Theorems in Higher Category Theory and Applications

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Introduction

In recent years, the theory of ∞-categories has seen spectacular applications in Algebraic Topology, Homotopy Theory and Algebraic Geometry amongst other fields. But not only for its applications, the theory of ∞-categories is also very appealing for its unified perspective on the study of classical category theory as well as the study of homotopy types. Thus, this provides us with motivation to further study the theory of ∞-categories.

The most developed framework to date to study the homotopy theory of ∞-categories is through the category of simplicial sets and thanks to the monumental foundational work of Jacob Lurie [**Lur09**], [**Lur17**] and André Joyal [**Joy02**], [**Joy08a**], [**Joy08b**], we have a rich and extensive toolkit for doing *coherent mathematics*. However, the approaches of Lurie and Joyal differ somewhat in philosophy. The road taken by Lurie is through a comparison with another model for higher categories, namely *simplicial categories*, whose homotopy theory was studied already by Bill Dwyer and Dan Kan in a series of articles, for example [**DK80b**], [**DK80a**], [**DK83**] and [**DK87**], and further developed by Julie Bergner [**Ber07**]. This presents a powerful approach and also provides us with examples of ∞-categories right away, through the homotopy coherent nerve functor from simplicial categories to simplicial sets, which was introduced by Jean-Marc Cordier [**Cor70**]. Using various comparison functors with simplicial categories, Lurie is able to lay the foundations for *higher topos theory* [**Lur09**], as well as to apply the theory to study *categorical algebra* from the perspective of higher category theory [**Lur17**]. This does not come for free and the constructions often tend to be rather complex. This complexity comes from the needed translation between simplicial sets and simplicial categories, thus requiring combinatorial knowledge as well as knowledge from the homotopy theory of simplicial categories.

On the other hand, the philosophy of Joyal is to literally interpret the language of category theory *inside* the category of simplicial sets and to observe that this leads to homotopically meaningful constructions. Thus, we are not relying on an external model and the constructions all use the basic language of simplicial sets. Recently, this point of view has been further developed by Denis-Charles Cisinski in his book [**Cis**]. As an example, Cisinski studies (amongst other things) the theory of *presheaves* on an ∞-category, which extends the classical theory of *discrete Grothendieck (op)fibrations*. He then constructs an ∞-category $\mathcal{S}$, which represents the ∞-category of small ∞-groupoids, such that any presheaf corresponds *tautologically* to a functor to $\mathcal{S}$, thereby solving a conjecture of Josh Nichols-Barrer [**NB07**].

This thesis aims to add some further pieces in the spirit of Joyal and Cisinski. We go one step further and study the theory of *presheaves of* ∞-*categories* on an ∞-category, extending the classical theory of *Grothendieck (op)fibrations*. Taking only basic combinatorial properties of *(co)Cartesian fibrations* (the higher categorical analogue of Grothendieck (op)fibrations) as input, we follow the pattern of Cisinski to proof the following theorem.

THEOREM (Proposition 3.4.5 and Theorem 3.4.7). *There exists an* ∞-*category* $\mathcal{Q}$ *together with a universal coCartesian fibration* $\mathcal{Q}_\bullet \to \mathcal{Q}$*, such that any coCartesian fibration* $X \to A$ *is classified by a map* $A \to Q$*, i.e. there is a pullback diagram of simplicial sets*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{Q}_\bullet \arrow[d] \\ A \arrow[r] & \mathcal{Q}. \end{tikzcd}$$

To this end, we set up a way to abstractly speak about *covariant* and *contravariant* homotopy theories in Chapter 2, which extends work of Cisinski [**Cis06**]. We then show in Chapter 3 that the homotopy theories of (co)Cartesian fibrations are instances of such covariant and contravariant homotopy theories (Theorem 3.1.6), and use this to derive the above theorem.

Another missing piece in the theory of ∞-categories is a *General Adjoint Functor Theorem*, generalizing the classical General Adjoint Functor Theorem of Freyd, see [**Mac71**] or [**Fre03**]. Though there exist Adjoint Functor Theorems for *presentable* ∞-categories by Lurie [**Lur09**], to our knowledge no Adjoint Functor Theorem for more general ∞-categories has been proven so far. In Chapter 4 we extend Freyd's theorems to the ∞-categorical setting.

THEOREM (Theorem 4.3.5). *Let* $G\colon D \to C$ *be a continuous functor. Suppose that* D *is locally small and complete and* C *is* 2*-locally small. Then* G *admits a left adjoint if and only if it satisfies the solution set condition.*

Since in higher category theory we have more degrees of freedom, we also find a second General Adjoint Functor Theorem.

THEOREM (Theorem 4.3.6). *Let* $G\colon D \to C$ *be a finitely continuous functor, where* D *is finitely complete. Then* G *admits a left adjoint if and only if it satisfies the h-initial object condition.*

As an interesting application, we find criteria on lifting adjunctions and equivalences which are only defined on the homotopy category.

THEOREM (Theorem 4.4.8 and Corollary 4.4.9). *Let* D *be an* ∞-*category admitting finite limits and let* $G\colon D \to C$ *be a functor between* ∞-*categories which preserves finite limits. Then* G *admits a left adjoint if and only if* hG *does. Furthermore, in this case* G *is an equivalence if and only if* hG *is.*

This is part of joint work with George Raptis and Christoph Schrade [**NRS18**].

Finally, higher category theory does not only provide us with tools, it also provides us with intuition on how we should think about certain mathematical objects. Thus, even in situations which do not immediately fit into our framework, we may use this intuition to guide our proofs. An example of this is given by the *cobordism category*, which incidentally was also one of the motivations in the development of higher category theory. The cobordism category can be realized as a topological category with objects embedded manifolds and morphisms embedded cobordisms in some ambient space. The classifying space of the cobordism category has been extensively studied and led to the solution of the Mumford conjecture [**GTMW09**]. However, the classifying space has more structure, it is an *infinite loop space*. This structure *should* intuitively be induced by the symmetric monoidal structure given by taking disjoint union of manifolds. This presents a problem; since the manifolds came with an embedding, we can not define a disjoint union operation which is associative on the nose, only up to homotopy. But using the intuition from higher category theory, we know how we should encode such a homotopy coherent monoidal product. In Chapter 5 we then use this to determine the homotopy type of the cobordism category as an *infinite loop space*.

Theorem (Theorem 5.6.6). *Let $B\mathrm{Cob}_\theta(d)$ be the classifying space of the d-dimensional cobordism category with tangential structure. Then this admits an infinite loop space structure induced by disjoint union and there is an equivalence of infinite loop spaces*

$$B\mathrm{Cob}_\theta(d) \simeq \Omega^\infty MT\theta(d)[1]$$

where the right hand side is the infinite loop space associated to the Madsen-Tillmann spectrum.

This chapter has appeared as [**Ngu17**].

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CHAPTER 1

Preliminaries

In this chapter we will review the basic technical tools for this thesis. The central technical tool will be Quillen's theory of model categories. This will form the basis of many of our constructions. A large part of this thesis concerns the application of the theory of model categories to study the category of simplicial sets and in particular the study of ∞-categories, hence we will also review some basic theory of simplicial sets in this first chapter.

1.1. Factorization systems & Model Categories

We review some basic theory of model categories, originally introduced by Daniel Quillen [**Qui67**]. Good modern references include Mark Hovey's book [**Hov99**] and the appendix of Jacob Lurie's book [**Lur09**]. A very useful source for many convenient results are the articles of Joyal [**Joy08b**] and Joyal and Tierney [**JT07**]. Our emphasis in this section is on weak factorization systems, which will play a central role throughout this thesis. The material of this section is standard and can be found in any of the above references.

Definition 1.1.1. Let C be a category and $i\colon A \to B$ and $p\colon X \to Y$ be morphisms of C. We say that i has the *left lifting property* with respect to p and equivalently that p has the *right lifting property* with respect to i, if for all commutative squares of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a lift as indicated.

More generally, we may speak of lifting properties against a class of morphisms.

Definition 1.1.2. Let $\mathcal{E}$ be a class of morphisms of C. A morphism has the *left lifting property*, (resp. *right lifting property*) with respect to $\mathcal{E}$ if it has the left lifting property (respectively right lifting property) with respect to any morphism in $\mathcal{E}$. We denote by $r(\mathcal{E})$ (respectively $l(\mathcal{E})$) the class of morphisms having the right lifting property (respectively left lifting property) with respect to $\mathcal{E}$.

The following is the central notion for most of this thesis.

Definition 1.1.3. A *weak factorization system* on a category C is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of C such that the following conditions hold.

- Any morphism f of C admits a factorization $f = pi$, where $i \in \mathcal{L}$ and $p \in \mathcal{R}$.
- We have $\mathcal{L} = l(\mathcal{R})$ and $\mathcal{R} = r(\mathcal{L})$.

In this case we say that $\mathcal{L}$ is the *left class* of a weak factorization system and that $\mathcal{R}$ is the *right class* of a weak factorization system.

Example 1.1.4. Suppose C is a category endowed with a weak factorization system $(\mathcal{L},\mathcal{R})$ and let $A \in C$ be an object. Then we have an induced factorization system $(\mathcal{L}_A,\mathcal{R}_A)$ on the slice category C/A in which the left (resp. right) class are those maps, whose image under the forgetful functor

$$C/A \to C$$

lie in the left (resp. right) class of the factorization system on C.

Definition 1.1.5. A class of morphism is called *saturated* if it is closed under retracts, pushouts and transfinite compositions.

Note that the left class in a weak factorization system is always saturated. A good source of weak factorization systems is given by the small object argument.

Proposition 1.1.6 (Small object argument). *Let C be a locally presentable category and I a small set of maps. Then $(l(r(I)), r(I))$ is a weak factorization system. Moreover, the class $l(r(I))$ is the smallest saturated class containing the set I.*

Proof. See for example [**Hov99**, Theorem 2.1.14]. □

Another useful lemma is the following.

Lemma 1.1.7 (Retract Lemma). *Let $f\colon X \to Y$ be a morphism and assume that we have a factorization*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & T \arrow[ur, "p"'] & \end{tikzcd}$$

If f has the right (resp. left) lifting property with respect to i (resp. p), then f is a retract of p (resp. of i).

Proof. This is [**Hov99**, Lemma 1.1.19]. □

In subsequent chapters we will encounter weak factorization systems of the following form.

Definition 1.1.8. Let C be a locally presentable category. A weak factorization system $(\mathcal{L},\mathcal{R})$ is called *tractable*, if there exists a set I such that $(\mathcal{L},\mathcal{R}) = (l(r(I)), r(I))$ and such that for any object $X \in C$, the canonical morphism $\emptyset \to X$ is in the left class $\mathcal{L}$.

Example 1.1.9. Suppose E is a topos. Let Mono be the class of monomorphisms in E. We call a morphism a *trivial fibration*, if it has the right lifting property with respect to the class of monomorphisms and denote the class consisting of these by Triv. Then (Mono, Triv) is a tractable weak factorization system. We will call any generating set for the monomorphisms a *cellular model*.

DEFINITION 1.1.10. Let $\mathcal{A}$ be a class of morphisms in a category. Then $\mathcal{A}$ satisfies the *right cancellation property* if for all composable morphisms f and g such that $f \in \mathcal{A}$, it follows that $gf \in \mathcal{A}$ if and only if $g \in \mathcal{A}$.

REMARK 1.1.11. Usually, the right cancellation property only requires the implication $gf \in \mathcal{A} \Rightarrow g \in \mathcal{A}$. In this sense, our cancellation property asserts that the class $\mathcal{A}$ is closed under composition and satisfies the more strict cancellation property. However, all the classes we will encounter satisfy the stronger right cancellation property and we chose to define it in this stronger sense for brevity.

The following principle will be used several times.

LEMMA 1.1.12. *Let E be a bicomplete category in which colimits are universal. Let $\mathcal{A}$ be a saturated class of maps and $\mathcal{F}$ be a class of maps which contains the isomorphisms and is closed under pushout. Let $\mathcal{B}$ be the class of maps whose pullback along any map in $\mathcal{F}$ is in the class $\mathcal{A}$. Then $\mathcal{B}$ is saturated. Moreover, if $\mathcal{A}$ has the right cancellation property, so does $\mathcal{B}$.*

PROOF. This is [**Joy08b**, Lemma D.2.17]. □

Finally, a model structure on a category is an interaction of two weak factorization systems.

DEFINITION 1.1.13. Let C be a category admitting (small) limits and colimits. A *model structure* on C consists of three classes of morphisms $(\mathcal{W},\mathcal{C},\mathcal{F})$, called *weak equivalences*, *cofibrations* and *fibrations* respectively, satisfying the following conditions.

- The class $\mathcal{W}$ satisfies the 2-out-of-3 property.
- The pair $(\mathcal{C},\mathcal{W}\cap\mathcal{F})$ is a weak factorization system.
- The pair $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ is a weak factorization system.

A category endowed with a model structure is called a *model category*.

DEFINITION 1.1.14. Let C and D be model categories and suppose we have an adjoint pair

$$F : C \rightleftarrows D : G.$$

Then the pair is a called a *Quillen adjunction* if the left adjoint F takes cofibrations in C to cofibrations in D and the right adjoint G takes fibrations in D to fibrations in C. We call the left adjoint in a Quillen adjunction a *left Quillen functor* and the right adjoint in a Quillen adjunction a *right Quillen functor*.

A useful simplification of a Quillen adjunction is the following.

PROPOSITION 1.1.15. *Suppose C and D are model categories and we have an adjunction*

$$F : C \rightleftarrows D : G$$

Then this defines a Quillen adjunction if and only if F takes cofibrations to cofibrations and G takes fibrations between fibrant objects to fibrations.

PROOF. This is [**JT07**, Proposition 7.15]. □

Definition 1.1.16. Let C be a model category and $X \in C$ be an object. A *fibrant replacement* of X is a weak equivalence $X \to RX$ with RX fibrant. A *cofibrant replacement* of X is a weak equivalence $LX \to X$ with LX cofibrant.

Definition 1.1.17. A Quillen adjunction $F : C \rightleftarrows D : G$ is called a *Quillen equivalence* if and only if, for all cofibrant $X \in C$ and fibrant $Y \in D$, a map $FX \to Y$ is a weak equivalence D if and only if the adjoint map $X \to GY$ is a weak equivalence in C.

A very useful criterion for checking Quillen equivalences is the following.

Proposition 1.1.18. *Suppose $F : C \rightleftarrows D : G$ is a Quillen adjunction. Then it is a Quillen equivalence if and only if G reflects weak equivalences between fibrant objects and, for every cofibrant $X \in C$, the map $X \to G(RFX)$ is a weak equivalence, where RFX is a fibrant replacement of FX.*

Proof. This is [**Hov99**, Corollary 1.3.16]. □

1.2. Simplicial sets

We will denote by Δ the simplex category whose objects are given by ordered sets of the form $[n] := \{0 < 1 < \ldots < n\}$ for $n \geq 0$ and whose morphisms are given by order preserving maps of sets.

Definition 1.2.1. A *simplicial set* is a presheaf on the category Δ. We will denote by **sSet** the category of simplicial sets and natural transformations.

The category of simplicial sets is the basic language we use for most parts of this thesis. Inside this category we will be able to interpret the theory of higher categories.

Definition 1.2.2. Let X be a simplicial set and consider an extension problem of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n. \arrow[ur, dashed] & \end{tikzcd}$$

Then X is called an ∞*-category* if the extension exists for $n \geq 2$ and $0 < k < n$ and is called an ∞*-groupoid* if the extension exists for $0 \leq k \leq n$. We denote by ∞**Cat** the full subcategory on ∞-categories and by ∞**Grpd** the full subcategory on ∞-groupoids.

Lemma 1.2.3. *The inclusion $\infty\mathbf{Grpd} \hookrightarrow \infty\mathbf{Cat}$ admits a left adjoint.*

Proof. An easy application of [**Cis**, Corollary 3.5.3]. □

Definition 1.2.4. We denote the left adjoint of the inclusion by

$$k\colon \infty\mathbf{Cat} \to \infty\mathbf{Grpd}$$

Given an ∞-category X, we refer to the ∞-groupoid $k(X)$ as the *maximal ∞-groupoid underlying X*.

Construction 1.2.5. To each ∞-category X we may associate a category hX as follows. The objects of hX are the vertices of X. The morphisms of hX are the 1-simplices of X modulo the following homotopy relation. Two 1-simplices $f, g\colon \Delta^1 \to X$ are *homotopic* if there exists a 2-simplex $\Delta^2 \to X$ of the form

$$\begin{tikzcd} & \cdot \arrow[dr, "id"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "g"'] & & \cdot \end{tikzcd}$$

where id denotes a degenerate 1-simplex. The inner horn filling conditions ensure that the homotopy relation is an equivalence relation and that hX is a category. If X is an ∞-groupoid, then hX is a groupoid.

We introduce particularly important classes of maps of simplicial sets.

Definition 1.2.6. A *right anodyne extension* is a map in the saturated class generated by the maps

$$\Lambda^n_k \to \Delta^n \quad \text{for } 0 < k \leq n.$$

A *left anodyne extension* is a map in the saturated class generated by the maps

$$\Lambda^n_k \to \Delta^n \quad \text{for } 0 \leq k < n.$$

An *inner anodyne extension* is a map in the saturated class generated by the maps

$$\Lambda^n_k \to \Delta^n \quad \text{for } n \geq 2, \text{ and } 0 < k < n.$$

Definition 1.2.7. A *right (resp. left, resp. inner) fibration* is a map of simplicial sets having the right lifting property with respect to right (resp. left, resp. inner) anodyne extensions.

An important operation on simplicial sets is given by the following construction. Let Δ_{aug} be the category obtained from Δ by adding the empty ordered set. This category comes equipped with a monoidal structure given by taking sums of ordinals

$$\begin{aligned} \Delta_{aug} \times \Delta_{aug} &\to \Delta_{aug} \\ ([m],[n]) &\mapsto [m+1+n]. \end{aligned}$$

Let us call a presheaf on Δ_{aug} an augmented simplicial set and denote the category by $\mathbf{sSet}_{aug}$. The monoidal product on Δ_{aug} induces via Day convolution a monoidal product on augmented simplicial sets, which we will call their *join*. In other words, given two augmented simplicial sets X and Y, their join, denoted by $X \star Y$, is computed as the left Kan extension

$$\begin{tikzcd} \Delta_{aug}^{op} \times \Delta_{aug}^{op} \arrow[r, "{(X,Y)}"] \arrow[d] & \mathbf{Set} \times \mathbf{Set} \arrow[d] \\ \Delta_{aug}^{op} \arrow[r, dashed, "X\star Y"] & \mathbf{Set}. \end{tikzcd}$$

where the left vertical map is given by the monoidal product on Δ_{aug} and the right vertical map is given by the cartesian product of sets.

We have an inclusion functor $i\colon \Delta \to \Delta_{aug}$. This induces a functor on presheaf categories

$$i^*\colon \mathbf{sSet}_{aug} \to \mathbf{sSet}$$

which admits both a right adjoint, denoted by i_*, and a left adjoint, denoted by $i_!$.

Definition 1.2.8. Let K and L be simplicial sets. The *join* of K and L, also denoted by $K \star L$, is defined as

$$K \star L := i^*(i_* K \star i_* L).$$

More explicitly, the join $K \star L$ can be described as follows. Its n-simplices are given by the formula

$$(K \star L)_n = \bigsqcup_{i+1+j=n} K_i \times L_j.$$

This defines a monoidal product on the category of simplicial sets, with unit object given by the empty simplicial set. We obtain canonical maps

$$K \sqcup L \to K \star L.$$

Given a simplicial set K, we obtain a functor

$$\begin{aligned}(\cdot) \star K\colon \mathbf{sSet} &\to K\backslash\mathbf{sSet}\\ X &\mapsto (\emptyset \sqcup K \to X \star K)\end{aligned}$$

and similarly a functor

$$\begin{aligned}K \star (\cdot)\colon \mathbf{sSet} &\to K\backslash\mathbf{sSet}\\ X &\mapsto (K \sqcup \emptyset \to K \star X),\end{aligned}$$

both admitting right adjoints.

Definition 1.2.9. Let $p\colon K \to X$ be a map of simplicial sets. We denote by $p\backslash X$ the image of the right adjoint to the functor $K \star (\cdot)$ and by X/p the image of the right adjoint to the functor $(\cdot) \star K$. More generally, suppose $p\colon K \to X$ and $F\colon Y \to X$ are maps of simplicial sets. We denote by F/p the pullback

$$\begin{tikzcd} F/p \arrow[r] \arrow[d] & X/p \arrow[d] \\ Y \arrow[r, "F"] & X \end{tikzcd}$$

and similarly we define $p\backslash F$.

Notation 1.2.10. Suppose we have a Cartesian square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & K \arrow[d] \\ B \arrow[r] & L \end{tikzcd}$$

in which each map is a monomorphism. Then the induced map from the pushout

$$B \sqcup_A K \to L$$

is a monomorphism and we denote its image by $B \cup A$. We thus have an inclusion $B \cup A \subset L$.

We have the following stability property of left and right anodyne extensions with respect to the join operation due to Joyal.

LEMMA 1.2.11. *Let $i\colon A \to B$ and $j\colon K \to L$ be monomorphisms. If i is right anodyne or j is left anodyne, then the induced map*

$$A \star L \cup B \star K \to B \star L$$

is inner anodyne.

PROOF. See [**Lur09**, Lemma 2.1.2.3]. □

An application of this stability property is the following important proposition.

PROPOSITION 1.2.12. *Let X be an ∞-category and consider an extension problem*

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[d] \arrow[dr, "f"] & \\ \Lambda^n_n \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

for $n \geq 2$. Then an extension exists if and only if f is an equivalence in X.

PROOF. See for example [**Lur09**, Proposition 1.2.4.3]. □

Another important class of maps of simplicial sets, whose homotopy theory will be studied in an entire chapter on its own, is the class of *Cartesian* and *coCartesian fibrations*, which we define below.

PROPOSITION 1.2.13. *Let $p\colon X \to A$ be an inner fibration of simplicial sets and let $f: x \to y \in X$ be an edge. Then the following are equivalent.*

- *The induced map*

$$X/f \to X/y \times_{A/p(y)} A/p(f)$$

  is a trivial fibration.
- *For all $n \geq 2$ and all lifting problems of the form*

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[d] \arrow[dr, "f"] & \\ \Lambda^n_n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[ur, dashed] \arrow[r] & A. \end{tikzcd}$$

  there exists a lift as indicated.

- *For all $n \geq 1$ and all lifting problems of the form*

$$\begin{tikzcd}
\Delta^1 \times \{1\} \arrow[d] \arrow[dr, "f"] & \\
\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dashed] & A
\end{tikzcd}$$

there exists a lift as indicated.

Proof. Combine [**Lur09**, Definition 2.4.1.1], [**Lur09**, Remark 2.4.1.4] and [**Lur09**, Proposition 2.4.1.8]. □

Definition 1.2.14. Let $p\colon X \to A$ be an inner fibration. Then an edge $f\colon \Delta^1 \to X$ is called *p-Cartesian* if it satisfies the equivalent conditions of the above Proposition.

Remark 1.2.15. One obtains a definition of *p-coCartesian* edges by duality. For instance, p-coCartesian edges satisfy the dual lifting property

$$\begin{tikzcd}
\Delta^1 \times \{1\} \arrow[d] \arrow[dr, "f"] & \\
\Delta^1 \times \partial\Delta^n \cup \{0\} \times \Delta^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dashed] & A
\end{tikzcd}$$

for $n \geq 0$. All of the results below then have their dual counterpart.

Examples of Cartesian edges are given by the following.

Proposition 1.2.16. *Let $p\colon C \to D$ be an inner fibration between ∞-categories and let $f\colon \Delta^1 \to C$ be an edge. Then the following are equivalent.*

- *The edge f is an equivalence in C.*
- *The edge f is p-Cartesian and its image $p(f)$ is an equivalence in D.*

Proof. See [**Lur09**, Proposition 2.4.1.5]. □

The following asserts cancellation properties of Cartesian edges.

Proposition 1.2.17. *Let $p\colon X \to A$ be an inner fibration between ∞-categories. Let $\sigma\colon \Delta^2 \to X$ be a 2-simplex depicted as*

$$\begin{tikzcd}
& \cdot \arrow[d, "g"] \\
\cdot \arrow[ur, "f"] \arrow[r, "h"'] & \cdot
\end{tikzcd}$$

Suppose that the edge g is p-Cartesian. Then f is p-Cartesian if and only if h is p-Cartesian.

Proof. See [**Lur09**, Proposition 2.1.4.7]. □

Definition 1.2.18. Let $p\colon X \to A$ be an inner fibration. Then p is called a *Cartesian fibration* if for all lifting problems of the form

$$\begin{tikzcd} \Delta^{\{1\}} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & A, \end{tikzcd}$$

there exists a lift as indicated, which is p-Cartesian. Dually, p is called a *coCartesian fibration* if for all lifting problems of the form

$$\begin{tikzcd} \Delta^{\{0\}} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & A, \end{tikzcd}$$

there exists a lift as indicated, which is p-coCartesian.

Finally, the following Proposition asserts that the property of being a Cartesian fibration can be checked on representables.

Proposition 1.2.19. *Let $p\colon X \to A$ be an inner fibration. Then p is a Cartesian fibration if and only if for all $n \geq 0$ every pullback $X \times_A \Delta^n \to \Delta^n$ is a Cartesian fibration.*

Proof. See [**Lur09**, Corollary 2.4.2.10]. $\square$

CHAPTER 2

Covariant & Contravariant homotopy theories

This chapter introduces a formalism which allows us to speak abstractly about 'covariant' and 'contravariant' homotopy theories. This formalism comes in the form of a model structure on a locally presentable category endowed with a tractable weak factorization system. Recall that a *Cisinski model structure* on a topos is a model structure in which the cofibrations are precisely the monomorphisms and which is cofibrantly generated. Given an elementary homotopical datum (which we will recall below), Cisinski constructs such a model structure in a very general way, see [**Cis02**] and [**Cis06**].

This construction has been generalized by Olschok [**Ols11**] to the setting of locally presentable categories observing that Cisinski's arguments work in a more general setting. We will revisit Cisinski's construction in the setting of Olschok in the first section and observe that one can drop one axiom. In this way, any elementary homotopical datum will give rise to two in general distinct model structures, which we will call *Covariant model structures* and *Contravariant model structures*, since a particular example will be the covariant and contravariant model structures in simplicial sets as introduced by Joyal [**Joy08a**]. The second section introduces the notion of (co)final maps and smooth and proper maps from an abstract perspective, but which arises very naturally in our context. Finally, the third section gives the aforementioned example for simplicial sets as well as a construction of the Joyal model structure. Both examples are not new, but we will view them from our perspective of co- and contravariant model structures and in case of the co- and contravariant model structures for simplicial sets, this gives rise to a fairly easy construction. Both examples also serve to lay some foundations for subsequent parts of this thesis.

We want to mention that the proofs in this chapter, and especially in the first two sections, are mostly due to Cisinski, although in a less general setting. Nevertheless we gave full proofs, just to verify that his arguments indeed carry over to our setting. Our main source of inspiration is [**Cis**, Section 2.4].

2.1. Covariant & contravariant model structures

We will first introduce the notion of an elementary homotopical datum, which is the basis for our construction. We fix a locally presentable category C together with a tractable weak factorization system $(\mathcal{L}, \mathcal{R})$ (see Definition 1.1.8).

Definition 2.1.1. Let $X \in C$ be an object. A *cylinder on* X is a commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "\partial_0"'] \arrow[drr, bend left, "id_X"] & & \\
& IX \arrow[r, "\sigma"] & X \\
X \arrow[ur, "\partial_1"] \arrow[urr, bend right, "id_X"'] & &
\end{tikzcd}$$

where the induced map $\partial_0 \sqcup \partial_1 \colon X \sqcup X \to IX$ is in the left class $\mathcal{L}$.

Consider the endomorphism category $End(C)$. This is a monoidal category with monoidal product given by composition. It acts on the left on C by

$$End(C) \times C \to C$$
$$(F, X) \mapsto F \otimes X = F(X).$$

In particular, for any natural transformation $\eta \colon F \Rightarrow G$ and any morphism $f \colon X \to Y \in C$ we obtain a morphism

$$\eta \otimes f \colon F \otimes X \to G \otimes Y.$$

Definition 2.1.2. A *functorial cylinder object* on the category C is an endofunctor $I \colon C \to C$ together with natural transformations

- $\partial_0 \sqcup \partial_1 \colon id_C \sqcup id_C \Rightarrow I$
- $\sigma \colon I \Rightarrow id_C$

such that for each $X \in C$, evaluation at X defines a cylinder on X.

Notation 2.1.3. Suppose we have a functorial cylinder $(I, \partial_0, \partial_1, \sigma)$ on C. We denote $\partial I := id_C \sqcup id_C$. We thus have natural transformations

- $\partial_0 \sqcup \partial_1 \colon \partial I \Rightarrow I$
- $\partial_i \otimes id \colon \{i\} \otimes id \cong id \Rightarrow I$ for $i = 0, 1$.

The cylinder induces three operations on the morphisms of C. Given a morphism $i \colon K \to L \in C$ we obtain a commutative square

$$\begin{tikzcd}
\partial I \otimes K \arrow[r] \arrow[d] & I \otimes K \arrow[d] \\
\partial I \otimes L \arrow[r] & I \otimes L.
\end{tikzcd}$$

We denote the induced map from the pushout

$$\partial I \boxtimes i \colon \partial I \otimes L \sqcup_{\partial I \otimes K} I \otimes K \to I \otimes L.$$

Similarly for $i = 0, 1$ we have a commutative square

$$\begin{tikzcd}
\{i\} \otimes K \arrow[r] \arrow[d] & I \otimes K \arrow[d] \\
\{i\} \otimes L \arrow[r] & I \otimes L
\end{tikzcd}$$

and we denote the induced map from the pushout

$$\partial_i \boxtimes i \colon \{i\} \otimes L \sqcup_{\{i\} \otimes K} I \otimes K \to I \otimes L.$$

Given a functorial cylinder, we impose additional compatibility conditions with respect to the weak factorization system $(\mathcal{L},\mathcal{R})$.

Definition 2.1.4. A functorial cylinder is called *exact* with respect to $(\mathcal{L},\mathcal{R})$ if the following hold.

- The functor I commutes with small colimits.
- For any morphism $i\colon K \to L \in \mathcal{L}$ the morphism $\partial I \boxtimes i$ is in $\mathcal{L}$.
- For any morphism $i\colon K \to L \in \mathcal{L}$ the morphism $\partial_1 \boxtimes i$ is in $\mathcal{L}$.

Example 2.1.5. Let A be a small category and consider its category of presheaves $\mathrm{PSh}(A)$. Let I be a presheaf together with two maps from the terminal presheaf $\partial_i\colon * \to I$, where $i = 0,1$, such that

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & * \arrow[d, "\partial_0"] \\ * \arrow[r, "\partial_1"'] & I \end{tikzcd}$$

is cartesian. Then the endofunctor

$$I \times (\cdot)\colon \mathrm{PSh}(A) \to \mathrm{PSh}(A)$$

defines an exact cylinder with respect to the weak factorization system (Mono, Triv). Indeed, for a presheaf X the structure maps are given by $\partial_i \times id_X\colon X \to I \times X$ and $\sigma\colon I \times X \to X$ is given by the projection to X. For any monomorphism of presheaves $i\colon K \to L$, we have a cartesian square

$$\begin{tikzcd} \partial I \times K \cong K \sqcup K \arrow[r] \arrow[d] & I \times K \arrow[d] \\ \partial I \times L \cong L \sqcup L \arrow[r] & I \times L \end{tikzcd}$$

since colimits are universal. It follows that the map $\partial I \boxtimes i$ is a monomorphism. Since the category of presheaves is cartesian closed, the functor $I \times (\cdot)$ commutes with colimits hence is exact with respect to (Mono, Triv).

Definition 2.1.6. A class of morphisms $An^r(I) \subseteq \mathcal{L}$ is called a class of *right I-anodyne extensions* if the following axioms are satisfied.

- There exists a (small) set of morphisms $\Lambda \subseteq \mathcal{L}$ such that we have $An^r(I) = l(r(\Lambda))$.
- For any $i\colon K \to L \in \mathcal{L}$, the induced map $\partial_1 \boxtimes i$ is in $An^r(I)$.
- For any $i\colon K \to L \in An^r(I)$, the map $\partial I \boxtimes i$ is also in $An^r(I)$

A *right homotopical structure* on C is the datum of an exact cylinder $(I, \partial_0, \partial_1, \sigma)$ together with a choice of right I-anodyne extensions $An^r(I)$. A *right I-fibration* is a morphism of C having the right lifting property with respect to the class of right I-anodyne extensions. An object is *right I-fibrant* if its canonical map to the terminal object is a right I-fibration.

Dually, we may define the following.

Definition 2.1.7. A class of morphisms $An^l(I) \subseteq \mathcal{L}$ is called a class of *left I-anodyne extensions* if the following axioms are satisfied.

- There exists a (small) set of morphisms $\Lambda \subseteq \mathcal{L}$ such that we have $An^r(I) = l(r(\Lambda))$.

- For any $i\colon K \to L \in \mathcal{L}$, the induced map $\partial_0 \boxtimes i$ is in $An^r(I)$.
- For any $i\colon K \to L \in An^r(I)$, the map $\partial I \boxtimes i$ is also in $An^r(I)$

A *left homotopical structure* on C is the datum of an exact cylinder $(I, \partial_0, \partial_1, \sigma)$ together with a choice of left I-anodyne extensions $An^l(I)$. A *left I-fibration* is a morphism of C having the right lifting property with respect to the class of left I-anodyne extensions. An object is *left I-fibrant* if its canonical map to the terminal object is a left I-fibration.

Remark 2.1.8. Our definition of *right* (and *left*) I-anodyne extension differs from Cisinski's notion of (plain) *I-anodyne extensions* in the following way. In Cisinski's axioms it is required that for any morphism $i\colon K \to L \in \mathcal{L}$ *both* morphisms

- $\partial_0 \boxtimes i$ and
- $\partial_1 \boxtimes i$

are I-anodyne extensions, while we only require the second one for our notion of right I-anodyne extensions. This gives a *direction* for right I-anodyne extensions. For example, for any object $K \in C$ the morphism $\{1\} \otimes K \to I \otimes K$ is right I-anodyne while the morphism $\{0\} \otimes K \to I \otimes K$ is not.

We will see that a class of right (or left) I-anodyne extensions always exists. For example we may take the class $\mathcal{L}$ to be a class of right I-anodyne extensions. At the end of this section, we will consider right I-anodyne extensions arising from an *elementary homotopical datum*. But first, our main goal of this section is to prove that any right and any left homotopical structure gives rise to a model structure on C.

Definition 2.1.9. Let $f, g\colon X \to Y$ be two morphisms. An *I-homotopy* from f to g is a morphism

$$h\colon I \otimes X \to Y$$

such that $h(\partial_0 \otimes id_X) = f$ and $h(\partial_1 \otimes id_X) = g$. We denote by $[X, Y]_I$ the quotient of $\hom_C(X, Y)$ by the equivalence relation generated by the notion of I-homotopy. We denote by $\mathrm{Ho}_I(C)$ the category having the same objects as C and morphism sets given by the quotients $[X, Y]_I$. We will refer to this category as the *I-homotopy category of C*. We have a canonical projection $C \to \mathrm{Ho}_I(C)$. A morphism is an *I-homotopy equivalence* if its image in the I-homotopy category is an isomorphism.

Remark 2.1.10. The functoriality of the cylinder ensures that $\mathrm{Ho}_I(C)$ is indeed a category.

We will prove the following pair of Theorems.

Theorem 2.1.11. *Suppose we have a right homotopical structure on C. Then there exists a unique model structure on C with the following description.*

- *The class of cofibrations is precisely the class $\mathcal{L}$.*
- *A morphism $f\colon A \to B$ is a weak equivalence if and only if for all right I-fibrant objects $W \in C$, the induced morphism*

$$f^*\colon [B, W]_I \to [A, W]_I$$

is bijective.

Furthermore, an object is fibrant if and only if it is right I-fibrant and a morphism between right I-fibrant objects is a fibration if and only if it is a right I-fibration.

Theorem 2.1.12. *Suppose we have a left homotopical structure on C. Then there exists a unique model structure on C with the following description.*

- *The class of cofibrations is precisely the class $\mathcal{L}$.*
- *A morphism $f\colon A \to B$ is a weak equivalence if and only if for all left I-fibrant objects $W \in C$, the induced morphism*
$$f^*\colon [B,W]_I \to [A,W]_I$$
is bijective.

Furthermore, an object is fibrant if and only if it is left I-fibrant and a morphism between left I-fibrant objects is a fibration if and only if it is a left I-fibration.

The proof requires several steps. We will only focus on right homotopical structures. The proof for left homotopical structures is entirely analogous, requiring only minor modifications in the direction of the homotopy. The basis is Jeff Smith's recognition theorem for combinatorial model categories. We will use the following variant due to Carlos Simpson.

Theorem 2.1.13. *Let $\mathbf{M}$ be a locally presentable category and S and Λ sets of morphisms such that $\Lambda \subset l(r(S))$. Define a morphism $f\colon A \to B$ to be a weak equivalence if and only if there exists a diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

such that the horizontal arrows are transfinite compositions of pushouts of morphisms in Λ and the right vertical arrow is in $r(S)$. Define the class of cofibrations to be $l(r(S))$ and suppose furthermore that

- *the domains of I and Λ are cofibrant,*
- *the class of weak equivalences above is closed under retracts and satisfies 2-out-of-3,*
- *the class of trivial cofibrations is closed under pushouts and transfinite compositions.*

Then there exists a cofibrantly generated model structure on $\mathbf{M}$ with the given class of cofibrations and weak equivalences.

Proof. This is [**Sim12**, Theorem 8.7.3] □

In our situation, the set Λ will be the generating set of right I-anodyne extensions and the set S will be a generating set for $\mathcal{L}$ (recall that $(\mathcal{L},\mathcal{R})$ was assumed to be cofibrantly generated). It is clear that the domains of S and Λ are cofibrant since $(\mathcal{L},\mathcal{R})$ was assumed to be tractable and that our class of weak equivalences is closed under retracts and satisfies 2-out-of-3. Thus, our task will be to show that our class of weak equivalences satisfy the description of Simpson's theorem and that the trivial cofibrations are closed under pushouts and transfinite compositions. Along the way, our proofs will also imply the description of fibrations we gave in our theorem.

We first show that any right I-anodyne is a weak equivalence in the sense of Theorem 2.1.11.

LEMMA 2.1.14. *If W is right I-fibrant , then I-homotopy is an equivalence relation on the set* $\mathrm{Hom}(X,W)$ *for any object* X.

PROOF. Consider three morphisms

$$u, v, w\colon X \to W.$$

Suppose we have homotopies

$$h\colon I \otimes X \to W \quad \text{such that } h(\partial_0 \otimes 1_X) = u,\ h(\partial_1 \otimes 1_X) = w$$

and

$$k\colon I \otimes X \to W \quad \text{such that } h(\partial_0 \otimes 1_X) = v,\ h(\partial_1 \otimes 1_X) = w.$$

We will show that there exists an I-homotopy from u to v.

We have a map

$$((h,k), \sigma \otimes w)\colon I \otimes \partial I \otimes X \sqcup_{\{1\}\otimes\partial I\otimes X} \{1\} \otimes I \otimes X \to W$$

and the map

$$I \otimes \partial I \otimes X \sqcup_{\{1\}\otimes\partial I\otimes X} \{1\} \otimes I \otimes X \to I \otimes I \otimes X$$

is a right I-anodyne extension since $\partial I \otimes X \to I \otimes X \in \mathcal{L}$. By assumption W is I-fibrant, thus we have a homotopy

$$H\colon I \otimes I \otimes X \to W$$

such that

$$H(1_I \otimes \partial_0 \otimes 1_X) = h$$

and

$$H(1_I \otimes \partial_1 \otimes 1_X) = k.$$

Moreover, we have

$$H(\partial_1 \otimes 1_I \otimes 1_X) = \sigma \otimes w$$

Now define an I-homotopy $\eta\colon I \otimes X \to W$ by the formula

$$\eta = H(\partial_0 \otimes 1_I \otimes 1_X).$$

We then have

$$\eta(\partial_0 \otimes 1_X) = H(\partial_0 \otimes \partial_0 \otimes 1_X) = h(\partial_0 \otimes 1_X) = u$$

and

$$\eta(\partial_1 \otimes 1_X) = H(\partial_0 \otimes \partial_1 \otimes 1_X) = k(\partial_0 \otimes 1_X) = v.$$

Thus η defines a homotopy from u to v.

Now if h is the constant homotopy at u and k is a homotopy from v to u, then η provides a homotopy from u to v showing that I-homotopy is symmetric. Transitivity follows from the above construction and symmetry. □

PROPOSITION 2.1.15. *Any right I-anodyne extension is a weak equivalence.*

Proof. Let $f: A \to B$ be a right I-anodyne extension and let W be right I-fibrant. It is enough to show that

$$f^*: [B, W]_I \to [A, W]_I$$

is injective. Thus let $\beta_0, \beta_1: B \to W$ be two morphisms such that $\beta_0 f$ is homotopic to $\beta_1 f$. By the above lemma, there exists a homotopy

$$h: I \otimes A \to W$$

such that $h_0 = \beta_0 f$ and $h_1 = \beta_1 f$. This gives rise to a lifting problem

$$\begin{tikzcd} I \otimes A \sqcup_{\partial I \otimes A} \partial I \otimes B \arrow[r, "(h,\beta_0 \sqcup \beta_1)"] \arrow[d] & W \\ I \otimes B \arrow[ur, dashed] & \end{tikzcd}$$

Since f is right I-anodyne, the vertical map is also right I-anodyne and hence, since W is right I-fibrant, the lifting problem admits a solution. This provides a homotopy from β_0 to β_1. □

Now suppose we have a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

in which the horizontal maps are transfinite compositions of pushouts of Λ and the map $X \to Y$ is in the class $\mathcal{R}$. In particular, the horizontal maps are right I-anodyne extensions and hence weak equivalences by the above proposition. To conclude that f is a weak equivalence, we need to show morphisms in the class $\mathcal{R}$ are weak equivalences. We can actually show a stronger statement. To this end we introduce a particularly nice class of I-homotopy equivalences (and hence weak equivalences).

Definition 2.1.16. A morphism $i: A \to X$ is called a *right deformation retract* if there exists a morphism $r: X \to A$ and a homotopy $h: I \otimes X \to X$ such that

(1) $ri = id_A$
(2) $h_0 = id_X$ and $h_1 = ir$
(3) $h(id_I \otimes i) = \sigma \otimes i$.

A morphism $r: X \to A$ is called a *dual of a right deformation retract* if there exists a map $i: A \to X$ and a homotopy $h: I \otimes X \to X$ such that

(1) $ri = id_A$
(2) $h_0 = id_X$ and $h_1 = ir$
(3) $rh = \sigma \otimes r$.

Proposition 2.1.17. *Any map $f: X \to Y \in \mathcal{R}$ is the dual of a right deformation retract.*

Proof. We find a section $s\colon Y \to X$ via the lifting problem

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "r"] \\ Y \arrow[r, "id"] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

since the factorization system is tractable. We have a lifting problem

$$\begin{tikzcd} \partial I \otimes X \arrow[r, "(id_X \sqcup sr)"] \arrow[d] & X \arrow[d, "r"] \\ I \otimes X \arrow[r, "\sigma \otimes r"] \arrow[ur, dashed] & Y \end{tikzcd}$$

which admits a lift since the left vertical map is in $\mathcal{L}$ by exactness of the cylinder, verifying that r is the dual of a right deformation retract. □

In conclusion we have shown that whenever we have a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

in which the horizontal maps are right I-anodyne extensions and the map $X \to Y$ is in the class $\mathcal{R}$, then f is a weak equivalence. In particular, any map satisfying Simpson's description is a weak equivalence in our sense. Conversely, suppose that $f\colon A \to B$ is a weak equivalence. By the small object argument we find a right I-anodyne extension $B \to Y$ such that Y is right I-fibrant. Again by the small object argument, we factorize the composition $A \to B \to Y$ as a right I-anodyne extension followed by a right I-fibration to obtain the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d] \\ B \arrow[r] & Y. \end{tikzcd}$$

By the 2-out-of-3 property, the morphism $X \to Y$ is a weak equivalence. By construction, it is also a right I-fibration with right I-fibrant domain. Thus we need to show that right I-fibrations with right I-fibrant domain, which are weak equivalences, are in the class $\mathcal{R}$.

Lemma 2.1.18. *A right I-fibration is in $\mathcal{R}$ if and only if it is the dual of a right deformation retract.*

Proof. We have already seen in Proposition 2.1.17, that morphisms in $\mathcal{R}$ are duals of deformation retracts. Thus consider a right I-fibration $p\colon X \to Y$ which is also the dual of a deformation retract. We have to show that for any morphism $i\colon K \to L \in \mathcal{L}$ a solution to the lifting problem

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ L \arrow[r, "b"] \arrow[ur, dashed] & Y \end{tikzcd}$$

exists. Since $p\colon X \to Y$ is the dual of a deformation retract, we have a retraction $s\colon Y \to X$ and a homotopy $h\colon I \otimes X \to X$ from the identity to sp. We obtain a solution for the lifting problem

$$\begin{tikzcd} I \otimes K \sqcup_{\{1\}\otimes K} \{1\} \otimes L \arrow[r, "(h(id_I \otimes a), sb)"] \arrow[d] & X \arrow[d, "p"] \\ I \otimes L \arrow[r, "\sigma \otimes b"'] \arrow[ur, dashed, "l"] & Y \end{tikzcd}$$

since the right vertical map is a right I-anodyne extension. One checks that this solution restricts to a solution of the original lifting problem. □

Proposition 2.1.19. *A right I-fibration with right I-fibrant codomain is a weak equivalence if and only if it is in the class $\mathcal{R}$.*

Proof. Suppose $p\colon X \to Y$ is a right I-fibration with right I-fibrant codomain, which is also a weak equivalence. We will show that in this case p is the dual of a right deformation retract, hence by the above lemma we may conclude that $p \in \mathcal{R}$. Since Y is right I-fibrant, p is an I-homotopy equivalence and we find a map $t\colon Y \to X$ and a homotopy $h\colon I \otimes Y \to Y$ from id_Y to pt. Consider the lifting problem

$$\begin{tikzcd} \{1\} \otimes Y \arrow[r, "t"] \arrow[d] & X \arrow[d, "p"] \\ I \otimes Y \arrow[r, "h"'] \arrow[ur, dashed, "h'"] & Y \end{tikzcd}$$

which admits the indicated lift h' since p is a right I-fibration and the left vertical map is a right I-anodyne extension. We define $s := h'_0\colon Y \to X$. Note that X is right I-fibrant and since p is an isomorphism in the I-homotopy category and s is a right inverse, and hence inverse to p, there is a homotopy $k\colon I \otimes X \to X$ from id_X to sp. In general, k does not necessarily exhibit p as a dual of a right deformation retract, since the assumption $pk = \sigma \otimes p$ need not be satisfied. However, we may consider the lifting problem

$$\begin{tikzcd} I \otimes \partial I \otimes X \sqcup_{\{1\}\otimes \partial I \otimes X} \{1\} \otimes I \otimes X \arrow[rr, "(k, spk)\cup(\sigma\otimes sp)"] \arrow[d] & & X \arrow[d, "p"] \\ I \otimes I \otimes X \arrow[r, "id_I \otimes \sigma"'] \arrow[urr, dashed, "K"] & I \otimes X \arrow[r, "pk"'] & Y. \end{tikzcd}$$

Now define $k' := K_0\colon I \otimes X \to X$. One readily checks that

- $k'_0 = k_0 = id_X$
- $k'_1 = k_1 = sp$
- $pk' = \sigma \otimes pk_0 = \sigma \otimes p$.

□

The only thing left to show to ensure the existence of our desired model structure is that trivial cofibrations are closed under pushouts and transfinite compositions. We will in fact show that they are saturated.

Lemma 2.1.20. *A morphism in the class $\mathcal{L}$ with right I-fibrant codomain is a weak equivalence if and only if it is a right I-anodyne extension.*

Proof. We already know that right I-anodyne extensions are weak equivalences by Proposition 2.1.15. Thus, let $i\colon K \to L \in \mathcal{L}$ with right I-fibrant codomain. We factorize $i = qj$ where j is right I-anodyne and q is a right I-fibration. Then i is a weak equivalence if and only if q is. Thus if i is a weak equivalence, it follows from Proposition 2.1.19 that $q \in \mathcal{R}$. It follows from the Retract Lemma 1.1.7 that i is a retract of j, hence a right I-anodyne extension. □

Proposition 2.1.21. *Let $i\colon K \to L \in \mathcal{L}$. Then i is a weak equivalence if and only if it has the left lifting property with respect to right I-fibrations with right I-fibrant codomain.*

Proof. Consider a right I-anodyne extension $j\colon L \to L'$, where L' is right I-fibrant. If i is a weak equivalence, it follows that ji is a weak equivalence and by the above lemma is in particular a right I-anodyne extension. Now consider a diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ L \arrow[r, "f"] \arrow[d, "j"'] & Y \\ L' & \end{tikzcd}$$

where p is a right I-fibration with right I-fibrant codomain. Then there exists a lift $\phi\colon L' \to Y$ such that $\phi j = f$. We obtain the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "ji"'] & X \arrow[d, "p"] \\ L' \arrow[r, "\phi"] & Y \end{tikzcd}$$

which admits a lift since ji is right I-anodyne. This lift restricts to a lift of the original diagram.

Conversely, consider a factorization of ji given by

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "k"'] & L \arrow[d, "j"] \\ X \arrow[r, "p"] & L' \end{tikzcd}$$

where k is right I-anodyne and p a right I-fibration. It follows from the Retract Lemma that ji is a retract of k, hence a right I-anodyne extension. Thus by the 2-out-of-3 property, i is a trivial cofibration. □

Proof of Theorem 2.1.11. It follows from Proposition 2.1.17 and Proposition 2.1.19 that a morphism $f\colon A \to B$ is a weak equivalence if and only if there exists a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

in which the horizontal maps are transfinite compositions of pushouts of Λ and the right vertical map is a trivial fibration. Furthermore, Proposition 2.1.21 implies that the class of trivial cofibrations is saturated, hence Simpson's Theorem guarantees the existence our desired model structure.

Proposition 2.1.21 also implies that right I-fibrations between right I-fibrant objects are fibrations and in particular the fibrant objects are precisely the right I-fibrant ones. □

We will finish this section with the definition of an *elementary homotopical datum*. Suppose we have fixed an exact functorial cylinder $(I, \partial_0, \partial_1, \sigma)$ on C with respect to $(\mathcal{L},\mathcal{R})$. For brevity, we will only speak of *right anodyne extensions* if the cylinder is clear from the context.

Construction 2.1.22. Suppose we have a set of morphisms S. Then there is a smallest class of right anodyne extensions containing S, which may be constructed as follows.

Given any set of morphisms $T \subset \mathcal{L}$, we define the set

$$\Lambda(T) := \{\partial I \boxtimes i \mid i \in T\}.$$

We now choose a generating set M of the class $\mathcal{L}$ and define the set $\Lambda_I(S, M)$ inductively by setting

$$\Lambda_I^{0,r} := S \cup \{\partial_1 \boxtimes i \mid i \in M\}$$

and

$$\Lambda_I^{n+1,r}(S, M) := \Lambda\left(\Lambda_I^{n,r}(S, M)\right).$$

Finally, we define

$$\Lambda_I^r(S, M) := \bigcup_n \Lambda_I^{n,r}(S, M).$$

Lemma 2.1.23. *The smallest saturated class generated by $\Lambda_I^r(S, M)$ is a class of right anodyne extensions.*

Proof. Since I commutes with colimits it has a right adjoint denoted by $(\cdot)^I$. Thus, lifting problems of the form

$$\begin{tikzcd} I \otimes K \sqcup_{\{1\}\otimes K} 1 \otimes L \arrow[r] \arrow[d] & X \arrow[d] \\ I \otimes L \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

correspond to lifting problems of the form

$$\begin{tikzcd} K \arrow[r] \arrow[d] & X^I \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & X \times_Y Y^I. \end{tikzcd}$$

We show that the smallest saturated class containing $\Lambda_I^r(S, M)$ is a class of right anodyne extensions. The above correspondence shows that whenever $X \to Y$ has the right lifting property with respect to $\Lambda_I^r(S, M)$, then

$$X^I \to X \times_Y \times Y^I$$

has the right lifting property with respect to any morphism in M and hence any morphism in $\mathcal{L}$. Thus, the saturated class is closed under the operation $\partial_1 \boxtimes (\cdot)$. A similar argument shows that it is also closed under the operation $\partial I \boxtimes (\cdot)$. Conversely, it is clear that any class of right anodyne extensions which contains S is contained in the weakly saturated class generated by $\Lambda_I^r(S,M)$. □

Definition 2.1.24. An *elementary homotopical datum* consists of an exact cylinder $(I,\partial_0,\partial_1,\sigma)$ together with a set of morphisms S.

Example 2.1.25. Let A be a small category and consider the exact cylinder $I\times(\cdot)$ on $\mathrm{PSh}(A)$ as in Example 2.1.5. Consider the elementary homotopical datum given by $(I,\emptyset)$. Then the right anodyne extensions have a particularly simple description. Let M be a cellular model for $\mathrm{PSh}(A)$ and consider the set of morphisms

$$I\times K \sqcup_{\{1\}\times K} \{1\}\times L \to I\times L$$

for $K\to L\in M$. An easy calculation shows that the saturated class generated by this set is the class of right anodyne extensions associated to $(I,\emptyset)$, see also [**Cis06**, Remarque 1.3.15].

Let us consider an elementary homotopical datum given by $(I,\partial_0,\partial_1,\sigma)$ and S, which we will denote by (I,S) for brevity. Then by Lemma 2.1.23 and its dual version, we obtain a right as well as a left homotopical structure and hence by Theorems 2.1.11 and 2.1.12 two model structures on the category of presheaves on C.

Definition 2.1.26. Let (I,S) be an elementary homotopical structure and denote by $r(I,S)$ the right homotopical structure generated by it and by $l(I,S)$ the left homotopical structure generated by it. We will call the model structure induced by $r(I,S)$ the *Contravariant model structure generated by* (I,S) and the model structure induced by $l(I,S)$ the *Covariant model structure generated by* (I,S). An equivalence in the Contravariant model structure is called a *contravariant equivalence* and an equivalence in the Covariant model structure is called a *covariant equivalence*.

2.2. Abstract cofinality

Suppose we have an elementary homotopical datum. In the previous section we have established two model structures arising from such a datum, the Covariant and the Contravariant model structure. In this section, we will discuss the notions of final and cofinal maps, which arise very naturally in this setting. To this end, we will consider Co- and Contravariant model structures for *families*.

Construction 2.2.1. Suppose we have an object $A\in C$. Then we have a weak factorization system $(\mathcal{L}_A,\mathcal{R}_A)$ on C/A. The cylinder I on C induces a cylinder I_A on the category of C/A whose action on objects $p\colon X\to A$ is given by the composition

$$I\otimes X \xrightarrow{\sigma} X \xrightarrow{p} A.$$

Starting with a class of right I-anodyne extensions $An^r(I)$, it is easy to check that the class $An^r(I_A)$ of those morphisms in C/A, whose underlying maps in C are right I-anodyne extension, defines a class of right I_A-anodyne extensions.

Thus applying Theorem 2.1.11, we obtain a relative version.

THEOREM 2.2.2. *There exists a unique model structure on the category C/A with cofibrations the class $\mathcal{L}_A$ and fibrant objects the right I-fibrations with target A. Dually, there exists a unique model structure on C/A with cofibrations the class $\mathcal{L}_A$ and fibrant objects the left fibrations with target A.*

Now fix an elementary homotopical datum $\mathcal{I} := (I, S)$. By the above theorem, we obtain for any object $A \in C$ a Contravariant and Covariant model structure on the category of C/A induced by $\mathcal{I}$.

DEFINITION 2.2.3. A morphism $f: X \to Y$ is called *$\mathcal{I}$-final* if for all objects A and all morphisms $p: Y \to A$ the induced morphism

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p\circ f"'] & Y \arrow[d, "p"] \\ & A \end{tikzcd}$$

is a contravariant equivalence in the category C/A. Dually, it is called *$\mathcal{I}$-cofinal* if the above morphism is a covariant equivalence in the category C/A.

Thus, the $\mathcal{I}$-final (resp. $\mathcal{I}$-cofinal) maps are precisely those, which are equivalences in the contravariant (resp. covariant) model structures for *all* families. We again drop the elementary homotopical datum from the notation for brevity. Thus, *final* will always mean $\mathcal{I}$-final.

LEMMA 2.2.4. *A morphism in the class $\mathcal{L}$ is final if and only if it is a right anodyne extension. A right fibration is final if and only if it is in the class $\mathcal{R}$.*

PROOF. By construction it is clear that right anodyne extensions are final. Conversely, if $i: X \to Y$ is final then it is in particular a trivial cofibration with fibrant domain in Contravariant model structure on C/Y. By Lemma 2.1.20, it is right anodyne.

It is also clear that any map in the class $\mathcal{R}$ is final. Conversely, if $p: X \to Y$ is a right fibration which is also final, then it is a right fibration with fibrant codomain in the Contravariant model structure on C/Y which is a weak equivalence. By Proposition 2.1.19 it is in the class $\mathcal{R}$. □

PROPOSITION 2.2.5. *The class of final maps satisfies the right cancellation property.*

PROOF. Suppose we have a composable sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and assume that f is final. Consider any morphism $Z \to A$, and consider g and gf as morphisms in C/A. Then by the 2-out-of-3 property of weak equivalences is is clear that g is a Contravariant equivalence in C/A if and only if gf is. Thus the final maps satisfy the right cancellation property. □

The next Proposition shows, that the $\mathcal{I}$-final and $\mathcal{I}$-cofinal maps are completely determined by the elementary homotopical datum $\mathcal{I}$.

Proposition 2.2.6. *A map is final if and only if it can be factorized as a right anodyne extension followed by a map in the class $\mathcal{R}$.*

Proof. By the right cancellation property the class of final maps is closed under composition. By Lemma 2.2.4 both right anodyne extensions and maps in the class $\mathcal{R}$ are final, hence their composition is final. Conversely, suppose f is a final map. We may factorize $f = pi$ with i a right anodyne extension and p a right fibration. By the right cancellation property, p is final thus by Lemma 2.2.4 $p \in \mathcal{R}$. □

Definition 2.2.7. Let $p\colon X \to Y$ be a morphism and consider a diagram of the form

$$\begin{tikzcd} A' \arrow[r, "j"] \arrow[d] & B' \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "i"] & B \arrow[r] & Y \end{tikzcd}$$

in which the squares are cartesian. Then p is called *$\mathcal{I}$-proper* if j is $\mathcal{I}$-final whenever i is $\mathcal{I}$-final. Dually, p is called *$\mathcal{I}$-smooth* if j is $\mathcal{I}$-cofinal whenever i is $\mathcal{I}$-cofinal.

In some cases, right (resp. left) fibrations provide examples of smooth (resp. proper) maps. Although it is not true general, that they are smooth (resp. proper), there is a particular class of left (resp. right) anodyne extensions, which are always preserved by pullback along a right (resp. left) fibration.

Lemma 2.2.8. *Any right deformation retract is a right anodyne extension and any left deformation retract is a left anodyne extension.*

Proof. We only show the case of a right deformation retract. Thus, let $i\colon K \to L$ be a right deformation retract with retraction $r\colon L \to K$ and homotopy $h\colon I \otimes L \to L$ from id_Y to ir which is constant on K. We obtain a commutative diagram

$$\begin{tikzcd} K \arrow[r, "\partial_0"] \arrow[d, "i"'] & I \otimes K \sqcup_{\{1\}\otimes K} \{1\} \otimes L \arrow[r, "(\sigma, r)"] \arrow[d] & K \arrow[d, "i"] \\ L \arrow[r, "\partial_0"] & I \otimes L \arrow[r, "h"] & L \end{tikzcd}$$

exhibiting i as a retract of a right anodyne extension. □

Proposition 2.2.9. *Consider a Cartesian square*

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r, "i"] & Y. \end{tikzcd}$$

If i is a right deformation retract and p is a left fibration, then j is a right deformation retract. Dually, if i is a left deformation retract and p is a right fibration, then j is a left deformation retract.

Proof. We only show the case when p is a left fibration and i is a right deformation retract. Suppose we have a retraction $r\colon Y \to B$ and a homotopy $h\colon I \otimes Y \to Y$ from id_Y to ir which is constant on B. We obtain a solution k in the following lifting problem

$$\begin{tikzcd} I \otimes A \sqcup_{\{0\}\otimes A} \{0\} \otimes X \arrow[r, "(\sigma\otimes j, id_X)"] \arrow[d] & X \arrow[d, "p"] \\ I \otimes X \arrow[r, "h(id_I \otimes p)"] \arrow[ur, dashed, "k"] & Y \end{tikzcd}$$

since p is a left fibration and the left vertical map is left anodyne. We claim that k exhibits j as a deformation retract. We have a map

$$X \xrightarrow{p} Y \xrightarrow{r} B \xrightarrow{i} Y$$

and also

$$X \xrightarrow{k_1} X \xrightarrow{p} Y.$$

Since $pk_1 = irp$ we get a unique map $s\colon X \to A$. Now we have $jsj = k_1 j = j$ and $qsj = rpj = q$ hence $sj = id_A$. Finally one checks that the homotopy k satisfies the right properties. □

2.3. First examples

We will consider two examples in this section. The Joyal model structure and the Co- and Contravariant model structure for simplicial sets. Both are originally due to Joyal, see for example [**Joy08a**], and are obtained using purely combinatorial methods. Lurie gives an alternative construction of these model structures in [**Lur09**], using a comparison to simplicial categories. Another approach is in Cisinski's book [**Cis**], using his theory of *anodyne extensions* which is our starting point. The construction of the Joyal model structure presented here is essentially the same as in [**Cis**]. Although we are using our theory of Contravariant model structures, it turns out that the right anodyne and left anodyne extensions coincide and hence are anodyne extensions in the sense of Cisinski. Our construction of the Contravariant model structure for simplicial sets is slightly more direct than Cisinski's construction in [**Cis**], and we hope this illustrates the use of 'directional homotopies'.

In this section we consider the category of simplicial sets **sSet** with the tractable weak factorization system given by (Mono,Triv) (see Example 1.1.9).

The Joyal model structure. Let J be the nerve of the category with two objects 0 and 1 and a unique isomorphism between them. Clearly, the inclusion of the objects is disjoint, hence by Example 2.1.5 we obtain an exact cylinder

$$J \times (\cdot)\colon \mathbf{sSet} \to \mathbf{sSet}$$

with respect to (Mono,Triv). The cylinder J comes with an extra structure, namely an involution which is defined as follows. We have a functor

$$\tau\colon J \to J$$

which exchanges the objects and sends a morphism to its inverse. It is then clear that $\tau^2 = id$. Moreover, the diagrams

$$\begin{tikzcd} & id \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\ J \arrow[rr, "\tau"] & & J, \end{tikzcd} \qquad \begin{tikzcd} & id \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_0"] & \\ J \arrow[rr, "\tau"] & & J \end{tikzcd}$$

commute.

We consider the elementary homotopical datum given by the pair $\mathcal{J} := (J, \mathrm{InnHorn})$, where InnHorn is the set of inner horn inclusions

$$\Lambda^n_k \to \Delta^n$$

for $n \geq 2$ and $0 < k < n$.

Definition 2.3.1. A *categorical anodyne extension* is a right anodyne extension with respect to the elementary homotopical datum $\mathcal{J}$. The *Joyal model structure* is the Contravariant model structure with respect to $\mathcal{J}$. We will refer to the Contravariant equivalences as *weak categorical equivalences* and to the right fibrations as *isofibrations*.

We can simplify the description of categorical anodyne extensions. We first have the following Proposition.

Proposition 2.3.2. *The following three sets of maps generate the same saturated class.*

- $\Lambda^n_k \to \Delta^n$ *where* $n \geq 2$ *and* $0 < k < n$.
- $\Lambda^2 \times \Delta^n \cup \Delta^2 \times \partial\Delta^n \to \Delta^2 \times \Delta^n$ *where* $n \geq 0$.
- $\Lambda^2 \times L \cup \Delta^2 \times K \to \Delta^2 \times L$ *where* $K \to L$ *is a monomorphism.*

Proof. This is classical and due to Joyal, see for example [**Lur09**, Proposition 2.3.2.1] or [**Cis**, Proposition 3.2.3]. □

Corollary 2.3.3. *For any inner anodyne extension* $A \to B$ *and any monomorphism* $K \to L$, *the induced map*

$$A \times L \cup B \times K \to B \times L$$

is inner anodyne.

Corollary 2.3.4. *The categorical anodyne extensions are generated as a saturated class by the inner horn inclusions and the set of maps*

$$J \times \partial\Delta^n \cup \{1\} \times \Delta^n \to J \times \Delta^n$$

for $n \geq 0$.

Proof. By the above corollary, for any inner anodyne map $i\colon A \to B$, the induced maps $\partial_1 \boxtimes i$ and $\partial J \boxtimes i$ are inner anodyne. The assertion then follows from Example 2.1.25. □

Because of the involution, this model structure is in fact also a Covariant model structure as the next lemma shows.

Lemma 2.3.5. *Any left anodyne extension with respect to* $\mathcal{J}$ *is a right anodyne extension.*

PROOF. Since left and right anodyne extensions are both inner anodyne, it suffices to show that for any $n \geq 0$, the map

$$J \times \partial\Delta^n \cup \{0\} \times \Delta^n \to J \times \Delta^n$$

is right anodyne. Since the class of right anodyne extensions is saturated and satisfies the right cancellation property, Proposition 2.2.6, it suffices to show that for any simplicial set K, the map

$$\{0\} \times K \to J \times K$$

is right anodyne. Consider the diagram

$$\begin{tikzcd} \{0\} \times K \arrow[r, "id"] \arrow[d] & \{1\} \times K \arrow[r, "id"] \arrow[d] & \{0\} \times K \arrow[d] \\ J \times K \arrow[r, "\tau \times id"] & J \times K \arrow[r, "\tau \times id"] & J \times K. \end{tikzcd}$$

This commutes and verifies that $\{0\} \times K \to J \times K$ is a retract of $\{1\} \times K \to J \times K$, which is right anodyne. □

In conclusion, the left anodynes and right anodynes with respect to $\mathcal{J}$ coincide. Thus, in this case we do in fact recover the classical Joyal model structure. The isofibrations between ∞-categories admit an easier description.

THEOREM 2.3.6. *Let $p\colon X \to Y$ be an inner fibration between ∞-categories. Then p is an isofibration if and only if for all diagrams of the form*

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ J \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a lift as indicated.

PROOF. This is a consequence of [**Cis**, Corollary 3.5.13]. □

COROLLARY 2.3.7. *A simplicial set is fibrant in the Joyal model structure if and only if it is an ∞-category.*

Finally, it will be important for us later on to also consider the Joyal model structure in families. Let A be a simplicial set. Using Theorem 2.2.2 we find the following.

THEOREM 2.3.8. *There exists a unique model structure on* **sSet**/A *where the cofibrations are the monomorphisms and the fibrant objects are the isofibrations with target A.*

DEFINITION 2.3.9. We will denote this model category by **IsoFib**(A).

The Contravariant model structure for simplicial sets. Consider the representable simplicial set Δ^1. Again, the inclusion of the endpoints is disjoint, hence by Example 2.1.5 we obtain the exact cylinder

$$\Delta^1 \times (\cdot)\colon \mathbf{sSet} \to \mathbf{sSet}$$

with respect to (Mono,Triv). Consider the elementary homotopical datum $\mathcal{I} := (\Delta^1, \emptyset)$. By Example 2.1.25 the right $\mathcal{I}$-anodyne extensions are precisely the saturated class generated by

$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \to \Delta^1 \times \Delta^n$$

for $n \geq 0$. In fact, this is a familiar class.

Lemma 2.3.10. *The following sets of morphisms generate the same saturated class.*

- $\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \to \Delta^1 \times \Delta^n$ *for* $n \geq 0$,
- $\Lambda^n_k \to \Delta^n$ *for* $0 < k \leq n$.

Proof. See for example [**Cis**, Lemma 3.1.3]. □

In other words, our right $\mathcal{I}$-anodyne extensions are precisely the right anodyne extensions of simplicial sets introduced earlier in this thesis. We again obtain Covariant and Contravariant model structures for the elementary homotopical datum given by $\mathcal{I}$. In this case, the model structures in families will be important for us. Applying Theorem 2.2.2 we obtain the following.

Theorem 2.3.11. *Let A be a simplicial set. There exists a unique model structure on* **sSet**/A *with cofibrations the monomorphisms and fibrant objects the right fibrations of simplicial sets with target A. Dually, there exists a unique model structure on* **sSet**/A *with cofibrations the monomorphisms and fibrant objects the left fibrations of simplicial sets with target A.*

Definition 2.3.12. We will denote the Contravariant model structure above by **RFib**(A) and the Covariant model structure by **LFib**(**A**).

Remark 2.3.13. It is possible to construct the Co- and Contravariant model structure using anodyne extensions, see Remark 2.1.8, instead of left and right ones. We can consider the cylinder $J \times (\cdot)$ on simplicial sets and the elementary homotopical datum given by J and the outer horn inclusions

$$\Lambda^n_k \to \Delta^n$$

where $0 < k \leq n$. One then has to check that the class of anodyne extensions associated to this elementary homotopical datum is indeed the class of right anodyne extensions. This is carried out in [**Cis**, Chapter 4].

The previous section shows that we obtain abstractly a notion of final and cofinal maps. We translate this to the following definition.

Definition 2.3.14. A map of simplicial sets $f\colon X \to Y$ is *final* if and only if for all simplicial sets A and all maps $Y \to A$, the morphism f is a Contravariant equivalence in **RFib**(A). Dually, the map f is called *cofinal* if it induces a Covariant equivalence in **LFib**(A).

We collect some useful results from [**Cis**], which we will need later. These results can also be found in [**Lur09**], though in general they are obtained by very different methods.

Definition 2.3.15. Let X be a simplicial set. Then an object $x \in X$ is *final* if the associated map $\Delta^0 \xrightarrow{x} X$ is final. Dually, $x \in X$ is *initial* if the associated map $\Delta^0 \xrightarrow{x} X$ is cofinal.

An example of initial objects is given by the following.

LEMMA 2.3.16. *For any simplicial set X and any object $x \in X$, the object id_x is initial in $x\backslash X$.*

PROOF. This is [**Cis**, Corollary 4.3.8]. □

REMARK 2.3.17. In particular this shows that for an ∞-category C and an object $x \in C$, a canonical fibrant replacement of the map $x\colon \Delta^0 \to C$ in the Covariant model structure can be computed as

$$\Delta^0 \to x\backslash C \to C.$$

As a consequence, we have the following characterization of initial objects in an ∞-category.

THEOREM 2.3.18. *An object $x \in C$ in an ∞-category is initial if and only if the map $x\backslash C \to C$ is a trivial fibration.*

PROOF. We have the diagram

$$\begin{tikzcd} & x\backslash C \arrow[d] \\ \Delta^0 \arrow[ur] \arrow[r, "x"] & C \end{tikzcd}$$

in which the vertical map is a left fibration since C is an ∞-category. The diagonal map is always cofinal. Hence the vertical map is cofinal if and only if the bottom map is cofinal. The fact that a left fibration is cofinal if and only if it is a trivial fibration, Proposition 2.2.4, proves the theorem. □

An important result on the recognition of final functors between ∞-categories is Quillen's Theorem A, the ∞-categorical version is originally due to Joyal.

THEOREM 2.3.19. *Let $F\colon C \to D$ be a functor between ∞-categories. Then F is final if and only if for all objects $d \in D$, the ∞-category $d\backslash F$ is weakly contractible.*

PROOF. See [**Cis**, Proposition 4.3.30] or [**Lur09**, Theorem 4.1.3.1]. □

The behavior of final functors with respect to slicing is given by the following Theorem.

THEOREM 2.3.20. *Suppose $v\colon A \to B$ is final and let $p\colon B \to C$ be a map where C is an ∞-category. Then the induced map $v\backslash C \to pv\backslash C$ is an equivalence of ∞-categories.*

PROOF. This is [**Lur09**, Proposition 4.1.1.7]. □

Finally, we may use the notion of finality to define limits and colimits in an ∞-category.

DEFINITION 2.3.21. Let C be an ∞-category and $p\colon K \to C$ be a map. A *limit* of p is a final object in the ∞-category C/p. Dually, a *colimit* of p is an initial object in $p\backslash C$.

As a consequence, we find that final maps induce the same colimits.

Proposition 2.3.22. *Suppose $v\colon A \to B$ is a final map. Let $p\colon B \to C$ be a map where C is an ∞-category. Then p admits a colimit if and only if pv admits a colimit. Moreover their colimits are equivalent.*

Proof. The induced map $p\backslash C \to pv\backslash C$ is an equivalence of ∞-categories. Thus it preserves and reflects initial objects. $\square$

We may also consider smooth and proper maps in this setting. Recall from the previous section that a map $p\colon X \to Y$ is proper if and only if for any diagram

$$\begin{tikzcd} A' \arrow[r, "w"] \arrow[d] & B' \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "v"] & B \arrow[r] & Y \end{tikzcd}$$

in which the squares are pullbacks, the map w is final if v is final. Dually, p is called smooth if the map w is cofinal whenever v is cofinal.

A good supply of smooth and proper morphisms is given by the following.

Proposition 2.3.23. *Left fibrations are proper and right fibrations are smooth.*

Proof. We only show the case of left fibrations. Since left fibrations are closed under pullback, it suffices to show that for any cartesian square

$$\begin{tikzcd} A' \arrow[r, "w"] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "v"] & Y \end{tikzcd}$$

in which v is final and p is a left fibration, the map w is final. Since any final map can be factorized as a right anodyne map followed by a trivial fibration and trivial fibrations are closed under pullback, it suffices to show that w is right anodyne whenever v is right anodyne. Let $\mathcal{A}$ be the class of morphisms whose pullbacks are right anodyne. Then this class is saturated and satisfies the right cancellation property, since this is true for right anodyne extensions. Thus it suffices to show that $\mathcal{A}$ contains the class of right anodyne extensions, and hence it suffices to show the assertion when v is an element of the generating set of right anodyne extension. We have already seen that the right anodyne extensions are the saturated class generated by

$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \to \Delta^1 \times \Delta^n$$

for $n \geq 0$. By the right cancellation property, it suffices to show the assertion for morphisms of the form

$$\{1\} \times K \to \Delta^1 \times K$$

for any simplicial set K. We now observe that the above map is a right deformation retract, hence by Proposition 2.2.9 its pullback along any left fibration is a right deformation retract, hence right anodyne. $\square$

Remark 2.3.24. As we shall see later, coCartesian fibrations are proper and Cartesian fibrations are smooth, which generalizes the above proposition.

Finally, we record the following important theorem for later use.

THEOREM 2.3.25. *Suppose we have a Cartesian square*

$$\begin{tikzcd} A' \arrow[r, "w"] \arrow[d] & X \arrow[d, "p"] \\ A \arrow[r, "v"] & Y \end{tikzcd}$$

in which v is a weak categorical equivalence and p is a left or right fibration. Then w is a weak categorical equivalence.

PROOF. See [**Cis**, Proposition 5.3.5]. □

CHAPTER 3

The universal coCartesian fibration

The goal of this chapter is to prove a correspondence between coCartesian fibrations $X \to A$ and maps of simplicial sets $A \to \mathcal{Q}$, where $\mathcal{Q}$ is an ∞-category whose objects are themselves small ∞-categories. In fact, we construct $\mathcal{Q}$ together with a *universal coCartesian fibration* $\mathcal{Q}_\bullet \to \mathcal{Q}$ such that any coCartesian fibration fits into a *strict* pullback diagram of simplicial sets

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{Q}_\bullet \arrow[d] \\ A \arrow[r] & \mathcal{Q} \end{tikzcd}$$

showing that each coCartesian fibration is classified by a map to $\mathcal{Q}$.

This has been proven by Lurie using his machinery of straightening and unstraightening [**Lur09**, 3.3.2]. Though powerful, the straightening and unstraightening constructions are rather complex and rely on an external model for higher categories in the form of categories enriched in (marked) simplicial sets. Using these constructions, he derives several useful properties of (co)Cartesian fibrations.

Our approach is opposite to the one of Lurie. We will construct the universal coCartesian fibration *intrinsically* using only the language of (marked) simplicial sets. We will take as input only basic properties of coCartesian fibrations, which are all combinatorial in nature and don't rely on Lurie's straightening functor. Moreover, we show that any coCartesian fibration can be realized as a set-theoretical pullback instead of a homotopy pullback as in [**Lur09**]. Our methods can be thought of as a continuation of Cisinski's method in [**Cis**] and in fact we will follow the same pattern of proof.

This chapter is organized as follows. In the first section we will apply our theory of Contravariant model structures to construct a model structure for Cartesian fibrations. In the second section, we will use our general theory to study smoothness properties of Cartesian fibrations. As a main application we will prove invariance of Contravariant model structures if the map on base marked simplicial sets is obtained from an inner horn inclusion. The third section takes a detour to minimal Cartesian fibrations, a technical tool to prove that the base of the universal Cartesian fibration is in fact an ∞-category. Finally, in the last section we construct the universal coCartesian fibration.

3.1. (co)Cartesian model structures

In this section we construct a model structure in which the fibrant objects will be precisely Cartesian fibrations. This construction uses the category of *marked simplicial sets*. The need of using marked simplicial sets can be explained by the following consideration. Suppose we have an inner fibration $X \to A$ and we would like to turn this into a Cartesian fibration, i.e. to take a fibrant replacement. However, it is not clear which morphism in X should be Cartesian and there is no obvious choice. The category of marked simplicial sets introduces extra structure on a simplicial set in the form of a distinguished subset of 1-simplices, which one might think of those morphisms which become Cartesian in a fibrant replacement.

The model structure has first been constructed by Lurie [**Lur09**, 3.1.3] and called the *Cartesian model structure* there. We give a new proof of the existence of this model structure. In fact, we find that this model structure is another instance of a Contravariant model structure. We also obtain the *coCartesian model structure* by duality, which is an instance of a Covariant model structure.

Definition 3.1.1. We denote by Δ^+ the category obtained from the simplex category by adding an object $[1^+]$ and a unique factorization

$$[1] \to [1^+] \to [0].$$

We denote by $\mathbf{sSet}^+$ the category of presheaves on Δ^+. This category admits a Grothendieck topology whose only non-trivial cover is given by the morphism $[1] \to [1^+]$. We denote by $\mathbf{mSet}$ the full subcategory of separated presheaves with respect to this topology and refer to it as the category of *marked simplicial sets*.

Remark 3.1.2. This description of marked simplicial sets implies that the category of marked simplicial sets inherits several nice properties from its ambient presheaf topos. For example it is cartesian closed, colimits are universal and coproducts are disjoint.

We identify the objects of $\mathbf{mSet}$ with pairs (K, E_K) where K is a simplicial set and $E_K \subseteq K_1$ is a subset of the set of 1-simplices of K containing all the degenerate edges, called the *marked edges*. A morphism of marked simplicial sets is thus a morphism of simplicial sets respecting the marked edges. There is a forgetful functor

$$\mathbf{mSet} \to \mathbf{sSet}$$

which admits a left adjoint denoted by $(\cdot)^\flat$ as well as a right adjoint denoted by $(\cdot)^\sharp$. Given a simplicial set K, the marked simplicial set $K^\flat$ has precisely the degenerate 1-simplices marked, while the marked simplicial set $K^\sharp$ has all 1-simplices marked.

Consider the functor

$$(\Delta^1)^\sharp \times (\cdot)\colon \mathbf{sSet}^+ \to \mathbf{sSet}^+$$

We claim that this is an exact cylinder for marked simplicial sets. We first observe that the class of morphisms of marked simplicial sets, whose underlying map of simplicial sets is a monomorphism, is a left class in a tractable weak factorization system.

Lemma 3.1.3. *Let $(K, E_K) \to (L, E_L)$ be a morphism of marked simplicial sets such that the map $K \to L$ is a monomorphism of simplicial sets. Then it is contained in the saturated class $\mathcal{L}$ generated by the set of morphisms*

- $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$ *for* $n \geq 0$ *and*
- $(\Delta^1)^\flat \to (\Delta^1)^\sharp$.

Proof. Since $K^\flat \to L^\flat$ is a monomorphism, it is contained in the weakly saturated class generated by $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$ for $n \geq 0$. This also implies that we can identify E_K with a subset $E_K \subseteq E_L$. We have a pushout diagram

$$\begin{tikzcd} \bigsqcup_{E_K}(\Delta^1)^\flat \arrow[r] \arrow[d] & K^\flat \arrow[d] \\ \bigsqcup_{E_K}(\Delta^1)^\sharp \arrow[r] & (K,E_K) \end{tikzcd}$$

and a pushout diagram

$$\begin{tikzcd} K^\flat \arrow[r] \arrow[d] & (K,E_K) \arrow[d] \\ L^\flat \arrow[r] & (L,E_K). \end{tikzcd}$$

Thus the morphism $(K,E_K) \to (L,E_K)$ is in the class $\mathcal{L}$. Finally, we have a pushout

$$\begin{tikzcd} \bigsqcup_{E_L \setminus E_K}(\Delta^1)^\flat \arrow[r] \arrow[d] & (L,E_K) \arrow[d] \\ \bigsqcup_{E_L \setminus E_K}(\Delta^1)^\sharp \arrow[r] & (L,E_L). \end{tikzcd}$$

Thus, the composition $(K,E_K) \to (L,E_K) \to (L,E_L)$ is in the class $\mathcal{L}$. $\square$

Thus, by the small object argument we have a factorization system on **mSet** given by $(\mathcal{L},\mathcal{R})$ where $\mathcal{R}$ is defined as the class having the right lifting property with respect to the generators of $\mathcal{L}$.

Lemma 3.1.4. *The cylinder*

$$(\Delta^1)^\sharp \times (\cdot)\colon \mathbf{mSet} \to \mathbf{mSet}$$

is an exact cylinder with respect to the factorization system $(\mathcal{L},\mathcal{R})$ *on* **mSet**.

Proof. It is clear that $(\Delta^1)^\sharp \times (\cdot)$ preserves colimits, since **mSet** is cartesian closed. Let $i\colon (K,E_K) \to (L,E_L) \in \mathcal{L}$. Consider the commutative diagram

$$\begin{tikzcd} \partial(\Delta^1)^\sharp \times (K,E_K) \arrow[r] \arrow[d] & \partial(\Delta^1)^\sharp \times (L,E_L) \arrow[d] \arrow[ddr, bend left] & \\ (\Delta^1)^\sharp \times (K,E_K) \arrow[r] \arrow[drr, bend right] & (\Delta^1)^\sharp \times (K,E_K) \cup \partial(\Delta^1)^\sharp \times (L,E_L) \arrow[dr] & \\ & & (\Delta^1)^\sharp \times (L,E_L) \end{tikzcd}$$

in which the square is a pushout. We need to show that the map $\partial(\Delta^1)^\sharp \boxtimes i$ is in the class $\mathcal{L}$, i.e. its underlying map of simplicial sets is a monomorphism. But the forgetful functor to simplicial sets is a left adjoint, hence the above diagram gives a diagram in simplicial sets in which the square is a pushout. Since the cylinder $\Delta^1 \times (\cdot)$ is exact, it follows that the underlying map of $\partial(\Delta^1)^\sharp \boxtimes i$, which is just $\partial\Delta^1 \boxtimes i$, is a monomorphism. □

Definition 3.1.5. Let $\mathcal{I}^+$ be the elementary homotopical datum associated to the exact cylinder $(\Delta^1)^\sharp \times (\cdot)$ with respect to $(\mathcal{L}, \mathcal{R})$, and the set of maps defined by

- $(\Lambda^n_k)^\flat \to (\Delta^n)^\flat$ for $n \geq 2$ and $0 < k < n$,
- $J^\flat \to J^\sharp$ where J is the nerve of the free walking isomorphism.

Now let (A, E_A) be a marked simplicial set. By Construction 2.2.1 we obtain an elementary homotopical datum $\mathcal{I}^+_{(A,E_A)}$. Thus applying Theorem 2.2.2 we obtain a Contravariant and Covariant model structure on **mSet**.

Theorem 3.1.6. *For any marked simplicial set (A, E_A), there is a Contravariant and Covariant model structure on* **mSet**$/(A, E_A)$ *induced by* $\mathcal{I}^+_{(A,E_A)}$.

Definition 3.1.7. We will call the Contravariant model structure on **mSet**$/(A, E_A)$ the *Cartesian model structure*. We denote this model category by **Cart**(A, E_A). To distinguish the right anodyne extensions and right fibrations from their counterparts in simplicial sets, we will refer to them as *marked right anodyne extensions* and *marked right fibrations.* Furthermore, we refer to the weak equivalences as *Cartesian equivalences*.

Definition 3.1.8. Dually, we will call the Covariant model structure on **mSet**$/(A, E_A)$ the *coCartesian model structure*. We denote this model category by **coCart**(A, E_A). We will refer to the left anodyne extensions and left fibrations as *marked left anodyne extensions* and *marked left fibrations.* Furthermore, we refer to the weak equivalences as *coCartesian equivalences*.

The rest of this section only considers the Cartesian model structure. The associated statements for the coCartesian model structure easily follow by duality.

We would like to have a finer control on the marked right anodyne extensions and marked right fibrations. To this end we first construct more explicit generators for the marked right anodyne extensions.

Definition 3.1.9. We define $\mathcal{A}$ to be the smallest saturated class containing the morphisms

(A1) $(\Lambda^n_k)^\flat \to (\Delta^n)^\flat$ for $n \geq 2$ and $0 < k < n$,
(A2) $J^\flat \to J^\sharp$,
(B1) $(\Delta^1)^\sharp \times (\Delta^1)^\flat \cup \{1\} \times (\Delta^1)^\sharp \to (\Delta^1)^\sharp \times (\Delta^1)^\sharp$,
(B2) $(\Delta^1)^\sharp \times (\partial\Delta^n)^\flat \cup \{1\} \times (\Delta^n)^\flat \to (\Delta^1)^\sharp \times (\Delta^n)^\flat$.

Lemma 3.1.10. *For all monomorphisms $K \to L$ and all $A \to B \in \mathcal{A}$ the morphism*

$$A \times L \cup B \times K \to L \times B$$

is also in $\mathcal{A}$.

Proof. It suffices to show this for the generators (A1) and (A2). Recall that the monomorphisms in **mSet** are generated by the morphisms

(1) $(\Delta^1)^\flat \to (\Delta^1)^\sharp$
(2) $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$.

We observe that the pushout product of (A1) and (1) as well as the pushout product of (A2) and (2) will yield isomorphisms. The pushout product of (A1) and (2) is an inner anodyne extension of simplicial sets and hence in $\mathcal{A}$. It remains to show that the pushout product

$$(\Delta^1)^\flat \times J^\sharp \cup (\Delta^1)^\sharp \times J^\flat \to (\Delta^1)^\sharp \times J^\sharp$$

is in $\mathcal{A}$. We observe that this map is an iterated pushout of maps in the class (B1). □

Lemma 3.1.11. *The class $\mathcal{A}$ is the class of marked right anodyne extensions.*

Proof. It is clear that the class of marked right anodyne extensions contains the class $\mathcal{A}$. Conversely, Construction 2.1.22 gave an explicit generating set for marked right anodyne extensions. Recall that this set of generators was constructed inductively and in our situation this takes the following form. The starting set is given by the set

$$(A1) \cup (A2) \cup \{\partial_1 \boxtimes i \mid i \in (B1) \cup (B2)\}.$$

We observe that the morphisms $\partial_1 \boxtimes i$ above are all in $\mathcal{A}$ by Lemma 3.1.10, hence the above set is contained in $\mathcal{A}$. To finish the proof we observe that, in the notation of Construction 2.1.22, we have $\Lambda(\mathcal{A}) \subseteq \mathcal{A}$ again by Lemma 3.1.10. Thus, any morphism in the generating set for marked right anodyne extensions is in $\mathcal{A}$ and hence $\mathcal{A}$ contains the marked right anodyne extensions. □

Lemma 3.1.12. *For any ∞-groupoid K, the morphism $K^\flat \to K^\sharp$ is a marked right anodyne extension.*

Proof. We have a pushout diagram

$$\begin{tikzcd} \sqcup J^\flat \arrow[r] \arrow[d] & K^\flat \arrow[d] \\ \sqcup J^\sharp \arrow[r] & K^\sharp \end{tikzcd}$$

where the coproduct is taken over all possible maps $J \to K$. □

The following proposition characterizes the marked right fibrations of **mSet** in the Cartesian model structure. Its proof is adapted from [**Lur09**, Proposition 3.1.1.6].

Proposition 3.1.13. *Let $p\colon (X, E_X) \to (A, E_A)$ be a morphism of marked simplicial sets. Then p is a marked right fibration if and only if the following conditions hold.*

(1) *The underlying map of simplicial sets is an inner fibration.*
(2) *For any $y \in X$ and any marked $\overline{f}\colon x \to p(y) \in E_A$, there exists a marked edge $f \in E_X$ such that $p(f) = \overline{f}$.*

(3) *An edge $f\colon \Delta^1 \to X$ is marked if and only if $p(f) \in E_A$ and f is p-Cartesian.*

Proof. We first show the 'only if' direction. Thus suppose the map $p\colon (X, E_X) \to (A, E_A)$ is a marked right fibration. Hence it satisfies the right lifting property with respect to the generators of Definition 3.1.9. The right lifting property with respect to (A1) implies that p is an inner fibration. The right lifting property with respect to (B2) for $n = 0$ implies that over each marked edge of the form $\overline{f}\colon x \to p(y) \in E_A$ there exists a marked edge $f \in E_X$ such that $p(f) = \overline{f}$. Moreover, the right lifting property with respect to (B2) for $n \geq 1$ shows that every marked edge is p-Cartesian by Proposition 1.2.13. It remains to show that an edge is marked only if it is p-Cartesian and its image is marked.

Suppose we have an edge $f\colon x \to y$ such that that $p(f)$ is marked and f is p-Cartesian. We have already seen that there exists a marked edge $f'\colon x' \to y$ such that $p(f') = p(f)$ and f' is p-Cartesian. In particular, we find a 2-simplex in X of the form

$$\begin{tikzcd} & x' \arrow[d, "f'"] \\ x \arrow[ur, "\alpha"] \arrow[r, "f"'] & y. \end{tikzcd}$$

Since f is also p-Cartesian, it follows by the right cancellation property of Cartesian edges, Lemma 1.2.17, that α is p-Cartesian. In particular, the edge α defines an equivalence in the fiber $X_{p(x)}$, which is an ∞-category. Consider the maximal ∞-groupoid $k(X_{p(x)})$. Since the map

$$X_{p(x)} \to *$$

is a marked right fibration, it has the right lifting property with respect to the map

$$k(X_{p(x)})^\flat \to k(X_{p(x)})^\sharp,$$

which implies that every equivalence of $k(X_{p(x)})$ and thus in particular α is marked. By the right lifting property with respect to (B1), it follows that f is also marked.

Now assume that $p\colon (X, E_X) \to (A, E_A)$ satisfies the assumptions of the proposition. We show that p is a marked right fibration. Thus we need to show the right lifting property against the generators of Definition 3.1.9. The right lifting property against (A1) follows since p is an inner fibration.

To show the right lifting property against (A2) it suffices to consider the case where $(A, E_A) = J^\sharp$. In this case p is an inner fibration over a Kan complex, hence the p-Cartesian edges are precisely the equivalences by Lemma 1.2.16, thus p has the right lifting property against (A2).

The right lifting property property against (B1) follows immediately from assumptions (2) and (3) and the right lifting property against (B2) follows since p-Cartesian edges satisfy the right cancellation property, by Lemma 1.2.17. □

Corollary 3.1.14. *A marked simplicial set (X, E_X) is marked right fibrant if and only if X is an ∞-category and E_X is precisely the set of equivalences in X.*

Corollary 3.1.15. *The Cartesian and coCartesian model structures over any ∞-groupoid coincide.*

Proof. The Cartesian and coCartesian model structures in this case have the same cofibrations and fibrant objects. □

Remark 3.1.16. The above Proposition also shows that our Cartesian model structure coincides with Lurie's [**Lur09**, Proposition 3.1.3.7], since they have the same cofibrations and fibrant objects. But we show a little more; we also characterize the fibrations between fibrant objects which is not evident in Lurie's treatment. On the other hand, we obtain the following characterization of weak equivalences.

Proposition 3.1.17. *Let $X^\natural \to Y^\natural$ be a map between fibrant objects of $\mathbf{mSet}/A^\sharp$. Then it is a Cartesian equivalence if and only if for all vertices $a \in A^\sharp$, the induced map on fibers $X_a^\natural \to Y_a^\natural$ is a Cartesian equivalence.*

Proof. This is [**Lur09**, Proposition 3.1.3.5]. □

Let $p\colon (X, E_X) \to (A, E_A)$ be a marked right fibration. Note that its underlying map of simplicial sets is not quite a Cartesian fibration. This is because we only require p-Cartesian lifts over the edges E_A, while a Cartesian fibration has lifts over *any* edge whose target is in the image of p. However, any marked right fibration of the form $p\colon (X, E_X) \to A^\sharp$ has an underlying Cartesian fibration, where E_X is precisely the set of *all* p-Cartesian edges.

Conversely, given a Cartesian fibration $p\colon X \to A$. We will denote $X^\natural$ the marked simplicial set with underlying simplicial set X and marked edges the p-Cartesian edges. Then it is clear that the map $X^\natural \to A^\sharp$ is a marked right fibration. Thus we will refer to marked right fibrations of the form $X^\natural \to A^\sharp$ also as *Cartesian fibrations*.

The rest of this section will compare the Cartesian model structure to the Joyal model structure. We will show that there is a Quillen equivalence between simplicial sets endowed with the Joyal model structure and marked simplicial sets endowed with the (co)Cartesian model structure.

Recall from Section 2.3 that the Joyal model structure is obtained from the cylinder J and the class of categorical anodyne extensions generated by the inner horn inclusions $\Lambda_k^n \to \Delta^n$. Moreover, for any simplicial set A we have constructed a Contravariant model structure on $\mathbf{sSet}/A$ which we denoted by $\mathbf{IsoFib}(A)$.

Proposition 3.1.18. *The forgetful functor*

$$\mathbf{Cart}(A^\sharp) \to \mathbf{IsoFib}(A)$$

is a right Quillen functor.

Proof. The left adjoint is given by sending a map $p\colon X \to A$ to the composition

$$X^\flat \to A^\flat \to A^\sharp.$$

It is clear that this maps cofibrations to cofibrations. Thus, according to Proposition 1.1.15, it suffices to show that the forgetful functor preserves fibrations between fibrant objects. We first observe that the forgetful functor preserves fibrant objects by Proposition 1.1.15. Indeed, any Cartesian fibration with target A is in particular an inner fibration. Thus it suffices to show that we can solve lifting problems of the form

$$\begin{tikzcd}
J^\flat \times (\partial\Delta^n)^\flat \cup \{1\} \times (\Delta^n)^\flat \arrow[r] \arrow[d] & X^\natural \arrow[d] \\
J^\flat \times (\Delta^n)^\flat \arrow[r] & A^\sharp.
\end{tikzcd}$$

The bottom map factorizes as

$$J^\flat \times (\Delta^n)^\flat \to J^\sharp \times (\Delta^n)^\sharp \to A^\sharp.$$

Thus, by taking pullbacks, we are reduced to solve lifting problems of the form

$$\begin{tikzcd}
J^\flat \times (\partial\Delta^n)^\flat \cup \{1\} \times (\Delta^n)^\flat \arrow[r] \arrow[d] & X^\natural \arrow[d] \\
J^\flat \times (\Delta^n)^\flat \arrow[r] & J^\sharp \times (\Delta^n)^\sharp.
\end{tikzcd}$$

In particular, X is an ∞-category and since the right vertical map is a Cartesian fibration, any equivalence in X is marked. Thus we have a factorization

$$\begin{tikzcd}
J^\flat \times (\partial\Delta^n)^\flat \cup \{1\} \times (\Delta^n)^\flat \arrow[r] \arrow[d] & J^\sharp \times (\partial\Delta^n)^\flat \cup \{1\} \times (\Delta^n)^\flat \arrow[r] \arrow[d] & X^\natural \arrow[d] \\
J^\flat \times (\Delta^n)^\flat \arrow[r] & J^\sharp \times (\Delta^n)^\flat \arrow[r] & J^\sharp \times (\Delta^n)^\sharp.
\end{tikzcd}$$

Since the middle vertical map is marked right anodyne, we can solve the lifting problem defined by the right square, which gives a solution to the lifting problem defined by the composite square. Thus $X^\natural \to A^\sharp$ is an isofibration.

Now a fibration between fibrant objects is given by a diagram

$$\begin{tikzcd}
Y^\natural \arrow[rr, "p"] \arrow[dr] & & X^\natural \arrow[dl] \\
& A^\sharp &
\end{tikzcd}$$

in which p is a marked right fibration. By the above, the structure maps are isofibrations, hence it suffices to show that p is an isofibration. Consider a lifting problem

$$\begin{tikzcd}
J^\flat \times (\partial\Delta^n)^\flat \cup \{1\} \times (\Delta^n)^\flat \arrow[r] \arrow[d] & X^\natural \arrow[d] & \\
J^\flat \times (\Delta^n)^\flat \arrow[r] \arrow[ur, dashed] & Y^\natural \arrow[r] & A^\sharp.
\end{tikzcd}$$

Since $Y^\natural \to A^\sharp$ is fibrant and $J^\flat \to J^\sharp$ is marked right anodyne, the lifting problem

$$\begin{tikzcd} J^\flat \times (\Delta^n)^\flat \arrow[r] \arrow[d] & Y^\natural \arrow[d] \\ J^\sharp \times (\Delta^n)^\flat \arrow[r] \arrow[ur, dashed] & A^\sharp \end{tikzcd}$$

admits a solution and this shows that we have a factorization

$$J^\flat \times (\Delta^n)^\flat \to J^\sharp \times (\Delta^n)^\flat \to Y^\natural.$$

Thus, again by taking pullbacks, we may reduce to lifting problems of the form

$$\begin{tikzcd} J^\flat \times (\partial\Delta^n)^\flat \cup \{1\} \times (\Delta^n)^\flat \arrow[r] \arrow[d] & (X, E_X) \arrow[d] \\ J^\flat \times (\Delta^n)^\flat \arrow[r] \arrow[ur, dashed] & J^\sharp \times (\Delta^n)^\flat, \end{tikzcd}$$

in which the right vertical map is a marked right fibration. Since the target of this map is an ∞-category, X is an ∞-category and in particular all equivalences are marked. Thus the same argument as above shows that we find a solution to this lifting problem, hence showing that p is an isofibration. $\square$

We now consider the case when $A = \Delta^0$. In this case we obtain a right Quillen functor

$$\mathbf{Cart}(\Delta^0) \to \mathbf{IsoFib}(\Delta^0) \cong \mathbf{sSet}_J$$

where the target is the category of simplicial sets with the Joyal model structure.

Theorem 3.1.19. *The functor*

$$\mathbf{Cart}(\Delta^0) \to \mathbf{IsoFib}(\Delta^0) \cong \mathbf{sSet}_J$$

is a Quillen equivalence.

Proof. We already know that we have a Quillen adjunction. Thus we need to show that the derived unit and counit are equivalences. To this end, we exhibit for any simplicial set A an explicit fibrant replacement of $A^\flat$ in marked simplicial sets. We may first consider an inner anodyne map $A \to A'$, where A' is an ∞-category, which induces a marked right anodyne map

$$A^\flat \to (A')^\flat.$$

Consider the maximal ∞-groupoid $k(A') \subseteq A'$ and take the pushout

$$\begin{tikzcd} k(A')^\flat \arrow[r] \arrow[d] & (A')^\flat \arrow[d] \\ k(A')^\sharp \arrow[r] & (A', E_{A'}). \end{tikzcd}$$

It follows that $E_{A'}$ is precisely the set of equivalences in A', thus the pair $(A', E_{A'})$ is marked right fibrant. By construction the composition

$$A^{\flat} \to (A')^{\flat} \to (A', E_{A'})$$

is marked right anodyne and hence $(A', E_{A'})$ is a fibrant replacement for $A^{\flat}$. This shows in particular that the derived unit is an equivalence for any marked simplicial set A, since it can be computed as above to be the map $A \to A'$, which is inner anodyne.

On the other hand, if (A, E_A) is fibrant, the derived counit is given by the map

$$A^{\flat} \to (A, E_A)$$

which is marked right anodyne by the above considerations. □

3.2. Smoothness & Properness of (co)Cartesian Fibrations

It is a classical result that Grothendieck opfibrations are proper. In this section we prove a refinement of this result for coCartesian fibrations. More precisely, we prove that the pullback of a right anodyne map along a coCartesian fibration yields a Cartesian equivalence. In particular this will imply properness of coCartesian fibrations.

If the coCartesian fibration is in fact a left fibration, we prove a stronger result, namely that the pullback along a left fibration induces a left Quillen functor of Cartesian model structures. As an application, we prove that any coCartesian fibration over an inner horn can be extended up to equivalence to a coCartesian fibration over a simplex.

Definition 3.2.1. A marked right anodyne extension is called *cellular* if it is in the smallest saturated class generated by

$$(\Delta^1)^{\sharp} \times (K, E_K) \cup \{1\} \times (L, E_L) \to (\Delta^1)^{\sharp} \times (L, E_L)$$

where $(K, E_K) \to (L, E_L)$ is a monomorphism.

Remark 3.2.2. In other words, a marked right anodyne extension is cellular if and only if it is in the smallest class of right anodyne extensions which contains the classes (B1) and (B2) of Remark 3.1.9.

Examples of cellular marked right anodyne extensions are provided by the following. Let (X, E_X) be a marked simplicial set. We define the simplicial set $\mu(X, E_X)$ as the simplicial subset of X generated by the marked edges. In other words we have a bijection

$$\mathbf{sSet}(K, \mu(X, E_X)) \cong \mathbf{mSet}(K^{\sharp}, (X, E_X))$$

for all simplicial sets K. This shows that μ is a functor

$$\mu\colon \mathbf{mSet} \to \mathbf{sSet}$$

which is right adjoint to $(\cdot)^{\sharp}$. In particular $(\cdot)^{\sharp}$ preserves colimits.

Lemma 3.2.3. *Let $K \to L$ be a right anodyne extension of simplicial sets. Then $K^{\sharp} \to L^{\sharp}$ is a cellular marked right anodyne extension.*

Proof. Since $(\cdot)^\sharp$ commutes with colimits, it suffices to show that the image of the generators of right anodyne extensions are cellular marked right anodyne. Thus we need to show that

$$(\Delta^1)^\sharp \times (\partial\Delta^n)^\sharp \cup \{1\} \times (\Delta^n)^\sharp \to (\Delta^1)^\sharp \times (\Delta^n)^\sharp$$

is cellular marked right anodyne, which is clear since $(\partial\Delta^n)^\sharp \to (\Delta^n)^\sharp$ is a monomorphism. $\square$

Theorem 3.2.4. *Consider a pullback square of marked simplicial sets*

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r, "i"'] & A \end{tikzcd}$$

where p is a marked left fibration and $i\colon B \to A$ is a cellular right anodyne extension. Then $j\colon Y \to X$ is marked right anodyne.

Proof. The proof is analogous to the proof for left fibrations of simplicial sets, see Proposition 2.3.23. The class of morphisms $B \to A$, for which the conclusion holds is saturated and satisfies the right cancellation property. Thus it suffices to show the assertion for pullback squares of the form

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d] & X \arrow[d, "p"] \\ \{1\} \times B \arrow[r, "i"'] & (\Delta^1)^\sharp \times B. \end{tikzcd}$$

We observe that in this case i is a right deformation retract, hence by Proposition 2.2.9 the map j is a right deformation retract and thus by Lemma 2.2.8 is a marked right anodyne extension. $\square$

Corollary 3.2.5. *Any coCartesian fibration of simplicial sets is proper with respect to the Contravariant model structure for simplicial sets.*

Proof. Consider a pullback square of simplicial sets

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r, "i"'] & A \end{tikzcd}$$

where i is right anodyne and p is a coCartesian fibration. We get a pullback square of marked simplicial sets

$$\begin{tikzcd} Y^\natural \arrow[r, "j"] \arrow[d] & X^\natural \arrow[d, "p"] \\ B^\sharp \arrow[r, "i"'] & A^\sharp. \end{tikzcd}$$

Thus by the above Theorem, j is marked right anodyne. By inspection of the generators of marked right anodyne extensions, it is easy to see that the underlying map of simplicial sets is right anodyne. $\square$

Let $p\colon X \to A$ be a left fibration of simplicial sets. Then it is in particular a coCartesian fibration and the map $p^\sharp\colon X^\sharp \to A^\sharp$ is a marked left fibration in the coCartesian model structure.

Proposition 3.2.6. *Let $p\colon X \to A$ be a left fibration of simplicial sets. Then p induces a left Quillen functor*

$$(p^\sharp)^*\colon \mathbf{mSet}/A^\sharp \to \mathbf{mSet}/X^\sharp$$

where both categories are endowed with the Cartesian model structure.

Proof. We need to show that for any diagram of Cartesian squares

$$\begin{tikzcd} (X'', E_{X''}) \arrow[r, "j"] \arrow[d] & (X', E_{X'}) \arrow[r] \arrow[d] & X^\sharp \arrow[d] \\ (K, E_K) \arrow[r, "i"] & (L, E_L) \arrow[r] & A^\sharp \end{tikzcd}$$

in which i is a marked right anodyne extension, the map j is also a marked right anodyne extension. We may check this on the generators for marked right anodyne extensions. This is clear for cellular marked right anodyne extensions by Theorem 3.2.4, hence we only need to check the morphisms $J^\flat \to J^\sharp$ and $(\Lambda^n_k)^\flat \to (\Delta^n)^\flat$. In the first case, we need to check that in the Cartesian square

$$\begin{tikzcd} (X, E_X) \arrow[r, "j"] \arrow[d] & X^\sharp \arrow[d] \\ J^\flat \arrow[r] & J^\sharp \end{tikzcd}$$

the morphism j is marked right anodyne. Since J is a Kan complex, X is an ∞-category. Consider $k(X) \subset X$, the maximal ∞-groupoid of X. We then have a pushout square

$$\begin{tikzcd} k(X)^\flat \arrow[r] \arrow[d] & (X, E_X) \arrow[d, "j"] \\ k(X)^\sharp \arrow[r] & X^\sharp, \end{tikzcd}$$

hence j is marked right anodyne. In case of the inner horn inclusion, we have a Cartesian square

$$\begin{tikzcd} (X', E_{X'}) \arrow[r, "j"] \arrow[d] & (X, E_X) \arrow[d, "p"] \\ (\Lambda^n_k)^\flat \arrow[r] & (\Delta^n)^\flat. \end{tikzcd}$$

In this case, $E_X = E_{X'}$ is precisely the set of equivalences in the ∞-category X. Moreover, p is a left fibration of simplicial sets. It follows that $X' \to X$ is a trivial cofibration of simplicial sets by Theorem 2.3.25. We obtain a commutative square

$$\begin{tikzcd} (X')^\flat \arrow[r] \arrow[d] & X^\flat \arrow[d] \\ (X', E_X) \arrow[r, "j"] & (X, E_X) \end{tikzcd}$$

where the upper horizontal map is a trivial cofibration of marked simplicial sets. We claim that the vertical maps are marked right anodyne. It then follows that j is a trivial cofibration of marked simplicial sets and since (X, E_X) is marked right fibrant, the map j is in fact marked right anodyne by Lemma 2.1.20.

To this end, we observe that for each object $k \in \Delta^n$ we have an isomorphism on fibers $(X'_k, E_{X_k}) \cong (X_k, E_{X_k})$. Now both vertical maps are pushouts of the map of Kan complexes

$$\coprod_k k(X_k)^\flat \to \coprod_k k(X_k)^\sharp$$

and hence are marked right anodyne. $\square$

Lemma 3.2.7. *Consider a commutative square of marked simplicial sets*

$$\begin{tikzcd} X^\natural \arrow[r, "j"] \arrow[d, "p"'] & Y^\natural \arrow[d, "q"] \\ (\Lambda^n_k)^\sharp \arrow[r, "i"] & (\Delta^n)^\sharp \end{tikzcd}$$

where p and q are coCartesian fibrations, j is marked left anodyne and i is an inner horn inclusion. Then the induced map

$$\begin{tikzcd} X^\natural \arrow[rr] \arrow[dr, "p"'] & & Y^\natural \times_{(\Delta^n)^\sharp} (\Lambda^n_k)^\sharp \arrow[dl] \\ & (\Lambda^n_k)^\sharp & \end{tikzcd}$$

is a coCartesian equivalence in $\mathbf{mSet}/(\Lambda^n_k)^\sharp$.

Proof. By Theorem 3.1.17 it suffices to show that for any object $k \in [n]$, the induced map on fibers $X^\natural_k \to Y^\natural_k$ is a coCartesian equivalence (over the point). Let us first assume that $k = n$. Since the inclusion $\{n\} \to \Lambda^n_k$ is right anodyne, the map $\{n\} \to (\Lambda^n_k)^\sharp$ is cellular marked right anodyne by Lemma 3.2.3. Hence we obtain a commutative square

$$\begin{tikzcd} X^\natural_n \arrow[r] \arrow[d] & Y^\natural_n \arrow[d] \\ X^\natural \arrow[r, "j"] & Y^\natural \end{tikzcd}$$

in which the vertical maps are marked right anodyne extensions by Theorem 3.2.4 and the lower horizontal map is marked left anodyne by assumption. In particular, they are trivial cofibrations in the coCartesian (and Cartesian) model structure over the point by Corollary 3.1.15. Thus by the 2-out-of-3 property, the induced map on fibers is a coCartesian equivalence.

Now assume that $k < n$. In this case we have a factorization

$$\{k\} \to (\Delta^k)^\sharp \to (\Lambda^n_k)^\sharp$$

where the first map is cellular marked right anodyne and the second map is a Cartesian fibration, hence its underlying map of simplicial sets is a right fibration. Let us denote $X^\natural_{\Delta^k} = X^\natural \times_{(\Lambda^n_k)^\sharp} (\Delta^k)^\sharp$ and similarly $Y^\natural_{\Delta^k}$. We get a commutative diagram

$$\begin{tikzcd}
X^\natural_k \arrow[r] \arrow[d] & Y^\natural_k \arrow[d] \\
X^\natural_{\Delta^k} \arrow[r] \arrow[d] & Y^\natural_{\Delta^k} \arrow[d] \\
X^\natural \arrow[r, "j"] & Y^\natural.
\end{tikzcd}$$

The middle horizontal map is a trivial cofibration in the coCartesian model structure over the point (in fact a marked left anodyne extension) being a pullback of j along a right fibration by Proposition 3.2.6. The inclusion $\{k\} \to (\Delta^k)^\sharp$ is cellular marked right anodyne, hence the upper vertical maps are marked right anodyne and it follows as above that $X^\natural_k \to Y^\natural_k$ is a coCartesian equivalence and this finishes the proof. □

THEOREM 3.2.8. *Let $i\colon \Lambda^n_k \to \Delta^n$ be an inner horn inclusion. Then this induces a Quillen equivalence*

$$i^\sharp_!\colon \mathbf{coCart}(\Lambda^n_k)^\sharp \to \mathbf{coCart}(\Delta^n)^\sharp.$$

PROOF. Since i is bijective on objects, the right adjoint $(i^\sharp)^*$ preserves weak equivalences between fibrant objects. Given a coCartesian fibration $X^\natural \to (\Lambda^n_k)^\sharp$, a fibrant replacement over $(\Delta^n)^\sharp$ can be obtained by factorizing the composition

$$X^\natural \to (\Lambda^n_k)^\sharp \to (\Delta^n)^\sharp$$

as a marked left anodyne extension followed by a coCartesian fibration. Hence by the previous lemma, the unit transformation is a weak equivalence. Thus, by Proposition 1.1.18 this shows that we have a Quillen equivalence. □

3.3. Minimal Cartesian fibrations

In this section we prove that any Cartesian fibration has a minimal model. To this end, we recollect some results on *minimal presheaves*. We fix a small Eilenberg-Zilber category A and denote by $\mathrm{PSh}(A)$ its category of presheaves. Our reference is [**Cis**, Section 5.1].

REMARK 3.3.1. The definition of an Eilenberg-Zilber will not be important for us, we refer to [**Cis**, Definition 1.3.1] for a precise definition. We only note that Δ is an Eilenberg-Zilber category and more generally for any simplicial set Δ/A is an Eilenberg-Zilber category.

Definition 3.3.2. Let $X \in \mathrm{PSh}(A)$ be a presheaf. Then X is called a *minimal presheaf* if for any presheaf $S \in \mathrm{PSh}(A)$, any trivial fibration $X \to S$ is an isomorphism.

Now assume that $\mathrm{PSh}(A)$ is endowed with a Cisinski model structure, i.e. a model structure in which the cofibrations are precisely the monomorphisms.

Definition 3.3.3. A *minimal complex* is a minimal presheaf which is fibrant. For X a fibrant presheaf, a *minimal model* of X is a trivial cofibration $S \to X$ where S is a minimal complex.

The following Theorem asserts that any fibrant presheaf has a minimal model.

Theorem 3.3.4. *Let $X \in \mathrm{PSh}(A)$ be a fibrant presheaf. Then there is a minimal complex S and a trivial cofibration $i\colon S \to X$. Moreover, any map $r\colon X \to S$ such that $ri = id_S$ (which always exists) is a trivial fibration.*

Proof. This is [**Cis**, Theorem 5.1.7] combined with [**Cis**, Proposition 5.1.8]. □

The purpose of introducing minimal complexes is that we can reduce questions about weak equivalences to questions about isomorphisms of presheaves as the following proposition shows.

Proposition 3.3.5. *Let X and Y be minimal complexes. Then any weak equivalence $X \to Y$ is an isomorphism of presheaves.*

Proof. This is [**Cis**, Proposition 5.1.10]. □

We will apply the theory of minimal complexes in the following situation. Let A be a simplicial set. Then the category Δ/A is an Eilenberg-Zilber category. We have constructed a Contravariant model structure on the category of presheaves $\mathrm{PSh}(\Delta/A) \simeq \mathbf{sSet}/A$, namely the model category $\mathbf{IsoFib}(A)$ in which the cofibrations are precisely the monomorphisms. In particular, this is an instance of a Cisinski model structure. We have seen that the fibrant objects are precisely the isofibrations with target A, hence by Theorem 3.3.4 any isofibration has a minimal model. More precisely, for any isofibration $X \to A$ there is a diagram

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr] & X \arrow[r, "r"] \arrow[d] & S \arrow[dl] \\ & A & \end{tikzcd}$$

in which $S \to A$ is a minimal isofibration, i is a trivial cofibration in $\mathbf{IsoFib}(A)$, the map r is a trivial fibration and $ri = id$. The following lemma is crucial.

Lemma 3.3.6. *The class of minimal isofibrations is stable under pullback.*

Proof. This is [**Cis**, Proposition 5.1.15]. □

Now suppose we have a Cartesian fibration $p\colon X^\natural \to A^\sharp$. By Proposition 3.1.18, the map of underlying simplicial sets $X \to A$ is an isofibration. Thus, we find a minimal isofibration $p'\colon S \to A$ and a trivial fibration $X \to S$ over A.

Proposition 3.3.7. *The minimal isofibration $p'\colon S \to A$ is a Cartesian fibration. Moreover, the trivial fibration of simplicial sets $r\colon X \to S$ over A induces a trivial fibration of marked simplicial sets $r^\natural\colon X^\natural \to S^\natural$ over $A^\sharp$.*

Proof. We first prove that $S \to A$ is a Cartesian fibration. Let E_X be the set of p-Cartesian edges of X. Then we define $E_S = r(E_X)$. We thus get a retraction diagram of marked simplicial sets over $A^\sharp$

$$\begin{tikzcd} (S,E_S) \arrow[r, "i"] \arrow[dr, "p'"'] & X^\natural \arrow[r, "r"] \arrow[d, "p"] & (S,E_S) \arrow[dl, "p'"] \\ & A^\sharp. & \end{tikzcd}$$

In particular, this shows that $p'\colon (S,E_S) \to A^\sharp$ is a marked right fibration, being a retract of p and hence a Cartesian fibration.

The underlying map of simplicial sets $X \to S$ is a trivial fibration. To show that $X^\natural \to S^\natural$ is a trivial fibration of marked simplicial sets it suffices to show the right lifting property with respect to the map $(\Delta^1)^\flat \to (\Delta^1)^\sharp$. But this is immediate by definition of E_S. □

Definition 3.3.8. A Cartesian fibration $X^\natural \to A^\sharp$ is called *minimal*, if its underlying isofibration is minimal.

Thus for any Cartesian fibration $X^\natural \to A^\sharp$ we find a Cartesian fibration $S^\natural \to A^\sharp$ whose underlying map of simplicial sets is a minimal isofibration and a trivial fibration of marked simplicial sets $X^\natural \to S^\natural$. In this sense any Cartesian fibration has a minimal model.

Proposition 3.3.9. *Let $X \to \Lambda^n_k$ be a minimal Cartesian fibration of simplicial sets, where $n \geq 2$ and $0 < k < n$. Then there exists a Cartesian fibration $Y \to \Delta^n$ and a pullback square of simplicial sets*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \Lambda^n_k \arrow[r] & \Delta^n. \end{tikzcd}$$

Proof. We consider the composition of marked simplicial sets

$$X^\natural \to (\Lambda^n_k)^\sharp \to (\Delta^n)^\sharp.$$

We factorize this composition into a marked right anodyne extension followed by a Cartesian fibration to obtain the square

$$\begin{tikzcd} X^\natural \arrow[r] \arrow[d] & \overline{Y}^\natural \arrow[d] \\ (\Lambda^n_k)^\sharp \arrow[r] & (\Delta^n)^\sharp. \end{tikzcd}$$

By Proposition 3.3.9, we find a minimal Cartesian fibration $Y^\natural \to (\Delta^n)^\sharp$ and a trivial fibration $\overline{Y}^\natural \to Y^\natural$ over $(\Delta^n)^\sharp$. Thus we obtain the square

$$\begin{tikzcd}
X^\natural \arrow[r] \arrow[d] & Y^\natural \arrow[d] \\
(\Lambda^n_k)^\sharp \arrow[r] & (\Delta^n)^\sharp,
\end{tikzcd}$$

in which both $X^\natural \to (\Lambda^n_k)^\sharp$ a d $Y^\natural \to (\Delta^n)^\sharp$ are minimal. We claim that the underlying diagram of simplicial sets is a pullback diagram.

Let us denote $\overline{Y}^\natural_{(\Lambda^n_k)^\sharp} = \overline{Y}^\natural \times_{(\Delta^n)^\sharp} (\Lambda^n_k)^\sharp$ and $Y^\natural_{(\Lambda^n_k)^\sharp} = Y^\natural \times_{(\Delta^n)^\sharp} (\Lambda^n_k)^\sharp$. We obtain the diagram

$$\begin{tikzcd}
X^\natural \arrow[r] \arrow[dr] & \overline{Y}^\natural_{(\Lambda^n_k)^\sharp} \arrow[r] \arrow[d] & Y^\natural_{(\Lambda^n_k)^\sharp} \arrow[dl] \\
& (\Lambda^n_k)^\sharp. &
\end{tikzcd}$$

The first horizontal map is a Cartesian equivalence by (the dual of) Lemma 3.2.7 and the second horizontal map is a trivial fibration. Hence the composition $X^\natural \to Y^\natural_{(\Lambda^n_k)^\sharp}$ is a Cartesian equivalence between fibrant objects over $(\Lambda^n_k)^\sharp$. By Lemma 3.3.6, the Cartesian fibration $Y^\natural_{(\Lambda^n_k)^\sharp}$ is minimal. In particular, the map of underlying simplicial sets $X \to Y_{(\Lambda^n_k)^\sharp}$ is a weak equivalence of minimal isofibrations over Λ^n_k by Proposition 3.1.18 and is thus an isomorphism by Proposition 3.3.5. This shows that

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
\Lambda^n_k \arrow[r] & \Delta^n
\end{tikzcd}$$

is a pullback square. □

3.4. The universal coCartesian fibration

We fix a Grothendieck universe $\mathbb{U}$. A set is called *small* if it belongs to $\mathbb{U}$. We suppose that the morphism set of Δ is small. Then we define a simplicial set $\mathcal{U}$ as follows. An element of $\mathcal{U}_n$ is a map $X \to \Delta^n$ where X is a *small* simplicial set. The simplicial operators are defined by a *choice* of a pullback square

$$\begin{tikzcd}
f^*X \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^m \arrow[r] & \Delta^n.
\end{tikzcd}$$

There is a pointed version $\mathcal{U}_\bullet$ whose n-simplices are given by maps $X \to \Delta^n$ with X small together with a section $\Delta^n \to X$. Forgetting the section defines a map $\mathcal{U}_\bullet \to \mathcal{U}$.

PROPOSITION 3.4.1. *Let $f: X \to Y$ be a map of simplicial sets. Then specifying a pullback square of the form*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathcal{U}_\bullet \arrow[d] \\ Y \arrow[r, "F"] & \mathcal{U} \end{tikzcd}$$

is equivalent to specifying for each n-simplex $\sigma: \Delta^n \to Y$ a choice of a pullback square

$$\begin{tikzcd} \sigma^* X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "\sigma"] & Y \end{tikzcd}$$

where $\sigma^* X$ is small.

PROOF. We may check if a square of simplicial sets is Cartesian on representables. The assertion that $\sigma^* X$ is small follows directly from the definition of $\mathcal{U}$. □

DEFINITION 3.4.2. In the situation above, we say that f is *classified* by F and that f has *small fibers*.

COROLLARY 3.4.3. *Let*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ A \arrow[r, "i"] & B \end{tikzcd}$$

be a Cartesian square of simplicial sets. Suppose that i is a monomorphism and that f is classified by a map $F: A \to \mathcal{U}$. If g has small fibers, then there exists a map $G: B \to \mathcal{U}$ which classifies g such that $F = Gi$.

DEFINITION 3.4.4. We define the subobject $\mathcal{Q} \subset \mathcal{U}$ to consist of coCartesian fibrations $X \to \Delta^n$. We define a morphism $q_{\mathrm{univ}}: \mathcal{Q}_\bullet \to \mathcal{Q}$ by the pullback diagram

$$\begin{tikzcd} \mathcal{Q}_\bullet \arrow[r] \arrow[d, "q_{\mathrm{univ}}"'] & \mathcal{U}_\bullet \arrow[d] \\ \mathcal{Q} \arrow[r] & \mathcal{U} \end{tikzcd}$$

Thus, the objects of $\mathcal{Q}$ are themselves small ∞-categories.

PROPOSITION 3.4.5. *The map $q_{\mathrm{univ}}: \mathcal{Q}_\bullet \to \mathcal{Q}$ is a coCartesian fibration. Moreover, any coCartesian fibration $X \to Y$ with small fibers arises from a pullback square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{Q}_\bullet \arrow[d, "q_{\mathrm{univ}}"] \\ Y \arrow[r] & \mathcal{Q} \end{tikzcd}$$

PROOF. This follows immediately from the definition of q_{univ} and Proposition 1.2.19. □

We will need the following lemma.

Lemma 3.4.6. *Let $X \to Y$ be a monomorphism of simplicial sets and let $X' \to X$ be a trivial fibration. Then there exists a trivial fibration $Y' \to Y$ and a pullback square*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Proof. See [**Cis**, Lemma 5.1.20]. □

Theorem 3.4.7. *The simplicial set $\mathcal{Q}$ is an ∞-category whose objects are small ∞-categories.*

Proof. It is clear that the objrects of $\mathcal{Q}$ are ∞-categories. Solving a lifting problem of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathcal{Q} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

is equivalent to finding for any coCartesian fibration $X \to \Lambda^n_k$ a coCartesian fibration $Y \to \Delta^n$ and a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \Lambda^n_k \arrow[r] & \Delta^n \end{tikzcd}$$

by Corollary 3.4.3. By Theorem 3.3.4, we may factorize $X \to \Lambda^n_k$ as

$$X \to X' \to \Lambda^n_k$$

where the first map is a trivial fibration and the second map is a minimal isofibration which by Proposition 3.3.7 is a coCartesian fibration. By Proposition 3.3.9 we have a diagram

$$\begin{tikzcd} X \arrow[d] & \\ X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ \Lambda^n_k \arrow[r] & \Delta^n \end{tikzcd}$$

in which the square is a pullback square and the map $Y' \to \Delta^n$ is a coCartesian fibration. By Lemma 3.4.6 we may complete this to a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ \Lambda^n_k \arrow[r] & \Delta^n \end{tikzcd}$$

in which $Y \to Y'$ is a trivial fibration and each square is a pullback. Now the composition $Y \to \Delta^n$ is a coCartesian fibration and this completes the proof. □

As a consequence, we can extend any coCartesian fibration along a trivial cofibration of the Joyal model structure.

Corollary 3.4.8. *Let $i\colon A \to B$ be a trivial cofibration of the Joyal model structure. Then for any coCartesian fibration $p\colon X \to A$ with small fibers, there exists a coCartesian fibration $q\colon Y \to B$ with small fibers and Cartesian square of the form*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "i"] & B \end{tikzcd}$$

Proof. Finding a Cartesian square as above corresponds to an extension

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \mathcal{Q} \\ B \arrow[ur, dashed] & \end{tikzcd}$$

Since $\mathcal{Q}$ is an ∞-category, it is fibrant in the Joyal model structure hence the required extension exists. □

Finally, we relate the universal coCartesian fibration with the universal left fibration and the universal Kan fibration.

Definition 3.4.9. Let $\mathcal{S}$ be the subobject of $\mathcal{Q}$ consisting of *left fibrations* over Δ^n. We have a map

$$p_{\mathrm{univ}}\colon \mathcal{S}_\bullet \to \mathcal{S}$$

defined as the pullback

$$\begin{tikzcd} \mathcal{S}_\bullet \arrow[r] \arrow[d, "p_{\mathrm{univ}}"'] & \mathcal{Q}_\bullet \arrow[d, "q_{\mathrm{univ}}"] \\ \mathcal{S} \arrow[r] & \mathcal{Q} \end{tikzcd}$$

Since left fibrations are recognized on representables, the map p_{univ} is a left fibration and classifies left fibrations with small fibers. In [**Cis**, Section 5.2] it is shown that $\mathcal{S}$ is also an ∞-category. We may consider the maximal ∞-groupoid $k(\mathcal{S}) \subset \mathcal{S}$. Since left fibrations are conservative we obtain the following diagram of pullbacks

$$\begin{tikzcd} k(\mathcal{S}_\bullet) \arrow[r] \arrow[d, "k(p_{\mathrm{univ}})"'] & \mathcal{S}_\bullet \arrow[r] \arrow[d, "p_{\mathrm{univ}}"] & \mathcal{Q}_\bullet \arrow[r] \arrow[d, "q_{\mathrm{univ}}"] & \mathcal{U}_\bullet \arrow[d] \\ k(\mathcal{S}) \arrow[r] & \mathcal{S} \arrow[r] & \mathcal{Q} \arrow[r] & \mathcal{U} \end{tikzcd}$$

Proposition 3.4.10. *Let $f\colon X \to A$ be a coCartesian fibration which is classified by $F\colon A \to \mathcal{Q}$. Then F factors through $\mathcal{S}$ if and only if f is a left fibration and F factors through $k(\mathcal{S})$ if and only if f is a Kan fibration.*

Proof. The first assertion is clear and the second assertion follows from [**Cis**, Proposition 5.2.13]. □

CHAPTER 4

Adjoint Functor Theorems

Adjoint functor theorems give necessary and sufficient conditions for a functor between appropriate categories to have an adjoint. They are fundamental results in category theory both for their theoretical value as well as for their applications. The most general and well-known adjoint functor theorems are Freyd's *General* and *Special Adjoint Functor Theorem* [**Fre03, Mac71**]. Other well-known adjoint functor theorems include those specialized to locally presentable categories – these can also be regarded as useful non-trivial specializations of Freyd's theorems.

The purpose of this chapter is to prove analogous adjoint functor theorems for functors between ∞-categories. The first one (Theorem 4.3.5) is an ∞-categorical generalization of Freyd's General Adjoint Functor Theorem and it provides a necessary and sufficient condition, in the form of Freyd's original *solution set condition*, for a limit-preserving functor between ∞-categories to admit a left adjoint. In addition, by employing a stronger form of the solution set condition, we find in this higher categorical setting a second and closely related adjoint functor theorem for functors which only preserve finite limits (Theorem 4.3.6). Both proofs of these theorems are quite elementary, and are based on some useful criteria for the existence of initial objects, very much in the spirit of the proof of Freyd's classical theorem.

The first section proves a characterization of initial objects as limits of the identity functor. It also introduces weakenings of the notion of initial objects in an ∞-category. These weakened initial objects are defined in the homotopy category of an ∞-category, and we give criteria when they determine an actual initial object. The second section states and proves General Adjoint Functor Theorems for ∞-categories. In contrast to classical category theory, we actually find two General Adjoint Functor Theorems. Finally, the third section gives criteria when adjunctions and equivalences on homotopy categories lift to adjunctions and equivalences on the level of ∞-categories.

This chapter is part of joint work with George Raptis and Christoph Schrade [**NRS18**].

4.1. Size

We will need to be more precise about size in this chapter. To this end, we choose to work in a model $\mathbb{V}$ of ZFC-set theory which contains an inaccessible cardinal. We fix the associated Grothendieck universe $\mathbb{U} \in \mathbb{V}$, which we use to distinguish between small and large sets.

In this chapter, a simplicial set will always be understood to be a functor $\Delta^{op} \to \mathbf{Set}_{\mathbb{V}}$.

Definition 4.1.1. A simplicial set $X\colon \Delta^{op} \to \mathbf{Set}_{\mathbb{V}}$ is called *small* if $K_n \in \mathbb{U}$ for each $[n] \in \Delta$. An ∞-category is *essentially small* if it is weakly categorical equivalent to a small simplicial set. An ∞-category is *locally small* if for any small set of objects, the full subcategory that it spans is essentially small.

Definition 4.1.2. An ∞-category is (finitely) *complete* (resp. *cocomplete* if it admits all limits (resp colimits) indexed by small (finite) simplicial sets. A functor is called (finitely) *continuous* (resp. *cocontinuous*) if it preserves all such limits (resp. colimits).

4.2. Criteria for the existence of initial objects

Recall that an initial object in a simplicial set C is a cofinal map $\Delta^0 \to C$. If C is an ∞-category, we have seen that this is equivalent to the map $x\backslash C \to C$ being a trivial fibration. As in classical category theory we also find a characterization of initial objects as certain limits.

Proposition 4.2.1. *Let C be an ∞-category. Then an object $x \in C$ is initial if and only if the identity functor $id\colon C \to C$ admits a limit whose cone object is $x \in C$.*

Proof. We first observe that for any pair of cones $\gamma, \delta\colon \Delta^0 \star C \to C$ over the identity functor there is a canonical morphism of cones $\gamma \to \delta$ which is given by

$$\Delta^1 \star C \cong \Delta^0 \star \Delta^0 \star C \xrightarrow{id\star\delta} \Delta^0 \star C \xrightarrow{\gamma} C$$

Now suppose we have a limiting cone over the identity functor

$$\lambda\colon \Delta^0 \star C \to C$$

with cone object $x := \lambda(-\infty) \in C$. Then we obtain a canonical endomorphism of cones $\varphi\colon \lambda \to \lambda$ as explained above. Since λ is a terminal object in the category of cones over the identity, this morphism is an equivalence in the slice C/id. In particular, the evaluation on cone objects $\varphi(-\infty)\colon x \to x$ is an equivalence in C. To show that x is initial, we need to show that each commutative diagram

$$\begin{tikzcd}
\Delta^{\{0\}} \arrow[d] \arrow[dr, "x"] & \\
\partial\Delta^n \arrow[r] \arrow[d] & C \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

admits an extension as indicated. Applying the functor $\Delta^0 \star (\cdot)$ to this diagram and composing with the limiting cone $\lambda\colon \Delta^0 \star C \to C$, we obtain a new diagram as follows

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[d] \arrow[dr, "\varphi(-\infty)"] & \\
\Lambda^{n+1}_0 \arrow[r] \arrow[d] & C \\
\Delta^{n+1}. \arrow[ur, dashed] &
\end{tikzcd}$$

Since $\varphi(-\infty)$ is an equivalence, it follows from Proposition 1.2.12 that the extension exists. Restricting this extension along the inclusion $\Delta^n \hookrightarrow \Delta^0 \star \Delta^n \cong \Delta^{n+1}$ gives an extension of the original diagram, showing that x is indeed an initial object.

Conversely, suppose that $x \in C$ is an initial object so that the map $x\backslash C \to C$ is a trivial fibration. First we find a cone over the identity with cone point $x \in C$ as a solution of the lifting problem

$$\begin{tikzcd}
\Delta^0 \arrow[r, "1_x"] \arrow[d, "x"] & x\backslash C \arrow[d] \\
C \arrow[r, "\mathrm{id}"] \arrow[ur, dashed, "\lambda"] & C.
\end{tikzcd}$$

We claim that λ defines a terminal object in C/id. For this, it suffices to show that for each commutative diagram

$$\begin{tikzcd}
\Delta^{\{n\}} \star C \arrow[d] \arrow[dr, "\lambda"] & \\
\partial\Delta^n \star C \arrow[r] \arrow[d] & C \\
\Delta^n \star C \arrow[ur, dashed] &
\end{tikzcd} \tag{1}$$

there is an extension as indicated by the dotted arrow for each $n \geq 1$. Here we have used the same notation λ for the map which is adjoint to the lift above. We extend this diagram to a new diagram as follows

$$\begin{tikzcd}
\Delta^1 \star C \cong \Delta^{\{n\}} \star \Delta^0 \star C \arrow[r, "\mathrm{id}\star\lambda"] \arrow[d] & \Delta^{\{n\}} \star C \arrow[d] \arrow[dr, "\lambda"] & \\
\Lambda^{n+1}_{n+1} \star C \cong \partial\Delta^n \star \Delta^0 \star C \arrow[r, "\mathrm{id}\star\lambda"] \arrow[d] & \partial\Delta^n \star C \arrow[r] \arrow[d] & C. \\
\Delta^{n+1} \star C \cong \Delta^n \star \Delta^0 \star C \arrow[r, "\mathrm{id}\star\lambda"] & \Delta^n \star C \arrow[ur, dashed] &
\end{tikzcd}$$

By adjunction, the composite extension problem corresponds to finding an extension in the diagram

$$\begin{tikzcd} \Delta^{\{n,n+1\}} \arrow[d] \arrow[dr] & \\ \Lambda^{n+1}_{n+1} \arrow[r] \arrow[d] & C/id \\ \Delta^{n+1}. \arrow[ur, dashed] & \end{tikzcd} \tag{2}$$

Note that by construction the morphism $\Delta^{\{n,n+1\}} \to C/id$ is an endomorphism of the cone λ. This is an equivalence since the underlying morphism on cone points is the identity of x and $C/id \to C$ is conservative (as a right fibration). Thus, again by Proposition 1.2.12 there exists an extension in (2) as required. The adjoint map of this extension restricts along $\Delta^n \subset \Delta^n \star \Delta^0 = \Delta^{n+1}$ to an extension for the original diagram (1). □

We will consider two weakenings of the notion of initial objects, which involve the homotopy category of an ∞-category. We first recall the notion of a weakly initial set in a category.

Definition 4.2.2. Let C be a category and let S be set of objects of C. Then S is called *weakly initial* if for all objects $d \in C$ there exists an object $s \in S$ such that $C(s,d)$ is non-empty.

Definition 4.2.3. Let C be an ∞-category and $x \in C$ an object. Then x is called *h-initial* if it determines an initial object in the homotopy category.

A set of objects $S \subset C_0$ is called *weakly initial*, if it determines a weakly initial set in the homotopy category.

Clearly any initial object in an ∞-category determines a weakly initial set as well as an initial object in the homotopy category. The goal of this section is to find criteria for the converse.

Proposition 4.2.4. *Let C be an ∞-category which admits finite limits. Then an object $x \in C$ is h-initial if and only if it is initial.*

Proof. The 'if' direction is clear. Suppose $x \in C$ is h-initial. Then for any object $y \in C$ the mapping space $\mathrm{map}_C(x,y)$ is non-empty and connected. By assumption, C admits finite limits, hence for any object $y \in C$ and any finite simplicial set K, there exists an object $y^K \in C$ such that there is a natural isomorphism in the homotopy category of spaces

$$\mathrm{Map}_C(x, y^K) \cong \mathrm{Map}_{\mathcal{S}}(K, \mathrm{Map}_C(x,y)).$$

In particular, since x is h-initial, these mapping spaces are non-empty and connected for any finite simplicial set K. It follows that $\mathrm{Map}_C(x,y)$ is contractible for any $y \in C$ and hence x is an initial object. □

In case C has a weakly initial set, we find the following.

Proposition 4.2.5. *Let C be an ∞-category which is locally small and complete. Then C admits an initial object if and only if it admits a small weakly initial set.*

The proof requires a little more work. We first need some lemmas.

Lemma 4.2.6. *Let C be an ∞-category and let $x \in C$ be an object. If C is complete, then so is C/x.*

Proof. Let $f: K \to C/x$ be a diagram and $f': K \star \Delta^0 \to C$ its adjoint. The adjoint limit cone $(K \star \Delta^0)^{\triangleleft} \to C$ for f' defines a limit cone for f. □

Lemma 4.2.7. *Let C be an ∞-category, $x \in C$ an object and let $\lambda: \Delta^0 \star K \to C$ be a cone. Then any morphism $u: x \to \lambda(\infty)$ determines a cone $\lambda': \Delta^0 \star K \to C$ with cone point $\lambda'(\infty) = x$ and a morphism of cones $\varphi: \lambda \to \lambda'$ with $\varphi(\infty) = u$.*

Proof. The morphism $u: x \to \lambda(\infty)$ determines a map

$$\Delta^1 \cup \left(\Delta^0 \star K\right) \xrightarrow{u \cup \lambda} C.$$

Since $\{1\} \to \Delta^1$ is right anodyne, the map

$$\Delta^1 \cup \left(\Delta^0 \star K\right) \to \Delta^1 \star K$$

is inner anodyne by Lemma 1.2.11, hence we find an extension

$$\begin{tikzcd} \Delta^1 \cup \left(\Delta^0 \star K\right) \arrow[r] \arrow[d] & C \\ \Delta^1 \star K \arrow[ur, dashed] & \end{tikzcd}$$

which has the desired properties. □

Proof of Proposition 4.2.5. Again it is clear that an initial object determines a weakly initial set. Thus, suppose $S \subset C_0$ is a small weakly initial set. By Proposition 4.2.1 we need to show that the identity functor $id: C \to C$ admits a limit. Let us denote by S also the full subcategory generated by the weakly initial set S. Without loss of generality we may assume that the subcategory S is small. We claim that the inclusion $i: S \hookrightarrow C$ is cofinal. By Quillen's Theorem A, Theorem 2.3.19, it suffices to show that the ∞-category i/c is weakly contractible for each $c \in C$.

Let K be a simplicial set and let $\lambda: K \to i/c$ a map. Consider the composition

$$\mu: K \to i/c \to C/c.$$

Since C is complete, so is C/c by Lemma 4.2.6. Hence there is an extension to a limit cone

$$\begin{tikzcd} K \arrow[r] \arrow[d] & i/c \arrow[r] & C/c \\ K^{\triangleleft}. \arrow[urr, dashed, "\overline{\mu}"'] & & \end{tikzcd}$$

The cone point $\overline{\mu}(\infty)$ corresponds to a morphism $l \to c \in C$. Since S is a weakly initial set, there is an object $s \in S$ and a morphism $\gamma: s \to l$, which determines a morphism $\overline{\mu}(\infty) \circ \gamma \to \overline{\mu}(\infty) \in C/c$. By Lemma 4.2.7, this extends to a morphism of cones

$$\Gamma: \Delta^1 \star K \to C/c$$

such that $\Gamma|_{\Delta^{\{1\}}\star K} = \overline{\mu}$ and $\Gamma|_{\Delta^{\{0\}}\star K} = \overline{\mu}\circ\gamma$. Let us denote $\Gamma_0 := \Gamma|_{\Delta^{\{0\}}\star K}$ and consider the composition

$$\Delta^0 \star K \xrightarrow{\Gamma_0} C/c \to C.$$

We observe that this composition sends every vertex of $\Delta^0 \star K$ to a vertex belonging to S and since S is a full subcategory, the functor Γ_0 factors through the inclusion $i/c \hookrightarrow C/c$,

$$\begin{tikzcd} & i/c \arrow[d] \\ \Delta^0 \star K \arrow[ur, dashed, "\Gamma_0'"] \arrow[r, "\Gamma_0"] & C/c. \end{tikzcd}$$

By construction, Γ_0' extends $\lambda\colon K \to i/c$. In conclusion, any map $K \to i/c$ admits an extension as follows

$$\begin{tikzcd} K \arrow[r] \arrow[d] & i/c \\ K^{\triangleleft}. \arrow[ur, dashed] & \end{tikzcd}$$

It follows by standard arguments that i/c is weakly contractible and therefore $i\colon S \to C$ is final as claimed. Since S is small, the inclusion $S \to C$ admits a limit and hence the identity functor admits a limit. By Proposition 4.2.1, C has an initial object. □

4.3. General adjoint functor theorems

Let us first recall the definition of an adjunction between ∞-categories. We will follow Lurie's treatment [**Lur09**, Section 5.2]. An alternative but equivalent approach is in [**Cis**, Section 6.1].

Definition 4.3.1. Let C and D be ∞-categories. An *adjunction* between C and D consists of a map $q\colon M \to \Delta^1$, which is both a Cartesian and a coCartesian fibration, together with weak categorical equivalences $C \simeq q^{-1}(0)$ and $D \simeq q^{-1}(1)$.

A useful criterion for recognizing adjunctions, mirroring the classical 1-categorical case, is the following description in terms of universal arrows.

Proposition 4.3.2. *Let $q\colon M \to \Delta^1$ be a Cartesian fibration corresponding to a functor $G\colon D \to C$ with $D = q^{-1}(1)$ and $C = q^{-1}(0)$. Then the following are equivalent.*

- *The functor G has a left adjoint.*
- *The ∞-category $c\backslash G$ has an initial object for each $c \in C$.*

Proof. This is a reformulation of [**Lur09**, Lemma 5.2.4.1]. Alternatively see [**Cis**, Proposition 6.1.11]. □

Freyd's classical *General Adjoint Functor Theorem* states that a limit preserving functor $G\colon D \to C$ from a locally small and complete category is a right adjoint if and only if it satisfies the solution set condition (see, for example, [**Mac71**, V.6, Theorem 2], or [**Fre03**, Ch. 3, Exercise J] for a little less general formulation). In general, this is a weakening of the condition that $c\backslash G$ has an initial object. In the previous section, we found that in the higher categorical setting, we may consider two weakenings of the notion of initial objects.

Definition 4.3.3. Let $G\colon D \to C$ be a functor between ∞-categories. Then G satisfies the *solution set condition* if the slice category $c\backslash G$ admits a small weakly initial set for each $c \in C$.

We say that G satisfies the *h-initial object condition* if the slice category $c\backslash G$ admits an h-initial object.

In Freyd's classical Adjoint Functor Theorem, there is no smallness assumption on the target category. However when generalizing to higher categories, we need a new notion of smallness for ∞-categories.

Definition 4.3.4. Let C be an ∞-category. We say that C is *2-locally small* if for every pair of objects $x, y \in C$, the mapping space $\mathrm{map}_C(x, y)$ is locally small.

Note that every ordinary category (not necessarily locally small) is always 2-locally small and every locally small ∞-category is also 2-locally small. We can now state our main adjoint functor theorems. The first one is a generalization of Freyd's General Adjoint Functor Theorem.

Theorem 4.3.5 (GAFT). *Let $G\colon D \to C$ be a continuous functor. Suppose that D is locally small and complete and C is 2-locally small. Then G admits a left adjoint if and only if it satisfies the solution set condition.*

Using instead the (stronger) h-initial object condition, we obtain our second adjoint functor theorem under weaker assumptions on the ∞-category D and no smallness assumption on C.

Theorem 4.3.6 ($\mathrm{GAFT}_{\mathrm{fin}}$). *Let $G\colon D \to C$ be a finitely continuous functor, where D is finitely complete. Then G admits a left adjoint if and only if it satisfies the h-initial object condition.*

Remark 4.3.7. Note that the finite General Adjoint Functor Theorem has no analogue in classical category theory since in this case it is a tautology.

For the proofs of these theorems, we will need the following lemmas.

Lemma 4.3.8. *Let $G\colon D \to C$ be a functor between ∞-categories and $c \in C$. Suppose that D is (finitely) complete and G is (finitely) continuous. Then $c\backslash G$ is (finitely) complete.*

Proof. This follows from [**Lur09**, Lemma 5.4.5.5] using that the functor $c\backslash C \to C$ preserves and reflects limits by [**Lur09**, Proposition 1.2.13.8]. □

Lemma 4.3.9. *Let $G\colon D \to C$ be a functor between ∞-categories, where D is locally small and C is 2-locally small. Then for every object $c \in C$, the ∞-category $c\backslash G$ is locally small.*

Proof. We need to show that for every pair of objects $(u\colon c \to G(d)) \in c\backslash G$ and $(u'\colon c \to G(d')) \in c\backslash G$, the mapping space

$$\mathrm{map}_{c/G}(u, u')$$

is essentially small (see [**Lur09**, Proposition 5.4.1.7]). The pullback square of ∞-categories

$$\begin{tikzcd} c\backslash G \arrow[r] \arrow[d, "q"] & c\backslash C \arrow[d, "p"] \\ D \arrow[r, "G"] & C \end{tikzcd}$$

yields a homotopy pullback square of mapping spaces

$$\begin{tikzcd} \mathrm{map}_{c\backslash G}(u,u') \arrow[r] \arrow[d, "q"] & \mathrm{map}_{c\backslash C}(u,u') \arrow[d, "p"] \\ \mathrm{map}_D(d,d') \arrow[r, "G"] & \mathrm{map}_C(G(d),G(d')). \end{tikzcd} \tag{3}$$

Since $c\backslash C \to C$ is a left fibration, the (homotopy) fiber of the right vertical map is either empty or can be identified using [**Lur09**, Proposition 2.4.4.2] with the mapping space

$$\mathrm{map}_{p^{-1}(G(d'))}(u',u'). \tag{4}$$

Since $p^{-1}(G(d')) \simeq \mathrm{map}_C(c, G(d'))$ is locally small by assumption, it follows that (4) is essentially small. Thus, the (homotopy) fibers of the left vertical map in (3) are essentially small. Then the result follows from [**Lur09**, Proposition 5.4.1.4] since $\mathrm{map}_D(d,d')$ is essentially small by assumption. □

Proof of Theorem 4.3.5. Suppose that G admits a left adjoint. Then by Proposition 4.3.2, the ∞-category $c\backslash G$ admits an initial object, which also defines a small weakly initial set.

Conversely, by Proposition 4.3.2, it is enough to show that the ∞-category $c\backslash G$ admits an initial object for each $c \in C$. By Lemma 4.3.9, $c\backslash G$ is locally small, and by Lemma 4.3.8, it is complete. The ∞-category $c\backslash G$ admits a small weakly initial set by assumption. Therefore it also admits an initial object by Proposition 4.2.5. □

Proof of Theorem 4.3.6. Suppose that G admits a left adjoint. Then for each $c \in C$, the ∞-category $c\backslash G$ has an initial object by Proposition 4.3.2, and therefore also an h-initial object.

Conversely, suppose that $c\backslash G$ has an h-initial object for each $c \in C$. By Lemma 4.3.8, the ∞-category $c\backslash G$ is finitely complete. Then Proposition 4.2.4 shows that $c\backslash G$ admits an initial object for each $c \in C$, and therefore the result follows from the characterization in Proposition 4.3.2. □

Remark 4.3.10. Our General Adjoint Functor Theorems imply, although non-trivially, the Adjoint Functor Theorems for presentable ∞-categories from [**Lur09**, Corollary 5.5.2.9]. We refer to our joint work with George Raptis and Christoph Schrade for a proof and statement of these results [**NRS18**]. We would like to emphasize that our proofs are much more elementary and require less machinery.

4.4. Adjunctions & homotopy categories

An adjunction $F\colon C \rightleftarrows D\colon G$ between ∞-categories induces an (ordinary) adjunction $hF\colon hC \rightleftarrows hD\colon hG$ between the homotopy categories. The converse statement, however, is false in general (for example, the canonical functor $C \to hC$ does not admit a left or a right adjoint in general).

On the other hand, we will see in this section that both the solution set and the h-initial object condition are really conditions on the functor hG. The obvious obstruction to lifting an adjunction, which is defined on the homotopy category, is the continuity of the functor G. We will find that this is the *only* obstruction.

The comparison of the solution set and h-initial object condition for the functors G and hG is really a comparison of the categories $h(c\backslash G)$ and $c\backslash hG$. These are in general *not* equivalent. However, we have a canonical functor

$$h(c\backslash G) \to c\backslash hG$$

which is almost an equivalence. Recall the following definition from [**RV15**].

Definition 4.4.1. Let $F\colon C \to D$ be a functor between ordinary categories. Then F is called *smothering*, if it is surjective on objects, full and conservative.

The following lemma shows that smothering functors reflect weakly initial sets.

Lemma 4.4.2. *Suppose $F\colon C \to D$ is smothering. Then C admits a small weakly initial set of objects if and only if D does.*

Proof. Let S_D be a small weakly initial set of D. Since F is surjective on objects, we may choose for each object in S_D one object in C in the preimage under F. We obtain a small set S_C and we claim that this is weakly initial. To this end, let $x \in C$ be an object. We find an object $d_S \in S_D$ and a morphism $f_D\colon d_S \to F(x) \in D$. By construction, there is an object $c_S \in S_C$ with $F(c_S) = d_S$ and since F is full, we find a morphism $f_C\colon c_S \to x$ such that $F(f_C) = f_D$ showing that S_C is weakly initial.

Conversely, it is clear that given a small weakly initial set S_C of C, its image $F(S_C)$ is a small weakly initial set of D. □

Under additional assumptions, smothering functors also reflect initial objects.

Lemma 4.4.3. *Let $G\colon D \to C$ be a smothering functor between (ordinary) categories. Suppose that for any pair of morphisms $f, g\colon d \to d'$ in D, there exists a morphism $u_{f,g}\colon w \to d$ such that $f \circ u_{f,g} = g \circ u_{f,g}$. Then $x \in D$ is initial if and only if $G(x)$ is initial in C.*

Proof. Since G is full and surjective on objects, it follows that it preserves initial objects. Conversely, suppose that $G(x)$ is initial in C for some object $x \in D$. We claim that x is initial in D. It is clear that x is weakly initial, since G is full. Suppose we have two morphisms $f, g\colon x \to d$ in D. By assumption, there exists a morphism $u_{f,g}\colon w \to x$ which equalizes f and g. The induced morphism $G(w) \to G(x)$ admits a section $s\colon G(x) \to G(w)$, since $G(x)$ is initial. Using that G is full, we find a morphism $v\colon x \to w$ such that $G(v) = s$. Since G is conservative, the composition $u_{f,g} \circ v \colon x \to x$ is an isomorphism. This means that $u_{f,g}$ is a (split) epimorphism which implies that $f = g$. □

Proposition 4.4.4. *The canonical functor*

$$h(c\backslash G) \to c\backslash hG$$

is smothering.

The proof requires a preparatory Lemma.

Lemma 4.4.5. *Let C be an ∞-category and consider a lifting problem*

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & C \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & hC. \end{tikzcd}$$

Then this admits a solution whenever $n = 0, 1, 2$.

Proof. This is clear by construction for $n = 0, 1$. For $n = 2$, a solution asserts that for any three morphisms, which compose in the homotopy category, there exists a 2-simplex verifying this composition. Thus, consider a 2-boundary in C depicted as

$$\begin{tikzcd} \cdot \arrow[d, "f"'] \arrow[dr, "h"] & \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

such that $[g] \circ [f] = [h]$ in the homotopy category hC. Since C is an ∞-category, we find a 2-simplex σ verifying composition of f and g in C of the form

$$\sigma = \begin{tikzcd} \cdot \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

Since $g \circ f$ and h have the same class in the homotopy category, there exists a 2-simplex τ of the form

$$\tau = \begin{tikzcd} \cdot \arrow[d, "g\circ f"'] \arrow[dr, "h"] & \\ \cdot \arrow[r, "id"'] & \cdot \end{tikzcd}$$

We may put these 2-simplices together to form a map

$$\Lambda^3_2 \to C$$

whose face opposite the vertex $\{0\}$ is degenerate on g, the face opposite the vertex $\{1\}$ is the 2-simplex τ and the face opposite $\{3\}$ is the 2-simplex σ. Since C is an ∞-category this extends to a 3-simplex, whose face opposite $\{2\}$ is a 2-simplex with the boundary we started with. □

Proof of Proposition 4.4.4. By definition of the slice categories, we have a canonical functor

$$c\backslash G \to c\backslash hG$$

such that there is a factorization

$$\begin{tikzcd} c\backslash G \arrow[r] \arrow[dr] & h(c\backslash G) \arrow[d] \\ & c\backslash hG. \end{tikzcd}$$

The upper horizontal arrow is surjective on objects and conservative and it is easy to see that the diagonal arrow is also surjective on objects and conservative so that the vertical arrow is also surjective on objects and conservative. Moreover, the upper horizontal arrow is full by construction, so we only need to show that the diagonal arrow is surjective on 2-simplices. A morphism of $c\backslash hG$ corresponds to a commutative diagram

$$\begin{tikzcd} \cdot \arrow[d] \arrow[dr] & \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

in the homotopy category hD. But (the proof of) Lemma 4.4.5 shows, that we may realize this morphism by an actual 2-simplex in D, which represents a morphism in $c\backslash G$, thus the diagonal arrow $c\backslash G \to c\backslash hG$ is surjective on 2-simplices. □

Thus we find that the solution set condition is really a 1-categorical condition.

Corollary 4.4.6. *A functor between ∞-categories $G\colon D \to C$ satisfies the solution set condition if and only if the functor $hG\colon hD \to hC$ does.*

Proof. The solution set condition for G asserts that the category $h(c\backslash G)$ admits a weakly initial set for all objects $c \in C$, while the solution set condition for hG asserts that the category $c\backslash(hG)$ admits a weakly initial set. But by Proposition 4.4.4, the canonical functor

$$h(c\backslash G) \to c\backslash hG$$

is smothering, so that $h(c\backslash G)$ admits a small weakly initial set if and only if $c\backslash hG$ does. □

We may also consider h-initial objects. We do not expect the h-initial object condition to be determined by the functor on homotopy categories in general, since smothering functors do not reflect initial objects in general. On the other hand, Lemma 4.4.3 gives a sufficient condition when this is the case.

Corollary 4.4.7. *Let $G\colon D \to C$ be a functor between ∞-categories and suppose that the slice categories $c\backslash G$ have equalizers for each $c \in C$. Then G satisfies the h-initial object condition if and only if hG does.*

Proof. We verify the conditions of Lemma 4.4.3. That is, for any pair of morphisms $[f], [g] \in h(c\backslash G)$ with the same source and target, we need to show that there exists a morphism equalizing them. Choose a representative

$f \in c\backslash G$ for $[f]$ and $g \in c\backslash G$ for $[g]$. Then f and g have the same source and target in $c\backslash G$. Since $c\backslash G$ has equalizers, we find an equalizer diagram

$$\cdot \xrightarrow{\varphi} \cdot \underset{g}{\overset{f}{\rightrightarrows}} \cdot \quad \in c\backslash G$$

In particular, $[f] \circ [\varphi] = [g] \circ [\varphi]$. Thus, since $h(c\backslash G) \to c\backslash hG$ is smothering, by Lemma 4.4.3 the category $h(c\backslash G)$ has an initial object if and only if $c\backslash hG$ has one. □

The fact that, under suitable circumstances, the h-initial condition is essentially 1-categorical leads to the following theorem on lifting adjunctions defined on the homotopy category.

Theorem 4.4.8. *Let D be an ∞-category admitting finite limits and let $G\colon D \to C$ be a functor between ∞-categories which preserves finite limits. Then G admits a left adjoint if and only if hG does.*

Proof. One direction is clear. Thus suppose hG has a left adjoint. By Proposition 4.3.2, the category $c\backslash hG$ has an initial object. Since D has finite limits and G preserves them, $c\backslash G$ has finite limits by Lemma 4.3.8. In particular, $c\backslash G$ has equalizers and hence by Corollary 4.4.7 the category $h(c\backslash G)$ has an initial object. By Proposition 4.2.4, $c\backslash G$ has an initial object and thus by Proposition 4.3.2 has a left adjoint. □

An interesting special case of Theorem 4.4.8 is the following result about equivalences of ∞-categories. This result is shown using different methods in [**Cis**, Theorem 7.6.10] and a weaker version of the result can also be found in [**Bar16**, Proposition 2.15]. Analogous results for Waldhausen categories are obtained in [**BM11**] and [**Cis10**].

Corollary 4.4.9. *Let C, D and $G\colon D \to C$ be as in Theorem 4.4.8. Then G is an equivalence of ∞-categories if and only if hG is an equivalence of (ordinary) categories.*

Proof. Note that for any ∞-category C the canonical functor

$$\mathrm{Fun}(C, C) \to \mathrm{Fun}(C, hC) \simeq \mathrm{Fun}(hC, hC)$$

is conservative. Hence by Theorem 4.4.8, if hG is an equivalence then G admits a left adjoint $F\colon C \to D$ such that the unit and counit transformations of the adjunction (F, G) are natural equivalences of functors. The converse is obvious. □

CHAPTER 5

The infinite loop space structure of the cobordism category

We switch gears to something more concrete. We compute the homotopy type of the classifying space of the cobordism category as an infinite loop space, whose infinite loop space structure is induced by a symmetric monoidal structure. The intuition for our proof is the following. The cobordism category defines an ∞-category which admits a symmetric monoidal structure given by taking disjoint union of manifolds, though we will not make this precise in this thesis. Following Lurie [**Lur17**], a symmetric monoidal structure on an ∞-category C is encoded by a coCartesian fibration

$$C^{\otimes} \to \Gamma^{op}$$

where Γ^{op} is the category of finite pointed sets (defined below), satisfying the Segal conditions and such that the fiber over the object 1_+ is equivalent to C. According to Theorem 3.4.7 this corresponds to a functor

$$\Gamma^{op} \to \mathcal{Q}$$

in other words a functor into the ∞-category of ∞-categories, and the Segal conditions translate to the preservation of products. We will use this point of view to encode coherent symmetric monoidal structures.

To be more precise, in this chapter we show that there is an equivalence of infinite loop spaces between the classifying space of the d-dimensional cobordism category $B\mathrm{Cob}_\theta(d)$ and the 0-th space of the shifted Madsen–Tillmann spectrum $MT\theta(d)[1]$. This extends a result by Galatius, Madsen, Tillmann and Weiss [**GTMW09**], who showed an equivalence of topological spaces

$$B\mathrm{Cob}_\theta(d) \simeq MT\theta(d)[1]_0.$$

Note that both spaces in the equivalence above admit infinite loop space structures. The symmetric monoidal structure on the cobordism category, given by disjoint union of manifolds, induces an infinite loop space structure on $B\mathrm{Cob}_\theta(d)$ as indicated above, while the infinite loop space structure on $MT\theta(d)[1]_0$ comes from it being the 0-th space of an Ω–spectrum. We will show that the equivalence of [**GTMW09**] actually extends to an equivalence of infinite loop spaces with the above mentioned infinite loop space structures.

In more detail, our proof will rely on certain spaces of manifolds introduced by Galatius and Randal-Williams [**GRW10**], which form an Ω–spectrum denoted here by ψ_θ. Using these spaces, they obtain a new proof of the result of [**GTMW09**], which we record as the following theorem.

THEOREM 5.0.1. *There are weak homotopy equivalences of spaces*

$$B\mathrm{Cob}_\theta(d) \simeq \psi_{\theta,0} \simeq MT\theta(d)[1]_0.$$

In this chapter, we will show that the equivalences of the above theorem come from equivalences of spectra.

Instead of directly constructing an equivalence of spectra, our strategy will be to construct Γ–spaces $\Gamma\mathrm{Cob}_\theta(d)$ and $\Gamma\psi_\theta$ with underlying spaces $B\mathrm{Cob}_\theta(d)$ and $\psi_{\theta,0}$ respectively, and we show that $\Gamma\psi_\theta$ is a model for the connective cover of the spectrum ψ_θ, denoted by $\psi_{\theta,\geq 0}$. This Γ-structure will be induced by taking disjoint union of manifolds, mirroring our intuition that this defines a symmetric monoidal structure. We then show that their associated spectra have the stable homotopy type of the connective cover of the shifted Madsen-Tillmann spectrum denoted by $MT\theta(d)[1]_{\geq 0}$, by constructing a Γ–space model for $MT\theta(d)[1]_{\geq 0}$ and exhibiting an equivalence of Γ–spaces. But more is true; we will see that the equivalences of 5.0.1 are the components of this equivalence of Γ–spaces and hence the main result of this chapter will be the following.

THEOREM 5.0.2. *There are stable equivalences of spectra*

$$\mathbf{B}\Gamma\mathrm{Cob}_\theta(d) \simeq \psi_{\theta,\geq 0} \simeq MT\theta(d)[1]_{\geq 0}$$

such that the induced weak equivalences of spaces

$$\Omega^\infty\mathbf{B}\Gamma\mathrm{Cob}_\theta(d) \simeq \Omega^\infty\psi_\theta \simeq \Omega^\infty MT\theta(d)[1]$$

are equivalent to the weak equivalences of 5.0.1.

Here, $\mathbf{B}\Gamma\mathrm{Cob}_\theta(d)$ is the spectrum associated to the symmetric monoidal category $\mathrm{Cob}_\theta(d)$. We would like to mention that a similar argument has been given by Madsen and Tillmann in [**MT01**] for the case $d = 1$.

This chapter is organized as follows. In the next section we recall some basic notions on spectra and Γ–spaces. This will also serve to fix notation and language. In Section 5.2 and Section 5.3 we review the proof of 5.0.1 of [**GRW10**]. In Section 5.4 we will construct Γ–space models for the spectra ψ_θ and $MT\theta(d)$, and in Section 5.5 we will show that these Γ–spaces are equivalent. Finally in Section 5.6, we will relate these Γ–spaces to the cobordism category with its infinite loop space structure induced by taking disjoint union of manifolds.

The contents of this chapter have appeared as [**Ngu17**].

5.1. Conventions on spectra and Γ–spaces

By a *space* we mean a compactly generated weak Hausdorff space. We denote by $\mathbf{S}$ the category of spaces and by $\mathbf{S}_*$ the category of based spaces. We fix a model for the circle by setting $S^1 := \mathbb{R} \cup \{\infty\}$.

We will work with the Bousfield–Friedlander model of sequential spectra, see Bousfield and Friedlander [**BF78**] or Mandell, May, Schwede and Shipley [**MMSS01**]. Recall that a *spectrum E* is a sequence of based spaces $E_n \in \mathbf{S}_*$, $n \in \mathbb{N}$ together with structure maps

$$s_n : S^1 \wedge E_n \to E_{n+1}.$$

A map of spectra $f : E \to F$ is a sequence of maps $f_n : E_n \to F_n$ commuting with the structure maps. We denote by **Spt** the category of spectra. A *stable equivalence* is a map of spectra inducing isomorphisms on stable homotopy groups. An Ω–*spectrum* is a spectrum E, where the adjoints of the structure maps $\Sigma E_n \to E_{n+1}$ are weak homotopy equivalences. There is a model structure on **Spt** with weak equivalences the stable equivalences and fibrant objects the Ω–spectra. Moreover, a stable equivalence between Ω–spectra is a levelwise weak homotopy equivalence. We obtain a Quillen adjunction

$$\Sigma^\infty : \mathbf{S}_* \leftrightarrow \mathbf{Spt} : \Omega^\infty$$

where Σ^∞ takes a based space to its suspension spectrum and Ω^∞ assigns to a spectrum its 0-th space.

A spectrum E is called *connective*, if its negative homotopy groups vanish. In case E is an Ω–spectrum this is equivalent to E_n being $(n-1)$-connected for all $n \in \mathbb{N}$. Note that a map $f : E \to F$ between connective Ω–spectra is a stable equivalence if and only if $f_0 : E_0 \to F_0$ is a weak homotopy equivalence. We denote by $\mathbf{Spt}_{\geq 0}$ the full subcategory of connective spectra. It is a reflective subcategory of **Spt** and we denote the left adjoint of the inclusion by

$$(-)_{\geq 0} : \mathbf{Spt} \to \mathbf{Spt}_{\geq 0}.$$

We will need two operations on spectra. The first one is the shift functor

$$(-)[1] : \mathbf{Spt} \to \mathbf{Spt}$$

defined on a spectrum E by setting $E[1]_n = E_{n+1}$ and obvious structure maps. The second operation is the loop functor

$$\Omega : \mathbf{Spt} \to \mathbf{Spt}$$

defined by $(\Omega E)_n = \Omega(E_n)$ and looping the structure maps.

We recall Segal's infinite loop space machine [**Seg88**], which provides many examples of connective spectra. We denote by Γ^{op} the skeleton of the category of finite pointed sets and pointed maps, i.e. its objects are the sets $m_+ := \{*, 1, \ldots, m\}$. A Γ-*space* is a functor

$$\Gamma^{op} \to \mathbf{S}_*$$

and we denote by $\Gamma\mathbf{S}_*$ the category of Γ–spaces and natural transformations.

There are distinguished maps $\rho_i : m_+ \to 1_+$ defined by $\rho_i(k) = *$ if $k \neq i$ and $\rho_i(i) = 1$. Let $A \in \Gamma\mathbf{S}_*$. The *Segal map* is the map

$$A(m_+) \xrightarrow{\prod_{i=1}^m \rho_i} \prod_m A(1_+).$$

A Γ–space is called *special* if the Segal map is a weak homotopy equivalence. If $A \in \Gamma\mathbf{S}_*$ is special, the set $\pi_0(A(1_+))$ is a monoid with multiplication induced by the span

$$A(1_+) \leftarrow A(2_+) \xrightarrow{\simeq} A(1_+) \times A(1_+)$$

where the left map is the map sending $i \mapsto 1$ for $i = 1, 2$ and the right map is the Segal map. A special Γ–space is called *very special* if this monoid is actually a group.

In [**BF78**], Bousfield and Friedlander construct a model structure on $\Gamma\mathbf{S}_*$ with fibrant objects the very special Γ–spaces and weak equivalences between fibrant objects levelwise weak equivalences.

There is a functor $\mathbf{B} : \Gamma\mathbf{S}_* \to \mathbf{Spt}$ defined as follows. We denote by $\mathbb{S} : \Gamma^{op} \to \mathbf{S}_*$ the inclusion of finite pointed sets into pointed spaces. Given $A \in \Gamma\mathbf{S}_*$ we have an (enriched) left Kan extension along $\mathbb{S}$

$$\begin{tikzcd} \Gamma^{op} \arrow[r, "A"] \arrow[d, "\mathbb{S}"'] & \mathbf{S}_* \\ \mathbf{S}_* \arrow[ur, dashed] & \end{tikzcd}$$

and we denote this left Kan extension by $L_{\mathbb{S}}A$. Now define $\mathbf{B}A_n := L_{\mathbb{S}}A(S^n)$. The structure maps are then given by the image of the identity morphism $S^1 \wedge S^n \to S^1 \wedge S^n$ under the composite map

$$\begin{aligned} \mathbf{S}_*(S^1 \wedge S^n, S^1 \wedge S^n) &\cong \mathbf{S}_*(S^1, \mathbf{S}_*(S^n, S^{n+1})) \\ &\to \mathbf{S}_*(S^1, \mathbf{S}_*(L_{\mathbb{S}}A(S^n), L_{\mathbb{S}}(A(S^{n+1})))) \\ &\cong \mathbf{S}_*(S^1 \wedge L_{\mathbb{S}}A(S^n), L_{\mathbb{S}}A(S^{n+1})). \end{aligned}$$

By the Barratt-Priddy-Quillen Theorem $L_{\mathbb{S}}\mathbb{S}$ is the sphere spectrum, hence the notation.

The functor $\mathbf{B}$ has a right adjoint $\mathbf{A} : \mathbf{Spt} \to \Gamma\mathbf{S}_*$ given by sending a spectrum $E \in \mathbf{Spt}$ to the Γ–space

$$n_+ \mapsto \mathbf{Spt}(\mathbb{S}^{\times n}, E)$$

using the topological enrichment of spectra. Moreover, the adjoint pair $\mathbf{B} \dashv \mathbf{A}$ is a Quillen pair which induces an equivalence of categories

$$\mathrm{Ho}(\Gamma\mathbf{S}_*) \simeq \mathrm{Ho}(\mathbf{Spt}_{\geq 0}).$$

In view of this equivalence we will say that a *Γ–space A* is a *model for a connective spectrum E*, if there is a stable equivalence $\mathbb{L}\mathbf{B}A \simeq E$, where $\mathbb{L}\mathbf{B}$ is the left derived functor. The main theorem of Segal [**Seg88**] states that $\mathbf{B}$ sends cofibrant-fibrant Γ–spaces to connective Ω–spectra.

Finally we make the following convention. We will refer to any zig-zag of equivalences (of spaces, spectra or Γ–spaces) as simply an *equivalence*.

5.2. Recollection on spaces of manifolds

We recall the spaces $\Psi_\theta(\mathbb{R}^n)$ of embedded manifolds with tangential structure from Galatius and Randall-Williams [**GRW10**]. Denote by $Gr_d(\mathbb{R}^n)$ the Grassmannian manifold of d-dimensional planes in $\mathbb{R}^n$ and denote $BO(d) := \mathrm{colim}_{n\in\mathbb{N}}\, Gr_d(\mathbb{R}^n)$ induced by the standard inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$. Let $\theta : X \to BO(d)$ be a Serre fibration and let $M \subset \mathbb{R}^n$ be a d-dimensional embedded smooth manifold. Then a *tangential θ-structure* on M is a lift

$$\begin{tikzcd} & X \arrow[d, "\theta"] \\ M \arrow[ur, dashed] \arrow[r, "\tau_M"'] & BO(d), \end{tikzcd}$$

where τ_M is the classifying map of the tangent bundle (determined by the embedding). The topological space $\Psi_\theta(\mathbb{R}^n)$ has as underlying set pairs (M,l), where M is a d-dimensional smooth manifold without boundary which is closed as a subset of $\mathbb{R}^n$ and $l : M \to X$ is a θ-structure. We refer to [**GRW10**] for a description of the topology. We will also in general suppress the tangential structure from the notation.

For $0 \le k \le n$, we have the subspaces $\psi_\theta(n,k) \subset \Psi_\theta(\mathbb{R}^n)$ of those manifolds $M \subset \mathbb{R}^n$ satisfying

$$M \subset \mathbb{R}^k \times (-1,1)^{n-k}.$$

In other words, $\psi_\theta(n,k)$ consists of manifolds with k possibly non-compact and $(n-k)$ compact directions. We denote

$$\Psi_\theta(\mathbb{R}^\infty) := \operatorname{colim}_{n\in\mathbb{N}} \Psi_\theta(\mathbb{R}^n)$$
$$\psi_\theta(\infty,k) := \operatorname{colim}_{n\in\mathbb{N}} \psi_\theta(n,k)$$

where the colimit is again induced by the standard inclusions. In [**BM14**] it is shown that the topological spaces $\Psi_\theta(\mathbb{R}^n)$ are metrizable and hence in particular compactly generated weak Hausdorff spaces.

For all $n \in \mathbb{N}$ and $1 \le k \le n-1$ we have a map

$$\mathbb{R} \times \psi_\theta(n,k) \to \psi_\theta(n,k+1)$$
$$(t,M) \mapsto M - t \cdot e_{k+1}$$

where e_{k+1} denotes the $(k+1)$-st standard basis vector. This descends to a map $S^1 \wedge \psi_\theta(n,k) \to \psi_\theta(n,k+1)$ when taking as basepoint the empty manifold.

Theorem 5.2.1. *The adjoint map*

$$\psi_\theta(n,k) \to \Omega\psi_\theta(n,k+1)$$

is a weak homotopy equivalence.

Proof. See Galatius and Randal-Williams [**GRW10**, Theorem 3.20]. □

Definition 5.2.2. Let ψ_θ be the spectrum with n-th space given by

$$(\psi_\theta)_n := \psi_\theta(\infty,n+1)$$

and structure maps given by the adjoints of the translations.

By the above theorem, the spectrum ψ_θ is an Ω–spectrum.

5.3. The weak homotopy type of $\psi_\theta(\infty,1)$

This section contains a brief review of the main theorem of Galatius, Madsen, Tillmann and Weiss [**GTMW09**] as proven by Galatius and Randal-Williams [**GRW10**]. Recall first the construction of the *Madsen–Tillmann spectrum* $MT\theta(d)$ associated to a Serre fibration $\theta : X \to BO(d)$. Denote by $X(\mathbb{R}^n)$ the pullback

$$\begin{tikzcd} X(\mathbb{R}^n) \arrow[r] \arrow[d, "\theta_n"'] & X \arrow[d, "\theta"] \\ Gr_d(\mathbb{R}^n) \arrow[r] & BO(d) \end{tikzcd}$$

and by $\gamma_{d,n}^{\perp}$ the orthogonal complement of the tautological bundle over $Gr_d(\mathbb{R}^n)$. Then define the spectrum $T\theta(d)$ to have as n-th space the Thom space of the pullback bundle $T\theta(d)_n := Th(\theta_n^*\gamma_{d,n}^{\perp})$. The structure maps are given by

$$S^1 \wedge Th(\theta_n^*\gamma_{d,n}^{\perp}) \cong Th(\theta_n^*\gamma_{d,n}^{\perp} \oplus \varepsilon) \to Th(\theta_{n+1}^*\gamma_{d,n+1}^{\perp})$$

where ε denotes the trivial bundle. Then define the Madsen-Tillmann spectrum $MT\theta(d)$ to be a fibrant replacement of the spectrum $T\theta(d)$. Since the adjoints of the structure maps of $T\theta(d)$ are inclusions, we can give an explicit construction of $MT\theta(d)$ as

$$MT\theta(d)_n := \mathrm{colim}_k\, \Omega^k T\theta(d)_{n+k}.$$

Hence we have $\Omega^\infty MT\theta(d) = \mathrm{colim}_k\, \Omega^k T\theta(d)_k$.

The passage from $MT\theta(d)$ to our spaces of manifolds is as follows. We have a map

$$Th(\theta_n^*\gamma_{d,n}^{\perp}) \to \Psi_\theta(\mathbb{R}^n)$$

given by sending an element (V,u,x), where $V \in Gr_d(\mathbb{R}^n)$, $u \in V^\perp$ and $x \in X$, to the translated plane $V - u \in \Psi_\theta(\mathbb{R}^n)$ with constant θ-structure at x and sending the basepoint to the empty manifold.

Theorem 5.3.1. *The map $Th(\theta_n^*\gamma_{d,n}^{\perp}) \to \Psi_\theta(\mathbb{R}^n)$ is a weak homotopy equivalence.*

Proof. See [**GRW10**, thm 3.22]. □

On the other hand, by 5.2.1 we also have a weak homotopy equivalence

$$\psi_\theta(n,1) \to \Omega^{n-1}\Psi_\theta(\mathbb{R}^n).$$

Combining the two equivalences, we obtain

$$\Omega^{n-1} Th(\theta_n^*\gamma_{d,n}^{\perp}) \xrightarrow{\simeq} \Omega^{n-1}\Psi_\theta(\mathbb{R}^n) \xleftarrow{\simeq} \psi_\theta(n,1).$$

Now we have a map

$$\begin{aligned} S^1 \wedge \Psi_\theta(\mathbb{R}^n) &\to \Psi_\theta(\mathbb{R}^{n+1}) \\ (t,M) &\mapsto M \times \{t\}, \end{aligned}$$

and we obtain the commutative diagram

$$\begin{tikzcd} \Omega^{n-1} Th(\theta_n^*\gamma_{d,n}^{\perp}) \arrow[r, "\simeq"] \arrow[d] & \Omega^{n-1}\Psi_\theta(\mathbb{R}^n) \arrow[d] & \psi_\theta(n,1) \arrow[l, "\simeq"'] \arrow[d] \\ \Omega^{n} Th(\theta_{n+1}^*\gamma_{d,n+1}^{\perp}) \arrow[r, "\simeq"] & \Omega^{n}\Psi_\theta(\mathbb{R}^{n+1}) & \psi_\theta(n+1,1). \arrow[l, "\simeq"'] \end{tikzcd}$$

Finally, letting $n \to \infty$ we can determine the weak homotopy type of $\psi_\theta(\infty,1)$.

Theorem 5.3.2. *There are weak equivalences of spaces*

$$\Omega^\infty MT\theta(d)[1] \xrightarrow{\simeq} \mathrm{colim}_{n\in\mathbb{N}}\, \Omega^{n-1}\Psi_\theta(\mathbb{R}^n) \xleftarrow{\simeq} \psi_\theta(\infty,1).$$

5.4. Γ–space models for $MT\theta(d)$ and ψ_θ

In this section we construct Γ–space models for the spectra $MT\theta(d)$ and ψ_θ. The comparison of these Γ–spaces to the respective spectra relies heavily on results of May and Thomason [**MT78**].

We will encounter the following situation.

Definition 5.4.1. A functor $E : \Gamma^{op} \to \mathbf{Spt}$ is called a Γ–*spectrum*. It is called a *special* Γ–spectrum if the Segal map

$$E(m_+) \to \prod_m E(1_+)$$

is a stable equivalence. Furthermore, we denote by $\Gamma^{(k)}E$ the Γ–space given by evaluating at the k-th space, that is

$$\Gamma^{(k)}E(m_+) := E(m_+)_k.$$

The key proposition for showing that we have constructed the right Γ–spaces will be the following.

Proposition 5.4.2. *Let $E : \Gamma^{op} \to \mathbf{Spt}$ be projectively fibrant and special. Then the Γ–space $\Gamma^{(k)}E$ is a model for the connective cover of $E(1_+)[k]$.*

Before we can prove the proposition, we will need some lemmas. The first one concerns the behavior of Segal's functor $\mathbf{B}$ with respect to the loop functor.

Lemma 5.4.3. *For $A \in \Gamma\mathbf{S}_*$ there is a natural map of spectra*

$$\mathbf{B}\Omega A \to \Omega\mathbf{B}A$$

which is the identity on 0-th spaces.

Proof. Since $\mathbb{S} : \Gamma^{op} \to \mathbf{S}_*$ is fully faithful, we have a strictly commutative diagram of functors

$$\begin{tikzcd} \Gamma^{op} \arrow[r, "\Omega A"] \arrow[d, "\mathbb{S}"'] & \mathbf{S}_* \\ \mathbf{S}_*. \arrow[ur, "L_{\mathbb{S}}\Omega A"'] & \end{tikzcd}$$

The composition of the loop functor with the left Kan extension $\Omega L_{\mathbb{S}}A$ also gives a strictly commutative diagram

$$\begin{tikzcd} \Gamma^{op} \arrow[r, "\Omega A"] \arrow[d, "\mathbb{S}"'] & \mathbf{S}_* \\ \mathbf{S}_*. \arrow[ur, "\Omega L_{\mathbb{S}}A"'] & \end{tikzcd}$$

Hence by the universal property of the left Kan extension we get a natural transformation $\gamma : L_{\mathbb{S}}\Omega A \Rightarrow \Omega L_{\mathbb{S}}A$. Now the components at the spheres assemble into a map of spectra $\mathbf{B}\Omega A \to \Omega\mathbf{B}A$, since by naturality we have a commutative diagram

$$\begin{array}{ccc} S^1 \wedge L_{\$}\Omega A(S^n) & \longrightarrow & L_{\$}\Omega A(S^{n+1}) \\ {\scriptstyle id\wedge\gamma}\downarrow & & \downarrow{\scriptstyle \gamma} \\ S^1 \wedge \Omega L_{\$} A(S^n) & \longrightarrow & \Omega L_{\$} A(S^{n+1}). \end{array}$$

Finally, since $S^0 = 1_+ \in \Gamma^{op}$, the map of spectra is the identity on 0-th spaces. □

In general, for any $A \in \Gamma\mathbf{S}_*$ the spectrum $\mathbf{B}A$ might not have the right stable homotopy type as the functor $\mathbf{B}$ only preserves weak equivalences between cofibrant objects. However, for very special Γ–spaces, there is a more convenient replacement, which gives the right homotopy type. As a second lemma we record the following fact from May-Thomason [**MT78**], which generalizes a construction of Segal in [**Seg88**].

Lemma 5.4.4. *There is a functor $W : \Gamma\mathbf{S}_* \to \Gamma\mathbf{S}_*$ such that the following holds for all very special $X \in \Gamma\mathbf{S}_*$.*

- *The spectrum $\mathbf{B}WX$ is a connective Ω–spectrum.*
- *The Γ–space WX is very special and there is a weak equivalence $WX \to X$.*
- *If X, Y are very special and there is a weak equivalence $X \simeq Y$, then $\mathbf{B}WX \simeq \mathbf{B}WY$.*
- *There is a weak equivalence $W\Omega X \to \Omega WX$.*

Proof. See [**MT78**, Appendix B]. □

The important thing for us will be that if $X \in \Gamma\mathbf{S}_*$ is very special, then $\mathbf{B}WX$ has the right stable homotopy type.

Lemma 5.4.5. *Let E^i, $i \in \mathbb{N}$ be a sequence of connective Ω–spectra together with stable equivalences $f^i : E^i \to \Omega E^{i+1}$. Let E_0 be the spectrum with $(E_0)_n := E_0^n$ and structure maps given by $f_0^n : E_0^n \to \Omega E_0^{n+1}$. Then there is a natural stable equivalence $E^0 \simeq E_0$.*

Proof. This is the 'up-and-across lemma' of May-Thomason [**MT78**] and Fiedorowicz [**Fie77**]. □

Note that in particular E_0 is connective. We are now ready to prove our key proposition.

Proof of Proposition 5.4.2. We prove the proposition for $k = 0$. The argument for higher k is completely analogous.

We first show that the Γ–space $\Gamma^{(0)}E$ is very special. Note that the Γ–spaces $\Gamma^{(k)}E$ are special, since E is projectively fibrant and thus the Segal map is a levelwise equivalence. It remains to show that $\pi_0\left(\Gamma^{(0)}E(1_+)\right)$ is a group. To this end, we compose with the functor $\mathbf{A} : \mathbf{Spt} \to \Gamma\mathbf{S}_*$ to obtain a functor

$$\Gamma^{op} \xrightarrow{E} \mathbf{Spt} \xrightarrow{\mathbf{A}} \Gamma\mathbf{S}_*$$

which is equivalently a functor

$$\widehat{A} := \Gamma^{op} \times \Gamma^{op} \to \mathbf{S}_*.$$

Fixing the first variable gives a Γ–space

$$\widehat{A}(k_+)(-) : \Gamma^{op} \to \mathbf{S}_*$$

which is obtained by first evaluating the Γ–spectrum E at k_+ and then applying the functor $\mathbf{A}$ to the spectrum $E(k_+)$. In particular, we have

$$\widehat{A}(1_+)(-) = \mathbf{A}\left(E(1_+)\right) : \Gamma^{op} \to \mathbf{S}_*$$

which is very special by construction.

Fixing the second variable gives a Γ–space

$$\widehat{A}(-)(k_+) : \Gamma^{op} \to \mathbf{S}_*$$

which is obtained as the composition

$$\Gamma^{op} \xrightarrow{E} \mathbf{Spt} \xrightarrow{\mathbf{A}} \Gamma\mathbf{S}_* \xrightarrow{ev_{k_+}} \mathbf{S}_*$$

where the last functor is given by evaluating a Γ–space at the object k_+. In particular, we have

$$\widehat{A}(-)(1_+) = \mathbf{A}(E(-))(1_+) = \Gamma^{(0)}E : \Gamma^{op} \to \mathbf{S}_*$$

which is special since $\Gamma^{(0)}E$ is special.

Now we have the following diagram, where the middle square commutes by functoriality

$$\begin{tikzcd}
 & & \widehat{A}(1_+)(1_+) \times \widehat{A}(1_+)(1_+) \\
 & \widehat{A}(2_+)(2_+) \arrow[r] \arrow[d] & \widehat{A}(2_+)(1_+) \arrow[u, "\simeq"'] \arrow[d] \\
\widehat{A}(1_+)(1_+) \times \widehat{A}(1_+)(1_+) & \widehat{A}(1_+)(2_+) \arrow[l, "\simeq"] \arrow[r] & \widehat{A}(1_+)(1_+).
\end{tikzcd}$$

By the above identification of the Γ–spaces $\widehat{A}(-)(1_+)$ and $\widehat{A}(1_+)(-)$ we see that the right vertical span represents the monoid structure of $\Gamma^{(0)}E$ and the lower horizontal span represents the monoid structure of $\mathbf{A}E(1_+)$. In other words, the maps into the products in the lower left and upper right corner are given by the Segal maps while the maps into the lower right corner are the respective multiplications induced by the non-trivial map $2_+ \to 1_+$, as are the remaining maps.

Hence we obtain two monoid structures on $\pi_0\left(\widehat{A}(1_+)(1_+)\right)$ induced by $\mathbf{A}E(1_+)$ and $\Gamma^{(0)}E$. The commutativity of the middle square is now precisely the statement that they are compatible, or in other words that one is a homomorphism for the other, thus they agree by the Eckmann-Hilton argument. We now observe that the monoid $\mathbf{A}E(1_+)$ is actually a group, since $\pi_0(\mathbf{A}E(1_+)(1_+))$ is the 0-th stable homotopy group of $E(1_+)$. It follows that $\Gamma^{(0)}E$ is very special.

As a next step, we compose with taking connective covers to obtain a special Γ–spectrum in connective Ω–spectra

$$E_{\geq 0} : \Gamma^{op} \to \mathbf{Spt}_{\geq 0}.$$

Note that $\Gamma^{(0)}E \simeq \Gamma^{(0)}E_{\geq 0}$ and hence $\Gamma^{(0)}E_{\geq 0}$ is very special. For $k \geq 1$ the Γ–spaces $\Gamma^{(k)}E_{\geq 0}$ will automatically be very special since $E_{\geq 0}(1_+)$ is connective and hence $\pi_0\left(\Gamma^{(k)}E(1_+)\right) \cong \pi_0\left(E_{\geq 0}(1_+)_k\right) = 0$.

We now consider the spectra associated to the very special Γ–spaces $\Gamma^{(k)}E_{\geq 0}$, i.e. we apply May-Thomason's replacement followed by Segal's functor to obtain a sequence of connective Ω–spectra

$$\mathbf{B}W\Gamma^{(k)}E_{\geq 0} \text{ for } k \in \mathbb{N}.$$

Now by Lemma 5.4.4 we have the following equivalence

$$\mathbf{B}W\Gamma^{(k)}E_{\geq 0} \xrightarrow{\simeq} \mathbf{B}W\Omega\Gamma^{(k+1)}E_{\geq 0} \xrightarrow{\simeq} \mathbf{B}\Omega W\Gamma^{(k+1)}E_{\geq 0}.$$

By Lemma 5.4.3 we have a map $\mathbf{B}\Omega W\Gamma^{(k)}E \to \Omega\mathbf{B}W\Gamma^{(k)}E$ which is the identity on 0-th spaces. In particular, since both spectra are Ω–spectra, this map is an equivalence on connective covers. We now observe that since $E(1_+)_{\geq 0}$ is a connective Ω–spectrum we have

$$\pi_0\left(\mathbf{B}W\Gamma^{(k)}E\right) = \pi_0\left(E(1^+)_{\geq 0,k}\right) = 0$$

for $k \geq 1$ and hence $\Omega\mathbf{B}W\Gamma^{(k)}E$ is connective. Thus we obtain a stable equivalence

$$\mathbf{B}\Omega W\Gamma^{(k)}E \to \Omega\mathbf{B}W\Gamma^{(k)}E$$

for $k \geq 1$. Putting all these maps together we obtain a sequence of connective Ω–spectra $\mathbf{B}W\Gamma^{(k)}E_{\geq 0}$ together with stable equivalences

$$\mathbf{B}W\Gamma^{(k)}E_{\geq 0} \xrightarrow{\simeq} \mathbf{B}W\Omega\Gamma^{(k+1)}E_{\geq 0} \xrightarrow{\simeq} \mathbf{B}\Omega W\Gamma^{(k+1)}E_{\geq 0} \xrightarrow{\simeq} \Omega\mathbf{B}W\Gamma^{(k+1)}E_{\geq 0},$$

that is, we have $\mathbf{B}W\Gamma^{(k)}E_{\geq 0} \xrightarrow{\simeq} \Omega\mathbf{B}W\Gamma^{(k+1)}E_{\geq 0}$. Thus we are in the situation of Lemma 5.4.5 and conclude that

$$\mathbf{B}W\Gamma^{(0)}E_{\geq 0} = \mathbf{B}W\Gamma^{(0)}E \simeq E(1_+)_{\geq 0}.$$

□

In light of obtaining the right stable homotopy type, we will from now on assume that we replace a Γ–space A by WA before applying the functor $\mathbf{B}$, i.e. in what follows $\mathbf{B}A$ will mean $\mathbf{B}WA$.

We start with constructing a Γ–space model for the (connective cover of the) spectrum ψ_θ. Recall that the spectrum ψ_θ has as n-th space the space $\psi_\theta(\infty, n+1)$ and structure maps given by translation of manifolds in the $(n+1)$-st coordinate. The idea is that the spaces $\psi_\theta(\infty, n)$ come with a preferred monoid structure, namely taking disjoint union of manifolds. To make this precise, we introduce the following notation.

Definition 5.4.6. Let $\theta : X \to BO(d)$ be a Serre fibration. We obtain for each $m \in \mathbb{N}$ the Serre fibration

$$\coprod_m \theta : \coprod_m X \to BO(d).$$

We denote this Serre fibration by $\theta(m_+)$.

We can now associate to each $m_+ \in \Gamma^{op}$ the space $\Psi_{\theta(m_+)}(\mathbb{R}^n)$. We think of elements of $\Psi_{\theta(m_+)}(\mathbb{R}^n)$ as manifolds with components labeled by non-basepoint elements of m_+ together with θ-structures on those labeled components.

Lemma 5.4.7. *For all $n \in \mathbb{N}$, the spaces $\Psi_{\theta(m_+)}(\mathbb{R}^n)$ assemble into a Γ–space.*

Proof. We have to define the induced maps. Let $\sigma : m_+ \to k_+$ be a map of based sets. We obtain a map

$$\coprod_{\sigma^{-1}(k_+\backslash\{*\})} X \to \coprod_{k_+\backslash\{*\}} X.$$

Now define the induced map $\Psi_{\theta(m_+)}(\mathbb{R}^n) \to \Psi_{\theta(k_+)}(\mathbb{R}^n)$ as follows. The image of a pair (M,l) is given by the manifold

$$M' := l^{-1}\left(\coprod_{\sigma^{-1}(k_+\backslash\{*\})} X\right)$$

together with $\theta(k_+)$-structure given by the composition

$$M' \xrightarrow{l|_{M'}} \coprod_{\sigma^{-1}(k_+\backslash\{*\})} X \to \coprod_{k_+\backslash\{*\}} X.$$

In other words, we relabel the components of M and forget about those components, which get labeled by the basepoint. Taking the empty manifold as basepoint, it is easy to see that this is functorial in Γ^{op}. □

Note that $\Psi_{\theta(0_+)} \cong *$ since it consists of only the empty manifold and that we have $\Psi_{\theta(1_+)}(\mathbb{R}^n) = \Psi_\theta(\mathbb{R}^n)$. Also note that we obtain by restriction for any $k \geq 1$ the Γ–spaces

$$m_+ \mapsto \psi_{\theta(m_+)}(\infty, k).$$

As mentioned above, the Γ-structure can be thought of as taking disjoint union of manifolds. Below we will see that, when stabilizing to $\mathbb{R}^\infty$, taking disjoint union gives a homotopy coherent multiplication on our spaces of manifolds.

Lemma 5.4.8. *The spectra $\psi_{\theta(m_+)}$ assemble into a projectively fibrant Γ–spectrum.*

Proof. By the above lemma we have for each $n \in \mathbb{N}$ and each map of finite pointed sets $\sigma : m_+ \to k_+$ a map

$$\sigma_*^n : \psi_{\theta(m_+)}(\infty, n+1) \to \psi_{\theta(k_+)}(\infty, n+1)$$

which is functorial in Γ^{op} for fixed n. Thus, we have to show that these maps commute with the structure maps, that is we need to show that the diagram

$$\begin{tikzcd} S^1 \wedge \psi_{\theta(m_+)}(\infty, n+1) \arrow[r] \arrow[d, "id\wedge\sigma_*^n"'] & \psi_{\theta(m_+)}(\infty, n+2) \arrow[d, "\sigma_*^{n+1}"] \\ S^1 \wedge \psi_{\theta(k_+)}(\infty, n+1) \arrow[r] & \psi_{\theta(k_+)}(\infty, n+2) \end{tikzcd}$$

commutes. But this is clear since the structure maps just translate the manifolds in the $(n+1)$-st coordinate, while the map σ_*^n relabels the components. □

DEFINITION 5.4.9. We denote by $\Gamma\psi_\theta$ the Γ–spectrum

$$m_+ \mapsto \psi_{\theta(m_+)}.$$

To avoid awkward notation, we will denote the induced Γ–spaces $\Gamma^{(k)}(\Gamma\psi_\theta)$ simply by $\Gamma^{(k)}\psi_\theta$.

PROPOSITION 5.4.10. *The Γ–space $\Gamma^{(0)}\psi_\theta$ is a model for the connective cover of ψ_θ, i.e. there is a stable equivalence*

$$\mathbf{B}\Gamma^{(0)}\psi_\theta \simeq \psi_{\theta,\geq 0}.$$

PROOF. We show that $\Gamma\psi_\theta$ is a special Γ–spectrum. The assertion then follows from Proposition 5.4.2. Since $\psi_{\theta(m_+)}$ is an Ω–spectrum for all $m_+ \in \Gamma^{op}$, it suffices to show that $\Gamma^{(k)}\psi_\theta$ is a special Γ–space for every k.

We observe that the Segal map for $\Gamma^{(k)}\psi_\theta$

$$\Gamma^{(k)}\psi_\theta(m_+) \to \prod_m \Gamma^{(k)}\psi_\theta(m_+)$$

is an embedding and we identify its image with a subspace of the product space. This subspace can be characterized as follows. A tuple $(M_1,\ldots,M_m)$ lies in this subspace if and only if $M_i \cap M_j = \emptyset \subset \mathbb{R}^\infty$ for all $i \neq j$. We show that this subspace is a weak deformation retract of the product space

$$\prod_m \Gamma^{(k)}\psi_\theta(m_+) = \prod_m \psi_\theta(\infty, k+1).$$

To this end, we need a map making manifolds (or more generally any subsets) disjoint inside $\mathbb{R}^\infty$. Consider the maps

$$F: \mathbb{R}^\infty \to \mathbb{R}^\infty$$
$$(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$$

as well as for any $a \in \mathbb{R}$ the map

$$G_a: \mathbb{R}^\infty \to \mathbb{R}^\infty$$
$$(x_1, x_2, \ldots) \mapsto (a + x_1, x_2, \ldots).$$

These maps are clearly homotopic to the identity via a straight line homotopy. Choosing $a \in (-1,1)$, the composition $G_a \circ F : \mathbb{R}^\infty \to \mathbb{R}^\infty$ induces a self-map

$$\psi_\theta(\infty, k+1) \to \psi_\theta(\infty, k+1)$$

which is homotopic to the identity. Using for each factor of the product space $\prod_m \psi_\theta(\infty, k+1)$ a different (fixed) real number gives a map

$$\prod_m \psi_\theta(\infty, k+1) \to \prod_m \psi_\theta(\infty, k+1)$$

which is our desired deformation retract; this is also illustrated in Figure 1. □

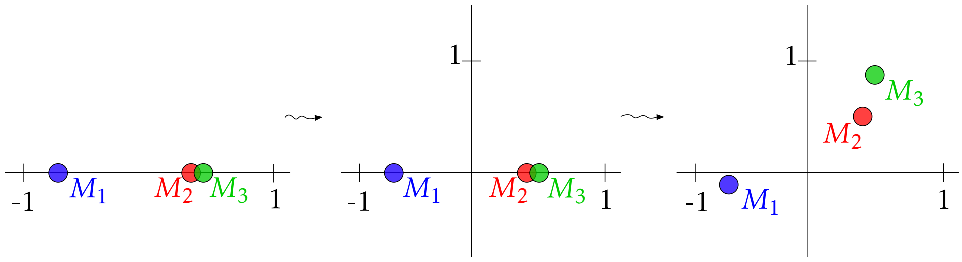

FIGURE 1. Making manifolds disjoint

Recall from Lemma 5.4.7 that the association

$$m_+ \mapsto \Psi_{\theta(m_+)}(\mathbb{R}^n)$$

defines a Γ–space for all $n \in \mathbb{N}$.

DEFINITION 5.4.11. Denote by $\Gamma\Psi_\theta$ the (levelwise) colimit of Γ–spaces

$$\Gamma\Psi_\theta(m_+) := \operatorname{colim}_{n\in\mathbb{N}} \Omega^{n-1}\Psi_{\theta(m_+)}(\mathbb{R}^n).$$

From Theorem 5.3.2 we obtain for each $m_+ \in \Gamma^{op}$ equivalences

$$\Gamma^{(0)}\psi_\theta(m_+) = \psi_{\theta(m_+)}(\infty, 1) \xrightarrow{\simeq} \operatorname{colim}_{n\in\mathbb{N}} \Omega^{n-1}\Psi_{\theta(m_+)}(\mathbb{R}^n) = \Gamma\Psi_\theta(m_+)$$

which are clearly functorial in Γ^{op}. Hence we obtain a levelwise equivalence of Γ–spaces $\Gamma^{(0)}\psi_\theta \xrightarrow{\simeq} \Gamma\Psi_\theta$.

COROLLARY 5.4.12. *The Γ–space $\Gamma\Psi_\theta$ is a model for the connective cover of the spectrum ψ_θ.*

We now construct a Γ–space model for the Madsen-Tilmann spectrum $MT\theta(d)$ and we will show in the next section that this Γ–space is equivalent to $\Gamma\Psi_\theta$. As before, we will use the Serre fibrations $\theta(m_+)$. First note that the construction of the Madsen-Tillmann spectrum commutes with coproducts over $BO(d)$, that is we have $MT\theta(m_+)(d) \cong \bigvee_m MT\theta(d)$.

DEFINITION 5.4.13. Define the Γ–spectrum $\Gamma MT\theta(d) : \Gamma^{op} \to \mathbf{Spt}$ by setting

$$\Gamma MT\theta(d)(m_+) := MT\theta(m_+)(d).$$

For any based map $\sigma : m_+ \to k_+$, define the induced map to be the fold map

$$\Gamma MT\theta(d)(m_+) \cong \bigvee_m MT\theta(d) \to \bigvee_k MT\theta(d) \cong \Gamma MT\theta(d)(k_+).$$

As before, we will denote the induced Γ–spaces by $\Gamma^{(k)}MT\theta(d)$ for all $k \in \mathbb{N}$.

PROPOSITION 5.4.14. *The Γ–space $\Gamma^{(1)}MT\theta(d)$ is a model for the connective cover of the spectrum $MT\theta(d)[1]$.*

PROOF. Again it suffices to show that $\Gamma MT\theta(d)$ is special. But this follows easily since in **Spt** we have a stable equivalence

$$MT\theta(m_+)(d) \cong \bigvee_m MT\theta(d) \simeq \prod_m MT\theta(d).$$

Thus by Proposition 5.4.2 we obtain a stable equivalence

$$\mathbf{B}\Gamma^{(1)}MT\theta(d) \simeq MT\theta(d)[1]_{\geq 0}.$$

□

5.5. Equivalence of Γ–space models

In the previous section we have constructed the Γ–space models $\Gamma\Psi_\theta$ for ψ_θ and $\Gamma^{(1)}MT\theta(d)$ for $MT\theta(d)[1]_{\geq 0}$. But more is true; by Theorem 5.3.2 we have for each $m_+ \in \Gamma^{op}$ a weak equivalence of spaces

$$\Gamma^{(1)}MT\theta(d)(m_+) = \Omega^\infty MT\theta(m_+)(d)[1] \xrightarrow{\simeq} \operatorname{colim}_{n\in\mathbb{N}} \Omega^{n-1}\Psi_{\theta(m_+)}(\mathbb{R}^n) = \Gamma\Psi_\theta(m_+).$$

The following lemma shows that these equivalences define a levelwise equivalence of Γ–spaces.

Lemma 5.5.1. *The weak equivalences of Theorem 5.3.1*

$$Th\left(\theta_n^*\gamma_{d,n}^\perp\right) \xrightarrow{\simeq} \Psi_\theta\left(\mathbb{R}^n\right)$$

assemble into a map of Γ*–spaces. In particular, we obtain a levelwise equivalence*

$$\Gamma^{(1)}MT\theta(d) \xrightarrow{\simeq} \Gamma\Psi_\theta.$$

Proof. We need to show that for any map of based sets $\sigma : m_+ \to k_+$ the diagram

$$\begin{tikzcd} Th\left(\theta_n(m_+)^*\gamma_{d,n}^\perp\right) \arrow[r] \arrow[d, "\sigma_*"'] & \Psi_{\theta(m_+)}(\mathbb{R}^n) \arrow[d, "\sigma_*"] \\ Th\left(\theta_n(k_+)^*\gamma_{d,n}^\perp\right) \arrow[r] & \Psi_{\theta(k_+)}(\mathbb{R}^n) \end{tikzcd}$$

commutes. But this follows easily since the left hand vertical map is just the fold map. In particular one can view this map as relabeling components of the wedge and mapping components labeled by $*$ to the basepoint. On the other hand this is precisely the description of the right hand vertical map. □

We can now prove the first part of our main theorem.

Theorem 5.5.2. *There is an equivalence of spectra*

$$MT\theta(d)[1]_{\geq 0} \simeq \psi_{\theta,\geq 0}.$$

Proof. By Lemma 5.5.1 we have an equivalence of Γ–spaces

$$\Gamma^{(1)}MT\theta(d) \xrightarrow{\simeq} \Gamma\Psi_\theta.$$

By Proposition 5.4.14, $\Gamma^{(1)}MT\theta(d)$ is a model for the spectrum $MT\theta(d)[1]_{\geq 0}$, while by Proposition 5.4.10 and its corollary, the Γ–space $\Gamma\Psi_\theta$ is a model for the connective cover of ψ_θ. Hence we obtain equivalences

$$MT\theta(d)[1]_{\geq 0} \simeq \mathbf{B}\Gamma^{(1)}MT\theta(d) \simeq \mathbf{B}\Gamma\Psi_\theta \simeq \psi_{\theta,\geq 0}.$$

□

5.6. The cobordism category

In the previous section we have exhibited an equivalence between the connective covers of the spectra $MT\theta(d)[1]$ and ψ_θ. It remains to relate these spectra to the (classifying space of the) topological cobordism category.

Classically, the d-dimensional cobordism category has as objects closed $(d-1)$-dimensional manifolds and morphisms given by diffeomorphism classes of cobordisms. It is a symmetric monoidal category with monoidal product given by taking disjoint union of manifolds. We will see that this is also true for the topological variant in a sense we will make precise below. In particular, having a symmetric monoidal structure endows the classifying space of the cobordism category with the structure of an infinite loop space and we will see that it is equivalent as such to the infinite loop space associated to $MT\theta(d)[1]$.

Recall that a *topological category* $\mathcal{C}$ has a space of objects $\mathcal{C}_0$ and a space of morphisms $\mathcal{C}_1$ together with source and target maps

$$s, t : \mathcal{C}_1 \to \mathcal{C}_0$$

a composition map

$$c : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$$

and a unit map

$$e : \mathcal{C}_0 \to \mathcal{C}_1$$

which satisfy the usual associativity and unit laws. There have appeared several definitions of the cobordism category as a topological category, which all have equivalent classifying spaces. The relevant model for us will be the topological poset model of Galatius and Randal-Williams [**GRW10**]. We recall its definition. Define D_θ to be the subspace

$$D_\theta \subset \mathbb{R} \times \psi_\theta(\infty, 1)$$

consisting of pairs (t, M) where $t \in \mathbb{R}$ is a regular value of the projection onto the first coordinate $M \subset \mathbb{R} \times (-1,1)^\infty \to \mathbb{R}$. Order its elements by $(t, M) \leq (t', M')$ if and only if $t \leq t'$ with the usual order on $\mathbb{R}$ and $M = M'$.

Definition 5.6.1. The *d-dimensional cobordism category* $\mathrm{Cob}_\theta(d)$ is the topological category associated to the topological poset D_θ. That is, its space of objects is given by $ob(\mathrm{Cob}_\theta(d)) = D_\theta$ and its space of morphisms is given by the subspace $mor(\mathrm{Cob}_\theta(d)) \subset \mathbb{R}^2 \times \psi_\theta(\infty, 1)$ consisting of triples (t_0, t_1, M) where $t_0 \leq t_1$. The source and target maps are simply given by forgetting regular values.

Given a topological category $\mathcal{C}$ we can take its internal nerve yielding a simplicial space

$$N_\bullet \mathcal{C} : \Delta^{op} \to \mathbf{S}$$

as follows. The space of 0-simplices and 1-simplices is given by $\mathcal{C}_0$ and $\mathcal{C}_1$ respectively. For $n \geq 2$ the space of n-simplices is given by the n-fold fiber product

$$N_n\mathcal{C} := \mathcal{C}_1 \times_{\mathcal{C}_0} \ldots \times_{\mathcal{C}_0} \mathcal{C}_1.$$

The face and the degeneracy maps are obtained from the structure maps of the topological category. The associativity and unit laws ensure that we indeed obtain a simplicial space. Applying this construction to the cobordism category now yields a simplicial space

$$N_\bullet \mathrm{Cob}_\theta(d) : \Delta^{op} \to \mathbf{S}.$$

We will also write $\mathrm{Cob}_\theta(d)$ for the simplicial space obtained from taking the nerve and write $\mathrm{Cob}_\theta(d)_k$ for the space of k-simplices.

Considering $\psi_\theta(\infty, 1)$ as a constant simplicial space, we have a forgetful map of simplicial spaces $\mathrm{Cob}_\theta(d) \to \psi_\theta(\infty, 1)$ defined on k-simplices by

$$\begin{aligned} \mathrm{Cob}_\theta(d)_k &\to \psi_\theta(\infty, 1) \\ (\underline{t}, M) &\mapsto M. \end{aligned}$$

Theorem 5.6.2. *The forgetful map induces a weak equivalence*

$$B\mathrm{Cob}_\theta(d) \xrightarrow{\simeq} \psi_\theta(\infty, 1)$$

where $B\mathrm{Cob}_\theta(d)$ *is the realization of the simplicial space* $\mathrm{Cob}_\theta(d)$.

Proof. See Galatius and Randal-Williams [**GRW10**, Theorem 3.10]. □

We now encode the symmetric monoidal structure of $\mathrm{Cob}_\theta(d)$ in terms of a Γ-structure.

Lemma 5.6.3. *The simplicial spaces* $\mathrm{Cob}_{\theta(m_+)}(d)$ *assemble into a* Γ*-object in simplicial spaces*

$$\mathrm{Cob}_{\theta(-)}(d) : \Gamma^{op} \to \mathbf{S}^{\Delta^{op}}.$$

Proof. For $m_+ \in \Gamma^{op}$ the k-simplices are given as subspaces

$$\mathrm{Cob}_{\theta(m_+)}(d)_k \subset \mathbb{R}^{k+1} \times \psi_{\theta(m_+)}(\infty, 1) = \mathbb{R}^{k+1} \times \Gamma^{(0)} \psi_\theta(m_+).$$

Thus for a map $\sigma : m_+ \to n_+$ we define the map

$$\mathrm{Cob}_{\theta(m_+)}(d) \to \mathrm{Cob}_{\theta(n_+)}(d)$$

on k-simplices to be induced by the map

$$id \times \sigma_* : \mathbb{R}^{k+1} \times \Gamma^{(0)} \psi_\theta(m_+) \to \mathbb{R}^{k+1} \times \Gamma^{(0)} \psi_\theta(n_+)$$

where σ_* comes from the functoriality in Γ^{op} of the Γ–space $\Gamma^{(0)}\psi_\theta$. From this description it is clear that the maps just defined are functorial in Δ^{op} and hence define a map of simplicial spaces. □

Definition 5.6.4. Denote by $\Gamma\mathrm{Cob}_\theta(d)$ the Γ-object in simplicial spaces

$$\begin{aligned} \Gamma\mathrm{Cob}_{\theta(m_+)}(d) &\to \mathbf{S}^{\Delta^{op}} \\ m_+ &\mapsto \mathrm{Cob}_{\theta(m_+)}(d). \end{aligned}$$

Composing with the realization of simplicial spaces we get a functor

$$B\Gamma\mathrm{Cob}_\theta(d) : \Gamma^{op} \to \mathbf{S}.$$

We obtain a Γ–space by choosing as basepoints the elements $(\underline{0}, \emptyset) \in \mathrm{Cob}_\theta(d)_k$ for all $k \in \mathbb{N}$.

Lemma 5.6.5. *The forgetful map induces a levelwise equivalence of* Γ*–spaces*

$$B\Gamma\mathrm{Cob}_\theta(d) \xrightarrow{\simeq} \Gamma^{(0)}\psi_\theta.$$

Proof. By construction it is clear that the forgetful maps are functorial in Γ^{op} so that they indeed define a map of Γ–spaces. By Theorem 5.6.2, these maps are weak equivalences and hence we obtain a levelwise equivalence of Γ–spaces. $\square$

In particular, the Γ–space $B\Gamma\mathrm{Cob}_\theta(d)$ is very special and applying Segal's functor we obtain a connective Ω–spectrum, which we denote by $\mathbf{B}\Gamma\mathrm{Cob}_\theta(d)$ to avoid awkward notation. In conclusion, we obtain an equivalence of spectra

$$\mathbf{B}\Gamma\mathrm{Cob}_\theta(d) \xrightarrow{\simeq} \mathbf{B}\Gamma^{(0)}\psi_\theta.$$

Combining with Theorem 5.5.2, we obtain our main theorem.

Theorem 5.6.6. *There are stable equivalences of spectra*

$$\mathbf{B}\Gamma\mathrm{Cob}_\theta(d) \simeq \mathbf{B}\Gamma^{(0)}\psi_\theta \simeq MT\theta(d)[1]_{\geq 0},$$

such that the induced equivalences

$$\Omega^\infty\mathbf{B}\Gamma\mathrm{Cob}_\theta(d) \simeq \Omega^\infty\psi_\theta \simeq \Omega^\infty MT\theta(d)[1]$$

are equivalent to the equivalences of Theorem 5.3.2 and Theorem 5.6.2.

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