The heat flow for Dirac-harmonic maps

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Abstract

This thesis consists of five chapters, each of which can be viewed as being independent in content from the others. Nonetheless, the chapters are interconnected and there are overarching ideas that unify them.

In the following, we will briefly describe the subject of each chapter and explain how they fit together.

The main result of Chapter 1 is short time existence of the heat flow for Dirac-harmonic maps on closed manifolds. Dirac-harmonic maps are the critical points of a functional motivated by the supersymmetric non-linear sigma model from quantum field theory. Finding non-trivial examples for Dirac-harmonic maps turned out to be a rather challenging task and not many examples were known. With the aim to get a general existence program for Dirac-harmonic maps, the heat flow for Dirac-harmonic maps was introduced by Chen, Jost, Sun, and Zhu. The flow consists of a second order harmonic map type system coupled with a first order Dirac type system. For source manifolds with boundary Chen, Jost, Sun, and Zhu obtained short time existence. This heat flow approach to obtain Dirac-harmonic maps was fully legitimized when the existence of a global weak solution was established by Jost, Liu, and Zhu, from which they deduced existence results for Dirac-harmonic maps (for source manifolds with boundary). Our strategy to show short time existence on closed manifolds roughly is as follows: first, we solve the first order Dirac type system, then we take its solution, plug it into the second order harmonic map type system and solve the latter with a contraction argument. A main ingredient for the contraction argument are estimates for Dirac operators along maps which we will develop.

The subject of Chapter 2 is the existence and genericness of minimal kernels of Dirac operators along maps. In particular, the existence results we achieve yield many suitable initial values for the short time existence result of Chapter 1.

In Chapter 3 and 4 we deal with a certain Banach bundle that has as base space the Banach manifold of k-times continuously differentiable maps between a closed manifold and a connected manifold without boundary. These results are the basis of our original ansatz to solve the first order Dirac type system.

The content of Chapter 5 is the computation of the curvature of the Bourguignon-Gauduchon connection in the semi-Riemannian case. The Bourguignon-Gauduchon connection is an important tool that allows to compare spinors for different metrics. We use it for example in Chapter 2.

Zusammenfassung

Die vorliegende Arbeit besteht aus fünf Kapiteln, wobei jedes davon als inhaltlich unabhängig von den anderen angesehen werden kann. Dennoch sind die Kapitel miteinander verbunden und es gibt überspannende Ideen, die sie thematisch vereinen.

Im Folgenden werden wir kurz den Inhalt jedes Kapitels beschreiben und wir erklären, wie die Kapitel zusammenhängen.

Das Hauptresultat von Kapitel 1 ist die Kurzzeitexistenz des Wärmeflusses für Dirac-harmonische Abbildungen auf geschlossenen Mannigfaltigkeiten. Dirac-harmonische Abbildungen sind die kritischen Punkte von einem Funktional, welches motiviert ist durch das supersymmetrische nicht-lineare Sigma-Modell aus der Quantenfeldtheorie. Nicht-triviale Beispiele für Dirac-harmonische Abbildungen zu finden erwies sich als herausfordernde Aufgabe und nicht viele waren bekannt. Mit dem Ziel ein allgemeines Existenzprogramm für Dirac-harmonische Abbildungen zu erhalten, wurde der Wärmefluss für Dirac-harmonische Abbildungen von Chen, Jost, Sun und Zhu eingeführt. Dieser Fluss besteht aus einem System zweiter Ordnung (genauer einem System das Ähnlichkeit hat mit einem Wärmefluss für harmonische Abbildungen), welches gekoppelt ist mit einem Dirac-artigen System erster Ordnung. Für Mannigfaltigkeiten mit Rand wurde die Kurzzeitexistenz von Chen, Jost, Sun und Zhu gezeigt. Dieser Wärmefluss-Ansatz wurde vollständig legitimiert, als die Existenz einer globalen schwachen Lösung von Jost, Liu und Zhu gezeigt wurde, aus der sie Existenzresultate für Dirac-harmonische Abbildungen folgerten (für Mannigfaltigkeiten mit Rand). Unsere Strategie, um Kurzzeitexistenz auf geschlossenen Mannigfaltigkeiten zu zeigen, lässt sich grob wie folgt beschreiben: zunächst lösen wir das Dirac-artige System erster Ordnung, setzen dessen Lösung in das System zweiter Ordnung ein und lösen dieses dann durch ein Kontraktionsargument. Ein wesentlicher Bestandteil des Kontraktionsarguments sind Abschätzungen für Dirac Operatoren entlang Abbildungen, welche wir zeigen werden.

Das Thema von Kapitel 2 ist die Existenz und Generizität von minimalen Kernen von Dirac Operatoren entlang Abbildungen. Insbesondere liefern die Existenzresultate viele passende Anfangswerte für die in Kapitel 1 gezeigte Kurzzeitexistenz.

In den Kapiteln 3 und 4 befassen wir uns mit einem gewissen Banachbündel, dessen Basisraum die Banachmannigfaltigkeit der k-mal stetig differenzierbaren Abbildungen zwischen einer geschlossenen Mannigfaltigkeit und einer zusammenhängenden Mannigfaltigkeit ohne Rand ist. Diese Resultate sind die Grundlage unseres ursprünglichen Ansatzes um das Dirac-artige System erster Ordnung zu lösen.

Der Inhalt von Kapitel 5 ist die Berechnung der Krümmung des Bourguignon-Gauduchon Zusammenhangs im semi-Riemannschen Fall. Der Bourguignon-Gauduchon Zusammenhang ist ein wichtiges Hilfsmittel, mit dem man Spinoren unterschiedlicher Metriken vergleichen kann. Wir verwenden ihn beispielsweise in Kapitel 2.

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Contents

Chapter 1

Short time existence of the heat flow for Dirac-harmonic maps on closed manifolds

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Abstract The heat flow for Dirac-harmonic maps on Riemannian spin manifolds is a modification of the classical heat flow for harmonic maps by coupling it to a spinor. For source manifolds with boundary it was introduced in [10] as a tool to get a general existence program for Dirac-harmonic maps, where also short time existence was obtained. The existence of a global weak solution was established in [20]. We prove short time existence of the heat flow for Dirac-harmonic maps on closed manifolds. This chapter is similar to [36] but significantly more detailed.

1.1 Introduction

1.1.1 Dirac-harmonic maps

Dirac-harmonic maps, introduced in [8], are the critical points of a functional motivated by the supersymmetric non-linear sigma model from quantum field theory.

More precisely, let M be a compact Riemannian spin manifold (with fixed spin structure) and N a compact Riemannian manifold. We denote by ΣM the complex spinor bundle of M. (We assume that the reader is familiar with the basics of spin geometry, see e.g. [23], [4], [18], [15], and [29].) For maps $f\colon M \to N$ and spinors $\psi \in \Gamma(\Sigma M \otimes_{\mathbb{R}} f^*TN)$ we consider the functional

$$(f, \psi) \mapsto \frac{1}{2} \int_M \left(\|df\|^2 + (\psi, \not{D}^f \psi) \right) dV. \tag{1.1.1}$$

Here, $\|df\|^2$ is to be understood as follows: for every $p \in M$, we have $df_p \colon T_pM \to (f^*TN)_p$, therefore we can view df_p as an element of $T_p^*M \otimes (f^*TN)_p$[1], i.e.,

$$df_p = d(x^j \circ u)_p \otimes \Big(\frac{\partial}{\partial x_j}(f(p))\Big)$$

for a chart x on N. (Here and in the following we use the usual Einstein summation convention, see e.g. [24, p. 18].) The Riemannian metrics on M and N induce a bundle metric $\langle .,. \rangle_{T^*M\otimes f^*TN}$ on $T^*M \otimes f^*TN$.[2] Then,

$$\|df\|^2|_p := \langle df_p, df_p \rangle_{T^*M\otimes f^*TN}.$$

Moreover, $(.,.)$ denotes is the inner product induced by the real part of the natural hermitian inner product on ΣM and the Riemannian metric on N. Finally, $\not{D}^f$ is the Dirac operator of the twisted Dirac bundle $\Sigma M \otimes f^*TN$. Locally,

$$\not{D}^f \psi = (\not{D}\psi^i) \otimes s_i + (e_\alpha \cdot \psi^i) \otimes \nabla^{f^*TN}_{e_\alpha} s_i$$

where $\psi = \psi^i \otimes s_i$, the ψ^i are local sections of ΣM, (s_i) is a local frame of f^*TN, (e_α) is a local orthonormal frame of TM, ∇^{f^*TN} is the pull-back of the Levi-Civita connection on TN, and $\not{D}$ is the usual Dirac operator acting on sections of ΣM. We say that $\not{D}^f$ is the *Dirac operator along the map f*.

The critical points of the above functional are called *Dirac-harmonic maps*. They are characterized by the equations

$$\begin{cases} \tau(f) = \mathcal{R}(f,\psi), \\ \not{D}^f \psi = 0. \end{cases} \tag{1.1.2}$$

Here, $\tau(f) = \mathrm{tr}\nabla(df) = \Big(\nabla_{e_\alpha}(df)\Big)(e_\alpha)$ is the tension of f and $\mathcal{R}(f,\psi)$ is given by

$$\mathcal{R}(f,\psi) = \frac{1}{2}(\psi^i, e_\alpha \cdot \psi^j) R^{TN}(\frac{\partial}{\partial x_i} \circ f, \frac{\partial}{\partial x_j} \circ f) df(e_\alpha)$$

for $\psi = \psi^i \otimes (\frac{\partial}{\partial x_i} \circ f)$, where R^{TN} denotes the curvature tensor of N, $(\frac{\partial}{\partial x_i})$ are local coordinates on N, and (e_α) is again a local orthonormal frame of TM. Moreover, $(.,.)$ denotes the real part of the natural hermitian inner product of ΣM.

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[1]Recall that if V and W are real vector spaces, we have an isomorphism $V^* \otimes W \to \mathrm{Hom}(V,W)$ which on elementary tensors is defined by $\varphi \otimes w \mapsto (v \mapsto \varphi(v)w)$

[2]Recall that if E and F are vector bundles over a manifold M with bundle metrics h_E and h_F on E and F, respectively, then we have an induced bundle metric $h_E \otimes h_F$ on $E \otimes F$ which on elementary tensors is defined by $(h_E \otimes h_F)(e \otimes f, \tilde{e} \otimes \tilde{f}) := h_E(e,\tilde{e})h_F(f,\tilde{f})$, $e, \tilde{e} \in E_x$, $f, \tilde{f} \in F_x$, $x \in M$.

Obvious examples (f, ψ) for Dirac-harmonic maps are the following: f is a harmonic map and $\psi = 0$, f is a constant map and $\psi \in \ker(\not{D})$ is a harmonic spinor. In that sense, Dirac-harmonic maps generalize the subject of harmonic maps and harmonic spinors.

Results concerning the regularity of Dirac-harmonic maps have been achieved in [8, 9, 38, 39, 37, 12, 30, 33, 11] (mainly in the case that M is 2-dimensional, since then the functional is conformally invariant).

1.1.2 The heat flow for Dirac-harmonic maps

Apart from the obvious examples explained above, not many concrete examples for Dirac-harmonic maps were known. For a general overview we refer to the discussion in [2, Section 2]. First examples for uncoupled Dirac-harmonic maps (i.e., the mapping part is harmonic) are constructed in [8, Proposition 2.2]. Other examples can be found in [21], [2]. For coupled Dirac-harmonic maps (i.e., the mapping part is not harmonic) even less was known [21], [3].

With the aim to get a general existence program for Dirac-harmonic maps, the *heat flow for Dirac-harmonic maps*,

$$\begin{cases} \partial_t u = \tau(u) - \mathcal{R}(u, \psi) & \text{on } (0,T) \times M, \quad (1.1.3) \\ \not{D}^u \psi = 0 & \text{on } [0,T] \times M, \quad (1.1.4) \end{cases}$$

was introduced in [10]. In the case that M has non-empty boundary, short time existence (and uniqueness) of (1.1.3)–(1.1.4) was shown in [10] under the presence of certain initial and boundary conditions. Moreover, the existence of a *global* weak solution of (1.1.3)–(1.1.4) was obtained in [20] (again for certain initial and boundary conditions) with some existence results for Dirac-harmonic maps as an application.

At this point we want to mention another approach, considered by Volker Branding in his PhD thesis [5], where he studied the evolution equations for so-called regularized Dirac-harmonic maps.

1.1.3 Main result and overview of the proof

Our main result is the short time existence of the heat flow for Dirac-harmonic maps on closed (i.e., compact and without boundary) manifolds.

Theorem 1.1.1. *Let M be a closed m-dimensional Riemannian spin manifold, $m \equiv 0, 1, 2, 4 \pmod 8$, and N a closed Riemannian manifold of arbitrary dimension. Let $u_0 \in C^{2+\alpha}(M, N)$ for some $0 < \alpha < 1$ with $\dim_{\mathbb{K}} \ker(\not{D}^{u_0}) = 1$, where*

$$\mathbb{K} = \begin{cases} \mathbb{C} & \text{if } m \equiv 0, 1 \pmod 8, \\ \mathbb{H} & \text{if } m \equiv 2, 4 \pmod 8. \end{cases}$$

Moreover, let $\psi_0 \in \ker(\not{D}^{u_0})$ *with* $\|\psi_0\|_{L^2} = 1$*. Then there exists* $T > 0$ *and a solution* $(u_t, \psi_t)_{t\in[0,T]}$*,*

$$u \in C^{1,2,\alpha}((0,T) \times M, N),$$
$$\psi_t \in \ker(\not{D}^{u_t}) \qquad \forall t \in [0,T],$$

of

$$\begin{cases} \partial_t u = \tau(u) - \mathcal{R}(u,\psi) & \text{on } (0,T) \times M, \\ \not{D}^u \psi = 0 & \text{on } [0,T] \times M, \\ \dim_{\mathbb{K}} \ker(\not{D}^{u_t}) = 1 & \text{for all } t \in [0,T], \\ \|\psi_t\|_{L^2} = 1 & \text{for all } t \in [0,T], \\ u|_{t=0} = u_0, \\ \psi|_{t=0} = \psi_0. \end{cases} \tag{1.1.5}$$

Furthermore, if we are given any $T > 0$ *and a solution* $(u_t, \psi_t)_{t\in[0,T]}$ *of* (1.1.5) *with* $u \in C^{1,2,\alpha}((0,T) \times M, N)$*, then this solution is unique up to multiplication of the* ψ_t *with elements of* $\mathbb{K}$ *whose norm is equal to one.*

Here, the space $C^{1,2,\alpha}((0,T)\times M, N)$ is to be understood as follows. Embedding N isometrically into some $\mathbb{R}^q$ we define $C^{1,2,\alpha}((0,T) \times M, N)$ to be the space of all maps $u\colon (0,T)\times M \to N$ s.t. the component functions of $u\colon (0,T)\times M \to N \hookrightarrow \mathbb{R}^q$ belong to $C^{1,2,\alpha}((0,T) \times M)$. A definition of $C^{1,2,\alpha}((0,T) \times M)$ can be found in e.g. [32]. Note that every $u \in C^{1,2,\alpha}((0,T) \times M)$ can be continuously extended to $[0,T] \times M$, hence the requirement $u|_{t=0} = u_0$ in (1.1.5) makes sense.

We want to remark that from our construction of the spinor part $\psi = \psi(u)$ of the solution we will get that $\psi(u)$ depends Lipschitz continuously on u (in the sense of the estimates we derive in Lemma 1.4.11).

For the existence of initial values, i.e., maps $f\colon M \to N$ with $\dim_{\mathbb{K}}\ker(\not{D}^f) = 1$, we expect something like this: assume M is 2-dimensional and $f\colon M \to N$ is a map with non-vanishing index $\operatorname{ind}_{f^*TN}(M) \neq 0$, c.f. Remark 1.4.8.[3] (Examples of such maps are constructed in [2].) Then for generic metrics on M and N, and generic maps $g\colon M \to N$ in the homotopy class of f it holds that $\dim_{\mathbb{H}}\ker(\not{D}^g) = 1$, c.f. [35] and Chapter 2.

In the following, we give an overview of the proof of Theorem 1.1.1. To show short time existence we use the general strategy from [10], i.e., we first solve the

[3]Note that $\operatorname{ind}_{f^*TN}(M)$ depends on the choice of a spin structure on M, but doesn't depend on the Riemannian metrics on M and N. Moreover, $\operatorname{ind}_{g^*TN}(M) = \operatorname{ind}_{f^*TN}(M)$ for any g in the homotopy class of f.

constraint equation (1.1.4) for any homotopy of the initial value u_0, then we take the solution of the constraint equation and plug it into (1.1.3). After that we use a contraction argument to solve (1.1.3) and get the mapping part of the solution.

For the contraction argument we will isometrically embed N into some $\mathbb{R}^q$, rewrite (1.1.3) as a heat type equation in $\mathbb{R}^q$, and then solve this rewritten equation. However, we will solve the constraint equation (1.1.4) in N. Note that in [10], also the constraint equation was rewritten and solved as an equation in $\mathbb{R}^q$.

Clearly we can't solve $\not{D}^u\psi = 0$ uniquely in the absence of a boundary. However, we can achieve the following: we start with a 1-dimensional kernel, $\dim_{\mathbb{K}}\ker(\not{D}^{u_0}) = 1$. Then we show that for homotopies u_t of u_0 the kernel will stay 1-dimensional for small times, $\dim_{\mathbb{K}}\ker(\not{D}^{u_t}) = 1$ for t small. (This is the only place where the restrictions on the dimension of M will play a role.) Then we impose the additional constraint $\|\psi_t\|_{L^2} = 1$ to deduce that we can uniquely solve $\not{D}^u\psi = 0$ up to multiplication with elements of $\mathbb{K}$ whose norm is equal to one. Now observe that $\mathcal{R}(u, \psi)$ is *invariant* under multiplication of ψ with elements of $\mathbb{K}$ that have norm one. Because of this we can use a contraction argument to show that the mapping part of the solution is in fact unique.

To make the contraction argument work, we need to estimate the solution $\psi = \psi(u)$ of $\not{D}^u\psi = 0$ in terms of u. More precisely, we will construct one such solution and derive estimates for it. To that end, we start with an initial value $\psi_0 = \psi(u_0) \in \ker(\not{D}^{u_0})$. Given a homotopy u_t of u_0, we then define $\sigma(u_t) \in \Gamma(\Sigma M \otimes u_t^*TN)$ by identifying the bundles u_0^*TN and u_t^*TN via parallel transport in N along the unique shortest geodesics connecting $u_0(x)$ and $u_t(x)$, $x \in M$. Note that while $\sigma(u_t)$ is in general not in the kernel of $\not{D}^{u_t}$, it still has some non-trivial part in the kernel. Hence the projection $\psi(u_t)$ of $\sigma(u_t)$ onto $\ker(\not{D}^{u_t})$ is non-zero. (In particular, we can normalize $\psi(u_t)$ s.t. $\|\psi(u_t)\|_{L^2} = 1$.) Writing the projection as a resolvent integral

$$\psi(u_t) = \int_\gamma (\mu I - \not{D}^{u_t})^{-1}\sigma(u_t)\, d\mu \tag{1.1.6}$$

combined with estimates for Dirac operators along maps (which we will derive in Section 1.4.1) we will deduce the necessary estimates for $\psi(u_t)$.

1.2 Preliminaries

1.2.1 Elliptic W_p^{k+1}-regularity for Dirac operators along non-smooth maps

Elliptic W_p^{k+1}-regularity for Dirac operators of smooth Dirac bundles is well known. It follows from the mapping properties of pseudo-differential operators with smooth coefficients or it can be shown directly as e.g. in [1, Theorem 3.2.3]. For Dirac operators of non-smooth Dirac bundles it is less known (just as the mapping properties of pseudo-differential operators with non-smooth coefficients are less known).

Equation (1.1.6) contains the resolvent of $\not{D}^{u_t}$, where $u_t\colon M \to N$ will be at least C^1, but not smooth in general. Therefore we can not apply elliptic regularity theory for smooth Dirac bundles in our setting.

In this section we will prove elliptic W_p^{k+1}-regularity for Dirac operators along C^{k+1}-maps. As a corollary we deduce basic facts about the spectrum of such operators.

Definition 1.2.1 (Dirac operator along a map)**.** Let M be a Riemannian spin manifold, N a Riemannian manifold, and $f \in C^k(M,N)$, $k \in \mathbb{N}_{>0}$, $\mathbb{N} := \{0,1,2,\ldots\}$. Then we define the *Dirac operator along* f,

$$\not{D}^f \colon \Gamma_{C^k}(\Sigma M \otimes_{\mathbb{R}} f^*TN) \to \Gamma_{C^{k-1}}(\Sigma M \otimes_{\mathbb{R}} f^*TN),$$

by

$$\not{D}^f \psi = (\not{D}\psi^i) \otimes s_i + (e_\alpha \cdot \psi^i) \otimes \nabla^{f^*TN}_{e_\alpha} s_i$$

where $\psi = \psi^i \otimes s_i$, the ψ^i are local sections of ΣM, (s_i) is a local frame of f^*TN, (e_α) is a local orthonormal frame of TM, ∇^{f^*TN} is the pull-back of the Levi-Civita connection on TN, and $\not{D}$ is the usual Dirac operator acting on sections of the complex spinor bundle ΣM.

Remark 1.2.2. Let us denote by $\nabla^{\Sigma M \otimes f^*TN}$ the connection on $\Sigma M \otimes_{\mathbb{R}} f^*TN$ that is induced by the spinorial Levi-Civita connection on ΣM and ∇^{f^*TN}. Moreover, let us define a Clifford-multiplication on $\Sigma M \otimes_{\mathbb{R}} f^*TN$ by

$$X \cdot (a \otimes b) := (X \cdot a) \otimes b$$

for $X \in T_xM$, $a \in \Sigma_x M$, and $b \in (f^*TN)_x$. This turns $\Sigma M \otimes_{\mathbb{R}} f^*TN$ into a Dirac bundle and $\not{D}^f$ is the associated Dirac operator, i.e.,

$$\not{D}^f = e_\alpha \cdot \nabla^{\Sigma M \otimes f^*TN}_{e_\alpha}$$

for a local orthonormal frame (e_α) of TM.

Given $f \in C^1(M,N)$, the Dirac operator along f is an elliptic first order differential operator and formally self-adjoint with respect to the L^2-inner product. We view $\not{D}^f$ as a bounded densely defined self-adjoint operator

$$\not{D}^f \colon \Gamma_{W_2^1}(\Sigma M \otimes f^*TN) \to \Gamma_{L^2}(\Sigma M \otimes f^*TN).$$

(We recall basic definitions from functional analysis in Appendix 1.A.) Note that if $f \in C^k(M,N)$, then f^*TN is a C^k-vector bundle. Hence we can define $\Gamma_{W_p^l}(\Sigma M \otimes f^*TN)$ for $l = 0, 1, \ldots, k$.

Lemma 1.2.3 (Elliptic W_p^{k+1}-regularity)**.** *Let M be a closed Riemannian spin manifold and N a closed Riemannian manifold. Let $f \in C^{k+1}(M,N)$, $k \in \mathbb{N}$, and $2 \le p < \infty$. Moreover let $\lambda \in \mathbb{C}$ be arbitrary. If*

$$(\lambda I - \not{D}^f)\psi = \varphi$$

for $\psi \in \Gamma_{W_2^1}(\Sigma M \otimes f^*TN)$, $\varphi \in \Gamma_{W_p^k}(\Sigma M \otimes f^*TN)$, then $\psi \in \Gamma_{W_p^{k+1}}(\Sigma M \otimes f^*TN)$ and

$$\|\psi\|_{W_p^{k+1}} \le C\Big(\|\varphi\|_{W_p^k} + \|\psi\|_{L^p}\Big)$$

where $C = C(\lambda, f, \Sigma M) > 0$ is independent of ψ, φ.[4]

The basic idea of the proof is to approximate both $\not{D}^f$ and the bundle $\Sigma M \otimes f^*TN$ by smooth objects.

Proof of Lemma 1.2.3. First we show the lemma for $\lambda = 0$. Given $f \in C^{k+1}(M,N)$ we choose $g \in C^\infty(M,N)$ with $d^N(f(x), g(x)) < c$ for all $x \in M$ where $0 < c < \frac{1}{2}\mathrm{inj}(N)$, where $\mathrm{inj}(N)$ denotes the injectivity radius of N.[5] In particular we can connect $g(x)$ and $f(x)$ by a unique shortest geodesic of N for every $x \in M$. The parallel transport in N along these geodesics induces C^{k+1}-isomorphisms of vector bundles

$$P \colon g^*TN \to f^*TN,$$
$$P \colon \Sigma M \otimes g^*TN \to \Sigma M \otimes f^*TN.$$

[4]Note that the requirement $p \ge 2$ comes from our point of view. More precisely we don't consider weak solutions (as we don't need them) but we view $\not{D}^f$ as an operator $\not{D}^f \colon \Gamma_{W_2^1} \to \Gamma_{L^2}$. Then we consider the equation $\not{D}^f\psi = \varphi$ for $\varphi \in \Gamma_{L^2}$, $\psi \in \Gamma_{W_2^1}$ and show that if φ is in the "better" space $\Gamma_{W_p^k}$ we have that ψ is in the "better" space $\Gamma_{W_p^{k+1}}$.

[5]The existence of such a g can be seen as follows: we choose an isometric embedding $i \colon N \to \mathbb{R}^q$ and a tubular neighborhood N_δ of N in $\mathbb{R}^q$ as in the beginning of Section 1.3.1. There exists $g_\delta \in C^\infty(M, \mathbb{R}^q)$ s.t. $\|f(x) - g_\delta(x)\|_2 < \frac{\delta}{2}$ for all $x \in M$ (see e.g. [24, Theorem 6.21]). In particular g_δ takes values in N_δ. Hence $\pi \circ g_\delta \in C^\infty(M,N)$. Applying Lemma 1.3.11 (and the global Lipschitz continuity of π) we deduce that there exists $\delta_0 > 0$ small enough with $d^N(f(x), (\pi \circ g_{\delta_0})(x)) < c$ for all $x \in M$. Set $g := \pi \circ g_{\delta_0}$.

We also get induced isomorphisms of Banach spaces

$$P\colon \Gamma_{W_p^l}(\Sigma M \otimes g^*TN) \to \Gamma_{W_p^l}(\Sigma M \otimes f^*TN)$$

for $l = 0, 1, \ldots, k+1$.[6] We consider

$$G := P^{-1} \not{D}^f P - \not{D}^g.$$

Note that G, acting on sections of $\Sigma M \otimes g^*TN$, is a differential operator of order zero. Heuristically this is the case because in the definition of G the difference of the ordinary Dirac operators $\not{D}$ acting on ΣM cancels out and we are left with the difference of two covariant derivatives. Any covariant derivative has the identity as principal symbol, hence the difference of two covariant derivatives has zero as principal symbol. Therefore G is of order zero. To make this precise we set

$$\nabla := \nabla^{g^*TN}, \qquad \tilde{\nabla} := P^{-1} \nabla^{f^*TN} P$$

(note that $\tilde{\nabla}$ is a (non-smooth) covariant derivative on g^*TN). Moreover we choose local frames (s_j) and (ψ^i) of g^*TN and ΣM, respectively. Given a section ψ of $\Sigma M \otimes g^*TN$ we write

$$\psi = \lambda_i^j (\psi^i \otimes s_j)$$

The local formula for Dirac operators along maps yields

$$\begin{aligned} G\psi &= \lambda_i^j (e_\alpha \cdot \psi^i) \otimes \left(\tilde{\nabla}_{e_\alpha} s_j - \nabla_{e_\alpha} s_j \right) \\ &= \lambda_i^j (e_\alpha \cdot \psi^i) \otimes \left(\tilde{\omega}_j^l (e_\alpha) s_l - \omega_j^l (e_\alpha) s_l \right) \\ &= \lambda_i^j (\tilde{\omega}_j^l (e_\alpha) - \omega_j^l (e_\alpha)) e_\alpha \cdot (\psi^i \otimes s_l). \end{aligned}$$

From this it is easy to see that G is a differential operator of order zero with C^k-coefficients. In particular G extends to a bounded linear map

$$G\colon \Gamma_{W_p^l}(\Sigma M \otimes g^*TN) \to \Gamma_{W_p^l}(\Sigma M \otimes g^*TN), \tag{1.2.1}$$

for $l = 0, 1, \ldots, k$. Now assume that we have

$$\not{D}^f \tilde{\psi} = \tilde{\varphi}$$

for some $\tilde{\psi} \in \Gamma_{W_2^1}$ and $\tilde{\varphi} \in \Gamma_{W_p^k}$. This is equivalent to

$$\not{D}^g \psi = \varphi - G\psi$$

[6]More precisely: $P\colon \Gamma_{W_p^l} \to \Gamma_{W_p^l}$ is a bijective map whose inverse is induced by the parallel transport along the unique shortest geodesics that connect $f(x)$ and $g(x)$. The boundedness of $P\colon \Gamma_{W_p^l} \to \Gamma_{W_p^l}$ follows from the fact that $P\colon \Sigma M \otimes g^*TN \to \Sigma M \otimes f^*TN$ is an isomorphism of vector bundles and the W_p^l-norm does not depend on the choice of covariant derivative (M is compact).

where $\psi := P^{-1}\tilde{\psi} \in \Gamma_{W_2^1}$, $\varphi := P^{-1}\tilde{\varphi} \in \Gamma_{W_p^k}$. Write $n = \dim(M)$. Now we use a bootstrap argument to show $\psi \in \Gamma_{W_p^{k+1}}$. From (1.2.1) we deduce that $\varphi - G\psi \in \Gamma_{W_2^1}$. Since g is smooth the elliptic W_2^2-regularity for smooth Dirac bundles [1, Theorem 3.2.3] yields that $\psi \in \Gamma_{W_2^2}$. Hence $\varphi - G\psi \in \Gamma_{W_2^2}$. The elliptic W_2^3-regularity for smooth Dirac bundles yields $\psi \in \Gamma_{W_2^3}$. Iteratively we get

$$\psi \in \Gamma_{W_2^{k+1}}.$$

Then the Sobolev embedding theorem yields

$$\varphi - G\psi \in \Gamma_{W_q^k}$$

where $q = \min\{p, \underbrace{(\frac{1}{2} - \frac{1}{n})^{-1}}_{>2}\}$ for $n \geq 3$ and $q = p$ for $n \leq 2$. The elliptic W_q^{k+1}-regularity for smooth Dirac bundles yields that

$$\psi \in \Gamma_{W_q^{k+1}}.$$

If $q < p$ (hence $n \geq 3$), we use the Sobolev embedding theorem again to get

$$\varphi - G\psi \in \Gamma_{W_{\tilde{q}}^k}$$

where $\tilde{q} = \min\{p, \underbrace{(\frac{1}{q} - \frac{1}{n})^{-1}}_{>q}\}$ if $n \geq q+1$ and $\tilde{q} = p$ for $n \leq q$. If $\tilde{q} < p$ we iterate this process to get

$$\varphi - G\psi \in \Gamma_{W_p^k}$$

after finitely many steps. Applying [1, Theorem 3.2.3] to $D\!\!\!/^g$ again we get $\psi \in W_p^{k+1}$ and the existence of some $\tilde{C} = \tilde{C}(p, g, \Sigma M) > 0$ s.t.

$$\|\psi\|_{W_p^{k+1}} \leq \tilde{C}(\|\psi\|_{L^p} + \|\varphi - G\psi\|_{W_p^k})$$

From (1.2.1) we get $\|\varphi - G\psi\|_{W_p^k} \leq \|\varphi\|_{W_p^k} + C\|\psi\|_{W_p^k}$ hence

$$\|\psi\|_{W_p^{k+1}} \leq C_1(\|\psi\|_{L^p} + \|\varphi\|_{W_p^k} + \|\psi\|_{W_p^k}).$$

We get rid of $\|\psi\|_{W_p^k}$ on the right hand side by applying [1, Theorem 3.2.3] finitely many times and we finally obtain

$$\|\psi\|_{W_p^{k+1}} \leq C_2(\|\psi\|_{L^p} + \|\varphi\|_{W_p^k}).$$

Since P is an isomorphism on $\Gamma_{W_p^{k+1}}$ we get $\tilde{\psi} = P\psi \in \Gamma_{W_p^{k+1}}$ and

$$\begin{aligned}\|\tilde{\psi}\|_{W_p^{k+1}} &\leq C_3 \|\psi\|_{W_p^{k+1}} \\ &\leq C_4(\|\psi\|_{L^p} + \|\varphi\|_{W_p^k}) \\ &\leq C_5(\|\tilde{\psi}\|_{L^p} + \|\tilde{\varphi}\|_{W_p^k})\end{aligned}$$

where we used in the last line that P^{-1} is an isomorphism on Γ_{L^p} and $\Gamma_{W_p^k}$. This proves the lemma for $\lambda = 0$. If $\lambda \neq 0$, we use the case $\lambda = 0$ and a bootstrap argument as above. □

Corollary 1.2.4 (Spectral properties)**.** *Let M be a closed Riemannian spin manifold and N a closed Riemannian manifold. Let $f \in C^1(M, N)$. Given any element μ of the resolvent set of $D\!\!\!/^f$ it holds that the resolvent $R(\mu, D\!\!\!/^f)\colon \Gamma_{L^2} \to \Gamma_{L^2}$ of $D\!\!\!/^f : \Gamma_{W_2^1} \to \Gamma_{L^2}$ is bounded as a map*

$$R(\mu, D\!\!\!/^f)\colon \Gamma_{L^2} \to \Gamma_{W_2^1}.$$

In particular, $D\!\!\!/^f$ has compact resolvent. Hence, the spectrum and the point spectrum of $D\!\!\!/^f$ agree,

$$\operatorname{spec}(D\!\!\!/^f) := \sigma(D\!\!\!/^f) = \sigma_p(D\!\!\!/^f)$$

and $\operatorname{spec}(D\!\!\!/^f) \subset \mathbb{R}$ *is discrete.*

Proof. The boundedness of $R(\mu, D\!\!\!/^f)\colon \Gamma_{L^2} \to \Gamma_{W_2^1}$ directly follows from Lemma 1.2.3. Since $\Gamma_{W_2^1} \hookrightarrow \Gamma_{L^2}$ is compact we get that $R(\mu, D\!\!\!/^f)\colon \Gamma_{L^2} \to \Gamma_{L^2}$ is compact. From Proposition 1.A.9 we get $\sigma(D\!\!\!/^f) \subset \mathbb{R}$. In particular $\rho(D\!\!\!/^f) \neq \varnothing$. Hence $D\!\!\!/^f$ has compact resolvent. Now we can apply Corollary 1.A.7. □

1.2.2 Quaternionic structures on spinor bundles

In this section we collect and recall some facts about quaternionic structures on spinor bundles.

We start by recalling the definition of a quaternionic structure on a complex vector space (see e.g. [15, p. 29]).

Definition 1.2.5 (Quaternionic structure)**.** Let V be a (not necessarily finite dimensional) $\mathbb{C}$-vector space. A *quaternionic structure on* V is a $\mathbb{R}$-linear map $j\colon V \to V$ s.t. $j^2 = -id_V$ and $j(iv) = -ij(v)$ for all $v \in V$.

Let $j\colon V \to V$ be a quaternionic structure on V. Then j induces the structure of a quaternionic vector space[7] on V as follows. First we view $\mathbb{H} = \mathbb{C}^2$ as a 2-dimensional $\mathbb{C}$-vector space with the basis $(\mathbf{1}, \mathbf{j})$ where $\mathbf{1} := (1, 0)$ and $\mathbf{j} := (0, 1)$. Then V turns into a quaternionic vector space by defining

$$vh := xv + yj(v)$$

for all $h = \mathbf{1}x + \mathbf{j}y \in \mathbb{H}$, $x, y \in \mathbb{C}$, and $v \in V$.

In this section let $m \equiv 2, 3, 4 \pmod 8$. Let $\rho\colon \mathbb{C}l_m \to \mathrm{End}_{\mathbb{C}}(\Sigma_m)$ be an irreducible complex algebra representation of $\mathbb{C}l_m$. (Recall that $\mathbb{C}l_m$ is the Clifford algebra of $\mathbb{C}^m$ with inner product given by the complex bilinear extension of the standard inner product of $\mathbb{R}^m$.) By [15, p. 31] and [17, Theorem 2.2.2.] there exists a quaternionic structure $j\colon \Sigma_m \to \Sigma_m$ on Σ_m s.t.

$$j \circ \rho(x) = \rho(x) \circ j \tag{1.2.2}$$

for all $x \in \mathbb{R}^m \subset \mathbb{C}^m \subset \mathbb{C}l_m$.[8]

For the remainder of the section let M be an m-dimensional closed Riemannian spin manifold with spin structure $\mathrm{Spin}(M)$. Then every fiber of the (complex) spinor bundle $\Sigma M = \mathrm{Spin}(M) \times_\rho \Sigma_m$ turns into a quaternionic vector space by defining

$$[p, v]h := [p, vh]$$

for all $p \in \mathrm{Spin}(M)$, $v \in \Sigma_m$, and $h \in \mathbb{H}$. Note that this is well-defined because of (1.2.2). Moreover, given a manifold N and $f \in C^1(M, N)$, every fiber of $\Sigma M \otimes_{\mathbb{R}} f^*TN$ turns into a quaternionic vector space by defining

$$(a \otimes b)h := (ah) \otimes b$$

[7] We say that a set W is a "quaternionic vector space" if W is a right $\mathbb{H}$-module.

[8] The existence of such a quaternionic structure is independent of the choice of ρ. To be more precise, let $\tilde{\rho}\colon \mathbb{C}l_m \to \mathrm{End}_{\mathbb{C}}(\tilde{\Sigma}_m)$ be another irreducible complex algebra representation of $\mathbb{C}l_m$. Then ρ and $\tilde{\rho}$ are equivalent, i.e., there exists an isomorphism of $\mathbb{C}$-vector spaces $f\colon \Sigma_m \to \tilde{\Sigma}_m$ s.t. $f \circ \rho(x) \circ f^{-1} = \tilde{\rho}(x)$ for all $x \in \Sigma_m$. Then $\tilde{j} := f \circ j \circ f^{-1}$ is a quaternionic structure on $\tilde{\Sigma}_m$ with $\tilde{j} \circ \tilde{\rho}(x) = \tilde{\rho}(x) \circ \tilde{j}$ for all $x \in \mathbb{R}^m$.

for all $a \in \Sigma_x M$, $b \in (f^*TN)_x$, and $h \in \mathbb{H}$. In particular the $\mathbb{C}$-vector spaces $\Gamma_{C^0}(\Sigma M)$ and $\Gamma_{C^0}(\Sigma M \otimes_{\mathbb{R}} f^*TN)$ are quaternionic vector spaces.

Proposition 1.2.6. *It holds that*

$$\not{D}(\varphi h) = (\not{D}\varphi)h$$

for all $\varphi \in \Gamma_{C^1}(\Sigma M)$ and all $h \in \mathbb{H}$. In particular all the eigenspaces of $\not{D}$ are quaternionic vector spaces.

Proof. Let $X = [p, v] \in T_x M$, $\sigma = [p, w] \in \Sigma_x M$, and $h = x\mathbf{1} + y\mathbf{j} \in \mathbb{H}$ (recall that TM is isomorphic to $\mathrm{Spin}(M) \times_{\tau\circ\theta} \mathbb{R}^m$ where $\theta\colon \mathrm{Spin}(m) \to \mathrm{SO}(m, \mathbb{R})$ is the connected twofold covering of $\mathrm{SO}(m, \mathbb{R})$ and τ is the standard representation of $\mathrm{SO}(m, \mathbb{R})$ on $\mathbb{R}^m$). Then we have

$$\begin{aligned}
(X \cdot \sigma)h &= [p, \rho(v)(w)]h \\
&= [p, \rho(v)(w)h] \\
&= [p, x\rho(v)(w) + yj(\rho(v)(w))] \\
&= [p, x\rho(v)(w) + y\rho(v)(j(w))] \\
&= [p, \rho(v)(xw + yj(w))] \\
&= [p, \rho(v)(wh)] \\
&= X \cdot (\sigma h).
\end{aligned}$$

Moreover, a short calculation shows that if $g\colon U \to \Sigma_m$, $U \subset M$ open, is a smooth function we have that

$$d(fh)_p X = (df_p X)h$$

for all $p \in U$, $X \in T_p M$, and $h \in \mathbb{H}$. Using the local formula for the spinorial Levi-Civita connection $\nabla^{\Sigma M}$ on ΣM this yields

$$\nabla^{\Sigma M}_X(\varphi h) = (\nabla^{\Sigma M}_X \varphi)h$$

for all $\varphi \in \Gamma_{C^1}(\Sigma M)$, $X \in TM$, $h \in \mathbb{H}$. Then we get from the local formula for the Dirac operator $\not{D}$ that

$$\not{D}(\varphi h) = (\not{D}\varphi)h$$

for all $\varphi \in \Gamma_{C^1}(\Sigma M)$ and all $h \in \mathbb{H}$. □

Corollary 1.2.7. *Let N be a closed Riemannian manifold and $f \in C^1(M, N)$. Then it holds that*

$$\not{D}^f(\varphi h) = (\not{D}^f\varphi)h$$

for all $\varphi \in \Gamma_{C^1}(\Sigma M \otimes_{\mathbb{R}} f^*TN)$ and all $h \in \mathbb{H}$. In particular all the eigenspaces of $\not{D}^f$ are quaternionic vector spaces.

Proof. This follows directly from Proposition 1.2.6 and the local formula for $\not{D}^f$. □

Lemma 1.2.8. *Denote the natural hermitian inner product on ΣM by $\langle .,. \rangle$ and write $(.,.) := \mathrm{Re}\langle .,. \rangle$ for the real part of $\langle .,. \rangle$. Then $(.,.)$ is invariant under multiplication by unit quaternions, i.e., it holds that*

$$(\varphi_1 h, \varphi_2 h) = (\varphi_1, \varphi_2)$$

for all $\varphi_1, \varphi_2 \in \Sigma_x M$, $x \in M$, $h \in S^3 \subset \mathbb{C}^2 = \mathbb{H}$.

Proof. First we recall the following: any hermitian inner product on Σ_m induces a hermitian inner product $\langle .,. \rangle_{\Sigma_m}$ on Σ_m with

$$\langle \rho(v)(x), y \rangle_{\Sigma_m} = -\langle x, \rho(v)(y) \rangle_{\Sigma_m} \tag{1.2.3}$$

for all $v \in \mathbb{R}^m$ (see [18]). Up to rescaling by positive constants there exists exactly one hermitian inner product on Σ_m s.t. (1.2.3) holds. Then the natural hermitian inner product $\langle .,. \rangle$ on ΣM is defined by

$$\langle \varphi_1, \varphi_2 \rangle := \langle w_1, w_2 \rangle_{\Sigma_m}$$

for all $\varphi_1 = [p, w_1], \varphi_2 = [p, w_2] \in \Sigma M$.

Now let g be an arbitrary hermitian inner product on Σ_m. We define the hermitian inner product $\tilde{g}$ on Σ_m by

$$\tilde{g}(x, y) := g(x, y) + \overline{g(j(x), j(y))}$$

for all $x, y \in \Sigma_m$. Then, as mentioned above, $\tilde{g}$ induces a hermitian inner product $\langle .,. \rangle_{\Sigma_m}$ on Σ_m with (1.2.3). From $\tilde{g}(j(x), j(y)) = \overline{\tilde{g}(x, y)}$ for all $x, y \in \Sigma_m$ it follows that

$$\langle j(x), j(y) \rangle_{\Sigma_m} = \overline{\langle x, y \rangle_{\Sigma_m}}$$

$x, y \in \Sigma_m$. In particular

$$\mathrm{Re}\langle j(x), j(y) \rangle_{\Sigma_m} = \mathrm{Re}\langle x, y \rangle_{\Sigma_m} \tag{1.2.4}$$

for all $x, y \in \Sigma_m$.

Let $\varphi_1 = [p, w_1], \varphi_2 = [p, w_2] \in \Sigma M$ and $h \in S^3 \subset \mathbb{C}^2 = \mathbb{H}$, i.e., $h = x\mathbf{1} + y\mathbf{j}$, $|x|^2 + |y|^2 = 1$. Then it holds that

$$\langle \varphi_1 h, \varphi_2 h \rangle = \langle w_1 h, w_2 h \rangle_{\Sigma_m} = x\overline{x}\langle w_1, w_2 \rangle_{\Sigma_m} + y\overline{y}\langle j(w_1), j(w_2) \rangle_{\Sigma_m}$$

With (1.2.4) we deduce

$$\begin{aligned}
(\varphi_1 h, \varphi_2 h) &= \mathrm{Re}\langle \varphi_1 h, \varphi_2 h \rangle \\
&= x\overline{x}\mathrm{Re}\langle w_1, w_2 \rangle_{\Sigma_m} + y\overline{y}\mathrm{Re}\langle j(w_1), j(w_2) \rangle_{\Sigma_m} \\
&= x\overline{x}\mathrm{Re}\langle w_1, w_2 \rangle_{\Sigma_m} + y\overline{y}\mathrm{Re}\langle w_1, w_2 \rangle_{\Sigma_m} \\
&= (x\overline{x} + y\overline{y})\mathrm{Re}\langle w_1, w_2 \rangle_{\Sigma_m} \\
&= \mathrm{Re}\langle w_1, w_2 \rangle_{\Sigma_m} \\
&= (\varphi_1, \varphi_2).
\end{aligned}$$

□

1.2.3 The heat kernel of a closed manifold

In this section we recall the definition and some basic properties of the heat kernel of a closed Riemannian manifold.

Definition 1.2.9. Let (M, g) be a closed Riemannian manifold. A function $p \in C^0(M \times M \times (0, \infty))$ is called *a heat kernel of M* if

i) $p(., y, t) \in C^2(M)$ for all $(y, t) \in M \times (0, \infty)$,

ii) $p(x, y, .) \in C^1((0, \infty))$ for all $x, y \in M$,

iii) $\partial_t p - \Delta_x p = 0$ on $M \times M \times (0, \infty)$,

iv) $\lim\limits_{t \to 0^+} p(., y, t) = \delta_y$ for all $y \in M$ where the limit is to be understood in the distributional sense, i.e., $\lim\limits_{t \to 0^+} \int_M p(x, y, t) f(x)\, dV(x) = f(y)$ for all $y \in M$ and all $f \in C^0(M)$. (We write dV for integration w.r.t. (M, g).)

Lemma 1.2.10. *Let (M, g) be a closed Riemannian manifold.*

i) There exists a unique heat kernel of M, called the heat kernel of M.

ii) It holds that $p \geq 0$ on $M \times M \times (0, \infty)$.[9]

iii) We have $p \in C^\infty(M \times M \times (0, \infty))$.

iv) For all $x, y \in M$ and $t \in (0, \infty)$ it holds that $p(x, y, t) = p(y, x, t)$.

v) For every $f \in C^0(M)$ it holds that

$$\int_M p(., y, t) f(y)\, dV(y) \to f$$

in $C^0(M)$ for $t \to 0^+$.

vi) For all $(x, t) \in M \times (0, \infty)$ it holds that

$$\int_M p(x, y, t)\, dV(y) = 1$$

vii) There exists $C > 0$ s.t.

$$\int_0^t \int_M |\nabla_x p(x, y, s)|\, dV(y) ds \leq C\sqrt{t}$$

for all $(x, t) \in M \times [0, 1]$ where ∇_x denotes the gradient w.r.t. the first variable.

[9] It even holds that $p > 0$ on $M \times M \times (0, \infty)$ but for our purposes non-negativity of p is sufficient.

Proof. The proofs of i)-vi) can be found in [7, Chapter VI]. To show vii) we briefly recall the construction of p given in [7, Chapter VI.4]. Since M is compact, we have $\varepsilon := \mathrm{inj}(M) > 0$. We choose $\rho \in C^\infty([0,\infty])$ with $0 \le \rho \le 1$, $\rho|_{[0,\frac{\varepsilon}{4}]} = 1$, and $\rho|_{[\frac{\varepsilon}{2},\infty]} = 0$. Then we define $\eta \in C^\infty(M \times M)$ by

$$\eta(x,y) := \rho(d(x,y))$$

where d denotes the metric on M induced by g. Moreover, let

$$\mathscr{E}(x,y,t) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x,y)}{4t}},$$

for all $(x,y,t) \in M \times M \times (0,\infty)$ where $n := \dim(M)$. For each $y \in M$ we define a sequence of functions $u_j(.,y)\colon B_\varepsilon(y) \to \mathbb{R}$, $j \in \mathbb{N}$, by

i) $u_0(y,y) = 0$,

ii) $\mathcal{H}_k(x,y,t) := \mathscr{E}(x,y,t) \sum_{j=0}^k t^j u_j(x,y)$ satisfies

$$-\partial_t \mathcal{H}_k + \Delta_x \mathcal{H}_k = \mathscr{E} t^k \Delta_x u_k$$

for all $k \in \mathbb{N}$.

It holds that u_j is smooth on $\{(x,y) \in M \times M \mid d(x,y) < \varepsilon\}$ for all j. Furthermore, we define

$$H_k\colon M \times M \times (0,\infty) \to \mathbb{R}$$

by $H_k := \eta \mathcal{H}_k$. For $k > \frac{n}{2}$ we have $H_k \in C^\infty(M \times M \times (0,\infty))$. In the following, the convolution $F * G$ of F and G is defined by

$$(F * G)(x,y,z) := \int_0^t \int_M F(x,z,\tau) G(z,y,t-\tau) dV(z) d\tau.$$

For $k > \frac{n}{2} + 2$

$$p(x,y,t) := H_k(x,y,t) + (H_k * F_k)(x,y,t)$$

is the heat kernel of M for an appropriate F_k. In the following, we will only need that $F_k \in C^0(M \times M \times [0,\infty))$. However, we remark that

$$F_k = \sum_{l=1}^\infty (\Delta_x H_k - \partial_t H_k)^{*l}$$

where $*l$ denotes the l-fold convolution.

We have that

$$\nabla_x p = \nabla_x H_k + (\nabla_x H_k) * F_k.$$

(Note that in this equation we view the integral $(\nabla_x H_k) * F_k$ as a $T_x M$-valued integral.) Therefore we need to estimate

$$I := \int_0^t \int_M |\nabla_x H_k(x,y,s)|\, dV(y) ds$$

and

$$II := \int_0^t \int_M |((\nabla_x H_k) * F_k)(x,y,s)|\, dV(y)ds$$

for all $(x,t) \in M \times [0,1]$. We have

$$\begin{aligned} I &= \int_0^t \int_M |\nabla_x H_k(x,y,s)|\, dV(y)ds \\ &= \int_0^t \int_M |\nabla_x(\eta \mathscr{E} \sum_{j=0}^k s^j u_j)|(x,y,s)\, dV(y)ds \\ &= I_1 + I_2 + I_3 \end{aligned}$$

with

$$\begin{aligned} I_1 &= \int_0^t \int_M |\eta \mathscr{E} \sum_{j=0}^k s^j \nabla_x u_j|(x,y,s)\, dV(y)ds, \\ I_2 &= \int_0^t \int_M |(\nabla_x \eta)\mathscr{E} \sum_{j=0}^k s^j u_j|(x,y,s)\, dV(y)ds, \\ I_3 &= \int_0^t \int_M |\eta(\nabla_x \mathscr{E}) \sum_{j=0}^k s^j u_j|(x,y,s)\, dV(y)ds. \end{aligned}$$

Since η vanishes outside of $\overline{B_{\frac{\varepsilon}{2}}(x)} \subset M$ and the u_j are smooth on $\{(x,y) \in M \times M \mid d(x,y) < \varepsilon\}$ it holds that

$$I_1(x,t) \le C_1 \int_0^t \int_{B_{\frac{\varepsilon}{2}(x)}} (4\pi s)^{-\frac{n}{2}} e^{-\frac{d^2(x,y)}{4s}}\, dV(y)ds$$

for all $(x,t) \in M \times [0,1]$. Using normal coordinates centered at $x \in M$ we calculate further

$$\begin{aligned} I_1(x,t) &\le C_1 \int_0^t \int_{B_{\frac{\varepsilon}{2}(x)}} (4\pi s)^{-\frac{n}{2}} e^{-\frac{d^2(x,y)}{4s}}\, dV(y)ds \\ &\le C_2 \int_0^t \underbrace{\int_{B_{\frac{\varepsilon}{2}(0)}} (4\pi s)^{-\frac{n}{2}} e^{-\frac{\|z\|_2^2}{4s}}\, dz}_{=1}\, ds \\ &= C_2 t \\ &\le C_2 \sqrt{t} \end{aligned}$$

for all $(x,t) \in M \times [0,1]$. Analogously we have

$$I_2(x,t) \le C_3 \sqrt{t}$$

for all $(x,t) \in M \times [0,1]$. Moreover,

$$I_3 \leq C_4 \int_0^t \int_{B_{\frac{\varepsilon}{2}}(x)} |\nabla_x \mathscr{E}|(x,y,s)\, dV(y) ds.$$

Using normal coordinates centered in $x \in M$ together with $|\nabla f|^2(x) = \sum_{i=1}^n (\frac{\partial f}{\partial x_i})^2$ and

$$\partial_{z_i}((4\pi s)^{-\frac{n}{2}} e^{-\frac{\|z\|_2^2}{4s}}) = -(4\pi s)^{-\frac{n}{2}} \frac{z_i}{2s} e^{-\frac{\|z\|_2^2}{4s}}$$

we get

$$\begin{aligned} I_3 &\leq C_4 \int_0^t \int_{B_{\frac{\varepsilon}{2}}(x)} |\nabla_x \mathscr{E}|(x,y,s)\, dV(y) ds \\ &\leq C_5 \int_0^t \sum_{i=1}^n \int_{B_{\frac{\varepsilon}{2}}(0)} (4\pi s)^{-\frac{n}{2}} \frac{|z_i|}{2s} e^{-\frac{\|z\|_2^2}{4s}}\, dz ds \\ &\leq C_6 \int_0^t (4\pi s)^{-\frac{n}{2}} \frac{1}{2s} \int_{\mathbb{R}^n} \|z\|_2 e^{-\frac{\|z\|_2^2}{4s}}\, dz ds \end{aligned}$$

Since[10]

$$\int_{\mathbb{R}^n} \|z\|_2 e^{-\frac{\|z\|_2^2}{4s}}\, dz \leq C_7 s^{\frac{n}{2}+\frac{1}{2}}$$

we conclude

$$\begin{aligned} I_3(x,t) &\leq C_8 \int_0^t s^{-\frac{n}{2}-1+\frac{n}{2}+\frac{1}{2}}\, ds \\ &= C_8 \int_0^t s^{-\frac{1}{2}}\, ds \\ &= 2C_8\sqrt{t} \end{aligned}$$

for all $(x,t) \in M \times [0,1]$. In summary we have shown

$$I(x,t) \leq C_9\sqrt{t}$$

[10] Recall that using spherical coordinates we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \|z\|_2 e^{-\frac{\|z\|_2^2}{4s}}\, dz \\ &= \int_0^\infty \int_0^{2\pi} \cdots \int_0^{2\pi} r e^{-\frac{r^2}{4s}} r^{n-1} \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \cdot \ldots \cdot \sin(\phi_{n-2})\, d\phi_1 \ldots d\phi_{n-1} dr \end{aligned}$$

and that for each $a > 0$ it holds that $\int_0^\infty r^n e^{-ar^2}\, dr = \frac{1}{2}\Gamma(\frac{n+1}{2}) a^{-\frac{n}{2}-\frac{1}{2}}$ where $\Gamma(\frac{n+1}{2}) = \int_0^\infty r^{\frac{n+1}{2}-1} e^{-r}\, dr$.

for all $(x,t) \in M \times [0,1]$. It remains to estimate II. Since $F_k \in C^0(M \times M \times [0,\infty))$, there exists $C_{10} > 0$ s.t. $|F_k(z,y,t)| \leq C_{10}$ for all $(z,y,t) \in M \times M \times [0,1]$. This implies that for every $(x,y,s) \in M \times M \times [0,1]$ we have

$$\begin{aligned} |((\nabla_x H_k) * F_k)(x,y,s)| &\leq C_{10} \int_0^s \int_M |\nabla_x H_k|(x,z,\tau)\, dV(z) d\tau \\ &\leq C_{11} \sqrt{s}. \end{aligned}$$

Hence

$$\begin{aligned} II(x,t) &= \int_0^t \int_M |((\nabla_x H_k) * F_k)(x,y,s)|\, dV(y) ds \\ &\leq C_{11} \int_0^t \sqrt{s}\, ds \\ &= C_{11} t^{\frac{3}{2}} \\ &\leq C_{11} \sqrt{t} \end{aligned}$$

for all $(x,t) \in M \times [0,1]$. $\square$

1.3 Setup for the contraction argument

In this section the setup for the contraction argument is developed. After we have stated the precise setting, we will take care of the constraint equation (1.1.4) in Section 1.4.

1.3.1 Translation of equation (1.1.3) into $\mathbb{R}^q$

In the following we will use that for $\mathcal{R}$ in (1.1.3) it holds that

$$\begin{aligned}\mathcal{R}(u,\psi) &= \frac{1}{2}(\psi^i, e_\alpha\cdot\psi^j)R^{TN}(\frac{\partial}{\partial x_i}\circ u, \frac{\partial}{\partial x_j}\circ u)(d(x^k\circ u)(e_\alpha))(\frac{\partial}{\partial x_k}\circ u)\\ &= \frac{1}{2}(\psi^i, (d(x^k\circ u)(e_\alpha)e_\alpha)\cdot\psi^j)(R_{ijk}\circ u)\\ &= \frac{1}{2}(\psi^i, \nabla(x^k\circ u)\cdot\psi^j)(R_{ijk}\circ u)\end{aligned}$$

where ∇ denotes the gradient and we write x for the chart on N that induces the local coordinates $(\frac{\partial}{\partial x_i})$. Moreover, recall that $(.,.)$ satisfies

$$(\varphi, X\cdot\sigma) = -(X\cdot\varphi,\sigma) \tag{1.3.1}$$

for all $X\in T_pM$, $\varphi,\sigma\in\Sigma_pM$, $p\in M$. In the following we denote the Riemannian metric on M by g.

Let $i\colon N\to\mathbb{R}^q$ be an isometric embedding of N in $\mathbb{R}^q$. In the following we view N as an embedded Riemannian submanifold of $\mathbb{R}^q$ via i and we rewrite the heat flow for Dirac-harmonic maps as an equation in $\mathbb{R}^q$. We choose $\delta>0$ s.t. the set

$$N_\delta := \{y\in\mathbb{R}^q \mid d(y,N)<\delta\}$$

is a tubular neighborhood of N in $\mathbb{R}^q$ and there exists a smooth map

$$\pi\colon N_\delta\to N$$

s.t.

i) we have $d\pi_x v = pr_{T_xN}v$ for all $x\in N$, $v\in\mathbb{R}^q$,

ii) for every $y\in N_\delta$ it holds that $\pi(y)$ is the unique point of N closest to y,

iii) $\pi\colon N_\delta\to N$ can be extended to a smooth map $\pi\colon\mathbb{R}^q\to\mathbb{R}^q$ with compact support.

Because of ii) we call π *nearest point projection.* Note that iii) can always be achieved by choosing δ smaller.

For $A, B \in \{1, \ldots, q\}$ and $z \in \mathbb{R}^q$ we write

$$\pi^A_B(z) := \frac{\partial \pi^A}{\partial z_B}(z)$$

for the B-th partial derivative of the A-th component function of $\pi\colon \mathbb{R}^q \to \mathbb{R}^q$. Similarly,

$$\pi^A_{BC}(z) := \frac{\partial^2 \pi^A}{\partial z_B \partial z_C}(z),$$

for $z \in \mathbb{R}^q$. Moreover, we write $(\partial_A)_{A=1,\ldots,q}$ for the standard basis of $T\mathbb{R}^q$, i.e., under the identification $T_z\mathbb{R}^q = \mathbb{R}^q$, $z \in \mathbb{R}^q$, we have $\partial_A(z) = e_A$ where e_A is the A-th standard basis vector of $\mathbb{R}^q$. For every $x \in N$ we have

$$\pi^A_B(x) = (d\pi^A)_x(\partial_B) = ((d\pi)_x(\partial_B))^A = (pr_{T_xN}(\partial_B))^A,$$

i.e., $\pi^A_B(x)$ is the A-th component of the projection of ∂_B onto T_xN.

For $A, B \in \{1, \ldots, q\}$ we define

$$\nu^A_B\colon \mathbb{R}^q \to \mathbb{R}$$

by $\nu^A_B := \delta^A_B - \pi^A_B$. Note that $\nu^A_B(x)$ is the A-th component of $pr_{T_x^\perp N}(\partial_B)$ for every $x \in N$.

Now we define

$$\nu\colon \mathbb{R}^q \to \mathbb{R}^{q\times q}$$

by $\nu(z) := (\nu^A_B(z))_{A,B=1,\ldots,q}$ for $z \in \mathbb{R}^q$.

For $u \in C^1(M, N)$ we have $d(\nu \circ u)\colon TM \to \mathbb{R}^{q\times q}$ and define

$$\Omega_p(X) := [\nu(u(p)), d(\nu \circ u)_p(X)]$$

for $p \in M$, $X \in T_pM$, where $[.,.]$ denotes the commutator of $q \times q$-matrices. Finally,

$$(\Omega^A_B)_p(X) := (\Omega_p(X))^A_B$$

denotes the (A, B)-th component of $\Omega_p(X)$.

Lemma 1.3.1. *Let $k, l \in \mathbb{N}$, $0 \le l \le k$. Let $u \in C^k(M, N)$ and $\psi \in \Gamma_{C^l}(\Sigma M \otimes u^*TN)$. For each $p \in M$ we have*

$$\psi(p) \in \Sigma_pM \otimes T_{u(p)}N \subset \Sigma_pM \otimes T_{u(p)}\mathbb{R}^q.$$

There exist uniquely determined spinors $\psi^A \in \Gamma_{C^l}(\Sigma M)$, $A = 1, \ldots, q$, s.t.

$$\psi = \psi^A \otimes (\partial_A \circ u)$$

on M. Moreover, the following holds: for each $v \in T^\perp_{u(p)}N$, $v = v^A\partial_A$, we have

$$v_A\psi^A(p) = 0.$$

In particular,

$$\nu_B^A(u)\psi^B = 0 \tag{1.3.2}$$

on M for $A = 1, \dots, q$.

Proof. We first show uniqueness. Let $p \in M$ and assume we have

$$\psi(p) = \varphi^A \otimes (\partial_A \circ u)(p) = \sigma^A \otimes (\partial_A \circ u)(p)$$

for some $\varphi^A, \sigma^A \in \Sigma_p M$. We show $\varphi^A = \sigma^A$ for each $A = 1, \dots, q$. To that end, choose a basis (b_i) of $\Sigma_p M$ and write

$$\varphi^A - \sigma^A = \lambda^{iA} b_i.$$

Therefore, we have

$$0 = \lambda^{iA} b_i \otimes (\partial_A(u(p))).$$

Since the $b_i \otimes (\partial_A(u(p)))$ form a basis of $\Sigma_p M \otimes T_{u(p)}\mathbb{R}^q$ we have

$$\lambda^{iA} = 0$$

for all i, A and therefore $\varphi^A = \sigma^A$ for all $A = 1, \dots, q$.

Now choose a local frame (s_i) of ΣM, defined on some open subset $U \subset M$. Then there exist C^l-functions $f^{iA} \colon U \to \mathbb{R}$ s.t.

$$\psi = f^{iA} s_i \otimes (\partial_A \circ u)$$

on U. We set

$$\psi^A|_U := f^{iA} s_i.$$

Together with the uniqueness, this shows the existence.

Let $v = v^A \partial_A \in T^\perp_{u(p)} N$. Let e_i be an orthonormal basis of $T_{u(p)}N$ and write

$$\psi(p) = \sigma^i \otimes e_i$$

Since $e_i = \langle e_i, \partial_A \rangle \partial_A$ we have

$$\psi(p) = (\sigma^i \langle e_i, \partial_A \rangle) \otimes (\partial_A(u(p))).$$

Thus,

$$\psi^A(p) = \sigma^i \langle e_i, \partial_A \rangle$$

and therefore

$$v_A \psi^A(p) = v_A \sigma^i \langle e_i, \partial_A \rangle = \langle v, \partial_A \rangle \langle e_i, \partial_A \rangle \sigma^i = \langle v, e_i \rangle \sigma^i = 0.$$

$\square$

For $A, G, D, F \in \{1, \ldots, q\}$ we define $R^A_{GDF}\colon \mathbb{R}^q \to \mathbb{R}$ by

$$R^A_{GDF} := \pi^A_B \pi^C_{BD} \pi^G_E \pi^C_{EF} - \pi^G_B \pi^C_{BD} \pi^A_E \pi^C_{EF}.$$

For $u \in C^1(M, N)$ and $\psi \in \Gamma_{C^1}(\Sigma M \otimes u^*TN)$, $\psi = \psi^A \otimes (\partial_A \circ u)$, $\psi^A \in \Gamma_{C^1}(\Sigma M)$, (e_α) an orthonormal basis of T_pM, and $X \in T_pM$ we define

$$(\tilde{\Omega}^A_G)_p X := \left(\frac{1}{2} R^A_{GDF}(u)(\psi^D, e_\alpha \cdot \psi^F) e^*_\alpha(X)\right)\Big|_p.$$

In [10] the following lemma was shown by deriving the Euler Lagrange equations of (1.1.1) in the setting provided by the tubular neighborhood.

Lemma 1.3.2. *A tuple (u, ψ) where $u\colon [0, T] \times M \to N$ and $\psi \in \Gamma(pr_2^* \Sigma M \otimes u^*TN)$ is a solution of the heat flow for Dirac harmonic maps* (1.1.3)-(1.1.4) *if and only if it is a solution of*

$$\begin{cases} \partial_t u^A - \Delta u^A = \langle \Omega^A_B, du^B \rangle - \langle \tilde{\Omega}^A_B, du^B \rangle & \text{on } (0, T) \times M,\ A = 1, \ldots, q, \\ \not{D} \psi^A + (\Omega^A_B)^\sharp \cdot \psi^B = 0 & \text{on } [0, T] \times M,\ A = 1, \ldots, q, \end{cases} \tag{1.3.3}$$

where we write $u^B\colon M \to \mathbb{R}$ for the B-th component function of $u\colon M \to N \subset \mathbb{R}^q$

In the following we show the equivalence of (1.1.3) and the first equation of (1.3.3) by direct calculations, providing and alternative proof of the above lemma. (The equivalence of (1.1.4) and the second equation of (1.3.3) can also be shown by direct calculations, but we don't need this equivalence.)

Proposition 1.3.3. *We have*

$$\pi^A_B(x) = \pi^B_A(x) \tag{1.3.4}$$

and

$$\pi^A_C(x) \pi^C_B(x) = \pi^A_B(x) \tag{1.3.5}$$

for all $x \in N$, $A, B \in 1, \ldots, q$. Moreover, given $u \in C^1(M, N)$, we have

$$\pi^A_B(u(p))(du^B)_p(X) = (du^A)_p X \tag{1.3.6}$$

for all $A \in 1, \ldots, q$, $p \in M$, and $X \in T_pM$.

Proof. Let $x \in N$ and $A, B \in 1, \ldots, q$. Let (u_i) be an orthonormal basis of T_xN. Since pr_{T_xN} is an orthogonal projection, we have

$$pr_{T_xN}(\partial_B) = \sum_i \langle u_i, \partial_B \rangle u_i$$

where $\langle .,. \rangle$ is the euclidean metric. Since $u_i = \sum_A \langle u_i, \partial_A \rangle \partial_A$ it follows directly that
$$\pi_B^A(x) = \sum_i \langle u_i, \partial_B \rangle \langle u_i, \partial_A \rangle.$$
In particular, $\pi_B^A(x) = \pi_A^B(x)$. Moreover,
$$\begin{aligned}
\pi_C^A(x)\pi_B^C(x) &= \sum_C \sum_{i,j} \langle u_i, \partial_C \rangle \langle u_i, \partial_A \rangle \langle u_j, \partial_B \rangle \langle u_j, \partial_C \rangle \\
&= \sum_{i,j} \langle u_i, u_j \rangle \langle u_i, \partial_A \rangle \langle u_j, \partial_B \rangle \\
&= \sum_{i,j} \delta_j^i \langle u_i, \partial_A \rangle \langle u_j, \partial_B \rangle \\
&= \sum_i \langle u_i, \partial_B \rangle \langle u_i, \partial_A \rangle \\
&= \pi_B^A(x).
\end{aligned}$$
Finally, we calculate (everything is evaluated at p)
$$\begin{aligned}
\pi_B^A(u) du^B(X) &= du^B(X) \Big(pr_{TN}(\partial_B(u)) \Big)^A \\
&= \Big(pr_{TN}(du^B(X)\partial_B(u)) \Big)^A \\
&= \Big(pr_{TN}(du(X)) \Big)^A \\
&= (du(X))^A \\
&= du^A(X).
\end{aligned}$$
□

Proposition 1.3.4. *Let $u \in C^1(M,N)$. Then it holds that*
$$\nu_B^A(u)\nabla(u^B) = 0$$
on M for every $A \in \{1, \ldots, q\}$.

Proof. We have $\nabla u = d(u^B)(e_\alpha)e_\alpha$ for an orthonormal basis (e_α) of M. If we have an arbitrary $w = w^B \partial_B \in \mathbb{R}^q$, it holds that
$$pr_{T^\perp N}(w) = \nu_B^A w^B \partial_A.$$
Since u takes values in N, we have $du^B(e_\alpha)\partial_B \in TN$. It follows that
$$0 = pr_{T^\perp N}((du^B)(e_\alpha)\partial_B) = \nu_B^A(u) du^B(e_\alpha)\partial_A.$$
In particular,
$$0 = \nu_B^A(u) du^B(e_\alpha)$$

for $A = 1, \dots, q$. Therefore,

$$0 = \nu^A_B(u) du^B(e_\alpha) e_\alpha = \nu^A_B(u) \nabla(u^B).$$

□

Lemma 1.3.5. *Let $u \in C^1(M, N)$. For every $p \in M$, $X \in T_pM$ we have*

$$(\Omega^A_B)_p X = \Big(-\nu^A_C(u) \pi^C_{DB}(u) du^D(X) + \nu^C_B(u) \pi^A_{DC}(u) du^D(X) \Big)\Big|_p$$

(where $u^B \colon M \to \mathbb{R}$ is the B-th component function of $u \colon M \to N \subset \mathbb{R}^q$).

Proof. We calculate

$$\begin{aligned} d(\nu^A_B \circ u)_p X &= (d\nu^A_B)_{u(p)} du(X) \\ &= d(\delta^A_B - \pi^A_B)_{u(p)} \Big((du^C)_p(X) \partial_C \Big) \\ &= -(d\pi^A_B)_{u(p)} \Big((du^C)_p(X) \partial_C \Big) \\ &= -(du^C)_p(X) \Big((d\pi^A_B)_{u(p)} (\partial_C) \Big) \\ &= -(du^C)_p(X) \pi^A_{CB}(u(p)). \end{aligned}$$

From this the lemma easily follows. □

Lemma 1.3.6. *Let II be the second fundamental form of N in $\mathbb{R}^q$, i.e.,*

$$II(X, Y) = pr_{T^\perp N}(\nabla^{\mathbb{R}^q}_X Y)$$

for $X, Y \in TN$. If $u \in C^1(M, N)$ and (e_α) is an orthonormal basis of T_pM, then it holds that

$$II(du_p(e_\alpha), du_p(e_\alpha)) = -\langle (\Omega^A_B)_p, (du^B)_p \rangle \partial_A(u(p))$$

Proof. Let $\frac{\partial}{\partial y_i}$ be local coordinates of N. Then $\frac{\partial}{\partial y_i} = \langle \frac{\partial}{\partial y_i}, \partial_A \rangle \partial_A$ implies

$$\frac{\partial}{\partial y_i} = pr_{TN}(\frac{\partial}{\partial y_i}) = \langle \frac{\partial}{\partial y_i}, \partial_A \rangle pr_{TN}(\partial_A) = \langle \frac{\partial}{\partial y_i}, \partial_A \rangle \pi^B_A \partial_B$$

and we get

$$\begin{aligned} II(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) &= \langle \frac{\partial}{\partial y_i}, \partial_A \rangle \langle \frac{\partial}{\partial y_j}, \partial_B \rangle II(pr_{TN}(\partial_A), pr_{TN}(\partial_B)) \\ &= \langle \frac{\partial}{\partial y_i}, \partial_A \rangle \langle \frac{\partial}{\partial y_j}, \partial_B \rangle pr_{T^\perp N}(\nabla^{\mathbb{R}^q}_{\pi^C_A \partial_C} \pi^D_B \partial_D) \\ &= \langle \frac{\partial}{\partial y_i}, \partial_A \rangle \langle \frac{\partial}{\partial y_j}, \partial_B \rangle \pi^C_A pr_{T^\perp N}(\pi^D_{CB} \partial_D) \\ &= \langle \frac{\partial}{\partial y_i}, \partial_A \rangle \langle \frac{\partial}{\partial y_j}, \partial_B \rangle \pi^C_A \pi^D_{CB} \nu^E_D \partial_E \end{aligned}$$

Now let (e_α) be an orthonormal basis of T_pM. Using $du(e_\alpha) = d(y^i \circ u)(e_\alpha)\frac{\partial}{\partial y_i}$ we get

$$\begin{aligned}
&II(du_p(e_\alpha), du_p(e_\alpha)) \\
&= d(y^i \circ u)_p(e_\alpha) d(y^j \circ u)_p(e_\alpha)\Big(\langle \frac{\partial}{\partial y_i}, \partial_A\rangle \langle \frac{\partial}{\partial y_j}, \partial_B\rangle \pi^C_A \pi^D_{CB} \nu^E_D \partial_E\Big)\Big|_{u(p)}
\end{aligned}$$

Noting that $d(y^i \circ u)_p(e_\alpha)\langle \frac{\partial}{\partial y_i}(u(p)), \partial_A(u(p))\rangle = (du^A)_p(e_\alpha)$ (since $du(e_\alpha) = d(y^i \circ u)(e_\alpha)\frac{\partial}{\partial y_i} = d(y^i \circ u)(e_\alpha)\langle \frac{\partial}{\partial y_i}, \partial_A\rangle \partial_A$ and on the other hand $du(e_\alpha) = du^A(e_\alpha)\partial_A$) the above equation reads

$$II(du_p(e_\alpha), du_p(e_\alpha)) = (du^A)_p(e_\alpha)(du^B)_p(e_\alpha)\Big(\pi^C_A \pi^D_{CB} \nu^E_D \partial_E\Big)\Big|_{u(p)}$$

and because of equation (1.3.6) we have

$$II(du_p(e_\alpha), du_p(e_\alpha)) = (du^C)_p(e_\alpha)(du^B)_p(e_\alpha)\Big(\pi^D_{CB} \nu^E_D \partial_E\Big)\Big|_{u(p)}.$$

Moreover, using Lemma 1.3.5 we calculate

$$\begin{aligned}
&-\langle (\Omega^A_B)_{u(p)}, d(u^B)_p\rangle \\
&= -\langle (\Omega^A_B)_p(e_\alpha)e_\alpha, (du^B)_p(e_\beta)e_\beta\rangle \\
&= -(\Omega^A_B)_p(e_\alpha)(du^B)_p(e_\alpha) \\
&= \Big(\nu^A_C(u)\pi^C_{DB}(u) du^D(e_\alpha) du^B(e_\alpha) - \nu^C_B(u)\pi^A_{DC}(u) du^D(e_\alpha) du^B(e_\alpha)\Big)\Big|_p \\
&= \Big(\nu^A_C(u)\pi^C_{DB}(u) du^D(e_\alpha) du^B(e_\alpha)\Big)\Big|_p,
\end{aligned}$$

where in the last line we used $\nu^C_B(u) du^B(e_\alpha) = 0$ (see the proof of Lemma 1.3.4). We have shown the lemma. □

Lemma 1.3.7. *Let $u \in C^1(M,N)$ and $\psi \in \Gamma_{C^0}(\Sigma M \otimes u^*TN)$. We have $\psi = \psi^A \otimes (\partial_A \circ u)$ for $\psi^A \in \Gamma_{C^0}(\Sigma M)$ (c.f. Lemma 1.3.1) . Then it holds that*

$$\pi^A_B(u)\pi^B_{DC}(u)\nabla(u^D) \cdot \psi^C = 0 \tag{1.3.7}$$

on M.

Proof. Let $p \in M$ and let (e_α) be a local orthonormal basis of M. Equation (1.3.5) together with the product rule yields

$$\begin{aligned}
d(\pi^A_B(u))(e_\alpha) &= d(\pi^A_C(u)\pi^C_B(u))(e_\alpha) \\
&= \pi^A_C(u)\Big(d(\pi^C_B(u))(e_\alpha)\Big) + \pi^C_B(u)\Big(d(\pi^A_C(u))(e_\alpha)\Big).
\end{aligned}$$

Combining this with

$$\begin{aligned} d(\pi^A_B(u))(e_\alpha) &= d(\pi^A_B)_u\Big(du(e_\alpha)\Big) \\ &= d(\pi^A_B)_u\Big(du^C(e_\alpha)\partial_C\Big) \\ &= du^C(e_\alpha)\pi^A_{CB}(u) \end{aligned}$$

implies

$$du^C(e_\alpha)\pi^A_{CB}(u) = \pi^A_C(u)\Big(du^D(e_\alpha)\pi^C_{DB}(u)\Big) + \pi^C_B(u)\Big(du^D(e_\alpha)\pi^A_{DC}(u)\Big).$$

Multiplying with e_α and summing over α yields

$$\pi^A_{CB}(u)\nabla(u^C) = \pi^A_C(u)\pi^C_{DB}(u)\nabla(u^D) + \pi^C_B(u)\pi^A_{DC}(u)\nabla(u^D). \tag{1.3.8}$$

Therefore,

$$\begin{aligned} \pi^A_C(u)\pi^C_{DB}(u)\nabla(u^D)\cdot\psi^B = \pi^A_{CB}(u)\nabla(u^C)\cdot\psi^B \\ - \pi^C_B(u)\pi^A_{DC}(u)\nabla(u^D)\cdot\psi^B. \end{aligned}$$

It remains to show that

$$\pi^A_{CB}(u)\nabla(u^C)\cdot\psi^B - \pi^C_B(u)\pi^A_{DC}(u)\nabla(u^D)\cdot\psi^B = 0.$$

This follows from

$$\begin{aligned} \pi^C_B(u)\pi^A_{DC}(u)\nabla(u^D)\cdot\psi^B &= \pi^A_{DC}(u)\nabla(u^D)\cdot\Big(\pi^C_B(u)\psi^B\Big) \\ &= \pi^A_{DC}(u)\nabla(u^D)\cdot\Big((\delta^C_B + \nu^C_B(u))\psi^B\Big) \\ &= \pi^A_{DC}(u)\nabla(u^D)\cdot\psi^C, \end{aligned}$$

where we used equation (1.3.2). □

Proposition 1.3.8.

$$\begin{aligned} &Rm^N(pr_{TN}(\partial_A), pr_{TN}(\partial_G), pr_{TN}(\partial_D), pr_{TN}(\partial_F)) \\ &= \pi^B_{CF}\pi^C_A\pi^E_{HD}\pi^H_G\nu^B_E - \pi^B_{CD}\pi^C_A\pi^E_{HF}\pi^H_G\nu^B_E. \end{aligned}$$

Proof. In the proof of Lemma 1.3.6 we showed that

$$II(pr_{TN}(\partial_A), pr_{TN}(\partial_B)) = \pi^C_A\pi^D_{CB}\nu^E_D\partial_E.$$

If we combine this with the Gauß equation, then the proposition follows from a short computation. □

Lemma 1.3.9. *For $u \in C^1(M,N)$ and $\psi \in \Gamma_{C^0}(\Sigma M \otimes u^*TN)$ it holds that*

$$\langle (\tilde{\Omega}^A_B)_p, (du^B)_p \rangle = \Big(\pi^A_B(u)\pi^C_{BD}(u)\pi^C_{EF}(u)(\psi^D, \nabla(u^E)\cdot\psi^F) \Big)\Big|_p, \tag{1.3.9}$$

and

$$\mathcal{R}(u,\psi)(p) = \langle (\tilde{\Omega}^A_B)_p, (du^B)_p \rangle \partial_A(u(p)) \tag{1.3.10}$$

for all $p \in M$.

Proof. Let (e_α) be an orthonormal basis of T_pM. We calculate

$$\begin{aligned}
&\langle \tilde{\Omega}^A_G, du^G \rangle \\
&= \langle \tilde{\Omega}^A_G(e_\alpha)e_\alpha, du^G(e_\beta)e_\beta \rangle \\
&= \tilde{\Omega}^A_G(e_\beta)du^G(e_\beta) \\
&= \frac{1}{2}R^A_{GDF}(u)(\psi^D, e_\alpha\cdot\psi^F)du^G(e_\alpha) \\
&= \frac{1}{2}R^A_{GDF}(u)(\psi^D, \nabla(u^G)\cdot\psi^F) \\
&= \frac{1}{2}(\pi^A_B\pi^C_{BD}\pi^G_E\pi^C_{EF})_u(\psi^D, \nabla(u^G)\cdot\psi^F) \\
&\quad - \frac{1}{2}(\pi^G_B\pi^C_{BD}\pi^A_E\pi^C_{EF})|_u(\psi^D, \nabla(u^G)\cdot\psi^F) \\
&= \frac{1}{2}(\pi^A_B\pi^C_{BD}\pi^C_{EF})_u(\psi^D, \nabla(u^E)\cdot\psi^F) \\
&\quad - \frac{1}{2}(\pi^C_{BD}\pi^A_E\pi^C_{EF})|_u(\psi^D, \nabla(u^B)\cdot\psi^F) \\
&= \frac{1}{2}(\pi^A_B\pi^C_{BD}\pi^C_{EF})_u(\psi^D, \nabla(u^E)\cdot\psi^F) \\
&\quad + \frac{1}{2}(\pi^C_{BD}\pi^A_E\pi^C_{EF})|_u(\psi^F, \nabla(u^B)\cdot\psi^D) \\
&= (\pi^A_B\pi^C_{BD}\pi^C_{EF})_u(\psi^D, \nabla(u^E)\cdot\psi^F),
\end{aligned}$$

where we used $\pi^G_E(u)\nabla(u^G) = \nabla(u^E)$ (follows from equation (1.3.6)) together with equation (1.3.1) and the fact that $(.,.)$ is symmetric. We have shown equation (1.3.9). To show (1.3.10), we choose local coordinates $\frac{\partial}{\partial x_i}$ on N and write as usual $\psi = \psi^i \otimes (\frac{\partial}{\partial x_i} \circ u)$ and $\psi = \psi^A \otimes (\partial_A \circ u)$. In the following calculation we will use that since the ψ^A are uniquely determined we have $\psi^A = \psi^i\langle\frac{\partial}{\partial x_i}, \partial_A\rangle$. We have

$$
\begin{aligned}
&\mathcal{R}(u,\psi) = \frac{1}{2}(\psi^i, \nabla(x^j \circ u)\cdot\psi^m)R_{imj}(u)\\
&= \frac{1}{2}(\psi^i, \nabla(x^j \circ u)\cdot\psi^m)\langle R_{imj}(u), \partial_A(u)\rangle\partial_A(u)\\
&= \frac{1}{2}(\psi^i, \nabla(x^j \circ u)\cdot\psi^m)\langle R_{imj}(u), pr_{TN}(\partial_A(u))\rangle\partial_A(u)\\
&= \frac{1}{2}(\psi^i, \nabla(x^j \circ u)\cdot\psi^m)\langle\frac{\partial}{\partial x_i}(u), \partial_D(u)\rangle\langle\frac{\partial}{\partial x_m}(u), \partial_F(u)\rangle\langle\frac{\partial}{\partial x_j}(u), \partial_G(u)\rangle\\
&\quad Rm^N(pr_{TN}(\partial_D), pr_{TN}(\partial_F), pr_{TN}(\partial_G), pr_{TN}(\partial_A))|_u\partial_A(u)\\
&= \frac{1}{2}(\psi^D, \nabla(x^G \circ u)\cdot\psi^F)\\
&\quad Rm^N(pr_{TN}(\partial_D), pr_{TN}(\partial_F), pr_{TN}(\partial_G), pr_{TN}(\partial_A))|_u\partial_A(u)\\
&= -\frac{1}{2}(\psi^D, \nabla(x^G \circ u)\cdot\psi^F)\\
&\quad Rm^N(pr_{TN}(\partial_A), pr_{TN}(\partial_G), pr_{TN}(\partial_D), pr_{TN}(\partial_F))|_u\partial_A(u)\\
&= -\frac{1}{2}(\psi^D, \nabla(x^G \circ u)\cdot\psi^F)(\pi^B_{CF}\pi^C_A\pi^E_{HD}\pi^H_G\nu^B_E - \pi^B_{CD}\pi^C_A\pi^E_{HF}\pi^H_G\nu^B_E)|_u\partial_A(u)\\
&= -\frac{1}{2}(\psi^D, \nabla(x^G \circ u)\cdot\psi^F)(\pi^B_{CF}\pi^C_A\pi^E_{HD}\pi^H_G\nu^B_E)|_u\partial_A(u)\\
&\quad + \frac{1}{2}(\psi^D, \nabla(x^G \circ u)\cdot\psi^F)(\pi^B_{CD}\pi^C_A\pi^E_{HF}\pi^H_G\nu^B_E)|_u\partial_A(u)\\
&\overset{(1.3.1)}{=} \frac{1}{2}(\psi^F, \nabla(x^G \circ u)\cdot\psi^D)(\pi^B_{CF}\pi^C_A\pi^E_{HD}\pi^H_G\nu^B_E)|_u\partial_A(u)\\
&\qquad + \frac{1}{2}(\psi^D, \nabla(x^G \circ u)\cdot\psi^F)(\pi^B_{CD}\pi^C_A\pi^E_{HF}\pi^H_G\nu^B_E)|_u\partial_A(u)\\
&= (\psi^D, \nabla(x^G \circ u)\cdot\psi^F)(\pi^B_{CD}\pi^C_A\pi^E_{HF}\pi^H_G\nu^B_E)|_u\partial_A(u)\\
&= (\psi^D, \nabla(x^G \circ u)\cdot\psi^F)(\pi^B_{CD}\pi^C_A\pi^E_{HF}\pi^H_G\delta^B_E)|_u\partial_A(u)\\
&\quad - (\psi^D, \nabla(x^G \circ u)\cdot\psi^F)(\pi^B_{CD}\pi^C_A\pi^E_{HF}\pi^H_G\pi^B_E)|_u\partial_A(u)\\
&\overset{(1.3.6)}{=} (\psi^D, \nabla(x^H \circ u)\cdot\psi^F)(\pi^B_{CD}\pi^C_A\pi^B_{HF})|_u\partial_A(u)\\
&\qquad - (\psi^D, \nabla(x^H \circ u)\cdot\psi^F)(\pi^B_{CD}\pi^C_A\pi^E_{HF}\pi^B_E)|_u\partial_A(u)\\
&\overset{(1.3.7)}{=} (\psi^D, \nabla(x^H \circ u)\cdot\psi^F)(\pi^B_{CD}\pi^C_A\pi^B_{HF})|_u\partial_A(u).
\end{aligned}
$$

This finishes the proof of the lemma. □

Now we show the equivalence of (1.1.3) and the first equation of (1.3.3). To that end, we use the formula

$$\tau^N(u) = \tau^{\mathbb{R}^q}(u) - II(du(e_\alpha), du(e_\alpha))$$

where $\tau^N(u)$ and $\tau^{\mathbb{R}^q}(u)$ denote the tension of u regarded as a map $M \to N$ and $M \to \mathbb{R}^q$, respectively. We have $\tau^{\mathbb{R}^q}(u) = (\Delta u^A)\partial_A(u)$. Lemma 1.3.6 yields

$$\tau^N(u) = \Big(\Delta u^A + \langle \Omega^A_B, du^B\rangle\Big)\partial_A(u).$$

Combining this with equation (1.3.10) yields

$$\tau^N(u) - \mathcal{R}(u,\psi) = \Big(\Delta u^A + \langle \Omega^A_B, du^B\rangle - \langle \tilde{\Omega}^A_B, du^B\rangle\Big)\partial_A(u).$$

This is what we wanted to show.

(At this point, it is worth pointing out that our identification of N with the image of the isometric embedding $i\colon N \to \mathbb{R}^q$ does not do any harm here. To see this, we will carefully distinguish between $u\colon M \to N$ and $i \circ u\colon M \to \mathbb{R}^q$ in the following. Note that the equation for the tensions above is

$$di(\tau^N(u)) = \tau^{\mathbb{R}^q}(i \circ u) - II(du(e_\alpha), du(e_\alpha)).$$

Moreover, if we are very precise, equation (1.3.10) actually reads

$$di(\mathcal{R}(u,\psi)) = \langle \tilde{\Omega}^A_B, d(i\circ u)^B\rangle \partial_A(i \circ u).$$

Then, since di is injective, we have

$$\begin{aligned}
&\partial_t u = \tau^N(u) - \mathcal{R}(u,\psi)\\
&\Leftrightarrow di(\partial_t u) = di(\tau^N(u)) - di(\mathcal{R}(u,\psi))\\
&\Leftrightarrow \partial_t(i\circ u) = \Big(\Delta(i\circ u)^A + \langle \Omega^A_B, d(i\circ u)^B\rangle - \langle \tilde{\Omega}^A_B, d(i\circ u)^B\rangle\Big)\partial_A(i\circ u)
\end{aligned}$$

where we also use $i \circ u$ instead of u in Ω^A_B and $\tilde{\Omega}^A_B$.)

The equations (1.3.3) already tell us how the heat flow for Dirac-harmonic maps can be rewritten as a system of equations in $\mathbb{R}^q$. For our purposes, we want to slightly rewrite the first equation of (1.3.3). This is the content of the following lemma.

Lemma 1.3.10. *A tuple (u,ψ) where $u\colon [0,T]\times M \to N$ and $\psi \in \Gamma(pr_2^*\Sigma M \otimes u^*TN)$ is a solution of* (1.1.3) *if and only if it is a solution of*

$$\partial_t u^A - \Delta u^A = F_1^A(u) + F_2^A(u,\psi) \text{ on } (0,T)\times M,\ A = 1,\dots,q,$$

where

$$\begin{aligned}
F_1^A(u) &= \langle \Omega^A_B, du^B\rangle\\
&= (\nu^C_B(u)\pi^A_{DC}(u) - \nu^A_C(u)\pi^C_{DB}(u))\langle \nabla_x u^B, \nabla_x u^D\rangle_g\\
&= -\pi^A_{BC}(u)\langle \nabla_x u^B, \nabla_x u^C\rangle_g,\\
F_2^A(u,\psi) &= -\langle \tilde{\Omega}^A_B, du^B\rangle\\
&= -\pi^A_B(u)\pi^C_{BD}(u)\pi^C_{EF}(u)(\psi^D, \nabla_x u^E \cdot \psi^F).
\end{aligned}$$

Here we write $\langle.,.\rangle_g$ for the Riemannian metric on M.

Proof. We only need to verify the identity for $\langle \Omega^A_B(u), du^B \rangle$. To that end we calculate with Lemma 1.3.5

$$
\begin{aligned}
\langle \Omega^A_B, du^B \rangle &= \langle g^{\alpha\beta} \Omega^A_B(\frac{\partial}{\partial x_\beta})\frac{\partial}{\partial x_\alpha}, g^{\lambda\mu} du^B(\frac{\partial}{\partial x_\mu})\frac{\partial}{\partial x_\lambda} \rangle \\
&= g^{\alpha\beta} \Omega^A_B(\frac{\partial}{\partial x_\beta}) du^B(\frac{\partial}{\partial x_\alpha}) \\
&= g^{\alpha\beta}\Big(- \nu^A_C(u)\pi^C_{DB}(u) + \nu^C_B(u)\pi^A_{DC}(u)\Big) du^D(\frac{\partial}{\partial x_\beta}) du^B(\frac{\partial}{\partial x_\alpha}) \\
&= (\nu^C_B(u)\pi^A_{DC}(u) - \nu^A_C(u)\pi^C_{DB}(u))\langle \nabla_x u^B, \nabla_x u^D \rangle
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&(\nu^C_B(u)\pi^A_{DC}(u) - \nu^A_C(u)\pi^C_{DB}(u))\langle \nabla_x u^B, \nabla_x u^D \rangle \\
&= \Big\langle \Big((\delta^C_B - \pi^C_B(u))\pi^A_{DC}(u) - (\delta^A_C - \pi^A_C(u))\pi^C_{DB}(u) \Big) \nabla_x u^B, \nabla_x u^D \Big\rangle \\
&= \Big\langle \underbrace{\Big(\delta^C_B \pi^A_{DC}(u) - \delta^A_C \pi^C_{DB}(u) \Big)}_{=\pi^A_{DB}(u) - \pi^A_{DB}(u) = 0} \nabla_x u^B, \nabla_x u^D \Big\rangle \\
&\quad + \Big\langle \Big(- \pi^C_B(u)\pi^A_{DC}(u) + \pi^A_C(u)\pi^C_{DB}(u) \Big) \nabla_x u^B, \nabla_x u^D \Big\rangle \\
&= \Big\langle \Big(- \pi^C_B(u)\pi^A_{DC}(u) + \pi^A_C(u)\pi^C_{DB}(u) \Big) \nabla_x u^D, \nabla_x u^B \Big\rangle \\
&= -\pi^A_{BC}(u)\langle \nabla_x u^B, \nabla_x u^C \rangle_g
\end{aligned}
$$

where the last identity follows from (1.3.8). □

1.3.2 The fixed point operator and the solution space

For every $T > 0$ we denote by X_T the Banach space of bounded measurable maps $[0, T] \to C^1(M, \mathbb{R}^q)$, i.e.,

$$X_T := B([0,T]; C^1(M, \mathbb{R}^q)),$$

$$\|u\|_{X_T} := \max_{A=1,\dots,q} \sup_{t\in[0,T]} \left(\|u^A(t,.)\|_{C^0(M)} + \|\nabla u^A(t,.)\|_{C^0(M)} \right).$$

We choose and fix an initial value for the mapping part $u_0 \in C^{2+\alpha}(M, N)$ for some $0 < \alpha < 1$. Moreover, we define $v_0 \in X_T$ by

$$v_0(t,x) := \int_M p(x,y,t) u_0(y)\, dV(y)$$

where p is the heat kernel of M and denote by

$$B_R^T(v_0) := \{u \in X_T \mid \|u - v_0\|_{X_T} \le R\}$$

the closed ball with center v_0 and radius R in X_T. Then we set

$$(Lu)(t,x) := v_0(t,x) + \int_0^t \int_M p(x,y,t-\tau) \left(F_1(u_\tau)(y) + F_2(u_\tau, \psi(u_\tau))(y)\right) dV(y) d\tau.$$

Short time existence then follows from Banach's fixed point theorem after we have shown that L is a contraction on $B_R^T(v_0)$ for R and T small enough. (Of course we have to show some additional things, e.g., that the fixed point takes values in N and has the desired regularity.)

Recalling the strategy of the proof we outlined in the introduction, we first have to solve the constraint equation (1.1.4). (In fact, the $\psi(u)$ in the definition of L will be the solution of the constraint equation.) As we mentioned, we will not transform (1.1.4) to $\mathbb{R}^q$ and solve it there, we rather solve it directly in N (in particular, the maps we consider have to be N-valued). At this point we run into a technical problem, since the elements of $B_R^T(v_0)$ are $\mathbb{R}^q$-valued. We remedy this by showing that for R and T small enough, every $u \in B_R^T(v_0)$ is N_δ-valued. Hence $\pi \circ u$ is N-valued. Then we solve the constraint equation for $\pi \circ u$ instead of u (i.e., we solve $\not{D}^{\pi\circ u} = 0$ instead of $\not{D}^u = 0$). This does not make a difference, since the fixed point u_* will be N-valued, hence $\pi \circ u_* = u_*$.

We also explained in the introduction that to get the necessary estimates for the solution of equation (1.1.4), we will use a construction that joins $u_0(x)$ and $(\pi \circ u_t)(x)$ by a unique shortest geodesic of N. To do this, we need the next lemma which states that locally we can bound distances in N by distances in $\mathbb{R}^q$.

Lemma 1.3.11. *Let $N \subset \mathbb{R}^q$ be a closed embedded submanifold of $\mathbb{R}^q$ with the induced Riemannian metric. Denote by A its Weingarten map. Choose $C > 0$ s.t. $\|A\| \le C$ where*

$$\|A\| := \sup\{\|A_v X\| \mid v \in T_p^\perp N,\ X \in T_p N,\ \|v\| = 1,\ \|X\| = 1,\ p \in N\}.$$

Then there exists $0 < \delta_0 < \frac{1}{C}$ *s.t. for all* $0 < \delta \leq \delta_0$ *and for all* $p, q \in N$ *with* $\|p - q\|_2 < \delta$ *it holds that*

$$d^N(p,q) \leq \frac{1}{1 - \delta C}\|p - q\|_2,$$

where $\|.\|_2$ *denotes the Euclidean norm.*

A proof can be found in Appendix 1.B.

In the following we will make some choices for the constants δ, R, and T (e.g. to ensure the existence of unique shortest geodesics). At this point it is worth being very precise, since the constants will also depend on each other and we want to avoid any unclarity in future arguments.

By Lemma 1.2.10 v) it holds that for every $R > 0$ there exists $T = T(R) > 0$ s.t.

$$\|v_0(t,.) - u_0\|_{C^0(M,\mathbb{R}^q)} < R \tag{1.3.11}$$

for all $t \in [0, T]$.

If $R < \frac{\delta}{2}$ and $T = T(R)$ is chosen s.t. (1.3.11) holds, then it holds for every $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$ that

$$u(t,x) \in N_\delta$$

for all $(t,x) \in [0,T] \times M$. (In particular $\pi \circ u$ is N-valued.) To see this, note that

$$\begin{aligned} d(u(t,x), N) &\leq \|u(t,x) - u_0(x)\|_2 \\ &\leq \|u(t,x) - v_0(t,x)\|_2 + \|v_0(t,x) - u_0(x)\|_2 \\ &< 2R \\ &< \delta \end{aligned}$$

for all $(t,x) \in [0,T] \times M$.

In order do deal with the constraint equation it will be important that we can connect $(\pi \circ u)(x,t)$ and $(\pi \circ v)(x,s)$ by a unique shortest geodesic of N. To that end we first note that for all $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$ it holds that

$$\begin{aligned} \|(\pi \circ u)(t,x) - u_0(x)\|_2 &\leq \|(\pi \circ u)(t,x) - u(t,x)\|_2 + \|u(t,x) - u_0(x)\|_2 \\ &= d(u(t,x), N) + \|u(t,x) - u_0(x)\|_2 \\ &< \delta + \delta \\ &= 2\delta. \end{aligned} \tag{1.3.12}$$

Now we choose $\varepsilon > 0$ with $2\varepsilon < \mathrm{inj}(N)$. Moreover, let $C > 0$ and $\delta_0 > 0$ be chosen as in Lemma 1.3.11 and assume

$$\delta < \min\{\frac{1}{4}\delta_0, \frac{1}{4}\varepsilon(1 - \delta_0 C)\}.$$

Using equation (1.3.12) we see that for all $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$ it holds that

$$\|(\pi \circ u)(t,x) - (\pi \circ v)(s,x)\|_2 < 4\delta < \delta_0.$$

Therefore Lemma 1.3.11 and the choice of δ yield

$$\begin{aligned} d^N((\pi \circ u)(t,x), (\pi \circ v)(s,x)) &\leq \frac{1}{1-\delta_0 C}\|(\pi \circ u)(t,x) - (\pi \circ v)(s,x)\|_2 \\ &< \frac{1}{1-\delta_0 C} 4\delta \\ &< \frac{1}{1-\delta_0 C} 4 \frac{1}{4}\varepsilon(1-\delta_0 C) \\ &= \varepsilon \\ &< \frac{1}{2}\operatorname{inj}(N). \end{aligned} \tag{1.3.13}$$

To summarize, we have chosen constants as follows:

$\varepsilon > 0$	s.t. $2\varepsilon < \operatorname{inj}(N)$,
$\delta = \delta(\varepsilon) > 0$	s.t. $\delta < \min\{\frac{1}{4}\delta_0, \frac{1}{4}\varepsilon(1-\delta_0 C)\}$,
$R = R(\delta, \varepsilon) > 0$	s.t. $R < \frac{\delta(\varepsilon)}{2}$,
$T = T(\delta, \varepsilon, R) > 0$	s.t. (1.3.11) holds

Table 1.1: Choices of constants.

where $\delta_0, C > 0$ are as in Lemma 1.3.11. We have shown that these choices imply

$$u(t,x) \in N_\delta$$

and

$$d^N((\pi \circ u)(t,x), (\pi \circ v)(s,x)) < \varepsilon < \frac{1}{2}\operatorname{inj}(N) \tag{1.3.14}$$

for all $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $x \in M$, $t, s \in [0,T]$. Note that ε, δ, R and T in Table 1.1 fulfill certain smallness assumptions. In particular, we are free to choose them even smaller. (We will do this in some of the following proofs.) In the following, constants appearing in inequalities might depend on M, N, and u_0, but we suppress this dependency in the notation since we view M, N, and u_0 as part of our fixed initial data.

1.4 The constraint equation

In this section we solve the constraint equation with the strategy outlined in the introduction. Until Section 1.4.3 we have no restrictions on the dimension of M.

Let $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$ and assume that the constants are chosen as in Table 1.1. In the following we denote by $P^{v_s,u_t} = P^{v_s,u_t}(x)$ the parallel transport of N along the unique[11] shortest geodesic from $\pi(v(s,x))$ to $\pi(u(t,x))$. We also denote by P^{v_s,u_t} the induced mappings

$$(\pi \circ v_s)^*TN \to (\pi \circ u_t)^*TN,$$

$$\Sigma M \otimes (\pi \circ v_s)^*TN \to \Sigma M \otimes (\pi \circ u_t)^*TN,$$

and

$$\Gamma_{C^1}(\Sigma M \otimes (\pi \circ v_s)^*TN) \to \Gamma_{C^1}(\Sigma M \otimes (\pi \circ u_t)^*TN).$$

1.4.1 Estimates for Dirac operators along maps

As mentioned in the introduction, we will use estimates for Dirac operators along maps to get estimates for the projection onto the kernels of such operators.

Lemma 1.4.1. *Choose ε, δ, R, and T as in Table 1.1. If $\varepsilon > 0$ and $T > 0$ are small enough, then there exists $C = C(R) > 0$ s.t.*

$$\| \left(\left(P^{v_s,u_t}\right)^{-1} \not{D}^{\pi\circ u_t} P^{v_s,u_t} - \not{D}^{\pi\circ v_s} \right) \psi(x) \| \leq C \|u_t - v_s\|_{C^0(M,\mathbb{R}^q)} \|\psi(x)\|$$

for all $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $\psi \in \Gamma_{C^1}(\Sigma M \otimes (\pi \circ v_s)^*TN)$, $x \in M$, $t, s \in [0,T]$.

We formulated the lemma in exactly the way we are going to use it later. However it is obvious from the proof that the assertion of the lemma holds in more general contexts (e.g. for arbitrary maps $f, g \in C^1(M,N)$ that are close enough in $C^0(M,N)$), provided the factors on the right hand side of the inequality are suitably adjusted (e.g. by $C(\|f\|_{C^1}, \|g\|_{C^1}) \sup_{y\in M} d^N(f(y), g(y)) \|\psi(x)\|$).

In the same way, most of the lemmas shown in Section 1.4 hold in more general situations with essentially the same proofs.

Proof of Lemma 1.4.1. We will prove the lemma by considering the curvature of F^*TN where $F\colon [0,1] \times M \to N$ is a certain C^1-mapping that we construct in the proof. Hence F^*TN is a C^1-vector bundle over a smooth manifold. To make sure we don't run into the issue that certain expressions involving the curvature of F^*TN are not well-defined we start the proof with a few general remarks.

[11] parametrized on $[0,1]$

If we define the curvature of a vector bundle $V \to \tilde{M}$ with connection ∇^V (where $\tilde{M}$ is supposed to be a smooth manifold) in the usual way, i.e.,

$$R^V(X,Y)s = \nabla^V_X \nabla^V_Y s - \nabla^V_Y \nabla^V_X s - \nabla^V_{[X,Y]} s \tag{1.4.1}$$

we need at least that $s\colon \tilde{M} \to V$ is a C^2-section (X and Y are smooth vector fields on $\tilde{M}$). Therefore V needs to be at least a C^2-vector bundle. Our F^*TN will only be a C^1-vector bundle. However in our special case of the pullback of a smooth vector bundle along a C^1-mapping we can still define the curvature. To that end we will work with connection and curvature forms.

Let $E \to \tilde{N}$ be a smooth vector bundle and $f\colon \tilde{M} \to \tilde{N}$ a C^1-mapping between smooth manifolds. Let ∇^E be a connection on E. Choose a local frame $(s_1, \dots, s_n)$ of E over some open subset $U \subset \tilde{N}$. We define the local 1-forms $\omega_i^j \in \Gamma(T^*M|_U)$ by

$$\nabla^E_X s_i := \omega_i^j(X) s_j$$

and call ω_i^j the *connection 1-form (of E) associated to* $(s_1, \dots, s_n)$. We define the curvature of (the smooth vector bundle) E by (1.4.1). Then it holds that

$$R^E(X,Y)s_i = \Omega_i^j(X,Y)s_j \tag{1.4.2}$$

where $\Omega_i^j := d\omega_i^j + \omega_k^j \wedge \omega_i^k \in \Gamma(\bigwedge^2 T^*M|_U)$ is the *curvature 2-form (of E) associated to* $(s_1, \dots, s_n)$.[12] The definition of the pullback-connection yields that the connection 1-form of f^*E associated to $(f^*s_1, \dots, f^*s_n)$ is given by $f^*\omega_i^j$. Now we *define* the curvature of (the C^1-vector bundle) f^*E by

$$R^{f^*E}(X,Y)f^*s_i := (f^*\Omega_i^j)(X,Y)f^*s_j.$$

This is a well-defined expression, since we can pull back 2-forms by C^1-maps.[13] More precisely, given $X, Y \in T_p\tilde{M}$, $v \in (f^*E)_p$, $p \in f^{-1}(U)$, $v = v^i(f^*s_i)|_p$ we define

$$R^{f^*E}(X,Y)v := v^i(f^*\Omega_i^j)(X,Y)(f^*s_j)|_p.$$

By this definition it trivially holds that

$$R^E(dfX, dfY)v|_{f(q)} = R^{f^*E}(X,Y)(f^*v)|_q \tag{1.4.3}$$

---

[12] Equation (1.4.2) easily follows from the definition of the curvature (1.4.1). We use the following convention: given to 1-forms α and β we define their wedge product by $(\alpha \wedge \beta)(X,Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$.

[13] If f is at least C^2, then the curvature of f^*E, defined by (1.4.1), indeed satisfies the equation $R^{f^*E}(X,Y)f^*s_i = (f^*\Omega_i^j)(X,Y)f^*s_j$, since in this case the curvature 2-form of f^*E associated to $(f^*s_1, \dots, f^*s_n)$ is given by $f^*\Omega_i^j$ where Ω_i^j is the curvature 2-form of E associated to $(s_1, \dots, s_n)$. This follow from $f^*\Omega_i^j = f^*(d\omega_i^j) + f^*(\omega_k^j \wedge \omega_i^k) = d(f^*\omega_i^j) + (f^*\omega_k^j) \wedge (f^*\omega_i^k)$. Note that $f^*(d\alpha) = d(f^*\alpha)$ only holds if f is at least C^2.

for all $X, Y \in T_q\tilde{M}$ and all rough sections $v\colon \tilde{N} \to E$. (A *rough* section is a mapping $v\colon \tilde{N} \to E$ s.t. $v(p) \in E_p$ for all $p \in \tilde{N}$. A rough section doesn't need to be continuous.)[14] In particular (1.4.3) implies that our definition of R^{f^*E} doesn't depend on any choices. Moreover our definition of R^{f^*E} yields that

$$R^{f^*E}(X,Y)s = \nabla_X^{f^*E}\nabla_Y^{f^*E}s - \nabla_Y^{f^*E}\nabla_X^{f^*E}s - \nabla_{[X,Y]}^{f^*E}s \tag{1.4.4}$$

if the expressions on the right hand side are well-defined in the sense that they are well-defined in local coordinates. We will explain in more detail what we mean by that: if $\frac{\partial}{\partial x_\alpha}$ are local coordinates of $\tilde{M}$ defined on $f^{-1}(U)$, $\frac{\partial}{\partial y_\beta}$ are local coordinates of $\tilde{N}$ defined on U, $X = X^\alpha \frac{\partial}{\partial x_\alpha}$, $Y = Y^\alpha \frac{\partial}{\partial x_\alpha}$, v a section of f^*E, $v = v^i(f^*s_i)$, then we have

$$\begin{aligned}
\nabla_X^{f^*E}\nabla_Y^{f^*E}v = X^\gamma\Bigg(& (L_{\frac{\partial}{\partial x_\gamma}}Y^\alpha)(L_{\frac{\partial}{\partial x_\alpha}}v^i)f^*s_i + Y^\alpha(L_{\frac{\partial}{\partial x_\gamma}}L_{\frac{\partial}{\partial x_\alpha}}v^i)f^*s_i \\
& + Y^\alpha(L_{\frac{\partial}{\partial x_\alpha}}v^i)\frac{(y\circ f\circ x^{-1})^\tau}{\partial x_\gamma}f^*(\nabla^E_{\frac{\partial}{\partial y_\tau}}s_i) \\
& + (L_{\frac{\partial}{\partial x_\gamma}}Y^\alpha)v^i\frac{\partial(y\circ f\circ x^{-1})^\beta}{\partial x_\alpha}f^*(\nabla^E_{\frac{\partial}{\partial y_\beta}}s_i) \\
& + Y^\alpha(L_{\frac{\partial}{\partial x_\gamma}}v^i)\frac{\partial(y\circ f\circ x^{-1})^\beta}{\partial x_\alpha}f^*(\nabla^E_{\frac{\partial}{\partial y_\beta}}s_i) \\
& + Y^\alpha v^i\frac{\partial^2(y\circ f\circ x^{-1})^\beta}{\partial x_\gamma\partial x_\alpha}f^*(\nabla^E_{\frac{\partial}{\partial y_\beta}}s_i) \\
& + Y^\alpha v^i\frac{\partial(x\circ f\circ x^{-1})^\beta}{\partial x_\alpha}\frac{\partial(y\circ f\circ x^{-1})^\tau}{\partial x_\gamma}f^*(\nabla^E_{\frac{\partial}{\partial y_\tau}}\nabla^E_{\frac{\partial}{\partial y_\beta}}s_i)\Bigg)
\end{aligned} \tag{1.4.5}$$

If we assume v to be C^1 and X, Y to be smooth, then we see that the terms on the right hand side that might prevent $\nabla_X^{f^*E}\nabla_Y^{f^*E}v$ from being well-defined are $L_{\frac{\partial}{\partial x_\gamma}}L_{\frac{\partial}{\partial x_\alpha}}v^i$ and $\frac{\partial^2(y\circ f\circ x^{-1})^\beta}{\partial x_\gamma\partial x_\alpha}$. This will play a role later in the proof. [15]

[14] One could come to the conclusion that the differentiability of sections doesn't matter at all for the curvature since the curvature is tensorial in all three entries. But this is of course wrong. The curvature is a tensorial expression, but to calculate it one needs extensions of class at least C^2. The tensoriality just says that the curvature doesn't depend on the choice of extensions.

[15] One could ask whether it ever happens that the expressions on the right hand side are well-defined if $f\colon \tilde{M} \to \tilde{N}$ is merely C^1. Consider for example the case that $\tilde{N} = \tilde{N}_1 \times \tilde{N}_2$. Then it may happen that while f is globally only C^1, some second order mixed derivatives of f exist. If one choses X and Y to be in the tangent spaces of $\tilde{N}_1$ and $\tilde{N}_2$ (no particular order) and s with the property that the coordinate functions of s have existing mixed derivatives, then the expression $\nabla_X^{f^*E}\nabla_Y^{f^*E}s$ is well-defined in the sense that it exists in local coordinates. We will see a concrete example later in the proof.

After these remarks we come to the actual proof of Lemma 1.4.1. We write $g := \pi \circ v_s$, $f := \pi \circ u_t$, and we define the C^1-mapping

$$F\colon [0,1] \times M \to N$$

by $F(t,x) := \exp_{g(x)}(t\exp^{-1}_{g(x)} f(x))$ where exp denotes the exponential map of the Riemannian manifold (N,h). Note that $F(0,.) = g$, $F(1,.) = f$, and $t \mapsto F(t,x)$ is the unique shortest geodesic from $g(x)$ to $f(x)$. (Recall that F is well-defined because of (1.3.14).) We denote by

$$\mathcal{P}_{t_1,t_2} = \mathcal{P}_{t_1,t_2}(x)\colon T_{F(t_1,x)}N \to T_{F(t_2,x)}N$$

the parallel transport in F^*TN w.r.t. ∇^{F^*TN} (pullback of the Levi-Civita connection on TN) along the curve $\gamma_x(t) := (t,x)$ from $\gamma_x(t_1)$ to $\gamma_x(t_2)$, $x \in M$, $t_1, t_2 \in [0,1]$. In particular,

$$\mathcal{P}_{0,1} = P^{v_s,u_t}.$$

Let $\psi \in \Gamma_{C^1}(\Sigma M \otimes g^*TN)$. We have

$$\begin{aligned}
&\left(\left(\mathcal{P}_{0,1}\right)^{-1} \not{D}^f \mathcal{P}_{0,1} - \not{D}^g\right)\psi \\
&= (e_\alpha \cdot \psi^i) \otimes \left(\left(\left(\left(\mathcal{P}_{0,1}\right)^{-1} \nabla^{f^*TN}_{e_\alpha} \mathcal{P}_{0,1}\right) - \nabla^{g^*TN}_{e_\alpha}\right)(b_i \circ g)\right)
\end{aligned} \tag{1.4.6}$$

where $\psi = \psi^i \otimes (b_i \circ g)$, (b_i) is a (smooth) local orthonormal frame of TN, ψ^i are local C^1-sections of ΣM, and (e_α) is a (smooth) local orthonormal frame of TM.

We define local C^1-sections Θ_i of F^*TN by

$$\Theta_i(t,x) := \mathcal{P}_{0,t}(x)(b_i \circ g)(x).$$

For each $t \in [0,1]$ we define the functions $T_{ij}(t,.) = T^\alpha_{ij}(t,.)$ by

$$\left(\mathcal{P}_{0,t}\right)^{-1}\left((\nabla^{F^*TN}_{e_\alpha}\Theta_i)(t,x)\right) = \sum_j T_{ij}(t,x)(b_j \circ g)(x). \tag{1.4.7}$$

A priori we only know that the T_{ij} are continuous. In the following we will do a few formal calculations and justify them afterwards. It holds that

$$\begin{aligned}
&\| \left(\left(\left(\mathcal{P}_{0,1}\right)^{-1} \nabla^{f^*TN}_{e_\alpha} \mathcal{P}_{0,1}\right) - \nabla^{g^*TN}_{e_\alpha}\right)(b_i \circ g)(x) \|_h^2 \\
&= \| \left(\mathcal{P}_{0,1}\right)^{-1}\left((\nabla^{F^*TN}_{e_\alpha}\Theta_i)(1,x)\right) - \left(\mathcal{P}_{0,0}\right)^{-1}\left((\nabla^{F^*TN}_{e_\alpha}\Theta_i)(0,x)\right)\|_h^2 \\
&= \| \Big(\sum_j T_{ij}(1,x)(b_j \circ g)(x)\Big) - \Big(\sum_j T_{ij}(0,x)(b_j \circ g)(x)\Big)\|_h^2 \\
&= \sum_j (T_{ij}(1,x) - T_{ij}(0,x))^2 \\
&= \sum_j \left(\int_0^1 \frac{d}{dt}\Big|_{t=r} T_{ij}(t,x)dr\right)^2.
\end{aligned} \tag{1.4.8}$$

Therefore we want to control the first time-derivative of the T_{ij}. Equation (1.4.7) implies that these time-derivatives are related to the curvature of F^*TN. More precisely, for all $X \in \Gamma_{C^\infty}(TM)$ we have

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=r} \left(\mathcal{P}_{0,t}^{-1}\left(\left(\nabla_X^{F^*TN}\Theta_i\right)(t,x)\right)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \left(\mathcal{P}_{0,t+r}^{-1}\left(\left(\nabla_X^{F^*TN}\Theta_i\right)(t+r,x)\right)\right) \\
&= \frac{d}{dt}\Big|_{t=0} \left(\mathcal{P}_{0,r}^{-1}\mathcal{P}_{r,t+r}^{-1}\left(\left(\nabla_X^{F^*TN}\Theta_i\right)(t+r,x)\right)\right) \\
&= \mathcal{P}_{0,r}^{-1}\frac{d}{dt}\Big|_{t=0} \left(\mathcal{P}_{r,t+r}^{-1}\left(\left(\nabla_X^{F^*TN}\Theta_i\right)(t+r,x)\right)\right) \\
&= \mathcal{P}_{0,r}^{-1}\left(\left(\nabla_{\frac{\partial}{\partial t}}^{F^*TN}\nabla_X^{F^*TN}\Theta_i\right)(r,x)\right).
\end{aligned} \tag{1.4.9}$$

Now we justify the formal calculations (1.4.8) and (1.4.9). Combining the definition of Θ_i as parallel transport and a careful examination of the regularity of F we deduce from (1.4.5) that $\left(\nabla_{\frac{\partial}{\partial t}}^{F^*TN}\nabla_X^{F^*TN}\Theta_i\right)(r,x)$ exists. Then (1.4.9) holds. In particular $\mathcal{P}_{0,t}^{-1}\left(\left(\nabla_X^{F^*TN}\Theta_i\right)(t,x)\right)$ is differentiable in t. Then (1.4.7) yields that the T_{ij} are differentiable in t. Therefore (1.4.8) holds. Applying (1.4.4) and (1.4.3) we further get

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t}}^{F^*TN}\nabla_X^{F^*TN}\Theta_i &= R^{F^*TN}(\frac{\partial}{\partial t}, X)\Theta_i + \nabla_X^{F^*TN}\nabla_{\frac{\partial}{\partial t}}^{F^*TN}\Theta_i - \nabla_{[\frac{\partial}{\partial t},X]}^{F^*TN}\Theta_i \\
&= R^{F^*TN}(\frac{\partial}{\partial t}, X)\Theta_i \\
&= R^{TN}(dF(\frac{\partial}{\partial t}), dF(X))\Theta_i,
\end{aligned}$$

since $\nabla_{\frac{\partial}{\partial t}}^{F^*TN}\Theta_i = 0$ by definition of Θ_i, and $[\frac{\partial}{\partial t}, X] = 0$.

This implies

$$\begin{aligned}
\sum_j \left(\frac{d}{dt}\Big|_{t=r} T_{ij}(t,x)\right)^2 &= \|\frac{d}{dt}\Big|_{t=r}\left(\mathcal{P}_{0,t}^{-1}\left(\left(\nabla_{e_\alpha}^{F^*TN}\Theta_i\right)(t,x)\right)\right)\|_h^2 \\
&= \|\left(\nabla_{\frac{\partial}{\partial t}}^{F^*TN}\nabla_{e_\alpha}^{F^*TN}\Theta_i\right)(r,x)\|_h^2 \\
&= \|R^{TN}(dF_{(r,x)}(\frac{\partial}{\partial t}), dF_{(r,x)}(e_\alpha))\Theta_i(r,x)\|_h^2 \\
&\le C_1\|dF_{(r,x)}(\frac{\partial}{\partial t})\|_h^2\|dF_{(r,x)}(e_\alpha)\|_h^2\|\Theta_i(r,x)\|_h^2 \\
&= C_1\|dF_{(r,x)}(\frac{\partial}{\partial t})\|_h^2\|dF_{(r,x)}(e_\alpha)\|_h^2
\end{aligned}$$

where we used Lemma 1.4.2 below.

In the following we estimate $\|dF_{(r,x)}(\frac{\partial}{\partial t})\|_h$ and $\|dF_{(r,x)}(e_\alpha)\|_h$.

Estimate for $\|dF_{(r,x)}(\frac{\partial}{\partial t})\|_h$: we have

$$\begin{aligned}
dF_{(r,x)}(\frac{\partial}{\partial t}\Big|_{(r,x)}) &= \frac{d}{dt}\Big|_{t=r}(\exp_{g(x)}(t\exp^{-1}_{g(x)}f(x)) \\
&= c'(r)
\end{aligned}$$

where $c(t) := \exp_{g(x)}(t\exp^{-1}_{g(x)}f(x))$ is a geodesic of N. In particular c' is parallel along c and thus $\|c'(r)\|_h = \|c'(0)\|_h = \|\exp^{-1}_{g(x)}f(x)\|_h$. Therefore we get

$$\begin{aligned}
\|dF_{(r,x)}(\frac{\partial}{\partial t}\Big|_{(r,x)})\|_h &= \|c'(r)\|_h \\
&= \|\exp^{-1}_{g(x)}f(x)\|_h \\
&\leq d^N(g(x), f(x)) \\
&\leq \frac{1}{1-\delta_0 C}\|f(x)-g(x)\|_2 \\
&\leq \frac{1}{1-\delta_0 C}\|f-g\|_{C^0(M,\mathbb{R}^q)} \\
&\leq C_2\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)}
\end{aligned}$$

where we used (1.3.13) and the (global) Lipschitz continuity of π.

Estimate for $\|dF_{(r,x)}(e_\alpha)\|_h$: our goal is to show that there exists some $C_3(R) > 0$ s.t. $\|dF_{(r,x)}(e_\alpha)\|_h \leq C_3(R)$ for all $(r,x) \in [0,1] \times M$.

First, we recall the following: define the open set $\mathcal{E} \subset TN$ by

$$\mathcal{E} := \{(p,v) \in TN \mid p \in N,\ v \in T_pN,\ \exp_p v \text{ exists}\}.$$

Then we define the smooth map

$$(pr_1, \exp)\colon \mathcal{E} \to N \times N$$

by $(pr_1,\exp)(p,v) := (p, \exp_p v)$. For each $p \in N$ there exists an open neighborhood W of $(p,0) \in TN$ s.t. the map

$$(pr_1,\exp)\colon W \to (pr_1,\exp)(W)$$

is a diffeomorphism (in particular $(pr_1,\exp)(W)$ is open in $N \times N$). One can always choose W to be of the form

$$W = \{(q,v) \in TN \mid q \in \tilde{W},\ \|v\|_h < c\}$$

for some $c > 0$ and an open neighborhood $\tilde{W}$ of p in N, see e.g. [27, p. 58]. Since N is compact there exists $c > 0$, an open covering $(V_i)_{i=1,\dots,k}$ of N with $\overline{V_i}$ compact and $\overline{V_i} \subset \tilde{W}_i$ s.t. $(pr_1,\exp)$ restricted to

$$W_i = \{(q,v) \in TN \mid q \in \tilde{W}_i,\ \|v\|_h < c\}$$

is a diffeomorphism.

We fix $r \in [0,1]$. The mapping

$$H\colon \bigcup_{i=1}^{k} (pr_1, \exp)(W_i) \to N$$

defined by

$$H(q_1, q_2) := \exp_{q_1}(r \exp_{q_1}^{-1} q_2).$$

Note that for each i it holds that

$$H|_{(pr_1,\exp)(W_i)}(q_1, q_2) = \exp_{q_1}\Big(r(pr_1, \exp)|_{W_i}^{-1}(q_1, q_2)\Big).$$

In particular H is smooth. Now we choose $\varepsilon > 0$ so small that $\varepsilon < \frac{c}{2}$. For each $x \in M$ there exists i s.t. $g(x) \in V_i$. Then it holds that $(g(x), f(x)) \in (pr_1, \exp)(W_i)$, since $d^N(f(x), g(x)) < \varepsilon$. In particular, we have

$$F(r, .) = H \circ (g, f)$$

on M. There exists $C > 0$ independent of the choice of $r \in [0,1]$ s.t.

$$\|(dH)_{(q_1,q_2)}(v, w)\|_h \leq C(\|v\|_h + \|w\|_h)$$

for all $(q_1, q_2) \in (pr_1, \exp)\Big(\{(q, v) \in TN \mid q \in \overline{V_i},\ \|v\|_h \leq \frac{c}{2}\}\Big)$, $i = 1, \ldots, k$. Therefore

$$\begin{aligned}
\|(dF)_{(r,x)} e_\alpha\|_h &= \|d(H \circ (g, f))_x e_\alpha\|_h \\
&= \|(dH)_{(g(x), f(x))}\Big((dg)_x e_\alpha, (df)_x e_\alpha\Big)\|_h \\
&\leq C(\|(dg)_x e_\alpha\|_h + \|(df)_x e_\alpha\|_h) \\
&\leq C2C(R)
\end{aligned}$$

for all $(r, x) \in [0,1] \times M$. We conclude that there exists some $C_3(R) > 0$ s.t. $\|dF_{(r,x)}(e_\alpha)\|_h \leq C_3(R)$ for all $(r, x) \in [0,1] \times M$.

We have shown

$$\begin{aligned}
\sum_j \left(\frac{d}{dt}\Big|_{t=r} T_{ij}(t, x)\right)^2 &\leq C_1 C_2^2 C_3(R)^2 \|f - g\|^2_{C^0(M,\mathbb{R}^q)} \\
&= C_4(R) \|f - g\|^2_{C^0(M,\mathbb{R}^q)}
\end{aligned}$$

for all (t, x). In particular,

$$-C_5(R)\|f - g\|_{C^0(M,\mathbb{R}^q)} \leq \frac{d}{dt}\Big|_{t=r} T_{ij}(t, x) \leq C_5(R)\|f - g\|_{C^0(M,\mathbb{R}^q)}$$

for each (i,j) and all (r,x). Therefore, (1.4.8) yields

$$\begin{aligned}
&\| \left(\left(\left(\mathcal{P}_{0,1}\right)^{-1} \nabla^{f^*TN}_{e_\alpha} \mathcal{P}_{0,1} \right) - \nabla^{g^*TN}_{e_\alpha} \right) (b_i \circ g)(x) \|_h \\
&\leq \sqrt{\dim(N)} C_5(R) \|f-g\|_{C^0(M,\mathbb{R}^q)} \\
&= C_6(R) \|f-g\|_{C^0(M,\mathbb{R}^q)}
\end{aligned}$$

for all $x \in M$. Using (1.4.6) we deduce

$$\| \left(\left(\mathcal{P}_{0,1}\right)^{-1} \not{D}^f \mathcal{P}_{0,1} - \not{D}^g \right) \psi(x) \| \leq C(R) \|f-g\|_{C^0(M,\mathbb{R}^q)} \|\psi(x)\|$$

for all $x \in M$ and all $\psi \in \Gamma_{C^1}(\Sigma M \otimes g^*TN)$. (Here we used that $\|e_\alpha(x) \cdot \psi^i(x)\| = \|\psi^i(x)\|$ and $\|\psi^i(x)\| \leq \left(\sum_j \|\psi^j(x)\|^2 \right)^{\frac{1}{2}} = \|\psi(x)\|$.) $\square$

Lemma 1.4.2. *Let (N,h) be a compact Riemannian manifold. Then there exists $C>0$ s.t.*

$$\|R^{TN}(X,Y)Z\|_h \leq C\|X\|_h \|Y\|_h \|Z\|_h$$

for all $X,Y,Z \in T_pN$ and all $p \in N$.

Proof. We use the local formula

$$R^{TN}(X,Y)Z = \left(L_{\frac{\partial}{\partial x_j}} \Gamma^l_{ik} - L_{\frac{\partial}{\partial x_k}} \Gamma^l_{ij} + \Gamma^l_{js}\Gamma^s_{ik} - \Gamma^l_{ks}\Gamma^s_{ij} \right) Z^i X^j Y^k \frac{\partial}{\partial x_l}.$$

To that end let $x\colon \tilde{U} \to \mathbb{R}^n$ be a chart of N and let $U \subset N$ be open with $\overline{U} \subset \tilde{U}$ and $\overline{U} \subset N$ compact. In particular, all the $L_{\frac{\partial}{\partial x_j}} \Gamma^l_{ik}$, Γ^l_{ij}, and $\frac{\partial}{\partial x_l}$ are bounded on U.

This implies that for any $q \in U$ and $X,Y,Z \in T_qN$ with $\|X\|_h = \|Y\|_h = \|Z\|_h = 1$ it holds that

$$\|R^{TN}(X,Y)Z\|_h \leq C(U,x).$$

Covering N with finitely many such charts and taking the maximum over the $C(U,x)$ we get that there exists $C>0$ s.t.

$$\|R^{TN}(X,Y)Z\|_h \leq C$$

for all $X,Y,Z \in T_pN$ with $\|X\|_h = \|Y\|_h = \|Z\|_h = 1$ and all $p \in N$. From this the lemma follows easily. $\square$

1.4.2 Estimates for the parallel transports

In this section we obtain estimates for the parallel transports which will be used later.

Lemma 1.4.3. *Choose ε, δ, R, and T as in Table 1.1. If $\varepsilon > 0$ is small enough, then there exists $C = C(\varepsilon) > 0$ s.t.*

$$\|P^{v_s,u_0}P^{u_t,v_s}P^{u_0,u_t}Z - Z\| \leq C\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)}\|Z\|$$

for all $Z \in T_{u_0(x)}N$, $x \in M$, $s,t \in [0,T]$, $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$.

Proof. We fix x, s, t, u, v and write $y := (\pi \circ v_s)(x)$, $z := (\pi \circ u_t)(x)$. Moreover, we denote by $\gamma_i \colon [0,1] \to N$ the unique shortest geodesics of N with

$$\gamma_1(0) = \gamma_3(1) = u_0(x),\ \gamma_1(1) = \gamma_2(0) = z,\ \gamma_2(1) = \gamma_3(0) = y.$$

Furthermore, we define $c := \gamma_3 * \gamma_2 * \gamma_1$, i.e., c is the curve obtained by first following γ_1, then γ_2, and then γ_3. Finally, we denote by P^c the induced parallel transport of N along c. (Hence $P^c = P^{v_s,u_0}P^{u_t,v_s}P^{u_0,u_t}$.)

We consider the geodesic variation

$$\alpha \colon [0,1] \times [0,1] \to N, \qquad (s,t) \mapsto \exp_{u_0(x)}(t\exp^{-1}_{u_0(x)}\gamma_2(s)),$$

where exp is the exponential map of N. Note that α is well-defined since

$$\begin{aligned} d^N(u_0(x), \gamma_2(s)) &\leq d^N(u_0(x), z) + d^N(z, \gamma_2(s)) \\ &< \varepsilon + d^N(z, y) \\ &< \varepsilon + \varepsilon \\ &< \operatorname{inj}(N) \end{aligned}$$

see (1.3.14).

We choose an arbitrary $Z \in T_{u_0(x)}N$. In the following we derive a formula that relates $P^cZ - Z$ to an integral over $R^{TN}(\frac{\partial}{\partial s}\alpha(s,t), \frac{\partial}{\partial t}\alpha(s,t))$ with a strategy inspired by [31, Section 7]. This formula is closely related to the general fact that "Deviation of parallel transport from the identity $\approx$ curvature $\cdot$ enclosed area".

Denote by $t \mapsto Z(t)$ the parallel vector field along γ_1 with $Z(0) = Z$. For every $t \in [0,1]$ let $s \mapsto Z(s,t)$ be the parallel vector field along $s \mapsto \alpha(s,t)$ with $Z(0,t) = Z(t)$. In particular we have

$$P^{\gamma_2 * \gamma_1}Z = Z(1,1).$$

Let $(E_0, \dots, E_n)$ be an orthonormal basis of $T_{u_0(x)}N$. Analogously we construct $E_i(s,t) \in T_{\alpha(s,t)}N$ s.t. $E_i(0,0) = E_i$, $t \mapsto E_i(1,t)$ is parallel along $t \mapsto \alpha(1,t)$, and $s \mapsto E_i(s,t)$ is parallel along $s \mapsto \alpha(s,t)$ for every $t \in [0,1]$.

We write $Z(s,t) = Z^i(s,t)E_i(s,t)$, i.e., $Z^i(s,t) = \langle Z(s,t), E_i(s,t)\rangle$ ($\langle .,.\rangle = h$ is the Riemannian metric on N). It holds that

$$\frac{d}{dt}\Big|_{t=t_0} Z^i(1,t) = \frac{d}{dt}\Big|_{t=t_0} \langle Z(1,t), E_i(1,t)\rangle = \langle \frac{D}{dt}\Big|_{t=t_0} Z(1,t), E_i(1,t_0)\rangle$$

and

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=s_0} \langle \frac{D}{dt}\Big|_{t=t_0} Z(s,t), E_i(s,t_0)\rangle \\
&= \langle \frac{D}{ds}\Big|_{s=s_0} \frac{D}{dt}\Big|_{t=t_0} Z(s,t), E_i(s_0,t_0)\rangle \\
&= \langle R^{TN}\Big(\frac{\partial}{\partial s}\Big|_{s=s_0} \alpha(s,t_0), \frac{\partial}{\partial t}\Big|_{t=t_0} \alpha(s_0,t)\Big) Z(s_0,t_0) \\
&\quad + \frac{D}{dt}\Big|_{t=t_0} \frac{D}{ds}\Big|_{s=s_0} Z(s,t), E_i(s_0,t_0)\rangle \\
&= \langle R^{TN}\Big(\frac{\partial}{\partial s}\Big|_{s=s_0} \alpha(s,t_0), \frac{\partial}{\partial t}\Big|_{t=t_0} \alpha(s_0,t)\Big) Z(s_0,t_0), E_i(s_0,t_0)\rangle.
\end{aligned}$$

Noting that $Z = Z(0,0) = Z(s,0)$ and $E_i = E_i(0,0) = E_i(s,0)$ for all $s \in [0,1]$ (since $s \mapsto \alpha(s,0)$ is constant) we get

$$\begin{aligned}
P^c Z - Z &= P^{\gamma_3}\left(P^{\gamma_2 * \gamma_1} Z\right) - Z \\
&= P^{\gamma_3}(Z(1,1)) - Z(1,0) \\
&= P^{\gamma_3}\Big(Z^i(1,1)E_i(1,1)\Big) - Z^i(1,0)E_i(1,0) \\
&= Z^i(1,1)E_i(1,0) - Z^i(1,0)E_i(1,0) \\
&= (Z^i(1,1) - Z^i(1,0))E_i(1,0) \\
&= \int_0^1 \langle \frac{D}{dt} Z(1,t), E_i(1,t)\rangle dt E_i \\
&= \int_0^1 \Big(\langle \frac{D}{dt} Z(1,t), E_i(1,t)\rangle - \langle \underbrace{\frac{D}{dt} Z(0,t)}_{=0}, E_i(0,t)\rangle\Big) dt E_i \\
&= \int_0^1 \int_0^1 \frac{d}{ds} \langle \frac{D}{dt} Z(s,t), E_i(s,t)\rangle dt ds E_i \\
&= \int_0^1 \int_0^1 \langle R^{TN}\Big(\frac{\partial}{\partial s}\alpha(s,t), \frac{\partial}{\partial t}\alpha(s,t)\Big) Z(s,t), E_i(s,t)\rangle dt ds E_i.
\end{aligned}$$

We have shown that

$$P^c Z - Z = \left(\int_0^1 \int_0^1 \langle R^{TN}\Big(\frac{\partial}{\partial s}\Big|_{s=\tilde{s}} \alpha(s,\tilde{t}), \frac{\partial}{\partial t}\Big|_{t=\tilde{t}} \alpha(\tilde{s},t)\Big) Z(\tilde{s},\tilde{t}), E_i(\tilde{s},\tilde{t})\rangle d\tilde{t} d\tilde{s}\right) E_i \tag{1.4.10}$$

holds for all $Z \in T_{u_0(x)}N$. In the next step we estimate $\|\frac{\partial}{\partial t}\alpha\|$ and $\|\frac{\partial}{\partial s}\alpha\|$. To that end, notice that

$$\|\frac{\partial}{\partial t}\alpha(s,t)\| = \|\exp_{u_0(x)}^{-1}\gamma_2(s)\| \leq d^N(u_0(x), \gamma_2(s)) < 2\varepsilon$$

see the calculation at the beginning of the proof. Therefore it remains to estimate $\|\frac{\partial}{\partial s}\alpha\|$. For each $s \in [0,1]$ we consider the Jacobi field

$$J_s(t) := \frac{\partial}{\partial s}\alpha(s,t).$$

It holds that

$$J_s(0) = 0 \text{ and } J_s(1) = \gamma_2'(s).$$

Equation (1.3.13) yields

$$\begin{aligned} \|J_s(1)\| &= \|\gamma_2'(s)\| \\ &= \|\gamma_2'(0)\| \\ &= \|\exp_z^{-1} y\| \\ &\leq d^N(z,y) \\ &\leq \frac{1}{1-\delta_0 C}\|\pi \circ u_t - \pi \circ v_s\|_{C^0(M,\mathbb{R}^q)} \\ &\leq \frac{1}{1-\delta_0 C} C_1 \|u_t - v_s\|_{C^0(M,\mathbb{R}^q)} \end{aligned} \tag{1.4.11}$$

In the last line we used that $\pi\colon \mathbb{R}^q \to \mathbb{R}^q$ is (globally) Lipschitz continuous. In the following we will use comparison theorems to show $\|J_s(t)\| \leq \|J_s(1)\|$ for all $t \in [0,1]$.

Since N is compact there exists $\mu > 0$ s.t.

$$K < \mu$$

where K is the sectional curvature of N. (More precisely, for every $p \in N$ and all linearly independent vectors $v, w \in T_pN$ we have $K(\mathrm{span}(v,w)) < \mu$. To show this, we can assume that (v,w) is an orthonormal basis of $\mathrm{span}(v,w)$. Then $K(\mathrm{span}(v,w)) = \langle R^{TN}(v,w)w, v\rangle \leq \|R^{TN}(v,w)w\|$. Now we apply Lemma 1.4.2.) Let ε be so small that (additionally to the assumptions in Table 1.1)

$$\varepsilon < \frac{\pi}{4\sqrt{\mu}}.$$

For each $s \in [0,1]$ we consider the geodesic

$$c_s(t) := \exp_{u_0(x)}\Big(t\frac{\exp_{u_0(x)}^{-1}\gamma_2(s)}{\|\exp_{u_0(x)}^{-1}\gamma_2(s)\|}\Big)$$

for $t \in [0, \|\exp^{-1}_{u_0(x)}\gamma_2(s)\|]$. (We assume $\|\exp^{-1}_{u_0(x)}\gamma_2(s)\| \neq 0$, since otherwise we have $\alpha(s,t) = u_0(x)$ for all $t \in [0,1]$ and therefore $\|J_s(t)\| = 0 \leq 0 = \|J_s(1)\|$ for every $t \in [0,1]$.) Then

$$\tilde{J}_s(t) := J_s\Big(\frac{t}{\|\exp^{-1}_{u_0(x)}\gamma_2(s)\|}\Big)$$

is a Jacobi field along c_s. Applying [19, equation (5.5.5) in Theorem 5.5.1 on p. 230] we get for every $t \in (0,1]$[16]

$$\begin{aligned}
\|J_s(t)\| &= \|\tilde{J}_s\Big(t\|\exp^{-1}_{u_0(x)}\gamma_2(s)\|\Big)\| \\
&\leq \frac{f_\mu(t\|\exp^{-1}_{u_0(x)}\gamma_2(s)\|)}{f_\mu(\|\exp^{-1}_{u_0(x)}\gamma_2(s)\|)}\|\tilde{J}_s\Big(\|\exp^{-1}_{u_0(x)}\gamma_2(s)\|\Big)\| \\
&= \frac{\sin(\sqrt{\mu}t\|\exp^{-1}_{u_0(x)}\gamma_2(s)\|)}{\sin(\sqrt{\mu}\|\exp^{-1}_{u_0(x)}\gamma_2(s)\|)}\|J_s(1)\| \\
&\leq \|J_s(1)\|
\end{aligned}$$

To apply the theorem we need $f_\mu(t) = \|\frac{D}{dt}\big|_{t=0}\tilde{J}_s(t)\|\frac{1}{\sqrt{\mu}}\sin(\sqrt{\mu}t) > 0$ for $t \in [0, \|\exp^{-1}_{u_0(x)}\gamma_2(s)\|]$. This holds since $\sqrt{\mu}\|\exp^{-1}_{u_0(x)}\gamma_3(s)\| < \sqrt{\mu}\varepsilon < \frac{\pi}{2}$.[17] In the last inequality above we used that sin is strictly increasing on $[0, \frac{\pi}{2}]$. If we combine this with (1.4.10), (1.4.11), and Lemma 1.4.2 we get

$$\|P^c Z - Z\| \leq C(\varepsilon)\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)}\|Z\|$$

for all $Z \in T_{u_0(x)}N$. $\square$

The operator norms of the induced maps

$$P^{v_s,u_t} \colon \Gamma_{W^1_p}(\Sigma M \otimes (\pi \circ v_s)^*TN) \to \Gamma_{W^1_p}(\Sigma M \otimes (\pi \circ u_t)^*TN)$$

are finite. However, we need that these operator norms are uniformly bounded in v_s and u_t. To that end we need the following lemma.

Lemma 1.4.4. *Choose $\varepsilon, \delta, R,$ and T as in Table 1.1. There exists $C = C(R) > 0$ s.t.*

$$\|\nabla^{(\pi\circ u_t)^*TN}_X\Big(P^{v_s,u_t}Z\Big)|_x\| \leq C\|Z\|_{\Gamma_{C^1}((\pi\circ v_s)^*TN)}\|X\|$$

[16] In the theorem choose $t_1 = t\|\exp^{-1}_{u_0(x)}\gamma_2(s)\|$, $t_2 = \|\exp^{-1}_{u_0(x)}\gamma_2(s)\|$, and τ a tiny amount larger than t_2.

[17] Note that we can assume $\frac{D}{dt}\big|_{t=0}\tilde{J}_s(t) \neq 0$. Otherwise we have $J_s(t) = 0$ for all $t \in [0,1]$ (uniqueness of Jacobi fields) and $\|J_s(t)\| \leq \|J_s(1)\|$ trivially holds.

for all $X \in T_xM$, $x \in M$, $Z \in \Gamma_{C^1}((\pi \circ v_s)^*TN)$, $s,t \in [0,T]$, $u,v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$.

Proof. We write $f := \pi \circ u_t$, $g := \pi \circ v_s$, $P := P^{v_s,u_t}$, and moreover $\nabla := \nabla^{(\pi \circ u_t)^*TN}$. Let $Z \in \Gamma_{C^1}((\pi \circ v_s)^*TN)$, $x \in M$, $X \in T_xM$, and $\gamma\colon (-c,c) \to M$ a smooth curve parametrized proportionally to arc length with $\gamma(0) = x$, $\gamma'(0) = X$. Let $(E_i(.))$ be a local orthonormal frame around x of f^*TN that is parallel along γ. Locally we have

$$P(y)Z(y) = f^i(y)E_i(y)$$

for suitable functions f^i. Then it holds that

$$\nabla_X\Big(PZ\Big)|_x = \Big(L_X f^i\Big)(x)E_i(x).$$

In the following, we estimate $(L_X f^i)E_i$. To that end we denote by P^γ the parallel transport in TN along $f \circ \gamma$ from $f(x)$ to $f(\gamma(\tau))$. We also denote by P^γ the parallel transport in TN along $g \circ \gamma$ from $g(x)$ to $g(\gamma(\tau))$. It should always be clear from the context which one we mean. We calculate

$$\begin{aligned}
&(L_X f^i)(x)E_i(x)\\
&= \lim_{\tau\to 0} \frac{f^i(\gamma(\tau)) - f^i(x)}{\tau} E_i(x)\\
&= \lim_{\tau\to 0} \frac{f^i(\gamma(\tau))E_i(x) - f^i(x)E_i(x)}{\tau}\\
&= \lim_{\tau\to 0} \frac{(P^\gamma)^{-1}\Big(f^i(\gamma(\tau))P^\gamma E_i(x)\Big) - PZ(x)}{\tau}\\
&= \lim_{\tau\to 0} \frac{(P^\gamma)^{-1}\Big(f^i(\gamma(\tau))E_i(\gamma(\tau))\Big) - PZ(x)}{\tau}\\
&= \lim_{\tau\to 0} \frac{(P^\gamma)^{-1}PZ(\gamma(\tau)) - PZ(x)}{\tau}\\
&= \lim_{\tau\to 0} \frac{(P^\gamma)^{-1}PZ(\gamma(\tau)) - (P^\gamma)^{-1}PP^\gamma Z(x) + (P^\gamma)^{-1}PP^\gamma Z(x) - PZ(x)}{\tau}
\end{aligned}$$

We will show

$$\|\Big((P^\gamma)^{-1}Z(\gamma(\tau))\Big) - Z(x)\| \le \tau\|X\|\|Z\|_{\Gamma_{C^1}(g^*TN)} \tag{1.4.12}$$

and

$$\begin{aligned}
&\|P^{-1}(P^\gamma)^{-1}PP^\gamma Z(x) - Z(x)\|\\
&\le \tau C(R)\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)}\|X\|\|Z(x)\|.
\end{aligned} \tag{1.4.13}$$

After that the lemma follows easily.

To show (1.4.12) we write $k(t) := \left(P^{\gamma|_{[0,t]}}\right)^{-1} Z(\gamma(t))$ and use the fundamental theorem of calculus to conclude

$$\begin{aligned}
\|\Big((P^\gamma)^{-1}Z(\gamma(\tau))\Big) - Z(x)\| &= \|k(\tau) - k(0)\| \\
&= \| \int_0^\tau k'(t)dt\| \\
&\le \tau \sup_t \|k'(t)\| \qquad\qquad (1.4.14)\\
&= \tau \sup_t \|(\nabla^{g^*TN}_{\gamma'(t)} Z)(\gamma(t))\| \\
&\le \tau \|X\| \|Z\|_{\Gamma_{C^1}(g^*TN)}.
\end{aligned}$$

It remains to show (1.4.13). To that end we recall that

$$P^\square := P^{-1}(P^\gamma)^{-1}PP^\gamma \colon T_{g(x)}N \to T_{g(x)}N$$

is the parallel transport along the following rectangle $\square$: first we follow $g \circ \gamma$ from $g(x)$ to $g(\gamma(\tau))$. Then we go along the unique shortest geodesic of N connecting $g(\gamma(\tau))$ and $f(\gamma(\tau))$. Afterwards we follow $f \circ g$ from $f(\gamma(\tau))$ to $f(x)$. Finally we go along the unique shortest geodesic of N connecting $f(x)$ and $g(x)$. We can estimate $\|P^\square - \mathrm{Id}\|$ with the same methods we used to show Lemma 1.4.3. More precisely we consider the (well-defined) geodesic variation

$$\alpha \colon [0,\tau] \times [0,1] \to N, \qquad (s_1, t_1) \mapsto \exp_{g(\gamma(s_1))}(t_1 \exp^{-1}_{g(\gamma(s_1))} f(\gamma(s_1))).$$

By definition the image of α is the filled rectangle $\square$. Analogously to the proof of (1.4.10) (the fact that we consider a rectangle now but in (1.4.10) we considered a triangle doesn't change the nature of the argument) we get

$$P^\square Z - Z =$$
$$\left(\int_0^\tau \int_0^1 \langle R^{TN}\Big(\frac{\partial}{\partial s_1}\Big|_{s_1=\tilde{s_1}} \alpha(s_1, \tilde{t_1}), \frac{\partial}{\partial t_1}\Big|_{t_1=\tilde{t_1}} \alpha(\tilde{s_1}, t_1)\Big) Z(\tilde{s_1}, \tilde{t_1}), \tilde{E}_i(\tilde{s_1}, \tilde{t_1})\rangle d\tilde{t_1} d\tilde{s_1} \right) \tilde{E}_i$$

(for the precise definition of $Z(.,.)$ and $\tilde{E}_i(.,.)$ we refer to the proof of (1.4.10)). Moreover, by (1.3.13) and the (global) Lipschitz continuity of π we have

$$\begin{aligned}
\|\frac{\partial}{\partial t_1}\alpha(s_1,t_1)\| &= \|\exp^{-1}_{g(\gamma(s_1))} f(\gamma(s_1))\| \\
&\le d^N(g(\gamma(s_1)), f(\gamma(s_1))) \\
&\le \frac{1}{1-\delta_0 C} \|v_s - u_t\|_{C^0(M,\mathbb{R}^q)}
\end{aligned}$$

for all s_1, t_1. We also get

$$\|\frac{\partial}{\partial s_1}\alpha(s_1,t_1)\| \le C(R)\|X\|$$

for all s_1, t_1. This can be shown analogously to the estimate for $\|dF_{(r,x)}e_\alpha\|$ in the proof of Lemma 1.4.1. We have shown (1.4.13).

$\square$

From Lemma 1.4.4 we directly get the following corollary.

Corollary 1.4.5. *Choose ε, δ, R, and T as in Table 1.1. For $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $s, t \in [0, T]$, the isometries*

$$P^{v_s,u_t}\colon \Gamma_{L^p}(\Sigma M \otimes (\pi \circ v_s)^*TN) \to \Gamma_{L^p}(\Sigma M \otimes (\pi \circ u_t)^*TN)$$

restrict to isomorphisms of Banach spaces

$$P^{v_s,u_t}\colon \Gamma_{W_p^1}(\Sigma M \otimes (\pi \circ v_s)^*TN) \to \Gamma_{W_p^1}(\Sigma M \otimes (\pi \circ u_t)^*TN)$$

with uniformly bounded operator norm, i.e., there exists $C = C(R, p)$ s.t.

$$\|P^{v_s,u_t}\|_{L(W_p^1, W_p^1)} \leq C$$

for all $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $s, t \in [0, T]$.

Proof. We have that

$$P^{v_s,u_t}\colon \Gamma_{C^1}(\Sigma M \otimes (\pi \circ v_s)^*TN) \to \Gamma_{C^1}(\Sigma M \otimes (\pi \circ u_t)^*TN)$$

is a bijective map with inverse P^{u_t,v_s}. By Lemma 1.4.4 there exists $C = C(R, p) > 0$ s.t.

$$\|P^{v_s,u_t}\|_{L((C^1,\|\cdot\|_{W_p^1}),(C^1,\|\cdot\|_{W_p^1}))}, \|P^{u_t,v_s}\|_{L((C^1,\|\cdot\|_{W_p^1}),(C^1,\|\cdot\|_{W_p^1}))} < C$$

for all $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $s, t \in [0, T]$. The corollary follows since C^1-sections form a dense subspace of $\Gamma_{W_p^1}$.[18]

$\square$

[18] More precisely: let $X \subset Y$ be a dense subspace of a Banach space Y and $F\colon X \to X$ be a bounded bijective map with bounded inverse $F^{-1}\colon X \to X$. If $\tilde{F}\colon Y \to Y$ denotes the extension of $F\colon X \to X \hookrightarrow Y$ and $\widetilde{F^{-1}}\colon Y \to Y$ denotes the extension of $F^{-1}\colon X \to X \hookrightarrow Y$, then $\tilde{F}\colon Y \to Y$ is an isomorphism of Banach spaces with inverse $\widetilde{F^{-1}}$.

1.4.3 The projection onto the kernel

When we write $\ker(\not{D}^{\pi\circ u_t})$ in the following we mean the kernel of

$$\not{D}^{\pi\circ u_t}\colon \Gamma_{W_2^1} \to \Gamma_{L^2}.$$

In this section we assume $m = \dim(M) \equiv 0, 1, 2, 4 \pmod 8$. In Remark 1.4.7 below it is explained why we restrict to these dimensions. Note that the dimension of N is still arbitrary.

Lemma 1.4.6. *Assume that* $\dim_{\mathbb{K}}\ker(\not{D}^{u_0}) = 1$*, where*

$$\mathbb{K} = \begin{cases} \mathbb{C} & \text{if } m \equiv 0, 1 \pmod 8, \\ \mathbb{H} & \text{if } m \equiv 2, 4 \pmod 8. \end{cases}$$

Choose $\varepsilon, \delta, R,$ *and* T *as in Lemma 1.4.1. If* $R > 0$ *is small enough, then it holds that*

$$\dim_{\mathbb{K}}\ker(\not{D}^{\pi\circ u_t}) = 1$$

for all $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$*,* $t \in [0, T]$*, and there exists* $\Lambda = \Lambda(R) > 0$ *s.t.*

$$\operatorname{spec}(\not{D}^{\pi\circ u_t}) \setminus \{0\} \subset \mathbb{R} \setminus (-\Lambda, \Lambda)$$

for all $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$*,* $t \in [0, T]$*.*

Proof. Since the spectrum of $\not{D}^{u_0}$ is a discrete subset of $\mathbb{R}$, we can choose $\tilde{\Lambda} > 0$ s.t. $\operatorname{spec}(\not{D}^{u_0}) \setminus \{0\} \subset \mathbb{R} \setminus (-\tilde{\Lambda}, \tilde{\Lambda})$. Let $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$. For any $\psi \in \ker(\not{D}^{u_0}) \setminus \{0\}$ we write $\tilde{\psi} := P^{u_0,u_t}\psi$. Using Lemma 1.4.1 we get

$$\begin{aligned}
\|\not{D}^{\pi\circ u_t}\tilde{\psi}\|_{L^2} &= \| \left(P^{u_0,u_t}\not{D}^{u_0}(P^{u_0,u_t})^{-1} - \not{D}^{\pi\circ u_t} \right) \tilde{\psi}\|_{L^2} \\
&\le C_1(R)\|u_t - u_0\|_{C^0(M,\mathbb{R}^q)}\|\tilde{\psi}\|_{L^2} \\
&\le C_1(R) \left(\|u(t,.) - v_0(t,.)\|_{C^0(M,\mathbb{R}^q)} + \|v_0(t,.) - u_0\|_{C^0(M,\mathbb{R}^q)} \right) \|\tilde{\psi}\|_{L^2} \\
&\le C_1(R)2R\|\tilde{\psi}\|_{L^2}
\end{aligned}$$

for all $\psi \in \ker(\not{D}^{u_0}) \setminus \{0\}$. Hence

$$\frac{\|\not{D}^{\pi\circ u_t}\tilde{\psi}\|_{L^2}}{\|\tilde{\psi}\|_{L^2}} \le C_1(R)2R$$

for all $\psi \in \ker(\not{D}^{u_0}) \setminus \{0\}$. Since $\not{D}^{\pi\circ u_t}\colon \Gamma_{W_2^1} \to \Gamma_{L^2}$ is self-adjoint we can estimate the Rayleigh quotient of $(\not{D}^{\pi\circ u_t})^2$ by

$$\frac{((\not{D}^{\pi\circ u_t})^2\tilde{\psi}, \tilde{\psi})_{L^2}}{(\tilde{\psi}, \tilde{\psi})_{L^2}} \le C_1(R)^2 4R^2$$

for all $\psi \in \ker(\not{D}^{u_0}) \setminus \{0\}$.[19] Applying the Min-Max principle (see e.g. [28, Corollary A.7.4.]), we get that $(\not{D}^{\pi\circ u_t})^2$ has at least one eigenvalue (we count eigenvalues by their $\mathbb{K}$-multiplicity) in the interval $[0, C_1(R)^2 4R^2]$. In particular, $\not{D}^{\pi\circ u_t}$ has at least one eigenvalue in $[-C_1(R)2R, C_1(R)2R]$.[20] Now set

$$\Lambda := \frac{1}{2}\tilde{\Lambda}$$

and choose $R > 0$ so small that $C_1(R)2R < \Lambda$. Hence we have shown that $\not{D}^{\pi\circ u_t}$ has at least one eigenvalue in $[-\Lambda, \Lambda]$. With the same methods we can show that $\not{D}^{\pi\circ u_t}$ has precisely one eigenvalue in $[-\Lambda, \Lambda]$. Suppose this is not the case. Choose two eigenvalues λ_1, λ_2 of $\not{D}^{\pi\circ u_t}$ in $[-\Lambda, \Lambda]$ with corresponding eigenvectors $\psi_1, \psi_2 \in \Gamma_{W_2^1}$. We calculate for $\tilde{\psi}_i := P^{u_0,u_t-1}\psi_i$

$$\begin{aligned}
\|(\lambda_i - \not{D}^{u_0})\tilde{\psi}_i\|_{L^2} &= \|P^{u_0,u_t-1}(\lambda_i - \not{D}^{\pi\circ u_t})P^{u_0,u_t}\tilde{\psi}_i - (\lambda_i - \not{D}^{u_0})\tilde{\psi}_i\|_{L^2} \\
&= \|P^{u_0,u_t-1}\not{D}^{\pi\circ u_t}P^{u_0,u_t}\tilde{\psi}_i - \not{D}^{u_0}\tilde{\psi}_i\|_{L^2} \\
&\leq C_1(R)2R\|\tilde{\psi}_i\|_{L^2} \\
&< \frac{1}{2}\tilde{\Lambda}\|\tilde{\psi}_i\|_{L^2}
\end{aligned}$$

Therefore,

$$\frac{\|\not{D}^{u_0}\tilde{\psi}_i\|_{L^2}}{\|\tilde{\psi}_i\|_{L^2}} \leq \frac{\|(\lambda_i - \not{D}^{u_0})\tilde{\psi}_i\|_{L^2}}{\|\tilde{\psi}_i\|_{L^2}} + \frac{\|\lambda_i\tilde{\psi}_i\|_{L^2}}{\|\tilde{\psi}_i\|_{L^2}} \leq \frac{1}{2}\tilde{\Lambda} + \Lambda = \tilde{\Lambda}.$$

As before, we conclude that $\not{D}^{u_0}$ has at least two eigenvalues in $[-\tilde{\Lambda}, \tilde{\Lambda}]$. Because of the choice of $\tilde{\Lambda}$ this is a contradiction to $\dim_{\mathbb{K}}\ker(\not{D}^{u_0}) = 1$.

We have shown that $\not{D}^{\pi\circ u_t}$ has precisely one eigenvalue in $[-\Lambda, \Lambda]$. The symmetry of the spectrum of $\not{D}^{\pi\circ u_t}$ yields that this eigenvalue has to be zero.[21] $\square$

---

[19] Note that $(\not{D}^{\pi\circ u_t})^2\tilde{\psi}$ is a well-defined expression. We have $\tilde{\psi} \in \Gamma_{W_2^1}$ and $\not{D}^{\pi\circ u_t}\tilde{\psi} = \left(P^{u_0,u_t}\not{D}^{u_0}(P^{u_0,u_t})^{-1} - \not{D}^{\pi\circ u_t}\right)\tilde{\psi}$. As in the proof of Lemma 1.2.3 it holds that the operator on the right hand side is of order zero, hence, $\left(P^{u_0,u_t}\not{D}^{u_0}(P^{u_0,u_t})^{-1} - \not{D}^{\pi\circ u_t}\right)\tilde{\psi} \in \Gamma_{W_2^1}$. Therefore $\not{D}^{\pi\circ u_t}\tilde{\psi} \in \Gamma_{W_2^1}$ and thus $(\not{D}^{\pi\circ u_t})^2\tilde{\psi} \in \Gamma_{L^2}$.

[20] This follows from the fact that if λ^2 is an eigenvalue of $(\not{D}^{u_0})^2$, then λ or $-\lambda$ is an eigenvalue of $\not{D}^{u_0}$. This can be shown for example as follows: denote by $E(\lambda^2)$ the (finite dimensional) eigenspace of $(\not{D}^{u_0})^2$ w.r.t. the eigenvalue λ^2. We consider $\not{D}^{u_0} : E(\lambda^2) \to E(\lambda^2)$ with associated matrix $A = (a_{ij})_{ij}$ w.r.t. an L^2-orthonormal basis (b_i) of $E(\lambda^2)$. Notice that $a_{ij} = \langle \not{D}^{u_0}b_i, b_j\rangle_{L^2} = \langle b_i, \not{D}^{u_0}b_j\rangle_{L^2} = \overline{a_{ji}}$, i.e., A is hermitian and in particular diagonalizable. Write $A = Q^{-1}DQ$ for a D a diagonal matrix. Then $A^2 = Q^{-1}D^2Q$ and the eigenvalues of A^2 are precisely the squares of the eigenvalues of A. Since A^2 only has the eigenvalue λ^2 (on the space $E(\lambda^2)$), it follows that λ or $-\lambda$ is an eigenvalue of $\not{D}^{u_0}$.

[21] If $m \not\equiv 3 \pmod 4$, then the spectrum of $\not{D}^f$ is symmetric w.r.t. zero. This can be shown analogously to [16, Theorem 1.3.7 iv)].

Remark 1.4.7. Lemma 1.4.6 is the only place where the restrictions on m in Theorem 1.1.1 play a role. In the proof of Lemma 1.4.6 we used that the spectrum of the Dirac operator along maps is symmetric, which holds if $m \not\equiv 3, 7 \pmod 8$. The dimensions $m \equiv 5, 6 \pmod 8$ were excluded, since in these dimensions there exists no quaternionic structure on Σ_m that commutes with Clifford-multiplication, however, there exists a quaternionic structure on Σ_m that *anticommutes* with Clifford-multiplication. This yields that the kernel of $\not{D}^f$ is a quaternionic vector space, *but not the other eigenspaces.*

Remark 1.4.8. Note that in dimensions $m \equiv 1, 2 \pmod 8$ we can use index theoretical informations to deduce that the dimension of the kernel of $\not{D}^{u_0}$ can not decrease along homotopies of u_0 if we have $\dim_{\mathbb{K}} \ker(\not{D}^{u_0}) = 1$. To be more precise, for $f \colon M \to N$ we have an index $\operatorname{ind}_{f^*TN}(M) \in KO^{-m}(pt)$, c.f. [23, p. 151]. Using the isomorphism $KO^{-m}(pt) \cong \mathbb{Z}_2$ if $m \equiv 1, 2 \pmod 8$ it holds that [23, Theorem 7.13. on p. 151]

$$\operatorname{ind}_{f^*TN}(M) = \begin{cases} [\dim_{\mathbb{C}} \ker(\not{D}^f)]_{\mathbb{Z}_2} & \text{if } m \equiv 1 \pmod 8, \\ [\dim_{\mathbb{H}} \ker(\not{D}^f)]_{\mathbb{Z}_2} & \text{if } m \equiv 2 \pmod 8. \end{cases}$$

Since the index is invariant under homotopies [2, Corollary 4.4.] and $\operatorname{ind}_{u_0^*TN}(M) \neq 0$ we have that $\operatorname{ind}_{g^*TN}(M) \neq 0$ for any $g \colon M \to N$ homotopic to u_0. Hence,

$$\dim_{\mathbb{K}} \ker(\not{D}^g) \geq 1 = \dim_{\mathbb{K}} \ker(\not{D}^{u_0}).$$

Lemma 1.4.9 (Uniform bounds for the resolvents)**.** *Assume we are in the situation of Lemma 1.4.6. We consider the resolvent* $R(\mu, \not{D}^{\pi \circ u_t}) \colon \Gamma_{L^2} \to \Gamma_{L^2}$ *of* $\not{D}^{\pi \circ u_t} \colon \Gamma_{W_2^1} \to \Gamma_{L^2}$. *By Lemma 1.2.3 we know that the restriction*

$$R(\mu, \not{D}^{\pi \circ u_t}) \colon \Gamma_{L^p} \to \Gamma_{W_p^1}$$

is well-defined and bounded for any $2 \leq p < \infty$. *If* $R > 0$ *is small enough, then there exists* $C = C(p, R) > 0$ *s.t.*

$$\sup_{|\mu| = \frac{\Lambda}{2}} \|R(\mu, \not{D}^{\pi \circ u_t})\|_{L(L^p, W_p^1)} < C$$

for all $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $t \in [0, T]$.

Proof. First we uniformly bound

$$\sup_{|\mu| = \frac{\Lambda}{2}} \|R(\mu, P^{u_t, u_0} \not{D}^{\pi \circ u_t} (P^{u_t, u_0})^{-1})\|_{L(L^p, W_p^1)}$$

(recall that P^{u_t, u_0} is an isometry when viewed as a mapping $\Gamma_{L^p} \to \Gamma_{L^p}$ and an isomorphism of Banach spaces when viewed as a mapping $\Gamma_{W_p^1} \to \Gamma_{W_p^1}$, see

Corollary 1.4.5). To be more precise, we will uniformly bound the above resolvents in terms of the resolvent of $\not{D}^{u_0}$ by using Lemma 1.A.4. To apply the lemma we first have to estimate

$$\|P^{u_t,u_0}\not{D}^{\pi\circ u_t}(P^{u_t,u_0})^{-1} - \not{D}^{u_0}\|_{L(W^1_p,L^p)}$$

appropriately. To that end we let $\psi \in \Gamma_{C^1}(\Sigma M \otimes u_0^*TN)$ be arbitrary. Using Lemma 1.4.1 and (1.3.11) we have

$$\begin{aligned}
&\|P^{u_t,u_0}\not{D}^{\pi\circ u_t}(P^{u_t,u_0})^{-1}\psi - \not{D}^{u_0}\psi\|_{L^p}\\
&\le C(R)\|u_t - u_0\|_{C^0(M,\mathbb{R}^q)}\|\psi\|_{L^p}\\
&\le C(R)\Big(\|u(t,.) - v_0(t,.)\|_{C^0(M,\mathbb{R}^q)} + \|v_0(t,.) - u_0\|_{C^0(M,\mathbb{R}^q)}\Big)\|\psi\|_{W^1_p}\\
&\le C(R)2R\|\psi\|_{W^1_p}
\end{aligned}$$

Choosing any $\theta \in (0,1)$ we thus have

$$\begin{aligned}
&\|P^{u_t,u_0}\not{D}^{\pi\circ u_t}(P^{u_t,u_0})^{-1} - \not{D}^{u_0}\|_{L(W^1_p(M),L^p(M))}\\
&\le \theta \min_{|\mu|=\frac{\Lambda}{2}} \frac{1}{\|R(\mu,\not{D}^{u_0})\|_{L(L^p(M),W^1_p(M))}}
\end{aligned}$$

for all $u, v \in B^T_R(v_0) \cap \{u|_{t=0} = u_0\}$, $t \in [0,T]$, $R > 0$ small enough. Applying Lemma 1.A.4 yields

$$\|R(\mu, P^{u_t,u_0}\not{D}^{\pi\circ u_t}(P^{u_t,u_0})^{-1}\|_{L(L^p,W^1_p)} \le \frac{1}{1-\theta}\|R(\mu,\not{D}^{u_0})\|_{L(L^p,W^1_p)}$$

for all $u, v \in B^T_R(v_0) \cap \{u|_{t=0} = u_0\}$, $t \in [0,T]$, $|\mu| = \frac{\Lambda}{2}$, $R > 0$ small enough. We have shown that for $R > 0$ small enough there exists $C = C(p,R)$ s.t.

$$\sup_{|\mu|=\frac{\Lambda}{2}} \|R(\mu, P^{u_t,u_0}\not{D}^{\pi\circ u_t}(P^{u_t,u_0})^{-1})\|_{L(L^p,W^1_p)} < C$$

for all $u \in B^T_R(v_0) \cap \{u|_{t=0} = u_0\}$, $t \in [0,T]$. The lemma now follows from

$$\begin{aligned}
&\|R(\mu,\not{D}^{\pi\circ u_t})\|_{L(L^p,W^1_p)}\\
&= \|R(\mu,(P^{u_t,u_0})^{-1}P^{u_t,u_0}\not{D}^{\pi\circ u_t}(P^{u_t,u_0})^{-1}P^{u_t,u_0})\|_{L(L^p,W^1_p)}\\
&= \|(P^{u_t,u_0})^{-1}R(\mu, P^{u_t,u_0}\not{D}^{\pi\circ u_t}(P^{u_t,u_0})^{-1})P^{u_t,u_0}\|_{L(L^p,W^1_p)}\\
&\le \|(P^{u_t,u_0})^{-1}\|_{L(W^1_p,W^1_p)}\|R(\mu, P^{u_t,u_0}\not{D}^{\pi\circ u_t}(P^{u_t,u_0})^{-1})\|_{L(L^p,W^1_p)}\|P^{u_t,u_0}\|_{L(L^p,L^p)}
\end{aligned}$$

together with the uniform bounds for $\|(P^{u_t,u_0})^{-1}\|_{L(W^1_p,W^1_p)}$ obtained in Corollary 1.4.5. $\square$

In the following, we will construct a particular solution of the constraint equation (1.1.4) with the strategy outlined in the introduction. For this solution we will show the estimates which are necessary for the contraction argument.

Lemma 1.4.10. *In the situation of Lemma 1.4.6 let* $\psi_0 \in \ker(\not{D}^{u_0})$ *with* $\|\psi_0\|_{L^2} = 1$. *We define*

$$\sigma(u_t) := P^{u_0,u_t}\psi_0.$$

Using the decomposition $\Gamma_{L^2} = \ker(\not{D}^{\pi\circ u_t}) \oplus (\ker(\not{D}^{\pi\circ u_t}))^\perp$ *we write*

$$\sigma(u_t) = \sigma_1(u_t) + \sigma_2(u_t).$$

Then it holds that

$$\sqrt{\frac{1}{2}} \leq \|\sigma_1(u_t)\|_{L^2(M)} \leq 1 \tag{1.4.15}$$

for all $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $t \in [0,T]$. *In particular* $\sigma_1(u_t) \neq 0$.

Proof. We write $\sigma := \sigma(u_t)$ and $\sigma_i := \sigma_i(u_t)$. In the following all constants are chosen as in the proof of Lemma 1.4.6 (in particular $C_1(R)2R < \frac{1}{4}\tilde{\Lambda} = \frac{1}{2}\Lambda$). Arguing as in the proof of Lemma 1.4.6 we have

$$\begin{aligned}
\|\not{D}^{\pi\circ u_t}(\sigma_1+\sigma_2)\|_{L^2}^2 &= \|P^{u_0,u_t-1}\not{D}^{\pi\circ u_t}(\sigma_1+\sigma_2)\|_{L^2}^2 \\
&= \|P^{u_0,u_t-1}\not{D}^{\pi\circ u_t}P^{u_0,u_t}\psi_0 - \not{D}^{u_0}\psi_0\|_{L^2}^2 \\
&\leq C_1(R)2R\|\psi_0\|_{L^2}^2 \\
&< \frac{1}{2}\Lambda\|\psi_0\|_{L^2}^2 \\
&= \frac{1}{2}\Lambda\|\sigma\|_{L^2}^2 \\
&= \frac{1}{2}\Lambda(\|\sigma_1\|_{L^2}^2 + \|\sigma_2\|_{L^2}^2).
\end{aligned}$$

Moreover it holds that

$$\|\not{D}^{\pi\circ u_t}(\sigma_1+\sigma_2)\|_{L^2}^2 = \|\not{D}^{\pi\circ u_t}\sigma_2\|_{L^2}^2 \geq \Lambda\|\sigma_2\|_{L^2}^2.$$

(The last inequality can be shown as follows: assume that $\sigma_2 \neq 0$ and $\|\not{D}^{\pi\circ u_t}\sigma_2\|_{L^2}^2 \leq \Lambda\|\sigma_2\|_{L^2}^2$. Then we conclude as in the proof of Lemma 1.4.6 that $\not{D}^{\pi\circ u_t}$ has at least $\dim_{\mathbb{K}}\ker(\not{D}^{\pi\circ u_t}) + 1$ eigenvalues in $[-\Lambda,\Lambda]$ which is a contradiction to Lemma 1.4.6.) Combining these two inequalities, we get

$$\Lambda\|\sigma_2\|_{L^2}^2 < \frac{1}{2}\Lambda(\|\sigma_1\|_{L^2}^2 + \|\sigma_2\|_{L^2}^2),$$

hence

$$\|\sigma_2\|_{L^2}^2 < \|\sigma_1\|_{L^2}^2.$$

This implies

$$1 = \|\sigma\|_{L^2}^2 = \|\sigma_1\|_{L^2}^2 + \|\sigma_2\|_{L^2}^2 < 2\|\sigma_1\|_{L^2}^2,$$

and

$$1 = \|\sigma\|_{L^2}^2 = \|\sigma_1\|_{L^2}^2 + \|\sigma_2\|_{L^2}^2 \geq \|\sigma_1\|_{L^2}^2.$$

□

Let us assume that we are in the situation of Lemma 1.4.10. Let $\Lambda > 0$ be as in Lemma 1.4.6. Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be defined by $\gamma(x) := \frac{\Lambda}{2} e^{ix}$. Then for every $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$ the mapping

$$\begin{aligned}\mathfrak{P}\colon \Gamma_{L^2}(\Sigma M \otimes (\pi \circ u_t)^* TN) &\to \Gamma_{L^2}(\Sigma M \otimes (\pi \circ u_t)^* TN),\\ s &\mapsto -\frac{1}{2\pi i}\int_\gamma R(\mu, \not{D}^{\pi\circ u_t}) s \, d\mu,\end{aligned}$$

where $R(\mu, \not{D}^{\pi\circ u_t})\colon \Gamma_{L^2} \to \Gamma_{L^2}$ is the resolvent of $\not{D}^{\pi\circ u_t}\colon \Gamma_{W_2^1} \to \Gamma_{L^2}$, is the orthogonal projection onto $\ker(\not{D}^{\pi\circ u_t})$. This follows by e.g. [28, Theorem A.4.5. and Corollary A.4.6.].[22]

Lemma 1.4.11. *In the situation of Lemma 1.4.10 we define*

$$\tilde{\psi}(u_t) := -\frac{1}{2\pi i}\int_\gamma R(\mu, \not{D}^{\pi\circ u_t}) \sigma(u_t) \, d\mu$$

for every $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$. *In particular* $\tilde{\psi}(u_t) \in \ker(\not{D}^{\pi\circ u_t}) \subset \Gamma_{C^0}(\Sigma M \otimes (\pi \circ u_t)^* TN)$ *and* $\tilde{\psi}(u_t) \neq 0$. *We write*

$$\psi(u_t) := \frac{\tilde{\psi}(u_t)}{\|\tilde{\psi}(u_t)\|_{L^2(M)}}.$$

Let $\psi^A(u_t)$, $A = 1, \ldots, q$, *be the uniquely determined (global) sections of* ΣM *s.t.*

$$\psi(u_t) = \psi^A(u_t) \otimes (\partial_A \circ \pi \circ u_t).$$

If $\varepsilon > 0$ *and* $T > 0$ *are small enough, then there exists* $C = C(R, \varepsilon, \psi_0) > 0$ *s.t.*

$$\|P^{u_t, v_s}\tilde{\psi}(u_t)(x) - \tilde{\psi}(v_s)(x)\| \leq C\|u_t - v_s\|_{C^0(M, \mathbb{R}^q)}, \tag{1.4.16}$$

and

$$\|\psi^A(u_t)(x) - \psi^A(v_s)(x)\|_{\Sigma_x M} \leq C\|u_t - v_s\|_{C^0(M, \mathbb{R}^q)} \tag{1.4.17}$$

for all $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $A = 1, \ldots, q$, $x \in M$, $s, t \in [0, T]$.

---

[22] To be more precise by [28, Theorem A.4.5.] applied for $\sigma_1 = \{0\}$ and $T = \not{D}^{\pi\circ u_t}\colon W_2^1 \subset L^2 \to L^2$ we get that there exists a decomposition $L^2 = X_1 \oplus X_2$ s.t. $\mathfrak{P}$ is the projection on X_1 along X_2. In particular, $\mathfrak{P}^2 = \mathfrak{P}$ and the image of $\mathfrak{P}$ is X_1. (The terminology "projection on ... along ..." is explained in e.g. [22, p. 20].) Using [28, Corollary A.4.6. i), ii), iii)] we deduce that $X_1 = \ker(\not{D}^{\pi\circ u_t})$. From the definition of $\mathfrak{P}$ we see that $\mathfrak{P}\colon L^2 \to L^2$ is self-adjoint, i.e., $(\mathfrak{P}s_1, s_2)_{L^2} = (s_1, \mathfrak{P}s_2)_{L^2}$ for all $s_1, s_2 \in L^2$. This yields that $\mathfrak{P}$ is the *orthogonal* projection onto $\ker(\not{D}^{\pi\circ u_t})$.

Proof. For the proof we write $\tilde{\psi}^A(u_t)$ for the uniquely determined (global) sections of ΣM s.t.

$$\tilde{\psi}(u_t) = \tilde{\psi}^A(u_t) \otimes (\partial_A \circ \pi \circ u_t).$$

We will prove the lemma in three steps. First we show (1.4.16). Then we use (1.4.16) to get

$$\|\tilde{\psi}^A(u_t)(x) - \tilde{\psi}^A(v_s)(x)\|_{\Sigma_x M} \leq C(R, \varepsilon, \psi_0)\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)} \tag{1.4.18}$$

for all $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $A = 1, \ldots, q$, $x \in M$, $s, t \in [0, T]$. From (1.4.16) and (1.4.18) the equation (1.4.17) will follow from a short computation.

Step 1: Proof of (1.4.16)**:** Using the fact that $P^{u_t,v_s} \colon \Gamma_{L^2} \to \Gamma_{L^2}$ is an isometry (in particular an isomorphism on the Banach space we integrate) and Lemma 1.A.3 we have

$$\begin{aligned}
&P^{u_t,v_s}\tilde{\psi}(u_t) - \tilde{\psi}(v_s) \\
&= -\frac{1}{2\pi i}\Big(P^{u_t,v_s}\int_\gamma R(\mu, \not{D}^{\pi\circ u_t})P^{u_0,u_t}\psi_0\, d\mu - \int_\gamma R(\mu, \not{D}^{\pi\circ v_s})P^{u_0,v_s}\psi_0\, d\mu\Big) \\
&= -\frac{1}{2\pi i}\Big(\int_\gamma P^{u_t,v_s}R(\mu, \not{D}^{\pi\circ u_t})\big(P^{u_t,v_s}\big)^{-1}P^{u_t,v_s}P^{u_0,u_t}\psi_0\, d\mu \\
&\qquad\qquad - \int_\gamma R(\mu, \not{D}^{\pi\circ v_s})P^{u_0,v_s}\psi_0\, d\mu\Big) \\
&= -\frac{1}{2\pi i}\Big(\int_\gamma R(\mu, P^{u_t,v_s}\not{D}^{\pi\circ u_t}\big(P^{u_t,v_s}\big)^{-1})P^{u_t,v_s}P^{u_0,u_t}\psi_0\, d\mu \\
&\qquad\qquad - \int_\gamma R(\mu, \not{D}^{\pi\circ v_s})P^{u_0,v_s}\psi_0\, d\mu\Big) \\
&= -\frac{1}{2\pi i}\int_\gamma R(\mu, P^{u_t,v_s}\not{D}^{\pi\circ u_t}\big(P^{u_t,v_s}\big)^{-1})\Big(P^{u_t,v_s}P^{u_0,u_t}\psi_0 - P^{u_0,v_s}\psi_0\Big)\, d\mu \\
&\quad - \frac{1}{2\pi i}\int_\gamma \Big(R(\mu, P^{u_t,v_s}\not{D}^{\pi\circ u_t}\big(P^{u_t,v_s}\big)^{-1}) - R(\mu, \not{D}^{\pi\circ v_s})\Big)P^{u_0,v_s}\psi_0\, d\mu \\
&= -\frac{1}{2\pi i}\int_\gamma R(\mu, P^{u_t,v_s}\not{D}^{\pi\circ u_t}\big(P^{u_t,v_s}\big)^{-1})\Big(P^{u_t,v_s}P^{u_0,u_t}\psi_0 - P^{u_0,v_s}\psi_0\Big)\, d\mu \\
&\quad - \frac{1}{2\pi i}\int_\gamma \Big(R(\mu, P^{u_t,v_s}\not{D}^{\pi\circ u_t}\big(P^{u_t,v_s}\big)^{-1}) \circ \Big(P^{u_t,v_s}\not{D}^{\pi\circ u_t}\big(P^{u_t,v_s}\big)^{-1} - \not{D}^{\pi\circ v_s}\Big) \circ \\
&\qquad\qquad R(\mu, \not{D}^{\pi\circ v_s})\Big)P^{u_0,v_s}\psi_0\, d\mu.
\end{aligned}$$

Therefore we get for p large enough

$$
\begin{aligned}
&\|P^{u_t,v_s}\tilde{\psi}(u_t)(x)-\tilde{\psi}(v_s)(x)\| \\
&\leq \|P^{u_t,v_s}\tilde{\psi}(u_t)-\tilde{\psi}(v_s)\|_{C^0(M)} \\
&= \|P^{v_s,u_0}P^{u_t,v_s}\tilde{\psi}(u_t)-P^{v_s,u_0}\tilde{\psi}(v_s)\|_{C^0(M)} \\
&\leq C(u_0)\|P^{v_s,u_0}P^{u_t,v_s}\tilde{\psi}(u_t)-P^{v_s,u_0}\tilde{\psi}(v_s)\|_{W^1_p(M)} \\
&\leq C_1\|P^{u_t,v_s}\tilde{\psi}(u_t)-\tilde{\psi}(v_s)\|_{W^1_p(M)} \\
&\leq C_2\left\|\int_\gamma R(\mu,P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1})\left(P^{u_t,v_s}P^{u_0,u_t}\psi_0-P^{u_0,v_s}\psi_0\right)d\mu\right\|_{W^1_p(M)} \\
&+C_2\left\|\int_\gamma\left(R(\mu,P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1})\circ\left(P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1}-\not{D}^{\pi\circ v_s}\right)\circ\right.\right. \\
&\qquad\qquad\left.\left.R(\mu,\not{D}^{\pi\circ v_s})\right)P^{u_0,v_s}\psi_0\,d\mu\right\|_{W^1_p(M)} \\
&\leq C_2\int_\gamma\left\|R(\mu,P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1})\left(P^{u_t,v_s}P^{u_0,u_t}\psi_0-P^{u_0,v_s}\psi_0\right)\right\|_{W^1_p(M)}d\mu \\
&+C_2\int_\gamma\left\|\left(R(\mu,P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1})\circ\left(P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1}-\not{D}^{\pi\circ v_s}\right)\circ\right.\right. \\
&\qquad\qquad\left.\left.R(\mu,\not{D}^{\pi\circ v_s})\right)P^{u_0,v_s}\psi_0\right\|_{W^1_p(M)}d\mu \\
&\leq C_3\sup_{\mu\in\mathrm{Im}(\gamma)}\|R(\mu,P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1})\|_{L(L^p,W^1_p)}\|P^{u_t,v_s}P^{u_0,u_t}\psi_0-P^{u_0,v_s}\psi_0\|_{L^p} \\
&\quad+C_3\sup_{\mu\in\mathrm{Im}(\gamma)}\|R(\mu,P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1})\|_{L(L^p,W^1_p)}\sup_{\mu\in\mathrm{Im}(\gamma)}\|R(\mu,\not{D}^{\pi\circ v_s})\|_{L(L^p,W^1_p)} \\
&\quad\|P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1}-\not{D}^{\pi\circ v_s}\|_{L(W^1_p,L^p)}\|P^{u_0,v_s}\psi_0\|_{L^p}
\end{aligned}
$$

Note that we can drag the W^1_p-norm inside the integral since the inclusion $\Gamma_{W^1_p}\hookrightarrow\Gamma_{L^2}$ is continuous and linear ($p\geq 2$).

In the following we estimate the terms step by step. First of all the norms of the resolvents are uniformly bounded by Lemma 1.4.9, the fact that

$$
\begin{aligned}
&\|R(\mu,P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1})\|_{L(L^p,W^1_p)} \\
&\leq \|P^{u_t,v_s}\|_{L(W^1_p,W^1_p)}\|R(\mu,\not{D}^{\pi\circ u_t})\|_{L(L^p,W^1_p)}\|\left(P^{u_t,v_s}\right)^{-1}\|_{L(L^p,L^p)},
\end{aligned}
$$

and Corollary 1.4.5. Next we estimate $\|P^{u_t,v_s}\not{D}^{\pi\circ u_t}\left(P^{u_t,v_s}\right)^{-1}-\not{D}^{\pi\circ v_s}\|_{L(W^1_p,L^p)}$.

Lemma 1.4.1 yields

$$\begin{aligned}
&\| \left(\left(P^{v_s,u_t} \right)^{-1} \not{D}^{\pi\circ u_t} P^{v_s,u_t} - \not{D}^{\pi\circ v_s} \right) \psi \|_{L^p(M)} \\
&\leq C(R) \| u_t - v_s \|_{C^0(M,\mathbb{R}^q)} \| \psi \|_{L^p(M)} \\
&\leq C(R) \| u_t - v_s \|_{C^0(M,\mathbb{R}^q)} \| \psi \|_{W^1_p(M)}
\end{aligned}$$

for all $\psi \in \Gamma_{C^1}(\Sigma M \otimes (\pi \circ v_s)^* TN)$. Hence

$$\| P^{u_t,v_s} \not{D}^{\pi\circ u_t} (P^{u_t,v_s})^{-1} - \not{D}^{\pi\circ v_s} \|_{L(W^1_p, L^p)} \leq C(R) \| u_t - v_s \|_{C^0(M,\mathbb{R}^q)}.$$

To estimate $\| P^{u_t,v_s} P^{u_0,u_t} \psi_0 - P^{u_0,v_s} \psi_0 \|_{L^p}$ we write $\psi_0 = \psi^i \otimes (b_i \circ u_0)$ for a local orthonormal frame (b_i) of TN. Then it holds that

$$\begin{aligned}
&\| P^{u_t,v_s} P^{u_0,u_t} \psi_0(x) - P^{u_0,v_s} \psi_0(x) \| \\
&= \| \underbrace{\left(P^{u_0,v_s} \right)^{-1} P^{u_t,v_s} P^{u_0,u_t}}_{=:P^c} \psi_0(x) - \psi_0(x) \| \\
&= \| \psi_0^i(x) \otimes \left(P^c(b_i(u_0(x))) - (b_i(u_0(x)) \right) \| \\
&\leq \| \psi_0^i(x) \| \| P^c(b_i(u_0(x))) - (b_i(u_0(x)) \| \\
&\leq \sqrt{\dim(N)} \| \psi_0(x) \| \sum_i \| P^c(b_i(u_0(x))) - (b_i(u_0(x)) \| \\
&\leq C(\varepsilon) \| u_t - v_s \|_{C^0(M,\mathbb{R}^q)} \| \psi_0(x) \|
\end{aligned}$$

where we used Lemma 1.4.3. In particular,

$$\| P^{u_t,v_s} P^{u_0,u_t} \psi_0 - P^{u_0,v_s} \psi_0 \|_{L^p(M)} \leq C(\varepsilon, \psi_0) \| u_t - v_s \|_{C^0(M,\mathbb{R}^q)}.$$

Putting everything together we have shown (1.4.16).

Step 2: Proof of (1.4.18)**:**[23]

[23]In this step it is important to keep track whether we view quantities like $\tilde{\psi}(u_t)(x)$ as an element of $\Sigma_x M \otimes T_{(\pi\circ u_t)(x)} N$, $\Sigma_x M \otimes T_{(\pi\circ u_t)(x)} \mathbb{R}^q$, or $\Sigma_x M \otimes \mathbb{R}^q$. Recall that $\Sigma_x M \otimes T_{(\pi\circ u_t)(x)} N \subset \Sigma_x M \otimes T_{(\pi\circ u_t)(x)} \mathbb{R}^q$ is realized by the isometric embedding $i\colon N \to \mathbb{R}^q$ and $\Sigma_x M \otimes T_{u_t(x)} \mathbb{R}^q = \Sigma_x M \otimes \mathbb{R}^q$ is induced by the identification $\partial_A(u_t(x)) = e_A$ where $\partial_A(\pi(u_t(x)))$ is the A-th standard basis vector of $T_{(\pi\circ u_t)(x)} \mathbb{R}^q$ and e_A is the A-th standard basis vector of $\mathbb{R}^q$. Of course, if we view $\tilde{\psi}(u_t)(x)$ as an element of $\Sigma_x M \otimes T_{(\pi\circ u_t)(x)} \mathbb{R}^q$ it is given by $\tilde{\psi}^A(u_t)(x) \otimes \partial_A(\pi(u_t(x)))$.

We have

$$
\begin{aligned}
&\|\tilde{\psi}^A(u_t)|_x - \tilde{\psi}^A(v_s)|_x\|_{\Sigma_x M} \\
&\leq \left(\sum_{A=1}^{q} \|\tilde{\psi}^A(u_t)|_x - \tilde{\psi}^A(v_s)|_x\|^2_{\Sigma_x M} \right)^{\frac{1}{2}} \\
&= \left(\sum_{A=1}^{q} \left(\|\tilde{\psi}^A(u_t)|_x - \tilde{\psi}^A(v_s)|_x\|^2_{\Sigma_x M} \|e_A\|^2_{\mathbb{R}^q} \right) \right)^{\frac{1}{2}} \\
&= \left(\sum_{A=1}^{q} \|\left(\tilde{\psi}^A(u_t)|_x - \tilde{\psi}^A(v_s)|_x \right) \otimes e_A\|^2_{\Sigma_x M \otimes \mathbb{R}^q} \right)^{\frac{1}{2}} \\
&= \|\tilde{\psi}(u_t)|_x - \tilde{\psi}(v_s)|_x\|_{\Sigma_x M \otimes \mathbb{R}^q} \\
&\leq \|P^{u_t,v_s}\tilde{\psi}(u_t)|_x - \tilde{\psi}(v_s)|_x\|_{\Sigma_x M \otimes \mathbb{R}^q} + \|P^{u_t,v_s}\tilde{\psi}(u_t)|_x - \tilde{\psi}(u_t)|_x\|_{\Sigma_x M \otimes \mathbb{R}^q}
\end{aligned}
$$

In the following we estimate the two summands separately. Using the fact that the differential of $i\colon N \to \mathbb{R}^q$ is an isometry and (1.4.16) we get

$$
\begin{aligned}
\|P^{u_t,v_s}\tilde{\psi}(u_t)|_x - \tilde{\psi}(v_s)|_x\|_{\Sigma_x M \otimes \mathbb{R}^q} &= \|P^{u_t,v_s}\tilde{\psi}(u_t)|_x - \tilde{\psi}(v_s)|_x\|_{\Sigma_x M \otimes T_{(\pi\circ v_s(x))}\mathbb{R}^q} \\
&= \|P^{u_t,v_s}\tilde{\psi}(u_t)|_x - \tilde{\psi}(v_s)|_x\|_{\Sigma_x M \otimes T_{(\pi\circ v_s(x))}N} \\
&\leq C(R,\varepsilon,\psi_0)\|u_t - v_s\|_{C^0(M,\mathbb{R}^q)}.
\end{aligned}
$$

It remains to find an appropriate estimate for $\|P^{u_t,v_s}\tilde{\psi}(u_t)|_x - \tilde{\psi}(u_t)|_x\|_{\Sigma_x M \otimes \mathbb{R}^q}$. To that end, let $\gamma(h) := \exp_{(\pi\circ u_t)(x)}(h\exp^{-1}_{(\pi\circ u_t)(x)}(\pi \circ v_s(x)),\ h \in [0,1]$, be the unique shortest geodesic of N from $(\pi \circ u_t)(x)$ to $(\pi \circ v_s)(x)$ (in particular exp is the exponential map of N).[24] Let $X \in T_{\gamma(0)}N$ be given and denote by $X(h)$ the unique parallel vector field (of N) along γ with $X(0) = X$. Then we have[25]

$$
P^{u_t,v_s}X - X = X(1) - X(0) = \int_0^1 \frac{d}{dh}\bigg|_{h=\tau} X(h)\, d\tau. \tag{1.4.19}
$$

If we denote by $\frac{\nabla^{\mathbb{R}^q}}{dh}$ and $\frac{\nabla^N}{dh}$ the covariant derivatives along γ in $\mathbb{R}^q$ and N,

[24] Recall that γ is well-defined because we chose our constants s.t. (1.3.14) holds.

[25] On the left hand side we have the difference of two vectors in $\mathbb{R}^q$ and the derivative on the right hand side under the integral is the usual derivative of a function $[0,1] \to \mathbb{R}^q$.

respectively, then it holds that

$$\begin{aligned}\frac{\nabla^{\mathbb{R}^q}}{dh}Y &= \frac{\nabla^N}{dh}Y + II(\gamma', Y),\\ \frac{\nabla^{\mathbb{R}^q}}{dh}\bigg|_{h_0} Z &= \frac{d}{dh}\bigg|_{h=h_0} Z\end{aligned}$$

for all vector fields Y of N along γ, all vector fields Z of $\mathbb{R}^q$ along γ, and all $h_0 \in [0,1]$. (On the right hand side of the second equation, we have the ordinary differential of a function $[0,1] \to \mathbb{R}^q$.) Plugging this into (1.4.19) yields

$$P^{u_t,v_s}X - X = \int_0^1 II(\gamma'(h), X(h))\, dh$$

and therefore[26]

$$\begin{aligned}\|P^{u_t,v_s}X - X\|_{\mathbb{R}^q} &\le C \sup_{h\in[0,1]} \|\gamma'(h)\|_N \sup_{h\in[0,1]} \|X(h)\|_N\\ &= C\|\gamma'(0)\|_N \|X\|_N.\end{aligned}$$

Using (1.3.13) and the fact that $\pi\colon \mathbb{R}^q \to \mathbb{R}^q$ is (globally) Lipschitz continuous we have that

$$\begin{aligned}\|\gamma'(0)\|_N &= \|\exp^{-1}_{(\pi\circ u_t)(x)}(\pi\circ v_s)(x)\|_N\\ &\le d^N((\pi\circ u_t)(x), (\pi\circ v_s)(x))\\ &\le C_1\|(\pi\circ u_t)(x) - (\pi\circ v_s)(x)\|_{\mathbb{R}^q}\\ &\le C_2\|u_t(x) - v_s(x)\|_{\mathbb{R}^q}.\end{aligned}$$

Hence,

$$\|P^{u_t,v_s}X - X\|_{\mathbb{R}^q} \le C_3\|u_t(x) - v_s(x)\|_{\mathbb{R}^q}\|X\|_N.$$

We write $\tilde\psi(u_t) = \tilde\psi^i(u_t) \otimes (b_i \circ \pi \circ u_t)$ where (b_i) is a local orthonormal frame of TN and the $\tilde\psi^i(u_t)$ are local sections of ΣM. We have

$$\begin{aligned}&\|P^{u_t,v_s}\tilde\psi(u_t)|_x - \tilde\psi(u_t)|_x\|_{\Sigma_x M\otimes\mathbb{R}^q}\\ &= \|\sum_i \tilde\psi^i(u_t)|_x \otimes \Big(P^{u_t,v_s}(b_i|_{(\pi\circ u_t)(x)}) - b_i|_{(\pi\circ u_t)(x)}\Big)\|_{\Sigma_x M\otimes\mathbb{R}^q}\\ &\le C_3 \max_i \|\tilde\psi^i(u_t)|_x\|_{\Sigma_x M}\|u_t(x) - v_s(x)\|_{\mathbb{R}^q}.\end{aligned}$$

[26]The existence of some $C > 0$ s.t. $\|II(X,Y)\|_{\mathbb{R}^q} \le C\|X\|_N\|Y\|_N$ for all $X, Y \in TN$ can be shown analogously to the proof of Lemma 1.4.2 and using that the differential of $i\colon N \to \mathbb{R}^q$ is an isometry.

Finally, we use (1.4.16) to get

$$
\begin{aligned}
\|\tilde{\psi}^i(u_t)|_x\|_{\Sigma_x M} &\leq \Big(\sum_j \|\tilde{\psi}^j(u_t)|_x\|^2_{\Sigma_x M}\Big)^{\frac{1}{2}} \\
&= \|\tilde{\psi}(u_t)|_x\|_{\Sigma_x M\otimes T_{(\pi\circ u_t)(x)}N} \\
&= \|P^{u_t,u_0}\tilde{\psi}(u_t)|_x\|_{\Sigma_x M\otimes T_{u_0(x)}N} \\
&\leq \|P^{u_t,u_0}\tilde{\psi}(u_t)|_x - \psi_0|_x\|_{\Sigma_x M\otimes T_{u_0(x)}N} + \|\psi_0|_x\|_{\Sigma_x M\otimes T_{u_0(x)}N} \\
&\leq C(R,\varepsilon,\psi_0)\|u_t - u_0\|_{C^0(M,\mathbb{R}^q)} + C_1(\psi_0) \\
&\leq C(R,\varepsilon,\psi_0)\Big(\|u(t,.) - v_0(t,.)\|_{C^0(M,\mathbb{R}^q)} \\
&\quad + \|v_0(t,.) - u_0\|_{C^0(M,\mathbb{R}^q)}\Big) + C_1(\psi_0) \\
&\leq C_2(R,\varepsilon,\psi_0)
\end{aligned}
$$

(recall that our choices of constants in Table 1.1 imply in particular that (1.3.11) holds). Therefore,

$$
\|P^{u_t,v_s}\tilde{\psi}(u_t)|_x - \tilde{\psi}(u_t)|_x\|_{\Sigma_x M\otimes\mathbb{R}^q} \leq C(R,\varepsilon,\psi_0)\|u_t(x) - v_s(x)\|_{\mathbb{R}^q}.
$$

Putting everything together we have shown (1.4.18).

Step 3: Proof of (1.4.17)**:**

We have

$$
\psi^A(u_t)(x) = \frac{\tilde{\psi}^A(u_t)(x)}{\|\tilde{\psi}(u_t)\|_{L^2(M)}}.
$$

This implies

$$\begin{aligned}
&\|\psi^A(u_t)(x) - \psi^A(v_s)(x)\| \\
&= \left\| \frac{\tilde{\psi}^A(u_t)(x)}{\|\tilde{\psi}(u_t)\|_{L^2(M)}} - \frac{\tilde{\psi}^A(v_s)(x)}{\|\tilde{\psi}(v_s)\|_{L^2(M)}} \right\| \\
&= \left\| \frac{\tilde{\psi}^A(u_t)(x)}{\|\tilde{\psi}(u_t)\|_{L^2(M)}} - \frac{\tilde{\psi}^A(u_t)(x)}{\|\tilde{\psi}(v_s)\|_{L^2(M)}} + \frac{\tilde{\psi}^A(u_t)(x)}{\|\tilde{\psi}(v_s)\|_{L^2(M)}} - \frac{\tilde{\psi}^A(v_s)(x)}{\|\tilde{\psi}(v_s)\|_{L^2(M)}} \right\| \\
&\leq \frac{\|\tilde{\psi}^A(u_t)(x)\|}{\|\tilde{\psi}(u_t)\|_{L^2(M)} \|\tilde{\psi}(v_s)\|_{L^2(M)}} \Big| \|\tilde{\psi}(v_s)\|_{L^2(M)} - \|\tilde{\psi}(u_t)\|_{L^2(M)} \Big| \\
&\quad + \frac{1}{\|\tilde{\psi}(v_s)\|_{L^2(M)}} \|\tilde{\psi}^A(u_t)(x) - \tilde{\psi}^A(v_s)(x)\| \\
&= \frac{\|\tilde{\psi}^A(u_t)(x)\|}{\|\tilde{\psi}(u_t)\|_{L^2(M)} \|\tilde{\psi}(v_s)\|_{L^2(M)}} \Big| \|\tilde{\psi}(v_s)\|_{L^2(M)} - \|P^{u_t,v_s} \tilde{\psi}(u_t)\|_{L^2(M)} \Big| \\
&\quad + \frac{1}{\|\tilde{\psi}(v_s)\|_{L^2(M)}} \|\tilde{\psi}^A(u_t)(x) - \tilde{\psi}^A(v_s)(x)\| \\
&\leq \frac{\|\tilde{\psi}^A(u_t)(x)\|}{\|\tilde{\psi}(u_t)\|_{L^2(M)} \|\tilde{\psi}(v_s)\|_{L^2(M)}} \|P^{u_t,v_s} \tilde{\psi}(u_t) - \tilde{\psi}(v_s)\|_{L^2(M)} \\
&\quad + \frac{1}{\|\tilde{\psi}(v_s)\|_{L^2(M)}} \|\tilde{\psi}^A(u_t)(x) - \tilde{\psi}^A(v_s)(x)\| \\
&\leq \Big(\frac{\|\tilde{\psi}^A(u_t)(x)\|}{\|\tilde{\psi}(u_t)\|_{L^2(M)} \|\tilde{\psi}(v_s)\|_{L^2(M)}} + \frac{1}{\|\tilde{\psi}(v_s)\|_{L^2(M)}} \Big) C(R, \varepsilon, \psi_0) \|u_t - v_s\|_{C^0(M,\mathbb{R}^q)}
\end{aligned}$$

where we used (1.4.16) and (1.4.18). Moreover, the L^2-norms in the denominators are uniformly bounded by Lemma 1.4.10. It remains to show that $\|\tilde{\psi}^A(u_t)(x)\|$ is uniformly bounded. To that end,

$$\begin{aligned}
\|\tilde{\psi}^A(u_t)(x)\| &\leq \|\tilde{\psi}^A(u_t)(x) - \tilde{\psi}^A(u_0)(x)\| + \|\tilde{\psi}^A(u_0)(x)\| \\
&\leq C(R, \varepsilon, \psi_0) \|u_t - u_0\|_{C^0(M,\mathbb{R}^q)} + C_1(\psi_0) \\
&\leq C(R, \varepsilon, \psi_0) \Big(\|u(t,.) - v_0(t,.)\|_{C^0(M,\mathbb{R}^q)} \\
&\quad + \|v_0(t,.) - u_0\|_{C^0(M,\mathbb{R}^q)} \Big) + C_1(\psi_0) \\
&\leq C_2(R, \varepsilon, \psi_0).
\end{aligned}$$

This completes the proof of the lemma. □

1.5 Short time existence

In this section we prove Theorem 1.1.1. As we already mentioned in the introduction the proof is inspired by [10]. A contraction argument with a similar structure can be found in [25, Proof of Theorem 5.2.1 on p. 111]. For the latter we also recommend [14] as a supplement.

Proof of Theorem 1.1.1. **Step 1: Solving the equation in $\mathbb{R}^q$:** In this step we want to find a solution $u\colon [0,T]\times M \to \mathbb{R}^q$, $\psi_t\colon M \to \Sigma M \otimes (\pi\circ u_t)^*TN$ of

$$\begin{cases} \partial_t u^A - \Delta u^A = F_1^A(u) + F_2^A(u,\psi) & \text{on } (0,T)\times M,\ A = 1,\dots,q,\\ \not{D}^{\pi\circ u_t}\psi_t = 0 & \text{on } [0,T]\times M,\\ u([0,T]\times M) \subset N_\delta, & \\ u|_{t=0} = u_0, & \\ \psi|_{t=0} = \psi_0, & \\ \|\psi_t\|_{L^2(M)} = 1 & \text{on } [0,T],\\ \dim_{\mathbb{K}}\ker(\not{D}^{\pi\circ u_t}) = 1 & \text{on } [0,T], \end{cases} \tag{1.5.1}$$

see Lemma 1.3.10, where $u_0 \in C^{2+\alpha}(M,N)$ with $\dim_{\mathbb{K}}\ker(\not{D}^{u_0}) = 1$, and $\psi_0 \in \ker(\not{D}^{u_0})$ with $\|\psi_0\|_{L^2(M)} = 1$ are given.

We choose ε, δ, R, and T as in Table 1.1. By making ε, R, and T smaller if necessary, Lemmas 1.4.6 and 1.4.11 hold. Recall that our choices imply in particular that $u([0,T]\times M) \subset N_\delta$ for all $u \in B_R^T(v_0)\cap\{u|_{t=0} = u_0\}$. Let $\psi(u_t)$ and $\psi^A(u_t)$, $A = 1,\dots,q$ be as in Lemma 1.4.11. In particular we have

$$\begin{aligned} \not{D}^{\pi\circ u_t}\psi(u_t) &= 0 \text{ on } [0,T]\times M,\\ \|\psi(u_t)\|_{L^2(M)} &= 1 \text{ on } [0,T],\\ \dim_{\mathbb{K}}\ker(\not{D}^{\pi\circ u_t}) &= 1 \text{ on } [0,T], \end{aligned}$$

and $\psi(u_t)|_{t=0} = \psi(u_0) = \psi_0$ for all $u \in B_R^T(v_0)\cap\{u|_{t=0} = u_0\}$, $t\in[0,T]$.

Plugging $\psi(u_t)$ into the first line of (1.5.1) it remains to find $u \in B_R^T(v_0)\cap\{u|_{t=0} = u_0\}$ that solves

$$\partial_t u^A - \Delta u^A = F_1^A(u) + F_2^A(u,\psi(u)) \text{ on } [0,T]\times M,\ A = 1,\dots,q. \tag{1.5.2}$$

To that end we define for $u \in B_R^T(v_0)\cap\{u|_{t=0} = u_0\}$

$$\begin{aligned} (Lu)(t,x) := &\int_M p(x,y,t)u_0(y)\,dV(y)\\ &+ \int_0^t\int_M p(x,y,t-\tau)\left(F_1(u_\tau)(y) + F_2(u_\tau,\psi(u_\tau))(y)\right)\,dV(y)d\tau \end{aligned}$$

where p is the heat kernel of M and the integrals are to be understood componentwise, i.e., $(Lu)(t,x)\in\mathbb{R}^q$.

In the following we show that if T is small enough, then it holds that

i) $L(B_R^T(v_0) \cap \{u|_{t=0} = u_0\}) \subset B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$,

ii) $\|Lu - Lv\|_{X_T} \leq \frac{1}{2}\|u - v\|_{X_T}$ for all $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$.

We start with i): Let $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$ and consider

$$(Lu - v_0)^A(t, x) = \int_0^t \int_M p(x, y, t - \tau) \left(F_1^A(u_\tau)(y) + F_2^A(u_\tau, \psi(u_\tau))(y)\right) dV(y)d\tau.$$

Note that

$$\begin{aligned}
&\nabla_x (Lu - v_0)^A(t, x) \\
&= \int_0^t \int_M (\nabla_x p)(x, y, t - \tau) \left(F_1^A(u_\tau)(y) + F_2^A(u_\tau, \psi(u_\tau))(y)\right) dV(y)d\tau.
\end{aligned}$$

Using Lemma 1.2.10 we have

$$|(Lu - v_0)^A(t, x)| \leq t \sup_{(s,z)\in[0,T]\times M} |F_1^A(u_s)(z) + F_2^A(u_s, \psi(u_s))(z)|$$

and

$$|\nabla_x (Lu - v_0)^A(t, x)| \leq C\sqrt{t} \sup_{(s,z)\in[0,T]\times M} |F_1^A(u_s)(z) + F_2^A(u_s, \psi(u_s))(z)|$$

for all $(t, x) \in [0, T] \times M$, $A = 1, \ldots, q$, provided that $T \leq 1$ is small enough. Since $u \in B_R^T(v_0)$ we have

$$\|u\|_{X_T} \leq \|u - v_0\|_{X_T} + \|v_0\|_{X_T} \leq R + \|v_0\|_{X_T} \tag{1.5.3}$$

hence

$$\sup_{(s,z)\in[0,T]\times M} |F_1^A(u_s)(z)| \leq C(R, \|v_0\|_{X_1})$$

(recall that $\pi \colon \mathbb{R}^q \to \mathbb{R}^q$ has compact support). Lemma 1.4.11 yields

$$\begin{aligned}
\|\psi^A(u_s)(z)\| &\leq \|\psi^A(u_s)(z) - \psi^A(u_0)(z)\| + \|\psi^A(u_0)(z)\| \\
&\leq C(R, \psi_0)\|u_t - u_0\|_{C^0(M,\mathbb{R}^q)} + \|\psi^A(u_0)(z)\| \\
&\leq C(R, \psi_0)\Big(\|u(t, .) - v_0(t, .)\|_{C^0(M,\mathbb{R}^q)} \\
&\quad + \|v_0(t, .) - u_0\|_{C^0(M,\mathbb{R}^q)}\Big) + \|\psi^A(u_0)(z)\| \\
&\leq C_1(R, \psi_0)
\end{aligned} \tag{1.5.4}$$

(recall that our choice of constants in Table 1.1 implies in particular that (1.3.11) holds). Therefore

$$\sup_{(s,z)\in[0,T]\times M} |F_2^A(u_s, \psi(u_s))(z)| \leq C_2(R, \psi_0).$$

We have shown that if $T > 0$ is small enough, then

$$|(Lu - v_0)^A(t,x)| \leq C_3(R,\psi_0)t,$$
$$|\nabla_x(Lu - v_0)^A(t,x)| \leq C_3(R,\psi_0)\sqrt{t}$$

for all $(t,x) \in [0,T] \times M$, $A = 1,\dots,q$, and for all $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$. Hence for $T > 0$ small enough we have $Lu \in B_R^T(v_0)$. (Note that since $v_0 \in X_T$ the above estimates show in particular that $Lu = (Lu - v_0) + v_0 \in X_T$ for $T > 0$ small enough.) This implies i) since $Lu|_{t=0} = u_0$.

Next we show ii): Let $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$. We have

$$\begin{aligned}
&(Lu - Lv)^A(t,x)\\
&= \int_0^t \int_M p(x,y,t-\tau)\left(F_1^A(u_\tau)(y) - F_1^A(v_\tau)(y)\right) dV(y)d\tau\\
&+ \int_0^t \int_M p(x,y,t-\tau)\left(F_2^A(u_\tau,\psi(u_\tau))(y) - F_2^A(v_\tau,\psi(v_\tau))(y)\right) dV(y)d\tau
\end{aligned}$$

As above Lemma 1.2.10 yields

$$\begin{aligned}
&|(Lu - Lv)^A(t,x)|\\
&\leq t \sup_{(s,z)\in[0,T]\times M} |F_1^A(u_s)(z) - F_1^A(v_s)(z)|\\
&+ t \sup_{(s,z)\in[0,T]\times M} |F_2^A(u_s,\psi(u_s))(z) - F_2^A(v_s,\psi(v_s))(z)|
\end{aligned} \tag{1.5.5}$$

and

$$\begin{aligned}
&|\nabla_x(Lu - Lv)^A(t,x)|\\
&\leq C\sqrt{t} \sup_{(s,z)\in[0,T]\times M} |F_1^A(u_s)(z) - F_1^A(v_s)(z)|\\
&+ C\sqrt{t} \sup_{(s,z)\in[0,T]\times M} |F_2^A(u_s,\psi(u_s))(z) - F_2^A(v_s,\psi(v_s))(z)|
\end{aligned} \tag{1.5.6}$$

for all $(t,x) \in [0,T] \times M$ provided that $T \leq 1$ is small enough. We calculate

$$\begin{aligned}
&|F_1^A(u) - F_1^A(v)|\\
&= |\pi_{BC}^A(u)\langle\nabla u^B, \nabla u^C\rangle - \pi_{BC}^A(v)\langle\nabla v^B, \nabla v^C\rangle|\\
&= |\pi_{BC}^A(u)\Big(\langle\nabla u^B, \nabla u^C\rangle - \langle\nabla v^B, \nabla v^C\rangle\Big)\\
&\quad + \Big(\pi_{BC}^A(u) - \pi_{BC}^A(v)\Big)\langle\nabla v^B, \nabla v^C\rangle|\\
&\leq |\pi_{BC}^A(u)|\Big(\|\nabla u^B\|\|\nabla u^C - \nabla v^C\| + \|\nabla v^C\|\|\nabla u^B - \nabla v^B\|\Big)|+\\
&\quad + |\pi_{BC}^A(u) - \pi_{BC}^A(v)|\|\nabla v^B\|\|\nabla v^C\|
\end{aligned}$$

Using the fact that $\pi^A_{BC}\colon \mathbb{R}^q \to \mathbb{R}$ has compact support and is (globally) Lipschitz continuous together with (1.5.3) we deduce

$$\sup_{(s,z)\in[0,T]\times M} |F^A_1(u_s)(z) - F^A_1(v_s)(z)| \leq C(R)\|u-v\|_{X_T}. \tag{1.5.7}$$

Now let

$$H^A_{DEF}(.) := -\pi^A_B(.)\pi^C_{BD}(.)\pi^C_{EF}(.).$$

We calculate

$$\begin{aligned}
&|F^A_2(u,\psi(u)) - F^A_2(v,\psi(v))| \\
&= |H^A_{DEF}(u)(\psi(u)^D, \nabla u^E \cdot \psi(u)^F) - H^A_{DEF}(v)(\psi(v)^D, \nabla v^E \cdot \psi(v)^F)| \\
&\leq |H^A_{DEF}(u) - H^A_{DEF}(v)|(\psi(u)^D, \nabla u^E \cdot \psi(u)^F) \\
&\quad + |H^A_{DEF}(v)||(\psi(u)^D, \nabla u^E \cdot \psi(u)^F) - (\psi(v)^D, \nabla v^E \cdot \psi(v)^F)| \\
&=: I_1 + I_2
\end{aligned}$$

Combining the fact that H^A_{DEF} is (globally) Lipschitz continuous with (1.5.3) and (1.5.4) yields

$$I_1 \leq C(R,\psi_0)\|u-v\|_{X_T}.$$

Using

$$\begin{aligned}
&(\psi(u)^D, \nabla u^E \cdot \psi(u)^F) - (\psi(v)^D, \nabla v^E \cdot \psi(v)^F) \\
&= (\psi(u)^D - \psi(v)^D, \nabla u^E \cdot \psi(u)^F) \\
&\quad + (\psi(v)^D, (\nabla u^E - \nabla v^E) \cdot \psi(u)^F \\
&\quad + (\psi(v)^D, \nabla v^E \cdot (\psi(u)^F - \psi(v)^F))
\end{aligned}$$

together with the fact that H^A_{DEF} has compact support, (1.5.3), (1.5.4), and Lemma 1.4.11 yields

$$I_2 \leq C(R,\psi_0)\|u-v\|_{X_T}.$$

We have shown

$$\sup_{(s,z)\in[0,T]\times M} |F^A_2(u_s,\psi(u_s))(z) - F^A_2(v_s,\psi(v_s))(z)| \leq C(R,\psi_0)\|u-v\|_{X_T}. \tag{1.5.8}$$

Plugging (1.5.7) and (1.5.8) into (1.5.5) and (1.5.6) yields

$$\begin{aligned}
|(Lu - Lv)^A(t,x)| &\leq tC(R,\psi_0)\|u-v\|_{X_T}, \\
|\nabla_x(Lu - Lv)^A(t,x)| &\leq \sqrt{t}C(R,\psi_0)\|u-v\|_{X_T}
\end{aligned}$$

for all $(t,x) \in [0,T] \times M$, $A = 1,\ldots,q$, and for all $u,v \in B^T_R(v_0) \cap \{u|_{t=0} = u_0\}$. Now ii) follows by choosing T small enough.

Applying the Banach fixed-point theorem we get a unique $u \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$ with $Lu = u$.[27]

Step 2: Regularity of the fixed point: Now we show that the fixed point u is an element of $C^{1,2,\alpha}((0,T) \times M, \mathbb{R}^q)$. Equation (1.5.4) implies that $F_1^A(u)$ and $F_2^A(u, \psi(u))$ are bounded on $[0,T] \times M$. In particular

$$F_1^A(u), F_2^A(u, \psi(u)) \in W^{0,0,p}((0,T) \times M)$$

for all $p \in [1, \infty]$. (A definition of the parabolic Hölder and Sobolev spaces $C^{k,l,\alpha}$ and $W^{k,l,p}$ can be found in e.g. [32].) From the L^p-regularity for the heat equation [32, p. 18, Theorem 3.4.] we get

$$u \in W^{1,2,p}((0,T) \times M)$$

for all $p \in (1, \infty)$. Hence we have[28]

$$u \in C^{0,1,\alpha}((0,T) \times M).$$

Claim: This implies $\psi^A(u) \in \Gamma_{C^{0,0,\alpha}}(\Sigma M \to (0,T) \times M)$, i.e.,

$$\sup_{x \in M} \|\psi^A(u_.)(x)\|_{C^{\frac{\alpha}{2}}((0,T) \to \Sigma_x M)} < \infty, \tag{1.5.9}$$

and

$$\sup_{t \in (0,T)} \|\psi^A(u_t)(.)\|_{\Gamma_{C^\alpha}(\Sigma M \to M)} < \infty. \tag{1.5.10}$$

Proof of the claim: By Lemma 1.4.11 we have

$$\begin{aligned} \|\psi^A(u_t)(x) - \psi^A(u_s)(x)\| &\leq C\|u_t - u_s\|_{C^0(M,\mathbb{R}^q)} \\ &\leq C\|u\|_{C^{0,0,\alpha}((0,T)\times M)} |t-s|^{\frac{\alpha}{2}}, \end{aligned}$$

hence we get (1.5.9). (Recall that we already have (1.5.4) and therefore we only need to consider the Hölder-seminorms.) It remains to show (1.5.10). Given $x, y \in M$ with $d^M(x,y) < \mathrm{inj}(M)$ we denote by $P_{y,x}^t$, $\tilde{P}_{y,x}$, and $\tilde{P}_{y,x}^t$ the parallel transports in $(\pi \circ u_t)^* TN$, ΣM, and $\Sigma M \otimes (\pi \circ u_t)^* TN$ along the unique shortest geodesic of M from y to x, respectively. We need to find an appropriate estimate

[27] Note that $B_R^T(v_0) \cap \{u|_{t=0} = u_0\} \subset X_T$ is closed.

[28] To show that $W^{1,2,p}((0,T) \times M) \subset C^{0,1,\alpha}((0,T) \times M)$ for p large enough one needs the Sobolev embedding and interpolation theory.

for

$$\begin{aligned}
&\|\psi^A(u_t)|_x - \tilde{P}_{y,x}\psi^A(u_t)|_y\|_{\Sigma_x M} \\
&\leq \|\psi(u_t)|_x - \sum_A \tilde{P}_{y,x}\psi^A(u_t)|_y \otimes e_A\|_{\Sigma_x M\otimes\mathbb{R}^q} \\
&\leq \|\psi(u_t)|_x - \tilde{P}^t_{y,x}\psi(u_t)|_y\|_{\Sigma_x M\otimes\mathbb{R}^q} \\
&\quad + \|\tilde{P}^t_{y,x}\psi(u_t)|_y - \sum_A \tilde{P}_{y,x}\psi^A(u_t)|_y \otimes e_A\|_{\Sigma_x M\otimes\mathbb{R}^q} \\
&=: J_1(t) + J_2(t).
\end{aligned} \tag{1.5.11}$$

Since the differential of $i\colon N \to \mathbb{R}^q$ is an isometry we get

$$\begin{aligned}
J_1(t) &= \|\psi(u_t)|_x - \tilde{P}^t_{y,x}\psi(u_t)|_y\|_{\Sigma_x M\otimes\mathbb{R}^q} \\
&= \|\psi(u_t)|_x - \tilde{P}^t_{y,x}\psi(u_t)|_y\|_{\Sigma_x M\otimes T_{(\pi\circ u_t)(x)}N} \\
&= \|P^{u_t,u_0}\psi(u_t)|_x - P^{u_t,u_0}\tilde{P}^t_{y,x}\psi(u_t)|_y\|_{\Sigma_x M\otimes T_{u_0(x)}N} \\
&\leq \|P^{u_t,u_0}\psi(u_t)|_x - \tilde{P}^0_{y,x}P^{u_t,u_0}\psi(u_t)|_y\|_{\Sigma_x M\otimes T_{u_0(x)}N} \\
&\quad + \|\tilde{P}^0_{y,x}P^{u_t,u_0}\psi(u_t)|_y - P^{u_t,u_0}\tilde{P}^t_{y,x}\psi(u_t)|_y\|_{\Sigma_x M\otimes T_{u_0(x)}N} \\
&=: J_{11}(t) + J_{12}(t).
\end{aligned}$$

Using the embedding $W^1_p \hookrightarrow C^\alpha$, for p large enough, and Lemma 1.4.9 we deduce

$$\sup_{t\in[0,T]} \|P^{u_t,u_0}\tilde{\psi}(u_t)\|_{\Gamma_{C^\alpha}(\Sigma M\otimes u_0^*TN)} \leq C_1$$

for some $C_1 > 0$. Hence

$$J_{11}(t) = \|P^{u_t,u_0}\psi(u_t)|_x - \tilde{P}^0_{y,x}P^{u_t,u_0}\psi(u_t)|_y\|_{\Sigma_x M\otimes T_{u_0(x)}N} \leq C_1 d^M(x,y)^\alpha$$

for all $t \in [0,T]$. In order to estimate $J_{12}(t)$ we write $\psi(u_t) = \psi^i(u_t) \otimes (b_i \circ \pi \circ u_t)$ where the $\psi^i(u_t)$ are local sections of ΣM and (b_i) is a local orthonormal frame of TN on some open subset $U \subset N$ s.t. $y \in (\pi \circ u_t)^{-1}(U)$. (Note that in particular the frame (b_i) depends on y and t.) Using the fact that the parallel transport in $\Sigma M \otimes (\pi \circ u_t)^*TM$ is the tensor product of the parallel transports in ΣM and

$(\pi \circ u_t)^* TN$, i.e., $\tilde{P}^t_{y,x} = \tilde{P}_{y,x} \otimes P^t_{y,x}$, we calculate

$$
\begin{aligned}
J_{12}(t) &= \|\tilde{P}^0_{y,x} P^{u_t,u_0} \psi(u_t)|_y - P^{u_t,u_0} \tilde{P}^t_{y,x} \psi(u_t)|_y\|_{\Sigma_x M \otimes T_{u_0(x)} N} \\
&= \|\left(\tilde{P}^t_{y,x}\right)^{-1} \left(P^{u_t,u_0}\right)^{-1} \tilde{P}^0_{y,x} P^{u_t,u_0} \psi(u_t)|_y - \psi(u_t)|_y\|_{\Sigma_y M \otimes T_{u_0(y)} N} \\
&= \|\psi^i(u_t)|_y \otimes \Bigg(\left(P^t_{y,x}\right)^{-1} \left(P^{u_t,u_0}\right)^{-1} P^0_{y,x} P^{u_t,u_0} (b_i \circ \pi \circ u_t)|_y \\
&\qquad\qquad - (b_i \circ \pi \circ u_t)|_y \Bigg) \|_{\Sigma_y M \otimes T_{u_0(y)} N} \\
&\leq \dim(N) \|\psi(u_t)|_y\| \\
&\quad \sum_i \|\left(P^t_{y,x}\right)^{-1} \left(P^{u_t,u_0}\right)^{-1} P^0_{y,x} P^{u_t,u_0} (b_i \circ \pi \circ u_t)|_y - (b_i \circ \pi \circ u_t)|_y\|
\end{aligned}
$$

Using (1.4.16) we see that

$$
\sup_{(t,y) \in [0,T] \times M} \|\psi(u_t)|_y\| \leq C_2 \tag{1.5.12}
$$

for some $C_2 > 0$. (The idea to show this is the same as in (1.5.4).) In the following we denote by $\gamma \colon [0,1] \to M$ the unique shortest geodesic of M from y to x. Notice that

$$
\|\left(P^t_{y,x}\right)^{-1} \left(P^{u_t,u_0}\right)^{-1} P^0_{y,x} P^{u_t,u_0} (b_i \circ \pi \circ u_t)|_y - (b_i \circ \pi \circ u_t)|_y\|
$$

is the deviation of the parallel transport (of N) along the "rectangle" Q from the identity, where Q is obtained by first following the unique shortest geodesic of N from $(\pi \circ u_t)(y)$ to $u_0(y)$, then following the curve $u_0 \circ \gamma$ from $u_0(y)$ to $u_0(x)$, then following the unique shortest geodesic of N from $u_0(x)$ to $(\pi \circ u_t)(x)$ and finally following the curve $\pi \circ u_t \circ \gamma$ from $(\pi \circ u_t)(x)$ to $(\pi \circ u_t)(y)$. The "filled rectangle" Q is given by the image of

$$
\begin{aligned}
&\alpha \colon [0,1] \times [0,1] \to N, \\
&\qquad (h,k) \mapsto \exp_{(\pi \circ u_t \circ \gamma)(h)} \Big(k \exp^{-1}_{(\pi \circ u_t \circ \gamma)(h)} (u_0 \circ \gamma)(h) \Big).
\end{aligned}
$$

Using the same arguments as in the estimate for $\|P^{\square} Z - Z\|$ in the proof of Lemma 1.4.4 we get

$$
\sup_{(h,k) \in [0,1] \times [0,1]} \|\frac{\partial}{\partial k} \alpha(h,k)\| \leq C_3,
$$

$$
\|\frac{\partial}{\partial h} \alpha(h,k)\| \leq C_4 \|\gamma'(0)\| \leq C_4 d^M(x,y),
$$

(where C_3 and C_4 depend on R) and conclude

$$
\sup_{t \in [0,T]} \|\left(P^t_{y,x}\right)^{-1} \left(P^{u_t,u_0}\right)^{-1} P^0_{y,x} P^{u_t,u_0} Z - Z\| \leq C_5 d^M(x,y) \|Z\|
$$

for all $Z \in T_{(\pi\circ u_t)(y)}N$. This yields

$$\begin{aligned} J_{12}(t) &\le C_6 d^M(x,y) \\ &\le C_6 d^M(x,y)^{1-\alpha} d^M(x,y)^\alpha \\ &\le C_6 \mathrm{inj}(M)^{1-\alpha} d^M(x,y)^\alpha. \end{aligned}$$

Putting together our estimates for $J_1(t)$, $J_{11}(t)$, and $J_{12}(t)$ we get

$$J_1(t) \le C_7 d^M(x,y)^\alpha \tag{1.5.13}$$

for all $t \in [0,T]$. It remains to estimate $J_2(t)$. We write $\psi(u_t) = \psi^i(u_t) \otimes (b_i \circ \pi \circ u_t)$ as before and recall that $\psi^A(u_t) = \langle b_i \circ \pi \circ u_t, \partial_A \circ \pi \circ u_t \rangle \psi^i(u_t)$. Using (1.5.12) again we get

$$\begin{aligned} J_2(t) &= \|\tilde{P}^t_{y,x}\psi(u_t)|_y - \sum_A \tilde{P}_{y,x}\psi^A(u_t)|_y \otimes e_A\|_{\Sigma_x M \otimes \mathbb{R}^q} \\ &\le \|\tilde{P}_{y,x}\psi^i(u_t)|_y \otimes \left(P^t_{y,x}(b_i \circ \pi \circ u_t)|_y - (b_i \circ \pi \circ u_t)|_y \right)\|_{\Sigma_x M \otimes T_{(\pi\circ u_t)(y)}N} \\ &\le \dim(N)\|\psi(u_t)|_y\| \sum_i \|P^t_{y,x}(b_i \circ \pi \circ u_t)|_y - (b_i \circ \pi \circ u_t)|_y\| \\ &\le C_8 \sum_i \|P^t_{y,x}(b_i \circ \pi \circ u_t)|_y - (b_i \circ \pi \circ u_t)|_y\| \end{aligned}$$

Analogously to (1.4.14)[29] we get

$$\begin{aligned} \|P^t_{y,x}(b_i \circ \pi \circ u_t)|_y - (b_i \circ \pi \circ u_t)|_y\| &\le C_9 \|\gamma'(0)\| \\ &\le C_9 d^M(x,y) \\ &\le C_9 \mathrm{inj}(M)^{1-\alpha} d^M(x,y)^\alpha \end{aligned}$$

(where $\gamma\colon [0,1] \to M$ still denotes the unique shortest geodesic of M from y to x). Hence

$$J_2(t) \le C_{10} d^M(x,y)^\alpha$$

for all $t \in [0,T]$. Combining this with (1.5.11) and (1.5.13) yields

$$\|\psi^A(u_t)|_x - \tilde{P}_{y,x}\psi^A(u_t)|_y\|_{\Sigma_x M} \le C_{11} d^M(x,y)^\alpha$$

for all $t \in [0,T]$ and all $x, y \in M$ with $d^M(x,y) < \mathrm{inj}(M)$. This shows (1.5.10). Therefore we have shown the claim. ✓

[29] To be precise one has to do the following: cover N by finitely many local orthonormal frames (b_i) whose local C^1-norms are bounded. Then the local C^1-norm of $b_i \circ \pi \circ u_t$ (viewed as a section of $(\pi \circ u_t)^* TN$) is bounded by some constant that depends on R. Hence C_9 depends only on R (and the choice of a covering of N by local orthonormal frames).

Combining the claim with $u \in C^{0,1,\alpha}((0,T) \times M)$ yields

$$F_1(u), F_2(u, \psi(u)) \in C^{0,0,\alpha}((0,T) \times M).$$

By the Hölder-regularity for the heat equation [32, p. 17, Theorem 3.3.] we deduce

$$u \in C^{1,2,\alpha}((0,T) \times M, \mathbb{R}^q).$$

This implies[30]

$$u \in C^{1,2,\alpha}((0,T) \times M, \mathbb{R}^q) \cap C^0([0,T] \times M).$$

Step 3: The fixed point takes values in N: First let $f\colon (0,T) \times M \to \mathbb{R}^q$ be an arbitrary function s.t. $f(t,.) \in C^2(M, \mathbb{R}^q)$ for all $t \in (0,T)$ and $f(.,p) \in C^1((0,T), \mathbb{R}^q)$ for all $p \in M$. In the following we write $\|.\|_2$ and $\langle .,. \rangle_2$ for the Euclidean metric and scalar product, respectively. Similarly we write $\|.\|_g$ and $\langle .,. \rangle_g$ for the norm and scalar product of (M,g), respectively. Then we define

$$\rho\colon \mathbb{R}^q \to \mathbb{R}^q$$

by $\rho(z) := z - \pi(z)$ and

$$\varphi\colon (0,T) \times M \to \mathbb{R}$$

by $\varphi(t,x) := \|\rho(f(t,x))\|_2^2 = \sum_{A=1}^q |\rho^A(f(t,x))|^2$.

We have

$$\frac{\partial}{\partial t}\varphi(t,x) = 2\langle \rho(f(t,x)), \frac{\partial}{\partial t}(\rho \circ f)(t,x)\rangle_2$$

and the product rule for the Laplace-Beltrami operator[31] yields

$$\Delta_x \varphi(t,x) = 2\langle \rho(f(t,x)), \Delta_x(\rho \circ f)(t,x)\rangle_2 + 2\sum_{A=1}^q \|\nabla_x(\rho^A \circ f)(t,x)\|_g^2$$

(where the Laplace-Beltrami operator on the right hand side is to be understood componentwise, i.e., $\Delta_x(\rho \circ f)(t,x) = (\Delta_x(\rho^A \circ f)(t,x))_A \in \mathbb{R}^q$). We have shown

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta_x\Big)\varphi(t,x) &= -2\sum_{A=1}^q \|\nabla_x(\rho^A \circ f)(t,x)\|_g^2 \\ &\quad + 2\Big\langle \rho(f(t,x)), \Big(\frac{\partial}{\partial t} - \Delta_x\Big)(\rho \circ f)(t,x)\Big\rangle_2.\end{aligned} \tag{1.5.14}$$

[30]Since $u \in C^{1,2,\alpha}((0,T) \times M) \subset C^{0,\frac{\alpha}{2}}((0,T); C^2(M))$ (c.f. [32]) we have in particular that $u\colon (0,T) \to C^0(M)$ is $\frac{\alpha}{2}$-Hölder continuous. Hence $u\colon (0,T) \to C^0(M)$ is uniformly continuous and can therefore be continuously extended to $u\colon [0,T] \to C^0(M)$.

[31]Namely, $\Delta(hk) = h\Delta k + k\Delta h + 2\langle \nabla h, \nabla k\rangle_g$.

Now we further calculate the time derivative and the Laplacian of $\rho \circ f$ on the right hand side. To that end, notice that

$$\begin{aligned}\frac{\partial}{\partial t}(\rho \circ f)(t,x) &= \frac{\partial}{\partial t}(f - \pi \circ f)(t,x) \\ &= \frac{\partial}{\partial t} f(t,x) - d\pi|_{f(t,x)} \frac{\partial}{\partial t} f(t,x) \\ &= \Big(\frac{\partial}{\partial t} f^A(t,x) - \pi^A_B(f(t,x)) \frac{\partial}{\partial t} f^B(t,x)\Big)_A \\ &= \Big(\nu^A_B(f(t,x)) \frac{\partial}{\partial t} f^B(t,x)\Big)_A .\end{aligned} \tag{1.5.15}$$

Moreover it holds that

$$\begin{aligned}&\Delta_x(\rho^A \circ f)(t,x) \\ &= \nu^A_B(f(t,x))\Delta_x f^B(t,x) - \pi^A_{CB}(f(t,x))\langle \nabla_x f^B(t,x), \nabla_x f^C(t,x)\rangle_g\end{aligned} \tag{1.5.16}$$

To prove this, we fix (t,x) and let ψ be normal coordinates of M centered in x. To shorten notation we write $f = f(t,.)\colon M \to \mathbb{R}^q$ in the proof of (1.5.16). We have

$$\Delta(\rho^A \circ f)|_x = \sum_i \partial_{x_i}\partial_{x_i}(\rho^A \circ f \circ \psi^{-1})|_{\psi(x)}.$$

For an arbitrary y in the Image of ψ it holds that

$$\begin{aligned}\partial_{x_i}(\rho^A \circ f \circ \psi^{-1})|_y &= \partial_{x_i}(f^A \circ \psi^{-1} - \pi^A \circ f \circ \psi^{-1})|_y \\ &= \partial_{x_i}(f^A \circ \psi^{-1})|_y - \pi^A_B(f(\psi^{-1}(y)))\partial_{x_i}(f^B \circ \psi^{-1})|_y\end{aligned}$$

hence

$$\begin{aligned}&\partial_{x_i}\partial_{x_i}(\rho^A \circ f \circ \psi^{-1})|_{\psi(x)} \\ &= \partial_{x_i}\partial_{x_i}(f^A \circ \psi^{-1})|_{\psi(x)} - \partial_{x_i}\Big(\pi^A_B(f \circ \psi^{-1})\partial_{x_i}(f^B \circ \psi^{-1})\Big)|_{\psi(x)} \\ &= \partial_{x_i}\partial_{x_i}(f^A \circ \psi^{-1})|_{\psi(x)} - \pi^A_{CB}(f(x))\partial_{x_i}(f^C \circ \psi^{-1})|_{\psi(x)}\partial_{x_i}(f^B \circ \psi^{-1})|_{\psi(x)} \\ &\quad - \pi^A_B(f(x))\partial_{x_i}\partial_{x_i}(f^B \circ \psi^{-1})|_{\psi(x)} \\ &= \nu^A_B(f(x))\partial_{x_i}\partial_{x_i}(f^B \circ \psi^{-1})|_{\psi(x)} \\ &\quad - \pi^A_{CB}(f(x))\partial_{x_i}(f^C \circ \psi^{-1})|_{\psi(x)}\partial_{x_i}(f^B \circ \psi^{-1})|_{\psi(x)}\end{aligned}$$

Thus

$$
\begin{aligned}
&\Delta(\rho^A \circ f)|_x \\
&= \sum_i \partial_{x_i}\partial_{x_i}(\rho^A \circ f \circ \psi^{-1})|_{\psi(x)} \\
&= \sum_i \Big(\nu^A_B(f(x))\partial_{x_i}\partial_{x_i}(f^B \circ \psi^{-1})|_{\psi(x)} \\
&\qquad\qquad - \pi^A_{CB}(f(x))\partial_{x_i}(f^C \circ \psi^{-1})|_{\psi(x)}\partial_{x_i}(f^B \circ \psi^{-1})|_{\psi(x)}\Big) \\
&= \nu^A_B(f(x))\Delta f^B|_x - \pi^A_{CB}(f(x))\langle \nabla f^C(x), \nabla f^B(x)\rangle_g
\end{aligned}
$$

(for the last line we used that ψ are normal coordinates centered in $x \in M$) and we have shown (1.5.16). Plugging (1.5.15) and (1.5.16) into (1.5.14) yields

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta_x\Big)\varphi(t,x) = &-2\sum_{A=1}^{q} \|\nabla_x(\rho^A \circ f)(t,x)\|_g^2 \\
&+ 2\Big\langle \rho(f(t,x)), \nu^A_B(f(t,x))\Big(\frac{\partial}{\partial t} - \Delta_x\Big) f^B(t,x)\Big\rangle_2 \\
&+ 2\Big\langle \rho(f(t,x)), \pi^A_{CB}(f(t,x))\langle \nabla_x f^C(t,x), \nabla_x f^B(t,x)\rangle_g\Big\rangle_2
\end{aligned}
$$

Now let $f = u$ be the solution constructed in the first step. Then we have

$$
\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta_x\Big)\varphi &= -2\sum_{A=1}^{q} \|\nabla_x(\rho^A \circ u)\|_g^2 \\
&\quad + 2\Big\langle \rho(u), \nu^A_B(u)\Big(F_1^B(u) + F_2^B(u,\psi(u))\Big)\Big\rangle_2 \\
&\quad + 2\Big\langle \rho(u), -F_1(u)\Big\rangle_2 \\
&= -2\sum_{A=1}^{q} \|\nabla_x(\rho^A \circ u)\|_g^2 \\
&\quad + 2\Big\langle \rho(u), -\pi^A_B(u)F_1^B(u) + \nu^A_B(u)F_2^B(u,\psi(u))\Big\rangle_2 \\
&= -2\sum_{A=1}^{q} \|\nabla_x(\rho^A \circ u)\|_g^2 \\
&\leq 0.
\end{aligned}
$$

Here we used that $\Big\langle \rho(u), -\pi^A_B(u)F_1^B(u) + \nu^A_B(u)F_2^B(u,\psi(u))\Big\rangle_2 = 0$. This holds because of the following: let (t,x) be arbitrary. Since $u(t,x) \in N_\delta$, we have that $\rho(u(t,x)) = u(t,x) - \pi(u(t,x)) \in T^\perp_{\pi(u(t,x))}N$. Moreover, $\Big(\pi^A_B(u(t,x))F_1^B(u)(t,x)\Big)_A \in$

$T_{\pi(u(t,x))}N$ since

$$\pi^A_B(u(t,x))F^B_1(u)(t,x) = (d\pi^A)_{u(t,x)}(F_1(u)(t,x)) = \Big((d\pi)_{u(t,x)}(F_1(u)(t,x)\Big)^A$$

and $(d\pi)_{u(t,x)}\colon \mathbb{R}^q \to T_{\pi(u(t,x))}N$. Hence,

$$\Big\langle \rho(u(t,x)), -\pi^A_B(u(t,x))F^B_1(u)(t,x)\Big\rangle_2 = 0.$$

To see that $\Big\langle \rho(u(t,x)), \nu^A_B(u(t,x))F^B_2(u,\psi(u))(t,x)\Big\rangle_2 = 0$ we write

$$\begin{aligned}&\Big\langle \rho(u(t,x)), \nu^A_B(u(t,x))F^B_2(u,\psi(u))(t,x)\Big\rangle_2\\ &= \Big\langle \rho(u(t,x)), F_2(u,\psi(u))(t,x) - \pi^A_B(u(t,x))F^B_2(u,\psi(u))(t,x)\Big\rangle_2\end{aligned}$$

and note that by definition of F_2 we have that $F_2(u,\psi(u))(t,x) \in T_{\pi(u(t,x))}N$ and as above we have

$$\Big(\pi^A_B(u(t,x))F^B_2(u,\psi(u))(t,x)\Big)_A = (d\pi)_{u(t,x)}\Big(F_2(u,\psi(u))(t,x)\Big) \in T_{\pi(u(t,x))}N.$$

Since $\left(\frac{\partial}{\partial t} - \Delta_x\right)\varphi(t,x) \leq 0$ for all (t,x) and $\varphi(0,.) = 0$ on M the maximum principle for the heat equation[32] yields $\varphi(t,x) \leq 0$ for all (t,x). The definition of φ implies $\varphi \geq 0$, hence $\varphi(t,x) = 0$ for all (t,x). This implies $u(t,x) \in N$ for all (t,x).

Step 4: Uniqueness of the solution: Let (u^1,ψ^1) and (u^2,ψ^2) be two solutions of the heat flow for Dirac harmonic maps as in Theorem 1.1.1.[33] In particular, $u^i\colon [0,T] \times M \to N \subset \mathbb{R}^q$, $\psi^i_t\colon M \to \Sigma M \otimes (u^i_t)^*TN$ solve (1.5.1) and $u^i \in C^{1,2,\alpha}((0,T) \times M, \mathbb{R}^q)$, $i = 1,2$.[34] Let $R > 0$ be as in the first step (i.e., L is a contraction on $B^{\hat{T}}_R(v_0) \cap \{u|_{t=0} = u_0\}$ for all $\hat{T} = \hat{T}(R) > 0$ small enough). We show that for $\widetilde{T} \leq T$ small enough, it holds that

$$u^1, u^2 \in B^{\widetilde{T}}_R(v_0).$$

We have that

$$\|u^i(t,.) - u_0\|_{C^0(M)}, \|\nabla u^i(t,.) - \nabla u_0\|_{C^0(M)} \to 0$$

[32] See e.g. [13, Theorem 4.4 on p. 96].

[33] $C^{1,2,\alpha}((0,T) \times M, N) = \{u \in C^{1,2,\alpha}((0,T) \times M, \mathbb{R}^q) \mid u(t,x) \in N$ for all $(t,x) \in (0,T) \times M\}$.

[34] Note that here $T > 0$ is just some T s.t. Theorem 1.1.1 holds. It does not have to do anything with the T we constructed in the first step.

for $t \to 0$.[35] Moreover we have

$$\|v_0(t,.) - u_0\|_{C^0(M)}, \|\nabla v_0(t,.) - \nabla u_0\|_{C^0(M)} \to 0$$

for $t \to 0$.[36] Therefore for $\widetilde{T} > 0$ small enough it holds that

$$\|u^i - v_0\|_{X_{\widetilde{T}}} < R,$$

and

$$u^i \in B_R^{\widetilde{T}}(v_0) \cap \{u|_{t=0} = u_0\}.$$

Since $\dim_{\mathbb{K}}(\not{D}^{\pi \circ u_t^i}) = 1$ on $[0,T]$ we have that

$$\psi_t^i = \psi(u_t^i) h_t^i$$

for all $t \in [0,\widetilde{T}]$ where $\psi(u_t^i)$ is defined as in Lemma 1.4.11 and $h_t^i \in \mathbb{K}$ for all $t \in [0,\widetilde{T}]$ and $i = 1,2$. Moreover, h_t^i is of unit length since $\|\psi_t^i\|_{L^2(M)} = \|\psi(u_t^i)\|_{L^2(M)} = 1$. Since the (real part of the) bundle metric on ΣM is invariant under multiplication with elements of $\mathbb{K}$ of unit length, c.f. Lemma 1.2.8 for the case $\mathbb{K} = \mathbb{H}$, we have that

$$F_2(u^i, \psi^i) = F_2(u^i, \psi(u^i))$$

on $[0,\widetilde{T}] \times M$.[37] In summary we have shown that u_1 and u_2 are elements of $B_R^{\widetilde{T}}(v_0)$ that solve (1.5.2). Since the fixed point we constructed in step 1 is unique, we have that $u_1 = u_2$ on $[0,\widetilde{T}] \times M$ for $\widetilde{T} > 0$ small enough. Next we define

$$T_0 := \sup\{t \in [\widetilde{T}, T] \mid u^1 = u^2 \text{ on } [0,t] \times M\}$$

---

[35] This can be seen as follows: we write $u = u^i$. Since $u \in C^{1,2,\alpha}((0,T) \times M) \subset C^{0,\frac{\alpha}{2}}((0,T); C^2(M))$ (c.f. [32]) we have in particular that $u, \nabla u\colon (0,T) \to C^0(M)$ are $(\alpha/2)$-Hölder continuous. (In the case of ∇u we write $C^0(M)$ as target space shortly for $\Gamma(TM)$ with the C^0-norm.) Hence $u, \nabla u\colon (0,T) \to C^0(M)$ are uniformly continuous and can therefore be continuously extended to $u, \nabla u\colon [0,T] \to C^0(M)$. Hence $u(t,.) \to u_0$ in $C^0(M)$ as $t \to 0^+$ and there exists a vector field $V \in \Gamma(TM)$ s.t. $\nabla u(t,.) \to V$ in $C^0(M)$ as $t \to 0^+$. We show $V = \nabla u_0$. To that end notice that for every $X \in \Gamma(TM)$ we have

$$\int_M \langle \nabla u(t,.), X\rangle = -\int_M u(t,.)\mathrm{div}(X) \xrightarrow{t \to 0^+} -\int_M u_0 \mathrm{div}(X) = \int_M \langle \nabla u_0, X\rangle,$$

and

$$\int_M \langle \nabla u(t,.), X\rangle \xrightarrow{t \to 0^+} \int_M \langle V, X\rangle.$$

[36] This can be shown analogously. Note that $v_0 \in C^{1,2,\alpha}((0,T) \times M)$ since $\partial_t v_0 - \Delta v_0 = 0$ and from Lemma 1.2.10 we have that $v_0(t,.) \to u_0$ in $C^0(M)$ as $t \to 0^+$.

[37] Here we used that $X \cdot (\varphi h) = (X \cdot \varphi) h$ for all $X \in T_pM$, $\varphi \in \Sigma_p M$, $h \in \mathbb{K}$, as shown in the proof of Proposition 1.2.6.

By the definition of T_0 and continuity we have $u^1 = u^2$ on $[0, T_0] \times M$. We show $T_0 = T$. To that end we argue by contradiction and suppose that $T_0 < T$. Then $(\hat{u}^i, \hat{\psi}^i)$ defined by $\hat{u}^i(t,x) := u^i(t + T_0, x)$, $(t,x) \in [0, T - T_0] \times M$, $\hat{\psi}^i_t := \psi^i_{t+T_0}$ are solutions of the heat flow for Dirac harmonic maps with T replaced by $T - T_0$, u_0 replaced by $\hat{u}_0$, where $\hat{u}_0 := u^1(T_0, .) = u^2(T_0, .)$, and ψ_0 replaced by $\hat{\psi}_0 := \psi^1_{T_0} = \psi^2_{T_0}$.[38] Using the preceding argument we get that there exists some $\widetilde{T} > T_0$ s.t. $u^1 = u^2$ on $[T_0, \widetilde{T}] \times M$. This contradicts the definition of T_0. Therefore $T_0 = T$. $\square$

[38]Since $\psi^1_{T_0}, \psi^2_{T_0} \in \ker(\not{D}^{u^1_{T_0}})$ and $\dim_{\mathbb{K}} \ker(\not{D}^{u^1_{T_0}}) = 1$, we can assume w.l.o.g. that $\psi^1_{T_0} = \psi^2_{T_0}$. Otherwise we replace ψ^2 by $\psi^2 h$, where $h \in \mathbb{K}$ has unit length with $\psi^1_{T_0} = \psi^2_{T_0} h$.

Appendix

1.A Results and definitions from functional analysis

In this section X denotes a complex Banach space. Recall that a mapping $T\colon D(T) \to X$ is called *operator* if $D(T) \subset X$ is a linear subspace and $T\colon D(T) \to X$ is a linear map.

Definition 1.A.1. Let $T\colon D(T) \to X$ be an operator.

i) T is called *densely defined* if $D(T)$ is a dense subset of X.

ii) T is called *closed* if the graph of T,

$$\operatorname{graph}(T) := \{(x, Tx) \mid x \in D(T)\},$$

is a closed subset of $X \times X$.

iii) Assume that T is closed (not necessarily densely defined). Then we define the *resolvent set of* T by

$$\begin{aligned}\rho(T) &:= \{\lambda \in \mathbb{C} \mid \lambda I - T\colon D(T) \to X \text{ is bijective}\}\\ &= \{\lambda \in \mathbb{C} \mid \lambda I - T\colon D(T) \to X \text{ is bijective and } (\lambda I - T)^{-1} \in L(X)\}\end{aligned}$$

where $L(X)$ denotes the set of bounded linear operators $X \to X$ and we used the closed graph theorem. For $\lambda \in \rho(T)$ we write

$$R(\lambda, T) := (\lambda I - T)^{-1}\colon X \to X$$

and call $R(\lambda, T)$ the *resolvent of* T. Moreover we define the *spectrum of* T by

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

Finally we define the *point spectrum of* T, the *continuous spectrum of* T, and the *rest spectrum of* T by

$$\begin{aligned}\sigma_p(T) &:= \{\lambda \in \mathbb{C} \mid \lambda I - T\colon D(T) \to X \text{ is not injective}\},\\ \sigma_c(T) &:= \{\lambda \in \mathbb{C} \mid \lambda I - T\colon D(T) \to X \text{ is injective,}\\ &\qquad\qquad \text{not surjective, but has dense image}\},\\ \sigma_r(T) &:= \{\lambda \in \mathbb{C} \mid \lambda I - T\colon D(T) \to X \text{ is injective}\\ &\qquad\qquad \text{and does not have dense image}\},\end{aligned}$$

respectively. If $\lambda \in \sigma_p(T)$, then we call λ *eigenvalue (of T)*.

iv) Assume that T is closed. We say that T *has compact resolvent* if $\rho(T) \neq \varnothing$ and for all $\lambda \in \rho(T)$ it holds that $R(\lambda, T) := (\lambda I - T)^{-1}\colon X \to X$ is a compact operator.

Remark 1.A.2. $\lambda \in \rho(T)$ means that for every $f \in X$, the equation $\lambda u - Tu = f$ has a unique solution $u \in D(T)$ which depends continuously on the right hand side f in the sense that the mapping $X \ni f \mapsto u \in X$ is continuous.

If we have an operator $T\colon D(T) \to X$ that is not closed, we could attempt to define the resolvent set of T analogously, i.e.,

$$\rho(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T\colon D(T) \to X \text{ is bijective and } (\lambda I - T)^{-1} \in L(X)\},$$

but then it can be shown that $\rho(T) = \emptyset$, so this is not a good definition.[39] Therefore we require T to be closed in the definition of the resolvent.

If $T\colon D(T) \to X$ is a so-called closable operator, then the resolvent set of T is often defined by $\rho(T) := \rho(\overline{T})$ where $\overline{T}$ is the closure of T (which is in particular a closed operator).

The statement of the following lemma can be found in [26, p. 408].

Lemma 1.A.3. *Let $T_0\colon D(T) \subset X \to X$ be an operator with the same domain of definition as T. Then for all $\lambda \in \rho(T) \cap \rho(T_0)$ we have*

$$R(\lambda, T) - R(\lambda, T_0) = R(\lambda, T) \circ (T - T_0) \circ R(\lambda, T_0).$$

Proof. We choose an arbitrary $x \in X$. Then there exists $y \in D(T)$ s.t. $x =$

[39] Defining $\rho(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T\colon D(T) \to X \text{ is bijective}\}$ for a non-closed operator $T\colon D(T) \to X$ is also not a good idea since we then lose the fact that the solution depends continuously on the right hand side.

$(\lambda I - T_0)y$. It holds that

$$\begin{aligned}
&R(\lambda,T)x - R(\lambda,T_0)x = R(\lambda,T)(T-T_0)R(\lambda,T_0)x \\
&\iff R(\lambda,T)\Big(I-(\lambda I-T)R(\lambda,T_0)\Big)x = R(\lambda,T)(Ty-T_0y) \\
&\iff \Big(I-(\lambda I-T)R(\lambda,T_0)\Big)x = Ty-T_0y \\
&\iff \Big(I-(\lambda I-T)R(\lambda,T_0)\Big)(\lambda I-T_0)y = Ty-T_0y \\
&\iff (\lambda I-T_0)y-(\lambda I-T)y = Ty-T_0y \\
&\iff Ty-T_0y = Ty-T_0y
\end{aligned}$$

□

Next we recall the following result from functional analysis that states in particular that the set of invertible operators in $L(X,Y)$ is open.

Lemma 1.A.4. *Let Y be a Banach space. Let $A \in L(X,Y)$ be invertible and $B \in L(X,Y)$. If*

$$\|A-B\|_{L(X,Y)} \le \theta \frac{1}{\|A^{-1}\|_{L(Y,X)}}$$

for some $\theta \in [0,1)$, then B is invertible with

$$\|B^{-1}\|_{L(Y,X)} \le \frac{1}{1-\theta}\|A^{-1}\|_{L(Y,X)}.$$

Proof. Assume that we have

$$\|A-B\|_{L(X,Y)} \le \theta \frac{1}{\|A^{-1}\|_{L(Y,X)}}.$$

We write $B = A(I - A^{-1}(A-B))$. The operator norm of $A^{-1}(A-B) \in L(X)$ can be estimated from above by

$$\|A^{-1}(A-B)\|_{L(X)} \le \theta < 1$$

It is a standard result (using the Neumann series) that this implies that $I - A^{-1}(A-B) \in L(X)$ is invertible with

$$\|(I-A^{-1}(A-B))^{-1}\|_{L(X)} \le (1-\|A^{-1}(A-B)\|_{L(X)})^{-1} \le \frac{1}{1-\theta}.$$

In particular B is invertible with

$$\|B^{-1}\|_{L(X)} = \|(I-A^{-1}(A-B))^{-1}A^{-1}\|_{L(X)} \le \frac{1}{1-\theta}\|A^{-1}\|_{L(Y,X)}.$$

□

In the following we collect some well known spectral properties of operators with compact resolvent. To that end we first relate the spectrum of T to the spectrum of the resolvents of T.

Lemma 1.A.5. *Let $T\colon D(T) \to X$ be a closed operator and let $\mu \in \rho(T)$. Then it holds that*

$$\sigma(R(\mu,T)) \setminus \{0\} = \{\frac{1}{\mu-\lambda} \mid \lambda \in \sigma(T)\}$$

and

$$\sigma_p(R(\mu,T)) \setminus \{0\} = \{\frac{1}{\mu-\lambda} \mid \lambda \in \sigma_p(T)\}.$$

Proof. We first show

$$\sigma(R(\mu,T)) \setminus \{0\} = \{\frac{1}{\mu-\lambda} \mid \lambda \in \sigma(T)\} =: A_\mu.$$

Let $\lambda_0 \in \sigma(R(\mu,T)) \setminus \{0\}$. Assume that $\lambda_0 \notin A_\mu$. Since $\lambda_0 = \frac{1}{\mu-(\mu-\frac{1}{\lambda_0})}$ we get that $\mu - \frac{1}{\lambda_0} \notin \sigma(T)$ hence $\mu - \frac{1}{\lambda_0} \in \rho(T)$. Moreover we have that

$$(\lambda_0 I - R(\mu,T))^{-1} = \frac{1}{\lambda_0}(\mu - T)R(\mu - \frac{1}{\lambda_0}, T)$$

which implies $\lambda_0 \in \rho(R(\mu,T))$. This contradicts our assumption that $\lambda_0 \notin A_\mu$ and we have shown "$\subseteq$".

Now let $\lambda_0 = \frac{1}{\mu-\lambda} \in A_\mu$, $\lambda \in \sigma(T)$. Assume that $\lambda_0 \in \rho(R(\mu,T))$. This implies $\lambda \in \rho(T)$ since we have that

$$(\lambda I - T)^{-1} = \lambda_0 R(\mu,T)(\lambda_0 I - R(\mu,T))^{-1}.$$

This contradicts $\lambda_0 \in \rho(R(\mu,T))$ and we have shown "$\supseteq$". The identity

$$\sigma_p(R(\mu,T)) \setminus \{0\} = \{\frac{1}{\mu-\lambda} \mid \lambda \in \sigma_p(T)\}$$

can be shown similarly. □

Lemma 1.A.6 (Spectral Theorem for compact operators)**.** *Let $K\colon X \to X$ be a compact operator. Then it holds that*

i) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$, i.e., every non-zero element of the spectrum of K is an eigenvalue and zero might or might not be an eigenvalue.

ii) Either $\sigma(K)$ consists only of finitely many elements or there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ in $\mathbb{C}$, $\lambda_i \neq 0$ for all $i \in \mathbb{N}$, s.t. $\lim\limits_{i\to\infty} \lambda_i = 0$ and

$$\sigma(K) = \{\lambda_i \mid i \in \mathbb{N}\} \cup \{0\}.$$

Corollary 1.A.7. *Let $T\colon D(T) \to X$ be a closed operator with compact resolvent. Then it holds that*

i) $\sigma(T) = \sigma_p(T)$.

ii) Either $\sigma(T)$ consists only of finitely many elements or there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ in $\mathbb{C}$ s.t. $\lim\limits_{i\to\infty} |\lambda_i| = \infty$ and

$$\sigma(T) = \{\lambda_i \mid i \in \mathbb{N}\}.$$

In particular $\sigma(T) \subset \mathbb{C}$ is discrete (i.e., for every $\lambda \in \sigma(T)$ there exists $\delta > 0$ s.t. for all $\tilde{\lambda} \in \sigma(T)$ with $|\tilde{\lambda} - \lambda| < \delta$ it holds that $\tilde{\lambda} = \lambda$).

Proof. By definition we have $\sigma(T) \supset \sigma_p(T)$. Choose an arbitrary $\mu \in \rho(T) \neq \varnothing$. Let $\lambda \in \sigma(T)$. Then $\frac{1}{\mu-\lambda} \in \sigma(R(\mu,T)) \setminus \{0\}$ by Lemma 1.A.5. Lemma 1.A.6 yields $\frac{1}{\mu-\lambda} \in \sigma_p(R(\mu,T)) \setminus \{0\}$. Applying Lemma 1.A.5 again we get $\lambda \in \sigma_p(T)$. We have shown i).

To show ii) we choose again an arbitrary $\mu \in \rho(T)$. Lemma 1.A.6 yields that either $\sigma(R(\mu,T))$ consists only of finitely many elements or there exists a sequence $(\lambda_i)_{i\in\mathbb{N}}$ in $\mathbb{C}$, $\lambda_i \neq 0$ for all $i \in \mathbb{N}$, s.t. $\lim\limits_{i\to\infty} \lambda_i = 0$ and $\sigma(R(\mu,T)) = \{\lambda_i \mid i \in \mathbb{N}\} \cup \{0\}$. Then by Lemma 1.A.5 we get that either $\sigma(T)$ is finite or it is given by

$$\sigma(T) = \{\frac{\lambda_i \mu - 1}{\lambda_i} \mid i \in \mathbb{N}\}.$$

□

Definition 1.A.8. Assume that H is a (complex) Hilbert space and $T\colon D(T) \to H$ is a densely defined operator. The *adjoint of A* is the operator

$$\begin{aligned} T^*\colon D(T^*) := \{y \in H \mid \exists z \in H \text{ s.t. } \langle Tx, y\rangle = \langle x, z\rangle\ \forall x \in D(T)\} &\to H, \\ y &\mapsto z. \end{aligned}$$

The operator T is called *self-adjoint* if $D(T) = D(T^*)$ and $T = T^*$.

Note that T^* is well-defined since T is densely defined.

Proposition 1.A.9. *Assume that H is a Hilbert space and $T\colon D(T) \to H$ is a densely defined self-adjoint operator.*[40] *Then $\sigma(T) \subset \mathbb{R}$.*

Proof. Let $\lambda = a + ib \in \mathbb{C}$ and assume that $b \neq 0$. Let $x \in D(T)$. It holds that

$$\langle (\lambda I - T)x, x\rangle = \lambda \|x\|^2 - \langle Tx, x\rangle.$$

[40] Since the adjoint is always closed we have in particular that T is closed.

Since T is self-adjoint we have $\langle Tx, x\rangle \in \mathbb{R}$ and therefore

$$\operatorname{Im}(\langle(\lambda I - T)x, x\rangle) = b\|x\|^2$$

where we write Im(.) for the imaginary part. The Cauchy-Schwarz inequality yields

$$|b|\|x\|^2 = |\operatorname{Im}(\langle(\lambda I - T)x, x\rangle)| \leq \|(\lambda I - T)x\|\|x\|$$

hence $\lambda I - T \colon D(T) \to H$ is injective. For the surjectivity we show that the image of $\lambda I - T$ is closed and dense in H. Let $(x_n) \in D(T)$ be a sequence s.t. $(\lambda I - T)x_n$ converges in H for $n \to \infty$. We have

$$|b|\|x_n - x_m\| \leq \|(\lambda I - T)(x_n - x_m)$$

hence (x_n) is a Cauchy sequence, $x_n \to x \in H$ for $n \to \infty$. Since $\lambda x_n - Tx_n$ converges it follows that Tx_n converges, $Tx_n \to y \in H$ for $n \to \infty$. Since T is closed we get $x \in D(T)$ and $y = Tx$. Hence $(\lambda I - T)x_n \to \lambda x - Tx$ for $n \to \infty$ and the image of $\lambda I - T$ is closed.

Assume that y is in the orthogonal complement of the range of $\lambda I - T$, i.e.,

$$\langle(\lambda I - T)x, y\rangle = 0$$

for all $x \in D(T)$. This implies that $y \in D(T^*) = D(T)$ and $(\bar{\lambda} I - T^*)y = 0$. Since $\bar{\lambda} I - T^* = \bar{\lambda} I - T$ is injective we get $y = 0$. Hence the image of $\lambda I - T$ is dense in H. □

1.B Proof of Lemma 1.3.11

Lemma 1.B.1. *Let $N \subset \mathbb{R}^q$ be a closed (i.e., compact and without boundary) embedded submanifold of $\mathbb{R}^q$, equipped with the induced Riemannian metric. Denote by A its Weingarten map. Choose $C > 0$ s.t. $\|A\| \le C$. Choose any $0 < \delta < \frac{1}{C}$ s.t.*

$$N_\delta := \{y \in \mathbb{R}^q \mid d(y, N) < \delta\}$$

is a tubular neighborhood of N in $\mathbb{R}^q$. Denote by $\pi\colon N_\delta \to N$ its nearest point projection. For all $x, y \in N$ s.t. $\|x - y\|_2 < \delta$ we define the curve $\gamma_{x,y}\colon [0,1] \to N$ by

$$\gamma_{x,y}(t) := \pi(ty + (1-t)x).$$

For all $x, y \in N$ s.t. $\|x - y\|_2 < \delta$ it holds that

$$L(\gamma_{x,y}) \le \frac{1}{1 - \delta C}\|x - y\|_2$$

where $L(\gamma_{x,y})$ denotes the length of $\gamma_{x,y}$.

After we have shown Lemma 1.B.1 we directly get Lemma 1.3.11. To prove Lemma 1.B.1 we use a version of the Rauch Comparison Theorem for submanifolds. More precisely, we will use a slightly modified version of [34, Theorem 4.3. (b)]. First we state the setting.

Let M be a d-dimensional Riemannian manifold with $d \ge 2$. Let K be a p-dimensional Riemannian submanifold of M with $0 \le p \le d-1$. Let $\sigma\colon [0,b] \to M$ be a geodesic of M parametrized by arc length, $\sigma(0) =: m \in K$, $\sigma'(0) \perp T_mK$. We denote by A the Weingarten map of K in M. In particular

$$A_{\sigma'(0)}\colon T_mK \to T_mK.$$

Moreover we set

$$\begin{aligned}\mathscr{L}(\sigma, b, K) := \{Y \mid Y &\text{ piecewise smooth vector field along } \sigma\colon [0,b] \to M,\\ &Y(0) \in T_mK,\ Y(t) \perp \sigma'(t) \text{ for all } t \in [0,b]\}\end{aligned}$$

A Jacobi field $Y \in \mathscr{L}(\sigma, b, K)$ is called a *K-Jacobi field*[41]

$$A_{\sigma'(0)}Y(0) - \frac{\nabla}{dt}Y(0) \in (T_mK)^\perp.$$

[41] Our definition of K-Jacobi field is the same as in [34]. It differs from the definition in [6] by the condition that a K-Jacobi field along σ has to be everywhere perpendicular to σ', i.e., our K-Jacobi fields are the same as normal K-Jacobi fields in [6]. Note, however, that the definition focal points in [34] and [6] coincide. The reason for this is the following: let Y be a Jacobi field along σ with $Y(0) \in T_mK$ and $Y(t_0) = 0$ for some t_0. We write $Y^\top(t) = (v + tw)\sigma'(t)$ for the tangential part of Y for some $v, w \in \mathbb{R}$. Because of $Y(0) \in T_mK \perp \sigma'(0)$ we have $v = 0$ and since $Y(t_0) = 0$ we have $b = 0$, i.e., Y is normal.

A *focal point on* σ is a point $\sigma(t)$, $t \neq 0$, at which a non-trivial K-Jacobi field along σ vanishes.

To state the version of the Rauch Comparison Theorem for submanifolds we need, let $\tilde{M}, \tilde{K}, \tilde{\sigma}$ be another such setup as above with $\dim(\tilde{M}) = \tilde{d} \geq 2$, $\dim(\tilde{K}) = \tilde{p}$, $0 \leq \tilde{p} \leq \tilde{d} - 1$ and domain of $\tilde{\sigma}$ is the interval $[0, b]$.

Theorem 1.B.2 (Rauch Comparison Theorem for submanifolds)**.** *Let* $\tilde{p} = \tilde{d} - 1$. *Let* $X \in \mathscr{L}(\sigma, b, K)$ *and* $Y \in \mathscr{L}(\tilde{\sigma}, b, \tilde{K})$ *be* K*- and* $\tilde{K}$*-Jacobi fields, respectively. Assume that*

i) $\|X(0)\| = \|Y(0)\| \neq 0$.

ii) There are no focal points on $\tilde{\sigma}$.

iii) For each $t \in [0, b]$ *and for all* 2*-planes* $P \subset T_{\sigma(t)}M$ *containing* $\sigma'(t)$ *and all* 2*-planes* $Q \subset T_{\tilde{\sigma}'(t)}\tilde{M}$ *containing* $\tilde{\sigma}'(t)$ *the sectional curvatures* $K(P)$ *and* $\tilde{K}(Q)$ *satisfy*
$$K(P) \leq \tilde{K}(Q).$$

iv) The minimum eigenvalue of $A_{\sigma'(0)}$ *is greater or equal than the maximum eigenvalue of* $\tilde{A}_{\tilde{\sigma}'(0)}$.

Then it holds that
$$\frac{\|X(t)\|}{\|Y(t)\|}$$
is monotonously increasing on $[0, b]$.

Proof. The statement of the theorem is essentially the same as [34, Theorem 4.3.(b)]. Only our conlcusion is different.[42] Taking a look at the proof in [34] we see that [34, Theorem 4.3.(b)] directly follows from [34, Theorem 3.3.]. The latter is proven by applying [34, Lemma 3.4.] for $f_1(t) := \|X(t)\|^2$, $f_2(t) := \|Y(t)\|^2$. In particular it is shown that [34, Lemma 3.4. (4)] holds, i.e.,
$$\frac{(\|X(t)\|^2)'}{\|X(t)\|^2} \geq \frac{(\|Y(t)\|^2)'}{\|Y(t)\|^2}$$
for all $t \in (0, b]$. This implies that
$$\left(\log\left(\frac{\|X(t)\|^2}{\|Y(t)\|^2}\right)\right)' \geq 0$$

[42] In [34, Theorem 4.3.(b)] the conclusion is that $\|X(t)\| \geq \|Y(t)\|$ for all $t \in [0, b]$ holds.

for all $t \in (0, b]$. Hence

$$\frac{\|X(t)\|^2}{\|Y(t)\|^2}$$

is monotonously increasing on $(0, b]$ and therefore

$$\frac{\|X(t)\|}{\|Y(t)\|}$$

also is monotonously increasing on $(0, b]$. Because of i) we have that $\frac{\|X(0)\|}{\|Y(0)\|} = 1$. Since [34, Lemma 3.4. (2)] holds we have that

$$\lim_{t \to 0^+} \frac{\|X(t)\|}{\|Y(t)\|} = 1$$

Summing up we have shown that

$$\frac{\|X(t)\|}{\|Y(t)\|}$$

is monotonously increasing on $[0, b]$. $\square$

Proof of Lemma 1.B.1. Our strategy is to apply Theorem 1.B.2 for $M = \tilde{M} = \mathbb{R}^q$, $K := N$, $\tilde{K}$ a sphere of a suitable radius, and suitable X,Y. First we construct X.

Choose $C > 0$ s.t.

$$\|A\| < C. \tag{1.B.1}$$

Choose $\delta > 0$ s.t. the set N_δ is a tubular neighborhood of N in $\mathbb{R}^q$ and $\delta < \frac{1}{C}$. Let $x, y \in N$ with $\|x - y\|_2 < \delta$. Define $c(s) := y + s(x - y)$, $s \in [0, 1]$, and

$$\alpha(t, s) := \pi(c(s)) + t(c(s) - \pi(c(s)),$$

$t, s \in [0, 1]$. Note that the image of c is contained in N_δ since $\|x - y\|_2 < \delta$. The curve $\alpha(s, .)$ is a geodesic of $\mathbb{R}^q$ that is perpendicular to N by definition of π. We have that $t \mapsto J_s(t) := \frac{\partial \alpha}{\partial s}(s, t)$ is a Jacobi field of $\mathbb{R}^q$ along $\alpha(., s)$. Now we fix an arbitrary $s \in [0, 1]$ with

$$a := \|c(s) - \pi(c(s))\|_2 \neq 0,$$

and

$$(\pi \circ c)'(s) \neq 0.$$

Then

$$t \mapsto \tilde{J}_s(t) := J_s(\frac{t}{a})$$

is a Jacobi field along $\sigma_s(t) := \alpha(\frac{t}{a}, s)$, $t \in [0, a]$. Note that σ_s is parametrized by arc length and $\sigma_s'(t) \perp T_{\sigma_s(t)}N$ for every $t \in [0, a]$. Moreover,

$$\tilde{J}_s(0) = J_s(0) = (\pi \circ c)'(s) \neq 0. \tag{1.B.2}$$

The definition of J_s implies that

$$\tilde{J}_s(0) \in T_{\sigma_s(0)}N,$$

and

$$A_{\sigma_s'(0)}\tilde{J}_s(0) - \frac{\nabla}{dt}\tilde{J}_s(0) \in (T_{\sigma_s(0)}N)^\perp \tag{1.B.3}$$

see [6, Chapter 10, 4.1 Lemma]. Now let

$$X_s(t) := \tilde{J}_s^\perp(t),$$

$t \in [0, a]$, be the normal component of $\tilde{J}_s$. We show that X_s is a N-Jacobi field along σ_s for every $s \in [0, 1]$. To that end notice that

$$\tilde{J}_s(0) = (\pi \circ c)'(s) \in T_{\sigma_s(0)}N \perp \sigma_s'(0)$$

hence

$$X_s(0) = \tilde{J}_s(0) \in T_{\sigma_s(0)}N. \tag{1.B.4}$$

We write the tangential part $\tilde{J}_s^\top$ of $\tilde{J}_s$ as

$$\tilde{J}_s^\top(t) = (a + tb)\sigma_s'(t)$$

for suitable $a, b \in \mathbb{R}$. Equation (1.B.4) yields $a = 0$. We have that

$$\begin{aligned} \frac{\nabla}{dt}X_s(0) &= \frac{\nabla}{dt}\big(\tilde{J}_s - tb\sigma_s'\big)(0) \\ &= \Big(\frac{\nabla}{dt}\tilde{J}_s(0)\Big) - b\sigma_s'(0), \end{aligned}$$

and

$$\frac{\nabla}{dt}X_s(0) + A_{\sigma_s'(0)}X_s(0) = \Big(\frac{\nabla}{dt}\tilde{J}_s(0)\Big) + \Big(A_{\sigma_s'(0)}\tilde{J}_s(0)\Big) - b\sigma_s'(0) \in (T_{\sigma_s(0)}N)^\perp$$

by (1.B.3) and $\sigma_s'(0) = c(s) - \pi(c(s)) \in (T_{\sigma_s(0)}N)^\perp$. Therefore X_s is a N-Jacobi field along σ_s.

Now we define

$$\tilde{K} := S^{q-1}_{\frac{1}{C}} := \{x \in \mathbb{R}^q \mid \|x\|_2 = \frac{1}{C}\}$$

where C is that of (1.B.1). Moreover we set

$$\beta(h,t) := (1-t)\frac{1}{C}\begin{pmatrix}\cos(hC\|X_s(0)\|)\\ \sin(hC\|X_s(0)\|)\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$h,t \in [0,1]$. Then we have that

$$t \mapsto \frac{\partial\beta}{\partial h}(0,t) = (1-t)\|X_s(0)\|\begin{pmatrix}0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix},$$

is a $\tilde{K}$-Jacobi field. This implies that

$$Y_s(t) := \frac{\partial\beta}{\partial h}(0,tC),$$

$t \in [0,\frac{1}{C})$ is a $\tilde{K}$-Jacobi field along $\tilde{\sigma}_s$ where

$$\tilde{\sigma}_s(t) := (1-tC)\frac{1}{C}\begin{pmatrix}1\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$t \in [0,\frac{1}{C})$, is a geodesic of $\mathbb{R}^q$ parametrized by arc length with $\tilde{\sigma}_s(0) \in \tilde{K}$ and $\tilde{\sigma}'(0) \perp T_{\tilde{\sigma}_s(0)}\tilde{K}$.

Now we want to apply Theorem 1.B.2 to X_s and Y_s for $b := a$. Note that $\tilde{\sigma}_s$ is defined on $[0,b]$ since $b = a = \|\sigma(s) - \pi(\sigma(s))\| < \delta < \frac{1}{C}$. We have to check i)-iv). Equations (1.B.2) and (1.B.4) yield $\|X_s(0)\| \neq 0$. The definition of Y_s implies $\|X_s(0)\| = \|X_s(0)\|$, hence we have i). The only focal point of $\tilde{K}$ in $\mathbb{R}^q$ is $0 \in \mathbb{R}^q$, hence ii) follows by our choice of b. iii) is clear since $M = \tilde{M} = \mathbb{R}^q$. For the Weingarten map $\tilde{A}$ of $\tilde{K}$ it holds that

$$\tilde{A}_{\tilde{\sigma}_s'(0)} = -C id_{T_{\tilde{\sigma}_s(0)}\tilde{K}}.$$

Because of $\|A\| < C$ we know that the minimal eigenvalue of A is greater or equal $-C$. This shows iv). Theorem 1.B.2 yields

$$\frac{\|X_s(0)\|}{\|Y_s(0)\|} \le \frac{\|X_s(b)\|}{\|Y_s(b)\|}.$$

Therefore we have

$$
\begin{aligned}
\|(\pi \circ c)'(s)\| &= \|\tilde{J}_s(0)\| \\
&= \|X_s(0)\| \\
&\leq \frac{\|X_s(b)\|}{\|Y_s(b)\|}\|Y_s(0)\| \\
&= \frac{\|Y_s(0)\|}{\|Y_s(b)\|}\|X_s(b)\| \\
&= \frac{\|X_s(0)\|}{|1-bC|\|X_s(0)\|}\|X_s(b)\| \\
&< \frac{1}{|1-\delta C|}\|X_s(b)\|
\end{aligned}
$$

and since $\|X_s(b)\| = \|\tilde{J}_s^{\perp}(b)\| \leq \|\tilde{J}_s(b)\| = \|J_s(1)\| = \|c'(s)\|$ by definition of J_s and since every orthogonal projection P satiesfies $\|Pv\| \leq \|v\|$ for all v, we get

$$
\|(\pi \circ c)'(s)\| < \frac{1}{1-\delta C}\|c'(s)\| = \frac{1}{1-\delta C}\|x-y\|_2. \tag{1.B.5}
$$

for all $s \in [0,1]$ s.t. $\|c(s) - \pi(c(s))\|_2 \neq 0$ and $(\pi \circ c)'(s) \neq 0$. Next we show that (1.B.5) holds for all $s \in [0,1]$. From this Lemma 1.B.1 directly follows. If $(\pi \circ c)'(s) = 0$, then (1.B.5) trivially holds. Now assume that $c(s) = (\pi \circ c)(s)$, i.e., $c(s) \in N$. Using that $(d\pi)_x v = pr_{T_xN} v$ for all $x \in N$ together with $\|(d\pi)_x v\| \leq \|v\|$ for all $x \in N$ (this again is due to the fact that $(d\pi)_x$ is an orthogonal projection for $x \in N$) we get

$$
\|(\pi \circ c)'(s)\| = \|(d\pi)_{c(s)} c'(s)\| \leq \|c'(s)\| \leq \frac{1}{1-\delta C}\|c'(s)\|.
$$

□

1.C Additional results about the parallel transports

Originally the following results were used to deal with the constraint equation. Later we were able to replace them by a more elegant approach. Nevertheless, the results of the original approach are still interesting in their own right and might find some independent applications.

The following lemma shows how the derivative of the parallel transport along a geodesic triangle that moves in space can be estimated with a very geometric argument. (From the proof it is evident that this argument also works in more general settings). As a corollary we get an estimate for the commutator of the Dirac operator along a map and such a parallel transport.

Lemma 1.C.1. *Choose ε, δ, R, and T as in Table 1.1. If these parameters are small enough, the following holds: let $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $x \in M$, $s, t \in [0, T]$. Denote by $\gamma_i = \gamma_{i,x} \colon [0,1] \to N$, $i = 1, 2, 3$, the unique shortest of N s.t.*

$$\gamma_1(0) = \gamma_3(1) = (\pi \circ u_t)(x),\ \gamma_1(1) = \gamma_2(0) = u_0(x),\ \gamma_2(1) = \gamma_3(0) = (\pi \circ v_s)(x).$$

Moreover, we define $c := c_x := \gamma_3 * \gamma_2 * \gamma_1$, i.e., c is the curve obtained by first following γ_1, then γ_2, and then γ_3. Finally, we write

$$P^c := P^{c_x} \colon T_{(\pi\circ u_t)(x)}N \to T_{(\pi\circ u_t)(x)}N$$

for the parallel transport (in N) induced by c.

We define

$$\begin{aligned} M(u_t, v_s) := \max\{&\|u_0 - \pi \circ u_t\|_{C^0(M,\mathbb{R}^q)}, \|u_0 - \pi \circ v_s\|_{C^0(M,\mathbb{R}^q)},\\ &\|\pi \circ u_t - \pi \circ v_s\|_{C^0(M,\mathbb{R}^q)}\}, \end{aligned}$$

Then there exits $C(R, \varepsilon) > 0$ independent of u, v, x, s and t s.t.

$$\|\Big(\nabla_X^{(\pi\circ u_t)^*TN}(P^c Z - Z)\Big)|_x\| \le C(R,\varepsilon) M(u_t, v_s)\|Z\|_{\Gamma_{C^1}((\pi\circ u_t)^*TN)}\|X\| \tag{1.C.1}$$

for all $X \in T_xM$, $x \in M$, $Z \in \Gamma_{C^1}((\pi \circ u_t)^*TN)$.

Proof. We write $f := \pi \circ u_t$, $g := \pi \circ v_s$, and moreover $\nabla := \nabla^{(\pi\circ u_t)^*TN}$. Let $x \in M$, $X \in T_xM$, and $\gamma \colon (-c, c) \to M$ a smooth curve parametrized proportionally to arclength with $\gamma(0) = x$, $\gamma'(0) = X$. Let $(E_i(.))$ be a local orthonormal frame of f^*TN that is parallel along γ. Locally we have

$$(P^{c_y}Z(y)) - Z(y) = f^i(y)E_i(y)$$

for suitable functions f^i. Then it holds that

$$\nabla_X\Big(P^cZ - Z\Big)|_x = \Big(L_X f^i\Big)(x)E_i(x).$$

In the following, we estimate $(L_X f^i)E_i$. To that end we denote by P^γ the parallel transport in TN along $f \circ \gamma$ from $f(x)$ to $f(\gamma(\tau))$ and we calculate

$$
\begin{aligned}
&(L_X f^i)(x)E_i(x) \\
&= \lim_{\tau\to 0} \frac{f^i(\gamma(\tau)) - f^i(x)}{\tau} E_i(x) \\
&= \lim_{\tau\to 0} \frac{f^i(\gamma(\tau))E_i(x) - f^i(x)E_i(x)}{\tau} \\
&= \lim_{\tau\to 0} \frac{(P^\gamma)^{-1}\Big(f^i(\gamma(\tau))P^\gamma E_i(x)\Big) - \Big(P^{c_x}Z(x) - Z(x)\Big)}{\tau} \\
&= \lim_{\tau\to 0} \frac{(P^\gamma)^{-1}\Big(f^i(\gamma(\tau))E_i(\gamma(\tau))\Big) - \Big(P^{c_x}Z(x) - Z(x)\Big)}{\tau} \\
&= \lim_{\tau\to 0} \frac{(P^\gamma)^{-1}\Big(P^{c_{\gamma(\tau)}}Z(\gamma(\tau)) - Z(\gamma(\tau)) - \Big(P^{c_x}Z(x) - Z(x)\Big)}{\tau} \\
&= \lim_{\tau\to 0} \frac{(P^\gamma)^{-1}P^{c_{\gamma(\tau)}}Z(\gamma(\tau)) - (P^\gamma)^{-1}Z(\gamma(\tau)) - P^{c_x}Z(x) + Z(x)}{\tau} \\
&= \lim_{\tau\to 0} \frac{1}{\tau}\Big((P^\gamma)^{-1}P^{c_{\gamma(\tau)}}Z(\gamma(\tau)) \\
&\qquad - (P^\gamma)^{-1}P^{c_{\gamma(\tau)}}P^\gamma Z(x) + (P^\gamma)^{-1}P^{c_{\gamma(\tau)}}P^\gamma Z(x) - (P^\gamma)^{-1}Z(\gamma(\tau)) \\
&\qquad - P^{c_x}Z(x) + Z(x)\Big) \\
&= \lim_{\tau\to 0} \frac{1}{\tau}\Bigg(\Big((P^\gamma)^{-1}P^{c_{\gamma(\tau)}}P^\gamma - Id\Big)\Big(\Big((P^\gamma)^{-1}Z(\gamma(\tau))\Big) - Z(x)\Big) \\
&\qquad + (P^\gamma)^{-1}P^{c_{\gamma(\tau)}}P^\gamma Z(x) - P^{c_x}Z(x)\Bigg)
\end{aligned}
$$

By Lemma 1.4.3 we have

$$
\begin{aligned}
&\|\Big((P^\gamma)^{-1}P^{c_{\gamma(\tau)}}P^\gamma - Id\Big)\Big(\Big((P^\gamma)^{-1}Z(\gamma(\tau))\Big) - Z(x)\Big)\| \\
&\leq C\|\pi\circ u_t - \pi\circ v_s\|_{C^0(M,\mathbb{R}^q)}\|\Big((P^\gamma)^{-1}Z(\gamma(\tau))\Big) - Z(x)\|.
\end{aligned}
$$

We will show

$$
\|\Big((P^\gamma)^{-1}Z(\gamma(\tau))\Big) - Z(x)\| \leq \tau\|X\|\|Z\|_{\Gamma_{C^1}(f^*TN)} \tag{1.C.2}
$$

and

$$
\begin{aligned}
&\|(P^\gamma)^{-1}P^{c_{\gamma(\tau)}}P^\gamma Z(x) - P^{c_x}Z(x)\| \\
&\leq \tau C_1 M(u_t, v_s)\|X\|\|Z(x)\|.
\end{aligned} \tag{1.C.3}
$$

After that, equation (1.C.1) follows easily.

To show (1.C.2) we write $k(t) := \left(P^{\gamma|_{[0,t]}}\right)^{-1} Z(\gamma(t))$ and use the fundamental theorem of calculus to conclude

$$\begin{aligned}
\|\Big((P^\gamma)^{-1}Z(\gamma(\tau))\Big) - Z(x)\| &= \|k(\tau)-k(0)\| \\
&= \|\int_0^\tau k'(t)dt\| \\
&\leq \tau \sup_t \|k'(t)\| \\
&= \tau \sup_t \|(\nabla^{(\pi\circ u_t)^*TN}_{\gamma'(t)} Z)(\gamma(t))\| \\
&\leq \tau \|X\| \|Z\|_{\Gamma_{C^1}(f^*TN)}.
\end{aligned} \tag{1.C.4}$$

It remains to show (1.C.3). To that end, we recall our setting. We have the geodesic triangles given by the images of c_x and $c_{\gamma(\tau)}$ with vertices $u_0(x)$, $f(x)$, $g(x)$ and $u_0(\gamma(\tau))$, $f(\gamma(\tau))$, $g(\gamma(\tau))$, respectively. The vertices can be joined by the curves $u_0 \circ \gamma$, $f \circ \gamma$, and $g \circ \gamma$. This is the situation of figure 1.C.1.

We will relate $(P^\gamma)^{-1} P^{c_{\gamma(\tau)}} P^\gamma Z(x) - P^{c_x} Z(x)$ to the parallel transports around the three "rectangles" of figure 1.C.1. Then we estimate the parallel transport around the rectangles with the same methods that we used to show Lemma 1.4.3. For $i = 1, 2, 3$ we denote by

$$P^{R_i} \colon T_{f(x)}N \to T_{f(x)}N$$

the parallel transports given by figure 1.C.2.

By definition of the P^{R_i} we have

$$P^{R_3} \circ P^{R_2} \circ P^{R_1} = (P^\gamma)^{-1} \circ (P^{c_{\gamma(\tau)}})^{-1} \circ P^\gamma \circ P^{c_x}$$

which implies

$$(P^\gamma)^{-1} \circ P^{c_{\gamma(\tau)}} \circ P^\gamma \circ P^{R_3} \circ P^{R_2} \circ P^{R_1} = P^{c_x}$$

and we conclude

$$P^{c_x} = (P^\gamma)^{-1} \circ P^{c_{\gamma(\tau)}} \circ P^\gamma - (P^\gamma)^{-1} \circ P^{c_{\gamma(\tau)}} \circ P^\gamma \circ (Id - P^{R_3} \circ P^{R_2} \circ P^{R_1}).$$

It follows that

$$\begin{aligned}
&\|(P^\gamma)^{-1} P^{c_{\gamma(\tau)}} P^\gamma Z(x) - P^{c_x} Z(x)\| \\
&= \|(P^\gamma)^{-1} P^{c_{\gamma(\tau)}} P^\gamma (Id - P^{R_3} P^{R_2} P^{R_1}) Z(x)\| \\
&= \|(Id - P^{R_3} P^{R_2} P^{R_1}) Z(x)\| \\
&= \|(Id - P^{R_3}) Z(x) + P^{R_3}(Id - P^{R_2} P^{R_1}) Z(x)\| \\
&\leq \|(Id - P^{R_3}) Z(x)\| + \|(Id - P^{R_2} P^{R_1}) Z(x)\| \\
&\leq \|(Id - P^{R_3}) Z(x)\| + \|(Id - P^{R_2}) Z(x)\| + \|(Id - P^{R_1}) Z(x)\|
\end{aligned} \tag{1.C.5}$$

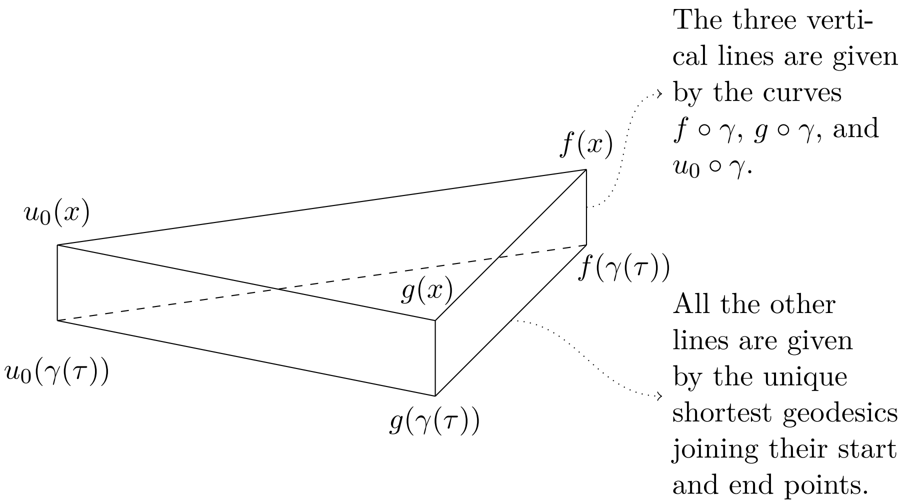

Figure 1.C.1: The setting of the proof of (1.C.3).

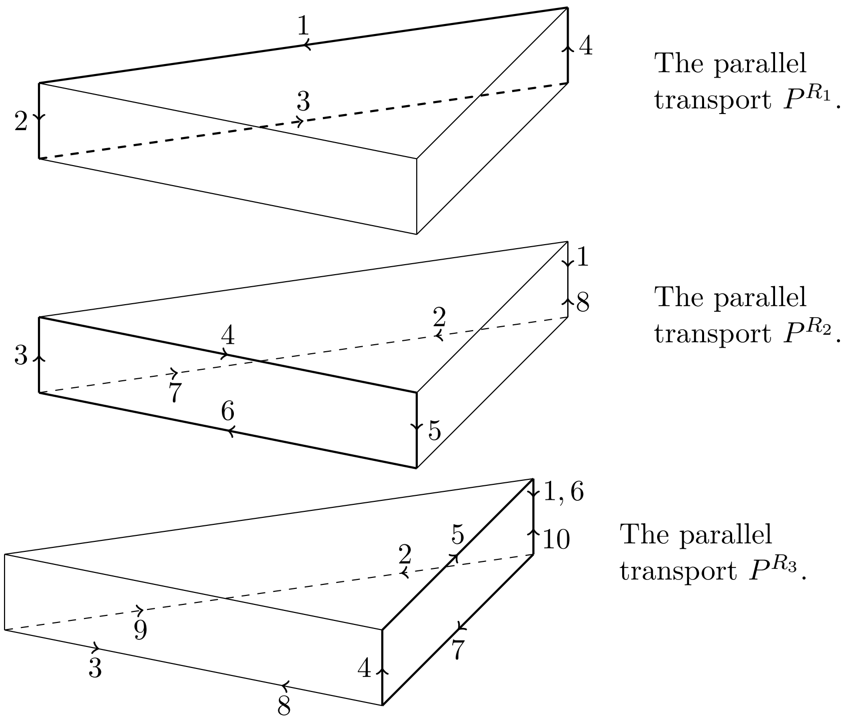

Figure 1.C.2: The parallel transports P^{R_i} are given by the above figures. They all start and end in the top right corner, i.e., in $f(x)$.

Therefore it remains to estimate the deviation of the P^{R_i} from the identity. Notice that each P^{R_i} is of the form

$$P^{R_i} = (P^i)^{-1} \circ P^{\square_i} \circ P^i$$

where $P^{\square_i}$ denotes the parallel transport along the rectangle highlighted in the definition of P^{R_i} by figure 1.C.2 (note that $P^1 = Id$). Therefore showing an inequality of the form

$$\|(Id - P^{R_i})Z\| \leq C\|Z\|$$

for all Z is equivalent to showing

$$\|(Id - P^{\square_i})Z\| \leq C\|Z\|$$

for all Z, i.e., we only have to estimate $Id - P^{\square_i}$ and this we can do with the same methods we used to show Lemma 1.4.3.

We start with $i = 1$. We consider the (well-defined) geodesic variation

$$\alpha\colon [0,\tau] \times [0,1] \to N, \qquad (s_1,t_1) \mapsto \exp_{u_0(\gamma(s_1))}(t_1 \exp^{-1}_{u_0(\gamma(s_1))} f(\gamma(s_1))).$$

By definition the image of α is the filled rectangle $\square_1$. Analogously to the proof of (1.4.10) (the fact that we consider a rectangle now but in (1.4.10) we considered a triangle doesn't change the nature of the argument) we get

$$P^{\square_1}Z - Z =$$
$$\left(\int_0^\tau \int_0^1 \langle R^{TN}\Big(\frac{\partial}{\partial s_1}\Big|_{s_1=\tilde{s_1}} \alpha(s_1,\tilde{t_1}), \frac{\partial}{\partial t_1}\Big|_{t_1=\tilde{t_1}} \alpha(\tilde{s_1},t_1)\Big) Z(\tilde{s_1},\tilde{t_1}), \tilde{E}_i(\tilde{s_1},\tilde{t_1})\rangle d\tilde{t_1} d\tilde{s_1}\right)\tilde{E}_i$$

Moreover, by (1.3.13) we have

$$\begin{aligned}\|\frac{\partial}{\partial t_1}\alpha(s_1,t_1)\| &= \|\exp^{-1}_{u_0(\gamma(s_1))} f(\gamma(s_1))\| \\ &\leq d^N(u_0(\gamma(s_1)), f(\gamma(s_1))) \\ &\leq \frac{1}{1-\delta_0 C}\|u_0 - \pi \circ u_t\|_{C^0(M,\mathbb{R}^q)}\end{aligned}$$

for all s_1, t_1. We also get

$$\|\frac{\partial}{\partial s_1}\alpha(s_1,t_1)\| \leq C(R)\|X\|$$

for all s_1, t_1. This can be shown analogously to the estimate for $\|dF_{(r,x)}e_\alpha\|$ in the proof of Lemma 1.4.1. We conclude

$$\|P^{\square_1}Z - Z\| \leq \tau C(R)\|u_0 - \pi \circ u_t\|_{C^0(M,\mathbb{R}^q)}\|Z\|\|X\|$$

for all $Z \in T_{u_0(x)}N$, $x \in M$. Analogously, we get

$$\|P^{\square_2} Z - Z\| \leq \tau C(R)\|u_0 - \pi \circ v_s\|_{C^0(M,\mathbb{R}^q)}\|Z\|\|X\|$$

and

$$\|P^{\square_3} Z - Z\| \leq \tau C(R)\|\pi \circ u_t - \pi \circ v_s\|_{C^0(M,\mathbb{R}^q)}\|Z\|\|X\|.$$

Using these estimates for $\|P^{\square_i} Z - Z\|$ together with (1.C.5) we get (1.C.3) and therefore (1.C.1). □

Corollary 1.C.2. *Assume we are in the situation of Lemma 1.C.1. Then there exists* $C(R,\varepsilon) > 0$ *s.t.*

$$\| \left((P^c)^{-1} \circ \not{D}^{\pi\circ u_t} \circ P^c - \not{D}^{\pi\circ u_t}\right) \psi(x)\| \leq C(R,\varepsilon) M(u_t, v_s)\|\psi(x)\|$$

for all $u, v \in B_R^T(v_0) \cap \{u|_{t=0} = u_0\}$, $\psi \in \Gamma_{C^1}(\Sigma M \otimes u_0^* TN)$, $x \in M$, $t, s \in [0,T]$.

Proof. Let γ_i, c, and P^c be as in the statement of Lemma 1.C.1 and $Z \in \Gamma_{C^1}((\pi \circ u_t)^*TN)$. Lemma 1.4.3 and Lemma 1.C.1 imply the following inequality for the commutator of P^c and $\nabla := \nabla^{(\pi\circ u_t)^*TN}$:

$$\begin{aligned}
&\|[P^c, \nabla_X]Z|_x\| \\
&= \|P^c(\nabla_X Z) - \nabla_X(P^c Z)\| \\
&= \|P^c(\nabla_X Z) - \nabla_X Z + \nabla_X Z - \nabla_X(P^c Z)\| \\
&= \|\nabla_X(P^c Z - Z) - (P^c - id)(\nabla_X Z)\| \\
&\leq C(R) M(u_t, v_s)\|X\|\|Z\|_{\Gamma_{C^1}((\pi\circ u_t)^*TN)}
\end{aligned}$$

for all $x \in M$, $X \in T_xM$. (Of course, if Z is a local C^1-section of $(\pi \circ u_t)^*TN$ around x with bounded local C^1-norm, then this inequality still holds if we replace the C^1-norm of Z by the local C^1-norm of Z on the right hand side.) Let (b_i) and (e_α) be local orthonormal frames of TN and TM, respectively. Let $\psi \in \Gamma_{C^1}(\Sigma M \otimes (\pi \circ u_t)^*TN)$ and write $\psi = \psi^i \otimes (b_i \circ \pi \circ u_t)$ for local C^1-sections ψ^i of ΣM. In the following we also write P^c for the induced mapping

$$P^c \colon \Sigma M \otimes (\pi \circ u_t)^*TN \to \Sigma M \otimes (\pi \circ u_t)^*TN.$$

It holds that

$$\begin{aligned}
[P^c, \not{D}^{\pi\circ u_t}]\psi &= (P^c \circ \not{D}^{\pi\circ u_t} - \not{D}^{\pi\circ u_t} \circ P^c)\psi \\
&= (e_\alpha \cdot \psi^i) \otimes ([P^c, \nabla_{e_\alpha}](b_i \circ \pi \circ u_t)) \,.
\end{aligned}$$

Hence we conclude[43]

$$\begin{aligned}\|[P^c, \not{D}^{\pi\circ u_t}]\psi(x)\| &\leq \|e_\alpha \cdot \psi^i\| \|[P^c, \nabla_{e_\alpha}](b_i \circ \pi \circ u_t)\| \\ &\leq \sum_i \sqrt{\dim(N)} \|\psi(x)\| \|[P^c, \nabla_{e_\alpha}](b_i \circ \pi \circ u_t)\| \\ &\leq C_1(R) M(u_t, v_s) \|\psi(x)\|.\end{aligned}$$

We have shown the corollary. □

[43]To be precise one has to do the following: cover N by finitely many local orthonormal frames (b_i) whose local C^1-norms are bounded. Then the local C^1-norm of $b_i \circ \pi \circ u_t$ (viewed as a section of $(\pi \circ u_t)^* TN$) is bounded by some constant that depends on R. Hence C_1 depends only on R (and the choice of a covering of N by local orthonormal frames).

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Chapter 2

Minimal kernels of Dirac operators along maps

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Abstract Let M be a closed spin manifold and let N be a closed manifold. For maps $f\colon M \to N$ and Riemannian metrics g on M and h on N, we consider the Dirac operator $\not{D}^f_{g,h}$ of the twisted Dirac bundle $\Sigma M \otimes_{\mathbb{R}} f^*TN$. To this Dirac operator one can associate an index in $KO^{-\dim(M)}(\mathrm{pt})$. If M is 2-dimensional, one gets a lower bound for the dimension of the kernel of $\not{D}^f_{g,h}$ out of this index. We investigate the question whether this lower bound is obtained for generic tupels (f, g, h). This chapter is similar to [23].

2.1 Introduction

Let M be a closed (i.e., compact and without boundary) 2-dimensional spin manifold with a fixed spin structure and let N be a closed manifold. We study the existence and genericness[1] of maps $f\colon M \to N$ and Riemannian metrics g on M and h on N, such that the kernel of the Dirac operator $\not{D}^f_{g,h}$ of the twisted Dirac bundle $\Sigma M \otimes_{\mathbb{R}} f^*TN$ has quaternionic dimension zero or one. Here, ΣM is the usual complex spinor bundle of M and $\not{D}^f_{g,h}$ is called the *Dirac operator along the map f*.

This problem is inherently tied to the vanishing of an index $\mathrm{ind}_{f^*TN}(M) \in KO^{-\dim(M)}(\mathrm{pt})$, see e.g. [19, eq. (7.24) on p. 151], which is a generalization of

[1] The term "generic" will be defined rigorously in Remark 2.3.2.

Hitchin's α-index [17, Section 4.2]. If M is 2-dimensional, then we have

$$\mathrm{ind}_{f^*TN}(M) = \left[\dim_{\mathbb{H}}\ker \not{D}^f_{g,h}\right]_{\mathbb{Z}_2}$$

under the isomorphism $KO^{-2}(\mathrm{pt}) \cong \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, where $[k]_{\mathbb{Z}_2}$ denotes the class of $k \in \mathbb{Z}$ in $\mathbb{Z}_2$. Note that $\mathrm{ind}_{f^*TN}(M)$ is independent of the choice of the Riemannian metrics on M and N. It is also invariant under homotopies of f. However, the index may depend on the choice of spin structure on M. As a consequence,

$$\dim_{\mathbb{H}}\ker \not{D}^f_{g,h} \geq \begin{cases} 1, & \text{if } \mathrm{ind}_{\tilde{f}^*TN}(M) \neq 0, \\ 0, & \text{if } \mathrm{ind}_{\tilde{f}^*TN}(M) = 0, \end{cases} \tag{2.1.1}$$

for any $f\colon M \to N$ homotopic to $\tilde{f}$ and any Riemannian metric g on M and h on N.

If equality holds in (2.1.1), then we call the kernel of $\not{D}^f_{g,h}$ *minimal*. We expect that for generic tupels (f, g, h) the kernel of $\not{D}^f_{g,h}$ is minimal. In an investigation for similar results (references are given in the next section) it turned out that often the strategy to prove such a result is the following: for a given $\tilde{f}\colon M \to N$ one has to find a map $f\colon M \to N$ homotopic to $\tilde{f}$ and Riemannian metrics g on M and h on N such that the kernel of $\not{D}^f_{g,h}$ is minimal. Genericness then follows from well known perturbation results. Finding "enough" examples for minimal kernels therefore seems to be the crucial part in proving the expectation.

Our first main theorem addresses the existence of tupels (f, g, h) such that the kernel of $\not{D}^f_{g,h}$ is 1-dimensional, c.f. Theorem 2.3.1. (Examples with 0-dimensional kernels are easy to construct.) In particular we show that if $\alpha(M) \neq 0$ (we denote by $\alpha(M)$ Hitchin's α-index), N is odd-dimensional and orientable, and $\tilde{f}\colon M \to N$ is null-homotopic, then there exists a map $f\colon M \to N$ homotopic to $\tilde{f}$ and Riemannian metrics g on M and h on N s.t.

$$\dim_{\mathbb{H}}\ker \not{D}^f_{g,h} = 1.$$

Our second main theorem addresses the genericness of minimal kernels, c.f. Theorem 2.3.4.

2.1.1 Motivation

Our motivation to study this problem is twofold.

On the one hand, there are many results in the literature concerning the genericness of minimal kernels under the presence of an index. In [2] it is shown that for generic metrics, the dimension of the kernel of the (untwisted) Dirac operator is as small as allowed by the index theorem of Atiyah and Singer (on a closed, connected manifold). This fact generalized results in [5] and [20]. In the latter article the author also considers spin^c-manifolds. The dependency of the kernel of the twisted Dirac operator, where one twists with hermitian vector bundles, is considered in [4]. Note that we twist with real vector bundles. This is one of the reasons why we were not able to apply the variational approach of [4] and [20] to our situation. Another article related to such problems is [17].

On the other hand, the existence of maps f with $\dim_{\mathbb{H}} \ker \not{D}^f_{g,h} = 1$ has a concrete application to the theory of Dirac-harmonic maps. Dirac-harmonic maps are the critical points of the supersymmetric analog of the classical Dirichlet energy functional. The supersymmetric analog is the underlying functional for the supersymmetric non-linear sigma model in quantum field theory, see e.g. [9], [10], [18, Chapter 10], and [12, Part 1, Supersolutions, Chapter 3]. The existence of maps $f\colon M \to N$ such that the kernel of $\not{D}^f_{g,h}$ is 1-dimensional is needed in order that the so-called heat flow for Dirac-harmonic maps, introduced in [11] for manifolds with non-empty boundary, is also well-posed on closed manifolds, c.f. [24] and Chapter 1.

2.2 Notation and preliminaries from spin geometry

In this section we introduce notation and recall some basics from spin geometry which will be relevant in the following, e.g. for understanding the precise meaning of our main theorems. For a more detailed introduction to spin geometry we refer to e.g. [19], [7], [16], [13], and [21].

Let M be an oriented m-dimensional manifold and denote by GL^+M the $GL^+(m)$-principal bundle of oriented frames for M. Moreover, we denote by $\theta\colon \widetilde{GL}^+(m) \to GL^+(m)$ the universal covering for $m \geq 3$ and the connected twofold covering for $m = 2$. A *topological spin structure on* M is a θ-reduction of GL^+M, i.e., a topological spin structure on M is a $\widetilde{GL}^+(m)$-principal bundle $\widetilde{\mathrm{GL}}^+M$ over M together with a twofold covering $\chi\colon \widetilde{\mathrm{GL}}^+M \to \mathrm{GL}^+M$ that commutes with the projections onto M and is compatible with the group actions of the principal bundles.

Now let (M,g) be an oriented Riemannian manifold and $\mathrm{SO}(M,g)$ the $SO(m)$-principal bundle of oriented orthonormal frames for M. Restricting θ to the *spin group* given by $\mathrm{Spin}(m) := \theta^{-1}(SO(m))$, we define a *metric spin structure on* (M,g) to be a $\theta|_{\mathrm{Spin}(m)}$-reduction of $\mathrm{SO}(M,g)$. Again, this means that a metric spin structure on M is a $\mathrm{Spin}(m)$-principal bundle $\mathrm{Spin}(M,g)$ over M together with a twofold covering $\eta\colon \mathrm{Spin}(M,g) \to \mathrm{SO}(M,g)$ that commutes with the projections onto M and is compatible with the group actions of the principal bundles.

Given a topological spin structure $\chi\colon \widetilde{\mathrm{GL}}^+M \to \mathrm{GL}^+M$ on an oriented manifold M, every Riemannian metric g on M defines a metric spin structure

$$\chi_g\colon \mathrm{Spin}(M,g) \to \mathrm{SO}(M,g)$$

on (M,g) by $\mathrm{Spin}(M,g) := \widetilde{\mathrm{GL}}^+M|_{\mathrm{SO}(M,g)}$. In the following, the term *spin structure* refers to a topological or metric spin structure and it should always be clear from the context which one we mean. A *spin manifold* is an oriented manifold that admits a spin structure.

On a Riemannian manifold (M,g) with metric spin structure η, we have the usual Dirac operator $\not{D} = \not{D}_\eta\colon \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ acting on sections of the complex spinor bundle ΣM. If we are given a map $f\colon M \to N$, where (N,h) is a Riemannian manifold, we define the *Dirac operator along* f

$$\not{D}^f_{g,h} = \not{D}^f_{\eta,h}\colon \Gamma(\Sigma M \otimes_{\mathbb{R}} f^*TN) \to \Gamma(\Sigma M \otimes_{\mathbb{R}} f^*TN)$$

to be the Dirac operator of the twisted Dirac bundle $\Sigma M \otimes_{\mathbb{R}} f^*TN$. In the notation for $\not{D}^f_{g,h} = \not{D}^f_{\eta,h}$ we highlight either the metric g on M or the spin structure η on M in the notation, depending on the context. Locally,

$$\not{D}^f_{\eta,h}\psi = (\not{D}_\eta \psi^i) \otimes s_i + (e_\alpha \cdot \psi^i) \otimes \nabla^{f^*TN}_{e_\alpha} s_i$$

where $\psi = \psi^i \otimes s_i$, the ψ^i are local sections of ΣM, (s_i) is a local frame of f^*TN, (e_α) is a local orthonormal frame of TM, and ∇^{f^*TN} is the pullback of the Levi-Civita connection on (N, h).

2.3 Statement of the results

In this section we state our main results about the existence and genericness of minimal kernels for Dirac operators along maps. We only consider manifolds that are non-empty and smooth.

Theorem 2.3.1. *Let M be a 2-dimensional closed manifold and N be a n-dimensional closed manifold. Moreover, assume that*

- *M is connected, oriented, and of positive genus.*
- *N is connected. If n is even, then we assume that N is non-orientable.*

Then the following holds:

Case** $n = 2$**:
Let h be a Riemannian metric on N. Then there exists a spin structure χ on M and a smooth map $f\colon M \to N$ s.t.

$$\dim_{\mathbb{H}} \ker \not{D}^f_{\chi_g,h} = 1$$

for generic Riemannian metrics g on M.

Case** $n \geq 2$**:
There exists a spin structure χ on M, a smooth map $f\colon M \to N$, and a Riemannian metric h on N s.t.

$$\dim_{\mathbb{H}} \ker \not{D}^f_{\chi_g,h} = 1$$

for generic Riemannian metrics g on M.

Remark 2.3.2.

i) By "generic" we mean C^∞-dense and C^1-open. More precisely, a statement $S = S(g)$ holds for generic Riemannian metrics g on M, if there exists a subset $\mathcal{G} \subset \mathrm{Riem}(M)$ of the space of Riemannian metrics on M which is dense in the C^∞-topology, open in the C^1-topology, and $S(g)$ is true for every $g \in \mathcal{G}$.

ii) The case $n = 1$ was not mentioned in the theorem, since for 1-dimensional N it is not difficult to find examples for 1-dimensional kernels. If we choose a spin structure on M for which the Dirac operator on ΣM has 1-dimensional kernel and f to be a constant map, then the kernel of $\not{D}^f_{g,h}$ is 1-dimensional.

iii) If N is 2-dimensional and orientable, or more general even dimensional and spin, then $\mathrm{ind}_{f^*TN}(M)$ always vanishes [3, Proposition 10.1], hence in this case the kernel of $\not{D}^f_{g,h}$ is never 1-dimensional.

iv) The above theorem gives information about the existence of minimal kernels if $\mathrm{ind}_{f^*TN}(M)$ does not vanish. If $\mathrm{ind}_{f^*TN}(M)$ vanishes, examples of minimal kernels are easy to construct. Just take a Riemannian metric and a spin structure on M for which the Dirac operator on ΣM has zero dimensional kernel (c.f. [2]) and twist with f^*TN where f is a constant map.

v) The proof of Theorem 2.3.1 is constructive. We will use differences of spin structures to construct maps $M \to S^1$ and then use certain closed geodesics $S^1 \to N$ s.t. the composition $M \to S^1 \to N$ is the desired map f.

From the proof of Theorem 2.3.1 we get the following corollary.

Corollary 2.3.3. *Let M be a 2-dimensional closed connected spin manifold with $\alpha(M) \neq 0$ and let N be an odd-dimensional orientable closed connected manifold. Let $\tilde{f}\colon M \to N$ be null-homotopic. Then there exists a map $f\colon M \to N$ homotopic to $\tilde{f}$ and Riemannian metrics g on M and h on N s.t.*

$$\dim_{\mathbb{H}} \ker \not{D}^f_{g,h} = 1.$$

The next theorem addresses the genericness of minimal kernels, assuming their existence.

Theorem 2.3.4. *Let M be a 2-dimensional closed spin manifold with spin structure χ and let N be an n-dimensional closed manifold. Assume that the kernel of $\not{D}^f_{\chi_g,h}$ is minimal for some smooth map $f\colon M \to N$ and some Riemannian metrics g on M and h on N.*

Then the following holds:

i) For generic Riemannian metrics $\tilde{h}$ on N the kernel of $\not{D}^f_{\chi_g,\tilde{h}}$ is minimal.

ii) For generic Riemannian metrics $\tilde{g}$ on M the kernel of $\not{D}^f_{\chi_{\tilde{g}},h}$ is minimal.

iii) If h is a real analytic Riemannian metric (and N is real analytic), then the kernel of $\not{D}^{\tilde{f}}_{\chi_g,h}$ is minimal for generic $\tilde{f} \in [f]$, i.e., for a C^∞-dense and C^1-open subset of $[f]$. (Here and in the following, $[f]$ denotes the homotopy class of $f\colon M \to N$.)

2.4 Differences of spin structures

In this section we consider differences of spin structures. These are also treated in [1, 7] and they are one of the main tools we use to construct the maps f of Theorem 2.3.1.

In this section, we let M be a m-dimensional connected spin manifold. Assume we are given a Riemannian metric g on M and two spin structures $\eta^i\colon \mathrm{Spin}(M,g)^i \to \mathrm{SO}(M,g)$, $i = 1, 2$.

Then we define the group homomorphism (c.f. [1, p. 15])

$$\delta = \delta_{\eta^1,\eta^2}\colon \pi_1(\mathrm{SO}(M,g)) \to \mathbb{Z}_2,$$
$$[\gamma] \mapsto \begin{cases} 1, & \text{if either } \gamma \text{ lifts to } \mathrm{Spin}(M,g)^1 \text{ and } \mathrm{Spin}(M,g)^2 \\ & \text{or it lifts to none of them.} \\ -1, & \text{if } \gamma \text{ lifts either to } \mathrm{Spin}(M,g)^1 \text{ or to } \mathrm{Spin}(M,g)^2. \end{cases}$$

We call δ *the difference of the spin structures* η^1 *and* η^2.

The name originates from the following: Let

$$\mathcal{M} := \{\text{spin structures on } (M,g)\}/\text{equivalence},$$
$$\mathcal{S} := \{\text{index} \leq 2 \text{ subgroups of } \pi_1(\mathrm{SO}(M,g))\},$$

and consider the maps

$$\Psi\colon \mathcal{M} \to \mathcal{S}, \qquad \big(\eta\colon \mathrm{Spin}(M,g) \to \mathrm{SO}(M,g)\big) \mapsto \eta_*(\pi_1(\mathrm{Spin}(M,g))),$$

$$\Omega\colon \mathcal{S} \to \mathrm{Hom}(\pi_1(\mathrm{SO}(M,g)), \mathbb{Z}_2), \qquad H \mapsto \Omega(H),$$

where the group homomorphism $\Omega(H)$ is defined by $\ker(\Omega(H)) = H$. Then it holds that

$$\delta = \Omega(\Psi(\eta^1)) - \Omega(\Psi(\eta^2)).$$

In particular, we have shown the following lemma.

Lemma 2.4.1. *If η_1 and η_2 are not equivalent, then δ_{η^1,η^2} is surjective.*

In the next lemma we show that δ descends to a group homomorphism $\pi_1(M) \to \mathbb{Z}_2$.

Lemma 2.4.2. *There exists a unique group homomorphism $\overline{\delta}\colon \pi_1(M) \to \mathbb{Z}_2$ s.t. the following diagram commutes*

$$\begin{tikzcd} \pi_1(\mathrm{SO}(M,g)) \arrow[rr] \arrow[dr, "\delta"'] & & \pi_1(M) \arrow[dl, "\overline{\delta}"] \\ & \mathbb{Z}_2 & \end{tikzcd}$$

where the horizontal map is induced by the bundle projection $\mathrm{SO}(M,g) \to M$.

Proof. There exists an exact sequence

$$\ldots \to \pi_1(\mathrm{SO}(m))) \xrightarrow{\iota_*} \pi_1(\mathrm{SO}(M,g)) \twoheadrightarrow \pi_1(M) \to \underbrace{\pi_0(\mathrm{SO}(m))}_{=\{1\}} \to \ldots$$

Hence it suffices to show that for every $[\gamma] \in \pi_1(\mathrm{SO}(m))$ we have $[\iota \circ \gamma] \in \ker(\delta)$. This directly follows from the commutative diagram

$$\begin{tikzcd} & \mathrm{Spin}(m) \arrow[r, hook] \arrow[d] & \mathrm{Spin}(M,g)^i \arrow[d, "\eta^i"] \\ S^1 \arrow[r, "\gamma"] & \mathrm{SO}(m) \arrow[r, hook, "\iota"] & \mathrm{SO}(M,g) \end{tikzcd}$$

□

2.4.1 Relation to spin structures and spinor bundles

Let $\overline{\delta} = \overline{\delta}_{\eta^1,\eta^2} : \pi_1(M) \to \mathbb{Z}_2$ be the group homomorphism of Lemma 2.4.2. Assume that η^1 and η^2 are not equivalent. Then $\overline{\delta}$ is surjective and hence $\ker(\overline{\delta}) \subset \pi_1(M)$ is an index 2 subgroup of $\pi_1(M)$. We denote by

$$p \colon P \to M$$

the connected twofold covering with $p_*(\pi_1(P)) = \ker(\overline{\delta})$.

Lemma 2.4.3. *There exists an isomorphism of* $\mathrm{Spin}(m)$*-principal bundles*

$$F \colon \mathrm{Spin}(M,g)^1 \times_{\mathbb{Z}_2} P \to \mathrm{Spin}(M,g)^2$$

where $\mathbb{Z}_2 = \ker(\mathrm{Spin}(m) \to \mathrm{SO}(m))$ *acts on* $\mathrm{Spin}(M,g)^1$ *from the right and* $\mathbb{Z}_2$ *acts on* P *from the left, and it holds that*

$$\eta^2(F([a,e])) = \eta^1(a) \tag{2.4.1}$$

for all $a \in \mathrm{Spin}(M,g)^1$, $e \in P$.

Due to its technical nature, the proof will be done in the appendix.

For the remainder of this section, let us additionally assume that M is closed, 2-dimensional, and of positive genus. In the following we want to relate the associated (complex) spinor bundles

$$\Sigma^i M = \mathrm{Spin}(M,g)^i \times_\rho \Sigma_2,$$

$i = 1, 2$, where $\rho \colon \mathrm{Spin}(2) \to \mathrm{Aut}(\Sigma_2)$ is the complex spinor representation.

Lemma 2.4.4. *There exists a smooth map*

$$f = f_\delta \colon M \to S^1$$

such that the following diagram commutes

$$\begin{tikzcd} & \pi_1(S^1) = \mathbb{Z} \arrow[d, "x \mapsto [x]"] \\ \pi_1(M) \arrow[ur, "f_*"] \arrow[r, "\bar{\delta}"] & \mathbb{Z}_2 \end{tikzcd}$$

and $f_* \colon \pi_1(M) \to \pi_1(S^1)$ is surjective.

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathbb{Z}^{2g} \arrow[dr, "h"] & \\ \frac{\pi_1(M)}{[\pi_1(M),\pi_1(M)]} \arrow[u, "I"] \arrow[dr, "\hat{\delta}"] & \mathbb{Z} \arrow[d, "\mathrm{pr}"] \\ \pi_1(M) \arrow[u, "\mathrm{pr}"] \arrow[r, "\bar{\delta}"] & \mathbb{Z}_2 \end{tikzcd}$$

where pr denote the respective projections, $\hat{\delta}$ is defined with the aid of the isomorphism $\mathrm{Hom}(\pi_1(M), \mathbb{Z}_2) \cong \mathrm{Hom}(\frac{\pi_1(M)}{[\pi_1(M),\pi_1(M)]}, \mathbb{Z}_2)$, the isomorphism I is given by

$$I \colon \frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]} \cong H_1(M; \mathbb{Z}_2) \cong \mathbb{Z}^{2g}$$

using the Hurewicz theorem and the fact that M is a closed orientable surface of genus $g \geq 1$, and h is defined by

$$h \colon \mathbb{Z}^{2g} \to \mathbb{Z}, \qquad h(e_i) := \begin{cases} -1, & \text{if } (\hat{\delta} \circ I^{-1})(e_i) = -1 \\ 1, & \text{if } (\hat{\delta} \circ I^{-1})(e_i) = 1 \end{cases}$$

where e_i is the i-th standard basis vector of $\mathbb{R}^{2g}$.

There exists a smooth map $f \colon M \to S^1$ such that the induced map $f_* \colon \pi_1(M) \to \pi_1(S^1)$ is given by $f_* = h \circ I \circ \mathrm{pr}$. (This follows e.g. from [15, Proposition 1B.9 on p. 90].) We have shown the lemma. $\square$

Lemma 2.4.5. *Let $E \to S^1$ be a Möbius bundle (i.e., $E \to S^1$ is a non-trivial real vector bundle of rank 1). Then there exists an isomorphism of complex vector bundles*

$$Q \colon \Sigma^1 M \otimes_{\mathbb{R}} f^* E \to \Sigma^2 M$$

where $f\colon M \to S^1$ *is the map from Lemma 2.4.2, such that*

$$Q \circ \not{D}^{f^*E}_{\eta^1} = \not{D}_{\eta^2} \circ Q. \tag{2.4.2}$$

Here, $\not{D}_{\eta^2}$ *is the usual Dirac operator on the bundle* $\Sigma^2 M$ *(with respect to the spin structure* η^2*) and* $\not{D}^{f^*E}_{\eta^1}$ *is the Dirac operator of the twisted Dirac bundle* $\Sigma^1 M \otimes_{\mathbb{R}} f^*E$*.*

Proof. Let $e\colon SE \to S^1$ be the unit sphere bundle of $E \to S^1$ (w.r.t. an arbitrary bundle metric on E). Then $e\colon SE \to S^1$ is a non-trivial twofold covering. Hence the pullback $f^*e\colon f^*(SE) \to M$ is also a twofold covering.

Step 1: $f^*e\colon f^*(SE) \to M$ is a connected covering with $(f^*e)_*\pi_1(f^*(SE)) = \ker(\overline{\delta})$.[2]

Proof of step 1: Since M and S^1 are connected, the induced map $f_*\colon \pi_0(M) \to \pi_0(S^1)$ is bijective. Moreover, $f_*\colon \pi_1(M) \to \pi_1(S^1)$ is surjective by Lemma 2.4.2. Then it follows from covering space theory that $f^*e\colon f^*(SE) \to M$ is connected. (See e.g. [8, Lemma 3.6].) Moreover, it holds that

$$\begin{aligned}(f^*e)_*\pi_1(f^*(SE)) &= (f_*)^{-1}(e_*\pi_1(SE))\\ &= (f_*)^{-1}(2\mathbb{Z})\\ &= \ker\overline{\delta},\end{aligned}$$

where the first equality again follows from covering space theory (see e.g. [8, Lemma 3.4]) and the third equality follows directly from the commutative diagram in Lemma 2.4.4. ✓

Step 2: The map

$$\begin{aligned}\alpha\colon \left(\mathrm{Spin}(M,g)^1 \times_{\mathbb{Z}_2} f^*(SE)\right) \times_\rho \Sigma_2 &\to \left(\mathrm{Spin}(M,g)^1 \times_\rho \Sigma_2\right) \otimes_{\mathbb{R}} f^*E,\\ [[a,b],v] &\mapsto [a,v] \otimes b,\end{aligned}$$

where $a \in \mathrm{Spin}(M,g)^1$, $b \in f^*(SE)$, $v \in \Sigma_2$, is an isomorphism of complex vector bundles.

Proof of step 2: Note that $\mathrm{Spin}(M,g)^1 \times_{\mathbb{Z}_2} f^*(SE)$ is a $(\mathrm{Spin}(2) \times_{\mathbb{Z}_2} \mathbb{Z}_2 \cong \mathrm{Spin}(2))$-principal bundle, hence the source of the map α is well-defined. It is not difficult to verify that α is well-defined. The (well-defined) inverse of α is given on elementary tensors by

$$\alpha^{-1}([a,v] \otimes \tilde{b}) = \left[\left[a, \frac{\tilde{b}}{\|\tilde{b}\|}\right], \|\tilde{b}\| v\right]$$

[2] In the notation of the beginning of Section 2.4.1 this means that $f^*(SE)$ and P are isomorphic as coverings.

where $a \in \mathrm{Spin}(M,g)^1$, $v \in \Sigma_2$, $\tilde{b} \in f^*E$, $b \neq 0$. We have shown step 2. ✓

Combining both steps with Lemma 2.4.3 we get

$$\begin{aligned}
\Sigma^1 M \otimes_{\mathbb{R}} f^*E &= \Big(\mathrm{Spin}(M,g)^1 \times_\rho \Sigma_2\Big) \otimes_{\mathbb{R}} f^*E \\
&\cong \Big(\mathrm{Spin}(M,g)^1 \times_{\mathbb{Z}_2} f^*(SE)\Big) \times_\rho \Sigma_2 \\
&\cong \mathrm{Spin}(M,g)^2 \times_\rho \Sigma_2 \\
&= \Sigma^2 M,
\end{aligned}$$

i.e., we define $Q := (F, \mathrm{id}_{\Sigma_2}) \circ \alpha^{-1}$. Using the construction of α and equation (2.4.1) one readily checks that Q commutes with Clifford-multiplications on $\Sigma^1 M \otimes_{\mathbb{R}} f^*E$ and $\Sigma^2 M$. Combining this with the local formulas for the covariant derivatives on the spinor bundles $\Sigma^i M$ it is straightforward to deduce (2.4.2).

□

2.5 Proof of the main theorems

Before we come to the proof of the main theorems we need one more lemma.

Lemma 2.5.1. *Let M be a 2-dimensional closed connected spin manifold of positive genus. Then there exist spin structures χ^i on M, $i = 1, 2$, such that*

$$\begin{aligned} \dim_{\mathbb{H}} \ker(\not{D}_{\chi_g^1}) &= 1, \\ \dim_{\mathbb{H}} \ker(\not{D}_{\chi_g^2}) &= 0, \end{aligned} \tag{2.5.1}$$

for generic Riemannian metrics g on M.

Proof. Since M has positive genus, there exist spin structures χ^1 and χ^2 on M such that $\alpha(M, \chi_1) = 0$ and $\alpha(M, \chi_2) \neq 0$. (We denote by $\alpha(M, \chi)$ Hitchin's α-index.) Now we apply [2, Theorem 1.1] for χ_1 and χ_2. The lemma follows since the intersection of two open and dense sets is again open and dense. □

Proof of Theorem 2.3.1, case $n = 2$. Let h be any Riemannian metric on N. We choose spin structures χ^1, χ^2 on M and a C^∞-dense and C^1-open set $\mathcal{G} \subset \mathrm{Riem}(M)$ s.t. (2.5.1) holds for every $g \in \mathcal{G}$. Let $g \in \mathcal{G}$ be arbitrary and let $f = f_\delta \colon M \to S^1$ be the map of Lemma 2.4.4 where $\delta = \delta_{\chi_g^1, \chi_g^2}$.

Since N is non-orientable, there exists an orientation reversing simple (i.e., without self-intersections) closed geodesic $\gamma \colon S^1 \to N$.[3] A proof of this fact can be found in the appendix. Viewing S^1 as a submanifold of N via γ, we have that

$$\gamma^* TN \cong TS^1 \oplus (TS^1)^\perp$$

where $TS^1 \cong \underline{\mathbb{R}}$ is trivial and $(TS^1)^\perp \to S^1$ is non-trivial (i.e., a Möbius bundle), since γ is orientation reversing. Since γ is a geodesic, we have

$$\nabla^{\gamma^* TN} \cong \begin{pmatrix} \nabla^{TS^1} & 0 \\ 0 & \nabla^{(TS^1)^\perp} \end{pmatrix} \tag{2.5.2}$$

under the above isomorphism. (Here, $\nabla^{\gamma^* TN}$ is the pullback of the Levi-Civita connection on (N, h) along γ, and ∇^{TS^1} and $\nabla^{(TS^1)^\perp}$ are the projections of $\nabla^{\gamma^* TN}$ on TS^1 and $(TS^1)^\perp$, respectively.) We set $\tilde{f} := \gamma \circ f \colon M \to N$. Applying Lemma 2.4.5 we find that

$$\begin{aligned} \Sigma^1 M \otimes_{\mathbb{R}} \tilde{f}^* TN &\cong \Sigma^1 M \otimes_{\mathbb{R}} f^*(\gamma^* TN) \\ &\cong \Sigma^1 M \otimes_{\mathbb{R}} f^*(\underline{\mathbb{R}} \oplus (TS^1)^\perp) \\ &\cong \left(\Sigma^1 M \otimes_{\mathbb{R}} \underline{\mathbb{R}}\right) \oplus \left(\Sigma^1 M \otimes_{\mathbb{R}} f^*(TS^1)^\perp\right) \\ &\cong \Sigma^1 M \oplus \Sigma^2 M. \end{aligned}$$

[3] Recall that a *closed curve* is a smooth map $S^1 \to N$ and a *closed geodesic* is a closed curve that is also a geodesic.

Using (2.5.2) it follows that under this isomorphism it holds that

$$\not{D}^{\tilde{f}}_{\chi^1_g,h} \cong \begin{pmatrix} \not{D}_{\chi^1_g} & 0 \\ 0 & \not{D}_{\chi^2_g} \end{pmatrix}. \tag{2.5.3}$$

In particular,

$$\ker(\not{D}^{\tilde{f}}_{\chi^1_g,h}) \cong \ker(\not{D}_{\chi^1_g}) \oplus \ker(\not{D}_{\chi^2_g}).$$

We conclude by using (2.5.1). □

Proof of Theorem 2.3.1, case $n \geq 2$. We choose spin structures χ^1, χ^2 on M and $\mathcal{G} \subset \mathrm{Riem}(M)$ as before. Let $g \in \mathcal{G}$ be arbitrary and let $f = f_\delta \colon M \to S^1$ be the map of Lemma 2.4.4 where $\delta = \delta_{\chi^1_g, \chi^2_g}$.

Let h_0 be a Riemannian metric on N s.t. there exists a simple closed geodesic $\gamma \colon S^1 \to N$. [4] Again, we view S^1 as a submanifold of N via γ.

In the case $n = 2$, the key ingredient was the identification (2.5.3), which followed from (2.5.2). If the dimension of N is greater than two, it is more complicated to deal with the complement $(TS^1)^\perp \subset TN$ in order to get a suitable higher dimensional analog of (2.5.2). For this reason we will modify the Riemannian metric h_0 in a neighborhood of $S^1 \subset N$. To that end, let

$$U_\varepsilon := \exp^\perp\{(p, v) \in TN \mid p \in S^1,\ v \in (T_pS^1)^\perp,\ \|v\|_{h_0} < \varepsilon\}$$

be a tubular neighborhood of S^1 in N, where $\varepsilon > 0$ is sufficiently small.

Moreover, let $(\nu_1, \ldots, \nu_{n-1})$ be an orthonormal basis of $(T_{\gamma(0)}S^1)^\perp$ where we think of S^1 as $[0, 2\pi]$ with 0 and 2π identified. We define

$$\nu_i(t) := P^\gamma_{0,t}\nu_i$$

where $P^\gamma_{0,t}$ denotes the parallel transport in (N, h_0) along $\gamma|_{[0,t]}$ from $\gamma(0)$ to $\gamma(t)$, $t \in [0, 2\pi]$. Since γ is a geodesic, $(\nu_1(t), \ldots, \nu_{n-1}(t))$ is an orthonormal basis of $(T_{\gamma(t)}S^1)^\perp$ for all $t \in [0, 2\pi]$. In the basis $(\nu_1, \ldots, \nu_{n-1})$ the map

$$P^\gamma_{0,2\pi} \colon (T_{\gamma(0)}S^1)^\perp \to (T_{\gamma(2\pi)}S^1)^\perp$$

is given by a matrix $A \in \mathrm{O}(n-1)$. Then we have a diffeomorphism

$$\begin{aligned} T_A := {}^{[0,2\pi]\times B_\varepsilon(0)} \big/_{(0,x)\sim(2\pi,Ax)} &\to U_\varepsilon, \\ [(t, \sum_{i=1}^{n-1} x_i e_i)] &\mapsto \exp(\gamma(t), \sum_{i=1}^{n-1} x_i\nu_i(t)), \end{aligned}$$

[4] Given any injective closed immersed curve $\gamma \colon S^1 \to N$, it is not hard to construct a Riemannian metric on N for which γ is a simple closed geodesic. One can do this e.g. by using a tubular neighborhood of the image of γ.

where $B_\varepsilon(0) \subset \mathbb{R}^{n-1}$ is the open ball of radius ε with center 0 and $(e_1, \dots, e_{n-1})$ is the standard basis of $\mathbb{R}^{n-1}$.

Note that if $A, B \in \mathrm{O}(n-1)$ are in the same connected component of $\mathrm{O}(n-1)$, then T_A and T_B are diffeomorphic. We will use this statement a few times below without further mentioning it.

Let $A \in \mathrm{O}(n-1)$. If we endow T_A with the quotient metric induced from the product metric on $[0, 2\pi] \times B_\varepsilon(0)$, then the parallel transport in T_A along the curve $c(t) := [(t, 0)]$ from $c(0)$ to $c(2\pi)$ is given by

$$P^c_{0,2\pi} \colon T_{c(0)} T_A \to T_{c(2\pi)} T_A, \qquad P^c_{0,2\pi} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

(with respect to the splitting $T_{c(0)} T_A = T_{c(0)} S^1 \oplus (T_{c(0)} S^1)^\perp$ where similar as before we write S^1 for the image of c).

Now we distinguish three cases.

Case 1: n is even and N is non-orientable:

We can choose γ to be orientation reversing (c.f. Lemma 2.B.1). Then $P^\gamma_{0,2\pi} \colon (T_{\gamma(0)} S^1)^\perp \to (T_{\gamma(2\pi)} S^1)^\perp$ is orientation reversing and hence the associated matrix is an element of $\mathrm{O}(n-1) \setminus \mathrm{SO}(n-1)$. Therefore,

$$U_\varepsilon \cong T_{-I_{n-1}}$$

where

$$I_{n-1} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

From the discussion above we see that we can choose a Riemannian metric on U_ε such that

$$P^\gamma_{0,2\pi} \colon (T_{\gamma(0)} S^1)^\perp \to (T_{\gamma(2\pi)} S^1)^\perp, \qquad P^\gamma_{0,2\pi} v = -v$$

is minus the identity. Using a partition of unity we have shown that there exists a Riemannian metric h on N such that $P^\gamma_{0,2\pi} \colon (T_{\gamma(0)} S^1)^\perp \to (T_{\gamma(2\pi)} S^1)^\perp$ is minus the identity. This means in particular that

$$\gamma^* TN \cong TS^1 \oplus (TS^1)^\perp \cong TS^1 \oplus E_1 \oplus \ldots \oplus E_{n-1}$$

where each $E_i \to S^1$ is a Möbius bundle. Moreover, under this isomorphism we have

$$\nabla^{\gamma^* TN} \cong \begin{pmatrix} \nabla^{TS^1} & & & \\ & \nabla^{E_1} & & \\ & & \ddots & \\ & & & \nabla^{E_{n-1}} \end{pmatrix} \tag{2.5.4}$$

where ∇^{γ^*TN} is the pullback of the Levi-Civita connection on (N, h) along γ, and ∇^{TS^1}, ∇^{E_i} are the projections of ∇^{γ^*TN}. Setting

$$\tilde{f} := \gamma \circ f$$

and using Lemma 2.4.5 we get

$$\begin{aligned}
\Sigma^1 M \otimes_{\mathbb{R}} \tilde{f}^*TN &\cong \Sigma^1 M \otimes_{\mathbb{R}} (f^*(\gamma^*TN)) \\
&\cong \Sigma^1 M \otimes_{\mathbb{R}} (\underline{\mathbb{R}} \oplus f^*(E_1) \oplus \ldots \oplus f^*(E_{n-1})) \\
&\cong \Sigma^1 M \oplus \Sigma^2 M \oplus \ldots \oplus \Sigma^2 M
\end{aligned}$$

and, similar to the proof of the case $n = 2$, under this isomorphism we have

$$\not{D}^{\tilde{f}}_{\chi^1_g, h} \cong \begin{pmatrix} \not{D}_{\chi^1_g} & & & \\ & \not{D}_{\chi^2_g} & & \\ & & \ddots & \\ & & & \not{D}_{\chi^2_g} \end{pmatrix} \tag{2.5.5}$$

and therefore

$$\ker(\not{D}^{\tilde{f}}_{\chi^1_g, h}) \cong \ker(\not{D}_{\chi^1_g}) \oplus \ker(\not{D}_{\chi^2_g}) \oplus \ldots \oplus \ker(\not{D}_{\chi^2_g}).$$

We conclude by using (2.5.1).

Case 2: n is odd and N is orientable:

Then γ is orientation preserving, hence $P^{\gamma}_{0,2\pi} \colon (T_{\gamma(0)}S^1)^{\perp} \to (T_{\gamma(2\pi)}S^1)^{\perp}$ is orientation preserving and the associated matrix is an element of $\mathrm{SO}(n-1)$. We get

$$U_\varepsilon \cong T_{-I_{n-1}}$$

since $-I_{n-1}$ is in the same connected component as the associated matrix (because both are orientation preserving). Now we can proceed as in case 1.

Case 3: n is odd and N is non-orientable:

Again we can assume that γ is orientation reversing. Then the tubular neighborhood U_ε is diffeomorphic to T_A for

$$A = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Then we can proceed analogous to the first two cases, but we have to switch the roles of the spin structures χ^1 and χ^2. $\square$

Proof of Theorem 2.3.4. Proof of i): Let $\tilde{h}$ be an arbitrary Riemannian metric on N. We define $h_t := th + (1-t)\tilde{h}$. Then $\not{D}^f_{\chi_g,h_t}$ depends analytically on t in the sense of [20, Section 11]. By [20, Proposition 11.4] the set

$$\{t \in [0,1] \mid \text{the kernel of } \not{D}^f_{\chi_g,h_t} \text{ is not minimal}\}$$

is finite. Hence the set of Riemannian metrics h on N such that the kernel of $\not{D}^f_{\chi_g,h}$ is minimal is C^∞-dense in $\mathrm{Riem}(N)$. Moreover, it is C^1-open.[5]

Proof of ii): The proof is similar to the proof of i), i.e., we use linear interpolation and [20, Proposition 11.4]. Note however, that if we vary the metric on M, then the space on which the Dirac operators are defined, also changes and we cannot apply the proposition directly. To get rid of this, we identify the spinor bundles on M as in [20, Section 2.2], [6] and then we are able to apply the proposition (compare also [20, Proof of Proposition 3.1]).

Proof of iii): We want to use the same strategy as before. The difficulty this time is to find a (piecewise) real analytic path between two homotopic elements in $C^\infty(M,N)$, since linear interpolation does no longer work. Let $\tilde{f} \in C^\infty(M,N)$ be any map with $d^N(f(x),\tilde{f}(x)) < \frac{1}{2}\mathrm{inj}(N)$ for all $x \in M$, where $\mathrm{inj}(N)$ denotes the injectivity radius of N. We define

$$f_t(x) := \exp_{\tilde{f}(x)}\left(t\exp^{-1}_{\tilde{f}(x)} f(x)\right),$$

$x \in M$, where exp denotes the exponential map of N.[6] Then we claim that for all but finitely many $t \in [0,1]$ it holds that the kernel of $\not{D}^{f_t}_{\chi_g,h}$ is minimal. To see this, we denote by $P^t\colon T_{f(x)}N \to T_{f_t(x)}N$ the parallel transport along unique shortest geodesics of N joining $f(x)$ and $f_t(x)$ and consider

$$D_t := (P^t)^{-1} \circ \not{D}^{f_t}_{\chi_g,h} \circ P^t.$$

The claim follows since the family of operators D_t depends analytically on t. (P^t depends analytically on t because of the analytic dependence of solutions of ordinary differential equations on parameters. f_t depends analytically on t, since the Riemannian metric on N is real analytic.)

Now let $\tilde{f} \in [f]$ be homotopic to f and let H be any homotopy between f and $\tilde{f}$. We view H as a path $H\colon [0,1] \to C^\infty(M,N)$ with $H(0) = f$ and $H(1) = \tilde{f}$. We

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[5] One way to see this is to use the Min-Max principle to show that the map $\mathrm{Riem}(N) \to \mathbb{N}$, $h \mapsto \dim_{\mathbb{H}}\ker\not{D}^f_{\chi_g,h}$, is upper semicontinuous where on $\mathrm{Riem}(N)$ we choose the C^1-topology.

[6] Conceptually, we take a chart of the manifold $C^\infty(M,N)$ around $\tilde{f}$, linearly interpolate between $\tilde{f}$ and f in the chart, and then use the inverse of the chart to go back to $C^\infty(M,N)$. The result is the map f_t.

can cover the image of H by finitely many C^0-balls U_i of radius less than $\frac{1}{2}\mathrm{inj}(N)$, $i = 1, \ldots, N$, such that $U_i \cap U_{i+1} \neq \varnothing$ for $i = 1, \ldots, N-1$, and $f \in U_1$, $\tilde{f} \in U_N$.

We choose $f_1 \in U_1 \cap U_2$ arbitrarily. From the beginning of the proof of iii), we get that there exists a homotopy H^1 between f and f_1 such that the kernel of $\not{D}^{H^1_t}_{\chi_g,h}$ is minimal for all but finitely many $t \in [0,1]$. We can assume that the kernel of $\not{D}^{f_1}_{\chi_g,h}$ is minimal. Continuing in that manner, we conclude that there exists $f_{N-1} \in U_{N-1} \cap U_N$ such that the kernel of $\not{D}^{f_{N-1}}_{\chi_g,h}$ is minimal and a homotopy H^{N-1} between f_{N-1} and $\tilde{f}$ such that the kernel of $\not{D}^{H^{N-1}_t}_{\chi_g,h}$ is minimal for all but finitely many $t \in [0,1]$. Hence the set of maps $f \in [\tilde{f}]$ such that the kernel of $\not{D}^{f}_{\chi_g,h}$ is minimal is C^∞-dense in $[f]$. As above, it is also C^1-open. $\square$

Appendix

2.A Proof of Lemma 2.4.3

Let us choose $x_0 \in M$ and $y_0 \in p^{-1}(x_0)$. Then we define a mapping

$$F\colon \mathrm{Spin}(M,g)^1 \times_{\mathbb{Z}_2} P \to \mathrm{Spin}(M,g)^2$$

as follows.

Let $a \in \mathrm{Spin}(M,g)^1_x$ and $b \in P_x$ be given.

1) Choose a path $\omega\colon [0,1] \to M$ s.t. $\omega(0) = x_0$ and $\omega(1) = x$. Moreover, denote by $\gamma^\omega\colon [0,1] \to P$ the lift of ω with $\gamma^\omega(0) = y_0$.

2) Choose a lift $\tilde{\omega}\colon [0,1] \to \mathrm{SO}(M,g)$ of ω.

3) Choose lifts $\gamma_i^{\tilde{\omega}}\colon [0,1] \to \mathrm{Spin}(M,g)^i$ of $\tilde{\omega}$ satisfying

$$\gamma_1^{\tilde{\omega}}(0) \cong \gamma_2^{\tilde{\omega}}(0)$$

where we identify $\mathrm{Spin}(M,g)^1_{x_0} \cong \mathrm{Spin}(M,g)^2_{x_0}$ with a fixed isomorphism (we fix the isomorphism for the whole proof).

4) Let $A = A^{\tilde{\omega}} \in \mathrm{Spin}(m)$ and $B = B^\omega \in \mathbb{Z}_2$ be the uniquely determined elements of $\mathrm{Spin}(m)$ and $\mathbb{Z}_2$, respectively, s.t.

$$\gamma_1^{\tilde{\omega}}(1) \cdot A = a,$$
$$\gamma^\omega(1) \cdot B = b.$$

Then we define

$$F([a,b]) := \gamma_2^{\tilde{\omega}}(1) \cdot A \cdot B.$$

The main task is to show that F is well-defined, i.e., doesn't depend on the choices made in 1)-3).

One easily verifys that the definition of F is independent of the choice of the $\gamma_i^{\tilde{\omega}}$, since for each $i = 1,2$ there exist exactly two such lifts and they differ only by $-1 \in \mathbb{Z}_2 = \ker(\mathrm{Spin}(m) \to \mathrm{SO}(m))$.

Therefore, it remains to show that F is independent of the choice of ω and $\tilde{\omega}$ in 1) and 2). To that end, we will show the following lemma.

Lemma 2.A.1. *Choose $x \in M$ and let $\omega, \sigma\colon [0,1] \to M$ be paths from x_0 to x. Moreover, let $\tilde{\omega}, \tilde{\sigma}\colon [0,1] \to \mathrm{SO}(M,g)$ be lifts of ω and σ, respectively. Then the following holds:*

i) If $\omega * \overline{\sigma} \in \ker(\overline{\delta})$, then $\gamma_2^{\tilde{\omega}}(1) \cdot A^{\tilde{\omega}} = \gamma_2^{\tilde{\sigma}}(1) \cdot A^{\tilde{\sigma}}$ and $B^{\omega} = B^{\sigma}$.

ii) If $\omega * \overline{\sigma} \notin \ker(\overline{\delta})$, then $\gamma_2^{\tilde{\omega}}(1) \cdot A^{\tilde{\omega}} = \gamma_2^{\tilde{\sigma}}(1) \cdot A^{\tilde{\sigma}} \cdot (-1)$ and $B^{\omega} = B^{\sigma} \cdot (-1)$.

In particular, F is well-defined.

Proof. Let us prove i) first. Notice that since $\omega * \overline{\sigma} \in \ker(\overline{\delta}) = p_*(\pi_1(P, y_0))$ (c.f. the beginning of Section 2.4.1) we have that $\omega * \overline{\sigma}$ can be lifted to a loop in P, and we directly get $B^{\omega} = B^{\sigma}$. Now we proceed in several steps.

Step 1: The assertion of i) holds if $\tilde{\omega}(0) = \tilde{\sigma}(0)$ and $\tilde{\omega}(1) = \tilde{\sigma}(1)$.
Since $\omega * \overline{\sigma} \in \ker(\overline{\delta})$ it follows from Lemma 2.4.2 that $\alpha := \tilde{\omega} * \overline{\tilde{\sigma}} \in \ker(\delta)$. We further distinguish two cases.
Case 1: α lifts to a loop in $\mathrm{Spin}(M,g)^i$, $i = 1,2$.
In this case we get lifts $\gamma_i^{\tilde{\omega}}, \gamma_i^{\tilde{\sigma}}\colon [0,1] \to \mathrm{Spin}(M,g)^i$ of $\tilde{\omega}$ and $\tilde{\sigma}$, respectively, s.t.

$$\begin{aligned} \gamma_i^{\tilde{\omega}}(0) &= \gamma_i^{\tilde{\sigma}}(0) \in \mathrm{Spin}(M,g)^i_{x_0}, \\ \gamma_i^{\tilde{\omega}}(1) &= \gamma_i^{\tilde{\sigma}}(1) \in \mathrm{Spin}(M,g)^i_{x}, \end{aligned}$$

$i = 1,2$ and step 1 directly follows. (Note that we already remarked that the definition of F is independent of the choices in 4).)
Case 2: α does not lift to a loop in $\mathrm{Spin}(M,g)^i$, $i = 1,2$.
In this case we get lifts $\gamma_1^{\tilde{\omega}}, \gamma_1^{\tilde{\sigma}}\colon [0,1] \to \mathrm{Spin}(M,g)^1$ of $\tilde{\omega}$ and $\tilde{\sigma}$, respectively, s.t. $\gamma_1^{\tilde{\omega}}(1) = \gamma_1^{\tilde{\sigma}}(1)$ and $\gamma_1^{\tilde{\omega}}(0) \neq \gamma_1^{\tilde{\sigma}}(0) \in \mathrm{Spin}(M,g)^1_{x_0}$, i.e.,

$$\{\gamma_1^{\tilde{\omega}}(0), \gamma_1^{\tilde{\sigma}}(0)\} = \mathrm{Spin}(M,g)^1_{x_0}. \tag{2.A.1}$$

Then we lift α to a path in $\mathrm{Spin}(M,g)^2$ with starting point $\gamma_1^{\tilde{\omega}}(0)$ and this lift gives us a choice for $\gamma_2^{\tilde{\omega}}$ and $\gamma_2^{\tilde{\sigma}}$. We have

$$\gamma_2^{\tilde{\omega}}(0) \neq \gamma_2^{\tilde{\sigma}}(0) \in \mathrm{Spin}(M,g)^1_{x_0} \cong \mathrm{Spin}(M,g)^2_{x_0}$$

and $\gamma_2^{\tilde{\omega}}(0) \cong \gamma_1^{\tilde{\omega}}(0)$. Combining with (2.A.1) we get $\gamma_2^{\tilde{\omega}}(0) \cong \gamma_1^{\tilde{\sigma}}(0)$ and we have shown step 1.

Step 2: The assertion of i) holds if $\tilde{\omega}(1) = \tilde{\sigma}(1)$.
Let $c\colon [0,1] \to \mathrm{SO}(M,g)_{x_0}$ be a path with $c(0) = \tilde{\omega}(0)$ and $c(1) = \tilde{\sigma}(0)$. Let $\hat{c}$ be the lift of c to $\mathrm{Spin}(M,g)^1$ with $\hat{c}(1) = \gamma_1^{\tilde{\sigma}}(0)$. Note that $\hat{c}$ only takes values in $\mathrm{Spin}(M,g)^1_{x_0} \cong \mathrm{Spin}(M,g)^2_{x_0}$, so we also think of $\hat{c}$ as lift of c to $\mathrm{Spin}(M,g)^2_{x_0}$. Now we can apply the result of step 1 to $\sigma_1 := \sigma * x_0$ (where x_0 denotes the constant path), ω, $\widetilde{\sigma_1} := \tilde{\sigma} * c$, and $\tilde{\omega}$.

Step 3: The assertion of i) holds.
Choose $X \in \mathrm{SO}(m)$ such that $\tilde{\sigma}(1) \cdot B = \tilde{\omega}(1)$. We conclude by using step 2.

For ii), we first observe the following: if $\omega * \overline{\sigma} \notin \ker(\overline{\delta}) = p_*(\pi_1(P, y_0))$, then $\omega * \overline{\sigma}$ does not lift to a loop in P. From this we easily get $\gamma^\omega(1) = \gamma^\sigma(1) \cdot (-1)$ and therefore $B^\omega = B^\sigma \cdot (-1)$. Moreover, $\gamma_2^{\tilde{\omega}}(1) \cdot A^{\tilde{\omega}} = \gamma_2^{\tilde{\sigma}}(1) \cdot A^{\tilde{\sigma}} \cdot (-1)$ can be shown similar to the proof of i) by splitting the proof into the same three steps. □

For the inverse of F, we define a mapping

$$G\colon \mathrm{Spin}(M,g)^2 \to \mathrm{Spin}(M,g)^1 \times_{\mathbb{Z}_2} P$$

by the following. Let $a \in \mathrm{Spin}(M,g)^2_x$, $x \in M$.

i) Choose a path $\omega\colon [0,1] \to M$ s.t. $\omega(0) = x_0$ and $\omega(1) = x$. Denote by $\gamma^\omega\colon [0,1] \to P$ the unique lift of ω to P with $\gamma^\omega(0) = y_0$.

ii) Choose a lift $\tilde{\omega}\colon [0,1] \to \mathrm{SO}(M,g)$ of ω to $\mathrm{SO}(M,g)$.

iii) For $i = 1, 2$ choose lifts $\gamma_i^{\tilde{\omega}}\colon [0,1] \to \mathrm{Spin}(M,g)^i$ with

$$\gamma_1^{\tilde{\omega}}(0) \cong \gamma_2^{\tilde{\omega}}(0).$$

iv) Denote by $A \in \mathrm{Spin}(m)$ the unique element of $\mathrm{Spin}(m)$ s.t.

$$\gamma_2^{\tilde{\omega}}(1) \cdot A = a.$$

Then we define

$$G(a) := [\gamma_1^{\tilde{\omega}}(1) \cdot A, \gamma^\omega(1)].$$

Using the same ideas as above one can show that G is well-defined. Directly from the definitions of F and G we get $F \circ G = id$ and $G \circ F = id$.

2.B Existence of orientation reversing simple closed geodesics

In the proof of Theorem 2.3.1 on page 121 we used the existence of an orientation reversing simple closed geodesic $\gamma\colon S^1 \to N$ where N is a non-orientable closed Riemannian manifold.

Starting with any closed curve $\gamma_0\colon S^1 \to N$ it is a standard result that one can find a closed geodesic in the homotopy class of γ_0. A direct proof can be found e.g. in [18, Theorem 1.5.1] or [14, 2.98 Theorem on p. 94] and a proof using the heat flow method is given in [18, Theorem 1.6.1]. This geodesic is orientation reversing provided that γ_0 is orientation reversing, but not necessarily without self-intersections.

Moreover, it is well known that if $\pi_1(N) \neq \{1\}$, then there exists a closed geodesic on N which minimizes length in the class of homotopically non-trivial closed curves on N and this geodesic has no self-intersections, see e.g. [22, Lemma 1.5. (2) and Exercise 3 on p. 197]. However, this geodesic is not necessarily orientation reversing.

We prove the following lemma.

Lemma 2.B.1. *Let N be a closed non-orientable Riemannian manifold (in particular this implies that $\pi_1(N) \neq \{1\}$). Then there exists an orientation reversing simple closed geodesic $\gamma\colon S^1 \to N$.*

Proof. Let $\gamma_0\colon S^1 \to N$ be an orientation reversing closed geodesic. If γ_0 has no self-intersections, we are done. So assume that γ_0 is not injective. Then we can split γ_0 into two geodesic loops[7] $\tilde{\gamma}_0, \hat{\gamma}_0\colon [0,1] \to N$ both based at the same point where $\tilde{\gamma}_0$ is orientation reversing (hence non-trivial). Let $c_0\colon S^1 \to N$ be a smooth approximation of $\tilde{\gamma}_0$ homotopic to $\tilde{\gamma}_0$ with

$$|L(\tilde{\gamma}_0) - L(c_0)| < \varepsilon$$

where $\varepsilon > 0$ is small and L denotes the length. Then there exists a closed orientation reversing geodesic $\gamma_1\colon S^1 \to N$ in the homotopy class of c_0 which also minimizes length in its homotopy class, see e.g. [22, Lemma 1.5. (1) on p. 197]. If γ_1 is injective, we are done. If not, we repeat the above process with γ_0 replaced by γ_1.

We have to ensure that this process stops after finitely many steps. This follows from the following two observations. Firstly, each γ_k has positive length, i.e., $L(\gamma_k) > 0$. Secondly, in each step, the length drops a fixed amount. To see the latter, we recall the following: if N is a closed Riemannian manifold and c is an arbitrary geodesic loop in N (the base point is allowed to vary), then the

[7] A *geodesic loop* is a geodesic $c\colon [0,1] \to N$ with $c(0) = c(1)$.

length of c is bounded from below by two times the injectivity radius of N,

$$L(c) \geq 2\mathrm{inj}(N) =: C.$$

Returning to the beginning of the proof, we choose $\varepsilon = \frac{1}{2}C$ to deduce

$$\begin{aligned} L(\gamma_0) &= L(\tilde{\gamma}_0) + L(\hat{\gamma}_0) \\ &\geq L(\tilde{\gamma}_0) + C \\ &\geq L(c_0) + \frac{1}{2}C \\ &\geq L(\gamma_1) + \frac{1}{2}C \end{aligned}$$

and entirely analogous

$$L(\gamma_{k+1}) \leq L(\gamma_k) - \frac{1}{2}C.$$

Hence in each step the length drops by at least $\frac{1}{2}C$.

□

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Chapter 3

The Banach manifold $C^k(M, N)$

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Abstract Let M be a compact manifold without boundary and let N be a connected manifold without boundary. For each $k \in \mathbb{N}$ the set of k times continuously differentiable maps between M and N has the structure of a smooth Banach manifold where the underlying manifold topology is the compact-open C^k topology. We provide a detailed and rigorous proof for this important statement. Our proof is already partially covered by existing literature. This chapter is similar to [11] but more detailed.

3.1 Introduction

Let M be a closed manifold[1] and let N be a connected manifold without boundary. For each $k \in \mathbb{N} := \{0, 1, 2, \ldots\}$ we denote by $C^k(M, N)$ the set of k times continuously differentiable maps between M and N.

It is well known that for each $k \in \mathbb{N}$ the set $C^k(M, N)$ has the structure of a smooth Banach manifold. The natural idea to turn $C^k(M, N)$ into a Banach manifold is to choose a Riemannian metric on N and then use the exponential map of N to construct the charts of $C^k(M, N)$. More precisely, for g close enough to f, the map

$$C^k(M, N) \ni g \mapsto (p \mapsto (\exp_{f(p)})^{-1} g(p)) \in \Gamma_{C^k}(f^*TN),$$

is a chart around f. Here, exp denotes the exponential map of the Riemannian manifold N. This idea can be found in many places in the literature (references are given below). Let us denote this chart by φ_f.

[1] By "manifold" we always mean a smooth ($= C^\infty$) and finite-dimensional manifold. All manifolds we consider are non-empty, second-countable, and Hausdorff. A closed manifold is a compact manifold without boundary.

Driven by applications, there are several natural requirements and questions: One needs a rigorous and detailed proof that these charts induce a smooth structure. Are the transition maps $\varphi_f \circ (\varphi_g)^{-1}$ only smooth for $f, g \in C^\infty(M,N)$ or are they also smooth in the case that f and g are precisely k times continuously differentiable? Is the manifold topology of $C^k(M,N)$ the compact-open C^k topology?

An investigation of literature regarding these questions only brought up partial answers and proofs [4, 1, 3, 5, 10, 9, 6, 2, 8]. Note that the case $k = \infty$ is better dealt with in the literature, in particular a very thorough treatment of the space $C^\infty(M,N)$ can be found in [8].

In this chapter we provide a detailed proof for the following theorem.

Theorem. Let $k \in \mathbb{N}$ and fix a Riemannian metric on N. Then the set $C^k(M,N)$ endowed with the compact-open C^k topology has the structure of a smooth Banach manifold with the following property: for any $f \in C^k(M,N)$ there is an open neighborhood U_f of f in $C^k(M,N)$ and an open neighborhood V_f of the zero section in $\Gamma_{C^k}(f^*TN)$ such that the map

$$\begin{aligned} \varphi_f \colon U_f &\to V_f, \\ g &\mapsto (p \mapsto (\exp_{f(p)})^{-1} g(p)) \end{aligned}$$

is a smooth diffeomorphism. Here, we endow the space $\Gamma_{C^k}(f^*TN)$ of C^k-sections of f^*TN with the usual C^k-norm. Moreover, this smooth structure on $C^k(M,N)$ does not depend on the choice of Riemannian metric on N.

The basic strategy to prove the theorem is as follows. We first show that the maps $\varphi_f \colon U_f \to V_f$ are homeomorphisms. Then we argue why the transition maps $\varphi_f \circ (\varphi_g)^{-1}$ are smooth provided that $U_f \cap U_g \neq \varnothing$. For this our arguments are inspired by [1]. The smoothness of the transition maps is the most delicate part, and one has to argue very carefully, since φ_f and φ_g are defined using not necessarily smooth maps f and g. The main input for this will be the Ω-lemma (using the terminology of [1, 2]) which we will first prove in a "local" version, see Lemma 3.2.4, and then "globalize" to maps between sections of vector bundles, see Lemma 3.4.2.

3.2 Preliminaries and the local Ω-lemma

We begin by recalling some basic definitions regarding differentiability of maps between normed vector spaces.

Let $U \subset \mathbb{R}^n$ be open and let $(Y, \|.\|_Y)$ be a normed vector space. We say that $f\colon U \to Y$ is *continuously differentiable* if for all $j = 1, \ldots, n$ and all $x_0 \in U$ the limit

$$\left(\partial_{x_j} f\right)(x_0) := \lim_{h \to 0} \frac{1}{h}\left(f(x_0 + he_j) - f(x_0)\right)$$

exists in Y and the maps $\partial_{x_j} f\colon U \to Y$ are continuous. Let $k \in \mathbb{N}_{>0}$. We say that $f\colon U \to Y$ is *k times continuously differentiable* (or *f is a C^k-map*) if for all $j = 1, \ldots, n$ the map $\partial_{x_j} f\colon U \to Y$ is continuous for $k = 1$, respectively $(k-1)$ times continuously differentiable for $k \geq 2$. We define

$$C^k(U, Y) := \{f\colon U \to Y \mid f \text{ is } k \text{ times continuously differentiable}\},$$
$$C^k(\overline{U}, Y) := \{f \in C^k(U, Y) \mid \partial_x^\alpha f \text{ has a continuous extension to } \overline{U} \text{ for all } |\alpha| \leq k\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multiindex, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$, and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. If $U \subset \mathbb{R}^n$ is open and bounded, we define

$$\|f\|_{C^k(\overline{U}, Y)} := \max_{|\alpha| \leq k} \sup_{x \in \overline{U}} \|\partial_x^\alpha f(x)\|_Y$$

for all $f \in C^k(\overline{U}, Y)$. If $(Y, \|.\|_Y)$ is a Banach space, then $(C^k(\overline{U}, Y), \|.\|_{C^k(\overline{U}, Y)})$ is a Banach space.

Details of the following definitions can be found in e.g. [2]. Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be normed vector spaces, $U \subset X$ open, and $f\colon U \to Y$ a map. We say that f is *differentiable* if for all $x_0 \in U$ there exists a continuous linear map $Df(x_0) := Df_{x_0}\colon X \to Y$ s.t. for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ s.t. whenever $0 < \|x - x_0\|_X < \delta$, we have

$$\frac{\|f(x) - f(x_0) - Df_{x_0}(x - x_0)\|_Y}{\|x - x_0\|_X} < \varepsilon.$$

We say that f is *continuously differentiable* if f is differentiable and the map

$$Df\colon U \to L(X, Y), \qquad x \mapsto Df_x,$$

is continuous. Here, $L(X, Y)$ denotes the space of continuous linear maps $X \to Y$. Similarly, $L^i(X, Y)$ denotes the space of i-multilinear continuous maps

$$\underbrace{X \times \ldots \times X}_{n \text{ times}} \to Y.$$

We endow $L^i(X, Y)$ with the norm

$$\|f\| := \sup\left\{\frac{\|f(x_1, \ldots, x_i)\|_Y}{\|x_1\|_X \cdot \ldots \cdot \|x_i\|_X} \mid x_1, \ldots, x_i \in X \setminus \{0\}\right\}.$$

Finally, we denote by $L^i_s(X,Y) \subset L^i(X,Y)$ the symmetric elements of $L^i(X,Y)$. Inductively, we define

$$D^i f := D(D^{i-1} f)\colon U \to L^i(X,Y)$$

if it exists, where we have identified $L(X, L^{i-1}(X,Y))$ with $L^i(X,Y)$. If $D^k f$ exists and is continuous, we say that f is *k times continuously differentiable* (or *f is a C^k-map*). In the case that $X = \mathbb{R}^n$ this definition coincides with the one given above. We define

$$C^k(U,Y) := \{f\colon U \to Y \mid f \text{ is } k \text{ times continuously differentiable}\}.$$

Note that if $f \in C^k(U,Y)$, then $D^k f(x) \in L^k_s(X,Y)$ for all $x \in U$.

The following technical lemma will be helpful to show e.g. that the maps that will later be the charts of $C^k(M,N)$ are homeomorphisms.

Lemma 3.2.1. *Let $U \subset \mathbb{R}^n$ be open and $K \subset U$ be compact. Moreover, let $\Phi\colon U \to \Phi(U)$ be a C^k-diffeomorphism, $\Psi\colon W \to \mathbb{R}^l$ a C^k-map, and $W \subset \mathbb{R}^m$ open.*

i) There exists a $C = C(\Phi, K) > 0$ s.t.

$$\begin{aligned}&\max_{|\alpha|\le k} \sup_{x\in\Phi(K)} \|\partial^\alpha_x (f_1 \circ \Phi^{-1})(x) - \partial^\alpha_x (f_2 \circ \Phi^{-1})(x)\| \\ &\le C \max_{|\alpha|\le k} \sup_{x\in K} \|\partial^\alpha_x f_1(x) - \partial^\alpha_x f_2(x)\|\end{aligned}$$

for all $U_i \subset \mathbb{R}^n$ open, $K \subset U_i \subset U$, and $f_i \in C^k(U_i, \mathbb{R}^m)$, $i = 1, 2$.

ii) Let $R > 0$ and $\tilde{K} \subset W$ be compact. Then there exists a $C = C(\Psi, K, \tilde{K}, R) > 0$ s.t.

$$\begin{aligned}&\max_{|\alpha|\le k} \sup_{x\in K} \|\partial^\alpha_x (\Psi \circ f_1)(x) - \partial^\alpha_x (\Psi \circ f_2)(x)\| \\ &\le C \max_{|\alpha|\le k} \sup_{x\in K} \|\partial^\alpha_x f_1(x) - \partial^\alpha_x f_2(x)\|\end{aligned}$$

for all $U_i \subset \mathbb{R}^n$ open, $f_i \in C^k(U_i, \mathbb{R}^m)$ with $f_i(K) \subset \tilde{K}$, $f_i(U_i) \subset W$, and

$$\max_{|\alpha|\le k} \sup_{x\in K} \|\partial^\alpha_x f_i(x)\| \le R$$

for $i = 1, 2$. Moreover, $C(\Psi, K, \tilde{K}, R)$ can be chosen s.t. $R \mapsto C(\Psi, K, \tilde{K}, R)$ is non-decreasing.

iii) Let $R > 0$, $U_1 \subset \mathbb{R}^n$ open, and $\tilde{K} \subset W$ be compact. Let $f_1 \in C^k(U_1, \mathbb{R}^m)$ with $f_1(K) \subset \tilde{K}$ and $f_1(U_1) \subset W$. Then there exists a $C = C(\Psi, K, \tilde{K}, R, f_1) > 0$ s.t.

$$\begin{aligned}&\max_{|\alpha|\le k} \sup_{x\in K} \|\partial^\alpha_x (\Psi \circ f_1)(x) - \partial^\alpha_x (\Psi \circ f_2)(x)\| \\ &\le C \max_{|\alpha|\le k} \sup_{x\in K} \|\partial^\alpha_x f_1(x) - \partial^\alpha_x f_2(x)\|\end{aligned}$$

for all $U_2 \subset \mathbb{R}^n$ open, $f_2 \in C^k(U_2, \mathbb{R}^m)$ with $f_2(K) \subset \tilde{K}$, $f_2(U_2) \subset W$, and

$$\max_{|\alpha|\leq k} \sup_{x\in K} \|\partial_x^\alpha f_1(x) - \partial_x^\alpha f_2(x)\| \leq R.$$

Moreover, $C(\Psi, K, \tilde{K}, R, f_1)$ can be chosen s.t. $R \mapsto C(\Psi, K, \tilde{K}, R, f_1)$ is non-decreasing.

The goal for the remainder of this section is to state and prove the so-called (local) Ω-lemma. As stated in the introduction, this lemma is the key to show that $C^k(M, N)$ carries a smooth structure. To that end, we recall the following version of Taylor's theorem.

Suppose that X is a Banach space and that $U \subset X$ is an open convex subset. An open subset $\tilde{U} \subset X \times X$ is *a thickening of U* if

i) $U \times \{0\} \subset \tilde{U}$,

ii) $u + th \in U$ for all $(u, h) \in \tilde{U}$ and $0 \leq t \leq 1$,

iii) $(u, h) \in \tilde{U}$ implies $u \in U$.

Note that there always exists a thickening of U.

Lemma 3.2.2 (Taylor's theorem)**.** *Let X and Y be Banach spaces, $U \subset X$ open and convex, $\tilde{U}$ a thickening of U. A map $f\colon U \to Y$ is r times continuously differentiable if and only if there are continuous maps*

$$\varphi_i\colon U \to L_s^i(X, Y), \qquad i = 1, \dots r,$$

and

$$R\colon \tilde{U} \to L_s^r(X, Y),$$

s.t. for all $(u, h) \in \tilde{U}$,

$$f(u + h) = f(u) + \left(\sum_{i=1}^{r} \frac{\varphi_i(u)}{i!} h^i\right) + R(u, h)h^r$$

where $h^i = (h, \dots, h)$ (i times) and $R(u, 0) = 0$. If f is r times continuously differentiable, then necessarily $\varphi_i = D^i f$ for all $i = 1, \dots, r$ and in addition

$$R(u, h) = \int_0^1 \frac{(1-t)^{r-1}}{(r-1)!} \left(D^r f(u + th) - D^r f(u)\right) dt$$

A proof can be found in e.g. [2, 2.4.15 Theorem].

Lemma 3.2.3 (Differentiating under the integral)**.** *Let Y be a Banach space, $U \subset \mathbb{R}^n$ open, $f \in C^r(\overline{(a,b) \times U}, Y)$. Then $h\colon U \to Y$ defined by*

$$h(x) = \int_a^b f(t,x)dt$$

is an element of $C^r(\overline{U}, Y)$ and

$$(\partial_x^\alpha h)(x) = \int_a^b (\partial_x^\alpha(f(t,.)))(x)dt$$

for all $x \in U$, $|\alpha| \leq r$.

Lemma 3.2.4 (local Ω-lemma)**.** *Let $r, l \in \mathbb{N}$. Let $U \subset \mathbb{R}^n$ be open and bounded and let $V \subset \mathbb{R}^m$ be open, bounded, and convex. Moreover, let Y be a Banach space and*

$$g\colon U \times V \to Y$$

a map s.t.

i) $g \in C^r(\overline{U \times V}, Y)$.

ii) For each $i \in \{0, \dots, l\}$ the map

$$D_2^i g\colon U \times V \to L_s^i(\mathbb{R}^m, Y),$$

defined by $(D_2^i g)(x,y) := (D^i(g(x,.))(y)$ for all $(x,y) \in U \times V$ exists and is an element of $C^r(\overline{U \times V}, L_s^i(\mathbb{R}^m, Y))$.

Then the map

$$\begin{aligned}\Omega_g\colon C^r(\overline{U}, V) &\to C^r(\overline{U}, Y)\\ f &\mapsto (x \mapsto g(x, f(x)))\end{aligned}$$

is an element of $C^l(C^r(\overline{U}, V), C^r(\overline{U}, Y))$. Here,

$$C^r(\overline{U}, V) := \{f \in C^r(\overline{U}, \mathbb{R}^m) \mid f(\overline{U}) \subset V\}$$

and $C^r(\overline{U}, V) \subset C^r(\overline{U}, \mathbb{R}^m)$ is open. Moreover, if $l > 0$, it holds that

$$D^i(\Omega_g) = A_i \circ \Omega_{D_2^i g} \tag{3.2.1}$$

for each $i = 1, \dots, l$, where A_i is the continuous map

$$A_i\colon C^r(\overline{U}, L_s^i(\mathbb{R}^m, Y)) \to L_s^i(C^r(\overline{U}, \mathbb{R}^m), C^r(\overline{U}, Y))$$

defined by

$$((A_i(H))(h_1, \dots, h_i))(x) := (H(x))(h_1(x), \dots, h_i(x))$$

The statement of Lemma 3.2.4 can be found in different versions in [1, 2]. Our proof is an adapted version of [2, Proof of 2.4.21 Proposition].

Proof of Lemma 3.2.4. First we prove that $C^r(\overline{U}, V) \subset C^r(\overline{U}, \mathbb{R}^m)$ is open. Choose $f_0 \in C^r(\overline{U}, V)$. Since $f_0(\overline{U})$ is compact, $\mathbb{R}^m \setminus V$ is closed, and $f_0(\overline{U}) \cap (\mathbb{R}^m \setminus V) = \varnothing$, we have

$$\varepsilon := \operatorname{dist}(f_0(\overline{U}), \mathbb{R}^m \setminus V) > 0.$$

Now assume that $\|f - f_0\|_{C^r(\overline{U}, \mathbb{R}^m)} < \varepsilon$. It follows that $\|f(x) - f_0(x)\|_Y < \varepsilon$ for all $x \in \overline{U}$. By definition of ε, this means $f(\overline{U}) \subset V$ and so $C^r(\overline{U}, V) \subset C^r(\overline{U}, \mathbb{R}^m)$ is open.

In the case "$l = 0$, $r \in \mathbb{N}$" the assertion of the lemma follows from a computation. Assume $l \in \mathbb{N}_{>0}$ and $r \in \mathbb{N}$. Let $\tilde{V} \subset \mathbb{R}^m \times \mathbb{R}^m$ be a thickening of V. From applying Lemma 3.2.2 to $g(x, .)$ (for x fixed) it follows that for all $(y_1, y_2) \in \tilde{V}$ and all $x \in U$ we have

$$g(x, y_1 + y_2) = g(x, y_1) + \left(\sum_{i=1}^{l} \frac{1}{i!} (D_2^i g)(x, y_1) y_2^i \right) + R(x, y_1, y_2) y_2^l \tag{3.2.2}$$

where the map

$$R\colon U \times \tilde{V} \to L_s^l(\mathbb{R}^m, Y)$$

is given by

$$R(x, y_1, y_2) = \int_0^1 \frac{(1-t)^{l-1}}{(l-1)!} \left(D_2^l g(x, y_1 + t y_2) - D_2^l g(x, y_1) \right) dt.$$

Define

$$F(t, x, y_1, y_2) := \frac{(1-t)^{l-1}}{(l-1)!} \left(D_2^l g(x, y_1 + t y_2) - D_2^l g(x, y_1) \right).$$

From ii) it follows that

$$F \in C^r(\overline{(0,1) \times U \times \tilde{V}}, L_s^l(\mathbb{R}^m, Y)).$$

From Lemma 3.2.3 it follows that

$$R \in C^r(\overline{U \times \tilde{V}}, L_s^l(\mathbb{R}^m, Y)).$$

Since we already proved the case "$l = 0$, $r \in \mathbb{N}$" we see that

$$\begin{aligned} \Omega_R\colon C^r(\overline{U}, \tilde{V}) &\to C^r(\overline{U}, L_s^l(\mathbb{R}^m, Y)), \\ h &\mapsto (x \mapsto R(x, h(x))), \end{aligned}$$

is continuous. In particular,

$$\tilde{R} := A_l \circ \Omega_R\colon C^r(\overline{U}, \tilde{V}) \to L_s^l(C^r(\overline{U}, \mathbb{R}^m), C^r(\overline{U}, Y))$$

is continuous. Analogously, we see that

$$\widetilde{\Omega_{D_2^i g}} := A_i \circ \Omega_{D_2^i g} \colon C^r(\overline{U},V) \to L_s^i(C^r(\overline{U},\mathbb{R}^m), C^r(\overline{U},Y))$$

is continuous for $i = 1, \ldots, l$. From (3.2.2) it follows that for all $(f,h) \in C^r(\overline{U},\tilde{V})$ we have

$$\Omega_g(f+h) = \Omega_g(f) + \left(\sum_{i=1}^{l} \frac{1}{i!}\widetilde{\Omega_{D_2^i g}}(f)h^i\right) + \tilde{R}(f,h)h^l.$$

From Lemma 3.2.2 we conclude that $\Omega_g \in C^l(C^r(\overline{U},V), C^r(\overline{U},Y))$ and

$$D^i\left(\Omega_g\right) = \widetilde{\Omega_{D_2^i g}} = A_i \circ \Omega_{D_2^i g}$$

for $i = 1, \ldots, l$. (Here we used that $C^r(\overline{U},\tilde{V})$, viewed as a subset of $C^r(\overline{U},\mathbb{R}^m) \times C^r(\overline{U},\mathbb{R}^m)$, is a thickening of $C^r(\overline{U},V)$.) $\square$

3.3 The topological space $C^k(M,N)$

In this section we recall the definitions of the compact-open C^k topology on $C^k(M,N)$ and the C^k-norm on sections of vector bundles. We are very precise when stating these definitions, so that no confusion arises when we use them later in technical proofs. Then we show that the maps which will be the charts of $C^k(M,N)$ are homeomorphisms.

The following definition is taken from [7, Chapter 2].

Definition 3.3.1 (compact-open C^k topology)**.** Let M and N be manifolds without boundary and $k \in \mathbb{N}$. For $f \in C^k(M,N)$, charts (φ, U) and (ψ, V) of M and N, respectively, $K \subset U$ compact with $f(K) \subset V$ and $\varepsilon > 0$ we define the set

$$\begin{aligned}\mathcal{N}^k(f,\varphi,U,\psi,V,K,\varepsilon) := \{&g \in C^k(M,N) \mid g(K) \subset V, \\ &\max_{|\alpha|\le k} \sup_{x\in\varphi(K)} \|\partial_x^\alpha(\psi\circ g\circ\varphi^{-1})(x) - \partial_x^\alpha(\psi\circ f\circ\varphi^{-1})(x)\| < \varepsilon\}\end{aligned}$$

where $\|.\|$ denotes the Euclidean norm. The *compact-open C^k topology* (or *weak topology*) *on* $C^k(M,N)$ is the topology generated by the set

$$\begin{aligned}\{\mathcal{N}^k(f,\varphi,U,\psi,V,K,\varepsilon) \mid f \in C^k(M,N),\ (\varphi,U) \text{ and } (\psi,V) \text{ charts of } M \text{ and } N,\\ \text{respectively, } K \subset U \text{ compact with } f(K) \subset V,\ \varepsilon > 0\}.\end{aligned}$$

From now on, we always assume $C^k(M,N)$ to be equipped with the compact-open C^k topology. The topological space $C^k(M,N)$ is second-countable and metrizable [7, p. 35]. In particular, it is Hausdorff.

For the next lemma, we recall two definitions. Let $\mathcal{A}$ be a set of open subsets of a topological space X. Then $\mathcal{A}$ is called a *neighborhood subbasis of* $x \in X$ if for every open set $U \subset X$ with $x \in U$ there exist $A_i \in \mathcal{A}$, $i = 1,\ldots,l$, $l \in \mathbb{N}$ s.t. $x \in A_1 \cap \ldots \cap A_l \subset U$. We call $\mathcal{A}$ a *neighborhood basis of* $x \in X$ if for every open set $U \subset X$ with $x \in U$ there exists $A \in \mathcal{A}$ s.t. $x \in A \subset U$.

We will use the following two lemmas later.

Lemma 3.3.2.

i) If $f \in \mathcal{N}^k(g,\varphi,U,\psi,V,K,\varepsilon)$, then there exists $\tilde\varepsilon > 0$ s.t.

$$\mathcal{N}^k(f,\varphi,U,\psi,V,K,\tilde\varepsilon) \subset \mathcal{N}^k(g,\varphi,U,\psi,V,K,\varepsilon)$$

ii) Let $f \in C^k(M,N)$. Then the set

$$\begin{aligned}\{\mathcal{N}^k(f,\varphi,U,\psi,V,K,\varepsilon) \mid (\varphi,U) \textit{ and } (\psi,V) \textit{ charts of } M \textit{ and } N\\ \textit{respectively, } K \subset U \textit{ compact with } f(K) \subset V,\ \varepsilon > 0\}\end{aligned}$$

is a neighborhood subbasis of f.

iii) *Assume M is closed. Let $f \in C^k(M,N)$, (φ_i, U_i) and (ψ_i, V_i) charts of M and N respectively, $K_i \subset U_i$ compact with $f(K_i) \subset V_i$, $i = 1, \dots r$, and $\bigcup_{i=1}^r K_i = M$. Then the set*

$$\{\bigcap_{i=1}^r \mathcal{N}^k(f, \varphi_i, U_i, \psi_i, V_i, K_i, \varepsilon) \mid \varepsilon > 0\}$$

is a neighborhood basis of f. In particular, a sequence $(f_n)_{n\in\mathbb{N}} \subset C^k(M,N)$ converges to f in $C^k(M,N)$ iff for all $\varepsilon > 0$ there exists some $N = N(\varepsilon)$ s.t. for all $n \geq N(\varepsilon)$ it holds that $f_n \in \bigcap_{i=1}^r \mathcal{N}^k(f, \varphi_i, U_i, \psi_i, V_i, K_i, \varepsilon)$.

Proof of Lemma 3.3.2. To prove i), we set

$$\varepsilon_\alpha := \sup_{x\in\varphi(K)} \|\partial_x^\alpha(\varphi \circ f \circ \varphi^{-1})(x) - \partial_x^\alpha(\varphi \circ g \circ \varphi^{-1})(x)\|$$

and choose $\tilde\varepsilon > 0$ s.t. $\tilde\varepsilon + \max_{|\alpha|\leq k} \varepsilon_\alpha < \varepsilon$. This is possible since $\varepsilon - \max_{|\alpha|\leq k} \varepsilon_\alpha > 0$. For $h \in \mathcal{N}^k(f, \varphi, U, \psi, V, K, \tilde\varepsilon)$ we calculate

$$\begin{aligned}
&\|\partial_x^\alpha(\psi \circ h \circ \varphi^{-1})(x) - \partial_x^\alpha(\psi \circ g \circ \varphi^{-1})(x)\| \leq \\
&\|\partial_x^\alpha(\psi \circ h \circ \varphi^{-1})(x) - \partial_x^\alpha(\psi \circ f \circ \varphi^{-1})(x)\| \\
&+ \|\partial_x^\alpha(\psi \circ f \circ \varphi^{-1})(x) - \partial_x^\alpha(\psi \circ g \circ \varphi^{-1})(x)\| \\
&\leq \tilde\varepsilon + \varepsilon_\alpha \\
&< \varepsilon
\end{aligned}$$

for all $x \in \varphi(K)$ and $|\alpha| \leq k$. We have shown $h \in \mathcal{N}^k(g, \varphi, U, \psi, V, K, \varepsilon)$.

ii) follows from i) and the definition of the weak topology. It remains to show iii). First we prove that

$$\{\bigcap_{i=1}^r \mathcal{N}^k(f, \varphi_i, U_i, \psi_i, V_i, K_i, \varepsilon) \mid \varepsilon > 0\}$$

is a neighborhood basis of f. From ii) we see that this follows after we showed the following claim.

Claim: If an arbitrary $\mathcal{N}^k(f, \varphi, U, \psi, V, K, \varepsilon)$ is given, then there exists some $\delta > 0$ s.t.

$$\bigcap_{i=1}^r \mathcal{N}^k(f, \varphi_i, U_i, \psi_i, V_i, K_i, \delta) \subset \mathcal{N}^k(f, \varphi, U, \psi, V, K, \varepsilon).$$

Proof of the claim: Assume that $K_i \cap K \neq \varnothing$. Since $\psi_i(V_i \cap V)^\complement$ is closed, $\psi_i(f(K_i \cap K))$ is compact, and $\psi_i(V_i \cap V)^\complement \cap \psi_i(f(K_i \cap K)) = \varnothing$ we have

$$\delta_i := \operatorname{dist}(\psi_i(V_i \cap V)^\complement, \psi_i(f(K_i \cap K))) > 0.$$

Now choose an arbitrary δ with

$$0 < \delta \leq \frac{1}{2} \min\{\delta_i \mid i \in \{1,\dots,r\} \text{ and } K_i \cap K \neq \varnothing\}$$

and let

$$g \in \bigcap_{i=1}^{r} \mathcal{N}^k(f, \varphi_i, U_i, \psi_i, V_i, K_i, \delta).$$

We show $g(K) \subset V$. Since $g(K_i) \subset V_i$ and because the K_i cover M, it is sufficient to show $g(K_i \cap K) \subset V_i \cap V$ whenever $K_i \cap K \neq \varnothing$. To that end, assume $K_i \cap K \neq \varnothing$. From $g \in \mathcal{N}^k(f, \varphi_i, U_i, \psi_i, V_i, K_i, \delta)$ it follows that

$$\max_{|\alpha| \leq k} \sup_{x \in \varphi_i(K_i \cap K)} \|\partial_x^\alpha(\psi_i \circ g \circ \varphi^{-1})(x) - \partial_x^\alpha(\psi_i \circ f \circ \varphi^{-1})(x)\| < \delta.$$

In particular, that means that for each $p \in K_i \cap K$ we have $\psi_i(g(p)) \in B_\delta(\psi_i(f(p)))$. From the definition of δ it follows that for all $p \in K_i \cap K$ we have $B_\delta(\psi_i(f(p))) \subset \psi_i(V_i \cap V)$. It follows that $\psi_i(g(K_i \cap K)) \subset \psi_i(V_i \cap V)$ and thus $g(K_i \cap K)) \subset V_i \cap V$. We have shown $g(K) \subset V$. Using Lemma 3.2.1i)+iii)[2] we calculate

$$\begin{aligned}
&\max_{|\alpha| \leq k} \sup_{x \in \varphi(K)} \|\partial_x^\alpha(\psi \circ g \circ \varphi^{-1})(x) - \partial_x^\alpha(\psi \circ f \circ \varphi^{-1})(x)\| \\
&= \max_{i=1,\dots,l} \max_{|\alpha| \leq k} \sup_{x \in \varphi(K_i \cap K)} \|\partial_x^\alpha(\psi \circ g \circ \varphi^{-1})(x) - \partial_x^\alpha(\psi \circ f \circ \varphi^{-1})(x)\| \\
&= \max_{i=1,\dots,l} \max_{|\alpha| \leq k} \sup_{x \in \varphi(K_i \cap K)} \|\partial_x^\alpha(\psi \circ \psi_i^{-1} \circ \psi_i \circ g \circ \varphi_i^{-1} \circ \varphi_i \circ \varphi^{-1})(x) \\
&\qquad\qquad - \partial_x^\alpha(\psi \circ \psi_i^{-1} \circ \psi_i \circ f \circ \varphi_i^{-1} \circ \varphi_i \circ \varphi^{-1})(x)\| \\
&\leq \max_{i=1,\dots,l} \left(C_i \max_{|\alpha| \leq k} \sup_{x \in \varphi_i(K_i \cap K)} \|\partial_x^\alpha(\psi_i \circ g \circ \varphi_i^{-1})(x) - \partial_x^\alpha(\psi_i \circ f \circ \varphi_i^{-1})(x)\| \right) \\
&\leq \left(\max_{i=1,\dots,l} C_i \right) \delta.
\end{aligned}$$

Now choose δ so small that $(\max_{i=1,\dots,l} C_i)\, \delta < \varepsilon$. This finishes the proof of the claim. The second statement of iii) follows directly from the fact that

$$\{\bigcap_{i=1}^{r} \mathcal{N}^k(f, \varphi_i, U_i, \psi_i, V_i, K_i, \varepsilon) \mid \varepsilon > 0\}$$

is a neighborhood basis of f. $\square$

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[2] For $f_1 = \psi_i \circ f \circ \varphi_i^{-1}$ defined on $\varphi_i(U_i \cap U \cap f^{-1}(V_i \cap V))$, $f_2 = \psi_i \circ g \circ \varphi_i^{-1}$ defined on $\varphi_i(U_i \cap U \cap g^{-1}(V_i \cap V))$, $\Psi = \psi \circ \psi_i^{-1}$ defined on $\psi_i(V_i \cap V)$, $\Phi = \varphi \circ \varphi_i^{-1}$ defined on $\varphi_i(U_i \cap U)$, and $\tilde{K} = \overline{B_\delta(\psi_i(f(K_i \cap K)))} \subset \psi_i(V_i \cap V)$.

Lemma 3.3.3. *Let $\mathcal{N}^k(f,\varphi,U,\psi,V,K,\varepsilon)$ be given. Assume additionally that $\psi(V)$ is convex, compact, and that N is a connected Riemannian manifold with induced distance function d. Then there exists $C > 0$ s.t. for all $g \in \mathcal{N}^k(f,\varphi,U,\psi,V,K,\varepsilon)$ and all $x \in K$ it holds that $d(g(x), f(x)) \leq C\varepsilon$.*

Definition 3.3.4 (C^k-norm on sections of a vector bundle)**.** Let $\pi\colon E \to M$ be a (smooth) vector bundle where E and M are manifolds without boundary and M is compact. Pick charts (U_i, φ_i) of M, $i = 1,\dots,l$, $\bigcup_{i=1}^l U_i = M$ s.t. $\overline{U_i} \subset M$ is compact, $\overline{U_i} \subset \tilde{U}_i$, $(\tilde{U}_i, \varphi_i)$ is still a chart of M and there are local trivializations $(\hat{U}_i, \Phi_i)$ of E with $\overline{U_i} \subset \hat{U}_i$ for each $i = 1,\dots,l$. For $k \in \mathbb{N}$ let

$$\Gamma_{C^k}(E) := \{s\colon M \to E \mid s \in C^k(M,E) \text{ and } \pi \circ s = id_M\}$$

be the space of C^k-sections of E. Define the C^k*-norm on* $\Gamma_{C^k}(E)$ by

$$\|s\|_{C^k} := \|s\|_{\Gamma_{C^k}(E)} := \max_{i=1,\dots,l} \max_{|\alpha|\leq k} \sup_{x \in \overline{\varphi_i(U_i)}} \|\partial_x^\alpha(pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1})\|$$

for $s \in \Gamma_{C^k}(E)$.

Lemma 3.3.5.

i) *The C^k-norm $\|.\|_{C^k}$ is a norm on $\Gamma_{C^k}(E)$.*

ii) $(\Gamma_{C^k}(E), \|.\|_{C^k})$ *is a Banach space.*

iii) *Up to equivalence of norms, the C^k-norm does not depend on the choices made. To be more precise, pick U_i, $\tilde{U}_i$, $\hat{U}_i$, φ_i and Φ_i, $i = 1,\dots,l$ as in Definition 3.3.4. For all $s \in \Gamma_{C^k}(E)$ define*

$$\|s\|_{(C^k,(U_i,\tilde{U}_i,\hat{U}_i,\varphi_i,\Phi_i)_{i=1,\dots,l})} := \max_{i=1,\dots,l} \max_{|\alpha|\leq k} \sup_{x \in \overline{\varphi_i(U_i)}} \|\partial_x^\alpha(pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1})\|.$$

Now choose charts $(V_j, \tilde{\varphi}_j)$ of M, $j = 1,\dots,r$, $\bigcup_{j=1}^r V_j = M$ s.t. $\overline{V_j} \subset M$ is compact, $\overline{V_j} \subset \tilde{V}_j$, $(\tilde{V}_j, \tilde{\varphi}_j)$ is still a chart of M and there are local trivializations $(\hat{V}_j, \tilde{\Phi}_j)$ with $\overline{V_j} \subset \hat{V}_j$ of E for each $j = 1,\dots,r$. For all $s \in \Gamma_{C^k}(E)$ define

$$\|s\|_{(C^k,(V_j,\tilde{V}_j,\hat{V}_j,\tilde{\varphi}_j,\tilde{\Phi}_j)_{j=1,\dots,r})} := \max_{j=1,\dots,r} \max_{|\alpha|\leq k} \sup_{x \in \overline{\tilde{\varphi}_j(V_j)}} \|\partial_x^\alpha(pr_2 \circ \tilde{\Phi}_j \circ s \circ \tilde{\varphi}_j^{-1})\|.$$

Then $\|.\|_{(C^k,(U_i,\tilde{U}_i,\hat{U}_i,\varphi_i,\Phi_i)_{i=1,\dots,l})}$ and $\|.\|_{(C^k,(V_j,\tilde{V}_j,\hat{V}_j,\tilde{\varphi}_j,\tilde{\Phi}_j)_{j=1,\dots,r})}$ are equivalent norms on $\Gamma_{C^k}(E)$.

Lemma 3.3.6. *Let $\langle .,. \rangle_E$ be a bundle metric on E with induced norm $\|.\|_E := \sqrt{\langle .,. \rangle_E}$ on the fibers. There exists $C > 0$ s.t. for all $s \in \Gamma_{C^k}(E)$ and all $x \in M$ we have*

$$\|s(x)\|_E \leq C\|s\|_{C^0}.$$

Proof. Define the compact set $K_i := \{v \in E \mid v \in E_x,\ x \in \overline{U_i},\ \|v\|_E = 1\} \subset E$. The function

$$\|(pr_2 \circ \Phi_i)(.)\| \colon K_i \to \mathbb{R}$$

is continuous and positive. Therefore there exists $C_i > 0$ s.t.

$$C_i \leq \|(pr_2 \circ \Phi_i)(v)\|$$

for all $v \in K_i$. Since $C_i\|v\|_E = C_i$ for all $v \in K_i$ it follows that

$$\|v\|_E \leq \frac{1}{C_i}\|(pr_2 \circ \Phi_i)(v)\|$$

for all $v \in K_i$ and since $(pr_2 \circ \Phi_i)$ is linear on each fiber E_x, $x \in \overline{U_i}$ we have

$$\|v\|_E \leq \frac{1}{C_i}\|(pr_2 \circ \Phi_i)(v)\|$$

for all $v \in E_x$, $x \in \overline{U_i}$. Define $C := \max_{i=1,\ldots,l} \frac{1}{C_i} > 0$, then we have

$$\|s(x)\|_E \leq C\|s\|_{C^0}$$

for all $s \in \Gamma_{C^k}(E)$ and all $x \in M$. □

For the definition of the charts of $C^k(M,N)$ the exponential map of N is the main input. For the convenience of the reader and to fix notation we recall some basic facts about the exponential map of a Riemannian manifold.

Lemma 3.3.7. *Let (N,h) be a Riemannian manifold. Define $\mathcal{E} \subset TN$ by*

$$\mathcal{E} := \{(p,v) \in TN \mid p \in N,\ v \in T_pN,\ \exp_p v \textit{ exists}\}.$$

i) $\mathcal{E} \subset TN$ is open and

$$\exp\colon \mathcal{E} \to N$$

defined by $\exp(p,v) := \exp_p v$ is smooth.

ii) Define the smooth map

$$(pr_1, \exp)\colon \mathcal{E} \to N \times N$$

by $(pr_1,\exp)(p,v) := (p, \exp_p v)$. For each $p \in N$ there exists a neighborhood W of $(p,0)$ in TN s.t. the map

$$(pr_1,\exp)\colon W \to (pr_1,\exp)(W)$$

is a diffeomorphism (in particular $(pr_1,\exp)(W)$ is open in $N \times N$).

iii) *For all $p \in N$ and $0 < \delta < \mathrm{inj}_p(N)$ where $\mathrm{inj}_p(N) > 0$ is the injectivity radius of N at p it holds that*

$$\exp_p \colon B_\delta(0_p) \to B_\delta(p)$$

is a diffeomorphism where $B_\delta(0_p) = \{v \in T_pM \mid \|v\|_h := \delta\}$, $B_\delta(p) = \{q \in N \mid d(p,q) < \delta\}$, and d is the distance function induced by h.

Now we define the maps that will later be the charts of $C^k(M,N)$ and show that they are homeomorphisms.

Lemma 3.3.8. *Let $k \in \mathbb{N}$. Let M and N be manifolds without boundary. Let M be compact and let N be connected. Choose a Riemannian metric h on N. Define*

$$U_{f,\varepsilon} := \bigcap_{i=1}^{l} \mathcal{N}^k(f, \varphi_i, \tilde{U}_i, \psi_i, V_i, \overline{U_i}, \varepsilon)$$

for (U_i, φ_i) charts of M, $i = 1, \dots, l$, $\bigcup_{i=1}^{l} U_i = M$, s.t. $\overline{U_i} \subset M$ is compact, $\overline{U_i} \subset \tilde{U}_i$, $(\tilde{U}_i, \varphi_i)$ is still a chart of M and charts (V_i, ψ_i) of N with $f(\overline{U_i}) \subset V_i$ for each $i = 1, \dots, l$, $\varepsilon > 0$. Define the map

$$\varphi_f \colon U_{f,\varepsilon} \to \varphi_f(U_{f,\varepsilon}) \subset \Gamma_{C^k}(f^*TN)$$

by

$$(\varphi_f(g))(p) := (\exp_{f(p)})^{-1} g(p)$$

for all $p \in M$, where $\exp$ is the exponential map of (N,h). Then, for $\varepsilon > 0$ small enough, it holds that

i) *There exists $\delta = \delta(\varepsilon) > 0$ s.t. for all $g \in U_{f,\varepsilon}$ and all $p \in M$ we have*

$$d(g(p), f(p)) < \delta < \inf_{p \in M} \mathrm{inj}_{f(p)}(N) \tag{3.3.1}$$

In particular, φ_f is well-defined.

ii) *$\varphi_f \colon U_{f,\varepsilon} \to \varphi_f(U_{f,\varepsilon})$ is continuous (where on $U_{f,\varepsilon}$ we have the subspace topology induced from the compact-open C^k topology and on $\varphi_f(U_{f,\varepsilon})$ we have the subspace topology induced from the C^k-norm on $\Gamma_{C^k}(f^*TN)$).*

iii) *$\varphi_f(U_{f,\varepsilon}) \subset \Gamma_{C^k}(f^*TN)$ is open. Moreover,*

$$\varphi_f(U_{f,\varepsilon}) \subset U := \{s \in \Gamma_{C^k}(f^*TN) \mid \|s(p)\|_h < \delta \text{ for all } p \in M\}$$

and U is open in $\Gamma_{C^k}(f^*TN)$.

iv) *$\varphi_f^{-1} \colon \varphi_f(U_{f,\varepsilon}) \to U_{f,\varepsilon}$ is continuous.*

Proof. We start by mentioning that since $C^k(M,N)$ and $\Gamma_{C^k}(f^*TN)$ are first-countable, it is sufficient to show that φ_f and φ_f^{-1} are sequentially continuous. To make the proofs of i) and ii) easier, we first choose the V_i s.t.

$$(A)\begin{cases} \psi_i(V_i) \text{ is convex and compact, } \overline{V_i} \subset \tilde{V}_i \text{ where } (\tilde{V}_i, \psi_i) \text{ is still a chart of } N, \\ \tilde{V}_i \times \tilde{V}_i \subset (pr_1, \exp)(W_i), \text{where } W_i \subset TN \text{ and} \\ (pr_1, \exp)(W_i) \subset N \times N \text{ are open s.t.} \\ (pr_1, \exp)\colon W_i \to (pr_1, \exp)(W_i) \text{ is a diffeomorphism,} \\ (\tilde{V}_i, \hat{\Phi}_i) \text{ are local trivializations of } TN \text{ with induced local} \\ \text{trivialization } (f^{-1}(\tilde{V}_i), \Phi_i) \text{ of } f^*TN \text{ for each } i = 1, \dots l. \end{cases}$$

(See Lemma 3.3.7 ii).)

In the following we prove i) and ii) with the additional assumption (A) and then show afterwards that we don't need it, provided that $\varepsilon > 0$ is small enough.

Proof of i): Due to Lemma 3.3.3, for every $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ s.t. for all $g \in U_{f,\varepsilon}$ and all $p \in M$ we have

$$d(g(p), f(p)) < C(\varepsilon)\varepsilon =: \delta(\varepsilon)$$

Choosing $\varepsilon > 0$ so small that $\delta < \inf_{p\in M} \operatorname{inj}_{f(p)}(N)$ we have shown (3.3.1). In particular, $(\exp_{f(p)})^{-1} g(p)$ exists for each $p \in M$. Moreover, $\varphi_f(g) \in \Gamma_{C^k}(f^*TN)$, since on U_i it holds that $\varphi_f(g) = ((pr_1, \exp)|_{W_i})^{-1} \circ (f, g)$. We have shown that φ_f is a well-defined map.

Proof of ii): Choose ε as in i). Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $U_{f,\varepsilon}$, $g \in U_{f,\varepsilon}$ with $g_n \xrightarrow{n\to\infty} g$ in $U_{f,\varepsilon}$. In particular, for each $r > 0$ there exists $N = N(r) \in \mathbb{N}$ s.t.

$$g_n \in \bigcap_{i=1}^{l} \mathcal{N}^k(g, \varphi_i, \tilde{U}_i, \psi_i, V_i, \overline{U_i}, r)$$

for all $n \geq N$. (We note that the $\varphi_i, \tilde{U}_i, \psi_i, V_i, U_i$ are the same as in the statement of the lemma where we additionally assume (A) as mentioned above.) That means, that for all $i = 1, \dots, l$ we have

$$\|\psi_i \circ g_n \circ \varphi_i^{-1} - \psi_i \circ g \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^n)} \xrightarrow{n\to\infty} 0$$

where $n = \dim(N)$. Using Lemma 3.2.1 iii) [3] we calculate for each $i = 1, \dots, l$

$$\begin{aligned}
&\|pr_2 \circ \Phi_i \circ \varphi_f(g_n) \circ \varphi_i^{-1} - pr_2 \circ \Phi_i \circ \varphi_f(g) \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n)} \\
&= \|pr_2 \circ \hat{\Phi}_i \circ (pr_1, \exp)|_{W_i}^{-1} \circ (f, g_n) \circ \varphi_i^{-1} \\
&\quad - pr_2 \circ \hat{\Phi}_i \circ (pr_1, \exp)|_{W_i}^{-1} \circ (f, g) \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n)} \\
&= \| \left(pr_2 \circ \hat{\Phi}_i \circ (pr_1, \exp)|_{W_i}^{-1} \circ (\psi_i^{-1} \times \psi_i^{-1})\right) \circ \left((\psi_i \times \psi_i) \circ (f, g_n) \circ \varphi_i^{-1}\right) \\
&\quad - \left(pr_2 \circ \hat{\Phi}_i \circ (pr_1, \exp)|_{W_i}^{-1} \circ (\psi_i^{-1} \times \psi_i^{-1})\right) \circ \left((\psi_i \times \psi_i) \circ (f, g) \circ \varphi_i^{-1}\right) \|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n)} \\
&\le C_i \|(\psi_i \times \psi_i) \circ (f, g_n) \circ \varphi_i^{-1} - (\psi_i \times \psi_i) \circ (f, g) \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n \times \mathbb{R}^n)} \\
&= C_i \|\psi_i \circ g_n \circ \varphi_i^{-1} - \psi_i \circ g \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n)} \xrightarrow{n \to \infty} 0,
\end{aligned}$$

i.e.,

$$\|\varphi_f(g_n) - \varphi_f(g)\|_{C^k} \xrightarrow{n \to \infty} 0.$$

Hence, $\varphi_f \colon U_{f,\varepsilon} \to \varphi_f(U_{f,\varepsilon})$ is continuous.

We have shown i) and ii) under the additional assumption (A). Now we show that we don't need the assumption (A), provided that $\varepsilon > 0$ is small enough. To that end, choose (U_i', φ_i') charts of M, $i = 1, \dots, m$, $\bigcup_{i=1}^m U_i' = M$, s.t. $\overline{U_i'} \subset M$ is compact, $\overline{U_i'} \subset \tilde{U}_i'$, $(\tilde{U}_i', \varphi_i')$ is still a chart of M and charts (V_i', ψ_i') of N with $f(\overline{U_i'}) \subset V_i'$ for each $i = 1, \dots, m$. Using Lemma 3.3.2 iii) we choose $\varepsilon' > 0$ s.t.

$$\bigcap_{i=1}^m \mathcal{N}^k(f, \varphi_i', \tilde{U}_i', \psi_i', V_i', \overline{U_i}', \varepsilon') \subset \bigcap_{i=1}^l \mathcal{N}^k(f, \varphi_i, \tilde{U}_i, \psi_i, V_i, \overline{U_i}, \varepsilon)$$

where the $\varphi_i, \tilde{U}_i, \psi_i, V_i, U_i$ are the same as in the statement of the lemma and satisfy (A). Since φ_f is well-defined and continuous on the set $\bigcap_{i=1}^l \mathcal{N}^k(f, \varphi_i, \tilde{U}_i, \psi_i, V_i, \overline{U_i}, \varepsilon)$ (that is what we have shown above) it is obviously well-defined and continuous on the subset $\bigcap_{i=1}^m \mathcal{N}^k(f, \varphi_i', \tilde{U}_i', \psi_i', V_i', \overline{U_i}', \varepsilon')$. Moreover, equation (3.3.1) is satisfied if we choose $\varepsilon' > 0$ small enough.

Proof of iii) and iv): Choose $\varepsilon > 0$ s.t. the statements i) and ii) of the lemma hold. From Lemma 3.3.7 iii) we see that $\varphi_f(U_{f,\varepsilon}) \subset U := \{s \in \Gamma_{C^k}(f^*TN) \mid \|s(p)\|_h < \delta \text{ for all } p \in M\}$. First we prove that U is open in $\Gamma_{C^k}(f^*TN)$. To that end, let $s_0 \in U$. Since the function $M \to \mathbb{R}$, $p \mapsto \|s_0(p)\|_h$, is continuous and M is compact, we have $\delta_0 := \max_{p \in M} \|s_0(p)\|_h < \delta$. Using Lemma 3.3.6 there exists $C > 0$ s.t.

$$\|s(p) - s_0(p)\|_h \le C \|s - s_0\|_{C^k}$$

[3] For $f_1 = (\psi_i \times \psi_i) \circ (f, g) \circ \varphi_i^{-1}$ defined on $\varphi_i(\tilde{U}_i \cap f^{-1}(\tilde{V}_i) \cap g^{-1}(\tilde{V}_i))$, $f_2 = (\psi_i \times \psi_i) \circ (f, g_n) \circ \varphi_i^{-1}$ defined on $\varphi_i(\tilde{U}_i \cap f^{-1}(\tilde{V}_i) \cap g_n^{-1}(\tilde{V}_i))$, $\Psi = pr_2 \circ \hat{\Phi}_i \circ (pr_1, \exp)|_{W_i}^{-1} \circ (\psi_i^{-1} \times \psi_i^{-1})$, defined on $\psi_i(\tilde{V}_i) \times \psi_i(\tilde{V}_i)$, $K = \overline{\varphi_i(U_i)}$, and $\tilde{K} = \psi_i(\overline{V_i}) \times \psi_i(\overline{V_i})$.

for all $s \in \Gamma_{C^k}(f^*TN)$ and all $p \in M$. Choose $r > 0$ s.t. $Cr < \delta - \delta_0$. If $\|s - s_0\|_{C^k} < r$, then

$$\|s(p)\|_h \leq \|s(p) - s_0(p)\|_h + \|s_0(p)\|_h \leq C\|s - s_0\|_{C^k} + \delta_0 < Cr + \delta_0 < \delta$$

for all $p \in M$, therefore U is open in $\Gamma_{C^k}(f^*TN)$.

Next we show that the well-defined map

$$H \colon U \to C^k(M, N),$$

$(H(s))(p) := \exp_{f(p)} s(p)$ is continuous. Then we have in particular that $\varphi_f^{-1} = H|_{\varphi_f(U_{f,\varepsilon})}$ is continuous and that $\varphi_f(U_{f,\varepsilon}) = H^{-1}(U_f)$ is open in U (and therefore also in $\Gamma_{C^k}(f^*TN)$).

To show continuity of H, choose charts (U_i, φ_i) of M, $i = 1, \ldots, l$, $\bigcup_{i=1}^l U_i = M$ s.t. $\overline{U_i} \subset M$ is compact, $\overline{U_i} \subset \tilde{U}_i$, $(\tilde{U}_i, \varphi_i)$ is still a chart of M and there are local trivializations $(\tilde{U}_i, \Phi_i)$ of f^*TN and charts (V_i, ψ_i) of N with $f(\overline{U_i}) \subset V_i$ and $(B_\delta(V_i), \psi_i)$ is still a chart of N for each $i = 1, \ldots, l$, where $B_\delta(V_i) = \{p \in N \mid \exists q \in V_i : \ d(p, q) < \delta\}$. (Note that the $\varphi_i, \tilde{U}_i, \psi_i, V_i, U_i$ here don't need to be the same as in the statement of the lemma.)

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in U, $s \in U$, with

$$\|s_n - s\|_{C^k} \xrightarrow{n\to\infty} 0,$$

i.e.,

$$\|pr_2 \circ \Phi_i \circ s_n \circ \varphi_i^{-1} - pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^n)} \xrightarrow{n\to\infty} 0$$

for each $i = 1, \ldots, l$. For showing $H(s_n) \xrightarrow{n\to\infty} H(s)$ in $C^k(M, N)$ it is sufficient to show that for all $r > 0$ there exists $N = N(r) \in \mathbb{N}$ s.t.

$$H(s_n) \in \bigcap_{i=1}^{l} \mathcal{N}^k(H(s), \varphi_i, \tilde{U}_i, \psi_i, B_\delta(V_i), \overline{U_i}, r)$$

for all $n \geq N$, see Lemma 3.3.2 iii). First of all, by definition of H and Lemma 3.3.7 iii) it holds that

$$d(H(s_n)(p), f(p)) < \delta \text{ for all } n \in \mathbb{N} \text{ and } d(H(s)(p), f(p)) < \delta$$

for each $p \in M$. Since $f(\overline{U_i}) \subset V_i$ it follows that $H(s_n)(\overline{U_i}) \subset B_\delta(V_i)$ and $H(s)(\overline{U_i}) \subset B_\delta(V_i)$ for each $n \in \mathbb{N}$ and $i = 1, \ldots, l$. Let $r > 0$. Using Lemma 3.2.1

iii) [4] we calculate for each $i = 1, \dots, l$ and n large enough

$$\begin{aligned}
&\|\psi_i \circ H(s_n) \circ \varphi_i^{-1} - \psi_i \circ H(s) \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n)} \\
&= \|\psi_i \circ f^*\mathrm{exp} \circ s_n \circ \varphi_i^{-1} - \psi_i \circ f^*\mathrm{exp} \circ s \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n)} \\
&= \| \left(\psi_i \circ f^*\mathrm{exp} \circ \Phi_i^{-1} \circ (\varphi_i^{-1} \times id)\right) \circ \left((\varphi_i \times id) \circ \Phi_i \circ s_n \circ \varphi_i^{-1}\right) \\
&- \left(\psi_i \circ f^*\mathrm{exp} \circ \Phi_i^{-1} \circ (\varphi_i^{-1} \times id)\right) \circ \left((\varphi_i \times id) \circ \Phi_i \circ s \circ \varphi_i^{-1}\right) \|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n)} \\
&\le C_i \|(\varphi_i \times id) \circ \Phi_i \circ s_n \circ \varphi_i^{-1} - (\varphi_i \times id) \circ \Phi_i \circ s \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n \times \mathbb{R}^n)} \\
&= C_i \|pr_2 \circ \Phi_i \circ s_n \circ \varphi_i^{-1} - pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)},\mathbb{R}^n)} \\
&< r.
\end{aligned}$$

We have shown

$$H(s_n) \in \bigcap_{i=1}^{l} \mathcal{N}^k(H(s), \varphi_i, \tilde{U}_i, \psi_i, B_\delta(V_i), \overline{U_i}, r)$$

for n large enough, so $H\colon U \to C^k(M,N)$ is continuous. $\square$

[4]For $f_1 = (\varphi_i \times id) \circ \Phi_i \circ s \circ \varphi_i^{-1}$ defined on $\varphi_i(\tilde{U}_i \cap f^{-1}(B_\delta(V_i)),$ $f_2 = (\varphi_i \times id) \circ \Phi_i \circ s_n \circ \varphi_i^{-1}$ also defined on $\varphi_i(\tilde{U}_i \cap f^{-1}(B_\delta(V_i))),$ $\Psi = \psi_i \circ f^*\mathrm{exp} \circ \Phi_i^{-1} \circ (\varphi_i^{-1} \times id)$ defined on $(\varphi_i \times id) \circ \Phi_i \left(\{v \in f^*TN \mid \|v\| < \delta\} \cap f^*TN|_{\tilde{U}_i \cap f^{-1}(B_\delta(V_i))}\right),$ $K = \overline{\varphi_i(U_i)},$ and $\tilde{K} = (\varphi_i \times id) \circ \Phi_i \left(\{v \in f^*TN \mid \|v\| \le \delta\} \cap f^*TN|_{\overline{U}_i \cap f^{-1}(\overline{V_i})}\right).$

3.4 The smooth structure on $C^k(M,N)$

In the following we "globalize" the local Ω-lemma (Lemma 3.2.4) to sections of vector bundles. This will be the main input for showing that $C^k(M,N)$ carries a *smooth* structure.

We start with a proposition that provides a criterion for a map with target $\Gamma_{C^k}(E)$ to be a C^r-map.

Proposition 3.4.1. *In the situation of Definition 3.3.4, we define*

$$R_i\colon \Gamma_{C^k}(E) \to C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^n)$$

by $R_i(s) := pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1}$ for $i = 1, \ldots, l$, where we assume that $\operatorname{rank}(E) = n$. *Let $r \in \mathbb{N}$, X a Banach space, $U \subset X$ open, and*

$$F\colon U \to \Gamma_{C^k}(E)$$

a map. Then $F \in C^r(U, \Gamma_{C^k}(E))$ if and only if $R_i \circ F \in C^r(U, C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^n))$ for $i = 1, \ldots, l$.

Proof. "$\Rightarrow$:" The R_i are linear and continuous, so they are smooth.
"$\Leftarrow$:" To make things easier, we first get rid of the Φ_i and φ_i in $R_i \circ F$ as follows: On the vector space

$$\Gamma_{C^k,\overline{U}_i}(E) := \{s\colon U_i \to E \mid s \in \Gamma_{C^k}(E|_{U_i}) \text{ and } pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1} \in C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^n)\}$$

we define the norm

$$\|s\|_i := \|pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1}\|_{C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^n)}.$$

We get an isomorphism of Banach spaces

$$\begin{aligned} J_i\colon \Gamma_{C^k,\overline{U}_i}(E) &\to C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^n) \\ s &\mapsto pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1}, \end{aligned}$$

with $J_i^{-1}(f) = \Phi_i^{-1}(id_{U_i}, f \circ \varphi_i)$. By assumption, we have that

$$\begin{aligned} F_i := J_i^{-1} \circ R_i \circ F\colon U &\to \Gamma_{C^k,\overline{U}_i}(E), \\ x &\mapsto F(x)|_{U_i} \end{aligned}$$

is an element of $C^r(U, \Gamma_{C^k,\overline{U}_i}(E))$ for $i = 1, \ldots, l$. Define

$$\tilde{D}^j F\colon U \to L^j_s(X, \Gamma_{C^k}(E))$$

by

$$\left(\tilde{D}^j F\right)_u (x_1, \ldots, x_j)|_{U_i} := \left(D^j F_i\right)_u (x_1, \ldots, x_j)$$

for $u \in U$, $x_1, \dots, x_j \in X$, $j = 1, \dots, r$.

In the following, we show that $\tilde{D}^j F$ is well-defined, continuous and F is r times continuously differentiable with $D^j F = \tilde{D}^j F$ for $j = 1, \dots, r$.

We start with the case $j = 1$:

$\tilde{D}F$ **is well-defined:** Assume $U_{i_1} \cap U_{i_2} \neq \varnothing$. Define the map

$$F_{i_1 i_2} \colon U \to \Gamma_{C^k, \overline{U_{i_1} \cap U_{i_2}}}(E), \qquad u \mapsto F(u)|_{U_{i_1} \cap U_{i_2}}.$$

Notice that we can define the Banach space $\Gamma_{C^k, \overline{U_{i_1} \cap U_{i_2}}}(E)$ with $\Phi_{i_1}, \varphi_{i_1}$ or with $\Phi_{i_2}, \varphi_{i_2}$ and we get the same sets and equivalent norms. Since the F_i are C^r, we have $F_{i_1 i_2} \in C^r(U, \Gamma_{C^k, \overline{U_{i_1} \cap U_{i_2}}}(E))$. We show

$$(DF_{i_1})_u \, x_1|_{U_{i_1} \cap U_{i_2}} = (DF_{i_1, i_2})_u \, x_1 = (DF_{i_2})_u \, x_1|_{U_{i_1} \cap U_{i_2}} \tag{3.4.1}$$

This implies that DF is well-defined. Equation (3.4.1) follows from

$$\begin{aligned}
&\|F_{i_1 i_2}(x) - F_{i_1 i_2}(u) - (DF_{i_1})_u(x-u)|_{U_{i_1} \cap U_{i_2}}\|_{\Gamma_{C^k, \overline{U_{i_1} \cap U_{i_2}}}(E)} \\
&= \|F(x)|_{U_{i_1} \cap U_{i_2}} - F(u)|_{U_{i_1} \cap U_{i_2}} - (DF_{i_1})_u(x-u)|_{U_{i_1} \cap U_{i_2}}\|_{\Gamma_{C^k, \overline{U_{i_1} \cap U_{i_2}}}(E)} \\
&\leq \|F(x)|_{U_{i_1}} - F(u)|_{U_{i_1}} - (DF_{i_1})_u(x-u)|_{U_{i_1}}\|_{\Gamma_{C^k, \overline{U_{i_1}}}(E)} \\
&= \|F_{i_1}(x) - F_{i_1}(u) - (DF_{i_1})_u(x-u)|_{U_{i_1}}\|_{\Gamma_{C^k, \overline{U_{i_1}}}(E)}
\end{aligned}$$

$\tilde{D}F$ **is continuous:** This follows from

$$\begin{aligned}
\|(\tilde{D}F)_{u_1} - (\tilde{D}F)_{u_2}\|_{L(X, \Gamma_{C^k}(E))} &= \sup_{\|x\|=1} \|(\tilde{D}F)_{u_1}x - (\tilde{D}F)_{u_2}x\|_{\Gamma_{C^k}(E)} \\
&= \sup_{\|x\|=1} \max_{i=1,\dots,l} \|(\tilde{D}F)_{u_1}x - (\tilde{D}F)_{u_2}x\|_i \\
&= \sup_{\|x\|=1} \max_{i=1,\dots,l} \|(\tilde{D}F)_{u_1}x|_{U_i} - (\tilde{D}F)_{u_2}x|_{U_i}\|_i \\
&= \sup_{\|x\|=1} \max_{i=1,\dots,l} \|(DF_i)_{u_1}x - (DF_i)_{u_2}x\|_i
\end{aligned}$$

together with the continuity of DF_i.

It holds that $\tilde{D}F = DF$: This follows from

$$\begin{aligned}
&\|F(x) - F(u) - (\tilde{D}F)_u(x-u)\|_{\Gamma_{C^r}(E)} \\
&= \max_{i=1,\dots,l} \|F(x)|_{U_i} - F(u)|_{U_i} - (\tilde{D}F)_u(x-u)|_{U_i}\|_i \\
&= \max_{i=1,\dots,l} \|F_i(x) - F_i(u) - (DF_i)_u(x-u)\|_i
\end{aligned}$$

together with the differentiability of F_i.

For $j=1$ we proved the following

$$A(j) = \begin{cases} (D^jF_{i_1i_2})_u(x_1,\dots,x_j) = (D^jF_{i_1})_u(x_1,\dots,x_j)|_{U_{i_1}\cap U_{i_2}} \\ = (D^jF_{i_2})_u(x_1,\dots,x_j)|_{U_{i_1}\cap U_{i_2}} \text{ for all } U_{i_1}\cap U_{i_2}\neq\varnothing, \\ F\in C^j(U,\Gamma_{C^k}(E)), \\ (D^jF)=\tilde{D}^jF. \end{cases}$$

Now we show: If $A(m)$ holds, then $A(m+1)$ also holds, provided that $m+1\leq r$. To that end, we calculate

$$\begin{aligned}
&\|(D^mF_{i_1i_2})_x-(D^mF_{i_1i_2})_u-(D^{m+1}F_{i_1})_u(x-u)|_{U_{i_1}\cap U_{i_2}}\|_{L^m_s(X,\Gamma_{C^k,\overline{U_{i_1}\cap U_{i_2}}}(E))}\\
&=\sup_{\|x_1\|=\ldots=\|x_m\|=1}\|(D^mF_{i_1i_2})_x(x_1,\dots,x_m)-(D^mF_{i_1i_2})_u(x_1,\dots,x_m)\\
&-(D^{m+1}F_{i_1})_u(x-u,x_1,\dots,x_m)|_{U_{i_1}\cap U_{i_2}}\|_{L^m_s(X,\Gamma_{C^k,\overline{U_{i_1}\cap U_{i_2}}}(E))}\\
&\overset{A(m)}{=}\sup_{\|x_1\|=\ldots=\|x_m\|=1}\|(D^mF_{i_1})_x(x_1,\dots,x_m)-(D^mF_{i_1})_u(x_1,\dots,x_m)\\
&-(D^{m+1}F_{i_1})_u(x-u,x_1,\dots,x_m)|_{U_{i_1}\cap U_{i_2}}\|_{L^m_s(X,\Gamma_{C^k,\overline{U_{i_1}\cap U_{i_2}}}(E))}\\
&\leq\sup_{\|x_1\|=\ldots=\|x_m\|=1}\|(D^mF_{i_1})_x(x_1,\dots,x_m)-(D^mF_{i_1})_u(x_1,\dots,x_m)\\
&-(D^{m+1}F_{i_1})_u(x-u,x_1,\dots,x_m)\|_{L^m_s(X,\Gamma_{C^k,\overline{U_{i_1}}}(E))}.
\end{aligned}$$

From that it follows that the first statement of $A(m+1)$ holds (and in particular $\tilde{D}^{m+1}F$ is well-defined). From

$$\begin{aligned}
&\|(D^mF)_x-(D^mF)_u-(\tilde{D}^{m+1}F)_u(x-u)\|_{L^m_s(X,\Gamma_{C^k}(E))}\\
&=\sup_{\|x_1\|=\ldots=\|x_m\|=1}\|(D^mF)_x(x_1,\dots,x_m)-(D^mF)_u(x_1,\dots,x_m)\\
&-(\tilde{D}^{m+1}F)_u(x-u,x_1,\dots,x_m)\|_{\Gamma_{C^k}(E)}\\
&=\sup_{\|x_1\|=\ldots=\|x_m\|=1}\max_{i=1,\dots,l}\|(D^mF)_x(x_1,\dots,x_m)-(D^mF)_u(x_1,\dots,x_m)\\
&-(\tilde{D}^{m+1}F)_u(x-u,x_1,\dots,x_m)\|_i\\
&\overset{A(m)}{=}\sup_{\|x_1\|=\ldots=\|x_m\|=1}\max_{i=1,\dots,l}\|(D^mF_i)_x(x_1,\dots,x_m)-(D^mF_i)_u(x_1,\dots,x_m)\\
&-(\tilde{D}^{m+1}F)_u(x-u,x_1,\dots,x_m)\|_i\\
&=\sup_{\|x_1\|=\ldots=\|x_m\|=1}\max_{i=1,\dots,l}\|(D^mF_i)_x(x_1,\dots,x_m)-(D^mF_i)_u(x_1,\dots,x_m)\\
&-(D^{m+1}F_i)_u(x-u,x_1,\dots,x_m)\|_i
\end{aligned}$$

and since the F_i are in C^r, it follows that D^mF is differentiable with $D^{m+1}F=\tilde{D}^{m+1}F$. It remains to show the second statement of $A(m+1)$. For this, it is sufficient to show the continuity of $\tilde{D}^{m+1}F$, which follows from

$$\begin{aligned}
&\|(\tilde{D}^{m+1}F)_{u_1} - (\tilde{D}^{m+1}F)_{u_2}\|_{L_s^{m+1}(X,\Gamma_{C^k}(E))} \\
&= \sup_{\|x_1\|=\ldots=\|x_{m+1}\|=1} \|(\tilde{D}^{m+1}F)_{u_1}(x_1,\ldots,x_{m+1}) - (\tilde{D}^{m+1}F)_{u_2}(x_1,\ldots x_{m+1})\|_{\Gamma_{C^k}(E)} \\
&= \sup_{\|x_1\|=\ldots=\|x_{m+1}\|=1} \max_{i=1,\ldots,l} \|(D^{m+1}F_i)_{u_1}(x_1,\ldots,x_{m+1}) - (D^{m+1}F_i)_{u_2}(x_1,\ldots x_{m+1})\|_i
\end{aligned}$$

together with the continuity of $D^{m+1}F_i$. □

Lemma 3.4.2 (global Ω-lemma)**.** *Let $r,k \in \mathbb{N}$. Let E and M be manifolds without boundary and let M be compact. Denote $m = \dim(M)$. Let $E \to M$ be a (smooth) vector bundle of rank n, and let h be a bundle metric on E. Choose $U_i, \tilde{U}_i, \hat{U}_i, \varphi_i, \Phi_i$, $i = 1,\ldots,l$ as in Definition 3.3.4 and s.t. the Φ_i are isometries on the fibers. Let $\delta > 0$ and define the open subset $U \subset E$ by*

$$U := \{v \in E \mid \|v\|_h < \delta\}.$$

Let $F \to M$ be a (smooth) vector bundle of rank d with local trivializations $(\hat{U}_i, \tilde{\Phi}_i)$, $i = 1,\ldots,l$, and

$$f\colon U \to F$$

a map s.t.

i) f is fiber-preserving and

ii) the maps

$$g_i\colon \varphi_i(U_i) \times B_\delta(0) \to \mathbb{R}^d$$

defined by

$$g_i(x,v) := \left(pr_2 \circ \tilde{\Phi}_i \circ f \circ \Phi_i^{-1} \circ (\varphi_i^{-1}, id)\right)(x,v)$$

for $i = 1,\ldots,l$ and $B_\delta(0) \subset \mathbb{R}^n$ the open ball in $\mathbb{R}^n$ of radius δ and center 0, satisfy

$$g_i \in C^k(\overline{\varphi_i(U_i) \times B_\delta(0)}, \mathbb{R}^d)$$

and for each $j = 0,\ldots,r$ the map

$$D_2^j g_i\colon \varphi_i(U_i) \times B_\delta(0) \to L_s^j(\mathbb{R}^m, \mathbb{R}^d)$$

defined by $(D_2^j g_i)(x,y) := (D^j(g_i(x,.)))(y)$ for all $(x,y) \in \varphi_i(U_i) \times B_\delta(0)$ exists and is an element of $C^k(\overline{\varphi_i(U_i) \times B_\delta(0)}, L_s^j(\mathbb{R}^m,\mathbb{R}^d))$.

Then the map

$$\begin{aligned}
\Omega_f\colon \Gamma_{C^k}(E)^U &\to \Gamma_{C^k}(F), \\
s &\mapsto f \circ s,
\end{aligned}$$

is an element of $C^r(\Gamma_{C^k}(E)^U, \Gamma_{C^k}(F))$ *where* $\Gamma_{C^k}(E)^U \subset \Gamma_{C^k}(E)$ *is the open subset of* C^k*-sections of* E *with image contained in* U*. If* $r \geq 1$*, then*

$$\left((D\Omega_f)_{s_0} s\right)(p) = (D(f|_{E_p \cap U}))_{s_0(p)} s(p) \tag{3.4.2}$$

for all $p \in M$*,* $s_0 \in \Gamma_{C^k}(E)^U$ *and all* $s \in \Gamma_{C^k}(E)$*.*

Remark 3.4.3.

i) Note that in the situation of Lemma 3.4.2 ii), the statement

$g_i \in C^k(\overline{\varphi_i(U_i) \times B_\delta(0)}, \mathbb{R}^d)$ and $D_2^j g_i \in C^k(\overline{\varphi_i(U_i) \times B_\delta(0)}, L_s^j(\mathbb{R}^m, \mathbb{R}^d))$

for $j = 0, \dots, r$ is equivalent to the statement that

$$\partial_y^\alpha \partial_x^\beta g_i \colon \varphi_i(U_i) \times B_\delta(0) \to \mathbb{R}^d$$

are continuous and continuously extendable to $\overline{\varphi_i(U_i) \times B_\delta(0)}$ for all $|\alpha| \leq k + r$, $|\beta| \leq k$, s.t. $|\alpha + \beta| \leq k + r$, where x denotes the "$\varphi_i(U_i)$-direction" and y denotes the "$B_\delta(0)$-direction".

ii) The assumptions of Lemma 3.4.2 ii) imply in particular that Ω_f is well-defined as a map: from ii) we see that $pr_2 \circ \tilde{\Phi}_i \circ f \colon U \cap E|_{U_i} \to \mathbb{R}^d$ is C^k. It follows that $f(v) = \tilde{\Phi}_i^{-1} \circ (\pi, pr_2 \circ \tilde{\Phi}_i)(v)$ for all $v \in U \cap E|_{U_i}$, where $\pi \colon E \to M$ is the projection of E, so $f \in C^k(U \cap E|_{U_i}, F)$. Since the U_i cover M, we have $f \in C^k(U, F)$ and thus $f \circ s \in \Gamma_{C^k}(F)$ for all $s \in \Gamma_{C^k}(E)^U$.

iii) We require the Φ_i to be isometries on the fibers, because then the domain of definition of the g_i is a product and we can apply Lemma 3.2.4. If we don't require the Φ_i to be isometries on the fibers, then the domain of definition of the g_i would be

$$\bigcup_{p \in U_i} \{\varphi_i(p)\} \times (pr_2 \circ \Phi_i)(E_p \cap U)$$

and we can't directly apply Lemma 3.2.4 on the g_i. However, the requirement that the Φ_i are isometries on the fibers is not too restrictive, since the following holds: given *any* local trivialization (W, Ψ) of E, there is a trivialization $(W, \tilde{\Psi})$ that is an isometry on the fibers (one can see this e.g. by applying the Gram-Schmidt process to the linearly independent sections $x \mapsto \Psi^{-1}(x, e_i)$ and afterwards putting them back to a local trivialization).

Proof of Lemma 3.4.2. For each $i = 1, \dots, l$ we have a commutative diagram

$$\begin{tikzcd}
\Gamma_{C^k}(E)^U \arrow[r, "\Omega_f"] \arrow[d, "R_i"'] & \Gamma_{C^k}(F) \arrow[r, "\tilde{R}_i"] & C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^d) \\
C^k(\overline{\varphi_i(U_i)}, B_\delta(0)) \arrow[urr, "\Omega_{g_i}"'] & &
\end{tikzcd}$$

where $R_i(s) := pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1}$, $\tilde{R}_i(s) := pr_2 \circ \tilde{\Phi}_i \circ s \circ \varphi_i^{-1}$, and $\Omega_{g_i}(h) = g \circ (id \times h)$. From Proposition 3.4.1 we see that Ω_f is C^r iff $\tilde{R}_i \circ \Omega_f$ is C^r. Moreover, $\tilde{R}_i \circ \Omega_f = \Omega_{g_i} \circ R_i$ is C^r because of Lemma 3.2.4, thus Ω_f is C^r.

Let $s_0 \in \Gamma_{C^k}(E)^U$. Differentiating the above commutative diagram yields the following commutative diagram

$$\begin{tikzcd}
\Gamma_{C^k}(E) \arrow[r, "(D\Omega_f)_{s_0}"] \arrow[d, "R_i"'] & \Gamma_{C^k}(F) \arrow[r, "\tilde{R}_i"] & C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^d) \\
C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^n) \arrow[urr, "(D\Omega_{g_i})_{R_i(s_0)}"'] & &
\end{tikzcd}$$

for $i = 1, \dots, l$, where

$$\left((D\Omega_{g_i})_{R_i(s_0)} h\right)(x) = (D_2 g_i)_{(x,(R_i(s_0))(x))} h(x) \tag{3.4.3}$$

for all $x \in \varphi_i(U_i)$ according to Lemma 3.2.4. Now we compute $D_2 g_i$. To that end, choose $p \in U_i$ and $v \in B_\delta(0)$. We have

$$(D_2 g_i)_{(\varphi_i(p),v)} = D(pr_2 \circ \tilde{\Phi}_i \circ f \circ \Phi_i^{-1} \circ h_p)_v$$

where $h_p(v) := (p, v)$. The mappings

$$\Phi_i^{-1} \circ h_p \colon B_\delta(0) \to E_p \cap U, \qquad v \mapsto \Phi_i^{-1}(p, v),$$

and

$$\tilde{\Phi}_i \circ pr_2 \colon F_p \to \mathbb{R}^d, \qquad w \mapsto pr_2(\tilde{\Phi}_i(w)),$$

are both the restrictions of continuous linear maps and thus we have

$$\begin{aligned}
&D(pr_2 \circ \tilde{\Phi}_i \circ f \circ \Phi_i^{-1} \circ h_p)_v v_1 \\
&= D(pr_2 \circ \tilde{\Phi}_i)_{(f(\Phi_i^{-1}(h_p(v))))} (D(f_p)_{\Phi_i^{-1}(h_p(v))} D(\Phi_i^{-1} \circ h_p)_v v_1 \\
&= \left(pr_2 \circ \tilde{\Phi}_i\right) \left((D(f_p))_{\Phi_i^{-1}(p,v)} \Phi_i^{-1}(p, v_1)\right)
\end{aligned}$$

for all $v \in B_\delta(0)$, $v_1 \in \mathbb{R}^m$, and $p \in U_i$ where $f_p \colon E_p \cap U \to F_p$ is given by $f_p(x) = f(x)$. We have shown

$$(D_2 g_i)_{(\varphi_i(p),v)} = \left(pr_2 \circ \tilde{\Phi}_i\right) \left((D(f_p))_{\Phi_i^{-1}(p,v)} \Phi_i^{-1}(p, v_1)\right)$$

for all $v \in B_\delta(0)$, $v_1 \in \mathbb{R}^m$, and $p \in U_i$. In particular, we have

$$\begin{aligned}
&(D_2 g_i)_{(x,(R_i s_0)(x))} (R_i s)(x) \\
&= \left(pr_2 \circ \tilde{\Phi}_i\right) \left((D(f_{\varphi_i^{-1}(x)}))_{\Phi_i^{-1}(\varphi_i^{-1}(x),(R_i s_0)(x))} \Phi_i^{-1}(\varphi_i^{-1}(x), (R_i s)(x))\right) \\
&= \left(pr_2 \circ \tilde{\Phi}_i\right) \left((D(f_{\varphi_i^{-1}(x)}))_{s_0(\varphi_i^{-1}(x))} s(\varphi_i^{-1}(x)\right)
\end{aligned}$$

for all $x \in \varphi_i(U_i)$, $s \in \Gamma_{C^k}(E)$. Combining this equation with (3.4.3) and the commutativity of the differentiated diagram, we have

$$\left(pr_2 \circ \tilde{\Phi}_i\right) \left(((D\Omega_f)_{s_0} s)\, \varphi_i^{-1}(x)\right) = \left(pr_2 \circ \tilde{\Phi}_i\right) \left((D(f_{\varphi_i^{-1}(x)}))_{s_0(\varphi_i^{-1}(x))} s(\varphi_i^{-1}(x)\right)$$

for all $x \in \varphi_i(U_i)$. Since $pr_2 \circ \tilde{\Phi}_i \colon F_{\varphi_i^{-1}(x)} \to \mathbb{R}^d$ is an isomorphism, we have that

$$((D\Omega_f)_{s_0} s)\, \varphi_i^{-1}(x) = (D(f_{\varphi_i^{-1}(x)}))_{s_0(\varphi_i^{-1}(x))} s(\varphi_i^{-1}(x))$$

for all $x \in \varphi_i(U_i)$. Since the U_i cover M, we get

$$\left((D\Omega_f)_{s_0} s\right)(p) = (D(f_p))_{s_0(p)} s(p)$$

for all $p \in M$, $s_0 \in \Gamma_{C^k}(E)^U$ and all $s \in \Gamma_{C^k}(E)$. □

Theorem 3.4.4 ($C^k(M,N)$ as a Banach manifold)**.** *Let $k \in \mathbb{N}$. Let M and N be manifolds without boundary. Let M be compact and let N be connected. Choose a Riemannian metric h on N. Then the topological space $C^k(M,N)$ (i.e., the set $C^k(M,N)$ equipped with the compact-open C^k topology) has the structure of a smooth Banach manifold such that the following holds: for any $f \in C^k(M,N)$ and $\varepsilon > 0$ small enough, the maps*

$$\varphi_f \colon U_{f,\varepsilon} \to \varphi_f(U_{f,\varepsilon}) \subset \Gamma_{C^k}(f^*TN),$$

defined in Lemma 3.3.8, are smooth diffeomorphisms. This smooth structure does not depend on the choice of the Riemannian metric h on N. Moreover, for all $f, g \in C^k(M,N)$ s.t. $U_{f,\varepsilon^f} \cap U_{g,\varepsilon^g} \neq \varnothing$ it holds that

$$\left(D(\varphi_f \circ \varphi_g^{-1})_{s_0} s\right)(p) = D(\exp_{f(p)}^{-1} \circ \exp_{g(p)})_{s_0(p)} s(p) \tag{3.4.4}$$

for all $p \in M$, $s_0 \in \varphi_g(U_{f,\varepsilon^f} \cap U_{g,\varepsilon^g})$, $s \in \Gamma_{C^k}(g^*TN)$.

Proof. First we show that for $U_{f,\varepsilon^f} \cap U_{g,\varepsilon^g} \neq \varnothing$ the transition map $\varphi_g \circ \varphi_f^{-1}$ is smooth. We use a strategy similar to the proofs of Lemma 3.3.8 i)-ii). To be more precise, we first show the statement holds for sets U_{f,ε^f} with some additional assumptions on the charts in the definition of U_{f,ε^f}. We will call these sets $U^{\text{add.}}_{f,\varepsilon^f}$. Then we show that we don't need these additional assumptions, provided that $\varepsilon^f > 0$ is small enough. We start by defining the sets $U^{\text{add.}}_{f,\varepsilon^f}$, that is, we formulate which additional assumptions we make on the charts in the definition of U_{f,ε^f}.

Let $f \in C^k(M,N)$. Choose charts (U_i^f, φ_i^f) of M, $i = 1, \ldots, l = l(f)$, $\bigcup_{i=1}^l U_i^f = M$, s.t. $\overline{U_i^f} \subset M$ is compact, $\overline{U_i^f} \subset \tilde{U}_i^f$, $(\tilde{U}_i^f, \underline{\varphi_i^f})$ is still a chart of M, $f(\overline{U_i^f}) \subset V_i^f$, (V_i^f, ψ_i) chart of N, $\overline{V}_i \subset N$ is compact, $\overline{V_i^f} \subset \tilde{V}_i^f$, $\overline{\tilde{V}_i^f} \subset N$ is compact, and $(\tilde{V}_i^f, \hat{\Phi}_i^f)$ is a local trivialization of TN which is an isometry on fibers for $i =$

$1,\dots,l$. Choose $0 < r^f < \inf_{p\in M} \mathrm{inj}_{f(p)}(N)$ s.t. $(pr_1, \exp)$ is a diffeomorphism on the set

$$X_i^f := \{(q,v) \in TN \mid q \in B_{r^f}(V_i^f),\ \|v\|_h < r^f\}$$

(To see that such a r^f exists, note that since $\overline{V}_i^f$ is compact, $N \setminus \tilde{V}_i^f$ is closed, and $\overline{V}_i^f \cap (N \setminus \tilde{V}_i^f) = \varnothing$ there exists $r > 0$ s.t. $B_r(V_i^f) \subset \tilde{V}_i^f$, thus $B_r(V_i^f)$ is contained in a compact set. Combine this with Lemma 3.3.7 to get existence of r^f.) Denote by $(f^{-1}(\tilde{V}_i^f), \Phi_i^f)$ the local trivialization of f^*TN induced by $(\tilde{V}_i^f, \hat{\Phi}_i^f)$. Now define the set

$$U^{\text{add.}}_{f,\varepsilon^f} := \bigcap_{i=1}^{l} \mathcal{N}^k(f, \varphi_i^f, \tilde{U}_i^f, \psi_i^f, V_i^f, \overline{U_i^f}, \varepsilon^f)$$

where $\varepsilon^f > 0$ is chosen s.t. Lemma 3.3.8 i)-iii) hold (for $U^{\text{add.}}_{f,\varepsilon^f}$) if we replace "$\inf_{p\in M} \mathrm{inj}_{f(p)}(N)$" by "$\frac{1}{6}\inf_{p\in M} \mathrm{inj}_{f(p)}(N)$" in (3.3.1) and choose $\delta(\varepsilon^f) < \frac{r^f}{6}$, where $\delta(\varepsilon^f)$ is that of (3.3.1).

Assume that $U^{\text{add.}}_{f,\varepsilon^f} \cap U^{\text{add.}}_{g,\varepsilon^g} \neq \varnothing$. Define

$$U := \{v \in f^*TN \mid \|v\|_h < 2\delta(\varepsilon^f)\}$$

and

$$F \colon U \to g^*TN$$

by

$$F(v) := \left((\exp_{g(p)})^{-1} \circ \exp_{f(p)}\right)(v)$$

for $v \in U \cap T_{f(p)}N$. Then F is well-defined: if $\delta(\varepsilon^f) \geq \delta(\varepsilon^g)$, then

$$F(v) = (pr_1, \exp)|^{-1}_{X_i^f}(g(p), \exp(f(p), v))$$

for all $v \in U \cap T_{f(p)}N$, where $p \in U_i^f \cap U_j^g$. If $\delta(\varepsilon^f) \leq \delta(\varepsilon^g)$, then

$$F(v) = (pr_1, \exp)|^{-1}_{X_j^g}(g(p), \exp(f(p), v))$$

for all $v \in U \cap T_{f(p)}N$, where $p \in U_i^f \cap U_j^g$. Hence, F is well-defined.

Now we want to use Lemma 3.4.2 to show that

$$\begin{aligned} \Omega_F \colon \Gamma_{C^k}(f^*TN)^U &\to \Gamma_{C^k}(g^*TN), \\ s &\mapsto F \circ s, \end{aligned}$$

is (well-defined and) smooth. If we have shown that, then we have in particular that $\varphi_g \circ \varphi_f^{-1} = \Omega_F|_{\varphi_f(U^{\text{add.}}_{f,\varepsilon^f} \cap U^{\text{add.}}_{g,\varepsilon^g})}$ is smooth. Now we show ii) of Lemma 3.4.2. To that end, we consider the maps

$$g_{ij} \colon pr_2 \circ \Phi_j^g \circ F \circ (\Phi_i^f)^{-1} \circ \left((\varphi_i^f)^{-1}, id\right) : \varphi_i^f(U_i^f \cap U_j^g) \times B_{2\delta(\varepsilon^f)}(0) \to \mathbb{R}^n,$$

where $n = \dim(N)$ and the maps

$$H_{ij}\colon Y_{ij} \to TN$$

where Y_{ij} is the non-empty open set

$$Y_{ij} := \{(q_1, q_2, y) \in V_i^f \times V_j^g \times B_{2\delta(\varepsilon^f)}(0) \mid q_2 \in B_{2\delta(\varepsilon^f)+\delta(\varepsilon^g)}(q_1)\}$$

and

$$H_{ij}(q_1, q_2, y) := \left((\exp_{q_2})^{-1} \circ \exp_{q_1}\right)\left((\hat{\Phi}_i^f)^{-1}(q_1, y)\right).$$

Note that H_{ij} is well-defined and

$$pr_2 \circ \Phi_j^g \circ H_{ij} \circ \left((f,g) \circ (\varphi_i^f)^{-1}, id\right) = g_{ij} \tag{3.4.5}$$

on $\varphi_i^f(U_i^f \cap U_j^g) \times B_{2\delta(\varepsilon^f)}(0)$. Moreover, H_{ij} is smooth on Y_{ij}, since for $r^f \geq r^g$ it holds that

$$H_{ij}(q_1, q_2, y) = (pr_1, \exp)|_{X_i^f}^{-1}(q_2, \exp(q_1, (\hat{\Phi}_i^f)^{-1}(q_1, y))$$

on Y_{ij} and for $r^g \geq r^f$ it holds that

$$H_{ij}(q_1, q_2, y) = (pr_1, \exp)|_{X_j^g}^{-1}(q_2, \exp(q_1, (\hat{\Phi}_i^f)^{-1}(q_1, y))$$

on Y_{ij} (see Lemma 3.3.7). Given any multiindex α we see from (3.4.5) that

$$\partial_y^\alpha g_{ij}(x,y) = (pr_2 \circ \Phi_j^g)\left(\left(\partial_y^\alpha H_{ij}\right)\left(f((\varphi_i^f)^{-1}(x)), g((\varphi_i^f)^{-1}(x)), y)\right)\right)$$

for all $(x,y) \in \varphi_i^f(U_i^f \cap U_j^g) \times B_{2\delta(\varepsilon^f)}(0)$, so $\partial_y^\alpha g_{ij}$ is C^k in (x,y). In particular, for $|\beta| \leq k$, we have that $\partial_x^\beta \partial_y^\alpha g_{ij}$ is continuous on $\overline{\varphi_i^f(U_i^f \cap U_j^g) \times B_{\delta(\varepsilon^f)}(0)}$. We have shown that Ω_F is smooth on $\{v \in f^*TN \mid \|v\|_h < \delta(\varepsilon^f)\}$.

Next we show that we don't need the additional assumptions we made in the definition of the sets $U_{f,\varepsilon^f}^{\text{add.}}$. For arbitrary $U_{f,\varepsilon}$ (defined as in Lemma 3.3.8) we choose ε so small, that there exists some $U_{f,\varepsilon^f}^{\text{add.}}$ with $U_{f,\varepsilon} \subset U_{f,\varepsilon^f}^{\text{add.}}$ (we can always do that, see Lemma 3.3.2). If $U_{f,\varepsilon} \cap U_{g,\tilde{\varepsilon}} \neq \varnothing$, then we have in particular $U_{f,\varepsilon^f}^{\text{add.}} \cap U_{g,\tilde{\varepsilon^g}}^{\text{add.}} \neq \varnothing$ (since we chose ε and $\tilde{\varepsilon}$ s.t. $U_{f,\varepsilon} \subset U_{f,\varepsilon^f}^{\text{add.}}$ and $U_{g,\tilde{\varepsilon}} \subset U_{g,\tilde{\varepsilon^g}}^{\text{add.}}$). We have shown that the transition map $\varphi_g \circ \varphi_f^{-1}$ is smooth on $\varphi_f(U_{f,\varepsilon^f}^{\text{add.}} \cap U_{g,\tilde{\varepsilon^g}}^{\text{add.}})$, so it is in particular smooth on $\varphi_f(U_{f,\varepsilon^f} \cap U_{g,\tilde{\varepsilon^g}})$.

A similar argument can be used to show that the above smooth structure does not depend on the choice of the Riemannian metric h on N.

Finally, equation (3.4.4) is a direct consequence of equation (3.4.2). □

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Chapter 4

The Banach bundle

$$\bigsqcup_{f \in C^k(M,N)} \Gamma_{C^k}(f^*TN \otimes_{\mathbb{R}} \Sigma M) \to C^k(M,N)$$

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Abstract In this chapter we are concerned with the bundle

$$\mathcal{E} := \bigsqcup_{f \in C^k(M,N)} \Gamma_{C^k}(f^*TN \otimes_{\mathbb{R}} \Sigma M) \to C^k(M,N)$$

where M is a closed spin manifold and N is a closed connected manifold. First we show that this is a Banach bundle. Essentially, it is the tangent bundle of $C^k(M,N)$ twisted with the complex spinor bundle ΣM. Afterwards we construct a natural connection on $\mathcal{E}$. As a preliminary step we show a criterion for smoothness of sections of $\mathcal{E}$.

Relation to Chapter 1: The results of this chapter are interesting in their own right and the methods can be easily applied to similar settings. Moreover, this chapter also provides the foundation for an alternative approach to showing Theorem 1.1.1 by solving the constraint equation (1.1.4) using an ansatz different from the resolvent integral (1.1.6). This alternative approach was our original ansatz for solving the constraint equation. We briefly explain the idea: suppose we are given some suitable initial data (u_0, ψ_0) for (1.1.5). In particular, $\dim_{\mathbb{H}} \ker \not{D}^{u_0} = 1$. Since the dimension of the kernel of $\not{D}^{u_0}$ is locally constant by Lemma 1.4.3, there exists a neighborhood $\mathcal{U} \subset C^k(M,N)$ of u_0 s.t. $\dim_{\mathbb{H}} \ker \not{D}^f = 1$ for all $f \in \mathcal{U}$ and we get a subbundle

$$\mathcal{K} := \bigsqcup_{f \in \mathcal{U}} \ker \not{D}^f \to \mathcal{U}.$$

Given any path $u_t\colon [0,T] \to \mathcal{U}$ starting at u_0, we can parallel transport ψ_0 in $\mathcal{K}$ along u_t with respect to the connection on $\mathcal{K}$ that we obtain from the connection on $\mathcal{E}$. This yields a family of spinors $\psi(u_t) \in \ker \not{D}^{u_t}$ and one could work with this family instead of (1.1.6). Note that the above parallel transport preserves the L^2-norm of spinors. Note also that one could formulate uniqueness of the solution in Theorem 1.1.1 by requiring the spinor part of the solution to be parallel for the connection on $\mathcal{K}$.

4.1 Construction of the bundle

We start with the following observation that we will use later.

Lemma 4.1.1. *Let E and F be vector bundles over a compact manifold M.*

i) There exists a $C_1 > 0$ s.t.

$$\|fs\|_{\Gamma_{C^k}(E)} \leq C_1 \|f\|_{C^k(M,\mathbb{R})} \|s\|_{\Gamma_{C^k}(E)}$$

for all $f \in C^k(M,\mathbb{R})$, $s \in \Gamma_{C^k}(E)$.

ii) There exists a $C_2 > 0$ s.t.

$$\|s_1 \otimes s_2\|_{\Gamma_{C^k}(E\otimes F)} \leq C_2 \|s_1\|_{\Gamma_{C^k}(E)} \|s_2\|_{\Gamma_{C^k}(F)}$$

for all $s_1 \in \Gamma_{C^k}(E)$, $s_2 \in \Gamma_{C^k}(F)$.

Proof. i): There exists $C_1 > 0$ s.t.

$$\|fs\|_{\Gamma_{C^k}(E)} \leq C_1$$

for all f, s with $\|f\|_{C^k(M,\mathbb{R})} = 1 = \|s\|_{\Gamma_{C^k}(E)}$. Now the statement of i) follows directly.

ii): There exists $C_1 > 0$ s.t.

$$\|s_1 \otimes s_2\|_{\Gamma_{C^k}(E\otimes F)} \leq C_2$$

for all s_1, s_2 with $\|s_1\|_{\Gamma_{C^k}(E)} = 1 = \|s_2\|_{\Gamma_{C^k}(F)}$. The statement of ii) directly follows from that. □

For the next definitions see [2, 1].

A *Banachable space* is a topological vector space X whose topology is induced by a complete norm on X.

Definition 4.1.2. Let E and M be C^k-manifolds. Let $\pi\colon E \to M$ be a C^k-mapping s.t. $E_x = \pi^{-1}(x)$ is a Banachable space for each $x \in M$. A *$(C^k$-$)$Banach bundle atlas (for π)* is a family $\mathcal{B} = \{(U_i, \varphi_i)\}_{i\in I}$ s.t. the following holds:

i) For each $i \in I$, $U_i \subset M$ is an open subset of M and $\bigcup_{i\in I} U_i = M$.

ii) For each $i \in I$ there exists a Banach space X_i and a C^k-diffeomorphism

$$\varphi_i \colon \pi^{-1}(U_i) \to U_i \times X_i$$

s.t. $pr_1 \circ \varphi_i = \pi$ and for each $x \in U_i$ the mapping

$$\varphi_{ix} := pr_2 \circ \varphi_i|_{E_x} \colon E_x \to X_i$$

is a continuous linear isomorphism.

iii) If $U_i \cap U_j \neq \varnothing$, then the mapping

$$\begin{aligned} \varphi_{jix} \colon U_i \cap U_j &\to L(X_i, X_j), \\ x &\mapsto \varphi_{jx} \circ (\varphi_{ix})^{-1} \end{aligned}$$

is a C^k-mapping.

Two C^k-Banach bundle atlases $\mathcal{B}_1$ and $\mathcal{B}_2$ are *equivalent*, if $\mathcal{B}_1 \cup \mathcal{B}_2$ is again a C^k-Banach bundle atlas. Given an equivalence class $[\mathcal{B}]$ of a C^k-Banach bundle atlas $\mathcal{B}$, we call $(\pi, [\mathcal{B}])$ a *(C^k-)Banach bundle*. Most of the time, we don't put $[\mathcal{B}]$ directly in the notation and call π a (C^k-)Banach bundle. Depending on the context, we also refer to E as (C^k-)Banach bundle. A *local trivialization of E* is a tuple (U, φ) s.t. $(U, \varphi) \in \tilde{\mathcal{B}}$ with $\tilde{\mathcal{B}} \in [\mathcal{B}]$.

Remark 4.1.3.

i) In the finite dimensional case, Definition 4.1.2 ii) implies Definition 4.1.2 iii).

ii) It holds that

$$\varphi_{kjx} \circ \varphi_{jix} = \varphi_{kix}$$

for all $k, i, j \in I$ and all $x \in M$.

Lemma 4.1.4. *Let M be a C^k-manifold, E a set, and $\pi \colon E \to M$ a mapping. Let $\mathcal{B} = \{(U_i, \varphi_i)\}_{i\in I}$ be a family s.t. the following holds:*

i) For each $i \in I$, $U_i \subset M$ is an open subset of M and $\bigcup_{i\in I} U_i = M$.

ii) For each $i \in I$ there exists a Banach space X_i and a bijection

$$\varphi_i \colon \pi^{-1}(U_i) \to U_i \times X_i$$

s.t. $pr_1 \circ \varphi_i = \pi$.

iii) If $U_i \cap U_j \neq \varnothing$, then the mapping

$$\begin{aligned} \varphi_{jix} \colon U_i \cap U_j &\to L(X_i, X_j), \\ x &\mapsto \varphi_{jx} \circ (\varphi_{ix})^{-1} \end{aligned}$$

is a C^k-mapping s.t. $\varphi_{kjx} \circ \varphi_{jix} = \varphi_{kix}$ for all $k, i, j \in I$ and all $x \in M$.

Then there exists a unique structure of a C^k-manifold on E and a unique structure of a Banachable space on each E_x s.t. $(\pi,[\mathcal{B}])$ becomes a C^k-Banach bundle.

Corollary 4.1.5. *If M is a C^k-Banach manifold, then $TM\to M$ is a C^{k-1}-Banach bundle: given any chart $\varphi\colon U\to X$ of M, a local trivialization of TM is given by*

$$\begin{aligned}\bigsqcup_{y\in U} T_yM &\to U\times X,\\ (y,v)&\mapsto (y,(d\varphi)_y v),\end{aligned}$$

where, for $v=[c]$, we identified $(d\varphi)_y v=(\varphi\circ c)'(0)$.

Our next goal is to show that

$$\bigsqcup_{f\in C^k(M,N)}\Gamma_{C^k}(f^*TN\otimes_{\mathbb{R}}\Sigma M)\to C^k(M,N)$$

is a C^∞-Banach bundle (for M a closed spin manifold and N a closed connected manifold). To prove that, we will use Lemma 4.1.4 and the following two lemmas.

Lemma 4.1.6. *Let E be a vector bundle over a closed manifold M. Then there exist $C>0$, $N\in\mathbb{N}$, and sections $s_1,\dots,s_N\in\Gamma_{C^k}(E)$ s.t. the following holds: for every section $s\in\Gamma_{C^k}(E)$ there exist $f_1,\dots,f_N\in C^k(M,\mathbb{R})$ s.t. $\|f_i\|_{C^k(M,\mathbb{R})}\leq C\|s\|_{\Gamma_{C^k}(E)}$ for $i=1,\dots,N$ and*

$$s(x)=\sum_{i=1}^N f_i(x)s_i(x)$$

for all $x\in M$. If $E=E_1\otimes_{\mathbb{R}}E_2$ is the tensor product of two vector bundles E_1 and E_2 over M, then each s_i can be chosen as $s_i=e_{i1}\otimes e_{i2}$ where $e_{ij}\in\Gamma_{C^k}(E_j)$.

Proof. Pick a bundle metric $\langle .,.\rangle_E$ on E. Choose $U_i,\tilde{U}_i,\hat{U}_i,\phi_i,\Phi_i$, $i=1,\dots,l$, as in Definition 3.3.4 and s.t. the Φ_i are isometries on the fibers. Then $\tilde{s}_{ij}(p):=\Phi_i^{-1}(p,e_j)$, $x\in U_i$, are local sections of E. Let $\phi_1,\dots,\phi_l$ be a smooth partition of unity subordinate to the U_i and define the global sections s_{ij} of E by

$$s_{ij}(p)=\begin{cases}\phi_i(p)\tilde{s}_{ij}(p), & \text{if } p\in U_i,\\ 0, & \text{if } p\in M\setminus\operatorname{supp}(\phi_i).\end{cases}$$

For each i, let $V_i\subset M$ be the open set on which $\phi_i>0$. Then the V_i also cover M. Let $s\in\Gamma_{C^k}(E)$. It holds that

$$s=\sum_j\frac{1}{\phi_i}\langle s,\tilde{s}_{ij}\rangle_E s_{ij}(p)$$

on V_i. Let $\psi_1, \dots, \psi_l$ be a smooth partition of unity subordinate to the V_i. Defining

$$f_{ij}(p) = \begin{cases} \psi_i(p) \frac{1}{\phi_i(p)} \langle s(p), \tilde{s}_{ij}(p) \rangle_E, & \text{if } p \in V_i, \\ 0, & \text{if } p \in M \setminus \operatorname{supp}(\psi_i). \end{cases}$$

we see that $f_{ij} \in C^k(M, \mathbb{R})$ with

$$s = \sum_{i,j} f_{ij} s_{ij}.$$

on M. Moreover, we compute

$$\langle s, \tilde{s}_{ij} \rangle_E = \langle pr_2 \circ \Phi_i \circ s, pr_2 \circ \Phi_i \circ \tilde{s}_{ij} \rangle = \langle pr_2 \circ \Phi_i \circ s, e_j \rangle = (pr_2 \circ \Phi_i \circ s)^j$$

and

$$\begin{aligned} \|f_{ij}\|_{C^k(M,\mathbb{R})} &= \max_{r=1,\dots,l} \max_{|\alpha| \le k} \sup_{\varphi_r(\overline{U_r})} \|\partial_x^\alpha (f_{ij} \circ \varphi_r^{-1})\| \\ &= \max_{r=1,\dots,l} \max_{|\alpha| \le k} \sup_{\varphi_r(\overline{U_r} \cap \operatorname{supp}(\psi_i))} \|\partial_x^\alpha (f_{ij} \circ \varphi_r^{-1})\| \\ &= \max_{r=1,\dots,l} \max_{|\alpha| \le k} \sup_{\varphi_r(\overline{U_r} \cap \operatorname{supp}(\psi_i))} \|\partial_x^\alpha \left(\left(\frac{\psi_i}{\phi_i} \circ \varphi_r^{-1} \right) \cdot \left(pr_2 \circ \Phi_i \circ s \circ \varphi_r^{-1} \right)^j \right) \| \\ &\le \tilde{C}_i \max_{r=1,\dots,l} \max_{|\alpha| \le k} \sup_{\varphi_r(\overline{U_r} \cap \operatorname{supp}(\psi_i))} \|\partial_x^\alpha \left(\left(pr_2 \circ \Phi_i \circ s \circ \varphi_r^{-1} \right)^j \right) \| \\ &= \tilde{C}_i \max_{r=1,\dots,l} \max_{|\alpha| \le k} \sup_{\varphi_r(\overline{U_r} \cap \operatorname{supp}(\psi_i))} \|\partial_x^\alpha \left(\left(pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1} \circ \varphi_i \circ \varphi_r^{-1} \right)^j \right) \| \\ &\le C_i \max_{r=1,\dots,l} \max_{|\alpha| \le k} \sup_{\varphi_i(\overline{U_r} \cap \operatorname{supp}(\psi_i))} \|\partial_x^\alpha \left(\left(pr_2 \circ \Phi_i \circ s \circ \varphi_i^{-1} \right)^j \right) \| \\ &\le C_i \|s\|_{\Gamma_{C^k}(E)} \end{aligned}$$

where we used $\operatorname{supp}(\psi_i) \subset V_i \subset U_i$ and Lemma 3.2.1 i). The statement regarding the tensor product follows from fact that every section $\gamma \in \Gamma_{C^k}(E_1 \otimes E_2)$ can be written as $\gamma = \sum_{i=1}^n e_{i1} \otimes e_{i2}$, see e.g. the proof of the following lemma.

□

Lemma 4.1.7. *Let X be a Banach space and $U \subset X$ an open subset. Let $E_1, E_2, F_1, F_2 \to M$ be real vector bundles over a closed manifold M. For $i = 1, 2$ let*

$$A_i \colon U \to L_{\mathbb{R}}(\Gamma_{C^k}(E_i), \Gamma_{C^k}(F_i))$$

be C^l-mappings s.t. $A_i(u)$ is $C^k(M, \mathbb{R})$-linear for every $u \in U$ and $(D^j A_i)(u)(x_1, \dots, x_j)$ are $C^k(M, \mathbb{R})$-linear for every $u \in U$, $x_1, \dots, x_j \in X$, $j = 1, \dots, l$.

Define the mapping

$$B\colon U \to L_{\mathbb{R}}\left(\Gamma_{C^k}(E_1 \otimes_{\mathbb{R}} E_2), \Gamma_{C^k}(F_1 \otimes_{\mathbb{R}} F_2)\right)$$

by

$$B(u)(\gamma) := \sum_{m=1}^{n} A_1(u)(e_{1m}) \otimes A_2(u)(e_{2m})$$

for $u \in U$, $\gamma \in \Gamma_{C^k}(E_1 \otimes E_2)$, $\gamma = \sum_{m=1}^{n} e_{1m} \otimes e_{2m}$, $e_{jm} \in \Gamma_{C^k}(E_j)$ *for* $j = 1, 2$. *Then B is a well-defined C^l-mapping ("$\otimes$" is the (pointwise) tensor product of sections) with*

$$\begin{aligned}(D^j B)_u(x_1, \ldots, x_j)&\left(\sum_{m=1}^{n} e_{1m} \otimes e_{2m}\right)\\ &= \sum_{m=1}^{n} (D^j A_1)_u(x_1, \ldots, x_j)(e_{1m}) \otimes (D^j A_2)_u(x_1, \ldots, x_j)(e_{2m})\end{aligned}$$

for $j = 1, \ldots, l$, $u \in U$, *and* $x_1, \ldots, x_j \in X$.

Proof. First, we show that B is well-defined. To that end, notice that every section $\gamma \in \Gamma_{C^k}(E_1 \otimes E_2)$ can be written as a sum $\gamma = \sum_{j=1}^{m} e_{1j} \otimes e_{2j}$ for some sections $e_{ij} \in \Gamma_{C^k}(E_i)$ for $i = 1, 2$, $j = 1, \ldots, m$. This follows from the fact that the mapping

$$\begin{aligned}\Gamma_{C^k}(E_1) \times \Gamma_{C^k}(E_2) &\to \Gamma_{C^k}(E_1 \otimes E_2),\\ (e_1, e_2) &\mapsto e_1 \otimes e_2,\end{aligned}$$

induces an isomorphism of $C^k(M, \mathbb{R})$-modules

$$I_{E_1,E_2}\colon \Gamma_{C^k}(E_1) \otimes_{C^k(M,\mathbb{R})} \Gamma_{C^k}(E_2) \cong \Gamma_{C^k}(E_1 \otimes E_2),$$

with $I_{E_1,E_2}(e_1 \otimes_{C^k} e_2) = e_1 \otimes e_2$, where "$\otimes_{C^k}$" denotes the tensor product in $\Gamma_{C^k}(E_1) \otimes_{C^k(M,\mathbb{R})} \Gamma_{C^k}(E_2)$.

For every $u \in U$ the universal property of the tensor product of modules gives us a $C^k(M, \mathbb{R})$-linear map

$$\begin{aligned}A_1(u) \otimes_{C^k} A_1(u)\colon \Gamma_{C^k}(E_1) \otimes_{C^k(M,\mathbb{R})} \Gamma_{C^k}(E_2) &\to \Gamma_{C^k}(F_1) \otimes_{C^k(M,\mathbb{R})} \Gamma_{C^k}(F_2),\\ e_1 \otimes_{C^k} e_2 &\mapsto A_1(u)(e_1) \otimes_{C^k} A_2(u)(e_2)\end{aligned}$$

Now assume that

$$\sum_{j=1}^{m} e_{1j} \otimes e_{2j} = \sum_{j=1}^{n} \tilde{e}_{1j} \otimes \tilde{e}_{2j}$$

for e_{ij}, $\tilde{e}_{ij} \in \Gamma_{C^k}(E_i)$. Then

$$\begin{aligned} B(\sum_{j=1}^{m} e_{1j} \otimes e_{2j}) &= \left(I_{F_1,F_2} \circ (A_1(u) \otimes_{C^k} A_1(u)) \circ I^{-1}_{E_1,E_2}\right) (\sum_{j=1}^{m} e_{1j} \otimes e_{2j}) \\ &= \left(I_{F_1,F_2} \circ (A_1(u) \otimes_{C^k} A_1(u)) \circ I^{-1}_{E_1,E_2}\right) (\sum_{j=1}^{n} \tilde{e}_{1j} \otimes \tilde{e}_{2j}) \\ &= B(\sum_{j=1}^{n} \tilde{e}_{1j} \otimes \tilde{e}_{2j}) \end{aligned}$$

and B is well-defined.

For the remainder of the proof we set $Y := L_{\mathbb{R}}\left(\Gamma_{C^k}(E_1 \otimes_{\mathbb{R}} E_2), \Gamma_{C^k}(F_1 \otimes_{\mathbb{R}} F_2)\right)$.

Now we prove the statement for $l = 0$: from Lemma 4.1.6 we get $C > 0$, $N \in \mathbb{N}$, and sections $s_{ij} \in \Gamma_{C^k}(E_j)$, $j = 1, 2$, s.t. for every section $\gamma \in \Gamma_{C^k}(E_1 \otimes E_2)$ there exist functions $\gamma^i \in C^k(M, \mathbb{R})$ with $\|\gamma^i\|_{C^k(M,\mathbb{R})} \leq C\|\gamma\|_{\Gamma_{C^k}(E_1 \otimes E_2)}$ and $\gamma = \sum_{i=1}^{N} \gamma^i s_{i1} \otimes s_{i2}$. Using this, we compute for $u, v \in U$

$$\begin{aligned} &\|B(u) - B(v)\|_Y \\ &= \sup_{\|\gamma\|_{\Gamma_{C^k}(E_1 \otimes E_2)}=1} \|B(u)(\gamma) - B(v)(\gamma)\|_{\Gamma_{C^k}(F_1 \otimes F_2)} \\ &= \sup_{\|\gamma\|_{\Gamma_{C^k}(E_1 \otimes E_2)}=1} \|\sum_{i=1}^{N} \gamma^i (A_1(u)(s_{i1}) \otimes A_2(u)(s_{i2}) \\ &\qquad\qquad - A_1(v)(s_{i1}) \otimes A_2(v)(s_{i2}))\|_{\Gamma_{C^k}(F_1 \otimes F_2)} \\ &\leq C_1 \sup_{\|\gamma\|_{\Gamma_{C^k}(E_1 \otimes E_2)}=1} \\ &\quad \sum_{i=1}^{N} \|\gamma^i\|_{C^k(M,\mathbb{R})} \|\left(A_1(u)(s_{i1}) \otimes A_2(u)(s_{i2}) - A_1(v)(s_{i1}) \otimes A_2(v)(s_{i2})\right)\|_{\Gamma_{C^k}(F_1 \otimes F_2)} \\ &\leq CC_1 \sum_{i=1}^{N} \|\left(A_1(u)(s_{i1}) \otimes A_2(u)(s_{i2}) - A_1(v)(s_{i1}) \otimes A_2(v)(s_{i2})\right)\|_{\Gamma_{C^k}(F_1 \otimes F_2)} \\ &= CC_1 \sum_{i=1}^{N} \|\left((A_1(u)(s_{i1}) - A_1(v)(s_{i1})) \otimes A_2(u)(s_{i2})\right) + \\ &\qquad (A_1(v)(s_{i1}) \otimes (A_2(u)(s_{i2}) - A_2(v)(s_{i2})))\|_{\Gamma_{C^k}(F_1 \otimes F_2)} \\ &\leq C_2 \sum_{i=1}^{N} (\|(A_1(u)(s_{i1}) - A_1(v)(s_{i1})\|_{\Gamma_{C^k}(F_1)} \cdot \|A_2(u)(s_{i2})\|_{\Gamma_{C^k}(F_2)} \\ &+ \|A_1(v)(s_{i1})\|_{\Gamma_{C^k}(F_1)} \cdot \|A_2(u)(s_{i2}) - A_2(v)(s_{i2})\|_{\Gamma_{C^k}(F_2)}) \end{aligned}$$

where we used Lemma 4.1.1. The continuity of B now follows from the continuity of A_1 and A_2. Now we prove the lemma by induction over l. We show "$l \rightsquigarrow l+1$":

Define the mapping $\tilde{D}^{l+1}B\colon U \to L_s^{l+1}(X,Y)$ by

$$
\begin{aligned}
&(\tilde{D}^{l+1}B)_u(x_1,\dots,x_j)\left(\sum_{m=1}^{n} e_{1m}\otimes e_{2m}\right)\\
&\quad := \sum_{m=1}^{n} (D^{l+1}A_1)_u(x_1,\dots,x_{l+1})(e_{1m}) \otimes (D^{l+1}A_2)_u(x_1,\dots,x_{l+1})(e_{2m})
\end{aligned}
$$

for $u \in U$, $x_1,\dots,x_{l+1} \in X$. First of all, $\tilde{D}^{l+1}B$ is a well-defined mapping. This can be shown like the well-definedness of B above. Moreover, analogously to the continuity of B, we see that $\tilde{D}^{l+1}B$ is continuous. For the following computation, we again write $\gamma = \sum_{i=1}^{N} \gamma^i s_{i1} \otimes s_{i2}$ with $\|\gamma^i\|_{C^k(M,\mathbb{R})} \le C\|\gamma\|_{\Gamma_{C^k}(E_1\otimes E_2)}$ for an arbitrary $\gamma \in \Gamma_{C^k}(E_1 \otimes E_2)$.

$$
\begin{aligned}
&\|(D^lB)_u - (D^lB)_{u_0} - (\tilde{D}^{l+1}B)_{u_0}(u-u_0)\|_{L_s^l(X,Y)}\\
&= \sup_{\|x_k\|=1}\sup_{\|\gamma\|=1} \|(D^lB)_u(x_1,\dots,x_l)(\gamma) - (D^lB)_{u_0}(x_1\dots,x_l)(\gamma)\\
&\quad - (\tilde{D}^{l+1}B)_{u_0}(u-u_0,x_1,\dots,x_l)(\gamma)\|_{\Gamma_{C^k}(F_1\otimes F_2)}\\
&= \sup_{\|x_k\|=1}\sup_{\|\gamma\|=1} \|\sum_{i=1}^{N} \gamma^i((D^lA_1)_u(x_1,\dots,x_l)(s_{i1}) \otimes (D^lA_2)_u(x_1,\dots,x_l)(s_{i2})\\
&\quad - (D^lA_1)_{u_0}(x_1,\dots,x_l)(s_{i1}) \otimes (D^lA_2)_{u_0}(x_1,\dots,x_l)(s_{i2})\\
&\quad - (D^{l+1}A_1)_{u_0}(u-u_0,x_1,\dots,x_l)(s_{i1}) \otimes (D^{l+1}A_2)_{u_0}(u-u_0,x_1,\dots,x_l)(s_{i2}))\|_{\Gamma_{C^k}(F_1\otimes F_2)}
\end{aligned}
$$

Now we define

$$
\begin{aligned}
\Delta_{ij} &:= (D^lA_j)_u(x_1,\dots,x_l)(s_{ij}),\\
\Lambda_{ij} &:= (D^lA_j)_{u_0}(x_1,\dots,x_l)(s_{ij}),\\
\Psi_{ij} &:= (D^{l+1}A_j)_{u_0}(u-u_0,x_1,\dots,x_l)(s_{ij}),
\end{aligned}
$$

and further compute

$$
\begin{aligned}
&\|(D^lB)_u - (D^lB)_{u_0} - (\tilde{D}^{l+1}B)_{u_0}(u-u_0)\|_{L^l_s(X,Y)} \\
&= \sup_{\|x_k\|=1} \sup_{\|\gamma\|=1} \|\sum_{i=1}^N \gamma^i(\Delta_{i1}\otimes\Delta_{i2} - \Lambda_{i1}\otimes\Lambda_{i2} - \Psi_{i1}\otimes\Psi_{i2})\|_{\Gamma_{C^k}(F_1\otimes F_2)} \\
&\le \sup_{\|x_k\|=1} \sum_{i=1}^N \|\Delta_{i1}\otimes\Delta_{i2} - \Lambda_{i1}\otimes\Lambda_{i2} - \Psi_{i1}\otimes\Psi_{i2}\|_{\Gamma_{C^k}(F_1\otimes F_2)} \\
&= \sup_{\|x_k\|=1} \sum_{i=1}^N \|(\Delta_{i1}-\Lambda_{i1}-\Psi_{i1})\otimes\Psi_{i2} + (\Delta_{i1}-\Lambda_{i1})\otimes(\Delta_{i2}-\Lambda_{i2}-\Psi_{i2}) \\
&\quad + (\Delta_{i1}-\Lambda_{i1})\otimes\Lambda_{i2} + \Lambda_{i1}\otimes(\Delta_{i2}-\Lambda_{i2})\|_{\Gamma_{C^k}(F_1\otimes F_2)} \\
&\le C \sup_{\|x_k\|=1} \sum_{i=1}^N (\|\Delta_{i1}-\Lambda_{i1}-\Psi_{i1}\|_{\Gamma_{C^k}(F_1)}\cdot\|\Psi_{i2}\|_{\Gamma_{C^k}(F_2)} \\
&\quad + \|\Delta_{i1}-\Lambda_{i1}\|_{\Gamma_{C^k}(F_1)}\cdot\|\Delta_{i2}-\Lambda_{i2}-\Psi_{i2}\|_{\Gamma_{C^k}(F_2)} \\
&\quad + \|\Delta_{i1}-\Lambda_{i1}\|_{\Gamma_{C^k}(F_1)}\cdot\|\Lambda_{i2}\|_{\Gamma_{C^k}(F_2)} + \|\Lambda_{i1}\|_{\Gamma_{C^k}(F_1)}\cdot\|\Delta_{i2}-\Lambda_{i2}\|_{\Gamma_{C^k}(F_2)})
\end{aligned}
$$

Each of the factors of each of the summands is either constant in u or converges to 0 as $u \to u_0$. Thus B is C^{l+1} with $D^{l+1}B = \tilde{D}^{l+1}B$. This finishes the proof. $\square$

Corollary 4.1.8. *Let M be a closed spin manifold and let N be a connected manifold without boundary. Then*

$$\coprod_{f\in C^k(M,N)} \Gamma_{C^k}(f^*TN \otimes_{\mathbb{R}} \Sigma M) \to C^k(M,N)$$

is a C^∞-Banach bundle with local trivializations

$$
\begin{aligned}
\Psi = \Psi_f \colon \coprod_{g\in U_{f,\varepsilon}} \Gamma_{C^k}(g^*TN\otimes_{\mathbb{R}}\Sigma M) &\to U_{f,\varepsilon}\times\Gamma_{C^k}(f^*TN\otimes_{\mathbb{R}}\Sigma M), \\
(g, s\otimes\varphi) &\mapsto \Big(g, \Big(d(\varphi_f\circ\varphi_g^{-1})_0 s\Big)\otimes\varphi\Big),
\end{aligned}
$$

where $(U_{f,\varepsilon}, \varphi_f)$ is a chart of $C^k(M,N)$ as in Theorem 3.4.4.

Proof. Essentially, $\coprod_{f\in C^k(M,N)}\Gamma_{C^k}(f^*TN\otimes_{\mathbb{R}}\Sigma M)$ is the tangent bundle of $C^k(M,N)$ twisted with ΣM:

Pick a chart

$$\varphi_f\colon U_{f,\varepsilon}\to\varphi_f(U_{f,\varepsilon})\subset\Gamma_{C^k}(f^*TN),$$

of $C^k(M,N)$. We then get a local trivialization of $TC^k(M,N)$ given by

$$
\begin{aligned}
\coprod_{g\in U_{f,\varepsilon}} T_gC^k(M,N) &\to U_{f,\varepsilon}\times\Gamma_{C^k}(f^*TN), \\
(g,v) &\mapsto \Big(g, d(\varphi_f)_g v\Big),
\end{aligned}
$$

compare Corollary 4.1.5. Using the continuous linear isomorphism

$$\begin{aligned}T_gC^k(M,N)&\to\Gamma_{C^k}(g^*TN),\\ v&\mapsto(d\varphi_g)_gv,\end{aligned}$$

together with Lemma 4.1.4 we see that

$$\bigsqcup_{f\in C^k(M,N)}\Gamma_{C^k}(f^*TN)\to C^k(M,N)$$

is a C^∞-Banach bundle with local trivializations

$$\begin{aligned}\bigsqcup_{g\in U_{f,\varepsilon}}\Gamma_{C^k}(g^*TN)&\to U_{f,\varepsilon}\times\Gamma_{C^k}(f^*TN),\\ (g,s)&\mapsto\Big(g,d(\varphi_f)_g\left(((d\varphi_g)_g)^{-1}s\right)\Big).\end{aligned}$$

By the chain rule,

$$d(\varphi_f)_g\left(((d\varphi_g)_g)^{-1}s\right)=d(\varphi_f\circ\varphi_g^{-1})_{\varphi_g(g)}s=d(\varphi_f\circ\varphi_g^{-1})_0s,$$

where 0 denotes the zero-section. Using Lemma 4.1.4 and Lemma 4.1.7, we see that

$$\bigsqcup_{f\in C^k(M,N)}\Gamma_{C^k}(f^*TN\otimes_{\mathbb{R}}\Sigma M)\to C^k(M,N)$$

is a C^∞-Banach bundle with local trivializations

$$\begin{aligned}\Psi=\Psi_f\colon\bigsqcup_{g\in U_{f,\varepsilon}}\Gamma_{C^k}(g^*TN\otimes_{\mathbb{R}}\Sigma M)&\to U_{f,\varepsilon}\times\Gamma_{C^k}(f^*TN\otimes_{\mathbb{R}}\Sigma M),\\ (g,s\otimes\varphi)&\mapsto\left(g,\left(d(\varphi_f\circ\varphi_g^{-1})_0s\right)\otimes\varphi\right),\end{aligned}$$

where $s\in\Gamma_{C^k}(g^*TN)$, $\varphi\in\Gamma_{C^k}(\Sigma M)$, and "$\otimes$" is the (pointwise) tensor product of sections. $\square$

4.2 A natural connection on the bundle

Let

$$\mathcal{E} := \bigsqcup_{f \in C^k(M,N)} \Gamma_{C^k}(f^*TN \otimes_{\mathbb{R}} \Sigma M) \to C^k(M,N)$$

be the C^∞-Banach bundle that we constructed in the previous section. We will construct a natural connection on this bundle. In order to do so, we first need to understand differentiability of section of $\mathcal{E}$.

4.2.1 Differentiability of sections

Let

$$U_{f,\varepsilon} = \bigcap_{i=1}^{l} \mathcal{N}^k(f, \varphi_i, \tilde{U}_i, \psi_i, V_i, \overline{U_i}, \varepsilon)$$

be as in Theorem 3.4.4 s.t. additionally for each $i = 1, \ldots, l$ there exists $W_i \subset M$ open, $\overline{W_i} \subset U_i$, $\bigcup_{i=1}^{l} W_i = M$, $\overline{V_i} \subset \tilde{V}_i$, $(\tilde{V}_i, \psi_i)$ is still a chart, there exist local trivializations $(\tilde{U}_i, \Phi_i)$ of $f^*TN \otimes \Sigma M$, and for each $i = 1, \ldots, l$ we have (smooth) local frames

$$\begin{aligned}(e_k^i)_k &\colon \tilde{V}_i \to TN,\\ (\sigma_l^i)_l &\colon U_i \to \Sigma M,\end{aligned}$$

for TN and ΣM, respectively.

Let $S \in \Gamma(\mathcal{E})$[1]. For every $g \in U_{f,\varepsilon}$ and every $i = 1, \ldots, l$ it holds that

$$S(g)|_{W_i} = \sum_{k,l} \lambda_{kl}^i(g)(g^* e_k^i) \otimes \sigma_l^i \tag{4.2.1}$$

for

$$\lambda_{kl}^i \colon U_{f,\varepsilon} \to C^k(\overline{W_i}, \mathbb{R}).$$

Our first goal is to understand differentiability of S in terms of differentiability of the λ_{kl}^i.

Lemma 4.2.1. *Let $r \in \mathbb{N}$. In the situation above, it holds that $S \in \Gamma_{C^r}(\mathcal{E})$ if and only if*

$$\lambda_{kl}^i \in C^r(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R}))$$

for all i, k, l and all $f \in C^k(M,N)$.

---

[1]Given a bundle $\pi\colon E \to M$, we define $\Gamma(E)$ to be the set of mappings $s\colon M \to E$ with $\pi \circ s = id_M$, where s is viewed as a mapping between sets, in particular s does not need to be differentiable or continuous.

For the proof of the lemma, we need two more ingredients.

Lemma 4.2.2. *Let $X\in\Gamma_{C^k}(TN)$ and $\varphi\in\Gamma_{C^k}(\Sigma M)$. Then the mapping*

$$S\colon C^k(M,N)\to\mathcal{E},\qquad g\mapsto(g^*X)\otimes\varphi,$$

is an element of $\Gamma_{C^\infty}(\mathcal{E})$.

Proposition 4.2.3. *Let X be a Banach space, $U\subset X$ open, $V\subset\mathbb{R}^n$ open and bounded, $r,k\in\mathbb{N}$, $f,g\in C^r(U,C^k(\overline{V},\mathbb{R}))$. Then it holds that*

i) $fg\in C^r(U,C^k(\overline{V},Y))$ where $(fg)(u)(v):=f(u)(v)g(u)(v)$ for all $(u,v)\in U\times\overline{V}$.

ii) If $g(u)(v)\neq 0$ for all $(u,v)\in U\times\overline{V}$, then $\frac{f}{g}\in C^r(U,C^k(\overline{V},Y))$ where $(\frac{f}{g})(u)(v):=\frac{f(u)(v)}{g(u)(v)}$ for all $(u,v)\in U\times\overline{V}$.

Proof of Lemma 4.2.1. We have that $S\in\Gamma_{C^r}(\mathcal{E})$ iff for all $f\in C^k(M,N)$ the mapping

$$U_{f,\varepsilon}\to\Gamma_{C^k}(f^*TN\otimes\Sigma M),\qquad g\mapsto(pr_2\circ\Psi_f\circ S)(g)$$

is of class C^r (where Ψ_f is the local trivialization of $\mathcal{E}$ from Corollary 4.1.8). By Proposition 3.4.1 (see also the beginning of its proof), this is the case iff for all $f\in C^k(M,N)$, $i=1,\ldots,l$, the mapping

$$U_{f,\varepsilon}\to\Gamma_{C^k,\overline{W_i}}(f^*TN\otimes\Sigma M),\qquad g\mapsto(pr_2\circ\Psi_f\circ S)(g)|_{W_i}$$

is of class C^r. (For the notation $\Gamma_{C^k,\overline{W}_i}$ see Proposition 3.4.1.) From (4.2.1) it follows that

$$(pr_2\circ\Psi_f\circ S)(g)|_{W_i}=\sum_{k,l}\lambda^i_{kl}(g)\left(d(\exp^{-1}_{f(.)})_{g(.)}(e^i_k(g(.)))\right)\otimes\sigma^i_l(.)$$

We have shown that $S\in\Gamma_{C^r}(\mathcal{E})$ if and only if for all $f\in C^k(M,N)$, $i=1,\ldots,l$, the mapping

$$\begin{aligned}F\colon U_{f,\varepsilon}&\to\Gamma_{C^k,\overline{W_i}}(f^*TN\otimes\Sigma M),\\ g&\mapsto\sum_{k,l}\lambda^i_{kl}(g)\left(d(\exp^{-1}_{f(.)})_{g(.)}(e^i_k(g(.)))\right)\otimes\sigma^i_l(.),\end{aligned}$$

is of class C^r. Next, we proof the following claim:

Claim: For all $f\in C^k(M,N)$ and each i,k,l, the mapping

$$\begin{aligned}F_{ikl}\colon U_{f,\varepsilon}&\to\Gamma_{C^k,\overline{W_i}}(f^*TN\otimes\Sigma M),\\ g&\mapsto\left(d(\exp^{-1}_{f(.)})_{g(.)}(e^i_k(g(.)))\right)\otimes\sigma^i_l(.),\end{aligned}$$

is smooth.

Proof of the claim: Fix f and i,k,l. Since e^i_k is C^k on $\tilde{V}_i$ and $\overline{V_i} \subset \tilde{V}_i$, there exists $X \in \Gamma_{C^k}(TN)$ s.t. $X|_{\overline{V_i}} = e^i_k$. Since σ^i_l is C^k on U_i and $\overline{W_i} \subset U_i$ there exists $\varphi \in \Gamma_{C^k}(\Sigma M)$ s.t. $\varphi|_{\overline{W_i}} = \sigma^i_l$ (see e.g. [3, Lemma 10.12]). In particular, for every $g \in U_{f,\varepsilon}$, it holds that

$$(g^*X \otimes \varphi) \in \Gamma_{C^k}(g^*TN \otimes \Sigma M), \qquad ((g^*X) \otimes \varphi)\,|_{\overline{W_i}} = (g^*e^i_k) \otimes \sigma^i_l.$$

Now the claim follows from Lemma 4.2.2 together with the following commutative diagram

$$\begin{tikzcd} U_{f,\varepsilon} \arrow[r, "g \mapsto g^*X\otimes\varphi"] \arrow[drr, "F_{ikl}"'] & \mathcal{E} \arrow[r, "pr_2 \circ \Psi_f"] & \Gamma_{C^k}(f^*TN \otimes \Sigma M) \arrow[d, "\text{restriction}"] \\ & & \Gamma_{C^k,\overline{W_i}}(f^*TN \otimes \Sigma M) \end{tikzcd}$$

This finishes the proof of the claim.

To finish the proof of Lemma 4.2.1, we show that F is of class C^r if and only if $\lambda^i_{kl} \in C^r(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R}))$ for all i,k,l.

"$\Rightarrow$": We have

$$F(g)(p) = \sum_{k,l} \lambda^i_{kl}(g)(p) F_{ikl}(g)(p)$$

for all $g \in U_{f,\varepsilon}$, $p \in \overline{W_i}$, with $F_{ikl} \in C^\infty(U_{f,\varepsilon}, \Gamma_{C^k,\overline{W_i}}(f^*TN \otimes \Sigma M))$ and $F \in C^r(U_{f,\varepsilon}, \Gamma_{C^k,\overline{W_i}}(f^*TN \otimes \Sigma M))$.

Note that for each $g \in U_{f,\varepsilon}$, $p \in \overline{W_i}$, the $(F_{ikl}(g)(p))_{k,l}$ are a $\mathbb{R}$-basis of $(g^*TN \otimes \Sigma M)_p$.

We now apply the local trivialization Φ_i of $f^*TN \otimes \Sigma M$ to get the following equation in $\mathbb{R}^n$, where $n = \operatorname{rank}(f^*TN \otimes \Sigma M)$:

$$\tilde{F}(g)(p) = \sum_{k,l} \lambda^i_{kl}(g)(p) \tilde{F}_{ikl}(g)(p) \tag{4.2.2}$$

for all $g \in U_{f,\varepsilon}$, $p \in \overline{W_i}$, where $\tilde{F}_{ikl} \in C^\infty(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R}^n))$ and $\tilde{F} \in C^r(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R}^n))$ are defined by

$$\tilde{F}(g)(p) := (pr_2 \circ \Phi_i)(F(g)(p)),$$

$$\tilde{F}_{ikl}(g)(p) := (pr_2 \circ \Phi_i)(F_{ikl}(g)(p)).$$

For each $g \in U_{f,\varepsilon}$, $p \in \overline{W_i}$ we define the invertible $n \times n$-matrix $A(g)(p)$ by

$$A(g)(p) = (\tilde{F}_{ikl}(g)(p))_{kl},$$

i.e., the columns of $A(g)(p)$ are given by the $\tilde{F}_{ikl}(g)(p)$ where k,l vary. Thus (4.2.2) is equivalent to

$$(\lambda^i_{kl}(g)(p))_{kl} = (A(g)(p))^{-1}\, \tilde{F}(g)(p)$$

(where we view $(\lambda^i_{kl}(g)(p))_{kl}$ as an element of $\mathbb{R}^n$ by varying k, l.) By definition, every entry of A can be viewed as an element of $C^\infty(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R}))$. Using Proposition 4.2.3 we conclude that every entry of A^{-1} can be viewed as an element of $C^\infty(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R}))$ and so we get $\lambda^i_{kl} \in C^r(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R}))$.

"$\Leftarrow$": This direction follows from the Leibniz rule (see e.g. [1, 2.4.4 Theorem]) applied to the multilinear and continuous "multiplication map"

$$\begin{aligned} C^k(\overline{W_i}, \mathbb{R}) \times \Gamma_{C^k,\overline{W_i}}(f^*TN \otimes \Sigma M) &\to \Gamma_{C^k,\overline{W_i}}(f^*TN \otimes \Sigma M) \\ (h, s) &\mapsto (p \mapsto h(p)s(p)). \end{aligned}$$

$\square$

4.2.2 Construction of the connection

We start by recalling the following definitions.

Definition 4.2.4. Let $\pi\colon E \to M$ be a C^k-Banach bundle, $k \in \mathbb{N} \cup \{\infty\}$, $1 \leq r \leq k$.

i) A *(C^r-)connection on E* (or *(C^r-)covariant derivative on E*) is a $\mathbb{R}$-bilinear map
$$\nabla\colon \Gamma_{C^1}(E) \times \Gamma_{C^0}(TM) \to \Gamma_{C^0}(E)$$
s.t.

- For all $f \in C^0(M,\mathbb{R})$, $g \in C^1(M,\mathbb{R})$, $\varphi \in \Gamma_{C^1}(E)$, and $X \in \Gamma_{C^0}(TM)$ it holds that $\nabla_{fX}\varphi = f\nabla_X\varphi$, and $\nabla_X(g\varphi) = (L_Xg)\varphi + g\nabla_X\varphi$.
- For all $1 \leq l \leq r$, if $X \in \Gamma_{C^{l-1}}(TM)$, $\varphi \in \Gamma_{C^l}(E)$, then $\nabla_X\varphi \in \Gamma_{C^{l-1}}(E)$. (Note that $\infty - 1 := \infty$.)

ii) A connection ∇ on E is *local*, if for all open sets $U \subset M$ and all $X, Y \in \Gamma_{C^0}(TM)$, $\varphi, \sigma \in \Gamma_{C^1}(E)$ with $X|_U = Y|_U$ and $\varphi|_U = \sigma|_U$ it holds that
$$(\nabla_X\varphi)|_U = (\nabla_Y\sigma)|_U.$$

iii) A connection ∇ on E is *directional pointwise* if for all $X, Y \in \Gamma_{C^0}(TM)$, $\varphi \in \Gamma_{C^1}(E)$, $p \in M$ it holds that, if $X(p) = Y(p)$, then
$$(\nabla_X\varphi)(p) = (\nabla_Y\varphi)(p).$$

iv) Let $c\colon [0,T] \to M$ be of class C^{k-1}. For arbitrary $0 \leq l \leq k$ we denote by $\Gamma_{c,l}(E)$ the space of C^l-sections of E along c, i.e. C^l-maps $\gamma\colon [0,T] \to E$ with $\pi \circ \gamma = c$. A *covariant derivative along c* is a $\mathbb{R}$-linear map
$$\frac{\nabla}{dt}\colon \Gamma_{c,k}(E) \to \Gamma_{c,k-1}(E)$$
s.t. for all $f \in C^k([0,T],\mathbb{R})$, $\gamma \in \Gamma_{c,k}(E)$ it holds, that
$$\frac{\nabla}{dt}(f\gamma) = \left(L_{\frac{\partial}{\partial t}}f\right)\gamma + f\frac{\nabla}{dt}\gamma.$$

Note that we could define covariant derivatives along curves more general (i.e., as maps $\frac{\nabla}{dt}\colon \Gamma_{c,1}(E) \to \Gamma_{c,0}(E)$), but we don't need this for our purposes.

In finite dimensions, every connection is local and directional pointwise. In the case of Banach bundles, this may not always be the case. The crucial point here is the existence of cut off functions [2, p. 202], that is not guaranteed for Banach spaces. In particular, it may not always be the case that a covariant derivative induces a covariant derivative along curves.

If M admits cut off functions, then every covariant derivative on $E\to M$ is local. If M admits cut off functions and the fibers of E are finite dimensional, then every covariant derivative on $E\to M$ is local and directional pointwise. These results can be found in [2, p. 202-203].

Remark 4.2.5. Let $E\to N$ be a smooth vector bundle (in particular, N and E are finite dimensional), M Banach manifold, $f\colon M\to N$ of class C^k, and ∇ a smooth connection on E. Then the pullback-bundle $f^*F\to M$ and the pullback-connection ∇^{f^*E} (which is a C^k-connection) can be defined analogous to the finite dimensional case. To be more precise,

$$\nabla^{f^*E}\colon \Gamma_{C^1}(f^*E)\times\Gamma_{C^0}(TM)\to\Gamma_{C^0}(f^*E)$$

is uniquely defined by the following: Let $\varphi\in\Gamma_{C^1}(f^*E)$, $X\in\Gamma_{C^0}(TM)$, $V\subset N$ open, $s_j\colon V\to E$ local frame of E, $U\subset M$ open, $f(U)\subset V$. Then there exist $\lambda^j\in C^1(U,\mathbb{R})$ s.t. $\varphi=\sum_j\lambda^j f^*s_j$ on U and for each $p\in U$ it holds that

$$\left(\nabla_X^{f^*E}\varphi\right)(p)=\sum_j\left(\left(L_X\lambda^j\right)(p)(f^*s_j)(p)+\lambda^j(p)\left(\nabla_{df_pX(p)}s_j\right)(f(p))\right). \tag{4.2.3}$$

In particular, ∇^{f^*E} is local and pointwise directional. Using (4.2.3) we also see that we are able to calculate $\left(\nabla_Y^{f^*E}\sigma\right)(p)$ for $Y=[c]\in T_pM$ and $\sigma\in\Gamma_{c,k}(f^*E)$.

Lemma 4.2.6. *The evaluation mapping*

$$\begin{aligned}\mathrm{ev}\colon C^k(M,N)\times M&\to N,\\ (f,p)&\mapsto f(p),\end{aligned}$$

is of class C^k, i.e., $\mathrm{ev}\in C^k(C^k(M,N)\times M,N)$.

Proof. Pick a chart $(U_{f,\varepsilon},\varphi_f)$ of $C^k(M,N)$. Define the evaluation mapping

$$\begin{aligned}\mathrm{ev}\colon \varphi_f(U_{f,\varepsilon})\times M&\to U,\\ (s,p)&\mapsto s(p),\end{aligned}$$

where $U\subset f^*TN$ is as in Lemma 3.3.8 iii). Then we have the commutative diagram

$$\begin{tikzcd} U_{f,\varepsilon}\times M \arrow[r, "\mathrm{ev}"] \arrow[d, "\varphi_f\times id"'] & N \\ \varphi_f(U_{f,\varepsilon})\times M \arrow[ur, "(f^*\exp)\circ\mathrm{ev}"'] & \end{tikzcd}$$

where $(f^*\exp)\circ\mathrm{ev}$ is well-defined, since the image of ev is contained in U. Therefore the statement of the lemma follows, once we have shown the following claim.

Claim: Let $E \to M$ be a vector bundle, M closed. Then the evaluation mapping

$$\begin{aligned} \mathrm{ev}\colon \Gamma_{C^k}(E) \times M &\to E, \\ (s,p) &\mapsto s(p), \end{aligned}$$

is of class C^k.

Proof of the claim: Using the notation of Proposition 3.4.1 we get the commutative diagram

$$\begin{tikzcd} \Gamma_{C^k}(E) \times U_i \arrow[rr, "\mathrm{ev}"] \arrow[d, "R_i \times \varphi_i"'] & & E|_{U_i} \\ C^k(\overline{\varphi_i(U_i)}, \mathbb{R}^n) \times \varphi_i(U_i) \arrow[rr] & & \varphi_i(U_i) \times \mathbb{R}^n \arrow[u, "\cong"'] \end{tikzcd}$$

where the lower horizontal arrow is given by $(f,x) \mapsto (x, f(x))$. Now the statement of the claim follows from the following well-known fact: Let $V \subset \mathbb{R}^m$ be open and bounded. Then the evaluation mapping

$$\begin{aligned} C^k(\overline{V}, \mathbb{R}^n) \times V &\to \mathbb{R}^n, \\ (f,x) &\mapsto f(x), \end{aligned}$$

is of class C^k. $\square$

Now assume that N is a closed connected Riemannian manifold and M is a closed spin manifold. We consider the projection

$$\begin{aligned} pr_2\colon C^k(M,N) \times M &\to M, \\ (f,p) &\mapsto p. \end{aligned}$$

Pulling back TN and ΣM along ev and pr_2, respectively, and taking the tensor product afterwards, we get the C^k-Banach bundle

$$\mathcal{F} \to C^k(M,N) \times M$$

given by

$$\mathcal{F} := \mathrm{ev}^* TN \otimes_{\mathbb{R}} pr_2^* \Sigma M.$$

Note that the we take the tensor product of Banach bundles with finite dimensional fibers. (In general, we cannot take the tensor product of two Banach bundles without further ado, because the tensor product of two Banach spaces is not "naturally" a Banach space. In our case, all the fibers are finite-dimensional and therefore we don't need to worry about this.)

Our first observation is that we have an isomorphism of $\mathbb{R}$-vector spaces

$$\begin{aligned} I\colon \Gamma(\mathcal{E}) &\to \Gamma(\mathcal{F}),\\ S &\mapsto ((f,p) \mapsto S(f)(p)). \end{aligned}$$

Now we construct a C^k-connection $\nabla^{\mathcal{F}}$ on $\mathcal{F}$ as follows: first, we pull back the Levi-Civita connection on TN and the spinorial Levi-Civita connection on ΣM by ev and pr_2, respectively (see Remark 4.2.5). We then take the tensor product of these connections, to get a C^k-connection

$$\nabla^{\mathcal{F}}\colon \Gamma_{C^1}(\mathcal{F}) \times \Gamma_{C^0}(T(C^k(M,N)\times M)) \to \Gamma_{C^0}(\mathcal{F})$$

on $\mathcal{F}$.

Let $S \in \Gamma_{C^1}(\mathcal{F})$ and $X \in \Gamma_{C^0}(T(C^k(M,N)\times M))$. Using the notation of the beginning of Section 4.2.1, we have

$$S = \sum_{kl} \mu^i_{kl}(ev^* e^i_k) \otimes (pr_2^* \sigma^i_l)$$

on $U_{f,\varepsilon} \times W_i$ for some $\mu^i_{kl} \in C^1(U_{f,\varepsilon}\times W_i, \mathbb{R})$. Then for each $(g,p) \in U_{f,\varepsilon}\times W_i$, it holds that

$$\begin{aligned} \left(\nabla^{\mathcal{F}}_X S\right)(g,p) = \sum_{kl} &\left(L_X \mu^i_{kl}\right)\left((\mathrm{ev}^* e^i_k) \otimes (pr_2^*\sigma^i_l)\right)(g,p)\\ &+ \mu^i_{kl}(g,p)\left((\nabla^{\mathrm{ev}^*TN}_X \mathrm{ev}^* e^i_k) \otimes (pr_2^*\sigma^i_l)\right)(g,p)\\ &+ \mu^i_{kl}(g,p)\left((\mathrm{ev}^* e^i_k) \otimes (\nabla^{pr_2^*\Sigma M}_X pr_2^*\sigma^i_l)\right)(g,p) \end{aligned}$$

In particular, $\nabla^{\mathcal{F}}$ is local, pointwise directional, and we are able to calculate $\left(\nabla^{\mathcal{F}}_X S\right)(g,p)$ for $X = [c] \in T_{(g,p)}(C^k(M,N)\times M)$ and $S \in \Gamma_{c,k}(\mathcal{F})$.

Arguing as in the finite dimensional case, we see that for any curve $c\colon [0,T] \to C^k(M,N)\times M$ of class C^{k-1}, $\nabla^{\mathcal{F}}$ defines a covariant derivative

$$\frac{\nabla^{\mathcal{F}}}{dt}\colon \Gamma_{c,k}(\mathcal{F}) \to \Gamma_{c,k-1}(\mathcal{F})$$

along c that is uniquely determined by the following:

Let $\gamma \in \Gamma_{c,k}(\mathcal{F})$ and $J \subset [0,T]$ open s.t. $c(J) \subset U_{f,\varepsilon}\times W_i$. Then there exist $\mu^i_{kl} \in C^k(J,\mathbb{R})$ s.t. $\gamma = \sum_{kl} \mu^i_{kl}\left((ev^* e^i_k) \otimes (pr_2^*\sigma^i_l)\right) \circ c$ on J and for each $t \in J$ it holds that

$$\begin{aligned} \left(\frac{\nabla^{\mathcal{F}}}{dt}\gamma\right)(t) = \sum_{kl}\Big(&\left(L_{\frac{\partial}{\partial t}} \mu^i_{kl}\right)(t)\left((ev^* e^i_k) \otimes (pr_2^*\sigma^i_l)\right)(c(t))\\ &+ \mu^i_{kl}(t)\left(\nabla^{\mathcal{F}}_{c'(t)}\left((ev^* e^i_k) \otimes (pr_2^*\sigma^i_l)\right)\right)(c(t))\Big). \end{aligned}$$

Theorem 4.2.7. *Fix $k \in \mathbb{N}$ and write $\mathcal{M} := C^k(M,N)$.*

i) The mapping

$$\nabla \colon \Gamma_{C^1}(\mathcal{E}) \times \Gamma_{C^0}(T\mathcal{M}) \to \Gamma_{C^1}(\mathcal{E}),$$
$$(S,X) \mapsto \nabla_X S := I^{-1}\left(\nabla^{\mathcal{F}}_X\left(I(S)\right)\right),$$

is well-defined and a connection on $\mathcal{E}$. If $X \in \Gamma_{C^0}(T\mathcal{M})$ and $S \in \Gamma_{C^1}(\mathcal{E})$ is given by $S(g)|_{W_i} = \sum_{k,l} \lambda^i_{kl}(g)(g^* e^i_k) \otimes \sigma^i_l$ (in the notation of the beginning of Section 4.2.1), then it holds that

$$(\nabla_X S)(g)(p) = \sum_{kl}\Big(\Big(L_X \lambda^i_{kl}\Big)(g)(p)\Big((g^* e^i_k) \otimes \sigma^i_l\Big)(p)$$
$$+ \lambda^i_{kl}(g)(p)\left(\left(\nabla^{TN}_{d(\mathrm{ev})X} e^i_k\right)(g(p))\right) \otimes \sigma^i_l(p)\Big)$$

for all $(g,p) \in U_{f,\varepsilon} \times W_i$. In particular, ∇ is local, pointwise directional, and we are able to calculate $(\nabla_X S)(g)|_{W_i}$ by only knowing $X(g)$ and $S(c(t))|_{W_i}$ for a curve c with $c(0) = g$ and $c'(0) = X(g)$.

ii) Let $c \colon [0,T] \to \mathcal{M}$ be a smooth curve. Then ∇ induces a covariant derivative

$$\frac{\nabla}{dt} \colon \Gamma_{c,\infty}(\mathcal{E}) \to \Gamma_{c,\infty}(\mathcal{E})$$

along c that is given as follows: Let $\gamma \in \Gamma_{c,\infty}(E)$, $J \subset [0,T]$ open s.t. $c(J) \subset U_{f,\varepsilon}$. Then there exist $\lambda^i_{kl} \in C^\infty(J, C^k(\overline{W_i},\mathbb{R}))$ s.t.

$$\gamma(t)|_{W_i} = \sum_{kl} \lambda^i_{kl}(t)(c(t)^* e^i_k) \otimes \sigma^i_l$$

for all $t \in I$. Then it holds that

$$(\frac{\nabla}{dt}\gamma)(t)(p) = \sum_{kl}\Big(\left(L_{\frac{\partial}{\partial t}} \lambda^i_{kl}\right)(t)(p)\Big((c(t)^* e^i_k) \otimes \sigma^i_l\Big)(p)$$
$$+ \lambda^i_{kl}(t)(p)\left(\nabla_{c'(t)}\left((c(t)^* e^i_k) \otimes \sigma^i_l\right)\right)(c(t))(p)\Big)$$

for all $(t,p) \in J \times U_{f,\varepsilon}$. In particular,

$$\frac{\nabla}{dt}\left(S \circ c\right)(t) = \left(\nabla_{c'(t)} S\right)(c(t))$$

for all $S \in \Gamma_{C^\infty}(\mathcal{E})$, $t \in [0,T]$.

iii) It holds that

$$\left(\frac{\nabla}{dt}\gamma\right)(t)(p) = \left(\frac{\nabla^{\mathcal{F}}}{dt}\left(\gamma(.)(p)\right)\right)(t)$$

where on the left hand side we take the covariant derivative of γ along c and on the right hand side we take the covariant derivative of $\gamma(.)(p)$ along $t \mapsto (c(t),p)$.

iv) Given $\Psi\in\mathcal{E}_{c(0)}$*, we have*

$$\|\left(P^{\mathcal{E},c}\Psi\right)(p)\| = \|\Psi(p)\|$$

where $P^{\mathcal{E},c}$ *denotes the parallel transport in* $\mathcal{E}$ *w.r.t.* ∇ *along* c *and* $\|.\|$ *denotes the pointwise norms, that are induced from the Riemannian metric on* N *and the bundle metric on* ΣM*.*

Proof. i): First we prove the following: For all $1\le r\le\infty$, $S\in\Gamma_{C^r}(\mathcal{E})$ and $X\in\Gamma_{C^{r-1}}(T\mathcal{M})$ it holds that $\nabla_X S\in\Gamma_{C^{r-1}}(\mathcal{E})$.

Step 1: The expression $\nabla^{\mathcal{F}}_X I(S)$ is well-defined.

We prove this step by showing that $I(S)\in\Gamma_{C^1}(\mathcal{F})$. Using Lemma 4.2.1 (and its notation) we have

$$S(g)|_{W_i} = \sum_{k,l}\lambda^i_{kl}(g)(g^*e^i_k)\otimes\sigma^i_l \tag{4.2.4}$$

for every $f\in\mathcal{M}$, $g\in U_{f,\varepsilon}$ and every i, where

$$\lambda^i_{kl}\in C^r(U_{f,\varepsilon},C^k(\overline{W_i},\mathbb{R})).$$

Defining

$$\begin{aligned}\tilde\lambda^i_{kl}\colon U_{f,\varepsilon}\times W_i &\to\mathbb{R},\\ (g,p)&\mapsto\lambda^i_{kl}(g)(p)\end{aligned}$$

We rewrite (4.2.4) to

$$I(S)(g,p) = \sum_{k,l}\tilde\lambda^i_{kl}(g,p)\left((\mathrm{ev}^*e^i_k)\otimes(pr_2^*\sigma^i_l)\right)(g,p) \tag{4.2.5}$$

for all $(g,p)\in U_{f,\varepsilon}\times W_i$. The identity

$$\tilde\lambda^i_{kl} = \mathrm{ev}\circ(\lambda^i_{kl}\times id_{W_i}) \tag{4.2.6}$$

on $U_{f,\varepsilon}\times W_i$ yields $\tilde\lambda^i_{kl}\in C^1(U_{f,\varepsilon}\times W_i,\mathbb{R})$. Therefore (4.2.5) implies that $I(S)\in\Gamma_{C^k}(\mathcal{F})$.

Step 2: $I^{-1}(\nabla^{\mathcal{F}}_X I(S))\in\Gamma_{C^{r-1}}(\mathcal{E})$.

The definition of $\nabla^{\mathcal{F}}$ combined with (4.2.5) yields

$$\begin{aligned}\left(\nabla^{\mathcal{F}}_X I(S)\right)(g,p) &= \sum_{k,l}\left(L_X\tilde\lambda^i_{kl}\right)(g,p)\left((\mathrm{ev}^*e^i_k)\otimes(pr_2^*\sigma^i_l)\right)(g,p)\\ &\quad+\sum_{k,l}\tilde\lambda^i_{kl}(g,p)\left(\left(\nabla^{\mathrm{ev}^*TN}_X(\mathrm{ev}^*e^i_k)\right)\otimes(pr_2^*\sigma^i_l)\right)(g,p)\end{aligned}$$

for all $(g,p) \in U_{f,\varepsilon} \times W_i$. This can be rewritten as

$$\begin{aligned} I^{-1}\left(\nabla^{\mathcal{F}}_X I(S)\right)(g)|_{W_i} = &\sum_{k,l}\left(L_X\tilde{\lambda}^i_{kl}\right)(g,.)\left((g^* e^i_k)\otimes \sigma^i_l\right) \\ &+ \sum_{k,l}\lambda^i_{kl}(g)\left(\left(\nabla^{TN}_{d(\mathrm{ev})X}e^i_k\right)(g(.))\right)\otimes\sigma^i_l. \end{aligned} \tag{4.2.7}$$

We want to use Lemma 4.2.1 on (4.2.7). First we consider the term $\left(L_X\tilde{\lambda}^i_{kl}\right)(g,.)$. Write $X(g) = [c]$ for a curve c in $U_{f,\varepsilon}$. Using (4.2.6), we compute

$$\begin{aligned} \left(L_X\tilde{\lambda}^i_{kl}\right)(g,p) &= L_X\left(\mathrm{ev}\circ(\lambda^i_{kl}\times id_{W_i})\right)(g,p) \\ &= \left.\frac{d}{dt}\right|_{t=0}\left(\mathrm{ev}\circ(\lambda^i_{kl}\times id_{W_i})\circ(c,p)\right)(t) \\ &= \left.\frac{d}{dt}\right|_{t=0}\left(\mathrm{ev}(.,p)\circ\lambda^i_{kl}\circ c\right)(t) \\ &= D(\mathrm{ev}(.,p))_{\lambda^i_{kl}(g)}\left((L_X\lambda^i_{kl})(g)\right) \\ &= \mathrm{ev}(.,p)\circ(L_X\lambda^i_{kl})(g) \\ &= (L_X\lambda^i_{kl})(g)(p) \end{aligned}$$

Combining that with $L_X\lambda^i_{kl} \in C^{r-1}(U_{f,\varepsilon}, C^k(\overline{W_i},\mathbb{R}))$ we get

$$[g \mapsto \left(L_X\tilde{\lambda}^i_{kl}\right)(g,.)] \in C^{r-1}(U_{f,\varepsilon}, C^k(\overline{W_i},\mathbb{R})).$$

Next, we deal with $\left(\nabla^{TN}_{d(\mathrm{ev})X}e^i_k\right)(g(.))$. We assume that $e^i_k = \frac{\partial}{\partial x_k}$ for a chart $(\tilde{V}_i, x)$ of N. (Note that x depends on i, but we don't put this dependency in the notation.) We then have

$$(d(ev))_{(g,p)}X(g) = \sum_k \left(L_X(x^k\circ\mathrm{ev})\right)(g,p)\frac{\partial}{\partial x_k}(g(p))$$

where $x^k\colon \tilde{V}_i \to \mathbb{R}$ is the k-th component of x. It follows that

$$\left(\nabla^{TN}_{d(\mathrm{ev})X}e^i_k\right)(g(p)) = \sum_{j,l}\left(L_X(x^j\circ\mathrm{ev})\right)(g,p)\Gamma^l_{jk}(g(p))(g^*\frac{\partial}{\partial x_l})(p)$$

where Γ^l_{jk} are the Christoffel symbols w.r.t. x. Now step 2 follows from Lemma 4.2.1 applied to (4.2.7), once we have shown the following claim:

Claim: It holds that

$$[g \mapsto \left(L_X(x^j\circ\mathrm{ev})\right)(g,.)\Gamma^l_{jk}(g(.))] \in C^{r-1}(U_{f,\varepsilon}, C^k(\overline{W_i},\mathbb{R})).$$

Proof of the claim: First we prove that

$$[g \mapsto \Gamma^l_{jk}(g(.))] \in C^\infty(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R})).$$

W.l.o.g. we assume that for each i, there exists a local trivialization $(\tilde{V}_i, \Phi_i)$ of TN that is an isometry on the fibers.[2] By $\tilde{\Phi}_i$ we denote the induced local trivialization on f^*TN. We have a commutative diagram

$$\begin{tikzcd} U_{f,\varepsilon} \arrow[r, "g\mapsto\Gamma^l_{jk}\circ g"] \arrow[d, "\varphi_f"'] & C^k(\overline{\varphi_i(W_i)}, \mathbb{R})) \\ \varphi_f(U_{f,\varepsilon}) \arrow[r, "R_i"] & C^k(\overline{\varphi_i(W_i)}, B_\delta(0)) \arrow[u, "\Omega"'] \end{tikzcd}$$

where $R_i(s) = pr_2 \circ \tilde{\Phi}_i \circ s \circ \varphi_i^{-1}$, $\delta > 0$ is chosen as in Lemma 3.3.8, and

$$\begin{aligned} \Omega(h) &= \Gamma^l_{jk} \circ f^*\exp \circ \tilde{\Phi}_i^{-1} \circ (\varphi_i^{-1}, h) \\ &= \Gamma^l_{jk} \circ \exp \circ \Phi_i^{-1} \circ (f \circ \varphi_i^{-1}, h). \end{aligned}$$

By the local Ω-lemma 3.2.4 we see that Ω is smooth and therefore $[g \mapsto \Gamma^l_{jk}(g(.))] \in C^\infty(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R}))$. Now we show that

$$[g \mapsto \left(L_X(x^j \circ \mathrm{ev})\right)(g, .)] \in C^{r-1}(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R})).$$

To that end, we consider the mapping

$$\begin{aligned} F\colon U_{f,\varepsilon} &\to C^k(\overline{W_i}, \mathbb{R}), \\ g &\mapsto (x^k \circ g). \end{aligned}$$

Arguing as above (with the local Ω-Lemma and "Γ^l_{jk}" replaced by "x^k") we see that F is smooth. In particular,

$$[g \mapsto (L_X F)(g)] \in C^{r-1}(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R})).$$

Using the identity

$$L_X(x^k \circ \mathrm{ev})(g, .) = (L_X F)(g)(.)$$

(this can be shown as the above identity regarding the Lie derivatives of $\tilde{\lambda}^i_{kl}$ and λ^i_{kl}) we conclude

$$[g \mapsto \left(L_X(x^j \circ \mathrm{ev})\right)(g, .)] \in C^{r-1}(U_{f,\varepsilon}, C^k(\overline{W_i}, \mathbb{R})).$$

[2] We can put that requirement in the definition of $U_{f,\varepsilon}$ at the beginning of the section, or we use the fact that smoothness is local and choose an appropriate $U_{g,\tilde{\varepsilon}} \subset U_{f,\varepsilon}$ for each $g \in U_{f,\varepsilon}$.

Finally, the claim follows by Proposition 4.2.3 i).

So far we have shown that for all $1 \leq r \leq \infty$, $S \in \Gamma_{C^r}(\mathcal{E})$ and $X \in \Gamma_{C^{r-1}}(T\mathcal{M})$ it holds that $\nabla_X S \in \Gamma_{C^{r-1}}(\mathcal{E})$. The fact that ∇ is a C^∞-connection on $\mathcal{E}$ now follows easily since $\nabla^{\mathcal{F}}$ is a connection on $\mathcal{F}$.

ii): Similar to Lemma 4.2.1 we have that for arbitrary $r \in \mathbb{N} \cup \{\infty\}$ it holds that $\gamma \in \Gamma_{c,r}(\mathcal{E})$ if and only if $\lambda^i_{kl} \in C^r(J, C^k(\overline{W_i}, \mathbb{R}))$. Then we just use the formula for $\left(\frac{\nabla}{dt}\gamma\right)(t)(p)$ as the definition of $\frac{\nabla}{dt}$.

iii): This follows from the local descriptions of $\frac{\nabla}{dt}$, $\frac{\nabla^{\mathcal{F}}}{dt}$, ∇, and $\nabla^{\mathcal{F}}$.

iv): Let $\Psi \in \mathcal{E}_{c(0)}$. From iii) it follows that

$$\left(P^{\mathcal{E},c}\Psi\right)(p) = P^{\mathcal{F},(c,p)}\Psi(p)$$

for all $p \in M$.

Since the Levi-Civita connection and the spinorial Levi-Civita connection are both metric connections, it follows that $\nabla^{\mathcal{F}}$ is a metric connection. In particular,

$$\|P^{\mathcal{F},(c,p)}\Psi(p)\| = \|\Psi(p)\|.$$

This finishes the proof. $\square$

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Chapter 5

The Bourguignon-Gauduchon connection in the semi-Riemannian case

Johannes Wittmann

Abstract Let V be a finite dimensional real vector space. In [2] the authors constructed a natural connection on the bundle $\mathcal{B}V \to \mathcal{M}V$, where $\mathcal{M}V$ denotes the set of positive definite inner products on V, $\mathcal{B}V$ is the set of bases of V, and the fiber over $g \in \mathcal{M}V$ consists of the set of g-orthonormal bases of V. One of the results of [2] is a formula for the curvature of this bundle. We generalize the formula for the curvature to the case were we replace $\mathcal{M}V$ by the set of inner products of a fixed signature. For the computation of the curvature in [2] square roots of certain endomorphisms play an important role. In the case of a general signature, it is a priori not clear that we can still take these square roots. We show that it is still possible, using the specific structure of the endomorphisms in question and a power series ansatz.

5.1 The bundle $\mathcal{B}^+V \to \mathcal{M}_{r,s}V$

Let $V \neq \{0\}$ be a n-dimensional oriented real vector space. Denote by $\mathcal{M}_{r,s}V$ the space of inner products of signature (r,s) on V. (In this chapter an inner product on a vector space V is a symmetric non-degenerate bilinear form on V.)

In the positive definite case $\mathcal{M}_{0,n}V$ is a convex cone in the space of symmetric bilinear forms, i.e., if $x, y \in \mathcal{M}_{0,n}V$, $\lambda > 0$, then $x + y$ and λx are elements of $\mathcal{M}_{0,n}V$. Note that in general $\mathcal{M}_{r,s}V$ is not convex: consider for example the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathcal{M}_{1,1}\mathbb{R}^2$.

Denote by $\mathcal{B}^+V$ the set of oriented bases of V. Formally, $\mathcal{B}^+V$ is the set of orientation preserving bijective linear maps $\mathbb{R}^n \to V$.[1] Given $g \in \mathcal{M}_{r,s}V$ we write $\mathcal{B}_g^+V$ for the set of oriented g-orthonormal bases of V.[2] Note that $\mathcal{B}_g^+V$ is non-empty by e.g. [4, 24. Lemma on p. 50].

Denote by $e_{r,s}$ the standard inner product of signature (r,s) on $\mathbb{R}^n$. Define a map

$$\begin{aligned}\pi = \pi_{r,s} \colon \mathcal{B}^+V &\to \mathcal{M}_{r,s}V,\\ f &\mapsto (f^{-1})^* e_{r,s},\end{aligned}$$

where $f \colon \mathbb{R}^n \to V$ is an oriented basis of V.[3] Note that the fiber of $\pi_{r,s}$ over $g \in \mathcal{M}_{r,s}V$ is $\mathcal{B}_g^+V$, i.e.,

$$\pi_{r,s}^{-1}(g) = \mathcal{B}_g^+V.$$

The special indefinite orthogonal group $\mathrm{SO}_{r,s}$ acts on the fibers $\mathcal{B}_g^+V$ by

$$(b_1, \dots, b_n) \cdot A := (\tilde{b}_1, \dots, \tilde{b}_n) \tag{5.1.1}$$

where $\tilde{b}_j := \sum_{i=1}^n b_i A_{ij}$. In the following proposition we show that this turns $\pi_{r,s}$ into a $\mathrm{SO}_{r,s}$-principal bundle.

Proposition 5.1.1. *$\pi_{r,s}$ is a $\mathrm{SO}_{r,s}$-principal bundle.*

Proof. Choose any oriented basis of V, i.e., an orientation preserving bijective linear map $f_0 \colon \mathbb{R}^n \to V$. The Lie group GL_n^+ acts smoothly and transitively[4] on $\mathcal{M}_{r,s}V$ by

$$g \cdot A := (f_0 A^{-1} f_0^{-1})^* g,$$

where $g \in \mathcal{M}_{r,s}V$, $A \in \mathrm{GL}_n^+$. This turns $\mathcal{M}_{r,s}V$ into a homogeneous space. The isotropy group of $(f_0^{-1})^* e_{r,s}$, which is a closed subgroup of GL_n^+, is given by

$$\begin{aligned}(\mathrm{GL}_n^+)_{(f_0^{-1})^* e_{r,s}} &= \{A \in \mathrm{GL}_n^+ \mid (A^{-1} f_0^{-1})^* e_{r,s} = (f_0^{-1})^* e_{r,s}\}\\ &= \{A \in \mathrm{GL}_n^+ \mid (A^{-1})^* e_{r,s} = e_{r,s}\}\\ &= \mathrm{SO}_{r,s}.\end{aligned}$$

[1] Every oriented basis $(b_1, \dots, b_n)$ of V defines an orientation preserving bijective linear map $f \colon \mathbb{R}^n \to V$ by $f(e_i) := b_i$ where $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$. Conversely, given an orientation preserving bijective linear map $f \colon \mathbb{R}^n \to V$ we get the oriented basis $(f(e_1), \dots, f(e_n)))$ of V.

[2] A basis $(b_1, \dots, b_n)$ of V is g-orthonormal if $g(b_i, b_j) = \varepsilon_i \delta_{ij}$ where $\varepsilon_i = 1$ for $i = 1, \dots, r$ and $\varepsilon_i = -1$ for $i = r+1, \dots, r+s = n$.

[3] Recall that $(f^{-1})^* e_{r,s}(v, w) = e_{r,s}(f^{-1}v, f^{-1}w)$.

[4] Given $g, \tilde{g} \in \mathcal{M}_{r,s}V$ we choose oriented orthonormal bases $f, \tilde{f} \colon \mathbb{R}^n \to V$ of g and $\tilde{g}$ respectively. Then we have $(f^{-1})^* e_{r,s} = g$ and $(\tilde{f}^{-1})^* e_{r,s} = \tilde{g}$. Now we choose $A \in \mathrm{GL}_n^+$ s.t. $A^{-1} = f_0^{-1} \circ \tilde{f}^{-1} \circ f \circ f_0$. Then it holds that $g \cdot A = \tilde{g}$.

The characterization Theorem for homogeneous spaces [3, Theorem 21.18 on p. 567] yields a diffeomorphism

$$I = I_{f_0}\colon \mathrm{GL}_n^+ \big/ \mathrm{SO}_{r,s} \to \mathcal{M}_{r,s}V,$$
$$[A] \mapsto \Big((f_0^{-1})^* e_{r,s}\Big) \cdot A.$$

The projection

$$p\colon \mathrm{GL}_n^+ \to \mathrm{GL}_n^+ \big/ \mathrm{SO}_{r,s}$$

is a $\mathrm{SO}_{r,s}$-principal bundle (the $\mathrm{SO}_{r,s}$-action is given by matrix multiplication). Moreover, we have the linear isomorphism

$$G\colon \mathcal{B}^+V \to \mathrm{GL}_n^+,$$
$$f \mapsto f_0^{-1} \circ f.$$

In particular,

$$\pi_{r,s} = I \circ p \circ G\colon \mathcal{B}^+V \to \mathcal{M}_{r,s}V$$

is a $\mathrm{SO}_{r,s}$-principal bundle whose $\mathrm{SO}_{r,s}$-action is given by (5.1.1). □

Given $k \in \mathrm{Sym}_2 V$, where $\mathrm{Sym}_2 V$ denotes the symmetric bilinear forms on V, and $g \in \mathcal{M}_{r,s}V$ we define the endomorphism $K_g\colon V \to V$ by

$$k(u,v) = g(K_g(u), v)$$

for all $u, v \in V$. (Since g is non-degenerate the linear map $F\colon V \to V^*$, $u \mapsto g(u,.)$, is an isomorphism. Hence $K_g(u) = F^{-1}(k(u,.))$.) The endomorphism K_g is self-adjoint w.r.t. g, i.e., $g(K_g(u), v) = g(u, K_g(v))$ for all $u, v \in V$.

Lemma 5.1.2. *For $k, k' \in \mathrm{Sym}_2 V$, $g \in \mathcal{M}_{r,s}V$, and $t \in \mathbb{R}$ it holds that*

$$K'_{g+tk} = (\mathrm{Id}_V + tK_g)^{-1} \circ K'_g$$

if the above expressions are well-defined.[5]

Proof. First note that since K_g commutes with $(\mathrm{Id}_V + tK_g)$ we also have that K_g

[5] For $|t|$ small, the above expressions are always well-defined since $\mathcal{M}_{r,s}V \subset \mathrm{Sym}_2 V$ is open and the invertible linear maps are open in $\mathrm{End}(V)$.

commutes with $(\mathrm{Id}_V + tK_g)^{-1}$. It holds that

$$\begin{aligned}
&K'_{g+tk} = (\mathrm{Id}_V + tK_g)^{-1} \circ K'_g \\
&\Leftrightarrow k'(u,v) = (g+tk)((\mathrm{Id}_V + tK_g)^{-1} K_g u, v)\ \forall u,v \in V \\
&\Leftrightarrow g(K_g u, v) = (g+tk)((\mathrm{Id}_V + tK_g)^{-1} K_g u, v)\ \forall u,v \in V \\
&\Leftrightarrow g(K_g u, v) = (g+tk)(K_g(\mathrm{Id}_V + tK_g)^{-1} u, v)\ \forall u,v \in V \\
&\Leftrightarrow g(K_g(\mathrm{Id}_V + tK_g)u, v) = (g+tk)(K_g u, v)\ \forall u,v \in V \\
&\Leftrightarrow g((\mathrm{Id}_V + tK_g)K_g u, v) = (g+tk)(K_g u, v)\ \forall u,v \in V \\
&\Leftrightarrow (g+tk)(K_g u, v) = (g+tk)(K_g u, v)\ \forall u,v \in V.
\end{aligned}$$

$\square$

5.2 The Bourguignon-Gauduchon connection

We write

$$\begin{aligned}\mathrm{Sym}_{r,s} &:= \{A \in \mathbb{R}^{n\times n} \mid I_{r,s}A^T = AI_{r,s}\},\\ \mathrm{Asym}_{r,s} &:= \{A \in \mathbb{R}^{n\times n} \mid I_{r,s}A^T = -AI_{r,s}\}.\end{aligned}$$

Recall that under the inclusion $\mathrm{SO}_{r,s} \subset \mathbb{R}^{n\times n}$ we have

$$T_{\mathbb{1}}\mathrm{SO}_{r,s} = \mathrm{Asym}_{r,s}.$$

Let $g \in \mathcal{M}_{r,s}V$ and $f \in \mathcal{B}_g^+V$. Since we have $\mathcal{B}_g^+V \subset \mathcal{B}^+V \subset \mathrm{Lin}(\mathbb{R}^n, V)$, where $\mathrm{Lin}(\mathbb{R}^n, V)$ denotes the linear maps $\mathbb{R}^n \to V$, it is natural to ask how $T_f\mathcal{B}_g^+V$ looks like as a subspace of $\mathrm{Lin}(\mathbb{R}^n, V)$. To that end we consider the inclusion

$$i\colon \mathcal{B}_g^+V \hookrightarrow \mathrm{Lin}(\mathbb{R}^n, V)$$

and show the following lemma.

Lemma 5.2.1.

$$(di)_fT_f\mathcal{B}_g^+V = \{h \in \mathrm{Lin}(\mathbb{R}^n, V) \mid f^{-1}\circ h \in \mathrm{Asym}_{r,s}\}$$

Hence the tangent space of $\mathcal{B}_g^+V$ *at* f *consists of those elements in* $\mathrm{Lin}(\mathbb{R}^n, V)$ *whose matrix relative to the basis* f *is in* $\mathrm{Asym}_{r,s}$.

Remark 5.2.2. To keep the notation simple we don't distinguish between linear maps $\mathbb{R}^n \to \mathbb{R}^n$ and matrices (elements of $\mathbb{R}^{n\times n}$). We identify them via the standard basis of $\mathbb{R}^n$.

Proof of Lemma 5.2.1. We have an isomorphism

$$\begin{aligned}F\colon \mathrm{Lin}(\mathbb{R}^n, V) &\to \mathbb{R}^{n\times n},\\ \tilde f &\mapsto f^{-1}\circ \tilde f,\end{aligned}$$

which restricts to an isomorphism $F\colon \mathcal{B}_g^+V \to \mathrm{SO}_{r,s}$. We also have the inclusion $j\colon \mathrm{SO}_{r,s} \hookrightarrow \mathbb{R}^{n\times n}$. We get the commutative diagram

$$\begin{tikzcd} T_f\mathcal{B}_g^+V \arrow[r, "(di)_f"] \arrow[d, "(dF)_f"'] & \mathrm{Lin}(\mathbb{R}^n, V) \arrow[d, "F"] \\ T_{\mathbb{1}}\mathrm{SO}_{r,s} \arrow[r, "(dj)_{\mathbb{1}}"] & \mathbb{R}^{n\times n} \end{tikzcd}$$

Hence,

$$\begin{aligned}(di)_fT_f\mathcal{B}_g^+V &= F^{-1}\Big((dj)_{\mathbb{1}}T_{\mathbb{1}}\mathrm{SO}_{r,s}\Big)\\ &= F^{-1}\mathrm{Asym}_{r,s}\\ &= \{h \in \mathrm{Lin}(\mathbb{R}^n, V) \mid f^{-1}\circ h \in \mathrm{Asym}_{r,s}\}.\end{aligned}$$

$\square$

We set

$$T_f^{\mathrm{hor}}\mathcal{B}^+V := \{h \in \mathrm{Lin}(\mathbb{R}^n, V) \mid f^{-1} \circ h \in \mathrm{Sym}_{r,s}\}.$$

Then we have that

$$T_f\mathcal{B}^+V = \mathrm{Lin}(\mathbb{R}^n, V) = T_f\mathcal{B}_g^+V \oplus T_f^{\mathrm{hor}}\mathcal{B}^+V$$

and this defines a connection on $\pi_{r,s}$, the so-called *Bourguignon-Gauduchon connection.*

5.3 The curvature of $\mathcal{B}^+V \to \mathcal{M}_{r,s}V$

Definition 5.3.1 (Curvature of a fiber bundle)**.** Let $\pi\colon E \to M$ be a fiber bundle with connection. For $X \in \Gamma(TM)$ we denote the horizontal lift of X by $X^* \in \Gamma(E)$. Let $VE \to E$ be the vertical tangent bundle of E. The curvature of E

$$\Omega \in \Gamma(\bigwedge^2 \pi^* T^*M \otimes VE)$$

is defined by

$$\Omega(X,Y) := \Omega_{X,Y} := [X,Y]^* - [X^*,Y^*]$$

for $X, Y \in \Gamma(TM)$. (If $e \in E$, then $\Omega(X,Y)(e) \in T_e^{\mathrm{vert}}E$.)

This definition can be found in e.g. [1, p. 21].

For the computation of the Lie bracket $[.,.]$ the following remark will be useful.

Remark 5.3.2. Let M be a manifold. Let $X, Y \in \Gamma(TM)$ and $p \in M$. Define a point $c(s,t) \in M$ as follows: first follow the integral curve of X with initial value p for time t. From there follow the integral curve of Y for time s. From there follow the integral curve of X backwards for time t. From there follow the integral curve of Y backwards for time s. We reached a point in M which we call $c(s,t)$.

It holds that

$$2[X,Y]|_p = \frac{d^2}{dt^2}\bigg|_{t=0} c(t,t),$$

c.f. [5, p. 162]. The chain rule yields

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} c(t,t) &= \frac{d^2}{dt^2}\bigg|_{t=0} c(0,t) + 2\frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} c(s,t) + \frac{d^2}{ds^2}\bigg|_{s=0} c(s,0)\\ &= 2\frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} c(s,t)\end{aligned}$$

hence,

$$[X,Y]|_p = \frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0} c(s,t).$$

In the following we want to compute the curvature of $\pi_{r,s}$ w.r.t. the Bourguignon-Gauduchon connection. To that end we choose

$$g_0 \in \mathcal{M}_{r,s}V$$

and

$$k, k' \in T_{g_0}\mathcal{M}_{r,s}V = \mathrm{Sym}_2 V.$$

We think of k and k' as vector fields on $\mathcal{M}_{r,s}V$ as follows. We have $k \in \Gamma(T\mathrm{Sym}_2 V)$ via

$$\begin{aligned} k\colon \mathrm{Sym}_2 V &\to T\mathrm{Sym}_2 V, \\ x &\mapsto (x, k). \end{aligned}$$

By restriction (and recalling that $\mathcal{M}_{r,s}V \subset \mathrm{Sym}_2 V$ is open) we have $k \in \Gamma(T\mathcal{M}_{r,s}V)$ via

$$\begin{aligned} k\colon \mathcal{M}_{r,s}V &\to T\mathcal{M}_{r,s}V = \mathcal{M}_{r,s}V \times \mathrm{Sym}_2 V, \\ x &\mapsto (x, k). \end{aligned}$$

Analogously for k'.

Lemma 5.3.3. *The horizontal lift*

$$k^*\colon \mathcal{B}^+V \to T\mathcal{B}^+V$$

of k is given by

$$k^*(f) = -\frac{1}{2} K_{\pi(f)} \circ f$$

for all $f \in \mathcal{B}^+V$. Let $g \in \mathcal{M}_{r,s}V$. Then the integral curve of k^* with initial value $f_0 \in \mathcal{B}^+_g V$ is given by

$$t \mapsto (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} f_0$$

for $|t|$ small, and for these t it holds that $(\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} f_0 \in \mathcal{B}^+_{g+tk}V$.

Proof. Let $f \in \mathcal{B}^+V$ and $\pi(f) =: g$. By definition we have

$$k^*(f) = \left(d\pi|_{T^{\mathrm{hor}}_f \mathcal{B}^+V}\right)^{-1} k(g).$$

Therefore we have to show

$$d\pi|_{T^{\mathrm{hor}}_f \mathcal{B}^+V}\left(-\frac{1}{2} K_g \circ f\right) = k(g). \tag{5.3.1}$$

To make sure that the left hand side of this equation is well-defined we first show

$$K_g \circ f \in T^{\mathrm{hor}}_f \mathcal{B}^+V. \tag{5.3.2}$$

Note that $f^{-1} \circ K_g \circ f \in \mathrm{Sym}_{r,s}V$ if $f^{-1} \circ K_g \circ f$ is self-adjoint w.r.t. $e_{r,s}$. The latter follows from

$$\begin{aligned} e_{r,s}(f^{-1}K_g f(v), w) &= e_{r,s}(f^{-1}K_g f(v), f^{-1}f(w)) \\ &= g(K_g f(v), f(w)) \\ &= k(f(v), f(w)) \\ &= k(f(w), f(v)) \\ &= \dots \\ &= e_{r,s}(f^{-1}K_g f(w), v) \\ &= e_{r,s}(v, f^{-1}K_g f(w)), \end{aligned}$$

and we have shown (5.3.2). Before we show (5.3.1) and the statement about the integral curves we have to fix some notation and give some definitions regarding powers of endomorphisms.[6]

Given $\alpha \in \mathbb{C}$ and $g \in \mathcal{M}_{r,s}V$ we set

$$(\mathrm{Id}_V + tK_g)^\alpha := \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n K_g^n. \tag{5.3.3}$$

We view the right hand side as the (limit of the) Banach space valued series of functions

$$h_n(t) := \binom{\alpha}{n} t^n K_g^n \in \mathrm{End}(V)$$

where we turn $\mathrm{End}(V)$ into a Banach space as follows: given any $\tilde{f} \in \mathcal{B}^+V$, we have an isomorphism $I\colon \mathrm{End}(V) \to \mathbb{R}^{n\times n}$, $I(x) := \tilde{f}^{-1} \circ x \circ \tilde{f}$. Choosing the Frobenius norm on $\mathbb{R}^{n\times n}$ and using I we turn $\mathrm{End}(V)$ into a Banach space and even into a Banach algebra. Then series on the right hand side of (5.3.3) converges uniformly for $|t| < \|K_g\|^{-1}$ by using the (Banach space valued) Weierstrass M-test and comparing the series to the binomial series

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha$$

which converges (absolutely) for $|x| < 1$ and any $\alpha \in \mathbb{C}$.

In the following we will need that for any $\alpha, \beta \in \mathbb{C}$ we have

$$(\mathrm{Id}_V + tK_g)^\alpha (\mathrm{Id}_V + tK_g)^\beta = (\mathrm{Id}_V + tK_g)^{\alpha+\beta}. \tag{5.3.4}$$

This follows from Vandermonde's identity $\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} = \binom{\alpha+\beta}{n}$, $\alpha, \beta \in \mathbb{C}$, $n \in \mathbb{N}$.

We will also need to differentiate (5.3.3). To that end we formally differentiate

[6]We will be mainly interested in taking square roots of endomorphisms of the form $\mathrm{Id}_V + tK_g$ for $g \in \mathcal{M}_{r,s}V$. Let us assume that we are in the positive definite case, i.e., $g \in \mathcal{M}_{n,0}$. Then K_g is diagonalizable since it is self-adjoint w.r.t. g, and g is positive definite. Hence $\mathrm{Id}_V + tK_g$ is diagonalizable and has positive eigenvalues for $|t|$ small. Therefore we can take the square root of $\mathrm{Id}_V + tK_g$. In the general case $g \in \mathcal{M}_{r,s}V$ this argument no longer works and we have use a different approach, namely power series.

the right hand side of (5.3.3) w.r.t. t term by term and get

$$\begin{aligned}\sum_{n=1}^{\infty}\binom{\alpha}{n}nt^{n-1}K_g^n &= \left(\sum_{n=1}^{\infty}\binom{\alpha}{n}nt^{n-1}K_g^{n-1}\right)K_g\\ &= \left(\sum_{n=0}^{\infty}\binom{\alpha}{n+1}(n+1)t^nK_g^n\right)K_g\\ &= \alpha\left(\sum_{n=0}^{\infty}\binom{\alpha-1}{n}t^nK_g^n\right)K_g\\ &= \alpha(\mathrm{Id}_V+tK_g)^{\alpha-1}K_g\end{aligned}$$

where we used $\binom{\alpha}{n+1}=\frac{\alpha}{n+1}\binom{\alpha-1}{n}$. Hence the series converges uniformly, the differentiation is justified, and we have shown

$$\frac{d}{dt}(\mathrm{Id}_V+tK_g)^{\alpha}=\alpha(\mathrm{Id}_V+tK_g)^{\alpha-1}K_g. \tag{5.3.5}$$

Now we show (5.3.1). To that end we consider the curve

$$\begin{aligned}c\colon(-\varepsilon,\varepsilon)&\to\mathcal{B}^+V,\\ t&\mapsto(\mathrm{Id}_V+tK_g)^{-\frac{1}{2}}f.\end{aligned}$$

First we need to check that c is well-defined for small $\varepsilon>0$, i.e., we need to verify $(\mathrm{Id}_V+tK_g)^{-\frac{1}{2}}f\in\mathcal{B}^+V$. Applying (5.3.4) for $\alpha=-\frac{1}{2}$, $\beta=\frac{1}{2}$ and vice versa we see that $(\mathrm{Id}_V+tK_g)^{-\frac{1}{2}}$ is invertible with inverse $(\mathrm{Id}_V+tK_g)^{\frac{1}{2}}$. Since

$$\det\colon\mathrm{End}(V)\to\mathbb{R}$$

is continuous and $\det(c(0))=1>0$ we have that for $|t|$ small it holds that $\det(c(t))>0$. Hence c is well-defined.

By (5.3.5) we have

$$c'(0)=-\frac{1}{2}K_g\circ f.$$

Since all the K_g^n are self-adjoint w.r.t. g we have that $(\mathrm{Id}_V+tK_g)^{\alpha}$ is self-adjoint w.r.t. g. Hence

$$\begin{aligned}(\pi\circ c)(t)(v,w)&=e_{r,s}(c(t)^{-1}v,c(t)^{-1}w)\\ &=e_{r,s}(f^{-1}(\mathrm{Id}_V+tK_g)^{\frac{1}{2}}v,f^{-1}(\mathrm{Id}_V+tK_g)^{\frac{1}{2}}w)\\ &=g((\mathrm{Id}_V+tK_g)^{\frac{1}{2}}v,(\mathrm{Id}_V+tK_g)^{\frac{1}{2}}w)\\ &=g((\mathrm{Id}_V+tK_g)v,w)\\ &=g(v,w)+tg(K_gv,w)\\ &=g(v,w)+tk(v,w).\end{aligned}$$

In particular

$$(d\pi)_f(-\frac{1}{2}K_g \circ f) = (d\pi)_f c'(0) = \frac{d}{dt}\bigg|_{t=0} (\pi \circ c)(t) = k = k(g)$$

and we have shown (5.3.1).

It remains to show the statement about the integral curves of k^*. To that end we consider again

$$c(t) := (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} f_0$$

where $f_0 \in \mathcal{B}^+_g V$. We will need that

$$K_g \circ (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} = (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} \circ K_g$$

which follows follows from (5.3.5). Combining this with Lemma 5.1.2 yields

$$\begin{aligned}
k^*(c(t)) &= -\frac{1}{2} K_{(\pi \circ c)(t)} \circ c(t) \\
&= -\frac{1}{2} K_{g+tk} \circ (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} \circ f_0 \\
&= -\frac{1}{2} (\mathrm{Id}_V + tK_g)^{-1} \circ K_g \circ (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} \circ f_0 \\
&= -\frac{1}{2} (\mathrm{Id}_V + tK_g)^{-1} \circ (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} \circ K_g \circ f_0 \\
&= -\frac{1}{2} (\mathrm{Id}_V + tK_g)^{-\frac{3}{2}} \circ K_g \circ f_0 \\
&= c'(t).
\end{aligned}$$

This finishes the proof of the lemma. $\square$

Theorem 5.3.4. *Let $g \in \mathcal{M}_{r,s}V$, $f \in \mathcal{B}^+_g V$, and $k, k' \in T_g\mathcal{M}_{r,s}V = \mathrm{Sym}_2(V)$. Then it holds that*

$$\Omega_{k,k'}(f) = -\frac{1}{4}[K_g, K'_g] \circ f$$

where $[.,.]$ denotes the commutator of endomorphisms.

Note that since K_g and K'_g are self-adjoint w.r.t. g, we get

$$e_{r,s}(f^{-1}[K_g, K'_g]fv, w) = -e_{r,s}(v, f^{-1}[K_g, K'_g]fw),$$

hence $f^{-1}[K_g, K'_g]f \in \mathrm{ASym}_{r,s}$ and Lemma 5.2.1 yields that $-\frac{1}{4}[K_{g_0}, K'_{g_0}]f$ lies indeed in the vertical tangent space $T_f\mathcal{B}^+_g V$.

Proof of Theorem 5.3.4. We view k and k' as vector fields of $\mathcal{M}_{r,s}V$ as before. In particular the Lie bracket of k and k' vanishes, $[k, k'] = 0$. Hence

$$\Omega_{k,k'}(f) = -[k^*, (k')^*]|_f.$$

As described above, the Lie bracket of the horizontal Lifts is given by

$$[k^*, (k')^*]|_f = \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} c(s,t)$$

where, due to Lemma 5.3.3, $c(s,t)$ is given by

$$\begin{aligned} c(s,t) =& (\mathrm{Id}_V - sK'_{g+sk'})^{-\frac{1}{2}} \circ (\mathrm{Id}_V - tK_{g+tk+sk'})^{-\frac{1}{2}} \circ (\mathrm{Id}_V + sK'_{g+tk})^{-\frac{1}{2}} \circ \\ & (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} \circ f. \end{aligned}$$

We set

$$\begin{aligned} D_s &:= (\mathrm{Id}_V - sK'_{g+sk'})^{-\frac{1}{2}}, \\ C_{s,t} &:= (\mathrm{Id}_V - tK_{g+tk+sk'})^{-\frac{1}{2}}, \\ B_{s,t} &:= (\mathrm{Id}_V + sK'_{g+tk})^{-\frac{1}{2}}, \\ A_t &:= (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}}, \end{aligned}$$

so that $c(s,t) = D_s(C_{s,t}(B_{s,t}(A_t(f))))$ and

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} c(s,t) &= \frac{d}{ds}\bigg|_{s=0} D_s(C_{0,t}(B_{0,t}(A_t(f)))) \\ &\quad + \frac{d}{ds}\bigg|_{s=0} D_0(C_{s,t}(B_{0,t}(A_t(f)))) \\ &\quad + \frac{d}{ds}\bigg|_{s=0} D_0(C_{0,t}(B_{s,t}(A_t(f)))) \\ &=: X(t) + Y(t) + Z(t). \end{aligned}$$

Using (5.3.5) we have

$$\begin{aligned} X(t) &= \frac{1}{2} K'_g (\mathrm{Id}_V - tK_{g+tk})^{-\frac{1}{2}} (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} f, \\ Y(t) &= \frac{d}{ds}\bigg|_{s=0} (\mathrm{Id}_V - tK_{g+tk+sk'})^{-\frac{1}{2}} (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} f, \\ Z(t) &= -\frac{1}{2} (\mathrm{Id}_V - tK_{g+tk})^{-\frac{1}{2}} K'_{g+tk} (\mathrm{Id}_V + tK_g)^{-\frac{1}{2}} f. \end{aligned}$$

Moreover,

$$\frac{d}{dt}\bigg|_{t=0} X(t) = 0.$$

Using Lemma 5.1.2 we further get

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} Z(t) &= -\frac{1}{2}\bigg(\frac{d}{dt}\bigg|_{t=0}(\mathrm{Id}_V - tK_{g+tk})^{-\frac{1}{2}}K'_g f \\
&\qquad + \frac{d}{dt}\bigg|_{t=0}K'_{g+tk}f + \frac{d}{dt}\bigg|_{t=0}K'_g(\mathrm{Id}_V + tK_g)^{-\frac{1}{2}}f\bigg) \\
&= -\frac{1}{2}\bigg(\frac{d}{dt}\bigg|_{t=0}(\mathrm{Id}_V - tK_{g+tk})^{-\frac{1}{2}}K'_g f \\
&\qquad + \frac{d}{dt}\bigg|_{t=0}(\mathrm{Id}_V + tK_g)^{-1}K'_g f \\
&\qquad + \frac{d}{dt}\bigg|_{t=0}K'_g(\mathrm{Id}_V + tK_g)^{-\frac{1}{2}}f\bigg) \\
&= -\frac{1}{2}\bigg(0 + \frac{1}{2}K_gK'_gf - K_gK'_gf - \frac{1}{2}K'_gK_gf\bigg) \\
&= \frac{1}{4}K_gK'_gf + \frac{1}{4}K'_gK_gf.
\end{aligned}$$

For $Y(t)$ we calculate

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} Y(t) &= \frac{d}{dt}\bigg|_{t=0}\frac{d}{ds}\bigg|_{s=0}(\mathrm{Id}_V - tK_{g+tk+sk'})^{-\frac{1}{2}}(\mathrm{Id}_V + tK_g)^{-\frac{1}{2}}f \\
&= \frac{d}{ds}\bigg|_{s=0}\frac{d}{dt}\bigg|_{t=0}(\mathrm{Id}_V - tK_{g+tk+sk'})^{-\frac{1}{2}}(\mathrm{Id}_V + tK_g)^{-\frac{1}{2}}f \\
&= \frac{d}{ds}\bigg|_{s=0}\bigg(\frac{1}{2}K_{g+sk'}f + 0 - \frac{1}{2}K_gf\bigg) \\
&= \frac{1}{2}\frac{d}{ds}\bigg|_{s=0}K_{g+sk'}f \\
&= \frac{1}{2}\frac{d}{ds}\bigg|_{s=0}(\mathrm{Id}_V + sK'_g)^{-1}K_gf \\
&= -\frac{1}{2}K'_gK_gf.
\end{aligned}$$

(We could also calculate $\frac{d}{dt}|_{t=0}Y(t)$ by using Lemma 5.1.2 instead of Schwarz's theorem, but then the calculation gets considerably longer.) Putting everything together we get

$$\Omega_{k,k'}(f) = -(0 - \frac{1}{2}K'_gK_gf + \frac{1}{4}K_gK'_gf + \frac{1}{4}K'_gK_gf) = -\frac{1}{4}[K_g, K'_g] \circ f.$$

□

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