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With an appendix by Jeremy Hahn

Abstract

We prove that the p-completed Brown–Peterson spectrum is a retract of a product of Morava E-theory spectra. As a consequence, we generalize results of Kashiwabara and of Ravenel, Wilson and Yagita from spaces to spectra and deduce that the notion of a good group is determined by Brown–Peterson cohomology. Furthermore, we show that rational factorizations of the Morava E-theory of certain finite groups hold integrally up to bounded torsion with height-independent exponent, thereby lifting these factorizations to the rationalized Brown–Peterson cohomology of such groups.

1. Introduction

Many important cohomology theories E are constructed from complex cobordism MU or Brown–Peterson cohomology BP via Landweber's exact functor theorem. Viewing this process as a simplification, one might wonder what kind of information about $BP^*(X)$ is retained in $E^*(X)$, for X a space or a spectrum. Motivated by this question, the goal of this paper is two-fold: in the first part, we show that many properties of the BP-cohomology of a spectrum are determined by the collection of its Morava K-theories $K(n)^*(X)$. In the second part, transchromatic character theory is used to factor the rationalized BP-cohomology of classifying spaces of certain finite groups by establishing height-independent bounds on the torsion in Morava E-theory. This has the curious consequence that the BP-cohomology of finite groups behaves more algebro-geometrically than one might expect.

Let E_n denote Morava E-theory of height n, which is a Landweber exact E_∞-ring spectrum with coefficients $E_n^* = W(\kappa)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$. The key observation of this paper is a natural extension of a theorem due to Hovey [Hov95, Theorem 3.1], realizing BP as a summand in a product of simpler cohomology theories.

THEOREM. *The p-completed Brown–Peterson spectrum BP_p is a retract of $\prod_{n>0} E_n$, the product over all Morava E-theories E_n.*

The general idea is as follows: the behavior of Morava E-theory at height n is closely connected to the behavior of Morava K-theory $K(n)$ at the same height, and these cohomology theories determine each other in many cases. Starting from knowledge about the Morava K-theories $K(n)^*(X)$ of a spectrum X, we can then deduce properties of $E_n^*(X)$, which collectively control $BP_p^*(X)$ by the above splitting. Conversely, the BP-cohomology of *spaces* determines their Morava K-theory.

The class of spectra amenable to such comparison results consists of those with evenly concentrated Morava K-theory for all heights or, more generally, those with Landweber flat BP-cohomology; examples abound. Using the previous theorem and certain base change formulas, we then obtain generalizations of the main structural results of [RWY98, Wil99] to spectra.

THEOREM. *Let X be a spectrum. If $K(n)^*X$ is even for all $n > 0$, then BP_p^*X is even and Landweber flat. Moreover, for a map $f : X \longrightarrow Y$ between spectra with even Morava K-theories, $BP_p^*(f)$ is injective or surjective if $K(n)^*(f)$ is injective or surjective for all n.*

Moreover, we give conditions on a spectrum X which imply that $BP_p^*(\mathbb{P}X)$ is Landweber flat, where $\mathbb{P}X$ denotes the free commutative algebra spectrum on X. Combining the retract theorem with work of Rezk [Rez09] as well as work of the first author and Frankland [BF15], we construct a functor

$$\mathbb{T}^{BP} : \mathrm{Mod}_{BP_p^*}^c \longrightarrow \mathrm{Mod}_{BP_p^*}.$$

For a large class of spaces, which includes spheres and higher Eilenberg–MacLane spaces, we then show that $BP_p^*(\mathbb{P}X)$ is functorially determined by $BP_p^*(X)$.

THEOREM. *There exists a functor $\mathbb{T}^{BP}$ on the category $\mathrm{Mod}_{BP_p^*}^c$ of (topological) BP_p^*-modules such that, for X a space with $K(n)_*(X)$ even and degreewise finite for all $n > 0$, there is a natural isomorphism*

$$\mathbb{T}^{BP} BP_p^*(X) \xrightarrow{\;\cong\;} BP_p^*(\mathbb{P}X)$$

of BP_p^*-modules.

Roughly speaking, this result says that the structure of the corresponding Künneth and homotopy orbit spectral sequences for $BP_p^*(\mathbb{P}X)$ is completely controlled by the topological module $BP_p^*(X)$.

As another consequence of our results, we deduce that various notions of good groups coincide and are controlled by the BP-cohomology of the group. This motivates the second part of the paper, where we take the above ideas one step further. Transchromatic character theory establishes a link between Morava E-theories of different heights, leading to the question of which properties of $E_n^*(X)$ and $BP^*(X)$ are detected by height 1 and height 0, i.e. by topological K-theory and rational cohomology. It turns out that the connection is surprisingly strong, an observation that has already been exploited in [SS15, BS16].

The second part of the paper focuses on the rationalization of the retract theorem. We apply the resulting map to classifying spaces of finite groups. Algebro-geometrically, $E_n^*(BA)$ for a finite abelian group A corepresents the scheme $\mathrm{Hom}(A^*, \mathbb{G}_{E_n})$ that parametrizes maps from the Pontryagin dual of A to the formal group $\mathbb{G}_{E_n}$ associated to Morava E-theory. Rationally, there is then a decomposition into schemes classifying level structures

$$\mathrm{Hom}(A^*, \mathbb{G}_{E_n}) \cong_{\mathbb{Q}} \coprod_{H \subseteq A} \mathrm{Level}(H^*, \mathbb{G}_{E_n}),$$

where H runs through the subgroups of A; see [Str97]. Up to torsion, the scheme of level structures $\mathrm{Level}(H^*, \mathbb{G}_{E_n})$ is corepresented by $E_n^*(BH)/I$, where I is the transfer ideal, i.e. the ideal generated by transfers from the maximal subgroups of H. We show that this statement holds integrally up to bounded integral torsion, where the exponent of the torsion is in fact independent of the height we work at. This uses a result about level structures proven by Jeremy Hahn, which is presented in the appendix to this paper.

Theorem. *Let A be a finite abelian group; then the exponent of the torsion in the cokernel of the natural map*

$$E_n^*(BA) \longrightarrow \prod_{H \subseteq A} E_n^*(BH)/I$$

is bounded independently of the height n.

In order to prove this, we construct a variant of the transchromatic character maps of [Sta13, BS16] from E-theory at height n to height 1, which enables tight control over the torsion. Since products of torsion abelian groups with a common torsion exponent are torsion as well, it follows that the natural map

$$\prod_n E_n^*(BA) \longrightarrow \prod_n \prod_{H \subseteq A} E_n^*(BH)/I$$

is a rational isomorphism. It is possible to apply the retract theorem to immediately deduce a similar decomposition for the BP-cohomology of finite abelian groups.

Corollary. *Let A be a finite abelian group and let I denote the transfer ideal; then the natural map*

$$BP_p^*(BA) \longrightarrow \prod_{H \subseteq A} BP_p^*(BH)/I$$

is a rational isomorphism.

Similarly, the scheme corepresented by $E_n^*(B\Sigma_m)$ decomposes rationally into a product of subgroup schemes $\Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_{E_n})$, indexed by partitions $\lambda \vdash m$ of m. Again, we prove that this statement holds integrally up to globally bounded integral torsion.

Theorem. *The exponent of the torsion in the cokernel of the natural map*

$$E_n^*(B\Sigma_m) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_{E_n})$$

is bounded independently of the height n. Consequently, there is a rational isomorphism

$$BP_p^*(B\Sigma_m) \longrightarrow \prod_{\lambda \vdash m} (BP_p^*(B\Sigma_\lambda)/I_\lambda)^{\Sigma_{\underline{a}}},$$

where I_λ is a certain transfer ideal.

Finally, we give a further illustration of the methods of the paper by proving a well-known version of Artin induction for the BP-cohomology of good groups.

1.1 Relation to the literature

The question of when and how $E^*(X)$ can be computed algebraically from $BP^*(X)$ or, conversely, what kind of information about $BP^*(X)$ is retained in $E^*(X)$ has a long history. After the pioneering work of Johnson and Wilson [JW73, JW75] and Landweber [Lan76], these problems were studied systematically in a series of papers by Ravenel, Wilson and Yagita [RWY98, Wil99] and Kashiwabara [Kas98, Kas01]. Their methods are based on a careful study of the associated Atiyah–Hirzebruch spectral sequences. As a consequence, the results contained in these papers are unstable, i.e. they are valid only for spaces rather than for arbitrary spectra.

The results of our paper generalize the main structural theorems of [RWY98, Wil99] from spaces to spectra, by replacing the Atiyah–Hirzebruch spectral sequence arguments with the

above retract result. This is remarkable, as such extensions had previously seemed impossible; see the remark following Theorem 1.8 in [RWY98, p. 2]. Furthermore, the retract result has the pleasant side effect of simplifying many of the proofs, at the cost of losing control over any explicit generators and relations descriptions. In particular, we are unable to recover the computations for spheres, Eilenberg–MacLane spaces, and other spaces appearing in certain Ω-spectra given in the aforementioned papers.

In contrast to this, passing from $BP^*(X)$ to $K(n)^*(X)$ is more subtle, due to the existence of counterexamples to the generalizations of certain base change formulas to arbitrary spectra. Here, our results do not improve upon the results in [RWY98, Wil99].

In [Kas01], Kashiwabara studied the following question: when X is a space, to what extent does $BP_p^*(X)$ control $BP_p^*(QX)$? He showed that $BP_p^*(QX)$ inherits Landweber flatness from $BP_p^*(X)$ and constructed a functor $\mathcal{D}$ such that

$$\mathcal{D}BP_p^*(X) \xrightarrow{\cong} BP_p^*(\Omega^\infty X)$$

is an isomorphism under certain algebraic conditions on X. Our description of $BP_p^*(\mathbb{P}X)$ is related to Kashiwabara's result via the Snaith splitting; the precise relation between Kashiwabara's theorem and ours is, however, not completely clear at the moment. It should nevertheless be noted that his functor allows for explicit calculations, whereas $\mathbb{T}^{BP}$ involves the idempotent given by the retract and is thus much less computable.

1.2 Outline

We begin in §2 by recalling the cohomology theories and basic concepts that will be used in this paper. These theories are related to each other by various base change formulas, most of which are collected from the literature. The first goal of §3 is to prove the result which the rest of our paper is based on, namely that BP_p is a retract of a product of Morava E-theories. We then deduce our main theorems about the structure of the BP-cohomology of spectra with even Morava K-theory. Moreover, we discuss the BP-cohomology of free commutative algebras and show that various notions of good groups are equivalent.

The second part of the paper starts in §4 with a toy example, obtaining the existence of a height-independent torsion exponent in the case of finite cyclic groups and Σ_p by an explicit calculation. A number of auxiliary commutative algebra results are proven before introducing a variant of the transchromatic character map. The rest of the section contains the proofs of the main theorems for Morava E-theory and their consequences for the rational BP-cohomology of abelian and symmetric groups. A key result for the case of abelian groups is deferred to an appendix, written by Jeremy Hahn.

1.3 Notation and conventions

Throughout this paper we fix a prime p and work in the category Sp of p-local spectra. By *space* we shall always mean a CW complex of finite type viewed as a suspension spectrum. The results in the first part of the paper are formulated and proven more generally for the theories $P(m)$, but the reader not familiar with these might want to specialize to $P(0) = BP_p$. We will also always write $\mathbb{Q}_p/\mathbb{Z}_p^n$ for $(\mathbb{Q}_p/\mathbb{Z}_p)^n$.

2. The Brown–Peterson spectrum and related cohomology theories

After introducing the complex oriented ring spectra that will be used throughout the paper, we recall the notion of Landweber flatness relative to the theory $P(m)$ and prove that Landweber flat

modules are closed under products. We then give base change formulas relating these cohomology theories. Although most of these results are well known, we hope the reader will find it helpful to see them collected in one place.

2.1 Preliminaries

We recall some basic terminology and facts that will be used in this paper; for more details, see for example [RWY98]. Let BP be the Brown–Peterson spectrum with coefficients $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where the v_i are Hazewinkel generators of degree $|v_i| = -2(p^i - 1)$ with $v_0 = p$; as usual, none of the constructions or results in this paper depend on this choice of generators. By [Lan73], the invariant prime ideals in BP^* are given by $I_n = (p, v_1, \ldots, v_{n-1})$ for $n \geqslant 0$.

Recall that $P(m)$ denotes the BP-module spectrum with coefficients

$$P(m)^* = BP^*/I_m,$$

which can be constructed using the Baas–Sullivan theory of singularities or the methods of [EKMM97]. We set $P(0) = BP_p$, the p-completion of BP. The ideals $I_{m,n} = (v_m, \ldots, v_{n-1})$ for $n \geqslant m$ are then precisely the invariant prime ideals of $P(m)^*$.

The Landweber filtration theorem [Lan76] for BP_*BP-comodules that are finitely presented as BP_*-modules admits the following generalization to $P(m)$, due independently to Yagita and Yoshimura.

THEOREM 2.1 [Yag76, Yos76]. *Suppose that M is a $P(m)^*P(m)$-module which is finitely presented as a $P(m)^*$-module. Then there exists a finite filtration of M by $P(m)^*P(m)$-modules,*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M,$$

with filtration quotients $M_{i+1}/M_i \cong P(m)^*/I_{m,n_i}$ up to a shift and for some $n_i \geqslant m$.

Remark 2.2. The classical Landweber filtration theorem is usually stated for BP, not for $P(0) = BP_p$. However, this will not affect any of the arguments or statements appearing later in this paper.

From $P(m)$ we can construct, for $n \geqslant m$, spectra $E(m, n)$ by taking the quotient by the ideals $(v_{n+1}, v_{n+2}, \ldots)$ and then inverting v_n; that is,

$$E(m, n)^* = \begin{cases} \mathbb{Z}_p[v_1, \ldots, v_n][v_n^{-1}] & \text{if } m = 0, \\ \mathbb{F}_p[v_m, \ldots, v_n][v_n^{-1}] & \text{if } m > 0. \end{cases}$$

In particular, if $m = n$, then $E(m, m) = K(m)$ is Morava K-theory; and if $m = 0$, then $E(0, n) = E(n)$ is p-complete Johnson–Wilson theory. Moreover, we define $\hat{E}(m, n)$ to be the $K(n)$-localization of $E(m, n)$,

$$\hat{E}(m, n) = L_{K(n)}E(m, n),$$

where $L_{K(n)}$ denotes Bousfield localization with respect to $K(n)$. Finally, we let E_n denote height n Morava E-theory, which is a 2-periodic finite free extension of $\hat{E}(0, n)$ given by adjoining an element u of degree 2 with $u^{1-p^n} = v_n$ and extending coefficients to the ring $W(\kappa)$ of Witt vectors over a perfect field κ of characteristic p.

2.2 Landweber flatness

Recall the following definition from [RWY98, Wil99].

DEFINITION 2.3. A module $M \in \mathrm{Mod}_{P(m)^*}$ is said to be Landweber flat if it is a flat module for the category of $P(m)^* P(m)$-modules which are finitely presented as $P(m)^*$-modules, i.e. if the functor

$$- \otimes_{P(m)^*} M : \mathrm{Mod}^{\mathrm{fp}}_{P(m)^* P(m)} \longrightarrow \mathrm{Mod}_{P(m)^*}$$

is exact.

The Landweber filtration theorem for $P(m)$ easily implies the following characterization of Landweber flatness; see for example [RWY98, Theorem 3.9].

PROPOSITION 2.4. A $P(m)^*$-module M is Landweber flat if and only if

$$v_n : P(n)^* \otimes_{P(m)^*} M \longrightarrow P(n)^* \otimes_{P(m)^*} M$$

is injective for all $n \geqslant m$.

In the next section, we need a closure property for the collection of Landweber flat modules. Since we do not know of a published reference for this fact, we include the proof.

LEMMA 2.5. *The collection of Landweber flat $P(m)^*$-modules is closed under products.*

Proof. Suppose that $(M_i)_i$ is a collection of Landweber flat BP^*-modules and let $n \geqslant m$. By Proposition 2.4, it suffices to show that the top map in the commutative diagram

$$
\begin{array}{ccc}
\left(\prod_i M_i\right) \otimes_{P(m)^*} P(m)^*/I_{m,n} & \xrightarrow{v_n} & \left(\prod_i M_i\right) \otimes_{P(m)^*} P(m)^*/I_{m,n} \\
\epsilon_m \downarrow & & \downarrow \epsilon_m \\
\prod_i (M_i \otimes_{P(m)^*} P(m)^*/I_{m,n}) & \xrightarrow{v_n} & \prod_i (M_i \otimes_{P(m)^*} P(m)^*/I_{m,n})
\end{array}
$$

is injective. Since $P(m)^*/I_{m,n}$ is a finitely presented $P(m)^*$-module, the vertical maps ϵ_m are isomorphisms. But, by assumption, the bottom map is injective, hence so is the top map. $\qquad\square$

Remark 2.6. This result should be compared to a theorem of Chase: for R a (not necessarily commutative) ring, the collection of flat R-modules is closed under arbitrary products if and only if R is left coherent. For a discussion of products of flat modules and a proof of Chase's theorem, see [Lam99].

2.3 Base change formulas

This section collects a number of base change formulas for the cohomology theories we use in this paper. While most of the results are well known, it is not always easy to locate references in the published literature.

LEMMA 2.7. *Suppose $n \geqslant m$ and let X be a spectrum such that $\hat{E}(m,n)^*(X)$ is evenly concentrated and flat as an $\hat{E}(m,n)^*$-module. Then*

$$K(n)^*(X) \cong \hat{E}(m,n)^*(X) \otimes_{\hat{E}(m,n)^*} K(n)^*,$$

which is even as well. Conversely, if $K(n)^*(X)$ is even, then $\hat{E}(m,n)^*(X)$ is even and flat as an $\hat{E}(m,n)^*$-module.

Proof. We give the proof for $m = 0$ and Morava E-theory E_n; the argument and referenced results generalize easily to the $m > 0$ case. First note that $E_n^*(X)$ is automatically complete with respect to the maximal ideal of E_n^*, because the function spectrum $F(X, E_n)$ is $K(n)$-local. By [BF15, Proposition A.15], this means that $E_n^*(X)$ is flat as an E_n^*-module if and only if it is pro-free, i.e. if it is the completion of some free E_n^*-module. This reduces the claim to [HS99b, Proposition 2.5]. $\square$

In order to relate $E(m, n)$-cohomology to $P(m)$-cohomology, we need to recall the construction of the completed tensor product; see for example [KY93].

DEFINITION 2.8. Let k be a cohomology theory and consider two complete topological algebras A and B over k^*, with filtrations given by systems of opens $\{A_r\}$ and $\{B_r\}$, respectively. The completed tensor product of A and B over k^* is then defined as

$$A \hat{\otimes}_{k^*} B = \lim_r (A \otimes_{k^*} B)/J_r,$$

where J_r is the ideal spanned by $A_r \otimes_{k^*} B + A \otimes_{k^*} B_r$.

If X is space with skeletal filtration $\{\mathrm{sk}_r X\}$, then its cohomology $k^*(X)$ can be topologized using the system of fundamental neighborhoods of 0 given by $F_r(X) = \ker(k^*(X) \to k^*(\mathrm{sk}_r X))$. In the following, we will always consider this topology when working with completed tensor products.

Remark 2.9. In particular, note that the completed tensor product of even cohomology groups is also concentrated in even degrees.

The next result is known as Morava's little structure theorem; a proof in the generality we need is given in [Wil99, Proposition 1.9]; see also [Str99].

PROPOSITION 2.10. *If X is a space, then*

$$E(m, n)^*(X) \cong P(m)^*(X) \hat{\otimes}_{P(m)^*} E(m, n)^*$$

for all $n \geqslant m > 0$ or for $m = 0$ and all $n > 0$, and similarly for the completed theories $\hat{E}(m, n)$. Moreover, if $P(m)^*(X)$ is Landweber flat, then $\hat{E}(m, n)^*(X)$ is flat over $\hat{E}(m, n)^*$ for all $n \geqslant m > 0$.

Proof. The base change formula for both $E(m, n)$ and $\hat{E}(m, n)$ is contained in the aforementioned references; see for example [Str99, Corollary 8.24]. To show the second part of the claim, observe that, by [Wil99, Proposition 2.1], the assumption implies that the ideal $I_{m,n}$ acts regularly on $\hat{E}(m, n)^*(X)$. Then the obvious extension of [HS99b, Theorem A.9] from E_n to arbitrary $\hat{E}(m, n)$ gives flatness. $\square$

Example 2.11. This result does not hold for arbitrary spectra instead of spaces. As a counterexample, consider the connective Morava K-theory spectrum $k(n)$. While $K(n)^* k(n) \neq 0$, it can be shown [RWY98, Remark 4.9] that $P(n)^* k(n) = 0$; hence the result fails for $m = n$.

LEMMA 2.12. *For any spectrum X with $P(m)^*(X)$ Landweber flat and $n \geqslant m$, there is a natural isomorphism*

$$P(n)^*(X) \cong P(n)^* \otimes_{P(m)^*} P(m)^*(X).$$

Proof. We prove this by induction on $n \geqslant m$, the statement for $n = m$ being trivial. Now let $n \geqslant m$ and recall that Proposition 2.4 shows that $P(m)^*(X)$ being Landweber flat implies that

$$v_n : P(n)^* \otimes_{P(m)^*} P(m)^*(X) \longrightarrow P(n)^* \otimes_{P(m)^*} P(m)^*(X)$$

is injective for all $n \geqslant m$. By construction of $P(n+1)$, there is an exact triangle

$$\text{(2.1)}$$

where the dashed arrow is the connecting homomorphism of degree 1. Since $P(n)^*(X) \cong P(n)^* \otimes_{P(m)} P(m)^*(X)$ by the induction hypothesis, our assumption combined with (2.1) shows that ρ_n is surjective and that

$$
\begin{aligned}
P(n+1)^*(X) &\cong \operatorname{coker}(v_n : P(n)^*(X) \to P(n)^*(X)) \\
&\cong \operatorname{coker}(v_n : P(n)^* \otimes_{P(m)^*} P(m)^*(X) \to P(n)^* \otimes_{P(m)^*} P(m)^*(X)) \\
&\cong P(n+1)^* \otimes_{P(m)^*} P(m)^*(X),
\end{aligned}
$$

thereby proving the claim. □

COROLLARY 2.13. *If X is a space with Landweber flat $P(m)$-cohomology, then*

$$K(n)^*(X) \cong K(n)^* \, \hat{\otimes}_{P(n)^*} (P(n)^* \otimes_{P(m)^*} P(m)^*(X))$$

for all $n \geqslant m > 0$ or for $m = 0$ and all $n > 0$. In particular, under the same assumptions, if $P(m)^*(X)$ is even, then $K(n)^*(X)$ is even as well.

Proof. Let X be a space such that $P(m)^*(X)$ is Landweber flat. From the previous lemma we get

$$P(n)^*(X) \cong P(n)^* \otimes_{P(m)^*} P(m)^*(X) \qquad \text{(2.3)}$$

for all $n \geqslant m$. But Proposition 2.10 with $m = n$ shows that

$$K(n)^*(X) \cong K(n)^* \, \hat{\otimes}_{P(n)^*} P(n)^*(X),$$

which together with (2.3) yields the claim. □

Remark 2.14. In [KY93, equation (1.8)], Kono and Yagita showed that $K(n)^*(X)$ is isomorphic to $K(n)^* \, \hat{\otimes}_{P(m)^*} P(m)^*(X)$ for a space X with $BP^*(X)$ Landweber flat. We do not know whether their methods generalize to imply that $K(n)^*(X) \cong K(n)^* \, \hat{\otimes}_{P(m)^*} P(m)^*(X)$ in the situation of Corollary 2.13.

3. The splitting and some consequences

In this section, we first prove that $P(m)$ is a retract of an infinite product of E-theories, following closely the argument given in [Hov95, Theorem 3.1]. Combined with the base change results of the previous section, this allows us to generalize the main structural results of [RWY98, Wil99] from spaces to spectra. Since we do not require any analysis of the Atiyah–Hirzebruch spectral sequence for this, our arguments are rather short in comparison. In § 3.3, we construct an analogue of Rezk's algebraic approximation functor $\mathbb{T}^{E_n}$ for BP_p. These functors are then used to prove a version of Kashiwabara's main result [Kas01] about the BP_p-cohomology of free commutative algebras. Finally, we give several equivalent characterizations of good groups, thereby showing that this notion is really a global one.

3.1 *BP* as a retract of a product of Morava *E*-theories

Let $I : \mathrm{Sp} \to \mathrm{Sp}$ denote Brown–Comenetz duality, which is a lift of Pontryagin duality for abelian groups to the category of spectra. To be precise, if X is a spectrum, then I represents the functor

$$X \mapsto \mathrm{Hom}(\pi_0 X, \mathbb{Q}/\mathbb{Z}_{(p)})$$

and we set $IX = F(X, I)$. In [Mar83, Proposition 5.2], Margolis proves a non-existence result for f-phantom maps with target of the form IX; recall that a map $Y \to X$ of spectra is said to be f-phantom if $Z \longrightarrow Y \longrightarrow X$ is null for all finite spectra Z.

LEMMA 3.1 (Margolis). *Any f-phantom map with target IX must be null.*

The next result is a natural generalization of [Hov95, Theorem 3.1].

THEOREM 3.2. *Suppose that D is a p-complete Landweber flat $P(m)$-module spectrum. If there is a morphism $f : P(m) \longrightarrow D$ such that the induced maps*

$$P(m)_*/I_{m,n} \longrightarrow D_*/I_{m,n}$$

are injective for all $n > m$, then f is a split inclusion of spectra.

Proof. We will prove the result for $m = 0$ using the classical Landweber filtration theorem. The argument in the $m > 0$ case uses Theorem 2.1, but is easier as $P(m)$ is already p-complete. Let F be the fiber of the map $BP_p \to D$ and let

$$F' = \mathrm{fib}(BP \longrightarrow D);$$

we need to check that $F \longrightarrow BP_p$ is null. Since BP has degreewise finitely generated homotopy groups, $BP_p = I^2 BP$. In order to show the claim, it thus suffices by Lemma 3.1 to check that $F \longrightarrow BP_p$ is f-phantom. There is a commutative diagram

$$
\begin{array}{ccccc}
F' & \longrightarrow & BP & \longrightarrow & D \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \simeq} \\
F & \longrightarrow & BP_p & \longrightarrow & D \\
\downarrow & & \downarrow & & \downarrow \\
C_1 & \longrightarrow & C_2 & \longrightarrow & C_3
\end{array}
$$

with all rows and columns being cofiber sequences. Since D is p-complete, we get $C_3 = 0$ and hence that the top row in the diagram

$$
\begin{array}{ccc}
F' & \longrightarrow F \longrightarrow & \mathrm{cof}(BP \longrightarrow BP_p) \simeq C_1 \\
\downarrow & \downarrow & \\
BP & \longrightarrow BP_p &
\end{array}
$$

is a cofiber sequence. If we can show that $F' \longrightarrow BP$ is f-phantom, then the composite $F' \longrightarrow BP \longrightarrow BP_p$ is also f-phantom, and hence is null by Lemma 3.1, so the indicated factorization in the right triangle exists. Because $\mathrm{cof}(BP \longrightarrow BP_p)$ is acyclic with respect to the mod-p Moore spectrum $M(p)$ and BP_p is $M(p)$-local, the dashed map must be null; hence so is $F \longrightarrow BP_p$.

Therefore, Spanier–Whitehead duality reduces the claim to proving that if X is a finite spectrum, the induced map

$$BP_*(X) \longrightarrow D_*(X)$$

is injective. By the Landweber filtration theorem, the BP_*BP-comodule $BP_*(X)$ admits a finite filtration by comodules F_i with filtration quotients $F_{i+1}/F_i \cong BP_*/I_{n_i}$ (up to suspensions). Since $BP_*/I_{n_i} \longrightarrow D_*/I_{n_i}$ is injective by assumption, Landweber exactness of D and the snake lemma applied to the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_i & \longrightarrow & F_{i+1} & \longrightarrow & BP_*/I_{n_i} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_i \otimes_{BP_*} D_* & \longrightarrow & F_{i+1} \otimes_{BP_*} D_* & \longrightarrow & BP_*/I_{n_i} \otimes_{BP_*} D_* & \longrightarrow & 0
\end{array}
$$

gives the claim inductively. $\qquad\square$

Remark 3.3. Note that the splitting constructed above is only additive, and is not a map of ring spectra.

COROLLARY 3.4. *The natural diagonal map*

$$P(m) \longrightarrow \prod_{n \in I} \hat{E}(m,n)$$

is a split inclusion for any infinite set I of integers greater than m. In particular, BP_p is a retract of $\prod_{n>0} E_n$.

Proof. It suffices to check that $D_I = \prod_{n \in I} \hat{E}(m,n)$ satisfies the conditions of Theorem 3.2. The spectra $\hat{E}(m,n)$ are Landweber exact and $K(n)$-local, hence p-complete. By Lemma 2.5, D_I is also Landweber flat, and because products of local objects are local, it is also p-complete. Since the standard maps $P(m) \longrightarrow \hat{E}(m,n)$ clearly make

$$P(m)_*/I_{m,k} \longrightarrow D_*/I_{m,k} = \prod_{n \in I} (\hat{E}(m,n)^*/I_{m,k})$$

injective for all $k \geqslant m$, the claim follows. $\qquad\square$

Remark 3.5. The infinite product in Corollary 3.4 cannot be replaced by a coproduct. In fact, every map

$$f : \bigvee_{n \in I} E_n \longrightarrow P(m)$$

is null, for any $m \geqslant 0$. To see this, note that $P(m)$ is $H\mathbb{F}_p$-local, while E_n is $H\mathbb{F}_p$-acyclic for all n because the Morava K-theories are; hence f must be null.

Remark 3.6. Combining [Hov95, Theorem 3.1] with [HS99a, Theorem B] yields a splitting similar to Corollary 3.4: the map

$$BP_p \lhook\joinrel\longrightarrow \prod_{n>0} L_{K(n)} \left(\bigvee_{r \in S(n)} \Sigma^r L_{K(n)} E(n) \right) \tag{3.1}$$

is a split inclusion, where the $S(n)$ are explicit indexing sets of even integers. The spectrum on the right-hand side is, however, much bigger than $\prod_{n>0} E_n$ and less convenient for taking cohomology due to the infinite wedge. In some sense, our splitting is as small as possible when using these techniques, but structural properties of BP_p could also be deduced from the spectrum in (3.1).

Furthermore, since MU splits p-locally into a wedge of even suspensions of BP, there are similar results for $MU_{(p)}$.

3.2 Brown–Peterson cohomology and Morava K-theory

In this section, we draw some inferences from Corollary 3.4 on the kind of information that the Morava K-theories detect about the $P(m)$-cohomology of spectra. These results generalize the main structural theorems of [RWY98] from spaces to spectra.

Remark 3.7. While the authors remark in [RWY98, p. 2] that their results are *strictly unstable*, their counterexample refers to Example 2.11 and hence only affects Proposition 2.10, which is not needed in our approach.

We start with a lemma that allows us to pass between ordinary and p-complete Brown–Peterson cohomology.

LEMMA 3.8. *Suppose that X is a spectrum of finite type which is bounded below and such that either:*

(i) *X is a suspension spectrum which is rationally equivalent to S^0; or*

(ii) *X is a spectrum which is rationally acyclic;*

then $BP^*(X)$ is even and Landweber flat if and only if $BP_p^*(X)$ is.

Proof. Since BP is connective, BP_p is the Bousfield localization of BP with respect to the mod-p Moore spectrum $M(p)$, and we get a fiber sequence

$$C_{M(p)}BP \longrightarrow BP \longrightarrow L_{M(p)}BP \simeq BP_p, \tag{3.2}$$

where $C_{M(p)}BP$ is rational and concentrated in odd degrees. On the one hand, if X is a rationally acyclic spectrum, then $C_{M(p)}BP^*(X) = 0$. On the other hand, if X is a space that is rationally equivalent to the sphere, then $C_{M(p)}BP^*(X) \cong C_{M(p)}BP^*(S^0)$ is concentrated in odd degrees. Moreover, the connecting homomorphism δ fitting in the commutative diagram

$$\begin{array}{ccc}
BP_p^*(X) & \xrightarrow{\delta} & C_{M(p)}BP^{*+1}(X) \\
\downarrow & & \downarrow{\scriptstyle\cong} \\
BP_p^*(S^0) & \longrightarrow & C_{M(p)}BP^{*+1}(S^0)
\end{array}$$

is surjective. Therefore, in either case the long exact sequence associated to (3.2) yields isomorphisms

$$BP^{\mathrm{odd}}(X) \xrightarrow{\cong} BP_p^{\mathrm{odd}}(X)$$

and a short exact sequence

$$0 \longrightarrow BP^{\mathrm{even}}(X) \longrightarrow BP_p^{\mathrm{even}}(X) \longrightarrow C_{M(p)}BP^{\mathrm{odd}}(X) \longrightarrow 0.$$

This implies the claim about evenness. To see that $BP^*(X)$ is Landweber flat if and only if $BP_p^*(X)$ is Landweber flat, note that the finite type condition together with the Atiyah–Hirzebruch spectral sequence show that $BP^*(X)$ is finitely generated in each degree. By [Bou79, Proposition 2.5], we thus get an isomorphism

$$BP^*(X) \otimes \mathbb{Z}_p \xrightarrow{\cong} BP_p^*(X),$$

so the claim follows from Proposition 2.4 by using flatness of $\mathbb{Z}_p$ and naturality. $\square$

In particular, this applies to finite spectra of type at least 1 and classifying spaces of finite groups. It will be used implicitly from now on, so whenever the assumptions are satisfied, BP refers to the p-complete Brown–Peterson spectrum BP_p. The following theorem was proven for spaces in [RWY98, Theorems 1.8 and 1.9].

THEOREM 3.9. *Let X be a spectrum. If $K(n)^*(X)$ is even for infinitely many n, then $P(m)^*(X)$ is even and Landweber flat for all m. In particular, (2.1) gives short exact sequences*

$$0 \longrightarrow P(m)^*(X) \xrightarrow{v_m} P(m)^*(X) \longrightarrow P(m+1)^*(X) \longrightarrow 0.$$

Proof. Let I be the infinite set of natural numbers n such that $K(n)^*(X)$ is even and $n \geqslant m$. By Lemma 2.7, the assumption implies that $\hat{E}(m,n)^*(X)$ is even and flat over $\hat{E}(m,n)^*$. Since the $P(m)^*$-module $\hat{E}(m,n)^*$ is Landweber flat, so is $\hat{E}(m,n)^*(X)$ for all $n \in I$. It follows from Lemma 2.5 that $\prod_{n \in I} \hat{E}(m,n)^*(X)$ is Landweber flat as well. Since Landweber flat modules are clearly closed under retracts, the claim now follows from Corollary 3.4. $\square$

Remark 3.10. More generally, the same proof shows that any spectrum X with $K(n)^*(X)$ concentrated in degrees divisible by a fixed integer d for infinitely many n has $P(m)^*X$ also concentrated in degrees divisible by d for all m. Similar observations apply to the next corollary; this should be compared to [Min02].

As an immediate consequence, we recover [RWY98, Theorem 1.2], which complements Theorem 3.9 for spaces. By Example 2.11, this result does not generalize to arbitrary spectra.

COROLLARY 3.11. *If X is a space with $K(n)^*(X)$ even for infinitely many n, then $K(m)^*(X)$ is even for all $m > 0$.*

Proof. By Theorem 3.9 with $m = 0$, $P(0)^*(X)$ is even, and hence so is $K(m)^*(X)$ for all $m > 0$ by Corollary 2.13. $\square$

Example 3.12. The spectrum $\Sigma H\mathbb{Q}$ has trivial Morava K-theory for all heights $n > 0$, but $K(0)^1 \Sigma H\mathbb{Q} \neq 0$. For a more interesting example, let $X = K(\mathbb{Z}, 3)$; then the rational cohomology $K(0)^*(X) \cong \mathbb{Q}[x]/x^2$ with x in degree 3, but $K(n)^*(X)$ is even and non-trivial for all $n \geqslant 2$ by [RW80]. Therefore, the conclusion of the corollary cannot be extended to $m = 0$.

COROLLARY 3.13. *If X is a space such that $P(m)^*(X)$ is even for some $m \geqslant 0$, then $K(n)^*(X)$ is even for all $n > 0$.*

Proof. If $P(m)^*(X)$ is even, then it is also Landweber flat as the connecting homomorphism in (2.1) must be zero for degree reasons. Therefore, Corollary 2.13 shows that $K(n)^*(X)$ is even for all $n \geqslant m + 1$, so Corollary 3.11 applies. $\square$

We will need the following lemma about maps of complete modules; a proof in the case of $m = n$ can be found in Hovey's unpublished notes [Hov04b] or [BF15], and it requires only minor modifications for the general case.

LEMMA 3.14. *Suppose that M and N are flat $\hat{E}(m,n)^*$-modules which are complete with respect to the maximal ideal $I_{m,n}$ of $\hat{E}(m,n)^*$. A map $f : M \longrightarrow N$ is injective or surjective if and only if $f \otimes_{\hat{E}(m,n)^*} K(n)^*$ is injective or surjective, respectively.*

The next theorem generalizes [RWY98, Theorems 1.17 and 1.18] to all spectra.

THEOREM 3.15. *Let $f : X \longrightarrow Y$ be a map of spectra such that both $K(n)^*(X)$ and $K(n)^*(Y)$ are even for $n \in I$ with $I \subset \mathbb{N}$ infinite.*

(i) *If $f^* : K(n)^*(Y) \longrightarrow K(n)^*(X)$ is injective (surjective) for all $n \in I$, then so is*

$$f^* : P(m)^*(Y) \longrightarrow P(m)^*(X)$$

for all m.

(ii) *Suppose that $g : Y \longrightarrow Z$ is another map with $K(n)^*(Z)$ even for all $n \in I$ and such that $g \circ f \simeq 0$. If*

$$K(n)^*(Z) \xrightarrow{g^*} K(n)^*(Y) \xrightarrow{f^*} K(n)^*(X) \longrightarrow 0$$

is an exact sequence for all $n \in I$, then so is

$$P(m)^*(Z) \xrightarrow{g^*} P(m)^*(Y) \xrightarrow{f^*} P(m)^*(X) \longrightarrow 0$$

for all $m \geqslant 0$.

Proof. Fix some integer $m \geqslant 0$. We will prove the surjectivity statement; the argument for injectivity is analogous. By assumption and Lemma 2.7, there is a commutative diagram

$$
\begin{array}{ccc}
K(n)^*(Y) & \xrightarrow{\quad K(n)^*f \quad} & K(n)^*(X) \\
\cong \downarrow & & \downarrow \cong \\
\hat{E}(m,n)^*(Y) \otimes_{\hat{E}(m,n)^*} K(n)^* & \xrightarrow{f^* \otimes_{\hat{E}(m,n)^*} K(n)^*} & \hat{E}(m,n)^*(X) \otimes_{\hat{E}(m,n)^*} K(n)^*.
\end{array}
$$

Since $K(n)^*f$ is surjective, Lemma 3.14 implies that $\hat{E}(m,n)^*f$ is surjective for $n \in I$ as well. Products in the category of modules are exact, so the bottom map in the commutative square is also surjective:

$$
\begin{array}{ccc}
P(m)^*(Y) & \xrightarrow{\quad f^* \quad} & P(m)^*(X) \\
\downarrow & & \downarrow \\
\prod_{n \in I} \hat{E}(m,n)^*(Y) & \xrightarrow{\quad f^* \quad} & \prod_{n \in I} \hat{E}(m,n)^*(X).
\end{array}
$$

Retracts of surjective maps are surjective, hence Corollary 3.4 yields the claim.

Assertion (ii) follows formally from (i) as in the proof of [RWY98, Theorem 1.18]. □

3.3 Applications to the cohomology of free commutative algebras

In [Kas98, Kas01], Kashiwabara studied the question of when and how $BP_p^*(\Omega^\infty X)$ is determined by $BP_p^*(X)$ for X a space or spectrum. In particular, he considered two variants $\mathcal{K}'_{0BP}$ and $\mathcal{M}'_{BP}$ of the category of augmented unstable BP-cohomology algebras and the category of stable BP_p-cohomology algebras, respectively, and showed that there is a left adjoint

$$\mathcal{D} : \mathcal{M}'_{BP} \xrightleftharpoons{\qquad} \mathcal{K}'_{0BP} : \mathcal{I}$$

to the augmention ideal functor $\mathcal{I}$. This adjunction should be thought of as being analogous to the adjunction $(\Sigma^\infty \dashv \Omega^\infty)$ between spaces and spectra. Kashiwabara then showed in [Kas01, Theorem 0.11] that if X is a space, $BP_p^*(QX)$ inherits Landweber flatness from $BP_p^*(X)$. Furthermore, if X is a connected space satisfying certain conditions, there is a natural isomorphism

$$\mathcal{D}BP_p^*(X) \xrightarrow{\cong} BP_p^*(QX);$$

see [Kas01, Theorem 0.12].

In this section, we will prove some extensions and variants of Kashiwabara's results. To this end, recall that if X is a connected space, the Snaith splitting provides an equivalence

$$\Sigma^\infty QX \simeq \mathbb{P}\Sigma_+^\infty X,$$

where $\mathbb{P}Y$ denotes the free commutative algebra on a spectrum Y. Thus our results, which are formulated in terms of $\mathbb{P}$, are readily translated and compared to Kashiwabara's.

Fix a height n and recall the completed algebraic approximation functor of [Rez09, BF15]: this is an endofunctor $\mathbb{T}^{E_n}$ on the category of complete E_n^*-modules together with a natural comparison map

$$\alpha_n(M) : \mathbb{T}^{E_n}\pi_* L_{K(n)}M \longrightarrow \pi_* L_{K(n)}\mathbb{P}^{E_n}M,$$

where M is an E_n-module and $\mathbb{P}^{E_n}$ denotes the free commutative E_n-algebra functor. By [Rez09, Proposition 4.9], $\alpha_n(M)$ is an isomorphism whenever M^* is flat over E_n^*. By [BF15, Propositions 3.9 and A.15], $\mathbb{T}^{E_n}$ preserves the category of flat E_n^*-modules, and it also preserves evenness; see [Rez12, 3.2(7)]. Moreover, there is natural decomposition of functors

$$\mathbb{T}^{E_n} \cong \bigvee_{d \geqslant 0} \mathbb{T}_d^{E_n},$$

corresponding to the analogous decomposition of $\mathbb{P}^{E_n}$.

We are now in a position to prove a generalization of [Kas01, Theorem 0.11] to spectra. Note, however, that Kashiwabara assumes that $BP_p^*(X)$ is Landweber flat (without evenness) to deduce that $BP_p^*(\mathbb{P}X)$ is also Landweber flat, so our result is only a partial generalization.

PROPOSITION 3.16. *If X is a spectrum with even Morava K-theory for infinitely many n, then $BP_p^*(\mathbb{P}X)$ is even and Landweber flat.*

Proof. Let $n > 0$ be such that $K(n)_*(X)$ is even. Recall that the completed E-homology of X is defined as $E_{n,*}^\vee(X) = \pi_* L_{K(n)}(E_n \wedge X)$. The assumption on X implies that $E_{n,*}^\vee(X)$ is flat and even by the homological version of Lemma 2.7 (see [HS99b, Proposition 8.4(f)]), and so the previous discussion gives an isomorphism

$$\alpha_n(E_n \wedge X) : \mathbb{T}^{E_n}E_{n,*}^\vee(X) \xrightarrow{\cong} E_{n,*}^\vee(\mathbb{P}X).$$

of flat and evenly concentrated E_n^*-modules, using the equivalence $E_n \wedge \mathbb{P}X \simeq \mathbb{P}^{E_n}(E_n \wedge X)$. It follows from Lemma 2.7 that $K(n)^*(\mathbb{P}X)$ is even as well, so Theorem 3.9 finishes the proof. □

Denote by

$$\epsilon : \prod_{n>0} E_n \to \prod_{n>0} E_n$$

the idempotent given by Corollary 3.4; note that ϵ is not necessarily unique, but we will fix a particular one throughout this section.

DEFINITION 3.17. The BP-based algebraic approximation functor $\mathbb{T}_d^{BP}$ of degree $d \geqslant 0$ is constructed as the functor

$$\mathbb{T}_d^{BP} := \epsilon \prod_{n>0} \mathbb{T}_d^{E_n}(E_n^* \,\hat{\otimes}_{BP_p^*} -) : \mathrm{Mod}_{BP_p^*}^c \longrightarrow \mathrm{Mod}_{BP_p^*}$$

on the category of BP_p^*-modules equipped with a complete topology as in Definition 2.8. We then define the total algebraic approximation functor as $\mathbb{T}^{BP} = \prod_{d \geqslant 0} \mathbb{T}_d^{BP}$.

This allows us to prove an analogue of Kashiwabara's result [Kas01, Theorem 0.12] exhibiting a class of spaces X for which the BP_p^*-cohomology of $\mathbb{P}X$ is functorially determined by the topological module $BP_p^*(X)$.

THEOREM 3.18. *Let X be a space with $K(n)_*(X)$ even and degreewise finite for infinitely many n; then there exists a natural isomorphism*

$$\mathbb{T}^{BP} BP_p^*(X) \xrightarrow{\cong} BP_p^*(\mathbb{P}X)$$

of BP_p^*-modules.

Proof. Let S be the set of natural numbers n such that $K(n)_*(X)$ is even and degreewise finite. The free commutative algebra functor decomposes into its degree-d constituents,

$$\mathbb{P}(-) \simeq \bigvee_{d \geqslant 0} \mathbb{P}_d(-) \simeq \bigvee_{d \geqslant 0} (-)_{h\Sigma_d}^{\wedge d},$$

so we obtain a natural commutative diagram

$$
\begin{array}{ccc}
BP_p^*(\mathbb{P}X) & \xrightarrow{\cong} & \prod_d BP_p^*(\mathbb{P}_d X) \\
\downarrow & & \downarrow \\
\prod_{n \in S} E_n^*(\mathbb{P}X) & \xrightarrow{\cong} & \prod_d \prod_{n \in S} E_n^*(\mathbb{P}_d X)
\end{array}
$$

for any spectrum X. Since $\mathbb{T}^{BP}$ is compatible with this decomposition, we can reduce to a fixed degree $d \geqslant 0$.

There are natural isomorphisms

$$\mathbb{T}_d^{BP} BP_p^*(X) = \epsilon \prod_{n \in S} \mathbb{T}_d^{E_n}(E_n^* \,\hat{\otimes}_{BP_p^*} BP_p^*(X))$$

$$\cong \epsilon \prod_{n \in S} \mathbb{T}_d^{E_n}(E_n^*(X)) \quad \text{by Proposition 2.10}$$

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$$\cong \epsilon \prod_{n \in S} \mathbb{T}_d^{E_n} \pi_* E_n^X$$

$$\cong \epsilon \prod_{n \in S} \pi_* L_{K(n)} \mathbb{P}_d^{E_n} E_n^X \quad \text{via} \quad \prod_{n \in S} \alpha_n(E_n^X),$$

so it is enough to understand $\pi_* L_{K(n)} \mathbb{P}_d^{E_n} E_n^X$ in terms of $E_n^*(\mathbb{P}_d X)$. By [HS99b, Theorem 8.6], a spectrum X has degreewise finite $K(n)_*(X)$ if and only if X is dualizable in the $K(n)$-local category; we write $D_{K(n)}$ for $K(n)$-local duality. Therefore, we obtain

$$\begin{aligned}
\pi_* L_{K(n)} \mathbb{P}_d^{E_n} E_n^X &\cong \pi_* L_{K(n)} \mathbb{P}_d^{E_n} (E_n \wedge D_{K(n)} X) \\
&\cong \pi_* L_{K(n)} (E_n \wedge \mathbb{P}_d D_{K(n)} X) \\
&\cong \pi_* L_{K(n)} (E_n \wedge D_{K(n)} \mathbb{P}_d X) \\
&\cong \pi_* L_{K(n)} E_n^{\mathbb{P}_d X} \\
&\cong E_n^*(\mathbb{P}_d X).
\end{aligned}$$

Here, the third isomorphism uses the fact [GS96] that homotopy orbits agree with homotopy fixed points with respect to a finite group G in the $K(n)$-local category, i.e. $Y_{hG} \xrightarrow{\simeq} Y^{hG}$ $K(n)$-locally. Moreover, the fourth isomorphism can be understood as follows: because X is dualizable, $K(n)_*(X)$ is degreewise finite, and hence so is $K(n)_*(\mathbb{P}_d X)$ as $K(n)_*(B\Sigma_d)$ is degreewise finite; using [HS99b, Theorem 8.6] again, we see that $\mathbb{P}_d X$ is also $K(n)$-locally dualizable, giving the fourth isomorphism above.

Putting the pieces together, we obtain

$$\begin{aligned}
\mathbb{T}_d^{BP} BP_p^*(X) &\cong \epsilon \prod_{n \in S} \pi_* L_{K(n)} \mathbb{P}_d^{E_n} E_n^X \\
&\cong \epsilon \prod_{n \in S} E_n^*(\mathbb{P}_d X) \\
&\cong BP_p^*(\mathbb{P}_d X),
\end{aligned}$$

hence the claim. $\qquad\qquad\square$

Note that having degreewise finite Morava K-theory is less restrictive than one might think. For example, all finite CW complexes satisfy this condition. For a different class of examples that includes classifying spaces of finite groups and Eilenberg–MacLane spaces $K(\mathbb{Z}/p, l)$, recall that a π-finite space is a space with only finitely many non-zero homotopy groups which are all finite. By [HS99b, Corollary 8.8], π-finite spaces are $K(n)$-locally dualizable and hence have degreewise finite Morava K-theory as well. Finite CW complexes with even cells and many π-finite spaces satisfy the evenness condition as well.

We should, however, mention that we do not know the precise relation between our condition and Kashiwabara's assumption of well-generated BP_p-cohomology. In fact, Kashiwabara's result is somewhat sharper, in that his functor $\mathcal{D}$ is computable in an entirely algebraic fashion, as demonstrated in [Kas01].

Remark 3.19. The hypotheses in Theorem 3.18 can be weakened if one is willing to work with a version of $\mathbb{T}^{BP}$ that incorporates a continuous dual. To be more precise, Hovey [Hov04a, Theorem 5.1] showed that for spectra X with flat completed E_n-homology $E_{*,n}^\vee(X)$, there is a natural isomorphism

$$E_n^*(X) \xrightarrow{\cong} \operatorname{Hom}_{E_n^*}(E_{n,*}^\vee(X), E_n^*).$$

Passing from cohomology to homology is more subtle, however, and requires that we take into account the topology on $E_n^*(X)$ coming from the skeletal filtration on a CW spectrum X. In [Str, § 7], Strickland proved that

$$E_{n,*}^\vee(X) \xrightarrow{\;\cong\;} \mathrm{Hom}^c_{E_n^*}(E_n^*(X), E_n^*)$$

when X is a CW spectrum with free $E_*(X)$, where Hom^c denotes the module of continuous homomorphisms. Together, these two isomorphisms can be used to translate the statement into one about completed homology, to which the algebraic approximation functor $\mathbb{T}^{E_n}$ applies directly.

We end this section by giving a different criterion for when $P(m)^*(X)$ is even and Landweber flat. Let $\Phi_n : \mathrm{Top} \longrightarrow \mathrm{Sp}$ be the Bousfield–Kuhn functor, which was constructed in [Bou87] for $n = 1$ and in [Kuh89] for arbitrary n by using the periodicity theorem [DHS88, HS98]. This functor gives a factorization

$$L_{K(n)}X \simeq \Phi_n \Omega^\infty X$$

for any spectrum X; in fact, this can be improved so as to factor the telescopic localization in an analogous way, but we will not need this here. For $X \in \mathrm{Sp}$, Kuhn [Kuh06] used Φ_n to produce a (weak) map

$$s_n(X) : L_{K(n)}\mathbb{P}X \longrightarrow L_{K(n)}\Sigma_+^\infty \Omega^\infty X$$

which induces a natural monomorphism

$$s_n(X)_* : K(n)_*(\mathbb{P}X) \longrightarrow K(n)_*(\Omega^\infty X).$$

As an easy consequence, we get the following result.

PROPOSITION 3.20. *Let X be a spectrum such that $K(n)_*\Omega^\infty X$ is even for infinitely many n; then $P(m)^*(\mathbb{P}X)$ is even and Landweber flat for all m.*

Proof. By assumption, Kuhn's map $s_n(X)_*$ shows that $K(n)_*(\mathbb{P}X)$ is even, so Theorem 3.9 applies. $\qquad\square$

3.4 Equivalent characterizations of good groups

Now let G be a finite group. For the purposes of this paper, a group G is said to be *good* if the Morava K-theories $K(n)^*(BG)$ are even for all $n \geqslant 0$. Note that this notion of good is *a priori* weaker than the one given in [HKR00], but is conjectured to be equivalent. The following definition first appeared in [SY01] for $m = 0$.

DEFINITION 3.21. A finite group G is said to be $P(m)$-*good* if the $P(m)$-cohomology of its classifying space BG is concentrated in even degrees and Landweber flat.

Note that it follows from Lemma 3.8 that a group is $P(0)$-good if and only if $BP^*(BG)$ is even and Landweber flat. We now see that the various notions of good groups coincide, thereby revealing the global nature of goodness for finite groups.

PROPOSITION 3.22. *For a finite group G, the following conditions are equivalent:*

(i) *G is good;*

(ii) *$K(n)^*(BG)$ is even for infinitely many $n \geqslant 0$;*

(iii) *G is BP_p-good;*

(iv) *G is $P(m)$-good for some $m \geqslant 0$.*

Proof. The implications (1) $\implies$ (2) and (3) $\implies$ (4) are trivial. Theorem 3.9 gives (2) $\implies$ (3), while (4) $\implies$ (1) follows from Corollary 3.13. □

This result generalizes work of Kono and Yagita in [KY93] for finite groups.

4. Bounded torsion results for the Morava E-cohomology of finite groups

In this section we study the rationalization of the BP-cohomology of certain finite groups by working with the rationalization of the split inclusion of Corollary 3.4:

$$\mathbb{Q} \otimes BP_p^*(X) \longrightarrow \mathbb{Q} \otimes \prod_n E_n^*(X).$$

There is a close analogy to number theory. The right-hand side is a kind of p-local adeles in the stable homotopy category. Just as in number theory, we use these p-local adeles to study more global phenomena (the left-hand side). The purpose of this section is to produce a factorization of $\mathbb{Q} \otimes BP_p^*(X)$ when $X = BA$ or $X = B\Sigma_m$ by lifting a factorization of $\mathbb{Q} \otimes \prod_n E_n^*(X)$. For instance, in the case where $X = BA$, we factor $\mathbb{Q} \otimes \prod_n E_n^*(BA)$ by proving that the cokernel of the canonical map

$$\prod_n E_n^*(BA) \longrightarrow \prod_n \prod_{H \subseteq A} E_n^*(BH)/I$$

has bounded torsion, where I is the image of a transfer map. In this section, we will always work with the p-complete version of the Brown–Peterson spectrum; for simplicity we shall write $BP = BP_p$ for this spectrum.

4.1 Cyclic groups and Σ_p: a toy case

In this subsection we present some elementary observations that provide evidence for general bounded torsion results. Therefore, this subsection works backwards from the subsections that follow it. It uses BP-cohomology to give bounded torsion results for Morava E-theory.

Recall that, after choosing a coordinate, there is an isomorphism

$$BP^*(B\mathbb{Z}/p^k) \cong BP^*[\![x]\!]/[p^k](x), \tag{4.1}$$

where $[p^k](x)$ is the p^k-series for the formal group law associated to BP.

For any formal group law, it is standard to set

$$\langle p^k \rangle(x) = [p^k](x)/[p^{k-1}](x) = \langle p \rangle([p^{k-1}](x))$$

so that

$$BP^*(B\mathbb{Z}/p^k)/I \cong BP^*[\![x]\!]/\langle p^k \rangle(x),$$

where $I \subset BP^*(B\mathbb{Z}/p^k)$ is the image of the transfer along $\mathbb{Z}/p^{k-1} \subset \mathbb{Z}/p^k$. This fact is a consequence of Quillen's result in [Qui71, Proposition 4.2]. There is a canonical map

$$BP^*(B\mathbb{Z}/p^k) \longrightarrow \prod_{0 \leqslant j \leqslant k} BP^*(B\mathbb{Z}/p^j)/I$$

given by sending 1 to $(1, \ldots, 1)$ and x to $(0, x, x, \ldots, x)$.

The following proposition is well known.

Proposition 4.1. *There is an isomorphism*

$$\mathbb{Q} \otimes BP^*(B\mathbb{Z}/p^k) \xrightarrow{\cong} \mathbb{Q} \otimes \prod_{0 \leqslant j \leqslant k} BP^*(B\mathbb{Z}/p^j)/I.$$

Proof. For any formal group law there is a factorization

$$[p^k](x) = \prod_{0 \leqslant j \leqslant k} \langle p^j \rangle(x).$$

We will show that the factors are coprime in the rationalization; the Chinese remainder theorem will then imply the claim. We may write $\langle p \rangle(x) = p + f(x)$ for some power series $f(x)$ with $x | f(x)$. This implies that

$$[p^j](x) = [p^{j-1}](x)(p + f([p^{j-1}](x)))$$

and hence

$$\langle p^j \rangle(x) = p + f([p^{j-1}](x)). \tag{4.2}$$

But now if $t < j$, we have $\langle p^t \rangle(x) | [p^{j-1}](x)$ by definition, so that

$$\langle p^t \rangle(x) | f([p^{j-1}](x)),$$

and using (4.2) then gives

$$p = \langle p^j \rangle(x) - \langle p^t \rangle(x) \frac{f([p^{j-1}](x))}{\langle p^t \rangle(x)}. \qquad \square$$

The Morava E-cohomology of $B\mathbb{Z}/p^k$ satisfies an isomorphism as in (4.1). Let $\mathbb{G}_{E_n}$ be the formal group associated to E_n. Given a coordinate x on $\mathbb{G}_{E_n}$, there is an isomorphism

$$E_n^*(B\mathbb{Z}/p^k) \cong E_n^*[\![x]\!]/[p^k](x),$$

where $[p^k](x)$ is the p^k-series for the formal group law $x +_{\mathbb{G}_{E_n}} y$ induced by the coordinate. It is worth noting that the Weierstrass preparation theorem implies that this ring is a free E_n^*-module of rank p^{kn}. This is very different from $BP^*(B\mathbb{Z}/p^k)$, which is infinitely generated as a BP^*-module.

Corollary 4.2. *There is an isomorphism*

$$\mathbb{Q} \otimes \prod_{n>0} E_n^*(B\mathbb{Z}/p^k) \xrightarrow{\cong} \mathbb{Q} \otimes \prod_{n>0} \prod_{0 \leqslant j \leqslant k} E_n^*[\![x]\!]/\langle p^j \rangle(x).$$

Proof. We will prove the easiest case for clarity. Fix a height n and let $k = 1$; then with a coordinate this is the map

$$E_n^*[\![x]\!]/[p](x) \longrightarrow E_n^* \times E_n^*[\![x]\!]/\langle p \rangle(x)$$

sending $x \mapsto (0, x)$ and where $[p](x)$ now indicates the p-series for the formal group law associated to E_n. We may write down a basis for each side. For the left-hand side we have the basis $\{1, x, \ldots, x^{p^n-1}\}$, and for the right-hand side we can take the basis

$$\{(1, 1), (0, 1), (0, x), \ldots, (0, x^{p^n-2})\}.$$

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It is clear that the basis elements $\{(1,1),(0,x),\ldots,(0,x^{p^n-2})\}$ are hit under this map. Dividing by a power of p is required to hit the basis element $(0,1)$. But BP provides a global element that hits $(0,1)$ at each height n. It is the element (using the notation of the proof of the previous proposition)

$$-f(x)/p.$$

Thus we see that we only need to divide by p once in order to establish an isomorphism for all n. For $k > 1$ the proof is similar, but we need to divide by p^k. For instance, when $k = 2$,

$$
\begin{aligned}
-f([p](x)) &\mapsto (0,0,p), \\
-f(x) &\mapsto (0,p,g(x)),
\end{aligned}
$$

where $g(x)$ is some polynomial. This implies that

$$-\frac{f(x)}{p} + \frac{f([p](x))g(x)}{p^2} \mapsto (0,1,0).$$

Finally, we see that the cokernel of the map

$$\prod_{n>0} E_n^*(B\mathbb{Z}/p^k) \longrightarrow \prod_{n>0} \prod_{0 \leqslant j \leqslant k} E_n^*[\![x]\!]/\langle p^j \rangle(x)$$

is all p^k-torsion independently of the height. Thus the map is a rational isomorphism. $\qquad\square$

To prove a similar result for Σ_p, we need to recall a few basic facts regarding $E_n^*(B\Sigma_p)$. Let $I \subset E_n^*(B\Sigma_p)$ be the image of the transfer along $e \subset \Sigma_p$. For each $n \in \mathbb{N}$, the natural map

$$E_n^*(B\Sigma_p) \longrightarrow E_n^* \times E_n^*(B\Sigma_p)/I$$

is injective and induces an isomorphism

$$\mathbb{Q} \otimes E_n^*(B\Sigma_p) \xrightarrow{\;\cong\;} \mathbb{Q} \otimes (E_n^* \times E_n^*(B\Sigma_p)/I).$$

Injectivity can be seen by considering Rezk's pullback square [Rez09, Proposition 10.5]

$$
\begin{array}{ccc}
E_n^*(B\Sigma_p) & \longrightarrow & E_n^*(B\Sigma_p)/I \\
\downarrow & & \downarrow \\
E_n^* & \longrightarrow & E_n^*/p
\end{array}
$$

and the isomorphism follows by applying $\mathbb{Q} \otimes -$ to the pullback square above using the fact that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

By considering the homotopy pullback of $Be \subset B\Sigma_p$ along $B\mathbb{Z}/p \to B\Sigma_p$ (the key point is that $|\Sigma_p/(\mathbb{Z}/p)|$ is coprime to p), one can see that there is a canonical map

$$E_n^*(B\Sigma_p)/I \to E_n^*(B\mathbb{Z}/p)/I.$$

LEMMA 4.3. *There is an isomorphism*

$$\mathbb{Q} \otimes \prod_n E_n^*(B\Sigma_p) \xrightarrow{\;\cong\;} \mathbb{Q} \otimes \prod_n E_n^* \times E_n^*(B\Sigma_p)/I,$$

where I is the ideal generated by the transfer from the trivial group.

Proof. Let $\mathrm{Cok}_{E_n}(\Sigma_p)$ be the cokernel

$$0 \longrightarrow E_n^*(B\Sigma_p) \longrightarrow E_n^* \times E_n^*(B\Sigma_p)/I \longrightarrow \mathrm{Cok}_{E_n}(\Sigma_p) \longrightarrow 0.$$

We will show that $\mathrm{Cok}_{E_n}(\Sigma_p)$ is all p-torsion with exponent 1, i.e. that it is annihilated by multiplication by p.

There is a stable splitting

$$\Sigma_+^\infty B\mathbb{Z}/p \simeq \Sigma_+^\infty B\Sigma_p \times Y$$

where Y is some spectrum. Applying E-theory to this yields a commutative diagram

$$\begin{array}{ccccc}
E_n^*(B\Sigma_p) & \lhook\joinrel\longrightarrow & E_n^* \times E_n^*(B\Sigma_p)/I & \longrightarrow & \mathrm{Cok}_{E_n}(\Sigma_p) \\
\;\; {\scriptstyle i} \big\uparrow\big\downarrow & & \big\uparrow\big\downarrow & & \big\downarrow \\
E_n^* B\mathbb{Z}/p & \lhook\joinrel\longrightarrow & E_n^* \times E_n^*(B\mathbb{Z}/p)/I & \longrightarrow\!\!\!\!\rightarrow & \mathrm{Cok}_{E_n}(\mathbb{Z}/p)
\end{array}$$

in which the first vertical map i is split injective (by the stable splitting above). Since $E_n^* B\mathbb{Z}/p$ is free, the cokernel of i is projective and hence p-torsion free. Since $\mathbb{Q} \otimes \mathrm{Cok}_{E_n}(\Sigma_p) = 0$, the kernel of the map $\mathrm{Cok}_{E_n}(\Sigma_p) \to \mathrm{Cok}_{E_n}(\mathbb{Z}/p)$ must be torsion. Therefore the snake lemma implies that the kernel of the right vertical map $\mathrm{Cok}_{E_n}(\Sigma_p) \to \mathrm{Cok}_{E_n}(\mathbb{Z}/p)$ must be a torsion subgroup of a torsion-free module and hence is zero. Thus $\mathrm{Cok}_{E_n}(\Sigma_p)$ is a subgroup of $\mathrm{Cok}_{E_n}(\mathbb{Z}/p)$. The claim now follows from Corollary 4.2. $\qquad\square$

4.2 Commutative algebra

In this subsection we prove some basic facts about integer torsion in modules.

The following lemma is part of a well-known collection of facts that go under the title 'fracture squares'.

LEMMA 4.4. *Let* $R = L_{K(t)} E_n^0 \cong W(\kappa)[\![u_1, \ldots, u_{n-1}]\!][u_t^{-1}]_{I_t}^\wedge$ *for* $t > 0$ *and let* S *be a subset of* $R - (p)$; *then there is a pullback square of commutative rings*

$$\begin{array}{ccc}
R & \longrightarrow & (S^{-1}R)_p^\wedge \\
\big\downarrow & & \big\downarrow \\
\mathbb{Q} \otimes R & \longrightarrow & \mathbb{Q} \otimes (S^{-1}R)_p^\wedge.
\end{array}$$

Moreover, for any finitely generated free R-module M there is a similar square given by tensoring the pullback square with M.

Proof. The square in the proposition is the composite of two squares:

$$\begin{array}{ccccc}
R & \longrightarrow & S^{-1}R & \longrightarrow & (S^{-1}R)_p^\wedge \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbb{Q} \otimes R & \longrightarrow & \mathbb{Q} \otimes S^{-1}R & \longrightarrow & \mathbb{Q} \otimes (S^{-1}R)_p^\wedge.
\end{array}$$

The right-hand square is an 'arithmetic square'. It is a pullback square by applying [DG02, 4.12 and 4.13]. In [DG02] it is shown that the derived functors of localization and completion fit into a homotopy pullback square. The result we want is recovered by noticing that $S^{-1}R$ is free

as an $S^{-1}R$-module so that $H_0(S^{-1}R) = S^{-1}R$, and that the derived completion and ordinary completion agree because $S^{-1}R$ is Noetherian.

The left-hand square is a pullback by direct computation. Let $r/s^k \in S^{-1}R$ and $r'/p^l \in \mathbb{Q} \otimes R$ such that

$$\frac{r}{s^k} = \frac{r'}{p^l} \in \mathbb{Q} \otimes S^{-1}R$$

where $r, r' \in R$. Thus $s^k r' = p^l r$ and

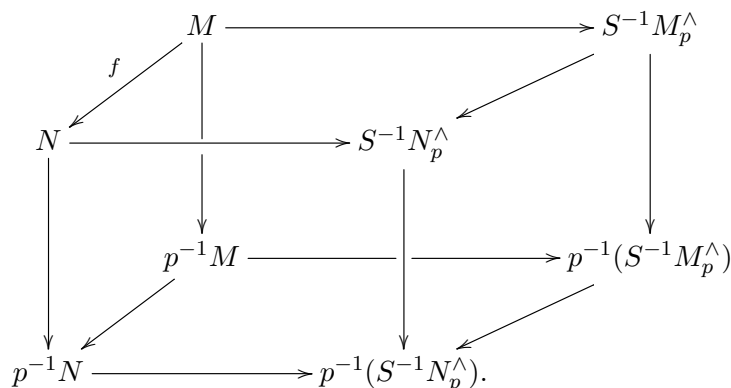$$r \mapsto \frac{s^k r'}{p^l} \in \mathbb{Q} \otimes R.$$

But this implies that $s^k r'/p^l \in R$, since the map is an inclusion. Since R is an integral domain and s is, by definition, not in the ideal generated by p, we have

$$p^l | r' \in R.$$

Now r'/p^l is the pullback. $\qquad \square$

LEMMA 4.5. *Let $M \xrightarrow{f} N$ be an injective map of finitely generated free R-modules. If $x \in N$ has the property that $p^l x \in \operatorname{im} f$ and $s^k x \in \operatorname{im} f$ for $s \in S$, then $x \in \operatorname{im} f$.*

Proof. It suffices to set $k = l = 1$. We will use Lemma 4.4. There are $m_1, m_2 \in M$ such that $m_1/s \in S^{-1}M_p^{\wedge}$ maps to x and $m_2/p \in p^{-1}M$ maps to x. Now consider the commutative cube

The front and back faces are pullback squares. The elements m_1/s and m_2/p must agree in $p^{-1}(S^{-1}M_p^{\wedge})$, and thus there is an element $m \in M$ that maps to both of these. By commutativity we must have $f(m) = x$. $\qquad \square$

By the torsion of an R-module M we will always mean the set of elements $x \in M$ such that $nx = 0$ for some non-zero $n \in \mathbb{N}$.

DEFINITION 4.6. For an abelian group M let $e(M)$ be the exponent of the torsion in M. That is, let K be the kernel of $M \longrightarrow \mathbb{Q} \otimes M$, and then $e(M)$ is the minimum number $n \in \mathbb{N} \cup \{\infty\}$ such that $nK = 0$.

PROPOSITION 4.7. *Let R be a Noetherian ring and M a finitely generated R-module. Then the torsion in M is bounded.*

Proof. Let K be the kernel

$$0 \longrightarrow K \longrightarrow M \longrightarrow \mathbb{Q} \otimes M = (\mathbb{Q} \otimes R) \otimes_R M.$$

The R-module K is finitely generated since R is Noetherian and M is finitely generated. Choose generators $k_1, \ldots, k_m$ of K; then there exists an n such that $nk_i = 0$ for all i. Now any element of K is of the form $\sum r_i k_i$, and this is also killed by n. $\square$

PROPOSITION 4.8. *Let R be a torsion-free Noetherian ring. Let $f : N \hookrightarrow M$ be an injection of finitely generated free R-modules and let $\Delta : M \hookrightarrow M \times M$ be the diagonal. Then $e(\mathrm{Cok}(f)) = e(\mathrm{Cok}(\Delta f))$.*

Proof. Consider the map of short exact sequences

$$
\begin{array}{ccccc}
N & \xrightarrow{\ f\ } & M & \longrightarrow & \mathrm{Cok}(f) \\
\Big\downarrow{=} & & \Big\downarrow{\Delta} & & \Big\downarrow{g} \\
N & \longrightarrow & M \times M & \longrightarrow & \mathrm{Cok}(\Delta f).
\end{array}
$$

The diagonal Δ is a section of the projection onto a factor and the left vertical arrow is an equality, so g is an injection. Thus there is an induced map $\mathrm{Cok}(\Delta f) \xrightarrow{\ r\ } \mathrm{Cok}(f)$ such that $rg = 1$. Since integer torsion is sent to integer torsion, this implies that the map induced by g on integer torsion is a split injection. It is an isomorphism since a torsion element must be of the form $(m, m) \in M \times M$ with $(nm, nm) \in \Delta f N$, and so the kernel of r is trivial. $\square$

PROPOSITION 4.9. *Let R be Noetherian and torsion free. For a flat R-algebra T and a finitely generated R-module M, $e(M) \geqslant e(T \otimes_R M)$. If T is a faithfully flat R-algebra, then equality holds, i.e. $e(M) = e(T \otimes_R M)$.*

Proof. Assume that $e(M) = n$ and let $[n] : M \longrightarrow M$ be the multiplication-by-n map. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\ [n]\ } \mathrm{im}[n] \longrightarrow 0,$$

where K is the torsion in M. Base change to T gives a short exact sequence and $T \otimes_R \mathrm{im}[n]$ is necessarily torsion free; thus

$$e(T \otimes_R K) = e(T \otimes_R M) \leqslant n.$$

If T is faithfully flat, then $K \longrightarrow T \otimes_R K$ is injective [Lam99, 4.74 and 4.83], so $e(T \otimes_R M) = e(M)$. $\square$

COROLLARY 4.10. *Let R be as in Lemma 4.4, let T be a faithfully flat $(S^{-1}R)_p^\wedge$-algebra, and let $M \xrightarrow{\ f\ } N$ be a map of finitely generated free R-modules; then*

$$e(N/\mathrm{im}\, f) = e(T \otimes_R (N/\mathrm{im}\, f)).$$

Proof. By Lemma 4.5, the torsion in $N/\mathrm{im}\, f$ is contained in the torsion of $(S^{-1}R)_p^\wedge \otimes_R N/\mathrm{im}\, f$, so

$$e(N/\mathrm{im}\, f) \leqslant e((S^{-1}R)_p^\wedge \otimes_R N/\mathrm{im}\, f).$$

Also, $(S^{-1}R)_p^\wedge$ is a flat extension of R, and this gives the reverse inequality. Since T is faithfully flat, base change to T is injective. Thus we have have equalities

$$e(N/\mathrm{im}\, f) = e((S^{-1}R)_p^\wedge \otimes_R N/\mathrm{im}\, f) = e(T \otimes_R (N/\mathrm{im}\, f))$$

by Proposition 4.9. $\square$

4.3 A few hand-crafted cohomology theories

In order to prove the bounded torsion results for E-theory we make use of several E_n-algebras. In this section we construct the E_n-algebras that we need and prove or recall some basic properties of them. We rely heavily on ideas from [BS16].

Let $L_1 = L_{K(1)}E_n$ (it does depend on n) and recall from [Hov97, Corollary 1.5.5] that

$$\pi_0 L_1 = W(\kappa)[\![u_1, \ldots, u_{n-1}]\!][u_1^{-1}]_p^{\wedge}.$$

This is a flat E_n^0-algebra. Recall that we may localize an E_∞-ring spectrum R at a prime ideal $\mathfrak{p} \subseteq \pi_0 R$ by inverting every element in $\pi_0 R - \mathfrak{p}$; we write $R_{\mathfrak{p}}$ for the localized ring spectrum. Define

$$F_1 = L_{K(1)}((E_n)_{(p)})$$

by inverting all of the elements away from $\mathfrak{p} = (p)$ and then $K(1)$-localizing the resulting spectrum. Note that this inverts $u_1, \ldots, u_{n-1}$. The ring spectrum F_1 is an even-periodic Landweber-exact E_∞-ring such that

$$\pi_0 F_1 = F_1^0 = ((E_n^0)_{(p)})_p^{\wedge}.$$

Since F_1 is $K(1)$-local, there is a canonical map

$$L_1 \longrightarrow F_1$$

inducing the obvious map on coefficients. This suggests another construction of F_1. It is clear that

$$((L_1)_{(p)})_p^{\wedge} \xrightarrow{\cong} F_1,$$

by considering the quotient of this ring by powers of p. This allows Corollary 4.10 to be used with $(R, S) = (E_n^*, E_n^* - (p))$ or $(R, S) = (L_1^*, L_1^* - (p))$.

Now let G be a finite group.

PROPOSITION 4.11. *The canonical map*

$$F_1 \wedge_{L_1} L_1^{BG} \xrightarrow{\simeq} F_1^{BG}$$

is an equivalence that is natural in G.

Proof. The proof follows the proof of Proposition 6.2 in [BS16]. It is clear that the map is natural. To prove that it is an equivalence, the main idea is to note that the smash product on the left is $K(1)$-local and then use the $K(1)$-local self-duality of classifying spaces of finite groups. $\square$

Following ideas in [BS16] based on work of Hopkins in [Hop], we further define

$$\bar{F}_1 = L_{K(1)}(F_1 \wedge K_p),$$

where K_p is p-adic K-theory. We do not have much control over the coefficients of this ring spectrum. However, following [BS16, Proposition 5.8], $\bar{F}_1$ does have a very desirable property as follows.

PROPOSITION 4.12. *The ring of coefficients of $\bar{F}_1$ is faithfully flat as a K_p^*-module and as an F_1^*-module.*

Proof. This follows from the proof of Proposition 5.8 in [BS16], which relies on the important (and somewhat surprising) Proposition 3.13 that extends an argument of Hovey's in [Hov04b]. Both module structures are faithfully flat because K_p^* and F_1^* are both complete local Noetherian. $\square$

PROPOSITION 4.13. *For a finite group G, there are natural equivalences*

$$\bar{F}_1 \wedge_{F_1} F_1^{BG} \xrightarrow{\simeq} \bar{F}_1^{BG} \xleftarrow{\simeq} \bar{F}_1 \wedge_{K_p} K_p^{BG}.$$

Proof. This also follows the proof of Proposition 6.2 in [BS16], using Proposition 4.12 above. $\square$

Now let $H \subset G$ so that we have a transfer map $E^{BH} \xrightarrow{\text{Tr}} E^{BG}$ for any cohomology theory E.

PROPOSITION 4.14. *Let E and F be E_∞-rings and let F be an E-algebra. There is a commutative square*

$$
\begin{array}{ccc}
F \wedge_E E^{BH} & \longrightarrow & F^{BH} \\
{\scriptstyle F \wedge_E \text{Tr}} \downarrow & & \downarrow {\scriptstyle \text{Tr}} \\
F \wedge_E E^{BG} & \longrightarrow & F^{BG}.
\end{array}
$$

Proof. The map of E_∞-rings $E \longrightarrow F$ and the transfer map $\Sigma_+^\infty BH \longrightarrow \Sigma_+^\infty BG$ induce a commutative square of function spectra

$$
\begin{array}{ccc}
E^{BH} & \longrightarrow & F^{BH} \\
{\scriptstyle \text{Tr}} \downarrow & & \downarrow {\scriptstyle \text{Tr}} \\
E^{BG} & \longrightarrow & F^{BG}
\end{array}
$$

and we can now base change the left-hand side to F over E. $\square$

Let E be any cohomology theory, let $H_1, \ldots, H_m \subset G$, and let $I \subset E^*(BG)$ be the ideal generated by the image of the transfers $E^*(BH_i) \xrightarrow{\text{Tr}} E^*(BG)$. This is the image of the map

$$\bigoplus_i E^*(BH_i) \xrightarrow{\oplus \text{Tr}} E^*(BG).$$

PROPOSITION 4.15. *Let $E = K_p, L_1, F_1$ or $\bar{F}_1$, let $H_1, \ldots, H_m \subset G$, and let $I \subset E^*(BG)$ be the associated transfer ideal. There are isomorphisms*

$$F_1^* \otimes_{L_1^*} L_1^*(BG)/I \xrightarrow{\cong} F_1^*(BG)/I$$

and

$$\bar{F}_1^* \otimes_{F_1^*} F_1^*(BG)/I \xrightarrow{\cong} \bar{F}_1^*(BG)/I \xleftarrow{\cong} \bar{F}_1^* \otimes_{K_p^*} K_p^*(BG)/I.$$

Proof. The first isomorphism follows from the previous proposition and Proposition 4.11, and the second isomorphism follows from the previous proposition and Proposition 4.13. $\square$

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In the next section we will make use of character theory for Morava E-theory. This will involve two more E_n-algebras. The first was introduced in [HKR00] and the second was introduced in [Sta13].

Let $\Lambda_{k,n} = (\mathbb{Z}/p^k)^n$. The E_n-algebra C_0 is constructed in [HKR00, §6.2]. It is a $p^{-1}E_n$-algebra and may be constructed as an E_n-algebra as a localization of the colimit

$$\operatorname*{colim}_{k} E_n^{B\Lambda_{k,n}}.$$

The commutative ring C_0^* is faithfully flat as a $\mathbb{Q} \otimes E_n^*$-algebra. The point of the construction (and this is [HKR00, Theorem C]) is that for any finite group G, there is a canonical equivalence

$$C_0 \wedge_{E_n} E_n^{BG} \xrightarrow{\ \sim\ } C_0 \wedge_{p^{-1}E_n} (p^{-1}E_n)^{\mathcal{L}^n BG},$$

where $\mathcal{L}(-) = \hom(B\mathbb{Z}_p, -)$ is the p-adic free loop space functor. Since $p^{-1}E_n$ is rational and C_0^* is faithfully flat as a $p^{-1}E_n^*$-algebra, the target is equivalent to

$$\prod_{\pi_0 \mathcal{L}^n BG} C_0.$$

The E_n-algebra C_1 is constructed (as a commutative ring) in [Sta13] and as an E_n-algebra in [BS16]. It is a $K(1)$-local L_1-algebra which is given as the p-completion of a localization of the colimit

$$\operatorname*{colim}_{k} L_1 \wedge_{E_n} E_n^{B\Lambda_{k,n-1}}.$$

The point of the construction (and this is a special case of the main theorem of [Sta13]) is that there is an equivalence

$$C_1 \wedge_{E_n} E_n^{BG} \xrightarrow{\ \sim\ } C_1 \wedge_{L_1} L_1^{\mathcal{L}^{n-1} BG}.$$

Since C_1^* is flat as an E_n^*-algebra and faithfully flat as an L_1^*-algebra, applying π_* to this equivalence gives the isomorphism

$$C_1^* \otimes_{E_n^*} E_n^*(BG) \xrightarrow{\ \cong\ } C_1^* \otimes_{L_1^*} L_1^*(\mathcal{L}^{n-1} BG). \tag{4.3}$$

The iterated p-adic free loop space of BG is a disjoint union of classifying spaces of certain centralizer subgroups of G, and this is how the codomain is most easily understood explicitly. Let

$$\hom(\mathbb{Z}_p^l, G)/\sim$$

be conjugacy classes of continuous maps from $\mathbb{Z}_p^l$ to G; then

$$C_1^* \otimes_{L_1^*} L_1^*(\mathcal{L}^{n-1} BG) \cong \prod_{[\alpha] \in \hom(\mathbb{Z}_p^{n-1}, G)/\sim} C_1^* \otimes_{L_1^*} L_1^*(\mathcal{L}^{n-1} BC(\operatorname{im}\alpha)),$$

where $C(\operatorname{im}\alpha)$ is the centralizer of the image of a choice $\alpha \in [\alpha]$.

The equivalences above behave well with respect to transfers. Formulas can be found in [HKR00, Theorem D] and [Sta15, Theorems 2.18 and 3.11].

4.4 Bounded torsion in E-theory

In this subsection we prove that rational factorizations of the Morava E-cohomology of finite abelian groups and symmetric groups hold integrally up to a torsion cokernel with a height-independent bounded exponent. We do this by putting together the tools developed in the preceding two subsections.

When we write $I \subset E^*(BA)$ for an abelian group A we will always mean the ideal generated by the image of the transfer along proper subgroups of A. When we write $I \subset E^*(B\Sigma_m)$ we will always mean the ideal generated by the image of the transfer along the proper partition subgroups $\Sigma_i \times \Sigma_j \longrightarrow \Sigma_m$, where $i + j = m$ with $i, j > 0$.

We will begin with symmetric groups. First we will set up the algebro-geometric objects that witness the rational factorization.

Let $\lambda \vdash m$ be a partition of m. We write

$$\lambda = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_k\lambda_k,$$

where $\lambda_i < \lambda_j$ when $i < j$, $a_i \in \mathbb{N}$, and the value of the sum is m. To simplify the notation, we write $\Sigma_\lambda = \Sigma_{\lambda_1}^{\times a_1} \times \cdots \times \Sigma_{\lambda_k}^{\times a_k}$ and $\Sigma_{\underline{a}} = \Sigma_{a_1} \times \cdots \times \Sigma_{a_k}$. Furthermore, let I_λ be the ideal in $E_n^*(B\Sigma_\lambda)$ generated by the individual transfer ideals $I \subset E_n^*(B\Sigma_{\lambda_i})$; that is, I_λ has the property that

$$E_n^*(B\Sigma_\lambda)/I_\lambda \cong \bigotimes_i (E_n^*(B\Sigma_{\lambda_i})/I)^{\otimes a_i}.$$

Furthermore, there is an action of Σ_{a_i} on the tensor power

$$(E_n^*(B\Sigma_{\lambda_i})/I)^{\otimes a_i}.$$

For each tensor factor, we will take the fixed points by this action and denote the result by

$$(E_n^*(B\Sigma_\lambda)/I_\lambda)^{\Sigma_{\underline{a}}} = \bigotimes_i ((E_n^*(B\Sigma_{\lambda_i})/I)^{\otimes a_i})^{\Sigma_{a_i}}.$$

This commutative ring can be understood algebro-geometrically by a theorem of Strickland's. In [Str97], Strickland proved that $E_n^*(B\Sigma_m)/I$ is a finitely generated free E_n^*-module and produces a canonical isomorphism

$$\mathrm{Spf}(E_n^*(B\Sigma_m)/I) \cong \mathrm{Sub}_m(\mathbb{G}_{E_n}),$$

where $\mathrm{Sub}_m(\mathbb{G}_{E_n})$ is the scheme classifying subgroup schemes of order m in $\mathbb{G}_{E_n}$. It is worth noting that this is the empty scheme if m is coprime to p. The scheme associated to the ring that we are interested in is built out of $\mathrm{Sub}_m(\mathbb{G}_{E_n})$. Thus we may define

$$\begin{aligned}
\mathrm{Sub}_{\lambda \vdash m}(\mathbb{G}_{E_n}) &= \mathrm{Spec}((E_n^*(B\Sigma_\lambda)/I_\lambda)^{\Sigma_{\underline{a}}}) \\
&\cong \prod_i \mathrm{Spec}(((E_n^*(B\Sigma_{\lambda_i})/I)^{\otimes a_i})^{\Sigma_{a_i}}).
\end{aligned}$$

The point of the notation $\mathrm{Sub}_{\lambda \vdash m}(\mathbb{G}_{E_n})$ is that this scheme represents unordered sets of $\sum_i a_i$ subgroups of $\mathbb{G}_{E_n}$ in which a_i of the subgroups have order λ_i. This scheme-theoretic description is not essential to the proof of the result. However, one of the goals of decomposing $BP^*(B\Sigma_m)$ rationally is to suggest that it may have a tractable algebro-geometric description.

Let $\Gamma(-)$ be the functor that takes a scheme to the ring of functions on the scheme. There is a canonical map

$$E_n^*(B\Sigma_m) \longrightarrow \Gamma\,\mathrm{Sub}_{\lambda \vdash m}(\mathbb{G}_{E_n})$$

induced by restriction. The only part to check here is that it actually lands in the fixed points, but this is clear as there is an inner automorphism of Σ_m that permutes the a_i groups of the form Σ_{λ_i} sitting inside Σ_m.

PROPOSITION 4.16. *The canonical map*

$$E_n^*(B\Sigma_m) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_{E_n})$$

is injective and a rational isomorphism.

Proof. This is a basic application of HKR-character theory [HKR00]. It is clear that the map is an isomorphism after base change to C_0^*. The resulting ring is the ring of C_0^*-valued functions on the set

$$\operatorname{Sum}_m(\mathbb{Q}_p/\mathbb{Z}_p^n) = \coprod_{\lambda \vdash m} \operatorname{Sub}_{\lambda \vdash m}(\mathbb{Q}_p/\mathbb{Z}_p^n).$$

But C_0^* is a faithfully flat $p^{-1}E_n^*$-algebra, so the map is a rational isomorphism.   □

Most of the results in §§ 4.2 and 4.3 will go into the proof of the next theorem. We will use the method of the proof twice more in Theorem 4.18 and Proposition 4.24. The idea of the proof is the following. We begin with a short exact sequence with a torsion cokernel. We will apply character theory to height 1 and use faithful flatness to get a short exact sequence in L_1-cohomology. In this case, the cokernel will be a product of cokernels of maps. The sorts of maps that occur are bounded by the number of centralizers of tuples of commuting elements in Σ_m. However, F_1 depends on n, so we do not have complete control over the cokernel as n varies. We will extend to $\bar{F}_1$ and then use faithful flatness to change the product of maps to a product of maps in K_p-cohomology. Now we are in a situation that is height independent, the number of maps that can appear in the product is bounded, and the cokernel of each map is bounded torsion.

THEOREM 4.17. *The cokernel of the map*

$$E_n^*(B\Sigma_m) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_{E_n})$$

is torsion with exponent bounded independently of the height n.

Proof. Note that both the source and the target are finitely generated free E_n^*-modules and the map is an injection. The cokernel $\operatorname{Cok}_{E_n}(\Sigma_m)$ is a torsion E_n^*-module. We will repeatedly base change the short exact sequence

$$0 \longrightarrow E_n^*(B\Sigma_m) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_{E_n}) \longrightarrow \operatorname{Cok}_{E_n}(\Sigma_m) \longrightarrow 0$$

to prove the result. Base change to C_1^* (a flat E_n^*-algebra) and (4.3) give the short exact sequence

$$\prod_{[\beta] \in \operatorname{Hom}(\mathbb{Z}_p^{n-1}, \Sigma_m)/\sim} C_1^* \otimes_{L_1^*} L_1^*(BC(\operatorname{im}\beta)) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_{C_1} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1})$$

$$\longrightarrow C_1^* \otimes_{L_1^*} L_1^* \otimes_{E_n^*} \operatorname{Cok}_{E_n}(\Sigma_m).$$

Here we have used the fact that the scheme classifying subgroups behaves well under base change ([Sta15, Theorem 3.11] is particularly relevant). Corollary 4.10 implies that the exponent of the torsion cannot have changed. By faithful flatness (Proposition 4.9), we may remove the C_1^* without changing the exponent of the torsion. Thus we arrive at the short exact sequence

$$\prod_{[\beta]} L_1^*(BC(\operatorname{im}\beta)) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_{L_1} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1}) \longrightarrow L_1^* \otimes_{E_n^*} \operatorname{Cok}_{E_n}(\Sigma_m). \qquad (4.4)$$

Because the first and second terms are finitely generated and free, we may apply Corollary 4.10 to base change (4.4) to F_1^* without changing the exponent of $L_1^* \otimes_{E_n^*} \operatorname{Cok}_{E_n}(A)$. We may then further base change to $\bar{F}_1^*$ (which is faithfully flat over F_1^* by Proposition 4.12), obtaining

$$\prod_{[\beta]} \bar{F}_1^*(BC(\operatorname{im}\beta)) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_{\bar{F}_1} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1}) \longrightarrow \bar{F}_1^* \otimes_{E_n^*} \operatorname{Cok}_{E_n}(\Sigma_m).$$

Now Proposition 4.13 gives the short exact sequence

$$\prod_{[\beta]} \bar{F}_1^* \otimes_{K_p^*} K_p^*(BC(\operatorname{im}\beta)) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_{\bar{F}_1} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1}) \longrightarrow \bar{F}_1^* \otimes_{E_n^*} \operatorname{Cok}_{E_n}(\Sigma_m).$$

By the naturality of the equivalence, the first map in the sequence is induced by restriction maps. Let

$$M_{A,n} = \operatorname{Cok}\left(\prod_{[\beta]} K_p^*(BC(\operatorname{im}\beta)) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1})\right);$$

then

$$\bar{F}_1^* \otimes_{K_p^*} M_{A,n} \cong \bar{F}_1^* \otimes_{E_n^*} \operatorname{Cok}_{E_n}(\Sigma_m).$$

By Proposition 4.9, it suffices to show that the torsion in $M_{A,n}$ is independent of n. The map

$$\prod_{[\beta]} K_p^*(BC(\operatorname{im}\beta)) \longrightarrow \prod_{\lambda \vdash m} \Gamma \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1})$$

is the ring of functions on the canonical map

$$\coprod_{\lambda \vdash m} \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1}) \xrightarrow{f} \operatorname{Spec}(K_p^*(\mathcal{L}^{n-1}B\Sigma_m)).$$

There is a commutative triangle

$$\begin{array}{ccc}
\coprod_{\lambda \vdash m} \operatorname{Sub}_{\lambda \vdash m}(\mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1}) & \longrightarrow & \operatorname{Spec}(K_p^*(\mathcal{L}^{n-1}B\Sigma_m)) \\
& \searrow & \downarrow \\
& & \operatorname{Sum}_{\leqslant m}(\mathbb{Q}_p/\mathbb{Z}_p^{n-1}),
\end{array}$$

where

$$\operatorname{Sum}_{\leqslant m}(\mathbb{Q}_p/\mathbb{Z}_p^{n-1}) = \left\{ \bigoplus_i H_i \ \middle|\ H_i \subset \mathbb{Q}_p/\mathbb{Z}_p^{n-1} \text{ and } \sum_i |H_i| \leqslant m \right\}$$

is the set of formal sums of subgroups in $\mathbb{Q}_p/\mathbb{Z}_p^{n-1}$ with order no greater than m. For each element $\alpha \in \operatorname{Sum}_{\leqslant m}(\mathbb{Q}_p/\mathbb{Z}_p^{n-1})$, the map that lives over α is a rational isomorphism (and an injection after applying $\Gamma(-)$). Thus it suffices to prove that, regardless of the choice of n, there are only

finitely many different maps that can appear as fibers. This is essentially a consequence of the fact that there are only finitely many groups that are centralizers of Σ_m.

To make this precise, note that there is an action of $\mathrm{Aut}(\mathbb{Q}_p/\mathbb{Z}_p^{n-1})$ on the triangle. This action must send the map over $\alpha \in \mathrm{Sum}_{\leqslant m}(\mathbb{Q}_p/\mathbb{Z}_p^{n-1})$ to a map with the same exponent of the torsion in the cokernel. The action of $\mathrm{Aut}(\mathbb{Q}_p/\mathbb{Z}_p^{n-1})$ is compatible with the inclusion of triangles induced by the inclusion of $\mathbb{Q}_p/\mathbb{Z}_p^{n-1} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p^n$ as the first $n-1$ summands. For n large enough, any sum of subgroups of size less than or equal to m in $\mathbb{Q}_p/\mathbb{Z}_p^n$ is isomorphic to a sum of subgroups in $\mathbb{Q}_p/\mathbb{Z}_p^{n-1}$ under this action, and this finishes the proof. $\qquad\square$

It is worth noting that this could probably be factored further using Strickland's schemes called Type and [Str97, Theorem 12.4].

Now we turn our attention to an analogous theorem for level structures. The theory of level structures for a formal group is developed in [Str97]. We make use of it now because there is a close relationship between $E_n^*(BH)/I$ and $\Gamma\mathrm{Level}(H^*, \mathbb{G}_{E_n})$, where H^* is the Pontryagin dual of H. In [AHS04], it is proved that the image of $E_n^*(BH)/I$ in $\mathbb{Q} \otimes E_n^*(BH)/I$ is canonically isomorphic to $\Gamma\mathrm{Level}(H^*, \mathbb{G}_{E_n})$.

Theorem 4.18. *For A a finite abelian group, the cokernel of the map*

$$E_n^*(BA) \longrightarrow \prod_{H \subseteq A} \Gamma\mathrm{Level}(H^*, \mathbb{G}_{E_n})$$

is torsion with exponent bounded independently of the height n.

Proof. Since the argument is similar to the proof of Theorem 4.17, we only point out the differences. We can again reduce to showing that the torsion in

$$M_{A,n} = \mathrm{Cok}\bigg(\prod_\beta K_p^*(BA) \longrightarrow \prod_{H \subseteq A} \Gamma\mathrm{Level}(H^*, \mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1})\bigg)$$

has exponent independent of n. To this end, note that the map

$$\prod_\beta K_p^*(BA) \longrightarrow \prod_{H \subseteq A} \Gamma\mathrm{Level}(H^*, \mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1})$$

is obtained by taking global sections of the canonical map

$$\coprod_{H \subset A} \mathrm{Level}(H^*, \mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1}) \xrightarrow{\ f\ } \mathrm{Hom}(A^*, \mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1}).$$

It is a product of ring maps since we have the commutative triangle

$$\coprod_{H \subset A} \mathrm{Level}(H^*, \mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1}) \longrightarrow \mathrm{Hom}(A^*, \mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1})$$
$$\mathrm{Hom}(A^*, \mathbb{Q}_p/\mathbb{Z}_p^{n-1})$$

and $\mathrm{Hom}(A^*, \mathbb{Q}_p/\mathbb{Z}_p^{n-1}) \cong \mathrm{Hom}(\mathbb{Z}_p^{n-1}, A)$ is a set and thus disconnects the horizontal map. The diagonal arrow sends a level structure $H^* \hookrightarrow \mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1}$ to the composite

$$A^* \twoheadrightarrow H^* \hookrightarrow \mathbb{G}_m \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-1} \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p^{n-1}.$$

Let f_β be the fiber over β. It is a map of schemes. It is clear that there is an action of $\mathrm{Aut}(\mathbb{Q}_p/\mathbb{Z}_p^{n-1})$ on the triangle above. This implies that the fibers over maps $\beta, \beta' \in \mathrm{Hom}(\mathbb{Z}_p^{n-1}, A)$ with $\mathrm{im}\,\beta = \mathrm{im}\,\beta'$ are isomorphic. Hence the exponent of the cokernel of $\Gamma(f_\beta)$ equals the exponent of the cokernel of $\Gamma(f_{\beta'})$. Adding another $\mathbb{Q}_p/\mathbb{Z}_p$ to the constant étale part does not affect this in the sense that the induced inclusion

$$\mathrm{Hom}(A^*, \mathbb{Q}_p/\mathbb{Z}_p^{n-1}) \longrightarrow \mathrm{Hom}(A^*, \mathbb{Q}_p/\mathbb{Z}_p^n)$$

lifts to an inclusion of triangles. As n increases above the number of generators of A, $e(M_{A,n})$ remains constant. The fibers over all of the maps in $\mathrm{Hom}(A^*, \mathbb{Q}_p/\mathbb{Z}_p^n)$ are isomorphic to the fibers over maps that were already there in $\mathrm{Hom}(A^*, \mathbb{Q}_p/\mathbb{Z}_p^{n-1})$. Thus the exponent of the torsion in $M_{A,n}$ is bounded independently of n. $\qquad\square$

The proof of the following result, due to Jeremy Hahn, is given in the appendix.

PROPOSITION 4.19. *Let H be a finite abelian group. The exponent of the torsion in*

$$E_n^*(BH)/I$$

is bounded independently of the height n.

COROLLARY 4.20. *If A is a finite abelian group, then the map*

$$\prod_n E_n^*(BA) \longrightarrow \prod_n \prod_{H \subseteq A} E_n^*(BH)/I$$

is a rational isomorphism.

Proof. Consider the composite

$$\prod_n E_n^*(BA) \longrightarrow \prod_n \prod_{H \subseteq A} E_n^*(BH)/I \longrightarrow \prod_n \prod_{H \subseteq A} \Gamma\mathrm{Level}(H^*, \mathbb{G}_{E_n}).$$

By Proposition 4.19 the second map is a rational isomorphism, and by Theorem 4.18 the composite is a rational isomorphism. Thus the first map is a rational isomorphism. $\qquad\square$

4.5 BP-cohomology of abelian and symmetric groups

We show that the rational decomposition of Theorem 4.17 extends to a rational decomposition of BP-cohomology.

LEMMA 4.21. *Let G be a finite group and let $H_1, \ldots, H_m \subset G$ be subgroups. Let I be the ideal generated by the image of the transfer maps along the inclusions $H_i \subset G$. There is a split inclusion*

$$BP^*(BG)/I \longrightarrow \prod_{n>0} E_n^*(BG)/I$$

which is compatible with the map of Corollary 3.4.

Proof. Let $H_1, \ldots, H_m \subseteq G$ be subgroups and consider the commutative diagram

$$
\begin{array}{ccccccc}
\bigoplus_i BP^*(BH_i) & \xrightarrow{\ \mathrm{tr}\ } & BP^*(BG) & \longrightarrow & BP^*(BG)/I & \longrightarrow & 0 \\
\Big\uparrow\Big\downarrow & & \Big\uparrow\Big\downarrow & & \Big\uparrow\Big\downarrow & & \\
\prod_{n>0} \bigoplus_i E_n^*(BH_i) & \xrightarrow{\ \mathrm{tr}\ } & \prod_{n>0} E_n^*(BG) & \longrightarrow & \prod_{n>0} E_n^*(BG)/I & \longrightarrow & 0,
\end{array}
$$

where I denotes the transfer ideal generated by $H_1, \ldots, H_m$. The dashed arrows are the retracts which exist by Corollary 3.4, and the transfers make the left square commute by naturality and the fact that the splitting exists on the level of spectra. Therefore, the maps between the quotients exist and the resulting retract is compatible with the rest of the diagram. $\quad\square$

Consequently, for any symmetric group Σ_m, we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{Q} \otimes BP^*(B\Sigma_m) & \lhook\joinrel\longrightarrow & \mathbb{Q} \otimes \prod_{\lambda \vdash m} \bigotimes_i ((BP^*(B\Sigma_{\lambda_i})/I)^{\otimes a_i})^{\Sigma_{a_i}} \\
\Big\uparrow\Big\downarrow & & \Big\uparrow\Big\downarrow \\
\mathbb{Q} \otimes \prod_{n>0} E_n^*(B\Sigma_m) & \lhook\joinrel\longrightarrow & \mathbb{Q} \otimes \prod_{n>0}\prod_{\lambda \vdash m} \bigotimes_i ((E_n^*(B\Sigma_{\lambda_i})/I)^{\otimes a_i})^{\Sigma_{a_i}} \\
\Big\downarrow & & \Big\downarrow \\
\prod_{n>0} \mathbb{Q} \otimes E_n^*(B\Sigma_m) & \xrightarrow{\ \cong\ } & \prod_{n>0} \mathbb{Q} \otimes \prod_{\lambda \vdash m} \bigotimes_i ((E_n^*(B\Sigma_{\lambda_i})/I)^{\otimes a_i})^{\Sigma_{a_i}}.
\end{array}
$$

In Theorem 4.17 we showed that the middle horizontal map is an isomorphism. Now it is an easy consequence that the top map is an isomorphism, stated in the next corollary.

COROLLARY 4.22. *There is a rational isomorphism*

$$
BP^*(B\Sigma_m) \longrightarrow \prod_{\lambda \vdash m} (BP^*(B\Sigma_\lambda)/I_\lambda)^{\Sigma_{\underline{a}}}.
$$

COROLLARY 4.23. *There is a rational isomorphism*

$$
BP^*(BA) \longrightarrow \prod_{H \subseteq A} BP^*(BH)/I.
$$

Proof. The argument is similar to the proof of Corollary 4.22. $\quad\square$

4.6 Artin induction for *BP*

It is not hard to use the methods of the previous subsections to prove an Artin induction theorem for BP. We do this by transferring the result for E_n (due to [HKR00]) to BP. Stronger results than this have been known for some time (e.g. [HKR00, Theorem A] applies to all complex oriented theories).

Let $\mathcal{A}(G)$ be the full subcategory of the orbit category of G consisting of quotients of the form G/A, where A is an abelian p-group.

Proposition 4.24. *If G is a good group, the canonical map*

$$BP^*(BG) \longrightarrow \lim_{A \in \mathcal{A}(\mathcal{G})} BP^*(BA)$$

is a rational isomorphism.

Proof. Because G is good there is no integral torsion in $BP^*(BG)$. This follows immediately from the injection

$$BP^*(BG) \hookrightarrow \prod_n E_n^*(BG).$$

Character theory for good groups [HKR00] implies that the natural map

$$\prod_n E_n^*(BG) \longrightarrow \prod_{n>0} \lim_{A \in \mathcal{A}(\mathcal{G})} E_n^*(BA)$$

is injective and, since limits of split injections are split injections, the canonical map

$$\lim_{A \in \mathcal{A}(\mathcal{G})} BP^*(BA) \longrightarrow \prod_{n>0} \lim_{A \in \mathcal{A}(\mathcal{G})} E_n^*(BA)$$

is injective.

We are interested in the commutative diagram

$$
\begin{array}{ccccccccc}
BP^*(BG) & \longrightarrow & \lim_{A \in \mathcal{A}(\mathcal{G})} BP^*(BA) & \longrightarrow & C_{BP} & \longrightarrow & 0 \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & \\
0 \longrightarrow \prod_{n>0} E_n^*(BG) & \longrightarrow & \prod_{n>0} \lim_{A \in \mathcal{A}(\mathcal{G})} E_n^*(BA) & \longrightarrow & \prod_n C_{E_n} & \longrightarrow & 0.
\end{array}
$$

Since the left vertical arrow is injective and the second horizontal arrow on the bottom is injective, the natural map

$$BP^*(BG) \longrightarrow \lim_{A \in \mathcal{A}(\mathcal{G})} BP^*(BA)$$

is injective. This implies that the top sequence of maps is short exact. Therefore it suffices to prove that C_{BP} is torsion.

This means that it suffices to show that C_{E_n} has exponent bounded independently of the height n. By [HKR00, Theorem A], the abelian group C_{E_n} is torsion.

Note the map of short exact sequences

$$
\begin{array}{ccccc}
E_n^*(BG) & \longrightarrow & \lim_{A \in \mathcal{A}(\mathcal{G})} E_n^*(BA) & \longrightarrow & C_{E_n} \\
\downarrow = & & \downarrow & & \downarrow \\
E_n^*(BG) & \longrightarrow & \prod_{A \subset G} E_n^*(BA) & \longrightarrow & D_{E_n},
\end{array}
$$

where D_{E_n} is the cokernel. The first two vertical arrows are injections (the first is an equality), so the right vertical arrow is an injection. Therefore it suffices to show that the exponent of the torsion in D_{E_n} is bounded independently of n.

To bound the torsion, we apply character theory to height 1. Since C_1^* is a flat E_n^*-module, we get a short exact sequence

$$C_1^* \otimes_{E_n^*} E_n^*(BG) \hookrightarrow \prod_{A \subset G} C_1^* \otimes_{E_n^*} E_n^*(BA) \longrightarrow C_1^* \otimes_{E_n^*} D_{E_n}.$$

Now we can follow the argument in Theorem 4.18 to reduce to p-adic K-theory. This leads to considering the cokernel of the injection

$$\prod_{[\alpha]} K_p^*(BC(\mathrm{im}\,\alpha)) \hookrightarrow \prod_{A \subset G} \prod_{\beta} K_p^*(BA).$$

If we fix a conjugacy class $[\alpha : \mathbb{Z}_p^{n-1} \to G]$, then there is a map $K_p^*(BC(\mathrm{im}\,\alpha)) \longrightarrow K_p^*(BA)$ when the composite

$$\mathbb{Z}_p^{n-1} \xrightarrow{\beta} A \subset G$$

is conjugate to α. In that case $A \subset C(\mathrm{im}\,\alpha)$. Let $i_H : A \subset G$ be the inclusion. The map out of the factor corresponding to $[\alpha]$ is

$$K_p^*(BC(\mathrm{im}\,\alpha)) \longrightarrow \prod_{A \subset G} \left(\prod_{\{\beta | [i_H \beta] = [\alpha]\}} K_p^*(BA) \right).$$

This map is an injection, although it may not be the case that every abelian subgroup of $C(\mathrm{im}\,\alpha)$ is represented in the codomain. Abelian subgroups of G that are conjugate to abelian subgroups of $C(\mathrm{im}\,\alpha)$ may appear as well. Also, subgroups may appear multiple times. However, Proposition 4.8 implies that repeat subgroups do not contribute to the exponent of the torsion in the cokernel. Finally, since there are only a finite number of abelian subgroups of $C(\mathrm{im}\,\alpha)$ (or abelian subgroups of G that may be conjugated into $C(\mathrm{im}\,\alpha)$), the exponent of the torsion in the cokernel must be bounded by the maximal exponent of this finite number of options. $\square$

This subsection is not just an exercise. Artin induction for E_n can be proved independently of [HKR00, Theorem A] by using character theory. The retract theorem now allows us to bootstrap to BP. It is not hard to imagine moving from there to any Landweber flat (or perhaps complex oriented) theory. One might hope that this could provide an independent proof of [HKR00, Theorem A], at least when G is good and $X = *$.

T. Barthel and N. Stapleton

Appendix. A Proof of Proposition 4.19

Jeremy Hahn

We recall our notational conventions. Let H be a finite abelian group, n a positive integer, and E_n a height n Morava E-theory at the prime p. We use H^* to denote the Pontryagin dual group $\mathrm{Hom}(H, S^1)$. In the ring $E_n^*(BH)$, denote by I the ideal generated by the images of transfers from proper subgroups of H. We let R_n denote the quotient ring $E_n^*(BH)/I$.

Algebro-geometrically, $E_n^*(BH)$ may be interpreted as global sections of the affine formal scheme $\mathrm{Hom}(H^*, \mathbb{G}_{E_n})$. The closed subscheme with global sections $R_n = E_n^*(BH)/I$ is not as well behaved as one might hope. In particular, for almost no values of H and n is R_n an integral domain. Let T_n denote the ideal of p-power torsion in R_n (i.e. $x \in T_n$ if and only if $p^a x = 0$ for some positive integer a). In [AHS04], Ando *et al.* identified the further quotient R_n/T_n with an object of algebro-geometric significance, namely the global sections of the formal scheme of Drinfeld level structures $\mathrm{Level}(H^*, \mathbb{G}_{E_n})$. In [Dri74], Drinfeld proved that $\Gamma\mathrm{Level}(H^*, \mathbb{G}_{E_n})$ is a regular local ring, so R_n/T_n is in fact very well behaved.

The Ando–Hopkins–Strickland result allows one to identify $\mathbb{Q} \otimes E_n^*(BH)/I$ with $\mathbb{Q} \otimes \Gamma\mathrm{Level}(H^*, \mathbb{G}_{E_n})$. In this appendix, our goal is to extend this to an isomorphism

$$\mathbb{Q} \otimes \prod_{n>0} E_n^*(BH)/I \cong \mathbb{Q} \otimes \prod_{n>0} \Gamma\mathrm{Level}(H^*, \mathbb{G}_{E_n}).$$

The isomorphism follows immediately from the following proposition, whose proof occupies the remainder of the appendix.

Proposition A.1. *There is a positive integer k, depending on H but not depending on n, such that $p^k T_n = 0 \subset R_n$.*

In other words, we claim that the exponent of the p-power torsion in $E_n^*(BH)/I_n$ may be bounded independently of the height n.

We can make two immediate reductions. First, note that H may be written as a direct sum of a p-group and a group of order prime to p, the prime-to-p part having no contribution to E-theory. Furthermore, since $E_n^*(BH)$ is Noetherian, the exponent of the p-power torsion in R_n is bounded for any fixed n. In light of these facts, for the remainder of the appendix we assume that H is a p-group and study $E_n^*(BH)$ for n larger than the p-rank of H.

Since $E_n^*(BH) \cong \Gamma\mathrm{Hom}(H^*, \mathbb{G}_{E_n})$, there is a natural inclusion of H^* into $E_n^*(BH)$. This is the map that takes a character $\chi : H \to S^1$ to the first Chern class of the induced map $BH \to BS^1$. Composing with the projection $E_n^*(BH) \to R_n$, we obtain a map $\phi : H^* \to R_n$.

Lemma A.2. *For each non-zero $\chi \in H^*$, $\phi(\chi)$ divides p in R_n.*

Proof. Suppose that $\chi : H \to S^1$ has kernel $A \subset H$. The quotient H/A is isomorphic to a non-trivial subgroup of S^1. Assume that H/A is cyclic of order p^k. It follows that the kernel of $p^{k-1}\chi$ is an index-p subgroup A' of H containing A. The composite of the transfer and projection

$$E_n^*(BA') \to E_n^*(BH) \to R_n$$

then sends 1 to

$$\langle p^k \rangle(\phi(\chi)) = \frac{[p^k](\phi(\chi))}{[p^{k-1}](\phi(\chi))},$$

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by the corresponding formula for the transfer $e \subset \mathbb{Z}/p$ and the naturality of transfer. In particular, $\langle p^k \rangle(\phi(\chi)) = 0$ in R_n. But $\langle p^k \rangle(\phi(\chi))$ is a power series in $\phi(\chi)$ with constant term p, so it follows that $\phi(\chi)$ divides p. $\qquad \square$

Suppose now that
$$H \cong \mathbb{Z}/p^{m_1}\mathbb{Z} \times \mathbb{Z}/p^{m_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{m_\ell}\mathbb{Z},$$
where $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_\ell$. We proceed to prove Proposition A.1 by induction on ℓ.

If $\ell = 1$, then $H \cong \mathbb{Z}/p^{m_1}\mathbb{Z}$. This case is covered in §4.1 of this paper, but see also [AHS04, Example 9.22].

For $\ell > 1$, consider
$$H' \cong \mathbb{Z}/p^{m_1}\mathbb{Z} \times \mathbb{Z}/p^{m_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{m_{\ell-1}}\mathbb{Z}$$
equipped with the natural inclusion $H' \subset H$. Let R'_n denote $E_n^*(BH')$ modulo its ideal of transfers, and let T'_n denote the ideal of p-power torsion in R'_n. Finally, we use the notation $\phi' : (H')^* \to R'_n$ to denote the obvious analogue of ϕ.

The power series ring $(R'_n)[\![z]\!]$ is an E_n^*-algebra, so one may consider the element $[p^{m_\ell}](z)$, which we denote by $q(z)$. We use $\bar{q}(z)$ to denote projection of $q(z)$ to $(R'_n/T'_n)[\![z]\!]$. For each $\chi \in (H')^*$ of order at most p^{m_ℓ}, one has that $[p^{m_\ell}](\phi'(\chi)) = 0$. Since R'_n/T'_n is a regular local ring, and since n is larger than the rank of H, it follows that $\bar{q}(z)$ is divisible by
$$\prod_\chi (z - \phi'(\chi)),$$
where the product runs across all $\chi \in (H')^*$ of order at most p^{m_ℓ}. The quotient of $\bar{q}(z)$ by this product may, by the Weierstrass preparation theorem, be written as a unit times a monic polynomial $g(z)$.

The following key lemma was proved by Drinfeld [Dri74, proof of Proposition 4.3] and was realized in topology by Strickland (see [Str, §26] and [Str97, §7]).

LEMMA A.3 (Drinfeld). *There is an isomorphism $R_n/T_n \cong (R'_n/T'_n)[z]/g(z)$.*

The coefficients of $g(z)$ lie in R'_n/T'_n and may be lifted arbitrarily to elements of R'_n. This defines a non-canonical lift $\tilde{g}(z) \in R'_n[z]$ of $g(z)$. Lemma A.3 then implies that R_n/T_n is isomorphic to the quotient of $R'_n[z]$ by the ideal $(\tilde{g}(z), T'_n)$ generated by $\tilde{g}(z)$ and the elements of T'_n. Note that R_n is already a quotient of $R'_n[z]$ by the Künneth isomorphism applied to $E_n^*(BH) \cong E_n^*(BH' \times B\mathbb{Z}/p^{m_\ell}\mathbb{Z})$ and elementary properties of transfers.

Example A.4. Suppose that H is the group $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$. Then, for sufficiently large n, Lemma A.3 implies that
$$R_n/T_n \cong E_n^*[\![x, y, z]\!]/(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3),$$
where:

- $\tilde{g}_1 = [2](x)/x$ is an element defined in $E_n^*[\![x]\!]$;
- $\tilde{g}_2 = [2](y)/(y(y - x))$ is a lift of an element uniquely defined in $E_n^*[\![x, y]\!]/\tilde{g}_1$; and
- $\tilde{g}_3 = [2](z)/(z(z - x)(z - y)(z - (x +_{\mathbb{G}} y)))$ is a lift of an element defined in $E_n^*[\![x, y, z]\!]/(\tilde{g}_1, \tilde{g}_2)$.

It follows that if $w \in T_n$, then $w = a_1 \tilde{g}_1 + a_2 \tilde{g}_2 + a_3 \tilde{g}_3$ for some $a_1, a_2, a_3 \in R_n$. The arguments below will show that, regardless of the height n, $16\tilde{g}_1 = 16\tilde{g}_2 = 16\tilde{g}_3 = 0$, and so $16w = 0$.

Remark A.5. The reader familiar with [AHS04] should be wary of a small typo in the statement of Proposition 9.15. The correct statement appears at the end of the proof of Proposition 9.15 on p. 29 of that paper. The difference between the stated and proved versions of [AHS04, Proposition 9.15] is elucidated by the existence of elements such as $\tilde{g}_2$ in the above example, which annihilates $y -_{\mathbb{G}_{E_n}} x$ in $E^*(BH)/(\mathrm{ann}(x), \mathrm{ann(y)})$ but is not defined in $E_n^*(BH)$ itself.

By the inductive assumption, there is an integer k' such that $p^{k'} T_n' = 0 \subset R_n'$. As a corollary of the comments preceding Example A.4, we see that the complete proof of Proposition A.1 rests only on the following lemma.

LEMMA A.6. *Let h denote the image of $\tilde{g}(z)$ under the projection $\pi : R_n'[z] \to R_n$. Then there is a positive integer r, independent of n, such that $p^r h = 0$.*

Proof. Let $\chi : H' \to S^1$ denote any element of $(H')^*$. We first claim that $\pi(z - \phi'(\chi))$ divides p in R_n.

- Let $\chi_1 : H \to S^1$ denote the composite of the projection $H \to \mathbb{Z}/p^{m_\ell}$ with the inclusion $\mathbb{Z}/p^{m_\ell} \to S^1$. Note that $\phi(\chi_1) = \pi(z)$.
- Let $\chi_2 : H \to S^1$ denote the composite of the projection $H \to H'$ with χ. Note that $\phi(\chi_2) = \pi(\phi'(\chi))$.

Now $(z - \phi'(\chi))$ is a unit times $z -_{\mathbb{G}_{E_n}} \phi'(\chi)$. Furthermore,

$$\pi(z -_{\mathbb{G}_{E_n}} \phi'(\chi)) = \phi(\chi_1) -_{\mathbb{G}_{E_n}} \phi(\chi_2) = \phi(\chi_1 - \chi_2),$$

and by Lemma A.2 the claim follows.

Turning to the proof of the lemma, recall that by definition $\tilde{g}(z) \prod_\chi (z - \phi'(\chi))$ has coefficients in T_n', where the product ranges over all $\chi : H' \to S^1$ of order at most p^{m_ℓ}. It follows that

$$\pi\left(p^{k'} \tilde{g}(z) \prod_\chi (z - \phi'(\chi)) \right) = 0$$

in R_n. Then, by the claim,

$$\pi\left(p^{k'} \tilde{g}(z) \prod_\chi p \right) = 0,$$

and so we may take r to be the sum of k' and the number of χ appearing in the product. $\square$

REFERENCES

AHS04   M. Ando, M. J. Hopkins and N. P. Strickland, *The sigma orientation is an H_∞ map*, Amer. J. Math. **126** (2004), 247–334; MR 2045503 (2005d:55009).

BF15   T. Barthel and M. Frankland, *Completed power operations for Morava E-theory*, Algebr. Geom. Topol. **15** (2015), 2065–2131; MR 3402336.

BS16   T. Barthel and N. Stapleton, *Centralizers in good groups are good*, Algebr. Geom. Topol. **16** (2016), 1453–1472.

Bou79   A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257–281; MR 551009 (80m:55006).

Bou87   A. K. Bousfield, *Uniqueness of infinite deloopings for K-theoretic spaces*, Pacific J. Math. **129** (1987), 1–31; MR 901254 (89g:55017).

DHS88   E. S. Devinatz, M. J. Hopkins and J. H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), 207–241; MR 960945 (89m:55009).

Dri74   V. G. Drinfeld, *Elliptic modules*, Math. USSR Sb. **23** (1974), 561–592.

DG02   W. G. Dwyer and J. P. C. Greenlees, *Complete modules and torsion modules*, Amer. J. Math. **124** (2002), 199–220; MR 1879003.

EKMM97   A. D. Elmendorf, I. Kriz, M. A. Mandell and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47 (American Mathematical Society, Providence, RI, 1997), with an appendix by M. Cole; MR 1417719 (97h:55006).

GS96   J. P. C. Greenlees and H. Sadofsky, *The Tate spectrum of v_n-periodic complex oriented theories*, Math. Z. **222** (1996), 391–405; MR 1400199 (97d:55010).

Hop   M. Hopkins, *Complex oriented cohomology theories and the language of stacks*, Course Notes, http://www.math.rochester.edu/people/faculty/doug/otherpapers/coctalos.pdf.

HKR00   M. J. Hopkins, N. J. Kuhn and D. C. Ravenel, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc. **13** (2000), 553–594.

HS98   M. J. Hopkins and J. H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), 1–49; MR 1652975 (99h:55009).

Hov95   M. Hovey, *Bousfield localization functors and Hopkins' chromatic splitting conjecture*, in *The Čech centennial (Boston, MA, 1993)*, Contemporary Mathematics, vol. 181 (American Mathematical Society, Providence, RI, 1995), 225–250; MR 1320994 (96m:55010).

Hov97   M. A. Hovey, *v_n-elements in ring spectra and applications to bordism theory*, Duke Math. J. **88** (1997), 327–356; MR 1455523 (98d:55017).

Hov04a   M. Hovey, *Operations and co-operations in Morava E-theory*, Homology, Homotopy Appl. **6** (2004), 201–236; MR 2076002 (2005f:55003).

Hov04b   M. Hovey, *Some spectral sequences in Morava E-theory*, Preprint (2004).

HS99a   M. Hovey and H. Sadofsky, *Invertible spectra in the $E(n)$-local stable homotopy category*, J. Lond. Math. Soc. (2) **60** (1999), 284–302; MR 1722151 (2000h:55017).

HS99b   M. Hovey and N. P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100pp; MR 1601906 (99b:55017).

JW73   D. C. Johnson and W. S. Wilson, *Projective dimension and Brown–Peterson homology*, Topology **12** (1973), 327–353; MR 0334257 (48 #12576).

JW75   D. C. Johnson and W. S. Wilson, *BP operations and Morava's extraordinary K-theories*, Math. Z. **144** (1975), 55–75; MR 0377856 (51 #14025).

Kas98   T. Kashiwabara, *Brown–Peterson cohomology of $\Omega^\infty \Sigma^\infty S^{2n}$*, Quart. J. Math. **49** (1998), 345–362; MR 1645564 (2000d:55013).

Kas01   T. Kashiwabara, *On Brown–Peterson cohomology of QX*, Ann. of Math. (2) **153** (2001), 297–328; MR 1829752 (2002f:55013).

KY93   A. Kono and N. Yagita, *Brown–Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups*, Trans. Amer. Math. Soc. **339** (1993), 781–798; MR 1139493 (93m:55006).

Kuh89   N. J. Kuhn, *Morava K-theories and infinite loop spaces*, in *Algebraic topology (Arcata, CA, 1986)*, Lecture Notes in Mathematics, vol. 1370 (Springer, Berlin, 1989), 243–257; MR 1000381 (90d:55014).

Kuh06   N. J. Kuhn, *Localization of André–Quillen–Goodwillie towers, and the periodic homology of infinite loopspaces*, Adv. Math. **201** (2006), 318–378; MR 2211532 (2007d:55006).

Lam99   T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189 (Springer, New York, 1999); MR 1653294 (99i:16001).

Lan73   P. S. Landweber, *Annihilator ideals and primitive elements in complex bordism*, Illinois J. Math. **17** (1973), 273–284; MR 0322874 (48 #1235).

Lan76   P. S. Landweber, *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*, Amer. J. Math. **98** (1976), 591–610; MR 0423332 (54 #11311).

Mar83   H. R. Margolis, *Modules over the Steenrod algebra and the stable homotopy category*, in *Spectra and the Steenrod algebra*, North-Holland Mathematical Library, vol. 29 (North-Holland, Amsterdam, 1983); MR 738973 (86j:55001).

Min02   N. Minami, *From $K(n+1)^*(X)$ to $K(n)^*(X)$*, Proc. Amer. Math. Soc. **130** (2002), 1557–1562; MR 1879983 (2003a:55009).

Qui71   D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Adv. Math. **7** (1971), 29–56; MR 0290382.

RW80   D. C. Ravenel and W. S. Wilson, *The Morava K-theories of Eilenberg–MacLane spaces and the Conner–Floyd conjecture*, Amer. J. Math. **102** (1980), 691–748; MR 584466 (81i:55005).

RWY98   D. C. Ravenel, W. S. Wilson and N. Yagita, *Brown–Peterson cohomology from Morava K-theory*, K-Theory **15** (1998), 147–199; MR 1648284 (2000d:55012).

Rez09   C. Rezk, *The congruence criterion for power operations in Morava E-theory*, Homology, Homotopy Appl. **11** (2009), 327–379; MR 2591924 (2011e:55021).

Rez12   C. Rezk, *Rings of power operations for Morava E-theories are Koszul*, Preprint (2012), arXiv:1204.4831.

SS15   T. Schlank and N. Stapleton, *A transchromatic proof of Strickland's theorem*, Adv. Math. **285** (2015), 1415–1447; MR 3406531.

SY01   B. Schuster and N. Yagita, *Transfers of Chern classes in BP-cohomology and Chow rings*, Trans. Amer. Math. Soc. **353** (2001), 1039–1054; MR 1804412 (2002b:55030).

Sta13   N. Stapleton, *Transchromatic generalized character maps*, Algebr. Geom. Topol. **13** (2013), 171–203; MR 3031640.

Sta15   N. Stapleton, *Subgroups of p-divisible groups and centralizers in symmetric groups*, Trans. Amer. Math. Soc. **367** (2015), 3733–3757; MR 3314822.

Str   N. Strickland, *Functorial philosophy for formal phenomena*, http://hopf.math.purdue.edu/Strickland/fpfp.pdf.

Str97   N. P. Strickland, *Finite subgroups of formal groups*, J. Pure Appl. Algebra **121** (1997), 161–208; MR 1473889 (98k:14065).

Str99   N. P. Strickland, *Formal schemes and formal groups*, in *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, Contemporary Mathematics, vol. 239 (American Mathematical Society, Providence, RI, 1999), 263–352; MR 1718087 (2000j:55011).

Wil99   W. S. Wilson, *Brown–Peterson cohomology from Morava K-theory. II*, K-Theory **17** (1999), 95–101; MR 1696426 (2000d:55014).

Yag76   N. Yagita, *The exact functor theorem for BP_*/I_n-theory*, Proc. Japan Acad. Ser. A Math. Sci. **52** (1976), 1–3; MR 0394631 (52 #15432).

Yos76   Z.-i. Yosimura, *Projective dimension of Brown–Peterson homology with modulo $(p, v_1, \ldots, v_{n-1})$ coefficients*, Osaka J. Math. **13** (1976), 289–309; MR 0415603 (54 #3686).

Brown–Peterson cohomology from Morava E-theory

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