Surface Diffusion Flow of Triple Junction Clusters in Higher Space Dimensions

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Abstract

We study the evolution of double bubbles driven by the surface diffusion flow. At the triple junction we use boundary conditions derived by Garcke and Novick-Cohen in [32] in the case of curves. These are concurrency of the triple junction, Young's law, that fixes the angles at which the three surfaces meet, continuity conditions for the chemical potentials and balance of flux conditions. In [32], Garcke and Novick-Cohen showed also short time existence in a Hölder setting and Arab proved in [5] stability of planar double bubbles moving due to surface diffusion flow. In this work, we generalize these results to arbitrary space dimensions. Hereby, we will first apply our techniques to closed hypersurfaces to illustrate them. The results for this situations were already proven by Escher, Mayer and Simonett in [23] but with different methods.
For the short time existence result we consider reference triple junction clusters for which each hypersurface is a submanifold of $\mathbb{R}^{n+1}$ of class $C^{5+\alpha}$. We then show that for triple junction clusters that can be described as graphs over the reference frame with a combination of a height function sufficiently small in the $C^{4+\alpha}$-norm and a tangential part, which is given as function in the height function, there exists a solution in the parabolic Hölder space $C^{4+\alpha,1+\frac{\alpha}{4}}$. To prove this we reduce the problem via direct mapping to a fourth order, parabolic partial differential equation on the reference frame. Hereby, the tangential part will contribute non-local terms of highest order. We then linearise the problem around the reference cluster and firstly consider only the highest order terms. For the reduced system we show existence of weak solutions with a Galerkin approach. Afterwards, we localize the equations both around points in the interior of the hypersurfaces and on the triple junction. For this problem we get well-posedness in a $C^{4+\alpha,1+\frac{\alpha}{4}}$-setting using classical results from Ladyzenskaja, Solonnikov and Uralceva, cf. [38]. With compactness arguments we then identify the weak solution locally as limit of solutions of the localized problem and thus get $C^{4+\alpha,1+\frac{\alpha}{4}}$-regularity for the weak solutions. Using perturbation techniques we conclude this result also for the complete linear problem. Finally, we get our existence result for the non-linear problem using a contraction mapping argument where technical difficulties arise due to the non-local tangential part and the fully non-linear angle conditions. Uniqueness of solutions remains an open problem.
In the second part of the work we show that if the reference surface is a stationary double bubble then there is a $\sigma > 0$ such that for all initial data with $C^{4+\alpha}$-norm less than σ the solution constructed above exists globally in time and converges to another stationary double bubble. This is done by verifying a Łojasiewicz-Simon gradient inequality for the surface area. During this the non-local tangential part causes crucial problems and has to be replaced by a local one. The proof of the gradient inequality itself uses then the results of Chill, see [13, Corollary 3.11]. Afterwards, we need to show parabolic regularization of the flow using the parameter trick to get bounds in the $C^{k,0}$-norm for arbitrary large k. With this the proof of stability can be carried out applying standard arguments.

Zusammenfassung

Wir betrachten Doppel-Blasen, die durch den Oberflächendiffusionsfluss evolviert werden. Auf der Tripellinie verwenden wir die Randbedingungen, die von Garcke und Novick-Cohen in [32] im Kurvenfall hergeleitet wurden. Dabei handelt es sich um die Erhaltung der Tripellinie, das Youngsche Gesetz, welches die Winkel, in denen die drei Flächen auf einander treffen, festlegt, Stetigkeitsbedingungen für die chemischen Potentiale und die Gleichheit der Ableitungen der mittleren Krümmungen in Richtung der äußeren Konormalen. Garcke und Novick-Cohen zeigten in [32] außerdem Kurzeitexistenz in Hölderräumen und in [5] wurde von Arab Stabilität planarer Doppel-Blasen, die sich auf Grund von Oberflächendiffusion bewegen, gezeigt. In unserer Arbeit verallgemeinern wir diese Resultate auf beliebige Raumdimensionen. Wir wenden unsere Methoden zuerst auf geschlossene Oberflächen an,

um deren Funktionsweise zu erklären. Die Aussagen wurden für diesen Fall von Escher, Mayer und Simonett in [23] mit anderen Techniken gezeigt.
Für die Kurzeitexistenz betrachten wir Referenzcluster, bei denen jede einzelne Oberfläche eine Untermannigfaltigkeit des $\mathbb{R}^{n+1}$ mit Regularität $C^{5+\alpha}$ ist. Wir zeigen, dass für alle Triplelinien-Cluster, die sich mittels einer Höhenfunktion, die klein genug in der $C^{4+\alpha}$-Norm ist, und eines Tangentialteils, der als Funktion in der Höhenfunktion gegeben ist, als Graph über dem Referenzcluster schreiben lassen, eine Lösung in dem parabolischen Hölderraum $C^{4+\alpha,1+\frac{\alpha}{4}}$ existiert. Für den Beweis reduzieren wir das Problem zu einer skalaren, parabolischen, partiellen Differentialgleichung vierter Ordnung auf dem Referenzcluster. Dabei liefert der Tangentialteil nichtlokale Terme höchster Ordnung. Danach linearisieren wir die Gleichungen im Referenzcluster und betrachten anfangs nur die Terme höchster Ordnung. Für dieses Problem zeigen wir die Existenz schwacher Lösungen mit einem Galerkinansatz. Danach lokalisieren wir das Problem sowohl um Punkte im Inneren der Flächen als auch um Punkte auf der Tripellinie. Für die Lokalisierung erhalten wir Wohlgestelltheit in $C^{4+\alpha,1+\frac{\alpha}{4}}$ durch Anwendung der Resultate von Ladyschenskaja, Solonnikov und Uralceva, siehe [51]. Mit einem Kompaktheitsargument identifizieren wir die schwache Lösung lokal als Grenzwert von Lösungen des lokalisierten Problems und erhalten damit auch $C^{4+\alpha,1+\frac{\alpha}{4}}$-Regularität für die schwache Lösunge. Durch ein Störungsargument folgern wir hieraus das gleiche Resultate auch für das komplette linearisierte Problem. Schließlich erhalten wir das Existenzresultat für das nichtlineare Problem mittels des Banachschen Fixpunktsatzes. Hierbei entstehen technische Schwierigkeiten durch den nichtlokalen Tangentialteil und die voll-nichtlinearen Winkelbedingungen. Eindeutigkeit für die Lösung des geometrischen Flusses bleibt ein offenes Problem.
Im zweiten Teil der Arbeit zeigen wir, dass es für Referenzcluster, die stationäre Doppel-Blasen sind, ein $\sigma > 0$ gibt, sodass für alle Anfangsdaten mit einer $C^{4+\alpha}$-Norm kleiner oder gleich σ die gefundene Lösung global in der Zeit existiert und gegen eine andere stationäre Doppel-Blase konvergiert. Hierbei nutzen wir einen Ansatz mit einer Łojasiewicz-Simon Gradientenungleichung für die Oberflächenenergie. Bei deren Beweis entpuppt sich der nichtlokale Tangentialteil als kritisches Problem, weshalb er durch einen lokalen ersetzt werden muss. Die Ungleichung selbst kann dann mit den Resultaten von Chill, siehe [13, Corollary 3.11], gezeigt werden. Danach muss parabolische Regularisierung des Flusses mit Hilfe des Parametertricks gezeigt werden, um Schranken in der $C^{k,0}$-Norm für beliebig große k zu zeigen. Mit diesen ist die Stabilitätsanalyse mit Standardargumenten möglich.

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1

Introduction

A geometric evolution equation is a law that either describes the evolution of a geometric object or a geometric quantity of a fixed object. These kind of problems have both motivations from applications in natural sciences and mathematics. With such equations one can describe for example crystal growth (see e.g. [8]), two-phase flows of two mixed liquids (see e.g. [47]), and flame propagation (see e.g. [50]). Also, they are used in image analysis, see e.g. [6]. Another very prominent application of geometric flows in mathematics is given by Perelman's proof of the Poincaré conjecture. In his work [46], the author used the Ricci flow together with so called surgery techniques to give a complete topological characterization of simply connected, closed 3-manifolds. This shows how broad the possibilities in this subject are.

For this thesis we are interested in evolution laws that describe evolution of geometric objects. Typical examples for such are the mean curvature flow

$$V_{\Gamma(t)} = H_{\Gamma(t)}, \tag{MCF}$$

the surface diffusion flow,

$$V_{\Gamma(t)} = -\Delta_{\Gamma(t)} H_{\Gamma(t)}, \tag{SDF}$$

and the Wilmore flow

$$V_{\Gamma(t)} = -\Delta_{\Gamma(t)} H_{\Gamma(t)} - H_{\Gamma(t)} |II_{\Gamma(t)}|^2 + \frac{1}{2} H^3_{\Gamma(t)}. \tag{WF}$$

Hereby, $\Gamma(t)$ denotes an evolving hypersurface, $V_{\Gamma(t)}$ its normal velocity, $H_{\Gamma(t)}$ its mean curvature, $\Delta_{\Gamma(t)}$ the Laplace-Beltrami operator and $II_{\Gamma(t)}$ the second fundamental form. From these evolution laws we see directly a general problem of these flows. They only fix the normal velocity and so these problems are degenerated as motion laws for the particles of the surface. So, to get a well-posed problem of the geometric object one has to restrict freedoms in the tangential motions. In [44, Proposition 1.3.4] it is proven that for manifolds without boundary solutions of the mean curvature flow with different tangential parts are equivalent up to reparametrisation. This result can be generalized to general flows but things are more complicated when manifolds with boundaries are involved. Near the boundary we cannot completely ignore tangential parts as a reparametrisation has to map boundary points to boundary points. Before we now move to our problem we want to mention that writing these problems in local coordinates results in quasi-linear systems which makes them more difficult then they look at first glance.

In this thesis we study motion by surface diffusion flow. This law was first proposed by Mullins [45] in 1957 to model the motion of grain boundaries of a heated polycrystal. This was identified by Cahn and Taylor in [11] for closed hypersurfaces as $\mathcal{H}^{-1}$-gradient flow of the surface energy and also linked by Cahn, Elliott and Novick-Cohen in [10] with the Cahn-Hilliard equation with degenerate mobility as its sharp-interface limit. Short time existence and stability of stationary points was discussed by Elliott and Garcke in [22] for closed, planar curves and generalized to the higher dimensional case by Escher, Mayer and Simonett in [23]. Finally, we want to give an overview on some results concerning long time behaviour. For mean curvature flow one can observe typical properties linked to maximum principles. For example, Grayson showed in [34] that a smooth embedded planar curve preserves this properties and becomes convex in time. This can be combined with the work of Gage and Hamilton [26] where the authors showed that a convex curve in the plane moving due to mean curvature flow remains convex and shrinks to a round point. This is both false for surface diffusion flow as it was proven by Giga and Ito in [27, 28].

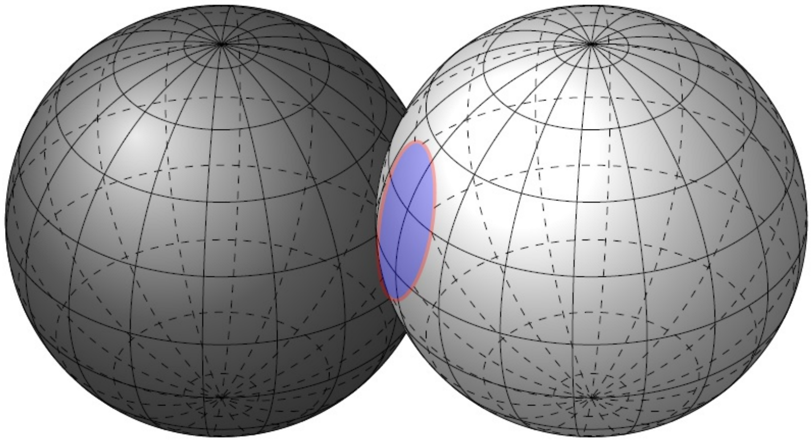

Figure 1.1: The picture shows the considered kind of triple junction cluster. In total, there are three hypersurfaces. In this illustration these are the two spherical caps and the flat blue area. The red line marks the triple junction, which is the boundary of all three hypersurfaces. Note that if the two enclosed volumes are unequal the blue surfaces will normally bend into direction of the larger volume.

Now we want to talk about the geometrical situation studied in this thesis. We will consider three embedded, oriented, compact, connected hypersurfaces $\Gamma^1, \Gamma^2, \Gamma^3$ in $\mathbb{R}^n$ that do not intersect with each other. Furthermore, their boundaries coincide, that is

$$\partial\Gamma^1 = \partial\Gamma^2 = \partial\Gamma^3 := \Sigma,$$

and so they meet each other in a triple junction Σ. The most prominent example for this kind of geometry is the so called standard double bubble that was proven in [36] to have the best ratio between its surface area and the enclosed volume. We will now consider an evolution

$$\Gamma(t) := \Gamma^1(t) \cup \Gamma^2(t) \cup \Gamma^3(t) \cup \Sigma(t),$$

that fulfils at every time (SDF) and in addition on $\Sigma(t)$ the boundary conditions

$$\partial\Gamma^1(t) = \partial\Gamma^2(t) = \partial\Gamma^3(t) = \Sigma(t), \tag{CC}$$

$$\angle(\nu_{\Gamma^i(t)}, \nu_{\Gamma^j(t)}) = \theta^k, \quad (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}, \tag{AC}$$

$$\gamma^1 H_{\Gamma^1(t)} + \gamma^2 H_{\Gamma^2(t)} + \gamma^3 H_{\Gamma^3(t)} = 0, \tag{CCP}$$

$$\nabla_{\Gamma^1(t)} H_{\Gamma^1(t)} \cdot \nu_{\Gamma^1(t)} = \nabla_{\Gamma^2(t)} H_{\Gamma^2(t)} \cdot \nu_{\Gamma^2(t)} = \nabla_{\Gamma^3(t)} H_{\Gamma^3(t)} \cdot \nu_{\Gamma^3(t)}. \tag{FB}$$

Here, $\nu_{\Gamma^i(t)}$ denotes the outer conormal of $\Gamma^i(t)$, $\gamma^1, \gamma^2, \gamma^3$ constants determining the energy density on the hypersurfaces $\Gamma^i(t)$ and $\theta^1, \theta^2, \theta^3 \in [0, 2\pi]$ given angles, that are actually given by the γ^i. Indeed, the condition (AC) is equivalent to Young's law

$$\frac{\sin(\theta^1)}{\gamma^1} = \frac{\sin(\theta^2)}{\gamma^2} = \frac{\sin(\theta^3)}{\gamma^3},$$

that was derived in [54] as balance of mechanical forces. The condition (CCP) results from continuity of the chemical potentials at the triple junction and (FB) are the flux balances. (CC) gives the concurrency of the triple junction during the flow. For the motivation of these boundary conditions as sharp interface limit of a Cahn-Hilliard equation with degenerate mobility see [32].

We will prove two main results in this thesis. The first one is short time existence in a Hölder setting for triple junction clusters that are for some $\alpha \in (0,1)$ close enough in the $C^{4+\alpha}$-norm to a $C^{5+\alpha}$-reference surface. Hereby, we follow the ideas from [32], which were also applied in [31]. In these works the authors linearised the problem over a fixed reference frame and then used directly the results from [38]. The non-linear analysis is then as usual based on a contraction argument using Banach's fixed-point theorem. In our situation we get more complications due to the higher space dimensions. In contrary to curves general manifolds cannot be written as parametrisation over one domain in $\mathbb{R}^n$. Thus, we can apply [38] only locally and we therefore need to construct a global weak solution and connect both problems using compactness arguments. Additionally, there are three other main problems in the analysis we want to explain. Firstly, as mentioned before, we have to reduce the tangential freedom to get a well-posed problem. Hereby, we use the ideas from [19] and as a consequence we get non-local terms in highest order. The authors observed that in the linearisation of their equations all non-local terms appear only in lower order. We want to emphasise that this is natural for the linearisation in the reference frame of expressions caused by tangential terms in curvature quantities. Thus, in the linear analysis of curvature flows we do not expect non-local terms to cause technical problems in general. For the non-linear analysis the authors of [19] modify theory for fully non-linear equations from [42]. In our work, we want to show that this is not necessary and we can use directly the quasi-linear structure of the non-local terms. The second difficult aspect are the angle conditions (AC). They will lead to a fully non-linear boundary condition for which we need techniques from [42]. Also, this will cause essential problems proving parabolic smoothing. Thirdly, in the weak analysis of the linearised problem we get an energetic problem with the inhomogeneities of all lower order boundary conditions. These have to be included in the end using perturbations techniques. As final comment we want to remark that we expect that the application of maximal L_p-regularity like they were used in [49] is in principle possible. But as we have boundary conditions of mixed orders we cannot apply the results directly.

The second main result of this thesis is that the evolution due to surface diffusion of triple junction clusters close to stationary double bubbles will exists globally in time. Furthermore, the flow converges to another stationary double bubble. This was already proven in [5] for planar double bubbles. The author used there the generalized principle of linearised stability which is also applied in related works. Depner proved in [16] linearised stability of mean curvature and surface diffusion flow with boundary contact and with and without triple lines. Abels, Garcke and Müller proved in [1] stability of spherical caps evolving due to Wilmore flow. A different approach was used in [23] where the authors used centre manifold analysis to prove stability of stationary points of surface diffusion flow on closed hypersurfaces. Both methods are difficult to apply in our situation as one needs a precise description of the set of equilibria of the flow. In [5] the author was able to give one in the case for planar double bubbles. But there the triple junctions are only points. In higher space dimensions they itself will be non-trivial geometrical objects causing more degree of freedoms in the set of equilibria. Thus, we used an approach with a Łojasiewicz-Simon gradient inequality. Hereby, one uses the fact that once such an inequality is proven we get estimates for the time derivative, cf. [14, Section 4] for a detailed explanation. In [13] the author gave a general result concerning prerequisites for a Łojasiewicz-Simon gradient inequality to be true. To use this in most situations a more practical version this results was written down by Feehan and Meridakis in [25]. This method is easy to apply and was for example also used in situations with higher codimensions, cf. [14] and [15]. In our work several aspects made the application more complicated than in the mentioned works. Firstly, most authors consider L^2-gradient flows. Our problem is related to a $\mathcal{H}^{-1}$-flow but the gradient flow structure itself is not clear. Thus, we have to do some modification in the stability argument. Secondly, due to the higher space dimensions one cannot work in the natural function spaces one would expect. In these spaces the geometric objects cannot guaranteed to be C^2-manifolds. To solve this complicated interpolation arguments are needed. Thirdly, the non-linear boundary conditions on the triple junctions are difficult to fit in the classical setting of [13, Corollary 3.11.]. In [15] the authors considered open curves with clamped boundaries. This results in linear boundary conditions which are much easier in their analysis. Finally and most critical is the tangential part of the flow. During our work we realized that a non-local tangential part will not be suitable to work with and so we have to replace it later in the work with a local version.

As an outlook we want to give some examples of further questions related to the topic which are not discussed in this thesis. The short time existence results remains open for general $C^{4+\alpha}$-surfaces. Here, suitable approximation results for triple junction clusters like they were proven for closed hypersurfaces in [49] are needed. Additionally, like in many higher dimensional situations the question for existence and uniqueness for the original geometric problem remains unanswered.

Lastly, we give a brief overview concerning the structure of this thesis. In Chapter 2 we will explain basic notation used in this thesis, recall necessary results from function analysis and give an overview of function spaces on manifolds we will use. In Chapter 3 we motivate our strategy to prove short time existence by applying it on closed hypersurfaces. This result was already proven in [23] with other methods. In Chapter 4 we will then use this methods on triple junction clusters. In Chapter 5 we will show stability of stationary points of surface diffusion flow on closed hypersurfaces. This results was also already proven in [23] with other methods. Finally, we will prove stability of standard double bubbles evolving due to surface diffusion flow in Chapter 6.

2

Preliminaries

2.1 Notation Conventions

Throughout this thesis we will work with two different kinds of time-evolving geometries: either closed (i.e. compact and without boundary), embedded, connected, orientable hypersurfaces or triple junction surface clusters of three compact, embedded, connected hypersurfaces. We will denote in both cases by $\Gamma(t)$ the geometric object at time t. In the case of closed hypersurfaces we will denote by Ω the volume enclosed by Γ. In case of triple junction manifolds we will denote by Γ^1, Γ^2 and Γ^3 the three hypersurfaces and by $\Sigma(t)$ the arising triple junction, that is

$$\Sigma(t) = \partial\Gamma^1(t) = \partial\Gamma^2(t) = \partial\Gamma^3(t).$$

Two of the hypersurfaces will always form a volume containing the third hypersurface, which we will choose to be Γ^1. By Ω_{12} and Ω_{13} we denote the volume enclosed by Γ^1 and Γ^2 resp. Γ^1 and Γ^3. We choose the normals of the hypersurfaces, which we will denote in both cases by N, such that the normal of Γ_1 points in the interior of Ω_{12}, the one of Γ^2 outside of Ω_{12} and the one of Γ^3 into the inside of Ω_{13}. The outer conormals will be denoted by ν.
Furthermore, we use the standard notation for quantities of differential geometry, see for example [37] or the second chapter of [16]. That includes the canonical basis $\{\partial_i\}_{i=1,\dots,n}$ of the tangent space $T_p\Gamma$ at a point $p \in \Gamma$ induced by a parametrisation φ, the entries g_{ij} of the metrical tensor g, the entries g^{ij} of the inverse metric tensor g^{-1}, the Christoffel symbols Γ^i_{jk} , the second fundamental form II, its squared norm $|II|^2$ and the entries h_{ij} of the shape operator. We use the usual differential operators on a manifold Γ, which are the surface gradient ∇_Γ, the surface divergence div_Γ and the Laplace-Beltrami operator Δ_Γ(cf. [16, Section 2.1]).
By ρ we will denote the evolution in normal direction and by μ the evolution in tangential direction, which we will use to track the evolution of $\Gamma(t)$ over Γ_* via a direct mapping approach. Γ_ρ resp. $\Gamma_{\rho,\mu}$ will denote the (triple junction) manifold that is given as graph over Γ_*, cf. (3.1) and (4.17). Sub- and superscripts ρ resp. μ on a quantity will indicate that the quantity refers to the manifold $\Gamma_{\rho,\mu}$. Hereby, we will normally omit μ as long it is given as function in ρ. An asterisk will denote an evaluation in the reference geometry. Both conventions are also used for differential operators. For example, we will write ∇_ρ for ∇_{Γ_ρ} and ∇_* for ∇_{Γ_*}. We will denote by J_ρ the transformation of the surface measure, that is,

$$d\mathcal{H}^n(\Gamma_\rho) = J_\rho d\mathcal{H}^n(\Gamma_*).$$

If we index a domain or a submanifold in $\mathbb{R}^n$ with a T or δ in the subscript, this indicates the corresponding parabolic set, e.g., $\Gamma_T = \Gamma \times [0, T]$. With an abuse of notation, in most parts of the work we will not differ between quantities on $\Gamma_{\rho,\mu}$ and the pullback of them on Γ_*. In the parts dealing with triple junction manifolds the index i will be used to indicate that a quantity refers to the hypersurface Γ^i. A quantity in bold characters will refer to the vector consisting of the quantity on the three hypersurfaces of a triple junction, e.g., $\boldsymbol{\rho} = (\rho^1, \rho^2, \rho^3)$.
For the used function spaces we want to clarify that a subscript (0) denotes in the case of closed hypersurfaces a mean value free function and for a function $\boldsymbol{\rho}$ on a triple junction manifold Γ the condition

$$\int_{\Gamma^1} \rho^1 d\mathcal{H}^n = \int_{\Gamma^2} \rho^2 d\mathcal{H}^n = \int_{\Gamma^3} \rho^3 d\mathcal{H}^n.$$

Also, we denote by $⨍_\Gamma f d\mathcal{H}^n$ the mean value of a function $f \in L^1(\Gamma)$, that is,

$$⨍_\Gamma f d\mathcal{H}^n := \mathrm{Area}(\Gamma)^{-1} \int_\Gamma f d\mathcal{H}^n.$$

If Γ is a triple junction manifold then the subscript TJ in a function space will indicate that the function space has to be read as product space on each hypersurface. For example, we write

$$L^2_{TJ}(\Gamma) := L^2(\Gamma^1) \times L^2(\Gamma^2) \times L^2(\Gamma^3).$$

In the chapter about stability we follow the notation of [13]. In particular, $E : V \to \mathbb{R}$ denotes an energy functional (in our case just the surface area) on a Banach spaces V, $\mathcal{M}$ its first derivative and $\mathcal{L}(0)$ its second derivative at point 0. Hereby, we will always consider $\mathcal{L}(0)$ as function on V with values (on a subset of) V'.
Finally, we will always use the convention of dynamical constants. This will also be used for coefficient functions of lower order terms. The latter will be introduced in Section 4.4.

2.2 Some Important Results from Functional Analysis

During this thesis, a lot of problems will be dealt with by using the implicit function theorem for functions between Banach spaces. Therefore we want to mention the following version from [55, Theorem 4B].

Proposition 2.1 (Implicit function theorem of Hildebrandt and Graves)**.**
Suppose that:

i.) the mapping : : $U(x_0, y_0) \subset X \times Y \to Z$ is defined on an open neighbourhood $U(x_0, y_0)$, and $F(x_0, y_0) = 0$, where X, Y, Z are Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $x_0 \in X, y_0 \in Y$.

ii.) $\partial_y F$ exists as a partial Fréchet-derivative on $U(x_0, y_0)$ and $\partial_y F(x_0, y_0) \colon Y \to Z$ is bijective.

iii.) F and $\partial_y F$ are continuous at (x_0, y_0).

Then, the following are true:

a.) Existence and uniqueness: There exist positive numbers r_0 and r such that for every $x \in X$ satisfying $\|x - x_0\|_X \leq r_0$ there is exactly one $y(x) \in Y$ for which $\|y(x) - y_0\|_Y \leq r$ and $F(x, y(x)) = 0$.

b.) Continuity: If F is continuous in a neighbourhood of (x_0, y_0), then $y(\cdot)$ is continuous in a neighbourhood of x_0.

c.) Continuous differentiability: If F is a C^m-map on a neighbourhood of (x_0, y_0), $1 \leq m \leq \infty$, then $y(.)$ is also a C^m-map on a neighbourhood of x_0.

For our work on stability, analyticity of functions between Banach spaces is an important concept. To define it, we first have to introduce so called power operators.

Definition 2.2 (Power operator)**.**
Let X and Y be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let there be given a k-linear, bounded operator $M : X \times \cdots \times X \to Y$ which is symmetric in all variables. A power operator of degree k is created from setting for all $m, n \in \{0, 1, ..., k\}$ with $m + n = k$ and $x, y \in X$

$$Mx^m y^n := M(\underbrace{x, ..., x}_{m\ times}, \underbrace{y, ..., y}_{n-times}).$$

Definition 2.3 (Analytic Operators between Banach Spaces)**.**
Let Z and Y be Banach spaces over $\mathbb{K}$ and $T : Z \supset D(T) \to Y$ defined on an open set $D(T)$.

i.) T is called analytic at $z_0 \in D(T)$, if there is a sequence $\{T_k\}_{k\in\mathbb{N}_0}$ of power operators of degree k together with an open neighbourhood U of $z_0 \in D(T)$ such that for all $z \in U$ the series

$$Sz := \sum_{k=0}^{\infty} T_k(z - z_0)^k \tag{2.1}$$

exists and we have $Sz = Tz$ for all $z \in U$.

ii.) T is called analytic on an open subset $V \subset D(T)$, if it is analytic at every point $z_0 \in V$.

Remark 2.4. *Note that if T is analytic at a point z_0, this implies that T is analytic in an open neighbourhood of z_0, cf. [55, p.98].*

Very important for our work will also be that the implicit function theorem inherits also analyticity, which is Corollary 4.23 in [55].

Corollary 2.5 (Analytic version of the implicit function theorem)**.**
If in the situation of Proposition 2.1 the function F is also analytic at (x_0, y_0), then the solution y is analytic at x_0 as well.

The last thing we want to mention is the following fact about compact perturbations of Fredholm operators, which is Proposition 8.14(3) in [55].

Proposition 2.6 (Compact perturbation of Fredholm operators)**.**
Let X, Y be Banach spaces, $B : X \to Y$ a Fredholm operator and $C : X \to Y$ a compact operator. Then the sum $B + C : X \to Y$ is also a Fredholm operator and the Fredholm index satisfies

$$ind(B + C) = ind(B). \tag{2.2}$$

2.3 Function Spaces on Manifolds

In this section we want to introduce the two most important function spaces on manifolds we will use. These are Sobolev and parabolic Hölder spaces. In this section, $(\Gamma, \mathcal{A})$ will always be a compact, orientable, embedded submanifold Γ of $\mathbb{R}^{n+1}$, either with or without boundary, together with a maximal atlas $\mathcal{A}$.

2.3.1 Sobolev-Spaces on Manifolds

Definition 2.7 (Sobolev spaces on manifolds)**.** *Let Γ be of class $C^j, j \in \mathbb{N}$. Then we define for $k \in \mathbb{N}, k < j, 1 \le p \le \infty$ the Sobolev space $W^{k,p}(\Gamma)$ as the set of all functions $f : \Gamma \to \mathbb{R}$, such that for any chart $\varphi \in \mathcal{A}, \varphi : V \to U$ with $V \subset \Gamma, U \subset \mathbb{R}^n$ the map $f \circ \varphi^{-1}$ is in $W^{k,p}(U)$. Hereby, $W^{k,p}(U)$ denotes the usual Sobolev space. We define a norm on $W^{k,p}(\Gamma)$ by*

$$\|f\|_{W^{k,p}(\Gamma)} := \sum_{i=1}^{s} \|f \circ \varphi_i^{-1}\|_{W^{k,p}(U_i)}, \tag{2.3}$$

where $\{\varphi_i : V_i \to U_i\}_{i=1,...,s} \subset \mathcal{A}$ is a family of charts that covers Γ.

Remark 2.8 (Equivalent norms on $W^{k,p}(\Gamma)$)**.**

i.) *The norm on $W^{k,p}(\Gamma)$ depends on the choice of the φ_i but for a different choice we will get an equivalent norm as the transitions maps are C^j.*

ii.) *For the space $W^{1,p}(\Gamma)$ we will use the norm*

$$\|f\|_{W^{1,p}(\Gamma)} = \left(\int_\Gamma |\nabla_\Gamma f|^p + |f|^p d\mathcal{H}^n\right)^{\frac{1}{p}}. \tag{2.4}$$

Equivalence to (2.3) follows directly from the representation of the surface gradient in local coordinates.

iii.) *As usual we will write $H^k(\Gamma)$ for $W^{k,2}(\Gamma)$.*

We want to make some further remarks on three properties of these spaces. The first one is a sufficient condition such that we get a Banach algebra structure.

Lemma 2.9 (Banach space property of Sobolev spaces)**.**
Let Γ be smooth, $k \in \mathbb{N}, 1 \leq p \leq \infty$, and assume that

$$p > \frac{2n}{k}. \tag{2.5}$$

Then, $W^{k,p}(\Gamma)$ is a Banach algebra. In particular, $H^k(\Gamma)$ is a Banach algebra for $k > n$.

Proof. It is enough to show the result in local coordinates. So, we consider for a bounded domain V two functions $f, g \in W^{k,p}(V)$. For any multi-index α with $|\alpha| \leq k$ the partial derivative $\partial_\alpha(fg)$ is due to the Leibnitz rule a sum of terms of the form $\partial^{\alpha_1} f \partial^{\alpha_2} g$ with $|\alpha_1|+|\alpha_2| = |\alpha|$. As each derivative of f and g is in $L^p(V)$ it is enough to guarantee that $W^{k,p}(V) \hookrightarrow C^{\left[\frac{k}{2}\right]}(V)$. Using the Sobolev embedding this is true as long as (2.5) is fulfilled. ☐

For the next two results we will consider a triple junction cluster Γ. For a differentiable manifold the Poincaré inequality is well known, cf. [35, Theorem 2.10]. This can also be used for each surface of Γ. But by imposing additional boundary conditions one also can guarantee a version for the whole cluster.

Proposition 2.10 (Poincaré-type inequality on triple junction manifolds)**.**
Let $\gamma^i > 0, i = 1, 2, 3$. Consider the space

$$\mathcal{E} := \left\{ \boldsymbol{\rho} \in H^1_{TJ}(\Gamma) \,\middle|\, \sum_{i=1}^{3} \gamma^i \rho^i = 0, \int_{\Gamma^1} \rho^1 d\mathcal{H}^n = \int_{\Gamma^2} \rho^2 d\mathcal{H}^n = \int_{\Gamma^3} \rho^3 d\mathcal{H}^n \right\}. \tag{2.6}$$

Then, there is a constant $C > 0$ such that for all $\boldsymbol{\rho} \in \mathcal{E}$ we have

$$\|\boldsymbol{\rho}\|_{L^2_{TJ}(\Gamma)} \leq C \|\nabla_\Gamma \boldsymbol{\rho}\|_{L^2_{TJ}(\Gamma)}. \tag{2.7}$$

Proof. This was proved in [16, Lemma 4.29] for the situation with boundary contact. The proof only uses the structure at the triple junction and therefore can also be used in our situation. ☐

For the study of weak solutions we will need an Ehrling-type lemma.

Proposition 2.11 (Ehrling-type lemma on triple junction)**.**
For every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ only depending on ε such that for all $\boldsymbol{u} \in H^1_{TJ}(\Gamma)$ it holds

$$\|\boldsymbol{u}\|_{L^2(\Sigma)^3} \leq \varepsilon \|\nabla_\Gamma \boldsymbol{u}\|_{L^2_{TJ}(\Gamma)} + C_\varepsilon \|\boldsymbol{u}\|_{L^2_{TJ}(\Gamma)}. \tag{2.8}$$

Proof. Assume by contradiction that for some $\varepsilon > 0$ there is a sequence $(\widetilde{\boldsymbol{u}}_n)_{n\in\mathbb{N}} \subset H^1_{TJ}(\Gamma)$ with

$$\|\widetilde{\boldsymbol{u}}_n\|_{L^2(\Sigma)^3} > \varepsilon \|\nabla_\Gamma \widetilde{\boldsymbol{u}}_n\|_{L^2_{TJ}(\Gamma)} + n \|\widetilde{\boldsymbol{u}}_n\|_{L^2_{TJ}(\Gamma)}. \tag{2.9}$$

Then consider for all $n \in \mathbb{N}$ the function $\boldsymbol{u}_n := \widetilde{\boldsymbol{u}}_n \cdot \|\widetilde{\boldsymbol{u}}_n\|^{-1}_{L^2(\Sigma)^3}$. Note that due to (2.9) this is well defined. Now, multiplying (2.9) with $\|\widetilde{\boldsymbol{u}}_n\|^{-1}_{L^2(\Sigma)^3}$ implies that

$$1 > \varepsilon \|\nabla_\Gamma \boldsymbol{u}_n\|_{L^2_{TJ}(\Gamma)} + n\|\boldsymbol{u}_n\|_{L^2_{TJ}(\Gamma)}. \tag{2.10}$$

From this we conclude that $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ has to converge to 0 in $L^2_{TJ}(\Gamma)$ and that $\|\nabla_\Gamma \boldsymbol{u}_n\|_{L^2_{TJ}(\Gamma)}$ is uniformly bounded by $\frac{1}{\varepsilon}$. Thus, $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ is bounded in $H^1_{TJ}(\Gamma)$ and consequently there is a subsequence $(\boldsymbol{u}_{n_k})_{k\in\mathbb{N}}$ converging weakly to a $\boldsymbol{u} \in H^1_{TJ}(\Gamma)$. Due to the compact embedding $H^1_{TJ}(\Gamma) \hookrightarrow L^2_{TJ}(\Gamma)$ this sequence converges strongly in $L^2_{TJ}(\Gamma)$ and by uniqueness of limits this shows that $\boldsymbol{u} \equiv 0$. Using compactness of the trace operator we see that $\boldsymbol{u}_{n_k}$ also converges strongly in $L^2_{TJ}(\Sigma)^3$ to 0. Now, this is a contradiction as we constructed $\boldsymbol{u}_n$ to be normalized in $L^2_{TJ}(\Sigma)^3$ and therefore we conclude our claim. □

The last thing we want to mention concerning Sobolev spaces is the space $\mathcal{H}^{-1}$. In the case of a closed hypersurface Γ this denotes the dual space of $\mathcal{E} = H^1_{(0)}(\Gamma)$ and in the case of triple junctions the dual space of $\mathcal{E}$ from (2.6). As we showed above we have a Poincaré inequality on both spaces and therefore an equivalent inner product on these spaces is given by the L^2-product of the surface gradients. Using the Riesz isomorphism we can identify the elements of $\mathcal{H}^{-1}$ with $\mathcal{E}$, that is, for every $f \in \mathcal{H}^{-1}$ there is a unique $\rho \in \mathcal{E}$ with

$$\int_\Gamma \nabla_\Gamma \rho \cdot \nabla_\Gamma \psi \, d\mathcal{H}^n = f(\psi) \quad \forall \psi \in \mathcal{E}. \tag{2.11}$$

But this is nothing else but the weak formulation of

$$-\Delta_\Gamma \rho = f \qquad \text{on } \Gamma_*, \tag{2.12}$$

in the closed case and otherwise the weak formulation of

$$\begin{aligned} -\Delta_{\Gamma^i} \rho^i &= f^i && \text{on } \Gamma^i_*, i = 1,2,3, && (2.13)\\ \rho^1 + \rho^2 + \rho^3 &= 0 && \text{on } \Sigma, && (2.14)\\ \partial_{\nu^1}\rho^1 = \partial_{\nu^2}\rho^2 &= \partial_{\nu^3}\rho^3 && \text{on } \Sigma. && (2.15) \end{aligned}$$

Therefore, we will write for the element from the Riesz identification $(-\Delta_\Gamma)^{-1} f$ and get the inner product on $\mathcal{H}^{-1}$ by

$$\langle f, g\rangle_{\mathcal{H}^{-1}} := \int_\Gamma \nabla_\Gamma ((-\Delta_\Gamma)^{-1} f) \cdot \nabla_\Gamma ((-\Delta_\Gamma)^{-1} g) d\mathcal{H}^n, \quad f, g \in \mathcal{H}^{-1}. \tag{2.16}$$

We will later need the following interpolation result.

Lemma 2.12 (Interpolation between $\mathcal{H}^{-1}$ and H^1)**.**
Let Γ be either a closed hypersurface or a triple junction cluster. Then we have for all $\boldsymbol{\rho} \in \mathcal{E}$ that

$$\|\boldsymbol{\rho}\|^2_{L^2(\Gamma)} \le \|\boldsymbol{\rho}\|_{\mathcal{H}^{-1}(\Gamma)} \|\boldsymbol{\rho}\|_{H^1(\Gamma)}. \tag{2.17}$$

.

Proof. We will only consider the case of triple junctions as the closed case works alike without boundary terms. Using the properties of $(-\Delta_\Gamma)^{-1}$ and the Cauchy-Schwarz inequality we get

$$\begin{aligned} \int_\Gamma \boldsymbol{\rho}^2 d\mathcal{H}^n &= \sum_{i=1}^3 \int_\Gamma -\Delta_{\Gamma^i} \left((-\Delta_\Gamma)^{-1}\boldsymbol{\rho}\right)^i \rho^i d\mathcal{H}^n \\ &= \sum_{i=1}^3 \int_{\Gamma^i} \nabla_{\Gamma^i} \left((-\Delta_\Gamma)^{-1}\boldsymbol{\rho}\right)^i \cdot \nabla_{\Gamma^i}\rho^i d\mathcal{H}^n - \int_\Sigma \sum_{i=1}^3 \nabla_{\Gamma^i} \left((-\Delta_\Gamma)^{-1}\boldsymbol{\rho}\right) \cdot \nu^i \rho^i d\mathcal{H}^{n-1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{3} \int_{\Gamma^i} \nabla_{\Gamma^i} \left((-\Delta_\Gamma)^{-1} \boldsymbol{\rho} \right)^i \cdot \nabla_{\Gamma^i} \rho^i d\mathcal{H}^n \\
&\leq \|\boldsymbol{\rho}\|_{\mathcal{H}^{-1}(\Gamma)} \|\boldsymbol{\rho}\|_{\mathcal{H}^1(\Gamma)}.
\end{aligned}
$$

This shows the claimed estimate. □

2.3.2 Parabolic Hölder-Spaces on Manifolds

Now, we want to move on to the second kind of important function spaces, the parabolic Hölder spaces. It is both possible to introduce these spaces on manifolds in local coordinates, e.g. [42, p.177], or without, e.g. [19]. We prefer the first approach as we want to use local results. We will first introduce these spaces on a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. For this, we first need for $\alpha \in (0,1), a, b \in \mathbb{R}$ the two semi-norms for a function $f : \bar{\Omega} \times [a,b] \to \mathbb{R}$ given by

$$
\begin{aligned}
\langle f \rangle_{x,\alpha} &:= \sup_{x_1, x_2 \in \bar{\Omega}, t \in [a,b]} \frac{|f(x_1,t) - f(x_2,t)|}{|x_1 - x_2|^\alpha}, \\
\langle f \rangle_{t,\alpha} &:= \sup_{x \in \bar{\Omega}, t_1, t_2 \in [a,b]} \frac{|f(x,t_1) - f(x,t_2)|}{|t_1 - t_2|^\alpha}.
\end{aligned}
$$

Now, we define for $k, k' \in \mathbb{N}, \alpha \in (0,1), m \in \mathbb{N}$ the spaces

$$
\begin{aligned}
C^{\alpha,0}(\bar{\Omega} \times [a,b]) &:= \{ f \in C(\bar{\Omega} \times [a,b]) | \langle f \rangle_{x,\alpha} < \infty \}, \\
\|f\|_{C^{\alpha,0}(\bar{\Omega} \times [a,b])} &:= \|f\|_\infty + \langle f \rangle_{x,\alpha}, \\
C^{0,\alpha}(\bar{\Omega} \times [a,b]) &:= \{ f \in C(\bar{\Omega} \times [a,b]) | \langle f \rangle_{t,\alpha} < \infty \}, \\
\|f\|_{C^{0,\alpha}(\bar{\Omega} \times [a,b])} &:= \|f\|_\infty + \langle f \rangle_{t,\alpha}, \\
C^{k+\alpha,0}(\bar{\Omega} \times [a,b]) &:= \{ f \in C(\bar{\Omega} \times [a,b]) | \forall t \in [a,b] : f \in C^k(\bar{\Omega}), \\
&\qquad \forall \beta \in \mathbb{N}_0^n, |\beta| \leq k : \partial_\beta^x f \in C^{\alpha,0}(\bar{\Omega} \times [a,b]) \}, \\
\|f\|_{C^{k+\alpha,0}(\bar{\Omega} \times [a,b])} &:= \sum_{|\beta| \leq k} \|\partial_\beta^x f\|_\infty + \sum_{|\beta| = k} \langle \partial_\beta^x f \rangle_{x,\alpha}, \\
C^{k+\alpha, \frac{k+\alpha}{m}}(\bar{\Omega} \times [a,b]) &:= \{ f \in C(\bar{\Omega} \times [a,b]) | \forall \beta \in \mathbb{N}_0^n, i \in \mathbb{N}_0, mi + |\beta| \leq k : \\
&\qquad \partial_t^i \partial_\beta^x f \in C^{\alpha,0}(\bar{\Omega} \times [a,b]) \cap C^{0, \frac{k+\alpha-mi-|\beta|}{m}}(\bar{\Omega} \times [a,b]) \}, \\
\|f\|_{C^{k+\alpha, \frac{k+\alpha}{m}}(\bar{\Omega} \times [a,b])} &:= \sum_{0 \leq mi + |\beta| \leq k} \left(\|\partial_t^i \partial_\beta^x f\|_\infty + \|\partial_t^i \partial_\beta^x f\|_{C^{0, \frac{k+\alpha-mi-|\beta|}{m}}(\bar{\Omega} \times [a,b])} \right) \\
&\quad + \sum_{mi + |\beta| = k} \|\partial_t^i \partial_\beta^x f\|_{C^{\alpha,0}(\bar{\Omega} \times [a,b])}.
\end{aligned}
$$

Hereby, we denote by ∂_β^x a partial derivative in space with respect to the multi-index β and ∂_t^i the i-th partial derivative in time. The parameter m corresponds to the order of the differential equation one is considering and in our work it will always be four. Now, we can also define parabolic Hölder spaces on submanifolds as follows.

Definition 2.13 (Parabolic Hölder spaces on submanifolds)**.**
Let Γ be a C^r-submanifold of $\mathbb{R}^n$, either with or without boundary. Then we define for $k \in \mathbb{N}_0, k < r, \alpha \in (0,1), a, b \in \mathbb{R}, m \in \mathbb{N}$ the space $C^{k+\alpha, \frac{k+\alpha}{m}}(\Gamma \times [a,b])$ as the set of all functions $f : \Gamma \to \mathbb{R}$ such that for any parametrisation $\varphi : \Omega \to V \subset \Gamma$ we have that $f \circ \varphi \in C^{k+\alpha, \frac{k+\alpha}{m}}(\bar{\Omega} \times [a,b])$.

Remark 2.14 (Traces of parabolic Hölder spaces)**.**
On the boundary Σ of Γ we may choose φ to be a parametrisation that flattens the boundary. From this we see that

$$
f \in C^{k+\alpha, k'+\alpha'}(\Gamma \times [a,b]) \Rightarrow f\big|_{\Sigma \times [a,b]} \in C^{k+\alpha, k'+\alpha'}(\Sigma \times [a,b]).
$$

Remark 2.15 (Hölder regularity in time for derivatives).
In some works these spaces are introduced with lower Hölder regularity in time for the lower order derivatives, cf. [42] and [19]. Actually, this approach is equivalent due to interpolation results for Hölder continuous functions, cf. [42, Proposition 1.1.4 and 1.1.5].

The following properties are proved only in the case of parabolic Hölder spaces on bounded domains of $\mathbb{R}^n$. Due to the definition in local coordinates they are also true for parabolic Hölder spaces on submanifolds.
Like in most works on well-posedness, product estimates will also be crucial in our work. Regarding this, we have very good properties in parabolic Hölder spaces.

Lemma 2.16 (Product estimates in parabolic Hölder spaces).
Let $k, m \in \mathbb{N}, \alpha \in (0,1)$ and $f, g \in C^{k+\alpha, \frac{k+\alpha}{m}}(\overline{\Omega} \times [0,T])$. Then we have

$$fg \in C^{k+\alpha, \frac{k+\alpha}{m}}(\overline{\Omega} \times [0,T]), \tag{2.18}$$

and furthermore we have that

$$\|fg\|_{C^{k+\alpha, \frac{k+\alpha}{m}}(\overline{\Omega}\times[0,T])} \le C \|f\|_{C^{k+\alpha, \frac{k+\alpha}{m}}(\overline{\Omega}\times[0,T])} \|g\|_{C^{k+\alpha, \frac{k+\alpha}{m}}(\overline{\Omega}\times[0,T])}, \tag{2.19}$$

$$\|fg\|_{C^{k+\alpha, \frac{k+\alpha}{m}}(\overline{\Omega}\times[0,T])} \le C \left(\|f\|_{C^{k+\alpha, \frac{k+\alpha}{m}}(\overline{\Omega}\times[0,T])} \|g\|_{C^{k,0}} + \|f\|_{C^{k,0}} \|g\|_{C^{k+\alpha, \frac{k+\alpha}{m}}(\overline{\Omega}\times[0,T])} \right). \tag{2.20}$$

Proof. Using again the Leibntz rule we see that all partial derivatives, that exist for f and g, exist also for fg. Furthermore, we note for $x, y \in \overline{\Omega}, s, t \in [0,T]$ and any $\bar{\alpha}$ that

$$\begin{aligned}
\|fg\|_\infty &\le \|f\|_\infty \|g\|_\infty, \\
\frac{|(fg)(x,t) - (fg)(y,t)|}{|x-y|^{\bar{\alpha}}} &\le \frac{|f(x,t)| \cdot |g(x,t) - g(y,t)| + |f(x,t) - f(y,t)| \cdot |g(y,t)|}{|x-y|^{\bar{\alpha}}} \\
&\le \|f\|_\infty \frac{|g(x,t) - g(y,t)|}{|x-y|^{\bar{\alpha}}} + \frac{|f(x,t) - f(y,t)|}{|x-y|^{\bar{\alpha}}} \cdot \|g\|_\infty, \\
\frac{|(fg)(x,s) - (fg)(x,t)|}{|s-t|^{\bar{\alpha}}} &\le \frac{|f(x,s)| \cdot |g(x,s) - g(x,t)| + |f(x,s) - f(y,t)| \cdot |g(x,t)|}{|s-t|^{\bar{\alpha}}} \\
&\le \|f\|_\infty \frac{|g(x,s) - g(x,t)|}{|s-t|^{\bar{\alpha}}} + \frac{|f(x,s) - f(x,t)|}{|s-t|^{\bar{\alpha}}} \cdot \|g\|_\infty.
\end{aligned}$$

These can be applied on all derivatives to derive (2.20). Then, (2.19) is just a weaker statement, which often will be enough for our calculations. ☐

A very important fact, which allows us in many situations to study only the highest order terms, is the following contractivity property of lower order terms. We will prove this only in local coordinates but due to the definition of parabolic Hölder spaces this is also true for submanifolds of $\mathbb{R}^n$.

Lemma 2.17 (Contractivity property of lower order terms in parabolic Hölder spaces).
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and

$$k, k' \in \{0,1,2,3,4\}, k' < k, \alpha \in (0,1), a, b \in \mathbb{R}.$$

Then, we have for any $f \in C^{k+\alpha, \frac{k+\alpha}{4}}(\bar{\Omega} \times [a,b])$ that

$$\|f\|_{C^{k'+\alpha, \frac{k'+\alpha}{4}}(\bar{\Omega}\times[a,b])} \le \|f\big|_{t=a}\|_{C^{k'+\alpha}(\bar{\Omega})} + C(b-a)^{\bar{\alpha}} \|f\|_{C^{k+\alpha, \frac{k+\alpha}{4}}(\bar{\Omega}\times[a,b])}. \tag{2.21}$$

Hereby, the constants C and $\bar{\alpha}$ depend on α, k, k' and $\bar{\Omega}$. Especially, if $f\big|_{t=a} \equiv 0$, we have

$$\|f\|_{C^{k'+\alpha, \frac{k'+\alpha}{4}}(\bar{\Omega}\times[a,b])} \le C(b-a)^{\bar{\alpha}} \|f\|_{C^{k+\alpha, \frac{k+\alpha}{4}}(\bar{\Omega}\times[a,b])}. \tag{2.22}$$

Proof. Note that due to $k' \neq 4$ the space $C^{k'+\alpha,\frac{k'+\alpha}{4}}(\bar{\Omega}\times[a,b])$ will not contain any partial derivatives in time. In the following, $\partial_\beta^x f$ will denote any derivative in space with respect to a multi-index β with $|\beta| \leq k'$. For the three different parts of the norm in $C^{k'+\alpha,\frac{k'+\alpha}{4}}(\bar{\Omega}\times[a,b])$ we get

$$
\begin{aligned}
\|\partial_\beta^x f\|_\infty &\leq \sup_{x\in\bar{\Omega}} |\partial_\beta^x f(x,a)| + (b-a)^{\frac{k-|\beta|+\alpha}{4}} \|\partial_\beta^x f\|_{C^{0,\frac{k-|\beta|+\alpha}{4}}} \\
&\leq \|\partial_\beta^x f(\cdot,a)\|_\infty + C(b-a)^{\frac{k-|\beta|+\alpha}{4}} \|f\|_{C^{k+\alpha,\frac{k+\alpha}{4}}(\bar{\Omega}\times[a,b])}, \\
\langle \partial_\beta^x f\rangle_{x,\alpha} &\leq \sup_{i=1,\dots,n} \|\partial_i\partial_\beta^x f\|_\infty \\
&\leq \sup_{i=1,\dots,n} \left(\|\partial_i\partial_\beta^x f(\cdot,a)\|_\infty + (b-a)^{\frac{k-|\beta|-1+\alpha}{4}} \|\partial_i\partial_\beta^x f\|_{C^{0,\frac{k-|\beta|-1+\alpha}{4}}(\bar{\Omega}\times[a,b])} \right) \\
&\leq \|f(\cdot,a)\|_{C^{k+\alpha}(\bar{\Omega})} + (b-a)^{\frac{k-|\beta|-1+\alpha}{4}} \|f\|_{C^{k+\alpha,\frac{k+\alpha}{4}}(\bar{\Omega}\times[a,b])}, \\
\langle \partial_\beta^x f\rangle_{t,\frac{k'-|\beta|+\alpha}{4}} &= \sup_{x\in\bar{\Omega},t_1,t_2\in[a,b]} \frac{|\partial_\beta^x f(x,t_1) - \partial_\beta^x f(x,t_2)|}{|t_1-t_2|^{\frac{k'-|\beta|+\alpha}{4}}} \\
&= \sup_{x\in\bar{\Omega},t_1,t_2\in[a,b]} (t_1-t_2)^{\frac{k-k'}{4}} \frac{|\partial_\beta^x f(x,t_1) - \partial_\beta^x f(x,t_2)|}{|t_1-t_2|^{\frac{k-|\beta|+\alpha}{4}}} \\
&\leq (b-a)^{\frac{k-k'}{4}} \langle \partial_\beta^x f\rangle_{t,\frac{k-|\beta|+\alpha}{4}} \leq (b-a)^{\frac{k-k'}{4}} \|f\|_{C^{k+\alpha,\frac{k+\alpha}{4}}(\bar{\Omega}\times[a,b])}.
\end{aligned}
$$

Together this shows the claim. □

As a final remark of this chapter we want to mention that we sometimes identify Sobolev and Hölder spaces (in local coordinates) with Besov spaces, especially to use interpolation and composition results. As we do not need them on manifolds we will not introduce them here but they can be found, e.g., in [53].

3

Short Time Existence for the Surface Diffusion Flow of Closed Hypersurfaces

In this chapter we want to prove a short time existence result in a Hölder setting for the motion of closed hypersurfaces evolving by the surface diffusion flow. The result itself was already proven in [23] but we want to demonstrate the ideas of [19] in an easier setting before moving on to a triple junction geometry. We prove the following result.

Theorem 3.1 (Short time existence for the surface diffusion flow on closed hypersurfaces)**.**
Let Γ_* be a closed, oriented, embedded hypersurface in $\mathbb{R}^{n+1}$. Then, there is an $\varepsilon_0 > 0$ and a $T > 0$ such that for all $h_0 \in C^{4+\alpha}(\Gamma_*)$ with $\|h_0\| \leq \varepsilon$ a solution of the surface diffusion flow with initial surface Γ_{h_0} exists up to time T.

3.1 Surface Diffusion as Gradient Flow

As we mentioned in Chapter 1, surface diffusion was suggested in the fifties by Mullins to describe the evolution of grain boundaries of heated polycrystals. From a mathematical point of view this motion law has the structure of a gradient flow of the surface area. We want to explain this in more detail. Consider the set $\mathcal{MH}^n$ of all C^2-hypersurfaces in $\mathbb{R}^{n+1}$. In [48] it was proven that this has the structure of a Banach manifold, where locally around a C^3-hypersurface Γ_* a parametrisation of $\mathcal{MH}^n$ is given by using C^2-distance functions on Γ_*. Precisely, the authors showed that for any other surface $\Gamma \in \mathcal{MH}^n$ that is close enough in a C^2-sense there is a $\rho \in C^2(\Gamma^*)$ with

$$\Gamma = \{x + \rho(x)N(x) | x \in \Gamma_*\}. \tag{3.1}$$

In this section, Γ_ρ will always refer to the hypersurface given as graph over Γ_* via (3.1). This shows that one can identify the tangent space of $\mathcal{MH}^n$ at Γ^* with normal velocity fields of class C^2. This is also true for a C^2-surface as by definition it is given as $C^2(\Gamma^*)$ and we can identify the normal velocity fields of Γ and Γ_* as long as

$$N_*(x) \cdot N_\Gamma(x + \rho(x)N(x)) \neq 0, \qquad \forall x \in \Gamma_*. \tag{3.2}$$

This is guaranteed for all ρ small enough in the C^1-norm as the normal field depends continuous on ρ and its first order derivatives, which we will see in more detail in Chapter 5, Lemma 5.9. Now restrict this set to the submanifold $\mathcal{NH}^n$ of all hypersurfaces enclosing the same fixed volume V_*. Due to Reynolds transport theorem we get that a motion in $\mathcal{NH}^n$ requires the normal velocity to be mean value free. On $C^2_{(0)}(\Gamma)$ we have a Poincaré-inequality and so the $\mathcal{H}^{-1}$-product is well-defined on $T_\Gamma\mathcal{NH}^n$, cf. Section 2.3.1. Consider now an evolution $\Gamma(t)$ in $\mathcal{NH}^n$. Then, we have for the evolution of the surface area Area($\Gamma(t)$) that

$$\begin{aligned}\frac{d}{dt}A(t) &= \int_{\Gamma(t)} -H_{\Gamma(t)}V_{\Gamma(t)}d\mathcal{H}^n = \int_{\Gamma(t)} -H_{\Gamma(t)}(-\Delta_{\Gamma(t)})((-\Delta^{-1}_{\Gamma(t)})V_{\Gamma(t)})d\mathcal{H}^n \\ &= \int_{\Gamma(t)} \nabla_{\Gamma(t)}(-H_{\Gamma(t)})\cdot\nabla_{\Gamma(t)}((-\Delta^{-1}_{\Gamma(t)})V_{\Gamma(t)}) = \langle \Delta_{\Gamma(t)}H_{\Gamma(t)}, V_{\Gamma(t)}\rangle_{\mathcal{H}^{-1}}. \end{aligned} \tag{3.3}$$

This shows now that in $\mathcal{NH}^n$ the $\mathcal{H}^{-1}$-gradient flow of the surface area is given by the evolution law

$$V_{\Gamma(t)} = -\Delta_{\Gamma(t)}H_{\Gamma(t)}. \tag{3.4}$$

3.2 Transformation and Linearisation

The equation we derived in the previous section is not suitable for classical analytical settings as it results in functions space that change in time. Also, the law itself is only well-posed as evolution in the set of hypersurfaces. As it only determines the normal velocity we get for the motion of a single point a degeneration in tangential directions. The classical way to overcome this problem is a direct mapping approach, i.e. we track the evolution over a fixed reference frame Γ_* using (3.2) and get a new problem on the reference geometry. This both fixes the function spaces and also makes the problem well-posed as we then basically assume that every material point only moves in normal direction. For a smooth flow one can always write the evolution in this form due to the results for the manifold $\mathcal{MH}^n$ proven in [49, Chapter 2] and so the unique existence of solutions of the following analytic problem actually proves the existence of a unique geometric solution of (3.4).
This means we are now searching for a function $\rho : \Gamma_* \times [0,T] \to \mathbb{R}$ such that for the evolving hypersurface of an initial surface Γ_0 moving due to (3.4) we have that

$$\Gamma(t) = \{x + \rho(x,t)\nu_*(x) | x \in \Gamma_*\} \quad \forall t \in [0,T]. \tag{3.5}$$

Thus, we get a parametrisation of $\Gamma(t)$ via the time depending function $\Phi_{\rho(t)} : \Gamma_* \to \Gamma(t)$ given by

$$x \mapsto x + \rho(x,t)\nu_*(x) \quad \forall x \in \Gamma_*.$$

Additionally, for a given local parametrisation $\varphi : U \to V$ with $U \subset \mathbb{R}^n, V \subset \Gamma_*$ we get a local parametrisation $\varphi_{\rho(t)}$ of $\Gamma(t)$ via $\varphi_{\rho(t)} := \Phi_{\rho(t)} \circ \varphi$.
To get a parabolic equation for ρ we observe that the normal velocity is given by

$$V(x,t) = \partial_t\rho(x,t)(N_*(x)\cdot N_\rho(x)), \quad (x,t) \in \Gamma_* \times [0,T]. \tag{3.6}$$

We recall that the product of the two normals is unequal to zero in every point as long as ρ is small enough in the C^1-norm. With this, we can rewrite (3.4) as

$$(SDFC)\begin{cases}\partial_t\rho &= -\frac{1}{N_*\cdot N_\rho}\Delta_\rho H_\rho \quad \text{on } \Gamma_* \times [0,T], \\ \rho(0) &= \rho_0 \quad \text{on } \Gamma_*.\end{cases}$$

Note that as we mentioned in Section 2.1 we will not differ between the quantities on the hypersurface $\Gamma(t)$ and their pullback on Γ_*.
We now want to linearise (SDFC) pointwise around $\rho \equiv 0$, i.e., we will calculate the linearisation in the reference frame. For this we want to note the following well-known results for purely normal evolutions.

Lemma 3.2 (Evolution of geometric quantities of closed hypersurfaces)**.**
Asume that $\Gamma(t)$ is an evolution that is purely normal at $t = 0$. Then we have

$$\partial_t g_{ij}\big|_{t=0} = 2Vh_{ij}, \tag{3.7}$$

$$\partial_t g^{ij}\big|_{t=0} = 2Vh^{ij}, \tag{3.8}$$
$$\partial_t N\big|_{t=0} = -\nabla_{\Gamma(0)} V, \tag{3.9}$$
$$\partial_t H\big|_{t=0} = \Delta_{\Gamma(0)} V + V|II_{\Gamma(0)}|^2, \tag{3.10}$$
$$\begin{aligned}\partial_t \Gamma^k_{ij}\big|_{t=0} = &\sum_{l=1}^n V h^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\ &- g^{kl}(\partial_i(Vh_{ij}) + \partial_j(Vh_{il}) - \partial_l(Vh_{ij})).\end{aligned} \tag{3.11}$$

Proof. In [44, Section 2.3], the first four identities are proven for the mean curvature flow and in all calculations one can replace H with V. The last identity follows from applying the others. □

With this we get for the linearisation[1] of $\Delta_{\varepsilon v} H_{\varepsilon v}$ for any $v \in C^4(\Gamma_*)$ that

$$\begin{aligned}
\frac{d}{d\varepsilon}\left(\Delta_{\Gamma_{\varepsilon v}} H_{\varepsilon v}\right)\big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\left(\sum_{i,j=1}^n g^{ij}_{\varepsilon v}\left(\partial_{ij} H_{\varepsilon v} - \sum_{k=1}^n \Gamma^k_{ij}\partial_k H_{\varepsilon v}\right)\right) \\
&= \underbrace{\sum_{i,j=1}^n g^{ij}_*\left(\partial_{ij}\frac{d}{d\varepsilon}H_{\varepsilon v}\big|_{\varepsilon=0} - \sum_{k=1}^n \Gamma^{k,*}_{i,j}\partial_k \frac{d}{d\varepsilon}H_{\varepsilon v}\big|_{\varepsilon=0}\right)}_{=\Delta_{\Gamma_*}(\frac{d}{d\varepsilon}H_{\varepsilon v}|_{\varepsilon=0}) = \Delta_{\Gamma_*}(\Delta_{\Gamma_*} v + v|II_*|^2)} \\
&+ \sum_{i,j=1}^n \left(\underbrace{\frac{d}{d\varepsilon}g^{ij}_{\varepsilon v}\big|_{\varepsilon=0}}_{=2vh^{ij}_*}\left(\partial_{ij}H_* - \sum_{k=1}^n \Gamma^{k,*}_{i,j}\partial_k H_*\right) - \sum_{k=1}^n \underbrace{\frac{d}{d\varepsilon}\Gamma^{k,\varepsilon v}_{i,j}\big|_{\varepsilon=0}}_{cf.(3.11)}\partial_k H_*\right) \\
&= \Delta_*\Delta_* v + \mathcal{A}_P(v),
\end{aligned}$$

where the lower order term $\mathcal{A}_P(v)$ contains derivatives of up to order two with coefficients in $C^\alpha(\Gamma_*)$. Due to the last fact we can take care of this term later using a perturbation argument. For the linearisation of $\frac{1}{N_* \cdot N_\rho}$ we get

$$\frac{d}{d\varepsilon}\left(\frac{1}{N_* \cdot N_{\varepsilon v}}\right)\Bigg|_{\varepsilon=0} = \frac{-\frac{d}{d\varepsilon}(N_{\varepsilon v})\cdot N_*}{(N_*\cdot N_*)^2} \overset{(3.9)}{=} \nabla_* v \cdot N_* = 0. \tag{3.12}$$

With the product rule and $N_* \cdot N_* \equiv 1$ we conclude the linearised problem

$$(LSDFC)\begin{cases}\partial_t v &= -\Delta_*\Delta_* v + \mathcal{A}_P(v) + \mathfrak{f} \quad \text{on } \Gamma_* \times [0,T],\\ v(\cdot,0) = v_0 & \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; \text{on } \Gamma_*.\end{cases} \tag{3.13}$$

Hereby, $\mathfrak{f} \in C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T})$ is an included inhomogeneity that we will need later.

3.3 Analysis of the Linearised Surface Diffusion Flow

We now want to discuss the analysis of (LSDFC), which will also give us the main strategy for our work on triple junction manifolds. At first, we will only consider the reduced system $(LSDFC)_P$ that is just $(LSDFC)$ without $\mathcal{A}_P$. We will introduce a concept of a weak solution of $(LSDFC)_P$ and will then show that for any $\mathfrak{f} \in L^2(0,T;L^2(\Gamma_*))$ a unique weak solution exists. In the next step we will apply locally Hölder theory to see that the localized problem has for any $T > 0$ and any $\mathfrak{f} \in C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T}), v_0 \in C^{4+\alpha}(\Gamma_*)$ a unique solution in $C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})$. Then we will check that we can approximate the weak solution constructed before with solutions of the localized problem and together with a compactness argument we conclude Hölder regularity for the weak solution. In the

[1] Note that for the linearization the normal velocity is just v!

end, we will include $\mathcal{A}(v)$ using a perturbation argument. Our final result of the section will be the following.

Theorem 3.3 (Short-time existence for (LSDFC)).
There is a $T > 0$ such that for all $\mathfrak{f} \in C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T})$ and $v_0 \in C^{4+\alpha}(\Gamma_*)$ there exists a solution v of $(LSDFC)$ in $C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})$ and we have the energy estimate

$$\|v\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})} \leq C(\|\mathfrak{f}\|_{C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T})} + \|v_0\|_{C^{4+\alpha}(\Gamma_*)}). \tag{3.14}$$

Hereby, the constant C does not depend on $\mathfrak{f}$ and v_0 and holds also for any smaller T.

To find a suitable concept of weak solution we observe that for any $\psi \in C^\infty(\Gamma_*)$ we have formally

$$\int_{\Gamma_*} (-\Delta_*\Delta_* v)\psi d\mathcal{H}^n = \int_{\Gamma_*} \nabla_* (\Delta_* v) \cdot \nabla_* \psi d\mathcal{H}^n = \int_{\Gamma_*} -\Delta_* v \Delta_* \psi d\mathcal{H}^n.$$

This motivates to define

$$\begin{aligned} \mathcal{L} &:= L^2(\Gamma_*), \\ \mathcal{L}_T &:= L^2(0,T;L^2(\Gamma_*)), \\ \mathcal{E} &:= H^2(\Gamma_*), \\ \mathcal{E}_T &:= L^2(0,T;H^2(\Gamma_*)), \\ B[v,\psi] &:= \int_{\Gamma_*} \Delta_* v \Delta_* \psi d\mathcal{H}^n, \quad \forall v,\psi \in \mathcal{E}, \end{aligned}$$

and introduce the following definition of weak solutions.

Definition 3.4 (Weak Solution of $(LSDFC)_P$).
We say that for $\mathfrak{f} \in \mathcal{L}_T$ and $v_0 \in \mathcal{L}$ a function

$$v \in \mathcal{E}_T, \text{ with } v' \in L^2(0,T;\mathcal{E}^{-1}),$$

is a weak solution of $(LSDFC)_P$ if we have

$$\langle v',\psi\rangle + B[v,\psi] = (\mathfrak{f},\psi), \tag{3.15}$$

for all $\psi \in \mathcal{E}$ and almost all $t \in [0,T]$ and additionally

$$v(0) = v_0. \tag{3.16}$$

Hereby, $\langle\cdot,\cdot\rangle$ denotes the usual dual pairing of $\mathcal{E}^{-1}$ and $\mathcal{E}$ and $(.,.)$ the L^2-product.

Like in the standard situation one can show that if a weak solution is in $C^{4,1}(\Gamma_* \times [0,T])$ it solves $(LSDFC)_P$ classically but we will skip the details here. We now want to show existence of such weak solutions.

Proposition 3.5 (Existence of weak solutions of $(LSDFC)_P$).
For any $\mathfrak{f} \in \mathcal{L}_T, v_0 \in L^2(\Gamma_*)$ there exists a unique weak solution v of $(LSDFC)_P$ and we have the energy estimate

$$\max_{0\leq t\leq T}\|v\|_{L^2(\Gamma_*)} + \|v\|_{L^2(0,T;\mathcal{E})} + \|v'\|_{L^2(0,T;\mathcal{E}^{-1})} \leq C(\|\mathfrak{f}\|_{L^2(0,T;L^2(\Gamma_*))} + \|v_0\|_{L^2(\Gamma_*)}). \tag{3.17}$$

Proof. We want to apply a Galerkin scheme. Firstly, we need an orthonormal basis $(u_m)_{m=1}^\infty$ of $\mathcal{L}$ that is also an orthogonal basis of $H^2(\Gamma_*)$. For this we consider the problem

$$(u,\psi)_\mathcal{E} = (f,\psi)_\mathcal{L} \quad \forall \psi \in \mathcal{E}, \tag{3.18}$$

for $f \in H^2(\Gamma_*)$. The Riesz representation theorem give us the existence of a unique solution $S(f)$. Additionally, the problem corresponds to a fourth order elliptic problem and thus by applying elliptic

regularity theory from [52, Theorem 11.1] we see that S is a continuous mapping from $\mathcal{E}$ to $H^4(\Gamma_*)$ and thus by composition with a compact Sobolev embedding a compact operator from $\mathcal{E}$ into itself. Additionally, by construction the operator is self-adjoint and so we can apply spectral theory to obtain an orthonormal basis $(\bar{u}_m)_{m=1}^{\infty}$ of $\mathcal{E}$. Furthermore, due to (3.18) this is also an orthogonal system of $\mathcal{L}$ and also a dense subset of $\mathcal{L}$ as it is dense in $\mathcal{E}$, which itself is dense in $\mathcal{L}$. So, by definition $(\bar{u}_m)_{m=1}^{\infty}$ is an orthogonal basis of $\mathcal{L}$ and after normalization we get the sought $(u_m)_{m=1}^{\infty}$.
Next, for a fixed $m \in \mathbb{N}$ we make the ansatz

$$v_m(t) := \sum_{k=1}^{m} d_m^k(t) u_k,$$

with unknown coefficient functions d_m^k. Supposing that these are differentiable, (3.15) and (3.16) transform to

$$\frac{d}{dt} d_m^k(t) + \sum_{l=1}^{m} e^{kl} d_m^l(t) = f^k(t), \qquad k = 1, ..., m, \tag{3.19}$$

$$d_m^k(0) = v_0^k, \qquad k = 1, ..., m, \tag{3.20}$$

where we used the abbreviations

$$e^{kl} := B[u_k, u_l], \quad f^k(t) := (f(t), u_k)_{L^2}, \quad v_0^k := (v_0, u_k)_{L^2}.$$

For this system the Caratheodory existence theorem, cf. [12, Chapter 2, Theorem 1.1], guarantees us the existence of a unique solution, which is differentiable almost everywhere.
We now need an energy estimate for the approximating solutions. We first note that we can apply locally[2] interior H^2-regularity (see e.g. [24, Section 6.3, Theorem 1]) on the Poisson's equation on Γ_* to get

$$\|u\|_{\mathcal{E}}^2 \leq C(\|\Delta_* u\|_{\mathcal{L}}^2 + \|u\|_{\mathcal{L}}^2), \quad \forall u \in \mathcal{E}. \tag{3.21}$$

Like in the flat situation we have for v_m that for almost all $t \in [0, T]$ it holds

$$(\partial_t v_m, v_m) + B[v_m, v_m] = (\mathfrak{f}, v_m). \tag{3.22}$$

As we have $B[v_m, v_m] = \|\Delta_* v_m\|_{\mathcal{L}}^2$, we may apply (3.21) to get

$$\frac{d}{dt}\|v_m\|_{\mathcal{L}}^2 + C\|v_m\|_{\mathcal{E}}^2 - C\|v_m\|_{\mathcal{L}}^2 \leq (\mathfrak{f}, v_m). \tag{3.23}$$

Now, using the weighted Young's inequality on the right-hand side leads to

$$\frac{d}{dt}\left(\|v_m\|_{L^2}^2\right) \leq \frac{d}{dt}\left(\|v_m\|_{L^2}^2\right) + 2\|v_m\|_{H^2}^2 \leq C_1\|v_m\|_{L^2}^2 + C_2\|f\|_{L^2}^2, \tag{3.24}$$

for almost all $t \in [0, T]$ for suitable constants C_1, C_2.We may apply Gronwall's inequality to get

$$\|v_m(t)\|_{L^2}^2 \leq e^{C_1 t}\left(\|v_m(0)\|_{L^2}^2 + C_2\|f\|_{L^2(0,T;L^2(\Gamma_*))}^2\right). \tag{3.25}$$

As $\|v_m(0)\|_{L^2}^2 \leq \|v_0\|_{L^2}^2$ due to the construction of v_m and the fact that u_m is an orthonormal basis of $\mathcal{L}$, this yields

$$\max_{t\in[0,T]}\|v_m(t)\|_{\mathcal{L}}^2 \leq C(\|f\|_{\mathcal{L}_T}^2 + \|v_0\|_{\mathcal{L}}^2). \tag{3.26}$$

[2] We will skip details how to connect the regularity of the localized problem with the original problem as this works very similarly to the arguments in the proof of Proposition 3.8

Now, integrating the second inequality in (3.24) in time we get

$$\begin{aligned}\|v_m\|_{\mathcal{E}_T}^2 &\leq \|v_m(0)\|_{\mathcal{L}}^2 - \|v_m(T)\|_{\mathcal{L}}^2 + C(\|v_m\|_{\mathcal{L}_T}^2 + \|\mathfrak{f}\|_{\mathcal{L}_T}^2) \\ &\leq CT(\|\mathfrak{f}\|_{\mathcal{L}_T}^2 + \|v_0\|_{\mathcal{L}}^2). \end{aligned} \tag{3.27}$$

Finally, we can argue as in the proof of [24, Section 7.1.2, Theorem 2] to get

$$\|v_m'\|_{L^2(0,T;\mathcal{E}^{-1})} \leq C(\|v_0\|_{\mathcal{L}}^2 + \|\mathfrak{f}\|_{\mathcal{L}_T}^2).$$

This shows (3.17) uniformly for all v_m and so we get the existence of $v \in \mathcal{E}_T, v' \in L^2(0,T;\mathcal{E}^{-1})$ and a subsequence $\{v_{m_l}\}_{l=1}^\infty \subset \{v_m\}_{m=1}^\infty$ with

$$\begin{cases} v_{m_l} \rightharpoonup v & \text{weakly in } \mathcal{E}_T, \\ v_{ml}' \rightharpoonup v' & \text{weakly in } L^2(0,T;\mathcal{E}^{-1}). \end{cases} \tag{3.28}$$

The rest of the proof can be carried out like in the proof of [24, Section 7.1.2, Theorem 3,4] □

For technical reasons, we will need higher H^k-regularity.

Corollary 3.6 (H^4-regularity for weak solutions of (LSDFC))**.**
Suppose that $v_0 \in H^2(\Gamma_*)$. Then, the solution from Proposition 3.5 is in $L^2(0,T;H^4(\Gamma_*))$ and we have

$$\begin{aligned}\|v\|_{L^2(0,T;H^4(\Gamma_*))} &\leq C(\|\mathfrak{f}\|_{L^2(0,T;L^2(\Gamma_*))} + \|v_0\|_{H^2(\Gamma_*)} + \|v\|_{L^2(0,T;\mathcal{L})}) \\ &\leq C(\|\mathfrak{f}\|_{L^2(0,T;L^2(\Gamma_*))} + \|v_0\|_{H^2(\Gamma_*)}). \end{aligned} \tag{3.29}$$

Proof. As in the proof of [24, Chapter 7, Theorem 5] we see that $v' \in L^2(0,T;L^2(\Gamma_*))$ and

$$\|v'\|_{L^2(0,T;L^2(\Gamma_*))} \leq C(\|\mathfrak{f}\|_{L^2(0,T;L^2(\Gamma_*))} + \|v_0\|_{H^2(\Gamma_*)}).$$

Applying [52, Chapter 5, Theorem 11.2] gives the first inequality and then together with (3.17) the sought result. □

We now want to verify Hölder regularity for the constructed weak solution for smooth enough initial data and right-hand sides. We have to study the localized problem first. For this we take any point $x \in \Gamma_*$ and a local parametrisation $\varphi : B_1(0) \to V$ with some open neighbourhood V of x on Γ_*. Then, we choose a smooth cut-off function $\chi : B_1(0) \to [0,1]$ with

$$\chi \equiv 1 \text{ on } B_{\frac{1}{2}}(0), \quad \operatorname{supp}(\chi) \subset B_{\frac{3}{4}}(0).$$

This induces now via pushforward a cutoff function on V, which we will also denote by χ with corresponding properties. Setting now $\widetilde{v} := \chi v$ we observe that the Bilaplacian of $\widetilde{v}$ in local coordinates is given by

$$\begin{aligned} &\sum_{i,j=1}^n g_*^{ij}\partial_{ij}\left(\sum_{k,l=1}^n g_*^{kl}\partial_{kl}\widetilde{v} - \sum_{m=1}^n \Gamma_{kl}^{m,*}\partial_m\widetilde{v}\right) + \sum_{m=1}^n \Gamma_{ij}^{m,*}\partial_m\left(\sum_{k,l=1}^n g_*^{kl}\partial_{kl}\widetilde{v} - \sum_{l=1}^n \Gamma_{kl}^{l,*}\partial_l\widetilde{v}\right) \\ = &\sum_{i,j,k,l=1}^n g_*^{ij}g_*^{kl}\partial_{ijkl}\widetilde{v} + \sum_{|\alpha|\leq 3} a_\alpha \partial_\alpha \widetilde{v}. \end{aligned}$$

The coefficient functions a_α are constant in time and at least $C^{1+\alpha}$ in space. Therefore, this motivates to consider the localized problem (LLSDFC) given by

$$\begin{cases} \partial_t \widetilde{v} + \sum_{i,j,k,l=1}^n g_*^{ij} g_*^{kl} \partial_{ijkl} \widetilde{v} - \sum_{|\alpha|\leq 3} a_\alpha \partial_\alpha \widetilde{v} = \widetilde{\mathfrak{f}} & \text{on } B_1(0) \times [0,T], \\ \widetilde{v} = 0 & \text{on } \partial B_1(0) \times [0,T], \\ \partial_\nu \widetilde{v} = 0 & \text{on } \partial B_1(0) \times [0,T], \\ \widetilde{v}\big|_{t=0} = \widetilde{v}_0 & \text{on } B_1(0). \end{cases}$$

Hereby, $\widetilde{v}_0$ denotes the cut-off of the initial data from before and the term $\widetilde{\mathfrak{f}}$ contains some perturbation terms resulting from the localization. We will discuss them later when we actually show regularity of the original weak solution from Proposition 3.5. Now, we want to see that this problem admits maximal Hölder regularity.

Proposition 3.7 (Hölder regularity for (LLSDFC)).
For any $\widetilde{\mathfrak{f}} \in C^{\alpha,\frac{\alpha}{4}}(B_1(0) \times [0,T])$ and $\widetilde{v}_0 \in C^{4+\alpha}(B_1(0))$ the system $(LLSDFC)$ has a unique solution $\widetilde{v}$ in $C^{4+\alpha,1+\frac{\alpha}{4}}(B_1(0) \times [0,T])$ and we have the energy estimate

$$\|\widetilde{v}\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(B_1(0)\times[0,T])} \leq C(\|\widetilde{\mathfrak{f}}\|_{C^{\alpha,\frac{\alpha}{4}}(B_1(0)\times[0,T])} + \|\widetilde{v}_0\|_{C^{4+\alpha}(B_1(0))}). \tag{3.30}$$

Additionally, for any domains $\Omega', \Omega'' \subset B_1(0)$ with $\Omega' \subset \Omega''$ and $d_H(\Omega', B_1(0) \backslash \Omega'') > 0$ we have

$$\|\widetilde{v}\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Omega'_T)} \leq C(\|\widetilde{\mathfrak{f}}\|_{C^{\alpha,\frac{\alpha}{4}}(\Omega''_T)} + \|\widetilde{v}_0\|_{C^{4+\alpha}(\Omega'')} + \|\widetilde{v}\|_{L^2(\Omega'')_T}) \tag{3.31}$$

Proof. We want to apply Theorem VI.21 of [21]. Firstly, we note that the system is uniform parabolic in the sense of Petrovsky as we have

$$L_0(x,t,i\zeta,p) = p + |\zeta|_g^4, \qquad \forall \zeta \in \mathbb{R}^n,$$

where $|.|_g$ denotes the norm induced by the inverse metric tensor g_*^{-1}. This polynom has just one p-zero, that is $-|\zeta|_g^4$. So, we may choose $s = 4, r = 1$ and $\delta_0 = \|g_*^{-1}\|$ to fulfil the parabolicity condition. Thus, the problem $(LLSDFC)$ fits in the setting introduced in [21, Section VI.3] with

$$b = 2, \quad m = 1, \quad r_1 = 0, \quad r_2 = 1, \quad \varphi_1 \equiv \varphi_2 \equiv 0, \quad b^1_{\alpha\alpha_0}(x,t) = 1, \quad b^2_{\alpha\alpha_0}(x,t) = x_\alpha.$$

As mentioned above all a_α are in $C^{1+\alpha}(B_1(0)\times[0,T])$ and the $b^q_{\alpha\alpha_0}, q = 1,2$, are in $C^\infty(\partial B_1(0)\times[0,T])$. Furthermore, $\partial B_1(0)$ is of class C^∞ and thus the smoothness conditions β_{23}, β_{24} and β_{25} are fulfilled for $l = \alpha$. Additionally, the condition (3.5) from [21, Section VI.3] is also fulfilled as $r_1 - 2b = -3$ and $r_2 - 2b = -2$. Finally, $\widetilde{v}_0$ equals zero in a neighbourhood of $\partial B_1(0)$ and consequently compatibility conditions of any order are fulfilled. Hence, we can apply Theorem VI.21 in [21] to get the sought result.
The estimate (3.31) follows from [51, Theorem 4.11]. □

Next we want to link the localized problem with local regularity of the solution v from Proposition 3.5. The idea is the following. First, we find a suitable problem for the cut-off of v on V that also has a unique weak solution that coincides with the cut-off of v. In local coordinates we will then basically have problem $(LLSDFC)$ with the only problem that the right-hand side $\mathfrak{f}$ is only in $L^2(B_1(0))$. Approximating $\mathfrak{f}$ and using a compactness argument will finish the proof.
We observe that formally we have

$$\begin{aligned} \Delta_*\Delta_*\widetilde{v} &= \Delta_*\left(\chi\Delta_* v + 2\nabla_* v \cdot \nabla_*\chi + v\Delta_*\chi\right) = \chi\Delta_*\Delta_* v + \mathfrak{f}_P, \\ \mathfrak{f}_P &:= \Delta_*\chi\Delta_* v + 2\nabla_*\chi \cdot \nabla_*\Delta_* v + \Delta_*\left(2\nabla_* v \cdot \nabla_*\chi + v\Delta_*\chi\right), \\ \partial_t\widetilde{v} &= \chi\partial_t v. \end{aligned}$$

Thus, it is natural to consider

$$\begin{cases} \partial_t \widetilde{v} + \Delta_* \Delta_* \widetilde{v} = \widetilde{\mathfrak{f}} & \text{on } V \times [0,T], \\ \widetilde{v} = 0 & \text{on } \partial V \times [0,T], \\ \partial_\nu v = 0 & \text{on } \partial V \times [0,T], \\ \widetilde{v}(\cdot,0) = \widetilde{v}_0 & \text{on } V, \end{cases} \tag{3.32}$$

where ν is understood as the normal of V as submanifold of Γ_* and we set

$$\widetilde{\mathfrak{f}} := \chi \mathfrak{f} + \mathfrak{f}_P. \tag{3.33}$$

Note that due to Corollary 3.6 the term $\widetilde{\mathfrak{f}}$ is well-defined and in $L^2(0,T;L^2(\Gamma_*))$. Similarly[3] to the proof of Lemma 3.5 we can show the existence of a unique weak solution $\widetilde{v}$ with the same energy estimates as in (3.17) and using definition 3.4 we see that $\widetilde{v} = \chi v$. So, if we can now link (3.32) with the localized problem we will get the desired Hölder regularity for v.

Proposition 3.8 (Hölder regularity for solutions of $(LSDFC)_P$**).**
Suppose that $\mathfrak{f} \in C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T}), v_0 \in C^{4+\alpha}(\Gamma_*)$. *Then, there is an open subset* $V' \subset V$ *with*

$$\widetilde{v}\big|_{V'} = v, \quad \widetilde{v}\big|_{V'} \in C^{4+\alpha,1+\frac{\alpha}{4}}(V' \times [0,T]),$$

on which the energy estimate

$$\|\widetilde{v}\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(V'_T)} \le C(\|\mathfrak{f}\|_{C^{\alpha,\frac{\alpha}{4}}(V'_T)} + \|v_0\|_{C^{4+\alpha}(V')}). \tag{3.34}$$

holds. Consequently, we have $v \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})$ *and the energy estimate*

$$\|v\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})} \le C(\|\mathfrak{f}\|_{C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T})} + \|v_0\|_{C^{4+\alpha}(\Gamma_*)}). \tag{3.35}$$

Proof. We will use the abbreviations

$$U_4 := B_1(0),\ U_3 := B_{\frac{1}{2}}(0),\ U_2 := B_{\frac{1}{4}}(0),\ U_1 := B_{\frac{1}{8}}(0),\ Q_i := \varphi(U_i).$$

We choose an approximating sequence $\{\widetilde{\mathfrak{f}}_n\}_{n\in\mathbb{N}} \subset C^{\alpha,\frac{\alpha}{4}}(U_4 \times [0,T])$ of $\widetilde{\mathfrak{f}}$ with

$$\|\widetilde{\mathfrak{f}}_n - \widetilde{\mathfrak{f}}\|_{L^2(U_1\times[0,T])} \to 0, \quad \|\widetilde{\mathfrak{f}}_n\|_{C^{\alpha,\frac{\alpha}{4}}(U_2\times[0,T])} \le \|\widetilde{\mathfrak{f}}\|_{C^{\alpha,\frac{\alpha}{4}}(U_3\times[0,T])} = \|\mathfrak{f}\|_{C^{\alpha,\frac{\alpha}{4}}(U_3\times[0,T])}. \tag{3.36}$$

The second property can be guaranteed for example in the following way. We choose any approximating sequence on $(U_4 \backslash U_3) \times [0,T]$, on $U_3 \times [0,T]$ just $\widetilde{\mathfrak{f}} = \mathfrak{f}$ and combine both approximations via a partition of unity. The problems $(LLSDFC)$ and (3.32) with inhomogeneity $\widetilde{\mathfrak{f}}_n$ we call now $(LLSDFC)_n$ resp. $(3.32)_n$. Due to Proposition 3.7 we get a unique solution $\widetilde{v}_n \in C^{4+\alpha,1+\frac{\alpha}{4}}(U_1\times[0,T])$ of $(LLSDFC)_n$ and $\widetilde{v}_n \circ \varphi^{-1}$(which we will also denote by $\widetilde{v}_n$) is the weak solution of $(3.32)_n$. Using (3.17) and (3.36) we get

$$\begin{aligned} \|\widetilde{v}_n\|_{L^2(0,T;H^2(Q_4))} &\le C\left(\|\widetilde{\mathfrak{f}}_n\|_{L^2(Q_4\times[0,T])} + \|\widetilde{v}_0\|_{L^2(Q_4)}\right) \\ &\le C\left(\|\widetilde{\mathfrak{f}}\|_{L^2(Q_4\times[0,T])} + \|v_0\|_{L^2(Q_4)}\right) \\ &\le C\left(\|\mathfrak{f}\|_{L^2(Q_4\times[0,T])} + \|v\|_{L^2(0,T;H^3(Q_4))} + \|v_0\|_{L^2(Q_4)}\right) \\ &\le C(\|\mathfrak{f}\|_{L^2(\Gamma_{*,T})} + \|v_0\|_{L^2(\Gamma_*)}). \end{aligned} \tag{3.37}$$

In the third inequality we used the identity (3.33) and in the fourth step (3.29). This implies now the existence of a subsequence $\{\widetilde{v}_{n_l}\}_{l\in\mathbb{N}}$ together with an $\bar{v} \in L^2(0,T;H^2(Q_4))$ such that

$$\widetilde{v}_{n_l} \rightharpoonup \bar{v}. \tag{3.38}$$

[3]The main difference is that we have to include the Dirichlet and Neumann boundary conditions in the space $\mathcal{E}$.

Due to the defintion of weak convergence and the construction of weak solutions we see that $\bar{v}$ is also a weak solution of (3.32) and by uniqueness of weak solutions we get $\bar{v} = \widetilde{v}$. We denote the sequence $\{\widetilde{v}_{n_l}\}$ by $\{\widetilde{v}_n\}$ and now calculate for its Hölder norm

$$\begin{aligned} \|\widetilde{v}_n\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(Q_{1,T})} &\le C(\|\widetilde{\mathfrak{f}}_n\|_{C^{\alpha,\frac{\alpha}{4}}(Q_2\times[0,T])} + \|\widetilde{v}_0\|_{C^{4+\alpha}(Q_2)} + \|\widetilde{v}_n\|_{L^2(Q_2\times[0,T])}) \\ &\le C(\|\mathfrak{f}\|_{C^{\alpha,\frac{\alpha}{4}}(Q_{3,T})} + \|v_0\|_{C^{4+\alpha}(Q_2)} + \|\mathfrak{f}\|_{L^2(\Gamma_{*,T})} + \|v_0\|_{L^2(\Gamma_*)}) \\ &\le C(\|\mathfrak{f}\|_{C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T})} + \|v_0\|_{C^{4+\alpha}(\Gamma_*)}). \end{aligned} \tag{3.39}$$

Here, we used in the first inequality the Hölder estimates (3.31) for the localized problem and in the second inequality (3.36) and (3.37). Now, as we have uniform bounds for the Hölder norms of all derivatives of up to order four this shows that the sequences of all derivatives are equicontinuous. Additionally, the bounds on the derivatives show that the sequences of all derivatives are bounded in C^0. Applying Arzela-Ascoli on all derivatives and using the fact that uniform convergence of derivatives implies differentiability of the limit, we see that a subsequence $\{\widetilde{v}_{n_l}\}_{l\in\mathbb{N}}$ converges in $C^{4,1}(\Gamma_{*,T})$ to $\widehat{v} \in C^{4,1}(\Gamma_{*,T})$. Additionally, the limit is also Hölder continuous with the same bounds (3.39) as we have for any $x, y \in V_1$ and any partial derivative $\partial_t\partial_k$ in space and time that

$$|\partial_t\partial_k\widehat{v}(x) - \partial_t\partial_k\widehat{v}(y)| = \lim_{l\to\infty} |\partial_t\partial_k\widehat{v}_{n_l}(x) - \partial_t\partial_x\widehat{v}_{n_l}(y)| \le Cd(x,y)^{\alpha(k,t)},$$

where $\alpha(k,t)$ denotes the Hölder exponent corresponding to the derivative. Using again uniqueness of limits we see $\widehat{v} = \bar{v} = \widetilde{v}$ and thus we get (3.34) on $V' = \varphi(B_{\frac{1}{8}}(0))$. Then, (3.35) is a direct consequence using compactness of Γ_*. □

We are now almost done with the analysis of the linear problem. In our last step, we have to include the lower order term $\mathcal{A}_P$ via a perturbartion argument.

Proof. (of Theorem 3.3) We fix $\mathfrak{f}$ and v_0 and consider the solution operator $S(\mathfrak{f}, v_0)$ given by Proposition 3.8. Setting for any $v \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})$

$$\bar{\mathfrak{f}}(v) := \mathfrak{f} - \mathcal{A}_P(v),$$

we get a mapping

$$\begin{aligned} \bar{S} : C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T}) &\to C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T}), \\ v &\mapsto S(\bar{\mathfrak{f}}(v), v_0). \end{aligned}$$

We claim that $\bar{S}$ is a contraction for T chosen small enough. To see this we first observe that for all $v_1, v_2 \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})$ we have due to the structure of $\mathcal{A}_P$ and Lemma 2.17 that

$$\|\mathcal{A}_P(v_1 - v_2)\|_{C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T})} \le C\|v_1 - v_2\|_{C^{3+\alpha,\frac{3+\alpha}{4}}(\Gamma_{*,T})} \le CT^{\bar{\alpha}}\|v_1 - v_2\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})},$$

for some $\bar{\alpha} \in (0,1)$. Thus, we conclude using linearity of S that

$$\begin{aligned} \|\bar{S}(v_1) - \bar{S}(v_2)\|_{C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T})} &= \|S(\mathcal{A}_P v_1 - \mathcal{A}_P v_2, 0)\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})} \\ &\le C\|\mathcal{A}_P(v_1 - v_2)\|_{C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T})} \\ &\le CT^{\bar{\alpha}}\|v_1 - v_2\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})}. \end{aligned} \tag{3.40}$$

Here, we used in the second step the energy estimates for S. Consequently, choosing T small enough we get a contraction and then Banach's fixed-point theorem gives us the existence of a unique fixed point, which is the sought solution of $(LSDFC)$. Note that the estimate (3.40) is independent of $\mathfrak{f}$ and v_0 and so the constructed T is. □

3.4 Analysis of the Non-Linear Problem

Before we start we want to make a comment on notation. Recall that according to Definition 2.13 the norm of parabolic Hölder spaces is defined in local coordinates. To keep notation simple, in the following section we will write the norm for a parabolic Hölder space on Γ_* even if we are working in local coordinates.
We now want to find a solution of $(SDFC)$ by connecting the non-linear problem with the linear one. For this, we introduce for $R, \delta > 0$ and $\rho_0 \in C^{4+\alpha}(\Gamma)$ the sets

$$\begin{aligned} X_{R,\delta} &:= \{\rho \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,\delta}) |\ \|\rho - \rho_0\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,\delta})} \leq R\}, \\ Y_\delta &:= C^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,\delta}). \end{aligned}$$

Hereby, $X_{R,\delta}$ is equipped with the $C^{4+\alpha,1+\frac{\alpha}{4}}$-norm. Observe now that with the inhomogeneity operator

$$\begin{aligned} S : X_{R,\delta} &\to Y_\delta, \\ \rho &\mapsto -\frac{1}{N_* \cdot N_\rho}\Delta_\rho H_\rho + \Delta_* \Delta_* v - \mathcal{A}_p(\rho), \end{aligned}$$

the solution of

$$\begin{aligned} \partial_t v + \Delta_* \Delta_* v - \mathcal{A}_P(v) &= S(\rho) && \text{on } \Gamma_{*,\delta}, \\ v(\cdot, 0) &= \rho_0 && \text{on } \Gamma_*, \end{aligned}$$

is just the solution of $(SDFC)$ with initial data ρ_0. Let L be the solution operator from Theorem 3.3 and set

$$\Lambda := L \circ S : X_{R,\delta} \to C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*).$$

We have to prove the existence of a unique fixed-point of Λ for δ and $\varepsilon := \|\rho_0\|_{C^{4+\alpha}(\Gamma_*)}$ sufficiently small and R sufficiently large. For this we will first verify that for any $R > 0$ we can choose δ small enough such that Λ is a $\frac{1}{2}$-contraction and afterwards that with this choice of δ we can choose R sufficiently large such that Λ is also a self mapping. But first we want to briefly check that Λ is well-defined.

Lemma 3.9 (Well-definedness of Λ).
There is an $\epsilon_W > 0$ such that for all $R > 0$ and all $\varepsilon < \varepsilon_W$ there exists a $\delta_W(\varepsilon, R)$ such that S (and consequently also Λ) is well-defined.

Proof. As we mentioned in Section 3.2 the operator S is well-defined as long we can guarantee that $\rho(t)$ is small enough in the C^1-norm. Due to the Hölder-regularity in time we have for all $\rho \in X_{R,\delta}$ that

$$\begin{aligned} \|\rho(t)\|_{C^4(\Gamma_*)} &\leq \|\rho_0\|_{C^4(\Gamma_*)} + \|\rho - \rho_0\|_{C^4(\Gamma_*)} \\ &\leq \varepsilon + C\left(t\|\partial_t(\rho - \rho_0)\|_\infty + \sum_{i=1}^{4} t^{\frac{4-i+\alpha}{4}} \sup_{|\beta|=i} \langle \partial_\beta^x(\rho - \rho_0)\rangle_{t,\frac{4-i+\alpha}{4}}\right) \\ &\leq \varepsilon + C\max(t, t^\alpha)\|\rho - \rho_0\|_{X_{R,\delta}} \\ &\leq \varepsilon + C(\max(\delta, \delta^\alpha)R). \end{aligned}$$

This shows that for any $\sigma > 0$ we will have $\|\rho(t)\|_{C^4(\Gamma_*)} \leq \sigma$ for $\varepsilon \leq \frac{\sigma}{2}$ and $\delta \leq \frac{\sigma}{2CR}$ if δ is larger than one or otherwise $\delta \leq \left(\frac{\sigma}{2CR}\right)^{\frac{1}{\alpha}}$. From this we conclude the claim. □

In the rest of this section we will always assume that $\varepsilon < \varepsilon_W$ and that δ is chosen small enough accordingly to Lemma 3.9. For technical reasons we will need the following observation before moving on to the contraction estimates.

Lemma 3.10 (Properties of lower order terms)**.**

i.) For any parametrisation $\varphi : V \to U \subset \Gamma_*$ and for all $i,j \in \{1,...,n\}$ we have that the g^ρ_{ij}, g^{ij}_ρ and N_ρ are Lipschitz continuous in ρ as maps

$$C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*) \to C^{1+\alpha,\frac{1+\alpha}{4}}(V) \text{ resp. } C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*) \to C^{1+\alpha,\frac{1+\alpha}{4}}(V,\mathbb{R}^n).$$

ii.) In particular, this is true for the mapping

$$\rho \mapsto g^{ij}_\rho g^{kl}_\rho \frac{1}{N_* \cdot N_\rho},$$

for all $i,j,k,l \in \{1,...,n\}$.

Proof. This can be proven as Lemma 4.22. As the situation here is even easier, we will skip the details. □

Lemma 3.11 (Contraction property of Λ)**.**
There is an $\varepsilon_0 < \min(\varepsilon_W, 1)$ with the following property: for any $R > 1$ and $\varepsilon < \varepsilon_0$ there is a $\delta(R,\varepsilon) > 0$ such that $\Lambda : X_{R,\delta} \to C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)$ is a $\frac{1}{2}$-contraction.

Proof. Let $\varepsilon < \min(\varepsilon_W, 1)$ and $R > 1$. Consider $\rho_1, \rho_2 \in X_{R,\delta}$ and observe that $\Lambda(\rho_1) - \Lambda(\rho_2)$ solves

$$\begin{aligned} \partial_t v + \Delta_*\Delta_* v - \mathcal{A}_P(v) &= S(\rho_1) - S(\rho_2) && \text{on } \Gamma_{*,\delta}, \\ v(\cdot,0) &= 0 && \text{on } \Gamma_*, \end{aligned}$$

Then, due to the energy estimates (3.14) we have that

$$\|\Lambda(\rho_1) - \Lambda(\rho_2)\|_{X_{R,\delta}} \leq C\|S(\rho_1) - S(\rho_2)\|_{Y,\delta}.$$

Thus, if we can show that S is a contraction mapping, we are finished. For this we first note that in local coordinates we have that

$$\begin{aligned} \Delta_\rho H_\rho &= \sum_{i,j,k,l=1}^n g^{ij}_\rho g^{kl}_\rho \partial_{ijkl}\rho + LOT, \\ \Delta_*\Delta_*\rho &= \sum_{i,j,k,l=1}^n g^{ij}_* g^{kl}_* \partial_{ijkl}\rho + LOT. \end{aligned}$$

This implies that the terms of highest order in $S(\rho_1) - S(\rho_2)$ write as

$$\begin{aligned} &\sum_{i,j,k,l=1}^n \left(-\frac{1}{N_* \cdot N_{\rho_1}} g^{ij}_{\rho_1} g^{kl}_{\rho_1} \partial_{ijkl}\rho_1 + g^{ij}_* g^{kl}_* \partial_{ijkl}\rho_1 + \frac{1}{N_* \cdot N_{\rho_2}} g^{ij}_{\rho_2} g^{kl}_{\rho_2} \partial_{ijkl}\rho_2 - g^{ij}_* g^{kl}_* \partial_{ijkl}\rho_2 \right) \\ = &\sum_{i,j,k,l=1}^n \partial_{ijkl}\rho_1 \left(g^{ij}_* g^{kl}_* - \frac{1}{N_* \cdot N_{\rho_1}} g^{ij}_{\rho_1} g^{kl}_{\rho_1} \right) - \partial_{ijkl}\rho_2 \left(g^{ij}_* g^{kl}_* - \frac{1}{N_* \cdot N_{\rho_2}} g^{ij}_{\rho_2} g^{kl}_{\rho_2} \right) \end{aligned}$$

Now, by subtracting and adding $\partial_{ijkl}\rho_2 \left(g^{ij}_* g^{kl}_* - \frac{1}{N_* \cdot N_{\rho_1}} g^{ij}_{\rho_1} g^{kl}_{\rho_1} \right)$ we get for each summand

$$\underbrace{(\partial_{ijkl}\rho_1 - \partial_{ijkl}\rho_2) \left(g^{ij}_* g^{kl}_* - \frac{1}{N_* \cdot N_{\rho_1}} g^{ij}_{\rho_1} g^{kl}_{\rho_1} \right)}_{:=(I)} + \underbrace{\partial_{ijkl}\rho_2 \left(\frac{1}{N_* \cdot N_{\rho_2}} g^{ij}_{\rho_2} g^{kl}_{\rho_2} - \frac{1}{N_* \cdot N_{\rho_1}} g^{ij}_{\rho_1} g^{kl}_{\rho_1} \right)}_{:=(II)}.$$

We will now discuss (I) and (II) separately. For the first term we have

$$\|(I)\|_{Y_\delta} \leq C\|\partial_{ijkl}(\rho_1 - \rho_2)\|_{Y_\delta} \|g^{ij}_* g^{kl}_* - \frac{1}{N_* \cdot N_{\rho_1}} g^{ij}_{\rho_1} g^{kl}_{\rho_1}\|_{Y_\delta}$$

$$
\begin{aligned}
&\le C\|\rho_1-\rho_2\|_{X_{R,\delta}}\left(\varepsilon+\delta^{\bar\alpha}\|g_*^{ij}g_*^{kl}-\frac{1}{N_*\cdot N_{\rho_1}}g_{\rho_1}^{ij}g_{\rho_1}^{kl}\|_{C^{1+\alpha,\frac{1+\alpha}{4}}(\Gamma_{*,\delta})}\right)\\
&\le C\|\rho_1-\rho_2\|_{X_{R,\delta}}\left(\varepsilon+\delta^{\bar\alpha}\|\rho_1\|_{X_{R,\delta}}\right)\\
&\le C\|\rho_1-\rho_2\|_{X_{R,\delta}}\left(\varepsilon+\delta^{\bar\alpha}(R+\varepsilon)\right)\\
&\le C\|\rho_1-\rho_2\|_{X_{R,\delta}}\left(\varepsilon+2R\delta^{\bar\alpha}\right).
\end{aligned}
$$

Here, firstly we used Lemma 2.16, then in the second inequality Lemma 3.10, in the third line Lemma 2.17 and in the last line the definition of $X_{R,\delta}$. On the other hand, we have

$$
\begin{aligned}
\|(II)\|_{Y_\delta}&\le C\|\partial_{ijkl}\rho_2\|_{Y_\delta}\|\frac{1}{N_*\cdot N_{\rho_2}}g_{\rho_2}^{ij}g_{\rho_2}^{kl}-\frac{1}{N_*\cdot N_{\rho_1}}g_{\rho_1}^{ij}g_{\rho_1}^{kl}\|_{Y_\delta}\\
&\le C\|\rho_2\|_{X_{R,\delta}}\delta^{\bar\alpha}\|\frac{1}{N_*\cdot N_{\rho_2}}g_{\rho_2}^{ij}g_{\rho_2}^{kl}-\frac{1}{N_*\cdot N_{\rho_1}}g_{\rho_1}^{ij}g_{\rho_1}^{kl}\|_{C^{1+\alpha,\frac{1+\alpha}{4}}(\Gamma_{*,\delta})}\\
&\le C(R+\varepsilon)\delta^{\bar\alpha}\|\rho_1-\rho_2\|_{X_{R,\delta}}\\
&\le 2CR\delta^{\bar\alpha}\|\rho_1-\rho_2\|_{X_{R,\delta}}
\end{aligned}
$$

Here, we used Lemma 2.16 for the first inequality, in the second inequality Lemma 2.17 and in the third inequality Lemma 3.10 and the definition of $X_{R,\delta}$. In total, combining the estimates for (I) and (II) shows that

$$
\|S(\rho_1)-S(\rho_2)\|_{Y_\delta}\le C\left(\varepsilon+R\delta^{\bar\alpha}\right)\|\rho_1-\rho_2\|_{X_{R,\delta}}. \tag{3.41}
$$

By choosing $\varepsilon\le\min\left(1,\varepsilon_W,\frac{1}{4C}\right)$ and $\delta\le\min\left(\delta_W(R,\varepsilon),(4RC)^{-\bar\alpha^{-1}}\right)$ we can guarantee that S is an $\frac{1}{2}$-contraction, which finishes the proof. □

With this result choosing R large enough will make Λ a self-mapping.

Lemma 3.12 (Self-mapping of Λ)**.**
For $R>1$ given, let $\varepsilon(R)$ and $\delta(R,\varepsilon(R))$ be chosen as in Lemma 3.11. Then, there is a $R_0>0$ such that for all $R>R_0$ the map Λ is a self-mapping.

Proof. We observe that for any $\rho\in X_{R,\delta}$ we have due to the contraction property of Λ that

$$
\begin{aligned}
\|\Lambda(\rho)-\rho_0\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)}&\le\|\Lambda(\rho)-\Lambda(\rho_0)+\Lambda(\rho_0)-\rho_0\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)}\\
&\le\|\Lambda(\rho)-\Lambda(\rho_0)\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)}+\|\Lambda(\rho_0)-\rho_0\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)}\\
&\le\frac{1}{2}\|\rho-\rho_0\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)}+\|\Lambda(\rho_0)-\rho_0\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)}\\
&\le\frac{1}{2}R+\|\Lambda(\rho_0)-\rho_0\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)}.
\end{aligned}
\tag{3.42}
$$

We now observe that as ρ_0 is constant in time it solves

$$
\partial_t\rho_0+\Delta_*\Delta_*\rho_0-\mathcal{A}_P(\rho_0)=\Delta_*\Delta_*\rho_0-\mathcal{A}_P(\rho_0).
$$

Therefore, due to the construction of Λ the function $\Lambda(\rho_0)-\rho_0$ solves the problem

$$
\begin{aligned}
\partial_t v+\Delta_*\Delta_* v-\mathcal{A}_P(v)&=-\frac{1}{N_*\cdot N_\rho}\Delta_{\rho_0}H_{\rho_0} &&\text{on } \Gamma_{*,\delta},\\
v(0)&=0 &&\text{on } \Gamma_*.
\end{aligned}
$$

Using the energy estimates for S we conclude

$$
\|\Lambda(\rho_0)-\rho_0\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)}\le C\left\|\frac{1}{N_\rho\cdot N_*}\Delta_{\rho_0}H_{\rho_0}\right\|\le C\|\rho_0\|_{C^{4+\alpha}(\Gamma_*)}\le C\varepsilon(R)\le C. \tag{3.43}
$$

Choosing now $R_0 = 2C$ we get with (3.42) for any $R > R_0$ that

$$\|\Lambda(\rho) - \rho_0\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)} \leq \frac{1}{2}R + C\varepsilon(R) \leq R. \tag{3.44}$$

So, with this choice of R we get that Λ is a self-mapping. □

With this work done we can now prove easily the main result of this chapter.

Lemma 3.13 (Fixed-point of Λ)**.**
There is a $R_0 > 0$ such that for all $R > R_0$ and $\varepsilon(R), \delta(R, \varepsilon(R))$ chosen as in Lemma 3.12 the map $\Lambda : X_{R,\delta} \to X_{R,\delta}$ is well-defined and has a unique fixed-point. In particular, this shows Theorem 3.1.

Proof. Due to the Lemmas 3.11 and 3.12 we can apply Banach's fixed-point theorem, which then proves the claim. □

Remark 3.14 (Uniqueness in the analytic setting)**.**
We want to emphasise that the solution just found is actually unique in $C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*)$. One might think that by enlarging R we get another fixed-point in a new function space. But we have for $R_2 > R_1$ and the existence times $\delta(R_1,\varepsilon), \delta(R_2,\varepsilon_0)$ from Lemma 3.13 that $\delta(R_2,\varepsilon_0) < \delta(R_1,\varepsilon)$ and consequently

$$X_{R_1,\delta(R_2,\varepsilon_0)} \subset X_{R_2,\delta(R_2,\varepsilon_0)}.$$

So if there was another fixed-point in $X_{R_2,\delta(R_2,\varepsilon_0)}$, we would have two fixed-points in $X_{R_1,\delta(R_2,\varepsilon_0)}$. On this set Banach's fixed-point theorem is still valid and so both solutions have to be equal on the time interval $[0, \delta(R_2, \varepsilon)]$. Fix now any $R_2 > R_1$ and observe that on $X_{R_2,\delta(R_2,\varepsilon_0)}$ we have a unique solution that is in $X_{R_1,\delta(R_2,\varepsilon_0)}$, which is in the interior of $X_{R_2,\delta(R_2,\varepsilon_0)}$. Now, if we had two solutions, they would have to to be equal on a small time interval by the argumentation we just used. Consider now the maximal time T' such that the two solutions are equal. Then, due to continuity reasons the second solution remains in the R_2-ball for a short time after T' as up to T' both solutions are in the interior of the R_2-ball. But on this set the solution is unique and thus we get a contradiction. This shows that our solution is indeed unique on the time interval $[0, \delta(R_2, \varepsilon_0)]$.

4

Short Time Existence for the Surface Diffusion Flow of Triple Junction Manifolds

After we understood our principal strategy to prove existence of solutions in a Hölder space setting in the last chapter we will now apply it to prove the first main result of this thesis, that is (analytic) existence of the surface diffusion flow on triple junction manifolds. We will first describe the considered geometry and derive suitable boundary conditions to guarantee a structure alike to a $\mathcal{H}^{-1}$-gradient flow of the surface area with volume preservation. Then we will introduce the parametrisation of the evolution over a reference surface and state the considered analytic problem. Afterwards we will discuss necessary compatibility conditions for the analytic and geometric version of the flow, show equivalence of both and state our result on short time existence. We may then start the procedure from the last chapter to prove it.

4.1 The Model and its Physical Properties

Let $\Gamma^1, \Gamma^2, \Gamma^3$ be three embedded, orientable, compact $C^{4+\alpha}$-hypersurfaces in $\mathbb{R}^{n+1}$ with a common boundary Σ, that is $\partial\Gamma^1 = \partial\Gamma^2 = \partial\Gamma^3 := \Sigma$, which is then an embedded submanifold of dimension $n-1$. The three hypersurfaces shall not intersect with each other. Every pair of two different hypersurfaces Γ^i and Γ^j forms then a closed volume, which we denote by Ω_{ij}, and there is precisely one choice such that the third hypersurface is included in this volume. Without loss of generality let Γ^1 be inside the domain bounded by Γ^2 and Γ^3. For Γ^1 we choose the unit normal field N^1 pointing inside of Ω_{12}, for Γ^2 we choose N^2 to point outside of Ω_{12} and for Γ^3 we choose N^3 to point inside of Ω_{13}. The corresponding outer conormals of the hypersurfaces will always be denoted by ν^i. In the introduction, we gave a physical modelling for the flow and the arising boundary conditions. Here, we would like to give a mathematical motivation. We want to construct a motion that both minimizes the surface areas and preserves the enclosed volumes Ω_{12} and Ω_{13}. Hereby, we consider a slightly generalized energy by assuming that each hypersurface has constant, positive surface density $\gamma^i, i = 1, 2, 3$. That is, we consider the surface energy function given by

$$E(\Gamma(t)) := \sum_{i=1}^{3} \int_{\Gamma^i(t)} \gamma^i d\mathcal{H}^n. \tag{4.1}$$

Motivated by the $\mathcal{H}^{-1}$-flow and its properties for closed hypersurfaces from the last chapter, we expect as a good candidate for the motion law

$$V_{\Gamma^i(t)} = -\Delta_{\Gamma^i(t)} H_{\Gamma^i(t)} \qquad \text{for } i = 1,2,3. \tag{4.2}$$

For this flow we derive for the change of the volume of Ω_{12} using Reynold's transport theorem that

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_{12}(t)} 1dx &= \int_{\partial\Omega_{12}(t)} V_{\partial\Omega_{12}(t)} d\mathcal{H}^n \\ &= \int_{\Gamma^1(t)} \Delta_{\Gamma^1(t)} H_{\Gamma^1(t)} d\mathcal{H}^n - \int_{\Gamma^2(t)} \Delta_{\Gamma^2(t)} H_{\Gamma^2(t)} d\mathcal{H}^n \\ &= \int_{\Sigma(t)} \nabla_{\Gamma^1(t)} H_{\Gamma^1(t)} \cdot \nu_{\Gamma^1(t)} - \nabla_{\Gamma^2(t)} H_{\Gamma^2(t)} \cdot \nu_{\Gamma^2(t)} d\mathcal{H}^n.\end{aligned}$$

One obtains a corresponding result for the evolution of the volume of Ω_{13} and so a sufficient condition to guarantee preservation of the bulk phase volume is

$$\nabla_{\Gamma^1(t)} H_{\Gamma^1(t)} \cdot \nu_{\Gamma^1(t)} = \nabla_{\Gamma^2(t)} H_{\Gamma^2(t)} \cdot \nu_{\Gamma^2(t)} = \nabla_{\Gamma^3(t)} H_{\Gamma^3(t)} \cdot \nu_{\Gamma^3(t)} \qquad \text{on } \Sigma(t). \tag{4.3}$$

Now, we get for the evolution of the surface energy using the surface transport theorem (cf. [49, Section 2.5, Paragraph 4]) that

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{i=1}^{3}\int_{\Gamma^i(t)} \gamma^i d\mathcal{H}^n\right) = \sum_{i=1}^{3}\int_{\Gamma^i(t)} \gamma^i H_{\Gamma^i(t)} \Delta_{\Gamma^i(t)} H_{\Gamma^i(t)} + \gamma^i (\nabla_{\Gamma^i(t)} \cdot v_{\Gamma^i(t)}) d\mathcal{H}^n \qquad (4.4)\\ &= \sum_{i=1}^{3}\int_{\Gamma^i(t)} -\gamma^i |\nabla_{\Gamma^i(t)} H_{\Gamma^i(t)}|^2 d\mathcal{H}^n + \sum_{i=1}^{3}\int_{\Sigma(t)} \gamma^i H_{\Gamma^i(t)} \partial_{\nu_{\Gamma^i(t)}} H_{\Gamma^i(t)} + \gamma^i v_{\Gamma^i(t)} \cdot \nu_{\Gamma^i(t)} d\mathcal{H}^{n-1}.\end{aligned}$$

Hereby, $v_{\Gamma_i(t)}$ denotes the undetermined tangential velocity of the flow. Due to (4.3) the second term will vanish, if we have that

$$\gamma^1 H_{\Gamma^1(t)} + \gamma^2 H_{\Gamma^2(t)} + \gamma^3 H_{\Gamma^3(t)} = 0 \tag{4.5}$$

on the boundary. For the third term we observe that due to the projection on the outer conormal we have for $i = 1,2,3$ that

$$v_{\Gamma_i(t)} \cdot \nu_{\Gamma^i(t)} = v^i(t) \cdot \nu_{\Gamma^i(t)}, \tag{4.6}$$

where $v^i(t)$ denote the complete velocity field of $\Gamma^i(t)$. To guarantee the concurrency of the triple junction the $v^i(t)$ have to match on $\Sigma(t)$. Thus, by postulating on $\Sigma(t)$ the force balance

$$\gamma^1 \nu_{\Gamma^1(t)} + \gamma^2 \nu_{\Gamma^2(t)} + \gamma^3 \nu_{\Gamma^3(t)} = 0 \tag{4.7}$$

we can guarantee that the third term in the second line of (4.4) will vanish. Condition (4.7) can only be fulfilled when the three conormals are in the same plane. Note that this is guaranteed due to the existence of the triple junction. The outer conormals are elements of the orthogonal complement of the tangent space of $\Sigma(t)$ that is a two dimensional space. (4.7) fixes actually the three contact angles $\theta^1, \theta^2, \theta^3$ given by

$$\angle(\nu_{\Gamma^i(t)}, \nu_{\Gamma^j(t)}) = \theta^k, (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}, \quad \theta^1 + \theta^2 + \theta^3 = 2\pi.$$

The three angles are determined by Young's law

$$\frac{\sin\theta^1}{\gamma^1} = \frac{\sin\theta^2}{\gamma^2} = \frac{\sin\theta^3}{\gamma^3}, \tag{4.8}$$

which is actually equivalent to (4.7), see, e.g., [16, Lemma 4.1]. In total, we get the system (SDFTJ) given by

$$\begin{aligned}
V_{\Gamma^i(t)} &= -\Delta_{\Gamma^i(t)} H_{\Gamma^i(t)} && \text{on } \Gamma^i(t),\ i=1,2,3, t\in[0,T] && (4.9)\\
\angle(\nu_{\Gamma^1(t)},\nu_{\Gamma^2(t)}) &= \theta^3, \angle(\nu_{\Gamma^2(t)},\nu_{\Gamma^3(t)}) = \theta^1 && \text{on } \Sigma(t), t\in[0,T], && (4.10)\\
0 &= \gamma^1 H_{\Gamma^1(t)} + \gamma^2 H_{\Gamma^2(t)} + \gamma^3 H_{\Gamma^3(t)} && \text{on } \Sigma(t), t\in[0,T], && (4.11)\\
\partial_{\nu_{\Gamma^2(t)}} H_{\Gamma^1(t)} &= \partial_{\nu_{\Gamma^2(t)}} H_{\Gamma^2(t)} = \partial_{\nu_{\Gamma^3(t)}} H_{\Gamma^3(t)} && \text{on } \Sigma(t), t\in[0,T], && (4.12)\\
\partial\Gamma^i(t) &= \Sigma(t) && \text{for } i=1,2,3, t\in[0,T], && (4.13)\\
\Gamma^i(0) &= \Gamma^i_0 && \text{for } i=1,2,3. && (4.14)
\end{aligned}$$

Here, we denoted by $\Gamma^i_0 = \Gamma^i(0)$ some initial triple junction cluster with triple junction Σ_0. We note that for a smooth solution $\Gamma(t)$ of (SDFTJ) the initial surface Γ_0 has to fulfil the boundary conditions (4.10)-(4.12) and

$$\gamma^1 \Delta_{\Gamma^1(t)} H_{\Gamma^1(t)} + \gamma^2 \Delta_{\Gamma^2(t)} H_{\Gamma^2(t)} + \gamma^3 \Delta_{\Gamma^3(t)} H_{\Gamma^3(t)} = 0 \quad \text{on } \Sigma_0. \tag{4.15}$$

The last condition follows by considering a smooth curve $c:[0,T]\to\mathbb{R}^{n+1}$ with $c(t)\in\Sigma(t), c(0)=\sigma$ for any $\sigma\in\Sigma_0$. Then we get $V_{\Gamma^i(0)}(\sigma) = \langle c'(0), N_{\Gamma^i(0)}\rangle$ for $i=1,2,3$. Note that due to the choices of the normals they are given as $R\nu_{\Gamma^i(t)}$, where R is a suitable rotation in the two dimensional plane spanned by the conormals, and so they also fulfil the balance law (4.7). Thus, we conclude

$$\sum_{i=1}^3 \gamma^i \Delta_{\Gamma^i(0)} H_{\Gamma^i(0)}(\sigma) = -\sum_{i=1}^3 \gamma^i V_{\Gamma^i(0)}(\sigma) = -\sum_{i=1}^3 \langle c'(0), \gamma^i N_{\Gamma^i(0)}(\sigma)\rangle = 0.$$

This gives us the geometrical compatibility conditions

$$(GCC) \begin{cases} \Gamma_0 \text{ fulfils } (4.10)-(4.12), \\ \gamma^1 \Delta_{\Gamma^1(0)} H_{\Gamma^1(0)} + \gamma^2 \Delta_{\Gamma^2(0)} H_{\Gamma^2(0)} + \gamma^3 \Delta_{\Gamma^3(0)} H_{\Gamma^3(0)} = 0 \quad \text{on } \Sigma_0. \end{cases} \tag{4.16}$$

To study this analytically we will use a parametrisation over a fixed reference frame following the idea of [19, Section 2.1].

4.2 Parametrisation and the Analytic Problem

Similar to our work in the case of closed hypersurfaces, we want to write the evolution of the hypersurface as a normal graph over a fixed reference triple junction manifold $\Gamma_* := \Gamma^1_* \cup \Gamma^2_* \cup \Gamma^3_* \cup \Sigma_*$, which we will need to be of class $C^{5+\alpha}$. But here we have to allow a tangential part near the triple junction Σ_*. Otherwise the parametrisation could only describe evolutions with a stationary boundary as otherwise concurrency of the triple junction would be not fulfilled. On the other hand, the parametrisation cannot have too much tangential freedom as otherwise this will result in a degenerated PDE system. Therefore, we follow the ideas of [19] and observe that as long as the tangential part on the triple junction is purely conormal we will get a linear dependence between the normal and the tangential part. This motivates to describe Γ as image of the diffeomorphism

$$\begin{aligned}
\Phi^i_{\boldsymbol{\rho},\boldsymbol{\mu}} : \Gamma^i_* \times [0,T] &\to \mathbb{R}^{n+1}, \\
(\sigma,t) &\mapsto \sigma + \rho^i(\sigma,t) N_{\Gamma^i_*}(\sigma) + \mu^i(\sigma,t)\tau^i_*(\sigma),
\end{aligned} \tag{4.17}$$

where τ^i_* are fixed, smooth tangential vector fields on Γ^i_* that equal $\nu_{\Gamma^i_*}$ on Σ_* and have a support in a neighbourhood of Σ_* in Γ^i_*. The tuple $(\boldsymbol{\rho},\boldsymbol{\mu})$ consists of the unknown functions for which we want to derive a PDE system. We know from the work of [18] that the condition

$$\Phi^1_{\boldsymbol{\rho},\boldsymbol{\mu}}(\sigma,t) = \Phi^2_{\boldsymbol{\rho},\boldsymbol{\mu}}(\sigma,t) = \Phi^3_{\boldsymbol{\rho},\boldsymbol{\mu}}(\sigma,t) \qquad \text{for } \sigma\in\Sigma^*, t\geq 0, \tag{4.18}$$

which guarantees concurrency of the triple junction, is equivalent to

$$\begin{cases} \gamma^1\rho^1 + \gamma^2\rho^2 + \gamma^3\rho^3 = 0 & \text{on } \Sigma_*, \\ \boldsymbol{\mu} = \mathcal{T}\boldsymbol{\rho} & \text{on } \Sigma_*. \end{cases} \tag{4.19}$$

Hereby, the matrix $\mathcal{T}$ is given by

$$\mathcal{T} = \begin{pmatrix} 0 & \frac{c^2}{s^1} & -\frac{c^3}{s^1} \\ -\frac{c^1}{s^2} & 0 & \frac{c^3}{s^2} \\ \frac{c^1}{s^3} & -\frac{c^2}{s^3} & 0 \end{pmatrix},$$

with $s^i = \sin(\theta^i)$ and $c^i = \cos(\theta^i)$. The second line in (4.19) implies that the tangential part $\boldsymbol{\mu}$ is uniquely determined on Σ_* by the values of $\boldsymbol{\rho}$. This motivated the authors of [19] to get rid of the degenerated degrees of freedom of $\boldsymbol{\mu}$ by setting $\mu^i(\sigma) := \mu(\mathrm{pr}^i_\Sigma(\sigma))$, where pr^i_Σ denote the projection from a point on Γ^i_* to the nearest point on Σ_*.[1] This choice will now lead to a non-local problem but for the concerns of the proof of short time existence this will not lead to technical difficulties. During the discussion of the stability result it will be a crucial obstacle and then we have to consider a different tangential part. But for the rest of this chapter $\boldsymbol{\mu}$ will always be given by this choice.
We now want to find a suitable PDE-setting for $\boldsymbol{\rho}$. To begin with, we retract the equations from $\Gamma^i(t)$ on Γ^i_*. From here on, we will write $\Gamma_{\boldsymbol{\rho}}$ resp. $\Sigma_{\boldsymbol{\rho}}$ when referring to the triple junction cluster and the triple junction given as image of $\Phi_{\boldsymbol{\rho},\boldsymbol{\mu}}$. Also, we will use sub- and superscripts i and $\boldsymbol{\rho}$ to denote pull-backs of quantities of the hypersurface $\Gamma_{\boldsymbol{\rho}}$ or of the triple junction $\Sigma_{\boldsymbol{\rho}}$. This will also be applied on differential operators. So for example we will write for $(\sigma, t) \in \Gamma^i_* \times [0,T], i = 1,2,3$,

$$\begin{aligned} H^i_{\boldsymbol{\rho}}(\sigma,t) &:= H_{\Gamma^i_{\boldsymbol{\rho}}}(\Phi^i_{\boldsymbol{\rho},\boldsymbol{\mu}}(\sigma,t)), \\ \Delta_{\boldsymbol{\rho}} H_{\boldsymbol{\rho}}(\sigma,t) &:= \left(\Delta_{\Gamma^i_{\boldsymbol{\rho}}} H_{\Gamma^i_{\boldsymbol{\rho}}}\right)(\Phi_{\boldsymbol{\rho},\boldsymbol{\mu}}(\sigma,t)). \end{aligned}$$

Later, we will also sometimes use this notation when we consider the quantities on $\Gamma_{\boldsymbol{\rho}}$ but it will be always clear what is meant.
The system (SDFTJ) rewrites now as the following problem on Γ_*.

$$\begin{cases} V^i_{\boldsymbol{\rho}} = -\Delta_{\boldsymbol{\rho}} H^i_{\boldsymbol{\rho}} & \text{on } \Gamma^i_*, t \in [0,T], i = 1,2,3, \\ \gamma^1\rho^1 + \gamma^2\rho^2 + \gamma^3\rho^3 = 0, & \text{on } \Sigma_*, t \in [0,T], \\ \langle N^1_{\boldsymbol{\rho}}, N^2_{\boldsymbol{\rho}} \rangle = \cos(\theta^3) & \text{on } \Sigma_*, t \in [0,T], \\ \langle N^2_{\boldsymbol{\rho}}, N^3_{\boldsymbol{\rho}} \rangle = \cos(\theta^1) & \text{on } \Sigma_*, t \in [0,T], \\ \gamma^1 H^1_{\boldsymbol{\rho}} + \gamma^2 H^2_{\boldsymbol{\rho}} + \gamma^3 H^3_{\boldsymbol{\rho}} = 0 & \text{on } \Sigma_*, t \in [0,T], \\ \nabla_{\boldsymbol{\rho}} H^1_{\boldsymbol{\rho}} \cdot \nu^1_{\boldsymbol{\rho}} = \nabla_{\boldsymbol{\rho}} H^2_{\boldsymbol{\rho}} \cdot \nu^2_{\boldsymbol{\rho}} & \text{on } \Sigma_*, t \in [0,T], \\ \nabla_{\boldsymbol{\rho}} H^2_{\boldsymbol{\rho}} \cdot \nu^2_{\boldsymbol{\rho}} = \nabla_{\boldsymbol{\rho}} H^3_{\boldsymbol{\rho}} \cdot \nu^3_{\boldsymbol{\rho}} & \text{on } \Sigma_*, t \in [0,T], \\ (\rho^i(\sigma,0), \mu^i(\sigma,0)) = (\rho^i_0, \mu^i_0) & \text{on } \Gamma^i_* \times \Sigma_*, i = 1,2,3. \end{cases} \tag{4.20}$$

Here, we assume that the initial surfaces are given as $\Gamma^i_0 = \Gamma^i_{\rho^i_0,\mu^i_0}, i = 1,2,3$ for $\boldsymbol{\rho}_0$ small enough in the $C^{4+\alpha}$-norm and $\boldsymbol{\mu}_0 = \mathcal{T}\boldsymbol{\rho}_0$. This will then guarantee that the Γ^i_0 are indeed embedded hypersurfaces, cf. [19, Remark 1] .
We want to see that (4.20) actually yields a fourth order PDE system for the functions (ρ^1, ρ^2, ρ^3). The argument is the same as in [19], only the operator $\mathcal{F}$ is in our case induced by the Laplacian of the mean curvature. We will state it for the sake of completeness. For a better readability we will omit the projection pr^i in the variable $\boldsymbol{\mu}$. Observe that the Laplacian of the mean curvature operator

[1] This map is only well-defined on a neighbourhood of Σ_* in Γ^i_* but we only need μ^i to be defined on the support of τ^i_* and this may be supposed to be small enough.

can be written in local coordinates as

$$\sum_{j,k,l,m=1}^{n} g^{jk}\partial_j\partial_k(g^{lm}h_{lm}) + l.o.t.$$

Thus, $\Delta_{\boldsymbol{\rho}} H_{\boldsymbol{\rho}}$ only depends on the values of ρ^i, μ^i and their covariant derivatives of up to order four and so we can write it as

$$\begin{aligned}-\Delta_{\boldsymbol{\rho}} H^i_{\boldsymbol{\rho}}(\sigma,t) = \widetilde{H}^i_\Delta\Big(&\sigma,\rho^i(\sigma,t),\nabla\rho^i(\sigma,t),\nabla^2\rho^i(\sigma,t),\nabla^3\rho^i(\sigma,t),\nabla^4\rho^i(\sigma,t),\\ &\boldsymbol{\mu}(\sigma,t),\overline{\nabla}\boldsymbol{\mu}(\sigma,t),\overline{\nabla}^2\boldsymbol{\mu}(\sigma,t),\overline{\nabla}^3\boldsymbol{\mu}(\sigma,t),\overline{\nabla}^4\boldsymbol{\mu}(\sigma,t)\Big) =: H^i_\Delta(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}),\end{aligned}$$

for a suitable function $\widetilde{H}^i_\Delta$. Here, we denote by ∇^k the k-th-covariant derivatives on Γ^i_* and by $\overline{\nabla}^k$ the k-th-covariant derivatives on Σ_*. Following the argumentation in [19], we can rewrite the first equation in (4.20) to

$$\partial_t\rho^i(\sigma,t) = a^i(\sigma,t,\rho^i,\mu^i)H^i_\Delta(\sigma,t,\rho^i,\mu^i) + a^i_\dagger(\sigma,t,\rho^i,\mu^i)\partial_t\mu^i, \tag{4.21}$$

where the functions

$$\begin{aligned}a^i(\sigma,t,\rho^i,\mu^i) &:= \widetilde{a}^i\left(\sigma,\rho^i(\sigma,t),\mu^i(\sigma,t),\nabla\rho^i(\sigma,t),\overline{\nabla}\mu^i(\sigma,t)\right) = \frac{1}{\langle N^i_*(\sigma),\mathcal{N}^i(\sigma,t,\rho^i,\mu^i)\rangle},\\ a^i_\dagger(\sigma,t,\rho^i,\mu^i) &:= \widetilde{a}^i_\dagger\left(\sigma,\rho^i(\sigma,t),\mu^i(\sigma,t),\nabla\rho^i(\sigma,t),\overline{\nabla}\mu^i(\sigma,t)\right) = -\frac{\langle\tau^i_*(\sigma),\mathcal{N}^i(\sigma,t,\rho^i,\mu^i)\rangle}{\langle N^i_*(\sigma),\mathcal{N}^i(\sigma,t,\rho^i,\mu^i)\rangle},\end{aligned}$$

are chosen as in [19] and the function $\widetilde{N}^i$ is - similarly to $\widetilde{H}^i_\Delta$ - chosen such that we have

$$N^i_{\boldsymbol{\rho}}(\sigma,t) = \mathcal{N}^i(\sigma,t,\rho^i,\mu^i) = \widetilde{N}^i(\sigma,\rho^i(\sigma,t),\mu^i(\sigma,t),\nabla\rho^i(\sigma,t),\overline{\nabla}\mu^i(\sigma,t)),$$

for a suitable function $\widetilde{N}^i$, cf. [19, p.309]. Now using (4.19) we get

$$\begin{aligned}\partial_t\rho^i &= \mathcal{F}^i(\rho^i,\boldsymbol{\rho}\big|_{\Sigma_*}) + a^i_\dagger(\rho^i,\boldsymbol{\rho}\big|_{\Sigma_*})\partial_t(\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i, \qquad (4.22)\\ \mathcal{F}^i(\rho^i,\boldsymbol{\rho}|_{\Sigma_*})(\sigma,t) &:= a^i\left(\sigma,t,\rho^i,(\mathcal{T}\boldsymbol{\rho}\big|_{\Sigma_*})^i\right)H^i_\Delta\left(\sigma,t,\rho^i,(\mathcal{T}\boldsymbol{\rho}\big|_{\Sigma_*})^i\right), \quad (\sigma,t)\in\Gamma^i_*\times[0,T], i=1,2,3,\\ a^i_\dagger(\rho^i,\boldsymbol{\rho}\big|_{\Sigma_*})(\sigma,t) &:= a^i_\dagger\left(\sigma,t,\rho^i,(\mathcal{T}\boldsymbol{\rho}|_{\Sigma_*})^i\right), \quad (\sigma,t)\in\Gamma^i_*\times[0,T], i=1,2,3.\end{aligned}$$

To get a parabolic equation in $\boldsymbol{\rho}$ we have to rewrite the time derivative of the non-local term. To this end it is enough to calculate $\partial_t(\mathcal{T}(\boldsymbol{\rho}\circ\mathrm{pr}))^i = \mathcal{T}(\partial_t(\boldsymbol{\rho}\circ\mathrm{pr}))^i$ on Σ_*. Using the notation

$$\begin{aligned}\boldsymbol{\mathcal{F}}(\boldsymbol{\rho},\boldsymbol{\rho}\big|_{\Sigma_*}) &:= \left(\mathcal{F}^i(\rho^i,\boldsymbol{\rho}\big|_{\Sigma_*})(\sigma,t)\right)_{i=1,2,3} && (\sigma,t)\in\Sigma_{*,T},\\ \mathcal{D}_\dagger(\boldsymbol{\rho},\boldsymbol{\rho}\big|_{\Sigma_*}) &:= \mathrm{diag}\left(\left(a^i_\dagger(\rho^i,\boldsymbol{\rho}\big|_{\Sigma_*})(\sigma,t)\right)_{i=1,2,3}\right) && (\sigma,t)\in\Sigma_{*,T},\end{aligned}$$

allows us to write (4.22) in the vector form

$$\partial_t\boldsymbol{\rho} = \boldsymbol{\mathcal{F}}(\boldsymbol{\rho},\boldsymbol{\rho}\big|_{\Sigma_*}) + \mathcal{D}_\dagger(\boldsymbol{\rho},\boldsymbol{\rho}\big|_{\Sigma_*})\mathcal{T}(\partial_t\boldsymbol{\rho}) \text{ on } \Sigma_*. \tag{4.23}$$

Rearranging this leads to

$$(\mathrm{Id} - \mathcal{D}_\dagger(\boldsymbol{\rho},\boldsymbol{\rho}\big|_{\Sigma_*})\mathcal{T})\partial_t\boldsymbol{\rho} = \boldsymbol{\mathcal{F}}(\boldsymbol{\rho},\boldsymbol{\rho}\big|_{\Sigma_*}) \text{ on } \Sigma_*. \tag{4.24}$$

Observe here that $\mathrm{Id} - \mathcal{D}_\dagger(\boldsymbol{\rho},\boldsymbol{\rho}\big|_{\Sigma_*})\mathcal{T}$ is invertible for $\boldsymbol{\rho}$ small enough in the C^1-norm as for $\boldsymbol{\rho}\equiv 0$ this is just the identity map due to $\boldsymbol{a}^i_\dagger(0) = 0$. Thus, we can define

$$\mathcal{P}(\boldsymbol{\rho},\boldsymbol{\rho}\big|_{\Sigma_*})(\sigma) := \mathcal{T}\left(\mathrm{Id} - \mathcal{D}_\dagger(\boldsymbol{\rho},\boldsymbol{\rho}\big|_{\Sigma_*})(\sigma)\mathcal{T}\right)^{-1} \text{ on } \Sigma_*, \tag{4.25}$$

getting

$$\mathcal{T}\partial_t\boldsymbol{\rho} = \mathcal{P}(\boldsymbol{\rho}, \boldsymbol{\rho}\big|_{\Sigma_*})\mathcal{F}(\boldsymbol{\rho}, \boldsymbol{\rho}\big|_{\Sigma_*}) \text{ on } \Sigma_*. \tag{4.26}$$

This gives us now in the neighbourhood of Σ_* in Γ^i_*, where pr^i is defined, the desired relation

$$\partial_t\mu^i(\mathrm{pr}^i(\sigma)) = \left(\left\{\mathcal{P}(\boldsymbol{\rho}, \boldsymbol{\rho}\big|_{\Sigma_*})\mathcal{F}(\boldsymbol{\rho}, \boldsymbol{\rho}\big|_{\Sigma_*})\right\} \circ \boldsymbol{pr}\right)^i(\sigma). \tag{4.27}$$

From this, we get the following, in the space variable non-local formulation of (4.21):

$$\partial_t\rho^i = \mathcal{F}^i(\rho^i, \boldsymbol{\rho}|_{\Sigma_*}) + \boldsymbol{a}^i_\dagger(\rho^i, \boldsymbol{\rho}\big|_{\Sigma_*})\left(\left\{\mathcal{P}(\boldsymbol{\rho}, \boldsymbol{\rho}\big|_{\Sigma_*})\boldsymbol{\mathcal{F}}(\boldsymbol{\rho}, \boldsymbol{\rho}\big|_{\Sigma_*})\right\} \circ \boldsymbol{pr}\right)^i \quad \text{on } \Gamma^i_* \times [0,T]. \tag{4.28}$$

Observe that like in [19] the non-local term appears in highest order. So, the analysis is much more involved than a simple perturbation argument for lower order terms.
To state the boundary conditions we use the analogous notation

$$\begin{aligned}
H^i_{\boldsymbol{\rho}}(\sigma,t) &= H^i_0(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}),\\
H^i_0(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) &:= \widetilde{H}^i_0\Big(\sigma, \rho^i(\sigma,t), (\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i(\sigma,t), \nabla\rho^i(\sigma,t), \overline{\nabla}(\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i(\sigma,t),\\
&\qquad \nabla^2\rho^i(\sigma,t), \overline{\nabla}^2(\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i(\sigma,t)\Big),\\
\nabla_{\boldsymbol{\rho}}H^i_{\boldsymbol{\rho}}(\sigma,t) &= H^i_\nabla(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}),\\
H^i_\nabla(\boldsymbol{\rho}) &:= \widetilde{H}^i_\nabla\Big(\sigma, \rho^i(\sigma,t)(\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i, \nabla\rho^i(\sigma,t), \overline{\nabla}(\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i(\sigma,t),\\
&\qquad \nabla^2\rho^i(\sigma,t), \overline{\nabla}^2(\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i(\sigma,t), \nabla^3\rho^i(\sigma,t), \overline{\nabla}^3(\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i(\sigma,t)\Big),\\
\nu^i_{\boldsymbol{\rho}}(\sigma,t) &= \mathcal{N}^i_\Sigma(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}),\\
\mathcal{N}^i_\Sigma(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) &:= \widetilde{\mathcal{N}}^i_\Sigma\left(\sigma, \rho^i(\sigma,t), (\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i(\sigma,t), \nabla\rho^i(\sigma,t), \overline{\nabla}(\mathcal{T}(\boldsymbol{\rho}\circ\boldsymbol{pr}))^i(\sigma,t)\right).
\end{aligned}$$

With this the boundary conditions on Σ_* are given by

$$\begin{aligned}
\mathcal{G}^1(\boldsymbol{\rho}) &:= \gamma^1\rho^1 + \gamma^2\rho^2 + \gamma^3\rho^3 = 0, \text{ on } \Sigma_* \times [0,T],\\
\mathcal{G}^2(\boldsymbol{\rho}) &:= \langle\mathcal{N}^1(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}), \mathcal{N}^2(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu})\rangle - \cos(\theta^3) = 0, \text{ on } \Sigma_* \times [0,T],\\
\mathcal{G}^3(\boldsymbol{\rho}) &:= \langle\mathcal{N}^2(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}), \mathcal{N}^3(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu})\rangle - \cos(\theta^1) = 0, \text{ on } \Sigma_* \times [0,T],\\
\mathcal{G}^4(\boldsymbol{\rho}) &:= \gamma^1H^1_0(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) + \gamma^2H^2_0(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) + \gamma^3H^3_0(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) = 0, \text{ on } \Sigma_* \times [0,T],\\
\mathcal{G}^5(\boldsymbol{\rho}) &:= H^1_\nabla(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) \cdot \mathcal{N}^1_\Sigma(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) - H^2_\nabla(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) \cdot \mathcal{N}^2_\Sigma(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) = 0, \text{ on } \Sigma_* \times [0,T],\\
\mathcal{G}^6(\boldsymbol{\rho}) &:= H^2_\nabla(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) \cdot \mathcal{N}^2_\Sigma(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) - H^3_\nabla(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) \cdot \mathcal{N}^3_\Sigma(\sigma,t,\boldsymbol{\rho},\boldsymbol{\mu}) = 0, \text{ on } \Sigma_* \times [0,T],\\
\Leftrightarrow \mathcal{G}(\boldsymbol{\rho}) &:= \left(\mathcal{G}^i(\boldsymbol{\rho})\right)_{i=1,\dots,6} = 0, \text{ on } \Sigma_* \times [0,T].
\end{aligned}$$

In total, (SDFTJ) rewrites to the following problem for (ρ^1, ρ^2, ρ^3)

$$\begin{cases}
\partial_t\rho^i = \mathcal{K}^i(\rho^i, \boldsymbol{\rho}|_{\Sigma_*}) & \text{on } \Gamma^i_* \times [0,T], i = 1,2,3,\\
\mathcal{G}(\boldsymbol{\rho}) = 0 & \text{on } \Sigma_* \times [0,T],\\
\rho^i(\cdot,0) = \rho^i_0 & \text{on } \Sigma_*,
\end{cases} \tag{4.29}$$

where we used the abbreviation

$$\mathcal{K}^i(\rho^i, \boldsymbol{\rho}|_{\Sigma_*}) := \mathcal{F}^i(\rho^i, \boldsymbol{\rho}|_{\Sigma_*}) + a^i_\dagger(\rho^i, \boldsymbol{\rho}|_{\Sigma_*})\left(\left\{\mathcal{P}(\boldsymbol{\rho}, \boldsymbol{\rho}\big|_{\Sigma_*})\boldsymbol{\mathcal{F}}(\boldsymbol{\rho}, \boldsymbol{\rho}\big|_{\Sigma_*})\right\} \circ \boldsymbol{pr}\right)^i. \tag{4.30}$$

4.3 The Compatibility Conditions and the Existence Result

For a solution of (4.29) that is smooth up to $t = 0$ we will need compatibility conditions similar to those we derived in (4.16) for the geometric version of the problem, which are the following.

$$(ACC)\begin{cases}\mathcal{G}(\boldsymbol{\rho}_0) = 0 & \text{on } \Sigma_*,\\ \mathcal{G}_0(\boldsymbol{\rho}_0) := \sum_{i=1}^3 \gamma^i \mathcal{K}^i(\rho_0^i, \boldsymbol{\rho}_0\big|_{\Sigma_*}) = 0 & \text{on } \Sigma_*.\end{cases} \tag{4.31}$$

Hereby, the first line is just the same as in (4.16) and only states that the boundary conditions are fulfilled by the initial surface. The second condition follows for a smooth solution of (4.29) by differentiating $\mathcal{G}^1(\rho) = 0$ in $t = 0$ and using the equation for $\partial_t \rho^i$. As we would like to start arguing from the geometric point of view we want (GCC) and (ACC) to be equivalent conditions which is indeed true. This is clearly true for $(4.16)_1$ and $(4.31)_1$ as both just state that the boundary conditions[2] are fulfilled. The rest is proven in the following Lemma.

Lemma 4.1 (Equivalence of geometric and analytic compatibility conditions)**.**
Assume that for the initial data $(\boldsymbol{\rho}_0, \boldsymbol{\mu}_0)$ we have $\mathcal{G}(\boldsymbol{\rho}) = 0$ and $\mathcal{T}\boldsymbol{\rho_0} = \boldsymbol{\mu}_0$. Then, for $\boldsymbol{\rho}_0$ small enough in the C^1-norm, $(4.16)_2$ and $(4.31)_2$ are equivalent.

Proof. The arguments are similar as in [19, Lemma 2] but for the sake of completeness we will state it here with comments on some details.
Using our usual notation convention every term with subscript 0 will indicate the evaluation at the initial triple junction, that is in the analytical setting just $(\rho_0^i, \boldsymbol{\rho}_0\big|_{\Sigma_*})$ respectively $(\boldsymbol{\rho}_0, \boldsymbol{\rho}_0\big|_{\Sigma_*})$. With the additional notation $\mathcal{L}_0^i = (\mathcal{T}\boldsymbol{\mathcal{K}}_0)^i$ and $\boldsymbol{\mathcal{L}}_0 = \mathcal{T}\boldsymbol{K}_0$ we want first to derive

$$\left\langle \left(\mathcal{K}_0^i N_*^i + \mathcal{L}_0^i \tau_*^i\right), N_0^i \right\rangle = -\Delta_{\rho_0} H_{\rho_0}^i \text{ on } \Sigma_*. \tag{4.32}$$

To do so, we notice that

$$\begin{aligned}&\mathcal{K}_0^i = a_0^i(\Delta_{\Gamma_{\rho_0}} H_{\boldsymbol{\rho}_0}^i) + a_{\dagger,0}^i \left(\mathcal{T}(\mathrm{Id} - \mathcal{D}_{\dagger,0}\mathcal{T})^{-1}\boldsymbol{\mathcal{F}}_0\right)^i = a_0^i(\Delta_{\boldsymbol{\rho}_0} H_{\boldsymbol{\rho}_0}^i) + \left(\mathcal{D}_{\dagger,0}\mathcal{T}(\mathrm{Id} - \mathcal{D}_{\dagger,0}\mathcal{T})^{-1}\boldsymbol{\mathcal{F}}_0\right)^i,\\ &\boldsymbol{\mathcal{K}}_0 = \boldsymbol{\mathcal{F}}_0 + \mathcal{D}_{\dagger,0}\mathcal{T}(\mathrm{Id} - \mathcal{D}_{\dagger,0}\mathcal{T})^{-1}\boldsymbol{\mathcal{F}}_0.\end{aligned}$$

Here, we used in the second equality that $\mathcal{D}_{\dagger,0}$ is just multiplication of the i-th component with $a_{\dagger,0}^i$. With the definitions of a^i and $a_\dagger^i$, the first equality gives us after multiplication with $\langle N_*^i, N_0^i\rangle$ that

$$\mathcal{K}_0^i \langle N_*^i, N_0^i\rangle = -\Delta_{\boldsymbol{\rho}_0} H_{\boldsymbol{\rho}_0}^i - \langle \tau_*^i, N_0^i\rangle \left(\mathcal{T}(\mathrm{Id} - \mathcal{D}_{\dagger,0}\mathcal{T})^{-1}\boldsymbol{\mathcal{F}}_0\right)^i.$$

This is just the first summand in the scalar product in (4.32) and so to prove this its enough to show

$$-\langle \tau_*^i, N_0^i\rangle \left(\mathcal{T}(\mathrm{Id} - \mathcal{D}_{\dagger,0}\mathcal{T})^{-1}\boldsymbol{\mathcal{F}}_0\right)^i = -\mathcal{L}_0^i \langle \tau_*^i, N_0^i\rangle.$$

Observe that this identity is true if $\langle \tau_*^i, N_0^i\rangle = 0$ and so w.l.o.g. we may suppose $\langle \tau_*^i, N_0^i\rangle \neq 0$. Then we can divide the equation by this term and get, again considering the corresponding vector problem,

$$\mathcal{T}(\mathrm{Id} - \mathcal{D}_{\dagger,0}\mathcal{T})^{-1}\boldsymbol{\mathcal{F}}_0 = \mathcal{T}\boldsymbol{\mathcal{K}}_0 \Leftrightarrow \mathcal{T}\left((\mathrm{Id} - \mathcal{D}_{\dagger,0}\mathcal{T})^{-1}\boldsymbol{F}_0 - \boldsymbol{\mathcal{K}}_0\right) = 0.$$

In [19] it was proven that

$$(\mathrm{Id} - \mathcal{D}_{\dagger,0}\mathcal{T})^{-1}\boldsymbol{\mathcal{F}}_0 - \boldsymbol{\mathcal{K}}_0 = 0,$$

which gives us (4.32).

[2]Recall the equivalence of (4.18) and (4.19)!

To derive the desired equivalence of the compatibility conditions we consider the matrix A given by

$$A = \begin{pmatrix} \langle N_*^1, N_0^1\rangle & \frac{c^2}{s^1}\langle \tau_*^1, N_0^1\rangle & -\frac{c^3}{s^1}\langle \tau_*^1, N_0^1\rangle \\ -\frac{c^1}{s^2}\langle \tau_*^2, N_0^2\rangle & \langle N_*^2, N_0^2\rangle & \frac{c^3}{s^2}\langle \tau_*^2, N_0^2\rangle \\ \frac{c^1}{s^3}\langle \tau_*^3, N_0^3\rangle & -\frac{c^2}{s^3}\langle \tau_*^3, N_0^3\rangle & \langle N_*^3, N_0^3\rangle \end{pmatrix}.$$

Observe that for $\boldsymbol{\rho}_0 \equiv 0$ we get $A = \mathrm{Id}$ and as all arising quantities are continuous functions in the values of $\boldsymbol{\rho}_0$ and its first order derivatives we see that A is invertible for $\boldsymbol{\rho}_0$ small enough in the C^1-norm. Analogously to the usual notation conventions we set

$$\boldsymbol{\gamma} = (\gamma^2, \gamma^2, \gamma^3), \qquad \boldsymbol{\gamma}^\perp = \{\boldsymbol{z} \in \mathbb{R}^3 | \langle z, \boldsymbol{\gamma}\rangle = 0\}, \qquad \boldsymbol{\Delta}_{\boldsymbol{\rho}_0} \boldsymbol{H}_{\boldsymbol{\rho}_0} = (\Delta_{\boldsymbol{\rho}_0} H^1_{\boldsymbol{\rho}_0}, \Delta_{\boldsymbol{\rho}_0} H^2_{\boldsymbol{\rho}_0}, \Delta_{\boldsymbol{\rho}_0} H^3_{\boldsymbol{\rho}_0}).$$

The identity (4.32) may now be rewritten as

$$A\boldsymbol{\mathcal{K}}_0 = -\boldsymbol{\Delta}_{\boldsymbol{\rho}_0} \boldsymbol{H}_{\boldsymbol{\rho}_0}. \tag{4.33}$$

As $(4.16)_2$ is equivalent to $-\boldsymbol{\Delta}_{\boldsymbol{\rho}_0} \boldsymbol{H}_{\boldsymbol{\rho}_0} \in \boldsymbol{\gamma}^\perp$, $(4.31)_2$ is equivalent to $\boldsymbol{\mathcal{K}}_0 \in \boldsymbol{\gamma}^\perp$ and A is invertible for $\boldsymbol{\rho}_0$ small enough, the problem reduces to $A\boldsymbol{\gamma}^\perp = \boldsymbol{\gamma}^\perp$. From here the proof is precisely the same as in [19] so we restrict ourselves to give a proof of the used identity

$$\eta^i N_*^i + (\mathcal{T}\eta)^i \tau_*^i = \eta^j N_*^j + (\mathcal{T}\eta)^j \tau_*^j \text{ for } \boldsymbol{\eta} \in \boldsymbol{\gamma}^\perp.$$

But as $\sum_{i=1}^3 \gamma^i \eta^i = 0$, the pair $(\boldsymbol{\eta}, \mathcal{T}\boldsymbol{\eta})$ fulfils the conditions (4.19) and then by [18, Lemma 2.3] for any point $\sigma \in \Sigma_*$ we have

$$\sigma + \eta^i N_*^i(\sigma) + (\mathcal{T}\eta)^i \tau_*^i(\sigma) = \sigma + \eta^j N_*^j(\sigma) + (\mathcal{T}\eta)^j \tau_*^j(\sigma),$$

which is equivalent to the identity above. $\square$

Now we are able to state the main result of this chapter.

Theorem 4.2 (Short time existence for surface diffusion flow with triple junctions)**.**
Let Γ_* be a $C^{5+\alpha}$-reference cluster. Then there exists an $\varepsilon_0 > 0$ and a $T > 0$ such that for all initial data $\boldsymbol{\rho}_0 \in C^{4+\alpha}_{TJ}(\Gamma_*)$ with $\|\boldsymbol{\rho}_0\|_{C^{4+\alpha}_{TJ}(\Gamma_*)} < \varepsilon_0$, which fulfil the analytic compatibility conditions (4.31), there exists a unique solution $\boldsymbol{\rho} \in C^{4+\alpha, 1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,T})$ of (4.29).

Remark 4.3 (Geometric existence and uniqueness)**.**

i.) We say that a geometric flow has geometric existence resp. geometric uniqueness if the geometric problem - that is in our situation $(SDFTJ)$ *- has a solution resp. a unique solution* $\Gamma(t)$*. In contrast, analytic existence resp. uniqueness refers to existence resp. uniqueness of solutions of the analytic formulation* (4.20)*. Naturally, we would like to deduce geometric results from the analytic ones.*

ii.) Theorem 4.2 guarantees geometric existence for $C^{5+\alpha}$-initial surfaces as in this case we may choose $\Gamma_* = \Gamma_0$ and thus $\boldsymbol{\rho}_0 \equiv 0$. For $C^{4+\alpha}$-initial surfaces one would need approximation results for hypersurfaces with boundary similar to those in [49, Section 2.3] for closed hypersurfaces.

iii.) Analytical uniqueness follows as in 3.14. Geometric uniqueness, though, remains like in many works on geometric flows an open problem.

4.4 Linearisation

We now want to linearise the non-linear problem around the $C^{5+\alpha}$-reference frame Γ_*. This is done pointwise in every $p \in \Gamma_* \cup \Sigma_*$ meaning the following. For any fixed point $\sigma \in \Gamma_*^i, i = 1, 2, 3$ or $\sigma \in \Sigma_*$, any term in (4.20), which are all given as functions in σ and the functions $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$, and any tuple[3]

$$(\boldsymbol{u}, \boldsymbol{\varphi}) \in \left(C^{4+\alpha}(\Gamma^1_*) \times C^{4+\alpha}(\Gamma^2_*) \times C^{4+\alpha}(\Gamma^3_*)\right) \times \left(C^{4+\alpha}(\Sigma_*)\right)^3,$$

fulfilling (4.19), we replace ρ^i with εu^i and μ^i with $\varepsilon\varphi^i$, differentiate the new expression in ε and evaluate this for $\varepsilon = 0$. We observe hereby that (4.19) due to its linear structure will also hold for $(\varepsilon\boldsymbol{u}, \varepsilon\boldsymbol{\varphi})$ and as before all terms are well defined for $(\boldsymbol{u}, \boldsymbol{\varphi})$ small enough in the $C^{4+\alpha}$-norm for every t so at least for ε small enough.

Remark 4.4. *The goal of this process is to derive a linear equation on Γ_*. Our procedure, though, will lead to equations in local coordinates. But as we linearise geometric quantities that are itself independent of local coordinates we conclude that the equations we derive have to represent a global equation on Γ_*. We will get this form for the terms of highest order and all other terms will be dealt with by using perturbation arguments.*

In the following, we will index a geometric quantity with $\varepsilon$to denote the quantity on $\Gamma_{\varepsilon u,\varepsilon\varphi}$ at the point $\Phi_{\varepsilon u,\varepsilon\varphi}(\sigma, t)$. We will omit the fixed time and space variable (σ, t) and also the projection in the $\boldsymbol{\varphi}$-terms. For the analysis we will do later it is not important to know the lower order terms precisely. Thus, we will denote them only in qualitative form using dynamical coefficient functions a^{k+s}, which denotes some function on the corresponding hypersurface Γ^i_* that has C^{k+s}-regularity on this surface. Also, like dynamical constants the a^{k+s} may adapt from line to line. Before we start with the equations we will calculate the linearisations of some basic geometric quantities we will need later.

Lemma 4.5 (Linearisation of basic geometric quantities)**.**

$$\begin{aligned}
\frac{d}{d\varepsilon}\partial^{i,\varepsilon}_k\big|_{\varepsilon=0} &= a^{3+\alpha}u^i + a^{3+\alpha}\varphi^i + a^{4+\alpha}\partial_k u^i + a^{4+\alpha}\partial_k\varphi^i,\\
\frac{d}{d\varepsilon}g^{i,\varepsilon}_{kl}\big|_{\varepsilon=0} &= a^{3+\alpha}u^i + a^{3+\alpha}\varphi^i + a^{4+\alpha}\partial_k\varphi^i + a^{4+\alpha}\partial_l\varphi^i,\\
\frac{d}{d\varepsilon}g^{kl}_{i,\varepsilon}\big|_{\varepsilon=0} &= a^{3+\alpha}u^i + a^{3+\alpha}\varphi^i + \sum_{j=1}^{n} a^{4+\alpha}\partial_j\varphi^i,\\
\frac{d}{d\varepsilon}(\Gamma^m_{kl})^{i,\varepsilon}\big|_{\varepsilon=0} &= a^{2+\alpha}u^i + a^{2+\alpha}\varphi^i + \sum_j a^{3+\alpha}\partial_j u^i + \sum_j a^{3+\alpha}\partial_j\varphi^i + \sum_{j,j'} a^{4+\alpha}\partial_{jj'}\varphi^i.
\end{aligned}$$

Proof. Differentiating the parametrisation of $\Gamma^i_{\varepsilon u^i,\varepsilon\boldsymbol{\varphi}}$ given as composition of the map Φ^i and a local parametrisation of Γ^i we get

$$\partial^{i,\varepsilon}_k = \partial^{i,*}_k + \varepsilon(\partial_k u^i)N^i_* + \varepsilon u^i\partial_k N^i_* + \varepsilon\varphi^i\partial_k\tau^i_* + \varepsilon\partial_k\varphi^i\tau^i_*, \tag{4.34}$$

$$\frac{d}{d\varepsilon}\partial^{i,\varepsilon}_k\big|_{\varepsilon=0} = (\partial_k u^i)N^i_* + u^i\partial_k N^i_* + \varphi^i\partial_k\tau^i_* + \partial_k\varphi^i\tau^i_*. \tag{4.35}$$

This shows the first equality. For the linearisation of the $g^{i,\varepsilon}_{kl}$ we observe

$$\begin{aligned}
\frac{d}{d\varepsilon}g^{i,\varepsilon}_{kl}\big|_{\varepsilon=0} &= \left\langle \frac{d}{d\varepsilon}\partial^{i,\varepsilon}_k, \partial^{i,*}_l\right\rangle\Big|_{\varepsilon=0} + \left\langle \partial^{i,*}_k, \frac{d}{d\varepsilon}\partial^{i,\varepsilon}_l\right\rangle\Big|_{\varepsilon=0}\\
&= \partial_k u^i \underbrace{\langle N^i_*, \partial^{i,*}_l\rangle}_{=0} + u^i \underbrace{\langle \partial_k N^i_*, \partial^{i,*}_l\rangle}_{C^{3+\alpha}} + \partial_k\varphi^i \underbrace{\langle \tau^i_*, \partial^{i,*}_l\rangle}_{C^{4+\alpha}} + \varphi^i \underbrace{\langle \partial_k\tau^i_*, \partial^{i,*}_l\rangle}_{C^{3+\alpha}}\\
&\quad + \partial_l u^i \underbrace{\langle \partial^{i,*}_k, N^i_*\rangle}_{=0} + u^i \underbrace{\langle \partial^{i,*}_k, \partial_l N^i_*\rangle}_{C^{3+\alpha}} + \partial_l\varphi^i \underbrace{\langle \partial^{i,*}_k, \tau^i_*\rangle}_{C^{4+\alpha}} + \varphi^i \underbrace{\langle \partial^{i,*}_k, \partial_l\tau^i_*\rangle}_{C^{3+\alpha}}\\
&= a^{3+\alpha}u^i + a^{3+\alpha}\varphi^i + a^{4+\alpha}\partial_k\varphi^i + a^{4+\alpha}\partial_k\varphi^i.
\end{aligned}$$

[3]We want to remark that in this section unlike our usual notation convention the term $\boldsymbol{\varphi}$ will refer to the tangential part in the linearisation and not to a local parametrisation.

For the g_i^{kl} we use the identity

$$\sum_{j=1}^{n} g_{kj}^{i,\varepsilon} g_{i,\varepsilon}^{jl} = \delta_k^l. \tag{4.36}$$

Differentiation, evaluation in $\varepsilon = 0$, multiplication with $g_{i,*}^{km}$ and summing over k then yields

$$\sum_{j,k} \left(\frac{d}{d\varepsilon} g_{kj}^{i,\varepsilon}\big|_{\varepsilon=0} \right) g_{i,*}^{km} g_{i,*}^{jl} = -\sum_j \underbrace{\left(\sum_k g_{jk}^{i,*} g_{i,*}^{km} \right)}_{=\delta_m^j} \frac{d}{d\varepsilon} g_{i,\varepsilon}^{jl}\big|_{\varepsilon=0} = \frac{d}{d\varepsilon} g_{i,\varepsilon}^{ml}\big|_{\varepsilon=0}. \tag{4.37}$$

Here, we used the symmetry of the metric tensor. As the $g_{i,*}^{kl}$ have $C^{4+\alpha}$-regularity, this proves the result above. Finally, we recall that the Christoffel symbols have the form

$$(\Gamma_{kl}^m)^{i,\varepsilon} = \sum_j \frac{1}{2} g_{i,\varepsilon}^{mj} (\partial_k g_{lj}^{i,\varepsilon} + \partial_l g_{kj}^{i,\varepsilon} - \partial_j g_{kl}^{i,\varepsilon}).$$

The claim then follows by applying product rule and the previous results. □

With this we can now start to linearise $(4.20)_1$. The results for the normal velocity and the mean curvature operator can be found in [19, Section 3]. With these and the lemma above we only have to take care of the surface Laplacian of the mean curvature. Using its formula in local coordinates and product rule we get

$$\frac{d}{d\varepsilon} \Delta_{\Gamma_\varepsilon^i} H_\varepsilon^i\big|_{\varepsilon=0} = \Delta_{\Gamma_*^i} \frac{d}{d\varepsilon} H_\varepsilon^i\big|_{\varepsilon=0} + \sum_{k,l} \left(\frac{d}{d\varepsilon} g_{i,\varepsilon}^{kl}\big|_{\varepsilon=0} \right) \underbrace{\partial_{kl} H_*^i}_{C^{1+\alpha}} - \sum_m \left(\frac{d}{d\varepsilon} (\Gamma_{kl}^m)^{i,\varepsilon}\big|_{\varepsilon=0} \right) \underbrace{\partial_m H_*^i}_{C^{2+\alpha}}.$$

Let us first consider the Δ_{Γ_*}-term. Using the results from [19, Section 3] for the linearisation of the mean curvature operator we derive

$$\begin{aligned}
\Delta_{\Gamma_*^i} \frac{d}{d\varepsilon} H_\varepsilon^i\big|_{\varepsilon=0} &= \Delta_{\Gamma_*^i} \Delta_{\Gamma_*^i} u^i + \underbrace{|II_*^i|^2}_{C^{3+\alpha}} \Delta_{\Gamma_*^i} u^i + \underbrace{\Delta_{\Gamma_*^i} |II_*^i|^2}_{C^{1+\alpha}} u^i + 2 \langle \underbrace{\nabla_{\Gamma_*^i} |II_*^i|^2}_{C^{2+\alpha}}, \nabla_{\Gamma_*^i} u^i \rangle \\
&\quad + \Delta_{\Gamma_*^i} \varphi^i \underbrace{\langle \nabla_{\Gamma_*^i} H_*^i, \tau_*^i \rangle}_{C^{2+\alpha}} + \underbrace{\Delta_{\Gamma_*^i} \langle \nabla_{\Gamma_*^i} H_*^i, \tau_*^i \rangle}_{C^{\alpha}} \varphi^i + 2 \langle \underbrace{\nabla_{\Gamma_*^i} (\langle \nabla_{\Gamma_*^i} H, \tau_*^i \rangle)}_{C^{1+\alpha}}, \nabla_{\Gamma_*^i} \varphi^i \rangle \\
&= \Delta_{\Gamma_*^i} \Delta_{\Gamma_*^i} u^i + \sum_{k,l} \left(a^{3+\alpha} \partial_{kl} u^i + a^{2+\alpha} \partial_{kl} \varphi^i \right) \\
&\quad + \sum_k \left(a^{2+\alpha} \partial_k u^i + a^{1+\alpha} \partial_k \varphi^i \right) + a^{1+\alpha} u^i + a^\alpha \varphi^i.
\end{aligned}$$

For the other two terms we just put in the results from (4.5) and correct them with the regularity of the new coefficients and get

$$a^{1+\alpha} u^i + a^{1+\alpha} \varphi^i + \sum_l a^{1+\alpha} \partial_l u^i + \sum_l a^{1+\alpha} \partial_l \varphi^i + \sum_{l,l'} a^{2+\alpha} \partial_{ll'} \varphi^i.$$

Adding up both we conclude finally

$$\frac{d}{d\varepsilon} \left(-\Delta_{\Gamma_\varepsilon^i} H_\varepsilon^i \right) \big|_{\varepsilon=0} = -\Delta_* \Delta_* u^i + \mathcal{A}^i u^i + \zeta^i (\boldsymbol{\varphi} \circ \boldsymbol{pr})^i,$$

where we used the abbreviations

$$\mathcal{A}^i u^i := \sum_{k,l} a^{3+\alpha} \partial_{kl} u^i + \sum_k a^{1+\alpha} \partial_k u^i + a^{1+\alpha} u^i,$$

$$\zeta^i\varphi^i := \sum_{k,l} a^{2+\alpha}\partial_{kl}\varphi^i + \sum_k a^{1+\alpha}\partial_k\varphi^i + a^\alpha\varphi^i.$$

In total, the linearisation of $(4.20)_1$ reads as

$$\partial_t u^i = -\Delta_*\Delta_* u^i + \mathcal{A}^i u^i + \zeta^i\varphi^i.$$

The angle conditions were linearised by the authors of [18] and can be found there and the $(4.20)_2$ is already linear. For $(4.20)_5$ we can just plug in our knowledge of the linearisation of the mean curvature operator from [19, Section 3]. Summing up this leads to

$$\begin{aligned} 0 &= \gamma^1\Delta_* u^1 + \gamma^2\Delta_* u^2 + \gamma^3\Delta_* u^3 + \gamma^1|II^1_*|^2u^1 + \gamma^2|II^2_*|^2u^2 + \gamma^3|II^3_*|^2u^3 \\ &\quad + \gamma^1\langle\nabla_* H^1_*, \tau^1_*\rangle\varphi^1 + \gamma^2\langle\nabla_* H^2_*, \tau^2_*\rangle\varphi^2 + \gamma^3\langle\nabla_* H^3_*, \tau^3_*\rangle\varphi^3. \end{aligned}$$

Observe that we can express $\boldsymbol{\varphi}$ as $\mathcal{T}u$, which is a local term on the boundary. So, we can write the equation above in the form

$$\gamma^1\Delta_* u^1 + \gamma^2\Delta_* u^2 + \gamma^3\Delta_* u^3 + \mathcal{A}^{CCP}_\Sigma \boldsymbol{u} = 0, \tag{4.38}$$

where we used the abbreviation

$$\mathcal{A}^{CCP}_\Sigma \boldsymbol{u} := \sum_{i=1}^3 \gamma^i|II^i_*|^2u^i + \gamma^i\langle\nabla_{\Gamma^i_*} H^i_*, \tau^i_*\rangle\varphi^i.$$

It remains to linearise $\mathcal{G}^5$ and $\mathcal{G}^6$. Before we can go on with this we need some qualitative results for the linearisation of the normal and the conormal.

Lemma 4.6 (Linearisation of the normal and the conormal)**.**

$$\begin{aligned} \frac{d}{d\varepsilon}N^i_\varepsilon\big|_{\varepsilon=0} &= \sum_{l=1}^n \left(a^{4+\alpha}\partial_l u^i + a^{4+\alpha}\partial_l\varphi^i\right) + a^{3+\alpha}u^i + a^{3+\alpha}\varphi^i, \\ \frac{d}{d\varepsilon}\nu^i_\varepsilon\big|_{\varepsilon=0} &= \sum_{l=1}^n \left(a^{4+\alpha}\partial_l u^i + a^{4+\alpha}\partial_l\varphi^i\right) + a^{3+\alpha}u^i + a^{3+\alpha}\varphi^i. \end{aligned}$$

Proof. Writing N^i_ε as

$$\frac{\partial^{i,\varepsilon}_1 \times \cdots \times \partial^{i,\varepsilon}_n}{\|\partial^{i,\varepsilon}_1 \times \cdots \times \partial^{i,\varepsilon}_n\|}, \tag{4.39}$$

the first part follows using the quotient rule from Lemma 4.5. If we choose ∂^ε_i such that $\{\partial^\varepsilon_1, ..., \partial^\varepsilon_{n-1}\}$ is a basis[4] of the tangent space of Σ_* we can write the conormal as

$$\frac{N^i_\varepsilon \times \partial^{i,\varepsilon}_1 \times \cdots \partial^{i,\varepsilon}_{n-1}}{\|N^i_\varepsilon \times \partial^{i,\varepsilon}_1 \times \cdots \partial^{i,\varepsilon}_{n-1}\|}, \tag{4.40}$$

and thus the second equality follows from Lemma 4.5 and the first equality. □

With this in mind, we can now as before write the surface gradient in local coordinates and then find the derivative using the product rules and the results for the geometric quantities. We have that

$$\frac{d}{d\varepsilon}\left(\nabla_{\Gamma^i_\varepsilon} H^i_\varepsilon \nu^i_\varepsilon\right)\big|_{\varepsilon=0} = \left(\frac{d}{d\varepsilon}\nabla_{\Gamma^i_\varepsilon} H^i_\varepsilon\big|_{\varepsilon=0}\right)\nu^i_* + \underbrace{\nabla_* H^i_*}_{C^{2+\alpha}}\frac{d}{d\varepsilon}\nu^i_\varepsilon\big|_{\varepsilon=0}$$

[4] If this holds for the reference frame, it will also be true for Σ_ε as this is parametrised via the composition of $\boldsymbol{\Phi}_{\varepsilon\boldsymbol{u},\varepsilon\boldsymbol{\varphi}}$ with the parametrisation of Σ_*.

$$
\begin{aligned}
&= \nabla_* \left(\Delta_* u^i + \underbrace{|II^i_*|^2}_{C^{3+\alpha}} u^i + \underbrace{\langle \nabla_* H^i_*, \tau^i_* \rangle}_{C^{2+\alpha}} \varphi^i \right) \cdot \nu^i_* \\
&\quad + a^{2+\alpha} u^i + \sum_l a^{2+\alpha} \partial_l u^i + \sum_l a^{2+\alpha} \partial_l u^i + a^{2+\alpha} u_i \\
&= \nabla_* (\Delta_* u^i) \cdot \nu^i_* + \sum_l a^{2+\alpha} \partial_l u^i + a^{2+\alpha} u^i .
\end{aligned}
$$

Note that in the second equality we directly write the terms arising from both the linearisation of the conormal and the gradient in the short notation for lower order rest terms. Linearising both flux terms we get

$$
\nabla_* (\Delta_* u^i) \cdot \nu^i_* - \nabla_* (\Delta_* u^j) \cdot \nu^j_* = \mathcal{A}^{BFC}_{ij}(u^i, u^j),
$$

where we used the abbreviation

$$
\mathcal{A}^{BFC}_{ij}(u^i, u^j) = \sum_{l=1}^n a^{2+\alpha} \partial_l u^i + a^{2+\alpha} \partial_l u^j + a^{2+\alpha} u^i + a^{2+\alpha} u^j .
$$

Summing up the work of this section we get the following linear system.

$$
(LSDFTJ): \begin{cases}
\partial_t u^i = -\Delta_* \Delta_* u^i + \mathcal{A}^i u^i + \zeta^i (\mathcal{T}(\boldsymbol{u} \circ \boldsymbol{pr}))^i & \text{on } \Gamma^i_* \times [0,T], i = 1,2,3, \\
\gamma^i u^i + \gamma^2 u^2 + \gamma^3 u^3 = 0 & \text{on } \Sigma_* \times [0,T], \\
\partial_{\nu^1_*} u^1 + II^1_*(\nu^1_*, \nu^1_*)(\mathcal{T}\boldsymbol{u})^1 = \partial_{\nu^2_*} u^2 + II^2_*(\nu^2_*, \nu^2_*)(\mathcal{T}\boldsymbol{u})^2 & \text{on } \Sigma_* \times [0,T], \\
\partial_{\nu^2_*} u^2 + II^2_*(\nu^2_*, \nu^2_*)(\mathcal{T}\boldsymbol{u})^2 = \partial_{\nu^3_*} u^3 + II^3_*(\nu^3_*, \nu^3_*)(\mathcal{T}\boldsymbol{u})^3 & \text{on } \Sigma_* \times [0,T], \\
\gamma^1 \Delta_* u^1 + \gamma^2 \Delta_* u^2 + \gamma^3 \Delta_* u^3 + \mathcal{A}^{CCP}_{\Sigma} \boldsymbol{u} = 0 & \text{on } \Sigma^* \times [0,T], \\
\partial_{\nu^1_*}(\Delta_* u^1) - \partial_{\nu^2_*}(\Delta_* u^2) = \mathcal{A}^{BFC}_{12}(u^1, u^2) & \text{on } \Sigma^* \times [0,T], \\
\partial_{\nu^2_*}(\Delta_* u^2) - \partial_{\nu^3_*}(\Delta_* u^3) = \mathcal{A}^{BFC}_{23}(u^2, u^3) & \text{on } \Sigma_* \times [0,T], \\
u^i \big|_{t=0} = \rho^i_0 & \text{on } \Gamma^i_* .
\end{cases}
$$

Here, we directly rewrote the $\boldsymbol{\varphi}$-terms via the linear relation with $\boldsymbol{u}$ to get an equation solely in $\boldsymbol{u}$. We note that the non-local terms only appear in second order and so we can do the linearised analysis without them and include them later using a perturbation argument. The same holds for the lower order terms in $\boldsymbol{u}$.
Thus, the linear equations write with regard to their analytic structure as

$$
\begin{cases}
\partial_t u^i = \mathcal{A}^i_{\mathrm{all}}(u^i, \boldsymbol{u}\big|_{\Sigma_*}) + \mathfrak{f}^i & \text{on } \Gamma^i_* \times [0,T], i = 1,2,3, \\
\mathcal{B}\boldsymbol{u} = \mathfrak{b} & \text{on } \Sigma_* \times [0,T], \\
u^i \big|_{t=0} = u^i_0 & \text{on } \Gamma^i_*, i = 1,2,3,
\end{cases} \tag{4.41}
$$

where we used the notation

$$
\begin{aligned}
\mathcal{A}^i_{\mathrm{all}}(u^i, \boldsymbol{u}\big|_{\Sigma_*}) &= \mathcal{A}^i_W(u^i) + \mathcal{A}^i_P(u^i, \boldsymbol{u}\big|_{\Sigma_*}), \\
\mathcal{A}^i_W(u^i) &= -\Delta_* \Delta_* u^i, \\
\mathcal{A}^i_P(u^i, \boldsymbol{u}\big|_{\Sigma_*}) &= \sum_{k,l} a^{3+\alpha} \partial_{kl} u^i + \sum_k a^{2+\alpha} \partial_k u^i + a^{1+\alpha} u^i + \sum_{k,l} a^{2+\alpha} \partial_{kl} (\mathcal{T}(\boldsymbol{u} \circ \boldsymbol{pr}))^i \\
&\quad + \sum_k a^{1+\alpha} \partial_k (\mathcal{T}(\boldsymbol{u} \circ \boldsymbol{pr}))^i + a^{\alpha} (\mathcal{T}(\boldsymbol{u} \circ \boldsymbol{pr}))^i, \\
\mathcal{B} &= \mathcal{B}_W + \mathcal{B}_P, \\
\mathcal{B}^1 \boldsymbol{u} = \mathcal{B}^1_W \boldsymbol{u} &= \gamma^1 u^1 + \gamma^2 u^2 + \gamma^3 u^3, \\
\mathcal{B}^2_W \boldsymbol{u} &= \partial_{\nu^1_*} u^1 - \partial_{\nu^2_*} u^2,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_P^2 \boldsymbol{u} &= II_*^1(\nu_*^1, \nu_*^1)(\mathcal{T}\boldsymbol{u})^1 - II_*^2(\nu_*^2, \nu_*^2(\mathcal{T}\boldsymbol{u})^2, \\
\mathcal{B}_W^3 \boldsymbol{u} &= \partial_{\nu_*^2} u^2 - \partial_{\nu_*^3} u^3, \\
\mathcal{B}_P^3 \boldsymbol{u} &= II_*^2(\nu_*^2, \nu_*^2)(\mathcal{T}\boldsymbol{u})^2 - II_*^3(\nu_*^3, \nu_*^3)(\mathcal{T}\boldsymbol{u})^3, \\
\mathcal{B}_W^4 \boldsymbol{u} &= \sum_{i=1}^3 \gamma^i \big(\Delta_* u^i\big), \\
\mathcal{B}_P^4 \boldsymbol{u} &= \sum_{i=1}^3 \gamma^i |II_*^i|^2 u^i + \gamma^i \partial_{\nu_*^i} H_*^i (\mathcal{T}\boldsymbol{u})^i, \\
\mathcal{B}_W^5 \boldsymbol{u} &= \partial_{\nu_*^1}\big(\Delta_* u^1\big) - \partial_{\nu_*^2}\big(\Delta_* u^2\big), \\
\mathcal{B}_P^5 \boldsymbol{u} &= \sum_{k=1}^n (a^{1+\alpha}\partial_k u^1 + a^{1+\alpha}\partial_k u^2) + a^\alpha u^1 + a^\alpha u^2, \\
\mathcal{B}_W^6 \boldsymbol{u} &= \partial_{\nu_*^2}\big(\Delta_* u^2\big) - \partial_{\nu_*^3}\big(\Delta_* u^3\big), \\
\mathcal{B}_P^6 \boldsymbol{u} &= \partial_{\nu_*^2} + \sum_{k=1}^n (a^{1+\alpha}\partial_k u^2 + a^{1+\alpha}\partial_k u^3) + a^\alpha u^2 + a^\alpha u^3.
\end{aligned}
$$

Additionally, the $\mathfrak{f}^i$ and $\mathfrak{b}^i$ denote possible inhomogeneities. Choosing them as

$$
\begin{aligned}
\mathfrak{f}^i(u^i, \boldsymbol{u}|_{\Sigma^*}) &:= -\mathcal{A}_{\text{all}}^i\left(u^i, \boldsymbol{u}\big|_{\Sigma_*}\right) + \mathcal{F}^i\left(u^i, \boldsymbol{u}^i\big|_{\Sigma_*}\right) + a_\dagger^i\left(u^i, \boldsymbol{u}\big|_{\Sigma_*}\right)\left(\left\{\mathcal{P}(\boldsymbol{u}, \boldsymbol{u}\big|_{\Sigma_*})\boldsymbol{\mathcal{F}}(\boldsymbol{u}, \boldsymbol{u}\big|_{\Sigma_*})\right\} \circ \boldsymbol{pr}\right)^i, \\
\mathfrak{b}^i(\boldsymbol{u}) &:= \mathcal{G}^i(\boldsymbol{u}) - \mathcal{B}^i(\boldsymbol{u}),
\end{aligned}
$$

we can rewrite problem (4.29) in the form for the fixed point problem we will study after the linearised analysis.

4.5 Analysis of the Linearised Problem

In this section we want to derive an existence result for the linearised system derived in the section before. We will first study the system without the non-local terms, zero initial data and and some other terms causing technical troubles. Once we have derived a solution theory of the reduced system with Hölder-regularity we can include the missing terms with a perturbation argument in the last section. We then get the following result.

Theorem 4.7 (Short time existence for (LSDFTJ) with zero initial data)**.**
Set $\sigma_1 = 0, \sigma_2 = \sigma_3 = 1, \sigma_4 = 2, \sigma_5 = \sigma_6 = 3$. For every $\alpha \in (0,1)$ there exists a $\delta_0 > 0$ such that for all

$$
\mathfrak{f}^i \in C_{TJ}^{\alpha, \frac{\alpha}{2}}(\Gamma_{*,\delta_0}), i = 1,2,3, \mathfrak{b}^i \in C^{4+\alpha-\sigma_i, \frac{4+\alpha-\sigma_i}{4}}(\Sigma_{*,\delta_0}), \quad i = 1, ..., 6,
$$

that fulfil the compatibility conditions

$$
\left(\gamma^1\mathfrak{f}^1 + \gamma^2\mathfrak{f}^3 + \gamma^3\mathfrak{f}^3\right)\big|_{t=0} = 0, \mathfrak{b}^i\big|_{t=0} = 0 \text{ on } \Sigma_*, \quad i = 1, ..., 6, \tag{4.42}
$$

problem (4.41) with zero initial data has a unique solution $(u^1, u^2, u^3) \in C_{TJ}^{4+\alpha, 1+\frac{\alpha}{4}}(\Gamma_{*,\delta_0})$. Moreover, there exists a constant $C > 0$ with

$$
\|\boldsymbol{u}\|_{C_{TJ}^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,\delta_0})} \le C\left(\|\mathfrak{f}\|_{C_{TJ}^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,\delta_0})} + \sum_{i=1}^6 \|\mathfrak{b}^i\|_{C^{4+\alpha-\sigma_i,\frac{4+\alpha-\sigma_i}{4}}(\Sigma_{*,\delta_0})}\right). \tag{4.43}
$$

In the last subsection we will also discuss that the last theorem implies short time existence for non-zero initial data and get the following consequence.

Corollary 4.8 (Short time existence for (LSDFTJ)).
Let $\sigma_i, i = 1, ..., 6$ be defined as in Theorem 4.7. For any $\alpha \in (0,1)$ and initial data $\boldsymbol{u}_0 \in C^{4+\alpha}_{TJ}(\Gamma_*)$ there exists a $\delta_0 > 0$ such that for every $\mathfrak{f} \in C^{\alpha,\frac{\alpha}{4}}_{TJ}(\Gamma_{*,\delta_0})$ and $\mathfrak{b}^i \in C^{4+\alpha-\sigma_i,\frac{4+\alpha-\sigma_i}{4}}(\Sigma_{*,\delta_0}), i = 1, ..., 6$ fulfilling the inhomogenous compatibility conditions

$$(CLP)\begin{cases}(\gamma^1\mathfrak{f}^1 + \gamma^2\mathfrak{f}^2 + \gamma^3\mathfrak{f}^3)\big|_{t=0} & = \mathcal{B}_0(\boldsymbol{u}_0) := -\sum_{i=1}^3 \gamma^i \mathcal{A}^i_{all} u^i_0,\\ \mathfrak{b}^i\big|_{t=0} & = -\mathcal{B}^i u^i_0, \quad i = 1, ..., 6,\end{cases} \tag{4.44}$$

on Σ_* the problem (4.41) with initial data $\boldsymbol{u}_0$ has a unique solution $\boldsymbol{u} \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,\delta_0})$. Moreover, there exists a $C > 0$ with

$$\|\boldsymbol{u}\|_{C^{4+\alpha,\frac{1+\alpha}{4}}_{TJ}(\Gamma_{*,\delta_0})} \le C\left(\|\mathfrak{f}\|_{C^{\alpha,\frac{\alpha}{4}}_{TJ}(\Gamma_{*,\delta_0})} + \|\boldsymbol{u}_0\|_{C^{4+\alpha}_{TJ}(\Gamma_*)} + \sum_{i=1}^{6}\|\mathfrak{b}^i\|_{C^{4+\alpha-\sigma_i,\frac{4+\alpha-\sigma_i}{4}}(\Sigma_{*,\delta_0})}\right). \tag{4.45}$$

4.5.1 Existence of a Weak Solution for the Principal Part

In the first step of the linear analysis we want to show existence of a unique weak solution. For this, we have to find a weak formulation of the problem such that sufficiently smooth solutions will fulfil the equations and all boundary conditions in a classical sense. Here we get a technical problem as the direct test procedure, which we successfully used in Section 3.3, will lead to a concept of weak solutions, where we search for solutions in $H^2_{TJ}(\Gamma_*)$. In this setting we can put the boundary condition $(LSDFTJ)_1, (LSDFTJ)_2$ and $(LSDFTJ)_3$ in the space of testfunctions but the sum condition for $\boldsymbol{u}$ on the boundary will then reduce the degree of freedom of the testfunctions on the boundary. But then we can only write two boundary conditions in the equation itself as otherwise we cannot retrieve them for a classical solution. Thus, with this approach we cannot fit all six boundary conditions in the weak formulation.
To solve this problem we need to split the fourth order parabolic problem in a coupled system of a second order parabolic and a second order elliptic equation. Then we can put one boundary condition in the space of testfunctions for each equation and two boundary conditions in both of the equations. For the split we need to introduce an artificial variable that we choose as

$$v^i := -\Delta_* u^i + C_v u^i, \quad i = 1, 2, 3,$$

which equals the linearisation of the mean curvature operator up to lower order terms. Originally, that was our candidate for v_i as the system basically splits in two subsystems for $\boldsymbol{u}$ and $\left(\frac{d}{d\varepsilon}H^i_\varepsilon\right)_{i=1,2,3}$ with symmetric boundary conditions. But this choice leads to non-local terms that cause energetic problems. Now, we consider in the formulation of problem (4.41) only the terms marked with a subscript W, which leads to

$$(LSDFTJ)_P\begin{cases}\partial_t u^i = \Delta_* v^i - C_u v^i + \mathfrak{f}^i & \text{on } \Gamma^i_* \times [0,T], i = 1,2,3,\\ v_i = -\Delta_* u_i + C_v u^i & \text{on } \Gamma^i_* \times [0,T], i = 1,2,3,\\ \gamma^1 u^1 + \gamma^2 u^2 + \gamma^3 u^3 = \mathfrak{b}^1 & \text{on } \Sigma_* \times [0,T],\\ \partial_{\nu^1_*} u^1 - \partial_{\nu^*} u^2 = \mathfrak{b}^2 & \text{on } \Sigma_* \times [0,T],\\ \partial_{\nu^2_*} u^2 - \partial_{\nu^3_*} u^3 = \mathfrak{b}^3 & \text{on } \Sigma_* \times [0,T],\\ \gamma^1 v^1 + \gamma^2 v^2 + \gamma^3 v^3 = \mathfrak{b}^4 & \text{on } \Sigma_* \times [0,T],\\ \partial_{\nu^1_*} v^1 - \partial_{\nu^2_*} v^2 = \mathfrak{b}^5 & \text{on } \Sigma_* \times [0,T],\\ \partial_{\nu^2_*} v^2 - \partial_{\nu^3_*} v^3 = \mathfrak{b}^6 & \text{on } \Sigma_* \times [0,T],\\ u\big|_{t=0} = 0 & \text{on } \Gamma^i_*, i = 1,2,3.\end{cases} \tag{4.46}$$

Hereby, the terms $-C_u v^i$ and $C_v u^i$ are included to guarantee coercivity for the weak differential operator. The constants C_u and C_v are chosen large enough such that these hold. During the proof of existence of weak solutions we will see that they only depend on the system. Once we haven proven Hölder regularity for this system they will disappear in the perturbation argument.

Remark 4.9 (The inhomogeneities $\mathfrak{b}^1 - \mathfrak{b}^4$).
In [19] a possible inhomogeneity $\mathfrak{b}^1$ was not discussed as neither for the non-linear analysis nor the localization procedure an inhomogeneity arises in this boundary condition. Indeed, one can include them with the standard procedure for Dirichlet-type boundary conditions. In our situation, we have to deal with supplementary technical problems. The split of the system causes energetic problems such that we cannot allow inhomogeneities $\mathfrak{b}^2, \mathfrak{b}^3$ for the weak analysis. Also, condition $(4.46)_6$ *is written in the space of testfunctions and thus has to be homogeneous, too. But all the inhomogeneities $\mathfrak{b}^2, \mathfrak{b}^3, \mathfrak{b}^4$ are needed for the non-linear analysis as the corresponding boundary conditions are non-linear. So our strategy is to omit them now and include them after proving Hölder-regularity for the system without these inhomogeneities. This will be carried out in Subsection 4.5.4 using perturbations arguments. As an additional technical problem we need to improve the localization procedure compared to [19] such that we get there no inhomogeneities in the corresponding boundary conditions when localizing the problem.*

We now search for a functional analytic setting for the weak solution theory. Formally, multiplying the equation for u^i with a smooth function $\gamma^i\zeta^i$ such that $\sum_{i=1}^3 \gamma^i\zeta^i = 0$ on Σ_*, integrating and summing up we see that

$$\begin{aligned}
\sum_{i=1}^3 \gamma^i \int_{\Gamma_*^i} \partial_t u^i \zeta^i d\mathcal{H}^n &= \sum_{i=1}^3 \int_{\Gamma_*^i} \gamma^i \left(\Delta_* v^i\right) \zeta^i - \gamma^i C_u v^i \zeta^i + \gamma^i \mathfrak{f}^i \zeta^i d\mathcal{H}^n \\
&= -\sum_{i=1}^3 \int_{\Gamma_*^i} \gamma^i \nabla_* v^i \cdot \nabla_* \zeta^i + \gamma^i C_u v^i \zeta^i - \gamma^i \mathfrak{f}^i \zeta^i d\mathcal{H}^n \\
&\quad + \sum_{i=1}^3 \int_{\Sigma_*} \gamma^i \partial_{\nu_*^i} v^i \zeta^i d\mathcal{H}^{n-1}.
\end{aligned}$$

For the boundary term we get using $(4.46)_7$ and $(4.46)_8$ that

$$\begin{aligned}
\sum_{i=1}^3 \int_{\Sigma_*} \gamma^i \partial_{\nu_*^i} v^i \zeta^i d\mathcal{H}^{n-1} &= \int_{\Sigma_*} \gamma^1 (\partial_{\nu_*^2} v^2 + \mathfrak{b}^5)\zeta^1 + \gamma^2 \partial_{\nu_*^2} v^2 \zeta^2 + \gamma^3 (\partial_{\nu_*^2} v^2 - \mathfrak{b}^6)\zeta^3 d\mathcal{H}^{n-1} \\
&= \int_{\Sigma_*} \left(\sum_{i=1}^3 \gamma^i \zeta^i\right) \partial_{\nu_*^2} v^2 + \gamma^1 \mathfrak{b}^5 \zeta^1 - \gamma^3 \mathfrak{b}^6 \zeta^3 d\mathcal{H}^{n-1} \\
&= \int_{\Sigma_*} \gamma^1 \mathfrak{b}^5 \zeta^1 - \gamma^3 \mathfrak{b}^6 \zeta^3 d\mathcal{H}^{n-1}.
\end{aligned}$$

Similarly, testing the equation for $\gamma^i v^i$ with ζ^i and summing up we get

$$\begin{aligned}
\sum_{i=1}^3 \int_{\Gamma_*^i} \gamma^i v^i \zeta^i d\mathcal{H}^n &= \sum_{i=1}^3 \int_{\Gamma_*^i} -\gamma^i \Delta_* u^i \zeta^i + \gamma^i C_v u^i \zeta^i d\mathcal{H}^n \\
&= \sum_{i=1}^3 \int_{\Gamma_*^i} \gamma^i \nabla_* u^i \cdot \nabla_* \zeta^i + \gamma^i C_v u^i \zeta^i d\mathcal{H}^n - \sum_{i=1}^3 \int_{\Sigma_*} \gamma^i \partial_{\nu_*^i} u^i \zeta^i d\mathcal{H}^{n-1},
\end{aligned}$$

where we see for the second term in the last line using $(4.46)_4$ and $(4.46)_5$, that

$$\sum_{i=1}^3 \int_{\Sigma_*} \gamma^i \partial_{\nu_*^i} u^i \zeta^i d\mathcal{H}^{n-1} = \int_{\Sigma_*} (\partial_{\nu_*^2} u^2) \sum_{i=1}^3 \gamma^i \zeta^i d\mathcal{H}^{n-1} = 0.$$

This motivates us to choose the following setting. We consider the function spaces

$$\begin{aligned}
\mathcal{L} &:= L^2(\Gamma_*^1) \times L^2(\Gamma_*^2) \times L^2(\Gamma_*^3), \\
\mathcal{L}_b &:= L^2(\Sigma_*)^3,
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^1 &:= H^1(\Gamma^1_*) \times H^1(\Gamma^2_*) \times H^1(\Gamma^3_*),\\
\mathcal{H}^{-1} &:= H^{-1}(\Gamma^1_*) \times H^{-1}(\Gamma^2_*) \times H^{-1}(\Gamma^3_*),\\
\mathcal{E} &:= \{\boldsymbol{u} \in \mathcal{H}^1 | \gamma^1 u^1 + \gamma^2 u^2 + \gamma^3 u^3 = 0 \text{ a.e. on } \Sigma^*\},
\end{aligned}$$

and the continuous operators given by

$$\begin{aligned}
\langle \partial_t \boldsymbol{u}, \boldsymbol{\zeta} \rangle_{dual} &:= \sum_{i=1}^3 \gamma^i \langle \partial_t u^i, \zeta^i \rangle_{dual} && \partial_t \boldsymbol{u} \in \mathcal{E}^{-1}, \boldsymbol{\zeta} \in \mathcal{E},\\
B_u[\boldsymbol{v}, \boldsymbol{\zeta}] &:= -\sum_{i=1}^3 \gamma^i \int_{\Gamma^i_*} \nabla_* v^i \cdot \nabla_* \zeta^i + C_u v^i \zeta^i d\mathcal{H}^n && \boldsymbol{v}, \boldsymbol{\zeta} \in \mathcal{E},\\
B_v[\boldsymbol{u}, \boldsymbol{\psi}] &:= \sum_{i=1}^3 \gamma^i \int_{\Gamma^i_*} \nabla_* u^i \cdot \nabla_* \psi^i + C_v u^i \psi^i d\mathcal{H}^n && \boldsymbol{u}, \boldsymbol{\psi} \in \mathcal{E},\\
b_u(\boldsymbol{\zeta}; t) &:= \int_{\Sigma_*} \gamma^1 \mathfrak{b}^5 \zeta^1 - \gamma^3 \mathfrak{b}^6 \zeta^3 d\mathcal{H}^n && \boldsymbol{\zeta} \in \mathcal{E},\\
f_u(\boldsymbol{\zeta}; t) &= \sum_{i=1}^3 \int_{\Gamma^i_*} \gamma^i \mathfrak{f}^i \zeta^i d\mathcal{H}^n && \boldsymbol{\zeta} \in \mathcal{E}.
\end{aligned}$$

For clarification we note that $\langle \cdot, \cdot \rangle_{\mathrm{dual}}$ in the first line on the right hand side is the usual duality pairing between $\mathcal{E}^{-1}$ and $\mathcal{E}$. Now we can define a suitable weak solution concept for our setting.

Definition 4.10 (Weak Solution of $(LSDFTJ)_P$)**.**
We call a tuple $(\boldsymbol{u}, \boldsymbol{v}) \in L^2(0, T; \mathcal{E}) \times L^2(0, T; \mathcal{E})$ with $\partial_t \boldsymbol{u} \in L^2(0, T; \mathcal{E}^{-1})$ a weak solution of (4.46) *if for all $\boldsymbol{\zeta}, \boldsymbol{\psi} \in \mathcal{E}$ and almost all $t \in [0, T]$ it holds*

$$\begin{cases} \langle \partial_t \boldsymbol{u}, \boldsymbol{\zeta} \rangle_{dual} = B_u[\boldsymbol{v}, \boldsymbol{\zeta}] + b_u(\boldsymbol{\zeta}; t) + f_u(\boldsymbol{\zeta}; t),\\ (\boldsymbol{\gamma} \boldsymbol{v}, \boldsymbol{\psi})_{\mathcal{L}} = B_v[\boldsymbol{u}, \boldsymbol{\psi}], \end{cases} \tag{4.47}$$

and additionally it holds

$$\boldsymbol{u}\big|_{t=0} = 0. \tag{4.48}$$

Before we go on we check that this is a reasonable solution concept.

Lemma 4.11. *A weak solution $(\boldsymbol{u}, \boldsymbol{v})$ of (4.46) with $\boldsymbol{u} \in C^{4,1}(\Gamma_* \times [0, T])$ is a classical solution of (4.46). Furthermore, every classical solution with $C^{4,1}$-regularity is a weak solution.*

Proof. Suppose that $(\boldsymbol{u}, \boldsymbol{v})$ is a weak solution of (4.46) with $\boldsymbol{u} \in C^{4,1}(\Gamma_* \times [0, T])$. Note that for $\zeta^1 \in C^\infty_0(\Gamma^1_*)$ the function $\boldsymbol{\zeta} = (\zeta^1, 0, 0)$ is a valid test function. Inserting this into the second line of (4.47) we get

$$\int_{\Gamma^1_*} \gamma^1 v^1 \zeta^1 d\mathcal{H}^n = \int_{\Gamma^1_*} \gamma^1 \nabla_{\Gamma^1_*} u^1 \nabla_{\Gamma^1_*} \zeta^1 + \gamma^1 C_v u^1 \zeta^1 d\mathcal{H}^n = \int_{\Gamma^1_*} \left(-\gamma^1 (\Delta_{\Gamma^1_*} u^1) + \gamma^1 C_v u^1\right) \zeta^1 d\mathcal{H}^n,$$

where we used that due to the choice of ζ^1 the terms b_v and the ones coming from integration by parts vanish. This implies that

$$v^1 = -\Delta_{\Gamma^1_*} u^1 + C_v u^1, \tag{4.49}$$

holds pointwise due to the fundamental theorem of calculus of variations. Analogously, this follows for v^2 and v^3. Inserting this in the first line of (4.47) and again testing with $(\zeta^1, 0, 0)$ we get

$$\int_{\Gamma^1_*} \gamma^1 \partial_t u^1 \zeta^1 d\mathcal{H}^n = \int_{\Gamma^1_*} -\gamma^1 \nabla_{\Gamma^1_*} (-\Delta_{\Gamma^1_*} u^1 + C_v u^1) \nabla_{\Gamma^1_*} \zeta^1 - \gamma^1 C_u (-\Delta_{\Gamma^1_*} u^1 + C_v u^1) \zeta^1 + \gamma^1 \mathfrak{f}^1 \zeta^1 d\mathcal{H}^n$$

$$= \int_{\Gamma_*^1} \left(-\gamma^1 \Delta_{\Gamma_*^1} \Delta_{\Gamma_*^1} u^1 + \gamma^1 C_v \Delta_* u^1 + \gamma^1 C_u \Delta_* u^1 - \gamma^1 C_u C_v u^1 + \gamma^1 \mathfrak{f}^1\right) \zeta^1 d\mathcal{H}^n,$$

again using the structure of ζ^1 and recalling that the classical time derivative of $\boldsymbol{u}$ corresponds with its distributive time derivative. Using this we get

$$\partial_t u^1 = -\gamma^1 \Delta_{\Gamma_*^1} \Delta_{\Gamma_*^1} u^1 + \gamma^1 C_v \Delta_* u^1 + \gamma^1 C_u \Delta_* u^1 - \gamma^1 C_u C_v u^1 + \gamma^1 \mathfrak{f}^1, \tag{4.50}$$

and by the same calculation we get the equation also on the other hypersurfaces.
Next, we want to see that $\boldsymbol{u}$ fulfils the boundary conditions 4.46_3-$(4.46)_8$ as well. For any $\zeta \in C^\infty(\Sigma_*)$ we choose a smooth continuation of $(\gamma^1)^{-1}\zeta$ on Γ_*^1 and a smooth continuation of $-(\gamma^2)^{-1}\zeta$ on Γ_*^2, which we also call $(\gamma^1)^{-1}\zeta$ resp. $-(\gamma^2)^{-1}\zeta$. Testing $(4.47)_2$ with $((\gamma^1)^{-1}\zeta, -(\gamma^2)^{-1}\zeta, 0)$, which is a valid testfunction, we get

$$\begin{aligned}\int_{\Gamma_*^1} \gamma^1 v^1 \frac{1}{\gamma^1}\zeta d\mathcal{H}^n - \int_{\Gamma_*^2} \gamma^2 v^2 \frac{1}{\gamma^2}\zeta d\mathcal{H}^n &= \int_{\Gamma_*^1} \gamma^1 \nabla_* u^1 \cdot \nabla_* \left(\frac{1}{\gamma^1}\zeta\right) + \gamma^1 C_v u^1 \frac{1}{\gamma^1}\zeta d\mathcal{H}^n \\ &\quad - \int_{\Gamma_*^2} \gamma^2 \nabla_* u^2 \cdot \nabla_* \left(\frac{1}{\gamma^2}\zeta\right) + \gamma^2 C_v u^2 \frac{1}{\gamma^2}\zeta d\mathcal{H}^n.\end{aligned}$$

Using integration by parts on the first summand in both integrals on the right hand side and using that we already know that $v^i = -\Delta_* u^i + C_v u^i$ holds pointwise, this reduces to

$$\int_{\Sigma_*} (\partial_{\nu_*^1} u^1 - \partial_{\nu_*^2} u^2)\zeta d\mathcal{H}^{n-1} = 0$$

Again using the fundamental theorem of calculus of variations this implies that $(4.46)_4$ holds classically. The conditions $(4.46)_5, (4.46)_7$ and $(4.46)_8$ follow with the same procedure and $(4.46)_3$ and $(4.46)_6$ are already part of the definition of the solution due to the choice of $\mathcal{E}$. □

Next, we show existence of a unique weak solution of $(LSDFTJ)_P$.

Proposition 4.12 (Existence of weak solutions of $(LSDFTJ)_P$).
For all $\mathfrak{f} \in L^2(0,T;\mathcal{L})$ and $\mathfrak{b}^5, \mathfrak{b}^6 \in L^2(0,T;L^2(\Sigma_*))$ there exists a unique weak solution $(\boldsymbol{u}, \boldsymbol{v})$ of (4.46) and we have the energy estimates

$$\begin{aligned}&\max_{0\le t\le T} \|\boldsymbol{u}(t)\|_{\mathcal{E}} + \|\boldsymbol{u}\|_{L^2(0,T;\mathcal{E})} + \|\boldsymbol{u'}\|_{L^2(0,T;\mathcal{E}^{-1})} + \|\boldsymbol{v}\|_{L^2(0,T;\mathcal{E})} \\ &\qquad \le C\left(\|\mathfrak{f}\|_{L^2(0,T;\mathcal{L})} + \sum_{i=5}^{6} \|\mathfrak{b}^i\|_{L^2(0,T;L^2(\Sigma_*))}\right).\end{aligned} \tag{4.51}$$

Proof. We want to apply the Galerkin scheme. As in [19] we can choose the eigenfunctions $(\boldsymbol{z}_j)_{j\in\mathbb{N}}$ of the eigenvalue problem

$$\begin{cases} -\gamma^i \Delta_* z^i = \lambda \gamma^i z^i & \text{on } \Gamma_*^i, i = 1,2,3, \\ \gamma^1 z^1 + \gamma^2 z^2 + \gamma^3 z^3 = 0 & \text{on } \Sigma_*, \\ \partial_{\nu_*^1} z^1 = \partial_{\nu_*^2} z^2 = \partial_{\nu_*^3} z^3 & \text{on } \Sigma_*. \end{cases} \tag{4.52}$$

as orthonormal basis of $\mathcal{L}$ equipped with the inner product

$$(\boldsymbol{f}, \boldsymbol{g})_{\gamma\mathcal{L}} := \sum_{i=1}^{3} \gamma^i \int_{\Gamma_*^i} f^i g^i d\mathcal{H}^n. \tag{4.53}$$

In [19], the authors proved also smoothness of these functions. Note also that due to the fact that the $\boldsymbol{z}_j$ are weak solutions of this eigenvalue problem, they are also orthogonal with respect to the products B_u and B_v. We now search for weak solutions $(\boldsymbol{u}_m, \boldsymbol{v}_m)$ of the problem projected to the

m-dimensional subspace $Z_m := \langle z_j \rangle_{j=1,...,m}$ of $\mathcal{E}$, i.e. we search for solutions of

$$\begin{aligned} \langle \partial_t \boldsymbol{u}_m, \boldsymbol{z}_j \rangle_{dual} &= B_u[\boldsymbol{v}_m, \boldsymbol{z}_j] + b_u(\boldsymbol{z}_j, t) + f_u(\boldsymbol{z}_j, t), \quad j = 1, ..., m, \\ (\boldsymbol{v}_m, \boldsymbol{z}_j)_{\gamma\mathcal{L}} &= B_v[\boldsymbol{u}_m, \boldsymbol{z}_j], \quad j = 1, ..., m, \\ u(\cdot, 0) &= 0. \end{aligned} \tag{4.54}$$

We will index (4.54) with an m to clarify on which space we are projecting. Solutions of $(4.54)_m$ are of the form

$$\begin{aligned} \boldsymbol{u}_m(x,t) &= \sum_{j=1}^m a_{mj}(t)\boldsymbol{z}_j(x), \\ \boldsymbol{v}_m(x,t) &= \sum_{j=1}^m b_{mj}(t)\boldsymbol{z}_j(x). \end{aligned}$$

Due to the orthonormality of the $\boldsymbol{z}_j$ with respect to the product (4.53) we get the following ODE-System for the coefficient functions a_{mj} and b_{mj}, where we use the orthogonality properties with respect to B_u and B_v discussed above.

$$\begin{aligned} a'_{mj}(t) &= b_{mj}(t)B_u[\boldsymbol{z}_j, \boldsymbol{z}_j] + b_u(\boldsymbol{z}_j; t) + f_u(\boldsymbol{z}_j; t) && j = 1, ..., m, \\ b_{mj}(t) &= a_{mj}(t)B_u[\boldsymbol{z}_j, \boldsymbol{z}_j] && j = 1, ..., m, \\ a_{mj}(0) &= 0 && j = 1, ..., m. \end{aligned}$$

Inserting the equations for $b_{mj}(t)$ in those for $a'_{mj}(t)$ we get a standard system of ODEs for which we can apply the Theorem of Caratheodory to show existence of absolute continuous solutions a_{mj}. Inserting this solution in the second equation we also get the b_{mj} as functions in $L^2(0,T;\mathbb{R})$.
The next thing to do is verifying an energy estimate for our problem. We notice that for the solution (u_m, v_m) of $(4.54)_m$, the functions $\partial_t u_m$ and v_m are valid testfunctions. Thus, we can test the first equation in $(4.54)_m$ with v_m and the second equation with $\partial_t u_m$ to get

$$\begin{aligned} \sum_{i=1}^3 \gamma^i \int_{\Gamma_*^i} \partial_t u_m^i v_m^i d\mathcal{H}^n &= -\sum_{i=1}^3 \gamma^i \int_{\Gamma_*^i} |\nabla_* v_m^i|^2 d\mathcal{H}^n - \gamma^i C_u \sum_{i=1}^3 \int_{\Gamma_*^i} |v_m^i|^2 d\mathcal{H}^n \\ &\quad + \sum_{i=1}^3 \gamma^i \int_{\Gamma_*^i} \mathfrak{f}^i v_m^i d\mathcal{H}^n + \int_{\Sigma_*} \gamma^1 \mathfrak{b}^5 v_m^1 - \gamma^3 \mathfrak{b}^6 v_m^3 d\mathcal{H}^{n-1}, \\ \sum_{i=1}^3 \gamma^i \int_{\Gamma_*^i} v_m^i \partial_t u_m^i d\mathcal{H}^n &= \sum_{i=1}^3 \gamma^i \partial_t \left(\frac{1}{2} \int_{\Gamma_*^i} |\nabla_* u^i|^2 d\mathcal{H}^n + \frac{1}{2} C_v \int_{\Gamma_*^i} |u_m^i|^2 d\mathcal{H}^n \right). \end{aligned} \tag{4.55}$$

Subtracting the first equation from the second and rearranging the terms leads to

$$\begin{aligned} \sum_{i=1}^3 \gamma^i \left(\int_{\Gamma_*^i} |\nabla_* v_m^i|^2 d\mathcal{H}^n + C_u \int_{\Gamma_*^i} |v_m^i|^2 d\mathcal{H}^n \right) + \gamma^i \partial_t \left(\frac{1}{2} \int_{\Gamma_*^i} |\nabla_* u_m^i|^2 + C_v |u_m^i|^2 d\mathcal{H}^n \right) \\ = \sum_{i=1}^3 \left(\gamma^i \int_{\Gamma_*^i} \mathfrak{f}^i v_m^i d\mathcal{H}^n \right) + \int_{\Sigma_*} \gamma^1 \mathfrak{b}^5 v_m^1 - \gamma^3 \mathfrak{b}^6 v_m^3 d\mathcal{H}^{n-1}. \end{aligned} \tag{4.56}$$

Applying the weighted Young-inequality gives us

$$\sum_{i=1}^3 \gamma^i \int_{\Gamma_*^i} \mathfrak{f}^i v_m^i d\mathcal{H}^n \le \sum_{i=1}^3 \frac{\gamma^i}{2\varepsilon} \int_{\Gamma_*^i} |\mathfrak{f}^i|^2 d\mathcal{H}^n + \frac{\varepsilon \gamma^i}{2} \int_{\Gamma_*^i} |v_m^i|^2 d\mathcal{H}^n, \tag{4.57}$$

$$\int_{\Sigma_*} \gamma^1 \mathfrak{b}^5 v_m^1 d\mathcal{H}^{n-1} \le \frac{1}{2\varepsilon'} \|\gamma^1 \mathfrak{b}^5\|^2_{L^2(\Sigma_*)} + \frac{\varepsilon'}{2} \|v_m^1\|^2_{L^2(\Sigma_*)}, \tag{4.58}$$

$$\int_{\Sigma_*} -\gamma^3 \mathfrak{b}^6 v_m^3 d\mathcal{H}^{n-1} \leq \frac{1}{2\varepsilon'} \|\gamma^3 \mathfrak{b}^6\|^2_{L^2(\Sigma_*)} + \frac{\varepsilon'}{2} \|v_m^3\|^2_{L^2(\Sigma_*)}. \tag{4.59}$$

Note now that Lemma 2.11 implies for any $\boldsymbol{g} \in \mathcal{H}^1$ that

$$\|g^i\|^2_{L^2(\Sigma_*)} \leq \overline{\varepsilon}^2 \|\nabla_* \boldsymbol{g}\|^2_{\mathcal{L}} + 2\overline{\varepsilon} C_{\overline{\varepsilon}} \|\nabla_* \boldsymbol{g}\|_{\mathcal{L}} \|\boldsymbol{g}\|_{\mathcal{L}} + C^2_{\overline{\varepsilon}} \|\boldsymbol{g}\|^2_{\mathcal{L}} \leq 2\overline{\varepsilon}^2 \|\nabla_* \boldsymbol{g}\|^2_{\mathcal{L}} + 2C^2_{\overline{\varepsilon}} \|\boldsymbol{g}\|^2_{\mathcal{L}}.$$

Therefore, we conclude

$$\frac{\varepsilon'}{2} \left(\|v_m^1\|^2_{L^2(\Sigma_*)} + \|v_m^3\|^2_{L^2(\Sigma_*)} \right) \leq \overline{\varepsilon}^2 \varepsilon' \|\nabla_* \boldsymbol{w}_m\|^2_{\mathcal{L}} + C^2_{\overline{\varepsilon}} \varepsilon' \|\boldsymbol{w}_m\|^2_{\mathcal{L}}. \tag{4.60}$$

Applying (4.57)-(4.60) on (4.56) we get

$$\begin{aligned}
\sum_{i=1}^{3} \gamma^i \left(\int_{\Gamma_*^i} |\nabla_* v_m^i|^2 d\mathcal{H}^n + C_u \int_{\Gamma_*^i} |v_m^i|^2 d\mathcal{H}^n \right) + \partial_t \left(\gamma^i \frac{1}{2} \int_{\Gamma_*^i} |\nabla_* u_m^i|^2 + C_v |u_m^i|^2 d\mathcal{H}^n \right) \\
\leq \sum_{i=1}^{3} \frac{\gamma^i}{2\varepsilon} \int_{\Gamma_*^i} |\mathfrak{f}^i|^2 d\mathcal{H}^n + \frac{1}{2\varepsilon'} \left(\|\gamma^1 \mathfrak{b}^5\|^2_{L^2(\Sigma_*)} + \|\gamma^3 \mathfrak{b}^6\|^2_{L^2(\Sigma_*)} \right) \\
+ \sum_{i=1}^{3} \frac{\varepsilon \gamma^i}{2} \int_{\Gamma_*^i} |v_m^i|^2 d\mathcal{H}^n + \overline{\varepsilon}^2 \varepsilon' \|\nabla_* \boldsymbol{w}_m\|^2_{\mathcal{L}} + C^2_{\overline{\varepsilon}} \varepsilon' \|\boldsymbol{w}_m\|^2_{\mathcal{L}}.
\end{aligned} \tag{4.61}$$

Now choosing $\overline{\varepsilon} = \frac{1}{2}, \varepsilon' = \min(\gamma_1, \gamma_2, \gamma_3), \varepsilon = 1$ and then C_u large enough we can absorb the $\boldsymbol{w}_m$- and $\nabla_* \boldsymbol{w}_m$-terms on the right-hand side by the ones on the left-hand side to get

$$\partial_t (\|\boldsymbol{u}_m\|^2_{\gamma\mathcal{L}} + \|\nabla_* \boldsymbol{u}_m\|^2_{\gamma\mathcal{L}}) + C(\|\boldsymbol{w}_m\|^2_{\mathcal{L}} + \|\nabla_* \boldsymbol{w}_m\|^2_{\mathcal{L}}) \leq C\big(\|\mathfrak{b}^5\|^2_{L^2(\Sigma_*)} + \|\mathfrak{b}^6\|^2_{L^2(\Sigma_*)} + \|\mathfrak{f}\|^2_{\mathcal{L}}\big). \tag{4.62}$$

In particular, we get for all $t \in [0, T]$ that

$$\partial_t (\|\boldsymbol{u}_m\|^2_{\gamma\mathcal{L}} + \|\nabla_* \boldsymbol{u}_m\|^2_{\gamma\mathcal{L}}) \leq C\big(\|\mathfrak{b}^5\|^2_{L^2(\Sigma_*)} + \|\mathfrak{b}^6\|^2_{L^2(\Sigma_*)} + \|\mathfrak{f}\|^2_{\mathcal{L}}\big) \tag{4.63}$$

Integrating this in time using $u(0) \equiv 0$ leads to

$$\begin{aligned}
\|\boldsymbol{u}_m(t)\|^2_{\gamma\mathcal{L}} + \|\nabla_* \boldsymbol{u}_m(t)\|^2_{\gamma\mathcal{L}} &\leq C \int_0^t \left(\|\mathfrak{b}^5(t)\|^2_{L^2(\Sigma_*)} + \|\mathfrak{b}^6(t)\|^2_{L^2(\Sigma_*)} + \|\mathfrak{f}(t)\|^2_{\mathcal{L}} \right) dt \\
&\leq C \left(\|\mathfrak{b}^5\|^2_{L^2(0,T;L^2(\Sigma_*))} + \|\mathfrak{b}^6\|^2_{L^2(0,T;L^2(\Sigma_*))} + \|\mathfrak{f}\|^2_{L^2(0,T;\mathcal{L})} \right)
\end{aligned} \tag{4.64}$$

for all $t \in [0, T]$. Integrating (4.62) from 0 to T implies for $\boldsymbol{v}_m$ that

$$\|\boldsymbol{v}_m\|^2_{L^2(0,T;\mathcal{E})} \leq C\big(\|\mathfrak{b}^5\|^2_{L^2(0,T;L^2(\Sigma_*))} + \|\mathfrak{b}^6\|^2_{L^2(0,T;L^2(\Sigma_*))} + \|\mathfrak{f}\|^2_{L^2(0,T;\mathcal{L})}\big). \tag{4.65}$$

It remains to study the norm of $\partial_t \boldsymbol{u}_m$ which can be carried out with a standard argument. We first introduce the space $\gamma\mathcal{E}$ that contains all elements of $\mathcal{E}$ and is equipped with the inner product

$$(\boldsymbol{v}, \boldsymbol{w})_{\gamma\mathcal{E}} := (\nabla_* \boldsymbol{v}, \nabla_* \boldsymbol{w})_{\gamma\mathcal{L}} + (\boldsymbol{v}, \boldsymbol{w})_{\gamma\mathcal{L}}. \tag{4.66}$$

Note that as the $\boldsymbol{z}_j$ are classical solutions of the eigenvalue problem (4.52) they are also orthogonal sytem in $\gamma\mathcal{E}$. Also, the norm induced by the $\gamma\mathcal{E}$-product is equivalent to the norm on $\mathcal{E}$. Now, choosing any $\boldsymbol{v} \in \mathcal{E}$ with $\|\boldsymbol{v}\|_{\mathcal{E}} \leq 1$ we write $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ with

$$\boldsymbol{v}_1 \in \operatorname{span}\{\boldsymbol{z}_j\}_{j=1,\dots,m}, \quad (\boldsymbol{v}_2, \boldsymbol{z}_j)_{\gamma\mathcal{L}} = 0, j = 1, ..., m. \tag{4.67}$$

Due to equivalence of norms we get $\|\boldsymbol{v}\| \leq C'$ for a constant independent of $\boldsymbol{v}$ and

$$\|\boldsymbol{v}_1\|_{\gamma\mathcal{E}} \leq \|\boldsymbol{v}\|_{\gamma\mathcal{E}} \leq C, \tag{4.68}$$

due to the orthogonality of the $\boldsymbol{z}_j$ in $\mathcal{E}_\gamma$. Then, again using equivalence of the norms on $\mathcal{E}$ and $\gamma\mathcal{E}$, we conclude $\|\boldsymbol{v}_1\|_{\mathcal{E}} \le C$ for a C independent of $\boldsymbol{v}$. With this in mind we get the estimate

$$
\begin{aligned}
|\langle \partial_t \boldsymbol{u}_m, \boldsymbol{v}\rangle_{dual}| = |(\partial_t \boldsymbol{u}_m, \boldsymbol{v}^1)_{\gamma\mathcal{L}}| \\
&\big(|B(\boldsymbol{v}_m, \boldsymbol{v}_1)| + |f_u(\boldsymbol{v}_1))| + |b_u(\boldsymbol{v}_1)|\big) \\
&\le C\big(\|\boldsymbol{v}_m\|_{\mathcal{E}} + \|\mathfrak{f}\|_{\mathcal{L}} + \|\mathfrak{b}^5\|_{L^2(\Sigma_*)} + \|\mathfrak{b}^6\|_{L^2(\Sigma_*)}\big) \\
&\le C\big(\|\mathfrak{f}\|_{\mathcal{L}} + \|\mathfrak{b}^5\|_{L^2(\Sigma_*)} + \|\mathfrak{b}^6\|_{L^2(\Sigma_*)}\big).
\end{aligned}
$$

Here, we used (4.62) in the last inequality and for $b_u(\boldsymbol{v}_1; t)$ we used the Cauchy-Schwarz inequality and continuity of the trace operator to derive

$$
\begin{aligned}
|b_u(\boldsymbol{v}_1; t)| &\le C\big(\|\mathfrak{b}^5(t)\|_{L^2(\Sigma_*)}\|v_1^1\|_{L^2(\Sigma_*)} + \|\mathfrak{b}^6(t)\|_{L^2(\Sigma_*)}\|v_1^3\|_{L^2(\Sigma_*)}\big) \\
&\le C\big(\|\mathfrak{b}^5(t)\|_{L^2(\Sigma_*)} + \|\mathfrak{b}^6(t)\|_{L^2(\Sigma_*)}\big)\|\boldsymbol{v}_1\|_{\mathcal{E}} \\
&\le C\big(\|\mathfrak{b}^5(t)\|_{L^2(\Sigma_*)} + \|\mathfrak{b}^6(t)\|_{L^2(\Sigma_*)}\big).
\end{aligned}
$$

As this holds for all $\boldsymbol{v} \in \mathcal{E}$ with $\|\boldsymbol{v}\|_{\mathcal{E}} \le 1$ we deduce for almost all $t \in [0, T]$ that

$$
\|\partial_t \boldsymbol{u}_m(t)\|_{\mathcal{E}^{-1}} \le C\big(\|\mathfrak{f}\|_{\mathcal{L}} + \|\mathfrak{b}^5(t)\|_{L^2(\Sigma_*)} + \|\mathfrak{b}^6(t)\|_{L^2(\Sigma_*)}\big), \tag{4.69}
$$

and thus

$$
\|\partial_t \boldsymbol{u}_m\|^2_{L^2(0,T,\mathcal{E}^{-1})} \le C\big(\|\mathfrak{f}\|^2_{L^2(0,T;\mathcal{L})} + \|\mathfrak{b}^5\|^2_{L^2(0,T,L^2(\Sigma_*))} + \|\mathfrak{b}^6\|^2_{L^2(0,T,L^2(\Sigma_*))}\big). \tag{4.70}
$$

So in total we get the energy estimate

$$
\begin{aligned}
\max_{t\in[0,T]} \big(\|\boldsymbol{u}_m(t)\|^2_{\mathcal{E}}\big) + \|\boldsymbol{u}_m\|^2_{L^2(0,T,\mathcal{E})} + \|\boldsymbol{v}_m\|^2_{L^2(0,T,\mathcal{E})} + \|\partial_t \boldsymbol{u}_m\|^2_{L^2(0,T,\mathcal{E}^{-1})} \\
\le C\big(\|\mathfrak{f}\|^2_{L^2(0,T;\mathcal{L})} + \|\mathfrak{b}^5\|^2_{L^2(0,T,L^2(\Sigma_*))} + \|\mathfrak{b}^6\|^2_{L^2(0,T,L^2(\Sigma_*))}\big)
\end{aligned} \tag{4.71}
$$

These energy estimates imply that there are $\boldsymbol{u} \in L^2(0,T;\mathcal{E}), \partial_t\boldsymbol{u} \in L^2(0,T;\mathcal{E}^{-1})$ and $\boldsymbol{v} \in L^2(0,T;\mathcal{E}))$ together with subsequences (which we will identify with the original sequence) of u_m, u_m' and v_m such that

$$
\begin{cases}
\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} & \text{in } L^2(0,T;\mathcal{E}), \\
\partial_t \boldsymbol{u_m} \rightharpoonup \partial_t \boldsymbol{u} & \text{in } L^2(0,T;\mathcal{E}^{-1}), \\
\boldsymbol{v}_m \rightharpoonup \boldsymbol{v} & \text{in } L^2(0,T;\mathcal{E}).
\end{cases}
$$

We note that like in the standard case we get that the weak limit of $\partial_t \boldsymbol{u_m}$ indeed corresponds with the weak time derivative of $\boldsymbol{u}$ by using the definition of a weak time derivative and weak convergence. Next we want to see that $(\boldsymbol{u}, \boldsymbol{v})$ is the sought weak solution. For any fixed $N \in \mathbb{N}$ and smooth functions $\{d_k\}_{k=1,\dots,N}$ we get for any $m > N$

$$
\begin{aligned}
\int_0^T \langle \partial_t \boldsymbol{u}_m, \boldsymbol{\zeta}\rangle_{dual,\gamma} dt &= \int_0^T B_u[\boldsymbol{v}_m, \boldsymbol{\zeta}; t] + b_u(\boldsymbol{\zeta}; t) + f_u(\boldsymbol{\zeta}, t) dt, \\
\int_0^T (\boldsymbol{v}_m, \boldsymbol{\zeta})_{\mathcal{L}_\gamma} dt &= \int_0^T B_v[\boldsymbol{u}_m, \boldsymbol{\zeta}; t],
\end{aligned}
$$

for $\boldsymbol{\zeta} = \sum_{j=1}^N d_j \boldsymbol{z}_j$. Note that for fixed $\boldsymbol{\zeta}$ each term gives an element of $(L(0,T;\mathcal{E}))'$ or $\big(L(0,T,\mathcal{E}^{-1})\big)'$ and so we can - using the definition of weak convergence - pass to the limit to get

$$
\begin{aligned}
\int_0^T \langle \partial_t \boldsymbol{u}, \boldsymbol{\zeta}\rangle_{dual,\gamma} dt &= \int_0^T B_u[\boldsymbol{v}, \boldsymbol{\zeta}; t] + b_u(\boldsymbol{\zeta}; t) + f_u(\boldsymbol{\zeta}, t) dt, \\
\int_0^T (\boldsymbol{v}, \boldsymbol{\zeta})_{\mathcal{L}_\gamma} dt &= \int_0^T B_v[\boldsymbol{u}, \boldsymbol{\zeta}; t].
\end{aligned}
$$

As the considered test functions $\boldsymbol{\zeta}$ are dense in $L^2(0,T;\mathcal{E})$ this holds for all such functions implying that (4.47) holds for allmost all $t \in [0,T]$. Thus, it remains to show that $u(0) = 0$, which follows as in the proof of [24, Theorem 3, p.378]. Finally, to derive uniqueness of the weak solution we observe that for two solutions $(\boldsymbol{u}_1, \boldsymbol{v}_1), (\boldsymbol{u}_2, \boldsymbol{v}_2)$ the difference $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}})$ solves (4.47) with $\mathfrak{f} \equiv 0, \mathfrak{b}^5 \equiv \mathfrak{b}^6 \equiv 0$ and then the energy estimates (4.62) imply $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) = (0,0)$ and the proof is finished. □

For technical reason in the localization argument we will need a result on higher Sobolev regularity.

Corollary 4.13 (H^3-regularity of $\boldsymbol{u}$).
The solution $\boldsymbol{u}$ found in Proposition 4.12 is in $L^2(0,T;H^3_{TJ}(\Gamma_*))$ and we have the estimate

$$\|\boldsymbol{u}\|_{L^2(0,T;H^3_{TJ}(\Gamma_*))} \leq C\left(\sum_{i=1}^{3}\|\mathfrak{f}^i\|_{L^2(0,T;\mathcal{L})} + \sum_{i=5}^{6}\|\mathfrak{b}^i\|_{L^2(0,T;L^2(\Sigma_*))}\right). \tag{4.72}$$

Proof. For every $t \in [0,T]$, the function $\boldsymbol{u}$ is a weak solution of the elliptic problem

$$\begin{aligned} -\Delta_* u^i + C_v u^i &= v^i && \text{on } \Gamma^i_*, \quad i = 1,2,3, \\ \gamma^1 u^1 + \gamma^2 u^2 + \gamma^3 u^3 &= 0 && \text{on } \Sigma_*, \\ \partial_{\nu^1_*} u^1 - \partial_{\nu^2_*} u^2 &= 0 && \text{on } \Sigma_*, \\ \partial_{\nu^2_*} u^2 - \partial_{\nu^3_*} u^3 &= 0 && \text{on } \Sigma_*. \end{aligned}$$

As we have $\boldsymbol{v} \in W^{1,2}_{TJ}(\Gamma)$ and it was shown in [19, Lemma 3] that the Lopatinskii-Shapiro conditions for these boundary conditions are satisfied, we may apply elliptic regularity theory[5] from [4] to get $\boldsymbol{u}(t) \in W^{3,2}_{TJ}(\Gamma)$ and

$$\|\boldsymbol{u}(t)\|_{H^3_{TJ}(\Gamma_*)} \leq \|\boldsymbol{v}(t)\|_{\mathcal{E}},$$

and then with (4.51)

$$\|\boldsymbol{u}\|_{L^2(0,T;H^3(\Gamma_*))} \leq \|\boldsymbol{v}\|_{L^2(0,T;\mathcal{E})} \leq C\left(\sum_{i=1}^{3}\|\mathfrak{f}^i\|_{L^2(0,T;\mathcal{L})} + \sum_{i=5}^{6}\|\mathfrak{b}^i\|_{L^2(0,T;L^2(\Sigma_*))}\right).$$

This shows the desired estimate. □

4.5.2 Schauder Estimates for the Localized System on the Boundary

In the next step we want to derive Schauder-estimates for the localized problem. For every point away from the triple junctions this follows as in Lemma 3.7 as regularity is a local property. Thus, we will discuss the analysis only around an arbitrary point $\sigma \in \Sigma_*$.
We want to apply the theory of [38] and for this we need to rewrite the problem in local coordinates around σ. Hereby, it is sufficient to prove the result for one parametrisation which allows us to choose a special parametrisation that will make the discussion easier. To do so we choose $\delta > 0$ sufficiently small such that for $i = 1,2,3$ the projection on Σ_* is well defined on $V^i = B_\delta(\sigma) \subset \Gamma^i_*$. For $V_\Sigma := B_\delta(\sigma) \subset \Sigma_*$ we choose $R > 0$ together with $U = B_R(0) \cap \mathbb{R}^n_+$ and any parametrisation $\varphi : U \cap \{x \in \mathbb{R}^n | x_n = 0\} \to V_\Sigma$. Now we extend this φ to diffeomorphisms $\varphi^i : U \to V^i$ by the distance function. To be precise this induces diffeomorphisms

$$\varphi^i : U \mapsto V_i, (x,d) \mapsto \gamma_{-\nu^i_*}(\varphi(x), d),$$

where $(x,0) \in U \cap \{x \in \mathbb{R}^n | x_n = 0\}, (x,d) \in U$ and $\gamma_{-\nu^i_*}(\sigma, d)$ denotes the evaluation of the geodesic through a point $\sigma \in \Sigma_*$ in direction $-\nu^i_*$ at distance d . Note that for these parametrisations it holds

[5] Actually, this is applied for a localized problem in the interior like in Lemma 3.7 and near the boundary like in Subsection 4.5.2. The connection to the original problem is made similarly.

that

$$g^i_{nn} = 1, g^i_{nl} = 0 \text{ for } l \neq n, \tag{4.73}$$

and the same holds for the inverse metric tensor due to to the inverse matrix formula for matrices in block form. We now want to study problem (4.46) localized on $B_R(0)$. Using the notation

$$C := \partial U \cap \{x_n = 0\}, \quad S := \partial U \backslash C,$$

we see that for the localization of (4.46) we only have boundary conditions on C. In order to get a well-posed problem we have to do a cut-off away from S so we choose $\varepsilon < \frac{R}{2}$ together with $\eta \in C^\infty(U)$ such that $\operatorname{supp}(\eta) \subset U \cap B_{R-\varepsilon}(0), \eta \equiv 1$ on $U \cap B_\varepsilon(0)$. In the following, we will write for this cut-off of the parametrised function $\boldsymbol{u}$ again to keep notation simple. We want now to consider the problem

$$(LLP) \begin{cases} \partial_t u^i + \sum\limits_{j,k,l,m=1}^n g_i^{jk} g_i^{lm} \partial_{jklm} u^i + \mathcal{A}^i_L(u_i) = f^i & \text{on } U \times [0,T], i = 1,2,3, \\ \gamma^1 u^1 + \gamma^2 u^2 + \gamma^3 u^3 = 0 & \text{on } C \times [0,T], \\ \partial_n u^1 - \partial_n u^2 + \mathfrak{B}_2(u^1, u^2) = 0 & \text{on } C \times [0,T], \\ \partial_n u^2 - \partial_n u^3 + \mathfrak{B}_3(u^2, u^3) = 0 & \text{on } C \times [0,T], \\ \sum\limits_{i=1}^3 \sum\limits_{j,k=1}^n \gamma^i g^{jk} \partial_{jk} u^i + \mathfrak{B}_4(u^1,u^2,u^3) = 0 & \text{on } C \times [0,T], \\ \sum\limits_{j,k=1}^n \left(g_1^{jk} \partial_{njk} u^1 - g_2^{jk} \partial_{njk} u^2\right) + \mathfrak{B}_5(u^1,u^2) = \mathfrak{b}^5 & \text{on } C \times [0,T], \\ \sum\limits_{j,k=1}^n \left(g_2^{jk} \partial_{njk} u^2 - g_3^{jk} \partial_{njk} u^3\right) + \mathfrak{B}_6(u^2,u^3) = \mathfrak{b}^6 & \text{on } C \times [0,T], \\ u^i = 0 & \text{on } S \times [0,T], \quad i = 1,2,3, \\ \Delta u^i = 0 & \text{on } S \times [0,T], \quad i = 1,2,3, \\ u^i\big|_{t=0} = u_0 & \text{on } U, i = 1,2,3. \end{cases}$$

We used here that due to the choice of our coordinates the derivative in direction of ν^i_* is given by $-\partial_n$. Also, we only need the highest order terms for the following discussion, which are given by the highest orders terms in the surface Laplacian and the surface Bilaplacian. All other terms are written in the $\mathcal{A}_L$ and $\mathcal{B}_j$. The inhomogeneities $\mathfrak{b}_1$ to $\mathfrak{b}_4$ were excluded from (LLP) as they also were for the weak analysis and so we cannot use them later when proving Hölder-regularity for the weak solution. Actually, the following analysis would also admit these inhomogeneities. The boundary conditions on S are relatively arbitrary as $\boldsymbol{u}$ vanishes near S and so fulfils any linear boundary condition. Note that the boundary condition on S and C are compatible as u solves both near $\partial S \cap \partial C$ due to the cut-off procedure. As a final remark we want to mention that we will need better properties for η later but we will discuss this in the next section. Now, we want to show that (LLP) fulfils the prerequisites to apply [38, Theorem 4.9].

Remark 4.14 (Regularity of ∂U).
To be precise we will need smoothness of ∂U for the analysis now. But as we cut off the problem away from S we may assume w.l.o.g. that the problem is indeed defined on such a domain.

Firstly, we want to see that for the system above the basic requirements described on page 8 of [38] hold. Setting $s_i = 0$ and $t_i = 4$ for $i = 1,2,3$, the degree conditions for the differential operators are fulfilled. For the parabolicity condition we see that $b = 2$ and

$$\begin{aligned} \mathcal{L}_0(x,t,\zeta,p) &= \operatorname{diag}(p + |\zeta|^4_{g_1}, p + |\zeta|^4_{g_2}, p + |\zeta|^4_{g_3}), \\ L(x,t,\zeta,p) &= \prod_{i=1}^3 (p + |\zeta|^4_{g_i}), \end{aligned}$$

for any $p \in \mathbb{C}, \zeta \in \mathbb{R}^n$, where $|\cdot|_{g_i}$ denotes the norm induced by the inverse metric tensor g^*_i. Thus, the roots of L with respect to p are precisely $p_i = -|\zeta|^4_{g_i}$, for which one has

$$p_i \leq -c^4|\zeta|^{2b},$$

for a suitable $c > 0$ fulfilling $c|\zeta| \leq |\zeta|_{g_i}$ for $i = 1, 2, 3$, which exists due to the equivalency of $|\cdot|$ and $|\cdot|_{g_i}$. Note that as the inverse metric tensor is bounded, c can be chosen independently of x and thus the equations is even uniformly parabolic with $b = 2$.
For the boundary conditions we get the following numbers.

β_{qj}	1	2	3	σ_q
1	0	0	0	-4
2	1	1	0	-3
3	0	1	1	-3
4	2	2	2	-2
5	3	3	0	-1
6	0	3	3	-1

This means that the number l used in Theorem 4.9 can be an arbitrary number larger than

$$\max\{0, \sigma_1, \cdots, \sigma_6\} = 0,$$

which works for us as we will need l to be the Hölder-continuity of the space derivatives, so an $\alpha \in (0, 1)$.
Hence, we see that (LLP) is indeed of the form covered by the theory of [51] and so we now have to check the complementary and compatibility conditions.
For the original version of the complementary conditions in [38] one has to check that a certain matrix has full rank. In our case, this would lead to a matrix with 12 rows and columns. A more elegant way is to check a version of the Lopatinski-Shapiro conditions where we have to study non-trivial solutions of an ODE-system. In [21, I.2] the authors proved equivalence of the both conditions and in [39] a very concrete formulation is given. The formulation is even easier in our situation as we have an already linearised system.
To state the corresponding ODE system for a function $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3) \in C_0(\mathbb{R}_+, \mathbb{C}^3)$, where the subscript zero denotes functions that vanish at infinity, $\zeta \in \mathbb{R}^n \cap \{x_n = 0\}$ and $\lambda \in \{z \in \mathbb{C}|\mathrm{Re}(z) \geq 0\}$, we first have to determine the principal parts $\mathcal{A}_\sharp$ of the equation itself and the principal parts $\mathcal{B}_{j\sharp}$ of the boundary conditions. For the first one, we note that $a_{jklm} = \mathrm{diag}\left(\left(g_i^{jk} g_i^{lm}\right)_{i=1}^3\right)$ is the coefficient function of $\partial_{jklm}\boldsymbol{u}$. But due to (4.73) we have $a_{jklm} \neq 0$ if and only if either $j = k = l = m = n$ or none of them equals n. So, this leads to the corresponding differential equation

$$\lambda\phi_i(y) + |\zeta'|^4_{g_i}\phi_i(y) + \phi_i''''(y) = 0, \quad y \geq 0, \quad i = 1, 2, 3.$$

For the boundary conditions the linearisations formed in [39] are just the boundary conditions itself as we already have linear equations. Therefore, we have to derive the induced conditions for y. The zero-order condition $(LLP)_2$ will just give the same conditions for y. For the first order conditions $(LLP)_3$ and $(LLP)_4$ we see that the coefficient functions are zero except for the one of $\partial_n\boldsymbol{u}$. So we get the condition

$$\phi_1' = \phi_2' = \phi_3', \quad y = 0.$$

For $(LLP)_5$ we again see that the coefficient functions are given by the g^{jk} and so again they will only be non-zero for either $j = k = n$ or both j and k unequal to n leading to

$$\sum_{i=1}^{3} \gamma^i\left(|\zeta'|^2_{g_i}\phi_i - \phi_i''\right) = 0, \quad y = 0.$$

Finally, for $(LLP)_6$ and $(LLP)_7$ we see that coefficient functions unequal to zero have to have one derivative in direction of e_n and the other two, with the same arguments as before, have to be both either equal to ∂_n or unequal to it giving us

$$\begin{aligned}
\phi_1''' - \phi_2''' - |\zeta'|_{g_1}^2 \phi_1' + |\zeta'|_{g_2}^2 \phi_2' &= 0, \quad y = 0, \\
\phi_2''' - \phi_3''' - |\zeta'|_{g_2}^2 \phi_2' + |\zeta'|_{g_3}^2 \phi_3' &= 0, \quad y = 0.
\end{aligned}$$

To sum up, the Lopatinski-Shapiro conditions read as follows. We need to verify that for all

$$\zeta' \in \mathbb{R}^{n-1}, \lambda \in \{z \in \mathbb{C} | \mathrm{Re}(z) \geq 0\} \text{ with } (\lambda, \zeta') \neq (0,0)$$

the only solution $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ in $C_0(\mathbb{R}, \mathbb{C}^3)$ of the ODE-system

$$\begin{aligned}
\lambda \phi_i + |\zeta'|_{g_i}^4 \phi_i + \phi_i'''' &= 0, && y > 0, i = 1,2,3, \\
\sum_i^3 \gamma^i \phi_i &= 0 && y = 0, \\
\phi_1' = \phi_2' &= \phi_3' && y = 0, \\
\sum_{i=1}^3 \gamma^i \left(|\zeta'|_{g_i}^2 \phi_i - \phi_i'' \right) &= 0 && y = 0, \\
\phi_1''' - \phi_2''' - |\zeta'|_{g_1}^2 \phi_1' + |\zeta'|_{g_2}^2 \phi_2' &= 0 && y = 0, \\
\phi_2''' - \phi_3''' - |\zeta'|_{g_2}^2 \phi_2' + |\zeta'|_{g_3}^2 \phi_3' &= 0 && y = 0,
\end{aligned} \tag{4.74}$$

is $\boldsymbol{\phi} \equiv 0$.

To prove this we use a straightforward energy method. We first note that due to the structure of the differential equation, all solutions are linear combinations of functions of the form $\exp(\sqrt[4]{-\lambda - |\zeta'|^4} y)$. If such functions converge to 0 for $y \to \infty$ then all their derivatives converge, too. Now, testing the first equation in (4.74) with $\gamma^i \overline{\phi_i}$ and summing over $i = 1,2,3$ gets us

$$\sum_{i=1}^3 \gamma^i (\lambda + |\zeta'|_{g_i}^4) \int_0^\infty |\phi_i|^2 dy + \sum_{i=1}^3 \gamma^i \int_0^\infty \phi_i'''' \overline{\phi_i} dy = 0. \tag{4.75}$$

The first sum in (4.75) has already a good structure, so we focus on the second term. Integration by parts twice yields

$$\begin{aligned}
\sum_{i=1}^3 \gamma^i \int_0^\infty \phi_i'''' \overline{\phi_i} dy &= \sum_{i=1}^3 \gamma^i \phi_i'''(0) \overline{\phi_i}(0) - \gamma^i \int_0^\infty \phi_i''' \overline{\phi_i'} dy \\
&= \underbrace{\sum_{i=1}^3 \gamma^i \phi_i'''(0) \overline{\phi_i}(0)}_{=(I)} + \underbrace{\sum_{i=1}^3 -\gamma^i \phi_i''(0) \overline{\phi_i'}(0)}_{=(II)} + \underbrace{\sum_{i=1}^3 \gamma^i \int_0^\infty |\phi_i''|^2 dy}_{=(III)}.
\end{aligned} \tag{4.76}$$

Here, we used that the solution and all their derivative vanish as $y \to \infty$. We now want to study these terms separately. For brevity we will omit the variable 0. The sum (III) is already non-negative, so we have nothing to do. For the term (I) we use the last two lines in (4.74) to express ϕ_1''' and ϕ_3''' and we derive

$$\begin{aligned}
(I) &= \gamma^1 \left(\phi_2''' - |\zeta'|_{g_2}^2 \phi_2' + |\zeta'|_{g_1}^2 \phi_1' \right) \overline{\phi}_1 + \gamma^2 \phi_2''' \overline{\phi}_2 + \gamma^3 \left(\phi_2''' - |\zeta'|_{g_2}^2 \phi_2' + |\zeta'|_{g_3}^2 \phi_3' \right) \overline{\phi_3} \\
&= \underbrace{\sum_{i=1}^3 \phi_2''' \gamma^i \overline{\phi}_i}_{=0} + \gamma^1 |\zeta'|_{g_1}^2 \overline{\phi}_i \phi_1' + \gamma^3 |\zeta'|_{g_3}^2 \overline{\phi}_3 \phi_3' + \phi_2' \underbrace{\left(-\gamma^1 |\zeta'|_{g_2}^2 \overline{\phi}_1 - \gamma^3 |\zeta'|_{g_2}^2 \overline{\phi}_3 \right)}_{=\gamma^2 |\zeta'|_{g_2}^2 \overline{\phi}_2} \\
&= \phi_1' \sum_{i=1}^3 |\zeta'|_{g_i}^2 \gamma^i \overline{\phi}_i.
\end{aligned} \tag{4.77}$$

Here, we used in the last two steps the second and third line in (4.74). We now have a look at the term (II). Due to the third and the fourth line in (4.74) we see

$$(II) = -\overline{\phi_1'}\sum_{i=1}^{3}\gamma^i\phi_i'' = -\overline{\phi_1'}\sum_{i=1}^{3}\gamma^i|\zeta'|_{g_i}^2\phi_i. \tag{4.78}$$

Putting (4.77) and (4.78) in (4.76) implies that

$$\sum_{i=1}^{3}\gamma^i\int_0^\infty \phi_i''''\overline{\phi_i}dy = \sum_{i=1}^{3}\gamma^i\int_0^\infty |\phi_i''|^2dy + 2\mathfrak{Im}\left(\phi_1'\sum_{i=1}^{3}|\zeta'|_{g_i}^2\gamma^i\overline{\phi}_i\right). \tag{4.79}$$

Combining this result with (4.75) we derive

$$\sum_{i=1}^{3}\gamma^i(\lambda+|\zeta'|_{g_i}^4)\int_0^\infty |\phi_i|^2dy + \sum_{i=1}^{3}\gamma^i\int_0^\infty |\phi_i''|^2dy + 2\mathfrak{Im}\left(\phi_1'\sum_{i=1}^{3}|\zeta'|_{g_i}^2\gamma^i\overline{\phi}_i\right) = 0. \tag{4.80}$$

As λ has a non-negative real part all terms of the left-hand-side have non-negative real part. For $\zeta' \neq 0$ we get therefore

$$\int_0^\infty |\phi_i|^2dy = 0, \quad i = 1,2,3, \tag{4.81}$$

which already implies $\boldsymbol{\phi} \equiv 0$. For $\zeta' = 0$ the equation above reduces to

$$\sum_{i=1}^{3}\gamma^i\lambda\int_0^\infty |\phi_i|^2dy + \sum_{i=1}^{3}\int_0^\infty |\phi_i''|^2dy = 0, \tag{4.82}$$

as $|\cdot|_{g_i}$ is also a norm on $\mathbb{R}^n$. If $\mathfrak{Re}(\lambda) \neq 0$ we can argue as before. Otherwise it has to hold that $\mathfrak{Im}(\lambda) \neq 0$ and as the second sum is real this again implies (4.81) and we hence showed $\boldsymbol{\phi} \equiv 0$. Consequently, the Lopatinskii-Shapiro conditions are fulfilled on ∂U.
It now remains to consider the compatibility conditions according to [38, p98]. As we choose $l = \alpha < 1$, we only need compatibility conditions of order 0. Due to the values of σ_q only for the first boundary equation i_q can take the values 0 and 1. For all other i_q we just get the trivial compatibility condition that the boundary condition has to be fulfilled for $\boldsymbol{u} = 0$. The only non-trivial compatibility condition is therefore

$$0 = \sum_{i=1}^{3}\gamma^i\partial_t u^i = \sum_{i=1}^{3}\gamma^i\mathfrak{f}^i,$$

using the first line in (LLP). Finally, we can use [38, Theorem 4.9] to get the following existence result for (LLP).

Proposition 4.15 (Schauder Estimates for (LLP))**.** *The system (LLP) has a unique solution* $\boldsymbol{u} \in C^{4+\alpha,1+\frac{\alpha}{4}}(U\times[0,T])^3$ *if and only if the compatibility conditions*

$$\mathfrak{b}^5 = \mathfrak{b}^6 = 0 \qquad \text{on } C\times\{0\}, \tag{4.83}$$

$$\sum_{i=1}^{3}\gamma^i\mathfrak{f}^i = 0 \qquad \text{on } C\times\{0\}. \tag{4.84}$$

are fulfilled.

4.5.3 Schauder Estimates for the Principal Part of the Linearised Problem

We now want to show local Hölder regularity and Schauder estimates for the weak solution constructed in Proposition 4.12 by connecting the weak problem and the localized problem. This we cannot

conclude from the results in the last section. So far, we do not know that the solution there is the same as the cut-off of the weak solution $\boldsymbol{u}$ constructed in Section 4.5.1 for a right-hand-side $\mathfrak{f} \in C_{TJ}^{\alpha,\frac{\alpha}{4}}(\Gamma_*)$ and $\mathfrak{b}_5, \mathfrak{b}_6 \in C^{1+\alpha,(1+\alpha)/4}(\Sigma^*)$. Our strategy for this is to multiply $\boldsymbol{u}$ with a cut-off function η, that is 1 locally in time and space around $(\sigma, t) \in \Sigma_* \times [0,T]$ and has compact support in $\mathrm{im}(\varphi^i)$. For this function $\eta\boldsymbol{u}$ we can derive a new PDE system that has ηu as unique solution. The arising inhomogeneities $\widetilde{\mathfrak{f}}$ and $\widetilde{\mathfrak{b}}^5$ and $\widetilde{\mathfrak{b}}^6$ are only in $L^2(0,T; L^2_{TJ}(\Gamma))$ resp. $L^2(0,T;L^2(\Sigma_*))$ and so to apply the results from Section 4.5.2 we have to consider a suitable approximating problem. Finally, we have to show that the solutions of this approximation converge on a subset to $\eta\boldsymbol{u}$ and that the limit has $C^{4+\alpha,1+\frac{\alpha}{4}}$-regularity and thus $\boldsymbol{u}$ has.
Consider now a fixed point $(\sigma,t) \in \Sigma_* \times [0,T]$. The choice of η has to be a bit more specific than in [19] as we have to guarantee that our cut-off procedure will not produce inhomogeneities in the linearised angle conditions or the linearisation of $\mathcal{B}_4$. We choose neighbourhoods Q_4^i of $\sigma \in \Gamma_*^i$ such that the projections $\mathrm{pr}^i_{\Sigma_*}$ are well defined and the intersections $Q_4^i \cap \Sigma_*$ equal for all $i=1,2,3$ a common set $Q_4^{\Sigma_*} \subset \Sigma_*$. We choose now a cut-off function $\eta_{\Sigma_*} \in C^\infty(Q_4^{\Sigma_*})$, such that η_{Σ_*} has compact support in $Q_4^{\Sigma_*}$ and $\eta_{\Sigma_*} \equiv 1$ on a neighbourhood of σ in $Q_4^{\Sigma_*}$. Now, we extend η_{Σ_*} via a cut-off of the distance function on Q_4^i. Precisely, we choose a cut-off function $\eta_d : [0,1] \to [0,1]$ with $\mathrm{supp}(\eta_d) \subset [0,\varepsilon)$ for some $\varepsilon < 1$, $\eta_d \equiv 1$ on a closed interval containing 0 and such that for all $i = 1,2,3$ the function

$$\begin{aligned} \eta^i := \eta_d \eta_\Sigma : Q_4^i &\to [0,1], \\ x &\mapsto \eta_d(\mathrm{dist}_{\Sigma_*}(x))\eta_\Sigma(\mathrm{pr}^i_{\Sigma_*}(x)), \end{aligned} \tag{4.85}$$

has compact support Q_3^i in Q_4^i. Note that for any point $x \in Q^\Sigma$ we have now

$$\partial_{\nu_*^i}\eta^i(x) = 0 \tag{4.86}$$

for $i = 1,2,3$. This is exactly what we need for our analysis. We want to note that one could multiply η^i with a cut-off funcion in time to attempt to prove parabolic regularization away from $t=0$ but here we do not need this.
We will need some additional notation now and set

$$C_l^i := \partial Q_l^i \cap \Sigma_*, \quad S_l^i := \partial Q_l^i \backslash C_l^i, \quad l = 1,2,3,4, i = 1,2,3.$$

Hereby, the sets Q_2^i and Q_1^i will be constructed later and as $Q_l^i \cap \Sigma_* = Q_l^{\Sigma_*}$ for $i=1,2,3$ we have $C_l^i = C_l$ for $i = 1,2,3$. Following our plan, we now want to derive a PDE for $\widetilde{\boldsymbol{u}} = \boldsymbol{\eta u}$. As $\boldsymbol{u}$ is a weak solution of (4.46) we calculate formally

$$\begin{aligned} \partial_t \widetilde{u}^i &= \eta^i \partial_t u^i + (\partial_t \eta^i) u^i, && \text{on } Q_4^i \times [0,T], i=1,2,3, \\ \Delta_* \widetilde{u}^i &= \eta^i \Delta_* u^i + 2\langle \nabla_* \eta^i, \nabla_* u^i \rangle + (\Delta_* \eta^i) u^i, && \text{on } Q_4^i \times [0,T], i=1,2,3, \\ -\Delta_*\Delta_* \widetilde{u}^i &= -\eta^i \Delta_*\Delta_* u^i - 4\Delta_* \langle \nabla_* \eta^i, \nabla_* u^i \rangle && \\ &\quad + 2\Delta_* u^i \Delta_* \eta^i - u^i \Delta_*\Delta_* \eta^i, && \text{on } Q_4^i \times [0,T], i=1,2,3, \\ \partial_{\nu_*^i} \widetilde{u}^i &= \eta^i \partial_{\nu_*^i} u^i + u^i \partial_{\nu_*^i} \eta^i = \eta^i \partial_{\nu_*^i} u^i, && \text{on } Q_4^{\Sigma_*} \times [0,T], i=1,2,3. \end{aligned} \tag{4.87}$$

From this we see directly that $\widetilde{\boldsymbol{u}}$ solves formally

$$\partial_t \widetilde{u}^i = -\Delta_{\Gamma_*^i}\Delta_{\Gamma_*^i}\widetilde{u}^i + C_v \Delta_{\Gamma_*^i}\widetilde{u}^i + C_u \Delta_{\Gamma_*^i}\widetilde{u}^i - C_v C_u \widetilde{u}^i + \widetilde{\mathfrak{f}}^i,$$

with the new inhomogeneity

$$\begin{aligned} \widetilde{\mathfrak{f}}^i = \eta^i \mathfrak{f}^i &+ (\partial_t \eta^i) u^i + 4\Delta_*(\langle \nabla_* \eta^i, \nabla_* u^i \rangle) - 2\Delta_* u^i \Delta_* \eta^i \\ &+ u \Delta_*\Delta_* \eta^i - 2(C_u + C_v)\langle \nabla_* \eta^i, \nabla_* u^i \rangle - (C_u + C_v)\Delta_* \eta^i \cdot u^i. \end{aligned}$$

Observe that due to the $L^2(0,T;H^3)$-regularity of u^i from Corollary 4.13, $\widetilde{\mathfrak{f}}^i$ is in $L^2(0,T;L^2)$. To get a formulation for which we can get unique existence of a weak solution we need to do a split again.

Therefore, we write the equation as

$$\begin{aligned}\partial_t \widetilde{u}^i &= \Delta_{\Gamma^i_*} \widetilde{v}^i - C_u \widetilde{v}^i + \widetilde{\mathfrak{f}}^i, \\ \widetilde{v}^i &= -\Delta_{\Gamma^i_*} \widetilde{u}^i + C_v \widetilde{u}^i.\end{aligned}$$

From (4.87) we deduce for the boundary conditions for $\widetilde{u}^i$ that

$$\partial_{\nu^i_*} \widetilde{u}^i - \partial_{\nu^j_*} \widetilde{u}^j = \eta_{\Sigma_*}(\partial_{\nu^i_*} u^i - \partial_{\nu^j_*} u^j). \tag{4.88}$$

Hence, we get the boundary conditions

$$\begin{aligned}\gamma^1 \widetilde{u}^1 + \gamma^2 \widetilde{u}^2 + \gamma^3 \widetilde{u}^3 &= 0 && \text{on } C_4 \times [0,T], \\ \partial_{\nu^1_*} \widetilde{u}^1 - \partial_{\nu^2_*} \widetilde{u}^2 &= 0 && \text{on } C_4 \times [0,T], \\ \partial_{\nu^2_*} \widetilde{u}^2 - \partial_{\nu^3_*} \widetilde{u}^3 &= 0 && \text{on } C_4 \times [0,T].\end{aligned}$$

Now we have to determine the boundary conditions fulfilled by $\widetilde{v}^i$. First we note that on $C_4 \times [0,T]$ it holds that

$$\begin{aligned}\sum_{i=1}^3 \gamma^i \widetilde{v}^i &= \sum_{i=1}^3 \gamma^i (-\Delta_* \widetilde{u}^i + C_v \widetilde{u}^i) \\ &= \sum_{i=1}^3 \gamma^i \left(-\eta^i \Delta_* u^i + \eta^i C_v u^i - 2\langle \nabla_* \eta^i, \nabla_* u^i \rangle - (\Delta^i_{\Gamma^\eta_*}) u^i \right) \\ &= \eta^1 \sum_{i=1}^3 \gamma^i v^i - \Delta_* \eta^1 \sum_{i=1}^3 \gamma^i u^i - 2 \left\langle \nabla_* \eta^1, \sum_{i=1}^3 \gamma^i \nabla_* u^i \right\rangle = 0.\end{aligned}$$

Here, we used in the third identity that the η^i, $\nabla_* \eta^i$ and $\Delta_* \eta^i$ equal on C_4. For the last identity observe that the first two summands vanish due to the boundary conditions for $\boldsymbol{u}$ and $\boldsymbol{v}$. For the last summand we note that by choosing the same local parametrisation as in the section before we can write

$$\nabla_* u^i = \nabla_{\Sigma_*} u^i + (\partial_{\nu^i_*} u^i) \nu^i_*.$$

Furthermore, $\sum_{i=1}^3 \gamma^i u^i = 0$ implies by differentiating also $\sum_{i=1}^3 \gamma^i \nabla_{C_4} u^i = 0$. Thus, using the boundary conditions for $\boldsymbol{u}$ and the angle conditions for the reference surface we calculate

$$\sum_{i=1}^3 \gamma^i \nabla_* u^i = \sum_{i=1}^3 \gamma^i \nabla_{C_4} u^i + \sum_{i=1}^3 \gamma^i (\partial_{\nu^i_*} u^i) \nu^i_* = \partial_{\nu^1_*} u^1 \sum_{i=1}^3 \gamma^i \nu^i_* = 0.$$

In total, this proves

$$\sum_{i=1}^3 \gamma^i \widetilde{v}^i = 0 \qquad \text{on } C_4 \times [0,T].$$

It remains to study the Neumann-type boundary conditions for $\widetilde{v}^i$. For this observe that

$$\widetilde{v}^i = \eta^i v^i - (\Delta_* \eta^i) u^i - 2\langle \nabla_* \eta^i, \nabla_* u^i \rangle,$$

and so

$$\begin{aligned}\partial_{\nu^i_*} \widetilde{v}^i = {}& \underbrace{(\partial_{\nu^i_*} \eta^i)}_{=0} v^i + \eta^i (\partial_{\nu^i_*} v^i) - (\partial_{\nu^i_*} \Delta_* \eta^i) u^i + (\Delta_* \eta^i) \partial_{\nu^i_*} u^i \\ & - 2\langle \partial_{\nu^i_*} \nabla_* \eta^i, \nabla_* u^i \rangle - 2\langle \nabla_* \eta^i, \partial_{\nu^i_*} \nabla_* u^i \rangle.\end{aligned}$$

There, we get on C_4 the boundary conditions

$$\begin{aligned} \partial_{\nu_*^1}\widetilde{v}^1 - \partial_{\nu_*^2}\widetilde{v}^2 &= \widetilde{\mathfrak{b}}^5, && \text{on } C_4\times[0,T],\\ \partial_{\nu_*^2}\widetilde{v}^2 - \partial_{\nu_*^3}\widetilde{v}^3 &= \widetilde{\mathfrak{b}}^6, && \text{on } C_4\times[0,T], \end{aligned}$$

with

$$\begin{aligned} \widetilde{\mathfrak{b}}^5 &= \eta\mathfrak{b}^5 - (\partial_{\nu_*^1}\Delta_*\eta^1)u^1 - 2\langle\partial_{\nu_*^1}\nabla_*\eta^1,\nabla_*u^1\rangle - 2\langle\nabla_*\eta^1,\partial_{\nu_*^1}\nabla_{\Gamma_*^1}u^1\rangle\\ &\quad + (\partial_{\nu_*^2}\Delta_*\eta^2)u^2 + 2\langle\partial_{\nu_*^2}\nabla_*\eta^2,\nabla_*u^2\rangle + 2\langle\nabla_*\eta^2,\partial_{\nu_*^2}\nabla_*u^2\rangle,\\ \widetilde{\mathfrak{b}}^6 &= \eta\mathfrak{b}^6 - (\partial_{\nu_*^2}\Delta_*\eta^2)u^2 - 2\langle\partial_{\nu_*^2}\nabla_*\eta^2,\nabla_*u^2\rangle - 2\langle\nabla_*\eta^2,\partial_{\nu_*^2}\nabla_*u^2\rangle\\ &\quad + (\partial_{\nu_*^3}\Delta_{\Gamma_*^3}\eta^3)u^3 + 2\langle\partial_{\nu_*^3}\nabla_*\eta^3,\nabla_*u^3\rangle + 2\langle\nabla_*\eta^3,\partial_{\nu_*^3}\nabla_*u^3\rangle. \end{aligned}$$

Here, we used that

$$(\Delta_*\eta)\,(\partial_{\nu_*^1}u^1 - \partial_{\nu_*^2}u^2) = (\Delta_*\eta)\,(\partial_{\nu_*^2}u^2 - \partial_{\nu_*^3}u^3) = 0 \qquad \text{on } C_4.$$

Therefore, we get the following problem for $\widetilde{\boldsymbol{u}}$

$$\begin{aligned} \partial_t\widetilde{u}^i - \Delta_*\widetilde{v}^i + C_u\widetilde{v}^i &= \widetilde{\mathfrak{f}}^i, && \text{on } Q_4^i\times[0,T], \quad i=1,2,3,\\ \widetilde{v}^i + \Delta_*\widetilde{u}^i - C_v\widetilde{u}^i &= 0 && \text{on } Q_4^i\times[0,T], \quad i=1,2,3,\\ \gamma^1\widetilde{u}^1 + \gamma^2\widetilde{u}^2 + \gamma^3\widetilde{u}^3 &= 0 && \text{on } C_4\times[0,T],\\ \partial_{\nu_*^1}\widetilde{u}^1 - \partial_{\nu_*^2}\widetilde{u}^2 &= 0 && \text{on } C_4\times[0,T],\\ \partial_{\nu_*^2}\widetilde{u}^2 - \partial_{\nu_*^3}\widetilde{u}^3 &= 0 && \text{on } C_4\times[0,T],\\ \gamma^1\widetilde{v}^1 + \gamma^2\widetilde{v}^2 + \gamma^3\widetilde{v}^3 &= 0 && \text{on } C_4\times[0,T],\\ \partial_{\nu_*^1}\widetilde{v}^1 - \partial_{\nu_*^2}\widetilde{v}^2 &= \widetilde{\mathfrak{b}}^5 && \text{on } C_4\times[0,T],\\ \partial_{\nu_*^2}\widetilde{v}^2 - \partial_{\nu_*^3}\widetilde{v}^3 &= \widetilde{\mathfrak{b}}^6 && \text{on } C_4\times[0,T],\\ \widetilde{u}^i &= 0 && \text{on } S_4\times[0,T], \quad i=1,2,3,\\ \widetilde{v}^i &= 0 && \text{on } S_4\times[0,T], \quad i=1,2,3,\\ \widetilde{u}^i &= 0 && \text{on } Q_4^i\times\{0\}, \quad i=1,2,3, \end{aligned} \tag{4.89}$$

with the inhomogeneities $\widetilde{\mathfrak{f}}^i, \widetilde{\mathfrak{b}}^5$ and $\widetilde{\mathfrak{b}}^6$ chosen as above. Note hereby that due to the chosen support of $\boldsymbol{\eta}$ we have

$$\begin{aligned} \widetilde{\mathfrak{f}}^i\big|_{Q_3^i\times[0,T]} &= \mathfrak{f}^i \in C^{\alpha,\frac{\alpha}{4}}(Q_3^i\times[0,T]), \quad i=1,2,3,\\ \widetilde{\mathfrak{b}}^i\big|_{C_3^i\times[0,T]} &= \mathfrak{b}^i \in C^{1+\alpha,(1+\alpha)/4}(C_3\times[0,T]), \quad i=5,6, \end{aligned} \tag{4.90}$$

and furthermore

$$\widetilde{\mathfrak{f}}^i \in L^2(Q_4\times[0,T]), \widetilde{\mathfrak{b}}^5 \in L^2(C_4\times[0,T]), \widetilde{\mathfrak{b}}^6 \in L^2(C_4\times[0,T]), \tag{4.91}$$

and additionally we have the estimates

$$\begin{aligned} \|\widetilde{\mathfrak{f}}^i\|_{L^2(Q_4^i\times[0,T])} &\le C\left(\|\mathfrak{f}^i\|_{L^2(Q_4^i\times[0,T])} + \|u^i\|_{L^2(0,T;H^3(Q_4))}\right),\\ \|\widetilde{\mathfrak{b}}^5\|_{L^2(C_4\times[0,T])} &\le C\left(\|\mathfrak{b}^5\|_{L^2(C_4\times[0,T])} + \sum_{i=1}^3\|u^i\|_{L^2(0,T;H^3(Q_4^i))}\right),\\ \|\widetilde{\mathfrak{b}}^6\|_{L^2(C_4\times[0,T])} &\le C\left(\|\mathfrak{b}^5\|_{L^2(C_4\times[0,T])} + \sum_{i=1}^3\|u^i\|_{L^2(0,T;H^3(Q_4^i))}\right), \end{aligned} \tag{4.92}$$

as all terms in $\widetilde{\mathfrak{f}}^i$ except for $\mathfrak{f}^i$ are products of derivatives of η^i and derivatives of u^i of order not larger than three and all terms in $\widetilde{\mathfrak{b}}^i$ except for $\mathfrak{b}^i$ are products of derivatives of η and derivatives of u^i of order not larger than 2 and the latter are due to compactness of the trace operator bounded by the H^3-norm of $\boldsymbol{u}$.
We observe that $\widetilde{\mathfrak{b}}^1 \equiv \widetilde{\mathfrak{b}}^2 \equiv \widetilde{\mathfrak{b}}^3 \equiv \widetilde{\mathfrak{b}}^4 \equiv 0$ and so we can derive the same weak existence theory as in Section 4.5.1. The only difference is that we include the boundary conditions on S_4 directly in the solution space and thus we do not want to repeat the procedure. Also, by construction

$$\widetilde{\boldsymbol{u}} = \boldsymbol{\eta}\boldsymbol{u}, \quad \widetilde{\boldsymbol{v}} = \boldsymbol{\eta}\boldsymbol{v} - (\Delta_* \boldsymbol{\eta})\boldsymbol{u} - 2\langle \nabla_* \boldsymbol{\eta}, \nabla_* \boldsymbol{u} \rangle,$$

is a weak and therefore the unique solution of this system. So if we can show appropriate Schauder estimates on an open subset of $(Q_4^1 \times Q_4^2 \times Q_4^3) \times [0,T]$ we are done.
For this we now want to apply the result of the section before on the parametrisation of (4.89) on a common domain. If $\widetilde{\mathfrak{f}}^i, \widetilde{\mathfrak{b}}^5$ and $\widetilde{\mathfrak{b}}^6$ were in $C^{\alpha,\frac{\alpha}{4}}$ resp. $C^{1+\alpha,(1+\alpha)/4}$ we could directly apply Proposition 4.15. But as these quantities have only L^2-regularity we have to approximate the problem. Precisely, we take sequences $\widetilde{\mathfrak{f}}_n^i, \widetilde{\mathfrak{b}}_n^5, \widetilde{\mathfrak{b}}_n^6$ fulfilling

$$\|\widetilde{\mathfrak{f}}_n^i - \widetilde{\mathfrak{f}}^i\|_{L^2(Q_4^i \times [0,T])} \to 0, \quad \|\widetilde{\mathfrak{b}}_n^5 - \widetilde{\mathfrak{b}}^6\|_{L^2(C_4 \times [0,T])} \to 0, \quad \|\widetilde{\mathfrak{b}}_n^6 - \widetilde{\mathfrak{b}}^6\|_{L^2(C_4 \times [0,T])} \to 0, \tag{4.93}$$

and additionally we need on subset $Q_2^i \subset Q_3^i$ as $n \to \infty$ that

$$\begin{aligned}
\|\widetilde{\mathfrak{f}}_n^i\|_{C^{\alpha,\frac{\alpha}{4}}(Q_2^i \times [0,T])} &\le \|\widetilde{\mathfrak{f}}^i\|_{C^{\alpha,\frac{\alpha}{4}}(Q_3^i \times [0,T])} = \|\mathfrak{f}^i\|_{C^{\alpha,\frac{\alpha}{4}}(Q_3^i \times [0,T])}, \\
\|\widetilde{\mathfrak{b}}_n^5\|_{C^{1+\alpha,(1+\alpha)/4}(C_2 \times [0,T])} &\le \|\widetilde{\mathfrak{b}}^5\|_{C^{1+\alpha,(1+\alpha)/4}(C_3 \times [0,T])} = \|\mathfrak{b}^5\|_{C^{1+\alpha,(1+\alpha)/4}(C_3 \times [0,T])}, \\
\|\widetilde{\mathfrak{b}}_n^6\|_{C^{1+\alpha,(1+\alpha)/4}(C_2 \times [0,T])} &\le \|\widetilde{\mathfrak{b}}^6\|_{C^{1+\alpha,(1+\alpha)/4}(C_3 \times [0,T])} = \|\mathfrak{b}^6\|_{C^{1+\alpha,(1+\alpha)/4}(C_3 \times [0,T])}.
\end{aligned} \tag{4.94}$$

One possibility to guarantee the condition $(4.94)_1$ is to choose one $C^{\alpha,\alpha/2}$-approximation of $\widetilde{\mathfrak{f}}_n^i$ on $Q_4 \backslash Q_3$ and take $\mathfrak{f}^i$, which equals $\widetilde{\mathfrak{f}}^i$ on Q_3, as $C^{\alpha,\frac{\alpha}{4}}$-approximation on Q_3. We then get a suitable approximation on Q_4 by connecting both approximations via partitions of unity. The same procedure can be done for $\widetilde{\mathfrak{b}}^5$ and $\widetilde{\mathfrak{b}}^6$.
We now call problem (4.89) with the approximating inhomogeneities $\widetilde{\mathfrak{f}}_n^i, \widetilde{\mathfrak{b}}_n^5$ and $\widetilde{\mathfrak{b}}_n^6$ problem $(4.89)_n$. The parametrisation of this problem has a unique solution $\widetilde{\boldsymbol{u}}_n \in C_{TJ}^{4+\alpha,1+\frac{\alpha}{4}}(Q_4 \times [0,T])$ due to Proposition 4.15. On the other hand, $(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{v}}_n)$ with $\widetilde{v}_n^i = -\Delta_{\Gamma_*^i} \widetilde{u}_n^i + C_v \widetilde{u}_n^i$ is also a weak solution of $(4.89)_n$ and so using the estimate (4.72) we get

$$\begin{aligned}
\sum_{i=1}^{3} \|\widetilde{u}_n^i\|_{L^2(0,T;H^3(Q_4^i))} &\le C \left(\sum_{i=1}^{3} \|\widetilde{\mathfrak{f}}_n^i\|_{L^2(0,T;L^2(Q_4^i))} + \sum_{i=5}^{6} \|\widetilde{\mathfrak{b}}_n^i\|_{L^2(0,T;L^2(C_4))} \right) \\
&\le C \left(\sum_{i=1}^{3} \|\widetilde{\mathfrak{f}}^i\|_{L^2(Q_4^i \times [0,T])} + \sum_{i=5}^{6} \|\widetilde{\mathfrak{b}}^i\|_{L^2(C_4 \times [0,T])} \right) \\
&\le C \left(\sum_{i=1}^{3} \|\mathfrak{f}^i\|_{L^2(Q_4^i \times [0,T])} + \sum_{i=5}^{6} \|\mathfrak{b}^i\|_{L^2(C_4 \times [0,T])} + \sum_{i=1}^{3} \|u^i\|_{L^2(0,T;H^3(Q_4^i))} \right) \\
&\le C \left(\sum_{i=1}^{3} \|\mathfrak{f}^i\|_{L^2(Q_4 \times [0,T])} + \sum_{i=5}^{6} \|\mathfrak{b}^i\|_{L^2(C_4 \times [0,T])} \right).
\end{aligned}$$

In the third inequality we used (4.92) and (4.72) in the last inequality. This yields now that both $\boldsymbol{u}_n$ and $\boldsymbol{v}_n$ are bounded in $L^2(0,T;H_{TJ}^1(Q_4))$ and so we can find subsequences $(\widetilde{\boldsymbol{u}}_{n_l})_{n_l}$ and $(\widetilde{\boldsymbol{v}}_{n_l})_{n_l}$ converging weakly to $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}} \in L^2(0,T;H_{TJ}^1(Q_4))$, which forms a weak solution of (4.89). Due to uniqueness of the weak solution it follows

$$\bar{\boldsymbol{u}} = \widetilde{\boldsymbol{u}} \qquad \text{in } Q_4 \times [0,T].$$

Now we need bounds for the Hölder-norms to get weak convergence in that space, too. For this, we use the local estimates from [38, Theorem 4.11] for $Q_1 \subset Q_2$ to derive

$$\begin{aligned}
\sum_{i=1}^{3} \|\widetilde{u}_n^i\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(Q_1^i\times[0,T])} &\leq C\left(\sum_{i=1}^{3}\|\widetilde{\mathfrak{f}}_n^i\|_{C^{\alpha,\frac{\alpha}{4}}(Q_2^i\times[0,T])} + \sum_{i=5}^{6}\|\widetilde{\mathfrak{b}}_n^i\|_{C^{1+\alpha,(1+\alpha)/4}(C_2\times[0,T])}\right)\\
&\quad + C\sum_{i=1}^{3}\|\widetilde{u}_n^i\|_{L^2(Q_2^i\times[0,T])}\\
&\overset{(4.94)}{\leq} C\left(\sum_{i=1}^{3}\|\widetilde{\mathfrak{f}}^i\|_{C^{\alpha,\frac{\alpha}{4}}(Q_3^i\times[0,T])} + \sum_{i=5}^{6}\|\widetilde{\mathfrak{b}}^i\|_{C^{1+\alpha,(1+\alpha)/4}(C_3\times[0,T])}\right)\\
&\quad + C\sum_{i=1}^{3}\|\widetilde{u}_n^i\|_{L^2(Q_4^i\times[0,T])} \qquad (4.95)\\
&\leq C\left(\sum_{i=1}^{3}\|\mathfrak{f}^i\|_{C^{\alpha,\frac{\alpha}{4}}(Q_3^i\times[0,T])} + \sum_{i=5}^{6}\|\mathfrak{b}^i\|_{C^{1+\alpha,(1+\alpha)/4}(C_3\times[0,T])}\right)\\
&\quad + C\left(\sum_{i=1}^{3}\|\mathfrak{f}^i\|_{L^2(Q_4^i\times[0,T])} + \sum_{i=1}^{5}\|\mathfrak{b}^i\|_{L^2(C_4\times[0,T])}\right)\\
&\leq C\left(\sum_{i=1}^{3}\|\mathfrak{f}^i\|_{C^{\alpha,\frac{\alpha}{4}}(Q_4^i\times[0,T])} + \sum_{i=5}^{6}\|\mathfrak{b}^i\|_{C^{1+\alpha,(1+\alpha)/4}(C_4\times[0,T])}\right).
\end{aligned}$$

But this implies now that the subsequence $\widetilde{\boldsymbol{u}}_{n_l}$ constructed above is bounded in $C^{4,1}(Q_1 \times [0,T])$ and as the Hölder-norms are bounded it is as well equicontinuous. Thus, the theorem of Arzela-Ascoli applied on every derivative gives us the existence of a subsequence, which we again call $\widetilde{\boldsymbol{u}}_{n_l}$ that converges to some $\widehat{\boldsymbol{u}} \in C^{4,1}_{TJ}(Q_1 \times [0,T])$. Indeed, we also have $\widehat{\boldsymbol{u}} \in C^{4+\alpha,1+\alpha}_{TJ}(Q_1 \times [0,T])$ as for any covariant derivative ∇^k of up to order 4 in space we have

$$|\nabla^k \widehat{u}^i(x,t) - \nabla^k \widehat{u}^i(y,t)| = \lim_{n_l\to\infty} |\nabla^k \widetilde{u}^i_{n_l}(x,t) - \nabla^k \widetilde{u}^i_{n_l}(y,t)| \leq C|x-y|^\alpha,$$

for any points $(x,t),(y,t) \in Q_4 \times [0,T]$ and we also have

$$|\partial_t \widehat{u}^i(x,t_1) - \partial_t \widehat{u}^i(x,t_2)| = \lim_{n_l\to\infty} |\widetilde{u}^i_{n_l}(x,t_1) - \widetilde{u}^i_{n_l}(x,t_2)| \leq C|t_1-t_2|^{\frac{\alpha}{4}},$$

for any points $(x,t_1),(x,t_2) \in Q_4 \times [0,T]$. The same argument can used for the Hölder regularity in time of the partial derivatives in space. Also, observe that we have for $\widehat{u}$ the estimate (4.95). Uniqueness of limits now implies

$$\widehat{\boldsymbol{u}} = \widetilde{\boldsymbol{u}} \qquad \text{on } Q_1 \times [0,T].$$

As we have $\widehat{u} = u$ on $Q_1 \times [0,T]$ this implies now the desired Schauder-estimate for $\boldsymbol{u}$, namely

$$\|\boldsymbol{u}\|_{C_{TJ}^{4+\alpha,1+\frac{\alpha}{4}}(Q_1\times[0,T])} \leq C\Big(\sum_{i=1}^{3}\|\mathfrak{f}^i\|_{C^{\alpha,\frac{\alpha}{4}}(Q_1\times[0,T])} + \sum_{i=5}^{6}\|\mathfrak{b}^i\|_{C^{1+\alpha,(1+\alpha)/4}(Q_1\times[0,T])}\Big),$$

holds. This finishes the study of $(LSDFTJ)_P$ and summing up the work of the last three subsection we get in total the following result.

Proposition 4.16 (Existence theory for $(LSDFTJ)_P$)**.**
For all $T > 0$ the problem $(LSDFTJ)_P$ has for all

$$\mathfrak{f} \in C_{TJ}^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T}), \quad \mathfrak{b}^5, \mathfrak{b}^6 \in C^{1+\alpha,\frac{1+\alpha}{4}}(\Sigma_* \times [0,T])$$

that fulfil the compatibility conditions

$$(\gamma^1\mathfrak{f}^1 + \gamma^2\mathfrak{f}^3 + \gamma^3\mathfrak{f}^3)\big|_{t=0} = 0, \mathfrak{b}^i\big|_{t=0} = 0 \text{ on } \Sigma_*, i = 5, 6,$$

a unique solution in $C_{TJ}^{4+\alpha,\frac{1+\alpha}{4}}(\Gamma_{*,T})$ *and we have the energy estimate*

$$\|\boldsymbol{u}\|_{C_{TJ}^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})} \leq C\left(\|\mathfrak{f}\|_{C_{TJ}^{\alpha,\frac{\alpha}{4}}(\Gamma_{*,T})} + \sum_{i=5}^{6}\|\mathfrak{b}^i\|_{C^{1+\alpha,\frac{1+\alpha}{4}}(\Sigma_{*,T})}\right) \tag{4.96}$$

4.5.4 The Analysis of the Full Linearised Problem

The system we studied so far differs in three aspects from the original problem. Firstly, we miss inhomogeneities in the first four boundary conditions. Secondly, both in the parabolic equation itself and the boundary conditions lower order terms are missing. And finally, we have to include general initial data. These final steps are carried out in the next three corollaries.

Corollary 4.17 (Inhomgeneities in $(LSDFTJ)_P$)**.**
For any

$$\mathfrak{b}^1 \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Sigma_{*,T}), \mathfrak{b}^2, \mathfrak{b}^3 \in C^{3+\alpha,(3+\alpha)/4}(\Sigma_{*,T}), \mathfrak{b}^4 \in C^{2+\alpha,(2+\alpha)/4}(\Sigma_{*,T}),$$

the system (4.46) has a unique solution $\boldsymbol{u} \in C_{TJ}^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_* \times [0,T])$ *and we have the energy estimate*

$$\|\boldsymbol{u}\|_{C_{TJ}^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})} \leq C\left(\sum_{i=1}^{3}\|\mathfrak{f}^i\|_{C^{\alpha,\frac{\alpha}{4}}(\Gamma^i_{*,T})} + \sum_{i=1}^{6}\|\mathfrak{b}^i\|_{C^{4-\sigma_i+\alpha,(4-\sigma_i+\alpha)/4}(\Sigma_{*,T})}\right). \tag{4.97}$$

Hereby, the σ_i *are chosen as in Theorem 4.7.*

Proof. The crucial problem here is that we are missing the Neumann inhomogeneities $\mathfrak{b}^2$ and $\mathfrak{b}^3$. The boundary condition linked to $\mathfrak{b}^1$ and $\mathfrak{b}^4$ are of Dirichlet type and so we could include them using a standard shifting argument by first solving a Bilaplacian equation for $\mathfrak{b}^4$ and then directly shift the equation with an extension of $\mathfrak{b}^1$. In our case, we have to do the shifting procedure more carefully to not influence the other boundary conditions. For this we use ideas of [29]. For any $\mathfrak{b}^4 \in C^{2+\alpha,(2+\alpha)/4}(\Sigma_{*,T})$ the system

$$\begin{aligned}
\partial_t \bar{b} + \Delta_{\Gamma_*^2}\Delta_{\Gamma_*^2}\bar{b} + \bar{b} &= 0 && \text{on } \Gamma_*^2 \times [0,T],\\
\bar{b} &= 0 && \text{on } \Sigma_* \times [0,T],\\
-\Delta_{\Gamma_*^2}\bar{b} &= \mathfrak{b}^4 && \text{on } \Sigma_* \times [0,T],\\
\bar{b}(x,0) &= 0 && \text{on } \Gamma_*^2,
\end{aligned}$$

has a unique solution $\bar{b} \in C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*^2)$ fulfilling the energy estimate

$$\|\bar{b}\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_*^2)} \leq C\|\mathfrak{b}^4\|_{C^{3+\alpha,(3+\alpha)/4}(\Sigma_*)}. \tag{4.98}$$

This can be proven with the same strategy as before except that we do not have to split the parabolic equation and so we really can include the inhomogeneity $\mathfrak{b}^4$. Now we define the auxiliary function $\chi : \Gamma_*^2 \to \mathbb{R}$ by

$$\chi(x) = \frac{1}{2}(\mathrm{dist}_{\Sigma_*}(x))^2\eta(\mathrm{dist}_{\Sigma_*}(x))\bar{b}(x),$$

where $\eta : [0,\infty] \to [0,1]$ is again a suitable cut-off function with $\eta \equiv 1$ on $[0,\varepsilon]$ for some sufficiently small ε. Note that $\mathrm{dist}^2_{\Sigma_*} \cdot \eta \in C^{5+\alpha,\infty}(\Gamma^2_{*,T})$ and so (4.98) holds also for χ. Define finally $\boldsymbol{\chi}$ by setting it zero on Γ^1_* and Γ^3_*. Now solving (4.46) with

$$\begin{aligned}
\widetilde{\mathfrak{f}}^1 &= \mathfrak{f}^1 - \partial_t \chi - \Delta_{\Gamma^1_*}\Delta_{\Gamma^1_*}\chi + (C_u + C_v)\Delta_{\Gamma^1_*}\chi + C_u C_v \chi,\\
\widetilde{\mathfrak{b}}^5 &= \mathfrak{b}^5 + \partial_{\nu^1_*}(-\Delta_{\Gamma^1_*}\chi),\\
\widetilde{\mathfrak{b}}^6 &= \mathfrak{b}^6,
\end{aligned}$$

the function $\boldsymbol{u} + \chi$ is the wished solution of (4.46) with included inhomogeneity $\mathfrak{b}^4$ and (4.98) implies together with the already known energy estimates the estimate (4.97).
To include the inhomogeneities $\mathfrak{b}^1, \mathfrak{b}^2$ and $\mathfrak{b}^3$ we can argue in a similar way. □

Corollary 4.18 (Lower order terms for $(LSDFTJ)_P$)**.** *Corollary 4.17 stays true if one includes the lower order perturbation terms $\mathcal{A}_P$ and $\mathcal{B}_P$. In particular, this shows Theorem 4.7.*

Proof. We consider (LSDFTJ) as a perturbation of (4.46). That is, given any

$$\boldsymbol{u} \in X := \left\{ \boldsymbol{u} \in C_{TJ}^{4+\alpha, 1+\frac{\alpha}{4}}(\Gamma_{*,T}) \,\middle|\, \boldsymbol{u}\big|_{t=0} \equiv 0 \right\}$$

we want to solve (4.46) with inhomogeneities

$$\widetilde{\mathfrak{f}}^i(\boldsymbol{u}) = \mathfrak{f}^i - \mathcal{A}^i_P \boldsymbol{u}, \quad \widetilde{\mathfrak{b}}^i(\boldsymbol{v}) = \mathfrak{b}^i - \mathcal{B}^i_P \boldsymbol{u}.$$

Note that $\widetilde{\mathfrak{f}}^i$ and $\widetilde{\mathfrak{b}}^i$ fulfil the compatibility conditions (CCP) as $\mathcal{A}^i_P$ and $\mathcal{B}^i_P$ vanish on X at $t = 0$. This implies that the solution $\Lambda(\boldsymbol{u})$ exists due to Corollary 4.17. We claim that for T small enough the map $\Lambda : X \to X$ is a contraction mapping. Then, Banach's fixed point theorem gives us the unique solution.
For $\boldsymbol{u}, \boldsymbol{v} \in X$ we note that the difference $\Lambda(\boldsymbol{u}) - \Lambda(\boldsymbol{v})$ solves (4.46) with inhomogeneities $\widetilde{\mathfrak{f}}^i = \mathcal{A}^i_P(\boldsymbol{u} - \boldsymbol{v})$ and $\widetilde{\mathfrak{b}}^i = \mathcal{B}^i_P(\boldsymbol{u} - \boldsymbol{v})$ and due to estimate (4.97) we have

$$\begin{aligned}
\|\Lambda(\boldsymbol{u}) - \Lambda(\boldsymbol{v})\|_X \leq C_1 &\sum_{i=1}^{3} \|\mathcal{A}^i_P(\boldsymbol{u} - \boldsymbol{v})\|_{C^{\alpha, \frac{\alpha}{4}}(\Gamma^i_{*,T})}\\
&+ C_2 \sum_{i=1}^{6} \|\mathcal{B}^i_P(\boldsymbol{u} - \boldsymbol{v})\|_{C^{4-\sigma_i+\alpha, (4-\sigma_i+\alpha)/4}(\Sigma_{*,T})}.
\end{aligned}$$

There are two different kind of terms that appear in the perturbation operators $\mathcal{A}_P$ and $\mathcal{B}_P$. One are lower order partial derivatives of $\boldsymbol{u} - \boldsymbol{v}$ with C^α-coefficients only depending on the reference geometry. For these we may directly apply Lemma 2.17 to get the sought contractivity property if T is sufficiently small. Note that the regularity of the coefficient functions is not a problem due to Lemma 2.16. The other terms are the non-local ones. But here we see that they are also of lower order (second order terms are the highest order arising) and by the chain rule all space and time derivatives are bounded by space and time derivatives on the boundary and so by the Hölder-norm of $\boldsymbol{u} - \boldsymbol{v}$ itself. So, this shows us that for T sufficiently small Λ is a $\frac{1}{2}$-contraction, which finishes the proof. □

Corollary 4.19 (Proof of Corollary 4.8)**.** *The short time existence result for (LSDFTJ) holds for any initial data fulfilling the compatibility conditions (CLP) with the energy estimates as in Corollary 4.8.*

Proof. We can do the same procedure as in [19] by shifting the equation by $\boldsymbol{\rho}_0$. The condition (CLP) will guarantee that the compatibility condition for Theorem 4.7 are fulfilled and the additional inhomogeneities give us the sought energy estimate. □

4.6 Analysis of the Non-Linear Problem

We now want to use the result for the linearised problem to get short time existence for our original problem (4.20). Here, we make a change in the strategy compared to the one applied in [19]. In their

work the authors used the analytic formulation (4.29) to formulate the fixed-point problem. But as we considered the parametrised, geometric version (4.20) for the linearisation, we think that it is more natural to use this version. Additionally, one sees with this approach directly that the contraction properties for most terms follow from the quasi-linear structure. This includes also the non-local terms, which in the end cause much less technical problems than one would expect at first glance. Actually, we will have to put the most work into the angle conditions as these are fully non-linear.
Our main strategy is now to write (4.20) as a fixed-point problem which we do in the following way. For $\boldsymbol{\rho}_0 \in C^{4+\alpha}(\Gamma_*)$, $\sigma_i, i = 1, ..., 6$ as in Theorem 4.7, $R, \varepsilon > 0$ and $\delta > 0$, which we always assume to be smaller than the δ_0 from Corollary 4.8, we consider the sets[6]

$$\begin{aligned} X^{\varepsilon}_{R,\delta} &:= \left\{ \boldsymbol{\rho} \in C^{4+\alpha, 1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,\delta}) \;\middle|\; \|\boldsymbol{\rho}(0)\|_{C^{4+\alpha}(\Gamma_*)} \leq \varepsilon, \|\boldsymbol{\rho} - \boldsymbol{\rho}(0)\|_{X_{R,\delta}} \leq R \right\}, \\ X^{\boldsymbol{\rho}_0}_{R,\delta} &:= \left\{ \boldsymbol{\rho} \in C^{4+\alpha, 1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,\delta}) \;\middle|\; \boldsymbol{\rho}(0) = \boldsymbol{\rho}_0, \sum_{i=1}^{3} \gamma^i \rho^i = 0 \text{ on } \Sigma_{*,\delta}, \|\boldsymbol{\rho} - \boldsymbol{\rho}_0\|_{X_{R,\delta}} \leq R \right\}, \\ Y_\delta &:= C^{\alpha, \frac{\alpha}{4}}_{TJ}(\Gamma_{*,\delta}) \times \left(\prod_{i=1}^{6} C^{4-\sigma_i+\alpha, \frac{4-\sigma_i+\alpha}{4}}(\Sigma_{*,\delta}) \right). \end{aligned}$$

We will denote by $\|\cdot\|_{X_{R,\delta}}$ and $\|\cdot\|_{Y_\delta}$ the canonical norms on $X^{\varepsilon}_{R,\delta}$ (resp. $X^{\boldsymbol{\rho}_0}_{R,\delta}$) and Y_δ. Observe that we have $\|\boldsymbol{\rho}_0\|_{X_{R,\delta}} = \|\boldsymbol{\rho}_0\|_{C^{4+\alpha}_{TJ}(\Gamma_*)}$ and therefore

$$\forall \boldsymbol{\rho} \in X_{R,\delta} : \|\boldsymbol{\rho}\|_{X_{R,\delta}} \leq R + \|\boldsymbol{\rho}_0\|_{C^{4+\alpha}_{TJ}(\Gamma_*)}. \tag{4.99}$$

On these sets we consider the inhomogeneities operator $S := (\mathfrak{f}, \mathfrak{b}) : X^{\boldsymbol{\rho}_0}_{R,\delta} \to Y_\delta$ given by

$$\mathfrak{f}^i(\boldsymbol{\rho}) := \partial_t \rho^i - V^i_{\boldsymbol{\rho}} + \Delta_{\boldsymbol{\rho}} H^i_{\boldsymbol{\rho}} - \mathcal{A}^i_{\text{all}}(\rho^i, \boldsymbol{\rho}|_{\Sigma_*}), \qquad i = 1, 2, 3, \tag{4.100}$$

$$\mathfrak{b}^i(\boldsymbol{\rho}) := \mathcal{B}^i(\boldsymbol{\rho}) - G^i(\boldsymbol{\rho}), \qquad i = 1, ..., 6. \tag{4.101}$$

Here, we used the notation from Section 4.2. Furthermore, we define $L : Y_\delta \to X^{\boldsymbol{\rho}_0}_{R,\delta}$ as the solution operator from Corollary 4.8 and $\Lambda := L \circ S : X^{\boldsymbol{\rho}_0}_{R,\delta} \to X^{\boldsymbol{\rho}_0}_{R,\delta}$. The main result of this section will now be the following.

Proposition 4.20 (Existence of a fixed-point of Λ)**.**
There exists $\varepsilon_0, R_0 > 0$ with the following property. For all $R > R_0$ and $\varepsilon < \varepsilon_0$ there exists a $\delta > 0$ such that for all $\boldsymbol{\rho}_0 \in C^{4+\alpha}(\Gamma_*)$, fulfilling $\|\boldsymbol{\rho}_0\| \leq \varepsilon_0$ and the geometric compatibility conditions (4.16), the map $\Lambda : X^{\boldsymbol{\rho}_0}_{R,\delta} \to X^{\boldsymbol{\rho}_0}_{R,\delta}$ is well-defined and there exists a unique fixed-point of Λ in $X_{R,\delta}$.

As in Section 3.4 the proof splits into three main parts. We will first verify that if we choose ε sufficiently small we can guarantee that Λ is well-defined as long as $\delta(R)$ is also sufficiently small. Then, we will check that for ε small and R large we can find a $\delta(R, \varepsilon)$ such that Λ is a $\frac{1}{2}$-contraction. Finally, we will see that with this choice of δ we can choose R large enough such that Λ is also a self-mapping on $X^{\boldsymbol{\rho}_0}_{R,\delta}$.

Lemma 4.21 (Well-definedness of Λ)**.**

i.) There is a $\varepsilon_W > 0$ such that for any $R > 0$ and $\varepsilon < \varepsilon_W$ there is a $\delta_W(\varepsilon, R) > 0$ such that S is a well-defined map $X^{\varepsilon}_{R,\delta} \to Y_\delta$.

ii.) For all initial data $\boldsymbol{\rho}_0$ fulfilling the geometric compatibility condition (4.16) and the bound from i.) we have that $S(\boldsymbol{\rho})$ fulfils the linear compatibility condition (4.44) *for all $\boldsymbol{\rho} \in X^{\boldsymbol{\rho}_0}_{R,\delta}$. .*

iii.) Choosing ε, R, δ and $\boldsymbol{\rho}_0$ as in i.),ii.) the map $\Lambda : X^{\boldsymbol{\rho}_0}_{R,\delta} \to X^{\boldsymbol{\rho}_0}_{R,\delta}$ is well-defined.

Proof. For i.) we have to check both well-definedness of the geometric quantities in $\mathfrak{f}$ and $\mathfrak{b}$ and the right regularity. For the first part recall that due to the Hölder-regularity in time for space derivatives

[6] Note that in $X^{\boldsymbol{\rho}_0}_{R,\delta}$ we include the sum condition for $\boldsymbol{\rho}$ on the parabolic boundary to guarantee compatibility conditions, which we will see in the proof of Lemma 4.21.

we have for a multi-index β with $1 \leq |\beta| \leq 4$ and any $\boldsymbol{\rho} \in X^{\varepsilon}_{R,\delta}$ with $\boldsymbol{\rho}(0) = \boldsymbol{\rho}_0 \in C^{4+\alpha}_{TJ}(\Gamma_*)$ that

$$\|\partial^x_\beta \boldsymbol{\rho}(t)\|_\infty \leq t^{\frac{4-|\beta|+\alpha}{4}} \langle \partial^x_\beta \boldsymbol{\rho} \rangle_{t,(4-|\beta|+\alpha)/4} + \|\partial^x_\beta \boldsymbol{\rho}(0)\|_\infty \leq t^{\frac{4-|\beta|+\alpha}{4}} (R+\varepsilon) + \|\boldsymbol{\rho}_0\|_{C^{4+\alpha}_{TJ}(\Gamma_*)}. \tag{4.102}$$

Additionally, we have that

$$\|\boldsymbol{\rho}(t)\|_\infty \leq \delta \|\partial_t \boldsymbol{\rho}\|_\infty + \|\boldsymbol{\rho}(0)\|_\infty \leq \delta(R+\varepsilon) + \|\boldsymbol{\rho}_0\|_{C^{4+\alpha}_{TJ}(\Gamma_*)}. \tag{4.103}$$

Together, this implies for $\delta < 1$ and all $t \in [0,\delta]$ that

$$\|\boldsymbol{\rho}(t)\|_{C^4_{TJ}(\Gamma_*)} \leq C\left(\delta(R+\varepsilon) + \|\boldsymbol{\rho}_0\|_{C^{4+\alpha}_{TJ}(\Gamma_*)}\right) \leq C(\delta(R+\varepsilon)+\varepsilon). \tag{4.104}$$

Now, for any $C' > 0$ we get for $\varepsilon \leq \frac{C'}{2C}$ and $\delta \leq \frac{C'}{2C(R+\varepsilon)}$ that

$$\|\boldsymbol{\rho}(t)\|_{C^4_{TJ}(\Gamma_*)} \leq C'. \tag{4.105}$$

Thus, we can get a bound for the C^2-norm of $\boldsymbol{\rho}$ sufficiently small such that [19, p. 326] implies that all geometric quantities - in particular the normal, the conormal and the inverse metric tensor - are well defined. It remains to show that these objects have the required regularity. For the normal and conormal this follows directly from the representations from Lemma 4.6. For the inverse metric tensor we observe that matrix inversion is a smooth operator $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$. This follows by applying the inverse function theorem on the map

$$GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R}), (A,B) \mapsto A \cdot B - \mathbb{E}_n.$$

Applying now composition operator theory for g^{-1} first as a function in time and then as function in space we get that g^{-1} has the same Hölder-regularity as g. This implies now $S(\boldsymbol{\rho}) \in Y_\delta$.
For part ii) we get from the geometric compatibility conditions (4.16)

$$\sum_{i=1}^3 \gamma^i \mathfrak{f}^i\big|_{t=0} = \sum_{i=1}^3 \gamma^i \left(\partial_t \rho^i(0) - V^i_{\boldsymbol{\rho}}(0)\right) - \sum_{i=1}^3 \gamma^i \mathcal{A}^i_{all}(\rho^i_0) \qquad \text{on } \Sigma_*. \tag{4.106}$$

So, it remains to see that the first sum vanishes. But as we included the sum condition for $\boldsymbol{\rho}$ on the triple junction on in $X^{\boldsymbol{\rho}_0}_{R,\delta}$ we get immediately

$$\sum_{i=1}^3 \gamma^i \partial_t \rho^i(0) = 0. \tag{4.107}$$

For the sum of the $V^i_{\boldsymbol{\rho}}$ we can argue like in the derivation of (4.16). The compatibility conditions for $\mathfrak{b}^i$ follow directly from $\mathcal{G}(\boldsymbol{\rho}_0) \equiv 0$, which in total shows ii.).
The last part follows from i.) and ii.) as L is well-defined as long as $S(\boldsymbol{\rho})$ fulfils the linear compatibility conditions (4.44). ☐

In the following we will always assume that ε and R are chosen such that Lemma 4.21 is fulfilled. Now we want to derive suitable contraction estimates for the operator S. Hereby, we will use the norm on Y_δ also when dealing with components of S. Before we can start with this we have to check some regularity results for the involved quantities.

Lemma 4.22 (Lipschitz continuity of geometrical quantities)**.**
Suppose that $\delta, \varepsilon < 1$ and $R > 1$.
i.) The mapping

$$X^{\varepsilon}_{R,\delta} \to C^{4+\alpha, 1+\frac{\alpha}{4}}_{TJ}(\Sigma_{*,\delta}), \boldsymbol{\rho} \mapsto \boldsymbol{\mu}(\boldsymbol{\rho}),$$

is linear and Lipschitz-continuous.

ii.) *For any local parametrisation* $\varphi : U \to V \subset \Gamma^i_*, i = 1,2,3$ *the* $g^{\boldsymbol{\rho}}_{jk}, g^{jk}_{\boldsymbol{\rho}}$ *are Lipschitz continuous as maps*

$$X^{\varepsilon}_{R,\delta} \to C^{3+\alpha,\frac{3+\alpha}{4}}(U_\delta).$$

and $N_{\boldsymbol{\rho}}$ *is a Lipschitz continuous function in* $\boldsymbol{\rho}$ *as map*

$$X^{\varepsilon}_{R,\delta} \to C^{3+\alpha,\frac{3+\alpha}{4}}(\Gamma_{*,\delta}, \mathbb{R}^n).$$

iii.) *For any local parametrisation* $\boldsymbol{\varphi} : U \to V \subset \Sigma_*$ *the* $g^{\boldsymbol{\rho}}_{jk}, g^{jk}_{\boldsymbol{\rho}}$ *and* $N_{\boldsymbol{\rho}}$ *are Lipschitz continuous functions in* $\boldsymbol{\rho}$ *as maps*

$$X^{\varepsilon}_{R,\delta} \to C^{3+\alpha,\frac{3+\alpha}{4}}(U_\delta).$$

Furthermore, all arising Lipschitz constants are independent of δ.

Proof. For part i.), the mapping is linear and then Lipschitz-continuity is equivalent to boundedness. For this, we see that due to the structure (4.19) for $\boldsymbol{\mu}$ we have that

$$\|\boldsymbol{\mu}(\boldsymbol{\rho})\|_{C^{4+\alpha,1+\frac{\alpha}{4}}_{TJ}(\Sigma_{*,\delta})} \le C(\boldsymbol{\gamma})\|\boldsymbol{\rho}\|_{C^{4+\alpha,1+\frac{\alpha}{4}}_{TJ}(\Sigma_{*,\delta})} \le C(\boldsymbol{\gamma})\|\boldsymbol{\rho}\|_{C^{4+\alpha,1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,\delta})}.$$

The constant $C(\boldsymbol{\gamma})$ only depends on $\boldsymbol{\gamma}$ and is therefore the sought time-independent Lipschitz constant. For the $g^{\boldsymbol{\rho}}_{jk}$ we recall the formula for the $\partial^{\boldsymbol{\rho}}_j$ (4.34). This shows that the $g^{\boldsymbol{\rho}}_{ij}$ can be written as sum of 25 terms of two different kinds. The first ones are linear terms in $\boldsymbol{\rho}$, which are obviously Lipschitz continuous as the requested maps with time independent Lipschitz constants. For the quadratic terms we observe that, for example, we have for $\boldsymbol{u}, \boldsymbol{v} \in X^{\varepsilon}_{R,\delta}, i = 1,2,3$ and $j,k \in \{1,..,n\}$

$$\begin{aligned}
\|\partial_j u^i \partial_k u^i - \partial_j v^i \partial_k v^i\|_{C^{1+\alpha,\frac{1+\alpha}{4}}} &\le \|(\partial_j u^i - \partial_j v^i)\partial_k u^i\|_{C^{1+\alpha,\frac{1+\alpha}{4}}} + \|\partial_j v^i(\partial_k u^i - \partial_k v^i)\|_{C^{1+\alpha,\frac{1+\alpha}{4}}} \\
&\le \|\partial_k u^i\|_{C^{1+\alpha,\frac{1+\alpha}{4}}}\|\boldsymbol{u}-\boldsymbol{v}\|_{X_{R,\delta}} + \|\partial_j v^i\|_{C^{1+\alpha,\frac{1+\alpha}{4}}}\|\boldsymbol{u}-\boldsymbol{v}\|_{X_{R,\delta}} \\
&\le 2(R+\varepsilon)\|\boldsymbol{u}-\boldsymbol{v}\|_{X_{R,\delta}} \le 4R\|\boldsymbol{u}-\boldsymbol{v}\|_{X_{R,\delta}}.
\end{aligned}$$

So, this shows that also this kind of terms is Lipschitz continuous with a constant only depending on R.
For the $g^{jk}_{\boldsymbol{\rho}}$ we recall that they result from composition of $g^{\boldsymbol{\rho}}$ with the matrix inversion. The latter is, at least on the image of $g^{\boldsymbol{\rho}}$, which is bounded on $X^{\varepsilon}_{R,\delta}$, Lipschitz continuous. This is also true for all derivatives of this map. Now we can explicit calculate the first order derivatives of g^{jk}_{ρ} in terms of derivative of $\boldsymbol{\rho}$ and derivatives of the matrix inversion and then use Lipschitz continuity of both to get Lipschitz continuity of $g^{jk}_{\boldsymbol{\rho}}$.
For $N_{\boldsymbol{\rho}}$ we recall (4.39) and see that for the numerator we can argue as for the $g^{\boldsymbol{\rho}}_{jk}$. Then, the rest follows with the same composition operator arguments as for the $g^{jk}_{\boldsymbol{\rho}}$. Finally, iii.) is a direct consequence of Remark 2.14. □

Lemma 4.23 (Contraction estimates for S)**.**
Suppose that $\delta, \varepsilon < 1$ *and* $R > 1$. *Then, for all* $\boldsymbol{u}, \boldsymbol{w} \in X^{\boldsymbol{\rho}_0}_{R,\delta}$ *there is an* $\bar{\alpha} \in (0,1)$ *such that the following contraction estimates hold:*

$$\|\partial_t \boldsymbol{u} - V_{\boldsymbol{u}} - \partial_t \boldsymbol{w} + V_{\boldsymbol{w}}\|_{Y_\delta} \le C(\Gamma_*)(\varepsilon + R\delta^{\bar{\alpha}})\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}, \tag{4.108}$$

$$\|\Delta_{\boldsymbol{u}} H_{\boldsymbol{u}} - \mathcal{A}_{all}(\boldsymbol{u}) - \Delta_{\boldsymbol{w}} H_{\boldsymbol{w}} + \mathcal{A}_{all}(\boldsymbol{w})\|_{Y_\delta} \le C(\Gamma_*)(\varepsilon + R\delta^{\bar{\alpha}})\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}, \tag{4.109}$$

$$\|\mathfrak{f}(\boldsymbol{u}) - \mathfrak{f}(\boldsymbol{w})\|_{Y_\delta} \le C(\Gamma_*)(\varepsilon + R\delta^{\bar{\alpha}})\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}, \tag{4.110}$$

$$\|\mathfrak{b}^1(\boldsymbol{u}) - \mathfrak{b}^1(\boldsymbol{w})\|_{Y_\delta} = 0, \tag{4.111}$$

$$\|\mathfrak{b}^2(\boldsymbol{u}) - \mathfrak{b}^2(\boldsymbol{w})\|_{Y_\delta} \le \left(C(\Gamma_*, R)\delta^{\bar{\alpha}} + C(\Gamma_*)\varepsilon\right)\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}, \tag{4.112}$$

$$\|\mathfrak{b}^3(\boldsymbol{u}) - \mathfrak{b}^3(\boldsymbol{w})\|_{Y_\delta} \le \left(C(\Gamma_*, R)\delta^{\bar{\alpha}} + C(\Gamma_*)\varepsilon\right)\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}, \tag{4.113}$$

$$\begin{aligned}
\|\mathfrak{b}^4(\boldsymbol{u})-\mathfrak{b}^4(\boldsymbol{w})\|_{Y_\delta} &\leq C(\Gamma_*)R\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}, && (4.114)\\
\|\mathfrak{b}^5(\boldsymbol{u})-\mathfrak{b}^5(\boldsymbol{w})\|_{Y_\delta} &\leq C(\Gamma_*)R\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}, && (4.115)\\
\|\mathfrak{b}^6(\boldsymbol{u})-\mathfrak{b}^6(\boldsymbol{w})\|_{Y_\delta} &\leq C(\Gamma_*)R\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}, && (4.116)\\
\|S(\boldsymbol{u})-S(\boldsymbol{w})\|_{Y_\delta} &\leq \left(C(\Gamma_*,R)\delta^{\bar{\alpha}}+C(\Gamma_*)\varepsilon\right)\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}. && (4.117)
\end{aligned}$$

Proof. For the first line we note that the term on the left-hand-side equals

$$\underbrace{\partial_t\boldsymbol{u}(1-N_{\boldsymbol{u}}\cdot N_*)-\partial_t\boldsymbol{w}(1-N_{\boldsymbol{w}}\cdot N_*)}_{=(I)}+\underbrace{\partial_t\boldsymbol{\mu}(\boldsymbol{w})\tau_*\cdot N_{\boldsymbol{w}}-\partial_t\boldsymbol{\mu}(\boldsymbol{u})\tau_*\cdot N_{\boldsymbol{u}}}_{=(II)}$$

We will discuss the terms (I) and (II) separately. Firstly, we rewrite (I) as

$$\partial_t\boldsymbol{u}(1-N_{\boldsymbol{u}}\cdot N_*)-\partial_t\boldsymbol{u}(1-N_{\boldsymbol{w}}\cdot N_*)+\partial_t\boldsymbol{u}(1-N_{\boldsymbol{w}}\cdot N_*)-\partial_t\boldsymbol{w}(1-N_{\boldsymbol{w}}\cdot N_*)$$

and observe using Lemma 2.16, Lemma 2.17, Lemma 4.22ii.) and (4.99) that

$$\begin{aligned}
\|\partial_t\boldsymbol{u}\,(N_{\boldsymbol{w}}-N_{\boldsymbol{u}}))\cdot N_*\|_{Y_\delta} &\leq \|\partial_t\boldsymbol{u}\|_{Y_\delta}\|(N_{\boldsymbol{w}}-N_{\boldsymbol{u}})\cdot N_*\|_{Y_\delta}\\
&\leq C(\Gamma_*)\|\boldsymbol{u}\|_{X_{R,\delta}}\delta^{\bar{\alpha}}\|\,(N_{\boldsymbol{w}}-N_{\boldsymbol{u}})\cdot N_*\|_{C_{TJ}^{3+\alpha,\frac{3+\alpha}{4}}(\Gamma_{*,\delta})}\\
&\leq C(\Gamma_*)(R+\varepsilon)\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}\\
&\leq C(\Gamma_*)2R\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}.\\
\|(\partial_t\boldsymbol{u}-\partial_t\boldsymbol{w})\,(1-N_{\boldsymbol{w}}\cdot N_*)\|_{Y_\delta} &\leq \|\partial_t\boldsymbol{u}-\partial_t\boldsymbol{w}\|_{Y_\delta}\|1-N_{\boldsymbol{w}}\cdot N_*\|_{Y_\delta}\\
&\leq \|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}C(\Gamma_*)\delta^{\bar{\alpha}}\|1-N_{\boldsymbol{w}}\cdot N_*\|_{C_{TJ}^{3+\alpha,\frac{3+\alpha}{4}}(\Gamma_{*,\delta})}\\
&\leq C(\Gamma_*)\delta^{\bar{\alpha}}\|\boldsymbol{w}\|_{X_{R,\delta}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}\\
&\leq C(\Gamma_*)(R+\varepsilon)\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}\\
&\leq C(\Gamma_*)2R\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}.
\end{aligned}$$

In total this implies

$$\|(I)\|_{Y_{\delta,R}} \leq C(\Gamma_*)R\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}. \tag{4.118}$$

Next, we write the term (II) as

$$\partial_t\boldsymbol{\mu}(\boldsymbol{w})\tau_*\cdot N_{\boldsymbol{w}}-\partial_t\boldsymbol{\mu}(\boldsymbol{w})\tau_*\cdot N_{\boldsymbol{u}}+\partial_t\boldsymbol{\mu}(\boldsymbol{w})\tau_*\cdot N_{\boldsymbol{u}}-\partial_t\boldsymbol{\mu}(\boldsymbol{u})\tau_*\cdot N_{\boldsymbol{u}},$$

and derive using Lemma 4.22i.) and ii.), Lemma 2.16, Lemma 2.17 and (4.99) that

$$\begin{aligned}
\|\partial_t\boldsymbol{\mu}(\boldsymbol{w})(N_{\boldsymbol{w}}-N_{\boldsymbol{u}})\cdot\tau_*\|_{Y_\delta} &\leq \|\partial_t\boldsymbol{\mu}(\boldsymbol{w})\|_{Y_\delta}\|(N_{\boldsymbol{w}}-N_{\boldsymbol{u}})\cdot\tau_*\|_{Y_\delta}\\
&\leq \|\boldsymbol{\mu}(\boldsymbol{w})\|_{X_{R,\delta}}C(\Gamma_*)\delta^{\bar{\alpha}}\|(N_{\boldsymbol{w}}-N_{\boldsymbol{u}})\cdot\tau_*\|_{C_{TJ}^{3+\alpha,\frac{3+\alpha}{4}}(\Gamma_{*,\delta})}\\
&\leq C(\Gamma_*)\|\boldsymbol{w}\|_{X_{R,\delta}}\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}\\
&\leq C(\Gamma_*)(R+\varepsilon)\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}\\
&\leq C(\Gamma_*)2R\delta^{\bar{\alpha}}\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}},\\
\|(\partial_t\boldsymbol{\mu}(\boldsymbol{w})-\partial_t\boldsymbol{\mu}(\boldsymbol{u}))N_{\boldsymbol{u}}\cdot\tau_*\|_{Y_\delta} &\leq \|\partial_t\boldsymbol{\mu}(\boldsymbol{u}-\boldsymbol{w})\|_{Y_\delta}\|N_{\boldsymbol{u}}\cdot\tau_*\|_{Y_\delta}\\
&\leq C(\Gamma_*)\|\boldsymbol{\mu}(\boldsymbol{u}-\boldsymbol{w})\|_{X_{R,\delta}}\left(\varepsilon+\delta^{\bar{\alpha}}\|N_{\boldsymbol{u}}\cdot\tau_*\|_{C_{TJ}^{3+\alpha,\frac{3+\alpha}{4}}(\Gamma_{*,\delta})}\right)\\
&\leq C(\Gamma_*)\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}\left(\varepsilon+\delta^{\bar{\alpha}}\|\boldsymbol{u}\|_{X_{R,\delta}}\right)\\
&\leq C(\Gamma_*)\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}\left(\varepsilon+\delta^{\bar{\alpha}}(R+\varepsilon)\right)\\
&\leq C(\Gamma_*)\|\boldsymbol{u}-\boldsymbol{w}\|_{X_{R,\delta}}\left(\varepsilon+2\delta^{\bar{\alpha}}R\right).
\end{aligned}$$

From this we conclude

$$\|(II)\|_{Y_\delta} \le C(\Gamma_*)(\varepsilon + \delta^{\bar\alpha} R)\|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}}. \tag{4.119}$$

The estimates (4.118) and (4.119) imply together (4.108).
For (4.109) we note that the highest order terms on the left-hand side are given in local coordinates by

$$\begin{aligned} &\left(g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot N_*)\right) \partial_{jklm}\boldsymbol{u} - \left(g^{jk}_{\boldsymbol{w}} g^{lm}_{\boldsymbol{w}} (N_{\boldsymbol{w}} \cdot N_*)\right) \partial_{jklm}\boldsymbol{w} - g^{jk}_* g^{lm}_* \partial_{jklm}(\boldsymbol{u} - \boldsymbol{w}) \\ &+ \left(g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot \tau_*)\right) \partial_{jklm}\boldsymbol{\mu}(\boldsymbol{u}) - \left(g^{jk}_{\boldsymbol{w}} g^{lm}_{\boldsymbol{w}} (N_{\boldsymbol{w}} \cdot \tau_*)\right) \partial_{jklm}\boldsymbol{\mu}(\boldsymbol{w}). \end{aligned}$$

We abbreviate the local terms in the first line by (I) and the non-local terms in the second line by (II). First, we rewrite (I) as

$$\begin{aligned} &- g^{jk}_* g^{lm}_* \partial_{jklm}(\boldsymbol{u} - \boldsymbol{w}) + g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot N_*) \partial_{jklm}(\boldsymbol{u} - \boldsymbol{w}) - g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot N_*) \partial_{jklm}(\boldsymbol{u} - \boldsymbol{w}) \\ &+ g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot N_*) \partial_{jklm}\boldsymbol{u} - g^{jk}_{\boldsymbol{w}} g^{lm}_{\boldsymbol{w}} (N_{\boldsymbol{w}} \cdot N_*) \partial_{jklm}\boldsymbol{w} \\ &= \underbrace{\left(-g^{jk}_* g^{lm}_* + g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot N_*)\right) \partial_{jklm}(\boldsymbol{u} - \boldsymbol{w})}_{(A)} + \underbrace{\left(g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot N_*) - g^{jk}_{\boldsymbol{w}} g^{lm}_{\boldsymbol{w}} (N_{\boldsymbol{w}} \cdot N_*)\right) \partial_{jklm}\boldsymbol{w}}_{(B)} \end{aligned}$$

Now, we observe that due to Lemma 4.22i.) the function

$$\begin{aligned} X^\varepsilon_{R,\delta} &\to C^{3+\alpha, \frac{3+\alpha}{4}}(\Gamma_{*,\delta}), \\ \boldsymbol{\rho} &\mapsto g^{jk}_{\boldsymbol{\rho}} g^{lm}_{\boldsymbol{\rho}} (N_{\boldsymbol{\rho}} \cdot N_*), \end{aligned}$$

is Lipschitz continuous and the evaluation at $\boldsymbol{\rho} \equiv 0$ equals $g^{jk}_* g^{lm}_*$. Thus, we get

$$\begin{aligned} \|(A)\|_{Y_\delta} &\le C(\Gamma_*)\delta^{\bar\alpha} \|g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot N_*) - g^{jk}_* g^{lm}_*\|_{C^{3+\alpha, \frac{3+\alpha}{4}}_{TJ}(\Gamma_{*,\delta})} \|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}} \\ &\le C(\Gamma_*)\delta^{\bar\alpha} \|\boldsymbol{u}\|_{X_{R,\delta}} \|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}} \\ &\le C(\Gamma_*)\delta^{\bar\alpha} (R + \varepsilon) \|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}} \\ &\le C(\Gamma_*)\delta^{\bar\alpha} 2R \|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}}, \\ \|(B)\|_{Y_\delta} &\le C(\Gamma_*)\delta^{\bar\alpha} \|g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot N_*) - g^{jk}_{\boldsymbol{w}} g^{lm}_{\boldsymbol{w}} (N_{\boldsymbol{w}} \cdot N_*)\|_{C^{3+\alpha, \frac{3+\alpha}{3}}_{TJ}(\Gamma_{*,\delta})} \|\boldsymbol{w}\|_{X_{R,\delta}} \\ &\le C(\Gamma_*)\delta^{\bar\alpha} \|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}} 2R. \end{aligned}$$

Together this implies

$$\|(I)\|_{Y_\delta} \le C(\Gamma_*)\delta^{\bar\alpha} R \|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}}. \tag{4.120}$$

Now, we write the term (II) as

$$\begin{aligned} &\underbrace{g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot \tau_*) \partial_{jklm}\boldsymbol{\mu}(\boldsymbol{u}) - g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot \tau_*) \partial_{jklm}\boldsymbol{\mu}(\boldsymbol{w})}_{(A)} \\ &+ \underbrace{g^{jk}_{\boldsymbol{u}} g^{lm}_{\boldsymbol{u}} (N_{\boldsymbol{u}} \cdot \tau_*) \partial_{jklm}\boldsymbol{\mu}(\boldsymbol{w}) - g^{jk}_{\boldsymbol{w}} g^{lm}_{\boldsymbol{w}} (N_{\boldsymbol{w}} \cdot \tau_*) \partial_{jklm}\boldsymbol{\mu}(\boldsymbol{w})}_{(B)}. \end{aligned}$$

Using Lemma 4.22i.) we see that the function

$$\begin{aligned} X_{R,\delta} &\to C^{3+\alpha, \frac{3+\alpha}{4}}_{TJ}(\Gamma_{*,\delta}) \\ \boldsymbol{\rho} &\mapsto g^{jk}_{\boldsymbol{\rho}} g^{lm}_{\boldsymbol{\rho}} (N_{\boldsymbol{\rho}} \cdot \tau_*), \end{aligned}$$

is Lipschitz continuous and the evaluation at $\boldsymbol{\rho} \equiv 0$ equals 0 . Thus, we can estimates similar to those applied in Lemma 3.11 for term (I) and then get

$$\|(II)\|_{Y_\delta} \leq C(\Gamma_*)(\varepsilon + R\delta^{\bar{\alpha}})\|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}}. \tag{4.121}$$

Combining the estimates (4.120) and (4.121) we get (4.109) and then together with (4.108) we conclude (4.110).
Next we have to deal with the boundary operator $\mathfrak{b}$. We start with $\mathfrak{b}^2$ resp. $\mathfrak{b}^3$ that correspond to the angle conditions. These are actually the technical most challenging as they are fully non-linear and so the arguments we used so far will fail. We will explicitly have to use the fact that $\mathcal{B}^2$ resp. $\mathcal{B}^3$ is the linearisation of $\mathcal{G}^2$ resp. $\mathcal{G}^3$ and follow the ideas of [42, Chapter 8].
Before we can go on we have to find representations of the function $N^1_{\boldsymbol{\rho}} \cdot N^2_{\boldsymbol{\rho}}$ and $N^2_{\boldsymbol{\rho}} \cdot N^3_{\boldsymbol{\rho}}$ that only depend on a finite number of values and show some regularity results for their partial derivatives.

Lemma 4.24 (Representation of $N^i_{\boldsymbol{\rho}} \cdot N^j_{\boldsymbol{\rho}}$)**.**
There is a function

$$N^{12} : C_\delta \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

such that we have for all $\boldsymbol{\rho} \in X_{R,\delta}$ and $(\sigma, t) \in \Sigma_\delta$

$$\left(N^1_{\boldsymbol{\rho}} \cdot N^2_{\boldsymbol{\rho}}\right)(\sigma, t) = N^{1,2}(x, t, \widehat{\rho}^1(x,t), \widehat{\rho}^2(x,t), \nabla\widehat{\rho}^1(x,t), \nabla\widehat{\rho}^2(x,t)).$$

Hereby, we denote by $\widehat{\rho}^1$ resp. $\widehat{\rho}^2$ the functions ρ^1 resp. ρ^2 in local coordinates with respect to a parametrisation $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$, where φ^i is a local parametrisation of Γ^i_* locally around σ, and $x = \varphi^{-1}(\sigma)$. Additionally, all partial derivatives of N^{12} with respect to the values of $\widehat{\rho}^i$ and $\partial_j\widehat{\rho}^i$ with $i = 1, 2$ and $j = 1, ..., n$, which we denote by $\partial_{[\rho^i]}$ resp. $\partial_{[\partial_j\rho^i]}$, are in $C^{3+\alpha, \frac{3+\alpha}{4}}$ and we have

$$\|\partial N^{12}\|_{C^{3+\alpha,\frac{3+\alpha}{4}}} \leq C(\Gamma_*, R), \tag{4.122}$$

for $\partial \in \{\partial_{[\rho^i]}, \partial_{[\partial_j\rho^i]} | i = 1, 2, j = 1, ..., n\}$.

Proof. We recall that the normal $N^i_{\boldsymbol{\rho}}$ is given as a normalized cross product of the transformed tangent vectors, cf. (4.39). By the definition of the cross product as formal determinant this implies immediately that the components of the numerator, which we will denote in the following by N^i_N, are of the form

$$a^{3+\alpha}(x)P^n(\widehat{\rho}^i(x), \nabla\widehat{\rho}^i(x)),$$

where the function a only depends on the reference geometry and P^n denotes polynoms in the values of $\widehat{\rho}^i(x)$ and $\nabla\widehat{\rho}^i(x)$ of up to order n which itself does not depend on the point x. This holds then also for the squared norm of N^i_N. As $N^i_{\boldsymbol{\rho}}$ is the composition of $\|N^i_N\|$ and N^i_N with smooth functions this shows the existence of N^{12}. For its partial derivatives at a fixed (x, t) we get by elementary calculation

$$\partial N^{12} = \frac{\sqrt{\|N^1_N\|^2 \cdot \|N^2_N\|^2}\,\partial(N^1_N \cdot N^2_N) - \left(N^1_N N^2_N\right)\frac{\partial(\|N^1_N\|^2 \cdot \|N^2_N\|^2)}{2\sqrt{\|N^1_N\|^2 \cdot \|N^2_N\|^2}}}{\|N^1_N\|^2 \cdot \|N^2_N\|^2}$$

Now, both $\partial(N^1_N \cdot N^2_N)$ and $\partial(\|N^1_N\|^2 \cdot \|N^2_N\|^2)$ are again polynomials of up to order $(n-1)n$ resp. $(n^2-1)n^2$ in $\widehat{\rho}^i$ and $\nabla\widehat{\rho}^i$ and are thus in $C^{3+\alpha,\frac{3+\alpha}{4}}$. Also, due to the Banach algebra property of Hölder-spaces their norms are bounded by powers of the $C^{4+\alpha,\frac{4+\alpha}{4}}$-norm of $\boldsymbol{\rho}$ and so they are bounded by a constant $C(\Gamma_*, R+\varepsilon)$. As before we can control $R + \varepsilon$ by $2R$ and so we choose the constant to be independent of ε. Furthermore, as the functions[7] $f : x \mapsto \frac{1}{x}$ and $g : x \mapsto \sqrt{x}$ are C^∞ on $\mathbb{R}_+$ away from 0, theory for composition operators for Besov spaces (see e.g. [53]) gives us the sought regularity for $\partial\widetilde{N}^{12}$. As the composition operator is also continuous it changes the bound for ∂N^{12} only by constant and so we are done. ☐

[7]Hereby, we formally replace f and g by $C^\infty(\mathbb{R})$- functions $\bar{f}, \bar{g}$ with $f = \bar{f}, g = \bar{g}$ on (ε, ∞) for ε small enough.

We now proceed with the contraction estimate for $\mathfrak{b}^2$. For any $\boldsymbol{u}, \boldsymbol{w} \in X_{R,\delta}^{\boldsymbol{\rho}_0}$ we write

$$\begin{aligned}\mathfrak{b}^2(\boldsymbol{u}) - \mathfrak{b}^2(\boldsymbol{w}) &= \sum_{i=1}^{2} \Theta^i(u^i - w^i) + \sum_{i=1}^{2}\sum_{j=1}^{n} \Theta^i_j \partial_j(u^i - w^i) \\ &\quad + \sum_{i=1}^{2} \Theta^i_0(u^i - w^i) + \sum_{i=1}^{2}\sum_{j=1}^{n} \Theta^i_{j;0} \partial_j(u^i - w^i),\end{aligned}$$

where we used the abbreviations

$$\begin{aligned}\Theta^i &:= \int_0^1 \partial_{[\rho^i]} N^{12}(s\boldsymbol{w} + (1-s)\boldsymbol{u}) - \partial_{[\rho^i]} N^{12}(\boldsymbol{\rho_0}) ds, \\ \Theta^i_j &:= \int_0^1 \partial_{[\partial_j \rho^i]} N^{12}(s\boldsymbol{w} + (1-s)\boldsymbol{u}) - \partial_{[\partial_j \rho^i]} N^{12}(\boldsymbol{\rho_0}) ds, \\ \Theta^i_0 &:= \int_0^1 \partial_{[\rho^i]} N^{12}(s\boldsymbol{\rho}_0) - \partial_{[\rho^i]} N^{12}(0) ds, \\ \Theta^i_{j;0} &:= \int_0^1 \partial_{[\partial_j \rho^i]} N^{12}(s\boldsymbol{\rho_0}) - \partial_{[\partial_j \rho^i]} N^{12}(0) ds.\end{aligned}$$

Hereby, we used that as $\mathcal{B}^2$ is the pointwise linearisation of $N^1_{\boldsymbol{\rho}} \cdot N^2_{\boldsymbol{\rho}}$ we can also write it in terms of ∂N^{12}. Furthermore, all the arising functions Θ are also in $C^{3+\alpha,\frac{3+\alpha}{4}}$ due to the theory of parameter integrals. Additionally, their norms are also bound by a constant $C(\Gamma_*, R)$. As

$$s\boldsymbol{w} + (1-s)\boldsymbol{u}\big|_{t=0} = \boldsymbol{\rho}_0$$

we get $\Theta\big|_{t=0} = 0$ for $\Theta \in \{\Theta^i, \Theta^i_j | i = 1,2, j = 1,...,n\}$ and therefore we conclude

$$\|\Theta\|_{C^{3,0}} \leq \delta^{\frac{\alpha}{4}} \langle\Theta\rangle_{t,\frac{\alpha}{4}} \leq \delta^{\frac{\alpha}{4}} C(\Gamma_*, R),$$

as all derivatives of Θ are at least in $C^{0,\frac{\alpha}{4}}$. Additionally, all Θ inherit the bound (4.122) as it holds uniformly in s. From this we deduce

$$\begin{aligned}\|\Theta D(u^i - w^i)\|_{C^{3+\alpha,\frac{3+\alpha}{4}}} &\leq \|\Theta^i\|_{C^{3+\alpha,\frac{3+\alpha}{4}}} \|D(u^i - w^i)\|_{C^{3,0}} + \|\Theta^i\|_{C^{3,0}} \|D(u^i - w^i)\|_{C^{3+\alpha,\frac{3+\alpha}{4}}} \\ &\leq C(\Gamma_*, R)\|D(u^i - w^i)\|_{C^{3,0}} + C(\Gamma_*, R)\delta^{\frac{\alpha}{4}} \|D(u^i - w^i)\|_{C^{3+\alpha,\frac{3+\alpha}{4}}} \\ &\leq C(\Gamma_*, R)\delta^{\frac{\alpha}{4}} \|\boldsymbol{u} - \boldsymbol{w}\|_{C^{4+\alpha,1+\frac{\alpha}{4}}},\end{aligned}$$

where D denotes the to Θ corresponding differential operator, which is the identity for Θ^i and ∂_j for Θ^i_j. For the Θ^i_0 and $\Theta^i_{j,0}$ we can use that due to the work in the proof of Lemma 4.24 we have Lipschitz continuity[8] of the ∂N^{12} as maps $C^{4+\alpha,1+\frac{\alpha}{4}} \to C^{3+\alpha,\frac{3+\alpha}{4}}$ and this implies then that

$$\|\Theta_0\|_{C^{3+\alpha,\frac{3+\alpha}{4}}} \leq C(\Gamma_*)\|\boldsymbol{\rho}_0\|_{C^{4+\alpha}} \leq C(\Gamma_*)\varepsilon,$$

for $\Theta_0 \in \{\Theta^i_0, \Theta^i_{j,0} | i = 1,2, j = 1,...,n\}$ and consequently

$$\|\Theta_0 D(u^i - w^i)\|_{C^{3+\alpha,\frac{3+\alpha}{4}}} \leq C(\Gamma_*)\varepsilon\|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}},$$

where D denotes again the differential operator matching to the choice of Θ_0. In total we deduce

$$\|\mathfrak{b}^2(\boldsymbol{u}) - \mathfrak{b}^2(\boldsymbol{w})\|_{Y_\delta} \leq (C(\Gamma_*, R)\delta^{\frac{\alpha}{4}} + C(\Gamma_*)\varepsilon)\|\boldsymbol{u} - \boldsymbol{w}\|_{X_{R,\delta}}.$$

For $\mathfrak{b}^3$ we can argue analogously and thus we conclude (4.112) and (4.113).

[8] Note that the Lipschitz constant here is independent of R as we need the Θ_0 only as a function in the initial data!

For $\mathfrak{b}^4$ the highest order terms of $\mathfrak{b}^4(\boldsymbol{u}) - \mathfrak{b}^4(\boldsymbol{w})$ are given by

$$\begin{aligned}&\sum_{i=1}^{3}\sum_{j,k=1}^{n}\gamma^i g^{jk}_{i,\boldsymbol{u}}\left(N^i_{\boldsymbol{u}}\cdot N^i_*\right)\partial_{jk}u^i - \gamma^i g^{jk}_{i,\boldsymbol{w}}\left(N^i_{\boldsymbol{w}}\cdot N^i_*\right)\partial_{jk}w^i - \gamma^i g^{jk}_{i,*}\partial_{jk}(u^i - w^i)\\ +&\sum_{i=1}^{3}\sum_{j,k=1}^{n}\gamma^i g^{jk}_{i,*}\left(N^i_{\boldsymbol{u}}\cdot \tau^i_*\right)\partial_{jk}\mu^i(\boldsymbol{u}) - \gamma^i g^{jk}_{i,*}\left(N^i_{\boldsymbol{w}}\cdot \tau^i_*\right)\partial_{jk}\mu^i(\boldsymbol{w}).\end{aligned}$$

Like before we will discuss the local terms in the first line and the non-local terms (which are actually local on the boundary) in the second line separately. In the first line we add and subtract

$$\sum_{i=1}^{3}\sum_{j,k=1}^{n}\left[\gamma^i g^{jk}_{i,\boldsymbol{u}}\left(N^i_{\boldsymbol{u}}\cdot N^i_*\right) - \gamma^i g^{jk}_{i,*}\right]\partial_{jk}w^i$$

and in the second line we add and subtract

$$\sum_{i=1}^{3}\sum_{j,k=1}^{n}\gamma^i g^{jk}_{i,*}\left(N^i_{\boldsymbol{u}}\cdot \tau^i_*\right)\partial_{jk}\mu^i(\boldsymbol{w}).$$

We then can argue as in the proof of (4.109), using this time Lemma 4.22iii.).
Finally, we have to consider the difference in $\mathfrak{b}^5$ resp. $\mathfrak{b}^6$. In highest order we have for $\mathfrak{b}^5(\boldsymbol{u}) - \mathfrak{b}^5(\boldsymbol{w})$

$$\begin{aligned}&\sum_{j,k=1}^{n} g^{jk}_{1,\boldsymbol{u}}\left(N^1_{\boldsymbol{u}}\cdot N^1_*\right)\partial_{njk}u^1 - g^{jk}_{1,\boldsymbol{w}}\left(N^1_{\boldsymbol{w}}\cdot N^1_*\right)\partial_{njk}w^1 - g^{jk}_{1,*}\partial_{njk}(u^1 - w^1)\\ -&\sum_{j,k=1}^{n} g^{jk}_{2,\boldsymbol{u}}\left(N^2_{\boldsymbol{u}}\cdot N^2_*\right)\partial_{njk}u^2 - g^{jk}_{2,\boldsymbol{w}}\left(N^2_{\boldsymbol{w}}\cdot N^2_*\right)\partial_{njk}w^2 - g^{jk}_{2,*}\partial_{njk}(u^2 - w^2)\\ +&\sum_{jk=1}^{n} g^{jk}_{1,\boldsymbol{u}}\left(N^1_{\boldsymbol{u}}\cdot \tau^1_*\right)\partial_{njk}\mu^1(\boldsymbol{u}) - g^{jk}_{1,\boldsymbol{w}}\left(N^1_{\boldsymbol{w}}\cdot \tau^1_*\right)\partial_{njk}\mu^1(\boldsymbol{w})\\ -&\sum_{jk=1}^{n} g^{jk}_{2,\boldsymbol{u}}\left(N^2_{\boldsymbol{u}}\cdot \tau^2_*\right)\partial_{njk}\mu^2(\boldsymbol{u}) - g^{jk}_{2,\boldsymbol{w}}\left(N^2_{\boldsymbol{w}}\cdot \tau^2_*\right)\partial_{njk}\mu^2(\boldsymbol{w}).\end{aligned}$$

Now by adding and subtracting the terms

$$\begin{aligned}&\sum_{j,k=1}^{n} g^{jk}_{1,\boldsymbol{u}}\left(N^1_{\boldsymbol{u}}\cdot N^1_*\right)\partial_{njk}(u^1 - w^1), -\sum_{j,k=1}^{n} g^{jk}_{2,\boldsymbol{u}}\left(N^2_{\boldsymbol{u}}\cdot N^2_*\right)\partial_{njk}(u^2 - w^2),\\ &\sum_{j,k=1}^{n} g^{jk}_{1,\boldsymbol{u}}\left(N^1_{\boldsymbol{u}}\cdot \tau^1_*\right)\partial_{njk}\mu^1(\boldsymbol{w}), -\sum_{j,k=1}^{n} g^{jk}_{2,\boldsymbol{u}}\left(N^2_{\boldsymbol{u}}\cdot \tau^2_*\right)\partial_{njk}\mu^2(\boldsymbol{w}),\end{aligned}$$

in this order to the four lines, using Lemma 4.22iii.)we can argue as before to get the sought estimate. The difference $\mathfrak{b}^6(\boldsymbol{u}) - \mathfrak{b}^6(\boldsymbol{w})$ is dealt with in the same way and then (4.117) is a direct consequence of the previous results. □

From this we may now easily conclude that Λ is a contraction mapping for suitable ε and δ.

Corollary 4.25 (Contraction property of Λ)**.**
There is an $\varepsilon_0 < \min(1, \varepsilon_W)$ with the following property: for any $R > 1$ and $\varepsilon < \varepsilon_0$ there is a $\delta(R,\varepsilon) > 0$ such that

$$\Lambda : X^{\boldsymbol{\rho}_0}_{R,\delta} \to C^{4+\alpha,1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,\delta})$$

is a $\frac{1}{2}$-contraction.

Proof. Let $\boldsymbol{v}, \boldsymbol{w} \in X_{R,\delta}^{\boldsymbol{\rho}_0}$. We observe that $\Lambda(\boldsymbol{v}) - \Lambda(\boldsymbol{w})$ solves (4.41) with

$$(\mathfrak{f}, \mathfrak{b}) = S(\boldsymbol{v} - \boldsymbol{w}), \quad \boldsymbol{u}_0 \equiv 0.$$

Suppose now that $\varepsilon, \delta < 1$. Then, we can apply the energy estimate (4.45) together with Lemma 4.23 to derive

$$\|\Lambda(\boldsymbol{v}) - \Lambda(\boldsymbol{w})\|_{X_{R,\delta}} \leq \left(C_1(\Gamma_*, R)\delta^{\bar{\alpha}} + C_2(\Gamma_*)\varepsilon\right) \|\boldsymbol{v} - \boldsymbol{w}\|_{X_{R,\delta}}, \tag{4.123}$$

with suitable constants $C_1(\Gamma_*, R)$ and $C_2(\Gamma_*)$. Now choosing

$$\varepsilon \leq \frac{1}{4_2 C(\Gamma_*)}, \quad \delta \leq \left(\frac{1}{4C_1(\Gamma_*, R)}\right)^{\bar{\alpha}^{-1}}, \tag{4.124}$$

we get the sought property for Λ. $\square$

Finally, we need that Λ is also a self-mapping, which we can guarantee as long as R is sufficiently large.

Lemma 4.26 (Self-mapping property of Λ)**.**
For given $R > 1$ and $\varepsilon < \varepsilon_0$ let $\delta(R, \varepsilon)$ be chosen as in Corollary 4.25. There is an $R_0 > 0$ such that for all $R > R_0$ the map Λ is a self-mapping on $X_{R,\delta}^{\boldsymbol{\rho}_0}$.

Proof. For any $\boldsymbol{u} \in X_{R,\delta}^{\boldsymbol{\rho}_0}$ we get

$$\begin{aligned}
\|\Lambda(\boldsymbol{u}) - \boldsymbol{\rho}_0\|_{C_{TJ}^{4+\alpha, \frac{1+\alpha}{4}}(\Gamma_{*,\delta})} &\leq \|\Lambda(\boldsymbol{u}) - \Lambda(\boldsymbol{\rho}_0)\|_{C_{TJ}^{4+\alpha, \frac{1+\alpha}{4}}(\Gamma_{*,\delta})} + \|\Lambda(\boldsymbol{\rho}_0) - \boldsymbol{\rho}_0\|_{C_{TJ}^{4+\alpha, \frac{1+\alpha}{4}}(\Gamma_{*,\delta})} \\
&\leq \frac{R}{2} + \|\Lambda(\boldsymbol{\rho}_0) - \boldsymbol{\rho}_0\|_{C_{TJ}^{4+\alpha, \frac{1+\alpha}{4}}(\Gamma_{*,\delta})},
\end{aligned}$$

where we used that Λ is a $\frac{1}{2}$-contraction on $X_{R,\delta}^{\boldsymbol{\rho}_0}$. As we want to R to be independent of $\boldsymbol{\rho}_0$ we now want to find an estimate for the second summand that is uniformly in ε. For this, we note now that the function $\boldsymbol{w} := \Lambda(\boldsymbol{\rho}_0) - \boldsymbol{\rho}_0$ solves the system

$$\begin{aligned}
\partial_t w^i &= \mathcal{A}_{all}^i w^i + \mathfrak{f}_0^i(\boldsymbol{\rho}_0) && \text{on } \Gamma_{*,\delta}^i, i = 1, 2, 3, \\
\mathcal{B}\boldsymbol{w} &= 0 && \text{on } \Sigma_{*,\delta}, \\
w^i\big|_{t=0} &= 0 && \text{on } \Gamma_*^i, i = 1, 2, 3,
\end{aligned} \tag{4.125}$$

with the inhomogeneity

$$\mathfrak{f}_0^i(\boldsymbol{\rho}_0) := \Delta_{\boldsymbol{\rho}_0} H_{\boldsymbol{\rho}_0}^i. \tag{4.126}$$

We want to give a short explanation of this. As $\Lambda(\boldsymbol{\rho}_0)$ and $\boldsymbol{\rho}_0$ have the same initial data their difference vanishes at $t = 0$. The boundary inhomogeneity operator from (4.101) does not explicitly depend on the time and so we get

$$\mathcal{B}(\Lambda(\boldsymbol{\rho}_0))(t) = \mathcal{B}(\Lambda(\boldsymbol{\rho}_0))(0) \quad \forall t \in [0, \delta]. \tag{4.127}$$

Furthermore, due to the compatibility conditions (4.16) for $\boldsymbol{\rho}_0$ we get

$$\mathcal{B}(\Lambda(\boldsymbol{\rho}_0))\big|_{t=0} = \mathcal{B}(\boldsymbol{\rho}_0)\big|_{t=0}. \tag{4.128}$$

Combining (4.127) and (4.128) we derive $(4.125)_2$.
Finally, recalling (4.100) we see that

$$\mathfrak{f}^i(\boldsymbol{\rho}_0) = \Delta_{\boldsymbol{\rho}_0} H_{\boldsymbol{\rho}_0}^i - \mathcal{A}_{all}^i(\rho_0^i, \boldsymbol{\rho}_0\big|_{\Sigma_*}) \tag{4.129}$$

Additionally, $\boldsymbol{\rho}_0$ solves $(4.41)_1$ with $\mathfrak{f}^i = \mathcal{A}^i_{all}(\boldsymbol{\rho}^i_0, \boldsymbol{\rho}\big|_{\Sigma_*})$. Together this shows now also $(4.125)_1$.
We now can apply Theorem 4.7 as due to the geometric compatibility conditions (4.16) the condition (4.42) is fulfilled for $\mathfrak{f}^i_0(\boldsymbol{\rho}_0)$ and therefore get using that (4.43)

$$\|\boldsymbol{w}\|_{C_{TJ}^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,\delta})} \leq C \sum_{i=1}^{3} \|\mathfrak{f}^i_0(\boldsymbol{\rho})\|_{C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma^i_{*,\delta})} \leq C'(\varepsilon).$$

Here, we used that $\Delta_{\boldsymbol{\rho}} H_{\boldsymbol{\rho}}$ depends continuously on derivatives of up to order four, cf. Section4.2. This leads us now to

$$\|\Lambda(\boldsymbol{u}) - \boldsymbol{\rho}_0\|_{C_{TJ}^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,\delta})} \leq \frac{R}{2} + C'(\varepsilon).$$

By choosing $R_0 > 2C'(\varepsilon)$ we get $\Lambda(\boldsymbol{u}) \in X^{\boldsymbol{\rho}_0}_{R,\delta}$ for all $R > R_0$ and so this shows that Λ is a self-mapping. $\square$

Proof. (of Proposition 4.20) Lemma 4.21 guarantees well-definedness of Λ. Then, Corollary 4.25 and Lemma 4.26 show that Λ is a self-mapping on $X^{\boldsymbol{\rho}_0}_{R,\delta}$ and a $\frac{1}{2}$-contraction. We then can apply Banach's fixed-point theorem to get the existence of a unique fixed-point of Λ in $X^{\boldsymbol{\rho}_0}_{R,\delta}$. $\square$

From this we conclude immediately Theorem 4.2. Observe that due to our efforts in the proof of Lemma 4.26 the existence time and the bound for the solution by the Radius R is uniformly in $\boldsymbol{\rho}_0$. This finishes our study on well-posedness of the problem and we now move on to the qualitative behaviour near stationary solutions.

5

Stability of the Surface Diffusion Flow of Closed Hypersurfaces

In the next two chapters we aim to derive global existence and stability results for the surface diffusion flow near stationary solutions. In this chapter we will consider the case of closed hypersurfaces. For this situation stability results are already well know, e.g. [23]. But the techniques used there are not suitable for the geometric difficulties of triple junctions in higher space dimensions. We will verify the results using an approach with a Łojasiewicz-Simon gradient inequality.

5.1 Technical Problems Proving a Łojasiewicz-Simon Gradient Inequality: Setting for $\mathcal{H}^{-1}$-flows, Banach Space Settings, Interpolation Problems

In 1962 Stanisław Łojasiewicz found in his study of analytic functions an estimate for the distance of a point to the set of the roots of the function. In the same work [41] he realised that his inequality can be used to show linear stability of equilibria of ordinary differential equations with a gradient flow structure. This idea was used by Simon in [51] to show the same results for PDEs with gradient flow structure and since then a lot of authors used this idea. The big advantage of this method is that one does not need to have a detailed knowledge of the set of equilibria and the stability argument is straight forward once a Łojasiewicz-Simon gradient inequality, which we will abbreviate in the remaining work with LSI, is proven. A very important and general result on this is [13], wherein the author showed that the critical condition of analyticity of the first derivative of the energy functional (even C^∞ is not enough to guarantee a LSI) needs only to hold on a subset called the critical manifold, which in most application is of finite dimension. Nevertheless, most authors use [13, Corollary 3.11], for whose application most authors normally verify that the first derivative is analytic (on the whole space!) and the second derivative evaluated in the equilibrium is a Fredholm operator of index 0. Hereby it is often suggested that the Hilbert structure is essential (see especially [15]) which strongly restricts the general result of [13]. But actually, this is not the case and in [25] an overview of different results about LSIs is given, from which we will use the following for our work, where we use the notation of [13] to state the result.

Proposition 5.1 (Feehan, Maridakis, 2015)**.**
Let V, W be real Banach spaces with continuous embeddings $V \subset W$ and $T : W \hookrightarrow V'$ such that the embedding $j : V \hookrightarrow V'$ is a definite embedding, that is the bilinear form $V \times V \to \mathbb{R}, (x, y) \mapsto j(y)(x)$ is definite.

Let $U \subset V$ be an open subset, $E : U \to \mathbb{R}$ a C^2-function such that $\mathcal{M} = E'$ is real analytic as a map $U \to W$. Furthermore, suppose that $x_\infty \in U$ is a critical point of E and $\mathcal{L} = E''(x_\infty)$ is a Fredholm operator with index zero as mapping $V \to W$. Then, there are constants $C \in (0,\infty), \sigma \in (0,1]$ and $\theta \in (0, \frac{1}{2}]$ such that for all $x \in U$ with

$$\|x - x_\infty\|_V \leq \sigma,$$

we have

$$|E(x) - E(x_\infty)|^{1-\theta} \leq C\|\mathcal{M}(x)\|_W. \tag{5.1}$$

Remark 5.2 ($\mathcal{M}$ and $\mathcal{L}$ as maps with values in W)**.**
We want to note that if we consider $\mathcal{M}$ and $\mathcal{L}$ as maps with values in W this is not completely precise. Formally, we are considering $T^{-1} \circ \mathcal{M}$ and $T^{-1} \circ \mathcal{L}$. When we introduce our setting later we will explain how the operators should read.

So, the only prerequisite we need, to use the standard procedure, is definiteness of the embedding $V \rightharpoonup V'$ induced by the choice of the embedding $W \rightharpoonup V'$ in [13, Hypothesis 3.4i)]. In our situation - and we expect this to be true for any application, that is motivated by a gradient flow structure - the embedding is induced by a inner product and in this case definiteness of the embedding follows from definiteness of the product.

Now we want to discuss how to choose the spaces V and W. For the application of the LSI to prove stability we need W to be corresponding to the gradient flow setting. In our situation, we have an H^{-1}-gradient. Recall that for a Hilbert space H and a linear map $F : H \to \mathbb{R}$, we have for any $x \in H$ that

$$\nabla_H F(x) \in H, \quad F'(x) \in H'.$$

From this we conclude that $\mathcal{M}(x)$ needs to be in the dual space of H^{-1} that is by reflexivity of the Sobolev spaces just H^1. Therefore, we expect W to be H^1. As we need $\mathcal{L}(0)$ to be a Fredholm operator the space V is induced naturally by W and the order of $\mathcal{L}(0)$. Up to lower order perturbation the second variation of the surface are is given by the Laplacian and so the expected space for V is H^3. We want to note that there are also other possible choices. For example, one could also work with $V = H^1$ and $W = H^{-1}$. This is linked to the formulation (3.3) and in some sense it is more natural. $\mathcal{M}$ will then be the negative Laplacian of the mean curvature which gives us a connection to the surface diffusion flow. But then we get in higher space dimensions surfaces that are only varifolds which brings new difficulties.

So far we have not used the fact that [13] is a general result in a Banach space setting. But as we just mentioned regularity of the surfaces is a more critical issue compared to the situation of the Wilmore flow. There, the operator $\mathcal{L}$ is a fourth order differential operator and as the Wilmore flow is a L^2-gradient flow this yields $V = H^4$ and thus $W = L^2$. Additionally, the works cited above restrict to curves and surfaces and for this space dimensions the involved Sobolev spaces will have Banach algebra structure and embed into C^2. As we want to work in arbitrary space dimensions we cannot guarantee this for H^3 any more and so we have to do a modification. Our first idea hereby was to choose the setting $V = W^{3,p}$ and $W = W^{1,p}$ for sufficiently large p such that the two properties mentioned hold for V. As we worked for our short time existence in a Hölder setting no additional regularity theory is needed. In the end this method has a problem in the application. Although we can prove a LSI in this setting yields an estimate by the L^p-norm of $\nabla_\Gamma H_\Gamma$. But we need an estimate by its L^2-norm as this is the quantity associated to the gradient flow structure of the surface diffusion flow[1]. This forces us to use interpolation arguments. Due to the Hölder-inequality we have for any $q > p$ that

$$\|\nabla_\Gamma H_\Gamma\|_{L^p(\Gamma)} \leq \|\nabla_\Gamma H_\Gamma\|_{L^2(\Gamma)}^{1-\bar{\theta}} \cdot \|\nabla_\Gamma H_\Gamma\|_{L^q(\Gamma)}^{\bar{\theta}}, \tag{5.2}$$

[1]Recall that $\|\nabla_\Gamma H_\Gamma\|_{L^2} = \| - \Delta_\Gamma H_\Gamma\|_{\mathcal{H}^{-1}}$!

where the interpolation exponent $\bar{\theta}$ is given by

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}. \tag{5.3}$$

In principal, this would lead to the desired bound as we have bounds for $\|\nabla_\Gamma H_\Gamma\|^{\bar{\theta}}_{L^q(\Gamma)}$ due to parabolic smoothing for the surface diffusion flow. But we see that as $q \to \infty$ we get

$$1 - \bar{\theta} \to \frac{2}{p}. \tag{5.4}$$

That means that we have an upper bound for $1-\bar{\theta}$. But if we apply this interpolation procedure on the right-hand side of (5.1) this increases the Łojasiewicz-Simon exponent $1-\theta$ by $\frac{1}{1-\bar{\theta}}$ and so by $\frac{p}{2}$. We then cannot guarantee that the LSI exponent is less than one. But we need this property to apply our stability argument and so this is not a possible solution. Using embedding theory for Besov spaces we were at least able to work with submanifolds in $\mathbb{R}^3$. But as we want to have results for any space dimension we have to find a new approach.
We then decided not to modify integrability but differentiability. The advantage in this approach is that on the scale of differentiability the interpolation exponent will get as good as we need it as long we can guarantee parabolic regularization and C^k-estimates for arbitrary large k, which again follow from parabolic smoothing properties of our flow.
One might now argue that one does not need a Banach space setting after all but we want to note the only problem in our situation are the interpolation properties of Sobolev spaces. There might be other situations where this idea is useful.

Remark 5.3 (The situation for the volume preserving mean curvature flow)**.**
Although we do not discuss it here we want to mention that our work basically shows also stability for the case of volume preserving mean curvature flow. One can use the same LSI and only has to see that one also has parabolic regularization for this flow which we expect to be true.

Now we use the discussed idea to prove the stability result from [23]. Precisely, we show the following:

Theorem 5.4 (Stability of spheres with respect to the surface diffusion flow)**.**
Let $\alpha > 0$. Any closed sphere Γ_* in $\mathbb{R}^{n+1}$ is stable with respect to surface diffusion flow in the following sense:

i.) The stationary solution $\rho \equiv 0$ of $(SDFC)$ is Lyapunov stable with respect to the $C^{4+\alpha}$-norm.

ii.) There is an $\varepsilon_S > 0$ such that for all initial data ρ_0 with $\|\rho_0\|_{C^{4+\alpha}(\Gamma_*)} \leq \varepsilon_S$ the solution of $(SDFC)$ converges for $t \to \infty$ to some $\rho_\infty \in C^{4+\alpha}(\Gamma_*)$. Furthermore, Γ_{ρ_∞} is a (possibly different) sphere enclosing the same volume like Γ_*.

5.2 Parametrisation of Volume Preserving Hypersurfaces and the Surface Energy

For the rest of this chapter let Γ_* be a closed sphere. We cover Γ_* with two parametrisations $\varphi_{1,2} : \widehat{Z}_{1,2} \to Z_{1,2} \subset \Gamma_*$, where we assume for technical reasons that

$$\det(g(p)) > \frac{1}{2} \quad \forall p \in Z_1, Z_2. \tag{5.5}$$

One can construct such parametrisations for example by using the stereographic projection.
As in Chapter 3 we denote by $\mathcal{MH}^n$ the Banach manifold of all closed, embedded, orientable hypersurfaces in $\mathbb{R}^{n+1}$ of class C^2, which was studied in [49, Section 2.4]. The natural energy functional E would now be the surface area of a surface $\Gamma \in \mathcal{MH}^n$. But as we need to have a suitable Banach space to work with, we parametrise as before the elements of $\mathcal{MH}^n$ via distance functions over Γ_*. But spheres will only be minimizers of the surface energy if we fix the enclosed volume, which is

guaranteed by the isoperimetric inequality. This is a non-linear constraint that we have to write in a Banach space using a suitable parametrisation. Such one is constructed in the following lemma.

Lemma 5.5 (Parametrisation of volume-preserving distance functions)**.**

i.) *There exists a neighbourhood $\widetilde{U}$ of $0 \in C^2_{(0)}(\Gamma_*)$ and a neighbourhood $\widehat{U}$ of $0 \in C^2(\Gamma_*)$ together with a unique C^2-diffeomorphism $\gamma := (\mathrm{Id} + \bar{\gamma}) : \widetilde{U} \to \widehat{U}$, where $\bar{\gamma}(\rho)$ is constant for all $\rho \in \widetilde{U}$, such that $\Gamma_{\gamma(\rho)}$ encloses the same volume as Γ_*.*

ii.) *For the first and second derivative of the function $\bar{\gamma}$ we have*

$$\bar{\gamma}'(\rho_0)\rho = -\frac{\int_{\bar{\gamma}(\rho_0)} \rho(N_{\Gamma_{\bar{\gamma}(\rho_0)}} \cdot N_*) d\mathcal{H}^n}{\int_{\Gamma_{\bar{\gamma}(\rho_0)}} N_{\bar{\gamma}(\rho_0)} \cdot N_* d\mathcal{H}^n},$$

$$\bar{\gamma}''(0)(\rho, \bar{\rho}) = \mathit{Area}(\Gamma_*)^{-1} \int_{\Gamma_*} H\rho\bar{\rho} d\mathcal{H}^n,$$

for all $\rho_0 \in \widetilde{U}$ and $\rho, \bar{\rho} \in C^2_{(0)}(\Gamma_*)$.

Proof. Note that $C^2_{(0)}(\Gamma_*)$ is a complemented subspace of $C^2(\Gamma_*)$ and a projection is given by

$$P : C^2(\Gamma_*) \to C^2_0(\Gamma_*), v \mapsto v - ⨍_{\Gamma_*} v d\mathcal{H}^n,$$

whose kernel is given by the subspace K of constant functions. Thus, we can write $C^2(\Gamma_*)$ as direct sum $C^2_{(0)}(\Gamma_*) \oplus K$. We consider the functional

$$\begin{aligned} G : C^2(\Gamma_*) &\to \mathbb{R}, \\ \rho &\mapsto \mathrm{Vol}(\Omega_\rho) - \mathrm{Vol}(\Omega_*). \end{aligned}$$

We want to apply the implicit function theorem on G and therefore we write $\rho \in C^2(\Gamma_*)$ as $\rho_0 + \rho_1 \in C^2_{(0)}(\Gamma_*) \oplus K$. By Reynold's transport theorem we have for all $\bar{\rho}_1 \in K$ that

$$\partial_2 G(\rho)\bar{\rho}_1 = \int_{\Gamma_\rho} \bar{\rho}_1 N_{\Gamma_\rho} \cdot N_* d\mathcal{H}^n,$$

$$\partial_2 G(0)\bar{\rho}_1 = \int_{\Gamma_*} \bar{\rho}_1 d\mathcal{H}^n = \bar{\rho}_1 \mathrm{Vol}(\Omega^*),$$

as the elements of K are constant functions. Hence, $\partial_2 G(0) : Z \to \mathbb{R}$ is non-zero and thus bijective as it is a linear map from an one dimensional vector space in another. We note that existence of the first variation guarantees Fréchet differentiability as G restricted on K is a map between one-dimensional Banach spaces. Furthermore, $\partial_2 G$ is continuous on a neighbourhood of $(0, 0)$ as both the unit normal N_{Γ_ρ} and the metric tensor are continuous functions $C^2(\Gamma_*) \to C^1(\Gamma_*)$ and additionally the integral operator $\int : C_0(\Gamma_*) \to \mathbb{R}$ is continuous. The function G can be written as

$$G(\rho) = \frac{1}{n+1} \int_{\Gamma_h} \mathrm{Id} \cdot N_\rho d\mathcal{H}^n = \frac{1}{n+1} \int_{\Gamma_*} (\mathrm{Id} \cdot N_\rho) J(\rho) d\mathcal{H}^n. \tag{5.6}$$

In Lemma 5.9 we will see that N_ρ and J_ρ are analytic in ρ and so G is. Thus, we can apply the implicit function theorem [55, Theorem 4.B] to get the existence of a neighbourhood $\widetilde{U} \subset C^2_{(0)}(\Gamma_*)$ of 0 together with a unique, analytic function $\bar{\gamma} : \widetilde{U} \to K$ with $G(\gamma(\rho)) = G(\rho, \bar{\gamma}(\rho)) = 0$ for all $\rho \in \widetilde{U}$. For the first order Fréchet derivative of γ we first note that we can identify K with $\mathbb{R}$ via the obvious linear isomorphism and as this does not influence the derivative we will consider γ as such map. Then, the method of implicit derivatives yields

$$0 = \frac{d}{d\rho}(G(\cdot, \gamma(\cdot)))(\rho_0) = \int_{\Gamma_{\gamma(\rho_0)}} (\rho + \bar{\gamma}'(\rho_0)\rho) N_* \cdot N_{\gamma(\rho_0)} d\mathcal{H}^n,$$

and by rearranging terms we conclude

$$\bar{\gamma}'(\rho_0)\rho = -\frac{\int_{\Gamma_{\gamma(\rho_0)}} \rho(N_{\Gamma_{\gamma(\rho_0)}} \cdot N_*) d\mathcal{H}^n}{\int_{\Gamma_{\gamma(\rho_0)}} N_{\Gamma_{\gamma(\rho_0)}} \cdot N_* d\mathcal{H}^n}, \quad \rho_0 \in U, \rho \in C^2_{(0)}(\Gamma_*). \tag{5.7}$$

For the second order Fréchet derivatives we can - again considering $\bar{\gamma}'(\rho_0)\rho$ as an element in $\mathbb{R}$ - apply the quotient rule and get

$$\begin{aligned}\gamma''(0)\rho\bar{\rho} &= -\frac{\int_{\Gamma_*} 1 d\mathcal{H}^n (\int_{\Gamma_*} \frac{D}{D\varepsilon}(\rho(N_{\Gamma_{\gamma(\varepsilon\bar{\rho})}} \cdot N_{\Gamma_*})\big|_{\varepsilon=0} - H\rho\bar{\rho}) d\mathcal{H}^n - \int_{\Gamma_*} \rho d\mathcal{H}^n \frac{d}{d\varepsilon} F(\varepsilon\rho_0)\big|_{\varepsilon=0}}{(\int_{\Gamma_*} 1 d\mathcal{H}^n)^2} \\ &= \mathrm{Area}^{-1}(\Gamma_*) \int_{\Gamma_*} H_* \rho\bar{\rho} d\mathcal{H}^n,\end{aligned}$$

for all $\rho, \bar{\rho} \in C^2_{(0)}(\Gamma_*)$, where we used the abbreviation

$$F(\rho_0) := \int_{\Gamma_{\gamma(\rho_0)}} N_{\Gamma_{\gamma(\rho_0)}} \cdot N_* d\mathcal{H}^n,$$

the fact that we have $\int_{\Gamma_*} \rho d\mathcal{H}^n = 0$ and that we have

$$\frac{D}{D\varepsilon}(\rho(N_{\Gamma_{\gamma(\bar{\rho})}} \cdot N_*)\Big|_{\varepsilon=0} = \rho \left(\frac{D}{D\varepsilon} N_{\Gamma_{\gamma(\bar{\rho})}} \right)\Bigg|_{\varepsilon=0} \cdot N_* = \rho \nabla_{\Gamma_*} \bar{\rho} \cdot N_* = 0,$$

as at $\rho_0 = 0$ variation in any direction corresponds to a purely normal movement at time zero and hN_* is constant along purely normal evolution. □

Remark 5.6 (Choice of the space V).
For the proof of the LSI we will work with Sobolev spaces that embed into $C^2(\Gamma_*)$ and thus we can use the result from the previous lemma. As we additionally want to have that V is a Banach algebra we choose $V = H^k_{(0)}(\Gamma_*)$ with $k > \max(\frac{n}{2}, 2 + \frac{n}{2}) = 2 + \frac{n}{2}$, which guarantees both properties due to Proposition 2.9. Additionally, we assume for technical reasons in the stability analysis k larger than 5.

We are now able to define the energy functionals $\widehat{E}$ and $\widetilde{E}$ via

$$\widehat{E} : \widehat{U} \to \mathbb{R}, \widehat{E}(\rho) := \mathrm{Area}(\Gamma_\rho), \tag{5.8}$$

$$\widetilde{E} : \widetilde{U} \to \mathbb{R}, \widetilde{E}(\rho) := \widehat{E}(\gamma(\rho)), \tag{5.9}$$

where we redefined the neighbourhoods $\widetilde{U} \subset H^k_{(0)}(\Gamma_*)$ and $\widehat{U} \subset H^k(\Gamma_*)$ of zero. Before we move on showing the prerequisites for [13, Corollary 3.11], we calculate the first derivative of $\widetilde{E}$ on $\widetilde{U}$ and the second derivative in 0, which we will need later.

Lemma 5.7 (Derivatives for $\widetilde{E}$).
For all $\rho_0 \in \widetilde{U}$ and $\rho, \bar{\rho} \in H^k_{(0)}(\Gamma_*)$ we have

$$\begin{aligned}\widetilde{E}'(\rho_0)\rho &= -\int_{\Gamma_{\gamma(\rho_0)}} H_{\gamma(\rho_0)}(\rho + \bar{\gamma}'(\rho_0)\rho)(N_{\gamma(\rho_0)} \cdot N_*) d\mathcal{H}^n, \\ &= \int_{\Gamma_*} \left(H_{\gamma(\rho_0)} - \frac{\int_{\Gamma_*} H_{\gamma(\rho_0)}(N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)} d\mathcal{H}^n}{\int_{\Gamma_*} N_{\Gamma_{\gamma(\rho_0)}} \cdot N_* J_{\gamma(\rho_0)} d\mathcal{H}^n} \right) (N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)} \rho d\mathcal{H}^n, \\ \widetilde{E}''(0)\rho\bar{\rho} &= \int_{\Gamma_*} (-\Delta_{\Gamma_*}\rho)\bar{\rho} - |II_*|^2 \rho\bar{\rho} d\mathcal{H}^n.\end{aligned}$$

Proof. We write $\widetilde{E} = \widehat{E} \circ \gamma$ and use the chain rule to get

$$\widetilde{E}'(\rho_0)h = \widehat{E}'(\gamma(\rho_0))\gamma'(\rho_0)\rho = \widehat{E}'(\gamma(\rho_0))(\rho + \bar{\gamma}'(\rho_0)\rho).$$

The well-know formula for the first variation of the surface area (see e.g. [16, Lemma 2.46]),

$$\frac{d}{dt}\int_{\Gamma(t)} 1 d\mathcal{H}^n = -\int_{\Gamma(t)} VHd\mathcal{H}^n, \tag{5.10}$$

implies the first identity for $\widetilde{E}'(\rho_0)$. For the second line we observe that as $\bar{\gamma}'(\rho_0)\rho$ is constant, we get

$$\begin{aligned}\int_{\Gamma_{\gamma(\rho_0)}} (\bar{\gamma}'(\rho_0)\rho)H_{\gamma(\rho_0)}(N_{\gamma(\rho_0)}\cdot N_*)d\mathcal{H}^n &= \bar{\gamma}'(\rho_0)\rho\int_{\Gamma_*} H_{\gamma(\rho_0)}(N_{\gamma(\rho_0)}\cdot N_*)d\mathcal{H}^n \\ = -\int_{\Gamma_*}\rho(N_{\gamma(\rho_0)}\cdot N_*)&\left(\frac{\int_{\Gamma_*} H_{\gamma(\rho_0)}(N_{\gamma(\rho_0)}\cdot N_*)J_{\gamma(\rho_0)}d\mathcal{H}^n}{\int_{\Gamma_{\gamma(\rho_0)}} N_{\gamma(\rho_0)}\cdot N_* d\mathcal{H}^n}\right) J_{\bar{\gamma}(\rho_0)}d\mathcal{H}^n\end{aligned} \tag{5.11}$$

From this we directly get the second identity for $\widetilde{E}'$.
For $\widetilde{E}''$ we observe that the first derivative has a product structure and thus we see

$$\widetilde{E}''(0)\rho\widetilde{\rho} = \left[\widehat{E}''(0)\gamma'(0)\bar{\rho}\right]\gamma'(0)\rho + \widehat{E}'(0)\gamma''(\rho)\bar{\rho}, \tag{5.12}$$

where we used that $\gamma(0) = 0$. For the second variation of $\widehat{E}$ we apply again the surface transport theorem to see

$$\begin{aligned}\widehat{E}''(0)\rho\bar{\rho} &= -\int_{\Gamma_*}\frac{D}{D\varepsilon}\left(H_{\varepsilon\bar{\rho}}\rho(N_{\Gamma_{\varepsilon\bar{\rho}}}\cdot N_*)\right)\big|_{\varepsilon=0} - H_*^2\rho\bar{\rho}d\mathcal{H}^n \\ &= \int_{\Gamma_*} H_*^2\rho\bar{\rho} - \rho(\Delta_{\Gamma_*}\bar{\rho} + (II_*)^2\bar{\rho})d\mathcal{H}^n.\end{aligned} \tag{5.13}$$

In the second equality we used the normal time derivative of the mean curvature operator (3.11) and as before the fact that the material derivative of the other factor vanishes. Due to (5.5)i) we have that $\gamma'(0)$ is the identity and together with ii) and (5.12) we obtain

$$\begin{aligned}\widetilde{E}''(0)\rho\bar{\rho} &= \int_{\Gamma_*} H_*^2\rho\bar{\rho} - \rho\Delta_{\Gamma_*}\bar{\rho} - (II_*)^2\rho\bar{\rho} - \underbrace{H_*\mathrm{Area}(\Gamma_*)^{-1}\left(\int_{\Gamma_*} H_*\rho\bar{\rho}d\mathcal{H}^n\right)}_{\text{constant on } \Gamma_*} d\mathcal{H}^n \\ &= \int_{\Gamma_*} H_*^2\rho\bar{\rho} - (\Delta_{\Gamma_*}\rho)\bar{\rho} + (II_*)^2\rho\bar{\rho} - H_*^2\rho\bar{\rho}d\mathcal{H}^n \\ &= \int_{\Gamma_*}(-\Delta_{\Gamma_*}\rho)\bar{\rho} - (II_*)^2\rho\bar{\rho}d\mathcal{H}^n.\end{aligned} \tag{5.14}$$

□

5.3 Proof of the Łojasiewicz-Simon Gradient Inequality

We now want to prove the LSI for $\widetilde{E}$ in 0, which corresponds to the stationary solution Γ_*. Hereby, we choose the setting $V := H^k_{(0)}(\Gamma_*), W := H^{k-2}_{(0)}(\Gamma_*)$ with k chosen as in Remark 5.6 and

$$T : W \hookrightarrow V' : \rho \mapsto \left(\bar{\rho} \to \int_{\Gamma_*}\rho\bar{\rho}d\mathcal{H}^n\right). \tag{5.15}$$

Note that by Remark 5.2 and Lemma 5.7 this means that we will consider $\widetilde{E}'$ and $\widetilde{E}''(0)$ as maps with values in W in the following sense:

$$\widetilde{E}'(\rho_0) = \left(H_{\gamma(\rho_0)} - \frac{\int_{\Gamma_*} H_{\gamma(\rho_0)}(N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)} d\mathcal{H}^n}{\int_{\Gamma_*} N_{\Gamma_{\gamma(\rho_0)}} \cdot N_* J_{\gamma(\rho_0)} d\mathcal{H}^n} \right) (N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)}, \quad \rho_0 \in V, \tag{5.16}$$

$$\widetilde{E}''(0)\rho = -\Delta_{\Gamma_*}\rho - |II_*|^2\rho, \quad \rho \in V. \tag{5.17}$$

With the general strategy of section 5.1 in mind we begin by showing the Fredholm property of $\widetilde{E}''(0)$ and verify afterwards analyticity of $\widetilde{E}'$.

Lemma 5.8 (Fredholm property of $\widetilde{E}''$)**.** $\widetilde{E}''(0) : V \to W$ *is a Fredholm operator of index 0.*

Proof. For $f \in L^2_{(0)}(\Gamma_*)$ the heat equation

$$-\Delta_{\Gamma_*} u = f \text{ on } \Gamma_*$$

admits a unique weak solution in $H^1_{(0)}(\Gamma_*)$, which follows from the standard Lax-Milgram procedure as we have a Poincaré-inequality on $H^1_{(0)}(\Gamma_*)$, see [35, Theorem 2.10]. For $f \in H^{k-2}_{(0)}(\Gamma_*)$ we can apply on the localized problem elliptic regularity theory (see e.g. [33]) and by the same procedure as in the proof of Proposition 3.8 that shows that $(-\Delta_{\Gamma_*})^{-1}$ is a continuous operator $W \to V$. From this we conclude that $-\Delta_{\Gamma_*} : V \to W$ is a continuous isomorphism and thus a Fredholm operator of index zero. On the other hand, the multiplication $\rho \mapsto -(II_*)^2\rho$ is obviously a compact operator $V \to W$. So, $\widetilde{E}''(0)$ is a compact perturbation of a Fredholm operator of index zero and by Propostion 2.6 $\widetilde{E}''(0)$ has the same property. □

Lemma 5.9 (Analyticity of $\widetilde{E}'$)**.**
Let $\varphi : \widehat{Z} \to Z$ be one of the two local coordinates chosen in 5.2. Then, the following maps are analytic on sufficiently small neighbourhoods $0 \in \widetilde{U} \subset H^k_{(0)}(\Gamma_*)$ resp. $0 \in \widehat{U} \subset H^k(\Gamma_*)$:

i.) $\widehat{U} \to H^{k-1}(Z, \mathbb{R}^{n+1}), \rho \mapsto \partial_i^\rho, i = 1, ..., n,$

ii.) $\widehat{U} \to H^{k-1}(Z), h \mapsto g_{ij}^\rho, \rho \mapsto g_\rho^{ij}, i, j \in \{1, ..., n\},$

iii.) $\widehat{U} \to H^{k-1}(Z), \rho \mapsto J_\rho,$

iv.) $\widehat{U} \to H^{k-1}(Z, \mathbb{R}^{n+1}), \rho \mapsto N_\rho,$

v.) $\widehat{U} \to H^{k-1}(Z), \rho \mapsto N_\rho \cdot N_*,$

vi.) $\widehat{U} \to H^{k-2}(Z), \rho \mapsto h_{ij}^\rho,$

vii.) $\widehat{U} \to H^{k-2}(Z), \rho \mapsto H_\rho,$

viii.) $\widetilde{U} \to H^k(\Gamma_*), \rho \mapsto \gamma(\rho),$

ix.) $\widetilde{U} \to W, \rho \mapsto \widetilde{E}'(\rho).$

Proof. We recall that the transformation of the ∂_i is given by

$$\partial_i^\rho = \partial_i^* + \partial_i\rho N_* - \rho R_*^{-1}\partial_i^*,$$

where we used that for the Weingarten map of the sphere we have $\partial_i N^* = -R_*^{-1}\partial_i^*$. Thus, ∂_i is an affin-linear map in ρ and therefore analytic. For the g_{ij}^ρ we conclude

$$g_{ij}^\rho = g_{ij}^*(1 - 2R_*^{-1}\rho + R_*^{-2}\rho^2) + \partial_i\rho\partial_j\rho.$$

The first summand is a polynom in ρ and thus analytic. The second can be written as evaluation of the symmetric 2-linear form

$$a(\rho_1, \rho_2) := \frac{1}{2}(\partial_i \rho_1 \partial_j \rho_2 + \partial_j \rho_1 \partial_i \rho_2),$$

and so is also a power operator. This implies analyticity of the g^{ρ}_{ij}. For the g^{ij}_{ρ} we note that the inverse matrix G_{ρ}^{-1} is by Cramers rule given by $(\det(G_\rho)\mathrm{adj}(G_\rho))^t$. As the determinant is a sum of n-linear functions, it is analytic and so the adjugate matrix. Then, G_ρ^{-1} is analytic as product of analytic functions and with G_ρ^{-1} also its entries are analytic.
For the transformation of the surface measure we have that (see [49, (2.43)])

$$J_\rho = \alpha(\rho)\mu(\rho), \quad d\mathcal{H}^n(\Gamma_\rho) = J_\rho d\mathcal{H}^n(\Sigma_*)$$

with

$$\alpha(\rho) = \prod_{i=1}^{n}(1 - \rho\kappa_i^*), \quad \mu(\rho) = (1 + \|(I - \rho L_{\Gamma_*})^{-1}\nabla_{\Gamma_*}\rho\|^2)^{\frac{1}{2}}.$$

In our situation Γ_* is a sphere of radius R_* and so $\kappa_i^* = \frac{1}{R_*}$ and $L_{\Gamma_*} = -\frac{1}{R}\mathrm{Id}$. The map $\rho \mapsto \alpha(\rho)$ takes values in $H^k(\Gamma_*)$ as κ_i^* is C^∞. Thus, the factors $(1 - \kappa_i^*\rho)$ are H^k and so is α due to the Banach algebra property of $H^k(\Gamma_*)$. Also, α is a polynom of degree n in ρ and therefore analytic. For $\mu(\rho)^2$ we have

$$\mu(\rho)^2 = 1 + \left\| \frac{R}{R-\rho}\nabla_{\Gamma_*}\rho \right\|^2.$$

By the geometric series we have $\frac{R}{R-\rho} = \sum_{i=0}^{\infty}\left(\frac{\rho}{R}\right)^i$ which is a convergent series in $H^k(\Gamma_*)$ for $\widehat{U}$ sufficiently small. This shows analyticity of

$$\frac{R}{R-\cdot} : H^k(\Gamma_*) \to H^k(\Gamma_*). \tag{5.18}$$

Furthermore, we have that

$$\nabla_{\Gamma_*}\cdot : H^k(\Gamma_*) \to H^{k-1}(\Gamma_*, \mathbb{R}^{n+1}), \tag{5.19}$$

is a linear, continuous map and thus analytic. This implies analyticity of

$$\frac{R}{R-\cdot}\nabla_{\Gamma_*}\cdot : H^k(\Gamma_*) \to H^{k-1}(\Gamma_*, \mathbb{R}^{n+1}), \tag{5.20}$$

as product of analytic functions. The squared norm is a quadratic form from which we conclude that $\mu(\rho)^2 : H^k(\Gamma_*) \to H^{k-1}(\Gamma_*)$ is analytic. Finally, we observe that $\mu^2(0) \equiv 1$. As the square root function is analytic in 1, this implies analyticity of μ and together with the analyticity of α we get that J_ρ is analytic.
For the normal N_ρ we choose a different representation. Following again [49] we have that

$$N_\rho = \frac{N_* + a_\rho}{\|N_* + a_\rho\|},$$

with $a_\rho(p) \in T_p\Gamma_*$. As we have $N_h \cdot \partial_i^h = 0$ for all $i = 1, ..., n$, it follows

$$0 = (N_* + a_\rho)(\partial_i^* + \partial_i\rho - \rho\partial_i^*) = \partial_i\rho + a_\rho\partial_i^*(1-\rho),$$

and hence

$$a_\rho \partial_i^* = -\frac{\partial_i h}{1-h}.$$

As seen above, $\frac{1}{1-\cdot}$ is analytic in $\rho \equiv 0$ and $\rho \mapsto -\partial_i \rho$ is also analytic as linear function. Thus, a_ρ is analytic. The scalar product induces a bilinear function and is therefore analytic. The square root function is analytic in $\rho \equiv 1$ and as $a_\rho \equiv 0$ for $\rho \equiv 0$, this holds for $N_* + a_\rho$ at $\rho \equiv 0$. So, the denominator is also analytic and therefore the whole function. Part v.) follows directly.
For part vi.) we compute the second order derivatives of φ_ρ

$$\partial_i \partial_j \varphi_\rho = \partial_i \partial_j^* - \rho \partial_i \partial_j^* - \partial_j \rho \partial_i^* - \partial_i \rho \partial_j^* + \partial_{ij} \rho N_*.$$

This is a linear, continuous map $H^k(Z) \to H^{k-2}(Z, \mathbb{R}^{n+1})$ and consequently analytic. Due to iv) this implies analyticity of h_{ij}^ρ. Part vii.) follows from the local representation of the mean curvature operator, ii.) and vi.).
For the parametrisation γ we recall formula (5.6) for the constraint G and by the previous results we see that G is analytic. From Corollary 2.5 it follows that γ is also analytic.
For the final step we recall that according to (5.16) we have that $\widetilde{E}'$ as element of $W \subset V'$ is given by

$$\left(H_{\gamma(\rho_0)} - \frac{\int_{\Gamma_*} H_{\gamma(\rho_0)} (N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)} d\mathcal{H}^n}{\int_{\Gamma_*} N_{\Gamma_{\gamma(\rho_0)}} \cdot N_* J_{\gamma(\rho_0)} d\mathcal{H}^n} \right) (N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)}. \tag{5.21}$$

Our results before imply the analyticity of this term. Here, we use that the pullback of functions in $H^{k-1}(Z)$ to functions in $H^{k-1}(\widehat{Z})$ via φ is linear and so analytic and that we have

$$\widetilde{E}'(\rho) = \widetilde{E}'_1(\rho)\chi_1 + \widetilde{E}'_2(\rho)\chi_2, \tag{5.22}$$

where $\widetilde{E}'_i(\rho)$ denotes the expression $\widetilde{E}'(\rho)$ in one of the two local coordinates and $\{\chi_1, \chi_2\}$ is a partition of unity. Both maps $\widetilde{E}'(\rho)_i \to \widetilde{E}'(\rho)_i \chi_i$ are linear and so analytic.
It remains to verify that (5.21) really is mean value free. This follows from

$$\begin{aligned}
&- \int_{\Gamma_*} \frac{\int_{\Gamma_*} H_{\gamma(\rho_0)} (N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)} d\mathcal{H}^n}{\int_{\Gamma_*} N_{\Gamma_{\gamma(\rho_0)}} \cdot N_* J_{\gamma(\rho_0)} d\mathcal{H}^n} (N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)} d\mathcal{H}^n \\
&= - \frac{\int_{\Gamma_*} H_{\gamma(\rho_0)} (N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)} d\mathcal{H}^n}{\int_{\Gamma_*} N_{\Gamma_{\gamma(\rho_0)}} \cdot N_* J_{\gamma(\rho_0)} d\mathcal{H}^n} \int_{\Gamma_*} (N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)} d\mathcal{H}^n \\
&= - \int_{\Gamma_*} H_{\gamma(\rho_0)} (N_{\gamma(\rho_0)} \cdot N_*) J_{\gamma(\rho_0)} d\mathcal{H}^n = 0.
\end{aligned}$$

□

With these two things proven we can now head on the sought gradient inequality.

Theorem 5.10 (Łojasiewicz-Simon gradient inequality for the parametrised surface area)**.**
Consider the energy functional $\widetilde{E} : \widetilde{U} \to R$ *from* (5.9)*. Additionally, let* $W = W^{k-2}_{(0)}(\Gamma_*)$ *and* $W \rightharpoonup V'$ *be chosen as in* (5.15)*. Then, there exists a* $\sigma, C > 0$ *and* $\bar{\theta} \in (0, \frac{1}{2}]$ *such that*

$$|\widetilde{E}(x) - \widetilde{E}(0)|^{1-\bar{\theta}} \leq C \|\widetilde{E}'(x)\|_W, \qquad x \in B_\sigma(0) \subset V. \tag{5.23}$$

Proof. The embedding $V \subset W \hookrightarrow V'$ is definite as the L^2-product is. Furthermore, we have due to Lemma 5.7 that

$$\widetilde{E}'(0)\rho = \int_{\Gamma_*} \left(H_* - \text{Area}(\Gamma_*)^{-1} \int_{\Gamma_*} H_* d\mathcal{H}^n \right) \rho d\mathcal{H}^n.$$

But as H_* is constant we have

$$\mathrm{Area}(\Gamma_*)^{-1} \int_{\Gamma_*} H_* d\mathcal{H}^n = H_*. \tag{5.24}$$

This implies $\widetilde{E}'(0) = 0$ and so 0 is a critical point of $\widetilde{E}$. The claim follows together with Lemma 5.8 and 5.9 from Proposition 5.1. □

5.4 Global Existence and Convergence Near Stationary Solutions

We now want to use the results of the section before to prove Theorem 5.4. Consider initial surfaces Γ_0 that can be written as graph over Γ_* with a $C^{4+\alpha}(\Gamma_*)$ distance function h_0. From Theorem 3.1 we already know that there is a $\varepsilon_0 > 0$ and a $T > 0$ such that for all $\rho_0 \in C^{4+\alpha}(\Gamma_*)$ with $\|\rho_0\| \leq \varepsilon_0$ there exists a solution of (3.4) in $C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})$. Additionally, from [23, Theorem 1.1] we get that this solution is smooth away from $t = 0$ and for all $k > 0$ there is a constant C_k with

$$\|\rho(t)\|_{C^k(\Gamma_*)} \leq \frac{C_k}{t} \|\rho(0)\|_{C^4(\Gamma_*)}, \qquad t \in (0, T]. \tag{5.25}$$

This will be crucial for some interpolation estimates later. Before we can start with the stability analysis itself we have to modify Theorem 5.10. In this version, the energy is defined on the set of distance functions. But it is more convenient to get an estimate that uses solely the geometry of the surfaces. Therefore, we want to prove the following.

Lemma 5.11 (Geometric LSI for the Surface Energy on Closed Hypersurfaces)**.**
Consider for $k' := \frac{2-\bar{\theta}}{\bar{\theta}} k + 2$ with $k, \bar{\theta}$ from Theorem 5.10 and any $R > 0$ the set

$$Z_R := B_R(0) \subset H^{k'}(\Gamma_*).$$

Then, there is a $\sigma > 0$ and a $C(R) > 0$ only depending on R such that for all

$$\rho \in H^k(\Gamma_*) \cap Z_R, \quad \|\rho\|_{H^k(\Gamma_*)} \leq \sigma, \tag{5.26}$$

such that Γ_ρ fulfils the volume constraint $Vol(\Omega_*) = Vol(\Omega_\rho)$, we have that

$$\|\widetilde{E}'(\widetilde{\rho})\|_W \leq C(R) \left(\int_{\Gamma_\rho} |\nabla_{\Gamma_\rho} H_\rho|^2 d\mathcal{H}^n \right)^{\frac{1}{2}\left(\frac{2-2\bar{\theta}}{2-\bar{\theta}}\right)}, \tag{5.27}$$

with

$$\widetilde{\rho} := \rho - \fint_{\Gamma_*} \rho d\mathcal{H}^n. \tag{5.28}$$

In particular, we get the following geometric LSI: there is a (possible smaller) $\sigma > 0$, $C > 0$ such that for $\theta := \frac{\bar{\theta}}{2}$ and all ρ fulfilling (5.26) and the volume constraint we have that $Vol(\Omega_*) = Vol(\Omega_\rho)$

$$|E(\Gamma_\rho) - E(\Gamma_*)|^{1-\theta} \leq C(X_2) \left(\int_{\Gamma_\rho} |\nabla_{\Gamma_\rho} H_\rho|^2 d\mathcal{H}^n \right)^{1/2}. \tag{5.29}$$

Proof. With the same argumentation as in Lemma 5.9 we see that $\widetilde{E}'$ is an analytic, and so a Lipschitz-continuous operator $H^{k'}(\Gamma_*) \to H^{k'-2}(\Gamma_*)$ and hence we have

$$\|\widetilde{E}'(\widetilde{\rho})\|_{H^{k'-2}} \leq C\|\widetilde{\rho}\|_{H^{k'}(\Gamma_*)} \leq CR.$$

This allows us to apply an interpolation argument to conclude

$$\|\widetilde{E}'(\widetilde{\rho})\|_W \leq C\|\widetilde{E}'(\widetilde{\rho})\|_{L^2}^{1-\frac{\bar{\theta}}{2-\bar{\theta}}} \cdot \|\widetilde{E}'(\widetilde{\rho})\|_{H^{k'-2}}^{\frac{\bar{\theta}}{2-\bar{\theta}}} \leq C(R)\|\widetilde{E}'(\widetilde{\rho})\|_{L^2}^{\frac{2-2\bar{\theta}}{2-\bar{\theta}}} \tag{5.30}$$

So, it remains to study $\|\widetilde{E}'(\widetilde{\rho})\|_{L^2}^2$. As $\widetilde{E}'(\widetilde{\rho})$ is mean value free, we can apply Poincaré's inequality to control its L^2-norm by

$$\int_{\Gamma_*} |\nabla_* H_\rho|^2 D_\rho^2 + 2(\nabla_* H_\rho \cdot \nabla_* D_\rho)D_\rho(H_\rho - M_\rho) + (H_\rho - M_\rho)^2|\nabla_* D_\rho|^2 d\mathcal{H}^n, \tag{5.31}$$

where we used the abbreviations

$$D_\rho := (N_\rho \cdot N_*)J_\rho, \quad M_\rho := \frac{\int_{\Gamma_*} H_{\gamma(\rho_0)}(N_{\gamma(\rho_0)} \cdot N_*)J_{\gamma(\rho_0)}d\mathcal{H}^n}{\int_{\Gamma_*} N_{\Gamma_{\gamma(\rho_0)}} \cdot N_* J_{\gamma(\rho_0)}d\mathcal{H}^n}.$$

We now want to see that we can control (5.31) by

$$\int_{\Gamma_*} |\nabla_* H_\rho|^2 d\mathcal{H}.$$

For this, we can estimate the first term using D_ρ is bounded due to the C^2-bound (5.25) we have for ρ. For the second term we can estimate the product of the gradients using Cauchy-Schwarz inequality and then use boundedness of $D_\rho, \nabla_* D_\rho$ and $H_\rho - M_\rho$ due to he C^2-bounds on ρ. For the third term, we first use boundedness of $\nabla_* D_\rho$ to get

$$\int_{\Gamma_*} (H_\rho - M_\rho)^2|\nabla_* D_\rho|^2 d\mathcal{H}^n \leq C\int_{\Gamma_*} (H_\rho - M_\rho)^2 d\mathcal{H}^n.$$

Furthermore, we observe with the abbreviation

$$MV_*(H_\rho) := \fint_{\Gamma_*} H_\rho d\mathcal{H}^n,$$

that

$$(H_\rho - M_\rho)^2 = H_\rho^2 - 2H_\rho M_\rho + M_\rho^2 \leq C(H_\rho^2 - 2H_\rho MV_*(H_\rho) + MV_*(H_\rho)^2) = C(H_\rho - MV_*(H_\rho))^2,$$

where we used in the second step that $M_0 = \fint_{\Gamma_*} H_* d\mathcal{H}^n$ is a negative constant and due to the C^2-bounds we have uniformly estimates

$$cMV_*(H_\rho) \leq M_\rho \leq CMV_*(H_\rho).$$

Now, we can apply Poincaré's inequality on $H_\rho - MV_*(H_\rho)$ to get

$$\int_{\Gamma_*} (H_\rho - M_\rho)^2|\nabla_* D_\rho|^2 d\mathcal{H}^n \leq C\int_{\Gamma_*} |\nabla_* H_\rho|^2 d\mathcal{H}^n.$$

In total we now have shown that

$$\|\widetilde{E}'(\widetilde{\rho})\|_W^2 \leq C\left(\int_{\Gamma_*} |\nabla_* H_\rho|^2 d\mathcal{H}^n\right)^{\frac{2-2\bar{\theta}}{2-\bar{\theta}}}.$$

Finally, we use continuous dependency of the surface gradient the surface measure in ρ to conclude (5.27). The geometric LSI (5.29) follows now from (5.27) and (5.23). □

Now we can proof Theorem 5.4. Recall that in Remark 5.6 we chose k larger than five. Consider the solution $\rho(\cdot)$ of the surface diffusion flow from Theorem 3.1. There, we also determined a bound ε_0 for the initial data such that we have short time existence on a time interval $[0,T]$. Now, due to (5.25) we can choose the initial data ρ_0 sufficiently small such that for $t \in [T_0, T]$ with $T_0 := \frac{T}{2}$ we have that

$$\|\rho(t)\|_{C^k(\Gamma_*)} \leq Z := \min(\sigma, \varepsilon_0), \tag{5.32}$$

where σ is as in Lemma 5.11. We define now $\widetilde{T}$ as the largest time such that (5.32) holds on $\widetilde{I} := [\frac{T}{2}, \widetilde{T})$. Note that on $\widetilde{I}$ the solution ρ exists as we can apply at every time t our short time existence result. Then, we can apply Lemma 5.11 to derive

$$\begin{aligned}
-\frac{d}{dt}(E(\Gamma(t)) - E(\Gamma_*))^\theta &= -\theta(E(\Gamma(t)) - E(\Gamma_*))^{\theta-1} \int_{\Gamma(t)} -VH_{\Gamma(t)} d\mathcal{H}^n \\
&= \theta(E(\Gamma(t)) - E(\Gamma_*))^{\theta-1} \int_{\Gamma(t)} (-\Delta_{\Gamma(t)} H_{\Gamma(t)}) H_{\Gamma(t)} d\mathcal{H}^n \\
&= \theta(E(\Gamma(t)) - E(\Gamma_*))^{\theta-1} \int_{\Gamma(t)} |\nabla_{\Gamma(t)} H_{\Gamma(t)}|^2 d\mathcal{H}^n \\
&\geq C \left(\int_{\Gamma(t)} |\nabla_{\Gamma(t)} H_{\Gamma(t)}|^2 d\mathcal{H}^n \right)^{1/2} \\
&= C \|\Delta_{\Gamma(t)} H_{\Gamma(t)}\|_{\mathcal{H}^{-1}(\Gamma(t))}
\end{aligned} \tag{5.33}$$

Hereby, in the fourth step we used the LSI from Lemma 5.11. Note now that we have due to the continuous transformation of the normal, the C^5-bound for ρ and Lemma 2.12 that

$$\begin{aligned}
\|\partial_t \rho(t)\|^2_{L^2(\Gamma_*)} \leq C \int_{\Gamma(t)} |\Delta_{\Gamma(t)} H_{\Gamma(t)}|^2 d\mathcal{H}^n &\leq C \|\Delta_{\Gamma(t)} H_{\Gamma(t)}\|_{\mathcal{H}^{-1}(\Gamma(t))} \|\Delta_{\Gamma(t)} H_{\Gamma(t)}\|_{H^1(\Gamma(t))} \\
&\leq C \|\Delta_{\Gamma(t)} H_{\Gamma(t)}\|_{\mathcal{H}^{-1}(\Gamma(t))}.
\end{aligned} \tag{5.34}$$

Combining (5.33) and (5.34) we get

$$\|\partial_t \rho(t)\|^2_{L^2(\Gamma_*)} \leq -C \frac{d}{dt} \left(E(\Gamma(t)) - E(\Gamma_*) \right)^\theta. \tag{5.35}$$

Integrating this in time yields for any $s \in \widetilde{I}$ that

$$\begin{aligned}
\|\rho(s)\|_{L^2(\Gamma_*)} &\leq \|\rho(s) - \rho(T_0)\|_{L^2(\Gamma_*)} + \|\rho(T_0)\|_{L^2(\Gamma_*)} \leq \int_{T_0}^s \|\partial_t \rho\|_{\mathcal{H}^{-1}} dt + \|\rho(T_0)\|_{L^2(\Gamma_*)} \\
&\leq \sqrt{\int_{T_0}^s \|\partial_t \rho\|^2_{L^2(\Gamma_*)} dt} + \|\rho(T_0)\|_{C^{\frac{\theta}{2}}(\Gamma_*)} \\
&\leq \sqrt{-C(E(\Gamma(s)) - E(\Gamma_*)^\theta + C(E(\Gamma(T_0)) - E(\Gamma_*))^\theta} + C\|\rho(T_0)\|^{\frac{\theta}{2}}_{C^0(\Gamma_*)} \\
&\leq C(E(\Gamma(T_0)) - E(\Gamma_*))^{\frac{\theta}{2}} + C\|\rho(T_0)\|^{\frac{\theta}{2}}_{C^{2+\alpha}(\Gamma_*)} \\
&\leq C\|\rho(T_0)\|^{\frac{\theta}{2}}_{C^{2+\alpha}(\Gamma_*)} + C\|\rho(T_0)\|^{\frac{\theta}{2}}_{C^{2+\alpha}} \\
&\leq C\|\rho_0\|^{\frac{\theta}{2}}_{C^{2+\alpha}(\Gamma_*)}.
\end{aligned} \tag{5.36}$$

Here, we used in the third inequality the Hölder inequality, in the fourth inequality (5.35), in the fifth inequality the fact that $-C(E(\Gamma(s)) - E(\Gamma_*))$ is negative as Γ_* is a minimum of the surface energy, in the sixth inequality Lipschitz-continuity of the surface area in the C^3-norm and in the seventh inequality Hölder continuity of $\boldsymbol{\rho}$ in time, where we used that due to Lemma 3.13 the Hölder coefficients in time are uniformly bounded. Additionally, we used in the fourth inequality that

$$\|\rho(T_0)\|_{C^{\frac{\theta}{2}}(\Gamma_*)} \leq C\|\rho(T_0)\|^{\frac{\theta}{2}}_{C^0(\Gamma_*)} \|\rho(T_0)\|^{1-\frac{\theta}{2}}_{C^1(\Gamma_*)} \leq C\|\rho(T_0)\|^{\frac{\theta}{2}}_{C^0(\Gamma_*)}, \tag{5.37}$$

where we used interpolation results for Hölder spaces and (5.25). Using (5.36) we derive for all $t \in \widetilde{I}$ for the $C^k(\Gamma_*)$-norm of $\rho(t)$ for some $\beta \in (0,1)$ that

$$\|\rho(t)\|_{C^k(\Gamma_*)} \leq C\|\rho(t)\|^{1-\beta}_{C^{k'}(\Gamma_*)}\|\rho(t)\|^{\beta}_{L^2(\Gamma_*)} \overset{(5.36)}{\leq} C\|\rho_0\|^{\frac{\beta\theta}{2}}_{C^{2+\alpha}(\Gamma_*)} \leq C\varepsilon^{\frac{\beta\theta}{2}}. \tag{5.38}$$

Now choosing ε sufficiently small we get hat

$$\|\rho(t)\|_{C^k(\Gamma_*)} \leq \frac{Z}{2}. \tag{5.39}$$

But then we can apply our short time existence result shortly before $\widetilde{T}$ to get a larger existence time and due to continuity reasons the new solution will fulfil $\|\rho(t)\|_{C^k(\Gamma_*)} \leq Z$ a short time interval after $\widetilde{T}$. As $\widetilde{T}$ was chosen to be the maximal time with this property, this implies that $\widetilde{T}$ cannot be finite and so $\widetilde{T} = \infty$ and $\rho(t)$ exists globally.
As $E(\Gamma(t)) - E(\Gamma_*)$ is strictly decreasing while bounded below by zero as Γ_* is a local minimum, we deduce from (5.35) that $\partial_t\rho$ is in $L^1(\mathbb{R}^+, L^2(\Gamma_*))$ and therefore the limit ρ_∞ exists in $L^2(\Gamma_*)$. Again doing interpolation and using (5.25) we get that $\rho(t)$ is a Cauchy-sequence in $C^{k+a}(\Gamma_*)$ for all $k \in \mathbb{N}$ and thus we have convergence in $C^{k+\alpha}(\Gamma_*)$ to ρ_∞. Finally, as ρ_∞ has a H^k-norm less than σ, we can apply the LSI to get

$$\begin{aligned} |E(\Gamma_\infty) - E(\Gamma_*)|^\theta &\leq C\left(\int_{\Gamma_\infty} |\nabla_{\Gamma_\infty} H_{\Gamma_\infty}|^2 d\mathcal{H}^n\right)^{1/2} \leq C\|\Delta_{\Gamma_\infty} H_{\Gamma_\infty}\|_{L^2(\Gamma_*)} \cdot \|H_{\Gamma_\infty}\|_{L^2(\Gamma_*)} \\ &\leq C\|\Delta_{\Gamma_\infty} H_{\Gamma_\infty}\|_{L^2(\Gamma_*)} = 0. \end{aligned} \tag{5.40}$$

In the last step we used that as $\partial_t\rho \in L^1(\mathbb{R}^+, L^2(\Gamma_*))$ and consequently $\|\partial_t\rho(t)\|_{L^2(\Gamma_*)} \to 0$ as $t \to \infty$, this implies using the motion law (SDFC) that also $\|\Delta_{\Gamma(t)} H_{\Gamma(t)}\|_{L^2(\Gamma_*)} \to 0$ as $t \to \infty$.
From (5.40) we conclude that Γ_∞ has to be a minimum of the surface energy with the volume constraint $\mathrm{Vol}(\Gamma_\infty) = \mathrm{Vol}(\Gamma_*)$ and consequently a sphere. Therefore, choosing ε_S such that (5.32) holds, finishes the proof of Theorem 5.4ii.).
For the proof of part i.) we want to notice that the argumentation we just used also works when we replace Z in (5.32) with any $0 < \tilde{Z} \leq Z$. Thus, for any $C > 0$ we can choose ρ_0 and T_0 small enough such that $\|\rho(t)\|_{C^{4+\alpha}(\Gamma_*)} \leq C$ on $[0, T_0]$ by (4.104). Then, by choosing ρ_0 possibly smaller we can also guarantee that $\|\rho(T_0)\|_{C^k(\Gamma_*)} \leq \min(C, Z)$, again using parabolic smoothing of the flow. Consequently, we will also have $\|\rho(t)\|_{C^k(\Gamma_*)} \leq \min(C, Z)$ on $[T_0, \infty)$. This shows Lyapunov stability of $(SDFC)$ and we are done.

6

Stability of the Surface Diffusion Flow of Double-Bubbles

Now, we consider the case of a triple junction geometry. Before we do so, we will redefine some notation for the rest of the section. Let $\Gamma_* \subset \mathbb{R}^{n+1}$ be a minimizer of the surface area energy functional (4.1) enclosing two given volumes $V_2^* := \mathrm{Vol}(\Omega_{12}), V_3^* := \mathrm{Vol}(\Omega_{13})$. Hereby, we restrict to the case $\gamma^1 = \gamma^2 = \gamma^3 = 1$ although we expect that our work can be generalized. By the work of [36] we know that Γ_* is a double bubble fulfilling the 120°- condition on the triple junction. If Γ_*^i is a spherical cap we will denote by R^i its radius. As before we want to parametrise triple junction manifolds near Γ_* via a pair of distance functions $\boldsymbol{\rho}$ in normal direction of Γ_* and a tangential part $\boldsymbol{\mu}$. During the calculations we realize that the tangential part chosen by the authors of [19] is not suitable to apply the results of [13]. The non-local form of $\boldsymbol{\mu}$ makes it impossible to choose suitable Banach spaces V, W together with an embedding $W \hookrightarrow V^*$ which we will discuss in detail in the first section. So before we can go on we have to do find an alternative formula for $\boldsymbol{\mu}$ which is discussed in the second section. Afterwards, we find an implicit parametrisation of the set of distance functions describing volume preserving triple junctions. With this geometric description we can find a suitable setting to prove a Łojasiewicz-Simon gradient inequality for the corresponding energy functional induced by the surface energy. Then, in the last section we are able to prove the second main result of this thesis, which is the following.

Theorem 6.1 (Stability of stationary double bubbles wit respect to the surface diffusion flow)**.** *Let $\alpha > 0$. Every stationary double bubble Γ_* is stable with respect to the surface diffusion flow in the following sense:*

i.) The stationary solution $\boldsymbol{\rho} \equiv 0$ of $(SDFTJ)$ is Lyapunov stable with respect to the $C^{4+\alpha}$-norm.

ii.) There is an $\varepsilon_S > 0$ such that for all $\boldsymbol{\rho}_0 \in C^{4+\alpha}_{TJ}(\Gamma_*)$ fulfilling $\|\boldsymbol{\rho}\| \leq \varepsilon_S$ and the compatibility conditions (4.31) the solution from Theorem 4.2 converges for $t \to \infty$ to some $\boldsymbol{\rho}_\infty$. Furthermore, $\Gamma_{\boldsymbol{\rho}_\infty}$ is a (possibly different) standard double bubble.

6.1 Technical Problems Proving a Łojasiewicz-Simon Gradient Inequality: Non-local Tangential Parts, Non-linear Boundary Conditions

In the last chapter we already discussed the basic idea of the choices for V and W in the context of the surface diffusion flow. Now, our geometric objects have a triple junction which causes two new difficulties to arise. Firstly, we will need to include some boundary conditions to guarantee the Fredholm property of the second derivative of the surface energy which is again the surface Laplacian.

This conditions should be connected to our flow as for the stability argument itself the solution $\boldsymbol{\rho}$ of (SDFTJ) should be an admissible function. As we have a second order differential operator we can only allow for three conditions and the natural choice are the angle conditions and the concurrency of the triple junction.
The application of the Łojasiewicz-Simon technique on the situation with non-linear boundary conditions was - according to our knowledge - not discussed, yet. This situation is much more complicated than linear boundary conditions, that, e.g. were discussed in [2] or[15] as there one can just write these conditions into the space V. Our first idea to overcome this was to include these non-linear boundary conditions in V by parametrising them over the linearised boundary conditions. This is actually possible and we will use this during the proof of Lemma 6.12. Also, we can include this in the parametrisation of the non-linear volume constraints we will need in the situation of double-bubbles to. But then, as in Lemma 5.7 we will get in the first derivative of the considered energy terms arising due to the parametrisation of the non-linear boundary conditions. These terms cannot be fitted in our setting[1] and so we have to find a different approach. The idea is then to write the boundary conditions in the energy $\widetilde{E}$ itself. The consequence of this is that the space W needs to have some trace parts for this boundary conditions. At first glance, that is bad for our stability analysis later as there these trace parts do not appear. But we just chose the boundary conditions for the surface diffusion flow such that these terms vanish, cf. (4.4) and its consequences. So, in the stability analysis we will indeed have the norm we want to work with.
The second technical difficulty is the tangential part $\boldsymbol{\mu}$. In (6.20) we see that in the first derivative of the surface energy at a point $\overline{\boldsymbol{\rho}} \in V$ we get a term of the form

$$\sum_{i=1}^{3} \int_{\Gamma_*^i} \mu^i(\boldsymbol{\rho}) g(\overline{\boldsymbol{\rho}}) d\mathcal{H}^n, \tag{6.1}$$

with some function g in $\overline{\boldsymbol{\rho}}$ and $\boldsymbol{\rho} \in V$. To apply our method, we need (6.1) to be of the form

$$\sum_{i=1}^{3} \int_{\Gamma_*^i} \rho^i \overline{g}(\overline{\boldsymbol{\rho}}) d\mathcal{H}^n, \tag{6.2}$$

with another function $\overline{g}$ in $\overline{\boldsymbol{\rho}}$. Observe now that if $\boldsymbol{\rho}$ vanishes on Σ_* we will have $\mu^i(\boldsymbol{\rho}) \equiv 0$ on Γ and thus (6.1) vanishes as well. Then, applying the fundamental lemma of calculus of variations on (6.2) we see that

$$\overline{g} \equiv 0. \tag{6.3}$$

But as the functional corresponding to (6.1) is not the zero functional, we cannot write (6.1) in the formulation (6.2). Thus, the non-local tangential part cannot be dealt with our LSI methods and so the only possible solution is to replace the tangential part by a local version that describes the same set of triple junction manifolds as long as $\boldsymbol{\rho}$ is small enough.

Remark 6.2 (Non-linear tangential parts)**.**
Although we did not study what happens in the case of non-linear tangential parts, we expect that these should in general not be a problem. One only needs to guarantee that their linearisation fits in the chosen setting for $W \hookrightarrow V^*$ for the analyticity of the first derivative. The evaluation of the second derivative at the reference frame normally depends only on the evolution in normal direction and so the tangential part does not matter at all.

6.2 Choice of the Tangential Part

As mentioned above we want to get rid of the non-local term of the tangential part. On the other hand, we still want the tangential part to be given as a function in $\boldsymbol{\rho}$ as otherwise we would have

[1] Observe that in the derivation of (5.11) we used the fact that $\overline{\gamma}'(\rho_0)$ takes values in a one dimensional space. This still works if $\overline{\gamma}'(\rho_0)$ takes values in a finite dimensional space, which we will see in Lemma 6.6. But if we put the boundary conditions into γ, this will not longer be the case!

different degrees of freedom for the first and second derivative of the surface area, which would be a problem in the proof of the LSI. The idea to achieve such a tangential part is surprisingly easy. We observe that the linear connection (4.19) between $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$ is the same along the triple junction. This suggests to use the matrix $\mathcal{T}$ also in the interior to get at every point the tangential part $\boldsymbol{\mu}$ as function of the normal part $\boldsymbol{\rho}$. The constructed function will then involve no projection on Σ_* and thus be purely local. The only problem hereby is that for the calculation we have to evaluate all the ρ_i in one point and technically each ρ_i only exists on Γ^i_*. But we can solve this by identifying the three hypersurfaces via diffeomorphisms.
More precisely, for $B = B_1(0) \subset \mathbb{R}^n$ we choose C^∞-parametrisations

$$\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3) : B \to (\mathbb{R}^{n+1})^3 \tag{6.4}$$

of $(\Gamma^1_*, \Gamma^2_*, \Gamma^3_*)$ such that for all $x \in \partial B$ we have that

$$\varphi_1(x) = \varphi_2(x) = \varphi_3(x). \tag{6.5}$$

We note here that it is indeed possible to parametrise each Γ^i_* with one parametrisation as Γ^i_* is either a spherical cap or a flat ball in a hypersurface. Now, we can define the new tangential coefficient field $\boldsymbol{\mu}_G$ on B via the linear connection from before and then do a pushfoward, that is,

$$\boldsymbol{\mu}_G(\boldsymbol{\rho}) := (\mathcal{T}(\boldsymbol{\rho} \circ \boldsymbol{\varphi}^{-1})) \circ \boldsymbol{\varphi}^{-1}. \tag{6.6}$$

By construction this function fulfils the necessary condition for the concurrency of the triple junctions.

Remark 6.3 (Linearisation results for the new tangential part)**.**
In the next section we will use the same linearisation results of [18] and [17] for the boundary conditions as in the chapter for short time existence. This is indeed possible as in these works the calculations were done for a general tangential part and so they work both for the original tangential part $(4.19)_2$ *and the new one* (6.6)*.*

Before we can use this new tangential term we have to check that by this procedure we describe the same triple junction manifolds as in the short time existence result which is carried out in the following lemma. There, we will denote by $\boldsymbol{\mu}_{DGK}$ the tangential part we used in Chapter 4 and by $\boldsymbol{\mu}_G$ the tangential given by (6.6).

Lemma 6.4 (Equivalence of the tangential parts)**.**
For every $k \geq 2$ we consider the space

$$C^k_{TJ,0}(\Gamma_*) := \{\boldsymbol{\rho} \in C^k_{TJ}(\Gamma_*) \big| \rho^i = 0 \text{ on } \Sigma_*, i = 1, 2, 3\}. \tag{6.7}$$

There exist $r, r' > 0$ (depending on k) together with a map

$$\bar{F} : C^k_{TJ}(\Gamma_*) \supset B_r(0) \to B_{r'}(0) \subset C^k_{TJ,0}(\Gamma_*), \tag{6.8}$$

such that the map $F = \bar{F} + \mathrm{Id}$ fulfils for all $\boldsymbol{\rho} \in B_r(0)$ that

$$\Gamma_{\boldsymbol{\rho}, \mu_{DGK}(\boldsymbol{\rho})} = \Gamma_{F(\boldsymbol{\rho}), \mu_G(F(\boldsymbol{\rho}))}. \tag{6.9}$$

Proof. Consider for $X = C^k_{TJ}(\Gamma_*), Y = Z = C^k_{TJ,0}(\Gamma_*)$ the map $G : X \times Y \to Z$

$$(\boldsymbol{\rho}, \overline{\boldsymbol{\rho}}) \mapsto (x \mapsto d_H(x + (\rho^i + \overline{\rho}^i)(x)\nu^i_*(x) + (\boldsymbol{\mu}_{\boldsymbol{G}}(\boldsymbol{\rho} + \overline{\boldsymbol{\rho}})(x))^i \tau^i_*(x), \Gamma^i_{\boldsymbol{\rho}, \boldsymbol{\mu}_{DGK}(\boldsymbol{\rho})}))_{i=1,2,3}.$$

Hereby, d_H denotes the usual Hausdorff distance from a point to a compact set. Note that G has indeed values in Z as $\boldsymbol{\mu}_{DGK}$ and $\boldsymbol{\mu}_G$ equal on Σ_*. Therefore, $\Gamma_{\boldsymbol{\rho}, \boldsymbol{\mu}_{DGK}(\boldsymbol{\rho})}$ and $\Gamma_{\boldsymbol{\rho}+\overline{\boldsymbol{\rho}}, \boldsymbol{\mu}_G(\boldsymbol{\rho}+\overline{\boldsymbol{\rho}})}$ have the same triple junction as $\boldsymbol{\rho} + \overline{\boldsymbol{\rho}} = \boldsymbol{\rho}$ on Σ_*. We want to use the implicit function theorem to find a map $\bar{F}$ with $G(\boldsymbol{\rho}, \bar{F}(\boldsymbol{\rho})) = 0$ for $\boldsymbol{\rho}$ small enough. Then, the surfaces $\Gamma^i_{F(\boldsymbol{\rho}), \mu_G(F(\boldsymbol{\rho}))}$ are subsets of $\Gamma^i_{\boldsymbol{\rho}, \mu_{DGK}(\boldsymbol{\rho})}$. Additionally, they are simply connected and have the same boundary as $\Gamma^i_{\boldsymbol{\rho}, \mu_{DGK}(\boldsymbol{\rho})}$ and consequently they have to be equal to $\Gamma^i_{\boldsymbol{\rho},\mu_{DGK}(\boldsymbol{\rho})}$. So, $\bar{F}$ fulfils (6.9). We claim that

$$\partial_2 G(0,0)\overline{\boldsymbol{\rho}} = \overline{\boldsymbol{\rho}}, \quad \overline{\boldsymbol{\rho}} \in C^2_{TJ}(\Gamma_*). \tag{6.10}$$

In order to see this we will calculate $G(0,\varepsilon\widetilde{\boldsymbol{\rho}})$ pointwise. There are two geometrical situations that could arise. The first is that $x + \overline{\rho}^i(x)\nu^i_* + (\mu_G(\overline{\boldsymbol{\rho}})(x))_i$ lies within the R^i-tube of Γ^i_*. The second possibility, which in theory could also arise, is that points near Σ_* could leave the R^i tube. We want to see that for $\overline{\boldsymbol{\rho}}$ small enough in Y only the first situation is possible. For this we note that for any $\sigma \in \Gamma^i_*$ the ball with radius $\min(R^i, \|\sigma - \mathrm{pr}_{\Sigma_*}(\sigma)\|_{\mathbb{R}^{n+1}})$ is completely contained in the R^i-tube of Γ^i_*. Hereby, pr_{Σ_*} denotes the projection the projection on the nearest point on Σ_*. This is at least near Σ_* well-defined. By elementary geometry we see that

$$\|\sigma - \mathrm{pr}_{\Sigma_*}(\sigma)\|_{\mathbb{R}^{n+1}} = 2R^i \sin\left(\frac{d_{\Gamma^i}(\sigma, \mathrm{pr}_{\Sigma_*}(\sigma))}{R^i}\right). \tag{6.11}$$

Thus, this quantity as function in $d_{\Gamma^i}(\sigma, \mathrm{pr}_{\Sigma_*}(\sigma))$ has a derivative larger than 1 close to zero. On the other hand, we know that

$$\begin{aligned}\|x - (x + \overline{\rho}^i(x)\nu^i_*(x) + \mu_G(\overline{\rho}(x))^i\tau^i_*(x))\|_{\mathbb{R}^{n+1}} &\leq C(|\overline{\rho}^i(x)| + \sum_{j=1}^3 |\overline{\rho}^j(x)|)\\ &\leq C d_{\Gamma^i_*}(x, \mathrm{pr}_{\Sigma_*}(x))\|\overline{\rho}\|_{C^1},\end{aligned}$$

withe the constant C only depending on the linear relation between $\boldsymbol{\mu}$ and $\boldsymbol{\rho}$ but not on the point x. So, by choosing r' small enough we can guarantee that the point $x + \overline{\rho}^i(x)\nu^i_*(x) + \mu_G(\overline{\rho}(x))_i\tau^i_*(x)$ lies within the ball within the ball around x with radius $\min(R^i, \|x - \mathrm{pr}_{\Sigma_*}(x)\|_{\mathbb{R}^{n+1}})$ and so within the R^i_*-tube of Γ^i_*. Note that this argumentation is also true if Γ^i_* is a flat ball.
We return now to the proof of (6.10) where we can now restrict to the first situation. If Γ^i_* is a flat ball this is clear as we then have for $\overline{\boldsymbol{\rho}} \in Z$ and $\varepsilon > 0$

$$d_H(x + \varepsilon\overline{\rho}^i(x)\nu^i_*(x) + (\boldsymbol{\mu}_G(\varepsilon\overline{\boldsymbol{\rho}})(x))^i\tau^i_*(x), \Gamma^i_*) = d_H(x + \varepsilon\overline{\rho}^i(x)\nu^i_*(x), \Gamma^i_*) = \varepsilon\overline{\rho}^i(x). \tag{6.12}$$

Here, we used that in this case the tangential movement is parallel to Γ^i_*. Thus, this will not change the distance to Γ^i_*. Taking the limit $\varepsilon \to 0$ we get easily (6.10).
We now consider the case that Γ^i_* is a spherical cap. From elementary geometry we know that the nearest point on Γ^i_* from $x + \overline{\rho}^i(x)\nu^i_*(x) + \mu_G(\overline{\rho}(x))_i\tau^i_*(x)$ is given by the intersection of the straight line between $x_{\overline{h},\mu_G(\overline{h})}$ and M^i_* and Γ^i_*. By Pythagoras' theorem we get that

$$\begin{aligned}G\left(0,\varepsilon\overline{\boldsymbol{\rho}}\right)(x) &= \sqrt{\left(R^i_* + \varepsilon\overline{\rho}^i\right)^2 + \|\mu^i_G(\varepsilon\overline{\boldsymbol{\rho}})(x)\tau^i_*(x)\|^2} - R^i_*\\ &= (R^i_* + \varepsilon\overline{\rho}^i) + \frac{1}{2(R^i_* + \varepsilon\overline{\rho}_i)}\varepsilon^2\|\mu^i_G(\overline{\boldsymbol{\rho}})(x)\tau^i_*(x)\|^2 + \sigma(\varepsilon^4) - R^*_i.\end{aligned}$$

Here, we used $\mu^i_G(\varepsilon\overline{\rho}) = \varepsilon\mu^i_G(\overline{\rho})$ and a Taylor expansion. This implies

$$\frac{G(0,\varepsilon\overline{\rho}) - G(0,0)}{\varepsilon} = \overline{\rho}_i + \frac{1}{2(R^*_i + \varepsilon\overline{\rho_i})}\varepsilon\|\mu_G(\overline{\boldsymbol{\rho}})\tau^i_*(x)\|^2 + \sigma(\varepsilon^4).$$

This converges uniformly in x to $\overline{\rho}^i(x)$ for $\varepsilon \to 0$ as $\mu^i_G(\overline{\boldsymbol{\rho}})(x)\tau^*(x)$ is bounded in x. This shows (6.10). Continuity of G and $\partial_2 G$ follows from the formulas in [49, Section 2.2], which we can apply as the points $x + (h_i + \overline{h}_i)(x)\nu^i_*(x) + (\boldsymbol{\mu_G}(h + \overline{h})(x))_i\tau^i_*(x)$ stay in the tubular neighbourhoods of $\Gamma^i_{h,\mu_{DGK}(h)}$ for r and r' small enough. Therefore, we can apply the implicit function theorem to show the claim. ☐

In the following, when we write $\boldsymbol{\mu}$ we will always refer to $\boldsymbol{\mu}_G$ unless said otherwise and if we omit the tangential part in the notation, the used tangential part will always be $\boldsymbol{\mu}_G$.

6.3 Parametrisation of the Set of Volume Preserving Triple Junction Manifolds

Next we need to rewrite the set of the volume preserving distance functions over a Banach space. The situation differs from the one in Section 5.2 for two reasons. We have additional non-linearities we would like to include in our setting. These are the angle conditions which would make the variational formulas and the analysis for the proof of the LSI easier. But as we argued in Section 6.1 this will lead to technical problems in the proof of the LSI and so we will not parametrise them. The second difference compared to the closed situation is that the space of (suitable) constant functions is not the best choice for a complementary space of the tangent space of volume preserving triple junctions. It is more convenient to work with functions that vanish near Σ_* to avoid additional tangential parts and so we will choose suitable bump functions.
To construct our setting we start with

$$U := \left\{ \boldsymbol{u} \in C^2_{TJ}(\Gamma_*) \middle| \gamma^1 u^1 + \gamma^2 u^2 + \gamma^3 u^3 = 0 \text{ on } \Sigma_* \right\}. \tag{6.13}$$

Note that we can put the condition for concurrency of the triple junction already in U as it is linear. Now, we have again to consider the tangent space of volume preserving evolutions, which leads to

$$U_1 := \left\{ \boldsymbol{v} \in U \middle| \int_{\Gamma^1_*} v^1 d\mathcal{H}^n = \int_{\Gamma^2_*} v^2 d\mathcal{H}^n = \int_{\Gamma^3_*} v^3 d\mathcal{H}^n \right\} \tag{6.14}$$

The equality condition for the integrals follows directly from applying Reynolds' transport theorem on the evolution of the enclosed volume and corresponds to the meanvalue freeness condition we got in the previous chapter.
Now we need to construct a suitable complemented space of U_1. To this end, we choose $f^1 \in C^\infty(\Gamma^1_*)$ and $f^2 \in C^\infty(\Gamma^2_*)$ with

$$\begin{aligned} &\operatorname{supp}(f^1) \subset\subset \Gamma^1_* \backslash \operatorname{supp}(\tau^1_*), \int_{\Gamma^1_*} f^1 d\mathcal{H}^n = 1, \\ &\operatorname{supp}(f^2) \subset\subset \Gamma^2_* \backslash \operatorname{supp}(\tau^2_*), \int_{\Gamma^2_*} f^2 d\mathcal{H}^n = 1 \end{aligned}$$

and set

$$U_2 := \operatorname{span}(f^1, f^2). \tag{6.15}$$

Here, we consider f^1, f^2 as functions in $C^\infty_{TJ}(\Gamma_*)$ extending them by zero. The space U_2 is closed being finite dimensional. Observe that the choice of the support of f^1 and f^2 guarantees that addition of linear combinations of them to other distance functions $\boldsymbol{\rho}$ will not change the tangential part. Before we go on we have to check that U_1 and U_2 are indeed complementary spaces in U.

Lemma 6.5 (Complementarity of U_1 and U_2)**.** *U_1 and U_2 are complementary spaces in U.*

Proof. For $a, b \in \mathbb{R}$ and $f = af^2 + bf^3$ we have that

$$\int_{\Gamma^1_*} f d\mathcal{H}^n = 0, \int_{\Gamma^2_*} f d\mathcal{H}^n = a, \int_{\Gamma^3_*} f^3 d\mathcal{H}^n = b.$$

Thus, only for $a = b = 0$ we have $f \in U_1$ and therefore $U_1 \cap U_2 = \{0\}$. On the other hand, for $\boldsymbol{\rho} \in U$ it holds that

$$\begin{aligned} &\boldsymbol{\rho} = \left(\boldsymbol{\rho} - \alpha f^2 - \beta f^3\right) + \alpha f^2 + \beta f^3, \\ &\alpha := \int_{\Gamma^2_*} \rho^2 d\mathcal{H}^n - \int_{\Gamma^1_*} \rho^1 d\mathcal{H}^n, \beta := \int_{\Gamma^3_*} \rho^3 d\mathcal{H}^n - \int_{\Gamma^1_*} \rho^1 d\mathcal{H}^n. \end{aligned}$$

Hereby, we have that $(\boldsymbol{\rho} - \alpha f^2 - \beta f^3) \in U_1$ and so we conclude $U = U_1 + U_2$. $\square$

Now we are able to prove the existence of the sought parametrisation of the volume constraint.

Lemma 6.6 (Parametrisation of the volume constraint for triple junction manifolds)**.**
Let $U = U_1 \oplus U_2$ be as above. Then, there exists an open neighbourhood $\widetilde{U}_1$ of 0 in U_1 and an open neighbourhood $\widetilde{U}$ of $0 \in U$ together with a unique map $\bar{\gamma} : \widetilde{U}_1 \to U_2$ such that the map

$$\gamma := \mathrm{Id} + \bar{\gamma} : \widetilde{U}_1 \to \widetilde{U} \tag{6.16}$$

parametrises the subset of all functions U that belong to triple junction manifolds fulfilling the volume constraints. Furthermore, $\bar{\gamma}$ is analytic and we have for the first derivative of $\bar{\gamma}$ for $\boldsymbol{v}_0 \in \widetilde{U}_1, \boldsymbol{v} \in U_1$ that

$$\bar{\gamma}'(\boldsymbol{v}_0)\boldsymbol{v} = \begin{pmatrix} \frac{\int_{\Gamma^1_{\gamma(\boldsymbol{v}_0)}} \left(v^1 N^1_* + \mu^1(\boldsymbol{v})\tau^1_*\right)\cdot N^1_{\gamma(\boldsymbol{v}_0)} d\mathcal{H}^n - \int_{\Gamma^2_{\gamma(\boldsymbol{v}_0)}} \left(v^2 N^2_* + \mu^2(\boldsymbol{v})\tau^2_*\right)\cdot N^2_{\gamma(\boldsymbol{v}_0)} d\mathcal{H}^n}{\int_{\Gamma^2_{\gamma(\boldsymbol{v}_0)}} N^2_* \cdot N^2_{\gamma(\boldsymbol{v}_0)} d\mathcal{H}^n} f^1 \\ \frac{\int_{\Gamma^3_{\gamma(\boldsymbol{v}_0)}} \left(v^3 N^3_* + \mu^3(\boldsymbol{v})\tau^3_*\right)\cdot N^3_{\gamma(\boldsymbol{v}_0)} d\mathcal{H}^n - \int_{\Gamma^1_{\gamma(\boldsymbol{v}_0)}} \left(v^1 N^1_* + \mu^1(\boldsymbol{v})\tau^1_*\right)\cdot N^1_{\gamma(\boldsymbol{v}_0)} d\mathcal{H}^n}{\int_{\Gamma^3_{\gamma(\boldsymbol{v}_0)}} N^3_* \cdot N^3_{\gamma(\boldsymbol{v}_0)} d\mathcal{H}^n} f^2 \end{pmatrix}. \tag{6.17}$$

Proof. Consider the functional

$$\begin{aligned} G : U = U_1 \oplus U_2 &\to \mathbb{R}^2, \\ (\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{u} &\mapsto \begin{pmatrix} \mathrm{Vol}(\Omega^{\boldsymbol{u}}_{13}) - V^*_1 \\ \mathrm{Vol}(\Omega^{\boldsymbol{u}}_{23}) - V^*_2 \end{pmatrix}. \end{aligned}$$

We want to apply the implicit function theorem and so we have to consider the partial derivative $\partial_2 G(0,0)$. For $\boldsymbol{w} = af^1 + bf^2, a, b \in \mathbb{R}$ we get by transport theorems

$$\partial_2 G(0,0)\boldsymbol{w} = \begin{pmatrix} \int_{\Gamma^1_*} af^1 d\mathcal{H}^n \\ -\int_{\Gamma^2_*} bf^2 d\mathcal{H}^n \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}.$$

This shows that $\partial_2 G(0,0) : W \to \mathbb{R}^2$ is bijective. Continuity of $\partial_2 G$ and analyticity of G follow as in the proof of Lemma 5.5 and so we get the existence of an analytic function $\bar{\gamma}$ with the desired properties.
Now, for $\bar{\gamma}'$ we do an implicit differentiation and rearrange the terms afterwards to get (6.17). We hereby used the fact that $\bar{\gamma}'$ can be seen as element in $\mathbb{R}^2$ and that $\bar{\gamma}(\boldsymbol{v})$ does not change the tangential part of $\boldsymbol{v}$. This yields then (6.17). $\square$

Remark 6.7. *One could also calculate $\bar{\gamma}''(0)$ using the same procedure as in the case of closed hypersurfaces. But for the application later we only need $\widetilde{E}''(0)$ and for its calculation we can also use the results from [36].*

6.4 Variation formulas for the Parametrised Surface Energy

We now derive the formulas for the first and second derivative of the surface energy. Before doing so we want to specify the notation for the different energies arising in the setting. By the plain E we denote the surface energy as a functional on the set of triple junction manifolds near the considered stationary point Γ_*. The functional $\widehat{E}$ arises from E by the parametrisation of these triple junction manifolds using distance functions $\boldsymbol{\rho}$ and the associated tangential part $\boldsymbol{\mu}_G(\boldsymbol{\rho})$. Finally, we restrict $\widehat{E}$ on the set of distance functions belonging to triple junction manifolds fulfilling the volume constraints, which we parametrise using the function γ constructed in the previous section. In total, we get the energy functional

$$\widetilde{E} : C^2_{TJ,C,(0)}(\Gamma_*) \supset \widetilde{V} \to \mathbb{R}, \boldsymbol{\rho} \mapsto \sum_{i=1}^{3} \mathrm{Area}\left(\Gamma^i_{\gamma(\boldsymbol{\rho})}\right). \tag{6.18}$$

on the space

$$C^2_{TJ,C,(0)}(\Gamma_*) = \left\{ \boldsymbol{\rho} \in C^2_{TJ}(\Gamma_*) \middle| \sum_{i=1}^{3} \rho^i = 0 \text{ on } \Sigma_*, \int_{\Gamma^1_*} \rho^1 d\mathcal{H}^n = \int_{\Gamma^2_*} \rho^2 d\mathcal{H}^n = \int_{\Gamma^3_*} \rho^3 d\mathcal{H}^n \right\}. \tag{6.19}$$

The variation of this can again be calculated using the surface transport theorem and for $\boldsymbol{\rho}_0 \in \widetilde{V}, \boldsymbol{\rho} \in C^2_{TJ,C,(0)}(\Gamma_*)$ this leads to

$$\begin{aligned} \widetilde{E}'(\boldsymbol{\rho}_0)\boldsymbol{\rho} = &- \sum_{i=1}^{3} \int_{\Gamma^i_{\gamma(\boldsymbol{\rho}_0)}} H^i_{\gamma(\boldsymbol{\rho}_0)} \left(\rho^i N^i_* + \mu^i(\boldsymbol{\rho}) \tau^i_* + [\gamma'(\boldsymbol{\rho}_0)\boldsymbol{\rho}]\, N^i_* \right) \cdot N^i_{\gamma(\boldsymbol{\rho}_0)} d\mathcal{H}^n \\ &- \int_{\Sigma_{\gamma(\boldsymbol{\rho}_0)}} \sum_{i=1}^{3} \left(\rho^i N^i_* + \mu^i(\boldsymbol{\rho}) \tau^i_* \right) \cdot \nu^i_{\gamma(\boldsymbol{\rho}_0)} d\mathcal{H}^{n-1}. \end{aligned} \tag{6.20}$$

Hereby, we used that $\gamma'(\boldsymbol{\rho}_0)\boldsymbol{\rho}$ vanishes on Σ_*. For later use we need to rewrite the boundary term in (6.20) such that it does not depend on ρ^1. Using $\rho^1 = -\rho^2 - \rho^3$ and

$$\begin{aligned} \mu^1 &= -\frac{1}{\sqrt{3}}(\rho^2 - \rho^3), \\ \mu^2 &= -\frac{1}{\sqrt{3}}(\rho^3 - \rho^1) = -\frac{1}{\sqrt{3}}(2\rho^3 + \rho^2), \\ \mu^3 &= -\frac{1}{\sqrt{3}}(\rho^1 - \rho^2) = \frac{1}{\sqrt{3}}(2\rho^2 + \rho^3), \end{aligned}$$

we can rewrite it as

$$\begin{aligned} -\int_{\Sigma_*} &\rho^2 (N^2_* \cdot \nu^2_{\gamma(\boldsymbol{\rho}_0)} - N^1_* \cdot \nu^1_{\gamma(\boldsymbol{\rho}_0)} - \frac{1}{\sqrt{3}} \tau^1_* \cdot \nu^1_{\gamma(\boldsymbol{\rho}_0)} - \frac{1}{\sqrt{3}} \tau^2_* \cdot \nu^2_{\gamma(\boldsymbol{\rho}_0)} \\ &+ \frac{2}{\sqrt{3}} \tau^3_* \cdot \nu^3_{\gamma(\boldsymbol{\rho}_0)}) J_{\Sigma_{\gamma(\boldsymbol{\rho}_0)}} + \rho^3 (N^3_* \cdot \nu^3_{\gamma(\boldsymbol{\rho}_0)} - N^1_* \cdot \nu^1_{\gamma(\boldsymbol{\rho}_0)} + \frac{1}{\sqrt{3}} \tau^1_* \cdot \nu^1_{\gamma(\boldsymbol{\rho}_0)} \\ &- \frac{2}{\sqrt{3}} \tau^2_* \cdot \nu^2_{\gamma(\boldsymbol{\rho}_0)} + \frac{1}{\sqrt{3}} \tau^3_* \cdot \nu^3_{\gamma(\boldsymbol{\rho}_0)}) J_{\Sigma_{\gamma(\boldsymbol{\rho}_0)}} d\mathcal{H}^{n-1} \end{aligned} \tag{6.21}$$

We also need to rewrite the tangential and the $\bar{\gamma}$-part in the first line in (6.20). For the first one we have

$$\begin{aligned} &- \sum_{i=1}^{3} \int_{\Gamma^i_{\gamma(\boldsymbol{\rho}_0)}} H^i_{\gamma(\boldsymbol{\rho}_0)} \mu^i(\boldsymbol{\rho}) \tau^i_* \cdot N^i_{\gamma(\boldsymbol{\rho}_0)} d\mathcal{H}^n = \int_{\Gamma^1_{\gamma(\boldsymbol{\rho}_0)}} H^1_{\gamma(\boldsymbol{\rho}_0)} \frac{1}{\sqrt{3}} (\rho^2 - \rho^3) \tau^1_* \cdot N^1_{\gamma(\boldsymbol{\rho}_0)} d\mathcal{H}^n \\ &+ \int_{\Gamma^2_{\gamma(\boldsymbol{\rho}_0)}} H^2_{\gamma(\boldsymbol{\rho}_0)} \frac{1}{\sqrt{3}} (\rho^3 - \rho^1) \tau^2_* N^2_{\gamma(\boldsymbol{\rho}_0)} d\mathcal{H}^n + \int_{\Gamma^3_{\gamma(\boldsymbol{\rho}_0)}} H^3_{\gamma(\boldsymbol{\rho}_0)} \frac{1}{\sqrt{3}} (\rho^1 - \rho^2) \tau^3_* \cdot N^3_{\gamma(\boldsymbol{\rho}_0)} d\mathcal{H}^n \\ &= \sum_{i=1}^{3} \int_{\Gamma^i_*} \rho^i \frac{1}{\sqrt{3}} (H^{i-1}_{\gamma(\boldsymbol{\rho}_0)} \tau^{i-1}_* \cdot N^{i-1}_{\gamma(\boldsymbol{\rho}_0)} - H^{i+1}_{\gamma(\boldsymbol{\rho}_0)} \tau^{i+1}_* \cdot N^{i+1}_{\gamma(\boldsymbol{\rho}_0)}) J_{\Gamma^i_{\gamma(\boldsymbol{\rho}_0)}} d\mathcal{H}^n. \end{aligned} \tag{6.22}$$

Hereby, an $i - 1 = -1$ has to be read as 3. Additionally, we did abuse of notation by just considering $H^{i-1}, \tau^{i-1}_*, N^{i-1}$ resp. $H^{i+1}, \tau^{i+1}_*, N^{i+1}$ as functions on Γ^i_*. To be precise one would need to include a pullback. For the $\bar{\gamma}'$-term we need to do the same trick as in the proof of the second identity for $\widetilde{E}'$ in Lemma 5.7 using the two dimensional structure of the range of $\bar{\gamma}$. We will skip this here for the sake of readability.
The second variation of $\widetilde{E}$ was already calculated in [36, Proposition 3.3] varying twice in the same direction. For the variation in two different directions $\boldsymbol{\rho}, \bar{\boldsymbol{\rho}} \in C^2_{TJ,C,(0)}(\Gamma_*)$ we then get by polarization

that

$$\widetilde{E}''(0)\boldsymbol{\rho}\bar{\boldsymbol{\rho}} = -\sum_{i=1}^{3}\int_{\Gamma_*^i}\left(\Delta_*\rho^i + |II_*^i|^2\rho^i\right)\bar{\rho}^i d\mathcal{H}^n - \int_{\Sigma_*}\sum_{i=1}^{3}(\partial_{\nu_*^i}\rho^i - q^i\rho^i)\bar{\rho}^i d\mathcal{H}^{n-1}. \tag{6.23}$$

with $q^1 = (\kappa_*^1 + \kappa_*^3)/\sqrt{3}, q^2 = (\kappa_*^1 - \kappa_*^3)/\sqrt{3}$ and $q^3 = (-\kappa_*^1 - \kappa_*^2)/\sqrt{3}$. Recall that the movement induced by $\boldsymbol{\mu}$ is purely tangential at the reference surface and therefore vanishes. Again with the next section in mind we rewrite the boundary term using $\bar{\rho}^1 = -\bar{\rho}^2 - \bar{\rho}^3$ as

$$\int_{\Sigma_*}(\partial_{\nu_*^2}\rho^2 - \partial_{\nu_*^1}\rho^1 - q^2\rho^2 + q^2\rho^1)\bar{\rho}^2 + (\partial_{\nu_*^3}\rho^3 - \partial_{\nu_*^1}\rho^1 - q^3\rho^3 + q^1\rho^1)\bar{\rho}^3 d\mathcal{H}^{n-1}. \tag{6.24}$$

6.5 The Łojasiewicz-Simon-Gradient Inequality for the Surface Energy on Triple Junction Manifolds

Now we can proceed to the proof of the LSI in the case of triple junction manifolds. We first need a setting for [25] and so similar to the previous chapter we set for $m > 2 + \frac{n}{2}, m \in \mathbb{N}$

$$V = \left\{\boldsymbol{\rho} \in H^m_{TJ}(\Gamma_*) \,\middle|\, \sum_{i=1}^{3}\rho^i = 0 \text{ on } \Sigma_*, \int_{\Gamma_*^1}\rho^1 d\mathcal{H}^n = \int_{\Gamma_*^2}\rho^2 d\mathcal{H}^n = \int_{\Gamma_*^3}\rho^3 d\mathcal{H}^n\right\}, \tag{6.25}$$

$$W = H^{m-2}_{TJ}(\Gamma_*) \times \left(H^{m-\frac{3}{2}}(\Sigma_*)\right)^2, \tag{6.26}$$

$$W \hookrightarrow V^* : (\mathfrak{f}, \mathfrak{b}^2, \mathfrak{b}^3) \mapsto \left(\boldsymbol{\rho} \mapsto \sum_{i=1}^{3}\int_{\Gamma_*^i}\mathfrak{f}^i\rho^i d\mathcal{H}^n + \int_{\Sigma_*}\mathfrak{b}^2\rho^2 + \mathfrak{b}^3\rho^3 d\mathcal{H}^{n-1}\right). \tag{6.27}$$

Here, we get only two copies of $H_{TJ}^{m-\frac{3}{2}}(\Sigma_*)$ as one degree of freedom is lost due to the sum condition. Note that due to our choice of m we have that $V \hookrightarrow C^2$. In particular, V and W are Banach algebras. Reminding Remark 5.2 we note that due to (6.20), (6.21) and (6.22) this means we consider $\widetilde{E}'$ as map with values in W in the following way:

$$\widetilde{E}'(\boldsymbol{\rho}_0) = \left(\mathfrak{f}(\boldsymbol{\rho}_0), \mathfrak{b}^2(\boldsymbol{\rho}_0), \mathfrak{b}^3(\boldsymbol{\rho}_0)\right), \quad \boldsymbol{\rho}_0 \in V, \tag{6.28}$$

$$\mathfrak{f}^i(\boldsymbol{\rho}_0) = \left(H^i_{\gamma(\boldsymbol{\rho}_0)}\left(N_*^i + \bar{\gamma}'(\boldsymbol{\rho}_0)^i N_*^i\right)\cdot N^i_{\gamma(\boldsymbol{\rho}_0)} + \frac{1}{\sqrt{3}}\sum_{j=1}^{2}(-1)^j(H^{i+j}_{\gamma(\boldsymbol{\rho}_0)}\tau_*^{i+j}\cdot N^{i+j}_{\gamma(\boldsymbol{\rho}_0)})\right)J_{\Gamma^i_{\gamma(\boldsymbol{\rho}_0)}},$$

$$\mathfrak{b}^2(\boldsymbol{\rho}_0) = N_*^2\cdot\nu^2_{\gamma(\boldsymbol{\rho}_0)} - N_*^1\cdot\nu^1_{\gamma(\boldsymbol{\rho}_0)} - \frac{1}{\sqrt{3}}\tau_*^1\cdot\nu^1_{\gamma(\boldsymbol{\rho}_0)} - \frac{1}{\sqrt{3}}\tau_*^2\cdot\nu^2_{\gamma(\boldsymbol{\rho}_0)} + \frac{2}{\sqrt{3}}\tau_*^3\cdot\nu^3_{\gamma(\boldsymbol{\rho}_0)}J_{\Sigma_{\gamma(\boldsymbol{\rho}_0)}},$$

$$\mathfrak{b}^3(\boldsymbol{\rho}_0) = N_*^3\cdot\nu^3_{\gamma(\boldsymbol{\rho}_0)} - N_*^1\cdot\nu^1_{\gamma(\boldsymbol{\rho}_0)} + \frac{1}{\sqrt{3}}\tau_*^1\cdot\nu^1_{\gamma(\boldsymbol{\rho}_0)} - \frac{2}{\sqrt{3}}\tau_*^2\cdot\nu^2_{\gamma(\boldsymbol{\rho}_0)} + \frac{1}{\sqrt{3}}\tau_*^3\cdot\nu^3_{\gamma(\boldsymbol{\rho}_0)}J_{\Sigma_{\gamma(\boldsymbol{\rho}_0)}}.$$

Using (6.23) and (6.24), we get for $\widetilde{E}''(0)$ as mapping $V \to W$ that

$$\boldsymbol{\rho} \mapsto \begin{pmatrix} -\Delta_{\Gamma_*}\boldsymbol{\rho} - |II_*|^2\boldsymbol{\rho} \\ \partial_{\nu_*^2}\rho^2 - \partial_{\nu_*^1}\rho^1 - q^2\rho^2 + q^1\rho^1 \\ \partial_{\nu_*^3}\rho^3 - \partial_{\nu_*^1}\rho^1 - q^3\rho^3 + q^1\rho^1 \end{pmatrix}, \quad \boldsymbol{\rho} \in V. \tag{6.29}$$

Now, we want to verify the prerequisites to apply Theorem 5.1 to $\widetilde{E} : V \to \mathbb{R}$. We begin with the analyticity of the first variation.

Lemma 6.8 (Analyticity of $\widetilde{E}'$)**.**
There are neighbourhoods

$$0 \in U \subset V, 0 \in \widetilde{U} \subset \left\{ \boldsymbol{\rho} \in H^m_{TJ}(\Gamma_*) \middle| \sum_{i=1}^{3} \rho^i = 0 \ \text{on} \ \Sigma_* \right\}$$

such that the following maps are analytic

i.) $\widetilde{U} \to H^{m-1}_{TJ}(\Gamma_*, \mathbb{R}^{n+1}), \boldsymbol{\rho} \mapsto \partial^{\boldsymbol{\rho}}_j$ *for all* $j = 1, ..., n$,
$\widetilde{U} \to \left(H^{m-\frac{3}{2}}(\Sigma_*, \mathbb{R}^{n+1})\right)^3, \boldsymbol{\rho} \mapsto \partial^{\boldsymbol{\rho}}_j$ *für all* $j = 1, ..., n-1$.

ii.) $\widetilde{U} \to H^{m-1}_{TJ}(\Gamma_*), \boldsymbol{\rho} \mapsto g^{\boldsymbol{\rho}}_{jk}, \boldsymbol{\rho} \mapsto g^{jk}_{\boldsymbol{\rho}}$ *for all pairs* $j, k \in \{1, ..., n\}$,
$\widetilde{U} \to \left(H^{m-\frac{3}{2}}(\Sigma_*)\right)^3, \boldsymbol{\rho} \mapsto g^{\boldsymbol{\rho}}_{jk}, \boldsymbol{\rho} \mapsto g^{jk}_{\boldsymbol{\rho}}$ *for all pairs* $j, k \in \{1, ..., n-1\}$.

iii.) $\widetilde{U} \to H^{m-1}_{TJ}(\Gamma_*), \boldsymbol{\rho} \mapsto J_{\Gamma_{\boldsymbol{\rho}}}$,
$\widetilde{U} \to \left(H^{m-\frac{3}{2}}(\Gamma_*)\right)^3, \boldsymbol{\rho} \mapsto J_{\Sigma_{\boldsymbol{\rho}}}$.

iv.) $\widetilde{U} \to H^{m-1}_{TJ}(\Gamma_*, \mathbb{R}^{n+1}), \boldsymbol{\rho} \mapsto N_{\boldsymbol{\rho}}$,
$\widetilde{U} \to H^{m-1}_{TJ}(\Gamma_*), \boldsymbol{\rho} \mapsto N_{\boldsymbol{\rho}} \cdot N_*$
$\widetilde{U} \to H^{m-1}_{TJ}(\Gamma_*), \boldsymbol{\rho} \mapsto N_{\boldsymbol{\rho}} \cdot \tau_*$.

v.) $\widetilde{U} \to \left(H^{m-\frac{3}{2}}(\Sigma_*, \mathbb{R}^{n+1})\right)^3, \boldsymbol{\rho} \mapsto \nu_{\boldsymbol{\rho}}$,
$\widetilde{U} \to \left(H^{m-\frac{3}{2}}(\Sigma_*)\right)^3, \boldsymbol{\rho} \mapsto \nu_{\boldsymbol{\rho}} \cdot N_*, \boldsymbol{\rho} \mapsto \nu_{\boldsymbol{\rho}} \cdot \tau_*$.

vi.) $\widetilde{U} \to H^{m-2}_{TJ}(\Gamma_*), \boldsymbol{\rho} \mapsto h^{\boldsymbol{\rho}}_{jk}$ *for all pairs* $j, k \in \{1, ..., n\}$.

vii.) $\widetilde{U} \to H^{m-2}_{TJ}(\Gamma_*), \boldsymbol{\rho} \mapsto H_{\boldsymbol{\rho}}$.

viii.) $U \to H^m_{TJ}(\Gamma_*), \boldsymbol{\rho} \mapsto \gamma(\boldsymbol{\rho})$.

ix.) $U \to W, \boldsymbol{\rho} \mapsto \widetilde{E}'(\boldsymbol{\rho})$.

Remark 6.9 (Argumentation in local coordinates)**.**
Remember that we parametrised all hypersurfaces over the same domain, cf. Section 6.2. Therefore, the $\partial_i, g_{ij}, g^{ij}$ *and* h_{ij} *are global quantities and we do not need a localization argument as in Lemma 5.9.*

Proof. Before we begin with the proof itself we want to remind that pullback and pushforward of function spaces on Γ_* to function spaces in local coordinates are analytic operators. So, we can work in the Sobolev spaces of local coordinates.
For the transformation of the ∂_i we recall from Section 4.4 that

$$\partial^{\boldsymbol{\rho}}_j = \partial^*_j + (\partial_j \boldsymbol{\rho})\, N_* + \boldsymbol{\rho} \partial_j N_* + \partial_j \boldsymbol{\mu}(\boldsymbol{\rho}) \tau_* + \boldsymbol{\mu}(\boldsymbol{\rho}) \partial_j \tau_*.$$

The first summand is constant, the second and third clearly linear in $\boldsymbol{\rho}$. As $\boldsymbol{\mu}$ is also linear in ρ and partial derivatives are linear operators this shows that the other terms are linear in $\boldsymbol{\rho}$. Thus, $\partial^{\boldsymbol{\rho}}_j$ is affin-linear in $\boldsymbol{\rho}$ and therefore analytic. As due to our choice of m we have that $H^{m-1}_{TJ}(\Gamma_*)$ is a Banach algebra, the first part of ii.) follows directly from i.) as products of analytic functions between Banach algebras are again analytic. For the g^{jk} we use as in the proof of 5.9 the fact that the inverse matrix is an analytic operator. For iii.) we use again that $J_{\Gamma_{\boldsymbol{\rho}}}$ resp. $J_{\Sigma_{\boldsymbol{\rho}}}$ are given by $\sqrt{g_{\Gamma_{\boldsymbol{\rho}}}}$ resp. $\sqrt{g_{\Sigma_{\boldsymbol{\rho}}}}$ and these quantities are analytic due to theory for composition operators and the fact that the determinant and the square root are analytic on suitable domains.
Using the multi-linear structure of the crossproduct and the analyticity of $\partial^{\boldsymbol{\rho}}_j$ we conclude analyticity of the crossproduct of the $\partial^{\boldsymbol{\rho}}_j$. Its normalization is then analytic due to composition operator theory

and so we get the first part of *iv.)* from which we get the results for the other functions. With analyticity of $N_{\boldsymbol{\rho}}$ we can argue for *v.)* in the same way as the outer conormal is given as normalized crossproduct of $N_{\boldsymbol{\rho}}$ and the $\partial_j^{\boldsymbol{\rho}}$ for $j = 1, ..., n-1$. For the $h_{jk}^{\boldsymbol{\rho}}$ we have to study the second derivatives for which we get

$$\begin{aligned}\partial_j\partial_k\varphi_{\boldsymbol{\rho}} = h_{jk}^* &+ (\partial_j\partial_k\boldsymbol{\rho})N_* + \partial_k\boldsymbol{\rho}\partial_j N_* + \partial_j\boldsymbol{\rho}\partial_k N_* + \boldsymbol{\rho}\partial_j\partial_k N_* \\ &+ \partial_j\partial_k\boldsymbol{\mu}(\boldsymbol{\rho})\tau_* + \partial_k\boldsymbol{\mu}(\boldsymbol{\rho})\partial_j\tau_* + \partial_j\boldsymbol{\mu}(\boldsymbol{\rho})\partial_k\tau_* + \boldsymbol{\mu}(\boldsymbol{\rho})\partial_j\partial_k\tau_*.\end{aligned}$$

Again using the fact that $\boldsymbol{\mu}$ is linear in $\boldsymbol{\rho}$ we see that this is an affin-linear function in $\boldsymbol{\rho}$ and thus analytic. Then, $h_{jk}^{\boldsymbol{\rho}}$ is analytic as product of analytic functions. Analyticity of $H_{\boldsymbol{\rho}}$ is a consequence of ii.) and vi.). For viii.) we can argue as in the proof of Lemma 5.5 to see that the function G from Lemma 6.6 is analytic and thus by the analytic version of the implicit function theorem (cf. Corollary 2.5) γ is.
Now we remind what $\widetilde{E}'$ as function with values in W actually is, cf. (6.28). Analyticity of these expressions follows now from the results i.)-viii.), which finishes the proof. $\square$

Now it remains to show that $\widetilde{E}''(0)$ is a Fredholm operator of index 0.

Lemma 6.10 (Fredholm propety of $\widetilde{E}''(0)$)**.**
The map $\widetilde{E}''(0) : V \to W$ is a Fredholm operator of index 0.

Proof. We remind here that (6.29) gives us $\widetilde{E}''(0)$ as map with values in W. As in the proof of Lemma 5.8 it is enough to prove bijectivity of the main part. Then, the claim follows as compact perturbations preserve the Fredholm index.
Therefore, we consider the elliptic problem

$$\begin{aligned}-\Delta_{\Gamma_*^i}\rho^i &= f^i && \text{on } \Gamma_*^i, i = 1,2,3, && (6.30)\\ \rho^1 + \rho^2 + \rho^3 &= 0 && \text{on } \Sigma_*, && (6.31)\\ \partial_{\nu_*^2}\rho^2 - \partial_{\nu_*^1}\rho^1 &= b^2 && \text{on } \Sigma_*, && (6.32)\\ \partial_{\nu_*^3}\rho^3 - \partial_{\nu_*^1}\rho^1 &= b^3 && \text{on } \Sigma_*, && (6.33)\end{aligned}$$

for $\boldsymbol{f} \in L^2_{TJ}(\Gamma_*), b^2, b^3 \in L^2(\Sigma_*)$. We observe that for a classical solution $\boldsymbol{\rho}$ and a testfunction $\boldsymbol{\psi} \in C^2_{TJ,C}(\Gamma_*)$ multiplication of 6.30^i with ψ^i, integrating over Γ_*^i and summing over i yields to

$$\begin{aligned}\sum_{i=1}^3\int_{\Gamma_*^i} f^i\psi^i d\mathcal{H}^n &= \sum_{i=1}^3\int_{\Gamma_*^i}(-\Delta_*\rho^i)\psi^i d\mathcal{H}^n = \sum_{i=1}^3\int_{\Gamma_*^i}\nabla_*\rho^i\cdot\nabla_*\psi^i d\mathcal{H}^n - \int_{\Sigma_*}\sum_{i=1}^3\left(\nabla_*\rho^i\cdot\nu_*^i\right)\psi^i d\mathcal{H}^{n-1}\\ &= \sum_{i=1}^3\int_{\Gamma_*^i}\nabla_*\rho^i\cdot\nabla_*\psi^i d\mathcal{H}^n - \int_{\Sigma_*}\partial_{\nu_*^1}\rho^1(-\psi^2-\psi^3) + \partial_{\nu_*^2}\rho^2\psi^2 + \partial_{\nu_*^3}\rho^3\psi^3 d\mathcal{H}^{n-1}\\ &= \sum_{i=1}^3\int_{\Gamma_*^i}\nabla_*\rho^i\cdot\nabla_*\psi^i d\mathcal{H}^n - \int_{\Sigma_*}\left(\partial_{\nu_*^2}\rho^2 - \partial_{\nu_*^1}\rho^1\right)\psi^2 + \left(\partial_{\nu_*^3}\rho^3 - \partial_{\nu_*^1}\rho^1\right)\psi^3 d\mathcal{H}^{n-1}\\ &= \sum_{i=1}^3\int_{\Gamma_*^i}\nabla_*\rho^i\cdot\nabla_*\psi^i d\mathcal{H}^n - \int_{\Sigma_*} b^2\psi^2 + b^3\psi^3 d\mathcal{H}^{n-1}.\end{aligned}$$

Therefore, we get the following weak formulation. We set

$$\begin{aligned}\mathcal{E} &:= \left\{\ \boldsymbol{\rho} \in H^1_{TJ}(\Gamma_*) \middle| \gamma^1\rho^1 + \gamma^2\rho^2 + \gamma^3\rho^3 = 0 \text{ on } \Sigma_*\right\},\\ B(\boldsymbol{\rho},\boldsymbol{\psi}) &:= \sum_{i=1}^3\int_{\Gamma_*^i}\nabla_{\Gamma_*^i}\rho^i\cdot\nabla_*\psi^i d\mathcal{H}^n, \quad \forall\boldsymbol{\rho},\boldsymbol{\psi}\in\mathcal{E},\\ F(\boldsymbol{\psi}) &:= \sum_{i=1}^3\int_{\Gamma_*^i} f^i\psi^i d\mathcal{H}^n + \int_{\Sigma_*} b^2\psi^2 + b^3\psi^3 d\mathcal{H}^{n-1}, \quad \forall\boldsymbol{\psi}\in\mathcal{E},\end{aligned}$$

and consider the problem

$$B(\boldsymbol{\rho}, \boldsymbol{\psi}) = F(\boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \mathcal{E}. \tag{6.34}$$

Due to Lemma 2.10 we get that B is a continuous, coercive bilinear form and then the Lax Milgram lemma gives the existence of a unique solution $\boldsymbol{\rho} \in \mathcal{E}$ to (6.34). For $\boldsymbol{f} \in H^{m-2}_{TJ}(\Gamma_*)$ and $b^2, b^3 \in H^{m-\frac{3}{2}}(\Sigma_*)$ we can apply locally elliptic regularity theory from [4] to get that $\boldsymbol{\rho}$ is actually in $H^m_{TJ}(\Gamma_*)$. Note that the necessary complementary conditions were proven in [19, Lemma 3]. From this we conclude that the operator

$$\boldsymbol{\rho} \mapsto \begin{pmatrix} -\Delta_{\Gamma_*}\boldsymbol{\rho} \\ \partial_{\nu_*^2}\rho^2 - \partial_{\nu_*^1}\rho^1 \\ \partial_{\nu_*^3}\rho^3 - \partial_{\nu_*^1}\rho^1 \end{pmatrix},$$

is bijective as a map from V to W, which finishes the proof. □

With this we deduce the desired LSI for the surface energy.

Theorem 6.11 (LSI for the parametrised surface area of triple junctions)**.**
The Łojasiewicz-Simon gradient inequality for $\widetilde{E}$ holds in $0 \in V$ for the setting (6.25)-(6.27) for V and W, i.e,. there exists $\sigma, C > 0$ and $\bar{\theta} \in (0, \frac{1}{2}]$ such that for all $x \in V$ with $\|x\| \leq \sigma$ it holds that

$$|\widetilde{E}(x) - \widetilde{E}(0)|^{1-\bar{\theta}} \leq C\|\widetilde{E}'(x)\|_W. \tag{6.35}$$

Proof. The embedding $V \subset W \hookrightarrow V'$ is definite as the L^2-product is. The point 0 is indeed a critical point of $\widetilde{E}$ as we have $\bar{\gamma}'(0) = 0$ due to (6.17) and then because of (6.20) for any $\boldsymbol{\rho} \in V$ that

$$\begin{aligned} \widetilde{E}(0)\boldsymbol{\rho} &= -\sum_{i=1}^{3} \int_{\Gamma_*^i} H_*^i \rho^i d\mathcal{H}^n - \int_{\Gamma_*^i} \sum_{i=1}^{3} \mu^i(\rho^i) d\mathcal{H}^{n-1} \\ &= -\int_{\Gamma_*^1} (-H_*^2 - H_*^3)\rho^1 d\mathcal{H}^n - \int_{\Gamma_*^2} H_*^2\rho^2 d\mathcal{H}^n - \int_{\Gamma_*^3} H_*^3\rho^3 d\mathcal{H}^n = 0, \end{aligned}$$

where we used in the second equality that both the μ^i and the H_*^i add up to zero and in the third equality the constraint of $\boldsymbol{\rho}$. The claims follows now from Theorem 5.1 using Lemma 6.8 and 6.10. □

6.6 Global Existence and Stability near Stationary Double Bubbles

We now want to use Theorem 6.11 to proof stability of stationary double bubbles evolving due to surface diffusion flow using the strategy from Chapter 5. Before we can do this we have to check that (4.20) admits parabolic smoothing for the solution found in Theorem 4.2. Hereby, the usual idea is to use the found solution to write the coefficient functions as fixed functions. This yields then a linear problem on which one could apply again theory from [51]. Unfortunately, the fully non-linear angle conditions prevent us from doing this. As the coefficient functions are of the same order as the boundary itself they will not have enough regularity to derive higher regularity.
Therefore, we will use the parameter trick instead. The smoothing result will only be true for reference frames that are stationary solutions. But this will be enough for the stability analysis.
The strategy of the proof splits into three steps. We will first use the parameter trick to show that away from $t = 0$ the time derivative $\partial_t\boldsymbol{\rho}$ inherits the space regularity of $\boldsymbol{\rho}$. From this we will get that the space regularity of $\boldsymbol{\rho}$ is increased by four orders. Finally, we can start a bootstrap procedure using the regularity we already have for $\boldsymbol{\rho}$ from Theorem 4.2.

Proposition 6.12 (Higher time regularity of solutions of (SDFTJ) near stationary double bubbles)**.**
Let $k \in \mathbb{N}_{\geq 4}, \alpha \in (0,1), T > 0, t_k \in (0,T]$. There are $\varepsilon_k, C_k > 0$ with the following property. For any initial data $\boldsymbol{\rho}_0 \in C^{4+\alpha}_{TJ}(\Gamma_*)$ with $\|\boldsymbol{\rho}_0\| \leq \varepsilon_k$ such that the solution $\boldsymbol{\rho}$ of (4.20) fulfils

$$\boldsymbol{\rho} \in C^{4+\alpha, 1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,T}) \cap C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T]), \tag{6.36}$$

we have the increased time regularity

$$\partial_t \boldsymbol{\rho} \in C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T]) \tag{6.37}$$

and we have the estimate

$$\|\partial_t \boldsymbol{\rho}\|_{C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k,T])} \leq \frac{C_k}{t_k} \|\boldsymbol{\rho}_0\|_{C^{4+\alpha}_{TJ}(\Gamma_*)}. \tag{6.38}$$

Proof. We first need to construct a parametrisation of the non-linear boundary and compatibility conditions over the linear ones. For this we consider the spaces

$$C^{4+\alpha,1+\frac{\alpha}{4}}_{TJ,LCC}(\Gamma_{*,T}) := \left\{ \boldsymbol{\rho} \in C^{4+\alpha,1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,T}) \middle| \, \mathcal{B}(\boldsymbol{\rho}) \equiv 0, \ \mathcal{B}_0(\boldsymbol{\rho}) \equiv 0 \text{ on } \Sigma_* \right\}, \tag{6.39}$$

$$X_k := C^{4+\alpha,1+\frac{\alpha}{4}}_{TJ,LCC}(\Gamma_{*,T}) \cap C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T]), \tag{6.40}$$

$$Y^1 := C^{\alpha}_{TJ}(\Sigma_*), \tag{6.41}$$

$$Y^2_k := \Pi_{i=1}^6 \left(C^{4+\alpha-\sigma_i, \frac{4+\alpha-\sigma_i}{4}}(\Sigma_{*,T}) \cap C^{k+\alpha,0}(\Sigma_* \times [t_k, T]) \right). \tag{6.42}$$

Recall that $\mathcal{B}$ denotes the linearised boundary operator and the sum condition in the first space is linked to the linearised compatibility conditions, cf. (4.44). Additionally, the σ_i correspond as before to the order of the boundary conditions, see Theorem 4.7. Now we consider on these spaces the operator

$$\begin{aligned} \widetilde{G} : X_k \oplus Z_k &\to Y^1 \times Y^2_k, \\ (\boldsymbol{u}, \bar{\boldsymbol{u}}) &\mapsto \left(\mathcal{G}_0((\boldsymbol{u} + \bar{\boldsymbol{u}})\big|_{t=0}), \mathcal{G}(\boldsymbol{u} + \bar{\boldsymbol{u}}) \right). \end{aligned}$$

Hereby, $\mathcal{G}$ denotes the non-linear boundary operator and $\mathcal{G}_0$ is corresponds to the non-linear compatibility conditions, see (4.31). Z_k is a suitable complementary space of X_k which we want to construct in the following. As we want to apply the inverse function theorem on G we want to have that $\partial_2 \widetilde{G}$ is a bijective mapping $Z_k \to Y^1 \times Y^2_k$. Thus, the space Z_k needs to contain exactly one representative for every possible value of the $\partial_2 \widetilde{G}$, which is given by the linearised boundary and compatibility operator. Note that actually every complementary space Z_k of X_k will have this property. If $x, y \in Z_k$ fulfil $\partial_2 \widetilde{G}(x) = \partial_2 \widetilde{G}(y)$ then their difference is in $\ker(\partial_2 \widetilde{G})$ and thus by construction in X_k. This implies now that $x - y = 0$.

For better readability we will write the construction of the space Z_k in the following lemma.

Lemma 6.13 (Existence of a complementary space of X_k)**.**
There is a closed complementary space of X_k in $C^{4+\alpha,1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,T}) \cap C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T])$.

Proof. We will construct Z_k as sum of two spaces, each fulfilling one of the two objectives. First, consider

$$\bar{Z}^k_1 := \left\{ \mathfrak{b} \in \Pi_{i=1}^6 C^{4+\alpha-\sigma_i,(4+\alpha-\sigma_i)/4}(\Gamma_{*,T}) : \mathfrak{b}\big|_{t=0} = 0 \right\} \cap \Pi_{i=1}^6 C^{k+\alpha-\sigma_i,0}(\Sigma_* \times [t_k, T]).$$

We can now apply the continuous solution operator from Theorem 4.7 with $\bar{Z}^k_1$ to get a subspace of $C^{4+\alpha,\frac{1+\alpha}{4}}_{TJ}(\Gamma_{*,T})$. Additionally, for (LSDFTJ) we can apply the standard localisation argument on $[t_k, T]$ to get that these functions are also in $C^{k+\alpha,0}(\Gamma_* \times [t_k, T])$ with corresponding energy estimates. Therefore, we get a continuous operator on $\bar{Z}^k_1$ with values in X_k and thus its image, which we will call Z^k_1 is a closed subspace and therefore a Banach space.

With the elements of Z^k_1 we can adjust the values of the linearised boundary operator away from $t = 0$. Now we need another space to control the boundary values at $t = 0$ and the compatibility condition. For this we consider the space

$$\bar{Z}_2 := C^{\alpha}(\Sigma_*) \times \Pi_{i=1}^6 C^{4+\alpha-\sigma_i}(\Sigma_*).$$

Now we construct for every $(\mathfrak{b}_0, \mathfrak{b}) \in \bar{Z}_2$ the solution $\boldsymbol{u}_0$ of the elliptic system

$$\begin{aligned} -\sum_{i=1}^{3} \gamma^i \mathcal{A}^i_{all} \boldsymbol{u}_0 &= \mathfrak{b}_0, && \text{on } \Gamma^i_*, i = 1, 2, 3, \\ \mathcal{B}(\boldsymbol{u}_0) &= \mathfrak{b}, && \text{on } \Sigma_*. \end{aligned}$$

Hereby, we formally extend $\mathbf{u}_0$ on Γ_*. Now extending these functions constantly in time we get a set of functions in $C^{4+\alpha,1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,T})$ which we call Z_2. Note that due to continuity of the solution operator of the elliptic problem the space Z_2 is closed.
Now we can set $Z_k := Z_1^k \times Z_2$. Obviously, we have $Z_1^k \cap Z_2 = \{0\}$. Additionally, Z is a closed subspace of $C^{4+\alpha,1+\frac{\alpha}{4}}(\Gamma_{*,T})$ which we see by the following. Suppose that we have a convergent sequence

$$(z_n)_{n\in\mathbb{N}} = (z_n^1 + z_n^2)_{n\in\mathbb{N}} \subset Z.$$

Then $(z_n^1 + z_n^2)\big|_{t=0} = z_n^2\big|_{t=0}$ converges in the $C^{4+\alpha}$-norm and as Z_2 is closed the (constantly in t extended) limit z^2 is in Z_2. This implies that

$$(z_n^1)_{n\in\mathbb{N}} = (z_n^1 + z_n^2)_{n\in\mathbb{N}} - (z_n^2)_{n\in\mathbb{N}}$$

converges also in the $C^{4+\alpha,1+\frac{\alpha}{4}}$-norm and as Z_1 is closed the limit z^1 is in Z_1. Hence, Z is a closed subspace and thus a Banach space. ☐

We continue now the proof of Proposition 6.12. Observe that $\partial_2 \widetilde{G}(0,0) = (\mathfrak{B}_0, \mathfrak{B})$ and due to the construction of Z this is now bijective. So, we can apply the implicit function theorem to get the existence of a unique function γ defined on a neighbourhood U of 0 with $\widetilde{G}(\boldsymbol{u}, \gamma(\boldsymbol{u})) = 0$. The set

$$\{\boldsymbol{u} + \gamma(\boldsymbol{u}) | \boldsymbol{u} \in U\} \tag{6.43}$$

describes all functions fulfilling the non-linear boundary and compatibility conditions near 0. In the following we will write $\bar{\gamma} := \mathrm{Id} + \gamma$ and P_{X_k} for the projection on X_k.
For some small $0 < \varepsilon < 1$ we now consider the map

$$\begin{aligned} G : (1-\varepsilon, 1+\varepsilon) \times C^{4+\alpha}_{TJ,LCC}(\Gamma_*) \times X_k &\to C^{4+\alpha}_{TJ,LCC}(\Gamma_*) \times \widetilde{X}_k, \\ (\lambda, \boldsymbol{u}_0, \boldsymbol{u}) &\mapsto \left(P_{X_k}(\bar{\gamma}(\boldsymbol{u})\big|_{t=0} - \bar{\gamma}(\boldsymbol{u}_0)), V_{\bar{\gamma}(\boldsymbol{u})} + \lambda \Delta_{\bar{\gamma}(\boldsymbol{u})} H_{\bar{\gamma}(\boldsymbol{u})} \right), \end{aligned}$$

where we use the spaces

$$\begin{aligned} C^{4+\alpha}_{TJ,LCC}(\Gamma_*) &:= \left\{ \boldsymbol{\rho} \in C^{4+\alpha}_{TJ}(\Gamma_*) | \mathcal{B}_0 \boldsymbol{\rho} \equiv 0, \mathcal{B}\boldsymbol{\rho} \equiv 0 \right\}, \\ \widetilde{X}_k &:= \left\{ \boldsymbol{\rho} \in C^{\alpha,\frac{\alpha}{4}}_{TJ}(\Gamma_{*,T}) \,\middle|\, \sum_{i=1}^{3} \rho^i\big|_{t=0} = 0 \text{ on } \Sigma_{*,T} \right\} \cap C^{k-4+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T]). \end{aligned}$$

Note that the second component G_2 indeed fulfils the compatibility condition of $\widetilde{X}_k$ as due to (4.16) we have

$$\lambda \left(\Delta^1_{\bar{\gamma}(\boldsymbol{u})} H^1_{\bar{\gamma}(\boldsymbol{u})} + \Delta^2_{\bar{\gamma}(\boldsymbol{u})} H^2_{\bar{\gamma}(\boldsymbol{u})} + \Delta^3_{\bar{\gamma}(\boldsymbol{u})} H^3_{\bar{\gamma}(\boldsymbol{u})} \right) = 0.$$

The fact that the normal velocities sum up to zero was proven in the proof of Lemma 4.21. We observe that as Γ_* is a stationary solution of (SDFTJ) we have that $G(1,0,0) = 0$. Furthermore, G is an analytic operator as it can be written as the sum of products of linear, continuous maps and parabolic Hölder-spaces have a Banach algebra structure. Finally, we have for the partial Fréchet-derivative $\partial_3 G(1,0,0)$ that

$$\partial_3 G(1,0,0)\boldsymbol{u} = (\boldsymbol{u}|_{t=0}, (\partial_t - \mathcal{A}_{all})\bar{\gamma}'(0)\boldsymbol{u}). \tag{6.44}$$

This follows from

$$\frac{d}{d\boldsymbol{u}} P_{X_k}(\bar{\gamma}(\boldsymbol{u})\big|_{t=0} - \bar{\gamma}(\boldsymbol{u}_0)) = P_{X_k}((\boldsymbol{u} + \gamma'(0)\boldsymbol{u})\big|_{t=0}) \\ = P_{X_k}(\boldsymbol{u})\big|_{t=0} + P_{X_k}(\gamma'(0)\boldsymbol{u})\big|_{t=0} = \boldsymbol{u}\big|_{t=0},$$

as $\gamma'(0)\boldsymbol{u} \in Z_k$ and thus its projection vanishes. Due to the result[2] from Corollary 4.8 we see that $\partial_3 G(1,0,0)$ is bijective and then the implicit function theorem yields the existence of neighbourhoods

$$(1,0) \in U \subset (1-\varepsilon, 1+\varepsilon) \times C^{4+\alpha}_{TJ,LCC}(\Gamma_*) \tag{6.45}$$

and

$$0 \in V \subset C^{4+\alpha, 1+\frac{\alpha}{4}}_{TJ,LCC}(\Gamma_{*,T}) \tag{6.46}$$

together with a unique, analytic function $\zeta : U \to V$ such that for all $(\lambda, \boldsymbol{u}_0) \in U$ we have that

$$G(\lambda, \boldsymbol{u}_0, \zeta(\lambda, \boldsymbol{u}_0)) = 0. \tag{6.47}$$

On the other hand, we may consider the solution of (SDFTJ) with initial data $\boldsymbol{\rho}_0$ denoted by $\boldsymbol{\rho}_{\boldsymbol{\rho}_0}$. Then the time-scaled function

$$\boldsymbol{\rho}_{\boldsymbol{\rho}_0, \lambda} := \boldsymbol{\rho}_{\boldsymbol{\rho}_\mathbf{0}}(x, \lambda t) \tag{6.48}$$

also solves (6.47) and by uniqueness we get

$$\zeta(\lambda, \boldsymbol{\rho}_0) = \boldsymbol{\rho}_{\boldsymbol{\rho}_0, \lambda}. \tag{6.49}$$

Consequently, this map is smooth in λ and we have

$$\partial_\lambda \boldsymbol{\rho}_{\boldsymbol{\rho}_0, \lambda}(t) = t \partial_t \boldsymbol{\rho}_{\boldsymbol{\rho}_0}(\cdot, \lambda t).$$

From this we conclude that $\partial_t \boldsymbol{\rho}_{\boldsymbol{\rho}_0}(\cdot, t) \in C^{k+\alpha}_{TJ}(\Gamma_*)$ for all $t \in [t_k, T]$. Finally, we observe that

$$\|\partial_\lambda \zeta(1, \boldsymbol{\rho}_0)\|_{X_k} \leq \int_0^1 \|\partial_2 \partial_\lambda \zeta(1, s\boldsymbol{\rho}_0)\boldsymbol{\rho}_0\|_{X_k} ds \leq \int_0^1 C\|\boldsymbol{\rho}_0\|_{C^{4+\alpha}} ds = C\|\boldsymbol{\rho}_0\|_{C^{4+\alpha}}. \tag{6.50}$$

Here, we used in the second step that ζ is analytic and thus its derivatives are bounded on U. This shows the claim. ☐

In the next step we want to use the gained time regularity to show additional regularity in space.

Proposition 6.14 (Higher space regularity of solutions of (SDFTJ) near stationary dubble-bubbles)**.** *For $T > 0, k \in \mathbb{N}_{\geq 4}, t_k \in (0, T]$, there is $D_k, C'_k > 0$ such that for all solutions*

$$\boldsymbol{\rho} \in C^{4+\alpha, 1+\frac{\alpha}{4}}_{TJ}(\Gamma_{*,T}) \cap C^{k+\alpha, 0}_{TJ}(\Gamma_* \times [t_k, T]) \tag{6.51}$$

with

$$\partial_t \boldsymbol{\rho} \in C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T]), \quad \|\partial_t \boldsymbol{\rho}(t)\|_{C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T])} \leq D_k, \tag{6.52}$$

we have that $\boldsymbol{\rho}(t) \in C^{k+3+\alpha}_{TJ}(\Gamma_*)$ for all $t \in [t_k, T]$ and

$$\|\boldsymbol{\rho}\|_{C^{k+3+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T])} \leq \frac{C'_k}{t_k} \|\partial_t \boldsymbol{\rho}\|_{C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T])}. \tag{6.53}$$

[2] As mentioned above we get higher regularity away from $t = 0$ for (LSDFTJ) for smooth enough data.

Proof. We consider the operator

$$G: C^{k-1+\alpha}_{TJ}(\Gamma_*) \times C^{k+3+\alpha}_{TJ}(\Gamma_*) \to C^{k-1+\alpha}_{TJ}(\Gamma_*) \times \Pi_{i=1}^{6} C^{k+3+\alpha-\sigma_i}(\Sigma_*),$$
$$(\mathfrak{f}, \boldsymbol{\rho}) \mapsto (-\Delta_{\boldsymbol{\rho}} H_{\boldsymbol{\rho}} - \mathfrak{f}, \mathcal{G}(\boldsymbol{\rho})).$$

Hereby, $\mathcal{G}$ denotes the non-linear boundary operator from (4.29) and the σ_i are chosen as in Theorem 4.7. We have that $G(0,0)=0$ as Γ_* is a stationary solution of $(SDFTJ)$ and additionally we observe that due to our results from Section 4.4 we have

$$\partial_2 G(0,0)\boldsymbol{\rho} = (\mathcal{A}_{all}(\boldsymbol{\rho}), \mathcal{B}(\boldsymbol{\rho})).$$

As we checked in Section 4.5.2 the Lopatinskii-Shapiro conditions for this system, we may apply the results from [4] to see that $\partial_2 G(0,0)$ is bijective. Hence, the implicit function theorem yields the existence of neighbourhoods 0 in $U \subset C^{k-1+\alpha}_{TJ}(\Gamma_*)$ and 0 in $V \subset C^{k+3+\alpha}_{TJ}(\Gamma_*)$ and a unique, smooth function $\zeta : U \to V$ fulfilling

$$G(\mathfrak{f}, \zeta(\mathfrak{f})) = 0. \tag{6.54}$$

Now, we want to connect this with the solution $\boldsymbol{\rho}$ of $(SDFTJ)$. For this we observe that for

$$\mathfrak{f}(\boldsymbol{\rho}(t)) := V_{\boldsymbol{\rho}}(t) = (\partial_t \boldsymbol{\rho}(t) N_* + \partial_t \boldsymbol{\mu}(\boldsymbol{\rho})(t)\tau_*) \cdot N_{\boldsymbol{\rho}(t)} \tag{6.55}$$

we have $\mathfrak{f}(\boldsymbol{\rho}(t)) \in C^{k-1+\alpha}(\Gamma_*)$ and

$$G(\mathfrak{f}(\boldsymbol{\rho}(t)), \boldsymbol{\rho}(t)) = 0. \tag{6.56}$$

Due to uniqueness of ζ this shows $\zeta(\mathfrak{f}(\boldsymbol{\rho}(t))) = \boldsymbol{\rho}(t)$ and thus $\boldsymbol{\rho}(t) \in C^{k+3+\alpha,0}_{TJ}(\Gamma_*)$. The estimate (6.53) can be proven as in (6.50). ☐

In the final step we start now a boot-strap procedure to get arbitrary high space regularity.

Proposition 6.15 (C^k-regularity in space near stationary double bubbles)**.**
Let $\boldsymbol{\rho}$ be the solution of (SDFTJ) from Theorem 4.2 with initial data $\boldsymbol{\rho}_0$ and existence time T. For every $k \in \mathbb{N}_{\geq 4}$ and $t_k \in (0,T]$ there are $\varepsilon_k > 0, C_k > 0$ such that for all $\boldsymbol{\rho}_0 \in C^{4+\alpha}_{TJ}(\Gamma_*)$ with $\|\boldsymbol{\rho}_0\| \leq \varepsilon_k$ we have

$$\boldsymbol{\rho} \in C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k, T]), \quad \|\boldsymbol{\rho}\|_{C^{k+\alpha,0}_{TJ}(\Gamma_* \times [t_k,T])} \leq \frac{C_k}{t_k} \|\boldsymbol{\rho}_0\|_{C^{4+\alpha}_{TJ}(\Gamma_*)}. \tag{6.57}$$

Remark 6.16 (C^∞-regularity)**.**
As we might have $\varepsilon_k \to 0$ for $k \to \infty$ this does not show C^∞-regularity. But for our stability analysis it will be enough to choose k big enough depending only on the dimension of the surrounding space.

Proof. Proposition 6.12 and Proposition 6.14 start a bootstrap procedure as for $k=4$ condition (6.36) is already fulfilled due to Theorem 4.2. Then, in every step we gain three orders of differentiability in space, which shows the claim. ☐

With this parabolic smoothing result we can now as before transform the analytic LSI from Theorem 6.11 to a more geometric version.

Lemma 6.17 (Geometric LSI for the Surface Energy on Triple Junctions)**.**
Consider for $k > \max(2+\frac{n}{2}, 5)$ and $k' := \frac{2-\bar\theta}{\bar\theta}k + 2$ with $\bar\theta$ from Theorem 6.11 and any $R > 0$ the set

$$Z_R := B_R(0) \subset H^{k'}_{TJ}(\Gamma_*).$$

Then, there is a$\sigma > 0$ and a $C(T) > 0$ only depending on R such that for all

$$\boldsymbol{\rho} \in \left\{ H^k_{TJ}(\Gamma_*) \cap Z_R \middle| \rho^1 + \rho^2 + \rho^3 = 0 \text{ on } \Sigma_* \right\}, \tag{6.58}$$

such that $\Gamma_{\boldsymbol{\rho}}$ fulfils the volume constraints

$$Vol(\Omega^{\boldsymbol{\rho}}_{12}) = V^2_*, \quad Vol(\Omega^{\boldsymbol{\rho}}_{13}) = V^2_*, \tag{6.59}$$

and the angle conditions (4.11) with $\theta^i = \frac{2\pi}{3}, i = 1, 2, 3,,$ we have that

$$\|\widetilde{E}'(\widetilde{\boldsymbol{\rho}})\|_W \le C(X_2) \sum_{i=1}^{3} \|\widetilde{E}'(\widetilde{\boldsymbol{\rho}})\|_{L^2(\Gamma^i_*)}^{\left(\frac{2-2\bar{\theta}}{2-\bar{\theta}}\right)}, \tag{6.60}$$

where $\widetilde{\boldsymbol{\rho}}$ is the projection of $\boldsymbol{\rho}$ on U_1 induced by the map γ from Lemma 6.6. In particular, we get the following geometric LSI: there is a (possible smaller) $\sigma > 0$, $C > 0$ such that we have for $\theta := \frac{\bar{\theta}}{2}$ and all $\boldsymbol{\rho} \in X_1 \cap X_2$ that

$$\begin{aligned} |E(\Gamma_{\boldsymbol{\rho}}) - E(\Gamma_*)|^{1-\theta} &\le C(X_2) \sum_{i=1}^{3} \left(\int_{\Gamma^i_{\boldsymbol{\rho}}} |\nabla_{\boldsymbol{\rho}} H^i_{\boldsymbol{\rho}}|^2 d\mathcal{H}^n \right)^{1/2} \\ &\le C(X_2) \sqrt{\sum_{i=1}^{3} \int_{\Gamma^i_{\boldsymbol{\rho}}} \left|\nabla_{\Gamma^i_{\boldsymbol{\rho}}} H_{\Gamma^i_{\boldsymbol{\rho}}}\right|^2 d\mathcal{H}^n}. \end{aligned} \tag{6.61}$$

Proof. We first observe that the angle conditions guarantee that the $H^{k-\frac{3}{2}}(\Sigma_*)^2$-part in the W-norm of $\widetilde{E}'(\boldsymbol{\rho})$ vanishes. We can apply on each of the remaining three terms the interpolation argument (5.30) to get (6.60). We apply this now on (6.35) to get

$$|E(\Gamma_{\boldsymbol{\rho}}) - E(\Gamma_*)|^{1-\theta} \le C(R) \left(\sum_{i=1}^{3} \|\widetilde{E}'(\widetilde{\boldsymbol{\rho}})\|_{L^2(\Gamma^i_*)}^{\left(\frac{2-2\bar{\theta}}{2-\bar{\theta}}\right)} \right)^{\frac{2-\bar{\theta}}{2-2\bar{\theta}}} \le C(R) \sum_{i=1}^{3} \|\widetilde{E}'(\widetilde{\boldsymbol{\rho}})\|_{L^2(\Gamma^i_*)}, \tag{6.62}$$

where we used Jensen's inequality in the last step. Now it remains to study the L^2-norm on the right-hand-side. As we have a Poincaré inequality on the space V from (6.25) with $m = 1$, we can control these terms by

$$C(X_2) \sum_{i=1}^{3} \int_{\Gamma^i_*} |\nabla_{\boldsymbol{\rho}} \mathfrak{f}^i|^2 d\mathcal{H}^n, \tag{6.63}$$

with

$$\mathfrak{f}^i := \left(H^i_{\gamma(\boldsymbol{\rho}_0)} \left(N^i_* + \bar{\gamma}'(\boldsymbol{\rho}_0)^i N^i_*\right) \cdot N^i_{\gamma(\boldsymbol{\rho}_0)} + \frac{1}{\sqrt{3}} \sum_{j=1}^{2} (-1)^j (H^{i+j}_{\gamma(\boldsymbol{\rho}_0)} \tau^{i+j}_* \cdot N^{i+j}_{\gamma(\boldsymbol{\rho}_0)}) \right) J_{\Gamma^i_{\gamma(\boldsymbol{\rho}_0)}}. \tag{6.64}$$

This can be taken care of similiar to (5.31) and leads to the first line in (6.61). For the second line we apply equivalence of norms on the vector $\boldsymbol{x} := \left(\sqrt{\int_{\Gamma^i(t)} |\nabla_{\Gamma^i(t)} H_{\Gamma^i(t)}|^2 d\mathcal{H}^n} \right)_{i=1,2,3}$ in $\mathbb{R}^3$. $\square$

As before we can now use this geometric version to prove stability. In the following, k, k', σ are chosen as in Lemma 6.17 and ε_0 and T are the bounds for the initial data and the existence time from Theorem 4.2). We consider for $\boldsymbol{\rho}_0 \in C^{4+\alpha}_{TJ}(\Gamma_*)$, that fulfils the compatibility conditions (4.16) and $\|\boldsymbol{\rho}_0\| \le \varepsilon_0$, the solution from Theorem 4.2. By choosing $\boldsymbol{\rho}_0$ sufficiently small we can guarantee due to Proposition 6.15 that the solution $\boldsymbol{\rho}$ fulfils for all $t \in [\frac{T_0}{2}, T]$

$$\|\boldsymbol{\rho}(t)\|_{C^k_{TJ}(\Gamma_*)} \le Z := \min(\sigma, \varepsilon_0). \tag{6.65}$$

Thus, we can apply (6.61) on this set. We consider now the largest $\widetilde{T}$ such that (6.65) is fulfilled on $\widetilde{I} := [\frac{T}{2}, \widetilde{T}]$. Note that due to Theorem 4.20 the solution $\boldsymbol{\rho}$ exists on this interval. Then, for all $t \in \widetilde{I}$

we have that

$$
\begin{aligned}
-\frac{d}{dt}(E(\Gamma(t)) - E(\Gamma_*))^\theta &= -\theta(E(\Gamma(t)) - E(\Gamma_*))^{\theta-1} \sum_{i=1}^{3} \int_{\Gamma(t)^i} -V_{\Gamma^i(t)} H_{\Gamma^i(t)} d\mathcal{H}^n \\
&= \theta(E(\Gamma(t)) - E(\Gamma_*))^{\theta-1} \sum_{i=1}^{3} \int_{\Gamma^i(t)} \left(-\Delta_{\Gamma^i(t)} H_{\Gamma^i(t)}\right) H_{\Gamma^i(t)} \\
&= \theta(E(\Gamma(t)) - E(\Gamma_*))^{\theta-1} \sum_{i=1}^{3} \int_{\Gamma^i(t)} |\nabla_{\Gamma^i(t)} H_{\Gamma^i(t)}|^2 d\mathcal{H}^n \\
&+ \int_{\Sigma(t)} \sum_{i=1}^{3} (\nabla_{\Gamma^i(t)} H_{\Gamma^i(t)} \cdot \nu_{\Gamma^i(t)}) H_{\Gamma^i(t)} d\mathcal{H}^{n-1} \\
&\geq C \sqrt{\sum_{i=1}^{3} \int_{\Gamma^i(t)} \left|\nabla_{\Gamma^i(t)} H_{\Gamma^i(t)}\right|^2 d\mathcal{H}^n} \qquad (6.66) \\
&= C\|\Delta_{\Gamma(t)} H_{\Gamma(t)}\|_{\mathcal{H}^{-1}(\Gamma(t))} \\
&\geq C\|\Delta_{\Gamma(t)} H_{\Gamma(t)}\|^2_{L^2(\Gamma(t))} \\
&\geq C\|\Delta_{\Gamma(t)} H_{\Gamma(t)}\|^2_{L^2(\Gamma_*)} \\
&= C\|V_{\Gamma(t)}\|^{\frac{1}{2}}_{L^2(\Gamma_*)} \\
&= C\|\partial_t \boldsymbol{\rho}(t)(N_{\boldsymbol{\rho}(t)} \cdot N_*) + \partial_t \boldsymbol{\mu}(t)(N_{\boldsymbol{\rho}(t)} \cdot \tau_*)\|^2_{L^2(\Gamma_*)} \\
&\geq C\Big|\|\partial_t \boldsymbol{\rho}(t)(N_{\boldsymbol{\rho}(t)} \cdot N_*)\|_{L^2(\Gamma_*)} - \|\partial_t \boldsymbol{\mu}(t)(N_{\boldsymbol{\rho}(t)} \cdot \tau_*)\|_{L^2(\Gamma_*)}\Big|^2 \\
&\geq C\|\partial_t \boldsymbol{\rho}(t)(N_{\boldsymbol{\rho}(t)} \cdot N_*)\|^2_{L^2(\Gamma_*)} \\
&\geq C\|\partial_t \boldsymbol{\rho}(t)\|^2_{L^2(\Gamma_*)}.
\end{aligned}
$$

Hereby, we used in the fifth line the boundary conditions for (SDFTJ) and (6.61), in the seventh line a combination of Lemma 2.12 and Proposition 6.15, cf. (5.34), in the eleventh line the inverse triangle inequality, in the twelfth line the fact that we can control the L^2-norm of the tangential part due to

$$
|N_{\boldsymbol{\rho}(t)} \cdot \tau_*| \leq \frac{1}{2} < \frac{3}{4} \leq |N_{\boldsymbol{\rho}(t)} \cdot N_*|, \qquad (6.67)
$$

and finally in the thirteenth line the bound (6.67) for $N_{\boldsymbol{\rho}(t)} \cdot N_*$. In total, this yields the estimate

$$
\|\partial_t \boldsymbol{\rho}(t)\|^2_{L^2(\Gamma_*)} \leq -C\frac{d}{dt}(E(\Gamma(t)) - E(\Gamma_*))^\theta. \qquad (6.68)
$$

Integrating this in time we get with the same argumentation as in(5.36) for all $s \in \widetilde{I}$ that

$$
\begin{aligned}
\|\boldsymbol{\rho}(s)\|_{L^2_{TJ}(\Gamma_*)} &\leq \|\boldsymbol{\rho}(s) - \boldsymbol{\rho}(T_0)\|_{L^2_{TJ}(\Gamma_*)} + \|\boldsymbol{\rho}(T_0)\|_{L^2_{TJ}(\Gamma_*)} \\
&\leq \int_{T_0}^{s} \|\partial_t \boldsymbol{\rho}\|_{L^2_{TJ}(\Gamma_*)} dt + \|\boldsymbol{\rho}(T_0)\|_{L^2_{TJ}(\Gamma_*)} \\
&\leq \sqrt{\int_{T_0}^{s} \|\partial_t \boldsymbol{\rho}\|^2_{L^2_{TJ}(\Gamma_*)} dt} + \|\boldsymbol{\rho}(T_0)\|_{C^{\frac{\theta}{2}}_{TJ}(\Gamma_*)} \\
&\leq \sqrt{-C(E(\Gamma(t)) - E(\Gamma_*)^\theta + C(E(\Gamma(T_0))) - E(\Gamma_*))^\theta} + \|\boldsymbol{\rho}(T_0)\|^{\frac{\theta}{2}}_{C^0_{TJ}(\Gamma_*)} \qquad (6.69) \\
&\leq C(E(\Gamma(T_0)) - E(\Gamma_*))^{\frac{\theta}{2}} + C\|\boldsymbol{\rho}(T_0)\|^{\frac{\theta}{2}}_{C^{2+\alpha}_{TJ}(\Gamma_*)} \\
&\leq C\|\boldsymbol{\rho}(T_0)\|^{\frac{\theta}{2}}_{C^{2+\alpha}_{TJ}(\Gamma_*)} + C\|\boldsymbol{\rho}(T_0)\|^{\frac{\theta}{2}}_{C^{2+\alpha}_{TJ}(\Gamma_*)}
\end{aligned}
$$

$$\leq C\|\boldsymbol{\rho}_0\|^{\frac{\theta}{2}}_{C^{2+\alpha}_{TJ}(\Gamma_*)}.$$

Finally, from this we derive for every $t \in \widetilde{I}$ and some $\beta \in (0,1)$ that

$$\|\boldsymbol{\rho}(t)\|_{C^k_{TJ}(\Gamma_*)} \leq C\|\boldsymbol{\rho}(t)\|^{1-\beta}_{C^{k'}_{TJ}(\Gamma_*)}\|\boldsymbol{\rho}(t)\|^{\beta}_{L^2_{TJ}(\Gamma_*)} \leq C\|\boldsymbol{\rho}_0\|^{\beta\theta}_{C^{2+\alpha}_{TJ}(\Gamma_*)} \leq C\varepsilon^{\frac{\beta\theta}{2}}. \tag{6.70}$$

Here, we used in the first step interpolation results for Besov spaces, in the next step (6.69) and Proposition 6.15 and finally the bounds for the initial data. By choosing

$$\varepsilon \leq e^{\frac{2}{\beta\theta}\ln\left(\frac{Z}{2C}\right)}, \tag{6.71}$$

we get for all $t \in \widetilde{I}$ that

$$\|\boldsymbol{\rho}(t)\|_{C^k_{TJ}(\Gamma_*)} \leq \frac{Z}{2}. \tag{6.72}$$

Applying close to $\widetilde{T}$ our short time existence Theorem 4.2 and using again Proposition 6.15 we see that $\widetilde{T}$ cannot be maximal such that (6.65) is fulfilled. This shows now that $\widetilde{T} = \infty$ and consequently global existence of $\boldsymbol{\rho}$. Now, like in Section 5.4 we get convergence to an energy minimum, which is another standard double bubble. Also, the argument for the Lyapunov stability is the same. In total, this finishes the proof of Theorem 6.1.

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