributed, integrable real-valued random variables. Then,*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} (X_i - \mathbb{E}(X_i)) = 0, \quad \textit{a.s.}$$

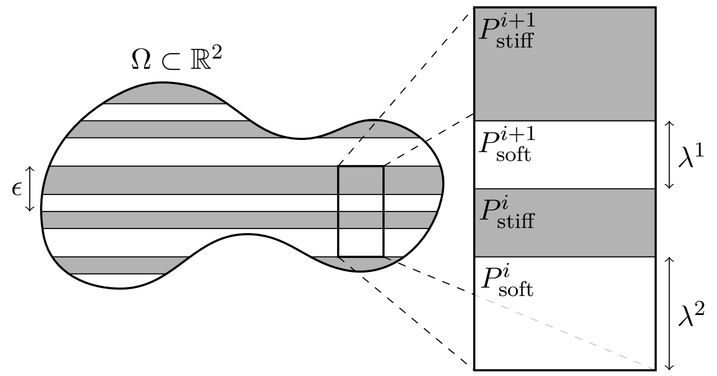

Figure 6.1: In contrast to the periodically layered materials considered previously, we assume that for each pair of a stiff and a soft layer, there are two possible layer ratios, $\lambda_1, \lambda_2 \in (0,1)$, both occuring with a certain probability. The overall configuration of the layered material is thus determined at random, with the randomness on each pair of a stiff and a soft layer modeled by independent Bernoulli distributed variables.

Remark 6.1.2. Notice that the result above directly generalizes to a family $(X_k)_{k\in\mathbb{Z}}$ with X_k as in the statement by splitting the sum over $\mathbb{Z}$ into one sum over $-\mathbb{N}$ and another sum over $\mathbb{N}$, and then applying the theorem twice.

The next proposition revisits the arguments made to prove Proposition 3.4.1 and is a simplified version of results usually obtained for more general stochastic processes such as Proposition 6.3.13. As it suffices for later applications, we restrict ourselves to the case of one space dimension.

Proposition 6.1.3 (Weak convergence via the law of large numbers in one variable)**.** *Let $f \in L^\infty(\mathbb{R}, L^1(\Xi, \mathbb{P}; \mathbb{R}))$ be such that $(\int_0^1 f(x+i)\,\mathrm{d}x)_{i\in\mathbb{Z}}$ is an independent and identically distributed family of random variables. Then, the sequence $(f_\epsilon)_\epsilon$ given by $f_\epsilon(x,\cdot\,) = f(\epsilon^{-1}x,\cdot\,)$ satisfies*

$$f_\epsilon \overset{*}{\rightharpoonup} \mathbb{E}\Big(\int_0^1 f(x,\cdot\,)\,\mathrm{d}x\Big) \quad \textit{in } L^\infty(\mathbb{R}) \textit{ as } \epsilon \to 0, \textit{ a.s.}$$

Proof. We again utilize Lemma 3.4.2. For $a \in \mathbb{R}$ and $h > 0$ let $[a, a+h] \subset \mathbb{R}$ be an interval and set $I_\epsilon = I_\epsilon^{[a,a+h]}$. Then, for small $\epsilon > 0$ we have

$$\int_a^{a+h} f_\epsilon(x,\cdot\,)\,\mathrm{d}x = \int_a^{\lfloor a\rfloor_\epsilon+\epsilon} f_\epsilon(x,\cdot\,)\,\mathrm{d}x + \int_{\lfloor a\rfloor_\epsilon+\epsilon}^{\lfloor a+h\rfloor_\epsilon} f_\epsilon(x,\cdot\,)\,\mathrm{d}x + \int_{\lfloor a+h\rfloor_\epsilon}^{a+h} f_\epsilon(x,\cdot\,)\,\mathrm{d}x.$$

For the first and analogously for the third term on the right hand side we argue

$$\int_a^{\lfloor a\rfloor_\epsilon+\epsilon} f_\epsilon(x,\cdot\,)\,\mathrm{d}x \le \|f\|_{L^\infty(\mathbb{R};L^1(\Xi,\mathbb{P};\mathbb{R}^2))}|\lfloor a\rfloor_\epsilon + \epsilon - a| \to 0 \quad \text{as } \epsilon \to 0.$$

For the second term, a change of variables leads to

$$\int_{\lfloor a\rfloor_\epsilon+\epsilon}^{\lfloor a+h\rfloor_\epsilon} f_\epsilon(x,\cdot\,)\,\mathrm{d}x = \epsilon \sum_{i\in I_\epsilon} \int_0^1 f(x+i,\cdot\,)\,\mathrm{d}x = \epsilon|I_\epsilon|\frac{1}{|I_\epsilon|}\sum_{i\in I_\epsilon}\int_0^1 f(x+i,\cdot\,)\,\mathrm{d}x.$$

Now, since $\epsilon|I_\epsilon| \to h$ as $\epsilon \to 0$ and by Kolmogorov's strong law of large numbers in the version of Theorem 6.1.1 we obtain for the family $(\int_0^1 f(x+i,\,\cdot\,)\,\mathrm{d}x)_{i\in\mathbb{Z}}$ that

$$\frac{1}{|I_\epsilon|}\sum_{i\in I_\epsilon}\int_0^1 f(x+i,\,\cdot\,)\,\mathrm{d}x \to \mathbb{E}\Big(\int_0^1 f(x,\,\cdot\,)\,\mathrm{d}x\Big) \quad \text{as } \epsilon \to 0.$$

Thus, we have for all intervals $[a, a+h] \subset \mathbb{R}$

$$\int_a^{a+h} f_\epsilon(x,\,\cdot\,)\,\mathrm{d}x \to h\mathbb{E}\Big(\int_0^1 f(x,\,\cdot\,)\,\mathrm{d}x\Big) = \int_a^{a+h} \mathbb{E}\Big(\int_0^1 f(x,\,\cdot\,)\,\mathrm{d}x\Big)\,\mathrm{d}y,$$

and Lemma 3.4.2 yields the claim. □

6.1.2 Homogenization of a Bernoulli model for layered materials

In this section, we want to develop the Bernoulli formulation for the geometry of layered materials of the previous section into a variational model for crystalline materials.

Example 6.1.4 (A Bernoulli model for layered materials)**.** Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain modeling the material. Recall the notation leading up to (6.1): We consider for $\rho \in (0,1)$ a family of independent random variables $(X_i)_{i\in\mathbb{Z}}$ that are Bernoulli distributed with parameter ρ. The underlying product probability space is denoted by $(\Xi, \mathcal{A}, \mathbb{P})$. Assuming that the ratio between a pair of stiff and soft layers can take the values $\lambda^1, \lambda^2 \in (0,1)$, we introduce the random variables $\lambda_i = \lambda^1 + (\lambda^2 - \lambda^1)X_i$. The stiff and soft layers are accordingly defined for $i \in \mathbb{Z}$ by

$$P^i_{\mathrm{soft}} = \mathbb{R} \times (i, i+\lambda_i) \quad \text{and} \quad P^i_{\mathrm{stiff}} = \mathbb{R} \times (i+\lambda_i, i+1).$$

Hence, in the following $\epsilon P^i_{\mathrm{soft}}$ denotes for each $i \in \mathbb{Z}$ the soft component of the i-th layer, which is with probability $\mathbb{P}(\lambda_i = \lambda^1) = \mathbb{P}(X_i = 0) = 1-\rho$ of thickness $\epsilon\lambda^1$.

It remains to adapt the energy functionals introduced for crystalline materials in Chapter 5 to the randomly layered structure, with the allocation of the energy densities now depending on the actual realization of the layered structure.

Accordingly, we consider for $\epsilon > 0$ the energy functional

$$E^\infty_\epsilon : L^2_0(\Omega;\mathbb{R}^2) \times \Xi \to [0,\infty]$$

given for $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ and $\omega \in \Xi$ by

$$E^\infty_\epsilon(u,\omega) = \int_\Omega W_{\mathrm{slip}}(\nabla u(x))\mathbb{1}_{\epsilon P_{\mathrm{soft}}\cap\Omega}(x,\omega) + W^\infty_{\mathrm{e}}(\nabla u(x))\mathbb{1}_{\epsilon P_{\mathrm{stiff}}\cap\Omega}(x,\omega)\,\mathrm{d}x, \tag{6.2}$$

where the energy densities W_{slip} and W^∞_{e} are defined as in (5.2) and (5.3), respectively. As for previous energies, we define E^∞_ϵ on $L^2_0(\Omega;\mathbb{R}^2) \times \Xi$.

Remark 6.1.5 (Γ-convergence in a stochastic setting)**.** In the following we aim to determine the Γ-limit $(E^\infty_\epsilon)_{\epsilon>0}$. Yet, since $(E^\infty_\epsilon)_{\epsilon>0}$ depends on the stochastic variable $\omega \in \Xi$, it remains to specify the meaning of Γ-convergence in this context.

The notion we want to consider is almost sure Γ-convergence in the sense that for almost every $\omega \in \Xi$, there is for each $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ a recovery sequence and for each sequence $u_\epsilon \to u$ in $L^2(\Omega;\mathbb{R}^2)$ a matching lower bound estimate, a notion that is for example also used in [117].

Next, we want to consider in analogy to Example 5.1.1 simple shear deformations in the Bernoulli model.

Example 6.1.6 (Shear deformations in the Bernoulli model)**.** The goal of this example is to show that simple shear deformations can be attained almost surely as limit functions. In contrast to the case of periodically layered materials, the construction can, and in general has to be tailored to each realization.

Let the macroscopic shear deformation $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ be given for $R \in SO(2)$ and $\gamma \in \mathbb{R}$ by

$$\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2).$$

As in the periodic case, the challenge is imposed by the rigid layers, and we apply the same strategy as there, which is shearing more on the soft layers to compensate for the presence of the rigid layers. In the periodic case where the component ratio for each pair of rigid and soft layer was identical, we scaled the amount of shear on the soft layer by a factor of $1/\lambda$. For random component ratios we scale by $1/\mathbb{E}(\lambda_i)$ instead, with the expected value for the thickness of the i-th soft layer given for each $i \in \mathbb{Z}$ by

$$\mathbb{E}(\lambda_i) = (1-p)\lambda^1 + p\lambda^2.$$

However, to tailor the approximating sequence to the specific realization, we introduce the function $\sigma \in L^\infty(\mathbb{R}^2; L^1(\Xi;\mathbb{P};\mathbb{R}))$ given by

$$\sigma(x,\omega) = \frac{1}{\mathbb{E}(\lambda_i)} \sum_{i\in\mathbb{Z}} \mathbb{1}_{P^i_{\text{soft}}}(x,\omega) = \frac{1}{\mathbb{E}(\lambda_i)} \sum_{i\in\mathbb{Z}} \mathbb{1}_{(i,(i+\lambda_i))}(x_2,\omega), \quad (x,\omega) \in \mathbb{R}^2 \times \Xi,$$

and set $\sigma_\epsilon(x,\omega) = \sigma(\epsilon^{-1}x,\omega)$ for $(x,\omega) \in \mathbb{R}^2 \times \Xi$.

Observe, that by definition of σ via $(X_i)_{i\in\mathbb{Z}}$ we have for $g \in L^\infty(P^0 \times \{0,1\};\mathbb{R})$ given by

$$g(x,\xi) = \mathbb{1}_{(0,\lambda^1]}(x_2)\, \mathbb{1}_{\{0\}}(\xi) + \mathbb{1}_{(0,\lambda^2]}(x_2)\, \mathbb{1}_{\{1\}}(\xi) \quad (x,\xi) \in P^0 \times \{0,1\},$$

the representation $\sigma(x,\omega) = g(x-i, X_i)(\omega)$, $(x,\omega) \in P^i \times \Xi$, $i \in \mathbb{Z}$. The representation also shows that we may identify σ with a function in the one variable x_2. Hence, $(\int_i^{i+1} \sigma \,\mathrm{d}x)_{i\in\mathbb{Z}}$ is a family of independent and identically distributed random variables, and thus Proposition 6.1.3 yields the existence of a set $\hat{\Xi} \subset \Xi$ of full measure, i.e. $\mathbb{P}(\hat{\Xi}) = 1$ such that for all $\omega_0 \in \hat{\Xi}$

$$\sigma_\epsilon(\,\cdot\,,\omega_0) \overset{*}{\rightharpoonup} \mathbb{E}\Big(\int_0^1 \sigma(x,\,\cdot\,)\,\mathrm{d}x\Big) = \tfrac{1}{\mathbb{E}(\lambda_i)}\mathbb{E}\Big(\int_0^1 \mathbb{1}_{(0,\lambda_i]}\,\mathrm{d}x_2\Big) = \tfrac{1}{\mathbb{E}(\lambda_i)}\mathbb{E}(\lambda_i) = 1 \quad \text{as } \epsilon \to 0.$$

Now, consider for fixed $\omega_0 \in \hat{\Xi}$ the sequence $(u_\epsilon)_\epsilon \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_0(\Omega;\mathbb{R}^2)$ given by

$$\nabla u_\epsilon(x) = R(\mathbb{I} + \gamma\sigma_\epsilon(\,\cdot\,,\omega_0) e_1 \otimes e_2).$$

To verify that $u_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$, firstly note that $\gamma\sigma_\epsilon(\,\cdot\,,\omega_0) \rightharpoonup \gamma$ in $L^2(\Omega)$, which yields $\nabla u_\epsilon \rightharpoonup \nabla u$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$ and since we required the mean value of u_ϵ to be zero we have $u_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$.

Finally, let us consider the asymptotic behavior of the energy of $(u_\epsilon)_\epsilon$. For this calculation, note that $\sigma_\epsilon^2 \in L^\infty(\mathbb{R}^2; L^1(\Xi,\mathbb{P},\mathbb{R}))$ as $\sigma_\epsilon^2 = 1/\mathbb{E}(\lambda^i) \cdot \sigma_\epsilon$ so that again Proposition 6.1.3 implies

$$\sigma_\epsilon^2(\,\cdot\,,\omega_0) \overset{*}{\rightharpoonup} \mathbb{E}\Big(\int_Y \sigma^2(x,\,\cdot\,)\,\mathrm{d}x\Big) = \frac{1}{\mathbb{E}(\lambda_i)^2}\mathbb{E}(\lambda_i) = \frac{1}{\mathbb{E}(\lambda_i)} = \frac{1}{(1-p)\lambda^1 + p\lambda^2} \quad \text{as } \epsilon \to 0.$$

Therefore, we obtain

$$E_\epsilon^\infty(u_\epsilon) = \int_\Omega \gamma^2 \sigma_\epsilon^2(\,\cdot\,,\omega_0)\,\mathrm{d}x = \frac{1}{\mathbb{E}(\lambda_i)}\int_\Omega \gamma^2 \sigma_\epsilon(x,\omega_0)\,\mathrm{d}x \to \frac{1}{\mathbb{E}(\lambda_i)}\int_\Omega \gamma^2\,\mathrm{d}x, \quad \text{as } \epsilon \to 0,$$

where again the difference to the periodic case is the scaling factor of $1/\mathbb{E}(\lambda^i)$ instead of λ.

The sequence constructed in the last example provides a recovery sequence in the following Γ-convergence result.

Theorem 6.1.7. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded Lipschitz domain which is $e_n^\perp$-connected in the sense of Definition 3.3.6. Then, the family of energy functionals $(E_\epsilon^\infty)_\epsilon$ defined in Example 6.1.4 by* (6.2) *converges almost surely in the sense of* Γ*-convergence with respect to the strong L^2-topology to a functional $E : L_0^2(\Omega;\mathbb{R}^2) \to [0,\infty]$ given for $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ a.e. in Ω, $R \in SO(2)$, $\gamma \in \mathbb{R}$ by*

$$E(u) = \frac{1}{\mathbb{E}(\lambda_i)}\int_\Omega \gamma^2\,\mathrm{d}x,$$

and $E(u) = \infty$ otherwise in $L^2(\Omega;\mathbb{R}^2)$.

Proof. Upper Bound. Suitable sequences have already been constructed in Example 6.1.6.

Lower Bound. Not to repeat all arguments of Section 5.2.5, we restrict our efforts to proving the case of affine limits, which takes the role of Lemma 5.2.23 in this context. With regard to asymptotic rigidity we apply techniques from [42], besides the more general arguments of Section 3. In particular, note that the application of Jensen's inequality is replaced by arguments utilizing the properties of $\mathbb{1}_{\epsilon P^i_{\text{stiff}}}$ and $\mathbb{1}_{\epsilon P^i_{\text{soft}}}$ that are well understood in this stochastic setting.

Indeed, since by $\mathbb{1}_{\epsilon P^i_{\text{stiff}}}(x,\omega) = \mathbb{1}_{P^i_{\text{stiff}}}(x/\epsilon,\omega) = \mathbb{1}_{(0,\lambda_i]}(x_2/\epsilon,\omega)$, $(x,\omega) \in \epsilon P^i \times \Xi$, $i \in \mathbb{Z}$, we can interpret $\mathbb{1}_{\epsilon P^i_{\text{stiff}}}$ as a one-dimensional function in space, and arguing as in Example 6.1.6, Proposition 6.1.3 yields that for a set of full measure $\hat{\Xi} \subset \Xi$, $P(\hat{\Xi}) = 1$ we have for all $\omega_0 \in \hat{\Xi}$

$$\mathbb{1}_{\epsilon P_{\text{stiff}}}(\,\cdot\,,\omega_0) \rightharpoonup \mathbb{E}\Big(\int_0^1 \mathbb{1}_{(0,\lambda_i]}(x_2)\,\mathrm{d}x_2\Big) = \mathbb{E}(\lambda_i) = (1-p)\lambda^1 + p\lambda^2 \quad \text{in } L^2(\Omega) \quad \text{as } \epsilon \to 0.$$

For the following arguments, let for $\omega_0 \in \hat{\Xi}$ fixed $(u_\epsilon) \subset W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ with $E_\epsilon^\infty(u_\epsilon,\omega_0) < C$ for a constant $C > 0$ converge to a limit function $u \in W^{1,2}(\Omega;\mathbb{R}^2) \cap L_0^2(\Omega;\mathbb{R}^2)$ that is affine.

Since $E_\epsilon^\infty(u_\epsilon,\omega_0) < C$ we know for each $\epsilon > 0$ that u_ϵ satisfies the pointwise differential inclusion constraints

$$\nabla u_\epsilon \in SO(2) \quad \text{on } \epsilon P_{\text{stiff}} \qquad \text{and} \qquad \nabla u_\epsilon \in \mathcal{M}_{e_1} \quad \text{on } \epsilon P_{\text{soft}} \quad \text{a.e.}$$

In particular we have $|\nabla u_\epsilon e_2| = 1$ on $\epsilon P_{\text{stiff}}$ a.e.

Furthermore, by Rešetnjak's Theorem 3.2.3 there is for each rigid layer $\epsilon P^i_{\text{stiff}}$, $i \in I_\epsilon$ a rotation $R^i_\epsilon \in SO(2)$ such that

$$\nabla u_\epsilon = R^i_\epsilon \in SO(2) \quad \text{on } \epsilon P^i_{\text{stiff}} \cap \Omega.$$

As in the proof of the rigidity result of Theorem 3.3.1 we gather all these rotations in the function $(\Sigma_\epsilon)_\epsilon \subset L^\infty(\Omega; SO(2))$, given for $\Omega_\epsilon = \bigcup_{i \in I_\epsilon} \epsilon P^i \cap \Omega$ by

$$\Sigma_\epsilon = \sum_{i \in I_\epsilon} R^i_\epsilon\,\mathbb{1}_{\epsilon P^i} + \mathbb{I}\,\mathbb{1}_{\Omega\setminus\Omega_\epsilon}.$$

Since $\lambda_1, \lambda_2 > \lambda > 0$ for some $\lambda > 0$, the estimate on rigid body motions on different rigid layers of Lemma 3.1.3 yields for all $i \in I_\epsilon \setminus \{i_{\max} - 1\}$

$$|R_\epsilon^{i+1} - R_\epsilon^i| < C\epsilon.$$

Summing over all layers, this entails that the variation of $(\Sigma_\epsilon)_\epsilon$ is uniformly bounded. Hence, by Helly's selection Theorem 3.3.16 (applied to Σ_ϵ as a one-dimensional function) we obtain the existence of an $R \in L^\infty(\Omega; SO(2))$ such that $\Sigma_\epsilon \to R$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$ as $\epsilon \to 0$.

Besides, as $\lambda_1, \lambda_2 > \lambda > 0$ for some $\lambda > 0$, we may assume by the characterization result of Theorem 3.3.1 that $u \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2(\Omega; \mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$, $\gamma \in \mathbb{R}$, in light of the fact that $\nabla u e_1 = R e_1$ uniquely determines the rotation R. Hence, we have

$$\nabla u_\epsilon \mathbb{1}_{\epsilon P_{\text{stiff}}}(\,\cdot\,, \omega_0) \rightharpoonup (1 - \mathbb{E}(\lambda_i))R \quad \text{in } L^2(\Omega, \mathbb{R}^{2\times 2}) \quad \text{as } \epsilon \to 0.$$

This, together with the fact that $\nabla u_\epsilon \rightharpoonup \nabla u$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$ as $\epsilon \to 0$ yields

$$\nabla u_\epsilon \mathbb{1}_{\epsilon P_{\text{soft}}}(\,\cdot\,, \omega_0) \rightharpoonup \nabla u - (1 - \mathbb{E}(\lambda_i))R \quad \text{in } L^2(\Omega, \mathbb{R}^{2\times 2}) \quad \text{as } \epsilon \to 0. \tag{6.3}$$

In particular, this convergence holds also in $L^1(\Omega, \mathbb{R}^{2\times 2})$.

As mentioned before, we now replace the application of Jensen's inequality which was used to conclude in Lemma 5.2.23 by techniques from [42] based on the properties $\mathbb{1}_{\epsilon P_{\text{soft}}}$, to show

$$\int_\Omega \frac{1}{\mathbb{E}(\lambda_i)} \big|\nabla u e_2 - (1 - \mathbb{E}(\lambda_i))R\big|^2 - \mathbb{E}(\lambda_i)\,\mathrm{d}x \leq \liminf_{\epsilon \to 0} \int_\Omega |\nabla u_\epsilon e_2|^2 - 1\,\mathrm{d}x.$$

By (6.3) and the lower semicontinuity of the L^1-norm we obtain

$$\int_\Omega \big|\nabla u e_2 - (1 - \mathbb{E}(\lambda_i))R e_2\big|\,\mathrm{d}x \leq \liminf_{\epsilon \to 0} \int_\Omega |\nabla u_\epsilon e_2| \mathbb{1}_{\epsilon P_{\text{soft}}}\,\mathrm{d}x. \tag{6.4}$$

With u affine, we have ∇u constant, and thus the left hand side can be estimated by

$$\begin{aligned} |\Omega| |\nabla u e_2 - (1 - \mathbb{E}(\lambda_i))R e_2| &= \Big|\int_\Omega \nabla u e_2 - (1 - \mathbb{E}(\lambda_i))R e_2\,\mathrm{d}x\Big| \\ &= \int_\Omega \big|\nabla u e_2 - (1 - \mathbb{E}(\lambda_i))R e_2\big|\,\mathrm{d}x. \end{aligned} \tag{6.5}$$

To estimate the right hand side of (6.4), we set $\Omega_\epsilon := \bigcup_{i \in I_\epsilon^\Omega} \epsilon P^i \cap \Omega$ and argue

$$\begin{aligned} \int_\Omega |\nabla u_\epsilon e_2|\, \mathbb{1}_{\epsilon P_{\text{soft}}}\,\mathrm{d}x &= \int_{\Omega_\epsilon} |\nabla u_\epsilon e_2|\, \mathbb{1}_{\epsilon P_{\text{soft}}}\,\mathrm{d}x + \int_{\Omega \setminus \Omega_\epsilon} |\nabla u_\epsilon e_2|\, \mathbb{1}_{\epsilon P_{\text{soft}}}\,\mathrm{d}x \\ &\leq \int_{\Omega_\epsilon} |\nabla u_\epsilon e_2\, \mathbb{1}_{\epsilon P_{\text{soft}}}|\, \mathbb{1}_{\epsilon P_{\text{soft}}}\,\mathrm{d}x + |\Omega \setminus \Omega_\epsilon|^{\frac{1}{2}} \|\nabla u_\epsilon e_2\|_{L^2(\Omega;\mathbb{R}^2)} \\ &\leq \|\nabla u_\epsilon e_2\, \mathbb{1}_{\epsilon P_{\text{soft}}}\|_{L^2(\Omega;\mathbb{R}^2)} \|\, \mathbb{1}_{\epsilon P_{\text{soft}}}\|_{L^2(\Omega)} + C|\Omega \setminus \Omega_\epsilon|^{\frac{1}{2}}. \end{aligned} \tag{6.6}$$

Overall, squaring (6.4) together with (6.5) and (6.6) entails

$$|\Omega| \int_\Omega |\nabla u e_2 - (1 - \mathbb{E}(\lambda_i))R e_2|^2\,\mathrm{d}x \tag{6.7}$$

$$\leq \liminf_{\epsilon \to 0} \|\nabla u_\epsilon e_2\, \mathbb{1}_{\epsilon P_{\text{soft}}}(\,\cdot\,, \omega_0)\|^2_{L^2(\Omega;\mathbb{R}^2)} \|\, \mathbb{1}_{\epsilon P_{\text{soft}}}(\,\cdot\,, \omega_0)\|^2_{L^2(\Omega;\mathbb{R}^2)}. \tag{6.8}$$

As on each rigid component $|\nabla u_\epsilon e_2| = 1$ we have for the first factor on the right hand side

$$\begin{aligned}\|\nabla u_\epsilon e_2 \, \mathbb{1}_{\epsilon P_{\text{soft}}}\|^2_{L^2(\Omega)} &= \int_\Omega |\nabla u_\epsilon e_2|^2 \, \mathbb{1}_{\epsilon P_{\text{soft}}} \, \mathrm{d}x \\ &= \int_\Omega |\nabla u_\epsilon e_2|^2 \, \mathrm{d}x - \int_\Omega |\nabla u_\epsilon e_2|^2 \, \mathbb{1}_{\epsilon P_{\text{stiff}}} \, \mathrm{d}x \\ &= \|\nabla u_\epsilon e_2\|^2_{L^2(\Omega;\mathbb{R}^2)} - \int_\Omega \mathbb{1}_{\epsilon P_{\text{stiff}}} \, \mathrm{d}x.\end{aligned}$$

Observe that

$$\| \, \mathbb{1}_{\epsilon P_{\text{soft}}}(\,\cdot\,, \omega_0)\|^2_{L^2(\Omega)} = \int_\Omega | \, \mathbb{1}_{\epsilon P_{\text{soft}}}(\,\cdot\,, \omega_0)|^2 \, \mathrm{d}x = \int_\Omega \mathbb{1}_{\epsilon P_{\text{soft}}}(\,\cdot\,, \omega_0) \, \mathrm{d}x \to |\Omega|\mathbb{E}(\lambda_i), \quad \text{as } \epsilon \to 0,$$

and similary

$$\| \, \mathbb{1}_{\epsilon P_{\text{stiff}}}(\,\cdot\,, \omega_0)\|^2_{L^2(\Omega)} \to |\Omega|(1 - \mathbb{E}(\lambda_i)), \quad \text{as } \epsilon \to 0.$$

Hence, (6.7) implies

$$|\Omega| \int_\Omega |\nabla u e_2 - (1 - \mathbb{E}(\lambda_i))Re_2|^2 \, \mathrm{d}x \le |\Omega|\mathbb{E}(\lambda_i)\Big(\liminf_{\epsilon \to 0} \|\nabla u_\epsilon e_2\|^2_{L^2(\Omega)} - |\Omega|(1 - \mathbb{E}(\lambda_i))\Big).$$

By rearranging the terms, we obtain the claimed

$$\int_\Omega \frac{1}{\mathbb{E}(\lambda_i)} |\nabla u e_2 - (1 - \mathbb{E}(\lambda_i))Re_2|^2 - \mathbb{E}(\lambda_i) \, \mathrm{d}x \le \liminf_{\epsilon \to 0} \int_\Omega |\nabla u_\epsilon e_2|^2 - 1 \, \mathrm{d}x.$$

□

Of course, basing the model on a sequence of independent identically distributed random variables is very restrictive and in the last century there has been great progress in generalizing the law of large numbers, spawning whole new fields of stochastic analysis, such as ergodic theory and renewal theory.

In the following we want to give a short introduction to ergodic theory and an overview on classic results in the theory of stochastic homogenization.

6.2 Elements of ergodic theory

6.2.1 The notion of ergodicity

This section is composed of selected results of ergodic theory with the aim to give a short introduction to the field and establish pointwise ergodic theorems for multiparameter and amenable semigroups, which provide a solid basis for a generalization of the Proposition 6.1.3 to a broad class of stochastic processes. We will follow the book of Krengel [94, Chapters 1, 2, 6], augmented by certain elements from other sources explicitly indicated below.

Definition 6.2.1 (Endomorphism [94, §1.1])**.** Let $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$ be measure spaces. We call a map $\tau : \Omega \to \Omega'$ a *homomorphism from* $(\Omega, \mathcal{A}, \mu)$ *to* $(\Omega', \mathcal{A}', \mu')$, if τ is measurable and the measure $\mu \circ \tau^{-1}$ on $\mathcal{A}'$ given by $(\mu \circ \tau^{-1})(A') = \mu(\tau^{-1}A')$ for all $A' \in \mathcal{A}'$ coincides with μ', i.e. $\mu \circ \tau^{-1} = \mu'$.

A map $\tau : \Omega \to \Omega$ is *measure preserving* if it is measurable and it holds that $\mu \circ \tau^{-1} = \mu$. Since τ is in this case a homomorphism from $(\Omega, \mathcal{A}, \mu)$ to $(\Omega, \mathcal{A}, \mu)$, we also say it is an *endomorphism.* A measure μ satisfying $\mu \circ \tau^{-1} = \mu$ for an endomorphism τ is call τ*-invariant.*

Example 6.2.2. Classical examples of endomorphisms are translations on $(\mathbb{R}^n, \mathcal{B}^n, \lambda^n)$, where for $n \in \mathbb{N}$ we denote by $\mathcal{B}^n$ the Borel-σ-algebra and λ^n the n-dimensional Lebesgue measure, as well as the rotation $x \mapsto x + \alpha \mod 1$ on $[0, 1)$ for $\alpha > 0$, also together with the Lebegue measure and the Borel-σ-algebra.

Definition 6.2.3 (Contraction [94, Definition 1.1])**.** Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ be normed vector spaces and $T : V \to W$ a linear operator and let $|||T||| = \|T\|_{\mathcal{L}(V,W)}$ denote the operator norm given by

$$|||T||| = \sup_{\|f\|_V \leq 1} \|Tf\|_W.$$

We call T *bounded* if $|||T||| < \infty$, a *contraction* if $|||T||| \leq 1$ and an *isometry* if $\|Tf\|_W = \|f\|_V$ for all $f \in V$.

If a partial order is defined on V, the *positive cone* is given by $V^+ := \{f \in V \mid f \geq 0\}$ and T is called *positive* if $T(V^+) = W^+$.

Example 6.2.4. Let τ be an endomorphism on $(\Omega, \mathcal{A}, \mu)$, then for $p \in [1, \infty]$, the operator $T : L^p(\Omega, \mathcal{A}, \mu) \to L^p(\Omega, \mathcal{A}, \mu)$ given by

$$T(f) = f \circ \tau$$

is a positive isometry, where $(L^p(\Omega, \mathcal{A}, \mu))^+ = \{f \in L^p \mid f \geq 0 \text{ a.e.}\}$.

Indeed, the operator is linear, preserves the sign of a function and since the measure μ is τ-invariant, we have

$$\int_\Omega f \circ \tau \, \mathrm{d}\mu = \int_{\tau^{-1}(\Omega)} f \circ \tau \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}(\mu \circ \tau^{-1}) = \int_\Omega f \, \mathrm{d}\mu.$$

Hence, for $1 \leq p < \infty$ we obtain

$$\begin{aligned} |||T||| = \sup_{\|f\|_{L^p(\Omega,\mathcal{A},\mu)} \leq 1} \|Tf\|_{L^p(\Omega,\mathcal{A},\mu)} &= \sup_{\|f \circ \tau\|_{L^p(\Omega,\mathcal{A},\mu)} \leq 1} \|Tf\|_{L^p(\Omega,\mathcal{A},\mu)} \\ &= \sup_{\|Tf\|_{L^p(\Omega,\mathcal{A},\mu)} \leq 1} \|Tf\|_{L^p(\Omega,\mathcal{A},\mu)} = 1, \end{aligned}$$

while the claim holds also true for $p = \infty$, as the essential supremum of $f \circ \tau$ with respect to the measure μ coincides with the essential supremum of f with respect to the measure $\mu \circ \tau^{-1} = \mu$.

Definition 6.2.5 (Invariant sets [94, Definition 1.1])**.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A measurable map $\tau : \Omega \to \Omega$ is called *null (set) preserving* if for all $A \in \mathcal{A}$ with $\mu(A) = 0$ it holds that $\mu \circ \tau^{-1}(A) = 0$.

For a null preserving map $\tau : \Omega \to \Omega$, a set $A \in \mathcal{A}$ is called τ-invariant if $\tau^{-1}(A) = A$. The σ-algebra of all invariant sets is denoted by $\mathcal{I}$.

Example 6.2.6. By definition, an endomorphism on a measure space is always null preserving.

Definition 6.2.7 (Conditional expectation [18, Satz 15.1])**.** Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$ such that the restriction of μ to $\mathcal{F}$ is σ-finite. Then, for each $f \in L^1(\Omega, \mathcal{A}, \mu)$ the $\mathcal{F}$-measurable function $f_0 \in L^1(\Omega, \mathcal{A}, \mu)$ that satisfies

$$\int_A f_0 \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu \quad \text{for all } A \in \mathcal{F}$$

and is uniquely determined up to a set of measure zero is called the *conditional expectation of f with respect to $\mathcal{F}$*. In reference to the notation for the expected value, we use for f_0 the notation $E(f \,|\, \mathcal{F})$.

Remark 6.2.8. The existence and uniqueness of f_0 up to a null set is a consequence of the Radon-Nikodym theorem. For more details see [18, Satz 15.1].

Definition 6.2.9 (Ergodicity [94, Definition 1.7])**.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\tau : \Omega \to \Omega$ a null preserving measurable map. We call τ *ergodic* if all τ-invariant sets A satisfy $\mu(A) = 0$ or $\mu(A^c) = 0$.

Remark 6.2.10. 1. Graphically speaking, τ is ergodic if the space cannot be decomposed into two non-trivial τ-invariant subsets.

2. If τ is ergodic, the σ-algebra of all invariant sets $\mathcal{I}$ is "almost trivial", in the sense that it may contain a set $A \in \mathcal{I}$ with $A \neq \emptyset, \Omega$, but A differs from these sets only by a null set.

Lemma 6.2.11 (Ergodicity of endomorphisms [94, Chapter 1, Section 4])**.** *An endomorphism* $\tau : \Omega \to \Omega$ *on a measure space* $(\Omega, \mathcal{A}, \mu)$ *is ergodic if and only if each measurable* τ*-invariant function* $f : \Omega \to \mathbb{R}$ *is constant* μ*-a.e.*

Proof. Firstly, observe that a real valued function f is τ-invariant if and only if $\{f = \alpha\}$ is τ-invariant for all $\alpha \in \mathbb{R}$, since depending on the assumption either the first or the last equality in

$$\{f = \alpha\} = \{f \circ \tau = \alpha\} = \tau^{-1}\{f = \alpha\} = \{f = \alpha\} \tag{6.9}$$

is given. Hence, $\{f = \alpha\} \in \mathcal{I}$. However, if τ is ergodic, it holds for all $A \in \mathcal{I}$ that $\mu(A) = 0$ or $\mu(A^c) = 0$. Thus, for each $\alpha \in \mathbb{R}$ we have $\mu(\{f = \alpha\}) = 0$ or $\mu(\{f = \alpha\}^c) = 0$, i.e. f is constant μ-a.e establishing sufficiency.

To show the necessity, notice that all characteristic functions $\mathbb{1}_A$ with $A \in \mathcal{I}$ are τ-invariant, as can be seen by (6.9) for $f = \mathbb{1}_A$ and $\alpha \in \{0, 1\}$. Thus, by assumption, $\mathbb{1}_A$ is constant μ-a.e. and hence $\mu(A) = 0$ or $\mu(A^c) = 0$, which implies that τ is ergodic. □

Remark 6.2.12. In the context of the ergodic theorems that are to be discussed now, Lemma 6.2.11 shows that in particular for ergodic τ the $\mathcal{I}$-measurable function $E(f \,|\, \mathcal{I})$ is constant and thus coincides with the constant function $\omega \mapsto E(f)$.

As the definition of ergodicity is for itself an abstract notion, we end this subsection with a sufficient condition for ergodicity, the so-call mixing condition.

Definition 6.2.13 (Mixing condition [94, §1.4])**.** An endomorphism τ on a finite measure space $(\Omega, \mathcal{A}, \mu)$ is called *mixing* if for all $A, B \in \mathcal{A}$ it holds that

$$\lim_{k\to\infty} \mu(A \cap \tau^{-k}B) = \frac{\mu(A)\mu(B)}{\mu(\Omega)}.$$

Lemma 6.2.14. *Every mixing endomorphism* τ *on a finite measure space* $(\Omega, \mathcal{A}, \mu)$ *is ergodic.*

Proof. Suppose that $A \in \mathcal{A}$ is a τ-invariant set with $0 < \mu(A) < \mu(\Omega)$. Then,

$$\mu(A) = \lim_{k\to\infty} \mu(A \cap \tau^{-k}A) = \frac{\mu^2(A)}{\mu(\Omega)},$$

which implies $\mu(A) = \mu(\Omega)$, contradicting the assumption. Hence, for every τ-invariant $A \in \mathcal{A}$ we have $\mu(A) = 0$ or $\mu(A^c) = 0$ which implies that τ is ergodic. □

6.2.2 Ergodic theorems

A classic result of ergodic theory is the mean ergodic theorem of von Neumann. Though it is not of particular interest for our intentions, it still ought to be cited for completeness.

Theorem 6.2.15 (Von Neumann's mean ergodic theorem [94, Theorem 1.5, Proposition 1.6])**.** *If τ is an endomorhpism on $(\Omega, \mathcal{A}, \mu)$ and μ is σ-finite on the σ-algebra $\mathcal{I}$ of τ-invariant sets, then, for all $f \in L^2(\Omega, \mathcal{A}, \mu)$ it holds that*

$$\frac{1}{k}\sum_{i=0}^{k} f \circ \tau^k \to E(f \,|\, \mathcal{I}) \quad \textit{in } L^2(\Omega) \quad \textit{as } k \to \infty.$$

To shorten the notation for sums and averages we introduce for a given linear operator T and $k \in \mathbb{N}$

$$S_k f = \sum_{i=0}^{k-1} T^i f, \quad \text{and} \quad A_k f = k^{-1} S_k f.$$

Also, for the maximum over the first k sums and averages, we write

$$M_k^S f = \max\{S_1 f, \dots, S_k f\}, \quad \text{and} \quad M_k f = \max\{A_1 f, \dots, A_k f\}.$$

Lastly, we set $M_\infty f = \sup_{k \in \mathbb{N}} M_k f$.

In the context of an endomorphism τ the notation introduced here always refers to the linear operator given by concatenation with τ from Example 6.2.4.

Theorem 6.2.16 (Maximal ergodic theorem (Hopf '54) [94, Theorem 2.1])**.** *Let T be a positive contraction in $L^1(\Omega)$. For $f \in L^1(\Omega)$ we set $E_k = \{M_k f \geq 0\}$ and $E_\infty = \bigcup_{k=1}^\infty E_k$.*

Then,

$$\int_{E_k} f \,\mathrm{d}\mu \geq 0 \quad \textit{and} \quad \int_{E_\infty} f \,\mathrm{d}\mu \geq 0.$$

Proof. The proof given follows Kregels presentation of arguments given originally by Garsia.

For a function $f : \Omega \to \mathbb{R}$, we denote by $f^+ = \max\{f, 0\}$ the non-negative part of f. For $\ell = 1, \dots, k$ the fact that $(M_k^S f)^+ \geq S_\ell f$ that T is positive implies

$$f + T(M_k^S f)^+ \geq f + T S_\ell f = S_{\ell+1} f.$$

This together with $f \geq f - T(M_k^S f)^+ = S_1 f - T(M_k^S f)^+$ yields

$$f \geq S_\ell f - T(M_k^S f)^+ \quad \text{for } \ell = 1, \dots, k.$$

By taking the maximum in $\{1, \dots, k\}$ we obtain $f \geq M_k^S f - T(M_k^S f)^+$. Observe that since T is a positive contraction, we have for all $g \in L^1(\Omega)$ with $g \geq 0$ the estimate

$$\int_\Omega g \,\mathrm{d}\mu = \|g\|_{L^1(\Omega)} \geq \|Tg\|_{L^1(\Omega)} = \int_\Omega Tg \,\mathrm{d}\mu.$$

Hence, integration over E_k entails by definition of E_k

$$\begin{aligned}
\int_{E_k} f \,\mathrm{d}\mu &\geq \int_{E_k} M_k^S f - T(M_k^S f)^+ \,\mathrm{d}\mu \\
&= \int_{E_k} (M_k^S f)^+ - T(M_k^S f)^+ \,\mathrm{d}\mu \\
&= \int_\Omega (M_k^S f)^+ \,\mathrm{d}\mu - \int_{E_k} T(M_k^S f)^+ \,\mathrm{d}\mu \\
&= \int_\Omega (M_k^S f)^+ \,\mathrm{d}\mu - \int_\Omega T(M_k^S f)^+ \,\mathrm{d}\mu \geq 0.
\end{aligned}$$

The second estimate follows by taking the limit $k \to \infty$. □

Remark 6.2.17. Notice that the statement also holds if we replace E_k by $\tilde{E}_k = \{M_k f > 0\}$. The reason is that if $M_k^S = 0$ for all $k \in \mathbb{N}$, then $f = S_1 f = 0$ and therefore the set $\{M_k f = 0\}$ does not affect the value of the integral as $\int_{\{M_k f = 0\}} f \, \mathrm{d}\mu = 0$.

As a corollary we obtain the maximal ergodic inequality, a result that was already known by Wiener.

Corollary 6.2.18 (Maximal ergodic inequality [94, Corollary 2.2])**.** *Let τ be an endomorhpism on the measure space $(\Omega, \mathcal{A}, \mu)$. Then, for all $\alpha > 0$ and all $f \in L^1(\Omega)$ we have the estimate*

$$\mu(M_k f \geq \alpha) \leq \alpha^{-1} \|f\|_{L^1(\Omega)}.$$

Proof. Observe that the triangle inequality implies for $f \in L^1(\Omega)$ that $(A_k f)_{k \in \mathbb{N}} \subset L^1(\Omega)$ is uniformly bounded and thus also $M_k f \in L^1(\Omega)$ yet without a uniform bound. Consequently, we obtain that $\{M_k f \geq \alpha\}$ has finite measure, and we have to show the explicit uniform bound. To shorten the notation, we set $A = \{M_k f > \alpha\}$ and set $E_{k,A} = \{M_k(f - \alpha \, \mathbb{1}_A) \geq 0\}$. Theorem 6.2.16 yields

$$\int_{E_{k,A}} (f - \alpha \, \mathbb{1}_A) \, \mathrm{d}\mu \geq 0.$$

Besides that, there exists for each $\omega \in A$ a $\ell \leq k$ with $A_\ell f(\omega) \geq \alpha$. Thus, it holds that $S_k(f - \alpha \, \mathbb{1}_A)(\omega) \geq 0$, which implies $\omega \in E_{k,A}$. Overall, we obtain $A \subset E_{k,A}$, which leads to

$$\|f\|_{L^1(\Omega)} \geq \int_{E_{k,A}} f \, \mathrm{d}\mu \geq \alpha \int_{E_{k,A}} \mathbb{1}_A \, \mathrm{d}\mu = \alpha \mu(A).$$

□

Theorem 6.2.19 (Birkhoff's ergodic theorem [94, Theorem 2.3])**.** *Let τ be an endomorphism on a measure space $(\Omega, \mathcal{A}, \mu)$ and $f \in L^1(\Omega)$. Then the averages $A_k f$ converge μ-a.e. to a τ-invariant function $\bar{f} \in L^1(\Omega)$ with $\|\bar{f}\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$ such that for each $A \in \mathcal{I}$ with $\mu(A) < \infty$ it holds that*

$$\int_A \bar{f} \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu.$$

Furthermore, if $\mu(\Omega) < \infty$ then $A_k f \to \bar{f} = E(f \,|\, \mathcal{I})$ in $L^1(\Omega)$.

Proof. Firstly, observe that due to

$$A_{k+1} f = (k+1)^{-1} S_{k+1} f = (k+1)^{-1} f + \frac{k}{k+1} (k^{-1} S_k f) \circ \tau = (k+1)^{-1} f + \frac{k}{k+1} A_k f \circ \tau,$$

the functions $f^u = \limsup_{k \to \infty} A_k f$ and $f^\ell = \liminf_{k \to \infty} A_k f$ are τ-invariant.

Next, we show that both functions are μ-a.e. finite. Indeed, for $\beta > 0$, arguing as in (6.9), the set $D_\beta = \{f^u > \beta\}$ is τ-invariant and satisfies

$$D_\beta = \{f^u > \beta\} \subset \{M_k f \geq \beta\}.$$

Thus, by the maximal ergodic inequality from Corollary 6.2.18 we obtain

$$\mu(D_\beta) \leq \mu(M_k f \geq \beta) \leq \beta^{-1} \|f\|_{L^1(\Omega)}.$$

Hence, since $\{f^u = \infty\} = \bigcap_{k\in\mathbb{N}}\{f^u > k\}$, the continuity of μ from above entails

$$\mu(\{f^u = \infty\}) = \lim_{k\to\infty} \mu(\{f^u > k\}) = \lim_{k\to\infty} k^{-1}\|f\|_{L^1(\Omega)} = 0.$$

This yields $f^u < \infty$ a.e. and by symmetry, $f^\ell > -\infty$ a.e. as

$$\mu(f^\ell < \alpha) = \mu\big(-\limsup_{k\to\infty} A_k(-f) < \alpha\big) \le |\alpha|^{-1}\|f\|_{L^1} \quad \text{for } \alpha < 0. \tag{6.10}$$

Now, we show that $A_n f$ converges μ-a.e. Suppose it does not, then we find rational numbers $\alpha < \beta$ such that the set $B := \{f^\ell < \alpha < \beta < f^u\}$ has positive measure. Since $\alpha < 0$ or $\beta > 0$ we find that $B \subset D_\beta = f^u > \beta$ or $B \subset \{f^\ell < \alpha\}$ and we have seen previously that in both cases the superset is of finite measure. Thus, $\mu(B)$ is finite. Furthermore, $B = \{f^\ell < \alpha\} \cap \{f^u > \beta\}$, which shows that B is τ-invariant. Thus, setting $f' = (f - \beta)\,\mathbb{1}_B$ we have for all $k \ge \mathbb{N}$ that $f' \circ \tau^k$ vanishes outside B and $B = \{\omega \in \Omega \mid \exists n \in \mathbb{N} \text{ such that } S_k f'(\omega) > 0\}$. By the maximal ergodic inequality from Corollary 6.2.18 in the variant of Remark 6.2.17 we obtain

$$\beta\mu(B) \le \int_B f \,\mathrm{d}\mu.$$

Arguing by symmetry for $f'' = (\alpha - f)\,\mathbb{1}_B$ yields $\int_B f\,\mathrm{d}\mu \le \alpha\mu(B)$. By adding both inequalities we see that

$$\alpha\mu(B) \le \beta\mu(B),$$

which can due to $\alpha < \beta$ only hold if $\mu(B) = 0$. Thus we have established that $f^u = f^\ell$ a.e. and together with the fact that these functions are finite μ-a.e. we conclude that there exists a measurable function $\bar{f} = f^u$ to which $(A_k f)_{k\in\mathbb{N}}$ converges μ-a.e. We proceed by showing that $\|\bar{f}\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}$. To that end, we decompose $f = f^+ + f^-$. By the Lemma of Fatou we see that

$$\int_\Omega \bar{f}^\pm \,\mathrm{d}\mu = \int_\Omega \liminf_{k\to\infty} A_k f^\pm \,\mathrm{d}\mu \le \liminf_{k\to\infty} \int_\Omega A_k f^\pm \,\mathrm{d}\mu = \int_\Omega f^\pm \,\mathrm{d}\mu,$$

where in the last step we have used that Ω is τ-invariant and therefore $\int_\Omega f \circ \tau^k \,\mathrm{d}\mu = \int_\Omega f\,\mathrm{d}\mu$. Since $|f| = f^+ + f^-$, this shows $\|\bar{f}\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}$.

For the characterization of the limit through (6.10), let $A \in \mathcal{I}$ be of finite measure. By using the decomposition $f = f^+ + f^-$ we may assume that $f|_A \ge 0$. Since $f \in L^1(A)$, there is for each $\epsilon > 0$ a $K_\epsilon \ge 0$ such that $g_\epsilon = f - (f - K_\epsilon)^+$ satisfies $\|g_\epsilon\|_{L^1(\Omega)} < \epsilon$. Since g_ϵ is non-negative and A τ-invariant it holds that

$$\int_A (A_k f - K_\epsilon)^+ \,\mathrm{d}\mu \le \int_A A_k (f - K_\epsilon)^+ \,\mathrm{d}\mu = \int_A A_k g_\epsilon \,\mathrm{d}\mu = \int_A g_\epsilon \,\mathrm{d}\mu = \int_A |g_\epsilon| \,\mathrm{d}\mu < \epsilon.$$

By Lemma 6.4.1 this shows that $(A_k f)_{k\in\mathbb{N}}$ is uniformly integrable. Since convergence in $L^1(\Omega)$ is equivalent to μ-a.e. convergence and uniform integrability of a sequence, the fact that $\int_A A_k f\,\mathrm{d}\mu = \int_A f\,\mathrm{d}\mu$ implies

$$\int_A \bar{f}\,\mathrm{d}\mu = \int_A f\,\mathrm{d}\mu. \tag{6.11}$$

Lastly, if $\mu(\Omega) < \infty$ equality (6.11) holds for all $A \in \mathcal{I}$, which implies that $\bar{f} = E(f\,|\,\mathcal{I})$ and by the uniform integrability of $(A_k f)_{k\in\mathbb{N}}$ on Ω the sequence also converges in $L^1(\Omega)$. □

Up to this point, we have considered the limits of sequences of additive structure in the sense of $(S_k f)_{k\in\mathbb{N}}$ for $f \in L^1(\Omega)$. In the context of lower bounds for Γ-convergence, the variational structure gives rise to subadditive rather than additive behavior of sequences.

Nevertheless, a subadditive structure also allows to establish convergence results, as the following lemma shows for sequences of real numbers

Lemma 6.2.20 (Convergence of subadditive real valued sequences [94, Lemma 5.1])**.** *Let $(g_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be a subadditive sequence of real numbers, i.e. for $i, j \in \mathbb{N}$ we have $g_{i+j} \leq g_i + g_j$. Then, the sequences $(k^{-1}g_k)_{k\in\mathbb{N}}$ converges to $\gamma = \inf_{k\in\mathbb{N}}\{k^{-1}g_k\}$.*

Proof. Let $N \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$ there are $n_k, r_k \in \mathbb{N}$ with $1 \leq r_k \leq N$ such that $k = n_k N + r_k$. In the limit $k \to \infty$, we have $k^{-1}r_k \leq k^{-1}N \to 0$ and thus $k^{-1}n_k \to N^{-1}$. Hence, by the subadditivity of $(g_k)_{k\in\mathbb{N}}$ we obtain

$$\gamma \leq k^{-1}g_k \leq k^{-1}(g_{n_k N} + g_{r_k}) \leq (k^{-1}n_k) \cdot N \cdot (N^{-1}g_N) + k^{-1} \max_{1\leq r\leq N} g_r.$$

Thus, passing to the limit $k \to \infty$ yields

$$\gamma \leq \liminf_{n\to\infty} n^{-1}g_n \leq N^{-1}g_N.$$

Finally, taking the infimum in N provides shows the claim. □

The applications call for a more involved notion of subadditivity, introduced in the next definition.

Definition 6.2.21 (Discrete parameter subadditive processes [94, §1.5, Section 1])**.** Let τ be an endomorphism on the measure space $(\Omega, \mathcal{A}, \mu)$ and set $Q = \{(i,j) \in \mathbb{Z}^2 \mid 0 \leq i < j\}$. A family $F = (F_{i,j})_{(i,j)\in Q} \subset L^1(\Omega)$ is called a *subadditive process* if

(i) $F_{i,j} \circ \tau = F_{i+1,j+1}$ for $(i,j) \in Q$;

(ii) $F_{i,h} \leq F_{i,j} + F_{j,h}$ for $(i,j), (j,h) \in Q$;

(iii) $\gamma(F) := \inf \{j^{-1} \int F_{0,j} \,\mathrm{d}\mu \mid k \in \mathbb{N}\} > -\infty$,

where $\gamma(F)$ is also known as the *time constant* of the process F. F is called a *superadditive process* if $(-F_{i,j})_{(i,j)\in Q}$ is subadditive and an *additive process* if it is both super- and subadditive.

Remark 6.2.22. a) If F is an additive process it has the form

$$F_{i,j} = \sum_{k=i}^{j-1} F_{k,k+1} = \sum_{k=i}^{j-1} F_{0,1} \circ \tau^k.$$

Thus, the a.e.-convergence of $k^{-1}F_{0,k}$ for additive processes is a consequence of Birkhoff's ergodic theorem.

b) Let us consider the sequence $(g_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ given by $g_j = \int_\Omega F_{0,j} \,\mathrm{d}\mu$. Since τ is measure preserving, condition (i) entails $\int_\Omega F_{i,j} \,\mathrm{d}\mu = \int_\Omega F_{i+1,j+1} \,\mathrm{d}\mu$. By condition (ii) we obtain for $i, j \in \mathbb{N}$

$$\begin{aligned} g_{i,j} = \int_\Omega F_{0,i+j} \,\mathrm{d}\mu &\leq \int_\Omega F_{0,i} + F_{i,i+j} \,\mathrm{d}\mu \\ &\leq \int_\Omega F_{0,i} \,\mathrm{d}\mu + \int_\Omega F_{0,j} \,\mathrm{d}\mu = g_i + g_j, \end{aligned}$$

which shows that $(g_j)_{j\in\mathbb{N}}$ is a subadditive sequence of real numbers.

The corresponding ergodic theorem for subadditive processes is known as Kingman's ergodic theorem.

Theorem 6.2.23 (Kingman's ergodic theorem [94, Theorem 5.3])**.** *Let τ be an endomorphism on the measure space $(\Omega, \mathcal{A}, \mu)$ and $F = (F_{i,j})_{(i,j)\in Q}$ a subadditive process for τ. Then, $(k^{-1}F_{0,k})_{k\in\mathbb{N}}$ converges a.e. to a τ-invariant function $\tilde{f} \in L^1(\Omega)$. If $\mu(\Omega) < \infty$, then $(k^{-1}F_{0,k})_{k\in\mathbb{N}}$ converges also in $L^1(\Omega)$ and $\gamma(F) = \int \tilde{f}\,\mathrm{d}\mu$.*

All ergodic theorems considered so far concerned one endomorphism on a measure space, for example a shift in one certain direction. For multi-dimensional homogenization problems this is insufficient, as also the periodic case requires typically periodicity in all space directions. Therefore we close this subsection by considering pointwise ergodic theorems for multiparameter semigroups which involve multiple endomorphisms, following [94, Chapter 6].

For $n \in \mathbb{N}$, we use for subsets of $\mathbb{V} = \mathbb{N}_0^n$ the notation for intervals in the sense that for $a, b \in \mathbb{V}$ such that $a < b$ with respect to the componentwise partial ordering we write $[a, b) = \{c \in \mathbb{V} \mid a \leq c < v\} \subset \mathbb{V}$. The set of all nonempty intervals in $\mathbb{V}$ is denoted by $\mathcal{J}$. For finite sets $A \subset \mathbb{V}$ we denote the number of elements of A by $\#A$. Furthermore, we will use multi-index notation, setting in particular for operators $T_1, \ldots, T_n$ and $k = (k_1, \ldots, k_n) \in \mathbb{V}$

$$T_k = T_1^{k_1} \ldots T_n^{k_n}, \qquad S_k = \sum_{\ell\in[0,k)} T_\ell, \qquad A_k = (\#[0, k))^{-1} S_k.$$

With more than one endomorphism involved in multiple dimensions, problems may arise concerning the interaction of different endomorphism. Firstly, the endomorphisms may not commute. In this work we restrict ourselves to commuting endomorphisms which define a semigroup $\tau = (\tau_k)_{k\in\mathbb{V}}$ by $\tau_k = \tau_1^{k_1} \ldots \tau_n^{k_n}$. A second problem arises from the interpretation of $k \to \infty$. While it will always mean that $k_i \to \infty$ for $i = 1, \ldots, n$, rates of convergence that are too unsimilar in different directions can hinder a meaningful convergence result. To that end we will only consider regular families in the sense of the following definition.

Definition 6.2.24 (Regular family [94, Chapter 6, Definition 2.4])**.** A family $(I_k)_{k\in\mathbb{N}}$, $I_k \subset \mathbb{V}$ for $k \in \mathbb{N}$ is called *regular (with constant $C < \infty$)* if there is an increasing sequence $(I'_k)_{k\in\mathbb{N}} \subset \mathcal{J}$ with $I_k \subset I'_k$ and $\#I'_k \leq C\#I_k$ for all $k \in \mathbb{N}$. If additionally $\bigcup_{k\in\mathbb{N}} I'_k = \mathbb{V}$, then we write $\lim_{k\to\infty} I_k = \mathbb{V}$.

An example of a regular family of constant 1 is any increasing sequence of set in $\mathcal{J}$, so this condition imposes that the sequence of sets does not deviate to much from increasing sequences rather than restricting the overall geometry of the sets.

Next, we generalize our notion of subadditive processes to the multidimensional context.

Definition 6.2.25 (Subadditive process [94, Chapter 6, Definition 2.1])**.** A *subadditive process with respect to a semigroup of endomorphisms* τ on a measure space $(\Omega, \mathcal{A}, \mu)$ is a family $F = (F_I)_{I\in\mathcal{J}} \subset L^1(\Omega)$ which satisfies

(i) $F_I \circ \tau_k = F_{I+k}$ for all $I \in \mathcal{J}$ and $k \in \mathbb{V}$;

(ii) $F_I \leq \sum_{i=1}^k F_{I_i}$ for disjoint sets $I_1, \ldots, I_k \subset \mathcal{J}$, $k \in \mathbb{N}$ such that $I = \bigcup_{i=1}^k I_i \in \mathcal{J}$;

(iii) $\gamma(F) := \inf \left\{(\#I)^{-1} \int_\Omega F_I \,\mathrm{d}\mu \mid I \in \mathcal{J}\right\} < \infty$.

The quantity $\gamma(F)$ in the third condition is also referred to as the *spatial constant* of F. If $-F$ is subadditive, then F is called *superadditive*, in which case the spacial constant is obtained by taking the supremum. F is called *additive* if both F and $-F$ are subadditive. Notice that additive processes are of the form $F_I = \sum_{k\in I} T_k f$ for $f \in L^1(\Omega)$.

	additive process	subadditive process
1-dim	Birkhoff	Kingsman
n-dim	Tempel'man	Akcoglu-Krengel

Figure 6.2: In this section, we discuss four major ergodic theorems, that can be distinguished by the categories that they either concern sub- or additive processes or either one- or multiparamter processes.

The multidimensional result corresponding to Birkhoff's ergodic theorem is Tempel'man's ergodic theorem, originally published in [127].

Theorem 6.2.26 (Tempel'man's ergodic theorem (1972) [94, Chapter 6, Theorem 2.8])**.** *Let $\tau_1, \dots, \tau_n$ be commuting endomorphisms on the measure space $(\Omega, \mathcal{A}, \mu)$ and $(I_k)_{k\in\mathbb{N}} \subset \mathcal{J}$ a regular family such that $\lim_{k\to\infty} I_k = \mathbb{V}$.*

Then, the sequence $(A_k f)_{k\in\mathbb{N}}$ given by $A_k f = (\#I_k)^{-1} S_{I_k} f$ converges a.e. for all $f \in L^p(\Omega)$, where $1 \leq p < \infty$.

For subadditive processes the multidimensional analogon to Kingman's ergodic theorem is the ergodic theorem by Akcoglu-Krengel, originally published in [2].

Theorem 6.2.27 (Akcoglu-Krengel's ergodic theorem [94, Chapter 6, Theorem 2.9])**.** *Let $(F_I)_{I\in\mathcal{J}}$ be a subadditive process and $(I_k)_{k\in\mathbb{N}} \subset \mathcal{J}$ a regular family such that $\lim_{k\to\infty} I_k = \mathbb{V}$. Then, the sequence $(A_k f)_{k\in\mathbb{N}}$ given by $A_k f = (\#I_k)^{-1} S_{I_k} f$ converges a.e. as $k \to \infty$.*

6.3 Stochastic homogenization via stationary ergodic processes

6.3.1 Applications of ergodic theory

In a stochastic context, it is preferable to focus on measurable functions rather than the underlying measure spaces. As usual, we refer to a measurable function defined on a probability space as a random variable. A sequence of random variables which is indexed by an ordered set (so that there is a notion of progression) is called a stochastic process. In the following we focus on stochastic processes indexed by $\mathbb{Z}$. We follow [94, §1.4].

Definition 6.3.1 (Probabilistic setting)**.** For a measurable space $(E, \mathcal{F})$ we denote by $(E^{\mathbb{Z}}, \mathcal{F}^{\mathbb{Z}})$ the product space with $\mathcal{F}^{\mathbb{Z}}$ the product-σ-algebra. The projection map on the j-th component is denoted by $\mathrm{proj}_j : (E^{\mathbb{Z}}, \mathcal{F}^{\mathbb{Z}}) \to (E, \mathcal{F})$.

Let $(\Omega', \mathcal{A}', P)$ be a probability space, then an E-valued *stochastic process* with *parameter space* $\mathbb{Z}$ is a family $Y = (Y_k)_{k\in\mathbb{Z}}$ of random variables $Y_k : \Omega' \to E$. The *distribution* of Y is the probability measure $P \circ Y^{-1}$ on $\mathcal{F}^{\mathbb{Z}}$.

For a family of probability measures $(\mu_k)_{k\in\mathbb{Z}}$ on $(E, \mathcal{F})$, the *product measure* $\mu = \prod_{k\in\mathbb{Z}} \mu_k$ is the unique probability measure on $\mathcal{F}^{\mathbb{Z}}$ characterized by the condition, that for all finite $J \subset \mathbb{Z}$ and all $A_j \in \mathcal{F}$, $j \in \mathbb{Z}$ it holds that

$$\mu\Big(\bigcap_{j\in J} \mathrm{proj}_j^{-1}(A_j)\Big) = \prod_{j\in J} \mu_j(A_j).$$

Furthermore, we call the random variables $(Y_k)_{k\in\mathbb{Z}}$ *independent* if for all finite $J \subset \mathbb{Z}$ and for all $A_k \in \mathcal{F}$, $k \in \mathbb{Z}$ it holds that

$$P\Big(\bigcap_{k\in J} Y_k^{-1}(A_k)\Big) = \prod_{k\in J} P(Y_k^{-1}A_k).$$

Definition 6.3.2 (Shift)**.** Let $(E,\mathcal{F})$ be a measurable space. The map $\theta : E^{\mathbb{Z}} \to E^{\mathbb{Z}}$ given by $\mathrm{proj}_k(\theta\omega) = \mathrm{proj}_{k+1}(\omega)$ is called *(bilateral) shift*.

Remark 6.3.3. The bilateral shift is a bijective map and both θ and θ^{-1} are measurable with respect to $\mathcal{F}^{\mathbb{Z}}$.

Example 6.3.4 (Bernoulli shift)**.** Let $(E,\mathcal{F})$ be a measurable space and let $(\mu_k)_{k\in\mathbb{N}}$ be a family of probability measures on E and denote the product measure by $\mu = \prod_{k\in\mathbb{N}} \mu_k$. Notice that μ is θ-invariant if and only if all μ_j are identical. In this case we call θ a *(bilateral) Bernoulli shift*.

Recall that a family of random variables $Y = (Y_k)_{k\in\mathbb{Z}}$ is called *identically distributed* if all distributions $P \circ Y_k^{-1}$ coincide. Thus, if Y is a sequence of independent identically distributed random variables, the distribution of Y coincides with the product measure of the distributions of Y_k, $k \in \mathbb{Z}$ and is therefore a Bernoulli shift.

Lemma 6.3.5 (Bernoulli shifts are mixing)**.** *Let $(E,\mathcal{F},\nu)$ be a probability space and denote the corresponding product space with $(E^{\mathbb{Z}},\mathcal{F}^{\mathbb{Z}},\mu)$, where $\mu = \prod_{k\in\mathbb{Z}} \nu$. Then, the Bernoulli shifting operator θ is mixing.*

Proof. We have to show that for all $A, B \in \mathcal{A}$ it holds that

$$\lim_{n\to\infty} \mu(A \cap \theta^{-n}B) = \frac{\mu(A)\mu(B)}{\mu(\Omega)}.$$

To that end we apply a Dynkin argument. First, consider the σ-algebras $\mathcal{A}(i,k)$ which are generated by the σ-algebras $\mathrm{proj}_k^{-1}\mathcal{F}$ for $i \le j \le k, k \ne \pm\infty$. Since θ is a Bernoulli shift, we find for $n \in \mathbb{N}$ that $\mu(A \cap \theta^{-n}B) = \mu(A)\mu(B)$ for all $A \in \mathcal{A}(k_1,k_2)$, $B \in \mathcal{A}(j_1,j_2)$ with $k_1 \le k_2 < \infty$ and $j_1 \le j_2 < \infty$ as soon as $j_1 + n > k_2$, because $\theta^n B \in \mathcal{A}(j_1+n, j_2+n)$ and for these n it holds that $\mathcal{A}(k_1,k_2)$ and $\mathcal{A}(j_1+n,j_2+n)$ are independent as μ is a product measure. Thus, A and $\theta^{-n}B$ are independent. Overall, it holds that

$$\lim_{n\to\infty} \mu(A \cap \theta^{-n}B) = \mu(A)\mu(B)$$

for all $A, B \in \mathcal{E} := \bigcup_{k,l\ne\pm\infty} \mathcal{A}(k,l)$. Next, we establish that $\mathcal{E}$ is closed under finite intersection. This is due to the fact, that if $A \in \mathcal{A}(k_1,k_2)$ and $B \in \mathcal{A}(j_1,j_2)$, then we have $A \cap B \in \mathcal{A}(\min(k_1,j_1), \max(k_2,j_2))$, which implies $A \cap B \in \mathcal{E}$.

Next, we show that the set $\mathcal{D}$ defined by

$$\mathcal{D} := \Big\{A \in \mathcal{A} \;\Big|\; \lim_{n\to\infty} \mu(A \cap \theta^{-n}B) = \frac{\mu(A)\mu(B)}{\mu(\Omega)} \text{ for all } B \in \mathcal{E}\Big\}$$

forms a Dynkin system. This is due to the fact that $\Omega \in \mathcal{D}$ as $\mu(\Omega \cap \theta^{-n}B) = \mu(\theta^{-n}B) = \mu(B)$ for all $B \in \mathcal{E}$. Furthermore,

$$\begin{aligned}\lim_{n\to\infty} \mu(A^c \cap \theta^{-n}B) &= \mu(\theta^{-n}B) - \lim_{n\to\infty} \mu(A \cap \theta^{-n}B) = \mu(B) - \frac{\mu(A)\mu(B)}{\mu(\Omega)} \\ &= (\mu(\Omega) - \mu(A))\frac{\mu(B)}{\mu(\Omega)} = \frac{\mu(A^c)\mu(B)}{\mu(\Omega)}\end{aligned}$$

for all $A \in \mathcal{D}$ and $B \in \mathcal{E}$. Finally for a family $(A_i)_{i\in\mathbb{N}}$ of disjoint elements of $\mathcal{D}$ and $B \in \mathcal{E}$ we have

$$\lim_{n\to\infty} \mu\Big(\bigcup_{i\in\mathbb{N}} A_i \cap \theta^{-n}B\Big) = \sum_{i\in\mathbb{N}} \lim_{n\to\infty} \mu(A_i \cap B) = \sum_{i\in\mathbb{N}} \mu(A_i)\frac{\mu(B)}{\mu(\Omega)} = \mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big)\frac{\mu(B)}{\mu(\Omega)}.$$

Thus, $\mathcal{D}$ forms indeed a Dynkin system. Now, we know that $\mathcal{E} \subset \mathcal{D} \subset \mathcal{A}$ and on the other hand, since $\mathcal{E}$ is closed under finite intersections and $\mathcal{A}$ is generated by the cylinder sets which are elements of the σ-algebras $\mathcal{A}(k,l), k,l \neq \pm\infty$ and hence is generated by $\mathcal{E}$ we see that

$$\mathcal{A} = \sigma(\mathcal{E}) = \delta(\mathcal{E}) \subset \mathcal{D},$$

where $\sigma(\mathcal{E})$ and $\delta(\mathcal{E})$ denote the σ-Algebra and the Dynkin-system generated by $\mathcal{E}$, respectively.

By the same arguments as before, we obtain that the set $\mathcal{D}'$ defined by

$$\mathcal{D}' := \Big\{B \in \mathcal{A} \;\Big|\; \lim_{n\to\infty} \mu(A \cap \theta^{-n}B) = \frac{\mu(A)\mu(B)}{\mu(\Omega)} \text{ for all } A \in \mathcal{A}\Big\}$$

forms a Dynkin-System. Furthermore, $\mathcal{E} \subset \mathcal{D}' \subset \mathcal{A}$, which leads as before to

$$\mathcal{A} = \sigma(\mathcal{E}) = \delta(\mathcal{E}) \subset \mathcal{D}',$$

which completes the proof. □

Remark 6.3.6 (Relation to Kolmogorov's strong law of large numbers)**.** Coming back to the setting of the introduction in Section 6.1, let us consider an independent and identically distributed family of E-valued random variables $Y = (Y_k)_{k\in\mathbb{N}}$. Then, Lemma 6.3.5 and Lemma 6.2.14 show ergodicity and thus we obtain by Birkhoff's ergodic theorem 6.2.19 applied to the endomorphism θ on $(E^{\mathbb{Z}}, \mathcal{F}^{\mathbb{Z}}, P \circ Y^{-1})$ the statement of Kolmogorov's strong law of large numbers.

Yet, note that Birkhoff's ergodic theorem is still applicable if we merely have that $P \circ Y^{-1}$ is θ-invariant, for which independence is not necessary. Since in this case it holds that $P \circ Y^{-1} = P \circ Y^{-1} \circ \theta^{-1} = P \circ (\theta \circ Y)^{-1}$, we introduce the following notation.

Definition 6.3.7 (Stationary process)**.** An E-valued stochastic process $Y = (Y_k)_{k\in\mathbb{Z}}$ is called *stationary* if Y and $\theta \circ Y$ have the same distribution, i.e. θ is an endomorphism on the probability space $(E^{\mathbb{Z}}, \mathcal{F}^{\mathbb{Z}}, P \circ Y^{-1})$.

The next proposition considers the converse, i.e. endomorphisms generating a stationary process.

Proposition 6.3.8 ([94, Chapter 1, Proposition 4.1])**.** *Let τ be an endomorphism on a probability space $(\Omega, \mathcal{A}, \mu)$, $(\tilde{E}, \tilde{F})$ a measurable space and $f : \Omega \to \tilde{E}$ a measurable function. Then the sequence $(Z_k)_{k\in\mathbb{N}}$ given by $Z_k = f \circ \tau^k$ is a stationary process.*

In the context of processes it is more convenient to express invariance and ergodicity with respect to the process.

Definition 6.3.9 (Invariant sets and ergodicity of processes)**.** If $Y = (Y_k)_{k\in\mathbb{N}}$ is defined on $(\Omega', \mathcal{A}', P)$ we call $B \in \mathcal{A}'$ *invariant* if there exists some $A \in \mathcal{A}$ such that

$$B = \{(Y_\ell, Y_{\ell+1}, \cdots) \in A\}$$

holds for all $\ell \in \mathbb{N}$. Note that this is equivalent to the existence of an $A^* \in \mathcal{A}$ with $B = Y^{-1}A^*$ and $\theta^{-1}A^* = A^*$. We denote the σ-algebra of all invariant sets by $\mathcal{I}'$.

Y is called *ergodic* if any invariant set $B \in \mathcal{A}'$ satisfies $P(B) = 0$ or $P(B^c) = 0$.

Furthermore, it can be shown that ergodicity of stationary processes is preserved by measurable maps.

Proposition 6.3.10 ([94, Chapter 1, Proposition 4.3])**.** *If $Y = (Y_i)_{i\in\mathbb{N}}$ is stationary and ergodic and $f : E^{\mathbb{N}} \to \tilde{E}$ is measurable, then $\tilde{Y} = (\tilde{Y}_i)_{i\in\mathbb{N}}$ defined by $\tilde{Y}_i = f(Y_i, Y_{i+1}, \dots)$ is ergodic.*

Finally, we can reformulate Birkhoff's ergodic theorem in the context of stationary processes. For the proof one may argue as in Remark 6.3.6.

Theorem 6.3.11 (Birkhoff's ergodic theorem for stationary processes [94, Chapter 1, Proposition 4.4])**.** *If $Y = (Y_k)_{k\in\mathbb{N}}$ is a stationary real valued process and Y_1 is integrable, then*

$$\lim_{k\to\infty} k^{-1} \sum_{\ell=1}^{k} Y_\ell = E(Y_1 \,|\, \mathcal{I}').$$

If Y is ergodic, then the limit is given bv $E(Y_1) = E(Y_1 \,|\, \mathcal{I}')$.

6.3.2 Homogenization of randomly layered materials with rigid components in crystal plasticity

First, let us revisit Proposition 6.1.3. Replacing the application of Kolmogorov's strong law of large numbers by Birkhoff's ergodic theorem in the formulation for stationary processes in Theorem 6.3.11, we obtain the following proposition.

Proposition 6.3.12 (Weak convergence via ergodic stationary processes in one variable)**.** *Let $f \in L^\infty(\mathbb{R}, L^1(\Xi, \mathbb{P}; \mathbb{R}))$ with the property that the family of random variables $(f|_{(i,i+1]})_{i\in\mathbb{Z}}$ is ergodic and stationary in the sense of Definition 6.3.7. Then, the sequence $(f_\epsilon)_\epsilon$ that is given by $f_\epsilon(x, \omega) = f(\epsilon^{-1}x, \omega)$ satisfies*

$$f_\epsilon \overset{*}{\rightharpoonup} \mathbb{E}\Big(\int_0^1 f(x, \cdot\,)\,\mathrm{d}x \Big) \quad \text{in } L^\infty(\mathbb{R}) \text{ as } \epsilon \to 0, \text{ a.s.}$$

This result is the one-dimensional case of the following multi-dimensional weak convergence result. For the proof, Birkhoff's ergodic theorem has to be replaced by multiparameter ergodic theorems such as Tempel'man's Ergodic Theorem 6.2.26.

Proposition 6.3.13 (Weak convergence via ergodic stationary processes [22, Theorem 1.1])**.** *For $n \in \mathbb{N}$ let $\tau = (\tau_k)_{k\in\mathbb{Z}^n}$ be a group action of $(\mathbb{Z}^n, +)$ on the probability space $(\Xi, \mathcal{A}, \mathbb{P})$. Assume that τ is ergodic and let $f \in L^\infty(\mathbb{R}^n, L^1(\Xi, \mathbb{P}; \mathbb{R}))$ be stationary in the sense that for all $k \in \mathbb{Z}^n$ we have almost surely*

$$f(x + k, \omega) = f(x, \tau_k\omega) \quad \text{a.e.}$$

Then, the sequence $(f_\epsilon)_\epsilon$ that is given by $f_\epsilon(x, \omega) = f(\epsilon^{-1}x, \omega)$ satisfies

$$f_\epsilon \overset{*}{\rightharpoonup} \mathbb{E}\Big(\int_{(0,1)^n} f(x, \cdot\,)\,\mathrm{d}x \Big) \quad \text{in } L^\infty(\mathbb{R}) \text{ as } \epsilon \to 0, \text{ a.s.}$$

Now, using the convergence results via stationary processes from Proposition 6.3.12 in place of the results for independent and identically distributed processes from Proposition 6.1.3, we obtain the following homogenization result.

Theorem 6.3.14 (Homogenization of randomly layered materials with rigid layers)**.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded Lipschitz domain which is $e_n^\perp$-connected in the sense of Definition 3.3.6. For $\lambda > 0$ let $(\lambda_i)_{i\in\mathbb{Z}}$ be a stationary and ergodic process with $\lambda_i > \lambda$ for all $i \in \mathbb{Z}$. Then, the family of energy functionals $E_\epsilon^\infty : L_0^2(\Omega;\mathbb{R}^n) \times \Xi \to [0,\infty]$ converges almost surely in the sense of Γ-convergence with respect to the strong L^2-topology to a functional $E : L_0^2(\Omega;\mathbb{R}^2) \to [0,\infty]$ given for $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ a.e. in Ω, $R \in SO(2)$, $\gamma \in \mathbb{R}$ by*

$$E(u) = \frac{1}{\mathbb{E}(\lambda_i)} \int_\Omega \gamma^2 \,\mathrm{d}x,$$

and $E(u) = \infty$ otherwise in $L_0^2(\Omega;\mathbb{R}^2)$.

We end this chapter with a short overview of the key contributions on stochastic homogenization.

Key contributions. For a more extensive literature review on the topic of stochastic homogenization see [76].

A first major contribution to the stochastic homogenization of nonlinear integral functionals was given by Dal Maso and Modica [60] in 1986, which was complemented in the same year by a publication by the same authors [61] that first made use of ergodic theorem by Akcoglu-Krengel which we cited in Theorem 6.2.27, which was originally published in 1981 [2].

Following the recent progress of the late 80s on the homogenization of non-convex integral functionals, such as [27, 112], Messaoudi and Michaille established a first stochastic homogenization results for this class of functionals [108].

Notice that these publications concern the analysis of nonlinear integral functions, while large parts of the literature study partial differential equations with stochastic coefficient fields, starting with the work of Kozlov [93].

In recent years, the theory of stochstic homogenization gained new momentum through the contributions of the schools of Gloria and Otto [77, 75] and Armstrong and Scott [12]. These results concern optimal convergence rates, which are a peculiarity of the stochastic homogenization problems as these are trivial in the periodic case.

6.4 Appendix

Lemma 6.4.1 (A criterium for uniform integrability)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, i.e. $\mu(\Omega) < \infty$. Let $(f_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ with $f_k \geq 0$ be a uniformly bounded sequence with the property that for each $\epsilon > 0$ there exists a $K_\epsilon > 0$ such that for all $k \in \mathbb{N}$ it holds that $\int_\Omega (f_k - K_\epsilon)^+ \,\mathrm{d}\mu < \epsilon$. Then, $(f_k)_{k\in\mathbb{N}}$ is uniformly integrable.*

Proof. Let $\epsilon > 0$ and K_ϵ such that for all $k \in \mathbb{N}$ we have $\int_\Omega (f_k - K_\epsilon)^+ \,\mathrm{d}\mu < \epsilon$ and let $M > 0$ be the uniform bound on the norm of $(f_k)_{k\in\mathbb{N}}$, i.e. for all $k \in \mathbb{N}$ it holds that $\|f_k\|_{L^1(\Omega)} < M$. Note that we may assume that $K_\epsilon \to \infty$ as $\epsilon \to 0$ as otherwise, $f \in L^\infty(\Omega)$ and the claim follows immediately.

Observe that for $C \geq 1$

$$\begin{aligned}\int_{\{f_k > C^2\}} f_k \,\mathrm{d}\mu &= \int_{\{f_k > C^2\}} f_k - C \,\mathrm{d}\mu + \int_{\{f_k > C^2\}} C \,\mathrm{d}\mu \\ &\leq \int_{\{f_k > C\}} f_k - C \,\mathrm{d}\mu + \int_{\{f_k > C^2\}} C \,\mathrm{d}\mu = \int_{\{f_k > C\}} f_k - C \,\mathrm{d}\mu + C\mu(\{f_k > C^2\}),\end{aligned}$$

and since the Markov inequality yields

$$\mu(\{f_k > C^2\}) \leq \frac{1}{C}\|f_k\|_{L^1(\Omega)},$$

we have

$$\begin{aligned}\int_{\{f_k>C^2\}} f_k \,\mathrm{d}\mu &\leq \int_{\{f_k>C\}} f_k - C \,\mathrm{d}\mu + \frac{1}{C}\|f_k\|_{L^1(\Omega)}\\ &\leq \int_{\{f_k>C\}} f_k - C \,\mathrm{d}\mu + \frac{M}{C}.\end{aligned}$$

Thus, if we set $C = K_\epsilon$ and send ϵ towards 0, we find that $\int_{\{f_k>C^2\}} f_k \,\mathrm{d}\mu$ converges uniformly to zero. This is for finite measure μ equivalent to uniform integrability. $\square$

7

Outlook

We end this work with a short recapitulation on the result obtained, discussing related open questions and pointing out paths for future research.

The first result proven in this work was Theorem 1.1.1 which gives a full characterization of possible macroscopic deformations of layered materials with a sufficient stiff component. In particular, the question of optimal scaling relations between stiffness and layer thickness was answered. This result is tailored to layered geometries, for which it provides new insights where previous work often focused on stiff inclusions [32, 39] or stood in the context of linearized elasticity [19, 20].

Future work could concern more complicated geometries, with the decisive property being that the stiff components spans throughout the material, for example periodic lattices or branching structures.

Other applications of the techniques developed may be models for polycrystals with the stiff material modeling the grain boundary. Besides a more general geometry, this would also require a careful analysis of the thickness of the boundary. A basis for this research is provided in this work with the study of ϵ-dependent layer ratio for Theorem 1.1.1 in Chapter 3.

Building on the asymptotic characterization of Theorem 1.1.1, Theorem 1.1.5 provided an explicit homogenization formula for variational material models with stiff components, where the soft component is given by an energy density that satisfies on the one hand p-growth and Lipschitz conditions. From a technical point of view, recent progress, such as [10, 7], may allow to lessen these conditions. On the other hand, we required that the quasiconvex envelope coincides with the polyconvex envelope, an assumption also found in the context of relaxation with determinant constraints [48]. In our case, as mentioned in Chapter 4, this condition is related to the question if fine oscillations between different stiff layers allow energetically lower deformations of the soft layers. If the answer is positive, the theorem presented would not generalize to the case without the condition on the convex envelopes. Yet, at this point, this question remains open.

In the context of crystal plasticity, we presented with Theorem 1.1.6 a homogenization result that characterized the macroscopic behavior of a stiff material with one active slip system in every other layer in terms of Γ-convergence. For the model considered, a physically rational generalization would be to assume more than one slip system in every other layer. Yet, the limiting factor to these generalizations is the fact that except recent progress for certain geometries [52, 49], the corresponding relaxation formulas are not known. Another possible variation would be to still consider single slip systems, but with different slip directions.

We concluded this work with the study of randomly layered crystalline materials, where we established with Theorem 1.1.7 a homogenization result concerning the rigid-plastic idealization of the model considered in Theorem 1.1.6 but with randomly varying layer thickness. In particular we showed that in this special case the characterization by Theorem 1.1.6 essentially holds true. To obtain this result we assumed that the layer thickness is always larger than a fixed lower bound. This was due to the fact that the deterministic asymptotic characterization result still had to be applicable. The natural question is if the asymptotic characterization itself can be transferred to the context of random layer thickness. Again, a first step in this direction has been taken with the study of ϵ-dependent layer ratio for Theorem 1.1.1 in Chapter 3. Though such results are yet to be obtained, an ambitious vision would be applications in the context of fiber reinforced materials with highly aligned fibers.

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