Sharp Interface Limit for a Stokes/ Cahn-Hilliard System

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Abstract

We rigorously show the sharp interface limit of a coupled Stokes/Cahn–Hilliard system in a two dimensional, bounded and smooth domain, i.e. we consider the limiting behavior of solutions when a parameter $\epsilon > 0$ corresponding to the thickness of the diffuse interface tends to zero. We show that for sufficiently short times the solutions to the Stokes/Cahn–Hilliard system converge to solutions of a sharp interface model, where the evolution of the interface is governed by a Mullins–Sekerka system with an additional convection term coupled to a two–phase stationary Stokes system with an extra contribution to the stress tensor, representing the capillary stress.

To show the sharp interface limit, we construct suitable approximate solutions to the Stokes/Cahn–Hilliard system, by devising an inductive scheme which allows for the construction of terms of arbitrarily high order in the formally matched asymptotic calculations. As a novelty, we also introduce fractional order terms, which are of significant importance. In order to estimate the difference between the exact and the approximate solutions, we make use of modifications of spectral estimates shown in [24] for the linearized Cahn-Hilliard operator. The treatment of the involved coupling terms poses several complications, which have to be overcome by intricate analysis.

Wir führen einen rigorosen Beweis für einen scharfen Grenzschicht-Limes eines gekoppelten Stokes/Cahn–Hilliard Systems in einem zweidimensionalen, beschränkten und glatten Gebiet. Dazu betrachten wir das Verhalten von Lösungen, wenn ein Parameter $\epsilon > 0$, welcher die Dicke der diffusen Grenzschichtregion beschreibt, gegen Null geht. Wir zeigen, dass Lösungen des Stokes/Cahn–Hilliard Systems für hinreichend kurze Zeiten gegen Lösungen eines scharfen Grenzschicht-Modells konvergieren, in welchem die Evolution der Grenzschicht durch ein Mullins–Sekerka System mit zusätzlichem Konvektionsterm bestimmt wird, welches an ein Zwei-Phasen Stokes System gekoppelt ist, das einen zusätzlichen, Kapillarkräfte repräsentierenden Term im Spannungstensor aufweist.

Um den scharfen Grenzschicht-Limes zu beweisen, konstruieren wir geeignete Approximationslösungen des Stokes/Cahn-Hilliard Systems mit Hilfe eines induktiven Schemas, welches es uns erlaubt, Terme beliebig hoher Ordnung in den Rechnungen zur formalen asymptotischen Entwicklung zu konstruieren. Als Neuerung führen wir zusätzlich Terme gebrochener Ordnung ein, die sich im Verlauf der Arbeit als zentrales Element herausstellen. Um die Differenz zwischen den exakten und approximativen Lösungen abschätzen zu können, nutzen wir eine Modifikation der Spektralabschätzung für den linearisierten Cahn–Hilliard Operator, welche in [24] gezeigt wurde. Die Behandlung der vorkommenden Kopplungsterme wirft mehrere Schwierigkeiten auf, welche einer aufwendigen Analyse bedürfen.

Contents

1. Introduction

> "According to Gibbs' theory, capillary phenomena are present only if there is a discontinuity between the portions of fluid that are face-to-face. [...] In contrast, the method that I propose to develop in the following pages is not a satisfactory treatment unless the density of the body varies continuously at and near its transition layer. It will not be without interest to show that the two apparently contradictory hypotheses lead to values of the same order of magnitude [...]."
>
> –Van der Waals, [45]

Classically, the transition between two immiscible fluids was considered to be abrupt, in the sense of an appearance of a lower-dimensional surface separating the phases. Famous historical figures such as Young, Laplace and Gauß were advocating this point of view and developing the theory behind it, see [15, 46]. Considering the transitional layer to have zero thickness, it is reasonable to take into account geometric quantities such as curvature and physical properties such as surface tension. The behavior of a multiphase system is then governed by the intricate interactions between the bulk regions and the interface, mathematically expressed as equations of motion, which hold in each fluid, complemented by boundary conditions at the (free) surface. Models incorporating these ideas – often called *sharp interface models* – and the corresponding free-boundary problems have been widely studied and used to great success in describing a multitude of physical and biological phenomena. These range from the classical Stefan Problem, over image development in electrophotography, the theory of two-phase bio membranes, fluid flow through porous media, up to tumor growth, see [18, 50, 30, 17, 32] and the references therein.

However, fundamental problems arise in the analysis and numerical simulation of such problems, whenever the considered interfaces develop singularities. In fluid dynamics, topological changes such as the pinch off of droplets or collisions are non-negligible features of many systems, having a significant impact on the flow.

Conversely, *diffuse interface models* turn out to provide a promising, alternative approach to describe such phenomena and overcome the associated difficulties. In these diffuse interface (or *phase field)* methods, a partial mixing of the two phases is taken into consideration, allowing for the quantities, which were localized to a surface in the free-boundary formulation, to be spread out throughout an interfacial region. To this end, an order parameter (potentially signifying concentration, density, velocity etc.) is introduced, which varies rapidly, but smoothly, throughout a thin interfacial layer, heuristically viewed to have a thickness proportional to a length scale parameter $\epsilon > 0$. As emphasized by the introductory epigraph, these ideas go back to the writings of Van der Waals and Lord Rayleigh and have gained considerable traction with the works of Cahn and Hilliard, see [15, 45, 23] with regard to the historical sources and e.g. [39, 1, 2, 20] for more recent results.

Naturally, together with the appearance of a transitional layer of thickness corresponding to $\epsilon > 0$ the question of the limit case $\epsilon \to 0$ arises. This so-called *sharp interface limit* is in fact a question about the connection of sharp and diffuse interface models. As phase field approaches may also be used as a tool, alongside level set methods and parametric techniques, to numerically solve free boundary problems and surface evolution equations (see also [27]), it is of paramount importance to know that they asymptotically approach the correct sharp interface models.

Concerning the flow of two macroscopically immiscible, viscous, incompressible Newtonian fluids with matched densities, a fundamental and broadly accepted diffuse interface model is the so-called *model H,* derived in [39, 36]. This model consists of a Navier-Stokes system coupled with the Cahn-Hilliard equation and is of the form

$$\rho\partial_t \mathbf{v}^\epsilon + \rho\mathbf{v}^\epsilon \cdot \nabla\mathbf{v}^\epsilon - \operatorname{div}\left(2\nu\left(c^\epsilon\right) D_s\mathbf{v}^\epsilon\right) + \nabla p^\epsilon = -\epsilon\operatorname{div}\left(\nabla c^\epsilon \otimes \nabla c^\epsilon\right) \qquad \text{in } \Omega_T, \tag{1.1}$$

$$\operatorname{div}\mathbf{v}^\epsilon = 0 \qquad \text{in } \Omega_T, \tag{1.2}$$

$$\partial_t c^\epsilon + \mathbf{v}^\epsilon \cdot \nabla c^\epsilon = m^\epsilon \Delta\mu^\epsilon \qquad \text{in } \Omega_T, \tag{1.3}$$

$$\mu^\epsilon = -\epsilon\Delta c^\epsilon + \epsilon^{-1} f'\left(c^\epsilon\right) \qquad \text{in } \Omega_T, \tag{1.4}$$

$$\left(\mathbf{v}^\epsilon, c^\epsilon\right)|_{t=0} = \left(\mathbf{v}_0^\epsilon, c_0^\epsilon\right) \qquad \text{in } \Omega. \tag{1.5}$$

Here $T > 0$, $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded and smooth domain, $\Omega_T = \Omega \times (0, T)$ and

$$\left(\mathbf{a} \otimes \mathbf{b}\right)_{i,j} = \mathbf{a}_i \mathbf{b}_j \tag{1.6}$$

for all $i, j \in \{1, \ldots, n\}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. $\mathbf{v}^\epsilon$ and p^ϵ represent the mean velocity and pressure,

$$D_s\mathbf{v}^\epsilon = \frac{1}{2}\left(\nabla\mathbf{v}^\epsilon + \left(\nabla\mathbf{v}^\epsilon\right)^T\right), \tag{1.7}$$

c^ϵ is an order parameter representing the concentration difference of the fluids and μ^ϵ is the chemical potential of the mixture. Moreover, $\mathbf{v}_0^\epsilon$ and c_0^ϵ are suitable initial values, ρ is the (supposedly constant) density of the fluids, ν is the viscosity of the mixture and $m^\epsilon > 0$ is a mobility coefficient related to the strength of the diffusion in the mixture. The function $f : \mathbb{R} \to \mathbb{R}$ is supposed to be a homogeneous free energy density of double-well shape, the exact specifications of which will be given later. This system is usually supplemented by a no-slip boundary condition for $\mathbf{v}^\epsilon$ and Neumann boundary conditions for c^ϵ and μ^ϵ, i.e.

$$\left(\mathbf{v}^\epsilon, \mathbf{n}_{\partial\Omega} \cdot \nabla c^\epsilon, \mathbf{n}_{\partial\Omega} \cdot \nabla\mu^\epsilon\right) = 0 \text{ on } \partial_T\Omega,$$

where $\partial_T\Omega = \partial\Omega \times (0, T)$ and $\mathbf{n}_{\partial\Omega}$ denotes the outer unit normal. To gain an inkling of the sharp interface limit of a system like (1.1)–(1.5) the so-called *method of formally matched asymptotics* has in recent years proven to be a very flexible and accessible approach, see e.g. [21, 33, 43]. The underlying idea of this method is to assume that the appearing variables may be expressed as power series or asymptotic expansions in ϵ, with different expansions close to and away from the interface, which allows for the consideration of different length scales. Additionally, the expansions are supposed to satisfy certain matching-properties, providing a connection between the bulk and interfacial regions (for more detailed assumptions and explanations, see the introduction of Chapter 5).

In the case of (1.1)–(1.5) such formal calculations have been done in [4] and yield in the case of a constant mobility $m^\epsilon = m_0 > 0$ the sharp interface system

$$\rho\partial_t \mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \nu^\pm\Delta\mathbf{v} + \nabla p = 0 \qquad \text{in } \Omega^\pm(t), t\in(0,T), \tag{1.8}$$
$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } \Omega^\pm(t), t\in(0,T), \tag{1.9}$$
$$\Delta\mu = 0 \qquad \text{in } \Omega^\pm(t), t\in(0,T), \tag{1.10}$$
$$\left[2\nu^\pm D_s\mathbf{v} - p\mathbf{I}\right]\mathbf{n}_{\Gamma_t} = -2\sigma H_{\Gamma_t}\mathbf{n}_{\Gamma_t} \qquad \text{on } \Gamma_t, t\in(0,T), \tag{1.11}$$
$$\mu = \sigma H_{\Gamma_t} \qquad \text{on } \Gamma_t, t\in(0,T), \tag{1.12}$$
$$-V_{\Gamma_t} + \mathbf{n}_{\Gamma_t}\cdot\mathbf{v} = \frac{m_0}{2}\left[\mathbf{n}_{\Gamma_t}\cdot\nabla\mu\right] \qquad \text{on } \Gamma_t, t\in(0,T), \tag{1.13}$$
$$[\mathbf{v}] = 0 \qquad \text{on } \Gamma_t, t\in(0,T), \tag{1.14}$$
$$\mathbf{v}|_{t=0} = \mathbf{v}_0 \qquad \text{in } \Omega, \tag{1.15}$$
$$\Gamma(0) = \Gamma_0, \tag{1.16}$$

closed by suitable boundary conditions on $\partial\Omega\times(0,T)$. Here Ω is the disjoint union of $\Omega^+(t)$, $\Omega^-(t)$ and Γ_t for every $t\in[0,T_0]$, where $\Gamma_t = \partial\Omega^+(t)$, $\mathbf{n}_{\Gamma_t}$ is the exterior normal with respect to $\Omega^-(t)$, and H_{Γ_t} and V_{Γ_t} denote the mean curvature and normal velocity of the interface Γ_t. Furthermore, we use the notations

$$[g](p,t) := \lim_{h\searrow 0}\left(g\left(p+\mathbf{n}_{\Gamma_t}(p)h\right) - g\left(p-\mathbf{n}_{\Gamma_t}(p)h\right)\right) \text{ for } p\in\Gamma_t,$$
$$\sigma := \frac{1}{2}\int_{-1}^{1}\sqrt{2f(s)}\mathrm{d}s, \tag{1.17}$$

and denote by Γ_0 a given initial surface and by $\nu^\pm$ the viscosity in the bulk phases.

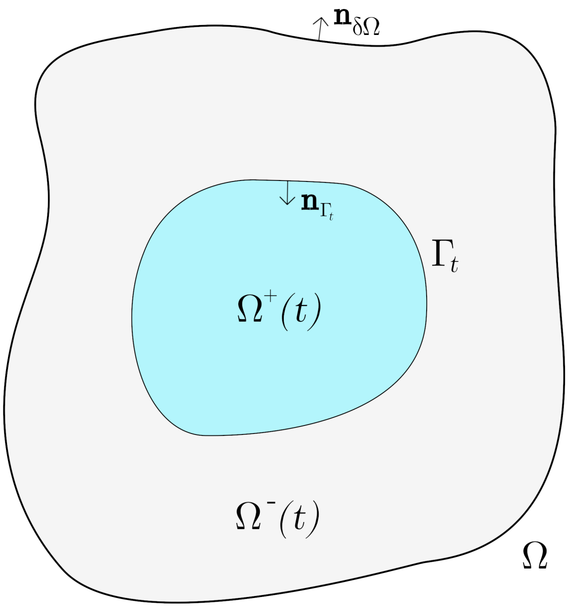

Figure 1.1.: A schematic representation of the general situation.

The identities (1.8), (1.9) correspond to the conservation of linear momentum and mass in the fluids, while (1.11) represents the jump in the stress tensor and (1.13) is a Stefan type condition for the evolution of the interface. If m_0 vanished, i.e. no diffusion of mass was taken into account, the latter would reduce to a pure transport equation. The system of equations (1.10), (1.12) and (1.13) (with $\mathbf{v} = 0$) are also known as the Mullins–Sekerka flow and may be obtained as the H^{-1} gradient flow of the area functional, see e.g. [34]. Hence, system (1.8)–(1.16) is commonly referred to as a Navier–Stokes/Mullins–Sekerka system.

Regarding the existence of solutions for (1.1)–(1.5) we refer to [2, 19]; short time existence of strong solutions of (1.8)–(1.16) was shown in [11] and existence of weak solutions for long times in [9]. Despite these analytic results and the formal findings for the sharp interface limit, there are only few attempts at rigorously showing the convergence of solutions of the diffuse interface system (1.1)–(1.5) to solutions of the sharp interface system (1.8)–(1.16) as $\epsilon \to 0$. This does not only hold true for the model H, but reflects the general situation in the theory of two-phase flows in fluid mechanics. One approach to rigorously proving sharp interface limits uses the notion of varifold solutions discussed in [25]; in the setting of two phase flows, such results for large times were shown in [9] for the model H and in [5] also for the more general situation of fluids with different densities. The pursued strategy in these works is to show (weak) compactness for the families of weak solutions to the corresponding diffuse interface system in suitable spaces, which then allows for the extraction of a convergent subsequence. It is then proven that the limit of such a subsequence is given by a varifold solution of the affiliated sharp interface system. The limitations of this technique are inherent to the underlying mathematical principles, as the notion of varifold solutions is rather weak and no convergence rates may be obtained from the compactness arguments.

Another approach is based on the works [42] and [14], where the method of formally matched asymptotics is used as a basis. It is assumed that both the considered diffuse and sharp interface model have smooth solutions in a short time interval $(0, T)$; in the case of [14] these systems consist of the Cahn-Hilliard equation and the Mullins-Sekerka (or Hele-Shaw) system. Using matched asymptotic expansions, an explicit approximate solution to the diffuse interface system is constructed, usually consisting of significantly more terms than needed for an only formal investigation. The key element of the argumentation is then to show that the difference between the real solutions and the approximate solutions tends to 0 in suitable (strong) norms as $\epsilon \to 0$. As the detailed structure of the approximate solutions is known, it may then easily be verified that they in turn converge to solutions of the underlying sharp interface system, yielding a result for the sharp interface limit. This strategy has been successfully adapted to a lot of different problems over the years: in [26] the mass conserving Allen-Cahn equation was connected to the volume preserving mean curvature flow, in [10] it was used to show the convergence of the Cahn–Larché system to a modified Hele–Shaw problem and in [22] several phase field models were considered. Most recently the approach has also been used to show the sharp interface limit for an Allen-Cahn system with 90°-contact angle, see [8].

However, in view of two-phase flow models in fluid mechanics and the arising difficulties therein, the first and so far only convergence result with convergence rates in strong norms is [6]. Considering a coupled Stokes/Allen-Cahn system in two dimensions, it is shown that smooth solutions of the diffuse interface system converge for short times to solutions of a sharp interface model, where the evolution of the free surface is governed by a convective mean curvature flow coupled to a two-phase Stokes system with a modified stress tensor, accounting for capillary forces. The Stokes/Allen-Cahn system is analyzed as it allows for the study of arising problems in the context of two phase flows within a simplified setting, neither having to take into account the instationary character and the nonlinearities of Navier-Stokes type equations, nor having to deal with a fourth order partial differential equation like the

Cahn-Hilliard equation and its more technically involved asymptotic expansion.

This contribution builds upon the ideas introduced in [6] and aims to establish the first rigorous result in strong norms for a sharp interface limit of a two phase flow model involving the Cahn-Hilliard equation with convergence rates. In doing so, we hope to build another cornerstone on the way to rigorously showing the sharp interface limit for model H.

More precisely we consider the Stokes/Cahn-Hilliard system

$$-\Delta \mathbf{v}^\epsilon + \nabla p^\epsilon = \mu^\epsilon \nabla c^\epsilon \quad \text{in } \Omega_T, \tag{1.18}$$
$$\operatorname{div}\mathbf{v}^\epsilon = 0 \quad \text{in } \Omega_T, \tag{1.19}$$
$$\partial_t c^\epsilon + \mathbf{v}^\epsilon \cdot \nabla c^\epsilon = \Delta \mu^\epsilon \quad \text{in } \Omega_T, \tag{1.20}$$
$$\mu^\epsilon = -\epsilon \Delta c^\epsilon + \frac{1}{\epsilon} f'(c^\epsilon) \quad \text{in } \Omega_T, \tag{1.21}$$
$$c^\epsilon|_{t=0} = c_0^\epsilon \quad \text{in } \Omega, \tag{1.22}$$
$$(-2D_s \mathbf{v}^\epsilon + p^\epsilon \mathbf{I}) \cdot \mathbf{n}_{\partial\Omega} = \alpha_0 \mathbf{v}^\epsilon \quad \text{on } \partial_T\Omega, \tag{1.23}$$
$$\mu^\epsilon = 0 \quad \text{on } \partial_T\Omega, \tag{1.24}$$
$$c^\epsilon = -1 \quad \text{on } \partial_T\Omega, \tag{1.25}$$

where $\Omega \subset \mathbb{R}^2$ is a domain with smooth boundary, $\alpha_0 > 0$ is fixed and c_0^ϵ is certain "well-chosen" initial data (see Theorem 4.1 for more details; we allow perturbations of some order of ϵ around a given value). Note that for $\psi \in C_{0,\sigma}^\infty(\Omega) := \left\{ \psi \in C_0^\infty(\Omega)^2 \middle| \operatorname{div}\psi = 0 \right\}$ we have by (1.21)

$$\int_\Omega \mu^\epsilon \nabla c^\epsilon \cdot \psi \mathrm{d}x = \int_\Omega \left(-\epsilon \Delta c^\epsilon \nabla c^\epsilon + \frac{1}{\epsilon} \nabla \left(f\left(c^\epsilon\right)\right) \right) \cdot \psi \mathrm{d}x = -\epsilon \int_\Omega \operatorname{div}\left(\nabla c^\epsilon \otimes \nabla c^\epsilon\right) \cdot \psi \mathrm{d}x,$$

where we used integration by parts and $\operatorname{div}(\nabla c^\epsilon \otimes \nabla c^\epsilon) - \frac{1}{2}\nabla |\nabla c^\epsilon|^2 = \nabla c^\epsilon \Delta c^\epsilon$ in the second equality. Thus, in the case of a no-slip boundary condition for $\mathbf{v}^\epsilon$ instead of (5.86), the right hand sides of (1.1) and (1.18) coincide in the weak formulation.

Existence of smooth solutions to (1.18)–(1.25) can be shown with similar methods as in [2], where the considered model is in fact way more complicated, as it involves the full Navier-Stokes equation. A word is in order about the unusual choice of boundary conditions (1.23)–(1.25). (1.23) can be thought of as a modified do-nothing boundary condition $(-2D_s \mathbf{v}^\epsilon + p^\epsilon \mathbf{I}) \cdot \mathbf{n}_{\partial\Omega} = 0$, which is equivalent to the case $\alpha_0 = 0$ or as an altered Navier boundary condition. Physically, it would be sensible to consider (1.23) when Ω is enclosed by a porous medium or a membrane, which allows for a flow in normal direction to the boundary, tied to the occurrence of certain stresses. The only reason we prescribe such boundary conditions instead of periodic, no-slip or Navier boundary conditions, are major difficulties which arise in the construction of the approximate solutions for $\mathbf{v}^\epsilon$. A more detailed account is given in Remark 5.23. Classically, the Cahn-Hilliard system is complemented with Neumann boundary conditions for c^ϵ and μ^ϵ. While it is rather unproblematic to adapt the present work to Neumann boundary conditions for c^ϵ, major issues arise when considering $\partial_{\mathbf{n}_{\partial\Omega}} \mu = 0$ instead of (1.24), see Remark 7.12. To circumvent these problems and as the focus of our interest and analysis lies in the obstacles and difficulties occurring close to the interface Γ_t, we decided on the present choice of boundary conditions.

We will show that the sharp interface limit of (1.18)–(1.25) is given by the system

$$
\begin{aligned}
-\Delta \mathbf{v} + \nabla p &= 0 && \text{in } \Omega^{\pm}(t), t \in [0, T_0], && (1.26)\\
\operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega^{\pm}(t), t \in [0, T_0], && (1.27)\\
\Delta \mu &= 0 && \text{in } \Omega^{\pm}(t), t \in [0, T_0], && (1.28)\\
(-2D_s \mathbf{v} + p\mathbf{I})\, \mathbf{n}_{\partial\Omega} &= \alpha_0 \mathbf{v} && \text{on } \partial_{T_0}\Omega, && (1.29)\\
\mu &= 0 && \text{on } \partial_{T_0}\Omega, && (1.30)\\
[2D_s \mathbf{v} - p\mathbf{I}]\, \mathbf{n}_{\Gamma_t} &= -2\sigma H_{\Gamma_t} \mathbf{n}_{\Gamma_t} && \text{on } \Gamma_t, t \in [0, T_0], && (1.31)\\
\mu &= \sigma H_{\Gamma_t} && \text{on } \Gamma_t, t \in [0, T_0], && (1.32)\\
-V_{\Gamma_t} + \mathbf{n}_{\Gamma_t} \cdot \mathbf{v} &= \frac{1}{2} [\mathbf{n}_{\Gamma_t} \cdot \nabla \mu] && \text{on } \Gamma_t, t \in [0, T_0], && (1.33)\\
[\mathbf{v}] &= 0 && \text{on } \Gamma_t, t \in [0, T_0], && (1.34)\\
\Gamma(0) &= \Gamma_0, && && (1.35)
\end{aligned}
$$

where we used the same notations as before and $T_0 > 0$. Regarding the existence of local strong solutions of (1.26)–(1.35), the proof in [11] may be adapted, where a coupled Navier-Stokes/Mullins-Sekerka system was treated. Regularity theory for parabolic equations and the Stokes equation may then be used to show smoothness of the solution for smooth initial values.

Through the course of this thesis, we will present an inductive scheme for the construction of approximate solutions $\{c_A^\epsilon, \mu_A^\epsilon, \mathbf{v}_A^\epsilon, p_A^\epsilon\}_{\epsilon>0}$ to (1.18)–(1.25) and show the existence of some $T_1 > 0$ such that the difference between c^ϵ and c_A^ϵ goes to zero in $L^\infty\left(0, T_1; H^{-1}(\Omega)\right)$ with $H^{-1}(\Omega) := \left(H_0^1(\Omega)\right)'$, $L^2(\Omega_{T_1})$, $L^2\left(0, T_1; H^1(\Omega)\right)$ and many other norms as $\epsilon \to 0$ with explicit convergence rates, for some small $T_1 > 0$. These rates will depend on the order up to which the approximate solutions have been constructed. Moreover, we will also present convergence rates for the error $\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon$ in $L^1(0, T_1; L^q(\Omega))$ for $q \in (1, 2)$. This result is stated in Theorem 4.1. The key to this endeavors will be a modification of the spectral estimate for the linearized Cahn-Hilliard operator as given in [24], see Theorem 3.12 in this thesis. As in [6], the main difficulties which arise in the treatment of the Stokes/Cahn-Hilliard system are due to the appearance of the capillary term $\mu^\epsilon \nabla c^\epsilon$ in (1.18) and the convective term $\mathbf{v}^\epsilon \cdot \nabla c^\epsilon$ in (1.20). Although we may build upon the insights gained in the cited article, several new and severe obstacles arise in the context of system (1.18)–(1.25) which have to be overcome with sophisticated techniques. Apart from the already mentioned improvement of the spectral estimates, we would like to highlight three of these ideas and approaches that are central to this thesis:

First, we need to get higher order terms in the construction of the approximate solutions than in [6] to ensure that the error estimates hold. Additionally, the outer expansion in the situation of a Cahn-Hilliard system is, in contrast to the Allen-Cahn case, not trivial. Thus, we devise an inductive scheme for the construction of arbitrarily high orders of the asymptotic expansion, which also includes the construction of a boundary layer expansion. This ensures that the approximate solutions also satisfy the boundary conditions (1.23)–(1.25). The construction scheme is based on a mixture of [14] and [26], with alterations and additions necessary to adapt to the coupling of the Stokes system.

Second, terms of fractional order are considered in the asymptotic expansions. The necessity of such terms is at its core a consequence of our treatment of the convective term $\mathbf{v}^\epsilon \cdot \nabla c^\epsilon$. Omitting them would result in insufficient estimates for the so-called remainder terms, which consist of the error that occurs when considering the approximate solutions in (1.18)–(1.21) instead of the real solution. A similar problem in [6] is solved by the intricate analysis of a second order, parabolic, degenerate partial differential equation, see Theorem 2.12 in the cited work. However, in the present situation a similar approach leads to a fourth order, parabolic, degenerate equation of Cahn-Hilliard type on an unbounded domain, presenting extreme difficulties. The introduction of fractional order terms renders such considerations unnecessary, with the caveat that while the produced terms are smooth, they may not be estimated uniformly in ϵ in arbitrarily strong norms. This is the cause for many technical subtleties in Chapter 6.

Third, we use a spectral decomposition as shown in [24] to gain a better structural understanding of the difference $R := c^\epsilon - c^\epsilon_A$ close to the interface. To be able to more accurately describe the decomposition, we introduce the so-called *optimal profile* $\theta_0 : \mathbb{R} \to \mathbb{R}$, which is the solution to the ordinary differential equation

$$\begin{aligned} -\theta_0'' + f'(\theta_0) &= 0 \quad \text{in } \mathbb{R} \\ \theta_0(0) = 0, \ \lim_{\rho \to \pm\infty} \theta_0(\rho) &= \pm 1 \end{aligned} \tag{1.36}$$

and appears frequently in the context of the Allen-Cahn and Cahn-Hilliard equation. With the help of this function, we will be able to show that in the leading order R resembles $\theta_0'\left(\frac{d_{\Gamma_t}}{\epsilon}\right) Z\left(Pr_{\Gamma_t}\right)$ in the interfacial region, where d_{Γ_t} denotes the signed distance function to the interface, $Z : \Gamma_t \to \mathbb{R}$ a suitable function and Pr_{Γ_t} the projection onto the interface (for precise information see Assumptions 1.1). This reflects well upon the intuition that c^ϵ acts like an optimal profile multiplied by tangential terms close to its zero-level set, since the leading order of c^ϵ_A turns out to be a scaled θ_0 and thus the above resemblance could be interpreted as the first order of a Taylor expansion. Throughout the course of this work, most notably in Subsection 5.2 and Chapter 6, this decomposition of R allows for many improved estimates without which we would not be able to show the main result of this thesis, Theorem 4.1.

This thesis is organized as follows: In Chapter 2, we give a short overview of the most important tools used throughout this thesis, which include existence results for certain ordinary differential equations arising in the process of the later performed inner expansion. Moreover, we review some differential geometric results in Section 2.3, which will be useful when working close to the interface Γ_t and discuss results for a class of functions with exponential decay in $\mathbb{R}$, referred to as *remainder terms*, in Section 2.4. While Section 2.2 is concerned with the existence of weak and strong solutions for inhomogeneous Stokes equations with boundary conditions akin to (1.23), Section 2.6 includes existence results for evolution equations on the interface coupled to certain two-phase systems. The analysis of the latter is important for the construction of the outer expansion in Chapter 5.

Chapter 3 consists of detailed adaptations and modifications of results from [24], Chapter 2. We need this adaptation since we work with a different stretched variable and need to ensure that all results, in particular the ones involving the decomposition of $c^\epsilon - c^\epsilon_A$ and the spectral estimate for the Cahn-Hilliard operator, also hold for our scaling. The main result of this part is the modified spectral estimate presented in Theorem 3.12, which is of paramount importance in Section 7.2. Corollary 3.11 together with Lemma 3.9 yield structural information which will be applied to our situation in Proposition 5.28, showing the aforementioned decomposition for R.

The centerpiece of this work is presented in Chapter 4, where Theorem 4.1 – the main result – is stated and a precise account of the properties of the approximate solutions is given in Theorem 4.3. In detail, we rigorously show the sharp interface limit for the coupled

Stokes/Cahn-Hilliard system (1.18)–(1.25), establishing the first non-formal result for strong solutions of a coupled Cahn-Hilliard system in the setting of two phase flows. We prove that during the time of existence of smooth solutions to (1.18)–(1.25) and (1.26)–(1.35) there is some $T > 0$ such that the errors $c^\epsilon - c_A^\epsilon$ and $\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon$ tend to 0 as $\epsilon \to 0$ in suitable norms. We show this under the assumption that the initial values c_0^ϵ are of some predefined form. To gain a first, weak control of the quantities c^ϵ, μ^ϵ and $\mathbf{v}^\epsilon$ we show some energy estimates in Section 4.1. All subsequent parts following Chapter 4 consist of auxiliary results needed to prove Theorem 4.1.

The construction of approximate solutions in Chapter 5 is the first of these. Based upon the approaches in [14, 26, 6] we devise an inductive scheme for the construction of inner, outer and boundary terms of arbitrarily high order of the asymptotic expansions for solutions of (1.18)–(1.25). At the end of Section 5.1, we shortly discuss necessary changes in the argumentation if an instationary Stokes system or the Navier-Stokes equations were considered instead of (1.18)–(1.19) or if the right hand side of (1.18) was replaced by $-\epsilon \operatorname{div}(\nabla c^\epsilon \otimes \nabla c^\epsilon)$. In Section 5.2, we introduce an auxiliary function $\tilde{\mathbf{w}}_1^\epsilon$, which turns out later to be the leading term of the error in the velocity $\mathbf{v}_A^\epsilon - \mathbf{v}^\epsilon$. The results in this section are already formulated in preparation of the following section, causing some rather complicated notations. These will however pay off in Section 5.3 and more precisely in one of the main results of this part, Theorem 5.32. At its core, this theorem proves the existence of certain fractional order terms in the asymptotic expansion, which are defined with the help of solutions to a nonlinear evolution equation involving $\tilde{\mathbf{w}}_1^\epsilon$ on the interface coupled to a two-phase Stokes and linearized Mullins-Sekerka system. Additionally, ϵ-independent control of certain norms is provided. This becomes an issue due to an implicit dependency of the fractional order terms on ∇c^ϵ, which blows up as $\epsilon \to 0$ (cf. Lemma 4.4).

To rigorously justify that the "approximate solutions" constructed up to this point in the work really are a good approximation of solutions, it is necessary to analyze the so-called remainder in Chapter 6. This remainder is nothing else than the error that occurs when the functions c_A^ϵ, μ_A^ϵ, $\mathbf{v}_A^\epsilon$, p_A^ϵ are plugged into the equations (1.18)–(1.21). The majority of Chapter 6 is thus made up of estimates for the different appearing terms, with Theorem 6.12 connecting the parts. The major reason for most of the technical and cumbersome analysis in this part of the thesis is that we have no ϵ-independent control of arbitrarily strong norms of the fractional terms. Furthermore, some terms appearing in the remainder are of a relatively low order in ϵ and thus demand for special techniques to be applied. The most prominent example of this is Lemma 6.9.

The last part of this thesis, Chapter 7, is dedicated to putting the different pieces together and proving Theorem 4.1. However, in an attempt to make the final proof more accessible, many auxiliary results are outsourced and shown before the actual "main proof". This is in particular true for the necessary estimates of the error in the velocity $\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon$, which is treated in detail in Subsection 7.1.1. The final proof in Section 7.2 is then again based upon the ideas presented in [6].

Throughout this contribution, we work under the following assumptions.

Assumption and Definition 1.1 (General Setting)**.**

1. *Let $M \geq 4$, $\alpha_0 > 0$ and $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary.*

2. *Let $\Gamma_0 \subset\subset \Omega$ be a given, smooth, non-intersecting, closed initial curve. Let moreover $(\mathbf{v}, p, \mu, \Gamma)$ be a smooth solution to (1.26)–(1.35) and $(c^\epsilon, \mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon)$ be a smooth solution to (1.18)–(1.25) for some $T_0 > 0$. We assume that $(\Gamma_t)_{t\in[0,T_0]}$ is a family of smoothly evolving, compact, non-intersecting, closed curves in Ω, such that $\Gamma = \cup_{t\in[0,T_0]}\Gamma_t \times \{t\}$.*

3. *We define* $\Omega^{+}(t)$ *to be the inside of* Γ_t *and set* $\Omega^{-}(t)$ *such that* Ω *is the disjoint union of* $\Omega^{+}(t)$, $\Omega^{-}(t)$ *and* Γ_t. *Moreover we define* $\Omega_T^{\pm} = \cup_{t\in[0,T]}\Omega^{\pm}(t)\times\{t\}$, $\Omega_T := \Omega\times(0,T)$ *and also* $\partial_T\Omega := \partial\Omega\times(0,T)$ *for* $T\in[0,T_0]$.

4. *We define* $\mathbf{n}_{\Gamma_t}(p)$ *for* $p\in\Gamma_t$ *as the exterior normal with respect to* $\Omega^{-}(t)$ *and* V_{Γ_t}, *and* H_{Γ_t} *as the normal velocity and mean curvature of* Γ_t *with respect to* $\mathbf{n}_{\Gamma_t}$, $t\in[0,T_0]$.

5. *Let*
$$d_\Gamma : \Omega_{T_0}\to\mathbb{R},\ (x,t)\mapsto\begin{cases} dist\left(\Omega^{-}(t),x\right) & \text{if } x\notin\Omega^{-}(t)\\ -dist\left(\Omega^{+}(t),x\right) & \text{if } x\in\Omega^{-}(t)\end{cases}$$
be the signed distance function to Γ *such that* d_Γ *is positive inside* $\Omega_{T_0}^{+}$.

6. *We write*
$$\Gamma_t(\alpha) := \{x\in\Omega|\,|d_\Gamma(x,t)|<\alpha\}$$
for $\alpha>0$ *and set*
$$\Gamma(\alpha;T) := \bigcup_{t\in[0,T]}\Gamma_t(\alpha)\times\{t\}$$
for $T\in[0,T_0]$.

7. *We assume that* $\delta>0$ *is a small positive constant such that* $dist(\Gamma_t,\partial\Omega)>5\delta$ *for all* $t\in[0,T_0]$ *and such that* $Pr_{\Gamma_t}:\Gamma_t(3\delta)\to\Gamma_t$ *is well-defined and smooth for all* $t\in[0,T_0]$ *(cf. Lemma 2.11 for existence of such* δ*). In the following we often use the notation*
$$\Gamma(2\delta) := \Gamma(2\delta;T_0)$$
as a simplification.

8. *We also define a tubular neighborhood around* $\partial\Omega$*: For this let* $d_{\mathbf{B}}:\Omega\to\mathbb{R}$ *be the signed distance function to* $\partial\Omega$ *such that* $d_{\mathbf{B}}<0$ *in* Ω. *As for* Γ_t *we define a tubular neighborhood by*
$$\partial\Omega(\alpha) = \{x\in\Omega\,|-\alpha<d_{\mathbf{B}}(x)<0\}$$
and
$$\partial_T\Omega(\alpha) = \{(x,t)\in\Omega_T|\,d_{\mathbf{B}}(x)\in(-\alpha,0)\}$$
for $\alpha>0$ *and* $T\in(0,T_0]$. *Moreover, we denote the outer unit normal to* Ω *by* $\mathbf{n}_{\partial\Omega}$ *and denote the normalized tangent by* $\tau_{\partial\Omega}$, *which is fixed by the relation*
$$\mathbf{n}_{\partial\Omega}(p) = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\tau_{\partial\Omega}(p)$$
for $p\in\partial\Omega$. *Finally we assume that* $\delta>0$ *is chosen small enough such that the projection* $Pr_{\partial\Omega}:\partial\Omega(\delta)\to\partial\Omega$ *along the normal* $\mathbf{n}_{\partial\Omega}$ *is also well-defined and smooth (existence of such* δ *may be shown as in Lemma 2.11).*

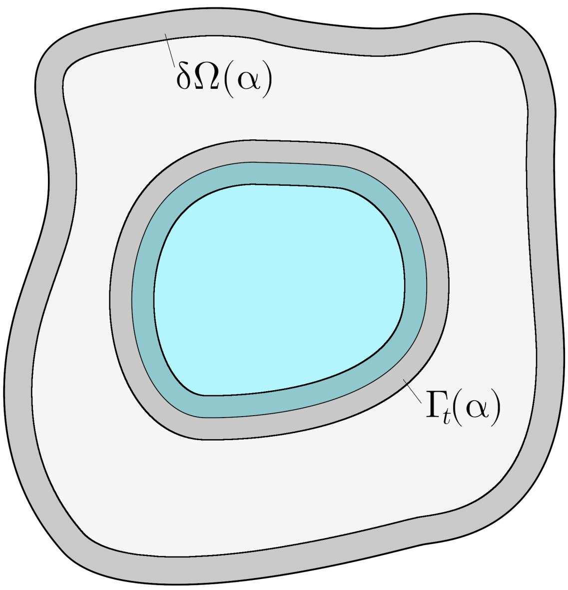

Figure 1.2.: The tubular neighborhoods around Γ_t and $\partial\Omega$.

We also consider the following properties for the potential f .

Assumption 1.2 (*Double Well Potential*)**.** *In the following we will consider a double well potential f which satisfies the following assumptions:*

$f:\mathbb{R}\to\mathbb{R}$ is a polynomial of fourth order satisfying $f(\pm1)=0,\ f'(\pm1)=0,\ f''(\pm1)>0$ and $f(s)=f(-s)>0$ for all $s\in(-1,1)$. Moreover we assume that there exists $C>0$ such that

$$sf^{(3)}(s)>0 \quad \text{if } |s|\geq C$$

and that $k_f:=f^{(4)}>0$.

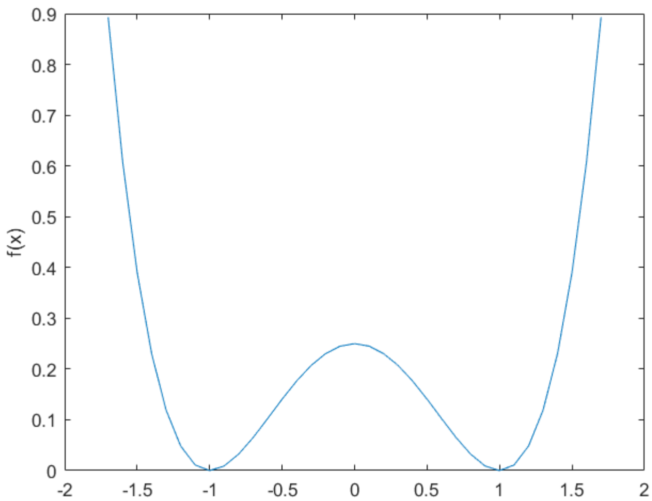

Figure 1.3.: Typical form of the double-well potential f represented by $f(x)=\frac{1}{4}\left(x^2-1\right)^2$.

2. Preliminaries

Since we will make extensive use of the following function later on, we will define it here and always reference back to this definition.

Definition 2.1 (A cut-off function)**.** Let $\delta > 0$ be given as in Assumption 1.1 and $\xi \in C^\infty(\mathbb{R})$ be a function such that

1. $\xi(s) = 1$ if $|s| \leq \delta$,

2. $\xi(s) = 0$ if $|s| > 2\delta$,

3. $0 \geq s\xi'(s) \geq -4$ if $\delta \leq |s| \leq 2\delta$.

We call this function **the cut-off function.**

2.1. Important Ordinary Differential Equations

The inner expansion that we construct in Subsection 5.1.2 will require intricate knowledge of certain ordinary differential equations and their solutions. This section is dedicated to the collection of results regarding these problems. The proofs of the statements in this section can be found in detail in [47], pages 14 pp., and will thus not be repeated here.

Lemma 2.2. *Let $f \in C^\infty(\mathbb{R})$ be given as in Assumption 1.2. Then the ordinary differential equation (1.36) allows for a unique, monotonically increasing solution $\theta_0 : \mathbb{R} \to (-1, 1)$. This solution furthermore satisfies the decay estimate*

$$\left|\theta_0^2(\rho) - 1\right| + \left|\theta_0^{(n)}(\rho)\right| \leq C_n e^{-\alpha|\rho|} \quad \forall \rho \in \mathbb{R},\ n \in \mathbb{N}\backslash\{0\} \tag{2.1}$$

for constants $C_n > 0$, $n \in \mathbb{N}\backslash\{0\}$ and fixed $\alpha \in \left(0, \min\left\{\sqrt{f''(-1)}, \sqrt{f''(1)}\right\}\right)$.

Proof. See [47], p. 14, Lemma 2.6.1. □

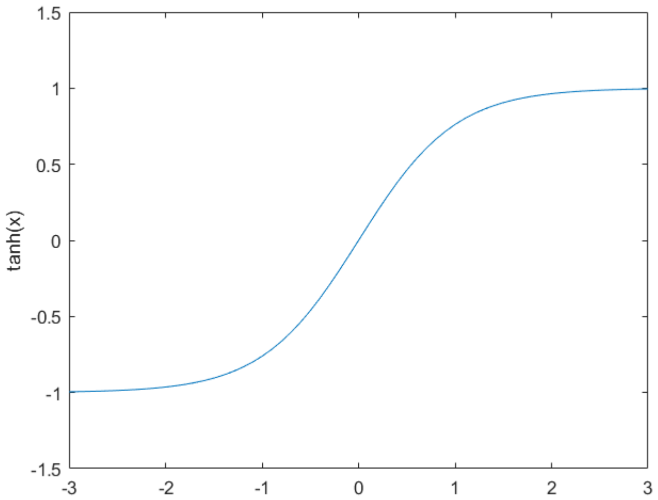

Figure 2.1.: Form of θ_0 in the case of $f(x) = \frac{1}{4}\left(x^2 - 1\right)^2$.

Lemma 2.3. *Let $U \subset \mathbb{R}^d$, θ_0 be given as in Lemma 2.2 and let $A : \mathbb{R} \times U \to \mathbb{R}$, $(\rho, x) \mapsto A(\rho, x)$ be given and smooth. Assume that for all $x \in U$ there exists $A^{\pm}(x)$ such that $A(\pm\rho, x) - A^{\pm}(x) = \mathcal{O}(e^{-\alpha\rho})$ as $\rho \to \infty$. Then for each $x \in U$ the system*

$$w_{\rho\rho}(\rho, x) - f''(\theta_0(\rho))\, w(\rho, x) = A(\rho, x) \;\; \forall \rho \in \mathbb{R}$$
$$w(0, x) = 0,$$

has a solution $w(., x) \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} A(\rho, x)\, \theta_0'(\rho)\, \mathrm{d}\rho = 0.$$

In addition, if the solution exists, then it is unique and satisfies for every $x \in U$ and $l \in \{0, 1, 2\}$

$$D_\rho^l \left[w(\pm\rho, x) + \frac{A^{\pm}(x)}{f''(\pm 1)} \right] = \mathcal{O}\left(e^{-\alpha\rho}\right) \;\; \text{as } \rho \to \infty,$$

where α is given as before. Furthermore, if there are some $M, L \in \mathbb{N}$ such that $A(\rho, x)$ satisfies for every $x \in U$

$$D_x^m D_\rho^l \left[A(\pm\rho, x) - A^{\pm}(x) \right] = \mathcal{O}\left(e^{-\alpha\rho}\right) \;\; \text{as } \rho \to \infty$$

for all $m \in \{0, \ldots, M\}$ and $l \in \{0, \ldots, L\}$, then

$$D_x^m D_\rho^l \left[w(\pm\rho, x) + \frac{A^{\pm}(x)}{f''(\pm 1)} \right] = \mathcal{O}\left(e^{-\alpha\rho}\right) \;\; \text{as } \rho \to \infty \tag{2.2}$$

for all $m \in \{0, \ldots, M\}$ and $l \in \{0, \ldots, L + 2\}$.

Proof. See [47], p. 16, Lemma 2.6.2. □

Lemma 2.4. *Let $U \subset \mathbb{R}^n$ be an open subset and let $B : \mathbb{R} \times U \to \mathbb{R}$, $(\rho, x) \mapsto B(\rho, x)$ be given and smooth. Assume that for all $x \in U$ the decay property $B(\pm\rho, x) = \mathcal{O}(e^{-\alpha\rho})$ as $\rho \to \infty$ is fulfilled.*

Then for each $x \in U$ the problem

$$w_{\rho\rho}(\rho, x) = B(\rho, x) \ \forall \rho \in \mathbb{R}$$

has a solution $w(., x) \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} B(\rho, x) \, \mathrm{d}\rho = 0. \tag{2.3}$$

Furthermore, if $w_*(\rho, x)$ is such a solution, then all the solutions can be written as

$$w(\rho, x) = w_*(\rho, x) + c(x),$$

where $c : U \to \mathbb{R}$ is an arbitrary function.

In particular, if (2.3) holds,

$$w_*(\rho, x) = \int_0^\rho \int_{-\infty}^r B(s, x) \, \mathrm{d}s \mathrm{d}\rho \tag{2.4}$$

is a solution.

Additionally, if $\int_{\mathbb{R}} B(\rho, x) \, \mathrm{d}\rho = 0$ for all $x \in U$ and there exist $M, L \in \mathbb{N}$ such that

$$D_x^m D_\rho^l B(\pm\rho, x) = \mathcal{O}\left(e^{-\alpha\rho}\right) \ \text{as } \rho \to \infty$$

for all $m \in \{0, \ldots, M\}$ and $l \in \{0, \ldots, L\}$ then there exist functions $w^+(x)$ and $w^-(x)$ such that

$$D_x^m D_\rho^l \left[w(\pm\rho, x) - w^\pm(x)\right] = \mathcal{O}\left(e^{-\alpha\rho}\right) \ \text{as } \rho \to \infty$$

for all $m \in \{0, \ldots, M\}$ and $l \in \{0, \ldots, L+2\}$.

Proof. See [47], p. 19, Lemma 2.6.3. □

2.2. Stationary Stokes Equation in One Phase

As the stationary Stokes equation will play an important role in later parts of this work we will give a short reminder of some results. For a thorough work on steady-state problems related to the Navier-Stokes equation, albeit with different boundary conditions, see [31]. Throughout this Section, we assume that Assumption 1.1 holds.

We consider the one-phase stationary Stokes equation

$$-\Delta \mathbf{v} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{2.5}$$

$$\operatorname{div}\mathbf{v} = g \qquad \text{in } \Omega, \tag{2.6}$$

$$\left(-2D_s\mathbf{v} + p\mathbf{I}\right)\mathbf{n}_{\partial\Omega} = \alpha_0 \mathbf{v} \qquad \text{on } \partial\Omega \tag{2.7}$$

for given $\mathbf{f} \in V_g'(\Omega)$ and $g \in L^2(\Omega)$. We denote $C_\sigma^\infty\left(\overline{\Omega}\right) := \left\{\mathbf{u} \in C^\infty\left(\overline{\Omega}\right)^2 \middle| \operatorname{div}\mathbf{u} = 0\right\}$, $H_\sigma^1(\Omega) := \overline{C_\sigma^\infty\left(\overline{\Omega}\right)}^{H_1(\Omega)}$ and set

$$V_g(\Omega) := \begin{cases} H_\sigma^1(\Omega) & \text{if } g = 0, \\ H^1(\Omega)^2 & \text{else,} \end{cases} \tag{2.8}$$

$$H_g(\Omega) := \begin{cases} L_\sigma^2(\Omega) & \text{if } g \equiv 0, \\ L^2(\Omega)^2 & \text{else.} \end{cases}$$

and let $V_g'(\Omega)$ denote the dual space of $V_g(\Omega)$.

We call $\mathbf{v} \in V_g(\Omega)$ a weak solution of (2.5)–(2.7) if

$$2\int_\Omega D_s\mathbf{v} : D_s\psi \,\mathrm{d}x + \alpha_0 \int_{\partial\Omega} \mathbf{v}\cdot\psi \,\mathrm{d}\mathcal{H}^1(s) = \langle \mathbf{f}, \psi\rangle_{V_g', V_g} \tag{2.9}$$

holds for all $\psi \in C_\sigma^\infty\left(\overline{\Omega}\right)$ and

$$\operatorname{div}\mathbf{v} = g \text{ in } L^2(\Omega). \tag{2.10}$$

Note that in the case $g = 0$ the condition (2.10) is already included in the definition of the space V_0 and can thus be omitted. Moreover, a classical solution to (2.5)–(2.7) is a weak solution, if $g \equiv 0$ almost everywhere on $\partial\Omega$. The following lemma immediately implies coercivity of the bilinear form induced by (2.9).

Lemma 2.5 (Modified Korn Inequality)**.** *Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1-boundary and let $\gamma \subset \partial\Omega$ be an open subset. Then there exist C_1, $C_2 > 0$, depending only on Ω and γ, such that*

$$\|\mathbf{u}\|_{H^1(\Omega)} \le C_1 \|D_s\mathbf{u}\|_{L^2(\Omega)} + C_2 \|\mathbf{u}\|_{L^2(\gamma)} \ \forall \mathbf{u} \in H^1(\Omega)^n.$$

Proof. See [13], p. 10, Corollary 5.8. □

Theorem 2.6. *For each $g \in L^2(\Omega)$ and $\mathbf{f} \in V_g'(\Omega)$ there is a unique weak solution $\mathbf{v} \in V_g(\Omega)$ of (2.5)–(2.7). Moreover there exists a constant $C(\Omega, \alpha_0) > 0$ which is independent of $\mathbf{f}$ such that*

$$\|\mathbf{v}\|_{H^1(\Omega)} \le C(\Omega, \alpha_0)\left(\|\mathbf{f}\|_{V_g'(\Omega)} + \|g\|_{L^2(\Omega)}\right). \tag{2.11}$$

Proof. We first consider the case $g = 0$. Then the statement is a direct consequence of the Lax-Milgram Lemma if we can show that

$$B : V_0 \times V_0 \to \mathbb{R},\ (\mathbf{u}, \mathbf{v}) \mapsto 2 \int_{\Omega} D_s\mathbf{u} : D_s\mathbf{v} \mathrm{d}x + \alpha_0 \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{v} \mathrm{d}\mathcal{H}^1(s)$$

is bounded and coercive. The boundedness of B follows immediately from the Trace Theorem, as $tr_{\partial\Omega} : H^1(\Omega) \to L^2(\partial\Omega)$ is continuous and the coercivity is a direct consequence of Lemma 2.5. Thus there exists a unique solution $\mathbf{v} \in H^1_\sigma(\Omega)$ of $B(u, \psi) = \mathbf{f}(\psi)$ for all $\psi \in H^1_\sigma(\Omega)$ and we have the estimate

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq C(\Omega, \alpha_0) \|\mathbf{f}\|_{V_0'(\Omega)}.$$

Let now $g \in L^2(\Omega)$ be arbitrary and $\mathbf{f} \in V_g'(\Omega)$. By standard elliptic theory there is a unique solution $q \in H^2(\Omega) \cap H^1_0(\Omega)$ of

$$\begin{aligned} -\Delta q &= g && \text{in } \Omega, \\ q &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with $\|q\|_{H^2(\Omega)} \leq C \|g\|_{L^2(\Omega)}$. As in the first part of the proof, there is a unique solution $\tilde{\mathbf{v}} \in H^1_\sigma(\Omega)$ to (2.9) for the right hand side

$$\tilde{\mathbf{f}}(\psi) := \mathbf{f}(\psi) - 2 \int_{\Omega} D_s(\nabla q) : D_s\psi \mathrm{d}x - \alpha_0 \int_{\partial\Omega} \nabla q \cdot \psi \mathrm{d}\mathcal{H}^1(s)$$

Now we define $\mathbf{v} := \tilde{\mathbf{v}} + \nabla q$ and immediately find $\mathrm{div}\mathbf{v} = g$ in $L^2(\Omega)$. Moreover

$$\begin{aligned} B(\mathbf{v}, \psi) &= \tilde{\mathbf{f}}(\psi) + 2 \int_{\Omega} D_s(\nabla q) : D_s\psi \mathrm{d}x + \alpha_0 \int_{\partial\Omega} \nabla q \cdot \psi \mathrm{d}\mathcal{H}^1(s) \\ &= \mathbf{f}(\psi) \end{aligned}$$

for all $\psi \in H^1_\sigma(\Omega)$. Now the definition of $\mathbf{v}$ and $\tilde{\mathbf{v}}$ together with the H^2–estimate for q and the Trace Theorem yield

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq C(\Omega, \alpha_0) \left(\|\mathbf{f}\|_{V_g'(\Omega)} + \|g\|_{L^2(\Omega)} \right).$$

□

The following corollary yields existence of a pressure term.

Corollary 2.7. *Let $g \in L^2(\Omega)$ and $\mathbf{f} \in L^2(\Omega)^2$. Then there is a unique weak solution $(\mathbf{v}, p) \in V_g \times L^2(\Omega)$ of (2.5)–(2.7) in the sense that*

$$2 \int_{\Omega} D_s\mathbf{v} : D_s\psi - p\mathrm{div}\psi \mathrm{d}x + \alpha_0 \int_{\partial\Omega} \mathbf{v} \cdot \psi \mathrm{d}\mathcal{H}^1(s) = \int_{\Omega} \mathbf{f} \cdot \psi \mathrm{d}x$$

for all $\psi \in H^1(\Omega)$ and (2.10) holds. Moreover, there is a constant $C > 0$, independent of $\mathbf{v}$ *and p, such that*

$$\|(\mathbf{v}, p)\|_{H^1(\Omega) \times L^2(\Omega)} \leq C \left(\|\mathbf{f}\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \right).$$

Proof. Let $\mathbf{v}$ be the weak solution to (2.9)–(2.10) as given by Theorem 2.6. Elliptic theory implies that $\Delta_D : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is bijective, where Δ_D denotes the Laplace operator supplemented with Dirichlet boundary conditions. Thus, the adjoint operator $(\Delta_D)' : L^2(\Omega)' \to \left(H^2(\Omega) \cap H_0^1(\Omega)\right)'$ is also bijective. Using the Trace Theorem and Hölder's inequality we find that the operator

$$F(\varphi) := \int\limits_{\Omega} 2D_s\mathbf{v} : D_s(\nabla\varphi) - \mathbf{f}\cdot\nabla\varphi \mathrm{d}x + \alpha_0 \int\limits_{\partial\Omega} \mathbf{v}\cdot\nabla\varphi \mathrm{d}\mathcal{H}^1(s) \quad \forall\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$$

is bounded and linear and thus the Riesz Representation Theorem yields the existence of $p \in L^2(\Omega)$ such that

$$(p, \Delta\varphi)_{L^2} = \left\langle \Delta_D'\left((p,.)_{L^2}\right), \varphi\right\rangle_{\left(H^2(\Omega)\cap H_0^1(\Omega)\right)', H^2(\Omega)\cap H_0^1(\Omega)} = F(\varphi) \tag{2.12}$$

for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. The operator $\left((\Delta_D)'\right)^{-1}$ is bounded and we find

$$\begin{aligned} \|p\|_{L^2(\Omega)} &\leq C\|F\|_{\left(H^2(\Omega)\cap H_0^1(\Omega)\right)'} \\ &\leq C\left(\|\mathbf{v}\|_{H^1(\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)}\right) \\ &\leq C\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\right), \end{aligned}$$

where we used (2.11) in the last line.

Let now $\psi \in H^1(\Omega)^2$ and let $q \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique solution to

$$\begin{aligned} \Delta q &= \mathrm{div}\psi && \text{in } \Omega, \\ q &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Moreover set $\psi_0 := \psi - \nabla q$, which satisfies $\mathrm{div}\psi_0 = 0$. Then,

$$\begin{aligned} \int\limits_{\Omega} 2D_s\mathbf{v} : D_s\psi - p\mathrm{div}\psi \mathrm{d}x + \alpha_0 \int\limits_{\partial\Omega} \mathbf{v}\cdot\psi \mathrm{d}\mathcal{H}^1(s) &= \int\limits_{\Omega} \mathbf{f}\cdot\psi_0 \mathrm{d}x + \int\limits_{\Omega} 2D_s\mathbf{v} : D_s(\nabla q) - p\Delta q \mathrm{d}x \\ &\quad + \alpha_0 \int\limits_{\partial\Omega} \mathbf{v}\cdot\nabla q \mathrm{d}\mathcal{H}^1(s) \\ &= \int\limits_{\Omega} \mathbf{f}\cdot\psi \mathrm{d}x, \end{aligned}$$

where we used (2.9) in the first equality and (2.12) in the second. As $\psi \in H^1(\Omega)^2$ was arbitrary, this yields the claim. ☐

Theorem 2.8 (Existence of Strong Solutions)**.** *Let $g \equiv 0$ and $\mathbf{f} \in L^2(\Omega)^2$. Then there exists a unique solution $(\mathbf{v}, p) \in H^2(\Omega)^2 \times H^1(\Omega)$ to (2.5)–(2.7), which satisfies the estimate*

$$\|\mathbf{v}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq C\|\mathbf{f}\|_{L^2(\Omega)}.$$

Moreover, if $\mathbf{f}$ is smooth, then $\mathbf{v}$ and p are smooth as well.

Proof. For $q \in (1,\infty)$, Theorem 3.1 in [49] (which was already shown in [35]) implies that there is $\lambda > 0$ such that for every $\mathbf{g} \in L^q(\Omega)^2$ and $\mathbf{a} \in W^1_q(\Omega)^2$ the problem

$$\begin{aligned} \lambda\mathbf{u} - \Delta\mathbf{u} + \nabla q &= \mathbf{g} && \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, \\ (-2D_s\mathbf{u} + q\mathbf{I})\,\mathbf{n}_{\partial\Omega} &= \mathbf{a} && \text{on } \partial\Omega \end{aligned} \tag{2.13}$$

admits for a unique solution $(\mathbf{u}, q) \in W^2_q(\Omega)^2 \times W^1_q(\Omega)$. Additionally, the estimate

$$\|\mathbf{u}\|_{W^2_q(\Omega)} + \|q\|_{W^1_q(\Omega)} \leq C\left(\|\mathbf{g}\|_{L^q(\Omega)} + \|\mathbf{a}\|_{W^1_q(\Omega)}\right) \tag{2.14}$$

holds. Considering a weak solution $(\mathbf{v}, p) \in V_0 \times L^2(\Omega)$ of (2.5)–(2.7) as given in Corollary 2.7 and defining $\mathbf{g} := \mathbf{f} + \lambda\mathbf{v} \in L^2(\Omega)^2$ and $\mathbf{a} := \alpha_0\mathbf{v} \in H^1(\Omega)^2$, we now introduce $(\mathbf{u}, q) \in H^2(\Omega) \times H^1(\Omega)$ as the strong solution to (2.13) regarding these data. Writing $\mathbf{w} := \mathbf{u} - \mathbf{v}$ and $r := q - p$ we find that $(\mathbf{w}, r)$ is a weak solution to

$$\begin{aligned} \lambda\mathbf{w} - \Delta\mathbf{w} + \nabla r &= 0 && \text{in } \Omega, \\ \operatorname{div}\mathbf{w} &= 0 && \text{in } \Omega, \\ (-2D_s\mathbf{w} + r\mathbf{I})\,\mathbf{n}_{\partial\Omega} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

since

$$\begin{aligned} \int_\Omega \lambda\mathbf{w}\psi + 2D_s\mathbf{w} : D_s\psi - r\operatorname{div}\psi \mathrm{d}x &= \int_\Omega (\lambda\mathbf{u} - \Delta\mathbf{u} + \nabla q)\cdot\psi - \lambda\mathbf{v}\cdot\psi - 2D_s\mathbf{v} : D_s\psi + p\operatorname{div}\psi\mathrm{d}x \\ &\quad - \int_{\partial\Omega} \mathbf{a}\cdot\psi \mathrm{d}\mathcal{H}^1 \\ &= \int_\Omega (\mathbf{g} - \lambda\mathbf{v} - \mathbf{f})\cdot\psi\mathrm{d}x \\ &= 0 \end{aligned}$$

for all $\psi \in H^1(\Omega)$. Choosing $\psi = \mathbf{w}$ we immediately find that $\mathbf{w} \equiv 0$ a.e. and thus $\mathbf{u} = \mathbf{v}$, in particular $\mathbf{v} \in H^2(\Omega)^2$. Furthermore, $\mathbf{w} = 0$ implies $\nabla r = 0$ in Ω and $r = 0$ on $\partial\Omega$, so that we can conclude $r \equiv 0$ a.e. in Ω leading to $p = q$ and $p \in H^1(\Omega)$. The estimate follows from (2.14) and (2.11). For higher regularity one may employ typical arguments used for elliptic partial differential equations, e.g. test with suitable difference quotients and locally transform to the half-space case when close to the boundary. □

Lemma 2.9. *Let $g \equiv 0$ and $\mathbf{f} \in V_0'$, and let $\mathbf{v} \in H^1_\sigma(\Omega)$ be the weak solution to (2.5)–(2.7). Then for all $q' \in (1,2)$*

$$\|\mathbf{v}\|_{L^{q'}(\Omega)} \leq C_q \sup_{\psi\in W^2_q(\Omega)^2, \psi\neq 0} \frac{|\mathbf{f}(\psi)|}{\|\psi\|_{W^2_q(\Omega)}}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $C_q > 0$ is independent of $\mathbf{v}$ and f.

Proof. For this we introduce $T(\mathbf{u}, p) := -2D_s\mathbf{u} + p\mathbf{I}$ for $\mathbf{u} \in W^1_q(\Omega)$, $p \in L^2(\Omega)$ and set

$$D(A_S) = \left\{ \mathbf{u} \in W^2_q(\Omega) \middle| \operatorname{div}\mathbf{u} = 0, \exists p \in W^1_q(\Omega) \text{ s.t. } T(\mathbf{u}, p)\,\mathbf{n}|_{\partial\Omega} = \alpha_0\mathbf{u}|_{\partial\Omega} \right\}.$$

We define the operator

$$A_S : D(A_S) \subset L^q_\sigma(\Omega) \to L^q_\sigma(\Omega), \, \mathbf{u} \mapsto P_\sigma(-\Delta \mathbf{u} + \nabla p),$$

for p as in the definition of $D(A_S)$ and where P_σ denotes the Leray-Helmholtz projection given by

$$P_\sigma : L^q(\Omega)^2 \to L^q_\sigma(\Omega), \, \psi \mapsto P_\sigma(\psi) = \psi - \nabla r,$$

where $r \in W^1_{q,0}(\Omega)$ is the unique weak solution to

$$\begin{aligned} \Delta r &= \operatorname{div}\psi && \text{in } \Omega, \\ r &= 0 && \text{on } \partial\Omega. \end{aligned}$$

See [3], Lemma 2.4, p. 6 for existence and uniqueness (as Ω is a bounded domain in our case, see also the remark after Definition 2.2 in the cited article). Here, div· is understood in the sense of distributions.

First, we show that A_S is well defined. For this, let $\mathbf{u} \in D(A_S)$ and $p_1, p_2 \in W^1_q(\Omega)$ such that $T(\mathbf{u}, p_i)\,\mathbf{n}|_{\partial\Omega} = \alpha_0 \mathbf{u}|_{\partial\Omega}$ for $i \in \{1, 2\}$. Then we have $p_1 = p_2$ on $\partial\Omega$. Now consider

$$P_\sigma(-\Delta \mathbf{u} + \nabla p_i) = -\Delta \mathbf{u} + \nabla p_i - \nabla r_i,$$

where $r_i \in W^1_{q,0}(\Omega)$ is the weak solution to

$$\begin{aligned} \Delta r_i &= \Delta p_i && \text{in } \Omega, \\ r_i &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Note that $\operatorname{div}(\Delta \mathbf{u}) = 0$ in $\mathcal{D}'(\Omega)$ as $\operatorname{div}\mathbf{u} = 0$, resulting in $P_\sigma(\Delta \mathbf{u}) = \Delta \mathbf{u}$. The uniqueness of the weak solution and $p_1 - p_2 = 0$ on $\partial\Omega$ imply $r_1 - r_2 = p_1 - p_2$. Thus,

$$\begin{aligned} P_\sigma(-\Delta \mathbf{u} + \nabla p_1) - P_\sigma(-\Delta \mathbf{u} + \nabla p_2) &= \nabla p_1 - \nabla p_2 - (\nabla r_1 - \nabla r_2) \\ &= 0 \end{aligned}$$

which implies that A_S is well defined. Moreover, A_S is positive regarding the L^2 scalar product, i.e.

$$\begin{aligned} \int_\Omega (A_S \mathbf{u}) \cdot \mathbf{u} \mathrm{d}x &= \int_\Omega (-\Delta \mathbf{u} + \nabla p) \cdot \mathbf{u} \mathrm{d}x \\ &= \int_\Omega 2 |D_s \mathbf{u}|^2 \mathrm{d}x + \alpha_0 \int_{\partial\Omega} \mathbf{u}^2 \mathrm{d}\mathcal{H}^1(s) \\ &\geq C \|\mathbf{u}\|^2_{L^2(\Omega)} \end{aligned} \tag{2.15}$$

for some $C > 0$ and $\mathbf{u} \in \mathcal{D}(A_S)$, where we used Lemma 2.5 in the last line. This immediately shows the injectivity of A_S. Concerning surjectivity, let $\tilde{\mathbf{f}} \in L^q_\sigma(\Omega)$. As $q > 2$, Theorem 2.8 implies that there is a unique strong solution $(\tilde{\mathbf{v}}, p) \in H^2(\Omega) \times H^1(\Omega)$ to (2.5)–(2.7) (with $\mathbf{f}$ replaced by $\tilde{\mathbf{f}}$ and $g \equiv 0$). Choosing $\lambda > 0$ as in the proof of Theorem 2.8, we find that $\mathbf{g} := \tilde{\mathbf{f}} + \lambda \tilde{\mathbf{v}}$ and $\mathbf{a} := \alpha_0 \tilde{\mathbf{v}}$ satisfy $\mathbf{g} \in L^q(\Omega)$ and $\mathbf{a} \in W^1_q(\Omega)$ as a consequence of the Sobolev Embedding theorem. Thus, Theorem 3.1. in [49] implies the existence of a unique solution $(\mathbf{u}, r) \in W^2_q(\Omega) \times W^1_q(\Omega)$ to (2.13) and an analogous argumentation as in the proof of Theorem 2.8 leads to the insight that $\tilde{\mathbf{v}} = \mathbf{u}$ and $p = r$ hold along with the estimate

$$\|\tilde{\mathbf{v}}\|_{W^2_q(\Omega)} + \|p\|_{W^1_q(\Omega)} \leq C \left\|\tilde{\mathbf{f}}\right\|_{L^q(\Omega)}. \tag{2.16}$$

In particular, $T(\tilde{\mathbf{v}}, p)\,\mathbf{n}|_{\partial\Omega} = \alpha_0 \tilde{\mathbf{v}}|_{\partial\Omega}$ is satisfied, so $\tilde{\mathbf{v}} \in \mathcal{D}(A_s)$, and since $-\Delta\tilde{\mathbf{v}} + \nabla p = \tilde{\mathbf{f}}$ holds in $L^q(\Omega)$, we have $A_s(\tilde{\mathbf{v}}) = \tilde{\mathbf{f}}$. In fact, this not only implies surjectivity, but also the existence of a bounded inverse A_S^{-1} as a result of (2.16). Consequently, $\left(\mathcal{D}(A_s), \|.\|_{A_s}\right)$ is a Banach space, where $\|.\|_{A_s}$ denotes the graph norm. All these considerations result in the fact that the adjoint $A_S' : (L^q_\sigma(\Omega))' \to (\mathcal{D}(A_S))'$ is an invertible, bounded operator.

Let now $\mathbf{v} \in H^1_\sigma(\Omega)$ be the given weak solution to (2.5)–(2.7) and fix $q > 2$. Then $\mathbf{v} \in L^{q'}_\sigma(\Omega) \cong (L^q_\sigma(\Omega))'$ and we have for $\psi \in \mathcal{D}(A_S)$

$$\begin{aligned}
\left\langle A_S'\mathbf{v}, \psi\right\rangle_{(\mathcal{D}(A_S))',\mathcal{D}(A_S)} &= \int_\Omega (-\Delta\psi + \nabla p)\cdot\mathbf{v}\mathrm{d}x \\
&= 2\int_\Omega D_s\mathbf{v} : D_s\psi\mathrm{d}x + \alpha_0\int_{\partial\Omega}\psi\cdot\mathbf{v}\mathrm{d}x \\
&= \langle\mathbf{f}, \psi\rangle_{(\mathcal{D}(A_S))',\mathcal{D}(A_S)},
\end{aligned}$$

where we used $\mathrm{div}\mathbf{v} = 0$ in the first line to get rid of the Helmholtz projection and the property of $\mathbf{v}$ as a weak solution combined with $\mathbf{V}_0' \subset \mathcal{D}(A_S)'$ in the last line. As a result $A_S'\mathbf{v} = \mathbf{f}$ in $(\mathcal{D}(A_S))'$ and thus $\mathbf{v} = (A_S')^{-1}\mathbf{f}$ in $(L^q_\sigma(\Omega))'$ which enables us to estimate

$$\|\mathbf{v}\|_{L^{q'}(\Omega)} = \left\|\left(A_S'\right)^{-1}\mathbf{f}\right\|_{\left(L^q_\sigma(\Omega)\right)'} \leq C\|\mathbf{f}\|_{(\mathcal{D}(A_S))'} \leq C\|\mathbf{f}\|_{\left(W^2_q(\Omega)\right)'}.$$

□

2.3. Differential-Geometric Background

Throughout this work we will deal with evolving hypersurfaces and tubular neighborhoods as those are the domains of the inner terms considered in the process of asymptotic matching, cf. Chapter 5. This section is thus dedicated to outlining the relevant results and notations we need in order to efficiently handle the differential-geometric difficulties. The results here are based on Chapter 4 in [26] and Chapter 2.1 in [6]. Details about the signed distance function and some other basic presented statements can be found in [38].

In this section we assume that Assumption 1.1 holds. We will use the following notations and conventions throughout this chapter:

We parameterize the curves $(\Gamma_t)_{t\in[0,T_0]}$ by choosing a family of smooth diffeomorphisms

$$X_0 : \mathbb{T}^1 \times [0, T_0] \to \Omega \tag{2.17}$$

such that $\partial_s X_0(s,t) \neq 0$ for all $s \in \mathbb{T}^1$, $t \in [0, T_0]$. In particular

$$\bigcup_{t\in[0,T_0]} X_0\left(\mathbb{T}^1 \times \{t\}\right) \times \{t\} = \Gamma.$$

Moreover, we define the tangent and normal vectors on Γ_t at $X_0(s,t)$ as

$$\tau(s,t) := \frac{\partial_s X_0(s,t)}{|\partial_s X_0(s,t)|} \text{ and } \mathbf{n}(s,t) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau(s,t) \tag{2.18}$$

for all $(s,t) \in \mathbb{T}^1 \times [0, T_0]$.

We choose X_0 (and thereby the orientation of Γ_t) such that $\mathbf{n}(.,t)$ is the exterior normal with respect to $\Omega^-(t)$. Thus, for a point $p \in \Gamma_t$ with $p = X_0(s,t)$ it holds $\mathbf{n}_{\Gamma_t}(p) = \mathbf{n}(s,t)$

Furthermore, we define $V(s,t) := V_{\Gamma_t}(X_0(s,t))$ and $H(s,t) := H_{\Gamma_t}(X_0(s,t))$ and note that

$$V(s,t) = \partial_t X_0(s,t) \cdot \mathbf{n}(s,t)$$

for all $(s,t) \in \mathbb{T}^1 \times [0, T_0]$ by definition of the normal velocity.

Notation 2.10. Let $d \in \mathbb{N}$. For a function $\mathbf{v} : \Gamma \to \mathbb{R}^d$ we define

$$(X_0^* \mathbf{v})(s,t) := \mathbf{v}(X_0(s,t), t) \tag{2.19}$$

for all $(s,t) \in \mathbb{T}^1 \times [0, T_0]$. On the other hand, we define for a function $h : \mathbb{T}^1 \times [0, T_0]$

$$\left(X_0^{*,-1} h\right)(p) := h\left(X_0^{-1}(p)\right) \tag{2.20}$$

for all $p \in \Gamma_t$, $t \in [0, T_0]$.

The following lemma guarantees that if we choose $\delta > 0$ small enough, we get a unique decomposition of every $x \in \Gamma_t(3\delta)$ into a surface and a normal part.

Lemma 2.11. *There exists $\delta > 0$ such that the orthogonal projection*

$$Pr_{\Gamma_t} : \Gamma_t(3\delta) \to \Gamma_t$$

is well defined and smooth for all $t \in [0, T_0]$ and the mapping

$$\phi_t : \Gamma_t(3\delta) \to (-3\delta, 3\delta) \times \Gamma_t,\ x \mapsto \left(d_\Gamma(x,t), Pr_{\Gamma_t}(x)\right)$$

is a diffeomorphism. Its inverse is given by $\phi_t^{-1}(r,p) = p + r\mathbf{n}_{\Gamma_t}(p)$.

Proof. See [44], Chapter 2.3.1. □

Although Pr_{Γ_t} and ϕ_t are well defined in $\Gamma_t(3\delta)$, almost all computations later on are performed in $\Gamma_t(2\delta)$, which is why, for the sake of readability, we work on $\Gamma_t(2\delta)$ in the following.

Combining ϕ_t^{-1} and X_0 we may define a diffeomorphism

$$\begin{aligned} X:\ &(-2\delta,2\delta)\times\mathbb{T}^1\times[0,T_0]\to\Gamma(2\delta),\\ &(r,s,t)\mapsto\left(\phi_t^{-1}(r,X_0(s,t)),t\right)=(X_0(s,t)+r\mathbf{n}(s,t),t). \end{aligned} \tag{2.21}$$

The inverse is given by

$$X^{-1}:\Gamma(2\delta)\to(-2\delta,2\delta)\times\mathbb{T}^1\times[0,T_0],\ (x,t)\mapsto(d_\Gamma(x,t),S(x,t),t), \tag{2.22}$$

where we define

$$S(x,t):=\left(X_0^{-1}\left(Pr_{\Gamma_t}(x)\right)\right)_1 \tag{2.23}$$

for $(x,t)\in\Gamma(2\delta)$ and where $(.)_1$ signifies that we take the first component. In particular it holds $S(x,t)=S(Pr_{\Gamma_t}(x),t)$.

Notation 2.12. We write $\mathbf{n}(x,t):=\mathbf{n}(S(x,t),t)$ and $\tau(x,t):=\tau(S(x,t),t)$ for $(x,t)\in\Gamma(3\delta)$.

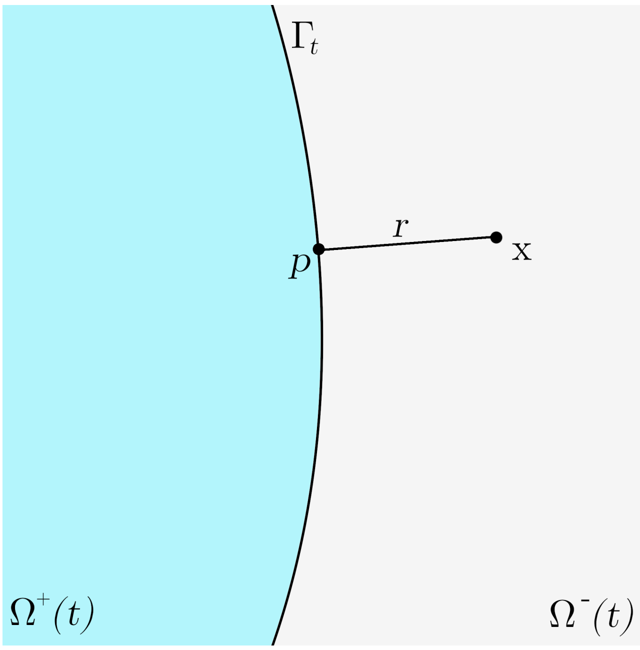

Figure 2.2.: Decomposition of $x\in\Gamma_t(2\delta)$ into $(r,p)\in(-2\delta,2\delta)\times\Gamma_t$ using ϕ_t.

The following lemma summarizes many important properties and connections between d_Γ, S, V, H and $\mathbf{n}$.

Proposition 2.13. *Let $t\in[0,T_0]$.*

1. *For all $x\in\Gamma_t(2\delta)$ the equality*

$$|\nabla d_\Gamma(x,t)|=1$$

holds.

2. *For all* $s \in \mathbb{T}^1$ *we have*

$$\Delta d_\Gamma \left(X_0 \left(s, t\right), t\right) = -H \left(s, t\right).$$

3. *For* $s \in \mathbb{T}^1$ *and all* $r \in (-2\delta, 2\delta)$ *we have*

$$-\partial_t d_\Gamma \left(X \left(r, s, t\right)\right) = V \left(s, t\right)$$

and

$$\nabla d_\Gamma \left(X \left(r, s, t\right)\right) = \mathbf{n} \left(s, t\right).$$

4. *For all* $x \in \Gamma_t \left(2\delta\right)$ *the identity*

$$\nabla S \left(x, t\right) \cdot \nabla d_\Gamma \left(x, t\right) = 0$$

holds.

Proof. Ad 1) This follows from 3.
Ad 2) See [44], Chapter 2.3.2.
Ad 3) For the first identity see [26], Chapter 4.1, for the second one see [44], Chapter 2.3.1.
Ad 4) See [26], Chapter 4.1. □

Notation 2.14. For a function $\phi : \Gamma \left(2\delta\right) \to \mathbb{R}$ we define

$$\tilde{\phi} \left(r, s, t\right) := \phi \left(X \left(r, s, t\right)\right).$$

We often write $\phi \left(r, s, t\right)$ instead of $\tilde{\phi} \left(r, s, t\right)$.

Lemma 2.15. *Let* $\phi : \Gamma \left(2\delta\right) \to \mathbb{R}$ *be twice continuously differentiable. Then the following formulas hold*

$$\begin{aligned}
\partial_t \phi \left(x, t\right) &= \left(-V \left(S \left(x, t\right), t\right) \partial_r + \partial_t^\Gamma\right) \tilde{\phi} \left(d_\Gamma \left(x, t\right), S \left(x, t\right), t\right), \\
\nabla \phi \left(x, t\right) &= \left(\mathbf{n} \left(S \left(x, t\right), t\right) \partial_r + \nabla^\Gamma\right) \tilde{\phi} \left(d_\Gamma \left(x, t\right), S \left(x, t\right), t\right), \\
\Delta \phi \left(x, t\right) &= \left(\partial_{rr} + \Delta d_\Gamma \left(x, t\right) \partial_r + \Delta^\Gamma\right) \tilde{\phi} \left(d_\Gamma \left(x, t\right), S \left(x, t\right), t\right),
\end{aligned}$$

for all $(x, t) \in \Gamma \left(2\delta\right)$*. Here we use for* $(r, s, t) \in (-2\delta, 2\delta) \times \mathbb{T}^1 \times [0, T_0]$ *the notations*

$$\begin{aligned}
\partial_t^\Gamma \tilde{\phi} \left(r, s, t\right) &= \left(\partial_t + \partial_t S \left(X \left(r, s, t\right)\right) \partial_s\right) \tilde{\phi} \left(r, s, t\right), \\
\nabla^\Gamma \tilde{\phi} \left(r, s, t\right) &= \nabla S \left(X \left(r, s, t\right)\right) \partial_s \tilde{\phi} \left(r, s, t\right), \\
\Delta^\Gamma \tilde{\phi} \left(r, s, t\right) &= \left(\Delta S \left(X \left(r, s, t\right)\right) \partial_s + \left(\nabla S \cdot \nabla S\right) \left(X \left(r, s, t\right)\right) \partial_{ss}\right) \tilde{\phi} \left(r, s, t\right).
\end{aligned} \tag{2.24}$$

Proof. This follows by using chain-rule and the identities stated in Proposition 2.13. □

In the situation of Lemma 2.15 we also define

$$D_\Gamma^2 \tilde{\phi} \left(r, s, t\right) = \left(D^2 S \partial_s + \nabla S \otimes \nabla S \partial_{ss}\right) \tilde{\phi} \left(r, s, t\right) \tag{2.25}$$

for future use.

Corollary 2.16. *Let* $\mathbf{v} : \Gamma \left(2\delta\right) \to \mathbb{R}^2$ *be continuously differentiable. As for scalar functions we write* $\tilde{\mathbf{v}} \left(r, s, t\right) := \mathbf{v} \circ X \left(r, s, t\right)$*. Then we have*

$$\operatorname{div} \mathbf{v} \left(x, t\right) = \left(\mathbf{n} \left(S \left(x, t\right), t\right) \partial_r + \operatorname{div}^\Gamma\right) \tilde{\mathbf{v}} \left(d_\Gamma \left(x, t\right), S \left(x, t\right), t\right)$$

for all $(x, t) \in \Gamma \left(2\delta\right)$*. Here we use for* $(r, s, t) \in (-2\delta, 2\delta) \times \mathbb{T}^1 \times [0, T_0]$ *the notation*

$$\operatorname{div}^\Gamma \tilde{\mathbf{v}} \left(r, s, t\right) = \nabla S \left(X \left(r, s, t\right)\right) \cdot \partial_s \tilde{\mathbf{v}} \left(r, s, t\right). \tag{2.26}$$

Proof. Again, this follows by chain-rule and using the identities from Proposition 2.13. □

Notation 2.17. Later on we will use the notation

$$\nabla^\Gamma \phi(x,t) := \nabla S(x,t)\,\partial_s \tilde{\phi}(d_\Gamma(x,t), S(x,t), t)$$

and

$$\mathrm{div}^\Gamma \mathbf{v}(x,t) := \nabla S(x,t)\,\partial_s \tilde{\mathbf{v}}(d_\Gamma(x,t), S(x,t), t)$$

for $(x,t) \in \Gamma(2\delta)$ and ϕ, $\mathbf{v}$ as in Lemma 2.15 and Corollary 2.16.

Using Notation 2.17, Lemma 2.15 and Corollary 2.16 we find that we have the decompositions

$$\nabla\phi(x,t) = \partial_{\mathbf{n}}\phi(x,t)\,\mathbf{n} + \nabla^\Gamma\phi(x,t), \tag{2.27}$$

$$\mathrm{div}\mathbf{v}(x,t) = \partial_{\mathbf{n}}\mathbf{v}(x,t)\cdot\mathbf{n} + \mathrm{div}^\Gamma\mathbf{v}(x,t) \tag{2.28}$$

for all $(x,t) \in \Gamma(2\delta)$, as

$$\frac{d}{dr}(\phi\circ X)|_{(r,s,t)=(d_\Gamma(x,t),S(x,t),t)} = \partial_{\mathbf{n}}\phi(x,t).$$

Proposition 2.18. *Let* $\mathbf{v}^\pm : \overline{\Omega^\pm_{T_0}} \to \mathbb{R}^2$ *be continuously differentiable and we denote in the following* $[\mathbf{v}](x,t) = \mathbf{v}^+(x,t) - \mathbf{v}^-(x,t)$ *for* $(x,t)\in\Gamma$ *(similarly for derivatives). Then it holds*

$$2[D_s\mathbf{v}]\cdot\mathbf{n} = [\partial_{\mathbf{n}}\mathbf{v}] - \nabla\mathbf{n}\cdot[\mathbf{v}] + \nabla^\Gamma[(\mathbf{v}\cdot\mathbf{n})] + \left([\mathrm{div}\mathbf{v}] - \mathrm{div}^\Gamma[\mathbf{v}]\right)\mathbf{n} \text{ on } \Gamma,$$

where $D_s\mathbf{v}$ *is defined as in (1.7).*

Proof. For $(x,t)\in\Gamma$ we have

$$\begin{aligned}
&2[D_s\mathbf{v}](x,t)\cdot\mathbf{n}(S(x,t),t)\\
&= \left(\nabla\mathbf{v}^+ + \left(\nabla\mathbf{v}^+\right)^T - \left(\nabla\mathbf{v}^- + \left(\nabla\mathbf{v}^-\right)^T\right)\right)(x,t)\cdot\mathbf{n}(S(x,t),t)\\
&= [\partial_{\mathbf{n}}\mathbf{v}](x,t) + [\nabla(\mathbf{v}(x,t)\cdot\mathbf{n}(S(x,t),t))] - \nabla(\mathbf{n}(S(x,t),t))[\mathbf{v}](x,t)\\
&= [\partial_{\mathbf{n}}\mathbf{v}](x,t) - \nabla(\mathbf{n}(S(x,t),t))\cdot[\mathbf{v}](x,t) + \left[\nabla^\Gamma(\mathbf{v}\cdot\mathbf{n})\right](x,t) + [\partial_{\mathbf{n}}(\mathbf{v}\cdot\mathbf{n})](x,t)\,\mathbf{n}(x,t)\\
&= [\partial_{\mathbf{n}}\mathbf{v}](x,t) - \nabla(\mathbf{n}(S(x,t),t))\cdot[\mathbf{v}](x,t) + \nabla^\Gamma[(\mathbf{v}\cdot\mathbf{n})](x,t)\\
&\quad + \left([\mathrm{div}\mathbf{v}](x,t) - \mathrm{div}^\Gamma[\mathbf{v}](x,t)\right)\mathbf{n}(x,t),
\end{aligned}$$

where we used (2.27) in the third equality and $\partial_{\mathbf{n}}\mathbf{n}(S(x,t),t) = 0$ (which is a consequence of Proposition 2.13) as well as (2.28) in the last equality. □

Remark 2.19. If $h : \mathbb{T}^1 \times [0,T_0] \to \mathbb{R}$ is a function that is independent of $r \in (-2\delta, 2\delta)$, the functions $\partial_t^\Gamma h, \nabla^\Gamma h$ and $\Delta^\Gamma h$ will nevertheless depend on r via the derivatives of S. To connect the presented concepts with the classical surface operators we introduce the following notations:

$$\begin{aligned}
D_{t,\Gamma}h(s,t) &= \partial_t^\Gamma h(0,s,t),\\
\nabla_\Gamma h(s,t) &= \nabla^\Gamma h(0,s,t),\\
\Delta_\Gamma h(s,t) &= \Delta^\Gamma h(0,s,t).
\end{aligned}$$

Later in this work (from Subsection 5.1.2 on forward) we will often consider $h(S(x,t),t)$ and thus will write for simplicity

$$\begin{aligned}\partial_t^\Gamma h(x,t) &:= (\partial_t + \partial_t S(x,t)\partial_s) h(S(x,t),t),\\ \nabla^\Gamma h(x,t) &:= (\nabla S(x,t)\partial_s) h(S(x,t),t),\\ \Delta^\Gamma h(x,t) &:= (\Delta S(x,t)\partial_s + \nabla S(x,t)\cdot\nabla S(x,t)\partial_{ss}) h(S(x,t),t)\end{aligned} \tag{2.29}$$

for $(x,t)\in\Gamma(2\delta)$. Using the definitions and notations from this chapter we gain the identity

$$\partial_t^\Gamma h(x,t) = X_0^*\left(\partial_t^\Gamma h\right)(s,t) = \partial_t^\Gamma h(0,s,t) = D_{t,\Gamma} h(s,t) \tag{2.30}$$

for $(s,t)\in\mathbb{T}^1\times[0,T_0]$ and $(X_0(s,t),t)=(x,t)\in\Gamma$. This might seem cumbersome but turns out to be convenient throughout this work.

In later parts of this thesis, we will introduce stretched coordinates of the form

$$\rho^\epsilon(x,t) = \frac{d_\Gamma(x,t) - \epsilon h(S(x,t),t)}{\epsilon} \tag{2.31}$$

for $(x,t)\in\Gamma(2\delta)$, $\epsilon\in(0,1)$ and for some smooth function $h:\mathbb{T}^1\times[0,T_0]\to\mathbb{R}$ (which will later on also depend on ϵ). Writing $\rho=\rho^\epsilon$, we have the equality

$$\rho(r,s,t) = \frac{r-\epsilon h(s,t)}{\epsilon}$$

for $(r,s,t)\in(-2\delta,2\delta)\times\mathbb{T}^1\times[0,T_0]$. The relation between the regular and the stretched variables can be expressed as

$$\hat{X}(\rho,s,t) := X(\epsilon(\rho+h(s,t)),s,t) = (X_0(s,t)+\epsilon(\rho+h(s,t))\mathbf{n}(s,t),t). \tag{2.32}$$

In subsequent chapters we will often consider concatenated functions $\phi(\rho(x,t),x,t)$ for $\phi:\mathbb{R}\times\Gamma(2\delta)\to\mathbb{R}$. In regard of the derivatives, we get the following lemma:

Lemma 2.20. *Let $\phi:\mathbb{R}\times\Gamma(2\delta)\to\mathbb{R}$ be two times continuously differentiable and let ρ be given as in (2.31). Then the following formulas hold for $(x,t)\in\Gamma(2\delta)$ and $\epsilon\in(0,1)$*

$$\begin{aligned}\partial_t(\phi(\rho(x,t),x,t)) &= \left(-\epsilon^{-1}V(S(x,t),t) - \partial_t^\Gamma h(x,t)\right)\partial_\rho\phi(\rho(x,t),x,t) + \partial_t\phi(\rho(x,t),x,t),\\ \nabla(\phi(\rho(x,t),x,t)) &= \left(\epsilon^{-1}\mathbf{n}(S(x,t),t) - \nabla^\Gamma h(x,t)\right)\partial_\rho\phi(\rho(x,t),x,t) + \nabla_x\phi(\rho(x,t),x,t),\\ \Delta(\phi(\rho(x,t),x,t)) &= \left(\epsilon^{-2} + \left|\nabla^\Gamma h(x,t)\right|^2\right)\partial_{\rho\rho}\phi(\rho(x,t),x,t)\\ &\quad + \left(\epsilon^{-1}\Delta d_\Gamma(x,t) - \Delta^\Gamma h(x,t)\right)\partial_\rho\phi(\rho(x,t),x,t)\\ &\quad + 2\left(\epsilon^{-1}\mathbf{n}(S(x,t),t) - \nabla^\Gamma h(x,t)\right)\cdot\nabla_x\partial_\rho\phi(\rho(x,t),x,t)\\ &\quad + \Delta_x\phi(\rho(x,t),x,t).\end{aligned}$$

Here ∇_x and Δ_x operate solely on the x-variable of ϕ.

Proof. This follows from the chain rule and Lemma 2.15 , Proposition 2.13 and the notations introduced in Remark 2.19. □

2.3.1. Divergence Theorem for Surface Operators

For certain considerations in this work it will be important to have formulae for integration by parts of the surface operators div^Γ and ∇^Γ. The following subsection is based on [6], Subsection 2.2.

By (2.27) and (2.28) we have

$$\nabla^\Gamma u(x,t) = (\mathbf{I} - \mathbf{n}(S(x,t),t) \otimes \mathbf{n}(S(x,t),t)) \nabla u(x,t) \tag{2.33}$$

and

$$\mathrm{div}^\Gamma \mathbf{v}(x,t) = (\mathbf{I} - \mathbf{n}(S(x,t),t) \otimes \mathbf{n}(S(x,t),t)) : \nabla \mathbf{v}(x,t) \tag{2.34}$$

for continuously differentiable $u : \Gamma(2\delta) \to \mathbb{R}$, $\mathbf{v} : \Gamma(2\delta) \to \mathbb{R}^2$. The following lemma is a consequence of this representation.

Lemma 2.21. *Let $t \in [0, T_0]$ and $\mathbf{v} \in H^1(\Gamma_t(\delta))^2$, $u \in H^1(\Gamma_t(\delta))$. Then it holds*

$$\int_{\Gamma_t(\delta)} u \mathrm{div}^\Gamma \mathbf{v} \mathrm{d}x = -\int_{\Gamma_t(\delta)} \nabla^\Gamma u \cdot \mathbf{v} \mathrm{d}x - \int_{\Gamma_t(\delta)} u \mathbf{v} \cdot \mathbf{n} \kappa \mathrm{d}x + \int_{\partial(\Gamma_t(\delta))} u ((\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{v}) \cdot \nu \mathrm{d}\mathcal{H}^1(s),$$

where $\kappa := -\mathrm{div}(\mathbf{n}(S(x,t),t))$ and $\nu(s)$ is the outer unit normal to $\Gamma_t(\delta)$ for $s \in \partial(\Gamma_t(\delta))$.

Proof. First off, we calculate

$$\begin{aligned} \mathrm{div}(\mathbf{I} - \mathbf{n}(S(x,t),t) \otimes \mathbf{n}(S(x,t),t)) &= -\mathrm{div}(\mathbf{n}(S(x,t),t)) \mathbf{n}(S(x,t),t) \\ &\quad - (D(\mathbf{n}(S(x,t),t))) \mathbf{n}(S(x,t),t) \\ &= \kappa(x,t) \mathbf{n}(S(x,t),t), \end{aligned}$$

where we used $(D(\mathbf{n}(S(x,t),t))) \mathbf{n}(S(x,t),t) = 0$, which is a consequence of Proposition 2.13 4). The claim now follows from (2.33), (2.34), and the divergence theorem. □

For later use we define

$$\left[\partial_\mathbf{n}, \nabla^\Gamma\right] u := \partial_\mathbf{n}((\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \nabla u) - (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \nabla(\partial_\mathbf{n} u) \tag{2.35}$$

and compute

$$\begin{aligned} \left[\partial_\mathbf{n}, \nabla^\Gamma\right] u &= -\sum_{i=1}^{2} ((\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \nabla \mathbf{n}_i) \partial_{x_i} u \\ &= -\sum_{i=1}^{2} \nabla^\Gamma \mathbf{n}_i \partial_{x_i} u \\ &= -\nabla S (\partial_s \mathbf{n} \cdot \nabla u) \end{aligned}$$

by the definition of ∇^Γ in (2.24). As $\partial_s \mathbf{n} \cdot \mathbf{n} = 0$ (which follows by deriving $|\mathbf{n}(s,t)|^2 = 1$ with respect to s) we find by (2.27)

$$\left[\partial_\mathbf{n}, \nabla^\Gamma\right] u = -\nabla S \left(\partial_s \mathbf{n} \cdot \nabla^\Gamma u\right). \tag{2.36}$$

2.4. Remainder Terms

To have the means for a systematic treatment of the appearing terms in Chapter 6, we will introduce concepts similar to those in [6], Section 2.5. The first of these are functions with mixed regularity in normal direction to Γ and along Γ. Let in the following Assumption 1.1 hold.

Definition 2.22.

1. Let $\tau \in [0, T_0]$ and $1 \leq p < \infty$ be given. We set

$$L^{p,\infty}\left(\Gamma_t\left(2\delta\right)\right) := \left\{ f : \Gamma_t\left(2\delta\right) \to \mathbb{R} \text{ measurable} \middle| \, \|f\|_{L^{p,\infty}(\Gamma_t(2\delta))} < \infty \right\},$$

where

$$\|f\|_{L^{p,\infty}(\Gamma_t(2\delta))} := \left(\int\limits_{\mathbb{T}^1} \operatorname{esssup}_{|r| \leq 2\delta} \left| f\left(\left(X\left(r,s,t\right)\right)_1\right) \right|^p \mathrm{d}s \right)^{\frac{1}{p}}.$$

Here $X_1\left(r,s,t\right) := X_0\left(s,t\right) + r\mathbf{n}\left(s,t\right)$ denotes the first component of X.

2. Let $T \in [0, T_0]$, $1 \leq p, q < \infty$ and $\alpha \in (0, 2\delta)$ be given.We set

$$L^q\left(0,T; L^p\left(\Gamma_t\left(\alpha\right)\right)\right) := \left\{ f : \Gamma\left(\alpha, T\right) \to \mathbb{R} \text{ measurable} \middle| \, \|f\|_{L^q(0,T;L^p(\Gamma_t(\alpha)))} < \infty \right\},$$

where

$$\|f\|_{L^q(0,T;L^p(\Gamma_t(\alpha)))} := \left(\int\limits_0^T \left(\int\limits_{\Gamma_t(\alpha)} \left| f\left(x,t\right) \right|^p \mathrm{d}x \right)^{\frac{q}{p}} \mathrm{d}t \right)^{\frac{1}{q}}.$$

In a similar way, we define $L^q\left(0,T; L^p\left(\Omega \backslash \Gamma_t\left(\alpha\right)\right)\right)$ and the according norm.

Lemma 2.23. *For $t \in [0, T]$ it holds*

$$H^1\left(\Gamma_t\left(2\delta\right)\right) \hookrightarrow L^{4,\infty}\left(\Gamma_t\left(2\delta\right)\right).$$

Proof. Let $I \subset \mathbb{R}$ be bounded and $g \in H^1\left(I\right)$. Then the Gagliardo Nirenberg interpolation theorem implies

$$\|g\|_{L^\infty(I)} \leq C\left(I\right) \|g\|_{L^2(I)}^{\frac{1}{2}} \|g\|_{H^1(I)}^{\frac{1}{2}}.$$

Let now $f \in H^1\left(\Gamma_t\left(2\delta\right)\right)$ and we write $f\left(r,s\right) := f\left(\left(X\left(r,s,t\right)\right)_1\right)$. Note that $H^1\left(\Gamma_t\left(2\delta\right)\right) \hookrightarrow L^2\left(\Gamma_t; H^1\left(-2\delta, 2\delta\right)\right) \cap H^1\left(\Gamma_t; L^2\left(-2\delta, 2\delta\right)\right)$. We may compute

$$\begin{aligned}
\int\limits_{\mathbb{T}^1} \|f\left(.,s\right)\|_{L^\infty(-2\delta,2\delta)}^4 \mathrm{d}s &\leq C \int\limits_{\mathbb{T}^1} \|f\|_{L^2(-2\delta,2\delta)}^2 \|f\|_{H^1(-2\delta,2\delta)}^2 \mathrm{d}s \\
&\leq C \|f\|_{L^\infty(\mathbb{T}^1;L^2(-2\delta,2\delta))}^2 \|f\|_{L^2(\mathbb{T}^1;H^1(-2\delta,2\delta))}^2 \\
&\leq C \|f\|_{H^1(\mathbb{T}^1;L^2(-2\delta,2\delta))}^2 \|f\|_{L^2(\mathbb{T}^1;H^1(-2\delta,2\delta))}^2 \\
&\leq C \|f\|_{H^1(\Gamma_t(2\delta))}^4.
\end{aligned}$$

Here we used $H^1\left(\mathbb{T}^1\right) \hookrightarrow L^\infty\left(\mathbb{T}^1\right)$ in the third line as $\mathbb{T}^1$ is one dimensional (in particular $\mathbb{T}^1 \backslash \{s\} \cong (0,1)$ for arbitrary $s \in \mathbb{T}^1$). □

Lemma 2.24. *Let $h : \mathbb{T}^1 \times [0, T_0] \to \mathbb{R}$ be continuous, $\epsilon \in (0,1)$, $t \in [0,T]$. Then there are constants $C_1, C_2 > 0$ independent of h,ϵ and t such that*

1. *for all $\psi \in L^{1,\infty}(\Gamma_t(2\delta))$, $\eta \in L^1(\mathbb{R})$*

$$\left\| \eta \left(\frac{d_\Gamma(.,t)}{\epsilon} - h(S(.,t),t) \right) \psi \right\|_{L^1(\Gamma_t(2\delta))} \leq C_1 \epsilon \|\eta\|_{L^1(\mathbb{R})} \|\psi\|_{L^{1,\infty}(\Gamma_t(2\delta))}.$$

2. *for all $\psi \in L^{2,\infty}(\Gamma_t(2\delta))$, $\eta \in L^2(\mathbb{R})$ and $u \in L^2(\Gamma_t(2\delta))$*

$$\left\| \eta \left(\frac{d_\Gamma(.,t)}{\epsilon} - h(S(.,t),t) \right) \psi u \right\|_{L^1(\Gamma_t(2\delta))} \leq C_2 \epsilon^{\frac{1}{2}} \|\eta\|_{L^2(\mathbb{R})} \|\psi\|_{L^{2,\infty}(\Gamma_t(2\delta))} \|u\|_{L^2(\Gamma_t(2\delta))}.$$

Proof. Ad 1) Doing two changes of variables we get

$$\begin{aligned}
&\left\| \eta \left(\frac{d_\Gamma(.,t)}{\epsilon} - h(S(.,t),t) \right) \psi \right\|_{L^1(\Gamma_t(2\delta))} \\
&= \int_{\mathbb{T}^1} \int_{-2\delta}^{2\delta} \left| \eta \left(\frac{r}{\epsilon} - h(s,t) \right) \psi(X_1(r,s,t)) \right| \left| \det(\nabla X_1(r,s,t)) \right| \mathrm{d}r \mathrm{d}s \\
&\leq C \int_{\mathbb{T}^1} \|\psi \circ X_1\|_{L^\infty(-2\delta,2\delta)} \int_{-\frac{2\delta}{\epsilon} - h(s,t)}^{\frac{2\delta}{\epsilon} - h(s,t)} \epsilon |\eta(\rho)| \mathrm{d}\rho \mathrm{d}s \\
&\leq C\epsilon \|\psi\|_{L^{1,\infty}(\Gamma_t(\delta))} \|\eta\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Here we used the uniform boundedness of $|\det(\nabla X_1)|$ in $(-2\delta, 2\delta) \times \mathbb{T}^1 \times [0, T_0]$ in the second inequality.

Ad 2) The same change of variables as before yields

$$\begin{aligned}
&\left\| \eta \left(\frac{d_\Gamma(.,t)}{\epsilon} - h(S(.,t),t) \right) \psi u \right\|_{L^1(\Gamma_t(2\delta))} \\
&\leq C \int_{\mathbb{T}^1} \|\psi \circ X_1\|_{L^\infty(-2\delta,2\delta)} \int_{-2\delta}^{2\delta} \left| \eta \left(\frac{r}{\epsilon} - h(s,t) \right) u(X_1) \right| \left| \det(\nabla X_1) \right| \mathrm{d}r \mathrm{d}s \\
&\leq C \int_{\mathbb{T}^1} \|\psi \circ X_1\|_{L^\infty(-2\delta,2\delta)} \left\| \eta \left(\frac{\cdot}{\epsilon} - h(s,t) \right) \right\|_{L^2(-2\delta,2\delta)} \left(\int_{-2\delta}^{2\delta} |u(X_1)|^2 \left| \det(\nabla X_1) \right| \mathrm{d}r \right)^{\frac{1}{2}} \mathrm{d}s \\
&\leq C\epsilon^{\frac{1}{2}} \|\eta\|_{L^2(\mathbb{R})} \|\psi\|_{L^{2,\infty}(\Gamma_t(2\delta))} \|u\|_{L^2(\Gamma_t(2\delta))}
\end{aligned}$$

where we used Hölder's inequality in lines two and three and again employ the uniform boundedness of $|\det(\nabla X_1)|$ in $(-2\delta, 2\delta) \times \mathbb{T}^1 \times [0, T_0]$. ☐

The second important concept we want to introduce in this section are remainder terms. These are families of functions depending on ϵ, which have exponential decay in their ρ component. All inner terms of the asymptotic expansion (cf. Subsection 5.1.2), when derived with respect to ρ, satisfy this property due to the inner-outer matching conditions. This exponential decay will allow for extra orders of ϵ to be produced when integrating.

Definition 2.25. Let $n \in \mathbb{N}$, $\epsilon_0 > 0$. For $\alpha > 0$ let $\mathcal{R}_\alpha$ denote the vector space of all families $(\hat{r}_\epsilon)_{\epsilon\in(0,\epsilon_0)}$ of continuous functions $\hat{r}_\epsilon : \mathbb{R} \times \Gamma(2\delta) \to \mathbb{R}^n$ which satisfy

$$|\hat{r}_\epsilon(\rho, x, t)| \leq Ce^{-\alpha|\rho|} \text{ for all } \rho \in \mathbb{R}, (x,t) \in \Gamma(2\delta), \epsilon \in (0,1).$$

Moreover, let $\mathcal{R}_\alpha^0$ be the subspace of all $(\hat{r}_\epsilon)_{\epsilon\in(0,\epsilon_0)} \in \mathcal{R}_\alpha$ such that

$$\hat{r}_\epsilon(\rho, x, t) = 0 \text{ for all } \rho \in \mathbb{R}, (x,t) \in \Gamma.$$

Let now a family $(h_\epsilon)_{\epsilon\in(0,\epsilon_0)}$ of continuous functions $h_\epsilon : \mathbb{T}^1 \times [0, T_0] \to \mathbb{R}$ be given which satisfies

$$K := \sup_{\epsilon\in(0,\epsilon_0),(s,t)\in\mathbb{T}^1\times[0,T_\epsilon]} |h_\epsilon(s,t)| < \infty, \tag{2.37}$$

with $\epsilon_0 \in (0,1)$ and $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset [0, T_0]$. In the following we define for $(\hat{r}_\epsilon)_{\epsilon\in(0,\epsilon_0)} \in \mathcal{R}_\alpha$

$$r_\epsilon(x,t) := \hat{r}_\epsilon\left(\frac{d_\Gamma(x,t)}{\epsilon} - h_\epsilon(S(x,t),t), x, t\right) \; \forall (x,t) \in \Gamma(2\delta), \epsilon \in (0,\epsilon_0).$$

Lemma 2.26. *Let $\epsilon_0 > 0$, $\alpha > 0$, $(\hat{r}_\epsilon)_{\epsilon\in(0,\epsilon_0)} \in \mathcal{R}_\alpha$ and $(h_\epsilon)_{\epsilon\in(0,\epsilon_0)}$ such that (2.37) holds.*

1. *There is a constant $C > 0$ independent of $\epsilon \in (0,\epsilon_0)$, such that*

$$\left\| \sup_{(x,t)\in\Gamma(2\delta)} \left| \hat{r}_\epsilon\left(\frac{\cdot}{\epsilon}, x, t\right)\right| \right\|_{L^p(\mathbb{R})} \leq C\epsilon^{\frac{1}{p}}$$

   for $1 \leq p \leq \infty$ and for all $\epsilon \in (0,\epsilon_0)$.

2. *If additionally $(\hat{r}_\epsilon)_{\epsilon\in(0,\epsilon_0)} \in \mathcal{R}_\alpha^0$ holds, then there is a constant $C > 0$ independent of K, T_ϵ and $\epsilon \in (0,\epsilon_0)$ such that*

$$\sup_{t\in(0,T_\epsilon),s\in\mathbb{T}^1} \|r_\epsilon(X(.,s,t),t)\|_{L^p(-2\delta,2\delta)} \leq C(1+K)\epsilon^{\frac{1}{p}+1}$$

   for all $1 \leq p \leq \infty$ and $\epsilon \in (0,\epsilon_0)$.

Proof. See [6], Lemma 2.6. □

Corollary 2.27. *Let $\alpha > 0$, $(\hat{r}_\epsilon)_{\epsilon\in(0,\epsilon_0)} \in \mathcal{R}_\alpha$ and $(h_\epsilon)_{\epsilon\in(0,\epsilon_0)}$ be given, such that (2.37) holds. Moreover, let $j = 1$ if $(\hat{r}_\epsilon)_{\epsilon\in(0,\epsilon_0)} \in \mathcal{R}_\alpha^0$ holds and $j = 0$ else. There is a constant $C > 0$ independent of ϵ, T_ϵ and K such that*

$$\|a(S(.,t))\, r_\epsilon \varphi\|_{L^1(\Gamma_t(2\delta))} \leq C(1+K)^j \epsilon^{1+j} \|\varphi\|_{H^1(\Omega)} \|a(S(.,t))\|_{L^2(\Gamma_t)} \tag{2.38}$$

$$\|a(S(.,t))\, r_\epsilon\|_{L^2(\Gamma_t(2\delta))} \leq C(1+K)^j \epsilon^{\frac{1}{2}+j} \|a\|_{L^2(\Gamma_t)} \tag{2.39}$$

for all $\varphi \in H^1(\Omega)$, $a \in L^2(\Gamma_t)$, $t \in [0, T_\epsilon]$ and $\epsilon \in (0,\epsilon_0)$.

Proof. See [6], Corollary 2.7 . □

2.5. Theory of Maximal Regularity

In the next chapter we will make heavy use of the theory of maximal regularity. Thus, we give a short overview of the basic definitions and results which we will use. These are taken from [16] and all the proofs of the statements can be found in that article.

In this chapter let X and D be two Banach spaces such that D is continuously and densely embedded in X.

Definition 2.28 (L^p-maximal regularity)**.** Let $p \in (1, \infty)$.

1. Let $A \in \mathcal{L}(D, X)$. Then A has **L^p-maximal regularity** and we write $A \in \mathcal{MR}_p$ if for some bounded interval $(t_1, t_2) \subset \mathbb{R}$ and all $f \in L^P(t_1, t_2; X)$ there exists a unique $u \in W^{1,p}(t_1, t_2; X) \cap L^p(t_1, t_2; D)$ such that
$$\begin{aligned} \partial_t u + Au &= f \quad \text{a.e. on } (t_1, t_2), \\ u(t_1) &= 0. \end{aligned}$$
2. Let $T > 0$ and $A : [0, T] \to \mathcal{L}(D, X)$ be a bounded and strongly measurable function. Then A has **L^p-maximal regularity** and we write $A \in \mathcal{MR}_p(0, T)$ if for all $f \in L^p(0, T; X)$ there exists a unique $u \in W^{1,p}(0, T; X) \cap L^p(0, T; D)$ such that
$$\begin{aligned} \partial_t u + A(t)u &= f \quad \text{a.e. on } (0, T), \\ u(0) &= 0. \end{aligned}$$

It can be shown that if $A \in \mathcal{MR}_p$ for some $p \in (1, \infty)$ then $A \in \mathcal{MR}_p$ for all $p \in (1, \infty)$. Hence, we often simply write $A \in \mathcal{MR}$.

Definition 2.29 (Relative Continuity)**.** We say that $A : [0, T] \to \mathcal{L}(D, X)$ is **relatively continuous** if for each $t \in [0, T]$ and all $\epsilon > 0$ there exist $\delta > 0$, $\eta \geq 0$ such that for all $x \in D$ and for all $s \in [0, T]$ with $|s - t| \leq \delta$ the inequality
$$\|A(t)x - A(s)x\|_X \leq \epsilon \|x\|_D + \eta \|x\|_X$$
holds.

Theorem 2.30. *Let $T > 0$ and $A : [0, T] \to \mathcal{L}(D, X)$ be a strongly measurable and relatively continuous function. If $A(t) \in \mathcal{MR}$ for all $t \in [0, T]$ then $A \in \mathcal{MR}_p(0, t)$ for every $0 < t \leq T$ and every $p \in (1, \infty)$.*

Proof. [16] page 9, Theorem 2.7. □

A very important tool for proving maximal regularity properties of different operators are perturbation techniques. Employing these can often help to show maximal regularity for a variety of operators by separating them into a main part (for which maximal regularity can be readily shown) and a perturbation.

In the following we give a perturbation result which is key to many results in the next chapter.

Definition 2.31 (Relatively Close)**.** Let Y be a Banach space such that
$$D \hookrightarrow Y \hookrightarrow X.$$
We say Y **is close to X compared with D**, if for each $\epsilon > 0$ there exists $\eta \geq 0$ such that
$$\|x\|_Y \leq \epsilon \|x\|_D + \eta \|x\|_X$$
for all $x \in D$.

Proposition 2.32. *Let Y be as in Definition 2.31 and let furthermore the inclusion $D \hookrightarrow Y$ be compact. Then Y is close to X compared with D.*

Proof. [16] page 11, Example 2.9 (d). $\square$

Theorem 2.33 (Perturbation)**.** *Let $T > 0$ and Y be a Banach space that is close to X compared with D. Furthermore, let $A : [0,T] \to \mathcal{L}(D,X)$ be relatively continuous and $B : [0,T] \to \mathcal{L}(Y,X)$ be strongly measurable and bounded. If $A(t) \in \mathcal{MR}$ for every $t \in [0,T]$ then $A + B \in \mathcal{MR}_p(0,T)$.*

Proof. [16] page 12, Theorem 2.11. $\square$

2.6. Parabolic Equations on Evolving Surfaces

The main goal of this subsection is showing existence of strong solutions for a stationary Stokes/linearized Mullins-Sekerka system, which will be of central importance in Chapter 5, more precisely in Subsections 5.1.5 and 5.1.6 and Theorem 5.32. Let Assumption 1.1 hold in the following.

We introduce the space

$$X_T \quad = \quad L^2\left(0,T;H^{\frac{7}{2}}\left(\mathbb{T}^1\right)\right)\cap H^1\left(0,T;H^{\frac{1}{2}}\left(\mathbb{T}^1\right)\right) \tag{2.40}$$

for $T\in\mathbb{R}^+\cup\{\infty\}$, where we equip X_T with the norm

$$\|h\|_{X_T} = \|h\|_{L^2\left(0,T;H^{\frac{7}{2}}(\mathbb{T}^1)\right)} + \|h\|_{H^1\left(0,T;H^{\frac{1}{2}}(\mathbb{T}^1)\right)} + \|h|_{t=0}\|_{H^2(\mathbb{T}^1)}.$$

Proposition 2.34. *Let $T\in(0,\infty)$. Then we have*

1. $\left(H^{\frac{7}{2}}\left(\mathbb{T}^1\right),H^{\frac{1}{2}}\left(\mathbb{T}^1\right)\right)_{\frac{1}{2},2} = H^2\left(\mathbb{T}^1\right)$,
2. $X_T\hookrightarrow C^0\left([0,T];H^2\left(\mathbb{T}^1\right)\right)$ *where the operator norm of the embedding is bounded independently of T,*
3. $X_T\hookrightarrow H^{\frac{1}{2}}\left(0,T;H^2\left(\mathbb{T}^1\right)\right)$,
4. $X_T\hookrightarrow H^{\frac{1}{3}}\left(0,T;H^{\frac{5}{2}}\left(\mathbb{T}^1\right)\right)$.

Proof. Ad 1) See e.g. [40] page 330, Theorem B.8.

Ad 2) See e.g. [7], Lemma A.8.

Ad 3) and 4) According to [41] Proposition 3.2 we have $X_T\hookrightarrow H^\sigma\left(0,T;H^{\frac{1}{2}+(1-\sigma)3}\left(\mathbb{T}^1\right)\right)$. Thus, 3. follows for $\sigma=\frac{1}{2}$ and 4. for $\sigma=\frac{1}{3}$. □

Theorem 2.35. *Let $T\in(0,T_0]$. Let $\mathbf{b}:\mathbb{T}^1\times[0,T]\to\mathbb{R}^2$ and $b_1,b_2:\mathbb{T}^1\times[0,T]\to\mathbb{R}$ be smooth given functions. For every $g\in L^2\left(0,T;H^{\frac{1}{2}}\left(\mathbb{T}^1\right)\right)$ and $h_0\in H^2\left(\mathbb{T}^1\right)$, there is a unique solution $h\in X_T$ of*

$$D_{t,\Gamma}h+\mathbf{b}\cdot\nabla_\Gamma h-b_1h+X_0^*\left(\left[\partial_{\mathbf{n}_{\Gamma_t}}\mu\right]\right)=g \qquad \text{on } \mathbb{T}^1\times(0,T), \tag{2.41}$$

$$h(.,0)=h_0 \qquad \text{on } \mathbb{T}^1,$$

where $\mu|_{\Omega^\pm(t)}\in H^2\left(\Omega^\pm(t)\right)$, for $t\in[0,T]$, is determined by

$$\Delta\mu^\pm = 0 \qquad \text{in } \Omega^\pm(t), \tag{2.42a}$$

$$\mu^\pm = X_0^{*,-1}\left(\sigma\Delta_\Gamma h\pm b_2h\right) \qquad \text{on } \Gamma_t, \tag{2.42b}$$

$$\mu^- = 0 \qquad \text{on } \partial\Omega. \tag{2.42c}$$

Furthermore, the estimates

$$\sum_\pm \left\|\mu^\pm\right\|_{L^2(0,T;H^2(\Omega^\pm(t)))} \le C\|h\|_{X_T}, \tag{2.43}$$

$$\sum_\pm \left\|\mu^\pm\right\|_{L^6(0,T;H^1(\Omega^\pm(t)))} \le C\|h\|_{X_T} \tag{2.44}$$

hold for some constant $C>0$ independent of μ and h.

Proof. We may write (2.41) in abstract form as

$$\begin{aligned} \partial_t h + \mathcal{A} h &= g && \text{in } \mathbb{T}^1 \times [0,T], \\ h(.,0) &= h_0 && \text{in } \mathbb{T}^1, \end{aligned}$$

where $\mathcal{A}$ depends on both variables $(s,t) \in \mathbb{T}^1 \times [0,T]$. Now we fix $t_0 \in [0,T]$ and analyze the operator $\mathcal{A}(t_0)$, where we replace t with the fixed t_0 in all time dependent coefficients. This is done in order to later on use Theorem 2.33 to show the maximal regularity of $\mathcal{A}$.

In order to understand this operator we define

$$\begin{aligned} \mathfrak{D}_{t_0} &: H^{\frac{7}{2}}\left(\mathbb{T}^1\right) \to H^{\frac{3}{2}}\left(\Gamma_{t_0}\right), && h \mapsto \left(X_0^{*,-1}\left(\sigma \Delta_\Gamma h\right)\right)(.,t_0), \\ S_{t_0}^N &: H^{\frac{3}{2}}\left(\Gamma_{t_0}\right) \to H^2\left(\Omega^+(t_0)\right) \times H^2\left(\Omega^-(t_0)\right), && f \mapsto \left(\Delta_N\right)^{-1}(f), \\ B_{t_0} &: H^2\left(\Omega^+(t_0)\right) \times H^2\left(\Omega^-(t_0)\right) \to H^{\frac{1}{2}}\left(\mathbb{T}^1\right), && \left(\mu^+,\mu^-\right) \mapsto \left(X_0^*\left(\left[\nabla\mu \cdot \mathbf{n}_{\Gamma_{t_0}}\right]\right)\right)(.,t_0), \end{aligned}$$

where $(\Delta_N)^{-1}(f)$ represents the unique solution $\left(\mu_N^+,\mu_N^-\right)$ to

$$\begin{aligned} \Delta\mu_N^\pm &= 0 && \text{in } \Omega^\pm(t_0), && \text{(2.45a)} \\ \mu_N^\pm &= f && \text{on } \Gamma_{t_0}, && \text{(2.45b)} \\ \nabla\mu_N^- \cdot \mathbf{n}_{\partial\Omega} &= 0 && \text{on } \partial\Omega. && \text{(2.45c)} \end{aligned}$$

In the literature the concatenation $B_{t_0} \circ S_{t_0}^N$ is often referred to as the *Dirichlet-to-Neumann* operator and $A_0(t_0) := B_{t_0} \circ S_{t_0}^N \circ \mathfrak{D}_{t_0}$ is called the *Mullins-Sekerka* operator. It can be shown that

$$A_0 : [0,T] \to \mathcal{L}\left(H^{\frac{7}{2}}\left(\mathbb{T}^1\right), H^{\frac{1}{2}}\left(\mathbb{T}^1\right)\right)$$

has L^p-maximal regularity, i.e. $A_0 \in \mathcal{MR}_p(0,T)$. We will not prove this in detail but just give a short sketch describing the essential ideas: first, a reference surface $\Sigma \subset\subset \Omega$ is fixed such that Γ_t can be expressed as a graph over Σ for t in some time interval $\left[\tilde{t},\tilde{t}+\epsilon\right] \subset [0,T]$. e.g. one may choose $\Sigma := \Gamma_0$ and then determine $\epsilon_0 > 0$ such that Γ_t may be written as graph over Γ_0 for all $t \in [0,\epsilon_0]$, which is possible since Γ is a smoothly evolving hypersurface. Next, a Hanzawa transformation is applied, enabling us to consider (2.45c) as a system on fixed domains $\Omega^\pm$ and Σ, but with time dependent coefficients (see e.g. [11], Chapter 2.2 and [47], Chapter 4). Here, Ω^+, Ω^- and Σ denote disjoint sets such that $\partial\Omega^+ = \Sigma$ and $\Omega = \Omega^+ \cup \Omega^- \cup \Sigma$ holds and we assume in the following that $t_0 \in [0,\epsilon_0]$. To be more specific, the Hanzawa transformation results in a system of the form

$$\begin{aligned} a(x,t,\nabla_x)\bar{\mu}^\pm &= 0 && \text{in } \Omega^\pm, \\ \bar{\mu}^\pm &= \tilde{f} && \text{on } \Sigma, \\ \nabla\bar{\mu}^- \cdot \mathbf{n}_{\partial\Omega} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where a is the transformed Laplacian, depending smoothly on t and $\tilde{f}$ is the transformation of f. Applying the Hanzawa transformation (and the diffeomorphism X_0 from Section 2.3) also to the operators $\mathfrak{D}_{t_0}$ and B_{t_0}, we end up with a transformed operator $\tilde{A}_0(t_0) \in \mathcal{L}\left(H^{\frac{7}{2}}(\Sigma), H^{\frac{1}{2}}(\Sigma)\right)$ and [44], Corollary 6.6.5, p. 301, implies that $\tilde{A}_0(t_0)$ has L^p-maximal regularity. As all involved differential operators and coefficients depend smoothly on t, it is possible to show that $\tilde{A}_0 : [0,\epsilon_0] \to \mathcal{L}\left(H^{\frac{7}{2}}(\Sigma), H^{\frac{1}{2}}(\Sigma)\right)$ is relatively continuous, wherefore Theorem 2.30 implies $\tilde{A}_0 \in \mathcal{MR}_p(0,\epsilon_0)$ and, transforming back, also

$A_0 \in \mathcal{MR}_p(0,\epsilon_0)$. Repeating this procedure with a new reference surface $\Sigma := \Gamma_{\epsilon_0}$ and iteratively continuing the argumentation, we end up with $A_0 \in \mathcal{MR}_p(0,T)$.

We proceed by showing that $\mathcal{A}(t_0) = A_0(t_0) + \mathcal{B}(t_0)$ holds for some lower order perturbation $\mathcal{B}$, so that we may use Theorem 2.33. The first step in that direction consists of correcting (2.45c) to a system with Dirichlet boundary data. We introduce

$$S^D_{t_0} : H^{\frac{3}{2}}(\Gamma_{t_0}) \to H^2\left(\Omega^+(t_0)\right) \times H^2\left(\Omega^-(t_0)\right),\ f \mapsto (\Delta_D)^{-1}(f),$$

where $\left(\mu^+_D, \mu^-_D\right) := (\Delta_D)^{-1}(f)$ is the unique solution to (2.45c), replacing $\nabla\mu^-_N \cdot \mathbf{n}_{\partial\Omega} = 0$ by $\mu^-_D = 0$ on $\partial\Omega$. Moreover, we write $S^\Delta_{t_0} := S^D_{t_0} - S^N_{t_0}$ and observe that the equality

$$B_{t_0} \circ S^D_{t_0} \circ \mathfrak{D}_{t_0} = A_0(t_0) + \mathcal{B}_0(t_0) \tag{2.46}$$

is satisfied, where $\mathcal{B}_0(t_0) := B_{t_0} \circ S^\Delta_{t_0} \circ \mathfrak{D}_{t_0}$. Let $f \in H^{\frac{3}{2}}(\Gamma_{t_0})$ be fixed, $\left(\mu^+_D, \mu^-_D\right) := S^D_{t_0}(f)$, $\left(\mu^+_N, \mu^-_N\right) := S^N_{t_0}(f)$ and $\tilde{\mu}^\pm := \mu^\pm_D - \mu^\pm_N$, implying $(\tilde{\mu}^+, \tilde{\mu}^-) = S^\Delta_{t_0}(f)$. Then $\tilde{\mu}^\pm \in H^2(\Omega^\pm(t_0))$ solves

$$\begin{aligned} \Delta\tilde{\mu}^\pm &= 0 && \text{in } \Omega^\pm(t_0), \\ \tilde{\mu}^\pm &= 0 && \text{on } \Gamma_{t_0}, \\ \tilde{\mu}^- &= \mu^-_N && \text{on } \partial\Omega \end{aligned}$$

and elliptic regularity theory implies

$$\left\|\tilde{\mu}^-\right\|_{H^2(\Omega^-(t_0))} \leq C\left\|\mu^-_N\right\|_{H^{\frac{3}{2}}(\partial\Omega)} \tag{2.47}$$

and $\tilde{\mu}^+ \equiv 0$ in $\Omega^+(t_0)$. To further our argumentation, we show

$$\left\|\mu^-_N\right\|_{H^{\frac{3}{2}}(\partial\Omega)} \leq C\left\|\mu^-_N\right\|_{H^{\frac{1}{2}}\left(\Gamma_{t_0}\right)}. \tag{2.48}$$

For this let $\gamma(x) := \xi(4d_{\mathbf{B}}(x))$ for all $x \in \Omega$, where ξ is the cut off function from Definition 2.1. In particular $\operatorname{supp}\gamma \cap \Gamma_t = \emptyset$ for all $t \in [0,T_0]$ by Assumption 1.1, and $\gamma \equiv 1$ in $\partial\Omega\left(\frac{\delta}{4}\right)$. Denoting $\hat{\mu} := \gamma\mu^-_N \in H^2(\Omega^-(t_0))$, we compute using $\Delta\mu^-_N = 0$ in $\Omega^-(t_0)$ that $\hat{\mu}$ is a solution to

$$\begin{aligned} \Delta\hat{\mu} &= 2\nabla\gamma \cdot \nabla\mu^-_N + \Delta\gamma\mu^-_N && \text{in } \Omega^-(t_0), \\ \hat{\mu} &= 0 && \text{on } \Gamma_{t_0}, \\ \nabla\hat{\mu} \cdot \mathbf{n}_{\partial\Omega} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

which, again regarding elliptic regularity theory, implies

$$\|\hat{\mu}\|_{H^2(\Omega^-(t_0))} \leq C\left\|\mu^-_N\right\|_{H^1(\Omega^-(t_0))}.$$

This is essential in view of (2.48) as it leads to

$$\begin{aligned} \left\|\mu^-_N\right\|_{H^{\frac{3}{2}}(\partial\Omega)} = \|\hat{\mu}\|_{H^{\frac{3}{2}}(\partial\Omega)} &\leq C\|\hat{\mu}\|_{H^2(\Omega^-(t_0))} \leq C\left\|\mu^-_N\right\|_{H^1(\Omega^-(t_0))} \\ &\leq C\left\|\mu^-_N\right\|_{H^{\frac{1}{2}}\left(\Gamma_{t_0}\right)}, \end{aligned}$$

where we used the continuity of the trace operator $tr : H^2(\Omega^-(t_0)) \to H^{\frac{3}{2}}(\partial\Omega^-(t_0))$ in the first inequality (cf. [40], Theorem 3.37, p. 102) and standard estimates for elliptic equations in the second and third inequality.

Let now $h \in H^{\frac{7}{2}}(\mathbb{T}_1)$ and $(\tilde{\mu}^+, \tilde{\mu}^-) := S^{\Delta}_{t_0} \circ \mathfrak{D}_{t_0}(h)$. Our prior considerations enable us to estimate

$$\begin{aligned}\left\|B_{t_0} \circ S^{\Delta}_{t_0} \circ \mathfrak{D}_{t_0}(h)\right\|_{H^{\frac{1}{2}}(\mathbb{T}^1)} &\leq C \left\|\tilde{\mu}^-\right\|_{H^2(\Omega^-(t_0))} \leq C \left\|\mu^-_N\right\|_{H^{\frac{3}{2}}(\partial\Omega)} \\ &\leq C \left\|\mu^-_N\right\|_{H^{\frac{1}{2}}(\Gamma_{t_0})} \leq C \left\|\sigma\Delta^{\Gamma} h\right\|_{H^{\frac{1}{2}}(\mathbb{T}^1)} \\ &\leq C \left\|h\right\|_{H^{\frac{5}{2}}(\mathbb{T}^1)},\end{aligned}$$

where we employed the continuity of the trace in the first line, (2.47) in the second, (2.48) in the third and the definition of μ^-_N in the fourth. As $H^{\frac{7}{2}}(\mathbb{T}^1)$ is dense in $H^{\frac{5}{2}}(\mathbb{T}^1)$, we may extend $\mathcal{B}_0(t_0)$ to an operator

$$\mathcal{B}_0(t_0) : H^{\frac{5}{2}}(\mathbb{T}^1) \to H^{\frac{1}{2}}(\mathbb{T}^1), \tag{2.49}$$

which shows in regard to (2.46) that we may view $B_{t_0} \circ S^D_{t_0} \circ \mathfrak{D}_{t_0}$ as a perturbed $A_0(t_0)$.

Next we take care of the term involving b_2 in (2.42c). For this we consider the operator

$$\mathcal{B}_1(t_0) : H^{\frac{7}{2}}(\mathbb{T}^1) \to H^{\frac{1}{2}}(\mathbb{T}^1),\ h \mapsto X^*_0\left(\left[\partial_{\mathbf{n}_{\Gamma_{t_0}}} \mu_1\right]\right),$$

where $\mu_1^{\pm} \in H^2(\Omega^{\pm}(t_0))$ is the solution to

$$\begin{aligned}\Delta\mu_1^{\pm} &= 0 && \text{in } \Omega^{\pm}(t_0), \\ \mu_1^{\pm} &= \pm b_2 h && \text{on } \Gamma_{t_0}, \\ \mu_1^{-} &= 0 && \text{on } \partial\Omega.\end{aligned}$$

We estimate

$$\begin{aligned}\left\|\mathcal{B}_1(t_0) h\right\|_{H^{\frac{1}{2}}(\mathbb{T}^1)} &\leq C \left\|\left[\partial_{\mathbf{n}_{\Gamma_{t_0}}} \mu_1\right]\right\|_{H^{\frac{1}{2}}(\Gamma_{t_0})} \leq C \left(\left\|\mu_1^+\right\|_{H^2(\Omega^+(t_0))} + \left\|\mu_1^-\right\|_{H^2(\Omega^-(t_0))}\right) \\ &\leq C \left\|h\right\|_{H^{\frac{3}{2}}(\mathbb{T}^1)},\end{aligned} \tag{2.50}$$

where $C > 0$ can be chosen independent of h and $t_0 \in [0, T]$. Here we again employed the continuity of the trace operator and elliptic theory.

Defining

$$\mathcal{B}(t_0) : H^{\frac{7}{2}}(\mathbb{T}^1) \to H^{\frac{1}{2}}(\mathbb{T}^1), h \mapsto \mathcal{B}(t_0) h := \tilde{b}(.,t_0)\,\partial_s h - b_1(.,t_0)\,h + (\mathcal{B}_0(t_0) + \mathcal{B}_1(t_0))\,h,$$

with $\tilde{b} := \mathbf{b} \cdot \nabla S$, and using (2.50) and (2.49), we find that

$$\left\|\mathcal{B}(t_0) h\right\|_{H^{\frac{1}{2}}(\mathbb{T}^1)} \leq C \left\|h\right\|_{H^{\frac{5}{2}}(\mathbb{T}^1)}.$$

Thus, we can extend $\mathcal{B}(t_0)$ to a bounded operator $\mathcal{B}(t_0) : H^{\frac{5}{2}}(\mathbb{T}^1) \to H^{\frac{1}{2}}(\mathbb{T}^1)$. Since $H^{\frac{5}{2}}(\mathbb{T}^1)$ is close to $H^{\frac{1}{2}}(\mathbb{T}^1)$ compared to $H^{\frac{7}{2}}(\mathbb{T}^1)$, as the embedding $H^{\frac{7}{2}}(\mathbb{T}^1) \hookrightarrow H^{\frac{5}{2}}(\mathbb{T}^1)$ is compact (see Proposition 2.32), we get due to the perturbation result Theorem 2.33 that $\mathcal{A} = A_0 + \mathcal{B}$ has L^p-maximal regularity for all $t \in [0, T]$.

Regarding the given estimates, (2.43) follows immediately by elliptic regularity theory and (2.44) by applying theory for weak solutions, since

$$\left\|\mu^{\pm}\right\|_{H^1(\Omega^{\pm}(t))} \leq C \left\|X_0^{*,-1}(\sigma\Delta_{\Gamma} h + b_2 h)\right\|_{H^{\frac{1}{2}}(\Gamma_t)} \leq C \left\|h\right\|_{H^{\frac{5}{2}}(\mathbb{T}^1)}$$

for almost all $t \in [0,T]$ and thus

$$\left\|\mu^{\pm}\right\|_{L^6(0,T;H^1(\Omega^{\pm}(t)))} \leq C \left\|h\right\|_{L^6\left(0,T;H^{\frac{5}{2}}(\mathbb{T}^1)\right)} \leq C \left\|h\right\|_{X_T}.$$

Here, the last inequality is a consequence of $H^{\frac{1}{3}}(0,T;Y) \hookrightarrow L^6(0,T;Y)$, as implied by the Sobolev embedding theorem, where Y is a Banach space, and Proposition 2.34 4). □

Theorem 2.36. *Let $T \in (0,T_0]$ and $t \in [0,T]$. For every $\mathbf{f} \in L^2(\Omega)^2, \mathbf{s} \in H^{\frac{3}{2}}(\Gamma_t)^2, \mathbf{a} \in H^{\frac{1}{2}}(\Gamma_t)^2$ and $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)^2$ the system*

$$\begin{aligned}
-\Delta \mathbf{v}^{\pm} + \nabla p^{\pm} &= \mathbf{f} && \text{in } \Omega^{\pm}(t), && (2.51)\\
\operatorname{div}\mathbf{v}^{\pm} &= 0 && \text{in } \Omega^{\pm}(t), && (2.52)\\
\left(-2D_s\mathbf{v}^- + p^-\mathbf{I}\right)\mathbf{n}_{\partial\Omega} &= \alpha_0 \mathbf{v}^- + \mathbf{g} && \text{on } \partial\Omega, && (2.53)\\
[\mathbf{v}] &= \mathbf{s} && \text{on } \Gamma_t, && (2.54)\\
\left[2D_s\mathbf{v} - p^-\mathbf{I}\right]\mathbf{n}_{\Gamma_t} &= \mathbf{a} && \text{on } \Gamma_t && (2.55)
\end{aligned}$$

has a unique solution $(\mathbf{v}^{\pm}, p^{\pm}) \in H^2(\Omega^{\pm}(t)) \times H^1(\Omega^{\pm}(t))$. Moreover, there is a constant $C > 0$ independent of $t \in [0,T_0]$ such that

$$\|(\mathbf{v},p)\|_{H^2(\Omega^{\pm}(t))\times H^1(\Omega^{\pm}(t))} \leq C\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{s}\|_{H^{\frac{3}{2}}(\Gamma_t)} + \|\mathbf{a}\|_{H^{\frac{1}{2}}(\Gamma_t)} + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\partial\Omega)}\right) \quad (2.56)$$

holds.

Proof. As a first step, we reduce the system (2.51)–(2.55) to the case $\mathbf{s} = 0$. Elliptic theory implies that the equation

$$\begin{aligned}
\Delta q &= 0 && \text{in } \Omega^-(t),\\
\nabla q \cdot \mathbf{n}_{\Gamma_t} &= \mathbf{s} \cdot \mathbf{n}_{\Gamma_t} && \text{on } \Gamma_t,\\
q &= 0 && \text{on } \partial\Omega
\end{aligned}$$

has a unique solution $q \in H^3(\Omega^-(t))$ since $\mathbf{s} \in H^{\frac{3}{2}}(\Gamma_t)$ and we have the estimate

$$\|q\|_{H^3(\Omega^-(t))} \leq C \|\mathbf{s}\|_{H^{\frac{3}{2}}(\Gamma_t)}.$$

Regarding the tangential part of $\mathbf{s}$, we may solve the stationary Stokes system

$$\begin{aligned}
-\Delta \mathbf{w} + \nabla \tilde{p} &= 0 && \text{in } \Omega^-(t),\\
\operatorname{div}\mathbf{w} &= 0 && \text{in } \Omega^-(t),\\
\mathbf{w} &= ((\mathbf{s} - \nabla q)\cdot \tau_{\Gamma_t})\,\tau_{\Gamma_t} && \text{on } \Gamma_t,\\
\mathbf{w} &= 0 && \text{on } \partial\Omega,
\end{aligned}$$

where $\tau_{\Gamma_t}(x) := \tau\left(X_0^{-1}(x)\right)$ for $x \in \Gamma_t$ and τ is given as in (2.18). We may find a solution $(\mathbf{w}, \tilde{p}) \in H^2(\Omega^-(t)) \times H^1(\Omega^-(t))$ (made unique by the normalization $\int_{\Omega^-(t)} \tilde{p}\,\mathrm{d}x = 0$) and also get the estimate

$$\|\mathbf{w}\|_{H^2(\Omega^-(t))} + \|\tilde{p}\|_{H^1(\Omega^-(t))} \leq C \|\mathbf{s}\|_{H^{\frac{3}{2}}(\Gamma_t)}.$$

Thus, defining

$$\tilde{\mathbf{w}} := \mathbf{w} + \nabla q, \quad (2.57)$$

the couple $(\tilde{\mathbf{w}}, \tilde{p})$ solves

$$\begin{aligned} -\Delta\tilde{\mathbf{w}} + \nabla\tilde{p} &= 0 && \text{in } \Omega^-(t), \\ \operatorname{div}\tilde{\mathbf{w}} &= 0 && \text{in } \Omega^-(t), \\ \tilde{\mathbf{w}} &= \mathbf{s} && \text{on } \Gamma_t, \\ \tilde{\mathbf{w}} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and may be estimated by $\mathbf{s}$ in strong norms. Next, let

$$\tilde{\mathbf{g}} := \mathbf{g} + (2D_s\tilde{\mathbf{w}} - \tilde{p}\mathbf{I}) \cdot \mathbf{n}_{\partial\Omega} + \alpha_0\tilde{\mathbf{w}} \in H^{\frac{1}{2}}(\partial\Omega)$$

and $\tilde{\mathbf{a}} := \mathbf{a} - (2D_s\tilde{\mathbf{w}} - \tilde{p}\mathbf{I}) \cdot \mathbf{n} \in H^{\frac{1}{2}}(\Gamma_t)$, where the regularity is due to the properties of the trace operator. Then, for every strong solution $(\hat{\mathbf{v}}^{\pm}, \hat{p}^{\pm})$ of (2.51)–(2.55), with $\mathbf{s} \equiv 0$ and $\mathbf{g}, \mathbf{a}$ substituted by $\tilde{\mathbf{g}}, \tilde{\mathbf{a}}$, the functions

$$\left(\mathbf{v}^+, p^+\right) := \left(\hat{\mathbf{v}}^+, \hat{p}^+\right) \text{ and } \left(\mathbf{v}^-, p^-\right) := \left(\hat{\mathbf{v}}^- + \tilde{\mathbf{w}}, \hat{p}^- + \tilde{p}\right)$$

are solutions to the original system (2.51)–(2.55). So, we will consider $\mathbf{s} \equiv 0$ in the following and show existence of strong solutions in that case.

As a starting point for that endeavor, we construct a solution $(\mathbf{v}, p) \in H^1_\sigma(\Omega) \times L^2(\Omega)$ to the weak formulation

$$\int_\Omega 2D_s\mathbf{v} : D_s\psi + p\operatorname{div}\psi \mathrm{d}x + \int_{\partial\Omega} \alpha_0\mathbf{v} \cdot \psi \mathrm{d}\mathcal{H}^1(s) = \int_\Omega \mathbf{f} \cdot \psi \mathrm{d}x + \int_{\Gamma_t} \mathbf{a} \cdot \psi \mathrm{d}\mathcal{H}^1(s) - \int_{\partial\Omega} \mathbf{g} \cdot \psi \mathrm{d}\mathcal{H}^1(s), \tag{2.58}$$

where $\psi \in H^1(\Omega)^2$. Considering first $\psi \in H^1_\sigma(\Omega)$ and the right hand side as a functional $F \in \left(H^1_\sigma(\Omega)\right)'$, the Lemma of Lax-Milgram implies the existence of a unique $\mathbf{v} \in H^1_\sigma(\Omega)$ solving (2.58) for all $\psi \in H^1_\sigma(\Omega)$, where the coercivity of the involved bilinear form is a consequence of Lemma 2.5. As in Corollary 2.7, we also get the existence of a unique pressure term $p \in L^2(\Omega)$ and the estimate

$$\|(\mathbf{v}, p)\|_{H^1(\Omega)\times L^2(\Omega)} \leq C\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{a}\|_{H^{\frac{1}{2}}(\Gamma_t)} + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\partial\Omega)}\right). \tag{2.59}$$

We now show higher regularity of $(\mathbf{v}, p)$ by localization and the appliance of results well-studied systems.

Let $\eta^{\pm} \in C^\infty\left(\overline{\Omega}\right)$ be a partition of unity of Ω, such that the inclusions $\Omega^+(t) \cup \Gamma_t(\delta) \subset \{x \in \Omega |\, \eta^+(x) = 1\}$ and $\partial\Omega(\delta) \subset \{x \in \Omega |\, \eta^-(x) = 1\}$ hold. We define $U^{\pm} := \operatorname{supp}(\eta^{\pm})$, $\partial U_0^- := \partial U^- \backslash \partial\Omega$ and

$$\dot{U} := \left\{x \in \Omega |\, \eta^+(x) \in (0,1)\right\} = \left\{x \in \Omega |\, \eta^-(x) \in (0,1)\right\}.$$

Moreover, we set $\dot{p}^- := p\eta^-$ and $\tilde{\mathbf{v}}^- := \mathbf{v}\eta^-$ in Ω and we correct the divergence of $\tilde{\mathbf{v}}^-$ with the help of the Bogovskii-operator: Let $\varphi \in C_c^\infty(\Omega)$ with $\operatorname{supp}(\varphi) \subset U^+ \backslash \dot{U}$ and $\int_\Omega \varphi \mathrm{d}x = 1$ and set

$$\hat{g} := \operatorname{div}\left(\tilde{\mathbf{v}}^-\right) - \varphi \int_{U^+} \operatorname{div}\left(\tilde{\mathbf{v}}^-\right) \mathrm{d}x$$

in U^+. As $\mathbf{v} \in H^1_\sigma(\Omega)$, we have $\operatorname{div}(\tilde{\mathbf{v}}^-) = \mathbf{v} \cdot \nabla \eta^-$ and thus $\hat{g} \in H^1_0(U^+)$, $\int_{U^+} \hat{g} \mathrm{d}x = 0$. Consequently, [31], Theorem III.3.3, p. 179, implies that there is $\hat{\mathbf{v}}^- \in H^2_0(U^+)$, which we extend onto Ω by 0, satisfying

$$\begin{aligned} \operatorname{div}\hat{\mathbf{v}}^- &= \hat{g} \text{ in } U^+, \\ \left\|\hat{\mathbf{v}}^-\right\|_{H^2(\Omega)} &\leq C \left\|\mathbf{v}\right\|_{H^1(\Omega)} . \end{aligned} \tag{2.60}$$

Therefore, $\dot{\mathbf{v}}^- := \tilde{\mathbf{v}}^- - \hat{\mathbf{v}}^-$ fulfills $\operatorname{div}\dot{\mathbf{v}}^- = 0$ in U^- since $\varphi \equiv 0$ in that domain. Let now $\psi \in \left\{ \mathbf{w} \in H^1(U^-) \middle| \mathbf{w} = 0 \text{ on } \partial U_0^- \right\}$, then

$$\begin{aligned} &\int_{U^-} 2D_s\dot{\mathbf{v}}^- : D_s\psi - \dot{p}^-\operatorname{div}\psi \mathrm{d}x + \int_{\partial\Omega} \alpha_0 \dot{\mathbf{v}}^- \cdot \psi \mathrm{d}\mathcal{H}^1(s) \\ &= \int_{U^-} 2D_s\tilde{\mathbf{v}}^- : D_s\psi - p\operatorname{div}\left(\psi\eta^-\right) + \left(p\nabla\eta^-\right)\cdot\psi \mathrm{d}x + \int_{\partial\Omega} \alpha_0 \mathbf{v}\cdot\psi \mathrm{d}\mathcal{H}^1(s) \\ &\quad - \int_{U^-} 2D_s\hat{\mathbf{v}}^- : D_s\psi \mathrm{d}x \\ &= \int_{U^-} \mathbf{f}\cdot\psi\eta^- \mathrm{d}x - \int_{\partial\Omega} \mathbf{g}\cdot\psi \mathrm{d}\mathcal{H}^1(s) + \left(p\nabla\eta^-\right)\cdot\psi \mathrm{d}x \\ &\quad + \int_{U^-} 2\operatorname{div}(D_s\hat{\mathbf{v}})\cdot\psi + \left(2D_s\mathbf{v}\nabla\eta^- - \operatorname{div}\left(\mathbf{v}\otimes\nabla\eta^- + \nabla\eta^-\otimes\mathbf{v}\right)\right)\cdot\psi \mathrm{d}x, \end{aligned}$$

where we used the definition of $\dot{\mathbf{v}}^-$ and $\dot{p}^-$ in the first equality and integration by parts together with $\hat{\mathbf{v}}^- \in H^2_0(U^+)$ and $\nabla\eta^- = 0$ on U^- in the second equality. Additionally, we employed the fact that $(\mathbf{v}, p)$ is the weak solution to (2.58). Hence, $(\dot{\mathbf{v}}^-, \dot{p}^-)$ are a weak solution to the system

$$\begin{aligned} -\Delta\dot{\mathbf{v}}^- + \nabla\dot{p}^- &= \tilde{\mathbf{f}} && \text{in } U^-, \\ \operatorname{div}\dot{\mathbf{v}}^- &= 0 && \text{in } U^-, \\ \dot{\mathbf{v}}^- &= \hat{\mathbf{v}}^- && \text{on } \partial U_0^-, \\ \left(-2D_s\dot{\mathbf{v}}^- + \dot{p}^-\mathbf{I}\right)\mathbf{n}_{\partial\Omega} &= \alpha_0\dot{\mathbf{v}}^- + \mathbf{g} && \text{on } \partial\Omega, \end{aligned} \tag{2.61}$$

where

$$\tilde{\mathbf{f}} := p\nabla\eta^- + 2\operatorname{div}(D_s\hat{\mathbf{v}}) + 2D_s\mathbf{v}\nabla\eta^- - \operatorname{div}\left(\mathbf{v}\otimes\nabla\eta^- + \nabla\eta^-\otimes\mathbf{v}\right) \in L^2\left(U^-\right)$$

and $\hat{\mathbf{v}}^- \in H^{\frac{3}{2}}\left(\partial U_0^-\right)$ by the properties of the trace operator. Writing $\tilde{\mathbf{g}} := \alpha_0\dot{\mathbf{v}}^- + \mathbf{g}$, using localization techniques and results for strong solutions of the stationary Stokes equation in one phase with inhomogeneous do-nothing boundary condition (cf. Theorem 3.1 in [49]) and with Dirichlet boundary condition (cf. [31]), we find that $(\dot{\mathbf{v}}^-, \dot{p}^-) \in H^2(U^-) \times H^1(U^-)$. Moreover, regarding (2.60), (2.59) and the definition of $\tilde{\mathbf{f}}$, we get

$$\left\|\left(\dot{\mathbf{v}}^-, \dot{p}^-\right)\right\|_{H^2(U^-)\times H^1(U^-)} \leq C\left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{a}\|_{H^{\frac{1}{2}}(\Gamma_t)} + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\partial\Omega)}\right).$$

Analogously, we define $\tilde{\mathbf{v}}^+ := \mathbf{v}\eta^+$ and $\hat{\mathbf{v}}^+ \in H^2_0\left(\dot{U}\right)$ as a solution to $\operatorname{div}\hat{\mathbf{v}}^+ = \operatorname{div}\tilde{\mathbf{v}}^+$. Here, we do not need to correct the mean value, since

$$\int_{\dot{U}} \operatorname{div}\tilde{\mathbf{v}}^+ \mathrm{d}x = \int_{\partial\dot{U}} \mathbf{v}\cdot\mathbf{n}_{\partial\dot{U}}\eta^+ \mathrm{d}\mathcal{H}^1(s) = -\int_{\{\eta^+=1\}} \operatorname{div}\mathbf{v} \mathrm{d}x = 0.$$

We set $\dot{\mathbf{v}}^+ := \tilde{\mathbf{v}}^+ - \hat{\mathbf{v}}^+$ and $\dot{p}^+ := p\eta^+$ and get after similar calculations as before that $(\dot{\mathbf{v}}^+, \dot{p}^+)$ is a weak solution to the two phase stationary Stokes system

$$
\begin{aligned}
-\Delta \dot{\mathbf{v}}^+ + \nabla \dot{p}^+ &= \hat{\mathbf{f}} && \text{in } U^+, && (2.62)\\
\operatorname{div} \dot{\mathbf{v}}^+ &= 0 && \text{in } U^+, && (2.63)\\
\dot{\mathbf{v}}^+ &= 0 && \text{on } \partial U^+, && (2.64)\\
\left[\dot{\mathbf{v}}^+\right] &= 0 && \text{on } \Gamma_t, && (2.65)\\
\left[2D_s \dot{\mathbf{v}}^+ - \dot{p}^+ \mathbf{I}\right] \cdot \mathbf{n}_{\Gamma_t} &= \mathbf{a} && \text{on } \Gamma_t, && (2.66)
\end{aligned}
$$

where $\hat{\mathbf{f}} \in L^2(U^+)$. Then, [48], Theorem 1.1, implies that $\dot{\mathbf{v}}^+|_{\Omega^+(t)} \in H^2(\Omega^+(t))$ and $\dot{\mathbf{v}}^+|_{U^+\backslash\Omega^+(t)} \in H^2(U^+\backslash\Omega^+(t))$, and also that the pressure satisfies $\dot{p}^+|_{\Omega^+(t)} \in H^1(\Omega^+(t))$ and $\dot{p}^+|_{U^+\backslash\Omega^+(t)} \in H^1(U^+\backslash\Omega^+(t))$ with estimates in associated norms. In particular, $\mathbf{v} = \dot{\mathbf{v}}^+$ in $\Omega^+(t)$ and $\mathbf{v} = \dot{\mathbf{v}}^+ + \dot{\mathbf{v}}^- + \hat{\mathbf{v}}^+ + \hat{\mathbf{v}}^-$ in $\Omega^-(t)$, yielding the desired regularity and (2.56). To show that $C > 0$ may be chosen independently of $t \in [0, T_0]$, one may make use of perturbation arguments, see e.g. the proof of Lemma 2.10, [6]. □

Theorem 2.37. *Let $T \in (0, T_0]$. Let $\mathbf{b} : \mathbb{T}^1 \times [0,T] \to \mathbb{R}^2$, $b : \mathbb{T}^1 \times [0,T] \to \mathbb{R}$, $a_1 : \Omega \times [0,T] \to \mathbb{R}$, $a_2, a_3, a_5 : \Gamma \to \mathbb{R}$, $a_4 : \partial_T\Omega \to \mathbb{R}$, $\mathbf{a}_1 : \Omega \times [0,T] \to \mathbb{R}^2$, $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5 : \Gamma \to \mathbb{R}^2$ and $\mathbf{a}_6 : \partial_T\Omega \to \mathbb{R}^2$ be smooth given functions. For every $g \in L^2\left(0, T; H^{\frac{1}{2}}\left(\mathbb{T}^1\right)\right)$ and $h_0 \in H^2\left(\mathbb{T}^1\right)$ there exists a unique solution $h \in X_T$ of*

$$
D_{t,\Gamma} h + \mathbf{b} \cdot \nabla_\Gamma h - bh + \frac{1}{2} X_0^* \left(\left(\mathbf{v}^+ + \mathbf{v}^-\right) \cdot \mathbf{n}_{\Gamma_t}\right) + \frac{1}{2} X_0^* \left(\left[\partial_{\mathbf{n}_{\Gamma_t}} \mu\right]\right) = g \quad \text{in } \mathbb{T}^1 \times (0,T),
$$
$$
h(.,0) = h_0 \quad \text{in } \mathbb{T}^1,
$$

where for every $t \in [0,T]$, the functions $\mathbf{v}^\pm = \mathbf{v}^\pm(x,t)$, $p^\pm = p^\pm(x,t)$ and $\mu^\pm = \mu^\pm(x,t)$ for $(x,t) \in \Omega_T^\pm$ with $\mathbf{v}^\pm \in H^2(\Omega^\pm(t))$, $p^\pm \in H^1(\Omega^\pm(t))$ and $\mu^\pm \in H^2(\Omega^\pm(t))$ are the unique solutions to

$$
\begin{aligned}
\Delta \mu^\pm &= a_1 && \text{in } \Omega^\pm(t), && (2.67)\\
\mu^\pm &= \sigma X_0^{*,-1}(\Delta_\Gamma h) \pm a_2 X_0^{*,-1}(h) + a_3 && \text{on } \Gamma_t, && (2.68)\\
\mu^- &= a_4 && \text{on } \partial\Omega, && (2.69)\\
-\Delta \mathbf{v}^\pm + \nabla p^\pm &= \mathbf{a}_1 && \text{in } \Omega^\pm(t), && (2.70)\\
\operatorname{div} \mathbf{v}^\pm &= 0 && \text{in } \Omega^\pm(t), && (2.71)\\
[\mathbf{v}] &= \mathbf{a}_2 && \text{on } \Gamma_t, && (2.72)\\
[2D_s \mathbf{v} - p\mathbf{I}]\, \mathbf{n}_{\Gamma_t} &= \mathbf{a}_3 X_0^{*,-1}(h) + \mathbf{a}_4 X_0^{*,-1}(\Delta_\Gamma h) && && \\
&\quad + a_5 X_0^{*,-1}(\nabla_\Gamma h) + \mathbf{a}_5 && \text{on } \Gamma_t, && (2.73)\\
\left(-2D_s \mathbf{v}^- + p^- \mathbf{I}\right) \mathbf{n}_{\partial\Omega} &= \alpha_0 \mathbf{v}^- + \mathbf{a}_6 && \text{on } \partial\Omega. && (2.74)
\end{aligned}
$$

Moreover, if g, h_0 and b, $\mathbf{b}$, a_i, and $\mathbf{a}_j$ are smooth on their respective domains for $i \in \{1, \ldots, 5\}$, $j \in \{1, \ldots, 6\}$ then h is smooth and $p^\pm$, $\mathbf{v}^\pm$ and $\mu^\pm$ are smooth on $\Omega^\pm(t)$.

Proof. We show this by a perturbation argument. First of all note that we may without loss of generality assume that a_1, a_3, a_4, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_5$, $\mathbf{a}_6 = 0$ on their respective domains. The above system may be reduced to this case by solving

$$
\begin{aligned}
\Delta \hat{\mu}^\pm &= a_1 && \text{in } \Omega^\pm(t),\\
\hat{\mu}^\pm &= a_3 && \text{on } \Gamma_t,\\
\hat{\mu}^- &= a_4 && \text{on } \partial\Omega,
\end{aligned}
$$

with the help of standard elliptic theory and

$$\begin{aligned}
-\Delta\hat{\mathbf{v}}^{\pm} + \nabla\hat{p}^{\pm} &= \mathbf{a}_1 && \text{in } \Omega^{\pm}(t)\,,\\
\operatorname{div}\hat{\mathbf{v}}^{\pm} &= 0 && \text{in } \Omega^{\pm}(t)\,,\\
[\hat{\mathbf{v}}] &= \mathbf{a}_2 && \text{on } \Gamma_t,\\
[2D_s\hat{\mathbf{v}} - p]\cdot\mathbf{n} &= \mathbf{a}_5 && \text{on } \Gamma_t,\\
\left(-2D_s\hat{\mathbf{v}}^- + \hat{p}^-\mathbf{I}\right)\cdot\mathbf{n}_{\partial\Omega} &= \alpha_0\hat{\mathbf{v}}^- + \mathbf{a}_6 && \text{on } \partial\Omega,
\end{aligned}$$

with the help of Theorem 2.36 and setting

$$\hat{g} = g - \frac{1}{2}X_0^*\left(\left[\partial_{\mathbf{n}_{\Gamma_t}}\hat{\mu}\right]\right) - \frac{1}{2}X_0^*\left(\left(\hat{\mathbf{v}}^+ + \hat{\mathbf{v}}^-\right)\cdot\mathbf{n}_{\Gamma_t}\right).$$

Now let $t\in[0,T]$, $h\in H^{\frac{7}{2}}\left(\mathbb{T}^1\right)$ and let $\mathbf{v}_h^{\pm}\in H^2\left(\Omega^{\pm}(t)\right)$, $p_h^{\pm}\in H^1\left(\Omega^{\pm}(t)\right)$ be the solution to (2.70)–(2.74). Multiplying (2.70) by $\mathbf{v}_h^{\pm}$ and integrating in $\Omega^{\pm}(t)$ together with integration by parts and the consideration of the boundary values (2.73) and (2.74) allows us to deduce

$$\begin{aligned}
&\int_{\Omega^+(t)} 2\left|D_s\mathbf{v}_h^+\right|^2 \mathrm{d}x + \int_{\Omega^-(t)} 2\left|D_s\mathbf{v}_h^-\right|^2 \mathrm{d}x + \alpha_0\int_{\partial\Omega}\left|\mathbf{v}_h^-\right|^2 \mathrm{d}\mathcal{H}^1(s)\\
&\quad= \int_{\Gamma_t}\left(\mathbf{a}_3 h + \mathbf{a}_4 X_0^{*,-1}\left(\Delta_\Gamma h\right) + a_5 X_0^{*,-1}\left(\nabla_\Gamma h\right)\right)\cdot\mathbf{v}_h^-\mathrm{d}\mathcal{H}^1(s)\,. \qquad (2.75)
\end{aligned}$$

Hence, by Lemma 2.5 and the continuity of the trace we find

$$\left\|\mathbf{v}_h^-\right\|_{H^1(\Omega^-(t))} \le C\left\|h\right\|_{H^2(\mathbb{T}^1)} \qquad (2.76)$$

for C independent of h and t. Lemma 2.5 also implies

$$\int_{\Omega^+(t)} 2\left|D_s\mathbf{v}_h^+\right|^2 \mathrm{d}x + \int_{\Gamma_t}\left|\mathbf{v}_h^+\right|^2 \mathrm{d}\mathcal{H}^1(s) \ge C\left\|\mathbf{v}_h^+\right\|^2_{H^1(\Omega^+(t))},$$

leading to

$$\left\|\mathbf{v}_h^+\right\|_{H^1(\Omega^-(t))} \le C\left\|h\right\|_{H^2(\mathbb{T}^1)} \qquad (2.77)$$

due to $\mathbf{v}_h^+ = \mathbf{v}_h^-$ on Γ_t, (2.76) and (2.75). Defining

$$\mathcal{B}(t) : H^{\frac{7}{2}}\left(\mathbb{T}^1\right)\to H^{\frac{1}{2}}\left(\mathbb{T}^1\right),\, h\mapsto\mathcal{B}(t)\,h = \frac{1}{2}X_0^*\left(\left(\mathbf{v}_h^+ + \mathbf{v}_h^-\right)\cdot\mathbf{n}_{\Gamma_t}\right),$$

we may use (2.76) and (2.77) to confirm

$$\left\|\mathcal{B}(t)\,h\right\|_{H^{\frac{1}{2}}(\mathbb{T}^1)} \le C\left\|h\right\|_{H^2(\mathbb{T}^1)}$$

for $C>0$ independent of h and t. As $H^{\frac{7}{2}}\left(\mathbb{T}^1\right)$ is dense in $H^2\left(\mathbb{T}^1\right)$ we can extend $\mathcal{B}(t)$ to an operator $\mathcal{B}(t) : H^2\left(\mathbb{T}^1\right)\to H^{\frac{1}{2}}\left(\mathbb{T}^1\right)$ and we conclude using Proposition 2.32 that $H^2\left(\mathbb{T}^1\right)$ is close to $H^{\frac{1}{2}}\left(\mathbb{T}^1\right)$ compared with $H^{\frac{7}{2}}\left(\mathbb{T}^1\right)$.

The existence of a unique solution $h\in X_T$ with the properties stated in the theorem is now a consequence of Theorem 2.33 and Theorem 2.35. Higher regularity may be shown by localization and e.g. the usage of difference quotients. □

3. Spectral Theory

The results in this chapter are adapted from [24]. They will be essential in the proof of the main theorem in Section 7.2, but we may not directly use the results obtained in [24] since we employ a different stretched variable. For this reason and for the sake of completeness, we present the results and detailed adaptations of the proofs to our situation again here. We consider the following situation:

We assume throughout the chapter that $\epsilon_0 > 0$ is fixed and f refers to a double well potential with properties as in Assumption 1.2. Moreover θ_0 is the optimal profile from Lemma 2.2 and $n \in \mathbb{N}$.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset, and let Γ be an $n-1$ dimensional closed, smooth submanifold of $\mathbb{R}^n$ such that $\Gamma \subset\subset \Omega$ and $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$ for disjoint sets $\Omega^\pm \subset \Omega$, satisfying $\partial\Omega^+ = \Gamma$. Note that we do not consider Γ to be an evolving hypersurface in this chapter.

We use similar notations to the ones introduced in Assumption 1.1 and Subsection 2.3. In particular we write d_Γ for the signed distance function (negative in Ω^-) and Pr_Γ for the orthogonal projection onto Γ, which is well-defined in

$$\Gamma(2\delta) = \{x \in \Omega | \, |d_\Gamma(x)| < 2\delta\}$$

for some small enough $\delta > 0$. We define

$$\phi : \Gamma(2\delta) \to (-2\delta, 2\delta) \times \Gamma, \; x \mapsto (d_\Gamma(x), Pr_\Gamma(x)) \tag{3.1}$$

to be the corresponding diffeomorphism with inverse

$$\phi^{-1} : (-2\delta, 2\delta) \times \Gamma \to \Gamma(2\delta), (r, s) \mapsto s + r\mathbf{n}_\Gamma(s).$$

Moreover, we define

$$J(r, s) := \det\left(d\left(\phi^{-1}\right)(r, s)\right) \tag{3.2}$$

to be the Jacobian of ϕ^{-1}. Here $d\left(\phi^{-1}\right)(r, s)$ denotes the differential of the mapping.

In contrast to Subsection 2.3, we use differential operators on Γ in this chapter (since we are working with arbitrary space dimensions). In the following let ∇_τ denote the usual surface gradient, i.e.

$$\nabla_\tau h(s) := \sum_{i=1}^{n-1} (dh(s)\tau_i)\tau_i,$$

where $h : \Gamma \to \mathbb{R}$ and $\{\tau_1, \ldots, \tau_{n-1}\}$ is an orthonormal basis of $T_s\Gamma$ – the tangent space of Γ at a point s. For a function $g : \Gamma(2\delta) \to \mathbb{R}$ we set

$$\nabla_\tau g(x) := \nabla g(x) - \mathbf{n}_\Gamma(Pr_\Gamma(x))(\nabla g(x) \cdot \mathbf{n}_\Gamma(Pr_\Gamma(x))). \tag{3.3}$$

Assumption 3.1. *Let $\epsilon \in (0,\epsilon_0)$ and ξ be a cut-off function as in Definition 2.1 for $\delta > 0$ as before. We assume that $c_A^\epsilon : \Omega \to \mathbb{R}$ is a smooth function, which has the structure*

$$c_A^\epsilon(x) = \xi(d_\Gamma(x))\left(\theta_0\left(\frac{d_\Gamma(x)}{\epsilon} - h^\epsilon(Pr_\Gamma(x))\right) + \epsilon p^\epsilon(Pr_\Gamma(x))\,\theta_1\left(\frac{d_\Gamma(x)}{\epsilon} - h^\epsilon(Pr_\Gamma(x))\right)\right)$$
$$+ \xi(d_\Gamma(x))\,\epsilon^2 q^\epsilon(x) + (1-\xi(d_\Gamma(x)))\left(c_A^{\epsilon,+}(x)\,\chi_{\Omega^+}(x) + c_A^{\epsilon,-}(x)\,\chi_{\Omega^-}(x)\right) \tag{3.4}$$

for all $x \in \Omega$. The occurring functions are supposed to be smooth and satisfy for some $C^* > 0$ the following properties:

$\theta_1 : \mathbb{R} \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\mathbb{R}} \theta_1 \left(\theta_0'\right)^2 f^{(3)}(\theta_0)\,\mathrm{d}\rho = 0. \tag{3.5}$$

Furthermore, $p^\epsilon, q^\epsilon : \Gamma(2\delta) \to \mathbb{R}$ satisfy

$$\sup_{\epsilon\in(0,\epsilon_0)} \sup_{x\in\Gamma(2\delta)} \left(|p^\epsilon(Pr_\Gamma(x))| + \frac{\epsilon}{\epsilon + |d_\Gamma(x) - \epsilon h^\epsilon(Pr_\Gamma(x))|}|q^\epsilon(x)|\right) \le C^*, \tag{3.6}$$

$h^\epsilon : \Gamma \to \mathbb{R}$ fulfills

$$\sup_{\epsilon\in(0,\epsilon_0)} \sup_{s\in\Gamma} \left(|h^\epsilon(s)| + |\nabla_\tau h^\epsilon(s)|\right) \le C^* \tag{3.7}$$

and $c_A^{\epsilon,\pm} : \Omega^\pm \to \mathbb{R}$ (which we may view as extended onto Ω by 0) satisfy

$$\pm\, c_A^{\epsilon,\pm} > 0 \text{ in } \Omega^\pm. \tag{3.8}$$

Additionally, we suppose that

$$\sup_{\epsilon\in(0,\epsilon_0)} \left(\sup_{x\in\Omega}|c_A^\epsilon(x)| + \sup_{x\in\Gamma(\delta)}|\nabla_\tau c_A^\epsilon(x)|\right) \le C^* \tag{3.9}$$

holds and that there exists a constant such that away from the interface Γ

$$\inf_{\epsilon\in(0,\epsilon_0)} \inf_{x\in\Omega\setminus\Gamma(\delta)} f''(c_A^\epsilon(x)) \ge \frac{1}{C^*} \tag{3.10}$$

holds.

Notation 3.2.

1. For notational simplicity we choose $\delta = 1$ throughout this chapter. For $s \in \Gamma$ we define

   $$I_\epsilon^s := \left(-\frac{1}{\epsilon} - h^\epsilon(s), \frac{1}{\epsilon} - h^\epsilon(s)\right)$$

   and set $I_1 := (-1,1)$. We furthermore introduce the stretched variable $\rho^\epsilon(x) = \frac{d_\Gamma(x)}{\epsilon} - h^\epsilon(Pr_\Gamma(x))$ for $x \in \Gamma(2)$, with the corresponding diffeomorphism

   $$F_\epsilon^s : I_\epsilon^s \to I_1,\ \rho \mapsto \epsilon(\rho + h^\epsilon(s)).$$

   In the following we write

   $$J^\epsilon(\rho,s) := J(F_\epsilon^s(\rho), s)$$

   for $(\rho,s) \in I_\epsilon^s \times \Gamma$.

2. We assume that $\epsilon_0 > 0$ is chosen small enough such that

$$(-1,1) \subset I_\epsilon^s \tag{3.11}$$

for all $s \in \Gamma$ and $\epsilon \in (0, \epsilon_0)$. This is possible since

$$\begin{aligned} \frac{1}{\epsilon} - h^\epsilon(s) \geq \frac{1}{\epsilon} - |h^\epsilon(s)| \geq \frac{1}{\epsilon} - C^* \\ \geq \frac{1}{2\epsilon} \end{aligned}$$

and

$$-\frac{1}{\epsilon} - h^\epsilon(s) \leq -\frac{1}{2\epsilon}$$

for $\epsilon \in (0, \epsilon_0)$ and $\epsilon_0 > 0$ small enough.

3. In order to simplify the notations and improve readabilty, we will in this chapter identify functions of x and of (r,s) in $\Gamma(2\delta)$ via the diffeomorphism ϕ introduced in (3.1), i.e. we write $g(r,s)$ instead of $g\left(\phi^{-1}(r,s)\right)$ for g defined on $\Gamma(2\delta)$ and $(r,s) \in (-2\delta, 2\delta) \times \Gamma$.

4. Let $\psi : \Omega \to \mathbb{R}$. Then we define

$$\Psi(\rho, s) := \sqrt{\epsilon}\psi\left(\epsilon\left(\rho + h^\epsilon(s)\right), s\right) \tag{3.12}$$

for $s \in \Gamma$ and $\rho \in I_\epsilon^s$. We will use this relationship between small and big greek letters throughout this chapter.

5. Let $\epsilon \in (0, \epsilon_0)$ and $s \in \Gamma$. Moreover, let $g_1, g_2 \in H^1(I_1)$ and $G_1, G_2 \in H^1(I_\epsilon^s)$. Then we define

$$\begin{aligned} \langle G_1, G_2 \rangle &:= \int_{I_\epsilon^s} G_1(\rho) G_2(\rho) \,\mathrm{d}\rho, & \|G_1\|^2 &= \langle G_1, G_1 \rangle, \\ \langle G_1, G_2 \rangle_J &:= \int_{I_\epsilon^s} G_1(\rho) G_2(\rho) J\left(F_\epsilon^s(\rho), s\right) \mathrm{d}\rho, & \|G_1\|_J^2 &= \langle G_1, G_1 \rangle_J, \\ (g_1, g_2)_J &:= \int_{I_1} g_1(r) g_2(r) J(r,s) \,\mathrm{d}r, & |g_1|_J^2 &= (g_1, g_1)_J, \end{aligned}$$

and write

$$\begin{aligned} G_1 \perp G_2 &:\Leftrightarrow \langle G_1, G_2 \rangle = 0, \\ G_1 \perp_J G_2 &:\Leftrightarrow \langle G_1, G_2 \rangle_J = 0, \\ g_1 \perp_J g_2 &:\Leftrightarrow (g_1, g_2)_J = 0. \end{aligned}$$

Furthermore, we define

$$\begin{aligned} L^0 \langle G_1, G_2 \rangle &:= \int_{I_\epsilon^s} \partial_\rho G_1(\rho) \partial_\rho G_2(\rho) + f''(\theta_0(\rho)) G_1(\rho) G_2(\rho) \,\mathrm{d}\rho, \\ L^J \langle G_1, G_2 \rangle &:= \int_{I_\epsilon^s} \left(\partial_\rho G_1(\rho) \partial_\rho G_2(\rho) + f''\left(c_A^\epsilon\left(F_\epsilon^s(\rho), s\right)\right) G_1(\rho) G_2(\rho)\right) J^\epsilon(\rho, s) \,\mathrm{d}\rho, \\ L^J(g_1, g_2) &:= \int_{I_1} \left(\epsilon \partial_r g_1(r) \partial_r g_2(r) + \epsilon^{-1} f''\left(c_A^\epsilon(r,s)\right) g_1(r) g_2(r)\right) J(r,s) \,\mathrm{d}r. \end{aligned}$$

A short motivation regarding the introduced notations: We will consider two related differential operators in this chapter. The first one is

$$\mathcal{L}_0 := -\partial_\rho^2 + f''(\theta_0)$$

in I_ϵ^s, complemented with the Neumann boundary condition

$$\frac{d}{d\rho}\cdot = 0 \text{ on } \left\{-\frac{1}{\epsilon} - h^\epsilon(s), \frac{1}{\epsilon} - h^\epsilon(s)\right\}$$

for fixed $s \in \Gamma$. We treat this as a reference operator and gain many important insights by studying it. The corresponding bilinear form, regarding the scalar product $\langle .,. \rangle$, is given by $L^0 \langle .,. \rangle$.

However, we will be more interested in the properties of

$$\mathcal{L}_J := -(J^\epsilon)^{-1}\partial_\rho(J^\epsilon\partial_\rho) + f''(c_A^\epsilon(.,s))$$

in I_ϵ^s, complemented with the Neumann boundary condition

$$\frac{d}{d\rho}\cdot = 0 \text{ on } \left\{-\frac{1}{\epsilon} - h^\epsilon(s), \frac{1}{\epsilon} - h^\epsilon(s)\right\}.$$

This is the operator which actually appears in Theorem 3.12 and we will gather results for this operator in Lemma 3.9. The corresponding bilinear form, regarding the scalar product $\langle .,. \rangle_J$ (see also Remark 3.4), is given by $L^J \langle .,. \rangle$.

Before we go into detail analyzing $\mathcal{L}_0$ and $\mathcal{L}_J$ we show some fundamental results:

Lemma 3.3. *For all $s \in \Gamma$ and $\rho \in I_\epsilon^s$ it holds*

$$J^\epsilon(\rho, s) := J(F_\epsilon^s(\rho), s) = \prod_{i=1}^{n-1}(1 + \epsilon(\rho + h^\epsilon(s))\kappa_i(s)),$$

where κ_i, $i \in \{1, \dots, n-1\}$, denote the principal curvatures of Γ.

Proof. See [26], Lemma 4. □

Remark 3.4. Lemma 3.3 implies that $\langle .,. \rangle_J$ and $(.,.)_J$ are in fact scalar products. This is a consequence of $J(0,s) = J(F_\epsilon^s(-h^\epsilon(s)), s) = 1$ and the fact that ϕ^{-1} is a diffeomorphism, which implies that there is some $c_0 > 0$ such that $\det\left(D\left(\phi^{-1}(r,s)\right)\right) > c_0$ for all $(r,s) \in I_1 \times \Gamma$. In particular, the induced norms are equivalent to the standard L^2–norm.

Proposition 3.5. *Let $\phi, \psi \in H^1(\Omega)$ and Φ, Ψ be given as in (3.12). Then*

1. $L^J\langle\Psi, \Phi\rangle = \epsilon L^J(\psi, \phi)$,
2. $(\psi, \phi)_J = \langle\Psi, \Phi\rangle_J$,
3. $\int_{\Gamma(1)} \epsilon|\nabla\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\psi^2 \mathrm{d}x = \int_\Gamma L^J(\psi(.,s), \psi(.,s))\,\mathrm{d}\mathcal{H}^{n-1}(s) + \epsilon\int_{\Gamma(1)}|\nabla_\tau\psi|^2\,\mathrm{d}x$,

hold for all $s \in \Gamma$.

Proof. Ad 1) and 2): the assertions follow directly from the corresponding definitions in Notations 3.2 by using a change of variables and noting that $\frac{d}{d\rho}F_\epsilon^s(\rho) = \epsilon$.

Ad 3): We remark that $\partial_r\left(\psi\left(\phi^{-1}(r,s)\right)\right) = \nabla\psi\left(\phi^{-1}(r,s)\right)\cdot\mathbf{n}_\Gamma(s)$ for $(r,s) \in (-2,2)\times\Gamma$ and thus get the statement by a change of variables and the decomposition of the gradient, see (3.3) □

3.1. Spectral properties of $\mathcal{L}_0$ and $\mathcal{L}_J$

Suppose that $s \in \Gamma$ is fixed. The bilinear forms L^0 and L^J are coercive, i.e. for $\Psi \in H^1(I_\epsilon^s)$ it holds

$$L^0 \langle \Psi, \Psi \rangle \geq \delta_0 \|\Psi\|_{H^1(I_\epsilon^s)}^2 - c_0 \|\Psi\|^2, \tag{3.13}$$

$$L^J \langle \Psi, \Psi \rangle \geq \delta_0 \left(\|\Psi\|_J^2 + \|\Psi'\|_J^2 \right) - \tilde{c}_0 \|\Psi\|_J^2, \tag{3.14}$$

for some fixed $\delta_0 > 0$ and

$$c_0 := \max_{\tau \in [-1,1]} |f''(\tau)| + \delta_0, \tag{3.15}$$

$$\tilde{c}_0 := \max_{\tau \in [-C^*, C^*]} |f''(\tau)| + \delta_0 \tag{3.16}$$

for C^* as in (3.9). Moreover, they are symmetric and the embedding $H^1(I_\epsilon^s) \hookrightarrow L^2(I_\epsilon^s)$ is compact, so there exist sequences of functions $\{\Psi_i^0(.,s)\}_{i\in\mathbb{N}}, \{\Psi_k(.,s)\}_{k\in\mathbb{N}} \subset H^1(I_\epsilon^s)$ and of real numbers $\{\lambda_i^0(s)\}_{i\in\mathbb{N}}, \{\lambda_k(s)\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

1. each $\Psi_i^0(.,s)$ is an eigenfunction of $\mathcal{L}_0$ with eigenvalue $\lambda_i^0(s)$, i.e. is a weak solution to

$$\begin{aligned} \mathcal{L}_0 \Psi_i^0(.,s) &= \lambda_i^0(s) \Psi_i^0(.,s) && \text{in } I_\epsilon^s, \\ \partial_\rho \Psi_i^0(.,s) &= 0 && \text{on } \partial I_\epsilon^s, \end{aligned} \tag{3.17}$$

   and each $\Psi_k(.,s)$ is an eigenfunction of $\mathcal{L}_J$ with eigenvalue $\lambda_i(s)$, i.e. is a weak solution to

$$\begin{aligned} \mathcal{L}_J \Psi_i(.,s) &= \lambda_i(s) \Psi_i(.,s) && \text{in } I_\epsilon^s, \\ \partial_\rho \Psi_i(.,s) &= 0 && \text{on } \partial I_\epsilon^s. \end{aligned} \tag{3.18}$$

2. $\{\lambda_i^0(s)\}_{i\in\mathbb{N}}, \{\lambda_k(s)\}_{k\in\mathbb{N}}$ are monotonously increasing sequences such that it holds $\lambda_i^0(s)$, $\lambda_k(s) \to \infty$ as $i, k \to \infty$.

3. The eigenfunctions $\{\Psi_i^0(.,s)\}_{i\in\mathbb{N}}$ form a complete orthonormal system in $L^2(I_\epsilon^s, \langle .,. \rangle)$ and the eigenfunctions $\{\Psi_k(.,s)\}_{k\in\mathbb{N}}$ form a complete orthonormal system in the space $L^2(I_\epsilon^s, \langle .,. \rangle_J)$.

As the coefficients of $\mathcal{L}_0$ and $\mathcal{L}_J$ are smooth, elliptic regularity theory also implies that $\{\Psi_i^0(.,s)\}_{i\in\mathbb{N}}$, $\{\Psi_k(.,s)\}_{k\in\mathbb{N}}$ are classical solutions to (3.17) and (3.18). See [40], Theorem 4.12 for the spectral decomposition (the result for $\mathcal{L}_J$ follows when considering the scalar product $\langle .,. \rangle_J$ on L^2 for the corresponding Gelfand triple) and [12], Theorem 8.1.5 for the elliptic regularity in the Neumann case. In the following, we will often drop the s-dependence of $\Psi_i^0, \Psi_k, \lambda_i^0$ and λ_i.

The next proposition reveals more details about the smallest eigenvalues and the corresponding eigenfunction:

Proposition 3.6. *Let $s \in \Gamma$. Then it holds for all $\epsilon \in (0, \epsilon_0)$*

1. *the eigenvalues satisfy*

$$\begin{aligned} -c_0 &< \lambda_1^0(s) \leq \lambda_2^0(s) \leq \ldots, \\ -\tilde{c}_0 &< \lambda_1(s) \leq \lambda_2(s) \leq \ldots. \end{aligned} \tag{3.19}$$

2. *The eigenfunction Ψ_1^0 corresponding to the smallest eigenvalue λ_1^0 can be chosen to be positive in I_ϵ^s. Moreover, λ_1^0 is simple, i.e. if Φ is any other solution to (3.17) for λ_1^0, then Φ is a multiple of Ψ_1^0.*
   The same statement holds true for Ψ_1 and λ_1.

3. *The* ***variational principles***

$$\lambda_1^0 = \inf \left\{ L^0 \left\langle \Psi, \Psi \right\rangle \middle| \, \Psi \in H^1 \left(I_\epsilon^s \right), \|\Psi\| = 1 \right\} \tag{3.20}$$

and

$$\lambda_2^0 = \inf \left\{ L^0 \left\langle \Psi, \Psi \right\rangle \middle| \, \Psi \in H^1 \left(I_\epsilon^s \right), \|\Psi\| = 1, \Psi \perp \Psi_1^0 \right\}$$

hold. The same equalities are satisfied for λ_1 and λ_2 if we replace L^0 by L^J, $\|.\|$ by $\|.\|_J$ and $\perp$ by $\perp_J$.

Proof. Ad 1) Let λ_i^0 be an eigenvalue with corresponding eigenfunction Ψ_i^0. Then it holds

$$\begin{aligned} \lambda_i^0 &= \left\langle \mathcal{L}_0 \Psi_i^0, \Psi_i^0 \right\rangle = L^0 \left\langle \Psi_i^0, \Psi_i^0 \right\rangle \geq C \left\| \Psi_i^0 \right\|_{H^1(I_\epsilon^s)}^2 - c_0 \left\| \Psi_i^0 \right\|^2 \\ &> -c_0 \end{aligned}$$

by (3.13). The same is true for λ_1 by (3.14) when we substitute the norms and scalar products correspondingly.

Ad 2) We set $X := H^3 \left(I_\epsilon^s \right)$ and replace $\mathcal{L}_0$ with $\mathcal{L}_0 + c_0$ in this proof (this does not change the eigenfunctions). Due to the Sobolev embedding theorem we have a continuous embedding $X \hookrightarrow C^2 \left(\bar{I}_\epsilon^s \right)$ and a compact embedding $H^5 \left(I_\epsilon^s \right) \hookrightarrow X$. Using elliptic regularity theory (see again [12], Theorem 8.1.5) we get that the unique weak solution of $\mathcal{L}_0 u = f$ with Neumann boundary conditions for $f \in X$ satisfies $u \in H^5 \left(I_\epsilon^s \right)$ and thus

$$S : X \to X, \; f \mapsto u$$

is compact. Moreover, S is self-adjoint and bounded. Now we define the cone

$$C := \left\{ f \in X \middle| \, f \geq 0 \text{ in } \bar{I}_\epsilon^s \right\}$$

and show that we have

$$S(f) > 0 \text{ in } \bar{I}_\epsilon^s \text{ for } f \in C \backslash \{0\} . \tag{3.21}$$

For $f \in C \backslash \{0\}$ we have $\mathcal{L}_0 u = f \geq 0$ in I_ϵ^s. It follows that u has no negative minimum inside of $\bar{I}_\epsilon^s$, since otherwise the strong maximum principle would imply that u is constant and non-positive in I_ϵ^s and thus

$$f = \mathcal{L}_0 u \leq 0 \leq f \text{ in } I_\epsilon^s,$$

contradicting the choice of f. Now assume that u has a non-positive minimum in some $x_0 \in \partial I_\epsilon^s$. Then $-u(x_0) > -u(x)$ for all $x \in I_\epsilon^s$ and Hopf's lemma (cf. [29], Chapter 6.4.2) implies $-u'(x_0) > 0$, which contradicts the boundary condition u satisfies and leads to (3.21).

Now Theorem A.4 implies that the spectral radius $r(S)$ is a simple eigenvalue of S and admits for a positive eigenfunction. As μ is an eigenvalue of S iff $\lambda = \frac{1}{\mu}$ is an eigenvalue of $\mathcal{L}_0$ and the according eigenfunctions coincide, this shows 2). An analogous proof holds for $\mathcal{L}_J$.

Ad 3) Let L be the bilinear form corresponding to $\mathcal{L}_0 + c_0$ with respect to $\langle .,. \rangle$. $\left\{\Psi_i^0\right\}_{i\in\mathbb{N}}$ is a complete orthonormal system of $L^2\left(I_\epsilon^s, \langle .,. \rangle\right)$ and $\left\{\left(\lambda_i^0 + c_0\right)^{-\frac{1}{2}} \Psi_i^0\right\}_{i\in\mathbb{N}}$ is a complete orthonormal system of $H^1\left(I_\epsilon^s, L\right)$ since on the one hand

$$L\left\langle \Psi_i^0, \Psi_j^0 \right\rangle = \left\langle \left(\mathcal{L}_0 + c_0\right) \Psi_i^0, \Psi_j^0 \right\rangle = \left(\lambda_i^0 + c_0\right) \delta_i^j,$$

which implies orthogonality. On the other hand, it holds for a function $\Phi \in H^1\left(I_\epsilon^s\right)$ satisfying $L\left\langle \Psi_i^0, \Phi \right\rangle = 0$ for all $i \in \mathbb{N}$ that

$$\left(\lambda_i^0 + c_0\right) \left\langle \Psi_i^0, \Phi \right\rangle = L\left\langle \Psi_i^0, \Phi \right\rangle = 0$$

for all $i \in \mathbb{N}$ and thus $\Phi \equiv 0$ in I_ϵ^s. This yields completeness of the system, see e.g. [52], Satz V.4.9. In both equalities we used the Neumann boundary condition for Ψ_i^0. Thus, we get for $\Phi \in H^1\left(I_\epsilon^s\right)$ with $\|\Phi\| = 1$, $\Phi \perp \Psi_i^0$ for all $i \in \{1, \dots, K\}$ and $\mu_i = L\left\langle \Phi, \left(\lambda_i^0 + c_0\right)^{-\frac{1}{2}} \Psi_i^0 \right\rangle$

$$L\left\langle \Phi, \Phi \right\rangle = \lim_{N\to\infty} \left(\sum_{i=0}^{N} \mu_i^2 \right) = \lim_{N\to\infty} \left(\sum_{i=0}^{N} \left(\lambda_i^0 + c_0\right) \left\langle \Psi_i^0, \Phi \right\rangle^2 \right) \geq \left(\lambda_{K+1}^0 + c_0\right) \|\Phi\|^2 .$$

Subtracting $c_0 \|\Phi\|$ from both sides, the statements follow immediately. The proof follows along the same lines as in the case of L^J when replacing the scalar products and norms. □

Before we may discuss the main results of this subsection, we show a very technical auxiliary result, which guarantees that eigenfunctions corresponding to "small" eigenvalues have exponential decay close to the boundary of I_ϵ^s.

Proposition 3.7. *Let* $m := \sqrt{\min\left\{f''(1), f''(-1)\right\}}$, $s \in \Gamma$ *and let* Ψ *be any* $\|.\|$*-normalized eigenfunction of* $\mathcal{L}_0$ *corresponding to an eigenvalue* $\lambda \leq \frac{m^2}{4}$. *Then there exist* $\epsilon_1 \in (0, \epsilon_0]$ *and* c_2, $C > 0$ *independent of* ϵ *and* s *and such that*

$$|\Psi(\rho, s)| \leq C e^{-\frac{m}{2}|\rho|} \text{ for all } \rho \in \left[\pm(c_2+1), \pm\epsilon^{-1} - h^\epsilon(s)\right], \tag{3.22}$$

for all $\epsilon \in (0, \epsilon_1)$.

This also holds (with different constants c_2, C *and* ϵ_1*) if* Ψ *is a* $\|.\|_J$*-normalized eigenfunction of* $\mathcal{L}_J$ *to an eigenvalue* $\lambda \leq \frac{m^2}{4}$.

Proof. Since $\theta_0(\tau) \to \pm 1$ for $\tau \to \pm\infty$ and θ_0 is monotonously increasing, there exists some $c_2 > 0$, such that $\inf_{|\tau|\geq c_2} f''(\theta_0(\tau)) \geq \frac{3m^2}{4}$. Using this, we want to show that for each $b \in \left[c_2, \epsilon^{-1} - h^\epsilon(s)\right]$ we may apply a comparison principle to deduce

$$|\Psi(\rho, s)| \leq \Phi^+(\rho) := |\Psi(b, s)| \frac{\cosh\left(\frac{m}{2}\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - \rho\right)\right)}{\cosh\left(\frac{m}{2}\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - b\right)\right)} \tag{3.23}$$

for all $\rho \in \left[b, \epsilon^{-1} - h^\epsilon(s)\right]$ and

$$|\Psi(\rho, s)| \leq \Phi^-(\rho) := |\Psi(-b, s)| \frac{\cosh\left(\frac{m}{2}\left(\left(\epsilon^{-1} + h^\epsilon(s)\right) + \rho\right)\right)}{\cosh\left(\frac{m}{2}\left(\left(\epsilon^{-1} + h^\epsilon(s)\right) - b\right)\right)} \tag{3.24}$$

for all $\rho \in \left[-\epsilon^{-1} - h^\epsilon(s), -b\right]$, for all $\epsilon \in (0, \epsilon_1)$. Here we choose $\epsilon_1 < \frac{1}{c_2 + C^*}$, where C^* is the constant from (3.7) and $\epsilon_1 \in (0, \epsilon_0]$.

We initially assume that Ψ is positive. In order to show (3.23) and (3.24) we observe that, since Ψ is an eigenfunction of the operator $\mathcal{L}_0$, we have

$$0 = \mathcal{L}_0(\Psi) - \lambda\Psi = -\partial_\rho^2\Psi + f''(\theta_0)\Psi - \lambda\Psi \geq -\partial_\rho^2\Psi + \Psi\left(\frac{3m^2}{4} - \frac{m^2}{4}\right)$$

on $\left[-\epsilon^{-1} - h^\epsilon(s), -b\right] \cup \left[b, \epsilon^{-1} - h^\epsilon(s)\right]$ and thus $\tilde{\mathcal{L}}(\Psi(.,s)) = \partial_\rho^2\Psi(.,s) - \frac{m^2}{2}\Psi(.,s) \geq 0$. On the other hand, we may compute

$$\partial_\rho^2\Phi^+ - \frac{m^2}{2}\Phi^+ = \Phi^+\left(\frac{m^2}{4} - \frac{m^2}{2}\right) \leq 0$$

and conclude $\tilde{\mathcal{L}}(\Psi(.,s)) \geq \tilde{\mathcal{L}}(\Phi^+)$ on $\left[b, \epsilon^{-1} - h^\epsilon(s)\right]$ and analogously $\tilde{\mathcal{L}}(\Psi(.,s)) \geq \tilde{\mathcal{L}}(\Phi^-)$ on $\left[-\epsilon^{-1} - h^\epsilon(s), -b\right]$. Since $\partial_\rho\Psi\left(\pm\epsilon^{-1} - h^\epsilon(s), s\right) = \partial_\rho\Phi^\pm\left(\pm\epsilon^{-1} - h^\epsilon(s)\right) = 0$ and it holds $\Psi(\pm b, s) = |\Psi(\pm b, s)| = \Phi^\pm(\pm b)$, Lemma A.1 from the Appendix implies the desired inequalities (3.23) and (3.24). The same ideas may be applied to the case that Ψ changes its sign throughout the intervals (or is negative); then the comparison principle has to be applied on the domains of constant sign for Ψ resp. $-\Psi$.

Since Ψ is normalized we have

$$1 = \left\|\Psi^2\right\| \geq \left(\int_{c_2}^{c_2+1} \Psi^2 \mathrm{d}\rho\right)^{\frac{1}{2}} = |\Psi(\xi)|$$

for some $\xi \in [c_2, c_2+1]$, by the mean value theorem. We may thus choose $b_0 \in [c_2, c_2+1]$ such that $|\Psi(\pm b_0, s)| \leq 1$ and get from (3.23) and (3.24)

$$|\Psi(\rho, s)| \leq \frac{\cosh\left(\frac{m}{2}\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - \rho\right)\right)}{\cosh\left(\frac{m}{2}\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - b_0\right)\right)} \text{ for all } \rho \in \left[c_2 + 1, \epsilon^{-1} - h^\epsilon(s)\right],$$
$$|\Psi(\rho, s)| \leq \frac{\cosh\left(\frac{m}{2}\left(\left(\epsilon^{-1} + h^\epsilon(s)\right) + \rho\right)\right)}{\cosh\left(\frac{m}{2}\left(\left(\epsilon^{-1} + h^\epsilon(s)\right) - b_0\right)\right)} \text{ for all } \rho \in \left[-\epsilon^{-1} - h^\epsilon(s), -(c_2 + 1)\right]$$

Using the definition of cosh, this leads to

$$\begin{aligned}
|\Psi(\rho, s)| &\leq \frac{e^{\frac{m}{2}\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - \rho\right)} + e^{-\frac{m}{2}\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - \rho\right)}}{e^{\frac{m}{2}\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - b_0\right)} + e^{-\frac{m}{2}\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - b_0\right)}} \\
&= e^{-\frac{m}{2}\rho}\left(e^{-\frac{m}{2}\left(\epsilon^{-1} - h^\epsilon(s)\right)} \frac{e^{\frac{m}{2}\left(\epsilon^{-1} - h^\epsilon(s)\right)} + e^{-\frac{m}{2}\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - 2\rho\right)}}{e^{-\frac{m}{2}b_0} + e^{-\frac{m}{2}\left(2\left(\epsilon^{-1} - h^\epsilon(s)\right) - b_0\right)}}\right) \\
&\leq e^{-\frac{m}{2}\rho}\left(\frac{1 + e^{-m\left(\left(\epsilon^{-1} - h^\epsilon(s)\right) - \rho\right)}}{e^{-\frac{m}{2}b_0}}\right) \\
&\leq e^{-\frac{m}{2}\rho}\left(2e^{\frac{m}{2}(c_2+1)}\right)
\end{aligned}$$

for all $\rho \in \left[c_2 + 1, \epsilon^{-1} - h^\epsilon(s)\right]$. Analogously we get a similar estimate for every $\rho \in \left[-\epsilon^{-1} - h^\epsilon(s), -(c_2 + 1)\right]$, leading to the assertion.

Now let Ψ be an eigenfunction of $\mathcal{L}_J$. We have $c_A^\epsilon(\rho, s) = \theta_0(\rho) + \epsilon p^\epsilon(s)\theta_1(\rho) + \epsilon^2 q^\epsilon(\rho, s)$ for $(\rho, s) \in I_\epsilon^s \times \Gamma$ and we can again find $c_2 > 0$ such that for ϵ small enough we have due to (3.6) the estimate $\inf_{\rho\in I_\epsilon^s\setminus(-c_2,c_2)} f''\left(c_A^\epsilon(\rho)\right)\geq\frac{3m^2}{4}$. Moreover, we can consider the same functions Φ^+ and Φ^- and find for positive Ψ that

$$\tilde{\mathcal{L}}\left(\Psi(.,s)\right):=\left(J^\epsilon\right)^{-1}\partial_\rho\left(J^\epsilon\partial_\rho\Psi(.,s)\right)-\frac{m^2}{2}\Psi(.,s)\geq 0.$$

On the other hand we get

$$\left(J^\epsilon\right)^{-1}\partial_\rho J^\epsilon\partial_\rho\Phi^+\leq C\epsilon\Phi^+$$

for some C independent of ϵ and s, as $\left|\left(J^\epsilon\right)^{-1}\right|\leq C$ for C independent of ϵ and s and $\partial_\rho J^\epsilon\in\mathcal{O}(\epsilon)$ (cf. Lemma (3.3)). Moreover $\left|\partial_\rho\Phi^+\right|\leq C\Phi^+$ as $\sinh(\rho)\leq\cosh(\rho)$ for all $\rho\in\mathbb{R}$. Thus,

$$\tilde{\mathcal{L}}\left(\Phi^+\right)\leq\Phi^+\left(-\frac{m^2}{4}+C\epsilon\right)\leq 0$$

for $\epsilon>0$ small enough. The rest of the proof then follows along the same lines, with the only difference that $|\Psi(\xi)|\leq C$ due to $\|\Psi\|_J=1$ and the uniform lower bound on J^ϵ. ☐

The following lemma is a modified version of Lemma 2.1 in [24] with an extended proof, adapted to our situation that the eigenvalues Ψ_i^0 depend on s.

Lemma 3.8. *Let Assumption 3.1 hold, let $s\in\Gamma$ be fixed and set $\beta(s):=\|\theta_0'\|^{-1}$. Then there are $C,C_1,C_2>0$ independent of s and $\epsilon_1\in(0,\epsilon_0]$, such that for all $\epsilon\in(0,\epsilon_1)$:*

1. *The principal eigenvalue $\lambda_1^0(s)$ and the corresponding positive, $\|.\|$–normalized eigenfunction $\Psi_1^0(.,s)$ of $\mathcal{L}_0$ satisfy*

$$\left|\lambda_1^0(s)\right|\leq C_1e^{-\frac{C_2}{\epsilon}},\tag{3.25}$$

where C_1 only depends on ϵ_0 and C_2 depends on α, which is given as in Lemma 2.2, and on $m:=\sqrt{\min\left\{f''(1),f''(-1)\right\}}$.

2. *It holds*

$$\lambda_2^0(s)=\inf_{\substack{\Psi\in H^1(I_\epsilon^s),\|\Psi\|=1\\ \Psi\perp\Psi_1^0(.,s)}}L^0\langle\Psi,\Psi\rangle\geq C.\tag{3.26}$$

3. *The function $\mathcal{R}(\rho,s):=\Psi_1^0(\rho,s)-\beta(s)\theta_0'(\rho)$ fulfills*

$$\|\mathcal{R}\|+\|\partial_\rho\mathcal{R}\|\leq C_1e^{-\frac{C_2}{\epsilon}}.\tag{3.27}$$

Proof. Due to Proposition 3.6 2) λ_1^0 is simple and Ψ_1^0 can be chosen to be positive. Due to 1) and 3) of the mentioned proposition, we find

$$\begin{aligned}-c_0 &< \lambda_1^0\leq L^0\left(\beta\theta_0',\beta\theta_0'\right)\\ &= \beta^2\left(\left\langle\mathcal{L}_0\theta_0',\theta_0'\right\rangle+\left[\theta_0''\theta_0'\right]_{-\frac{1}{\epsilon}-h^\epsilon(s)}^{\frac{1}{\epsilon}-h^\epsilon(s)}\right)\\ &\leq C_1e^{-\frac{C_2}{\epsilon}}\end{aligned}\tag{3.28}$$

for c_0 as in (3.15) and some constants $C_1,C_2>0$ independent of ϵ and s. In the last step we employed $\mathcal{L}_0\theta_0'=0$ and the fact that $\left[\theta_0''\theta_0'\right]_{-\frac{1}{\epsilon}-h^\epsilon(s)}^{\frac{1}{\epsilon}-h^\epsilon(s)}\leq C_1e^{-\frac{C_2}{\epsilon}}$ holds for all $\epsilon\in(0,\epsilon_0)$.

Both follow from Lemma 2.2 and C_2 depends only on α and m since $\left|\frac{1}{\epsilon}-h^{\epsilon}(s)\right|\geq\frac{1}{2\epsilon}$ for all $s\in\Gamma$, see Notation 3.2 2). Moreover, we used

$$\beta(s)\leq\beta_0:=\left(\int_{-1}^{1}\left(\theta_0'\right)^2\mathrm{d}\rho\right)^{-\frac{1}{2}}, \tag{3.29}$$

cf. (3.11).

Now we only have to take care of the lower bound of λ_1^0. We first observe that we may choose $\epsilon_1>0$ small enough such that $\lambda_1^0\leq\frac{m^2}{4}$ and thus Proposition 3.7 implies

$$\left|\Psi_1^0(\rho,s)\right|\leq Ce^{-\frac{m}{2}|\rho|}\text{ for all }\rho\in\left[\pm(c_2+1),\pm\epsilon^{-1}-h^{\epsilon}(s)\right] \tag{3.30}$$

for all $\epsilon\in(0,\epsilon_1)$, where C and c_2 are independent of ϵ and s. Using this, we get

$$\begin{aligned}\lambda_1^0\left\langle\Psi_1^0,\theta_0'\right\rangle &= \left\langle\mathcal{L}_0\Psi_1^0,\theta_0'\right\rangle=\left\langle\Psi_1^0,\mathcal{L}_0\theta_0'\right\rangle-\left[\theta_0''\Psi_1^0\right]_{-\frac{1}{\epsilon}-h^{\epsilon}(s)}^{\frac{1}{\epsilon}-h^{\epsilon}(s)}\\ &\geq -C_1e^{-\frac{C_2}{\epsilon}},\end{aligned}$$

where we used the same properties of θ_0' as above and where C_2 only depends on α and m. Thus, we may reduce the problem of proving the lower estimate for λ_1^0 to proving that $\left\langle\Psi_1^0,\theta_0'\right\rangle$ has a uniform positive lower bound. As $\theta_0'>0$ in $\mathbb{R}$ and Ψ_1^0 is positive, the claim follows if we can show that there exists some $c^*>0$ independent of ϵ such that Ψ_1^0 has a uniform and positive bound from below in $(-c^*,c^*)$.

Since (3.30) holds and Ψ_1^0 is normalized, we may find some $c^*\geq 1>0$ independent of $\epsilon>0$ such that

$$\int_{-c^*}^{c^*}\left|\Psi_1^0(\rho)\right|^2\mathrm{d}\rho>\frac{1}{2}$$

holds. Due to the positivity of Ψ_1^0 and the uniform bound on λ_1^0 we may use Harnack's inequality (cf. [29], Chapter 6.4, Theorem 5) for the equation $\left(\mathcal{L}_0-\lambda_1^0\right)\Psi_1^0=0$ in the interval $(-c^*-\tilde{\epsilon},c^*+\tilde{\epsilon})$ for small $\tilde{\epsilon}>0$ and get

$$\inf_{(-c^*,c^*)}\Psi_1^0\geq C\sup_{(-c^*,c^*)}\Psi_1^0\geq C\left(\frac{1}{2c^*}\int_{-c^*}^{c^*}\left|\Psi_1^0(\rho)\right|^2\mathrm{d}\rho\right)^{\frac{1}{2}}>C\left(4c^*\right)^{-\frac{1}{2}},$$

where C depends only on c^* and the coefficients of $\mathcal{L}_0-\lambda_1^0$, which are uniformly bounded (in ϵ and s). As we remarked above, this proves the assertion 1).

In order to prove 2), we first observe that if $\lambda_2^0(s)>\frac{m^2}{4}$ holds, there is nothing to show. We therefore assume $\lambda_2^0(s)\leq\frac{m^2}{4}$ holds and may thus use the results of Proposition 3.7 for the $\|.\|$-normalized eigenfunction Ψ_2^0. Furthermore, there exists exactly one $x_0\in I_\epsilon^s$ with $\Psi_2^0(x_0)=0$ (cf. [37] Theorem 1.3.2). Since it is shown in Proposition 3.7 that for every $b\in\left[c_2,\epsilon^{-1}-h^{\epsilon}(s)\right]$ the estimates (3.23) and (3.24) (found below) hold, we may deduce $x_0\in(-c_2,c_2)$ (otherwise Ψ_2^0 would be zero in an entire interval). Note that $x_0=x_0(s)$ depends on s. We assume without loss of generality that Ψ_2^0 is positive in $\left(x_0,\epsilon^{-1}-h^{\epsilon}(s)\right]$.

We get a first estimate on λ_2^0 by using the equality $\mathcal{L}_0\Psi_2^0 = \lambda_2^0\Psi_2^0$ in I_ϵ^s and calculating

$$\begin{aligned}\lambda_2^0 \int_{x_0}^{\epsilon^{-1}-h^\epsilon(s)} \Psi_2^0\theta_0'\mathrm{d}\rho &= \int_{x_0}^{\epsilon^{-1}-h^\epsilon(s)} \mathcal{L}_0\Psi_2^0\theta_0'\mathrm{d}\rho \\ &= \int_{x_0}^{\epsilon^{-1}-h^\epsilon(s)} \Psi_2^0\mathcal{L}_0\theta_0'\mathrm{d}\rho + \left[\Psi_2^0\theta_0'' - \partial_\rho\left(\Psi_2^0\right)\theta_0'\right]_{x_0}^{\epsilon^{-1}-h^\epsilon(s)} \\ &\geq \partial_\rho\left(\Psi_2^0\right)(x_0,s)\,\theta_0'(x_0) - C_1 e^{-\frac{C_2}{\epsilon}}.\end{aligned}$$

In the last inequality we again used the properties of θ_0' as we did before in the proof of 1) together with $\left|\Psi_2^0(\rho,s)\right| \leq Ce^{-\frac{m}{2}|\rho|}$ (see again Proposition 3.7) and

$$\Psi_2^0(x_0,s) = \partial_\rho\left(\Psi_2^0\right)\left(\epsilon^{-1} - h^\epsilon(s), s\right) = 0.$$

Thus, we have

$$\lambda_2^0 \geq \frac{\partial_\rho\left(\Psi_2^0\right)(x_0,s)\,\theta_0'(x_0) - C_1 e^{-\frac{C_2}{\epsilon}}}{\int_{x_0}^{\epsilon^{-1}-h^\epsilon(s)} \Psi_2^0\theta_0'\mathrm{d}\rho}. \tag{3.31}$$

As Ψ_2^0 is normalized we may estimate

$$\int_{x_0}^{\epsilon^{-1}-h^\epsilon(s)} \Psi_2^0\theta_0'\mathrm{d}\rho \leq \left\|\Psi_2^0\right\|\left\|\theta_0'\right\| \leq \left\|\theta_0'\right\|_{L^2(\mathbb{R})} = C,$$

where C is again independent of ϵ and s. So we may prove the desired lower bound for λ_2^0 by showing that $\partial_\rho\left(\Psi_2^0\right)(x_0,s)$ is strictly positive, independent of $\epsilon > 0$. For this we will employ Hopf's lemma and carefully consider the constants appearing in the proof (cf. [29] Chapter 6.4.2).

First off, note that since Ψ_2^0 is normalized, we can choose $c^* \geq \max\{1, c_2\} + 1$ independent of ϵ and s such that

$$\left(\int_{-c^*}^{c^*} \left(\Psi_2^0\right)^2 \mathrm{d}\rho\right)^{\frac{1}{2}} > \frac{1}{2},$$

if ϵ is small enough. Moreover, $-c_0 < \lambda_2^0 \leq \frac{m^2}{4}$ and we have

$$\left(\mathcal{L}_0 + c_0\right)\Psi_2^0 = \left(\lambda_2^0 + c_0\right)\Psi_2^0 < 0 \tag{3.32}$$

in $(-c^*, x_0)$, as Ψ_2^0 is negative in that domain. Assume now that $\left(\int_{-c^*}^{x_0}\left(\Psi_2^0\right)^2\mathrm{d}\rho\right)^{\frac{1}{2}} \geq \frac{1}{4}$ (if this does not hold, then one may repeat the following argumentation on (x_0, c^*) with $-\Psi_2^0$). Then it follows

$$\left|\Psi_2^0(p_{max})\right| = \max_{\rho\in[-c^*,x_0]}\left|\Psi_2^0\right| \geq c > 0$$

for some constant c independent of s and ϵ and for some $p_{max} \in [-c^*, x_0]$. Before we may now go into the details of the application of Hopf's lemma, we need some more information on Ψ_2^0 . Due to (3.32) we have

$$\partial_{\rho\rho}\Psi_2^0 > \left(f''(\theta_0) + c_0\right)\Psi_2^0 \geq -2c_0\left|\Psi_2^0\right|$$

as Ψ_2^0 is negative, and thus for all $\rho_1, \rho_2 \in [-c^*, x_0]$, $\rho_1 \leq \rho_2$, we have

$$\begin{aligned}\int_{\rho_1}^{\rho_2} \partial_{\rho\rho}\Psi_2^0 \mathrm{d}\rho &> -\int_{-c^*}^{c^*} 2c_0 \left|\Psi_2^0\right| \mathrm{d}\rho \\ &\geq -\tilde{c}\left\|\Psi_2^0\right\| \\ &\geq -\tilde{c}, \end{aligned} \tag{3.33}$$

where $\tilde{c} > 0$ only depends on c_0 and c^* and is thus independent of ϵ and s.

First we show that we may assume that $p_{max} \in [-c^*, x_0 - \delta_1]$ for some fixed $\delta_1 > 0$ independent of ϵ and s. For this let $p_{max} \in (x_0 - \delta_1, x_0)$ for some δ_1 which we choose below. Then

$$0 < c \leq \Psi_2^0(x_0) - \Psi_2^0(p_{max}) = (x_0 - p_{max})\,\partial_\rho \Psi_2^0(\xi)$$

and thus

$$\partial_\rho \Psi_2^0(\xi) \geq \frac{c}{\delta_1} \tag{3.34}$$

for a suitable point $\xi \in (p_{max}, x_0)$ by the mean value theorem. Thus,

$$\begin{aligned}\partial_\rho \Psi_2^0(x_0) &= \int_{\xi}^{x_0} \partial_{\rho\rho}\Psi_2^0 \mathrm{d}\rho + \partial_\rho \Psi_2^0(\xi) \\ &\geq -\tilde{c} + \frac{c}{\delta_1} \\ &\geq 1 \end{aligned}$$

by choosing δ_1 small enough, depending only on $\tilde{c}$ and c. Here we used the fundamental theorem of calculus in the first step and (3.33), (3.34) in the second.

Thus, we may assume that $p_{max} \in [-c^*, x_0 - \delta_1]$. Let now $z := -c^* - 1$, $R := x_0 - z$ and $r := p_{max} - z$. Moreover, we define

$$v(\rho) := e^{-\lambda(\rho - z)^2} - e^{-\lambda R^2}$$

for $\rho \in (z + r, z + R) =: J$ and for some λ which we choose in the following way:

It holds

$$\begin{aligned}(\mathcal{L}_0 + c_0)\, v(\rho) &= e^{-\lambda(\rho - R)^2}\left(-4\lambda^2(\rho - z)^2 + 2\lambda + \left(f''(\theta_0(\rho)) + c_0\right)\right) \\ &\leq e^{-\lambda(\rho - R)^2}\left(-\lambda^2 r^2 + 2\lambda + 2c_0\right) \\ &\leq e^{-\lambda(\rho - R)^2}\left(-\lambda^2 + 2\lambda + 2c_0\right) \\ &\leq 0 \end{aligned}$$

for $\lambda > 0$ big enough, independent of ϵ and s, where we used $r \geq 1$ in the third step.

Moreover, for $\tilde{\epsilon} > 0$ it holds

$$\Psi_2^0(p_{max}) + \tilde{\epsilon} v(z + r) \leq -c + \tilde{\epsilon}\left(e^{-\lambda r^2} - e^{-\lambda R^2}\right)$$

and since $1 \leq r \leq R - \delta_1$ and

$$1 \leq R \leq 2c^* \tag{3.35}$$

we can choose $\tilde{\epsilon} > 0$ independent of ϵ and s such that

$$0 = \Psi_2^0(x_0) \geq \Psi_2^0(p_{max}) + \tilde{\epsilon} v(p_{max})$$

and of course

$$\Psi_2^0(x_0) = \Psi_2^0(x_0) + \tilde{\epsilon} v(x_0)$$

as $v(x_0) = 0$ by definition. Thus, we have

$$(\mathcal{L}_0 + c_0)\left(\Psi_2^0 + \tilde{\epsilon} v - \Psi_2(x_0)\right) \leq 0$$

in J and

$$\Psi_2^0 + \tilde{\epsilon} v - \Psi_2(x_0) \leq 0$$

on ∂J and thus the weak maximum principle implies

$$\partial_\rho\left(\Psi_2^0\right)(x_0) \geq -\tilde{\epsilon} v'(x_0) = 2\lambda R \tilde{\epsilon} e^{-\lambda R^2} \geq C > 0$$

for some $C > 0$ independent of ϵ and s, due to the choices of λ and $\tilde{\epsilon}$ and (3.35). In view of (3.31), this proves 2).

To obtain assertion 3) we introduce a technique that will also be helpful in the proof of Lemmata 3.9 and 3.10 (although we will use it there in a slightly different way). The idea is to analyze a decomposition of Ψ_1^0 into a multiple of θ_0' and a part that is orthogonal to Ψ_1^0, which will be shown to have exponential decay.

To get this decomposition, we define $a(s) := \beta \int_{I_\epsilon^s} \Psi_1^0 \theta_0' \mathrm{d}\rho$ and $\left(\Psi_1^0\right)^\perp(\rho, s) := \beta\theta_0'(\rho) - a(s)\Psi_1^0(\rho, s)$ for $\rho \in I_\epsilon^s$, $s \in \Gamma$. Note that $a(s) > 0$ since $\Psi_1^0, \theta_0' > 0$. From the definitions we immediately get $\Psi_1^0 \perp \left(\Psi_1^0\right)^\perp$ and

$$\begin{aligned} 1 = \beta^2 \int_{I_\epsilon^s} \left(\theta_0'\right)^2 \mathrm{d}\rho &= \left\|\left(\Psi_1^0\right)^\perp\right\|^2 + a(s)\left\langle \Psi_1^0, \left(\Psi_1^0\right)^\perp\right\rangle + a(s)^2 \left\|\Psi_1^0\right\|^2 \\ &= \left\|\left(\Psi_1^0\right)^\perp\right\|^2 + a(s)^2, \end{aligned} \tag{3.36}$$

which in particular implies $a(s)^2 \leq 1$ for all $s \in \Gamma$. Regarding (3.28), we also get

$$\begin{aligned} C_1 e^{-\frac{C_2}{\epsilon}} &\geq L^0\left\langle \beta\theta_0', \beta\theta_0'\right\rangle = a(s)^2 L^0\left\langle \Psi_1^0, \Psi_1^0\right\rangle + L^0\left\langle \left(\Psi_1^0\right)^\perp, \left(\Psi_1^0\right)^\perp\right\rangle \\ &\geq a(s)^2 \lambda_1^0 + \lambda_2^0 \left\|\left(\Psi_1^0\right)^\perp\right\|^2, \end{aligned}$$

where the last inequality is implied by 1) and 2). The second identity is an auxiliary result which is shown at the end of this proof.

Using the estimates on λ_1^0, λ_2^0 and $a(s)^2 \leq 1$ we find

$$\left\|\left(\Psi_1^0\right)^\perp\right\|^2 \leq C_1 e^{-\frac{C_2}{\epsilon}}$$

and thus also

$$a(s)^2 \geq 1 - C_1 e^{-\frac{C_2}{\epsilon}}$$

by (3.36). As $a(s)$ is strictly positive due to its definition, this implies

$$a(s) \geq \left(1 - C_1 e^{-\frac{C_2}{\epsilon}}\right)^{\frac{1}{2}} \geq 1 - \tilde{C}_1 e^{-\frac{\tilde{C}_2}{\epsilon}}$$

for $\epsilon > 0$ small enough. Now we have

$$\begin{aligned}\left\|\beta\theta_0' - \Psi_1^0\right\|^2 &= \int\limits_{I_\epsilon^s} \beta^2 \left(\theta_0'\right)^2 - 2\Psi_1^0\beta\theta_0' + \left(\Psi_1^0\right)^2 \mathrm{d}\rho \\ &= 2 - 2a\left(s\right) \le C_1 e^{-\frac{C_2}{\epsilon}},\end{aligned}$$

showing

$$\|\mathcal{R}\| \le C_1 e^{-\frac{C_2}{\epsilon}}.$$

In order to estimate $\partial_\rho \mathcal{R}$ we calculate

$$\left\|\beta\theta_0'' - \partial_\rho\Psi_1^0\right\|^2 = \int\limits_{I_\epsilon^s} \left(\beta\theta_0''\right)^2 - 2\beta\theta_0''\partial_\rho\Psi_1^0 + \left(\partial_\rho\Psi_1^0\right)^2 \mathrm{d}\rho. \tag{3.37}$$

Moreover,

$$\begin{aligned}\int\limits_{I_\epsilon^s} \left(\partial_\rho\Psi_1^0\right)^2 \mathrm{d}\rho &= \lambda_1^0 - \int\limits_{I_\epsilon^s} f''\left(\theta_0\right)\left(\Psi_1^0\right)^2 \mathrm{d}\rho \\ &\le -\int\limits_{I_\epsilon^s} f''\left(\theta_0\right)\left(\beta\theta_0' + \mathcal{R}\right)^2 \mathrm{d}\rho + C_1 e^{-\frac{C_2}{\epsilon}} \\ &\le -\int\limits_{I_\epsilon^s} f''\left(\theta_0\right)\left(\theta_0'\right)^2 \beta^2 d\rho + C_1 e^{-\frac{C_2}{\epsilon}} \\ &\le \int\limits_{I_\epsilon^s} \left(\theta_0''\right)^2 \beta^2 \mathrm{d}\rho + C_1 e^{-\frac{C_2}{\epsilon}},\end{aligned}$$

where we used (3.25) in the first inequality, the decay of $\|\mathcal{R}\|$ in the second inequality and $f''\left(\theta_0\right)\theta_0' = \frac{d}{d\rho}\left(f'\left(\theta_0\right)\right) = \theta_0'''$ due to (1.36) as well as integration by parts and the exponential decay of θ_0' in the last inequality. On the other hand, we have

$$\begin{aligned}-\int\limits_{I_\epsilon^s} 2\beta\theta_0''\partial_\rho\Psi_1^0 \mathrm{d}\rho &= -\int\limits_{I_\epsilon^s} \left(\lambda_1^0 - f''\left(\theta_0\right)\right) 2\Psi_1^0\beta\theta_0' \\ &= -\int\limits_{I_\epsilon^s} \left(\lambda_1^0 - f''\left(\theta_0\right)\right) 2\left(\left(\beta\theta_0'\right)^2 + \mathcal{R}\beta\theta_0'\right) \\ &\le -2\lambda_1^0 \int\limits_{I_\epsilon^s} \left(\beta^2\theta_0'\right)^2 \mathrm{d}\rho + 2\int\limits_{I_\epsilon^s} f''\left(\theta_0\right)\left(\theta_0'\right)^2 \beta^2 \mathrm{d}\rho + C_1 e^{-\frac{C_2}{\epsilon}} \\ &\le -2\int\limits_{I_\epsilon^s} \left(\theta_0''\right)^2 \beta^2 + C_1 e^{-\frac{C_2}{\epsilon}},\end{aligned}$$

where we again used the decay of $\|\mathcal{R}\|$ in the third line and (3.25), $f''\left(\theta_0\right)\theta_0' = \theta_0'''$ and (2.1) in the last step. Plugging those two estimates into (3.37) yields the claim.

As mentioned before, we still have to show

$$L^0\left\langle\beta\theta_0', \beta\theta_0'\right\rangle = a\left(s\right)^2 L^0\left\langle\Psi_1^0, \Psi_1^0\right\rangle + L^0\left\langle\left(\Psi_1^0\right)^\perp, \left(\Psi_1^0\right)^\perp\right\rangle. \tag{3.38}$$

Plugging in the definition of $\left(\Psi_1^0\right)^\perp$ gives

$$\begin{aligned} L^0 \left\langle \beta\theta_0', \beta\theta_0' \right\rangle &= a(s)^2 L^0 \left\langle \Psi_1^0, \Psi_1^0 \right\rangle + L^0 \left\langle \left(\Psi_1^0\right)^\perp, \left(\Psi_1^0\right)^\perp \right\rangle \\ &\quad + 2a(s) \int_{I_\epsilon^s} \partial_\rho \left(\Psi_1^0\right) \partial_\rho \left(\left(\Psi_1^0\right)^\perp\right) + f'(\theta_0)\, \Psi_1^0 \left(\Psi_1^0\right)^\perp \mathrm{d}\rho \end{aligned}$$

and since Ψ_1^0 is an eigenfunction with Neumann boundary condition and $\Psi_1^0 \perp \left(\Psi_1^0\right)^\perp$ we get

$$\begin{aligned} &\int_{I_\epsilon^s} a(s)\, \partial_\rho \left(\Psi_1^0\right) \partial_\rho \left(\left(\Psi_1^0\right)^\perp\right) + f''(\theta_0)\, \Psi_1^0 \left(\Psi_1^0\right)^\perp \mathrm{d}\rho \\ &\qquad = a(s) \left\langle \mathcal{L}_0 \Psi_1^0, \left(\Psi_1^0\right)^\perp \right\rangle + a(s) \left[\partial_\rho \left(\Psi_1^0\right) \left(\Psi_1^0\right)^\perp\right]_{\partial I_\epsilon^x} \\ &\qquad = a(s)\, \lambda_1(s) \left\langle \Psi_1^0, \left(\Psi_1^0\right)^\perp \right\rangle \\ &\qquad = 0, \end{aligned}$$

completing the proof. □

Next we prove similar properties as in Lemma 3.8 for the eigenvalues of the operator $\mathcal{L}_J$.

Lemma 3.9. *Let Assumptions 3.1 hold, let $s \in \Gamma$ and let $\beta(s) = \|\theta_0'\|^{-1}$. Then there are $\epsilon_1 \in (0, \epsilon_0]$ and $C > 0$ independent of s, such that for all $\epsilon \in (0, \epsilon_1)$:*

1. *The principal eigenvalue $\lambda_1(s)$ and its $\|.\|_J$-normalized positive eigenfunction $\Psi_1(\rho, s)$ of $\mathcal{L}_J$ satisfy*

$$|\lambda_1(s)| \le C\epsilon^2 \tag{3.39}$$

for all $s \in \Gamma$ and

$$\Psi_1^{\mathbf{R}}(\rho, s) := \Psi_1(\rho, s) - \beta(s)\, \theta_0'(\rho)$$

fulfills

$$\sup_{s\in\Gamma} \left(\|\Psi_1^{\mathbf{R}}\|_J + \left\| \left(\Psi_1^{\mathbf{R}}\right)_\rho \right\|_J \right) \le C\epsilon. \tag{3.40}$$

2. *It holds*

$$\lambda_2(s) \ge \frac{1}{C}$$

for all $s \in \Gamma$.

3. *It holds*

$$\sup_{s\in\Gamma} \left(\|\nabla_\tau \Psi_1(\cdot, s)\|_J \right) \le C \left(\epsilon + \|\nabla_\tau c_A^\epsilon\|_{L^\infty(\Gamma(1))} \right).$$

Proof. Ad 1) First of all, we show that we may express L^J in terms of L^0 plus some perturbation of higher order in ϵ. This will allow us to use results from Lemma 3.8.

Writing $J^\epsilon(\rho,s) := J(F_\epsilon^s(\rho),s)$ and $\tilde\Psi := (J^\epsilon)^{\frac{1}{2}}\Psi$, for $\Psi \in H^1(I_\epsilon^s)$, we calculate

$$\begin{aligned}
L^J\langle\Psi,\Psi\rangle &= \int_{I_\epsilon^s} \left|\tilde\Psi_\rho(\rho)\right|^2 + f''\left(c_A^\epsilon\left(F_\epsilon^s(\rho),s\right)\right)\tilde\Psi(\rho)^2\,\mathrm{d}\rho \\
&\quad - \int_{I_\epsilon^s} \partial_\rho J^\epsilon(\rho,s)\,\Psi(\rho)\,\Psi_\rho(\rho) - \frac{1}{4}\frac{\left(\partial_\rho J^\epsilon(\rho,s)\,\Psi(\rho)\right)^2}{J^\epsilon(\rho,s)}\mathrm{d}\rho \\
&= \int_{I_\epsilon^s} \left|\tilde\Psi_\rho(\rho)\right|^2 + f''\left(c_A^\epsilon\left(F_\epsilon^s(\rho),s\right)\right)\tilde\Psi(\rho)^2\,\mathrm{d}\rho \\
&\quad - \int_{I_\epsilon^s} \frac{1}{2}\partial_\rho\left(\Psi(\rho)^2\right)\partial_\rho J^\epsilon(\rho,s) + \frac{1}{4}\frac{\left(\partial_\rho J^\epsilon(\rho,s)\,\Psi(\rho)\right)^2}{J^\epsilon(\rho,s)}\mathrm{d}\rho \\
&= \int_{I_\epsilon^s} \left|\tilde\Psi_\rho(\rho)\right|^2 + f''\left(c_A^\epsilon\left(F_\epsilon^s(\rho),s\right)\right)\tilde\Psi(\rho)^2\,\mathrm{d}\rho \\
&\quad + \int_{I_\epsilon^s} \frac{1}{4}\tilde\Psi(\rho)^2\left(\left(\frac{2\partial_\rho^2 J^\epsilon(\rho,s)}{J^\epsilon(\rho,s)}\right) - \frac{\left(\partial_\rho J^\epsilon(\rho,s)\right)^2}{J^\epsilon(\rho,s)^2}\right)\mathrm{d}\rho \\
&\quad - \frac{1}{2}\left[\frac{\tilde\Psi^2\partial_\rho J^\epsilon(.,s)}{J^\epsilon(.,s)}\right]_{-\frac{1}{\epsilon}-h^\epsilon(s)}^{\frac{1}{\epsilon}-h^\epsilon(s)},
\end{aligned}$$

where we used integration by parts in the last step. Thus,

$$\begin{aligned}
L^J\langle\Psi,\Psi\rangle &= L^0\left\langle\tilde\Psi,\tilde\Psi\right\rangle + \epsilon p^\epsilon(s)\int_{I_\epsilon^s} f^{(3)}\left(\theta_0(\rho)\right)\theta_1(\rho)\,\tilde\Psi(\rho)^2\,\mathrm{d}\rho \\
&\quad + \epsilon^2\int_{I_\epsilon^s}\tilde q^\epsilon(\rho,s)\,\tilde\Psi(\rho)^2\,\mathrm{d}\rho - \frac{\epsilon}{2}\left[\frac{\tilde\Psi(.,s)^2\,\partial_r J\left(F_\epsilon^s(.),s\right)}{J\left(F_\epsilon^s(.),s\right)}\right]_{-\frac{1}{\epsilon}-h^\epsilon(s)}^{\frac{1}{\epsilon}-h^\epsilon(s)} \qquad (3.41)
\end{aligned}$$

holds, with

$$\begin{aligned}
\tilde q^\epsilon(\rho,s) &:= \epsilon^{-2}\left(f''\left(c_A^\epsilon\left(F_\epsilon^s(\rho),s\right)\right) - f''\left(\theta_0(\rho)\right)\right) - \epsilon^{-1}f^{(3)}\left(\theta_0(\rho)\right)p^\epsilon(s)\,\theta_1(\rho) \\
&\quad + \frac{1}{4}\left(\frac{2\left(\partial_r^2 J\left(F_\epsilon^s(\rho),s\right)\right)}{J\left(F_\epsilon^s(\rho),s\right)} - \frac{\left(\partial_r J\left(F_\epsilon^s(\rho),s\right)\right)^2}{J\left(F_\epsilon^s(\rho),s\right)^2}\right).
\end{aligned}$$

Due to the structure of c_A^ϵ as noted in (3.4), the smoothness of f and Γ (and thus also of J), (3.6) and the boundedness of θ_0, θ_1, we find using a Taylor expansion

$$\begin{aligned}
\left|\tilde q^\epsilon(\rho,s)\right| &\le C + \epsilon^{-2}\left(\left|f^{(3)}\left(\theta_0(\rho)\right)\epsilon^2 q^\epsilon\left(s + F_\epsilon^s(\rho)\,\mathbf{n}_\Gamma(s)\right)\right|\right) \\
&\quad + \epsilon^{-2}\frac{1}{2}\max_{\tau\in[-1,1]}\left|f^{(4)}(\tau)\right|\left(\epsilon p^\epsilon(s)\,\theta_1(\rho) + \epsilon^2 q^\epsilon\left(s + F_\epsilon^s(\rho)\,\mathbf{n}_\Gamma(s)\right)\right)^2 \\
&\le C_1\left(\left|q^\epsilon\left(s + F_\epsilon^s(\rho)\,\mathbf{n}_\Gamma(s)\right)\right|\right) + C_2 \\
&\le C\left(1 + |\rho|\right). \qquad (3.42)
\end{aligned}$$

Here, C, C_1, $C_2 > 0$ are independent of both ϵ and s. Now we are able to find an upper bound for λ_1 : using (3.42) and the fact that we have $\left\|\beta J^{-\frac{1}{2}}\left(F_\epsilon^s(.), s\right)\theta_0'\right\|_J = \|\beta\theta_0'\| = 1$, we may employ (3.41) to calculate

$$\begin{aligned}\lambda_1(s) &\le L^J\left\langle\beta\left(J^\epsilon\right)^{-\frac{1}{2}}\theta_0', \beta\left(J^\epsilon\right)^{-\frac{1}{2}}\theta_0'\right\rangle \\ &\le \beta^2 L^0\left\langle\theta_0', \theta_0'\right\rangle + \beta\epsilon p^\epsilon(s)\int_{I_\epsilon^s} f^{(3)}(\theta_0)\,\theta_1\left(\theta_0'\right)^2 \mathrm{d}\rho + C\epsilon^2, \end{aligned}\tag{3.43}$$

where we used Proposition 3.6 in the first step and the exponential decay of θ_0 and its derivatives (cf. (2.1)) in the second. Now it holds

$$L^0\left(\theta_0', \theta_0'\right) = \left[\theta_0''\theta_0'\right]_{-\frac{1}{\epsilon}-h^\epsilon(s)}^{\frac{1}{\epsilon}-h^\epsilon(s)} \le C_1 e^{-C_2\frac{1}{\epsilon}}$$

for ϵ small enough and C_1, $C_2 > 0$ independent of ϵ and s, see also (3.28). Furthermore, (3.5) implies

$$\begin{aligned}\left|\beta\int_{I_\epsilon^s} f^{(3)}(\theta_0)\,\theta_1\left(\theta_0'\right)^2 \mathrm{d}\rho\right| &\le C\int_{\mathbb{R}\setminus I_\epsilon^s}\left|f''(\theta_0)\,\theta_1\right|\left(\theta_0'\right)^2 \mathrm{d}\rho \\ &\le C_1 e^{-C_2\frac{1}{\epsilon}}, \end{aligned}\tag{3.44}$$

where we used the uniform boundedness of β (for ϵ_1 small enough, cf. (3.29)) in the first inequality, as well as $\theta_1 \in L^\infty(\mathbb{R})$ and the decay of θ_0' in the second. Thus, we get the desired upper bound $\lambda_1(s) \le \mathcal{O}\left(\epsilon^2\right)$ from (3.43), as p^ϵ is uniformly bounded due to (3.6).

Next we will find a lower bound for λ_1. First off, we note that we may choose ϵ small enough to ensure $\lambda_1(s) \le \frac{m^2}{4}$. Corollary 3.7 thus implies that the normalized positive eigenfunction $\Psi_1(., s)$, $s \in \Gamma$, corresponding to λ_1 satisfies (3.22).

We define $\tilde{\Psi}_1 = \left(J^\epsilon\right)^{\frac{1}{2}}\Psi_1$ and use a decomposition similar to the one in the proof for Lemma 3.8 3). That is, we define $a(s) := \left\langle\tilde{\Psi}_1(., s), \Psi_1^0(., s)\right\rangle$ and $\left(\Psi_1^0\right)^\perp(\rho, s) := \tilde{\Psi}_1(\rho, s) - a(s)\,\Psi_1^0(\rho, s)$, where we again have $\Psi_1^0 \perp \left(\Psi_1^0\right)^\perp$ and $a(s) > 0$ as both eigenfunctions and J^ϵ are positive. The main idea now is to use (3.41) to establish a connection between λ_1 and λ_1^0 and showing that the perturbations of L^0 are only of order ϵ^2.

Using $\Psi_1^0(\rho, s) = \beta\theta_0'(\rho) + \mathcal{R}(\rho, s)$, where $\|\mathcal{R}\| \le C_1 e^{-C_2\frac{1}{\epsilon}}$ (cf. Lemma 3.8 3)), we may calculate

$$\begin{aligned}\left|\int_{I_\epsilon^s} f^{(3)}(\theta_0(\rho))\,\theta_1(\rho)\left(\tilde{\Psi}_1(\rho, s)\right)^2 \mathrm{d}\rho\right| \le{}& \left|a(s)^2\int_{I_\epsilon^s} f^{(3)}(\theta_0(\rho))\,\theta_1(\rho)\left(\Psi_1^0(\rho, s)\right)^2 \mathrm{d}\rho\right| \\ &+ \left|\int_{I_\epsilon^s} f^{(3)}(\theta_0(\rho))\,\theta_1(\rho)\left(\left(\Psi_1^0\right)^\perp(\rho, s)\right)^2 \mathrm{d}\rho\right| \\ &+ \left|2a(s)\int_{I_\epsilon^s} f^{(3)}(\theta_0(\rho))\,\theta_1(\rho)\,\Psi_1^0(\rho, s)\left(\Psi_1^0\right)^\perp(\rho, s)\,\mathrm{d}\rho\right|\end{aligned}$$

$$
\begin{aligned}
&\leq C_1 \left(\beta^2 \left| \int_{I_\epsilon^s} f^{(3)}(\theta_0)\, \theta_1 \left(\theta_0'\right)^2 \mathrm{d}\rho \right| + e^{-C_2 \frac{1}{\epsilon}} \right) \\
&\quad + C_2 \left(\left\| \left(\Psi_1^0\right)^\perp \right\|^2 + \left\|\Psi_1^0\right\| \left\| \left(\Psi_1^0\right)^\perp \right\| \right) \\
&\leq C_1 \left(e^{-C_2 \frac{1}{\epsilon}} + \left\| \left(\Psi_1^0\right)^\perp \right\|^2 + \left\| \left(\Psi_1^0\right)^\perp \right\| \right), \qquad (3.45)
\end{aligned}
$$

where we used Hölder's inequality and (3.44) in the third line and $\left\|\Psi_1^0\right\| = 1$ in the last. We also employed the uniform boundedness of $a(s)$, which follows from

$$
a(s)^2 = 1 - \left\| \left(\Psi_1^0\right)^\perp \right\|^2 \leq 1.
$$

Now we calculate a lower bound for λ_1: using (3.41) we get

$$
\begin{aligned}
\lambda_1(s) &= L^J \langle \Psi_1, \Psi_1 \rangle \\
&= L^0 \left\langle \tilde{\Psi}_1, \tilde{\Psi}_1 \right\rangle + \epsilon p^\epsilon(s) \int_{I_\epsilon^s} f^{(3)}(\theta_0(\rho))\, \theta_1(\rho)\, \tilde{\Psi}_1(\rho, s)^2 \mathrm{d}\rho + \epsilon^2 \int_{I_\epsilon^s} \tilde{q}^\epsilon(\rho, s)\, \tilde{\Psi}_1(\rho, s)^2 \mathrm{d}\rho \\
&\quad - \frac{\epsilon}{2} \left[\frac{\tilde{\Psi}_1(., s)^2\, \partial_r J\left(F_\epsilon^s(.), s\right)}{J\left(F_\epsilon^s(.), s\right)} \right]_{-\frac{1}{\epsilon} - h^\epsilon}^{\frac{1}{\epsilon} - h^\epsilon} \\
&\geq L^0 \left\langle \tilde{\Psi}_1, \tilde{\Psi}_1 \right\rangle - C_1 \epsilon \left(e^{-C_2 \frac{1}{\epsilon}} + \left\| \left(\Psi_1^0\right)^\perp \right\|^2 + \left\| \left(\Psi_1^0\right)^\perp \right\| \right) \\
&\quad - C_3 \epsilon^2 \int_{I_\epsilon^s} (1 + |\rho|)\, \tilde{\Psi}_1(\rho, s)^2 \mathrm{d}\rho \\
&\geq a(s)^2 L^0 \left\langle \Psi_1^0, \Psi_1^0 \right\rangle + L^0 \left\langle \left(\Psi_1^0\right)^\perp, \left(\Psi_1^0\right)^\perp \right\rangle - C_1 \epsilon \left(\left\| \left(\Psi_1^0\right)^\perp \right\|^2 + \left\| \left(\Psi_1^0\right)^\perp \right\| \right) \\
&\quad - C_2 \epsilon^2. \qquad (3.46)
\end{aligned}
$$

Here we used (3.45) and (3.6) in the third step and the exponential decay of Ψ_1 in the third and the fourth step. Furthermore, we used an analogon to (3.38) for $L^0 \left\langle \tilde{\Psi}_1, \tilde{\Psi}_1 \right\rangle$ in the last inequality, which may again be justified by the circumstance that Ψ_1^0 is an Eigenfunction of $\mathcal{L}_0$ and $\Psi_1^0 \perp \left(\Psi_1^0\right)^\perp$ holds.

We can further the above estimate by computing

$$
\begin{aligned}
\lambda_1(s) &\geq a(s)^2 \lambda_1^0 + \lambda_2^0 \left\| \left(\Psi_1^0\right)^\perp \right\|^2 - C_1 \epsilon \left(\left\| \left(\Psi_1^0\right)^\perp \right\|^2 + \left\| \left(\Psi_1^0\right)^\perp \right\| \right) - C_2 \epsilon^2 \\
&\geq -C_1 e^{-C_2 \frac{1}{\epsilon}} + \left\| \left(\Psi_1^0\right)^\perp \right\|^2 (C_3 - \epsilon C_4) - C_5 \epsilon \left\| \left(\Psi_1^0\right)^\perp \right\| - C_6 \epsilon^2 \\
&\geq C_1 \left\| \left(\Psi_1^0\right)^\perp \right\|^2 - C_2 \epsilon \left\| \left(\Psi_1^0\right)^\perp \right\| - C_3 \epsilon^2 \\
&\geq C_1 \left\| \left(\Psi_1^0\right)^\perp \right\|^2 - C_2 \epsilon^2
\end{aligned}
$$

for ϵ_1 – and thus ϵ – sufficiently small. Here we used the definition of λ_1^0, λ_2^0 in the first inequality, $a(s)^2 \leq 1$, (3.25), and (3.26) in the second inequality and the smallness of ϵ in the third. The last estimate holds due to Young's inequality.

The upper bound on λ_1 now implies

$$\left\| \left(\Psi_1^0\right)^{\perp} \right\| \leq C\epsilon, \tag{3.47}$$

leading to

$$\left|\lambda_1(s)\right| \leq C\epsilon^2 \text{ and } a(s)^2 = 1 - \left\| \left(\Psi_1^0\right)^{\perp} \right\|^2 \geq 1 - C\epsilon^2 \tag{3.48}$$

and in particular

$$a(s) \geq \left(1 - C\epsilon^2\right)^{\frac{1}{2}} \geq 1 - C\epsilon^2. \tag{3.49}$$

Thus, we have finished our estimate of λ_1. It remains to show (3.40).

In order to do that, we remark that (3.46) also implies $L^0 \left\langle \left(\Psi_1^0\right)^{\perp}, \left(\Psi_1^0\right)^{\perp} \right\rangle \in \mathcal{O}\left(\epsilon^2\right)$. In conjunction with the definition of L^0 and the estimate for $\left\| \left(\Psi_1^0\right)^{\perp} \right\|$, this leads to

$$\left\| \partial_\rho \left(\Psi_1^0\right)^{\perp} \right\| \in \mathcal{O}(\epsilon). \tag{3.50}$$

Combining $\tilde{\Psi}_1 = a\Psi_1^0 + \left(\Psi_1^0\right)^{\perp}$ and $\Psi_1^0 = \beta\theta_0' + \mathcal{R}$, we find

$$\begin{aligned} \Psi_1^{\mathbf{R}} &= \left(J^\epsilon\right)^{-\frac{1}{2}} \left(a\left(\beta\theta_0' + \mathcal{R}\right) + \left(\Psi_1^0\right)^{\perp}\right) - \beta\theta_0' \\ &= \left(J^\epsilon\right)^{-\frac{1}{2}} \left(\beta\theta_0' \left(a - \left(J^\epsilon\right)^{\frac{1}{2}}\right) + a\mathcal{R} + \left(\Psi_1^0\right)^{\perp}\right). \end{aligned} \tag{3.51}$$

As $1 - C\epsilon^2 < a \leq 1$ and $\left(J^\epsilon\right)^{\frac{1}{2}} \geq 1 - \mathcal{O}\left(\epsilon\left|\rho + h^\epsilon(s)\right|\right)$ due to Lemma 3.3 we have

$$a - \left(J^\epsilon\right)^{\frac{1}{2}} = \mathcal{O}\left(\epsilon\left|\rho + h^\epsilon(s)\right|\right) \ \forall \rho \in I_\epsilon^s, s \in \Gamma.$$

Thus, we have

$$\begin{aligned} \left\|\Psi_1^{\mathbf{R}}(., s)\right\|_J^2 &= \int_{I_\epsilon^s} \left(\left(J^\epsilon\right)^{-\frac{1}{2}} \left(\beta\theta_0' \left(a - \left(J^\epsilon\right)^{\frac{1}{2}}\right) + a\mathcal{R} + \left(\Psi_1^0\right)^{\perp}\right)\right)^2 J^\epsilon \mathrm{d}\rho \\ &\leq C \left(\epsilon^2 \int_{I_\epsilon^s} \left(\beta\theta_0'\left(\rho + h^\epsilon(s)\right)\right)^2 \mathrm{d}\rho + \|\mathcal{R}\|^2 + \left\| \left(\Psi_1^0\right)^{\perp} \right\|^2\right) \\ &\leq C\epsilon^2. \end{aligned} \tag{3.52}$$

Here we used (3.51) in the first step and (3.27), (3.47), as well as the decay of θ_0' in the last step.

To estimate the derivative we calculate

$$\begin{aligned} \partial_\rho \Psi_1^{\mathbf{R}} = &\left(J^\epsilon\right)^{-\frac{1}{2}} \left(\beta\theta_0'' \left(a - \left(J^\epsilon\right)^{\frac{1}{2}}\right) + a\partial_\rho \mathcal{R} + \partial_\rho \left(\Psi_1^0\right)^{\perp} - \frac{\beta}{2}\theta_0' \left(J^\epsilon\right)^{-\frac{1}{2}} \partial_\rho J^\epsilon\right) \\ &- \frac{1}{2} \left(J^\epsilon\right)^{-1} \partial_\rho J^\epsilon \Psi_1^{\mathbf{R}}. \end{aligned}$$

Due to Lemma 3.3 we have $\partial_\rho J^\epsilon = \mathcal{O}(\epsilon)$ and $C \geq J^\epsilon(\rho, s) \geq c$ for all $\rho \in I_\epsilon^s$, $s \in \Gamma$. Thus, we can estimate $\partial_\rho \Psi_1^{\mathbf{R}}$ in a similar fashion as in (3.52) using (3.50) and the estimate for $\partial_\rho \mathcal{R}$ in (3.27). This proves the claim.

Ad 2) and Ad 3): As there arise no new difficulties, we refer to [24], pp. 1381 for the proofs. □

3.2. Useful Decompositions

As $\{\Psi_i\}_{i\in\mathbb{N}}$ form a complete orthonormal system of $L^2\left(I_\epsilon^s, \langle .,.\rangle_J\right)$, we can write every $\Phi \in L^2\left(I_\epsilon^s\right)$ as $\Phi = \langle \Phi, \Psi_1\rangle_J \Psi_1 + \Psi^{\mathbf{R}}$ with $\Psi^{\mathbf{R}} \perp_J \Psi_1$. If we can additionally control $L^J\langle\Phi,\Phi\rangle$, we will also be able to control $L^J\left\langle \Psi^{\mathbf{R}}, \Psi^{\mathbf{R}}\right\rangle$ as $L^J\langle\Psi_1,\Psi_1\rangle = \lambda_1 \in \mathcal{O}\left(\epsilon^2\right)$ by (3.39). Note that this is essentially how we proved (3.40), where we used Ψ_1^0 and $\langle .,.\rangle$ instead of Ψ_1 and $\langle .,.\rangle_J$. These ideas are the cornerstones of the following lemma.

Lemma 3.10. *Let Assumption 3.1 hold and let $\epsilon_1 > 0$ be given as in Lemma 3.9. Let $\psi \in H^1(\Gamma(1))$ and $\Lambda_\epsilon \in \mathbb{R}$ such that*

$$\|\psi\|_{L^2(\Gamma(1))} = 1 \text{ and } \int_{\Gamma(1)} \epsilon\,|\nabla\psi|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)\psi^2 \mathrm{d}x \le \Lambda_\epsilon. \tag{3.53}$$

Then there exist constants $C, C_1, C_2 > 0$ and functions $Z \in H^1(\Gamma)$ and $\psi^{\mathbf{R}} \in H^1(\Gamma(1))$ such that

$$\psi(r,s) = \epsilon^{-\frac{1}{2}} Z(s)\,\Psi_1\left(\frac{r}{\epsilon} - h^\epsilon(s), s\right) + \psi^{\mathbf{R}}(r,s) \tag{3.54}$$

for almost every $(r,s) \in (-1,1)\times\Gamma$, satisfying

$$\left\|\psi^{\mathbf{R}}\right\|^2_{L^2(\Gamma(1))} \le C\left(\epsilon\Lambda_\epsilon + \epsilon^2\right) \tag{3.55}$$

and

$$\|Z\|^2_{H^1(\Gamma)} + \|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} + \left\|\psi^{\mathbf{R}}\right\|^2_{H^1(\Gamma(1))} \le C\left(1 + \frac{\Lambda_\epsilon}{\epsilon}\right). \tag{3.56}$$

Proof. We set

$$\begin{aligned} \psi_1(r,s) &:= \epsilon^{-\frac{1}{2}}\Psi_1\left(\frac{r}{\epsilon} - h^\epsilon(s), s\right), \\ Z(s) &:= (\psi_1, \psi)_J \end{aligned}$$

and define

$$\psi^{\mathbf{R}}(r,s) := \psi(r,s) - Z(s)\,\psi_1(r,s) \tag{3.57}$$

for almost all $(r,s) \in (-1,1)\times\Gamma$. Then the identity (3.54) is fulfilled and for $s \in \Gamma$ we compute

$$\left(\psi_1, \psi^{\mathbf{R}}\right)_J = (\psi_1,\psi)_J - Z(s)\,|\psi_1|_J^2 = 0 \tag{3.58}$$

due to the definition of Z, Proposition 3.5 and $|\psi_1|_J = \|\Psi_1\|_J = 1$. Thus, $\psi_1 \perp_J \psi^{\mathbf{R}}$ for all $s \in \Gamma$ and

$$\begin{aligned} 1 &= \int_\Gamma (\psi,\psi)_J \,\mathrm{d}\mathcal{H}^{n-1}(s) \\ &= \int_\Gamma \left|\psi^{\mathbf{R}}\right|_J^2 + Z^2(s)\,\mathrm{d}\mathcal{H}^{n-1}(s). \end{aligned}$$

Furthermore, we have

$$L^J(\psi(.,s), \psi(.,s)) = L^J\left(\psi^{\mathbf{R}}(.,s), \psi^{\mathbf{R}}(.,s)\right) + Z^2(s)\,L^J(\psi_1(.,s), \psi_1(.,s)). \tag{3.59}$$

We may deduce this by calculating

$$\begin{aligned}
L^J(\psi(.,s),\psi(.,s)) &= \int_{I_1} \left(\epsilon {\psi_r}^2 + \epsilon^{-1} f''(c_A^\epsilon)\psi^2\right) J\mathrm{d}r \\
&= L^J\left(\psi^{\mathbf{R}}(.,s),\psi^{\mathbf{R}}(.,s)\right) + Z^2(s) L^J(\psi_1(.,s),\psi_1(.,s)) \\
&\quad + \frac{2}{\epsilon} Z(s) \int_{I_\epsilon^s} \left(\partial_\rho \Psi^{\mathbf{R}} \partial_\rho \Psi_1 + f''(c_A^\epsilon)\Psi^{\mathbf{R}}\Psi_1\right) J^\epsilon \mathrm{d}\rho, \qquad (3.60)
\end{aligned}$$

where we define $\Psi^{\mathbf{R}}(\rho,s) := \sqrt{\epsilon}\psi^{\mathbf{R}}(\epsilon(\rho + h^\epsilon(s)), s)$. Due to (3.58) and Proposition 3.5 we have $\Psi^{\mathbf{R}} \perp_J \Psi_1$ and as Ψ_1 is an eigenfunction of $\mathcal{L}_J$ fulfilling a Neumann boundary condition we get

$$\begin{aligned}
\int_{I_\epsilon^s} \partial_\rho \Psi^{\mathbf{R}} \partial_\rho \Psi_1 J^\epsilon \mathrm{d}\rho &= -\int_{I_\epsilon^s} \Psi^{\mathbf{R}} \partial_\rho\left(J^\epsilon \partial_\rho \Psi_1\right)\mathrm{d}\rho \\
&= \int_{I_\epsilon^s} \lambda_1(s)\Psi^{\mathbf{R}}\Psi_1 J^\epsilon - f''(c_A^\epsilon)\Psi^{\mathbf{R}}\Psi_1 J^\epsilon \mathrm{d}\rho \\
&= -\int_{I_\epsilon^s} f''(c_A^\epsilon)\Psi^{\mathbf{R}}\Psi_1 J^\epsilon \mathrm{d}\rho,
\end{aligned}$$

where we used integration by parts in the first equality. Plugging this result into (3.60) immediately leads to (3.59).

In order to prove (3.55), we now utilize Lemma 3.9. Using the assumption (3.53) and the decomposition of the gradient as discussed in (3.3) we find

$$\begin{aligned}
\Lambda_\epsilon &\geq \int_{\Gamma(1)} \epsilon|\nabla\psi|^2 + \epsilon^{-1} f''(c_A^\epsilon)\psi^2 \mathrm{d}x \\
&= \int_\Gamma L^J(\psi,\psi)\,\mathrm{d}\mathcal{H}^{n-1}(s) + \epsilon \int_{\Gamma(1)} |\nabla_\tau \psi|^2 \mathrm{d}x \\
&\geq \int_\Gamma L^J\left(\psi^{\mathbf{R}},\psi^{\mathbf{R}}\right) + Z^2(s) L^J(\psi_1,\psi_1)\,\mathrm{d}\mathcal{H}^{n-1}(s) + \epsilon\|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} \\
&\geq \epsilon^{-1}\left(\int_\Gamma L^J\left\langle\Psi^{\mathbf{R}},\Psi^{\mathbf{R}}\right\rangle + Z(s)^2 L^J\langle\Psi_1,\Psi_1\rangle\,\mathrm{d}\mathcal{H}^{n-1}(s)\right) + \epsilon\|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} \\
&\geq \epsilon^{-1}\int_\Gamma C\left|\psi^{\mathbf{R}}\right|_J^2 + Z^2(s)\lambda_1(s)\,\mathrm{d}\mathcal{H}^{n-1}(s) + \epsilon\|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} \\
&\geq \epsilon^{-1} C_1 \left\|\psi^{\mathbf{R}}\right\|^2_{L^2(\Gamma(1))} + \epsilon\|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} - \epsilon C_2, \qquad (3.61)
\end{aligned}$$

where we used (3.59) in the third line and Lemma 3.9 in the fifth line. Moreover, we used

$$\int_\Gamma Z^2(s)\,\mathrm{d}\mathcal{H}^{n-1}(s) = 1 - \left\|\psi^{\mathbf{R}}\right\|^2_{L^2(\Gamma(1))} \leq 1 \qquad (3.62)$$

in the last inequality. This leads to

$$\epsilon^{-1}\left\|\psi^{\mathbf{R}}\right\|_{L^2(\Gamma(1))}^2+\epsilon\left\|\nabla_\tau\psi\right\|_{L^2(\Gamma(1))}^2\leq C\left(\Lambda_\epsilon+\epsilon\right) \tag{3.63}$$

and in particular proves (3.55) as well as the estimate for $\nabla_\tau\psi$ in (3.56).

Next, we prove the estimate for $\|Z\|_{H^1(\Gamma)}^2$ in (3.56): The L^2-norm of Z has already been taken care of in (3.62) leaving us with the task of estimating the derivative. We proceed by first estimating

$$\begin{aligned}
|\nabla_\tau\psi_1|_J^2&=\int_{I_1}\left|\nabla_\tau\epsilon^{-\frac{1}{2}}\Psi_1\left(\frac{r}{\epsilon}-h^\epsilon(s),s\right)\right|^2 J(r,s)\,\mathrm{d}r\\
&=\int_{I_1}\epsilon^{-1}\left|-\partial_\rho\Psi_1\left(\frac{r}{\epsilon}-h^\epsilon(s),s\right)\nabla_\tau h^\epsilon(s)+\nabla_\tau\Psi_1\left(\frac{r}{\epsilon}-h^\epsilon(s),s\right)\right|^2 J(r,s)\,\mathrm{d}r\\
&\leq C\int_{I_\epsilon^s}\left(|\partial_\rho\Psi_1(\rho,s)\nabla_\tau h^\epsilon(s)|^2+|\nabla_\tau\Psi_1(\rho,s)|^2\right)J^\epsilon(\rho,s)\,\mathrm{d}\rho\\
&\leq C_1\int_{I_\epsilon^s}(\partial_\rho\Psi_1(\rho,s))^2 J^\epsilon(\rho,s)\,\mathrm{d}\rho+C_2\left(\epsilon+\|\nabla_\tau c_A^\epsilon\|_{L^\infty(\Gamma(1))}\right)\\
&\leq C\left(\|\partial_\rho\Psi_1\|_J^2+1\right),
\end{aligned} \tag{3.64}$$

where we used Lemma 3.9 3) and assumption (3.7) in the fourth line and assumption (3.9) in the last. Using Lemma 3.9 1) we furthermore find

$$\|\partial_\rho\Psi_1\|_J^2+\int_{I_\epsilon^s}f''\left(c_A^\epsilon\left(F_\epsilon^s(\rho),s\right)\right)\Psi_1(\rho,s)^2 J^\epsilon(\rho,s)\,\mathrm{d}\rho\leq C\epsilon^2$$

leading to

$$\|\partial_\rho\Psi_1\|_J^2\leq\max_{x\in\Gamma(1)}\left|f''\left(c_A^\epsilon(x)\right)\right|\|\Psi_1\|_J^2+C\epsilon^2\leq C$$

as c_A^ϵ is uniformly bounded by our assumptions and $\|\Psi_1\|_J^2=1$. Plugging this into (3.64) leads to

$$\sup_{s\in\Gamma}|\nabla_\tau\psi_1|_J^2\leq C \tag{3.65}$$

By the definition of Z we have

$$\begin{aligned}
&\int_\Gamma|\nabla_\tau Z(s)|^2\,\mathrm{d}\mathcal{H}^{n-1}(s)\\
&\quad\leq\int_\Gamma\left((|\nabla_\tau\psi|,|\psi_1|)_J+(|\psi|,|\nabla_\tau\psi_1|)_J+\left(|\psi|,\left|\psi_1(\nabla_\tau J)J^{-1}\right|\right)_J\right)^2\mathrm{d}\mathcal{H}^{n-1}(s)\\
&\quad\leq\|\nabla_\tau\psi\|_{L^2(\Gamma(1))}^2+\int_\Gamma\left(|\psi|_J\left(\sup_{s\in\Gamma}|\nabla_\tau\psi_1|_J+\left|\psi_1(\nabla_\tau J)J^{-1}\right|_J\right)\right)^2\mathrm{d}\mathcal{H}^{n-1}(s)\\
&\quad\leq\|\nabla_\tau\psi\|_{L^2(\Gamma(1))}^2+C_1\int_\Gamma|\psi|_J^2\int_{I_\epsilon^s}\Psi_1(\rho,s)^2\left|\nabla_\tau J\left(\epsilon\left(\rho+h^\epsilon(s)\right),s\right)\right|^2\mathrm{d}\rho\mathrm{d}\mathcal{H}^{n-1}(s)\\
&\qquad+C_2\|\psi\|_{L^2(\Gamma(1))}^2,
\end{aligned} \tag{3.66}$$

where we used $|\psi_1|_J = 1$ in the first step and (3.65) in the last. Since

$$|\Psi_1(\rho,s)| \leq Ce^{-\frac{m}{2}|\rho|} \text{ for all } \rho \in \left[\pm(c_2+1), \pm\epsilon^{-1} - h^\epsilon(s)\right]$$

for all $\epsilon \in (0,\epsilon_1)$ by Proposition 3.7 we can use Lemma 3.3 to get

$$\int_{I_\epsilon^s} \Psi_1(\rho,s)^2 |\nabla_\tau J(\epsilon(\rho + h^\epsilon(s)), s)|^2 \,\mathrm{d}\rho \leq C$$

for some $C > 0$ independent of ϵ and s. Thus, (3.66) implies

$$\begin{aligned}\int_\Gamma |\nabla_\tau Z(s)|^2 \,\mathrm{d}\mathcal{H}^{n-1}(s) &\leq \|\nabla_\tau \psi\|^2_{L^2(\Gamma(1))} + C \\ &\leq C\left(\epsilon^{-1}\Lambda_\epsilon + 1\right), \qquad (3.67)\end{aligned}$$

as $\|\psi\|_{L^2(\Gamma(1))} = 1$, where we used (3.63) in the the second inequality.

Since the estimates for $\|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))}$ and $\left\|\psi^{\mathbf{R}}\right\|_{L^2(\Gamma(1))}$ have already been shown in (3.63), we only need to consider $\nabla\psi^{\mathbf{R}} = \partial_r\psi^{\mathbf{R}}\mathbf{n}_\Gamma + \nabla_\tau\psi^{\mathbf{R}}$. We note that we may use the estimates in (3.61) and the previously found inequalities to show

$$\int_\Gamma L^J\left(\psi^{\mathbf{R}}, \psi^{\mathbf{R}}\right) \mathrm{d}\mathcal{H}^{n-1}(s) \leq \Lambda_\epsilon + C\epsilon$$

thus leading to

$$\begin{aligned}\epsilon\left\|\partial_r\psi^{\mathbf{R}}\right\|^2_{L^2(\Gamma(1))} &\leq \Lambda_\epsilon + C\epsilon + c_2 \max_{x\in\Gamma(1)}\left|f''(c_A^\epsilon(x))\right|\epsilon^{-1}\int_\Gamma \left|\psi^{\mathbf{R}}\right|_J^2 \mathrm{d}\mathcal{H}^{n-1}(s) \\ &\leq C(\Lambda_\epsilon + \epsilon)\end{aligned}$$

by (3.63) and the uniform boundedness of c_A^ϵ.

To estimate the surface gradient, we observe that the definition of $\psi^{\mathbf{R}}$ implies

$$\nabla_\tau\psi^{\mathbf{R}}(r,s) = \nabla_\tau\psi(r,s) - \nabla_\tau Z(s)\,\psi_1(r,s) - Z(s)\,\nabla_\tau\psi_1(r,s)$$

for almost all $(r,s) \in (-1,1)\times\Gamma$. Thus, we have

$$\begin{aligned}\left\|\nabla_\tau\psi^{\mathbf{R}}\right\|^2_{L^2(\Gamma(1))} &\leq C_1\left(\|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} + \int_\Gamma |\nabla_\tau Z(s)|^2 \int_{I_1} \psi_1(r,s)^2 J(r,s)\,\mathrm{d}r\mathrm{d}s\right) \\ &\quad + C_2 \sup_{s\in\Gamma} |\nabla_\tau\psi_1|_J^2 \|Z\|^2_{H^1(\Gamma)} \\ &\leq C\left(\epsilon^{-1}\Lambda_\epsilon + 1\right)\end{aligned}$$

due to the estimate (3.63), the estimate on $\|Z\|_{H^1(\Gamma)}$ and the estimate on $\nabla_\tau\psi_1$ in (3.65), and the identity $|\psi_1(.,s)|_J^2 = 1$. This proves the assertion. □

A similar result also holds if ψ is not normalized. This will in fact be essential later on, as it yields important structural information on the difference between the exact solution c^ϵ (of (1.18)–(1.25)) and its approximation. The most important applications of the following corollary can be found in Lemma 5.29 and in Theorem 6.12 (in particular in the auxiliary results Proposition 6.8 and Lemma 6.9).

Corollary 3.11. *Let Assumptions 3.1 hold and let $\epsilon_1 > 0$ be given as in Lemma 3.9. Let $\psi \in H^1(\Gamma(1))$ and $\Lambda_\epsilon \in \mathbb{R}$ such that*

$$\int_{\Gamma(1)} \epsilon |\nabla\psi|^2 + \epsilon^{-1} f''(c_A^\epsilon)\psi^2 \mathrm{d}x \leq \Lambda_\epsilon.$$

Then there exist functions $Z \in H^1(\Gamma)$ and $\psi^{\mathbf{R}} \in H^1(\Gamma(1))$ such that

$$\psi(r,s) = \epsilon^{-\frac{1}{2}} Z(s)\Psi_1\left(\frac{r}{\epsilon} - h^\epsilon(s), s\right) + \psi^{\mathbf{R}}(r,s) \tag{3.68}$$

for almost every $(r,s) \in (-1,1) \times \Gamma$, satisfying

$$\left\|\psi^{\mathbf{R}}\right\|^2_{L^2(\Gamma(1))} \leq C\left(\epsilon\Lambda_\epsilon + \epsilon^2 \|\psi\|^2_{L^2(\Gamma(1))}\right) \tag{3.69}$$

and

$$\|Z\|^2_{H^1(\Gamma)} + \|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} + \left\|\psi^{\mathbf{R}}\right\|^2_{H^1(\Gamma(1))} \leq C\left(\|\psi\|^2_{L^2(\Gamma(1))} + \frac{\Lambda_\epsilon}{\epsilon}\right). \tag{3.70}$$

Proof. We define

$$\tilde{\psi} := \frac{\psi}{\|\psi\|_{L^2(\Gamma(1))}}.$$

Then we have

$$\int_{\Gamma(1)} \epsilon\left|\nabla\tilde{\psi}\right|^2 + \epsilon^{-1} f''(c_A^\epsilon)\tilde{\psi}^2 \mathrm{d}x \leq \frac{\Lambda_\epsilon}{\|\psi\|^2_{L^2(\Gamma(1))}}$$

and may use Lemma 3.10 to obtain the existence of some functions $\tilde{Z} \in H^1(\Gamma)$ and $\tilde{\psi}^{\mathbf{R}} \in H^1(\Gamma(1))$ such that

$$\tilde{\psi}(r,s) = \epsilon^{-\frac{1}{2}}\tilde{Z}(s)\Psi_1\left(\frac{r}{\epsilon} - h^\epsilon(s), s\right) + \tilde{\psi}^{\mathbf{R}}(r,s) \tag{3.71}$$

with

$$\left\|\tilde{Z}\right\|^2_{H^1(\Gamma)} + \left\|\nabla_\tau\tilde{\psi}\right\|^2_{L^2(\Gamma(1))} + \left\|\tilde{\psi}^{\mathbf{R}}\right\|^2_{H^1(\Gamma(1))} \leq C\left(1 + \frac{\Lambda_\epsilon}{\epsilon\|\psi\|^2_{L^2(\Gamma(1))}}\right) \tag{3.72}$$

and

$$\left\|\tilde{\psi}^{\mathbf{R}}\right\|^2_{L^2(\Gamma(1))} \leq C\left(\epsilon\frac{\Lambda_\epsilon}{\|\psi\|^2_{L^2(\Gamma(1))}} + \epsilon^2\right). \tag{3.73}$$

Furthermore, if we define $\psi_1(r,s) := \epsilon^{-\frac{1}{2}}\Psi_1\left(\frac{r}{\epsilon} - h^\epsilon(s), s\right)$, $Z(s) := (\psi_1, \psi)_J$ and $\psi^{\mathbf{R}}(r,s) := \psi(r,s) - Z(s)\psi_1(r,s)$ we have the identities

$$Z(s) = (\psi_1, \psi)_J = \left(\psi_1, \tilde{\psi}\|\psi\|_{L^2(\Gamma(1))}\right) = \tilde{Z}(s)\|\psi\|_{L^2(\Gamma(1))}$$

and

$$\psi^{\mathbf{R}}(r,s) = \tilde{\psi}(r,s)\|\psi\|_{L^2(\Gamma(1))} + \tilde{Z}(s)\|\psi\|_{L^2(\Gamma(1))}\psi_1(r,s) = \tilde{\psi}^{\mathbf{R}}(r,s)\|\psi\|_{L^2(\Gamma(1))}$$

for almost all $(r,s) \in (-1,1)\times\Gamma$. Thus, it holds (3.68) as a result of (3.71) and the inequalities (3.70) and (3.69) follow after multiplying (3.72) and (3.73) by $\|\psi\|^2_{L^2(\Gamma(1))}$. □

3.3. The Spectral Estimate

Now we show the main theorem of this section, a spectral estimate for the Cahn-Hilliard operator. The proof is based on [6], Theorem 2.13 and [24], Theorem 2.3. This spectral estimate builds the foundation of the proof of Theorem 4.1, which is the main result of this work. In the following we consider $H_0^1(\Omega)$ equipped with the scalar product $(u,v)_1 = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$. The induced norm $|.|_1$ is equivalent to the usual H^1-norm by Poincaré's inequality. Moreover, we set

$$H^{-1} := \left(H_0^1(\Omega)\right)'.$$

Theorem 3.12 (Spectral Estimate)**.** *Let Assumption 3.1 hold. There exist constants $C_1 > 0$, $C_2 \geq 0$ and $\epsilon_1 > 0$ such that for all $\psi \in H_0^1(\Omega)$ it holds*

$$\begin{aligned}\int_\Omega \epsilon |\nabla\psi|^2 + \epsilon^{-1} f''(c_A^\epsilon)\psi^2 \mathrm{d}x \geq\; & C_1\left(\epsilon \|\psi\|_{L^2(\Omega)}^2 + \epsilon^{-1}\|\psi\|_{L^2(\Omega\setminus\Gamma(1))} + \epsilon\|\nabla_\tau\psi\|_{L^2(\Gamma(1))}^2\right) \\ & + C_1\left(\epsilon^3\|\nabla\psi\|_{L^2(\Omega)}^2 + \epsilon\|\nabla\psi\|_{L^2(\Omega\setminus\Gamma(1))}^2\right) \\ & - C_2\|\psi\|_{H^{-1}(\Omega)}^2 . \end{aligned} \tag{3.74}$$

for all $\epsilon \in (0, \epsilon_1)$.

Proof. Let $\psi \in H_0^1(\Omega)$. First of all we will show that there exist $C_1, C_2 > 0$ such that

$$\begin{aligned}\int_\Omega \epsilon|\nabla\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\psi^2\mathrm{d}x \geq & \int_{\Omega\setminus\Gamma(1)} \epsilon|\nabla\psi|^2 + C_1\epsilon^{-1}|\psi|^2\mathrm{d}x \\ & + \epsilon\int_{\Gamma(1)} |\nabla_\tau\psi|^2\mathrm{d}x - C_2\epsilon\|\psi\|_{L^2(\Omega)}^2 \end{aligned} \tag{3.75}$$

holds for ϵ small enough.

Due to (3.10) we may estimate

$$\begin{aligned}\int_\Omega \epsilon|\nabla\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\psi^2\mathrm{d}x \geq & \int_{\Omega\setminus\Gamma(1)} \epsilon|\nabla\psi|^2 + C\epsilon^{-1}|\psi|^2\mathrm{d}x \\ & + \int_\Gamma\int_{I_1}\left(\epsilon|\psi_r|^2 + \epsilon|\nabla_\tau\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\psi^2\right)J\mathrm{d}r\mathrm{d}s \\ \geq & \int_{\Omega\setminus\Gamma(1)} \epsilon|\nabla\psi|^2 + C\epsilon^{-1}|\psi|^2\mathrm{d}x + \epsilon\int_{\Gamma(1)}|\nabla_\tau\psi|^2\mathrm{d}x \\ & + \int_\Gamma L^J(\psi,\psi)\mathrm{d}s. \end{aligned}$$

Setting $\Psi(\rho, s) = \sqrt{\epsilon}\psi(\epsilon(\rho + h^\epsilon(s)), s)$ we may estimate

$$\begin{aligned}\int\limits_\Gamma L^J(\psi,\psi)\,\mathrm{d}s &= \epsilon^{-1}\int\limits_\Gamma L^J(\Psi,\Psi)\,\mathrm{d}s \\ &\geq \epsilon^{-1}\int\limits_\Gamma \lambda_1(s)\,\|\Psi\|_J^2\,\mathrm{d}s \\ &\geq -C\epsilon\int\limits_\Omega \psi^2\mathrm{d}x,\end{aligned}$$

where we used Proposition 3.5 1) in the first line, Proposition 3.6 3) in the second line and Lemma 3.9 1) together with Proposition 3.5 2) in the last line. This proves (3.75).

We observe that we may now use (3.75) to derive

$$\begin{aligned}\int\limits_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\,\psi^2\mathrm{d}x \geq{}& C_1\left(\epsilon\,\|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} + \epsilon^{-1}\,\|\psi\|^2_{L^2(\Omega\backslash\Gamma(1))} + \epsilon\,\|\nabla\psi\|^2_{L^2(\Omega\backslash\Gamma(1))}\right) \\ &+ C_1\epsilon^3\,\|\nabla\psi\|^2_{L^2(\Omega)} - C_2\epsilon\,\|\psi\|^2_{L^2(\Omega)} \qquad (3.76)\end{aligned}$$

for $C_1, C_2 > 0$ and all $\epsilon \in (0, \epsilon_1)$, after choosing ϵ_1 so small that $\epsilon_1 \leq \frac{1}{2}$ is fulfilled. The inequality (3.76) follows from the estimate

$$\begin{aligned}\int\limits_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\,\psi^2\mathrm{d}x ={}& \left(1-\epsilon^2\right)\int\limits_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\,\psi^2\mathrm{d}x \\ &+ \epsilon^3\int\limits_\Omega |\nabla\psi|^2\,\mathrm{d}x + \epsilon\int\limits_\Omega f''(c_A^\epsilon)\,\psi^2\mathrm{d}x \\ \geq{}& C_1\left(\epsilon\,\|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} + \epsilon^{-1}\,\|\psi\|_{L^2(\Omega\backslash\Gamma(1))} + \epsilon\,\|\nabla\psi\|^2_{L^2(\Omega\backslash\Gamma(1))}\right) \\ &+ \epsilon^3\,\|\nabla\psi\|^2_{L^2(\Omega)} - C_2\epsilon\,\|\psi\|^2_{L^2(\Omega)}\,.\end{aligned}$$

Now, in order to prove (3.74) we fix a constant $c > C_2$ and $\epsilon \in (0, \epsilon_0)$ and consider two different cases:

First, we assume

$$\int\limits_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\,\psi^2\mathrm{d}x > c\epsilon\,\|\psi\|^2_{L^2(\Omega)}\,.$$

This leads to the claim immediately, since adding that inequality to (3.76) gives

$$\begin{aligned}\int\limits_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\,\psi^2\mathrm{d}x \geq{}& C\left(\epsilon\,\|\psi\|^2_{L^2(\Omega)} + \epsilon\,\|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))} + \epsilon\,\|\nabla\psi\|^2_{L^2(\Omega\backslash\Gamma(1))}\right) \\ &+ C\epsilon^3\,\|\nabla\psi\|^2_{L^2(\Omega)}\end{aligned}$$

for $C > 0$, which is equivalent to (3.74) with $C_2 = 0$.

In the opposite case, that is, if

$$\int\limits_\Omega \epsilon\,|\nabla\psi|^2 + \epsilon^{-1}f''(c_A^\epsilon)\,\psi^2\mathrm{d}x \leq c\epsilon\,\|\psi\|^2_{L^2(\Omega)}$$

holds, we have to invest a little more work. Let $w \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique solution to

$$\begin{aligned} -\Delta w &= \psi && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then [24], Theorem 3.1, implies

$$\tilde{C}\epsilon \|\psi\|^2_{L^2(\Omega)} \leq \|\nabla w\|^2_{L^2(\Omega)} . \tag{3.77}$$

Moreover,

$$\|\psi\|^2_{H^{-1}(\Omega)} = \|\nabla w\|^2_{L^2(\Omega)} \tag{3.78}$$

which follows from

$$\left| \int_\Omega \psi\eta \mathrm{d}x \right| = \left| \int_\Omega \nabla w \cdot \nabla\eta \mathrm{d}x \right| \leq \|\nabla w\|_{L^2(\Omega)} \|\nabla\eta\|_{L^2(\Omega)} \tag{3.79}$$

for all $\eta \in H_0^1(\Omega)$, $\eta \neq 0$, where equality is realized for $\eta = w$. Thus, we get by (3.77) and (3.78)

$$-\epsilon \|\psi\|^2_{L^2(\Omega)} \geq \epsilon \|\psi\|^2_{L^2(\Omega)} - C \|\psi\|^2_{H^{-1}(\Omega)} ,$$

which yields

$$\begin{aligned} \int_\Omega \epsilon |\nabla\psi|^2 + \epsilon^{-1} f'(c_A^\epsilon)\psi^2 \mathrm{d}x \geq{}& C\left(\epsilon \|\psi\|^2_{L^2(\Omega)} + \epsilon^{-1} \|\psi\|_{L^2(\Omega\setminus\Gamma(1))} + \epsilon \|\nabla_\tau\psi\|^2_{L^2(\Gamma(1))}\right) \\ &+ C\left(\epsilon \|\nabla\psi\|^2_{L^2(\Omega\setminus\Gamma(1))} + \epsilon^3 \|\nabla\psi\|^2_{L^2(\Omega)}\right) - \tilde{C} \|\psi\|^2_{H^{-1}(\Omega)} \end{aligned}$$

when plugged into (3.76). This proves the assertion. □

4. The Main Result

The following result is the central theorem of this thesis.

Theorem 4.1 (The Main Result)**.** *Let Assumption 1.1 hold, let f satisfy Assumption 1.2 and let ξ be a cut-off function for $\delta > 0$ as in Definition 2.1. Let moreover $\gamma(x) := \xi(4d_{\mathbf{B}}(x))$ for all $x \in \Omega$ and let for $\epsilon \in (0,1)$ a smooth function $\psi_0^\epsilon : \Omega \to \mathbb{R}$ be given, which satisfies $\|\psi_0^\epsilon\|_{C^1(\Omega)} \leq C_{\psi_0}\epsilon^M$ for some $C_{\psi_0} > 0$ independent of ϵ.*

Then there are smooth functions $c_A^\epsilon : \Omega \times [0,T_0] \to \mathbb{R}, \mathbf{v}_A^\epsilon : \Omega \times [0,T_0] \to \mathbb{R}^2$ for $\epsilon \in (0,1)$ such that the following holds:

If $(\mathbf{v}^\epsilon, p^\epsilon, c^\epsilon, \mu^\epsilon)$ are smooth solutions to (1.18)–(1.25) with initial value

$$c_0^\epsilon(x) = c_A^\epsilon(x,0) + \psi_0^\epsilon(x) \tag{4.1}$$

for all $x \in \Omega$, then there are some $\epsilon_0 \in (0,1]$, $K > 0$, $T \in (0,T_0]$ such that

$$\|c^\epsilon - c_A^\epsilon\|_{L^2(0,T;L^2(\Omega))} + \left\|\nabla^\Gamma(c^\epsilon - c_A^\epsilon)\right\|_{L^2(0,T;L^2(\Gamma_t(\delta)))} \leq K\epsilon^{M-\frac{1}{2}}, \tag{4.2a}$$

$$\epsilon\|\nabla(c^\epsilon - c_A^\epsilon)\|_{L^2(0,T;L^2(\Omega\setminus\Gamma_t(\delta)))} + \|c^\epsilon - c_A^\epsilon\|_{L^2(0,T;L^2(\Omega\setminus\Gamma_t(\delta)))} \leq K\epsilon^{M+\frac{1}{2}}, \tag{4.2b}$$

$$\epsilon^{\frac{3}{2}}\|\partial_{\mathbf{n}}(c^\epsilon - c_A^\epsilon)\|_{L^2(0,T;L^2(\Gamma_t(\delta)))} + \|c^\epsilon - c_A^\epsilon\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq K\epsilon^M, \tag{4.2c}$$

$$\int_{\Omega_T} \epsilon|\nabla(c^\epsilon - c_A^\epsilon)|^2 + \epsilon^{-1}f''(c_A^\epsilon)(c^\epsilon - c_A^\epsilon)^2 \,\mathrm{d}(x,t) \leq K^2\epsilon^{2M}, \tag{4.2d}$$

$$\|\gamma(c^\epsilon - c_A^\epsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \epsilon^{\frac{1}{2}}\|\gamma\Delta(c^\epsilon - c_A^\epsilon)\|_{L^2(\Omega_T)} \leq K\epsilon^{M-\frac{1}{2}}, \tag{4.2e}$$

$$\|\gamma\nabla(c^\epsilon - c_A^\epsilon)\|_{L^2(\Omega_T)} + \|\gamma(c^\epsilon - c_A^\epsilon)\nabla(c^\epsilon - c_A^\epsilon)\|_{L^2(\Omega_T)} \leq K\epsilon^M, \tag{4.2f}$$

and for $q \in (1,2)$

$$\|\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon\|_{L^1(0,T;L^q(\Omega))} \leq C(K,q)\epsilon^{M-\frac{1}{2}}, \tag{4.3}$$

hold for all $\epsilon \in (0,\epsilon_0)$ and some $C(K,q) > 0$. Moreover, we have

$$\lim_{\epsilon\to 0} c_A^\epsilon = \pm 1 \text{ in } L^\infty\left(\Omega_T'\right) \tag{4.4}$$

and

$$\lim_{\epsilon\to 0} \mathbf{v}_A^\epsilon = \mathbf{v}^\pm \text{ in } L^6\left(0,T;H^2\left(\Omega'(t)\right)^2\right) \tag{4.5}$$

where $\Omega_T' \subset\subset \Omega_T^\pm$ and $\Omega_T' = \cup_{t\in[0,T]}\Omega'(t) \times \{t\}$.

Throughout this work we will often consider the following assumptions. They allow us to separately consider many steps of the proof of Theorem 4.1 such that the final proof in Chapter 7 can be kept rather short.

Assumption 4.2. *Let $\gamma(x) := \xi(4d_{\mathbf{B}}(x))$ for all $x \in \Omega$. We assume that $c_A : \Omega\times[0,T_0] \to \mathbb{R}$ is a smooth function and that there are $\epsilon_0 \in (0,1)$, $K \geq 1$ and a family $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset (0,T_0]$ such that the following holds: if c^ϵ is given as in Theorem 4.1 with $c_0^\epsilon(x) = c_A(x,0)$, then it holds for $R := c^\epsilon - c_A^\epsilon$*

$$\|R\|_{L^2(\Omega_{T_\epsilon})} + \left\|\nabla^\Gamma R\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))} + \left\|\left(\frac{1}{\epsilon}R, \nabla R\right)\right\|_{L^2(0,T_\epsilon;L^2(\Omega\backslash\Gamma_t(\delta)))} \leq K\epsilon^{M-\frac{1}{2}}, \quad (4.6a)$$

$$\epsilon^{\frac{3}{2}} \left\|\partial_{\mathbf{n}} R\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))} + \|R\|_{L^\infty(0,T_\epsilon;H^{-1}(\Omega))} \leq K\epsilon^M, \quad (4.6b)$$

$$\int\limits_{\Omega_{T_\epsilon}} \epsilon\left|\nabla R\right|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right)(R)^2 \,\mathrm{d}\left(x,t\right) \leq K^2\epsilon^{2M}, \quad (4.6c)$$

$$\epsilon^{\frac{1}{2}} \|\gamma R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} + \|(\epsilon\gamma\Delta R, \gamma\nabla R, \gamma R\left(\nabla R\right))\|_{L^2(\Omega_{T_\epsilon})} \leq K\epsilon^M \quad (4.6d)$$

for all $\epsilon \in (0, \epsilon_0)$.

A major part of this work lies in the construction of suitable approximate solutions and showing that they solve the system (1.18)–(1.25) up to a sufficiently high order.

Theorem 4.3. *Let Assumption 1.1 be satisfied. Then for every* $\epsilon \in (0,1)$ *there are*

$$\mathbf{v}_A^\epsilon, \mathbf{w}_1^\epsilon : \Omega_{T_0} \to \mathbb{R}^2,\ c_A^\epsilon, \mu_A^\epsilon, p_A^\epsilon : \Omega_{T_0} \to \mathbb{R}$$

and

$$\mathbf{r}_{\mathrm{S}}^\epsilon : \Omega_{T_0} \to \mathbb{R}^2,\ r_{\mathrm{div}}^\epsilon, r_{\mathrm{CH1}}^\epsilon, r_{\mathrm{CH2}}^\epsilon : \Omega_{T_0} \to \mathbb{R}$$

such that

$$-\Delta\mathbf{v}_A^\epsilon + \nabla p_A^\epsilon = \mu_A^\epsilon \nabla c_A^\epsilon + \mathbf{r}_{\mathrm{S}}^\epsilon, \quad (4.7)$$

$$\mathrm{div}\mathbf{v}_A^\epsilon = r_{\mathrm{div}}^\epsilon, \quad (4.8)$$

$$\partial_t c_A^\epsilon + \left(\mathbf{v}_A^\epsilon + \epsilon^{M-\frac{1}{2}}\ \mathbf{w}_1^\epsilon|_\Gamma\, \xi\left(d_\Gamma\right)\right)\cdot\nabla c_A^\epsilon = \Delta\mu_A^\epsilon + r_{\mathrm{CH1}}^\epsilon, \quad (4.9)$$

$$\mu_A^\epsilon = -\epsilon\Delta c_A^\epsilon + \epsilon^{-1} f'\left(c_A^\epsilon\right) + r_{\mathrm{CH2}}^\epsilon, \quad (4.10)$$

in Ω_{T_0}. *Furthermore, the boundary conditions*

$$c_A^\epsilon = -1,\ \mu_A^\epsilon = 0,\ \left(-2D_s\mathbf{v}_A^\epsilon + p_A^\epsilon\mathbf{I}\right)\mathbf{n}_{\partial\Omega} = \alpha_0\mathbf{v}_A^\epsilon \quad (4.11)$$

are satisfied on $\partial_{T_0}\Omega$. *If additionally Assumption 4.2 holds for* $\epsilon_0 \in (0,1)$, $K \geq 1$ *and a family* $\left(T_\epsilon\right)_{\epsilon\in(0,\epsilon_0)} \subset (0, T_0]$, *then there are some* $\epsilon_1 \in (0,\epsilon_0]$, $C\left(K\right) > 0$ *depending on* K *and* $C_K : (0,T_0]\times(0,1] \to (0,\infty)$ *(also dependent on* K*), which satisfies* $C_K\left(T,\epsilon\right) \to 0$ *as* $(T,\epsilon) \to 0$, *such that*

$$\int\limits_0^{T_\epsilon}\left|\int\limits_\Omega r_{\mathrm{CH1}}^\epsilon\left(x,t\right)\varphi\left(x,t\right)\mathrm{d}x\right|\mathrm{d}t \leq C_K\left(T_\epsilon,\epsilon\right)\epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))},$$

$$\int\limits_0^{T_\epsilon}\left|\int\limits_\Omega r_{\mathrm{CH2}}^\epsilon\left(x,t\right)\left(c^\epsilon\left(x,t\right) - c_A^\epsilon\left(x,t\right)\right)\mathrm{d}x\right|\mathrm{d}t \leq C_K\left(T_\epsilon,\epsilon\right)\epsilon^{2M},$$

$$\|\mathbf{r}_{\mathrm{S}}^\epsilon\|_{L^2\left(0,T_\epsilon;(H^1(\Omega))'\right)} + \|r_{\mathrm{div}}^\epsilon\|_{L^2(\Omega_{T_\epsilon})} \leq C\left(K\right)\epsilon^M,$$

for all $\epsilon \in (0,\epsilon_1)$ *and* $\varphi \in L^\infty\left(0,T_\epsilon;H^1\left(\Omega\right)\right)$.

To gain an idea of the proof of Theorem 4.1, consider the following heuristics: assume that c_A^ϵ, μ_A^ϵ, p_A^ϵ, $\mathbf{v}_A^\epsilon$ are given, such that (4.7)–(4.10) hold, where $\mathbf{w}_1^\epsilon \equiv 0$ for now. While these functions may be good approximate solutions to (1.18)–(1.25) in the sense that $\mathbf{r}_{\mathrm{S}}^\epsilon$, $r_{\mathrm{div}}^\epsilon$, $r_{\mathrm{CH1}}^\epsilon$ and $r_{\mathrm{CH2}}^\epsilon$ are "small", in order to establish estimates of the kind (4.2), we need to connect the approximations to the real solutions c^ϵ, μ^ϵ, p^ϵ and $\mathbf{v}^\epsilon$. At the core of this endeavor lies Theorem 3.12: Denoting $R := c^\epsilon - c_A^\epsilon$, we test the difference between (4.7)–(4.10) and (1.18)–(1.21) by $\varphi := -\Delta_D^{-1} R$, where Δ_D is the Dirichlet Laplacian, which results due to (3.74) and the Gronwall inequality in an estimate similar to

$$\|R\|_{L^\infty(0,T;H^1(\Omega))} + \epsilon \|R\|_{L^2(\Omega_T)} \leq \int_0^T \left| \int_\Omega r_{\mathrm{CH1}}^\epsilon \varphi + r_{\mathrm{CH2}}^\epsilon R + \mathbf{v}^\epsilon \cdot \nabla c^\epsilon - \mathbf{v}_A^\epsilon \cdot \nabla c_A^\epsilon \mathrm{d}x \right| \mathrm{d}t.$$

Of course, in the actual proof in Section 7.2 the estimate will involve more different norms of R and be significantly more complex, but this simplified presentation allows for two major insights: first, it is imperative to construct approximate solutions in a way that enables us to establish suitable estimates for $r_{\mathrm{CH1}}^\epsilon$ and $r_{\mathrm{CH2}}^\epsilon$, as these correspond to the convergence rates we aim to find. For this, we make use of the method of matched asymptotic expansions, devising an inductive scheme for the construction of terms of arbitrarily high order. Second, we need to control the difference $\mathbf{v}^\epsilon \cdot \nabla c^\epsilon - \mathbf{v}_A^\epsilon \cdot \nabla c_A^\epsilon$, the appearance of which is a consequence of the convection term in (1.20). To gain said control, we need to invest a lot of technical work and introduce new ideas, which are mainly based on the procedures in [6]. A cornerstone to that approach is the introduction of an exact solution $\overline{\mathbf{v}^\epsilon}$ to equation (1.18), where the right hand side is substituted by the approximation $\mu_A^\epsilon \nabla c_A^\epsilon$. With the help of the results for the stationary Stokes equation shown in Section 2.2, we will be able to find estimates for both $\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}$ and $\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon$.

These basic ideas dictate the structure of the following work: Chapters 5 and 6 are concerned with the construction of approximate solutions and the derivation of suitable estimates for the terms $\mathbf{r}_{\mathrm{S}}^\epsilon$, $r_{\mathrm{div}}^\epsilon$, $r_{\mathrm{CH1}}^\epsilon$ and $r_{\mathrm{CH2}}^\epsilon$. These considerations are made significantly more difficult due to the introduction of $\mathbf{w}_1^\epsilon$ in (4.9), which can be interpreted as the leading term of the error $\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}$. Nevertheless, this insertion is of utmost importance in the proof of Theorem 4.1. In Chapter 7, we first delve deeper into analyzing the errors $\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}$ and $\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon$ in Subsection 7.1.1 before putting all the pieces together in Section 7.2.

4.1. An Energy Estimate

In the following we will derive an energy estimate for (1.18)–(1.25). We consider for $\epsilon > 0$ the Ginzburg Landau energy

$$E^\epsilon(c^\epsilon)(t) = \frac{\epsilon}{2} \int_\Omega |\nabla c^\epsilon(x,t)|^2 \mathrm{d}x + \frac{1}{\epsilon} \int_\Omega f(c^\epsilon(x,t)) \mathrm{d}x \text{ for } t \in [0, T_0]. \tag{4.12}$$

Moreover, we assume that there exist $\epsilon_0 > 0$ and a constant $C_0 > 0$ independent of ϵ, such that

$$E^\epsilon(c_0^\epsilon) \leq C_0 \tag{4.13}$$

and

$$\|c_0^\epsilon\|_{L^\infty(\Omega)} \leq C_0 \tag{4.14}$$

for all $\epsilon \in (0, \epsilon_0)$. We will show in Proposition 7.2 that these conditions hold for the initial condition (4.1) and c_A^ϵ as in Theorem 4.3.

Lemma 4.4. *Let* $(c^\epsilon, \mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon)$ *be a classical solution to (1.18)–(1.25) and let* $\epsilon_0 > 0$ *and* $C_0 > 0$ *be given such that (4.13) and (4.14) hold. Then there is some* $\epsilon_1 \in (0, \epsilon_0)$ *and some constant* $C > 0$ *, depending only on* T_0, C_0 *and* ϵ_0*, such that*

$$\epsilon^7 \|\Delta c^\epsilon\|^2_{L^2(\Omega_t)} + \epsilon \sup_{\tau\in[0,t]} \|\nabla c^\epsilon(.,\tau)\|^2_{L^2(\Omega)} + \|(\nabla\mu^\epsilon, \nabla\mathbf{v}^\epsilon)\|^2_{L^2(\Omega_t)} + \alpha_0 \|\mathbf{v}^\epsilon\|^2_{L^2(\partial_t\Omega)} \leq C$$

for all $t \in [0, T_0]$ *and* $\epsilon \in (0, \epsilon_1)$.

Proof. Let $t \in [0, T_0]$. By the fundamental theorem of calculus we get

$$\begin{aligned} E(c^\epsilon)(t) - E(c^\epsilon)(0) &= \int_0^t \int_\Omega \epsilon \nabla c^\epsilon \cdot \nabla \partial_t c^\epsilon + \frac{1}{\epsilon} f'(c^\epsilon)\, \partial_t c^\epsilon \mathrm{d}x\mathrm{d}t \\ &= \int_0^t \int_\Omega -\epsilon \Delta c^\epsilon \partial_t c^\epsilon + \frac{1}{\epsilon} f'(c^\epsilon)\, \partial_t c^\epsilon \mathrm{d}x\mathrm{d}t \\ &= \int_0^t \int_\Omega \mu^\epsilon (\Delta\mu^\epsilon - \mathbf{v}^\epsilon \cdot \nabla c^\epsilon)\, \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.15}$$

where we used integration by parts in the second equality and (1.20), (1.21) in the third. There do not appear boundary terms since we assume Dirichlet boundary conditions for μ^ϵ and c^ϵ (cf. (1.24)–(1.25)).

Moreover, we have

$$\begin{aligned} \int_0^t \int_\Omega \mu^\epsilon (\mathbf{v}^\epsilon \cdot \nabla c^\epsilon)\, \mathrm{d}x\mathrm{d}t &= \int_0^t \int_\Omega \mathbf{v}^\epsilon (-\Delta \mathbf{v}^\epsilon + \nabla p^\epsilon)\, \mathrm{d}x\mathrm{d}t \\ &= \int_0^t \int_\Omega |\nabla \mathbf{v}^\epsilon|^2 \mathrm{d}x\mathrm{d}t + \int_0^t \int_{\partial\Omega} \alpha_0 |\mathbf{v}^\epsilon|^2 \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t, \end{aligned} \tag{4.16}$$

where we again used integration by parts in the last equality, together with the boundary condition (1.23) and $\mathrm{div}\mathbf{v}^\epsilon = 0$ in Ω.

Together we get by (4.15), (4.16), integration by parts and the dirichlet boundary condition satisfied by μ^ϵ

$$E(c^\epsilon)(t) + \int_0^t \int_\Omega |\nabla\mu^\epsilon|^2 + |\nabla \mathbf{v}^\epsilon|^2 \mathrm{d}x\mathrm{d}t + \int_0^t \int_{\partial\Omega} \alpha_0 |\mathbf{v}^\epsilon|^2 \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t = E(c_0^\epsilon),$$

which implies the claimed estimate (without the $\|\Delta c^\epsilon\|_{L^2(\Omega_t)}$ term) due to the positivity of f.

Considering the Dirichlet boundary condition of μ^ϵ we get

$$\begin{aligned}
\|\Delta c^\epsilon\|_{L^2(\Omega_t)} &\leq \frac{1}{\epsilon}\|\mu^\epsilon\|_{L^2(\Omega_t)} + \frac{1}{\epsilon^2}\left\|f'(c^\epsilon)\right\|_{L^2(\Omega_t)} \\
&\leq \frac{1}{\epsilon}C\|\nabla\mu^\epsilon\|_{L^2(\Omega_t)} + \frac{1}{\epsilon^2}\left\|f'(c^\epsilon)\right\|_{L^2(\Omega_t)} \\
&\leq \frac{C}{\epsilon^2}\left(1+\|c^\epsilon\|^3_{L^6(\Omega_t)}\right) \\
&\leq \frac{C}{\epsilon^2}\left(1+\|\nabla c^\epsilon\|^3_{L^\infty(0,t;L^2(\Omega))}\right) \\
&\leq \frac{C}{\epsilon^2}\left(1+\epsilon^{-\frac{3}{2}}\right)
\end{aligned}$$

for ϵ small enough, where we used Poincaré's inequality in the second inequality, and the fact that f is a polynomial of fourth order by Assumption 1.2 in the third inequality. The fourth inequality is due to $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and again Poincaré's inequality (applicable since c^ϵ satisfies Dirichlet boundary conditions) and the last inequality is justified by the energy estimates shown before. □

5. Construction of Approximate Solutions

In the following we use the method of matched asymptotic expansions to construct approximate solutions $(c_A^\epsilon, \mu_A^\epsilon, \mathbf{v}_A^\epsilon, p_A^\epsilon)$ of (1.18)–(1.25). Throughout this chapter the formalism "≈" will represent a formal asymptotic expansion ansatz, that is, writing $u^\epsilon \approx \sum_{k\geq 0} \epsilon^k u_k$ means that for every integer $K \in \mathbb{N}$ we have

$$u^\epsilon = \sum_{k=0}^{K} \epsilon^k u_k + \tilde{u}_{K+1}\epsilon^{K+1}, \tag{5.1}$$

where $\tilde{u}_{K+1}$ is uniformly bounded in ϵ.

Whenever the method of asymptotic expansions and asymptotic matching is used in the literature, two different ways of presenting these techniques and the corresponding findings are frequently encountered:

The first is to just present the results, that is, give some rather explicit form of the expansion terms involved and show that everything works out fine for this choice of terms. For a reader experienced in the field of asymptotic expansions, this yields the most useful information in as little space as possible. The caveat is that for the inexperienced reader, the formulae for the approximate solutions seem to appear by "magic", with no hint as how to apply the technique to a different set of problems.

The second approach is to write down the whole process of asymptotic matching, analyzing the different orders in detail and discussing the compatibility conditions and steps for solving(which also has to be done for the first way at some point). This tends to be cumbersome and technical, but also more educational than the first approach.

Thus, in this work we decide to stick to the latter path and show the reader all necessary involved steps. The logic of the construction of approximate solutions can be laid out as follows:

First, we assume that the solutions $(c^\epsilon, \mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon)$ of (1.18)–(1.25) are of a form similar to (5.1) for some $K \in \mathbb{N}$ (depending on the needed order of approximation). Then we derive necessary partial differential equations and ordinary differential equations which have to hold under these assumptions and also conditions which have to be satisfied for the differential equations in order to be solvable. These steps will be performed in the *inner, boundary* and *outer region*, i.e. close to the interface Γ, close to the outer boundary $\partial\Omega$ (both in stretched coordinate systems) and in the intermediate domain. The details are given in Subsections 5.1.1, 5.1.2, 5.1.3, and 5.1.4.

Subsequently, we drop the assumption of existence of an expansion and explicitly construct solutions to the before derived differential equations, which is done in Subsections 5.1.5 and 5.1.6, leading to Lemmata 5.19 and 5.22.

This chapter consists of three parts: First, we construct $M+1$ terms of the approximate solutions, where the analysis (regarding the Cahn-Hilliard part at least) is based on [14], [47] and [26]. The inductive construction of the terms for the stationary Stokes part incorporates ideas from [6]. In the second section, which is based on [6], we introduce a function $\tilde{\mathbf{w}}_1^\epsilon$ which will turn out to be the leading term in the error $\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon$. This term is both central to proving Theorem 4.1 and to showing suitable error estimates for the velocity. As $\tilde{\mathbf{w}}_1^\epsilon$ can be considered to be a term of order $M - \frac{1}{2}$ appearing in the asymptotic expansion, the third part of this chapter deals with constructing terms of precisely that order. An essential part of that construction is Theorem 5.32, which is also based on [6].

The work in this chapter is done under Assumption 1.1.

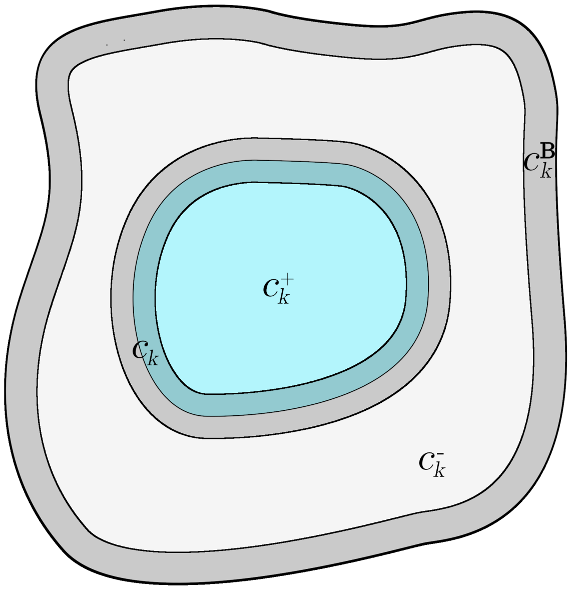

Figure 5.1.: The different zones of the asymptotic expansion.

5.1. The First $M+1$ Terms

5.1.1. The Outer Expansion

We assume that in $\Omega_{T_0}^\pm \backslash \Gamma(2\delta)$ the solutions of (1.18)–(1.25) have the expansions

$$\begin{aligned} c^\epsilon(x,t) &\approx c_0^\pm(x,t) + \epsilon c_1^\pm(x,t) + \epsilon^2 c_2^\pm(x,t) + \dots, \\ \mu^\epsilon(x,t) &\approx \mu_0^\pm(x,t) + \epsilon \mu_1^\pm(x,t) + \epsilon^2 \mu_2^\pm(x,t) + \dots, \\ \mathbf{v}^\epsilon(x,t) &\approx \mathbf{v}_0^\pm(x,t) + \epsilon \mathbf{v}_1^\pm(x,t) + \epsilon^2 \mathbf{v}_2^\pm(x,t) + \dots, \\ p^\epsilon(x,t) &\approx p_0^\pm(x,t) + \epsilon p_1^\pm(x,t) + \epsilon^2 p_2^\pm(x,t) + \dots, \end{aligned} \tag{5.2}$$

where $c_k^\pm$, $\mu_k^\pm$, $\mathbf{v}_k^\pm$ and $p_k^\pm$ are smooth functions defined in $\Omega_{T_0}^\pm$. Plugging this ansatz into (1.18), (1.19), (1.20) and (1.21) yields

$$-\sum_{k\geq 0} \epsilon^k \Delta \mathbf{v}_k^\pm + \sum_{k\geq 0} \epsilon^k \nabla p_k^\pm = \sum_{k,j\geq 0} \epsilon^k \mu_k^\pm \epsilon^j \nabla c_j^\pm \tag{5.3}$$

$$\sum_{k\geq 0} \epsilon^k \operatorname{div} \mathbf{v}_k^\pm = 0 \tag{5.4}$$

$$\sum_{k\geq 0} \epsilon^k \partial_t c_k^\pm + \left(\sum_{k\geq 0} \epsilon^k \mathbf{v}_k^\pm\right) \cdot \left(\sum_{k\geq 0} \epsilon^k \nabla c_k^\pm\right) = \left(\sum_{k\geq 0} \epsilon^k \Delta \mu_k^\pm\right) \tag{5.5}$$

and

$$\sum_{k\geq 0}\epsilon^k\mu_k^\pm = -\epsilon\sum_{k\geq 0}\epsilon^k\Delta c_k^\pm + \frac{1}{\epsilon}f'\left(\sum_{k\geq 0}\epsilon^k c_k^\pm\right) \tag{5.6}$$

for all $(x,t)\in\Omega_{T_0}^\pm$. Using a Taylor expansion we get

$$\begin{aligned} f'\left(\sum_{k\geq 0}\epsilon^k c_k^\pm\right) &= f'\left(c_0^\pm\right) + \epsilon f''\left(c_0^\pm\right)\sum_{k\geq 1}\epsilon^{k-1}c_k^\pm + \sum_{j\geq 2}\epsilon^j\frac{f^{(j+1)}\left(c_0^\pm\right)}{j!}\left(\sum_{k\geq 1}\epsilon^{k-1}c_k^\pm\right)^j \\ &= f'\left(c_0^\pm\right) + \epsilon f''\left(c_0^\pm\right)\sum_{k\geq 1}\epsilon^{k-1}c_k^\pm + \epsilon^2\sum_{k\geq 1}\epsilon^{k-1}f_k\left(c_0^\pm,\dots,c_k^\pm\right), \end{aligned} \tag{5.7}$$

where for fixed $c_0^\pm$ the functions f_k are polynomial in $\left(c_1^\pm,\dots,c_k^\pm\right)$. Moreover, $f_k\left(c_0^\pm,\dots,c_k^\pm\right)$ are chosen such that they do not depend on ϵ. Plugging this into (5.6), we get

$$\begin{aligned} \sum_{k\geq 0}\epsilon^k\mu_k^\pm = -\epsilon\sum_{k\geq 0}\epsilon^k\Delta c_k^\pm + \frac{1}{\epsilon}f'\left(c_0^\pm\right) + f''\left(c_0^\pm\right)\sum_{k\geq 1}\epsilon^{k-1}c_k^\pm \\ + \epsilon\sum_{k\geq 1}\epsilon^{k-1}f_k\left(c_0^\pm,\dots,c_k^\pm\right) \end{aligned} \tag{5.8}$$

and matching the $\mathcal{O}\left(\epsilon^{-1}\right)$ terms yields $f'\left(c_0^\pm\right)=0$. In view of the Dirichlet boundary data for c^ϵ we set

$$c_0^\pm = \pm 1. \tag{5.9}$$

This simplifies all equations that involve derivatives of $c_0^\pm$, e.g. (5.5) becomes

$$\sum_{k\geq 1}\epsilon^k\partial_t c_k^\pm + \left(\sum_{k\geq 0}\epsilon^k\mathbf{v}_k^\pm\right)\cdot\left(\sum_{k\geq 1}\epsilon^k\nabla c_k^\pm\right) = \left(\sum_{k\geq 0}\epsilon^k\Delta\mu_k^\pm\right), \tag{5.10}$$

thus we get by matching the $\mathcal{O}(1)$ terms

$$\Delta\mu_0^\pm = 0 \text{ in } \Omega_{T_0}^\pm. \tag{5.11}$$

Doing the same in (5.3) and (5.4) we find

$$\begin{aligned} -\Delta\mathbf{v}_0^\pm + \nabla p_0^\pm &= 0 && \text{in } \Omega_{T_0}^\pm, \\ \operatorname{div}\mathbf{v}_0^\pm &= 0 && \text{in } \Omega_{T_0}^\pm. \end{aligned} \tag{5.12}$$

Comparing the higher order terms $\mathcal{O}\left(\epsilon^k\right)$, where $k\geq 1$, yields:

1. For $c_k^\pm$ the identity

$$c_k^\pm = \frac{\mu_{k-1}^\pm + \Delta c_{k-2}^\pm - f_{k-1}\left(c_0^\pm,\dots,c_{k-1}^\pm\right)}{f''(\pm 1)} \text{ in } \Omega_{T_0}^\pm \tag{5.13}$$

   by rewriting (5.8), where we set $c_{-1}^\pm = f_0\left(c_0^\pm\right) = 0$.

2. For $\mu_k^\pm$

$$\Delta\mu_k^\pm = \partial_t c_k^\pm + \sum_{j=0}^{k}\mathbf{v}_j^\pm\cdot\nabla c_{k-j}^\pm \text{ in } \Omega_{T_0}^\pm \tag{5.14}$$

   by (5.10) (where we include the vanishing $\mathbf{v}_k^\pm\cdot\nabla c_0^\pm$ term just for notational coherency with later considerations).

3. For $\mathbf{v}_k^\pm$ and $p_k^\pm$

$$\begin{aligned} -\Delta \mathbf{v}_k^\pm + \nabla p_k^\pm &= \sum_{j=0}^{k-1} \mu_j^\pm \nabla c_{k-j}^\pm && \text{in } \Omega_{T_0}^\pm \\ \operatorname{div} \mathbf{v}_k^\pm &= 0 && \text{in } \Omega_{T_0}^\pm \end{aligned} \tag{5.15}$$

by (5.3) and (5.4).

Remark 5.1. Concerning the outer expansion, we remark the following:

1. As we will only construct $c_0^\pm, \ldots, c_{M+1}^\pm$, we need to consider the remainder of the Taylor expansion of f'. In this case, we choose to expand f' up to order $M+2$ and get

$$\begin{aligned} f'\left(\sum_{k=0}^{M+1} \epsilon^k c_k^\pm\right) = {} & f'\left(c_0^\pm\right) + \epsilon f''\left(c_0^\pm\right) \sum_{k=1}^{M+1} \epsilon^{k-1} c_k^\pm + \epsilon^2 \sum_{k=1}^{M} \epsilon^{k-1} f_k\left(c_0^\pm, \ldots, c_k^\pm\right) \\ & + \epsilon^{M+2} \tilde{f}_\epsilon\left(c_0^\pm, \ldots, c_{M+1}^\pm\right). \end{aligned}$$

Here $\tilde{f}_\epsilon\left(c_0^\pm, \ldots, c_{M+1}^\pm\right)$ consists of polynomials in $\left(c_1^\pm, \ldots, c_{M+1}^\pm\right)$, which may be of even higher order in ϵ and which are either multiplied by $f^{(j)}\left(c_0^\pm\right)$ for $j \in \{2, \ldots, M+1\}$ or by $f^{(M+2)}\left(\xi\left(c_0^\pm, \ldots, c_{M+1}^\pm\right)\right)$ for suitable $\xi \in \left[c_0^\pm, \sum_{k=0}^{M+1} \epsilon^k c_k^\pm\right]$. If $c_k^\pm \in L^\infty\left(\Omega_{T_0}^\pm\right)$ for all $k \in \{0, \ldots, M+1\}$, there is a constant $C > 0$ independent of ϵ, such that for all $\epsilon \in (0,1)$

$$\left\| f^{(M+2)}\left(\xi\left(c_0^\pm, \ldots, c_{M+1}^\pm\right)\right)\right\|_{L^\infty\left(\Omega_{T_0}^\pm\right)} \leq C$$

holds. In this situation, it holds in particular

$$\left\| \tilde{f}_\epsilon\left(c_0^\pm, \ldots, c_{M+1}^\pm\right)\right\|_{L^\infty\left(\Omega_{T_0}^\pm\right)} \leq C$$

for all $\epsilon \in (0,1)$.

2. In view of the definition of $c_0^\pm$ we argued on basis of the Dirichlet boundary condition for c^ϵ. We could also choose $c_0^\pm \equiv -1$, but this case corresponds to a one-phase system.

3. In order to find unique solutions $\left(\mu_k^\pm, p_k^\pm, \mathbf{v}_k^\pm\right)$ to (5.11), (5.14) respectively (5.12), (5.15) we need to prescribe boundary conditions. We use the solutions of the inner expansion and the inner-outer matching conditions to gain the necessary boundary conditions on Γ. In particular, we get Dirichlet data on Γ for (5.14) and prescribe the values of $[2D_s\mathbf{v}_k - p_k] \cdot \mathbf{n}$ and $[\mathbf{v}_k]$ on Γ for (5.15). The boundary condition on $\partial\Omega$ will be derived from the solutions of the boundary layer expansion and the outer-boundary matching conditions. These yield Dirichlet data on $\partial\Omega \times [0, T_0]$ for (5.14) and also values for a boundary condition as in (1.23).

4. Note that we assume that the expansions in (5.2) hold true only in $\Omega_{T_0}^\pm \backslash \Gamma(2\delta; T_0)$, but we want all the single terms of the expansion to be well-defined and smooth in all of $\Omega_{T_0}^\pm$ (and even in $\Omega_{T_0}^\pm \cup \Gamma(2\delta; T_0)$, see below). Furthermore, we demand that the formulae (5.3), (5.4), (5.5), and (5.4) hold in $\Omega_{T_0}^\pm$.

5. As mentioned above, we will need $\left(c_k^\pm, \mu_k^\pm, \mathbf{v}_k^\pm, p_k^\pm\right)$, for $k \geq 0$, to not only be defined in $\Omega_{T_0}^\pm$, but we have to extend them onto $\Omega_{T_0}^\pm \cup \Gamma\left(2\delta; T_0\right)$. For $\mu_k^\pm$ and $p_k^\pm$ we may use any smooth extension, so one possibility is to use the extension operator defined in [51], part VI, Theorem 5. It is trivial to extend $c_0^\pm$ and if all $\left(c_i^\pm, \mu_i^\pm\right)$ for $i \leq k-1$ have been defined on $\Omega_{T_0}^\pm \cup \Gamma\left(2\delta; T_0\right)$, then $c_k^\pm$ is as well, by (5.13).
   For $\mathbf{v}_k^\pm$ the extension process is slightly more involved, as we would like to find a divergence free extension. For this we use a similar approach as was used in [6], Section 3.1. That is, we first use the extension operators $\mathfrak{E}^\pm : H^i\left(\Omega^\pm(t)\right) \to H^i\left(\mathbb{R}^2\right)$ (componentwise) for $i \in \mathbb{N}$ to extend $\mathbf{v}_k^\pm$ onto $\mathbb{R}^2$ as a H^i function. Then we define smooth cut-off functions $\xi^\pm : \Omega \to \mathbb{R}$, such that $\xi^\pm \equiv 0$ in $\Omega^\mp(t) \backslash \Gamma_t(3\delta)$, $\xi^\pm \equiv 1$ in $\Omega \backslash \left(\Omega^\mp(t)\right)$ and $\xi \geq 0$ in $\Omega^\mp(t) \cap \Gamma_t(3\delta)$. Moreover, we choose smooth functions $\psi^\pm$, which satisfy $\operatorname{supp}\psi^\pm \subset \left(\Omega^\mp(t) \cap \Gamma_t(3\delta)\right) \backslash \Gamma_t(2\delta)$ and $\int_\Omega \psi^\pm \mathrm{d}x = 1$. Then $\tilde{g}^\pm := \operatorname{div}\left(\xi^\pm \mathfrak{E}^\pm\left(\mathbf{v}_k^\pm\right)\right) \in H_0^i\left(\Gamma_t(3\delta) \backslash \Omega^\pm(t)\right)$ due to $\operatorname{div}\mathbf{v}_k^\pm = 0$ in $\Omega^\pm(t)$ and the definition of the cut-off function $\xi^\pm$. Moreover,

$$\hat{g}^\pm := \tilde{g}^\pm - \psi \int_{\Gamma_t(3\delta)\backslash\Omega^\pm(t)} \tilde{g}^\pm \mathrm{d}x \in H_0^i\left(\Gamma_t(3\delta) \backslash \Omega^\pm(t)\right)$$

   and

$$\int_{\Gamma_t(3\delta)\backslash\Omega^\pm(t)} \hat{g}^\pm \mathrm{d}x = 0.$$

   By [31] Chapter III, Theorem 3.3, there is a solution $\tilde{\mathbf{v}}^\pm \in H_0^{i+1}\left(\Gamma_t(3\delta) \backslash \Omega^\pm(t)\right)$ of $\operatorname{div}\tilde{\mathbf{v}}^\pm = \hat{g}^\pm$. Extending $\tilde{\mathbf{v}}^\pm$ onto $\Omega^\pm(t)$ by zero, we find that $\mathcal{E}^\pm\left(\mathbf{v}_k^\pm\right) := \xi^\pm \mathfrak{E}^\pm\left(\mathbf{v}_k^\pm\right) - \tilde{\mathbf{v}}^\pm$ is a divergence free extension of $\mathbf{v}_k^\pm$ in $\Gamma_t(2\delta)$. In particular we have the identity $\mathcal{E}^\pm\left(\mathbf{v}_k^\pm\right)|_{\Omega^\pm(t)} = \mathbf{v}_k^\pm$ in $\Omega^\pm(t)$ and

$$\left\|\mathcal{E}^\pm\left(\mathbf{v}_k^\pm\right)\right\|_{H^2(\Omega^\pm(t)\cup\Gamma_t(2\delta))} \leq C\left\|\mathbf{v}_k^\pm\right\|_{H^2(\Omega^\pm(t))}. \tag{5.16}$$

   The last inequality is due to

$$\left\|\tilde{\mathbf{v}}^\pm\right\|_{H^2(\Gamma_t(3\delta)\backslash\Omega^\pm(t))} \leq C_1\left\|\hat{g}^\pm\right\|_{H^1(\Gamma_t(3\delta)\backslash\Omega^\pm(t))} \leq C_2\left\|\mathbf{v}_k^\pm\right\|_{H^2(\Omega^\pm(t))}.$$

For later use we define

$$\begin{aligned} U_k^\pm(x,t) &= \Delta\mu_k^\pm(x,t) - \partial_t c_k^\pm(x,t) - \sum_{j=0}^{k} \mathbf{v}_j^\pm(x,t) \cdot \nabla c_{k-j}^\pm(x,t), \\ U^\pm &= \sum_{k\geq 0} \epsilon^k U_k^\pm, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} \mathbf{W}_k^\pm(x,t) &= -\Delta\mathbf{v}_k^\pm(x,t) + \nabla p_k^\pm(x,t) - \sum_{j=0}^{k-1} \mu_j^\pm(x,t)\nabla c_{k-j}^\pm(x,t), \\ \mathbf{W}^\pm &= \sum_{k\geq 0} \epsilon^k \mathbf{W}_k^\pm, \end{aligned} \tag{5.18}$$

for $(x,t) \in \Omega_{T_0}^\pm \cup \Gamma(2\delta)$. Note that by (5.14) and (5.15) we have $\mathbf{W}_k^\pm(x,t) = U_k^\pm(x,t) = 0$ for all $(x,t) \in \overline{\Omega_{T_0}^\pm}$.

5.1.2. The Inner Expansion

Close to the interface Γ we introduce a stretched variable

$$\rho^{\epsilon}(x,t) := \frac{d_{\Gamma}(x,t) - \epsilon h^{\epsilon}(S(x,t),t)}{\epsilon} \text{ for all } (x,t) \in \Gamma(2\delta) \tag{5.19}$$

for $\epsilon \in (0,1)$. Here $h^{\epsilon} : \mathbb{T}^1 \times [0, T_0] \to \mathbb{R}$ is a given smooth function and can heuristically be interpreted as the distance of the zero level set of c^{ϵ} to Γ, see also [26], Chapter 4.2. From a more practical viewpoint, regarding the construction of the inner terms in Subsection 5.1.6, the presence of h^{ϵ} and more precisely its expansion enables us to satisfy the compatibility condition of Lemma 5.9. In the following, we will often drop the ϵ–dependence and write $\rho(x,t) = \rho^{\epsilon}(x,t)$.

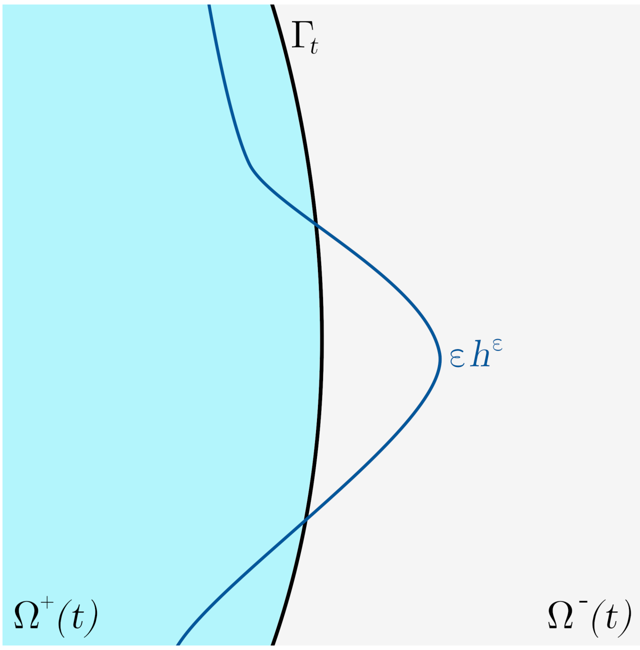

Figure 5.2.: Heuristics of h^{ϵ} close to the surface Γ_t.

Now assume that, in $\Gamma(2\delta)$, the identities

$$\begin{aligned} c^{\epsilon}(x,t) &= \tilde{c}^{\epsilon}\left(\frac{d_{\Gamma}(x,t)}{\epsilon} - h^{\epsilon}(S(x,t),t), x, t\right), \\ \mu^{\epsilon}(x,t) &= \tilde{\mu}^{\epsilon}\left(\frac{d_{\Gamma}(x,t)}{\epsilon} - h^{\epsilon}(S(x,t),t), x, t\right), \\ p^{\epsilon}(x,t) &= \tilde{p}^{\epsilon}\left(\frac{d_{\Gamma}(x,t)}{\epsilon} - h^{\epsilon}(S(x,t),t), x, t\right), \\ \mathbf{v}^{\epsilon}(x,t) &= \tilde{\mathbf{v}}^{\epsilon}\left(\frac{d_{\Gamma}(x,t)}{\epsilon} - h^{\epsilon}(S(x,t),t), x, t\right) \end{aligned} \tag{5.20}$$

hold for the solutions of (1.18)–(1.25) and some smooth functions $\tilde{c}^{\epsilon}, \tilde{\mu}^{\epsilon}, \tilde{p}^{\epsilon} : \mathbb{R} \times \Gamma(2\delta) \to \mathbb{R}$,

$\tilde{\mathbf{v}}^\epsilon : \mathbb{R} \times \Gamma(2\delta) \to \mathbb{R}^2$. Furthermore, we assume that we have the expansions

$$\begin{aligned}
\tilde{c}^\epsilon(\rho, x, t) &\approx \sum_{k\geq 0} \epsilon^k c_k(\rho, x, t), \\
\tilde{\mu}^\epsilon(\rho, x, t) &\approx \sum_{k\geq 0} \epsilon^k \mu_k(\rho, x, t), \\
\tilde{p}^\epsilon(\rho, x, t) &\approx \sum_{k\geq 0} \epsilon^k p_k(\rho, x, t), \\
\tilde{\mathbf{v}}^\epsilon(\rho, x, t) &\approx \sum_{k\geq 0} \epsilon^k \mathbf{v}_k(\rho, x, t)
\end{aligned} \tag{5.21}$$

for all $(\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta)$ and also

$$h^\epsilon(s,t) \approx \sum_{k\geq 0} \epsilon^k h_{k+1}(s,t), \tag{5.22}$$

where $c_k, \mu_k, p_k : \mathbb{R} \times \Gamma(2\delta) \to \mathbb{R}$, $\mathbf{v}_k : \mathbb{R} \times \Gamma(2\delta) \to \mathbb{R}^2$ and $h_k : \mathbb{T}^1 \times [0, T_0] \to \mathbb{R}$ are smooth functions for all $k \geq 0$.

Using Lemma 2.20 and the ansatz (5.20) we may compute

$$\begin{aligned}
\partial_t c^\epsilon(x,t) &= \partial_\rho \tilde{c}^\epsilon(\rho(x,t), x, t)\left(\frac{1}{\epsilon}\partial_t d_\Gamma(x,t) - \partial_t^\Gamma h^\epsilon(x,t)\right) + \partial_t \tilde{c}^\epsilon(\rho(x,t), x, t), \\
\nabla c^\epsilon(x,t) &= \partial_\rho \tilde{c}^\epsilon(\rho(x,t), x, t)\left(\frac{1}{\epsilon}\mathbf{n}(S(x,t),t) - \nabla^\Gamma h^\epsilon(x,t)\right) + \nabla \tilde{c}^\epsilon(\rho(x,t), x, t), \\
\Delta c^\epsilon(x,t) &= \partial_{\rho\rho} \tilde{c}^\epsilon(\rho(x,t), x, t)\left(\frac{1}{\epsilon^2} + \left|\nabla^\Gamma h^\epsilon(x,t)\right|^2\right) \\
&\quad + \partial_\rho \tilde{c}^\epsilon(\rho(x,t), x, t)\left(\frac{1}{\epsilon}\Delta d_\Gamma(x,t) - \Delta^\Gamma h^\epsilon(x,t)\right) \\
&\quad + 2\nabla\partial_\rho \tilde{c}^\epsilon(\rho(x,t), x, t) \cdot \left(\frac{1}{\epsilon}\mathbf{n}(S(x,t),t) - \nabla^\Gamma h^\epsilon(x,t)\right) \\
&\quad + \Delta \tilde{c}^\epsilon(\rho(x,t), x, t).
\end{aligned} \tag{5.23}$$

When referring to $\tilde{c}, \tilde{\mu}, \tilde{p}, \tilde{\mathbf{v}}$ and the expansion terms we write $\nabla = \nabla_x$ and $\Delta = \Delta_x$. The expressions $\partial_t^\Gamma h^\epsilon(x,t)$, $\nabla^\Gamma h^\epsilon(x,t)$, $\Delta^\Gamma h^\epsilon(x,t)$ and $D_\Gamma^2 h^\epsilon(x,t)$ are for $(x,t) \in \Gamma(2\delta)$ to be understood in the sense of Remark 2.19. Equivalent formulae to the ones above also hold for μ^ϵ and p^ϵ.

The derivatives of $\mathbf{v}^\epsilon$ which are necessary for the inner expansion are given by

$$\begin{aligned}
\operatorname{div}\mathbf{v}^\epsilon &= \partial_\rho \tilde{\mathbf{v}}^\epsilon \cdot \left(\frac{1}{\epsilon}\mathbf{n} - \nabla^\Gamma h^\epsilon\right) + \operatorname{div}\tilde{\mathbf{v}}^\epsilon, \\
\Delta \mathbf{v}^\epsilon &= \partial_{\rho\rho}\tilde{\mathbf{v}}^\epsilon\left(\frac{1}{\epsilon^2} + \left|\nabla^\Gamma h^\epsilon\right|^2\right) + \partial_\rho \tilde{\mathbf{v}}^\epsilon\left(\frac{1}{\epsilon}\Delta d_\Gamma - \Delta^\Gamma h^\epsilon\right) + 2\left(\nabla\partial_\rho \tilde{\mathbf{v}}^\epsilon\right)^T \cdot \left(\frac{1}{\epsilon}\mathbf{n} - \nabla^\Gamma h^\epsilon\right) \\
&\quad + \Delta \tilde{\mathbf{v}}^\epsilon.
\end{aligned}$$

In order to match the inner and outer expansions, we require that for all k the so-called

inner-outer matching conditions

$$\sup_{(x,t)\in\Gamma(2\delta)} \left|\partial_x^m \partial_t^n \partial_\rho^l \left(c_k\left(\pm\rho,x,t\right) - c_k^{\pm}\left(x,t\right)\right)\right| \le Ce^{-\alpha\rho}, \tag{5.24}$$

$$\sup_{(x,t)\in\Gamma(2\delta)} \left|\partial_x^m \partial_t^n \partial_\rho^l \left(\mu_k\left(\pm\rho,x,t\right) - \mu_k^{\pm}\left(x,t\right)\right)\right| \le Ce^{-\alpha\rho}, \tag{5.25}$$

$$\sup_{(x,t)\in\Gamma(2\delta)} \left|\partial_x^m \partial_t^n \partial_\rho^l \left(\mathbf{v}_k\left(\pm\rho,x,t\right) - \mathbf{v}_k^{\pm}\left(x,t\right)\right)\right| \le Ce^{-\alpha\rho}, \tag{5.26}$$

$$\sup_{(x,t)\in\Gamma(2\delta)} \left|\partial_x^m \partial_t^n \partial_\rho^l \left(p_k\left(\pm\rho,x,t\right) - p_k^{\pm}\left(x,t\right)\right)\right| \le Ce^{-\alpha\rho} \tag{5.27}$$

hold for constants $\alpha, C > 0$ and all $\rho > 0$, $m,n,l \ge 0$.

Remark 5.2. A deeper insight into the importance of the matching conditions will only be possible after the gluing of the inner and outer solutions and the subsequent analysis of the remainder in Chapter 6, see in particular Theorem 6.12 and Remark 6.13. In those sections we will only use the matching conditions for $m,n,l \in \{0,1,2\}$. However, since the ordinary differential equations for $\left(c_k, c_k^{\pm}, \mu_k, \mu_k^{\pm}, \mathbf{v}_k, \mathbf{v}_k^{\pm}, p_k, p_k^{\pm}\right)$ (cf. (5.40), (5.42), (5.44), (5.46)) are dependent on derivatives of lower order terms, it is necessary and sufficient for the matching conditions to hold for $m,n,l \in \{0,\ldots,C(M)\}$ for some $C(M) \in \mathbb{N}$ depending on the general number of terms in the expansion.

As stated before, we interpret $\{(x,t) \in \Gamma(2\delta) |\, d_\Gamma(x,t) = \epsilon h^\epsilon(S(x,t),t)\}$ to be the 0-level set of c^ϵ. Thus, we normalize c^k such that

$$c^k(0,x,t) = 0$$

holds for all $(x,t) \in \Gamma(2\delta)$ and $k \ge 0$.

In view of the above given equations for derivatives, we may rewrite (1.18) and (1.19) as

$$\begin{aligned}
-\partial_{\rho\rho}\tilde{\mathbf{v}}^\epsilon + \epsilon\partial_\rho\tilde{p}^\epsilon\mathbf{n} &= \epsilon\left(\partial_\rho\tilde{\mathbf{v}}^\epsilon\Delta d_\Gamma + 2\left(\nabla\partial_\rho\tilde{\mathbf{v}}^\epsilon\right)^T\mathbf{n} + \tilde{\mu}^\epsilon\partial_\rho\tilde{c}^\epsilon\mathbf{n}\right)\\
&\quad + \epsilon^2\Big(-2\left(\nabla\partial_\rho\tilde{\mathbf{v}}^\epsilon\right)^T\cdot\nabla^\Gamma h^\epsilon + \partial_{\rho\rho}\tilde{\mathbf{v}}^\epsilon\left|\nabla^\Gamma h^\epsilon\right|^2 - \partial_\rho\tilde{\mathbf{v}}^\epsilon\Delta^\Gamma h^\epsilon - \tilde{\mu}^\epsilon\partial_\rho\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon\\
&\qquad + \partial_\rho\tilde{p}^\epsilon\nabla^\Gamma h^\epsilon + \Delta\tilde{\mathbf{v}}^\epsilon - \nabla\tilde{p}^\epsilon + \tilde{\mu}^\epsilon\nabla\tilde{c}^\epsilon\Big)\\
&=: \mathbf{R}^\epsilon
\end{aligned} \tag{5.28}$$

$$\partial_\rho\tilde{\mathbf{v}}^\epsilon\cdot\mathbf{n} = \epsilon\left(\partial_\rho\tilde{\mathbf{v}}^\epsilon\cdot\nabla^\Gamma h^\epsilon - \operatorname{div}\tilde{\mathbf{v}}^\epsilon\right), \tag{5.29}$$

where the equalities are only assumed to hold in

$$S^\epsilon := \left\{(\rho,x,t) \in \mathbb{R}\times\Gamma(2\delta) \,\middle|\, \rho = \frac{d_\Gamma(x,t)}{\epsilon} - h^\epsilon(S(x,t),t)\right\}.$$

Regarding (1.20) and (1.21) we get the equations

$$\begin{aligned}
\partial_{\rho\rho}\tilde{c}^\epsilon - f'(c^\epsilon) &= \epsilon\left(-\tilde{\mu}^\epsilon - \partial_\rho\tilde{c}^\epsilon\Delta d_\Gamma - 2\nabla\partial_\rho\tilde{c}^\epsilon\cdot\mathbf{n}\right)\\
&\quad + \epsilon^2\left(-\partial_{\rho\rho}\tilde{c}^\epsilon\left|\nabla^\Gamma h^\epsilon\right|^2 + \partial_\rho\tilde{c}^\epsilon\Delta^\Gamma h^\epsilon + 2\nabla\partial_\rho\tilde{c}^\epsilon\cdot\nabla^\Gamma h^\epsilon - \Delta\tilde{c}^\epsilon\right)
\end{aligned} \tag{5.30}$$

$$\begin{aligned}
\partial_{\rho\rho}\tilde{\mu}^\epsilon &= \epsilon\left(\partial_\rho\tilde{c}^\epsilon\left(\partial_t d_\Gamma + \tilde{\mathbf{v}}^\epsilon\cdot\mathbf{n}\right) - \partial_\rho\tilde{\mu}^\epsilon\Delta d_\Gamma - 2\nabla\partial_\rho\tilde{\mu}^\epsilon\cdot\mathbf{n}\right)\\
&\quad + \epsilon^2\Big(-\partial_\rho\tilde{c}^\epsilon\left(\partial_t^\Gamma h^\epsilon + \tilde{\mathbf{v}}^\epsilon\cdot\nabla^\Gamma h^\epsilon\right) + \partial_\rho\tilde{\mu}^\epsilon\Delta^\Gamma h^\epsilon\\
&\qquad - \partial_{\rho\rho}\tilde{\mu}^\epsilon\left|\nabla^\Gamma h^\epsilon\right|^2 + 2\nabla\partial_\rho\tilde{\mu}^\epsilon\cdot\nabla^\Gamma h^\epsilon + \tilde{\mathbf{v}}^\epsilon\cdot\nabla\tilde{c}^\epsilon + \partial_t\tilde{c}^\epsilon - \Delta\tilde{\mu}^\epsilon\Big)
\end{aligned} \tag{5.31}$$

in S^ϵ. In the above identities, the operators ∇, Δ, div, and D^2 are only operating on the spatial variable x, with ρ fixed.

Note that (5.28)–(5.31) are only derived and supposed to be fulfilled on the set S^ϵ, but we may view these equations as ordinary differential equations in $\rho \in \mathbb{R}$, where x and t are considered to be fixed parameters. Moreover, we may add extra terms into the equations which vanish on S^ϵ, as those do not change the original problem. However, these “artificial” terms are essential when it comes to ensuring the matching or compatibility conditions (cf. Subsection 5.1.3).

Thus, we introduce the functions $g^\epsilon(x,t)$, $j^\epsilon(x,t)$ and $l^\epsilon(x,t)$ as well as $\mathbf{u}^\epsilon(x,t)$ and $\mathbf{q}^\epsilon(x,t)$ for $(x,t) \in \Gamma(2\delta)$. As a rough guideline, the functions g^ϵ, j^ϵ, and $\mathbf{q}^\epsilon$ will enable us to fulfill the compatibility conditions in $\Gamma(2\delta)\backslash\Gamma$. l^ϵ and $\mathbf{u}^\epsilon$ on the other hand are of importance when it comes to fulfilling the matching conditions in $\Gamma(2\delta)\backslash\Gamma$. Furthermore, we introduce the following essential function:

Proposition 5.3. *Let θ_0 be chosen as in Lemma 2.2. Then there is a smooth function $\eta : \mathbb{R} \to [0,1]$ such that $\eta = 0$ in $(-\infty,-1]$, $\eta = 1$ in $[1,\infty)$ and $\eta' \geq 0$ in $\mathbb{R}$, which satisfies*

$$\int_{\mathbb{R}} \left(\eta(\rho) - \frac{1}{2}\right) \theta_0'(\rho)\, \mathrm{d}\rho = 0. \tag{5.32}$$

Proof. By Lemma 2.2 the identity (5.32) is equivalent to $\int_{\mathbb{R}} \eta\theta_0' \mathrm{d}\rho = 1$ which is again equivalent to

$$\int_{-1}^{1} \eta\theta_0' \mathrm{d}\rho = \theta_0(1) \tag{5.33}$$

for smooth functions η satisfying $\eta = 0$ in $(-\infty,-1]$ and $\eta = 1$ in $[1,\infty)$. The last equivalence is due to (2.1). Since moreover $\int_{-1}^{1} \theta_0' \mathrm{d}\rho = \theta_0(1) - \theta_0(-1) > \theta_0(1)$, as $\theta_0(0) = 0$ and θ_0 is monotonically increasing, we can always find a smooth function $\varphi : [-1,1] \to [0,1]$ with $\varphi' \geq 0$ and $\varphi(-1) = 0$, $\varphi(1) = 1$, $\varphi^{(k)}(\pm 1) = 0$ for all $k \in \mathbb{N}$ such that (5.33) is satisfied for φ (e.g. by modifying a standard cutoff function). This yields the result by choosing $\eta|_{(-1,1)}$ accordingly. ☐

In the following, we let η be a fixed function satisfying the properties of Proposition 5.3. It will serve a variety of purposes in the upcoming analysis: its derivative serves as a cutoff function around 0, ensuring that the matching conditions may all be satisfied. Furthermore, it ensures that the right hand side terms $\mathbf{V}^{k-1}$, W^{k-1}, A^{k-1} and B^{k-1} of (5.40), (5.42), (5.44), (5.46), which are introduced below, still have exponential decay after the addition of extra terms. The normalization (5.32) plays an important role in Step 3 of Subsection 5.1.6. In essence, multiplying the newly introduced functions g^ϵ, l^ϵ etc. by derivatives of η yields proper behaviour in ρ.

For later use we also define

$$\eta^{C,\pm}(\rho) = \eta(-C \pm \rho)$$

for an arbitrary constant $C > 0$ and $\rho \in \mathbb{R}$.

Using these auxiliary functions, we assume from now on that the following equalities are fulfilled in $\mathbb{R} \times \Gamma(2\delta)$:

$$\begin{aligned} -\partial_{\rho\rho}\tilde{\mathbf{v}}^\epsilon = \mathbf{R}^\epsilon &+ \left(-\mathbf{u}^\epsilon \eta''(\rho) + \mathbf{q}^\epsilon \eta'(\rho)\right)\left(d_\Gamma - \epsilon(\rho + h^\epsilon)\right) \\ &+ \epsilon^2 \left(\mathbf{W}^+ \eta^{C_S,+} + \mathbf{W}^- \eta^{C_S,-}\right), \end{aligned} \tag{5.34}$$

$$\partial_\rho \tilde{\mathbf{v}}^\epsilon \cdot \mathbf{n} = \epsilon \left(\partial_\rho \tilde{\mathbf{v}}^\epsilon \nabla^\Gamma h^\epsilon - \mathrm{div}\tilde{\mathbf{v}}^\epsilon\right) + \left(\mathbf{u}^\epsilon \cdot \left(\mathbf{n} - \epsilon \nabla^\Gamma h^\epsilon\right)\right) \eta'(\rho) \left(d_\Gamma - \epsilon(\rho + h^\epsilon)\right), \tag{5.35}$$

$$\begin{aligned}
\partial_{\rho\rho}\tilde{c}^\epsilon - f'(c^\epsilon) &= \epsilon \left(-\tilde{\mu}^\epsilon - \partial_\rho \tilde{c}^\epsilon \Delta d_\Gamma - 2\nabla \partial_\rho \tilde{c}^\epsilon \cdot \mathbf{n}\right) \\
&\quad + \epsilon^2 \left(-\partial_{\rho\rho}\tilde{c}^\epsilon \left|\nabla^\Gamma h^\epsilon\right|^2 + \partial_\rho \tilde{c}^\epsilon \Delta^\Gamma h^\epsilon + 2\nabla \partial_\rho \tilde{c}^\epsilon \cdot \nabla^\Gamma h^\epsilon - \Delta \tilde{c}^\epsilon\right) \\
&\quad + g^\epsilon \eta'(\rho) \left(d_\Gamma - \epsilon(\rho + h^\epsilon)\right), \qquad (5.36)\\
\partial_{\rho\rho}\tilde{\mu}^\epsilon &= \epsilon \left(\partial_\rho \tilde{c}^\epsilon \left(\partial_t d_\Gamma + \tilde{\mathbf{v}}^\epsilon \cdot \mathbf{n}\right) - \partial_\rho \tilde{\mu}^\epsilon \Delta d_\Gamma - 2\nabla \partial_\rho \tilde{\mu}^\epsilon \cdot \mathbf{n}\right) \\
&\quad + \epsilon^2 \left(-\partial_\rho \tilde{c}^\epsilon \left(\partial_t^\Gamma h^\epsilon + \tilde{\mathbf{v}}^\epsilon \cdot \nabla^\Gamma h^\epsilon\right) + \partial_\rho \tilde{\mu}^\epsilon \Delta^\Gamma h^\epsilon \right.\\
&\quad \left. - \partial_{\rho\rho}\tilde{\mu}^\epsilon \left|\nabla^\Gamma h^\epsilon\right|^2 + 2\nabla \partial_\rho \tilde{\mu}^\epsilon \cdot \nabla^\Gamma h^\epsilon + \tilde{\mathbf{v}}^\epsilon \cdot \nabla \tilde{c}^\epsilon + \partial_t \tilde{c}^\epsilon - \Delta \tilde{\mu}^\epsilon\right) \\
&\quad + \left(l^\epsilon \eta''(\rho) + j^\epsilon \eta'(\rho)\right) \left(d_\Gamma - \epsilon(\rho + h^\epsilon)\right) \\
&\quad + \epsilon^2 \left(U^+ \eta^{C_S,+} + U^- \eta^{C_S,-}\right). \qquad (5.37)
\end{aligned}$$

The terms $U^\pm$ and $\mathbf{W}^\pm$ (cf. (5.17), (5.18)) are used here in order to ensure the exponential decay of the right hand sides, which is necessary for the compatibility conditions in Subsection 5.1.3. Although most of the appearing terms in (5.34) and (5.37) already have exponential decay due to the matching conditions and the properties of η, the terms involving only space or time derivatives (and none in ρ) only get the decay due to the outer equations (5.14) and (5.15). But those outer equations do not hold for the extensions of the involved functions. To deal with this problem, we introduce $U^\pm$ and $\mathbf{W}^\pm$. For more details, see the proofs of Lemmata 5.9, 5.10 and 5.11. In this context $C_S > 0$ is a constant which will be determined later on (see Remark 5.5). Now we plug the expansions (5.21), (5.22) into the derived equations and equate the ϵ^k terms. We also assume that the newly introduced functions have expansions of the form

$$\begin{aligned}
\mathbf{u}^\epsilon(x,t) &\approx \sum_{k\geq 0} \mathbf{u}_k(x,t)\,\epsilon^k, & l^\epsilon(x,t) &\approx \sum_{k\geq 0} l_k(x,t)\,\epsilon^k, \\
\mathbf{q}^\epsilon(x,t) &\approx \sum_{k\geq 0} \mathbf{q}_k(x,t)\,\epsilon^{k+1}, & j^\epsilon(x,t) &\approx \sum_{k\geq 0} j_k(x,t)\,\epsilon^{k+1}, \\
g^\epsilon(x,t) &\approx \sum_{k\geq 0} g_k(x,t)\,\epsilon^{k+1}, & & \qquad (5.38)
\end{aligned}$$

for $(x,t) \in \Gamma(2\delta)$ and $(s,t) \in \mathbb{T}^1 \times [0, T_0]$. We gain the following ordinary differential equations in ρ:

1. For (5.28) we get

$$-\partial_{\rho\rho}\left(\mathbf{v}_0 - \mathbf{u}_0 \eta d_\Gamma\right) = 0, \tag{5.39}$$

$$-\partial_{\rho\rho}\left(\mathbf{v}_k - \left(\mathbf{u}_k d_\Gamma - \mathbf{u}_0 h_k\right)\eta\right) + \partial_\rho p_{k-1}\mathbf{n} = \mathbf{V}^{k-1} \tag{5.40}$$

   for $k \geq 1$, $\mathbf{V}^{k-1} = \mathbf{V}^{k-1}(\rho, x, t)$ as defined below and $\rho \in \mathbb{R}$, $(x,t) \in \Gamma(2\delta)$.

2. For (5.29) we get

$$\partial_\rho \left(\mathbf{v}_0 \cdot \mathbf{n} - \mathbf{u}_0 \cdot \mathbf{n} d_\Gamma \eta\right) = 0, \tag{5.41}$$

$$\partial_\rho \left(\mathbf{v}_k \cdot \mathbf{n} - \left(\mathbf{u}_k d_\Gamma - \mathbf{u}_0 h_k\right) \cdot \mathbf{n}\eta\right) = W^{k-1} + \nabla^\Gamma h_k \cdot \left(\partial_\rho \mathbf{v}_0 - \mathbf{u}_0 d_\Gamma \eta'\right) \tag{5.42}$$

   for $k \geq 1$, $W^{k-1} = W^{k-1}(\rho, x, t)$ as defined below and $\rho \in \mathbb{R}$, $(x,t) \in \Gamma(2\delta)$.

3. For (5.30) we get

$$\partial_{\rho\rho}c_0 - f'(c_0) = 0, \tag{5.43}$$

$$\partial_{\rho\rho}c_k - f''(c_0)\,c_k = A^{k-1} \tag{5.44}$$

for $k \geq 1$, $A^{k-1} = A^{k-1}(\rho, x, t)$ as defined below and $\rho \in \mathbb{R}$, $(x,t) \in \Gamma(2\delta)$.

4. For (5.31) we get

$$\partial_{\rho\rho}(\mu_0 - l_0\eta d_\Gamma) = 0, \tag{5.45}$$

$$\partial_{\rho\rho}(\mu_k - (l_k d_\Gamma - l_0 h_k)\,\eta) = B^{k-1} \tag{5.46}$$

for $k \geq 1$, $B^{k-1} = B^{k-1}(\rho, x, t)$ as defined below and $\rho \in \mathbb{R}$, $(x,t) \in \Gamma(2\delta)$.

In the above systems we used

$$\begin{aligned}\mathbf{V}^{k-1} &= \partial_\rho \mathbf{v}_{k-1}\Delta d_\Gamma + 2\left(\nabla\partial_\rho\mathbf{v}_{k-1}\right)^T\mathbf{n} - 2\left(\nabla\partial_\rho\mathbf{v}_0\right)^T\nabla^\Gamma h_{k-1} - \partial_\rho\mathbf{v}_0\Delta^\Gamma h_{k-1}\\ &\quad + \partial_\rho p_0\nabla^\Gamma h_{k-1} + \beta_2^k 2\partial_{\rho\rho}\mathbf{v}_0\nabla^\Gamma h_{k-1}\cdot\nabla^\Gamma h_1 + \beta_1^k\left(\mu_0\partial_\rho c_{k-1} + \mu_{k-1}\partial_\rho c_0\right)\mathbf{n}\\ &\quad - \mu_0\partial_\rho c_0\nabla^\Gamma h_{k-1} + \mathbf{q}_{k-1}\eta' d_\Gamma - \mathbf{q}_0\eta' h_{k-1} + \left(\rho + \delta_1^k h_1\right)\mathbf{u}_{k-1}\eta'' + \mathbf{u}_1\eta'' h_{k-1}\\ &\quad + \mathcal{V}^{k-2},\end{aligned} \tag{5.47}$$

$$\begin{aligned}\mathcal{V}^{k-2} &= \sum_{i=1}^{k-2}\mu_i\partial_\rho c_{k-1-i}\mathbf{n} + \sum_{\substack{i,j,l=0\\ i+j+l=k-1}}^{k-2}\mu_i\partial_\rho c_j\nabla^\Gamma h_l + \sum_{i=2}^{k-2}\mathbf{u}_i\eta'' h_{k-i}\\ &\quad - \sum_{i=1}^{k-2}\mathbf{q}_i\eta' h_{k-1-i} - \mathbf{q}_{k-2}\eta'\rho + \Delta\mathbf{v}_{k-2} - \nabla p_{k-2} + \sum_{i=0}^{k-2}\mu_i\nabla c_{k-2-i}\\ &\quad + \mathbf{W}_{k-2}^+\eta^{C_S,+} + \mathbf{W}_{k-2}^-\eta^{C_S,-},\end{aligned} \tag{5.48}$$

$$\begin{aligned}W^{k-1} &= \delta_1^k\partial_\rho\mathbf{v}_{k-1}\nabla^\Gamma h_1 + \partial_\rho\mathbf{v}_1\nabla^\Gamma h_{k-1} - \operatorname{div}\mathbf{v}_{k-1} - \mathbf{u}_{k-1}\cdot\mathbf{n}\eta'\rho - \delta_1^k\mathbf{u}_{k-1}h_1\cdot\mathbf{n}\eta'\\ &\quad - \mathbf{u}_1\cdot\mathbf{n}\eta' h_{k-1} - \delta_1^k\left(\mathbf{u}_{k-1}\cdot\nabla^\Gamma h_1 + \mathbf{u}_1\cdot\nabla^\Gamma h_{k-1}\right)d_\Gamma\eta'\\ &\quad + \mathbf{u}_0\cdot\left(\nabla^\Gamma h_{k-1}\rho + \beta_2^k\left(\nabla^\Gamma h_{k-1}h_1 + \nabla^\Gamma h_1 h_{k-1}\right)\right)\eta' + \mathcal{W}^{k-2},\end{aligned} \tag{5.49}$$

$$\begin{aligned}\mathcal{W}^{k-2} &= \sum_{i=2}^{k-2}\partial_\rho\mathbf{v}_i\nabla^\Gamma h_{k-i} - \sum_{i=2}^{k-2}\mathbf{u}_i\cdot\mathbf{n}\eta' h_{k-i} - \sum_{i=2}^{k-2}\mathbf{u}_i\cdot\nabla^\Gamma h_{k-i}d_\Gamma\eta'\\ &\quad + \eta'\left(\sum_{i=1}^{k-2}\mathbf{u}_i\cdot\nabla^\Gamma h_{k-i-1}\rho + \sum_{i,j,l=1, i+j+l=k}^{k-2}\mathbf{u}_i\cdot\nabla^\Gamma h_j h_l\right),\end{aligned} \tag{5.50}$$

and

$$\begin{aligned}A^{k-1} &= -\mu_{k-1} - \partial_\rho c_{k-1}\Delta d_\Gamma - 2\nabla\partial_\rho c_{k-1}\cdot\mathbf{n} + f_{k-1}(c_0,\ldots,c_{k-1}) + g_{k-1}\eta' d_\Gamma\\ &\quad - \beta_2^k 2\partial_{\rho\rho}c_0\nabla^\Gamma h_{k-1}\cdot\nabla^\Gamma h_1 + \partial_\rho c_0\Delta^\Gamma h_{k-1} + 2\nabla\partial_\rho c_0\cdot\nabla^\Gamma h_{k-1} - g_0 h_{k-1}\eta'\\ &\quad + \mathcal{A}^{k-2},\end{aligned} \tag{5.51}$$

$$\mathcal{A}^{k-2} = \left(\sum_{i=0,j,l=1,i+j+l=k}^{k-2} -\partial_{\rho\rho} c_i \nabla^\Gamma h_j \cdot \nabla^\Gamma h_l \right) - g_{k-2}\eta'\rho - \sum_{i=1}^{k-2} g_i h_{k-1-i}\eta'$$
$$+ \left(\sum_{i=1}^{k-2} \partial_\rho c_i \Delta^\Gamma h_{k-1-i} + 2\nabla \partial_\rho c_i \cdot \nabla^\Gamma h_{k-1-i} \right) - \Delta c_{k-2}, \tag{5.52}$$

and

$$B^{k-1} = \partial_\rho c_{k-1}\partial_t d_\Gamma + \beta_1^k \left(\partial_\rho c_{k-1}\mathbf{v}_0 + \partial_\rho c_0 \mathbf{v}_{k-1} \right) \cdot \mathbf{n} - \partial_\rho \mu_{k-1}\Delta d_\Gamma - 2\nabla\partial_\rho \mu_{k-1} \cdot \mathbf{n}$$
$$- l_{k-1}\eta''\rho - \delta_1^k l_{k-1} h_1 \eta'' + j_{k-1}\eta' d_\Gamma - \partial_\rho c_0 \mathbf{v}_0 \cdot \nabla^\Gamma h_{k-1} - \partial_\rho c_0 \partial_t^\Gamma h_{k-1}$$
$$- \beta_2^k 2\partial_{\rho\rho}\mu_0 \nabla^\Gamma h_{k-1} \cdot \nabla^\Gamma h_1 + \partial_\rho \mu_0 \Delta^\Gamma h_{k-1} + 2\nabla\partial_\rho \mu_0 \cdot \nabla^\Gamma h_{k-1}$$
$$- l_1 h_{k-1}\eta'' - j_0 h_{k-1}\eta' + \mathcal{B}^{k-2}, \tag{5.53}$$

$$\mathcal{B}^{k-2} = \sum_{i=1}^{k-2} \partial_\rho c_{k-1-i}\mathbf{v}_i \cdot \mathbf{n} - \left(\sum_{i,j=0,l=1,i+j+l=k-1}^{k-2} \partial_\rho c_i \mathbf{v}_j \cdot \nabla^\Gamma h_l \right)$$
$$- \left(\sum_{i=0,j,l=1,i+j+l=k}^{k-2} \partial_{\rho\rho}\mu_i \nabla^\Gamma h_j \cdot \nabla^\Gamma h_l \right) - \sum_{i=2}^{k-2} l_i h_{k-i}\eta'' - j_{k-2}\eta'\rho$$
$$+ \left(\sum_{i=0}^{k-3} -\partial_\rho c_{k-2-i}\partial_t^\Gamma h_{i+1} + \partial_\rho \mu_{k-2-i}\Delta^\Gamma h_i + 2\nabla\partial_\rho \mu_{k-2-i} \cdot \nabla^\Gamma h_i \right)$$
$$- \sum_{i=1}^{k-2} j_i h_{k-1-i}\eta' + \partial_t c_{k-2} - \Delta \mu_{k-2} + \sum_{i=0}^{k-2} \mathbf{v}_i \nabla c_{k-2-i}$$
$$+ U_{k-2}^{+}\eta^{C_S,+} + U_{k-2}^{-}\eta^{C_S,-}. \tag{5.54}$$

In all of the above identities we used the following conventions:

Notation 5.4.

1. If the upper limit of the summation is less than the lower limit, then the sum is to be understood as zero.

2. All functions with negative index are supposed to be zero. In particular $\mathcal{V}^{-1} = \mathcal{W}^{-1} = \mathcal{A}^{-1} = \mathcal{B}^{-1} = 0$. Moreover, $h_0 := 0$.

3. We introduced the notation
$$\beta_i^k = \begin{cases} \frac{1}{2} & \text{if } i = k, \\ 1 & \text{else} \end{cases}$$
and δ_i^k is an "inverse" Kronecker delta, i.e.
$$\delta_i^k = \begin{cases} 0 & \text{if } i = k, \\ 1 & \text{else.} \end{cases}$$

4. We define $f_{k-1}(c_0, \ldots, c_{k-1})$ (appearing in (5.51)) in a completely similar fashion to (5.7). In particular, we will later on also use a remainder term $\tilde{f}$ as discussed in Remark 5.1 for the inner solutions. We use the convention $f_0(c_0) = 0$.

We will see after the construction of the zeroth order terms that the term h_k appearing on the right hand side of (5.42) is actually multiplied by 0.

Remark 5.5. Note that $\mathbf{W}^{\pm}$ and $U^{\pm}$, which we inserted in (5.34) and (5.37), are not multiplied by terms of the kind $(d_\Gamma - \epsilon(\rho + h^\epsilon))$. So we have to make sure that they do not appear in the actual equations for $\tilde{\mathbf{v}}^\epsilon$ and $\tilde{\mu}^\epsilon$, that is, we have to make sure they vanish on the set S^ϵ. This is accomplished by choosing the constant $C_S > 0$ in a suitable way.

In particular we set

$$C_S := \|h_1\|_{C^0(\mathbb{T}^1\times[0,T_0])} + 2$$

and assume that

$$\left|\sum_{k\geq 1} \epsilon^k h_{k+1}\left(S(x,t),t\right)\right| \leq 1 \tag{5.55}$$

holds for all $\epsilon > 0$ small enough. We will see later on that h_1 does not depend on the term $\epsilon^2\left(U^+\eta^{C_S,+} + U^-\eta^{C_S,-}\right)$ and $\epsilon^2\left(\mathbf{W}^+\eta^{C_S,+} + \mathbf{W}^-\eta^{C_S,-}\right)$, so this choice of C_S does not cause problems.

To understand why the choice of C_S prevents the aforementioned problems, let $(x,t) \in \Gamma(2\delta)$ and

$$\rho = \frac{d_\Gamma(x,t)}{\epsilon} - h^\epsilon\left(S(x,t),t\right) = \frac{d_\Gamma(x,t)}{\epsilon} - \sum_{k\geq 0}\epsilon^k h_{k+1}\left(S(x,t),t\right).$$

In particular $(\rho, x, t) \in S^\epsilon$. By choosing $\epsilon > 0$ small enough, we may ascertain that

$$\left|\rho - \left(\frac{d_\Gamma(x,t)}{\epsilon} - h_1\left(S(x,t),t\right)\right)\right| = \left|\sum_{k\geq 1}\epsilon^k h_{k+1}\left(S(x,t),t\right)\right| \leq 1 \tag{5.56}$$

holds for all $(x,t) \in \Gamma(2\delta)$. If $d_\Gamma(x,t) \geq 0$, it follows per definitionem that $(x,t) \in \overline{\Omega^+}$ and from (5.56) that

$$\begin{aligned}\rho &\geq \frac{d_\Gamma(x,t)}{\epsilon} - h_1\left(S(x,t),t\right) - 1 \geq -\left(h_1\left(S(x,t),t\right) + 1\right) \geq -\left(\|h_1\|_{C^0(\mathbb{T}^1\times[0,T_0])} + 1\right)\\ &= -C_S + 1.\end{aligned}$$

Thus, $\eta^{C_S,-}(\rho) = 0$ and since $(x,t) \in \overline{\Omega^+}$ we have $\mathbf{W}^+(x,t) = U^+(x,t) = 0$ and so

$$\epsilon^2\left(U^+\eta^{C_S,+} + U^-\eta^{C_S,-}\right) = \epsilon^2\left(\mathbf{W}^+\eta^{C_S,+} + \mathbf{W}^-\eta^{C_S,-}\right) = 0.$$

By an analogous procedure we get

$$\rho \leq C_S - 1$$

if $d_\Gamma(x,t) < 0$, which shows that the equations for $\tilde{\mathbf{v}}^\epsilon$ and $\tilde{\mu}^\epsilon$ are not influenced by the addition of $\epsilon^2\left(U^+\eta^{C_S,+} + U^-\eta^{C_S,-}\right)$ and $\epsilon^2\left(\mathbf{W}^+\eta^{C_S,+} + \mathbf{W}^-\eta^{C_S,-}\right)$.

5.1.3. Compatibility Conditions

Up to this point we have assumed that close to Γ the exact solutions $(c^\epsilon, \mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon)$ have expansions as in (5.20) and (5.21). Under that assumption we were able to derive the necessary ordinary differential equations given in (5.40)–(5.46).

Now in order for those differential equations to be well-posed, the respective right-hand sides have to satisfy certain conditions, the so-called **compatibility conditions**. These are given in the following results, which build on Lemma 2.3 and Lemma 2.4. On the other hand, these results also allow for an explicit construction of the terms $(c_k, \mu_k, \mathbf{v}_k, p_k)$ for certain k, which we will use in the inductive scheme presented in Subsection 5.1.6.

The strategy of the proofs for the following lemmata is similar to [47, 14]. In this subsection, it is assumed for all results for order $k \geq 2$ that the zeroth order terms are of the form presented in Lemma 5.19 and for order $k = 1$ that $c_0 = \theta_0$ and $c_0^{\pm} = \pm 1$. We introduce the following notation:

Notation 5.6. For $k \geq 0$ and $\beta^{\pm} = c_k^{\pm}, \mu_k^{\pm}, \mathbf{v}_k^{\pm}, p_k^{\pm}$ we denote

$$[\beta](x,t) := \beta^{+}(x,t) - \beta^{-}(x,t) \ \forall (x,t) \in \Gamma(2\delta).$$

Lemma 5.7. *Let $k \geq 1$ be an integer and A^{k-1}, $\mathcal{A}^{k-2}$ be defined as in (5.51) and (5.52). Moreover, assume that the matching conditions (5.24)–(5.27) are satisfied for (c_l, μ_l) for all $l \in \{0, \ldots, k-1\}$.*

Then (5.44) coupled with the normalization $c_k(0,x,t) = 0$ for $(x,t) \in \Gamma(2\delta)$ has a unique bounded and smooth solution c_k if and only if, in the case $k = 1$,

$$\tilde{\mu}_0 + \sigma \Delta d_\Gamma - \tilde{\eta} g_0 d_\Gamma = 0 \quad \text{in } \Gamma(2\delta) \tag{5.57}$$

or, in the case $k \geq 2$,

$$\tilde{\mu}_{k-1} - \tilde{\eta}\left(g_{k-1} d_\Gamma - g_0 h_{k-1}\right) - \sigma \Delta^\Gamma h_{k-1} = \tilde{\mathcal{A}}^{k-2} \quad \text{in } \Gamma(2\delta), \tag{5.58}$$

with

$$\tilde{\mu}_{k-1}(x,t) = \frac{1}{2} \int_{\mathbb{R}} \mu_{k-1}(\rho,x,t)\, \theta_0'(\rho)\, \mathrm{d}\rho, \tag{5.59}$$

$$\tilde{\eta} = \frac{1}{2} \int_{\mathbb{R}} \eta'(\rho)\, \theta_0'(\rho)\, \mathrm{d}\rho, \tag{5.60}$$

$$\sigma = \frac{1}{2} \int_{\mathbb{R}} \left(\theta_0'(\rho)\right)^2 \mathrm{d}\rho, \tag{5.61}$$

$$\begin{aligned}\tilde{\mathcal{A}}^{k-2}(x,t) = \frac{1}{2} \int_{\mathbb{R}} \Big(& -\partial_\rho c_{k-1}(\rho,x,t)\, \Delta d_\Gamma(x,t) - 2\nabla \partial_\rho c_{k-1}(\rho,x,t) \cdot \mathbf{n}(x,t) \\ & + f_{k-1}(c_0, \ldots, c_{k-1})(\rho,x,t) + \mathcal{A}^{k-2}(\rho,x,t) \Big)\, \theta_0'(\rho)\, \mathrm{d}\rho.\end{aligned}$$

Additionally, c_k satisfies the matching condition (5.24) with $c_k^{\pm}$ given as in (5.13).

Proof. We prove this by using Lemma 2.3. Equation (5.57) follows directly from that result, when using $c_0 = \theta_0$. Regarding the case $k \geq 2$: Since the matching conditions hold, we get for

$$A^{k-1,\pm} := -\mu_{k-1}^{\pm} - \Delta c_{k-2}^{\pm} + f_{k-1}\left(c_0^{\pm}, \ldots, c_{k-1}^{\pm}\right)$$

the estimate

$$\begin{aligned} A^{k-1}(\rho,x,t) - A^{k-1,+}(x,t) &= -\mu_{k-1} + \mu_{k-1}^{+} - \Delta c_{k-2} + \Delta c_{k-2}^{+} \\ &\quad + f_{k-1}(c_0, \ldots, c_{k-1}) - f_{k-1}\left(c_0^{+}, \ldots, c_{k-1}^{+}\right) + \mathcal{O}\left(e^{-C\rho}\right) \\ &= \mathcal{O}\left(e^{-C\rho}\right) \end{aligned}$$

as $\rho \to \infty$. Here we also used $\eta' \equiv 0$ in $\mathbb{R}\backslash(-1,1)$. The case $\rho \to -\infty$ can be treated similarly.

Thus, Lemma 2.3 implies that (5.44) with normalization $c_k(0,x,t) = 0$ for all $(x,t) \in \Gamma(2\delta)$ has a unique bounded solution if and only if

$$\int_{\mathbb{R}} A^{k-1}(\rho,x,t)\,\theta_0'(\rho)\,\mathrm{d}\rho = 0 \text{ for all } (x,t) \in \Gamma(2\delta).$$

This condition is equivalent to (5.58) if we note that $c_0(\rho,x,t) = \theta_0(\rho)$ and thus

$$\int_{\mathbb{R}} \nabla\partial_\rho c_0\theta_0'\mathrm{d}\rho = 0, \int_{\mathbb{R}} \partial_{\rho\rho}c_0\theta_0'\mathrm{d}\rho = \int_{\mathbb{R}} \frac{1}{2}\partial_\rho\left(\theta_0'\right)^2\mathrm{d}\rho = 0.$$

By a slight abuse of notation we also plug the terms involving c_{k-1} (but no other terms of order $k-1$) into $\tilde{\mathcal{A}}^{k-2}$, as they only depend on terms of order $k-2$ or lower and can thus be treated as terms of order $k-2$ (cf. the discussion in Step 1 of Subsection 5.1.6). Now (2.2) implies the matching condition since $c_k^\pm = -\frac{A^{k-1,\pm}}{f''(\pm 1)}$ by (5.13). □

Remark 5.8. The definition of σ in (5.61) coincides with the one given in (1.17). To gain this insight, first remark that $\frac{1}{2}(\theta_0')^2 = f(\theta_0)$ as a consequence of integrating (1.36). Thus, we find

$$\frac{1}{2}\int_{\mathbb{R}}\left(\theta_0'\right)^2\mathrm{d}\rho = \int_{\mathbb{R}} f(\theta_0)\,\mathrm{d}\rho = \int_{\mathbb{R}}\sqrt{\frac{1}{2}f(\theta_0)}\theta_0'\mathrm{d}\rho = \frac{1}{2}\int_{-1}^{1}\sqrt{2f(s)}\mathrm{d}s.$$

Lemma 5.9. *Let $k \geq 1$ be any integer and B^{k-1}, $\mathcal{B}^{k-2}$ be defined as in (5.53) and (5.54). Moreover, assume that the matching conditions (5.24)-(5.27) are satisfied for $(c_l, \mu_l, \mathbf{v}_l)$ for all $l \in \{0,\ldots,k-1\}$.*

Then (5.46) has a (smooth) solution if and only if, in the case $k=1$

$$2\partial_t d_\Gamma + \int_{\mathbb{R}} \theta_0'\mathbf{v}_0\cdot\mathbf{n}\mathrm{d}\rho - [\mu_0]\,\Delta d_\Gamma - 2\,[\nabla\mu_0]\cdot\mathbf{n} + l_0 + j_0 d_\Gamma = 0 \tag{5.62}$$

or, in the case $k \geq 2$,

$$\begin{aligned}\bar{\mathcal{B}}^{k-2} = &\int_{\mathbb{R}} \theta_0'\left(-\mathbf{v}_{k-1}\cdot\mathbf{n} + \mathbf{v}_0\cdot\nabla^\Gamma h_{k-1}\right)\mathrm{d}\rho + 2\partial_t^\Gamma h_{k-1} - [\mu_0]\,\Delta^\Gamma h_{k-1} - 2\,[\nabla\mu_0]\cdot\nabla^\Gamma h_{k-1}\\ &+ [\mu_{k-1}]\,\Delta d_\Gamma + 2\,[\nabla\mu_{k-1}]\cdot\mathbf{n} - l_{k-1} - (j_{k-1}d_\Gamma - j_0 h_{k-1})\end{aligned} \tag{5.63}$$

in $\Gamma(2\delta)$, with

$$\begin{aligned}\bar{\mathcal{B}}^{k-2}(x,t) = &\int_{-\infty}^{\infty} \partial_\rho c_{k-1}(\rho,x,t)\,\mathbf{v}_0(\rho,x,t)\cdot\mathbf{n}(x,t) + \mathcal{B}^{k-2}(\rho,x,t)\,\mathrm{d}\rho.\\ &+ [c_{k-1}](x,t)\,\partial_t d_\Gamma(x,t).\end{aligned}$$

Furthermore, if (5.63) holds, then the solution is of the form

$$\begin{aligned}\mu_k(\rho,x,t) = &\bar{\mu}_k(x,t) + \left(l_k(x,t)\,d_\Gamma(x,t) - l_0(x,t)\,h_k(S(x,t),t)\right)\left(\eta(\rho) - \frac{1}{2}\right)\\ &+ \mu_{k-1}^*(\rho,x,t),\end{aligned} \tag{5.64}$$

where $\bar{\mu}_k$ is an arbitrary (smooth) function and μ^*_{k-1} depends only on functions of order $k-1$ or lower and is uniquely determined by the normalization

$$\int_{\mathbb{R}} \mu^*_{k-1}(\rho, x, t)\, \theta_0'(\rho)\, \mathrm{d}\rho = 0 \quad \forall (x,t) \in \Gamma(2\delta). \tag{5.65}$$

Additionally, μ^*_{k-1} satisfies

$$D_x^m D_t^n D_\rho^l \left(\mu^*_{k-1}(\pm\rho, x, t) - \mu^{*,\pm}_{k-1}(x,t)\right) = \mathcal{O}\left(e^{-\alpha\rho}\right) \text{ as } \rho \to \infty \tag{5.66}$$

for all $m, n, l \geq 0$ for some $\mu^{*,\pm}_{k-1}$ depending only on functions of order $k-1$ or lower.

Proof. We want to use Lemma 2.4 and first need to make sure that B^{k-1} has exponential decay in ρ. For B^0 (the case $k=1$) this follows directly from the matching conditions and the properties of η. In the case $k \geq 2$ we additionally use

$$\partial_t c_{k-2} - \Delta\mu_{k-2} + \sum_{i=0}^{k-2} \mathbf{v}_i \cdot \nabla c_{k-2-i} + U^+_{k-2}\eta^{C_{S,+}} + U^-_{k-2}\eta^{C_{S,-}} = \mathcal{O}\left(e^{-C|\rho|}\right) \text{ as } \rho \to \pm\infty$$

in $\Gamma(2\delta)$, since (5.14) holds in $\Omega^\pm_{T_0}$ and $\partial_t c^\pm_{k-2} - \Delta\mu^\pm_{k-2} + \sum_{i=0}^{k-2} \mathbf{v}^\pm_i \nabla c^\pm_{k-2-i} + U^\pm_{k-2} = 0$ in $\Omega^\mp_{T_0} \cap \Gamma(2\delta)$.

Due to the matching conditions, the requirement $\int_{\mathbb{R}} B^{k-1}\mathrm{d}\rho = 0$ is equivalent to (5.62) for $k=1$, or (5.63) for $k \geq 2$, if we note

$$\int_{\mathbb{R}} \eta''(\rho)\, \mathrm{d}\rho = \int_{\mathbb{R}} \partial_{\rho\rho}\mu_0(\rho, x, t)\, \mathrm{d}\rho = 0 \text{ and } \int_{\mathbb{R}} \eta''(\rho)\, \rho \mathrm{d}\rho = -1$$

for all $(x,t) \in \Gamma(2\delta)$. As in Lemma 5.58, we consider the c_{k-1} terms as terms of order $k-2$. Then the claim follows from Lemma 2.4 and the normalization (5.65) can be achieved by just adding a suitable constant. □

Lemma 5.10. *Let $k \geq 1$ be any integer and V^{k-1} and $\mathcal{V}^{k-2}$ be defined as in (5.47), (5.48). Moreover, assume that the matching conditions (5.24)–(5.27) are satisfied for $(c_l, \mu_l, \mathbf{v}_l, p_{l-1})$ for all $l \in \{0, \ldots, k-1\}$.*

Then

$$-\partial_{\rho\rho}\left(\mathbf{v}_k - \left(\mathbf{u}_k d_\Gamma - \mathbf{u}_0 h_k\right)\eta\right) \cdot \tau = \mathbf{V}^{k-1} \cdot \tau, \; k \geq 1 \tag{5.67}$$

has a solution for all $(x,t) \in \Gamma(2\delta)$ if and only if, in the case $k=1$

$$[\mathbf{v}_0] \cdot \tau \Delta d_\Gamma + 2\left([\nabla \mathbf{v}_0]^T \mathbf{n}\right) \cdot \tau + \mathbf{q}_0 \cdot \tau d_\Gamma - \mathbf{u}_0 \cdot \tau = 0 \tag{5.68}$$

or, if $k \geq 2$,

$$\begin{aligned} \bar{\mathcal{V}}^{k-2,\tau} &= [\mathbf{v}_{k-1}] \cdot \tau \Delta d_\Gamma - [\mathbf{v}_0] \cdot \tau \Delta^\Gamma h_{k-1} + 2\left(\left([\nabla \mathbf{v}_{k-1}]\right)^T \mathbf{n}\right) \cdot \tau - 2\left([\nabla \mathbf{v}_0]\right)^T \nabla^\Gamma h_{k-1} \cdot \tau \\ &\quad + \left(-\left(\mu_0^+ + \mu_0^-\right) + [p_0]\right) \nabla^\Gamma h_{k-1} \cdot \tau + \mathbf{q}_{k-1} \cdot \tau d_\Gamma - \mathbf{q}_0 \cdot \tau h_{k-1} \\ &\quad - \mathbf{u}_{k-1} \cdot \tau \end{aligned} \tag{5.69}$$

in $\Gamma(2\delta)$, where σ is defined as in Lemma 5.7 and

$$\bar{\mathcal{V}}^{k-2,\tau}(x,t) := -\int_{\mathbb{R}} \mathcal{V}^{k-2}(\rho, x, t) \cdot \tau(x,t)\, \mathrm{d}\rho.$$

Furthermore, if (5.69) holds and $k \geq 2$, then the solution is of the form

$$\mathbf{v}_k(\rho,x,t)\cdot\tau = v_k^\tau(x,t) + (\mathbf{u}_k(x,t)\,d_\Gamma(x,t) - \mathbf{u}_0(x,t)\,h_k(S(x,t),t))\cdot\tau\left(\eta(\rho)-\frac{1}{2}\right) + v_{k-1}^{\tau,*}(\rho,x,t) \tag{5.70}$$

where v_k^τ is an arbitrary function and $v_{k-1}^{\tau,*}$ depends only on functions of order lower or equal to $k-1$ and is uniquely determined by the normalization

$$\int_{\mathbb{R}} v_{k-1}^{\tau,*}(\rho,x,t)\,\theta_0'(\rho)\,\mathrm{d}\rho = 0 \quad \forall (x,t)\in\Gamma(2\delta).$$

Additionally, $v_{k-1}^{\tau,*}$ satisfies

$$D_x^m D_t^n D_\rho^l\left(v_{k-1}^{\tau,*}(\pm\rho,x,t) - v_{k-1}^{\tau,*,\pm}(x,t)\right) = \mathcal{O}\left(e^{-\alpha\rho}\right) \text{ as } \rho\to\infty \tag{5.71}$$

for all $m,n,l\geq 0$, for some $v_{k-1}^{\tau,*,\pm}$ depending only on functions of order lower or equal to $k-1$.

Proof. In the case $k=1$, the statement follows from the matching condition for $\mathbf{v}_0$ and the properties of η. For $k\geq 2$, the exponential decay of $\mathbf{V}^{k-1}\cdot\tau$ is due to the matching conditions and the fact that

$$\Delta\mathbf{v}_{k-2} - \nabla p_{k-2} - \sum_{i=0}^{k-2}\mu_i\nabla c_{k-2-i} + \mathbf{W}_{k-2}^+\eta^{C_S,+} + \mathbf{W}_{k-2}^-\eta^{C_S,-} = \mathcal{O}\left(e^{-C|\rho|}\right)$$

as $\rho\to\pm\infty$ in $\Gamma(2\delta)$. This may be shown as in the proof of Lemma 5.9. Note in particular that $\nabla c_0 = 0$.

Concerning the equivalence of $\int_{\mathbb{R}}\mathbf{V}^{k-1}\cdot\tau\mathrm{d}\rho = 0$ and (5.69), we note that the matching conditions imply

$$\int_{\mathbb{R}}\partial_{\rho\rho}\mathbf{v}_0\mathrm{d}\rho = 0, \tag{5.72}$$

and it holds

$$\mu_0 = \frac{1}{2}\left(\mu_0^+ + \mu_0^-\right) + \left(\mu_0^+ - \mu_0^-\right)\left(\eta - \frac{1}{2}\right)$$

by Lemma 5.19 below. Thus, using (5.32) we find

$$\int_{\mathbb{R}}\mu_0\partial_\rho c_0\nabla^\Gamma h_{k-1}\mathrm{d}\rho = \left(\mu_0^+ + \mu_0^-\right)\nabla^\Gamma h_{k-1}. \tag{5.73}$$

Furthermore, $\int_{\mathbb{R}}\eta''\mathrm{d}\rho = 0$ and $\int_{\mathbb{R}}\eta''\rho\mathrm{d}\rho = -1$ as well as $\mathbf{n}\cdot\tau = 0$. The statement then follows from Lemma 2.4. □

Lemma 5.11. *Let $k\geq 1$ be any integer and $\mathbf{V}^{k-1}$ and $\mathcal{V}^{k-2}$ be defined as in (5.47) and (5.48). Moreover, assume that the matching conditions (5.24)–(5.27) are satisfied for $\mathbf{v}_l,\mu_l,c_l,p_{l-1}$ for all $l\in\{0,\dots,k-1\}$ and also for p_{k-1}. Then*

$$-\partial_{\rho\rho}\left(\mathbf{v}_k - (\mathbf{u}_k d_\Gamma - \mathbf{u}_0 h_k)\,\eta\right)\cdot\mathbf{n} = \mathbf{V}^{k-1}\cdot\mathbf{n} - \partial_\rho p_{k-1},\; k\geq 1 \tag{5.74}$$

has a solution for all $(x,t) \in \Gamma(2\delta)$ *if and only if, in the case* $k=1$,

$$[\mathbf{v}_0]\cdot \mathbf{n}\Delta d_\Gamma + 2\left([\nabla \mathbf{v}_0]^T \mathbf{n}\right)\cdot \mathbf{n} - [p_0] + \left(\mu_0^+ + \mu_0^-\right) + \mathbf{q}_0\cdot \mathbf{n} d_\Gamma - \mathbf{u}_0 \cdot \mathbf{n} = 0 \tag{5.75}$$

or, in the case $k \geq 2$,

$$\begin{aligned}\bar{\mathcal{V}}^{k-2,\mathbf{n}} = & -[p_{k-1}] + [\mathbf{v}_{k-1}]\cdot \mathbf{n}\Delta d_\Gamma - [\mathbf{v}_0]\cdot \mathbf{n}\Delta^\Gamma h_{k-1} + \int_{\mathbb{R}} \mu_{k-1}\partial_\rho c_0 \mathrm{d}\rho \\ & + 2\left(([\nabla \mathbf{v}_{k-1}])^T \mathbf{n} - ([\nabla \mathbf{v}_0])^T \nabla^\Gamma h_{k-1}\right)\cdot \mathbf{n} + \mathbf{q}_{k-1}\cdot \mathbf{n} d_\Gamma - \mathbf{q}_0 \cdot \mathbf{n} h_{k-1} \\ & - \mathbf{u}_{k-1}\cdot \mathbf{n},\end{aligned} \tag{5.76}$$

where

$$\bar{\mathcal{V}}^{k-2,\mathbf{n}}(x,t) := -\int_{\mathbb{R}} \mu_0 \partial_\rho c_{k-1} + \mathcal{V}^{k-2}\cdot \mathbf{n} \mathrm{d}\rho.$$

Furthermore, if (5.76) holds, then the solution is of the form

$$\begin{aligned}\mathbf{v}_k(\rho,x,t)\cdot \mathbf{n} = & v_k^{\mathbf{n}}(x,t) + \left(\mathbf{u}_k(x,t) d_\Gamma(x,t) - \mathbf{u}_0(x,t) h_k(S(x,t),t)\right)\cdot \mathbf{n}\left(\eta(\rho) - \frac{1}{2}\right) \\ & + v_{k-1}^{\mathbf{n},*}(\rho,x,t),\end{aligned} \tag{5.77}$$

where $v_k^{\mathbf{n}}$ *is an arbitrary function and* $v_{k-1}^{\mathbf{n},*}$ *depends only on functions of order lower or equal to* $k-1$ *and is uniquely determined by the normalization*

$$\int_{\mathbb{R}} v_{k-1}^{\mathbf{n},*}(\rho,x,t)\,\theta_0'(\rho)\,\mathrm{d}\rho = 0 \quad \forall (x,t)\in \Gamma(2\delta). \tag{5.78}$$

Additionally, $v_{k-1}^{\mathbf{n},*}$ *satisfies*

$$D_x^m D_t^n D_\rho^l \left(v_{k-1}^{\mathbf{n},*}(\pm\rho,x,t) - v_{k-1}^{\mathbf{n},*,\pm}(x,t)\right) = \mathcal{O}\left(e^{-\alpha\rho}\right) \text{ as } \rho\to\infty \tag{5.79}$$

for all $m,n,l \geq 0$*, for some* $v_{k-1}^{\mathbf{n},*,\pm}$ *depending only on functions of order lower or equal to* $k-1$.

Proof. The proof is very similar to the proof of Lemma 5.10. In particular, the exponential decay of the right hand side of (5.74) follows as before, considering the assumed matching conditions. To get the condition (5.76), we use $\nabla^\Gamma h_i \cdot \mathbf{n} = 0$ for arbitrary i, see Proposition 2.13, and the properties of η. Moreover, we may use the properties of η. Lemma 2.4 again implies the statement. Note that in order to get the term $\mu_0^+ + \mu_0^-$ in (5.75) we use an argument similar to (5.73).

As before, we view c_{k-1} as a term of order $k-2$ and thus the right hand side of (5.74) and then also $v_{k-1}^{\mathbf{n},*}$ only depend on terms of order $k-1$ or lower. □

Remark 5.12. Note that we cannot use Lemma 5.11 if we have not verified that the matching conditions for p_{k-1} are satisfied. For this reason, we will in the following induction in Subsection 5.1.6 only assume that (5.76) is satisfied, which does **not** guarantee the existence of a solution $\mathbf{v}_k$ unless the aforementioned properties of p_{k-1} are shown.

In particular (5.76) implies

$$\int_{\mathbb{R}} \mathbf{V}^{k-1} \cdot \mathbf{n} \mathrm{d}\rho = [p_{k-1}] \text{ for all } (x,t) \in \Gamma(2\delta)$$

if the matching conditions (5.24)–(5.27) are satisfied for the functions $\mathbf{v}_l, \mu_l, c_l, p_{l-1}$ for all $l \in \{0, \dots, k-1\}$.

5.1.4. The Boundary Layer Expansion

To be able to guarantee that the approximate solutions satisfy boundary conditions akin to (1.23)–(1.25), we also need to consider a separate expansion close to the boundary of Ω. The process of finding the terms of the boundary expansion resembles the one for the inner expansion, as we also introduce a stretched variable. Note that due to the general Assumption 1.1 the projection

$$Pr_{\partial\Omega} : \{x \in \Omega |\, d_{\mathbf{B}}(x) \in (-\delta, 0)\} \to \partial\Omega$$

along the normal of $\partial\Omega$ is well-defined and smooth.

Notation. In the following we write $\mathbf{n}_{\partial\Omega}(x) := \mathbf{n}_{\partial\Omega}(Pr_{\partial\Omega}(x))$ and $\tau_{\partial\Omega}(x) := \tau_{\partial\Omega}(Pr_{\partial\Omega}(x))$ for $x \in \partial\Omega(\delta)$.

We assume that for $(x,t) \in \overline{\partial_T \Omega(\delta)}$ the identities

$$\begin{aligned}
c^{\epsilon}(x,t) &= c^{\epsilon}_{\mathbf{B}}\left(\frac{d_{\mathbf{B}}(x)}{\epsilon}, x, t\right),\\
\mu^{\epsilon}(x,t) &= \mu^{\epsilon}_{\mathbf{B}}\left(\frac{d_{\mathbf{B}}(x)}{\epsilon}, x, t\right),\\
p^{\epsilon}(x,t) &= p^{\epsilon}_{\mathbf{B}}\left(\frac{d_{\mathbf{B}}(x)}{\epsilon}, x, t\right),\\
\mathbf{v}^{\epsilon}(x,t) &= \mathbf{v}^{\epsilon}_{\mathbf{B}}\left(\frac{d_{\mathbf{B}}(x)}{\epsilon}, x, t\right)
\end{aligned} \tag{5.80}$$

hold for the solutions of (1.18)–(1.25) and smooth functions $c^{\epsilon}_{\mathbf{B}}, \mu^{\epsilon}_{\mathbf{B}}, p^{\epsilon}_{\mathbf{B}} : \mathbb{R} \times \overline{\partial_{T_0}\Omega(\delta)} \to \mathbb{R}$, $\mathbf{v}^{\epsilon}_{\mathbf{B}} : \mathbb{R} \times \overline{\partial_{T_0}\Omega(\delta)} \to \mathbb{R}^2$. Furthermore, we assume that the expansions

$$\begin{aligned}
c^{\epsilon}_{\mathbf{B}}(z,x,t) &\approx -1 + \sum_{k\geq 1} \epsilon^k c^{\mathbf{B}}_k(z,x,t),\\
\mu^{\epsilon}_{\mathbf{B}}(z,x,t) &\approx \sum_{k\geq 0} \epsilon^k \mu^{\mathbf{B}}_k(z,x,t),\\
p^{\epsilon}_{\mathbf{B}}(z,x,t) &\approx \sum_{k\geq 0} \epsilon^k p^{\mathbf{B}}_k(z,x,t),\\
\mathbf{v}^{\epsilon}_{\mathbf{B}}(z,x,t) &\approx \sum_{k\geq 0} \epsilon^k \mathbf{v}^{\mathbf{B}}_k(z,x,t)
\end{aligned} \tag{5.81}$$

are given for all $(z,x,t) \in (-\infty, 0] \times \overline{\partial_{T_0}\Omega(\delta)}$. As in the case of the inner expansion, we also assume that the **outer-boundary matching conditions**

$$\sup_{(x,t)\in\overline{\partial_{T_0}\Omega(\delta)}} \left|\partial_x^m \partial_t^n \partial_z^l \left(c^{\mathbf{B}}_k(z,x,t) - c^-_k(x,t)\right)\right| \leq Ce^{\alpha z}, \tag{5.82}$$

$$\sup_{(x,t)\in\overline{\partial_{T_0}\Omega(\delta)}} \left|\partial_x^m \partial_t^n \partial_z^l \left(\mu^{\mathbf{B}}_k(z,x,t) - \mu^-_k(x,t)\right)\right| \leq Ce^{\alpha z}, \tag{5.83}$$

$$\sup_{(x,t)\in\overline{\partial_{T_0}\Omega(\delta)}} \left|\partial_x^m \partial_t^n \partial_z^l \left(\mathbf{v}^{\mathbf{B}}_k(z,x,t) - \mathbf{v}^-_k(x,t)\right)\right| \leq Ce^{\alpha z}, \tag{5.84}$$

$$\sup_{(x,t)\in\overline{\partial_{T_0}\Omega(\delta)}} \left|\partial_x^m \partial_t^n \partial_z^l \left(p^{\mathbf{B}}_k(z,x,t) - p^-_k(x,t)\right)\right| \leq Ce^{\alpha z} \tag{5.85}$$

hold for some constants $\alpha, C > 0$ and all $z \leq 0$, $m, n, l \geq 0$. Plugging the assumed form of the exact solutions (5.80) into the equations (1.18)–(1.21) we obtain for $(x,t) \in \overline{\partial_{T_0}\Omega(\delta)}$ and $z = \frac{d_{\mathbf{B}}(x)}{\epsilon}$ the identities

$$\begin{aligned} -\partial_{zz}\mathbf{v}^{\epsilon}_{\mathbf{B}} + \partial_z p^{\epsilon}_{\mathbf{B}} \nabla d_{\mathbf{B}} &= \epsilon \left(2\partial_z D\mathbf{v}^{\epsilon}_{\mathbf{B}} \nabla d_{\mathbf{B}} + \partial_z \mathbf{v}^{\epsilon}_{\mathbf{B}} \Delta d_{\mathbf{B}} + \mu^{\epsilon} \partial_z c^{\epsilon}_{\mathbf{B}} \nabla d_{\mathbf{B}}\right) \\ &\quad + \epsilon^2 \left(\Delta \mathbf{v}^{\epsilon}_{\mathbf{B}} - \nabla p^{\epsilon}_{\mathbf{B}} + \mu^{\epsilon}_{\mathbf{B}} \nabla c^{\epsilon}_{\mathbf{B}}\right), \\ \partial_z \mathbf{v}^{\epsilon}_{\mathbf{B}} \cdot \nabla d_{\mathbf{B}} &= -\epsilon \operatorname{div} \mathbf{v}^{\epsilon}_{\mathbf{B}}, \end{aligned}$$

where ∇, div and Δ only operate on the spatial variable x, not on z. In the calculations we used $|\nabla d_{\mathbf{B}}|^2 = 1$ for $(x,t) \in \overline{\partial_{T_0}\Omega(\delta)}$.

Moreover, we have

$$\begin{aligned} \partial_{zz} c^{\epsilon}_{\mathbf{B}} - f'\left(c^{\epsilon}_{\mathbf{B}}\right) &= -\epsilon \left(\mu^{\epsilon}_{\mathbf{B}} + 2\partial_z \nabla c^{\epsilon}_{\mathbf{B}} \cdot \nabla d_{\mathbf{B}} + \partial_z c^{\epsilon}_{\mathbf{B}} \Delta d_{\mathbf{B}}\right) - \epsilon^2 \Delta c^{\epsilon}_{\mathbf{B}}, \\ \partial_{zz} \mu^{\epsilon}_{\mathbf{B}} &= \epsilon \left(-2\partial_z \nabla \mu^{\epsilon}_{\mathbf{B}} \cdot \nabla d_{\mathbf{B}} - \partial_z \mu^{\epsilon}_{\mathbf{B}} \Delta d_{\mathbf{B}} + \mathbf{v}^{\epsilon}_{\mathbf{B}} \cdot \nabla d_{\mathbf{B}} \partial_z c^{\epsilon}_{\mathbf{B}}\right) \\ &\quad + \epsilon^2 \left(\partial_t c^{\epsilon}_{\mathbf{B}} + \mathbf{v}^{\epsilon} \cdot \nabla c^{\epsilon}_{\mathbf{B}} - \Delta \mu^{\epsilon}_{\mathbf{B}}\right). \end{aligned}$$

Using the expansions as given in (5.81) and equating similar orders of ϵ – as before in the inner and outer expansions – we get the ordinary differential equations

$$-\partial_{zz}\mathbf{v}^{\mathbf{B}}_k + \partial_z p^{\mathbf{B}}_{k-1} \nabla d_{\mathbf{B}} = \mathbf{V}^{k-1}_{\mathbf{B}} \qquad k \geq 0, \tag{5.86}$$

$$\partial_z \mathbf{v}^{\mathbf{B}}_k \cdot \nabla d_{\mathbf{B}} = -\operatorname{div} \mathbf{v}^{\mathbf{B}}_{k-1} \qquad k \geq 0, \tag{5.87}$$

$$\partial_{zz} c^{\mathbf{B}}_k - f''(-1)\, c^{\mathbf{B}}_k = A^{k-1}_{\mathbf{B}} \qquad k \geq 1, \tag{5.88}$$

$$\partial_{zz} \mu^{\mathbf{B}}_k = B^{k-1}_{\mathbf{B}} \qquad k \geq 0, \tag{5.89}$$

for $(z,x,t) \in (-\infty, 0] \times \overline{\partial_{T_0}\Omega(\delta)}$, where $\mathbf{V}^{k-1}_{\mathbf{B}} = \mathbf{V}^{k-1}_{\mathbf{B}}(z,x,t)$, $A^{k-1}_{\mathbf{B}} = A^{k-1}_{\mathbf{B}}(z,x,t)$ and $B^{k-1}_{\mathbf{B}} = B^{k-1}_{\mathbf{B}}(z,x,t)$. In detail, we have

$$\begin{aligned} \mathbf{V}^{k-1}_{\mathbf{B}} &:= 2\partial_z D\mathbf{v}^{\mathbf{B}}_{k-1} \nabla d_{\mathbf{B}} + \partial_z \mathbf{v}^{\mathbf{B}}_{k-1} \Delta d_{\mathbf{B}} + \mu^{\mathbf{B}}_0 \partial_z c^{\mathbf{B}}_{k-1} \nabla d_{\mathbf{B}} \\ &\quad + \Delta \mathbf{v}^{\mathbf{B}}_{k-2} - \nabla p^{\mathbf{B}}_{k-2} + \sum_{i=0}^{k-2} \mu^{\mathbf{B}}_i \nabla c^{\mathbf{B}}_{k-2-i}. \end{aligned} \tag{5.90}$$

Here we made use of the fact that $c^{\mathbf{B}}_0 = -1$ and thus $\partial_z^l \partial_x^k c^{\mathbf{B}}_0 = 0$ for $k \geq 1$ or $l \geq 1$.

Moreover,

$$A^{k-1}_{\mathbf{B}} := -\mu^{\mathbf{B}}_{k-1} - 2\partial_z \nabla c^{\mathbf{B}}_{k-1} \cdot \nabla d_{\mathbf{B}} - \partial_z c^{\mathbf{B}}_{k-1} \Delta d_{\mathbf{B}} - \Delta c^{\mathbf{B}}_{k-2} + f_{k-1}\left(c^{\mathbf{B}}_0, \ldots, c^{\mathbf{B}}_{k-1}\right) \tag{5.91}$$

and

$$\begin{aligned} B^{k-1}_{\mathbf{B}} &:= -2\partial_z \nabla \mu^{\mathbf{B}}_{k-1} \cdot \nabla d_{\mathbf{B}} - \partial_z \mu^{\mathbf{B}}_{k-1} \Delta d_{\mathbf{B}} + \sum_{\substack{i,j \geq 0 \\ i+j=k-1}} \mathbf{v}^{\mathbf{B}}_i \cdot \nabla d_{\mathbf{B}} \partial_z c^{\mathbf{B}}_j + \partial_t c^{\mathbf{B}}_{k-2} \\ &\quad + \sum_{\substack{i,j \geq 0 \\ i+j=k-2}} \mathbf{v}^{\mathbf{B}}_i \cdot \nabla c^{\mathbf{B}}_j - \Delta \mu^{\mathbf{B}}_{k-2}. \end{aligned} \tag{5.92}$$

We used the convention that all terms with negative index are supposed to be zero, i.e. $\mu_{-2} = \mu_{-1} = 0$.

To enforce the Dirichlet boundary condition we suppose that

$$c_k^{\mathbf{B}}(0,x,t) = \frac{\mu_{k-1}^{\mathbf{B}}(0,x,t)}{f''(-1)} \qquad \text{for all } (x,t) \in \overline{\partial_{T_0}\Omega(\delta)}, k \geq 1, \tag{5.93}$$

$$\mu_k^{\mathbf{B}}(0,x,t) = 0 \qquad \text{for all } (x,t) \in \partial_{T_0}\Omega, k \geq 0. \tag{5.94}$$

Regarding the boundary condition of the Stokes system we calculate

$$\begin{aligned} 2D_s\left(\mathbf{v}_k^{\mathbf{B}}\left(\frac{d_{\mathbf{B}}(x)}{\epsilon},x,t\right)\right)\mathbf{n}_{\partial\Omega}(x) &= \frac{1}{\epsilon}\left(\mathbf{I}+\mathbf{n}_{\partial\Omega}(x)\otimes\mathbf{n}_{\partial\Omega}(x)\right)\partial_z\mathbf{v}_k^{\mathbf{B}}\left(\frac{d_{\mathbf{B}}(x)}{\epsilon},x,t\right) \\ &\quad + 2D_s\mathbf{v}_k^{\mathbf{B}}\left(\frac{d_{\mathbf{B}}(x)}{\epsilon},x,t\right)\mathbf{n}_{\partial\Omega}(x) \end{aligned}$$

and thus impose

$$\begin{aligned} -\left(\mathbf{I}+\mathbf{n}_{\partial\Omega}(x)\otimes\mathbf{n}_{\partial\Omega}(x)\right)\partial_z\mathbf{v}_k^{\mathbf{B}}(0,x,t) &= 2D_s\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)\,\mathbf{n}_{\partial\Omega}(x) \\ &\quad - p_{k-1}^{\mathbf{B}}(0,x,t)\,\mathbf{n}_{\partial\Omega}(x) + \alpha_0\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t) \end{aligned} \tag{5.95}$$

for all $(x,t) \in \partial_{T_0}\Omega, k \geq 0$.

Remark 5.13. Let us comment on the boundary condition (5.93). The reason for choosing this is threefold:

First and foremost, we want the equality $c_k^{\mathbf{B}}(0,x,t) = 0$ to hold for $(x,t) \in \partial_{T_0}\Omega$ and $k \geq 1$, which corresponds to a Dirichlet boundary condition for the approximate solution c_A^ϵ (cf. Definition 6.2). Equation (5.93) ensures this equality, if (5.94) holds for $k-1$.

Second, we want to get a unique solution $c_k^{\mathbf{B}}$ on $\overline{\partial_{T_0}\Omega(\delta)}$ and thus need a normalization like (5.93) to hold on all of $\overline{\partial_{T_0}\Omega(\delta)}$. Moreover, choosing (5.93) we may ensure that $c_k^{\mathbf{B}}$ only depends on terms of order $k-1$ or lower.

Third, (5.93) allows us to immediately deduce that $c_1^{\mathbf{B}}(z,x,t) = c_1^-(x,t)$ for all $(x,t) \in \overline{\partial_{T_0}\Omega(\delta)}$, see Corollary 5.18. Choosing (5.93) differently for higher orders, one may show that even more terms of the boundary layer expansion "vanish" in that way. As we do not need this result in the present context and it would needlessly complicate matters, we skip it here. See [14], Lemma 4.7 for the argumentation in the Neumann case.

Lemma 5.14. *Let $k \geq 1$ and assume that for all $i \in \{0,\ldots,k-1\}$ the functions c_i^-, μ_i^-, $\mathbf{v}_i^-$, p_i^-, $c_i^{\mathbf{B}}$, $\mu_i^{\mathbf{B}}$, $\mathbf{v}_i^{\mathbf{B}}$, $p_{i-1}^{\mathbf{B}}$ are known and satisfy the matching conditions (5.82)–(5.85). Then equation (5.88) together with the boundary condition (5.93) has a unique bounded solution $c_k^{\mathbf{B}}$ in $(-\infty,0]$ for all $(x,t) \in \overline{\partial_{T_0}\Omega(\delta)}$. Moreover, the solution $c_k^{\mathbf{B}}$ satisfies the matching condition (5.82) with c_k^- as in (5.13) and the boundary condition (5.93).*

Proof. The proof is very similar to the one given in [47], Lemma 3.2.14, p. 59 and we will only show the existence of a unique bounded solution for the different boundary condition (5.93). For the matching condition (5.82) we refer to [47] (note that f' transfers to f'' in the present work).

We fix $(x,t) \in \overline{\partial_{T_0}\Omega(\delta)}$, rewrite (5.88) into

$$\mathbf{x}'(z) = A\mathbf{x}(z) + \mathbf{g}(z)$$

and use the boundary condition (5.93) to gain an initial value $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}_1(z) = c_k^{\mathbf{B}}(z,x,t)$, $\mathbf{x}_2(z) = \mathbf{x}_1'(z)$, $A = \begin{pmatrix} 0 & 1 \\ f''(-1) & 0 \end{pmatrix}$, $\mathbf{g}(z) = \begin{pmatrix} 0 \\ A_{\mathbf{B}}^{k-1}(z,x,t) \end{pmatrix}$, and $\mathbf{x}_0 =$

$\begin{pmatrix} \frac{1}{f''(-1)}\mu_{k-1}^{\mathbf{B}}(0,x,t) \\ C_0 \end{pmatrix}$. Here $C_0 \in \mathbb{R}$ is an arbitrary constant. Since $\pm\sqrt{f''(-1)}$ are the eigenvalues of A, we can choose a transformation matrix $T \in \mathbb{R}^{2\times 2}$ such that

$$T^{-1}AT = \begin{pmatrix} -\sqrt{f''(-1)} & 0 \\ 0 & \sqrt{f''(-1)} \end{pmatrix} =: D.$$

Setting $\mathbf{w}(z) = T^{-1}\mathbf{x}(z)$ we find

$$\mathbf{w}'(z) = D\mathbf{w}(z) + \mathbf{h}(z) \tag{5.96}$$

with initial value $\mathbf{w}(0) = T^{-1}\mathbf{x}_0$, where $\mathbf{h}(z) = T^{-1}\mathbf{g}(z)$. The general solution of (5.96) is given by

$$\mathbf{w}(z) = C_1 e^{-\sqrt{f''(-1)}z}\mathbf{e}_1 + C_2 e^{\sqrt{f''(-1)}z}\mathbf{e}_2 + \int_0^z \operatorname{diag}\left(e^{-\sqrt{f''(-1)}(z-s)}, e^{\sqrt{f''(-1)}(z-s)}\right)\mathbf{h}(s)\,\mathrm{d}s$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are the standard unit vectors in $\mathbb{R}^2$ and $C_1, C_2 \in \mathbb{R}$ are constants which will be defined in the following. As all lower order terms are supposed to be known and satisfy the matching conditions, we find that $\mathbf{h}$ is bounded in $(-\infty, 0]$. Thus, we may show that the solution $\mathbf{w}$ is bounded in $(-\infty, 0]$ if and only if

$$C_1 = -\int_0^{-\infty} e^{\sqrt{f''(-1)}s}\mathbf{h}_1(s)\,\mathrm{d}s.$$

Considering the initial value, we have

$$C_1\mathbf{e}_1 + C_2\mathbf{e}_2 = \mathbf{w}(0) = T^{-1}\left(\frac{\mu_{k-1}^{\mathbf{B}}(0,x,t)}{f''(-1)}\mathbf{e}_1 + C_0\mathbf{e}_2\right)$$

and may therefore determine

$$C_0 = \frac{1}{\mathbf{e}_1 T^{-1}\mathbf{e}_2}\left(C_1 - \mathbf{e}_1 T^{-1}\mathbf{e}_1 \frac{\mu_{k-1}^{\mathbf{B}}(0,x,t)}{f''(-1)}\right)$$

and

$$C_2 = \mathbf{e}_2 T^{-1}\left(\frac{\mu_{k-1}^{\mathbf{B}}(0,x,t)}{f''(-1)}\mathbf{e}_1 + C_0\mathbf{e}_2\right).$$

Note that $\mathbf{e}_1 T^{-1}\mathbf{e}_2 \neq 0$ since

$$\mathbf{e}_1 T^{-1}\mathbf{e}_2 = -\frac{\mathbf{e}_1 D T^{-1}\mathbf{e}_2}{\sqrt{f''(-1)}} = -\frac{\mathbf{e}_1 T^{-1} A\mathbf{e}_2}{\sqrt{f''(-1)}} = -\frac{1}{\sqrt{f''(-1)}}\mathbf{e}_1 T^{-1}\mathbf{e}_1,$$

which would imply $\mathbf{e}_1 T^{-1}\mathbf{e}_2 = \mathbf{e}_1 T^{-1}\mathbf{e}_1 = 0$ otherwise – a contradiction to the invertibility of T^{-1}. This shows that there is a unique bounded solution to (5.88) satisfying (5.93). □

Lemma 5.15. *Let $k \geq 0$ and assume that for all $i \in \{0, \ldots, k-1\}$ the functions c_i^-, μ_i^-, $\mathbf{v}_i^-$, p_i^-, $c_i^{\mathbf{B}}$, $\mu_i^{\mathbf{B}}$, $\mathbf{v}_i^{\mathbf{B}}$, $p_{i-1}^{\mathbf{B}}$ are known and satisfy the matching conditions (5.82)–(5.85) as well as the outer equation*

$$\Delta \mu_i^- = \partial_t c_i^- + \sum_{j=0}^{i} \mathbf{v}_j^- \cdot \nabla c_{i-j}^- \ \text{in}\ \Omega_{T_0}^-.$$

Moreover, assume that μ_k^- is known and satisfies

$$\mu_k^-(x,t) = -\int_{-\infty}^{0} \int_{-\infty}^{\tilde{z}} B_{\mathbf{B}}^{k-1}\left(z', x, t\right) \mathrm{d}z' \mathrm{d}\tilde{z} \tag{5.97}$$

for all $(x,t) \in \partial_{T_0}\Omega$. Then

$$\mu_k^{\mathbf{B}}(z,x,t) = \mu_k^-(x,t) + \int_{-\infty}^{z} \int_{-\infty}^{\tilde{z}} B_{\mathbf{B}}^{k-1}\left(z', x, t\right) \mathrm{d}z' \mathrm{d}\tilde{z}, \quad (z,x,t) \in (-\infty, 0] \times \overline{\partial_{T_0}\Omega(\delta)} \tag{5.98}$$

satisfies the boundary equation (5.89), the boundary condition (5.94), and the matching condition (5.83).

Proof. First, we ensure that $\mu_{\mathbf{B}}^k$ and the right hand side of (5.97) are well-defined. This can be verified as follows: as we suppose that the matching conditions and the outer equation (5.14) hold for all $i \in \{0, \ldots, k-1\}$, we have

$$\sup_{(x,t)\in\overline{\partial_{T_0}\Omega(\delta)}} \left| \partial_t c_{k-2}^{\mathbf{B}} + \sum_{\substack{i,j\geq 0 \\ i+j=k-2}} \mathbf{v}_i^{\mathbf{B}} \cdot \nabla c_j^{\mathbf{B}} - \Delta \mu_{k-2}^{\mathbf{B}} \right| \leq C e^{\alpha z}$$

for all $z \in (-\infty, 0]$. All other appearing terms in $B_{\mathbf{B}}^{k-1}$ include derivatives in z-direction, which vanish exponentially fast as $z \to -\infty$ due to the matching conditions. Thus,

$$\sup_{(x,t)\in\overline{\partial_{T_0}\Omega(\delta)}} \left| B_{\mathbf{B}}^{k-1} \right| \leq C e^{\alpha z}$$

for all $z \in (-\infty, 0]$, which implies that $\mu_{\mathbf{B}}^k$ and (5.97) are well-defined. The boundary condition (5.89) and the matching condition (5.83) now follow immediately from the definition of $\mu_{\mathbf{B}}^k$ and the condition (5.97). That $\mu_{\mathbf{B}}^k$ satisfies equation (5.89) follows by differentiating the defining equation twice with respect to z. □

To improve the readability of the following lemma we define

$$\begin{aligned} \mathbf{U}^{k-2} := &\int_{-\infty}^{0} \mu_0^{\mathbf{B}} \partial_z c_{k-1}^{\mathbf{B}} \mathbf{n}_{\partial\Omega} + \Delta \mathbf{v}_{k-2}^{\mathbf{B}} - \nabla p_{k-2}^{\mathbf{B}} - \sum_{i=0}^{k-2} \mu_i^{\mathbf{B}} \nabla c_{k-2-i}^{\mathbf{B}} \mathrm{d}\tilde{z} \\ &+ \left(D\left(\tilde{\mathbf{U}}^{k-2}\right) - \nabla\left(\tilde{\mathbf{U}}^{k-2}\right)\right) \mathbf{n}_{\partial\Omega} + \tilde{\mathbf{U}}^{k-2}\left(\Delta d_{\mathbf{B}} - \alpha_0\right) + \operatorname{div}\left(\tilde{\mathbf{U}}^{k-2}\right) \mathbf{n}_{\partial\Omega}, \end{aligned} \tag{5.99}$$

where

$$\tilde{\mathbf{U}}^{k-2}(x,t) := -\left(\int_{-\infty}^{0} \operatorname{div}\mathbf{v}_{k-2}^{\mathbf{B}}(\tilde{z},x,t)\,\mathrm{d}\tilde{z}\right)\mathbf{n}_{\partial\Omega} - \left(\int_{-\infty}^{0}\int_{-\infty}^{\tilde{z}} \mathbf{V}_{\mathbf{B}}^{k-2}(z',x,t)\cdot\tau_{\partial\Omega}\mathrm{d}z'\mathrm{d}\tilde{z}\right)\tau_{\partial\Omega}$$

for $k \geq 0$, with $\mathbf{U}^{-2} = \mathbf{U}^{-1} = 0$.

Lemma 5.16. *Let $k \geq 0$ and assume that for all $i \in \{0,\ldots,k-1\}$ the functions c_i^-, μ_i^-, $\mathbf{v}_i^-$, p_i^-, $c_i^{\mathbf{B}}$, $\mu_i^{\mathbf{B}}$, $\mathbf{v}_i^{\mathbf{B}}$, $p_{i-1}^{\mathbf{B}}$ are known and satisfy the matching conditions (5.82)–(5.85) as well as the outer equations*

$$\begin{aligned}-\Delta\mathbf{v}_i^- + \nabla p_i^- &= \sum_{j=0}^{k-1}\mu_j^-\nabla c_{i-j}^- && \text{in } \Omega_{T_0}^-,\\ \operatorname{div}\mathbf{v}_i^- &= 0 && \text{in } \Omega_{T_0}^-.\end{aligned}$$

Moreover, we assume that

$$\left(-2D_s\mathbf{v}_{k-1}^- + p_{k-1}^-\mathbf{I}\right)\cdot\mathbf{n}_{\partial\Omega} = \alpha_0\mathbf{v}_{k-1}^- - \mathbf{U}^{k-2} \text{ on } \partial_{T_0}\Omega \tag{5.100}$$

for $\mathbf{U}^{k-2}$ as in (5.99) and that for $i \in \{0,\ldots,k-1\}$ the terms $\mathbf{v}_i^{\mathbf{B}}$ and $p_{i-1}^{\mathbf{B}}$ are given by

$$\begin{aligned}\mathbf{v}_i^{\mathbf{B}}(z,x,t) = &-\left(\int_{-\infty}^{z}\int_{-\infty}^{\tilde{z}} \mathbf{V}_{\mathbf{B}}^{i-1}(z',x,t)\cdot\tau_{\partial\Omega}(x,t)\,\mathrm{d}z'\mathrm{d}\tilde{z}\right)\tau_{\partial\Omega}(x,t)\\ &-\left(\int_{-\infty}^{z}\operatorname{div}\mathbf{v}_{i-1}^{\mathbf{B}}(\tilde{z},x,t)\,\mathrm{d}\tilde{z}\right)\mathbf{n}_{\partial\Omega}(x,t) + \mathbf{v}_i^-(x,t)\end{aligned} \tag{5.101}$$

and

$$p_{i-1}^{\mathbf{B}}(z,x,t) = \int_{-\infty}^{z}\mathbf{V}_{\mathbf{B}}^{i-1}(\tilde{z},x,t)\cdot\mathbf{n}_{\partial\Omega}(x,t)\,\mathrm{d}\tilde{z} - \operatorname{div}\mathbf{v}_{i-1}^{\mathbf{B}}(z,x,t) + p_{i-1}^-(x,t) \tag{5.102}$$

for $(z,x,t) \in (-\infty,0] \times \overline{\partial_{T_0}\Omega(\delta)}$. Then, for any smooth $\mathbf{v}_k^- : \Omega_{T_0}^- \to \mathbb{R}^2$ the terms $\mathbf{v}_k^{\mathbf{B}}$, $p_{k-1}^{\mathbf{B}}$ defined by (5.101), (5.102) (for $i = k$) satisfy the boundary equations (5.86), (5.87), the boundary condition (5.95) and the matching conditions (5.84), (5.85).

Proof. Note that the assumed matching conditions and outer equations imply that all involved integrals exist, which can be seen similar to the argumentation in the proof of Lemma 5.15. Thus, $\mathbf{U}^{k-2}$ and $\mathbf{v}_i^{\mathbf{B}}$, $p_i^{\mathbf{B}}$, $i \in \{0,\ldots,k\}$, are well-defined and smooth. Note also that the definition of $p_{i-1}^{\mathbf{B}}$ in (5.102) does not lead to circular reasoning, as $\mathbf{V}_{\mathbf{B}}^{i-1}$ only depends on $p_{i-2}^{\mathbf{B}}$ and not on $p_{i-1}^{\mathbf{B}}$. Additionally, it follows from the definition that $\mathbf{v}_k^{\mathbf{B}}$, $p_{k-1}^{\mathbf{B}}$ satisfy the matching conditions (5.84), (5.85). Remark here that due to our assumptions

$$\lim_{z\to-\infty}\operatorname{div}\mathbf{v}_{k-1}^{\mathbf{B}}(z,x,t) = \operatorname{div}\mathbf{v}_{k-1}^-(x,t) = 0$$

for all $(x,t) \in \overline{\partial_{T_0}\Omega(\delta)}$.

To validate that the boundary equation (5.86) is satisfied, we differentiate (5.101) twice with respect to z and get

$$\begin{aligned}\partial_{zz}\mathbf{v}_k^{\mathbf{B}} &= -\left(\left(\partial_z \mathrm{div}\mathbf{v}_{k-1}^{\mathbf{B}}\right)\mathbf{n}_{\partial\Omega} + \left(\mathbf{V}_{\mathbf{B}}^{k-1}\cdot\tau_{\partial\Omega}\right)\tau_{\partial\Omega}\right)\\ &= -\left(\left(\mathbf{V}_{\mathbf{B}}^{k-1}\cdot\mathbf{n}_{\partial\Omega}\right)\mathbf{n}_{\partial\Omega} - \partial_z p_{k-1}^{\mathbf{B}}\mathbf{n}_{\partial\Omega} + \left(\mathbf{V}_{\mathbf{B}}^{k-1}\cdot\tau_{\partial\Omega}\right)\tau_{\partial\Omega}\right)\end{aligned}$$

in $(-\infty,0]\times\overline{\partial_{T_0}\Omega(\delta)}$, which implies (5.86). Here we used (5.102) differentiated with respect to z in the second line. (5.87) follows immediately by differentiating (5.101) once with respect to z and taking its normal component.

Next we verify that the boundary condition (5.95) is satisfied. For that we compute

$$\begin{aligned}p_{k-1}^{\mathbf{B}}(0,x,t) &= \int_{-\infty}^{0}\mathbf{V}_{\mathbf{B}}^{k-1}(\tilde z,x,t)\cdot\mathbf{n}_{\partial\Omega}\mathrm{d}\tilde z - \mathrm{div}\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t) + p_{k-1}^{-}(x,t)\\
&= \left(2\left(D\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t) - D\mathbf{v}_{k-1}^{-}(x,t)\right)\mathbf{n}_{\partial\Omega} + \left(\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-\mathbf{v}_{k-1}^{-}(x,t)\right)\Delta d_{\mathbf{B}}\right)\cdot\mathbf{n}_{\partial\Omega}\\
&\quad + \partial_z\mathbf{v}_k^{\mathbf{B}}(0,x,t)\cdot\mathbf{n}_{\partial\Omega} + p_{k-1}^{-}(x,t) + \mathbf{R}_{k-2}^{\mathbf{B}}(x,t)\cdot\mathbf{n}_{\partial\Omega}\\
&= \left(2\left(D_s\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t) - D_s\mathbf{v}_{k-1}^{-}(x,t)\right)\mathbf{n}_{\partial\Omega} + \alpha_0\left(\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-\mathbf{v}_{k-1}^{-}(x,t)\right)\right)\cdot\mathbf{n}_{\partial\Omega}\\
&\quad + \left(\left(D\left(\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-\mathbf{v}_{k-1}^{-}(x,t)\right)\right)\mathbf{n}_{\partial\Omega} + \left(\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-\mathbf{v}_{k-1}^{-}(x,t)\right)(\Delta d_{\mathbf{B}}-\alpha_0)\right)\cdot\mathbf{n}_{\partial\Omega}\\
&\quad - \left(\left(\nabla\left(\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-\mathbf{v}_{k-1}^{-}(x,t)\right)\right)\mathbf{n}_{\partial\Omega}\right)\cdot\mathbf{n}_{\partial\Omega} + 2\partial_z\mathbf{v}_k^{\mathbf{B}}(0,x,t)\cdot\mathbf{n}_{\partial\Omega} + \mathrm{div}\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)\\
&\quad + p_{k-1}^{-}(x,t) + \mathbf{R}_{k-2}^{\mathbf{B}}(x,t)\cdot\mathbf{n}_{\partial\Omega},\end{aligned}\tag{5.103}$$

for $(x,t)\in\partial_{T_0}\Omega$, where we used $-\mathrm{div}\mathbf{v}_{k-1}^{\mathbf{B}} = \partial_z\mathbf{v}_k^{\mathbf{B}}\cdot\mathbf{n}_{\partial\Omega}$, which is due to (5.101), and the notation

$$\mathbf{R}_{k-2}^{\mathbf{B}} := \int_{-\infty}^{0}\mu_0^{\mathbf{B}}\partial_z c_{k-1}^{\mathbf{B}}\mathbf{n}_{\partial\Omega} + \Delta\mathbf{v}_{k-2}^{\mathbf{B}} - \nabla p_{k-2}^{\mathbf{B}} - \sum_{i=0}^{k-2}\mu_i^{\mathbf{B}}\nabla c_{k-2-i}^{\mathbf{B}}\mathrm{d}\tilde z.$$

Here we consider $c_{k-1}^{\mathbf{B}}$ as a term of order $k-2$, which is justified, as it only depends on terms of order $k-2$ or lower (cf. Lemma 5.14 and Subsection 5.1.6, Step 1). Now by (5.101) it holds

$$\begin{aligned}\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t) - \mathbf{v}_{k-1}^{-}(x,t) &= -\left(\int_{-\infty}^{0}\mathrm{div}\mathbf{v}_{k-2}^{\mathbf{B}}(\tilde z,x,t)\,\mathrm{d}\tilde z\right)\mathbf{n}_{\partial\Omega}\\ &\quad -\left(\int_{-\infty}^{0}\int_{-\infty}^{\tilde z}\mathbf{V}_{\mathbf{B}}^{k-2}\left(z',x,t\right)\cdot\tau_{\partial\Omega}\mathrm{d}z'\mathrm{d}\tilde z\right)\tau_{\partial\Omega}\end{aligned}\tag{5.104}$$

and

$$\begin{aligned}\mathrm{div}\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t) &= -\mathrm{div}\left(\left(\int_{-\infty}^{0}\mathrm{div}\mathbf{v}_{k-2}^{\mathbf{B}}(\tilde z,x,t)\,\mathrm{d}\tilde z\right)\mathbf{n}_{\partial\Omega}(x,t)\right)\\ &\quad - \mathrm{div}\left(\left(\int_{-\infty}^{0}\int_{-\infty}^{\tilde z}\mathbf{V}_{\mathbf{B}}^{k-2}\left(z',x,t\right)\cdot\tau_{\partial\Omega}(x,t)\,\mathrm{d}z'\mathrm{d}\tilde z\right)\tau_{\partial\Omega}(x,t)\right),\end{aligned}$$

since it holds $\operatorname{div}\mathbf{v}_{k-1}^{-}=0$ as a consequence of (5.15). Using this in (5.103) we get

$$\begin{aligned}
&p_{k-1}^{\mathbf{B}}(0,x,t)-\left(2D_s\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)\,\mathbf{n}_{\partial\Omega}+\alpha_0\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)+2\partial_z\mathbf{v}_k^{\mathbf{B}}(0,x,t)\right)\cdot\mathbf{n}_{\partial\Omega}\\
&\quad=p_{k-1}^{-}-\left(D_s\mathbf{v}_{k-1}^{-}\mathbf{n}_{\partial\Omega}+\alpha_0\mathbf{v}_{k-1}^{-}\right)\cdot\mathbf{n}_{\partial\Omega}+\mathbf{U}^{k-2}\cdot\mathbf{n}_{\partial\Omega}\\
&\quad=0
\end{aligned}$$

for all $(x,t)\in\partial_{T_0}\Omega$, due to (5.100). This shows the assertion for the normal part of (5.95), as

$$\left(\left(\mathbf{I}+\mathbf{n}_{\partial\Omega}(x)\otimes\mathbf{n}_{\partial\Omega}(x)\right)\partial_z\mathbf{v}_k^{\mathbf{B}}(0,x,t)\right)\cdot\mathbf{n}_{\partial\Omega}=2\partial_z\mathbf{v}_k^{\mathbf{B}}(0,x,t)\cdot\mathbf{n}_{\partial\Omega}.$$

For the tangential part we may use a similar strategy and consider

$$\begin{aligned}
&-\partial_z\mathbf{v}_k^{\mathbf{B}}(0,x,t)\cdot\tau_{\partial\Omega}=\int_{-\infty}^{0}\mathbf{V}_{\mathbf{B}}^{k-1}(\tilde z,x,t)\cdot\tau_{\partial\Omega}\mathrm{d}\tilde z\\
&=\left(2\left(D_s\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-D_s\mathbf{v}_{k-1}^{-}(x,t)\right)\mathbf{n}_{\partial\Omega}+\alpha_0\left(\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-\mathbf{v}_{k-1}^{-}(x,t)\right)\right)\cdot\tau_{\partial\Omega}\\
&\quad+\left(\left(D\left(\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-\mathbf{v}_{k-1}^{-}(x,t)\right)-\nabla\left(\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-\mathbf{v}_{k-1}^{-}(x,t)\right)\right)\mathbf{n}_{\partial\Omega}\right)\cdot\tau_{\partial\Omega}\\
&\quad+\left(\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)-\mathbf{v}_{k-1}^{-}(x,t)\right)(\Delta d_{\mathbf{B}}-\alpha_0)\cdot\tau_{\partial\Omega}+\mathbf{R}_{k-2}^{\mathbf{B}}(x,t)\cdot\tau_{\partial\Omega}
\end{aligned}$$

for $(x,t)\in\partial_{T_0}\Omega$. Again using (5.104), we find

$$\begin{aligned}
&-\left(2D_s\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)\,\mathbf{n}_{\partial\Omega}+\alpha_0\mathbf{v}_{k-1}^{\mathbf{B}}(0,x,t)+\partial_z\mathbf{v}_k^{\mathbf{B}}(0,x,t)\right)\cdot\tau_{\partial\Omega}\\
&=-\left(2D_s\mathbf{v}_{k-1}^{-}\mathbf{n}_{\partial\Omega}+\alpha_0\mathbf{v}_{k-1}^{-}\right)\cdot\tau_{\partial\Omega}+\mathbf{U}^{k-2}\cdot\tau_{\partial\Omega}\\
&=0
\end{aligned}$$

for $(x,t)\in\partial_{T_0}\Omega$ due to (5.100). □

Remark 5.17. Lemmata 5.15 and 5.16 provide boundary conditions on $\partial_{T_0}\Omega$, which we will refer to as **boundary compatibility conditions** in the following. These complete the system (5.157)–(5.166), which will be introduced in Subsection 5.1.6, allowing for outer and inner expansion terms of order k to be constructed. More precisely, the strategy pursued in Subsection 5.1.6 will be the following:

Assume that for all $i\in\{0,\dots,k-1\}$ the functions c_i^{-}, μ_i^{-}, $\mathbf{v}_i^{-}$, p_i^{-}, $c_i^{\mathbf{B}}$, $\mu_i^{\mathbf{B}}$, $\mathbf{v}_i^{\mathbf{B}}$, $p_{i-1}^{\mathbf{B}}$ are known and satisfy all the usual conditions (matching conditions, outer equations, boundary compatibility conditions, ...). Then Lemma 5.14 immediately yields $c_k^{\mathbf{B}}$ and (5.97) implies a Dirichlet datum for μ_k^{-} on $\partial_{T_0}\Omega$ (note that $B_{\mathbf{B}}^{k-1}$ does not depend on $p_{k-1}^{\mathbf{B}}$ and is thus known). Next, we may construct $p_{k-1}^{\mathbf{B}}$ via formula (5.102), as p_{k-1}^{-} is known, and thus gain knowledge of $\mathbf{U}^{k-1}$. This allows us to impose boundary condition (5.100) for $\mathbf{v}_k^{-}$ and p_k^{-}, leading to a closed system for the outer terms, which may then be solved. Having c_k^{-}, μ_k^{-}, $\mathbf{v}_k^{-}$, p_k^{-} at our disposal, we may use Lemma 5.15 and Lemma 5.16 (more precisely (5.101)) to construct $\mu_k^{\mathbf{B}}$ and $\mathbf{v}_k^{\mathbf{B}}$. Then we may iteratively continue this process, cf. Subsection 5.1.6.

Corollary 5.18. *Let $c_0^{-}=-1$, μ_0^{-} be a solution to (5.11) satisfying the boundary compatibility condition (5.97) and c_1^{-} be given as in (5.13) in $\overline{\Omega_{T_0}^{-}}$. Then $\mu_0^{\mathbf{B}}(z,x,t)=\mu_0^{-}(x,t)$ and $c_1^{\mathbf{B}}(z,x,t)=c_1^{-}(x,t)$ for all $(z,x,t)\in(-\infty,0]\times\overline{\partial_{T_0}\Omega(\delta)}$.*

Proof. By (5.13) we have

$$c_1^{-}=\frac{\mu_0^{-}}{f''(-1)}$$

and (5.98) implies $\mu_0^{\mathbf{B}} \equiv \mu_0^-$ in $(-\infty, 0] \times \overline{\partial_{T_0}\Omega\left(\delta\right)}$. Thus, (5.88) for $k = 1$ implies

$$\partial_{zz} c_1^{\mathbf{B}} - f''\left(-1\right) c_1^{\mathbf{B}} = -\mu_0^-$$

and with regard to (5.93), $c_1^{\mathbf{B}} = \frac{\mu_0^-}{f''(-1)} = c_1^-$ is the unique solution in $(-\infty, 0] \times \overline{\partial_{T_0}\Omega\left(\delta\right)}$. □

5.1.5. The Zeroth Order Expansion

In the following, we present an explicit scheme for constructing the lowest order terms

$$\mathfrak{S}_0 = \left(\mathbf{v}_0, \mathbf{v}_0^\pm, \mathbf{v}_0^\mathbf{B}, \mathbf{u}_0, \mathbf{q}_0, \mu_0, \mu_0^\pm, \mu_0^\mathbf{B}, c_0, c_0^\pm, c_0^\mathbf{B}, l_0, j_0, g_0, p_0^\pm\right).$$

Since it will be part of the inductive hypothesis in Subsection 5.1.6, we also make sure that the compatibility conditions (5.57), (5.62), (5.68), and (5.75) are satisfied.

As most steps will be revisited in Subsection 5.1.6 in more generality and are also carried out in more detail, we refer to that part for more explanations on technical details.

1. We have already established in (5.9) that $c_0^\pm = \pm 1$ in $\Omega_{T_0}^\pm$ and have shown in Lemma 2.2 that $c_0(\rho, x, t) = \theta_0(\rho)$ is the unique solution to (5.43) in $\mathbb{R} \times \Gamma(2\delta)$. In the boundary layer expansion we have explicitly chosen $c_\mathbf{B}^0(z, x, t) = -1$ for $(z, x, t) \in (-\infty, 0] \times \overline{\partial_{T_0}\Omega(\delta)}$.

2. In order to satisfy the compatibility condition (5.57) for $k = 1$, the identity

$$\tilde{\mu}_0(x, t) = -\sigma \Delta d_\Gamma(x, t)$$

has to be satisfied for all $(x, t) \in \Gamma$.

3. All bounded solutions on $\mathbb{R}$ to (5.45) are constant in ρ and so we can write

$$\mu_0(\rho, x, t) = \bar{\mu}_0(x, t) + l_0(x, t)\, d_\Gamma(x, t) \left(\eta(\rho) - \frac{1}{2}\right) \quad \forall (\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta) \quad (5.105)$$

for some yet to be determined function $\bar{\mu}_0$. Regarding Step 2 and using (5.32) we set

$$\bar{\mu}_0 = -\sigma \Delta d_\Gamma \quad \text{on } \Gamma \quad (5.106)$$

and thus have for $(x, t) \in \Gamma$

$$\mu_0(\rho, x, t) = \bar{\mu}_0(x, t) = -\sigma \Delta d_\Gamma(x, t). \quad (5.107)$$

4. By letting ρ go to $\pm\infty$ in (5.105) we may deduce that it is necessary and sufficient to set

$$\mu_0^\pm(x, t) = \lim_{\rho \to \infty} \mu_0(\pm\rho, x, t) = -\sigma \Delta d_\Gamma(x, t) \quad \forall (x, t) \in \Gamma \quad (5.108)$$

in order for the matching condition (5.25) to hold. As a consequence of (5.11) and (5.97) we may now solve the outer equation

$$\begin{aligned} \Delta \mu_0^\pm &= 0 && \text{in } \Omega_{T_0}^\pm \\ \mu_0^\pm &= -\sigma \Delta d_\Gamma && \text{on } \Gamma \\ \mu_0^- &= 0 && \text{on } \partial_{T_0}\Omega. \end{aligned}$$

In particular, we have

$$[\mu_0] = 0 \text{ on } \Gamma. \quad (5.109)$$

5. To enforce the matching condition in (5.105), the equation

$$\mu_0^{\pm}(x,t) = \bar{\mu}_0(x,t) \pm \frac{1}{2} l_0(x,t)\, d_\Gamma(x,t) \tag{5.110}$$

has to hold for all $(x,t) \in \Gamma(2\delta)$. It is therefore necessary and sufficient to define

$$\bar{\mu}_0(x,t) := \frac{1}{2}\left(\mu_0^+(x,t) + \mu_0^-(x,t)\right), \quad \text{for } (x,t) \in \Gamma(2\delta), \tag{5.111}$$

$$l_0(x,t) := \begin{cases} \frac{1}{d_\Gamma(x,t)}\left(\mu_0^+(x,t) - \mu_0^-(x,t)\right) & (x,t) \in \Gamma(2\delta)\backslash\Gamma, \\ \nabla d_\Gamma(x,t) \cdot \nabla\left(\mu_0^+(x,t) - \mu_0^-(x,t)\right) & (x,t) \in \Gamma. \end{cases} \tag{5.112}$$

Note that l_0 is a smooth function in $\Gamma(2\delta)$ since $\mu_0^+ - \mu_0^-$ vanishes on Γ.

6. As the right hand side of (5.39) is zero, all bounded solutions have to be constant in ρ, implying the form

$$\mathbf{v}_0(\rho,x,t) \cdot \mathbf{n}(x,t) = v_0^{\mathbf{n}}(x,t) + \mathbf{u}_0(x,t) \cdot \mathbf{n}(x,t)\, d_\Gamma(x,t)\left(\eta(\rho) - \frac{1}{2}\right) \tag{5.113}$$

for all $(\rho,x,t) \in \mathbb{R} \times \Gamma(2\delta)$ and for some function $v_0^{\mathbf{n}}$. For the matching condition (5.26) it is necessary and sufficient that

$$\mathbf{v}_0^{\pm} \cdot \mathbf{n} = v_0^{\mathbf{n}} \pm \frac{1}{2}\mathbf{u}_0 \cdot \mathbf{n} d_\Gamma$$

holds in $\Gamma(2\delta)$. In particular, we get

$$[\mathbf{v}_0] \cdot \mathbf{n} = 0 \tag{5.114}$$

on Γ and it is necessary and sufficient to define

$$v_0^{\mathbf{n}} := \frac{1}{2}\left(\mathbf{v}_0^+ + \mathbf{v}_0^-\right) \cdot \mathbf{n} \quad \text{in } \Gamma(2\delta), \tag{5.115}$$

$$\mathbf{u}_0 \cdot \mathbf{n} := \begin{cases} \frac{1}{d_\Gamma}\left(\left(\mathbf{v}_0^+ - \mathbf{v}_0^-\right) \cdot \mathbf{n}\right) & \text{in } \Gamma(2\delta)\backslash\Gamma, \\ \nabla d_\Gamma \cdot \nabla\left(\left(\mathbf{v}_0^+ - \mathbf{v}_0^-\right) \cdot \mathbf{n}\right) & \text{on } \Gamma. \end{cases} \tag{5.116}$$

By this definition, $v_0^{\mathbf{n}}$ and $\mathbf{u}_0 \cdot \mathbf{n}$ are smooth functions if $\mathbf{v}_0^{\pm}$ is smooth.

7. Considering the tangential part of (5.39) we find that every bounded solution has to be of the form

$$\mathbf{v}_0(\rho,x,t) \cdot \tau(x,t) = v_0^{\tau}(x,t) + \mathbf{u}_0(x,t) \cdot \tau(x,t)\, d_\Gamma(x,t)\left(\eta(\rho) - \frac{1}{2}\right) \tag{5.117}$$

for all $(\rho,x,t) \in \mathbb{R} \times \Gamma(2\delta)$ and some function v_0^{τ}. Due to the matching conditions, the identity

$$\mathbf{v}_0^{\pm} \cdot \tau = v_0^{\tau} \pm \frac{1}{2}\mathbf{u}_0 \cdot \tau d_\Gamma$$

has to be satisfied in $\Gamma(2\delta)$, yielding

$$[\mathbf{v}_0] \cdot \tau = 0 \tag{5.118}$$

on Γ and the definitions

$$v_0^\tau := \frac{1}{2}\left(\mathbf{v}_0^+ + \mathbf{v}_0^-\right)\cdot\tau \quad \text{in } \Gamma\left(2\delta\right), \tag{5.119}$$

$$\mathbf{u}_0\cdot\tau := \begin{cases} \frac{1}{d_\Gamma}\left(\left(\mathbf{v}_0^+ - \mathbf{v}_0^-\right)\cdot\tau\right) & \text{in } \Gamma\left(2\delta\right)\backslash\Gamma, \\ \nabla d_\Gamma\cdot\nabla\left(\left(\mathbf{v}_0^+ - \mathbf{v}_0^-\right)\cdot\tau\right) & \text{on } \Gamma. \end{cases} \tag{5.120}$$

The functions v_0^τ and $\mathbf{u}_0\cdot\tau$ are again smooth if $\mathbf{v}_0^\pm$ is smooth.

8. This and the next step guarantee that the compatibility conditions of Lemmata 5.10 and 5.11 are satisfied on Γ for $k=1$. This will yield a condition for the jump of the stress tensor $[2D_s\mathbf{v}_0 - p_0]\cdot\mathbf{n}$.
   In order for (5.68) to hold on Γ the identity

   $$[\mathbf{v}_0]\cdot\tau\Delta d_\Gamma + 2\left(\left([\nabla\mathbf{v}_0]\right)^T\mathbf{n}\right)\cdot\tau - \mathbf{u}_0\cdot\tau = 0$$

   has to be satisfied on Γ. Using (5.118) and (5.120) we deduce that this is equivalent to

   $$[\partial_{\mathbf{n}}\mathbf{v}_0\cdot\tau] = 0 \tag{5.121}$$

   on Γ.

9. The identity (5.75) is equivalent to

   $$[\mathbf{v}_0]\cdot\mathbf{n}\Delta d_\Gamma - [p_0] + 2\left(\left([\nabla\mathbf{v}_0]\right)^T\mathbf{n}\right)\cdot\mathbf{n} - \mathbf{u}_0\cdot\mathbf{n} = -\left(\mu_0^+ + \mu_0^-\right)$$

   on Γ. Using (5.108), (5.114), and (5.116) we find that

   $$[\partial_{\mathbf{n}}\mathbf{v}_0\cdot\mathbf{n} - p_0] = 2\sigma\Delta d_\Gamma$$

   has to be fulfilled in order for the compatibility condition to hold on Γ.

10. By Proposition 2.18 we find that $2\left[D_s\mathbf{v}_0\right]\cdot\mathbf{n} = [\partial_{\mathbf{n}}\mathbf{v}_0]$ holds as a consequence of $[\mathbf{v}_0]=0$ on Γ and $\operatorname{div}\mathbf{v}_0^\pm = 0$ in $\Omega_{T_0}^\pm$. Hence, we have by steps 9 and 10

    $$[2D_s\mathbf{v}_0 - p_0]\cdot\mathbf{n} = 2\sigma\Delta d_\Gamma\mathbf{n}.$$

11. We also need to make sure that the compatibility condition (5.62) is satisfied on Γ, i.e.

    $$2\partial_t d_\Gamma + \int_{\mathbb{R}}\theta_0'\mathbf{v}_0\cdot\mathbf{n}\,\mathrm{d}\rho - [\mu_0]\,\Delta d_\Gamma - 2\left[\nabla\mu_0\right]\cdot\mathbf{n} + l_0 = 0$$

    on Γ. By (5.113), (5.115), (5.109) and (5.112) this is equivalent to

    $$2\partial_t d_\Gamma = -\left(\mathbf{v}_0^+ + \mathbf{v}_0^-\right)\cdot\mathbf{n} - [\nabla\mu_0]\cdot\mathbf{n}.$$

    Summarizing previous steps and taking into account the outer equations (5.12) and (5.11), as well as the boundary compatibility conditions (5.97) and (5.100) (where the right hand sides vanish), we end up with the closed system

$$
\begin{aligned}
-\Delta \mathbf{v}_0^\pm + \nabla p_0^\pm &= 0 && \text{in } \Omega_{T_0}^\pm,\\
\operatorname{div} \mathbf{v}_0^\pm &= 0 && \text{in } \Omega_{T_0}^\pm,\\
\Delta \mu_0^\pm &= 0 && \text{in } \Omega_{T_0}^\pm,\\
\left(-2D_s\mathbf{v}_0^- + p_0^-\mathbf{I}\right)\mathbf{n}_{\partial\Omega} &= \alpha_0 \mathbf{v}_0^- && \text{on } \partial_{T_0}\Omega,\\
\mu^- &= 0 && \text{on } \partial_{T_0}\Omega,\\
\left[2D_s\mathbf{v}_0 - p_0\mathbf{I}\right]\mathbf{n} &= -2\sigma H_\Gamma \mathbf{n} && \text{on } \Gamma,\\
\mu_0^\pm &= \sigma H_\Gamma && \text{on } \Gamma,\\
-V_\Gamma + \frac{1}{2}\mathbf{n}\cdot\left(\mathbf{v}_0^+ + \mathbf{v}_0^-\right) &= \frac{1}{2}\left[\partial_{\mathbf{n}}\mu_0\right] && \text{on } \Gamma,\\
\left[\mathbf{v}_0\right] &= 0 && \text{on } \Gamma,
\end{aligned}
$$

for $\mu_0^\pm$,$\mathbf{v}_0^\pm$ and $p_0^\pm$. Here we used Proposition 2.13 2) and 3) for $\Delta d_\Gamma = -H_\Gamma$, and $\partial_t d_\Gamma = -V_\Gamma$. By construction this immediately yields $\mathbf{u}_0$ and $v_0^\tau, v_0^{\mathbf{n}}$ and also $\mathbf{v}_0\cdot\tau$ which then also satisfies (5.26). Note that this system coincides with (1.26)–(1.35).

12. Now we have to make sure that the compatibility conditions for $k=1$ are also satisfied in $\Gamma(2\delta)\setminus\Gamma$. In order for (5.57) to hold in $\Gamma(2\delta)\setminus\Gamma$ we define

$$
g_0 := \begin{cases} \frac{1}{2\tilde{\eta} d_\Gamma}\left(\mu_0^+ + \mu_0^- + 2\sigma\Delta d_\Gamma\right) & \text{in } \Gamma(2\delta)\setminus\Gamma,\\ \frac{1}{2\tilde{\eta}}\nabla d_\Gamma\cdot\nabla\left(\mu_0^+ + \mu_0^- + 2\sigma\Delta d_\Gamma\right) & \text{on } \Gamma, \end{cases} \tag{5.122}
$$

which is a smooth function in $\Gamma(2\delta)$. Analogously in order for (5.62), (5.68) and (5.75) to hold in $\Gamma(2\delta)\setminus\Gamma$ we set

$$
j_0 := \frac{1}{d_\Gamma}\left(-2\partial_t d_\Gamma - \left(\mathbf{v}_0^+ + \mathbf{v}_0^-\right)\cdot\mathbf{n} + \left(\mu_0^+ - \mu_0^-\right)\Delta d_\Gamma + 2\left(\partial_{\mathbf{n}}\mu_0^+ - \partial_{\mathbf{n}}\mu_0^-\right) + l_0\right)
$$

in $\Gamma(2\delta)\setminus\Gamma$, and

$$
\begin{aligned}
j_0 := {} & \nabla d_\Gamma\cdot\nabla\left(-2\partial_t d_\Gamma - \left(\mathbf{v}_0^+ + \mathbf{v}_0^-\right)\cdot\mathbf{n} + \left(\mu_0^+ - \mu_0^-\right)\Delta d_\Gamma\right)\\
& + \nabla d_\Gamma\cdot\nabla\left(2\left(\partial_{\mathbf{n}}\mu_0^+ - \partial_{\mathbf{n}}\mu_0^-\right) + l_0\right)
\end{aligned} \tag{5.123}
$$

on Γ,

$$
\mathbf{q}_0\cdot\tau := \begin{cases} \frac{1}{d_\Gamma}\left(-\left[\mathbf{v}_0\right]\cdot\tau\Delta d_\Gamma - 2\left(\partial_{\mathbf{n}}\mathbf{v}_0^+ - \partial_{\mathbf{n}}\mathbf{v}_0^-\right)\cdot\tau + \mathbf{u}_0\cdot\tau\right) & \text{in } \Gamma(2\delta)\setminus\Gamma,\\ \nabla d_\Gamma\cdot\nabla\left(-\left[\mathbf{v}_0\right]\cdot\tau\Delta d_\Gamma - 2\left(\partial_{\mathbf{n}}\mathbf{v}_0^+ - \partial_{\mathbf{n}}\mathbf{v}_0^-\right)\cdot\tau + \mathbf{u}_0\cdot\tau\right) & \text{on } \Gamma, \end{cases} \tag{5.124}
$$

and

$$
\mathbf{q}_0\cdot\mathbf{n} := \frac{1}{d_\Gamma}\left(-\left[\mathbf{v}_0\right]\cdot\mathbf{n}\Delta d_\Gamma - 2\left(\partial_{\mathbf{n}}\mathbf{v}_0^+ - \partial_{\mathbf{n}}\mathbf{v}_0^-\right)\cdot\mathbf{n} + \left[p_0\right] + \mathbf{u}_0\cdot\mathbf{n} + \left(\mu_0^+ + \mu_0^-\right)\right)
$$

in $\Gamma(2\delta)\setminus\Gamma$, and

$$
\begin{aligned}
\mathbf{q}_0\cdot\mathbf{n} := {} & \nabla d_\Gamma\cdot\nabla\left(-\left[\mathbf{v}_0\right]\cdot\mathbf{n}\left(\Delta d_\Gamma - \partial_t d_\Gamma\right) - 2\left(\partial_{\mathbf{n}}\mathbf{v}_0^+ - \partial_{\mathbf{n}}\mathbf{v}_0^-\right)\cdot\mathbf{n} + \left[p_0\right]\right)\\
& + \nabla d_\Gamma\cdot\nabla\left(\mathbf{u}_0\cdot\mathbf{n} + 2\sigma\Delta d_\Gamma\right)
\end{aligned} \tag{5.125}
$$

on Γ.

13. With the knowledge of $\mathbf{v}_0^-$ and μ_0^- we may construct the remaining zeroth order terms $\mathbf{v}_0^{\mathbf{B}}$ and $\mu_0^{\mathbf{B}}$ of the boundary layer expansion by using Lemmata 5.15 and 5.16, where $p_{-1}^{\mathbf{B}} = 0$. In particular, (5.98) and (5.101) imply $\mu_0^{\mathbf{B}} = \mu_0^-$ and $\mathbf{v}_0^{\mathbf{B}} = \mathbf{v}_0^-$. Additionally, anticipating the construction of $p_0^{\mathbf{B}}$ in Subsection 5.1.6, we have $\operatorname{div}\mathbf{v}_0^{\mathbf{B}} = \operatorname{div}\mathbf{v}_0^- = 0$ and $\mathbf{V}_{\mathbf{B}}^0 \cdot \mathbf{n}_{\partial\Omega} = 0$ and thus also $p_0^{\mathbf{B}} = p_0^-$ by (5.102).

Lemma 5.19 (The zeroth order terms). *Let Assumption 1.1 hold and let $(\mathbf{v}^\pm, p^\pm, \mu^\pm)$ be extended onto $\Omega_{T_0}^\pm \cup \Gamma(2\delta; T_0)$. Let moreover η be given as in Remark 5.3 and θ_0 as in Lemma 2.2. We define the terms of the outer expansion $\left(c_0^\pm, \mu_0^\pm, \mathbf{v}_0^\pm, p_0^\pm\right)$ for $(x,t) \in \Omega_{T_0}^\pm \cup \Gamma(2\delta; T_0)$ as*

$$c_0^\pm(x,t) = \pm 1,\ \mu_0^\pm(x,t) = \mu^\pm(x,t),\ \mathbf{v}_0^\pm(x,t) = \mathbf{v}^\pm(x,t),\ p_0^\pm(x,t) = p^\pm(x,t),$$

the terms of the inner expansion $(c_0, \mu_0, \mathbf{v}_0)$ for $(\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta; T_0)$ as

$$\begin{aligned} c_0(\rho,x,t) &= \theta_0(\rho), \\ \mu_0(\rho,x,t) &= \mu_0^+(x,t)\,\eta(\rho) - \mu_0^-(x,t)\,(\eta(\rho)-1), \\ \mathbf{v}_0(\rho,x,t) &= \mathbf{v}_0^+(x,t)\,\eta(\rho) - \mathbf{v}_0^-(x,t)\,(\eta(\rho)-1), \end{aligned}$$

and the terms of the boundary expansion $\left(c_0^{\mathbf{B}}, \mu_0^{\mathbf{B}}, \mathbf{v}_0^{\mathbf{B}}, p_0^{\mathbf{B}}\right)$ for $(z,x,t) \in (-\infty, 0] \times \overline{\partial_{T_0}\Omega(\delta)}$ as

$$c_0^{\mathbf{B}}(z,x,t) = -1,\ \mu_0^{\mathbf{B}}(z,x,t) = \mu_0^-(x,t),\ \mathbf{v}_0^{\mathbf{B}}(z,x,t) = \mathbf{v}_0^-(x,t),\ p_0^{\mathbf{B}}(z,x,t) = p_0^-(x,t).$$

Moreover, we define $\bar{\mu}_0$ by (5.111), l_0 by (5.112), $v_0^{\mathbf{n}}$ by (5.115), v_0^τ by (5.119), $\mathbf{u}_0$ by (5.116) and (5.120), g_0 by (5.122), j_0 by (5.123) and $\mathbf{q}_0$ by (5.124) and (5.125).

Then the outer equations (5.9), (5.11), (5.12), the inner equations (5.39), (5.41), (5.43), (5.45), the boundary equations (5.86)–(5.89) (for $k=0$), the inner-outer matching conditions (5.24)–(5.27), the outer-boundary matching conditions (5.82)–(5.85) and the boundary conditions (5.94) and (5.95) (for $k=0$) are satisfied. Moreover, the compatibility conditions (5.57), (5.62), (5.68) and (5.75) and the boundary compatibility conditions (5.97) (for $k=0$) and (5.100) (for $k=1$) are satisfied.

Proof. This follows directly from construction. □

Remark 5.20. It holds

$$\mathbf{u}_0 = 0 \text{ on } \Gamma \tag{5.126}$$

since $\mathbf{u}_0 \cdot \tau = [\partial_{\mathbf{n}}\mathbf{v}_0 \cdot \tau] = 0$ on Γ due to (5.121) and $\mathbf{u}_0 \cdot \mathbf{n} = [\partial_{\mathbf{n}}\mathbf{v}_0 \cdot \mathbf{n}] = \operatorname{div}^\Gamma[\mathbf{v}_0] = 0$ on Γ, which is due to $\operatorname{div}\mathbf{v}_0^\pm = 0$ on $\overline{\Omega_{T_0}^\pm}$ and $[\mathbf{v}_0] = 0$ on Γ.

Remark 5.21. Note that we gave the form of $p_0^{\mathbf{B}}$ for the sake of completeness in Lemma 5.19, although it is actually not part of $\mathfrak{S}_0$. The strategy for constructing p_0 will be given in the next subsection, Subsection 5.1.6. It is constructed along with the terms of order 1, although it turns out that it only depends on terms of order 0 and may thus be treated as such.

5.1.6. Basic Strategy for Solving Each Order

Now we assume that for all $i \in \{0, \ldots, k-1\}$ the system

$$\mathfrak{S}_i := \left(\mathbf{v}_i, \mathbf{v}_i^\pm, \mathbf{v}_i^{\mathbf{B}}, \mathbf{u}_i, \mathbf{q}_i, \mu_i, \mu_i^\pm, \mu_i^{\mathbf{B}}, c_i, c_i^\pm, c_i^{\mathbf{B}} h_i, l_i, j_i, g_i, p_{i-1}, p_i^\pm, p_{i-1}^{\mathbf{B}}\right) \tag{5.127}$$

is known and satisfies the inner-outer matching conditions (5.24)–(5.27), the outer-boundary matching conditions (5.82)–(5.85), the boundary compatibility conditions (5.97) (for $k = i$), (5.100) (for $k - 1 = i$), as well as the compatibility conditions (5.58), (5.63), (5.69), and (5.76) (cf. Remark 5.12 regarding the last one). We solve for order k by following these steps (note that for the pressure term, we have to construct p_{k-1}, $p_{k-1}^{\mathbf{B}}$ and $p_k^{\pm}$):

1. $c_k^{\pm} : \Omega_{T_0}^{\pm} \to \mathbb{R}$ may be directly calculated using formula (5.13) for which only terms of order $i \leq k-1$ are needed. Since the compatibility conditions are supposed to be fulfilled, we may use Lemma 5.7 to get $c_k : \mathbb{R} \times \Gamma(2\delta) \to \mathbb{R}$. Regarding boundary expansion terms, the function $c_k^{\mathbf{B}} : (-\infty, 0] \times \overline{\partial_{T_0}\Omega(\delta)} \to \mathbb{R}$ can be found using Lemma 5.14 and $p_{k-1}^{\mathbf{B}} : (-\infty, 0] \times \overline{\partial_{T_0}\Omega(\delta)} \to \mathbb{R}$ can be defined by (5.102). Note that in this entire process, we only needed $\mathfrak{S}_i$ for $i \leq k-1$ (and the compatibility condition which is provided by the mathematical induction) and may thus regard $c_k^{\pm}$, $c_k^{\mathbf{B}}$, $p_{k-1}^{\mathbf{B}}$ and c_k to be depending only on order $i \leq k-1$.

2. In the following steps, we assume that h_k is known. In order to satisfy the compatibility condition (5.58) restricted to Γ for order $k+1$ we have to demand
$$\tilde{\mu}_k = \tilde{\mathcal{A}}^{k-1} - \tilde{\eta} g_0 h_k + \sigma \Delta^{\Gamma} h_k \tag{5.128}$$
on Γ.

3. By the induction hypothesis, the compatibility condition (5.63) is satisfied and we have
$$\begin{aligned}\mu_k(\rho, x, t) = \bar{\mu}_k(x, t) + \left(l_k(x, t)\, d_\Gamma(x, t) - l_0(x, t)\, h_k(S(x, t), t)\right)\left(\eta(\rho) - \frac{1}{2}\right) \\ + \mu_{k-1}^{*}(\rho, x, t)\end{aligned} \tag{5.129}$$
for all $(\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta)$. By (5.32) and (5.65) we furthermore find
$$\bar{\mu}_k(x, t) = \frac{1}{2}\int_{\mathbb{R}} \theta_0'(\rho)\, \mathrm{d}\rho \bar{\mu}_k(x, t) = \frac{1}{2}\int_{\mathbb{R}} \theta_0'(\rho)\, \mu_k(\rho, x, t)\, \mathrm{d}\rho = \tilde{\mu}_k(x, t) \tag{5.130}$$
for all $(x, t) \in \Gamma(2\delta)$. Restricting (5.129) to Γ we may conclude using step 2
$$\begin{aligned}\mu_k(\rho, x, t) = \tilde{\mathcal{A}}^{k-1}(x, t) - \tilde{\eta} g_0(x, t)\, h_k(S(x, t), t) + \sigma \Delta^{\Gamma} h_k(S(x, t), t) \\ - l_0(x, t)\, h_k(S(x, t), t)\left(\eta(\rho) - \frac{1}{2}\right) + \mu_{k-1}^{*}(\rho, x, t)\end{aligned} \tag{5.131}$$
for all $(\rho, x, t) \in \mathbb{R} \times \Gamma$.

4. By letting ρ go to $\pm\infty$ in (5.131) we find that in order for the matching condition (5.25) to hold, it is necessary and sufficient to have
$$\begin{aligned}\mu_k^{\pm}(x, t) &= \lim_{\rho \to \pm\infty} \mu_k(\rho, x, t) \\ &= \tilde{\mathcal{A}}^{k-1}(x, t) - \tilde{\eta} g_0(x, t)\, h_k(S(x, t), t) + \sigma \Delta^{\Gamma} h_k(S(x, t), t) \\ &\quad \mp \frac{1}{2} l_0(x, t)\, h_k(S(x, t)) + \mu_{k-1}^{*,\pm}(x, t)\end{aligned} \tag{5.132}$$
for all $(x, t) \in \Gamma$. With this knowledge we may find the solution $\mu_k^{\pm}$ to equation (5.14) of the outer expansion and extend it as discussed in Remark 5.1 onto $\Omega^{\pm} \cup \Gamma(2\delta)$. By construction, $\mu_k^{\pm}$ only depends on h_k and terms of order lower or equal to $k-1$.

5. Using the matching condition again and letting ρ go to $\pm\infty$ in (5.129) yields by the properties of η that

$$\mu_k^{\pm}(x,t) = \bar{\mu}_k(x,t) \pm \frac{1}{2}\left(l_k(x,t)\, d_\Gamma(x,t) - l_0(x,t)\, h_k(S(x,t),t)\right) + \mu_{k-1}^{*,\pm}(x,t)$$

has to hold for all $(x,t) \in \Gamma(2\delta)$. It is thus necessary and sufficient to define l_k and $\bar{\mu}_k$ as

$$\bar{\mu}_k := \frac{1}{2}\left(\mu_k^+ + \mu_k^- - \mu_{k-1}^{*,+} - \mu_{k-1}^{*,-}\right) \text{ in } \Gamma(2\delta), \tag{5.133}$$

$$l_k := \begin{cases} \frac{1}{d_\Gamma}\left(\mu_k^+ - \mu_{k-1}^{*,+} - \mu_k^- + \mu_{k-1}^{*,-} + l_0 h_k\right) & \text{in } \Gamma(2\delta)\setminus\Gamma, \\ \nabla d_\Gamma \cdot \nabla\left(\mu_k^+ - \mu_{k-1}^{*,+} - \mu_k^- + \mu_{k-1}^{*,-} + l_0 h_k\right) & \text{on } \Gamma. \end{cases} \tag{5.134}$$

Note that l_k is a smooth function by this definition as the enumerator vanishes on Γ. Furthermore, it depends only on h_k and some lower order terms. At this point it is easy to see why we had to introduce l^ϵ in the first place, as fulfilling the matching conditions in $\Gamma(2\delta)\setminus\Gamma$ would otherwise be impossible. The same observation holds true for $\mathbf{u}^\epsilon$ which we will define in a similar way in Steps 7 and 8.

6. Next we construct p_{k-1}. Multiplying (5.40) by $\mathbf{n}$, we get that if there exists a solution to (5.40), (5.42) satisfying the matching conditions, it is necessary that p_{k-1} fulfills the ordinary differential equation

$$\partial_\rho p_{k-1} = \partial_{\rho\rho}\left(\mathbf{v}_k - (\mathbf{u}_k d_\Gamma - \mathbf{u}_0 h_k)\,\eta\right) \cdot \mathbf{n} + \mathbf{V}^{k-1} \cdot \mathbf{n} \tag{5.135}$$

in $\Gamma(2\delta)$ and that

$$\partial_\rho\left(\mathbf{v}_k - (\mathbf{u}_k d_\Gamma - \mathbf{u}_0 h_k)\,\eta\right) \cdot \mathbf{n} = W^{k-1} + \nabla^\Gamma h_k \cdot \left(\partial_\rho \mathbf{v}_0 - \mathbf{u}_0 d_\Gamma \eta'\right) \tag{5.136}$$

in $\Gamma(2\delta)$ by (5.42). Note that by (5.113) and (5.117) it holds $\partial_\rho \mathbf{v}_0 = \mathbf{u}_0 d_\Gamma \eta'$ and thus the term h_k on the right hand side of (5.136) vanishes. Assuming that p_{k-1} satisfies the matching conditions, integrating (5.135) and using (5.136) we get that it is necessary to choose

$$\begin{aligned} p_{k-1}(\rho,x,t) = {} & \frac{1}{2}\left(\int_{-\infty}^{\rho} \mathbf{V}^{k-1}(\tilde{\rho},x,t)\cdot\mathbf{n}(x,t)\,\mathrm{d}\tilde{\rho} - \int_{\rho}^{\infty} \mathbf{V}^{k-1}(\tilde{\rho},x,t)\cdot\mathbf{n}(x,t)\,\mathrm{d}\tilde{\rho}\right) \\ & + \frac{1}{2}\left(p_{k-1}^-(x,t) + p_{k-1}^+(x,t)\right) + W^{k-1}(\rho,x,t). \end{aligned} \tag{5.137}$$

Moreover, $W^{k-1} \in \mathcal{O}\left(e^{-C|\rho|}\right)$ as $\rho \to \pm\infty$ due to the matching conditions of the lower order terms and $\int_\rho^{\pm\infty} \mathbf{V}^{k-1}\cdot\mathbf{n}\,\mathrm{d}\tilde{\rho} \in \mathcal{O}\left(e^{-C|\rho|}\right)$ for $\rho \to \pm\infty$, since $\mathbf{V}^{k-1}\cdot\mathbf{n} \in \mathcal{O}\left(e^{-C|\rho|}\right)$ for $\rho \to \pm\infty$. Defining p_{k-1} in this way, the matching conditions are satisfied, as

$$\int_{\mathbb{R}} \mathbf{V}^{k-1}\cdot\mathbf{n}\,\mathrm{d}\rho = p_{k-1}^+ - p_{k-1}^- \text{ in } \Gamma(2\delta)$$

by the induction assumption, see Remark 5.12. Furthermore, this form of p_{k-1} ensures that every solution to (5.40) also satisfies (5.42) and p_{k-1} depends only on terms of order $k-1$ or lower.

In order to find $\mathbf{v}_k, \mathbf{v}_k^\pm, p_k^\pm$ we will in the following deduce a closed system for the outer equations, which involves finding $[2D_s\mathbf{v}_k - p_k\mathbf{I}]\,\mathbf{n}$ and $[\mathbf{v}_k]$.

7. By the induction hypothesis, the equality (5.76) is satisfied and by step 6 the term p_{k-1} fulfills the matching conditions so that Lemma 5.11 is applicable. Then (5.77) yields

$$\mathbf{v}_k(\rho,x,t)\cdot\mathbf{n}(x,t) = v_k^{\mathbf{n}}(x,t) + (\mathbf{u}_k d_\Gamma - \mathbf{u}_0 h_k)\cdot\mathbf{n}\left(\eta(\rho)-\frac{1}{2}\right) + v_{k-1}^{\mathbf{n},*}(\rho,x,t) \quad (5.138)$$

for all $(\rho,x,t)\in\mathbb{R}\times\Gamma(2\delta)$. Using the definition of η and (5.79), we get

$$\begin{aligned}\lim_{\rho\to\infty}\mathbf{v}_k(\pm\rho,x,t)\cdot\mathbf{n}(x,t) = {}& \pm\frac{1}{2}(\mathbf{u}_k(x,t)d_\Gamma(x,t)-\mathbf{u}_0(x,t)h_k(S(x,t),t))\cdot\mathbf{n}(x,t)\\ &+v_k^{\mathbf{n}}(x,t)+v_{k-1}^{\mathbf{n},*,\pm}(x,t)\end{aligned}$$

for all $(x,t)\in\Gamma(2\delta)$. To satisfy the matching condition, we require

$$\mathbf{v}_k^{\pm}\cdot\mathbf{n} = v_k^{\mathbf{n}} \mp \frac{1}{2}\mathbf{u}_0\cdot\mathbf{n}h_k + v_{k-1}^{\mathbf{n},*,\pm} \quad \text{on } \Gamma \quad (5.139)$$

and thus

$$[\mathbf{v}_k]\cdot\mathbf{n} = \left(\mathbf{v}_k^{+}-\mathbf{v}_k^{-}\right)\cdot\mathbf{n} = v_{k-1}^{\mathbf{n},*,+} - v_{k-1}^{\mathbf{n},*,-} \quad (5.140)$$

on Γ since $\mathbf{u}_0=0$ on Γ due to (5.126). Furthermore, to satisfy the matching condition on $\Gamma(2\delta)\setminus\Gamma$, it is necessary and sufficient to define

$$v_k^{\mathbf{n}} := \frac{1}{2}\left(\left(\mathbf{v}_k^{+}+\mathbf{v}_k^{-}\right)\cdot\mathbf{n} - v_{k-1}^{\mathbf{n},*,+} - v_{k-1}^{\mathbf{n},*,-}\right) \text{ in } \Gamma(2\delta), \quad (5.141)$$

$$\mathbf{u}_k\cdot\mathbf{n} := \begin{cases}\frac{1}{d_\Gamma}\left(\left(\mathbf{v}_k^{+}-\mathbf{v}_k^{-}\right)\cdot\mathbf{n} - v_{k-1}^{\mathbf{n},*,+} + v_{k-1}^{\mathbf{n},*,-} + \mathbf{u}_0\cdot\mathbf{n}h_k\right) & \text{in } \Gamma(2\delta)\setminus\Gamma,\\ \nabla d_\Gamma\cdot\nabla\left(\left(\mathbf{v}_k^{+}-\mathbf{v}_k^{-}\right)\cdot\mathbf{n} - v_{k-1}^{\mathbf{n},*,+} + v_{k-1}^{\mathbf{n},*,-} + \mathbf{u}_0\cdot\mathbf{n}h_k\right) & \text{on } \Gamma.\end{cases} \quad (5.142)$$

Note that by this definition, the normal component of $\mathbf{u}_k$ is a smooth function in $\Gamma(2\delta)$ as the numerator vanishes on Γ. Furthermore, remark that this procedure differs from the one in Steps 4 and 5 since at this point we have not found $\mathbf{v}_k^{\pm}$ (not even on Γ) but just rewritten $v_k^{\mathbf{n}}$ and $\mathbf{u}_k\cdot\mathbf{n}$ in terms of $\mathbf{v}_k^{\pm}$ (and of course h_k).

8. Using the compatibility condition (5.69) allows us to derive similar equalities as above for the tangential components. That is, due to (5.70) we have

$$\begin{aligned}\mathbf{v}_k(\rho,x,t)\cdot\tau = {}&(\mathbf{u}_k(x,t)d_\Gamma(x,t)-\mathbf{u}_0(x,t)h_k(S(x,t),t))\cdot\tau\left(\eta(\rho)-\frac{1}{2}\right)\\ &+v_k^{\tau}(x,t)+v_{k-1}^{\tau,*}(\rho,x,t)\end{aligned} \quad (5.143)$$

for all $(\rho,x,t)\in\mathbb{R}\times\Gamma(2\delta)$. The definition of η and (5.71) yields

$$\begin{aligned}\lim_{\rho\to\infty}\mathbf{v}_k(\pm\rho,x,t)\cdot\tau = {}&\pm\frac{1}{2}(\mathbf{u}_k(x,t)d_\Gamma(x,t)-\mathbf{u}_0(x,t)h_k(S(x,t),t))\cdot\tau(x,t)\\ &+v_k^{\tau}(x,t)+v_{k-1}^{\tau,*,\pm}(x,t)\end{aligned}$$

for all $(x,t)\in\Gamma(2\delta)$. Again, we need

$$\mathbf{v}_k^{\pm}\cdot\tau = v_k^{\tau} \mp \frac{1}{2}\mathbf{u}_0\cdot\tau h_k + v_{k-1}^{\tau,*,\pm} \quad \text{on } \Gamma \quad (5.144)$$

in order to satisfy the matching condition and thus

$$[\mathbf{v}_k]\cdot\tau = \left(\mathbf{v}_k^+ - \mathbf{v}_k^-\right)\cdot\tau = v_{k-1}^{\tau,*,+} - v_{k-1}^{\tau,*,-} \tag{5.145}$$

on Γ, where we again used $\mathbf{u}_0 = 0$ on Γ. Also, on $\Gamma(2\delta)\setminus\Gamma$ it is necessary and sufficient to define

$$v_k^\tau := \frac{1}{2}\left(\left(\mathbf{v}_k^+ + \mathbf{v}_k^-\right)\cdot\tau - v_{k-1}^{\tau,*,+} - v_{k-1}^{\tau,*,-}\right) \text{ in } \Gamma(2\delta), \tag{5.146}$$

$$\mathbf{u}_k\cdot\tau := \begin{cases} \frac{1}{d_\Gamma}\left(\left(\mathbf{v}_k^+ - \mathbf{v}_k^-\right)\cdot\tau - v_{k-1}^{\tau,*,+} + v_{k-1}^{\tau,*,-} + \mathbf{u}_0\cdot\tau h_k\right) & \text{in } \Gamma(2\delta)\setminus\Gamma, \\ \nabla d_\Gamma\cdot\nabla\left(\left(\mathbf{v}_k^+ - \mathbf{v}_k^-\right)\cdot\tau - v_{k-1}^{\tau,*,+} + v_{k-1}^{\tau,*,-} + \mathbf{u}_0\cdot\tau h_k\right) & \text{on } \Gamma. \end{cases} \tag{5.147}$$

As the numerator vanishes on Γ, the tangential component of $\mathbf{u}_k$ is also a smooth function in $\Gamma(2\delta)$.

9. This and the next step will yield $[2D_s\mathbf{v}_k - p_k\mathbf{I}]\,\mathbf{n}$. As it is part of the inductive claim that (5.69) has to be satisfied for order $k+1$, we restrict the equation to Γ and get that

$$\begin{aligned} 2\left(([\nabla\mathbf{v}_k])^T\mathbf{n}\right)\cdot\tau - \mathbf{u}_k\cdot\tau = &-[\mathbf{v}_k]\cdot\tau\Delta d_\Gamma + [\mathbf{v}_0]\cdot\tau\Delta^\Gamma h_k \\ &+ 2\left(([\nabla\mathbf{v}_0])^T\nabla^\Gamma h_k\right)\cdot\tau - [p_0]\,\nabla^\Gamma h_k\cdot\tau \\ &+ \left(\mu_0^+ + \mu_0^-\right)\nabla^\Gamma h_k\cdot\tau + \mathbf{q}_0\cdot\tau h_k + \bar{\mathcal{V}}^{k-1,\tau} \end{aligned} \tag{5.148}$$

has to hold on Γ. Now since $\nabla\tau(S(x,t),t)\cdot\mathbf{n}(S(x,t),t) = 0$ for all $(x,t)\in\Gamma(2\delta)$, cf. Proposition 2.13 4), we have

$$[\partial_\mathbf{n}\mathbf{v}_k]\cdot\tau = \left(([\nabla\mathbf{v}_k])^T\mathbf{n}\right)\cdot\tau = \nabla d_\Gamma\cdot\nabla\left(\left(\mathbf{v}_k^+ - \mathbf{v}_k^-\right)\cdot\tau\right) \text{ on } \Gamma. \tag{5.149}$$

Plugging (5.145), (5.147), (5.149), (5.108) and $[\mathbf{v}_0]\cdot\tau = 0$ on Γ into (5.148), it turns out that we have to set

$$\begin{aligned} [\partial_\mathbf{n}\mathbf{v}_k\cdot\tau] = &\nabla d_\Gamma\cdot\nabla\left(-v_{k-1}^{\tau,*,+} + v_{k-1}^{\tau,*,-} + \mathbf{u}_0\cdot\tau h_k\right) - [p_0]\,\nabla^\Gamma h_k\cdot\tau \\ &- \left(v_{k-1}^{\tau,*,+} - v_{k-1}^{\tau,*,-}\right)\Delta d_\Gamma + \mathbf{q}_0\cdot\tau h_k \\ &+ 2\left(([\nabla\mathbf{v}_0])^T\nabla^\Gamma h_k\right)\cdot\tau - 2\sigma\Delta d_\Gamma\nabla^\Gamma h_k\cdot\tau + \bar{\mathcal{V}}^{k-1,\tau} \end{aligned} \tag{5.150}$$

in order for the compatibility condition to hold on Γ. Note that on the right hand side only terms of order $k-1$ or lower and h_k appear.

10. Restricting (5.76) to Γ for order $k+1$, we deduce that

$$\begin{aligned} 2\left(([\nabla\mathbf{v}_k])^T\mathbf{n}\right)\cdot\mathbf{n} - \mathbf{u}_k\cdot\mathbf{n} = &[p_k] - [\mathbf{v}_k]\cdot\mathbf{n}\Delta d_\Gamma \\ &+ 2\left(([\nabla\mathbf{v}_0])^T\nabla^\Gamma h_k\right)\cdot\mathbf{n} + \mathbf{q}_0\cdot\mathbf{n}h_k \\ &+ [\mathbf{v}_0]\cdot\mathbf{n}\Delta^\Gamma h_k - \int_\mathbb{R}\mu_k\partial_\rho c_0\,\mathrm{d}\rho + \bar{\mathcal{V}}^{k-1,\mathbf{n}} \end{aligned} \tag{5.151}$$

has to hold on Γ. Similar as in (5.149), we find that due to the equality $\nabla\mathbf{n}\left(S\left(x,t\right),t\right)\cdot\mathbf{n}\left(S\left(x,t\right),t\right)=0$ (cf. Proposition 2.13 4)) we get the identity

$$[\partial_{\mathbf{n}}\mathbf{v}_k]\cdot\mathbf{n}=\left(\left([\nabla\mathbf{v}_k]\right)^T\mathbf{n}\right)\cdot\mathbf{n}=\nabla d_\Gamma\cdot\nabla\left(\left(\mathbf{v}_k^+-\mathbf{v}_k^-\right)\cdot\mathbf{n}\right) \tag{5.152}$$

on Γ. Moreover, the form of μ_k as given in (5.129) and the identity (5.130) imply

$$\int_{\mathbb{R}}\mu_k\theta_0'\mathrm{d}\rho=2\tilde{\mu}_k=2\left(\tilde{\mathcal{A}}^{k-1}-\tilde{\eta}g_0h_k+\sigma\Delta^\Gamma h_k\right).$$

Combining this, (5.140), (5.142), (5.152) and $[\mathbf{v}_0]\cdot\mathbf{n}=0$ on Γ with (5.151) we get that

$$\begin{aligned}[\partial_{\mathbf{n}}\mathbf{v}_k\cdot\mathbf{n}-p_k]=&\nabla d_\Gamma\cdot\nabla\left(-v_{k-1}^{\mathbf{n},*,+}+v_{k-1}^{\mathbf{n},*,-}+\mathbf{u}_0\cdot\mathbf{n}h_k\right)\\&-\left(v_{k-1}^{\mathbf{n},*,+}-v_{k-1}^{\mathbf{n},*,-}\right)(\Delta d_\Gamma)+\mathbf{q}_0\cdot\mathbf{n}h_k+2\tilde{\eta}g_0h_k\\&+2\left(\left([\nabla\mathbf{v}_0]\right)^T\nabla^\Gamma h_k\right)\cdot\mathbf{n}-2\sigma\Delta^\Gamma h_k+\bar{\mathcal{V}}^{k-1,\mathbf{n}}-2\tilde{\mathcal{A}}^{k-1}\end{aligned} \tag{5.153}$$

has to be satisfied in order for (5.76) to hold on Γ. Again, remark that on the right side only terms of order $k-1$ and h_k appear.

11. Having (5.150) and (5.153), we found an explicit representation of $[\partial_{\mathbf{n}}\mathbf{v}_k-p_k\mathbf{n}]=[\partial_{\mathbf{n}}\mathbf{v}_k\cdot\mathbf{n}-p_k]\cdot\mathbf{n}+[\partial_{\mathbf{n}}\mathbf{v}_k\cdot\tau]\,\tau$. Using Proposition 2.18 together with the facts that $\mathrm{div}\mathbf{v}_k^\pm=0$ and that $[\mathbf{v}]$ is known on Γ via (5.140) and (5.145) and only depends on terms of lower order, we get

$$2\left[D_s\mathbf{v}_k\right]\cdot\mathbf{n}=[\partial_{\mathbf{n}}\mathbf{v}_k]+\mathbf{s}_{k-1}.$$

Here $\mathbf{s}_{k-1}$ is a term that depends only on expansion terms of order $k-1$ and lower. This gives us a complete system for the outer equation, which may be solved for $\mathbf{v}_k^\pm$ and $p_k^\pm$. Using the definitions in Steps 7 and 8 we immediately obtain $\mathbf{u}_k$ and $\mathbf{v}_k$ with all needed matching conditions.

12. In this step we we will find h_k. The last compatibility condition, which is not yet satisfied on Γ, is (5.63). Restricting (5.63) of order $k+1$ on Γ and utilizing the definition of μ_k and l_k, we get

$$\begin{aligned}\bar{\mathcal{B}}^{k-1}=&\int_{\mathbb{R}}\theta_0'\left(-\mathbf{v}_k\cdot\mathbf{n}+\mathbf{v}_0\cdot\nabla^\Gamma h_k\right)\mathrm{d}\rho+2\partial_t^\Gamma h_k-2\left[\nabla\mu_0\right]\cdot\nabla^\Gamma h_k+[\mu_k]\,\Delta d_\Gamma\\&+2\left[\nabla\mu_k\right]\cdot\mathbf{n}-l_k+j_0h_k,\end{aligned}$$

where we used that on Γ we have $[\mu_0]=0$, cf. (5.109). We may rewrite

$$\begin{aligned}2\partial_t^\Gamma h_k=&\int_{\mathbb{R}}\theta_0'\left(v_k^{\mathbf{n}}-\mathbf{u}_0h_k\cdot\mathbf{n}\left(\eta\left(\rho\right)-\frac{1}{2}\right)+v_{k-1}^{\mathbf{n},*}-\mathbf{v}_0\cdot\nabla^\Gamma h_k\right)\mathrm{d}\rho\\&+2\left[\nabla\mu_0\right]\cdot\nabla^\Gamma h_k-\left(-l_0h_k+\mu_{k-1}^{*,+}-\mu_{k-1}^{*,-}\right)\Delta d_\Gamma\\&-2\nabla\left(\mu_k^+-\mu_k^-\right)\cdot\mathbf{n}+\nabla\left(\mu_k^+-\mu_{k-1}^{*,+}-\mu_k^-+\mu_{k-1}^{*,-}+l_0h_k\right)\cdot\mathbf{n}\\&-j_0h_k+\bar{\mathcal{B}}^{k-1}\end{aligned} \tag{5.154}$$

by employing the identities

$$\begin{aligned}
\mathbf{v}_k \cdot \mathbf{n} &= v_k^{\mathbf{n}} - \mathbf{u}_0 h_k \cdot \mathbf{n} \left(\eta(\rho) - \frac{1}{2} \right) + v_{k-1}^{\mathbf{n},*}(\rho, x, t), \\
[\mu_k] &= -l_0 h_k + \mu_{k-1}^{*,+} - \mu_{k-1}^{*,-}, \\
l_k &= \nabla \left(\mu_k^+ - \mu_{k-1}^{*,+} - \mu_k^- + \mu_{k-1}^{*,-} + l_0 h_k \right) \cdot \mathbf{n}, \\
[\nabla \mu_k] &= \nabla \left(\mu_k^+ - \mu_k^- \right)
\end{aligned} \tag{5.155}$$

on Γ, where the first three equations are due to (5.77), (5.132) and (5.134).
Using (5.32), (5.78) and (5.141) in the first line of (5.154) and combining all terms of order $k-1$ or lower in a new variable denoted by h_{k-1}^* yields

$$\begin{aligned}
2\partial_t^\Gamma h_k = &\left(\mathbf{v}_k^+ + \mathbf{v}_k^- \right) \cdot \mathbf{n} - \int_{\mathbb{R}} \theta_0' \mathbf{v}_0 \cdot \nabla^\Gamma h_k \mathrm{d}\rho + 2 [\nabla \mu_0] \cdot \nabla^\Gamma h_k \\
&+ (l_0 \Delta d_\Gamma - j_0 + \nabla l_0 \cdot \mathbf{n}) h_k - \nabla \left(\mu_k^+ - \mu_k^- \right) \cdot \mathbf{n} + l_0 \nabla^\Gamma h_k \cdot \mathbf{n} + h_{k-1}^*.
\end{aligned}$$

Because of $[\nabla \mu_0] = \mathbf{n} l_0$ (see (5.110)), $\mathbf{v}_0(\rho, x, t) = \mathbf{v}_0(x, t) = \frac{1}{2}\left(\mathbf{v}_0^+ + \mathbf{v}_0^- \right)$ on Γ (see (5.117) and (5.113)) and since $\nabla^\Gamma h_k \cdot \mathbf{n} = 0$ due to Proposition 2.13, we have

$$\begin{aligned}
2\partial_t^\Gamma h_k = &\left(\mathbf{v}_k^+ + \mathbf{v}_k^- \right) \cdot \mathbf{n} - 2\mathbf{v}_0 \cdot \nabla^\Gamma h_k \\
&+ (l_0 \Delta d_\Gamma - j_0 + \nabla l_0 \cdot \mathbf{n}) h_k - \nabla \left(\mu_k^+ - \mu_k^- \right) \cdot \mathbf{n} + h_{k-1}^*.
\end{aligned} \tag{5.156}$$

Taking into account the boundary compatibility conditions (5.97) and (5.100) (for k instead of $k-1$) on $\partial_{T_0}\Omega$, we have the following system for $\left(\mu_k^\pm, \mathbf{v}_k^\pm, p_k^\pm, h_k \right)$

$$\Delta \mu_k^\pm = a_{k-1}^1 \qquad \text{in } \Omega_{T_0}^\pm, \tag{5.157}$$

$$\mu_k^\pm = a_{k-1}^{2,\pm} + \left(\mp \frac{1}{2} l_0 - \tilde{\eta} g_0 \right) h_k + \sigma \Delta_\Gamma h_k \qquad \text{on } \Gamma, \tag{5.158}$$

$$-\Delta \mathbf{v}_k^\pm + \nabla p_k^\pm = \mathbf{a}_{k-1}^1 \qquad \text{in } \Omega_{T_0}^\pm, \tag{5.159}$$

$$\operatorname{div} \mathbf{v}_k^\pm = 0 \qquad \text{in } \Omega_{T_0}^\pm, \tag{5.160}$$

$$\mu_k^- = a_{k-1}^3 \qquad \text{on } \partial_{T_0}\Omega, \tag{5.161}$$

$$\left(-2D_s \mathbf{v}_k^- + p_k^- \mathbf{I} \right) \mathbf{n}_{\partial\Omega} = \alpha_0 \mathbf{v}_k^- - \mathbf{a}_{k-1}^2 \qquad \text{on } \partial_{T_0}\Omega, \tag{5.162}$$

$$[\mathbf{v}_k] = \mathbf{a}_{k-1}^3 \qquad \text{on } \Gamma, \tag{5.163}$$

$$[2D_s \mathbf{v}_k - p_k] \mathbf{n} = \mathbf{a}_{k-1}^4 h_k + \mathbf{a}_{k-1}^5 \Delta^\Gamma h_k + a_{k-1}^4 \nabla^\Gamma h_k + \mathbf{a}_{k-1}^6 \qquad \text{on } \Gamma, \tag{5.164}$$

$$\begin{aligned}
\partial_t^\Gamma h_k = &-\mathbf{v}_0 \cdot \nabla^\Gamma h_k + a_{k-1}^5 h_k - a_{k-1}^6 \\
&+ \frac{1}{2} \left(\mathbf{v}_k^+ + \mathbf{v}_k^- \right) \cdot \mathbf{n} - \frac{1}{2} \left(\partial_{\mathbf{n}} \mu_k^+ - \partial_{\mathbf{n}} \mu_k^- \right) \qquad \text{on } \Gamma,
\end{aligned} \tag{5.165}$$

$$h_k(x, 0) = 0 \qquad \text{on } \Gamma_0. \tag{5.166}$$

Here the terms a_{k-1}^i, $\mathbf{a}_{k-1}^l$ for $i \in \{1, \ldots, 6\}$, $l \in \{1, \ldots, 6\}$ only depend on expansions of order $k-1$ or lower. In particular, as we have seen in the construction of the outer system (5.14) and (5.15), we have

$$a_{k-1}^1 = \partial_t c_k^\pm + \sum_{j=0}^{k-1} \mathbf{v}_j^\pm \cdot \nabla c_{k-j}^\pm, \ \mathbf{a}_{k-1}^1 = \sum_{j=0}^{k-1} \mu_j^\pm \nabla c_{k-j}^\pm,$$

where $c_k^{\pm}$ can be seen as a term of order $k-1$, see Step 1. With regard to (5.97) and (5.100), we have

$$a_{k-1}^3 = -\int_{-\infty}^{0}\int_{-\infty}^{\tilde{z}} B_{\mathbf{B}}^{k-1}\left(z',x,t\right)\mathrm{d}z'\mathrm{d}\tilde{z},$$
$$\mathbf{a}_{k-1}^2 = \mathbf{U}^{k-1}$$

for $B_{\mathbf{B}}^{k-1}$ as in (5.92) and $\mathbf{U}^{k-1}$ as in (5.99). Here $\mathbf{U}^{k-1}$ is known, since $c_k^{\mathbf{B}}$ and $p_{k-1}^{\mathbf{B}}$ have already been constructed in step 1.
Note that in order to find the equations (5.157)–(5.166), we have used the compatibility conditions (5.63), (5.69) and (5.76) for order $k+1$ only on Γ, where it holds (due to the smoothness of the outer solutions)

$$\mathbf{W}_{k-1}^{+} = \mathbf{W}_{k-1}^{-} = 0,\ U_{k-1}^{+} = U_{k-1}^{-} = 0.$$

Hence, for $k=1$ we note that h_1 is independent of the terms $\epsilon^2\left(\mathbf{W}^{+}\eta^{C_S,+} + \mathbf{W}^{-}\eta^{C_S,-}\right)$ and $\epsilon^2\left(U^{+}\eta^{C_S,+} + U^{-}\eta^{C_S,-}\right)$, which we added in (5.34) and (5.37). In particular, h_1 is independent of C_S and we may proceed as in Remark 5.5 once h_1 is constructed. Theorem 2.37 now yields the existence of a smooth solution $\left(\mu_k^{\pm}, \mathbf{v}_k^{\pm}, p_k^{\pm}, h_k\right)$.

13. We still have to make sure that the compatibility conditions (5.58), (5.63), (5.69) and (5.76) also hold in $\Gamma(2\delta)\setminus\Gamma$ for $k+1$ as this is part of the assumptions of the induction. Note that we have yet to choose the functions g_k, j_k and $\mathbf{q}_k$ in $\Gamma(2\delta)$. We may now define g_k uniquely so that the compatibility condition (5.58) for $k+1$ is fulfilled for all $(x,t)\in\Gamma(2\delta)\setminus\Gamma$. Furthermore, by step 2, (5.58) is already satisfied for $k+1$ on Γ and all involved functions are smooth. Thus, we may extend g_k smoothly to $\Gamma(2\delta)$.
Following a similar line of argumentation we may define j_k, $\mathbf{q}_k\cdot\tau$ and $\mathbf{q}_k\cdot\mathbf{n}$ as smooth functions such that the compatibility conditions (5.63), (5.69) and (5.76) are fulfilled.

14. To find $\mu_k^{\mathbf{B}}$ and $\mathbf{v}_k^{\mathbf{B}}$ we may now use Lemmata 5.15 and 5.16 (more precisely (5.101) for $\mathbf{v}_k^{\mathbf{B}}$), which immediately guarantee that the outer-boundary matching conditions (5.83), (5.84) are satisfied, along with the boundary conditions (5.94) and (5.95).

Lemma 5.22 (The k-th order terms)**.** *Let $k\in\{1,\dots,M+1\}$ be given, let Assumption 1.1 hold and assume that $\mathfrak{S}_0,\dots,\mathfrak{S}_{k-1}$ (as defined in (5.127)) are given and satisfy the matching conditions (5.24)–(5.27). Moreover, assume that the compatibility conditions (5.58), (5.63), (5.69) and (5.76) hold for k. Then there exists*

$$\mathfrak{S}_k = \left(\mathbf{v}_k, \mathbf{v}_k^{\pm}, \mathbf{v}_k^{\mathbf{B}}, \mathbf{u}_k, \mathbf{q}_k, \mu_k, \mu_k^{\pm}, \mu_k^{\mathbf{B}}, c_k, c_k^{\pm}, c_k^{\mathbf{B}}, h_k, l_k, j_k, g_k, p_{k-1}, p_k^{\pm}, p_{k-1}^{\mathbf{B}}\right)$$

such that for k-th order the outer equations (5.13), (5.14) and (5.15), the inner equations (5.40), (5.42), (5.44) and (5.46), the boundary equations (5.86)–(5.89), the inner-outer matching conditions (5.24)–(5.27), the outer-boundary matching conditions (5.82)–(5.85) and the boundary conditions (5.93)–(5.95) are satisfied. Furthermore, the compatibility conditions (5.58), (5.63), (5.69) and (5.76) for $k+1$ and the outer compatibility conditions (5.97) and (5.100) (for k instead of $k-1$) are satisfied.

Proof. First we may define $\left(c_k^{\pm}, c_k, c_k^{\mathbf{B}}, p_k^{\mathbf{B}}\right)$ as in Step 1 of Subsection 5.1.6. Next, we choose the functions $\left(h_k, \mu_k^{\pm}, \mathbf{v}_k^{\pm}, p_{k-1}^{\pm}\right)$ to be the solutions to (5.157)–(5.166), for which existence and uniqueness is shown in (2.37). Then the remaining functions in $\mathfrak{S}_k$ can be defined as in the different steps of Subsection 5.1.6: μ_k as in (5.129) with $\bar{\mu}_k$ and l_k as in (5.133), (5.134) and p_{k-1} as in (5.137). Furthermore, we define $\mathbf{v}_k$ as in (5.138) and (5.143) with $v_k^{\mathbf{n}}$, v_k^{τ} and $\mathbf{u}_k$ as in (5.138), (5.143) and (5.142), (5.147). $\mathbf{q}_k$,j_k and g_k are defined as in Step 13 and $\mu_k^{\mathbf{B}}$, $\mathbf{v}_k^{\mathbf{B}}$ as in Step 14.

Regarding the claims, we first of all note that c_k satisfies the inner-outer matching condition (5.24) due to Lemma 5.7. The term p_{k-1} satisfies the inner-outer matching condition (5.27) since $\mathbf{V}^{k-1}\cdot\mathbf{n}$ and W^{k-1} have exponential decay in ρ and it holds $\int_{\mathbb{R}}\mathbf{V}^{k-1}\cdot\mathbf{n}\mathrm{d}\rho=[p_{k-1}]$ in $\Gamma(2\delta)$ due to (5.76) (cf. Remark 5.12).

Now all the outer equations and the boundary compatibility conditions (5.97) and (5.100) are satisfied by the definition of $\left(c_k^{\pm},\mu_k^{\pm},\mathbf{v}_k^{\pm},p_k^{\pm}\right)$ and the inner equations (5.40), (5.44) and (5.46) are also satisfied due to the Lemmata 5.7, 5.9, 5.10 and 5.11 and the definition of $(c_k,\mu_k,\mathbf{v}_k)$. The inner-outer matching conditions (5.25) and (5.26) are satisfied by $(\mu_k,\mathbf{v}_k)$ due to the definition of $\bar{\mu}_k$, l_k and (5.66), respectively the definition of $v_k^{\mathbf{n}},v_k^{\tau},\mathbf{u}_k$ and (5.71), (5.79).

Note in particular that the definition of p_{k-1} implies that every solution of (5.40) fulfills

$$-\partial_{\rho\rho}\left(\mathbf{v}_k-\left(\mathbf{u}_k d_\Gamma-\mathbf{u}_0 h_k\right)\eta\right)\cdot\mathbf{n}=-\partial_\rho W^{k-1}. \tag{5.167}$$

Thus, (5.167) implies that the last inner equation, (5.42), is also satisfied.

In regard to the boundary equations (5.86)–(5.89), the outer-boundary matching conditions (5.82)–(5.85) and the boundary conditions (5.93)–(5.95), we refer to the Lemmata 5.14, 5.15 and 5.16.

It remains to show that the compatibility conditions for order $k+1$ are all fulfilled. First, we show that they hold on Γ: for (5.58) this follows from the condition (5.158), which implies due to the definition of μ_k (in (5.129)) that (5.128) and thus (5.58) holds. Moreover, (5.63) is equivalent to (5.165) as discussed in Step 12 and (5.69), (5.76) are equivalent to (5.164) as discussed in Steps 9–11.

By the definition of $\mathbf{q}_k,j_k$ and g_k the compatibility conditions then also hold on $\Gamma(2\delta)\setminus\Gamma$. □

Remark 5.23. This is a good place to remark upon the difficulties that would arise if we considered e.g. no-slip boundary conditions for $\mathbf{v}^\epsilon$. In that case, we would demand for $\mathbf{v}_A^\epsilon$ to also satisfy $\mathbf{v}_A^\epsilon=0$ on $\partial_{T_0}\Omega$, which may be achieved by suitable changes to the presented boundary layer expansion. However, if $\mathbf{v}_k^{\mathbf{B}}(0,x,t)=0$ was supposed to hold for $(x,t)\in\partial_{T_0}\Omega$, we would need to prescribe inhomogeneous boundary data of the form $\mathbf{v}_k^-(x,t)=\hat{\mathbf{U}}^{k-1}(x,t)$ for $(x,t)\in\partial_{T_0}\Omega$, for a suitable function $\hat{\mathbf{U}}^{k-1}$. This can be verified by a similar argumentation as presented in Subsection 5.1.4, more precisely as in Lemma 5.16 and (5.100). As a consequence of $\mathrm{div}\mathbf{v}_k^{\pm}=0$ in $\Omega_{T_0}^{\pm}$, we may then calculate

$$\begin{aligned}0&=\int_{\Omega^+(t)}\mathrm{div}\mathbf{v}_k^+\mathrm{d}x+\int_{\Omega^-(t)}\mathrm{div}\mathbf{v}_k^-\mathrm{d}x=-\int_{\Gamma_t}[\mathbf{v}_k]\cdot\mathbf{n}_{\Gamma_t}\mathrm{d}\mathcal{H}^1(p)+\int_{\partial\Omega}\mathbf{v}_k^-\cdot\mathbf{n}_{\partial\Omega}\mathrm{d}\mathcal{H}^1(p)\\&=-\int_{\Gamma_t}\mathbf{a}_{k-1}^3\cdot\mathbf{n}_{\Gamma_t}\mathrm{d}\mathcal{H}^1(p)+\int_{\partial\Omega}\hat{\mathbf{U}}^{k-1}\cdot\mathbf{n}_{\partial\Omega}\mathrm{d}\mathcal{H}^1(p)\end{aligned}$$

for $t\in[0,T_0]$, where we used the divergence theorem in the second equality and (5.163) in the last. But this equality does not have to be satisfied for arbitrary k. To avoid such issues, we restricted ourselves to the case of the boundary condition (1.23).

Now we "glue" together the inner and outer expansions of c^ϵ in order to get an approximate solution. We will repeat this later for approximate solutions of $\mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon$, cf. Definition 6.2.

Definition 5.24 (A First Approximate Solution)**.** Let $\mathfrak{S}_0, \ldots, \mathfrak{S}_{M+1}$ be the expansions up to order $M+1$ as given in Lemmata 5.19 and 5.22. Let furthermore some $\epsilon_0 > 0$, $T' \in (0, T_0]$ and a family of functions $\left(\tilde{h}^\epsilon\right)_{\epsilon\in(0,\epsilon_0)} \subset X_{T'}$ be given, with $\tilde{h}^\epsilon|_{t=0} = 0$ (cf. (2.40) for the definition of $X_{T'}$). In the following, we write $H := \left(\tilde{h}^\epsilon\right)_{\epsilon\in(0,\epsilon_0)}$.

Now we define

$$h_A^{\epsilon,H}(s,t) := \sum_{i=0}^{M} \epsilon^i h_{i+1}(s,t) + \epsilon^{M-\frac{3}{2}} \tilde{h}^\epsilon(s,t) \tag{5.168}$$

for $(s,t) \in \mathbb{T}^1 \times [0,T']$. Note that $h^\epsilon(s,t)$ is well-defined for all $(s,t) \in \mathbb{T}^1 \times [0,T']$ since $X_{T'} \hookrightarrow C^0\left([0,T']; C^1\left(\mathbb{T}^1\right)\right)$ due to Proposition 2.34 2) and the Sobolev embeddings. Furthermore, we set

$$\tilde{c}_I(\rho,x,t) := \sum_{i=0}^{M+1} \epsilon^i c_i(\rho,x,t), \tag{5.169}$$
$$c_I^H(x,t) := \sum_{i=0}^{M+1} \epsilon^i c_i\left(\rho^H(x,t), x, t\right)$$

for $\rho \in \mathbb{R}$, $(x,t) \in \Gamma(2\delta; T')$ and

$$\rho^H(x,t) := \frac{d_\Gamma(x,t)}{\epsilon} - h_A^{\epsilon,H}(S(x,t),t). \tag{5.170}$$

For the outer part we set

$$c_O(x,t) := \sum_{i=0}^{M+1} \epsilon^i \left(c_i^+(x,t)\chi_{\overline{\Omega^+}}(x,t) + c_i^-(x,t)\chi_{\Omega^-}(x,t)\right)$$

for $(x,t) \in \Omega_{T'}$ and for the boundary part we define

$$c_{\mathbf{B}}(x,t) := \sum_{i=0}^{M+1} \epsilon^i c_i^{\mathbf{B}}\left(\frac{d_{\mathbf{B}}(x,t)}{\epsilon}, x, t\right)$$

for $(x,t) \in \overline{\partial_{T'}\Omega(\delta)}$.

Let $\xi \in C^\infty(\mathbb{R})$ be the cutoff function as defined in Definition 2.1. We now define the approximate solution

$$c_A^{\epsilon,H} := \begin{cases} c_{\mathbf{B}} & \text{in } \overline{\partial_{T'}\Omega\left(\frac{\delta}{2}\right)}, \\ \xi(2d_{\mathbf{B}})c_{\mathbf{B}} + (1-\xi(2d_{\mathbf{B}}))c_O & \text{in } \partial_{T'}\Omega(\delta)\setminus\partial_{T'}\Omega\left(\frac{\delta}{2}\right), \\ c_O & \text{in } \Omega_{T'}\setminus\left(\partial_{T'}\Omega(\delta)\cup\Gamma(2\delta;T')\right), \\ \xi(d_\Gamma)c_I^H + (1-\xi(d_\Gamma))c_O & \text{in } \Gamma(2\delta;T')\setminus\Gamma(\delta;T'), \\ c_I^H & \text{in } \Gamma(\delta;T') \end{cases} \tag{5.171}$$

in $\Omega_{T'}$. In the following, we will also use the alternative way of writing

$$c_A^{\epsilon,H} = \xi(d_\Gamma)c_I^H + (1-\xi(d_\Gamma))(1-\xi(2d_{\mathbf{B}}))c_O + \xi(2d_{\mathbf{B}})c_{\mathbf{B}}.$$

Later on, the family H will be replaced by the terms of correct order $h^{\epsilon}_{M-\frac{1}{2}}$, which will then depend on ϵ. But in order to find those terms we need some preparations first, which will turn out to be more flexible and notationally consistent when they are done with an arbitrary family of functions H.

At the end of this section, we would like to shortly discuss the effects of different choices in equation (1.18) on the ordinary differential equation for $\mathbf{v}_k$ as discussed in (5.40).

Remark 5.25.

Substituting $\mu^\epsilon \nabla c^\epsilon$ by $-\epsilon\mathbf{div}\left(\nabla c^\epsilon \otimes \nabla c^\epsilon\right)$: First note that $\operatorname{div}\left(\nabla c^\epsilon \otimes \nabla c^\epsilon\right) = \nabla c^\epsilon \Delta c^\epsilon + \frac{1}{2}\nabla\left|\nabla c^\epsilon\right|^2$ and that

$$\begin{aligned}\frac{1}{2}\nabla\left|\nabla c^\epsilon\right|^2 &= \partial_{\rho\rho}\tilde{c}^\epsilon\left(\frac{1}{\epsilon}\mathbf{n}-\nabla^\Gamma h^\epsilon\right)\left(\partial_\rho\tilde{c}^\epsilon\left(\frac{1}{\epsilon^2}+\left|\nabla^\Gamma h^\epsilon\right|^2\right)+\nabla\tilde{c}^\epsilon\cdot\left(\frac{1}{\epsilon}\mathbf{n}-\nabla^\Gamma h^\epsilon\right)\right)\\
&\quad+\partial_\rho\tilde{c}^\epsilon\left(\nabla\partial_\rho\tilde{c}^\epsilon\left(\frac{1}{\epsilon^2}+\left|\nabla^\Gamma h^\epsilon\right|^2\right)+D^2\tilde{c}^\epsilon\left(\frac{1}{\epsilon}\mathbf{n}-\nabla^\Gamma h^\epsilon\right)\right)\\
&\quad+\partial_\rho\tilde{c}^\epsilon\left(\frac{1}{\epsilon}D^2 d_\Gamma\nabla\tilde{c}^\epsilon - D^2_\Gamma h^\epsilon\nabla\tilde{c}^\epsilon+\partial_\rho\tilde{c}^\epsilon D^2_\Gamma h^\epsilon\nabla^\Gamma h^\epsilon\right)\\
&\quad+\left(\frac{1}{\epsilon}\mathbf{n}-\nabla^\Gamma h^\epsilon\right)\left(\nabla\tilde{c}^\epsilon\cdot\nabla\partial_\rho\tilde{c}^\epsilon\right)+\nabla\partial_\rho\tilde{c}^\epsilon\left(\nabla\tilde{c}^\epsilon\cdot\left(\frac{1}{\epsilon}\mathbf{n}-\nabla^\Gamma h^\epsilon\right)\right)\\
&\quad+\partial_\rho\tilde{c}^\epsilon\left(\frac{1}{\epsilon}\mathbf{n}-\nabla^\Gamma h^\epsilon\right)\left(\nabla\partial_\rho\tilde{c}^\epsilon\cdot\left(\frac{1}{\epsilon}\mathbf{n}-\nabla^\Gamma h^\epsilon\right)\right)+D^2\tilde{c}^\epsilon\nabla\tilde{c}^\epsilon,\end{aligned}$$

where we used the form of $\tilde{c}^\epsilon$ as given in (5.20), the identity $\nabla d_\Gamma(x,t)\cdot\nabla^\Gamma h^\epsilon(x,t)=0$ and the notation D^2_Γ as discussed in (2.25). Here we skipped the explicit notation of (x,t), $\rho(x,t)$, $S(x,t)$ in favor of the brevity of presentation. Using this and the formulae for ∇c^ϵ, Δc^ϵ in (5.23) we get

$$\begin{aligned}-\partial_{\rho\rho}\tilde{\mathbf{v}}^\epsilon &= -2\partial_{\rho\rho}\tilde{c}^\epsilon\partial_\rho\tilde{c}^\epsilon\mathbf{n}\\
&\quad+\epsilon\Big(\partial_{\rho\rho}\tilde{c}^\epsilon\left(2\partial_\rho\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon-\nabla\tilde{c}^\epsilon\right)+\partial_\rho\tilde{\mathbf{v}}^\epsilon\Delta d_\Gamma+2\left(\nabla\partial_\rho\tilde{\mathbf{v}}^\epsilon\right)^T\mathbf{n}\\
&\qquad-\Big(\partial_\rho\tilde{p}^\epsilon+\left(\partial_\rho\tilde{c}^\epsilon\right)^2\Delta d_\Gamma+3\left(\nabla\partial_\rho\tilde{c}^\epsilon\cdot\mathbf{n}\right)\partial_\rho\tilde{c}^\epsilon\Big)\mathbf{n}-\partial_\rho\tilde{c}^\epsilon\nabla\partial_\rho\tilde{c}^\epsilon-\left(\nabla\tilde{c}^\epsilon\cdot\mathbf{n}\right)\partial_{\rho\rho}\tilde{c}\mathbf{n}\Big)\\
&\quad+\epsilon^2\Big(-2\left(\nabla\partial_\rho\tilde{\mathbf{v}}^\epsilon\right)^T\cdot\nabla^\Gamma h^\epsilon+\Delta\tilde{\mathbf{v}}^\epsilon+\partial_{\rho\rho}\tilde{\mathbf{v}}^\epsilon\left|\nabla^\Gamma h^\epsilon\right|^2-\partial_\rho\tilde{\mathbf{v}}^\epsilon\Delta^\Gamma h^\epsilon\\
&\qquad+\partial_\rho\tilde{p}^\epsilon\nabla^\Gamma h^\epsilon+\nabla\tilde{c}^\epsilon\cdot\mathbf{n}\left(\partial_{\rho\rho}\tilde{c}\nabla^\Gamma h^\epsilon-\nabla\partial_\rho\tilde{c}\right)-\nabla\tilde{p}^\epsilon-\partial_\rho\tilde{c}^\epsilon D^2\left(\tilde{c}^\epsilon\right)\mathbf{n}\\
&\qquad+\left(\partial_\rho\tilde{c}^\epsilon\Delta d_\Gamma+2\nabla\partial_\rho\tilde{c}^\epsilon\cdot\mathbf{n}\right)\cdot\partial_\rho\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon+\partial_\rho\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon\left(\nabla\partial_\rho\tilde{c}^\epsilon\cdot\mathbf{n}\right)\\
&\qquad+\Big(-2\partial_{\rho\rho}\tilde{c}^\epsilon\left|\nabla^\Gamma h^\epsilon\right|^2+\partial_\rho\tilde{c}^\epsilon\Delta^\Gamma h^\epsilon+3\nabla\partial_\rho\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon-\Delta\tilde{c}^\epsilon\Big)\left(\partial_\rho\tilde{c}^\epsilon\cdot\mathbf{n}\right)\\
&\qquad-D^2\left(d_\Gamma\right)\nabla\tilde{c}^\epsilon\partial_\rho\tilde{c}^\epsilon-\left(\nabla\tilde{c}^\epsilon\cdot\nabla\partial_\rho\tilde{c}^\epsilon\right)\mathbf{n}+\nabla\tilde{c}^\epsilon\cdot\nabla^\Gamma h^\epsilon\partial_{\rho\rho}\tilde{c}\mathbf{n}\Big)\\
&\quad+\epsilon^3\Big(\Big(-\partial_{\rho\rho}\tilde{c}^\epsilon\left|\nabla^\Gamma h^\epsilon\right|^2+\partial_\rho\tilde{c}^\epsilon\Delta^\Gamma h^\epsilon+2\nabla\partial_\rho\tilde{c}^\epsilon\cdot\nabla^\Gamma h^\epsilon\Big)\left(-\partial_\rho\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon+\nabla\tilde{c}^\epsilon\right)\\
&\qquad+\Delta\tilde{c}^\epsilon\partial_\rho\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon+\left|\nabla^\Gamma h^\epsilon\right|^2\partial_\rho\tilde{c}^\epsilon\partial_{\rho\rho}\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon-\left|\nabla^\Gamma h^\epsilon\right|^2\partial_\rho\tilde{c}^\epsilon\nabla\partial_\rho\tilde{c}^\epsilon\\
&\qquad-\left(\partial_\rho\tilde{c}^\epsilon\right)^2 D^2_\Gamma\left(h^\epsilon\right)\cdot\nabla^\Gamma h^\epsilon+\nabla\tilde{c}^\epsilon\cdot\nabla^\Gamma h^\epsilon\left(-\partial_{\rho\rho}\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon+\nabla\partial_\rho\tilde{c}^\epsilon\right)\\
&\qquad+\partial_\rho\tilde{c}^\epsilon D^2\left(\tilde{c}^\epsilon\right)\nabla^\Gamma h^\epsilon-\partial_\rho\tilde{c}^\epsilon\nabla^\Gamma h^\epsilon\left(\nabla\partial_\rho\tilde{c}^\epsilon\cdot\nabla^\Gamma h^\epsilon\right)+\partial_\rho\tilde{c}^\epsilon D^2_\Gamma\left(h^\epsilon\right)\nabla\tilde{c}^\epsilon\\
&\qquad+\left(\nabla\tilde{c}^\epsilon\cdot\nabla\partial_\rho\tilde{c}^\epsilon\right)\nabla^\Gamma h^\epsilon-\operatorname{div}\left(\nabla\tilde{c}^\epsilon\otimes\nabla\tilde{c}^\epsilon\right)\Big)\end{aligned}\tag{5.172}$$

instead of (5.28). Some remarks:

- In this case, terms of the expansion of $\tilde{c}^\epsilon$ of the same order as of $\tilde{\mathbf{v}}^\epsilon$ appear in (5.40). In particular, the term $\partial_\rho \left(\partial_\rho c_k \partial_\rho c_0\right)$ shows up on the right hand side of the ordinary differential equation (5.40). But this does not cause problems in the construction scheme, as c_k can be viewed as a term of order $k-1$.
- Nearly all terms that are multiplied by ϵ^2 and ϵ^3 are harmless, the sole exception being $\epsilon^2 \left(\Delta\tilde{\mathbf{v}}^\epsilon - \nabla\tilde{p}^\epsilon\right) - \epsilon^3 \mathrm{div}\left(\nabla\tilde{c}^\epsilon \otimes \nabla\tilde{c}^\epsilon\right)$ which corresponds to the outer system and needs to be taken care of with the help of a function $\mathbf{W}^\pm$ as in (5.18). In view of the estimates for the remainder terms, all second derivatives of h^ϵ deserve a second glance, as we will see later on that $h^\epsilon_{M-\frac{1}{2}}$ can only be controlled independently of ϵ in the space X_T (see (2.40) for the definition). Nevertheless, it is possible to estimate these terms using the same strategy as in Lemma 6.7 1).

Considering the Instationary Stokes Equation: Having $\partial_t \mathbf{v}^\epsilon - \Delta\mathbf{v}^\epsilon + \nabla p^\epsilon$ on the left hand side of (1.18) results in additional, higher order (in ϵ) terms related to the time derivative. More specifically, we get

$$\tilde{\mathbf{R}}^\epsilon := \mathbf{R}^\epsilon - \epsilon\partial_t d_\Gamma \partial_\rho \tilde{\mathbf{v}}^\epsilon - \epsilon^2 \left(\partial_t \tilde{\mathbf{v}}^\epsilon - \partial_\rho \tilde{\mathbf{v}}^\epsilon \partial_t^\Gamma h^\epsilon\right) \tag{5.173}$$

as the right hand side of (5.28) (or the same terms added to (5.172) if one chooses to use the alternative right hand side). Thus, in the ordinary differential equation for $\mathbf{v}_k$ no new terms of order k appear, but only new terms of order $k-1$ and lower. It is however noteworthy that the term $\partial_\rho \tilde{\mathbf{v}}^\epsilon \partial_t^\Gamma h^\epsilon$ leads to the appearance of $[\mathbf{v}_0] \cdot \tau \partial_t^\Gamma h_k$ and $[\mathbf{v}_0] \cdot \mathbf{n} \partial_t^\Gamma h_k$ in (5.148) and (5.151) respectively. But as $[\mathbf{v}_0] \equiv 0$ on Γ this does not lead to the appearance of $\partial_t^\Gamma h_k$ in (5.164). Hence, the general strategy of constructing the k–th order terms can easily be adapted to this situation; the only real difference lies in showing existence for solutions of the adapted system (5.157)–(5.166) where the outer Stokes equation is now instationary. The structure of $\mathbf{V}^{k-1}$ if we considered the Instationary Stokes Equation with right hand side $-\epsilon \mathrm{div}\left(\nabla c^\epsilon \otimes \nabla c^\epsilon\right)$ (as discussed above) is given in Appendix A.2.

Considering the full Navier-Stokes Equation: Having $\partial_t \mathbf{v}^\epsilon + \mathbf{v}^\epsilon \cdot \nabla\mathbf{v}^\epsilon - \Delta\mathbf{v}^\epsilon + \nabla p^\epsilon$ on the left hand side of (1.18) results in

$$\hat{\mathbf{R}}^\epsilon := \tilde{\mathbf{R}}^\epsilon - \epsilon\tilde{\mathbf{v}}^\epsilon \cdot \partial_\rho \tilde{\mathbf{v}}^\epsilon \otimes \mathbf{n} + \epsilon^2 \left(\tilde{\mathbf{v}}^\epsilon \partial_\rho \tilde{\mathbf{v}}^\epsilon \otimes \nabla^\Gamma h^\epsilon - \tilde{\mathbf{v}}^\epsilon \cdot \nabla\tilde{\mathbf{v}}^\epsilon\right)$$

as the right hand side of (5.28), with $\tilde{\mathbf{R}}$ as in (5.173).

5.2. A First Estimate of the Error in the Velocity

Let the assumptions and notations of Definition 5.24 hold in this subsection. Moreover, we will use the function spaces introduced in Section 2.2 and introduce:

Notation 5.26. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}^n$, then we denote

$$a \otimes_s b := a \otimes b + b \otimes a.$$

For $T \in (0, T_0]$, $\epsilon \in (0, \epsilon_0)$ and $H = \left(\tilde{h}^\epsilon\right)_{\epsilon \in (0,\epsilon_0)} \subset X_T$ with $\tilde{h}^\epsilon|_{t=0} = 0$ we consider weak solutions $\tilde{\mathbf{w}}_1^{\epsilon,H} : \Omega_T \to \mathbb{R}^2$ and $q_1^{\epsilon,H} : \Omega_T \to \mathbb{R}$ of

$$-\Delta \tilde{\mathbf{w}}_1^{\epsilon,H} + \nabla q_1^{\epsilon,H} = -\epsilon \mathrm{div}\left(\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right) \otimes_s \nabla R^H\right) \qquad \text{in } \Omega_T, \tag{5.174}$$

$$\mathrm{div} \tilde{\mathbf{w}}_1^{\epsilon,H} = 0 \qquad \text{in } \Omega_T, \tag{5.175}$$

$$\left(-2D_s \tilde{\mathbf{w}}_1^{\epsilon,H} + q_1^{\epsilon,H}\mathbf{I}\right) \cdot \mathbf{n}_{\partial\Omega} = \alpha_0 \tilde{\mathbf{w}}_1^{\epsilon,H} \qquad \text{on } \partial_T\Omega \tag{5.176}$$

in the sense of (2.9). Here we denote $R^H := c^\epsilon - c_A^{\epsilon,H}$, where $c^\epsilon : \Omega_{T_0} \to \mathbb{R}$ is a smooth solution to (1.18)–(1.25) with c_0^ϵ defined as in (4.1), for $c_A^\epsilon = c_A^{\epsilon,H}$ and fixed ψ_0^ϵ. Note that c^ϵ does not depend on H, as

$$c_I^H(x,0) = \sum_{i=0}^{M+1} \epsilon^i c_i\left(\rho^H(x,0), x, 0\right) = \sum_{i=0}^{M+1} \epsilon^i c_i\left(\frac{d_\Gamma(x,0)}{\epsilon}, x, t\right)$$

due to $h_i|_{t=0} = 0$ by construction for $i \in \{1, \ldots, M+1\}$ and $\tilde{h}^\epsilon|_{t=0} = 0$. Moreover, we define $\mathbf{h}^H$ by

$$\mathbf{h}^H(x,t) := -\xi\left(d_\Gamma(x,t)\right) \partial_\rho \tilde{c}_I\left(\rho^H(x,t), x, t\right) \epsilon^{M-\frac{3}{2}} \nabla^\Gamma \tilde{h}^\epsilon(x,t). \tag{5.177}$$

Introducing $\mathbf{h}^H$, we avoid considering a term which is quadratic in $\tilde{h}^\epsilon$ on the right hand side of (5.174), enabling us to show Proposition 5.30 – an essential part of Theorem 5.32. By definition, we have

$$\begin{aligned}
&\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right)\left(\rho^H(x,t), x, t\right) \\
&= \xi'\left(d_\Gamma(x,t)\right) \nabla d_\Gamma(x,t)\, c_I^H(x,t) + \xi\left(d_\Gamma(x,t)\right)\left(\nabla \tilde{c}_I\left(\rho^H(x,t), x, t\right)\right) \\
&\quad + \xi\left(d_\Gamma(x,t)\right)\left(\partial_\rho \tilde{c}_I\left(\rho^H(x,t), x, t\right)\left(\frac{1}{\epsilon}\nabla d_\Gamma(x,t) - \sum_{i=0}^{M} \epsilon^i \nabla^\Gamma h_{i+1}(x,t)\right)\right) \\
&\quad + \nabla\left(\left(1 - \xi\left(d_\Gamma(x,t)\right)\right)\left(1 - \xi\left(2d_{\mathbf{B}}(x,t)\right)\right) c_O(x,t) + \xi\left(2d_{\mathbf{B}}(x,t)\right) c_{\mathbf{B}}(x,t)\right)
\end{aligned} \tag{5.178}$$

for $(x,t) \in \Omega_T$. As we consider $\tilde{\mathbf{w}}_1^{\epsilon,H}$ to be a weak solution, we understand the right hand side of equation (5.174) as a functional in $(V_0)'$ given by

$$\mathbf{f}^{\epsilon,H}(\psi) := \int_\Omega \epsilon\left(\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right) \otimes \nabla R^H + \nabla R^H \otimes \left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right)\right) : \nabla\psi \mathrm{d}x \tag{5.179}$$

for $\psi \in V_0$ and fixed $t \in [0,T]$. As $H \subset X_T$, Theorem 2.6 implies the existence of a unique weak solution.

Although – from a logical point of view – it makes sense to introduce $\tilde{\mathbf{w}}_1^{\epsilon,H}$ at this point in the thesis, it is difficult to give a thorough explanation of why it is necessary to introduce

$\tilde{\mathbf{w}}_1^{\epsilon,H}$ without the notion of approximate solutions and remainder terms as constructed in Chapter 6. So we relocate the information of the precise use of $\tilde{\mathbf{w}}_1^{\epsilon,H}$ to the beginning of Section 7.1.1. For now it suffices to know that $\tilde{\mathbf{w}}_1^{\epsilon,H}$ plays a key role in estimating $\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon$ (where $\mathbf{v}_A^\epsilon$ is a – yet to be defined – approximation of $\mathbf{v}^\epsilon$) as it turns out to be the leading term in the error of the velocity.

To gain an intuition as to what order of ϵ we might expect when estimating $\tilde{\mathbf{w}}_1^{\epsilon,H}$, we can use Hölder's inequality in (5.179) to get

$$\left|\mathbf{f}^{\epsilon,H}(\psi)\right| \leq \epsilon \left\|\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right\|_{L^\infty(\Omega)} \left\|\nabla R^H\right\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}$$

and as $\nabla c_A^{\epsilon,H} \in \mathcal{O}\left(\epsilon^{-1}\right)$ in L^∞ close to Γ and $\left\|\nabla R^H\right\|_{L^2(\Omega)} \in \mathcal{O}\left(\epsilon^{M-\frac{3}{2}}\right)$ if (4.6a) and (4.6b) hold, we expect $\left\|\tilde{\mathbf{w}}_1^{\epsilon,H}\right\|_{L^2(0,T)(H^1(\Omega))} \in \mathcal{O}\left(\epsilon^{M-\frac{3}{2}}\right)$. In fact we will improve this first estimate by a power of ϵ in the main result of this chapter, Lemma 5.29.

An important tool for that result will be a decomposition of R^H close to the interface as suggested by Corollary 3.11. In order to use this, we have to make sure that c_A^ϵ has the form needed to use the results in Chapter 3.

5.2.1. Decomposition of R^H

Lemma 5.27. *Let $\epsilon_0 > 0$, $T' \in (0, T_0]$ and families $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset (0, T']$, $H := \left(\tilde{h}^\epsilon\right)_{\epsilon\in(0,\epsilon_0)} \subset X_{T'}$ with $\tilde{h}^\epsilon|_{t=0} = 0$ be given. We assume that there is $\bar{C} > 0$ such that*

$$\sup_{\epsilon\in(0,\epsilon_0)} \left\|\tilde{h}^\epsilon\right\|_{X_{T_\epsilon}} \leq \bar{C} \tag{5.180}$$

holds. Then there is $\epsilon_1 \in (0, \epsilon_0]$ such that $c_A^{\epsilon,H}(.,t)$ satisfies Assumption 3.1 for all $t \in [0, T_\epsilon]$ and $\epsilon \in (0, \epsilon_1)$, where the appearing constant C^* does not depend on ϵ, T_ϵ, H or $\bar{C}$.

Proof. First of all, we note that there exists $\epsilon_1 \in (0, \epsilon_0]$, which depends on $\bar{C}$, such that

$$\left|\frac{d_\Gamma(x,t)}{\epsilon} - h_A^{\epsilon,H}(S(x,t),t)\right| \geq \frac{\delta}{2\epsilon} \tag{5.181}$$

for all $(x,t) \in \Gamma(2\delta; T_\epsilon) \backslash \Gamma(\delta; T_\epsilon)$ and $\epsilon \in (0, \epsilon_1)$. This is due to the fact that $X_T \hookrightarrow C^0\left([0,T]; C^1\left(\mathbb{T}^1\right)\right)$ and that (5.180) holds. After possibly choosing $\epsilon_1 > 0$ smaller, we may ensure that

$$\left|\theta_0\left(\rho^H(x,t)\right) - \chi_{\Omega^+}(x,t) + \chi_{\Omega^-}(x,t)\right| + \left|\theta_0'(\rho(x,t))\right| \leq C_1 e^{-C_2\frac{\delta}{2\epsilon}} \tag{5.182}$$

holds for all $(x,t) \in \Gamma(2\delta; T_\epsilon) \backslash \Gamma(\delta; T_\epsilon)$ and $\epsilon \in (0, \epsilon_1)$, as a consequence of (2.1), where $C_1, C_2 > 0$ can be chosen independently of ϵ_1. As a last condition on ϵ_1 we impose that $\epsilon_1^{M-\frac{3}{2}} \leq \frac{1}{\bar{C}}$ such that

$$\epsilon^{M-\frac{3}{2}} \left\|\tilde{h}^\epsilon\right\|_{X_{T_\epsilon}} \leq 1 \tag{5.183}$$

for all $\epsilon \in (0, \epsilon_1)$. In particular, this implies

$$\left\|h_A^{\epsilon,H}\right\|_{C^0([0,T_\epsilon];C^1(\mathbb{T}^1))} \leq C^* \tag{5.184}$$

for all $\epsilon \in (0, \epsilon_1)$, where C^* is independent of $\bar{C}$, H and T_ϵ, since the operator norm of the embedding $X_{T_\epsilon} \hookrightarrow C^0\left([0,T_\epsilon]\,;H^2\left(\mathbb{T}^1\right)\right)$ is independent of T_ϵ, cf. Proposition 2.34 and since $h_i \in C^0\left([0,T_0]\,;C^1\left(\mathbb{T}^1\right)\right)$. Thus, assumption (3.7) follows.

By definition in (5.171), we have

$$c_A^{\epsilon,H} = \xi\left(d_\Gamma\right) c_I^H + \left(1-\xi\left(d_\Gamma\right)\right)\left(1-\xi\left(2d_{\mathbf{B}}\right)\right) c_O + \xi\left(2d_{\mathbf{B}}\right) c_{\mathbf{B}}.$$

Moreover, $c_A^{\epsilon,H} \in L^\infty\left(\Omega_{T'}\right)$ by construction and

$$\begin{aligned}\nabla_\tau c_A^{\epsilon,H}(x,t) &= \nabla^\Gamma c_A^{\epsilon,H}(x,t)\\ &= \partial_\rho \tilde{c}_I\left(\rho^H(x,t),x,t\right)\nabla^\Gamma h_A^{\epsilon,H} + \left(\nabla^\Gamma \tilde{c}_I\right)\left(\rho^H(x,t),x,t\right)\end{aligned}$$

for $(x,t) \in \Gamma\left(\delta,T'\right)$ and $\partial_\rho \tilde{c}_I, \nabla\tilde{c}_I \in L^\infty\left(\Gamma\left(2\delta\right)\right)$ as well as $\left\|\nabla^\Gamma h_A^{\epsilon,H}\right\|_{L^\infty(\Gamma(2\delta;T_\epsilon))} \le C^*$ due to (5.184), which implies assumption (3.9). Assumption (3.8) follows immediately from the definitions of c_O and $c_{\mathbf{B}}$, as $c_O = \pm 1 + \mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega^\pm_{T_0}\right)$ and $c_{\mathbf{B}} = -1 + \mathcal{O}(\epsilon)$ in $L^\infty\left(\partial_{T_0}\Omega\left(\delta\right)\right)$. Next, we show

$$c_I^H = \theta_0\left(\rho^H\right) + \epsilon p^\epsilon\left(Pr_{\Gamma_t}\right)\theta_1\left(\rho^H\right) + \epsilon^2 q^\epsilon,$$

where θ_1 satisfies (3.5) and p^ϵ, q^ϵ satisfy (3.6). As $c_0 = \theta_0$ by Lemma 5.19 and $c_2,\ldots,c_{M+1} \in L^\infty\left(\mathbb{R}\times\Gamma\left(2\delta\right)\right)$, the only thing we need to show is that c_1 can be decomposed suitably. By (5.44) and (5.51) c_1 satisfies

$$\partial_{\rho\rho} c_1 - f''\left(\theta_0\right) c_1 = -\mu_0 - \theta_0'\Delta d_\Gamma + g_0\eta' d_\Gamma \text{ for all } (\rho,x,t) \in \mathbb{R}\times\Gamma\left(2\delta\right).$$

Thus, we find by (5.105) and (5.106) that

$$\partial_{\rho\rho} c_1 - f''\left(\theta_0\right) c_1 = \Delta d_\Gamma\left(\sigma - \theta_0'\right) \text{ for all } (\rho,x,t) \in \mathbb{R}\times\Gamma$$

for $\sigma = \frac{1}{2}\int_{\mathbb{R}}\left(\theta_0'\right)^2 \mathrm{d}\rho$ and thus $c_1(\rho,x,t) = \Delta d_\Gamma(x,t)\,\theta_1(\rho)$ for all $(\rho,x,t) \in \mathbb{R}\times\Gamma$, where θ_1 is the unique solution to

$$\begin{aligned}\theta_1'' - f''\left(\theta_0\right)\theta_1 &= \sigma - \theta_0',\\ \theta_1(0) &= 0\end{aligned}$$

on $\mathbb{R}$ with $\theta_1 \in L^\infty(\mathbb{R})$. θ_1 exists since $\int_{\mathbb{R}}\left(\sigma - \theta_0'\right)\theta_0'\mathrm{d}\rho = 0$ by the definition of σ (cf. Lemma 2.3) . Moreover, we have

$$\begin{aligned}0 &= \int_{\mathbb{R}} \theta_0''\left(\sigma - \theta_0'\right)\mathrm{d}\rho = \int_{\mathbb{R}} \theta_0''\left(\theta_1'' - f''\left(\theta_0\right)\theta_1\right)\mathrm{d}\rho\\ &= \int_{\mathbb{R}} \theta_1'\left(-\theta_0''' + f''\left(\theta_0\right)\theta_0'\right)\mathrm{d}\rho + \int_{\mathbb{R}} f^{(3)}\left(\theta_0\right)\left(\theta_0'\right)^2\theta_1\mathrm{d}\rho\\ &= \int_{\mathbb{R}} f^{(3)}\left(\theta_0\right)\left(\theta_0'\right)^2\theta_1\mathrm{d}\rho,\end{aligned}$$

where we used the equation for θ_1 in the first line, integration by parts in the second line and (1.36) in the third. Thus, θ_1 satisfies (3.5). Setting $p^\epsilon = \Delta d_\Gamma$ in $\Gamma\left(2\delta\right)$ and

$$\tilde{q}^\epsilon(x,t) := \frac{1}{\epsilon}\left(c_1\left(\rho^H(x,t),x,t\right) - p^\epsilon\left(Pr_{\Gamma_t}(x),t\right)\theta_1\left(\rho^H(x,t)\right)\right),$$

we can write

$$c_1\left(\rho^H(x,t),x,t\right)=p^\epsilon\left(Pr_{\Gamma_t}(x),t\right)\theta_1\left(\rho^H(x,t)\right)+\epsilon\tilde{q}^\epsilon(x,t).$$

Now we estimate

$$\begin{aligned}\epsilon\left|\tilde{q}^\epsilon(x,t)\right|&=\left|c_1\left(\rho^H(x,t),x,t\right)-c_1\left(\rho^H(x,t),Pr_{\Gamma_t}(x),t\right)\right|\\&=\left|\nabla_x c_1\left(\rho^H(x,t),\xi(x),t\right)\cdot\left(x-Pr_{\Gamma_t}(x)\right)\right|\\&\leq C\left|d_\Gamma(x,t)\right|\\&\leq\epsilon\left(C\left|\rho^H(x,t)\right|+C^*\right),\end{aligned}$$

where we used a Taylor expansion in the second line and the definition of ρ^H as well as (5.184) in the last line. Here $C>0$ only depends on c_1, as $|\nabla_x c_1|\in L^\infty(\mathbb{R}\times\Gamma(2\delta))$. This shows assumption (3.6).

Finally, assumption (3.10) follows by choosing $\epsilon_1>0$ small enough, since $f''(\pm1)>0$ by our assumptions and since $c_A^{\epsilon,H}=\pm1+\mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega_{T'}\backslash\Gamma(\delta,T')\right)$. This can be derived from the fact that $c_O=\pm1+\mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega^\pm_{T_0}\right)$ and $c_{\mathbf{B}}=-1+\mathcal{O}(\epsilon)$ in $L^\infty\left(\partial_{T_0}\Omega(\delta)\right)$ as remarked before, and $c_I=\theta_0\left(\rho^H\right)+\mathcal{O}(\epsilon)$ in $L^\infty\left(\Gamma(2\delta;T')\right)$, where

$$\left|\theta_0\left(\rho^H(x,t)\right)-\chi_{\Omega^+}(x,t)+\chi_{\Omega^-}(x,t)\right|\leq C_1e^{-C_2\frac{\delta}{2\epsilon}}$$

by (5.182) for $(x,t)\in\Gamma(2\delta;T_\epsilon)\backslash\Gamma(\delta;T_\epsilon)$. This shows the claim. □

The following proposition appears very technical at first glance, but at its core lies the idea that across the interface R^H should resemble θ_0' plus some perturbation terms of higher order in ϵ.

Proposition 5.28. *Let $\epsilon_0>0$, $T'\in(0,T_0]$ and families $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)}\subset(0,T']$, $H:=\left(\tilde{h}^\epsilon\right)_{\epsilon\in(0,\epsilon_0)}\subset X_{T'}$ with $\tilde{h}^\epsilon|_{t=0}=0$ be given. Let Assumption 4.2 hold for $c_A=c_A^{\epsilon,H}$ and we assume that there is $\bar{C}\geq1$ such that*

$$\sup_{\epsilon\in(0,\epsilon_0)}\left\|\tilde{h}^\epsilon\right\|_{X_{T_\epsilon}}\leq\bar{C}$$

holds. We write

$$I_\epsilon^{s,t}:=\left(-\frac{\delta}{\epsilon}-h_A^{\epsilon,H}(s,t),\frac{\delta}{\epsilon}-h_A^{\epsilon,H}(s,t)\right)$$

for $\epsilon\in(0,\epsilon_0)$, $s\in\mathbb{T}^1$ and $t\in[0,T_\epsilon]$. Then there is $\epsilon_1\in(0,\epsilon_0]$ and there exist $Z\in L^2\left(0,T_\epsilon;H^1\left(\mathbb{T}^1\right)\right)$, $F_2^{\mathbf{R}}\in L^2\left(0,T_\epsilon;H^1\left(\Gamma_t(\delta)\right)\right)$ and smooth $F_1^{\mathbf{R}}:\Gamma(\delta;T_\epsilon)\to\mathbb{R}$ such that

$$R^H(x,t)=\epsilon^{-\frac{1}{2}}Z\left(S(x,t),t\right)\left(\beta(s,t)\theta_0'\left(\rho^H(x,t)\right)+F_1^{\mathbf{R}}(x,t)\right)+F_2^{\mathbf{R}}(x,t)\tag{5.185}$$

for almost all $(x,t)\in\Gamma(\delta;T_\epsilon)$ and all $\epsilon\in(0,\epsilon_1)$. Here $\beta(s,t):=\|\theta_0'\|^{-1}_{L^2\left(I_\epsilon^{s,t}\right)}$.

Furthermore, there exist $C(K)$, $C>0$ independent of ϵ, T_ϵ, H and $\bar{C}$ such that

$$\|\beta\|_{L^\infty(\mathbb{T}^1\times(0,T_\epsilon))}\leq C$$

and

$$\left\|F_2^{\mathbf{R}}\right\|^2_{L^2(\Gamma(\delta;T_\epsilon))}\leq C(K)\,\epsilon^{2M+1}\tag{5.186}$$

and

$$\|Z\|^2_{L^2(0,T_\epsilon;H^1(\mathbb{T}^1))} + \left\|F_2^{\mathbf{R}}\right\|^2_{L^2(0,T_\epsilon;H^1(\Gamma_t(\delta)))} \le C(K)\,\epsilon^{2M-1} \tag{5.187}$$

for all $\epsilon \in (0,\epsilon_1)$.

Lastly, it holds

$$\sup_{t\in[0,T_\epsilon]}\sup_{s\in\mathbb{T}^1}\int\limits_{I_\epsilon^{s,t}} \left(\left|F_1^{\mathbf{R}}(\rho,s,t)\right|^2 + \left|\partial_\rho F_1^{\mathbf{R}}(\rho,s,t)\right|^2\right) J^\epsilon(\rho,s,t)\,\mathrm{d}\rho \le C(K)\,\epsilon^2 \tag{5.188}$$

for all $\epsilon \in (0,\epsilon_1)$. *Here,*

$$F_1^{\mathbf{R}}(\rho,s,t) := F_1^{\mathbf{R}}\left(X\left(\epsilon\left(\rho + h_A^{\epsilon,H}(s,t)\right),s,t\right)\right)$$

for X *as in (2.21) and*

$$J^\epsilon(\rho,s,t) := J\left(\epsilon\left(\rho + h_A^{\epsilon,H}(s,t)\right),s,t\right)$$

with $J(r,s,t) := det\left(D_{(r,s)}X\right)(r,s,t)$ *(see further (3.3)).*

Proof. The proof essentially relies on Lemma 3.9 and Corollary 3.11. Let ϵ_1 be chosen as in Lemma 5.27. First of all, we note that (4.6c) implies

$$\int\limits_{\Omega_{T_\epsilon}} \epsilon\left|\nabla R^H\right|^2 + \epsilon^{-1} f''\left(c_A^{\epsilon,H}\right)\left(R^H\right)^2 \mathrm{d}(x,t) \le CK^2\epsilon^{2M}$$

and $c_A^{\epsilon,H}$ satisfies Assumption 3.1 by Lemma 5.27 for all $\epsilon \in (0,\epsilon_1)$. Thus, we have by (3.10)

$$\int\limits_{\Omega\backslash\Gamma_t(\delta)} \epsilon\left|\nabla R^H\right|^2 + \epsilon^{-1} f''\left(c_A^{\epsilon,H}\right)\left(R^H\right)^2 \mathrm{d}x \ge 0$$

and get for

$$\Lambda_\epsilon(t) := \int\limits_{\Gamma_t(\delta)} \epsilon\left|\nabla R^H\right|^2 + \epsilon^{-1} f''\left(c_A^{\epsilon,H}\right)\left(R^H\right)^2 \mathrm{d}x$$

the estimate

$$\int\limits_0^{T_\epsilon} \Lambda_\epsilon(t) \le CK^2\epsilon^{2M}. \tag{5.189}$$

Hence for each $t \in [0,T_\epsilon]$, Lemma 3.11 implies the existence of functions $Z(.,t) \in H^1\left(\mathbb{T}^1\right)$ and $F_2^{\mathbf{R}}(.,t) \in H^1(\Gamma_t(\delta))$ such that

$$R^H(x,t) = \epsilon^{-\frac{1}{2}} Z(S(x,t),t)\,\Psi_1\left(\rho^H(x,t), Pr_{\Gamma_t}(x), t\right) + F_2^{\mathbf{R}}(x,t)$$

for almost all $x \in \Gamma_t(\delta)$ and all $\epsilon \in (0,\epsilon_1)$, where Ψ_1 is the same eigenfunction as in Lemma 3.9. Here we possibly had to choose ϵ_1 smaller than before. Moreover,

$$\left\|F_2^{\mathbf{R}}(.,t)\right\|^2_{L^2(\Gamma_t(\delta))} \le C\left(\epsilon\Lambda_\epsilon(t) + \epsilon^2\left\|R^H(.,t)\right\|^2_{L^2(\Gamma_t(\delta))}\right)$$

$$\|Z(.,t)\|^2_{H^1(\mathbb{T}^1)} + \left\|F_2^{\mathbf{R}}(.,t)\right\|^2_{H^1(\Gamma_t(\delta))} \le C\left(\left\|R^H(.,t)\right\|^2_{L^2(\Gamma_t(\delta))} + \frac{\Lambda_\epsilon(t)}{\epsilon}\right)$$

for all $\epsilon \in (0, \epsilon_2)$. Note in particular that $C > 0$ is independent of ϵ, T_ϵ, H and $\bar{C}$ as we only used C^* – which is independent of these variables as well, by our choice in Lemma 5.27 – in the estimates in Chapter 3. Since $\left\|R^H\right\|^2_{L^2(\Omega_{T_\epsilon})} \leq CK^2\epsilon^{2M-1}$ and (5.189) hold due to (4.6), integration over $(0, T_\epsilon)$ yields (5.186) and (5.187).

Moreover, we can use Lemma 3.9 1) for $t \in [0, T_\epsilon]$ to get the existence of a smooth $F_1^{\mathbf{R}}$ satisfying $F_1^{\mathbf{R}}(x,t) = \Psi_1\left(\rho^H(x,t), Pr_{\Gamma_t}(x), t\right) - \beta(S(x,t),t)\,\theta_0'\left(\rho^H(x,t)\right)$ which fulfills (5.188). Lastly, $\beta(s,t) = \|\theta_0'\|^{-1}_{L^2\left(I_\epsilon^{s,t}\right)} \leq \|\theta_0'\|^{-1}_{L^2(-1,1)}$ for ϵ_1 small enough. □

5.2.2. Estimates concerning $\tilde{\mathbf{w}}_1^{\epsilon,H}$

Now we show the main result of this subsection.

Lemma 5.29. *Let $\epsilon_0 > 0$, $T' \in (0, T_0]$ and families $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset (0, T']$, $H := \left(\tilde{h}^\epsilon\right)_{\epsilon\in(0,\epsilon_0)} \subset X_{T'}$ with $\tilde{h}^\epsilon|_{t=0} = 0$ be given. Let Assumption 4.2 hold for $c_A = c_A^{\epsilon,H}$ and we assume that there is $\bar{C} \geq 1$ such that*

$$\sup_{\epsilon\in(0,\epsilon_0)} \left\|\tilde{h}^\epsilon\right\|_{X_{T_\epsilon}} \leq \bar{C} \tag{5.190}$$

holds. Then there exists a constant $C(K) > 0$, which is independent of ϵ, T_ϵ, H and $\bar{C}$, and some $\epsilon_1 \in (0, \epsilon_0)$ such that

$$\left\|\tilde{\mathbf{w}}_1^{\epsilon,H}\right\|_{L^2(0,T;H^1(\Omega))} \leq C(K)\,\epsilon^{M-\frac{1}{2}}, \tag{5.191}$$

for all $\epsilon \in (0, \epsilon_1)$ and $T \in (0, T_\epsilon]$.

Proof. As H is a fixed family of functions, we drop the explicit notation throughout this proof, i.e. we write $c_A^\epsilon, h_A^\epsilon, \mathbf{h}, \ldots$ instead of $c_A^{\epsilon,H}, h_A^{\epsilon,H}, \mathbf{h}^H, \ldots$. Let ϵ_1 be chosen as in the proof of Lemma 5.27, i.e. such that (5.181), (5.182) and (5.183) holds.

Since $\tilde{\mathbf{w}}_1^\epsilon$ is a weak solution to (5.174)–(5.176) in Ω_{T_ϵ}, we have due to Theorem 2.6

$$\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T;H^1(\Omega))} \leq C\,\|\mathbf{f}^\epsilon\|_{L^2\left(0,T;V_0'(\Omega)\right)}$$

for all $T \in (0, T_\epsilon)$, where $\mathbf{f}^\epsilon$ is given as in (5.179). Thus, we estimate $\mathbf{f}^\epsilon(\psi)$ for an arbitrary $\psi \in L^2(0, T_\epsilon; V_0(\Omega))$, $\psi \neq 0$, as $\left(L^2(0,T;V_0(\Omega))\right)' = L^2(0,T;V_0'(\Omega))$. Let in the following $T \in (0, T_\epsilon]$.

As a starting point, we decompose

$$\begin{aligned}\int_{\Omega_T} \epsilon\left(\nabla c_A^\epsilon - \mathbf{h}\right) \otimes \nabla R : \nabla\psi\,\mathrm{d}(x,t) &= \int_{\Gamma(\delta,T)} \epsilon\left(\nabla c_A^\epsilon - \mathbf{h}\right) \otimes \nabla R : \nabla\psi\,\mathrm{d}(x,t) \\ &\quad + \int_{\Omega_T\setminus\Gamma(\delta;T)} \epsilon\left(\nabla c_A^\epsilon - \mathbf{h}\right) \otimes \nabla R : \nabla\psi\,\mathrm{d}(x,t)\end{aligned} \tag{5.192}$$

and estimate the two integrals on the right hand side separately. The second summand in $\mathbf{f}^\epsilon$ (due to $\otimes_s$) may then be treated completely similar to the approach below.

To estimate the second integral in (5.192), note that $c_I, \nabla\tilde{c}_I, \partial_\rho\tilde{c}_I, \nabla^\Gamma h_i \in L^\infty(\Gamma(2\delta))$, $i \in \{1, \ldots, M+1\}$, $c_O, \nabla c_O \in L^\infty(\Omega_{T_0})$ and $c_{\mathbf{B}}, \nabla c_{\mathbf{B}} \in L^\infty(\partial_{T_0}\Omega(\delta))$ by construction and that we may employ (5.182). Thus, $|\nabla c_A^\epsilon(x,t) - \mathbf{h}(x,t)| \le C_1 \left(1 + \frac{1}{\epsilon} e^{-C_2 \frac{\delta}{2\epsilon}}\right)$ for all $(x,t) \in \Omega_{T_\epsilon} \backslash \Gamma(\delta; T_\epsilon)$ and $\epsilon \in (0, \epsilon_1)$ and we may estimate

$$\int_0^T \int_{\Omega\backslash\Gamma_t(\delta)} |\epsilon(\nabla c_A^\epsilon - \mathbf{h}) \otimes \nabla R : \nabla\psi| \,\mathrm{d}x\mathrm{d}t \le C\epsilon \|\nabla R\|_{L^2(0,T;L^2(\Omega\backslash\Gamma_t(\delta)))} \|\psi\|_{L^2(0,T;H^1(\Omega))}$$
$$\le C(K)\,\epsilon^{M+\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Omega))}$$

for $T \in (0, T_\epsilon)$, where we used (4.6a) in the last inequality.

Dealing with the first integral on the right hand side of (5.192) will be more complicated. We compute

$$\int_{\Gamma(\delta;T)} \epsilon(\nabla c_A^\epsilon - \mathbf{h}) \otimes \nabla R : \nabla\psi \mathrm{d}(x,t)$$
$$= \int_{\Gamma(\delta;T)} \theta_0'(\rho) \left(\mathbf{n} - \epsilon\left(\sum_{i=0}^{M} \epsilon^i \nabla^\Gamma h_{i+1}\right)\right) \otimes \nabla R : \nabla\psi \mathrm{d}(x,t)$$
$$+ \int_{\Gamma(\delta;T)} \epsilon\left(\nabla(c_A^\epsilon - \theta_0(\rho)) - \left(\mathbf{h} + \theta_0'(\rho)\,\epsilon^{M-\frac{3}{2}} \nabla^\Gamma \tilde{h}^\epsilon\right)\right) \otimes \nabla R : \nabla\psi \mathrm{d}(x,t), \quad (5.193)$$

where we employ the shortened notations $\rho = \rho(x,t)$ and $\mathbf{n} = \mathbf{n}(S(x,t), t)$.

As $(c_A^\epsilon - \theta_0 \circ \rho)(x,t) = \sum_{i=1}^{M+1} \epsilon^i c_i(\rho(x,t), x, t)$ for all $(x,t) \in \Gamma(\delta; T_\epsilon)$ we find by the same L^∞-estimates as before, that there exists some $C > 0$ independent of K and ϵ such that

$$\left|\nabla(c_A^\epsilon - \theta_0(\rho)) - \left(\mathbf{h} + \theta_0'(\rho)\,\epsilon^{M-\frac{3}{2}} \nabla^\Gamma \tilde{h}^\epsilon\right)\right| \le C$$

for all $(x,t) \in \Gamma(\delta; T_\epsilon)$ and thus

$$\int_0^T \int_{\Gamma_t(\delta)} \left|\epsilon\left(\nabla(c_A^\epsilon - \theta_0(\rho)) - \left(\mathbf{h} + \theta_0'(\rho)\,\epsilon^{M-\frac{3}{2}} \nabla^\Gamma \tilde{h}^\epsilon\right)\right) \otimes \nabla R : \nabla\psi\right| \mathrm{d}x\mathrm{d}t$$
$$\le C\epsilon \|\nabla R\|_{L^2(0,T;L^2(\Gamma_t(\delta)))} \|\psi\|_{L^2(0,T;H^1(\Omega))}$$
$$\le C(K)\,\epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Omega))}$$

for $T \in (0, T_\epsilon]$ and $\epsilon \in (0, \epsilon_1)$, by (4.6).

Using the boundedness of θ_0' in $L^\infty(\mathbb{R})$ and that of $\nabla^\Gamma h_i$ in $L^\infty(\Gamma(2\delta))$, $i \in \{1, \ldots, M+1\}$, we also find

$$\int_0^T \int_{\Gamma_t(\delta)} \left|\theta_0'(\rho)\,\mathbf{n} \otimes \nabla^\Gamma R : \nabla\psi\right| \mathrm{d}x\mathrm{d}t \le C \left\|\nabla^\Gamma R\right\|_{L^2(0,T;L^2(\Gamma_t(\delta)))} \|\psi\|_{L^2(0,T;H^1(\Omega))}$$
$$\le C(K)\,\epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Omega))},$$

$$\int_0^T \int_{\Gamma_t(\delta)} \left|\epsilon\theta_0'(\rho)\left(\sum_{i=0}^{M} \epsilon^i \nabla^\Gamma h_{i+1}\right) \otimes \nabla R : \nabla\psi\right| \mathrm{d}x\mathrm{d}t \le C\epsilon \|\nabla R\|_{L^2(0,T;L^2(\Gamma_t(\delta)))} \|\psi\|_{L^2(0,T;H^1(\Omega))}$$
$$\le C(K)\,\epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Omega))}$$

by (4.6). Hence, plugging these results into (5.193), we obtain

$$\left|\int_{\Gamma(\delta;T)} \epsilon\left(\nabla c_A^\epsilon - \mathbf{h}\right)\otimes\nabla R : \nabla\psi \mathrm{d}\left(x,t\right)\right| \leq \left|\int_{\Gamma(\delta;T)} \theta_0'\left(\rho\right)\mathbf{n}\otimes\mathbf{n}\partial_{\mathbf{n}} R : \nabla\psi\mathrm{d}\left(x,t\right)\right| + C\left(K\right)\epsilon^{M-\frac{1}{2}}\left\|\psi\right\|_{L^2(0,T;H^1(\Omega))}$$

for $T\in(0,T_\epsilon)$ and $\epsilon\in(0,\epsilon_1)$.

In order to treat the remaining term

$$\mathcal{I} := \left|\int_{\Gamma(\delta;T)} \theta_0'\left(\rho\right)\mathbf{n}\otimes\mathbf{n}\partial_{\mathbf{n}} R : \nabla\psi\mathrm{d}\right|\left(x,t\right),$$

we have to use a more sophisticated technique since we would not get a sufficiently high order of ϵ in the estimate if we just used L^∞ bounds and (4.6) as before. The strategy is to make use of the additional information about R provided by Proposition 5.28. Since $\psi\in V_0$, we have $\operatorname{div}\psi = 0$ which implies by (2.28) that

$$\operatorname{div}^\Gamma\psi = -\partial_{\mathbf{n}}\psi\cdot\mathbf{n} = -\mathbf{n}\otimes\mathbf{n} : \nabla\psi$$

holds. As the assumptions of Proposition 5.28 are satisfied, we may estimate $\mathcal{I}$ using (5.185) so that

$$\begin{aligned}
\mathcal{I} &= \left|\int_{\Gamma(\delta;T)} \theta_0'\left(\rho\right)\partial_{\mathbf{n}}\left(\epsilon^{-\frac{1}{2}}Z\left(S\left(x,t\right),t\right)\left(\beta\left(S\left(x,t\right),t\right)\theta_0'\left(\rho\right) + F_1^{\mathbf{R}}\right) + F_2^{\mathbf{R}}\right)\operatorname{div}^\Gamma\psi\mathrm{d}\left(x,t\right)\right| \\
&\leq \left|\int_{\Gamma(\delta;T)} \theta_0'\left(\rho\right)\left(\epsilon^{-\frac{1}{2}}Z\left(S\left(x,t\right),t\right)\left(\beta\left(S\left(x,t\right),t\right)\theta_0''\left(\rho\right)\frac{1}{\epsilon} + \partial_{\mathbf{n}}F_1^{\mathbf{R}}\right)\right)\operatorname{div}^\Gamma\psi\mathrm{d}\left(x,t\right)\right| \\
&\quad + \int_{\Gamma(\delta;T)} \left|\theta_0'\left(\rho\right)\partial_{\mathbf{n}}F_2^{\mathbf{R}}\left(x,t\right)\operatorname{div}^\Gamma\psi\right|\mathrm{d}\left(x,t\right) \\
&\leq \left|\int_0^T\int_{\Gamma_t(\delta)} \frac{1}{2}\partial_{\mathbf{n}}\left(\theta_0'\left(\rho\right)^2\right)\epsilon^{-\frac{1}{2}}Z\left(S\left(x,t\right),t\right)\beta\left(S\left(x,t\right),t\right)\operatorname{div}_\Gamma\psi\mathrm{d}x\mathrm{d}t\right| \\
&\quad + C_1\left|\int_0^T\int_{\mathbb{T}^1}\int_{-\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)}^{\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)} \theta_0'\left(\rho\right)\epsilon^{-\frac{1}{2}}Z\left(s,t\right)\partial_\rho F_1^{\mathbf{R}}\left(\rho,s,t\right)\operatorname{div}_\Gamma\psi J^\epsilon\left(\rho,s,t\right)\mathrm{d}\rho\mathrm{d}s\mathrm{d}t\right| \\
&\quad + C_2\left\|F_2^{\mathbf{R}}\right\|_{L^2(0,T;H^1(\Gamma_t(\delta)))}\left\|\psi\right\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \\
&=: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}$$

Here we used the same notations as in Proposition 5.28 and in the first lines the short notation $\rho = \rho\left(x,t\right)$. Now (5.187) implies

$$\mathcal{J}_3 \leq C\left(K\right)\epsilon^{M-\frac{1}{2}}\left\|\psi\right\|_{L^2(0,T;H^1(\Gamma_t(\delta)))}$$

and we may estimate $\mathcal{J}_2$ by

$$\begin{aligned}\mathcal{J}_2 &\leq C\epsilon^{-1}\|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))}\left(\int_0^T\int_{\mathbb{T}^1} Z(s,t)^2 \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)}^{\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)} \left(\partial_\rho F_1^{\mathbf{R}}(\rho,s,t)\right)^2 J^\epsilon \mathrm{d}\rho\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{2}}\\
&\leq C\|Z\|_{L^2(0,T;H^1(\mathbb{T}^1))}\|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))}\\
&\leq C(K)\epsilon^{M-\frac{1}{2}}\|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))},\end{aligned}$$

where we used (5.188) in the last line. To treat the remaining integral, we may use Lemma 2.21 to get

$$\begin{aligned}\mathcal{J}_1 \leq &\left|\int_0^T\int_{\Gamma_t(\delta)} \frac{1}{2}\nabla^\Gamma\left(\partial_{\mathbf{n}}\left(\theta_0'(\rho)^2\right)\epsilon^{-\frac{1}{2}}Z(S(x,t),t)\,\beta(S(x,t),t)\right)\cdot\psi\mathrm{d}x\mathrm{d}t\right|\\
&+\left|\int_0^T\int_{\Gamma_t(\delta)} \partial_{\mathbf{n}}\left(\theta_0'(\rho)^2\right)\epsilon^{-\frac{1}{2}}Z(S(x,t),t)\,\beta(S(x,t),t)\,\psi\cdot\mathbf{n}\kappa(x,t)\,\mathrm{d}x\mathrm{d}t\right|\\
&+C\int_0^T\int_{\mathbb{T}^1}\left|\partial_\rho\left(\theta_0'\left(\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)\right)^2\right)\epsilon^{-\frac{3}{2}}Z(s,t)\,\beta(s,t)\,\psi(\delta,s,t)\right|\mathrm{d}s\mathrm{d}t\\
&+C\int_0^T\int_{\mathbb{T}^1}\left|\partial_\rho\left(\theta_0'\left(\frac{-\delta}{\epsilon}-h_A^\epsilon(s,t)\right)^2\right)\epsilon^{-\frac{3}{2}}Z(s,t)\,\beta(s,t)\,\psi(-\delta,s,t)\right|\mathrm{d}s\mathrm{d}t\\
:=&\ \mathcal{J}_1^1+\mathcal{J}_1^2+\mathcal{J}_1^{3,+}+\mathcal{J}_1^{3,-}.\end{aligned}$$

Now

$$\begin{aligned}\mathcal{J}_1^{3,+} &\leq C_1\epsilon^{-\frac{3}{2}}e^{-C_2\frac{\delta}{2\epsilon}}\int_0^T\int_{\mathbb{T}^1}|Z(s,t)|\sup_{r\in(-\delta,\delta)}|\psi(r,s,t)|\,\mathrm{d}s\mathrm{d}t\\
&\leq C_1\epsilon^{-\frac{3}{2}}e^{-C_2\frac{\delta}{2\epsilon}}\|Z\|_{L^2(0,T;H^1(\mathbb{T}^1))}\|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))}\\
&\leq C(K)\epsilon^{M-\frac{1}{2}}\|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))},\end{aligned}$$

where we used (5.182) and the uniform bound on β in the first step and $H^1(\Gamma_t(\delta)) \hookrightarrow L^{2,\infty}(\Gamma_t(\delta))$ (cf. Lemma 2.23) in the second step. $\mathcal{J}_1^{3,-}$ can be estimated analogously. For $\mathcal{J}_1^2$, we use integration by parts and get

$$\begin{aligned}\mathcal{J}_1^2 \leq &\left|\int_0^T\int_{\Gamma_t(\delta)}\left(\theta_0'(\rho)\right)^2\epsilon^{-\frac{1}{2}}Z(S(x,t),t)\,\beta(S(x,t),t)\,\partial_{\mathbf{n}}\psi\cdot\mathbf{n}(S(x,t),t)\,\kappa(x,t)\,\mathrm{d}x\mathrm{d}t\right|\\
&+C\int_0^T\int_{\Gamma_t(\delta)}\left|\left(\theta_0'(\rho)\right)^2\epsilon^{-\frac{1}{2}}Z(S(x,t),t)\,\beta(S(x,t),t)\,\psi\right|\mathrm{d}x\mathrm{d}t\\
&+C(K)\,e^{-C_2\frac{\delta}{2\epsilon}}\|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))}\end{aligned}$$

$$
\begin{aligned}
&\leq C\epsilon^{-\frac{1}{2}} \|Z\|_{L^2(0,T;H^1(\mathbb{T}^1))} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \epsilon^{\frac{1}{2}} \left\| \left(\theta_0'\right)^2 \right\|_{L^2(\mathbb{R})} \\
&\quad + C(K) e^{-C_2 \frac{\delta}{2\epsilon}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \\
&\leq C(K) \epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))},
\end{aligned}
$$

where the exponential decaying term in the first inequality is a consequence of the appearing boundary integral, which may be estimated as in the case of $\mathcal{J}_1^{3,\pm}$. Moreover, we used a change of variables $r \mapsto \frac{r}{\epsilon} - h_A^\epsilon$ in the second step and (5.187) in the last step.

Now we discuss $\mathcal{J}_1^1$ – the last term we need to estimate. Note that by the definition of β in Proposition 5.28, we have

$$
\begin{aligned}
\nabla^\Gamma \beta(s,t) &= -\frac{1}{\|\theta_0'\|^2_{L^2\left(I_\epsilon^{s,t}\right)}} \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)}^{\frac{\delta}{\epsilon}-h_A^\epsilon(s,t)} \frac{1}{2}\frac{d}{d\rho}\left(\theta_0'(\rho)^2\right) \mathrm{d}\rho \left(-\nabla^\Gamma h_A^\epsilon\right) \\
&\leq C_1 e^{-C_2 \frac{\delta}{2\epsilon}}
\end{aligned}
$$

for all $\epsilon \in (0,\epsilon_1)$, due to (5.182) and $\epsilon^{M-\frac{3}{2}} \left\|\tilde{h}^\epsilon\right\|_{X_{T_\epsilon}} \leq 1$, cf. (5.183). Thus, we compute

$$
\begin{aligned}
\mathcal{J}_1^1 &\leq \left| \int_0^T \int_{\Gamma_t(\delta)} \frac{1}{2} \partial_{\mathbf{n}} \nabla^\Gamma \left(\theta_0'(\rho(x,t))^2\right) \epsilon^{-\frac{1}{2}} Z(S(x,t),t)\, \beta(S(x,t),t) \cdot \psi \mathrm{d}x\mathrm{d}t \right| \\
&\quad + \left| \int_0^T \int_{\Gamma_t(\delta)} \frac{1}{2} \left[\partial_{\mathbf{n}}, \nabla^\Gamma\right] \left(\theta_0'(\rho(x,t))^2\right) \epsilon^{-\frac{1}{2}} Z(S(x,t),t)\, \beta(S(x,t),t) \cdot \psi \mathrm{d}x\mathrm{d}t \right| \\
&\quad + \left| \int_0^T \int_{\Gamma_t(\delta)} \frac{1}{2} \partial_{\mathbf{n}} \left(\theta_0'(\rho(x,t))^2\right) \epsilon^{-\frac{1}{2}} \nabla^\Gamma \left(Z(S(x,t),t)\, \beta(S(x,t),t)\right) \cdot \psi \mathrm{d}x\mathrm{d}t \right| \\
&\leq C_1 \int_0^T \int_{\Gamma_t(\delta)} \left| \partial_\rho \left(\theta_0'(\rho(x,t))^2\right) \nabla^\Gamma h_A^\epsilon \epsilon^{-\frac{1}{2}} Z(S(x,t),t)\, \beta(S(x,t),t) \cdot \partial_{\mathbf{n}} \psi \right| \mathrm{d}x\mathrm{d}t \\
&\quad + C_2 \int_0^T \int_{\Gamma_t(\delta)} \left| \partial_\rho \left(\theta_0'(\rho(x,t))^2\right) \nabla^\Gamma h_A^\epsilon \epsilon^{-\frac{1}{2}} Z(S(x,t),t)\, \beta(S(x,t),t) \cdot \psi \right| \mathrm{d}x\mathrm{d}t \\
&\quad + C_3 \int_0^T \int_{\Gamma_t(\delta)} \left| \left(\theta_0'(\rho(x,t))^2\right) \epsilon^{-\frac{1}{2}} \nabla^\Gamma Z(S(x,t),t)\, \beta(S(x,t),t) \cdot \partial_{\mathbf{n}} \psi \right| \mathrm{d}x\mathrm{d}t \\
&\quad + C_4 \int_0^T \int_{\Gamma_t(\delta)} \left| \left(\theta_0'(\rho(x,t))^2\right) \epsilon^{-\frac{1}{2}} \partial_s Z(S(x,t),t)\, \beta(S(x,t),t)\, \psi \right| \mathrm{d}x\mathrm{d}t \\
&\quad + C_5 e^{-C_6 \frac{\delta}{2\epsilon}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \|Z\|_{L^2(0,T;H^1(\mathbb{T}^1))} \\
&\leq C_1 \|Z\|_{L^2(0,T;H^1(\mathbb{T}^1))} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} + C_2(K) e^{-C_6 \frac{\delta}{2\epsilon}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))} \\
&\leq C(K) \epsilon^{M-\frac{1}{2}} \|\psi\|_{L^2(0,T;H^1(\Gamma_t(\delta)))}.
\end{aligned}
$$

Here we used the definition of $\left[\partial_{\mathbf{n}}, \nabla^\Gamma\right]$ in the first estimate (cf. (2.35)) and integration by parts, (2.36) and the exponential decay of $\nabla^\Gamma \beta$ and the boundary terms in the second step. In the third step we again used $\epsilon^{M-\frac{3}{2}} \left\|\tilde{h}^\epsilon\right\|_{X_{T_\epsilon}} \leq 1$. This concludes the proof. □

The following proposition is rather technical but will be useful in the proof of existence for the $\left(M - \frac{1}{2}\right)$-th order of the expansion of h^ϵ, see Theorem 5.32.

Proposition 5.30. *Let $\epsilon_0 \in (0,1)$ and $T' \in (0, T_0]$ be fixed. Furthermore, let for a given family $H = \left(\tilde{h}^\epsilon\right)_{\epsilon\in(0,\epsilon_0)} \subset X_{T'}$ with $\tilde{h}^\epsilon|_{t=0} = 0$ the function $\tilde{\mathbf{w}}_1^{\epsilon,H}$ be defined as the weak solution to (5.174)–(5.176) for $\epsilon \in (0,\epsilon_0)$. Then the following statements hold:*

1. *For all $\epsilon \in (0,\epsilon_0)$, there exists a constant $C(\epsilon) > 0$ such that*
$$\left\|\tilde{\mathbf{w}}_1^{\epsilon,H}\right\|_{L^2(0,T';H^1(\Omega))} \leq C(\epsilon)\left(\left(T'\right)^{\frac{1}{2}} + \left\|\tilde{h}^\epsilon\right\|_{L^2(0,T';H^1(\mathbb{T}^1))}\right).$$
2. *Let $H_1 = \left(h_1^\epsilon\right)_{\epsilon\in(0,\epsilon_0)}, H_2 = \left(h_2^\epsilon\right)_{\epsilon\in(0,\epsilon_0)} \subset X_{T'}$ be given. For all $\epsilon \in (0,\epsilon_0)$, there exists a constant $\tilde{C}(\epsilon) > 0$ such that*
$$\left\|\tilde{\mathbf{w}}_1^{\epsilon,H_1} - \tilde{\mathbf{w}}_1^{\epsilon,H_2}\right\|_{L^2(0,T';H^1(\Omega))} \leq \tilde{C}(\epsilon)\left(T'\right)^{\frac{1}{2}}\left(1 + \|h_2^\epsilon\|_{X_{T'}}\right)\|h_1^\epsilon - h_2^\epsilon\|_{X_{T'}}.$$

Proof. Ad 1) By Theorem 2.6, there is a constant $C > 0$ such that
$$\left\|\tilde{\mathbf{w}}_1^{\epsilon,H}\right\|_{L^2(0,T',H^1(\Omega))} \leq C\left\|\epsilon\left(\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right) \otimes_s \nabla R^H\right)\right\|_{L^2(0,T';L^2(\Omega))}. \tag{5.194}$$
Now in order to further estimate the right hand side, we first note that
$$\sup_{(x,t)\in\Omega\times(0,T')}\left|\nabla c_A^{\epsilon,H}(x,t) - \mathbf{h}^H(x,t)\right| \leq \frac{C}{\epsilon}, \tag{5.195}$$
with a constant $C > 0$ that does not depend on H. This can be deduced from representation (5.178) and the fact that $c_{\mathbf{B}}$ and its appearing derivatives are in $L^\infty\left(\partial_{T_0}\Omega(\delta)\right)$, c_O and its derivatives are in $L^\infty\left(\Omega_{T_0}\right)$ and $\tilde{c}_I$ and its appearing derivatives are in $L^\infty\left(\mathbb{R}\times\Gamma(2\delta;T_0)\right)$. So we compute
$$\begin{aligned}
&\left\|\epsilon\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right)\otimes\nabla R^H\right\|_{L^2(\Omega_{T'})}\\
&\leq \left\|\epsilon\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right)\otimes\nabla c^\epsilon\right\|_{L^2(\Omega_{T'})} + \left\|\epsilon\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right)\otimes\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right)\right\|_{L^2(\Omega_{T'})}\\
&\quad + \left\|\epsilon\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right)\otimes\mathbf{h}^H\right\|_{L^2(\Omega_{T'})}\\
&\leq C_1(\epsilon)\left(T'\right)^{\frac{1}{2}} + C_2(\epsilon)\left\|\nabla^\Gamma\tilde{h}^\epsilon\right\|_{L^2(\Gamma(2\delta;T'))}\\
&\leq C(\epsilon)\left(\left(T'\right)^{\frac{1}{2}} + \|h^\epsilon\|_{(L^2(0,T';H^1(\mathbb{T}^1)))}\right),
\end{aligned}$$
where we used that c^ϵ is a known function and thus
$$\sup_{t\in(0,T')}\|\nabla c^\epsilon\|_{L^2(\Omega)} \leq C(\epsilon) \tag{5.196}$$

holds for some ϵ-dependent constant $C(\epsilon)$.[1] Moreover, we employed the smoothness of the mapping $S : \Gamma(2\delta) \to \mathbb{T}^1$ in the last line. A completely similar approach for $\nabla R^H \otimes \left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right)$ yields 1).

Ad 2) We write

$$\mathbf{f}^{\epsilon,H} := \epsilon\left(\left(\nabla c_A^{\epsilon,H} - \mathbf{h}^H\right) \otimes_s \nabla\left(c^\epsilon - c_A^{\epsilon,H}\right)\right)$$

and get using Theorem 2.6 that

$$\left\|\tilde{\mathbf{w}}_1^{\epsilon,H_1} - \tilde{\mathbf{w}}_1^{\epsilon,H_2}\right\|_{L^2(0,T';H^1(\Omega))} \leq C\left\|\mathbf{f}^{\epsilon,H_1} - \mathbf{f}^{\epsilon,H_2}\right\|_{L^2(0,T';L^2(\Omega))} \tag{5.197}$$

holds, since $\tilde{\mathbf{w}}_1^{\epsilon,H_1} - \tilde{\mathbf{w}}_1^{\epsilon,H_2}$ is a weak solution to (5.174)–(5.176) with right hand side given by $\operatorname{div}\left(\mathbf{f}^{\epsilon,H_1} - \mathbf{f}^{\epsilon,H_2}\right)$. Now in order to show 2) we first note that

$$D_\rho^k D_x^l\left(\tilde{c}_I\left(\rho^{H_1}(x,t), x, t\right) - \tilde{c}_I\left(\rho^{H_2}(x,t), x, t\right)\right) = D_\rho^{k+1} D_x^l c_I\left(\xi(x,t), x, t\right)\epsilon^{M-\frac{3}{2}}\left(h_2^\epsilon - h_1^\epsilon\right)$$

for all $(x,t) \in \Gamma(2\delta, T')$ and $k, l \in \{0,1\}$ due to Taylor's theorem. Here $\xi : \Gamma(2\delta, T') \to \mathbb{R}$ is a suitable function depending on H_1 and H_2. Since all the terms which do not depend on H_1, H_2 cancel, we may estimate

$$\epsilon\left\|\left(\left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) - \left(\nabla c_A^{\epsilon,H_2} - \mathbf{h}^{H_2}\right)\right) \otimes \nabla c^\epsilon\right\|_{L^2(\Omega_{T'})} \leq C(\epsilon)\left(T'\right)^{\frac{1}{2}}\left\|h_1^\epsilon - h_2^\epsilon\right\|_{X_{T'}}$$

by (5.196), a Taylor expansion and $X_{T'} \hookrightarrow C^0\left([0,T']; C^1\left(\mathbb{T}^1\right)\right)$. Next, we may estimate

$$\begin{aligned}
&\epsilon\left\|\left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) \otimes \left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) - \left(\nabla c_A^{\epsilon,H_2} - \mathbf{h}^{H_2}\right) \otimes \left(\nabla c_A^{\epsilon,H_2} - \mathbf{h}^{H_2}\right)\right\|_{L^2(\Omega_{T'})} \\
&\leq \epsilon\left\|\left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) \otimes \left(\left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) - \left(\nabla c_A^{\epsilon,H_2} - \mathbf{h}^{H_2}\right)\right)\right\|_{L^2(\Omega_{T'})} \\
&\quad + \epsilon\left\|\left(\left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) - \left(\nabla c_A^{\epsilon,H_2} - \mathbf{h}^{H_2}\right)\right) \otimes \left(\nabla c_A^{\epsilon,H_2} - \mathbf{h}^{H_2}\right)\right\|_{L^2(\Omega_{T'})} \\
&\leq C\left(T'\right)^{\frac{1}{2}}\left\|h_1^\epsilon - h_2^\epsilon\right\|_{X_{T'}}
\end{aligned}$$

by (5.195), a Taylor expansion and $X_{T'} \hookrightarrow C^0\left([0,T']; C^1\left(\mathbb{T}^1\right)\right)$. Here, the constant $C > 0$ may in fact be chosen independent of $\epsilon > 0$ since $M \geq 4$.

Lastly, we compute

$$\begin{aligned}
&\left\|\left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) \otimes \mathbf{h}^{H_1} - \left(\nabla c_A^{\epsilon,H_2} - \mathbf{h}^{H_2}\right) \otimes \mathbf{h}^{H_2}\right\|_{L^2(\Omega_{T'})} \\
&\leq \left\|\left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) \otimes \left(\mathbf{h}^{H_1} - \mathbf{h}^{H_2}\right)\right\|_{L^2(\Omega_{T'})} \\
&\quad + \left\|\left(\left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) - \left(\nabla c_A^{\epsilon,H_2} - \mathbf{h}^{H_2}\right)\right) \otimes \mathbf{h}^{H_2}\right\|_{L^2(\Omega_{T'})}.
\end{aligned}$$

and note

$$\begin{aligned}
\left\|\mathbf{h}^{H_1} - \mathbf{h}^{H_2}\right\|_{L^2(\Omega_{T'})} &\leq C\epsilon^{M-\frac{3}{2}}\left\|\partial_\rho \tilde{c}_I\left(\rho^{H_1}\right)\nabla^\Gamma h_1^\epsilon - \partial_\rho \tilde{c}_I\left(\rho^{H_2}\right)\nabla^\Gamma h_2^\epsilon\right\|_{L^2(\Omega_{T'})} \\
&\leq C\epsilon^{M-\frac{3}{2}}\left\|\partial_\rho \tilde{c}_I\left(\rho^{H_1}\right)\left(\nabla^\Gamma h_1^\epsilon - \nabla^\Gamma h_2^\epsilon\right)\right\|_{L^2(\Omega_{T'})} \\
&\quad + C\epsilon^{M-\frac{3}{2}}\left\|\left(\partial_\rho \tilde{c}_I\left(\rho^{H_1}\right) - \partial_\rho \tilde{c}_I\left(\rho^{H_2}\right)\right)\nabla^\Gamma h_2^\epsilon\right\|_{L^2(\Omega_{T'})}.
\end{aligned}$$

[1] More precisely, we could use Proposition 7.2 and Lemma 4.4 at this point to find that we may choose $C(\epsilon) = C\epsilon^{-\frac{1}{2}}$ for some $C > 0$. However, as this more accurate constant yields no advantage in the following, we decided to skip it in favor of the brevity of presentation.

By similar arguments as made before, this yields

$$\epsilon \left\| \left(\nabla c_A^{\epsilon,H_1} - \mathbf{h}^{H_1}\right) \otimes \mathbf{h}^{H_1} - \left(\nabla c_A^{\epsilon,H_2} - \mathbf{h}^{H_2}\right) \otimes \mathbf{h}^{H_2} \right\|_{L^2(0,T';L^2(\Omega))} \leq C \left(T'\right)^{\frac{1}{2}} \left(1 + \|h_2^\epsilon\|_{X_{T'}}\right) \cdot \|h_1^\epsilon - h_2^\epsilon\|_{X_{T'}} .$$

Plugging these results into (5.197) and noting that the exact same observations hold for the transposed matrices, we have proven 2). □

5.3. Constructing the $\left(M-\frac{1}{2}\right)$-th Terms

Our goal is to construct approximate solutions $(\mathbf{v}_A^\epsilon, p_A^\epsilon, c_A^\epsilon, \mu_A^\epsilon)$ which fulfill (4.7)–(4.10) in Ω_{T_0}, where $\mathbf{r}_{\mathrm{S}}^\epsilon, r_{\mathrm{div}}^\epsilon, r_{\mathrm{CH1}}^\epsilon$ and $r_{\mathrm{CH2}}^\epsilon$ are suitable error terms, which will be discussed in detail in Chapter 6. In (4.9) we consider

$$\mathbf{w}_1^{\epsilon,H} = \frac{\tilde{\mathbf{w}}_1^{\epsilon,H}}{\epsilon^{M-\frac{1}{2}}} \tag{5.198}$$

instead of $\mathbf{w}_1^\epsilon$, where $\tilde{\mathbf{w}}_1^{\epsilon,H}$ is the weak solution to (5.174)–(5.176). Moreover, we write $\mathbf{w}_1^{\epsilon,H}|_\Gamma(x,t) := \mathbf{w}_1^{\epsilon,H}(Pr_{\Gamma_t}(x),t)$ for $(x,t) \in \Gamma(2\delta;T_0)$ and we use a suitable family $H = (h^\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset X_{T_0}$.

The definition of the normalized function $\mathbf{w}_1^{\epsilon,H}$ in (5.198) is motivated by the result in (5.191), which allows for $\tilde{\mathbf{w}}_1^{\epsilon,H}$ to be viewed as a term of order $M-\frac{1}{2}$. Due to this appearance of a non-integer order term, it is natural to also consider non-integer order terms in the expansion of $(c^\epsilon, \mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon)$. More specifically, we assume that terms

$$\epsilon^{M-\frac{1}{2}}(\mathbf{v}^\pm_{M-\frac{1}{2}}, p^\pm_{M-\frac{1}{2}}, c^\pm_{M-\frac{1}{2}}, \mu^\pm_{M-\frac{1}{2}})$$

appear in the outer expansion, which are defined in $\Omega^\pm_{T_0}$, and that terms

$$\epsilon^{M-\frac{1}{2}}(\mathbf{v}_{M-\frac{1}{2}}, p_{M-\frac{1}{2}}, c_{M-\frac{1}{2}}, \mu_{M-\frac{1}{2}})$$

appear in the inner expansion, which are defined in $\mathbb{R}\times\Gamma(2\delta;T_0)$. Moreover, we assume that there is a term $\epsilon^{M-\frac{3}{2}}h_{M-\frac{1}{2}} : \mathbb{T}^1 \times [0,T_0] \to \mathbb{R}$ appearing in the expansion of h^ϵ and further that there are $\epsilon^{M-\frac{1}{2}}\mathbf{u}_{M-\frac{1}{2}}$ and $\epsilon^{M-\frac{1}{2}}l_{M-\frac{1}{2}}$ appearing in the expansions of $\mathbf{u}^\epsilon$ and l^ϵ. We assume that all these functions are smooth in their respective domains; thus we can also consider $\mathbf{w}_1^{\epsilon,H}$ and $\mathbf{w}_2^{\epsilon,H}$ to be smooth, due to regularity theory. Note that we will not ensure any compatibility conditions on $\Gamma(2\delta)\setminus\Gamma$ for higher orders and consequently do not introduce $\mathbf{q}_{M-\frac{1}{2}}$, $j_{M-\frac{1}{2}}$ or $g_{M-\frac{1}{2}}$.

A crucial detail in the following construction is that we will not get uniform control in ϵ of higher norms of $h_{M-\frac{1}{2}}$. This is a consequence of the appearance of ∇c^ϵ in the definition of $\mathbf{w}_1^{\epsilon,H}$, cf. Theorem 5.32 for the technical details. As all other terms of order $M-\frac{1}{2}$ depend on $h_{M-\frac{1}{2}}$ (the sole exception being $c_{M-\frac{1}{2}}$ which will be shown to vanish), we will in turn only have limited control of them. This fact will magnify the amount of work needed to estimate the remainder terms $\mathbf{r}_{\mathrm{S}}^\epsilon$, $r_{\mathrm{div}}^\epsilon$, $r_{\mathrm{CH1}}^\epsilon$ and $r_{\mathrm{CH2}}^\epsilon$. Due to this ϵ-dependence we will sometimes write $h^\epsilon_{M-\frac{1}{2}}$ instead of $h_{M-\frac{1}{2}}$ for emphasis.

In the following, we will fix $\overline{H} = \left(h^\epsilon_{M-\frac{1}{2}}\right)_{\epsilon\in(0,\epsilon_0)}$ and drop the explicit dependence on a family H in the notations when referring to $\overline{H}$, i.e. we write $\mathbf{h} = \mathbf{h}^{\overline{H}}$, $\tilde{\mathbf{w}}_1^\epsilon = \tilde{\mathbf{w}}_1^{\epsilon,\overline{H}}$ and so forth.

As before in Subsection 5.1 we will now first deduce which equations have to be satisfied, if those fractional order terms exist, and then construct them using the same results for ordinary differential equations as before. As we introduced no other non-integer order terms so far, most of the terms up to order $M+1$ constructed before will not be relevant. In the following, we only assume that $\mathfrak{S}_0$ and $\mathfrak{S}_1$ are given as in Subsections 5.1.5 and 5.1.6, resp. Lemmata 5.19 and 5.22.

5.3.1. The Outer Expansion

Applying the same Taylor expansion as before in (1.21), see (5.7), we explicitly get in $\Omega^{\pm}_{T_0}$

$$c^{\pm}_{M-\frac{1}{2}} = 0, \tag{5.199}$$

which can be derived similarly to (5.13). From (1.18)–(1.19), we deduce that the equations

$$-\Delta \mathbf{v}^{\pm}_{M-\frac{1}{2}} + \nabla p^{\pm}_{M-\frac{1}{2}} = 0 \qquad \text{in } \Omega^{\pm}_{T_0}, \tag{5.200}$$

$$\operatorname{div} \mathbf{v}^{\pm}_{M-\frac{1}{2}} = 0 \qquad \text{in } \Omega^{\pm}_{T_0}, \tag{5.201}$$

have to hold, as $\nabla c^{\pm}_{M-\frac{1}{2}} = \nabla c^{\pm}_0 = 0$.

Using $c^{\pm}_{M-\frac{1}{2}} = 0$ in (1.20), we get

$$\Delta \mu^{\pm}_{M-\frac{1}{2}} = \partial_t c^{\pm}_{M-\frac{1}{2}} + \mathbf{v}^{\pm}_{M-\frac{1}{2}} \cdot \nabla c^{\pm}_0 + \mathbf{v}^{\pm}_0 \cdot \nabla c^{\pm}_{M-\frac{1}{2}} = 0 \qquad \text{in } \Omega^{\pm}_{T_0}. \tag{5.202}$$

As in the construction of the lower order terms, we get corresponding boundary conditions for (5.200)–(5.201) and (5.202) on Γ from the inner expansion. These boundary conditions will turn out to not be trivial. But note that, since $c^{\pm}_{M-\frac{1}{2}} = 0$, we do not have to construct a boundary layer expansion, as we may explicitly prescribe the boundary values

$$\left(-2D_s \mathbf{v}^{-}_{M-\frac{1}{2}} + p^{-}_{M-\frac{1}{2}} \mathbf{I}\right) \mathbf{n}_{\partial\Omega} = \alpha_0 \mathbf{v}^{-}_{M-\frac{1}{2}} \text{ on } \partial_{T_0}\Omega$$

for (5.200)–(5.201) and the Dirichlet datum

$$\mu^{-}_{M-\frac{1}{2}} = 0$$

for (5.202).

In the following, we assume that $\left(\mathbf{v}^{\pm}_{M-\frac{1}{2}}, p^{\pm}_{M-\frac{1}{2}}, c^{\pm}_{M-\frac{1}{2}}, \mu^{\pm}_{M-\frac{1}{2}}\right)$ are smoothly extended unto $\Omega^{\pm}_{T_0} \cup \Gamma(2\delta; T_0)$, as discussed in Remark 5.1 for the integer order terms.

5.3.2. The Inner Expansion

We assume that the matching conditions (5.24)–(5.27) hold for the inner terms $\mathbf{v}_{M-\frac{1}{2}}$, $p_{M-\frac{1}{2}}$, $c_{M-\frac{1}{2}}$, $\mu_{M-\frac{1}{2}}$. As these are the first terms of fractional order which we introduce, it can be derived from the general form (5.34)–(5.37) that the inner terms satisfy the following equations:

$$-\partial_{\rho\rho}\left(\mathbf{v}_{M-\frac{1}{2}} - \left(\mathbf{u}_{M-\frac{1}{2}} d_\Gamma - \mathbf{u}_0 h_{M-\frac{1}{2}}\right)\eta\right) = 0, \tag{5.203}$$

$$\partial_\rho\left(\mathbf{v}_{M-\frac{1}{2}} \cdot \mathbf{n} - \left(\mathbf{u}_{M-\frac{1}{2}} d_\Gamma - \mathbf{u}_0 h_{M-\frac{1}{2}}\right) \cdot \mathbf{n}\eta\right) = 0, \tag{5.204}$$

$$\partial_{\rho\rho} c_{M-\frac{1}{2}} - f''(c_0)\, c_{M-\frac{1}{2}} = 0, \tag{5.205}$$

$$\partial_{\rho\rho}\left(\mu_{M-\frac{1}{2}} - \left(l_{M-\frac{1}{2}} d_\Gamma - l_0 h_{M-\frac{1}{2}}\right)\eta\right) = 0 \tag{5.206}$$

in $\mathbb{R} \times \Gamma(2\delta; T_0)$. Note that we have used $\nabla^\Gamma h_{M-\frac{1}{2}} \cdot (\partial_\rho \mathbf{v}_0 - \mathbf{u}_0 d_\Gamma \eta') = 0$ in $\mathbb{R} \times \Gamma(2\delta; T_0)$ on the right hand side of (5.204), which is a consequence of the definition of $\mathbf{v}_0$ and $\mathbf{u}_0$, cf. Lemma 5.19.

As before, we complement (5.205) with the normalization $c_{M-\frac{1}{2}}(0,x,t)=0$ for all $(x,t)\in\Gamma(2\delta;T_0)$. Then we immediately find that $c_{M-\frac{1}{2}}=0$ is the unique solution to (5.205) by Lemma 2.3.

Now we introduce terms $V^{M-\frac{1}{2}}$, $W^{M-\frac{1}{2}}$, $A^{M-\frac{1}{2}}$, $B^{M-\frac{1}{2}}$ which correspond to the respective terms in (5.40)–(5.46) for order $k=M+\frac{1}{2}$, i.e. right hand sides for fictive terms $\left(\mathbf{v}_{M+\frac{1}{2}},p_{M+\frac{1}{2}},c_{M+\frac{1}{2}},\mu_{M+\frac{1}{2}}\right)$ which we will not construct. Nevertheless, we will hold on to part of the construction scheme of the terms of integer order in that we assume that certain "compatibility conditions" are satisfied for the next order. We define

$$A^{M-\frac{1}{2}}=-\mu_{M-\frac{1}{2}}-2\partial_{\rho\rho}c_0\nabla^\Gamma h_{M-\frac{1}{2}}\cdot\nabla^\Gamma h_1+\partial_\rho c_0\Delta^\Gamma h_{M-\frac{1}{2}}-g_0h_{M-\frac{1}{2}}\eta', \tag{5.207}$$

$$\begin{aligned}B^{M-\frac{1}{2}}&=\partial_\rho c_0\mathbf{v}_{M-\frac{1}{2}}\cdot\mathbf{n}-\partial_\rho\mu_{M-\frac{1}{2}}\Delta d_\Gamma-2\nabla\partial_\rho\mu_{M-\frac{1}{2}}\cdot\mathbf{n}-l_{M-\frac{1}{2}}\eta''(\rho+h_1)\\&\quad-\partial_\rho c_0\mathbf{v}_0\cdot\nabla^\Gamma h_{M-\frac{1}{2}}-2\partial_{\rho\rho}\mu_0\nabla^\Gamma h_{M-\frac{1}{2}}\cdot\nabla h_1-\partial_\rho c_0\partial_t^\Gamma h_{M-\frac{1}{2}}+\partial_\rho\mu_0\Delta^\Gamma h_{M-\frac{1}{2}}\\&\quad+2\nabla\partial_\rho\mu_0\cdot\nabla^\Gamma h_{M-\frac{1}{2}}-h_{M-\frac{1}{2}}\left(l_1\eta''+j_0\eta'\right)+\mathbf{w}_1^\epsilon|_\Gamma\cdot\mathbf{n}\partial_\rho c_0,\end{aligned} \tag{5.208}$$

$$\begin{aligned}\mathbf{V}^{M-\frac{1}{2}}&=\partial_\rho\mathbf{v}_{M-\frac{1}{2}}\Delta d_\Gamma+2\left(\left(\nabla\partial_\rho\mathbf{v}_{M-\frac{1}{2}}\right)^T\mathbf{n}-\left(\nabla\partial_\rho\mathbf{v}_0\right)^T\nabla^\Gamma h_{M-\frac{1}{2}}\right)-\partial_\rho\mathbf{v}_0\Delta^\Gamma h_{M-\frac{1}{2}}\\&\quad-\partial_\rho p_{M-\frac{1}{2}}\mathbf{n}+\partial_\rho p_0\nabla^\Gamma h_{M-\frac{1}{2}}+2\partial_{\rho\rho}\mathbf{v}_0\nabla^\Gamma h_{M-\frac{1}{2}}\cdot\nabla^\Gamma h_1+\mu_{M-\frac{1}{2}}\partial_\rho c_0\mathbf{n}\\&\quad-\mu_0\partial_\rho c_0\nabla^\Gamma h_{M-\frac{1}{2}}+(\rho+h_1)\,\mathbf{u}_{M-\frac{1}{2}}\eta''+h_{M-\frac{1}{2}}\left(\mathbf{u}_1\eta''-\mathbf{q}_0\eta'\right)\end{aligned} \tag{5.209}$$

and

$$\begin{aligned}W^{M-\frac{1}{2}}&=\partial_\rho\mathbf{v}_{M-\frac{1}{2}}\nabla^\Gamma h_1+\partial_\rho\mathbf{v}_1\nabla^\Gamma h_{M-\frac{1}{2}}-\operatorname{div}\mathbf{v}_{M-\frac{1}{2}}-\mathbf{u}_{M-\frac{1}{2}}\cdot\mathbf{n}\eta'(\rho+h_1)\\&\quad-\mathbf{u}_1\cdot\mathbf{n}\eta' h_{M-\frac{1}{2}}-\left(\mathbf{u}_{M-\frac{1}{2}}\cdot\nabla^\Gamma h_1+\mathbf{u}_1\cdot\nabla^\Gamma h_{M-\frac{1}{2}}\right)d_\Gamma\eta'\\&\quad+\mathbf{u}_0\cdot\left(\nabla^\Gamma h_{M-\frac{1}{2}}\rho+\left(\nabla^\Gamma h_{M-\frac{1}{2}}h_1+\nabla^\Gamma h_1h_{M-\frac{1}{2}}\right)\right)\eta'.\end{aligned} \tag{5.210}$$

These are all terms of order $\epsilon^{M+\frac{1}{2}}$ appearing in the accordingly adapted (5.34)–(5.37) (as $c_{M-\frac{1}{2}}$ vanishes). Note in particular the appearance of $\mathbf{w}_1^\epsilon|_\Gamma\cdot\mathbf{n}\partial_\rho c_0$ in (5.208) which is due to the fact that we want to approximate (4.9).

Corollary 5.31. *Let $\epsilon>0$, $\mathfrak{S}_0$ and $\mathfrak{S}_1$ be given as in Lemmata 5.19 and 5.22 and assume that $\left(\mathbf{v}_{M-\frac{1}{2}},p_{M-\frac{1}{2}},c_{M-\frac{1}{2}},\mu_{M-\frac{1}{2}}\right)$ satisfy the matching conditions (5.24)–(5.27) for $k=M-\frac{1}{2}$. Then it holds*

1. $\int_{\mathbb{R}}A^{M-\frac{1}{2}}\theta_0'\mathrm{d}\rho=0$ *for all* $(x,t)\in\Gamma$ *iff*

$$\tilde{\mu}_{M-\frac{1}{2}}=\sigma\Delta^\Gamma h_{M-\frac{1}{2}}-g_0h_{M-\frac{1}{2}}\tilde{\eta}\ \text{on}\ \Gamma \tag{5.211}$$

   where $\tilde{\eta}$, $\tilde{\mu}_{M-\frac{1}{2}}$ are given as in (5.59) and (5.60).

2. $\int_{\mathbb{R}}B^{M-\frac{1}{2}}\mathrm{d}\rho=0$ *for all* $(x,t)\in\Gamma$ *iff*

$$\begin{aligned}0=&\int_{\mathbb{R}}\theta_0'\left(\mathbf{v}_{M-\frac{1}{2}}\cdot\mathbf{n}-\mathbf{v}_0\cdot\nabla^\Gamma h_{M-\frac{1}{2}}\right)\mathrm{d}\rho-\left[\mu_{M-\frac{1}{2}}\right]\Delta d_\Gamma-2\left[\nabla\mu_{M-\frac{1}{2}}\right]\cdot\mathbf{n}+l_{M-\frac{1}{2}}\\&-2\partial_t^\Gamma h_{M-\frac{1}{2}}+[\mu_0]\,\Delta^\Gamma h_{M-\frac{1}{2}}+2\left[\nabla\mu_0\right]\cdot\nabla^\Gamma h_{M-\frac{1}{2}}-j_0h_{M-\frac{1}{2}}+2\mathbf{w}_1^\epsilon|_\Gamma\cdot\mathbf{n}.\end{aligned} \tag{5.212}$$

3. $\int_{\mathbb{R}} \mathbf{V}^{M-\frac{1}{2}} \cdot \mathbf{n} \mathrm{d}\rho = 0$ *for all* $(x,t) \in \Gamma$ *iff*

$$\begin{aligned} 0 = &- \left[p_{M-\frac{1}{2}}\right] + \left[\mathbf{v}_{M-\frac{1}{2}}\right] \cdot \mathbf{n} \Delta d_\Gamma + 2\left(\left(\left[\nabla \mathbf{v}_{M-\frac{1}{2}}\right]\right)^T \mathbf{n} - \left([\nabla \mathbf{v}_0]\right)^T \nabla^\Gamma h_{M-\frac{1}{2}}\right) \cdot \mathbf{n} \\ &+ \int_{\mathbb{R}} \mu_{M-\frac{1}{2}} \theta_0' \mathrm{d}\rho - \mathbf{q}_0 \cdot \mathbf{n} h_{M-\frac{1}{2}} - \mathbf{u}_{M-\frac{1}{2}} \cdot \mathbf{n}. \end{aligned} \tag{5.213}$$

4. $\int_{\mathbb{R}} \mathbf{V}^{M-\frac{1}{2}} \cdot \tau \mathrm{d}\rho = 0$ *for all* $(x,t) \in \Gamma$ *iff*

$$\begin{aligned} 0 = &\left[\mathbf{v}_{M-\frac{1}{2}}\right] \cdot \tau \Delta d_\Gamma + 2\left(\left(\left[\nabla \mathbf{v}_{M-\frac{1}{2}}\right]\right)^T \mathbf{n} - \left([\nabla \mathbf{v}_0]\right)^T \nabla^\Gamma h_{M-\frac{1}{2}}\right) \cdot \tau \\ &+ [p_0] \nabla^\Gamma h_{M-\frac{1}{2}} \cdot \tau + 2\sigma \Delta d_\Gamma \nabla^\Gamma h_{M-\frac{1}{2}} \cdot \tau - \mathbf{q}_0 \cdot \tau h_{M-\frac{1}{2}} - \mathbf{u}_{M-\frac{1}{2}} \cdot \tau. \end{aligned} \tag{5.214}$$

Proof. This can be shown by direct calculations, where similar properties of zero–order terms are used as in the proofs of Lemma 5.7 – Lemma 5.11. □

5.3.3. Construction Scheme for the $\left(M-\frac{1}{2}\right)$-th Order Terms

Now we will construct

$$\mathfrak{S}_{M-\frac{1}{2}} := \left(\mathbf{v}_{M-\frac{1}{2}}, \mathbf{v}^\pm_{M-\frac{1}{2}}, \mathbf{u}_{M-\frac{1}{2}}, \mu_{M-\frac{1}{2}}, \mu^\pm_{M-\frac{1}{2}}, c_{M-\frac{1}{2}}, c^\pm_{M-\frac{1}{2}}, h_{M-\frac{1}{2}}, l_{M-\frac{1}{2}}, p^\pm_{M-\frac{1}{2}}\right).$$

Many steps are similar in kind to the ones in Subsection 5.1.6. But we will not enforce compatibility conditions for a higher order in $\Gamma(2\delta) \backslash \Gamma$, we will just make sure that (5.211)–(5.214) are satisfied.

1. As we assume $c_{M-\frac{1}{2}}(0,x,t) = 0$ for all $(x,t) \in \Gamma(2\delta; T_0)$, $c_{M-\frac{1}{2}} \equiv 0$ is the unique solution to (5.205) in $\mathbb{R} \times \Gamma(2\delta; T_0)$ and by (5.199) also $c^\pm_{M-\frac{1}{2}} \equiv 0$. Thus, there is no $M-\frac{1}{2}$ order term in the expansion of c^ϵ.

2. Assume from now on that $h_{M-\frac{1}{2}}$ is known. In order to satisfy (5.211), we demand

$$\tilde{\mu}_{M-\frac{1}{2}} = \sigma \Delta^\Gamma h_{M-\frac{1}{2}} - g_0 h_{M-\frac{1}{2}} \tilde{\eta}$$

on Γ.

3. Lemma 2.4 implies the existence of a bounded, smooth solution to (5.206). Moreover, this solution is then of the form

$$\begin{aligned} \mu_{M-\frac{1}{2}}(\rho,x,t) = &\left(l_{M-\frac{1}{2}}(x,t)\, d_\Gamma(x,t) - l_0(x,t)\, h_{M-\frac{1}{2}}(S(x,t),t)\right)\left(\eta(\rho) - \frac{1}{2}\right) \\ &+ \bar{\mu}_{M-\frac{1}{2}}(x,t) \end{aligned} \tag{5.215}$$

for all $(\rho,x,t) \in \mathbb{R} \times \Gamma(2\delta; T_0)$. Due to the properties of η as discussed in Proposition 5.3 we get

$$\bar{\mu}_{M-\frac{1}{2}}(x,t) = \tilde{\mu}_{M-\frac{1}{2}}(x,t)$$

in $\Gamma(2\delta; T_0)$ and thus by Step 2 (5.215) reads for $(\rho,x,t) \in \mathbb{R} \times \Gamma$

$$\begin{aligned} \mu_{M-\frac{1}{2}}(\rho,x,t) = &\sigma \Delta^\Gamma h_{M-\frac{1}{2}} - l_0(x,t)\, h_{M-\frac{1}{2}}(S(x,t),t)\left(\eta(\rho) - \frac{1}{2}\right) \\ &- g_0 h_{M-\frac{1}{2}} \tilde{\eta}. \end{aligned} \tag{5.216}$$

4. In order for the matching condition to hold, (5.216) implies that it is necessary and sufficient to set

$$\mu^{\pm}_{M-\frac{1}{2}}(x,t) = \sigma\Delta^{\Gamma} h_{M-\frac{1}{2}} - g_0 h_{M-\frac{1}{2}}\tilde{\eta} \mp \frac{1}{2} l_0(x,t)\, h_{M-\frac{1}{2}}(S(x,t),t) \tag{5.217}$$

for $(x,t) \in \Gamma$, which together with (5.202) yields a closed system for the outer term. Thus, we assume in the following that $\mu^{\pm}_{M-\frac{1}{2}}$ are known (depending on $h_{M-\frac{1}{2}}$) and extended unto $\Omega^{\pm}_{T_0} \cup \Gamma(2\delta; T_0)$.

5. It is necessary and sufficient for the matching conditions in $\Gamma(2\delta;T_0)\setminus\Gamma$ to hold, to set

$$\bar{\mu}_{M-\frac{1}{2}} := \frac{1}{2}\left(\mu^{+}_{M-\frac{1}{2}} + \mu^{-}_{M-\frac{1}{2}}\right) \text{ in } \Gamma(2\delta;T_0). \tag{5.218}$$

$$l_{M-\frac{1}{2}} := \begin{cases} \frac{1}{d_\Gamma}\left(\mu^{+}_{M-\frac{1}{2}} - \mu^{-}_{M-\frac{1}{2}} + l_0 h_{M-\frac{1}{2}}\right) & \text{in } \Gamma(2\delta;T_0)\setminus\Gamma, \\ \nabla d_\Gamma \cdot \nabla\left(\mu^{+}_{M-\frac{1}{2}} - \mu^{-}_{M-\frac{1}{2}} + l_0 h_{M-\frac{1}{2}}\right) & \text{on } \Gamma. \end{cases} \tag{5.219}$$

6. As the right hand sides of (5.203) and (5.204) are 0 since $c_{M-\frac{1}{2}} \equiv 0$, we do not need to construct a term $p_{M-\frac{3}{2}}$, as every bounded solution on $\mathbb{R}\times\Gamma(2\delta)$ to (5.203) automatically satisfies (5.204).

7. As mentioned in Step 6, the right hand side of (5.203) equals 0 and thus Lemma 2.4 implies

$$\begin{aligned} \mathbf{v}_{M-\frac{1}{2}}(\rho,x,t) = {} & \left(\mathbf{u}_{M-\frac{1}{2}}(x,t)\, d_\Gamma(x,t) - \mathbf{u}_0(x,t)\, h_{M-\frac{1}{2}}(x,t)\right)\left(\eta(\rho) - \frac{1}{2}\right) \\ & + \bar{\mathbf{v}}_{M-\frac{1}{2}}(x,t) \end{aligned} \tag{5.220}$$

and in order to satisfy the matching conditions it is necessary and sufficient that

$$\mathbf{v}^{\pm}_{M-\frac{1}{2}} = \bar{\mathbf{v}}_{M-\frac{1}{2}} \pm \left(\mathbf{u}_{M-\frac{1}{2}} d_\Gamma - \mathbf{u}_0 h_{M-\frac{1}{2}}\right)$$

holds in $\Gamma(2\delta;T_0)$. In particular

$$\left[\mathbf{v}_{M-\frac{1}{2}}\right] = 0 \text{ on } \Gamma, \tag{5.221}$$

since $\mathbf{u}_0 = 0$ on Γ due to (5.126). In order for the matching condition to hold on $\Gamma(2\delta)\setminus\Gamma$ we set

$$\bar{\mathbf{v}}_{M-\frac{1}{2}} := \frac{1}{2}\left(\mathbf{v}^{+}_{M-\frac{1}{2}} + \mathbf{v}^{-}_{M-\frac{1}{2}}\right) \text{ in } \Gamma(2\delta;T_0), \tag{5.222}$$

$$\mathbf{u}_{M-\frac{1}{2}} := \begin{cases} \frac{1}{d_\Gamma}\left(\left(\mathbf{v}^{+}_{M-\frac{1}{2}} - \mathbf{v}^{-}_{M-\frac{1}{2}}\right) + \mathbf{u}_0 h_{M-\frac{1}{2}}\right) & \text{in } \Gamma(2\delta;T_0)\setminus\Gamma, \\ \nabla d_\Gamma \cdot \nabla\left(\left(\mathbf{v}^{+}_{M-\frac{1}{2}} - \mathbf{v}^{-}_{M-\frac{1}{2}}\right) + \mathbf{u}_0 h_{M-\frac{1}{2}}\right) & \text{on } \Gamma. \end{cases} \tag{5.223}$$

8. We want to ensure that $\int_{\mathbb{R}} \mathbf{V}^{M-\frac{1}{2}} \mathrm{d}\rho = 0$ holds on Γ. By (5.214), for the tangential component this is equivalent to

$$\left[\partial_{\mathbf{n}} \mathbf{v}_{M-\frac{1}{2}} \cdot \tau\right] = \left(\partial_{\mathbf{n}} \mathbf{u}_0 h_{M-\frac{1}{2}} + 2\left([\nabla \mathbf{v}_0]\right)^T \nabla^\Gamma h_{M-\frac{1}{2}} - [p_0]\, \nabla^\Gamma h_{M-\frac{1}{2}}\right) \cdot \tau \\ + \left(\mathbf{q}_0 h_{M-\frac{1}{2}} - 2\sigma \Delta d_\Gamma \nabla^\Gamma h_{M-\frac{1}{2}}\right) \cdot \tau$$

on Γ, where we used (5.223), (5.149) and (5.221).

9. Due to (5.213), we require

$$\left[\partial_{\mathbf{n}} \mathbf{v}_{M-\frac{1}{2}} \cdot \mathbf{n} - p_{M-\frac{1}{2}}\right] = \left(\partial_{\mathbf{n}} \mathbf{u}_0 h_{M-\frac{1}{2}} + 2\left([\nabla \mathbf{v}_0]\right)^T \nabla^\Gamma h_{M-\frac{1}{2}} + \mathbf{q}_0 h_{M-\frac{1}{2}}\right) \cdot \mathbf{n} \\ - 2\left(\sigma \Delta^\Gamma h_{M-\frac{1}{2}} - g_0 h_{M-\frac{1}{2}} \tilde{\eta}\right)$$

on Γ, where we used the same techniques as in Step 8 and

$$\int_{\mathbb{R}} \mu_{M-\frac{1}{2}} \theta_0' \mathrm{d}\rho = 2\left(\sigma \Delta^\Gamma h_{M-\frac{1}{2}} - g_0 h_{M-\frac{1}{2}} \tilde{\eta}\right)$$

by (5.216).

10. Using Proposition 2.18 and $\left[\mathbf{v}_{M-\frac{1}{2}}\right] = 0$ on Γ we get

$$2\left[D_s \mathbf{v}_{M-\frac{1}{2}}\right] \cdot \mathbf{n} = \left[\partial_{\mathbf{n}} \mathbf{v}_{M-\frac{1}{2}}\right].$$

So together with (5.200)–(5.201) we now have a closed system for the outer terms.

11. We next state the complete outer system. Due to (5.212) $\int_{\mathbb{R}} B^{M-\frac{1}{2}} \mathrm{d}\rho = 0$ is equivalent to

$$0 = \int_{\mathbb{R}} \theta_0' \left(\mathbf{v}_{M-\frac{1}{2}} \cdot \mathbf{n} - \mathbf{v}_0 \cdot \nabla^\Gamma h_{M-\frac{1}{2}}\right) \mathrm{d}\rho - \left[\mu_{M-\frac{1}{2}}\right] \Delta d_\Gamma - 2\left[\nabla \mu_{M-\frac{1}{2}}\right] \cdot \mathbf{n} + l_{M-\frac{1}{2}} \\ - 2\partial_t^\Gamma h_{M-\frac{1}{2}} + [\mu_0]\, \Delta^\Gamma h_{M-\frac{1}{2}} + 2\,[\nabla \mu_0] \cdot \nabla^\Gamma h_{M-\frac{1}{2}} - j_0 h_{M-\frac{1}{2}} + 2\mathbf{w}_1^\epsilon|_\Gamma \cdot \mathbf{n}$$

on Γ. Using that on Γ we have $\int_{\mathbb{R}} \theta_0' \mathbf{v}_{M-\frac{1}{2}} \cdot \mathbf{n} \mathrm{d}\rho = \left(\mathbf{v}^+_{M-\frac{1}{2}} + \mathbf{v}^-_{M-\frac{1}{2}}\right) \cdot \mathbf{n}$, $\mathbf{v}_0 = \frac{1}{2}\left(\mathbf{v}_0^+ + \mathbf{v}_0^-\right)$, $\left[\mu_{M-\frac{1}{2}}\right] = -l_0 h_{M-\frac{1}{2}}$, $l_{M-\frac{1}{2}} = \left[\partial_{\mathbf{n}} \mu_{M-\frac{1}{2}}\right] + \partial_{\mathbf{n}} l_0 h_{M-\frac{1}{2}}$ $[\mu_0] = 0$, $[\nabla \mu_0] = \mathbf{n} l_0$, by (5.220), Lemma 5.19, (5.217), (5.219), (5.108), (5.110), respectively, we get the equivalent equation

$$2\partial_t^\Gamma h_{M-\frac{1}{2}} = \left(\mathbf{v}^+_{M-\frac{1}{2}} + \mathbf{v}^-_{M-\frac{1}{2}}\right) \cdot \mathbf{n} - 2\mathbf{v}_0 \cdot \nabla^\Gamma h_{M-\frac{1}{2}} - \partial_{\mathbf{n}}\left(\mu^+_{M-\frac{1}{2}} - \mu^-_{M-\frac{1}{2}}\right) \\ + \left(l_0 \Delta d_\Gamma - j_0 + \partial_{\mathbf{n}} l_0\right) h_{M-\frac{1}{2}} + 2\mathbf{w}_1^\epsilon|_\Gamma \cdot \mathbf{n}. \tag{5.224}$$

Taking all gathered information together, we get the following system for the functions $\left(\mathbf{v}^{\pm}_{M-\frac{1}{2}}, \mu^{\pm}_{M-\frac{1}{2}}, p^{\pm}_{M-\frac{1}{2}}, h_{M-\frac{1}{2}}\right)$:

$$\Delta\mu^{\pm}_{M-\frac{1}{2}} = 0 \qquad \text{in } \Omega^{\pm}_{T_0}, \tag{5.225}$$

$$-\Delta\mathbf{v}^{\pm}_{M-\frac{1}{2}} + \nabla p^{\pm}_{M-\frac{1}{2}} = 0 \qquad \text{in } \Omega^{\pm}_{T_0}, \tag{5.226}$$

$$\operatorname{div}\mathbf{v}^{\pm}_{M-\frac{1}{2}} = 0 \qquad \text{in } \Omega^{\pm}_{T_0}, \tag{5.227}$$

$$\mu^{-}_{M-\frac{1}{2}} = 0 \qquad \text{on } \partial_{T_0}\Omega, \tag{5.228}$$

$$\left(-2D_s\mathbf{v}^{-}_{M-\frac{1}{2}} + p^{-}_{M-\frac{1}{2}}\mathbf{I}\right)\mathbf{n}_{\partial\Omega} = \alpha_0\mathbf{v}^{-}_{M-\frac{1}{2}} \qquad \text{on } \partial_{T_0}\Omega, \tag{5.229}$$

in the bulk,

$$\mu^{\pm}_{M-\frac{1}{2}} = \sigma\Delta^{\Gamma}h_{M-\frac{1}{2}} + \left(\mp\frac{1}{2}l_0 - \tilde{\eta}g_0\right)h_{M-\frac{1}{2}} \qquad \text{on } \Gamma, \tag{5.230}$$

$$\begin{aligned}\left[2D_s\mathbf{v}_{M-\frac{1}{2}} - p_{M-\frac{1}{2}}\right]\mathbf{n} = {} & \nabla\mathbf{u}_0\mathbf{n}h_{M-\frac{1}{2}} - [p_0]\nabla^{\Gamma}h_{M-\frac{1}{2}} + \mathbf{q}_0h_{M-\frac{1}{2}} \\ & + 2\left([\nabla\mathbf{v}_0]\right)^T\nabla^{\Gamma}h_{M-\frac{1}{2}} - 2\sigma\Delta d_{\Gamma}\nabla^{\Gamma}h_{M-\frac{1}{2}} \\ & - 2\left(\sigma\Delta^{\Gamma}h_{M-\frac{1}{2}} - g_0h_{M-\frac{1}{2}}\tilde{\eta}\right)\mathbf{n} \qquad \text{on } \Gamma,\end{aligned} \tag{5.231}$$

$$\left[\mathbf{v}_{M-\frac{1}{2}}\right] = 0 \qquad \text{on } \Gamma, \tag{5.232}$$

$$\begin{aligned}\partial^{\Gamma}_t h_{M-\frac{1}{2}} = {} & \frac{1}{2}\left(l_0\Delta d_{\Gamma} - j_0 + \partial_{\mathbf{n}}l_0\right)h_{M-\frac{1}{2}} + \mathbf{w}^{\epsilon}_1\cdot\mathbf{n} \\ & + \frac{1}{2}\left(\mathbf{v}^{+}_{M-\frac{1}{2}} + \mathbf{v}^{-}_{M-\frac{1}{2}}\right)\cdot\mathbf{n} - \mathbf{v}_0\cdot\nabla^{\Gamma}h_{M-\frac{1}{2}} \\ & - \frac{1}{2}\left(\partial_{\mathbf{n}}\mu^{+}_{M-\frac{1}{2}} - \partial_{\mathbf{n}}\mu^{-}_{M-\frac{1}{2}}\right) \qquad \text{on } \Gamma,\end{aligned} \tag{5.233}$$

$$h_{M-\frac{1}{2}}|_{t=0} = 0 \qquad \text{on } \Gamma_0, \tag{5.234}$$

at the interface.
Note that the evolution equation for $h_{M-\frac{1}{2}}$ is not linear, as $\mathbf{w}^{\epsilon}_1\cdot\mathbf{n}$ depends on $h_{M-\frac{1}{2}}$ via (5.174). Consequently, proving solvability is more involved for (5.225)–(5.234) than it was for (5.157)–(5.166). Moreover, due to the dependency of $\mathbf{w}^{\epsilon}_1$ on ∇c^{ϵ}, we will not get estimates uniform in $\epsilon > 0$ for $h_{M-\frac{1}{2}}$ in arbitrary norms. We will show the existence of solutions for (5.225)–(5.234) in the following theorem.

Theorem 5.32. *Let $\epsilon_0 \in (0,1)$ and the space V_0 be chosen as in (2.8).*

1. *There exist unique solutions $h^{\epsilon}_{M-\frac{1}{2}} \in X_{T_0}$, $\mu^{\pm,\epsilon}_{M-\frac{1}{2}} \in L^2\left(0,T_0;H^2\left(\Omega^{\pm}(t)\right)\right)$ and*

$$\left(\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}, p^{\pm,\epsilon}_{M-\frac{1}{2}}\right) \in L^2\left(0,T_0;H^2\left(\Omega^{\pm}(t)\right)\right) \times L^2\left(0,T_0;H^1\left(\Omega^{\pm}(t)\right)\right)$$

of (5.225)–(5.234) for all $\epsilon \in (0,\epsilon_0)$, where

$$\epsilon^{M-\frac{1}{2}}\mathbf{w}^{\epsilon}_1 = \tilde{\mathbf{w}}^{\epsilon}_1 \in L^2\left(0,T_0;V_0\right)$$

is the weak solution of (5.174)–(5.176) with $H = \left(h^{\epsilon}_{M-\frac{1}{2}}\right)_{\epsilon\in(0,\epsilon_0)}$.

2. *If Assumption 4.2 holds for $c_A = c_A^{\epsilon,H}$, there exist $\epsilon_1 \in (0,\epsilon_0]$ and a constant $C(K) > 0$ independent of ϵ such that*
$$\left\| h^\epsilon_{M-\frac{1}{2}} \right\|_{X_{T_\epsilon}} \leq C(K) \tag{5.235}$$
and, writing $Z_{T_\epsilon} := L^2\left(0,T_\epsilon; H^2\left(\Omega^\pm(t)\right)\right) \cap L^6\left(0,T_\epsilon; H^1\left(\Omega^\pm(t)\right)\right)$,
$$\left\| \mu^{\pm,\epsilon}_{M-\frac{1}{2}} \right\|_{Z_{T_\epsilon}} + \left\| \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}} \right\|_{L^6(0,T_\epsilon;H^2(\Omega^\pm(t)))} + \left\| p^{\pm,\epsilon}_{M-\frac{1}{2}} \right\|_{L^6(0,T_\epsilon;H^1(\Omega^\pm(t)))} \leq C(K) \tag{5.236}$$
for all $\epsilon \in (0,\epsilon_1)$.

Proof. It is important to be aware that $\mathbf{w}_1^\epsilon$ depends on $h^\epsilon_{M-\frac{1}{2}}$ since it is a solution to (5.174) where c_A^ϵ depends on
$$\rho(x,t) = \frac{d_\Gamma(x,t)}{\epsilon} - h_A^\epsilon\left(S(x,t),t\right)$$
inside of $\Gamma(2\delta)$ and $h^\epsilon_{M-\frac{1}{2}}$ is a summand in h_A^ϵ, see (5.168), where $\tilde{h}^\epsilon := h^\epsilon_{M-\frac{1}{2}}$. Hence we cannot simply use Theorem 2.37 to get a solution for (5.225)–(5.234) but we have to use a fixed point argument. Hereinafter, we will drop the ϵ-dependence in the notation, i.e. write $\mu^\pm_{M-\frac{1}{2}}$ instead of $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}$ etc.

Simplifying the notation, we consider the system
$$\begin{aligned}
\Delta\mu^\pm_{M-\frac{1}{2}} &= 0 && \text{in } \Omega^\pm_{T_0}, && (5.237)\\
\mu^\pm_{M-\frac{1}{2}} &= \sigma\Delta^\Gamma h_{M-\frac{1}{2}} + a^{1,\pm} h_{M-\frac{1}{2}} && \text{on } \Gamma, && (5.238)\\
-\Delta\mathbf{v}^\pm_{M-\frac{1}{2}} + \nabla p^\pm_{M-\frac{1}{2}} &= 0 && \text{in } \Omega^\pm_{T_0}, && (5.239)\\
\operatorname{div}\mathbf{v}^\pm_{M-\frac{1}{2}} &= 0 && \text{in } \Omega^\pm_{T_0}, && (5.240)\\
\left[\mathbf{v}_{M-\frac{1}{2}}\right] &= 0 && \text{on } \Gamma, && (5.241)\\
\left[2D_s\mathbf{v}_{M-\frac{1}{2}} - p_{M-\frac{1}{2}}\right]\cdot\mathbf{n} &= \mathbf{a}^1 h_{M-\frac{1}{2}} + \mathbf{a}^2\Delta^\Gamma h_{M-\frac{1}{2}} + a^2\nabla^\Gamma h_{M-\frac{1}{2}} && \text{on } \Gamma, && (5.242)\\
\mathbf{v}^\pm_{M-\frac{1}{2}}|_{t=0} &= 0 && \text{in } \Omega^\pm(0), && (5.243)\\
\partial_t^\Gamma h_{M-\frac{1}{2}} &= \mathbf{a}^3\cdot\nabla^\Gamma h_{M-\frac{1}{2}} + a^3 h_{M-\frac{1}{2}} && &&\\
&\quad + \frac{1}{2}\left(\mathbf{v}^+_{M-\frac{1}{2}} + \mathbf{v}^-_{M-\frac{1}{2}}\right)\cdot\mathbf{n} && &&\\
&\quad - \frac{1}{2}\left(\partial_{\mathbf{n}}\mu^+_{M-\frac{1}{2}} - \partial_{\mathbf{n}}\mu^-_{M-\frac{1}{2}}\right) + \mathbf{w}_1^\epsilon\cdot\mathbf{n} && \text{on } \Gamma, && (5.244)\\
h_{M-\frac{1}{2}}|_{t=0} &= 0 && \text{on } \Gamma_0 && (5.245)
\end{aligned}$$
for some smooth functions $a^{1,\pm}, a^2, a^3 : \Gamma \to \mathbb{R}$, $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3 : \Gamma \to \mathbb{R}^2$ which are defined by (5.225)–(5.234).

Now we introduce the auxiliary operators
$$\mathcal{S}^{\mathbf{v}} : X_{T_0} \to L^2\left(0,T_0; H^{\frac{1}{2}}(\Gamma_t)\right), h \mapsto tr_\Gamma\left(\mathbf{v}^+ + \mathbf{v}^-\right)\cdot\mathbf{n}$$
and
$$\mathcal{S}^\mu : X_{T_0} \to L^2\left(0,T_0; H^{\frac{1}{2}}(\Gamma_t)\right), h \mapsto tr_\Gamma\left(\partial_{\mathbf{n}}\mu^+ - \partial_{\mathbf{n}}\mu^-\right),$$

where $\mathbf{v}^{\pm}$ and $\mu^{\pm}$ are the unique h-dependent solutions to (5.239)–(5.243) and (5.237)–(5.238) and tr_Γ denotes the trace. The unique solvability for $\mathbf{v}$ follows from Theorem 2.36 and for μ from general elliptic theory.

Thus, we may write the evolution equation (5.244)–(5.245) as

$$\begin{aligned}
&D_{t,\Gamma} h_{M-\frac{1}{2}} + \mathbf{a}\cdot\nabla_\Gamma h_{M-\frac{1}{2}} - a h_{M-\frac{1}{2}} \\
&+\frac{1}{2} X_0^* \left(\mathcal{S}^{\mathbf{v}}\left(h_{M-\frac{1}{2}}\right) - \mathcal{S}^{\mu}\left(h_{M-\frac{1}{2}}\right)\right) = X_0^*\left(\mathbf{w}_1^\epsilon\cdot\mathbf{n}\right) \qquad \text{in } \mathbb{T}^1\times(0,T_0), \\
&h|_{t=0} = 0 \qquad \text{in } \mathbb{T}^1
\end{aligned} \tag{5.246}$$

for some smooth coefficients $a : \mathbb{T}^1\times[0,T_0]\to\mathbb{R}$, $\mathbf{a} : \mathbb{T}^1\times[0,T_0]\to\mathbb{R}$, where we used (2.30). We now define

$$\begin{aligned}
L(h) &:= 2D_{t,\Gamma}h + \mathbf{a}\cdot\nabla_\Gamma h - ah + \frac{1}{2}X_0^*\left(\mathcal{S}^{\mathbf{v}}(h) + \mathcal{S}^{\mu}(h)\right), \\
\mathcal{N}(h) &:= X_0^*\left(\mathbf{w}_1^{\epsilon,h}\cdot\mathbf{n}\right)
\end{aligned}$$

for $h\in X_{T_0}$, where $\mathbf{w}_1^{\epsilon,h} := \mathbf{w}_1^{\epsilon,H}$ as in (5.198) with $H = \{h\}_{\epsilon\in(0,\epsilon_0)}$. With this we may write (5.246) as a fixed point equation via

$$h = L^{-1}(\mathcal{N}(h)) \text{ for } h\in X_{T_0} \text{ with } h|_{t=0} = 0 \text{ in } \mathbb{T}^1. \tag{5.247}$$

Note that the bijectivity of the operator $L : X_{T'} \to L^2\left(0,T';H^{\frac{1}{2}}(\mathbb{T}^1)\right)$ for every $T'\in(0,T_0]$ follows from Theorem 2.37 and thus L^{-1} is bounded by the inverse mapping theorem. Moreover, the norm of L^{-1} is bounded by a constant $c_L > 0$ independent of $T'\in(0,T_0]$. Remark furthermore that we do not let L^{-1} depend on initial values, as we will in the following always consider L^{-1} to be the solution operator corresponding to initial value 0.

Now we want to show that (5.247) really has a fixed point in $X_{T'}$ for some $T'\in(0,T_0]$. Using the uniform bound of L^{-1} and the continuity of the trace operator together with Proposition 5.30 2) yields for $h_1,h_2\in X_{T'}$ and $T'\in(0,T_0]$

$$\begin{aligned}
\left\|L^{-1}(\mathcal{N}(h_1)-\mathcal{N}(h_2))\right\|_{X_{T'}} &\le C\left\|X_0^*\left(\left(\mathbf{w}_1^{\epsilon,h_1}-\mathbf{w}_1^{\epsilon,h_2}\right)\cdot\mathbf{n}\right)\right\|_{L^2\left(0,T';H^{\frac{1}{2}}(\mathbb{T}^1)\right)} \\
&\le C\left\|\mathbf{w}_1^{\epsilon,h_1}-\mathbf{w}_1^{\epsilon,h_2}\right\|_{L^2(0,T';H^1(\Omega))} \\
&\le C(\epsilon)\left(T'\right)^{\frac{1}{2}}\left(1+\|h_2\|_{X_{T'}}\right)\|h_1-h_2\|_{X_{T'}}.
\end{aligned} \tag{5.248}$$

Now we define $R = R(\epsilon,T_0) := 2\left\|L^{-1}(\mathcal{N}(0))\right\|_{X_{T_0}}$. For $T'\in(0,T_0]$ and $h,h_1,h_2\in\overline{B_{X_{T'}}(0;R)}$ we have due to (5.248)

$$\left\|L^{-1}(\mathcal{N}(h_1)-\mathcal{N}(h_2))\right\|_{X_{T'}} \le C(\epsilon)\left(T'\right)^{\frac{1}{2}}(1+R)\|h_1-h_2\|_{X_{T'}}$$

and

$$\begin{aligned}
\left\|L^{-1}(\mathcal{N}(h))\right\|_{X_{T'}} &\le \left\|L^{-1}(\mathcal{N}(h)-\mathcal{N}(0))\right\|_{X_{T'}} + \left\|L^{-1}(\mathcal{N}(0))\right\|_{X_{T_0}} \\
&\le C(\epsilon)\left(T'\right)^{\frac{1}{2}}(1+R)\|h\|_{X_{T'}} + \frac{R}{2}.
\end{aligned}$$

Hence, by choosing $T(\epsilon) \leq \min\left\{\left(\frac{1}{2C(\epsilon)(1+R(\epsilon,T_0))}\right)^2, T_0\right\}$, we get that $L^{-1}\mathcal{N}$ is a contraction and a self-mapping in $\overline{B_{X_{T(\epsilon)}}(0;R)}$ and thus the Banach fixed-point theorem implies that there exists a $h^* \in X_{T(\epsilon)}$ satisfying (5.247).

Now we assume that the maximal existence time T^* of h^* is smaller than T_0 and want to lead this to a contradiction. For this, we first show an a priori estimate: it holds for $T' \in (0,T^*]$

$$\begin{aligned}\sup_{t\in(0,T')} \|h^*(.,t)\|^2_{H^2(\mathbb{T}^1)} &\leq C\|h^*\|^2_{X_{T'}}\\ &\leq C\|\mathcal{N}(h^*)\|^2_{L^2\left(0,T';H^{\frac{1}{2}}(\mathbb{T}^1)\right)}\\ &\leq C\left(\left\|X_0^*\left(\mathbf{w}_1^{\epsilon,h^*}\cdot\mathbf{n}\right)\right\|^2_{L^2\left(0,T';H^{\frac{1}{2}}(\mathbb{T}^1)\right)}\right)\\ &\leq C(\epsilon)\left(T' + \int_0^{T'} \|h^*(.,t)\|^2_{H^2(\mathbb{T}^1)}\,\mathrm{d}t\right),\end{aligned}$$

where we used the embedding $X_{T'} \hookrightarrow C^0\left([0,T'];H^2\left(\mathbb{T}^1\right)\right)$ in the second inequality, the uniform bound on L^{-1} in the third inequality and the continuity of the trace operator together with Proposition 5.30 1) in the last step. Using Gronwall's inequality we get

$$\sup_{t\in(0,T^*)} \|h^*(.,t)\|_{H^2(\mathbb{T}^1)} \leq c_1(\epsilon,T_0)$$

for some constant $c_1 = c_1(\epsilon,T_0) > 0$. Now let $h_0 \in H^2\left(\mathbb{T}^1\right)$ with $\|h_0\|_{H^2(\mathbb{T}^1)} \leq c_1$. We are interested in solving

$$L(h) = \mathcal{N}(h) \qquad \text{in } \mathbb{T}^1 \times \left(0,T'\right), \tag{5.249}$$

$$h_{|t=0} = h_0 \qquad \text{in } \mathbb{T}^1 \tag{5.250}$$

for some $T' \in (0,T_0]$. It is possible to show that there exists $\tilde{h} \in X_\infty$ with $\left\|\tilde{h}\right\|_{X_\infty} \leq 2\|h_0\|_{H^2(\mathbb{T}^1)} \leq 2c_1$ and $\tilde{h}|_{t=0} = h_0$ (see e.g. [7]) and we may thus consider the problem

$$\begin{aligned} L(v) &= \tilde{\mathcal{N}}(v) && \text{in } \mathbb{T}^1 \times \left(0,T'\right),\\ v_{|t=0} &= 0 && \text{in } \mathbb{T}^1 \end{aligned}$$

with $h = v + \tilde{h}$ and $\tilde{\mathcal{N}}(v) = \mathcal{N}\left(v+\tilde{h}\right) - L\left(\tilde{h}\right)$. Now let

$$\tilde{R} := 2\max\left\{c_1, \left\|L^{-1}\left(\tilde{\mathcal{N}}(0)\right)\right\|_{X_{T_0}}\right\},$$

$T' \in (0,T_0]$ and $v, v_1, v_2 \in \overline{B_{X_{T'}}\left(0;\tilde{R}\right)}$ and we get again by (5.248)

$$\begin{aligned}\left\|L^{-1}\left(\tilde{\mathcal{N}}(v_1)-\tilde{\mathcal{N}}(v_2)\right)\right\|_{X_{T'}} &\leq C(\epsilon)\left(T'\right)^{\frac{1}{2}}\left(1+\left\|v_2+\tilde{h}\right\|_{X_{T'}}\right)\|v_1-v_2\|_{X_{T'}}\\ &\leq C(\epsilon)\left(T'\right)^{\frac{1}{2}}\left(1+2\tilde{R}\right)\|v_1-v_2\|_{X_{T'}}\end{aligned}$$

and

$$\begin{aligned}\left\|L^{-1}\left(\tilde{\mathcal{N}}(v)\right)\right\|_{X_{T'}} &\leq \left\|L^{-1}\left(\tilde{\mathcal{N}}(v)-\tilde{\mathcal{N}}(0)\right)\right\|_{X_{T'}}+\left\|L^{-1}\left(\tilde{\mathcal{N}}(0)\right)\right\|_{X_{T_0}}\\ &\leq C(\epsilon)\left(T'\right)^{\frac{1}{2}}\left(1+\left\|\tilde{h}\right\|_{X_{T'}}\right)\|v\|_{X_{T'}}+\frac{\tilde{R}}{2}\\ &\leq C(\epsilon)\left(T'\right)^{\frac{1}{2}}\left(1+\tilde{R}\right)\|v\|_{X_{T'}}+\frac{\tilde{R}}{2}.\end{aligned}$$

Thus, there exists $\hat{T}=\hat{T}(T_0,\epsilon)$ such that (5.249), (5.250) has a unique solution $h\in X_{\hat{T}}$. Thus, solving (5.249), (5.250) with $h_0=h^*|_{t=T^*-\frac{\hat{T}}{2}}$ (w.l.o.g. $\hat{T}\leq T^*$) we get a continuation of h^* onto $\left(0,T^*+\frac{\hat{T}}{2}\right)$. This is a contradiction to the maximality of T^* and thus $h_{M-\frac{1}{2}}:=h^*\in X_{T_0}$ for all $\epsilon\in(0,\epsilon_0]$

Hence it remains to prove the second statement. Let $T_\epsilon>0$ be given for $\epsilon\in(0,\epsilon_0)$ as in the assumptions. We have shown

$$\begin{aligned}L\left(h^\epsilon_{M-\frac{1}{2}}\right)&=X_0^*\left(\mathbf{w}_1^{\epsilon,H}\cdot\mathbf{n}\right) && \text{in } \mathbb{T}^1\times(0,T),\\ h^\epsilon_{M-\frac{1}{2}}|_{t=0}&=0 && \text{on } \mathbb{T}^1,\end{aligned}\tag{5.251}$$

for $H=\left(h^\epsilon_{M-\frac{1}{2}}\right)_{\epsilon\in(0,\epsilon_0)}$. Thus,

$$\left\|h^\epsilon_{M-\frac{1}{2}}\right\|_{X_{T'}}\leq C\left\|X_0^*\left(\mathbf{w}_1^{\epsilon,H}\cdot\mathbf{n}\right)\right\|_{L^2\left(0,T';H^{\frac{1}{2}}(\mathbb{T}^1)\right)}\leq C_1\frac{1}{\epsilon^{M-\frac{1}{2}}}\left\|\tilde{\mathbf{w}}_1^{\epsilon,H}\right\|_{L^2(0,T';H^1(\Omega))}\tag{5.252}$$

for all $T'\in(0,T_0)$. Here C_1 can be chosen independently of T'. Now we choose $C(K)$ as in Lemma 5.29 (note that this constant is independent of the choice of $\tilde{h}^\epsilon$ in the lemma) and define $\hat{c}(K):=2C_1C(K)$. Then we find that

$$T'_\epsilon:=\sup\left\{t\in(0,T_\epsilon)|\left\|h^\epsilon_{M-\frac{1}{2}}\right\|_{X_t}\leq\hat{c}(K)\right\}$$

satisfies $T'_\epsilon>0$, due to the continuity of the norm $\|.\|_{X_t}$ in $t>0$ and since $h^\epsilon_{M-\frac{1}{2}}|_{t=0}=0$ in $H^2\left(\mathbb{T}^1\right)$.

Using Lemma 5.29 again (with T'_ϵ instead of T_ϵ), we get the existence of $\epsilon_1\in(0,\epsilon_0]$ such that

$$\left\|\tilde{\mathbf{w}}_1^{\epsilon,H}\right\|_{L^2(0,T'_\epsilon;H^1(\Omega))}\leq C(K)\,\epsilon^{M-\frac{1}{2}}$$

for all $\epsilon\in(0,\epsilon_1)$ with the same constant $C(K)$ as above. Thus, by (5.252) we have

$$\left\|h^\epsilon_{M-\frac{1}{2}}\right\|_{X_{T'_\epsilon}}\leq\frac{\hat{c}(K)}{2}<\hat{c}(K)$$

for all $\epsilon\in(0,\epsilon_1)$. By the definition of T'_ϵ this already implies $T'_\epsilon=T_\epsilon$.

(5.236) then follows from (2.43), (2.44) and (2.56) taken together with the embedding $H^{\frac{1}{3}}(0,T_\epsilon;Y)\hookrightarrow L^6(0,T_\epsilon;Y)$ for a Banach space Y and Proposition 2.34 4). This concludes the proof. □

Remark 5.33. Let $\left(h^{\epsilon}_{M-\frac{1}{2}}, \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}, p^{\pm,\epsilon}_{M-\frac{1}{2}}, \mu^{\pm,\epsilon}_{M-\frac{1}{2}}\right)$ be given as in Theorem 5.32 for some $\epsilon \in (0, \epsilon_0)$.

1. As $h^{\epsilon}_{M-\frac{1}{2}} \in X_{T_0}$ the right hand side of (5.174) is already in $L^2\left(\Omega_{T_0}\right)$, so by regularity theory and a bootstrap argumentation, we see that $h^{\epsilon}_{M-\frac{1}{2}}$ and $\tilde{\mathbf{w}}^{\epsilon}_1$ are smooth functions, which transfers to $\left(\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}, p^{\pm,\epsilon}_{M-\frac{1}{2}}, \mu^{\pm,\epsilon}_{M-\frac{1}{2}}\right)$. So the true difficulty in the following is not the missing regularity, but the missing control of higher norms uniformly in ϵ.
2. As for lower order terms, we may also extend $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}$, $p^{\pm,\epsilon}_{M-\frac{1}{2}}$, $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$ onto $\Omega^{\pm}_{T_0} \cup \Gamma\left(2\delta\right)$ by using the same extension as discussed in Remark 5.1. As the extension operator $\mathfrak{E}^{\pm} : W^k_p\left(\Omega^{\pm}\left(t\right)\right) \to W^k_p\left(\mathbb{R}^2\right)$ is continuous, we get in particular
$$\left\|\mathfrak{E}^{\pm}\left(\mu^{\pm,\epsilon}_{M-\frac{1}{2}}\right)\right\|_{H^k(\Omega^{\pm}(t)\cup\Gamma_t(2\delta))} \leq C\left\|\mu^{\pm,\epsilon}_{M-\frac{1}{2}}\right\|_{H^k(\Omega^{\pm}(t))}$$
for $k \in \mathbb{N}$, where we can choose C independently of $t \in [0, T_0]$. Similar estimates hold for $p^{\epsilon}_{M-\frac{1}{2}}$ and $\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}$ (for the latter see (5.16)).
3. In the following we write $c^{\epsilon}_A := c^{\epsilon,H}_A$ for $H = \left(h^{\epsilon}_{M-\frac{1}{2}}\right)_{\epsilon\in(0,\epsilon_0)}$, where $c^{\epsilon,H}_A$ is defined in Definition 5.24.

In the following we will use $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}, \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$ and $p^{\pm,\epsilon}_{M-\frac{1}{2}}$ as functions on $\Omega^{\pm} \cup \Gamma\left(2\delta\right)$ and not explicitly mention the extension operators. After these considerations we may define the $\left(M-\frac{1}{2}\right)$–th order terms.

Lemma 5.34 (The $\left(M-\frac{1}{2}\right)$–th order terms)**.** *Let Assumption 1.1 hold and let η be given as in Proposition 5.3 and θ_0 as in Lemma 2.2. Let moreover $\mathfrak{S}_i$ be given as in Lemmata 5.19, 5.22 for all $i \in \{0, \ldots, M+1\}$ and let $\epsilon \in (0,1)$.*

Then we define the terms of the outer expansion $\left(h^{\epsilon}_{M-\frac{1}{2}}, \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}, p^{\pm,\epsilon}_{M-\frac{1}{2}}, \mu^{\pm,\epsilon}_{M-\frac{1}{2}}\right)$ as the unique solution to (5.225)–(5.234) as given by Theorem 5.32 1) and we consider $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$, $p^{\pm,\epsilon}_{M-\frac{1}{2}}$, $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}$ to be extended onto $\Omega^{\pm}_{T_0} \cup \Gamma\left(2\delta; T_0\right)$ (cf. Remark 5.33). Moreover, we set $c^{\pm,\epsilon}_{M-\frac{1}{2}} \equiv 0$ in $\Omega^{\pm}_{T_0}$. We define the terms of the inner expansion given by the functions $\left(c^{\epsilon}_{M-\frac{1}{2}}, \mu^{\epsilon}_{M-\frac{1}{2}}, \mathbf{v}^{\epsilon}_{M-\frac{1}{2}}, p^{\epsilon}_{M-\frac{1}{2}}, h^{\epsilon}_{M-\frac{1}{2}}\right)$ for $(\rho, x, t) \in \mathbb{R} \times \Gamma\left(2\delta; T_0\right)$ as

$$c^{\epsilon}_{M-\frac{1}{2}} \equiv 0,$$
$$\mu^{\epsilon}_{M-\frac{1}{2}}\left(\rho, x, t\right) := \mu^{+,\epsilon}_{M-\frac{1}{2}}\left(x,t\right)\eta\left(\rho\right) + \mu^{-,\epsilon}_{M-\frac{1}{2}}\left(x,t\right)\left(1-\eta\left(\rho\right)\right), \tag{5.253}$$
$$\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}\left(\rho, x, t\right) := \mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}}\left(x,t\right)\eta\left(\rho\right) + \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\left(x,t\right)\left(1-\eta\left(\rho\right)\right), \tag{5.254}$$
$$p^{\epsilon}_{M-\frac{1}{2}}\left(\rho, x, t\right) := p^{+,\epsilon}_{M-\frac{1}{2}}\left(x,t\right)\eta\left(\rho\right) + p^{-,\epsilon}_{M-\frac{1}{2}}\left(x,t\right)\left(1-\eta\left(\rho\right)\right). \tag{5.255}$$

Further, we define $l^{\epsilon}_{M-\frac{1}{2}}$ and $\mathbf{u}^{\epsilon}_{M-\frac{1}{2}}$ as in (5.219) and (5.223) (substituting $\mu^{\pm}_{M-\frac{1}{2}}$, $\mathbf{v}^{\pm}_{M-\frac{1}{2}}$, $h_{M-\frac{1}{2}}$ in these formulae by $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}$, $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$, $h^{\epsilon}_{M-\frac{1}{2}}$).

Then the outer equations (5.200)–(5.202), the inner equations (5.203)–(5.206) and the identities (5.211)–(5.214) are all satisfied.

Proof. As $\left(h^{\epsilon}_{M-\frac{1}{2}}, \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}, p^{\pm,\epsilon}_{M-\frac{1}{2}}, \mu^{\pm,\epsilon}_{M-\frac{1}{2}}\right)$ solves (5.225)–(5.234) it is immediately clear that the outer equations (5.200)–(5.202) are satisfied.

Concerning (5.211), we compute

$$\frac{1}{2}\int_{\mathbb{R}} \theta_0' \mu^{\epsilon}_{M-\frac{1}{2}} \mathrm{d}\rho = \frac{1}{2}\int_{\mathbb{R}} \theta_0' \frac{1}{2}\left(\mu^{+,\epsilon}_{M-\frac{1}{2}} + \mu^{-,\epsilon}_{M-\frac{1}{2}}\right) \mathrm{d}\rho = \sigma\Delta^{\Gamma} h^{\epsilon}_{M-\frac{1}{2}} - g_0 h^{\epsilon}_{M-\frac{1}{2}}\tilde{\eta},$$

where we used (5.32) in the first equality and (5.230) in the second. The validity of (5.213), (5.214) and (5.212) then follow by the same arguments as made in Steps 8–12 of Subsection 5.3.3.

Regarding the inner equations (5.203)–(5.206) we compute exemplarily

$$\begin{aligned} \mu^{\epsilon}_{M-\frac{1}{2}} - \left(l^{\epsilon}_{M-\frac{1}{2}} d_{\Gamma} - l_0 h^{\epsilon}_{M-\frac{1}{2}}\right)\eta &= \mu^{-,\epsilon}_{M-\frac{1}{2}} && \text{in } \Gamma(2\delta)\backslash\Gamma, \\ \mu^{\epsilon}_{M-\frac{1}{2}} + l_0 h^{\epsilon}_{M-\frac{1}{2}}\eta &= \mu^{-,\epsilon}_{M-\frac{1}{2}} && \text{on } \Gamma, \end{aligned}$$

where we used the definition of $l^{\epsilon}_{M-\frac{1}{2}}$ in the first equality and $\left[\mu^{\epsilon}_{M-\frac{1}{2}}\right] = -l_0 h^{\epsilon}_{M-\frac{1}{2}}$ on Γ in the second equality. The latter is a consequence of (5.230). This implies (5.206). Equations (5.203) and (5.204) follow in the same way, remarking $\mathbf{u}_0 = 0$ on Γ and $\left[\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}\right] = 0$ on Γ. (5.205) is a direct consequence of $c^{\epsilon}_{M-\frac{1}{2}} = 0$. □

Notation 5.35. Assume that the assumptions of Lemma 5.34 holds. In the following, we collect the terms of order $M-\frac{1}{2}$ and write

$$\mathfrak{S}_{M-\frac{1}{2}} := \left(\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}, \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}, \mathbf{u}^{\epsilon}_{M-\frac{1}{2}}, \mu^{\epsilon}_{M-\frac{1}{2}}, \mu^{\pm,\epsilon}_{M-\frac{1}{2}}, h^{\epsilon}_{M-\frac{1}{2}}, l^{\epsilon}_{M-\frac{1}{2}}, p^{\epsilon}_{M-\frac{1}{2}}, p^{\pm,\epsilon}_{M-\frac{1}{2}}\right).$$

For simplicity, we often write $\mathbf{v}_{M-\frac{1}{2}} = \mathbf{v}^{\epsilon}_{M-\frac{1}{2}}$, $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}} = \mathbf{v}^{\pm}_{M-\frac{1}{2}}$ etc., especially if we consider fractional and integer expansion orders together, as in Section 6.1.

Note that, although we assumed it during the construction, we may not give estimates similar to the matching conditions for lower order (5.24)–(5.27), i.e. estimates of the kind

$$\sup_{(x,t)\in\Gamma(2\delta)} \left| D^k_\rho D^l_x \left(\mu^{\epsilon}_{M-\frac{1}{2}}(\pm\rho, x, t) - \mu^{\pm}_{M-\frac{1}{2}}(x,t)\right)\right| \leq C_1 e^{-C_2|\rho|} \text{ for all } \rho \geq 0$$

for the terms $\mu^{\epsilon}_{M-\frac{1}{2}}$, $\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}$ and $p^{\epsilon}_{M-\frac{1}{2}}$. This follows since e.g. there is no embedding $L^2\left(0,T;H^2\left(\Omega^+(t)\right)\right) \hookrightarrow C^0\left([0,T];C^0\left(\Omega^+(t)\right)\right)$ and we have thus no ϵ independent control over the appearing terms for $\rho \in (0,1)$.

Nevertheless, if $\rho \geq 1$ we have $\mu^{\epsilon}_{M-\frac{1}{2}}(\pm\rho, x, t) \equiv \mu^{\pm,\epsilon}_{M-\frac{1}{2}}(x,t)$ for $(x,t) \in \Gamma(2\delta)$, by its definition in (5.253) (this also holds true of course for $\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}$ and $p^{\epsilon}_{M-\frac{1}{2}}$). Consequently, we may later on show Proposition 6.10, which is a suitable substitute for the matching conditions.

The following lemma allows for a better understanding of the structure of the terms $A^{M-\frac{1}{2}}$, $B^{M-\frac{1}{2}}$ and $\mathbf{V}^{M-\frac{1}{2}}$ and is the main reason why we assumed (5.211)–(5.214). The importance of the lemma will become apparent when showing Lemma 6.9, where it is the key ingredient.

Lemma 5.36. *Let the $\left(M-\frac{1}{2}\right)$–th order terms be given as in Lemma 5.34 and let the assumptions of Theorem 5.32 2) hold and let $\epsilon \in (0, \epsilon_1)$.*

1. *There are $L_1, L_2 \in \mathbb{N}$ such that*

$$A^{M-\frac{1}{2}}(\rho, x, t) = \sum_{k=1}^{L_1} \mathtt{A}_k^1(x,t)\,\mathtt{A}_k^2(\rho) \text{ for } (\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta)$$

and

$$A^{M-\frac{1}{2}}(\rho, x, t) = \sum_{j=1}^{L_2} \mathtt{A}_j^{1,\Gamma}(x,t)\,\mathtt{A}_j^{2,\Gamma}(\rho) \text{ for } (\rho, x, t) \in \mathbb{R} \times \Gamma,$$

where $\left\|\mathtt{A}_k^2\right\|_{L^\infty(\mathbb{R})} + \left\|\mathtt{A}_j^{2,\Gamma}\right\|_{L^\infty(\mathbb{R})} \leq C$ for some $C > 0$ independent of ϵ, and

$$\left\|\mathtt{A}_k^1\right\|_{L^6(0,T_\epsilon;L^2(\Gamma_t(2\delta)))} + \left\|\mathtt{A}_j^{1,\Gamma}\right\|_{L^6(0,T_\epsilon;L^2(\Gamma_t))} \leq C(K) \tag{5.256}$$

for all $k \in \{1, \ldots, L_1\}$, $j \in \{1, \ldots, L_2\}$. Moreover, there are $C, \alpha > 0$ independent of ϵ such that

$$\left|\int_{-\tau_1}^{\tau_2} \mathtt{A}_j^{2,\Gamma}\theta_0' \mathrm{d}\rho\right| \leq Ce^{-\alpha\min\{\tau_1,\tau_2\}} \tag{5.257}$$

for $\tau_1, \tau_2 > 0$ large enough and all $j \in \{1, \ldots, L_2\}$.

2. *There are $K_1, K_2 \in \mathbb{N}$ such that*

$$B^{M-\frac{1}{2}}(\rho, x, t) = \sum_{k=1}^{K_1} \mathtt{B}_k^1(x,t)\,\mathtt{B}_k^2(\rho) \text{ for } (\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta)\backslash\Gamma$$

and

$$B^{M-\frac{1}{2}}(\rho, x, t) = \sum_{j=1}^{K_2} \mathtt{B}_j^{1,\Gamma}(x,t)\,\mathtt{B}_j^{2,\Gamma}(\rho) \text{ for } (\rho, x, t) \in \mathbb{R} \times \Gamma,$$

where $\mathtt{B}_k^2, \mathtt{B}_j^{2,\Gamma} \in \mathcal{O}\left(e^{-\alpha|\rho|}\right)$ for $\rho \to \pm\infty$ and

$$\left\|\mathtt{B}_k^1\right\|_{L^2(\Gamma(2\delta;T_\epsilon))} + \left\|\mathtt{B}_j^{1,\Gamma}\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t))} \leq C(K) \tag{5.258}$$

for all $k \in \{1, \ldots, K_1\}$, $j \in \{1, \ldots, K_2\}$. Moreover, there are $C, \alpha > 0$ independent of ϵ such that

$$\left|\int_{-\tau_1}^{\tau_2} \mathtt{B}_j^{2,\Gamma} \mathrm{d}\rho\right| \leq Ce^{-\alpha\min\{\tau_1,\tau_2\}} \tag{5.259}$$

for $\tau_1, \tau_2 > 0$ large enough and all $j \in \{1, \ldots, K_2\}$.

3. *There are $N_1, N_2 \in \mathbb{N}$ such that*

$$\mathbf{V}^{M-\frac{1}{2}}(\rho, x, t) = \sum_{k=1}^{N_1} \mathtt{V}_k^1(x,t)\,\mathtt{V}_k^2(\rho, x, t) \text{ for } (\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta)$$

and

$$\mathbf{V}^{M-\frac{1}{2}}(\rho,x,t)=\sum_{j=1}^{N_2}\mathtt{V}_j^{1,\Gamma}(x,t)\,\mathtt{V}_j^{2,\Gamma}(\rho,x,t)\ \text{ for }\ (\rho,x,t)\in\mathbb{R}\times\Gamma,$$

where $\mathtt{V}_k^2,\mathtt{V}_j^{2,\Gamma}\in\mathcal{R}_\alpha$ *and*

$$\left\|\mathtt{V}_k^1\right\|_{L^2(\Gamma(2\delta;T_\epsilon))}+\left\|\mathtt{V}_j^{1,\Gamma}\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t))}\le C(K)$$

for all $k\in\{1,\dots,N_1\}$, $j\in\{1,\dots,N_2\}$. *Moreover, there are* $C,\alpha>0$ *independent of* ϵ *such that*

$$\sup_{(x,t)\in\Gamma}\left|\int_{-\tau_1}^{\tau_2}\mathtt{V}_j^{2,\Gamma}\mathrm{d}\rho\right|\le Ce^{-\alpha\min\{\tau_1,\tau_2\}}\tag{5.260}$$

for $\tau_1,\tau_2>0$ *large enough and all* $j\in\{1,\dots,N_2\}$.

Proof. Ad 1) Plugging the explicit structure of $\mu^\epsilon_{M-\frac{1}{2}}$ as given in (5.253) into the definition of $A^{M-\frac{1}{2}}$ (see (5.207)) we get

$$\begin{aligned}A^{M-\frac{1}{2}}&=-\frac{1}{2}\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}+\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)-\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)\left(\eta-\frac{1}{2}\right)\\&\quad-2\partial_{\rho\rho}c_0\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\cdot\nabla^\Gamma h_1+\partial_\rho c_0\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}-g_0h^\epsilon_{M-\frac{1}{2}}\eta' \qquad (5.261)\\&=\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}(\partial_\rho c_0-\sigma)+g_0h^\epsilon_{M-\frac{1}{2}}(-\eta'+\tilde{\eta})-\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)\left(\eta-\frac{1}{2}\right)\\&\quad-2\partial_{\rho\rho}c_0\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\cdot\nabla^\Gamma h_1 \qquad (5.262)\end{aligned}$$

on $\mathbb{R}\times\Gamma$, where we used (5.230) in the second line (which is satisfied due to construction in Lemma 5.34). Since (5.261) also holds on $\mathbb{R}\times\Gamma(2\delta)$, we get the first decomposition by setting $\mathtt{A}_1^1:=-\frac{1}{2}\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}+\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)$, $\mathtt{A}_1^2:=1$, $\mathtt{A}_2^1:=-\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)$, $\mathtt{A}_2^2:=\left(\eta-\frac{1}{2}\right)$, etc. and noting that $c_0(\rho,x,t)=\theta_0(\rho)$. Setting $\mathtt{A}_1^{1,\Gamma}=\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}$, $\mathtt{A}_1^{2,\Gamma}=(\partial_\rho c_0-\sigma)$, etc. we get the desired splitting on Γ (with $L_2=4$). It is clear by the properties of c_0 and η that all terms $\mathtt{A}_k^2$, $\mathtt{A}_j^{2,\Gamma}$ are bounded on $\mathbb{R}$. Now

$$\int_{\mathbb{R}}\left((\partial_\rho c_0-\sigma)+(-\eta'+\tilde{\eta})-\left(\eta-\frac{1}{2}\right)-(\partial_{\rho\rho}c_0)\right)\theta_0'\mathrm{d}\rho=0$$

by (5.61), (5.60), (5.32) and the fact that $\partial_{\rho\rho}c_0\theta_0'=\frac{1}{2}\frac{d}{d\rho}(\theta_0')^2$. Since θ_0' has exponential decay by (2.1) we get (5.257).

Now note that by definition in Remark 2.19 we have e.g.

$$\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}(x,t)=\left(\Delta S(x,t)\,\partial_s+|\nabla S(x,t)|^2\,\partial_{ss}\right)h^\epsilon_{M-\frac{1}{2}}(S(x,t),t),$$

where S is a smooth function and can be uniformly estimated in $\Gamma(2\delta;T_0)$ along with its derivatives. Thus, by (5.235) and Proposition 2.34 3) it follows

$$\left\|\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}+g_0h^\epsilon_{M-\frac{1}{2}}-2\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\cdot\nabla^\Gamma h_1\right\|_{L^6(0,T_\epsilon;L^2(\Gamma_t(2\delta)))}\le C(K)$$

and the same estimate also holds true if we exchange $\Gamma_t(2\delta)$ for Γ_t. On the other hand, the continuity of the trace operator implies

$$\left\|\left[\mu^{\epsilon}_{M-\frac{1}{2}}\right]\right\|_{L^6(0,T_\epsilon;L^2(\Gamma_t))} \leq C\left(\left\|\mu^{+,\epsilon}_{M-\frac{1}{2}}\right\|_{L^6(0,T_\epsilon;H^1(\Omega^+(t)))}+\left\|\mu^{-,\epsilon}_{M-\frac{1}{2}}\right\|_{L^6(0,T_\epsilon;H^1(\Omega^-(t)))}\right)$$

and the continuity of the extension operator as discussed in Remark 5.33 implies

$$\left\|\left[\mu^{\epsilon}_{M-\frac{1}{2}}\right]\right\|_{L^6(0,T_\epsilon;L^2(\Gamma_t(2\delta)))} \leq C\left(\left\|\mu^{+,\epsilon}_{M-\frac{1}{2}}\right\|_{L^6(0,T_\epsilon;H^1(\Omega^+(t)))}+\left\|\mu^{-,\epsilon}_{M-\frac{1}{2}}\right\|_{L^6(0,T_\epsilon;H^1(\Omega^-(t)))}\right),$$

where $\left[\mu^{\epsilon}_{M-\frac{1}{2}}\right]=\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}$. Since (5.236) holds, the claim follows.

Ad 2) We have by definition of $B^{M-\frac{1}{2}}$ in (5.208)

$$\begin{aligned}B^{M-\frac{1}{2}} = {}&\partial_\rho c_0\left(\left(\frac{1}{2}\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}}+\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)+\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}}-\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)\left(\eta-\frac{1}{2}\right)\right)\cdot\mathbf{n}\right)\\
&+\partial_\rho c_0\left(-\mathbf{v}_0\cdot\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}-\partial_t^\Gamma h^\epsilon_{M-\frac{1}{2}}+\mathbf{w}_1^\epsilon|_\Gamma\cdot\mathbf{n}\right)-\left(l^\epsilon_{M-\frac{1}{2}}\right)\eta''\rho\\
&-\eta'\left(\Delta d_\Gamma\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)+2\partial_{\mathbf{n}}\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)+j_0h^\epsilon_{M-\frac{1}{2}}\right)\\
&+\eta''\left(-l^\epsilon_{M-\frac{1}{2}}h_1-h^\epsilon_{M-\frac{1}{2}}l_1\right)-2\partial_{\rho\rho}\mu_0\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\cdot\nabla^\Gamma h_1\\
&+\partial_\rho\mu_0\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}+2\nabla\partial_\rho\mu_0\cdot\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\end{aligned}$$

on $\Gamma(2\delta)$, where we used (5.254) and (5.253). This makes the decomposition on $\mathbb{R}\times\Gamma(2\delta)$ obvious if we note that by (5.105) we have

$$\nabla_x^i\partial_\rho^l\mu_0=\nabla_x^i([\mu_0])\,\partial_\rho^l\eta \text{ in } \mathbb{R}\times\Gamma(2\delta)\,, i\in\{0,1\}\,, l\in\{1,2\}$$

and it is again clear by the properties of $c_0=\theta_0$ and η that all terms $\mathtt{B}^2_k$ exhibit exponential decay.

Now for the decomposition on Γ: By (5.233) it holds

$$\begin{aligned}&\partial_t^\Gamma h^\epsilon_{M-\frac{1}{2}}-\frac{1}{2}\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}}+\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)\cdot\mathbf{n}\\
&\qquad-\mathbf{w}_1^\epsilon\cdot\mathbf{n}+\mathbf{v}_0\cdot\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}=\frac{1}{2}\left(\left(l_0\Delta d_\Gamma-j_0+\partial_{\mathbf{n}}l_0\right)h^\epsilon_{M-\frac{1}{2}}-\partial_{\mathbf{n}}\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)\right)\end{aligned}$$

on Γ and thus

$$\begin{aligned}B^{M-\frac{1}{2}} = {}&\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}}-\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)\cdot\mathbf{n}\left(\eta-\frac{1}{2}\right)\partial_\rho c_0+j_0h^\epsilon_{M-\frac{1}{2}}\left(\frac{1}{2}\partial_\rho c_0-\eta'\right)\\
&-\left(\eta'\Delta d_\Gamma\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)+\frac{1}{2}l_0\Delta d_\Gamma\partial_\rho c_0h^\epsilon_{M-\frac{1}{2}}\right)\\
&+\partial_{\mathbf{n}}\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)\left(\frac{1}{2}\partial_\rho c_0-\eta''\rho-2\eta'\right)+\partial_{\mathbf{n}}l_0h^\epsilon_{M-\frac{1}{2}}\left(-\eta''\rho-\frac{1}{2}\partial_\rho c_0\right)\\
&+\eta''\left(-l^\epsilon_{M-\frac{1}{2}}h_1-h^\epsilon_{M-\frac{1}{2}}l_1\right)-2\partial_{\rho\rho}\mu_0\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\cdot\nabla^\Gamma h_1+\partial_\rho\mu_0\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}\\
&+2\nabla\partial_\rho\mu_0\cdot\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\end{aligned}$$

on $\mathbb{R}\times\Gamma$, where we used the structure of $l^\epsilon_{M-\frac{1}{2}}$ on Γ as given in (5.219). Using $\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}} = -l_0 h^\epsilon_{M-\frac{1}{2}}$ on Γ due to (5.230), $\partial_{\rho\rho}\mu_0 = \partial_\rho\mu_0 = 0$ on Γ due to (5.105) and $\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\cdot\nabla\partial_\rho\mu_0 = \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\cdot\mathbf{n}l_0\eta' = 0$ on Γ also by (5.105), we arrive at

$$\begin{aligned}B^{M-\frac{1}{2}} &= \left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} - \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)\cdot\mathbf{n}\left(\eta(\rho)-\frac{1}{2}\right)\partial_\rho c_0 + h^\epsilon_{M-\frac{1}{2}}\left(j_0 - l_0\Delta d_\Gamma\right)\left(\frac{1}{2}\partial_\rho c_0 - \eta'\right)\\ &\quad + \partial_{\mathbf{n}}\left(\mu^{+,\epsilon}_{M-\frac{1}{2}} - \mu^{-,\epsilon}_{M-\frac{1}{2}}\right)\left(\frac{1}{2}\partial_\rho c_0 - \eta''\rho - 2\eta'\right) + \partial_{\mathbf{n}} l_0 h^\epsilon_{M-\frac{1}{2}}\left(-\eta''\rho - \frac{1}{2}\partial_\rho c_0\right)\\ &\quad + \left(-l^\epsilon_{M-\frac{1}{2}}h_1 - h^\epsilon_{M-\frac{1}{2}}l_1\right)\eta''\end{aligned}$$

on $\mathbb{R}\times\Gamma$. This implies the sought after decomposition if we set $\mathtt{B}^{1,\Gamma}_1 = \left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} - \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)\cdot\mathbf{n}$, $\mathtt{B}^{2,\Gamma}_1 = \left(\eta(\rho)-\frac{1}{2}\right)\partial_\rho c_0$, etc. As before the $\mathtt{B}^{2,\Gamma}_k$ terms show exponential decay. The integral over the $\mathtt{B}^{2,\Gamma}_k$ terms has exponential decay due to the properties of η (see Proposition 5.3) and c_0, since e.g.

$$\int_{\mathbb{R}}\left(\frac{1}{2}\partial_\rho c_0 - \eta''\rho - 2\eta'\right)\mathrm{d}\rho = 1 + \int_{\mathbb{R}}\eta\,\mathrm{d}\rho - 2 = 0,\quad \int_{\mathbb{R}}\eta''\mathrm{d}\rho = 0.$$

This implies (5.259).

The $L^2 - L^2$ estimate for the terms of kind $\mathtt{B}^{1,\Gamma}_k$ and $\mathtt{B}^1_k$ now follows from (5.235), (5.236) and the continuity of the trace operator $H^1(\Omega^\pm(t)) \to L^2(\Gamma_t)$ as well as from the continuity of the extension operators for $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}$ and $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$.

Ad 3) We use the definition of $\mathbf{V}^{M-\frac{1}{2}}$ in (5.209)

$$\begin{aligned}\mathbf{V}^{M-\frac{1}{2}} &= \eta'\left(2\left(\nabla\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} - \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)\right)^T\mathbf{n} - \left(p^{+,\epsilon}_{M-\frac{1}{2}} - p^{-,\epsilon}_{M-\frac{1}{2}}\right)\mathbf{n}\right)\\ &\quad + \eta'\left(\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} - \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)\Delta d_\Gamma - h^\epsilon_{M-\frac{1}{2}}\mathbf{q}_0\right) - 2\left(\nabla\partial_\rho\mathbf{v}_0\right)^T\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}} - \partial_\rho\mathbf{v}_0\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}\\ &\quad + \partial_\rho c_0\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}\eta + \mu^{-,\epsilon}_{M-\frac{1}{2}}(1-\eta)\right)\cdot\mathbf{n} - \partial_\rho c_0\mu_0\nabla^\Gamma h_{M-\frac{1}{2}} + \partial_\rho p_0\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\\ &\quad + 2\partial_{\rho\rho}\mathbf{v}_0\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\cdot\nabla^\Gamma h_1 + \eta''\left((\rho+h_1)\mathbf{u}^\epsilon_{M-\frac{1}{2}} + h^\epsilon_{M-\frac{1}{2}}\mathbf{u}_1\right),\end{aligned}$$

where we used (5.253), (5.255) and (5.254). Noting the matching conditions for $\mathbf{v}_0$, p_0, μ_0 as well as the properties of $c_0 = \theta_0$ and η, this representation immediately yields a decomposition into terms of kind $\mathtt{V}^1_k$ and $\mathtt{V}^2_k$, where $\mathtt{V}^2_k \in \mathcal{R}_\alpha$.

Now on Γ we observe that $\partial_\rho\mathbf{v}_0 = \partial_{\rho\rho}\mathbf{v}_0 = 0$ by (5.113), (5.117) and $\mu_0 = -\sigma\Delta d_\Gamma$ by (5.107). Since $\left[\mathbf{v}^\epsilon_{M-\frac{1}{2}}\right] = 0$ on Γ by (5.232) and $\operatorname{div}\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}} = 0$ by (5.227), we have $\left[\partial_{\mathbf{n}}\mathbf{v}^\epsilon_{M-\frac{1}{2}}\right] = 2\left[D_s\mathbf{v}^\epsilon_{M-\frac{1}{2}}\cdot\mathbf{n}\right]$ by Proposition 2.18 and thus using (5.231) for the structure of $\left[2D_s\mathbf{v}^\epsilon_{M-\frac{1}{2}}-p^\epsilon_{M-\frac{1}{2}}\right]\cdot\mathbf{n}$, we get

$$\begin{aligned}\mathbf{V}^{M-\frac{1}{2}}=&\left(\eta'+\eta''\rho\right)\left(\partial_{\mathbf{n}}\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}}-\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)+\partial_{\mathbf{n}}\mathbf{u}_0h^\epsilon_{M-\frac{1}{2}}\right)\\&+2\left(\eta'\left([\nabla\mathbf{v}_0]\right)^T-\left(\nabla\partial_\rho\mathbf{v}_0\right)^T\right)\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}+\left(\partial_\rho p_0-\eta'\left[p_0\right]\right)\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\\&+\left(\frac{1}{2}\partial_\rho c_0-\eta'\right)\left(2\sigma\left(\Delta d_\Gamma\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}+\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}\mathbf{n}\right)-g_0h_{M-\frac{1}{2}}\tilde{\eta}\mathbf{n}\right)\\&-\partial_\rho c_0\left(\eta-\frac{1}{2}\right)l_0+\eta''\left(h_1\mathbf{u}^\epsilon_{M-\frac{1}{2}}+h^\epsilon_{M-\frac{1}{2}}\mathbf{u}_1\right).\end{aligned}$$

Here we also used the structure of $\mathbf{u}^\epsilon_{M-\frac{1}{2}}$ on Γ as given in (5.223) and

$$\mu^{+,\epsilon}_{M-\frac{1}{2}}\eta+\mu^{-,\epsilon}_{M-\frac{1}{2}}\left(1-\eta\right)=\sigma\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}-g_0h_{M-\frac{1}{2}}\tilde{\eta}-l_0\left(\eta-\frac{1}{2}\right)$$

due to (5.230). Now we set $\mathtt{V}^{2,\Gamma}_1=\eta'+\eta''\rho$, $\mathtt{V}^{2,\Gamma}_2=\eta'\left([\nabla\mathbf{v}_0]\right)^T-\left(\nabla\partial_\rho\mathbf{v}_0\right)^T$, etc. and $\mathtt{V}^{1,\Gamma}_1=\left(\partial_{\mathbf{n}}\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}}-\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right)+\partial_{\mathbf{n}}\mathbf{u}_0h^\epsilon_{M-\frac{1}{2}}\right)$, $\mathtt{V}^{1,\Gamma}_2=2\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}$ etc. We have

$$\int_{\mathbb{R}}\eta'+\eta''\rho\mathrm{d}\rho=\int_{\mathbb{R}}\frac{1}{2}\partial_\rho c_0-\eta'\mathrm{d}\rho=\int_{\mathbb{R}}\partial_\rho c_0\left(\eta-\frac{1}{2}\right)\mathrm{d}\rho=\int_{\mathbb{R}}\eta''\mathrm{d}\rho=0$$

by the properties of η and θ_0 and all integrands have exponential decay. For $\tau_1,\tau_2>0$ large enough we also get

$$\begin{aligned}\left|\int_{-\tau_1}^{\tau_2}\partial_\rho p_0-\eta'\left[p_0\right]\mathrm{d}s\right|&\le\left|p_0\left(\tau_2,x,t\right)-p_0^+\left(x,t\right)\right|+\left|p_0\left(-\tau_1,x,t\right)-p_0^-\left(x,t\right)\right|\\&\le Ce^{-\alpha\min\{\tau_1,\tau_2\}}\end{aligned}$$

due to the matching condition for p_0 for all $(x,t)\in\Gamma$. The same argumentation holds for $\eta'\left([\nabla\mathbf{v}_0]\right)^T-\left(\nabla\partial_\rho\mathbf{v}_0\right)^T$, so we get (5.260).

Again the L^2-L^2 estimate for the terms of kind $\mathtt{V}^{1,\Gamma}_k$ and $\mathtt{V}^1_k$ follows from (5.235), (5.236), (5.223), the continuity of the trace operator $H^1\left(\Omega^\pm_t\right)\to L^2\left(\Gamma_t\right)$ and the properties of the extension operators for $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$ and $p^{\pm,\epsilon}_{M-\frac{1}{2}}$, see again Remark 5.33. □

Remark 5.37.

1. We will not construct terms of order $M+\frac{1}{2}$ as the right hand sides of the according ordinary differential equations (similar to (5.203)–(5.206)) would depend on derivatives of the kind $\partial^\Gamma_t h^\epsilon_{M-\frac{1}{2}}$ and $\Delta h^\epsilon_{M-\frac{1}{2}}$ among others. As a result, the already tenuous control (independent of ϵ) we have over the terms of order $M-\frac{1}{2}$ would only get worse for terms of order $M+\frac{1}{2}$. On the other hand, terms like $\Delta\mu_{M+\frac{1}{2}},\partial_t\mathbf{v}_{M+\frac{1}{2}}$, etc. would appear in the remainder and have to be estimated suitably, which the missing estimates prohibit.

2. Note that we may construct arbitrarily high integer orders of the expansion in the way described in Section 5.1 without knowledge of the expansion of fractional order $M - \frac{1}{2}$. This is true as long as we do not consider products of fractional order terms in the right hand sides of the ordinary differential equations (5.40), (5.42), (5.44), (5.46) (and also in the outer equations) which – taken together – yield an integer order of ϵ. In other words: all higher order terms, which come up due to the construction of the fractional order of the expansion, appear in the remainder terms (cf. Section 6.1) and are not taken into account when constructing the integer orders of the expansion.
Nevertheless, if one wanted to include the higher order terms appearing due to the fractional order in the construction of the integer order terms, the following should be considered: in order to appear in A^k, B^k, $\mathbf{V}^k$ or W^k for an integer k, (at least) two terms of fractional order need to be multiplied; the lowest integer order of the expansion which would be influenced by such a product is $2M - 1$ (which can be readily verified by considering (5.28)–(5.31)). As (in our case) $M \geq 4$, we may construct expansions up to order $M + 1$ without having to worry about the appearance of fractional order terms.

6. Estimates for the Remainder

It is of great interest to what extent the constructed approximate solutions "solve" the original system (1.18)–(1.25). More precisely, it is important to show that the remainder terms $\mathbf{r}_{\mathrm{S}}^{\epsilon}$, $r_{\mathrm{div}}^{\epsilon}$, $r_{\mathrm{CH1}}^{\epsilon}$, $r_{\mathrm{CH2}}^{\epsilon}$, which are given as in (4.7)–(4.10), are perturbations of a high order of ϵ when considered in suitable norms. This will be a key element in proving the main result, Theorem 4.1, as this is a guarantee that the approximate solutions really approximate the original equations properly. The exact statement of the result we need is given in Theorem 6.12, which is also the fundamental result of this chapter.

This chapter is made up of three sections: first we properly define the approximate solutions by gluing together the constructed inner, outer and boundary terms and take a closer look at the structure of the remainder terms. This structural knowledge allows for auxiliary results to be shown in the next segment. These will significantly shorten the amount of work we have to invest in the last third, where we show the main result of this part, Theorem 6.12. Taken together, the definition of the approximate solutions in Definition 6.2 and the estimate for the remainder terms in Theorem 6.12 prove the main result for the approximate solutions, Theorem 4.3.

As in Chapter 5 we assume that Assumption 1.1 holds throughout this chapter. Moreover, we introduce some additional notation, which will considerably improve the readability of this chapter.

Notation 6.1. Let $k \in \mathbb{N}$ and $q \in \mathbb{R}$. Then we define

$$I_q^k := \{0, \ldots, k\} \cup \{q\}. \tag{6.1}$$

The following definition is central to this work.

Definition 6.2 (The approximate solutions). Let $\epsilon \in (0,1)$, let Assumption 1.1 hold, let $\mathfrak{S}_0, \ldots, \mathfrak{S}_{M+1}$ be given as in Lemmata 5.19 and 5.22 and let $\mathfrak{S}_{M-\frac{1}{2}}$ be given as in Notation 5.35. Furthermore, let ξ be the cut-off function from Definition 2.1. We define

$$h_A^{\epsilon}(s,t) := \sum_{i \in I_{M-\frac{3}{2}}^{M}} \epsilon^i h_{i+1}(s,t)$$

for $(s,t) \in \mathbb{T}^1 \times [0, T_0]$ and

$$\begin{aligned} \rho(x,t) &:= \frac{d_\Gamma(x,t)}{\epsilon} - h_A^{\epsilon}(S(x,t),t), \\ z(x,t) &:= \frac{d_{\mathbf{B}}(x,t)}{\epsilon}. \end{aligned}$$

1. We define the **inner solutions** as

$$c_I(x,t) := \sum_{i=0}^{M+1} \epsilon^i c_i(\rho(x,t),x,t),$$
$$\mu_I(x,t) := \sum_{i \in I_{M-\frac{1}{2}}^{M+1}} \epsilon^i \mu_i(\rho(x,t),x,t),$$
$$\mathbf{v}_I(x,t) := \sum_{i \in I_{M-\frac{1}{2}}^{M+1}} \epsilon^i \mathbf{v}_i(\rho(x,t),x,t),$$
$$p_I(x,t) := \sum_{i \in I_{M-\frac{1}{2}}^{M}} \epsilon^i p_i(\rho(x,t),x,t),$$

for all $(x,t) \in \Gamma(2\delta)$ and write

$$c_{I,k}(x,t) := c_k(\rho(x,t),x,t) \qquad \forall (x,t) \in \Gamma(2\delta), \tag{6.2}$$

and similarly $\mu_{I,k}$ etc.

2. We define the **outer solutions** as

$$c_O(x,t) := \sum_{i=0}^{M+1} \epsilon^i \left(c_i^+(x,t)\chi_{\overline{\Omega_{T_0}^+}}(x,t) + c_i^-(x,t)\chi_{\Omega_{T_0}^-}(x,t) \right),$$
$$\mu_O(x,t) := \sum_{i \in I_{M-\frac{1}{2}}^{M+1}} \epsilon^i \left(\mu_i^+(x,t)\chi_{\overline{\Omega_{T_0}^+}}(x,t) + \mu_i^-(x,t)\chi_{\Omega_{T_0}^-}(x,t) \right),$$
$$\mathbf{v}_O(x,t) := \sum_{i \in I_{M-\frac{1}{2}}^{M+1}} \epsilon^i \left(\mathbf{v}_i^+(x,t)\chi_{\overline{\Omega_{T_0}^+}}(x,t) + \mathbf{v}_i^-(x,t)\chi_{\Omega_{T_0}^-}(x,t) \right),$$
$$p_O(x,t) := \sum_{i \in I_{M-\frac{1}{2}}^{M}} \epsilon^i \left(p_i^+(x,t)\chi_{\overline{\Omega_{T_0}^+}}(x,t) + p_i^-(x,t)\chi_{\Omega_{T_0}^-}(x,t) \right),$$

for $(x,t) \in \Omega_{T_0}$ and write

$$c_{O,k}(x,t) := c_k^+(x,t)\chi_{\overline{\Omega^+}}(x,t) + c_k^-(x,t)\chi_{\Omega^-}(x,t), \qquad \forall (x,t) \in \Omega_{T_0}, \tag{6.3}$$

and similarly $\mu_{O,k}$ etc.

3. We define the **boundary solutions** as

$$c_{\mathbf{B}}(x,t) := -1 + \sum_{i=1}^{M+1} \epsilon^i c_i^{\mathbf{B}}(z(x,t),x,t),$$
$$\mu_{\mathbf{B}}(x,t) := \sum_{i \in I_{M-\frac{1}{2}}^{M+1}} \epsilon^i \mu_i^{\mathbf{B}}(z(x,t),x,t),$$
$$\mathbf{v}_{\mathbf{B}}(x,t) := \sum_{i \in I_{M-\frac{1}{2}}^{M+1}} \epsilon^i \mathbf{v}_i^{\mathbf{B}}(z(x,t),x,t) - \epsilon^{M+1}\mathbf{v}_{M+1}^{\mathbf{B}}(0,x,t),$$
$$p_{\mathbf{B}}(x,t) := \sum_{i \in I_{M-\frac{1}{2}}^{M}} \epsilon^i p_i^{\mathbf{B}}(z(x,t),x,t),$$

for $(x,t) \in \overline{\partial_{T_0}\Omega(\delta)}$, where we set

$$\mu^{\mathbf{B}}_{M-\frac{1}{2}}(z,x,t) := \mu^{-}_{M-\frac{1}{2}}(x,t)\,, \mathbf{v}^{\mathbf{B}}_{M-\frac{1}{2}}(z,x,t) := \mathbf{v}^{-}_{M-\frac{1}{2}}(x,t)\,,$$
$$p^{\mathbf{B}}_{M-\frac{1}{2}}(z,x,t) := p^{-}_{M-\frac{1}{2}}(x,t) \tag{6.4}$$

for $(z,x,t) \in (-\infty,0] \times \overline{\partial_{T_0}\Omega(\delta)}$ and write

$$c_{\mathbf{B},k}(x,t) := c^{\mathbf{B}}_{k}(z(x,t),x,t) \qquad (x,t) \in \overline{\partial_{T_0}\Omega(\delta)} \tag{6.5}$$

and similarly $\mu_{\mathbf{B},k}$ etc. Here we include the peculiarity in $\mathbf{v}_{\mathbf{B}}$, i.e. we write

$$\mathbf{v}_{\mathbf{B},M+1}(x,t) = \mathbf{v}^{\mathbf{B}}_{M+1}(z(x,t),x,t) - \mathbf{v}^{\mathbf{B}}_{M+1}(0,x,t)\,.$$

4. We define the **approximate solutions**

$$c^{\epsilon}_{A} := \xi(d_\Gamma)\, c_I + (1-\xi(d_\Gamma))(1-\xi(2d_{\mathbf{B}}))\, c_O + \xi(2d_{\mathbf{B}})\, c_{\mathbf{B}},$$
$$\mu^{\epsilon}_{A} := \xi(d_\Gamma)\, \mu_I + (1-\xi(d_\Gamma))(1-\xi(2d_{\mathbf{B}}))\, \mu_O + \xi(2d_{\mathbf{B}})\, \mu_{\mathbf{B}},$$
$$\mathbf{v}^{\epsilon}_{A} := \xi(d_\Gamma)\, \mathbf{v}_I + (1-\xi(d_\Gamma))(1-\xi(2d_{\mathbf{B}}))\, \mathbf{v}_O + \xi(2d_{\mathbf{B}})\, \mathbf{v}_{\mathbf{B}},$$
$$p^{\epsilon}_{A} := \xi(d_\Gamma)\, p_I + (1-\xi(d_\Gamma))(1-\xi(2d_{\mathbf{B}}))\, p_O + \xi(2d_{\mathbf{B}})\, p_{\mathbf{B}}, \tag{6.6}$$

in Ω_{T_0} and write

$$c_{A,k}(x,t) := \xi(d_\Gamma)\, c_{I,k} + (1-\xi(d_\Gamma))(1-\xi(2d_{\mathbf{B}}))\, c_{O,k} + \xi(2d_{\mathbf{B}})\, c_{\mathbf{B},k} \tag{6.7}$$

for all $(x,t) \in \Omega_{T_0}$ and similarly $\mu_{A,k}$ etc.

This definition implies in particular $\mu_{A,M-\frac{1}{2}} = \xi(d_\Gamma)\, \mu_{I,M-\frac{1}{2}} + (1-\xi(d_\Gamma))\, \mu_{O,M-\frac{1}{2}}$ and a similar structure for $\mathbf{v}_{A,M-\frac{1}{2}}, p_{A,M-\frac{1}{2}}$.

In the following, we will need the boundedness of $h^{\epsilon}_{M-\frac{1}{2}}$ (cf. Theorem 5.32 for existence) and thus work under the following assumptions.

Assumption 6.3. *Throughout this chapter we assume that Assumption 4.2 holds for $c_A = c^{\epsilon}_{A}$ and $\epsilon_0 \in (0,1)$, the family $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)}$ and $K \geq 1$. Moreover, we assume $\epsilon_1 \in (0,\epsilon_0]$ is given as in Theorem 5.32 2) and such that (5.191) holds for $\tilde{\mathbf{w}}^{\epsilon}_{1}$, the weak solution to (5.174)–(5.176) with $H = \left(h^{\epsilon}_{M-\frac{1}{2}}\right)_{\epsilon\in(0,\epsilon_0)}$.*

Note in particular that the assumptions of Lemma 5.27 are satisfied in this situation and we may thus access the results of Chapter 3.

Remark 6.4.

1. There is some $C > 0$ such that

$$\|\nabla c^{\epsilon}_{A}\|_{L^\infty\left(\Omega_{T_0}\setminus\Gamma(2\delta)\right)} \leq C\epsilon \tag{6.8}$$

for all $\epsilon \in (0,1)$ small enough. This is the case, since $c^{\pm}_{0} = \pm 1$ in $\Omega^{\pm}_{T_0}$ (cf. (5.9)) and since $c^{\mathbf{B}}_{0} = -1$ and $c^{\mathbf{B}}_{1} = c^{-}_{1}$ in $\overline{\partial_{T_0}\Omega(\delta)}$ due to Corollary (5.18). Thus, $\nabla c_O = \epsilon \nabla c^{\pm}_{1} + \mathcal{O}(\epsilon^2)$ in $L^\infty\left(\Omega^{\pm}_{T_0}\setminus\Gamma(2\delta)\right)$ and $\nabla c_{\mathbf{B}}(x,t) = \epsilon\nabla c^{-}_{1}(x,t) + \epsilon\partial_z c^{\mathbf{B}}_{2}(z(x,t),x,t)\,\mathbf{n}_{\partial\Omega}(x) + \mathcal{O}(\epsilon^2)$ for $(x,t) \in \partial_{T_0}\Omega(\delta)$.

2. It holds
$$\|h_A^\epsilon\|_{C^0(0,T_\epsilon;C^1(\Gamma_t(2\delta)))} \leq C(K) \tag{6.9}$$
for some $C(K) > 0$ and all $\epsilon \in (0, \epsilon_1)$. This is a consequence of the uniform boundedness of h_k, $k \in \{1, \dots, M+1\}$, and (5.235) for $h^\epsilon_{M-\frac{1}{2}}$ since $X_{T_\epsilon} \hookrightarrow C^0\left([0, T_\epsilon]; C^1\left(\mathbb{T}^1\right)\right)$ by Proposition 2.34 and the Sobolev embedding theorem.

3. The choice $\mu^{\mathbf{B}}_{M-\frac{1}{2}}(z, x, t) := \mu^-_{M-\frac{1}{2}}(x, t)$ (and also for the other $\left(M - \frac{1}{2}\right)$–th terms) in Definition 6.2 reflects the fact that we do not need an explicit boundary layer expansion for the $\left(M - \frac{1}{2}\right)$–th order, as $c^\pm_{M-\frac{1}{2}}$ vanishes anyway and the other boundary data may be explicitly prescribed.

6.1. The Structure of the Remainder Terms

6.1.1. The Inner Remainder Terms

In the following, let Assumption 6.3 hold and we work under the notations and assumptions of Definition 6.2. We now analyze up to which order the equations (1.18)–(1.21) are fulfilled by the inner solutions $c_I, \mu_I, \mathbf{v}_I, p_I$. For this we use the ordinary differential equations satisfied by $(c_k, \mu_k, \mathbf{v}_k, p_{k-1})$ for $k \in \{0, \ldots, M+1\}$ as constructed for the inner terms and evaluate them at

$$\rho(x,t) = \frac{d_\Gamma(x,t)}{\epsilon} - h_A^\epsilon(S(x,t),t) \tag{6.10}$$

for $(x,t) \in \Gamma(2\delta; T_\epsilon)$ and $\epsilon \in (0, \epsilon_1)$. Before we give the explicit formula, note that there is $\epsilon_2 \in (0, \epsilon_1]$ such that for all $\epsilon \in (0, \epsilon_2)$ we have

$$\left| \sum_{i \in I^M_{M-\frac{3}{2}}} h_{i+1}\epsilon^i - h_1 \right| \leq 1$$

due to (5.235). Thus, (5.55) is satisfied and using Remark 5.5 we get

$$\epsilon^2 \left(U^+ \eta^{C_S,+} + U^- \eta^{C_S,-}\right)|_{\rho = \frac{d_\Gamma}{\epsilon} - h_A^\epsilon} = \epsilon^2 \left(\mathbf{W}^+ \eta^{C_S,+} + \mathbf{W}^- \eta^{C_S,-}\right)|_{\rho = \frac{d_\Gamma}{\epsilon} - h_A^\epsilon} = 0.$$

Let in the following $\epsilon \in (0, \epsilon_2)$. Using the inner equations derived in Chapter 5 we get

$$\begin{aligned}
&\partial_t c_I + \mathbf{v}_I \cdot \nabla c_I + \epsilon^{M-\frac{1}{2}} \mathbf{w}_1^\epsilon|_\Gamma \cdot \nabla c_I - \Delta \mu_I \\
&= \epsilon^M \left(\partial_\rho c_{M+1} \partial_t d_\Gamma - \partial_\rho \mu_{M+1} \Delta d_\Gamma - 2\nabla \partial_\rho \mu_{M+1} \cdot \mathbf{n} - j_M \eta' \rho - l_{M+1} \eta'' \rho\right) \\
&\quad + \epsilon^{M-\frac{1}{2}} \mathbf{w}_1^\epsilon|_\Gamma \cdot \left(\sum_{i=1}^{M+1} \epsilon^{i-1} \partial_\rho c_i \mathbf{n} + \left(\sum_{i=0}^{M+1} - \left(\sum_{j \in I^M_{M-\frac{3}{2}}} \epsilon^{i+j} \partial_\rho c_i \nabla^\Gamma h_{j+1} \right) + \epsilon^i \nabla c_i \right)\right) \\
&\quad - \sum_{\substack{0 \leq i \leq M+1 \\ j \in I^M_{M-\frac{3}{2}} \\ i+j \geq M-\frac{1}{2}}} \epsilon^{i+j} \partial_\rho c_i \partial_t^\Gamma h_{j+1} + \sum_{\substack{i \in I^{M+1}_{M-\frac{1}{2}}, j \in I^M_{M-\frac{3}{2}} \\ i+j \geq M-\frac{1}{2}}} \epsilon^{i+j} \left(2\nabla \partial_\rho \mu_i \cdot \nabla^\Gamma h_{j+1} + \partial_\rho \mu_i \Delta^\Gamma h_{j+1}\right) \\
&\quad + \sum_{i \in I^{M+1}_{M-\frac{1}{2}}} \mathbf{v}_i \cdot \left(\frac{1}{\epsilon} \sum_{\substack{0 \leq j \leq M+1 \\ i+j \geq M+\frac{1}{2}}} \left(\epsilon^{i+j} \partial_\rho c_j \mathbf{n}\right) - \sum_{\substack{0 \leq j \leq M+1 \\ l \in I^M_{M-\frac{3}{2}} \\ i+j+l \geq M-\frac{1}{2}}} \left(\epsilon^{i+j+l} \partial_\rho c_j \nabla^\Gamma h_{l+1}\right) \right) \\
&\quad - \sum_{\substack{i \in I^{M+1}_{M-\frac{1}{2}}, j,l \in I^M_{M-\frac{3}{2}} \\ i+j+l \geq M-\frac{1}{2}}} \left(\epsilon^{i+j+l} \partial_{\rho\rho} \mu_i \nabla^\Gamma h_{j+1} \cdot \nabla^\Gamma h_{l+1}\right) - \frac{1}{\epsilon} \sum_{\substack{i \in I^{M+1}_{M-\frac{1}{2}}, j \in I^M_{M-\frac{3}{2}} \\ i+j \geq M+\frac{1}{2}}} \left(\epsilon^{i+j} l_i \eta'' h_{j+1}\right) \\
&\quad - \sum_{\substack{0 \leq i \leq M \\ k \in I^M_{M-\frac{3}{2}} \\ i+j \geq M-\frac{1}{2}}} \left(\epsilon^{i+k} j_i \eta' h_{k+1}\right) + \sum_{i=M}^{M+1} \epsilon^i \left(\partial_t c_i - \Delta \mu_i\right) - \epsilon^{M-\frac{1}{2}} \Delta \mu_{M-\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}}\\ 0\le j\le M\\ i+j\ge M-\frac{1}{2}}} \epsilon^{i+j}\mathbf{v}_i\cdot\nabla c_j + \epsilon^{M-\frac{3}{2}}B^{M-\frac{1}{2}}\\
&=: r^{\epsilon}_{\mathrm{CH1},I},
\end{aligned}
\tag{6.11}
$$

in $\Gamma(2\delta;T_\epsilon)$, where $\mathbf{w}^{\epsilon}_1$ is given as in Theorem 5.32. We also get

$$
\begin{aligned}
&\epsilon\Delta c^I - \epsilon^{-1}f'\left(c^I\right) + \mu^I\\
&\quad = \epsilon^{M+1}\Big(\partial_\rho c_{M+1}\Delta d_\Gamma + 2\nabla\partial_\rho c_{M+1}\cdot\mathbf{n} + g_M\eta'\rho - \tilde{f}_{M+1}\left(c_0,\ldots,c_{M+1}\right)\Big)\\
&\quad\quad - \epsilon \sum_{\substack{0\le i\le M+1\\ j\in I^M_{M-\frac{3}{2}}\\ i+j\ge M-\frac{1}{2}}} \epsilon^{i+j}\left(\partial_\rho c_i\Delta^\Gamma h_{j+1} + 2\nabla\partial_\rho c_i\cdot\nabla^\Gamma h_{j+1}\right)\\
&\quad\quad + \epsilon \sum_{\substack{0\le i\le M\\ j,l\in I^M_{M-\frac{3}{2}}\\ i+j+l\ge M-\frac{1}{2}}} \epsilon^{i+j+l}\partial_{\rho\rho}c_i\nabla^\Gamma h_{j+1}\cdot\nabla^\Gamma h_{l+1} + \sum_{\substack{0\le i\le M\\ j\in I^M_{M-\frac{3}{2}}\\ i+j\ge M+\frac{1}{2}}} \epsilon^{i+j}g_i\eta' h_{j+1}\\
&\quad\quad + \epsilon\sum_{i=M}^{M+1}\epsilon^i\Delta c_i + \epsilon^{M+1}\mu_{M+1} - \epsilon^{M-\frac{1}{2}}A^{M-\frac{1}{2}}\\
&\quad =: r^{\epsilon}_{\mathrm{CH2},I}
\end{aligned}
\tag{6.12}
$$

in $\Gamma(2\delta;T_\epsilon)$, where $\tilde{f}_{M+1}(c_0,\ldots,c_{M+1})$ is defined as in Notation 5.4 (resp. Remark 5.1 1)). Furthermore,

$$
\begin{aligned}
\mathrm{div}\mathbf{v}_I &= \epsilon^{M+1}\mathrm{div}\mathbf{v}_{M+1} - \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}},j\in I^M_{M-\frac{3}{2}}\\ i+j\ge M+\frac{1}{2}}} \epsilon^{i+j}\partial_\rho\mathbf{v}_i\cdot\nabla^\Gamma h_{j+1} + \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}},j\in I^M_{M-\frac{3}{2}}\\ i+j\ge M+\frac{1}{2}}} \epsilon^{i+j}\mathbf{u}_i\cdot\mathbf{n}\eta' h_{j+1}\\
&\quad + \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}},j\in I^M_{M-\frac{3}{2}}\\ i+j\ge M+\frac{1}{2}}} \epsilon^{i+j}\mathbf{u}_i\cdot\nabla^\Gamma h_{j+1}\eta' d_\Gamma - \epsilon \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}},j\in I^M_{M-\frac{3}{2}}\\ i+j\ge M-\frac{1}{2}}} \epsilon^{i+j}\mathbf{u}_i\cdot\nabla^\Gamma h_{j+1}\eta'\rho\\
&\quad - \epsilon \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}},j,k\in I^M_{M-\frac{3}{2}}\\ i+j+k\ge M-\frac{1}{2}}} \epsilon^{i+j+k}\mathbf{u}_i\cdot\nabla^\Gamma h_{j+1}h_{k+1}\eta' - \epsilon^{M-\frac{1}{2}}W^{M-\frac{1}{2}}\\
&=: r^{\epsilon}_{\mathrm{div},I}
\end{aligned}
\tag{6.13}
$$

and

$$
\begin{aligned}
&-\Delta\mathbf{v}_I + \nabla p_I - \mu_I\nabla c_I\\
&\quad = \epsilon^M\Big(-\partial_\rho\mathbf{v}_{M+1}\Delta d_\Gamma - 2\left(\nabla\partial_\rho\mathbf{v}_{M+1}\right)^T\mathbf{n} + \mathbf{q}_M\eta'\rho - \mathbf{u}_{M+1}\eta''\rho\Big)\\
&\quad\quad - \frac{1}{\epsilon}\sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}}\\ 0\le j\le M+1\\ i+j\ge M+\frac{1}{2}}} \epsilon^{i+j}\mu_i\partial_\rho c_j\mathbf{n} - \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}},j,l\in I^M_{M-\frac{3}{2}}\\ i+j+l\ge M-\frac{1}{2}}} \epsilon^{i+j+l}\partial_{\rho\rho}\mathbf{v}_i\nabla^\Gamma h_{j+1}\nabla^\Gamma h_{l+1}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{i\in I^M_{M-\frac{1}{2}}, j\in I^M_{M-\frac{3}{2}}\\ i+j\geq M-\frac{1}{2}}} \epsilon^{i+j}\partial_\rho p_i \nabla^\Gamma h_{j+1} + \sum_{\substack{0\leq j\leq M+1\\ i\in I^{M+1}_{M-\frac{1}{2}}, l\in I^M_{M-\frac{3}{2}}\\ i+j+l\geq M-\frac{1}{2}}} \epsilon^{i+j+l}\mu_i \partial_\rho c_j \nabla^\Gamma h_{l+1}\\
&+ \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}}, j\in I^M_{M-\frac{3}{2}}\\ i+j\geq M-\frac{1}{2}}} \epsilon^{i+j}\left(\partial_\rho \mathbf{v}_i \Delta^\Gamma h_{j+1} + 2\left(\nabla\partial_\rho \mathbf{v}_i\right)^T \nabla^\Gamma h_{j+1}\right)\\
&+ \frac{1}{\epsilon}\left(- \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}}, j\in I^M_{M-\frac{3}{2}}\\ i+j\geq M+\frac{1}{2}}} \epsilon^{i+j}\mathbf{u}_i \eta'' h_{j+1} + \sum_{\substack{0\leq i\leq M\\ j\in I^M_{M-\frac{3}{2}}\\ i+j\geq M-\frac{1}{2}}} \epsilon^{i+j+1}\mathbf{q}_i \eta' h_{j+1}\right)\\
&- \sum_{i=M}^{M+1} \epsilon^i \Delta \mathbf{v}_i + \epsilon^M \nabla p_M - \epsilon^{M-\frac{1}{2}}\left(\Delta \mathbf{v}_{M-\frac{1}{2}} - \nabla p_{M-\frac{1}{2}}\right)\\
&- \sum_{\substack{0\leq j\leq M\\ i\in I^{M+1}_{M-\frac{1}{2}}\\ i+j\geq M-\frac{1}{2}}} \epsilon^{i+j}\mu_i \nabla c_j - \epsilon^{M-\frac{3}{2}}\mathbf{V}^{M-\frac{1}{2}}\\
&=: \mathbf{r}^\epsilon_{\mathrm{S},I} \qquad (6.14)
\end{aligned}
$$

in $\Gamma(2\delta; T_\epsilon)$.

6.1.2. The Outer Remainder Terms

By definition of the outer terms and the outer equations considered in Chapter 5 we get the identities

$$
\begin{aligned}
\partial_t c_O + \mathbf{v}_O \cdot \nabla c_O - \Delta \mu_O &= \epsilon^{M+\frac{1}{2}} \mathbf{v}_{O,M-\frac{1}{2}} \cdot \nabla c_{O,1} + \sum_{\substack{i\in I^{M+1}_{M-\frac{1}{2}}\\ 0\leq j\leq M+1\\ i+j\geq M+\frac{3}{2}}} \epsilon^{i+j}\mathbf{v}_{O,i}\cdot \nabla c_{O,j}\\
&=: r^\epsilon_{\mathrm{CH1},O} \qquad (6.15)
\end{aligned}
$$

and

$$
\begin{aligned}
\epsilon \Delta c_O - \epsilon^{-1} f'(c_O) + \mu_O &= \epsilon \sum_{i=M}^{M+1} \epsilon^i \Delta c_{O,i} - \epsilon^{M+1}\tilde{f}_{M+1}\left(c_0^\pm, \ldots, c_{M+1}^\pm\right) + \epsilon^{M+1}\mu_{O,M+1}\\
&\quad + \epsilon^{M-\frac{1}{2}}\mu_{O,M-\frac{1}{2}}\\
&=: r^\epsilon_{\mathrm{CH2},O} \qquad (6.16)
\end{aligned}
$$

in $\Omega^+_{T_0} \cup \Omega^-_{T_0}$ (see Remark 5.1 1) for the definition of $\tilde{f}_{M+1}\left(c_0^\pm, \dots, c_{M+1}^\pm\right)$). Furthermore,

$$-\Delta \mathbf{v}_O + \nabla p_O - \mu_O \nabla c_O = -\epsilon^{M+\frac{1}{2}} \mu_{O,M-\frac{1}{2}} \nabla c_{O,1} - \sum_{\substack{i \in I^{M+1}_{M-\frac{1}{2}} \\ 0 \le j \le M+1 \\ i+j \ge M+\frac{3}{2}}} \epsilon^{i+j} \mu_{O,i} \nabla c_{O,j}$$
$$=: \mathbf{r}^\epsilon_{\mathrm{S},O} \tag{6.17}$$

and

$$\mathrm{div} \mathbf{v}_O = 0 =: r^\epsilon_{\mathrm{div},O} \tag{6.18}$$

in $\Omega^+_{T_0} \cup \Omega^-_{T_0}$.

6.1.3. The Boundary Remainder Terms

To gain the form of the remainder close to the boundary of Ω, we consider the ordinary differential equations (5.86)–(5.89) satisfied by $\left(c_k^{\mathbf{B}}, \mu_k^{\mathbf{B}}, \mathbf{v}_k^{\mathbf{B}}, p_{k-1}^{\mathbf{B}}\right)$ for $k \in \{0, \dots, M+1\}$ and evaluated at

$$z(x,t) = \frac{d_{\mathbf{B}}(x,t)}{\epsilon}$$

for $(x,t) \in \overline{\partial_{T_0}\Omega(\delta)}$ and $\epsilon \in (0, \epsilon_1)$. Furthermore, we may use the outer equations as discussed in (5.225)–(5.227) for $\left(\mu^{\mathbf{B}}_{M-\frac{1}{2}}, \mathbf{v}^{\mathbf{B}}_{M-\frac{1}{2}}, p^{\mathbf{B}}_{M-\frac{1}{2}}\right)$. Then we find

$$\begin{aligned}
&\partial_t c_{\mathbf{B}} + \mathbf{v}_{\mathbf{B}} \cdot \nabla c_{\mathbf{B}} - \Delta \mu_{\mathbf{B}} \\
&\quad= \epsilon^M \left(-2\partial_z \nabla \mu^{\mathbf{B}}_{M+1} \cdot \nabla d_{\mathbf{B}} - \partial_z \mu^{\mathbf{B}}_{M+1} \Delta d_{\mathbf{B}}\right) + \frac{1}{\epsilon} \sum_{\substack{0 \le i,j \le M+1 \\ i+j \ge M+1}} \epsilon^{i+j} \mathbf{v}_i^{\mathbf{B}} \cdot \nabla d_{\mathbf{B}} \partial_z c_j^{\mathbf{B}} \\
&\qquad + \sum_{i=M}^{M+1} \epsilon^i \left(\partial_t c_i^{\mathbf{B}} - \Delta \mu_i^{\mathbf{B}}\right) + \sum_{\substack{0 \le i,j \le M+1 \\ i+j \ge M}} \epsilon^{i+j} \mathbf{v}_i^{\mathbf{B}} \cdot \nabla c_j^{\mathbf{B}} \\
&\qquad - \epsilon^{M+1} \mathbf{v}^{\mathbf{B}}_{M+1}|_{z=0} \cdot \sum_{1 \le j \le M+1} \left(\epsilon^j \nabla c_j^{\mathbf{B}} + \epsilon^{j-1} \nabla d_{\mathbf{B}} \partial_z c_j^{\mathbf{B}}\right) \\
&\quad + \epsilon^{M+\frac{1}{2}} \sum_{1 \le j \le M+1} \epsilon^{j-1} \mathbf{v}^-_{M-\frac{1}{2}} \cdot \nabla c_j^{\mathbf{B}} + \epsilon^{M+\frac{1}{2}} \sum_{2 \le j \le M+1} \epsilon^{j-2} \mathbf{v}^-_{M-\frac{1}{2}} \cdot \nabla d_{\mathbf{B}} \partial_z c_j^{\mathbf{B}} \\
&\quad =: r^\epsilon_{\mathrm{CH1},\mathbf{B}},
\end{aligned} \tag{6.19}$$

where we used $\partial_z c_0^{\mathbf{B}} = \partial_z c_1^{\mathbf{B}} = 0$, see Corollary 5.18. Moreover, we calculate

$$\begin{aligned}
\epsilon \Delta c_{\mathbf{B}} - \epsilon^{-1} f'(c_{\mathbf{B}}) + \mu_{\mathbf{B}} &= \epsilon^{M+1} \left(\mu^{\mathbf{B}}_{M+1} + 2\partial_z \nabla c^{\mathbf{B}}_{M+1} \cdot \nabla d_{\mathbf{B}} + \partial_z c^{\mathbf{B}}_{M+1} \Delta d_{\mathbf{B}}\right) \\
&\quad + \epsilon \sum_{i=M}^{M+1} \epsilon^i \Delta c_i^{\mathbf{B}} - \epsilon^{M+1} \tilde{f}_{M+1}\left(c_0^{\mathbf{B}}, \dots, c^{\mathbf{B}}_{M+1}\right) + \epsilon^{M-\frac{1}{2}} \mu^-_{M-\frac{1}{2}} \\
&=: r^\epsilon_{\mathrm{CH2},\mathbf{B}},
\end{aligned} \tag{6.20}$$

$$
\begin{aligned}
-\Delta \mathbf{v}_{\mathbf{B}} + \nabla p_{\mathbf{B}} - \mu_{\mathbf{B}} \nabla c_{\mathbf{B}} = {} & \epsilon^{M} \left(-2\partial_z D \mathbf{v}^{\mathbf{B}}_{M+1} \cdot \nabla d_{\mathbf{B}} - \partial_z \mathbf{v}^{\mathbf{B}}_{M+1} \right) - \frac{1}{\epsilon} \sum_{\substack{0 \le i,j \le M+1 \\ i+j \ge M+1}} \mu^{\mathbf{B}}_i \partial_z c^{\mathbf{B}}_j \nabla d_{\mathbf{B}} \\
& - \sum_{i=M}^{M+1} \epsilon^i \left(\Delta \mathbf{v}^{\mathbf{B}}_i \right) + \epsilon^M \nabla p^{\mathbf{B}}_M - \sum_{\substack{0 \le i,j \le M+1 \\ i+j \ge M}} \epsilon^{i+j} \mu^{\mathbf{B}}_i \nabla c^{\mathbf{B}}_j \\
& + \epsilon^{M+1} \Delta \mathbf{v}^{\mathbf{B}}_{M+1}|_{z=0} - \epsilon^{M+\frac{1}{2}} \mu^{-}_{M-\frac{1}{2}} \sum_{1 \le j \le M+1} \epsilon^{j-1} \nabla c^{\mathbf{B}}_j \\
& - \epsilon^{M+\frac{1}{2}} \mu^{-}_{M-\frac{1}{2}} \sum_{2 \le j \le M+1} \epsilon^{j-2} \nabla d_{\mathbf{B}} \partial_z c^{\mathbf{B}}_j && (6.21) \\
=: {} & \mathbf{r}^{\epsilon}_{\mathrm{S},\mathbf{B}} && (6.22)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{div} \mathbf{v}_{\mathbf{B}} &= \epsilon^{M+1} \left(\mathrm{div} \mathbf{v}^{\mathbf{B}}_{M+1} - \mathrm{div} \mathbf{v}^{\mathbf{B}}_{M+1}|_{z=0} \right) \\
&=: r^{\epsilon}_{\mathrm{div},\mathbf{B}} && (6.23)
\end{aligned}
$$

in $\overline{\partial_{T_0} \Omega(\delta)}$. Moreover, the boundary conditions

$$
\begin{aligned}
\mu_{\mathbf{B}} &= 0, && (6.24) \\
c_{\mathbf{B}} &= -1, && (6.25) \\
\left(-2D_s \mathbf{v}_{\mathbf{B}} + p_{\mathbf{B}} \mathbf{I} \right) \cdot \mathbf{n}_{\partial\Omega} &= \alpha_0 \mathbf{v}_{\mathbf{B}} && (6.26)
\end{aligned}
$$

are satisfied on $\partial_{T_0} \Omega$.

Remark 6.5.

1. That (6.26) is satisfied is a consequence of the modification in the definition of $\mathbf{v}_{\mathbf{B}}$ in Definition 6.2.

2. The approximate solutions $c^{\epsilon}_A, \mu^{\epsilon}_A, \mathbf{v}^{\epsilon}_A, p^{\epsilon}_A$ satisfy (4.11) by construction, i.e. as a consequence of (5.93)–(5.95) and (5.228), (5.229).

3. As the appearance of $\epsilon^{M-\frac{1}{2}} \mu^{-,\epsilon}_{M-\frac{1}{2}}$ "drags down" the order of ϵ of $r^{\epsilon}_{\mathrm{CH2},\mathbf{B}}$, we introduce the notation
$$\tilde{r}^{\epsilon}_{\mathrm{CH2},\mathbf{B}} := r^{\epsilon}_{\mathrm{CH2},\mathbf{B}} - \epsilon^{M-\frac{1}{2}} \mu_{O,M-\frac{1}{2}} \quad (6.27)$$
in $\overline{\partial_{T_0} \Omega(\delta)}$, which will later on simplify the proof for a slightly better estimate of $r^{\epsilon}_{\mathrm{CH2},\mathbf{B}}$, cf. (7.68). Note that $\tilde{r}^{\epsilon}_{\mathrm{CH2},\mathbf{B}} \in \mathcal{O}\left(\epsilon^{M+1}\right)$ in $L^{\infty}\left(\partial_{T_0} \Omega(\delta)\right)$.

4. One may be tempted to simply cut off the $\left(M - \frac{1}{2}\right)$–th order terms close to the boundary in order to gain boundary remainder terms which are independent of the more complicated $\left(M - \frac{1}{2}\right)$–th order terms. But this approach is ill-advised, as it results in the appearance of terms of relatively low order. For example, consider in $\Omega^{-}_{T_0}$ the function $\tilde{\mathbf{v}}_{O,M-\frac{1}{2}} := \left(1 - \xi(d_{\mathbf{B}})\right) \mathbf{v}^{-}_{M-\frac{1}{2}}$. Then $\Delta \tilde{\mathbf{v}}_{O,M-\frac{1}{2}} = \left(1 - \xi(d_{\mathbf{B}})\right) \Delta \mathbf{v}^{-}_{M-\frac{1}{2}} - 2\xi'(d_{\mathbf{B}}) \mathbf{n}_{\partial\Omega} \left(\nabla \mathbf{v}^{-}_{M-\frac{1}{2}} \right)^T - \xi''(d_{\mathbf{B}}) \Delta d_{\mathbf{B}} \mathbf{v}^{-}_{M-\frac{1}{2}}$, but the terms including derivatives of ξ do not solve any outer equation and thus would appear with only order $\epsilon^{M-\frac{1}{2}}$ in (6.22) resp. (6.17), depending on the support of the cutoff function.

6.2. First Estimates

There are a variety of terms appearing in the inner remainders (6.11)–(6.14), which all need to be estimated in order to prove Theorem 6.12. We outsource these estimates into auxiliary results which we show in this subsection.

In order to streamline the results a little, we define

$$\mathcal{T}_G := \bigcup_{i\in\{0,\dots,M+1\}} \{c_i, \mu_i, l_i\eta, j_i\eta\},$$
$$\hat{\mathcal{T}}_G := \bigcup_{i\in\{0,\dots,M+1\}} \{\mathbf{v}_i, \mathbf{u}_i\eta, \mathbf{q}_i\eta\},$$

where we collected all orders of c_I and μ_I which are bounded independently of ϵ. Moreover, we introduce

$$\mathcal{T}_h := \bigcup_{i,j\in I^{M+1}_{M-\frac{1}{2}}\setminus\{0\}} \left\{h_j, \nabla^\Gamma h_j, \Delta^\Gamma h_j, \partial_t^\Gamma h_j, \nabla^\Gamma h_j \cdot \nabla^\Gamma h_i\right\}.$$

As before, let Assumption 6.3 hold. The following lemma will yield estimates for almost every term in (6.11), except for $B^{M-\frac{1}{2}}$, which is treated in Lemma 6.9.

Lemma 6.6 (Estimates for $r^\epsilon_{\mathrm{CH1},I}$)**.** *Let $f \in \mathcal{T}_G$, $g \in \mathcal{T}_h$ or $g \in L^\infty(\Gamma(2\delta; T_0))$ and $\varphi \in L^\infty\left(0, T_0; H^1(\Gamma_t(2\delta))\right)$. Then there is some constant $C(K) > 0$ such that for*

$$\mathfrak{R}(K, \epsilon, T_\epsilon, \varphi) := C(K)\,\epsilon T_\epsilon^{\frac{1}{2}} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))}$$

and all $\epsilon \in (0, \epsilon_1)$

1. *it holds*
$$\left\|D^l_\rho D^k_x f \cdot g\varphi\right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \mathfrak{R}(K, \epsilon, T_\epsilon, \varphi)$$
for $l \in \{1, 2\}$, $k \in \{0, 1\}$.

2. *it holds*
$$\epsilon^{\frac{1}{2}} \left\|l^\epsilon_{M-\frac{1}{2}} \eta'' h_i \varphi\right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \mathfrak{R}(K, \epsilon, T_\epsilon, \varphi)$$
for $i \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$.

3. *it holds*
$$\left\|\left(\mathbf{w}^\epsilon_1|_\Gamma \cdot \partial_\rho c_i \mathbf{n}, \mathbf{w}^\epsilon_1|_\Gamma \cdot \partial_\rho c_l \nabla^\Gamma h_j, \mathbf{w}^\epsilon_1|_\Gamma \cdot \epsilon\nabla c_l\right)\varphi\right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \mathfrak{R}(K, \epsilon, T_\epsilon, \varphi)$$
for $i \in \{1, \dots, M+1\}$, $j \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$, $l \in \{0, \dots, M+1\}$.

4. *it holds*
$$\left\|\left(\partial_\rho c_j \mathbf{v}_i \cdot \mathbf{n}, \partial_\rho c_j \mathbf{v}_i \cdot \nabla^\Gamma h_k\right)\varphi\right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \mathfrak{R}(K, \epsilon, T_\epsilon, \varphi)$$
for $j \in \{0, \dots, M+1\}$, $i \in I^{M+1}_{M-\frac{1}{2}}$, $k \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$.

5. *it holds*
$$\left\|\left(\nabla\partial_\rho \mu^\epsilon_{M-\frac{1}{2}} \cdot \nabla h_i, \partial_{\rho\rho}\mu^\epsilon_{M-\frac{1}{2}} \nabla h_i \cdot \nabla h_j, \partial_\rho \mu^\epsilon_{M-\frac{1}{2}} \Delta h_i\right)\varphi\right\|_{L^1(\Gamma(2\delta);T_\epsilon)} \leq \mathfrak{R}(K, \epsilon, T_\epsilon, \varphi)$$
for $i, j \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$.

6. it holds

$$\left\| \left(\epsilon^{M-\frac{1}{2}} \Delta \mu^{\epsilon}_{M-\frac{1}{2}}, \epsilon^{i+j} \mathbf{v}_i \cdot \nabla c_j \right) \varphi \right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \epsilon^{M-1} \mathfrak{R} \left(K, \epsilon, T_\epsilon, \varphi \right)$$

for $i \in I^{M+1}_{M-\frac{1}{2}}, j \in \{0, \dots, M+1\}$ *such that* $i + j \geq M - \frac{1}{2}$.

Proof. The proof makes heavy use of the fact that (5.235) and (5.236) hold under Assumption 6.3.

Ad 1) Due to the matching conditions (5.24) and (5.25) and the definition of η in Proposition 5.3, all $f \in \mathcal{T}_L$ satisfy $D^l_\rho D^k_x f \in \mathcal{R}_\alpha$ for $l \in \{1, 2\}$, $k \in \{0, 1\}$ and some $\alpha > 0$. For a reminder of the space $\mathcal{R}_\alpha$ see Section 2.4. Now let $g \in \mathcal{T}_h$. Since $S : \Gamma(2\delta) \to \mathbb{T}^1$ (as defined in (2.23)) and its derivatives are bounded in $\Gamma(2\delta)$ we have

$$|g(x,t)| \leq C \left| a \left(S(x,t), t \right) \right|$$

for some function $a : \mathbb{T}^1 \times [0, T_0] \to \mathbb{R}$ (where a is given by a suitable derivative of the corresponding h_i, $i \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$, or h_i itself, see Remark 2.19).

Thus, we may use (2.38) to get

$$\begin{aligned} \int_{\Gamma(2\delta;T_\epsilon)} \left| D^l_\rho D^k_x f \cdot g \varphi \right| \mathrm{d}(x,t) &\leq C \epsilon \int_0^{T_\epsilon} \|a(.,t)\|_{L^2(\mathbb{T}^1)} \|\varphi\|_{H^1(\Gamma_t(2\delta))} \, \mathrm{d}t \\ &\leq C \epsilon T_\epsilon^{\frac{1}{2}} \|a\|_{L^2((0,T_\epsilon)\times\mathbb{T}^1)} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))}. \end{aligned}$$

Now if g corresponds to h_l or its derivatives for $l \in \{1, \dots, M+1\}$, then a may be uniformly (in ϵ) estimated in $L^\infty\left((0,T_0) \times \mathbb{T}^1\right)$. In case g corresponds to $h^\epsilon_{M-\frac{1}{2}}$ or its derivatives, we use

$$\left\| \left(h^\epsilon_{M-\frac{1}{2}}, \partial_s h^\epsilon_{M-\frac{1}{2}}, \partial^2_s h^\epsilon_{M-\frac{1}{2}}, \partial_t h^\epsilon_{M-\frac{1}{2}}, \left(\partial_s h^\epsilon_{M-\frac{1}{2}} \right)^2 \right) \right\|_{L^2((0,T_\epsilon)\times\mathbb{T}^1)} \leq C \left\| h^\epsilon_{M-\frac{1}{2}} \right\|_{X_{T_\epsilon}},$$

which follows from the definition of X_T and the fact that $X_T \hookrightarrow L^2\left(0,T; W^1_4\left(\mathbb{T}^1\right)\right)$, due to the Sobolev embeddings theorem. Since (5.235) holds due to Assumption 6.3, this proves the claim. If $g \in L^\infty\left(\Gamma(2\delta;T_0)\right)$ similar estimates follow with $a \equiv 1$.

Ad 2) Since $X_{T_\epsilon} \hookrightarrow C^0\left([0,T_\epsilon] \times \mathbb{T}^1\right)$ due to Proposition 2.34 2), we get by Lemma 2.24 2)

$$\epsilon^{\frac{1}{2}} \int_{\Gamma(2\delta;T_\epsilon)} \left| l^\epsilon_{M-\frac{1}{2}} \eta'' h_i \varphi \right| \mathrm{d}(x,t) \leq C(K) \epsilon T_\epsilon^{\frac{1}{2}} \left\| l^\epsilon_{M-\frac{1}{2}} \right\|_{L^2(\Gamma(2\delta;T_\epsilon))} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))}.$$

Here we also used $H^1(\Gamma_t(2\delta)) \hookrightarrow L^{2,\infty}(\Gamma_t(2\delta))$ as implied by Lemma 2.23 and again the bound on $h^\epsilon_{M-\frac{1}{2}}$, cf. (5.235). Considering $l^\epsilon_{M-\frac{1}{2}}$ as given in (5.219), we note that the numerator in the definition, $\mu^{+,\epsilon}_{M-\frac{1}{2}} - \mu^{+,\epsilon}_{M-\frac{1}{2}} + l_0 h^\epsilon_{M-\frac{1}{2}}$, vanishes on Γ due to (5.230). Thus, the mean value theorem implies for a function $\gamma : (-2\delta, 2\delta) \to (-2\delta, 2\delta)$

$$\begin{aligned}
\left\| l^{\epsilon}_{M-\frac{1}{2}} \right\|^2_{L^2(\Gamma(2\delta;T_\epsilon))} &\leq C \int_0^{T_\epsilon} \int_{\mathbb{T}^1} \left(\int_{-2\delta}^{2\delta} \left| \partial_{\mathbf{n}} \left(\left[\mu^{\epsilon}_{M-\frac{1}{2}} \right] + l_0 h^{\epsilon}_{M-\frac{1}{2}} \right) (X(\gamma(r), s, t)) \right|^2 \mathrm{d}r \right) \mathrm{d}s\mathrm{d}t \\
&\leq C_1 \left(\int_0^{T_\epsilon} \int_{\mathbb{T}^1} \sup_{r \in (-2\delta, 2\delta)} \left| \left[\partial_{\mathbf{n}} \mu^{\epsilon}_{M-\frac{1}{2}} \right] (X(r,s,t)) \right|^2 \mathrm{d}s\mathrm{d}t \right) \\
&\quad + C_2 \left\| h^{\epsilon}_{M-\frac{1}{2}} \right\|^2_{L^2((0,T_\epsilon)\times\mathbb{T}^1)} \\
&\leq C \left(\left\| \mu^{+,\epsilon}_{M-\frac{1}{2}} \right\|^2_{L^2(0,T_\epsilon;H^2(\Omega^+(t)))} + \left\| \mu^{-,\epsilon}_{M-\frac{1}{2}} \right\|^2_{L^2(0,T_\epsilon;H^2(\Omega^-(t)))} \right) \\
&\quad + \left\| h^{\epsilon}_{M-\frac{1}{2}} \right\|^2_{L^2((0,T_\epsilon)\times\mathbb{T}^1)}. \qquad (6.28)
\end{aligned}$$

Here we again used $H^1(\Gamma_t(2\delta)) \to L^{2,\infty}(\Gamma_t(2\delta))$ as shown in Lemma 2.23 and the continuity of the extension operator, cf. Remark 5.33. Moreover, we used the notation $\left[\mu^{\epsilon}_{M-\frac{1}{2}}\right] = \mu^{+,\epsilon}_{M-\frac{1}{2}} - \mu^{-,\epsilon}_{M-\frac{1}{2}}$, as before. Now (5.235) and (5.236) imply the claim.

Ad 3) As $\partial_\rho c_i \in \mathcal{R}_\alpha$ for $i \in \{1, \ldots, M+1\}$ we may again use (2.38) to get

$$\int_{\Gamma(2\delta;T_\epsilon)} \left| \mathbf{w}^{\epsilon}_1|_\Gamma \cdot \mathbf{n} \partial_\rho c_i \varphi \right| \mathrm{d}(x,t) \leq C\epsilon T_\epsilon^{\frac{1}{2}} \left\| \mathbf{w}^{\epsilon}_1 \right\|_{L^2(0,T;L^2(\Gamma_t))} \left\| \varphi \right\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))}.$$

Since $\mathbf{w}^{\epsilon}_1 = \frac{\tilde{\mathbf{w}}^{\epsilon}_1}{\epsilon^{M-\frac{1}{2}}}$ (cf. Theorem 5.32 1)), we get due to Lemma 5.29 and the continuity of the trace operator

$$\left\| \mathbf{w}^{\epsilon}_1 \right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t))} \leq \frac{C}{\epsilon^{M-\frac{1}{2}}} \left\| \tilde{\mathbf{w}}^{\epsilon}_1 \right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \leq C(K)$$

and thus

$$\int_{\Gamma(2\delta;T_\epsilon)} \left| \mathbf{w}^{\epsilon}_1|_\Gamma \cdot \mathbf{n} \partial_\rho c_i \varphi \right| \mathrm{d}(x,t) \leq C(K)\, \epsilon T_\epsilon^{\frac{1}{2}} \left\| \varphi \right\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))}.$$

For $j \in \{1, \ldots, M+1\}$ we have $\sup_{(x,t)\in\Gamma(2\delta;T_0)} \left| \nabla^\Gamma h_j(x,t) \right| \leq C$ by construction and for $j = M - \frac{1}{2}$ we have $\sup_{(x,t)\in\Gamma(2\delta;T_\epsilon)} \left| \nabla^\Gamma h^{\epsilon}_{M-\frac{1}{2}}(x,t) \right| \leq C(K)$ since $X_{T_\epsilon} \hookrightarrow C^0\left([0,T_\epsilon]; C^1\left(\mathbb{T}^1\right)\right)$ as before. Thus, we get

$$\int_{\Gamma(2\delta;T_\epsilon)} \left| \mathbf{w}^{\epsilon}_1|_\Gamma \partial_\rho c_l \nabla^\Gamma h_j \varphi \right| \mathrm{d}(x,t) \leq C(K)\, \epsilon T_\epsilon^{\frac{1}{2}} \left\| \varphi \right\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))}$$

by similar arguments as above. Last, we have $\nabla c_0 = 0$ (as $c_0(\rho,x,t) = \theta_0(\rho)$, cf. Lemma 5.19) and may use the uniform boundedness of ∇c_l for $l \in \{1, \ldots, M+1\}$ to get

$$\begin{aligned}
\int_{\Gamma(2\delta;T_\epsilon)} \left| \mathbf{w}^{\epsilon}_1|_\Gamma \cdot \epsilon \nabla c_l \varphi \right| \mathrm{d}(x,t) &\leq C\epsilon T_\epsilon^{\frac{1}{2}} \left\| \mathbf{w}^{\epsilon}_1 \right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t))} \left\| \varphi \right\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))} \\
&\leq C(K)\, \epsilon T^{\frac{1}{2}} \left\| \varphi \right\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))}
\end{aligned}$$

which proves 3).

Ad 4) Again we note $\partial_\rho c_j \in \mathcal{R}_\alpha$ for all $j \in \{0, \dots, M+1\}$ and $\left\| \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}} \right\|_{L^\infty(\Gamma(2\delta;T_\epsilon))} \leq C(K)$ due to (5.235) and $X_{T_\epsilon} \hookrightarrow C^0\left([0,T_\epsilon];C^1\left(\mathbb{T}^1\right)\right)$. Thus, the only interesting terms are those involving $\mathbf{v}^\epsilon_{M-\frac{1}{2}}$. Using the explicit form of $\mathbf{v}^\epsilon_{M-\frac{1}{2}}$ as given in (5.254) and Lemma 2.24 2), we get

$$\int\limits_{\Gamma(2\delta;T_\epsilon)} \left|\partial_\rho c_j \mathbf{v}^\epsilon_{M-\frac{1}{2}} \cdot \left(\mathbf{n} - \nabla^\Gamma h_k\right)\varphi\right| \mathrm{d}(x,t) \leq C T_\epsilon^{\frac{1}{2}} \epsilon \left\| \left| \mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} \right| + \left| \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}} \right| \right\|_{L^2(0,T_\epsilon;L^{2,\infty}(\Gamma_t(2\delta)))} \cdot \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))}.$$

By $H^1(\Gamma_t(2\delta)) \hookrightarrow L^{2,\infty}(\Gamma_t(2\delta))$, (5.236) and the continuity of the extension operator we get the claim.

Ad 5) To show this, we use the explicit structure of $\mu^\epsilon_{M-\frac{1}{2}}$ as given in (5.253) and estimate

$$\begin{aligned}
&\int\limits_{\Gamma(2\delta;T_\epsilon)} \left|\nabla \partial_\rho \mu^\epsilon_{M-\frac{1}{2}} \cdot \nabla^\Gamma h_i \varphi\right| \mathrm{d}(x,t) \\
&\leq C \int_0^{T_\epsilon} \int_{\mathbb{T}^1} |\partial_s h_i(s,t)| \int_{-2\delta}^{2\delta} \left| \nabla \partial_\rho \mu^\epsilon_{M-\frac{1}{2}} \left(\frac{r}{\epsilon} - h^\epsilon_A(s,t), X(r,s,t)\right) \varphi \circ X \right| \mathrm{d}r \mathrm{d}s \mathrm{d}t \\
&\leq C\epsilon \int_0^{T_\epsilon} \int_{\mathbb{T}^1} |\partial_s h_i(s,t)| \sup_{r\in(-2\delta,2\delta)} \left| \left[\nabla \mu^\epsilon_{M-\frac{1}{2}}\right] (X(r,s,t)) \varphi \circ X \right| \mathrm{d}s\mathrm{d}t \int_{\mathbb{R}} \left|\eta'(\rho)\right| \mathrm{d}\rho \\
&\leq C\epsilon \int_0^{T_\epsilon} \left\| \left[\nabla \mu^\epsilon_{M-\frac{1}{2}}\right] \right\|_{L^{4,\infty}(\Gamma_t(2\delta))} \|\varphi\|_{L^{4,\infty}(\Gamma_t(2\delta))} \mathrm{d}t \, \|\partial_s h_i\|_{L^\infty(0,T_\epsilon;L^2(\mathbb{T}^1))} \\
&\leq C(K)\epsilon T_\epsilon^{\frac{1}{2}} \left(\left\| \mu^{+,\epsilon}_{M-\frac{1}{2}} \right\|_{L^2(0,T_\epsilon;H^2(\Omega^+(t)))} + \left\| \mu^{-,\epsilon}_{M-\frac{1}{2}} \right\|_{L^2(0,T_\epsilon;H^2(\Omega^-(t)))} \right) \\
&\quad \cdot \|\varphi\|_{L^\infty(0,T;H^1(\Gamma_t(2\delta)))} \\
&\leq C(K)\epsilon T_\epsilon^{\frac{1}{2}} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(2\delta)))}.
\end{aligned} \tag{6.29}$$

Here we again used $H^1(\Gamma_t(2\delta)) \hookrightarrow L^{4,\infty}(\Gamma_t(2\delta))$ together with the continuity of the extension of $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}$ onto $\Omega^\pm \cup \Gamma(2\delta)$ in the fourth inequality and (5.236). Moreover, we used $\sup_{(x,t)\in\Gamma(2\delta;T_\epsilon)} \left|\nabla^\Gamma h_j(x,t)\right| \leq C(K)$ for $j \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$, as already noted in 2) and 4). The same procedure yields an estimate for $\partial_{\rho\rho}\mu^\epsilon_{M-\frac{1}{2}} \nabla h_i \cdot \nabla h_j$ and $\partial_\rho \mu^\epsilon_{M-\frac{1}{2}} \Delta h_i$, $i,j \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$. For the latter, it is necessary to use $X_{T_\epsilon} \hookrightarrow C^0\left([0,T_\epsilon];H^2\left(\mathbb{T}^1\right)\right)$.

Ad 6) We define

$$C_K := \sup_{\epsilon\in(0,\epsilon_1)} \sup_{(s,t)\in\mathbb{T}^1\times[0,T_\epsilon]} \left|h^\epsilon_A(s,t)\right| \tag{6.30}$$

which is well defined due to Remark 6.4. As $\Delta\mu^\epsilon_{M-\frac{1}{2}} = \eta\left(\Delta\mu^{+,\epsilon}_{M-\frac{1}{2}}\right) + (1-\eta)\left(\Delta\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)$

and $\Delta\mu^{\pm,\epsilon}_{M-\frac{1}{2}} = 0$ in $\Omega^{\pm}_{T_\epsilon}$ by (5.225), we find

$$\begin{aligned}
&\int_{\Omega^{+}_{T_\epsilon}\cap\Gamma(2\delta;T_\epsilon)} \left|\Delta\mu^{\epsilon}_{M-\frac{1}{2}}\varphi\right| \mathrm{d}\left(x,t\right) \\
&\leq C\int_0^{T_\epsilon}\int_{\mathbb{T}^1} \left\|\varphi\left(.,s,t\right)\right\|_{L^\infty(-2\delta,2\delta)} \int_0^{2\delta} \left|\Delta\mu^{-,\epsilon}_{M-\frac{1}{2}}\left(1-\eta\left(\rho\left(r,s,t\right)\right)\right)\right| \mathrm{d}r\mathrm{d}s\mathrm{d}t \\
&\leq CT_\epsilon^{\frac{1}{2}} \left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \left\|\Delta\mu^{-,\epsilon}_{M-\frac{1}{2}}\right\|_{L^2(\Gamma(2\delta;T_\epsilon))} \epsilon^{\frac{1}{2}} \left\|(1-\eta)\right\|_{L^2(-C_K,\infty)} \\
&\leq C\left(K\right)T_\epsilon^{\frac{1}{2}}\epsilon^{\frac{1}{2}} \left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \qquad (6.31)
\end{aligned}$$

where $\rho\left(r,s,t\right) = \frac{r}{\epsilon} - h^{\epsilon}_{A}\left(s,t\right)$ in the second line and where we used $\eta - 1 \equiv 0$ in $(1,\infty)$, the continuity of the extension operator for $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}$ and (5.236) in the last line. A similar estimate holds on $\Omega^{-}_{T_\epsilon}\cap\Gamma\left(2\delta;T_\epsilon\right)$.

Since $\nabla c_0 \equiv 0$ the remaining terms are now of order ϵ^M and can be estimated by simply using Hölder's inequality and L^∞ bounds on the functions $\mathbf{v}_i, c_i$ for $i\in\{0,\ldots,M+1\}$, respectively the L^2-estimates for $\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}$ in (5.236) as before. □

Lemma 6.7 (Estimates for $\mathbf{r}^{\epsilon}_{\mathrm{S},I}$)**.** *Let $\mathbf{v}\in\hat{\mathcal{T}}_G$ and $g\in\mathcal{T}_h$ or $g\in L^\infty\left(\Gamma\left(2\delta;T_0\right)\right)$ and let a function $\mathbf{z}\in L^2\left(0,T_\epsilon;H^1\left(\Omega\right)^2\right)$ be given. Then there is some constant $C\left(K\right)>0$ such that for*

$$\mathcal{E}\left(K,\epsilon,\mathbf{z}\right) := C\left(K\right)\epsilon\left\|\mathbf{z}\right\|_{L^2(0,T_\epsilon;H^1(\Omega))}$$

and all $\epsilon\in(0,\epsilon_1)$

1. *it holds*

$$\int_{\Gamma(2\delta;T_\epsilon)} \left|D^l_\rho D^k_x\mathbf{v}\right|\left|g\right|\left|\mathbf{z}\right| \mathrm{d}\left(x,t\right) \leq \mathcal{E}\left(K,\epsilon,\mathbf{z}\right)$$

   for $l\in\{1,2\}$, $k\in\{0,1\}$.

2. *it holds*

$$\left\|\partial_\rho p_i \nabla^\Gamma h_j \mathbf{z}\right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \mathcal{E}\left(K,\epsilon,\mathbf{z}\right)$$

   for all $i\in I^M_{M-\frac{1}{2}}$, $j\in I^{M+1}_{M-\frac{1}{2}}\setminus\{0\}$.

3. *it holds*

$$\left\|\left(\partial_\rho\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}\Delta^\Gamma h_i, \left(\nabla\partial_\rho\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}\right)^T\cdot\nabla^\Gamma h_i\right)\mathbf{z}\right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \mathcal{E}\left(K,\epsilon,\mathbf{z}\right)$$

   and

$$\left\|\partial_{\rho\rho}\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}\nabla^\Gamma h_i\cdot\nabla^\Gamma h_j\mathbf{z}\right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \mathcal{E}\left(K,\epsilon,\mathbf{z}\right)$$

   for all $i,j\in I^{M+1}_{M-\frac{1}{2}}\setminus\{0\}$.

4. *it holds*
$$\epsilon^{\frac{1}{2}} \left\| \mathbf{u}_{M-\frac{1}{2}} \eta'' h_i \mathbf{z} \right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \mathcal{E}\left(K, \epsilon, \mathbf{z}\right)$$
for all $i \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$.

5. *it holds*
$$\left\| \left(\epsilon^{M-\frac{1}{2}} \left(\Delta \mathbf{v}^{\epsilon}_{M-\frac{1}{2}} - \nabla p^{\epsilon}_{M-\frac{1}{2}} \right), \epsilon^{i+j} \mu_i \nabla c_j \right) \mathbf{z} \right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \epsilon^{M-1} \mathcal{E}\left(K, \epsilon, \mathbf{z}\right)$$
for all $i \in I^{M+1}_{M-\frac{1}{2}}$, $j \in \{0, \ldots, M+1\}$ *with* $i + j \geq M - \frac{1}{2}$.

6. *it holds*
$$\left\| \left(\mu_i \partial_\rho c_j \mathbf{n}, \mu_i \partial_\rho c_j \nabla^{\Gamma} h_l \right) \mathbf{z} \right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq \mathcal{E}\left(K, \epsilon, \mathbf{z}\right)$$
for all $i \in I^{M+1}_{M-\frac{1}{2}}$, $j \in \{0, \ldots, M+1\}$, $l \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$.

Proof. The proof is very similar to the proof of Lemma 6.6. In particular, it relies heavily on the fact that $h_{M-\frac{1}{2}}$ is uniformly bounded in the X_{T_ϵ} norm due to (5.235), which holds under the Assumption 6.3, so we will not repeat this fact in every step.

Ad 1) Since $\left| D^l_\rho D^k_x \mathbf{v} \right| \in \mathcal{R}_\alpha$, we may argue as in the proof of Lemma 6.6 1) that for $g \in \mathcal{T}_h$ it holds due to (2.38)
$$\begin{aligned} \int_{\Gamma(2\delta;T_\epsilon)} \left| D^l_\rho D^k_x \mathbf{v} \right| |g| \, |\mathbf{z}| \, \mathrm{d}\left(x,t\right) &\leq C\epsilon \left\| \mathbf{z} \right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \left\| a \right\|_{L^2((0,T_\epsilon)\times\mathbb{T}^1)} \\ &\leq C\left(K\right) \epsilon \left\| \mathbf{z} \right\|_{L^2(0,T_\epsilon;H^1(\Omega))}, \end{aligned}$$
where a corresponds to $h_j, \partial_s h_j, \partial_t h_j$ etc. depending on the form of $g \in \mathcal{T}_h$. If $g \in L^\infty\left(\Gamma\left(2\delta; T_0\right)\right)$ the estimate follows by setting $a \equiv 1$.

Ad 2) The only interesting terms are $\partial_\rho p^{\epsilon}_{M-\frac{1}{2}} \nabla^{\Gamma} h_j \mathbf{z}$ for $j \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$, since all other terms can be treated as in 1). We use the explicit form of $p^{\epsilon}_{M-\frac{1}{2}}$ as given in (5.255) to compute
$$\begin{aligned} \left\| \partial_\rho p^{\epsilon}_{M-\frac{1}{2}} \nabla^{\Gamma} h_j \mathbf{z} \right\|_{L^1(\Gamma(2\delta;T_\epsilon))} &\leq C\left(K\right) \epsilon \left\| \left[p^{\epsilon}_{M-\frac{1}{2}} \right] \mathbf{z} \right\|_{(L^1(0,T_\epsilon);L^{1,\infty}(\Gamma_t(2\delta)))} \left\| \eta' \right\|_{L^1(\mathbb{R})} \\ &\leq C\left(K\right) \epsilon \left\| \left[p^{\epsilon}_{M-\frac{1}{2}} \right] \right\|_{L^2(0,T_\epsilon;H^1(\Gamma_t(2\delta)))} \left\| \mathbf{z} \right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \\ &\leq C\left(K\right) \epsilon \left\| \mathbf{z} \right\|_{L^2(0,T_\epsilon;H^1(\Omega))}, \end{aligned}$$
where we used Lemma 2.24 1) in the first inequality, $H^1\left(\Gamma_t\left(2\delta\right)\right) \hookrightarrow L^{2,\infty}\left(\Gamma_t\left(2\delta\right)\right)$ (cf. Lemma 2.23) in the second inequality and (5.236) together with the continuity of the extension operator for $p^{\pm,\epsilon}_{M-\frac{1}{2}}$ (cf. Remark 5.33) in the last inequality. Here we again used the notation $\left[p^{\epsilon}_{M-\frac{1}{2}} \right] = p^{+,\epsilon}_{M-\frac{1}{2}} - p^{-,\epsilon}_{M-\frac{1}{2}}$.

Ad 3) Using the explicit form of $\mathbf{v}^{\epsilon}_{M-\frac{1}{2}}$ as given in (5.254), we exemplarily calculate

$$\begin{aligned}
&\int\limits_{\Gamma(2\delta;T_\epsilon)} \left| \left(\nabla \partial_\rho \mathbf{v}^{\epsilon}_{M-\frac{1}{2}}\right)^T \cdot \nabla^\Gamma h_i \mathbf{z} \right| \mathrm{d}\,(x,t) \\
&\quad \leq C\epsilon \int_0^{T_\epsilon} \|\partial_s h_i\|_{L^2(\mathbb{T}^1)} \left\| \left[\nabla \mathbf{v}^{\epsilon}_{M-\frac{1}{2}} \right] \right\|_{L^{4,\infty}(\Gamma_t(2\delta))} \|\mathbf{z}\|_{L^{4,\infty}(\Gamma_t(2\delta))} \mathrm{d}t \left\|\eta'\right\|_{L^1(\mathbb{R})} \\
&\quad \leq C\epsilon \|h_i\|_{L^4(0,T_\epsilon;H^2(\mathbb{T}^1))} \left\| \left[\nabla \mathbf{v}^{\epsilon}_{M-\frac{1}{2}} \right] \right\|_{L^4(0,T_\epsilon;H^1(\Gamma_t(2\delta)))} \|\mathbf{z}\|_{L^2(0,T_\epsilon;H^1(\Omega))} \\
&\quad \leq C\,(K)\,\epsilon \|\mathbf{z}\|_{L^2(0,T_\epsilon;H^1(\Omega))}
\end{aligned}$$

for all $i \in I^{M-\frac{1}{2}}_{M+1} \setminus \{0\}$, where we used $H^1(\Gamma_t(2\delta)) \hookrightarrow L^{4,\infty}(\Gamma_t(2\delta))$ in the second inequality and the continuity of the trace operator, (5.236) and $X_{T_\epsilon} \hookrightarrow H^{\frac{1}{2}}\left(0, T_\epsilon; H^2\left(\mathbb{T}^1\right)\right)$ in the last inequality. The same procedure can be used to estimate $\partial_{\rho\rho}\mathbf{v}^{\epsilon}_{M-\frac{1}{2}} \nabla^\Gamma h_i \cdot \nabla^\Gamma h_j$ and $\partial_\rho \mathbf{v}^{\epsilon}_{M-\frac{1}{2}} \Delta^\Gamma h_i$ for all $i, j \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$.

Ad 4) Again, the only terms which cannot simply be estimated by the same means used in 1) are the ones of the form $\mathbf{u}^{\epsilon}_{M-\frac{1}{2}} \eta'' h_i \mathbf{z}$ for $i \in I^{M+1}_{M-\frac{1}{2}} \setminus \{0\}$. Lemma 2.24 2) implies

$$\epsilon^{\frac{1}{2}} \left\| \mathbf{u}^{\epsilon}_{M-\frac{1}{2}} \eta'' h_i \mathbf{z} \right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \leq C\,(K)\,\epsilon \left\| \mathbf{u}^{\epsilon}_{M-\frac{1}{2}} \right\|_{L^2(\Gamma(2\delta;T_\epsilon))} \|\mathbf{z}\|_{L^2(0,T_\epsilon;H^1(\Omega))}$$

and in a similar fashion to (6.28) it follows by the estimates for $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$ in (5.236) that

$$\left\| \mathbf{u}^{\epsilon}_{M-\frac{1}{2}} \right\|_{L^2(\Gamma(2\delta;T_\epsilon))} \leq C\,(K) \tag{6.32}$$

holds, which yields the claim.

Ad 5) Let C_K be given as in (6.30). Since

$$\Delta \mathbf{v}^{\epsilon}_{M-\frac{1}{2}} - \nabla p^{\epsilon}_{M-\frac{1}{2}} = \left(\Delta \mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} - \nabla p^{+,\epsilon}_{M-\frac{1}{2}} \right) \eta + \left(\Delta \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}} - \nabla p^{-,\epsilon}_{M-\frac{1}{2}} \right) (1-\eta)$$

and $\Delta \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}} - \nabla p^{\pm,\epsilon}_{M-\frac{1}{2}} = 0$ in $\Omega^{\pm}_{T_\epsilon}$ by (5.226), we have

$$\begin{aligned}
&\int\limits_{\Omega^+_{T_\epsilon} \cap \Gamma(2\delta;T_\epsilon)} \left| \left(\Delta \mathbf{v}^{\epsilon}_{M-\frac{1}{2}} - \nabla p^{\epsilon}_{M-\frac{1}{2}} \right) \mathbf{z} \right| \mathrm{d}\,(x,t) \\
&\quad \leq C \int_0^{T_\epsilon} \int_{\mathbb{T}^1} \|\mathbf{z}\|_{L^\infty(-2\delta,2\delta)} \int_0^{2\delta} \left| \Delta \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}} - \nabla p^{-,\epsilon}_{M-\frac{1}{2}} \right| |(1-\eta)| \,\mathrm{d}\,(x,t) \\
&\quad \leq C\,(K) \|\mathbf{z}\|_{L^2(0,T_\epsilon;H^1(\Omega))} \epsilon^{\frac{1}{2}}.
\end{aligned}$$

Here we used the continuity of the expansion together with (5.236) in the last line.

Now the products $\mu_i \nabla c_j$ are multiplied by ϵ^M (or even higher powers of ϵ) as $\nabla c_0 = 0$ and can thus be estimated in $L^\infty(\Gamma(2\delta;T_0))$ or in $L^2(\Gamma(2\delta;T_\epsilon))$ with the help of (5.236) which yields the claim.

Ad 6) Considering only the case $i = M - \frac{1}{2}$ and noting $\partial_\rho c_i \in \mathcal{R}_\alpha$ for all $i \in \{0, \ldots, M+1\}$, we immediately get

$$\begin{aligned}
&\left\| \left(\mu^\epsilon_{M-\frac{1}{2}} \partial_\rho c_j \left(\mathbf{n} - \nabla^\Gamma h_l \right) \right) \mathbf{z} \right\|_{L^1(\Gamma(2\delta;T_\epsilon))} \\
&\leq C\left(K\right) \epsilon \left\| \mathbf{z} \right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \left(\left\| \mu^{+,\epsilon}_{M-\frac{1}{2}} \right\|_{L^2(0,T_\epsilon;H^1(\Omega^+(t)))} + \left\| \mu^{-,\epsilon}_{M-\frac{1}{2}} \right\|_{L^2(0,T_\epsilon;H^1(\Omega^-(t)))} \right) \\
&\leq C\left(K\right) \epsilon \left\| \mathbf{z} \right\|_{L^2(0,T_\epsilon;H^1(\Omega))}
\end{aligned}$$

by the same means as before. □

The following proposition greatly simplifies the estimates for the remainder terms occurring in (6.12), as it gives better control of the error in the approximation $c^\epsilon - c^\epsilon_A$ in the L^1-norm. This is a consequence of Proposition 5.28.

Proposition 6.8. *Let $R = c^\epsilon - c^\epsilon_A$. It holds*

$$\left\| R \right\|_{L^1(\Gamma(\delta;T_\epsilon))} \leq C\left(K\right) T_\epsilon^{\frac{1}{2}} \epsilon^M$$

for all $\epsilon \in (0, \epsilon_1)$.

Proof. Due to Assumption 6.3, we may use the results of Proposition 5.28 and find that R can be decomposed as in (5.185) with according estimates. Thus, we get

$$\begin{aligned}
\int_0^{T_\epsilon} \int_{\Gamma_t(\delta)} |R| \, \mathrm{d}x\mathrm{d}t &\leq C \int_0^{T_\epsilon} \int_{\mathbb{T}^1} \int_{-\frac{\delta}{\epsilon} - h^\epsilon_A}^{\frac{\delta}{\epsilon} - h^\epsilon_A} \epsilon^{\frac{1}{2}} \left| Z\left(s,t\right) \left(\beta\left(s,t\right) \theta_0'\left(\rho\right) + F^{\mathbf{R}}_1\left(\rho,s,t\right) \right) \right| \left| J^\epsilon\left(\rho,s,t\right) \right| \mathrm{d}\rho\mathrm{d}s\mathrm{d}t \\
&\quad + C T_\epsilon^{\frac{1}{2}} \left\| F^{\mathbf{R}}_2 \right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))} \\
&\leq C T_\epsilon^{\frac{1}{2}} \left(\epsilon^{\frac{1}{2}} \left\| Z \right\|_{L^2(0,T_\epsilon;L^2(\mathbb{T}^1))} \left(1 + \epsilon^{\frac{1}{2}} \right) + C\left(K\right) \epsilon^{M+\frac{1}{2}} \right) \\
&\leq C\left(K\right) T_\epsilon^{\frac{1}{2}} \epsilon^M
\end{aligned}$$

for all $\epsilon \in (0, \epsilon_1)$. Here we used (5.186)–(5.188) in the second and third inequality. □

When inspecting the remainder terms (6.44)–(6.47), it catches the eye that the terms $A^{M-\frac{1}{2}}$, $B^{M-\frac{1}{2}}$, $\mathbf{V}^{M-\frac{1}{2}}$ and $W^{M-\frac{1}{2}}$ are multiplied by a lower power of ϵ than the rest. Gaining these missing powers of ϵ needs delicate work; the main ingredient for this is that we have intricate structural knowledge of $A^{M-\frac{1}{2}}$ etc. due to Lemma 5.36.

Lemma 6.9. *Let $\varphi \in L^\infty\left(0, T_\epsilon; H^1\left(\Omega\right)\right)$, $\mathbf{z} \in L^2\left(0, T_\epsilon; H^1\left(\Omega\right)^2\right)$ and $R = c^\epsilon - c^\epsilon_A$. Then* *there is* $\epsilon_2 \in (0, \epsilon_1]$ *such that for all* $\epsilon \in (0, \epsilon_2)$

$$\epsilon^{M-\frac{3}{2}} \int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} B^{M-\frac{1}{2}} \varphi \mathrm{d}x \right| \mathrm{d}t \leq C(K)\, \epsilon^M \left(T_\epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{2}} \right) \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \tag{6.33}$$

$$\epsilon^{M-\frac{3}{2}} \int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} \left(\mathbf{V}^{M-\frac{1}{2}} \right) \cdot \mathbf{z} \mathrm{d}x \right| \mathrm{d}t \leq C(K)\, \epsilon^M \|\mathbf{z}\|_{L^2(0,T_\epsilon;H^1(\Omega))}, \tag{6.34}$$

$$\epsilon^{M-\frac{1}{2}} \int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} A^{M-\frac{1}{2}} R \mathrm{d}x \right| \mathrm{d}t \leq C(K)\, \epsilon^{2M} \left(T_\epsilon^{\frac{1}{3}} + \epsilon^{\frac{1}{2}} \right), \tag{6.35}$$

$$\epsilon^{M-\frac{1}{2}} \left\| W^{M-\frac{1}{2}} \right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(2\delta)))} \leq C(K)\, \epsilon^M. \tag{6.36}$$

Proof. For the sake of readability we will write throughout this proof

$$(f)^\Gamma(\rho, x, t) := f(\rho, x, t) - f(\rho, Pr_{\Gamma_t}(x), t)$$

for an arbitrary function f depending on $(\rho, x, t) \in \mathbb{R} \times \Gamma(\delta; T_\epsilon)$ (and similarly for functions depending only on (x, t)). Moreover, for functions $\psi : \Gamma(\delta; T_\epsilon) \to \mathbb{R}$ we use the usual notation $\psi(r, s, t) := \psi(X(r, s, t))$ for $(r, s, t) \in (-\delta, \delta) \times \mathbb{T}^1 \times [0, T_\epsilon]$ and write

$$J^\epsilon(\rho, s, t) := J(\epsilon(\rho + h_A^\epsilon(s, t)), s, t) \ \forall (\rho, s, t) \in I_\epsilon^{s,t} \times \mathbb{T}^1 \times [0, T_0]$$

with $J(r, s, t) := \det\left(D_{(r,s)}X\right)(r, s, t)$ for $(r, s, t) \in (-\delta, \delta) \times \mathbb{T}^1 \times [0, T_\epsilon]$ and

$$I_\epsilon^{s,t} := \left(-\frac{\delta}{\epsilon} - h_A^\epsilon(s, t), \frac{\delta}{\epsilon} - h_A^\epsilon(s, t) \right).$$

Proof of (6.33): We use a splitting

$$B^{M-\frac{1}{2}}(\rho, x, t) = B^{M-\frac{1}{2}}(\rho, Pr_{\Gamma_t}(x), t) + \left(B^{M-\frac{1}{2}}(\rho, x, t) - B^{M-\frac{1}{2}}(\rho, Pr_{\Gamma_t}(x), t) \right),$$

denote $B^{M-\frac{1}{2}}|_\Gamma(\rho, x, t) := B^{M-\frac{1}{2}}(\rho, Pr_{\Gamma_t}(x), t)$ and get

$$\begin{aligned} \int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} B^{M-\frac{1}{2}} \varphi \mathrm{d}x \right| \mathrm{d}t &\leq \int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} B^{M-\frac{1}{2}}|_\Gamma \varphi \mathrm{d}x \right| \mathrm{d}t \\ &\quad + \int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} \left(B^{M-\frac{1}{2}} - B^{M-\frac{1}{2}}|_\Gamma \right) \varphi \mathrm{d}x \right| \mathrm{d}t \\ &=: \mathcal{J}_1 + \mathcal{J}_2. \end{aligned}$$

The fundamental theorem of calculus implies $\varphi(r, s, t) = \varphi(0, s, t) + \int_0^r \partial_\mathbf{n} \varphi(\tilde{r}, s, t) \mathrm{d}\tilde{r}$ for $(r, s, t) \in (-\delta, \delta) \times \mathbb{T}^1 \times [0, T]$ and we write

$$\mathcal{J}_1^1 := \int_0^{T_\epsilon} \left| \int_{\mathbb{T}^1} \int_{-\delta}^{\delta} B^{M-\frac{1}{2}}|_\Gamma \varphi(0, s, t) J(r, s, t) \mathrm{d}r \mathrm{d}s \right| \mathrm{d}t$$

$$\mathcal{J}_1^2 := \int_0^{T_\epsilon} \left| \int_{\mathbb{T}^1} \int_{-\delta}^{\delta} B^{M-\frac{1}{2}}|_\Gamma \int_0^r \partial_\mathbf{n} \varphi(\tilde{r}, s, t) \mathrm{d}\tilde{r} J(r, s, t) \mathrm{d}r \mathrm{d}s \right| \mathrm{d}t.$$

Concerning $\mathcal{J}_1^1$ we use the splitting of $B^{M-\frac{1}{2}}$ on $\mathbb{R}\times\Gamma$ as in Lemma 5.36 2) and get

$$\mathcal{J}_1^1 \leq \sum_{k=1}^{K_2}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}\left|\mathtt{B}_k^{1,\Gamma}(0,s,t)\,\varphi(0,s,t)\right|\left|\epsilon\int_{-\frac{\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon}\mathtt{B}_k^{2,\Gamma}(\rho)\,J^\epsilon(\rho,s,t)\,\mathrm{d}\rho\right|\mathrm{d}s\mathrm{d}t.$$

Since $\sup_{\epsilon\in(0,\epsilon_1)}\|h_A^\epsilon\|_{L^\infty((0,T_\epsilon)\times\mathbb{T}^1)} < C(K)$ due to (6.9) it holds

$$\left|\frac{\delta}{\epsilon}-h_A^\epsilon\right| \geq \frac{\delta}{\epsilon}-C(K) \geq \frac{\delta}{2\epsilon} \text{ for } \epsilon>0 \text{ small enough.} \tag{6.37}$$

Moreover, we have

$$J^\epsilon(\rho,s,t) = 1+\epsilon\left(\rho+h_A^\epsilon(s,t)\right)\kappa(s,t) \tag{6.38}$$

by Lemma 3.3, where $\kappa(s,t)=\kappa(X_0(s,t))$ denotes the (principal) curvature of Γ_t at a point $X_0(s,t)=p\in\Gamma_t$. Thus, we may use that $\mathtt{B}_k^{2,\Gamma}$ satisfies (5.259) and $H^1(\Gamma_t(\delta))\hookrightarrow L^{2,\infty}(\Gamma_t(\delta))$ for φ to get

$$\begin{aligned}\mathcal{J}_1^1 &\leq CT_\epsilon^{\frac{1}{2}}\sum_{k=1}^{K_2}\left\|\mathtt{B}_k^{1,\Gamma}\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t))}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\,\epsilon\left(e^{-\frac{\alpha\delta}{\epsilon}}+\epsilon C(K)\right)\\ &\leq C(K)\,\epsilon^2\,\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\end{aligned}$$

for $\epsilon>0$ small enough. Here we also used that $\left|\mathtt{B}_j^{2,\Gamma}(\rho)\right| \leq C_1 e^{-C_2|\rho|}$ for $\rho\in\mathbb{R}$ (cf. Lemma 5.36 2)) and thus

$$\int_{I_\epsilon^{s,t}}\mathtt{B}_k^{2,\Gamma}(\rho)\left(\rho+h_A^\epsilon(s,t)\right)\mathrm{d}(\rho) \leq C(K)$$

for all $(s,t)\in\mathbb{T}^1\times[0,T_\epsilon]$.

To treat $\mathcal{J}_1^2$ we again use the fact that all terms of kind $\mathtt{B}_k^{2,\Gamma}$ exhibit exponential decay and thus

$$\begin{aligned}\mathcal{J}_1^2 &\leq C\sum_{k=1}^{K_2}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}\|\partial_{\mathbf{n}}\varphi(.,s,t)\|_{L^2(-\delta,\delta)}\left|\mathtt{B}_k^{1,\Gamma}(0,s,t)\right|\int_{-\delta}^{\delta}|r|^{\frac{1}{2}}\left|\mathtt{B}_k^{2,\Gamma}(\rho(r,s,t))\right|\mathrm{d}r\mathrm{d}s\mathrm{d}t\\ &\leq C\sum_{k=1}^{K_2}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}\|\partial_{\mathbf{n}}\varphi(.,s,t)\|_{L^2(-\delta,\delta)}\left|\mathtt{B}_k^{1,\Gamma}(0,s,t)\right|\epsilon^{\frac{3}{2}}\int_{\frac{-\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon}|\rho+h_A^\epsilon|^{\frac{1}{2}}\left|\mathtt{B}_k^{2,\Gamma}(\rho)\right|\mathrm{d}r\mathrm{d}s\mathrm{d}t\\ &\leq C\epsilon^{\frac{3}{2}}\sum_{k=1}^{K_2}T_\epsilon^{\frac{1}{2}}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(\delta)))}\left\|\mathtt{B}_k^{1,\Gamma}\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t))}\\ &\leq C(K)\,\epsilon^{\frac{3}{2}}T_\epsilon^{\frac{1}{2}}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\end{aligned}$$

where we used (5.258) in the last inequality.

Now we consider $\mathcal{J}_2$: here we use the explicit form of $B^{M-\frac{1}{2}}$ as given in (5.208) and separately estimate the occurring terms. First, note that there appears no term involving

$\mathbf{w}_1^\epsilon|_\Gamma$ in $\left(B^{M-\frac{1}{2}}\right)^\Gamma$ as it cancels out. In order to estimate the term $\left(\nabla\partial_\rho\mu^\epsilon_{M-\frac{1}{2}}\cdot\mathbf{n}\right)^\Gamma =$ $\eta'\left(\left[\nabla\mu^\epsilon_{M-\frac{1}{2}}\right]\right)^\Gamma\cdot\mathbf{n}$ (where the equality follows from (5.253)), we compute

$$\begin{aligned}
&\int\limits_{\Gamma(\delta;T_\epsilon)}\left|\eta'\left(\left[\nabla\mu^\epsilon_{M-\frac{1}{2}}\right]\right)^\Gamma\cdot\mathbf{n}\varphi\right|\mathrm{d}x\mathrm{d}t\\
&\quad\le C\int\limits_0^{T_\epsilon}\int\limits_{\mathbb{T}^1}\int\limits_{-\delta}^{\delta}\left|\eta'\left(\left[\nabla\mu^\epsilon_{M-\frac{1}{2}}\right](r,s,t)-\left[\nabla\mu^\epsilon_{M-\frac{1}{2}}\right](0,s,t)\right)\cdot\mathbf{n}\varphi\right|\mathrm{d}r\mathrm{d}s\mathrm{d}t\\
&\quad\le C\int\limits_0^{T_\epsilon}\int\limits_{\mathbb{T}^1}\int\limits_{-\delta}^{\delta}\left|\eta'\int\limits_0^r\partial^2_{\mathbf{n}}\left[\mu^\epsilon_{M-\frac{1}{2}}\right](\tilde r,s,t)\,\mathrm{d}\tilde r\varphi\right|\mathrm{d}r\mathrm{d}s\mathrm{d}t\\
&\quad\le C\int\limits_0^{T_\epsilon}\int\limits_{\mathbb{T}^1}\left\|\partial^2_{\mathbf{n}}\left[\mu^\epsilon_{M-\frac{1}{2}}\right]\right\|_{L^2(-\delta,\delta)}\|\varphi\|_{L^\infty(-\delta,\delta)}\int\limits_{\frac{-\delta}{\epsilon}-h^\epsilon_A}^{\frac{\delta}{\epsilon}-h^\epsilon_A}\left|\eta'(\rho)\left(\rho+h^\epsilon_A\right)^{\frac{1}{2}}\right|\epsilon^{\frac{3}{2}}\mathrm{d}\rho\mathrm{d}s\mathrm{d}t\\
&\quad\le C(K)\,T_\epsilon^{\frac{1}{2}}\left\|\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right\|_{L^2(0,T_\epsilon;H^2(\Gamma_t(\delta)))}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(\delta)))}\epsilon^{\frac{3}{2}}\\
&\quad\le C(K)\,T_\epsilon^{\frac{1}{2}}\epsilon^{\frac{3}{2}}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Gamma_t(\delta)))},
\end{aligned}\tag{6.39}$$

where we used $H^1(\Gamma_t(\delta))\hookrightarrow L^{2,\infty}(\Gamma_t(\delta))$ in the fourth inequality and (5.236) together with the continuity of the extension operator for $\mu^{\pm,\epsilon}_{M-\frac{1}{2}}$ in the last inequality. $\left(\partial_\rho c_0\mathbf{v}^\epsilon_{M-\frac{1}{2}}\right)^\Gamma$ and $\left(\partial_\rho\mu^\epsilon_{M-\frac{1}{2}}\Delta d_\Gamma\right)^\Gamma$ may be treated in a very similar fashion. For $\left(l^\epsilon_{M-\frac{1}{2}}\eta''(\rho+h_1)\right)^\Gamma$ note that by Taylor's theorem used with integral form of the remainder, we get by the definition of $l^\epsilon_{M-\frac{1}{2}}$ in (5.219)

$$\begin{aligned}
\left|\left(l^\epsilon_{M-\frac{1}{2}}\right)^\Gamma(r,s,t)\right| &= \left|\int\limits_0^r\frac{(r-\tilde r)}{r}\left(\partial^2_{\mathbf{n}}\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)+\partial^2_{\mathbf{n}}l_0h^\epsilon_{M-\frac{1}{2}}\right)(\tilde r,s,t)\,\mathrm{d}\tilde r\right|\\
&\le Cr^{\frac{1}{2}}\left\|\partial^2_{\mathbf{n}}\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}-\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)+\partial^2_{\mathbf{n}}l_0h^\epsilon_{M-\frac{1}{2}}\right\|_{L^2(-\delta,\delta)}
\end{aligned}\tag{6.40}$$

for $(r,s,t)\in(-\delta,\delta)\times\mathbb{T}^1\times(0,T_\epsilon)$. This allows for the same strategy to be used as in (6.39).

By Remark 2.19, we have

$$\begin{aligned}
\left(\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}(x,t)\right)^\Gamma &= \left(\Delta S(x,t)\right)^\Gamma\partial_s h^\epsilon_{M-\frac{1}{2}}(S(x,t),t)\\
&\quad+\left(|\nabla S(x,t)|^2\right)^\Gamma\partial_s^2 h^\epsilon_{M-\frac{1}{2}}(S(x,t),t).
\end{aligned}\tag{6.41}$$

Thus, (2.38), $\partial_\rho\mu_0\in\mathcal{R}_\alpha$ and (5.235) imply

$$\begin{aligned}
\int\limits_0^{T_\epsilon}\left|\int\limits_{\Gamma_t(\delta)}\left(\partial_\rho\mu_0\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}\right)^\Gamma\varphi\mathrm{d}x\right|\mathrm{d}t &\le C(K)\,T_\epsilon^{\frac{1}{2}}\epsilon^2\left\|h^\epsilon_{M-\frac{1}{2}}\right\|_{L^2(0,T_\epsilon;H^2(\mathbb{T}^1))}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\\
&\le C(K)\,T_\epsilon^{\frac{1}{2}}\epsilon^2\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}.
\end{aligned}\tag{6.42}$$

The remaining terms

$$\left(\partial_\rho c_0 \mathbf{v}_0 \cdot \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}, 2\partial_{\rho\rho}\mu_0 \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}} \cdot \nabla^\Gamma h_1, \partial_\rho c_0 \partial_t^\Gamma h^\epsilon_{M-\frac{1}{2}}, 2\nabla\partial_\rho \mu_0 \cdot \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\right)^\Gamma$$

and

$$\left(l_1 h^\epsilon_{M-\frac{1}{2}} \eta'', j_0 h^\epsilon_{M-\frac{1}{2}} \eta'\right)^\Gamma$$

may then be treated in a similar fashion, which proves (6.33).

Proof of (6.34): This can be shown analogously to (6.33) due to Lemma 5.36 3), if we remark that $\mathbf{z}$ is only in L^2 in time and thus we may not expect $T_\epsilon^{\frac{1}{2}}$ to appear on the right hand side. Due to the similarities we shorten the proof: we again consider

$$\begin{aligned}\int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} \mathbf{V}^{M-\frac{1}{2}} \cdot \mathbf{z} \mathrm{d}x \right| \mathrm{d}t &\leq \int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} \mathbf{V}^{M-\frac{1}{2}}|_\Gamma \cdot \mathbf{z} \mathrm{d}x \right| \mathrm{d}t + \int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} \left(\mathbf{V}^{M-\frac{1}{2}}\right)^\Gamma \cdot \mathbf{z} \mathrm{d}x \right| \mathrm{d}t \\ &=: \mathcal{U}_1 + \mathcal{U}_2.\end{aligned}$$

Then we use Lemma 5.36 3) and (5.260) to estimate

$$\begin{aligned}&\int_0^{T_\epsilon} \left| \int_{\mathbb{T}^1} \int_{-\delta}^{\delta} \left(\mathbf{V}^{M-\frac{1}{2}}\right)|_\Gamma \cdot \mathbf{z}(0,s,t) J(r,s,t) \mathrm{d}r\mathrm{d}s \right| \mathrm{d}t \\ &\leq \sum_{k=1}^{N_2} \int_0^{T_\epsilon} \int_{\mathbb{T}^1} \left| \mathtt{V}_k^{1,\Gamma}(0,s,t) \cdot \mathbf{z}(0,s,t) \right| \epsilon \sup_{(x,\tau)\in\Gamma} \left| \int_{-\frac{\delta}{\epsilon}-h^\epsilon_A}^{\frac{\delta}{\epsilon}-h^\epsilon_A} \mathtt{V}_k^{2,\Gamma}(\rho,x,\tau) J^\epsilon(\rho,x,\tau) \mathrm{d}\rho \right| \mathrm{d}s\mathrm{d}t \\ &\leq C(K)\epsilon \|\mathbf{z}\|_{L^2(0,T_\epsilon;H^1(\Omega))} \left(e^{-\frac{\alpha\delta}{\epsilon}} + \epsilon C(K)\right) \\ &\leq C(K)\epsilon^{\frac{3}{2}} \|\mathbf{z}\|_{L^2(0,T_\epsilon;H^1(\Omega))}.\end{aligned}$$

As before, we also get

$$\begin{aligned}&\int_0^{T_\epsilon} \left| \int_{\mathbb{T}^1} \int_{-\delta}^{\delta} \left(\mathbf{V}^{M-\frac{1}{2}}\right)|_\Gamma \cdot \int_0^r \partial_{\mathbf{n}} \mathbf{z}(\tilde{r},s,t) \mathrm{d}\tilde{r} J(r,s,t) \mathrm{d}r\mathrm{d}s \right| \mathrm{d}t \\ &\leq C(K) \sum_{k=1}^{N_2} \|\mathbf{z}\|_{L^2(0,T_\epsilon;H^1(\Omega))} \epsilon \int_{\mathbb{R}} \epsilon^{\frac{1}{2}}(\rho+1) \sup_{(x,t)\in\Gamma} \left| \mathtt{V}_k^{2,\Gamma}(\rho,x,t) \right| \mathrm{d}\rho \\ &\leq C(K)\epsilon^{\frac{3}{2}} \|\mathbf{z}\|_{L^2(0,T_\epsilon;H^1(\Omega))}.\end{aligned}$$

Using these two estimates and the fundamental theorem of calculus implies the estimate for $\mathcal{U}_1$. To estimate $\mathcal{U}_2$, we again use the explicit form of $\mathbf{V}^{M-\frac{1}{2}}$ as given in (5.209) and take a similar approach to (6.39) for the terms

$$\left(-\partial_\rho \mathbf{v}^\epsilon_{M-\frac{1}{2}} \Delta d_\Gamma - 2\left(\left(\nabla\partial_\rho \mathbf{v}^\epsilon_{M-\frac{1}{2}}\right)^T \mathbf{n}\right) + \mu_{M-\frac{1}{2}} \partial_\rho c_0 \mathbf{n} - \mathbf{u}^\epsilon_{M-\frac{1}{2}} \eta''(\rho+h_1) + \partial_\rho p^\epsilon_{M-\frac{1}{2}} \mathbf{n}\right)^\Gamma.$$

Note that for $\left(\mathbf{u}^\epsilon_{M-\frac{1}{2}}\right)^\Gamma$ a similar estimate to (6.40) holds. The other terms in $\left(\mathbf{V}^{M-\frac{1}{2}}\right)^\Gamma$ then consist of some derivative of $h^\epsilon_{M-\frac{1}{2}}$ (which we may estimate in $L^2\left(0,T_\epsilon;L^2\left(\mathbb{T}^1\right)\right)$)

multiplied by an element in $\mathcal{R}^0_\alpha$. These terms may thus be estimated as in (6.42).
Proof of (6.35): To prove (6.35) we first use the decomposition of R as in (5.185) and the decomposition of $A^{M-\frac{1}{2}}$ as in Lemma 5.36 1) to get

$$
\begin{aligned}
&\int_0^{T_\epsilon}\left|\int_{\Gamma_t(\delta)} A^{M-\frac{1}{2}} R \mathrm{d}x\right| \mathrm{d}t \\
&\leq C\epsilon^{-\frac{1}{2}} \int_0^{T_\epsilon}\int_{\mathbb{T}^1} |Z(s,t)\,\beta(s,t)| \left|\int_{-\delta}^{\delta} \theta_0'(\rho(r,s,t))\, A^{M-\frac{1}{2}} J(r,s,t)\,\mathrm{d}r\right| \mathrm{d}s\mathrm{d}t \\
&\quad + C\epsilon^{-\frac{1}{2}} \sum_{k=1}^{L_1} \int_0^{T_\epsilon}\int_{\mathbb{T}^1} |Z(s,t)| \left\|\mathtt{A}^1_k(.,s,t)\right\|_{L^2(-\delta,\delta)} \left(\int_{\frac{-\delta}{\epsilon}-h^\epsilon_A}^{\frac{\delta}{\epsilon}-h^\epsilon_A} \epsilon \left|F_1^{\mathbf{R}}\right|^2 J^\epsilon \mathrm{d}\rho\right)^{\frac{1}{2}} \left\|\mathtt{A}^2_k\right\|_{L^\infty(\mathbb{R})} \mathrm{d}s\mathrm{d}t \\
&\quad + C \sum_{k=1}^{L_1} T_\epsilon^{\frac{1}{3}} \left\|F_2^{\mathbf{R}}\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))} \left\|\mathtt{A}^1_k\right\|_{L^6(0,T_\epsilon;L^2(\Gamma_t(\delta)))} \left\|\mathtt{A}^2_k\right\|_{L^\infty(\mathbb{R})} \\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}
$$

We start with analyzing $\mathcal{I}_1$: we split

$$
A^{M-\frac{1}{2}}(\rho,x,t) = A^{M-\frac{1}{2}}|_\Gamma(\rho,x,t) + \left(A^{M-\frac{1}{2}}(\rho,x,t) - A^{M-\frac{1}{2}}|_\Gamma(\rho,x,t)\right)
$$

and write

$$
\begin{aligned}
\mathcal{I}_1^1 &:= \epsilon^{-\frac{1}{2}} \int_0^{T_\epsilon}\int_{\mathbb{T}^1} |Z(s,t)| \left|\int_{-\delta}^{\delta} \theta_0'(r,s,t)\, A^{M-\frac{1}{2}}|_\Gamma J(r,s,t)\,\mathrm{d}r\right| \mathrm{d}s\mathrm{d}t \\
\mathcal{I}_1^2 &:= \epsilon^{-\frac{1}{2}} \int_0^{T_\epsilon}\int_{\mathbb{T}^1} |Z(s,t)| \left|\int_{-\delta}^{\delta} \theta_0'(r,s,t) \left(A^{M-\frac{1}{2}}\right)^\Gamma J(r,s,t)\,\mathrm{d}r\right| \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Now for $\mathcal{I}_1^1$, we use the decomposition in Lemma 5.36 1) on $\mathbb{R}\times\Gamma$ to get

$$
\mathcal{I}_1^1 \leq \sum_{k=1}^{L_2} \epsilon^{-\frac{1}{2}} \int_0^{T_\epsilon}\int_{\mathbb{T}^1} |Z(s,t)| \left|\mathtt{A}^{1,\Gamma}_k(0,s,t)\right| \epsilon \left|\int_{-\frac{\delta}{\epsilon}-h^\epsilon_A}^{\frac{\delta}{\epsilon}-h^\epsilon_A} \mathtt{A}^{2,\Gamma}_k(\rho)\,\theta_0'(\rho)\, J^\epsilon(\rho,s,t)\,\mathrm{d}\rho\right| \mathrm{d}s\mathrm{d}t.
$$

The estimate in (6.37), (6.38), the properties of $\mathtt{A}^{2,\Gamma}_k$ as shown in (5.257) and the exponential decay of θ_0' imply

$$
\mathcal{I}_1^1 \leq C\epsilon^{\frac{1}{2}} \sum_{k=1}^{L_2} \|Z\|_{L^2(0,T_\epsilon;L^2(\Gamma_t))} \left\|\mathtt{A}^{1,\Gamma}_k\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t))} \left(e^{-\frac{\alpha\delta}{\epsilon}} + C(K)\,\epsilon\right) \leq C(K)\,\epsilon^{M+1}
$$

for $\epsilon > 0$ small enough, were we used (5.187) and (5.256) for $\mathtt{A}^{1,\Gamma}_k$.

In order to estimate $\mathcal{I}_1^2$, we again use the explicit structure of $A^{M-\frac{1}{2}}$ and first of all analyze the term

$$\left(\mu^{\epsilon}_{M-\frac{1}{2}}\right)^{\Gamma}(\rho,x,t)=\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}\right)^{\Gamma}(x,t)\,\eta(\rho)+\left(\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)^{\Gamma}(x,t)\,(1-\eta(\rho)) \tag{6.43}$$

which appears in $\left(A^{M-\frac{1}{2}}\right)^{\Gamma}$. We estimate

$$\begin{aligned}
&\epsilon^{-\frac{1}{2}}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}|Z(s,t)|\left|\int_{-\delta}^{\delta}\theta_0'(\rho(r,s,t))\left(\mu^{+,\epsilon}_{M-\frac{1}{2}}\right)^{\Gamma}\eta(\rho(r,s,t))\,J(r,s,t)\,\mathrm{d}r\right|\mathrm{d}s\mathrm{d}t\\
&\leq C\epsilon^{-\frac{1}{2}}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}|Z(s,t)|\int_{-\delta}^{\delta}\left|\theta_0'(\rho(r,s,t))\int_0^{r}\partial_{\mathbf{n}}\mu^{+,\epsilon}_{M-\frac{1}{2}}(\tilde{r},s,t)\,\mathrm{d}\tilde{r}\right|\mathrm{d}r\mathrm{d}s\mathrm{d}t\\
&\leq C\epsilon^{\frac{3}{2}}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}|Z(s,t)|\sup_{r\in(-\delta,\delta)}\left(\left|\partial_{\mathbf{n}}\mu^{+,\epsilon}_{M-\frac{1}{2}}(r,s,t)\right|\right)\int_{\frac{-\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon}\left|\theta_0'(\rho)\right|\left|\rho+h_A^\epsilon\right|\mathrm{d}\rho\mathrm{d}s\mathrm{d}t\\
&\leq C(K)\,\epsilon^{M+1}\left(\left\|\mu^{+,\epsilon}_{M-\frac{1}{2}}\right\|_{L^2(0,T_\epsilon;H^2(\Gamma_t(\delta)))}\right)\\
&\leq C(K)\,\epsilon^{M+1}.
\end{aligned}$$

Here we used the fundamental theorem of calculus in the first inequality and Lemma 2.23, the exponential decay of θ_0' and (5.187) in the third inequality. The last inequality now follows from (5.236) and the properties of the extension of $\mu^{+,\epsilon}_{M}$. We may treat the term $\left(\mu^{-,\epsilon}_{M-\frac{1}{2}}\right)^{\Gamma}(x,t)\,(1-\eta(\rho(x,t)))$ completely analogously, which finishes the sought after estimate for $\left(\mu^{\epsilon}_{M-\frac{1}{2}}\right)^{\Gamma}$.

Due to (6.41), we will now only consider the term $\left(|\nabla S(x,t)|^2\right)^{\Gamma}\partial_s^2 h^{\epsilon}_{M-\frac{1}{2}}(S(x,t),t)$ in $A^{M-\frac{1}{2}}$, the other occurring terms only involve derivatives of lower order and can be treated in the same manner. Applying similar techniques as above, we get

$$\begin{aligned}
\epsilon^{-\frac{1}{2}}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}|Z(s,t)|&\left|\int_{-\delta}^{\delta}\theta_0'(r,s,t)\left(|\nabla S|^2\right)^{\Gamma}\partial_s^2 h^{\epsilon}_{M-\frac{1}{2}}J(r,s,t)\,\mathrm{d}r\right|\mathrm{d}s\mathrm{d}t\\
&\leq CT_\epsilon^{\frac{1}{3}}\left\|Z\right\|_{L^2(0,T_\epsilon;L^2(\mathbb{T}^1))}\left\|h^{\epsilon}_{M-\frac{1}{2}}\right\|_{L^6(0,T_\epsilon;H^2(\mathbb{T}^1))}\epsilon^{\frac{3}{2}}.
\end{aligned}$$

Now (5.187), (5.235) and Proposition 2.34 3) again together with $H^{\frac{1}{2}}(0,T_\epsilon)\hookrightarrow L^6(0,T_\epsilon)$ yield the claim.

Concerning $\mathcal{I}_2$ and $\mathcal{I}_3$: using Hölder's inequality, (5.187), (5.188), the uniform boundedness of $\mathtt{A}_k^2$ in $\mathbb{R}$ and (5.256) for $\mathtt{A}_k^1$ we get

$$\mathcal{I}_2\leq C(K)\,T_\epsilon^{\frac{1}{3}}\epsilon^{-\frac{1}{2}}\epsilon^{M-\frac{1}{2}}\epsilon^{\frac{1}{2}}\epsilon=C(K)\,T_\epsilon^{\frac{1}{3}}\epsilon^{M+\frac{1}{2}}.$$

Noting the estimate for $F_2^{\mathbf{R}}$ in (5.186), we also get

$$\mathcal{I}_3 \leq C(K)\, T_\epsilon^{\frac{1}{3}} \epsilon^{M+\frac{1}{2}} = C(K)\, T_\epsilon^{\frac{1}{3}} \epsilon^{M+\frac{1}{2}}.$$

Combining the estimates for $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$, we get (6.35).
Proof of (6.36): Concerning (6.36), we first note that

$$\operatorname{div}\mathbf{v}^\epsilon_{M-\frac{1}{2}}(\rho,x,t) = \operatorname{div}\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}}(x,t)\,\eta(\rho) + \operatorname{div}\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}(x,t)\,(1-\eta(\rho)) = 0$$

by construction (cf. Lemma 5.34) and the properties of the extension operator for $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$. We show the estimate by using the explicit form of $\mathbf{W}^{M-\frac{1}{2}}$: we compute

$$\begin{aligned}
\left\|\partial_\rho \mathbf{v}^\epsilon_{M-\frac{1}{2}} \nabla^\Gamma h_1\right\|^2_{L^2(\Gamma(2\delta;T_\epsilon))} &\leq C\epsilon \int_0^{T_\epsilon}\int_{\mathbb{T}^1} \sup_{r\in(-2\delta,2\delta)} \left|\left[\mathbf{v}^\epsilon_{M-\frac{1}{2}}\right](r,s,t)\right|^2 \int_{-\frac{2\delta}{\epsilon}-h^\epsilon_A}^{\frac{2\delta}{\epsilon}-h^\epsilon_A} \left|\eta'(\rho)\right|^2 \mathrm{d}\rho \mathrm{d}s \mathrm{d}t \\
&\leq C\epsilon \left\|\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} - \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right\|^2_{L^2(0,T_\epsilon;H^2(\Gamma_t(2\delta)))} \\
&\leq C(K)\,\epsilon
\end{aligned}$$

where we again used $H^1(\Gamma_t(2\delta)) \hookrightarrow L^{2,\infty}(\Gamma_t(2\delta))$. This argumentation can easily be adapted for $\partial_\rho \mathbf{v}_1 \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}$, $\mathbf{u}_1 \cdot \mathbf{n}\eta' h^\epsilon_{M-\frac{1}{2}}$, $\mathbf{u}_1 \cdot \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}} d_\Gamma \eta'$ and $\mathbf{u}_0 \cdot \nabla^\Gamma h^\epsilon_{M-\frac{1}{2}}\rho\eta'$, $\mathbf{u}_0 \cdot \left(\nabla^\Gamma h^\epsilon_{M-\frac{1}{2}} h_1 + \nabla^\Gamma h_1 h^\epsilon_{M-\frac{1}{2}}\right)\eta'$ since $X_{T_\epsilon} \hookrightarrow C^0\left([0,T_\epsilon];C^1\left(\mathbb{T}^1\right)\right)$. To treat the $\mathbf{u}^\epsilon_{M-\frac{1}{2}} \cdot \mathbf{n}\eta'(\rho+h_1)$ term, we note that we may employ a similar strategy as we did when estimating $l^\epsilon_{M-\frac{1}{2}}$ in (6.28). We use the mean value theorem and the definition of $\mathbf{u}^\epsilon_{M-\frac{1}{2}}$ in (5.223) to estimate

$$\begin{aligned}
&\left\|\mathbf{u}^\epsilon_{M-\frac{1}{2}} \cdot \mathbf{n}\eta'(\rho+h_1)\right\|^2_{L^2(\Gamma(2\delta;T_\epsilon))} \\
&\leq C \int_0^{T_\epsilon}\int_{\mathbb{T}^1}\int_{-2\delta}^{2\delta} \left(\partial_{\mathbf{n}}\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} - \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}} + \mathbf{u}_0 h^\epsilon_{M-\frac{1}{2}}\right)(X(\gamma(r),s,t))\,\eta' \cdot (\rho(r,s,t)+h_1)\right)^2 \mathrm{d}r\mathrm{d}s\mathrm{d}t \\
&\leq C\epsilon \left\|\partial_{\mathbf{n}}\left(\mathbf{v}^{+,\epsilon}_{M-\frac{1}{2}} - \mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}} + \mathbf{u}_0 h^\epsilon_{M-\frac{1}{2}}\right)\right\|^2_{L^2(0,T_\epsilon;L^{2,\infty}(\Gamma_t(2\delta)))} \int_{\mathbb{R}} \left|\eta'(\rho+1)\right|^2 \mathrm{d}\rho \\
&\leq C(K)\,\epsilon,
\end{aligned}$$

where $\gamma(r)$ is a suitable point in $(0,r)$ and where we used (5.236), (5.235) and $H^1(\Gamma_t(2\delta)) \hookrightarrow L^{2,\infty}(\Gamma_t(2\delta))$ in the last estimate. A similar estimate can be used for $\mathbf{u}^\epsilon_{M-\frac{1}{2}} \cdot \nabla^\Gamma h_1 d_\Gamma \eta'$, which proves the claim. □

Before we formulate and prove the main result of this section, we show the following proposition, which will be used as a substitute for the matching conditions (5.25)–(5.27) for $\mu^\epsilon_{M-\frac{1}{2}}$, $\mathbf{v}^\epsilon_{M-\frac{1}{2}}$ and $p^\epsilon_{M-\frac{1}{2}}$.

Proposition 6.10. *There is $\epsilon_2 \in (0, \epsilon_1]$ such that for all $\epsilon \in (0, \epsilon_2)$*

$$D_\rho^k D_x^l \left(\mu_{M-\frac{1}{2}}^\epsilon (\rho, x, t) - \left(\mu_{M-\frac{1}{2}}^{+,\epsilon} \chi_{\overline{\Omega_{T_0}^+}} + \mu_{M-\frac{1}{2}}^{-,\epsilon} \chi_{\Omega_{T_0}^-} \right)(x,t) \right)\Big|_{\rho=\rho(x,t)} = 0$$

$$D_t^m D_\rho^k D_x^l \left(\mathbf{v}_{M-\frac{1}{2}}^\epsilon (\rho, x, t) - \left(\mathbf{v}_{M-\frac{1}{2}}^{+,\epsilon} \chi_{\overline{\Omega_{T_0}^+}} + \mathbf{v}_{M-\frac{1}{2}}^{-,\epsilon} \chi_{\Omega_{T_0}^-} \right)(x,t) \right)\Big|_{\rho=\rho(x,t)} = 0$$

$$p_{M-\frac{1}{2}}^\epsilon (\rho(x,t), x, t) - \left(p_{M-\frac{1}{2}}^{+,\epsilon} \chi_{\overline{\Omega_{T_0}^+}} + p_{M-\frac{1}{2}}^{-,\epsilon} \chi_{\Omega_{T_0}^-} \right)(x,t) = 0$$

for all $(x,t) \in \Gamma(2\delta; T_\epsilon) \setminus \Gamma(\delta; T_\epsilon)$ and $m, k, l \geq 0$.

Proof. Let $(x,t) \in \Gamma(2\delta; T_\epsilon) \setminus \Gamma(\delta; T_\epsilon)$ such that $x \in \Omega^+(t)$. Since $\|h_A^\epsilon\|_{L^\infty((0,T_\epsilon)\times\mathbb{T}^1)} < C(K)$ for all $\epsilon \in (0, \epsilon_0)$ due to (6.9), we have

$$\rho(x,t) = \frac{d_\Gamma(x,t)}{\epsilon} - h_A^\epsilon(x,t) \geq \frac{\delta}{2\epsilon} \geq 1$$

for $\epsilon > 0$ small enough. Then we have due to the explicit form of $\mu_{M-\frac{1}{2}}^\epsilon$ in (5.253)

$$\begin{aligned} D_\rho^k D_x^l \left(\mu_{M-\frac{1}{2}}^\epsilon (\rho, x, t) - \mu_{M-\frac{1}{2}}^{+,\epsilon}(x,t) \right)\Big|_{\rho=\rho(x,t)} &= D_\rho^k (\eta(\rho(x,t)) - 1) D_x^l \left(\mu_{M-\frac{1}{2}}^{+,\epsilon} - \mu_{M-\frac{1}{2}}^{-,\epsilon} \right) \\ &= 0 \end{aligned}$$

for $l, k \geq 0$, as $\eta \equiv 1$ in $(1, \infty)$. As $\eta \equiv 0$ in $(-\infty, -1)$, we may use a similar approach to gain an analogous identity if $x \in \Omega^-(t)$. The results for $\mathbf{v}_{M-\frac{1}{2}}^\epsilon$ and $p_{M-\frac{1}{2}}^\epsilon$ follow in the same way. □

The next corollary is a direct result of Proposition 6.10 and the matching conditions for the integer orders.

Corollary 6.11. *There is $\epsilon_2 \in (0, \epsilon_1]$ such that for all $\epsilon \in (0, \epsilon_2)$*

$$\begin{aligned}
\left\| D_x^l (\mu_I - \mu_O) \right\|_{L^\infty(\Gamma(2\delta;T_\epsilon)\setminus\Gamma(\delta;T_\epsilon))} + \left\| D_x^l (\mu_O - \mu_{\mathbf{B}}) \right\|_{L^\infty\left(\partial_{T_\epsilon}\Omega(\delta)\setminus\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} &\leq C(K) e^{-\frac{\tilde{C}}{\epsilon}}, \\
\left\| D_x^l (c_I - c_O) \right\|_{L^\infty(\Gamma(2\delta;T_\epsilon)\setminus\Gamma(\delta;T_\epsilon))} + \left\| D_x^l (c_O - c_{\mathbf{B}}) \right\|_{L^\infty\left(\partial_{T_\epsilon}\Omega(\delta)\setminus\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} &\leq C(K) e^{-\frac{\tilde{C}}{\epsilon}}, \\
\left\| D_x^l (\mathbf{v}_I - \mathbf{v}_O) \right\|_{L^\infty(\Gamma(2\delta;T_\epsilon)\setminus\Gamma(\delta;T_\epsilon))} + \left\| D_x^l (\mathbf{v}_O - \mathbf{v}_{\mathbf{B}}) \right\|_{L^\infty\left(\partial_{T_\epsilon}\Omega(\delta)\setminus\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} &\leq C(K) \epsilon^{M+1}, \\
\| p_I - p_O \|_{L^\infty(\Gamma(2\delta;T_\epsilon)\setminus\Gamma(\delta;T_\epsilon))} + \| p_O - p_{\mathbf{B}} \|_{L^\infty\left(\partial_{T_\epsilon}\Omega(\delta)\setminus\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} &\leq C(K) e^{-\frac{\tilde{C}}{\epsilon}}
\end{aligned}$$

for $l \in \{0, 1\}$ and constants $C(K), \tilde{C} > 0$.

Proof. Due to the matching conditions (5.25) and Proposition 6.10 it holds

$$\sup_{(x,t)\in\Omega_{T_\epsilon}^+ \cap \Gamma(2\delta;T_\epsilon)\setminus\Gamma(\delta;T_\epsilon)} \left| \partial_x^m \partial_\rho^i \left(\mu_k(\rho, x, t) - \mu_k^+(x,t) \right)\big|_{\rho=\rho(x,t)} \right| \leq C e^{-\alpha\rho(x,t)}$$

for $m, i \in \{0, 1\}$ and $k \in I_{M-\frac{1}{2}}^{M+1}$. As in the proof of Proposition 6.10, we have $\rho(x,t) \geq \frac{\delta}{2\epsilon}$ for ϵ small enough, which yields the sought after estimate as we have a similar estimate in

$\Omega^-_{T_0}$. Note that the term $\nabla^\Gamma h^\epsilon_A$, which occurs when considering $\nabla\mu_{I,k}$, can be uniformly controlled, cf. Remark 6.4. Close to $\partial\Omega$ we have by (5.83)

$$\sup_{(x,t)\in\partial_{T_\epsilon}\Omega(\delta)\setminus\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)}\left|\partial_x^m\partial_z^i\left(\mu_k^{\mathbf{B}}(z,x,t)-\mu_k^-(x,t)\right)\big|_{z=z(x)}\right|\leq Ce^{-\alpha\frac{\delta}{2\epsilon}}$$

for $z(x)=\frac{d_{\mathbf{B}}(x)}{\epsilon}$ and $k\in I^{M+1}_{M-\frac{1}{2}}$; note in particular $\mu^{\mathbf{B}}_{M-\frac{1}{2}}=\mu^{-,\epsilon}_{M-\frac{1}{2}}$ as defined in (6.4). This shows the estimate for μ_I,μ_O and $\mu_{\mathbf{B}}$. The other estimates follow completely analogously. Note that for $k=M+1$, we have

$$\begin{aligned}\epsilon^{M+1}&\left\|D_x^l\left(\mathbf{v}_{O,M+1}-\mathbf{v}_{\mathbf{B},M+1}\right)\right\|_{L^\infty\left(\partial_{T_\epsilon}\Omega(\delta)\setminus\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)}\\ &\leq Ce^{-\alpha\frac{\delta}{2\epsilon}},+\epsilon^{M+1}\left\|\mathbf{v}^{\mathbf{B}}_{M+1}(0,.)\right\|_{L^\infty\left(\partial_{T_0}\Omega(\delta)\right)}\end{aligned}$$

accounting for the special case. □

6.3. Main Theorem

The following theorem is the main result of this section and at the same time proves Theorem 4.3.

Theorem 6.12 (Remainder Terms)**.** *Let Assumptions 1.1 and 6.3 hold and let for $\epsilon\in(0,\epsilon_0)$ the functions c^ϵ_A, μ^ϵ_A, $\mathbf{v}^\epsilon_A$, p^ϵ_A, h^ϵ_A be defined as in Definition 6.2 and $\mathbf{r}^\epsilon_{\mathrm{S}}$, $r^\epsilon_{\mathrm{div}}$, $r^\epsilon_{\mathrm{CH1}}$, $r^\epsilon_{\mathrm{CH2}}$ be given as in (4.7)–(4.10), for $\mathbf{w}^\epsilon_1:=\frac{1}{\epsilon^{M-\frac{1}{2}}}\tilde{\mathbf{w}}^\epsilon_1$. Here $\tilde{\mathbf{w}}^\epsilon_1$ is the weak solution to (5.174)–(5.176) with $H=\left(h^\epsilon_{M-\frac{1}{2}}\right)_{\epsilon\in(0,\epsilon_0)}$. Moreover, let $\varphi\in L^\infty\left(0,T_\epsilon;H^1(\Omega)\right)$ and $R=c^\epsilon-c^\epsilon_A$. Then there is $\epsilon_2\in(0,\epsilon_1]$ and a constant $C(K)>0$ such that for all $\epsilon\in(0,\epsilon_2)$*

$$\int_0^{T_\epsilon}\left|\int_\Omega r^\epsilon_{\mathrm{CH1}}\varphi\mathrm{d}x\right|\mathrm{d}t\leq C(K)\left(T_\epsilon^{\frac{1}{2}}+\epsilon^{\frac{1}{2}}\right)\epsilon^M\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))},\tag{6.44}$$

$$\int_0^{T_\epsilon}\left|\int_\Omega r^\epsilon_{\mathrm{CH2}}R\mathrm{d}x\right|\mathrm{d}t\leq C(K)\left(T_\epsilon^{\frac{1}{3}}+\epsilon^{\frac{1}{2}}\right)\epsilon^{2M}.\tag{6.45}$$

$$\|\mathbf{r}^\epsilon_{\mathrm{S}}\|_{L^2\left(0,T_\epsilon;(H^1(\Omega))'\right)}\leq C(K)\epsilon^M,\tag{6.46}$$

$$\|r^\epsilon_{\mathrm{div}}\|_{L^2(\Omega_{T_\epsilon})}\leq C(K)\epsilon^M.\tag{6.47}$$

Proof. As before, we will use the notation $\psi(r,s,t):=\psi(X(r,s,t))$ for $(r,s,t)\in(-2\delta,2\delta)\times\mathbb{T}^1\times[0,T_\epsilon]$ for functions $\psi:\Gamma(2\delta;T_\epsilon)\to\mathbb{R}$. Let in the following $\tilde{\epsilon}_2\in(0,\epsilon_1]$ be chosen such that the results of Section 6.2 hold and let $\epsilon\in(0,\tilde{\epsilon}_2)$.

Proof of (6.44): Since $\xi(d_\Gamma)\equiv1$ in $\Gamma(\delta;T_0)$, we have $r^\epsilon_{\mathrm{CH1}}=r^\epsilon_{\mathrm{CH1},I}$ in $\Gamma(\delta;T_\epsilon)$ with $r^\epsilon_{\mathrm{CH1},I}$ as in (6.11). Now

$$\int_0^{T_\epsilon}\left|\int_{\Gamma_t(\delta)}r^\epsilon_{\mathrm{CH1},I}\varphi\mathrm{d}x\right|\mathrm{d}t\leq C(K)\left(T_\epsilon^{\frac{1}{2}}+\epsilon^{\frac{1}{2}}\right)\epsilon^M\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

holds due to Lemma 6.6 and Lemma 6.9, more precisely (6.33).

Moreover, we have $(1-\xi(d_\Gamma))(1-\xi(2d_{\mathbf{B}}))\equiv 1$ in $\Omega_{T_\epsilon}\backslash(\Gamma(2\delta;T_\epsilon)\cup\partial_{T_\epsilon}\Omega(\delta))$ and thus $r^\epsilon_{\mathrm{CH1}}=r^\epsilon_{\mathrm{CH1},O}$ in that domain, with $r^\epsilon_{\mathrm{CH1},O}$ as in (6.15). Now all terms in $r^\epsilon_{\mathrm{CH1},O}$ which do not involve $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$ can be estimated in $L^\infty(\Omega_{T_0}\backslash\Gamma(2\delta;T_0))$, yielding the sought-after estimate. The terms involving $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$ can be treated by using Hölder's inequality and (5.236), i.e.

$$
\begin{aligned}
\epsilon^{M+\frac{1}{2}}\int_0^{T_\epsilon}\int_{\Omega^+(t)\backslash\Gamma_t(2\delta)}\left|\mathbf{v}^{\epsilon,+}_{M-\frac{1}{2}}\cdot\nabla c^+_j\varphi\right|\mathrm{d}x\mathrm{d}t &\le CT_\epsilon^{\frac{1}{2}}\epsilon^{M+\frac{1}{2}}\left\|\mathbf{v}^{\epsilon,+}_{M-\frac{1}{2}}\right\|_{L^2\left(\Omega^+_{T_\epsilon}\right)}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\\
&\le C(K)\,T_\epsilon^{\frac{1}{2}}\epsilon^{M+\frac{1}{2}}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \qquad (6.48)
\end{aligned}
$$

for $j\in\{1,\dots,M+1\}$. The same argumentation also holds in $\Omega^-(t)$.

Close to the boundary, in $\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)$, we have $\xi(2d_{\mathbf{B}})\equiv 1$ and thus $r^\epsilon_{\mathrm{CH1}}=r^\epsilon_{\mathrm{CH1},\mathbf{B}}$. As in the outer case, all terms not involving $\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}$ may be estimated in $L^\infty(\partial_{T_0}\Omega(\delta))$, the rest can be estimated as in (6.48).

Next, we give estimates for $r^\epsilon_{\mathrm{CH1}}$ in $\Gamma(2\delta;T_\epsilon)\backslash\Gamma(\delta;T_\epsilon)$: By definition of c^ϵ_A and μ^ϵ_A in (6.6) we have

$$
\begin{aligned}
r^\epsilon_{\mathrm{CH1}} &= \xi(d_\Gamma)\,r^\epsilon_{\mathrm{CH1},I}+(1-\xi(d_\Gamma))\,r^\epsilon_{\mathrm{CH1},O}\\
&\quad+\xi'(d_\Gamma)\left(\partial_t d_\Gamma+\mathbf{v}^\epsilon_A\cdot\mathbf{n}+\epsilon^{M-\frac{1}{2}}\mathbf{w}^\epsilon_1|_\Gamma\cdot\mathbf{n}\xi(d_\Gamma)\right)(c_I-c_O)\\
&\quad+\left(\mathbf{v}^\epsilon_A\cdot(\xi(d_\Gamma)\nabla c_I+(1-\xi(d_\Gamma))\nabla c_O)-\xi(d_\Gamma)\mathbf{v}_I\cdot\nabla c_I-(1-\xi(d_\Gamma))\mathbf{v}_O\cdot\nabla c_O\right)\\
&\quad-(\mu_I-\mu_O)\left(\xi''+\xi'\Delta d_\Gamma\right)+\epsilon^{M-\frac{1}{2}}\mathbf{w}^\epsilon_1|_\Gamma\cdot\nabla c_O\xi(d_\Gamma)(1-\xi(d_\Gamma))\\
&\quad-2\xi'\mathbf{n}\cdot\nabla(\mu_I-\mu_O) \qquad (6.49)
\end{aligned}
$$

The term $(1-\xi(d_\Gamma))\,r^\epsilon_{CH1,O}$ may be treated in the same fashion as in the outer domain $\Omega_{T_\epsilon}\backslash(\Gamma(2\delta;T_\epsilon)\cup\partial_{T_\epsilon}\Omega(\delta))$, estimating $|1-\xi(d_\Gamma)|\le 1$. Regarding $\xi(d_\Gamma)\,r^\epsilon_{\mathrm{CH1},I}$, there is a subtlety we have to deal with: all appearing terms in the explicit structure of the difference $\xi(d_\Gamma)\left(r^\epsilon_{\mathrm{CH1},I}-\epsilon^{M-\frac{3}{2}}B^{M-\frac{1}{2}}\right)$ can be estimated with the help of Lemma 6.6. But we may not simply use (6.33) for $\xi(d_\Gamma)\,\epsilon^{M-\frac{3}{2}}B^{M-\frac{1}{2}}\varphi$ in $\Gamma(2\delta)$.

To remedy that situation let $J=(-2\delta,-\delta)\cup(\delta,2\delta)$ and we compute, using Lemma 5.36 2),

$$
\begin{aligned}
&\int_0^{T_\epsilon}\int_{\Gamma_t(2\delta)\backslash\Gamma_t(\delta)}\left|\xi(d_\Gamma)\,\epsilon^{M-\frac{3}{2}}B^{M-\frac{1}{2}}\varphi\right|\mathrm{d}(x,t)\\
&\quad\le C\epsilon^{M-\frac{3}{2}}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}\int_J\left|B^{M-\frac{1}{2}}\varphi\right|\mathrm{d}r\mathrm{d}s\mathrm{d}t\\
&\quad\le C\epsilon^{M-\frac{3}{2}}\sum_{k=1}^{K_1}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}\|\varphi(.,s,t)\|_{L^\infty(-2\delta,2\delta)}\int_J\left|\mathsf{B}^1_k(r,s,t)\,\mathsf{B}^2_k(\rho(r,s,t))\right|\mathrm{d}r\mathrm{d}s\mathrm{d}t\\
&\quad\le C\epsilon^{M-\frac{3}{2}}\sum_{k=1}^{K_1}\int_0^{T_\epsilon}\int_{\mathbb{T}^1}\|\varphi(.,s,t)\|_{L^\infty(-2\delta,2\delta)}\left\|\mathsf{B}^1_k(.,s,t)\right\|_{L^2(-2\delta,2\delta)}\left\|\mathsf{B}^2_k(\rho)\right\|_{L^2(J)}\mathrm{d}s\mathrm{d}t. \qquad (6.50)
\end{aligned}
$$

Now since $\sup_{\epsilon\in(0,\epsilon_0)} \|h_A^\epsilon\|_{L^\infty((0,T_\epsilon)\times\mathbb{T}^1)} =: c_K < \infty$ holds (cf. (6.9)), we get

$$\frac{\delta}{\epsilon} - h_A^\epsilon \geq \frac{\delta}{2\epsilon} \tag{6.51}$$

for $\epsilon > 0$ small enough. Thus, we may calculate for $k \in \{1, \dots, K_1\}$

$$\begin{aligned}\int_\delta^{2\delta} \left|\mathtt{B}_k^2\left(\rho\left(r,p,t\right)\right)\right|^2 \mathrm{d}r &\leq \epsilon \int_{\frac{\delta}{\epsilon}-h_A^\epsilon(p,t)}^{\frac{2\delta}{\epsilon}-h_A^\epsilon(p,t)} \left|\mathtt{B}_k^2\left(\rho\right)\right|^2 \mathrm{d}\rho \leq \epsilon \int_{\frac{\delta}{2\epsilon}}^{\infty} \left|\mathtt{B}_k^2\left(\rho\right)\right|^2 \mathrm{d}\rho \\ &\leq \epsilon C_1 e^{-\frac{C_2}{\epsilon}} \end{aligned}\tag{6.52}$$

for some constants $C_1, C_2 > 0$, where we used $\mathtt{B}_k^2 \in \mathcal{O}\left(e^{-\alpha|\rho|}\right)$ due to Lemma 5.36 2). A similar estimate holds on $(-2\delta, -\delta)$ and we may thus continue estimating (6.50) and get

$$\begin{aligned}&\int_0^{T_\epsilon} \int_{\Gamma_t(2\delta)\setminus\Gamma_t(\delta)} \left|\xi\left(d_\Gamma\right)\epsilon^{M-\frac{3}{2}} B^{M-\frac{1}{2}}\varphi\right| \mathrm{d}\left(x,t\right) \\ &\qquad \leq C T_\epsilon^{\frac{1}{2}} \epsilon^{M-1} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \sum_{k=1}^{K_1} \left\|\mathtt{B}_k^1\right\|_{L^2(\Gamma(2\delta;T_\epsilon))} e^{-\frac{C_2}{\epsilon}} \\ &\qquad \leq C\left(K\right) T_\epsilon^{\frac{1}{2}} \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \end{aligned}\tag{6.53}$$

for $\epsilon > 0$ small enough, where we used $H^1\left(\Gamma_t\left(2\delta\right)\right) \hookrightarrow L^{2,\infty}\left(\Gamma_t\left(2\delta\right)\right)$ in the first inequality and (5.258) in the second inequality. This yields the desired estimate.

Concerning $\xi'\left(d_\Gamma\right)\left(\partial_t d_\Gamma + \mathbf{v}_A^\epsilon \cdot \mathbf{n} + \epsilon^{M-\frac{1}{2}} \mathbf{w}_1^\epsilon|_\Gamma \cdot \mathbf{n}\xi\left(d_\Gamma\right)\right)\left(c_I - c_O\right)$ in (6.49), we exemplarily compute

$$\begin{aligned}&\int_0^{T_\epsilon} \int_{\Gamma_t(2\delta)\setminus\Gamma_t(\delta)} \left|\epsilon^{M-\frac{1}{2}} \mathbf{w}_1^\epsilon|_\Gamma \cdot \mathbf{n}\left(c_I - c_O\right)\varphi\right| \mathrm{d}x\mathrm{d}t \\ &\qquad \leq C T_\epsilon^{\frac{1}{2}} \left\|\epsilon^{M-\frac{1}{2}} \mathbf{w}_1^\epsilon\right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \|c_I - c_O\|_{L^\infty(\Gamma(2\delta;T_\epsilon)\setminus\Gamma(\delta;T_\epsilon))} \\ &\qquad \leq C\left(K\right) T_\epsilon^{\frac{1}{2}} \epsilon^{M-\frac{1}{2}} e^{-\frac{C_2}{\epsilon}} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \\ &\qquad \leq C\left(K\right) T_\epsilon^{\frac{1}{2}} \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \end{aligned}\tag{6.54}$$

where we used the continuity of the trace operator $H^1\left(\Omega^+\left(t\right)\right) \hookrightarrow L^2\left(\Gamma_t\right)$ and the embedding $H^1\left(\Gamma_t\left(2\delta\right)\right) \hookrightarrow L^{2,\infty}\left(\Gamma_t\left(2\delta\right)\right)$ in the first inequality. In the second inequality we used Lemma 5.29 and Corollary 6.11. An analogous (but simpler) argumentation may be used for $\partial_t d_\Gamma \in L^\infty\left(\Gamma\left(2\delta; T_0\right)\right)$ and $\left(\mathbf{v}_A^\epsilon - \epsilon^{M-\frac{1}{2}} \mathbf{v}_{A,M-\frac{1}{2}}\right) \in L^\infty\left(\Omega_{T_0}\right)$ (cf. Definition 6.2 for notations). The estimate for $\epsilon^{M-\frac{1}{2}} \mathbf{v}_{A,M-\frac{1}{2}}^\epsilon$ then follows by using (5.236).

The terms $2\xi'\mathbf{n} \cdot \nabla\left(\mu_I - \mu_O\right) + \left(\mu_I - \mu_O\right)\left(\xi'' + \xi'\Delta d_\Gamma\right)$ in (6.49) may be treated by using Corollary 6.11.

For the third line of (6.49), we calculate

$$\begin{aligned}\mathbf{v}_A^\epsilon \cdot \nabla c_I - \mathbf{v}_I \cdot \nabla c_I &= \left(1 - \xi\left(d_\Gamma\right)\right)\left(\mathbf{v}_O - \mathbf{v}_I\right) \cdot \nabla c_I \\ \mathbf{v}_A^\epsilon \cdot \nabla c_O - \mathbf{v}_O \cdot \nabla c_O &= \xi\left(d_\Gamma\right)\left(\mathbf{v}_I - \mathbf{v}_O\right) \cdot \nabla c_I \end{aligned}$$

and Corollary 6.11 yields the estimate as before.

The only remaining, not estimated term in (6.49) can be treated by

$$\int_0^{T_\epsilon} \int_{\Gamma_t(2\delta)\setminus\Gamma_t(\delta)} \left|\epsilon^{M-\frac{1}{2}} \mathbf{w}_1^\epsilon|_\Gamma \cdot \nabla c_O \varphi\right| \mathrm{d}x\mathrm{d}t \leq C\epsilon T_\epsilon^{\frac{1}{2}} \left\|\tilde{\mathbf{w}}_1^\epsilon\right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$
$$\leq C(K)\,\epsilon^{M+\frac{1}{2}} \left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))},$$

where we again used the continuity of the trace combined with Lemma 5.29 and the fact that $\nabla c_O = \mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega_{T_0}^\pm\right)$.

Thus, we need only consider $r_{\mathrm{CH1}}^\epsilon$ in $\partial_{T_\epsilon}\Omega(\delta)\setminus\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)$: here we get a structure very similar to (6.49), as we have

$$\begin{aligned} r_{\mathrm{CH1}}^\epsilon &= (1-\xi(2d_{\mathbf{B}}))\, r_{\mathrm{CH1},O}^\epsilon + \xi(2d_{\mathbf{B}})\, r_{\mathrm{CH1},\mathbf{B}}^\epsilon + 2\xi'(2d_{\mathbf{B}})\left(\partial_t d_{\mathbf{B}} + \mathbf{v}_A^\epsilon\cdot \mathbf{n}_{\partial\Omega}\right)(c_{\mathbf{B}} - c_O) \\ &\quad + \left(\mathbf{v}_A^\epsilon \cdot \left((1-\xi(2d_{\mathbf{B}}))\nabla c_O + \xi(2d_{\mathbf{B}})\nabla c_{\mathbf{B}}\right) - (1-\xi(2d_{\mathbf{B}}))\,\mathbf{v}_O\cdot\nabla c_O\right) \\ &\quad - \xi(2d_{\mathbf{B}})\,\mathbf{v}_{\mathbf{B}}\cdot\nabla c_{\mathbf{B}} - 4\xi'\mathbf{n}_{\partial\Omega}\cdot\nabla(\mu_{\mathbf{B}}-\mu_O) - (\mu_{\mathbf{B}}-\mu_O)\left(4\xi'' + 2\xi'\Delta d_{\mathbf{B}}\right). \end{aligned}$$

The proof now follows in a very similar fashion to the one for (6.49), considering the already shown estimates for $r_{\mathrm{CH1},O}^\epsilon$ and $r_{\mathrm{CH1},\mathbf{B}}^\epsilon$ as well as the estimates close to the boundary in Corollary 6.11. This shows (6.44).

Proof of (6.45): We use a similar approach as before: in $\Gamma(\delta;T_\epsilon)$ we have $r_{\mathrm{CH2}}^\epsilon = r_{\mathrm{CH2},I}^\epsilon$, where $r_{\mathrm{CH2},I}^\epsilon$ is defined in (6.12). For all terms in $r_{\mathrm{CH2},I}^\epsilon$, which can be estimated in $L^\infty(\Gamma(2\delta;T_\epsilon))$ (uniformly in ϵ), we may use Proposition 6.8 to gain the claim. Noting (6.9), the only terms that may not be treated in this fashion are the ones involving $\Delta^\Gamma h_{M-\frac{1}{2}}^\epsilon$ and $A^{M-\frac{1}{2}}$. Regarding $\epsilon^{M-\frac{1}{2}}A^{M-\frac{1}{2}}$, we may use Lemma 6.9, more precisely (6.35). Concerning $\Delta^\Gamma h_{M-\frac{1}{2}}^\epsilon$, we compute

$$\epsilon^{M+\frac{1}{2}} \int_{\Gamma(\delta;T_\epsilon)} \left|\Delta^\Gamma h_{M-\frac{1}{2}}\partial_\rho c_1 R\right| \mathrm{d}(x,t) \leq C\epsilon^{M+1}\left\|\left(\partial_s^2 h_{M-\frac{1}{2}}^\epsilon, \partial_s h_{M-\frac{1}{2}}^\epsilon\right)\right\|_{L^\infty(0;T_\epsilon;L^2(\mathbb{T}^1))}\|R\|_{L^2(\Omega_{T_\epsilon})}$$
$$\cdot\left\|\sup_{(x,t)\in\Gamma(2\delta;T_0)}\left|\partial_\rho c_1(.,x,t)\right|\right\|_{L^2(\mathbb{R})}$$
$$\leq C(K)\,\epsilon^{2M+\frac{1}{2}},$$

where we used $\partial_\rho c_1 \in \mathcal{R}_\alpha$, $X_T \hookrightarrow C^0\left([0,T];H^2\left(\mathbb{T}^1\right)\right)$ (cf. Proposition 2.34 2)) and the L^2-estimate for R in (4.6).

In $\Omega_{T_\epsilon}\setminus(\Gamma(2\delta;T_\epsilon)\cup\partial_{T_\epsilon}\Omega(\delta))$, we have $r_{\mathrm{CH2}}^\epsilon = r_{\mathrm{CH2},O}^\epsilon$ with $r_{\mathrm{CH2},O}^\epsilon$ as in (6.16). For that, we compute (exemplarily in $\Omega^+(t)$ to shorten the calculations)

$$\int_0^{T_\epsilon}\int_{\Omega^+(t)\setminus\Gamma_t(2\delta)} \left|\epsilon^{M-\frac{1}{2}}\mu_{M-\frac{1}{2}}^{+,\epsilon}R\right|\mathrm{d}x\mathrm{d}t \leq CT_\epsilon^{\frac{1}{3}}\epsilon^{M-\frac{1}{2}}\left\|\mu_{M-\frac{1}{2}}^{+,\epsilon}\right\|_{L^6(0,T_\epsilon;L^2(\Omega^+(t)))}\|R\|_{L^2(L^2(\Omega_{T_\epsilon}\setminus\Gamma(\delta;T_\epsilon)))}$$
$$\leq C(K)\,T_\epsilon^{\frac{1}{3}}\epsilon^{2M}$$

where we used (5.236) and (4.6). As $c_i^\pm \in L^\infty\left(\Omega_{T_0}^\pm\right)$ for all $i\in\{0,\ldots,M+1\}$, a similar estimate follows by again using (4.6) for the remaining terms in $r_{\mathrm{CH2},O}^\epsilon$ (cf. Remark 5.1 for the $\tilde{f}$ term).

In $\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)$, it holds $r^\epsilon_{\mathrm{CH2}} = r^\epsilon_{\mathrm{CH2},\mathbf{B}}$ and we may proceed as in $\Omega_{T_\epsilon}\backslash\left(\Gamma\left(2\delta;T_\epsilon\right)\cup\partial_{T_\epsilon}\Omega\left(\delta\right)\right)$. In $\Gamma\left(2\delta;T_\epsilon\right)\backslash\Gamma\left(\delta;T_\epsilon\right)$, we have

$$\begin{aligned} r^\epsilon_{\mathrm{CH2}} &= \xi\left(d_\Gamma\right)\left(\epsilon\Delta c_I + \mu_I\right) + \left(1-\xi\left(d_\Gamma\right)\right)\left(\epsilon\Delta c_O + \mu_O\right) - \epsilon^{-1} f'\left(c^\epsilon_A\right) \\ &\quad + \epsilon\left(\left(c_I - c_O\right)\left(\xi''\left(d_\Gamma\right) + \xi'\left(d_\Gamma\right)\Delta d_\Gamma\right) + \epsilon 2\xi'\left(d_\Gamma\right)\mathbf{n}\cdot\nabla\left(c_I - c_O\right)\right). \end{aligned} \tag{6.55}$$

The estimate for the second line in (6.55) follows by similar techniques as in the proof of (6.44), by using Corollary 6.11.

For the first line, we use a Taylor expansion to get

$$\begin{aligned} f'\left(c^\epsilon_A\right) &= f'\left(c_I\right) + f''\left(\sigma_1\left(c^\epsilon_A, c_I\right)\right)\left(c_O - c_I\right)\left(1-\xi\left(d_\Gamma\right)\right), \\ f'\left(c^\epsilon_A\right) &= f'\left(c_O\right) + f''\left(\sigma_2\left(c^\epsilon_A, c_O\right)\right)\left(c_O - c_I\right)\left(-\xi\left(d_\Gamma\right)\right), \end{aligned}$$

where $\sigma_1\left(c^\epsilon_A, c_I\right)$, $\sigma_2\left(c^\epsilon_A, c_O\right)$ are suitable intermediate points. Thus, the first line of (6.55) reads

$$\begin{aligned} &\xi\left(d_\Gamma\right)\left(r^\epsilon_{\mathrm{CH2},I}\right) + \left(1-\xi\left(d_\Gamma\right)\right)\left(r^\epsilon_{\mathrm{CH2},O}\right) \\ &\quad + \epsilon^{-1}\left(c_O - c_I\right)\xi\left(d_\Gamma\right)\left(1-\xi\left(d_\Gamma\right)\right)\left(-f''\left(\sigma_2\left(c^\epsilon_A, c_O\right)\right) + f''\left(\sigma_1\left(c^\epsilon_A, c_I\right)\right)\right). \end{aligned} \tag{6.56}$$

Now $c^\epsilon_A, c_O, c_I \in L^\infty\left(\Gamma\left(2\delta;T_0\right)\backslash\Gamma\left(\delta;T_0\right)\right)$ uniformly in ϵ and thus

$$\left|f''\left(\sigma_1\right)\right|, \left|f''\left(\sigma_2\right)\right| < \sup_{x\in[-C_1,C_1]}\left|f''\left(x\right)\right| \leq C.$$

Since $\left(c_O - c_I\right) = \mathcal{O}\left(e^{-\frac{C}{\epsilon}}\right)$ in $L^\infty\left(\Gamma\left(2\delta;T_\epsilon\right)\backslash\Gamma\left(\delta;T_\epsilon\right)\right)$ for small ϵ (see Corollary 6.11), we may estimate the last part in (6.56) as before and the term involving $r^\epsilon_{\mathrm{CH2},O}$ as in the case of $\Omega_{T_\epsilon}\backslash\left(\Gamma\left(2\delta;T_\epsilon\right)\cup\partial_{T_\epsilon}\Omega\left(\delta\right)\right)$. Regarding $r^\epsilon_{\mathrm{CH2},I}$, although we may not use the decomposition of R anymore (Proposition 6.8 only holds in $\Gamma\left(\delta;T_\epsilon\right)$), we may now use $\|R\|_{L^2(0,T_\epsilon;L^2(\Omega\backslash\Gamma_t(\delta)))} \leq C\left(K\right)\epsilon^{M+\frac{1}{2}}$ due to (4.6a). Thus, all terms in $r^\epsilon_{\mathrm{CH2},I}$, which can be estimated in $L^\infty\left(\Gamma\left(2\delta;T_\epsilon\right)\right)$ (uniformly in ϵ), are of no concern. This leaves us with terms involving $\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}$ (which may be treated as before) and $\xi\left(d_\Gamma\right)\epsilon^{M-\frac{1}{2}}A^{M-\frac{1}{2}}$, since (6.35) only holds inside $\Gamma\left(\delta;T_\epsilon\right)$. According to (4.6) and Lemma 5.36 1) we may estimate

$$\begin{aligned} \epsilon^{M-\frac{1}{2}}\int_{\Gamma(2\delta;T_\epsilon)\backslash\Gamma(\delta;T_\epsilon)}\left|A^{M-\frac{1}{2}}R\right|\mathrm{d}\left(x,t\right) &\leq C\left(K\right)\epsilon^{2M}\sum_{k=1}^{L_1}\left\|\mathtt{A}^1_k\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(2\delta)))} \\ &\leq C\left(K\right)\epsilon^{2M}T_\epsilon^{\frac{1}{3}}. \end{aligned}$$

Now we only have to consider the situation in $\partial_{T_\epsilon}\Omega\left(\delta\right)\backslash\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)$: The structure heavily resembles (6.55) in that we get

$$\begin{aligned} r^\epsilon_{\mathrm{CH2}} &= \xi\left(2d_{\mathbf{B}}\right)\left(\epsilon\Delta c_{\mathbf{B}} + \mu_{\mathbf{B}}\right) + \left(1-\xi\left(2d_{\mathbf{B}}\right)\right)\left(\epsilon\Delta c_O + \mu_O\right) - \epsilon^{-1}f'\left(c^\epsilon_A\right) \\ &\quad + \epsilon\left(\left(c_{\mathbf{B}} - c_O\right)\left(4\xi''\left(2d_{\mathbf{B}}\right) + 2\xi'\left(2d_{\mathbf{B}}\right)\Delta d_\Gamma\right) + \epsilon 4\xi'\left(d_{\mathbf{B}}\right)\mathbf{n}_{\partial\Omega}\cdot\nabla\left(c_{\mathbf{B}} - c_O\right)\right) \end{aligned}$$

and the estimate follows in a similar way as for (6.55). Thus, we have completely estimated $r^\epsilon_{\mathrm{CH2}}$.

Proof of (6.46): The approach to showing (6.46) is very similar to the one used for (6.44):

we have $\mathbf{r}_{\mathrm{S}}^{\epsilon}=\mathbf{r}_{\mathrm{S},I}^{\epsilon}$ in $\Gamma\left(\delta;T_{\epsilon}\right)$ with $\mathbf{r}_{\mathrm{S},I}^{\epsilon}$ as in (6.14) and may then use Lemma 6.7 and Lemma 6.9 (more precisely (6.34)) to get

$$\left|\int\limits_{\Gamma(\delta;T_{\epsilon})}\mathbf{r}_{\mathrm{S},I}^{\epsilon}\cdot\mathbf{z}\mathrm{d}\left(x,t\right)\right|\leq C\left(K\right)\epsilon^{M}\left\|\mathbf{z}\right\|_{L^{2}(0,T_{\epsilon};H^{1}(\Omega))}$$

for all $\mathbf{z}\in L^{2}\left(0,T_{\epsilon};H^{1}\left(\Omega\right)\right)$, implying the estimate in $\Gamma\left(\delta;T_{\epsilon}\right)$.

In $\Omega_{T_{\epsilon}}\backslash\left(\Gamma\left(2\delta;T_{\epsilon}\right)\cup\partial_{T_{\epsilon}}\Omega\left(\delta\right)\right)$ it holds $\mathbf{r}_{\mathrm{S}}^{\epsilon}=\mathbf{r}_{\mathrm{S},O}^{\epsilon}$ and we may simply estimate the occurring terms in $L^{\infty}\left(\Omega_{T_{0}}\right)$ (as they are already multiplied by ϵ^{M} or even higher powers of ϵ) or with the help of (5.236), which leads to the desired inequality.

In $\partial_{T_{\epsilon}}\Omega\left(\frac{\delta}{2}\right)$ it holds $\mathbf{r}_{\mathrm{S}}^{\epsilon}=\mathbf{r}_{\mathrm{S},\mathbf{B}}^{\epsilon}$, allowing for a similar approach as for the outer remainder.

In $\Gamma\left(2\delta;T_{\epsilon}\right)\backslash\Gamma\left(\delta,T_{\epsilon}\right)$, we have

$$\begin{aligned}\mathbf{r}_{\mathrm{S}}^{\epsilon}&=\xi\left(d_{\Gamma}\right)\mathbf{r}_{\mathrm{S},I}^{\epsilon}+\left(1-\xi\left(d_{\Gamma}\right)\right)\mathbf{r}_{\mathrm{S},O}^{\epsilon}-\left(\xi'\left(d_{\Gamma}\right)\Delta d_{\Gamma}+\xi''\left(d_{\Gamma}\right)\right)\left(\mathbf{v}_{I}-\mathbf{v}_{O}\right)\\&\quad-2\xi'\left(d_{\Gamma}\right)D\left(\mathbf{v}_{I}-\mathbf{v}_{O}\right)\mathbf{n}+\xi'\left(d_{\Gamma}\right)\mathbf{n}\left(p_{I}-p_{O}\right)-\mu_{A}^{\epsilon}\xi'\left(d_{\Gamma}\right)\mathbf{n}\left(c_{I}-c_{O}\right)\\&\quad+\left(-\mu_{A}^{\epsilon}\left(\xi\left(d_{\Gamma}\right)\nabla c_{I}+\left(1-\xi\left(d_{\Gamma}\right)\right)\nabla c_{O}\right)+\xi\left(d_{\Gamma}\right)\mu_{I}\nabla c_{I}+\left(1-\xi\left(d_{\Gamma}\right)\right)\mu_{O}\nabla c_{O}\right).\end{aligned}\quad(6.57)$$

To estimate $\mathbf{r}_{\mathrm{S},I}^{\epsilon}$, we may again use Lemma 6.7 as inside $\Gamma\left(\delta;T_{\epsilon}\right)$, but have to be careful when estimating $\epsilon^{M-\frac{3}{2}}\xi\left(d_{\Gamma}\right)\left(\mathbf{V}^{M-\frac{1}{2}}\right)\mathbf{z}$, since (6.34) cannot be used. But, as for $r_{\mathrm{CH1},I}^{\epsilon}$, we can get the sought-after inequality in $\Gamma\left(2\delta;T_{\epsilon}\right)\backslash\Gamma\left(\delta,T_{\epsilon}\right)$ by using an approach analogous to (6.53), which is possible since Lemma 5.36 3) guarantees $\mathbb{V}_{k}^{2}\in\mathcal{R}_{\alpha}$. $\mathbf{r}_{\mathrm{S},O}^{\epsilon}$ may be treated as in $\Omega_{T_{\epsilon}}\backslash\left(\Gamma\left(2\delta;T_{\epsilon}\right)\cup\partial_{T_{\epsilon}}\Omega\left(\delta\right)\right)$ and due to Corollary 6.11 we get the right estimate for the terms involving $\left(\mathbf{v}_{I}-\mathbf{v}_{O}\right)$, $\nabla\left(\mathbf{v}_{I}-\mathbf{v}_{O}\right)$, $\left(p_{I}-p_{O}\right)$ and $\left(c_{I}-c_{O}\right)$.

Regarding the last line of (6.57), we have

$$\begin{aligned}\left(-\mu_{A}^{\epsilon}+\mu_{I}\right)\nabla c_{I}&=\left(1-\xi\left(d_{\Gamma}\right)\right)\left(\mu_{I}-\mu_{O}\right)\nabla c_{I}\\\left(-\mu_{A}^{\epsilon}+\mu_{O}\right)\nabla c_{O}&=\xi\left(d_{\Gamma}\right)\left(\mu_{O}-\mu_{I}\right)\nabla c_{O},\end{aligned}$$

allowing for the usage of Corollary 6.11.

As in the proofs before, the estimates in $\partial_{T_{\epsilon}}\Omega\left(\delta\right)\backslash\partial_{T_{\epsilon}}\Omega\left(\frac{\delta}{2}\right)$ may be shown as in in the case $\Gamma\left(2\delta;T_{\epsilon}\right)\backslash\Gamma\left(\delta,T_{\epsilon}\right)$.

Proof of (6.47): We observe that in $\Omega_{T_{\epsilon}}\backslash\left(\Gamma\left(2\delta;T_{\epsilon}\right)\cup\partial_{T_{\epsilon}}\Omega\left(\delta\right)\right)$ it holds $r_{\mathrm{div},O}^{\epsilon}=0$ by (6.18) and thus in particular $r_{\mathrm{div}}^{\epsilon}=0$ in $\Omega_{T_{\epsilon}}\backslash\left(\Gamma\left(2\delta;T_{\epsilon}\right)\cup\partial_{T_{\epsilon}}\Omega\left(\delta\right)\right)$. In $\Gamma\left(2\delta;T_{\epsilon}\right)$ we have

$$r_{\mathrm{div}}^{\epsilon}=\xi\left(d_{\Gamma}\right)r_{\mathrm{div},I}^{\epsilon}+\xi'\left(d_{\Gamma}\right)\mathbf{n}\cdot\left(\mathbf{v}_{I}-\mathbf{v}_{O}\right).$$

As before, we can treat the term $\xi'\left(d_{\Gamma}\right)\mathbf{n}\cdot\left(\mathbf{v}_{I}-\mathbf{v}_{O}\right)$ by using Corollary 6.11.

For $r_{\mathrm{div},I}^{\epsilon}$, as defined in (6.13), we first note that we may use (6.36) to suitably estimate $\epsilon^{M-\frac{1}{2}}W^{M-\frac{1}{2}}$. Moreover, $\mathrm{div}\mathbf{v}_{M+1}\in L^{\infty}\left(\mathbb{R}\times\Gamma\left(2\delta;T_{0}\right)\right)$ by construction and to estimate the products $\partial_{\rho}\mathbf{v}_{i}\cdot\nabla^{\Gamma}h_{j+1}$, where $i+j\geq M+\frac{1}{2}$, we use that

$$\left\|\partial_{\rho}\mathbf{v}_{i}\right\|_{L^{2}(\Gamma(2\delta;T_{\epsilon}))}\left\|\nabla^{\Gamma}h_{j+1}\right\|_{L^{\infty}(\Gamma(2\delta;T_{\epsilon}))}\leq C\left(K\right)$$

for all $i\in I_{M-\frac{1}{2}}^{M+1}$, $j\in I_{M-\frac{3}{2}}^{M}$, due to construction in the case of $i,j\in\{0,\ldots,M\}$ and $i=M+1$ and due to (5.235) resp. (5.236) in the case of $j=M-\frac{3}{2}$ resp. $i=M-\frac{1}{2}$. Similarly, we get

$$\left\|\mathbf{u}_{i}\cdot\mathbf{n}\right\|_{L^{2}(\Gamma(2\delta;T_{\epsilon}))}\left\|h_{j+1}\right\|_{L^{\infty}((0,T_{\epsilon})\times\mathbb{T}^{1})}\leq C\left(K\right),$$

where we receive an $L^2 - L^2$ estimate for $\mathbf{u}^\epsilon_{M-\frac{1}{2}}$ in the same fashion as in (6.32). The other terms appearing in the definition of $r^\epsilon_{\mathrm{div},I}$ can then be treated in the same way.

In $\partial_{T_\epsilon}\Omega(\delta)$, we finally have

$$r^\epsilon_{\mathrm{div}} = \xi(2d_{\mathbf{B}})\, r^\epsilon_{\mathrm{div},\mathbf{B}} + 2\xi'(2d_{\mathbf{B}})\,\mathbf{n}_{\partial\Omega}(\mathbf{v}_{\mathbf{B}} - \mathbf{v}_O)$$

and the form of $r^\epsilon_{\mathrm{div},\mathbf{B}}$ together with Corollary 6.11 implies the estimate. Thus, we have proven the claim. □

Remark 6.13. In light of the technique used to estimate the remainder terms, this is a good place to remark upon the differences of our approach of constructing approximate solutions to the one used e.g. in [26]. There, the terms c_k are functions in $(\rho, S(x,t), t)$ instead of (ρ, x, t) as in our case, where $(\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta; T)$. While using $(\rho, S(x,t), t)$ intuitively seems to make more sense, as it can be interpreted to be a stretched representation of coordinates $(x,t) \in \Gamma(2\delta; T)$, this ansatz should only be chosen if it is clear that the outer terms are constant in $\Omega^\pm_{T_0}$ (which is true in the case of the Allen-Cahn equation). If we considered expansions of $c^\epsilon, \mu^\epsilon, \dots$ in coordinates $(\rho, S(x), t)$, we would only be able to enforce matching conditions on Γ. In particular, it is in general not possible to satisfy

$$\sup_{(x,t)\in\Gamma(2\delta;T)} \left|c_k(\pm\rho, S(x), t) - c^\pm_k(x,t)\right| \in \mathcal{O}\left(e^{-C\rho}\right), \text{ as } \rho \to \infty,$$

if $c^\pm_k$ is not constant. Thus, terms of the form $\xi'(d_\Gamma)(c_I - c_O)$, which appear due to the "gluing" of inner and outer terms with the help of the cut off function ξ, resulting in integrals of the form

$$\int_{\Gamma(2\delta;T)\setminus\Gamma(\delta;T)} \left|c_k(\rho, S(x,t), t) - c^\pm_k(x,t)\right| \mathrm{d}(x,t)$$

cannot be estimated with a high enough order of ϵ.

While Theorem 6.12 gave important estimates for the remainder terms, we will also need estimates in other norms, which we are given in the following lemma.

Lemma 6.14. *Let the assumptions of Theorem 6.12 hold. Then there are $\epsilon_2 \in (0, \epsilon_1]$ and a constant $C(K) > 0$ such that for all $\epsilon \in (0, \epsilon_2)$*

$$\|r^\epsilon_{\mathrm{CH2}} \nabla c^\epsilon_A\|_{L^2\left(0,T_\epsilon;\left(H^1(\Omega)^2\right)'\right)} \leq C(K)\, C(T_\epsilon, \epsilon)\, \epsilon^M \tag{6.58}$$

$$\|r^\epsilon_{\mathrm{CH1}}\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \leq C(K)\,\epsilon^M \tag{6.59}$$

where $C(T,\epsilon) \to 0$ as $(T,\epsilon) \to 0$.

Proof. We start by showing (6.58). For $\psi \in H^1(\Omega)^2$, we consider

$$\left|\int_\Omega r^\epsilon_{\mathrm{CH2}} \nabla c^\epsilon_A \cdot \psi \mathrm{d}x\right| \leq \left|\int_{\Gamma_t(\delta)} r^\epsilon_{\mathrm{CH2},I} \nabla c^\epsilon_A \cdot \psi \mathrm{d}x\right| + \left|\int_{\Omega\setminus\Gamma_t(\delta)} r^\epsilon_{\mathrm{CH2}} \nabla c^\epsilon_A \cdot \psi \mathrm{d}x\right| \tag{6.60}$$

and begin with analyzing the integral over $\Gamma_t(\delta)$.

First off note that

$$\nabla c^\epsilon_A = \theta_0'\left(\frac{1}{\epsilon}\mathbf{n} - \nabla^\Gamma h^\epsilon_A\right) + \partial_\rho c_1 \mathbf{n} + \mathcal{O}(\epsilon) \tag{6.61}$$

in $L^\infty(\Gamma(\delta;T_\epsilon))$ by construction and the fact that $\left\|\nabla^\Gamma h_A^\epsilon\right\|_{L^\infty(\Gamma(2\delta;T_\epsilon))} \leq C(K)$ by (6.9). Thus, for all terms $g:\Gamma(2\delta)\to\mathbb{R}$ appearing in $r^\epsilon_{\mathrm{CH2},I}$, which are multiplied by at least ϵ^M and which may be estimated in $L^\infty(\Gamma(2\delta;T_\epsilon))$ uniformly in ϵ, we may use the estimate

$$\begin{aligned}\left\|\int_{\Gamma_t(\delta)} g\theta_0'\left(\left(\frac{1}{\epsilon}\mathbf{n}-\nabla^\Gamma h_A^\epsilon\right)\cdot\psi\right)\mathrm{d}x\right\|_{L^2(0,T_\epsilon)} &\leq T_\epsilon^{\frac{1}{2}}\|g\|_{L^\infty(\Gamma(2\delta;T_\epsilon))}\|\psi\|_{H^1(\Omega)}\epsilon\left(\frac{1}{\epsilon}+C(K)\right)\\ &\leq C(K)T_\epsilon^{\frac{1}{2}}\|\psi\|_{H^1(\Omega)},\end{aligned}$$

where we used $H^1(\Gamma_t(2\delta))\hookrightarrow L^{2,\infty}(\Gamma_t(2\delta))$ and the exponential decay of θ_0' in the first line. As discussed in the proof of Theorem 6.12, this works for all terms in $r^\epsilon_{\mathrm{CH2},I}$ except those involving $A^{M-\frac{1}{2}}$ and $\Delta^\Gamma h_{M-\frac{1}{2}}$. For the latter we get

$$\begin{aligned}\left\|\int_{\Gamma_t(\delta)}\Delta^\Gamma h^\epsilon_{M-\frac{1}{2}}\partial_\rho c_i\theta_0'\left(\frac{1}{\epsilon}\mathbf{n}\cdot\psi\right)\mathrm{d}x\right\|_{L^2(0,T_\epsilon)} &\leq CT_\epsilon^{\frac{1}{2}}\left\|h^\epsilon_{M-\frac{1}{2}}\right\|_{L^\infty(0,T_\epsilon;H^2(\mathbb{T}^1))}\|\psi\|_{H^1(\Omega)}\\ &\leq C(K)T_\epsilon^{\frac{1}{2}}\|\psi\|_{H^1(\Omega)}\end{aligned}$$

where we used the same techniques as before together with Proposition 2.34 2).

The most difficult part of the proof is to show

$$\epsilon^{M-\frac{1}{2}}\left\|\int_{\Gamma_t(\delta)}A^{M-\frac{1}{2}}\nabla c_A^\epsilon\cdot\psi\mathrm{d}x\right\|_{L^2(0,T_\epsilon)}\leq C(K)C(T_\epsilon,\epsilon)\epsilon^M\|\psi\|_{H^1(\Omega)}. \tag{6.62}$$

For this, we will employ similar techniques as used in the proof of (6.35) and (6.33), in particular we will use the same notations as discussed right at the beginning of the proof of Lemma 6.9. We will first consider $\frac{1}{\epsilon}\theta_0'\mathbf{n}$ instead of ∇c_A^ϵ. Using the fundamental theorem of calculus we have $\psi(r,s)=\psi(0,s)+\int_0^r\partial_\mathbf{n}\psi(\tilde{r},s)\,\mathrm{d}r$ for $(r,s,t)\in(-\delta,\delta)\times\mathbb{T}^1$. Thus, we may write

$$\begin{aligned}\left|\int_{\Gamma_t(\delta)}A^{M-\frac{1}{2}}\frac{1}{\epsilon}\theta_0'\mathbf{n}\cdot\psi\mathrm{d}x\right| &\leq\frac{1}{\epsilon}\int_{\mathbb{T}^1}|\psi(0,s)|\left|\int_{-\delta}^{\delta}A^{M-\frac{1}{2}}|_\Gamma\theta_0'J(r,s,t)\,\mathrm{d}r\right|\mathrm{d}s\\ &\quad+\frac{C_1}{\epsilon}\int_{\mathbb{T}^1}\int_{-\delta}^{\delta}\left|A^{M-\frac{1}{2}}|_\Gamma\theta_0'\int_0^r\partial_\mathbf{n}\psi(\tilde{r},s,t)\,\mathrm{d}\tilde{r}\right|\mathrm{d}r\mathrm{d}s\\ &\quad+\frac{C_2}{\epsilon}\int_{\mathbb{T}^1}\int_{-\delta}^{\delta}\left|\left(A^{M-\frac{1}{2}}\right)^\Gamma\theta_0'\psi\right|\mathrm{d}r\mathrm{d}s\\ &=:\mathcal{I}_1^1+\mathcal{I}_1^2+\mathcal{I}_2.\end{aligned}$$

By Lemma 5.36 (after choosing $\epsilon > 0$ small enough such that (6.37) holds), we may estimate

$$\begin{aligned}\mathcal{I}_1^1 &\leq \sum_{k=1}^{L_2} \int_{\mathbb{T}^1} \left|\psi(0,s)\,\mathtt{A}_k^{1,\Gamma}\right| \left| \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon} \mathtt{A}_k^{2,\Gamma}\theta_0' J^\epsilon \mathrm{d}\rho \right| \mathrm{d}s \\ &\leq C_1 \|\psi\|_{H^1(\Omega)} \left\|\mathtt{A}_k^{1,\Gamma}\right\|_{L^2(\Gamma_t)} \left(e^{-C_2\frac{\delta}{\epsilon}} + C(K)\,\epsilon\right)\end{aligned}$$

and thus

$$\left\|\mathcal{I}_1^1\right\|_{L^2(0,T_\epsilon)} \leq C(K)\,\epsilon\,\|\psi\|_{H^1(\Omega)}$$

due to (5.256).

Concerning $\mathcal{I}_1^2$, we have

$$\begin{aligned}\left\|\mathcal{I}_1^2\right\|_{L^2(0,T_\epsilon)} &\leq \left\| \frac{1}{\epsilon}\int_{\mathbb{T}^1} \|\psi\|_{H^1(-\delta,\delta)} \int_{-\delta}^{\delta} \left| A^{M-\frac{1}{2}}|_\Gamma \theta_0' r^{\frac{1}{2}}\right| \mathrm{d}r\mathrm{d}s \right\|_{L^2(0,T_\epsilon)} \\ &\leq C(K)\,T_\epsilon^{\frac{1}{3}}\epsilon^{\frac{1}{2}} \sum_{k=1}^{L_2} \|\psi\|_{H^1(\Omega)} \left\|\mathtt{A}_k^{1,\Gamma}\right\|_{L^6(0,T_\epsilon;L^2(\Gamma_t))} \\ &\leq C(K)\,T_\epsilon^{\frac{1}{3}}\epsilon^{\frac{1}{2}} \|\psi\|_{H^1(\Omega)}\end{aligned}$$

by (5.256). Additionally, we used $\left\|\mathtt{A}_k^{2,\Gamma}\right\|_{L^\infty(\mathbb{R})} \leq C$ for all $k \in \{1,\ldots,L_2\}$ in the second line.

For $\mathcal{I}_2$, we need to consider the explicit structure of $A^{M-\frac{1}{2}}$ and show two exemplary estimates, all others follow along the same lines. First, we consider the term $\left(\mu_{M-\frac{1}{2}}^\epsilon\right)^\Gamma$ appearing in $\left(A^{M-\frac{1}{2}}\right)^\Gamma$ (see also (6.43) for the detailed structure):

$$\begin{aligned}&\frac{1}{\epsilon}\int_{\mathbb{T}^1}\int_{-\delta}^{\delta} \left| \left(\mu_{M-\frac{1}{2}}^{+,\epsilon}\right)^\Gamma \eta\theta_0'\psi \right| \mathrm{d}r\mathrm{d}s \\ &\quad\leq C\frac{1}{\epsilon}\int_{\mathbb{T}^1} \sup_{r\in(-\delta,\delta)} |\psi(r,s)| \int_{-\delta}^{\delta} \left| \int_0^r \partial_{\mathbf{n}}\mu_{M-\frac{1}{2}}^{+,\epsilon}(r,s,t)\,\mathrm{d}\tilde{r}\theta_0' \right| \mathrm{d}r\mathrm{d}s \\ &\quad\leq C\int_{\mathbb{T}^1} \sup_{r\in(-\delta,\delta)} |\psi(r,s)| \sup_{r\in(-\delta,\delta)} \left|\partial_{\mathbf{n}}\mu_{M-\frac{1}{2}}^{+,\epsilon}(r,s,t)\right| \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon} \left|\epsilon\left(\rho+h_A^\epsilon\right)\right| \left|\theta_0'\right| \mathrm{d}\rho\mathrm{d}s \\ &\quad\leq C(K)\,\epsilon\,\|\psi\|_{H^1(\Omega)} \left\|\mu_{M-\frac{1}{2}}^{+,\epsilon}\right\|_{H^2(\Omega^+(t))},\end{aligned}$$

where we used the fundamental theorem of calculus in the first line and $H^1(\Gamma_t(\delta)) \hookrightarrow L^{2,\infty}(\Gamma_t(\delta))$ together with the continuity of the extension operator in the last line. The estimate for $\left(\mu_{M-\frac{1}{2}}^{-,\epsilon}\right)^\Gamma \eta$ follows analogously.

Second, we consider the term $\left(|\nabla S(x,t)|^2\right)^\Gamma \partial_s^2 h^\epsilon_{M-\frac{1}{2}}(S(x,t),t)$, as all other occurring terms in $\left(A^{M-\frac{1}{2}}\right)^\Gamma$ consist of lower derivatives of $h^\epsilon_{M-\frac{1}{2}}$ and can be treated in the same way. Using similar techniques as in the estimate above, we get

$$\frac{1}{\epsilon}\int\limits_{\mathbb{T}^1}\int\limits_{-\delta}^{\delta}\left|\left(|\nabla S|^2\right)^\Gamma \partial_s^2 h^\epsilon_{M-\frac{1}{2}}\theta_0'\psi\right| \mathrm{d}r\mathrm{d}s \leq C(K)\,\epsilon\,\|\psi\|_{H^1(\Omega)}\left\|h^\epsilon_{M-\frac{1}{2}}\right\|_{H^2(\mathbb{T}^1)}.$$

Thus, we get by (5.235) and (5.236)

$$\|\mathcal{I}_2\|_{L^2(0,T_\epsilon)} \leq C(K)\,\epsilon\,\|\psi\|_{H^1(\Omega)}.$$

Taking the estimates for $\mathcal{I}_1^1$, $\mathcal{I}_1^2$ and $\mathcal{I}_2$ together, we have

$$\epsilon^{M-\frac{1}{2}}\left\|\int\limits_{\Gamma_t(\delta)} A^{M-\frac{1}{2}}\frac{1}{\epsilon}\theta_0'\mathbf{n}\cdot\psi\mathrm{d}x\right\|_{L^2(0,T_\epsilon)} \leq C(K)\,C(T_\epsilon,\epsilon)\,\epsilon^M\|\psi\|_{H^1(\Omega)}.$$

Regarding (6.62), we next compute

$$\left|\int\limits_{\Gamma_t(\delta)} A^{M-\frac{1}{2}}\theta_0'\nabla^\Gamma h^\epsilon_A\cdot\psi\mathrm{d}x\right| \leq C(K)\sum_{k=1}^{L_1}\left\|\mathtt{A}_k^1\right\|_{L^2(\Gamma_t(2\delta))}\|\psi\|_{L^{2,\infty}(\Gamma_t(2\delta))}\,\epsilon^{\frac{1}{2}}\left\|\theta_0'\right\|_{L^2(\mathbb{R})}$$

where we again used Lemma 5.36. Thus,

$$\epsilon^{M-\frac{1}{2}}\left\|\int\limits_{\Gamma_t(\delta)} A^{M-\frac{1}{2}}\theta_0'\nabla^\Gamma h^\epsilon_A\cdot\psi\mathrm{d}x\right\|_{L^2(0,T_\epsilon)} \leq C(K)\,T_\epsilon^{\frac{1}{3}}\epsilon^M\|\psi\|_{H^1(\Omega)}$$

and a similar estimate holds for $\partial_\rho c_1\cdot\mathbf{n}$, since $\partial_\rho c_1\in\mathcal{R}_\alpha$. As all other terms appearing in ∇c^ϵ_A are already of order ϵ "better" (see (6.61)), this proves (6.62) and as a consequence also

$$\left\|\int\limits_{\Gamma_t(\delta)} r^\epsilon_{\mathrm{CH2},I}\nabla c^\epsilon_A\cdot\psi\mathrm{d}x\right\|_{L^2(0,T_\epsilon)} \leq C(K)\,C(T,\epsilon)\,\epsilon^M\|\psi\|_{H^1(\Omega)}.$$

In view of (6.60), we still need to consider $\left|\int_{\Omega\backslash\Gamma_t(\delta)} r^\epsilon_{\mathrm{CH2}}\nabla c^\epsilon_A\cdot\psi\mathrm{d}x\right|$. But this term may be treated with similar techniques as used in the proof of (6.45), taking into account matching conditions and the fact that

$$|\partial_\rho c_k(\rho(x,t),x,t)| \leq C_1 e^{-C_2\frac{\delta}{2\epsilon}}\ \forall (x,t)\in\Gamma(2\delta;T_\epsilon)\backslash\Gamma(\delta;T_\epsilon)$$

for ϵ small enough and all $k\in\{0,\dots,M+1\}$, together with $\nabla c^\epsilon_A\in\mathcal{O}(\epsilon)$, which holds in $L^\infty(\Omega_{T_0}\backslash\Gamma(2\delta;T_0))$ (cf. Remark 6.4). This shows (6.58).

(6.59) follows immediately by noting that $r^\epsilon_{\mathrm{CH1}} = r^\epsilon_{\mathrm{CH1},\mathbf{B}}$ in $\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)$, the form of the boundary remainder terms (6.19) and the fact that all occurring terms in those boundary remainders are either uniformly bounded in $L^\infty(\partial_{T_0}\Omega(\delta))$ or may be estimated in $L^2\left(\Omega^-_{T_\epsilon}\right)$ with the help of (5.236). □

7. The Proof of Theorem 4.1

In the final chapter of this work, we will combine all the results shown so far to prove Theorem 4.1. We do this in two parts: first, in Section 7.1, we take care of additional, rather technical issues. Many of these revolve around the error in the velocity $\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon$ and how to estimate it; we outsource the corresponding results into their own subsection, so that the actual proof of Theorem 4.1 is a little less cluttered. The second part, Section 7.2, is then only concerned with showing the main theorem.

7.1. Auxiliary Results

Without repeating it, we will consider the following assumptions throughout this section.

Assumption 7.1. *We assume that Assumption 1.1 holds, that $c_A^\epsilon, \mu_A^\epsilon, \mathbf{v}_A^\epsilon, p_A^\epsilon, h_A^\epsilon$ are defined as in Definition 6.2, that $r_{\text{CH1}}^\epsilon, r_{\text{CH2}}^\epsilon, \mathbf{r}_{\text{S}}^\epsilon, r_{\text{div}}^\epsilon$ are given as in (4.7)–(4.10), that $\tilde{\mathbf{w}}_1^\epsilon$ is given as in Section 5.2 and that Assumption 4.2 holds for $c_A = c_A^\epsilon$, $\epsilon_0 \in (0,1)$, $K \geq 1$ and a family $(T_\epsilon)_{\epsilon\in(0,\epsilon_0)} \subset (0, T_0]$. Moreover, we assume that $\epsilon_1 \in (0, \epsilon_0]$ is chosen small enough, such that Theorem 5.32 2), (5.191) and Theorem 6.12 hold. We denote $R := c^\epsilon - c_A^\epsilon$.*

The following proposition guarantees that the energy estimates in Section 4.4 may be used.

Proposition 7.2. *Let $\epsilon_0 \in (0,1)$ and $\psi_0^\epsilon : \Omega \to \mathbb{R}$ be a smooth function satisfying the inequality $\|\psi_0^\epsilon\|_{C^1(\Omega)} \leq C_{\psi_0}\epsilon^M$ for $\epsilon \in (0,\epsilon_0)$. Moreover let*

$$c_0^\epsilon(x) := c_A^\epsilon(x,0) + \psi_0^\epsilon(x) \ \forall x \in \Omega.$$

Then there is some $\tilde{\epsilon} \in (0, \epsilon_0]$ and a constant $C_0 > 0$ which only depends on $\tilde{\epsilon}$, C_{ψ_0} and $\sup_{\epsilon\in(0,\epsilon_0)} \|c_A^\epsilon(x,0)\|_{L^\infty(\Omega)}$, such that

$$E^\epsilon(c_0^\epsilon) \leq C_0, \ \|c_0^\epsilon\|_{L^\infty(\Omega)} \leq C_0$$

for all $\epsilon \in (0, \tilde{\epsilon})$, where E^ϵ is given as in (4.12).

Proof. For simplicity we consider $c_0^\epsilon(x) = c_A^\epsilon(x,0)$ and highlight the situations where ψ_0^ϵ would play a role. The estimate on $\|c_0^\epsilon\|_{L^\infty(\Omega)}$ follows immediately by the construction of c_A^ϵ. Considering $\frac{\epsilon}{2}\int_\Omega |\nabla c_A^\epsilon(x,0)|^2 \,\mathrm{d}x$ we note that $\|\nabla c_A^\epsilon\|_{L^\infty(\Omega_{T_0}\setminus\Gamma(2\delta))} \leq C\epsilon$ by (6.8) and we estimate

$$\begin{aligned}\frac{\epsilon}{2}\int_{\Gamma_0(2\delta)} |\nabla c_A^\epsilon(x,0)|^2 \,\mathrm{d}x \leq{}& \frac{\epsilon}{2}\int_{\Gamma_0(2\delta)} |(1-\xi(d_\Gamma))\nabla c_O(x,0) + \nabla(\xi(d_\Gamma))(c_I - c_O)(x,0)|^2 \,\mathrm{d}x \\ &+ \frac{\epsilon}{2}\int_{\Gamma_0(2\delta)} |\xi(d_\Gamma)\nabla c_I(x,0)|^2 \,\mathrm{d}x. \end{aligned} \tag{7.1}$$

Now we have $\nabla c_0(.,0) \in \mathcal{O}(\epsilon)$ in $L^\infty(\Omega)$ and $c_I - c_O \in \mathcal{O}(1)$ in $L^\infty(\Gamma_0(2\delta))$. Moreover, $\rho(x,0) = \frac{d_\Gamma(x,0)}{\epsilon}$, as $h_A^\epsilon(x,0) = 0$ by construction, and thus

$$\nabla c_{I,0}(x,0) = \frac{1}{\epsilon}\theta_0'(\rho(x,0)) \cdot \mathbf{n}(x,0).$$

In particular

$$\begin{aligned}\frac{\epsilon}{2}\int\limits_{\Gamma_0(2\delta)} |\xi(d_\Gamma)\nabla c_{I,0}(x,0)|^2 \,\mathrm{d}x &\le C\frac{1}{\epsilon}\int\limits_{\Gamma_0(2\delta)} \left(\theta_0'(\rho(x,0))\right)^2 \mathrm{d}x \\ &\le C\int\limits_{\mathbb{T}^1}\int\limits_{-\frac{2\delta}{\epsilon}}^{\frac{2\delta}{\epsilon}} \theta_0'(\rho)^2 \,\mathrm{d}\rho \mathrm{d}s \le C.\end{aligned}$$

As $\epsilon^k \nabla c_{I,k} \in \mathcal{O}(1)$ in $L^\infty(\Gamma_0(2\delta))$ for $k \ge 1$, we may use (7.1) to find

$$\frac{\epsilon}{2}\int\limits_{\Gamma_0(2\delta)} |\nabla c_A^\epsilon(x,0)|^2 \,\mathrm{d}x \le C_1.$$

Note that ψ_0^ϵ can be uniformly estimated in $C^1(\Omega)$ and is multiplied by ϵ^M, so would cause no troubles in these estimates. For the second term in $E^\epsilon(c_0^\epsilon)$, we compute

$$\frac{1}{\epsilon}\int\limits_{\Omega^+(0)\setminus\Gamma_0(2\delta)} f(c_0^\epsilon)\,\mathrm{d}x = \frac{1}{\epsilon}\int\limits_{\Omega^+(0)\setminus\Gamma_0(2\delta)} f'(\beta(x))\left(c_A^\epsilon(x,0) - 1\right)\mathrm{d}x \le C$$

for some suitable $\beta(x) \in (1, c_A^\epsilon(x,0))$, where we used a Taylor expansion around 1 in the first step and the fact that $c_A^\epsilon = \pm 1 + \mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega_{T_0}^\pm \setminus \Gamma(2\delta)\right)$. A similar estimate holds in $\Omega^-(0)\setminus\Gamma_0(2\delta)$.

In $A^+ := \Omega^+(0) \cap (\Gamma_0(2\delta)\setminus\Gamma_0(\delta))$, we find

$$\begin{aligned}\frac{1}{\epsilon}\int\limits_{A^+} f(c_0^\epsilon)\,\mathrm{d}x &= \frac{1}{\epsilon}\int\limits_{A^+} f'\left(\tilde\beta(x)\right)\left(\xi(d_\Gamma)c_I(x,0) + (1-\xi(d_\Gamma))c_O(x,0) - 1\right)\mathrm{d}x \\ &\le C + \frac{1}{\epsilon}\int\limits_{A^+} f'\left(\tilde\beta(x)\right)\xi(d_\Gamma)\left(c_I(x,0) - c_O(x,0)\right)\mathrm{d}x \le C\end{aligned}$$

for all ϵ small enough, where we again used a Taylor expansion in the first line, $c_O = \pm 1 + \mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega_{T_0}^\pm\right)$ in the second line and the matching conditions in the last line. Again, a similar estimate holds in $\Omega^-(0) \cap (\Gamma_0(2\delta)\setminus\Gamma_0(\delta))$.

Finally, inside $\Gamma_0^+(\delta) := \Omega^+(0) \cap \Gamma_0(\delta)$, we have

$$\begin{aligned}\frac{1}{\epsilon}\int\limits_{\Gamma_0^+(\delta)} f(c_0^\epsilon)\,\mathrm{d}x &= \frac{1}{\epsilon}\int\limits_{\Gamma_0^+(\delta)} f'\left(\hat\beta(x)\right)\left(c_I(x,0) - 1\right)\mathrm{d}x \\ &\le C + \frac{1}{\epsilon}\int\limits_{\Gamma_0^+(\delta)} f'\left(\hat\beta(x)\right)\left(\theta_0(\rho(x,0)) - 1\right)\mathrm{d}x \\ &\le C_1 + C_2\int\limits_{\mathbb{T}^1}\int\limits_0^{\frac{\delta}{\epsilon}} |\theta_0(\rho) - 1|\,\mathrm{d}\rho\mathrm{d}s \le C\end{aligned}$$

where we used $c_I = \theta_0 + \mathcal{O}(\epsilon)$ in $L^\infty(\Gamma(2\delta; T_0))$ in the second step and (2.1) in the last step. Again, a similar estimate holds in $\Omega^-(0) \cap \Gamma_0(\delta)$. This shows

$$\frac{1}{\epsilon}\int f(c_0^\epsilon)\,\mathrm{d}x \le C_2,$$

where the appearance of ψ_0^ϵ would have changed nothing in the argumentation. This proves the claim. □

Lemma 7.3. *For all $\epsilon \in (0, \epsilon_1)$*

1. *it holds for all $\alpha \in (0,1)$*

$$\|R\|_{L^2(0,T_\epsilon;L^\infty(\Omega))} \le C(K,\alpha)\,\epsilon^{M-\frac{3}{2}}\epsilon^{-(M+2)\alpha},$$

2. *it holds*

$$\|\nabla R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \le C(K)\,\epsilon^{-\frac{1}{2}}, \tag{7.2}$$

3. *it holds for all $\kappa \in (0,1)$*

$$\|\gamma R\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} \le C(K)\,\epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M},$$

4. *it holds*

$$\|R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \le C(K)\,\epsilon^{\frac{1}{2}\left(M-\frac{1}{2}\right)}.$$

Proof. Ad 1) For $\alpha \in (0,1)$ it holds

$$\|R\|_{L^\infty(\Omega)} \le C(\alpha)\,\|R\|_{H^{1+\alpha}(\Omega)} \le C(\alpha)\,\|R\|_{H^1(\Omega)}^{1-\alpha}\,\|R\|_{H^2(\Omega)}^{\alpha}\,. \tag{7.3}$$

Here the first inequality is due to the Sobolev embeddings, as $1+\alpha-\frac{2}{2} > 0$ by the choice of α, and the second inequality is a consequence of interpolation theory, as $H^{1+\alpha}(\Omega)$ is the interpolation space with respect to $\left(H^1(\Omega), H^2(\Omega)\right)$ of exponent α, cf. [40] p. 330, Theorem B.8. In order to control the H^2-norm of R we use elliptic regularity theory to get

$$\|R\|_{H^2(\Omega)} \le C\,\|\Delta R\|_{L^2(\Omega)}\,,$$

which holds since $R|_{\partial_{T_0}\Omega} = 0$. Due to the construction and since h_A^ϵ is uniformly bounded in X_{T_ϵ} (cf. (5.235)), it can be easily verified by direct calculations that

$$\|\Delta c_A^\epsilon\|_{L^2(\Omega_{T_\epsilon})} \le C(K)\,\frac{1}{\epsilon^2}$$

(see (5.23) for the concrete form of the derivatives of the single terms close to Γ) and we may thus use Lemma 4.4 to get

$$\|\Delta R\|_{L^2(\Omega_{T_\epsilon})} \le C(K)\,\epsilon^{-\frac{7}{2}}, \tag{7.4}$$

where $C(K)$ depends – apart from K – only on T_0 and C_0 (where C_0 is the constant from (4.13)). Using this and (4.6) in (7.3), we find

$$\begin{aligned}\|R\|_{L^2(0,T_\epsilon;L^\infty(\Omega))} &\le C(K)\left(\epsilon^{M-\frac{3}{2}}\right)^{1-\alpha}\left(\epsilon^{-\frac{7}{2}}\right)^{\alpha}\\ &\le C(K)\,\epsilon^{M-\frac{3}{2}}\epsilon^{-(M+2)\alpha}.\end{aligned}$$

Ad 2) We employ Lemma 4.4, which yields

$$\epsilon^{\frac{1}{2}} \|\nabla R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \leq \epsilon^{\frac{1}{2}} \left(\|\nabla c^\epsilon\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} + \|\nabla c_A^\epsilon\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \right) \leq C(K).$$

Here we used

$$\epsilon^{\frac{1}{2}} \|\nabla c_A^\epsilon\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \leq \epsilon^{\frac{1}{2}} \left\| \theta_0'(\rho) \frac{1}{\epsilon} \right\|_{L^\infty(0,T_\epsilon;L^2(\Gamma_t(2\delta)))} + C(K) \leq C(K),$$

where the uniform bound on $c_k, c_k^\pm, c_k^{\mathbf{B}}$ and their derivatives for $k \in \{0,\ldots,M+1\}$ and the boundedness of $h^\epsilon_{M-\frac{1}{2}}$ in X_{T_ϵ} was employed, together with (2.39) for $a \equiv 1$.

Ad 3) First, note that for $\kappa > 0$ and $U \subset \Omega$, we have

$$\|R\|_{L^{2+\kappa}(U)} \leq C_1 \|R\|_{L^2(U)}^{1-\frac{\kappa}{2+\kappa}} \|\nabla R\|_{L^2(U)}^{\frac{\kappa}{2+\kappa}} + C_2 \|R\|_{L^2(U)} \tag{7.5}$$

for $C_1, C_2 > 0$ due to the Gagliardo-Nirenberg interpolation inequality. Moreover, (7.5) implies

$$\begin{aligned}\|\gamma R\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} &\leq C_1 \|\gamma R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}^{1-\frac{\kappa}{2+\kappa}} \|\nabla R\|_{L^\infty\left(0,T_\epsilon;L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)\right)}^{\frac{\kappa}{2+\kappa}} \\ &\quad + C_2 \|\gamma R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \\ &\leq C(K)\, \epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M} \end{aligned} \tag{7.6}$$

by (7.2) and (4.6d).

Ad 4) We have

$$\begin{aligned}\|R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}^2 &\leq \|R\|_{L^\infty(0,T_\epsilon;H^{-1}(\Omega))} \|\nabla R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \\ &\leq C(K)\, \epsilon^{M-\frac{1}{2}},\end{aligned}$$

where the first line is due to an estimate similar to (3.79) and the second line follows from (4.6b) and (7.2). □

The following lemma is an adapted version of [6], Lemma 5.4.

Lemma 7.4. *Let $u \in H^1(\Omega)$. Then there is some constant $C > 0$ such that*

$$\begin{aligned}\|u\|_{L^3(\Gamma_t(\delta))}^3 \leq C &\left(\|u\|_{L^2(\Gamma_t(\delta))} + \left\|\nabla^\Gamma u\right\|_{L^2(\Gamma_t(\delta))} \right)^{\frac{1}{2}} \left(\|u\|_{L^2(\Gamma_t(\delta))} + \|\partial_{\mathbf{n}} u\|_{L^2(\Gamma_t(\delta))} \right)^{\frac{1}{2}} \\ &\cdot \left(\|u\|_{L^2(\Gamma_t(\delta))} \right)^2\end{aligned}$$

holds for all $t \in [0, T_0]$.

Proof. In this proof we write $u(p,r) = u(p + r\mathbf{n}_{\Gamma_t}(p))$ for $(p,r) \in \Gamma_t \times (-\delta,\delta)$ and for fixed $t \in [0,T]$. First of all, we note

$$\|u\|_{L^3(\Gamma_t(\delta))}^3 \leq C \int_{-\delta}^{\delta} \int_{\Gamma_t} |u(p,r)|^3 \, \mathrm{d}\mathcal{H}^1(p) \, \mathrm{d}r = C \left\| \|u\|_{L^3(\Gamma_t)} \right\|_{L^3(-\delta,\delta)}^3$$

and deduce by using the Gagliardo Nirenberg interpolation theorem, that

$$\|u\|_{L^3(\Gamma_t)} \leq C \|u\|_{H^1(\Gamma_t)}^{\frac{1}{6}} \|u\|_{L^2(\Gamma_t)}^{\frac{5}{6}}$$

holds, as Γ_t is one-dimensional.

Combining these results and using Hölder's inequality leads to

$$\begin{aligned} \|u\|_{L^3(\Gamma_t(\delta))}^3 &\leq C \left\| \|u\|_{H^1(\Gamma_t)}^{\frac{1}{6}} \|u\|_{L^2(\Gamma_t)}^{\frac{5}{6}} \right\|_{L^3(-\delta,\delta)}^3 \\ &\leq C \left\| \|u\|_{H^1(\Gamma_t)} \right\|_{L^2(-\delta,\delta)}^{\frac{1}{2}} \left\| \int_{\Gamma_t} |u(p,.)|^2 \,\mathrm{d}\mathcal{H}^1(p) \right\|_{L^{\frac{5}{3}}(-\delta,\delta)}^{\frac{5}{4}} \\ &\leq C \left\| \|u\|_{H^1(\Gamma_t)} \right\|_{L^2(-\delta,\delta)}^{\frac{1}{2}} \left\| \|u\|_{L^{\frac{10}{3}}(-\delta,\delta)} \right\|_{L^2(\Gamma_t)}^{\frac{5}{2}}. \end{aligned}$$

Using Gagliardo-Nirenberg again leads to

$$\|u\|_{L^{\frac{10}{3}}(-\delta,\delta)} \leq C \|u\|_{H^1(-\delta,\delta)}^{\frac{1}{5}} \|u\|_{L^2(-\delta,\delta)}^{\frac{4}{5}}$$

and thus to

$$\begin{aligned} \|u\|_{L^3(\Gamma_t(\delta))}^3 &\leq C \left\| \|u\|_{H^1(\Gamma_t)} \right\|_{L^2(-\delta,\delta)}^{\frac{1}{2}} \left\| \|u\|_{H^1(-\delta,\delta)}^{\frac{1}{5}} \|u\|_{L^2(-\delta,\delta)}^{\frac{4}{5}} \right\|_{L^2(\Gamma_t)}^{\frac{5}{2}} \\ &\leq C \left\| \|u\|_{H^1(\Gamma_t)} \right\|_{L^2(-\delta,\delta)}^{\frac{1}{2}} \left\| \|u\|_{H^1(-\delta,\delta)} \right\|_{L^2(\Gamma_t)}^{\frac{1}{2}} \left\| \|u\|_{L^2(-\delta,\delta)} \right\|_{L^2(\Gamma_t)}^{2}, \end{aligned}$$

where we have used Hölder's inequality in the last step. □

Lemma 7.5. *Let $C_0 > 0$ be given, $f : \mathbb{R} \to \mathbb{R}$ satisfy Assumption 1.2 and let*

$$\mathcal{N}\left(u, \tilde{R}\right) := f'\left(u + \tilde{R}\right) - f'(u) - f''(u)\tilde{R}$$

for $\left(u, \tilde{R}\right) \in [-C_0, C_0] \times \mathbb{R}$. Then for every $p \in [2,3]$ there exists a constant $C_p > 0$ depending only on p, the Norm $\|f\|_{C^3([-3C_0,3C_0])}$ and C_0, such that

$$\tilde{R}\mathcal{N}\left(u, \tilde{R}\right) \geq -C_p \left|\tilde{R}\right|^p$$

holds for all $u \in [-C_0, C_0]$ and $\tilde{R} \in \mathbb{R}$.

Proof. See [14], Lemma 2.2. □

7.1.1. The Error in the Velocity

For $\epsilon \in (0, \epsilon_0)$ we consider strong solutions $\overline{\mathbf{v}^\epsilon} : \Omega_{T_0} \to \mathbb{R}^2$ and $\overline{p^\epsilon} : \Omega_{T_0} \to \mathbb{R}$ of the system

$$-\Delta\overline{\mathbf{v}^\epsilon} + \nabla\overline{p^\epsilon} = \mu_A^\epsilon \nabla c_A^\epsilon \qquad \text{in } \Omega_{T_0}, \tag{7.7}$$

$$\operatorname{div}\overline{\mathbf{v}^\epsilon} = 0 \qquad \text{in } \Omega_{T_0}, \tag{7.8}$$

$$\left(-2D_s\overline{\mathbf{v}^\epsilon} + \overline{p^\epsilon}\mathbf{I}\right)\mathbf{n}_{\partial\Omega} = \alpha_0\overline{\mathbf{v}^\epsilon} \qquad \text{on } \partial_{T_0}\Omega \tag{7.9}$$

(cf. Theorem 2.8) and weak solutions – in the sense of (2.9) – $\tilde{\mathbf{w}}_2^\epsilon : \Omega_{T_0} \to \mathbb{R}^2$ and $q_2^\epsilon : \Omega_{T_0} \to \mathbb{R}$ of

$$-\Delta\tilde{\mathbf{w}}_2^\epsilon + \nabla q_2^\epsilon = -\epsilon\left(\mathrm{div}\left(\mathbf{h} \otimes_s \nabla R\right) + \mathrm{div}\left(\nabla R \otimes \nabla R\right)\right) \qquad \text{in } \Omega_{T_0}, \tag{7.10}$$

$$\mathrm{div}\tilde{\mathbf{w}}_2^\epsilon = 0 \qquad \text{in } \Omega_{T_0}, \tag{7.11}$$

$$\left(-2D_s\tilde{\mathbf{w}}_2^\epsilon + q_2^\epsilon\mathbf{I}\right)\mathbf{n}_{\partial\Omega} = \alpha_0\tilde{\mathbf{w}}_2^\epsilon \qquad \text{in } \partial_{T_0}\Omega, \tag{7.12}$$

where $\mathbf{h}$ is defined as in (5.177) for $H = \left(h^\epsilon_{M-\frac{1}{2}}\right)_{\epsilon\in(0,\epsilon_0)}$. We consider the right hand side of (7.10) as a functional in V_0' given by

$$\mathbf{g}^\epsilon\left(\psi\right) := \epsilon\int_\Omega \left(\left(\mathbf{h} \otimes_s \nabla R\right) + \left(\nabla R \otimes \nabla R\right)\right) : \nabla\psi \mathrm{d}x. \tag{7.13}$$

To gain an understanding of why we introduce $\overline{\mathbf{v}^\epsilon}$, $\tilde{\mathbf{w}}_1^\epsilon$ (cf. Section 5.2) and $\tilde{\mathbf{w}}_2^\epsilon$ consider the following: a fundamental part of the strategy of showing Theorem 4.1 is to consider the difference between the “exact” equations (1.20), (1.21) and the equations (4.9), (4.10) that are satisfied by the approximate solutions. In the process, there appears an error term of the form $\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon$ which has to be estimated suitably. Now the difference between $\mathbf{v}^\epsilon$ and $\overline{\mathbf{v}^\epsilon}$ stems from the right hand side of (7.7), where $\mu_A^\epsilon\nabla c_A^\epsilon$ appears instead of $\mu^\epsilon\nabla c^\epsilon$ as in (1.18). Nevertheless, this distinction allows us to easily estimate the error $\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon$, see Theorem 7.8. What is still missing is a handle on the error $\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}$. In [6], Section 3.2., similar functions $\tilde{\mathbf{w}}_1^\epsilon, \tilde{\mathbf{w}}_2^\epsilon$ are introduced with the advantage in mind that $\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon} = \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon$. This equality does **not** hold in our case due to the different forms of the right hand sides of (1.18), (7.7) and (5.174), (7.10) and the different boundary conditions. Introducing

$$\mathbf{v}_{err}^\epsilon := \mathbf{v}^\epsilon - \left(\overline{\mathbf{v}^\epsilon} + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right) \tag{7.14}$$

we have $\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon} = \mathbf{v}_{err}^\epsilon + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon$, so if we can control $\mathbf{v}_{err}^\epsilon$, $\tilde{\mathbf{w}}_1^\epsilon$, and $\tilde{\mathbf{w}}_2^\epsilon$, we can control the actually sought after error $\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}$. The foundation for this was already set in Section 5.2 with Lemma 5.29; the estimates for $\tilde{\mathbf{w}}_2^\epsilon$ and $\mathbf{v}_{err}^\epsilon$ are shown in the different results of this subsection.

Lemma 7.6. *Let $\tilde{\mathbf{w}}_2^\epsilon$ be the unique weak solution to (7.10)–(7.12) in Ω_{T_0} for $\epsilon \in (0,\epsilon_1)$. Then it holds for all $r \in [1,2]$ and $q \in (1,2)$*

$$\|\tilde{\mathbf{w}}_2^\epsilon\|_{L^r(0,T_\epsilon;L^q(\Omega))} \leq C\left(K,r,q\right)\epsilon^{\frac{2(M-1)}{r}} \tag{7.15}$$

for all $\epsilon \in (0,\epsilon_1)$.

Proof. For unique, weak solvability of (7.10)–(7.12), see Theorem 2.6. Since $\Omega \subset \mathbb{R}^2$, we have $W^1_{q'}\left(\Omega\right) \hookrightarrow C^0\left(\Omega\right)$, where $\frac{1}{q'} + \frac{1}{q} = 1$ and thus for $\psi \in W^2_{q'}$ and g^ϵ as in (7.13)

$$g^\epsilon\left(\psi\right) \leq C\left(q\right)\epsilon\left(\|\mathbf{h} \otimes_s \nabla R\|_{L^1(\Omega)} + \|\nabla R \otimes \nabla R\|_{L^1(\Omega)}\right)\|\psi\|_{W^2_{q'}(\Omega)},$$

so Lemma 2.9 implies

$$\|\tilde{\mathbf{w}}_2^\epsilon\|_{L^r(0,T_\epsilon;L^q(\Omega))} \leq C\left(q\right)\epsilon\left(\|\nabla R \otimes \mathbf{h}\|_{L^r(0,T_\epsilon;L^1(\Omega))} + \|\nabla R \otimes \nabla R\|_{L^r(0,T_\epsilon;L^1(\Omega))}\right).$$

Concerning $\nabla R \otimes \mathbf{h}$, we use $X_{T_\epsilon} \hookrightarrow C^0\left([0,T_\epsilon];C^1\left(\mathbb{T}^1\right)\right)$ and $\partial_\rho \tilde{c}_I \in \mathcal{R}_\alpha$ (cf. (5.169) for the definition of $\tilde{c}_I$) and get

$$\begin{aligned}\epsilon\left\|\nabla R\otimes \mathbf{h}\right\|_{L^r(0,T_\epsilon;L^1(\Omega))} &\leq C\epsilon^{M-\frac{1}{2}}\epsilon^{\frac{1}{2}}\left\|\partial_\rho\tilde{c}_I\right\|_{L^\infty(\Gamma(2\delta;T_0);L^2(\mathbb{R}))}\left\|h^\epsilon_{M-\frac{1}{2}}\right\|_{C^0([0,T_\epsilon];C^1(\mathbb{T}^1))}\\ &\quad\cdot\left\|\nabla R\right\|_{L^2(0,T_\epsilon;L^2(\Omega))}\\ &\leq C(K)\,\epsilon^{2M-\frac{3}{2}}\end{aligned}$$

for $\epsilon \in (0,\epsilon_1)$ by (5.235) and (4.6). Concerning $\nabla R\otimes\nabla R$, we compute

$$\begin{aligned}\epsilon\left\|\nabla R\otimes\nabla R\right\|_{L^r(0,T_\epsilon;L^1(\Omega))} &\leq \epsilon\left\|\nabla R\right\|^{\frac{2}{r}}_{L^2(0,T_\epsilon;L^2(\Omega))}\left\|\nabla R\right\|^{2-\frac{2}{r}}_{L^\infty(0,T_\epsilon;L^2(\Omega))}\\ &\leq C(K)\,\epsilon\left(\epsilon^{M-\frac{3}{2}}\right)^{\frac{2}{r}}\left(\epsilon^{\frac{1}{2}}\left\|\nabla R\right\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}\right)^{2-\frac{2}{r}}\epsilon^{-1+\frac{1}{r}}\\ &\leq C(K)\,\epsilon^{\frac{2M}{r}-\frac{2}{r}}\end{aligned}$$

for $\epsilon\in(0,\epsilon_1)$, again by (4.6) and (7.2). Putting the estimates together, we get

$$\left\|\tilde{\mathbf{w}}^\epsilon_2\right\|_{L^r(0,T_\epsilon;L^q(\Omega))}\leq C(K,r,q)\left(\epsilon^{\frac{2M-2}{r}}+\epsilon^{2M-\frac{3}{2}}\right)$$

and since $r>1$ the claim follows. □

Lemma 7.7. *Let $\varphi\in L^\infty\left(0,T_\epsilon;H^1(\Omega)\right)$ and let the assumptions of Lemma 7.6 hold. Then there is some $r'>0$ such that*

$$\int_0^{T_\epsilon}\left|\int_\Omega\left(\tilde{\mathbf{w}}^\epsilon_2\cdot\nabla c^\epsilon_A\right)\varphi\,\mathrm{d}x\right|\mathrm{d}t\leq C(K)\,T_\epsilon^{r'}\epsilon^M\left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))},$$

for all $\epsilon\in(0,\epsilon_1)$.

Proof. Let $r\in(1,2)$. We have $\nabla c^\epsilon_A\in\mathcal{O}(\epsilon)$ in $L^\infty\left(\Omega_{T_0}\backslash\Gamma(2\delta)\right)$ and thus

$$\begin{aligned}\int_0^{T_\epsilon}\left|\int_{\Omega\backslash\Gamma_t(2\delta)}\left(\tilde{\mathbf{w}}^\epsilon_2\cdot\nabla c^\epsilon_A\right)\varphi\,\mathrm{d}x\right|\mathrm{d}t &\leq C\epsilon\left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\int_0^{T_\epsilon}\left\|\tilde{\mathbf{w}}^\epsilon_2\right\|_{L^q(\Omega)}\mathrm{d}t\\ &\leq C\epsilon T_\epsilon^{\frac{1}{r'}}\left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\left\|\tilde{\mathbf{w}}^\epsilon_2\right\|_{L^r(0,T_\epsilon;L^q(\Omega))}\\ &\leq C(K)\,T_\epsilon^{\frac{1}{r'}}\left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\epsilon^{\frac{2(M-1)}{r}+1}\end{aligned}\tag{7.16}$$

by (7.15) for $q\in(1,2)$. Here we also used $H^1(\Omega)\hookrightarrow L^s(\Omega)$ for all $s\geq 1$ in the first line. The same estimate holds for $\left(\nabla\xi(d_\Gamma)(c_I-c_O)+(1-\xi(d_\Gamma))\nabla c_O\right)$ in $\Gamma_t(2\delta)\backslash\Gamma_t(\delta)$, where one may use Corollary 6.11.

Next we consider the “worst” term in $\Gamma(2\delta;T_\epsilon)$, $\nabla\left(c_{I,0}(x,t)\right)=\nabla\left(\theta_0(\rho(x,t))\right)$ (for a reminder of the notation $c_{I,k}$ see (6.2)): we compute

$$\begin{aligned}
&\int_0^{T_\epsilon} \left| \int_{\Gamma_t(2\delta)} \left(\tilde{\mathbf{w}}_2^\epsilon \cdot \nabla\left(\theta_0\left(\rho\left(x,t\right)\right)\right)\xi\left(d_\Gamma\right)\varphi \mathrm{d}x\right| \mathrm{d}t \\
&\leq \int_0^{T_\epsilon} \int_{\Gamma_t(2\delta)} \left| \left(\tilde{\mathbf{w}}_2^\epsilon \cdot \left(\mathbf{n} - \epsilon\nabla^\Gamma h_A^\epsilon\left(x,t\right)\right)\frac{1}{\epsilon}\theta_0'\left(\rho\left(x,t\right)\right)\right)\varphi\right| \mathrm{d}x\mathrm{d}t \\
&\leq C\left(K\right)\left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\epsilon^{-1}\int_0^{T_\epsilon}\left\|\tilde{\mathbf{w}}_2^\epsilon\right\|_{L^q(\Omega)}\mathrm{d}t\left\|\theta_0'\right\|_{L^\infty(\mathbb{R})} \\
&\leq C\left(K\right)T_\epsilon^{\frac{1}{r'}}\left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\epsilon^{\frac{2(M-1)}{r}-1}.
\end{aligned}$$

Here we used (6.9) in the second estimate and (7.15) in the last line. Since $\nabla\left(c_I - c_{I,0}\right) \in \mathcal{O}\left(1\right)$ in $L^\infty\left(\Gamma\left(2\delta; T_\epsilon\right)\right)$, we immediately get

$$\int_0^{T_\epsilon}\left|\int_{\Gamma_t(2\delta)}\left(\tilde{\mathbf{w}}_2^\epsilon \cdot \nabla\left(c_I - c_{I,0}\right)\right)\varphi\mathrm{d}x\right|\mathrm{d}t \leq C\left(K\right)T_\epsilon^{\frac{1}{r'}}\left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\epsilon^{\frac{2(M-1)}{r}}$$

by similar arguments as in (7.16).

Thus, we ultimately obtain

$$\int_0^{T_\epsilon}\left|\int_\Omega\left(\tilde{\mathbf{w}}_2^\epsilon \cdot \nabla c_A^\epsilon\right)\varphi\mathrm{d}x\right|\mathrm{d}t \leq C\left(K\right)T_\epsilon^{\frac{1}{r'}}\left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}\epsilon^{\frac{2(M-1)}{r}-1}.$$

Now for $r \in (1,2)$ it holds $\frac{2(M-1)}{r} - 1 > M$ iff $r < \frac{2(M-1)}{M+1}$ and we have $\frac{2(M-1)}{M+1} \geq \frac{6}{5}$ since $M \geq 4$. Hence, there always exists $r \in (1,2)$ (and with it $r' \in (2,\infty)$) such that $\epsilon^{\frac{2(M-1)}{r}-1} < \epsilon^M$ which concludes the proof. □

Theorem 7.8 (Error in the Velocity)**.** *Let* $\overline{\mathbf{v}^\epsilon}$ *be a strong solution to (7.7)–(7.9), let the assumptions of Lemma 7.6 hold and let* $\mathbf{v}_{err}^\epsilon := \mathbf{v}^\epsilon - \left(\overline{\mathbf{v}^\epsilon} + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right)$.

1. *There is a constant* $C\left(K\right) > 0$ *such that*

$$\left\|\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \leq C\left(K\right)\epsilon^M$$

for all $\epsilon \in (0,\epsilon_1)$.

2. *There are constants* $C_1, C_2, C\left(K\right) > 0$ *such that for all* $\beta \in \left(0,\frac{1}{2}\right)$

$$\begin{aligned}
\left\|\mathbf{v}_{err}^\epsilon\right\|_{H^1(\Omega)} \leq C_1&\left(\left\|r_{\mathrm{CH2}}^\epsilon\nabla c_A^\epsilon\right\|_{(H_\sigma^1(\Omega))'} + \epsilon\left\|\nabla R\right\|_{L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)}^{1-2\beta}\left\|\gamma\nabla R\right\|_{H^1\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)}^{1+2\beta}\right) \\
&+ C_2\epsilon^2\left\|\nabla R\right\|_{L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)}^{\frac{1}{2}-\beta}\left\|\gamma\nabla R\right\|_{H^1\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)}^{\frac{1}{2}+\beta}
\end{aligned} \tag{7.17}$$

and

$$\left\|\mathbf{v}_{err}^\epsilon\right\|_{L^1(0,T_\epsilon;H^1(\Omega))} \leq C\left(K\right)C\left(T_\epsilon,\epsilon\right)\epsilon^M \tag{7.18}$$

for all $\epsilon \in (0,\epsilon_1)$*, where* $C\left(T,\epsilon\right) \to 0$ *as* $\left(T,\epsilon\right) \to 0$.

Proof. Ad 1) By definition, $\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}$ satisfies

$$\begin{aligned} -\Delta\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) + \nabla\left(p_A^\epsilon - \overline{p^\epsilon}\right) &= \mathbf{r}_{\mathrm{S}}^\epsilon && \text{in } \Omega_{T_\epsilon}, \\ \operatorname{div}\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) &= r_{\mathrm{div}}^\epsilon && \text{in } \Omega_{T_\epsilon}, \\ \left(-2D_s\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) + \left(p_A^\epsilon - \overline{p^\epsilon}\right)\mathbf{I}\right)\mathbf{n}_{\partial\Omega} &= \alpha_0\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) && \text{on } \partial_{T_\epsilon}\Omega. \end{aligned}$$

Thus, we have by Theorem 2.6 and since $r_{\mathrm{div}}^\epsilon = 0$ on $\partial_{T_0}\Omega$ (cf. (6.23))

$$\|\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\|_{L^2(0,T_\epsilon;H^1(\Omega))} \leq C\left(\|\mathbf{r}_{\mathrm{S}}^\epsilon\|_{L^2\left(0,T_\epsilon;(H^1(\Omega))'\right)} + \|r_{\mathrm{div}}^\epsilon\|_{L^2(\Omega_{T_\epsilon})}\right)$$

and the claim follows from Theorem 6.12, in particular (6.46) and (6.47).

Ad 2) Using integration by parts, we find that for $\psi \in H_\sigma^1(\Omega)$, we have

$$\int_\Omega 2D_s\left(\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}\right) : D_s\psi \mathrm{d}x + \alpha_0 \int_{\partial\Omega}\left(\mathbf{v}^\epsilon - \overline{\mathbf{v}^\epsilon}\right)\cdot\psi \mathrm{d}\mathcal{H}^1(s) = \int_\Omega \left(\mu^\epsilon \nabla c^\epsilon - \mu_A^\epsilon \nabla c_A^\epsilon\right)\cdot\psi \mathrm{d}x. \quad (7.19)$$

Plugging in (1.21) and (4.10) we get

$$\begin{aligned} &\int_\Omega \left(\mu^\epsilon \nabla c^\epsilon - \mu_A^\epsilon \nabla c_A^\epsilon\right)\cdot\psi \mathrm{d}x \\ &= \int_\Omega \left(-\epsilon\left(\Delta c^\epsilon \nabla c^\epsilon - \Delta c_A^\epsilon \nabla c_A^\epsilon\right) + \frac{1}{\epsilon}\nabla\left(f\left(c^\epsilon\right) - f\left(c_A^\epsilon\right)\right) - r_{\mathrm{CH2}}^\epsilon \nabla c_A^\epsilon\right)\cdot\psi \mathrm{d}x \\ &= \epsilon \int_\Omega \left(-\operatorname{div}\left(\nabla c^\epsilon \otimes \nabla c^\epsilon\right) + \operatorname{div}\left(\nabla c_A^\epsilon \otimes \nabla c_A^\epsilon\right)\right)\cdot\psi \mathrm{d}x - \int_\Omega r_{\mathrm{CH2}}^\epsilon \nabla c_A^\epsilon \cdot \psi \mathrm{d}x \\ &\quad + \epsilon \int_\Omega \frac{1}{2}\nabla\left(|\nabla c^\epsilon|^2 - |\nabla c_A^\epsilon|^2\right)\cdot\psi \mathrm{d}x \\ &= \epsilon \int_\Omega \left(\nabla c^\epsilon \otimes \nabla c^\epsilon - \nabla c_A^\epsilon \otimes \nabla c_A^\epsilon\right) : \nabla\psi \mathrm{d}x - \int_\Omega r_{\mathrm{CH2}}^\epsilon \nabla c_A^\epsilon \cdot \psi \mathrm{d}x \\ &\quad + \epsilon \int_{\partial\Omega}\left(\left(\nabla c_A^\epsilon \otimes \nabla c_A^\epsilon - \nabla c^\epsilon \otimes \nabla c^\epsilon\right)\mathbf{n}_{\partial\Omega} + \frac{1}{2}\left(|\nabla c^\epsilon|^2 - |\nabla c_A^\epsilon|^2\right)\mathbf{n}_{\partial\Omega}\right)\cdot\psi \mathrm{d}\mathcal{H}^1(s). \quad (7.20) \end{aligned}$$

In the first step, we used (1.18) and (4.10), in the second step we used integration by parts, the fact that $c^\epsilon = c_A^\epsilon = -1$ on $\partial_T\Omega$ (due to (1.25) and the construction of $c_{\mathbf{B}}$) together with $f(-1) = 0$, and the formula $\operatorname{div}(\nabla c \otimes \nabla c) = \Delta c \nabla c + \frac{1}{2}\nabla\left(|\nabla c|^2\right)$ for smooth $c : \Omega \to \mathbb{R}$. In the third step, we again used integration by parts; note that $\psi \in H_\sigma^1(\Omega)$.

Regarding the definitions of $\tilde{\mathbf{w}}_1^\epsilon$ and $\tilde{\mathbf{w}}_2^\epsilon$ (cf. (5.174), (7.10)) as weak solutions, we have

$$\begin{aligned} &\int_\Omega 2D_s\left(\tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right) : D_s\psi \mathrm{d}x + \alpha_0 \int_{\partial\Omega}\left(\tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right)\cdot\psi \mathrm{d}\mathcal{H}^1(s) \\ &\quad = \epsilon \int_\Omega \left(\left(\nabla c_A^\epsilon - \mathbf{h}\right)\otimes_s \nabla R + \mathbf{h} \otimes_s \nabla R + \nabla R \otimes \nabla R\right) : \nabla\psi \mathrm{d}x \\ &\quad = \epsilon \int_\Omega \left(\nabla c^\epsilon \otimes \nabla c^\epsilon - \nabla c_A^\epsilon \otimes \nabla c_A^\epsilon\right) : \nabla\psi \mathrm{d}x, \end{aligned} \quad (7.21)$$

where $\mathbf{h}$ is defined as in (5.177) (with $H = \left(h^{\epsilon}_{M-\frac{1}{2}}\right)_{\epsilon\in(0,\epsilon_0)}$ as always in this chapter) and $\otimes_s$ as given in Notation 5.26. Here we used the calculations

$$\begin{aligned}\nabla c^{\epsilon}_A \otimes_s \nabla R &= \nabla c^{\epsilon}_A \otimes_s \nabla c^{\epsilon} - 2\nabla c^{\epsilon}_A \otimes \nabla c^{\epsilon}_A\\ \nabla R \otimes \nabla R &= \nabla c^{\epsilon} \otimes \nabla c^{\epsilon} - \nabla c^{\epsilon}_A \otimes_s \nabla c^{\epsilon} + \nabla c^{\epsilon}_A \otimes \nabla c^{\epsilon}_A.\end{aligned}$$

So, defining $\mathbf{v}^{\epsilon}_{err}$ as in (7.14) and taking into account (7.19), (7.20), and (7.21), we find that $\mathbf{v}^{\epsilon}_{err}$ solves

$$\begin{aligned}\int_{\Omega} 2D_s\mathbf{v}^{\epsilon}_{err} : D_s\psi \mathrm{d}x + \alpha_0 \int_{\partial\Omega} \mathbf{v}^{\epsilon}_{err} \cdot \psi \mathrm{d}\mathcal{H}^1(s)&\\ =&\epsilon\int_{\partial\Omega}\left(\left(\nabla c^{\epsilon}_A \otimes \nabla c^{\epsilon}_A - \nabla c^{\epsilon} \otimes \nabla c^{\epsilon}\right)\mathbf{n}_{\partial\Omega} + \frac{1}{2}\left(|\nabla c^{\epsilon}|^2 - |\nabla c^{\epsilon}_A|^2\right)\mathbf{n}_{\partial\Omega}\right)\cdot\psi \mathrm{d}\mathcal{H}^1(s)\\ &- \int_{\Omega} r^{\epsilon}_{\mathrm{CH2}}\nabla c^{\epsilon}_A \cdot \psi \mathrm{d}x\\ =:&\mathcal{F}^{\epsilon}(\psi)\end{aligned} \tag{7.22}$$

for all $\psi \in H^1_{\sigma}(\Omega)$. Thus, in order to get an estimate in $L^1\left(0, T_{\epsilon}; H^1(\Omega)\right)$ (or only $H^1(\Omega)$) for $\mathbf{v}^{\epsilon}_{err}$ we need to find a suitable upper bound for $\|\mathcal{F}^{\epsilon}\|_{L^1\left(0,T_{\epsilon};(H^1_{\sigma}(\Omega))'\right)}$ (or only $\|\mathcal{F}^{\epsilon}\|_{(H^1_{\sigma}(\Omega))'}$).

Due to Lemma 6.14 we have

$$\begin{aligned}\int_0^{T_{\epsilon}}\left|\int_{\Omega} r^{\epsilon}_{\mathrm{CH2}}\nabla c^{\epsilon}_A \cdot \psi \mathrm{d}x\right|\mathrm{d}t &\leq \int_0^{T_{\epsilon}} \|r^{\epsilon}_{\mathrm{CH2}}\nabla c^{\epsilon}_A\|_{(H^1_{\sigma}(\Omega))'}\,\mathrm{d}t\,\|\psi\|_{H^1(\Omega)}\\ &\leq C(K)\,C(T_{\epsilon},\epsilon)\,\epsilon^M,\end{aligned} \tag{7.23}$$

where $C(T,\epsilon) \to 0$ as $(T,\epsilon) \to 0$. Thus, we only need to estimate the appearing boundary terms in (7.22).

For this, let $\beta \in \left(0, \frac{1}{2}\right)$ and we compute

$$\begin{aligned}\epsilon\int_0^{T_{\epsilon}}\int_{\partial\Omega}&\left|\left(|\nabla c^{\epsilon}|^2 - |\nabla c^{\epsilon}_A|^2\right)\psi\right|\mathrm{d}\mathcal{H}^1(s)\,\mathrm{d}t\\ &\leq \epsilon\int_0^{T_{\epsilon}}\int_{\partial\Omega}\left(|\nabla R|^2 + 2|\nabla R|\,|\nabla c^{\epsilon}_A|\right)|\psi|\,\mathrm{d}\mathcal{H}^1(s)\,\mathrm{d}t\\ &\leq C\int_0^{T_{\epsilon}}\left(\epsilon\|\nabla R\|^2_{L^{2+\beta}(\partial\Omega)} + \epsilon^2\|\nabla R\|_{L^{2+\beta}(\partial\Omega)}\right)\|\psi\|_{H^{\frac{1}{2}}(\partial\Omega)}\,\mathrm{d}t\\ &\leq C\int_0^{T_{\epsilon}}\left(\epsilon\|\gamma\nabla R\|^2_{H^{\frac{1}{2}+\beta}(\Omega)} + \epsilon^2\|\gamma\nabla R\|_{H^{\frac{1}{2}+\beta}(\Omega)}\right)\|\psi\|_{H^1(\Omega)}\,\mathrm{d}t\\ &\leq C_1\int_0^{T_{\epsilon}}\left(\epsilon\|\nabla R\|^{1-2\beta}_{L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)}\|\gamma\nabla R\|^{1+2\beta}_{H^1(\Omega)}\right)\|\psi\|_{H^1(\Omega)}\,\mathrm{d}t\end{aligned}$$

$$+ C_2 \int_0^{T_\epsilon} \left(\epsilon^2 \|\nabla R\|^{\frac{1}{2}-\beta}_{L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)} \|\gamma \nabla R\|^{\frac{1}{2}+\beta}_{H^1(\Omega)} \right) \|\psi\|_{H^1(\Omega)} \, \mathrm{d}t \tag{7.24}$$

$$\leq C_1 \left(\epsilon \|\nabla R\|^{1-2\beta}_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \|\gamma \nabla R\|^{1+2\beta}_{L^2(0,T_\epsilon;H^1(\Omega))} \right) \|\psi\|_{H^1(\Omega)}$$
$$+ C_2 T_\epsilon^{\frac{1}{2}} \left(\epsilon^2 \|\nabla R\|^{\frac{1}{2}-\beta}_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \|\gamma \nabla R\|^{\frac{1}{2}+\beta}_{L^2(0,T_\epsilon;H^1(\Omega))} \right) \|\psi\|_{H^1(\Omega)} , \tag{7.25}$$

where we used in the second inequality that $\nabla c_A^\epsilon = \mathcal{O}(\epsilon)$ in $L^\infty\left(\overline{\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)}\right)$ due to (6.8) and that $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^s(\partial\Omega)$ for all $s \geq 1$ by the Sobolev-Embeddings Theorem, as $\partial\Omega$ is one dimensional. In the third inequality we used the Trace Theorem (cf. [40], Theorem 3.37, p. 102), which implies the existence of a continuous mapping $H^{\frac{1}{2}+\beta}(\Omega) \to H^\beta(\partial\Omega)$. Moreover, we used that $H^\beta(\partial\Omega) \hookrightarrow L^{2+\beta}(\partial\Omega)$, which is again due to the Sobolev-Embeddings Theorem, since $\beta - \frac{1}{2} \geq -\frac{1}{2+\beta}$ for $\beta \geq 0$. In the fourth estimate, we used that for a bounded domain $U \subset \mathbb{R}^2$, $H^{\frac{1}{2}+\beta}(U)$ is an interpolation space with respect to $\left(L^2(U), H^1(U)\right)$ of exponent $\frac{1}{2}+\beta$. Now we may estimate

$$\|\gamma \nabla R\|_{H^1(\Omega)} \leq \|\nabla(\gamma R)\|_{H^1(\Omega)} + \|\nabla\gamma R\|_{H^1(\Omega)}$$
$$\leq C \|(\gamma \Delta R, |\nabla R|, R)\|_{L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)} \tag{7.26}$$

due to elliptic regularity theory and the definition of γ. Using this in (7.25) together with (4.6a) and (4.6d), we find

$$\epsilon \int_0^{T_\epsilon} \int_{\partial\Omega} \left| \left(|\nabla c^\epsilon|^2 - |\nabla c_A^\epsilon|^2 \right) \psi \right| \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t \leq \|\psi\|_{H^1(\Omega)} C(K) \left(\epsilon^{2M-\frac{1}{2}-\beta} + T_\epsilon^{\frac{1}{2}} \epsilon^{M+\frac{5}{4}-\frac{1}{2}\beta} \right)$$
$$\leq \|\psi\|_{H^1(\Omega)} C(K) \left(\epsilon^{\frac{1}{2}} + T_\epsilon^{\frac{1}{2}} \right) \epsilon^M$$

as $M \geq 4$ and $\beta > 0$ can be chosen sufficiently small.

For the remaining, not estimated term in (7.22), we note

$$\epsilon \int_0^{T_\epsilon} \int_{\partial\Omega} \left| \left(-\nabla c^\epsilon \otimes \nabla c^\epsilon + \nabla c_A^\epsilon \otimes \nabla c_A^\epsilon \right) \mathbf{n}_{\partial\Omega} \cdot \psi \right| \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t$$
$$\leq \int_0^{T_\epsilon} \int_{\partial\Omega} \left| \left(\nabla R \otimes \nabla R + \nabla c_A^\epsilon \otimes_s \nabla R \right) \right| |\psi| \, \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t$$
$$\leq \int_0^{T_\epsilon} \int_{\partial\Omega} \left(|\nabla R|^2 + 2 |\nabla R| |\nabla c_A^\epsilon| \right) |\psi| \, \mathrm{d}\mathcal{H}^1(s)\, \mathrm{d}t$$

and may then proceed as in (7.25). This proves (7.18) and also (7.17), if we drop the time integrals in (7.23) and (7.24). □

An immediate consequence of this theorem is the following corollary.

Corollary 7.9. *Let the assumptions of Theorem 7.8 hold and let* $\varphi \in L^\infty\left(0, T_\epsilon; H^1(\Omega)\right)$. *Then it holds*

$$\int_0^{T_\epsilon} \left| \int_\Omega \left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) \cdot \nabla c_A^\epsilon \varphi \mathrm{d}x \right| \mathrm{d}t \leq C(K)\, T_\epsilon^{\frac{1}{2}} \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \tag{7.27}$$

$$\int_0^{T_\epsilon} \left| \int_\Omega \mathbf{v}_{err}^\epsilon \cdot \nabla c_A^\epsilon \varphi \mathrm{d}x \right| \mathrm{d}t \leq C(K)\, C(\epsilon, T_\epsilon)\, \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \tag{7.28}$$

$$\int_0^{T_\epsilon} \left| \int_\Omega R \mathbf{v}_{err}^\epsilon \cdot \nabla R \gamma^2 \mathrm{d}x \right| \mathrm{d}t \leq C(K)\, C(\epsilon, T_\epsilon)\, \epsilon^{2M-1}, \tag{7.29}$$

$$\int_0^{T_\epsilon} \left| \int_\Omega \mathbf{v}_{err}^\epsilon \cdot \nabla R \varphi \mathrm{d}x \right| \mathrm{d}t \leq C(K)\, C(\epsilon, T_\epsilon)\, \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \tag{7.30}$$

for all $\epsilon \in (0, \epsilon_1)$ *and* $C(\epsilon, T) \to 0$ *if* $(\epsilon, T) \to 0$.

Proof. In $\Omega_{T_\epsilon} \backslash \Gamma(2\delta; T_\epsilon)$, we have $\nabla c_A^\epsilon \in \mathcal{O}(\epsilon)$ in L^∞ by (6.8) and thus get the estimate by simply using Hölder's inequality and Theorem 7.8 1). It remains to give an estimate for

$$\begin{aligned}\int_{\Gamma(2\delta;T_\epsilon)} \left|\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) \cdot \nabla c_A^\epsilon \varphi\right| \mathrm{d}(x,t) \leq &\int_{\Gamma(2\delta;T_\epsilon)} \left|\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) \cdot \left(\xi \nabla c_I + \xi' \mathbf{n}(c_I - c_O)\right) \varphi\right| \mathrm{d}(x,t). \\ &+ \int_{\Gamma(2\delta;T_\epsilon)} \left|\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) \cdot \left((1-\xi) \nabla c_O\right) \varphi\right| \mathrm{d}(x,t)\end{aligned}$$

We have $\nabla c_O \in \mathcal{O}(\epsilon)$ in L^∞ and the term involving $(c_I - c_O)$ can be handled by using the Corollary (6.11), Hölder's inequality and Theorem 7.8 1) as before. Moreover, we estimate

$$\begin{aligned}\int_{\Gamma(2\delta;T_\epsilon)} \left|\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) \xi \nabla c_{I,0} \varphi\right| \mathrm{d}(x,t) &\leq C \int_0^{T_\epsilon} \int_{\mathbb{T}^1} \left\|\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) \varphi\right\|_{L^\infty(-2\delta,2\delta)} \int_{\mathbb{R}} \left|\theta_0' \left(\mathbf{n} + \nabla^\Gamma h_A^\epsilon\right)\right| \mathrm{d}\rho \mathrm{d}s \mathrm{d}t \\ &\leq C(K) \int_0^{T_\epsilon} \left\|\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right\|_{L^{2,\infty}(\Gamma_t(2\delta))} \|\varphi\|_{L^{2,\infty}(\Gamma_t(2\delta))} \mathrm{d}t \\ &\leq C(K)\, T_\epsilon^{\frac{1}{2}} \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))},\end{aligned}$$

where we used $H^1(\Gamma_t(2\delta)) \hookrightarrow L^{2,\infty}(\Gamma_t(2\delta))$ together with Theorem 7.8 1) in the last step. For $k \geq 1$ we can use $\nabla c_{I,k} \in \mathcal{O}(1)$ in $L^\infty(\Gamma(2\delta; T_\epsilon))$ uniformly in ϵ. This proves (7.27).

(7.28) follows in the same way by using (7.18) and noticing that we may not generate a term $T_\epsilon^{\frac{1}{2}}$ as we only control $\|\mathbf{v}_{err}^\epsilon\|_{L^1(0,T_\epsilon;H^1(\Omega))}$.

For (7.29) we need to invest more work: Since $H^1(\Omega) \hookrightarrow L^s(\Omega)$ for all $s \geq 1$ we have

$$\int_0^{T_\epsilon} \left| \int_\Omega R \mathbf{v}_{err}^\epsilon \cdot \nabla R \gamma^2 \mathrm{d}x \right| \mathrm{d}t \leq C \int_0^{T_\epsilon} \|\mathbf{v}_{err}^\epsilon\|_{H^1(\Omega)} \|\gamma R\|_{L^{2+\kappa}(\Omega)} \|\gamma \nabla R\|_{L^2(\Omega)} \mathrm{d}t \tag{7.31}$$

for $\kappa > 0$. Regarding (7.17), we need to show three estimates:

First, we have

$$\begin{aligned}
&\int_0^{T_\epsilon} \left\| r^\epsilon_{\mathrm{CH2}} \nabla c^\epsilon_A \right\|_{(H^1(\Omega))'} \left\| \gamma R \right\|_{L^{2+\kappa}(\Omega)} \left\| \gamma \nabla R \right\|_{L^2(\Omega)} \mathrm{d}t \\
&\leq \left\| r^\epsilon_{\mathrm{CH2}} \nabla c^\epsilon_A \right\|_{L^2\left(0,T_\epsilon;(H^1_\sigma(\Omega))'\right)} \left\| \gamma R \right\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} \left\| \gamma \nabla R \right\|_{L^2(\Omega_{T_\epsilon})} \\
&\leq C\left(K\right) C\left(T_\epsilon, \epsilon\right) \epsilon^{2M} \left(\epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M} \right) \\
&\leq C\left(K\right) C\left(T_\epsilon, \epsilon\right) \epsilon^{2M-1} && (7.32)
\end{aligned}$$

where we used Lemma 6.14, (4.6d) and Lemma 7.3 3) in the second line and the fact that $M \geq 4$ and $\kappa > 0$ can be chosen arbitrarily small in the last step.

Second, we estimate for $\beta \in \left(0, \frac{1}{2}\right)$

$$\begin{aligned}
&\int_0^{T_\epsilon} \epsilon \left\| \nabla R \right\|^{1-2\beta}_{L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)} \left\| \gamma \nabla R \right\|^{1+2\beta}_{H^1(\Omega)} \left\| \gamma R \right\|_{L^{2+\kappa}(\Omega)} \left\| \gamma \nabla R \right\|_{L^2(\Omega)} \mathrm{d}t \\
&\leq C\epsilon \left\| \nabla R \right\|^{1-2\beta}_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \left\| \gamma \nabla R \right\|^{1+2\beta}_{L^2(0,T_\epsilon;H^1(\Omega))} \left\| \gamma R \right\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} \left\| \nabla R \right\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \\
&\leq C\left(K\right) \left(\epsilon^{2M-\frac{1}{2}-\beta} \epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M} \epsilon^{-\frac{1}{2}} \right) \\
&\leq C\left(K\right) \epsilon^{2M-\frac{1}{2}}, && (7.33)
\end{aligned}$$

where we used (7.26), (4.6a), (4.6d), Lemma 7.3 3) and (7.2) in the third line and the fact that $M \geq 4$ and that $\beta > 0$,$\kappa > 0$ can be chosen arbitrarily small in the last line.

Third, we have

$$\begin{aligned}
&\int_0^{T_\epsilon} \epsilon^2 \left\| \nabla R \right\|^{\frac{1}{2}-\beta}_{L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)} \left\| \gamma \nabla R \right\|^{\frac{1}{2}+\beta}_{H^1(\Omega)} \left\| \gamma R \right\|_{L^{2+\kappa}(\Omega)} \left\| \gamma \nabla R \right\|_{L^2(\Omega)} \mathrm{d}t \\
&\leq C\epsilon^2 \left\| \nabla R \right\|^{\frac{1}{2}-\beta}_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \left\| \gamma \nabla R \right\|^{\frac{1}{2}+\beta}_{L^2(0,T_\epsilon;H^1(\Omega))} \left\| \gamma R \right\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} \left\| \gamma \nabla R \right\|_{L^2(0,T_\epsilon;L^2(\Omega))} \\
&\leq C\left(K\right) \left(\epsilon^{M+\frac{5}{4}-\frac{\beta}{2}} \epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M} \epsilon^{M-\frac{1}{2}} \right) \\
&\leq C\left(K\right) \epsilon^{2M-\frac{1}{2}}, && (7.34)
\end{aligned}$$

where we again used (7.26), (4.6a), (4.6d) and Lemma 7.3 3) in the third step and the fact that $M \geq 4$ and that $\beta > 0$, $\kappa > 0$ can be chosen arbitrarily small in the last line. Now (7.31)–(7.34) together with (7.17) yield (7.29).

Concerning the last estimate, (7.30) we note

$$\left| \int_\Omega \mathbf{v}^\epsilon_{err} \cdot \nabla R \varphi \mathrm{d}x \right| \leq \left\| \mathbf{v}^\epsilon_{err} \right\|_{H^1(\Omega)} \left\| \nabla R \right\|_{L^2(\Omega)} \left\| \varphi \right\|_{L^4(\Omega)}.$$

Regarding (7.17), we again consider three different terms:

First,

$$\int_0^{T_\epsilon} \|r^\epsilon_{\mathrm{CH2}} \nabla c^\epsilon_A\|_{(H^1(\Omega))'} \|\nabla R\|_{L^2(\Omega)} \|\varphi\|_{L^4(\Omega)} \,\mathrm{d}t$$
$$\le \|r^\epsilon_{\mathrm{CH2}} \nabla c^\epsilon_A\|_{L^2\left(0,T_\epsilon;H^1(\Omega)'\right)} \|\nabla R\|_{L^2(\Omega_{T_\epsilon})} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$
$$\le C(K)\, C(T_\epsilon,\epsilon)\, \epsilon^M \epsilon^{M-\frac{3}{2}} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$
$$\le C(K)\, C(T_\epsilon,\epsilon)\, \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

where we used (6.58) and (4.6) in the third line and $M \ge 4$ in the last line.

Second,

$$\int_0^{T_\epsilon} \epsilon \|\nabla R\|^{1-2\beta}_{L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)} \|\gamma \nabla R\|^{1+2\beta}_{H^1(\Omega)} \|\nabla R\|_{L^2(\Omega)} \|\varphi\|_{L^4(\Omega)} \,\mathrm{d}t$$
$$\le C\epsilon \|\nabla R\|^{1-2\beta}_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \|\gamma \nabla R\|^{1+2\beta}_{L^2(0,T_\epsilon;H^1(\Omega))} \|\nabla R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$
$$\le C(K)\, \epsilon^{2M-\frac{1}{2}-\beta} \epsilon^{-\frac{1}{2}} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$
$$\le C(K)\, C(T_\epsilon,\epsilon)\, \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

for $\beta \in \left(0, \frac{1}{2}\right)$, where we used (4.6) and (7.2) in the third line and $M \ge 4$ in the last.

Third,

$$\int_0^{T_\epsilon} \epsilon^2 \|\nabla R\|^{\frac{1}{2}-\beta}_{L^2\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)} \|\gamma \nabla R\|^{\frac{1}{2}+\beta}_{H^1(\Omega)} \|\nabla R\|_{L^2(\Omega)} \|\varphi\|_{L^4(\Omega)} \,\mathrm{d}t$$
$$\le C\epsilon^2 \|\nabla R\|^{\frac{1}{2}-\beta}_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \|\gamma \nabla R\|^{\frac{1}{2}+\beta}_{L^2(0,T_\epsilon;H^1(\Omega))} \|\nabla R\|_{L^2(\Omega_{T_\epsilon})} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$
$$\le C(K)\, \epsilon^{M+\frac{5}{4}-\frac{\beta}{2}} \epsilon^{M-\frac{3}{2}} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$
$$\le C(K)\, C(T_\epsilon,\epsilon)\, \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

for $\beta \in \left(0, \frac{1}{2}\right)$, where we used (4.6) in the third line and $M \ge 4$ in the last line. This shows (7.30). ☐

Lemma 7.10. *Let $\varphi \in L^\infty\left(0,T_\epsilon;H^1(\Omega)\right)$. Then it holds for $(\tilde{\mathbf{w}}^\epsilon_1)^\Gamma = \tilde{\mathbf{w}}^\epsilon_1 - \tilde{\mathbf{w}}^\epsilon_1|_\Gamma$*

$$\int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} \frac{1}{\epsilon} (\tilde{\mathbf{w}}^\epsilon_1)^\Gamma \cdot \mathbf{n}\theta_0'(\rho)\, \varphi \,\mathrm{d}x \right| \mathrm{d}t \le C(K)\, (T_\epsilon)^{\frac{1}{2}} \epsilon^M \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \tag{7.35}$$

$$\int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} (\tilde{\mathbf{w}}^\epsilon_1)^\Gamma \cdot \nabla^\Gamma h^\epsilon_A \theta_0'(\rho)\, \varphi \,\mathrm{d}x \right| \mathrm{d}t \le C(K)\, (T_\epsilon)^{\frac{1}{2}} \epsilon^{M+1} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}, \tag{7.36}$$

$$\int_0^{T_\epsilon} \left| \int_{\Gamma_t(\delta)} (\tilde{\mathbf{w}}^\epsilon_1)^\Gamma \cdot \left(\frac{\mathbf{n}}{\epsilon} - \nabla^\Gamma h^\epsilon_A\right) \epsilon \partial_\rho c_1 \varphi \,\mathrm{d}x \right| \mathrm{d}t \le C(K)\, (T_\epsilon)^{\frac{1}{2}} \epsilon^{M+1} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \tag{7.37}$$

for all $\epsilon \in (0, \epsilon_1)$.

Proof. In order to prove the first inequality, we fix $t \in (0, T_\epsilon)$ and let $\phi_t : \Gamma_t(2\delta) \to (-2\delta, 2\delta) \times \Gamma_t$ be defined as in Lemma 2.11. Moreover, we define $J(r,p,t) := \det\left(d\left(\phi_t^{-1}\right)(r,p)\right)$, where $d\left(\phi_t^{-1}\right)(r,p)$ denotes the differential. To simplify the following presentation, we write $\psi(r,p,t) := \psi\left(\phi_t^{-1}(r,p), t\right)$ for functions ψ defined on $\Gamma_t(2\delta)$. Now we may compute

$$
\begin{aligned}
&\int_{\Gamma_t(\delta)} \frac{1}{\epsilon}\left(\tilde{\mathbf{w}}_1^\epsilon - \tilde{\mathbf{w}}_1^\epsilon|_\Gamma\right) \cdot \mathbf{n}\theta_0'(\rho)\varphi \mathrm{d}x \\
&= \int_{-\delta}^{\delta}\int_{\Gamma_t} \left(\frac{1}{\epsilon}\left(\tilde{\mathbf{w}}_1^\epsilon(r,p,t) - \tilde{\mathbf{w}}_1^\epsilon(0,p,t)\right)\cdot \mathbf{n}_{\Gamma_t}(p)\right)\theta_0'(\rho(r,p,t))\varphi(r,p,t)J(r,p,t)\mathrm{d}\mathcal{H}^1(p)\mathrm{d}r \\
&= \int_{-\delta}^{\delta}\int_{\Gamma_t}\int_0^r \frac{1}{\epsilon}\partial_{\mathbf{n}_{\Gamma_t}(p)}\tilde{\mathbf{w}}_1^\epsilon(\sigma,p,t)\cdot\mathbf{n}_{\Gamma_t}(p)\mathrm{d}\sigma\theta_0'(\rho(r,p,t))\varphi(r,p,t)J(r,p,t)\mathrm{d}\mathcal{H}^1(p)\mathrm{d}r \\
&= \int_{-\delta}^{\delta}\int_{\Gamma_t}\int_0^r -\frac{1}{\epsilon}\mathrm{div}_\tau\tilde{\mathbf{w}}_1^\epsilon(\sigma,p,t)\theta_0'(\rho(r,p,t))\varphi(r,p,t)J(r,p,t)\mathrm{d}\sigma\mathrm{d}\mathcal{H}^1(p)\mathrm{d}r,
\end{aligned}
\tag{7.38}
$$

where we used the fundamental theorem of calculus in the second step and

$$
0 = \mathrm{div}\tilde{\mathbf{w}}_1^\epsilon = \partial_{\mathbf{n}_{\Gamma_t}}\tilde{\mathbf{w}}_1^\epsilon \cdot \mathbf{n}_{\Gamma_t} + \mathrm{div}_\tau\tilde{\mathbf{w}}_1^\epsilon
$$

in the last step, where div_τ denotes the surface divergence. Using Fubini's and Gauß theorem, we may further estimate (7.38)

$$
\begin{aligned}
&\int_{\Gamma_t(\delta)} \frac{1}{\epsilon}\left(\tilde{\mathbf{w}}_1^\epsilon - \tilde{\mathbf{w}}_1^\epsilon|_\Gamma\right) \cdot \mathbf{n}\theta_0'(\rho)\varphi \mathrm{d}x \\
&= \int_{-\delta}^{\delta}\int_0^r\int_{\Gamma_t} \frac{1}{\epsilon}\tilde{\mathbf{w}}_1^\epsilon(\sigma,p,t)\cdot\nabla_\tau\left(\theta_0'(\rho(r,p,t))\varphi(r,p,t)J(r,p,t)\right)\mathrm{d}\mathcal{H}^1(p)\mathrm{d}\sigma\mathrm{d}r \\
&\quad + \int_{-\delta}^{\delta}\int_0^r\int_{\Gamma_t} \frac{1}{\epsilon}\tilde{\mathbf{w}}_1^\epsilon(\sigma,p,t)\cdot\mathbf{n}_{\Gamma_t}(p)\kappa(p)\theta_0'(\rho(r,p,t))\varphi(r,p,t)J(r,p,t)\mathrm{d}\mathcal{H}^1(p)\mathrm{d}\sigma\mathrm{d}r \\
&= \int_{-\delta}^{\delta}\int_0^r\int_{\Gamma_t} \frac{1}{\epsilon}\tilde{\mathbf{w}}_1^\epsilon(\sigma,p,t)\cdot\nabla_\tau(\rho(r,p,t))\theta_0''(\rho(r,p,t))\varphi(r,p,t)J(r,p,t)\mathrm{d}\mathcal{H}^1(p)\mathrm{d}\sigma\mathrm{d}r \\
&\quad + \int_{-\delta}^{\delta}\int_0^r\int_{\Gamma_t} \frac{1}{\epsilon}\tilde{\mathbf{w}}_1^\epsilon(\sigma,p,t)\cdot\nabla_\tau(\varphi(r,p,t)J(r,p,t))\theta_0'(\rho(r,p,t))\mathrm{d}\mathcal{H}^1(p)\mathrm{d}\sigma\mathrm{d}r \\
&\quad + \int_{-\delta}^{\delta}\int_0^r\int_{\Gamma_t} \frac{1}{\epsilon}\tilde{\mathbf{w}}_1^\epsilon(\sigma,p,t)\cdot\mathbf{n}_{\Gamma_t}(p)\kappa(p)\theta_0'(\rho(r,p,t))\varphi(r,p,t)J(r,p,t)\mathrm{d}\mathcal{H}^1(p)\mathrm{d}\sigma\mathrm{d}r \\
&=: I_1 + I_2 + I_3,
\end{aligned}
$$

where κ denotes the mean curvature and ∇_τ the surface gradient. To estimate the occurring integrals, we note that

$$\left|\int_0^r \tilde{\mathbf{w}}_1^\epsilon(\sigma,p,t)\,\mathrm{d}\sigma\right| \leq r\,\|\tilde{\mathbf{w}}_1^\epsilon(.,p,t)\|_{L^\infty(-\delta,\delta)} \leq Cr\,\|\tilde{\mathbf{w}}_1^\epsilon(.,p,t)\|_{H^1(-\delta,\delta)} \tag{7.39}$$

holds for all $p \in \Gamma_t$ and $r \in (-\delta,\delta)$.

After a change of variables, we get

$$\begin{aligned}
|I_2| &\leq C\epsilon \int_{\Gamma_t} \|\tilde{\mathbf{w}}_1^\epsilon(.,p,t)\|_{H^1(-\delta,\delta)} \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon} \left|\left(\nabla_\tau(\varphi J)\left(\epsilon\left(\rho+h_A^\epsilon\right),p,t\right)\right)\left(\rho+h_A^\epsilon\right)\theta_0'(\rho)\right| \mathrm{d}\rho\mathrm{d}\mathcal{H}^1(p) \\
&\leq C(K)\,\epsilon^{\frac{1}{2}}\,\|\tilde{\mathbf{w}}_1^\epsilon(.,t)\|_{L^2(\Gamma_t;H^1(-\delta,\delta))}\left\|(\rho+1)\,\theta_0'\right\|_{L^2(\mathbb{R})}\left(\|\varphi\|_{H^1(\Omega)}+\epsilon^{\frac{1}{2}}\,\|\varphi\|_{L^{2,\infty}(\Gamma_t(\delta))}\right) \\
&\leq C(K)\,\epsilon^{\frac{1}{2}}\,\|\tilde{\mathbf{w}}_1^\epsilon(.,t)\|_{H^1(\Omega)}\,\|\varphi\|_{H^1(\Omega)},
\end{aligned}$$

where we used (7.39), $\|h_A^\epsilon\|_{C^0([0,T];C^1(\mathbb{T}^1))} \leq C(K)$ as in (6.9) and the uniform boundedness of $\nabla_\tau J$ in the second step. In the last step, we used Lemma 2.23 and the exponential decay of θ_0', cf. (2.1).

Thus, we get

$$\int_0^{T_\epsilon} |I_2|\,\mathrm{d}t \leq C(K)\,(T_\epsilon)^{\frac{1}{2}}\,\epsilon^{\frac{1}{2}}\,\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T_\epsilon;H^1(\Omega))}\,\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}.$$

In a similar manner, we may take care of I_1. Performing the same change of variables as above and observing $\nabla_\tau(\rho(r,p,t)) = \nabla_\tau(h_A^\epsilon(S(p,t),t))$ leads to

$$\begin{aligned}
|I_1| &\leq C\epsilon \int_{\Gamma_t} \|\tilde{\mathbf{w}}_1^\epsilon\|_{H^1(-\delta,\delta)}\,\|\varphi\|_{L^\infty(-\delta,\delta)} \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon} \left|\partial_s h_A^\epsilon(S(p,t),t)\right| \left|\left(\rho-h_A^\epsilon\right)\theta_0''(\rho)\right| \mathrm{d}\rho\mathrm{d}\mathcal{H}^1(p) \\
&\leq C(K)\,\epsilon\,\|\tilde{\mathbf{w}}_1^\epsilon(.,t)\|_{L^2(\Gamma_t;H^1(-\delta,\delta))}\,\|\varphi\|_{L^{2,\infty}(\Gamma_t(\delta))} \\
&\leq C(K)\,\epsilon\,\|\tilde{\mathbf{w}}_1^\epsilon(.,t)\|_{H^1(\Omega)}\,\|\varphi\|_{H^1(\Gamma_t(\delta))},
\end{aligned}$$

where we again used the uniform bound on h_A^ϵ in the second step and Lemma 2.23 in the last. Thus, we have

$$\int_0^{T_\epsilon} |I_1|\,\mathrm{d}t \leq C(K)\,(T_\epsilon)^{\frac{1}{2}}\,\epsilon\,\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T_\epsilon;H^1(\Omega))}\,\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}.$$

A completely analogue procedure, together with the fact that $|\kappa(p)| \leq C$ for all $p \in \Gamma_t$, yields

$$\int_0^{T_\epsilon} |I_3|\,\mathrm{d}t \leq C(K)\left(T'\right)^{\frac{1}{2}}\,\epsilon\,\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T_\epsilon;H^1(\Omega))}\,\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}.$$

Lemma 5.29 together with the estimates on I_1, I_2 and I_3 completes the proof for (7.35).

To prove (7.36), we calculate

$$
\begin{aligned}
&\left|\int_{\Gamma_t(\delta)} \left(\left(\tilde{\mathbf{w}}_1^\epsilon - \tilde{\mathbf{w}}_1^\epsilon|_{\Gamma_t}\right)\right) \cdot \nabla^\Gamma h_A^\epsilon \left(S\left(x,t\right),t\right) \theta_0'\left(\rho\right) \varphi \mathrm{d}x\right| \\
&\leq C \int_{\mathbb{T}^1} \int_{-\delta}^{\delta} \left|\left(\tilde{\mathbf{w}}_1^\epsilon \left(X\left(r,s,t\right)\right) - \tilde{\mathbf{w}}_1^\epsilon \left(X\left(0,s,t\right)\right)\right) \cdot \nabla^\Gamma h_A^\epsilon \left(s,t\right) \theta_0'\left(\rho\left(X\left(r,s,t\right)\right)\right) \varphi\right| \mathrm{d}r \mathrm{d}s \\
&\leq C \int_{\mathbb{T}^1} \int_{-\delta}^{\delta} \int_0^r \left|\left(\partial_{\mathbf{n}} \tilde{\mathbf{w}}_1^\epsilon\right)\left(X\left(\sigma,s,t\right)\right)\right| \mathrm{d}\sigma \left|\nabla^\Gamma h_A^\epsilon \left(s,t\right) \theta_0'\left(\rho\left(X\left(r,s,t\right)\right)\right) \varphi\right| \mathrm{d}r \mathrm{d}s.
\end{aligned}
$$

Using that

$$
\left|\int_0^r \left(\partial_{\mathbf{n}} \tilde{\mathbf{w}}_1^\epsilon\right)\left(X\left(r,s,t\right)\right) \mathrm{d}\sigma\right| \leq \left\|\tilde{\mathbf{w}}_1^\epsilon \left(X\left(.,s,t\right)\right)\right\|_{H^1(-\delta,\delta)} \sqrt{|r|}
$$

holds for all $r \in (-\delta, \delta)$, $s \in \mathbb{T}^1$, $t \in [0, T_\epsilon]$, we may proceed to estimate

$$
\begin{aligned}
&\left|\int_{\Gamma_t(\delta)} \left(\left(\tilde{\mathbf{w}}_1^\epsilon - \tilde{\mathbf{w}}_1^\epsilon|_{\Gamma_t}\right)\right) \cdot \nabla^\Gamma h^\epsilon \left(S\left(x,t\right),t\right) \theta_0'\left(\rho\right) \varphi \mathrm{d}x\right| \\
&\leq C\left(K\right) \int_{\mathbb{T}^1} \left\|\tilde{\mathbf{w}}_1^\epsilon \left(X\left(.,s,t\right)\right)\right\|_{H^1(-\delta,\delta)} \left\|\varphi\right\|_{L^\infty(-\delta,\delta)} \int_{-\frac{\delta}{\epsilon}-h_A^\epsilon}^{\frac{\delta}{\epsilon}-h_A^\epsilon} \epsilon^{\frac{3}{2}} \left|\rho+1\right| \left|\theta_0'\left(\rho\right)\right| \mathrm{d}\rho \mathrm{d}s \\
&\leq C\left(K\right) \epsilon^{\frac{3}{2}} \left\|\tilde{\mathbf{w}}_1^\epsilon \left(.,t\right)\right\|_{H^1(\Omega)} \left\|\varphi\right\|_{H^1(\Omega)},
\end{aligned}
$$

where we used the same techniques as before. Integration from 0 to T_ϵ and Lemma 5.29 yield the assertion.

The proof of (7.37) follows analogously to the proof of (7.36) since $\partial_\rho c_1 \in \mathcal{R}_\alpha$. □

Lemma 7.11. *Let $\varphi \in L^\infty\left(0, T_\epsilon; H^1\left(\Omega\right)\right)$ and $\mathbf{w}_1^\epsilon = \frac{\tilde{\mathbf{w}}_1^\epsilon}{\epsilon^{M-\frac{1}{2}}}$. Then it holds*

$$
\int_0^{T_\epsilon} \left|\int_\Omega \epsilon^{M-\frac{1}{2}} \left(\mathbf{w}_1^\epsilon - \mathbf{w}_1^\epsilon|_\Gamma \xi\right) \cdot \nabla c_A^\epsilon \varphi \mathrm{d}x\right| \mathrm{d}t \leq C\left(K\right) C\left(T_\epsilon, \epsilon\right) \epsilon^M \left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))},
$$

for all $\epsilon \in (0, \epsilon_1)$, where $C\left(T, \epsilon\right) \to 0$ as $\left(T, \epsilon\right) \to 0$.

Proof. In this proof, we denote $\mathcal{R}^\epsilon = \epsilon^{M-\frac{1}{2}} \left(\mathbf{w}_1^\epsilon - \mathbf{w}_1^\epsilon|_\Gamma \xi\right) \cdot \nabla c_A^\epsilon$. Since $\nabla c_A^\epsilon \in \mathcal{O}\left(\epsilon\right)$ in $L^\infty\left(\Omega_{T_0} \backslash \Gamma\left(2\delta\right)\right)$ (cf. (6.8)), we have

$$
\begin{aligned}
\int_0^{T_\epsilon} \left|\int_{\Omega\backslash\Gamma_t(2\delta)} \mathcal{R}^\epsilon \varphi \mathrm{d}x\right| \mathrm{d}t &\leq C\epsilon \int_0^{T_\epsilon} \int_{\Omega\backslash\Gamma_t(2\delta)} \left|\tilde{\mathbf{w}}_1^\epsilon \varphi\right| \mathrm{d}x \mathrm{d}t \\
&\leq C\epsilon \left\|\tilde{\mathbf{w}}_1^\epsilon\right\|_{L^2(0,T_\epsilon;H^1(\Omega))} \left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \\
&\leq C\left(K\right) \epsilon^{M+\frac{1}{2}} \left\|\varphi\right\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \qquad (7.40)
\end{aligned}
$$

due to Lemma 5.29. Next we estimate

$$\int_0^{T_\epsilon}\left|\int_{\Gamma_t(2\delta)}\mathcal{R}^\epsilon\varphi\mathrm{d}x\right|\mathrm{d}t\leq\int_{\Gamma(2\delta;T_\epsilon)}\left|\epsilon^{M-\frac{1}{2}}\left(\mathbf{w}_1^\epsilon-\mathbf{w}_1^\epsilon|_\Gamma\,\xi\right)\cdot\left(\xi\nabla c_I+\xi'\mathbf{n}\left(c_I-c_O\right)\right)\varphi\right|\mathrm{d}\left(x,t\right)$$
$$+\int_{\Gamma(2\delta;T_\epsilon)}\left|\epsilon^{M-\frac{1}{2}}\left(\mathbf{w}_1^\epsilon-\mathbf{w}_1^\epsilon|_\Gamma\,\xi\right)\cdot\left(\left(1-\xi\right)\nabla c_O\right)\varphi\right|\mathrm{d}\left(x,t\right)\qquad(7.41)$$

The term involving $c_I - c_O$ may be suitably estimated by using the matching conditions in Corollary 6.11, Hölder's inequality and the fact that $\|\mathbf{w}_1^\epsilon|_\Gamma\|_{L^2(\Gamma_t(2\delta))}\leq C\,\|\mathbf{w}_1^\epsilon\|_{H^1(\Gamma_t(2\delta))}$ due to the continuity of the trace operator. Now

$$\nabla c_I\left(x,t\right)=\sum_{i=0}^{M+1}\epsilon^i\left(\partial_\rho c_i\left(\rho\left(x,t\right),x,t\right)\left(\frac{\mathbf{n}\left(S\left(x,t\right),t\right)}{\epsilon}-\nabla^\Gamma h_A^\epsilon\left(x,t\right)\right)+\nabla_x c_i\left(\rho\left(x,t\right),x,t\right)\right)$$

by definition. Since $\nabla_x c_0\equiv 0$, we have $\sum_{i=0}^{M+1}\epsilon^i\nabla_x c_i\in\mathcal{O}\left(\epsilon\right)$ in $L^\infty\left(\Gamma\left(2\delta\right)\right)$ and we get a similar estimate to (7.40) for the terms of kind $\nabla_x c_i$. Choosing ϵ small enough, we have

$$\left|\frac{d_\Gamma\left(x,t\right)}{\epsilon}-h_A^\epsilon\left(S\left(x,t\right),t\right)\right|\geq\frac{\delta}{2\epsilon}$$

for all $(x,t)\in\Gamma\left(2\delta;T_\epsilon\right)\backslash\Gamma\left(\delta;T_\epsilon\right)$ due to the uniform bound on h_A^ϵ. As $\partial_\rho c_i\in\mathcal{R}_\alpha$, this leads to

$$\sup_{(x,t)\in\Gamma(2\delta;T_\epsilon)\backslash\Gamma(\delta;T_\epsilon)}\left|\partial_\rho c_i\left(\rho\left(x,t\right),x,t\right)\right|\leq C_1e^{-C_2\frac{\delta}{2\epsilon}}.$$

Thus, we have

$$\int_{\Gamma(2\delta;T_\epsilon)\backslash\Gamma(\delta;T_\epsilon)}\epsilon^{M-\frac{1}{2}}\left|\mathbf{w}_1^\epsilon-\mathbf{w}_1^\epsilon|_\Gamma\,\xi\right|\left|\partial_\rho c_i\left(\rho\left(x,t\right),x,t\right)\left(\frac{\mathbf{n}}{\epsilon}-\nabla^\Gamma h_A^\epsilon\right)\right|\left|\varphi\right|\mathrm{d}\left(x,t\right)$$
$$\leq C\left(K\right)\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T_\epsilon;H^1(\Omega))}\|\varphi\|_{L^2(0,T_\epsilon;H^1(\Omega))}\frac{1}{\epsilon}C_1e^{-C_2\frac{\delta}{2\epsilon}}$$
$$\leq C\left(K\right)\epsilon^{M+\frac{1}{2}}\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

for all $i\in\{0,\ldots,M+1\}$, where we have used $\left\|h_{M-\frac{1}{2}}^\epsilon\right\|_{C^0(0,T_\epsilon;C^1(\mathbb{T}^1))}\leq C\left(K\right)$ due to (6.9). So we have reduced the task of estimating (7.41) to showing

$$\int_{\Gamma(\delta;T_\epsilon)}\epsilon^{M-\frac{1}{2}}\left|\left(\mathbf{w}_1^\epsilon-\mathbf{w}_1^\epsilon|_\Gamma\right)\cdot\left(\epsilon^i\partial_\rho c_i\left(\rho\left(x,t\right),x,t\right)\left(\frac{\mathbf{n}}{\epsilon}-\nabla^\Gamma h_A^\epsilon\right)\right)\varphi\right|\mathrm{d}\left(x,t\right)$$
$$\leq C\left(T,\epsilon\right)C\left(K\right)\epsilon^M\|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))}$$

for $i\in\{0,\ldots,M+1\}$, where $C\left(T,\epsilon\right)\to 0$ as $\left(T,\epsilon\right)\to 0$. For $i\in\{0,1\}$ this is a consequence of Lemma 7.10. For $i\geq 2$ this is a consequence of $\partial_\rho c_i\in L^\infty\left(\mathbb{R}\times\Gamma\left(2\delta\right)\right)$, allowing for a similar estimate as in (7.40). This shows the claim.

□

7.2. The Proof of the Main Result

Let the assumptions of Theorem 4.1 hold. Let $c_A^\epsilon, \mu_A^\epsilon, \mathbf{v}_A^\epsilon, p_A^\epsilon, h_A^\epsilon$ be defined as in Definition 6.2 and let $\tilde{\mathbf{w}}_1^\epsilon$ and $\tilde{\mathbf{w}}_2^\epsilon$ be weak solutions to (5.174)–(5.176) (with $H = \left(h_{M-\frac{1}{2}}^\epsilon\right)_{\epsilon>0}$), and (7.10)–(7.12) and let $\overline{\mathbf{v}^\epsilon}$ be a strong solution to (7.7)–(7.9). We denote $\mathbf{w}_1^\epsilon = \frac{\tilde{\mathbf{w}}_1^\epsilon}{\epsilon^{M-\frac{1}{2}}}$. Additionally, let $(\mathbf{v}^\epsilon, p^\epsilon, c^\epsilon, \mu^\epsilon)$ be smooth solutions to (1.18)–(1.25) such that

$$c_0^\epsilon(x) = c_A^\epsilon(x,0) + \psi_0^\epsilon(x) \tag{7.42}$$

is satisfied. Note that Proposition 7.2 implies that Lemma 4.4 is applicable in this situation. We define $R := c^\epsilon - c_A^\epsilon$ in Ω_{T_0} and let $\varphi(.,t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for $t \in [0,T_0]$ be the unique solution of the problem

$$\begin{aligned} -\Delta\varphi(.,t) &= R(.,t) && \text{in } \Omega, \\ \varphi(.,t) &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Standard regularity theory yields the smoothness of φ and we have

$$\|\varphi(.,0)\|_{H^1(\Omega)} \leq \|R(.,0)\|_{L^2(\Omega)} \leq C_{\psi_0}\epsilon^M$$

for all $\epsilon \in (0,1)$. This implies the existence of some family $(\tau_\epsilon)_{\epsilon\in(0,1)} \subset (0,T_0]$ and $K \geq 1$ such that Assumption 4.2 is satisfied (and in particular (4.6) holds for τ_ϵ) and such that

$$\|\varphi(.,0)\|_{H^1(\Omega)} \leq \|R(.,0)\|_{L^2(\Omega)} \leq \frac{K}{2}\epsilon^M. \tag{7.43}$$

Moreover, we may choose $\epsilon_0 \in (0,1)$ small enough, such that (5.191), Theorem 5.32 2) and Theorem 6.12 hold. This implies in particular that Assumption 7.1 is satisfied and that we may use all the results shown in Section 7.1. Now let $T \in (0,T_0]$ and for $\epsilon \in (0,\epsilon_0)$ we set

$$T_\epsilon := \sup\{t \in (0,T] \,|\, (4.6) \text{ holds true for } t\}. \tag{7.44}$$

We will show in the following that we may choose $T \in (0,T_0]$ (independent of ϵ) and ϵ_0 small enough, such that $T_\epsilon = T$ for all $\epsilon \in (0,\epsilon_0)$.

Testing Procedure with φ
Let $T' \in (0,T_0]$ be fixed. Multiplying the difference of the differential equations (1.20) and (4.9) by φ and integrating the result over Ω yields

$$\begin{aligned} 0 &= \int_\Omega \varphi\left(\partial_t R + (\mathbf{v}^\epsilon \cdot \nabla c^\epsilon - \mathbf{v}_A^\epsilon \cdot \nabla c_A^\epsilon - \tilde{\mathbf{w}}_1^\epsilon|_\Gamma \cdot \nabla c_A^\epsilon \xi(d_\Gamma)) - \Delta(\mu^\epsilon - \mu_A^\epsilon) + r_{\mathrm{CH1}}^\epsilon\right) \mathrm{d}x \\ &= \int_\Omega \varphi\partial_t(-\Delta\varphi) + \varphi\left((\mathbf{v}^\epsilon \cdot \nabla R) - (\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}) \cdot \nabla c_A^\epsilon + (\tilde{\mathbf{w}}_1^\epsilon - \tilde{\mathbf{w}}_1^\epsilon|_\Gamma \xi(d_\Gamma)) \cdot \nabla c_A^\epsilon\right) \mathrm{d}x \\ &\quad + \int_\Omega \varphi\left(\mathbf{v}_{err}^\epsilon \cdot \nabla c_A^\epsilon + \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla c_A^\epsilon - \Delta(\mu^\epsilon - \mu_A^\epsilon)\right) + \varphi r_{\mathrm{CH1}}^\epsilon \mathrm{d}x \end{aligned} \tag{7.45}$$

for all $t \in (0,T)$. In the second equality we used the definition of φ and the identity

$$\mathbf{v}^\epsilon \cdot \nabla c^\epsilon - \mathbf{v}_A^\epsilon \cdot \nabla c_A^\epsilon = \mathbf{v}^\epsilon \cdot \nabla R + (\tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon) \cdot \nabla c_A^\epsilon - (\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}) \cdot \nabla c_A^\epsilon + \mathbf{v}_{err}^\epsilon \cdot \nabla c_A^\epsilon, \quad (7.46)$$

which is a consequence of the definition of $\mathbf{v}_{err}^\epsilon$ (cf. (7.14)). In order to shorten the notation, we now write

$$\mathcal{E}\left(R, T'\right) := \int_{\Omega_{T'}} \epsilon \left|\nabla R\right|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right) R^2 \mathrm{d}\left(x,t\right),$$

$$\mathcal{N}\left(c_A^\epsilon, R\right) := f'\left(c_A^\epsilon + R\right) - f'\left(c_A^\epsilon\right) - f''\left(c_A^\epsilon\right) R$$

and

$$\mathcal{R}^\epsilon := \left(\epsilon^{M-\frac{1}{2}}\left(-\mathbf{w}_1^\epsilon + \mathbf{w}_1^\epsilon|_\Gamma \xi\left(d_\Gamma\right)\right) \cdot \nabla c_A^\epsilon\right)$$

which – regarding the formulae for μ^ϵ and μ_A^ϵ in (1.21) and (4.10) – leads us to

$$\begin{aligned} 0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|\nabla\varphi\right|^2 \mathrm{d}x + \int_\Omega \varphi\left(\mathbf{v}^\epsilon \cdot \nabla R\right) + \epsilon\left|\nabla R\right|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right) R^2 + \epsilon^{-1}\mathcal{N}\left(c_A^\epsilon, R\right) R \mathrm{d}x \\ - \int_\Omega \varphi\left(\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) \cdot \nabla c_A^\epsilon - \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla c_A^\epsilon - \mathbf{v}_{err}^\epsilon \cdot \nabla c_A^\epsilon - r_{\mathrm{CH1}}^\epsilon + \mathcal{R}^\epsilon\right) + R r_{\mathrm{CH2}}^\epsilon \mathrm{d}x \qquad (7.47) \end{aligned}$$

for all $t \in (0,T')$. We obtained this equality by using integration by parts in (7.45) and noting that the boundary integrals vanish due to the Dirichlet boundary conditions satisfied by φ, μ_A^ϵ and μ^ϵ.

Using Theorem 3.12, we obtain

$$\begin{aligned} \int_\Omega \epsilon\left|\nabla R\right|^2 + \epsilon^{-1} f''\left(c_A^\epsilon\right) R^2 \mathrm{d}x \geq C_1\left(\epsilon\left\|R\right\|_{L^2(\Omega)}^2 + \epsilon^{-1}\left\|R\right\|_{L^2(\Omega\setminus\Gamma_t(\delta))} + \epsilon\left\|\nabla^\Gamma R\right\|_{L^2(\Gamma_t(\delta))}^2\right) \\ + C_2\left(\epsilon^3\left\|\nabla R\right\|_{L^2(\Omega)}^2 + \epsilon\left\|\nabla R\right\|_{L^2(\Omega\setminus\Gamma_t(\delta))}^2\right) \\ - C_3\left\|\nabla\varphi\right\|_{L^2(\Omega)}^2 \qquad (7.48) \end{aligned}$$

and due to Lemma 7.5 and the assumptions on f, we get

$$\frac{1}{\epsilon}\int_\Omega \mathcal{N}\left(c_A^\epsilon, R\right) R \mathrm{d}x \geq -\frac{C}{\epsilon}\int_\Omega \left|R\right|^3 \mathrm{d}x.$$

Plugging these observations into (7.47) enables us to get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|\nabla\varphi\right|^2 \mathrm{d}x + C_1\left(\left\|\left(\epsilon R, \epsilon^3\nabla R\right)\right\|_{L^2(\Omega)}^2 + \left\|\left(\epsilon^{-1}R, \epsilon\nabla R\right)\right\|_{L^2(\Omega\setminus\Gamma_t(\delta))} + \epsilon\left\|\nabla^\Gamma R\right\|_{L^2(\Gamma_t(\delta))}^2\right) \\ \leq \left|\int_\Omega \left(\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon}\right) \cdot \nabla c_A^\epsilon + r_{\mathrm{CH1}}^\epsilon - \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla c_A^\epsilon + \mathcal{R}^\epsilon - \mathbf{v}^\epsilon \cdot \nabla R - \mathbf{v}_{err}^\epsilon \cdot \nabla c_A^\epsilon\right)\varphi \mathrm{d}x\right| \\ + C_2\left\|\nabla\varphi\right\|_{L^2(\Omega)}^2 + \frac{C_3}{\epsilon}\int_\Omega \left|R\right|^3 \mathrm{d}x + \left|\int_\Omega R r_{\mathrm{CH2}}^\epsilon \mathrm{d}x\right|. \qquad (7.49) \end{aligned}$$

To shorten the notations, we introduce one last notation,

$$\mathcal{RS} := \left| \int_\Omega \left(\left(\mathbf{v}_A^\epsilon - \overline{\mathbf{v}^\epsilon} \right) \cdot \nabla c_A^\epsilon + r_{\text{CH1}}^\epsilon - \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla c_A^\epsilon + \mathcal{R}^\epsilon - \mathbf{v}^\epsilon \cdot \nabla R - \mathbf{v}_{err}^\epsilon \cdot \nabla c_A^\epsilon \right) \varphi \mathrm{d}x \right|.$$
$$+ \frac{C_3}{\epsilon} \int_\Omega |R|^3 \, \mathrm{d}x + \left| \int_\Omega R r_{\text{CH2}}^\epsilon \mathrm{d}x \right|$$

Gronwall's inequality now yields

$$\sup_{0 \le \tau \le T'} \|\nabla \varphi(.,\tau)\|_{L^2(\Omega)}^2 \le e^{C_2 T'} \left(\|\nabla \varphi(.,0)\|_{L^2(\Omega)}^2 + \int_0^{T'} \mathcal{RS} \mathrm{d}t \right). \tag{7.50}$$

Integrating (7.49) over $(0,T')$ and using (7.50), we get

$$\begin{aligned} &\sup_{0 \le \tau \le T'} \|\nabla \varphi\|_{L^2(\Omega)}^2 + \left\| \left(\epsilon R, \epsilon^3 |\nabla R| \right) \right\|_{L^2(\Omega_{T'})}^2 + \left\| \left(\epsilon^{-1} R, \epsilon |\nabla R| \right) \right\|_{L^2(\Omega \setminus \Gamma(\delta;T'))}^2 \\ &+ \epsilon \left\| \nabla^\Gamma R \right\|_{L^2(\Gamma(\delta;T'))}^2 \le C(T_0) \left(\|\nabla \varphi(.,0)\|_{L^2(\Omega)}^2 + \int_0^{T'} \mathcal{RS} \mathrm{d}t \right) \end{aligned} \tag{7.51}$$

for some positive constant $C(T_0) > 0$. On the other hand, (7.47) together with (7.50) and (7.43) also implies

$$\mathcal{E}\left(R, T'\right) \le C(T_0) \left(\|\nabla \varphi(.,0)\|_{L^2(\Omega)}^2 + \int_0^{T'} \mathcal{RS} \mathrm{d}t \right). \tag{7.52}$$

In regard to our definition of T_ϵ, the idea now is the following: as (4.6) and (7.43) hold for T_ϵ by construction, we will show that we may choose $T \in (0, T_0]$ in the definition of T_ϵ and $\epsilon_0 > 0$ so small, that

$$\begin{aligned} C(T_0) \left(\|\nabla \varphi(.,0)\|_{L^2(\Omega)}^2 + \int_0^{T_\epsilon} \mathcal{RS} \mathrm{d}t \right) &\le \frac{K^2}{4} \epsilon^{2M} + \frac{K^2}{2} \epsilon^{2M} \\ &< K^2 \epsilon^{2M}. \end{aligned}$$

for all $\epsilon \in (0, \epsilon_0)$. By (7.51) and (7.52) and the definitition of T_ϵ, this implies (4.2a)–(4.2d).

Estimating $\mathcal{RS}$:
Due to Theorem 6.12, in particular due to (6.44)–(6.45), and since (4.6) holds for T_ϵ, we get

$$\int_0^{T_\epsilon} \left| \int_\Omega R r_{\text{CH2}}^\epsilon \mathrm{d}x \right| \mathrm{d}t + \int_0^{T_\epsilon} \left| \int_\Omega r_{\text{CH1}}^\epsilon \varphi \mathrm{d}x \right| \mathrm{d}t \le C(T,\epsilon) \, C(K) \, \epsilon^{2M}$$

with $C(T,\epsilon)\to 0$ as $(T,\epsilon)\to 0$. Moreover, by Corollary 7.9 we immediately get

$$\int_0^{T_\epsilon}\left|\int_\Omega (\mathbf{v}_A^\epsilon-\overline{\mathbf{v}^\epsilon})\cdot\nabla c_A^\epsilon\varphi \mathrm{d}x\right|\mathrm{d}t\le C(K)\,T_\epsilon^{\frac{1}{2}}\epsilon^{2M},$$

$$\int_0^{T_\epsilon}\left|\int_\Omega \mathbf{v}_{err}^\epsilon\cdot\nabla c_A^\epsilon\varphi \mathrm{d}x\right|\mathrm{d}t\le C(K)\,C(T,\epsilon)\,\epsilon^{2M}.$$

Lemma 7.11 yields

$$\int_0^{T_\epsilon}\left|\int_\Omega \mathcal{R}^\epsilon\varphi \mathrm{d}x\right|\mathrm{d}t\le C(K)\,C(T,\epsilon)\,\epsilon^{2M}$$

and Lemma 7.7 implies

$$\int_0^{T_\epsilon}\left|\int_\Omega (\tilde{\mathbf{w}}_2^\epsilon\cdot\nabla c_A^\epsilon)\,\varphi \mathrm{d}x\right|\mathrm{d}t\le C(K)\,C(T,\epsilon)\,\epsilon^{2M}$$

with $C(T,\epsilon)\to 0$ as $(T,\epsilon)\to 0$.

Next we consider $\frac{1}{\epsilon}\int_{\Omega_{T_\epsilon}}|R|^3\,\mathrm{d}x$: As a consequence of Lemma 7.4 we have

$$\begin{aligned}
&\int_0^{T_\epsilon}\|R\|^3_{L^3(\Gamma_t(\delta))}\,\mathrm{d}t\\
&\le C\int_0^{T_\epsilon}\left(\|R\|_{L^2(\Gamma_t(\delta))}+\left\|\nabla^\Gamma R\right\|_{L^2(\Gamma_t(\delta))}\right)^{\frac{1}{2}}\left(\|R\|_{L^2(\Gamma_t(\delta))}+\|\partial_{\mathbf{n}}R\|_{L^2(\Gamma_t(\delta))}\right)^{\frac{1}{2}}\left(\|R\|_{L^2(\Gamma_t(\delta))}\right)^2\mathrm{d}t\\
&\le C\left(\|R\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))}+\left\|\nabla^\Gamma R\right\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))}\right)^{\frac{1}{2}}\\
&\quad\cdot\left(\|R\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))}+\|\partial_{\mathbf{n}}R\|_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))}\right)^{\frac{1}{2}}\|R\|^2_{L^4(0,T_\epsilon;L^2(\Gamma_t(\delta)))}\,,
\end{aligned}\tag{7.53}$$

where we used Hölder's inequality in the last line.

The only term we have no control over yet is the last one, so we need to find an appropriate estimate. Since

$$\|R\|^2_{L^2(\Omega)}=\int_\Omega -\Delta\varphi R\mathrm{d}x=\int_\Omega\nabla\varphi\cdot\nabla R\mathrm{d}x\le\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla R\|_{L^2(\Omega)}\,,$$

we may deduce

$$\begin{aligned}
\|R\|^2_{L^4(0,T_\epsilon;L^2(\Omega))}&\le\left(\int_0^{T_\epsilon}\|\nabla\varphi\|^2_{L^2(\Omega)}\|\nabla R\|^2_{L^2(\Omega)}\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\le\sup_{\tau\in(0,T_\epsilon)}\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla R\|_{L^2(\Omega_{T_\epsilon})}\,.
\end{aligned}\tag{7.54}$$

Regarding (4.6), (7.53) and the definition of T_ϵ, this implies

$$\frac{1}{\epsilon}\int_0^{T_\epsilon} \|R\|^3_{L^3(\Gamma_t(\delta))}\,\mathrm{d}t < \frac{1}{\epsilon}CK^3\epsilon^{\frac{1}{2}M-\frac{1}{4}}\epsilon^{\frac{1}{2}M-\frac{3}{4}}\epsilon^{M}\epsilon^{M-\frac{3}{2}} = CK^3\epsilon^{3M-\frac{7}{2}} \leq CK^3\epsilon^{2M+\frac{1}{2}}$$

as $M \geq 4$.

On the other hand, we have, for $\epsilon > 0$ small enough,

$$\begin{aligned}\frac{1}{\epsilon}\int_0^{T_\epsilon} \|R\|^3_{L^3(\Omega\backslash\Gamma_t(\delta))}\,\mathrm{d}t &\leq \frac{1}{\epsilon}C\int_0^{T_\epsilon} \|R\|_{H^1(\Omega\backslash\Gamma_t(\delta))}\|R\|^2_{L^2(\Omega\backslash\Gamma_t(\delta))}\,\mathrm{d}t\\ &\leq \frac{1}{\epsilon}C\|R\|_{L^2(0,T_\epsilon;H^1(\Omega\backslash\Gamma_t(\delta)))}\|R\|^2_{L^4(0,T_\epsilon;L^2(\Omega\backslash\Gamma_t(\delta)))}\\ &\leq \frac{1}{\epsilon}CK^3\epsilon^{M-\frac{1}{2}}\epsilon^{2M-\frac{3}{2}}\\ &\leq CK^3\epsilon^{2M+1},\end{aligned} \tag{7.55}$$

where we used the Gagliardo Nirenberg interpolation theorem, (7.54) and (4.6).

The last term we have to estimate is $\int_0^{T_\epsilon}\left|\int_\Omega \mathbf{v}^\epsilon\cdot\nabla R\varphi\mathrm{d}x\right|\mathrm{d}t$: We compute

$$\begin{aligned}\int_0^{T_\epsilon}\left|\int_\Omega \mathbf{v}^\epsilon\cdot\nabla R\varphi\mathrm{d}x\right|\mathrm{d}t &= \int_0^{T_\epsilon}\left|\int_\Omega \mathbf{v}^\epsilon\cdot\nabla\varphi R\mathrm{d}x\right|\mathrm{d}t\\ &\leq \int_0^{T_\epsilon}\left|\int_\Omega \mathbf{v}^\epsilon_A\cdot\nabla\varphi R\mathrm{d}x\right| + \left|\int_\Omega \left(\mathbf{v}^\epsilon - \mathbf{v}^\epsilon_A\right)\cdot\nabla\varphi R\mathrm{d}x\right|\mathrm{d}t.\end{aligned} \tag{7.56}$$

Before we continue with the estimates, we introduce

$$\hat{\mathbf{v}}^\epsilon_A := \mathbf{v}^\epsilon_A - \epsilon^{M-\frac{1}{2}}\mathbf{v}^\epsilon_{A,M-\frac{1}{2}} \in L^\infty\left(\Omega_{T_0}\right)$$

(see Definition 6.2 for notations). We first off compute

$$\begin{aligned}\int_0^{T_\epsilon}\left|\int_\Omega \hat{\mathbf{v}}^\epsilon_A\cdot\nabla\varphi R\mathrm{d}x\right|\mathrm{d}t \leq{}& \int_0^{T_\epsilon}\int_\Omega \left|\nabla\left((1-\gamma)\hat{\mathbf{v}}^\epsilon_A\right):(\nabla\varphi\otimes\nabla\varphi)\right| + \left|(1-\gamma)\hat{\mathbf{v}}^\epsilon_A\cdot\nabla\left(\frac{|\nabla\varphi|^2}{2}\right)\right|\mathrm{d}x\mathrm{d}t\\ &+ \int_0^{T_\epsilon}\left|\int_\Omega \gamma\hat{\mathbf{v}}^\epsilon_A\cdot\nabla\varphi R\mathrm{d}x\right|\mathrm{d}t,\end{aligned} \tag{7.57}$$

where we used $-\Delta\varphi = R$ and introduced γ, as $\hat{\mathbf{v}}^\epsilon_A$ does not satisfy Dirichlet boundary conditions (nor does φ satisfy Neumann boundary conditions).

Now $\left|\nabla\hat{\mathbf{v}}^\epsilon_A(x,t)\right| \leq \left|\xi\left(d_\Gamma(x,t)\right)\partial_\rho\mathbf{v}_0\left(\rho(x,t),x,t\right)\frac{1}{\epsilon}\right| + C(K)$, which is a consequence of the uniform boundedness of the terms $\mathbf{v}_k$, $\mathbf{v}^\pm_k$ and $\mathbf{v}^{\mathbf{B}}_k$ and of $\|h^\epsilon_A\|_{C^0(0,T_\epsilon;C^1(\Gamma_t(2\delta)))} \leq C(K)$ (see (6.9)). Moreover, by (5.113) and (5.117), and since $d_\Gamma(x,t) = \epsilon\left(\rho(x,t) + h^\epsilon_A(x,t)\right)$ for $(x,t)\in\Gamma(2\delta)$, we have

$$\left|\partial_\rho\mathbf{v}_0\left(\rho(x,t),x,t\right)\right| \leq \epsilon\eta'\left(\rho(x,t)\right)\left(\rho(x,t) + h^\epsilon_A(x,t)\right)\mathbf{u}_0(x,t) \tag{7.58}$$

for $(x,t) \in \Gamma(2\delta)$, which results due to $\eta'(\rho)\rho < C$ for all $\rho \in \mathbb{R}$ and $\mathbf{u}_0 \in L^\infty(\Gamma(2\delta; T_0))$ in

$$\int_0^{T_\epsilon} \left| \int_\Omega \nabla\left((1-\gamma)\hat{\mathbf{v}}_A^\epsilon\right) : (\nabla\varphi \otimes \nabla\varphi)\, \mathrm{d}x \right| \mathrm{d}t \leq C(K)\, T_\epsilon \|\nabla\varphi\|^2_{L^\infty(0,T_\epsilon;L^2(\Omega))} \leq C(K)\, T_\epsilon \epsilon^{2M},$$

by (4.6b) and the facts that $\hat{\mathbf{v}}_A^\epsilon \in L^\infty(\Omega_{T_0})$ and γ, γ' are bounded.

Concerning the second term on the right hand side of (7.57), we note that

$$\left|\operatorname{div}\left(\mathbf{v}_{A,0}^\epsilon\right)\right| \leq \left| \xi(d_\Gamma)\, \partial_\rho \mathbf{v}_0 \cdot \mathbf{n}\frac{1}{\epsilon} \right| + C(K) \leq C(K)$$

in Ω_{T_ϵ} for some $C > 0$ independent of ϵ, where we used (7.58) and (6.9) again. Thus, $\|\operatorname{div}(\hat{\mathbf{v}}_A^\epsilon)\|_{L^\infty(\Omega_{T_\epsilon})} \leq C(K)$ and

$$\int_0^{T_\epsilon} \left| \int_\Omega \left((1-\gamma)\hat{\mathbf{v}}_A^\epsilon\right) \cdot \nabla\left(\frac{1}{2}|\nabla\varphi|^2\right) \mathrm{d}x \right| \mathrm{d}t \leq C(K)\, T_\epsilon \|\nabla\varphi\|^2_{L^\infty(0,T_\epsilon;L^2(\Omega))} \leq C(K)\, T_\epsilon \epsilon^{2M}.$$

For the third term on the right hand side of (7.57), we may directly use (4.6) to get

$$\begin{aligned} \int_0^{T_\epsilon} \left| \int_\Omega \gamma \hat{\mathbf{v}}_A^\epsilon \cdot \nabla\varphi R \mathrm{d}x \right| \mathrm{d}t &\leq C T_\epsilon^{\frac{1}{2}} \|\nabla\varphi\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \|R\|_{L^2(\Omega_{T_\epsilon}\setminus\Gamma(2\delta;T_\epsilon))} \\ &\leq C(K)\, T_\epsilon^{\frac{1}{2}} \epsilon^{2M+\frac{1}{2}}. \end{aligned}$$

Regarding (7.56), we next estimate

$$\begin{aligned} \int_0^{T_\epsilon} \left| \int_\Omega \epsilon^{M-\frac{1}{2}} \mathbf{v}^\epsilon_{A,M-\frac{1}{2}} \cdot \nabla\varphi R \mathrm{d}x \right| \mathrm{d}t &\leq \epsilon^{M-\frac{1}{2}} \left\| \mathbf{v}^\epsilon_{A,M-\frac{1}{2}} \right\|_{L^2(0,T_\epsilon;L^\infty(\Omega))} \|\nabla\varphi\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \\ &\quad \cdot \|R\|_{L^2(0,T_\epsilon;L^2(\Omega))} \\ &\leq C(K)\, \epsilon^{M-\frac{1}{2}} \epsilon^M \epsilon^{M-\frac{1}{2}} \\ &< C(K)\, \epsilon^{2M+\frac{1}{2}} \end{aligned}$$

as $M \geq 4$. Here we used in the second inequality the explicit form of $\mathbf{v}^\epsilon_{M-\frac{1}{2}}$ in (5.254) and the fact that $\left\| \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}} \right\|_{L^2(0,T_\epsilon;L^\infty(\Omega^\pm(t)\cup\Gamma_t(2\delta)))} \leq C(K)$ due to (5.236) and the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. Moreover, we employed (4.6). To finish off estimating

$\int_0^{T_\epsilon} \left| \int_\Omega \mathbf{v}^\epsilon \cdot \nabla R \varphi \mathrm{d}x \right| \mathrm{d}t$ we consider

$$\begin{aligned}
&\int_0^{T_\epsilon} \left| \int_\Omega (\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon) \cdot \nabla \varphi R \mathrm{d}x \right| \mathrm{d}t \\
&= \int_0^{T_\epsilon} \left| \int_\Omega \left(\mathbf{v}_{err}^\epsilon + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon + (\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon)\right) \cdot \nabla \varphi R \mathrm{d}x \right| \mathrm{d}t \\
&\leq \left(\|\tilde{\mathbf{w}}_1^\epsilon\|_{L^2(0,T_\epsilon;L^4(\Omega))} + \|\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon\|_{L^2(0,T_\epsilon;L^4(\Omega))} \right) \|\nabla \varphi\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \|R\|_{L^2(0,T_\epsilon;L^4(\Omega))} \\
&\quad + \int_0^{T_\epsilon} \left| \int_\Omega \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla R \varphi \mathrm{d}x \right| \mathrm{d}t + \int_0^{T_\epsilon} \left| \int_\Omega \mathbf{v}_{err}^\epsilon \cdot \nabla R \varphi \mathrm{d}x \right| \mathrm{d}t \\
&\leq C(K) \left(\left(\epsilon^{M-\frac{1}{2}} + \epsilon^M \right) \epsilon^M \epsilon^{M-\frac{3}{2}} + \int_0^{T_\epsilon} \left| \int_\Omega \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla R \varphi \mathrm{d}x \right| \mathrm{d}t + C(T_\epsilon, \epsilon) \epsilon^{2M} \right) \\
&\leq C(K) \left(\epsilon^{2M+\frac{1}{2}} + \epsilon^{2M+\frac{1}{2}} + C(T, \epsilon) \epsilon^{2M} \right) + \int_0^{T_\epsilon} \left| \int_\Omega \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla R \varphi \mathrm{d}x \right| \mathrm{d}t, \qquad (7.59)
\end{aligned}$$

where we used Hölder's inequality and $\mathbf{v}_{err}^\epsilon = \mathbf{v}^\epsilon - (\overline{\mathbf{v}^\epsilon} + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon)$ in the first step, Theorem 7.8 1), (5.191), (4.6) and (7.30) in the third step and the fact that $M \geq 4$ in the last step. Regarding the $\tilde{\mathbf{w}}_2^\epsilon$ term we first note that for $\kappa > 0$ we have

$$\begin{aligned}
\|\nabla R\|_{L^2(0,T_\epsilon;L^{2+\kappa}(\Omega))} &\leq C \left(\|\nabla R\|_{L^2(\Omega_{T_\epsilon})}^{1-\frac{\kappa}{2+\kappa}} \|\Delta R\|_{L^2(\Omega_{T_\epsilon})}^{\frac{\kappa}{2+\kappa}} + \|\nabla R\|_{L^2(\Omega_{T_\epsilon})} \right) \\
&\leq C(K) \left(\epsilon^{M-\frac{3}{2}} \epsilon^{-(M+2)\frac{\kappa}{2+\kappa}} \right) \qquad (7.60)
\end{aligned}$$

where we used the Gagliardo Nirenberg interpolation theorem in the first line together with $\|R\|_{H^2(\Omega)} \leq C \|\Delta R\|_{L^2(\Omega)}$ due to elliptic regularity theory and $\|\Delta R\|_{L^2(\Omega_{T_\epsilon})} \leq C(K) \epsilon^{-\frac{7}{2}}$ as in (7.4). Thus, we may estimate for $\kappa > 0$ and $2 > q > \frac{2+\kappa}{(2+\kappa)-1}$

$$\begin{aligned}
\int_0^{T_\epsilon} \left| \int_\Omega \tilde{\mathbf{w}}_2^\epsilon \cdot \nabla R \varphi \mathrm{d}x \right| \mathrm{d}t &\leq \|\tilde{\mathbf{w}}_2^\epsilon\|_{L^2(0,T_\epsilon;L^q(\Omega))} \|\nabla R\|_{L^2(0,T_\epsilon;L^{2+\kappa}(\Omega))} \|\varphi\|_{L^\infty(0,T_\epsilon;H^1(\Omega))} \\
&\leq C(K) \epsilon^{3M-\frac{5}{2}} \epsilon^{-(M+2)\frac{\kappa}{2+\kappa}} \\
&\leq C(K) \epsilon^{2M+\alpha}
\end{aligned}$$

for some $\alpha > 0$, where we used (7.15), (4.6b) and (7.60) in the second line together with the fact that $M \geq 4$ and that $\kappa > 0$ may be chosen arbitrarily small.

With regard to (7.59) we get

$$\int_0^{T_\epsilon} \left| \int_\Omega (\mathbf{v}^\epsilon - \mathbf{v}_A^\epsilon) \cdot \nabla \varphi R \mathrm{d}x \right| \mathrm{d}t \leq C(K) C(T, \epsilon) \epsilon^{2M},$$

which concludes the estimates for $\int_0^{T_\epsilon} \left| \int_\Omega \mathbf{v}^\epsilon \cdot \nabla R \varphi \mathrm{d}x \right| \mathrm{d}t$.

This shows the estimate for $\mathcal{RS}$. Nevertheless, (7.51) and (7.52) do not imply estimates of the kind (4.2e) and (4.2f). For those, we need to apply another strategy.

Testing with $\gamma^2 R$:
Let again $T' \in (0, T_0]$. Multiplying the difference of the differential equations (1.20) and (4.9) by $\gamma^2 R$ and integrating the result over $\Omega_{T'}$ yields

$$
\begin{aligned}
0 &= \int_\Omega \gamma^2 R \left(\partial_t R + \mathbf{v}^\epsilon \cdot \nabla c^\epsilon - \mathbf{v}_A^\epsilon \cdot \nabla c_A^\epsilon - \Delta \left(\mu^\epsilon - \mu_A^\epsilon\right) + r_{\mathrm{CH1}}^\epsilon\right) \mathrm{d}x \\
&= \frac{1}{2} \int_\Omega \frac{d}{dt} \left(R^2\right) \gamma^2 \mathrm{d}x + \int_{\Omega_T} \gamma^2 R \left(\mathbf{v}^\epsilon \cdot \nabla R + \mathbf{v}_{err}^\epsilon \cdot \nabla c_A^\epsilon + \left(\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right) \cdot \nabla c_A^\epsilon\right) \mathrm{d}x \\
&\quad + \int_\Omega \gamma^2 R r_{\mathrm{CH1}}^\epsilon - \Delta \left(\gamma^2 R\right) \left(-\epsilon \Delta R + \frac{1}{\epsilon} \left(f''\left(c_A^\epsilon\right) R + \mathcal{N}\left(c_A^\epsilon, R\right)\right) - r_{\mathrm{CH2}}^\epsilon\right) \mathrm{d}x,
\end{aligned} \tag{7.61}
$$

where we used $\mathrm{supp}\gamma \cap \mathrm{supp}\eta = \emptyset$ in the first line and (7.46), integration by parts and $R = \mu^\epsilon = \mu_A^\epsilon = 0$ on $\partial_{T_0}\Omega$ in the second line.

As $c_A^\epsilon = -1 + \mathcal{O}(\epsilon)$ in $L^\infty\left(\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)\right)$, we have $f''\left(c_A^\epsilon(x,t)\right) = f''(-1) + \epsilon \tilde{f}(x,t)$ for $(x,t) \in \partial_{T_0}\Omega\left(\frac{\delta}{2}\right)$ by a Taylor expansion, where $\tilde{f} \in L^\infty\left(\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)\right)$. Moreover,

$$
\begin{aligned}
\nabla\left(\gamma^2 R\right) &= 2\gamma R \nabla\gamma + \gamma^2 \nabla R, \\
\Delta\left(\gamma^2 R\right) &= \Delta\left(\gamma^2\right) R + 4\gamma \nabla\gamma \cdot \nabla R + \gamma^2 \Delta R
\end{aligned}
$$

in $\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)$ and we may calculate

$$
\begin{aligned}
\frac{1}{\epsilon} \int_\Omega -\Delta\left(\gamma^2 R\right) f''\left(c_A^\epsilon\right) R \mathrm{d}x &= \frac{1}{\epsilon} \int_\Omega \nabla\left(\gamma^2 R\right) f''(-1) \cdot \nabla R - \epsilon \Delta\left(\gamma^2 R\right) \tilde{f} R \mathrm{d}x \\
&= \frac{1}{\epsilon} f''(-1) \left\|\gamma \nabla R\right\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \int_\Omega f''(-1) R \nabla\left(\gamma^2\right) \cdot \nabla R \mathrm{d}x \\
&\quad - \int_\Omega \Delta\left(\gamma^2 R\right) \tilde{f} R \mathrm{d}x,
\end{aligned} \tag{7.62}
$$

where we used integration by parts in the first step and the fact that the boundary terms vanish due to $R = 0$ on $\partial_{T_0}\Omega$.

Moreover, we have

$$
\begin{aligned}
\nabla \mathcal{N}\left(c_A^\epsilon, R\right) &= \nabla c_A^\epsilon \left(f''\left(c_A^\epsilon + R\right) - f''\left(c_A^\epsilon\right) - f^{(3)}\left(c_A^\epsilon\right) R\right) + \left(f''\left(c_A^\epsilon + R\right) - f''\left(c_A^\epsilon\right)\right) \nabla R \\
&= \nabla c_A^\epsilon f^{(4)}(\xi) R^2 + \left(f^{(3)}\left(c_A^\epsilon\right) R + f^{(4)}(\xi) R^2\right) \nabla R \\
&= k_f \nabla c_A^\epsilon R^2 + \left(f^{(3)}\left(c_A^\epsilon\right) R + k_f R^2\right) \nabla R,
\end{aligned}
$$

due to Assumption 1.2. This yields

$$\begin{aligned}\int_\Omega -\Delta\left(\gamma^2 R\right)\frac{1}{\epsilon}\mathcal{N}\left(c_A^\epsilon, R\right)\mathrm{d}\left(x,t\right) &= \frac{1}{\epsilon}\int_\Omega k_f\left(\left|\gamma\left(\nabla R\right)R\right|^2 + \nabla\left(\gamma^2\right)R^3\cdot\nabla R\right)\mathrm{d}x \\ &\quad + \frac{1}{\epsilon}\int_\Omega \nabla\left(\gamma^2 R\right)\cdot\left(k_f\nabla c_A^\epsilon R^2 + f^{(3)}\left(c_A^\epsilon\right)R\nabla R\right)\mathrm{d}x \\ &= \frac{1}{\epsilon}k_f\left\|\gamma\left|\nabla R\right|R\right\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\int_\Omega \mathcal{N}^\nabla\left(c_A^\epsilon, R\right)\mathrm{d}x, \qquad (7.63)\end{aligned}$$

where the boundary terms due to integration by parts vanish since $f'(-1) = R(x,t) = 0$ and $c_A^\epsilon(x,t) = -1$ for $(x,t) \in \partial_{T_0}\Omega$. Here we used the notation

$$\mathcal{N}^\nabla\left(c_A^\epsilon, R\right) := k_f\nabla\left(\gamma^2\right)R^3\cdot\nabla R + \nabla\left(\gamma^2 R\right)\cdot\left(k_f\nabla c_A^\epsilon R^2 + f^{(3)}\left(c_A^\epsilon\right)R\nabla R\right). \qquad (7.64)$$

Additionally, we compute

$$\int_\Omega -\Delta\left(\gamma^2 R\right)\left(-\epsilon\Delta R\right)\mathrm{d}x = \epsilon\left\|\gamma\Delta R\right\|^2_{L^2(\Omega)} + \epsilon\int_\Omega 4\gamma\nabla\gamma\cdot\nabla R\Delta R + \Delta\left(\gamma^2\right)R\Delta R\mathrm{d}x. \quad (7.65)$$

Plugging (7.62), (7.63) and (7.65) (noting that $k_f, f''(-1) > 0$) into (7.61) and integrating in time yields

$$\begin{aligned}&\sup_{t\in(0,T')}\left\|\gamma R\left(.,t\right)\right\|^2_{L^2(\Omega)} + \epsilon\left\|\gamma\Delta R\right\|^2_{L^2(\Omega_{T'})} + \frac{1}{\epsilon}\left\|\left(\gamma\nabla R, \gamma R\nabla R\right)\right\|^2_{L^2(\Omega_{T'})} \\ &\leq \left\|\gamma R\left(.,0\right)\right\|^2_{L^2(\Omega)} + C_1\int_0^{T'}\left|\int_\Omega \gamma^2 R\left(\mathbf{v}^\epsilon\cdot\nabla R + \left(\mathbf{v}^\epsilon_{err} + \overline{\mathbf{v}^\epsilon} - \mathbf{v}^\epsilon_A + \tilde{\mathbf{w}}^\epsilon_1 + \tilde{\mathbf{w}}^\epsilon_2\right)\cdot\nabla c_A^\epsilon\right)\mathrm{d}x\right|\mathrm{d}t \\ &\quad + C_2\int_0^{T'}\left|\int_\Omega \gamma^2 R r^\epsilon_{\mathrm{CH1}} + \epsilon\left(\Delta\left(\gamma^2\right)R + 4\gamma\nabla\gamma\cdot\nabla R\right)\Delta R + \frac{1}{\epsilon}\mathcal{N}^\nabla\left(c_A^\epsilon, R\right)\mathrm{d}x\right|\mathrm{d}t \\ &\quad + C_3\int_0^{T'}\left|\int_\Omega \Delta\left(\gamma^2 R\right)\left(\tilde{f}R - r^\epsilon_{\mathrm{CH2}}\right) + R\nabla\left(\gamma^2\right)\cdot\nabla R\frac{1}{\epsilon}f''\left(-1\right)\mathrm{d}x\right|\mathrm{d}t. \qquad (7.66)\end{aligned}$$

If we may now give suitable estimates for the right hand side of (7.66), replacing T' by T_ϵ, we get (4.2e) and (4.2f).

Estimating the Right Hand Side of (7.66):
Starting from the bottom (7.66), we have

$$\begin{aligned}\int_0^{T_\epsilon}\left|\int_\Omega \nabla\left(\gamma^2\right)R\nabla R\frac{1}{\epsilon}f''\left(-1\right)\mathrm{d}x\right|\mathrm{d}t &\leq \frac{C}{\epsilon}\left\|\gamma\nabla R\right\|_{L^2(\Omega_{T_\epsilon})}\left\|\nabla\gamma R\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \\ &\leq C\left(K\right)\epsilon^{2M-\frac{1}{2}} \qquad (7.67)\end{aligned}$$

due to (4.6a) and (4.6d). For the next term, we note $r^\epsilon_{\mathrm{CH2}} = r^\epsilon_{\mathrm{CH2},\mathbf{B}}$ in $\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)$ and use Remark 6.5 3) to write

$$\begin{aligned}
\int_0^{T_\epsilon} \left| \int_\Omega \Delta\left(\gamma^2 R\right) r^\epsilon_{\mathrm{CH2}} \mathrm{d}x \right| \mathrm{d}t &\leq C \left\|(\gamma \Delta R, \nabla R, R)\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \left\|\tilde{r}^\epsilon_{\mathrm{CH2},\mathbf{B}}\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \\
&\quad + C_2 \int_0^{T_\epsilon} \left| \epsilon^{M-\frac{1}{2}} \int_\Omega \nabla\left(\gamma^2 R\right) \nabla \mu^-_{M-\frac{1}{2}} \mathrm{d}x \right| \mathrm{d}t \\
&\leq C(K)\,\epsilon^{2M} + \epsilon^{M-\frac{1}{2}} \left\| \nabla \mu^-_{M-\frac{1}{2}} \right\|_{L^2\left(\Omega^-_{T_\epsilon}\right)} \left\|(\gamma \nabla R, R)\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \\
&\leq C(K)\,\epsilon^{2M-\frac{1}{2}}, \qquad (7.68)
\end{aligned}$$

where we used integration by parts in the first step and the boundary terms vanish since $\mu^-_{M-\frac{1}{2}} = 0$ on $\partial_{T_\epsilon}\Omega$ by (5.228). In the second step, we used (4.6a) and (4.6d) and in the last step (5.236) and again (4.6a), (4.6d). Similarly, we get

$$\begin{aligned}
\int_0^{T_\epsilon} \left| \int_\Omega \Delta\left(\gamma^2 R\right) \tilde{f} R \mathrm{d}x \right| \mathrm{d}t &\leq C \left\|(\gamma \Delta R, \nabla R, R)\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \left\|R\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \\
&\leq C(K)\,\epsilon^{2M-\frac{1}{2}}. \qquad (7.69)
\end{aligned}$$

Skipping $\mathcal{N}^\nabla\left(c^\epsilon_A, R\right)$ for now, we next estimate

$$\begin{aligned}
\int_0^{T_\epsilon} \left| \int_\Omega \epsilon 4 \left(\nabla\gamma \cdot \nabla R\right) \gamma \Delta R \mathrm{d}x \right| \mathrm{d}t &\leq C\epsilon \left\|\gamma \Delta R\right\|_{L^2(\Omega_{T_\epsilon})} \left\|\nabla R\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \\
&\leq C_1 \epsilon^{2M-\frac{1}{2}} \qquad (7.70)
\end{aligned}$$

due to (4.6a) and (4.6d). Additionally,

$$\begin{aligned}
\int_0^{T_\epsilon} \left| \int_\Omega \epsilon \Delta\left(\gamma^2\right) R \Delta R \mathrm{d}x \right| \mathrm{d}t &= \int_0^{T_\epsilon} \left| \int_\Omega \epsilon \left(\nabla \Delta\left(\gamma^2\right) R + \Delta\left(\gamma^2\right) \nabla R\right) \cdot \nabla R \mathrm{d}x \right| \mathrm{d}t \\
&\leq C\epsilon \left(\left\|R\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \left\|\nabla R\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} + \left\|\nabla R\right\|^2_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \right) \\
&\leq C(K)\,\epsilon^{2M}, \qquad (7.71)
\end{aligned}$$

where we used integration by parts in the first line and the boundary terms vanish due to the Dirichlet boundary condition for R. In the last step, we used (4.6a). Now

$$\begin{aligned}
\int_0^{T_\epsilon} \left| \int_\Omega \gamma^2 R r^\epsilon_{CH1} \mathrm{d}x \right| \mathrm{d}t &\leq C \left\|R\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \left\|r^\epsilon_{\mathrm{CH1}}\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \\
&\leq C(K)\,\epsilon^{2M+\frac{1}{2}} \qquad (7.72)
\end{aligned}$$

due to (4.6a) and (6.59) and

$$\begin{aligned}
&\left|\int\limits_{\Omega_{T_\epsilon}} \gamma^2 R\left(\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon + \tilde{\mathbf{w}}_1^\epsilon + \tilde{\mathbf{w}}_2^\epsilon\right)\cdot \nabla c_A^\epsilon \mathrm{d}\left(x,t\right)\right| \\
&\le C\epsilon \left\|R\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \left(\left\|\overline{\mathbf{v}^\epsilon} - \mathbf{v}_A^\epsilon\right\|_{L^2(0,T_\epsilon;H^1(\Omega))} + \left\|\tilde{\mathbf{w}}_1^\epsilon\right\|_{L^2(0,T_\epsilon;H^1(\Omega))}\right) \\
&\quad + C\epsilon \left\|R\right\|_{L^2\left(0,T_\epsilon;L^{q'}\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)\right)} \left\|\tilde{\mathbf{w}}_2^\epsilon\right\|_{L^2(0,T_\epsilon;L^q(\Omega))} \\
&\le C\left(K\right)\epsilon^{2M-\frac{1}{2}},
\end{aligned} \tag{7.73}$$

where $q \in (1,2)$ and $\frac{1}{q'} + \frac{1}{q} = 1$ and where we used $\nabla c_A^\epsilon = \mathcal{O}(\epsilon)$ in $L^\infty\left(\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)\right)$ (cf. Remark 6.4 1)) in the first step. In the second step we used (5.191), Theorem 7.8 1) and (7.15) together with $H^1\left(\partial\Omega\left(\frac{\delta}{2}\right)\right) \hookrightarrow L^{q'}\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)$ and (4.6a).

Now using (7.67)–(7.73) in (7.66), we find

$$\begin{aligned}
&\sup_{t\in(0,T_\epsilon)} \left\|\gamma R\left(.,t\right)\right\|_{L^2(\Omega)}^2 + \epsilon\left\|\gamma\Delta R\right\|_{L^2(\Omega_{T_\epsilon})}^2 + \frac{1}{\epsilon}\left\|\left(\gamma\nabla R, \gamma R\nabla R\right)\right\|_{L^2(\Omega_{T_\epsilon})}^2 \\
&\le C\left(\int\limits_0^{T_\epsilon}\left|\int\limits_\Omega \gamma^2 R\left(\mathbf{v}^\epsilon\cdot\nabla R + \mathbf{v}_{err}^\epsilon\cdot\nabla c_A^\epsilon\right)\mathrm{d}x\right| + \left|\int\limits_\Omega \frac{1}{\epsilon}\mathcal{N}^\nabla\left(c_A^\epsilon, R\right)\mathrm{d}x\right|\mathrm{d}t\right) \\
&\quad + C\left(K\right)\epsilon^{2M-\frac{1}{2}} + \frac{K^2}{4}\epsilon^{2M},
\end{aligned} \tag{7.74}$$

where we also used $\left\|\gamma R\left(.,0\right)\right\|_{L^2(\Omega_{T_\epsilon})}^2 \le \frac{K^2}{4}\epsilon^{2M}$, cf. (7.43). So we have three more estimates to show:

1. Estimating $\int_0^{T_\epsilon}\left|\int_\Omega \gamma^2 R\mathbf{v}_{err}^\epsilon\cdot\nabla c_A^\epsilon \mathrm{d}x\right|\mathrm{d}t$*:*

$$\begin{aligned}
\int\limits_0^{T_\epsilon}\left|\int\limits_\Omega \gamma^2 R\mathbf{v}_{err}^\epsilon\cdot\nabla c_A^\epsilon\mathrm{d}x\right|\mathrm{d}t &\le \epsilon\left\|\gamma R\right\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}\left\|\mathbf{v}_{err}^\epsilon\right\|_{L^1(0,T_\epsilon;H^1(\Omega))} \\
&\le C\left(K\right)\epsilon^{2M+\frac{1}{2}},
\end{aligned}$$

where we again used $\nabla c_A^\epsilon = \mathcal{O}(\epsilon)$ in $L^\infty\left(\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)\right)$ in the first line and (7.18), (4.6d) in the second line.

2. Estimating $\int_0^{T_\epsilon}\left|\int_\Omega \frac{1}{\epsilon}\mathcal{N}^\nabla\left(c_A^\epsilon, R\right)\mathrm{d}x\right|\mathrm{d}t$*:*

Using the explicit form of $\mathcal{N}^\nabla$ given in (7.64), we calculate

$$\begin{aligned}
&\frac{1}{\epsilon}\int\limits_0^{T_\epsilon}\left|\int\limits_\Omega \mathcal{N}^\nabla\left(c_A^\epsilon, R\right)\mathrm{d}x\right|\mathrm{d}t \\
&\quad= \frac{1}{\epsilon}\int\limits_0^{T_\epsilon}\left|\int\limits_\Omega k_f\nabla\left(\gamma^2\right)R^3\cdot\nabla R + \nabla\left(\gamma^2 R\right)\cdot\left(k_f\nabla c_A^\epsilon R^2 + f^{(3)}\left(c_A^\epsilon\right)R\nabla R\right)\mathrm{d}x\right|\mathrm{d}t \\
&\quad\le \frac{1}{\epsilon}\int\limits_0^{T_\epsilon}\left|\int\limits_\Omega k_f\nabla\left(\gamma^2\right)R^3\cdot\nabla R\mathrm{d}x\right|\mathrm{d}t + C_1\left\|R\right\|_{L^3(\Omega_{T_\epsilon}\setminus\Gamma(2\delta;T_\epsilon))}^3 + C_2\int\limits_0^{T_\epsilon}\left|\int\limits_\Omega \gamma^2\nabla R R^2\mathrm{d}x\right|\mathrm{d}t
\end{aligned}$$

$$+ C_3 \frac{1}{\epsilon} \int_0^{T_\epsilon} \left| \int_\Omega \nabla \left(\gamma^2 R\right) R \nabla R \mathrm{d}x \right| \mathrm{d}t, \tag{7.75}$$

where we again used $\nabla c_A^\epsilon = \mathcal{O}(\epsilon)$ in $L^\infty\left(\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)\right)$ in the last step. Now we have

$$\begin{aligned}
&\frac{1}{\epsilon} \int_0^{T_\epsilon} \left| \int_\Omega k_f \nabla\left(\gamma^2\right) R^3 \cdot \nabla R \mathrm{d}x \right| \mathrm{d}t \\
&\leq \frac{1}{\epsilon} C \int_0^{T_\epsilon} \|\gamma R |\nabla R|\|_{L^2(\Omega)} \|R\|^2_{L^4\left(\partial\Omega\left(\frac{\delta}{2}\right)\right)} \mathrm{d}t \\
&\leq \frac{1}{\epsilon} C \|\gamma R |\nabla R|\|_{L^2(\Omega_{T_\epsilon})} \|R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \|R\|_{L^2\left(0,T_\epsilon;H^1\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)\right)} \\
&\leq C(K) \epsilon^{-1} \epsilon^M \epsilon^{\frac{M}{2}-\frac{1}{4}} \epsilon^{M-\frac{1}{2}} \\
&\leq C(K) \epsilon^{2M},
\end{aligned}$$

where we used the Gagliardo Nirenberg interpolation theorem in the second line and (4.6) together with Lemma 7.3 4) in the third line. The final estimate holds true since $M \geq 4$. Next we have

$$\|R\|_{L^3(\Omega_{T_\epsilon} \setminus \Gamma(2\delta;T_\epsilon))} \leq C(K) \epsilon^{2M+1}$$

due to (7.55) and

$$\int_0^{T_\epsilon} \left| \int_\Omega \gamma^2 \nabla R R^2 \mathrm{d}x \right| \mathrm{d}t \leq C \|\gamma R \nabla R\|_{L^2(\Omega_{T_\epsilon})} \|R\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \leq C(K) \epsilon^{2M+\frac{1}{2}}$$

due to (4.6). Regarding the last term in (7.75) we have on the one hand

$$\frac{1}{\epsilon} \int_0^{T_\epsilon} \left| \int_\Omega \left(\nabla \gamma^2\right) R^2 \nabla R \mathrm{d}x \right| \mathrm{d}t \leq C \frac{1}{\epsilon} \|\gamma R \nabla R\|_{L^2(\Omega_{T_\epsilon})} \|R\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)} \leq C(K) \epsilon^{2M-\frac{1}{2}}$$

as before and on the other hand

$$\begin{aligned}
\frac{1}{\epsilon} \int_0^{T_\epsilon} \left| \int_\Omega \gamma^2 (\nabla R)^2 R \mathrm{d}x \right| \mathrm{d}t &\leq \frac{1}{\epsilon} \int_0^{T_\epsilon} \|R\|_{L^2(\Omega)} \|\gamma \nabla R\|^2_{L^4(\Omega)} \mathrm{d}t \\
&\leq C \frac{1}{\epsilon} \|R\|_{L^\infty(0,T_\epsilon;L^2(\Omega))} \|\gamma \nabla R\|_{L^2(\Omega_{T_\epsilon})} \|\gamma \nabla R\|_{L^2(0,T_\epsilon;H^1(\Omega))} \\
&\leq C(K) \epsilon^{\frac{M}{2}-\frac{1}{4}} \epsilon^M \epsilon^{M-1} \epsilon^{-1} \\
&\leq C(K) \epsilon^{2M-\frac{1}{2}},
\end{aligned}$$

where we again used the Gagliardo Nirenberg interpolation theorem in the second line and 7.3 4) together with (4.6) in the third line. Note that

$$\begin{aligned}
\|\gamma \nabla R\|_{H^1(\Omega)} &\leq \|\gamma R\|_{H^2(\Omega)} \leq C \|\Delta(\gamma R)\|_{L^2(\Omega)} \\
&\leq C \left(\|\gamma \Delta R\|_{L^2(\Omega)} + \|(|\nabla R|, R)\|_{L^2(\partial\Omega(\delta))} \right)
\end{aligned}$$

by elliptic regularity theory. All in all, we have with (7.75)

$$\frac{1}{\epsilon}\int_0^{T_\epsilon}\left|\int_\Omega \mathcal{N}^\nabla\left(c_A^\epsilon, R\right)\mathrm{d}x\right|\mathrm{d}t \le \epsilon^{2M-\frac{1}{2}}.$$

3. Estimating $\int_0^{T_\epsilon}\left|\int_\Omega \gamma^2 R\left(\mathbf{v}^\epsilon\cdot\nabla R\right)\mathrm{d}x\right|\mathrm{d}t$*:*

We have

$$\begin{aligned}
&\int_0^{T_\epsilon}\left|\int_\Omega \gamma^2 R\left(\mathbf{v}^\epsilon\cdot\nabla R\right)\mathrm{d}x\right|\mathrm{d}t\\
&\le \int_0^{T_\epsilon}\left|\int_\Omega \gamma^2 R\left(\mathbf{v}_{err}^\epsilon\cdot\nabla R\right)\mathrm{d}x\right| + \left|\int_\Omega \left(\mathbf{v}^\epsilon-\mathbf{v}_{err}^\epsilon\right)\cdot\frac{1}{2}\nabla\left(R^2\gamma^2\right)\mathrm{d}x\right|\mathrm{d}t\\
&\quad+\int_0^{T_\epsilon}\left|\int_\Omega \left(\mathbf{v}^\epsilon-\mathbf{v}_{err}^\epsilon\right)\cdot\frac{1}{2}\nabla\left(\gamma^2\right)R^2\mathrm{d}x\right|\mathrm{d}t\\
&\le C(K)\,C(\epsilon,T_\epsilon)\,\epsilon^{2M-1}+\frac{1}{2}\int_{\Omega_{T_\epsilon}}\left|\left(\overline{\mathbf{v}^\epsilon}-\mathbf{v}_A^\epsilon+\tilde{\mathbf{w}}_1^\epsilon+\tilde{\mathbf{w}}_2^\epsilon\right)\cdot\nabla\left(\gamma^2\right)R^2\right|+\left|\mathbf{v}_A^\epsilon\cdot\nabla\left(\gamma^2\right)R^2\right|\mathrm{d}(x,t)\\
&\le C(K)\,C(\epsilon,T_\epsilon)\,\epsilon^{2M-1}+\int_{\Omega_{T_\epsilon}}\left|\left(\overline{\mathbf{v}^\epsilon}-\mathbf{v}_A^\epsilon+\tilde{\mathbf{w}}_1^\epsilon+\tilde{\mathbf{w}}_2^\epsilon\right)\cdot\nabla\left(\gamma^2\right)R^2\right|\mathrm{d}(x,t)\\
&\quad+\epsilon^{M-\frac{1}{2}}\int_0^{T_\epsilon}\int_\Omega\left|\mathbf{v}_{M-\frac{1}{2}}^{-,\epsilon}\cdot\nabla\left(\gamma^2\right)R^2\right|\mathrm{d}t,
\end{aligned}$$

where we used integration by parts together with the facts that $\mathbf{v}^\epsilon-\mathbf{v}_{err}^\epsilon$ is divergence free and $R=0$ on $\partial_{T_0}\Omega$, as well as (7.29) and the definition of $\mathbf{v}_{err}^\epsilon$ in (7.14) in the second step. In the third inequality, we used $\mathbf{v}_A^\epsilon-\epsilon^{M-\frac{1}{2}}\mathbf{v}_{A,M-\frac{1}{2}}^\epsilon\in L^\infty\left(\Omega_{T_0}\right)$ and (4.6a). Note that $\mathbf{v}_{A,M-\frac{1}{2}}^\epsilon=\mathbf{v}_{M-\frac{1}{2}}^{-,\epsilon}$ in $\partial_{T_0}\Omega\left(\frac{\delta}{2}\right)$. We may continue estimating

$$\begin{aligned}
\int_{\Omega_{T_\epsilon}}\left|\left(\overline{\mathbf{v}^\epsilon}-\mathbf{v}_A^\epsilon+\tilde{\mathbf{w}}_1^\epsilon\right)\cdot\nabla\left(\gamma^2\right)R^2\right|\mathrm{d}(x,t) &\le \left(\left\|\overline{\mathbf{v}^\epsilon}-\mathbf{v}_A^\epsilon\right\|_{L^2(0,T_\epsilon;H^1(\Omega))}+\left\|\tilde{\mathbf{w}}_1^\epsilon\right\|_{L^2(0,T_\epsilon;H^1(\Omega))}\right)\\
&\qquad\cdot\left\|\gamma R\right\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))}\left\|R\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)}\\
&\le C(K)\left(\epsilon^M+\epsilon^{M-\frac{1}{2}}\right)\epsilon^{M-\frac{1}{2}-\frac{\kappa}{2+\kappa}M}\epsilon^{M+\frac{1}{2}}\\
&\le C(K)\,\epsilon^{2M},
\end{aligned}$$

where we used $H^1(\Omega)\hookrightarrow L^s(\Omega)$ for all $s\ge 1$ in the first inequality, Theorem 7.8 1), Lemma 5.29 (in particular (5.191)), Lemma 7.3 3) and (4.6a) in the second inequality. In the final step, we made use of the facts that $M\ge 4$ and that $\kappa>0$ may be chosen arbitrarily small.

Regarding $\tilde{\mathbf{w}}_2^\epsilon$, we choose $\kappa > 0$ and $q = \frac{2+\kappa}{(2+\kappa)-1}$ and compute

$$\int_{\Omega_{T_\epsilon}} \left|\tilde{\mathbf{w}}_2^\epsilon \cdot \nabla\left(\gamma^2\right) R^2\right| \mathrm{d}\left(x,t\right) \leq \left\|\tilde{\mathbf{w}}_2^\epsilon\right\|_{L^2(0,T_\epsilon;L^q(\Omega))} \left\|\gamma R\right\|_{L^\infty(0,T_\epsilon;L^{2+\kappa}(\Omega))} \left\|R\right\|_{L^2(0,T_\epsilon;L^\infty(\Omega))}$$
$$\leq C\left(K,\alpha\right)\epsilon^{M-1}\epsilon^{M-\frac{1}{2}}\epsilon^{-\frac{\kappa}{2+\kappa}M}\epsilon^{M-\frac{3}{2}}\epsilon^{-(M+2)\alpha}$$
$$\leq C\left(K,\alpha\right)\epsilon^{2M-\frac{1}{2}}$$

for $\alpha > 0$, where we used (7.15), (4.6d) and Lemma 7.3 1) in the first line as well as the fact that $M \geq 4$ and that we may choose $\alpha > 0$ and $\kappa > 0$ arbitrarily small in the last line. For the term involving $\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}$ we compute

$$\epsilon^{M-\frac{1}{2}}\int_{\Omega_{T_\epsilon}} \left|\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}} \cdot \nabla\left(\gamma^2\right) R^2\right| \mathrm{d}\left(x,t\right) \leq C\epsilon^{M-\frac{1}{2}} \left\|\mathbf{v}^{-,\epsilon}_{M-\frac{1}{2}}\right\|_{L^2(0,T_\epsilon;L^\infty(\Omega^-(t)))} \left\|\gamma R\right\|_{L^\infty(0,T_\epsilon;L^2(\Omega))}$$
$$\cdot \left\|R\right\|_{L^2\left(\partial_{T_\epsilon}\Omega\left(\frac{\delta}{2}\right)\right)}$$
$$\leq C\left(K\right)\epsilon^{3M-\frac{1}{2}}$$
$$\leq C\left(K\right)\epsilon^{2M},$$

where we used (5.236) (together with $H^2\left(\Omega^-\left(t\right)\right) \hookrightarrow L^\infty\left(\Omega^-\left(t\right)\right)$) and (4.6) in the second estimate and $M \geq 4$ in the last line.

Thus, we have shown

$$\int_0^{T_\epsilon} \left|\int_\Omega \gamma^2 R\left(\mathbf{v}^\epsilon \cdot \nabla R\right)\mathrm{d}x\right| \mathrm{d}t \leq C\left(K\right)C\left(T_\epsilon,\epsilon\right)\epsilon^{2M-1}.$$

and with that may conclude using (7.74) that

$$\sup_{t\in(0,T_\epsilon)} \left\|\gamma R\left(.,t\right)\right\|^2_{L^2(\Omega)} + \epsilon\left\|\gamma\Delta R\right\|^2_{L^2(\Omega_{T_\epsilon})}$$
$$+\frac{1}{\epsilon}\left\|\left(\gamma\nabla R, \gamma R\nabla R\right)\right\|^2_{L^2(\Omega_{T_\epsilon})} \leq C\left(K\right)C\left(T,\epsilon\right)\epsilon^{2M-1}. \tag{7.76}$$

Putting the Pieces Together

Altogether we have shown

$$\int_0^{T_\epsilon} \mathcal{RS}\mathrm{d}t \leq C\left(T,\epsilon\right)C\left(K\right)\epsilon^{2M}$$

and may now choose $\epsilon_0 > 0$ and $T \in (0, T_0]$ sufficiently small, to ensure

$$\sup_{0\leq\tau\leq T_\epsilon} \left\|\nabla\varphi\right\|^2_{L^2(\Omega)} + \left\|\left(\epsilon R, \epsilon^3\left|\nabla R\right|\right)\right\|^2_{L^2(\Omega_{T_\epsilon})} + \left\|\left(\epsilon^{-1}R, \epsilon\left|\nabla R\right|\right)\right\|^2_{L^2(0,T_\epsilon;L^2(\Omega\backslash\Gamma_t(\delta)))}$$
$$+\,\epsilon\left\|\nabla^\Gamma R\right\|^2_{L^2(0,T_\epsilon;L^2(\Gamma_t(\delta)))} \leq \frac{K^2}{2}\epsilon^{2M}$$

by (7.51),

$$\mathcal{E}\left(R,T_\epsilon\right) \leq \frac{K^2}{2}\epsilon^{2M}$$

by (7.52) and

$$\sup_{t\in(0,T_\epsilon)} \|\gamma R(.,t)\|^2_{L^2(\Omega)} + \epsilon \|\gamma\Delta R\|^2_{L^2(\Omega_{T_\epsilon})} + \frac{1}{\epsilon}\|(\gamma\nabla R, \gamma R\nabla R)\|^2_{L^2(\Omega_{T_\epsilon})} \leq \frac{K^2}{2}\epsilon^{2M-1}$$

by (7.76). By the definition of T_ϵ in (7.44), this implies $T_\epsilon = T$. This shows (4.2).

Regarding (4.3), we have by the definition of $\mathbf{v}^\epsilon_{err}$ in (7.14) for $q \in (1,2)$

$$\begin{aligned}\|\mathbf{v}^\epsilon - \mathbf{v}^\epsilon_A\|_{L^1(0,T;L^q(\Omega))} &\leq \|\mathbf{v}^\epsilon_{err} + \tilde{\mathbf{w}}^\epsilon_1 + \tilde{\mathbf{w}}^\epsilon_2\|_{L^1(0,T;L^q(\Omega))} + C\|\overline{\mathbf{v}^\epsilon} - \mathbf{v}^\epsilon_A\|_{L^1(0,T;L^2(\Omega))}\\ &\leq C(K,q)\,\epsilon^{M-\frac{1}{2}}\end{aligned}$$

by (5.191), (7.15) and Theorem 7.8. The convergence results (4.4) and (4.5) are then due to the construction of c^ϵ_A and $\mathbf{v}^\epsilon_A$, more precisely Definition 6.2, where it is important to note (5.236) for $\mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}$. Thus, all claims are shown.

Remark 7.12. In this final remark, we want to discuss the consequences of considering Neumann boundary conditions $\partial_{\mathbf{n}_{\partial\Omega}}\mu^\epsilon = 0$ on $\partial_{T_0}\Omega$ instead of (1.24). Of course, in this case we would construct μ^ϵ_A such that $\partial_{\mathbf{n}_{\partial\Omega}}\mu^\epsilon_A = 0$ is satisfied on $\partial_{T_0}\Omega$. To gain (7.47), which is a vital point of the proof, we need to ensure that

$$\int_\Omega \varphi\Delta(\mu^\epsilon - \mu^\epsilon_A)\,\mathrm{d}x = \int_\Omega \Delta\varphi(\mu^\epsilon - \mu^\epsilon_A)\,\mathrm{d}x$$

holds, which is satisfied if we choose Neumann boundary conditions for φ. In particular, φ should be the solution to

$$-\Delta\varphi(.,t) = R(.,t) \text{ in } \Omega,\ \partial_{\mathbf{n}_{\partial\Omega}}\varphi = 0 \text{ on } \partial\Omega, \tag{7.77}$$

made unique by a normalization of the kind $\int_\Omega \varphi(.,t)\,\mathrm{d}x = 0$. However, in order for (7.77) to be well-posed, $\int_\Omega R(.,t)\,\mathrm{d}x = 0$ needs to be satisfied, where

$$\begin{aligned}\int_\Omega R(x,t)\,\mathrm{d}x &= \int_0^t\int_\Omega \partial_t(c^\epsilon - c^\epsilon_A)\,\mathrm{d}x\mathrm{d}\tau + \int_\Omega c^\epsilon_0 - c^\epsilon_A|_{t=0}\mathrm{d}x\\ &= \int_0^t\int_\Omega \operatorname{div}(\mathbf{v}^\epsilon_A)\,c^\epsilon_A + \tilde{\mathbf{w}}^\epsilon_1|_\Gamma\cdot\nabla c^\epsilon_A\xi - r^\epsilon_{\mathrm{CH1}}\mathrm{d}x\mathrm{d}\tau + \int_\Omega c^\epsilon_0 - c^\epsilon_A|_{t=0}\mathrm{d}x\end{aligned}$$

in the case of no-slip boundary conditions for $\mathbf{v}^\epsilon$. This expression does not vanish and is also not of a high enough order of ϵ to make a different strategy viable[1]. A similar problem arises in the case of periodic boundary conditions. To circumvent this difficulty, we decided to stick to Dirichlet boundary values.

[1]Such as the the one suggested in [14], Remark 2.1.

A. Appendix

Lemma A.1 (A Comparison Principle)**.** *Let $I = (\alpha, \beta) \subset \mathbb{R}$ and we consider the differential operator L given by*

$$Lu(x) = a(x)u''(x) + b(x)u'(x) + c(x)u(x)$$

for all $x \in I$ and $u \in C^2(I)$, where $a, b, c \in C^0(\bar{I})$ with $a > 0$ and $c < 0$ on $\bar{I}$. Let furthermore $v_1, v_2 \in C^2(\bar{I})$ be functions fulfilling

$$Lv_1(x) \geq Lv_2(x) \text{ for all } x \in I$$

and

$$v_1(\alpha) \leq v_2(\alpha), \ v_1'(\beta) \leq v_2'(\beta).$$

Then

$$v_1(x) \leq v_2(x) \text{ for all } x \in I.$$

Proof. The proof is a modified version of the usual proof presented for the comparison principle of elliptic operators. Let $\tilde{v} = v_1 - v_2$. Then we have $L\tilde{v} \geq 0$ in I and

$$\tilde{v}(\alpha) \leq 0, \tilde{v}'(\beta) \leq 0. \tag{A.1}$$

If $\tilde{v}(x) \leq 0$ holds for all $x \in I$ there is nothing to show. Thus, we assume that there exists $x_0 \in I$ with $\tilde{v}(x_0) > 0$ and define $I^+ := \{x \in I | \tilde{v}(x) > 0\}$, which is consequently open, bounded and non-empty. $L\tilde{v} \geq 0$ in I implies

$$L_0\tilde{v}(x) := a\tilde{v}''(x) + bv'(x) \geq -c(x)\tilde{v}(x) > 0 \tag{A.2}$$

for all $x \in I^+$. Hence, the weak maximum principle for elliptic operators used on L_0 and I^+ implies

$$\max_{x \in \overline{I^+}} \tilde{v}(x) = \max_{x \in \partial I^+} \tilde{v}(x) \tag{A.3}$$

and $\tilde{v}(x) = 0$ for $\partial I^+ \cap I$. If $\partial I^+ \cap \partial I = \{\alpha\}$ or $\partial I^+ \cap \partial I = \emptyset$ holds, this already implies

$$\max_{x \in \bar{I}} \tilde{v}(x) \leq \max_{x \in \overline{I^+}} \tilde{v}(x) = \max_{x \in \partial I^+} \tilde{v}(x) = 0$$

in contradiction to the existence of a x_0 with $\tilde{v}(x_0) > 0$.

So we only have to consider the case $\beta \in \partial I^+ \cap \partial I$. Assume $\tilde{v}(\beta) > 0$, otherwise there is nothing to show. In particular, $\max_{x \in \overline{I^+}} \tilde{v}(x) = \tilde{v}(\beta)$, which is implied by $\tilde{v}(\alpha) \leq 0$. Due to (A.3) it has to hold $\tilde{v}'(\beta) \geq 0$ and thus it follows $\tilde{v}'(\beta) = 0$ by (A.1). As $\tilde{v} \in C^2(\bar{I})$, we can use (A.2) on $\overline{I^+}$ and get

$$\tilde{v}''(\beta) \geq -\frac{c(\beta)}{a(\beta)}\tilde{v}(\beta) > 0.$$

This implies that β is a local minimum in contradiction to (A.3). Thus, $\tilde{v}(\beta) \leq 0$, which shows the assertion. □

A.1. Krein-Rutmann Theorem

Definition A.2 (Cone)**.** Let X be a real Banach space.

1. $K \subset X$ is called a **cone**, if K is a closed, convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$.

2. If in addition K has nonempty interior K^o, then K is called a **solid cone**.

Proposition A.3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $n \in \mathbb{N}$ and $X := C^2\left(\bar{\Omega}\right)$ with the norm $\|f\|_{C^2(\bar{\Omega})} := \sum_{|s|\leq 2} \|\partial^s f\|_{C^0(\bar{\Omega})}$ for $f \in C^2\left(\bar{\Omega}\right)$, and let $K := \left\{ f \in X | \, f \geq 0 \text{ in } \bar{\Omega} \right\}$. Then K is a solid cone.*

Proof. This follows immediately from the definition. □

Theorem A.4. *Let X be a real Banach space, $K \subset X$ a solid cone and $T : X \to X$ a compact linear operator which fulfills $Tu \in K^o$ for $u \in K \setminus \{0\}$. Then*

1. *The spectral radius $r(T)$ satisfies $r(T) > 0$ and is a simple eigenvalue with an eigenvector $v \in K^o$. There is no other eigenvalue with an eigenvector in K.*

2. *$|\lambda| < r(T)$ for all eigenvalues $\lambda \neq r(T)$.*

Proof. See [28] Theorem 1.2. □

A.2. Expansion for an Instationary Stokes Equation

As mentioned in Remark 5.25, we will give the structure of the inner expansion equation (5.40) in case we considered an Instationary Stokes Equation with right hand side $-\epsilon \mathrm{div}\left(\nabla c^\epsilon \otimes \nabla c^\epsilon\right)$ instead of (1.18). The equivalent to (5.40) would read

$$\begin{aligned}
-\partial_{\rho\rho}\left(\mathbf{v}_0 - \mathbf{u}_0 \eta d_\Gamma\right) &= -\partial_\rho p_{-1}\mathbf{n} - 2\partial_{\rho\rho} c_0 \partial_\rho c_0 \mathbf{n} \\
-\partial_{\rho\rho}\left(\mathbf{v}_k - \left(\mathbf{u}_k d_\Gamma - \mathbf{u}_0 h_k\right)\eta\right) &= -\partial_\rho p_{k-1}\mathbf{n} - 2\left(\partial_\rho\left(\partial_\rho c_k \partial_\rho c_0\right)\right)\mathbf{n} \\
&\quad + \left(\partial_\rho p_{-1} + \partial_\rho\left(\partial_\rho c_0\right)^2\right)\nabla^\Gamma h_k + \tilde{\mathbf{V}}^{k-1}
\end{aligned}$$

for $k \geq 1$, where $\tilde{\mathbf{V}}^{k-1} = \tilde{\mathbf{V}}^{k-1}(\rho, x, t)$, $(\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta)$, is given by

$$\begin{aligned}
\tilde{\mathbf{V}}^{k-1} = &- \delta_1^k \beta_2^k 2\partial_\rho\left(\partial_\rho c_{k-1}\partial_\rho c_1\right)\mathbf{n} + \delta_1^k\beta_2^k 2\left(\partial_\rho\left(\partial_\rho c_{k-1}\partial_\rho c_0\right)\nabla^\Gamma h_1 + \partial_\rho\left(\partial_\rho c_1 \partial_\rho c_0\right)\nabla^\Gamma h_{k-1}\right) \\
&- \partial_{\rho\rho} c_{k-1}\nabla c_0 - \partial_{\rho\rho} c_0 \nabla c_{k-1} + \partial_\rho \mathbf{v}_{k-1}\left(\Delta d_\Gamma - \partial_t d_\Gamma\right) + 2\left(\nabla\partial_\rho \mathbf{v}_{k-1}\right)^T\mathbf{n} \\
&- 2\left(\nabla\partial_\rho\mathbf{v}_0\right)^T\nabla^\Gamma h_{k-1} - \partial_\rho\mathbf{v}_0\left(\Delta^\Gamma h_{k-1} - \partial_t^\Gamma h_{k-1}\right) + \partial_\rho p_0 \nabla^\Gamma h_{k-1} \\
&+ \beta_2^k 2\partial_{\rho\rho}\mathbf{v}_0 \nabla^\Gamma h_{k-1}\cdot\nabla^\Gamma h_1 + \left(\partial_\rho c_0 \Delta d_\Gamma + 2\nabla\partial_\rho c_0\cdot\mathbf{n}\right)\partial_\rho c_0 \nabla^\Gamma h_{k-1} \\
&- \beta_1^k\left(2\partial_\rho c_0 \partial_\rho c_{k-1}\Delta d_\Gamma + 3\left(\nabla\partial_\rho c_0\cdot\mathbf{n}\right)\partial_\rho c_{k-1} + 3\left(\nabla\partial_\rho c_{k-1}\cdot\mathbf{n}\right)\partial_\rho c_0\right)\mathbf{n} \\
&+ \left(\nabla c_0\cdot\mathbf{n}\right)\partial_{\rho\rho}c_0\nabla^\Gamma h_{k-1} + \left(\partial_\rho c_0\Delta^\Gamma h_{k-1} + 3\nabla\partial_\rho c_0 \nabla^\Gamma h_{k-1}\right)\left(\partial_\rho c_0\mathbf{n}\right) \\
&+ \partial_\rho c_0\nabla^\Gamma h_{k-1}\left(\nabla\partial_\rho c_0\cdot\mathbf{n}\right) + \nabla c_0\cdot\nabla^\Gamma h_{k-1}\partial_{\rho\rho}c_0\mathbf{n} - \partial_\rho c_0 \nabla\partial_\rho c_{k-1} \\
&- \beta_1^k\left(\left(\nabla c_{k-1}\cdot\mathbf{n}\right)\partial_{\rho\rho}c_0 + \left(\nabla c_0\cdot\mathbf{n}\right)\partial_{\rho\rho}c_{k-1}\right)\mathbf{n} - \beta_2^k 4\partial_{\rho\rho}c_0\nabla^\Gamma h_{k-1}\cdot\nabla^\Gamma h_1\left(\partial_\rho c_0\cdot\mathbf{n}\right) \\
&- \partial_\rho c_{k-1}\nabla\partial_\rho c_0 + \mathbf{q}_{k-1}\eta' d_\Gamma - \mathbf{q}_0\eta' h_{k-1} + \left(\rho + \delta_1^k h_1\right)\mathbf{u}_{k-1}\eta'' + \mathbf{u}_1\eta'' h_{k-1} \\
&+ \tilde{\mathcal{V}}^{k-2},
\end{aligned}$$

$$
\begin{aligned}
\tilde{\mathcal{V}}^{k-2} = & -2\left(\sum_{i=2}^{k-2}\partial_{\rho\rho}c_{k-i}\partial_\rho c_i\mathbf{n}\right) + \left(\sum_{i,j=0,l=1,i+j+l=k}^{k-2} 2\partial_{\rho\rho}c_i\partial_\rho c_j\nabla^\Gamma h_l\right)\\
& - \left(\sum_{i=1}^{k-2}\partial_{\rho\rho}c_{k-1-i}\nabla c_i\right) - \sum_{i=1}^{k-2}\left(\partial_\rho c_{k-1-i}\partial_\rho c_i\Delta d_\Gamma + 3\left(\nabla\partial_\rho c_{k-1-i}\cdot\mathbf{n}\right)\partial_\rho c_i\right)\mathbf{n}\\
& + \left(\sum_{i=1}^{k-2} -\partial_\rho c_{k-1-i}\nabla\partial_\rho c_i - \left(\nabla c_{k-1-i}\cdot\mathbf{n}\right)\partial_{\rho\rho}c_i\mathbf{n}\right)\\
& + \left(\sum_{i=0}^{k-3} -2\left(\nabla\partial_\rho\mathbf{v}_{k-2-i}\right)^T\cdot\nabla^\Gamma h_{i+1} - \partial_\rho\mathbf{v}_{k-2-i}\left(\Delta^\Gamma h_{i+1} - \partial_t^\Gamma h_{i+1}\right)\right)\\
& + \sum_{i=0}^{k-3}\partial_\rho p_{k-2-i}\nabla^\Gamma h_{i+1} - \sum_{i=0}^{k-2}\left(\left(\partial_\rho c_{k-2-i}\Delta d_\Gamma + 2\nabla\partial_\rho c_{k-2-i}\cdot\mathbf{n}\right)\nabla c_i\right)\\
& - \sum_{i=0}^{k-2}\left(D^2\left(d_\Gamma\right)\nabla c_{k-2-i}\partial_\rho c_i + \left(\nabla c_{k-2-i}\cdot\nabla\partial_\rho c_i\right)\mathbf{n}\right)\\
& + \left(\sum_{i=0}^{k-2} -\left(\nabla c_{k-2-i}\cdot\mathbf{n}\right)\nabla\partial_\rho c_i - \Delta c_{k-2-i}\partial_\rho c_i\mathbf{n} - \partial_\rho c_{k-2-i}D^2\left(c_i\right)\mathbf{n}\right)\\
& + \left(\sum_{i=0,,j,l=1,i+j+l=k}^{k-2}\partial_{\rho\rho}\mathbf{v}_i\nabla^\Gamma h_j\cdot\nabla^\Gamma h_l\right)\\
& + \left(\sum_{i,j=0,l=1,i+j+l=k-1}^{k-2}\left(\partial_\rho c_i\Delta d_\Gamma + 2\nabla\partial_\rho c_i\cdot\mathbf{n}\right)\partial_\rho c_j\nabla^\Gamma h_l + \left(\nabla c_i\cdot\mathbf{n}\right)\partial_{\rho\rho}c_j\nabla^\Gamma h_l\right)\\
& + \left(\sum_{i,l=0,j=1,i+j+l=k-1}^{k-2}\left(\partial_\rho c_i\Delta^\Gamma h_j + 3\nabla\partial_\rho c_i\nabla^\Gamma h_j\right)\left(\partial_\rho c_l\mathbf{n}\right)\right)\\
& + \left(\sum_{i,l=0,j=1,i+j+l=k-1}^{k-2}\partial_\rho c_i\nabla^\Gamma h_j\left(\nabla\partial_\rho c_l\cdot\mathbf{n}\right) + \nabla c_i\cdot\nabla^\Gamma h_j\partial_{\rho\rho}c_l\mathbf{n}\right)\\
& + \left(\sum_{i,m=0,j,l=1,i+j+l+m=k}^{k-2} -2\partial_{\rho\rho}c_i\nabla^\Gamma h_j\cdot\nabla^\Gamma h_l\left(\partial_\rho c_m\cdot\mathbf{n}\right)\right) + \sum_{i=2}^{k-2}\mathbf{u}_i\eta'' h_{k-i}\\
& - \sum_{i=1}^{k-2}\mathbf{q}_i\eta' h_{k-1-i} - \mathbf{q}_{k-2}\eta'\rho - \partial_t\mathbf{v}_{k-2} + \Delta\mathbf{v}_{k-2} - \nabla p_{k-2}\\
& - \sum_{i=0}^{k-3}\operatorname{div}\left(\nabla c_i\otimes\nabla c_{k-3-i}\right) + \mathbf{W}_{k-2}^{+}\eta^{C_S,+} + \mathbf{W}_{k-2}^{-}\eta^{C_S,-} + \mathfrak{V}^{k-3},
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{V}^{k-3} = & + \left(\sum_{i,j,l=0,i+j+l=k-3}^{k-3}\left(\partial_\rho c_i\Delta^\Gamma h_{j+1} + 2\nabla\partial_\rho c_i\nabla^\Gamma h_{j+1}\right)\nabla c_l + \Delta c_i\partial_\rho c_j\nabla^\Gamma h_{l+1}\right)\\
& + \left(\sum_{i,j,l=0,i+j+l=k-3}^{k-3}\left(\nabla c_i\cdot\nabla^\Gamma h_{j+1}\right)\nabla\partial_\rho c_l\partial_\rho c_i D^2\left(c_j\right)\nabla^\Gamma h_{l+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\sum_{i,j,l=0,i+j+l=k-3}^{k-3} \partial_\rho c_i D^2_\Gamma\left(h_{j+1}\right)\nabla c_l + \left(\nabla c_i \cdot \nabla\partial_\rho c_j\right)\nabla^\Gamma h_{l+1}\right)\\
&-\left(\sum_{i,j,l,m=0,i+j+l+m=k-3}^{k-3} \partial_{\rho\rho} c_i\left(\nabla^\Gamma h_{j+1}\cdot\nabla^\Gamma h_{l+1}\right)\nabla c_m\right)\\
&-\left(\sum_{i,j,l,m=0,i+j+l+m=k-3}^{k-3} \partial_\rho c_i \Delta^\Gamma h_{j+1}\partial_\rho c_l \nabla^\Gamma h_{m+1}\right)\\
&-\left(\sum_{i,j,l,m=0,i+j+l+m=k-3}^{k-3} 2\left(\nabla\partial_\rho c_i\cdot\nabla^\Gamma h_{j+1}\right)\partial_\rho c_l\nabla^\Gamma h_{m+1}\right)\\
&+\left(\sum_{i,j,l,m=0,i+j+l+m=k-3}^{k-3} \left(\nabla^\Gamma h_{i+1}\cdot\nabla^\Gamma h_{j+1}\right)\partial_\rho c_l\nabla\partial_\rho c_m\right)\\
&-\left(\sum_{i,j,l,m=0,i+j+l+m=k-3}^{k-3} \partial_\rho c_i\partial_\rho c_j D^2_\Gamma\left(h_{l+1}\right)\nabla^\Gamma h_{m+1}\right)\\
&-\left(\sum_{i,j,l,m=0,i+j+l+m=k-3}^{k-3} \left(\nabla c_i\cdot\nabla^\Gamma h_{j+1}\right)\partial_{\rho\rho} c_l\nabla^\Gamma h_{m+1}\right)\\
&+\left(\sum_{i,j,l,m=0,i+j+l+m=k-3}^{k-3} -\partial_\rho c_i\nabla^\Gamma h_{j+1}\left(\nabla\partial_\rho c_l\cdot\nabla^\Gamma h_{m+1}\right)\right)\\
&+\left(\sum_{i,j,l,m,\alpha=0,i+j+l+m+\alpha=k-3}^{k-3} \partial_{\rho\rho} c_i\left(\nabla^\Gamma h_{j+1}\cdot\nabla^\Gamma h_{l+1}\right)\partial_\rho c_m\nabla^\Gamma h_{\alpha+1}\right)\\
&+\left(\sum_{i,j,l,m,\alpha=0,i+j+l+m+\alpha=k-3}^{k-3} \left(\nabla^\Gamma h_{i+1}\cdot\nabla^\Gamma h_{j+1}\right)\partial_\rho c_l\partial_{\rho\rho} c_m\nabla^\Gamma h_{\alpha+1}\right).
\end{aligned}
$$

List of Notations

D	Jacobian Matrix
∇	Gradient
D_s	(1.7)
$\partial_t^\Gamma, \nabla^\Gamma, \Delta^\Gamma$	(2.24), Notation 2.17
D_Γ^2	(2.25)
$\operatorname{div}^\Gamma$	(2.26)
$D_{t,\Gamma}$, ∇_Γ, Δ_Γ	Remark 2.19
$\left[\partial_{\mathbf{n}}, \nabla^\Gamma\right]$	(2.36)
$\mathbf{a}$ (bold letter)	Element in $\mathbb{R}^2$ or $\mathbb{R}^2$-valued function
$\cdot$	Euclidean scalar product on $\mathbb{R}^2$, e.g. $\mathbf{a} \cdot \mathbf{b}$
$\otimes$	(1.6)
$\otimes_s$	Notation 5.26
Ω	smooth domain in $\mathbb{R}^2$
Ω_T, $\partial_T \Omega$	$\Omega_T := \Omega \times (0,T)$, $\partial_T \Omega := \partial\Omega \times (0,T)$
δ	Assumption 1.1 7) and 8)
$\Omega^{\pm}(t), \Omega_T^{\pm}$	Assumption 1.1 3)
Γ_t, Γ	Assumption 1.1 2)
$\Gamma_t(\alpha), \Gamma(\alpha; T), \Gamma(2\delta)$	Assumption 1.1 3)
$\partial\Omega(\alpha), \partial_T \Omega(\alpha)$	Assumption 1.1 8)
$\mathbf{n}_{\Gamma_t}$, $\mathbf{n}_{\partial\Omega}$	Assumption 1.1 4) and 8)
$\mathbf{n}, \tau$	(2.18), Notation 2.12
V_{Γ_t}, H_{Γ_t}	Assumption 1.1 4)
$Pr_{\Gamma_t}, Pr_{\partial\Omega}$	Assumption 1.1 7) and 8)
$d_\Gamma, d_{\mathbf{B}}$	Assumption 1.1 5) and 8)
$S : \Gamma(2\delta) \to \mathbb{T}^1$	(2.23)
X_0	(2.17)
X_0^*, $X_0^{*,-1}$	(2.19),(2.20)
X	(2.21)
$\mathcal{R}_\alpha$	Definition 2.25
$L^{p,\infty}(\Gamma_t(\alpha))$	Definition 2.22 1)
$L^q(0,T; L^p(\Gamma_t(\alpha)))$	Definition 2.22 2)
X_T	(2.40)
$H^s, s > 0$	$H^s := W_2^s$, Sobolev-Slobodeckij space
H^{-1}	$H^{-1} = \left(H_0^1\right)'$
$\left(W_{p_1}^{k_1}, W_{p_2}^{k_2}\right)_{\theta,p}$	Real interpolation space of exponent θ

f	double-well potential, Assumption 1.2
θ_0	Lemma 2.2
σ	(1.17),(5.61)
$\mu, \mathbf{v}, p$	Assumption 1.1 1)
$c^\epsilon, \mu^\epsilon, \mathbf{v}^\epsilon, p^\epsilon$	Assumption 1.1 1)
ρ^ϵ	(5.19)
$c_k^\pm, \mu_k^\pm, \mathbf{v}_k^\pm, p_k^\pm$	(5.2)
$\mu^{\pm,\epsilon}_{M-\frac{1}{2}}, \mathbf{v}^{\pm,\epsilon}_{M-\frac{1}{2}}, p^{\pm,\epsilon}_{M-\frac{1}{2}}$	Lemma 5.34
$\tilde{c}^\epsilon, \tilde{\mu}^\epsilon, \tilde{\mathbf{v}}^\epsilon, \tilde{p}^\epsilon$	(5.20)
$c_k, \mu_k, \mathbf{v}_k, p_k$	(5.21)
h_k	(5.22)
$[c_k]$, $[\mu_k]$, $[\mathbf{v}_k]$, $[p_k]$	Notation5.6
$c^\epsilon_{M-\frac{1}{2}}, \mu^\epsilon_{M-\frac{1}{2}}.\mathbf{v}^\epsilon_{M-\frac{1}{2}}, p^\epsilon_{M-\frac{1}{2}}, h^\epsilon_{M-\frac{1}{2}}$	Lemma 5.34
$\mathbf{u}_k, \mathbf{q}_k, l_k, j_k, g_k$	(5.38)
$\mathbf{u}^\epsilon_{M-\frac{1}{2}}, l^\epsilon_{M-\frac{1}{2}}$	Lemma 5.34
$c^\epsilon_{\mathbf{B}}, \mu^\epsilon_{\mathbf{B}}, \mathbf{v}^\epsilon_{\mathbf{B}}, p^\epsilon_{\mathbf{B}}$	(5.80)
$c^{\mathbf{B}}_k, \mu^{\mathbf{B}}_k, \mathbf{v}^{\mathbf{B}}_k, p^{\mathbf{B}}_k$	(5.81)
$c^\epsilon_A, \mu^\epsilon_A, \mathbf{v}^\epsilon_A, p^\epsilon_A, h^\epsilon_A$	Definition 6.2
$c_I, \mu_I, \mathbf{v}_I, p_I$ and $c_{I,k}$, ...	Definition 6.2 1)
$c_O, \mu_O, \mathbf{v}_O, p_O$ and $c_{O,k}$, ...	Definition 6.2 2)
$c_{\mathbf{B}}, \mu_{\mathbf{B}}, \mathbf{v}_{\mathbf{B}}, p_{\mathbf{B}}$ and $c_{\mathbf{B},k}$,...	Definition 6.2 3)
$\mu^{\mathbf{B}}_{M-\frac{1}{2}}, \mathbf{v}^{\mathbf{B}}_{M-\frac{1}{2}}, p^{\mathbf{B}}_{M-\frac{1}{2}}$	Definition 6.2 3)
η	Proposition 5.3
$U_k^\pm, U^\pm, \mathbf{W}_k^\pm, \mathbf{W}^\pm$	(5.17),(5.18)
$f_k\left(c_0^\pm, \ldots, c_k^\pm\right), f_k\left(c_0, \ldots, c_k\right)$	(5.7), Notation 5.4 4)
$h_A^{\epsilon,H}$	(5.168)
$\tilde{c}_I$	(5.169)
$c_I^H, c_A^{\epsilon,H}$	Definition 5.24
ρ^H	(5.170)
$\rho\,(x,t)$	(6.10)
β_i^k, δ_i^k	Notation 5.4 3)
I_q^k	(6.1)
$\mathfrak{S}_0, \mathfrak{S}_k, \mathfrak{S}_{M-\frac{1}{2}}$	Subsections 5.1.5, 5.1.6, 5.3.3

$\tilde{\mathbf{w}}_1^{\epsilon,H}, q_1^{\epsilon,H}$	(5.174)–(5.176)
$\mathbf{w}_1^{\epsilon,H}$	$\mathbf{w}_1^{\epsilon,H} := \frac{\tilde{\mathbf{w}}_1^{\epsilon,H}}{\epsilon^{M-\frac{1}{2}}}$
$\mathbf{h}^H$	(5.177)
$\overline{\mathbf{v}^\epsilon}, \overline{p^\epsilon}$	(7.7)–(7.9)
$\tilde{\mathbf{w}}_2^\epsilon, q_2^\epsilon$	(7.10)–(7.12)
$\mathbf{v}^\epsilon_{err}$	(7.14)

$r^\epsilon_{\mathrm{CH1}}, r^\epsilon_{\mathrm{CH2}}, \mathbf{r}^\epsilon_{\mathrm{S}}, r^\epsilon_{\mathrm{div}}$	(4.7)–(4.10)
$r^\epsilon_{\mathrm{CH1},I}, r^\epsilon_{\mathrm{CH2},I}, r^\epsilon_{\mathrm{div},I}, \mathbf{r}^\epsilon_{\mathrm{S},I}$	(6.11)–(6.14)
$r^\epsilon_{\mathrm{CH1},O}, r^\epsilon_{\mathrm{CH2},O}, r^\epsilon_{\mathrm{div},O}, \mathbf{r}^\epsilon_{\mathrm{S},O}$	(6.15)–(6.18)
$r^\epsilon_{\mathrm{CH1},\mathbf{B}}, r^\epsilon_{\mathrm{CH2},\mathbf{B}} r^\epsilon_{\mathrm{div},\mathbf{B}}, \mathbf{r}^\epsilon_{\mathrm{S},\mathbf{B}}$	(6.19)–(6.23)

$\cdot|_\Gamma$ $$g|_\Gamma\,(x,t) = g\left(Pr_{\Gamma_t}(x)\,,t\right)$$

Bibliography

[1] H. Abels, *Existence of Weak Solutions for a Diffuse Interface Model for Viscous, Incompressible Fluids with General Densities*, Communications in Mathematical Physics **289** (2009), 45–73.

[2] ———, *On a Diffuse Interface Model for Two-Phase Flows of Viscous, Incompressible Fluids with Matched Densities*, Archives of Rational Mechanics and Analysis **194** (2009), 463–506.

[3] ———, *Nonstationary Stokes System with Variable Viscosity in Bounded and Unbounded Domains*, Discrete & Continuous Dynamical Systems **3** (2010), no. 2, 141–157.

[4] H. Abels, H. Garcke, and G. Grün, *Thermodynamically Consistent, Frame Indifferent Diffuse Interface Models for Incompressible Two-Phase Flows With Different Densities*, Mathematical Models and Methods in Applied Sciences **22** (2012), no. 3, 1150013 (40 pages).

[5] H. Abels and D. Lengeler, *On Sharp Interface Limits for Diffuse Interface Models for Two-Phase Flows*, Interfaces and Free Boundaries **16** (2012), no. 3.

[6] H. Abels and Y. Liu, *Sharp Interface Limit for a Stokes/Allen-Cahn System*, Archives for Rational Mechanics and Analysis **229** (2018), no. 1, 417–502.

[7] H. Abels and M. Moser, *Well-Posedness of a Navier-Stokes/Mean Curvature Flow system*, June 2017.

[8] ———, *Convergence of the Allen-Cahn Equation to the Mean Curvature Flow with* $90°$*-Contact Angle in 2D*, tba (2018).

[9] H. Abels and M. Röger, *Existence of Weak Solutions for a Non-Classical Sharp Interface Model for a Two-Phase Flow of Viscous, Incompressible Fluids*, Annales de l'Institut Henri Poincaré (C) Non Linear Analysis **26** (2009), no. 6, 2403–2424.

[10] H. Abels and S. Schaubeck, *Sharp Interface Limit for the Cahn-Larché System*, Asymptotic Analysis **91** (2014), no. 3-4, 283–340.

[11] H. Abels and M. Wilke, *Well-Posedness and Qualitative Behaviour of Solutions for a Two-Phase Navier-Stokes/Mullins-Sekerka System*, Interfaces and Free Boundaries **15** (2013), 39–75.

[12] M. S. Agranovich, *Sobolev Spaces, Their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains*, Springer International Publishing, 2015.

[13] G. Alessandrini, A. Morassi, and E. Rosset, *The linear constraint in Poincaré and Korn type inequalities*, Forum Mathematicum **20** (2006), no. 3.

[14] N. Alikakos, P. Bates, and X. Chen, *Convergence of the Cahn-Hilliard Equation to the Hele-Shaw Model*, Archive for Rational Mechanics and Analysis **128** (1994), no. 2, 165–205.

[15] D. Anderson, G. McFadden, and A. Wheeler, *Diffuse-Interface Methods in Fluid Mechanics*, Annual Review of Fluid Mechanics **30** (1998), 139–165.

[16] W. Arendt, R. Chill, S. Fornaro, and C. Poupaud, *L^p-Maximal Regularity for Non-Autonomous Evolution Equations*, Journal of Differential Equations **237** (2007), no. 1, 1–26.

[17] C. Baiocchi, V. Comincioli, E. Magenes, and G. Pozzi, *Free Boundary Problems in the Theory of Fluid Flow Through Porous Media: Existence and Uniqueness Theorems*, Annali di Matematica Pura ed Applicata **97** (1973), no. 1, 1–82.

[18] J. Barrett, H. Garcke, and R. Nürnberg, *Gradient Flow Dynamics of Two-Phase Biomembranes: Sharp Interface Variational Formulation and Finite Element Approximation*, SMAI Journal of Computational Mathematics **4** (2018), 151–195.

[19] F. Boyer, *Mathematical Study of Multiphase Flow Under Shear through Order Parameter Formulation*, Asymptotic Analysis **20** (1999), 175–212.

[20] ______, *Nonhomogeneous Cahn-Hilliard Fluids*, Annales de l'Institut Henri Poincaré (C) Non Linear Analysis **18** (2001), no. 2, 225–259.

[21] G. Caginalp, *Stefan and Hele-Shaw Type Models as Asymptotic Limits of the Phase-Field Equations*, Physical Review A **39** (1989), no. 11, 5887–5896.

[22] G. Caginalp and X. Chen, *Convergence of the Phase Field Model to its Sharp Interface Limits*, European Journal of Applied Mathematics **9** (1998), no. 4, 417–445.

[23] J. Cahn and J. Hilliard, *Free Energy of a Nonuniform System*, The Journal of Chemical Physics **28** (1958), no. 2, 258–267.

[24] X. Chen, *Spectrum for the Allen-Cahn, Cahn-Hilliard, and Phase-Field Equations for Generic Interfaces*, Communications in Partial Differential Equations **19** (1994), no. 7–8, 1371–1395.

[25] ______, *Global Asymptotic Limit of Solutions of the Cahn-Hilliard Equation*, Journal of Differential Geometry **44** (1996), no. 2, 262–311.

[26] X. Chen, D. Hilhorst, and E. Logak, *Mass Conserving Allen-Cahn Equation and Volume Preserving Mean Curvature Flow*, Interfaces and Free Boundaries **12** (2010), 527–549.

[27] K. Deckelnick, G. Dziuk, and C. Elliott, *Computation of Geometric Partial Differential Equations and Mean Curvature Flow*, Acta Numerica **14** (2005), 1–94.

[28] Yihong Du, *Order Structure and Topological Methods in Nonlinear Partial Differential Equations*, World Scientific, 2006.

[29] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.

[30] A. Friedman, *Free Boundary Problems in Science and Technology*, Notices of the AMS **47** (2000), no. 8, 854–861.

[31] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*, second ed., Springer Monographs in Mathematics, 2011.

[32] H. Garcke, K. Lam, E. Sitka, and V. Styles, *A Cahn-Hilliard-Darcy Model for Tumour Growth with Chemotaxis and Active Transport*, Mathematical Models and Methods in Applied Sciences **26** (2016), no. 6, 1095–1148.

[33] H. Garcke, K. F. Lam, and B. Stinner, *Diffuse Interface Modelling of Soluble Surfactants in Two-Phase Flow*, Communications in Mathematical Science **12** (2014), no. 8, 1475–1522.

[34] Harald Garcke, *Curvature Driven Interface Evolution*, Jahresbericht der Deutschen Mathematik-Vereinigung **115** (2013), no. 2.

[35] G. Grubb and V. A. Solonnikov, *Boundary Value Problems for the Nonstationary Navier-Stokes Equations treated by Pseudo-Differential Methods*, Mathematica Scandinavica **69** (1991), 217–290.

[36] M. Gurtin, D. Polignone, and J. Vinals, *Two-Phase Binary Fluids and Immiscible Fluids Described by an Order Parameter*, Mathematical Models and Methods in Applied Sciences **6** (1996), no. 6, 815–831.

[37] A. Henrot, *Extremum Problems for Eigenvalues of Elliptic Operators*, Birkhäuser, 2006.

[38] Stefan Hildebrandt, *Analysis 2*, Springer-Verlag Berlin Heidelberg, 2003.

[39] P. Hohenberg and B. Halperin, *Theory of Dynamic Critical Phenomena*, Reviews of Modern Physics **49** (1977), no. 3, 463–475.

[40] William McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000.

[41] M. Meyries and R. Schnaubelt, *Interpolation, Embeddings and Traces of Anisotropic Fractional Sobolev Spaces With Temporal Weights*, Journal of Functional Analysis **262** (2012), no. 3, 1200–1229.

[42] P. Mottoni and M. Schatzmann, *Geometrical Evolution of Developed Interfaces*, Transactions of the American Mathematical Society **347** (1995), 1533–1589.

[43] R. L. Pego, *Front Migration in the Nonlinear Cahn-Hilliard Equation*, Proc. R. Soc. Lond. A **422** (1989), 261–278.

[44] J. Pruess and G. Simonett, *Moving Interfaces and Quasilinear Parabolic Evolution Equations*, Birkhäuser Basel, 2016.

[45] J. Rowlinson, *Translation of J.D. van der Waals' "The Thermodynamic Theory of Capillarity under the Hypothesis of a Continuous Variation of Density"*, Journal of Statistical Physics **20** (1979).

[46] J. Rowlinson and B. Widom, *Molecular Theory of Capillarity*, Dover Publications, Inc., 1982.

[47] Stefan Schaubeck, *Sharp Interface Limits for Diffuse Interface Models*, Ph.D. thesis, University of Regensburg, 2014.

[48] Y. Shibata and S. Shimizu, *On a Resolvent Estimate of the Interface Problem for the Stokes System in a Bounded Domain*, Journal of Differential Equations **191** (2003), no. 2, 408–444.

[49] ———, *On the Lp-Lq Maximal Regularity of the Neumann Problem for the Stokes Equations in a Bounded Domain*, Journal für die reine und angewandte Mathematik **615** (2007), 1–53.

[50] J. Stefan, *Ueber die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere*, Annalen der Physik **278** (1891), no. 2, 269–286.

[51] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Hall Press, 1970.

[52] D. Werner, *Funktionalanalysis*, seventh ed., Springer Verlag Berlin Heidelberg, 2011.

Acknowledgments

> “Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it’s dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it’s all illuminated. You can see exactly where you were.”
>
> –Andrew Wiles, *Fermat’s Last Theorem, Horizon BBC, 1996*

To me, this quote of famous mathematician Andrew Wiles is an excellent description of the process of writing a mathematical thesis. I would however add two things: First, when you enter the mansion, you cannot be sure that there even is a light switch anywhere, which can make the whole adventure seem more frightening and discouraging. Second, to find the right way, treat the bruises from bumping into furniture and generally have a better time exploring, it needs people offering advice, friendship and support. In this part, I would like to honor all those persons.

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