Strong Coupling and non-Markovian Effects in the Statistical Notion of Temperature

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We investigate the emergence of temperature T in the system-plus-reservoir paradigm starting from the fundamental microcanonical scenario at total fixed energy E where, contrary to the canonical approach, $T = T(E)$ is not a control parameter but a derived auxiliary concept. As shown by Schwinger for the regime of weak coupling γ between system and environment, $T(E)$ emerges from the saddle-point analysis that leads to the usual ensemble equivalence in the thermodynamic limit. By extending these ideas for finite γ, we provide a consistent generalization of temperature $T(E,\gamma)$ in strongly coupled systems and we illustrate its main features for the specific model of Quantum Brownian Motion where it leads to consistent microcanonical thermodynamics. Interestingly, while this $T(E,\gamma)$ is a monotonically increasing function of the total energy E, its dependence with γ is a purely quantum effect drastically different for Markovian and non-Markovian regimes.

In the context of statistical physics there are two ways to explain how a system $\mathcal{A}$ acquires a property associated with the thermodynamic notion of temperature [1, 2]. In the first approach, one considers the system as *weakly* coupled with a thermal bath $\mathcal{B}$ that is initially in a canonical state at temperature T. If we wait long enough, $\mathcal{A}$ will equilibrate (in the sense of stationarity of macroscopic observables) and acquire itself a canonical distribution at the same T. Here, therefore, the idea of temperature is pre-assumed from the beginning. In the second approach, one considers instead that the global system $\mathcal{A}+\mathcal{B}$ is in a microcanonical distribution at total energy E (we agree with [3, 4] that this is the conceptually foundational starting point to understand the meaning of temperature). Here $\mathcal{A}$ and $\mathcal{B}$ equilibrate due to the presence of a *weak* interaction term, and the temperature will emerge as a parameter that fixes the condition of equilibrium. The temperature $T = T(E)$ is then a derived rather than a fundamental quantity.

As it has been shown when going beyond the assumption of weak interactions, for strongly coupled $\mathcal{A}$ and $\mathcal{B}$ deviations from the standard thermodynamics emerge [5–8], as well as problems defining local temperature [9–12]. Also, the equivalence between the microcanonical and the canonical approach does not hold, correlations between system and bath become important, and the system is non-extensive by nature [13–15]. In this context it is well known that when $\mathcal{A}$ is strongly coupled to a thermal bath the long time steady state of the system, contrary to the weak coupling scenario, does not take the Boltzmann form, neither in the open-quantum system approach [16–18], in the global closed thermal state scenario [19], nor in the pure state setup [20, 21].

In this paper we will provide a consistent definition of temperature T in the system-plus-bath scenario with arbitrary coupling strength γ by starting with a global microcanonical state at energy E and generalizing the saddle point analysis of ensemble equivalence pioneered by Schwinger [22, 23]. As an experimental relevant application, we will present the main features of this definition of temperature $T(E,\gamma)$ in the solvable case of Quantum Brownian Motion.

Let us first review the weak coupling case by considering two initially isolated systems $\mathcal{A}$ and $\mathcal{B}$ with Hamiltonians $\hat{H}_\mathcal{A}$ and $\hat{H}_\mathcal{B}$. They have fixed energies $E_\mathcal{A}$, $E_\mathcal{B}$ and each of them is described by a microcanonical state

$$\hat{\rho}_i = \frac{\delta(E_i - \hat{H}_i)}{\mathcal{G}_i(E_i)}, \qquad i = \mathcal{A}, \mathcal{B}, \tag{1}$$

where $\mathcal{G}_i(E_i) = \mathrm{Tr}_i\, \delta(E_i - \hat{H}_i)$ denotes the density of states, and Tr_i the trace over each Hilbert subspace $\mathcal{H}_i$. $\mathcal{G}(E)$ is the central quantity connecting statistics and thermodynamics through the Boltzmann equation

$$S_i(E_i) = K \log \mathcal{G}_i(E_i), \tag{2}$$

where K is the Boltzmann constant and S_i the thermodynamic entropy of each subsystem.
When we bring $\mathcal{A}$ and $\mathcal{B}$ into *weak* thermal contact, allowing them to interchange energy through a small interaction term such that $E_{int}(\gamma) \ll E_\mathcal{A} + E_\mathcal{B}$, in the long time limit, $t \gg \gamma^{-1}$, the resulting global state is microcanonical

$$\hat{\rho}_{\mathcal{AB}} = \frac{\delta(E - \hat{H}_{\mathcal{AB}})}{\mathcal{G}_{\mathcal{AB}}(E)}, \tag{3}$$

with total energy $E = E_\mathcal{A} + E_\mathcal{B} + \mathcal{O}(\gamma)$. Here $\hat{H}_{\mathcal{AB}} = \hat{H}_\mathcal{A} + \hat{H}_\mathcal{B} + \mathcal{O}(\gamma)$ acts in $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ and $\mathcal{G}_{\mathcal{AB}}(E) = \mathrm{Tr}_{\mathcal{AB}}\, \delta(E - \hat{H}_{\mathcal{AB}})$. Equation (3) can be now written as

$$\begin{aligned} \mathcal{G}_{\mathcal{AB}}(E) &= \mathrm{Tr}_{\mathcal{AB}}\ \delta(E - \hat{H}_{\mathcal{AB}}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\, e^{iE\tau} \mathrm{Tr}_{\mathcal{AB}} \left[e^{-i\tau \hat{H}_{\mathcal{AB}}} \right], \end{aligned} \tag{4}$$

where τ is an integration variable with units of inverse energy. Defining $\mathrm{Tr}_{\mathcal{AB}}[e^{-i\tau \hat{H}_{\mathcal{AB}}}] = Z_{\mathcal{AB}}(i\tau)$, Eq. (4) takes the form

$$\mathcal{G}_{\mathcal{AB}}(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\, e^{iE\tau} e^{\log Z_{\mathcal{AB}}(i\tau)}, \tag{5}$$

with an associated action ϕ

$$\begin{aligned} \phi(E,\tau) &= iE\tau + \log\, Z_{\mathcal{AB}}(i\tau) \\ &= iE\tau + \log\, Z_\mathcal{A}(i\tau) + \log\, Z_\mathcal{B}(i\tau), \end{aligned} \tag{6}$$

where the decomposition $Z_{\mathcal{AB}} = Z_{\mathcal{A}} + Z_{\mathcal{B}}$ with $Z_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{A}}[e^{-i\tau\hat{H}_{\mathcal{A}}}]$ and $Z_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{B}}[e^{-i\tau\hat{H}_{\mathcal{B}}}]$ is possible because the interaction term is small enough to be neglected. Following an idea due to Schwinger [22], for a large number of degrees of freedom, the quantities E and $\log Z_{\mathcal{AB}}$ are large (allowing rapid oscillations in the exponential) and we can solve Eq. (5) using the Saddle-Point-Approximation (SPA). The saddle-point condition

$$\frac{d}{d\tau}\phi(E,\tau)\bigg|_{\tau=\tau^*} \stackrel{!}{=} 0 \tag{7}$$

admits an analytical continuation over the lower half of the complex τ plane where we find the sole saddle-point $\tau^* = -i\beta$, with β satisfying

$$iE + i\frac{d}{d\beta}\log Z_{\mathcal{A}}(\beta)\bigg|_{\beta=i\tau^*} + i\frac{d}{d\beta}\log Z_{\mathcal{B}}(\beta)\bigg|_{\beta=i\tau^*} \stackrel{!}{=} 0. \tag{8}$$

By interpreting the real solution $\beta = 1/KT$ as the inverse temperature and $Z_i(\beta) = Z_i(i\tau^*)$ as the canonical partition function, then $\bar{E}_i = -\frac{d}{d\beta}\log Z_i(\beta)$ is the mean internal energy of each subsystem, and the relation

$$E = \bar{E}_{\mathcal{A}}(\beta = i\tau^*) + \bar{E}_{\mathcal{B}}(\beta = i\tau^*) \tag{9}$$

gives a condition on how the total energy is distributed between systems $\mathcal{A}$ and $\mathcal{B}$ when they are brought into contact. This microscopic analysis is thus used as the definition of both thermal equilibrium and of the inverse temperature that fixes this condition. To make this microscopic construction complete and see how $\beta(E)$ can indeed be interpreted as the thermodynamic temperature, let us use Eqns. (2) and (5) with the solution established by Eq. (8) to obtain

$$\mathcal{G}_{\mathcal{AB}}(E) = \mathrm{e}^{S(E)/K} = \frac{\mathrm{e}^{\beta E + \log Z_{\mathcal{AB}}(\beta)}}{\sqrt{2\pi\frac{\partial^2 \log Z_{\mathcal{AB}}(\beta)}{\partial\beta^2}}}, \tag{10}$$

which for a large number of degrees of freedom and by introducing the Helmholtz free energy $F(\beta) = -\frac{1}{\beta}\log Z_{\mathcal{AB}}(\beta)$, immediately gives the well known thermodynamic relation [23]

$$S(E)/K = \beta E - \beta F(\beta) \iff F = E - TS(E). \tag{11}$$

In the approach of Schwinger, the focus of the SPA analysis was to provide a way to justify the ensemble equivalence as follows. Following a similar procedure as the one leading to Eq. (5), we write down the microcanonical density matrix $\hat{\rho}_{\mathcal{AB}}$ as

$$\begin{aligned}\hat{\rho}_{\mathcal{AB}}(E) = &\frac{1}{2\pi\mathcal{G}_{\mathcal{AB}}(E)} \\ &\times \int_{-\infty}^{\infty} d\tau\, \mathrm{e}^{i\tau E}\mathrm{e}^{\log Z_{\mathcal{AB}}(i\tau)}\frac{\mathrm{e}^{-i\tau\hat{H}_{\mathcal{AB}}}}{Z_{\mathcal{AB}}(i\tau)},\end{aligned} \tag{12}$$

where we have multiplied and divided by $Z_{\mathcal{AB}}(i\tau)$. The expectation value of an observable $\hat{O}$ acting on the global system can then be written in the form

$$\langle\hat{O}\rangle_E = \frac{1}{2\pi\mathcal{G}_{\mathcal{AB}}(E)}\int_{-\infty}^{\infty} d\tau\, \mathrm{e}^{i\tau E}\mathrm{e}^{\log Z_{\mathcal{AB}}(i\tau)}\langle\hat{O}\rangle_{i\tau}, \tag{13}$$

where $\langle\hat{O}\rangle_{i\tau} = \mathrm{Tr}_{\mathcal{AB}}[\hat{O}\mathrm{e}^{-i\tau\hat{H}_{\mathcal{AB}}}/Z_{\mathcal{AB}}(i\tau)]$. In the thermodynamic limit, provided $\langle\hat{O}\rangle_{i\tau}$ varies slowly with respect to τ, Eq. (13) can also be solved by SPA, resulting in $\langle\hat{O}\rangle_E \approx \langle\hat{O}\rangle_{\beta(E)}$, where $\beta(E)$ is the solution of Eq. (8). This is the meaning of equivalence of ensembles, according to Schwinger, in the weak coupling approach.

After this revision of the key aspects of the emergence of temperature in composite weakly interacting systems, we now proceed to extend these ideas to systems with non-negligible interaction Hamiltonian $\hat{H}_{\mathrm{int}}$.

While the global equilibrium state in the case of finite interaction energy is

$$\hat{\rho}_{\mathcal{AB}} = \frac{\delta(E - \hat{H}_{\mathcal{A}} - \hat{H}_{\mathcal{B}} - \hat{H}_{\mathrm{int}})}{\mathcal{G}_{\mathcal{AB}}(E)}, \tag{14}$$

and the density of states is still given by Eq. (5), now $Z_{\mathcal{AB}}(i\tau)$ can not be unambiguously decomposed in general in terms of the bare Hamiltonians $\hat{H}_{\mathcal{A}}$ and $\hat{H}_{\mathcal{B}}$ [25]. However, our key observation is that *as long as we can solve Eq. (5) by SPA, the resulting real solution for β, which now depends not only on the total energy E but also on the parameters of the interaction, characterizes the condition of thermal equilibrium between $\mathcal{A}$ and $\mathcal{B}$, thus providing the statistical definition of temperature for systems with finite coupling.* To support this claim we will now study the consistency and consequences of this definition in a solvable example.

As a specific microscopic model that allows for almost full analytical treatment and remains of high experimental relevance, we consider now a microcanonical modification of the widely used open-system approach to Quantum Brownian Motion (QBM)[24]. Here $\mathcal{A}$ consists of a quantum harmonic oscillator linearly coupled to a bath $\mathcal{B}$ of N non-interacting harmonic oscillators. The total Hamiltonian reads

$$\begin{aligned}\hat{H}_{\mathcal{AB}} = &\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{q}^2 \\ &+ \frac{1}{2}\sum_{n=1}^{N}\left[\frac{\hat{p}_n^2}{m_n} + m_n\omega_n^2\left(\hat{q}_n - \frac{c_n}{m_n\omega_n^2}\hat{q}\right)^2\right],\end{aligned} \tag{15}$$

where $\hat{p}$ and $\hat{q}$ are momentum and position operators of the coupled harmonic oscillator with bare frequency ω_0 and mass m, and $\hat{p}_n, \hat{q}_n$ the momentum and position operators of the nth bath oscillator with frequency ω_n and mass m_n coupled with the central system through c_n. The bath and interactions are characterized by the bare and coupled spectral densities that distribute the

frequencies ω_n like

$$I(\omega) = \pi \sum_n \delta(\omega - \omega_n) = \kappa\omega^2 e^{-\omega/\omega_D} \tag{16a}$$

$$J(\omega) = \pi \sum_{n=1}^{N} \frac{c_n^2}{2m\omega_n}\delta(\omega - \omega_n) = m\gamma\omega e^{-\omega/\omega_D}, \tag{16b}$$

with cut-off Drude frequency ω_D and so called damping parameter γ, which is a function of the parameters c_n^2 characterizing the system-bath coupling strength. The parameter κ is a characteristic of the bath with units of ω^{-3} such that $\int_0^\infty d\omega I(\omega)/\pi = N$.

In order to use Eq. (5), we construct $Z_{\mathcal{AB}}(i\tau)$ of the QBM model by analytical continuation of the Matsubara frequencies $\nu_n = \frac{2\pi n}{\hbar i\tau}$ from the known result [26], to get

$$Z_{\mathcal{AB}}(i\tau) = Z_{\mathcal{B}}(i\tau) \times \frac{1}{\hbar i\tau\omega_0}\prod_{n=1}^{\infty}\frac{\nu_n^2(\omega_D+\nu_n)}{(\omega_0^2+\nu_n^2)(\omega_D+\nu_n)+\nu_n\gamma\omega_D}. \tag{17}$$

Here the imaginary temperature partition function $Z_{\mathcal{B}}(i\tau)$ of $\mathcal{B}$, using the spectral density from Eq. (16a), reads

$$\log Z_{\mathcal{B}}(i\tau) = -i\tau E_0 + \frac{2\kappa\zeta(4)}{(\hbar i\tau)^3}, \tag{18}$$

where $E_0 = 3\kappa\hbar\omega_D^4$ is the zero point energy of the bath, and $\zeta(x)$ is the Riemann zeta function. In this way we arrive at

$$\log Z_{\mathcal{AB}}(i\tau) = \log Z_{\mathcal{B}}(i\tau) + \log\tilde{Z}(i\tau), \tag{19}$$

where the effective $\tilde{Z}$, related to the coupled harmonic oscillator, has an explicit form in terms of Gamma functions Γ [26],

$$\tilde{Z}(i\tau) = \frac{\hbar i\tau\omega_0\Gamma(\hbar i\tau\lambda_1/2\pi)\Gamma(\hbar i\tau\lambda_2/2\pi)\Gamma(\hbar i\tau\lambda_3/2\pi)}{4\pi^2\Gamma(\hbar i\tau\omega_D/2\pi)}, \tag{20}$$

with λ_1, λ_2 and λ_3 being the roots of the polynomial expression in ν_n that appears in the denominator of Eq. (17) and which carry the dependence on γ, ω_D and ω_0. Interestingly, as shown in [27], for systems that have an interaction that involves only relative coordinates, like in Eq. (15), the *classical* partition function does not depend at all on the coupling

$$\log Z^{\text{classic}}_{\mathcal{AB}}(i\tau) = \log Z^{\text{classic}}_{\mathcal{B}}(i\tau) + \log Z^{\text{classic}}_{\mathcal{A}}(i\tau),$$

and therefore the temperature β is independent of γ, regardless how strong the interaction is. This means that for the model in Eq. (15) *the dependence of the temperature on the coupling strength is purely a quantum effect.*

In order to obtain the equilibrium condition, we now use Eqns. (5, 20) to identify the action

$$\phi(i\tau) = i(E-E_0)\tau + \frac{2\kappa\zeta(4)}{(\hbar i\tau)^3} + \log\tilde{Z}(i\tau), \tag{21}$$

where we have used Eqns. (18) and (19). Solving Eq. (5) by SPA, using the Saddle-point condition in Eq. (7), and again, looking for real solutions $\beta = i\tau^*$, we get

$$E - E_0 - \frac{2\kappa\zeta(4)}{3\hbar^3}\frac{1}{\beta^4} = \frac{1}{\beta} + \frac{\hbar}{2\pi}\left\{\omega_D\psi(1+\hbar\beta\omega_D/2\pi) - \sum_{i=1}^{3}\lambda_i\psi(1+\hbar\beta\lambda_i/2\pi)\right\}, \tag{22}$$

where we have inserted Eqns. (19) and (20) into (21). Here $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ denotes the Digamma function.

Equation (22) establishes the equilibrium relation for the total energy E between systems $\mathcal{A}$ and $\mathcal{B}$, where the *l.h.s* is related with the energy of the bath and the *r.h.s* accounts for the energy of the coupled oscillator. The divergences affecting Eq. (22) for $\omega_D \to \infty$ arise from the well known [28] divergences of the ground state energy of the coupled harmonic oscillator ϵ_0,

$$\epsilon_0 = \frac{\hbar}{2\pi}\left[\lambda_1\log[\omega_D/\lambda_1] + \lambda_2\log[\omega_D/\lambda_2] + \lambda_3\log[\omega_D/\lambda_3]\right],$$

but are readily renormalized by $\tilde{Z}\times e^{\beta\epsilon_0}$ to obtain a global zero ground state energy. The new relation (22) for renormalized energy finally reads

$$E - \frac{2\kappa\zeta(4)}{3\hbar^3}\frac{1}{\beta^4} = \frac{1}{\beta} + \frac{\hbar}{2\pi}\left\{\omega_D\psi(1+\hbar\beta\omega_D/2\pi) - \omega_D\log\omega_D\right\} - \frac{\hbar}{2\pi}\sum_{i=1}^{3}\left(\lambda_i\psi(1+\hbar\beta\lambda_i/2\pi) - \lambda_i\log\lambda_i\right), \tag{23}$$

where we have made use of the Vieta relation $\lambda_1+\lambda_2+\lambda_3 = \omega_D$ [26]. The solution of Eq. (23) for β (fixing the energy equilibrium condition for our model) accordingly defines the inverse temperature in the finite coupling regime. This solution $\beta(E,\gamma)$, our main result, depends on the interaction γ, the total energy E, but also the bath parameters κ and ω_D, and the frequency ω_0. Note that the infinite number of degrees of freedom in the subsystem $\mathcal{B}$ is what makes the SPA justified.

In Fig. 1 we show the numerical solution $\beta(E,\gamma)$ of Eq. (23) for given values of κ and ω_D. As it can be observed, β is a monotonous function of the total energy E, with a decreasing tendency that is more pronounced close to the renormalized ground state $E < \hbar\omega_0/2$. From Eq. (23), and using the asymptotic expansion of Digamma functions, we obtain that for $\beta \to \infty$, $E \to 0$ as expected. Fig. 1 shows also the growing behavior of β for increasing γ, that is again more evident in the regime of small E.

Remarkably, in the regime for large E the dependence of β with γ can be modulated to the extend of inverting the decrease observed in Fig. 1. The tunning parameter is the ratio ω_D/ω_0, that also quantifies the degree of memory of the bath or non-Markovianity [29]. This surprising connection between the dependence of the temperature on the coupling and the time scale $2\pi/\omega_D$ associated to memory effects in the environment is shown in Fig. 2,

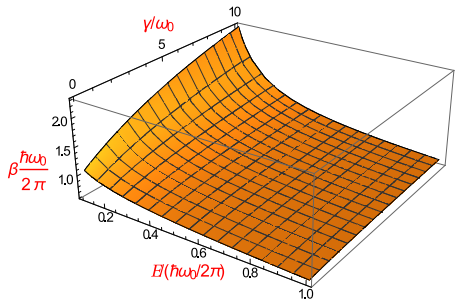

FIG. 1. Inverse temperature $\beta(E,\gamma)$ for given values $\kappa\omega_0^3 = 5$ and $\omega_D/\omega_0 = 10$, showing a smooth increase from weak to strong coupling γ and smooth decrease from low to large renormalized energy E.

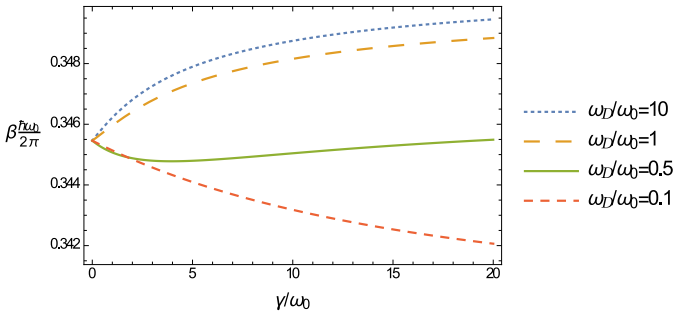

FIG. 2. $\beta(\gamma)$ for parameters $\kappa\omega_0^3 = 5$ and $E/\left(\frac{\hbar\omega_0}{2\pi}\right) = 10$, showing the increasing of β in the Markovian regime and its decreasing behavior in the non-Markovian case.

and it is further explored by the explicit dependence of β on the Drude frequency in Fig. 3.

We can provide a physical interpretation of the peculiar dependence of $d\beta/d\gamma$ on the Drude frequency ω_D for our results in the following way: If we focus on the right hand side of Eq. (23), which is related with the contribution to the condition of equilibrium of energy

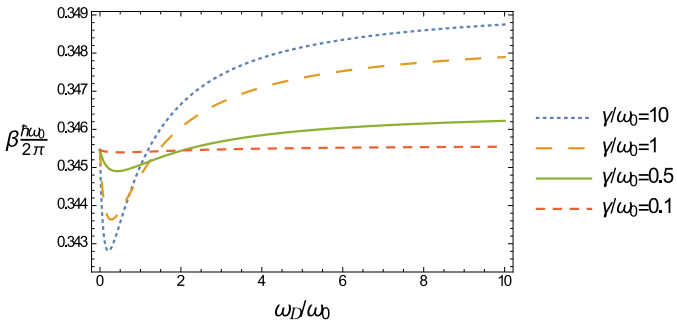

FIG. 3. $\beta(\omega_D)$ for parameters $\kappa\omega_0^3 = 5$ and $E/\left(\frac{\hbar\omega_0}{2\pi}\right) = 10$, showing the explicit dependence of β with the Drude frequency and its change of behavior from Markovian to non-Markovian regime.

for the coupled harmonic oscillator, it can be shown that this term *grows* with increasing γ in the Markovian regime (e.g., $\omega_D/\omega_0 = 10$), at least in the range of values where we find a solution for β and, on the other hand, the term *decreases* in the non-Markovian case (e.g., $\omega_D/\omega_0 = 0.1$). Now, Eq. (23) is an energy equilibrium condition between bath and the coupled oscillator fixed by the equilibrium parameter $\beta(E,\gamma)$. Therefore, it is not difficult to realize that in the Markovian regime, where the contribution of the *r.h.s* of Eq. (23) increases with γ, the equilibrium temperature is smaller compared with that obtained in the case for $\gamma = 0$, and that this equilibrium temperature actually decreases when γ increases. That explains the growing behavior of β in this regime. On the other hand, in the non-Markovian case the contribution of the *r.h.s* of (23) becomes less important and decreases when γ increases. In this case then the equilibrium temperature is greater than that obtained for $\gamma = 0$ and *grows* with γ.

Having obtained the inverse temperature $\beta(E,\gamma)$ we can calculate thermodynamic potentials for finite coupling. We start with Eq. (10), from which the entropy of the *global* system can be calculated by

$$\log Z_{\mathcal{AB}}(\beta) = S(E) - \beta E. \tag{24}$$

Recognizing that $\beta(E,\gamma)$ is a function of E, we get

$$\frac{\partial S}{\partial E} = \beta + \frac{\partial\beta}{\partial E}\left(E + \frac{\partial}{\partial\beta}\log Z_{\mathcal{AB}}\right), \tag{25}$$

and since $\frac{\partial}{\partial\beta}\log Z_{\mathcal{AB}} = -E$, we finally obtain the thermodynamic relation

$$\frac{\partial S}{\partial E} = \beta(E,\gamma). \tag{26}$$

Following Ref. [31] we may also calculate the entropy for the coupled oscillator as

$$\frac{S_{\mathcal{A}}}{K} = \log\tilde{Z}(\beta) - \beta\frac{\partial}{\partial\beta}\log\tilde{Z}(\beta), \tag{27}$$

where $\tilde{Z}$ is taken from Eq. (20) and evaluated at the solution $\beta(E,\gamma)$ given by the SPA condition. The term $-\frac{\partial}{\partial\beta}\log\tilde{Z}(\beta)$ is the thermodynamic mean energy of the coupled oscillator given by the *r.h.s* of Eq. (23) and of course also evaluated at the solution $\beta(E,\gamma)$.

Fig. 4 illustrates the behavior of $S_{\mathcal{A}}$ as a function of the total energy E for various values of the coupling γ. As can be observed, $S_{\mathcal{A}}$ is a positive quantity that becomes zero for $E = 0$, in nice accordance with the third law of thermodynamics. The entropy is also a monotonically increasing function of E and γ. This latter feature accounts for the decrease in purity of the reduced density matrix $\mathrm{Tr}_{\mathcal{B}}\,\hat{\rho}_{\mathcal{AB}}$ with increasing γ.

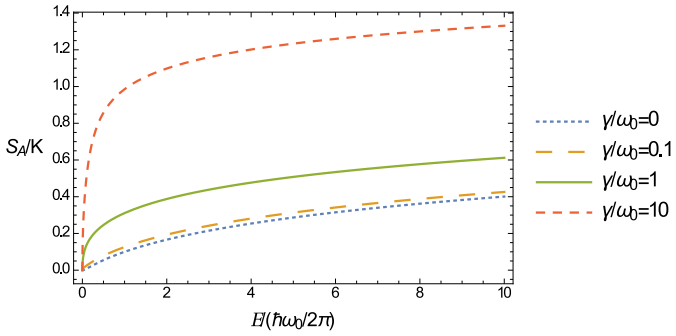

FIG. 4. Subsystem entropy $S_{\mathcal{A}}(E,\gamma)$ for parameters $\kappa\omega_0^3 = 5$ and $\omega_D/\omega_0 = 10$. The entropy of the subsystem $\mathcal{A}$ increases with the damping parameter γ and also with the energy, and becomes zero for $E = 0$, in accordance with the laws of thermodynamics.

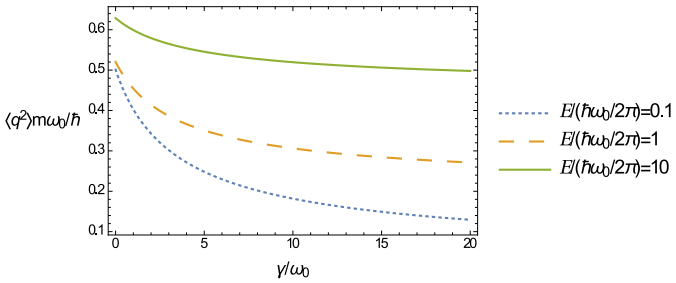

FIG. 5. $\langle q^2\rangle(\gamma)$ for parameters $\kappa\omega_0^3 = 5$ and $\omega_D/\omega_0 = 10$. As expected the particle gets more localized with increasing γ and its squared position expectation value increases with E.

Finally, let us consider the finite coupling version of the ensemble equivalence. Whenever the solution of the integral in Eq. (13) is justified by SPA, the saddle-point condition will give a relation between the expectation value of any smooth operator calculated in the microcanonical ensemble and the one calculated in the canonical case, *but for a temperature given by the solution* $\beta(E,\gamma)$ *in the finite coupling regime*. The same considerations also hold for the reduced density matrix describing the subsystem $\mathcal{A}$ and, in that case, the relation for the expectation value of an observable $\hat{O}_{\mathcal{A}}$ is given as

$$\langle \hat{O}_{\mathcal{A}} \rangle_E \approx \langle \hat{O}_{\mathcal{A}} \rangle_{\beta(E,\gamma)}\,, \tag{28}$$

that provides the sought extension of the equivalence of ensembles for systems with finite coupling in the thermodynamic limit. Accordingly, in Fig. 5 we show the expectation value of the squared position operator of the coupled oscillator $\hat{q}^2$ evaluated at the solution $\beta(E,\gamma)$. As expected $\langle q^2\rangle$ grows with the energy E and the particle is getting more localized with the increase of the damping parameter γ, as the bath monitors the position of the central particle [32].

Conclusions. Starting from the fundamental microcanonical distribution we have studied the emergence of temperature T in the finite coupling regime of open quantum systems. Following the approach pioneered by Schwinger resulting in $T = T(E)$ as an emergent quantity which establishes the condition of equilibrium between two weakly coupled subsystems at energy E, we have shown that in the finite and strong coupling regime $T = T(E,\gamma)$ also depends on the parameter γ that characterize the strength of the interaction. We have applied this idea to the paradigmatic Quantum Brownian Motion model and study the main features of this notion of temperature, confirming that $T(E,\gamma)$ is a monotonically decreasing function of the total energy E, while the entropy of the coupled oscillator, which now depends on γ, is a positive quantity that starts from zero for $E = 0$ and increases monotonically with E and γ. Remarkably, we found also an unexpected dependence on the memory properties of the bath: while $T(E,\gamma)$ decreases as a function of the interaction parameter γ in the Markovian regime, this behavior is inverted in the non-Markovian case.

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