The Curve Diffusion Flow with a Contact Angle

DISSERTATION ZUR ERLANGUNG DES DOKTORGRADES
DER NATURWISSENSCHAFTEN (DR. RER. NAT.)
AN DER FAKULTÄT FÜR MATHEMATIK
DER UNIVERSITÄT REGENSBURG

vorgelegt von

Julia Butz

aus Regensburg

im Jahr 2018

Promotionsgesuch eingereicht am: 04.04.2018

Die Arbeit wurde angeleitet von: Prof. Dr. Helmut Abels

Prüfungsausschuss:	Vorsitzender:	Prof. Dr. Bernd Ammann
	Erst-Gutachter:	Prof. Dr. Helmut Abels
	Zweit-Gutachter:	Prof. Dr. Anna Dall'Acqua
	weiterer Prüfer:	Prof. Dr. Harald Garcke
	Ersatzprüfer:	Prof. Dr. Georg Dolzmann

Abstract

We consider the evolution of open curves driven by curve diffusion flow. This geometric evolution equation arises in problems of phase separation in material science and is the one-dimensional analogue of the surface diffusion flow. The evolving family of curves has free boundary points, which are supported on a line and it has a fixed contact angle $\alpha \in (0, \pi)$ with that line. Moreover, it satisfies a no-flux condition.

First, we discuss a result on well-posedness locally in time for curves which can be described by a sufficiently small height function of class W_2^γ, $\gamma \in (\frac{3}{2}, 2]$, over a reference curve. In order to proof the result, we reduce the geometric evolution equation to a fourth order quasilinear, parabolic partial differential equation for the height function on a fixed interval. The proof of the well-posedness of this problem is based on a contraction mapping argument: A result on maximal L_p-regularity with temporal weights by Meyries and Schnaubelt enables us to solve the linearized problem with optimal regularity. By establishing multiplication results in time weighted anisotropic L_2-Sobolev spaces of low regularity, we can to show that the non-linearities are contractive for small times.

Furthermore, we show the existence of a suitable reference curve for every admissible initial curve: We smoothen the initial curve by evolving it by a parabolic equation. Afterwards, we establish conditions on the distance of two curves which guarantee that one curve can be used as a reference curve for the other one. By C_0-semigroup and interpolation theory, we confirm that the solution of the aforementioned parabolic equation is in fact a viable reference curve. Combining this with the first result, we obtain that the flow starts for every admissible initial curve of class W_2^γ, $\gamma \in (\frac{3}{2}, 2]$.

By exploiting this result, we can give a blow-up criterion in terms of a L_2-bound of the curvature: If a solution of the curve diffusion flow subject to the previously mentioned boundary conditions exists only for a maximal time $T_{max} < \infty$, then the L_2-norm of the curvature tends to ∞ as $t \to T_{max}$. For the proof, we assume, contrary to our claim, that the L_2-norm of the curvature remains bounded for a sequence in time approaching $T_{\max}$. A compactness argument combined with the short time existence result enables us to extend the flow beyond T_{max}, which contradicts the maximality of the solution.

Zusammenfassung

Wir betrachten offene Kurven, die durch den Kurvendiffusionsfluss evolviert werden. Diese geometrische Evolutionsgleichung tritt bei Phasenseparationsphänomenen in den Materialwissenschaften auf und ist das eindimensionale Analogon des Oberflächendiffusionsflusses. Die Familie evolvierender Kurven hat Randpunkte, die sich frei auf einer Linie bewegen und die Kurven bilden einen festen Winkel $\alpha \in (0, \pi)$ mit der Linie. Außerdem ist die Bogenlängenableitung der skalaren Krümmung der Kurven am Rand null.

Zunächst diskutieren wir lokale Wohlgestelltheit für Kurven, die durch eine genügend kleine Höhenfunktion der Klasse W_2^γ, $\gamma \in (\frac{3}{2}, 2]$, über einer Referenzkurve dargestellt werden können. Um das Resultat zu zeigen, reduzieren wir die geometrische Evolutionsgleichung auf eine quasilineare, parabolische partielle Differentialgleichung vierter Ordnung für die Höhenfunktion. Der Beweis dafür basiert auf dem Banachschen Fixpunktsatz: Eine Arbeit von Meyries und Schnaubelt über

maximale L_p-Regularität mit Zeitgewichten ermöglicht es, das linearisierte Problem mit optimaler Regularität zu lösen. Danach müssen Multiplikationsresultate in zeitgewichteten anisotropen L_2-Sobolevräumen mit niedriger Regularität hergeleitet werden, um zu beweisen, dass die auftretenden Nichtlineritäten für kleine Zeiten kontrahieren.

Darüber hinaus zeigen wir, dass geeignete Referenzkurven für sämtliche zulässige Anfangskurven existieren: Dazu glätten wir die Anfangskurven, indem wir sie mittels einer parabolischen Gleichung evolvieren. Danach finden wir Bedingungen an den Abstand der Kurven, die garantieren, dass eine Kurve als Referenzkurve der anderen dienen kann. Mit Hilfe von C_0-Halbgruppen und Interpolationstheorie kann bestätigt werden, dass die Kurven, die durch parabolische Regularisierung der Anfangskurve erzeugt wurden, als Referenzkurven genutzt werden können. Durch Kombination dieser Aussage und des Existenzresultats erhalten wir, dass der Fluss für alle zulässigen Anfangskurven der Klasse W_2^{γ}, $\gamma \in (\frac{3}{2}, 2]$, startet.

Unter Verwendung der vorherigen Ergebnisse gelingt es ein Blow-up-Kriterium anhand einer L_2-Schranke der Krümmung zu zeigen: Falls eine Lösung des Kurvendiffusionsflusses mit den zugehörigen Randbedingungen nur für eine maximale Zeit $T_{max} < \infty$ existiert, dann wird die L_2-Norm der Krümmung der Lösung unbeschränkt für $t \to T_{max}$. Um das zu beweisen, nimmt man umgekehrt an, dass die L_2-Norm der Krümmung für eine Folge von Zeitpunkten, welche gegen T_{max} konvergiert, beschränkt bleibt. Ein Kompaktheitsschluss erlaubt die Lösung durch das Kurzzeitexistenzresultat über T_{max} hinaus fortzusetzen. Dies steht im Widerspruch zur Maximalität der Lösung.

Acknowledgements

First and foremost, I want to express my special thanks to my supervisor Prof. Dr. Helmut Abels for giving me the opportunity to work on the interesting field of geometric evolution equations. I am grateful for the helpful discussions and that his door was open whenever I needed advice.

I would like to thank Prof. Dr. Harald Garcke for always being open for answering my questions and helping me gain a better understanding of curvature flows.

I want to thank Prof. Dr. Anna Dall'Acqua and Prof. Dr. Glen Wheeler for the interesting discussions on establishing a blow-up rate.

I am also grateful to Christopher Brand for numerous fruitful discussions.

Many thanks go to my office mate Andreas Marquardt for the enriching conversations about mathematics and for encouraging me.

I would like to thank all my colleagues at the chair for providing a good working atmosphere. I thank Fabian Christowiak, Michael Gößwein, Johannes Kampmann, Julia Menzel, Alessandra Pluda, Ph.D., and Dr. Mathias Wilke for their interest in my work and for the pleasant time.

I thank Johannes Stigloher for linguistic proofreading.

I gratefully acknowledge the financial support by the DFG graduate school GRK 1692 Curvature, Cycles, and Cohomology in Regensburg.

Contents

1 Introduction

Geometric evolution equations govern a large variety of models in different fields of science ranging from grain boundary motions and crystal growth to image analysis. For an overview on their applications, we refer to [14]. Yet, while their analysis has been an active field of mathematical research through the last decades, geometric evolution equations are still challenging, as singularities can often occur in finite time and the corresponding partial differential equations obtained after a suitable parametrization are quasilinear.

In this thesis, we contribute new results for a time dependent family of regular open curves $\{\Gamma_t\}_{t\geq 0}$ moving according to curve diffusion flow, i.e.

$$V = -\partial_{ss}\kappa_{\Gamma_t} \qquad \text{on } \Gamma_t, t > 0, \tag{CDF}$$

where V is the scalar normal velocity, κ_{Γ_t} is the scalar curvature of Γ_t, and s denotes the arc length parameter. We complement the evolution law with the boundary conditions

$$\begin{aligned} \partial\Gamma_t &\subset \mathbb{R}\times\{0\} && \text{for } t > 0, \\ \measuredangle\left(n_{\Gamma_t}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) &= \pi - \alpha && \text{at } \partial\Gamma_t \text{ for } t > 0, \\ \partial_s\kappa_{\Gamma_t} &= 0 && \text{at } \partial\Gamma_t \text{ for } t > 0, \end{aligned} \tag{BC}$$

where n_{Γ_t} is the unit normal vector of Γ_t and $\alpha \in (0, \pi)$. A sketch is given in Figure 1.1. The main goal of this work is to investigate the behavior of this curvature flow.

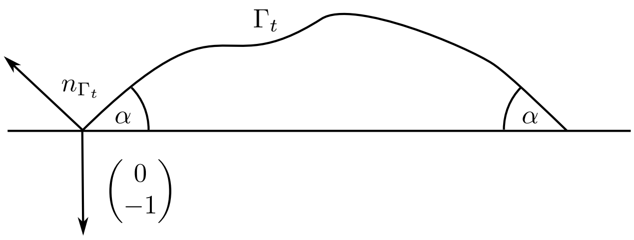

Figure 1.1: Evolution by curve diffusion flow with α-angle condition for $\alpha < \frac{\pi}{2}$.

The curve diffusion flow is the one-dimensional analogue to surface diffusion flow, which describes the motion of interfaces in the case that it is governed purely by diffusion within the interface. It was originally derived by Mullins to model the development of surface grooves at the grain boundaries of a heated polycrystal in 1957, see [26]. It turns out, that curve diffusion flow is the H^{-1}-gradient flow of the length of the curves, see [14]. Thus, the flow is clearly related to the mean curvature flow, which is the L_2-gradient flow of the length functional, cf. [14]. Mean curvature flow evolves curves by the law

$$V = \kappa \qquad \text{on } \Gamma_t, t > 0. \tag{MCF}$$

Although, both curvature flows decrease the length of curves, there are some significant differences in their behavior: For the mean curvature flow, Grayson showed that the curves evolving from a smooth embedded curve in the plane keep these properties and become convex in finite time, see [19].

Moreover, Gage and Hamilton proved that a convex curve in the plane, which is moving according to mean curvature flow, remains convex and shrinks to a round point, see [13]. Both statements are not true for the curve diffusion flow: For evolution by curve diffusion flow, Giga and Ito gave an example for an embedded initial curve such that the evolving curves fail to be embedded at some point, [17]. Additionally, they have proven that the flow does not preserve convexity in [18], like it was conjectured by Garcke and Elliot in [11]. These properties of the curve diffusion flow reflect that the evolution law (CDF) leads to a forth order parabolic equation, while (MCF) corresponds to a second order parabolic equation: The proofs of the mentioned results on the mean curvature flow are based on the maximum principle, which is not available for fourth order equations.

However, in contrast to the mean curvature flow, the curve diffusion flow preserves the signed area which is enclosed by the initial curve during the flow, see [10]. This makes curve diffusion flow also attractive for applications, as conservation laws can result in a preservation of volume over time. For instance, the flow is related to the Cahn-Hilliard equation for a degenerate mobility. This equation arises in material science and models the phase separation of a binary alloy, which separates and forms domains mainly filled by a single component. Formal asymptotic expansions suggest that surface diffusion flow is the singular limit of the Cahn-Hilliard equation with a degenerate mobility for the case that the interfacial layer does not intersect the boundary of the domain, see [5]. Garcke and Novick-Cohen considered also the situation of an intersection of the interfacial layer with the external boundary and identified formally the sharp interface model, where the interfaces evolve in the two-dimensional case according to (CDF) and subject to an attachment condition, a $\frac{\pi}{2}$-angle condition, and a no flux-condition, see [16]. This is related to the subject of this thesis in the case $\alpha = \frac{\pi}{2}$.

Even though Garcke, Ito, and Kohsaka proved a global existence result for initial data sufficiently close to an equilibrium for a $\frac{\pi}{2}$-angle condition in [15], Escher, Mayer, and Simonett gave numerical evidence that closed curves in the plane, which are moving according to (CDF) can develop singularities in finite time, cf. [12]. Indeed, for smooth closed curves driven by (CDF), Chou provided a sharp criterion for a finite lifespan of the flow in [6]. Additionally, Chou, see [6], and Dzuik, Kuwert, and Schätzle, see [10], showed that if a solution has a maximal lifespan $T_{max} < \infty$, then the L_2-norm of the curvature with respect to the arc length parameter tends to infinity as T_{max} is approached. Moreover, they gave a rate for the blow-up.

In this thesis, we want to establish a blow-up criterion for the geometric evolution equation (CDF) with (BC). More precisely, we will show that if a solution has a maximal lifespan $T_{max} < \infty$, then the L_2-norm of its curvature with respect to the arc length parameter tends to infinity as t approaches T_{max}, see Theorem 4.1.4.

Since our strategy of the proof is inspired by the blow-up criterion in [10], we will explain the cited result in the following: By assumption, $f : [0, T_{max}) \times \mathbb{S}^1 \to \mathbb{R}^n$, $T_{max} < \infty$, is a smooth solution of CDF for closed curves, which cannot be extended in time. The authors assume, contrary to their claim, that $\|\kappa(t)\|_{L_2}$ is uniformly bounded in $t < T_{max}$. Here $\|\cdot\|_{L^2}$ denotes the L_2-norm with respect to the arc length parameter. Carrying on, they consider the normal component of the derivative, i.e.

$$\nabla_s \vec{\kappa} := \partial_s \vec{\kappa} - \langle \partial_s \vec{\kappa}, \tau \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the euclidean inner product on $\mathbb{R}^n$ and τ is the unit tangent vector. Using the motion law, they obtain differential inequalities for $\|\nabla_s^m \vec{\kappa}(t)\|_{L_2}$ for all $t < T_{max}$ and for all $m \in \mathbb{N}$. By the curvature bound, they iteratively establish bounds on $\|\nabla_s^m \vec{\kappa}(t)\|_{L_2}$ by Gagliardo-Nirenberg-type inequalities. Comparing the full arc length derivatives $\partial_s^m \vec{\kappa}(t)$ to the projected ones $\nabla_s^m \vec{\kappa}(t)$, they can prove bounds on the L_2-norms of the full spatial derivatives of the curvature vector for all $t < T_{max}$. This permits for an extension the flow beyond T_{max}, which contradicts the maximality of the solution.

The authors use the same approach also in another context: Considering the L_2-gradient flow of an energy which provides a bound on $\|\kappa(t)\|_{L_2}$ the strategy allows for proving global existence of solutions and subconvergence results as $t \to \infty$, i.e. convergence of a subsequence. In [10], Dziuk, Kuwert, and Schätzle also inspect the bending energy with length penalization for closed curves, which is for a smooth, regular $f : \mathbb{S}^1 \to \mathbb{R}^n$, $n \geq 2$ given by

$$\mathcal{B}[f] := \int_{\mathbb{S}^1} \left(\frac{1}{2} |\vec{\kappa}|^2 + \lambda \right) ds \qquad \text{for } \lambda \in \mathbb{R}. \tag{B}$$

They obtain by the previously described technique that for smooth, regular initial data the L_2-gradient flow of (B) with $\lambda \in \mathbb{R}_0^+$ has a smooth global solution. In the case $\lambda > 0$, they deduce that it subconverges to an equilibrium after reparametrization to arc length and a suitable translation. Moreover, they give an analogous result for the L_2-gradient flow of (B) for $\lambda = 0$ with a length constraint.

The same strategy was also adapted to the case of open curves: In [21], [8], and [7], the authors consider L_2-gradient flows of the bending energy, either with length penalization or with length constraint, for open curves. For a smooth, regular function $f : I \to \mathbb{R}^n$, $n \geq 2$, I a closed bounded interval, they look for different parameters ξ and λ at the energy

$$\mathcal{E}[f] := \int_I \left(\frac{1}{2} |\vec{\kappa} - \xi|^2 + \lambda \right) ds \qquad \text{for } \xi \in \mathbb{R}^n \text{ and } \lambda \in \mathbb{R}. \tag{E}$$

Here, the vector ξ is called spontaneous curvature, see [8].

Lin proved a global existence result for the L^2-flow of (E) for fixed $\lambda \in \mathbb{R}^+$ and $\xi = 0$ for open curves with clamped boundary conditions in [21]. The gradient flow is considered among curves with fixed boundary points and fixed tangent vectors at the boundary points. Additionally, the initial datum is supposed to be smooth with positive, finite length and satisfying certain compatibility conditions. Again, it is assumed that $f : [0, T_{max}) \times I \to \mathbb{R}^n$, $T_{max} < \infty$, is a maximal, smooth solution. The author gains control over $\nabla_t^m f(t)$ for all $t < T_{max}$, instead of $\nabla_s^m \vec{\kappa}(t)$ as in [10], where

$$\nabla_t f := \partial_t f - \langle \partial_t f, \tau \rangle.$$

He obtains bounds on $\|\nabla_t^m f(t)\|_{L_2}$ for all $m \in \mathbb{N}$ in terms of $\nabla_s^p \vec{\kappa}(t)$, $p \in \mathbb{N}$, for all $t < T_{max}$. In contrast to the setting in [10], attention has to be paid to the boundary terms, which occur due to integration by parts. Thus, the quantities $\nabla_t^m f(t)$ are a clever choice, as they vanish at the boundary points due to the boundary conditions. Additionally, from the global existence of the flow it is deduced that the family of curves subconverges after reparametrization by arc length to an equilibrium. Dall'Acqua, Pozzi, and Spener strengthened the result of Lin in [21] by showing that up to a time dependent reparametrization $\phi(t, \cdot) : I \to I$, $t \in [0, \infty)$, the whole solution $f(t, \phi(t, \cdot))$ converges to a critical point of the energy in L_2 for $t \to \infty$, see [9].

In [8], Dall'Acqua and Pozzi proved a global existence and subconvergence result for the L^2-flow of the energy (E) for $\lambda \in \mathbb{R}_0^+$ and $\xi \in \mathbb{R}^n$, with fixed boundary points and such that the curvature vector equals the normal component of ξ at the boundary. In this setting, the authors also control the quantities $\nabla_t^m f(t)$, but those do not vanish at the boundary and thus have to be analyzed carefully.

Moreover, Dall'Acqua, Lin, and Pozzi obtained an analogous result for the L^2-flow of (E) with $\xi = 0$, which is complemented with hinged boundary conditions, i.e. fixed boundary points and zero curvature at the boundary points, see [7]. Additionally, a special time dependent λ is chosen to preserve the length of the curve during the flow.

The strategy of the proof of the blow-up criterion Theorem 4.1.4 in our case is similar: For a given maximal solution $f : [0, T_{max}) \times \bar{I} \to \mathbb{R}^2$, $I := (0, 1)$, $T_{max} < \infty$, we assume, contrary to the claim, that $\|\kappa(t_l)\|_{L_2}$ is uniformly bounded for a sequence $(t_l)_{l\in\mathbb{N}}$. In contrast to the reasoning in [10] we do not work with smooth solutions: For $\alpha \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ the considered flow also has a tangential component, as the boundary points are attached to the x-axis during the flow. This makes the control of the boundary terms complicated. By just using the bound on the curvature, we obtain a uniform W_2^2-bound for $\tilde{f}_l$, which denotes the reparametrized and translated solution at time t_l. This motivates us to consider solutions in the space $W_2^1((0, T_{max}); L_2(I; \mathbb{R}^2)) \cap L_2((0, T_{max}); W_2^4(I; \mathbb{R}^2))$, as its temporal trace space is $W_2^2(I; \mathbb{R}^2))$. The idea is to restart the flow at these times t_l in order to extend the flow beyond T_{max}, since this contradicts the maximality of the solution. But to achieve this, we need a uniform lower bound on the existence time of the solutions.

We pursue the following strategy: We represent the initial curves $\tilde{f}_l$ by height functions over suitable reference curves Φ_l^*. By deriving the equation for the evolution of the height functions corresponding to (CDF) and (BC), we obtain a short time existence result such that the time of existence is determined by the reference curve and the norm of the initial height function. Consequently, we obtain a uniform lower bound on the existence time, if we can reduce to the case of finitely many reference curves.

To this end, we observe that by the compact embedding $W_2^2(I; \mathbb{R}^2) \hookrightarrow W_2^\gamma(I; \mathbb{R}^2)$, $\gamma < 2$, the set $\{\tilde{f}_l : l \in \mathbb{N}\}$ is precompact in $W_2^\gamma(I; \mathbb{R}^2)$. Next, we cover this set by W_2^γ-balls around the initial curves $\tilde{f}_l$. By compactness, there exists a finite set of balls, corresponding to finitely many reference curves, such that $\{\tilde{f}_l : l \in \mathbb{N}\}$ is still covered. Thus, we need to be able to restart the flow for initial height functions merely of class $W_2^\gamma(I)$, $\gamma < 2$, and to come back to the regularity of the original solution space.

To this end, we use the time weighted parabolic space $W_{2,\mu}^1((0, T); L_2(I)) \cap L_{2,\mu}((0, T); W_2^4(I))$ as solution space, where for a Banach space E

$$L_{2,\mu}((0, T); E) := \left\{ u : (0, T) \to E \text{ is strongly measurable} : \left\| \left[t \mapsto t^{1-\mu} u(t) \right] \right\|_{L_2((0,T);E)} < \infty \right\},$$

and $W_{2,\mu}^1((0, T); E)$ is defined accordingly, see Chapter 2. This setting allows for starting the flow for admissible initial data in the space $W_2^{4(\mu - 1/2)}(I; \mathbb{R}^2)$, $\mu \in (\frac{7}{8}, 1]$. The time weight μ is chosen such that

$$W_2^{4(\mu-1/2)}(I; \mathbb{R}^2) \hookrightarrow C^1(\bar{I}; \mathbb{R}^2).$$

Note that

$$W_2^{4(\mu-1/2)}(I; \mathbb{R}^2) \not\hookrightarrow C^2(\bar{I}; \mathbb{R}^2) \qquad \text{for } \mu \in \left(\frac{7}{8}, 1\right].$$

Additionally, the solution space allows for exploiting the effect of parabolic smoothing, as the time weight does not play a role away from zero.

A related result for the curve diffusion flow with a free boundary was proven by Wheeler and Wheeler in [32]: They consider immersed curves supported on two parallel lines moving according to curve diffusion flow, such that the evolving curves are orthogonal to the boundary and satisfy a no flux condition. A blow-up criterion is given in terms of the sum of the position vector and the L_2-norm of the arc length derivative of the curvature. Moreover, they establish criteria for global existence of the flow.

Even though a short time existence result for (CDF) and (BC) in the case $\alpha = \frac{\pi}{2}$ has been established for initial data of class $C^{4+\alpha}$, see [16], it is not sufficient here due to the needed low initial regularity. In order to prove the previously mentioned short time existence result for initial data in $W_2^{4(\mu-1/2)}(I)$, $\mu \in (\frac{7}{8}, 1]$, we will pursue the following strategy: We consider a fixed reference curve $\Phi^* : \bar{I} \to \mathbb{R}^2$ of class C^5, which fulfills suitable boundary conditions, and fixed curvilinear coordinates. Thus, we can represent evolving curves which are C^1-close to the reference curve by a height function. This enables us to reduce the geometric evolution equation (CDF) and (BC) to a quasilinear fourth order parabolic partial differential equation on a fixed interval for the height function $\rho : [0,T) \times \bar{I} \to \mathbb{R}$, as long as the height function is small enough. We give a sketch of the situation in Figure 1.2.

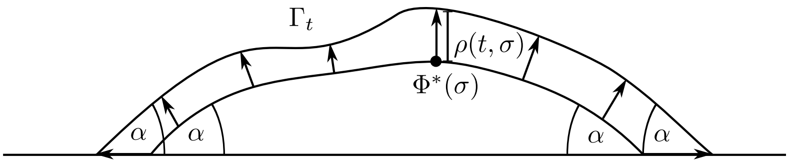

Figure 1.2: Representation of a curve by a reference curve $\Phi*$, curvilinear coordinates, and a height function $\rho(t,\sigma)$.

The equations are of the form

$$\begin{aligned} \rho_t + a(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^4\rho &= f(\rho,\partial_\sigma\rho,\partial_\sigma^2\rho,\partial_\sigma^3\rho) && \text{for } (t,\sigma) \in (0,T)\times[0,1], \\ b_1(\sigma)\partial_\sigma\rho &= 0 && \text{for } \sigma \in \{0,1\} \text{ and } t \in (0,T), \\ b_2(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho &= -g_2(\rho,\partial_\sigma\rho,\partial_\sigma^2\rho) && \text{for } \sigma \in \{0,1\} \text{ and } t \in (0,T), \\ \rho|_{t=0} &= \rho_0 && \text{in } [0,1], \end{aligned} \tag{PDE}$$

where $\rho_0 : [0,1] \to \mathbb{R}$ is the height function corresponding to the initial curve. Moreover, the non-linearities on the right-hand side have the structure

$$\begin{aligned} f(\sigma,\rho,\partial_\sigma\rho,\partial_\sigma^2\rho,\partial_\sigma^3\rho) &:= S(\rho)\partial_\sigma^3\rho\partial_\sigma^2\rho + S(\rho)\partial_\sigma^3\rho + S(\rho)\left(\partial_\sigma^2\rho\right)^3 + S(\rho)\left(\partial_\sigma^2\rho\right)^2 + S(\rho)\partial_\sigma^2\rho + S(\rho), \\ g_2(\sigma,\rho,\partial_\sigma\rho,\partial_\sigma^2\rho) &:= T(\rho)\left(\partial_\sigma^2\rho\right)^2 + T(\rho)\partial_\sigma^2\rho + T(\rho), \end{aligned} \tag{NL}$$

where the prefactors $S(\rho) = S(\sigma,\rho,\partial_\sigma\rho)$ and $T(\rho) = T(\sigma,\rho,\partial_\sigma\rho)$ are terms of lower order.

The equation (PDE) is now solved by a contraction mapping argument: We linearize the equation around the initial datum and obtain the problem

$$\mathcal{L}(\rho) = (\mathcal{F}(\rho),\rho_0), \tag{LP}$$

for a linear operator $\mathcal{L}$ which is given by the left-hand side of (PDE), where the coefficient are evaluated at ρ_0 instead of ρ. The non-linear operator $\mathcal{F}$ corresponds to the sum of the left-hand side terms of equation (PDE) and the terms which have to be compensated to make the problem (LP) equivalent to (PDE). In order to derive a fixed point equation, we have to solve the linear problem for the right-hand side $(\mathcal{F}(\bar{\rho}),\rho_0)$ with optimal regularity, where $\bar{\rho}$ is an arbitrary element of a suitable ball in the solution space $W^1_{2,\mu}((0,T);L_2(I)) \cap L_{2,\mu}((0,T);W^4_2(I))$. The key ingredient for this step is a result by Meyries and Schnaubelt on maximal L_p-regularity with temporal weights, [25]: We deduce that for admissible initial data in $W_2^{4(\mu-1/2)}(I)$, $\mu \in (\frac{7}{8},1]$, there exists a unique solution of the linear problem in $W^1_{2,\mu}((0,T);L_2(I) \cap L_{2,\mu}((0,T);W^4_2(I))$ for a sufficiently small $T>0$. By inverting the linear operator, we can express the partial differential equation (PDE) by the fixed point equation

$$\rho = \mathcal{L}^{-1}(\mathcal{F}(\rho),\rho_0). \tag{FP}$$

In order to show the existence of a unique fixed point by Banach's fixed point theorem, it is crucial to study the structure of the non-linearities, cf. (NL). We have to establish suitable product estimates in time weighted anisotropic L_2-Sobolev spaces of low regularity to guarantee that $\mathcal{F}(\rho)$

is contractive for a suitably small time $T > 0$. Finally, we deduce by applying Banach's fixed point theorem to (FP) that for every admissible initial height function there exists a $T > 0$ such that there exists a solution to (PDE) in $W^1_{2,\mu}((0,T); L_2(I)) \cap L_{2,\mu}((0,T); W^4_2(I))$. Moreover, the curve corresponding to the height function is regular. In order to use this result for the blow up criterion, we have to keep track on which quantities T depends.

In the next step, we have to assure that for every admissible regular initial curve there exists a suitable reference curve. By evolving the initial curve by a linear parabolic equation, we obtain a smoothened curve close to the initial curve. In the following, we use C_0-semigroup and interpolation theory to carry out technical estimates, which provide control on the distance of the two curves. Moreover, we find conditions on the distance of two curves which guarantee that one curve is a reference curve of the other one. Combining those steps enables us to confirm that the solution of the aforementioned parabolic equation is in fact a viable reference curve.

Lastly, we give a brief overview concerning the structure of this thesis. In Chapter 2, we present preliminary results on fractional Sobolev spaces, maximal L_2-regularity, and a geometrical estimate. We prove embeddings and multiplication results, which are crucial for the proof of the theorem on short time existence. In Chapter 3, we introduce the general setting of the geometric problem of curve diffusion flow with an angle condition. Additionally, we give some properties of smooth solutions of the problem. The main results are stated in Chapter 4: We present a local well-posedness result for rough initial data in $W_2^{4(\mu-1/2)}(I)$ for $\mu \in (\frac{7}{8}, 1]$, cf. Theorem 4.1.3. Secondly, we give a blow-up criterion for solutions to the curve diffusion flow which only exists for a finite time, see Theorem 4.1.4. The corresponding proofs are done in the following sections: Chapter 5 is devoted to the proof of a short time existence result for initial curves which are represented via a sufficiently small height function over a reference curve. Afterwards, we construct reference curves to general admissible initial data in Chapter 6. Finally, we give a proof of the blow-up criterion Theorem 4.1.4 in Chapter 7.

2 Preliminaries and Fundamental Mathematical Tools

In this chapter, we want to present some preliminary results on fractional Sobolev spaces and maximal L_2-regularity. Additionally, a geometrical estimate is given.

In Section 2.1, we introduce fractional Sobolev spaces and some useful properties. Moreover, we prove an embedding theorem for these spaces. This will allow to deduce product estimates.

In Section 2.2, we present a result on maximal regularity with temporal weights, see [25], which will be an important tool to prove short time existence. Afterwards we discuss embeddings of the involved parabolic spaces.

In the last section of this Chapter 2.3, we prove an upper bound for the reciprocal length of the curve by its maximal curvature.

2.1 Fractional Sobolev Spaces and Some Properties

The following spaces will be crucial for our setting. In large part the facts and definitions stated in the first part of this section are derived in [24]. For more results about these spaces, e.g. dense subsets, extension operators, trace theorems and embeddings, the reader is referred to [23] and [24]. General properties of real and complex interpolation theory can be found in [22] or [30].

Definition 2.1.1 (Weighted Lebesgue Space)
Let $J = (0,T)$, $0 < T \leq \infty$ and E be a Banach space. For $1 < p < \infty$ and $\mu \in \left(\frac{1}{p}, 1\right]$ the ***weighted Lebesgue space*** *is given by*

$$L_{p,\mu}(J;E) := \left\{u : J \to E \text{ is strongly measurable} : \|u\|_{L_{p,\mu}(J;E)} < \infty\right\},$$

where

$$\|u\|_{L_{p,\mu}(J;E)} := \left\| \left[t \mapsto t^{1-\mu}u(t)\right] \right\|_{L_p(J;E)} = \left(\int_J t^{(1-\mu)p} \|u(t)\|_E^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

Remark 2.1.2 *1. $(L_{p,\mu}(J;E), \|u\|_{L_{p,\mu}(J;E)})$ is a Banach space.*

2. One easily sees that for $T < \infty$ it follows

$$L_p(J;E) \hookrightarrow L_{p,\mu}(J;E).$$

This does not hold true for $T = \infty$.

3. We have $L_{p,\mu}((0,T);E) \subset L_p((\tau,T);E)$ for $\tau \in (0,T)$.

4. For $\mu = 1$ it holds $L_{p,1}(J;E) = L_p(J;E)$.

Moreover, we define associated weighted Sobolev spaces.

Definition 2.1.3 (Weighted Sobolev Space)
Let $J=(0,T)$, $0<T\leq\infty$ and E be a Banach space. For $1\leq p<\infty$, $k\in\mathbb{N}_0$, and $\mu\in\left(\frac{1}{p},1\right]$ the ***weighted Sobolev space*** *is given by*

$$W^k_{p,\mu}(J;E)=H^k_{p,\mu}(J;E):=\left\{u\in W^k_{1,loc}(J;E):u^{(j)}\in L_{p,\mu}(J;E)\text{ for }\{0,\cdots,k\}\right\}$$

for $k\neq 0$, where $u^{(j)}:=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j u$, and we set $W^0_{p,\mu}(J;E):=L_{p,\mu}(J;E)$. We equip it with the norm

$$\|u\|_{W^k_{p,\mu}(J;E)}:=\left(\sum_{j=0}^k\left\|u^{(j)}\right\|^p_{L_{p,\mu}(J;E)}\right)^{\frac{1}{p}}.$$

Remark 2.1.4 *$(W^k_{p,\mu}(J;E),\|u\|_{W^k_{p,\mu}(J;E)})$ is a Banach space, see Theorem in Section 3.2.2 of [30].*

In the following, we introduce a generalization of the usual Sobolev spaces by the means of interpolation theory. By $(\cdot,\cdot)_{\theta,p}$ and $(\cdot,\cdot)_{[\theta]}$ we denote real and complex interpolation functor, respectively.

Definition 2.1.5 (Weighted Slobodetskii Space, Weighted Bessel Potential Space)
Let $J=(0,T)$, $0<T\leq\infty$ and E be a Banach space. For $1\leq p<\infty$, $s\in\mathbb{R}^+\backslash\mathbb{N}$, and $\mu\in\left(\frac{1}{p},1\right]$ the ***weighted Slobodetskii space*** *and the* ***weighted Bessel potential space****, respectively, are given by*

$$\begin{aligned}W^s_{p,\mu}(J;E)&:=\left(W^{\lfloor s\rfloor}_{p,\mu}(J;E),W^{\lfloor s\rfloor+1}_{p,\mu}(J;E)\right)_{s-\lfloor s\rfloor,p},\\ H^s_{p,\mu}(J;E)&:=\left(W^{\lfloor s\rfloor}_{p,\mu}(J;E),W^{\lfloor s\rfloor+1}_{p,\mu}(J;E)\right)_{[s-\lfloor s\rfloor]}.\end{aligned}$$

Remark 2.1.6

1. *Both spaces are Banach spaces by interpolation theory, cf. Proposition 1.2.4 in [22] and the Theorem in 1.9.1 in [30].*
2. *We have $W^s_{p,1}(J;E)=W^s_p(J;E)$ and $H^s_{p,1}(J;E)=H^s_p(J;E)$ for all $s\geq 0$.*
3. *For $p\in(1,\infty)$ we obtain by Lemma 2.1 in [24] that the trace $u\mapsto u^{(j)}(0)$ is continuous from $W^k_{p,\mu}(J;E)$ to E for all $j\in\{0,\dots,k-1\}$. Thus, for $k\in\mathbb{N}$ we can define*

   $${}_0W^k_{p,\mu}(J;E)={}_0H^k_{p,\mu}(J;E):=\left\{u\in W^k_{p,\mu}(J;E):u^{(j)}(0)=0\text{ for }j\in\{0,\dots,k-1\}\right\},$$

   which are Banach spaces with the norm of $W^k_{p,\mu}(J;E)$. Moreover, we set for convenience

   $${}_0W^0_{p,\mu}(J;E)={}_0H^0_{p,\mu}(J;E):=L_{p,\mu}(J;E).$$

4. *By Proposition 2.10 in [24], it follows for $k+1-\mu+1/p<s<k+2-\mu+1/p$ with $k\in\mathbb{N}_0$*

   $$\begin{aligned}W^s_{p,\mu}(J;E)&\hookrightarrow BUC^k(\bar{J};E),\\ H^s_{p,\mu}(J;E)&\hookrightarrow BUC^k(\bar{J};E).\end{aligned}$$

   If additionally one replaces the spaces $W^s_{p,\mu}(J;E)$ and $H^s_{p,\mu}(J;E)$ by the ${}_0W^s_{p,\mu}(J;E)$ and ${}_0H^s_{p,\mu}(J;E)$, respectively, and $s\in[0,2]$, then the operator norms of the embeddings do not depend on J.

5. *We can define the corresponding fractional order spaces* ${}_0W^s_{p,\mu}(J;E)$ *and* ${}_0H^s_{p,\mu}(J;E)$ *analogously to Definition 2.1.5. By Proposition 2.10 in [24], we have for* $p \in (1,\infty)$ *the characterization*

$$
\begin{aligned}
{}_0W^s_{p,\mu}(J;E) &= \left\{u \in W^s_{p,\mu}(J;E) : u^{(j)}(0) = 0 \text{ for } j \in \{0,\dots,k\}\right\},\\
{}_0H^s_{p,\mu}(J;E) &= \left\{u \in W^s_{p,\mu}(J;E) : u^{(j)}(0) = 0 \text{ for } j \in \{0,\dots,k\}\right\},
\end{aligned}
$$

if $k+1-\mu+1/p < s < k+1+(1-\mu+1/p)$, $k \in \mathbb{N}_0$.

6. *By equation (2.7) and (2.8) in [24], we have for* $s = \lfloor s \rfloor + s^*$

$$
\begin{aligned}
{}_0W^s_{p,\mu}(J;E) &= \left\{u \in {}_0W^{\lfloor s \rfloor}_{p,\mu}(J;E) : u^{(\lfloor s \rfloor)} \in {}_0W^{s^*}_{p,\mu}(J;E)\right\},\\
W^s_{p,\mu}(J;E) &= \left\{u \in W^{\lfloor s \rfloor}_{p,\mu}(J;E) : u^{(\lfloor s \rfloor)} \in W^{s^*}_{p,\mu}(J;E)\right\},
\end{aligned}
$$

where the natural norms are equivalent with constants independent of J.

7. *By Proposition 2.10 in [24], we obtain that* $W^s_{p,\mu}(J;E) = {}_0W^s_{p,\mu}(J;E)$ *for* $1-\mu+1/p > s > 0$.

8. *By interpolation theory, see (2.6) in [24], we have the following representation of the Slobodetskii space: For* $s \in (0,1)$ *it holds*

$$
W^s_{p,\mu}(J;E) = \left\{u \in L_{p,\mu}(J;E) : [u]_{W^s_{p,\mu}(J;E)} < \infty\right\},
$$

where

$$
[u]_{W^s_{p,\mu}(J;E)} := \left(\int_0^T \int_0^t \tau^{(1-\mu)p} \frac{\|u(t)-u(\tau)\|_E^p}{|t-\tau|^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}t\right)^{\frac{1}{p}}.
$$

Then, the norm given by

$$
\|u\|_{W^s_{p,\mu}(J;E)} := \|u\|_{L_{p,\mu}(J;E)} + [u]_{W^s_{p,\mu}(J;E)}
$$

is equivalent to the one induced by interpolation.

9. *If* $E = \mathbb{R}$ *is the image space, we omit it, e.g.* $W^s_{p,\mu}(J) := W^s_{p,\mu}(J;\mathbb{R})$.

In the following we introduce another generalization of Sobolev spaces for the scalar valued case, see Theorem of Section 3.3.1 in [30] for the cone $\mathbb{R}^+$ and restricted to the subset $(0,T)$.

Definition 2.1.7 (Weighted Besov Space)
Let $J = (0,T)$, $0 < T \le \infty$. *Let* $m_1, m_2 \in \mathbb{N}$ *such that* $0 \le m_1 < m_2 < \infty$, $1 \le p < \infty$, $1 \le q < \infty$ *and* $\mu \in \left(\frac{1}{p}, 1\right]$. *Moreover, let* $\theta \in (0,1)$ *and* $s = (1-\theta)m_1 + \theta m_2$. *The* ***weighted Besov space*** *is given by*

$$
B^s_{p,q,\mu}(J) := \left(W^{m_1}_{p,\mu}(J), W^{m_2}_{p,\mu}(J)\right)_{\theta,q}.
$$

Remark 2.1.8
1. *The Besov spaces are independent of the choice of* m_1 *and* m_2, *see Section 3.3.1, Remark 1, in [30]. This justifies the notation.*

2. *We have the representation*

$$
B^s_{p,q,\mu}(J) = \left\{u \in W^{\lfloor s \rfloor}_{p,\mu}(J) : [u]_{B^s_{p,q,\mu}(J)} < \infty\right\},
$$

where

$$[u]^q_{B^s_{p,q,\mu}(J)} := \int_J \left(h^{-(s-\lfloor s \rfloor)} \|\tau_h u^{(\lfloor s \rfloor)}(\cdot) - u^{(\lfloor s \rfloor)}(\cdot)\|_{L_{p,\mu}(J_h)} \right)^q \frac{\mathrm{d}h}{h},$$

for

$$J_h := \begin{cases} (0, T-h) & \text{for } h < T \\ \emptyset & \text{for } h \geq T. \end{cases}$$

The norm given by

$$\|u\|_{B^s_{p,q,\mu}(J)} := \|u\|_{L_{p,\mu}(J)} + [u]_{B^s_{p,q,\mu}(J)}$$

is equivalent to the norm induced by interpolation. This can be proven using Theorem in Section 3.3.1 of [30] for the cone $\mathbb{R}^+$. The case $T < \infty$ can be justified by an extension argument like in step 1 of the proof of Theorem 1 in Section 3.6.4 of [30].

3. *For $E = \mathbb{R}$ the previously introduced weighted Slobodetskii spaces are Besov spaces for $p = q$. By changing the order of integration, variable transformation $\tau = t + h$, and changing the order of integration again, we obtain*

$$\begin{aligned} [f]^p_{B^s_{p,p,\mu}(J)} &= \int_0^T \int_0^{T-h} t^{p(1-\mu)} \frac{|f(t+h) - f(t)|^p}{h^{1+sp}} \,\mathrm{d}t\,\mathrm{d}h \\ &= \int_0^T \int_0^{T-t} t^{p(1-\mu)} \frac{|f(t+h) - f(t)|^p}{h^{1+sp}} \,\mathrm{d}h\,\mathrm{d}t \\ &= \int_0^T \int_0^{\tau} t^{p(1-\mu)} \frac{|f(\tau) - f(t)|^p}{|\tau - t|^{1+sp}} \,\mathrm{d}t\,\mathrm{d}\tau = [f]^p_{W^s_{p,\mu}(J)}. \end{aligned}$$

2.1.1 An Embedding Theorem for Fractional Sobolev Spaces

We will need to improve the first part of the following embedding result, see Proposition 2.11 in [24], to deduce multiplication results:

Proposition 2.1.9
Let $J = (0, T)$ be finite, $1 < p < q < \infty$, $\mu \in \left(\frac{1}{p}, 1\right]$ and $s > \tau \geq 0$ and E be a Banach space. Then

$$W^s_{p,\mu}(J; E) \hookrightarrow W^\tau_{q,\mu}(J; E) \qquad \textit{holds if} \qquad s - (1-\mu) - \frac{1}{p} > \tau - \frac{p\left(1 - \mu + \frac{1}{p}\right)}{q}, \tag{2.1.1}$$

$$W^s_{p,\mu}(J; E) \hookrightarrow W^\tau_{q}(J; E) \qquad \textit{holds if} \qquad s - (1-\mu) - \frac{1}{p} > \tau - \frac{1}{q}. \tag{2.1.2}$$

These embeddings remain true if one replaces the W-spaces by the H-, the ${}_0W$- and the ${}_0H$-spaces, respectively. In the two latter cases, restricting to $s \in [0, 2]$, for given $T_0 > 0$ the embeddings hold with a uniform constant for all $0 < T \leq T_0$.

In the following, we prove a refinement of embedding (2.1.1).

Theorem 2.1.10
Let $J = (0, T)$ be finite, $1 < p < q < \infty$, $\mu \in \left(\frac{1}{p}, 1\right]$ and $s > \tau \geq 0$. Then,

$$W^s_{p,\mu}(J) \hookrightarrow W^\tau_{q,\mu}(J) \qquad \textit{holds if} \qquad s - \frac{1}{p} > \tau - \frac{1}{q}. \tag{2.1.3}$$

This embedding remains true if one replaces the W-spaces by the H-, the ${}_0W$- and the ${}_0H$-spaces, respectively. In the two latter cases, restricting to $s \in [0,2]$, for given $T_0 > 0$ the embeddings holds with a uniform constant for all $0 < T \leq T_0$.

In order to prove this statement, we need the following lemmas to represent the function in a different way, see Lemma 2.1.11, and modified Young's inequality, see Lemma 2.1.12 for $E = \mathbb{R}$. These lemmas are adapted versions of Lemma 7 and Lemma 5 in [29].

Lemma 2.1.11
Let $f \in L_{p,\mu}(J;E)$, E a Banach space, $J = (0,T)$ for $T < \infty$, $1 \leq p < \infty$, and $a > 0$. Assume that

$$\int_0^a \|f - \tau_h f\|_{L_{p,\mu}(J_h;E)} \frac{\mathrm{d}h}{h} < \infty.$$

Then

$$f = \frac{1}{a}\int_0^a \tau_h f \,\mathrm{d}h + \int_0^a \int_0^{a-h} (I - \tau_h)\tau_s f \frac{\mathrm{d}s}{(s+h)^2}\,\mathrm{d}h \qquad \text{in } L_{p,\mu}(J_a;E).$$

Proof. First of all, we show that the function $h \mapsto \tau_h f$ is continuous from $[0,a]$ to $L_{p,\mu}(J_a;E)$. Let $0 \leq b \leq c \leq a$. Then, we have via the transformation formula

$$\begin{aligned}
\|\tau_c f - \tau_b f\|^p_{L_{p,\mu}(J_a;E)} &= \|\tau_b(\tau_{c-b} f - f)\|^p_{L_{p,\mu}(J_a;E)} = \int_0^{T-a} \left[t^{1-\mu}\|\tau_b(\tau_{c-b}f - f)\|_E(t)\right]^p \mathrm{d}t \\
&= \int_0^{T-a} \left[t^{1-\mu}\|\tau_{c-b}f - f\|_E(t+b)\right]^p \mathrm{d}t \\
&\leq \int_0^{T-a} \left[(t+b)^{1-\mu}\|\tau_{c-b}f - f\|_E(t+b)\right]^p \mathrm{d}t \\
&\leq \int_b^{T-(a-b)} \left[s^{1-\mu}\|\tau_{c-b}f - f\|_E(s)\right]^p \mathrm{d}s \\
&= \|\tau_{c-b}f - f\|^p_{L_{p,\mu}(J_{a-b};E)}.
\end{aligned} \tag{2.1.4}$$

Using the density of $C_c^\infty(\bar{J}\backslash\{0\};E)$ in $L_{p,\mu}(J;E)$, see Lemma 2.4 in [24], we find for each $\epsilon > 0$ a $g \in C_c^\infty(\bar{J}\backslash\{0\};E)$ fulfilling $\|f - g\|_{L_{p,\mu}(I;E)} \leq \epsilon/4$. Thus, it follows for $h := c - b$

$$\begin{aligned}
\|\tau_h f - f\|_{L_{p,\mu}(J_{a-b};E)} &\leq \|f(\cdot + h) - g(\cdot + h)\|_{L_{p,\mu}(J_{a-b})} + \|g(\cdot + h) - g\|_{L_{p,\mu}(J_{a-b};E)} \\
&\quad + \|g - f\|_{L_{p,\mu}(J_{a-b};E)} \\
&\leq 2\|f - g\|_{L_{p,\mu}(J;E)} + \|g(\cdot + h) - g\|_{L_{p,\mu}(J_{a-b};E)} \\
&\leq \frac{\epsilon}{2} + \|g(\cdot + h) - g\|_{L_{p,\mu}(J_{a-b};E)}.
\end{aligned}$$

By choosing h small enough, we obtain $\|g(\cdot+h) - g\|_{L_{p,\mu}(J_{a-b};E)} \leq \epsilon/2$ and thereby the claim. Hence, the function

$$\left[t \mapsto Y(a,t) := \frac{1}{a}\int_0^a \tau_h f(t)\,\mathrm{d}h\right] \in L_{p,\mu}(J_a;E).$$

The next step is to show that

$$\left[t \mapsto X(a,t) := \int_0^a \int_0^{a-h} (I - \tau_h)\tau_s f(t) \frac{\mathrm{d}s}{(s+h)^2}\,\mathrm{d}h\right] \in L_{p,\mu}(J_a;E).$$

The function $[(h,s) \in [0,a] \times [0,a-h] \mapsto (I-\tau_h)\tau_s f] \in L_{p,\mu}(J_a;E)$ is continuous and therefore, strongly measurable. Thus, it follows

$$\int_0^a \int_0^{a-h} \|(I-\tau_h)\tau_s f(\cdot)\|_{L_{p,\mu}(J_a;E)} \frac{\mathrm{d}s}{(s+h)^2}\,\mathrm{d}h \\ \leq \int_0^a \int_0^{a-h} \|(I-\tau_h) f(\cdot)\|_{L_{p,\mu}(J_h;E)} \frac{\mathrm{d}s}{(s+h)^2}\,\mathrm{d}h,$$

where we used the estimate in (2.1.4) again for $c := s+h$, $b := s$. By integrating with respect to s, we obtain

$$\int_0^a \int_0^{a-h} \|(I-\tau_h) f(\cdot)\|_{L_{p,\mu}(J_h)} \frac{\mathrm{d}s}{(s+h)^2}\,\mathrm{d}h \leq \int_0^a \|(I-\tau_h) f(\cdot)\|_{L_{p,\mu}(J_h)} \left(\frac{1}{h} - \frac{1}{a}\right)\mathrm{d}h,$$

which is finite by hypothesis. Consequently,

by the theorem on differentiability of parameter integrals for separable Banach spaces, see Theorem 3.18 in [3], also the integral $X(a,t)$ exists in $L_{p,\mu}(J_a;E)$.

It remains to prove that the sum of $X(a,\cdot)$ and $Y(a,\cdot)$ equals f. By changing the variables $u = s$, $k = s+h$, we obtain

$$X(a,\cdot) = \int_0^a \int_0^k (\tau_u f - \tau_k f)\,\mathrm{d}u \frac{\mathrm{d}k}{k^2} \qquad \text{in } L_{p,\mu}(J_a;E). \tag{2.1.5}$$

Now, let α, with $\alpha \geq a > 0$, be given. Then, it holds $J_\alpha \subset J_a$. Therefore, by restriction the previous equality holds as well in $L_{p,\mu}(J_\alpha;E)$ and we can consider the differential quotient of $a \mapsto X(a,\cdot)$ for $a \in (0,\alpha]$. For $a \in (0,\alpha)$ and a sufficiently small $\tilde{h}$, we have

$$\frac{X(a+\tilde{h},\cdot) - X(a,\cdot)}{\tilde{h}} = \frac{1}{\tilde{h}} \int_a^{a+\tilde{h}} \int_0^k (\tau_u f - \tau_k f)\,\mathrm{d}u \frac{\mathrm{d}k}{k^2} \qquad \text{in } L_{p,\mu}(J_\alpha;E).$$

We obtain by the estimate (2.1.4) for $0 \leq u \leq k < \alpha$

$$\frac{1}{k}\int_0^k \|\tau_u f - \tau_k f\|_{L_{p,\mu}(J_\alpha;E)}\,\mathrm{d}u \leq \frac{1}{k}\int_0^k \|f - \tau_{k-u} f\|_{L_{p,\mu}(J_{\alpha-u};E)}\,\mathrm{d}u.$$

By transformation formula for $v = k-u$ and $J_{\alpha-(k-v)} \subset J_v$, for $k \leq a < \alpha$, it holds

$$\frac{1}{k}\int_0^k \|\tau_u f - \tau_k f\|_{L_{p,\mu}(J_\alpha;E)}\,\mathrm{d}u \leq \int_0^\alpha \|f - \tau_v f\|_{L_{p,\mu}(J_v;E)} \frac{\mathrm{d}v}{v},$$

where we additionally used that $v \leq k$. This expression is finite by the assumption on f. For $a = \alpha$, we argue analogously for the left-hand side limit. Consequently, as $[k \mapsto \frac{1}{k^2}\int_0^k (\tau_u f - \tau_k f)\,\mathrm{d}u]$ is continuous from $(0,\alpha]$ to $L_{p,\mu}(J_\alpha;E)$, the mapping $a \mapsto X(a,\cdot)$ is differentiable from $(0,\alpha]$ into $L_{p,\mu}(J_\alpha;E)$. Additionally, we have by restriction $[a \mapsto Y(a,\cdot)] \in L_{p,\mu}(J_\alpha;E)$ for $a \in (0,\alpha]$. By the Banach space valued version of the fundamental theorem of calculus, see Proposition 3.7 in [33], it follows that $[a \mapsto Y(a,\cdot)]$ is differentiable from $(0,\alpha]$ into $L_{p,\mu}(J_\alpha;E)$. Therefore, straight forward calculations show

$$\frac{\partial X}{\partial a}(a) = \frac{1}{a^2}\int_0^a (\tau_u f - \tau_a f)\,\mathrm{d}u = -\frac{\partial}{\partial a}\left(\frac{1}{a}\int_0^a \tau_h f\,\mathrm{d}h\right)(a) = -\frac{\partial Y}{\partial a}(a) \qquad \text{in } L_{p,\mu}(J_\alpha;E).$$

Thus,

$$\frac{\partial X}{\partial a}(a) + \frac{\partial Y}{\partial a}(a) = 0 \qquad\qquad \text{in } L_{p,\mu}(J_\alpha; E),$$

and $X(a) + Y(a)$ restricted to J_α does not depend on a. Using the identity (2.1.5), we obtain by the dominated convergence theorem that $X(a) \to 0$ in $L_{p,\mu}(J_\alpha; E)$ for $a \to 0$. Combining this with $Y(a) \to f$ in $L_{p,\mu}(J_\alpha; E)$ for $a \to 0$, we obtain

$$X(a) + Y(a) = f \qquad\qquad \text{in } L_{p,\mu}(J_\alpha; E) \text{ for all } \alpha \geq a.$$

In particular, this is satisfied for $\alpha = a$, which proves the lemma. $\square$

Lemma 2.1.12
Let $f \in L_{p,\mu}(J; E)$, E a Banach space, $J = (0,T)$ for $T < \infty$, $1 \leq p < \infty$, and $g \in L_r(0,a)$, $1 \leq r < \infty$, where $a > 0$ and $\frac{1}{p} + \frac{1}{r} > 1$. Let $F : J_a \to E$ be given by

$$F(x) := \int_0^a f(x+t) g(t)\, \mathrm{d}t.$$

Then, $F \in L_{q,\mu}(J_a; E)$, where $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, and

$$\|F\|_{L_{q,\mu}(J_a;E)} \leq \|f\|_{L_{p,\mu}(J;E)} \|g\|_{L_r(0,a)}.$$

Proof. First, we assume that f is in $C_0^\infty(\bar{J}\backslash\{0\}; E)$. By the assumptions on the exponents, we obtain

$$\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r} \geq 0 \quad \Leftrightarrow \quad q \geq p \qquad \text{and} \qquad \frac{1}{r} - \frac{1}{q} = 1 - \frac{1}{p} \geq 0 \quad \Leftrightarrow \quad q \geq p,$$

as $1 \leq r < \infty$, $1 \leq p < \infty$, respectively. In this proof, we denote by $\|\cdot\|$ the norm on E. Given $x \in J_a$, we have

$$\begin{aligned}\left\|F(x)x^{1-\mu}\right\| &\leq \int_0^a \|f(x+t)\| x^{1-\mu} |g(t)|\, \mathrm{d}t \\ &\leq \int_0^a \left(\|f(x+t)\|x^{1-\mu}\right)^{\frac{p}{q}} |g(t)|^{\frac{r}{q}} \left(\|f(x+t)\|x^{1-\mu}\right)^{1-\frac{p}{q}} |g(t)|^{1-\frac{r}{q}}\, \mathrm{d}t.\end{aligned}$$

Using Hölder's inequality with $1/q + (1/p - 1/q) + (1/r - 1/q) = 1$, we obtain

$$\begin{aligned}\left\|F(x)x^{1-\mu}\right\| \leq{}& \left(\int_0^a \left(\|f(x+t)\|x^{1-\mu}\right)^p |g(t)|^r\, \mathrm{d}t\right)^{\frac{1}{q}} \left(\int_0^a \left(\|f(x+t)\|x^{1-\mu}\right)^p\, \mathrm{d}t\right)^{\frac{1}{p}-\frac{1}{q}} \\ &\times \left(\int_0^a |g(t)|^r\, \mathrm{d}t\right)^{\frac{1}{r}-\frac{1}{q}}.\end{aligned}$$

By transformation formula, the second factor can be estimated by

$$\begin{aligned}\int_0^a \left(\|f(x+t)\|x^{1-\mu}\right)^p\, \mathrm{d}t &= \int_x^{a+x} \left(\|f(y)\|y^{1-\mu}\right)^p (x/y)^{(1-\mu)p}\, \mathrm{d}y \leq \int_x^{a+x} \left(\|f(y)\|y^{1-\mu}\right)^p\, \mathrm{d}y \\ &\leq \|f\|^p_{L_{p,\mu}(J;E)},\end{aligned}$$

where we used $({}^{x}\!/\!{}_{y})^{(1-\mu)p} \leq 1$, as $y \geq x$, and $a + x < T$ for $x \in J_a$. Thus,

$$\left\|F(x)x^{1-\mu}\right\|^q \leq \int_0^a \left(\|f(x+t)\|x^{1-\mu}\right)^p |g(t)|^r \,\mathrm{d}t \,\|f\|_{L_{p,\mu}(J;E)}^{(\frac{1}{p}-\frac{1}{q})pq} \left(\int_0^a |g(t)|^r \,\mathrm{d}t\right)^{(\frac{1}{r}-\frac{1}{q})q}$$

and by integrating with respect to x over J_a and extracting the q-th root

$$\begin{aligned}\left(\int_{I_a} \|F(x)x^{1-\mu}\|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leq &\left(\int_{I_a}\int_0^a \left(\|f(x+t)\|x^{1-\mu}\right)^p |g(t)|^r \,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{q}} \\ &\times \|f\|_{L_{p,\mu}(J;E)}^{(\frac{1}{p}-\frac{1}{q})p} \|g\|_{L_r(0,a)}^{(\frac{1}{r}-\frac{1}{q})r}.\end{aligned}$$

Lastly, we can estimate the first factor by

$$\begin{aligned}\int_{I_a}\int_0^a \left(\|f(x+t)\|x^{1-\mu}\right)^p |g(t)|^r \,\mathrm{d}t\,\mathrm{d}x &= \int_0^a \int_{I_a} \left(\|f(x+t)\|x^{1-\mu}\right)^p \,\mathrm{d}x |g(t)|^r \,\mathrm{d}t \\ &= \int_0^a \int_t^{T-a+t} \left(\|f(y)\|(y-t)^{1-\mu}\right)^p \,\mathrm{d}y |g(t)|^r \,\mathrm{d}t \\ &\leq \int_0^a |g(t)|^r \,\mathrm{d}t \int_t^{T-a+t} \left(\|f(y)\|y^{1-\mu}\right)^p \,\mathrm{d}y \\ &\leq \|g\|_{L_r(0,a)}^r \|f\|_{L_{p,\mu}(J;E)}^p,\end{aligned}$$

where we used Fubini's Theorem, transformation formula, and the fact that $y - t \leq y$ and $t < a$. Putting everything together, we obtain

$$\|F\|_{L_{q,\mu}(J_a;E)} \leq \left(\|g\|_{L_r(0,a)}^r \|f\|_{L_{p,\mu}(J;E)}^p\right)^{\frac{1}{q}} \|f\|_{L_{p,\mu}(J;E)}^{(\frac{1}{p}-\frac{1}{q})p} \|g\|_{L_r(0,a)}^{(\frac{1}{r}-\frac{1}{q})r} \leq \|g\|_{L_r(0,a)} \|f\|_{L_{p,\mu}(J;E)}.$$

By the density of $C_0^\infty(\bar{J}\backslash\{0\};E)$ in $L_{p,\mu}(J;E)$, cf. Lemma 2.4 in [24], we obtain the result. ☐

Now we are ready to prove Theorem 2.1.10:

Proof of Theorem 2.1.10. For this proof, let $T_0 > 0$ be given.
We may assume that $s \notin \mathbb{N}$, as we have strict inequalities in (2.1.3): For $q \in (p, \infty)$, $p \in (1, \infty)$ and $s \in \mathbb{N}$ with $s > \tau \geq 0$ fulfilling the assumption, it is always possible to find an $\tilde{s} \notin \mathbb{N}$, $s - 1 < \tilde{s} < s$ such that the assumptions still hold true for $\tilde{s}$ instead of s. By the properties of interpolation spaces, see Proposition 1.2.3 in [22], we obtain

$$W_{p,\mu}^s(J) \hookrightarrow \left(W_{p,\mu}^{\lfloor s\rfloor-1}(J), W_{p,\mu}^{\lfloor s\rfloor}(J)\right)_{\tilde{s}-\lfloor\tilde{s}\rfloor,p} = W_{p,\mu}^{\tilde{s}}(J). \tag{2.1.6}$$

We distinguish between the cases $\tau = 0$ and $\tau > 0$.

First, we focus on the case $\tau = 0$: If $s > 1$, we can reduce to the case $\tilde{s} < 1$ in the following way: For $q \in (p, \infty)$ and $p \in (1, \infty)$, there exists an $\tilde{s} < 1$ such that we have

$$s > 1 > \tilde{s} = \frac{1}{p} - \frac{1}{q}.$$

As $s \notin \mathbb{N}$, we have by the properties of interpolation spaces, see Proposition 1.2.3 in [22],

$$W_{p,\mu}^s(J) = \left(W_{p,\mu}^{\lfloor s\rfloor}(J), W_{p,\mu}^{\lfloor s\rfloor+1}(J)\right)_{s-\lfloor s\rfloor,p} \hookrightarrow W_{p,\mu}^{\lfloor s\rfloor}(J) \hookrightarrow W_{p,\mu}^1(J) \hookrightarrow W_{p,\mu}^{\tilde{s}}(J).$$

Thus, it suffices to prove that

$$W^{\tilde{s}}_{p,\mu}(J) \hookrightarrow B^{\bar{s}}_{p,1,\mu}(J) \hookrightarrow L_{q,\mu}(J) \qquad \text{if } \tilde{s} > \bar{s} = \frac{1}{p} - \frac{1}{q}. \tag{2.1.7}$$

This is done in the following two claims.

Claim 2.1.13 *For $0 < \epsilon < s < 1$, we have*

$$W^{s}_{p,\mu}(J) \hookrightarrow B^{s-\epsilon}_{p,1,\mu}(J),$$

with the estimate

$$[f]_{B^{s-\epsilon}_{p,1,\mu}(J)} \leq \frac{T^{\epsilon p'}}{\epsilon p'}[f]_{W^{s}_{p,\mu}(J)} \qquad \textit{for } \frac{1}{p'} + \frac{1}{p} = 1.$$

Proof of the claim. Since we have $W^s_{p,\mu}(J) = B^s_{p,p,\mu}(J)$ by the remark following Definition 2.1.7, it suffices to show

$$B^{s}_{p,p,\mu}(J) \hookrightarrow B^{s-\epsilon}_{p,1,\mu}(J)$$

with the corresponding semi-norm estimates. By straight forward computations, it follows

$$\begin{aligned}
[f]_{B^{s-\epsilon}_{p,1,\mu}(J)} &= \int_J h^{-(s-\epsilon)}\|\tau_h f(\cdot) - f(\cdot)\|_{L_{p,\mu}(J_h)}\frac{\mathrm{d}h}{h} \\
&= \left\|h^{-(s-\epsilon)}\|\tau_h f(\cdot) - f(\cdot)\|_{L_{p,\mu}(J_h)}\right\|_{L_1(J;\mathbb{R};\frac{dh}{h})} \\
&\leq \left\|h^{-s}\|\tau_h f(\cdot) - f(\cdot)\|_{L_{p,\mu}(J_h)}\right\|_{L_p(J;\mathbb{R};\frac{dh}{h})} \|h^{\epsilon}\|_{L_{p'}(J;\mathbb{R};\frac{dh}{h})} \\
&\leq \frac{T^{\epsilon p'}}{\epsilon p'}[f]_{B^s_{p,p,\mu}(J)},
\end{aligned}$$

where we used Hölder's inequality. This gives the result. □

Now we choose ϵ such that $\bar{s} := s - \epsilon = 1/p - 1/q$. Therefore, it remains to show:

Claim 2.1.14 *For $\bar{s} \in (0,1)$, $p \in [1,\infty)$ and $q \in (p,\infty)$ it holds*

$$B^{\bar{s}}_{p,1,\mu}(J) \hookrightarrow L_{q,\mu}(J) \qquad \textit{if } \bar{s} - \frac{1}{p} = -\frac{1}{q}$$

with the estimate

$$\|f\|_{L_{q,\mu}(J)} \leq C(q,\mu)\left[(T/2)^{-\bar{s}}\|f\|_{L_{p,\mu}(J)} + [f]_{B^{\bar{s}}_{p,1,\mu}(J)}\right].$$

Proof of the claim: For the proof, we adapt the proof of Lemma 8 in [29] for the scalar valued, time weighted case.
Let $f \in B^{\bar{s}}_{p,1,\mu}(J)$ and $a > 0$. Since we have

$$\int_0^a \|f - \tau_h f\|_{L_{p,\mu}(J_h)}\frac{\mathrm{d}h}{h} \leq a^{\bar{s}}\int_0^a h^{-\bar{s}}\|f - \tau_h f\|_{L_{p,\mu}(J_h)}\frac{\mathrm{d}h}{h} \leq a^{\bar{s}}[f]_{B^{\bar{s}}_{p,1,\mu}(J)} < \infty,$$

we know by Lemma 2.1.11 that

$$f = \frac{1}{a}\int_0^a \tau_h f\,\mathrm{d}h + \int_0^a\int_0^{a-h}(I-\tau_h)\tau_s f\frac{\mathrm{d}s}{(s+h)^2}\,\mathrm{d}h \qquad \text{in } L_{p,\mu}(J_a). \tag{2.1.8}$$

In the following, we want to show that f is in $L_{q,\mu}(J_a)$. To this end, we estimate the first summand of (2.1.8) by a modified Young's inequality, cf. Lemma 2.1.12, where we set $g(t) = 1$ and $1/r = 1 - \bar{s}$ in the assumptions, thus $1/q = 1/p + 1/r - 1$. We deduce $1/a \int_0^a \tau_h f(\cdot)\,\mathrm{d}h \in L_{q,\mu}(J_a)$ with the inequality

$$\left\| \frac{1}{a} \int_0^a \tau_h f(\cdot)\,\mathrm{d}h \right\|_{L_{q,\mu}(J_a)} \leq \frac{1}{a} \|f\|_{L_{p,\mu}(J)} \|1\|_{L_r(0,a)} \leq a^{-\bar{s}} \|f\|_{L_{p,\mu}(J)}.$$

It remains to consider the second summand of (2.1.8): Setting f in Lemma 2.1.12 to $f := f - \tau_h f \in L_{p,\mu}(J_h)$ and $g = (\cdot + h)^2$, we obtain

$$\left\| \int_0^{a-h} \tau_s \left[(I - \tau_h)\right] f(\cdot) \frac{\mathrm{d}s}{(s+h)^2} \right\|_{L_{q,\mu}(J_a)} \leq \|f - \tau_h f\|_{L_{p,\mu}(J_h)} \left\|(\cdot + h)^{-2}\right\|_{L_r(0,a-h)}.$$

Moreover, direct calculations yield

$$\left\|(\cdot + h)^{-2}\right\|_{L_r(0,a-h)} = \left(\frac{h^{1-2r} - a^{1-2r}}{2r-1}\right)^{\frac{1}{r}} \leq C h^{\frac{1}{r} - 2} = C h^{-\bar{s}-1},$$

where $C = (2r-1)^{-\frac{1}{r}} \leq 1$. Thus, we have

$$\begin{aligned} \int_0^a \left\| \int_0^{a-h} \tau_s \left[(I - \tau_h)\right] f(\cdot) \frac{\mathrm{d}s}{(s+h)^2} \right\|_{L_{q,\mu}(J_a)} \mathrm{d}h &\leq \int_0^a h^{-\bar{s}} \|f - \tau_h f\|_{L_{p,\mu}(J_h)} \frac{\mathrm{d}h}{h} \\ &\leq \|f\|_{B^{\bar{s}}_{p,1,\mu}(J)}, \end{aligned}$$

and consequently the second summand of (2.1.8) exists in $L_{q,\mu}(J_a)$. The combination of the estimates for both summands of (2.1.8) leads to

$$\|f\|_{L_{q,\mu}(J_a)} \leq a^{-\bar{s}} \|f\|_{L_{p,\mu}(J)} + [f]_{B^{\bar{s}}_{p,1,\mu}(J)}. \tag{2.1.9}$$

For $a = T/2$, it follows that

$$\begin{aligned} \|f\|_{L_{q,\mu}(J)} &\leq \left(\int_0^{T/2} \left(t^{1-\mu}|f(t)|\right)^q \,\mathrm{d}t + \int_{T/2}^T \left(t^{1-\mu}|f(t)|\right)^q \,\mathrm{d}t \right)^{\frac{1}{q}} \\ &\leq 2^{\frac{1}{q}} \left(\|f\|_{L_{q,\mu}(J_{T/2})} + T^{1-\mu} \|f\|_{L_q(T/2,T)} \right). \end{aligned} \tag{2.1.10}$$

We can directly use the statement of Lemma 8 in [29] for second summand of (2.1.10) and obtain for $a := T/4$

$$\|f\|_{L_q(T/2,T)} \leq 2^{\frac{1}{q}} \left(\left(\frac{T}{4}\right)^{-\bar{s}} \|f\|_{L_p(T/2,T)} + [f]_{B^{\bar{s}}_{p,1,1}(T/2,T)} \right).$$

Combining this with the estimate

$$\|f\|_{L_p(T/2,T)} \leq \left(\frac{T}{2}\right)^{-(1-\mu)} \|f\|_{L_{p,\mu}(J)},$$

which holds true as the time weight does not matter away from zero, we obtain

$$\|f\|_{L_q(T/2,T)} \leq 2^{\frac{1}{q}} \left(\frac{T}{2}\right)^{-(1-\mu)} \left(\left(\frac{T}{4}\right)^{-\bar{s}} \|f\|_{L_{p,\mu}(J)} + [f]_{B^{\bar{s}}_{p,1,\mu}(J)}\right).$$

Putting the estimates (2.1.10), (2.1.9), and the previous one together, we end up with

$$\begin{aligned}
\|f\|_{L_{q,\mu}(J)} &\leq 2^{\frac{1}{q}} \left(\|f\|_{L_{q,\mu}(J_{T/2})} + T^{1-\mu}\|f\|_{L_q(T/2,T)}\right) \\
&\leq 2^{\frac{1}{q}} \Bigg[\left(\frac{T}{2}\right)^{-\bar{s}} \|f\|_{L_{p,\mu}(J)} + [f]_{B^{\bar{s}}_{p,1,\mu}(J)} \\
&\quad + 2^{\frac{1}{q}} T^{1-\mu} \left(\frac{T}{2}\right)^{-(1-\mu)} \left(\left(\frac{T}{4}\right)^{-\bar{s}} \|f\|_{L_{p,\mu}(J)} + [f]_{B^{\bar{s}}_{p,1,\mu}(J)}\right)\Bigg] \\
&\leq C(q,\mu) \left[\left(\frac{T}{2}\right)^{-\bar{s}} \|f\|_{L_{p,\mu}(J)} + [f]_{B^{\bar{s}}_{p,1,\mu}(J)}\right].
\end{aligned}$$

□

Therefore, we obtain (2.1.7) and consequently the embedding (2.1.3) for $\tau = 0$ with the estimate

$$\begin{aligned}
\|f\|_{L_{q,\mu}(J)} &\leq C(q,\mu) \left[\left(\frac{T}{2}\right)^{-\bar{s}} \|f\|_{L_{p,\mu}(J)} + [f]_{B^{\bar{s}}_{p,1,\mu}(J)}\right], \\
&\leq C(q,\mu) \left(\left(\frac{2}{T}\right)^{s-\epsilon} \|f\|_{L_{p,\mu}(J)} + \frac{T^{\epsilon p'}}{\epsilon p'} (2T)^{\tilde{s}-s} [f]_{W^{\tilde{s}}_{p,\mu}(J)}\right) \\
&\leq C(q,\mu,T) \|f\|_{W^{\tilde{s}}_{p,\mu}(J)}.
\end{aligned}$$

Now we proceed with the case $\tau > 0$: W.l.o.g. we assume that $\tau \notin \mathbb{N}$, as we have strict inequalities in (2.1.3). It is always possible to find an $\tau < \tilde{\tau} \notin \mathbb{N}$, such that the assumption still holds true. For $\tau \in \mathbb{N}$, we obtain

$$W^{\tilde{\tau}}_{p,\mu}(J) = \left(W^{\tau}_{p,\mu}(J), W^{\tau+1}_{p,\mu}(J)\right)_{\tilde{\tau}-\lfloor \tilde{\tau} \rfloor, p} \hookrightarrow W^{\tau}_{p,\mu}(J)$$

by the properties of interpolation spaces, see Proposition 1.2.3 in [22]. The proof follows the idea of the proof of Proposition 2.11 in [24]. To this end, we set $\alpha := 1/p - 1/q > 0 \in (0,1)$ and fix $\epsilon > 0$, such that

$$s - \epsilon > \tau + \alpha, \tag{2.1.11}$$

which is possible due to the strict inequality in (2.1.3). Moreover, we define

$$k := \lfloor s - \epsilon - \alpha \rfloor \in \mathbb{N}_0 \qquad \text{and} \qquad \kappa := k + \epsilon + \alpha \in (s-1, s),$$

as $k < s - \epsilon - \alpha < k + 1$. W.l.o.g. we can assume that $\kappa \notin \mathbb{N}$, otherwise we decrease ϵ again. Then, there exists a $\theta \in (0,1)$, such that

$$s = \kappa(1-\theta) + (\kappa+1)\theta = \kappa + \theta$$

and, by Lemma 2.8 in [24], we have

$$W^{s}_{p,\mu}(J) = \left(W^{\kappa}_{p,\mu}(J), W^{\kappa+1}_{p,\mu}(J)\right)_{s-\kappa,p}.$$

Using the identity

$$W^s_{p,\mu}(J) = \left\{ u \in W^{[s]}_{p,\mu}(J;E) : u^{[s]} \in W^{s-[s]}_{p,\mu}(J) \right\},$$

cf. (2.8) in [24], the question, whether the embeddings

$$\begin{aligned} W^{\kappa}_{p,\mu}(J) &\hookrightarrow W^{k}_{q,\mu}(J) \\ W^{\kappa+1}_{p,\mu}(J) &\hookrightarrow W^{k+1}_{q,\mu}(J) \end{aligned} \tag{2.1.12}$$

hold true is, reduced to showing

$$W^{\kappa-k}_{p,\mu}(J) \hookrightarrow L_{q,\mu}(J), \tag{2.1.13}$$

which is covered by the first case. Due to the choice of α and κ, we have the equivalence

$$\epsilon > 0 \qquad \Leftrightarrow \qquad \alpha + \epsilon + k - k - \frac{1}{p} > -\frac{1}{q} \qquad \Leftrightarrow \qquad (\kappa - k) - \frac{1}{p} > -\frac{1}{q},$$

where the last inequality shows that the assumptions for the $\tau = 0$ case are fulfilled. Therefore, the embedding (2.1.13) and thus the embeddings in (2.1.12) hold true, and by the properties of real interpolation spaces, see the Proposition in 1.2.3 in [22], we obtain

$$\begin{aligned} W^s_{p,\mu}(J) &= \left(W^{\kappa}_{p,\mu}(J), W^{\kappa+1}_{p,\mu}(J)\right)_{s-\kappa,p} \hookrightarrow \left(W^{k}_{q,\mu}(J), W^{k+1}_{q,\mu}(J)\right)_{s-\kappa,p} \\ &\overset{q>p}{\hookrightarrow} \left(W^{k}_{q,\mu}(J), W^{k+1}_{q,\mu}(J)\right)_{s-\kappa,q} = W^{\tilde{\tau}}_{q,\mu}(J), \end{aligned}$$

where

$$\tilde{\tau} = k(1-(s-\kappa)) + (k+1)(s-\kappa) = k + s - \kappa = k + s - (k + \alpha + \epsilon) = s - \alpha - \epsilon.$$

Due to (2.1.11), we have $\tilde{\tau} > \tau$ and, therefore like in (2.1.6),

$$W^{\tilde{\tau}}_{q,\mu}(J) \hookrightarrow W^{\tau}_{q,\mu}(J).$$

This shows the embedding (2.1.3).

Furthermore, we only have to consider the W-case, as the general properties of real and complex interpolation spaces imply that one has the scale of dense embeddings

$$W^{s_1}_{p,\mu}(J) \hookrightarrow H^{s_2}_{p,\mu}(J) \hookrightarrow W^{s_3}_{p,\mu}(J) \hookrightarrow H^{s_4}_{p,\mu}(J), \qquad \text{for } s_1 > s_2 > s_3 > s_4 \geq 0, \tag{2.1.14}$$

cf. (2.1) and (2.2) in [24]. More precisely, if we want to show (2.1.3) in the H-case, we find $\epsilon_1, \epsilon_2 > 0$, such that

$$s - \frac{1}{p} > (s - \epsilon_1) - \frac{1}{p} > (\tau + \epsilon_2) - \frac{1}{q} > \tau - \frac{1}{q}$$

is fulfilled. Then the claim follows by the previously proven result, if

$$H^s_{p,\mu}(J) \hookrightarrow W^{s-\epsilon_1}_{p,\mu}(J) \tag{2.1.15}$$

$$W^{\tau+\epsilon_2}_{q,\mu}(J) \hookrightarrow H^{\tau}_{q,\mu}(J) \tag{2.1.16}$$

holds true, since we have consequently

$$H^s_{p,\mu}(J) \hookrightarrow W^{s-\epsilon_1}_{p,\mu}(J) \hookrightarrow W^{\tau+\epsilon_2}_{q,\mu}(J) \hookrightarrow H^{\tau}_{q,\mu}(J).$$

We come to embedding (2.1.15): Since $s \notin \mathbb{N}$, w.l.o.g. we can assume that $\lfloor s \rfloor = \lfloor s - \epsilon_1 \rfloor$ and $s - \epsilon_1 \notin \mathbb{N}$. Otherwise, we decrease ϵ_1. By the definition of Bessel potential spaces, see Definition 2.1.5, we know that $H^s_{p,\mu}(J) = (H^{\lfloor s \rfloor}_{p,\mu}(J), H^{\lfloor s \rfloor+1}_{p,\mu}(J))_{[s-\lfloor s \rfloor]}$ is a complex interpolation space with $\theta := s - \lfloor s \rfloor \in (0,1)$ and thus belongs to the class $K(\theta)$, see Theorem 1, Section 1.10.3 in [30]. But this implies that

$$H^s_{p,\mu}(J) \hookrightarrow \left(H^{\lfloor s \rfloor}_{p,\mu}(J), H^{\lfloor s \rfloor+1}_{p,\mu}(J)\right)_{\theta,\infty}.$$

The Slobodetskii space $W^{s-\epsilon_1}_{p,\mu}(J)$, see Definition 2.1.5, is a real interpolation space with $\theta - \epsilon_1 \in (0,1)$. Therefore, it belongs to the class $J(\theta)$, since

$$\left(H^{\lfloor s \rfloor}_{p,\mu}(J), H^{\lfloor s \rfloor+1}_{p,\mu}(J)\right)_{\theta-\epsilon_1,1} \hookrightarrow W^s_{p,\mu}(J),$$

see Theorem 1, Section 1.10.3 in [30]. Using the properties of real interpolation spaces, see item (e) of the Theorem in Section 1.3.3. in [30], we obtain that

$$(X_0, X_1)_{\theta,\infty} \hookrightarrow (X_0, X_1)_{\theta-\epsilon_1,1}$$

for an interpolation couple (X_0, X_1). This proves (2.1.15) and embedding (2.1.16) follows analogously.

It remains to show that the estimate holds with a uniform constant for all $0 < T \leq T_0$ in the case of a ${}_0W$-space and for $s \in [0,2]$. To this end, let $f \in {}_0W^s_{p,\mu}(0,T)$ and let the assumption of (2.1.3) be true. We use the extension operator $\mathcal{E}^0_T \in \mathcal{L}({}_0W^s_{p,\mu}(0,T), {}_0W^s_{p,\mu}(\mathbb{R}_+))$ from Lemma 2.5 in [24]. The restriction operator $\mathcal{R}_{T_0}$ is trivially an element of $\mathcal{L}({}_0W^s_{p,\mu}(\mathbb{R}_+), {}_0W^s_{p,\mu}(0,T_0))$, where the constant in the norm estimate does not depend on T_0. Then, it holds

$$\left\|\mathcal{R}_{T_0}\mathcal{E}^0_T(f)\right\|_{W^s_{p,\mu}(0,T_0)} \leq \left\|\mathcal{E}^0_T(f)\right\|_{W^s_{p,\mu}(\mathbb{R}_+)} \leq C\|f\|_{W^s_{p,\mu}(0,T)},$$

where we used that the constant C in the estimate for the extension operator does not depend on T for $s \in [0,2]$. Therefore, we obtain for $0 < T \leq T_0$ the estimate

$$\|f\|_{L_{q,\mu}(0,T)} \leq \left\|\mathcal{R}_{T_0}\mathcal{E}^0_T(f)\right\|_{L_{q,\mu}(0,T_0)} \leq C(T_0)\left\|\mathcal{R}_{T_0}\mathcal{E}^0_T(f)\right\|_{W^s_{p,\mu}(0,T_0)} \leq C(T_0)\|f\|_{W^s_{p,\mu}(0,T)},$$

which shows the claim of the addendum in the ${}_0W$-case.

In the ${}_0H$-case, we use the scale of embeddings

$${}_0W^{s_1}_{p,\mu}(J) \hookrightarrow {}_0H^{s_2}_{p,\mu}(J) \hookrightarrow {}_0W^{s_3}_{p,\mu}(J) \hookrightarrow {}_0H^{s_4}_{p,\mu}(J) \qquad \text{for } s_1 > s_2 > s_3 > s_4 \geq 0,$$

which can be proven analogously to (2.1.14): additionally, one uses the extension operator $\mathcal{E}^0_T \in \mathcal{L}({}_0W^s_{p,\mu}(0,T), {}_0W^s_{p,\mu}(\mathbb{R}_+;E))$ from Lemma 2.5 in [24] to show that the the operator norms are uniform in $0 < T \leq T_0$. $\square$

2.1.2 Embeddings with Uniform Operator Norms

The following proposition shows that the operator norms in Theorem 2.1.10 are uniform in time for all $0 < T \leq T_0 < \infty$ if one uses a suitable norm.

Proposition 2.1.15
Let $0 < T_0 < \infty$ be fixed and $J = (0,T)$ for $0 < T \leq T_0$. Moreover, let $\mu \in \left(\frac{1}{p}, 1\right]$ and E be a

Banach space. We set for $s > 1-\mu+\frac{1}{p}$

$$\|\rho\|'_{W^s_{p,\mu}(J;E)} := \|\rho\|_{W^s_{p,\mu}(J;E)} + \|\rho_{|t=0}\|_E. \tag{2.1.17}$$

1. *Let* $1 < p < q < \infty$, $2 \geq s > \tau \geq 0$, *and* $s - \frac{1}{p} > \tau - \frac{1}{q}$.
   Then $W^s_{p,\mu}(J;E) \hookrightarrow W^\tau_{q,\mu}(J;E)$ *with the estimate*

$$\|\rho\|_{W^\tau_{q,\mu}(J;E)} \leq C(T_0)\|\rho\|'_{W^s_{p,\mu}(J;E)} \qquad \text{for } s > 1-\mu+\frac{1}{p},$$
$$\|\rho\|_{W^\tau_{q,\mu}(J;E)} \leq C(T_0)\|\rho\|_{W^s_{p,\mu}(J;E)} \qquad \text{for } s < 1-\mu+\frac{1}{p}.$$

2. *Let* $1 < p < q < \infty$, $2 \geq s > \tau \geq 0$, *and* $s-(1-\mu)-\frac{1}{p} > \tau - \frac{1}{q}$.
   Then $W^s_{p,\mu}(J;E) \hookrightarrow W^\tau_{q}(J;E)$ *with the estimate*

$$\|\rho\|_{W^\tau_{q}(J;E)} \leq C(T_0)\|\rho\|'_{W^s_{p,\mu}(J;E)} \qquad \text{for } s > 1-\mu+\frac{1}{p},$$
$$\|\rho\|_{W^\tau_{q}(J;E)} \leq C(T_0)\|\rho\|_{W^s_{p,\mu}(J;E)} \qquad \text{for } s < 1-\mu+\frac{1}{p}.$$

3. *Let* $1 < p < \infty$, $2 \geq s > 1-\mu+\frac{1}{p}$, *and* $\alpha \in (0,1)$.
   Then $W^s_{p,\mu}(J;E) \hookrightarrow C^\alpha(\bar{J};E)$ *for* $s-(1-\mu)+\frac{1}{p} > \alpha > 0$ *with the estimate*

$$\|\rho\|_{C^\alpha(\bar{J};E)} \leq C(T_0)\|\rho\|'_{W^s_{p,\mu}(J;E)}.$$

Each of the constants C *does not depend on* T.

Remark 2.1.16 *Based on item 3, we can also prove the following statement: Let* $1 < p < \infty$, $k \in \mathbb{N}$. *Then* $W^s_p(J;E) \hookrightarrow C^{k,\alpha}(\bar{J};E)$ *for* $s - 1/p > k+\alpha > 0$ *with the estimate*

$$\|\rho\|_{C^{k,\alpha}(\bar{J};E)} \leq C\|\rho\|_{W^s_p(J;E)},$$

where C *depends on* J.

In order to show this, we use the characterization of Slobodetskii spaces in Lemma 1.1.8 in [23]. Thus, we can apply the reasoning of the proof of 3 for $\mu = 1$ *to* $\partial^m_\sigma f \in W^{s-k}_p(J;E)$ *for* $m < s$, *since* $s-k > 0$.

Proof. Ad 1:
Let $u \in W^s_{p,\mu}(J;E)$ and $s > 1-\mu+1/p$. Using Proposition 2.10 in [24], we can evaluate u for $t=0$ and set $v(t) = u(0)$ for $t \in [0,T_0]$. Then, $v \in W^\tau_{q,\mu}((0,T_0);E)$ is an extension of $\rho(0)$ and

$$\begin{aligned}\|v\|^q_{W^\tau_{q,\mu}((0,T_0);E)} = \|v\|^q_{L_{q,\mu}((0,T_0);E)} &= \int_0^{T_0} \left(t^{1-\mu}\|u(0)\|_E\right)^q \,\mathrm{d}t = \|u(0)\|^q_E \int_0^{T_0} t^{(1-\mu)q}\,\mathrm{d}t \\ &\leq C(T_0)\|u(0)\|^q_E.\end{aligned}$$

Furthermore, we set $\bar{v} := u - v$ and obtain that $\bar{v} \in {}_0W^\tau_{q,\mu}(J;E)$, since $\bar{v}(0) := u(0)-v(0) = 0$. We deduce

$$\|u\|_{W^\tau_{q,\mu}(J;E)} \leq \|v\|_{W^\tau_{q,\mu}(J;E)} + \|\bar{v}\|_{W^\tau_{q,\mu}(J;E)} \leq \|v\|_{W^\tau_{q,\mu}(J;E)} + C\|\bar{v}\|_{W^s_{p,\mu}(J;E)},$$

where we used Theorem 2.1.10 for elements of ${}_0W^s_{p,\mu}(J;E)$. For $s \in [0,2]$ the embedding holds with a uniform constant C for all T fulfilling $0 < T \leq T_0$. Moreover, we have by construction of v

$$\begin{aligned}
\|u\|_{W^\tau_{q,\mu}(J;E)} &\leq C(T_0)\|u(0)\|_E + C\|\bar{v}\|_{W^s_{p,\mu}(J;E)} \\
&\leq C(T_0)\|u(0)\|_E + C\left(\|u\|_{W^s_{p,\mu}(J;E)} + C(T_0)\|\rho(0)\|_E\right) \\
&\leq C(T_0)\left(\|u(0)\|_E + \|u\|_{W^s_{p,\mu}(J;E)}\right),
\end{aligned}$$

which proves the claim in the first case.

If $s < 1 - \mu + {}^1/_p$, then it follows that ${}_0W^s_{p,\mu}(J;E) = W^s_{p,\mu}(J;E)$ and ${}_0W^\tau_{q,\mu}(J;E) = W^\tau_{q,\mu}(J;E)$ by Proposition 2.10 in [24], since

$$0 > s - (1-\mu) - \frac{1}{p} > \tau - (1-\mu) - \frac{1}{q} \qquad \Leftrightarrow \qquad \tau < 1 - \mu + \frac{1}{q}.$$

Therefore, the claim follows directly by Theorem 2.1.10 for elements of ${}_0W^s_{p,\mu}(J;E)$ and ${}_0W^\tau_{q,\mu}(J;E)$.

Ad 2:
The assertion follows by the same strategy as item 1.

Ad 3:
Firstly, let $f \in {}_0W^s_{p,\mu}(J;E)$. Then, by Lemma 2.5 in [24], there is an extension $\mathcal{E}^0_T f \in W^s_{p,\mu}(\mathbb{R}_+;E) \subset W^s_{p,\mu}((0,T_0);E)$, whose operator norm is independent of T. Moreover, for $1 < p < \infty$, $s > 0$ and $s - 1 + \mu - {}^1/_p > 0$, we find a $\tau \in (0, \min\{s,1\})$ and a $q \in (p,\infty)$, such that the assumption of item 2, i.e. $s - 1 + \mu - {}^1/_p > \tau - {}^1/_q > \alpha$, is fulfilled. We obtain $\mathcal{E}^0_T f \in W^\tau_q((0,T_0);E)$ with the estimate

$$\left\|\mathcal{E}^0_T f\right\|_{W^\tau_q((0,T_0);E)} \leq C(T_0)\left\|\mathcal{E}^0_T f\right\|_{W^s_{p,\mu}((0,T_0);E)},$$

where the constant does not depend on T. Now, we use

$$W^\tau_q\big((0,T_0);E\big) \hookrightarrow C^\alpha\big([0,T_0];E\big) \qquad \text{for } \tau - \frac{1}{q} > \alpha \in (0,1),$$

see Corollary 26 in [29], which applies for $\tau < 1$. Here, the operator norm depends on T_0. Combining both embeddings, we end up with

$$\|f\|_{C^\alpha([0,T];E)} \leq \left\|\mathcal{E}^0_T f\right\|_{C^\alpha([0,T_0];E)} \leq C(T_0)\left\|\mathcal{E}^0_T f\right\|_{W^s_{p,\mu}((0,T_0);E)} \leq C(T_0)\|f\|_{W^s_{p,\mu}(J;E)},$$

where the constant does not depend on T. This shows the claim in the ${}_0W^s_{p,\mu}(J;E)$-case.
The general case follows by the same strategy as the proof of item 1. □

2.1.3 Multiplication in Slobodetskii Spaces

Lemma 2.1.17
Let $0 < T_0 < \infty$ be fixed and $J = (0,T)$ for $0 < T \leq T_0$. Moreover, let $\mu \in \left(\frac{7}{8}, 1\right]$.

1. *Let $f \in {}_0W^{5/8}_{2,\mu}(J)$ and $g \in W^{1/8}_{2,\mu}(J)$, then $fg \in W^{1/8}_{2,\mu}(J)$ and*

$$\|fg\|_{W^{1/8}_{2,\mu}(J)} \leq C(T)\|f\|_{W^{5/8}_{2,\mu}(J)}\|g\|_{W^{1/8}_{2,\mu}(J)},$$

for a constant $C(T) \to 0$ monotonically as $T \to 0$.

2. *Let $f, g \in {}_0W^{5/8}_{2,\mu}(J)$, then $fg \in {}_0W^{5/8}_{2,\mu}(J)$ and*

$$\|fg\|_{W^{5/8}_{2,\mu}(J)} \leq C(T)\|f\|_{W^{5/8}_{2,\mu}(J)}\|g\|_{W^{5/8}_{2,\mu}(J)},$$

for a constant $C(T) \to 0$ monotonically as $T \to 0$, i.e. the space ${}_0W^{5/8}_{2,\mu}(J)$ is a Banach algebra up to a constant in the norm estimate for the product.

3. *Let $f, g \in W^{3/8}_{2,\mu}(J)$, then $fg \in W^{1/8}_{2,\mu}(J)$ and*

$$\|fg\|_{W^{1/8}_{2,\mu}(J)} \leq C(T)\|f\|_{W^{3/8}_{2,\mu}(J)}\|g\|_{W^{3/8}_{2,\mu}(J)},$$

for a constant $C(T) \to 0$ monotonically as $T \to 0$.

4. *Let $f \in W^s_{2,\mu}(J)$, $1 > s > \frac{1}{2} - \mu$ such that there exists a $\tilde{C} > 0$ with $|f| \geq \tilde{C}$. Then $\frac{1}{f} \in W^s_{2,\mu}(J)$ with*

$$\left\|\tfrac{1}{f}\right\|_{W^s_{2,\mu}(J)} \leq C\left(\|f\|_{W^s_{2,\mu}(J)}, \tilde{C}, T_0\right).$$

Remark 2.1.18 *Using Lemma 2.1.17, items 1 and 2, we can state similar claims for functions which do not have trace zero: Let $0 < T_0 < \infty$ be fixed and $J = (0, T)$ for $0 < T \leq T_0$. Furthermore, let $f \in W^{5/8}_{2,\mu}(J)$ and $g \in W^{1/8}_{2,\mu}(J)$, then $fg \in W^{1/8}_{2,\mu}(J)$ and*

$$\|fg\|^2_{W^{1/8}_{2,\mu}(J)} \leq C(T_0)\|f\|^2_{W^{5/8}_{2,\mu}(J)}\|g\|^2_{W^{1/8}_{2,\mu}(J)} + C(T_0)|f(0)|^2\|g\|^2_{W^{1/8}_{2,\mu}(J)},$$

for a uniform constant $C(T_0)$ for all $0 < T \leq T_0$.
The proof uses Lemma 2.1.17, item 1: Due to Proposition 2.10 in [24], we can evaluate f pointwise. We set $v(t) = f(0)$ for $t \in [0, T_0]$. Then, $v \in W^{5/8}_{2,\mu}((0, T_0))$ is an extension of $f(0)$ fulfilling

$$\|v\|_{W^{5/8}_{2,\mu}((0,T_0))} \leq C(T_0)|f(0)|.$$

Furthermore, we set $\bar{v} := f - v$ and obtain that $\bar{v} \in {}_0W^{5/8}_{2,\mu}(J)$, since $\bar{v}(0) := f(0) - v(0) = 0$, cf. the proof of item 1 of Lemma 2.1.15. Then, we conclude by the previous case

$$\begin{aligned}
\|fg\|^2_{W^{1/8}_{2,\mu}(J)} &\leq 2\left(\|(f-v)g\|^2_{W^{1/8}_{2,\mu}(J)} + \|f(0)g\|^2_{W^{1/8}_{2,\mu}(J)}\right) \\
&\leq C(T_0)\|f - v\|^2_{W^{5/8}_{2,\mu}(J)}\|g\|^2_{W^{1/8}_{2,\mu}(J)} + 2|f(0)|^2\|g\|^2_{W^{1/8}_{2,\mu}(J)} \\
&\leq C(T_0)\|f\|^2_{W^{5/8}_{2,\mu}(J)}\|g\|^2_{W^{1/8}_{2,\mu}(J)} + C(T_0)\|f(0)\|^2_{W^{5/8}_{2,\mu}(J)}\|g\|^2_{W^{1/8}_{2,\mu}(J)} + 2|f(0)|^2\|g\|^2_{W^{1/8}_{2,\mu}(J)} \\
&\leq C(T_0)\|f\|^2_{W^{5/8}_{2,\mu}(J)}\|g\|^2_{W^{1/8}_{2,\mu}(J)} + C(T_0)|f(0)|^2\|g\|^2_{W^{1/8}_{2,\mu}(J)} + 2|f(0)|^2\|g\|^2_{W^{1/8}_{2,\mu}(J)}.
\end{aligned}$$

By this, we obtain a uniform constant $C(T_0)$ for the product estimate for all $0 < T \leq T_0$.

Proof of Lemma 2.1.17. Ad 1:
Let $f \in {}_0W^{5/8}_{2,\mu}(J)$ and $g \in W^{1/8}_{2,\mu}(J) = {}_0W^{1/8}_{2,\mu}(J)$, where the equality is due to Proposition 2.10 in [24]. We want to show that the product fg is again in $W^{1/8}_{2,\mu}(J)$ and fulfills the corresponding norm-estimate. Using the embeddings (2.1.2), cf. Proposition 2.11 in [24], and Theorem 2.1.10, we

have

$$
\begin{aligned}
&{}_0W^{5/8}_{2,\mu}(J) \hookrightarrow L_p(J) && \text{if } \frac{5}{8} - (1-\mu) - \frac{1}{2} > -\frac{1}{p} \\
&W^{1/8}_{2,\mu}(J) \hookrightarrow L_{q,\mu}(J) && \text{if } \frac{1}{8} - \frac{1}{2} > -\frac{1}{q},
\end{aligned}
$$

both of the operator norms of the embeddings are independent of T. We can choose $p, q > 2$ such that $1/p + 1/q < 1/2$ for $\mu \in (7/8, 1]$. Thus, by Hölder's inequality, we obtain

$$
\|fg\|_{L_{2,\mu}(J)} \leq T^{\frac{1}{2}-\left(\frac{1}{p}+\frac{1}{q}\right)} \|f\|_{L_p(J)} \|g\|_{L_{q,\mu}(J)} \leq C(T) \|f\|_{W^{5/8}_{2,\mu}(J)} \|g\|_{W^{1/8}_{2,\mu}(J)},
$$

where $C(T) \to 0$ monotonically as $T \to 0$.

In order to verify the estimate for the semi-norm $[\cdot]_{W^{1/8}_{2,\mu}(J)}$, we use the expansion

$$
f(t)g(t) - f(\tau)g(\tau) = f(t)(g(t) - g(\tau)) + (f(t) - f(\tau))g(\tau).
$$

Consequently, it follows

$$
\begin{aligned}
[fg]^2_{W^{1/8}_{2,\mu}(J)} &= \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|f(t)g(t) - f(\tau)g(\tau)|^2}{|t-\tau|^{1+2\frac{1}{8}}} \,\mathrm{d}\tau \,\mathrm{d}t \\
&\leq C \Bigg(\int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|f(t)(g(t) - g(\tau))|^2}{|t-\tau|^{1+2\frac{1}{8}}} \,\mathrm{d}\tau \,\mathrm{d}t \\
&\quad + \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|(f(t) - f(\tau))g(\tau)|^2}{|t-\tau|^{1+2\frac{1}{8}}} \,\mathrm{d}\tau \,\mathrm{d}t \Bigg) = C(I + II).
\end{aligned}
$$

Using Theorem 2.1.15, item 3, on the first summand, we obtain

$$
{}_0W^{5/8}_{2,\mu}(J) \hookrightarrow C^\alpha(J) \qquad \text{for } \frac{5}{8} - (1-\mu) - \frac{1}{2} > \alpha > 0,
$$

where the operator norm of the embedding does not depend on T. As $f(0) = 0$, this yields

$$
\begin{aligned}
I &\leq \sup_{t \in J} |f(t)|^2 \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|g(t) - g(\tau)|^2}{|t-\tau|^{1+2\frac{1}{8}}} \,\mathrm{d}\tau \,\mathrm{d}t \\
&\leq \sup_{t \in J} |f(t) - f(0)|^2 \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|g(t) - g(\tau)|^2}{|t-\tau|^{1+2\frac{1}{8}}} \,\mathrm{d}\tau \,\mathrm{d}t \\
&\leq CT^{2\alpha} \|f\|^2_{W^{5/8}_{2,\mu}(J)} \|g\|^2_{W^{1/8}_{2,\mu}(J)},
\end{aligned}
$$

where $C(T) := T^{2\alpha} \to 0$ as $T \to 0$.

Expanding the integrand of II, we have for an $s = 1/8 + \epsilon$ with a suitably small $\epsilon > 0$,

$$
\begin{aligned}
II &= \int_0^T \int_0^t \left(\tau^{1-\mu} |g(\tau)| |t-\tau|^{s-\frac{1}{8}} \right)^2 \left(\frac{|(f(t) - f(\tau))|}{|t-\tau|^s} \right)^2 \frac{\mathrm{d}\tau \,\mathrm{d}t}{|t-\tau|} \\
&\leq \left(\int_0^T \int_0^t \left(\tau^{1-\mu} |g(\tau)| |t-\tau|^{s-\frac{1}{8}} \right)^p \frac{\mathrm{d}\tau \,\mathrm{d}t}{|t-\tau|} \right)^{\frac{2}{p}} \left(\int_0^T \int_0^t \left(\frac{|(f(t) - f(\tau))|}{|t-\tau|^s} \right)^q \frac{\mathrm{d}\tau \,\mathrm{d}t}{|t-\tau|} \right)^{\frac{2}{q}},
\end{aligned}
$$

$$\leq \left(\int_0^T \int_0^t \left(\tau^{1-\mu} |g(\tau)| |t-\tau|^{s-\frac{1}{8}} \right)^p \frac{\mathrm{d}\tau \, \mathrm{d}t}{|t-\tau|} \right)^{\frac{2}{p}} \|f\|^2_{W^s_q(J)},$$

where the second estimate follows by Hölder's inequality for the measure ${}^{d\tau dt}/_{|t-\tau|}$ and ${}^1/_p + {}^1/_q = {}^1/_2$. By changing the order of integration in the second factor, we have

$$\begin{aligned} \int_0^T \int_0^t \left(\tau^{1-\mu} |g(\tau)| |t-\tau|^{\epsilon} \right)^p \frac{\mathrm{d}\tau \, \mathrm{d}t}{|t-\tau|} &= \int_0^T \tau^{p(1-\mu)} |g(\tau)|^p \int_\tau^T |t-\tau|^{p\epsilon - 1} \, \mathrm{d}t \, \mathrm{d}\tau \\ &\leq \left[\frac{|t-0|^{p\epsilon}}{p\epsilon} \right]_0^T \int_0^T \tau^{p(1-\mu)} |g(\tau)|^p \, \mathrm{d}\tau \leq C(T) \|g\|^p_{L_{p,\mu}(J)}, \end{aligned}$$

for $C(T) := {}^{T^{p\epsilon}}/_{p\epsilon} \to 0$ as $T \to 0$. Thus, we have

$$[fg]^2_{W^{1/8}_{2,\mu}(J)} \leq C(T) \|f\|^2_{W^s_q(J)} \|g\|^2_{L_{p,\mu}(J)}$$

for $C(T) \to 0$ as $T \to 0$. It remains to show that there are p and q, ${}^1/_p + {}^1/_q = {}^1/_2$, such that

$$\begin{aligned} {}_0W^{5/8}_{2,\mu}(J) &\hookrightarrow W^s_q(J), && \text{for } s = \frac{1}{8} + \epsilon \text{ with } \frac{1}{2} > \epsilon > 0, \\ W^{1/8}_{2,\mu}(J) &\hookrightarrow L_{p,\mu}(J), \end{aligned}$$

where the operator norms of the embeddings do not depend on T. By Theorem 2.1.10, p and q have to fulfill the conditions

$$\begin{aligned} \frac{5}{8} - (1-\mu) - \frac{1}{2} &> \frac{1}{8} + \epsilon - \frac{1}{q}, && \text{with } \frac{1}{2} > \epsilon > 0, \\ \frac{1}{8} - \frac{1}{2} &> -\frac{1}{p}. \end{aligned}$$

Direct calculations confirm that for $\mu \in ({}^7/_8, 1]$ suitable p and q can be chosen. Therefore, we have

$$[fg]^2_{W^{1/8}_{2,\mu}(J)} \leq C(T) \|f\|^2_{W^s_q(J)} \|g\|^2_{L_{p,\mu}(J)} \leq C(T) \|f\|^2_{W^{5/8}_{2,\mu}(J)} \|g\|^2_{W^{1/8}_{2,\mu}(J)}$$

for a $C(T) \to 0$ monotonically as $T \to 0$, as the operator norms of the embeddings do not depend on T. This proves the claim.

Ad 2:
We want to show that ${}_0W^{5/8}_{2,\mu}(J)$ is a Banach algebra up to a constant in the norm-estimate. The proof of the $\|\cdot\|_{L_{2,\mu}(J)}$-part of the norm is done in the proof of item 1. For the $[\cdot]_{W^{5/8}_{2,\mu}(J)}$-part, we add a zero like in the proof of item 1. We can now estimate both summands analogously to the first summand in the previous proof and obtain

$$\begin{aligned} \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|f(t)(g(t)-g(\tau))|^2}{|t-\tau|^{1+2\frac{5}{8}}} \, \mathrm{d}\tau \, \mathrm{d}t &\leq \sup_{t \in J} |f(t)|^2 \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|g(t)-g(\tau)|^2}{|t-\tau|^{1+2\frac{5}{8}}} \, \mathrm{d}\tau \, \mathrm{d}t \\ &\leq \sup_{t \in J} |f(t) - f(0)|^2 \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|g(t)-g(\tau)|^2}{|t-\tau|^{1+2\frac{5}{8}}} \, \mathrm{d}\tau \, \mathrm{d}t \\ &\leq C T^{2\alpha} \|f\|^2_{W^{5/8}_{2,\mu}(J)} \|g\|^2_{W^{5/8}_{2,\mu}(J)}, \end{aligned}$$

where $C(T) := CT^{2\alpha} \to 0$ as $T \to 0$.

Ad 3:
Let $f,g \in W^{3/8}_{2,\mu}(J) = {}_0W^{3/8}_{2,\mu}(J)$, where the equality is due to Proposition 2.10 in [24]. We want to prove that $fg \in W^{1/8}_{2,\mu}(J)$. In order to do this, we begin with the $\|\cdot\|_{L_{2,\mu}(J)}$-part of the norm: For $3/8 < 1-\mu+1/2$, $\mu \in (7/8;1]$ we have by Lemma 2.1.15, item 1 and 2,

$$W^{3/8}_{2,\mu}(J) \hookrightarrow L_{p,\mu}(J) \qquad \text{for } \frac{3}{8}-\frac{1}{2} > -\frac{1}{p},$$

$$W^{3/8}_{2,\mu}(J) \hookrightarrow L_{q}(J) \qquad \text{for } \frac{3}{8}-(1-\mu)-\frac{1}{2} > -\frac{1}{q},$$

where the operator norms of the embeddings do not depend on T. By direct computation, we see that we can choose p and $\tilde{q} < q$ fulfilling $1/p + 1/\tilde{q} = 1/2$ for $\mu \in (7/8;1]$. Therefore, Hölder's inequality for p and $\tilde{q}$ yields

$$\begin{aligned}\|fg\|^2_{L_{2,\mu}(J)} &= \int_J (f(t))^2 \left(g(t)t^{(1-\mu)}\right)^2 \,\mathrm{d}t = \left(\int_J (f(t))^{\tilde{q}}\,\mathrm{d}t\right)^{\frac{2}{\tilde{q}}} \left(\int_J \left(g(t)t^{(1-\mu)}\right)^p \mathrm{d}t\right)^{\frac{2}{p}} \\ &\le C(T)\,\|f\|^2_{L_q(J)}\,\|g\|^2_{L_{p,\mu}(J)}\end{aligned}$$

for some $C(T) \to 0$ monotonically as $T \to 0$.

The next step is to control the semi-norm $[\cdot]_{W^{1/8}_{2,\mu}(J)}$: We use again item 1 and 2 of Lemma 2.1.15 for $3/8 < 1-\mu+1/2$, $\mu \in (7/8;1]$, and obtain

$$W^{3/8}_{2,\mu}(J) \hookrightarrow W^{1/8+\epsilon}_{p,\mu}(J) \qquad \text{for } \frac{3}{8}-\frac{1}{2} > \frac{1}{8}+\epsilon-\frac{1}{p}$$

$$W^{3/8}_{2,\mu}(J) \hookrightarrow L_{q}(J) \qquad \text{for } \frac{3}{8}-(1-\mu)-\frac{1}{2} > -\frac{1}{q}$$

for a suitably small $\epsilon > 0$. By direct calculations, it turns out that we can choose p and q, such that $1/p + 1/q = 1/2$, as long as ϵ is small enough. Using Hölder's inequality for the measure $d\tau dt/|t-\tau|$, we deduce

$$\begin{aligned}[fg]^2_{W^{1/8}_{2,\mu}(J)} &\le \int_0^T\int_0^t t^{2(1-\mu)} \frac{\left|f(t)g(t)-f(\tau)g(\tau)\right|^2}{|t-\tau|^{1+2\frac{1}{8}}}\,\mathrm{d}\tau\,\mathrm{d}t \\ &\le 2\left[\int_0^T\int_0^t \left(t^{(1-\mu)}\frac{|f(t)-f(\tau)|}{|t-\tau|^{\frac{1}{8}+\epsilon}}\frac{|g(t)|}{|t-\tau|^{-\epsilon}}\right)^2 \frac{\mathrm{d}\tau\,\mathrm{d}t}{|t-\tau|}\right. \\ &\quad \left.+\int_0^T\int_0^t \left(t^{(1-\mu)}\frac{|g(t)-g(\tau)|}{|t-\tau|^{\frac{1}{8}+\epsilon}}\frac{|f(\tau)|}{|t-\tau|^{-\epsilon}}\right)^2 \frac{\mathrm{d}\tau\,\mathrm{d}t}{|t-\tau|}\right] \\ &\le 2\left(\int_0^T\int_0^t \left(t^{(1-\mu)}\frac{|f(t)-f(\tau)|}{|t-\tau|^{\frac{1}{8}+\epsilon}}\right)^p \frac{\mathrm{d}\tau\,\mathrm{d}t}{|t-\tau|}\right)^{\frac{2}{p}} \left(\int_0^T\int_0^t \left(\frac{|g(t)|}{|t-\tau|^{-\epsilon}}\right)^q \frac{\mathrm{d}\tau\,\mathrm{d}t}{|t-\tau|}\right)^{\frac{2}{q}} \\ &\quad +2\left(\int_0^T\int_0^t \left(t^{(1-\mu)}\frac{|g(t)-g(\tau)|}{|t-\tau|^{\frac{1}{8}+\epsilon}}\right)^p \frac{\mathrm{d}\tau\,\mathrm{d}t}{|t-\tau|}\right)^{\frac{2}{p}} \left(\int_0^T\int_0^t \left(\frac{|f(\tau)|}{|t-\tau|^{-\epsilon}}\right)^q \frac{\mathrm{d}\tau\,\mathrm{d}t}{|t-\tau|}\right)^{\frac{2}{q}} \\ &\le 2\,\|f\|^2_{W^{\frac{1}{8}+\epsilon}_{p,\mu}} \left(\int_0^T\int_0^t \frac{|g(t)|^q}{|t-\tau|^{1-\epsilon q}}\,\mathrm{d}\tau\,\mathrm{d}t\right)^{\frac{2}{q}}\end{aligned}$$

$$+ 2 \, \|g\|^2_{W^{\frac{1}{8}+\epsilon}_{p,\mu}} \left(\int_0^T \int_0^t \frac{|f(\tau)|^q}{|t-\tau|^{1-\epsilon q}} \, \mathrm{d}\tau \, \mathrm{d}t \right)^{\frac{2}{q}} .$$

We take a closer look at the second factor of the second summand: By a change of variables and Fubini's theorem, we obtain

$$\begin{aligned}\int_0^T \int_0^t \frac{|f(\tau)|^q}{|t-\tau|^{1-\epsilon q}} \, \mathrm{d}\tau \, \mathrm{d}t &= \int_0^T |f(\tau)|^q \int_\tau^T \frac{1}{|t-\tau|^{1-\epsilon q}} \, \mathrm{d}t \, \mathrm{d}\tau = \int_0^T |f(t)|^q \left[\frac{|t-\tau|^{\epsilon q}}{\epsilon q} \right]_\tau^T \mathrm{d}t \\ &\leq C(q,T) \|f\|^q_{L_q(J)},\end{aligned}$$

where $C(q,T) \to 0$ monotonically as $T \to 0$. For the second factor of the first summand, it follows similarly

$$\int_0^T \int_0^t \frac{|g(t)|^q}{|t-\tau|^{1-\epsilon q}} \, \mathrm{d}\tau \, \mathrm{d}t \leq C(q,T) \|g\|^q_{L_q(J)}$$

for $C(q,T) \to 0$ monotonically as $T \to 0$. Thus, we have

$$\begin{aligned}[fg]^2_{W^{1/8}_{2,\mu}(J)} &\leq C(q,T) \left[\|f\|^2_{W^{1/8+\epsilon}_{p,\mu}(J)} \|g\|^2_{L_q(J)} + \|g\|^2_{W^{1/8+\epsilon}_{p,\mu}(J)} \|f\|^2_{L_q(J)} \right] \\ &\leq C(q,T) \, \|f\|^2_{W^{3/8}_{2,\mu}} \|g\|^2_{W^{3/8}_{2,\mu}} ,\end{aligned}$$

for $C(q,T) \to 0$ monotonically as $T \to 0$. Combining this with the estimate of $\|fg\|_{L_{2,\mu}}$, we deduce the claim.

Ad 4:
By item 3 of Lemma 2.1.15, we obtain $f \in C^0(\bar{J})$. W.l.o.g. we can assume $f(t) \geq C > 0$ for all $t \in J$. Otherwise we replace f by $-f$ and $[C, \infty)$ by $(-\infty; -C]$. By Lipschitz-continuity of $1/x$ on the interval $[C, \infty)$, we obtain

$$\begin{aligned}\|(f)^{-1}\|^2_{L_{2,\mu}(J)} &= \int_J t^{2(1-\mu)} \left|(f(t))^{-1}\right|^2 \mathrm{d}t \\ &\leq 2 \left(\int_J t^{2(1-\mu)} \left|(f(t))^{-1} - C^{-1}\right|^2 \mathrm{d}t + \int_J t^{2(1-\mu)} C^{-2} \, \mathrm{d}t \right) \\ &\leq 2 \left(L^2 \int_J t^{2(1-\mu)} \left|f(t) - C\right|^2 \mathrm{d}t + C(T_0) \right) \leq C \int_J t^{2(1-\mu)} |f(t)|^2 \, \mathrm{d}t + C(T_0) \\ &\leq C \|f\|^2_{L_{2,\mu}(J)} + C(T_0),\end{aligned}$$

where L is the Lipschitz constant of $1/x$ on the interval $[C, \infty)$. For the semi-norm part, we derive

$$\begin{aligned}\left[(f)^{-1}\right]^2_{W^s_{2,\mu}(J)} &\leq \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{\left|(f(t))^{-1} - (f(\tau))^{-1}\right|^2}{|t-\tau|^{1+2s}} \, \mathrm{d}\tau \, \mathrm{d}t \\ &\leq L^2 \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|f(t) - f(\tau)|^2}{|t-\tau|^{1+2s}} \, \mathrm{d}\tau \, \mathrm{d}t = L^2 \|f\|^2_{W^s_{2,\mu}(J)}.\end{aligned}$$

This proves the regularity and the estimate follows directly by the previous calculations. □

2.2 Maximal L_2-Regularity with Temporal Weights and Related Embeddings

At several points in this work we need to solve linear problems with optimal regularity. The following statements are based on the results in [24], [25], and [23].

2.2.1 A Maximal L_2-Regularity Result with Temporal Weights for Parabolic Problems

We will introduce a simplified version of a maximal L_p-regularity result with temporal weights for parabolic problems for $p = 2$, see Theorem 2.1 in [25]. Before stating the theorem, we need some notation:

Let E be a complex Banach space of class $\mathcal{HT}$ and $\mu \in (1/2, 1]$. Let $J = (0, T)$ be a finite interval and let $I = (0, c)$, $c \in \mathbb{R}$. Furthermore, let $\mathcal{A}$ be an operator of order $2m$, $m \in \mathbb{N}$, given by

$$\mathcal{A}(t, \sigma, D) = a(t, \sigma) D^{2m}, \qquad \text{for } \sigma \in I, t \in J,$$

where $D = -i\partial_\sigma$ and $a(t, \sigma) \in \mathbb{R}$. Let the boundary operators $\mathcal{B}_j$ be given by

$$\mathcal{B}_j(t, \sigma, D) = b_j(t, \sigma) \operatorname{tr}_{\partial I} D^{m_j}, \qquad \text{for } \sigma \in I, t \in J, j = 1, \ldots, m,$$

where $m_j \in \{0, \ldots, 2m-1\}$ is the order of $\mathcal{B}_j$ and $b_j(t, \sigma) \in \mathbb{R}$. We assume that each of these operators is non-trivial. Moreover, we define for $\sigma = 0, c$ the rotated operators

$$\mathcal{A}^\nu(t, 0, D) := \mathcal{A}(t, 0, D) \qquad \mathcal{A}^\nu(t, c, D) := \mathcal{A}(t, \sigma, -D).$$

Furthermore, we will use the space

$$C_0\left([0, \infty); E\right) := \left\{ f : [0, \infty) \to E \text{ is continuous with } \lim_{t \to \infty} f(t) = 0 \right\}. \tag{2.2.1}$$

We look for a solution ρ of

$$\begin{aligned} \rho_t + \mathcal{A}(t, \sigma, D)\rho &= F(t, \sigma) && \text{for } \sigma \in I, t \in J, \\ \mathcal{B}_j(t, \sigma, D)\rho &= G_j(t, \sigma) && \text{for } \sigma \in \partial I, t \in J, j = 1, \ldots, m, \\ \rho(0, \sigma) &= \rho_0(\sigma) && \text{for } \sigma \in I. \end{aligned} \tag{2.2.2}$$

To this end, we define the spaces

$$\begin{aligned} \mathbb{E}_{\mu,T,E} &:= W^1_{2,\mu}\big(J; L_2(I; E)\big) \cap L_{2,\mu}\big(J; W^{2m}_2(I; E)\big), \\ \mathbb{E}_{0,\mu,E} &:= L_{2,\mu}\big(J; L_2(I; E)\big), \\ X_{\mu,E} &:= W^{2m(\mu - 1/2)}_2(I; E), \\ \mathbb{F}_{j,\mu,E} &:= W^{\omega_j}_{2,\mu}\big(J; L_2(\partial I; E)\big) \cap L_{2,\mu}\big(J; W^{2m\omega_j}_2(\partial I; E)\big), \end{aligned} \tag{2.2.3}$$

where $\omega_j := 1 - m_j/2m - 1/4m$, $j = 1, \ldots, m$, and for convenience

$$\tilde{\mathbb{F}}_{\mu,E} := \mathbb{F}_{1,\mu,E} \times \cdots \times \mathbb{F}_{m,\mu,E}.$$

All the spaces are equipped with their natural norms. We will omit the subscript $\cdot_E$ in the spaces if $E = \mathbb{R}$, e.g. $\mathbb{E}_{\mu,T} := \mathbb{E}_{\mu,T,\mathbb{R}}$.

Now we can state the maximal L_2-regularity result, which is a simplified version of Theorem 2.1 in [25].

Theorem 2.2.1
Let E be a complex Banach space of class $\mathcal{HT}$ and $\mu \in \left(\frac{1}{2}, 1\right]$. Let $J = (0,T)$ be a finite interval and let $I = (0,c)$, $c \in \mathbb{R}$. Let the operators $\mathcal{A}$ and $\mathcal{B}_j$, $j = 1,\dots,m$ be non-trivial. Furthermore, let the following conditions hold true:

(SD) *It holds $a \in BUC(\bar{J} \times \bar{I})$.*

(SB) *For $j = 1,\dots,m$ it holds*
either $b_j \in C^{\tau_j, 2m\tau_j}(\bar{J} \times \partial I)$ with some $\tau_j > \omega_j$,
or $b_{m_j} \in \mathbb{F}_{j,\mu}$ and $\omega_j > 1 - \mu + \frac{1}{2}$.

(E) *For all $t \in \bar{J}$, $x \in \bar{I}$, and $|\xi| = 1$, it holds for the spectrum $\Sigma(\mathcal{A}(t,\sigma,\xi)) \subset \mathbb{C}_+ := \{\Re z > 0\}$. (normal ellipticity)*

(LS) *For each fixed $t \in \bar{J}$ and $\sigma \in \partial I$, for each $\lambda \in \overline{\mathbb{C}_+}$ with $|\lambda| \neq 0$, and each $h \in E^m$ the ordinary initial value problem*

$$\begin{aligned} \lambda v(y) + \mathcal{A}^\nu(t,\sigma,D_y)v(y) &= 0 && y > 0 \\ \mathcal{B}_j^\nu(t,\sigma,D_y)v(y)|_{y=0} &= h_j && j = 1,\dots,m \end{aligned}$$

has a unique solution $v \in C_0([0,\infty);E)$. (Lopatinskii-Shapiro-condition)

Furthermore, assume that $\omega_j \neq 1 - \mu + \frac{1}{2}$ for $j = 1,\dots,m$. Then the problem (2.2.2) has a unique solution $\rho := \mathcal{L}^{-1}(F,\tilde{G},\rho_0) \in \mathbb{E}_{\mu,T,E}$, $\tilde{G} := (G_1,\dots,G_m)$, if and only if $(F,\tilde{G},\rho_0) \in \mathcal{D}$, where

$$\mathcal{D} := \Big\{ \Big(F,\tilde{G},\rho_0\Big) \in \mathbb{E}_{0,\mu,E} \times \tilde{\mathbb{F}}_{\mu,E} \times X_{\mu,E} : \text{ for } j = 1,\dots,m \text{ it holds} \\ \mathcal{B}_j(0,\cdot,D)\rho_0 = G_j(\cdot,0) \text{ on } \partial I \text{ if } \omega_j > 1 - \mu + \frac{1}{2} \Big\}.$$

The corresponding solution operator $\mathcal{L}^{-1} : \mathcal{D} \to \mathbb{E}_{\mu,T,E}$ is continuous. If $\mathcal{L}^{-1}$ is restricted to

$$\mathcal{D}_0 := \Big\{ \Big(F,\tilde{G},\rho_0\Big) \in \mathcal{D} : G_j(0,\cdot) = 0 \text{ on } \partial I \text{ if } \omega_j > 1 - \mu + \frac{1}{2} \text{ for } j = 1,\dots,m \Big\}$$

for any given $T_0 > 0$ the operator norm of the restriction is uniformly bounded for $T \in (0,T_0]$.

Note that in the case that the data are real-valued, then the solution is real-valued as well due to the remark before Theorem 2.2 in [25].

In the following, we want to give some mapping properties of the involved spaces:
By the combination of Proposition 1.4.2 in Chapter III.1 of [2] and Lemma 2.6 in [24], it follows for the temporal trace space

$$\mathbb{E}_{\mu,T,E} \hookrightarrow BUC\big(\bar{J}; X_{\mu,E}\big). \tag{2.2.4}$$

Additionally, there exists a continuous right inverse

$$X_{\mu,E} \to \mathbb{E}_{\mu,\infty,E},$$

cf. [24] Lemma 4.3. By [24] Lemma 3.4, the pointwise realization of $\partial_\sigma^{m_j}$ is a continuous map

$$\mathbb{E}_{\mu,T,E} \to H_{2,\mu}^{1-m_j/2m}\big(J; L_2(I;E)\big) \cap L_{2,\mu}\big(J; W_2^{2m-m_j}(I;E)\big)$$

for $m_j \in \{1, \dots, 2m\}$. Its operator norm is independent of T, if we restrict to ${}_0\mathbb{E}_{\mu,T,E}$, where

$${}_0\mathbb{E}_{\mu,T,E} := {}_0W^1_{2,\mu}\big(J; L_2(I;E)\big) \cap L_{2,\mu}\big(J; W^{2m}_2(I;E)\big).$$

Moreover, by [24] Lemma 4.5, the spatial trace tr_I is a continuous map

$$H^{1-m_j/2m}_{2,\mu}\big(J; L_2(I;E)\big) \cap L_{2,\mu}\big(J; W^{2m-m_j}_2(I;E)\big) \to W^{\omega_j}_{2,\mu}\big(J; L_2(\partial I;E)\big) \cap L_{2,\mu}\big(J; W^{2m\omega_j}_2(\partial I;E)\big).$$

with $\omega_j := 1 - {}^{m_j}/{}_{2m} - {}^{1}/{}_{4m}$, $m_j \in \{1, \dots, 2m\}$. The operator norm is independent of the length of J, if we restrict to the space

$${}_0H^{1-m_j/2m}_{2,\mu}\big(J; L_2(I;E)\big) \cap L_{2,\mu}\big(J; W^{2m-m_j}_2(I;E)\big).$$

We will often use that

$$W^s_2\big(I;E\big) \hookrightarrow C^{k,\alpha}\big(\bar{I};E\big) \hookrightarrow C^k\big(\bar{I};E\big) \tag{2.2.5}$$

for $s - {}^1/{}_2 > k + \alpha > 0$, $k \in \mathbb{N}$, and $\alpha \in (0,1)$, with the estimate

$$\|\rho\|_{C^k(\bar{I};E)} \leq \|\rho\|_{C^{k,\alpha}(\bar{I};E)} \leq C\|\rho\|_{W^s_2(I;E)},$$

cf. the Remark 2.1.16. In particular,

$$X_{\mu,E} = W^{2m(\mu-1/2)}_2\big(I;E\big) \hookrightarrow C^k\left(\bar{I};E\right) \tag{2.2.6}$$

for $2m\left(\mu - {}^1/{}_2\right) - {}^1/{}_2 > k$.

Remark 2.2.2 *1. The previous well-posedness result is optimal in the following sense: Applying the operators on the left-hand side of the system* (2.2.2) *to the solution gives by the previously mentioned mappings the same regularity as demanded for the right-hand side terms.*

2. Compared to the unweighted approach, this result has some advantages, see Chapter 2 in [25]. One is that the theorem allows for initial data in flexible spaces, i.e. $X_{\mu,E}$, $\mu \in ({}^1/{}_2, 1]$. Moreover, one can directly exploit the inherent smoothing effect of parabolic equations: Let the data $(F, \tilde{G}, \rho_0) \in \mathbb{E}_{0,\mu,E} \times \tilde{\mathbb{F}}_{\mu,E} \times X_{\mu,E}$, $\mu \in ({}^1/{}_2, 1)$ be in $\mathcal{D}$. For the solution $\rho = \mathcal{L}^{-1}(F, \tilde{G}, \rho_0) \in \mathbb{E}_{\mu,T,E}$, we obtain for each $0 < \epsilon < T$

$$\|\rho\|_{W^1_{2,\mu}((\epsilon,T);L_2(I;E)) \cap L_{2,\mu}((\epsilon,T);W^{2m}_2(I;E))} \leq C(T) \left\| \left(F, \tilde{G}, \rho_0\right) \right\|_{\mathbb{E}_{0,\mu,E} \times \tilde{\mathbb{F}}_{\mu,E} \times X_{\mu,E}}.$$

Combining item 3 of Remark 2.1.2 and (2.2.4)*, we deduce immediately*

$$\|\rho(t)\|_{X_1} \leq C(T,\epsilon) \left\| \left(F, \tilde{G}, \rho_0\right) \right\|_{\mathbb{E}_{0,\mu,E} \times \tilde{\mathbb{F}}_{\mu,E} \times X_{\mu,E}} \qquad \text{for } t \in (\epsilon, T].$$

This means that we can control a strong norm of the solution at time $t \neq 0$ by a weaker norm at an earlier time and the corresponding data.

2.2.2 Some Useful Embeddings for Parabolic Spaces

Lemma 2.2.3
Let T_0 be fixed, $J = (0,T)$, $0 < T \leq T_0$, and I a bounded open interval. Let E be a Banach space.

1. *Then*

$$\mathbb{E}_{\mu,T,E} \hookrightarrow BUC\big(\bar{J}, X_{\mu,E}\big)$$

with the estimate

$$\|\rho\|_{BUC(\bar{J},X_{\mu,E})} \leq C(T_0)\left(\|\rho\|_{\mathbb{E}_{\mu,T,E}} + \|\rho_{|t=0}\|_{X_{\mu,E}}\right).$$

2. *Let* $m = 2$ *in* (2.2.3) *and* $\mu \in \left(\frac{7}{8}, 1\right]$. *Then there exists an* $\bar{\alpha} \in (0,1)$ *such that*

$$\mathbb{E}_{\mu,T,\mathbb{R}^n} \hookrightarrow C^{\bar{\alpha}}\left(\bar{J}; C^1(\bar{I};\mathbb{R}^n)\right)$$

with the estimate

$$\|\rho\|_{C^{\bar{\alpha}}([0,T];C^1(\bar{I};\mathbb{R}^n))} \leq C(T_0)\left(\|\rho\|_{\mathbb{E}_{\mu,T,\mathbb{R}^n}} + \|\rho_{|t=0}\|_{X_{\mu,\mathbb{R}^n}}\right).$$

3. *Let* $m = 2$ *in* (2.2.3) *and* $\mu \in \left(\frac{7}{8}, 1\right]$. *Then the pointwise realization of the* k*-th spatial derivative* ∂_σ^k, $k = 1, \ldots, 4$, *is a continuous map*

$$\mathbb{E}_{\mu,T,E} \hookrightarrow H_{2,\mu}^{(4-k)/4}\big(J; L_2(I;E)\big) \cap L_{2,\mu}\big(J; H_2^{4-k}(I;E)\big)$$

with the estimate

$$\left\|\partial_\sigma^k \rho\right\|_{H_{2,\mu}^{(4-k)/4}(J;L_2(I;E))\cap L_{2,\mu}(J;H_2^{4-k}(I;E))} \leq C(T_0)\left(\|\rho\|_{\mathbb{E}_{\mu,T,E}} + \|\rho_0\|_{X_{\mu,E}}\right).$$

4. *Let* $m = 2$ *in* (2.2.3) *and* $\mu \in \left(\frac{7}{8}, 1\right]$. *Then the spatial trace operator applied to the* k*-th spatial derivative* $tr_{|I}\partial_\sigma^k$, $k = 0,1,2,3$, *is a continuous map*

$$\mathbb{E}_{\mu,T,E} \to W_{2,\mu}^{(8-2k-1)/8}\big(J; L_2(\partial I;E)\big) \cap L_{2,\mu}\big(J; W_2^{(8-2k-1)/2}(\partial I;E)\big)$$

with the estimate

$$\begin{aligned}\left\|tr_{|I}\partial_\sigma^k \rho\right\|_{W_{2,\mu}^{(8-2k-1)/8}(J;L_2(\partial I;E))\cap L_{2,\mu}(J;W_2^{(8-2k-1)/2}(\partial I;E))}& \\ \leq C(T_0)\left(\|\rho\|_{\mathbb{E}_{\mu,T,E}} + \|\rho_{|t=0}\|_{X_{\mu,E}}\right)&.\end{aligned}$$

Proof. Ad 1:
Let $\rho \in \mathbb{E}_{\mu,T,E}$. Using the embedding (2.2.4), we can evaluate ρ at $t = 0$ with $\rho_{|t=0} \in X_{\mu,E}$, and, by Lemma 4.3 in [24], we find an extension $E\rho \in \mathbb{E}_{\mu,\infty,E} = W_{2,\mu}^1(\mathbb{R}^+; L_2(I;E)) \cap L_{2,\mu}(\mathbb{R}^+; W_2^4(I;E))$ of ρ_0, such that $E\rho_{|t=0} = \rho_{|t=0}$ in I and

$$\|E\rho\|_{\mathbb{E}_{\mu,\infty,E}} \leq C\|\rho_{|t=0}\|_{X_{\mu,E}},$$

where the constant C does not depend on T. An argumentation analogous to the proof of Lemma 2.1.15 proves the claim.

Ad 2:
In the following, we want to prove that

$$\mathbb{E}_{\mu,T,\mathbb{R}^n} \hookrightarrow C^{\bar{\alpha}}\left(\bar{J}; W_2^s(I;\mathbb{R}^n)\right) \hookrightarrow C^{\bar{\alpha}}\left(\bar{J}; C^1(\bar{I};\mathbb{R}^n)\right)$$

for a suitable s, and the corresponding norm estimate.

First of all, we have

$$\mathbb{E}_{\mu,T,\mathbb{R}^n} \hookrightarrow W^1_{2,\mu}(J; L_2(I;\mathbb{R}^n)) \hookrightarrow C^\alpha(\bar{J}; L_2(I;\mathbb{R}^n)) \tag{2.2.7}$$

for $\alpha \in (0, \mu - 1/2)$ by Lemma 2.1.15, item 3. The operator norm of the embedding depends on T_0, but not on T, if one uses a suitable norm, cf. (2.1.17). Moreover, by the definition of Slobodetskii spaces combined with the properties of interpolation spaces, see Proposition 1.2.3 in [22], it holds

$$X_{\mu,\mathbb{R}^n} = W_2^{4(\mu-1/2)}(I;\mathbb{R}^n) \hookrightarrow W_2^s(I;\mathbb{R}^n)$$

for $4(\mu - 1/2) > s$. Additionally, we have by (2.2.5)

$$W_2^s(I;\mathbb{R}^n) \hookrightarrow C^1(\bar{I};\mathbb{R}^n),$$

if $s > 1 + 1/2$. For $\mu \in (7/8, 1]$, it is possible to choose s such that $4(\mu - 1/2) > s > 1 + 1/2$, thus, we can combine the previous embeddings. By Theorem 1 in Section 4.3.1 of [30], we know that

$$W_2^s(I;\mathbb{R}^n) = \Big(L_2(I;\mathbb{R}^n), W_2^{4(\mu-1/2)}(I;\mathbb{R}^n)\Big)_{\theta,2}$$

for θ fulfilling $s = \theta 4(\mu - 1/2)$. Thus, we have for $u \in X_{\mu,\mathbb{R}^n}$

$$\|u\|_{C^1(\bar{I};\mathbb{R}^n)} \leq C\|u\|_{W_2^s(I;\mathbb{R}^n)} \leq C\|u\|^{1-\theta}_{L_2(I;\mathbb{R}^n)}\|u\|^{\theta}_{X_{\mu,\mathbb{R}^n}},$$

for a constant C independent of T. Replacing u by $\rho(t) - \rho(s)$, $t \neq s$, for $\rho \in \mathbb{E}_{\mu,T,\mathbb{R}^n}$, we obtain

$$\|\rho(t) - \rho(s)\|_{C^1(\bar{I};\mathbb{R}^n)} \leq C\|\rho(t) - \rho(s)\|^{1-\theta}_{L_2(I;\mathbb{R}^n)}\|\rho(t) - \rho(s)\|^{\theta}_{X_{\mu,\mathbb{R}^n}}. \tag{2.2.8}$$

We take a look at the factors of (2.2.8) separately: using the embedding (2.2.7), we obtain

$$\|\rho(t) - \rho(s)\|_{L_2(I;\mathbb{R}^n)} \leq C(T_0)|t-s|^\alpha \left(\|\rho\|_{\mathbb{E}_{\mu,T,\mathbb{R}^n}} + \|\rho_{|t=0}\|_{L_2(I;\mathbb{R}^n)}\right),$$

where the constant does not depend on T. For the second factor, we obtain

$$\|\rho(t) - \rho(s)\|_{X_{\mu,\mathbb{R}^n}} \leq 2\|\rho\|_{BUC(\bar{J};X_{\mu,\mathbb{R}^n})}.$$

Combining these estimates and using item 1 of Lemma 2.2.3, we deduce from (2.2.8)

$$\begin{aligned}\|\rho(t) - \rho(s)\|_{C^1(\bar{I};\mathbb{R}^n)} &\leq C(T_0)|t-s|^{\alpha(1-\theta)} \left(\|\rho\|_{\mathbb{E}_{\mu,T,\mathbb{R}^n}} + \|\rho_{|t=0}\|_{L_2(I;\mathbb{R}^n)}\right)^{1-\theta} \|\rho\|^{\theta}_{BUC(\bar{J};X_{\mu,\mathbb{R}^n})} \\ &\leq C(T_0)|t-s|^{\alpha(1-\theta)} \left(\|\rho\|_{\mathbb{E}_{\mu,T,\mathbb{R}^n}} + \|\rho_{|t=0}\|_{X_{\mu,\mathbb{R}^n}}\right)\end{aligned}$$

with $C(T_0)$ independent of T. Additionally, we obtain by $X_{\mu,\mathbb{R}^n} \hookrightarrow C^1(\bar{I};\mathbb{R}^n)$, see (2.2.6), and item 1 of Lemma 2.2.3

$$\|\rho\|_{C(\bar{J};C^1(\bar{I};\mathbb{R}^n))} \leq C\|\rho\|_{C(\bar{J};X_{\mu,\mathbb{R}^n})} \leq C(T_0)\left(\|\rho\|_{\mathbb{E}_{\mu,T,\mathbb{R}^n}} + \|\rho_{|t=0}\|_{X_{\mu,\mathbb{R}^n}}\right).$$

Therefore, it follows

$$\begin{aligned}\|\rho\|_{C^{\alpha(1-\theta)}([0,T];C^1(\bar{I};\mathbb{R}^n))} &\leq C(T_0)\left(\|\rho\|_{\mathbb{E}_{\mu,T,\mathbb{R}^n}} + \|\rho_{|t=0}\|_{X_{\mu,\mathbb{R}^n}}\right) + \|\rho\|_{C(\bar{J};C^1(\bar{I};\mathbb{R}^n))} \\ &\leq C(T_0)\left(\|\rho\|_{\mathbb{E}_{\mu,T,\mathbb{R}^n}} + \|\rho_{|t=0}\|_{X_{\mu,\mathbb{R}^n}}\right),\end{aligned}$$

the constant $C(T_0)$ does not depend on T.

Ad 3:
Let $\rho \in {}_0\mathbb{E}_{\mu,T,E} := {}_0H^1_{2,\mu}(J; H^0_2(I;E)) \cap H^0_{2,\mu}(J; H^4_2(I;E))$, cf. item 3 of Remark 2.1.6. Then the claim follows directly by applying k times Lemma 3.4 in [24].

We proceed with the general case: We know by (2.2.4) that $\rho \in \mathbb{E}_{\mu,T,E}$ has its temporal trace in $W_2^{4(\mu-1/2)}(I;E)$. By Lemma 4.3 in [24], we obtain an extension $E\rho \in \mathbb{E}_{\mu,T,E}$, where the operator norm of the extension operator E does not depend on T. The function $\rho - E\rho$ has again trace zero and we can proceed as before in the proof of Lemma 2.1.15.

Ad 4:
Let $\rho \in {}_0\mathbb{E}_{\mu,T,E} = {}_0H^1_{2,\mu}(J; H^0_2(I;E)) \cap H^0_{2,\mu}(J; H^4_2(I;E))$. Combining item 3 with Theorem 4.5 in [24], we directly obtain the result. For the general case, we again apply the same strategy as before in the proof of Lemma 2.1.15. □

The following proposition will be needed to estimate non-linearities.

Proposition 2.2.4
Let $J = (0,T)$ let I be a bounded open interval. Further let $\mu \in \left(\frac{1}{2}, 1\right]$, $k \in \mathbb{N}$ and $q \in [2,\infty]$. Then

$$L_\infty\left(J, W_2^{4(\mu-1/2)}(I)\right) \cap L_{2,\mu}\left(J, W_2^4(I)\right) \hookrightarrow L_{l,\tilde{\mu}}\left(J, W_q^k(I)\right)$$

for $\tilde{\mu} = \mu + (1-\theta)(1-\mu) \in [\mu, 1]$, if $k + \frac{1}{2} - \frac{1}{q} = 4\left(\mu - \frac{1}{2}\right)(1-\theta) + 4\theta$ and $l = \frac{2}{\theta}$ for a $\theta \in (0,1)$. The operator norm does not depend on T.

Proof. To show the embedding, we use

$$W_2^s(I) \hookrightarrow W_q^k(I) \qquad \text{for } s - \frac{1}{2} \geq k - \frac{1}{q} \text{ with } q \geq 2, \tag{2.2.9}$$

which follows by the definition of the spaces and by a standard embedding theorem for Besov spaces, see for example Theorem 6.5.1 in [4]. Furthermore, by Theorem 1 in Section 4.3.1 of [30], we have

$$W_2^s(I) = \left(W_2^{4(\mu-1/2)}(I), W_2^4(I)\right)_{\theta,2} \qquad \text{for } s = 4\left(\mu - \frac{1}{2}\right)(1-\theta) + 4\theta, \theta \in (0,1)$$

with

$$\|\phi\|_{W_2^s(I)} \leq C\|\phi\|^{1-\theta}_{W_2^{4(\mu-1/2)}(I)}\|\phi\|^\theta_{W_2^4(I)} \qquad \text{for } \phi \in W_2^{4(\mu-1/2)}(I) \cap W_2^4(I) = W_2^4(I). \tag{2.2.10}$$

In the following, we consider $\|\rho\|_{L_{l,\tilde{\mu}}(J,W_q^k(I))}$ for a $\rho \in L_\infty(J, W_2^{4(\mu-1/2)}(I)) \cap L_{2,\mu}(J, W_2^4(I))$ for $l = 2/\theta$ and $\tilde{\mu} = \mu + (1-\theta)(1-\mu)$. To this end, we use $s = k + 1/2 - 1/q = 4(\mu - 1/2)(1-\theta) + 4\theta$ and combine (2.2.9) and (2.2.10). We obtain

$$\|\rho\|_{L_{l,\tilde{\mu}}(J,W_q^k(I))} \leq C\|\rho\|_{L_{l,\tilde{\mu}}(J,W_2^s(I))} \leq C\left\|\|\rho(t)\|^{1-\theta}_{W_2^{4(\mu-1/2)}(I)}\|\rho(t)\|^\theta_{W_2^4(I)}\right\|_{L_{l,\tilde{\mu}}(J)}.$$

Taking care of the time weight, we use $1 - \tilde{\mu} = \theta(1-\mu)$ to deduce

$$\|\rho\|_{L_{l,\tilde{\mu}}(J,W_q^k(I))} \leq C\left\|\|\rho(t)\|^{1-\theta}_{W_2^{4(\mu-1/2)}(I)}\left(t^{1-\mu}\|\rho(t)\|_{W_2^4(I)}\right)^\theta\right\|_{L_l(J)}.$$

By Hölder's inequality for $\tilde{p} = \infty/1-\theta$ and $\tilde{q} = 2/\theta$, it follows

$$\begin{aligned}\|\rho\|_{L_{l,\tilde{\mu}}(J,W_q^k(I))} &\leq C \left\| \|\rho(t)\|_{W_2^{4(\mu-1/2)}(I)} \right\|_{L_\infty(J)}^{1-\theta} \left\| t^{1-\mu} \|\rho(t)\|_{W_2^4(I)} \right\|_{L_2(J)}^{\theta} \\ &\leq C \left(\|\rho\|_{L_\infty(J,W_2^{4(\mu-1/2)}(I))} + \|\rho\|_{L_{2,\mu}(J,W_2^4(I))} \right),\end{aligned}$$

where Young's inequality yields the last estimate. Here, the constant does not depend on T. This shows the claim. $\square$

2.3 An Estimate for the Reciprocal Length of the Curve by its Curvature

We will often use the following lemma.

Lemma 2.3.1
Let $\alpha \in (0,\pi)$. Furthermore, let $c : [0,1] \to \mathbb{R}^2, \sigma \mapsto c(\sigma)$, be a regular curve of class C^2 parametrized proportional to arc length. Moreover, let the unit tangent $\tau := \frac{\partial_\sigma c}{\mathcal{L}[c]}$ fulfill

$$\tau(\sigma) = \begin{pmatrix} \cos\alpha \\ \pm\sin\alpha \end{pmatrix} \qquad \text{for } \sigma = 0, 1,$$

where $\mathcal{L}[c]$ denotes the length of the curve. Then it holds

$$\frac{1}{\mathcal{L}[c]} \le \frac{1}{\sqrt{2}\sin\alpha} \|\kappa[c]\|_{C([0,1])}.$$

Proof. We denote by $\vec{\kappa} = \partial_\sigma^2 c/(\mathcal{L}[c])^2$ and $\kappa = \langle \partial_\sigma^2 c/(\mathcal{L}[c])^2, R\tau\rangle$ the curvature vector and the scalar curvature, respectively. Here, $\langle\cdot,\cdot\rangle$ denotes the euclidean inner product on $\mathbb{R}^2$ and R the matrix which rotates vectors counterclockwise in $\mathbb{R}^2$ by the angle $\pi/2$. We deduce by the fundamental theorem of calculus

$$|\tau(1) - \tau(0)| = \left|\int_0^1 \partial_\sigma \tau(x)\,\mathrm{d}x\right| = \left|\int_0^1 \mathcal{L}[c]\vec{\kappa}[c](x)\,\mathrm{d}x\right| \le \mathcal{L}[c]\|\kappa[c]\|_{C([0,1])},$$

where we used that $\langle \vec{\kappa}, \tau\rangle = 0$. This follows by

$$0 = \partial_\sigma 1 = \partial_\sigma\langle\tau,\tau\rangle = \mathcal{L}[c]\langle\vec{\kappa},\tau\rangle.$$

Moreover, we have

$$|\tau(1) - \tau(0)|^2 = 1 - 2\langle\tau(1),\tau(0)\rangle + 1 = 2 - 2(\cos\alpha)^2 + 2(\sin\alpha)^2 \ge 2(\sin\alpha)^2 > 0,$$

for $\alpha \in (0,\pi)$. Combining both estimates, we deduce the claim. $\square$

3 The Curve Diffusion Flow

In this chapter, we introduce the general setting of the geometric problem of curve diffusion flow with an angle condition. Additionally, we give some properties of smooth solutions for the problem.

To this end, we will use the following notation: Let $f : \bar{I} \to \mathbb{R}^2$, $\sigma \mapsto f(\sigma)$, $I = (0,1)$ be a smooth regular curve. The arc length variable is denoted by s and $\partial_s = {}^{\partial_\sigma}/_{|\partial_\sigma f|}$ denotes the arc length differentiation. Thus, $\tau := \partial_s f$ is a unit tangent and $\vec{\kappa} := \partial_s^2 f$ is the curvature vector. The Euclidean inner product for vectors a, b in $\mathbb{R}^2$ is denoted by $\langle a, b \rangle$ or $a \cdot b$. Furthermore, we denote the normal component of $\partial_s \phi$ by

$$\nabla_s \phi := \partial_s \phi - \langle \partial_s \phi, \tau \rangle \tau.$$

We use the same notation for the derivative with respect to t. Furthermore, let $\phi : \bar{I} \to \mathbb{R}^2$ be a differentiable normal field along f. Then it holds $\langle \phi, \tau \rangle = 0$, thus $\langle \partial_s \phi, \tau \rangle = -\langle \phi, \vec{\kappa} \rangle$. Consequently, it follows

$$\nabla_s \phi = \partial_s \phi + \langle \phi, \vec{\kappa} \rangle \, \tau \qquad \text{for } \phi \text{ normal field along } f.$$

In this chapter, vector fields with an arrow on top denote normal vector fields along f, e.g. $\vec{\phi}$. Moreover, we use integration by parts for ∇_s. This is possible for differentiable normal fields $\vec{\phi}$ and $\vec{\psi}$ since

$$\partial_s \left\langle \vec{\phi}, \vec{\psi} \right\rangle = \left\langle \nabla_s \vec{\phi}, \vec{\psi} \right\rangle + \left\langle \vec{\phi}, \nabla_s \vec{\psi} \right\rangle \qquad \text{for } \vec{\phi}, \vec{\psi} \text{ differentiable normal fields along } f.$$

3.1 The Geometrical Setting

We consider the curve diffusion flow

$$\nabla_t f = -\nabla_s^2 \vec{\kappa}, \qquad \text{for } \sigma \in \bar{I}, t \in (0,T) \tag{3.1.1}$$

for a time dependent regular curve $f : [0,T) \times \bar{I} \to \mathbb{R}^2$, $(t,\sigma) \mapsto f(t,\sigma)$, $I = (0,1)$ subject to the conditions

$$\begin{aligned} f(t,0), f(t,1) &\in \mathbb{R} \times \{0\} && \text{for } t \in (0,T) && (3.1.2)\\ \measuredangle \left(\vec{n}_{\Gamma_t}(\sigma), \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) &= \pi - \alpha && \text{for } \sigma \in \{0,1\}, t \in (0,T) && (3.1.3)\\ \nabla_s \vec{\kappa}_{\Gamma_t}(\sigma) &= 0 && \text{for } \sigma \in \{0,1\}, t \in (0,T) && (3.1.4)\\ f(0,\sigma) &= f_0(\sigma) && \text{for } \sigma \in [0,1], && (3.1.5) \end{aligned}$$

where $\Gamma_t := f(t, \bar{I})$, $t \in (0,T)$, and $\alpha \in (0,\pi)$. Moreover, we denote by $\tau_{\Gamma_t}(\sigma) := {}^{\partial_\sigma f(t,\sigma)}/_{|\partial_\sigma f(t,\sigma)|}$ the unit tangent vector and by $\vec{n}_{\Gamma_t}(\sigma) := R\tau_{\Gamma_t}(\sigma)$ the unit normal vector of Γ_t at $f(t,\sigma)$ for $\sigma \in \bar{I}$ and $t \in (0,T)$, respectively. Here, R is the counterclockwise rotation by angle ${}^{\pi}/_{2}$, i.e.

$$Rv = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} v \qquad \text{for all } v \in \mathbb{R}^2.$$

Furthermore, the curvature vector $\vec{\kappa}_{\Gamma_t}$ of the curve Γ_t at $f(t,\sigma)$ is given by $\vec{\kappa}_{\Gamma_t}(\sigma) := \partial_s^2 f(t,\sigma)$ for $\sigma \in \bar{I}$ and $t \in (0,T)$. By $f_0 : \bar{I} \to \mathbb{R}^2$ we denote the initial datum, which is a regular function, such that

$$\begin{aligned} f_0(\sigma) &\in \mathbb{R} \times \{0\} && \text{for } \sigma \in \{0,1\}, \\ \measuredangle \left(\vec{n}_{\Gamma_0}(\sigma), \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) &= \pi - \alpha && \text{for } \sigma \in \{0,1\}, \end{aligned} \tag{3.1.6}$$

where $\Gamma_0 := f_0(\bar{I})$, and $\tau_{\Gamma_0}(\sigma)$ and $\vec{n}_{\Gamma_0}(\sigma)$, $\sigma \in \bar{I}$ are defined analogously. We give a sketch of the geometrical situation in Figure 3.1.

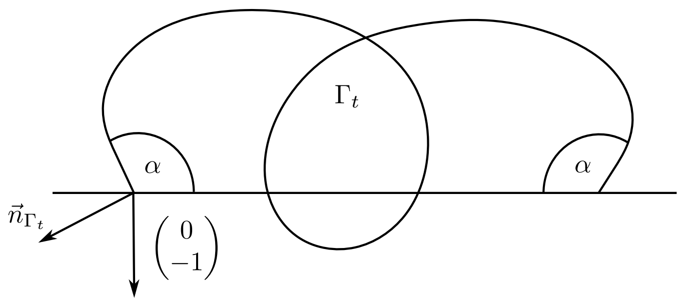

Figure 3.1: Evolution by curve diffusion flow with α-angle condition for $\alpha > \frac{\pi}{2}$.

Remark 3.1.1 *By the equation* (3.1.1) *the tangential movement of the curve is not prescribed. Regardless, there has to exists tangential movement for $\alpha \neq \pi/2$, since we immediately violate condition* (3.1.2) *for $t > 0$, if $\nabla_t f = \partial_t f$ holds true.*

In some arguments, we will use a different representation of (3.1.1), (3.1.3), and (3.1.4).

Remark 3.1.2 *We can express* (3.1.1) *differently, with the help of the calculation*

$$\nabla_t f = \langle \nabla_t f, \vec{n}_{\Gamma_t} \rangle \, \vec{n}_{\Gamma_t} = \langle -\nabla_s^2 \vec{\kappa}_{\Gamma_t}, \vec{n}_{\Gamma_t} \rangle \vec{n}_{\Gamma_t} = -\partial_s \langle \nabla_s \vec{\kappa}_{\Gamma_t}, \vec{n}_{\Gamma_t} \rangle \, \vec{n}_{\Gamma_t} = -\partial_s^2 \kappa_{\Gamma_t} \vec{n}_{\Gamma_t},$$

where $\kappa_{\Gamma_t} := \langle \vec{\kappa}_{\Gamma_t}, \vec{n}_{\Gamma_t} \rangle$ is the scalar curvature. Here, we used that $\langle \nabla_s \vec{\phi}, \vec{n}_{\Gamma_t} \rangle = \partial_s \langle \vec{\phi}, \vec{n}_{\Gamma_t} \rangle$ for normal fields $\vec{\phi}$, as $\partial_s \vec{n}_{\Gamma_t}$ is purely tangential by $0 = \partial_s |\vec{n}_{\Gamma_t}|^2 = 2\langle \partial_s \vec{n}_{\Gamma_t}, \vec{n}_{\Gamma_t} \rangle$.
Moreover, (3.1.3) *is equivalent to*

$$\tau_{\Gamma_t}(\sigma) = \begin{pmatrix} \cos\alpha \\ \pm \sin\alpha \end{pmatrix} \qquad \textit{for } \sigma = 0,1 \textit{ and } t \in (0,T).$$

Furthermore, we obtain by the boundary condition (3.1.4)

$$\partial_s \kappa_{\Gamma_t} \vec{n}_{\Gamma_t} = \partial_s \langle \vec{\kappa}, \vec{n}_{\Gamma_t} \rangle \vec{n}_{\Gamma_t} = \langle \nabla_s \vec{\kappa}_{\Gamma_t}, \vec{n}_{\Gamma_t} \rangle \vec{n}_{\Gamma_t} = 0 \qquad \textit{for } \sigma \in \{0,1\} \textit{ and } t \in (0,T),$$

Thus, (3.1.4) *is equivalent to*

$$\partial_s \kappa_{\Gamma_t} = 0 \qquad \textit{for } \sigma \in \{0,1\} \textit{ and } t \in (0,T). \tag{3.1.7}$$

3.2 Some Basic Properties of Smooth Solutions

In order to derive basic properties of the curve diffusion flow, we study the variation of some geometrical quantities considering solutions $f : [0,T) \times \bar{I} \to \mathbb{R}^2$ of the more general flow

$$\partial_t f = \vec{V} + \varphi \tau_{\Gamma_t},$$

where $\vec{V}$ is the normal velocity and $\varphi = \langle \partial_t f, \tau_{\Gamma_t} \rangle$ is the tangential component of the velocity.

The proofs to the statements given in the next lemma, can be found in Lemma 7 in [10] or Lemma 2.1 in [8].

Lemma 3.2.1
Let $f : [0,T) \times \bar{I} \to \mathbb{R}^2$, $(t,\sigma) \mapsto f(t,\sigma)$, be a smooth time dependent curve, such that $f(t,\cdot)$ is regular and it fulfills $\partial_t f = \vec{V} + \varphi \tau_{\Gamma_t}$ for $t \in (0,T)$, $\sigma \in I$, and with $\vec{V}$ the normal velocity and $\varphi = \langle \partial_t f, \tau_{\Gamma_t} \rangle$ the tangential component of the velocity. Given any smooth normal field $\vec{\phi}$ along f, the following formulas hold

$$\partial_t(\,\mathrm{d}s_f) = (\partial_s \varphi - \langle \vec{\kappa}_{\Gamma_t}, \vec{V} \rangle)\,\mathrm{d}s_f, \tag{3.2.1}$$

$$\partial_t \partial_s - \partial_s \partial_t = (\langle \vec{\kappa}_{\Gamma_t}, \vec{V} \rangle - \partial_s \varphi)\partial_s, \tag{3.2.2}$$

$$\partial_t \tau_{\Gamma_t} = \nabla_s \vec{V} + \varphi \vec{\kappa}_{\Gamma_t}, \tag{3.2.3}$$

$$\partial_t \vec{\phi} = \nabla_t \vec{\phi} - \langle \nabla_s \vec{V} + \varphi \vec{\kappa}_{\Gamma_t}, \vec{\phi} \rangle \tau_{\Gamma_t}, \tag{3.2.4}$$

$$\partial_t \vec{\kappa}_{\Gamma_t} = \partial_s \nabla_s \vec{V} + \langle \vec{\kappa}_{\Gamma_t}, \vec{V} \rangle \vec{\kappa}_{\Gamma_t} + \varphi \partial_s \vec{\kappa}_{\Gamma_t}, \tag{3.2.5}$$

$$\nabla_t \vec{\kappa}_{\Gamma_t} = \nabla_s^2 \vec{V} + \langle \vec{\kappa}_{\Gamma_t}, \vec{V} \rangle \vec{\kappa}_{\Gamma_t} + \varphi \nabla_s \vec{\kappa}_{\Gamma_t}, \tag{3.2.6}$$

$$(\nabla_t \nabla_s - \nabla_s \nabla_t)\vec{\phi} = \big(\langle \vec{\kappa}_{\Gamma_t}, \vec{V} \rangle - \partial_s \varphi\big)\nabla_s \vec{\phi} + \big[\langle \vec{\kappa}_{\Gamma_t}, \vec{\phi} \rangle \nabla_s \vec{V} - \langle \nabla_s \vec{V}, \vec{\phi} \rangle \vec{\kappa}_{\Gamma_t}\big]. \tag{3.2.7}$$

We will use the following notation for integrals with respect to arc length.

Notation
Let $f : \mathbb{R}^2 \to \mathbb{R}^n$, $g : \bar{I} \to \mathbb{R}^2$ be a regular curve of class C^1. We use the notation

$$\int_I h \,\mathrm{d}s_f := \int_I h(f(\sigma)) |\partial_\sigma f(\sigma)| \,\mathrm{d}\sigma.$$

In the following, we will omit the subscript f in $\mathrm{d}s_f$, when it is clear from the context.

For the further steps, we will need the following geometric quantities.

Definition 3.2.2 (Length, Energy, Signed Area)
Let $g : \bar{I} \to \mathbb{R}^2$, be a smooth regular curve. Then we define the ***length of the curve*** *g and the* ***energy of the curve*** *g by*

$$\mathcal{L}[g] := \int_I \mathrm{d}s_g,$$

$$\mathcal{E}[g] := \mathcal{L}[g] + \cos\alpha [g(0) - g(1)]_1.$$

Moreover, let $g : \bar{I} \to \mathbb{R}^2$ and $\tilde{g} : \bar{I} \to \mathbb{R}^2$, $I = (0,1)$, be smooth regular curves, such that $g(0) = \tilde{g}(1)$ and $g(1) = \tilde{g}(0)$. Then, we define the ***signed area enclosed by the curves*** *g* ***and*** *$\tilde{g}$ by*

$$\mathcal{A}[g,\tilde{g}] := -\frac{1}{2}\left[\int_I \langle g, \vec{n}_{g(\bar{I})} \rangle \,\mathrm{d}s_g + \int_I \langle \tilde{g}, \vec{n}_{\tilde{g}(\bar{I})} \rangle \,\mathrm{d}s_{\tilde{g}}\right].$$

Remark 3.2.3 *1. If we study the signed area enclosed by a curve and the real axis $\mathbb{R}\times\{0\}$, the formula simplifies to*

$$\mathcal{A}[g] := \mathcal{A}\left[g, \mathbb{R}\times\{0\}\right] = -\frac{1}{2}\int_I \left\langle g, \vec{n}_{g(\bar{I})}\right\rangle \mathrm{d}s_g.$$

2. Note that the previous definition of the signed area enclosed by the curve g coincides with the usual definition for curves without self intersection. More precisely, we have for a smooth, regular, closed, and embedded curve $g: S^1 \to \mathbb{R}^2$ by Gauss's theorem

$$\mathcal{A}[g] = -\frac{1}{2}\int_{S^1} \left\langle g, \vec{n}_{g(\bar{I})}\right\rangle |\partial_\sigma g| \,\mathrm{d}\sigma = -\int_{\partial\Omega} \left\langle \sigma, \vec{n}_{\partial\Omega}\right\rangle \mathrm{d}s = \frac{1}{2}\int_\Omega \nabla\cdot\sigma \,\mathrm{d}\sigma = \int_\Omega \mathrm{d}\sigma,$$

where Ω is the area enclosed by the curve g.

The next lemma provides information on the behavior of the length of the curve and the signed area enclosed by the curve and $\mathbb{R}\times\{0\}$. The bounds are crucial for our analysis.

Lemma 3.2.4 (Energy Reduction, Area Preservation, Bounds on the Length)
Let $f : [0,T)\times\bar{I} \to \mathbb{R}^2$ be smooth, such that $f(t,\cdot)$ is a regular curve for all $[0,T)$, and let it fulfill (3.1.1)–(3.1.4) *for all $t\in[0,T)$.*

1. Then it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[f](t) \le 0 \qquad \textit{for all } t\in[0,T).$$

2. Then it holds for every $\tilde{t}\in[0,T)$

$$\mathcal{L}[f(t)] \le \frac{\mathcal{E}[f(\tilde{t})]}{1-|\cos\alpha|} \qquad \textit{for all } t\in[\tilde{t},T).$$

3. Then it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}[f(t)] = 0 \qquad \textit{for all } t\in[0,T).$$

4. Let $\mathcal{A}[f(0)]\neq 0$. Then it holds

$$\sqrt{\mathcal{A}[f](0)\pi} \le \mathcal{L}[f(t)] \qquad \textit{for all } t\in[0,T).$$

Proof. In this proof, we will omit the arguments of the functions if the situation is contextually clear.

Ad 1:
We consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[f(t)] = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}[f(t)] + \cos\alpha\frac{\mathrm{d}}{\mathrm{d}t}[f(t,0)-f(t,1)]_1 = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}[f(t)] + \cos\alpha[\partial_t f(t,0)-\partial_t f(t,1)]_1.$$

Taking a closer look at the first summand, we obtain by $\partial_t|\partial_\sigma f| = \langle\tau_{\Gamma_t}, \partial_t\partial_\sigma f\rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}[f(t)] = \frac{\mathrm{d}}{\mathrm{d}t}\int_I \mathrm{d}s = \int_I \partial_t|\partial_\sigma f|\,\mathrm{d}x = \int_I \langle\tau_{\Gamma_t}, \partial_t\partial_\sigma f\rangle\,\mathrm{d}x = \int_I \langle\tau_{\Gamma_t}, \partial_\sigma\partial_t f\rangle\,\mathrm{d}x.$$

Integration by parts yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}[f(t)] = [\langle \tau_{\Gamma_t}, \partial_t f\rangle]_0^1 - \int_I \langle \partial_\sigma \tau_{\Gamma_t}, \partial_t f\rangle \,\mathrm{d}x = [\langle \tau_{\Gamma_t}, \partial_t f\rangle]_0^1 - \int_I \langle \vec{\kappa}_{\Gamma_t}, \nabla_t f\rangle \,\mathrm{d}s.$$

Thus, we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[f(t)] = [\langle \tau_{\Gamma_t}, \partial_t f\rangle]_0^1 - \int_I \langle \vec{\kappa}_{\Gamma_t}, \nabla_t f\rangle \,\mathrm{d}s + \cos\alpha[\partial_t f(t,0) - \partial_t f(t,1)]_1.$$

Moreover, as $\partial_t f(t,\sigma) = \begin{pmatrix} \partial_t f_1(t,\sigma) \\ 0 \end{pmatrix}$ by (3.1.2) and $\tau_{\Gamma_t}(\sigma) = \begin{pmatrix} \cos\alpha \\ \pm\sin\alpha \end{pmatrix}$ by (3.1.3) for $\sigma = 0, 1$ and all $t \in [0,T)$, it follows

$$\langle \partial_t f(t,\sigma), \tau_{\Gamma_t}(\sigma)\rangle = \cos\alpha\, \partial_t f_1(t,\sigma) \qquad \text{for } \sigma \in \{0,1\} \text{ and } t \in [0,T).$$

This shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[f(t)] = -\int_I \langle \vec{\kappa}_{\Gamma_t}, \nabla_t f\rangle \,\mathrm{d}s \qquad \text{for all } t \in [0,T). \tag{3.2.8}$$

Using $\nabla_t f = \vec{V} = -\nabla_s^2 \vec{\kappa}_{\Gamma_t}$ and integration by parts with (3.1.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[f(t)] = -\int_I |\nabla_s \vec{\kappa}_{\Gamma_t}|^2 \,\mathrm{d}s \le 0 \qquad \text{for all } t \in [0,T),$$

which proves the claim.

Ad 2:
Due to the previous claim, we have $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[f(t)] \le 0$, thus, for all $t \in [\tilde{t}, T)$

$$\mathcal{L}[f(t)] + \cos\alpha[f(t,0) - f(t,1)]_1 = \mathcal{E}[f(t)] \le \mathcal{E}[f(\tilde{t})].$$

Moreover, we notice for $L(f(t)) := [f(t,0) - f(t,1)]_1$ that

$$|L(f(t))| \le \mathcal{L}[f(t)],$$

which yields

$$-|\cos\alpha|\mathcal{L}[f(t)] \le -|\cos\alpha||L(f(t))| \le \cos\alpha L(f(t))$$

for all $\alpha \in (0,\pi)$. It follows

$$\mathcal{L}[f(t)](1 - |\cos\alpha|) \le \mathcal{L}[f(t)] + \cos\alpha L(f(t)) = \mathcal{E}[f(t)] \le \mathcal{E}[f(\tilde{t})] \qquad \text{for all } t \in [\tilde{t}, T),$$

which proves the claim.

Ad 3:
Using (3.2.1) and the equation (3.1.1), we infer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}[f(t)] &= -\frac{1}{2}\int_I \langle \partial_t f, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s - \frac{1}{2}\int_I \langle f, \partial_t \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s - \frac{1}{2}\int_I \langle f, \vec{n}_{\Gamma_t}\rangle \left(\partial_s\varphi - \langle \vec{\kappa}_{\Gamma_t}, \vec{V}\rangle\right) \mathrm{d}s \\ &= -\frac{1}{2}\int_I \langle \vec{V}, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s - \frac{1}{2}\int_I \langle f, \partial_t \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s - \frac{1}{2}\int_I \langle f, \vec{n}_{\Gamma_t}\rangle (\partial_s\varphi - \langle \vec{\kappa}_{\Gamma_t}, \vec{V}\rangle) \,\mathrm{d}s.\end{aligned}$$

We take care of the second summand separately: Since $\partial_t \vec{n}_{\Gamma_t}$ is tangential, formula (3.2.4) yields

$$\begin{aligned}
-\frac{1}{2}\int_I \langle f, \partial_t \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s &= \frac{1}{2}\int_I \langle f, \tau_{\Gamma_t}\rangle\langle \nabla_s \vec{V}, \vec{n}_{\Gamma_t}\rangle + \varphi\langle f, \tau_{\Gamma_t}\rangle\langle \vec{\kappa}_{\Gamma_t}, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s \\
&= \frac{1}{2}\int_I \langle f, \tau_{\Gamma_t}\rangle \partial_s\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle + \varphi\langle f, \tau_{\Gamma_t}\rangle\langle \vec{\kappa}_{\Gamma_t}, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s \\
&= \Big[\langle f, \tau_{\Gamma_t}\rangle\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle\Big]_0^1 - \frac{1}{2}\int_I (1 + \langle f, \vec{\kappa}_{\Gamma_t}\rangle)\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s + \frac{1}{2}\int_I \varphi\langle f, \tau_{\Gamma_t}\rangle\langle \vec{\kappa}_{\Gamma_t}, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s.
\end{aligned}$$

Here, the second equality follows by $\langle \nabla_s \vec{V}, \vec{n}_{\Gamma_t}\rangle = \partial_s\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle$ and the third by integration by parts. Plugging this into the original calculation, we have the representation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}[f(t)] = [\langle f, \tau_{\Gamma_t}\rangle\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle]_0^1 - \int_I \langle \vec{V}, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s + \frac{1}{2}\int_I \varphi\langle f, \tau_{\Gamma_t}\rangle\langle \vec{\kappa}_{\Gamma_t}, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s \\
- \frac{1}{2}\int_I \partial_s\varphi\langle f, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s,
\end{aligned}$$

as $\langle f, \vec{\kappa}_{\Gamma_t}\rangle\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle = \langle f, \vec{n}_{\Gamma_t}\rangle\langle \vec{\kappa}_{\Gamma_t}, \vec{V}\rangle$. Considering the last term separately, we deduce by integration by parts

$$-\frac{1}{2}\int_I \langle f, \vec{n}_{\Gamma_t}\rangle \partial_s\varphi \,\mathrm{d}s = -\,[\langle f, \vec{n}_{\Gamma_t}\rangle\varphi]_0^1 - \frac{1}{2}\int_I \langle f, \kappa\tau_{\Gamma_t}\rangle\varphi \,\mathrm{d}s.$$

Thus, by $\langle \vec{\kappa}_{\Gamma_t}, \vec{n}_{\Gamma_t}\rangle = \kappa_{\Gamma_t}$ and $\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle = -\partial_s^2\kappa$ it follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}[f(t)] &= -\int_I \langle \vec{V}, \vec{n}_{\Gamma_t}\rangle \,\mathrm{d}s + [\langle f, \tau_{\Gamma_t}\rangle\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle]_0^1 - [\langle f, \vec{n}_{\Gamma_t}\rangle\varphi]_0^1 \\
&= \int_I \partial_s^2\kappa \,\mathrm{d}s + [\langle f, \tau_{\Gamma_t}\rangle\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle]_0^1 - [\langle f, \vec{n}_{\Gamma_t}\rangle\varphi]_0^1 .
\end{aligned}$$

By using $\partial_s\kappa = 0$ at the boundary, see (3.1.7), we obtain that the first term vanishes. It just remains to show

$$[\langle f, \tau_{\Gamma_t}\rangle\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle]_0^1 - [\langle f, \vec{n}_{\Gamma_t}\rangle\varphi]_0^1 = 0 \qquad \text{for all } t \in [0, T).$$

By boundary condition (3.1.2), we observe that $[f(t,\sigma)]_2 = 0$ and $[\partial_t f(t,\sigma)]_2 = 0$ for $\sigma \in \{0, 1\}$ and all $t \in [0, T)$. Moreover, by the (3.1.3), we have $[\tau_{\Gamma_t}(\sigma)]_1 = \cos\alpha$, and $[\vec{n}_{\Gamma_t}(\sigma)]_1 = \mp\sin\alpha$ for $\sigma = 0, 1$ and all $t \in [0, T)$. The combination of these identities leads to

$$\begin{aligned}
\langle f, \tau_{\Gamma_t}\rangle\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle &= f_1 \cos\alpha\, \partial_t f_1(\mp\sin\alpha) &&\text{for } \sigma = 0, 1 \text{ and all } t \in [0, T), \\
\langle f, \vec{n}_{\Gamma_t}\rangle\varphi &= f_1(\mp\sin\alpha)\partial_t f_1 \cos\alpha &&\text{for } \sigma = 0, 1 \text{ and all } t \in [0, T),
\end{aligned}$$

where we used $\varphi = \langle \partial_t f, \tau_{\Gamma_t}\rangle$ and $\langle \vec{V}, \vec{n}_{\Gamma_t}\rangle = \langle \partial_t f, \vec{n}_{\Gamma_t}\rangle$. This proves the claim.

Ad 4:
The isoperimetric inequality states for the length of a closed curve and the enclosed area that

$$4\pi \text{Area} \leq (\text{Length})^2,$$

where the equality holds if and only if the curve is a circle with radius $R = \sqrt{\text{Area}/\pi}$.

Due to the statement before, the included area $\mathcal{A}[f(t)]$ has to be preserved over time. First, we take care of the case that $\mathcal{A}[f(t)] = \mathcal{A}[f(0)] > 0$. In our case, the total length of the curve is given by

$$\overline{\mathcal{L}}[f(t)] := \mathcal{L}[f(t)] + |[f(t,1) - f(t,0)]_1| .$$

We deduce by the isoperimetric inequality

$$0 < 4\pi\mathcal{A}[f](0) = 4\pi\mathcal{A}[f(t)] \leq (\overline{\mathcal{L}}[f(t)])^2 < (2\mathcal{L}[t])^2.$$

Therefore, the half of the circumference of the optimal circle, i.e.

$$\sqrt{\mathcal{A}[f](0)\pi} \leq \mathcal{L}[f(t)]$$

provides a lower bound for $\mathcal{L}[f(t)]$ for $t \in [0,T)$.
In the case $\mathcal{A}[f(t)] = \mathcal{A}[f](0) < 0$, we can do the same calculations with $|\mathcal{A}[f](0)|$. □

Remark 3.2.5 *Note that we did not use that f solves the equations* (3.1.1)-(3.1.5) *to deduce*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[f(t)] = -\int_I \langle \vec{\kappa}_{\Gamma_t}, \nabla_t f \rangle \,\mathrm{d}s \qquad \textit{for all } t \in [0,T),$$

cf. (3.2.8)*. Thus, it is fulfilled for every function $f \in C^\infty([0,T];C^\infty(\bar{I}))$ satisfying*

$$[f(t,\sigma)]_2 = 0 \qquad \textit{and} \qquad \tau_{f(t,\bar{I})}(\sigma) = \begin{pmatrix} \cos\alpha \\ \pm\sin\alpha \end{pmatrix} \qquad \textit{for } \sigma = 0,1 \textit{ and } t \in [0,T].$$

By the density of this set of functions in $f \in W_2^1((0,T);L_2(I;\mathbb{R}^2)) \cap L_2((0,T);W_2^4(I;\mathbb{R}^2))$ fulfilling (3.1.2) *and* (3.1.3)*, we deduce that* (3.2.8) *holds also true for almost every $t \in (0,T)$ for solutions of* (3.1.1)-(3.1.5) *in the space $W_2^1((0,T);L_2(I;\mathbb{R}^2)) \cap L_2((0,T);W_2^4(I;\mathbb{R}^2))$. Integrating* (3.2.8) *with respect to time and using $\nabla_t f = \vec{V} = -\nabla_s^2\vec{\kappa}_{\Gamma_t}$, we obtain for every $\tilde{t} \in [0,T)$*

$$\mathcal{E}[f(t)] - \mathcal{E}[f(\tilde{t})] = \int_{\tilde{t}}^t \int_I \langle \vec{\kappa}_{\Gamma_t}, \nabla_s^2\vec{\kappa}_{\Gamma_t} \rangle \,\mathrm{d}s \,\mathrm{d}\bar{t} \qquad \textit{for all } t \in [\tilde{t},T).$$

Integration by parts combined with $\nabla_s\kappa = 0$ for almost every $t \in (\tilde{t},T)$ yields

$$\mathcal{E}[f(t)] - \mathcal{E}[f(\tilde{t})] = -\int_{\tilde{t}}^t \int_I |\nabla_s\vec{\kappa}_{\Gamma_t}|^2 \,\mathrm{d}s \,\mathrm{d}\bar{t} \leq 0.$$

Due to this bound, we obtain analogously to the proof of item 2 for $\alpha \in (0,\pi)$

$$L[f(t)] \leq \frac{\mathcal{E}[f(\tilde{t})]}{1 - |\cos\alpha|} \qquad \textit{for all } t \in [\tilde{t},T).$$

4 The Main Results

In the following, we present our main results, a local well-posedness theorem and a blow-up criterion. To this end, we will use the following type of solution.

Definition 4.1.1 (Strong Solution, Maximal Solution)
Let $f_0 : \bar{I} \to \mathbb{R}^2$, $I := (0,1)$, be a regular curve in $W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)$, $\mu \in \left(\frac{7}{8}, 1\right]$. Furthermore, let it fulfill the boundary conditions

$$\begin{aligned} f_0(\sigma) &\in \mathbb{R} \times \{0\} && \text{for } \sigma \in \{0,1\}, \\ \measuredangle\left(n_{\Gamma_0}(\sigma), \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) &= \pi - \alpha && \text{for } \sigma \in \{0,1\}, \end{aligned}$$

*where $\Gamma_0 := f_0(\bar{I})$ and $\alpha \in (0,\pi)$, cf. (3.1.6). We call $f : [0,T) \times \bar{I} \to \mathbb{R}^2$, $I = (0,1)$ a **strong solution** of the curve diffusion flow, if the following holds true:*

1. $f \in \mathbb{E}_{\mu,T,\mathbb{R}^2,loc} := W^1_{2,\mu,loc}\left([0,T); L_2(I;\mathbb{R}^2)\right) \cap L_{2,\mu,loc}\left([0,T); W_2^4(I;\mathbb{R}^2)\right)$, *where*

$$\begin{aligned} W^k_{2,\mu,loc}([0,T);E) := \big\{ u : [0,T) \to E \text{ is strongly measurable } : u_{|K} \in W^k_{2,\mu}(K;E) \\ \text{for all compact } K \subset [0,T) \big\} \end{aligned}$$

   for $k \in \{0,1\}$,

2. *f fulfills (3.1.1)-(3.1.4) and there exists a regular C^1-reparametrization $\varphi : [0,1] \to [0,1]$ such that $f_0(\varphi(\sigma)) = f(0,\sigma)$ for all $\sigma \in [0,1]$,*

3. *$f(t,\cdot)$ is for each $t \in [0,T)$ a regular parametrization of the curve $f(t,\bar{I})$.*

*If T is the largest time such that there is a strong solution on $[0,T)$, we set $T_{max} = T$ and call it a **maximal solution** of curve diffusion flow.*

Remark 4.1.2 *Strong solutions are invariant under translation: Let $f : [0,T) \times \bar{I} \to \mathbb{R}^2$ be a strong solution to (3.1.1)-(3.1.4) with $f(0,\bar{I}) = f_0(\bar{I})$. Then $f_h(t,\sigma) := f(t,\sigma) + (h,0)^T$ is a strong solution to (3.1.1)-(3.1.4) with $f_h(0,\bar{I}) = f_0(\bar{I}) + (h,0)^T$.*

We can prove that the flow starts for a fixed initial curve:

Theorem 4.1.3 (Local Well-Posedness for a Fixed Initial Curve)
Let $f_0 : \bar{I} \to \mathbb{R}^2$, $I := (0,1)$, be a regular curve in $W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)$, $\mu \in \left(\frac{7}{8}, 1\right]$. Furthermore, let it fulfill the boundary conditions

$$\begin{aligned} f_0(\sigma) &\in \mathbb{R} \times (0,\infty) && \text{for } \sigma \in \{0,1\}, \\ \angle\left(n_{\Gamma_0}(\sigma), \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) &= \pi - \alpha && \text{for } \sigma \in \{0,1\}, \end{aligned}$$

where $\Gamma_0 := f_0(\bar{I})$ and $\alpha \in (0,\pi)$, cf. (3.1.6). Then, there exists a $T > 0$, such that $f \in \mathbb{E}_{\mu,T,\mathbb{R}^2}$ is a strong solution to curve diffusion flow.

This result will be proven in two steps: In the first step, see Chapter 5, we will prove a first well-posedness result by describing the evolving curves via a fixed reference curve and curvilinear coordinates. This will allow to start the flow for initial curves which are in some sense close enough to the reference curve. In Chapter 6, we generate potential reference curves by evolving the initial curve f_0 by a parabolic equation. Then, we provide conditions for curves which guarantee that they can be used as a reference curve for a certain initial curve f_0. By technical estimates involving properties of C^0-semigroups and interpolation theory, we confirm that the previously generated curves can in fact be used as reference curves in the proof of the first step.

The short time existence result enables us to deduce a blow-up criterion for maximal solutions of curve diffusion flow if $T_{max} < \infty$.

Theorem 4.1.4 (Blow-up Criterion)
Let $f : [0, T_{max}) \times \bar{I} \to \mathbb{R}^2$, $I := (0, 1)$, $T_{max} < \infty$, be a maximal solution of (3.1.1)-(3.1.5). *Then,* $\lim_{t \to T_{max}} \|\kappa[f(t)]\|_{L_2(0,\mathcal{L}[f(t)])} = \infty$.

The proof of the theorem is done in Chapter 7. The strategy is based on the assumption that the L_2-norm of the curvature with respect to the arc length parameter remains bounded uniformly for a sequence $t_l \to T_{max}$ for $l \to \infty$. We will see that this allows for an extension of the solution f beyond T_{max}, which is contradictory to the maximality of T_{max}. This proves our assumption wrong.

5 Short Time Existence for the Curve Diffusion Flow

We want to derive a short time existence and uniqueness result for the surface diffusion flow introduced in 3.1. More precisely, we look for a time dependent family of regular curves $\Gamma := \{\Gamma_t\}_{t\geq 0}$ satisfying

$$V = -\partial_{ss}\kappa_{\Gamma_t} \qquad \text{on } \Gamma_t, t > 0, \tag{5.0.1}$$

where V is the scalar normal velocity, subject to the boundary conditions

$$\partial\Gamma_t \subset \mathbb{R} \times \{0\} \qquad \text{for } t > 0, \tag{5.0.2}$$

$$\measuredangle\left(n_{\Gamma_t}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = \pi - \alpha \qquad \text{at } \partial\Gamma_t \text{ for } t > 0, \tag{5.0.3}$$

$$\partial_s\kappa_{\Gamma_t} = 0 \qquad \text{at } \partial\Gamma_t \text{ for } t > 0, \tag{5.0.4}$$

with the same notation as before, cf. Remark 3.1.2. Furthermore, an initial datum Γ_0 fulfilling (5.0.2) and (5.0.3) will be specified later.

In order to obtain a short time existence result, the following strategy is pursued: For a fixed reference curve and coordinates, we can represent the evolving curves, which are "close" to the reference curve by a height function. Thus, the geometrical problem (5.0.1)-(5.0.4) is reduced to a quasilinear parabolic partial differential equation on a fixed interval, at least as long as the curve is sufficiently close to the initial curve. The standard approach to attack these kind of problems is a contraction mapping argument: First, the equation is linearized and the function spaces for the solution and the data of the linearized system are chosen such that the linear problem can be solved with optimal regularity and the nonlinear terms are contractive for small times. In the next step, the original partial differential equation can be expressed by an equivalent fixed point problem due to the invertibility of the linear operator. By proving that the nonlinear terms are contractive if the time of existence is small enough, we can apply Banach's fixed point theorem and obtain a unique solution to the partial differential equation. In order to achieve this, it will be crucial to keep track of the dependencies of the constants.

Note that in this chapter, we will work with fixed reference curves and coordinates. Thus, we obtain short time existence for curves which can be described as a graph over the reference curve with a height function, which is small in some sense. At a later point, we will also obtain a result which allows for starting the flow for a fixed initial curve. For this, it will be important that the result in this chapter is not achieved by diminishing the norm of the initial datum.

5.1 Reduction of the Geometric Evolution Equation to a PDE

In order to reduce the geometric evolution equation to a partial differential equation on a fixed interval, we employ a parametrization which is similar to the one established in [31]. To this end, let $\Phi^* : [0, 1] \to \mathbb{R}^2$ be a regular C^5-curve parametrized proportional to arc length. Moreover, let

$\Lambda := \Phi^*([0,1])$ fulfill the conditions

$$\begin{aligned} \Phi^*(\sigma) &\in \mathbb{R} \times \{0\} && \text{for } \sigma \in \{0,1\}, \\ \measuredangle \left(n_\Lambda(\sigma), \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) &= \pi - \alpha && \text{for } \sigma \in \{0,1\}, \\ \kappa_\Lambda(\sigma) &= 0 && \text{for } \sigma \in \{0,1\}, \end{aligned} \tag{5.1.1}$$

where $\tau_\Lambda(\sigma) := \partial_\sigma \Phi^*(\sigma)/\mathcal{L}[\Phi^*]$ and $n_\Lambda(\sigma) := R\tau_\Lambda(\sigma)$ are the unit tangent and unit normal vector of Λ at the point $\Phi^*(\sigma)$ for $\sigma \in [0,1]$, respectively. Again, R is the counterclockwise $\pi/2$-rotation matrix. Furthermore, the curvature vector of Λ at $\Phi^*(\sigma)$ is given by $\vec{\kappa}_\Lambda(\sigma) := \partial_\sigma^2 \Phi^*(\sigma)/(\mathcal{L}[\Phi^*])^2$ for $\sigma \in [0,1]$.

For a sufficiently small d, curvilinear coordinates are defined as

$$\begin{aligned} \Psi : [0,1] \times (-d,d) &\to \mathbb{R}^2 \\ (\sigma, q) &\mapsto \Phi^*(\sigma) + q\left(n_\Lambda(\sigma) + \cot\alpha\,\eta(\sigma)\tau_\Lambda(\sigma)\right), \end{aligned} \tag{5.1.2}$$

where the function $\eta : [0,1] \to [-1,1]$ is given by

$$\eta(x) := \begin{cases} -1 & \text{for } 0 \le x < \frac{1}{6} \\ 0 & \text{for } \frac{2}{6} \le x < \frac{4}{6} \\ 1 & \text{for } \frac{5}{6} \le x \le 1 \\ \text{arbitrary} & \text{else,} \end{cases} \tag{5.1.3}$$

such that it is monotonically increasing and smooth. If $\alpha = \pi/2$, then $\cot\alpha = 0$ and the second summand in the definition of Ψ vanishes, cf. (5.1.2). Figure 5.1 sketches the reference curve and the corresponding coordinates.

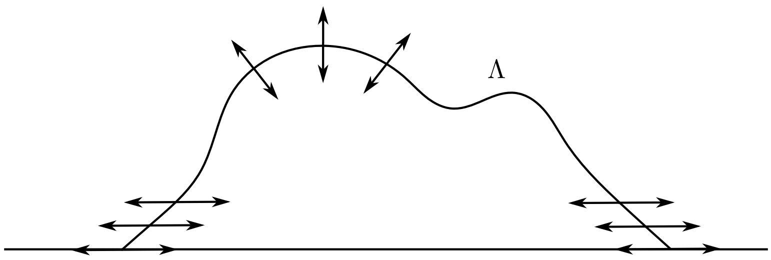

Figure 5.1: A reference curve and curvilinear coordinates.

Remark 5.1.1 *1. It is trivial that $q \mapsto \Psi(\sigma,q)$ is smooth for $\sigma \in [0,1]$. Since Φ^* and η are functions of class C^5, it follows that $\Psi \in C^4([0,1] \times (-d,d))$ and*

$$\|\Psi\|_{C^4([0,1]\times(-d,d))} \le C(\alpha, \Phi^*, \eta, |d|).$$

2. The tangential part is weighted by the function η, which assures that $[\Psi(\sigma,q)]_2 = 0$ for $\sigma \in \{0,1\}$ and each $q \in (-d,d)$. This is important, since we want the solution to have its boundary points on the real axis, cf. (5.0.2). The tangential part is constant in a neighborhood of the boundary points and it vanishes in the middle of the curve.

In the following, we consider functions

$$\begin{aligned} \rho : [0,T) \times [0,1] &\to (-d,d) \\ (t,\sigma) &\mapsto \rho(t,\sigma) \end{aligned}$$

and define

$$\Phi(t,\sigma) := \Psi(\sigma, \rho(t,\sigma)). \tag{5.1.4}$$

An evolving curve is now given by

$$\Gamma_t := \{\Phi(t,\sigma) |\ \sigma \in [0,1]\}, \tag{5.1.5}$$

where we obtain $\Phi(t,\sigma) \in \mathbb{R} \times \{0\}$ for $\sigma \in \{0,1\}$, $t \in [0,T)$, cf. (5.0.2), by construction.

Next, we want to express (5.0.1)–(5.0.4) with the help of the parametrization induced by (5.1.5). In order to improve readability, we omit the arguments at some points, e.g. $\Psi_\sigma = \Psi_\sigma(\sigma, \rho(t,\sigma))$ and $\rho = \rho(t,\sigma)$. Assuming $|\Phi_\sigma(t,\sigma)| \neq 0$, we derive for the arc length parameter s of Γ_t

$$\frac{ds}{d\sigma} = |\Phi_\sigma| = \sqrt{|\Psi_\sigma|^2 + 2\langle \Psi_\sigma, \Psi_q \rangle \partial_\sigma \rho + |\Psi_q|^2 (\partial_\sigma \rho)^2} =: J(\rho) = J(\sigma, \rho, \partial_\sigma \rho),$$

where $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and the inner product in $\mathbb{R}^2$, respectively. Thus, the unit tangent τ_{Γ_t} and the outer unit normal n_{Γ_t} of the curve Γ_t are given by

$$\begin{aligned} \tau_{\Gamma_t} &= \frac{1}{J(\rho)} \Phi_\sigma = \frac{1}{J(\rho)} (\Psi_\sigma + \Psi_q \partial_\sigma \rho), \\ n_{\Gamma_t} &= R \tau_{\Gamma_t} = \frac{1}{J(\rho)} R \Phi_\sigma. \end{aligned}$$

For the scalar normal velocity V of Γ_t, we have

$$V = \langle \Phi_t, n_{\Gamma_t} \rangle = \frac{1}{J(\rho)} \langle \Phi_t, R\Phi_\sigma \rangle = \frac{1}{J(\rho)} \langle \Psi_q \rho_t, R(\Psi_\sigma + \Psi_q \partial_\sigma \rho) \rangle = \frac{1}{J(\rho)} \langle \Psi_q, R\Psi_\sigma \rangle \rho_t,$$

where we used that $\langle v, Rv \rangle = 0$ for all $v \in \mathbb{R}^2$. Moreover, the Laplace-Beltrami operator on Γ_t as a function in ρ is defined by

$$\Delta(\rho) := \partial_s^2 = \frac{1}{J(\rho)} \partial_\sigma \left(\frac{1}{J(\rho)} \partial_\sigma \right) = \frac{1}{J(\rho)} \partial_\sigma \left(\frac{1}{J(\rho)} \right) \partial_\sigma + \frac{1}{(J(\rho))^2} \partial_\sigma^2.$$

Thereby, the scalar curvature of Γ_t as function in ρ can be expressed by

$$\kappa(\rho) = \frac{1}{(J(\rho))^3} \langle \Psi_q, R\Psi_\sigma \rangle \partial_\sigma^2 \rho + U(\sigma, \rho, \partial_\sigma \rho), \tag{5.1.6}$$

cf. Appendix A.1, where $U(\sigma, \rho, \partial_\sigma \rho)$ denotes terms of the form

$$U(\sigma, \rho, \partial_\sigma \rho) = C\, (J(\rho))^k \left(\prod_{i=0}^{p} \left\langle \Psi^{\beta_i}_{(\sigma,q)}, R\Psi^{\gamma_i}_{(\sigma,q)} \right\rangle (\sigma, \rho) \right) (\partial_\sigma \rho)^r, \tag{5.1.7}$$

with $C \in \mathbb{R}$, $k \in \mathbb{Z}$, $p, r \in \mathbb{N}_0$, and $\beta_i, \gamma_i \in \mathbb{N}_0^2$, such that $|\beta_i|, |\gamma_i| \geq 1$ and $|\beta_i| + |\gamma_i| \leq 3$ for all $i \in \{0, \dots, p\}$. Here, the leading order prefactor is kept explicitly, since it will be important for the analysis of the equation. Consequently, the first derivative of the curvature is given by

$$\begin{aligned} \partial_s \kappa(\rho) = {} & \frac{1}{(J(\rho))^4} \langle \Psi_q, R\Psi_\sigma \rangle \partial_\sigma^3 \rho + T(\sigma, \rho, \partial_\sigma \rho) \left(\partial_\sigma^2 \rho \right)^2 + T(\sigma, \rho, \partial_\sigma \rho) \partial_\sigma^2 \rho \\ & + T(\sigma, \rho, \partial_\sigma \rho), \end{aligned} \tag{5.1.8}$$

where the prefactors $T(\sigma,\rho,\partial_\sigma\rho)$ denote terms of the form

$$T(\sigma,\rho,\partial_\sigma\rho) = C\,(J(\rho))^k \left(\prod_{i=0}^{p}\left\langle \Psi^{\beta_i}_{(\sigma,q)}, R\Psi^{\gamma_i}_{(\sigma,q)}\right\rangle(\sigma,\rho)\right)(\partial_\sigma\rho)^r, \tag{5.1.9}$$

with $C\in\mathbb{R}$, $k\in\mathbb{Z}$, $p,r\in\mathbb{N}_0$, and $\beta_i,\gamma_i\in\mathbb{N}_0^2$, such that $|\beta_i|,|\gamma_i|\geq 1$ and $|\beta_i|+|\gamma_i|\leq 4$ for all $i\in\{0,\dots,p\}$, see cf. Appendix A.3 for the derivation. Moreover, we have

$$\begin{aligned}\partial_s^2\kappa(\rho) &= \frac{1}{(J(\rho))^5}\langle\Psi_q,R\Psi_\sigma\rangle\partial_\sigma^4\rho + \tilde S(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho\partial_\sigma^2\rho + \tilde S(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho + \tilde S(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^3\\ &\quad + \tilde S(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^2 + \tilde S(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + \tilde S(\sigma,\rho,\partial_\sigma\rho),\end{aligned} \tag{5.1.10}$$

with the prefactors of the form

$$\tilde S(\sigma,\rho,\partial_\sigma\rho) := C\,(J(\rho))^k \left(\prod_{i=0}^{p}\left\langle \Psi^{\beta_i}_{(\sigma,q)}, R\Psi^{\gamma_i}_{(\sigma,q)}\right\rangle(\sigma,\rho)\right)(\partial_\sigma\rho)^r, \tag{5.1.11}$$

with $C\in\mathbb{R}$, $k\in\mathbb{Z}$, $p,r\in\mathbb{N}_0$, and $\beta_i,\gamma_i\in\mathbb{N}_0^2$, such that $|\beta_i|,|\gamma_i|\geq 1$ and $|\beta_i|+|\gamma_i|\leq 5$ for all $i\in\{0,\dots,p\}$, cf. Appendix A.3. We assume that $1/J(\rho)\langle\Psi_q,R\Psi_\sigma\rangle\neq 0$ and obtain by (5.0.1) the equation

$$\rho_t = -\frac{J(\rho)}{\langle\Psi_q,R\Psi_\sigma\rangle}\Delta(\rho)\kappa(\rho) \qquad \text{for } \sigma\in(0,1) \text{ and } t>0. \tag{5.1.12}$$

Furthermore, the boundary condition (5.0.3) is represented by

$$\cos(\pi-\alpha) = \left\langle n_{\Gamma_t}, \begin{pmatrix}0\\-1\end{pmatrix}\right\rangle = \left\langle \tau, R^T\begin{pmatrix}0\\-1\end{pmatrix}\right\rangle = \frac{1}{J(\rho)}\left\langle(\Psi_\sigma+\Psi_q\partial_\sigma\rho), \begin{pmatrix}-1\\0\end{pmatrix}\right\rangle. \tag{5.1.13}$$

Straightforward calculations together with the assumptions on the reference curve Φ^* show that this is fulfilled if and only if

$$\partial_\sigma\rho(t,\sigma) = 0 \qquad \text{for } \sigma\in\{0,1\} \text{ and } t>0. \tag{5.1.14}$$

This gives the reformulation of the angle condition.

In summary, we have deduced that the problem (5.0.1)-(5.0.4) can be expressed by

$$\begin{aligned}\rho_t &= -\frac{J(\rho)}{\langle\Psi_q,R\Psi_\sigma\rangle}\Delta(\rho)\kappa(\rho) && \text{for } \sigma\in(0,1) \text{ and } t>0,\\ \partial_\sigma\rho(t,\sigma) &= 0 && \text{for } \sigma\in\{0,1\} \text{ and } t>0,\\ \partial_\sigma\kappa(\rho) &= 0 && \text{for } \sigma\in\{0,1\} \text{ and } t>0.\end{aligned} \tag{5.1.15}$$

Before we state the local well-posedness result for this partial differential equation, we want to give a condition to guarantee that $\Phi_\sigma(\sigma,t)\neq 0$.

Lemma 5.1.2
Let $\rho:[0,1]\to\mathbb{R}$ satisfy the bound

$$\|\rho\|_{C([0,1])} < \frac{1}{2\|\kappa_\Lambda\|_{C(\bar I)}\left(1+(\cot\alpha)^2+\hat C|\cot\alpha|\|\eta'\|_{C([0,1])}\right)} =: K_0(\alpha,\Phi^*,\eta), \tag{5.1.16}$$

and in the case $\alpha \neq \frac{\pi}{2}$ *additionally*

$$\|\partial_\sigma \rho\|_{C([0,1])} < \frac{\mathcal{L}[\Phi^*]}{12|\cot\alpha|} =: K_1(\alpha, \Phi^*), \tag{5.1.17}$$

where $\hat{C} := \sqrt{2}\sin\alpha > 0$. *Then* $J(\rho) > 0$ *and* $[0,1] \ni \sigma \mapsto \Psi(\sigma, \rho(\sigma))$ *is a regular parametrization. In particular, if* (5.1.16) *and* (5.1.17) *are fulfilled for* $\frac{2}{3}K_0$ *and* $\frac{2}{3}K_1$, *respectively, then there exists a* $C(\alpha, \Phi^*, \eta) > 0$, *such that*

$$J(\rho) > C(\alpha, \Phi^*, \eta) > 0. \tag{5.1.18}$$

Proof. We have to show that $|\Phi_\sigma(\sigma)| > 0$. To this end, we consider

$$|\Phi_\sigma(\sigma)|^2 = |\Psi_\sigma|^2(\sigma, \rho(\sigma)) + 2\langle \Psi_\sigma, \Psi_q\rangle(\sigma, \rho(\sigma))\partial_\sigma\rho(\sigma) + |\Psi_q|^2(\sigma)(\partial_\sigma\rho(\sigma))^2.$$

Using the identities

$$\begin{aligned}
\partial_\sigma \Phi^*(\sigma) &= \mathcal{L}[\Phi^*]\tau_\Lambda(\sigma)\\
\partial_\sigma \tau_\Lambda(\sigma) &= \mathcal{L}[\Phi^*]\kappa n_\Lambda(\sigma)\\
\partial_\sigma n_\Lambda(\sigma) &= -\mathcal{L}[\Phi^*]\kappa\tau_\Lambda(\sigma),
\end{aligned}$$

we obtain

$$\begin{aligned}
\Psi_\sigma(\sigma, q) &= \left(\mathcal{L}[\Phi^*] - q\mathcal{L}[\Phi^*]\kappa_\Lambda + q\cot\alpha\eta'\right)\tau_\Lambda + q\mathcal{L}[\Phi^*]\kappa_\Lambda\cot\alpha\eta n_\Lambda,\\
\Psi_q(\sigma, q) &= \Psi_q(\sigma) = n_\Lambda + \cot\alpha\eta\tau_\Lambda,
\end{aligned} \tag{5.1.19}$$

where the arguments of the functions are omitted for the sake of readability. Clearly, it holds

$$\langle \Psi_\sigma, \Psi_q\rangle(\sigma, \rho) = \mathcal{L}[\Phi^*]\cot\alpha\eta + \rho(\cot\alpha)^2\eta\eta'. \tag{5.1.20}$$

For $\alpha \neq \pi/2$, we derive

$$\begin{aligned}
|\Phi_\sigma(t,\sigma)|^2 &\geq |\Psi_\sigma|^2(\sigma, \rho(\sigma)) + 2\langle \Psi_\sigma, \Psi_q\rangle(\sigma, \rho(\sigma))\partial_\sigma\rho(\sigma)\\
&\geq \left[\mathcal{L}[\Phi^*] - \rho\mathcal{L}[\Phi^*]\kappa_\Lambda + \rho\cot\alpha\eta'\right]^2 - 2|\partial_\sigma\rho| \left|\mathcal{L}[\Phi^*]\cot\alpha\eta + \rho(\cot\alpha)^2\eta\eta'\right|.
\end{aligned}$$

Using $(a+b)^2 \geq (a/2 + b)^2 + a^2/4$ for $(a/2+b)a/2 \geq 0$, we obtain

$$\begin{aligned}
|\Phi_\sigma(t,\sigma)|^2 \geq{}& \left(\frac{\mathcal{L}[\Phi^*]}{2} - \rho\mathcal{L}[\Phi^*]\kappa_\Lambda + \rho\cot\alpha\eta'\right)^2\\
&+ \frac{(\mathcal{L}[\Phi^*])^2}{4} - 2|\partial_\sigma\rho| \left|\mathcal{L}[\Phi^*]\cot\alpha\eta + \rho(\cot\alpha)^2\eta\eta'\right|.
\end{aligned} \tag{5.1.21}$$

Here $a := \mathcal{L}[\Phi^*] > 0$ and $a/2 + b := \mathcal{L}[\Phi^*]/2 - \rho\mathcal{L}[\Phi^*]\kappa_\Lambda + \rho\cot\alpha\eta' > 0$ holds true due to the condition (5.1.16), since

$$\begin{aligned}
\frac{\mathcal{L}[\Phi^*]}{2} - q\mathcal{L}[\Phi^*]\kappa_\Lambda + \rho\cot\alpha\eta' &\geq \frac{\mathcal{L}[\Phi^*]}{2} - |q|\left(\mathcal{L}[\Phi^*]\|\kappa_\Lambda\|_{C([0,1])} + |\cot\alpha\eta'|\right)\\
&\geq \mathcal{L}[\Phi^*]\left[\frac{1}{2} - |\rho|\left(\|\kappa_\Lambda\|_{C([0,1])} + \frac{1}{\mathcal{L}[\Phi^*]}|\cot\alpha\eta'|\right)\right]\\
&\geq \mathcal{L}[\Phi^*]\left[\frac{1}{2} - |\rho|\|\kappa_\Lambda\|_{C([0,1])}\left(1 + \hat{C}|\cot\alpha\eta'|\right)\right] > 0,
\end{aligned}$$

where $\hat{C} := (\sqrt{2}\sin\alpha)^{-1} > 0$, see Lemma 2.3.1. Thus, it only remains to prove that the sum of the second and third summand of (5.1.21) is positive. By the bound (5.1.16) and Lemma 2.3.1, it follows

$$
\begin{aligned}
&\frac{(\mathcal{L}[\Phi^*])^2}{4} - 2|\partial_\sigma\rho| \,\big(\mathcal{L}[\Phi^*]|\cot\alpha| + |\rho|(\cot\alpha)^2|\eta'|\big)\\
&\qquad > \frac{(\mathcal{L}[\Phi^*])^2}{4} - 2|\partial_\sigma\rho| \left(\mathcal{L}[\Phi^*]|\cot\alpha| + \frac{(\cot\alpha)^2|\eta'|}{2\|\kappa_\Lambda\|_{C([0,1])}\left(1+\hat{C}|\cot\alpha\eta'|\right)}\right)\\
&\qquad > \mathcal{L}[\Phi^*]\left[\frac{\mathcal{L}[\Phi^*]}{4} - 2|\cot\alpha|\,|\partial_\sigma\rho| \left(1 + \frac{\hat{C}|\cot\alpha\eta'|}{2\left(1+\hat{C}|\cot\alpha\eta'|\right)}\right)\right]\\
&\qquad > \mathcal{L}[\Phi^*]\left[\frac{\mathcal{L}[\Phi^*]}{4} - 2|\cot\alpha|\,|\partial_\sigma\rho|\frac{3}{2}\right] > 0,
\end{aligned}
$$

where we additionally used the bound (5.1.17) in the last line.

In the case $\alpha = \pi/2$, we have by (5.1.19) and (5.1.20)

$$
|\Phi_\sigma(\sigma)| = (\mathcal{L}[\Phi^*])^2((1-\rho[\Phi^*]\kappa_\Lambda)^2 + (\partial_\sigma\rho(\sigma))^2 \geq (\mathcal{L}[\Phi^*])^2\,(1-\rho[\Phi^*]\kappa_\Lambda)^2\,.
$$

Thus, the bound (5.1.16) suffices to guarantee $|\Phi_\sigma(\sigma)| > 0$. By inspection of the previously derived estimates, the addendum follows directly. ☐

We give a sketch of the representation in Figure 5.2.

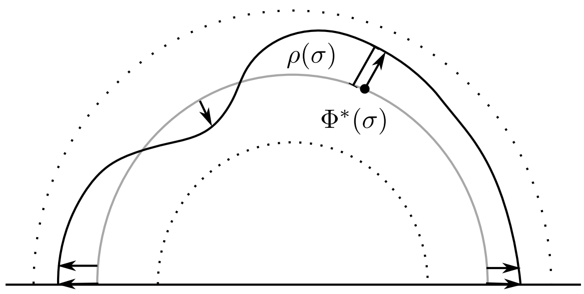

Figure 5.2: Representation of a curve by curvilinear coordinates and a height function.

In the following, the notation from Section 2.2 with $m = 2$ is used to denote the spaces appearing in the local well-posedness result:

Notation
For $J = (0,T)$ and $I = (0,1)$, we have

$$
\begin{aligned}
X_\mu &= W_2^{4(\mu-1/2)}(I),\\
\mathbb{E}_{\mu,T} &= W^1_{2,\mu}\big(J;L_2(I)\big) \cap L_{2,\mu}\big(J;W_2^4(I)\big),\\
\mathbb{E}_{0,\mu} &= L_{2,\mu}\big(J;L_2(I)\big),\\
\mathbb{F}_{j,\mu} &= W^{\omega_j}_{2,\mu}\big(J;L_2(\partial I)\big) \cap L_{2,\mu}\big(J;W_2^{4\omega_j}(\partial I)\big),
\end{aligned}
$$

where $\omega_j = 1 - m_j/4 - 1/8$, $j = 1,2$, and

$$
\tilde{\mathbb{F}}_\mu = \mathbb{F}_{1,\mu} \times \mathbb{F}_{2,\mu}.
$$

All the spaces are equipped with their natural norms. The definitions of the appearing time weighted spaces have been given in Section 2.1.

Theorem 5.1.3 (Local Well-Posedness for Data Close to a Reference Curve)
Let Φ^* and η be given such that (5.1.1) *and* (5.1.3) *are fulfilled, respectively. Furthermore, let $\rho_0 \in X_\mu$, $\mu \in \left(\frac{7}{8}, 1\right]$, fulfill the conditions*

$$\|\rho_0\|_{C(\bar{I})} < \frac{K_0}{3} \qquad \text{and} \qquad \|\partial_\sigma \rho_0\|_{C(\bar{I})} < \frac{K_1}{3}, \tag{5.1.22}$$

and the compatibility condition

$$\partial_\sigma \rho_0(\sigma) = 0 \qquad \text{for } \sigma \in \{0,1\}, \tag{5.1.23}$$

where K_0, K_1 are specified in Lemma 5.1.2.
Then there exists a $T = T(\alpha, \Phi^*, \eta, R_1, R_2) > 0$, $\|\rho_0\|_{X_\mu} \le R_1$ and $\|\mathcal{L}^{-1}\|_{L(\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu\times X_\mu;\mathbb{E}_{\mu,T})} \le R_2$, cf. (5.2.11) *for the Definition of $\mathcal{L}^{-1}$, such that the problem* (5.1.15) *possesses a unique solution $\rho \in \mathbb{E}_{\mu,T}$, such that $\rho(t)$ satisfies the bounds* (5.1.16) *and* (5.1.17) *for all $t \in [0,T]$, and $\rho(\cdot,0) = \rho_0$ in X_μ.*

The theorem is designed in a way, such that it can be applied at a later point:

Remark 5.1.4 1. *We use the time weighted approach, as it enables us to apply the result to initial data with flexible regularity. Moreover, we can directly exploit the smoothing properties of parabolic equations, see Remark 2.2.2, item 2. This property will be crucial in the proof of the blow-up criterion stated in Theorem 4.1.4.*

2. *Note that Theorem 5.1.3 guarantees the same existence time T for initial data ρ_l, $l \in \mathbb{N}$, which fulfill the demanded conditions and additionally satisfy the uniform bounds*

$$\|\rho_l\|_{X_\mu} \le C_1 \text{ and } \|\mathcal{L}^{-1}(\rho_l)\|_{L(\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu\times X_\mu;\mathbb{E}_{\mu,T})} \le C_2 \qquad \text{for all } l \in \mathbb{N}.$$

3. *It is possible to start with an initial datum, which fulfills the condition*

$$\|\rho_0\|_{C(\bar{I})} < K_0 \qquad \text{and} \qquad \|\partial_\sigma \rho_0\|_{C(\bar{I})} < K_1$$

instead of (5.1.22)*, but we lose the previously stated uniformity of the existence time.*

It is a direct consequence of our choice of spaces that the initial curve fulfills in fact the expected conditions in (3.1.6).

Remark 5.1.5 1. *Let ρ_0 fulfill the assumptions of Theorem 5.1.3. Then $\rho_0 \in X_\mu$, $\mu \in (7/8, 1]$, implies $\rho_0 \in C^1(\bar{I})$ by the embedding* (2.2.6)*, as $4(\mu - 1/2) - 1/2 > 1$. It follows by Lemma 5.1.2 that $J(\rho_0) > C(\alpha, \Phi^*, \eta) > 0$ and that*

$$[0,1] \ni \sigma \mapsto \Phi(0,\sigma) = \Psi(\sigma, \rho_0(\sigma))$$

is a regular parametrization of the initial curve corresponding to ρ_0. It holds by construction that $[\Phi(0,\sigma)]_2 = [\Psi(\sigma,\rho_0(\sigma))]_2 = 0$ for $\sigma \in \{0,1\}$.

2. *Note that by* (5.1.14) *the compatibility condition* (5.1.23) *implies*

$$\measuredangle\left(n_{\Gamma_0}(\sigma), \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = \pi - \alpha \qquad \text{for } \sigma \in \{0,1\},$$

where $\Gamma_0 := \Phi(0, \bar{I})$. *Thus, the* α*-angle condition holds true for the initial curve.*

Using the previous considerations, we are able to reduce the geometric evolution equation to a quasilinear partial differential equation on a fixed interval: To this end, the representations derived in (5.1.10), (5.1.8), and (5.1.14) are used and plugged into (5.1.15). Putting the highest order terms on the left-hand side and remaining ones on the right-hand side, we formally obtain that problem (5.1.15) is an initial value problem for a quasilinear parabolic partial differential equation of the form

$$\begin{aligned} \rho_t + a(\sigma, \rho, \partial_\sigma \rho)\partial_\sigma^4 \rho &= f(\rho, \partial_\sigma\rho, \partial_\sigma^2\rho, \partial_\sigma^3\rho) && \text{for } (t,\sigma) \in (0,T)\times[0,1], \\ b_1(\sigma)\partial_\sigma\rho &= -g_1 && \text{for } \sigma \in \{0,1\} \text{ and } t \in (0,T), \\ b_2(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho &= -g_2(\rho,\partial_\sigma\rho,\partial_\sigma^2\rho) && \text{for } \sigma \in \{0,1\} \text{ and } t \in (0,T), \\ \rho|_{t=0} &= \rho_0 && \text{in } [0,1]. \end{aligned} \tag{5.1.24}$$

Here, the coefficients on the left-hand side are given by

$$\begin{aligned} a(\sigma,\rho,\partial_\sigma\rho) &:= \frac{1}{(J(\rho))^4} \\ b_1(\sigma) &:= 1 \\ b_2(\sigma,\rho,\partial_\sigma\rho) &:= \frac{1}{(J(\rho))^4}\langle \Psi_q, R\Psi_\sigma\rangle(\sigma,\rho). \end{aligned} \tag{5.1.25}$$

The right-hand side of the first equation is defined by

$$\begin{aligned} f(\sigma,\rho,\partial_\sigma\rho,\partial_\sigma^2\rho,\partial_\sigma^3\rho) &:= S(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho\partial_\sigma^2\rho + S(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho + S(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^3 \\ &\quad + S(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^2 + S(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + S(\sigma,\rho,\partial_\sigma\rho), \end{aligned} \tag{5.1.26}$$

where the prefactors $S(\sigma,\rho,\partial_\sigma\rho)$ denote terms of the form

$$S(\sigma,\rho,\partial_\sigma\rho) := \frac{-J(\rho)}{\langle\Psi_q, R\Psi_\sigma\rangle(\sigma,\rho)}\tilde{S}(\sigma,\rho,\partial_\sigma\rho) \qquad \text{for } \tilde{S} \text{ of the form (5.1.11).} \tag{5.1.27}$$

Moreover, the right-hand sides of the second and third equation are given by

$$\begin{aligned} g_1(\sigma) &:= 0 \\ g_2(\sigma,\rho,\partial_\sigma\rho,\partial_\sigma^2\rho) &:= T(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^2 + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho), \end{aligned} \tag{5.1.28}$$

where the prefactors $T(\sigma,\rho,\partial_\sigma\rho)$ are introduced in (5.1.9).

Deriving the motion law, we assumed $1/J(\rho)\langle\Psi_q, R\Psi_\sigma\rangle(\sigma,\rho) \neq 0$, see (5.1.12). As confirmation, the following lemma is proven:

Lemma 5.1.6
Let $\rho_0 \in X_\mu$*, such that the condition* (5.1.22) *is satisfied. Furthermore, let* $K \in \mathbb{R}^+$*. Then, there exists a* $\tilde{T} = \tilde{T}(\alpha, \Phi^*, \eta, K, R)$ *with* $\|\rho_0\|_{X_\mu} \leq R$*, such that for* $\rho \in \mathbb{E}_{\mu,T}$ *fulfilling* $\rho_{|t=0} = \rho_0$ *and* $\|\rho\|_{\mathbb{E}_{\mu,T}} \leq K$ *for* $0 < T < \tilde{T}$ *it holds:* ρ *fulfills the bounds* (5.1.16) *and* (5.1.17) *for* $\frac{2}{3}K_0$ *and* $\frac{2}{3}K_1$*. In particular,*

$$J(\rho) > C(\alpha,\Phi^*,\eta) > 0 \qquad \textit{for } \sigma \in [0,1] \textit{ and all } 0 \leq t \leq T \textit{ with } 0 < T < \tilde{T}, \tag{5.1.29}$$

where $C(\alpha,\Phi^*,\eta)$ *is given by* (5.1.18)*. Additionally,* $[0,1] \ni \sigma \mapsto \Psi(\sigma, \rho(t,\sigma))$ *is a regular parametri-*

zation for all $0 \leq t \leq T$ *with* $0 < T < \tilde{T}$.

Proof. By the embedding

$$\mathbb{E}_{\mu,T} \hookrightarrow C^{\bar{\alpha}}\left(\bar{J}; C^1(\bar{I})\right),$$

item 2 of Lemma 2.2.3, we have

$$\|\partial_\sigma^i \rho(t) - \partial_\sigma^i \rho_0\|_{C(\bar{I})} \leq T^{\bar{\alpha}} C\left(\|\rho\|_{\mathbb{E}_{\mu,T}} + \|\rho_{|t=0}\|_{X_\mu}\right),$$

where $\bar{\alpha} > 0$, $i \in \{0,1\}$, and $t \in [0,T]$. Note that the constant C does not depend on T. Thus, by choosing $\tilde{T}$ small enough, it holds for $i = 0,1$

$$\|\partial_\sigma^i \rho(t) - \partial_\sigma^i \rho_0\|_{C(\bar{I})} \leq \tilde{T}^{\bar{\alpha}} C\left(K + \|\rho_{|t=0}\|_{X_\mu}\right) < \frac{\min\{K_0, K_1\}}{3}.$$

Consequently,

$$\|\partial_\sigma^i \rho(t)\|_{C(\bar{I})} \leq \|\partial_\sigma^i \rho(t) - \partial_\sigma^i \rho_0\|_{C(\bar{I})} + \|\partial_\sigma^i \rho_0\|_{C(\bar{I})} < \frac{2K_i}{3}$$

is obtained for $i \in \{0,1\}$. The addendum follows directly by (5.1.18) in Lemma 5.1.2. □

Definition 5.1.7
For fixed K, *we consider the corresponding* $\tilde{T}(\alpha, \Phi^*, \eta, K, R) > 0$ *with* $\|\rho_0\|_{X_\mu} \leq R$, *which is given in Lemma 5.1.6. We set*

$$\mathcal{B}_{K,T} := \left\{\rho \in \mathbb{E}_{\mu,T} : \|\rho\|_{\mathbb{E}_{\mu,T}} \leq K, \rho_{|t=0} = \rho_0\right\} \qquad \text{for } 0 < T < \tilde{T}.$$

Hence, we obtain:

Remark 5.1.8 *1. There exists a* $C(\alpha, \Phi^*, \eta, K) > 0$, *such that*

$$\langle \Psi_q, R\Psi_\sigma\rangle(\sigma,\rho) > C(\alpha, \Phi^*, \eta, K) > 0 \qquad \text{for } \sigma \in [0,1] \text{ and } \rho \in \mathcal{B}_{K,T} \text{ with } 0 < T < \tilde{T}.$$

By direct calculations, we obtain

$$\begin{aligned}\langle \Psi_q, R\Psi_\sigma\rangle(\sigma,\rho) &= \langle n_\Lambda, \left[\mathcal{L}[\Phi^*] - \rho\mathcal{L}[\Phi^*]\kappa_\Lambda + \rho \cot\alpha\eta'\right] R\tau_\Lambda\rangle \\ &\quad + \langle \cot\alpha\eta\tau_\Lambda, \rho\mathcal{L}[\Phi^*]\kappa_\Lambda \cot\alpha\eta R n_\Lambda\rangle \\ &= \mathcal{L}[\Phi^*]\left[1 - \rho\left(\kappa_\Lambda - \frac{1}{\mathcal{L}[\Phi^*]}\cot\alpha\eta' - (\cot\alpha)^2\eta^2\kappa_\Lambda\right)\right],\end{aligned}$$

where we used (5.1.19) *and the identities*

$$\langle n_\Lambda, R\tau_\Lambda\rangle = 1 \qquad \text{and} \qquad \langle \tau_\Lambda, R n_\Lambda\rangle = -1.$$

Exploiting Lemma 2.3.1 again, we deduce for $\rho \in \mathcal{B}_{K,T}$

$$\begin{aligned}\langle \Psi_q, R\Psi_\sigma\rangle(\sigma,\rho) &\geq \mathcal{L}[\Phi^*]\left[1 - \rho\left(\kappa_\Lambda - \hat{C}\kappa_\Lambda \cot\alpha\eta' - (\cot\alpha)^2\eta^2\kappa_\Lambda\right)\right] \\ &\geq \mathcal{L}[\Phi^*]\left[1 - |\rho|\,\|\kappa_\Lambda\|_{C(\bar{I})}\left(1 + \hat{C}|\cot\alpha\eta'| + (\cot\alpha)^2\right)\right] \\ &> C(\alpha, \Phi^*, \eta, K) > 0.\end{aligned}$$

2. *By the expressions for the derivatives of* Ψ, *cf.* (5.1.19), *we additionally derive*

$$J(\rho) < C(\alpha, \Phi^*, \eta, K) \qquad \textit{for } \sigma \in [0,1] \textit{ and } \rho \in \mathcal{B}_{K,T} \textit{ with } 0 < T < \tilde{T}.$$

3. *The previous statements hold also true if* $\rho \in \mathcal{B}_{K,T}$ *is replaced by an initial datum* ρ_0 *fulfilling the assumptions of Theorem 5.1.3.*

In the following, we want to solve the partial differential equation (5.1.24). As a first step, the equation is linearized and the generated linear problem is solved with optimal regularity.

5.2 The Linear Problem

In order to solve the problem (5.1.24), we linearize it around the initial datum ρ_0. In the following, we want to show that, for a suitable choice of the function spaces, the linearized problem

$$\begin{aligned} \rho_t + \mathcal{A}\rho &= F(t,\sigma) && \text{for } (t,\sigma) \in (0,T) \times [0,1], \\ \mathcal{B}_1\rho &= G_1(t,\sigma) && \text{for } \sigma \in \{0,1\} \text{ and } t \in (0,T), \\ \mathcal{B}_2\rho &= G_2(t,\sigma) && \text{for } \sigma \in \{0,1\} \text{ and } t \in (0,T), \\ \rho(0,\sigma) &= \rho_0(\sigma) && \text{for } \sigma \in [0,1], \end{aligned} \tag{5.2.1}$$

has a unique solution ρ with optimal regularity. The linear operators $\mathcal{A}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$\begin{aligned} \mathcal{A} &:= a(\sigma, \rho_0, \partial_\sigma \rho_0) D^4 && \text{for } \sigma \in [0,1], \\ \mathcal{B}_1 &:= i b_1(\sigma) \operatorname{tr}_{\partial I} D^1 && \text{for } \sigma \in \{0,1\}, \\ \mathcal{B}_2 &:= i^3 b_2(\sigma, \rho_0, \partial_\sigma \rho_0) \operatorname{tr}_{\partial I} D^3 && \text{for } \sigma \in \{0,1\}, \end{aligned} \tag{5.2.2}$$

where $D = -i\partial_\sigma$ and a, b_1, and b_2 are given by (5.1.25). Moreover, for a fixed but arbitrary $\bar{\rho} \in \mathcal{B}_{K,T}$, cf. Definition 5.1.7, we set

$$\begin{aligned} F(t,\sigma) &:= -\left(\frac{1}{(J(\bar{\rho}))^4} - \frac{1}{(J(\rho_0))^4}\right) \partial_\sigma^4 \bar{\rho} + f(\bar{\rho}, \partial_\sigma \bar{\rho}, \partial_\sigma^2 \bar{\rho}, \partial_\sigma^3 \bar{\rho}), \\ G_1(t,\sigma) &:= 0, \\ G_2(t,\sigma) &:= -\left(b_2(\bar{\rho}, \partial_\sigma \bar{\rho}) - b_2(\rho_0, \partial_\sigma \rho_0)\right) \partial_\sigma^3 \bar{\rho} - g_2(\bar{\rho}, \partial_\sigma \bar{\rho}, \partial_\sigma^2 \bar{\rho}), \end{aligned}$$

and $\tilde{G} := (G_1, G_2)$. After solving the linear problem (5.2.1) for a general $\bar{\rho} \in \mathcal{B}_{K,T}$, we will prove the well-posedness of the partial differential equation (5.1.24) by a fixed point argument.

The existence of a unique solution for the linearized problem (5.2.1) will be proven by applying the maximal regularity result stated in Theorem 2.1 of [25] to this problem. A simplified version of this local existence theorem is stated in Theorem 2.2.1. Applying this, we deduce the following theorem:

Theorem 5.2.1 (Existence for the Linear Problem (5.2.1))
Let the assumptions of Theorem 5.1.3 hold true and let K *and* $\tilde{T}$ *be given by Definition 5.1.7. Then the linearized problem* (5.2.1) *possesses a unique solution* $\rho \in \mathbb{E}_{\mu,T}$ *for* $0 < T < \tilde{T}$, *if and only if*

$$F \in \mathbb{E}_{0,\mu}, \ \tilde{G} \in \tilde{\mathbb{F}}_\mu,$$

and the compatibility condition

$$\mathcal{B}_1(0,\cdot,D)\rho_0 = G_1(0,\cdot) = 0 \qquad \textit{for } \sigma \in \{0,1\} \tag{5.2.3}$$

is fulfilled.

Remark 5.2.2 *Note that the compatibility condition* (5.2.3) *is fulfilled immediately if* (5.1.23) *holds true.*

Proof of Theorem 5.2.1. We have to check the assumptions of Theorem 2.2.1 for the operators given in (5.2.2) and $E := \mathbb{C}$, which is a Banach space of class $\mathcal{HT}$, as it is a Hilbert space.

First of all, by Remark 5.1.5, item 1, a lower bound on $J(\rho_0)$ for $\sigma \in [0,1]$ is obtained. Thus, $a(\sigma, \rho_0, \partial_\sigma \rho_0)$ is well-defined and positive for $\sigma \in [0,1]$. Consequently, the operator $\mathcal{A}$ is well-defined and non-trivial.

Now, set $m_1 = 1$ and $m_2 = 3$. It is obvious that the boundary operator $\mathcal{B}_1$ is non-trivial as well, as its coefficient is $ib_1 = i$. We proceed with the non-triviality of $\mathcal{B}_2$: Like previously mentioned, we have $J(\rho_0) > 0$ for $\sigma \in \{0,1\}$. Moreover, by Remark 5.1.8, item 3, is obtained that $\langle \Psi_q, R\Psi_\sigma \rangle(\sigma, \rho_0) > 0$ for $\sigma \in \{0,1\}$. Thus, the coefficient $i^3 b_2(\sigma, \rho_0, \partial_\sigma \rho_0)$ is well-defined and non-zero for $\sigma \in \{0,1\}$, and $\mathcal{B}_2$ is non-trivial.

By definition, we have

$$\omega_1 = \frac{5}{8} \qquad \omega_2 = \frac{1}{8}. \tag{5.2.4}$$

We begin by verifying condition (SD), i.e. $a \in BUC(\bar{J} \times \bar{I})$. As the initial datum ρ_0 and accordingly the coefficient of $\mathcal{A}$, given by $a(\sigma, \rho_0, \partial_\sigma \rho_0)$, does not depend on t, the time regularity is trivial. Remark 5.1.5, item 1 and Remark 5.1.8, item 3, give a lower and upper bound on $J(\rho_0)$ for $\sigma \in [0,1]$, respectively. Moreover, we observe that for $0 < a < b < \infty$, the mapping $[a,b] \ni x \mapsto (1/x)^4$ is continuous. Additionally, as a sum, product and concatenation of continuous functions, the mapping $[0,1] \ni \sigma \mapsto J(\rho_0)$ is continuous. Thus, $[0,1] \ni a(\sigma, \rho_0, \partial_\sigma \rho_0)$ is continuous on a compact interval, hence uniformly continuous and bounded.

Next, we want to verify condition (SB): For $j = 1$ we show that the "or"-case is fulfilled, i.e. $ib_1 \in \mathbb{F}_{1,\mu}$. The regularity is clear, as the coefficient $ib_1 = i$ is constant. Moreover, we observe for $\mu \in (7/8, 1]$

$$\omega_1 = \frac{5}{8} = 1 - \frac{7}{8} + \frac{1}{2} > 1 - \mu + \frac{1}{2}.$$

For $j = 2$ we prove that the assumptions of the "either"-case hold true, i.e. $i^3 b_2 \in C^{\tau_2, 4\tau_2}(\bar{J} \times \partial I)$ with some $\tau_2 > \omega_2 = 1/8$. We fix τ_2 such that $1 > 4\tau_2 > 4\omega_2$ is fulfilled. The time-regularity is trivial, as the coefficient does not depend on t. As ∂I consists only of two points, the spatial regularity follows directly as well.

Moreover, we have to verify the ellipticity condition (E), i.e. for all $t \in \bar{J}$, $x \in \bar{I}$ and $|\xi| = 1$, it holds for the spectrum $\Sigma(\mathcal{A}(t, \sigma, \xi)) \subset \mathbb{C}_+ := \{\Re z > 0\}$. The symbol of $\mathcal{A}$ is given by

$$\mathcal{A}(t, \sigma, \xi) = i^4 \frac{1}{(J(\rho_0))^4} \xi^4 = \frac{1}{(J(\rho_0))^4} > 0.$$

Thus, it fulfills the condition by Remark 5.1.5, item 1.

Lastly, the Lopatinskii-Shapiro-condition (LS) is verified: We have to show that for each fixed

$t \in \bar{J}$ and $\sigma \in \{0,1\}$, for each $\lambda \in \overline{\mathbb{C}_+}\backslash\{0\}$ and each $h \in \mathbb{C}^2$ the ordinary boundary value problem

$$\lambda v(y) + a(\sigma)\partial_y^4 v(y) = 0 \qquad y > 0, \tag{5.2.5}$$
$$\tilde{b}_2(\sigma)\partial_y^3 v(y)|_{y=0} = h_2, \tag{5.2.6}$$
$$\tilde{b}_1(\sigma)\partial_y v(y)|_{y=0} = h_1, \tag{5.2.7}$$
$$\lim_{y\to\infty} v(y) = 0, \tag{5.2.8}$$

with

$$\left.\begin{aligned} \tilde{b}_2(\sigma) &:= \tfrac{\pm 1}{(J(\rho_0(\sigma)))^3}\langle \Psi_q, R\Psi_\sigma\rangle(\sigma,\rho_0(\sigma)) \\ \tilde{b}_1(\sigma) &:= \pm 1, \end{aligned}\right\} \qquad \text{for } \sigma = 0,1,$$

has a unique solution $v \in C_0([0,\infty);\mathbb{C})$, see (2.2.1) for a definition. Note that we already proved before, that $\tilde{b}_2$ and $\tilde{b}_1$ do not vanish due to our assumptions on ρ_0. Solving the corresponding characteristic equation to (5.2.5), we obtain that a solution is given by

$$v(y) = c_0 e^{\mu_0 y} + c_1 e^{\mu_1 y} + c_2 e^{\mu_2 y} + c_3 e^{\mu_3 y},$$

where $c_0, \dots, c_3 \in \mathbb{C}$ and

$$\mu_0 = \sqrt[4]{r}(\bar{a} + i\bar{b}), \qquad \mu_1 = \sqrt[4]{r}(-\bar{b} + i\bar{a}),$$
$$\mu_2 = \sqrt[4]{r}(-\bar{a} + i(-\bar{b})), \qquad \mu_3 = \sqrt[4]{r}(\bar{b} + i(-\bar{a})).$$

Here the numbers $r, \bar{a}$, and $\bar{b}$ are determined by the identity

$$\frac{-\lambda}{a(\sigma)} = r[\cos\theta + i\sin\theta] \qquad \text{for } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right],$$

where $\lambda \in \overline{\mathbb{C}_+}\backslash\{0\}$ and $\sigma = 0,1$. More precisely, it holds

$$r := \frac{|\lambda|}{a(\sigma)} > 0, \qquad \bar{a} := \cos(\theta/4) > 0, \qquad \bar{b} := \sin(\theta/4) > 0,$$

for $\theta/4 \in [\pi/8, 3\pi/8]$. Due to condition (5.2.8) we see immediately that $c_0 = c_3 = 0$ and the solution simplifies to

$$v(y) = c_1 e^{\mu_1 y} + c_2 e^{\mu_2 y}.$$

The first and third derivative is given by

$$v'(0) = c_1\mu_1 + c_2\mu_2 \qquad \text{and} \qquad v'''(0) = c_1\mu_1^3 + c_2\mu_2^3.$$

Combining this with (5.2.6) and (5.2.7), we have to show that for each $(h_2, h_1) \in \mathbb{C}^2$

$$M\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} h_2/\tilde{b}_2 \\ h_1/\tilde{b}_1 \end{pmatrix}, \qquad \text{with } M := \begin{pmatrix} \mu_1^3 & \mu_2^3 \\ \mu_1 & \mu_2 \end{pmatrix},$$

has a unique solution (c_1, c_2). We observe that $\det M = \mu_1\mu_2(\mu_1^2 - \mu_2^2) \neq 0$, as $\mu_1 \neq 0 \neq \mu_2$ and

$$\mu_1^2 - \mu_2^2 = \sqrt[2]{r}\left[\left(2\bar{b}^2 - 2\bar{a}^2\right) + i\left(-4\bar{a}\bar{b}\right)\right] \neq 0, \qquad \text{as } \bar{a}, \bar{b} > 0.$$

This proves condition (LS).

Finally, we observe by (5.2.4) for $\mu \in \left(\frac{7}{8}, 1\right]$

$$\begin{aligned}\omega_1 &= \frac{5}{8} = 1 - \frac{7}{8} + \frac{1}{2} > 1 - \mu + \frac{1}{2},\\ \omega_2 &= \frac{1}{8} < 1 - 1 + \frac{1}{2} \leq 1 - \mu + \frac{1}{2},\end{aligned} \tag{5.2.9}$$

and therefore, $1 - \mu + 1/p \neq \omega_1, \omega_2$. $\square$

Remark 5.2.3 *By* (5.2.9) *it follows directly that the data space, see Theorem 2.2.1, is given by*

$$\mathcal{D} = \left\{ \left(F, \tilde{G}, \rho_0\right) \in \mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu \times X_\mu : \mathcal{B}_1(0,\cdot,D)\rho_0 = G_1(0,\cdot) \text{ on } \partial I \right\}.$$

The initial datum ρ_0 does not have to fulfill further compatibility conditions than (5.2.3)*. Moreover, we observe that the compatibility condition is fulfilled for right-hand side zero, i.e. $G_1(0,\cdot) = 0$, hence $\mathcal{D} = \mathcal{D}_0$, cf. Theorem 2.2.1. Consequently, by the addendum in Theorem 2.2.1, we immediately get a uniform estimate for the solution operator for all $T \in (0, T_0]$, T_0 given, since we are automatically in the case of $\mathcal{D}_0$.*

In order to obtain a solution to the quasilinear problem (5.1.15), cf. Theorem 5.1.3, a contraction mapping argument is applied. We consider the linear operator $\mathcal{L} : \mathbb{E}_{\mu,T} \to \mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu \times X_\mu$, which is given by the left-hand side of (5.2.1). Moreover, we set

$$\mathcal{F}(\rho) := \begin{pmatrix} F(t,\sigma) \\ G_1(t,\sigma) \\ G_2(t,\sigma) \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{(J(\rho))^4} - \frac{1}{(J(\rho_0))^4}\right)\partial_\sigma^4 \rho + f(\rho, \partial_\sigma \rho, \partial_\sigma^2 \rho, \partial_\sigma^3 \rho) \\ 0 \\ -\left(b_2(\rho, \partial_\sigma \rho) - b_2(\rho_0, \partial_\sigma \rho_0)\right)\partial_\sigma^3 \rho - g_2(\rho, \partial_\sigma \rho, \partial_\sigma^2 \rho) \end{pmatrix}, \tag{5.2.10}$$

which corresponds to the right-hand side of (5.2.1). The equation (5.2.1) is now represented by

$$\mathcal{L}(\rho) = (\mathcal{F}(\rho), \rho_0) \qquad \text{for } \rho \in \mathcal{B}_{K,T} \text{ with } 0 < T < \tilde{T}. \tag{5.2.11}$$

In order to receive a fixed point equation, Theorem 5.2.1 is used for inverting the linear operator $\mathcal{L}$. Thus, it remains to show that $(F, \tilde{G}) \in \mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu$.

Lemma 5.2.4
Let the assumptions of Theorem 5.1.3 hold true and let K and $\tilde{T}$ be given by Definition 5.1.7. Then it holds

$$\mathcal{F}(\rho) \in \mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu \qquad \text{for } \rho \in \mathcal{B}_{K,T} \text{ with } 0 < T < \tilde{T}.$$

Remark 5.2.5 *Even though it is not important for the proof of this claim, we will keep track of the dependencies of the constants, especially when they depend on the time T. This will enable us to use the derived estimates in the proof of the contraction property of the operator $\mathcal{F}$.*

Proof of Lemma 5.2.4. Firstly, we want to take care of the first component of $\mathcal{F}(\rho)$, cf. (5.2.10): More precisely, we want to show that $F \in \mathbb{E}_{0,\mu} = L_{2,\mu}(J; L_2(I))$ for $\rho \in \mathcal{B}_{K,T}$ with $0 < T < \tilde{T}$. To this end, the following claim is proven.

Claim 5.2.6 *Let K and $\tilde{T}$ be given by Definition 5.1.7. For $\rho \in \mathcal{B}_{K,T}$ with $0 < T < \tilde{T}$, there exists a constant $C(\alpha, \Phi^*, \eta, K) > 0$, such that*

$$\left\| \left(\frac{1}{(J(\rho))^4} - \frac{1}{(J(\rho_0))^4} \right) \right\|_{C(\bar{J}, C(\bar{I}))} \leq C(\alpha, \Phi^*, \eta, K) \| J(\rho) - J(\rho_0) \|_{C(\bar{J}, C(\bar{I}))},$$

Proof of the claim: By (5.1.29), there exists a $C(\alpha, \Phi^*, \eta, K)$, such that

$$\left\| \frac{1}{J(\rho)} \right\|_{C(\bar{J}, C(\bar{I}))} \leq C(\alpha, \Phi^*, \eta, K) \qquad \text{for } \rho \in \mathcal{B}_{K,T} \text{ with } 0 < T < \tilde{T}. \tag{5.2.12}$$

Using Remark 5.1.8, item 2, we have a $\bar{C}(\alpha, \Phi^*, \eta, K) > 0$ such that

$$\|J(\rho)\|_{C(\bar{J}, C(\bar{I}))} \leq \bar{C}(\alpha, \Phi^*, \eta, K) \qquad \text{for } \rho \in \mathcal{B}_{K,T} \text{ with } 0 < T < \tilde{T}. \tag{5.2.13}$$

Now, the Lipschitz continuity of $x \mapsto 1/x^4$ on the interval $[C^{-1}, \bar{C}]$ is exploited: We denote the corresponding Lipschitz constant by $C(\alpha, \Phi^*, \eta, K)$ and the claim follows directly. □

For the first summand of F, we have by Claim 5.2.6 and Remark 5.1.8, item 2 and 3,

$$\begin{aligned} \left\| -\left(\frac{1}{(J(\rho))^4} - \frac{1}{(J(\rho_0))^4} \right) \partial_\sigma^4 \rho \right\|_{L_{2,\mu}(J; L_2(I))} &\leq L \|J(\rho) - J(\rho_0)\|_{C(\bar{J}, C(\bar{I}))} \|\rho\|_{L_{2,\mu}(J; W_2^4(I))} \\ &\leq C(\alpha, \Phi^*, \eta, K), \end{aligned}$$

for all $\rho \in \mathcal{B}_{K,T}$ with $0 < T < \tilde{T}$.

We proceed with the estimate of $\|f\|_{L_{2,\mu}(J;L_2(I))}$ by $\|\rho\|_{\mathbb{E}_{\mu,T}}$, cf. (5.1.26) for the representation of f. First, we take care of the prefactors $S(\sigma, \rho, \partial_\sigma \rho)$: We want to prove that they are bounded in $L_\infty(\bar{J} \times \bar{I})$ by a constant depending on α, Φ^*, η, and K. To this end, we consider the structure

$$S(\sigma, \rho, \partial_\sigma \rho) := \frac{1}{\langle \Psi_q, R\Psi_\sigma \rangle(\sigma, \rho)} C \left(J(\rho) \right)^k \left(\prod_{i=0}^{p} \left\langle \Psi_{(\sigma,q)}^{\beta_i}, R\Psi_{(\sigma,q)}^{\gamma_i} \right\rangle (\sigma, \rho) \right) (\partial_\sigma \rho)^r,$$

with $C \in \mathbb{R}$, $k \in \mathbb{Z}$, $p, r \in \mathbb{N}_0$, and $\beta_i, \gamma_i \in \mathbb{N}_0^2$, such that $|\beta_i|, |\gamma_i| \geq 1$ and $|\beta_i| + |\gamma_i| \leq 5$ for all $i \in \{0, \dots, p\}$, see (5.1.27) and (5.1.11). By Remark 5.1.8, item 1, we obtain a bound on $1/\langle \Psi_q, R\Psi_\sigma \rangle(\sigma, \rho)$. For the factor $(J(\rho))^n$, $n \in \mathbb{Z}$ we can directly use the bounds established in (5.2.12) and (5.2.13), respectively. Moreover, we control ρ and $\partial_\sigma \rho$ by Lemma 5.1.6. Finally, we take care of the scalar products: Note that there are at most four derivatives on Ψ. Combining the C^4-bound on $[\sigma \mapsto \Psi(\sigma, q)]$ established in Remark 5.1.1, item 1, with the previously discussed bound on ρ, we obtain

$$\left\| \left[(t, \sigma) \mapsto \Psi_{(\sigma,q)}^{\gamma}(\sigma, \rho(t, \sigma)) \right] \right\|_{L_\infty(\bar{J} \times \bar{I})} \leq C(\alpha, \Phi^*, \eta, K),$$

for $|\gamma| \leq 4$. In summary, a suitable bound is given by

$$\|S(\sigma, \rho, \partial_\sigma \rho)\|_{L_\infty(\bar{J} \times \bar{I})} \leq C(\alpha, \Phi^*, \eta, K).$$

Next, we find $L_{2,\mu}(J; L_2(I))$-bounds for the summands of $f(\rho, \partial_\sigma \rho, \partial_\sigma^2 \rho, \partial_\sigma^3 \rho)$, where the prefactors can be neglected due to the previously established bound. In the following, we will use ρ_1 and ρ_2 to keep our calculations general. We have

$$\begin{aligned} \|\partial_\sigma^2 \rho_1 \partial_\sigma^3 \rho_2\|_{L_{2,\mu}(J; L_2(I))} &\leq \left\| \|\partial_\sigma^2 \rho_1(t)\|_{L_{q_1}(I)} \|\partial_\sigma^3 \rho_2(t)\|_{L_{q_2}(I)} \right\|_{L_{2,\mu}(J)} \\ &\leq \left\| \|\rho_1(t)\|_{W_{q_1}^2(I)} \|\rho_2(t)\|_{W_{q_2}^3(I)} \right\|_{L_{2,\mu}(J)}, \end{aligned}$$

where the estimates follow for $q_1, q_2 \in [2, \infty]$ fulfilling $1/q_1 + 1/q_2 = 1/2$ by Hölder's inequality. By expanding the integrand, we obtain for $\tilde{\mu}_i \in [\mu, 1]$, $i = 1, 2$,

$$\begin{aligned}
\left\|\partial_\sigma^2 \rho_1 \partial_\sigma^3 \rho_2\right\|_{L_{2,\mu}(J;L_2(I))} &\le \left\| \left(t^{1-\tilde{\mu}_1} \|\rho_1(t)\|_{W^2_{q_1}(I)} t^{1-\tilde{\mu}_2} \|\rho_2(t)\|_{W^3_{q_2}(I)} \right) t^{1-\mu-(1-\tilde{\mu}_1+1-\tilde{\mu}_2)} \right\|_{L_2(J)} \\
&\le C_1 \left\| \left(t^{1-\tilde{\mu}_1} \|\rho_1(t)\|_{W^2_{q_1}(I)} \right) \left(t^{1-\tilde{\mu}_2} \|\rho_2(t)\|_{W^3_{q_2}(I)} \right) \right\|_{L_2(J)} \\
&\le C_1 C_2 \|\rho_1\|_{L_{l_1,\tilde{\mu}_1}(J;W^3_{q_1}(I))} \|\rho_2\|_{L_{l_2,\tilde{\mu}_2}(J;W^3_{q_2}(I))}, \qquad (5.2.14)
\end{aligned}$$

for $1/l_1 + 1/l_2 \le 1/2$. Here

$$C_1 = \begin{cases} C(T) \to 0 \text{ as } T \to 0 & \text{if } 1 - \mu - (1 - \tilde{\mu}_1 + 1 - \tilde{\mu}_2) > 0, \\ 1 & \text{if } 1 - \mu - (1 - \tilde{\mu}_1 + 1 - \tilde{\mu}_2) = 0, \end{cases}$$

$$C_2 = \begin{cases} C(T) \to 0 \text{ as } T \to 0 & \text{if } \frac{1}{l_1} + \frac{1}{l_2} < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{l_1} + \frac{1}{l_2} = \frac{1}{2}. \end{cases}$$

Our aim is to control $\|\partial_\sigma^2 \rho_1 \partial_\sigma^3 \rho_2\|_{L_{2,\mu}(J;L_2(I))}$ in terms of $\|\rho_i\|_{\mathbb{E}_{\mu,T}}$. To this end, Proposition 2.2.4 is used to estimate the right-hand side of (5.2.14), where $1/q_1 + 1/q_2 = 1/2$, $k_1 = 2, k_2 = 3$, and l_i and the time-weights $\tilde{\mu}_i$ have to be chosen carefully.

By the assumptions of the Proposition 2.2.4, we obtain by $\theta_i/2 = 1/l_i$, $i = 1, 2$,

$$\theta_1 + \theta_2 < 1 \quad \Leftrightarrow \quad \frac{1}{l_1} + \frac{1}{l_2} < \frac{1}{2} \qquad \text{and} \qquad \theta_1 + \theta_2 = 1 \quad \Leftrightarrow \quad \frac{1}{l_1} + \frac{1}{l_2} = \frac{1}{2}.$$

Moreover, we have by $\tilde{\mu}_i = \mu + (1 - \theta_i)(1 - \mu) \in [\mu, 1]$ the equality

$$\begin{aligned}
1 - \mu - (1 - \tilde{\mu}_1 + 1 - \tilde{\mu}_2) &= \tilde{\mu}_1 + \tilde{\mu}_2 - 1 - \mu \\
&= \mu + (1 - \theta_1)(1 - \mu) + \mu + (1 - \theta_2)(1 - \mu) - 1 - \mu \\
&= -(1 - \mu) + (2 - (\theta_1 + \theta_2))(1 - \mu) = (1 - (\theta_1 + \theta_2))(1 - \mu), \qquad (5.2.15)
\end{aligned}$$

thus, for $\mu \in (7/8, 1)$

$$\begin{aligned}
\theta_1 + \theta_2 < 1 &\Leftrightarrow 1 - \mu - (1 - \tilde{\mu}_1 + 1 - \tilde{\mu}_2) > 0, \\
\theta_1 + \theta_2 = 1 &\Leftrightarrow 1 - \mu - (1 - \tilde{\mu}_1 + 1 - \tilde{\mu}_2) = 0,
\end{aligned}$$

and $1 - \mu - (1 - \tilde{\mu}_1 + 1 - \tilde{\mu}_2) = 0$ for $\mu = 1$. Consequently, to produce a constant $C(T) \to 0$ for $T \to 0$, it remains to prove that for $1/q_1 + 1/q_2 = 1/2$, $k_1 = 2, k_2 = 3$ and $\mu \in (7/8, 1]$, it is possible to fulfill $\theta_1 + \theta_2 < 1$. Using $s_i := k_i + 1/2 - 1/q_i = 4\,(\mu - 1/2)\,(1 - \theta_i) + 4\theta_i$, we directly calculate

$$\begin{aligned}
\theta_1 + \theta_2 &= 1 - \frac{4 - s_1}{4\left(1 - \mu + \frac{1}{2}\right)} + 1 - \frac{4 - s_2}{4\left(1 - \mu + \frac{1}{2}\right)} = 2 - \frac{8 - \left(k_1 + k_2 + 1 - \left(\frac{1}{q_1} + \frac{1}{q_2}\right)\right)}{4\left(1 - \mu + \frac{1}{2}\right)} \\
&= 2 - \frac{2,5}{4\left(1 - \mu + \frac{1}{2}\right)} = 2 - \frac{5}{8\left(1 - \mu + \frac{1}{2}\right)} < 2 - \frac{5}{8\left(1 - \frac{7}{8} + \frac{1}{2}\right)} = 1.
\end{aligned}$$

Hence, applying Proposition 2.2.4 to (5.2.14), it follows

$$\left\|\partial_\sigma^2 \rho_1 \partial_\sigma^3 \rho_2\right\|_{L_{2,\mu}(J;L_2(I))} \le C(T) \prod_{i=1}^{2} \left(\|\rho_i\|_{L_\infty(J, W_2^{4(\mu - 1/2)}(I))} + \|\rho_i\|_{L_{2,\mu}(J, W_2^4(I))} \right), \qquad (5.2.16)$$

where $C(T) \to 0$ as $T \to 0$, as the constant of the embedding in Proposition 2.2.4 does not depend on T. Setting $\rho_i = \rho$, $i = 1, 2$, in (5.2.16) and using the embedding of the solution space into the temporal trace space, see (2.2.4), we deduce

$$\left\|S(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho\partial_\sigma^3\rho\right\|_{L_{2,\mu}(J;L_2(I))} \leq C(\alpha,\Phi^*,\eta,K,T) \qquad \text{for } \rho\in\mathcal{B}_{K,T} \text{ with } 0<T<\tilde{T}.$$

Note that the constant on the right-hand side does in general no longer fulfill $C(T)\to 0$ for $T\to 0$, as the operator norm of (2.2.4) also depends on T.

By inspection of the remaining summands, we obtain that all terms besides $S(\sigma,\rho,\partial_\sigma\rho)(\partial_\sigma^2\rho)^3$ can be treated as in (5.2.16), where an additional T^s with $s>0$ is possibly produced. Thus, it just remains to show that $(\partial_\sigma^2\rho)^3\in L_{2,\mu}(J;L_2(I))$ for $\rho\in\mathcal{B}_{K,T}$ with $0<T<\tilde{T}$. To keep the calculations general for employing them later, we use again ρ_i, $i=1,2,3$. By Hölder's inequality and direct estimates, we deduce

$$\left\|\prod_{i=1}^3\partial_\sigma^2\rho_i\right\|_{L_{2,\mu}(J;L_2(I))} \leq \left\|t^{1-\mu}\prod_{i=1}^3\left\|\partial_\sigma^2\rho_i(t)\right\|_{L_6(I)}\right\|_{L_2(J)} \leq \left\|t^{1-\mu}\prod_{i=1}^3\|\rho_i(t)\|_{W_6^2(I)}\right\|_{L_2(J)}.$$

Again, we want to find bounds for the right-hand side by Proposition 2.2.4, where $k=k_i=2$ and $q=q_i=6$, $i=1,2,3$. A direct calculation shows that $\theta=\theta_i$ is given by

$$\theta=\frac{k+\frac{1}{2}-\frac{1}{q}-4\left(\mu-\frac{1}{2}\right)}{4\left(\frac{3}{2}-\mu\right)}=\frac{4+\frac{1}{3}-4\mu}{4\left(\frac{3}{2}-\mu\right)}\in\left[\frac{1}{6},\frac{1}{3}\right) \qquad \text{for } \mu\in\left(\frac{7}{8},1\right]. \tag{5.2.17}$$

Moreover, by $\theta/2={1}/{l}$ and analogously to (5.2.15), we have for $\mu\in({7}/{8},1)$

$$\frac{3}{l}<\frac{1}{2} \qquad\Leftrightarrow\qquad 3\theta<1 \qquad\Leftrightarrow\qquad 1-\mu-3(1-\tilde{\mu})>0,$$

and $1-\mu-3(1-\tilde{\mu})=0$ for $\mu=1$, where $\tilde{\mu}=\tilde{\mu}_i=\mu+(1-\theta_i)(1-\mu)\in[\mu,1]$. Using (5.2.17), we deduce $3\theta<1$ for $\mu\in({7}/{8},1]$ and consequently we obtain for $l>6$

$$\begin{aligned}\left\|\prod_{i=1}^3\partial_\sigma^2\rho_i\right\|_{L_{2,\mu}(J;L_2(I))} &\leq \left\|t^{1-\mu-3(1-\tilde{\mu})}\left(t^{1-\tilde{\mu}}\right)^3\prod_{i=1}^3\|\rho_i(t)\|_{W_6^2(I)}\right\|_{L_2(J)}\\ &\leq C(T)\prod_{i=1}^3\left\|t^{1-\tilde{\mu}}\|\rho_i(t)\|_{W_6^2(I)}\right\|_{L_l(J)}=C(T)\prod_{i=1}^3\|\rho_i\|_{L_{l,\tilde{\mu}}(J;W_6^2(I))},\end{aligned}$$

where $C(T)\to 0$ as $T\to 0$. By Proposition 2.2.4, it follows

$$\left\|\prod_{i=1}^3\partial_\sigma^2\rho_i\right\|_{L_{2,\mu}(J;L_2(I))} \leq C(T)\prod_{i=1}^3\left(\|\rho_i\|_{L_\infty(J,W_2^{4(\mu-1/2)}(I))}\| + \|\rho_i\|_{L_{2,\mu}(J,W_2^4(I))}\right), \tag{5.2.18}$$

where the constant still fulfills $C(T)\to 0$ for $T\to 0$, since the operator norm of the embedding in Proposition 2.2.4 does not depend on T. Setting $\rho_i=\rho$, $i=1,2,3$, and using again the embedding (2.2.4), we obtain

$$\left\|S(\sigma,\rho,\partial_\sigma\rho)(\partial_\sigma^2\rho)^3\right\|_{L_{2,\mu}(J;L_2(I))} \leq C(\alpha,\Phi^*,\eta,K,T) \qquad \text{for } \rho\in\mathcal{B}_{K,T} \text{ with } 0<T<\tilde{T}.$$

Next, we take care of the other components of $\mathcal{F}(\rho)$: It will be proven that $(G_1,G_2)\in\tilde{\mathbb{F}}_\mu$ for $\rho\in\mathcal{B}_{K,T}$, i.e.

$$G_1\in W_{2,\mu}^{5/8}\Big(J;L_2(\partial I)\Big)\cap L_{2,\mu}\left(J;W_2^{5/2}(\partial I)\right),$$

$$G_2 \in W^{1/8}_{2,\mu}\Big(J; L_2(\partial I)\Big) \cap L_{2,\mu}\left(J; W^{1/2}_2(\partial I)\right).$$

We observe that it suffices to verify $G_1 \in W^{5/8}_{2,\mu}(J)$ and $G_2 \in W^{1/8}_{2,\mu}(J)$ for $\sigma \in \{0,1\}$, as the boundary ∂I only consists of two points. Note that the claim for $G_1 \equiv 0$ is trivially fulfilled.

First, we consider the regularity of ρ and its derivatives at the boundary ∂I: Using Lemma 2.2.3, item 3, for $\rho \in W^1_{2,\mu}(J; L_2(I)) \cap L_{2,\mu}(J; W^4_2(I))$, we obtain

$$\begin{aligned}
\partial_\sigma \rho &\in H^{3/4}_{2,\mu}(J; L_2(I)) \cap L_{2,\mu}\left(J; H^3_2(I)\right), \\
\partial^2_\sigma \rho &\in H^{1/2}_{2,\mu}(J; L_2(I)) \cap L_{2,\mu}\left(J; H^2_2(I)\right), \\
\partial^3_\sigma \rho &\in H^{1/4}_{2,\mu}(J; L_2(I)) \cap L_{2,\mu}\left(J; H^1_2(I)\right).
\end{aligned}$$

Furthermore, Lemma 2.2.3, item 4 provides

$$\begin{aligned}
\operatorname{tr}_{|\partial I}\rho &\in W^{7/8}_{2,\mu}\Big(J; L_2(\partial I)\Big) \cap L_{2,\mu}\left(J; W^{7/2}_2(\partial I)\right), \\
\operatorname{tr}_{|\partial I}\partial_\sigma\rho &\in W^{5/8}_{2,\mu}\Big(J; L_2(\partial I)\Big) \cap L_{2,\mu}\left(J; W^{5/2}_2(\partial I)\right), \\
\operatorname{tr}_{|\partial I}\partial^2_\sigma\rho &\in W^{3/8}_{2,\mu}\Big(J; L_2(\partial I)\Big) \cap L_{2,\mu}\left(J; W^{3/2}_2(\partial I)\right), \\
\operatorname{tr}_{|\partial I}\partial^3_\sigma\rho &\in W^{1/8}_{2,\mu}\Big(J; L_2(\partial I)\Big) \cap L_{2,\mu}\left(J; W^{1/2}_2(\partial I)\right).
\end{aligned}$$

Therefore, we have

$$\partial^k_\sigma \rho(\cdot,\sigma) \in W^{(7-2k)/8}_{2,\mu}(J) \qquad \text{for } k = 0,\dots,3 \text{ and } \sigma \in \{0,1\}. \tag{5.2.19}$$

This enables us to prove $G_2 \in W^{1/8}_{2,\mu}(J)$ for $\sigma \in \{0,1\}$: We recall the structure

$$G_2(t,\sigma) = -\left(b_2(\sigma,\rho,\partial_\sigma\rho) - b_2(\sigma,\rho_0,\partial_\sigma\rho_0)\right)\partial^3_\sigma\rho - g_2(\rho,\partial_\sigma\rho,\partial^2_\sigma\rho),$$

see (5.2.10). The summand $b_2(\sigma,\rho_0,\partial_\sigma\rho_0)\partial^3_\sigma\rho$ is clearly an element of $W^{1/8}_{2,\mu}(J)$, since the coefficient does not depend on t. In order to prove the regularity for the other summands, Lemma 2.1.17, item 1, and Remark 2.1.18 are used: The previously mentioned results state that the product of an element of $W^{5/8}_{2,\mu}(J)$ with one of $W^{1/8}_{2,\mu}(J)$ is again in $W^{1/8}_{2,\mu}(J)$. Therefore, it is helpful to prove

$$\left.\begin{aligned}
b_2(\sigma,\rho,\partial_\sigma\rho) &\in W^{5/8}_{2,\mu}(J), \\
T(\sigma,\rho,\partial_\sigma\rho) &\in W^{5/8}_{2,\mu}(J)
\end{aligned}\right\} \qquad \text{for } \sigma \in \{0,1\}, \tag{5.2.20}$$

see (5.1.25) for the definition of b_2 and (5.1.28) for the representation of g_2 with the coefficients T, cf. (5.1.9). In order to achieve this, we show the following claims.

Claim 5.2.7 *For $\rho \in \mathcal{B}_{K,T}$ with $0 < T < \tilde{T}$, it holds $J(\rho) \in W^{5/8}_{2,\mu}(J)$ and $\frac{1}{J(\rho)} \in W^{5/8}_{2,\mu}(J)$ for $\sigma \in \{0,1\}$ with the estimates*

$$\left.\begin{aligned}
\|J(\rho)\|_{W^{5/8}_{2,\mu}(J)} &\le C(\alpha,\Phi^*,\eta,K,\tilde{T}), \\
\left\|\tfrac{1}{J(\rho)}\right\|_{W^{5/8}_{2,\mu}(J)} &\le C(\alpha,\Phi^*,\eta,K,\tilde{T})
\end{aligned}\right\} \qquad \textit{for } \rho \in \mathcal{B}_{K,T} \textit{ with } 0 < T < \tilde{T} \textit{ and } \sigma \in \{0,1\}.$$

Proof of the claim: First, we show $J(\rho) \in W^{5/8}_{2,\mu}(J)$ for $\sigma \in \{0,1\}$. Recall that $J(\rho)$ is given by

$$J(\rho) = |\Phi_\sigma| = |\Psi_\sigma + \Psi_q \partial_\sigma \rho| = \sqrt{|\Psi_\sigma|^2 + 2\langle \Psi_\sigma, \Psi_q\rangle \partial_\sigma \rho + |\Psi_q|^2 (\partial_\sigma \rho)^2},$$

where Ψ_σ and Ψ_q can be found in (5.1.19). We notice that $\Psi_\sigma(\sigma, \rho(\cdot,\sigma)) \in W^{5/8}_{2,\mu}(J)$ for $\sigma \in \{0,1\}$, since its summands are either independent of t or are given by $\rho(t,\sigma)v$ for $v \in \mathbb{R}^2$, which does not depend on t. Due to the regularity of ρ in (5.2.19), the claim follows directly by the definition of the norm. Besides, $\Psi_q(\sigma)\partial_\sigma \rho(\cdot,\sigma) \in W^{5/8}_{2,\mu}(J)$ for $\sigma \in \{0,1\}$ as well by (5.2.19), since Ψ_q does not depend on t. Moreover, we have

$$|\langle \Psi_\sigma, \Psi_q\rangle \partial_\sigma \rho| \leq |\Psi_\sigma||\Psi_q|\partial_\sigma \rho| \leq |\Psi_\sigma|^2 + (|\Psi_q|\partial_\sigma \rho)^2$$

by Cauchy-Schwartz-inequality and Young's inequality. Using this, we obtain directly

$$\begin{aligned}\|J(\rho)\|^2_{L_{2,\mu}(J)} &= \int_J t^{2(1-\mu)} \left(|\Psi_\sigma|^2 + 2|\langle \Psi_\sigma, \Psi_q\rangle \partial_\sigma \rho| + |\Psi_q|^2(\partial_\sigma\rho)^2\right) \mathrm{d}t \\ &\leq 2\int_J t^{2(1-\mu)} \left(|\Psi_\sigma|^2 + |\Psi_q|^2(\partial_\sigma\rho)^2\right) \mathrm{d}t\end{aligned}$$

for $\sigma \in \{0,1\}$. For the semi-norm, the properties of the scalar product for $\Psi_\sigma(t) = \Psi_\sigma(\sigma, \rho(t,\sigma))$ yield

$$\begin{aligned}[J(\rho)]^2_{W^{5/8}_{2,\mu}(J)} &\leq \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{\left||\Phi_\sigma|(t) - |\Phi_\sigma|(\tau)\right|^2}{|t-\tau|^{1+2\frac{5}{8}}} \,\mathrm{d}\tau\,\mathrm{d}t \\ &\leq \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|\Phi_\sigma(t) - \Phi_\sigma(\tau)|^2}{|t-\tau|^{1+2\frac{5}{8}}} \,\mathrm{d}\tau\,\mathrm{d}t \\ &\leq \int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|\Psi_\sigma(t) + \Psi_q\partial_\sigma\rho(t) - \Psi_\sigma(\tau) - \Psi_q\partial_\sigma\rho(\tau)|^2}{|t-\tau|^{1+2\frac{5}{8}}} \,\mathrm{d}\tau\,\mathrm{d}t \\ &\leq C\int_0^T \int_0^t \tau^{2(1-\mu)} \frac{|\Psi_\sigma(t) - \Psi_\sigma(\tau)|^2 + |\Psi_q(\partial_\sigma\rho(t) - \partial_\sigma\rho(\tau))|^2}{|t-\tau|^{1+2\frac{5}{8}}} \,\mathrm{d}\tau\,\mathrm{d}t,\end{aligned}$$

where we used $(a+b)^2 \leq 2(a^2+b^2)$. Note that Ψ_q does not depend on t, as it is independent of ρ. This shows the first claim and the first estimate. Combining this result with Lemma 5.1.6, the second claim and the estimate follows directly by item 4 of Lemma 2.1.17. □

Claim 5.2.8 *For $\rho \in \mathcal{B}_{K,T}$ with $0 < T < \tilde{T}$, it holds for $\sigma \in \{0,1\}$ that*

$$\left\langle \Psi^\beta_{(\sigma,q)}, R\Psi^\gamma_{(\sigma,q)} \right\rangle (\sigma,\rho) \in W^{5/8}_{2,\mu}(J)$$

for $\beta, \gamma \in \mathbb{N}_0^2$, such that $|\beta|, |\gamma| \geq 1$ and $|\beta| + |\gamma| \leq 4$. The $W^{5/8}_{2,\mu}(J)$-norm of the quantity is bounded by a constant $C(\alpha, \Phi^*, \eta, K, \tilde{T})$.

Proof of the claim. By Lemma 2.1.17, item 2, and Remark 2.1.18, we know that $W^{5/8}_{2,\mu}(J)$ is a Banach algebra up to a constant in the norm estimate. Thus, it suffices to prove $\Psi^\beta_{(\sigma,q)}(\sigma, \rho(\cdot,\sigma)) \in W^{5/8}_{2,\mu}(J)$ for $\sigma \in \{0,1\}$, where $1 \leq |\beta| \leq 4$. Considering the structure of the mapping $[(\sigma,q) \mapsto \Psi(\sigma,q)]$, we obtain that the term has the form

$$\Psi^\beta_{(\sigma,q)}(\sigma, \rho(t,\sigma)) = \begin{cases} v_1 & \text{if } \beta_2 \geq 1, \\ v_1 + v_2\rho(t,\sigma), & \text{else,}\end{cases}$$

where $v_1, v_2 \in \mathbb{R}^2$ are independent of t. The claim follows by the regularity properties of ρ, see (5.2.19) and item 2 of Lemma 2.1.17 together with Remark 2.1.18. $\square$

Now, we are ready to prove (5.2.20): Combining the results of Claim 5.2.7 and Claim 5.2.8 with the Banach algebra property of $W^{5/8}_{2,\mu}(J)$, see Lemma 2.1.17, item 2 and Remark 2.1.18, we obtain

$$b_2(\sigma, \rho, \partial_\sigma \rho) = \frac{1}{(J(\rho))^4} \langle \Psi_q, R\Psi_\sigma \rangle (\sigma, \rho) \in W^{5/8}_{2,\mu}(J) \qquad \text{for } \sigma \in \{0, 1\},$$

with a suitable estimate. We recall that $\partial^3_\sigma \rho(\cdot, \sigma) \in W^{1/8}_{2,\mu}(J)$ for $\sigma \in \{0, 1\}$ by (5.2.19). As the product of an element in $W^{5/8}_{2,\mu}(J)$ with one in $W^{1/8}_{2,\mu}(J)$ is again in $W^{1/8}_{2,\mu}(J)$, see item 2 of Lemma 2.1.17, we deduce that the first summand of G_2 is an element of $W^{1/8}_{2,\mu}(J)$ for $\sigma \in \{0, 1\}$.

It remains to consider

$$g_2(\sigma, \rho, \partial_\sigma \rho, \partial^2_\sigma \rho) = T(\sigma, \rho, \partial_\sigma \rho) \left(\partial^2_\sigma \rho\right)^2 + T(\sigma, \rho, \partial_\sigma \rho) \partial^2_\sigma \rho + T(\sigma, \rho, \partial_\sigma \rho),$$

cf. (5.1.28). We want to prove that the prefactors $T(\sigma, \rho, \partial_\sigma \rho)$ are in $W^{5/8}_{2,\mu}(J)$ and the terms $(\partial^2_\sigma \rho)^2$ and $\partial^2_\sigma \rho$ are elements of $W^{1/8}_{2,\mu}(J)$ for $\sigma \in \{0, 1\}$. Then, the claim follows by Lemma 2.1.17, item 1, and Remark 2.1.18, which state that the products of these functions are again in $W^{1/8}_{2,\mu}(J)$ for $\sigma \in \{0, 1\}$ with a corresponding estimate. Recall that

$$T(\sigma, \rho, \partial_\sigma \rho) := C\,(J(\rho))^n \left(\prod_{i=0}^{p} \left\langle \Psi^\beta_{(\sigma,q)}, R\Psi^\gamma_{(\sigma,q)} \right\rangle (\sigma, \rho) \right) (\partial_\sigma \rho)^r,$$

with $C \in \mathbb{R}$, $n \in \mathbb{Z}$, $p, r \in \mathbb{N}_0$, and $\beta, \gamma \in \mathbb{N}^2_0$, such that $|\beta|, |\gamma| \geq 1$ and $|\beta| + |\gamma| \leq 4$. Hence, the claim follows directly by combining Claim 5.2.7, Claim 5.2.8, and the regularity of ρ, see (5.2.19), with the fact that $W^{5/8}_{2,\mu}(J)$ is a Banach algebra up to a constant in the norm estimate, see Lemma 2.1.17, item 3 and Remark 2.1.18.

Finally, we have to take care of $(\partial^2_\sigma \rho)^2$ and $\partial^2_\sigma \rho$ in the first and second summand of g_2, respectively: By (5.2.19), we obtain $\partial^2_\sigma \rho \in W^{3/8}_{2,\mu}(J)$ for $\sigma \in \{0, 1\}$. Thus, it holds clearly

$$T(\sigma, \rho, \partial_\sigma \rho) \partial^2_\sigma \rho \in W^{1/8}_{2,\mu}(J) \qquad \text{for } \sigma \in \{0, 1\},$$

by Lemma 2.1.17, item 1 and Remark 2.1.18. Additionally, it follows by item 3 of Lemma 2.1.17 that $(\partial^2_\sigma \rho)^2 \in W^{1/8}_{2,\mu}(J)$ for $\sigma \in \{0, 1\}$. Consequently, we obtain

$$T(\sigma, \rho, \partial_\sigma \rho) \left(\partial^2_\sigma \rho\right)^2 \in W^{1/8}_{2,\mu}(J) \qquad \text{for } \sigma \in \{0, 1\},$$

by Lemma 2.1.17, item 1. This completes the proof. $\square$

Remark 5.2.9 *Combining Theorem 5.2.1 and Lemma 5.2.4, we obtain that the linear problem* (5.2.1) *possesses a unique solution $\rho \in \mathbb{E}_{\mu,T}$, for $\bar{\rho} \in B_{K,T}$ for $0 < T < \tilde{T}$, if the conditions on the initial datum are fulfilled. This enables us to invert the linear operator in equation* (5.2.11) *and consequently we receive the fixed point problem*

$$\rho = \mathcal{L}^{-1}(\mathcal{F}(\rho), \rho_0) \qquad \text{for } \rho \in \mathcal{B}_{K,T} \text{ with } 0 < T < \tilde{T},$$

which is to be solved by Banach's fixed point theorem.

5.3 The Contraction Mapping

The next step is to show that the nonlinear operator is contractive. To this end, we state the following lemma:

Lemma 5.3.1
Let the assumptions of Theorem 5.1.3 hold true and let K and $\tilde{T}$ be given by Definition 5.1.7. Then

$$\mathcal{F} : \mathcal{B}_{K,T} \to \mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu \qquad \text{for } 0 < T < \tilde{T},$$

given by (5.2.10)*, is Lipschitz continuous with a constant $C_{\mathcal{F}_T} = C(\alpha, \Phi^*, \eta, K, R, \tilde{T}, T)$ with $\|\rho_0\|_{X_\mu} \leq R$ satisfying $C_{\mathcal{F}_T} \to 0$ monotonically as $T \to 0$.*

Proof. Let ρ_1 and ρ_2 be in $\mathcal{B}_{K,T}$ for $0 < T < \tilde{T}$.
First, we concentrate on the first line of $\mathcal{F}$. By adding zeros, it holds

$$\begin{aligned}
F(\rho_1) - F(\rho_2) &= -\left(\frac{1}{(J(\rho_1))^4} - \frac{1}{(J(\rho_0))^4}\right) \partial_\sigma^4 \rho_1 + f(\rho_1, \partial_\sigma \rho_1, \partial_\sigma^2 \rho_1, \partial_\sigma^3 \rho_1) \\
&\quad + \left(\frac{1}{(J(\rho_2))^4} - \frac{1}{(J(\rho_0))^4}\right) \partial_\sigma^4 \rho_2 - f(\rho_2, \partial_\sigma \rho_2, \partial_\sigma^2 \rho_2, \partial_\sigma^3 \rho_2) \\
&= \underbrace{-\left(\frac{1}{(J(\rho_1))^4} - \frac{1}{(J(\rho_0))^4}\right) (\partial_\sigma^4 \rho_1 - \partial_\sigma^4 \rho_2)}_{=:I} \underbrace{- \left(\frac{1}{(J(\rho_1))^4} - \frac{1}{(J(\rho_2))^4}\right) \partial_\sigma^4 \rho_2}_{=:II} \\
&\quad + \underbrace{\left(f(\rho_1, \partial_\sigma \rho_1, \partial_\sigma^2 \rho_1, \partial_\sigma^3 \rho_1) - f(\rho_2, \partial_\sigma \rho_2, \partial_\sigma^2 \rho_2, \partial_\sigma^3 \rho_2)\right)}_{=:III}.
\end{aligned}$$

We begin with the first factors of I and II, which can be estimated similarly. For II, we obtain analogously to Claim 5.2.6

$$\left\| \frac{1}{(J(\rho_1))^4} - \frac{1}{(J(\rho_2))^4} \right\|_{C(\bar{J};C(\bar{I}))} \leq L \|J(\rho_1) - J(\rho_2)\|_{C(\bar{J};C(\bar{I}))},$$

where L does not depend on T. We recall that for $\rho \in \mathcal{B}_{K,T}$, $0 < T < \tilde{T}$, by Lemma 5.1.6 it follows that ρ and $\partial_\sigma \rho$ are bounded in $C^0([0,1] \times [0,T])$ by ${}^{2K_0}/_3$ and ${}^{2K_1}/_3$, respectively. Thus, we can exploit the Lipschitz continuity of

$$[0,1] \times [-{}^{2K_0}/_3, {}^{2K_0}/_3] \times [-{}^{2K_1}/_3, {}^{2K_1}/_3] \ni (\sigma, \rho, \partial_\sigma \rho) \mapsto J(\rho)$$

for $\sigma \in \bar{I}$, $\rho_i \in \mathcal{B}_{K,T}$, which follows by the fact that $J(\rho)$ is continuously differentiable with respect to the variables $(\sigma, \rho, \partial_\sigma \rho)$. The Lipschitz constant depends on α, Φ^*, η, and K, but not on T. We obtain

$$\begin{aligned}
\left\| \frac{1}{(J(\rho_1))^4} - \frac{1}{(J(\rho_2))^4} \right\|_{C(\bar{J};C(\bar{I}))} &\leq C(\alpha, \Phi^*, \eta, K) \|(\rho_1, \partial_\sigma \rho_1) - (\rho_2, \partial_\sigma \rho_2)\|_{C(\bar{J};C(\bar{I}))} \\
&\leq C(\alpha, \Phi^*, \eta, K) \|\rho_1 - \rho_2\|_{C(\bar{J};C^1(\bar{I}))} \\
&= C(\alpha, \Phi^*, \eta, K) \sup_{t \in \bar{J}} \|(\rho_1 - \rho_2)(t) - (\rho_1 - \rho_2)(0)\|_{C^1(\bar{I})} \\
&\leq C(\alpha, \Phi^*, \eta, K) T^\alpha \|\rho_1 - \rho_2\|_{C^\alpha(\bar{J};C^1(\bar{I}))},
\end{aligned}$$

where $C(\alpha, \Phi^*, \eta, K)$ does not depend on T. Here we additionally used that $\rho_{i|t=0} = \rho_0$ for $i = 1, 2$.

Replacing ρ_2 by ρ_0, we obtain for summand I

$$\left\| \frac{1}{(J(\rho_1))^4} - \frac{1}{(J(\rho_0))^4} \right\|_{C(\bar{J};C(\bar{I}))} \leq C(\alpha, \Phi^*, \eta, K) T^\alpha \|\rho_1 - \rho_0\|_{C^\alpha(\bar{J};C^1(\bar{I}))}.$$

Next, the embedding

$$\mathbb{E}_{\rho,\mu,T} \hookrightarrow C^\alpha\left(\bar{J}; C^1(\bar{I})\right)$$

is employed, where the operator norm does not depend on T, if a suitable norm is used, cf. Lemma 2.2.3, item 2. Therefore, it holds

$$\begin{aligned} \|I\|_{L_{2,\mu}(J;L_2(I))} &\leq C(\alpha, \Phi^*, \eta, K) T^\alpha \|\rho_1\|_{C^\alpha(\bar{J};C^1(\bar{I}))} \|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}} \\ &\leq C(\alpha, \Phi^*, \eta, K) T^\alpha \tilde{C}(\|\rho_1\|_{\mathbb{E}_{\mu,T}} + \|\rho_{1|t=0}\|_{X_\mu}) \|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}} \\ &\leq C(\alpha, \Phi^*, \eta, K, \|\rho_0\|_{X_\mu}, T) \|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}}, \end{aligned}$$

where we used that $\rho_1 \in \mathcal{B}_{K,T}$. Note that $C(\alpha, \Phi^*, \eta, K, \|\rho_0\|_{X_\mu}, T) \to 0$ monotonically as $T \to 0$. For the second summand II, it follows analogously

$$\begin{aligned} \|II\|_{L_{2,\mu}(J;L_2(I))} &\leq C(\alpha, \Phi^*, \eta, K) T^\alpha \|\rho_1 - \rho_2\|_{C^\alpha(\bar{J};C^1(\bar{I}))} \|\rho_2\|_{\mathbb{E}_{\mu,T}} \\ &\leq C(\alpha, \Phi^*, \eta, K, T) \|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}}. \end{aligned}$$

In the following, we take care of summand III: By adding zeros, we obtain

$$\begin{aligned} &f(\rho_1, \partial_\sigma \rho_1, \partial_\sigma^2 \rho_1, \partial_\sigma^3 \rho_1) - f(\rho_2, \partial_\sigma \rho_2, \partial_\sigma^2 \rho_2, \partial_\sigma^3 \rho_2) \\ &\quad = S(\rho_1)\left(\partial_\sigma^2 \rho_1 \partial_\sigma^3 \rho_1 - \partial_\sigma^2 \rho_2 \partial_\sigma^3 \rho_2\right) + (S(\rho_1) - S(\rho_2))\, \partial_\sigma^2 \rho_2 \partial_\sigma^3 \rho_2 \\ &\quad\quad + S(\rho_1)\left(\partial_\sigma^3 \rho_1 - \partial_\sigma^3 \rho_2\right) + (S(\rho_1) - S(\rho_2)) \partial_\sigma^3 \rho_2 + S(\rho_1)\left((\partial_\sigma^2 \rho_1)^3 - (\partial_\sigma^2 \rho_2)^3\right) \\ &\quad\quad + (S(\rho_1) - S(\rho_2))\left(\partial_\sigma^2 \rho_2\right)^3 + S(\rho_1)\left((\partial_\sigma^2 \rho_1)^2 - (\partial_\sigma^2 \rho_2)^2\right) \\ &\quad\quad + (S(\rho_1) - S(\rho_2))(\partial_\sigma^2 \rho_2)^2 + S(\rho_1)\left(\partial_\sigma^2 \rho_1 - \partial_\sigma^2 \rho_2\right) + (S(\rho_1) - S(\rho_2)) \partial_\sigma^2 \rho_2 \\ &\quad\quad + (S(\rho_1) - S(\rho_2)). \end{aligned} \tag{5.3.1}$$

where we denote by $S(\rho_i) = S(\sigma, \rho_i, \partial_\sigma \rho_i)$, see (5.1.27).

First, we inspect the summands which have a factor $(S(\rho_1) - S(\rho_2))$. To this end, we consider the representation of $S(\rho)$, i.e.

$$S(\sigma, \rho, \partial_\sigma \rho) := \frac{1}{\langle \Psi_q, R\Psi_\sigma \rangle(\sigma, \rho)} C\left(J(\rho)\right)^k \left(\prod_{i=0}^{p} \left\langle \Psi_{(\sigma,q)}^{\beta_i}, R\Psi_{(\sigma,q)}^{\gamma_i} \right\rangle(\sigma, \rho) \right) (\partial_\sigma \rho)^q,$$

with $C \in \mathbb{R}$, $k \in \mathbb{Z}$, $p, q \in \mathbb{N}_0$, and $\beta_i, \gamma_i \in \mathbb{N}_0^2$, such that $|\beta_i|, |\gamma_i| \geq 1$ and $|\beta_i| + |\gamma_i| \leq 5$ for all $i \in \{0, \ldots, p\}$, see (5.1.27) and (5.1.11). We observe that for every $\sigma \in [0,1]$ the mapping

$$[-{}^{2K_0}/_3, {}^{2K_0}/_3] \times [-{}^{2K_1}/_3, {}^{2K_1}/_3] \ni (\rho, \partial_\sigma \rho) \mapsto S(\sigma, \rho, \partial_\sigma \rho)$$

is Lipschitz-continuous with a Lipschitz constant depending on α, Φ^*, η, and K. By this, we obtain by the same strategy as for the summands I and II

$$\begin{aligned} \|S(\rho_1) - S(\rho_2)\|_{C(\bar{J};C(\bar{I}))} &\leq C(\alpha, \Phi^*, \eta, K) \|(\rho_1, \partial_\sigma \rho_1) - (\rho_2, \partial_\sigma \rho_2)\|_{C(\bar{J};C(\bar{I}))} \\ &\leq C(\alpha, \Phi^*, \eta, K) \|\rho_1 - \rho_2\|_{C(\bar{J};C^1(\bar{I}))} \leq C(K) T^\alpha \|\rho_1 - \rho_2\|_{C^\alpha(\bar{J};C^1(\bar{I}))} \\ &= C(\alpha, \Phi^*, \eta, K, T) \|\rho_1 - \rho_2\|_{C^\alpha(\bar{J};C^1(\bar{I}))}, \end{aligned} \tag{5.3.2}$$

where $C(\alpha, \Phi^*, \eta, K, T) \to 0$ monotonically for $T \to 0$. Furthermore, we have

$$\begin{aligned}
\|S(\rho_1)\|_{C(\bar{J};C((\bar{I}))} &\leq \|S(\rho_1) - S(\rho_0)\|_{C(\bar{J};C((\bar{I}))} + \|S(\rho_0)\|_{C(\bar{J};C(\bar{I}))} \\
&\leq C(\alpha, \Phi^*, \eta, K)\|\rho_1 - \rho_0\|_{C(\bar{J};C^1((\bar{I}))} + \|S(\rho_0)\|_{C(\bar{J};C(\bar{I}))} \\
&\leq C(\alpha, \Phi^*, \eta, K)\left(\|\rho_1\|_{\mathbb{E}_{\mu,T}} + \|\rho_0\|_{X_\mu} + \|\rho_0\|_{C^1(\bar{I})} + 1\right) \\
&\leq C(\alpha, \Phi^*, \eta, K, \|\rho_0\|_{X_\mu}),
\end{aligned} \tag{5.3.3}$$

where $C(\alpha, \Phi^*, \eta, K, \|\rho_0\|_{X_\mu})$ does not depend on T. By adding a zero, the first summand of (5.3.1) reads

$$S(\rho_1)\left(\partial_\sigma^2\rho_1\partial_\sigma^3\rho_1 - \partial_\sigma^2\rho_2\partial_\sigma^3\rho_2\right) = S(\rho_1)\partial_\sigma^2\rho_1(\partial_\sigma^3\rho_1 - \partial_\sigma^3\rho_2) + S(\rho_1)(\partial_\sigma^2\rho_1 - \partial_\sigma^2\rho_2)\partial_\sigma^3\rho_2$$

and can be estimated by (5.2.16): Combining this with Lemma 2.2.3, item 1, and (5.3.3) we obtain

$$\begin{aligned}
\left\|S(\rho_1)\partial_\sigma^2\rho_j\ (\partial_\sigma^3\rho_1 - \partial_\sigma^3\rho_2)\right\|_{L_{2,\mu}(J;L_2(I))} &\leq C(T)\|S(\rho_1)\|_{C(\bar{J};C((\bar{I}))}\Big(\|\rho_j\|_{L_\infty(J,W_2^{4(\mu-1/2)}(I))} \\
&\quad + \|\rho_j\|_{L_{2,\mu}(J,W_2^4(I))}\Big)\left(\|\rho_1 - \rho_2\|_{L_\infty(J,W_2^{4(\mu-1/2)}(I))}\| + \|\rho_1 - \rho_2\|_{L_{2,\mu}(J,W_2^4(I))}\right) \\
&\leq C(\alpha, \Phi^*, \eta, K, T)\left[C\left(\|\rho_j\|_{\mathbb{E}_{\mu,T}} + \|\rho_0\|_{X_\mu}\right) + \|\rho_j\|_{\mathbb{E}_{\mu,T}}\right]\|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}} \\
&\leq C(\alpha, \Phi^*, \eta, K, \|\rho_0\|_{X_\mu}, T)\|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}},
\end{aligned}$$

for $j = 1, 2$ and a $C(\alpha, \Phi^*, \eta, K, T, \|\rho_0\|_{X_\mu}) \to 0$ monotonically for $T \to 0$. The second summand of (5.3.1) can be treated analogously. Moreover, similar estimates hold true for all the summands except for

$$S(\rho_1)\left((\partial_\sigma^2\rho_1)^3 - (\partial_\sigma^2\rho_2)^3\right) + (S(\rho_1) - S(\rho_2))\left(\partial_\sigma^2\rho_2\right)^3.$$

We observe that they can be estimated by using (5.2.18) instead of (5.2.16) together with the estimate on the prefactors (5.3.2) and (5.3.3), respectively.

There is nothing to show for the second component of $\mathcal{F}_T$. Finally, we take a look at the last component of $\mathcal{F}_T$: By adding a zero, we have

$$\begin{aligned}
G_2(\rho_1) - G_2(\rho_2) &= -\left(b_2(\rho_1) - b_2(\rho_0)\right)\partial_\sigma^3\rho_1 - g_2(\rho_1) + \left(b_2(\rho_2) - b_2(\rho_0)\right)\partial_\sigma^3\rho_2 + g_2(\rho_2) \\
&= \underbrace{-\left(b_2(\rho_1) - b_2(\rho_0)\right)\left(\partial_\sigma^3\rho_1 - \partial_\sigma^3\rho_2\right)}_{=:I}\underbrace{-\left(b_2(\rho_1) - b_2(\rho_2)\right)\partial_\sigma^3\rho_2}_{II} \\
&\quad \underbrace{-\left(g_2(\rho_1) - g_2(\rho_2)\right)}_{=:III},
\end{aligned}$$

where $b_2(\rho) = b_2(\sigma, \rho, \partial_\sigma\rho)$ and $g_2(\rho) = g_2(\sigma, \rho, \partial_\sigma\rho, \partial_\sigma^2\rho)$, respectively. By Lemma 2.1.17, item 1, we deduce for the first summand I

$$\|I\|_{W_{2,\mu}^{1/8}(J)} \leq C(T)\left\|b_2(\rho_1) - b_2(\rho_0)\right\|_{W_{2,\mu}^{5/8}(J)}\left\|\partial_\sigma^3\rho_1 - \partial_\sigma^3\rho_2\right\|_{W_{2,\mu}^{1/8}(J)},$$

where $C(T) \to 0$ monotonically for $T \to 0$, since $(b_2(\rho_1) - b_2(\rho_0))_{|t=0} = 0$ by $\rho_{1|t=0} = \rho_0$. Considering the representation of $b_2(\sigma, \rho, \partial_\sigma\rho)$, cf. (5.1.25), we can estimate the first factor by combining the results of Claim 5.2.7 and Claim 5.2.8 with the Banach-algebra property of $W_{2,\mu}^{5/8}(J)$, see Lemma 2.1.17, item 2, and Remark 2.1.18. Hence, we obtain

$$\|b_2(\rho_1)\|_{W_{2,\mu}^{5/8}(J)} \leq C(\alpha, \Phi^*, \eta, K, \tilde{T}).$$

Clearly, it holds

$$\|b_2(\rho_0)\|_{W^{5/8}_{2,\mu}(J)} \leq C(\alpha, \Phi^*, \eta, K, \tilde{T}),$$

as the term does depend on time. Thus, it follows

$$\|I\|_{W^{1/8}_{2,\mu}(J)} \leq C(\alpha, \Phi^*, \eta, K, \tilde{T}, T)\,\|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}}$$

for $C(\alpha, \Phi^*, \eta, K, \tilde{T}, T) \to 0$ monotonically for $T \to 0$. Here, we additionally used the mapping Lemma 2.2.3, item 4, to estimate $\|\partial^3_\sigma \rho_1 - \partial^3_\sigma \rho_2\|_{W^{1/8}_{2,\mu}(J)}$.

We proceed with the second summand II: In order to estimate the first factor, we use that for $\sigma = 0, 1$ the mapping

$$[-2K_0/3, 2K_0/3] \times [-2K_1/3, 2K_1/3] \ni (\rho, \partial_\sigma \rho) \mapsto b_2(\sigma, \rho, \partial_\sigma \rho)$$

is Lipschitz-continuous with a Lipschitz constant depending on α, Φ^*, η, and K. Thus, the estimate

$$\begin{aligned}\|b_2(\rho_1) - b_2(\rho_2)\|_{W^{5/8}_{2,\mu}(J)} &\leq C(\alpha, \Phi^*, \eta, K)\,\|(\rho_1, \partial_\sigma \rho_1) - (\rho_2, \partial_\sigma \rho_2)\|_{W^{5/8}_{2,\mu}(J)} \\ &\leq C(\alpha, \Phi^*, \eta, K)\,\|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}}\end{aligned}$$

is obtained, where we again used the mapping Lemma 2.2.3, item 4. It follows similarly to the argumentation for the first summand I

$$\|II\|_{W^{1/8}_{2,\mu}(J)} \leq C(\alpha, \Phi^*, \eta, K, \|\rho_0\|_{X_\mu}, \tilde{T}, T)\,\|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}},$$

where $C(\alpha, \Phi^*, \eta, K, \|\rho_0\|_{X_\mu}, \tilde{T}, T) \to 0$ monotonically for $T \to 0$.

For part III, we consider the representation of g_2, see (5.1.28): By adding a zero

$$\begin{aligned}g_2(\rho_1,) - g_2(\rho_2) &= T(\rho_1)(\partial^2_\sigma \rho_1)^2 - T(\rho_2)(\partial^2_\sigma \rho_2)^2 + T(\rho_1)\partial^2_\sigma \rho_1 - T(\rho_2)\partial^2_\sigma \rho_2 + T(\rho_1) - T(\rho_2) \\ &= \underbrace{T(\rho_1)\left((\partial^2_\sigma \rho_1)^2 - (\partial^2_\sigma \rho_2)^2\right)}_{=:I_A} + \underbrace{(T(\rho_1) - T(\rho_2))\,(\partial^2_\sigma \rho_2)^2}_{=:II_A} \\ &\quad + \underbrace{T(\rho_1)\left(\partial^2_\sigma \rho_1 - \partial^2_\sigma \rho_2\right)}_{=:I_B} + \underbrace{(T(\rho_1) - T(\rho_2))\,\partial^2_\sigma \rho_2}_{=:II_B} + \underbrace{T(\rho_1) - T(\rho_2)}_{=:II_C},\end{aligned}$$

where $T(\rho) := T(\sigma, \rho, \partial_\sigma \rho)$, see (5.1.9). The prefactors $T(\rho_i)$, $i = 1, 2$ and $(T(\rho_1) - T(\rho_2))$ can be estimated analogously to $b_2(\rho_i)$ and $(b_2(\rho_1) - b_2(\rho_2))$, respectively. It follows

$$\|T(\rho_1) - T(\rho_2)\|_{W^{5/8}_{2,\mu}(J)} \leq C(\alpha, \Phi^*, \eta, K)\,\|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}}, \tag{5.3.4}$$

$$\|T(\rho_1)\|_{W^{5/8}_{2,\mu}(J)} \leq C(\alpha, \Phi^*, \eta, K, \tilde{T}). \tag{5.3.5}$$

For the summands II_A, II_B, and II_C the constant $C(T)$, which tends to zero monotonically for $T \to 0$, is directly generated by using estimate Lemma 2.1.17, item 1, since $(T(\rho_1) - T(\rho_2))_{|t=0} = 0$ by $\rho_{i|t=0} = \rho_0$ for $i = 1, 2$. Using (5.3.4), we deduce

$$\begin{aligned}\left\|(T(\rho_1) - T(\rho_2))\,(\partial^2_\sigma \rho_2)^2\right\|_{W^{1/8}_{2,\mu}(J)} &\leq C(\alpha, \Phi^*, \eta, K, T)\,\|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}}\left\|(\partial^2_\sigma \rho_2)^2\right\|_{W^{1/8}_{2,\mu}(J)}, \\ \left\|(T(\rho_1) - T(\rho_2))\,\partial^2_\sigma \rho_2\right\|_{W^{1/8}_{2,\mu}(J)} &\leq C(\alpha, \Phi^*, \eta, K, T)\,\|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}}\left\|\partial^2_\sigma \rho_2\right\|_{W^{1/8}_{2,\mu}(J)}, \\ \|T(\rho_1) - T(\rho_2)\|_{W^{1/8}_{2,\mu}(J)} &\leq C(T)\,\|T(\rho_1) - T(\rho_2)\|_{W^{5/8}_{2,\mu}(J)}\,\|1\|_{W^{1/8}_{2,\mu}(J)}\,\|\rho_1 - \rho_2\|_{\mathbb{E}_{\mu,T}}\end{aligned}$$

$$\leq C(\alpha, \Phi^*, \eta, K, \tilde{T}, T) \left\| \rho_1 - \rho_2 \right\|_{\mathbb{E}_{\mu,T}},$$

where the constants $C(\alpha, \Phi^*, \eta, K, \tilde{T}, T) \to 0$ monotonically for $T \to 0$. Furthermore, the bounds on $\|(\partial_\sigma^2 \rho_2)^2\|_{W_{2,\mu}^{1/8}(J)}$ and $\|\partial_\sigma^2 \rho_2\|_{W_{2,\mu}^{1/8}(J)}$ follow directly by combining Lemma 2.2.3, item 4, and Lemma 2.1.17, item 3.

In order to estimate I_A and I_B, we exploit the uniform estimate stated in Remark 2.1.18 together with (5.3.5) and deduce

$$\begin{aligned} \left\|T(\rho_1)\left((\partial_\sigma^2\rho_1)^2 - (\partial_\sigma^2\rho_2)^2\right)\right\|_{W_{2,\mu}^{1/8}(J)} &\leq C(\alpha, \Phi^*, \eta, K, \tilde{T}) \left\|(\partial_\sigma^2\rho_1 - \partial_\sigma^2\rho_2)(\partial_\sigma^2\rho_1 + \partial_\sigma^2\rho_2)\right\|_{W_{2,\mu}^{1/8}(J)} \\ \left\|T(\rho_1)\left(\partial_\sigma^2\rho_1 - \partial_\sigma^2\rho_2\right)\right\|_{W_{2,\mu}^{1/8}(J)} &\leq C(\alpha, \Phi^*, \eta, K, \tilde{T}) \left\|\partial_\sigma^2\rho_1 - \partial_\sigma^2\rho_2\right\|_{W_{2,\mu}^{1/8}(J)}. \end{aligned}$$

Now, we can use the estimate of Lemma 2.1.17, item 3, to generate a constant $C(T) \to 0$ monotonically for $t \to 0$. This proves the claim. $\square$

Remark 5.3.2 *Note that the formulation of the boundary condition $\partial_\sigma\rho(t,\sigma) = 0$ for $\sigma \in \{0,1\}$ and $t > 0$, (5.1.14), is very useful for our analysis. It is induced by condition $\kappa_\Lambda(\sigma) = 0$ for $\sigma \in \{0,1\}$ on the reference curve. If $\kappa_\Lambda(\sigma) \neq 0$, we would have to use the boundary condition*

$$\frac{1}{J(\rho)} \left\langle \Psi_q, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\rangle \partial_\sigma \rho = -\frac{1}{J(\rho)} \left\langle \Psi_\sigma, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\rangle + \cos(\pi - \alpha),$$

cf. (5.1.13). *We observe that the coefficient in front of $\partial_\sigma\rho$ itself depends on the first derivative of ρ. Since the estimate for the Lipschitz continuity has to be done in the space $W_{2,\mu}^{5/8}(J)$ and both factors can be expected to be elements of this space but do not have higher regularity, we would miss the regularity gap, which we exploited to treat some of the other non-linearities.*

Now, all the tools are available to solve equation (5.2.11).

Lemma 5.3.3
Let the assumptions of Theorem 5.1.3 hold true and let K and $\tilde{T}$ be given by Definition 5.1.7. Then there exists a $T = T(\alpha, \Phi^*, \eta, R_1, R_2) \in (0, \tilde{T})$, $\|\rho_0\|_{X_\mu} \leq R_1$ and $\|\mathcal{L}^{-1}\|_{L(\mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu \times X_\mu; \mathbb{E}_{\mu,T})} \leq R_2$, such that there exists a unique solution $\rho \in \mathbb{E}_{\mu,T}$ for the equation $\mathcal{L}(\rho) = (\mathcal{F}(\rho), \rho_0)$.

Proof. We consider equation (5.2.11)

$$\mathcal{L}(\rho) = (\mathcal{F}(\rho), \rho_0), \tag{5.2.11}$$

where $\mathcal{F} : \mathbb{E}_{\mu,T} \to \mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu$ is again the right-hand side of (5.2.1),

$$\mathcal{F}(\rho) := \begin{pmatrix} F(t,\sigma) \\ G_1(t,\sigma) \\ G_2(t,\sigma) \end{pmatrix},$$

and $\mathcal{L} : \mathbb{E}_{\mu,T} \to \mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu \times X_\mu$ is defined as the left-hand side of (5.2.1). Consequently, (5.2.11) is equivalent to the fixed point problem $K(\rho) = \rho$, where

$$\begin{aligned} &\mathcal{K} : \mathcal{B}_{K,T} \to \mathbb{E}_{\mu,T}, \\ &\qquad \rho \mapsto \mathcal{K}(\rho) := \mathcal{L}^{-1}(\mathcal{F}(\rho), \rho_0). \end{aligned} \tag{5.3.6}$$

In order to solve the problem using Banach's fixed point theorem, we find an extension of the initial datum ρ_0 in the following way: We consider the linearized problem in (5.2.1) for the right-hand side $(0,0,0,\rho_0)$, i.e.

$$
\begin{aligned}
\rho_t + \mathcal{A}\rho &= 0 && \text{for } (t,x) \in (0,T)\times[0,1],\\
\mathcal{B}_1\rho &= 0 && \text{for } \sigma \in \{0,1\} \text{ and } t \in (0,T),\\
\mathcal{B}_2\rho &= 0 && \text{for } \sigma \in \{0,1\} \text{ and } t \in (0,T),\\
\rho_{t=0} &= \rho_0 && \text{for } \sigma \in [0,1].
\end{aligned}
$$

By Theorem 5.2.1, there exists a solution $\widetilde{\rho_0} \in \mathbb{E}_{\mu,T}$ for $0 < T < T_0$ and we find a constant $C(T_0) \geq \|\mathcal{L}^{-1}\|_{L(\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu\times X_\mu;\mathbb{E}_{\mu,T})} > 0$, such that

$$\|\widetilde{\rho_0}\|_{\mathbb{E}_{\mu,T}} \leq C(T_0)\|\rho_0\|_{X_\mu}, \tag{5.3.7}$$

for $0 < T < T_0$, cf. the Remark 5.2.2 and Remark 5.2.3.

For showing that $\mathcal{K}: \mathcal{B}_{K,T} \to \mathbb{E}_{\mu,T}$ is a self-mapping, we consider

$$\|\mathcal{K}(\rho)\|_{\mathbb{E}_{\mu,T}} \leq \left\|\mathcal{L}^{-1}\right\| \; \|(\mathcal{F}(\rho),\rho_0)\|_{\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu\times X_\mu} \leq \left\|\mathcal{L}^{-1}\right\| \; \left(\|\mathcal{F}(\rho)\|_{\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu} + \|\rho_0\|_{X_\mu}\right)$$

for every $\rho \in \mathcal{B}_{K,T}$ and $\|\mathcal{L}^{-1}\| := \|\mathcal{L}^{-1}\|_{L(\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu\times X_\mu;\mathbb{E}_{\mu,T})}$. By adding the zero, it follows

$$\|\mathcal{K}(\rho)\|_{\mathbb{E}_{\mu,T}} \leq \left\|\mathcal{L}^{-1}\right\| \; \left(\|\mathcal{F}(\rho)-\mathcal{F}(\widetilde{\rho_0})\|_{\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu} + \|\mathcal{F}(\widetilde{\rho_0})\|_{\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu} + \|\rho_0\|_{X_\mu}\right).$$

Now, we choose $K > 0$, such that

$$\max\left\{C(T_0)\|\rho_0\|_{X_\mu}, \left\|\mathcal{L}^{-1}\right\| \|\mathcal{F}(\widetilde{\rho_0})\|_{\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu}, \left\|\mathcal{L}^{-1}\right\| \|\rho_0\|_{X_\mu}\right\} \leq \frac{K}{4} \tag{5.3.8}$$

for the constant $C(T_0)$ in (5.3.7). Note that this implies $\|\widetilde{\rho_0}\|_{\mathbb{E}_{\mu,T}} \leq \frac{K}{4}$ by (5.3.7). Thus, Lemma 5.3.1 yields

$$
\begin{aligned}
\|\mathcal{K}(\rho)\|_{\mathbb{E}_{\mu,T}} &\leq \left\|\mathcal{L}^{-1}\right\| \left(C_{\mathcal{F}_T}\|\rho-\widetilde{\rho_0}\|_{\mathbb{E}_{\mu,T}} + \|\mathcal{F}(\widetilde{\rho_0})\|_{\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu} + \|\rho_0\|_{X_\mu}\right)\\
&\leq \left\|\mathcal{L}^{-1}\right\| C_{\mathcal{F}_T}\|\rho-\widetilde{\rho_0}\|_{\mathbb{E}_{\mu,T}} + \left\|\mathcal{L}^{-1}\right\| \|\mathcal{F}(\widetilde{\rho_0})\|_{\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu} + \left\|\mathcal{L}^{-1}\right\| \|\rho_0\|_{X_\mu}\\
&\leq \left\|\mathcal{L}^{-1}\right\| C_{\mathcal{F}_T}K + \left\|\mathcal{L}^{-1}\right\| C_{\mathcal{F}_T}\|\widetilde{\rho_0}\|_{\mathbb{E}_{\mu,T}} + \frac{K}{4} + \frac{K}{4}\\
&\leq \left\|\mathcal{L}^{-1}\right\| C_{\mathcal{F}_T}K + C_{\mathcal{F}_T}\frac{K}{4} + \frac{K}{4} + \frac{K}{4},
\end{aligned}
$$

where we used $\rho \in \mathcal{B}_{K,T}$ for $0 < T < \max\{T_0,\tilde{T}\}$, cf. Definition 5.1.7, and (5.3.8). Moreover, Lemma 5.3.1 guarantees

$$\left\|\mathcal{L}^{-1}\right\| C_{\mathcal{F}_T} < \frac{1}{4} \quad \text{and} \quad C_{\mathcal{F}_T} \leq 1,$$

by optionally making T smaller, where we used that $\|\mathcal{L}^{-1}\|$ and K do not depend on T, cf. Remark 5.2.3. Consequently, we obtain $\mathcal{K}(\mathcal{B}_{K,T}) \subset \mathcal{B}_{K,T}$.

It remains to show that $\mathcal{K}$ is contractive on $\mathcal{B}_{K,T}$. Employing a similar strategy again, we have for all $\rho_1,\rho_2 \in \mathcal{B}_{K,T}$

$$\|\mathcal{K}(\rho_1)-\mathcal{K}(\rho_2)\|_{\mathbb{E}_{\mu,T}} = \left\|\mathcal{L}^{-1}(\mathcal{F}(\rho_1),\rho_0) - \mathcal{L}^{-1}(\mathcal{F}(\rho_2,\rho_0))\right\|_{\mathbb{E}_{\mu,T}}$$

$$\begin{aligned}
&= \left\|\mathcal{L}^{-1}(\mathcal{F}(\rho_1) - \mathcal{F}(\rho_2)), 0)\right\|_{\mathbb{E}_{\mu,T}} \\
&\leq \left\|\mathcal{L}^{-1}\right\| \left\|\mathcal{F}(\rho_1) - \mathcal{F}(\rho_2)\right\|_{\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu} \\
&\leq \left\|\mathcal{L}^{-1}\right\| C_{\mathcal{F}_T} \left\|\rho_1 - \rho_2\right\|_{\mathbb{E}_{\mu,T}} \leq \frac{1}{4} \left\|\rho_1 - \rho_2\right\|_{\mathbb{E}_{\mu,T}} .
\end{aligned}$$

Thus $\mathcal{K} : \mathcal{B}_{K,T} \to \mathcal{B}_{K,T}$ is a contraction and by Banach's fixed point theorem follows the existence of a unique fixed point ρ of (5.3.6) in $\mathcal{B}_{K,T}$ for a small enough $T = T(\alpha, \Phi^*, \eta, R_1, R_2) > 0$, $\|\rho_0\|_{X_\mu} \leq R_1$ and $\|\mathcal{L}^{-1}\|_{L(\mathbb{E}_{0,\mu}\times\tilde{\mathbb{F}}_\mu\times X_\mu;\mathbb{E}_{\mu,T})} \leq R_2$.

Assume that $\bar{\rho} \in \mathbb{E}_{\mu,T}$ is another solution to problem (5.2.11). Then, we choose $\bar{K} \geq K$, such that $\bar{K} \geq \|\bar{\rho}\|_{\mathbb{E}_{\mu,T}}$. Next, we replace K in the definition of the ball in Definition 5.1.7 by $\bar{K}$ and $\tilde{T} = \tilde{T}(K)$ by $\tilde{T}_* = \tilde{T}_*(\bar{K})$. Then there exists a $T_* \in (0,T)$ such that $\mathcal{K} : \mathcal{B}_{\bar{K},T_*} \to \mathcal{B}_{\bar{K},T_*}$ is again a contraction. Since both ρ and $\bar{\rho}$ are fixed points of $\mathcal{K} : \mathcal{B}_{\bar{K},T_*} \to \mathcal{B}_{\bar{K},T_*}$, it follows that $\rho_{|[0,T_*]} = \bar{\rho}_{|[0,T_*]}$. Let now

$$T_0 = \sup\left\{t \in [T_*, T] : \rho(t) = \bar{\rho}(t) \text{ for all } \tau \leq t\right\}.$$

If it holds $T_0 < T$, we replace ρ_0 by $\bar{\rho}(T_0, \cdot)$ and t by $t - T_0$ in (5.2.11). Here, we can use $\bar{\rho}(T_0, \cdot)$ as initial value, since the solution spaces embeds continuously into the temporal trace space, see (2.2.4), and $\rho(T_0, \cdot)$ fulfills the bounds (5.1.16) and (5.1.17) for $2/3K_0$ and $2/3K_1$, cf. Remark 5.1.4, item 3. By repeating the previous argument, we obtain $\rho_{|[T_0,T_{**}]} = \bar{\rho}_{|[T_0,T_{**}]}$, for $T_{**} \in (T_0, T]$, which contradicts the maximality of T_0. Thus, it holds $T_0 = T$ and $\rho \equiv \bar{\rho}$. □

The well-posedness-result Theorem 5.1.3 follows immediately from Lemma 5.3.3. We deduce the following Corollary as a direct consequence of Theorem 5.1.3:

Corollary 5.3.4
Let the assumptions of Theorem 5.1.3 hold true and let $\rho \in \mathbb{E}_{\mu,T}$ be the unique solution to (5.1.15) *given by Theorem 5.1.3, which fulfills $\rho(\cdot, 0) = \rho_0$ in X_μ. Then the function $(t,\sigma) \mapsto \Phi(t,\sigma) = \Psi(\sigma, \rho(t,\sigma))$, see* (5.1.4)*, which is an element of $\mathbb{E}_{\mu,T,\mathbb{R}^2}$, is a solution to* (3.1.1)-(3.1.4) *with $\Phi(0,\cdot) = \Phi(\rho_0)$.*

Remark 5.3.5 *1. Note that we did not prove a statement about uniqueness of the geometric problem.*

2. We notice that for a regular initial curve $f_0 \in C^5(\bar{I}; \mathbb{R}^2)$ fulfilling the boundary conditions (3.1.6) *and, additionally*

$$\kappa_{\Gamma_0}(\sigma) = 0 \qquad\qquad \textit{for } \sigma \in \{0,1\},$$

we can use f_0 as a reference curve with $\rho_0 = 0$. Then the existence time of the solution depends on α, f_0, η, and $1/\|\kappa_{\Gamma_0}\|_{C^0}$.

6 Construction of Reference Curves

We already proved a short time existence result, see Theorem 5.1.3, which guarantees that the curve diffusion flow starts for initial curves given by a certain sufficiently small height function of class $W_2^{4(\mu-1/2)}(I)$ of a reference curve in $C^5(\bar{I};\mathbb{R}^2)$. But in order to establish a blow-up criterion for the flow, it will be crucial to make sure that it starts for every admissible $W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)$-curve f_0. To this end, we generate in Section 6.1 potential reference curves f_ϵ, $\epsilon > 0$, by evolving the initial curve f_0 by a parabolic equation. Afterwards, we find criteria on the distance of two curves, which permit to use one curve as a reference curve for the other one in Section 6.3. In order to confirm that f_ϵ are reference curves, in Section 6.4, we deduce some technical estimates by the properties of the C^0-semigroup related to the parabolic equation in Section 6.3. The idea of the proof is inspired by the argumentation for the approximation of C^2-hypersurfaces in Section 2.3 in [27].

6.1 Generation of Potential Reference Curves

Let $f_0 : \bar{I} \to \mathbb{R}^2$, $I := (0,1)$, be parametrized proportional to arc length and in $W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)$, $\mu \in (7/8, 1]$. Moreover, let f_0 fulfill the boundary conditions given in (3.1.6), i.e.

$$\begin{aligned} f_0(x) &\in \mathbb{R} \times (0,\infty) && \text{for } x \in \{0,1\}, \\ \measuredangle\left(n_{\Gamma_0}(x), \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) &= \pi - \alpha && \text{for } x \in \{0,1\}, \end{aligned} \tag{3.1.6}$$

where $\Gamma_0 := f_0(\bar{I})$.

In order to apply the short time existence result, Theorem 5.1.3, to the curve f_0, we have to construct a reference curve for f_0. More precisely, we look for a regular function $\Phi^* : [0,1] \to \mathbb{R}^2$ of class C^5, such the boundary conditions in (5.1.1) are fulfilled, i.e.

$$\begin{aligned} \Phi^*(x) &\in \mathbb{R} \times \{0\} && \text{for } x \in \{0,1\}, \\ \measuredangle\left(n_{\Lambda}(x), \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) &= \pi - \alpha && \text{for } x \in \{0,1\}, \\ \kappa_\Lambda(x) &= 0 && \text{for } x \in \{0,1\}, \end{aligned} \tag{5.1.1}$$

where $\Lambda := \Phi^*([0,1])$. We use the same notations as before. Later on, we will take care of the smallness condition of the corresponding initial height function, see (5.1.22) in Theorem 5.1.3.

We find a family of such curves by solving a parabolic equation subject to the previously mentioned boundary conditions (5.1.1) with the initial value f_0. The required regularity will be proven by the regularizing effects of parabolic equations. We consider

$$\begin{aligned} \partial_t f - \partial_x^6 f &= 0 && \text{for } x \in (0,1) \text{ and } t > 0, \\ f &= f_0 && \text{for } x \in \{0,1\} \text{ and } t > 0, \\ \partial_x f &= \partial_x f_0 && \text{for } x \in \{0,1\} \text{ and } t > 0, \\ \partial_x^2 f &= 0 && \text{for } x \in \{0,1\} \text{ and } t > 0, \\ f_{|t=0} &= f_0 && \text{for } x \in (0,1), \end{aligned} \tag{6.1.1}$$

where the first order condition is a representation of the α-angle condition and the second order one implies that $\vec{\kappa}[f(t)] = 0$ at the boundary for $t > 0$.

Remark 6.1.1 *In general it is not possible to use f_0 as boundary data by reasons of the regularity of the function. It works in this case, since the boundary consists only of two points, which means that the spatial regularity can be neglected.*

The following lemma provides a local well-posedness result for (6.1.1) with initial data f_0 of class $W_2^{4(\mu-1/2)}$. Again, we will use the notation from Section 2.2 with $m = 3$ to denote the spaces appearing in the local well-posedness result.

Notation
For $J = (0,T)$ and $I = (0,1)$, we have

$$\begin{aligned}
X_{\tilde{\mu},\mathbb{R}^2} &= W_2^{6(\tilde{\mu}-1/2)}(I;\mathbb{R}^2) = W_2^{4(\mu-1/2)}(I;\mathbb{R}^2) && \text{for } \tilde{\mu}(\mu) = \tfrac{2}{3}\mu + \tfrac{1}{6} \in \left(\tfrac{3}{4}, \tfrac{5}{6}\right],\\
\mathbb{E}_{\tilde{\mu},T,\mathbb{R}^2} &= W^1_{2,\tilde{\mu}}\big(J; L_2(I;\mathbb{R}^2)\big) \cap L_{2,\tilde{\mu}}\big(J; W_2^6(I;\mathbb{R}^2)\big),\\
\mathbb{E}_{0,\tilde{\mu},\mathbb{R}^2} &= L_{2,\tilde{\mu}}\big(J; L_2(I;\mathbb{R}^2)\big),\\
\mathbb{F}_{j,\tilde{\mu},\mathbb{R}^2} &= W^{\omega_j}_{2,\tilde{\mu}}\big(J; L_2(\partial I;\mathbb{R}^2)\big) \cap L_{2,\tilde{\mu}}\big(J; W_2^{4\omega_j}(\partial I;\mathbb{R}^2)\big),
\end{aligned}$$

where $\omega_j = 1 - {}^{m_j}/_6 - {}^1/_{12}$ with $m_j = j-1$ for $j = 1,2,3$, and

$$\tilde{\mathbb{F}}_{\tilde{\mu},\mathbb{R}^2} = \mathbb{F}_{1,\tilde{\mu},\mathbb{R}^2} \times \mathbb{F}_{2,\tilde{\mu},\mathbb{R}^2} \times \mathbb{F}_{3,\tilde{\mu},\mathbb{R}^2}.$$

All the spaces are equipped with their natural norms. The definitions of the appearing time weighted spaces have been given in Section 2.1.

Lemma 6.1.2
Let $J = (0,T)$ be finite and let $f_0 : \bar{I} \to \mathbb{R}^2$, $I := (0,1)$, be parametrized proportional to arc length and in $W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)$ for $\mu \in \left(\frac{7}{8}, 1\right]$. Then the problem (6.1.1) *possesses a unique solution $f \in \mathbb{E}_{\tilde{\mu},T,\mathbb{R}^2}$ with $\tilde{\mu}(\mu) = \frac{2}{3}\mu + \frac{1}{6} \in \left(\frac{3}{4}, \frac{5}{6}\right]$.*

Proof. Again, we employ the maximal regularity result, Theorem 2.1 in [25], see the simplified version Theorem 2.2.1. To this end, we set $E := \mathbb{C}^2$ and observe that is a Banach space of class $\mathcal{HT}$, as it is a Hilbert space. Moreover, we set for $D = -i\partial_x$

$$\mathcal{A}(D)u := -\partial_x^6 u = D^6 u.$$

Hence, the order of the operator is $6 = 2m$ for $m = 3$. We use the following three boundary operators

$$\mathcal{B}_1(D)u = \operatorname{tr}_{\partial I} D^0 u, \qquad \mathcal{B}_2(D)u = i\operatorname{tr}_{\partial I} D^1 u, \qquad \mathcal{B}_3(D)u = -\operatorname{tr}_{\partial I} D^2 u$$

and consequently the corresponding coefficients are given by

$$a = 1, \qquad b_1 = 1, \qquad b_2 = i, \qquad b_3 = -1.$$

In the following, we prove that the assumptions of Theorem 2.2.1, which is a simplified version of Theorem 2.1 in [25], hold true in our case: Firstly, we look at the requirements for the coefficients of the operators: For the operator $\mathcal{A}$, we just have to check that $a \in BUC(\bar{J} \times \bar{I})$, cf. (SD) in Theorem 2.1 in [25]. This is trivially fulfilled since a is just constant.

For the boundary operators, we show the "either"-case in condition (SB), i.e. $b_j \in C^{\tau_j, 6\tau_j}(\bar{J} \times \partial I)$ for some $\tau_j > \omega_j$, $j \in \{1, 2, 3\}$, where

$$\omega_j := 1 - \frac{m_j}{6} - \frac{1}{12}, \tag{6.1.2}$$

with $m_j = j - 1$ the order of the boundary operator. The claim is again trivial, as the coefficients of the boundary operators are constant.

Next, we prove the normal ellipticity (E) of the operator $\mathcal{A}$, i.e. for all $t \in \bar{J}$, $x \in \bar{I}$ and $|\xi| = 1$, it holds that the spectrum $\Sigma(\mathcal{A}(t, x, \xi)) \subset \mathbb{C}_+ := \{\Re z > 0\}$. We have for $\xi = \pm 1$

$$\mathcal{A}(t, x, \xi) = \xi^6 = 1.$$

Therefore, $\Sigma(\mathcal{A}(t, x, \xi)) = 1$, which proves the claim.

Moreover, we have to prove a condition of Lopatinskii-Shapiro-type (LS). Using coordinates which are associated to the boundary points and the corresponding rotated operators, one easily calculates that it suffices to show the following claim: For $x = 0, 1$, $\lambda \in \overline{\mathbb{C}_+} \backslash \{0\}$ and each $h \in (\mathbb{C}^2)^3$ the ordinary boundary value problem

$$\begin{aligned} \lambda v(y) + a(\mp i\partial_y)^6 v(y) &= 0 && \text{for } y > 0, \\ b_j(\mp i\partial_y)^j v(y)_{|y=0} &= h_j && \text{for } j = 1, 2, 3, \end{aligned}$$

has a unique solution $v \in C_0([0, \infty); \mathbb{C}^2)$, see Definition 2.2.1. Since $\mathbb{R}^2$ is finite dimensional, it suffices to consider $h_j = 0$ in the previous condition. This means, we have to show that for each $\lambda \in \overline{\mathbb{C}_+} \backslash \{0\}$ the function $v \equiv 0$ is the only solution in $C_0([0, \infty); \mathbb{C}^2)$ of the problem

$$\begin{aligned} \lambda v(y) - \partial_y^6 v(y) &= 0 && \text{for } y > 0, \\ v(y)_{|y=0} &= 0, \\ (\pm)\partial_y v(y)_{|y=0} &= 0, \\ \partial_y^2 v(y)_{|y=0} &= 0. \end{aligned} \tag{6.1.3}$$

To this end, we use the following claim.

Claim 6.1.3 *The solution v of* (6.1.3) *and its derivatives decay exponentially for $y \to \infty$.*

Proof of the claim: Considering the corresponding characteristic equation for the first line of (6.1.3), we obtain that the components of the solutions $v_n(y) = [v(y)]_n$, $n = 1, 2$, are of the form

$$v_n(y) = \sum_{k=0}^{5} c_k e^{\mu_k y}, \tag{6.1.4}$$

where $c_k \in \mathbb{C}$ and μ_k are the complex roots of the characteristic polynomial $\lambda = \mu^6$ for $k \in \{0, \dots, 5\}$. As v is required to be in $C_0([0, \infty); \mathbb{C}^2)$, we know that $c_k = 0$ for each k with $\Re\mu_k > 0$. Moreover, there cannot be a purely imaginary root: If $\mu_{k_a} := ie$ for $e \in \mathbb{R}$ is a solution of $\lambda = \mu^6$, then the complex conjugate $\mu_{k_b} = \overline{\mu_{k_a}} = -ie = -\mu_{k_a}$ is also a root. Since the roots have the form

$$\mu_k = |\lambda| \left[\cos\frac{\theta + 2\pi k}{6} + i \sin\frac{\theta + 2\pi k}{6}\right] \qquad \text{for a } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } k \in \{0, \dots, 5\},$$

we obtain by $\Re\mu_{k_a} = 0 = \Re\mu_{k_b}$ that $\cos{}^{(\theta + 2\pi k_{a,b})}/_6 = 0$. Thus, there exists $j_{a,b} \in \mathbb{Z}$ corresponding to $k_{a,b}$ such that it holds

$$(2j_{a,b}+1)\pi = \frac{\theta + 2\pi k_{a,b}}{6} \in \left[\frac{\theta}{6}, \frac{\theta + 2\pi \cdot 5}{6}\right].$$

As the interval has length $2\pi \cdot 5/6$, this implies that $j_a = j_b$ and, thus, $k_a = k_b$ has to hold. Therefore, the only possible choice for a purely imaginary root is $\mu_{k_b} = \mu_{k_a} = 0$, but this case is excluded by $\lambda \neq 0$. Thus, the roots cannot by purely imaginary and the claim follows by the form of the solution in (6.1.4). □

This result enables us to multiply the single components of the first equation of (6.1.3) by $\overline{v_n(y)}$ and integrate with respect to y in $(0,\infty)$, and obtain for $n = 1, 2$

$$0 = \lambda \int_0^\infty |v_n(y)|^2 \,\mathrm{d}y - \int_0^\infty \partial_y^6 v_n(y) \cdot \overline{v_n(y)} \,\mathrm{d}y. \tag{6.1.5}$$

Using integration by parts three times on the second summand, we obtain

$$\begin{aligned}
\int_0^\infty \partial_y^6 v_n(y) \cdot \overline{v(y)} \,\mathrm{d}y &= \left[\partial_y^5 v_n(y) \cdot \overline{v_n(y)}\right]_0^\infty - \int_0^\infty \partial_y^5 v_n(y) \cdot \overline{\partial_y v_n(y)} \,\mathrm{d}y \\
&= -\left[\partial_y^4 v_n(y) \cdot \overline{\partial_y v_n(y)}\right]_0^\infty + \int_0^\infty \partial_y^4 v_n(y) \cdot \overline{\partial_y^2 v_n(y)} \,\mathrm{d}y \\
&= \left[\partial_y^3 v_n(y) \cdot \overline{\partial_y^2 v_n(y)}\right]_0^\infty - \int_0^\infty \left|\partial_y^3 v_n(y)\right|^2 \,\mathrm{d}y \\
&= -\int_0^\infty \left|\partial_y^3 v_n(y)\right|^2 \,\mathrm{d}y,
\end{aligned}$$

where all the boundary terms vanish due to the boundary conditions in (6.1.3). Plugging the result into (6.1.5), we obtain for $n = 1, 2$

$$0 = \lambda \int_0^\infty |v_n(y)|^2 \,\mathrm{d}y + \int_0^\infty \left|\partial_y^3 v_n(y)\right|^2 \,\mathrm{d}y. \tag{6.1.6}$$

In order to deduce that $v \equiv 0$, we start with the case $\Im\lambda \neq 0$: By taking the imaginary part of (6.1.6), it follows for $n = 1, 2$

$$0 = \Im\lambda \int_0^\infty |v_n(y)|^2 \,\mathrm{d}y.$$

Thus, $v \equiv 0$ holds true, as v is smooth. We proceed with the case $\{\Im\lambda = 0 \cap \Re\lambda > 0\}$: From (6.1.6) we have for $n = 1, 2$

$$0 = \Re\lambda \int_0^\infty |v_n(y)|^2 \,\mathrm{d}y + \int_0^\infty \left|\partial_y^3 v_n(y)\right|^2 \,\mathrm{d}y.$$

Since both of the summands are non-negative, but their sum equals zero, both integrals have to vanish. Again, by smoothness we have $v \equiv 0$. We conclude that in the space $C_0([0,\infty);\mathbb{C}^2)$ the trivial solution is the only admissible solution to the problem (6.1.3).

It remains to check that $\omega_j \neq 1 - \tilde{\mu} + 1/2$, cf. (6.1.2), holds true for $j = 1, 2, 3$: It follows

$$\omega_1 = 1 - \frac{0}{6} - \frac{1}{12} = \frac{11}{12}, \qquad \omega_2 = 1 - \frac{1}{6} - \frac{1}{12} = \frac{3}{4}, \qquad \omega_3 = 1 - \frac{2}{6} - \frac{1}{12} = \frac{7}{12}.$$

By $1 - \tilde{\mu} + 1/2 \in [2/3, 3/4)$, we obtain

$$\omega_1 > 1 - \tilde{\mu} + \frac{1}{2}, \qquad \omega_2 > 1 - \tilde{\mu} + \frac{1}{2}, \qquad \omega_3 < 1 - \tilde{\mu} + \frac{1}{2}.$$

Finally, by applying Theorem 2.2.1 to problem (6.1.1) we deduce that it possesses a unique solution $u = \mathcal{L}^{-1}(f, g, u_0)$, $g = (g_1, g_2, g_3)$, if and only if the right-hand side of the equation, given by (f, g, u_0), is an element of a suitable product space. More precisely, we have to show that

$$\begin{aligned}
g_2 &:= \partial_x f_0 \in W^{\omega_2}_{2,\mu}\left(J; L_2\left(\partial I; \mathbb{R}^2\right)\right) \cap L_{2,\mu}\left(J; W^{6\omega_2}_2\left(\partial I; \mathbb{R}^2\right)\right), \\
g_1 &:= f_0 \in W^{\omega_1}_{2,\mu}\left(J; L_2\left(\partial I; \mathbb{R}^2\right)\right) \cap L_{2,\mu}\left(J; W^{6\omega_1}_2\left(\partial I; \mathbb{R}^2\right)\right), \\
u_0 &:= f_0 \in W^{6(\tilde{\mu}-1/2)}_2\left(I; \mathbb{R}^2\right),
\end{aligned}$$

and that the initial data fulfill some compatibility conditions.

We start with the regularity: As the boundary of ∂I just consists of the two points $x = 0, 1$, the conditions simplify to

$$g_2 := \partial_x f_0 \in W^{\omega_2}_{2,\mu}\left(J; \mathbb{R}^2\right), \qquad g_1 := f_0 \in W^{\omega_1}_{2,\mu}\left(J; \mathbb{R}^2\right), \qquad u_0 := f_0 \in W^{6(\tilde{\mu}-1/2)}_2\left(J; \mathbb{R}^2\right).$$

The conditions on g_1 and g_2 are trivially fulfilled, since the initial datum f_0 does not depend on time. Furthermore, the condition on the initial data was an assumption. We proceed with the compatibility conditions: The initial data have to fulfill the boundary conditions

$$\mathcal{B}_j(D) f_0 = g_j \qquad\qquad \text{for } x \in \{0,1\}, \text{ if } \omega_j > 1 - \mu + \frac{1}{2},$$

thus, for $j = 1, 2$. This is trivially fulfilled by the construction of the problem. □

In summary, we found a curve $f \in \mathbb{E}_{\tilde{\mu},T,\mathbb{R}^2}$, which satisfies the desired boundary conditions (5.1.1) and is arbitrarily $W^{4(\mu-1/2)}_2(I; \mathbb{R}^2)$-close to the initial datum f_0. This can be easily seen by

$$\mathbb{E}_{\tilde{\mu},T,\mathbb{R}^2} \hookrightarrow BUC\left(\bar{J}; W^{6(\tilde{\mu}-1/2)}_2\left(I; \mathbb{R}^2\right)\right) = BUC\left(\bar{J}; W^{4(\mu-1/2)}_2\left(I; \mathbb{R}^2\right)\right),$$

where the embedding follows from (2.2.4). It remains to show that $f(t, \cdot)$, $t > 0$, is smooth enough to use it as a reference curve.

Lemma 6.1.4
Let f be the solution of problem (6.1.1) *given by Lemma 6.1.2. Then $f(t, \cdot) \in C^5(\bar{I}; \mathbb{R}^2)$ for all $t \in (0, T]$.*

Proof. The proof is split into the following parts: We find a homogeneous problem equivalent to (6.1.1) and use the regularization effects, which follow by the fact that the corresponding operator generates an analytic semigroup. Then we show that the regularity transfers to the solution of the original problem.

First, we choose a function $\xi \in C^\infty(\bar{I}; \mathbb{R}^2)$ fulfilling

$$\left.\begin{aligned}
\xi(x) &= f_0(x) \\
\partial_x \xi(x) &= \partial_x f_0(x) \\
\partial_x^2 \xi(x) &= 0
\end{aligned}\right\} \qquad \text{for } x \in \{0,1\} \tag{6.1.7}$$

and set $u_0 := f_0 - \xi$ and $h := \partial_x^6 \xi$. If $f \in \mathbb{E}_{\tilde{\mu},T,\mathbb{R}^2}$ is the unique solution of (6.1.1), then the function

$v := f - \xi$ fulfills

$$\begin{aligned}
\partial_t v = \partial_t f = \partial_x^6 f = \partial_x^6 v + \partial_x^6 \xi \qquad & \text{for } x \in (0,1) \text{ and } t > 0,\\
v = f - \xi = f_0 - f_0 = 0 \qquad & \text{for } x \in \{0,1\} \text{ and } t > 0,\\
\partial_x v = \partial_x f - \partial_x \xi = \partial_x f_0 - \partial_x f_0 = 0 \qquad & \text{for } x \in \{0,1\} \text{ and } t > 0,\\
\partial_x^2 v = \partial_x^2 f - \partial_x^2 \xi = 0 \qquad & \text{for } x \in \{0,1\} \text{ and } t > 0,\\
v_{|t=0} = f_0 - \xi \qquad & \text{for } x \in (0,1).
\end{aligned}$$

Thus, $v \in \mathbb{E}_{\tilde{\mu},T,\mathbb{R}^2}$ solves

$$\begin{aligned}
\partial_t v - \partial_x^6 v = \partial_x^6 \xi \qquad & \text{for } x \in (0,1) \text{ and } t > 0,\\
\partial_x^2 v = 0 \qquad & \text{for } x \in \{0,1\} \text{ and } t > 0,\\
\partial_x v = 0 \qquad & \text{for } x \in \{0,1\} \text{ and } t > 0,\\
v = 0 \qquad & \text{for } x \in \{0,1\} \text{ and } t > 0,\\
v_{|t=0} = u_0 \qquad & \text{for } x \in (0,1),
\end{aligned} \tag{6.1.8}$$

and problem (6.1.8) can be written as the abstract Cauchy problem

$$\begin{aligned}
\partial_t u(t) + Au(t) = h(t) \qquad & \text{for } t \in (0,T),\\
u_{|t=0} = u_0, \qquad &
\end{aligned} \tag{6.1.9}$$

where $A = -\partial_x^6 : D(A) \to X$ with $D(A) := \{u \in W_2^6(I;\mathbb{R}^2) |\ u_{|\partial I} = 0, \partial_x u_{|\partial I} = 0, \partial_x^2 u_{|\partial I} = 0\}$ and $X := L_2(I;\mathbb{R}^2)$. The following claim enables us to exploit the properties of analytic C_0-semigroups.

Claim 6.1.5 *$-A$ is the generator of an analytic C_0-semigroup.*

Proof of the claim: We observe by the proof of Lemma 6.1.2 that there exists a unique solution $u = \mathcal{L}^{-1}(h,(0,0,0),u_0) \in \mathbb{E}_{\tilde{\mu},T,\mathbb{R}^2}$ of problem (6.1.9) if and only if $u_0 \in X_{\tilde{\mu}} := W_2^{6(\tilde{\mu}-1/2)}(I;\mathbb{R}^2)$ and $h \in \mathbb{E}_{0,\tilde{\mu}} := L_{2,\tilde{\mu}}(J;L_2(I;\mathbb{R}^2))$ and the compatibility conditions for the initial data are fulfilled, i.e.

$$\left.\begin{aligned} u_0 &= 0\\ \partial_x u_0 &= 0 \end{aligned}\right\} \qquad \text{for } x \in \{0,1\}.$$

As this holds for each $h \in \mathbb{E}_{0,\tilde{\mu}}$ and $u_0 = 0$, the operator A belongs to the class $\mathcal{MR}_{p,\tilde{\mu}}(J;X)$, i.e. the operator has maximal $L_{p,\tilde{\mu}}$-regularity, see Definition 3.5.1 in [27]. By Theorem 3.5.4 in [27] it follows that $A : D(A) \to X$ belongs to the class $\mathcal{MR}_{p,\tilde{\mu}}(J;X)$ if and only if it belongs to $\mathcal{MR}_p(J;X)$. Thus, we deduce by Proposition 3.5.2 in [27] that there exists a $\lambda_0 \geq 0$, such that the operator $\lambda_0 + A$ is sectorial with spectral angle less than $\pi/2$. We observe that the operator A is closed and densely defined by the property of maximal regularity. Hence, by Theorem 12.31 in [28], $-A$ generates an analytic C_0-semigroup. □

Now we can show higher regularity for the solution of the homogeneous problem.

Claim 6.1.6 $v \in C^\infty\big((0,T];C^5(\bar{I};\mathbb{R}^2)\big)$.

Proof of the claim: For an $h \in X$, which is an element of $L_1(J;X) \cap C(\bar{J};X)$ for J bounded, and a $u_0 \in X$, we consider the mild solution of the Cauchy problem (6.1.9) given by

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} h(s)\,\mathrm{d}s,$$

cf. Definition 4.1.4 in [22]. As h does not depend on time, the expression simplifies to

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-sA} h \,\mathrm{d}s.$$

Using the basic properties of the analytic semigroup generated by $-A$, see Chapter 2.1 in [22], we conclude that for an $x \in X$ the mapping $[t \mapsto e^{-tA}x] \in C^\infty((0,T], D(A))$ for $T < \infty$. Combining this fact with the theorem on parameter integrals for separable Banach spaces, see Theorem 3.17 in [3], it holds

$$u \in C^\infty\big((0,T]; D(A)\big) \hookrightarrow C^\infty\left((0,T]; C^5\left(\bar{I};\mathbb{R}^2\right)\right).$$

This proves the claim. □

It follows by direct calculations that $\tilde{f} := v + \xi \in \mathbb{E}_{\tilde{\mu},T,\mathbb{R}^2}$ solves the original problem (6.1.1) and that the function is in $C^\infty((0,T]; C^5(\bar{I};\mathbb{R}^2))$. Due to the fact that the solution is unique, we obtain $f(t) \in C^5(\bar{I};\mathbb{R}^2)$ for $t \in (0,T]$. □

The following lemma is needed for estimates in the next chapter.

Lemma 6.1.7
$\|\cdot\|_{W_2^6(I;\mathbb{R}^2)}$ and $\|\cdot\|_{D(A)}$ are equivalent norms on $D(A)$.

Proof. By the sectoriality of $\lambda_0 + A$ the following holds true for $\lambda \geq \lambda_0$: For each $h \in L_2(I;\mathbb{R}^2)$ the elliptic problem

$$\begin{aligned} \lambda f - \partial_x^6 f &= h && \text{for } x \in (0,1), \\ f &= 0 && \text{for } x \in \{0,1\}, \\ \partial_x f &= 0 && \text{for } x \in \{0,1\}, \\ \partial_x^2 f &= 0 && \text{for } x \in \{0,1\}, \end{aligned}$$

has a unique solution $f \in W_2^6(I;\mathbb{R}^2)$. This implies that the linear operator

$$\lambda + A : \Big(D(A), \|\cdot\|_{W_2^6(I;\mathbb{R}^2)}\Big) \to X$$

is invertible and bounded, as

$$\|(\lambda + A)f\|_{L_2(I;\mathbb{R}^2)} \leq \lambda \|f\|_{L_2(I;\mathbb{R}^2)} + \|Af\|_{L_2(I;\mathbb{R}^2)} \leq C(\lambda)\|f\|_{W_2^6(I;\mathbb{R}^2)}.$$

Therefore, we obtain

$$\|f\|_{D(A)} \leq C(\lambda)\|f\|_{W_2^6(I;\mathbb{R}^2)}. \tag{6.1.10}$$

As $D(A)$ is a closed subspace of the Banach space $W_2^6(I;\mathbb{R}^2)$, it is - equipped with the norm $\|\cdot\|_{W_2^6(I;\mathbb{R}^2)}$ - a Banach space. Thus, the open mapping theorem yields that the inverse

$$(\lambda + A)^{-1} : L_2(I;\mathbb{R}^2) \to \Big(D(A), \|\cdot\|_{W_2^6(I;\mathbb{R}^2)}\Big)$$

is bounded. It follows for $f \in D(A)$ that

$$\|f\|_{W_2^6(I;\mathbb{R}^2)} = \left\|(\lambda + A)^{-1}h\right\|_{W_2^6(I;\mathbb{R}^2)} \leq C(\lambda)\|h\|_{L_2(I;\mathbb{R}^2)},$$

and hence

$$\|f\|_{W_2^6(I;\mathbb{R}^2)} \leq C(\lambda)\|(\lambda + A)f\|_X \leq C(\lambda)\|f\|_{D(A)}. \tag{6.1.11}$$

The estimates (6.1.10) and (6.1.11) show the claim. $\square$

6.2 Characterization of Reference Curves

This section aims at to establish certain conditions which enable us to prove that the smoothed versions of the initial curve $f_\epsilon := f(\epsilon, \cdot)$, $\epsilon > 0$, see Chapter 6.1, are convenient to use as reference curves in the short time existence result, Theorem 5.1.3.

A good starting point is the formulation of conditions for the admissible initial curves for a fixed reference curve denoted by Φ^*: Let $\Phi^* : [0,1] \to \mathbb{R}^2$ be a regular C^5-curve parametrized proportional to arc length. Moreover, let it fulfill the conditions in (5.1.1).
Again, we use the curvilinear coordinates in (5.1.2), i.e.

$$\begin{aligned}\Psi : [0,1] \times (-d,d) &\to \mathbb{R}^2\\ (\sigma, q) &\mapsto \Phi^*(\sigma) + q(n_\Lambda(\sigma) + \cot\alpha\eta(\sigma)\tau_\Lambda(\sigma)),\end{aligned}$$

where η is given by (5.1.3). Like before, we denote by $\tau_\Lambda(\sigma) = {}^{\partial_\sigma\Phi^*(\sigma)}/_{\mathcal{L}[\Phi^*]}$ and $n_\Lambda(\sigma) = R\tau_\Lambda(\sigma)$ for $\sigma \in [0,1]$ the unit tangent and unit normal vector of Λ at $\Phi^*(\sigma)$, respectively, for $\sigma \in [0,1]$.

We begin by specifying the requested properties of a reference curve.

Definition 6.2.1 (Reference Curve Φ^* for the Initial Curve f_0)
Let $\alpha \in (0,\pi)$ be fixed. Furthermore, let $\Phi^* : [0,1] \to \mathbb{R}^2$ be a regular C^5-curve fulfilling the boundary conditions (5.1.1) *and let $f_0 : [0,1] \to \mathbb{R}^2$ be a regular W_2^β-curve, $\beta \in \left(\frac{3}{2}, 2\right]$, fulfilling the boundary conditions* (3.1.6)*, i.e.*

$$\begin{aligned} f_0(\sigma) &\in \mathbb{R} \times (0,\infty) && \text{for } \sigma \in \{0,1\},\\ \measuredangle\left(n_{\Gamma_0}(\sigma), \begin{pmatrix}0\\-1\end{pmatrix}\right) &= \pi - \alpha && \text{for } \sigma \in \{0,1\},\end{aligned} \tag{3.1.6}$$

for $\Gamma_0 := f_0([0,1])$. We call Φ^* ***a reference curve for the initial curve*** *f_0, if the following conditions hold true:*

1. *There exists a regular C^1-reparametrization $\varphi : [0,1] \to [0,1]$ and a function $\rho : [0,1] \to (-d,d)$ of class W_2^β, such that*

$$f_0(\varphi(\sigma)) = \Phi^*(\sigma) + \rho(\sigma)(n_\Lambda(\sigma) + \cot\alpha\eta(\sigma)\tau_\Lambda(\sigma)). \tag{6.2.1}$$

2. *The function ρ satisfies the bounds* (5.1.22)*, i.e.*

$$\|\rho\|_{C([0,1])} < \frac{1}{6\|\kappa_\Lambda\|_{C(\bar{I})}\left(1 + (\cot\alpha)^2 + \hat{C}|\cot\alpha|\|\eta'\|_{C([0,1])}\right)} = \frac{K_0(\alpha,\Phi^*)}{3},$$

and in the case $\alpha \neq \frac{\pi}{2}$ additionally

$$\|\partial_\sigma\rho\|_{C([0,1])} < \frac{\mathcal{L}[\Phi^*]}{36|\cot\alpha|} = \frac{K_1(\alpha,\Phi^*)}{3},$$

and there exists a constant $C = C\left(\alpha, \Phi^*, \eta, \|f_0\|_{W_2^\beta((0,1);\mathbb{R}^2)}\right)$ *such that* $\|\rho\|_{W_2^\beta((0,1))} \leq C$.

This enables us to state the main result of this section.

Theorem 6.2.2
Let $\alpha \in (0, \pi)$ *be fixed. Furthermore, let* $\Phi^* : [0,1] \to \mathbb{R}^2$ *be a regular* C^5*-curve parametrized proportional to arc length and let it fulfill the boundary conditions* (5.1.1). *Let* $f_0 : [0,1] \to \mathbb{R}^2$ *be a regular* W_2^β*-curve,* $\beta \in \left(\frac{3}{2}, 2\right]$, *fulfilling the boundary conditions* (3.1.6). *Moreover, let* $\lambda \in (0,1)$ *be given such that the conditions*

$$\lambda C_\alpha \sqrt{1 + (\cot\alpha)^2} < \min\left\{ \sin\left(\frac{\sqrt{(\cot\alpha)^2+1} - |\cot\alpha|}{4} \right), \begin{cases} \sin\left(\frac{1}{2(4\cdot 144)^2 (\cot\alpha)^2} \right) & \text{for } \alpha \neq \frac{\pi}{2} \\ 1 & \text{for } \alpha = \frac{\pi}{2} \end{cases} \right\} \tag{6.2.2}$$

$$\lambda < \min\left\{ \frac{1}{6\sqrt{1 + (\cot\alpha)^2}}, \begin{cases} \frac{1}{144|\cot\alpha|} & \text{for } \alpha \neq \frac{\pi}{2} \\ 1 & \text{for } \alpha = \frac{\pi}{2} \end{cases} \right\} \tag{6.2.3}$$

hold true. If $f_0 \in B_{\xi_0}^{C^0}(\Phi^*)$ *and* $\partial_\sigma f_0 \in B_{\xi_1}^{C^0}(\partial_\sigma \Phi^*)$, *for*

$$\xi_0 = \min\left\{ \overline{C_\alpha(\lambda)}, \frac{(\sin\alpha)^2}{2} \right\} \frac{1}{\|\kappa_\Lambda\|_{C([0,1])}},$$

$$\xi_1 = \min\left\{ \frac{\sqrt{(\cot\alpha)^2 + 1} - |\cot\alpha|}{4}, \begin{cases} \frac{1}{2(4\cdot 144)^2|\cot\alpha|^2} & \text{for } \alpha \neq \frac{\pi}{2} \\ 1 & \text{for } \alpha = \frac{\pi}{2} \end{cases}, \frac{\sin\alpha}{2} \right\} \mathcal{L}[\Phi^*],$$

where

$$\overline{C_\alpha(\lambda)} := \left(1 - \sqrt{(\lambda C_\alpha \cot\alpha)^2 + (1 - \lambda C_\alpha)^2}\right) \in (0, \lambda]$$

for C_α *given by*

$$C_\alpha := \left[1 + (\cot\alpha)^2 + \hat{C}|\cot\alpha| \|\eta'\|_{C([0,1])}\right]^{-1} \in (0,1]$$

and $\hat{C} := (\sqrt{2}\sin\alpha)^{-1} > 0$, *cf. Lemma 2.3.1. Then* Φ^* *is a reference curve for the initial curve* f_0.

The first step to prove this theorem, is to show that Ψ is a local diffeomorphism in a suitably small neighborhood of Φ^*.

Lemma 6.2.3
Let $\lambda \in (0,1]$ *and* d *be given by*

$$d := \frac{C_\alpha}{\|\kappa_\Lambda\|_{C([0,1])}}, \tag{6.2.4}$$

where C_α *is given like in Theorem 6.2.2. Then* Ψ *is a local diffeomorphism on* $[0,1] \times (-\lambda d, \lambda d)$ *with*

$$|D\Psi|(\sigma, q) > (1 - \lambda)\mathcal{L}[\Phi^*] \qquad \text{for } (\sigma, q) \in [0,1] \times (-\lambda d, \lambda d).$$

Proof. Using the calculations in (5.1.19), we obtain for the derivatives of Ψ

$$\partial_\sigma \Psi = \underbrace{\mathcal{L}[\Phi^*]\left(1 - q\kappa_\Lambda + q\cot\alpha\frac{\eta'(\sigma)}{\mathcal{L}[\Phi^*]}\right)}_{=:a}\tau_\Lambda(\sigma) + \underbrace{\mathcal{L}[\Phi^*]q\cot\alpha\eta(\sigma)\kappa_\Lambda}_{=:b} n_\Lambda(\sigma),$$
$$\partial_q \Psi = \underbrace{\cot\alpha\eta(\sigma)}_{=:c}\tau_\Lambda(\sigma) + \underbrace{1}_{=:e} n_\Lambda(\sigma).$$

Due to the fact that $\tau_\Lambda(\sigma)$ and $n_\Lambda(\sigma)$ are linearly independent for each $\sigma \in [0,1]$, we deduce by changing the basis that

$$|D\Psi|(\sigma,q) > (1-\lambda)\mathcal{L}[\Phi^*] \quad \Leftrightarrow \quad \begin{vmatrix} a & c \\ b & e \end{vmatrix} = \mathcal{L}[\Phi^*]\left[1 - q\kappa_\Lambda + q\cot\alpha\frac{\eta'(\sigma)}{\mathcal{L}[\Phi^*]} - q(\cot\alpha\eta(\sigma))^2\kappa_\Lambda\right] > (1-\lambda)\mathcal{L}[\Phi^*]. \tag{6.2.5}$$

Combining this with the estimate

$$\frac{1}{\mathcal{L}[\Phi^*]} \leq \hat{C}\|\kappa_\Lambda\|_{C([0,1])},$$

for $\hat{C} := (\sqrt{2}\sin\alpha)^{-1} > 0$, cf. Lemma 2.3.1, allows for stating the sufficient condition

$$1 - |q|\|\kappa_\Lambda\|_{C([0,1])}\left[1 + (\cot\alpha)^2 + \hat{C}|\cot\alpha|\|\eta'\|_{C([0,1])}\right] > 1-\lambda \quad \Rightarrow \quad \text{(6.2.5) holds true.}$$

The condition on the left-hand side is fulfilled for $0 \leq |q| < \lambda d$. Thus, we conclude that the function Ψ is a local diffeomorphism on $[0,1] \times (-\lambda d, \lambda d)$ for $\lambda \in (0,1]$. ☐

Now, we first criterion for reference curves can be formulated.

Lemma 6.2.4
Let $\alpha \in (0,\pi)$ be fixed. Furthermore, let $\Phi^* : [0,1] \to \mathbb{R}^2$ be a regular C^5-curve parametrized proportional to arc length and let it fulfill the boundary conditions (5.1.1)*. Let $f_0 : [0,1] \to \mathbb{R}^2$ be a regular W_2^β-curve, $\beta \in \left(\frac{3}{2}, 2\right]$, fulfilling the boundary conditions* (3.1.6)*. Moreover, let the following conditions hold true for a $\lambda \in (0,1)$ fulfilling the assumptions* (6.2.2) *and* (6.2.3)*:*

1. *The initial curve $\Gamma_0 := f_0(\bar{I})$ is contained in $\Psi([0,1] \times (-\lambda d, \lambda d))$.*
2. *The curves f_0 and Φ^* fulfill the conditions*

$$|f_0(\sigma) - \Phi^*(\sigma)| < \frac{\overline{C_\alpha(\lambda)}}{\|\kappa_\Lambda\|_{C([0,1])}},$$
$$|\partial_\sigma f_0(\sigma) - \partial_\sigma \Phi^*(\sigma)| < \min\left\{\frac{\sqrt{(\cot\alpha)^2+1} - |\cot\alpha|}{4}, \begin{cases} \frac{1}{2(4\cdot 144)^2|\cot\alpha|^2} & \text{for } \alpha \neq \frac{\pi}{2} \\ 1 & \text{for } \alpha = \frac{\pi}{2} \end{cases}\right\}\mathcal{L}[\Phi^*].$$

Then Φ^* is a reference curve for the initial curve f_0.

Proof. The proof is done in two steps: In the first step, we show that there exist functions φ and ρ fulfilling item 1 in Definition 6.2.1. Then, we prove the bounds given in item 2.

<u>*Step 1:*</u> *Finding φ and ρ fulfilling* (6.2.1)
In order to prove this, we want to use the implicit function theorem. To this end, we need the local invertibility of the function Φ^* on a slightly larger open set: We extend the function Φ^* to

$(-\tilde{\delta}, 1+\tilde{\delta})$ for a small $\tilde{\delta} > 0$: We define the extension of Φ^*, which we again denote by Φ^*, by

$$\Phi^*(\sigma) := \begin{cases} \Phi^*(0) + \sigma\partial_\sigma\Phi^*(0) & \text{for } \sigma \in (-\tilde{\delta}, 0), \\ \Phi^*(\sigma) & \text{for } \sigma \in [0,1], \\ \Phi^*(1) + (\sigma-1)\partial_\sigma\Phi^*(1) & \text{for } \sigma \in (1, 1+\tilde{\delta}). \end{cases}$$

Furthermore, we extend the function η by

$$\eta(\sigma) := \begin{cases} \eta(0) & \text{for } \sigma \in (-\tilde{\delta}, 0), \\ \eta(\sigma) & \text{for } \sigma \in [0,1], \\ \eta(1) & \text{for } \sigma \in (1, 1+\tilde{\delta}), \end{cases}$$

and call the extension η again. We observe that for a sufficiently small $\tilde{\delta} > 0$ the function Ψ is still a local diffeomorphism on $(-\tilde{\delta}, 1+\tilde{\delta}) \times (-d, d)$, where we used Lemma 6.2.3. For the following argumentation, it is convenient to decompose the local inverse of Ψ at the point $\Psi(\tilde{\sigma}, q) = p$ into $\Psi^{-1} = (\Pi_\Lambda, d_\Lambda)$, such that

$$\Pi_\Lambda \in C^1\big(U; (-\tilde{\delta}, 1+\tilde{\delta})\big) \qquad \text{and} \qquad d_\Lambda \in C^1\big(U; (-d, d)\big), \tag{6.2.6}$$

where U is a suitable neighborhood of p in $\mathbb{R}^2$. Note that the inverse is not necessarily unique.

Moreover, we need to extend the initial curve f_0 as well. We denote by $\tau_{\Gamma_0}(\sigma) = {\partial_\sigma f_0(\sigma)}/{|\partial_\sigma f_0(\sigma)|}$ and $n_{\Gamma_0}(\sigma) = R\tau_{\Gamma_0}(\sigma)$ the unit tangent vector and the unit normal vector of Γ_0 at $f_0(\sigma)$, respectively, for $\sigma \in [0,1]$. Then we set

$$f_0(\tilde{\sigma}) := \begin{cases} f_0(0) + \tilde{\sigma}\tau_{\Gamma_0}(0) & \text{for } \tilde{\sigma} \in (-\delta, 0), \\ f_0(\tilde{\sigma}) & \text{for } \tilde{\sigma} \in [0,1], \\ f_0(1) + (1-\tilde{\sigma})\tau_{\Gamma_0}(1) & \text{for } \tilde{\sigma} \in (1, 1+\delta), \end{cases}$$

and denote the extension by f_0 again. Hence, we can define

$$\begin{aligned} H : (-\delta, 1+\delta) \times (-\tilde{\delta}, 1+\tilde{\delta}) \times (-d, d) &\to \mathbb{R}^2 \\ (\tilde{\sigma}, \sigma, q) &\mapsto f_0(\tilde{\sigma}) - \Psi(\sigma, q) \\ &= f_0(\tilde{\sigma}) - \Phi^*(\sigma) - q(n_\Lambda(\sigma) + \cot\alpha\,\eta(\sigma)\tau_\Lambda(\sigma)). \end{aligned}$$

For a fixed but arbitrary $\tilde{\sigma}_0 \in [0,1]$ it follows by (6.2.6) that

$$H(\tilde{\sigma}_0, \underbrace{\Pi_\Lambda(f_0(\tilde{\sigma}_0))), d_\Lambda(f_0(\tilde{\sigma}_0)))}_{=:y_0}) = 0.$$

Since Ψ is a local diffeomorphism on $(-\tilde{\delta}, 1+\tilde{\delta}) \times (-d, d)$, the function $\partial_y H$ is invertible in $(\tilde{\sigma}_0, y_0)$. Moreover, H and $\partial_y H$ are continuous in $(\tilde{\sigma}_0, y_0)$. Thus, we obtain by the implicit function theorem, cf. Theorem 4.B in [33], that there exists a neighborhood $U_0 \subset (-\delta, 1+\delta)$ of $\tilde{\sigma}_0$ and $V_0 \subset (-\tilde{\delta}, 1+\tilde{\delta})$ of y_0 and a function $g : U_0 \to V_0$ such that $H(\tilde{\sigma}, g(\tilde{\sigma})) = 0$. As f_0 and Ψ are C^1-maps, g is as well of class C^1 in a neighborhood of x_0.

In this way, we obtain for each $\tilde{\sigma}_0 \in [0,1]$ a function g, which is just locally defined in U_0. As Ψ is a local C^1-diffeomorphism on the domain, g is uniquely determined. Thus, two functions g_a and g_b, for $a, b \in [0,1]$ with $a \neq b$, have to coincide on $U_a \cap U_b$. Additionally, it follows by construction that the first component of g, which is denoted by g_1, fulfills $g_1(0) = g_1(0) = 0$ and $g_1(1) = 1$. By gluing together the locally defined functions, we obtain a C^1-function $g(\cdot)$ with $g_1([0,1]) = [0,1]$.

In order to define the function φ, cf. Definition 6.2.1, we show that g_1 is injective. Consequently, it is invertible on $[0,1]$ and we can define

$$\varphi(\sigma) := g_1^{-1}(\sigma) \qquad \text{and} \qquad \rho(\sigma) := g_2(\varphi(\sigma)) = g_2 \circ g_1^{-1}(\sigma).$$

By the differential rule for inverse mappings, cf. Corollary 4.37 in [33], both functions are of class C^1.

Claim 6.2.5 *The function $g_1 : [0,1] \to [0,1]$ is injective.*

Proof of the claim: We use the chain rule and obtain for $\tilde{\sigma} \in [0,1]$

$$0 = \partial_{\tilde{\sigma}}(H(\tilde{\sigma}, g(\tilde{\sigma}))) = (\partial_{\tilde{\sigma}} H)(\tilde{\sigma}, g(\tilde{\sigma})) + (\partial_y H)(\tilde{\sigma}, g(\tilde{\sigma}))\partial_{\tilde{\sigma}} g(\tilde{\sigma}).$$

By Lemma 6.2.3, it follows that $\partial_{\tilde{\sigma}}\Psi$ and $\partial_q\Psi$ are linearly independent for $(\sigma, q) \in [0,1] \times (-d, d)$, where d is given by (6.2.4). Using Condition 1, we obtain from the previous calculations that

$$\partial_{\tilde{\sigma}} g_1(\tilde{\sigma}) = 0 \qquad \Leftrightarrow \qquad \partial_\sigma f_0(\tilde{\sigma}) = \partial_q \Psi(g_1(\tilde{\sigma}))\partial_{\tilde{\sigma}} g_2(\tilde{\sigma}).$$

Thus, we want to rule out that it holds $\tau_{\Gamma_0}(\tilde{\sigma}) \parallel \partial_q\Psi(g_1(\tilde{\sigma}))$, or equivalently $n_{\Gamma_0}(\tilde{\sigma}) \perp \partial_q\Psi(g_1(\tilde{\sigma}))$. Direct calculations provide

$$\begin{aligned}\langle n_{\Gamma_0}(\tilde{\sigma}), \partial_q\Psi(g_1(\tilde{\sigma}))\rangle &= \frac{1}{2}\left(|n_{\Gamma_0}(\tilde{\sigma})|^2 + |\partial_q\Psi(g_1(\tilde{\sigma}))|^2 - |n_{\Gamma_0}(\tilde{\sigma}) - \partial_q\Psi(g_1(\tilde{\sigma}))|^2\right) \\ &= \frac{1}{2}\left(1 + 1 + (\cot\alpha)^2(\eta(g_1(\tilde{\sigma})))^2 - |n_{\Gamma_0}(\tilde{\sigma}) - \partial_q\Psi(g_1(\tilde{\sigma}))|^2\right).\end{aligned}$$

This implies that $\langle n_{\Gamma_0}(\tilde{\sigma}), \partial_q\Psi(g_1(\tilde{\sigma}))\rangle > 1/2$, if

$$|n_{\Gamma_0}(\tilde{\sigma}) - \partial_q\Psi(g_1(\tilde{\sigma}))|^2 < 1 + (\cot\alpha)^2(\eta(g_1(\tilde{\sigma})))^2 \tag{6.2.7}$$

holds true. We deduce

$$\begin{aligned}|n_{\Gamma_0}(\tilde{\sigma}) - \partial_q\Psi(g_1(\tilde{\sigma}))| &\le |n_{\Gamma_0}(\tilde{\sigma}) - n_\Lambda(g_1(\tilde{\sigma}))| + |n_\Lambda(g_1(\tilde{\sigma})) - \partial_q\Psi(g_1(\tilde{\sigma}))| \\ &\le \underbrace{|n_{\Gamma_0}(\tilde{\sigma}) - n_\Lambda(\tilde{\sigma})| + |n_\Lambda(\tilde{\sigma}) - n_\Lambda(g_1(\tilde{\sigma}))|}_{=:I+II} + |\cot\alpha\eta(g_1(\tilde{\sigma}))|,\end{aligned}$$

where we used triangle inequality and the definition of $\partial_q\Psi$. Comparing this with (6.2.7), it suffices to show that

$$(I + II + |\cot\alpha\eta(g_1(\tilde{\sigma}))|)^2 < 1 + (\cot\alpha)^2(\eta(g_1(\tilde{\sigma})))^2,$$

which is equivalent to

$$(I + II)^2 + 2|\cot\alpha|(I + II) < 1.$$

Solving the quadratic inequality and taking the positive solution, we deduce the condition

$$I + II < \sqrt{(\cot\alpha)^2 + 1} - |\cot\alpha| \qquad \Rightarrow \qquad g_1 \text{ is injective.} \tag{6.2.8}$$

In the following, we show that (6.2.8) is fulfilled by estimating I and II separately: For the term $I = |n_{\Gamma_0}(\tilde{\sigma}) - n_\Lambda(\tilde{\sigma})|$, we obtain by adding a zero

$$I = \frac{\big||\partial_\sigma f_0(\tilde{\sigma})|\partial_\sigma\Phi^*(\tilde{\sigma})| - \partial_\sigma\Phi^*(\tilde{\sigma})|\partial_\sigma f_0(\tilde{\sigma})|\big|}{|\partial_\sigma f_0(\tilde{\sigma})||\partial_\sigma\Phi^*(\tilde{\sigma})|}$$

$$= \frac{\big|\partial_\sigma f_0(\tilde\sigma)|\partial_\sigma \Phi^*(\tilde\sigma)| - \partial_\sigma f_0(\tilde\sigma)|\partial_\sigma f_0(\tilde\sigma)|\big|}{|\partial_\sigma f_0(\tilde\sigma)||\partial_\sigma \Phi^*(\tilde\sigma)|} + \frac{\big|\partial_\sigma f_0(\tilde\sigma)|\partial_\sigma f_0(\tilde\sigma)| - \partial_\sigma \Phi^*(\tilde\sigma)|\partial_\sigma f_0(\tilde\sigma)|\big|}{|\partial_\sigma f_0(\tilde\sigma)||\partial_\sigma \Phi^*(\tilde\sigma)|}$$
$$\leq \frac{\big||\partial_\sigma \Phi^*(\tilde\sigma)| - |\partial_\sigma f_0(\tilde\sigma)|\big|}{|\partial_\sigma \Phi^*(\tilde\sigma)|} + \frac{|\partial_\sigma f_0(\tilde\sigma) - \partial_\sigma \Phi^*(\tilde\sigma)|}{|\partial_\sigma \Phi^*(\tilde\sigma)|} \leq \frac{2}{|\partial_\sigma \Phi^*(\tilde\sigma)|}|\partial_\sigma f_0(\tilde\sigma) - \partial_\sigma \Phi^*(\tilde\sigma)|, \quad (6.2.9)$$

where $|\partial_\sigma \Phi^*(\tilde\sigma)| = \mathcal{L}[\Phi^*]$, as Φ^* is parametrized proportional to arc length. Thus, we have by Condition 2

$$I < \frac{\sqrt{(\cot\alpha)^2+1} - |\cot\alpha|}{2}. \quad (6.2.10)$$

Considering the second summand II, we deduce by the fundamental theorem of calculus

$$II = |\tau_\Lambda(\tilde\sigma) - \tau_\Lambda(g_1(\tilde\sigma))| \leq \left|\int_{\tilde\sigma}^{g_1(\tilde\sigma)} \mathcal{L}[\Phi^*]\kappa_\Lambda(\sigma) n_\Lambda(\sigma)\,\mathrm{d}\sigma\right| \leq |g_1(\tilde\sigma) - \tilde\sigma|\;\mathcal{L}[\Phi^*]\|\kappa_\Lambda\|_{C([0,1])}$$
$$= \mathcal{L}[\Phi^*([g_1(\tilde\sigma),\tilde\sigma])]\|\kappa_\Lambda\|_{C([0,1])}, \quad (6.2.11)$$

since $|g_1(\tilde\sigma) - \tilde\sigma|\mathcal{L}[\Phi^*] = \mathcal{L}[\Phi^*([g_1(\tilde\sigma),\tilde\sigma])]$ by the proportional-to-arc-length-parametrization of Φ^*. Here, we assumed w.l.o.g. that $g_1(\tilde\sigma) \leq \tilde\sigma$. Thus, it remains to estimate $\mathcal{L}[\Phi^*([g_1(\tilde\sigma),\tilde\sigma])]$ by geometric considerations. By the bound on the curvature, it follows, that $\mathcal{L}[\Phi^*([g_1(\tilde\sigma),\tilde\sigma])]$ cannot be larger than the circle arc with radius $r = 1/\|\kappa_\Lambda\|_{C([0,1])}$ that connects the points $\Phi^*(g_1(\tilde\sigma))$ and $\Phi^*(\tilde\sigma)$. It is denoted by $\mathcal{L}_{max}[\Phi^*([g_1(\tilde\sigma),\tilde\sigma])]$, cf. Figure 6.1.

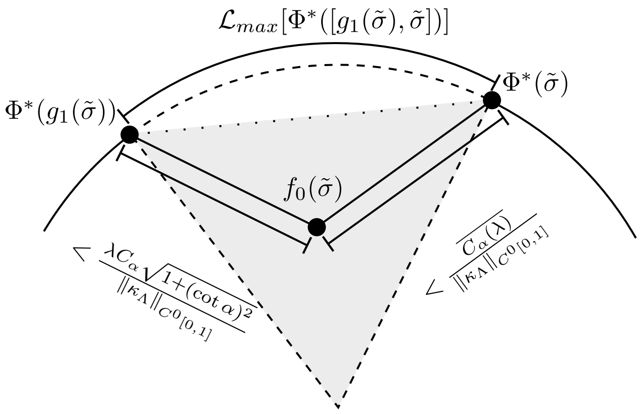

Figure 6.1: The estimate of $\mathcal{L}_{max}[\Phi^*([g_1(\tilde\sigma),\tilde\sigma])]$ for $|\kappa_\Lambda(\sigma)| = \|\kappa_\Lambda\|_{C([0,1])}$ for $\sigma \in [g_1(\tilde\sigma),\tilde\sigma]$.

By Condition 1 of Lemma 6.2.4 it holds

$$|\Phi^*(g_1(\tilde\sigma)) - f_0(\tilde\sigma)| = |d_\Lambda(f_0(\tilde\sigma))| < \lambda d\sqrt{1+(\cot\alpha)^2} \leq \frac{\lambda C_\alpha\sqrt{1+(\cot\alpha)^2}}{\|\kappa_\Lambda\|_{C^0[0,1]}}. \quad (6.2.12)$$

Moreover, we have a bound on $|\Phi^*(\tilde\sigma) - f_0(\tilde\sigma)|$ for $\tilde\sigma \in [0,1]$ by Condition 2 of Lemma 6.2.4. Thus, we observe that

$$|\Phi^*(g_1(\tilde\sigma)) - \Phi^*(\tilde\sigma)| \leq |\Phi^*(g_1(\tilde\sigma)) - f_0(\tilde\sigma)| + |f_0(\tilde\sigma) - \Phi^*(\tilde\sigma)|$$
$$< \left[\lambda C_\alpha\sqrt{1+(\cot\alpha)^2} + \overline{C_\alpha(\lambda)}\right]\frac{1}{\|\kappa_\Lambda\|_{C^0[0,1]}}$$

and the estimated term corresponds to the length of the doted line in Figure 6.1. The inequality

$$\overline{C_\alpha(\lambda)} = \left(1 - \sqrt{(\lambda C_\alpha \cot\alpha)^2 + (1-\lambda C_\alpha)^2}\right) < 1 - \sqrt{(1-\lambda C_\alpha)^2} = \lambda C_\alpha,$$

provides

$$|\Phi^*(g_1(\tilde\sigma)) - \Phi^*(\tilde\sigma)| < \frac{2\lambda C_\alpha\sqrt{1+(\cot\alpha)^2}}{\|\kappa_\Lambda\|_{C([0,1])}}.$$

Now, we can use basic geometry on the symmetric rectangular triangles, which arise by splitting the grey area in Figure 6.1, to estimate $\mathcal{L}[\Phi^*([g_1(\tilde\sigma),\tilde\sigma])]$. It follows

$$\begin{aligned}\mathcal{L}_{max}[\Phi^*([g_1(\tilde\sigma),\tilde\sigma])] &\le 2\arcsin\left(\frac{|\Phi^*(g_1(\tilde\sigma)) - \Phi^*(\tilde\sigma)|\|\kappa_\Lambda\|_{C([0,1])}}{2}\right)\frac{1}{\|\kappa_\Lambda\|_{C([0,1])}}\\ &< 2\arcsin\left(\lambda C_\alpha\sqrt{1+(\cot\alpha)^2}\right)\frac{1}{\|\kappa_\Lambda\|_{C([0,1])}}.\end{aligned}$$

By assumption (6.2.2), we have

$$\lambda C_\alpha\sqrt{1+(\cot\alpha)^2} < \sin\left(\frac{\sqrt{(\cot\alpha)^2+1} - |\cot\alpha|}{4}\right).$$

Hence,

$$\mathcal{L}_{max}[\Phi^*([g_1(\tilde\sigma),\tilde\sigma])] < \frac{\sqrt{(\cot\alpha)^2+1} - |\cot\alpha|}{2\|\kappa_\Lambda\|_{C([0,1])}}$$

is deduced. By plugging this into (6.2.11), we obtain for $II = |n_\Lambda(\tilde\sigma) - n_\Lambda(g_1(\tilde\sigma))|$

$$II < 2\arcsin\left(\lambda C_\alpha\sqrt{1+(\cot\alpha)^2}\right) < \frac{\sqrt{(\cot\alpha)^2+1} - |\cot\alpha|}{2}. \tag{6.2.13}$$

Combining this with (6.2.10) shows that condition (6.2.8) is fulfilled. This proves the claim. □

By construction we obtain for $\sigma \in [0,1]$

$$0 = H(\varphi(\sigma),\sigma,\rho(\sigma)) = f_0(\varphi(\sigma)) - \Phi^*(\sigma) - \rho(\sigma)(n_\Lambda(\sigma) + \cot\alpha\eta(\sigma)\tau_\Lambda(\sigma)),$$

hence, the identity (6.2.1) is fulfilled. Differentiation with respect to σ yields

$$0 = \partial_\sigma H(\varphi(\sigma),\sigma,\rho(\sigma)) = \partial_\sigma f_0(\varphi(\sigma))\varphi'(\sigma) - \Psi_\sigma(\sigma,\rho(\sigma)) - \Psi_q(\sigma)\partial_\sigma\rho(\sigma). \tag{6.2.14}$$

Step 2: Proof of the bounds given in item 2 of Definition 6.2.1
It remains to show that the bounds on ρ and $\partial_\sigma\rho$ stated in Definition 6.2.1 hold true. To this end, the following claims are proven:

Claim 6.2.6 *It holds* $\|\rho\|_{C([0,1])} < \frac{K_0(\alpha,\Phi^*,\eta)}{3}$.

Proof of the claim. By the identity (6.2.1), we obtain

$$\|\rho\|_{C([0,1])} \le \|\rho(n_\Lambda + \cot\alpha\eta\tau_\Lambda)\|_{C([0,1])} = \|f_0\circ\varphi - \Phi^*\|_{C([0,1])}.$$

As reparametrization does not change the $\|\cdot\|_{C([0,1])}$-norm, it follows

$$\|\rho\|_{C([0,1])} \leq \|f_0 - \Phi^* \circ g_1\|_{C([0,1])} < \frac{\lambda C_\alpha \sqrt{1+(\cot\alpha)^2}}{\|\kappa_\Lambda\|_{C^0[0,1]}}, \tag{6.2.15}$$

where (6.2.12) is used for the last inequality. By assumption (6.2.3), the inequality

$$\lambda C_\alpha \sqrt{1+(\cot\alpha)^2} < \frac{1}{6\left(1+(\cot\alpha)^2 + \hat{C}|\cot\alpha|\|\eta'\|_{C([0,1])}\right)},$$

is inferred and therefore the claim. $\square$

Claim 6.2.7 *In the case $\alpha \neq \frac{\pi}{2}$, it holds $\|\partial_\sigma \rho\|_{C([0,1])} < \frac{K_1(\alpha,\Phi^*)}{3}$.*

Proof. By taking the inner product of identity (6.2.14) with $R\partial_\sigma f_0(\varphi(\sigma))$, where R is the matrix which rotates vectors by $\pi/2$ counterclockwise, we have

$$\langle R\partial_\sigma f_0(\varphi(\sigma)), \Psi_q(\sigma,\rho(\sigma))\rangle \partial_\sigma\rho(\sigma) = -\langle R\partial_\sigma f_0(\varphi(\sigma)), \Psi_\sigma(\sigma,\rho(\sigma))\rangle.$$

Taking into account $\langle n_{\Gamma_0}(\tilde{\sigma}), \partial_q \Psi(g_1(\tilde{\sigma}))\rangle > 1/2$, see proof of Claim 6.2.5, we obtain

$$\partial_\sigma\rho(\sigma) = -\frac{\langle R\partial_\sigma f_0(\varphi(\sigma)), \Psi_\sigma(\sigma,\rho(\sigma))\rangle}{\langle R\partial_\sigma f_0(\varphi(\sigma)), \Psi_q(\sigma,\rho(\sigma))\rangle} = -\frac{\langle n_{\Gamma_0}(\varphi(\sigma)), \Psi_\sigma(\sigma,\rho(\sigma))\rangle}{\langle n_{\Gamma_0}(\varphi(\sigma)), \Psi_q(\sigma,\rho(\sigma))\rangle}, \tag{6.2.16}$$

thus,

$$\|\partial_\sigma\rho\|_{C([0,1])} \leq 2\left\|\langle n_{\Gamma_0}(\varphi(\cdot)), \Psi_\sigma(\cdot,\rho(\cdot))\rangle\right\|_{C([0,1])}. \tag{6.2.17}$$

In the following, we use

$$\Psi_\sigma(\sigma,\rho(\sigma)) = \mathcal{L}[\Phi^*]\left(1 - \rho\kappa_\Lambda + \rho\cot\alpha\frac{\eta'(\sigma)}{\mathcal{L}[\Phi^*]}\right)\tau_\Lambda(\sigma) + \left(\mathcal{L}[\Phi^*]\rho\cot\alpha\eta(\sigma)\kappa_\Lambda\right)n_\Lambda(\sigma),$$

see the calculation in the proof of Lemma 6.2.3. First, we show that

$$\langle n_{\Gamma_0}(\varphi(\sigma)), \tau_\Lambda(\sigma)\rangle = \sqrt{1 - \langle n_{\Gamma_0}(\varphi(\sigma)), n_\Lambda(\sigma)\rangle^2} \tag{6.2.18}$$

is small. To this end, we use again the representation

$$\langle n_{\Gamma_0}(\varphi(\sigma)), n_\Lambda(\sigma)\rangle = \frac{1}{2}\big(2 - |n_{\Gamma_0}(\varphi(\sigma)) - n_\Lambda(\sigma)|\big). \tag{6.2.19}$$

By triangle inequality, it follows

$$|n_{\Gamma_0}(\varphi(\sigma)) - n_\Lambda(\sigma)| \leq |n_{\Gamma_0}(\varphi(\sigma)) - n_\Lambda(\varphi(\sigma))| + |n_\Lambda(\varphi(\sigma)) - n_\Lambda(\sigma)| = I + II.$$

We observe that we already derived bounds on both summands previously, cf. (6.2.9) and (6.2.13). More precisely, we obtain by Condition 2 in Lemma 6.2.4 and assumption (6.2.2) that

$$I < \frac{1}{(4\cdot 144)^2(\cot\alpha)^2} \qquad \text{and} \qquad II < \frac{1}{(4\cdot 144)^2(\cot\alpha)^2}.$$

Consequently, it holds

$$|n_{\Gamma_0}(\varphi(\sigma)) - n_\Lambda(\sigma)| < \frac{1}{2(4\cdot 144)^2(\cot\alpha)^2} =: \gamma$$

and by plugging this into (6.2.19)

$$\langle n_{\Gamma_0}(\varphi(\sigma)), n_\Lambda(\sigma)\rangle > \frac{1}{2}\,(2-2\gamma) = 1-\gamma.$$

Finally, we deduce by (6.2.18)

$$|\langle n_{\Gamma_0}(\varphi(\sigma)), \tau_\Lambda(\sigma)\rangle| < \sqrt{1-(1-\gamma)^2} = \sqrt{2\gamma-\gamma^2} < \sqrt{2\gamma} = \frac{1}{2\cdot 144|\cot\alpha|}.$$

This yields

$$|\langle n_{\Gamma_0}(\varphi(\sigma)), \Psi_\sigma(\sigma,\rho(\sigma))\rangle| < \left|\mathcal{L}[\Phi^*]\left(1-\rho\kappa_\Lambda+\rho\cot\alpha\frac{\eta'(\sigma)}{\mathcal{L}[\Phi^*]}\right)\right|\sqrt{2\gamma} + |\mathcal{L}[\Phi^*]\rho\cot\alpha\eta(\sigma)\kappa_\Lambda|$$

Using the bound on ρ in (6.2.15), we infer that

$$|\langle n_{\Gamma_0}(\varphi(\sigma)), \Psi_\sigma(\sigma,\rho(\sigma))\rangle| < \mathcal{L}[\Phi^*]2\sqrt{2\gamma} + \mathcal{L}[\Phi^*]\lambda = \frac{\mathcal{L}[\Phi^*]}{144|\cot\alpha|} + \mathcal{L}[\Phi^*]\lambda.$$

The claim follows by the assumption (6.2.3). $\square$

Finally, the bound on $\|\rho\|_{W_2^\beta((0,1))}$ is shown.

Claim 6.2.8 *$\|\rho\|_{W_2^\beta((0,1))}$ is bounded by a constant, which depends on α, Φ^*, η and $\|f_0\|_{W_2^\beta((0,1);\mathbb{R}^2)}$.*

Proof of the claim. First, we observe that $4(\mu - 1/2) \in (3/2, 2]$ for $\mu \in (7/8, 1]$. Using the characterization of Slobodetskii spaces in item 6 of Remark 2.1.6, we have for $s \in (1,2)$ that

$$u \in W_2^s(I) \qquad\Leftrightarrow\qquad u \in W_2^1(I) \wedge u' \in W_2^{s^*}(I) \text{ for } s^* := s - \lfloor s\rfloor \in (0,1).$$

By the Claims 6.2.6 and 6.2.7, we obtain bounds on $\|\rho\|_{C([0,1])}$ for arbitrary $\alpha \in (0,\pi)$ and on $\|\partial_\sigma\rho\|_{C([0,1])}$ for $\alpha \in (0,\pi)\backslash\{\pi/2\}$. Note that in the case $\alpha = \pi/2$, we obtain by (6.2.17) the estimate for $\|\partial_\sigma\rho\|_{C([0,1])}$. Thus it suffices to show that

- for $4(\mu - 1/2) \in (3/2, 2)$ the semi-norm $[\partial_\sigma\rho]_{W_2^{s^*}((0,1))}$ is bounded by a suitable constant,
- for $4(\mu - 1/2) = 2$ the norm $\left\|\partial_\sigma^2\rho\right\|_{L_2((0,1))}$ is bounded by a suitable constant.

To this end, we use equality (6.2.16): We already proved that the denominator is bounded from below, see proof of Claim 6.2.5. Moreover, we observe that $W_2^a((0,1)) \hookrightarrow C([0,1])$ for $a \in (1/2, 1]$ by Proposition 2.10 in [24]. Then it follows analogously to the proof of item 2 in Lemma 2.1.17, that $W_2^{s^*}(I)$, $s^* \in (1/2, 1)$ is closed under multiplication. Moreover, $W_2^1((0,1))$ is also a Banach algebra. Additionally, by item 4 in Lemma 2.1.17, it holds $1/f \in W_2^{s^*}(I)$, $s^* \in (1/2,1)$, if $f \in W_2^{s^*}(I)$ and f is bounded away from zero. The analogous result holds for $f \in W_2^1(I)$. The claim follows by combining these results. $\square$

This proves Lemma 6.2.4. $\square$

There is one last step to conclude Theorem 6.2.2 from Lemma 6.2.4: We have to show that we can substitute Condition 1 and 2 in Lemma 6.2.4 by conditions on the C^0-difference of f_0 and Φ^*, and on the one of their first derivatives. Note that Condition 2 is already in a convenient form. We observe that Condition 1 cannot be achieved directly by choosing the difference small enough, as it cannot rule out that the initial curve is negative in a neighborhood of the boundary points, cf. 6.2.

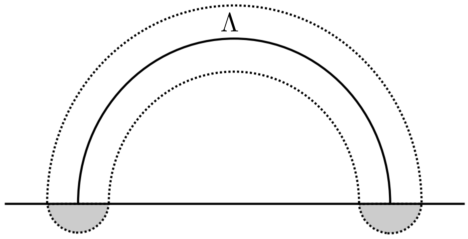

Figure 6.2: We have to rule out that the initial curve is in the grey areas of the C^0-neighborhood of the reference curve Λ.

The next lemma solves this problem:

Lemma 6.2.9
Let Φ^* and f_0 fulfill the assumptions of Theorem 6.2.2. Then $[f_0(\sigma)]_2 > 0$ for $\sigma \in (0, x) \cup (y, 1)$, where $x := \min\{\sigma \in [0,1] \mid [\Phi^*(\sigma)]_2 = \xi_0\}$ and $y := \max\{\sigma \in [0,1] \mid [\Phi^*(\sigma)]_2 = \xi_0\}$. In particular, the Conditions 1 and 2 in Lemma 6.2.4 are fulfilled.

Proof of Lemma 6.2.9. The proof of the lemma is split into two parts: First we show that a condition on the L_∞-distance of f_0 and Φ^* in terms of $\|\kappa_\Lambda\|_{C([0,1])}$ is sufficient to guarantee $f_0([0,1]) \subset \Psi([0,1] \times (-\lambda d, \lambda d)) \cup \bigcup_{i=0,1} B_{\bar{d}}^{C^0}(\Phi^*(i))$, which is a first step to replace Condition 1 of Lemma 6.2.4. The remaining part is done afterwards: it can be observed in Figure 6.2 that we have to assure that the initial curve is not contained in the grey parts in Figure 6.2.

W.l.o.g. we can assume that $\alpha \in (0, \pi/2)$: the handling of the case $\alpha \in (\pi/2, \pi)$ will be the same as the first one, as they are symmetric. Moreover, in the case $\alpha = \pi/2$ the bound is just given by λd, as $\cot \pi/2 = 0$.

Now let $\sigma \in [0,1]$ be arbitrary but fixed. By rotation, we can assume that the tangent vector of $\Phi^*(\sigma)$ is horizontal. Due to the curvature bounds on the reference curve Φ^*, we deduce that the curve is in the complement of two circles with radius $r := (\|\kappa_\Lambda\|_{C([0,1])})^{-1}$ touching at $\Phi^*(\sigma)$, cf. Figure 6.3.

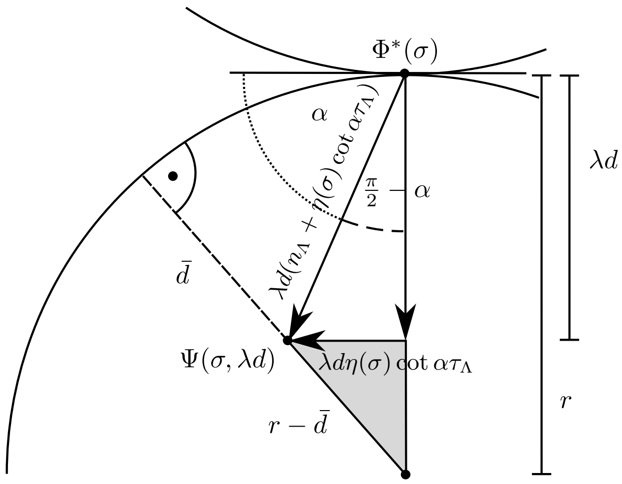

Figure 6.3: Estimation of the C^0-neighborhood.

This implies that the distance of $\Psi(\sigma, \lambda d)$ to the curve $\Phi^*([0,1])$ can be bounded from below by the distance of $\Psi(\sigma, \lambda d)$ to the circle around, which we denote by $\bar{d}$. In order to quantify $\bar{d}$, we use elementary geometry on the grey triangle in Figure 6.3. By Pythagoras' Theorem, we have

$$(r-\bar{d})^2 = (\lambda d\eta(\sigma)\cot\alpha)^2 + (r-\lambda d)^2 \qquad \Leftrightarrow \qquad \bar{d} = r - \sqrt{(\lambda d\eta(\sigma)\cot\alpha)^2 + (r-\lambda d)^2}.$$

Using the definition of d in (6.2.4), we obtain

$$\bar{d} \geq r\left(1 - \sqrt{(\lambda C_\alpha \cot\alpha)^2 + (1-\lambda C_\alpha)^2}\right) = r\overline{C_\alpha(\lambda)}.$$

These considerations are a starting point to replace Condition 1 by an C^0-condition: If the f_0 and Φ^* fulfill $|f_0(\sigma) - \Phi^*(\sigma)| < \bar{d}$ for all $\sigma \in [0,1]$, which holds due to Condition 2, then it follows $f_0([0,1]) \subset \Psi([0,1] \times (-\lambda d, \lambda d)) \cup \bigcup_{i=0,1} B_{\bar{d}}^{C^0}(\Phi^*(i))$.

In order to satisfy Condition 1 it remains to prove that $[f_0(\sigma)]_2$ is non negative in a neighbourhood of the boundary points, more precisely for $\sigma \in (0,x) \cup (y,1)$, where $x := \min\{\sigma \in [0,1] \mid [\Phi^*(\sigma)]_2 = \xi_0\}$ and $y := \max\{\sigma \in [0,1] \mid [\Phi^*(\sigma)]_2 = \xi_0\}$. Since the situations at the boundary points are the same, we concentrate on the boundary point $\sigma = 0$. By direct estimates it follows

$$\begin{aligned}
[\Phi^*(\sigma)]_2 &= \int_0^\sigma [\partial_\sigma \Phi^*(\tilde{\sigma})]_2 \,\mathrm{d}\tilde{\sigma} = \int_0^\sigma \int_0^{\tilde{\sigma}} \left[\partial_\sigma^2 \Phi^*(\bar{\sigma})\right]_2 \mathrm{d}\bar{\sigma} + [\partial_\sigma \Phi^*(0)]_2 \,\mathrm{d}\tilde{\sigma} \\
&= \int_0^\sigma \int_0^{\tilde{\sigma}} [\partial_\sigma^2 \Phi^*(\bar{\sigma})]_2 \,\mathrm{d}\bar{\sigma}\,\mathrm{d}\tilde{\sigma} + \sin\alpha \mathcal{L}[\Phi^*]\sigma \\
&\geq -\int_0^\sigma \int_0^{\tilde{\sigma}} \|\kappa_\Lambda\|_{C([0,1])} (\mathcal{L}[\Phi^*])^2 \,\mathrm{d}\bar{\sigma}\,\mathrm{d}\tilde{\sigma} + \sin\alpha \mathcal{L}[\Phi^*]\sigma \\
&= -\frac{\sigma^2}{2} \|\kappa_\Lambda\|_{C([0,1])} (\mathcal{L}[\Phi^*])^2 + \sin\alpha \mathcal{L}[\Phi^*]\sigma,
\end{aligned}$$

where it was used that

$$\partial_\sigma \Phi^*(0) = \mathcal{L}[\Phi^*] \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}.$$

We deduce

$$[\Phi^*(\sigma)]_2 > \xi_0, \qquad\qquad \text{if } -\frac{\sigma^2}{2}\|\kappa_\Lambda\|_{C([0,1])} (\mathcal{L}[\Phi^*])^2 + \sin\alpha \mathcal{L}[\Phi^*]\sigma > \xi_0.$$

By using the quadratic formula, the previous inequality holds true for $\sigma \in (x_-, x_+)$, where

$$x_\mp = \frac{\sin\alpha \mathcal{L}[\Phi^*] \mp \sqrt{(\sin\alpha \mathcal{L}[\Phi^*])^2 - 2\|\kappa_\Lambda\|_{C([0,1])} (\mathcal{L}[\Phi^*])^2 \xi_0}}{\|\kappa_\Lambda\|_{C([0,1])} (\mathcal{L}[\Phi^*])^2}.$$

For $\xi_0 < (\sin\alpha)^2 (2\|\kappa_\Lambda\|_{C([0,1])})^{-1}$, we obtain by using $(a^2 - b^2) > (a-b)^2$ for $a > b > 0$ to the argument of the square root that

$$x_- < \frac{\sqrt{2\xi_0}}{\sqrt{\|\kappa_\Lambda\|_{C([0,1])}} \mathcal{L}[\Phi^*]} := \bar{x}.$$

Thus, it suffices to show $[f(\sigma)]_2 > 0$ for $\sigma \in (0, \bar{x})$: By the fundamental theorem of calculus, we have

$$\begin{aligned}[f(\sigma)]_2 &= \int_0^\sigma [\partial_\sigma f_0(\tilde\sigma)]_2 \,\mathrm{d}\tilde\sigma \geq \int_0^\sigma \left([\partial_{\tilde\sigma}\Phi^*(\tilde\sigma)]_2 - \xi_1\right) \mathrm{d}\tilde\sigma \\ &\geq -\frac{\sigma^2}{2}\|\kappa_\Lambda\|_{C([0,1])}(\mathcal{L}[\Phi^*])^2 + \sin\alpha\mathcal{L}[\Phi^*]\sigma - \xi_1\sigma,\end{aligned}$$

where we used $[f_0(0)]_2 = 0$, $\partial_\sigma f_0 \in B^{C^0}_{\xi_1}(\partial_\sigma\Phi^*)$, and for the last line the same argument as for $[\Phi^*(\sigma)]_2$. The roots of the equation

$$-\frac{\sigma^2}{2}\|\kappa_\Lambda\|_{C([0,1])}(\mathcal{L}[\Phi^*])^2 + \sin\alpha\mathcal{L}[\Phi^*]\sigma - \xi_1\sigma = 0$$

are given by

$$z_- = 0 \qquad \text{and} \qquad z_+ = \frac{2(\mathcal{L}[\Phi^*]\sin\alpha - \xi_1)}{\|\kappa_\Lambda\|_{C([0,1])}(\mathcal{L}[\Phi^*])^2}.$$

Note that z_+ is positive as $\xi_1 < \mathcal{L}[\Phi^*]\sin\alpha$. Thus, $[f(\sigma)]_2 > 0$ for $\sigma \in (z_-, z_+) = (0, z_+)$ and it remains to prove that

$$\begin{aligned} z+ > \bar{x} \qquad &\Leftrightarrow \qquad \frac{2(\mathcal{L}[\Phi^*]\sin\alpha - \xi_1)}{\|\kappa_\Lambda\|_{C([0,1])}(\mathcal{L}[\Phi^*])^2} > \frac{\sqrt{2\xi_0}}{\sqrt{\|\kappa_\Lambda\|_{C([0,1])}}\mathcal{L}[\Phi^*]} \\ &\Leftarrow \qquad \left(\mathcal{L}[\Phi^*]\sin\alpha > \sqrt{2\xi_0\|\kappa_\Lambda\|_{C([0,1])}}\mathcal{L}[\Phi^*]\right) \wedge \left(\mathcal{L}[\Phi^*]\sin\alpha > 2\xi_1\right).\end{aligned}$$

But these inequalities follow by the choice of ξ_0 and ξ_1.

The combination of this argument with the result from the first part shows that Condition 1 holds true. As Condition 2 is fulfilled by assumption, the proof is complete. □

Proof of Theorem 6.2.2. The claim follows from the Lemmas 6.2.4 and 6.2.9. □

6.3 Some Technical Estimates

In order to prove that the smoothed curves constructed in Section 6.1 can be used as reference curves, we need the following estimates:

Lemma 6.3.1
Let $f_0 : \bar{I} \to \mathbb{R}^2$, $I := (0,1)$, be parametrized proportional to arc length and in $W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)$ for $\mu \in \left(\frac{7}{8}, 1\right]$. Moreover, let $f(t) = f(t,\cdot) \in C^5(\bar{I};\mathbb{R}^2)$, $t \in (0,T)$, be the curves given by Lemma 6.1.2, cf. Lemma 6.1.4 for the regularity. Then it holds for $\frac{1}{6} > \delta > 0$

$$\|f(\epsilon) - f_0\|_{C(\bar{I};\mathbb{R}^2)} \leq C\left(\epsilon^{\frac{2}{3}\mu - \frac{5}{12} - \delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{11}{12}-\delta}\|h\|_{L_2(I;\mathbb{R}^2)}\right), \tag{6.3.1}$$

$$\|f(\epsilon)\|_{C^2(\bar{I};\mathbb{R}^2)} \leq C\left(\epsilon^{-\frac{3}{4} + \frac{2}{3}\mu - \delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{7}{12}-\delta}\|h\|_{L_2(I;\mathbb{R}^2)} + \|\xi\|_{C^2(\bar{I};\mathbb{R}^2)}\right). \tag{6.3.2}$$

Additionally, we have for a sufficiently small $\delta > 0$

$$\|f(\epsilon) - f_0\|_{C^1(\bar{I};\mathbb{R}^2)} \leq C\left(\epsilon^{\frac{2}{3}\mu - \frac{7}{12} - \delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{3}{4}-\delta}\|h\|_{L_2(I;\mathbb{R}^2)}\right). \tag{6.3.3}$$

Here, ξ, $u_0 := f_0 - \xi$ and $h := \partial_x^6\xi$ are quantities determined by f_0, see (6.1.7).

Proof. Since we want to take advantage of the properties of the analytic C_0-semigroup generated by $-A$, we switch again to the Cauchy problem which is equivalent to (6.1.1), cf. the proof of Lemma 6.1.4. More precisely, we consider the abstract Cauchy problem

$$\begin{aligned} \partial_t u(t) + Au(t) &= h(t) && \text{for } t \in (0,T) \\ u_{|t=0} &= u_0, \end{aligned} \tag{6.1.9}$$

where $A = -\partial_x^6 : D(A) \to X$ with $X := L_2(I;\mathbb{R}^2)$ and the domain

$$D(A) := \left\{ u \in W_2^6(I;\mathbb{R}^2) : u_{|\partial I} = 0, \partial_x u_{|\partial I} = 0, \partial_x^2 u_{|\partial I} = 0 \right\},$$

for $u_0 = f_0 - \xi$ and $h(t) = h = \partial_x^6 \xi$, $\xi \in C^\infty(\bar{I};\mathbb{R}^2)$ is given by (6.1.7). As $-A$ generates an analytic C_0-semigroup, cf. Claim 6.1.5, the mild solution formula can be exploited, see proof of Claim 6.1.6,

$$v(t) = e^{-tA}u_0 + \int_0^t e^{-sA} h \, \mathrm{d}s.$$

Additionally, we will use the following characterization of $D(A)$: By Lemma 6.1.7, we know that the norms $\|\cdot\|_{W_2^6(I;\mathbb{R}^2)}$ and $\|\cdot\|_{D(A)}$ are equivalent on $D(A)$. We can consider the components of u separately, as they are not coupled by the equations. Applying Section 4.3.3 in [30] on both components of u, we obtain the representation

$$D(A) = B^6_{2,2,\{B_j\}}(I;\mathbb{R}^2) := \left\{ f \in B^6_{2,2}(I;\mathbb{R}^2) : B_j f_{|\partial I} = 0 \text{ for } j < s - \frac{1}{2} \right\}, \tag{6.3.4}$$

where B_j are the differential operators given by

$$B_j f := b_j \partial_x^j f \qquad \text{for } b_j = \begin{cases} 1 & \text{for } j = 0,1,2, \\ 0 & \text{for } j = 3,4,5. \end{cases}$$

In the following, our main tool will be Proposition 2.2.9(i) in [22], which is stated here in a notation adjusted to our problem, as $-A$ (and not A) is the generator of the semigroup:
Let (α,p), $(\beta,p) \in (0,1) \times [1,\infty] \cup \{(1,\infty)\}$, and let $n \in \mathbb{N}$. Then there are constants $C = C(n,p,\alpha,\beta)$ such that

$$\left\| t^{n-\alpha+\beta}(-A)^n e^{-tA} \right\|_{L(D_A(\alpha,p),D_A(\beta,p))} \le C \qquad \text{for } 0 < t \le 1. \tag{6.3.5}$$

The statement also holds for $n = 0$, provided $\alpha \le \beta$. Moreover, we will use an inequality in the proof of the latter proposition: For $\alpha = 0$ - we set $D_A(\alpha,p) = X$, cf. Remark before Proposition 2.2.9 in [22] - we have

$$\left\| t^{n+\beta}(-A)^n e^{-tA} \right\|_{L(X,D_A(\beta,p))} \le C \qquad \text{for } 0 < t \le 1. \tag{6.3.6}$$

We start proving the estimate (6.3.1). Clearly, it holds

$$f(\epsilon) - f_0 = (f(\epsilon) - \xi) - (f_0 - \xi) = v(\epsilon) - u_0.$$

Applying item 3 of Lemma 2.1.15, we have for $\delta > 0$

$$W_2^{1/2+6\delta}\left(I;\mathbb{R}^2\right) \hookrightarrow C^\gamma\left(\bar{I};\mathbb{R}^2\right) \hookrightarrow C\left(\bar{I};\mathbb{R}^2\right),$$

for $0 < \gamma < 6\delta$. Thus, it follows

$$\|f(\epsilon) - f_0\|_{C(\bar{I};\mathbb{R}^2)} \le C \|v(\epsilon) - u_0\|_{W_2^{1/2+6\delta}(I;\mathbb{R}^2)}.$$

Choosing $\beta = 1/12 + \delta$, for an arbitrary $1/6 > \delta > 0$, we obtain

$$D_A(\beta,2) = (X, D(A))_{\beta,2} = \left\{ u \in W_2^{1/2+6\delta}\left(I;\mathbb{R}^2\right) : u_{|\partial I} = 0 \right\} \tag{6.3.7}$$

with equivalent norms. Here we used Proposition 2.2.2 in [22] for the first equality and the combination of (6.3.4) and the Theorem in Section 4.3.3 of [30] for the second identity. Consequently, we have

$$\|f(\epsilon) - f_0\|_{C(\bar{I};\mathbb{R}^2)} \leq C\|v(\epsilon) - u_0\|_{D_A(\beta,2)}.$$

Using the mild solution formula and the triangle inequality, we obtain

$$\begin{aligned} \|f(\epsilon) - f_0\|_{C(\bar{I};\mathbb{R}^2)} &\leq C\left(\left\|e^{-\epsilon A}u_0 - u_0\right\|_{D_A(\beta,2)} + \left\| \int_0^\epsilon e^{-sA} h \,\mathrm{d}s \right\|_{D_A(\beta,2)} \right) \\ &= C\left(\left\| \int_0^\epsilon -Ae^{-sA}u_0 \,\mathrm{d}s \right\|_{D_A(\beta,2)} + \left\| \int_0^\epsilon e^{-sA} h \,\mathrm{d}s \right\|_{D_A(\beta,2)} \right) = C(I + II), \end{aligned}$$

where we used $e^{-\epsilon A}u_0 - u_0 = \int_0^\epsilon -Ae^{-sA}u_0 \,\mathrm{d}s$, cf. Proposition 2.1.4 (ii) in [22]. For the first summand, we obtain by (6.3.5) in case $n = 1$, $\beta = 1/12 + \delta$, $\alpha = \tilde{\mu} - 1/2$

$$\begin{aligned} I \leq \int_0^\epsilon \left\|Ae^{-sA}u_0\right\|_{D_A(\beta,2)} \mathrm{d}s &\leq \int_0^\epsilon \left\|Ae^{-sA}\right\|_{L(D_A(\alpha,2),D_A(\beta,2))} \|u_0\|_{D_A(\alpha,2)} \,\mathrm{d}s \\ &\leq \int_0^\epsilon \frac{C}{s^{1-(\tilde{\mu}-1/2)+(1/12+\delta)}} \|u_0\|_{D_A(\alpha,2)} \,\mathrm{d}s. \end{aligned}$$

Here, it follows analogously to the explanation of (6.3.7) that

$$\begin{aligned} &\left\{ u \in W_2^{4(\mu-1/2)}(I;\mathbb{R}^2) : \ u_{|\partial I} = 0, \partial_x u_{|\partial I} = 0 \right\} \\ &\qquad = \left\{ u \in W_2^{6(\tilde{\mu}-1/2)}(I;\mathbb{R}^2) : \ u_{|\partial I} = 0, \partial_x u_{|\partial I} = 0 \right\} = D_A(\alpha, 2) \end{aligned}$$

with equivalent norms for $\mu \in (7/8, 1]$ and $\tilde{\mu} \in (3/4, 5/6]$ with $\tilde{\mu} = 2/3\mu + 1/6$, respectively. Since we have

$$\left(\tilde{\mu} - \frac{1}{2}\right) - \left(\frac{1}{12} + \delta\right) = \tilde{\mu} - \frac{7}{12} - \delta > \frac{3}{4} - \frac{7}{12} - \frac{1}{6} = 0 \qquad \text{for } 0 < \delta < 1/6,$$

and equivalently $1 - (\tilde{\mu} - 1/2) + (1/12 + \delta) < 1$, we can integrate with respect to s and obtain

$$I \leq C\epsilon^{\tilde{\mu}-\frac{7}{12}-\delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)}.$$

Using (6.3.6) in the case $n = 0$, $\beta = 1/12 + \delta$, $\alpha = 0$, we deduce for the second summand

$$\begin{aligned} II &\leq \int_0^\epsilon \left\|e^{-sA}h\right\|_{D_A(\beta,2)} \mathrm{d}s \leq \int_0^\epsilon \left\|e^{-sA}\right\|_{L(X,D_A(\beta,2))} \|h\|_X \,\mathrm{d}s \leq \int_0^\epsilon \frac{C}{s^\beta}\|h\|_X \,\mathrm{d}s \\ &\leq C\epsilon^{\frac{11}{12}-\delta}\|h\|_X. \end{aligned}$$

In summary, we obtain

$$\|f(\epsilon) - f_0\|_{C(\bar{I};\mathbb{R}^2)} \leq C\left(\epsilon^{\tilde{\mu}-\frac{7}{12}-\delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{11}{12}-\delta}\|h\|_X\right),$$

and, by $\tilde{\mu}(\mu) = 2/3\mu + 1/6$,

$$\|f(\epsilon) - f_0\|_{C(\bar{I};\mathbb{R}^2)} \leq C\left(\epsilon^{\frac{2}{3}\mu - \frac{5}{12} - \delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{11}{12}-\delta}\|h\|_X\right),$$

where $2/3\mu - 5/12 - \delta > 2/3 \cdot 7/8 - 5/12 - \delta = 1/6 - \delta > 0$ for $\mu \in (7/8, 1]$.

We proceed similarly for the second estimate (6.3.3): Again, we use

$$D_A(\beta, 2) = \left\{u \in W_2^{3/2+6\delta}\left(I;\mathbb{R}^2\right) : \ u_{|\partial I} = 0, \partial_x u_{|\partial I} = 0\right\} \hookrightarrow C^1\left(\bar{I};\mathbb{R}^2\right) \tag{6.3.8}$$

for $\beta = 1/4+\delta$, where $1/6 > \delta > 0$ arbitrary. The first identity follows like (6.3.7) and the embedding holds due to the Remark 2.1.16. Consequently, we have

$$\begin{aligned}\|f(\epsilon) - f_0\|_{C^1(\bar{I};\mathbb{R}^2)} &< C\left(\left\|\int_0^\epsilon -Ae^{-sA}u_0 \,\mathrm{d}s\right\|_{D_A(\beta,2)} + \left\|\int_0^\epsilon e^{-sA}h \,\mathrm{d}s\right\|_{D_A(\beta,2)}\right)\\ &= C\left(\overline{I} + \overline{II}\right).\end{aligned}$$

Using (6.3.5) in case $n = 1$, $\beta = 1/4 + \delta$, $\alpha = \tilde{\mu} - 1/2$, we obtain

$$\begin{aligned}\overline{I} &\leq \int_0^\epsilon \left\|Ae^{-sA}u_0\right\|_{D_A(\beta,2)} \,\mathrm{d}s \leq \int_0^\epsilon \left\|Ae^{-sA}\right\|_{L(D_A(\alpha,2),D_A(\beta,2))} \|u_0\|_{D_A(\alpha,2)} \,\mathrm{d}s\\ &\leq \int_0^\epsilon \frac{C}{s^{1-(\tilde{\mu}-1/2)+(1/4+\delta)}}\|u_0\|_{D_A(\alpha,2)} \,\mathrm{d}s.\end{aligned}$$

Since we have

$$\left(\tilde{\mu} - \frac{1}{2}\right) - \frac{1}{4} = \tilde{\mu} - \frac{9}{12} > \frac{3}{4} - \frac{3}{4} = 0,$$

a sufficiently small $\delta > 0$ can be chosen, such that $1 - (\tilde{\mu} - 1/2) + 1/4 + \delta < 1$ holds true. Thus, we can integrate with respect to s and obtain

$$\overline{I} \leq C\epsilon^{\tilde{\mu} - \frac{3}{4} - \delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)},$$

where $\tilde{\mu} - 3/4 - \delta > 0$. Using (6.3.6) in case $n = 0$, $\beta = 1/4 + \delta$, $\alpha = 0$, we infer for the second summand

$$\begin{aligned}\overline{II} &\leq \int_0^\epsilon \left\|e^{-sA}h\right\|_{D_A(\beta,2)} \,\mathrm{d}s \leq \int_0^\epsilon \left\|e^{-sA}\right\|_{L(X,D_A(\beta,2))} \|h\|_X \,\mathrm{d}s \leq \int_0^\epsilon \frac{C}{s^\beta}\|h\|_X \,\mathrm{d}s\\ &\leq C\epsilon^{\frac{3}{4}-\delta}\|h\|_X.\end{aligned}$$

Thus, it follows

$$\|f(\epsilon) - f_0\|_{C^1(\bar{I};\mathbb{R}^2)} \leq C\left(\epsilon^{\tilde{\mu} - \frac{3}{4} - \delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{3}{4}-\delta}\|h\|_X\right),$$

and, by $\tilde{\mu}(\mu) = \frac{2}{3}\mu + \frac{1}{6}$,

$$\|f(\epsilon) - f_0\|_{C^1(\bar{I};\mathbb{R}^2)} \leq C\left(\epsilon^{\frac{2}{3}\mu - \frac{7}{12} - \delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{3}{4}-\delta}\|h\|_X\right).$$

Note that the power of ϵ in the first summand is positive for our choice of δ.

For the proof of the inequality (6.3.2) we employ the same techniques: By adding a zero it holds

$$f(\epsilon) = (f(\epsilon) - \xi) + \xi = v(\epsilon) + \xi.$$

Thus, we consider

$$\begin{aligned} \|f(\epsilon)\|_{C^2(\bar{I};\mathbb{R}^2)} = \|v(\epsilon) + \xi\|_{C^2(\bar{I};\mathbb{R}^2)} &\leq \left\|e^{-\epsilon A}u_0\right\|_{C^2(\bar{I};\mathbb{R}^2)} + \left\|\int_0^\epsilon e^{-sA} h \,\mathrm{d}s\right\|_{C^2(\bar{I};\mathbb{R}^2)} \\ &+ \|\xi\|_{C^2(\bar{I};\mathbb{R}^2)} = \widetilde{I} + \widetilde{II} + \|\xi\|_{C^2(\bar{I};\mathbb{R}^2)}. \end{aligned}$$

In order to estimate the summand $\widetilde{I}$, we choose $\beta = 5/12 + \delta$, such that

$$D_A(\beta,2) = \left\{u \in W_2^{6\beta}\left(I;\mathbb{R}^2\right) : u_{|\partial I} = 0, \partial_x u_{|\partial I} = 0, \partial_x^2 u_{|\partial I} = 0\right\} \hookrightarrow C^2\left(\bar{I};\mathbb{R}^2\right)$$

with equivalent norms for arbitrary $1/6 > \delta > 0$, which follows again like (6.3.8). As before, we set $\alpha = \tilde{\mu} - 1/2$, which implies $\alpha \in (1/4, 1/3]$. Using (6.3.5) for $n = 0$ and $\beta > \alpha$, we obtain

$$\begin{aligned} \widetilde{I} &= \left\|e^{-\epsilon A}u_0\right\|_{C^2(\bar{I};\mathbb{R}^2)} \leq C\left\|e^{-\epsilon A}u_0\right\|_{D_A(\beta,2)} \\ &\leq C\left\|e^{-\epsilon A}\right\|_{L(D_A(\alpha,2),D_A(\beta,2))}\|u_0\|_{D_A(\alpha,2)} \leq C\epsilon^{-\beta+\alpha}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} \\ &\leq C\epsilon^{-\frac{5}{12}-\delta+\tilde{\mu}+\frac{1}{2}}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} \leq C\epsilon^{-\frac{11}{12}+\tilde{\mu}-\delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)}. \end{aligned}$$

The second summand $\widetilde{II}$ can be treated analogously to II and we obtain

$$\widetilde{II} \leq C\epsilon^{1-\beta}\|h\|_X \leq C\epsilon^{\frac{7}{12}-\delta}\|h\|_X.$$

Finally, we deduce

$$\|f(\epsilon)\|_{C^2(\bar{I};\mathbb{R}^2)} \leq C\left(\epsilon^{-\frac{11}{12}+\tilde{\mu}-\delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{7}{12}-\delta}\|h\|_X + \|\xi\|_{C^2(\bar{I};\mathbb{R}^2)}\right),$$

and again by $\tilde{\mu}(\mu) = 2/3\mu + 1/6$,

$$\|f(\epsilon)\|_{C^2(\bar{I};\mathbb{R}^2)} \leq C\left(\epsilon^{-\frac{3}{4}+\frac{2}{3}\mu-\delta}\|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{7}{12}-\delta}\|h\|_X + \|\xi\|_{C^2(\bar{I};\mathbb{R}^2)}\right).$$

This concludes the proof. □

6.4 f_ϵ is a Reference Curve

The technical estimates from the previous section enable us to apply the results of Section 6.2 to the curves $f_\epsilon = f(\epsilon,\cdot)$, $\epsilon > 0$, derived in Section 6.1.

The main result reads as follows:

Theorem 6.4.1 (f_ϵ is a reference curve for f_0 provided ϵ is small enough)
Let $f_0 : \bar{I} \to \mathbb{R}^2$, $I := (0,1)$, have the properties as in Chapter 6.1, i.e. let f_0 be parametrized proportional to arc length, let it be in $W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)$, $\mu \in \left(\frac{7}{8},1\right]$, and let it fulfill the boundary conditions given in (3.1.6)*, i.e.*

$$f_0(x) \in \mathbb{R}\times(0,\infty) \qquad \textit{for } x \in \{0,1\},$$

$$\measuredangle\left(n_{\Gamma_0}(x), \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = \pi - \alpha \qquad \textit{for } x \in \{0,1\}, \tag{3.1.6}$$

where $\Gamma_0 := f_0(\bar{I})$. Moreover, let $f(\epsilon,\cdot) := f_\epsilon : \bar{I} \to \mathbb{R}^2$, $\epsilon > 0$, be the smoothed curves generated by evolving f_0 by a parabolic equation, see Lemma 6.1.2 and Lemma 6.1.4 in Chapter 6.1.
There exists an $\tilde{\epsilon}$, such that for f_ϵ, $0 < \epsilon < \tilde{\epsilon}$, the conditions in Theorem 6.2.2 are fulfilled for the parameterizations $f_\epsilon \circ \beta_\epsilon$ and $f_0 \circ \beta_\epsilon$. Here, $\beta_\epsilon : \bar{I} \to \bar{I}$ is the orientation preserving reparametrization such that $f_\epsilon \circ \beta_\epsilon$ is parametrized proportional to arc length. In particular, f_ϵ is a reference curve for the initial curve f_0.

Note that it is not trivial that f_ϵ is a reference curve for the initial curve f_0 for some $\epsilon > 0$: Although, by making ϵ smaller, we can diminish the C^0-distance between f_0 and f_ϵ, see (6.3.1), but the curvature κ of f_ϵ may explode, see (6.3.2). Therefore, we have to be careful, since the bound on the C^0-distance is proportional to the reciprocal of the C^0-norm of the curvature of the reference curve, cf. (6.2.4). In order to compare those two effects, we give a formulation of Lemma 6.2.2 in the "initial curve perspective", i.e. in the proportional-to-arc-length-parametrization on $[0,1]$ of the initial curve.

Lemma 6.4.2
Let f_0 and f_ϵ be as in Theorem 6.4.1. Then there exists an $\bar{\epsilon}$ such that f_ϵ, $0 < \epsilon < \bar{\epsilon}$, is a regular curve. Let $\lambda \in (0,1)$ be given such that the conditions (6.2.2) and (6.2.3) are fulfilled. If $f_\epsilon \in B^{C^0}_{\xi_0}(f_0)$ and $\partial_\sigma f_\epsilon \in B^{C^0}_{\xi_1}(\partial_\sigma f_0)$ for $0 < \epsilon < \bar{\epsilon}$ and

$$\xi_0 = \min\left\{\overline{C_\alpha(\lambda)}, \frac{(\sin\alpha)^2}{2}\right\} \frac{1}{\|\kappa[f_\epsilon]\|_{C([0,1])}},$$
$$\xi_1 = \min\left\{\frac{\sqrt{(\cot\alpha)^2+1} - |\cot\alpha|}{4}, \begin{cases} \frac{1}{2(4\cdot 144)^2|\cot\alpha|^2} & \textit{for } \alpha \neq \frac{\pi}{2} \\ 1 & \textit{for } \alpha = \frac{\pi}{2} \end{cases}, \frac{\sin\alpha}{2}\right\} K(\bar{\epsilon}, f_0),$$

where

$$K(\bar{\epsilon}, f_0) := \mathcal{L}[f_0] - C(\bar{\epsilon}) > 0$$

for $C(\epsilon) \to 0$ monotonically for $\epsilon \to 0$, and $\overline{C_\alpha(\lambda)}$ is defined in Lemma 6.2.2, then f_ϵ is a reference curve for the initial curve f_0.

Proof. The strategy is to go through the proof of Theorem 6.2.2, to reparametrize the inequalities and to replace the conditions. We recall that reparametrization of the condition for f_ϵ and f_0 does not affect the radius ξ_0. Furthermore, we set

$$\widetilde{\xi_1} := \min\left\{\frac{\sqrt{(\cot\alpha)^2+1} - |\cot\alpha|}{4}, \begin{cases} \frac{1}{2(4\cdot 144)^2|\cot\alpha|^2} & \text{for } \alpha \neq \frac{\pi}{2} \\ 1 & \text{for } \alpha = \frac{\pi}{2} \end{cases}, \frac{\sin\alpha}{2}\right\}$$

Then, the condition for the derivatives of f_ϵ and f_0 is given by

$$|\partial_\sigma(f_0 \circ \beta_\epsilon)(\sigma) - \partial_\sigma(f_\epsilon \circ \beta_\epsilon)(\sigma)| < \widetilde{\xi_1}|\partial_\sigma(f_\epsilon \circ \beta_\epsilon)(\sigma)|,$$

supposed $\beta_\epsilon : \bar{I} \to \bar{I}$ is a regular orientation preserving reparametrization such that $f_\epsilon \circ \beta_\epsilon$ is parametrized proportional to arc length. By chain rule, this is equivalent to

$$|\partial_\sigma f_0(\beta_\epsilon(\sigma)) - \partial_\sigma f_\epsilon(\beta_\epsilon(\sigma))| < \widetilde{\xi_1}|\partial_\sigma f_\epsilon(\beta_\epsilon(\sigma))|, \tag{6.4.1}$$

as $\beta'_\epsilon(\sigma) > 0$ for $\sigma \in [0,1]$. Thus, it remains to prove that f_ϵ is a regular curve and that there exists a uniform lower bound for $|\partial_\sigma f_\epsilon|$ for every $0 < \epsilon < \bar{\epsilon}$.

Claim 6.4.3 *There exists an $\bar{\epsilon}$ such that for every $0 < \epsilon < \bar{\epsilon}$ the following holds true: The function $f_\epsilon(\cdot)$ is regular with the bound*

$$K(\bar{\epsilon}, f_0) \leq |\partial_\sigma f_\epsilon(\sigma)| \tag{6.4.2}$$

and there exists a uniform lower bound on $\mathcal{L}[f_\epsilon]$

$$K(\bar{\epsilon}, f_0) \leq \mathcal{L}[f_\epsilon], \tag{6.4.3}$$

where $K(\bar{\epsilon}, f_0) := \mathcal{L}[f_0] - C(\bar{\epsilon}) > 0$ with $C(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof of the Claim: We want to use estimate (6.3.3) in Lemma 6.3.1: For $\mu \in (7/8, 1]$ it holds

$$\frac{2}{3}\mu - \frac{7}{12} > \frac{2}{3} \cdot \frac{7}{8} - \frac{7}{12} = \frac{7}{12} - \frac{7}{12} = 0,$$

thus, the first summand of the right-hand side in (6.3.3) has a positive ϵ power if δ is chosen sufficiently small. This implies that

$$\mathcal{L}[f_0] - C(\epsilon) = \min_{\sigma\in[0,1]} |\partial_\sigma f_0(\sigma)| - C(\epsilon) \leq \min_{\sigma\in[0,1]} |\partial_\sigma f_\epsilon(\sigma)|$$

with a $C(\epsilon) \to 0$ monotonically as $\epsilon \to 0$. Choosing $\bar{\epsilon}$ sufficiently small, we obtain

$$0 < \mathcal{L}[f_0] - C(\bar{\epsilon}) \leq \mathcal{L}[f_0] - C(\epsilon) \leq \min_{\sigma\in[0,1]} |\partial_\sigma f_\epsilon(\sigma)| \leq |\partial_\sigma f_\epsilon(\sigma)|$$

for $\sigma \in [0, 1]$ and all $0 < \epsilon < \bar{\epsilon}$, which shows the estimate (6.4.2) and that the parametrization is regular. Integrating the last inequality with respect to the parameter σ over $[0, 1]$, we deduce estimate (6.4.3)

$$0 < \mathcal{L}[f_0] - C(\bar{\epsilon}) = \int_{[0,1]} \mathcal{L}[f_0] - C(\bar{\epsilon}) \,\mathrm{d}\sigma \leq \int_{[0,1]} |\partial_\sigma f_\epsilon(\sigma)| \,\mathrm{d}\sigma = \mathcal{L}[f_\epsilon]$$

for all $0 < \epsilon < \bar{\epsilon}$. □

Thus, the condition on ξ_1 in Lemma 6.4.2 is stronger than (6.4.1) and the lemma is proven. □

We proceed with the proof of Theorem 6.4.1:

Proof of Theorem 6.4.1. By estimate (6.3.3), we see that $\|f(\epsilon) - f_0\|_{C^1(\bar{I};\mathbb{R}^2)} \to 0$ as $\epsilon \to 0$. Thus, by choosing $\bar{\epsilon}$ small enough, the conditions on the derivatives of f_ϵ and f_0 in Lemma 6.4.2 are fulfilled for $0 < \epsilon < \bar{\epsilon}$. It just remains to show the following claim:

Claim 6.4.4 *Let $\lambda \in (0, 1)$ be given such that the conditions (6.2.2) and (6.2.3) are fulfilled. Then there exists an $\tilde{\epsilon}$ with $0 < \tilde{\epsilon} \leq \bar{\epsilon}$, such that $f_\epsilon \in B^{C^0}_{\xi_0}(f_0)$ for $0 < \epsilon < \tilde{\epsilon}$, where ξ_0 is as in Lemma 6.4.2.*

Proof of the claim: We have to show that there exists an $\tilde{\epsilon}$, such that

$$\|f_0 - f_\epsilon\|_{C([0,1];\mathbb{R}^2)} < \min\left\{\overline{C_\alpha(\lambda)}, \frac{(\sin\alpha)^2}{2}\right\} \frac{1}{\|\kappa[f_\epsilon]\|_{C([0,1])}}$$

for $0 < \epsilon < \tilde{\epsilon}$. To this end, we use the estimate (6.3.1), i.e.

$$\|f_0 - f_\epsilon\|_{C([0,1];\mathbb{R}^2)} \leq C_1 \left(\epsilon^{\frac{2}{3}\mu - \frac{5}{12} - \delta_1} \|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{11}{12} - \delta_1} \|h\|_X \right)$$

for $0 < \delta < \frac{1}{6}$ and a rearranged version of (6.3.2) given by

$$\frac{1}{\|\kappa[f_\epsilon]\|_{C([0,1];\mathbb{R}^2)}} \geq C_2^{-1} \left(\epsilon^{-\frac{3}{4} + \frac{2}{3}\mu - \delta_2} \|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{7}{12} - \delta_2} \|h\|_X + \|\xi\|_{C^2(I;\mathbb{R}^2)} \right)^{-1},$$

where the $\delta_2 > 0$ is a sufficiently small number and the norms are finite and do not depend on ϵ. Thus, it is enough to show that there exists an $\tilde{\epsilon} \in (0, \bar{\epsilon}]$ such that the inequality

$$\left(\epsilon^{\frac{2}{3}\mu - \frac{5}{12} - \delta_1} \|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{11}{12} - \delta_1} \|h\|_X \right) \times \left(\epsilon^{-\frac{3}{4} + \frac{2}{3}\mu - \delta_2} \|u_0\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)} + \epsilon^{\frac{7}{12} - \delta_2} \|h\|_X + \|\xi\|_{C^2(I;\mathbb{R}^2)} \right) < C \tag{6.4.4}$$

is fulfilled for each $0 < \epsilon < \tilde{\epsilon}$, where $C := C_1^{-1} \min\left\{ \overline{C_\alpha(\lambda)}, {}^{(\sin\alpha)^2}/_2 \right\} C_2^{-1}$. Direct calculations show that the powers in ϵ of the first factor are both positive, and that $2/3\mu - 5/12 - \delta_1$ is the smaller one. Concerning the second factor, the first summand is the only critical one, as its power in ϵ is negative for $\mu \in (7/8, 1]$. We want to make sure that the product of these worst factors has in total a positive power. Thus, we calculate

$$\frac{2}{3}\mu - \frac{5}{12} - \left(\frac{3}{4} - \frac{2}{3}\mu \right) = \frac{4}{3}\mu - \frac{7}{6} > \frac{4}{3} \cdot \frac{7}{8} - \frac{7}{6} = 0,$$

and deduce the existence of $\delta_1, \delta_2 > 0$, such that

$$\frac{2}{3}\mu - \frac{5}{12} - \delta_1 - \left(\frac{3}{4} - \frac{2}{3}\mu + \delta_2 \right) > 0.$$

This implies that the smallest power of ϵ of the summands on the left-hand side of (6.4.4) is positive for a $\mu \in (7/8, 1]$ provided $\delta_1, \delta_2 > 0$ are sufficiently small. Consequently, there exists $\tilde{\epsilon} > 0$ such that the inequality (6.4.4) is fulfilled for each $0 < \epsilon < \tilde{\epsilon}$. □

By Claim 6.4.4, it follows that both ball conditions in Lemma 6.4.2 are fulfilled for $0 < \epsilon < \bar{\epsilon}$, if $\bar{\epsilon}$ is chosen small enough. Since the conditions in Lemma 6.4.2 are stronger than the ones from Theorem 6.2.2, the latter are also satisfied. It is a direct consequence that f_ϵ is a reference curve for the initial curve f_0. □

This enables us to prove the theorem on local well-posedness for a fixed initial curve.

Proof of Theorem 4.1.3. Let $f_0 : \bar{I} \to \mathbb{R}^2$, $I := (0,1)$ fulfill the assumption of Theorem 4.1.3. Then, we obtain by Theorem 6.4.1 a reference curve $\Phi^* = f_\epsilon \circ \beta : [0,1] \to \mathbb{R}^2$, which is parametrized proportional to arc length, and a corresponding initial height function ρ_0, which fulfill the conditions given in Definition 6.2.1. In particular, there exists a regular C^1-reparametrization $\varphi : [0,1] \to [0,1]$ and a function $\rho_0 : [0,1] \to (-d, d)$ in $W_2^{4(\mu-1/2)}(I)$, $\mu \in (7/8, 1]$, such that

$$f_0(\varphi(\sigma)) = \Phi^*(\sigma) + \rho_0(\sigma)(n_\Lambda(\sigma) + \cot\alpha\, \eta(\sigma)\tau_\Lambda(\sigma)),$$

where $\Lambda = \Phi^*([0,1])$. By Theorem 5.1.3 and Corollary 5.3.4, we obtain a solution $(t,\sigma) \mapsto \Phi(t,\sigma) := \Psi(\sigma, \rho(t,\sigma))$ to (3.1.1)-(3.1.4) with $\Phi(0,\cdot) = \Psi(\cdot, \rho_0(\cdot))$ and $\Phi \in \mathbb{E}_{\mu,T,\mathbb{R}^2}$. By construction, compare the coordinates (5.1.2) to the formula (6.2.1), we observe that $f_0(\varphi(\cdot)) = \Psi(\cdot, \rho_0(\cdot)) = \Phi(0, \rho_0(\cdot))$. This shows the existence of a solution. $\square$

7 The Proof of the Blow-up Criterion Theorem 4.1.4

In the following, we give a proof of the main result.

Proof of Theorem 4.1.4. The proof is done in three steps. By assuming, contrary to the claim, that $\|\kappa[f(t_l)]\|_{L_2(0,\mathcal{L}[f(t_l)])}$ is uniformly bounded for a sequence $(t_l)_{l\in\mathbb{N}}$, we find a uniform in time W_2^2-bound for the reparametrized and translated solution in the first step. In the second step, we use these points in the temporal trace space of the solution space as initial data in order to restart the flow over finitely many reference curves. This is achieved by a compactness argument. In this way, we establish a lower bound on the existence time of the solution, which enables us to extend the original solution in the third step. This provides a contradiction to the maximality of the solution.

Step 1: Finding a W_2^2-bound for the reparametrized and translated solution
Conversely, we assume that there exists a sequence in time $(t_l)_{l\in\mathbb{N}}$ with $t_l \to T_{max}$ as $l \to \infty$, such that $\kappa[f(t_l)] : [0, \mathcal{L}[f(t_l)]] \to \mathbb{R}$ satisfies an L_2-bound with respect to the arc length, which is uniform in $l \in \mathbb{N}$, i.e.

$$\|\kappa[f(t_l)]\|_{L_2(0,\mathcal{L}[f(t_l)])} \leq C \qquad \text{for all } l \in \mathbb{N}. \tag{7.0.1}$$

Here, $f : [0, T_{max}) \times \bar{I} \to \mathbb{R}^2$, $I = (0,1)$, $T_{max} < \infty$, is the maximal solution of (3.1.1)–(3.1.5), which is given by assumption. In the following, we want to establish a reparametrization of the solution $f(t_l, \cdot)$, which is translated by $f(t_l, 0)$, such that the we can control the $W_2^2(I;\mathbb{R}^2)$-norm uniformly in $l \in \mathbb{N}$. To this end, let

$$\bar{I} \ni s \mapsto \sigma_l(s) \in \bar{I}$$

be the orientation preserving reparametrization such that $f(t_l, \sigma_l(s)) : \bar{I} \to \mathbb{R}^2$ is parametrized proportional to arc length. Then, we denote by

$$\tau(t_l, \sigma_l(s)) := \frac{\partial_s f(t_l, \sigma_l(s))}{\mathcal{L}[f(t_l)]} \qquad \text{and} \qquad \vec{\kappa}[f(t_l)](\sigma_l(s)) := \frac{\partial_s^2 f(t_l, \sigma_l(s))}{(\mathcal{L}[f(t_l)])^2}$$

the tangent vector and the curvature vector of $f(t_l, \bar{I})$ at $f(t_l, \sigma(s))$, respectively. Now, we define

$$\begin{aligned} &\tilde{f}(t_l, \cdot) : \bar{I} \to \mathbb{R}^2, \\ &\qquad s \mapsto \tilde{f}(t_l, s) := \mathcal{L}[f(t_l)] \left(\int_0^s \int_0^{\tilde{s}} \vec{\kappa}[f(t_l)](\sigma_l(y))\mathcal{L}[f(t_l)]\, \mathrm{d}y + \tau(t_l, 0)\, \mathrm{d}\tilde{s} \right). \end{aligned}$$

By the identities

$$\begin{aligned} \int_0^{\tilde{s}} \vec{\kappa}[f(t_l)](\sigma_l(y))\, \mathrm{d}y &= \frac{1}{\mathcal{L}[f(t_l)]} \left(\tau(t_l, \sigma_l(\tilde{s})) - \tau(t_l, 0)\right), \\ \int_0^s \tau(t_l, \sigma_l(\tilde{s}))\, \mathrm{d}\tilde{s} &= \frac{1}{\mathcal{L}[f(t_l)]} \left(f(t_l, \sigma_l(s)) - f(0, t_l)\right), \end{aligned}$$

we observe that

$$\tilde{f}(t_l, s) = f(t_l, \sigma_l(s)) - f(t_l, 0). \tag{7.0.2}$$

By substitution with $z = \mathcal{L}[f(t_l)]y$, we have

$$\begin{aligned}\int_0^1 |\vec{\kappa}[f(t_l)](\sigma_l(y))|^2 \mathcal{L}[f(t_l)]\,\mathrm{d}y &= \int_0^{\mathcal{L}[f(t_l)]} \left|\vec{\kappa}[f(t_l)](\sigma_l(z/\mathcal{L}[f(t_l)]))\right|^2 \mathrm{d}z \\ &= \|\kappa[f(t_l)]\|^2_{L_2(0,\mathcal{L}[f(t_l)])},\end{aligned}$$

since $[0, \mathcal{L}[f(t_l)]] \ni z \mapsto f(t_l, (\sigma_l(z/\mathcal{L}[f(t_l)])))$ is the orientation preserving arc length parametrization of $f(t_l, \cdot)$. By Corollary 7.15 in [20], an absolutely continuous function whose integrand is in $L_2(I)$ is an element of $W^1_2(I)$. Using this result twice, we obtain $\tilde{f}(t_l, \cdot) \in W^2_2(I; \mathbb{R}^2)$. Thus, we can employ the embedding $W^2_2(I; \mathbb{R}^2) \hookrightarrow C^1(\bar{I}; \mathbb{R}^2)$, cf. 10.13 (2) in [1] and consequently we have

$$\begin{aligned}\partial_s \tilde{f}(t_l, s) &= \mathcal{L}[f(t_l)] \left(\int_0^s \vec{\kappa}[f(t_l)](\sigma_l(y))\mathcal{L}[f(t_l)]\,\mathrm{d}y + \tau(t_l, 0)\right) \\ &= \mathcal{L}[f(t_l)] \left([\tau(t_l, \sigma_l(y))]_0^s + \tau(t_l, 0)\right) = \mathcal{L}[f(t_l)]\tau(t_l, \sigma_l(s)),\end{aligned}$$

which implies $|\partial_s \tilde{f}(t_l, \sigma_l(s))| = \mathcal{L}[f(t_l)]$ for $s \in \bar{I}$. Hence, $s \mapsto \tilde{f}(t_l, s)$ is - up to translation by $f(t_l, 0)$ - for each $l \in \mathbb{N}$ the orientation preserving reparametrization of $f(t_l, \cdot)$ on $\bar{I}$ which is proportional to arc length. Moreover, we deduce the bounds

$$\begin{aligned}\left\|[s \mapsto \partial_s \tilde{f}(t_l, s)]\right\|_{L_2(I;\mathbb{R}^2)} &= \mathcal{L}[f(t_l)]^{\frac{1}{2}}, \\ \left\|[s \mapsto \partial_s^2 \tilde{f}(t_l, s)]\right\|_{L_2(I;\mathbb{R}^2)} &= \mathcal{L}[f(t_l)]\, \|[s \mapsto \vec{\kappa}[f(t_l)](\sigma_l(s))\mathcal{L}[f(t_l)]]\|_{L_2(I;\mathbb{R}^2)} \\ &= \mathcal{L}[f(t_l)]^{\frac{3}{2}}\, \|\kappa[f(t_l)]\|_{L_2(0,\mathcal{L}[f(t_l)])}.\end{aligned}$$

By enlarging the domain of integration, a change of variables with $z = \mathcal{L}[f(t_l)]y$, and Hölder's inequality, we obtain

$$\begin{aligned}\left\|\left[s \mapsto \int_0^s \int_0^{\tilde{s}} \vec{\kappa}[f(t_l)](\sigma_l(y))\mathcal{L}[f(t_l)]\,\mathrm{d}y\,\mathrm{d}\tilde{s}\right]\right\|_{L_2(I;\mathbb{R}^2)} &\leq \int_0^1 |\vec{\kappa}[f(t_l)](\sigma_l(y))|\, \mathcal{L}[f(t_l)]\,\mathrm{d}y \\ = \int_0^{\mathcal{L}[f(t_l)]} |\vec{\kappa}[f(t_l)](\sigma_l(z/\mathcal{L}[f(t_l)]))|\,\mathrm{d}z &\leq \mathcal{L}[f(t_l)]^{\frac{1}{2}}\, \|\kappa[f(t_l)]\|_{L_2(0,\mathcal{L}[f(t_l)])}.\end{aligned}$$

Additionally, we deduce

$$\left\|\left[s \mapsto \int_0^s \tau(t_l, 0)\,\mathrm{d}\tilde{s}\right]\right\|_{L_2(I;\mathbb{R}^2)} \leq 1.$$

Combining these bounds with the bound on the length of the curve in Remark 3.2.5, we obtain

$$\|\tilde{f}(t_l, \cdot)\|_{W^2_2(I;\mathbb{R}^2)} \leq C_* \qquad \text{for each } l \in \mathbb{N}, \tag{7.0.3}$$

where we used the uniform in time bound (7.0.1).

Step 2: Restarting the flow for translated initial data

We set $\tilde{f}_l := \tilde{f}(t_l, \cdot)$. The bound (7.0.3) implies,

$$\|\tilde{f}_l\|_{W^2_2(I;\mathbb{R}^2)} \leq C_* \qquad \text{for all } l \in N. \tag{7.0.4}$$

Thus, we observe that by

$$M := \left\{ \tilde{f}_l : \; l \in \mathbb{N} \right\},$$

we have a bounded set in $W_2^2(I;\mathbb{R}^2)$. By combining the statements in Theorem 2 (b) in Section 1.16.4 and Theorem 1 in 4.3.1, both in [30], with Theorem 10.9 (2) in [1], we have the compact embedding

$$W_2^2\left(I;\mathbb{R}^2\right) \hookrightarrow W_2^\gamma\left(I;\mathbb{R}^2\right) \qquad \text{for } \gamma < 2. \tag{7.0.5}$$

Consequently, the set M is precompact in $W_2^\gamma\left(I;\mathbb{R}^2\right)$. Note, that for a fixed $\gamma \in ({}^3\!/_2, 2)$ we find a $\mu \in ({}^7\!/_8, 1)$ such that $\gamma = 4(\mu - {}^1\!/_2)$.

In the following, we want to find a covering for the closure of M with respect to $\|\cdot\|_{W_2^\gamma(I;\mathbb{R}^2)}$. By Theorem 6.4.1, there exists for each $\tilde{f}_l$ a reference curve $\Phi_l^* : \bar{I} = [0,1] \to \mathbb{R}^2$ with the following properties:

- Φ_l^* is a regular curve and in $C^5(\bar{I};\mathbb{R}^2)$, see Lemma 6.1.4.
- Φ_l^* fulfills (5.1.1), i.e.

$$\begin{aligned} \Phi_l^*(\sigma) &\in \mathbb{R} \times \{0\} && \text{for } \sigma \in \{0,1\}, \\ \measuredangle\left(n_{\Lambda_l}(\sigma), \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) &= \pi - \alpha && \text{for } \sigma \in \{0,1\}, \\ \kappa_{\Lambda_l}(\sigma) &= 0 && \text{for } \sigma \in \{0,1\}, \end{aligned} \tag{5.1.1}$$

  where $\Lambda_l = \Phi_l^*(\bar{I})$.

- Let $\beta_l : \bar{I} \to \bar{I}$ be the orientation preserving reparametrization such that $\Phi_l^* \circ \beta_l$ is parametrized proportional to arc length. Then, it holds that

$$\begin{aligned} \|\tilde{f}_l \circ \beta_l - \Phi_l^* \circ \beta_l\|_{C^0(\bar{I};\mathbb{R}^2)} &< \xi_{0,l}, \\ \|\partial_\sigma(\tilde{f}_l \circ \beta_l) - \partial_\sigma(\Phi_l^* \circ \beta_l)\|_{C^0(\bar{I};\mathbb{R}^2)} &< \xi_{1,l}, \end{aligned}$$

  see Theorem 6.2.2 for the definitions of ξ_0 and ξ_1.

Now, we set

$$\begin{aligned} \delta_{0,l} &:= \xi_{0,l} - \|\tilde{f}_l \circ \beta_l - \Phi_l^* \circ \beta_l\|_{C^0(\bar{I};\mathbb{R}^2)} > 0, \\ \delta_{1,l} &:= \xi_{1,l} - \|\partial_\sigma(\tilde{f}_l \circ \beta_l) - \partial_\sigma(\Phi_l^* \circ \beta_l)\|_{C^0(\bar{I};\mathbb{R}^2)} > 0, \end{aligned}$$

and consider the balls $B^{W_2^\gamma(I;\mathbb{R}^2)}(\tilde{f}_l \circ \beta_l, {}^{\min_{i=1,2}\delta_{i,l}}\!/_{2C})$, which are balls in $W_2^\gamma(I;\mathbb{R}^2)$ around $\tilde{f}_l \circ \beta_l$ with radius ${}^{\min_{i=1,2}\delta_{i,l}}\!/_{2C}$. Here, the constant denoted by C is the operator norm of the embedding

$$i : W_2^\gamma\left(I;\mathbb{R}^2\right) \hookrightarrow C^1\left(\bar{I};\mathbb{R}^2\right).$$

We can cover $\overline{M}^{W_2^\gamma(I;\mathbb{R}^2)}$ by the union of all these balls. By compactness, there exists a finite set $S \subset \mathbb{N}$ such that it holds

$$\overline{M}^{W_2^\gamma(I;\mathbb{R}^2)} \subset \bigcup_{l \in S} B^{W_2^\gamma(I;\mathbb{R}^2)}\left(\tilde{f}_l \circ \beta_l, \frac{\min_{i=1,2}\delta_{i,l}}{2C}\right).$$

Therefore, for each $k \in \mathbb{N}$ there exists an $l \in S$ with $\tilde{f}_k \circ \beta_k \in B^{W_2^\gamma(I;\mathbb{R}^2)}(\tilde{f}_l \circ \beta_l, {}^{\min_{i=1,2}\delta_{i,l}}/_{2C})$ again. Consequently, we have

$$\left\|\tilde{f}_k \circ \beta_k - \tilde{f}_l \circ \beta_l\right\|_{C(\bar{I};\mathbb{R}^2)} < C \left\|\tilde{f}_k \circ \beta_k - \tilde{f}_l \circ \beta_l\right\|_{W_2^\gamma(I;\mathbb{R}^2)} < \frac{\delta_{0,l}}{2}$$

and analogously

$$\left\|\partial_\sigma(\tilde{f}_k \circ \beta_k) - \partial_\sigma(\tilde{f}_l \circ \beta_l)\right\|_{C(\bar{I};\mathbb{R}^2)} < \frac{\delta_{1,l}}{2}.$$

These estimates imply

$$\begin{aligned}\|\tilde{f}_k \circ \beta_k - \Phi_l^* \circ \beta_l\|_{C(\bar{I};\mathbb{R}^2)} &\leq \left\|\tilde{f}_k \circ \beta_k - \tilde{f}_l \circ \beta_l\right\|_{C(\bar{I};\mathbb{R}^2)} + \left\|\tilde{f}_l \circ \beta_l - \Phi_l^* \circ \beta_l\right\|_{C(\bar{I};\mathbb{R}^2)} \\ &\leq \frac{1}{2}\left(\xi_{0,l} - \left\|\tilde{f}_l \circ \beta_l - \Phi_l^* \circ \beta_l\right\|_{C(\bar{I};\mathbb{R}^2)}\right) + \left\|\tilde{f}_l \circ \beta_l - \Phi_l^* \circ \beta_l\right\|_{C(\bar{I};\mathbb{R}^2)} \\ &\leq \frac{1}{2}\left(\xi_{0,l} + \left\|\tilde{f}_l \circ \beta_l - \Phi_l^* \circ \beta_l\right\|_{C(\bar{I};\mathbb{R}^2)}\right) < \xi_{0,l}\end{aligned}$$

and

$$\left\|\partial_\sigma(\tilde{f}_l \circ \beta_l) - \partial_\sigma(\Phi_l^* \circ \beta_l)\right\|_{C(\bar{I};\mathbb{R}^2)} < \xi_{1,l},$$

respectively. The combination of the established inequalities with Theorem 6.2.2 shows that $\Phi_l^* \circ \beta_l$ is a reference curve for the initial curve $\tilde{f}_k \circ \beta_k$: There exists a regular C^1-reparametrization $\varphi_k : \bar{I} \to \bar{I}$ and a function $\rho_{k,0} : \bar{I} \to (-d, d)$ of class C^1, such that

$$\tilde{f}_k \circ \beta_k(\varphi_k(\sigma)) = \Phi_l^* \circ \beta_l(\sigma) + \rho_{k,0}(\sigma)(n_{\Lambda_l}(\sigma) + \cot\alpha\,\eta(\sigma)\tau_{\Lambda_l}(\sigma)), \qquad \text{for } \sigma \in \bar{I} \tag{7.0.6}$$

where $\Lambda_l = \Phi_l^* \circ \beta_l(\bar{I})$. Moreover, by Condition 2 in Definition 6.2.1, $\rho_{k,0}$ satisfies the bounds (5.1.22) and the bound

$$\|\rho_{k,0}\|_{W_2^\gamma(I)} \leq C\left(\alpha, \Phi_l^* \circ \beta_l, \eta, \|\tilde{f}_k\|_{W_2^\gamma(I;\mathbb{R}^2)}\right), \tag{7.0.7}$$

which are required in the short time existence result Theorem 5.1.3. Consequently, we obtain by (7.0.4) and the fact that S is a finite set that

$$\|\rho_{k,0}\|_{W_2^\gamma(I)} \leq \max_{l \in S} C\left(\alpha, \Phi_l^* \circ \beta_l, \eta, C_*\right) \qquad \text{for all } k \in \mathbb{N}.$$

We note that the time of existence T in Theorem 5.1.3 is determined by α, the reference curve $\Phi_l^* \circ \beta_l$, the coordinates η, and the constants R_1 and R_2, where

$$\|\rho_0\|_{X_\mu} \leq R_1 \quad \text{and} \quad \left\|\mathcal{L}^{-1}\right\|_{L(\mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu \times X_\mu; \mathbb{E}_{\mu,T})} \leq R_2.$$

We recall that $\mathcal{L}^{-1}$ depends on the reference curve $\Phi_l^* \circ \beta_l$ and also on the initial curve $\rho_{k,0}$. Since we only need finitely many reference curves to be able to represent the initial curves, it just remains to prove that for each reference curve $\Phi_l^* \circ \beta_l$ there exists a constant $C > 0$ such that

$$\left\|\mathcal{L}^{-1}(\rho_{k,0})\right\|_{L(\mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_\mu \times X_\mu; \mathbb{E}_{\mu,T})} < C \tag{7.0.8}$$

for each $\rho_{k,0} \in W_2^\gamma(I)$, which corresponds to an $\tilde{f}_k \circ \beta_k \in B^{W_2^\gamma(I;\mathbb{R}^2)}(\tilde{f}_l \circ \beta_l, {}^{\min_{i=1,2}\delta_{i,l}}/_{2C})$. In the following, the set of those $\rho_{k,0} \in W_2^{\gamma}(I)$ is denoted by M_l. By the compact embedding

$$W_2^{\gamma}(I) \hookrightarrow W_2^{\bar{\gamma}}(I) \qquad \text{for } \gamma > \bar{\gamma} > 3/2,$$

which is proven like in (7.0.5), we observe that the set $\overline{M_l}^{W_2^{\bar{\gamma}}(I)}$ is compact in $W_2^{\bar{\gamma}}(I)$. By direct calculations, it follows that $\mathcal{L}(\rho_{k,0}) \in L(\mathbb{E}_{\mu,T}; \mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_{\mu} \times X_{\mu})$ depends continuously on $\rho_{k,0} \in W_2^{\bar{\gamma}}(I)$ fulfilling the bounds (5.1.22). Thus, we obtain by a Neumann series argument that for each $\rho_{k,0} \in M_l$ there exists a $\delta(\rho_{k,0}) > 0$, such that for all $\rho \in B_0^{W_2^{\bar{\gamma}}(I)}(\rho_{k,0}, \delta(\rho_{k,0}))$, where

$$B_0^{W_2^{\bar{\gamma}}(I)}(\rho_{k,0}, \delta(\rho_{k,0})) = \left\{\rho \in W_2^{\bar{\gamma}}(I) : \partial_{\sigma}\rho(\sigma) = 0 \text{ for } \sigma = 0,1 \text{ and } \|\rho_{k,0} - \rho\|_{W_2^{\bar{\gamma}}(I)} < \delta(\rho_{k,0})\right\},$$

the operator $\mathcal{L}(\rho) \in L(\mathbb{E}_{\mu,T}; \mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_{\mu} \times X_{\mu})$ is invertible with

$$\left\|\mathcal{L}^{-1}(\rho)\right\|_{L(\mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_{\mu} \times X_{\mu}; \mathbb{E}_{\mu,T})} \leq 2 \left\|\mathcal{L}^{-1}(\rho_{k,0})\right\|_{L(\mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_{\mu} \times X_{\mu}; \mathbb{E}_{\mu,T})}.$$

Moreover, we can cover $\overline{M_l}^{W_2^{\bar{\gamma}}(I)}$ by the union of $B_0^{W_2^{\bar{\gamma}}(I)}(\rho_{k,0}, \delta(\rho_{k,0}))$ with $k \in M_l$. By compactness of $\overline{M_l}^{W_2^{\bar{\gamma}}(I)}$, there exists a finite set $S_l \subset \mathbb{N}$ such that

$$\overline{M_l}^{W_2^{\bar{\gamma}}(I)} \subset \bigcup_{S_l \subset \mathbb{N}} B^{W_2^{\bar{\gamma}}(I)}(\rho_{k,0}, \delta(\rho_{k,0})).$$

Consequently, we deduce

$$\left\|\mathcal{L}^{-1}(q)\right\|_{L(\mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_{\mu} \times X_{\mu}; \mathbb{E}_{\mu,T})} \leq 2 \max_{k \in S_l} \left\|\mathcal{L}^{-1}(\rho_{k,0})\right\|_{L(\mathbb{E}_{0,\mu} \times \tilde{\mathbb{F}}_{\mu} \times X_{\mu}; \mathbb{E}_{\mu,T})} =: C$$

for all $\rho_{k,0} \in M$, cf. (7.0.8). Thus, it makes sense to set $\tilde{T} := \min_{l \in S} T_l$. As $t_k \to T_{max}$ for $k \to \infty$, we can choose a sufficiently large $k \in \mathbb{N}$ such that $t_k + \tilde{T} > T_{max}$, cf. Remark 5.1.4 item 2.

We fix a $k \in \mathbb{N}$ with this property. Let $\rho_{k,0} \in W_2^{4(\mu-1/2)}(I)$ be the height function over $\Phi_l^* \circ \beta_l$, $l \in S$, which corresponds to $\tilde{f}_k \circ \beta_k$. By the short time existence result, Theorem 5.1.3, we obtain for the initial datum $\rho_{k,0} \in W_2^{4(\mu-1/2)}(I)$ a solution

$$\begin{aligned} \rho : [0, \tilde{T}) \times I &\to (-d, d), \\ (t, x) &\mapsto \rho(t, x), \end{aligned}$$

such that $\rho \in \mathbb{E}_{\mu,\tilde{T}} := W_2^1([0,\tilde{T}); L_2(I)) \cap L_{2,loc}([0,\tilde{T}); W_2^4(I))$ and $\rho(0,\cdot) = \rho_{l,0}$. This implies that for

$$\tilde{f}(t,\sigma) := \Phi_l^* \circ \beta_l(\sigma) + \rho(t,\sigma)(n_{\Lambda_l}(\sigma) + \cot\alpha\eta(\sigma)\tau_{\Lambda_l}(\sigma))$$

the following holds true:

1. $\tilde{f} \in \mathbb{E}_{\mu,\tilde{T},\mathbb{R}^2} := W_{2,\mu}^1\left([0,\tilde{T}); L_{2,\mu}(I;\mathbb{R}^2)\right) \cap L_2\left([0,\tilde{T}); W_2^4(I;\mathbb{R}^2)\right)$,
2. $\tilde{f}$ fulfills (3.1.1)-(3.1.4) and there exists a regular C^1-reparametrization $\varphi_k : \bar{I} \to \bar{I}$ such that $\tilde{f}(0,\sigma) = \tilde{f}_k \circ \beta_k(\varphi_k(\sigma))$ for all $\sigma \in \bar{I}$, cf. (7.0.6),
3. $\tilde{f}(t,\cdot)$ is for each $t \in [0,\tilde{T})$ a regular parametrization of the curve $\tilde{f}(t,\bar{I})$.

Step 3: Extension of the original solution
It remains to show that we can extend the original solution f beyond $T_{max} < \infty$, which was assumed to be maximal. To this end, we want to translate $\tilde{f}$ by $f(t_k, 0)$, as $\tilde{f}(t_k,0) + f(t_k,0) = f(t_k,0)$ cf.

(7.0.2). By Lemma 2.2.3, item 1, we have

$$f \in \mathbb{E}_{\mu,T,\mathbb{R}^2,loc} \hookrightarrow \mathbb{E}_{\mu,T-\epsilon,\mathbb{R}^2} \hookrightarrow BUC\left([0,T-\epsilon], W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)\right) \qquad \text{for each } 0<\epsilon<T.$$

Consequently, it follows by (2.2.6)

$$|f(t_l,0)| \leq \|f(t_l,\cdot)\|_{C(\bar{I};\mathbb{R}^2)} \leq C\|f(t_l,\cdot)\|_{W_2^{4(\mu-1/2)}(I;\mathbb{R}^2)}.$$

We recall that by Remark 4.1.2 the flow is invariant under translation. Therefore, $\tilde{f}+f(t_k,0)$ fulfills properties analogous to $\tilde{f}$, see the previous step of the proof. By concatenating the "old" part of the solution for $t\in[0,t_k]$ and the new one for $t\in[t_k,t_k+\tilde{T})$ at $t=t_k$, we obtain an extension of the original solution as $t_k+\tilde{T}>T_{max}$, which contradicts the maximality of the original solution. Thus, the assumption (7.0.1) on the curvature cannot be true. As the sequence $(t_l)_{l\in\mathbb{N}}$ was arbitrary, the claim is proven. □

A Appendix

We need the following formal calculations to derive the structure of problem (5.1.15).

A.1 Calculation of $\kappa(\rho)$

We assume that $J(\rho) > 0$ for $\sigma \in [0,1]$ and suitable ρ. For a sufficiently smooth ρ, the scalar curvature as a function of ρ is given by

$$\begin{aligned}
\kappa(\rho) &= \langle \Delta(\rho)\Phi, n_{\Gamma_t} \rangle \\
&= \left\langle \frac{1}{J(\rho)}\partial_\sigma \left(\frac{1}{J(\rho)}\right)\Phi_\sigma + \frac{1}{(J(\rho))^2}\Phi_{\sigma\sigma}, \frac{1}{J(\rho)}R\Phi_\sigma \right\rangle = \left\langle \frac{1}{(J(\rho))^2}\Phi_{\sigma\sigma}, \frac{1}{J(\rho)}R\Phi_\sigma \right\rangle \\
&= \frac{1}{(J(\rho))^3}\big\langle \partial_\sigma(\Psi_\sigma + \Psi_q\partial_\sigma\rho), R(\Psi_\sigma + \Psi_q\partial_\sigma\rho)\big\rangle \\
&= \frac{1}{(J(\rho))^3}\big\langle \Psi_{\sigma\sigma} + 2\Psi_{\sigma q}\partial_\sigma\rho + \underbrace{\Psi_{qq}}_{=0}(\partial_\sigma\rho)^2 + \Psi_q\partial_\sigma^2\rho, R(\Psi_\sigma + \Psi_q\partial_\sigma\rho)\big\rangle
\end{aligned}$$

Thus, we have the representation

$$\kappa(\rho) = \frac{1}{(J(\rho))^3}\langle \Psi_q, R\Psi_\sigma\rangle \partial_\sigma^2\rho + U(\sigma,\rho,\partial_\sigma\rho), \tag{5.1.6}$$

where the prefactors $U(\sigma,\rho,\partial_\sigma\rho)$ denote terms of the form

$$U(\sigma,\rho,\partial_\sigma\rho) := C\,(J(\rho))^k \left(\prod_{i=0}^{p} \left\langle \Psi^{\beta_i}_{(\sigma,q)}, R\Psi^{\gamma_i}_{(\sigma,q)} \right\rangle(\sigma,\rho)\right)(\partial_\sigma\rho)^r, \tag{5.1.7}$$

with $C \in \mathbb{R}$, $k \in \mathbb{Z}$, $p, r \in \mathbb{N}_0$, and $\beta_i, \gamma_i \in \mathbb{N}_0^2$, such that $|\beta_i|, |\gamma_i| \geq 1$ and $|\beta_i| + |\gamma_i| \leq 3$ for all $i \in \{0,\dots,p\}$. Here, we keep the leading order prefactor explicitly, since it will be important for the analysis of the equation.

In order to differentiate $\kappa(\rho)$ twice with respect to the arc length element, we need to calculate the form of the derivatives of $\partial_\sigma^i J(\rho)$, $i = 1, 2$.

A.2 Calculation of $\partial_\sigma J(\rho)$, $(\partial_\sigma J(\rho))^2$, and $\partial_\sigma^2 J(\rho)$

We recall that by definition

$$\begin{aligned}
J(\rho) &= |\partial_\sigma(\Psi(\sigma,\rho(t,\sigma)))| = |\partial_\sigma\Psi(\sigma,\rho(t,\sigma)) + \partial_q\Psi(\sigma,\rho(t,\sigma))\partial_\sigma\rho(t,\sigma)| \\
&= \sqrt{|\Psi_\sigma|^2 + 2\langle\Psi_\sigma,\Psi_q\rangle\partial_\sigma\rho + |\Psi_q|^2(\partial_\sigma\rho)^2}.
\end{aligned}$$

Assuming that $J(\rho) > 0$ for $\sigma \in [0,1]$ and ρ is sufficiently smooth, we obtain

$$\partial_\sigma J(\rho) = \frac{1}{2}\frac{1}{J(\rho)}\partial_\sigma\left(|\Psi_\sigma|^2 + 2\langle\Psi_\sigma,\Psi_q\rangle\partial_\sigma\rho + |\Psi_q|^2(\partial_\sigma\rho)^2\right)$$

$$= \frac{1}{2}\frac{1}{J(\rho)}\left[\partial_\sigma\left(|\Psi_\sigma|^2\right) + 2\partial_\sigma\left(\langle\Psi_\sigma,\Psi_q\rangle\right)\partial_\sigma\rho + 2\langle\Psi_\sigma,\Psi_q\rangle\partial_\sigma^2\rho + \partial_\sigma\left(|\Psi_q|^2\right)(\partial_\sigma\rho)^2 \right.$$
$$\left. + 2|\Psi_q|^2\partial_\sigma\rho\partial_\sigma^2\rho\right].$$

Thus, we have the representation

$$\partial_\sigma J(\rho) = U(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + U(\sigma,\rho,\partial_\sigma\rho), \tag{A.2.1}$$

where the prefactors $U(\sigma,\rho,\partial_\sigma\rho)$ are given by (5.1.7).

We observe the following properties:

Remark A.2.1 *1. For products of U, it holds*

$$(U(\sigma,\rho,\partial_\sigma\rho))^s = U(\sigma,\rho,\partial_\sigma\rho) \qquad \textit{for } s\in\mathbb{N}.$$

2. Moreover, we obtain

$$\partial_\sigma\left(\Psi^{\beta}_{(\sigma,q)}(\sigma,\rho)\right) = \Psi^{\tilde\beta_1}_{(\sigma,q)}(\sigma,\rho) + \Psi^{\tilde\beta_2}_{(\sigma,q)}(\sigma,\rho)\partial_\sigma\rho,$$

where $\tilde\beta_i\in\mathbb{N}_0^2$ with $|\beta|+1=|\tilde\beta_i|$ for $i=1,2$. Combining this with (A.2.1)*, we deduce*

$$\partial_\sigma\left(U(\sigma,\rho,\partial_\sigma\rho)\right) = T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho),$$

where the prefactors $T(\sigma,\rho,\partial_\sigma\rho)$ denote terms of the form

$$T(\sigma,\rho,\partial_\sigma\rho) := C\,(J(\rho))^k\left(\prod_{i=0}^{p}\left\langle\Psi^{\beta_i}_{(\sigma,q)}, R\Psi^{\gamma_i}_{(\sigma,q)}\right\rangle(\sigma,\rho)\right)(\partial_\sigma\rho)^r, \tag{5.1.9}$$

with $C\in\mathbb{R}$, $k\in\mathbb{Z}$, $p,r\in\mathbb{N}_0$, and $\beta_i,\gamma_i\in\mathbb{N}_0^2$, such that $|\beta_i|,|\gamma_i|\geq 1$ and $|\beta_i|+|\gamma_i|\leq 4$ for all $i\in\{0,\dots,p\}$.

This yields the representation

$$(\partial_\sigma J(\rho))^2 = U(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^2 + U(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + U(\sigma,\rho,\partial_\sigma\rho).$$

Moreover, by Remark A.2.1, we obtain from (A.2.1)

$$\begin{aligned}\partial_\sigma^2 J(\rho) &= \partial_\sigma\left(U(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + U(\sigma,\rho,\partial_\sigma\rho)\right)\\ &= \left(T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho)\right)\partial_\sigma^2\rho + U(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho\\ &\quad + T(\sigma,\rho,\partial_\sigma\rho)\\ &= T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho + T(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^2 + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho).\end{aligned}$$

Now, we are ready to give representations for $\partial_s\kappa(\rho)$ and $\partial_s^2\kappa(\rho)$.

A.3 Calculation of $\partial_s\kappa(\rho)$ and $\partial_s^2\kappa(\rho)$

Using (5.1.6), we obtain for a sufficiently smooth ρ

$$\partial_s\kappa(\rho) = \frac{\partial_\sigma\kappa(\rho)}{J(\rho)} = \frac{1}{J(\rho)}\partial_\sigma\left(\frac{1}{(J(\rho))^3}\langle\Psi_q,R\Psi_\sigma\rangle\partial_\sigma^2\rho + U(\sigma,\rho,\partial_\sigma\rho)\right)$$

$$
\begin{aligned}
&= \frac{1}{(J(\rho))^4}\langle\Psi_q, R\Psi_\sigma\rangle\partial_\sigma^3\rho + \frac{1}{J(\rho)}\partial_\sigma\left(U(\sigma,\rho,\partial_\sigma\rho)\right)\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho)\\
&= \frac{1}{(J(\rho))^4}\langle\Psi_q, R\Psi_\sigma\rangle\partial_\sigma^3\rho + \left(T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho)\right)\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho\\
&\quad + T(\sigma,\rho,\partial_\sigma\rho)\\
&= \frac{1}{(J(\rho))^4}\langle\Psi_q, R\Psi_\sigma\rangle\partial_\sigma^3\rho + T(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^2 + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho\\
&\quad + T(\sigma,\rho,\partial_\sigma\rho),
\end{aligned} \tag{5.1.8}
$$

where we used Remark A.2.1, item 2. The prefactors $T(\sigma,\rho,\partial_\sigma\rho)$ are given in (5.1.9).

For the representation of $\partial_s^2\kappa(\rho)$, we need to consider derivatives of $T(\sigma,\rho,\partial_\sigma\rho)$.

Remark A.3.1 *By the same reasoning as in item 2, we infer*

$$\partial_\sigma\left(T(\sigma,\rho,\partial_\sigma\rho)\right) = \tilde{S}(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + \tilde{S}(\sigma,\rho,\partial_\sigma\rho),$$

where the prefactors $\tilde{S}(\sigma,\rho,\partial_\sigma\rho)$ denote terms of the form

$$\tilde{S}(\sigma,\rho,\partial_\sigma\rho) := C\left(J(\rho)\right)^k\left(\prod_{i=0}^{p}\left\langle\Psi_{(\sigma,q)}^{\beta_i}, R\Psi_{(\sigma,q)}^{\gamma_i}\right\rangle(\sigma,\rho)\right)(\partial_\sigma\rho)^r, \tag{5.1.11}$$

with $C\in\mathbb{R}$, $k\in\mathbb{Z}$, $p,r\in\mathbb{N}_0$, and $\beta_i,\gamma_i\in\mathbb{N}_0^2$, such that $|\beta_i|,|\gamma_i|\geq 1$ and $|\beta_i|+|\gamma_i|\leq 5$ for all $i\in\{0,\dots,p\}$.

Furthermore, we formally deduce

$$
\begin{aligned}
\partial_s^2\kappa(\rho) &= \frac{\partial_\sigma\left(\partial_s\kappa(\rho)\right)}{J(\rho)} = \frac{1}{J(\rho)}\partial_\sigma\left(\frac{1}{(J(\rho))^4}\langle\Psi_q, R\Psi_\sigma\rangle\partial_\sigma^3\rho + T(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^2\right.\\
&\quad \left. + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho)\right)\\
&= \frac{1}{J(\rho)}\left(\frac{1}{(J(\rho))^4}\langle\Psi_q, R\Psi_\sigma\rangle\partial_\sigma^4\rho + \partial_\sigma\left(U(\sigma,\rho,\partial_\sigma\rho)\right)\partial_\sigma^3\rho + 2T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho\partial_\sigma^2\rho\right.\\
&\quad + \partial_\sigma\left(T(\sigma,\rho,\partial_\sigma\rho)\right)\left(\partial_\sigma^2\rho\right)^2 + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho + \partial_\sigma\left(T(\sigma,\rho,\partial_\sigma\rho)\right)\partial_\sigma^2\rho\\
&\quad \left. + \partial_\sigma\left(T(\sigma,\rho,\partial_\sigma\rho)\right)\right)\\
&= \frac{1}{(J(\rho))^5}\langle\Psi_q, R\Psi_\sigma\rangle\partial_\sigma^4\rho + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho\partial_\sigma^2\rho + T(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho + \tilde{S}(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^3\\
&\quad + \tilde{S}(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^2 + \tilde{S}(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + \tilde{S}(\sigma,\rho,\partial_\sigma\rho).
\end{aligned}
$$

Replacing the remaining $T(\sigma,\rho,\partial_\sigma\rho)$ by $\tilde{S}(\sigma,\rho,\partial_\sigma\rho)$, we obtain

$$
\begin{aligned}
\partial_s^2\kappa(\rho) &= \frac{1}{(J(\rho))^5}\langle\Psi_q, R\Psi_\sigma\rangle\partial_\sigma^4\rho + \tilde{S}(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho\partial_\sigma^2\rho + \tilde{S}(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^3\rho + \tilde{S}(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^3\\
&\quad + \tilde{S}(\sigma,\rho,\partial_\sigma\rho)\left(\partial_\sigma^2\rho\right)^2 + \tilde{S}(\sigma,\rho,\partial_\sigma\rho)\partial_\sigma^2\rho + \tilde{S}(\sigma,\rho,\partial_\sigma\rho).
\end{aligned} \tag{5.1.10}
$$

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