Adams Operations in Differential Algebraic K-theory

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Contents

1. Introduction

Adams operations first appeared in the context of K-theory of topological spaces. For a finite CW-complex X there is the zeroth topological K-group $K_0(X)$ constructed by group completion of the monoid $\mathrm{Vect}(X)$ of isomorphism classes of finite rank vector bundles over X with the direct sum operation. In fact there is even a ring structure on $K_0(X)$ which is induced by the tensor product operation of vector bundles. For a natural number k, the k-th Adams operation is then a ring homomorphism

$$\psi^k : K_0(X) \to K_0(X)$$

which is characterized by the property that it sends the class of a line bundle to the class of its k-th tensor power. These ring homomorphisms were considered by Frank Adams in his solution of the vector fields on spheres proplem.
There is an analogous story for algebraic K_0 of schemes. If X is a scheme (having certain good properties), one can again consider the monoid $\mathrm{Vect}(X)$ of isomorphism classes of vector bundles with the direct sum operation. Since short exact sequences of vector bundles over a scheme do not split in general, the zeroth K-group $K_0(X)$ of X is now defined as the universal abelian group that comes equipped with a monoid homomorphism $\mathrm{Vect}(X) \to K_0(X)$ such that every short exact sequence of vector bundles over X splits in $K_0(X)$. Again there is a ring structure on $K_0(X)$ induced by the tensor product operation and one can show that for every natural number k there is a uniquely defined ring homomorphism ψ^k on $K_0(X)$ with the property that the class of a line bundle is sent to the class of its k-th power.

For a scheme X there is not just the zeroth K-group $K_0(X)$ but there are also all the higher K-groups $K_i(X)$ (first defined by Quillen). They are the homotopy groups of a certain connective spectrum $\mathcal{K}_{st}(X)$ with the property that the zeroth stable homotopy group is isomorphic to the classical $K_0(X)$. It turns out that the spectrum $\mathcal{K}_{st}(X)$ can be naturally equipped with the structure of an $\mathbb{E}_\infty$-algebra object in spectra (i.e. an $\mathbb{E}_\infty$-ring spectrum structure). This means that there is a multiplication map

$$\mathcal{K}_{st}(X) \wedge \mathcal{K}_{st}(X) \to \mathcal{K}_{st}(X)$$

which is commutative and associative up to coherent homotopy. A natural question is then if the classical k-th Adams operation on $K_0(X)$ lifts to a multiplicative stable map on $\mathcal{K}_{st}(X)$, i.e. if there is an $\mathbb{E}_\infty$-ring spectrum map

$$\mathcal{K}_{st}(X) \to \mathcal{K}_{st}(X)$$

which induces the classical operation on π_0. The first result of this thesis is that such a map exists if one inverts k in the spectrum $\mathcal{K}_{st}(X)$. We show that there is an $\mathbb{E}_\infty$-map

$$\psi^k : \mathcal{K}_{st}(X)[k^{-1}] \to \mathcal{K}_{st}(X)[k^{-1}]$$

whose π_0 is precisely the map

$$K_0(X)[k^{-1}] \to K_0(X)[k^{-1}]$$

that is induced by the classical Adams operation.

If one considers a smooth scheme X over the complex numbers, there is a Chern character map from the algebraic K-theory of X into the absolute Hodge cohomology

$$K_i(X) \to \mathrm{H}^{2p-i}_{\mathrm{aH}}(X, \mathbb{R}(p)).$$

This Chern character map is called the Beilinson regulator. There is a refined version of the K-theory of X that takes this Beilinson regulator map into account; it is called differential algebraic K-theory. The differential algebraic K-theory groups are constructed as the homotopy groups of an $\mathbb{E}_\infty$-ring spectrum $\widehat{\mathcal{K}}(X)$ which by construction comes with an $\mathbb{E}_\infty$-ring map

$$I : \widehat{\mathcal{K}}(X) \to \bigvee_{n \in \mathbb{Z}} \mathcal{K}_{st}(X).$$

The second result of this thesis is the construction of an $\mathbb{E}_\infty$-map

$$\widehat{\psi}^k : \widehat{\mathcal{K}}(X)[k^{-1}] \to \widehat{\mathcal{K}}(X)[k^{-1}]$$

(for each k) that makes the diagram

$$\begin{tikzcd}
\widehat{\mathcal{K}}(X)[k^{-1}] \arrow[r, "\widehat{\psi}^k"] \arrow[d, "I"'] & \widehat{\mathcal{K}}(X)[k^{-1}] \arrow[d, "I"] \\
\bigvee_{n\in\mathbb{Z}} \mathcal{K}_{st}(X)[k^{-1}] \arrow[r, "\bigvee \psi^k"] & \bigvee_{n\in\mathbb{Z}} \mathcal{K}_{st}(X)[k^{-1}]
\end{tikzcd}$$

commutative. In other words, the multiplicative stable Adams operations lift to differential algebraic K-theory.

We now want to give an overview on the structure of the thesis:
Our constructions of the stable Adams operations and the differential refinement of the stable Adams operations use higher algebra in a crucial way. We do higher algebra in the language of ∞-categories and

the aim of the second section of this thesis is to review the required parts of this theory. We begin with an introduction to Joyal's theory of quasi-categories. The main sources for this are Joyal's notes [Joy08] and Lurie's book [Lur09]. We recall how notions from ordinary category theory are intrinsically defined in quasi-categories and how one can get examples of quasi-categories from Dwyer-Kan localizations and nerve functors. We continue by reviewing the definitions of symmetric monoidal ∞-categories and commutative algebra and module objects in them. Our source here is Lurie's book Higher Algebra [Lur17]. At the end of the section also stable ∞-categories will be treated.

In the third section we will construct the stable Adams operations (Theorem 3.42). The strategy of the construction is to use the fact, that over a regular base scheme algebraic K-theory is representable in the stable ∞-category of motivic spectra by the Snaith spectrum

$$\mathbf{K} := \Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}[\beta^{-1}].$$

The stable Adams operations are then induced by an operation

$$\psi^k_{mot} : \mathbf{K}[k^{-1}] \to \mathbf{K}[k^{-1}]$$

which we construct in Theorem 3.40. It should be mentioned that, independently to our work, Arndt constructed the same maps in his thesis ([Arn16]). Since motivic methods are applied in the third section, we will also recall some things about motivic homotopy theory in the ∞-categorical context.

The fourth section is about the lift of the stable Adams operations to differential algebraic K-theory. As an input we use the description of the Beilinson regulator as an $\mathbb{E}_\infty$-ring map

$$reg_X : \mathcal{K}_{st}(X) \to H(\mathrm{IDR}(X))$$

where $\mathrm{IDR}(X)$ is a certain dg-algebra whose cohomology is absolute Hodge cohomology of X and H is the Eilenberg-MacLane functor. This description is due to Bunke, Nikolaus and Tamme ([BNT15]) and we explain the main results of their article at the beginning of the section. The main result of the fourth section is Corollary 4.58 which provides the lift of the Adams operations.

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2. ∞-CATEGORICAL BACKGROUND

Throughout the whole thesis, the language we use for doing homotopy theory is the language of ∞-categories. More precisely we use the theory of quasi-categories which was developed by Joyal in [Joy08] and massively extended by Lurie in [Lur09]. In the present chapter we list all the objects and statements we will need from the theory of ∞-categories. Most of the things in this chapter can be found in Joyal's notes [Joy08], Lurie's books [Lur09] and [Lur17] and Groth's notes [Gro15]. Things that cannot be found in these sources will be singled out explicitly.

In some of the constructions that follow one has to be careful about set theoretical issues. We use one of the standard ways to handle these issues, namely the usage of Grothendieck universes. For our purposes one Grothendieck universe will be enough. For this we fix a model $\mathbb{V}$ for ZFC-set theory in which an unaccessible cardinal κ exists and we denote the associated Grothendieck universe by

$$\mathbb{U} \in \mathbb{V}.$$

A simplicial set is a simplicial object in $\mathbb{V}$ and we say that a simplicial set is small if the collection of all simplices lies in $\mathbb{U}$. We always assume that for a category $\mathcal{C}$ the nerve $\mathrm{N}(\mathcal{C})$ is a simplicial set, i.e. that it is a simplicial object in $\mathbb{V}$. A category $\mathcal{C}$ is called small if $\mathrm{N}(\mathcal{C})$ is small.

2.1. **General theory.**

2.1.1. *Basic definitions and constructions.* As we already said, an ∞-category for us will always be a quasi-category in the sense of Joyal and Lurie. We use the symbol **sSet** to denote the 1-category of simplicial sets.

Definition 2.1. *(∞-category)*
An ∞-category $\mathcal{C}$ is a simplicial set, which has fillers for all inner horns. This means, that for any diagram in **sSet** *of the form*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n & \end{tikzcd}$$

with $0 < i < n$, there exists a map $\Delta^n \to \mathcal{C}$ such that

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, bend right, "\exists"'] & \end{tikzcd}$$

is commutative.

The objects in an ∞-category are precisely the 0-simplices and the morphisms are the 1-simplices. If X is an object of an ∞-category $\mathcal{C}$ we will simply write $X \in \mathcal{C}$. For each object $X \in \mathcal{C}$ of an ∞-category the degenerate 1-simplex $s_0(X)$ is called identity on X and is denoted by id_X. The choice of a composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ is given by a 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr] & & Z. \end{tikzcd}$$

Often, we will just say that the morphism $X \to Z$ in the above 2-simplex is a composition of f and g. If $\mathcal{C}$ is an ∞-category, it is well known (see for example Theorem 2.18. in [Joy08]) that the map

$$\mathcal{C}^{\Delta^2} \longrightarrow \mathcal{C}^{\Lambda^2_1} \tag{2.2}$$

is a trivial fibration of simplicial sets and therefore the fibers are contractible Kan complexes. In other words, composition of morphisms in ∞-categories is well-defined up to a contractible space of choices. In fact one can show that ∞-categories are characterized by this property: Every simplicial set, for which the map from 2.2 is a trivial fibration, is an ∞-category.

We say that a morphism $f : X \to Y$ in an ∞-category is invertible, or that f is an equivalence from X to Y, if there are a morphism $g : Y \to X$ and two 2-simplices of the form

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "s_0(X)"] & & X \end{tikzcd}$$

and

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ Y \arrow[ur, "g"] \arrow[rr, "s_0(Y)"] & & Y. \end{tikzcd}$$

We call g an inverse of f. The existence of the 2-simplices exactly says, that $s_0(X) = \mathrm{id}_X$ is a choice for a composition $g \circ f$ and $s_0(Y) = \mathrm{id}_Y$ is a choice for a composition $f \circ g$.

If $\mathcal{C}$ is an ordinary 1-category, then the nerve construction gives an ∞-category. More precisely there is the nerve functor

$$\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$$

from the category of small categories to simplicial sets, whose image lies in the full sub-category of ∞-categories in **sSet**. One can show that N has a left adjoint

$$\tau_1 : \mathbf{sSet} \to \mathbf{Cat},$$

the fundamental category functor (see [Joy08], p. 209). If $\mathcal{C}$ is an ∞-category, then $\tau_1(\mathcal{C})$ has an easy description in terms of the homotopy relation on morphisms (see [Joy08], Proposition 1.11.). In this case we set

$$\mathrm{Ho}(\mathcal{C}) := \tau_1(\mathcal{C}).$$

If $\mathcal{C}$ is an ∞-category, the unit of the adjunction $\tau_1 \dashv \mathrm{N}$ gives a canonical map

$$\mathcal{C} \to \mathrm{N}(\mathrm{Ho}(\mathcal{C})). \tag{2.3}$$

Definition 2.4. *(Full subcategories of ∞-categories)*
Let $\mathcal{C}$ be an ∞-category and $M \subset \mathcal{C}_0$ a collection of objects of $\mathcal{C}$. By definition, the objects of Ho($\mathcal{C}$) *are also the* 0*-simplices of $\mathcal{C}$. Let $\mathcal{Z}$ be the full subcategory of* Ho($\mathcal{C}$) *on M, then the full subcatgeory $\mathcal{C}_M$ of $\mathcal{C}$ on M is defined as the following pullback in* **sSet***:*

$$\begin{tikzcd}
\mathcal{C}_M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d] \\
\mathrm{N}(\mathcal{Z}) \arrow[r] & \mathrm{N}(\mathrm{Ho}(\mathcal{C})).
\end{tikzcd} \tag{2.5}$$

Remark 2.6. The right vertical map in 2.5 is an inner fibration since it is a map from an ∞-category to the nerve of an ordinary 1-category (see [Joy08], Proposition 2.2.). Therefore also the left vertical map is an inner fibration because inner fibrations are stable under pullback. Since the target of this left vertical map is the nerve of something and in particular an ∞-category, also $\mathcal{C}_M$ is an ∞-category.

A functor

$$F : \mathcal{C} \to \mathcal{D},$$

from an ∞-category $\mathcal{C}$ to an ∞-category $\mathcal{D}$ is just a map of simplicial sets. Note that such a map sends objects to objects, morphisms to morphisms, identities to identities and compositions to compositions. A functor therefore induces a functor on the homotopy category which we denote by

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D}).$$

The ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$ is defined to be

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) := \underline{\mathrm{hom}}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{D}),$$

where $\underline{\mathrm{hom}}_{\mathbf{sSet}}$ denotes internal Hom-objects in the cartesian closed category of simplicial sets **sSet**. We will also consider functors into ∞-categories whose domain is a general simplicial set K. A functor in this case is also just a map of simplicial sets and we define as above:

$$\mathrm{Fun}(K, \mathcal{D}) := \underline{\mathrm{hom}}_{\mathbf{sSet}}(K, \mathcal{D}).$$

Lemma 2.7. *(see* [Joy08]*, Theorem 2.18.)*
For any simplicial set K and any ∞-category $\mathcal{C}$ the simplicial set $\mathrm{Fun}(K, \mathcal{C})$ *is an ∞-category.*

As in ordinary 1-category theory one can check objectwise if a morphism between functors is invertible or not.

Theorem 2.8. *(see* [Joy08]*, Chapter 5, Theorem C)*
Let $F, G : K \to \mathcal{C}$ be two functors from a simplicial set to an ∞-category and

$$a : F \to G$$

a natural transformation (i.e. a morphism in the ∞-category $\mathrm{Fun}(K, \mathcal{C})$*). Then a is invertible if and only if $a(X)$ is invertible in $\mathcal{C}$ for all* 0*-simplices X of K.*

In ∞-categories we do not just have Hom-sets but mapping spaces in which homotopical information is encoded, as for example the existence of homotopies between morphisms. There are different constructions of these mapping spaces all of which give equivalent Kan-complexes (for a comparison of three different constructions see [Lur09], Corollary 4.2.1.8.). We use as a definition the following intuitive construction:

Definition 2.9. *(Mapping spaces in ∞-categories)*
Let X, Y be two objects in an ∞-category $\mathcal{C}$. Then the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ *from X to Y is defined to be the following pullback in* **sSet**

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "d_1 \times d_0"] \\
\Delta^0 \arrow[r, "{(X,Y)}"] & \mathcal{C} \times \mathcal{C},
\end{tikzcd}$$

where d_1 is induced by the inclusion of the 0*-vertex into* Δ^1 *and d_0 by the inclusion of the* 1*-vertex.*

These mapping spaces really play the analogous role that is played by Hom-sets in ordinary 1-categories. For example, there are the Yoneda embedding and Yoneda lemma. Moreover, in the theory of ∞-categories universal properties are described in terms of the mapping spaces. A first example of this is the following:

Definition 2.10. *(Initial and terminal objects)*
Let $\mathcal{C}$ be an ∞-category and $X \in \mathcal{C}$. The object X is called initial, if for any object $Y \in \mathcal{C}$ the mapping space $\operatorname{Map}_{\mathcal{C}}(X, Y)$ is contractible. Dually, X is called terminal, if for any Y the space $\operatorname{Map}_{\mathcal{C}}(Y, X)$ is contractible.

Remark 2.11. Note that, as one would expect, a functor between ∞-categories

$$F : \mathcal{C} \to \mathcal{D}$$

induces maps on mapping spaces. To see this, let X, Y be any objects in $\mathcal{C}$. Then the following diagram of simplicial sets commutes:

$$\begin{tikzcd}
\Delta^0 \arrow[r, "{(X,Y)}"] \arrow[d, equal] & \mathcal{C} \times \mathcal{C} \arrow[d, "F\times F"] & \operatorname{Fun}(\Delta^1, \mathcal{C}) \arrow[l, "d_1\times d_0"'] \arrow[d, "\circ F"] \\
\Delta^0 \arrow[r, "{(F(X),F(Y))}"] & \mathcal{D} \times \mathcal{D} & \operatorname{Fun}(\Delta^1, \mathcal{D}). \arrow[l, "d_1\times d_0"']
\end{tikzcd}$$

Therefore we get an induced map on the associated pullbacks which are precisely the mapping spaces in $\mathcal{C}$ and $\mathcal{D}$:

$$F_{X,Y} : \operatorname{Map}_{\mathcal{C}}(X, Y) \to \operatorname{Map}_{\mathcal{D}}(F(X), F(Y)).$$

We say that a functor F is fully faithful if the maps $F_{X,Y}$ are homotopy equivalences for all X, Y.

We have already seen the adjunction $N \dashv \tau_1$ between the nerve functor and the fundamental category functor. There is also a nerve construction for simplicial categories, the homotopy coherent nerve functor

$$\mathrm{N}_\Delta : \mathbf{sCat} \to \mathbf{sSet},$$

where **sCat** is the category of small simplicial categories. And this functor also has a left adjoint, which is denoted by

$$\mathfrak{C} : \mathbf{sSet} \to \mathbf{sCat}.$$

Remark 2.12. Let $\mathcal{C}$ be an ordinary 1-category. Using the adjunction

$$\pi_0 : \mathbf{sSet} \rightleftarrows \mathbf{Set} : \mathrm{disc},$$

where disc is the functor that associates to a set the corresponding discrete simplicial set, one can give $\mathcal{C}$ the discrete simplicial enrichment. We denote the resulting simplicial category by $\mathcal{C}^{\mathrm{disc}}$. It is then a fact that there is an isomorphism of simplicial sets

$$\mathrm{N}_\Delta(\mathcal{C}^{\mathrm{disc}}) \cong \mathrm{N}(\mathcal{C}),$$

see [Lur09], Example 1.1.5.8.

In general the homotopy coherent nerve of a simplicial category is not an ∞-category. But one can prove that N_Δ is right Quillen with respect to the Joyal model structure on **sSet** and the Bergner model structure on **sCat** (In fact, $\mathrm{N}_\Delta \dashv \mathfrak{C}$ is a Quillen equivalence, see [Lur09], Theorem 2.2.5.1.). In particular, the homotopy coherent nerve sends a Bergner-fibrant simplicial category to an ∞-category. Since the fibrant objects in the Bergner model structure are precisely the Kan-complex-enriched categories, N_Δ sends such to ∞-categories.

Definition 2.13. *(Underlying ∞-category of a simplicial model category)*
Let $\mathcal{M}$ be a simplicial model category (for a definition see [Lur09]*, Definition A.3.1.5.). Then the full subcategory*

$$\mathcal{M}^{cf} \xhookrightarrow{\text{full}} \mathcal{M}$$

of fibrant-cofibrant objects is Kan-complex-enriched (this follows directly from the definition of a simplicial model category, see [Lur09]*, Remark A.3.1.6.). Therefore the homotopy coherent nerve of $\mathcal{M}^{cf}$*

$$\mathrm{N}_\Delta(\mathcal{M}^{cf})$$

is an ∞-category which we call the underlying ∞-category of $\mathcal{M}$.

Example 2.14. (∞-category of spaces)
Consider the simplicial model category of simplicial $\mathbb{U}$-small sets $\mathbf{sSet}_\mathbb{U}$ with the Kan-Quillen model structure. The full subcategory of fibrant-cofibrant objects is precisely the full subcategory of $\mathbb{U}$-small Kan-complexes

$$\mathbf{Kan}_\mathbb{U} \xhookrightarrow{\text{full}} \mathbf{sSet}_\mathbb{U}.$$

Therefore the homotopy coherent nerve $\mathrm{N}_\Delta(\mathbf{Kan}_\mathbb{U})$ is an ∞-category. Note that we have to go to the universe $\mathbb{U}$, since the formation of all Kan-complexes is a construction that cannot be carried out in a single universe.
We call the ∞-category $\mathrm{N}_\Delta(\mathbf{Kan}_\mathbb{U})$ the ∞-category of spaces and denote it by

$$\mathcal{S} := \mathrm{N}_\Delta(\mathbf{Kan}_\mathbb{U}). \tag{2.15}$$

Example 2.16. (∞-category of small ∞-categories)
Consider the following Bergner-fibrant simplicial category $\mathcal{N}$: The objects are the small ∞-categories and for two such small ∞-categories $\mathcal{C}$ and $\mathcal{D}$ the mapping simplicial set $\mathrm{Map}_\mathcal{N}(\mathcal{C}, \mathcal{D})$ is defined to be the maximal Kan-subcomplex of the small ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Since the functor that sends an ∞-category to its maximal Kan-subcomplex

preserves products (it is right adjoint to the inclusion of the category of Kan-complexes into the category of ∞-categories, see [Joy08], Theorem 4.19.) we get a composition on $\mathcal{N}$. The ∞-category of all small ∞-categories is then defined by

$$\mathcal{C}\mathrm{at}_\infty := \mathrm{N}_\Delta(\mathcal{N}).$$

Another source of ∞-categories is given through Dwyer-Kan localization.

Definition 2.17. *(Dwyer-Kan localization)*
Let $\mathcal{C}$ be an ∞-category and $\mathcal{W} \subset \mathcal{C}_1$ a subset of the set of morphisms in $\mathcal{C}$. A Dwyer-Kan localization of $\mathcal{C}$ with respect to $\mathcal{W}$ is an ∞-category $\mathcal{C}[\mathcal{W}^{-1}]$ together with a functor $l : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$, satisfying the following universal property: For every ∞-category $\mathcal{D}$ the map

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \xrightarrow{\circ l} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

that is induced by l is fully faithful and the essential image consists of the full sub-∞-category of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *on all functors that send all members of $\mathcal{W}$ to invertible morphisms in $\mathcal{D}$.*

Remark 2.18. Using the model structure on marked simplicial sets Lurie shows that Dwyer-Kan localizations always exist (see [Lur17], Remark 1.3.4.2.).

Example 2.19. Let $\mathcal{M}$ be a model category (not necessarily a simplicial one). We let $\mathcal{W} \subset \mathrm{N}(\mathcal{M})_1$ be the set of all weak equivalences. Then we call

$$\mathrm{N}(\mathcal{M})[\mathcal{W}^{-1}]$$

the underlying ∞-category of $\mathcal{M}$.

If $\mathcal{M}$ is a simplicial model category, we now have two notions of an underlying ∞-catgeory of $\mathcal{M}$. But in fact they agree if the model category has functorial factorizations.

Theorem 2.20. *(see* [Lur17]*, Theorem 1.3.4.20. + Remark 1.3.4.16.)*
Let $\mathcal{M}$ be a simplicial model category with functorial factorizations and $\mathcal{W}$ the set of all weak equivalences. Then there is an equivalence of ∞-categories

$$\mathrm{N}_\Delta(\mathcal{M}^{cf}) \simeq \mathrm{N}(\mathcal{M})[\mathcal{W}^{-1}].$$

Remark 2.21. The last theorem in particular says that $\mathrm{N}_\Delta(\mathcal{M}^{cf})$ does not depend on the simplicial structure of $\mathcal{M}$.

The opposite of an ∞-category $\mathcal{C}$ is defined to be just the opposite simplicial set (see [Lur09], 1.2.1) and we denote it by

$$\mathcal{C}^{op}.$$

Definition 2.22. *(Presheaf ∞-category)*
Let $\mathcal{C}$ be an ∞-category. Then the ∞-category of presheaves on $\mathcal{C}$ is defined to be

$$\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}).$$

There is the ∞-categorical analog of the Yoneda embedding ([Lur09], 5.1.3) which we denote by

$$\mathrm{Yo}_\infty : \mathcal{C} \to \mathcal{P}(\mathcal{C}).$$

It has the property that for each object X in $\mathcal{C}$ the image $\mathrm{Yo}_\infty(X)$ is a presheaf, whose evaluation on an object Y is homotopy equivalent to the mapping space $\mathrm{Map}_{\mathcal{C}}(Y, X)$. Therefore the existence of such a Yoneda embedding already implicitly implies a functoriality of the mapping space construction which is not directly clear from the definition 2.9. We will often abuse notation and write

$$\mathrm{Map}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \mathcal{S}$$

instead of $\mathrm{Yo}_\infty(X)$. We do the same for the Yoneda embedding of $\mathcal{C}^{op}$ and use the notation

$$\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}.$$

A presheaf on $\mathcal{C}$ which is equivalent to $\mathrm{Yo}_\infty(X)$ for some object X is called representable.
As in ordinary category theory, we have the following Yoneda lemmas:

Lemma 2.23. *(Weak Yoneda lemma; see* [Lur09]*, Proposition 5.1.3.1.) The ∞-categorical Yoneda embedding*

$$\mathrm{Yo}_\infty : \mathcal{C} \to \mathcal{P}(\mathcal{C})$$

is fully faithful, i. e. the induced map on mapping spaces

$$\mathrm{Yo}_\infty : \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(\mathrm{Yo}_\infty(X), \mathrm{Yo}_\infty(Y))$$

is a homotopy equivalence of Kan-complexes for all objects $X, Y \in \mathcal{C}$.

Lemma 2.24. *(Strong Yoneda lemma; see* [Lur09]*, Lemma 5.1.5.2.) Let $X \in \mathcal{C}$ be an object of the ∞-category $\mathcal{C}$. Then the functors*

$$\mathcal{P}(\mathcal{C}) \xrightarrow{ev_X} \mathrm{Fun}(\Delta^0, \mathcal{S}) \xrightarrow{\simeq} \mathcal{S}$$

and

$$\mathcal{P}(\mathcal{C}) \xrightarrow{\mathrm{Yo}_\infty} \mathcal{P}(\mathcal{P}(\mathcal{C})) \xrightarrow{ev_{\mathrm{Yo}_\infty(X)}} \mathcal{S}$$

are equivalent in the ∞-category of functors $\mathrm{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{S})$.

Remark 2.25. On objects, the first of the two functors in the last lemma is given by

$$F \mapsto F(X)$$

and the second by

$$F \mapsto \mathrm{Map}_{\mathcal{P}(\mathcal{C})}(\mathrm{Map}_{\mathcal{C}}(-, X), F).$$

Recall that the join of two simplicial sets K and L is defined as the simplicial set whose n-simplices are

$$(K \star L)_n = K_n \amalg L_n \amalg \coprod_{i+j=n-1} K_i \times L_j$$

and whose face and degeneracy maps are the obvious ones (see for example [Joy08], chapter 3.2). The join construction is functorial in each variable. In particular for each K we have functors

$$- \star K : \mathbf{sSet} \to \mathbf{sSet}_{K/} : X \mapsto X \star K$$

and

$$K \star - : \mathbf{sSet} \to \mathbf{sSet}_{K/} : X \mapsto K \star X.$$

Both functors, $- \star K$ and $K \star -$, have right adjoints (see [Joy08], chapters 3.1.1 and 3.1.2).

Definition 2.26. *(Slice ∞-categories)*
Let $b : K \to X$ be a map of simplicial sets. The image of b under the right adjoint of $-\star K$ is denoted by $X_{/b}$ and is called the slice simplicial set over b. The image of b under the right adjoint of $K \star -$ is denoted by $X_{b/}$ and is called the slice simplicial set under b.

Theorem 2.27. *(see* [Joy08]*, Corollary 3.20.)*
If $F : K \to \mathcal{C}$ is a functor into an ∞-category, then $\mathcal{C}_{/F}$ and $\mathcal{C}_{F/}$ are ∞-categories.

Next, we want to recall the definitions of colimits and limits for ∞-categories. As in ordinary 1-category theory they are defined to be universal cocones and cones, respectively. For a functor

$$F : K \to \mathcal{C}$$

from a simplicial set K to an ∞-category $\mathcal{C}$ a cocone is a functor

$$\bar{F} : K^{\triangleright} := K \star \Delta^0 \to \mathcal{C}$$

which coincides with F on K and similar a cone is a functor

$$\bar{F} : K^{\triangleleft} := \Delta^0 \star K \to \mathcal{C}$$

which coincides with F on K. The ∞-category of cocones over F is defined to be $\mathcal{C}_{F/}$ and the ∞-category of cones over F is $\mathcal{C}_{/F}$. Note that the 0-simplices of $\mathcal{C}_{F/}$ and $\mathcal{C}_{/F}$ are really cocones over F and cones over F, respectively. We want to point out here that there are canonical maps (see [Lur09], p. 241)

$$K \times \Delta^1 \longrightarrow (K \times \Delta^1)/(K \times \{1\}) \longrightarrow K \star \Delta^0 \tag{2.28}$$

and

$$K \times \Delta^1 \longrightarrow (K \times \Delta^1)/(K \times \{0\}) \longrightarrow \Delta^0 \star K. \tag{2.29}$$

Definition 2.30. *(Colimits and limits)*
A colimit of a functor $F : K \to \mathcal{C}$ into an ∞-category is an initial object in the ∞-category $\mathcal{C}_{F/}$ of cocones over F. Dually, a limit of F is a terminal object in the ∞-category $\mathcal{C}_{/F}$ of cones over F.

Remark 2.31. We will often be a bit imprecise and refer to the cocone point

$$\Delta^0 \hookrightarrow K^{\triangleright} \longrightarrow \mathcal{C}$$

of an initial cocone of a functor F as the colimit of F and denote it by $\mathrm{colim}(F) \in \mathcal{C}$. We will proceed in the same way for limits.

Lemma 2.32. *(Universal property of colimits and limits in terms of mapping spaces; see* [Lur09]*, Lemma 4.2.4.3. (ii))*
Let $F : K \to \mathcal{C}$ be a functor into an ∞-category and $\bar{F} : K^{\triangleright} \to \mathcal{C}$ a cocone of F with cocone point $\bar{F}|_{\Delta^0}$. $\bar{F}$ and the map from 2.28 induce

$$K \times \Delta^1 \longrightarrow K \star \Delta^0 \xrightarrow{\bar{F}} \mathcal{C}. \tag{2.33}$$

Let furthermore

$$\mathrm{const} : \mathcal{C} \cong \mathrm{Fun}(\Delta^0, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$$

be the functor which associates to an object X of $\mathcal{C}$ the constant diagram at X. The map 2.33 is the same as a morphism

$$F \longrightarrow \mathrm{const}(\bar{F}|_{\Delta^0}) \tag{2.34}$$

in the ∞-category $\mathrm{Fun}(K, \mathcal{C})$*. Then* const *and 2.34 induce the following map on mapping spaces for any object $X \in \mathcal{C}$:*

$$\mathrm{Map}_{\mathcal{C}}(\bar{F}|_{\Delta^0}, X) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(F, \mathrm{const}(X)). \tag{2.35}$$

The statement of the lemma is now that 2.35 is a homotopy equivalence for all objects $X \in \mathcal{C}$ if and only if $\bar{F}$ is a colimit of F.
The dual statement for limits is that if $\bar{F} : K^{\triangleleft} \to \mathcal{C}$ is a cone of F then

$$\mathrm{Map}_{\mathcal{C}}(X, \bar{F}|_{\Delta^0}) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\mathrm{const}(X), F)$$

is a homotopy equivalence for all objects $X \in \mathcal{C}$ *if and only if* $\bar{F}$ *is a limit of* F*.*

As in ordinary 1-category theory we have special names for limits and colimits of diagrams of special shape:

Examples 2.36.

- $K = \coprod_{i \in I} \{*\}$
  A colimit in this case is called coproduct and a limit is called product. If I is empty, then the colimit is an initial object and the limit a terminal object.
- $K = \cdot \longrightarrow \cdot \longleftarrow \cdot$
  A limit in this case is called a pullback.
- $K = \cdot \longleftarrow \cdot \longrightarrow \cdot$
  A colimit in this case is called a pushout.
- K κ-filtered
  Let κ be any cardinal number. A simplicial set K is called κ-filtered if any map $g : D \to K$ from a κ-small simplicial set D to K has a cocone, i.e. if any such g extends to a map $\bar{g} : D^{\triangleright} \to K$. The colimit of a functor whose domain is κ-filtered is called κ-filtered colimit.

As in ordinary 1-category theory a (co)limit of a diagram of functors can be computed objectwise in the following sense.

Theorem 2.37. *(see* [Lur09]*, Corollary 5.1.2.3.)*
Let K *and* L *be simplicial sets and* $\mathcal{C}$ *an* ∞*-category and let*

$$F : K \to \mathrm{Fun}(L, \mathcal{C})$$

be a diagram of functors such that for every 0*-simplex* l *of* L *the induced diagram*

$$F_l : K \to \mathcal{C}$$

has a (co)limit in $\mathcal{C}$*. Then also* F *has a (co)limit. Moreover a (co)cone*

$$\bar{F} : (K^{\triangleright})K^{\triangleleft} \to \mathrm{Fun}(L, \mathcal{C})$$

is a (co)limit if and only if for every 0*-simplex* l *of* L *the induced (co)cone*

$$\bar{F}_l : (K^{\triangleright})K^{\triangleleft} \to \mathcal{C}$$

is a (co)limit in $\mathcal{C}$*.*

We call an ∞-category complete if all diagrams whose shape K is small have a limit and we call it cocomplete if all such diagrams have a colimit.

Examples 2.38.

- Let $\mathcal{M}$ be a combinatorial simplicial model category. Then the associated ∞-categroy $\mathrm{N}_{\Delta}(\mathcal{M}^{cf})$ is both complete and cocomplete (see [Lur09], Corollary 4.2.4.8.).

- The ∞-category of spaces $\mathcal{S}$ is both complete and cocomplete. This is a special case of the first example, since **sSet** with the Kan-Quillen model structure is a combinatorial simplicial model category.
- Let $\mathcal{C}$ be an ∞-category. Then the ∞-category $\mathcal{P}(\mathcal{C})$ of presheaves on $\mathcal{C}$ is both complete and cocomplete. This follows from the last example and 2.37.

We have already stated the Yoneda lemmas above. Now we want to state two further properties of the Yoneda embedding Yo_∞ which are very important.

Theorem 2.39. *(Universal property of* Yo_∞*; see* [Lur09]*, Theorem 5.1.5.6.)*
Let $\mathcal{C}$ *be a small and* $\mathcal{D}$ *a cocomplete* ∞*-category. Then precomposition with the Yoneda embedding,*

$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\circ \mathrm{Yo}_\infty} \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

(Fun^L *is the* ∞*-category of colimit preserving functors) is an equivalence of* ∞*-categories, which means a weak equivalence in the Joyal model structure on* **sSet***.*

Because of the last theorem and since $\mathcal{P}(\mathcal{C})$ is cocomplete, we say that the Yoneda embedding exhibits $\mathcal{P}(\mathcal{C})$ as the free cocompletion of $\mathcal{C}$.
Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and $p : K \to \mathcal{C}$ a diagram in $\mathcal{C}$ with colimit $\bar{p} : K^{\triangleright} \to \mathcal{C}$. We say that F preserves the colimit $\bar{p}$ if the cocone

$$K^{\triangleright} \xrightarrow{\bar{p}} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a colimit of the diagram $F \circ p$. Preservation of limits is defined analogously. As in ordinary 1-category theory one has the following result:

Proposition 2.40. *(see* [Lur09]*, Proposition 5.1.3.2.)*
Let $\mathcal{C}$ *be an* ∞*-category. The Yoneda embedding* $\mathrm{Yo}_\infty : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ *preserves all limits that exist in* $\mathcal{C}$*.*

Next, we want to recall the concept of adjoint functors in the ∞-categorical setting:

Definition 2.41. *(Adjoint functors)*
Let

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

be two functors between ∞-categories. We say that F is left adjoint to G (and write $F \dashv G$) if there is a unit transformation, i.e. a morphism $u : \mathrm{id}_{\mathcal{C}} \to G \circ F$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ *such that for all objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ the composition*

(2.42)
$$\mathrm{Map}_{\mathcal{D}}(F(X), Y) \xrightarrow{G} \mathrm{Map}_{\mathcal{C}}(G(F(X)), G(Y)) \xrightarrow{\circ u_X} \mathrm{Map}_{\mathcal{C}}(X, G(Y))$$

is a homotopy equivalence of Kan-complexes.

Remark 2.43. There is an alternative definition: Two functors
$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$
are defined to be adjoint if there is the following data: An ∞-category $\mathcal{M}$ together with a map
$$\mathcal{M} \to \Delta^1$$
which is cartesian and cocartesian at the same time with the following additional properties: The fibers $\mathcal{M}_0$ and $\mathcal{M}_1$ are equivalent to $\mathcal{C}$ and $\mathcal{D}$, respectively, and the two functors between $\mathcal{M}_0$ and $\mathcal{M}_1$ that are induced by the ∞-categorical Grothendieck constructions are equivalent to F and G. The concept of the ∞-categorical Grothendieck construction will be recalled below and the equivalence of the two definitions of adjoint functors can be found in [Lur09], proposition 5.2.2.8.

Some of the properties of adjoints one is used to have in ordinary 1-category theory carry over to ∞-category theory:

Proposition 2.44. *(Essential uniqueness of adjoints; see* [Lur09]*, Proposition 5.2.6.2.)*
Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Denote by $\mathrm{Fun}^{le}(\mathcal{C}, \mathcal{D})$ *the ∞-category of left adjoint functors from $\mathcal{C}$ to $\mathcal{D}$ and by* $\mathrm{Fun}^{ri}(\mathcal{D}, \mathcal{C})$ *the ∞-category of right adjoint functors from $\mathcal{D}$ to $\mathcal{C}$. Then there is an equivalence of ∞-categories*
$$\mathrm{Fun}^{le}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{ri}(\mathcal{D}, \mathcal{C})^{op}$$
which on objects sends a functor F to a choice of a right adjoint of F.

Remark 2.45. Since the homotopy fiber (with respect to the Joyal model structure) of an equivalence of ∞-categories is a contractible Kan-complex, the last proposition says that adjoints are unique up to a contractible space of choices.

Proposition 2.46. *(Adjoints preserve (co)limits; see* [Lur09]*, Proposition 5.2.3.5.)*
Let
$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

be an adjoint pair of functors. Then F preserves all colimits that exist in $\mathcal{C}$ and G preserves all limits that exist in $\mathcal{D}$.

Another useful feature of adjointness is that it descends to the homotopy category.

Proposition 2.47. *(see* [Lur09]*; Proposition 5.2.2.9.)*
An adjoint pair of functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

between ∞-categories induces an adjunction

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathrm{Ho}(G)$$

on the homotopy categories.

2.1.2. *∞-categorical Grothendieck construction.* Recall that in ordinary category theory there is the correspondence between Grothendieck opfibrations and pseudofunctors into the 2-category of small categories. This correspondence is called the Grothendieck construction. The general idea behind the Grothendieck construction is that a family of categories which is parametrized by $\mathcal{C}$ should be completely describable through a functor $p : \mathcal{E} \to \mathcal{C}$ such that the members of the family are precisely the fibers of p. And, in fact, if one has a Grothendieck opfibration $p : \mathcal{E} \to \mathcal{C}$, then the Grothendieck construction associates to it a pseudofunctor

$$\mathcal{C} \to \mathbf{Cat}$$

which is given on objects by $X \mapsto p^{-1}(X)$.
As an application of the Grothendieck construction for ordinary categories, it is for example possible, to encode all the data of a monoidal category $\mathcal{A}$ in a single Grothendieck opfibration $p : \mathcal{E} \to \mathrm{N}(\Delta^{op})$ such that $p^{-1}(\{1\})$ is equivalent to $\mathcal{A}$. Something similar is possible for a symmetric monoidal category which one can describe as a Grothendieck opfibration over $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ where $\mathcal{F}\mathrm{in}_*$ is the category of finite pointed sets. In ∞-category theory, the description of (symmetric) monoidal categories through single functors is turned into a definition. We will see this in the chapter about higher commutative algebra.
The analog of a Grothendieck opfibration for ∞-categories is a cocartesian fibration and in fact Lurie showed that there is an equivalence between the ∞-category of cocartesian fibrations into a fixed ∞-category $\mathcal{C}$ and the ∞-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty)$ (see theorem 2.53 below).
We want to recall here the definition of a cocartesian fibration and therefore we have to recall the notion of a cocartesian edge first:

Definition 2.48. *(Cocartesian edge)*
Let $p : \mathcal{C} \to \mathcal{D}$ be functor between ∞-categories. A morphism $f : X_1 \to X_2$ in $\mathcal{C}$ is called p-cocartesian if the canonical map

$$\mathcal{C}_{f/} \to \mathcal{C}_{X_1/} \times_{\mathcal{D}_{p(X_1)/}} \mathcal{D}_{p(f)/} \tag{2.49}$$

is a trivial fibration of simplicial sets. In this case we also call f a p-cocartesian lift of $p(f)$.

Remark 2.50. In this remark we want to make the definition of a p-cocartesian edge a little bit more clear. We want to do this by writing out what it means for 2.49 to being a trivial fibration by checking what happens on the level of 0-simplices. The set of 0-simplices of the left handside of 2.49 consists of 2-simplices in $\mathcal{C}$ of the form

$$\begin{tikzcd}
\overset{X_1}{\bullet} \arrow[rr, "f"] \arrow[dr] & & \overset{X_2}{\bullet} \arrow[dl] . \\
& \bullet &
\end{tikzcd}$$

The set of 0-simplices of the right hand side is given by pairs (g, σ), where

$$g : \overset{X_1}{\bullet} \to \bullet$$

is a 1-simplex in $\mathcal{C}$ and σ is a 2-simplex in $\mathcal{D}$ of the form

$$\begin{tikzcd}
\overset{p(X_1)}{\bullet} \arrow[rr, "p(f)"] \arrow[dr, "p(g)"'] & & \overset{p(X_2)}{\bullet} \arrow[dl] . \\
& \bullet &
\end{tikzcd} \tag{2.51}$$

Therefore, if f is p-cocartesian, the following holds: Whenever we have a diagram of the form

$$\begin{tikzcd}
\overset{X_1}{\bullet} \arrow[rr, "f"] \arrow[dr, "g"'] & & \overset{X_2}{\bullet} \\
& \bullet &
\end{tikzcd}$$

in $\mathcal{C}$ such that there exists a 2-simplex in $\mathcal{D}$ of the form 2.51 then there exists also a 2-simplex σ'

$$\begin{tikzcd}
\overset{X_1}{\bullet} \arrow[rr, "f"] \arrow[dr, "g"'] & & \overset{X_2}{\bullet} \arrow[dl] \\
& \bullet &
\end{tikzcd}$$

in $\mathcal{C}$ with $p(\sigma') = \sigma$. Moreover the space of such σ''s is a contractible Kan-complex.

Definition 2.52. *(Cocartesian fibration)*
A map $p : \mathcal{C} \to \mathcal{D}$ between ∞-categories is called cocartesian fibration if it is an inner fibration and if the following condition holds: For every $X_1 \in \mathcal{C}$ and every morphism $h : p(X_1) \to Y_2$ in $\mathcal{D}$ there exists a p-cocartesian lift $X_1 \to X_2$ of h.
For a small ∞-category $\mathcal{C}$ the full sub-∞-category of $(\mathcal{C}\mathrm{at}_\infty)_{/\mathcal{C}}$ on all cocartesian fibrations with target $\mathcal{C}$ will be denoted by

$$\mathrm{CoCart}(\mathcal{C}).$$

Now here is the analog of the Grothendieck construction for ∞-categories:

Theorem 2.53. *(∞-categorical Grothendieck construction; follows directly from* [Lur09]*, Theorem 3.2.0.1.)*
Let $\mathcal{C}$ be a small ∞-category. Then there is an equivalence of ∞-categories

$$\mathrm{CoCart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty).$$

The functor $F_p : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ that corresponds to a cocartesian fibration p satisfies $F_p(X) \simeq p^{-1}(X)$ for every object $X \in \mathcal{C}$.

There are also the dual notions of cartesians edges and cartesian fibrations. The ∞-categorical Grothendieck construction for cartesian fibrations then gives an equivalence

$$\mathrm{Cart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{C}\mathrm{at}_\infty).$$

2.1.3. *Presentable ∞-categories.* In this section we want to recall the notion of presentable ∞-categories (in the sense of Lurie, [Lur09], chapter 5.5). To avoid confusion about the term "presentable" we want to stress right away that presentability is a direct generalization of the 1-categorical property of being locally presentable which is studied in the book [AR94]. ∞-categories that are presentable are not small themselves, but they are generated by a small sub-∞-category in a certain sense. A small ∞-category that is complete or cocomplete must be equivalent as an ∞-category to the nerve of a partially ordered set. In this sense small ∞-categories are almost never complete or cocomplete. In contrast, presentable ∞-categories are always cocomplete (even complete). This interplay of small generation and cocompleteness allows a good structure theory for presentable ∞-categories (see Theorem 2.63 below). This structure theory then can be used to prove important statements about existence of adoints of functors between presentable ∞-categories.

We begin with Ind-completions of small ∞-categories.

Definition 2.54. *(*Ind_κ*-completion)*
Let κ *be a regular cardinal and* $\mathcal{C}$ *an* ∞*-category. Then the* Ind_κ*-completion of* $\mathcal{C}$*, which we denote by* $\mathrm{Ind}_\kappa(\mathcal{C})$*, is defined as the full sub-*∞*-category of* $\mathcal{P}(\mathcal{C})$ *on all presheaves that can be written as a small* κ*-filtered colimit of representable presheaves.*

Remark 2.55. Obviously, the Yoneda embedding factors through the Ind_κ-completion:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{Yo}_\kappa"] \arrow[rr, bend left, "\mathrm{Yo}_\infty"] & \mathrm{Ind}_\kappa(\mathcal{C}) \arrow[r, hook, "full"] & \mathcal{P}(\mathcal{C}).
\end{tikzcd}$$

It is also clear that Yo_κ is fully faithful.

Recall that $\mathcal{P}(\mathcal{C})$ should be thought of as the free cocompletion of $\mathcal{C}$. Now the Ind_κ-completion of a small ∞-category is an analogous construction: Not all small colimits but all small κ-filtered colimits get freely adjoined. The next two statements make this precise:

Proposition 2.56. *(see* [Lur09]*, Proposition 5.3.5.3.)*
Let $\mathcal{C}$ *be a small* ∞*-category. Then* $\mathrm{Ind}_\kappa(\mathcal{C})$ *has all small* κ*-filtered colimits. Moreover, the inclusion*

$$\mathrm{Ind}_\kappa(\mathcal{C}) \xhookrightarrow{\mathit{full}} \mathcal{P}(\mathcal{C})$$

preserves all these κ*-filtered colimits.*

Proposition 2.57. *(Universal property of* Yo_κ*; see* [Lur09]*, Proposition 5.3.5.10.)*
Let $\mathcal{C}$ *be an* ∞*-category and* $\mathcal{D}$ *an* ∞*-category that has all small* κ*-filtered colimits. Then precomposition with* Yo_κ*,*

$$\mathrm{Fun}^\kappa(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \xrightarrow{\circ \mathrm{Yo}_\kappa} \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

(Fun^κ *is the* ∞*-category of functors that preserve all* κ*-filtered colimits) is an equivalence of* ∞*-categories.*

In general, the Yoneda embedding Yo_∞ preserves almost no colimits. For the Ind_κ-completion we have the following statement:

Proposition 2.58. *(see* [Lur09]*, Proposition 5.3.5.14.)*
Let $\mathcal{C}$ *be an* ∞*-category. Then* $\mathrm{Yo}_\kappa : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ *preserves all* κ*-small colimits that exist in* $\mathcal{C}$*.*

Definition 2.59. *(Presentable* ∞*-category)*
An ∞*-category* $\mathcal{D}$ *is called presentable if it is cocomplete and if there*

exists a small ∞-category $\mathcal{C}$ such that $\mathcal{D}$ is equivalent as an ∞-category to $\mathrm{Ind}_\kappa(\mathcal{C})$.

As we already pointed out above, presentable ∞-categories have the following property:

Proposition 2.60. *(see* [Lur09]*, Corollary 5.5.2.4.)*
Presentable ∞-categories are complete.

To formulate the structure theorem for presentable ∞-categories we need the notion of localization functors.

Definition 2.61. *(Localization functor)*

1. *A functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is called a localization functor if it has a right adjoint which is fully faithful. In this case $\mathcal{D}$ is called a localization of $\mathcal{C}$.*
2. *A localization $\mathcal{D}$ of $\mathcal{C}$ is called accessible if there exists a regular cardinal κ such that the fully faithful right adjoint of the localization functor preserves all small κ-filtered colimits which exist in $\mathcal{D}$.*

Remark 2.62. We want to point out that localizations in the sense of the last definition can be described as certain Dwyer-Kan localizations (see definition 2.17). Let $F : \mathcal{C} \to \mathcal{D}$ be a localization functor with fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$. We define $\mathcal{W} \subset \mathcal{C}_1$ to consist of all morphisms that get sent to an equivalence by F. Then the composition

$$\mathcal{D} \to \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

is an equivalence of ∞-categories (see [Lur17], Example 1.3.4.3.).

Now we are ready to state the structure theorem for presentable ∞-catgegories:

Theorem 2.63. *(Simpson's theorem; see* [Lur09]*, Theorem 5.5.1.1.)*
An ∞-category $\mathcal{C}$ is presentable if and only if there exists a small ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is equivalent to an accessible localization of $\mathcal{P}(\mathcal{D})$.

This structure theorem can be used to prove the following theorem about existence of adjoint functors which is very useful in practice:

Theorem 2.64. *(Adjoint functor theorem; see* [Lur09]*, Corollary 5.5.2.9.)*
A functor between presentable ∞-categories is a left adjoint if and only if it preserves all small colimits; it is a right adjoint if and only if it preserves all small limits and all small κ-filtered colimits for some regular cardinal κ.

Remark 2.65. In [NRS] the authors show that there is an ∞-categorical analog of Freyd's general adjoint functor theorem ("GAFT") and they deduce the adjoint functor theorem 2.64 for presentable ∞-categories from GAFT without using the structure theorem 2.63.

2.2. **Higher commutative algebra.**

Later in this thesis, we will deal with symmetric monoidal ∞-categories and we will do commutative algebra in symmetric monoidal ∞-categories. Therefore we want to list now all the definitions and statements from higher commutative algebra we will need.

2.2.1. *Symmetric monoidal ∞-categories.*

As we already pointed out earlier, the definition of a symmetric monoidal ∞-category makes use of the ∞-categorical Grothendieck construction. This makes it possible to encode all the data within one single object, namely a certain cocartesian fibration over $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$, where $\mathcal{F}\mathrm{in}_*$ is the standard skeleton of the category of finite pointed sets. The objects of $\mathcal{F}\mathrm{in}_*$ are the finite pointed sets of the form $\langle n \rangle := \{*, 1, ..., n\}$ and the morphisms are the maps that send $*$ to $*$. The one-point set $\{*\}$ is denoted by $\langle 0 \rangle$.
Recall from theorem 2.53 that a cocartesian fibration $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ corresponds to a functor $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \to \mathbf{Cat}_\infty$ which on objects is given by sending $\langle n \rangle$ to the fiber $\mathcal{C}^{\otimes}_{\langle n \rangle} := p^{-1}(\langle n \rangle)$. In particular, every map in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ induces a functor between fibers. For example, the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ that send everything to the point except i induce functors $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle}$. These functors together induce a functor

$$\rho_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n \tag{2.66}$$

which is called a Segal map.

Definition 2.67. *(Symmetric monoidal ∞-category)*
A symmetric monoidal ∞-category is a cocartesian fibration

$$\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$$

satisfying the following property: For every natural number n the Segal map $\rho_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$ from 2.66 induces an equivalence of ∞-categories. This includes the case $n = 0$ which says that the fiber over $\langle 0 \rangle$ is equivalent to the trivial one-point ∞-category Δ^0.
The fiber over $\langle 1 \rangle$ is called the underlying ∞-category of the symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and is denoted by $\mathcal{C}$. We often call an ∞-category symmetric monoidal if it is the fiber over $\langle 1 \rangle$ of a symmetric monoidal ∞-category.

Remark 2.68. We want to recall how the unit and the tensor product functor are encoded in a cocartesian fibration that defines a symmetric monoidal ∞-category.
The unique map $\iota : \langle 0\rangle \to \langle 1\rangle$ induces a functor

(2.69)
$$\begin{tikzcd} \iota_! : \Delta^0 \arrow[r, dash, "\simeq"] & \mathcal{C}^{\otimes}_{\langle 0\rangle} \arrow[r] & \mathcal{C}^{\otimes}_{\langle 1\rangle} = \mathcal{C} \end{tikzcd}$$

that picks out a unit object $\mathbf{1}$ for the symmetric monoidal structure.
The map $\phi : \langle 2\rangle \to \langle 1\rangle$ that sends everything to 1 except $*$ induces the tensor product functor

(2.70)
$$\begin{tikzcd} \otimes : \mathcal{C} \times \mathcal{C} \arrow[r, dash, "\simeq"] & \mathcal{C}^{\otimes}_{\langle 2\rangle} \arrow[r, "\phi_!"] & \mathcal{C}^{\otimes}_{\langle 1\rangle} = \mathcal{C} \end{tikzcd}$$

of the symmetric monoidal ∞-category.
Let $(12) : \langle 2\rangle \to \langle 2\rangle$ be the map in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ that swaps 1 and 2. Since the diagram

$$\begin{tikzcd} \langle 2\rangle \arrow[rr, "(12)"] \arrow[dr, "\phi"'] & & \langle 2\rangle \arrow[dl, "\phi"] \\ & \langle 1\rangle & \end{tikzcd}$$

commutes, we get an associated symmetry 2-cell in $\mathcal{C}\mathrm{at}_\infty$

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[rr, "(12)_!"] \arrow[dr, "\otimes"'] & & \mathcal{C} \times \mathcal{C} \arrow[dl, "\otimes"] \\ & \mathcal{C}. & \end{tikzcd}$$

Of course, we can do the same for any permutation $\tau : \langle n\rangle \to \langle n\rangle$.
We also want to point out that the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a symmetric monoidal ∞-category carries a symmetric monoidal structure in which the tensor product functor is induced by 2.70 and the unit is induced by 2.69 (see[Lur17], Remark 2.1.2.20.).

Example 2.71. (Cartesian and cocartesian symmetric monoidal structure; see [Lur17], Proposition 2.4.1.5.)
Let $\mathcal{C}$ be an ∞-category with finite products. Then there exists a symmetric monoidal ∞-category

$$\mathcal{C}^{\times} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$$

whose underlying ∞-category is equivalent to $\mathcal{C}$ und whose tensor product functor is given on objects by

$$(X, Y) \mapsto X \times Y.$$

Analogously, for an ∞-category $\mathcal{C}$ with finite coproducts there is a symmetric monoidal ∞-category

$$\mathcal{C}^{\amalg} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$$

whose tensor product functor is given on objects by taking the coproduct.

Example 2.72. (Objectwise symmetric monoidal structure on functor ∞-categories; see [Lur17], Remark 2.1.3.4.)
Let K be any simplicial set and $\mathcal{D}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ a symmetric monoidal ∞-category. Then there is a canonical symmetric monoidal structure on $\mathrm{Fun}(K, \mathcal{D})$ which is called the objectwise symmetric monoidal structure. More precisely, the pullback in simplicial sets

$$\begin{tikzcd}
\mathrm{Fun}(K, \mathcal{D})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(K, \mathcal{D}^{\otimes}) \arrow[d] \\
\mathrm{N}(\mathcal{F}\mathrm{in}_*) \arrow[r, "\mathrm{const}"] & \mathrm{Fun}(K, \mathrm{N}(\mathcal{F}\mathrm{in}_*))
\end{tikzcd}$$

defines a symmetric monoidal ∞-category $\mathrm{Fun}(K, \mathcal{D})^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ whose underlying ∞-category is $\mathrm{Fun}(K, \mathcal{D})$. Moreover, there is an equivalence

$$\mathrm{CAlg}(\mathrm{Fun}(K, \mathcal{D})) \simeq \mathrm{Fun}(K, \mathrm{CAlg}(\mathcal{D})).$$

Definition 2.73. *(Presentably symmetric monoidal ∞-category)*
A symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ whose underlying ∞-category is presentable is called presentably symmetric monoidal if the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits in each variable separately.

Although we will mainly be interested in symmetric monoidal ∞-categories, we want to also recall the notion of a monoidal ∞-category. One reason for that is that there is an easy definition for ∞-categories of module objects in monoidal ∞-categories and we will define ∞-categories of module objects in symmetric monoidal ∞-categories using the underlying monoidal ∞-category of a symmetric monoidal one (see definition 2.101 below). We want to point out here that the definition of monoidal ∞-category that we use is the same as the definition in Derived algebraic geometry II ([Lur07], Definition 1.1.2.) and Groth's notes ([Gro15], Definition 4.14.), while Lurie calls these objects $\mathbb{A}_\infty$-monoidal ∞-categories in Higher algebra ([Lur17], Definition 4.1.3.6.) and has a different definition for a monoidal ∞-category that suits better to his language of ∞-operads (*loc.cit.* Definition 4.1.1.10.). But in any case, both definitions give the same theory of monoidal ∞-categories in the sense that one can construct for both definitions an associated ∞-category of monoidal ∞-categories and these are equivalent ([Lur17], Theorem 4.1.3.14.).

Definition 2.74. *(Monoidal ∞-category)*
A monoidal ∞-category is a cocartesian fibration

$$p : \mathcal{C}^{\otimes} \to \mathrm{N}(\Delta^{op})$$

fulfilling a Segal condition analogous to the symmetric case above: Let n be fixed. Consider for each $1 \leq i \leq n$ the unique map $f_i : \{0,1\} \to \{0,1,...,n\}$ in Δ with image $\{i-1, i\}$. These f_i's induce corresponding maps $f_i^{op} : \{0,1,...,n\} \to \{0,1\}$ in Δ^{op}. Since p is a cocartesian fibration, the $(f_i^{op})'$s induce functors on the fibers

$$(f_i)_! : \mathcal{C}^{\otimes}_{\{0,1,\dots,n\}} \to \mathcal{C}^{\otimes}_{\{0,1\}}$$

and therefore also a functor

$$f_! : \mathcal{C}^{\otimes}_{\{0,1,\dots,n\}} \to (\mathcal{C}^{\otimes}_{\{0,1\}})^n. \tag{2.75}$$

The requirement for p being a monoidal ∞-category now is that 2.75 is an equivalence for all n.
The fiber $\mathcal{C}^{\otimes}_{\{0,1\}}$ is called the underlying ∞-category of $\mathcal{C}^{\otimes}$ and it is denoted by $\mathcal{C}$.

Now we want to explain how one can associate an underlying monoidal ∞-category to a symmetric monoidal ∞-category. For this we consider the following functor

$$\psi : \Delta^{op} \to \mathcal{F}\mathrm{in}_*,$$

which is given on objects by $\psi(\{0,1,...,n\}) := \langle n \rangle$ and which sends a map $f : \{0,1,...,k\} \to \{0,1,...,n\}$ in Δ to

$$\psi(f)(i) := \begin{cases} j, & \text{if there exists } j \text{ with } i \in [f(j-1)+1, f(j)], \\ *, & \text{else} \end{cases}$$

for all $i \in \{0,1,...,n\}$.

Definition 2.76. *(Underlying monoidal ∞-category of a symmetric monoidal ∞-category)*
Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a symmetric monoidal ∞-category. Then the underlying monoidal ∞-category of p is defined via the following

pullback in **sSet***:*

$$\begin{tikzcd}
\mathrm{U}(\mathcal{C}^{\otimes}) \arrow[r] \arrow[d, "U(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\
\mathrm{N}(\Delta^{op}) \arrow[r, "\mathrm{N}(\psi)"] & \mathrm{N}(\mathcal{F}\mathrm{in}_*).
\end{tikzcd}$$

Recall that a morphism $f : \{0, 1, ..., n\} \to \{0, 1, ..., k\}$ in Δ is convex if it is injective and the image is an interval without gaps and that a morphism $g : \langle n\rangle \to \langle m\rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is inert if for each $i \in \langle m\rangle \setminus \{*\}$ the preimage $p^{-1}(\{i\})$ consists of precisely one element. These two notions of convex and inert morphisms are related by the fact that a morphism f in Δ is convex if and only if $\psi(f)$ is inert in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$.

Definition 2.77. *((Commutative) algebra objects)*

1. *An algebra object in a monoidal* ∞*-category* $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\Delta^{op})$ *is a section of* p *sending convex morphisms to* p*-cocartesian edges. We denote the* ∞*-category of algebra objects by* $\mathcal{A}\mathrm{lg}(\mathcal{C})$*.*
2. *A commutative algebra object in a symmetric monoidal* ∞*-category* $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *is a section of* p *sending inert morphisms to* p*-cocartesian morphisms. We denote the* ∞*-category of commutative algebra objects by* $\mathrm{CAlg}(\mathcal{C})$*.*

In both cases, the map that evaluates at $\langle 1\rangle$ *gives a forgetful functor to* $\mathcal{C}$*.*

If one has a symmetric monoidal ∞-category and a commutative algebra object F in it then the unique map $\langle 0\rangle \to \langle 1\rangle$ in $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ induces a map from the unit object $\mathbf{1}$ in $\mathcal{C}$ to the underlying object in $\mathcal{C}$ of F.

Example 2.78. If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category, the ∞-category $\mathrm{CAlg}(\mathcal{C})$ has an initial object. Moreover an object F in $\mathrm{CAlg}(\mathcal{C})$ is initial if and only if the map from the unit in $\mathcal{C}$ to the underlying object of F is an equivalence (see [Lur17], Corollary 3.2.1.9.). In this sense the unit object of a symmetric monoidal ∞-category is a commutative algebra object and it is the initial one.

We will later need the existence of a symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ for a symmetric monoidal ∞-category $\mathcal{C}$:

Theorem 2.79. *(see* [Lur17]*, Example 3.2.4.4.)*
Let $\mathcal{C}$ *be a symmetric monoidal* ∞*-category. Then the* ∞*-category* $\mathrm{CAlg}(\mathcal{C})$ *carries a symmetric monoidal structure such that the forgetful functor*

$$V : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$$

has a refinement to a symmetric monoidal functor.

As we said above, a morphism in Δ is convex if and only if it gets sent by ψ to an inert one. One can check that this implies that the formation of the underlying monoidal ∞-category of a symmetric monoidal ∞-category (2.76) induces a functor

$$V : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{A}\mathrm{lg}(\mathrm{U}(\mathcal{C})). \tag{2.80}$$

Definition 2.81. *((Lax) symmetric monoidal functor)*
Let $p, q : \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *be two symmetric monoidal ∞-catgeories.*

1. *A symmetric monoidal functor from* $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *to* $q : \mathcal{D}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *is a map* $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *which makes the diagram*

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\ & \mathrm{N}(\mathcal{F}\mathrm{in}_*) & \end{tikzcd}$$

commutative and which sends p-cocartesian edges to q-cocartesian edges.

2. *A lax symmetric monoidal functor from* $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *to* $q : \mathcal{D}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *is a map* $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *which makes the diagram*

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\ & \mathrm{N}(\mathcal{F}\mathrm{in}_*) & \end{tikzcd}$$

commutative and which sends p-cocartesian lifts of inert morphisms to q-cocartesian edges.

The ∞-category of symmetric monoidal functors from p to q is denoted by

$$\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$$

and the ∞-category of lax symmetric monoidal functors is denoted by

$$\mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D}).$$

There are analogous notions in the case of monoidal ∞-categories, but we won't need them.

Example 2.82. If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories with finite products, then they both can be equipped with the cartesian symmetric monoidal structure (example 2.71). A functor $\mathcal{C} \to \mathcal{D}$ that preserves finite products now determines up to equivalence a unique symmetric monoidal functor and in fact one can show that the ∞-category of symmetric monoidal functors $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ is equivalent to the ∞-category of finite products preserving functors from $\mathcal{C}$ to $\mathcal{D}$ in this case (see [Lur17], Corollary 2.4.1.8.).

Remark 2.83. Note that a commutative algebra object in a symmetric monoidal ∞-category $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is the same thing as a lax symmetric monoidal functor from the trivial symmetric monoidal ∞-category $\mathrm{N}(\mathcal{F}\mathrm{in}_*) = \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ to $\mathcal{C}^{\otimes}$. We want to point out that in this sense the definition of commutative algebra objects in a symmetric monoidal ∞-category is really analogous to the classical situation. If one has an ordinary symmetric monoidal category $\mathcal{C}$, then one can describe commutative algebra objects in there as lax symmetric monoidal functors from the one-point category with the trivial symmetric monoidal structure to $\mathcal{C}$.

Recall from Definition 2.61 that a localization is a functor that has a fully faithful right adjoint. We now want to study localizations which behave well with respect to symmetric monoidal structures:

Definition 2.84. *(Symmetric monoidal localization)*
Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and $L : \mathcal{C} \to \mathcal{D}$ a localization functor where $\mathcal{C}$ is the underlying ∞-category of $\mathcal{C}^{\otimes}$. L is called symmetric monoidal if the following condition is satisfied: If $f : X \to Y$ is a morphism of $\mathcal{C}$ such that $L(f)$ is an equivalence then $L(f \otimes id) : L(X \otimes Z) \to L(X \otimes Z)$ is an equivalence for every object $Z \in \mathcal{C}$.

We will need the following statements about symmetric monoidal localizations:

Lemma 2.85. *(see* [Lur17]*, Proposition 2.2.1.9.)*
Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and $L : \mathcal{C} \to \mathcal{D}$ a symmetric monoidal localization with right adjoint $R : \mathcal{D} \to \mathcal{C}$ where $\mathcal{C}$ is the underlying ∞-category of $\mathcal{C}^{\otimes}$. Then there exists a preferred symmetric monoidal structure $\mathcal{D}^{\otimes}$ with underlying ∞-category $\mathcal{D}$ and a lift of L to a symmetric monoidal functor

$$L^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}.$$

Moreover the right adjoint R refines to a lax symmetric monoidal functor

$$R^{\otimes} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}.$$

Corollary 2.86. *(see* [GGN15]*, Lemma 3.6.)*
Let $L : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal localization where $\mathcal{C}$ is the

underlying ∞-category of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$. Then there is a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{C}) \arrow[r, "\bar{L}"] \arrow[d, "V"'] & \mathrm{CAlg}(\mathcal{D}) \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "L"] & \mathcal{D},
\end{tikzcd}$$

where the symmetric monoidal structure on $\mathcal{D}$ is the one from lemma 2.85. Furthermore, the functor $\bar{L}$ is also a localization functor.

In the later chapters of this thesis concrete examples of symmetric monoidal ∞-categories will come from symmetric monoidal 1-categories (via the symmetric monoidal nerve) and Dwyer-Kan localizations of such.

Construction 2.87. *(Symmetric monoidal nerve)*
Let $\mathcal{C}$ be a symmetric monoidal category. Then we define the following category $\mathcal{C}^{\otimes}$: Its objects are tuples $(\langle n\rangle, X_1, ..., X_n)$ where $\langle n\rangle \in \mathcal{F}\mathrm{in}_*$ is the finite pointed set with $n+1$ elements and the X_i are objects of $\mathcal{C}$. The Hom*-sets are defined by the following formula:*

$$\mathrm{Hom}_{\mathcal{C}^{\otimes}}((\langle n\rangle, X_1, ..., X_n), (\langle m\rangle, Y_1, ..., Y_m)) :=$$
$$:= \coprod_{f:\langle n\rangle \to \langle m\rangle} \prod_{1\leq j\leq m} \mathrm{Hom}_{\mathcal{C}}(\bigotimes_{f(i)=j} X_i, Y_j).$$

There is an obvious functor

$$\mathcal{C}^{\otimes} \to \mathcal{F}\mathrm{in}_*$$

which induces a map of simplicial sets

$$\mathrm{N}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*). \tag{2.88}$$

Proposition 2.89. *(see* [Lur17]*, Example 2.1.2.21.)*
If $\mathcal{C}$ is a symmetric monoidal category, then the map from 2.88

$$\mathrm{N}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$$

is a symmetric monoidal ∞-category.

The next thing we want to discuss is symmetric monoidal Dwyer-Kan localization.

Definition 2.90. *(Symmetric monoidal Dwyer-Kan localization)*
Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a symmetric monoidal ∞-category and $\mathcal{W}$ a subset of morphisms of the underlying ∞-category $\mathcal{C}$. A symmetric

monoidal Dwyer-Kan localization of p *is a symmetric monoidal* ∞*-category* $q : \mathcal{E}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *together with a symmetric monoidal functor*

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[rr, "l"] \arrow[dr, "p"'] & & \mathcal{E}^{\otimes} \arrow[dl, "q"] \\
& \mathrm{N}(\mathcal{F}\mathrm{in}_*) &
\end{tikzcd}$$

satisfying the following universal property: For every symmetric monoidal ∞*-category* $\mathcal{D}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *the induced map*

$$\mathrm{Fun}^{\otimes}(\mathcal{E}, \mathcal{D}) \to \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$$

is fully faithful and the essential image consists of all symmetric monoidal functors that send elements of $\mathcal{W}$ *to equivalences.*

Proposition 2.91. *(see* [Hin16]*, Proposition 3.2.2.)*
Let $\mathcal{C}^{\otimes}$ *be a symmetric monoidal* ∞*-category with underlying* ∞*-category* $\mathcal{C}$ *and let* $\mathcal{W} \subset \mathcal{C}_1$ *be a subset of the morphisms of* $\mathcal{C}$ *such that the following condition is fulfilled: For all objects* X *in* $\mathcal{C}$ *and all morphisms* $f : Y \to Y'$ *in* $\mathcal{W}$*, the morphism*

$$X \otimes Y \to X \otimes Y'$$

belongs to $\mathcal{W}$*. Then a symmetric monoidal Dwyer-Kan localization exists and is unique up to equivalence of symmetric monoidal* ∞*-categories. We denote the symmetric monoidal Dwyer-Kan localization by*

$$\mathcal{C}^{\otimes}[\mathcal{W}^{-1}] \to \mathrm{N}(\mathcal{F}\mathrm{in}_*).$$

In fact the underlying ∞*-category of* $\mathcal{C}^{\otimes}[\mathcal{W}^{-1}]$ *is the Dwyer-Kan localization* $\mathcal{C}[\mathcal{W}^{-1}]$ *from Definition 2.17.*

2.2.2. *Modules and algebras over commutative algebra objects.* In the last section we recalled the formalism of symmetric monoidal ∞-categories and how one can get examples of such. In the present section we want to recall the notions of modules and algebras over a commutative algebra object in a symmetric monoidal ∞-category.
We start with the definition of modules. In Lurie's Higher Algebra ([Lur17]) several definitions can be found. For modules over a commutative algebra object in a symmetric monoidal ∞-category all of these definitions agree. We want to give a definition that enables us to avoid ∞-operads. In order to do so, we make use of the underlying monoidal ∞-category of a symmetric monoidal ∞-category to being able to use one of Lurie's definitions of module ∞-categories for the monoidal case. We begin with the following definition:

Definition 2.92. *(∞-categories tensored over a monoidal ∞-category) Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category. An ∞-category that is tensored over $\mathcal{C}^{\otimes}$ consists of an ∞-category $\mathcal{M}^{\otimes}$ together with a categorical fibration (i.e. a fibration with respect to the Joyal model structure) $r : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ such that:*

1. *(1) The composition*

$$\mathcal{M}^{\otimes} \xrightarrow{r} \mathcal{C}^{\otimes} \xrightarrow{p} \mathrm{N}(\Delta^{op})$$

   is a cocartesian fibration.

2. *(2) The map r carries $(p \circ r)$-cocartesian edges to p-cocartesian edges.*
3. *(3) The canonical map*

$$\mathcal{M}^{\otimes}_{\{0,1,\dots,n\}} \to \mathcal{C}^{\otimes}_{\{0,1,\dots,n\}} \times \mathcal{M}^{\otimes}_{\{0\}} \tag{2.93}$$

   which on the second factor is induced by the inclusion $\{0\} \cong \{n\} \subset \{0,1,\dots,n\}$, is an equivalence.

The fiber $\mathcal{M}^{\otimes}_{\{0\}}$ is called the underlying ∞-category of $\mathcal{M}^{\otimes}$ and is denoted by $\mathcal{M}$ and we will say that $\mathcal{M}$ is tensored over $\mathcal{C}$.

Remark 2.94. Consider the two maps

$$\mathcal{M} \longleftarrow \mathcal{M}^{\otimes}_{\{0,1\}} \longrightarrow \mathcal{C}^{\otimes}_{\{0,1\}} \times \mathcal{M},$$

where the right map is the map from 2.93 for $n = 1$ and the left map is induced by $\{0\} \subset \{0,1\}$. Since the right map is an equivalence by definition, we get a tensor functor (well defined up to equivalence)

$$\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}. \tag{2.95}$$

In fact, a monoidal ∞-category $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\Delta^{op})$ is tensored over itself and the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that is induced by the monoidal structure is equivalent to the one from remark 2.93:

Proposition 2.96. *(see* [Lur07], *Example 2.1.3.)*
Let $\mathcal{C}^{\otimes} \to \mathrm{N}(\Delta^{op})$ be a monoidal ∞-category. There exists an ∞-category $\mathcal{C}^{\otimes,L}$ together with a categorical fibration $\mathcal{C}^{\otimes,L} \to \mathcal{C}^{\otimes}$ such that $\mathcal{C}^{\otimes,L}$ is tensored over $\mathcal{C}^{\otimes}$ and such that the following two things hold:

1. *(1) The fiber $\mathcal{C}^{\otimes,L}_{\{0\}}$ is equivalent to $\mathcal{C}$.*
2. *(2) The tensor product functor that is induced by the symmetric monoidal structure, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, is equivalent to the one that is constructed using the fact that $\mathcal{C}^{\otimes,L}$ is tensored over $\mathcal{C}^{\otimes}$ (see remark 2.94).*

In other words: $\mathcal{C}$ is tensored over itself.

We now give Lurie's definition (from DAGII, [Lur07]) of module objects in ∞-categories that are tensored over monoidal ∞-categories. Since by the last proposition 2.96 every monoidal ∞-category is tensored over itself, this gives us a definition for module objects in monoidal ∞-categories.

Definition 2.97. *(Module objects)*
Let $r : \mathcal{M}^{\otimes} \to \mathcal{C}^{\otimes}$ be an ∞-category, that is tensored over the monoidal ∞-category $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\Delta^{op})$. A module object in $\mathcal{M}^{\otimes}$ is a map $M : \mathrm{N}(\Delta^{op}) \to \mathcal{M}^{\otimes}$ satisfying the following condition:

(1) The composition $\mathrm{N}(\Delta^{op}) \xrightarrow{M} \mathcal{M}^{\otimes} \xrightarrow{r} \mathcal{C}^{\otimes}$ is an algebra object in $\mathcal{C}^{\otimes}$ (see 2.77 for the definition of algebra object).

(2) M carries each convex morphism $\{0, 1, ..., n\} \to \{0, 1, ..., m\}$ that sends n to m to a $p \circ r$-cocartesian edge.

We denote the ∞-category of module objects in $\mathcal{M}^{\otimes}$ by

$$\mathrm{Mod}(\mathcal{M}).$$

It comes equipped with a forgetful functor

$$\mathrm{Mod}(\mathcal{M}) \to \mathcal{M} \tag{2.98}$$

which is given by evaluation at $\{0\}$.
Furthermore by condition (1) *above, there is a forgetful functor*

$$\mathrm{Mod}(\mathcal{M}) \to \mathcal{A}\mathrm{lg}(\mathcal{C})$$

which in fact is a categorical fibration (see [Lur07]*, Remark 2.1.8.).*
Let $R \in \mathcal{A}\mathrm{lg}(\mathcal{C})$. Then the ∞-category of R-module objects or R-modules is defined by the following pullback in **sSet***:*

$$\begin{tikzcd}
\mathrm{Mod}_R(\mathcal{M}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Mod}(\mathcal{M}) \arrow[d] \\
\Delta^0 \arrow[r, "R"] & \mathcal{A}\mathrm{lg}(\mathcal{C}).
\end{tikzcd}$$

Remark 2.99. A module object in particular encodes the information of an underlying object in $M(\{0\}) \in \mathcal{M}$ (which one gets by applying the forgetful functor $\mathrm{Mod}(\mathcal{M}) \to \mathcal{M}$) and an algebra object $R \in \mathcal{A}\mathrm{lg}(\mathcal{C})$ (which one gets by applying the forgetful functor $\mathrm{Mod}(\mathcal{M}) \to \mathcal{A}\mathrm{lg}(\mathcal{C})$. We now want to explain how a multiplication map

$$R_0 \otimes M(\{0\}) \to M(\{0\})$$

is also encoded, where $R_0 \in \mathcal{C}$ is the underlying object of R.
Consider the morphisms $\alpha : \{0\} \subset \{0, 1\}$ and $\beta : \{0\} \cong \{1\} \subset \{0, 1\}$ in

Δ. Since the fiber $\mathcal{M}^{\otimes}_{\{0,1\}}$ is equivalent to $\mathcal{C} \times \mathcal{M}$, the image $M(\{0,1\})$ corresponds to a pair

$$M(\{0,1\}) \leftrightsquigarrow (R_0, M_0)$$

where $R_0 \in \mathcal{C}$ and $M_0 \in \mathcal{M}$. Let us now consider the following diagram in $\mathcal{M}^{\otimes}$:

$$\begin{tikzcd} M(\{0,1\}) \leftrightsquigarrow (R_0, M_0) \arrow[d, "M(\alpha)"] \arrow[r, "f"] & R_0 \otimes M_0 \\ M(\{0\}) & \end{tikzcd}$$

where the horizontal map f is a $p \circ r$-cocartesian lift of α and the vertical map is just M applied to α. Note that the target of f is $R_0 \otimes M_0$ by definition of the tensor functor 2.95. Since both maps in the diagram lie over α, by the universal property of the cocartesian edge f there is a 2-cell

$$\begin{tikzcd} M(\{0,1\}) \leftrightsquigarrow (R_0, M_0) \arrow[d, "M(\alpha)"] \arrow[r, "f"] & R_0 \otimes M_0 \arrow[dl] \\ M(\{0\}) & \end{tikzcd}$$

in $\mathcal{M}^{\otimes}$ and in particular a map

$$R_0 \otimes M_0 \to M(\{0\}).$$

We now just have to check, that $M(\{0\})$ is indeed equivalent to M_0. For this we use that by definition of a module object the map $M(\beta) : (R_0, M_0) \to M(\{0\})$ is a $p \circ r$-cocartesian lift of β. But since $p \circ r$-cocartesian lifts of β are used to construct the equivalence $\mathcal{M}^{\otimes}_{\{0,1\}} \simeq \mathcal{C}^{\otimes}_{\{0,1\}} \times \mathcal{M}^{\otimes}_{\{0\}}$ from 2.93, it follows that $M(\{0\})$ must be equivalent to M_0.

Definition 2.100. *(Module objects in a monoidal ∞-category)* *A module object in a monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathrm{N}(\Delta^{op})$ is a module object (in the sense of definition 2.97) in the ∞-category $\mathcal{C}^{\otimes,L}$ which is tensored over $\mathcal{C}^{\otimes}$ (see Proposition 2.96).*

We now go back to symmetric monoidal ∞-categories:

Definition 2.101. *(Module objects in symmetric monoidal ∞-categories)* *Let $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a symmetric monoidal ∞-category. We define a module object in $\mathcal{C}^{\otimes}$ to be a module object in the underlying monoidal ∞-category $U(\mathcal{C}^{\otimes}) \to \mathrm{N}(\Delta^{op})$ (Definition 2.76). The ∞-category of module objects in $\mathcal{C}^{\otimes}$ will be denoted by*

$$\mathrm{Mod}(\mathcal{C}).$$

If $R \in \mathrm{CAlg}(\mathcal{C})$ *is a commutative algebra object in* $\mathcal{C}^{\otimes}$*, then it has an underlying algebra object* $U(R) \in \mathcal{A}\mathrm{lg}(U(\mathcal{C}^{\otimes}))$ *(see 2.80). We define the* ∞*-category of* R*-module objects (or* R*-modules) by*

$$\mathrm{Mod}_R(\mathcal{C}) := \mathrm{Mod}_{U(R)}(U(\mathcal{C}^{\otimes})).$$

Remark 2.102. We have already said that there are several different definitions of module objects and ∞-categories of module objects in symmetric monoidal ∞-categories in Lurie's Higher Algebra [Lur17]. We want to compare the definition, we have chosen, to the other ones: The definition we use appears as the definition of left module objects in Derived Algberaic Geometry II ([Lur07], Definition 2.1.4.). In Higher Algebra, our definition appears under the name of $\mathbb{A}_\infty$-module objects (see Definition 4.2.2.10. and Remark 4.2.2.19. in [Lur17]) and is denoted there by $\mathrm{LMod}^{\mathbb{A}_\infty}(\mathcal{C})$. For a commutative algebra object $R \in \mathrm{CAlg}(\mathcal{C})$ the ∞-category of $R - \mathbb{A}_\infty$-module objects is denoted by $\mathrm{LMod}_R^{\mathbb{A}_\infty}(\mathcal{C})$.

Then there is the construction of modules in the third chapter of Higher Algebra ([Lur17], Definition 3.3.3.8.) which is used to construct symmetric monoidal structures on ∞-categories of module objects in certain cases. If $R \in \mathrm{CAlg}(\mathcal{C})$ is a commutative algebra object in the symmetric monoidal ∞-category $\mathcal{C}$, the ∞-category of R-module objects in this sense comes equipped with a map to $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$ and is denoted by

$$\gamma : (\mathrm{Mod}_R^{\mathrm{N}(\mathcal{F}\mathrm{in}_*)})^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*).$$

Lurie then shows that this map γ exhibits $(\mathrm{Mod}_R^{\mathrm{N}(\mathcal{F}\mathrm{in}_*)})^{\otimes}$ as a symmetric monoidal ∞-category if $\mathcal{C}$ fulfills certain assumptions (see below, Theorem 2.104). In any case, Lurie denotes the fiber of γ over $\langle 1 \rangle$ by $\mathrm{Mod}_R(\mathcal{C})$ and shows (Corollary 4.5.1.5. in [Lur17]) that there is an equivalence of ∞-categories

$$\mathrm{Mod}_R(\mathcal{C}) \simeq \mathrm{LMod}_R^{\mathbb{A}_\infty}(\mathcal{C}). \tag{2.103}$$

Another definition for an ∞-category of module objects that Lurie gives in Higher algebra makes use of the ∞-operads $\mathcal{LM}^{\otimes}$ and $\mathcal{RM}^{\otimes}$. Using these, he defines the ∞-category of R-left module objects $\mathrm{LMod}_R(\mathcal{C})$ (Definition 4.2.1.13. in [Lur17]) and R-right module objects $\mathrm{RMod}_R(\mathcal{C})$ (Variant 4.2.1.36. in [Lur17]) in a symmetric monoidal ∞-category $\mathcal{C}$. In Corollary 4.2.2.16. he constructs an equivalence of the ∞-categories of R-left module objects and $R - \mathbb{A}_\infty$-module objects

$$\mathrm{LMod}_R^{\mathbb{A}_\infty}(\mathcal{C}) \simeq \mathrm{LMod}_R(\mathcal{C}),$$

so that together with 2.103 there is a chain of equivalences

$$\mathrm{Mod}_R(\mathcal{C}) \simeq \mathrm{LMod}_R^{\mathbb{A}_\infty}(\mathcal{C}) \simeq \mathrm{LMod}_R(\mathcal{C}).$$

In the introduction to chapter 4.5 of Higher Algebra it is then also explained how one shows that also the ∞-category of R-right module objects $\mathrm{RMod}_R(\mathcal{C})$ in a symmetric monoidal ∞-category $\mathcal{C}$ can be added to this chain of equivalences.
In summary it can be said that all definitions that Lurie gives for module objects in symmetric monoidal ∞-categories are equivalent, and that in particular our definition 2.101 is the right one in the sense that it is equivalent to all definitions that Lurie gives. An important consequence of that is that the symmetric monoidal structure that can be constructed on $\mathrm{Mod}_R(\mathcal{C})$ for certain symmetric monoidal ∞-categories $\mathcal{C}$ can be transferred to our definition 2.101.

In the last remark we used notations from Higher Algebra [Lur17] in order to explain certain notions of module ∞-categories in *loc.cit.*. From now on we are not going to use these notations again and will solely stick to the notations we have introduced before.
We already pointed out that for $R \in \mathrm{CAlg}(\mathcal{C})$ under certain circumstances the ∞-category of R-module objects $\mathrm{Mod}_R(\mathcal{C})$ in a symmetric monoidal ∞-category $\mathcal{C}$ carries also a symmetric monoidal structure. We want to make this precise now.
A simplicial object in an ∞-category $\mathcal{C}$ is a functor

$$F : \mathrm{N}(\Delta^{op}) \to \mathcal{C}$$

and a geometric realization of such a simplicial object F is a colimit

$$\bar{F} : \mathrm{N}(\Delta^{op})^{\triangleright} \to \mathcal{C}.$$

Theorem 2.104. *(Existence of a symmetric monoidal structure on* $\mathrm{Mod}_R(\mathcal{C})$*; see* [Lur17]*, Theorem 4.5.2.1.)*
Let $\mathcal{C}$ be a symmetric monoidal ∞-category which has geometric realizations for all simplicial objects in it and whose tensor product functor 2.70

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

preserves geometric realizations in each variable separately. Then for each commutative algebra object $R \in \mathrm{CAlg}(\mathcal{C})$ the ∞-category of R-module objects $\mathrm{Mod}_R(\mathcal{C})$ carries a symmetric monoidal structure $\mathrm{Mod}_R(\mathcal{C})^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ *and the forgetful functor (see 2.98)* $\mathrm{Mod}_R(\mathcal{C}) \to \mathcal{C}$ *can in fact be lifted to a lax symmetric monoidal functor*

$$\begin{tikzcd}
\mathrm{Mod}_R(\mathcal{C})^{\otimes} \arrow[rr] \arrow[dr] & & \mathcal{C}^{\otimes} \arrow[dl] \\
& \mathrm{N}(\mathcal{F}\mathrm{in}_*). &
\end{tikzcd}$$

Remark 2.105. The fact, that the forgetful functor from R-modules to $\mathcal{C}$ is really lax symmetric monoidal is not included in the reference we gave for the last theorem 2.104. Therefore we have to say some words about how one can extract this result from Higher Algebra [Lur17].
In fact, in chapter 3.3.3 Lurie constructs a map $\mathrm{Mod}(\mathcal{C})^{\otimes} \to \mathrm{CAlg}(\mathcal{C}) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ such that the symmetric monoidal structure on R-modules from the above theorem is obtained via the following pullback in **sSet**:

$$\begin{tikzcd}
\mathrm{Mod}_R(\mathcal{C})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mod}(\mathcal{C})^{\otimes} \arrow[d] \\
\Delta^0 \times \mathrm{N}(\mathcal{F}\mathrm{in}_*) \arrow[r, "\{R\}\times \mathrm{id}"] & \mathrm{CAlg}(C) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*).
\end{tikzcd}$$

Corollary 3.4.3.4. in [Lur17] tells us that the map $\mathrm{Mod}(\mathcal{C})^{\otimes} \to \mathrm{CAlg}(\mathcal{C})$ is a cartesian fibration while Theorem 4.5.3.1. tells us that $\mathrm{Mod}(\mathcal{C})^{\otimes} \to \mathrm{CAlg}(\mathcal{C}) \times \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ is a cocartesian fibration. This implies the following: The map $\mathrm{Mod}(\mathcal{C})^{\otimes} \to \mathrm{CAlg}(\mathcal{C})$ is cartesian and cocartesian. In particular, if we take a morphism $f : S \to R$ in $\mathrm{CAlg}(\mathcal{C})$ and consider the pullback in **sSet**

$$\begin{tikzcd}
\mathrm{Pb} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mod}(\mathcal{C})^{\otimes} \arrow[d] \\
\Delta^1 \arrow[r, "f"] & \mathrm{CAlg}(\mathcal{C}),
\end{tikzcd}$$

the left vertical map is also cartesian and cocartesian. This means, that this left vertical map encodes precisely an adjunction (see 2.43)

$$\mathrm{Mod}_S(\mathcal{C})^{\otimes} \rightleftarrows \mathrm{Mod}_R(\mathcal{C})^{\otimes}.$$

Furthermore by Remark 4.5.3.2. of [Lur17] the left adjoint of this adjunction is symmetric monoidal. Now the right adjoint of such a symmetric monoidal functor must be lax symmetric monoidal (follows by Corollary 7.3.2.7. in [Lur17]. It should be mentioned that this Corollary as it is stated is not correct. But it is correct if one adds the assumption that the functor F preserves cocartesian arrows.). Therefore, in our case the restriction functor

$$\mathrm{Mod}_R(\mathcal{C})^{\otimes} \to \mathrm{Mod}_S(\mathcal{C})^{\otimes}$$

is lax symmetric monoidal. If one chooses for the map of commutative algebra objects f the unit morphism $\mathbf{1} \to R$ (see 2.78) then one gets that the restriction functor

$$\mathrm{Mod}_R(\mathcal{C})^{\otimes} \to \mathrm{Mod}_{\mathbf{1}}(\mathcal{C})^{\otimes}$$

is lax symmetric monoidal. But modules over the unit are the same as objects of $\mathcal{C}$: There is an equivalence of symmetric monoidal ∞-categories

$$\mathrm{Mod}_{\mathbf{1}}(\mathcal{C})^{\otimes} \simeq \mathcal{C}^{\otimes}$$

by Proposition 3.4.2.1. in [Lur17]. In particular the forgetful functor

$$\mathrm{Mod}_R(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$$

is lax symmetric monoidal.

We now want to come to the definition of R-algebras in a symmetric monoidal ∞-category and their relation to R-modules.

Definition 2.106. *(R-algebras)*
Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $R \in \mathrm{CAlg}(\mathcal{C})$ a commutative algebra object. An R-algebra is a commutative algebra object under R. More precisely, the ∞-category of R-algebras is defined to be

$$R\text{-}\mathrm{CAlg}(\mathcal{C}) := \mathrm{CAlg}(\mathcal{C})_{R/}.$$

This definition copies the definition of R-algebras in classical algebra: In classical algebra, R-algebras are rings with a structure morphism from R. Another viewpoint in the very classical setting is that R-algebras are R-modules with a multiplication map that is R-linear in both variables. That this is also true in the setting of symmetric monoidal ∞-categories, is the content of the following proposition:

Proposition 2.107. *(see* [Lur17]*, Corollary 3.4.1.7.)*
Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $R \in \mathrm{CAlg}(\mathcal{C})$ a commutative algebra object in it. Further assume that $\mathcal{C}$ has geometric realizations of all simplicial objects and that the tensor product functor preserves these realizations in each variable separately. (These are the conditions on $\mathcal{C}$ one needs to ensure, that R-modules carry a symmetric monoidal structure.)
Then there is an equivalence of ∞-categories

$$\mathrm{CAlg}(\mathrm{Mod}_R(\mathcal{C})) \simeq \mathrm{CAlg}(\mathcal{C})_{R/}.$$

If one has a classical module M over a commutative ring R, one can localize M with respect to elements $r \in R$. This means that one can construct a universal R-module $M[r^{-1}]$ together with a map $M \to M[r^{-1}]$ with the property that r acts invertibly on $M[r^{-1}]$. Moreover, if one has a classical commutative R-algebra A, one can construct a universal R-algebra $A[r^{-1}]$ with a map from A such that r acts invertibly on $A[r^{-1}]$; and the underlying module of $A[r^{-1}]$ is precisely the r-localization of the underlying module of A. In chapter 3 about the construction of the stable Adams operations on algebraic K-theory we need an analog of this sort of localization of R-algebra and R-module objects in symmetric monoidal ∞-categories with respect to elements of commutative algebra objects. Of course, for this we need at first a notion of an element of a commutative algebra object.

Definition 2.108. *(Elements in commutative algebra objects)*
Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and $I \in \mathcal{C}$ a tensor invertible object, i.e. an object that is tensor invertible in the symmetric monoidal homotopy category $\mathrm{Ho}(\mathcal{C})$*. Let $R \in \mathrm{CAlg}(\mathcal{C})$ be a commutative algebra object. An element in R is a morphism*

$$r : I \to R$$

in $\mathcal{C}$. If $M \in \mathrm{Mod}_R(\mathcal{C})$ is an R-module object with underlying object M_0 in $\mathcal{C}$, then we say that an element $r : I \to R$ acts invertibly on M if the composition

$$\bar{\mu}_r : I \otimes M_0 \xrightarrow{r \otimes \mathrm{id}} R \otimes M_0 \xrightarrow{\nu_M} M_0 \tag{2.109}$$

is an equivalence. Here ν_M is the R-action morphism which is encoded in the R-module structure of M (see Remark 2.99).

Definition 2.110. *(Module localization)*
Let M be an R-module object and $r : I \to R$ an element in R. Then a localization of M with respect to r is an R-module $M[r^{-1}]$ on which r acts invertibly and a map of R-modules $M \to M[r^{-1}]$ such that the induced map

$$\mathrm{Map}_{\mathrm{Mod}_R(\mathcal{C})}(M[r^{-1}], N) \to \mathrm{Map}_{\mathrm{Mod}_R(\mathcal{C})}(M, N)$$

is a homotopy equivalence of Kan complexes for every R-module N on which the element r acts invertibly.

As said above, localizations of modules and algebras at elements always exist in classical commutative algebra. Under certain conditions this is also true in higher commutative algebra. For us it will be important that localizations at elements always exist if the underlying ∞-category $\mathcal{C}$ is additive. We will recall this fact in lemma 2.117 below.

2.3. **Additive and stable ∞-categories.**

In upcoming chapters of this thesis we will deal for example with the ∞-categories of spectra, of motivic $\mathbb{P}^1$-spectra and of chain complexes over some ring. All of these ∞-categories are stable. In this section we want to recall the definitions of additive and stable ∞-categories and we will discuss porperties of such ∞-categories that we will need.

We begin with the notions of pre-additive and additive ∞-categories. The reference we use here is [GGN15].

Definition 2.111. *(Pre-additive ∞-category)*
We call an ∞-category $\mathcal{C}$ pre-additive if it has the following three properties:

(1) $\mathcal{C}$ has a zero-object (i.e. an object which is both initial and terminal).
(2) $\mathcal{C}$ has finite products and finite coproducts.
(3) For all objects $X_1, X_2 \in \mathcal{C}$ the canonical map

$$X_1 \amalg X_2 \to X_1 \times X_2$$

is an equivalence.

By the last condition the coproduct and the product of two objects X_1, X_2 can be identified via the canoncial map and we write $X_1 \oplus X_2$ for both constructions. In particular one has a diagonal and codiagonal map

$$X \xrightarrow{\triangle} X \oplus X \xrightarrow{\triangledown} X$$

for every object $X \in \mathcal{C}$.
If one uses the last definition in the setting of ordinary 1-categories, the diagonal and codiagonal maps induce a commutative monoid structure on each Hom-set $\mathrm{Hom}(X, Y)$ by sending two maps $f, g : X \to Y$ to the composition

$$X \xrightarrow{\triangle} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\triangledown} Y. \tag{2.112}$$

A pre-additive 1-category is additive precisely when this commutative monoid structure on the Hom-sets is always a group and there is a slick way to impose this. In fact, a pre-additive 1-category is an additive category if and only if for all objects X the so called shear map

$$\mathrm{sh}_X : X \oplus X \to X \oplus X$$

which is the projection $pr_1 : X \oplus X \to X$ on the first factor and the codiagonal $X \oplus X \to X$ on the second factor is an isomorphism. This observation motivates the definition of an additive ∞-category:

Definition 2.113. *(Additive ∞-category)*
A pre-additive ∞-category $\mathcal{C}$ is called additive if for all objects $X \in \mathcal{C}$ the shear map

$$\mathrm{sh}_X : X \oplus X \to X \oplus X \tag{2.114}$$

which is the projection $pr_1 : X \oplus X \to X$ on the first factor and the codiagonal $X \oplus X \to X$ on the second factor is an equivalence.

Recall that for any ∞-category $\mathcal{C}$ there is a canonical functor $\mathcal{C} \to \mathrm{N}(\mathrm{Ho}(\mathcal{C}))$ (see 2.3). One can show that this functor always preserves all coproducts and all products that exist in $\mathcal{C}$. In fact this just follows from the definition of products and coproducts in ordinary 1-categories and ∞-categories. Therefore it follows directly that an ∞-category with finite products and finite coproducts is pre-additive if and only if its homotopy category $\mathrm{Ho}(\mathcal{C})$ is pre-additive. And even more is true:

Proposition 2.115. *(see* [GGN15]*, Proposition 2.8.)*
Let $\mathcal{C}$ be an ∞-category with finite products and finite coproducts. Then $\mathcal{C}$ is additive if and only if its homotopy category $\mathrm{Ho}(\mathcal{C})$ *is additive.*

This result also directly follows from the fact that the functor $\mathcal{C} \to \mathrm{N}(\mathrm{Ho}(\mathcal{C}))$ preserves products and coproducts, since one just has to check that the canonical maps $X \amalg X \to X \times X$ and the shear maps are all equivalences.
If one works in additive 1-categories, one can express the negative of a morphism $f \in \mathrm{Hom}(X, Y)$ using the shear map: One can show that the negative of f with respect to the commutative monoid structure which is given by 2.112 can be written as

$$(-f) = \quad X \xrightarrow{(f,0)} Y \oplus Y \xrightarrow{\mathrm{sh}_Y^{-1}} Y \oplus Y \xrightarrow{pr_2} Y.$$

We use this observation (together with 2.112) to make the following definition:

Definition 2.116. *(Sum and negatives of morphisms in additive ∞-categories)*
Let $f, g : X \to Y$ be two morphisms in an additive ∞-category. We define their sum to be a choice of a composition

$$X \xrightarrow{\triangle} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\triangledown} Y$$

and denote any such sum by $f + g$.
A negative of a morphism $f : X \to Y$ is defined to be a choice of the composition

$$X \xrightarrow{(f,0)} Y \oplus Y \xrightarrow{\mathrm{sh}_Y^{-1}} Y \oplus Y \xrightarrow{pr_2} Y$$

and we use the symbol $(-f)$ for any such choice.

By the explanations above it follows that the induced constructions on morphisms in the additive homotopy category describe precisely the abelian group structure on the Hom-sets.
In view of the localization with respect to elements in commutative algebra objects additive ∞-categories behave well. By this we mean

that such localizations always exist and that they can be computed by an explicit formula.

Lemma 2.117. *(see* [BNT15]*, Lemma C.2. and Proposition C.3.)* *Let $\mathcal{C}^\otimes \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a cocomplete symmetric monoidal ∞-category such that the tensor product functor preserves colimits in both variables separately and such that the underlying category $\mathcal{C}$ is additive. Let $I \to R$ be an element in the commutative algebra object $R \in \mathrm{CAlg}(\mathcal{C})$. Then, if M is an R-module object, the map*

$$\mu_r := I^{-1} \otimes \bar{\mu}_r : M_0 \to I^{-1} \otimes M_0$$

(where $\bar{\mu}_r$ is the map from 2.109) can be lifted canonically to a map of modules and the R-module

(2.118)

$$M[r^{-1}] := \mathrm{colim}(\ M \xrightarrow{\mu_r} I^{-1} \otimes M \xrightarrow{\mathrm{id}\otimes\mu_r} I^{-1} \otimes (I^{-1} \otimes M) \longrightarrow ...\ )$$

is a localization of M with respect to r.

Since we assumed in the last lemma that the tensor product functor preserves colimits in both variables separately, it is clear that we have an equivalence of R-modules

$$M[r^{-1}] \simeq M \otimes_R R[r^{-1}].$$

Therefore localization of R-modules with respect to r in fact defines a functor

$$L_r : \mathrm{Mod}_R(\mathcal{C}) \to \mathrm{Mod}_R(\mathcal{C}),\ M \mapsto M[r^{-1}].$$

By the universal property of the localization with respect to elements (see Definition 2.110) we also have that $R[r^{-1}][r^{-1}] \simeq R[r^{-1}]$. Therefore L_r is a localization functor in the sense of Definition 2.61 and because it is given by tensoring with a certain object it is clearly a symmetric monoidal localization functor. Using Proposition 2.107 and Corollary 2.86 we therefore get:

Proposition 2.119. *Let $\mathcal{C}^\otimes$ be a cocomplete symmetric monoidal ∞-category such that the tensor product functor preserves colimits in both variables separately. Moreover assume that the underlying ∞-category is additive. Then the localization functor*

$$L_r : \mathrm{Mod}_R(\mathcal{C}) \to \mathrm{Mod}_R(\mathcal{C})$$

(for an element $r : I \to R$) lifts to R-algebras, i.e. there is a commutative diagram of ∞-categories

$$\begin{tikzcd}
R\text{-CAlg}(\mathcal{C}) \arrow[r, "\bar{L}_r"] \arrow[d] & R\text{-CAlg}(\mathcal{C}) \arrow[d] \\
\mathrm{Mod}_R(\mathcal{C}) \arrow[r, "L_r"] & \mathrm{Mod}_R(\mathcal{C})
\end{tikzcd}$$

in which $\bar{L}_r$ is also a localization functor.

One particular case of an element in an additive symmetric monoidal ∞-category are natural numbers. These are defined to be elements in the unit object (which is a commutative algebra by example 2.78):

Definition 2.120. *(Natural numbers in additive symmetric monoidal ∞-categories)*
Let $\mathcal{C}$ be an additive symmetric monoidal ∞-category with unit object $\mathbf{1} \in \mathcal{C}$ and $n \in \mathbb{N}$ a natural number. Then we can view n as an element in the unit object via the following map:

$$n : \mathbf{1} \xrightarrow{\triangle} \underbrace{\mathbf{1} \oplus ... \oplus \mathbf{1}}_{n \text{ times}} \xrightarrow{\triangledown} \mathbf{1}.$$

Remark 2.121. In the above situation, the map $n : \mathbf{1} \to \mathbf{1}$ induces n times the identity of $\mathbf{1}$ in the additive homotopy category $\mathrm{Ho}(\mathcal{C})$. Now assume that the tensor product functor of $\mathcal{C}$ commutes with finite coproducts. If $X \in \mathcal{C}$ is any object (and therefore also an $\mathbf{1}$-module), then n acts invertibly on X precisely if the map

$$X \xrightarrow{\triangle} \underbrace{X \oplus ... \oplus X}_{n \text{ times}} \xrightarrow{\triangledown} X$$

is an equivalence, i.e. if n times the identity of X in the homotopy category $\mathrm{Ho}(\mathcal{C})$ is an isomorphism.
Lemma 2.117 tells us that we can compute the localization of an object $X \in \mathcal{C}$ with respect to the natural number n as

$$X[n^{-1}] := \; X \xrightarrow{n} X \xrightarrow{n} X \xrightarrow{n} ... \; .$$

We now want to recall the definition of a stable ∞-category and some properties of stable ∞-categories that will be used later. We begin with the following:

Definition 2.122. *(Triangles/fiber and cofiber sequences in a pointed ∞-category)*

Let $\mathcal{C}$ be a pointed ∞-category (in other words $\mathcal{C}$ has a zero object). A triangle in $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr] & B \arrow[d] \\ 0 \arrow[r] & C. \end{tikzcd}$$

A triangle is called fiber sequence if it is a pullback, it is called cofiber sequence if it is a pushout.
If $f : A \to B$ is a morphism in $\mathcal{C}$, then a fiber of f is a fiber sequence of the form

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr] & A \arrow[d, "f"] \\ 0 \arrow[r] & B; \end{tikzcd}$$

a cofiber of f is a cofiber sequence of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr] & B \arrow[d] \\ 0 \arrow[r] & C. \end{tikzcd}$$

Definition 2.123. *(Stable ∞-category)*
A pointed ∞-category is called stable if the following two conditions are fulfilled:

(1) All morphisms have a fiber and a cofiber.
(2) A triangle is a cofiber sequence if and only if it is a fiber sequence.

One can show that the homotopy category of a stable ∞-category carries a canonical triangulated structure ([Lur17], Theorem 1.1.2.14.). We don't want to recall how one constructs this triangulated structure but we want to point out that this in particular implies that the homotopy category of a stable ∞-category is additive. Since additivity is a property of an ∞-category which can be tested on the level of homotopy categories (Proposition 2.115) this implies that stable ∞-categories are additive. If one has a stable symmetric monoidal ∞-category $\mathcal{C}$ (with enough colimits and compatibility of the tensor product with colimits), localizations with respect to elements therefore always exist and it makes sense to invert natural numbers of objects in $\mathcal{C}$ as in remark 2.121. These conclusions will be used later on.
If $\mathcal{C}$ is an ∞-category with finite limits, it is possible to construct a universal stable ∞-category with a functor to $\mathcal{C}$. This construction goes

under the name of stabilization or formation of spectrum objects. We want to make this precise in the following and for this we firstly recall the notion of reduced and of excisive functors as well as the notion of a pointed object.

Definition 2.124. *(Reduced and excisive functors)*
Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Let us assume that $\mathcal{C}$ has a final object and pushouts. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is called

(1) reduced, if it sends the final object to a final object in $\mathcal{D}$.
(2) excisive, if it sends pushouts to pullbacks in $\mathcal{D}$.

The ∞-category of reduced excisive functors is denoted by

$$\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}) \xhookrightarrow{full} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Definition 2.125. *(Pointed objects)*
Let $\mathcal{C}$ be an ∞-category with a final object. A pointed object in $\mathcal{C}$ is a map $* \to X$ in $\mathcal{C}$ where $*$ is a final object. The ∞-category of pointed objects is defined to be the full sub-∞-category of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ *on the pointed objects and it is denoted by $\mathcal{C}_*$. Evaluation at the* 1*-vertex defines a forgetful functor*

$$\mathcal{C}_* \to \mathcal{C}.$$

Lemma 2.126. *(see* [Lur09]*, Lemma 7.2.2.9.)*
If $\mathcal{C}$ is an ∞-category with a final object then $\mathcal{C}_*$ is pointed. A zero object is given by the identity $* = *$. If $\mathcal{C}$ itself has a zero object then the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ is an equivalence.

Remark 2.127. If $\mathcal{C}$ is presentable then so is $\mathcal{C}_*$ and the forgetful functor $\mathcal{C}_* \to \mathcal{C}$ is accessible and preserves all limits. Therefore in this case the forgetful functor has a left adjoint

$$(-)_+ : \mathcal{C} \to \mathcal{C}_*. \tag{2.128}$$

If $\mathcal{C}^\otimes$ is presentably symmetric monoidal then there is a presentably symmetric monoidal structure on $\mathcal{C}_*$ such that this left adjoint $(-)_+$ refines to a symmetric monoidal functor (see [Nik], Proposition 5.6.).

Recall that the ∞-category of finite spaces which is denoted by $\mathcal{S}^{fin} \subseteq \mathcal{S}$ is the full sub-∞-category of the ∞-category of spaces $\mathcal{S}$ which contains the terminal space $\{*\}$ and which is closed under finite colimits in $\mathcal{S}$. $\mathcal{S}^{fin}_*$ is the ∞-category of pointed finite spaces.

Definition 2.129. *(Spectrum objects)*
Let $\mathcal{C}$ be an ∞-category with finite limits. A spectrum object in $\mathcal{C}$ is

a reduced excisive functor $\mathcal{S}_*^{fin} \to \mathcal{C}$ *and the* ∞*-category of spectrum objects in* $\mathcal{C}$ *is denoted by* $\mathrm{Sp}(\mathcal{C})$*. There is the canonical functor*

$$\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$$

which is given by evaluation at the 0*-sphere* $S^0 \in \mathcal{S}_*^{fin}$*.*
The ∞*-category of spectrum objects in pointed spaces is called the* ∞*-category of spectra and we denote it by*

$$\mathrm{Sp} := \mathrm{Sp}(\mathcal{S}_*).$$

Remark 2.130. The last definition is an intrinsic ∞-categorical definition of the ∞-category of spectra. Another approach to defining such an ∞-category would be to taking one of the simplicial model categories of spectra and taking the underlying ∞-category of that (i.e. taking the simplicial nerve of the full subcategory of fibrant-cofibrant objects). Of course both definitions should give equivalent ∞-categories. In fact they do: Robalo gives the desired comparison result in [Rob13], Proposition 4.15. What he shows there is that the underlying ∞-category of the simplicial stable model category of Bousfield-Friedlander spectra is equivalent to $\mathrm{Sp}(\mathcal{S}_*)$.

The importance of the formation of spectrum objects stems from the following result which unformally says that $\mathrm{Sp}(\mathcal{C})$ is the stable ∞-category that is as close to $\mathcal{C}$ as possible:

Proposition 2.131. *(see* [Lur17]*, Corollary 1.4.2.17., Proposition 1.4.2.21. and Corollary 1.4.2.23.)*
Let $\mathcal{C}$ *be an* ∞*-category with finite limits. Then the* ∞*-category of spectrum objects* $\mathrm{Sp}(\mathcal{C})$ *is stable. Moreover* $\mathcal{C}$ *is stable if and only if the infinite loop functor*

$$\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$$

is an equivalence.
If $\mathcal{D}$ *is a stable* ∞*-category then composition with* Ω^∞ *induces an equivalence*

$$\mathrm{Fun}^{lex}(\mathcal{D}, \mathrm{Sp}(\mathcal{C})) \to \mathrm{Fun}^{lex}(\mathcal{D}, \mathcal{C})$$

where Fun^{lex} *stands for the* ∞*-category of those functors that preserve finite limits (in other words left exact functors).*

The last proposition implies that mapping spaces in stable ∞-categories can be refined to mapping spectra:

Definition 2.132. *(Mapping spectra in stable* ∞*-categories)*
Let X *be an object in a stable* ∞*-category* $\mathcal{C}$*. By proposition 2.40 we know that* $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}$ *preserves all limits and is in particular*

left exact. Therefore the last proposition 2.131 implies that we have an essentially unique factorization

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{Map}_{\mathcal{C}}(X,-)"] \arrow[d, dashed, "\mathrm{map}_{\mathcal{C}}(X,-)"'] & \mathcal{S}. \\
\mathrm{Sp} \arrow[ur, "\Omega^\infty"'] &
\end{tikzcd}$$

The evaluation $\mathrm{map}_{\mathcal{C}}(X,Y)$ *of the functor* $\mathrm{map}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathrm{Sp}$ *at an object* Y *is called the mapping spectrum in* $\mathcal{C}$ *from* X *to* Y*.*

The infinite loop functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) = \mathrm{Exc}_*(\mathcal{S}_*^{fin}, \mathcal{C}) \to \mathcal{C}$ commutes by definition of a reduced excisive functor with all limits, since it is given by evaluation on a fixed object. If $\mathcal{C}$ is a presentable one can show even more:

Proposition 2.133. *(see* [Lur17]*, Proposition 1.4.4.4.)*
Let $\mathcal{C}$ *be a presentable* ∞*-category. Then* $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ *has a left adjoint*

$$\Sigma_+^\infty : \mathcal{C} \rightleftarrows \mathrm{Sp}(\mathcal{C}) : \Omega^\infty. \tag{2.134}$$

The proof of this uses the fact that for presentable $\mathcal{C}$ also $\mathrm{Sp}(\mathcal{C})$ is presentable and that in this case the infinite loop functor Ω^∞ is accessible (i.e. commutes with κ-filtered colimits for some regular cardinal κ), so that one can use the adjoint functor theorem (2.64) to deduce the existence of a left adjoint.

By Remark 1.4.2.18. of Higher Algebra [Lur17] there is an equivalence $\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Sp}(\mathcal{C}_*)$ for any ∞-category $\mathcal{C}$ with finite limits. Therefore Ω^∞ factors as

$$\mathrm{Sp}(\mathcal{C}) \longrightarrow \mathcal{C}_* \longrightarrow \mathcal{C}.$$

If $\mathcal{C}$ is presentable, one has the adjunction (see remark 2.127)

$$(-)_+ : \mathcal{C} \rightleftarrows \mathcal{C}_* : \mathrm{Forget}.$$

It follows that the functor Σ_+^∞ from 2.134 can be factored as

$$\mathcal{C} \xrightarrow{(-)_+} \mathcal{C}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}(\mathcal{C}).$$

This explains why one denotes the left adjoint of Ω^∞ by Σ_+^∞ and not just by Σ^∞.

3. Multiplicative Adams operations on algebraic K-theory

On the 0-th (say complex) topological K-theory $K_0(X)$ of a finite CW-complex X for each $k \geq 0$ there is the k-th Adams operation which is the unique (natural) ring homomorphism

$$\psi^k : K_0(X) \to K_0(X)$$

which takes classes of line bundles to their k-th power. Now $K_0(X)$ is naturally isomorphic to homotopy classes of pointed maps from X_+ into the infinite loop space $\mathbb{Z} \times BU \simeq \Omega^\infty(\mathbf{KU})$ and in fact one can show that ψ^k lifts to a map $\psi^k : \mathbb{Z} \times BU \to \mathbb{Z} \times BU$. Therefore the Adams operations are at least unstable cohomology operations. One can show that after inversion of k the k-th Adams operation is even a stable cohomology operation, that is we can lift even further to a map of spectra

$$\psi^k : \mathbf{KU}[k^{-1}] \to \mathbf{KU}[k^{-1}].$$

Adams operations also exist in algebraic K-theory. If for example X is a scheme with an ample family of line bundles, so that algebraic K-theory is computed using the category of algebraic vector bundles over X, then we have an operation on the 0-th K-group $K_0(X)$ which is defined completely analogously to the above situation: There is a unique natural ring homomorphism

$$\psi^k : K_0(X) \to K_0(X)$$

that sends classes of line bundles to their k-th power. But of course there are also all the higher algebraic K-groups. By definition they come as the homotopy groups of the connective K-theory spectrum $\mathcal{K}_{st}(X)$ which is defined by a group completion procedure applied to the symmetric monoidal groupoid of vector bundles over X. Now the obvious question is: Is there a natural transformation of spectral valued presheaves

$$\psi^k : \mathcal{K}_{st}(-) \to \mathcal{K}_{st}(-)$$

which induces the Adams operations on π_0? As in topology the answer turns out to be that there is such a transformations after inverting k in $\mathcal{K}_{st}(-)$.

Another thing one can do is to go to the unstable or stable motivic homotopy category. If one works over a regular base scheme, algebraic K-theory is representable in these categories and one could ask for a construction of Adams operations on these representing objects. Note that, as we said above, in topological K-theory analogous constructions exist: There we have Adams operations on the representing objects.

In his thesis Riou shows that one can construct unstable Adams operations on the representing object of algebraic K-theory and he further shows that also stable operations exist after inversion of certain natural numbers (see [Rio06], Définition iv.59).
In the present chapter of the thesis we will also construct such motivic Adams operations using completely different methods than Riou. The new result here is that we show that these motivic Adams operations are multiplicative in the following sense: The motivic K-theory spectrum carries the structure of an $\mathbb{E}_\infty$-algebra in the symmetric monoidal ∞-category of motivic $\mathbb{P}^1$-spectra. We show that Riou's Adams operations can be refined to $\mathbb{E}_\infty$-maps. Peter Arndt got similar results in his thesis independently to our work (see [Arn16]).

3.1. **The Snaith model for the motivic K-theory spectrum.**

3.1.1. *Motivic homotopy theory.* Let S be a noetherian scheme of finite Krull dimension. At first we recall the construction of the ∞-categories of motivic spaces and motivic spectra over S. By $\mathbf{Sm}_S$ we denote the essentially small category of smooth separated schemes of finite type over S; morphisms in $\mathbf{Sm}_S$ are just morphisms of schemes over S, i.e. $\mathbf{Sm}_S$ is a full subcategory of all schemes over S. The ∞-category of motivic spaces Spc^{mot} is now defined as a certain localization of the ∞-category of presheaves $\mathcal{P}(\mathbf{Sm}_S)$. More precisely it is defined as the full reflective sub-∞-category of $\mathbb{A}^1$-local Nisnevich sheaves:

$$\mathrm{Spc}^{mot} := \mathrm{Fun}^{Nis,\mathbb{A}^1}(\mathbf{Sm}_S^{op}, \mathcal{S}) \subset \mathcal{P}(\mathbf{Sm}_S).$$

The corresponding localization functor is called

$$L^{mot} : \mathcal{P}(\mathbf{Sm}_S) \to \mathrm{Spc}^{mot}.$$

In fact, Spc^{mot} is an accessible localization of $\mathcal{P}(\mathbf{Sm}_S)$ and in particular it is a presentable ∞-category (cf. Theorem 2.63).
We want to point out that the construction of Spc^{mot} is in fact a two step localization: L^{mot} can be factored as the composition of Nisnevich localization followed by $\mathbb{A}^1$-localization

$$\begin{tikzcd}
\mathcal{P}(\mathbf{Sm}_S) \arrow[rr, "L^{mot}"] \arrow[dr, "L^{Nis}"'] & & \mathrm{Spc}^{mot}. \\
& \mathrm{Fun}^{Nis}(\mathbf{Sm}_S^{op}, \mathcal{S}) \arrow[ur, "L^{\mathbb{A}^1}"'] &
\end{tikzcd}$$

As a sheafification functor, L^{Nis} preserves finite limits. It turns out that in contrast the $\mathbb{A}^1$-localization functor $L^{\mathbb{A}^1}$ is not right exact (and hence Spc^{mot} is not an ∞-topos). But it still preserves finite products and thus L^{mot} preserves them as well. If we equip $\mathcal{P}(\mathbf{Sm}_S)$ and Spc^{mot} with the cartesian symmetric monoidal structure, then it follows (see 2.82) that L^{mot} can be essentially uniquely refined to a symmetric monoidal functor. Since $\mathcal{P}(\mathbf{Sm}_S)$ with the cartesian symmetric monoidal structure is obviously a presentably symmetric monoidal ∞-category (for a definition of this notion, see 2.73) this shows that Spc^{mot} is also presentably symmetric monoidal. Now, because the Yoneda embedding preserves limits, the canonical functor

$$\mathbf{Sm}_S \xrightarrow{\mathrm{Yo}_\infty} \mathcal{P}(\mathbf{Sm}_S) \xrightarrow{L^{mot}} \mathrm{Spc}^{mot} \tag{3.1}$$

also has an essentially unique refinement to a symmetric monoidal functor when $\mathbf{Sm}_S$ is equipped with the cartesian symmetric monoidal structure. We want to introduce the following notational convention: The image of an object $X \in \mathbf{Sm}_S$ under the functor 3.1 will always be denoted by the same symbol X.

There is the left adjoint functor

$$(-)_+ : \mathrm{Spc}^{mot} \to \mathrm{Spc}_*^{mot},$$

that adds a disjoint base point and there is a canonical symmetric monoidal structure on Spc_*^{mot} which makes it a presentably symmetric monoidal ∞-category such that the functor $(-)_+$ can be refined to a symmetric monoidal functor (see remark 2.127). The tensor product functor associated to this symmetric monoidal structure on Spc_*^{mot} is called the smash product and is denoted by the symbol $\wedge$. The ∞-category of motivic spectra over S is now constructed by formally inverting the pointed motivic space $(\mathbb{P}^1, \infty)$:

Definition/Proposition 3.2. *(∞-category of motivic spectra over S) Up to symmetric monoidal equivalence there is a uniquely determined presentably symmetric monoidal ∞-category $\mathrm{Sp}^{\mathbb{P}^1}$ together with a colimit preserving (and therefore left adjoint) symmetric monoidal functor*

$$\Sigma^\infty_{\mathbb{P}^1} : \mathrm{Spc}_*^{mot} \to \mathrm{Sp}^{\mathbb{P}^1}, \tag{3.3}$$

such that for every presentably symmetric monoidal ∞-category $\mathcal{D}$ the induced functor

$$\mathrm{Fun}^{\otimes,L}(\mathrm{Sp}^{\mathbb{P}^1}, \mathcal{D}) \to \mathrm{Fun}^{\otimes,L}_{\mathbb{P}^1}(\mathrm{Spc}_*^{mot}, \mathcal{D})$$

is an equivalence where the target is the ∞-catgeory of colimit preserving symmetric monoidal functors that send the pointed motivic space $(\mathbb{P}^1, \infty)$ to an invertible object in $\mathcal{D}$. $\mathrm{Sp}^{\mathbb{P}^1}$ is called the ∞-category of motivic spectra over S or of $\mathbb{P}^1$-spectra over S. The right adjoint of $\Sigma^\infty_{\mathbb{P}^1}$ is called $\Omega^\infty_{\mathbb{P}^1}$.

Let us denote the functor that takes the smash product with the pointed motivic space $(\mathbb{P}^1, \infty)$ by

$$\Sigma_{\mathbb{P}^1} := (-) \wedge (\mathbb{P}^1, \infty) : \mathrm{Spc}_*^{mot} \to \mathrm{Spc}_*^{mot}.$$

Note that the fact that $\Sigma_{\mathbb{P}^1}^\infty$ is symmetric monoidal implies that there is a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Spc}_*^{mot} \arrow[r, "\Sigma_{\mathbb{P}^1}^\infty"] \arrow[d, "\Sigma_{\mathbb{P}^1}"'] & \mathrm{Sp}^{\mathbb{P}^1} \arrow[d, "{\Sigma_{\mathbb{P}^1}, \simeq}"] \\
\mathrm{Spc}_*^{mot} \arrow[r, "\Sigma_{\mathbb{P}^1}^\infty"] & \mathrm{Sp}^{\mathbb{P}^1}
\end{tikzcd}$$

where the right vertical map is given by smashing with $\Sigma_{\mathbb{P}^1}^\infty(\mathbb{P}^1, \infty)$ which is an equivalence since $\Sigma_{\mathbb{P}^1}^\infty(\mathbb{P}^1, \infty)$ is an invertible object. And there is an associated commutative diagram which one gets by taking the right adjoints to any functor in the above diagram:

$$\begin{tikzcd}
\mathrm{Spc}_*^{mot} & \mathrm{Sp}^{\mathbb{P}^1} \arrow[l, "\Omega_{\mathbb{P}^1}^\infty"'] \\
\mathrm{Spc}_*^{mot} \arrow[u, "\Omega_{\mathbb{P}^1}"] & \mathrm{Sp}^{\mathbb{P}^1}. \arrow[l, "\Omega_{\mathbb{P}^1}^\infty"] \arrow[u, "{\Omega_{\mathbb{P}^1}, \simeq}"']
\end{tikzcd} \tag{3.4}$$

The existence of $\mathrm{Sp}^{\mathbb{P}^1}$ follows from [Rob13], Proposition 4.10. There Robalo discusses the existence of the formal inversion of objects for a general presentably symmetric monoidal ∞-category. In our specific situation one can give a concrete formula:

Proposition 3.5. *(see* [Rob13]*, Corollary 4.24)*
Let $\mathcal{P}\mathrm{r}^L$ be the (co)complete ∞-category of presentable ∞-categories and left adjoint functors. Then $\mathrm{Sp}^{\mathbb{P}^1}$ is equivalent to the colimit of the diagram

$$\Delta^1 \amalg_{\Delta^0} \Delta^1 \amalg_{\Delta^0} ... \to \mathcal{P}\mathrm{r}^L$$

which is given by

$$\mathrm{Spc}_*^{mot} \xrightarrow{\wedge(\mathbb{P}^1,\infty)} \mathrm{Spc}_*^{mot} \xrightarrow{\wedge(\mathbb{P}^1,\infty)} ...$$

and

$$\Sigma_{\mathbb{P}^1}^\infty : \mathrm{Spc}_*^{mot} \to \mathrm{Sp}^{\mathbb{P}^1}$$

is the canonical functor that is induced by the inclusion of the first vertex

$$\Delta^0 \hookrightarrow \Delta^1 \amalg_{\Delta^0} \Delta^1 \amalg_{\Delta^0} ... .$$

Remark 3.6. It is shown in [Lur09], Corollary 5.5.3.4., that $\mathcal{P}\mathrm{r}^L$ and $(\mathcal{P}\mathrm{r}^R)^{op}$ ($\mathcal{P}\mathrm{r}^R$ the ∞-category of presentable ∞-categories and right adjoint functors) are equivalent, and that the equivalence is given by the identity on objects and the choice of a right adjoint on morphisms. If one denotes the right adjoint of the colimit preserving functor

$$- \wedge (\mathbb{P}^1, \infty) : \mathrm{Spc}_*^{mot} \to \mathrm{Spc}_*^{mot}$$

by $\Omega_{\mathbb{P}^1}$, then this means that $\mathrm{Sp}^{\mathbb{P}^1}$ is also equivalent to the limit of the diagram

$$... \xrightarrow{\Omega_{\mathbb{P}^1}} \mathrm{Spc}_*^{mot} \xrightarrow{\Omega_{\mathbb{P}^1}} \mathrm{Spc}_*^{mot}.$$

in $\mathcal{P}\mathrm{r}^R$. Therefore the objects in $\mathrm{Sp}^{\mathbb{P}^1}$ can be described by a sequence of pointed motivic spaces $(X_i)_{i\geq 0}$ together with equivalences

$$X_i \simeq \Omega_{\mathbb{P}^1} X_{i+1}.$$

Using this description, it is clear that $\Omega^\infty_{\mathbb{P}^1} : \mathrm{Sp}^{\mathbb{P}^1} \to \mathrm{Spc}_*^{mot}$ can be described on objects by sending a sequence $(X_i)_i$ to X_0. It also follows that the functor $\Omega_{\mathbb{P}^1} : \mathrm{Sp}^{\mathbb{P}^1} \to \mathrm{Sp}^{\mathbb{P}^1}$ can be computed on objects by

$$\Omega_{\mathbb{P}^1}((M_n)_n) \simeq (\Omega_{\mathbb{P}^1} M_n)_n \simeq (..., M_1, M_0, \Omega_{\mathbb{P}^1}(M_0))$$

and can be therefore understood as a shift functor. Since $\Sigma_{\mathbb{P}^1} : \mathrm{Sp}^{\mathbb{P}^1} \to \mathrm{Sp}^{\mathbb{P}^1}$ is inverse to $\Omega_{\mathbb{P}^1}$ it therefore follows that it can be described on objects as a shift in the other direction:

$$\Sigma_{\mathbb{P}^1}((M_n)_n) \simeq (..., M_2, M_1).$$

Remark 3.7. In [Rob13] Robalo explains, why $\mathrm{Sp}^{\mathbb{P}^1}$ is even a stable ∞-category (right before Corollary 5.11.). The argument is essentially as follows: Since

$$\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$$

in pointed motivic spaces Spc_*^{mot}, one can construct $\mathrm{Sp}^{\mathbb{P}^1}$ by firstly formally inverting S^1 in Spc_*^{mot} and then formally inverting $\mathbb{P}^1$ in $\mathrm{Spc}_*^{mot}[(S^1)^{-1}]$. But because

$$\mathrm{Spc}_*^{mot}[(S^1)^{-1}] \simeq \mathrm{Sp}(\mathrm{Spc}_*^{mot})$$

is the stabilization of Spc_*^{mot} (see [Rob13], Remark 4.28.) it is a stable ∞-category. Therefore the ∞-category of $\mathbb{P}^1$-spectra can be constructed by formally inverting the object $\mathbb{P}^1$ in a stable ∞-category and hence is also stable (by [Rob13], Corollary 4.25.).

3.1.2. *K-theory.* Let S be a regular noetherian scheme of finite Krull dimension. Then the algebraic K-theory of schemes over S is representable in the ∞-category of motivic spaces. In this section we want to recall that algebraic K-theory is also representable in the ∞-category of motivic spectra and that a concrete model is given by the Snaith construction.

At first we recall the construction of the motivic K-theory space. The concrete ∞-categorical construction we use is the same as the one which is used in [BT15b] and [BT15a].

Let $\mathcal{C}^{\otimes}$ be a closed symmetric monoidal ∞-category (i.e. tensoring with an object is a left adjoint) such that the underlying ∞-category $\mathcal{C}$ has finite products; in particular there is also the cartesian monoidal structure $\mathcal{C}^{\times}$. By [GGN15], Theorem 5.1., there is a uniquely determined symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C}^{\times})$ such that the free functor

$$\mathcal{C} \to \mathrm{CAlg}(\mathcal{C}^{\times}),$$

i.e. the left adjoint to the forgetful functor, is symmetric monoidal with respect to the symmetric monoidal structure $\otimes$ on the domain. Following *loc.cit.* we call a commutative algebra object in the symmetric monoidal ∞-category $\mathrm{CAlg}(\mathcal{C}^{\times})$ a semiring object in $\mathcal{C}$ and denote the ∞-category of semi-ring objects by

$$\mathbf{Rig}(\mathcal{C}).$$

For $X \in \mathbf{Sm}_S$ let now $\mathcal{V}\mathrm{ect}(X)$ be the groupoid of vector bundles over X. Direct sum and tensor product of vector bundles define the structure of a semi-ring object in $\mathbf{Cat}[\mathcal{W}^{-1}]$ on $\mathcal{V}\mathrm{ect}(X)$. Here $\mathbf{Cat}[\mathcal{W}^{-1}]$ is the symmetric monoidal Dwyer-Kan localization (see 2.91 for existence) of the 1-category $\mathbf{Cat}$ of small categories with the cartesian symmetric monoidal structure with respect to the equivalences of categories $\mathcal{W}$. This defines a functor

$$\mathcal{V}\mathrm{ect} \in \mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathbf{Rig}(\mathbf{Cat}[\mathcal{W}^{-1}])).$$

The nerve functor $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ preserves finite products and is therefore symmetric monoidal with respect to the cartesian monoidal structures. Moreover it sends equivalences between categories to homotopy equivalences of simplicial sets. This shows that the nerve refines to a functor

$$\mathrm{CAlg}(\mathbf{Cat}[\mathcal{W}^{-1}]) \xrightarrow{\mathrm{N}} \mathrm{CAlg}(\mathbf{sSet}[\mathcal{W}^{-1}]) \xrightarrow{\simeq, 2.20} \mathrm{CAlg}(\mathcal{S}).$$

In [GGN15] (in the proof of Proposition 8.2.) it is shown that this refinement can furtherly be refined to a lax symmetric monoidal functor and therefore gives a functor

$$\mathrm{N} : \mathbf{Rig}(\mathbf{Cat}[\mathcal{W}^{-1}]) \to \mathbf{Rig}(\mathcal{S}). \tag{3.8}$$

We want to point out that $\mathrm{CAlg}(\mathcal{S})$ is precisely the ∞-category of $\mathbb{E}_\infty$-spaces. As usual an $\mathbb{E}_\infty$-space X is called grouplike if the monoid $\pi_0(X)$ is a group. We denote the full sub-∞-category of $\mathrm{CAlg}(\mathcal{S})$ generated by the grouplike $\mathbb{E}_\infty$-spaces by

$$\mathcal{G}\mathrm{rp}_{\mathrm{E}_\infty}(\mathcal{S}) \subset \mathrm{CAlg}(\mathcal{S}). \tag{3.9}$$

In [GGN15], Proposition 4.1. and Corollary 4.4., the authors show that $\mathcal{G}\mathrm{rp}_{\mathrm{E}_\infty}(\mathcal{S})$ and $\mathrm{CAlg}(\mathcal{S})$ are both presentable and that the full inclusion 3.9 preserves limits. The inclusion moreover preserves filtered colimits and is therefore accessible. This follows from the fact that taking π_0 preserves filtered colimits. By Lurie's adjoint functor theorem for presentable ∞-categories 2.64 it follows that we have an adjunction

$$\Omega\mathrm{B} : \mathrm{CAlg}(\mathcal{S}) \rightleftarrows \mathcal{G}\mathrm{rp}_{\mathrm{E}_\infty}(\mathcal{S}) : \mathrm{incl}$$

and the left adjoint $\Omega\mathrm{B}$ admits a unique symmetric monoidal structure ([GGN15], Theorem 5.1.). The group completion functor therefore also descends to (semi)-ring objects:

$$\Omega\mathrm{B} : \mathbf{Rig}(\mathrm{CAlg}(\mathcal{S})) \to \mathbf{Ring}(\mathcal{S})$$

where $\mathbf{Ring}(\mathcal{S}) := \mathrm{CAlg}(\mathcal{G}\mathrm{rp}_{\mathrm{E}_\infty}(\mathcal{S}))$ is the ∞-category of ring objects in spaces. Now the algebraic K-theory presheaf $\bar{\mathcal{K}}$ on $\mathbf{Sm}_S$ is defined as the image of $\mathcal{V}\mathrm{ect}$ under the following composition:

$$\mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathbf{Rig}(\mathbf{Cat}[\mathcal{W}^{-1}])) \xrightarrow{\mathrm{N}\circ} \mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathbf{Rig}(\mathcal{S})) \xrightarrow{\Omega\mathrm{B}\circ} \mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathbf{Ring}(\mathcal{S})).$$

Remark 3.10. The ∞-category $\mathcal{G}\mathrm{rp}_{\mathrm{E}_\infty}(\mathcal{S})$ with its symmetric monoidal structure is equivalent to the symmetric monoidal ∞-category of connective spectra. Therefore there is canonical functor

$$\mathbf{Ring}(\mathcal{S}) \to \mathrm{CAlg}(\mathrm{Sp}),$$

see [GGN15], Theorem 5.1. In particular there is a K-theory presheaf of spectra

$$\mathcal{K}_{st} \in \mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathrm{CAlg}(\mathrm{Sp})) \tag{3.11}$$

that associates to a scheme its connective K-theory spectrum.

Definition 3.12. *(Motivic K-theory space)*
The motivic K-theory space

$$\mathcal{K} \in \mathrm{Spc}^{mot}$$

is defined as the motivic localization of the algebraic K-theory presheaf $\bar{\mathcal{K}}$*. Note that since* L^{mot} *preserves finite products it is symmetric monoidal with respect to the cartesian symmetric monoidal structures and therefore refines to a functor*

$$L^{mot} : \mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathrm{CAlg}(\mathcal{S})) \xrightarrow{\simeq, 2.72} \mathrm{CAlg}(\mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathcal{S})) \longrightarrow \mathrm{CAlg}(\mathrm{Spc}^{mot})$$

which itself again carries a symmetric monoidal structure ([GGN15]*, Corollary 5.5.) and therefore in fact we have*

$$\mathcal{K} := L^{mot}(\bar{\mathcal{K}}) \in \mathbf{Rig}(\mathrm{Spc}^{mot}).$$

Since the ∞-category $\mathrm{CAlg}(\mathrm{Spc}^{mot})$ is pointed, the free functor $\mathrm{Spc}^{mot} \to \mathrm{CAlg}(\mathrm{Spc}^{mot})$ factors essentially uniquely through Spc_*^{mot}

$$\begin{tikzcd}
\mathrm{Spc}^{mot} \arrow[r, "(-)_+"] \arrow[d] & \mathrm{Spc}_*^{mot} \arrow[dl, dashed, "\exists ! F"] \\
\mathrm{CAlg}(\mathrm{Spc}^{mot}) &
\end{tikzcd}$$

and in fact $F : \mathrm{Spc}_* \to \mathrm{CAlg}(\mathrm{Spc})$ is again a left adjoint ([GGN15], Corollary 4.10.). As explained above there is a canonical symmetric monoidal structure on $\mathrm{CAlg}(\mathrm{Spc}^{mot})$ and it turns out that the functor F can be lifted to a symmetric monoidal functor with respect to this canonical symmetric monoidal structure on the target and the smash product on the domain (again by [GGN15], Theorem 5.1.). In particular the right adjoint of F

$$\mathrm{CAlg}(\mathrm{Spc}^{mot}) \to \mathrm{Spc}_*^{mot}$$

is lax symmetric monoidal ([Lur17], Corollary 7.3.2.7.) and defines therefore a functor

$$V : \mathbf{Rig}(\mathrm{Spc}^{mot}) \to \mathrm{CAlg}(\mathrm{Spc}_*^{mot,\wedge}). \tag{3.13}$$

Theorem 3.14. *The motivic K-theory space* $\mathcal{K}$ *represents algebraic K-theory in the following sense: For every* $X \in \mathbf{Sm}_S$ *and every* $i \geq 0$ *there is a natural group isomorphism:*

$$\pi_i(\mathrm{Map}_{\mathrm{Spc}^{mot}}(X, \mathcal{K})) \cong K_i(X), \tag{3.15}$$

where $K_i(X)$ *is Quillen's higher algebraic K-theory of schemes.*

Proof. If $X \cong \mathrm{Spec}(R)$ is affine, then we have isomorphisms

$$K_i(\mathrm{Spec}(R)) \cong \pi_i(\bar{\mathcal{K}}(\mathrm{Spec}(R))) \cong \pi_i(\mathrm{Map}_{\mathcal{P}(\mathbf{Sm}_S)}(\mathrm{Yo}_\infty(\mathrm{Spec}(R)), \bar{\mathcal{K}})).$$

Here the first isomorphism is a result due to Quillen (see [Wei13], IV. Corollary 4.11.1.) which says that the group completion procedure gives K-theory; and the second is just the Yoneda lemma. The theorem then follows from the Nisnevich descent result for algebraic K-theory and $\mathbb{A}^1$-invariance of algebraic K-theory of regular schemes ([TT90], Theorem 6.8.). Note that $\mathbf{Sm}_S$ consists solely of regular schemes since the base S is assumed to be regular. $\square$

Now we are going to define the motivic $\mathbb{P}^1$-spectrum $\mathcal{K}\mathrm{Gl}$ that represents algebraic K-theory in the stable ∞-category of $\mathbb{P}^1$-spectra. The motivic algebraic K-theory space is (using the functor V from 3.13) canonically an object in $\mathrm{CAlg}(\mathrm{Spc}_*^{mot,\wedge})$ and we therefore have a multiplication map

$$m : \mathcal{K} \wedge \mathcal{K} \to \mathcal{K} \tag{3.16}$$

that classifies the operation of taking tensor products of bundles. Now consider the map $\gamma : (\mathbb{P}^1, \infty) \to \mathcal{K}$ in Spc_*^{mot} that classifies the virtual bundle $L - \mathtt{1}$ of rank 0, where L is the tautological line bundle and $\mathtt{1}$ is the trivial line bundle over $\mathbb{P}^1$. Using the multiplication from 3.16 one can consider the multiplication by γ map:

$$\mu(\gamma) : \mathbb{P}^1 \wedge \mathcal{K} \xrightarrow{\gamma \wedge \mathrm{id}} \mathcal{K} \wedge \mathcal{K} \xrightarrow{m} \mathcal{K}. \tag{3.17}$$

We get a map

$$\overline{\mu(\gamma)} : \mathcal{K} \to \Omega_{\mathbb{P}^1}\mathcal{K} \tag{3.18}$$

that is adjoint to the multiplication by γ map.

Theorem 3.19. *(Motivic Bott periodicity; see for example* [GS09]*, Proposition 4.1.)*
The map $\overline{\mu(\gamma)}$ from 3.18 is an equivalence.

Now we are ready to define the motivic K-theory spectrum $\mathcal{K}\mathrm{Gl}$. By remark 3.6 an object in $\mathrm{Sp}^{\mathbb{P}^1}$ can be described by a sequence $(X_i)_i$ of pointed motivic spaces, indexed by the natural numbers, together with equivalences $X_i \simeq \Omega_{\mathbb{P}^1} X_{i+1}$ for each $i \geq 0$.

Definition 3.20. *(*$\mathcal{K}\mathrm{Gl}$*)*
The motivic spectrum $\mathcal{K}\mathrm{Gl}$ is defined by the sequence $(X_i)_i$ of pointed motivic spaces where $X_i := \mathcal{K}$ for each i, together with the equivalences

$$\overline{\mu(\gamma)} : X_i \to \Omega_{\mathbb{P}^1} X_{i+1}$$

from 3.18.

Corollary 3.21. $\mathcal{K}\mathrm{Gl}$ *represents algebraic K-theory in the sense, that for each $X \in \mathbf{Sm}_S$ we have the following natural isomorphism of groups:*

$$\pi_i(\mathrm{Map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathcal{K}\mathrm{Gl})) \cong K_i(X).$$

Proof. This follows directly from $\Omega^\infty_{\mathbb{P}^1}\mathcal{K}\mathrm{Gl} \simeq \mathcal{K}$ and Theorem 3.14. $\square$

By the motivic Snaith theorem (see [GS09], Theorem 4.17. and [SØ09], Theorem 1.1.), another model for a motivic spectrum that represents algebraic K-theory is the Bott inverted infinite projective space $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+[\beta^{-1}]$. In our construction of Adams operations on K-theory we use this fact in a crucial way.
There are the two maps in Spc^{mot}

$$L, \mathtt{1} : \mathbb{P}^1 \to \mathbb{P}^\infty$$

where L is induced by the canonical closed inclusion of the scheme $\mathbb{P}^1$ into the schemes $\mathbb{P}^n$ for $n \geq 1$ and the map $\mathtt{1}$ is induced by the map of schemes which is constant at the point ∞. Since $\mathrm{Sp}^{\mathbb{P}^1}$ is stable and therefore additive we can form the map

$$\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1_+ \xrightarrow{\Sigma^\infty_{\mathbb{P}^1}L_+ - \Sigma^\infty_{\mathbb{P}^1}\mathtt{1}_+} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$$

which in fact factors essentially uniquely as

$$\begin{tikzcd} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^1_+ \arrow[rr] \arrow[dr] & & \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ . \\ & \Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1, \infty) \arrow[ur, "\exists!\beta"'] & \end{tikzcd} \tag{3.22}$$

We want to view the map β in the diagram 3.22 as an abstract element in $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$ in the sense of definition 2.108. Note that by definition of the symmetric monoidal ∞-category $\mathrm{Sp}^{\mathbb{P}^1}$ the object $\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1, \infty)$ is tensor invertible. Therefore we just have to give $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$ the structure of a commutative algebra object in $\mathrm{Sp}^{\mathbb{P}^1}$. We get this structure from the following two facts: The composition of functors

$$\mathrm{Spc}^{mot} \xrightarrow{(-)_+} \mathrm{Spc}^{mot}_* \xrightarrow{\Sigma^\infty_{\mathbb{P}^1}} \mathrm{Sp}^{\mathbb{P}^1}$$

is symmetric monoidal and the motivic space $\mathbb{P}^\infty$ carries the structure of a commutative algbera object in Spc^{mot}. This structure on $\mathbb{P}^\infty$ is transferred from the classifying space $\mathrm{B}\mathbb{G}_m$ by a canonical equivalence $\mathbb{P}^\infty \simeq \mathrm{B}\mathbb{G}_m$ (see [BNT15], Lemma 5.1.).

Remark 3.23. The existence of an equivalence $\mathrm{B}\mathbb{G}_m \simeq \mathbb{P}^\infty$ implies that for each $X \in \mathbf{Sm}_S$ there is a natural isomorphism

$$\pi_0(\mathrm{Map}_{\mathrm{Spc}^{mot}}(X, \mathbb{P}^\infty)) \cong \mathrm{Pic}(X) \tag{3.24}$$

where $\mathrm{Pic}(X)$ is the Picard group of X. The transferred commutative algebra object structure on $\mathbb{P}^\infty$ equips the left hand side of 3.24 with a monoid structure such that the natural isomorphism is a monoid isomorphism. In other words $\mathbb{P}^\infty$ carries the structure of a commutative algebra object in Spc^{mot} such that the associated multiplication map classifies the tensor product of line bundles.

Definition 3.25. *The map*

$$\beta : \Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1, \infty) \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$$

from 3.22 is called the Bott element in the commutative algebra object $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$*.*

Since $\mathrm{Sp}^{\mathbb{P}^1}$ is presentably symmetric monoidal and stable, localizations of $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$-modules with respect to the element β always exist and localizations of $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$-algebras are canonically algebras (see Lemma 2.117 and Proposition 2.119). Therefore we can define:

Definition 3.26. *(Motivic Snaith spectrum)*
The motivic Snaith spectrum is defined to be

$$\mathbf{K} := \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+[\beta^{-1}] \in \mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1}).$$

Theorem 3.27. *(Motivic Snaith theorem; see* [GS09] *Theorem 4.17. or* [SØ09]*, Theorem 1.1.)*
There is an equivalence

$$\psi : \mathbf{K} \xrightarrow{\simeq} \mathcal{K}\mathrm{Gl} \tag{3.28}$$

in $\mathrm{Sp}^{\mathbb{P}^1}$*.*

In fact in [BNT15] the authors show that one can refine this statement.

Theorem 3.29. *(see* [BNT15]*, Theorem 5.3.)*
There is an equivalence in $\mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathrm{CAlg}(\mathrm{Sp}))$ *of the form*

$$\mathcal{K}_{st} \simeq \mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \mathbf{K}).$$

Remark 3.30. The right handside in the last theorem carries canonically the structure of a commutative algebra object. This follows since X is a commutative algebra object in $\mathrm{CAlg}((\mathrm{Spc}^{mot})^{op})$ via the diagonal (as is always the case for an object in a cartesian symmetric monoidal ∞-category) and $\mathbf{K}$ is in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$. The mapping spectrum always refines to an object in $\mathrm{CAlg}(\mathrm{Sp})$ in this case (see [BNT15], Corollary B.4.).

Theorem 3.29 is our starting point for the construction of multiplicative Adams operations on the connective K-theory spectra of schemes.

3.2. **Construction of the Adams operations.** Now we are going to construct Adams operations on algebraic K-theory. For each $k \in \mathbb{N}$ we will construct a map of $\mathbb{E}_\infty$-$\mathbb{P}^1$-spectra

$$\psi^k : \mathbf{K}[k^{-1}] \to \mathbf{K}[k^{-1}]$$

where $\mathbf{K}$ is the Snaith-type $\mathbb{P}^1$-spectrum representing algebraic K-theory from 3.26. We use the fact that $\mathbf{K}$ has a universal property since it can be written as a certain localization of $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$.
The ∞-category of motivic spaces carries the cartesian symmetric monoidal structure and we already saw that the infinite projective space $\mathbb{P}^\infty$ is a commutative algebra object with respect to this structure. In particular we have the k-th power maps

$$(-)^k : \mathbb{P}^\infty \to \mathbb{P}^\infty$$

in Spc^{mot}. This map is defined as the composition

$$\mathbb{P}^\infty \xrightarrow{\Delta} \underbrace{\mathbb{P}^\infty \times ... \times \mathbb{P}^\infty}_{k \text{ times}} \xrightarrow{m_k} \mathbb{P}^\infty$$

where the first map is the diagonal and the second map is the multiplication coming from the commutative algebra structure on $\mathbb{P}^\infty$. Note that this map classifies the k-fold tensor product of line bundles. Now in ordinary symmetric monoidal categories it is clear that these k-th power maps are always maps of algebras and this is also the case here. For commutative algebra objects in cartesian symmetric monoidal ∞-categories the k-th power maps canocially refine to maps of algebras (see [Arn16], section 2.6.). In *loc.cit.* it is also shown that one has canonical homotopies between the composition of the k-th and l-th power maps and the kl-th power map. In particular $(-)^k : \mathbb{P}^\infty \to \mathbb{P}^\infty$ is canonically a map in $\mathrm{CAlg}(\mathrm{Spc}^{mot})$ and there are homotopies $(-)^k \circ (-)^l \simeq (-)^{kl}$ for all natural numbers k and l. Since $\Sigma^\infty_{\mathbb{P}^1}(-)_+$ is symmetric monoidal, we have an induced map

$$\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \xrightarrow{\Sigma^\infty_{\mathbb{P}^1}(-)^k_+} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$$

in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$. Localizing the target of the last map with respect to the Bott element and with respect to the element k (since $\mathrm{Sp}^{\mathbb{P}^1}$ is additive, natural numbers can be considered as elements in commutative algebra objects, see 2.120), we get

$$f : \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \xrightarrow{\Sigma^\infty_{\mathbb{P}^1}(-)^k_+} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \xrightarrow{can} \mathbf{K}[k^{-1}] \tag{3.31}$$

in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$. Therefore the $\mathbb{P}^1$-spectrum $\mathbf{K}[k^{-1}]$ carries - apart from the canonical one - a structure of a $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$-algebra which is induced from f. We call this algebra

$$\mathbf{K}[k^{-1}]_f \in \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+\text{-}\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1}).$$

Proposition 3.32. *The Bott element* $\beta : \Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1,\infty) \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$ *acts invertibly on the* $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$*-algebra* $\mathbf{K}[k^{-1}]_f$.

Proof. We have to show that the composition

$$\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1,\infty)\wedge\mathbf{K}[k^{-1}] \xrightarrow{\beta\wedge\mathrm{id}} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+\wedge\mathbf{K}[k^{-1}] \xrightarrow{\nu_f} \mathbf{K}[k^{-1}]$$

is an equivalence in $\mathrm{Sp}^{\mathbb{P}^1}$ where $\nu_f : \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+\wedge\mathbf{K}[k^{-1}] \to \mathbf{K}[k^{-1}]$ comes from the $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$-algebra structure on $\mathbf{K}[k^{-1}]_f$. This composition written out in greater detail is given by
(3.33)

$$\begin{tikzcd}
\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1,\infty)\wedge\mathbf{K}[k^{-1}] \arrow[r, "\beta\wedge id"] & \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+\wedge\mathbf{K}[k^{-1}] \arrow[r, "\Sigma^\infty_{\mathbb{P}^1}(-)^k_+\wedge id"] & \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+\wedge\mathbf{K}[k^{-1}] \arrow[d, "can\wedge id"] \\
 & \mathbf{K}[k^{-1}] & \mathbf{K}[k^{-1}]\wedge\mathbf{K}[k^{-1}] \arrow[l, "m_{\mathbf{K}[k^{-1}]}"']
\end{tikzcd}$$

where $m_{\mathbf{K}[k^{-1}]}$ is the multiplication map that comes from the commutative algebra structure on k-inverted K-theory. We already know that the composition

$$\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1,\infty)\wedge\mathbf{K} \xrightarrow{\beta\wedge id} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+\wedge\mathbf{K} \xrightarrow{can\wedge id} \mathbf{K}\wedge\mathbf{K} \xrightarrow{m_{\mathbf{K}}} \mathbf{K} \tag{3.34}$$

is an equivalence since by definition the Bott element β acts invertibly on $\mathbf{K}$. The claim now is that the composition

$$\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1,\infty) \xrightarrow{\beta} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \xrightarrow{\Sigma^\infty_{\mathbb{P}^1}(-)^k_+} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \xrightarrow{can} \mathbf{K} \tag{3.35}$$

is homotopic to k-times the composition

$$\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1,\infty) \xrightarrow{\beta} \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \xrightarrow{can} \mathbf{K}. \tag{3.36}$$

From this claim the proposition obviously follows.
The map 3.36 precisely classifies the virtual bundle $1-\mathcal{O}(-1)$ (where $\mathcal{O}(-1)$ is the tautological line bundle) of rank 0 in

$$\pi_0(\mathrm{Map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(\mathbb{P}^1,\infty),\mathbf{K})) \cong \ \ker(\mathrm{K}_0(\mathbb{P}^1) \xrightarrow{\infty^*} \mathrm{K}_0(S)) \tag{3.37}$$

where S is the base scheme; whereas the map 3.34 classifies the virtual bundle $1-\mathcal{O}(-1)^k$ of rank 0. The claim that k times the map 3.36 is homotopic to the map 3.34 therefore follows if we can show that there is an equality of K_0-classes

$$[1] - [\mathcal{O}(-1)^k] = k\left([1] - [\mathcal{O}(-1)]\right). \tag{3.38}$$

For this we use the short exact sequence of line bundles

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O}(0) \to 0$$

which is a special case of the Euler sequence for the 1-dimensional projective space. Note that this sequence does not split but we get the relation

$$2[\mathcal{O}(-1)] = [1] + [\mathcal{O}(-2)]. \tag{3.39}$$

in K_0. To show the equality 3.38 we do an easy induction on k using the relation 3.39. The base case $k = 1$ is trivial and the inductive step is carried out in the following computation (where we leave out the square brackets for simplicity):

$$\begin{aligned}
1 - \mathcal{O}(-1)^k &= 1 - \mathcal{O}(-1)\mathcal{O}(-1)^{k-1} \\
&= 1 - \mathcal{O}(-1)\left(1 - (k-1)(1 - \mathcal{O}(-1))\right) \\
&= 1 - \mathcal{O}(-1) + (k-1)\mathcal{O}(-1) - (k-1)\mathcal{O}(-1)^2 \\
&= 1 - \mathcal{O}(-1) + (k-1)\mathcal{O}(-1) - (k-1)(2\mathcal{O}(-1) - 1) \\
&= k(1 - \mathcal{O}(-1)).
\end{aligned}$$

Here the second equality follows from the induction hypothesis and the fourth equality from 3.39. $\square$

As a corollary we get:

Theorem 3.40. *For every natural number k there exists an essentially uniquely defined map $\mathbf{K}[k^{-1}] \to \mathbf{K}[k^{-1}]$ in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$ such that the diagram*

$$\begin{tikzcd}
\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \arrow[r, "\Sigma^\infty_{\mathbb{P}^1}(-)^k_+"] \arrow[d] & \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \arrow[d] \\
\mathbf{K}[k^{-1}] \arrow[r] & \mathbf{K}[k^{-1}]
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$ commutes.

Proof. This follows directly from the universal property of localizations and proposition 3.32. $\square$

Definition 3.41. *(Motivic Adams operations) We call the map from the last theorem 3.40 the k-th stable motivic Adams operation and denote it by*

$$\psi^k_{mot} : \mathbf{K}[k^{-1}] \to \mathbf{K}[k^{-1}].$$

Using the refinement of the motivic Snaith theorem (see 3.29) we get the following corollary:

Corollary 3.42. *For any $X \in \mathbf{Sm}_S$ there is a map*

$$\psi^k_X : \mathcal{K}_{st}(X)[k^{-1}] \to \mathcal{K}_{st}(X)[k^{-1}]$$

in CAlg(Sp) *that induces the classical Adams operations on K_0, i.e. the unique ring homomorphisms*

$$K_0(X)[k^{-1}] \to K_0(X)[k^{-1}]$$

that send a class of a line bundle $\mathcal{L}$ over X to the class of its k-th power $\mathcal{L}^{\otimes k}$. Moreover the ψ^k_X are natural in X.

Proof. The only thing which doesn't follow directly from what we have done so far is that one has an equivalence

$$\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathbf{K}[k^{-1}]) \simeq \mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathbf{K})[k^{-1}],$$

i.e. that we can pull out filtered colimits out of the mapping spectrum functor. But this is true since the objects $\Sigma^\infty_{\mathbb{P}^1} X_+$ are all compact in $\mathrm{Sp}^{\mathbb{P}^1}$ by work of Dugger and Isaksen. More precisely they show (see [DI05], Theoprem 9.1.) that these objects are compact objects in the triangulated homotopy category $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{P}^1})$, i.e. mapping out of these objects in $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{P}^1})$ preserves coproducts. But by a result from Lurie (see [Lur17], Proposition 1.4.4.1. (3)) this implies that these objects are compact in the ∞-categorical sense, i.e. that mapping out of these objects in $\mathrm{Sp}^{\mathbb{P}^1}$ preserves filtered colimits. ☐

Remark 3.43. Since for any natural numbers k, l the maps

$$\mathbb{P}^\infty \xrightarrow{(-)^k} \mathbb{P}^\infty \xrightarrow{(-)^l} \mathbb{P}^\infty$$

and

$$\mathbb{P}^\infty \xrightarrow{(-)^{kl}} \mathbb{P}^\infty$$

in $\mathrm{CAlg}(\mathrm{Spc}^{mot})$ are homotopic, we also get homotopies

$$\psi^l_{mot} \circ \psi^k_{mot} \simeq \psi^{kl}_{mot}$$

by the universal property of localizations that was used to construct the motivic stable Adams operations. Therefore there are homotopies

$$\psi^l_X \circ \psi^k_X \simeq \psi^{kl}_X,$$

too.

We want to finish the chapter by comparing the construction of the maps $\psi^k_{mot} : \mathbf{K}[k^{-1}] \to \mathbf{K}[k^{-1}]$ to other constructions of motivic Adams operations that are in the literature. As already mentioned, Arndt in his thesis [Arn16] also constructed multiplicative stable Adams operations on $\mathbf{K}[k^{-1}]$ (independently to our work). His approach is similar to ours. In particular he proves an analogue of our proposition 3.32 but his proof of this proposition is different.
Without considering multiplicative structures Riou constructed motivic Adams operations in his thesis [Rio06]. His construction is based on the observation that in the case of $S = \mathrm{Spec}(\mathbb{Z})$ ones has an injection

$$(3.44) \qquad \mathrm{End}_{\mathrm{Ho}(\mathrm{Sp}^{\mathbb{P}^1})}(\mathbf{K}[k^{-1}]) \hookrightarrow \mathrm{End}_{\mathrm{Fun}(\mathbf{Sm}^{op}_{\mathrm{Spec}(\mathbb{Z})},\mathrm{Ab})}(K_0(-)[k^{-1}]),$$

i.e. there is up to homotopy a unique endomorphism of $\mathbf{K}[k^{-1}]$ that lifts the Adams operations on K_0. This observation can be found as a corollary of Proposition 5.1.1. and Corollary 5.2.7. and Proposition 5.2.8. of the published article [Rio10]. He then constructs operations on $\mathbf{K}[k^{-1}]$ over an arbitrary base scheme S (still noetherian, of finite dimension and regular) by changing the base along the unique morphism of schemes from S to $\mathrm{Spec}(\mathbb{Z})$.
By injectivity of the map 3.44 his operations for $S = \mathrm{Spec}(\mathbb{Z})$ have to be the same as the maps in $\mathrm{Sp}^{\mathbb{P}^1}$ that underlie our multiplicative maps ψ^k_{mot} up to homotopy. In order to conclude that they are equal up to homotopy for a general base scheme S we have to check that our maps ψ^k_{mot} are compatible with respect to base change. But this is true because the base change functors

$$\mathrm{Spc}^{mot}_{\mathrm{Spec}(\mathbb{Z})} \to \mathrm{Spc}^{mot}_S$$

and

$$\mathrm{Sp}^{\mathbb{P}^1}_{\mathrm{Spec}(\mathbb{Z})} \to \mathrm{Sp}^{\mathbb{P}^1}_S$$

are symmetric monoidal left adjoints and fit into a commutative diagram of symmetric monoidal ∞-categories and left adjoint functors

$$\begin{tikzcd}
\mathrm{Spc}^{mot}_{\mathrm{Spec}(\mathbb{Z})} \arrow[r] \arrow[d, "\Sigma^\infty_+"'] & \mathrm{Spc}^{mot}_S \arrow[d, "\Sigma^\infty_+"] \\
\mathrm{Sp}^{\mathbb{P}^1}_{\mathrm{Spec}(\mathbb{Z})} \arrow[r] & \mathrm{Sp}^{\mathbb{P}^1}_S
\end{tikzcd}$$

(see [Hoy17], Proposition 4.1 and the discussion right before Lemma 6.2). Thus the basechange functor preserves the k-th power map and the Snaith type spectrum $\mathbf{K} = \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+[\beta^{-1}]$ and therefore also preserves the motivic stable Adams operations ψ^k_{mot}.

4. ADAMS OPERATIONS ON DIFFERENTIAL ALGEBRAIC K-THEORY

In the last section we explained how one can use the motivic Snaith theorem to construct $\mathbb{E}_\infty$-Adams operations ψ^k on the algebraic K-theory (localized at the natural number k) of smooth separated schemes of finite type over a regular base. In the present section we want to show that it is possible to lift these Adams operations to $\mathbb{E}_\infty$-maps on differential algebraic K-theory.
If the base scheme is the spectrum of the complex numbers one can define differential algebraic K-theory of smooth separated schemes. This is a differential refinement of algebraic K-theory using the Beilinson regulator

$$r_X : K_*(X) \to \bigoplus_{i\in\mathbb{N}} \mathrm{H}^{2i-*}_{\mathrm{aH}}(X, \mathbb{R}(i)) \tag{4.1}$$

from algebraic K-theory to absolute Hodge cohomology. In [BNT15] Bunke, Nikolaus and Tamme realize the Beilinson regulator as a natural $\mathbb{E}_\infty$-map between spectra

$$reg_X : \mathcal{K}_{st}(X) \to H(\mathrm{IDR}(X)),$$

i.e. $\pi_*(reg_X) = r_X$. Here H is the Eilenberg-MacLane functor and $\mathrm{IDR}(X)$ is a certain dg-algebra over the real numbers that computes absolute Hodge cohomology of X. For us the n-th differential algebraic K-theory spectrum of X is defined as the pullback

$$\begin{tikzcd}
\widehat{\mathcal{K}}^n(X) \arrow[r] \arrow[d] & H(\sigma^{\geq n}\mathrm{IDR}(X)) \arrow[d] \\
\mathcal{K}_{st}(X) \arrow[r, "reg_X"] & H(\mathrm{IDR}(X))
\end{tikzcd}$$

in the ∞-category of spectra; $\sigma^{\geq n}$ denotes the stupid truncation functor that truncates a complex below degree n. We should mention that in the papers [BT15a] and [BT15b] the authors construct differential algebraic K-theory as a functor from the product category $\mathbf{Mf}\times\mathbf{Sm}_{\mathbb{C}}$ of the category of smooth manifolds and smooth schemes over the complex numbers. In their context our definition of the differential algebraic K-theory functor is the evaluation at the one-point manifold $\{*\}$ in the manifold direction.
The goal of this section is to show that one can lift the $\mathbb{E}_\infty$-Adams operations on algebraic K-theory to differential algebraic K-Theory. More precisely we will construct a commutative diagram (functorial in

$X)$

(4.2)
$$\begin{tikzcd}
\mathcal{K}_{st}(X)[k^{-1}] \arrow[r, "reg_X"] \arrow[d, "\psi^k"'] & H(\mathrm{IDR}(X)) \arrow[d] & H(\sigma^{\geq n}\mathrm{IDR}(X)) \arrow[l] \arrow[d] \\
\mathcal{K}_{st}(X)[k^{-1}] \arrow[r, "reg_X"] & H(\mathrm{IDR}(X)) & H(\sigma^{\geq n}\mathrm{IDR}(X)) \arrow[l]
\end{tikzcd}$$

in spectra with the stable Adams operations ψ^k_X from Corollary 3.42 as the left vertical morphism. This diagram then induces a natural map of spectra

$$\widehat{\psi}^{k,n} : \widehat{\mathcal{K}}^n(X)[k^{-1}] \to \widehat{\mathcal{K}}^n(X)[k^{-1}].$$

The main difficulty here is to find a map of dg-algebras $\mathrm{IDR}(X) \to \mathrm{IDR}(X)$ such that the left square commutes. The commutativity of the right square will then be a triviality.

We want to point out right away that the n-th differential algebraic K-theory $\widehat{\mathcal{K}}^n(X)$ does not canonically carry the structure of an $\mathbb{E}_\infty$-ring spectrum since the stupid truncation $\sigma^{\geq n}\mathrm{IDR}(X)$ is not a dg-algebra. But, as the authors show in [BT15a], there is a canonical structure of an $\mathbb{E}_\infty$-ring spectrum on the direct sum

$$\widehat{\mathcal{K}}(X) := \bigoplus_{n\in\mathbb{Z}} \widehat{\mathcal{K}}^n(X)$$

and we will show that the direct sum map

$$\widehat{\psi}^k := \bigoplus_n \widehat{\psi}^{k,n} : \widehat{\mathcal{K}}(X)[k^{-1}] \to \widehat{\mathcal{K}}(X)[k^{-1}]$$

refines to a natural map of $\mathbb{E}_\infty$-ring spectra.

4.1. **The Beilinson regulator as a map of ring spectra.** To pin down the setting we firstly want to recall results from [BNT15] about the Beilinson regulator. We consider schemes over the complex numbers. The category of smooth separated schemes of finite type over $\mathrm{Spec}(\mathbb{C})$ is denoted by $\mathbf{Sm}_{\mathbb{C}}$.

By the refined version of the motivic Snaith theorem 3.29 the motivic spectrum $\mathbf{K} := \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+[\beta^{-1}]$ represents algebraic K-theory in the stable ∞-category of motivic spectra, i.e. there is a natural equivalence

$$\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathbf{K}) \simeq \mathcal{K}_{st}(X).$$

Now in [BNT15] the authors construct an object $\mathbf{H} \in \mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$, which they call the motivic absolute Hodge spectrum, that represents absolute Hodge cohomology. More precisely they show an equivalence of $\mathbb{E}_\infty$-spectra

$$\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathbf{H}) \simeq H(\mathrm{IDR}(X))$$

where on the right handside H is the Eilenberg-MacLane functor and $\mathrm{IDR}(X)$ is a certain dg-algebra whose cohomology is absolute Hodge cohomology of X. The definition of the dg-algebra $\mathrm{IDR}(X)$ will be recalled below.

Having motivic $\mathbb{E}_\infty$-spectra $\mathbf{K}$ and $\mathbf{H}$ representing algebraic K-theory and absolute Hodge cohomology, the natural question is if there exists an $\mathbb{E}_\infty$-map $\mathbf{K} \to \mathbf{H}$ which, evaluated at an object $X \in \mathbf{Sm}_\mathbb{C}$, induces the Beilinson regulator map

$$r_X : K_*(X) \to \bigoplus_{i \in \mathbb{N}} \mathrm{H}^{2i-*}_{\mathrm{aH}}(X, \mathbb{R}(i))$$

on homotopy groups. In [BNT15] this question is answered positively. In what follows we are going to give the definition of the object $\mathbf{H} \in \mathrm{Sp}^{\mathbb{P}^1}$ and the construction of the multiplicative motivic Beilinson regulator map $\mathbf{K} \to \mathbf{H}$.

Since Beilinson's absolute Hodge cohomology ([Bei86], chapter 5) is defined in terms of Hodge theory we firstly have to recall some definitions and results about Hodge structures and Hodge complexes. In what follows all filtrations will assumed to be Hausdorff, exhaustive and finite.

Definition 4.3. *(Pure Hodge structures)*
A pure $\mathbb{R}$-Hodge structure of weight $n \in \mathbb{Z}$ is a pair $(H, \mathcal{F})$ consisting of a finite dimensional $\mathbb{R}$-vector space H and a decreasing filtration $\mathcal{F}$ of complex subspaces on the complexification $H_\mathbb{C} := H \otimes_\mathbb{R} \mathbb{C}$ such that for every integer $p \in \mathbb{Z}$

$$H_\mathbb{C} = \mathcal{F}^p H_\mathbb{C} \oplus \overline{\mathcal{F}^{n-p+1} H_\mathbb{C}}$$

where $\overline{(-)}$ takes the complex conjugated subspace inside $H_\mathbb{C}$ ($H_\mathbb{C}$ carries the canonical real structure). The filtration $\mathcal{F}$ is called the Hodge filtration. A morphism between pure $\mathbb{R}$-Hodge structures is an $\mathbb{R}$-vector space homomorphism which respects the filtration on the complexification. The category of pure $\mathbb{R}$-Hodge structures is denoted by $\mathbf{PHS}_\mathbb{R}$.

The main motivating example of a pure $\mathbb{R}$-Hodge structure is the deRham cohomology with real coefficients of a smooth projective variety over the complex numbers with its Hodge filtration. This is the content of the Hodge decomposition theorem for Kähler manifolds. Much simpler examples are the following:

Example 4.4. For each $n \in 2\mathbb{Z}$ there is a pure Hodge structure $(\mathbb{R}, \mathcal{F}_n)$ of weight n on the one dimensional real vector space $\mathbb{R}$. Here the Hodge filtration $\mathcal{F}_n$ is defined by

$$\mathcal{F}^{n/2}\mathbb{C} = \mathbb{C}, \ \mathcal{F}^{n/2+1}\mathbb{C} = \{0\}.$$

We denote $(\mathbb{R}, \mathcal{F}_n)$ by $\mathbb{R}(-n/2)$. $\mathbb{R}(0)$ is called the trivial pure $\mathbb{R}$-Hodge structure on $\mathbb{R}$ and $\mathbb{R}(1)$ is called the Tate structure.

Definition 4.5. *(Mixed Hodge structures)*
A mixed Hodge structure is a triple $(H, \mathcal{W}, \mathcal{F})$ consisting of a finite dimensional real vector space H, an increasing filtration of real vector spaces on H and a decreasing filtration $\mathcal{F}$ of complex subspaces on the complexification $H_{\mathbb{C}} := H \otimes_{\mathbb{R}} \mathbb{C}$ which satisfy the following condition: For each $n \in \mathbb{Z}$ the pair $(\mathrm{Gr}_n^{\mathcal{W}} H, \mathrm{Gr}_n^{\mathcal{W}_{\mathbb{C}}} \mathcal{F})$ is a pure Hodge structure of weight n. Here $\mathrm{Gr}_n^{\mathcal{W}} H := \mathcal{W}_n H / \mathcal{W}_{n-1} H$ is the n-th piece of the associated graded of the filtration $\mathcal{W}$, $\mathcal{W}_{\mathbb{C}}$ is the increasing filtration on $H_{\mathbb{C}}$ which is induced by $\mathcal{W}$ and $\mathrm{Gr}_n^{\mathcal{W}_{\mathbb{C}}} \mathcal{F}$ is the decreasing filtration on $\mathrm{Gr}_n^{\mathcal{W}_{\mathbb{C}}} H_{\mathbb{C}}$ which is induced by $\mathcal{F}$. A morphism between mixed $\mathbb{R}$-Hodge structures is an $\mathbb{R}$-vector space homomorphism which is compatible with both filtrations. The category of mixed $\mathbb{R}$-Hodge structures is denoted by $\mathbf{MHS}_{\mathbb{R}}$.

Recall that a morphism $f : (A, \mathcal{F}) \to (B, \mathcal{G})$ between (decreasingly) filtered vector spaces is called strictly compatible with the filtrations if

$$\mathrm{im}(f) \cap \mathcal{G}^p = \mathrm{im}(f|_{\mathcal{F}^p}).$$

If $(C, \mathcal{F})$ is a (decreasingly) filtered cochain complex, then a straightforward calculation shows that the differential is strictly compatible with the filtration if and only if the spectral sequence associated to $(C, \mathcal{F})$ degenerates at the E_1-term (see [Del71], Proposition 1.3.2.).

Definition 4.6. *(Mixed Hodge complexes)*
A mixed $\mathbb{R}$-Hodge complex is a triple $(C, \mathcal{W}, \mathcal{F})$ consisting of a bounded cochain complex of (not necessarily finite dimensional) $\mathbb{R}$-vector spaces C, an increasing filtration of bounded cochain complexes of $\mathbb{R}$-vector spaces $\mathcal{W}$ on C and a decreasing filtration of bounded cochain complexes of complex vector spaces $\mathcal{F}$ on the complexification $C_{\mathbb{C}} := C \otimes_{\mathbb{R}} \mathbb{C}$ such that the following conditions are fulfilled:

1. *The cohomology groups $\mathrm{H}^k(C)$ are finite dimensional $\mathbb{R}$-vector spaces in each degree.*
2. *The differential of the filtered complex $(\mathrm{Gr}_n^{\mathcal{W}_{\mathbb{C}}} C_{\mathbb{C}}, \mathrm{Gr}_n^{\mathcal{W}_{\mathbb{C}}} \mathcal{F})$ is strictly compatible with the filtration for every n.*
3. *For every k the pair $(\mathrm{H}^k(\mathrm{Gr}_n^{\mathcal{W}} C), \overline{\mathcal{F}})$, where $\overline{\mathcal{F}}$ is the decreasing filtration on $\mathrm{H}^k(\mathrm{Gr}_n^{\mathcal{W}_{\mathbb{C}}} C_{\mathbb{C}})$, which is induced by $\mathcal{F}$, is a pure Hodge structure of weight n.*

A morphism between mixed $\mathbb{R}$-Hodge complexes is a morphism of cochain complexes of $\mathbb{R}$-vector spaces which is compatible with both the filtrations. The category of mixed $\mathbb{R}$-Hodge complexes is denoted by $\mathbf{MHC}_{\mathbb{R}}$.

We have a chain of full inclusions

$$\mathbf{PHS}_{\mathbb{R}} \hookrightarrow \mathbf{MHS}_{\mathbb{R}} \hookrightarrow \mathbf{MHC}_{\mathbb{R}}, \tag{4.7}$$

where the first inclusion is given by sending a pure structure $(H, \mathcal{F})$ of weight n to the trivial mixed structure $(H, \mathcal{W}, \mathcal{F})$ with

$$\mathcal{W}_{n-1}H = \{0\}, \ \mathcal{W}_n H = H$$

and where the second inclusion is given by sending the mixed structure $(H, \mathcal{W}, \mathcal{F})$ to the complex which is trivial outside degree 0 and which has $(H, \mathcal{W}, \mathcal{F})$ sitting in degree 0. Note that each of the categories $\mathbf{PHS}_{\mathbb{R}}$, $\mathbf{MHS}_{\mathbb{R}}$ and $\mathbf{MHC}_{\mathbb{R}}$ has a symmetric monoidal structure which essentially comes from the tensor product of $\mathbb{R}$-vector spaces, respectively tensor product of cochain complexes. The full inclusions from 4.7 are symmetric monoidal functors with respect to these structures. Another fact we will use later is that the symmetric monoidal category $\mathbf{MHC}_{\mathbb{R}}$ is even symmetric monoidal closed (see [BNT15], Construction 2.7.).

Example 4.8. Let $X \in \mathbf{Sm}_{\mathbb{C}}$. Then X has an associated complex manifold and we use the same symbol X to denote that manifold. An example of a mixed $\mathbb{R}$-Hodge complex which will be important for us is the complex of smooth $\mathbb{R}$-valued differential forms with logarithmic singularities along infinity on X. We give a detailed definition of this complex.

By existence of compactifications ([Nag62]) and resolution of singularities in characteristic zero ([Hir64]) there always exists a so called good compactification of X, i.e. an open embedding $X \hookrightarrow \overline{X}$ of X into a smooth and proper $\overline{X}$, such that the complement $D := \overline{X} \setminus X$ is a normal crossing divisor. For any complex manifold Y let $\mathrm{A}(Y)_{\mathbb{R}}$ be the algebra of smooth $\mathbb{R}$-valued differential forms on Y. For a chosen good compactification we define $\mathrm{A}_{\overline{X},\mathbb{R}}(X, \log D)$ to be the unital sub-$\mathrm{A}(\overline{X})_{\mathbb{R}}$-algebra of $\mathrm{A}(X)_{\mathbb{R}}$ which is generated by

$$\log(z_i \overline{z_i}), \ \mathrm{Re}(\mathrm{d}z_i/z_i) \text{ and } \mathrm{Im}(\mathrm{d}z_i/z_i)$$

where $(z_i)_i$ are local coordinates of $\overline{X}$ such that D is locally the zero locus of the product $\prod_i z_i$.

The weight filtration $\mathcal{W}$ on $\mathrm{A}_{\overline{X},\mathbb{R}}(X, \log D)$ is defined to be the multiplicative increasing filtration with respect to which all elements of $\mathrm{A}(\overline{X})$ have weight 0 and the elements $\log(z_i \overline{z_i})$, $\mathrm{Re}(\mathrm{d}z_i/z_i)$ and $\mathrm{Im}(\mathrm{d}z_i/z_i)$ have weight 1.

The decreasing Hodge filtration $\mathcal{F}$ on the complexification

$$\mathrm{A}_{\overline{X}}(X, \log D) := \mathrm{A}_{\overline{X},\mathbb{R}}(X, \log D) \otimes_{\mathbb{R}} \mathbb{C}$$

is defined in the usual way, i.e. $\mathcal{F}^p\mathrm{A}_{\overline{X}}(X, \log D)$ consists of elements which can locally be written as an $\mathbb{R}$-linear combination of forms

$$\mathrm{d}v_{i_1} \wedge \mathrm{d}v_{i_2} \wedge ... \wedge \mathrm{d}v_{i_p} \wedge \omega,$$

where the $(v_i)_i$ are local holomorphic coordinates of X. Burgos' main result in [Bur94] (Corollary 2.2. in *loc.cit*) states that the triple

$$(\mathrm{A}_{\overline{X},\mathbb{R}}(X, \log D), \mathcal{W}, \mathcal{F})$$

is a mixed $\mathbb{R}$-Hodge complex. Of course this construction depends on the chosen good compactification $X \hookrightarrow \overline{X}$ but in fact, if one has a commutative diagram in $\mathbf{Sm}_{\mathbb{C}}$ of the form

(4.9)

$$\begin{tikzcd} & X \arrow[dl, hook] \arrow[dr, hook] & \\ \overline{X} \arrow[rr] & & \overline{X}' \end{tikzcd}$$

where $\overline{X}$ and $\overline{X}'$ are good compactifications of X with complements $D := \overline{X} \setminus X$ and $D' := \overline{X}' \setminus X$, then the induced map

$$\mathrm{A}_{\overline{X}',\mathbb{R}}(X, \log D') \to \mathrm{A}_{\overline{X},\mathbb{R}}(X, \log D)$$

is a quasi-isomorphism which respects both the weight and the Hodge filtration (follows from [Del71], Proposition 3.1.8. and the discussion before Définition 3.2.12.). To make the construction fully functorial in X we now take the colimit of the $\mathrm{A}_{\overline{X},\mathbb{R}}(X, \log D)$ over all good compactifications (as it is also done in [BT15b]). For this let I_X be the category whose objects are good compactifications of X and morphisms are commutative diagrams as in 4.9. Note that I_X is cofiltered ([Del71], discussion before Définition 3.2.12.). There is a functor

$$\mathrm{I}_X^{op} \to \mathbf{MHC}_{\mathbb{R}}, \tag{4.10}$$

that sends a good compactification $X \hookrightarrow \overline{X}$ to the mixed $\mathbb{R}$-Hodge complex
$\mathrm{A}_{\overline{X},\mathbb{R}}(X, \log D)$. We define

$$\mathrm{A}_{\log}(X)_{\mathbb{R}} := \mathrm{colim}_{\mathrm{I}_X^{op}} \mathrm{A}_{\overline{X},\mathbb{R}}(X, \log D) \tag{4.11}$$

which comes equipped with an induced weight filtration $\mathcal{W}$ and an induced Hodge filtration $\mathcal{F}$ on the complexification and the triple

$$(\mathrm{A}_{\log}(X)_{\mathbb{R}}, \mathcal{W}, \mathcal{F})$$

is a mixed $\mathbb{R}$-Hodge complex. Note also, that by filteredness of I_X^{op} the canonical morphisms

$$\mathrm{A}_{\overline{X},\mathbb{R}}(X, \log D) \to \mathrm{A}_{\log}(X)_{\mathbb{R}} \tag{4.12}$$

in $\mathbf{MHC}_{\mathbb{R}}$ are all quasi-isomorphisms. This construction defines a functor

$$\mathrm{A}_{\log} : \mathbf{Sm}_{\mathbb{C}}^{op} \to \mathbf{MHS}_{\mathbb{R}}, \tag{4.13}$$

since pulling back differential forms along morphisms $X \to Y$ in $\mathbf{Sm}_{\mathbb{C}}$ preserves both the weight and the Hodge filtrations.

The category $\mathbf{MHC}_{\mathbb{R}}$ has a natural notion of weak equivalences: We call those morphisms in $\mathbf{MHC}_{\mathbb{R}}$ weak equivalence which are quasi-ismorphisms on the underlying cochain complex of $\mathbb{R}$-vector spaces. Since quasi-isomorphisms between cochain complexes of vector spaces over a field are homotopy equivalences it follows easily that the class of so defined quasi-isomorphisms W in $\mathbf{MHC}_{\mathbb{R}}$ is compatible in the sense of proposition 2.91 with the tensor product of mixed Hodge complexes in the sense of proposition 2.91. Therefore by the same proposition the symmetric monoidal Dwyer-Kan localization $\mathbf{MHC}_{\mathbb{R}}$ exists and we have a canonical symmetric monoidal functor

$$\iota : \mathbf{MHC}_{\mathbb{R}} \to \mathbf{MHC}_{\mathbb{R}}[W^{-1}]. \tag{4.14}$$

The next two statements contain the structural properties of $\mathbf{MHC}_{\mathbb{R}}[W^{-1}]$:

Lemma 4.15. *(see* [BNT15]*, Lemma 2.6.)*
The Dwyer-Kan localization $\mathbf{MHC}_{\mathbb{R}}[W^{-1}]$ *is a stable symmetric monoidal* ∞*-category.*

Theorem 4.16. *(see* [BNT15]*, discussion after Lemma 2.6. and* [Bei86]*, Theorem 3.4.)*
The canonical symmetric monoidal functor

$$\mathrm{Ch}^b(\mathbf{MHS}_{\mathbb{R}}) \hookrightarrow \mathbf{MHC}_{\mathbb{R}}$$

where Ch^b *denotes the category of bounded chain complexes, descends to an equivalence of stable symmetric monoidal* ∞*-categories*

$$\mathrm{Ch}^b(\mathbf{MHS}_{\mathbb{R}})[W^{-1}] \xrightarrow{\simeq} \mathbf{MHC}_{\mathbb{R}}[W^{-1}]. \tag{4.17}$$

Definition 4.18. *(Absolute Hodge cohomology)*
Let $X \in \mathbf{Sm}_{\mathbb{C}}$ *then absolute Hodge cohomology of* X *is defined as*
(4.19)

$$\mathrm{H}^n_{\mathrm{aH}}(X, \mathbb{R}(i)) := \pi_{-n}\left(\mathrm{map}_{\mathbf{MHC}_{\mathbb{R}}[W^{-1}]}\left(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \iota(\mathbb{R}(i))\right)\right).$$

We now want to recall from [BNT15] how one can construct an object in CAlg(Sp) such that the homotopy ring of that object is precisely the target

$$\bigoplus_{i \in \mathbb{N}} \mathrm{H}^{2i-*}_{\mathrm{aH}}(X, \mathbb{R}(i))$$

of the Beilinson regulator 4.1 evaluated at X.
We consider the 1-category $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}})$. Note that there is no notational ambiguity here because for a 1-category the usual Ind-completion and the ∞-categorical Ind-completion are equivalent, i.e. there is a categorical equivalence $\mathrm{Ind}(\mathrm{N}(\mathcal{C})) \simeq \mathrm{N}(\mathrm{Ind}(\mathcal{C}))$ for any 1-category $\mathcal{C}$. Furthermore the symmetric monoidal structures on $\mathbf{MHC}_{\mathbb{R}}$ and $\mathbf{MHC}_{\mathbb{R}}[W^{-1}]$ can be extended to the Ind-completions canonically and the functor

$$\iota : \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}) \to \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]), \tag{4.20}$$

which is induced from 4.14 is symmetric monoidal (see [BNT15], Appendix A). On the object

$$\bigoplus_{i \geq 0} \mathbb{R}(i)[2i] \in \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}) \tag{4.21}$$

there is the structure of a commutative algebra object since it is the underlying object of the symmetric algebra on $\mathbb{R}(1)[2]$ (see [Bra14], Lemma 4.4.5.).
By definition of the forgetful functor V (see Definition 2.77) there is the following commutative diagram of ∞-categories:

$$\begin{tikzcd} \mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}})) \arrow[r, "V"] \arrow[d, "\iota"'] & \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}) \arrow[d, "\iota"] \\ \mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])) \arrow[r, "V"] & \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]) \end{tikzcd} \tag{4.22}$$

where the left vertical functor exists since the functor 4.20 is symmetric monoidal. From these observations we can conclude the existence of the structure of a commutative algebra object on

$$\bigoplus_{i \geq 0} \iota\left(\mathbb{R}(i)[2i]\right) \in \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$$

if we show the following lemma:

Lemma 4.23. *The functor*

$$\iota : \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}) \to \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$$

preserves countably infinite coproducts.

Proof. It is enough to show that the localization functor

$$\iota : \mathbf{MHC}_{\mathbb{R}} \to \mathbf{MHC}_{\mathbb{R}}[W^{-1}]$$

preserves finite coproducts. Using the equivalence from 4.17 this follows from the fact that the localization $\mathrm{Ch}^b(\mathbf{MHS}_{\mathbb{R}}) \to \mathrm{Ch}^b(\mathbf{MHS}_{\mathbb{R}})[W^{-1}]$

preserves finite coproducts and that the coproduct of a zigzag of quasi-ismorphisms of mixed Hodge complexes is a again a zigzag of quasi-isomorphisms. ☐

We get an object

$$\mathbb{T} \in \mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$$

such that

$$V(\mathbb{T}) \simeq \bigoplus_{i \geq 0} \iota\left(\mathbb{R}(i)[2i]\right) \in \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]).$$

Remark 4.24. It is well known that $\mathbb{T} \in \mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$ is also the ∞-categorical free commutative algebra on $\iota\left(\mathbb{R}(1)[2]\right)$, i.e. for each $X \in \mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$ there is an equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))}(\mathbb{T}, X) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota\left(\mathbb{R}(1)[2]\right), X).$$

We will use this later in the construction of the $\mathbb{E}_\infty$-differential Adams operations.

We are now able to give the construction of an $\mathbb{E}_\infty$-ring spectrum such that the associated homotopy ring is the target of the Beilinson regulator.

Lemma 4.25. *The mapping spectrum*

$$\mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes V(\mathbb{T}))$$

has a canonical refinement to a commutative algebra object in spectra and the homotopy ring is

$$\bigoplus_{i \in \mathbb{N}} \mathrm{H}^{2i-*}_{\mathrm{aH}}(X, \mathbb{R}(i)).$$

Proof. The object $\iota(\mathbb{R}(0))$ is the unit of the symmetric monoidal structure on $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$ and therefore carries the structure of a coalgebra. Using [BNT15], Corollary B.4. there is a canonical refinement of the functor

$$\mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)), -) : \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]) \to \mathrm{Sp}$$

to a lax symmetric monoidal functor which therefore sends commutative algebra objects to such. Since $\iota(\mathrm{A}_{\log}(X))$ and $V(\mathbb{T})$ carry the structure of commutative algebra objects also their tensor product does canonically by theorem 2.79. This proves the first part of the lemma. For the second part we compute

$$\begin{aligned}
&\pi_* \left(\mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])} \left(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes V(\mathbb{T})\right)\right) \\
\cong &\pi_* \left(\mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])} \left(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \bigoplus_{i\geq 0} \iota(\mathbb{R}(i)[2i])\right)\right) \\
\cong &\pi_* \left(\mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])} \left(\iota(\mathbb{R}(0)), \mathrm{colim}_n \iota(\mathrm{A}_{\log}(X)) \otimes \bigoplus_{i=0}^{n} \iota(\mathbb{R}(i)[2i])\right)\right) \\
\cong &\mathrm{colim}_n \pi_* \left(\mathrm{map}_{\mathbf{MHC}_{\mathbb{R}}[W^{-1}]} \left(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \bigoplus_{i=0}^{n} \iota(\mathbb{R}(i)[2i])\right)\right) \\
\cong &\mathrm{colim}_n \bigoplus_{i=0}^{n} \pi_* \left(\mathrm{map}_{\mathbf{MHC}_{\mathbb{R}}[W^{-1}]} \left(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \iota(\mathbb{R}(i)[2i])\right)\right) \\
\cong &\bigoplus_{i\geq 0} H_{\mathrm{aH}}^{2i-*}(X, \mathbb{R}(i)).
\end{aligned}$$

Here we have used that tensoring with a fixed object in the symmetric monoidal ∞-category $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$ preserves filtered colimits and that

$$\iota(A) \otimes (\iota(B) \oplus \iota(C)) \simeq (\iota(A) \otimes \iota(B)) \oplus (\iota(A) \otimes \iota(C))$$

which follows from the fact that $\iota : \mathbf{MHC}_{\mathbb{R}} \to \mathbf{MHC}_{\mathbb{R}}[W^{-1}]$ preserves finite coproducts and that tensoring with a fixed object in the symmetric monoidal closed category $\mathbf{MHC}_{\mathbb{R}}$ preserves finite coproducts (even all colimits). □

Using the internal Hom-objects in $\mathbf{MHC}_{\mathbb{R}}$ we get a symmetric monoidal duality functor

$$\underline{\mathrm{hom}}(-, \mathbb{R}(0)) : \mathbf{MHC}_{\mathbb{R}}^{op} \to \mathbf{MHC}_{\mathbb{R}}$$

which induces a symmetric monoidal functor on the Dwyer-Kan localization

$$(-)^{\vee} : \mathbf{MHC}_{\mathbb{R}}^{op}[W^{-1}] \to \mathbf{MHC}_{\mathbb{R}}[W^{-1}]. \tag{4.26}$$

The next result will be technically important for the construction of the motivic Hodge spectrum $\mathbf{H}$:

Proposition 4.27. *(see* [BNT15], *Proposition 6.1.) There is an essentially uniquely determined symmetric monoidal left adjoint functor*

$$\tilde{\mathrm{A}} : \mathrm{Sp}^{\mathbb{P}^1} \to \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$$

such that the diagram

$$\begin{tikzcd}
\mathbf{Sm}_{\mathbb{C}} \arrow[r, "\mathrm{A}_{\log}"] \arrow[d, "\Sigma^{\infty}_{\mathbb{P}^1}(-)_+"'] & \mathbf{MHC}_{\mathbb{R}}^{op} \arrow[d, "\mathrm{Yo}_{\omega}\circ(-)^{\vee}\circ\iota"] \\
\mathrm{Sp}^{\mathbb{P}^1} \arrow[r, "\tilde{\mathrm{A}}"] & \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])
\end{tikzcd}$$

commutes.

We denote the right adjoint of $\tilde{\mathrm{A}}$ by

$$R : \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]) \to \mathrm{Sp}^{\mathbb{P}^1}.$$

As a functor which is right adjoint to a symmetric monoidal functor it has a canonical lax symmetric monoidal refinement (see [Lur17], Corollary 7.3.2.7.) and therefore carries commutative algebra objects to such.
The commutative algebra object $\mathbb{T} \in \mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$ from above is canonical an algebra over itself via the canonical structure morphism

$$\mathbb{T} \xrightarrow{id} \mathbb{T}.$$

Since the underlying object is

$$V(\mathbb{T}) \simeq \bigoplus_{i\geq 0} \iota\left(\mathbb{R}(i)[2i]\right) \in \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$$

there is a canonical map

$$\hat{\beta} : \iota\left(\mathbb{R}(1)[2]\right) \to V(\mathbb{T}).$$

Since $\iota\left(\mathbb{R}(1)[2]\right)$ is obviously a tensor-invertible object, the map $\hat{\beta}$ is an element in the algebra $\mathbb{T}$ in the sense of Definition 2.108.

Definition 4.28. *(Motivic absolute Hodge spectrum)*
We define

$$\mathbf{H} := R(\mathbb{T}[\hat{\beta}^{-1}]) \in \mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1}).$$

Remark 4.29. Note that the underlying object

$$V(\mathbb{T}[\hat{\beta}^{-1}]) \in \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$$

is canonically equivalent to the direct sum $\bigoplus_{p\in\mathbb{Z}} \iota\left(\mathbb{R}(p)[2p]\right)$ and since R preserves colimits (see [BNT15], Lemma 6.3.), the underlying object $V(\mathbf{H}) \in \mathrm{Sp}^{\mathbb{P}^1}$ is canonically equivalent to the direct sum $\bigoplus_{p\in\mathbb{Z}} R\left(\iota\left(\mathbb{R}(p)[2p]\right)\right)$.

To simplify the notation we drop the forgetful functor V from now on.

Proposition 4.30. *For each* $X \in \mathbf{Sm}_{\mathbb{C}}$ *there is an equivalence of* $\mathbb{E}_\infty$*-ring spectra*

$$\operatorname{map}_{\operatorname{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \mathbb{T}) \simeq \operatorname{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathbf{H}), \tag{4.31}$$

which is natural in X.

Proof. We firstly check that there is such an equivalence of underlying spectra. Let $X \in \mathbf{Sm}_{\mathbb{C}}$. Then we have:

$$\begin{aligned} \operatorname{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathbf{H}) &\simeq \operatorname{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1} X_+, R(\mathbb{T}[\hat{\beta}^{-1}])) \\ &\simeq \operatorname{map}_{\operatorname{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1} X_+), \mathbb{T}[\hat{\beta}^{-1}]) \\ &\simeq \operatorname{map}_{\operatorname{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathrm{A}_{\log}(X))^\vee, \mathbb{T}[\hat{\beta}^{-1}]) \\ &\simeq \operatorname{map}_{\operatorname{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \mathbb{T}[\hat{\beta}^{-1}]) \end{aligned} \tag{4.32}$$

It is therefore enough to see that the canonical map of spectra

$$\operatorname{map}_{\operatorname{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \mathbb{T}) \to \tag{4.33}$$

$$\to \operatorname{map}_{\operatorname{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \mathbb{T}[\hat{\beta}^{-1}]) \tag{4.34}$$

which is induced by the localization map $\mathbb{T} \to \mathbb{T}[\hat{\beta}^{-1}]$ is an equivalence. But this follows from the fact that for $i < 0$ the absolute Hodge cohomology

$$\mathrm{H}^*_{\mathrm{aH}}(X, \mathbb{R}(i))$$

vanishes in each cohomological degree for a smooth variety X over the complex numbers.

We still have to argue why the equivalence takes place even in CAlg(Sp). But this follows from the following general fact, which is established in Proposition B.5. [BNT15]: If one has an adjunction

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

between presentably symmetric monoidal ∞-categories such that L is symmetric monoidal, then for $X \in \mathrm{CAlg}(\mathcal{C}^{op})$ and $Y \in \mathrm{CAlg}(\mathcal{D})$ the adjunction equivalence

$$\operatorname{map}_{\mathcal{C}}(X, R(Y)) \simeq \operatorname{map}_{\mathcal{D}}(L(X), Y),$$

naturally lifts to CAlg(Sp). ☐

Corollary 4.35. *For* $X \in \mathbf{Sm}_{\mathbb{C}}$ *there is a natural isomorphism*

$$\pi_*\left(\operatorname{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathbf{H})\right) \cong \bigoplus_{i \in \mathbb{N}} \mathrm{H}^{2i-*}_{\mathrm{aH}}(X, \mathbb{R}(i)),$$

in other words the motivic absolute Hodge spectrum $\mathbf{H}$ *represents the target of the Beilinson regulator 4.1 in the* ∞*-category of motivic spectra.*

Let $\mathbf{K}$ be the motivic Snaith spectrum (3.26). By adjunction, a morphism $\mathbf{K} \to \mathbf{H}$ in $\mathrm{Sp}^{\mathbb{P}^1}$ corresponds essentially uniquely to a morphism $\tilde{A}(\mathbf{K}) \to \mathbb{T}[\hat{\beta}^{-1}]$ in $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$. Since in the adjunction $\tilde{\mathrm{A}} \vdash R$ the functor $\tilde{\mathrm{A}}$ is symmetric monoidal and R is lax symmetric monoidal, there is an induced adjunction

$$\tilde{\mathrm{A}} : \mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1}) \leftrightarrows \mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])) : R$$

(cf. [Lur17], Remark 7.3.2.13.). In particular a map $\mathbf{K} \to \mathbf{H}$ in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$ corresponds essentially uniquely to a map $\tilde{\mathrm{A}}(\mathbf{K}) \to \mathbb{T}[\hat{\beta}^{-1}]$ in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$.
The main result of [BNT15] is the following theorem:

Theorem 4.36. *There are choices of equivalences*

$$\delta : \tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \simeq \mathbb{T}$$

and

$$\delta' : \tilde{\mathrm{A}}(\mathbf{K}) \simeq \mathbb{T}[\hat{\beta}^{-1}]$$

in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$ *such that the diagram*

$$\begin{tikzcd}
\tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \arrow[r, "{\delta,\simeq}"] \arrow[d, "\tilde{\mathrm{A}}(\mathrm{can})"] & \mathbb{T} \arrow[d, "\mathrm{can}"] \\
\tilde{\mathrm{A}}(\mathbf{K}) \arrow[r, "{\delta',\simeq}"] & \mathbb{T}[\hat{\beta}^{-1}]
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$ *commutes and such that the map* $\mathbf{reg} : \mathbf{K} \to \mathbf{H}$ *in* $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$*, which is adjoint to* δ'*, represents the Beilinson regulator, i.e. for any* $X \in \mathbf{Sm}_{\mathbb{C}}$ *the induced map*

$$\pi_* \left(\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^{\infty}_{\mathbb{P}^1}X_+, \mathbf{K})\right) \to \pi_* \left(\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^{\infty}_{\mathbb{P}^1}X_+, \mathbf{H})\right)$$

is the Beilinson regulator 4.1.

This theorem will be crucial for our construction of the $\mathbb{E}_\infty$-Adams operations on differential algebraic K-theory. The map $\mathbf{reg} : \mathbf{K} \to \mathbf{H}$ from the theorem will be called motivic Beilinson regulator.

Remark 4.37. The equivalence $\delta : \tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \to \mathbb{T}$ in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$ is constructed using the universal property of $\mathbb{T}$ being the free commutative algebra on $\iota\left(\mathbb{R}(1)[2]\right)$. More precisely it is constructed as the adjoint of the composition

$$\iota\left(\mathbb{R}(1)[2]\right) \xrightarrow{(\delta_1)^{-1}} \iota\left(\mathrm{A}_{\log}(\mathbb{P}^1, \infty)\right)^\vee \xrightarrow{\mathrm{can}} \tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+)$$

in $\mathrm{Ind}(\mathbf{MHC}_\mathbb{R}[W^{-1}])$ for a certain chosen equivalence

$$\iota\left(\mathbb{R}(1)[2]\right) \xrightarrow{(\delta_1)^{-1}, \simeq} \iota\left(\mathrm{A}_{\log}(\mathbb{P}^1, \infty)\right)^\vee .$$

4.2. **Construction of the differential refinement of the Adams operations.** We want to begin with the definition of the differential refinement of algebraic K-theory of smooth schemes over $\mathbb{C}$. The main idea here is that for $X \in \mathbf{Sm}_\mathbb{C}$ there is a certain dg-algebra $\mathrm{IDR}(X) \in \mathrm{CAlg}(\mathrm{Ch}(\mathbb{R}))$ such that there is an equivalence

$$\mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_\mathbb{R}[W^{-1}])}(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes V(\mathbb{T})) \simeq H(\mathrm{IDR}(X)),$$

where $H : \mathrm{Ch}(\mathbb{R})[W^{-1}] \to \mathrm{Ch}[W^{-1}] \to \mathrm{Sp}$ is the symmetric monoidal Eilenberg-MacLane functor. Note, that the left-hand side of this equivalence is precisely the $\mathbb{E}_\infty$-spectrum that computes the target of the Beilinson regulator (see lemma 4.25). If one has differential form data computing the target of the Beilinson regulator for X one can use these data to refine the algebraic K-theory of X. More precisely the n-th differential algebraic K-theory spectrum of X is defined as the following pullback in the ∞-category of spectra

$$\begin{tikzcd}
\widehat{\mathcal{K}}^n(X) \arrow[r] \arrow[d] & H(\sigma^{\geq n}\mathrm{IDR}(X)) \arrow[d] \\
\mathcal{K}_{st}(X) \arrow[r, "reg_X"] & H(\mathrm{IDR}(X))
\end{tikzcd}$$

where $\sigma^{\geq n}$ is the functor that stupidly cuts off everything below degree n. Note that the n-th cohomology of a cochain complex which is truncated below n consists precisely of the n-cocycles of this complex.

We want to begin with the definition of the complex $\mathrm{IDR}(X)$ from [BNT15] and for this we have to explain how one can extend the functor $\mathrm{A}_{\log}$ (see example 4.8) from the category $\mathbf{Sm}_\mathbb{C}^{op}$ to the product category $\mathbf{Mf}^{op} \times \mathbf{Sm}_\mathbb{C}^{op}$. For this let $M \times X \in \mathbf{Mf} \times \mathbf{Sm}_\mathbb{C}$. For a fixed good compactification $X \hookrightarrow \overline{X}$ with complement D one defines

$$\mathrm{A}_{M \times \overline{X}, \mathbb{R}}(M \times X, \log\, D)$$

as the commutative unital sub-$\mathrm{A}(M \times \overline{X})_\mathbb{R}$-dg-algebra of $\mathrm{A}(M \times X)_\mathbb{R}$ which is generated locally by elements of the form

$$\log(z_i\overline{z_i}),\ \mathrm{Re}(\mathrm{d}z_i/z_i) \text{ and } \mathrm{Im}(\mathrm{d}z_i/z_i),$$

where $(z_i)_i$ are local coordinates of $\overline{X}$ such that D is locally the zero locus of the product $\prod_i z_i$. This dg-algebra $\mathrm{A}_{M\times\overline{X},\mathbb{R}}(M\times X,\log D)$ carries an increasing filtration $\mathcal{W}$ which coincides with the weight filtration on $\mathrm{A}_{\overline{X},\mathbb{R}}(X,\log D)$ for $M=\{*\}$ and the complexification

$$\mathrm{A}_{M\times\overline{X}}(M\times X,\log D):=\mathrm{A}_{M\times\overline{X},\mathbb{R}}(M\times X,\log D)\otimes_{\mathbb{R}}\mathbb{C}$$

carries a decreasing filtration $\mathcal{F}$ that coincides with the Hodge filtration on $\mathrm{A}_{\overline{X}}(X,\log D)$ for $M=\{*\}$. Let us recall the definitions of these two filtrations:
The increasing filtration $\mathcal{W}'$ on $\mathrm{A}_{M\times\overline{X},\mathbb{R}}(M\times X,\log D)$ is defined as the multiplicative filtration of $\mathrm{A}(M\times\overline{X})_{\mathbb{R}}$-modules with

$$\mathrm{A}(M\times\overline{X})_{\mathbb{R}}\subset\mathcal{W}'_0$$

and

$$\log(z_i\overline{z_i}),\ \mathrm{Re}(\mathrm{d}z_i/z_i),\ \mathrm{Im}(\mathrm{d}z_i/z_i)\in\mathcal{W}'_1.$$

Then there is the decreasing filtration $\mathcal{L}$ on $\mathrm{A}_{M\times\overline{X},\mathbb{R}}(M\times X,\log D)$ where $\mathcal{L}^p$ contains all elements which are locally $C^\infty_{\mathbb{R}}(M\times X)$-linear combinations of elements of the form

$$dx^I\wedge\mathrm{Re}(dz^J)\wedge\mathrm{Im}(dz^K)$$

with $|I|\geq p$. We can then define the increasing filtration $\mathcal{W}''$ on $\mathrm{A}_{M\times\overline{X},\mathbb{R}}(M\times X,\log D)$ by

$$\mathcal{W}''_k:=\sum_p\mathcal{W}'_{k+p}\cap\mathcal{L}^p$$

and our filtration $\mathcal{W}$ is defined as the décalage of $\mathcal{W}''$, i. e.

$$\begin{aligned}\mathcal{W}_k\mathrm{A}^m_{M\times\overline{X},\mathbb{R}}(M\times X,\log D):=&\{\alpha\in\mathcal{W}''_{k-m}\mathrm{A}^m_{M\times\overline{X},\mathbb{R}}(M\times X,\log D)|\\ &|d\alpha\in\mathcal{W}''_{k-m-1}\mathrm{A}^{m+1}_{M\times\overline{X},\mathbb{R}}(M\times X,\log D)\}.\end{aligned}$$

The decreasing filtration $\mathcal{F}$ on $\mathrm{A}_{M\times\overline{X}}(M\times X,\log D)$ is defined as usual such that $\mathcal{F}^p$ contains all elements which are locally $C^\infty(M\times X)$-linear combinations of elements of the form

$$dx^I\wedge dz^J\wedge d\overline{z}^K$$

with $|J|\geq p$. To make the definition of $\mathrm{A}_{\log,\mathbb{R}}(M\times X)$ and $\mathrm{A}_{\log}(M\times X)$ independent of the chosen good compactification $X\hookrightarrow\overline{X}$, one finally defines:

$$\mathrm{A}_{\log,\mathbb{R}}(M\times X):=\mathrm{colim}_{I_X}\mathrm{A}_{M\times\overline{X},\mathbb{R}}(M\times X,\log D)$$

and

$$\mathrm{A}_{\log}(M\times X):=\mathrm{colim}_{I_X}\mathrm{A}_{M\times\overline{X}}(M\times X,\log D),$$

where I_X is the category of good compactifications of X (cf. example 4.8) and

$$\mathrm{A}_{\log}(M \times X) \cong \mathrm{A}_{\log,\mathbb{R}}(M \times X) \otimes_{\mathbb{R}} \mathbb{C}.$$

One gets induced filtrations $\mathcal{W}$ and $\mathcal{F}$ on the colimits.

Definition 4.38. (IDR) *Let $X \in \mathbf{Sm}_{\mathbb{C}}$ and $p \in \mathbb{Z}$. Then one defines the p-part of* IDR(X) *as*

$$\mathrm{IDR}(p)(X) :=$$

$$:= \{\alpha \in \mathcal{W}_{2p}\mathrm{A}_{\log}(I \times X)[2p] | \alpha|_0 \in (2\pi i)^p \mathrm{A}_{\log}(X)_{\mathbb{R}}, \alpha|_1 \in \mathcal{F}^p\mathrm{A}_{\log}(X)\}$$

and

$$\mathrm{IDR}(X) := \bigoplus_{p \geq 0} \mathrm{IDR}(p)(X).$$

Remark 4.39. The wedge product of forms defines a map

$$\mathrm{IDR}(p)(X) \otimes \mathrm{IDR}(q)(X) \to \mathrm{IDR}(p+q)(X),$$

so that IDR(X) becomes a dg-algebra with the grading given by the p-parts.

The importance of the dg-algebra IDR(X) lies in the following fact:

Proposition 4.40. *For $X \in \mathbf{Sm}_{\mathbb{C}}$ there is a natural equivalence*

$$\mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \mathbb{T}) \simeq H(\mathrm{IDR}(X))$$

in CAlg(Sp). *In particular, the homotopy ring of the $\mathbb{E}_\infty$-spectrum $H(\mathrm{IDR}(X))$ is the target of the Beilinson regulator. Moreover, the underlying equivalence in* Sp *is a direct sum of equivalences*

$$\mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)), \iota(\mathrm{A}_{\log}(X)) \otimes \iota\,(\mathbb{R}(p)[2p])) \simeq H(\mathrm{IDR}(X)(p))$$

where p runs over all natural numbers.

Remark 4.41. We want to sketch the line of arguments that is used for the proof of proposition 4.40 in [BNT15], since we will need one of the intermediate steps later on. We consider the lax symmetric monoidal functor

$$\mathcal{E} : \mathbf{MHC}_{\mathbb{R}} \to \mathrm{Ch}(\mathbb{R}),$$

that sends a mixed $\mathbb{R}$-Hodge complex M to the object

$$\{\alpha \in \mathrm{A}(I)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{W}_0(\underline{\mathrm{hom}}(\mathbb{R}(0), M) \otimes \mathbb{C}) |\ \alpha|_0 \in \underline{\mathrm{hom}}(\mathbb{R}(0), M)_{\mathbb{R}},$$
$$\alpha|_1 \in \mathcal{F}^0(\underline{\mathrm{hom}}(\mathbb{R}(0), M) \otimes \mathbb{C})\}.$$

(I is the unit interval). Since Ch$(\mathbb{R})$ has filtered colimits we get an induced lax symmetric monoidal functor

$$\mathcal{E} : \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}) \to \mathrm{Ch}(\mathbb{R})$$

which we also denote by $\mathcal{E}$. The proposition is then implied by the following two statements:

1. There is an equivalence of lax symmetric monoidal functors from $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}})$ to Sp
$$H \circ \mathcal{E} \simeq \mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)), \iota(-))$$
where $H : \mathrm{Ch}(\mathbb{R}) \to \mathrm{Sp}$ is the Eilenberg-MacLane functor and ι is the canonical symmetric monoidal functor $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}) \to \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$.
2. There is a natural map of commutative dga's
$$\mathcal{E}(\bigoplus_{p\geq 0} \mathbb{R}(p)[2p] \otimes \mathrm{A}_{\log}(X))) \to \mathrm{IDR}(X),$$
which is a quasi-isomorphims and which is given by a sum of quasi-ismorphisms of the form
$$\mathcal{E}(\mathbb{R}(p)[2p] \otimes \mathrm{A}_{\log}(X))) \to \mathrm{IDR}(p)(X).$$

Corollary 4.42. *In the ∞-category $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{CAlg}(\mathrm{Sp}))$ there is a commutative diagram of the form*

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \mathbf{K}) \arrow[r, "\mathbf{reg}_*"] \arrow[d, "\simeq"'] & \mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \mathbf{H}) \arrow[d, "\simeq"] \\
\mathcal{K}_{st}(-) \arrow[r, "reg"] & H(\mathrm{IDR}(-)),
\end{tikzcd}$$

such that the lower horizontal map induces the Beilinson regulator in homotopy.

This last corollary is now used for the definition of differential algebraic K-theory:

Definition 4.43. *(n-th differential algebraic K-theory)*
The n-th differential algebraic K-theory presheaf
$$\widehat{\mathcal{K}}^n \in \mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{Sp})$$
is defined as the pullback

$$\begin{tikzcd}
\widehat{\mathcal{K}}^n \arrow[r, "R"] \arrow[d, "I"'] & H(\sigma^{\geq n}\mathrm{IDR}) \arrow[d] \\
\mathcal{K}_{st} \arrow[r, "reg"] & H(\mathrm{IDR}),
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{Sp})$, where $\sigma^{\geq n}$ is the functor that stupidly truncates a cochain complex below degree n.

As already said before, the n-th differential algebraic K-theory presheaf does not canonically have values in CAlg(Sp) since the stupid truncation $\sigma^{\geq n}\mathrm{IDR}$ is not a dg-algebra (it does not have a unit). In [BT15a], the authors show that the direct sum

$$\bigoplus_{n\in\mathbb{Z}} \widehat{\mathcal{K}}^n$$

has canonically values in CAlg(Sp). We want to shortly explain this construction of multiplicative differential algebraic K-theory.
The main point here is the observation, that the direct sum

$$\bigoplus_{n\in\mathbb{Z}} \sigma^{\geq n}\mathrm{IDR}(X) \tag{4.44}$$

carries canonically the structure of a commutative dg-algebra. In fact, the underlying chain complex of the commutative dg-algebra

$$\sigma^{\geq \bullet}\mathrm{IDR}(X) := \bigoplus_{n\in\mathbb{Z}} z^n\sigma^{\geq n}\mathrm{IDR}(X) \subseteq \mathrm{IDR}(X)[z,z^{-1}]$$

is precisely the direct sum from 4.44.
Since the functors

$$\mathbf{Set} \xrightarrow{\text{const}} \mathbf{sSet} \xrightarrow{\text{can}} \mathcal{S} \xrightarrow{\Sigma^\infty_+} \mathrm{Sp}$$

are all symmetric monoidal (here can is the localization functor from **sSet** to the ∞-category of spaces $\mathcal{S}$), we get an object

$$\Sigma^\infty_+ \left(\mathrm{can}\left(\mathrm{const}\left(\mathbb{Z}\right)\right)\right) \in \mathrm{CAlg}(\mathrm{Sp})$$

which we denote by just $\Sigma^\infty_+\mathbb{Z}$. For the construction of multiplicative differential algebraic K-theory we then need the following lemma:

Lemma 4.45. *(see* [Bun13]*, Problem 4.115.)*
There is an equivalence

$$H(\mathrm{IDR}[z,z^{-1}]) \simeq H(\mathrm{IDR}) \wedge \Sigma^\infty_+\mathbb{Z}$$

in $\mathrm{Fun}(\mathbf{Sm}^{op}_{\mathbb{C}}, \mathrm{CAlg}(\mathrm{Sp}))$.

Definition 4.46. *(Multiplicative differential algebraic K-theory)*
The multiplicative differential algebraic K-theory presheaf

$$\widehat{\mathcal{K}} \in \mathrm{Fun}(\mathbf{Sm}^{op}_{\mathbb{C}}, \mathrm{CAlg}(\mathrm{Sp}))$$

is defined via the following pullback in the ∞*-category* $\mathrm{Fun}(\mathbf{Sm}^{op}_{\mathbb{C}}, \mathrm{CAlg}(\mathrm{Sp}))$*:*

$$\begin{tikzcd}
\widehat{\mathcal{K}} \arrow[r, "R"] \arrow[d, "I"'] & H(\sigma^{\geq\bullet}\mathrm{IDR}) \arrow[d] \\
\mathcal{K}_{st} \wedge \Sigma^\infty_+\mathbb{Z} \arrow[r, "reg\wedge\Sigma^\infty_+\mathbb{Z}"] & H(\mathrm{IDR}) \wedge \Sigma^\infty_+\mathbb{Z}.
\end{tikzcd}$$

Here the right vertical map is the composition of the canonical map

$$H(\sigma^{\geq \bullet}\mathrm{IDR}) \to H(\mathrm{IDR}[z, z^{-1}])$$

and the equivalence

$$H(\mathrm{IDR}[z, z^{-1}]) \simeq H(\mathrm{IDR}) \wedge \Sigma^{\infty}_{+}\mathbb{Z}$$

from lemma 4.45.

Note that the presheaf of spectra that underlies $\widehat{\mathcal{K}}$ is really the direct sum

$$\bigoplus_{n \in \mathbb{Z}} \widehat{\mathcal{K}}^n.$$

The goal of the present section is now to lift the spectral valued Adams operations from the last chapter to multiplicative differential algebraic K-theory.

By Theorem 3.40 there is a commutative diagram

$$\begin{tikzcd}
\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+} \arrow[r, "\Sigma^{\infty}_{\mathbb{P}^1}(-)^k"] \arrow[d, "\mathrm{can}"] & \Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+} \arrow[d, "\mathrm{can}"] \\
\mathbf{K}[k^{-1}] \arrow[r, "\psi^k_{mot}"] & \mathbf{K}[k^{-1}]
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$. The main result which is needed for the construction of the differential refinement of the Adams operations on algebraic K-theory is the following:

Theorem 4.47. *There exists a map*

$$\psi^k_{\mathbf{H}} : \mathbf{H} \to \mathbf{H}$$

in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$ *such that the diagram*

$$\begin{tikzcd}
\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+} \arrow[r, "\Sigma^{\infty}_{\mathbb{P}^1}(-)^k_+"] \arrow[d, "\mathrm{can}"] & \Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+} \arrow[d, "\mathrm{can}"] \\
\mathbf{K}[k^{-1}] \arrow[r, "\psi^k_{mot}"] \arrow[d, "\mathbf{reg}"'] & \mathbf{K}[k^{-1}] \arrow[d, "\mathbf{reg}"] \\
\mathbf{H} \arrow[r, "\psi^k_{\mathbf{H}}"] & \mathbf{H}
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$ *commutes. Moreover, the map in* $\mathrm{Sp}^{\mathbb{P}^1}$ *that underlies* $\psi_{\mathbf{H}}$ *is*

$$\bigoplus k^p : \bigoplus_{p\in\mathbb{Z}} R\left(\iota(\mathbb{R}(p)[2p])\right) \to \bigoplus_{p\in\mathbb{Z}} R\left(\iota(\mathbb{R}(p)[2p])\right),$$

(cf. remark 4.29 for the decomposition of the underlying $\mathbb{P}^1$*-spectrum of* H*).*

As a preparation for the proof of theorem 4.47 we have to understand the map

$$\phi := \tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1}(-)^k_+) : \tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+) \to \tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+)$$

first. We already said that there is an equivalence

$$\delta : \tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+) \simeq \mathbb{T}$$

in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$ (cf. 4.36). The goal now is to identify the map $\mathbb{T} \to \mathbb{T}$ such that the diagram

$$\begin{tikzcd}
\tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+) \arrow[r, "\phi"] \arrow[d, "\delta"'] & \tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+) \arrow[d, "\delta"] \\
\mathbb{T} \arrow[r] & \mathbb{T}
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$ commutes.

Let $k \in \mathbb{N}$. In the 1-category $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}})$ we then consider the map

$$\textstyle\bigoplus_{p\geq 0} \mathbb{R}(p)[2p] \xrightarrow{\bigoplus_p k^p} \bigoplus_{p\geq 0} \mathbb{R}(p)[2p]$$

which is given by multiplication with k^p on the p-th summand. This is obviously a map in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}))$ if we equip source and target with the canonical commutative algebra structure. Since the functor

$$\iota : \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}) \to \mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$$

is symmetric monoidal we get an induced map

$$\bigoplus_{p\geq 0} k^p : \mathbb{T} \to \mathbb{T} \tag{4.48}$$

in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$. A crucial step in the proof of theorem 4.47 is the following proposition:

Proposition 4.49. *The diagram*

(4.50)
$$\begin{tikzcd}
\tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+) \arrow[r, "\phi"] \arrow[d, "\delta"'] & \tilde{\mathrm{A}}(\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+) \arrow[d, "\delta"] \\
\mathbb{T} \arrow[r, "\bigoplus_{p\geq 0} k^p"] & \mathbb{T}
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_\mathbb{R}[W^{-1}]))$ *commutes.*

In order to being able to proof this proposition we first have to say something about the commutative algebra structure on $\Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$. More precisely we will study the map $(-)^k_+ : \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+ \to \Sigma^\infty_{\mathbb{P}^1}\mathbb{P}^\infty_+$. Here we use the result that the commutative algebra struture on the motivic space $\mathbb{P}^\infty$ is induced from the Segre embeddings (cf. proposition A.7). For this let

$$\mathrm{Seg}_n : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^{n^2+2n}$$

be the map in $\mathbf{Sm}_\mathbb{C}$ which is the concrete Segre embedding from construction A.1. Then the following holds:

Lemma 4.51. *Let* $n \in \mathbb{N}$. *Then in* Spc^{mot} *there is a commutative diagram*

$$\begin{tikzcd}
\mathbb{P}^n \times \mathbb{P}^n \arrow[r, "\mathrm{Seg}_n"] \arrow[d] & \mathbb{P}^{n^2+2n} \arrow[d] \\
\mathbb{P}^\infty \times \mathbb{P}^\infty \arrow[r, "m"] & \mathbb{P}^\infty,
\end{tikzcd}$$

where m *is the multiplication map which is induced from the commutative algebra structure on* $\mathbb{P}^\infty$.

Corollary 4.52. *Let* $n, k \in \mathbb{N}$. *Then there is a commutative diagram*

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "\epsilon_{n,k}"] \arrow[d] & \mathbb{P}^{(n+1)^{2^{k-1}}-1} \arrow[d] \\
\mathbb{P}^\infty \arrow[r, "(-)^k"] & \mathbb{P}^\infty
\end{tikzcd}$$

in Spc^{mot} *where the map* $\epsilon_{n,k}$ *is induced from the map in* $\mathbf{Sm}_\mathbb{C}$ *that is given as the composition*

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "\Delta"] & \underbrace{\mathbb{P}^n \times \mathbb{P}^n \times ... \times \mathbb{P}^n}_{\mathrm{k-times}} \arrow[r, "\mathrm{id}\times\mathrm{Seg}_n"] & \mathbb{P}^n \times ...\mathbb{P}^{(n+1)^2-1} \arrow[d, hook, "incl"] \\
\mathbb{P}^{(n+1)^{2^{k-1}}-1} & ... \arrow[l, "\mathrm{Seg}_{(n+1)^{2^{k-2}}-1}"'] & \mathbb{P}^{(n+1)^2-1} \times ... \times \mathbb{P}^{(n+1)^2-1}, \arrow[l, "\mathrm{id}\times\mathrm{Seg}_{(n+1)^2-1}"']
\end{tikzcd}$$

i.e. a composition of successive application of the Segre embedding to the last two factors and canonical inclusions of projective spaces into higher dimensional projective spaces.

We are now ready to prove proposition 4.49.

Proof. (of proposition 4.49)
Since $\mathbb{T}$ is the free commutative algebra on $\iota\left(\mathbb{R}(1)[2]\right)$ in $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$ it is enough to prove the commutativity of the following diagram in $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$:

$$\begin{tikzcd}
\tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \arrow[rr, "\phi"] & & \tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \\
\iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right)^{\vee} \arrow[u, "\mathrm{can}"] & & \\
\iota\left(\mathbb{R}(1)[2]\right) \arrow[u, "(\delta_1)^{-1},\simeq"] \arrow[r, "\cdot k"] & \iota\left(\mathbb{R}(1)[2]\right) \arrow[r, "\mathrm{can}"] & \bigoplus_{p\geq 0}\iota\left(\mathbb{R}(p)[2p]\right), \arrow[uu, "\delta^{-1},\simeq"']
\end{tikzcd}$$

cf. remark 4.37 for the meaning of the maps δ and δ'. For this it is obviously enough to show commutativity of

$$\begin{tikzcd}
\tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \arrow[r, "\phi"] & \tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \\
 & \iota\left(\mathrm{A}_{\log}(\mathbb{P}^{2^{2^{k-1}}-1},\infty)\right)^{\vee} \arrow[u, "\mathrm{can}"'] \\
\iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right)^{\vee} \arrow[uu, "\mathrm{can}"] & \iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right)^{\vee} \arrow[u, "\mathrm{incl}_*"'] \\
\iota\left(\mathbb{R}(1)[2]\right) \arrow[u, "(\delta_1)^{-1},\simeq"] \arrow[r, "\cdot k"] & \iota\left(\mathbb{R}(1)[2]\right) \arrow[u, "(\delta_1)^{-1},\simeq"']
\end{tikzcd}$$

in $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])$. But by corollary 4.52 the square

$$\begin{tikzcd}
\tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \arrow[r, "\phi"] & \tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \\
\iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right)^{\vee} \arrow[u, "\mathrm{can}"] \arrow[r, "(\epsilon_{1,k})_*"] & \iota\left(\mathrm{A}_{\log}(\mathbb{P}^{2^{2^{k-1}}-1},\infty)\right)^{\vee} \arrow[u, "\mathrm{can}"']
\end{tikzcd}$$

commutes, so we are reduced to proving commutativity of

$$\begin{tikzcd}
\iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right)^\vee \arrow[r, "(\epsilon_{1,k})_*"] & \iota\left(\mathrm{A}_{\log}(\mathbb{P}^{2^{2^{k-1}}-1},\infty)\right)^\vee \\
 & \iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right)^\vee \arrow[u, "\mathrm{incl}_*"'] \\
\iota\left(\mathbb{R}(1)[2]\right) \arrow[r, "\cdot k"] \arrow[uu, "(\delta_1)^{-1},\simeq"] & \iota\left(\mathbb{R}(1)[2]\right). \arrow[u, "(\delta_1)^{-1},\simeq"']
\end{tikzcd}$$

The commutativity of the last diagram is equivalent to the commutativity of

$$\begin{tikzcd}
\iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right) \arrow[dd, "(\delta_1^{-1})^\vee,\simeq"'] & \iota\left(\mathrm{A}_{\log}(\mathbb{P}^{2^{2^{k-1}}-1},\infty)\right) \arrow[l, "\epsilon_{1,k}^*"'] \arrow[d, "\mathrm{incl}^*"] \\
 & \iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right) \arrow[d, "(\delta_1^{-1})^\vee,\simeq"] \\
\iota\left(\mathbb{R}(-1)[-2]\right) & \iota\left(\mathbb{R}(-1)[-2]\right). \arrow[l, "\cdot k"']
\end{tikzcd}$$

But this now follows from the commutativity of

$$\begin{tikzcd}
\iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right) & \iota\left(\mathrm{A}_{\log}(\mathbb{P}^{2^{2^{k-1}}-1},\infty)\right) \arrow[l, "\epsilon_{1,k}^*"'] \arrow[d, "\mathrm{incl}^*"] \\
 & \iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right) \arrow[ul, "\cdot k"]
\end{tikzcd}$$

which we are going to prove now: $\iota\left(\mathrm{A}_{\log}(\mathbb{P}^1,\infty)\right)$ is generated by the Kähler 2-form associated to the Fubini-Study metric on $\mathbb{P}^1$. Therefore the map incl^* cuts off the part of cohomological degree different from 2. Since the degree 2 part of $\iota\left(\mathrm{A}_{\log}(\mathbb{P}^{2^{2^{k-1}}-1},\infty)\right)$ is also generated by the Kähler form $\omega \in \Omega^2_{\mathrm{cl}}(\mathbb{P}^{2^{2^{k-1}}-1})$ associated to the Fubini-Study metric we just have to compute the effect of $\epsilon_{1,k}^*$ on ω. For simplicity we just carry out this for the case $k = 2$. Recall that in this case the map $\epsilon_{1,k}$ is the composition

$$\mathbb{P}^1 \xrightarrow{\Delta} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\mathrm{Seg}_1} \mathbb{P}^3.$$

We compute using the explicit formula for ω in terms of homogeneous coordinates:

$$\begin{aligned}
\epsilon_{1,2}^*(\omega) &= \Delta^*\mathrm{Seg}_1^*\left(\frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{i=0}^{3}|Z_i|^2\right)\right)\\
&= \Delta^*\left(\frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{0\leq i,j\leq 1}|X_iY_j|^2\right)\right)\\
&= \Delta^*\left(\frac{i}{2\pi}\partial\overline{\partial}\log\left(\left(\sum_{i=0}^{1}|X_i|^2\right)\left(\sum_{j=0}^{1}|Y_j|^2\right)\right)\right)\\
&= \Delta^*\left(\frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{i=0}^{1}|X_i|^2\right)+\frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{i=0}^{1}|Y_i|^2\right)\right)\\
&= 2\cdot\omega,
\end{aligned}$$

where we have used the letter ω for the Kähler form on $\mathbb{P}^3$ as well as for the Kähler form on $\mathbb{P}^1$. By the same sort of computation for general k one gets

$$\epsilon_{1,k}^*\omega = k\cdot\omega.$$

Note here that in the definition of the map $\epsilon_{1,k}$ for general k there appear canonical inclusions $\mathbb{P}^r \hookrightarrow \mathbb{P}^s$ which are given in homogeneous coordinates by inserting zeroes. Along such maps the Kähler form associated to the Fubini-Study metric on the target gets pulled back to the Kähler form associated to the same metric in the domain.

□

Proof. (of theorem 4.47)
The diagram

$$\begin{tikzcd}
\bigoplus_{p\geq 0}\mathbb{R}(p)[2p] \arrow[r, "\bigoplus k^p"] \arrow[d, "\mathrm{can}"'] & \bigoplus_{p\geq 0}\mathbb{R}(p)[2p] \arrow[d, "\mathrm{can}"] \\
\bigoplus_{p\in\mathbb{Z}}\mathbb{R}(p)[2p] \arrow[r, "\bigoplus k^p"] & \bigoplus_{p\in\mathbb{Z}}\mathbb{R}(p)[2p]
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}))$ commutes and induces a commutative diagram

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "\bigoplus k^p"] \arrow[d, "\mathrm{can}"'] & \mathbb{T} \arrow[d, "\mathrm{can}"] \\
\mathbb{T}[\hat{\beta}^{-1}] \arrow[r, "\bigoplus k^p"] & \mathbb{T}[\hat{\beta}^{-1}]
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$. By this observation, Proposition 4.49 and Theorem 3.40 we get the cube

$$\begin{tikzcd}
& \tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \arrow[rr, "{\delta,\simeq}"] \arrow[dd] & & \mathbb{T} \arrow[dd] \\
\tilde{\mathrm{A}}(\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{P}^{\infty}_{+}) \arrow[ur, "\phi"] \arrow[rr, "{\delta,\simeq}"] \arrow[dd] & & \mathbb{T} \arrow[ur, "\bigoplus k^p"'] \arrow[dd] & \\
& \tilde{\mathrm{A}}(\mathbf{K}[k^{-1}]) \arrow[rr, dashed, "{\delta',\simeq}"] & & \mathbb{T}[\hat{\beta}^{-1}] \\
\tilde{\mathrm{A}}(\mathbf{K}[k^{-1}]) \arrow[ur, dashed, "\tilde{\mathrm{A}}(\psi^k_{mot})"] \arrow[rr, "{\delta',\simeq}"'] & & \mathbb{T}[\hat{\beta}^{-1}] \arrow[ur, "\bigoplus k^p"'] &
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}]))$ in which everything is commutative except possibly the bottom face. But by the universal property of $\mathbb{T}[\hat{\beta}^{-1}]$ as a localization and the fact, that δ and δ' are equivalences, also the bottom face commutes. Applying the adjunction $\tilde{\mathrm{A}} \vdash R$ (cf. 4.27) to this commutative bottom face we get a commutative square

$$\begin{tikzcd}
\mathbf{K}[k^{-1}] \arrow[r, "\mathbf{reg}"] \arrow[d, "\psi^k_{mot}"'] & R\left(\mathbb{T}[\hat{\beta}^{-1}]\right) \arrow[r, equal] \arrow[d, "R(\bigoplus k^p)"] & \mathbf{H} \arrow[d, "\psi^k_{\mathbf{H}}"] \\
\mathbf{K}[k^{-1}] \arrow[r, "\mathbf{reg}"'] & R\left(\mathbb{T}[\hat{\beta}^{-1}]\right) \arrow[r, equal] & \mathbf{H}
\end{tikzcd} \tag{4.53}$$

in $\mathrm{CAlg}(\mathrm{Sp}^{\mathbb{P}^1})$. Since R preserves all colimits we have that the map of $\mathbb{P}^1$-spectra that underlies $\psi_{\mathbf{H}}$ is $R(\bigoplus k^p) \simeq \bigoplus k^p$, which proves the claim of the theorem. ☐

Now we are ready to show that the multiplicative Adams operations on algebraic K-theory lift to differential algebraic K-theory. Let $k \in \mathbb{N}$. On the object

$$\mathrm{IDR} = \bigoplus_{p\in\mathbb{Z}} \mathrm{IDR}(p) \in \mathrm{Fun}(\mathbf{Sm}^{op}_{\mathbb{C}}, \mathrm{CAlg}(\mathrm{Ch}(\mathbb{R})))$$

we then define an endomorphism given by multiplication with k^p on the p-th summand of $\mathrm{IDR} \cong \bigoplus_p \mathrm{IDR}(p)$:

$$\bigoplus k^p : \mathrm{IDR} = \bigoplus_{p\in\mathbb{Z}} \mathrm{IDR}(p) \to \bigoplus_{p\in\mathbb{Z}} \mathrm{IDR}(p) = \mathrm{IDR}.$$

Applying the Eilenberg-MacLane functor we get an induced map

$$H(\bigoplus k^p) : H(\mathrm{IDR}) \to H(\mathrm{IDR})$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{CAlg}(\mathrm{Sp}))$, whose underlying map of spectra is also given by summandwise multiplication with k^p:

(4.54)

$$H(\bigoplus k^p) \simeq \bigoplus k^p : H(\mathrm{IDR}) \simeq \bigoplus_{p\geq 0} H(\mathrm{IDR}(p)) \to \bigoplus_{p\geq 0} H(\mathrm{IDR}(p)) \simeq H(\mathrm{IDR}).$$

Theorem 4.55. *Let $k \in \mathbb{N}$. Then the diagram*

(4.56)

$$\begin{tikzcd}
\mathcal{K}_{st}[k^{-1}] \arrow[r, "reg"] \arrow[d, "\psi^k"'] & H(\mathrm{IDR}) \arrow[d, "\bigoplus k^p"] \\
\mathcal{K}_{st}[k^{-1}] \arrow[r, "reg"] & H(\mathrm{IDR})
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{CAlg}(\mathrm{Sp}))$ *commutes.*

Proof. By theorem 4.47 we get a commutative diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \mathbf{K}[k^{-1}]) \arrow[r, "\mathbf{reg}_*"] \arrow[d, "\psi^k_{mot,*}"'] & \mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \mathbf{H}) \arrow[d, "\psi^k_{\mathbf{H}}"] \\
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \mathbf{K}[k^{-1}]) \arrow[r, "\mathbf{reg}_*"] & \mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, \mathbf{H})
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{CAlg}(\mathrm{Sp}))$. By corollary 4.42 and since suspension spectra of schemes are compact objects in $\mathrm{Sp}^{\mathbb{P}^1}$ this commutative diagram is equivalent to a commutative diagram

(4.57)

$$\begin{tikzcd}
\mathcal{K}_{st}[k^{-1}] \arrow[r, "reg"] \arrow[d, "\psi^k"'] & H(\mathrm{IDR}) \arrow[d] \\
\mathcal{K}_{st}[k^{-1}] \arrow[r, "reg"] & H(\mathrm{IDR})
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{CAlg}(\mathrm{Sp}))$, in which the right vertical map is induced by $\psi^k_{\mathbf{H}}$. Using again the compactness of suspension spectra of schemes, we get canonical equivalences in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op},\mathrm{Sp})$ of the form

$$\begin{aligned}
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+,\mathbf{H}) &\simeq \mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+,\bigoplus_{p\in\mathbb{Z}} R\,(\iota(\mathbb{R}(p)[2p])))\\
&\simeq \bigoplus_{p\in\mathbb{Z}} \mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, R\,(\iota(\mathbb{R}(p)[2p]))\\
&\simeq \bigoplus_{p\geq 0} \mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+, R\,(\iota(\mathbb{R}(p)[2p])))\\
&\simeq \bigoplus_{p\geq 0} \mathrm{map}_{\mathrm{Ind}(\mathbf{MHC}_{\mathbb{R}}[W^{-1}])}(\iota(\mathbb{R}(0)),\iota(\mathrm{A}_{\log}(-))\otimes\iota\,(\mathbb{R}(p)[2p]))\\
&\simeq \bigoplus_{p\geq 0} \mathrm{IDR}(p).
\end{aligned}$$

Since the underlying map of $\psi^k_{\mathbf{H}}:\mathbf{H}\to\mathbf{H}$ in $\mathrm{Sp}^{\mathbb{P}^1}$ is given by multiplication with k^p in the p-th summand, we therefore have a commutative square

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+,\mathbf{H}) \arrow[r, dash, "\simeq"] \arrow[d, "\psi^k_{\mathbf{H}}"'] & H(\mathrm{IDR}) \arrow[r, dash, "\simeq"] \arrow[d] & \bigoplus_{p\geq 0} H(\mathrm{IDR}(p)) \arrow[d, "\bigoplus k^p"] \\
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+,\mathbf{H}) \arrow[r, dash, "\simeq"] & H(\mathrm{IDR}) \arrow[r, dash, "\simeq"] & \bigoplus_{p\geq 0} H(\mathrm{IDR}(p))
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op},\mathrm{Sp})$. So, the map of presheaves of spectra that underlies the right vertical map in 4.57 is precisely

$$H(\bigoplus k^p)\simeq\bigoplus k^p : H(\mathrm{IDR})\simeq\bigoplus_{p\in\mathbb{N}} H(\mathrm{IDR}(p))\to\bigoplus_{p\in\mathbb{N}} H(\mathrm{IDR}(p))\simeq H(\mathrm{IDR})$$

from 4.54. But we still have to argue that also the multiplicative structures are compatible, i.e. that the diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+,\mathbf{H}) \arrow[r, dash, "\simeq"] \arrow[d, "\psi^k_{\mathbf{H}}"'] & H(\mathrm{IDR}) \arrow[d, "\bigoplus k^p"] \\
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+,\mathbf{H}) \arrow[r, dash, "\simeq"] & H(\mathrm{IDR})
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op},\mathrm{CAlg}(\mathrm{Sp}))$ commutes. For this we use remark 4.41. Consider the diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+,\mathbf{H}) \arrow[r, "\simeq"] \arrow[d, "\psi_{\mathbf{H}}"'] & H(\mathcal{E}(\bigoplus_{p\geq 0}\mathbb{R}(p)[2p]\otimes\mathrm{A}_{\log})) \arrow[r, "\simeq"] \arrow[d, "H(\bigoplus k^p)"] & H(\mathrm{IDR}) \arrow[d, "H(\bigoplus k^p)"] \\
\mathrm{map}_{\mathrm{Sp}^{\mathbb{P}^1}}(\Sigma^\infty_{\mathbb{P}^1}(-)_+,\mathbf{H}) \arrow[r, "\simeq"] & H(\mathcal{E}(\bigoplus_{p\geq 0}\mathbb{R}(p)[2p])\otimes\mathrm{A}_{\log}) \arrow[r, "\simeq"] & H(\mathrm{IDR})
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{R}}, \mathrm{CAlg}(\mathrm{Sp}))$. By definition of $\psi_{\mathbf{H}}$ in 4.2 and point (1) of remark 4.41 the left square commutes and by point (2) of the same remark also the right square commutes. This finally shows commutativity of 4.56. ☐

The first application of the last theorem 4.55 is the construction of Adams operations on n-th differential algebraic K-theory for each n:

Corollary 4.58. *For each $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ there is a map in* $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{Sp})$

$$\widehat{\psi}^{k,n} : \widehat{\mathcal{K}}^n[k^{-1}] \to \widehat{\mathcal{K}}^n[k^{-1}]$$

such that the diagrams

$$\begin{tikzcd}
\widehat{\mathcal{K}}^n[k^{-1}] \arrow[r, "I"] \arrow[d, "\widehat{\psi}^{k,n}"'] & \mathcal{K}_{st}[k^{-1}] \arrow[d, "\psi^k"] \\
\widehat{\mathcal{K}}^n[k^{-1}] \arrow[r, "I"] & \mathcal{K}_{st}[k^{-1}]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\widehat{\mathcal{K}}^n[k^{-1}] \arrow[r, "R"] \arrow[d, "\widehat{\psi}^{k,n}"'] & H(\sigma^{\geq n}\mathrm{IDR}) \arrow[d, "\bigoplus k^p"] \\
\widehat{\mathcal{K}}^n[k^{-1}] \arrow[r, "R"] & H(\sigma^{\geq n}\mathrm{IDR})
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{Sp})$ *commute.*

Proof. By theorem 4.55 we have a commutative diagram

$$\begin{tikzcd}
\mathcal{K}_{st}[k^{-1}] \arrow[r, "reg"] \arrow[d, "\psi^k"'] & H(\mathrm{IDR}) \arrow[d, "\bigoplus k^p"] & H(\sigma^{\geq n}\mathrm{IDR}) \arrow[l] \arrow[d, "\bigoplus k^p"] \\
\mathcal{K}_{st}[k^{-1}] \arrow[r, "reg"] & H(\mathrm{IDR}) & H(\sigma^{\geq n}\mathrm{IDR}) \arrow[l]
\end{tikzcd} \tag{4.59}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{Sp})$. Here the commutativity of the right square is a triviality. ☐

Furthermore it is also possible to construct multiplicative Adams operations on multiplicative differential algebraic K-theory:

Corollary 4.60. *For each $k \in \mathbb{N}$ there is a self map*

$$\widehat{\psi}^k : \widehat{\mathcal{K}}[k^{-1}] \to \widehat{\mathcal{K}}[k^{-1}]$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{CAlg}(\mathrm{Sp}))$ *on k-inverted multiplicative differential algebraic K-theory, such that the underlying map in* $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}^{op}, \mathrm{Sp})$ *is the* *direct sum*

$$\widehat{\psi}^k \simeq \bigoplus_{n\in\mathbb{Z}} \widehat{\psi}^{k,n}.$$

Proof. This follows directly from the commutative diagrams

$$\begin{tikzcd}
\mathcal{K}_{st}[k^{-1}] \wedge \Sigma^\infty_+\mathbb{Z} \arrow[r, "reg\wedge\Sigma^\infty_+\mathbb{Z}"] \arrow[d, "\psi^k\wedge\Sigma^\infty_+\mathbb{Z}"'] & H(\mathrm{IDR}) \wedge \Sigma^\infty_+\mathbb{Z} \arrow[d, "(\oplus k^p)\wedge\Sigma^\infty_+\mathbb{Z}"] \\
\mathcal{K}_{st}[k^{-1}] \wedge \Sigma^\infty_+\mathbb{Z} \arrow[r, "reg\wedge\Sigma^\infty_+\mathbb{Z}"] & H(\mathrm{IDR}) \wedge \Sigma^\infty_+\mathbb{Z}
\end{tikzcd}$$

and

$$\begin{tikzcd}
H(\sigma^{\geq\bullet}\mathrm{IDR}) \arrow[r] \arrow[d, "H(\sigma^{\geq\bullet}\oplus k^p)"'] & H(\mathrm{IDR}[z,z^{-1}]) \arrow[d, "H(\oplus k^p[z,z^{-1}])"] \\
H(\sigma^{\geq\bullet}\mathrm{IDR}) \arrow[r] & H(\mathrm{IDR}[z,z^{-1}])
\end{tikzcd}$$

in $\mathrm{Fun}(\mathbf{Sm}_{\mathbb{C}}, \mathrm{CAlg}(\mathrm{Sp}))$. Here again we get the first commutative diagram from theorem 4.55 and the commutativity of the second square is trivial. Note also, that we implicitely use lemma 4.45 for the construction of $\widehat{\psi}^k$. $\square$

Definition 4.61. *The map*

$$\widehat{\psi}^k : \widehat{\mathcal{K}}[k^{-1}] \to \widehat{\mathcal{K}}[k^{-1}]$$

is called the multiplicative differential refinement of the k-th Adams operation on algebraic K-theory of smooth schemes over $\mathbb{C}$ or just the k-th differential Adams operation.

Remark 4.62. Since the stable Adams operations

$$\psi^k : \mathcal{K}_{st}[k^{-1}] \to \mathcal{K}_{st}[k^{-1}]$$

fulfill the relation

$$\psi^k \circ \psi^l \simeq \psi^{kl}$$

and since one obviously also has a canonical homotopy

$$\bigoplus k^p \circ \bigoplus l^p \simeq \bigoplus (kl)^p,$$

the differential refinement of the k-th stable Adams operation satisfies the same relation

$$\widehat{\psi}^k \circ \widehat{\psi}^l \simeq \widehat{\psi}^{kl}.$$

APPENDIX A. SEGRE EMBEDDINGS AND COMMUTATIVE ALGEBRA STRUCTURE ON $\mathbb{P}^\infty$

In this appendix all schemes of considerstions lie over a fixed but arbitrary regular base scheme S. Since we will explicitly distinguish between space-valued presheaves on $\mathbf{Sm}_S$ and their motivic localizations we will be very strict about the notation here. While in earlier chapters we denoted both the space-valued presheaf represented by a scheme and its motivic localization by the same letter, we will explicitly keep track of the localization functor in this appendix. For example, we will write $\mathbb{P}^\infty$ for the space valued presheaf and $L^{mot}(\mathbb{P}^\infty)$ for the associated motivic space.

The motivic space $L^{mot}(\mathbb{P}^\infty)$ carries the structure of a commutative algebra object with respect to the cartesian monoidal structure. This algebra structure stems from the fact, that there is an equivalence $L^{mot}(\mathbb{P}^\infty) \simeq \mathrm{B}\mathbb{G}_m$ of motivic spaces. As a commutative algebra object $L^{mot}(\mathbb{P}^\infty)$ has in particular a multiplication

$$m : L^{mot}(\mathbb{P}^\infty) \times L^{mot}(\mathbb{P}^\infty) \to L^{mot}(\mathbb{P}^\infty).$$

It is a well-known fact that this multiplication map is induced from the Segre embeddings. In this appendix we want to recall how one proves this statement. At first we want to make the statement precise. For this we have to explain how the Segre embeddings for finite dimensional projective spaces induce a map on $L^{mot}(\mathbb{P}^\infty) \in \mathrm{Spc}^{mot}$.

A Segre embedding is a map of the form

$$\begin{tikzcd} \mathbb{P}^n \times \mathbb{P}^m \arrow[rr] & & \mathbb{P}^{nm+n+m} \\ ([X_0 : ... : X_n], [Y_0 : ... : Y_m]) \arrow[r, mapsto] & {[X_iY_j]_{i,j}.} \end{tikzcd}$$

There is a freedom of choice for the ordering of the X_iY_j. The goal is to choose these orderings in the case $n = m$ such that we get an induced map on infinite projective spaces

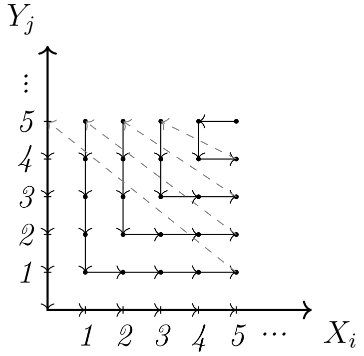
$$\mathrm{Seg} : L^{mot}(\mathbb{P}^\infty) \times L^{mot}(\mathbb{P}^\infty) \to L^{mot}(\mathbb{P}^\infty).$$

Note that the localization functor $L^{mot} : \mathcal{P}(\mathbf{Sm}_S) \to \mathrm{Spc}^{mot}$ preserves finite products.

Construction A.1. *We construct concrete Segre embeddings*

$$\mathrm{Seg}_n : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^{n^2+2n}$$

now. The choice of the ordering of the X_iY_j is made in a way such that we get an induced map after taking the colimit over the n's. We will *not give an explicit formula for the ordering but rather give a picture for the ordering in the case $n = 5$. It is obvious how to generalize it to general n.*



Here the (i, j)-th grid point stands for the product X_iY_j and the order of the products in $[X_iY_j]_{0\leq i,j\leq n}$ is given by the indicated path starting at the right upper corner.

The construction of the Segre embeddings Seg_n is made in a way such that the following lemma is true. The proof of this lemma is made by an easy bookkeeping of indices.

Lemma A.2. *Let $m \geq n$, then the diagram of schemes*

$$\begin{tikzcd}
\mathbb{P}^n \times \mathbb{P}^n \arrow[r, "\mathrm{Seg}_n"] \arrow[d, hook, "i_1 \times i_1"'] & \mathbb{P}^{n^2+2n} \arrow[d, hook, "i_2"] \\
\mathbb{P}^m \times \mathbb{P}^m \arrow[r, "\mathrm{Seg}_m"] & \mathbb{P}^{m^2+2m}
\end{tikzcd} \tag{A.3}$$

where i_1 is the canonical embedding that sends $[X_0 : ... : X_r]$ to $[0 : ... : 0 : X_0 : ... : X_r]$ and i_2 is the canonical embedding that sends $[X_0 : ... : X_r]$ to $[X_0 : ... : X_r : 0 : ... : 0]$, commutes.

Example A.4. We want to illustrate the commutativity of diagram A.3 in the case $n = 1$ and $m = 2$. In this case the diagram is

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\mathrm{Seg}_1"] \arrow[d, hook, "i_1"'] & \mathbb{P}^3 \arrow[d, hook, "i_2"] \\
\mathbb{P}^2 \times \mathbb{P}^2 \arrow[r, "\mathrm{Seg}_2"] & \mathbb{P}^8,
\end{tikzcd}$$

where

$$\begin{aligned}
&\mathrm{Seg}_1([X_0, X_1], [Y_0, Y_1]) \\
=&[X_1Y_1 : X_0Y_1 : X_0Y_0 : X_1Y_0]
\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Seg}_2([A_0 : A_1 : A_2],[B_0 : B_1 : B_2])\\ =&[A_2B_2 : A_1B_2 : A_1B_1 : A_2B_1 : A_0B_2 : A_0B_1 : A_0B_0 : A_1B_0 : A_2B_0].\end{aligned}$$

One checks that both compositions send $([X_0 : X_1],[Y_0 : Y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ to $[X_1Y_1 : X_0Y_1 : X_0Y_0 : X_1Y_0 : 0 : 0 : 0 : 0 : 0] \in \mathbb{P}^8$.

Definition A.5. *(Segre map on $\mathbb{P}^\infty$)*
By lemma A.2 the maps Seg_n *induce a map of motivic spaces*

$$\sigma : L^{mot}(\mathbb{P}^\infty) \times L^{mot}(\mathbb{P}^\infty) \to L^{mot}(\mathbb{P}^\infty),$$

which we call the Segre map on the infinite projective space.

Remark A.6. The infinite projective space $L^{mot}(\mathbb{P}^\infty) \in \mathrm{Spc}^{mot}$ is defined as colimit of the finite dimensional projective spaces $L^{mot}(\mathbb{P}^n)$. Note that in the colimit system of successive embeddings of projective spaces, that one uses to define $L^{mot}(\mathbb{P}^\infty)$, one can take the inclusions i_1 as well as the inclusions i_2. This follows from the fact, that $i_1, i_2 : \mathbb{P}^n \hookrightarrow \mathbb{P}^m$ are homotopic maps. We used this implicitly in the last definition.

Proposition A.7. *The two maps*

$$m, \sigma : L^{mot}(\mathbb{P}^\infty) \times L^{mot}(\mathbb{P}^\infty) \to L^{mot}(\mathbb{P}^\infty)$$

are homotopic.

The proof of this proposition is the actual aim of this appendix.

For $X \in \mathbf{Sm}_S$ we now consider the symmetric monoidal Picard groupoid of X

$$\underline{\mathrm{Pic}}(X).$$

As usual this is the groupoid of line bundles over X with the symmetric monoidal structure coming from the tensor product operation. Pullback of line bundles turns $\underline{\mathrm{Pic}}$ into a presheaf of symmetric monoidal groupoids and applying the nerve functors therefore defines a presheaf of $\mathbb{E}_\infty$-spaces

$$\mathrm{N}(\underline{\mathrm{Pic}}) \in \mathrm{Fun}(\mathbf{Sm}_S, \mathrm{CAlg}(\mathcal{S})) \simeq \mathrm{CAlg}(\mathrm{Fun}(\mathbf{Sm}_S, \mathcal{S})).$$

In fact $\mathrm{N}(\underline{\mathrm{Pic}})$ is already Nisnevich local and since S is regular it is also $\mathbb{A}^1$-local and therefore we already have

$$\mathrm{N}(\underline{\mathrm{Pic}}) \in \mathrm{CAlg}(\mathrm{Spc}^{mot}).$$

In fact it is well known, that this object in $\mathrm{CAlg}(\mathrm{Spc}^{mot})$ is a concrete model for $\mathrm{B}\mathbb{G}_m$. The fact, that $L^{mot}(\mathbb{P}^\infty) \simeq \mathrm{B}\mathbb{G}_m$ can then be formulated in the following way:

Proposition A.8. *(see* [NSØ09], *Proposition 2.1)*
There exists an equivalence

$$c : \mathrm{N}(\underline{\mathrm{Pic}}) \to \mathrm{Map}_{\mathrm{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty))$$

in $\mathcal{P}(\mathbf{Sm}_S)$ *such that there is a commutative diagram*

$$\begin{tikzcd}
\mathrm{N}(\underline{\mathrm{Pic}}) \arrow[rr, "{c,\simeq}"] & & \mathrm{Map}_{\mathrm{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty)) \\
& \mathrm{Map}_{\mathcal{P}(\mathbf{Sm}_S)}(\mathrm{Yo}_\infty(-), \mathbb{P}^\infty) \arrow[ul, "\gamma"] \arrow[ur, "L^{mot}"'] &
\end{tikzcd}$$

where γ *is given by taking the pullback of the tautological bundle* $\mathcal{O}(-1)$.

Remark A.9. We want to make some short remarks on the definition of the map γ in the last proposition.
Note, that since $\mathbf{Sm}_S$ is a 1-category the ∞-categorical Yoneda embedding for $\mathbf{Sm}_S$ can be factored as

$$\begin{tikzcd}
\mathbf{Sm}_S \arrow[r, "\mathrm{Yo}"] \arrow[rr, bend right, "\mathrm{Yo}_\infty"'] & \mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathbf{Set}) \arrow[r, hook, "\mathrm{full}"] & \mathcal{P}(\mathbf{Sm}_S).
\end{tikzcd}$$

Therefore, the mapping space $\mathrm{Map}_{\mathcal{P}(\mathbf{Sm}_S)}(\mathrm{Yo}_\infty(-), \mathbb{P}^\infty)$ is in fact discrete and equal to the Hom-set $\mathrm{Hom}_{\mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathbf{Set})}(\mathrm{Yo}(-), \mathbb{P}^\infty)$. If $X \in \mathbf{Sm}_S$, then $\mathrm{Yo}(X)$ is a compact object in $\mathrm{Fun}(\mathbf{Sm}_S^{op}, \mathbf{Set})$ and so every morphism $\mathrm{Yo}(X) \to \mathbb{P}^\infty$ factors through $\mathrm{Yo}(\mathbb{P}^n)$ for some n. Since Yo is fully faithful this factorization corresponds uniquely to a morphism $X \to \mathbb{P}^n$ and it is then possible to pull back the tautological line bundle $\mathcal{O}(-1)$ along this morphism to X.

Since the commutative algebra structure on $L^{mot}(\mathbb{P}^\infty)$ is transferred from $\mathrm{B}\mathbb{G}_m$ through the equivalence $L^{mot}(\mathbb{P}^\infty) \simeq \mathrm{B}\mathbb{G}_m$, we immediately get the following result:

Lemma A.10. *Let*

$$\otimes : \mathrm{N}(\underline{\mathrm{Pic}}) \times \mathrm{N}(\underline{\mathrm{Pic}}) \to \mathrm{N}(\underline{\mathrm{Pic}})$$

be the multiplication map that corresponds to the commutative algebra structure on $\mathrm{N}(\underline{\mathrm{Pic}})$. *Then there is the following commutative diagram in* $\mathcal{P}(\mathbf{Sm}_S)$*:*

$$\begin{tikzcd}
\mathrm{N}(\underline{\mathrm{Pic}}) \times \mathrm{N}(\underline{\mathrm{Pic}}) \arrow[r, "{c\times c,\simeq}"] \arrow[d, "\otimes"'] & \mathrm{Map}_{\mathrm{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty) \times L^{mot}(\mathbb{P}^\infty)) \arrow[d, "m_*"] \\
\mathrm{N}(\underline{\mathrm{Pic}}) \arrow[r, "{c,\simeq}"] & \mathrm{Map}_{\mathrm{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty)).
\end{tikzcd}$$

In order to proof proposition A.7 it is therefore enough to show the following:

Lemma A.11. *The diagram*

$$\begin{tikzcd}
\mathrm{N}(\underline{\mathrm{Pic}}) \times \mathrm{N}(\underline{\mathrm{Pic}}) \arrow[r, "c\times c{,}\simeq"] \arrow[d, "\otimes"'] & \mathrm{Map}_{\mathsf{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty) \times L^{mot}(\mathbb{P}^\infty)) \arrow[d, "\sigma_*"] \\
\mathrm{N}(\underline{\mathrm{Pic}}) \arrow[r, "c{,}\simeq"] & \mathrm{Map}_{\mathsf{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty))
\end{tikzcd}$$

in $\mathcal{P}(\mathbf{Sm}_S)$ is commutative.

For the proof of Lemma A.11 we need the following geometric input:

Lemma A.12. *(see the proof of Lemma 15 in section 9.4. of* [Bos13]*) There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Sm}_S}(-, \mathbb{P}^n \times \mathbb{P}^m) \arrow[r, "\mathrm{can}{,}\cong"] \arrow[dr, "\mathrm{Segre}_*"'] & \mathrm{Hom}_{\mathbf{Sm}_S}(-, \mathbb{P}^n) \times \mathrm{Hom}_{\mathbf{Sm}_S}(-, \mathbb{P}^m) \arrow[r, "\gamma_n \times \gamma_m"] \arrow[d] & \mathrm{N}(\underline{\mathrm{Pic}}) \times \mathrm{N}(\underline{\mathrm{Pic}}) \arrow[d, "\otimes"] \\
 & \mathrm{Hom}_{\mathbf{Sm}_S}(-, \mathbb{P}^{nm+n+m}) \arrow[r, "\gamma_{nm+n+m}"] & \mathrm{N}(\underline{\mathrm{Pic}})
\end{tikzcd}$$

in the category $\mathcal{P}(\mathbf{Sm}_S)$ where γ_i is the map that associates to a morphism $f : X \to \mathbb{P}^i$ the line bundle $f^*\mathcal{O}(-1)$, and Segre $: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ *is any Segre embedding.*

Proof. (of Proposition A.11)
By lemma A.12 also the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{P}(\mathbf{Sm}_S)}(\mathrm{Yo}_\infty(-), \mathbb{P}^\infty \times \mathbb{P}^\infty) \arrow[r, "\gamma\times\gamma"] \arrow[d, "\sigma_*"'] & \mathrm{N}(\underline{\mathrm{Pic}}) \times \mathrm{N}(\underline{\mathrm{Pic}}) \arrow[d, "\otimes"] \\
\mathrm{Map}_{\mathcal{P}(\mathbf{Sm}_S)}(\mathrm{Yo}_\infty(-), \mathbb{P}^\infty) \arrow[r, "\gamma"] & \mathrm{N}(\underline{\mathrm{Pic}})
\end{tikzcd}$$

in $\mathcal{P}(\mathbf{Sm}_S)$ commutes. Using this observation together with proposition A.8 we get that in the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathsf{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty) \times L^{mot}(\mathbb{P}^\infty)) \arrow[r, "\sigma_*"] \arrow[dd, bend right=60, "c^{-1}\times c^{-1}"'] & \mathrm{Map}_{\mathsf{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty)) \arrow[dd, bend left=60, "c^{-1}"] \\
\mathrm{Map}_{\mathcal{P}(\mathbf{Sm}_S)}(\mathrm{Yo}_\infty(-), \mathbb{P}^\infty \times \mathbb{P}^\infty) \arrow[r, "\sigma_*"] \arrow[u, "L^{mot}"] \arrow[d, "\gamma\times\gamma"'] & \mathrm{Map}_{\mathcal{P}(\mathbf{Sm}_S)}(\mathrm{Yo}_\infty(-), \mathbb{P}^\infty) \arrow[u, "L^{mot}"'] \arrow[d, "\gamma"] \\
\mathrm{N}(\underline{\mathrm{Pic}}) \times \mathrm{N}(\underline{\mathrm{Pic}}) \arrow[r, "\otimes"'] & \mathrm{N}(\underline{\mathrm{Pic}})
\end{tikzcd}$$

everything is commutative except possibly the outer rectangle. But now the morphism

$$\mathrm{Map}_{\mathcal{P}(\mathbf{Sm}_S)}(\mathrm{Yo}_\infty(-), \mathbb{P}^\infty \times \mathbb{P}^\infty) \to \mathrm{Map}_{\mathsf{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty) \times L^{mot}(\mathbb{P}^\infty))$$

in $\mathcal{P}(\mathbf{Sm}_S)$ has the universal property with respect to maps from $\mathrm{Map}_{\mathcal{P}(\mathbf{Sm}_S)}(\mathrm{Yo}_\infty(-), \mathbb{P}^\infty \times \mathbb{P}^\infty)$ to objects of $\mathcal{P}(\mathbf{Sm}_S)$ that are Nisnevich- and $\mathbb{A}^1$-local. Since $\mathrm{N}(\underline{\mathrm{Pic}})$ is both Nisnevich and $\mathbb{A}^1$-local, the commutativity of

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty) \times L^{mot}(\mathbb{P}^\infty)) \arrow[r, "c^{-1} \times c^{-1}"] \arrow[d, "\sigma_*"'] & \mathrm{N}(\underline{\mathrm{Pic}}) \times \mathrm{N}(\underline{\mathrm{Pic}}) \arrow[d, "\otimes"] \\
\mathrm{Map}_{\mathrm{Spc}^{mot}}(L^{mot} \circ \mathrm{Yo}_\infty(-), L^{mot}(\mathbb{P}^\infty)) \arrow[r, "c^{-1}"] & \mathrm{N}(\underline{\mathrm{Pic}})
\end{tikzcd}$$

follows, which proves the claim. $\square$

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