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SPLITTING NUMBERS OF LINKS

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Abstract The splitting number of a link is the minimal number of crossing changes between different components required, on any diagram, to convert it to a split link. We introduce new techniques to compute the splitting number, involving covering links and Alexander invariants. As an application, we completely determine the splitting numbers of links with nine or fewer crossings. Also, with these techniques, we either reprove or improve upon the lower bounds for splitting numbers of links computed by Batson and Seed using Khovanov homology.

Keywords: splitting numbers of links; covering links; Alexander polynomial

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1. Introduction

Any link in S^3 can be converted to the split union of its component knots by a sequence of crossing changes between different components. Following Batson and Seed [**2**], we define the *splitting number* of a link L, denoted by $\mathrm{sp}(L)$, as the minimal number of crossing changes in such a sequence.

We present two new techniques for obtaining lower bounds for the splitting number. The first approach uses covering links, and the second method arises from the multivariable Alexander polynomial of a link.

Our general covering link theorem is stated as Theorem 3.2. Theorem 1.1 gives a special case that applies to two-component links L with unknotted components and odd linking number. Note that the splitting number is equal to the linking number modulo 2. If we take the two-fold branched cover of S^3 with branching set a component of L, then the preimage of the other component is a knot in S^3, which we call a *two-fold covering knot*

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of L. Also recall that the *slice genus* of a knot K in S^3 is defined to be the minimal genus of a surface F smoothly embedded in D^4 such that $\partial(D^4, F) = (S^3, K)$.

Theorem 1.1. *Suppose that L is a two-component link with unknotted components. If $\operatorname{sp}(L) = 2k + 1$, then any two-fold covering knot of L has slice genus at most k.*

Theorem 3.2 also has other useful consequences, given in Corollaries 3.5 and 3.6, dealing with the case of even linking numbers, for example. Three covering link arguments that use these corollaries are given in § 7.

Our Alexander polynomial method is efficacious for two-component links when the linking number is 1 and at least one component is knotted. By looking at the effect of a crossing change on the Alexander module, we obtain the following result.

Theorem 1.2. *Suppose that L is a two-component link with Alexander polynomial $\Delta_L(s, t)$. If $\operatorname{sp}(L) = 1$, then $\Delta_L(s, 1) \cdot \Delta_L(1, t)$ divides $\Delta_L(s, t)$.*

We will use elementary methods explained in Lemma 2.1 and our techniques from covering links and Alexander polynomials to obtain lower bounds on the splitting number for links with nine or fewer crossings. Together with enough patience with link diagrams, this is sufficient to determine the splitting number for all of these links. Our results for links with up to nine crossings are summarized by Table 3 in § 6.

In [**2**], Batson and Seed defined a spectral sequence from the Khovanov homology of a link L that converges to the Khovanov homology of the split link with the same components as L. They showed that this spectral sequence gives rise to a lower bound on $\operatorname{sp}(L)$, and by computing it for links with up to 12 crossings, they gave many examples for which this lower bound is strictly stronger than the lower bound coming from linking numbers. They determined the splitting number of some of these examples, while some were left undetermined.

We revisit the examples of Batson and Seed and show that our methods are strong enough to recover their lower bounds. Furthermore, we show that for several cases our methods give more information. In particular, we completely determine the splitting numbers of all the examples of Batson and Seed. We refer the reader to § 5 for more details.

Organization of the paper

We start out, in § 2, with some basic observations on the splitting number of a link. In § 3.1 we prove Theorem 3.2, which is a general result on the effect of crossing changes on covering links, and then we provide an example in § 3.2. We give a proof of Theorem 1.2 in §§ 4.1 and 4.2 and we illustrate its use with an example in § 4.3. The examples of Batson and Seed are discussed in § 5, with § 5.1 focusing on examples that use Theorem 1.2, and § 5.2 on examples that require Theorem 1.1. A three-component example of Batson and Seed is discussed in § 5.3. Next, our results on the splitting numbers of links with nine crossings or fewer are given in § 6, with some particular arguments used to obtain these results described in § 7.

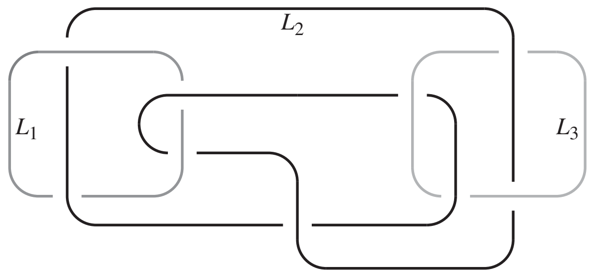

Figure 1. The link $L9a54$.

2. Basic observations

A link is *split* if it is a split union of knots. We recall from the introduction that the *splitting number* $\mathrm{sp}(L)$ of a link L is defined to be the minimal number of crossing changes that one needs to make on L, each crossing change between different components, in order to obtain a split link.

We note that this differs from the definition of 'splitting number' that occurs in [**1**,**17**]; in these papers, crossing changes of a component with itself are permitted.

Given a link L we say that a non-split sublink with all of the linking numbers zero is *obstructive*. (All obstructive sublinks that occur in the applications of this paper will be Whitehead links.) We then define $c(L)$ to be the maximal size of a collection of distinct obstructive sublinks of L such that any two sublinks in the collection have at most one component in common. Note that c is zero for trivial links.

As another example consider the link $L9a54$ shown in Figure 1. The sublink $L_1 \sqcup L_3$ is an unlink, while both $L_1 \sqcup L_2$ and $L_2 \sqcup L_3$ are Whitehead links, and hence are obstructive. Thus, $c(L) = 2$.

Finally, we discuss the link J in Figure 17. It has four components J_1, J_2, J_3, J_4, and $J_1 \cup J_3$ and $J_2 \cup J_4$ each form a Whitehead link. It follows that $c(J) = 2$.

In practice it is straightforward to obtain lower bounds for $c(L)$. In most cases it is also not too hard to determine $c(L)$ precisely.

Now we have the following elementary lemma.

Lemma 2.1. *Let $L = L_1 \sqcup \cdots \sqcup L_m$ be a link. Then*

$$\mathrm{sp}(L) \equiv \sum_{i>j} \mathrm{lk}(L_i, L_j) \mod 2$$

and

$$\mathrm{sp}(L) \geqslant \sum_{i>j} |\mathrm{lk}(L_i, L_j)| + 2c(L).$$

Proof. Given a link L we write

$$a(L) = \sum_{i>j} |\mathrm{lk}(L_i, L_j)|.$$

Note that a crossing change between two different components always changes the value of a by precisely 1. Since a of the unlink is zero we immediately obtain the first statement.

If we do a crossing change between two components with non-zero linking number, then a goes down by at most 1, whereas c stays the same or increases by 1. On the other hand, if we do a crossing change between two components with zero linking number, then a goes up by 1 and c decreases by at most 1, since the two components belong to at most one obstructive sublink in any maximal collection whose cardinality realizes $c(L)$. It now follows that $a(L) + 2c(L)$ decreases with each crossing change between different components by at most 1. $\square$

The right-hand side of the second inequality is greater than or equal to the lower bound $b_{\mathrm{lk}}(L)$ of [**2**, § 5]. In some cases the lower bound coming from Lemma 2.1 is stronger. For example, let L be two split copies of the Borromean rings. For this L we have $c(L) = 2$, giving a sharp lower bound on the splitting number of 4, whereas $b_{\mathrm{lk}}(L) = 2$.

3. Covering link calculus

In this section, first we prove our main covering link result, Theorem 3.2, showing that covering links can be used to give lower bounds on the splitting number. Then we show how to extract Theorem 1.1 and three other useful corollaries from Theorem 3.2. In § 3.2 we present an example of this approach.

3.1. Crossing changes and covering links

The following definition is a special case of the notion of a covering link occurring, for example, in [**14**, Method 5] and [**5**].

Definition 3.1. Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link with L_i unknotted. We denote the double branched cover of S^3 with branching set the unknot L_i by $p\colon S^3 \to S^3$. We refer to $p^{-1}(L \setminus L_i)$ as the *two-fold covering link of L with respect to L_i*.

We note that a choice of orientation of a link induces an orientation of its covering links.

In the theorem below we use the term *internal band sum* to refer to the operation on an oriented link L described as follows. The data for the move is an embedding $f\colon D^1 \times D^1 \subset S^3$ such that $f(D^1 \times D^1) \cap L = f(\{-1, 1\} \times D^1)$, the orientation of $f(\{-1\} \times D^1)$ agrees with that of L, and the orientation of $f(\{1\} \times D^1)$ is opposite to that of L. The output is a new oriented link given by $(L \setminus f(\{-1, 1\} \times D^1)) \cup f(D^1 \times \{-1, 1\})$, after rounding corners. The new link has the orientation induced from L.

Theorem 3.2. *Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link and suppose that L_i is unknotted for some fixed i. Fix an orientation of L. Suppose that L can be transformed to a split link by $\alpha + \beta$ crossing changes involving distinct components, where α of these involve L_i and β of these do not involve L_i. Then the two-fold covering link J of L with respect to L_i can be altered by performing α internal band sums and 2β crossing changes between different components to the split union of $2(m-1)$ knots comprising two copies of L_j for each $j \neq i$.*

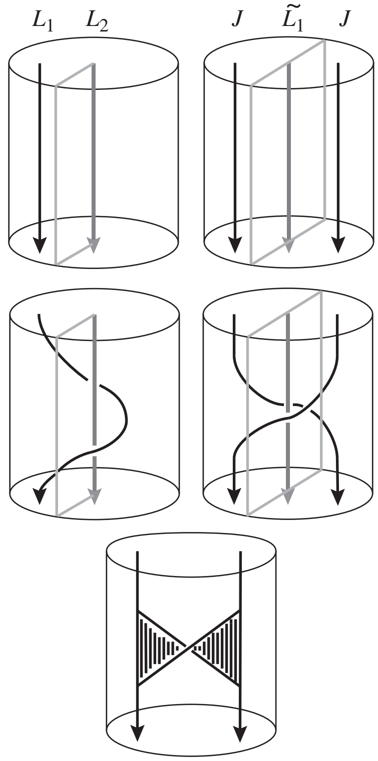

Figure 2. The effect of a crossing change on a two-fold covering link where one component is the branching set.

Proof. We may assume that $i = 1$. We begin by investigating the effect of crossing changes on the two-fold covering link with respect to the first component L_1 of a link L.

Type A. First we consider crossing changes between the branching component L_1 and another component, say L_2. Such a crossing change lifts to a rotation of the preimage J of L_2 around the lift $\widetilde{L_1}$ of L_1, as shown in Figure 2. The top-left and middle-left diagrams show a link before and after a crossing change, in a cylindrical neighbourhood that contains an interval from each of L_1 and L_2. To branch over L_1, which is the component running down the centre of the cylinders, cut along the surface that is shown in the diagrams. The results of taking the top-left and middle-left diagrams, cutting, and gluing two copies together, are shown in the top-right and middle-right diagrams, respectively.

After forgetting the branching set, the same effect on the lift of L_2 can be achieved by adding a band to J (see the bottom diagram of Figure 2). By ignoring the band, we obtain the top-right diagram with the branching component removed. If we instead use the band to make an internal band sum, we obtain the middle-right diagram with the branching set removed. Note that this band is attached to J in such a way that orientations are preserved. This holds no matter what choice of orientations was made for L. Thus, we see that a crossing change between L_1 and L_2 corresponds to an internal band sum on the covering link.

Type B. Consider a crossing change that does not involve L_1, say between L_2 and L_3. Such a crossing change can be realized by ± 1 Dehn surgery on a circle that has zero linking number with L and that bounds an embedded disc, say D, in S^3 that intersects L in two points of opposite signs, one point of L_2 and one point of L_3. By performing the Dehn surgeries, and then taking the branched cover over L_1, we produce the covering link of the link obtained by the crossing change.

Note that the preimage of the disc D in the double branched cover consists of two disjoint discs, each of which intersects the covering link transversally at two points with opposite signs, one point of the preimage of L_2 and one point of the preimage of L_3. As an alternative construction, we can take the branched cover and then perform ± 1 Dehn surgeries along the boundary circles of the preimage discs. This gives the same covering link. From this it follows that a single crossing change between L_2 and L_3 corresponds to two crossing changes on the covering link.

Note that when there is more than one crossing change, of either type, the corresponding surgery discs and bands associated with the covering link are disjoint.

Recall that the link L can be altered to become the split union of m knots $L_1, \dots, L_m$ by α crossing changes of Type A and β crossing changes of Type B. By the above arguments, the two-fold covering link of L with respect to the first component L_1 can be altered to become the corresponding covering link of the split link, which is the split union $L_2 \sqcup L_2 \sqcup \cdots \sqcup L_m \sqcup L_m$, by α internal band sums and 2β crossing changes. $\square$

In the following result, $g_4(K)$ denotes the slice genus of a knot K in S^3, namely, the minimal genus of a smoothly embedded connected oriented surface in D^4 whose boundary is K.

Corollary 3.3. *Under the same hypotheses as Theorem 3.2, the two-fold covering link of L with respect to L_i bounds a smoothly embedded oriented surface F in D^4 that has no closed components and has Euler characteristic*

$$\chi(F) = 2(m-1) - \alpha - 4\beta - 4\sum_{k\neq i} g_4(L_k).$$

In addition, if there is some $j \neq i$ such that each L_k with $k \neq j$ is involved in some crossing change with L_j, then F is connected.

Proof. Once again we may assume that $i = 1$. Let J be the two-fold covering link of L with respect to L_1.

An internal band sum can be inverted by performing another band sum, while the inverse of a crossing change is also a crossing change. Hence, by Theorem 3.2 we can also obtain the covering link J from the split union $L_2 \sqcup L_2 \sqcup \cdots \sqcup L_m \sqcup L_m$ by performing α internal band sums and 2β crossing changes.

Choose surfaces V_j embedded in D^4 with $\partial V_j = L_j$ and genus $g_4(L_j)$. Take a split union $V_2 \sqcup V_2 \sqcup \cdots \sqcup V_m \sqcup V_m$ in D^4. The boundary of these surfaces is the split union

$L_2 \sqcup L_2 \sqcup \cdots \sqcup L_m \sqcup L_m$. The covering link J can be realized as the boundary of a surface obtained from the split union of the surfaces by attaching α bands and 2β clasps in S^3. As pointed out in the proof of Theorem 3.2, the surgery discs and bands associated with crossing changes are disjoint. Pushing slightly into D^4, we obtain an immersed surface in D^4 bounded by J; each clasp gives a transverse intersection. As usual, we remove the intersections by cutting out a disc neighbourhood of the intersection point from each sheet and gluing a twisted annulus that is a Seifert surface for the Hopf link. This gives a smoothly embedded oriented surface F in D^4 bounded by the covering link J. Note that each band attached changes the Euler characteristic of the surface by -1, while each twisted annulus used to remove an intersection point changes the Euler characteristic by -2. Therefore, the resulting surface F has Euler characteristic

$$\chi(F) = \sum_{k=2}^{m} 2 \cdot (1 - 2g_4(V_k)) - \alpha - 4\beta,$$

which is equal to the claimed value.

The final conclusion of the corollary states (when $i = 1$) that F is connected if there is some $j \neq 1$ such that each L_k with $k \neq j$ is involved in some crossing change with L_j. To see this, observe that a crossing change involving L_j and L_1 joins the two copies of V_j; a crossing change involving L_j and L_k with $j, k \geqslant 2$ joins one of the two copies of V_j to one of the two copies of V_k and joins the other copy of V_j to the other copy of V_k. Under the hypothesis, it follows that F is connected. □

Corollary 3.3 has some useful consequences of its own. Considering the case in which $m = 2$, $\alpha = 2k + 1$, $\beta = 0$ and $g_4(L_k) = 0$, we obtain Theorem 1.1.

Theorem 1.1. *Suppose that L is a two-component link with unknotted components. If $\mathrm{sp}(L) = 2k + 1$, then any two-fold covering knot of L has slice genus at most k.*

Remark 3.4.

(1) In the proof of Corollary 3.3, when $\beta = 0$, we construct an embedded surface F without local maxima. Therefore, in order to show, using Theorem 1.1, that a link of linking number 1 with unknotted components has splitting number at least 3, it suffices to show that the covering link is not a ribbon knot.

(2) Different choices of orientation on a link J can change the minimal genus of a connected surface that J bounds in D^4. Since the splitting number is independent of orientations, in applications we will choose the orientation that gives the strongest lower bound. This remark will be relevant in § 5.3.

(3) If L is a non-split two-component link, then the surface F of Corollary 3.3 is automatically connected, by the last sentence of that corollary.

The following is another useful consequence of Corollary 3.3.

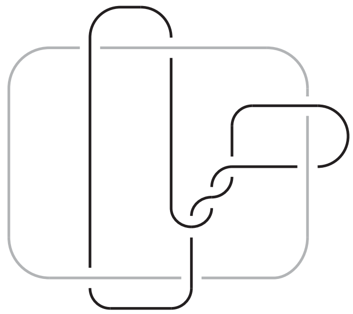

Figure 3. The link $L9a30$.

Corollary 3.5. *Suppose that L is a two-component link with unknotted components and $\operatorname{sp}(L) = 2$. Then any two-fold covering link of L is weakly slice; that is, bounds an annulus smoothly embedded in D^4.*

Proof. First note that a two-fold covering link has two components, since the linking number is even by Lemma 2.1. Applying Corollary 3.3 with $m = 2$, $\alpha = 2$, $\beta = 0$ and $g_4(L_k) = 0$, the conclusion follows. □

We state one more corollary to Theorem 3.2. Let $\operatorname{sp}_i(L)$ be the minimal number of crossing changes between distinct components not involving L_i required to transform L to a split link. By convention, $\operatorname{sp}_i(L)$ is infinite if we must make a crossing change involving L_i in order to split L.

Corollary 3.6 (see Kohn [14, Method 5]). *For a link $L = L_1 \sqcup \cdots \sqcup L_m$ and its two-fold covering link J with respect to L_i, we have $\operatorname{sp}_i(L) \geqslant \frac{1}{2}\operatorname{sp}(J)$.*

Proof. This follows from Theorem 3.2 with $\alpha = 0$. □

We remark that the above results generalize to n-fold covering links in a reasonably straightforward manner. One can also draw analogous conclusions when the branching component is knotted. We do not address these generalizations here, since the results stated above are sufficient for the applications considered in this paper.

3.2. An example of the covering link technique

To illustrate the use of the method developed in § 3.1, we now apply it to prove that the splitting number of the two-component link $L9a30$ is 3. More applications of Theorem 1.1 and Corollaries 3.5 and 3.6 will be discussed later (see, for example, §§ 5.2, 5.3, 6 and 7). In this paper, we use the link names employed in the LinkInfo database [**7**]. The link $L9a30$ is shown in Figure 3. It is a two-component link of linking number 1 with unknotted components. Recall that the splitting number is determined modulo 2 by the linking number by Lemma 2.1. It is easy to see from Figure 3 that three crossing changes suffice, so the splitting number is either 1 or 3.

To see that $\operatorname{sp}(L9a30) \neq 1$, we take a two-fold cover branched over one of the components, and check that the resulting knot is not slice. Figure 4 shows the result of an

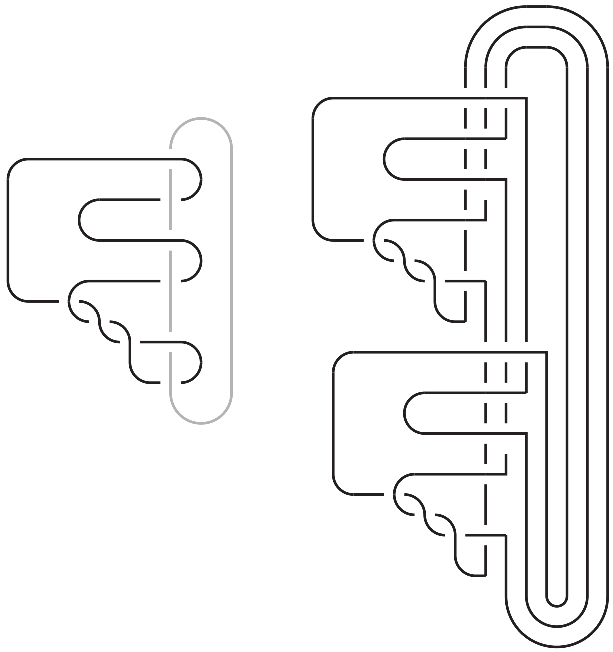

Figure 4. Left: the link $L9a30$ after an isotopy to prepare for taking the cover branching over the most obviously unknotted component. Right: the knot that arises as the covering link after taking a two-fold branched cover and deleting the branching set.

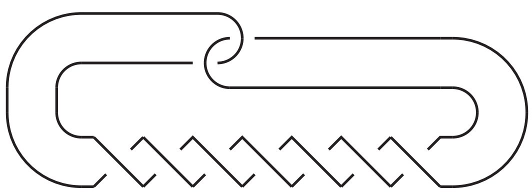

Figure 5. The covering knot on the right of Figure 4 after an isotopy.

isotopy that was made in preparation for taking a branched cover on the left, and the knot obtained as the preimage of $L9a30$ after deleting the preimage of the branching component on the right.

The knot on the right of Figure 4 after a simplifying isotopy is shown in Figure 5; it is a twist knot with a negative clasp and seven positive half-twists. This knot is well known not to be a slice knot, a fact that was first proved by Casson and Gordon [**3**, **4**]. Therefore, by Theorem 1.1, the splitting number of $L9a30$ is at least 3, as claimed.

4. Alexander invariants

In this section we will recall the definition of Alexander modules and polynomials of oriented links. We then show how Alexander modules are affected by a crossing change, which then allows us to prove Theorem 1.2.

4.1. Crossing changes and the Alexander module

Throughout this section, given an oriented m-component link L, the oriented meridians are denoted by $\mu_1, \ldots, \mu_m$. Note that $\mu_1, \ldots, \mu_m$ give rise to a basis for $H_1(S^3 \setminus \nu L; \mathbb{Z})$. We will henceforth use this basis to identify $H_1(S^3 \setminus \nu L; \mathbb{Z})$ with $\mathbb{Z}^m$. Let R be a subring of $\mathbb{C}$ and let $\psi\colon \mathbb{Z}^m \to F$ be a homomorphism to a free abelian group. We denote the induced map

$$\pi_1(S^3 \setminus \nu L) \to H_1(S^3 \setminus \nu L; \mathbb{Z}) = \mathbb{Z}^m \xrightarrow{\psi} F$$

by ψ as well. We can then consider the corresponding Alexander module

$$H_1^{\psi}(S^3 \setminus \nu L; R[F])$$

and the order of the Alexander module is denoted by

$$\Delta_L^{\psi} \in R[F] = \operatorname{ord}_{R[F]}(H_1^{\psi}(S^3 \setminus \nu L; R[F])).$$

(We refer the reader to [**9**] for the definition of the order of an $R[F]$-module.) If ψ is the identity, then we drop ψ from the notation and we obtain the usual multivariable Alexander polynomial Δ_L.

Note that what we term the Alexander module has also been called the 'link module' in the literature (see, for example, [**11**]). The following proposition relates the Alexander modules of two oriented links that differ by a crossing change.

Proposition 4.1. *Let L and L' be two oriented m-component links that differ by a single crossing change. Let R be a subring of $\mathbb{C}$ and let $\psi\colon \mathbb{Z}^m \to F$ be a homomorphism to a free abelian group. Then there exists a diagram*

$$\begin{tikzcd}
 & R[F] \arrow[dr] & & R[F] \arrow[dl] & \\
 & & M \arrow[dl] \arrow[dr] & & \\
 & H_1^{\psi}(S^3 \setminus \nu L; R[F]) \arrow[dl] & & H_1(S^3 \setminus \nu L'; R[F]) \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}$$

where M is some $R[F]$-module and where the diagonal sequences are exact.

The formulation of this proposition is somewhat more general than what is strictly needed in the proof of Theorem 1.2. We hope that this more general formulation will be applicable, in future work, to the computation of unlinking numbers; see the beginning of § 6 for the definition of the unlinking number of a link.

Proof. We write $X = S^3 \setminus \nu L$ and $X' = S^3 \setminus \nu L'$. We pick two open disjoint discs D_1 and D_2 in the interior of D^2 and we write

$$\begin{aligned}
B &= (D^2 \setminus (D_1 \cup D_2)) \times [0,1], \\
S &= (D^2 \setminus (D_1 \cup D_2)) \times \{0,1\} \cup_{\partial D^2 \times \{0,1\}} S^1 \times [0,1].
\end{aligned}$$

Put differently, S is the 'top and bottom boundary' of B together with the outer cylinder $S^1 \times [0,1]$.

Since L and L' are related by a single crossing change, there exists a subset Y of X and continuous injective maps $f\colon B \to X$ and $f'\colon B' \to X'$ with the following properties:

(1) $X = Y \cup f(B)$ and $Y \cap f(B) = f(S)$,

(2) $X' = Y \cup f'(B)$ and $Y \cap f'(B) = f'(S)$.

We can now state the following claim.

Claim 4.2. *There exists a short exact sequence*

$$R[F] \to H_1^{\psi}(Y; R[F]) \to H_1^{\psi}(X; R[F]) \to 0.$$

By a slight abuse of notation we now write $B = f(B)$ and $S = f(S)$. We then consider the Mayer–Vietoris sequence

$$\begin{aligned} \cdots \to H_1^{\psi}(S; R[F]) \xrightarrow{i_* \oplus j_*} H_1^{\psi}(B; R[F]) \oplus H_1^{\psi}(Y; R[F]) \to H_1^{\psi}(X; R[F]) \\ \to H_0^{\psi}(S; R[F]) \xrightarrow{i_* \oplus j_*} H_0^{\psi}(B; R[F]) \oplus H_0^{\psi}(Y; R[F]), \end{aligned}$$

where $i\colon S \to B$ and $j\colon S \to Y$ are the two inclusion maps. We need to study the relationships between the homology groups of S and B. We make the following observations. By [**10**, § VI.3] we have the following commutative diagram:

$$\begin{tikzcd} H_0^{\psi}(S; R[F]) \arrow[r] \arrow[d, "\cong"'] & H_0^{\psi}(B; R[F]) \arrow[d, "\cong"'] \\ R[F]/\{\psi(g)v - v\}_{v \in R[F],\, g \in \pi_1(S)} \arrow[r] & R[F]/\{\psi(g)v - v\}_{v \in R[F],\, g \in \pi_1(B)} \end{tikzcd}$$

Here the horizontal maps are induced by the inclusion $S \to B$ and the vertical maps are isomorphisms. The map $i_*\colon \pi_1(S) \to \pi_1(B)$ is surjective; it follows that the bottom horizontal map is an isomorphism. Hence, the top horizontal map is also an isomorphism. The above Mayer–Vietoris sequence thus simplifies to the following sequence:

$$H_1^{\psi}(S; R[F]) \xrightarrow{i_* \oplus j_*} H_1^{\psi}(B; R[F]) \oplus H_1^{\psi}(Y; R[F]) \to H_1^{\psi}(X; R[F]) \to 0.$$

We note that the space B is homotopy equivalent to a wedge of two circles m and n. Furthermore, S is homotopy equivalent to the wedge of m, n and another curve m' that is homotopic to m in B. By another slight abuse of notation we now replace B and S by these wedges of circles and we view B and S as CW-complexes with precisely one 0-cell in the obvious way. We denote by $p\colon \tilde{S} \to S$ and $p\colon \tilde{B} \to B$ the coverings corresponding to the homomorphisms $\pi_1(S) \to \pi_1(B) \to \pi_1(X) \xrightarrow{\psi} F$. Note that we can and will view $\tilde{B}$ as a subset of $\tilde{S}$. We now pick preimages $\tilde{m}$, $\tilde{n}$ and $\tilde{m}'$ of m, n and m' under the covering map $p\colon \tilde{S} \to S$. Note that $\{\tilde{m}, \tilde{n}\}$ is a basis for $C_1(B; R[F]) = C_1(\tilde{B})$ and $\{\tilde{m}, \tilde{n}, \tilde{m}'\}$ is a basis for $C_1(S; R[F]) = C_1(\tilde{S})$. The kernel of the map $C_1(S; R[F]) \to C_1(B; R[F])$ is given by $R[F]\cdot(m-m')$. We thus obtain the following commutative diagram of chain complexes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R[F]\cdot(m-m') \arrow[r] \arrow[d] & C_1(S;R[F]) \arrow[r, "i_*"] \arrow[d] & C_1(B;R[F]) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & C_0(S;R[F]) \arrow[r, "i_*"] & C_0(B;R[F]) \arrow[r] & 0
\end{tikzcd}$$

It now follows easily from the diagram, or more formally from the snake lemma, that

$$\ker(i_*\colon H_1^\psi(S;R[F]) \to H_1^\psi(B;R[F])) \cong R[F]\cdot(m-m') \tag{4.1}$$

and that

$$\operatorname{coker}(i_*\colon H_1^\psi(S;R[F]) \to H_1^\psi(B;R[F])) = 0. \tag{4.2}$$

Finally, we consider the following commutative diagram:

$$\begin{tikzcd}
R[F]\cdot(m-m') \arrow[r] \arrow[d] & H_1^\psi(Y;R[F]) \arrow[r] \arrow[d, "\begin{pmatrix}0\\ \mathrm{Id}\end{pmatrix}"'] & H_1^\psi(X;R[F]) \arrow[r] \arrow[d, "="] & 0 \\
H_1^\psi(S;R[F]) \arrow[r] & H_1^\psi(B;R[F]) \oplus H_1^\psi(Y;R[F]) \arrow[r] & H_1^\psi(X;R[F]) \arrow[r] & 0
\end{tikzcd}$$

We have already seen above that the bottom horizontal sequence is exact. It now follows from (4.1), (4.2) and a straightforward diagram chase, that the top horizontal sequence is also exact. This concludes the proof of the claim.

Precisely the same proof shows that there exists a short exact sequence

$$R[F] \to H_1^\psi(Y;R[F]) \to H_1^\psi(X';R[F]) \to 0.$$

(Use $B = f'(B)$, $S = f'(S)$ instead of $B = f(B)$, $S = f(S)$.) Combining these two short exact sequences now gives the desired result, by taking $M := H_1^\psi(Y;R[F])$. $\square$

4.2. The Alexander polynomial obstruction

Using Proposition 4.1 we can prove the following obstruction to the splitting number being equal to 1.

Theorem 4.3. *Let L be a two-component oriented link. We denote the Alexander polynomial of L by $\Delta_L(s,t)$. If the splitting number of L equals 1, then $\Delta_L(s,1)\cdot\Delta_L(1,t)$ divides $\Delta_L(s,t)$.*

Let $L = J \cup K$ be an oriented link with splitting number equal to 1. We denote the Alexander polynomials of J and K by Δ_J and Δ_K, respectively. It follows from Lemma 2.1 that the linking number satisfies $|\mathrm{lk}(J,K)| = 1$. Therefore, by the Torres condition, $|\Delta_L(1,1)| = 1$ and we have that

$$\Delta_L(s,1) = \Delta_J(s) \quad \text{and} \quad \Delta_L(1,t) = \Delta_K(t).$$

We can thus reformulate the statement of the theorem as saying that if $L = J \cup K$ is an oriented link with splitting number equal to 1, then $\Delta_J(s)$ and $\Delta_K(t)$ both divide $\Delta_L(s,t)$.

Proof. Let $L = J \cup K$ be an oriented link with splitting number equal to 1. We denote by $\psi\colon H_1(S^3 \setminus L;\mathbb{Z}) \to \langle s,t | [s,t] = 1\rangle$ the map that is given by sending the meridian of J to s and by sending the meridian of K to t. We write $\Lambda := \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$.

In the following we also denote by ψ the map $H_1(S^3 \setminus J;\mathbb{Z}) \to \langle s,t | [s,t] = 1\rangle$, which is given by sending the meridian of J to s. Note that with this convention we have an isomorphism

$$H_1^\psi(S^3 \setminus J;\Lambda) = H_1(S^3 \setminus J;\mathbb{Z}[s^{\pm 1}]) \otimes_{\mathbb{Z}[s^{\pm 1}]} \Lambda$$

and we obtain that

$$\operatorname{ord}_\Lambda(H_1^\psi(S^3 \setminus J;\Lambda)) = \Delta_J(s). \tag{4.3}$$

Similarly, we define a map $H_1(S^3 \setminus K;\mathbb{Z}) \to \langle s,t | [s,t] = 1\rangle$ by sending the meridian of K to t. We see that

$$\operatorname{ord}_\Lambda(H_1^\psi(S^3 \setminus K;\Lambda)) = \Delta_K(t). \tag{4.4}$$

We denote the split link with components J and K by $J \sqcup K$. The Mayer–Vietoris sequence for $S^3 \setminus (J \sqcup K)$, which comes from splitting along the separating 2-sphere S, gives rise to an exact sequence

$$\begin{aligned} 0 \to H_1^\psi(S^3 \setminus J;\Lambda) \oplus H_1^\psi(S^3 \setminus K;\Lambda) &\to H_1^\psi(S^3 \setminus (J \sqcup K);\Lambda) \\ &\xrightarrow{h} H_0(S;\Lambda) \to H_0^\psi(S^3 \setminus J;\Lambda) \oplus H_0^\psi(S^3 \setminus K;\Lambda). \end{aligned}$$

We recall that by [**10**, § VI], for any connected space X with a ring homomorphism $\psi\colon \pi_1(X) \to \Lambda$, we have

$$H_0^\psi(X;\Lambda) = \Lambda / \{\psi(g)v - v \mid v \in \Lambda,\ g \in \pi_1(X)\}.$$

It follows easily that $H_0^\psi(S;\Lambda) \cong \Lambda$ and that $H_0^\psi(S^3 \setminus J;\Lambda)$ and $H_0^\psi(S^3 \setminus K;\Lambda)$ are Λ-torsion. In particular, we see that the last map in the above long exact sequence has a non-trivial kernel. By the exactness of the Mayer–Vietoris sequence above, it follows that the map h has non-trivial image.

Since L has splitting number 1, we can do one crossing change involving both J and K to turn L into $J \sqcup K$. The conclusion of Proposition 4.1 together with the above Mayer–Vietoris sequence gives rise to a diagram of maps as follows:

$$\begin{tikzcd} & \Lambda \arrow[r, "f"] & M \arrow[r, two heads, "p"] \arrow[d, two heads, "g"] & H_1^\psi(S^3 \setminus L;\Lambda) \\ H_1^\psi(S^3 \setminus J;\Lambda) \oplus H_1^\psi(S^3 \setminus K;\Lambda) \arrow[rr, hook] & & H_1^\psi(S^3 \setminus (J \sqcup K);\Lambda) \arrow[r, "h"] & \Lambda \end{tikzcd}$$

where the top and bottom horizontal sequences are exact, and where the map h is non-trivial. In particular, note that p gives rise to an isomorphism $M/f(\Lambda) \cong H_1^\psi(S^3 \setminus L; \Lambda)$, and that g gives rise to an epimorphism

$$H_1^\psi(S^3 \setminus L; \Lambda) \cong M/f(\Lambda) \to H_1^\psi(S^3 \setminus (J \sqcup K); \Lambda)/(g \circ f)(\Lambda). \tag{4.5}$$

Next we will prove the following claim.

Claim 4.4. *The map*

$$H_1^\psi(S^3 \setminus J; \Lambda) \oplus H_1^\psi(S^3 \setminus K; \Lambda) \to H_1^\psi(S^3 \setminus J \sqcup K; \Lambda)/(g \circ f)(\Lambda)$$

is a monomorphism.

We consider the commutative diagram

$$\begin{tikzcd}
 & & \Lambda \arrow[d, "g\circ f"] & \\
0 \arrow[r] & H_1^\psi(S^3 \setminus J; \Lambda) \oplus H_1^\psi(S^3 \setminus K; \Lambda) \arrow[r, hook] & H_1^\psi(S^3 \setminus (J \sqcup K); \Lambda) \arrow[r, "h"] \arrow[d] & \Lambda \arrow[d] \\
 & & \dfrac{H_1^\psi(S^3 \setminus (J \sqcup K); \Lambda)}{(g \circ f)(\Lambda)} \arrow[r, "h"] & \Lambda/(h \circ g \circ f)(\Lambda)
\end{tikzcd}$$

where the bottom vertical maps are the obvious projection maps. Furthermore, as above, the map h in the middle sequence is non-trivial.

We first note that the bottom-left group is Λ-torsion. Indeed, in the discussion preceding the proof we saw that $\Delta_L(s,t) \neq 0$. This implies that the homology group $H_1^\psi(S^3 \setminus L; \Lambda)$ is Λ-torsion. But by (4.5) this also implies that the bottom-left group of the diagram is Λ-torsion.

It follows that in the square the composition of maps given by going down and then right factors through a Λ-torsion group. On the other hand, we have seen that the map $h\colon H_1^\psi(S^3 \setminus (J \sqcup K); \Lambda) \to \Lambda$ is non-trivial. By the commutativity of the square and by the fact that the down-right composition of maps factors through a Λ-torsion group, it now follows that the projection map $\Lambda \to \Lambda/(h \circ g \circ f)(\Lambda)$ cannot be an isomorphism. But this just means that the composition

$$\Lambda \xrightarrow{g\circ f} H_1^\psi(S^3 \setminus (J \sqcup K); \Lambda) \xrightarrow{h} \Lambda$$

is non-trivial and, in particular, injective. Put differently, we have

$$\mathrm{im}(g \circ f\colon \Lambda \to H_1^\psi(S^3 \setminus (J \sqcup K); \Lambda)) \cap \ker(h) = 0.$$

By the exactness of the middle horizontal sequence, we thus see that the intersection of the images of $(g \circ f)(\Lambda)$ and of $H_1^\psi(S^3 \setminus J; \Lambda) \oplus H_1^\psi(S^3 \setminus K; \Lambda)$ in $H_1^\psi(S^3 \setminus (J \sqcup K); \Lambda)$ is trivial. It follows that the map

$$H_1^\psi(S^3 \setminus J; \Lambda) \oplus H_1^\psi(S^3 \setminus K; \Lambda) \to H_1^\psi(S^3 \setminus J \sqcup K; \Lambda)/(g \circ f)(\Lambda)$$

is indeed a monomorphism. This concludes the proof of the claim.

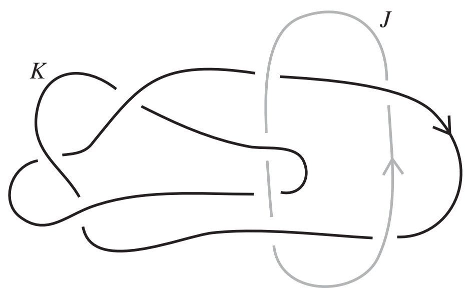

Figure 6. The link $L9a29$, with splitting number 3.

Before we continue with the proof we recall that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of Λ-modules, then by [**9**, Part 1.3.3] the orders of the modules are related by the equality

$$\operatorname{ord}_\Lambda(B) = \operatorname{ord}_\Lambda(A) \cdot \operatorname{ord}_\Lambda(C). \tag{4.6}$$

We thus see, from the claim and from (4.6), (4.3) and (4.4), that

$$\Delta_J(s) \cdot \Delta_K(t) \mid \operatorname{ord}_\Lambda(H_1^\psi(S^3 \setminus J \sqcup K; \Lambda)/(g \circ f)(\Lambda)).$$

On the other hand, it follows from (4.5) and again from (4.6) that

$$\operatorname{ord}_\Lambda(H_1^\psi(S^3 \setminus (J \sqcup K); \Lambda)/(g \circ f)(\Lambda)) \mid \operatorname{ord}_\Lambda(H_1^\psi(S^3 \setminus L; \Lambda)). \qquad \square$$

4.3. An example of the Alexander polynomial technique

We consider the oriented link $L = K \sqcup J = L9a29$ from Figure 6.

It has linking number 1 and it is not hard to see that one can turn it into a split link using three crossing changes between the two components. The multivariable Alexander polynomial of L is

$$\Delta_L(s,t) = s - s^2 + t - st + s^2t - t^2 + st^2 - s^2t^2 + t^3 - st^3 + s^2t^3 - t^4 + st^4.$$

It is straightforward to see that $\Delta_J(s) \cdot \Delta_K(t) = 1 - t + t^2$ does not divide $\Delta_L(s,t)$. It thus follows from Theorem 4.3 that the splitting number of L is 3.

This is one of the instances of the use of the Alexander polynomial that is cited in § 6, in Table 3 (Method (4)). The other computations listed in that table as using this method are performed in a similar fashion; see the LinkInfo tables [**7**] for the multivariable Alexander polynomials of the other nine-crossing links, which are $L9a24$, $L9n13$, $L9n14$ and $L9n17$. Since these are two-component links of linking number 1, the Alexander polynomials of the components can be obtained by substituting either $t = 1$ or $s = 1$ into the multivariable Alexander polynomial in $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$.

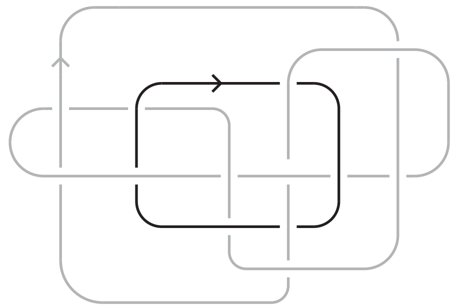

Figure 7. A two-component link of linking number 1 whose splitting number equals 3.

5. The examples of Batson and Seed

In [**2**], Batson and Seed constructed a spectral sequence from the Khovanov homology of a link L to the Khovanov homology of the split link with the same components as L. This spectral sequence gives rise to a lower bound on the splitting number, given by the lowest page on which their spectral sequence collapses.

Batson and Seed computed the lower bound for all links up to 12 crossings and they showed that it provides more information than basic linking number observations (see our Lemma 2.1) for 17 links. The lower bound they computed will be denoted by $b(L)$. One of the 17 links is a three-component link with 12 crossings, for which $b(L) = 3$, while the sum of the absolute values of the linking numbers is 1. The remaining 16 links have two components and satisfy $\mathrm{lk}(L) = \pm 1$ and $b(L) = 3$. One of these has 11 crossings, and 15 of these have 12 crossings. Batson and Seed determined the splitting numbers for seven links among these 17 links, while for the other 10 cases the splitting numbers are listed as being either 3 or 5. This information is given in [**2**, Table 3].

In this section we revisit these links to reprove or improve the results in [**2**]. In particular, we completely determine the splitting numbers by using our methods.

5.1. Using the Alexander polynomial

We first apply our Alexander polynomial method to the examples of [**2**] with at least one knotted component. This will reprove their splitting number results for these links. Before we turn to the links of [**2**, Table 3], we will discuss a link with 13 crossings in detail, which is also discussed in [**2**].

A 13-crossing example

Consider the two-component link L shown in Figure 7. It is the link denoted by ${}^2n^{13}_{8862}$ in [**2**].

Note that one component is an unknot and the other is a trefoil. We refer to the unknotted component as J and to the knotted component as K. It is not hard to see that L can be turned into a split link using three crossing changes. On the other hand, the linking number is 1, so it follows from Lemma 2.1 that the splitting number is either 1 or 3. The invariant $b(L)$ shows that the splitting number of L is in fact 3.

We will now use Theorem 4.3 to give another proof that the splitting number of L equals 3. We used Kodama's program KNOTGTK to show that

$$\begin{aligned}\Delta_L(s,t) = &-s^8t^4 + s^7t^5 + 4s^8t^3 - 5s^6t^5 - 6s^8t^2 - 9s^7t^3 + 13s^6t^4 + 11s^5t^5 + 4s^8t \\ &+ 17s^7t^2 - 6s^6t^3 - 37s^5t^4 - 14s^4t^5 - s^8 - 12s^7t - 10s^6t^2 + 45s^5t^3 - 4s^6 \\ &+ 52s^4t^4 + 11s^3t^5 + 3s^7 + 12s^6t - 24s^5t^2 - 74s^4t^3 - 44s^3t^4 - 5s^2t^5 \\ &+ 2s^5t + 51s^4t^2 + 67s^3t^3 + 23s^2t^4 + st^5 + 3s^5 - 13s^4t - 46s^3t^2 - 39s^2t^3 \\ &- 7st^4 - s^4 + 11s^3t + 25s^2t^2 + 15st^3 + t^4 - 3s^2t - 9st^2 - 3t^3 + 2t^2.\end{aligned}$$

It is straightforward to see (we used MAPLE) that

$$\Delta_L(s,1)\cdot\Delta_L(1,t) = \Delta_J(s)\cdot\Delta_K(t) = 1 - t + t^2$$

does not divide $\Delta_L(s,t)$. Thus, it follows from Theorem 4.3 that the splitting number of L is not 1. By the above observations we therefore see that the splitting number of L is equal to 3.

Seven 12-*crossing examples*

In [**2**, Table 3], Batson and Seed give seven examples of two-component 12-crossing links that have linking number equal to 1 and for which $b(L)$ detects that the splitting number is 3.

In Table 1 we list the links together with their Dowker–Thistlethwaite (DT) codes and multivariable Alexander polynomials. The translation between the names we use (following LinkInfo [**7**]) and the convention used in [**2**] is given by $L12nX = {}^2a^{12}_{X+4196}$. All these Alexander polynomials are irreducible. For each link, both components are trefoils, so $\Delta_L(s,1) = 1 - s + s^2$ and $\Delta_L(1,t) = 1 - t + t^2$ do not divide the multivariable Alexander polynomial. Thus, it follows from Theorem 4.3 that the splitting number of each of these links is at least 3, which recovers the results of Batson and Seed. Inspection of the diagrams shows that the splitting numbers are indeed equal to 3.

5.2. Using the covering link technique

Batson and Seed [**2**, Table 3] gave nine further examples of links that have two unknotted components and linking number ± 1. They list these links as having splitting number either 3 or 5. Translating notation again, we have: $L11a372 = {}^2a^{11}_{739}$, $L12aX = {}^2a^{12}_{X+1288}$ and $L12nY = {}^2n^{12}_{Y+4196}$.

Table 2 lists the results of our computations, giving the slice genus of the knot obtained by taking a two-fold branched cover of S^3, branched over one of the components, the method that we use to compute the slice genus, and the resulting splitting number obtained by the methods of § 3.1.

The methods we use to compute the slice genus of the covering knot are as follows. First, the slice genus of a knot is bounded below by half the absolute value of its signature $\sigma(K) = \mathrm{sign}(A + A^{\mathrm{T}})$, where A is a Seifert matrix of K, by [**15**, Theorem 9.1]. We used a PYTHON software package of the first author to compute $\sigma(K)$. The Rasmussen

Table 1. *Seven 12-crossing links and their Alexander polynomials.*

link	DT code	Alexander polynomial
$L12n1342$	$(14, \overline{6}, \overline{10}, 16, \overline{4}, \overline{18})$, $(\overline{20}, 22, 8, \overline{24}, \overline{2}, 12)$	$s^2t^4 - st^4 - s^2t^2 + s^2t + st^2 + t^3 - t^2 - s + 1$
$L12n1350$	$(14, \overline{6}, \overline{10}, \overline{16}, \overline{4}, 20)$, $(12, 22, \overline{8}, 24, 2, 18)$	$-s^4t^3 + s^3t^4 + s^4t^2 + s^3t^3 - 2s^2t^4 - 2s^3t^2 + s^2t^3$ $+st^4 + s^2t^2 + s^3 + s^2t - 2st^2 - 2s^2 + st + t^2 + s - t$
$L12n1357$	$(14, \overline{6}, \overline{10}, 16, \overline{4}, 22)$, $(20, 2, 8, 24, 12, 18)$	$-s^4t^4 + s^4t^3 + s^3t^4 - s^4t^2 + s^4t - s^2t^3$ $+s^2t^2 - s^2t + t^3 - t^2 + s + t - 1$
$L12n1363$	$(14, \overline{6}, \overline{10}, \overline{18}, \overline{4}, 22)$, $(2, 20, \overline{8}, 24, 12, 16)$	$2s^2t^2 - 3s^2t - 3st^2 + 2s^2 + 5st + 2t^2 - 3s - 3t + 2$
$L12n1367$	$(14, \overline{6}, \overline{10}, \overline{18}, \overline{4}, 24)$, $(2, 12, 22, \overline{8}, 16, 20)$	$s^4t^2 + s^3t^3 - s^2t^4 - s^4t - 2s^3t^2 + st^4 + s^3t$ $+s^2t^2 + st^3 + s^3 - 2st^2 - t^3 - s^2 + st + t^2$
$L12n1374$	$(14, \overline{6}, \overline{10}, 20, \overline{4}, 16)$, $(2, 12, 22, 24, 8, 18)$	$s^4t^3 + s^3t^4 - s^4t^2 - s^3t^3 - s^2t^4 + s^4t + st^4$ $-s^4 + s^2t^2 - t^4 + s^3 + t^3 - s^2 - st - t^2 + s + t$
$L12n1404$	$(14, \overline{8}, \overline{18}, \overline{12}, \overline{22}, \overline{4})$, $(20, 2, 24, \overline{6}, 16, \overline{10})$	$2s^2t^3 - st^4 - 2s^2t^2 - st^3 + t^4$ $+3st^2 + s^2 - st - 2t^2 - s + 2t$

Table 2. *Nine links and their covering knot slice genus.*

link	DT code	covering knot slice genus	method for slice genus	splitting number
$L11a372$	$(12, 14, 16, 20, 18)$, $(10, 2, 4, 22, 8, 6)$	2	$\sigma = -4$	5
$L12a1233$	$(12, 14, 16, 18, 20)$, $(2, 24, 4, 6, 22, 8, 10)$	2	$\sigma = -4$	5
$L12a1264$	$(12, 14, 16, 20, 18)$, $(2, 24, 4, 22, 8, 6, 10)$	2	$\sigma = -4$	5
$L12a1384$	$(14, 8, 16, 24, 18, 20)$, $(2, 22, 4, 10, 12, 6)$	2	$\sigma = -4$	5
$L12n1319$	$(12, 14, 16, 24, \overline{18})$, $(2, 10, 22, \overline{20}, \overline{8}, 4, 6)$	1	$\sigma = -2$	3
$L12n1320$	$(12, \overline{14}, \overline{16}, 24, \overline{18})$, $(6, 22, \overline{20}, \overline{8}, \overline{4}, \overline{2}, 10)$	1	$p = 3$, $q = 7$, $\Delta^\chi(s) = 7s^2 + 15s + 7$	3
$L12n1321$	$(12, 14, 16, 24, \overline{18})$, $(10, 2, 22, \overline{20}, \overline{8}, 4, 6)$	2	$s = 4$	5
$L12n1323$	$(12, 14, 18, 16, \overline{20})$, $(10, 2, 24, 6, \overline{22}, \overline{8}, 4)$	2	$\sigma = -4$	5
$L12n1326$	$(12, 14, \overline{18}, 24, \overline{20})$, $(8, 2, 4, \overline{22}, \overline{10}, \overline{6}, \overline{16})$	1	$p = 3$, $q = 7$, $\Delta^\chi(s) = 7s^2 - 71s + 7$	3

s-invariant [**16**] also gives a lower bound by $|s(K)| \leqslant 2g_4(K)$. We used JavaKh of Green and Morrison to compute $s(K)$.

We can also prove that a knot is not slice using the twisted Alexander polynomial [**12**], denoted by $\Delta_K^\chi(s) \in \mathbb{Q}(\zeta_q)[s^{\pm 1}]$, where ζ_q is a primitive qth root of unity, associated with the p-fold cyclic cover X_p of the knot exterior X and a character $\chi\colon TH_1(X_p;\mathbb{Z}) \to \mathbb{Z}_q$. For slice knots, there exists a metaboliser of the $\mathbb{Q}/\mathbb{Z}$-valued linking form on $H_1(X_p;\mathbb{Z})$ such that, for characters χ that vanish on the metaboliser, the twisted Alexander polynomial factorizes (up to a unit) as $g(s)\overline{g(s)}$ for some $g \in \mathbb{Q}(\zeta_q)[s^{\pm 1}]$. By checking that this condition does not hold for all metabolisers, we can prove that a knot is not slice. (For each covering knot K to which we apply this, all metabolisers give the same polynomial Δ_K^χ.) Our computations of twisted Alexander polynomials were performed using a Maple program written by Herald *et al*. [**8**].

The invariants discussed above give us lower bounds on the slice genera of the covering knots. We do not need to know the precise slice genera in order to obtain lower bounds. Nevertheless, we point out that we are able to determine them. In each case we found the requisite crossing changes to split the link, so an application of Theorem 1.1 gives us an upper bound on the slice genus of the covering knot, which implies that the above lower bounds are sharp.

For the links $L11a372$, $L12a1233$, $L12a1264$, $L12a1384$, $L12n1321$ and $L12n1323$, we are able to show that the splitting numbers of these links are 5. Amusingly, we use Khovanov homology, in the guise of the s-invariant, to compute that the slice genus of the covering knot of $L12n1321$ is 2. We remark that this knot has $\sigma = -2$, which is only sufficient to show that the splitting number is at least 3.

For the other links, as in § 5.1, our obstruction gives the same information as the Batson–Seed lower bound, namely, that the splitting number is at least 3. For the links $L12n1319$, $L12n1320$ and $L12n1326$ we looked at the diagrams and found the crossing changes to verify that the splitting number is indeed 3.

We present one example in detail, the link $L11a372$, which is shown as given by LinkInfo on the left of Figure 8, while on the right the link $L11a372$ is shown after an isotopy, to prepare for making a diagram of a covering link. It is easy to see from the diagram that the splitting number is at most 5. The link $L11a372$ has $b(L11a372) = 3$.

The two-fold covering link obtained by branching over the left-hand component is shown in Figure 9. This turns out to be the knot 7_5, which according to KnotInfo [**6**] has $|\sigma| = 4$ and slice genus 2. Therefore, by Theorem 1.1, the splitting number is 5.

5.3. A three-component example

There is one final link listed in [**2**, Table 3] as having splitting number either 3 or 5, namely, the three-component link $L := L12a1622$, which is shown in Figure 10. In the notation of [**2**], L is the link ${}^3a^{12}_{2910}$.

We show that the splitting number of L is in fact 5. Note that the components are unknotted, and the only non-zero linking number is between L_2 and L_3, which have $|\mathrm{lk}(L_2, L_3)| = 1$. Thus, the splitting number is odd by Lemma 2.1. It is easy to find five crossing changes that suffice.

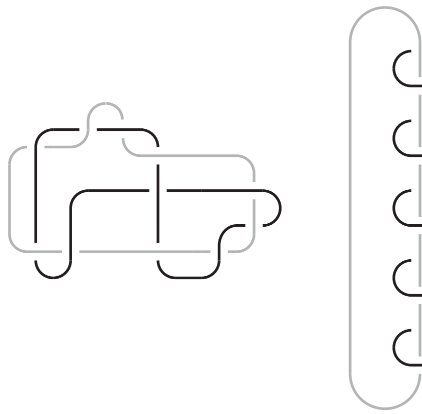

Figure 8. Left: the link $L11a372$. Right: after an isotopy.

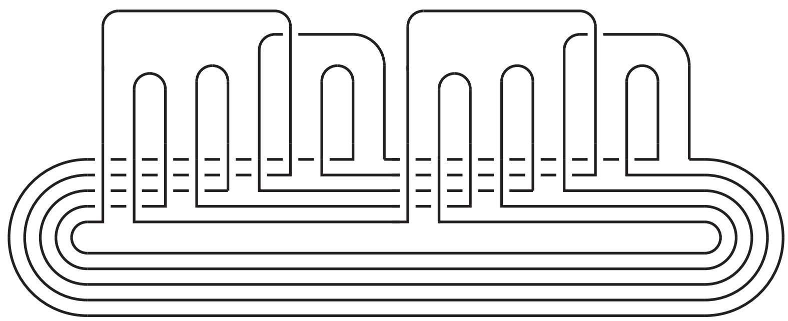

Figure 9. The covering link obtained by taking a two-fold branched covering over the left-hand component of the link in Figure 8.

We begin by showing that three crossing changes involving just L_2 and L_3 do not suffice to split the link. We take the two-fold covering link J with respect to L_1. The result of an isotopy to prepare for taking such a covering is shown on the right of Figure 10. The resulting covering link J is shown in Figure 11. The link J has splitting number 10 by Lemma 2.1, with a sharp lower bound given by the sum of the absolute values of the linking numbers between the components. By Corollary 3.6, we have that $\operatorname{sp}_1(L) \geqslant 5$.

Combining $\operatorname{sp}_1(L) \geqslant 5$ with the linking number, it follows that if $\operatorname{sp}(L) \leqslant 3$, then exactly one crossing change involving (L_2, L_3) is required to split the link, and there can be either two additional (L_1, L_2) crossing changes, or two (L_1, L_3) crossing changes. We will give the argument to show that the first possibility cannot happen; the argument discounting the second possibility is analogous.

Suppose that two (L_1, L_2) crossing changes and one (L_2, L_3) crossing change yields the unlink. Applying Corollary 3.3 (with $m = 3$, $\alpha = 2$, $\beta = 1$, $g_4(L_k) = 0$), it follows that the covering link $J \subset S^3$ bounds an oriented surface F of Euler characteristic $2(3-1) - 2 - 4 = -2$ that is smoothly embedded in D^4 and has no closed component. Also, F is connected by the last part of Corollary 3.3 since both L_1 and L_3 are involved

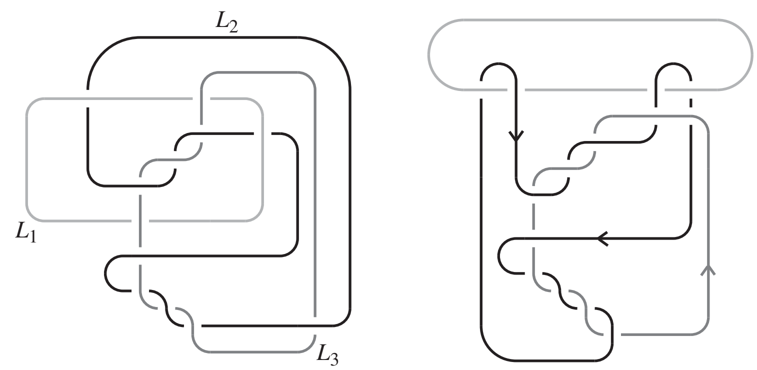

Figure 10. Left: the link $L12a1622$. Right: the same link, after an isotopy to prepare for taking a covering link by branching over the top component.

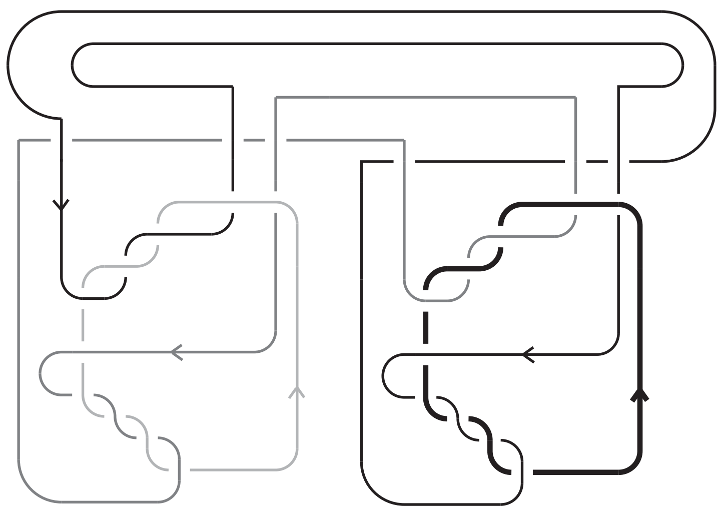

Figure 11. The two-fold covering link of the link in Figure 10, branched over L_1.

in some crossing change with L_2. Since J has four components, F is a three-punctured disc. That is, J is weakly slice.

To show that this cannot be the case for J, we use the link signature invariant, which is defined similarly to the knot signature: for a link J, choose a surface V in S^3 bounded by J (V may be disconnected), define the Seifert pairing on $H_1(V)$ and an associated Seifert matrix A as usual. Then the *link signature* of J is defined by $\sigma(J) = \operatorname{sign}(A + A^{\mathrm{T}})$. Due to Murasugi [**15**], if an m-component link J bounds a smoothly embedded oriented surface F in D^4, we have $|\sigma(J)| \leqslant 2g(F) + m - b_0(F)$, where $g(F)$ is the genus and $b_0(F)$ is the zeroth Betti number of F. For our covering link J, since it bounds a three-punctured disc in D^4, we have $|\sigma(J)| \leqslant 3$. Here we orient J as in Figure 11; this orientation is obtained using the orientations of L_2 and L_3 shown on the right of Figure 10. On the other hand, a computation aided by a Python software package of the first author shows that $\sigma(J) = -7$. From this contradiction it follows that one (L_2, L_3) crossing change and two (L_1, L_2) crossing changes never split L.

6. Links with nine or fewer crossings

In Table 3 we give the splitting numbers for the links of nine crossings or fewer, together with the method that is used to give a sharp lower bound for the splitting number. The entry in the method column of the table refers to the list below.

In the case of two-component links with unknotted components and linking number 1, knowing that the *unlinking number* is greater than 1 implies that the splitting number is at least 3. Recall that by definition the unlinking number of an m-component link L is the minimal number of crossing changes required to convert L to the m-component unlink. Note that for this link invariant, crossing changes of a component with itself are in general permitted.

In Method (3), below, we will make use of computations of unlinking numbers made by Kohn in [**14**], where, making considerable use of his earlier work [**13**], he computed the unlinking numbers of two-component links with nine or fewer crossings, in all but five cases.

(1) Using Lemma 2.1, the linking numbers determine the lower bound for the splitting number by providing a lower bound or by fixing the splitting number modulo 2.

(2) A combination of linking numbers and either one or two Whitehead links as a sublink determine a lower bound for the splitting number. That is, Lemma 2.1 provides a sharp lower bound, with $c(L) \neq 0$.

(3) This is a link where the sum of the linking numbers is 1 and the components are unknotted, but which does not have unlinking number 1, and so cannot have splitting number 1. Therefore, the splitting number is at least 3.

    For the two-component case (all that use this method have two components apart from $L9a46$ and $L8a16$), we know that this link does not have unlinking number 1 by [**14**]. Kohn did not explicitly give an argument that the unlinking number of $L9a30$ is at least 2, but we computed the splitting number of $L9a30$ in § 3.2.

    For the three-component links $L8a16$ and $L9a46$, we show that the splitting number (and unlinking number) is not 1 in §§ 7.2 and 7.3, respectively.

(4) A two-component link of linking number 1, with at least one component knotted. The Alexander polynomials of the components do not divide the multivariable Alexander polynomial of the link, so by Theorem 4.3 the splitting number must be at least 3. See § 4.3 for an example of this argument in action, for the link $L9a29$.

(5) A two-component link with unknotted components and linking number zero modulo 2. For the link $L9a36$, we note that the unlinking number is not 2 by [**14**]. Thus, the splitting number must be at least 4. For the link $L9a40$, we show that the splitting number is not 2 in § 7.1.

We remark that some of the splitting numbers in the table are also given in [**2**].

Table 3. *Splitting numbers of links with nine or fewer crossings.*

link L	sp(L)	method	link L	sp(L)	method	link L	sp(L)	method
$L2a1$	1	(1)	$L8n6$	4	(1)	$L9a42$	2	(1)
$L4a1$	2	(1)	$L8n7$	4	(1)	$L9a43$	3	(1)
$L5a1$	2	(1)	$L8n8$	4	(1)	$L9a44$	3	(1)
$L6a1$	2	(1)	$L9a1$	2	(1)	$L9a45$	3	(1)
$L6a2$	3	(1)	$L9a2$	2	(1)	$L9a46$	3	(3)
$L6a3$	3	(1)	$L9a3$	2	(1)	$L9a47$	4	(2)
$L6a4$	2	(1)	$L9a4$	2	(1)	$L9a48$	4	(1)
$L6a5$	3	(1)	$L9a5$	2	(1)	$L9a49$	4	(1)
$L6n1$	3	(1)	$L9a6$	2	(1)	$L9a50$	4	(2)
$L7a1$	2	(1)	$L9a7$	2	(1)	$L9a51$	4	(1)
$L7a2$	2	(1)	$L9a8$	2	(1)	$L9a52$	4	(2)
$L7a3$	2	(1)	$L9a9$	2	(1)	$L9a53$	2	(1)
$L7a4$	2	(1)	$L9a10$	2	(1)	$L9a54$	4	(2)
$L7a5$	1	(1)	$L9a11$	2	(1)	$L9a55$	4	(1)
$L7a6$	3	(3)	$L9a12$	2	(1)	$L9n1$	2	(1)
$L7a7$	3	(1)	$L9a13$	2	(1)	$L9n2$	2	(1)
$L7n1$	2	(1)	$L9a14$	2	(1)	$L9n3$	2	(1)
$L7n2$	2	(1)	$L9a15$	2	(1)	$L9n4$	2	(1)
$L8a1$	2	(1)	$L9a16$	2	(1)	$L9n5$	2	(1)
$L8a2$	2	(1)	$L9a17$	2	(1)	$L9n6$	2	(1)
$L8a3$	2	(1)	$L9a18$	2	(1)	$L9n7$	2	(1)
$L8a4$	2	(1)	$L9a19$	2	(1)	$L9n8$	2	(1)
$L8a5$	2	(1)	$L9a20$	3	(3)	$L9n9$	2	(1)
$L8a6$	2	(1)	$L9a21$	1	(1)	$L9n10$	2	(1)
$L8a7$	2	(1)	$L9a22$	3	(3)	$L9n11$	2	(1)
$L8a8$	3	(3)	$L9a23$	3	(1)	$L9n12$	2	(1)
$L8a9$	3	(3)	$L9a24$	3	(4)	$L9n13$	3	(4)
$L8a10$	3	(1)	$L9a25$	3	(1)	$L9n14$	3	(4)
$L8a11$	3	(1)	$L9a26$	3	(3)	$L9n15$	3	(1)
$L8a12$	4	(1)	$L9a27$	1	(1)	$L9n16$	3	(1)
$L8a13$	4	(1)	$L9a28$	3	(1)	$L9n17$	3	(4)
$L8a14$	4	(1)	$L9a29$	3	(4)	$L9n18$	4	(1)
$L8a15$	3	(1)	$L9a30$	3	(3)	$L9n19$	4	(1)
$L8a16$	3	(3)	$L9a31$	1	(1)	$L9n20$	3	(1)
$L8a17$	4	(1)	$L9a32$	3	(1)	$L9n21$	3	(1)
$L8a18$	4	(1)	$L9a33$	3	(1)	$L9n22$	3	(1)
$L8a19$	2	(1)	$L9a34$	2	(1)	$L9n23$	4	(2)
$L8a20$	4	(1)	$L9a35$	2	(1)	$L9n24$	4	(2)
$L8a21$	4	(1)	$L9a36$	4	(5)	$L9n25$	2	(2)
$L8n1$	2	(1)	$L9a37$	2	(1)	$L9n26$	4	(2)
$L8n2$	2	(1)	$L9a38$	2	(1)	$L9n27$	4	(2)
$L8n3$	4	(1)	$L9a39$	2	(1)	$L9n28$	4	(2)
$L8n4$	4	(1)	$L9a40$	4	(5)	—	—	—
$L8n5$	2	(1)	$L9a41$	2	(1)	—	—	—

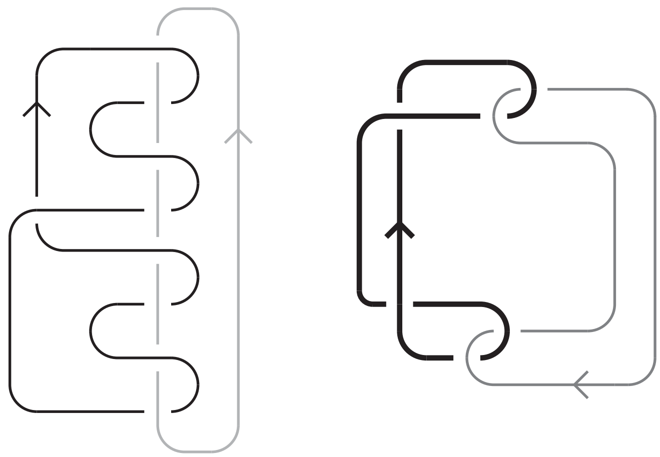

Figure 12. Left: the link $L9a40$. Right: the two-fold branched cover with respect to the right-hand component.

7. Arguments for the splitting number of particular links

7.1. The link $L9a40$

The link $L9a40$ is shown on the left of Figure 12. We claim that the splitting number of $L9a40$ is 4. Note that the linking number is zero, so the splitting number is either 2 or 4, since it is easy to see from the diagram that four crossing changes suffice to split the link.

To show that the splitting number cannot be 2, we consider the two-component link obtained by taking the two-fold covering link with respect to the right-hand component, which is shown on the right of Figure 12. This is the link $L6a1$. By Corollary 3.5, if $\mathrm{sp}(L9a40) = 2$, then $L6a1$ would bound an annulus smoothly embedded in D^4. Thus, any internal band sum of $L6a1$, which is a knot, would have slice genus at most 1. But the band sum of $L6a1$ shown in Figure 13 is the knot 7_5, which has signature 4 and smooth slice genus 2. It follows that the splitting number of $L9a40$ is 4 as claimed.

7.2. The link $L8a16$

The link $L8a16$ is shown in Figure 14. The components are labelled L_1, L_2 and L_3. The linking number $|\mathrm{lk}(L_1, L_2)| = 1$, and the other linking numbers are trivial. We claim that $\mathrm{sp}(L8a16) = 3$. It is not hard to find three crossing changes that work; for example, change all three of the crossings where L_2 passes over L_1 in Figure 14. By this observation and Lemma 2.1, the splitting number is either 1 or 3. We therefore need to show that it is not possible to split the link with a single crossing change. (We remark that this is the same as showing that the unlinking number is greater than 1, since the components are unknotted.)

By linking number considerations, a single crossing change would have to involve L_1 and L_2. To discard this eventuality, we will take a two-fold covering link branched over L_3.

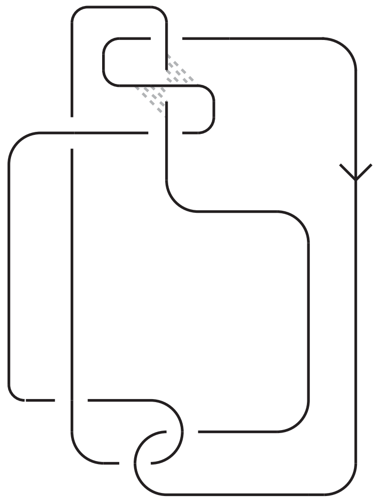

Figure 13. A band sum of $L6a1$ to produce the knot 7_5. The band that was added is indicated by dotted lines; it has a half twist in it.

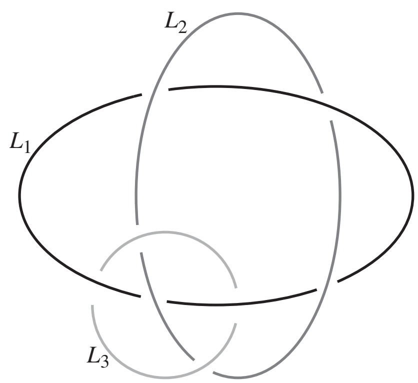

Figure 14. The link $L8a16$.

The left of Figure 15 shows the link $L8a16$ after an isotopy; the right-hand picture shows the 2-fold cover branched over L_3. Call this link J. The sum of linking numbers $\sum_{i<j} |\mathrm{lk}(J_i, J_j)| = 6$, so $\mathrm{sp}(J) \geqslant 6$ by Lemma 2.1 (in fact $\mathrm{sp}(J) = u(J) = 6$). Therefore, by Corollary 3.6, we see that $\mathrm{sp}_3(L8a16) \geqslant 3$. (Recall that $\mathrm{sp}_i(L)$ denotes the splitting number of L, where the component L_i is not involved in any crossing changes.) Thus, as claimed, it is not possible to split the link in a single crossing.

7.3. The link $L9a46$

The link $L9a46$ is shown on the left of Figure 16. We claim that $\mathrm{sp}(L9a46) = 3$. It is not hard to find three changes that suffice. For example, in Figure 16, change the crossings where L_1 passes under L_2.

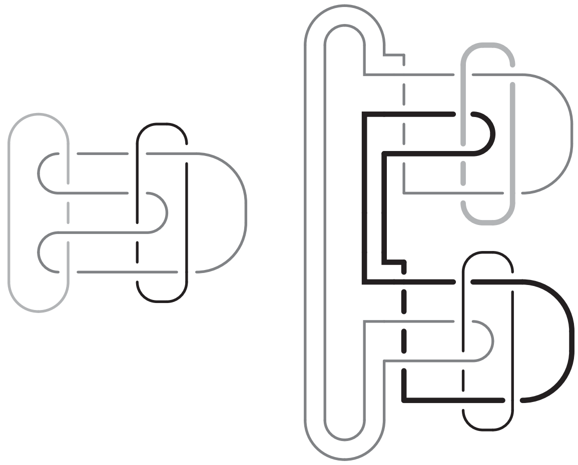

Figure 15. Left: the link $L8a16$ after an isotopy. Right: the two-fold cover branched over L_3.

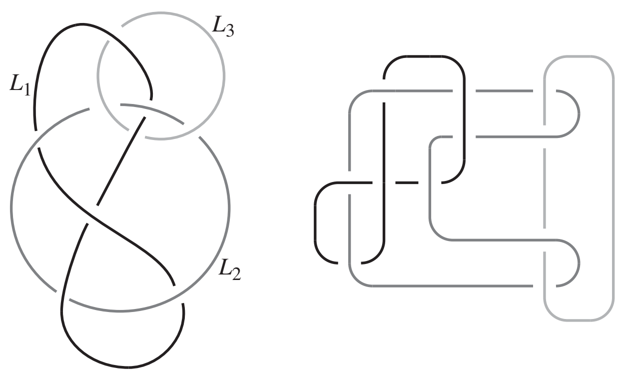

Figure 16. Left: the link $L9a46$. Right: the link $L9a46$ after an isotopy.

Note that $|\mathrm{lk}(L_1, L_2)| = 1$. Therefore, if one crossing change suffices, it must be between L_1 and L_2. We need to show that this is not possible. For this purpose we apply Corollary 3.6 again. We will take a two-fold covering link branched over L_3. In preparation for this, the link from the left of Figure 16 is shown, after an isotopy, on the right of Figure 16.

Taking the cover branched over the right-hand component of the link on the right of Figure 16, we obtain the two-fold covering link J shown in Figure 17.

We need to see that the link J of Figure 17 has splitting number at least 6. Observe that $|\mathrm{lk}(J_1, J_4)| = |\mathrm{lk}(J_2, J_3)| = 1$. Moreover, the sublinks $J_1 \sqcup J_3$ and $J_2 \sqcup J_4$ are Whitehead links. It now follows from Lemma 2.1 that $\mathrm{sp}(J) \geqslant 6$. By Corollary 3.6, we obtain that $\mathrm{sp}_3(L9a46) \geqslant 3$. It thus follows from the above discussion that $\mathrm{sp}(L9a46) = 3$.

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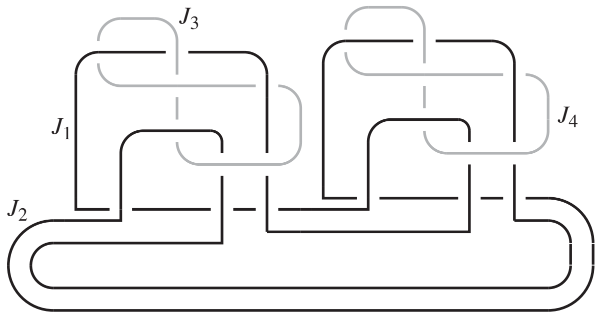

Figure 17. The two-fold covering link of the link $L9a46$ from the right of Figure 16.

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