Coupled Evolution Equations for Immersions of Closed Manifolds and Vector Fields

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Zusammenfassung

Eine mögliche Beschreibung der elastischen Energie einer Biomembran ist durch den L^2-Abstand der mittleren Krümmung von einer spontanen Krümmung plus einer topologischen Konstante gegeben. Ein solches Energiefunktional wird oft als Helfrich Energie bezeichnet.

Wir untersuchen eine Erweiterung dieser Modellierung, bei der die spontane Krümmung durch die Oberflächendivergenz eines Vektorfeldes entlang der Fläche gegeben ist. Eine abstrakte Formulierung für Immersionen von Mannigfaltigkeiten beliebiger Dimension wird hergeleitet.

Für Kurven in der euklidischen Ebene zeigen wir für dieses Funktional Existenz von globalen Minimierern und Regularität von stationären Punkten unter den verschiedenen Nebenbedingungen. Mögliche Nebenbedingungen sind die Länge der Kurve, der von der Kurve eingeschlossenen Flächeninhalt und der Bildbereich des Vektorfeldes.

Außerdem leiten wir für Immersionen von Mannigfaltigkeiten beliebiger Raumdimension eine Gradientenflussdynamik her, die auf ein gekoppeltes System partieller Differentialgleichungen führt. Für dieses gekoppelte System zeigen wir lokale Wohlgestelltheit auch im Fall, dass der Fluss die Nebenbedingungen erhält.

Weiterhin zeigen wir für das uneingeschränkte Funktional sowie unter Berücksichtigung der Nebenbedingungen eine Łojasiewicz-Simon Gradientenungleichung, aus welcher man dann Rückschlüsse auf das asymptotische Verhalten des Flusses nahe lokaler Minimierer ziehen kann.

Für Kurven und Vektorfelder in der Ebene geben wir eine geometrische Größe an, deren Kleinheit Glattheit des Flusses garantiert. Durch Reskalieren erreichen wir, dass es bereits ausreicht, dass diese Größe endlich ist, um Singularitäten auszuschließen.

Abstract

A possible description of the elastic energy of a biological membrane is given by the L^2-distance of its mean curvature from a spontaneous curvature plus a topological constant. Such energy functional is often referred to as Helfrich energy.

We study a generalization of this model, where the spontaneous curvature arises as the divergence of a vector field along the surface. An abstract formulation for immersions of manifolds of arbitrary dimension is derived.

For plane curves we prove for this energy functional the existence of global minimizers and regularity of stationary points subject to different constraints. The constraints we considered are the length and enclosed signed area of the curve and the range of the vector field.

Furthermore, we derive a gradient-flow equation in the general situation of immersions of manifolds of arbitrary space dimension which leads to a coupled system of partial differential equations. For this coupled system we show local well-posedness even for a constraint preserving adaption of the flow.

Moreover, we show a Łojasiewicz-Simon gradient inequality for the unrestricted functional as well as in the presence of constraints. From this we draw conclusions about the asymptotic behavior of the flow close to a local minimizer.

For curves and vector fields in the euclidean plane we introduce a geometric quantity whose smallness guarantees smoothness of the flow. By a rescaling argument we achieve that even finiteness of this quantity suffices to exclude the formation of singularities.

Contents

Introduction

With the range of applications reaching from deep questions of topology and geometry to models for complex physical processes and even image segmentation, geometric evolution equations form a common basis of an extremely wide range of mathematical research areas.

Intrinsic geometric evolutions deform the metric of a Riemannian manifold in a smooth way, without consideration of any embedding or immersion. Their most prominent instance is by all accounts the Ricci flow, profoundly investigated by Hamilton. It was central in Perelman's proof of Thurston's geometrization conjecture, which implies the Poincaré conjecture, and the proof of the differentiable sphere theorem by Brendle and Schoen after 30 years of intensive research (cf. [11, 45, 74, 75, 86]).

Extrinsic evolutions smoothly vary mappings or immersions of a smooth manifold according to a law involving geometric quantities of this immersion. A large variety of different flows of this kind has been studied, including the harmonic-map heatflow, the Willmore flow and the mean-curvature flow, the latter being the most famous example of a whole range of curvature flows. Some of them have been introduced to improve the understanding of existence and properties of solutions of stationary geometric problems, while others are part of models for physical mechanisms, for example for phase transitions or biological membranes.

Roughly summarized, the setting of extrinsic geometric flows can be summarized as follows. The word *flow* usually refers to a semi-group action. In analysis, a (local) flow is usually given implicitly through an initial value problem. The actual flow is then the mapping that relates an element x of a set—a subset of a euclidean space or some function space—to a solution of the initial value problem, given for example by an ordinary or partial differential equation, for the initial datum x. In this context, *local* means that the admissible parameter set—that commonly is referred to as the *time variable*—depends on the element in question.

This concept generalizes to the setting of two Riemannian manifolds. When (M, g) is a smooth closed and orientable Riemannian manifold of dimension $d \in \mathbb{N}$ and $(N, \bar{g})$ is another smooth Riemannian manifold of dimension $k \in \mathbb{N}$, then a geometric flow takes an initial map $\varphi_0 : M \to N$ and time t_0 and maps it to a solution $(\varphi, T) \in \{M \times [t_0, t_0 + T) \to N\} \times (0, \infty]$ of an initial value problem of the form

$$\begin{aligned} \partial_t \psi(p,t) &= F(p, t, \psi(p,t), \nabla\psi(p,t), \ldots, \nabla^m \psi(p,t)) \text{ on } M \times [t_0, t_0 + T) \text{ and} \\ \psi(p, t_0) &= \varphi_0(p) \text{ on } M, \end{aligned} \tag{1}$$

where F is a smooth function that takes for all $p \in M$ values in $T_{\psi(p)}N$, and $m \in \mathbb{N}$ is called the order of the equation. Observe that, in this context, the time of existence of the solution, if a solution exists at all, is also to be determined. If $T = \infty$, the solution is called a global solution.

This generic form of a partial differential equation also includes cases that are not well-posed in the sense of Hadamard, i.e. that an initial value problem should have a unique solution that depends continuously on the initial value. A specific class of equations that are well-behaved in this sense are parabolic equations. Note that there is a whole range of notions of parabolicity (cf. [32, Chapter 7] and [77, Chapter 6]), the archetypical example being the heat equation. The heat equation can also be seen as the flow in the direction of steepest descent, or gradient flow, of the Dirichlet energy for functions $u : \mathbb{R}^d \supset \Omega \to \mathbb{R}$ reading

$$E_D(u) = \int_\Omega |\mathrm{D}u|^2 \,\mathrm{d}x.$$

Here, the gradient of E_D has to be taken with respect to the usual L^2 scalar product. The concept of L^2-gradient flows is also central for many geometric flows. However, the scalar-product that is used to identify the gradient may vary with the immersion, much as in Riemannian geometry.

For every geometric flow—as in general for initial value problems—one is mainly interested in the following questions.

- In what setting is the problem well-posed?
- What are possible obstructions to the existence of global solutions, i.e. singularities?
- Can these singularities be characterized and possibly overcome by a different notion of solution?
- Can we determine the asymptotic behavior of global solutions as t tends to ∞?

The answer to the first and last question can be often treated by reformulating the problem to a strictly parabolic evolution equation and adaption of special techniques from their theory. We will follow this general approach for a specific geometric evolution equation also in this work, as will be discussed later. Also the second and fourth question will be treated in this thesis for a particular problem. With their introduction to the field of geometric evolutions often attributed to Hamilton, typical techniques include integral and interpolation estimates, maximum principles, and monotonicity formulae and constitute the heart of a large share of theorems in geometric analysis. The third question requires techniques, that are different from those used in this work. Overcoming of singularities without losing track of topological changes or uniqueness of solutions led to famous theorems (see e.g. [74], [51]).

A large class of flows is also geometric in the more specific sense that the evolution is only defined for immersions and is invariant under reparametrization of the manifold M. With the exception of the harmonic-map heat flow, this is the case for all above-mentioned flows. Let us now take a closer look on some specific evolution equations.

The mean-curvature flow arises for $k > d$ when we define for immersions $\varphi : M \to N$ the map F in (1) to be given by $F = \overrightarrow{H}$, where $\overrightarrow{H}$ is the mean curvature vector at $\varphi(p,t)$ of the submanifold given by $\varphi(\cdot,t)$ around $\varphi(p,t)$. It was suggested in the 1950s as a model for the motion of interfaces in soap froth by von Neumann [89], and for the motion of grain boundaries in an annealed, recrystallized metal by Mullins [71], who also formulated it as a partial differential equation for closed curves in the plane and found some self-similar and a translating solution, often called the grim reaper.

For compact surfaces the mean-curvature flow decreases the area of the given surface as fast as possible. When the manifold M is immersed into N by an immersion φ, we pull back the metric of N and obtain a surface measure $\mathrm{d}\mu_\varphi$. The area functional A is then simply

$$A(\varphi) = \int_M 1 \,\mathrm{d}\mu_\varphi.$$

Comparison with the flow of a round sphere shows that any solution with compact initial datum can only exist for finite time. Most results consider the case where $N = \mathbb{R}^{d+1}$.

For the case of the mean curvature flow of closed curves in $\mathbb{R}^2$, often referred to as the curve shortening flow, Gage, Hamilton and Grayson were able to show that any embedded curve will first become convex [42] and then asymptotically round while shrinking to a point in finite time, that is proportional to the initially enclosed area [35, 36, 38].

Independently, for convex $d \geq 2$ and d-dimensional hypersurfaces in $\mathbb{R}^{d+1}$ Huisken showed that solutions also become asymptotically round, before they disappear in a single point [48]. Relaxing the condition of convexity to mean-convexity of the initial surface, Huisken and Sinestrari [51] described a surgery procedure, that allows one to extend the flow beyond singularities, by cutting out the parts of highest curvature, which turn out to be cylindrical, while controlling the topology. This led to a topological classification result for two-convex hypersurfaces.

Of course, over a timespan of more than 30 years, a great number of authors has contributed new results and new proofs. Therefore, for further reading on the subject we refer to introductory works of Mantegazza [68] and Colding and Minicozzi [17] and the references therein.

The Willmore flow is the flow of immersions along the steepest descent of the Willmore functional W [93], that was originally introduced for immersion of a 2-dimensional surface M in $\mathbb{R}^3$. For $\varphi : M \to \mathbb{R}^3$ it is given by

$$W(\varphi) = \frac{1}{2\pi} \int_M |H|^2 \,\mathrm{d}\mu_\varphi,$$

with the mean curvature H of $\varphi(M)$. Willmore was in particular interested in the infimum of W among all possible immersions of surfaces of a fixed genus. The normalization factor is in the literature mostly chosen according to the preferences of the respective author.

A study of the actual flow was initiated by Kuwert and Schätzle [56]. Denoting the Laplace Beltrami operator, the second fundamental form and the choice of unit normal by Δ, A, ν, respectively, the evolution law reads

$$\partial_t \varphi = \frac{1}{\pi}\big(\Delta H + H|A|^2 - \frac{1}{2}H^3\big)\nu.$$

Kuwert and Schätzle gave a lower bound for the time of smooth existence of the flow in terms of the initial spatial concentration of $|A_0|^2$, where A_0 denotes the trace-free second fundamental form. In two subsequent papers [55, 57], they proved for any codimension that if M is a sphere and initially $\int_M |A_0|^2 \,\mathrm{d}\mu_\varphi < 16\pi$,then the flow exists for all times and converges to a round sphere. Moreover, they established a results on the structure of point singularities.

While Willmore was interested in topological invariants, for a regular curve γ in the plane with curvature κ, parametrized over a real interval I, the one dimensional analogue of Willmore's functional, namely the quantity

$$\int_I \kappa^2 \,\mathrm{d}s, \tag{2}$$

was already discussed by Daniel Bernoulli and Euler [31] as an important quantity while aiming to determine the shape of an elastic rod whose endpoints are fixed, but that can move freely otherwise. This question is known as the problem of the elastica and the quantity (2) is—as we will see—still an object of mathematical research. The report on the history of elasticae by Levien [61] is a nice overview and includes historical drawings of Euler and Bernoulli of astonishing precision.

The study of elastic properties of lipid bilayers [47] led Helfrich to a model for the elastic energy of biological membranes. When the shape of the membrane is represented by an embedding φ of a 2-dimensional closed manifold M to $\mathbb{R}^3$, its elastic energy is described by

$$E(\varphi) = \int_M \frac{1}{2}k_c(H - c_0)^2 + \bar{k}_c K \,\mathrm{d}\mu, \tag{3}$$

where K is the Gauß curvature of the embedded surface, k_c and $\bar{k}_c$ are the relevant curvature-elastic moduli and c_0 is called spontaneous curvature and was originally introduced to allow for chemically different sides of the bilayer. By the Gauß-Bonnet Theorem, the quantity $\int K \,\mathrm{d}\mu$ is a topological invariant and does not change, when φ varies continuously, that is in particular in one topological class. Considering the flow in direction of the steepest descent for E, one obtains an evolution law that is in highest order the same as that for of the Willmore flow. However, the spontaneous curvature breaks the scaling symmetry of the energy. For immersions of 2-dimensional manifolds this flow has been analytically studied by several authors [54, 65, 72].

The one-dimensional analogue, that is the gradient flow of (2), is often called the elastic flow and has also been subject to recent investigations. For closed curves, even in arbitrary co-dimension, the behavior of this flow is now well-understood. Dziuk, Kuwert and Schätzle [28] proved global existence and convergence (up to subsequence) to a stationary point of (2), that is an elastica.

The harmonic-map heat flow is of different nature, as it is not only depending on the shape of the immersed manifold, but also takes the particular parametrization into account. It was introduced by Eells and Sampson [29] in order to find stationary points of a generalized Dirichlet energy. For a closed, smooth Riemannian manifold (M, g) and another smooth Riemannian manifold (N, h) the energy of a smooth map $u : M \to N$ is given by

$$E(u) = \int_M |\mathrm{d}u|^2 \,\mathrm{d}\mu_g.$$

In their work Eells and Sampson provide conditions on intrinsic curvature quantities of N under which the flow converges indeed to a stationary point of E, called a harmonic map, independent of the initial datum. Therefore, they concluded that under such suitable assumptions on N, every smooth map from M to N is homotopic to a harmonic map. For an introduction to the theory of harmonic maps and their heat flow we refer to the book of Lin and Wang [63] who also provide a great many useful references.

An augmentation of the usual setting of extrinsic geometric flows is the abandonment of symmetries. Studying the evolution of immersions into a (non-flat) Riemannian manifold is a first step in this direction. Interesting questions emerge when the target manifold has non-trivial topology. For the curve-shortening flow, Grayson solved the case of curves in general 2-dimensional manifolds [43]. Huisken studied the mean-curvature flow of convex immersions of higher dimensional manifolds [49]. To find suitable surgery procedures is subject of very recent research [9, 10]. The Willmore energy of immersions into spheres was considered by White [92] showing that stationary points are preserved under stereographic projection. Lamm, Metzger and Schulze [58] and Jachan [53] considered the Willmore energy and flow in manifolds that are asymptotically Schwarzschild.

A further generalization is the concept of anisotropic flows. The Russian material scientist Wulff [94] had, already at the beginning of the 20th century, evidence that the energy of the boundary of a crystalline material depends on the direction of the surface normal to the lattice structure of the crystal. He also gave a method how to determine the shape of minimal energy for a given direction dependent energy density.

Models for interfaces often incorporate directional dependence of the surface energy density. Such energies can be described by a positive, 1-homogenous function $\eta : \mathbb{R}^n \to \mathbb{R}$. For a smooth immersion $\varphi : M \to \mathbb{R}^{d+1}$ with a unit normal ν the energy is then defined as

$$E(\varphi) = \int_M \eta(\nu) \,\mathrm{d}\mu_\varphi.$$

Analogously to the mean curvature flow, the gradient flow of anisotropic energies has been subject of mathematical research since the 1970s. Taylor [82] distinguished between the case of smooth and crystalline surface energies. The latter ones, arising from non convex anisotropies η, lead to polyhedral minimizing surfaces. In the case that η is strictly convex and smooth, also the minimizing shape, often referred to as the Wulff shape, is convex and smooth.

Ambient vector fields in $\mathbb{R}^{d+1}$ were taken into account by Wheeler [91]. For $d \in \mathbb{N}$, curves $\gamma : \mathbb{S}^1 \to \mathbb{R}^{d+1}$, a vector field $\mathfrak{c} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ and a function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ he generalized the Helfrich energy (3) for curves by establishing the spontaneous curvature

$$c_0 = \mathfrak{c} \circ \gamma + (f \circ \gamma)\tau,$$

where τ is the unit tangent to γ. After a short discussion of general properties of this energy, he proves that under suitable assumptions on $\mathfrak{c}$ and f the gradient flow equation has a global solution for all initial curves and that there is a sequence of times $t_j \to \infty$ such that the curves $\gamma(t_j)$ converge to a stationary point of the energy after suitable translation and reparametrization.

In Chapter 2 of this work we will discuss, how the idea of an ambient vector field can be used to construct an anisotropic and homogeneous energy for curves in the plane, and review existing literature to ascertain existence of global solutions and their asymptotic behavior.

A different generalization of the Helfrich energy (3) is proposed by Bartels, Dolzmann, Nochetto and Raisch [7]. They trace back the spontaneous curvature term to the local orientation of the rod-shaped lipid molecules. This is modeled for two-dimensional membranes by the introduction of a vector field $n : M \to \mathbb{R}^3$. With a physical constant $\delta \in \mathbb{R}$ the quantity

$$\delta \operatorname{div} n$$

is then interpreted as the spontaneous curvature. Here div is the divergence with respect to the metric on M that is induced by the immersion $\varphi : M \to \mathbb{R}^3$ that represents the membrane. This situation is depicted in Figure 1 for a curve and vector field in $\mathbb{R}^2$. The energy for the whole system also takes into account molecular forces between the lipid molecules and is given by

$$E(\varphi, n) = \frac{1}{2} \int_M (\operatorname{div}_\varphi \nu_\varphi - \delta \operatorname{div}_\varphi n)^2 \, \mathrm{d}\mu_\varphi + \frac{\lambda}{2} \int_M |\nabla_\varphi n|^2 \, \mathrm{d}\mu_\varphi, \tag{4}$$

with another physical constant $\lambda > 0$. In their work, the L^2-gradient flow of this energy is derived which leads to a system of coupled partial differential equations. Moreover, they derive a parametric finite elements scheme and present some numerical experiments.

The equation for the evolution of the immersion shares some structure with the Willmore flow. Indeed, when $\delta = \lambda = 0$ the flows coincide. But for λ positive while $\delta = 0$ the equations are coupled.

The evolution of the vector field n is governed by a law that shares its leading term with the harmonic map heat flow. This relation is of additional importance, when a supplementary length condition for n is imposed. Since φ represents a biological membrane, it is reasonable to assume that the area of the represented surface and its enclosed volume are constant during the evolution.

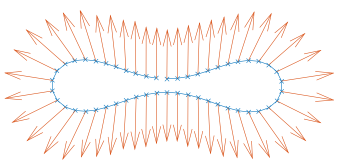

Figure 1: A plane curve with an attached vector field.

The main topic of this thesis is the evaluation of the analytical properties of this generalized Helfrich energy in combinations with the above-mentioned constraints

$$\begin{aligned} &\text{Fixed signed volume enclosed by } \varphi(M): && \int_M \varphi \cdot \nu_\varphi \, \mathrm{d}\mu_\varphi = V_0, \\ &\text{Fixed surface area of } \varphi(M): && \int_M 1 \, \mathrm{d}\mu_\varphi = A_0, \\ &\text{Length constraint for } n: && \forall_{p \in M} \;\; \|n(p)\|_{\mathbb{R}^{d+1}} = 1, \end{aligned} \tag{5}$$

for suitable choices of A_0 and V_0 especially not violating the isoperimetric inequality.

Considering (4) for curves and vector fields with values in $\mathbb{R}^2$, inspired by arguments of [19,20], we show existence and smoothness of minimizers of the energy (4) when a length penalization term is added using variational techniques and elliptic regularity theory.

Theorem 1
For $\delta \geq 0$, $\lambda > 0$ the energy E given by

$$E(\gamma, n) = \int_\gamma \frac{1}{2}(\kappa + \delta \operatorname{div}(n))^2 + \frac{\lambda}{2}|\partial_s n|^2 + 1 \, \mathrm{d}s$$

has a smooth global minimizer. Furthermore, a global minimizer exists for any combination of the constraints in (5) *imposed.*

For a closed orientable manifold of dimension d, we consider the gradient flow of the energy (4) as derived by [7] for an immersion and a vector field with values in $\mathbb{R}^{d+1}$. Due to the strong coupling of the evolution equations and due to the fact that one of them is of fourth, the other of second order, even the well-posedness of the evolution equation is not covered by standard theory. We use energy methods to solve the corresponding linearized problem and employ a typical parabolic approach to obtain a short-time existence result for initial data in a Sobolev space of sufficient regularity.

Theorem 2
For $d \in \mathbb{N}$ let M be a closed d-dimensional smooth orientable manifold. Let $k > d/2 + 3$ be a natural number, $\varphi_0 \in H^k(M, \mathbb{R}^{d+1})$ be an immersion and $n_0 \in H^k(M, \mathbb{R}^{d+1})$ be a vector field with $\|n_0\|_{\mathbb{R}^{d+1}} \equiv 1$.

Then, there is a $T > 0$, such that the area and volume preserving gradient-flow equation of the energy (4) *has a unique solution (φ, n) on $M \times [0, T)$ and $\|n\|_{\mathbb{R}^{d+1}} \equiv 1$. Moreover, for positive times the surface and the vector field are smooth in space and time.*

The result remains valid, if we impose at most two of the three constraints in (5)*. If the unit-length constraint is disregarded, the condition on n_0 is obsolete.*

For the questions concerning the development of singularities and the asymptotic properties of the flow we provide first partial answers. In view of the very tame behavior of the elastic flow and the harmonic map heat flow for curves, that both admit global solutions and converge to stationary points, one might expect that a similar result can be established for the coupled flow. However, the coupling terms make the situation rather involved. Using interpolation and Sobolev inequalities we find a geometric quantity that cannot remain bounded, when a singularity occurs.

Theorem 3
Let $(\gamma, n) : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2 \times \mathbb{R}^2$ *be a smooth gradient flow of the energy* (4)*, that cannot be smoothly extended beyond* T*. Then, for* $z = \kappa + \delta \operatorname{div}(n)$ *it holds that*

$$\lim_{t \to T} \int_{S^1} |\partial_s z|^2 + |\partial_s^2 n|^2 \, \mathrm{d}s = \infty.$$

The Łojasiewicz-Simon inequality is a rather resilient tool for the analysis of global solutions of evolution equations. In the 1960s, Łojasiewicz proved the following result [66, Theorème 4; Proposition 1, p. 92].

Theorem 4 (Łojasiewicz gradient inequality)
Let $U \subset \mathbb{R}^d$ *be open,* $f : U \to \mathbb{R}$ *be real analytic, and let* $a \in U$*: Then there exist constants* $\theta \in (0, 1/2], c, \sigma > 0$ *such that for every* $z \in U$*,* $\|z - a\| \leq \sigma$

$$|f(a) - f(z)|^{1-\theta} \leq c \|\nabla f(z)\|.$$

Note that the stated inequality holds in particular when $\nabla f(a) = 0$. It can be used to guarantee convergence to a local minimizer x^* for solutions of gradient flow equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} u = \nabla E(u)$$

for analytic energies $E : \mathbb{R}^d \to \mathbb{R}$ by considering $\frac{\mathrm{d}}{\mathrm{d}t} |E(u(t)) - E(x^*)|^{\theta}$ [67].

Simon [80] was able to extend this result to energies defined on Hilbert spaces and obtained a convergence result for a class of parabolic or hyperbolic evolutions based on the gradient of such energies. Thus, the result in the infinite dimensional setting is often referred to as the Łojasiewicz-Simon inequality. Simon's result however, cannot directly be applied to energies involving curvature, since it assumes that the energy only depends on first derivatives with main applications being the harmonic map heat flow and area minimizing submanifolds.

Only recently, different authors either contributed to the development of an abstract functional analytic setting, in which Łojasiewicz-Simon-type gradient inequalities can be derived, or applied this framework to geometric evolution equations to prove convergence of global solutions to stationary points.

Wheeler [91] gave an argument how to adapt Simon's methods to Helfrich-type energies and obtained full convergence for global solutions of a Helfrich flow with externally given spontaneous curvature. Chill [15] and Feehan and Maridakis [33] formalized the problem further to energies on Banach spaces and showed that the analyticity assumption on the energy can be relaxed. It suffices that it is analytic on a critical manifold that is finite dimensional in applications. Formally, the resulting estimate looks very similar to the original estimate of Łojasiewicz, however it is a delicate question in what norm the gradient has to be measured in the Banach space setting.

Chill, Fasangova and Schätzle used this abstract framework to prove that Willmore blow-ups are never compact [16]. For elastic curves in $\mathbb{R}^d$ subject to different boundary conditions Lin [62] proved global existence of solutions and convergence up to translation of a subsequence to a minimizer. Here, Dall'Acqua, Pozzi and Spener [22] were able to obtain smooth convergence of the whole flow by means of Chill's result on the Łojasiewicz-Simon inequality. For a Helfrich-type model for 2-dimensional surfaces in $\mathbb{R}^3$, Lengeler [60] proved stability of local minimizers and global existence for solutions starting close by.

We discuss in Chapter 5 how constraints can be incorporated in this setting and use the results of Feehan and Maridakis to obtain a Łojasiewicz-Simon inequality for the generalized Helfrich energy E from (4) and infer stability of local minimizers of E.

Theorem 5
For $d \in \mathbb{N}$ let M be an orientable, d-dimensional, smooth, closed manifold. For dimension $d = 1$ we set $k = 3$, else let $k > d/2 + 3$ be an integer, and let $(\varphi^*, n^*) \in C^\infty(M, \mathbb{R}^{d+1}) \times C^\infty(M, \mathbb{R}^{d+1})$ be a smooth local minimizer of the energy (4) with respect to any combination of constraints (5). Then there exists $\varepsilon > 0$ such that for all initial data $(\varphi_0, n_0) \in H^k(m, \mathbb{R}^{d+1}) \times H^k(m, \mathbb{R}^{d+1})$ with

$$\|(\varphi_0 - \varphi^*, n_0 - n^*)\|_{H^k(M,\mathbb{R}^{d+1}) \times H^{k-1}(M,\mathbb{R}^{d+1})} < \varepsilon$$

the gradient flow has a global solution $(\varphi, n) : M \times [0, \infty) \to \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$, smooth away from time 0, that converges smoothly to a possibly different local minimizer $(\tilde{\varphi}, \tilde{n})$ as $t \to \infty$ and $E(\varphi^*, n^*) = E(\tilde{\varphi}, \tilde{n})$.

To conclude this introduction to geometric evolution equations and the topic of this thesis, we want to comment briefly on some other aspects of modern research in geometric evolution equations, that however are beyond the scope of this work.

Weak notions and approximate solutions for geometric flows include approaches based on techniques as level-set methods, varifolds, viscosity solutions and phase field approximations. The advantage of weak formulations lies in the easier treatment of singularities. The main challenge consists in characterization of situations, where regularity of solutions can be recovered and non-uniqueness avoided. Also one possible approach to numerical treatment of geometric evolutions uses these weak notions of solutions. For hints on literature concerning weak formulations we refer to the introductions of [68] and [40].

However, **developing numerical methods for geometric flows** is a very subtle and challenging problem on its own. First efforts have been made e.g. by Dziuk [27]. Numerical approaches that share the view on geometric evolutions of this work, are called parametric methods, as they parametrize the evolving surface in each time step. One challenging question is then to find an efficient way to redistribute the mesh points on the surface to avoid mesh degeneration. Different ideas in this directions have been tested, among many other problems the Willmore flow was numerically treated by Barrett, Garcke and Nürnberg [6]. A Recent approach by Elliot and Fritz [30] used a trick, originally due to DeTurck [25], regularizing the mesh with help of the harmonic map heat flow.
We mention this idea in particular, since it was also used in the numerical experiments presented in Appendix of this work. To get a better feeling for the behavior of different geometric flows, we used a very straight forward finite differences scheme and Matlab's routines for ordinary differential equations to simulate the flow of curves for different evolution equations. Since mesh degeneracy becomes a problem almost immediately, we tried to account for it by the coupling to the harmonic maps heat flow.

Problems with boundary and geometric flows of networks are already present in the works of Mullins and von Neumann [71, 89] and also an import component in Euler's [31] study of elasticae. Strict analytical treatment of this class of problem is a main subject of a large number of recent research projects and the analogous of long-established results for closed manifolds are still wide open in the case of manifolds with boundaries and networks. First analytical results were obtained by Bronsard and Reitich [13] and Mantegazza, Novaga and Tortorelli [69].

Organization of this Work. In the first chapter we fix some notation and introduce important mathematical concepts that appear at different points in the later chapters.

In the second chapter, we explore second-order flows that are derived from weighted surface area functionals and couple to a vector field. This will lead to a discussion of results concerning smooth anisotropic and inhomogeneous curvature flows.

In Chapter 3 we discuss important properties of the generalized Helfrich energy (3). We prove Theorem 1 and recall the gradient flow equation.

The fourth chapter is dedicated to the proof the short-time existence result as stated in Theorem 2. We start by considering the linearized equation and prove estimates for the non-linear remainder term. Together these considerations suffice to prove a first local well-posedness result. Afterwards we discuss the situation in the presence of constraints. Lastly, we employ parabolic techniques to obtain smoothness of solutions away from the initial data.

The discussion of long-time behavior of solutions in the fifth chapter is split in two parts. For curves we prove the blow-up result as stated in Theorem 3. In the general situation of the short-time existence result, we deduce stability of local minimizers as stated in Theorem 5 by the use of the Łojasiewicz-Simon inequality.

In the Appendix, we present some numerical experiments.

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1

Notation and Prerequisites

In this chapter, we will introduce notation and state important definitions and theorems from geometry and analysis. Moreover, we will derive some results already tailored to the needs of later chapters, the proofs being mostly adaptions of related results from the literature.

1.1 Analysis

We adapt the usual notations of (functional) analysis. When we consider a normed space E, we denote the corresponding norm by $\|\cdot\|_E$. If E, F are Banach space, we denote the space of linear maps from E to F by $L(E;F) := \{A : E \to F \text{ linear and continuous}\}$ with the usual operator norm, turning it into a Banach space as well. The dual space $L(E, \mathbb{R})$ is usually denoted by E^*. For $x \in E$ and $\varphi \in E^*$ we use the notations $\varphi(x)$ and $\langle \varphi, x \rangle_{E^* \times E}$ to denote the application of the linear map φ to the element x. For an operator $A \in L(E, F)$ we denote the range and kernel of A by $R(A)$ and $\ker(A)$, respectively. Moreover, for $A \in L(E, F)$ we introduce the adjoint operator $A^* \in L(F^*, E^*)$ given by $\varphi \mapsto \varphi \circ A$.

If a map $G : E \to F$ has a Fréchet derivative it will be denoted by $\mathrm{d}G$ or simply G'. For $d, k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ open, we denote the Jacobian of a continuously differentiable function $f : \Omega \to \mathbb{R}^k$ by $\mathrm{D}f$.

For a sequence x_n in E, we denote strong and weak convergence by $\to$ and $\rightharpoonup$, respectively, and if H is a Hilbert space, we denote the scalar product by $\langle \cdot, \cdot \rangle_H$.

1.2 Basic Properties of Riemannian Manifolds and Hypersurfaces

Let M be a d-dimensional, orientable, smooth, closed manifold, that is compact without boundary. As all manifolds in this work will be compact, we only consider connected manifolds, because otherwise, we would do everything for all connected components separately. We denote its tangent bundle by TM and for $p \in M$ we write T_pM and T_p^*M for the tangent and cotangent space at p. For a vector bundle E over M we will denote the smooth sections of this bundle by $\Gamma(E)$. For open sets $V \subset M$ and $U \subset \mathbb{R}^d$ and a chart $x : V \to U$ we denote the canonical basis vectors of T_pM as ∂_{x_i} and that of T_p^*M by $\mathrm{d}x^i$. For $k \in \mathbb{N}$ and a smooth function $f : M \to \mathbb{R}^k$, we write $\frac{\partial f}{\partial x_i}$ or shorter $\partial_i f$ to denote the derivatives with respect to the chart x. That is,

$$\frac{\partial}{\partial x_i} f(p) = \partial_i f(p) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(x^{-1}(x(p) + te_i))$$

and we write $\mathrm{d}f = f_i \mathrm{d}x^i$ to denote the differential of f.

If g is a Riemannian metric on M, i.e. (M, g) is a Riemannian manifold, we denote the volume element, gradient, divergence and Laplace-Beltrami operator as $\mathrm{d}\mu_g, \nabla_g, \mathrm{div}_g, \Delta_g$. In particular we have for $1 \leq p < \infty$ the Banach spaces

$$L^p(M) = \left\{ f : M \to \mathbb{R} \,\middle|\, f \text{ measurable and } \|f\|_{L^p(M)} := \left(\int_M |f|^p \mathrm{d}\mu_g \right)^{1/p} < \infty \right\}.$$

The Levi-Civita covariant derivative will be denoted by ∇ and for $1 \leq i \leq n$ we write ∇_i to mean $\nabla_{\partial_{x_i}}$ and for $k \in \mathbb{N}$ we denote the k-th covariant derivative by ∇^k. The Christoffel symbols are denoted by Γ_{ij}^r and they are defined by $\nabla_i X_j = \Gamma_i j^r X_r$.

For $k, r, s \in \mathbb{N}$ and an (r, s)-tensor field T the quantity $\nabla^k T$ is, without further specification of vector fields with respect to which the covariant derivative is taken, an $(r, s + k)$-tensor. The metric induces a scalar product and norm also for tensors by

$$\langle T, S \rangle = g^{i_1 k_1} \dots g^{i_r k_r} g_{j_1 \ell_1} \dots g_{j_s \ell_s} T^{i_1 \dots i_r}_{j_1 \dots j_s} S^{k_1 \dots k_r}_{\ell_1 \dots \ell_s}$$

and $|T| = \sqrt{\langle T, T \rangle}$ for (r, s)-tensors T and S. With this we can explain the L^p-norm also for a section X of a tensor bundle E over M by

$$\|X\|_{L^p(M)} := \left(\int_M |X|^p \mathrm{d}\mu_g \right)^{1/p} < \infty$$

as well as the L^2 scalar product.

For two vectors $u, v \in \mathbb{R}^{d+1}$, we will use the notation

$$u \otimes v := uv^T.$$

Therefore, $u \otimes v$ is a matrix s with entries $s_{ij} = u_i v_j$ and corresponds to the linear map $\mathbb{R}^{d+1} \to \mathbb{R}^{d+1} : x \mapsto u \langle v, x \rangle_{\mathbb{R}^{d+1}}$.

We will often consider the case that M is immersed to $\mathbb{R}^{d+1}$ by $\varphi : M \to \mathbb{R}^{d+1}$. In this case the tangent space to $\varphi(M)$ at a point $\varphi(p) \in \mathbb{R}^{d+1}$ is given by the image of the differential $T_{\varphi(p)}\varphi(M) = \mathrm{d}\varphi(T_pM)$ and we write

$$X_i := \partial_i \varphi$$

for the induced basis vectors and after choosing an orientation we write $\nu : M \to \mathbb{S}^d$ for the positively oriented unit normal to $\varphi(M)$. We can pull back the metric on $\mathbb{R}^{d+1}$ via φ by setting

$$g_{ij} = \langle X_i, X_j \rangle_{\mathbb{R}^{d+1}}.$$

That is, we consider the metric given by the pullback of the metric on $\mathbb{R}^{d+1}$ via φ. The pullback is usually denoted by a $*$ symbol, hence we might write $g = \varphi^* \langle \cdot, \cdot \rangle_{\mathbb{R}^{d+1}}$. In this case, we will also use the notation $\mathrm{d}\mu_\varphi, \nabla_\varphi, \mathrm{div}_\varphi, \Delta_\varphi$ to mean the quantities related to the metric induced by the immersion. For immersions, we will consider the signed area functional A and the signed volume functional V at various points of this work. They are given by

$$A(\varphi) = \int_M \mathrm{d}\mu_\varphi \qquad \text{and} \qquad V(\varphi) = \int_M \langle \varphi, \nu_\varphi \rangle \, \mathrm{d}\mu_\varphi.$$

Since $T_{\varphi(p)}\varphi(M)$ and $T_{\nu(p)}\mathbb{S}^d$ consist of all vectors $v \in \mathbb{R}^{d+1}$such that $v \perp \nu(p)$ and since $T_{\varphi(p)}\varphi(M) \cong T_pM$, we can identify these spaces. The second fundamental form is the map

$$A : T_pM \times T_pM \to \mathbb{R},$$

that is given in local coordinates by

$$h_{ij} = \langle \frac{\partial^2 \varphi}{\partial_i \partial_j}, \nu \rangle_{\mathbb{R}^{d+1}} = -\langle \partial_i \nu, X_j \rangle_{\mathbb{R}^{d+1}} = -g(d\nu X_i, X_j).$$

Working with Riemannian manifolds, we will in general use the Einstein summation convention. For example a tangential vector field $v \in \Gamma(TM)$ will be denoted in local coordinates as $v = v^i \partial_{x_i}$.Finally we introduce the mean curvature

$$H = g^{ij} h_{ij}.$$

When M is immersed to $\mathbb{R}^{d+1}$ via φ and $v : M \to \mathbb{R}^{d+1}$ is a smooth vector valued map, we define $\nabla_\varphi v$ and $\Delta_\varphi v$ component-wise as

$$\nabla_\varphi v = \sum_{\alpha=1}^{d+1} e_\alpha \otimes \nabla_\varphi v^\alpha \quad \text{and} \quad \Delta_\varphi v = \sum_{\alpha=1}^{d+1} e_\alpha \otimes \Delta_\varphi v^\alpha \quad .$$

The tangential divergence is defined as usual by

$$\operatorname{div}_\varphi v = g^{ij} \langle \partial_i v, X_j \rangle$$

yielding the important identity $H = -\operatorname{div}_\varphi \nu$.

1.3 Plane Curves

Following the conventions in related literature, we will use different notation when we discuss curves. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth map. We identify $\mathbb{S}^1$ with $[0, 1]$ if convenient. The parameter in $[0, 1]$ will mostly be denoted by x and we write $\dot{\gamma}(x) := \frac{\mathrm{d}}{\mathrm{d}x}\gamma(x)$. The length of a differentiable curve is given by $L(\gamma) = \int_0^1 |\dot{\gamma}(x)| \mathrm{d}x$.

Definition 1.1 (Regular curve)
The map γ is called a *regular curve*, if for all $x \in [0, 1]$ we have $\dot{\gamma}(x) \neq 0$.

In this case we denote the unit tangent vector by $\tau := \frac{\dot{\gamma}}{\|\dot{\gamma}\|}$ and the unit normal by ν, where we choose the direction of ν so that $(\tau, \ \nu)$ has positive orientation. For plane curves, we denote the curvature by κ.

If γ is regular, it is possible to reparametrize γ by arc length. That is, there exists a strictly increasing, bijective, and smooth map $\alpha : [0, 1] \to [0, L]$ such that $\|\frac{\mathrm{d}}{\mathrm{d}s}\gamma(\alpha^{-1}(s))\| \equiv 1$. We call s the arc length parameter. For a function $f : \mathbb{S}^1 \to \mathbb{R}$ smoothness means smooth in charts. Moreover, for $y \in \mathbb{S}^1$ we define the arc length derivative as

$$\frac{\mathrm{d}}{\mathrm{d}s} f \bigg|_y := \frac{1}{\|\dot{\gamma}\|} \frac{\mathrm{d}}{\mathrm{d}x} f \bigg|_y$$

and observe

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} f = \frac{1}{\|\dot{\gamma}(x)\|} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{\|\dot{\gamma}(x)\|} \frac{\mathrm{d}}{\mathrm{d}x} f(y) \right) = \frac{1}{\|\dot{\gamma}(x)\|^2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f - \frac{\langle \dot{\gamma}, \ddot{\gamma} \rangle}{\|\dot{\gamma}(x)\|^4} \frac{\mathrm{d}}{\mathrm{d}x} f. \tag{1.1}$$

In this notation $\tau = \frac{\mathrm{d}}{\mathrm{d}s}\gamma$ and

$$\kappa = \langle \frac{\mathrm{d}}{\mathrm{d}s}\tau, \nu \rangle = \langle \frac{\mathrm{d}^2}{\mathrm{d}s^2}\gamma, \nu \rangle.$$

The curvature κ, τ and ν are related by the Frenet equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\tau = \kappa\nu, \quad \frac{\mathrm{d}}{\mathrm{d}s}\nu = -\kappa\tau.$$

In the manifold notation from Section 1.2 we have

$$\begin{aligned} X =& \frac{\mathrm{d}}{\mathrm{d}x}\gamma = \|\dot{\gamma}\|\tau, \\ g =& \langle X, X \rangle = \langle \frac{\mathrm{d}}{\mathrm{d}x}\gamma, \frac{\mathrm{d}}{\mathrm{d}x}\gamma \rangle = \|\dot{\gamma}\|^2, \\ \nabla_g f =& g^{-1} \frac{\mathrm{d}}{\mathrm{d}x} f X = (\frac{\mathrm{d}}{\mathrm{d}s} f)\tau =: \nabla_s f. \end{aligned}$$

If $v : \mathbb{S}^1 \to \mathbb{R}^2$is a vector field, the divergence is given by

$$\operatorname{div}(v) = \langle \frac{\mathrm{d}}{\mathrm{d}s} v, \tau \rangle.$$

Applying this to the tangential gradient of f we find

$$\operatorname{div}(\nabla_s f) = \langle \frac{\mathrm{d}}{\mathrm{d}s} \left[\left(\frac{\mathrm{d}}{\mathrm{d}s} f \right) \tau \right], \tau \rangle = \frac{\mathrm{d}^2}{\mathrm{d}s^2} f.$$

1.4 Function Spaces on Manifolds

Theory of Sobolev spaces on open domains in euclidean space is explained in the books by Evans [32] or Adams [1] and the most important results carry over to the manifold setting that is e.g. treated in the Books by Aubin [5], Hebey [46] and Taylor [83–85]. For the following definitions and theorems we follow Hebey [46, Chapter 2].

Definition 1.2 (Sobolev and Hölder Spaces)
Let (M, g) be an orientable smooth closed Riemannian manifold of dimension $d \in \mathbb{N}$. For $1 \leq p < \infty$, $k \in \mathbb{N}$ and $f \in C^\infty(M)$ we set

$$\|f\|_{W^{k,p}(M)} = \sum_{j=0}^{k} \|\nabla^j f\|_{L^p(M)}.$$

The spaces $W^{k,p}$ are defined as closures of $C^\infty(M)$ with respect to these norms.

We write $H^k(M)$ to denote the Hilbert spaces $W^{k,2}(M)$ with

$$\langle f_1, f_2 \rangle_{H^k(M)} = \sum_{j=0}^{k} \langle \nabla^j f_1, \nabla^j f_2 \rangle_{L^2(M)}.$$

Let $d_g : M \times M \to \mathbb{R}$ denote the geodesic distance on (M, g). For $f \in C^0(M)$, $k \in \mathbb{N}$and $\alpha \in (0, 1)$ we set

$$\|f\|_{C^{k,\alpha}(M)} = \sum_{j=0}^{k} \sup_{p \in M} |\nabla^j f(p)| + \sup_{r \neq s \in M} \frac{|\nabla^j f(r) - \nabla^j f(s)|}{d_g(r, s)^\alpha}$$

and

$$C^{k,\alpha}(M) = \left\{ f : M \to \mathbb{R} \text{ continuous } \middle| \|f\|_{C^{k,\alpha}(M)} < \infty \right\}$$

For the definition of fractional Sobolev spaces on manifolds, i.e. Besov and Bessel-potential spaces, we refer to the work of Triebel [87, 88]. In the subsequent chapters we will also use the embedding theorems of Sobolev and Morrey and the Rellich-Kondrakov theorem.

Theorem 1.3 (Theorems 2.6, 2.7, 2.8, and 2.9 from [46])
Let (M, g) be a smooth closed Riemannian manifold of dimension $d \in \mathbb{N}$. Let $k, \ell \in \mathbb{N} \cup \{0\}$ with $k \geq \ell$, $1 \leq p, q < \infty$ and $\alpha \in (0, 1)$ be fixed. If $k - \frac{d}{p} \geq \ell - \frac{d}{q}$, then

$$W^{k,p}(M) \hookrightarrow W^{\ell,q}(M)$$

and if $k - \frac{d}{p} \geq \ell + \alpha$, then

$$W^{k,p}(M) \hookrightarrow C^{\ell,\alpha}(M).$$

These embeddings are compact, if $k - \frac{d}{p} > \ell - \frac{d}{q}$ or $k - \frac{d}{p} > \ell + \alpha$, respectively.

Since we assume M to have no boundary, we state the mean value version of Poincaré's inequality.

Theorem 1.4 (Theorem 2.10 from [46])
Let (M, g) be a smooth closed Riemannian manifold of dimension $d \in \mathbb{N}$ and fix $1 < p < \infty$. Then there is $C > 0$ such that for all $f \in W^{1,p}(M)$ setting $\bar{f} := \frac{1}{Vol(M,g)} \int f \mathrm{d}\mu_g$ it holds that

$$\left(\int_M |f - \bar{f}|^p \mathrm{d}\mu_g \right)^{1/p} \leq C \left(\int_M |\nabla f|^p \mathrm{d}\mu_g \right)^{1/p}.$$

We will often consider immersions of smooth manifolds. For them we introduce the following notation.

Definition 1.5
Let M be a smooth, closed and orientable manifold of dimension $d \in \mathbb{N}$, then the set of immersions is open in $C^1(M, \mathbb{R}^{d+1})$,we denote it by

$$C^1_{\mathrm{imm}}(M, \mathbb{R}^{d+1}) = \left\{ \varphi \in C^1(M, \mathbb{R}^{d+1}) \middle| \forall p \in M \text{ the map } \mathrm{d}\varphi(p) : T_pM \to \mathbb{R}^{d+1} \text{ is injective} \right\}.$$

Moreover, we have for $2 \leq k \in \mathbb{N}$ the Banach space

$$C^1(M, \mathbb{R}^{d+1}) \cap H^k(M, \mathbb{R}^{d+1})$$

which we equip with the usual norm $\| \cdot \|_{C^1 \cap H^k} = \| \cdot \|_{C^1} + \| \cdot \|_{H^k}$. Then, the set

$$H^k_{\mathrm{imm}}(M, \mathbb{R}^{d+1}) = H^k(M, \mathbb{R}^{d+1}) \cap C^1_{\mathrm{imm}}(M, \mathbb{R}^{d+1}).$$

is open in $C^1 \cap H^k$.

1.5 Moving Hypersurfaces

For studying the evolution of hypersurfaces, we will follow the usual approach as explained by Huisken, Polden [50, 76] or Mantegazza [68]. In the following, let M be a d-dimensional, smooth, closed, orientable manifold and $\varphi : M \to \mathbb{R}^{d+1}$ a smooth immersion. We equip M with a Riemannian metric by pulling back the metric of $\mathbb{R}^{d+1}$ along φ as in Section 1.2. Thus, a smooth family φ_t of immersions yields a smooth family of Riemannian manifolds (M, g_t). In the following we will discuss some formulas for the evolution of geometric quantities of (M, g_t) as φ_t varies in time. These formulas are well established in literature. Huisken and Polden [50] consider the more general case where φ is an immersion to a higher dimensional manifold N rather than $\mathbb{R}^{d+1}$, Mantegazza [68] considers essentially the situation described here and Kuwert and Schätzle [55, 56] derive their formulas in the context of immersions in euclidean space but arbitrary codimension.

Definition 1.6
Let φ_t be a smooth family of immersions and let $\Psi_t := \partial_t \varphi_t$ denote its time derivative. We can decompose Ψ_t in its normal and tangential parts as

$$\Psi_t = \Psi_t^N \nu_t + \Psi_t^k X_{k,t}.$$

We say that φ_t is a normal family of immersions if and only if the tangential part of Ψ_t vanishes, i.e. for all $k = 1, ..., d$ we have $\Psi_t^k \equiv 0$.

The next lemma is a slight modification of a well known result, we follow the proof of Mantegazza [68, Proposition 1.3.4].

Lemma 1.7
For $T_1, T_2 \in \mathbb{R}$ let $(\varphi_t)_{t\in[T_1,T_2]}$ be a smooth family of immersions and let $\Psi_t := \partial_t \varphi_t$ denote its time derivative. Then, for every $s \in [T_1, T_2]$ there exists a family $F_{t,s} : M \to M$ of diffeomorphisms such that $F_{s,s} = \mathrm{Id}_M$ and $\partial_t(\varphi_t \circ F_{t,s})$ is normal along $\varphi_t(M)$, that is there exists a family of functions $V_{t,s} : M \to \mathbb{R}$ such that $\partial_t(\varphi_t \circ F_{t,s}) = V_{t,s}\nu_t \circ F_{t,s})$.

Proof. We split Ψ_t in its normal and tangential part setting $\Psi_t = \Psi_t^N \nu_t + \Psi_t^k X_{k,t}$. Hence, $\Psi_t^k X_{k,t}$ is a tangential vector field along $\varphi_t(M)$. Since φ_t are immersions, $\mathrm{d}\varphi_t$ is invertible on its image and therefore, we can define a smooth family of vector fields on M by

$$Y_t = -[\mathrm{d}\varphi_t]^{-1} \Psi_t^k X_{k,t}.$$

Since M is compact, by the theory of ordinary differential equations [59, Theorem 9.12], for every $s \in [T_1, T_2]$ we find a smooth family of maps $F_{t,s} : M \to M$ satisfying

$$\partial_t F_{t,s}(p) = Y_t(F(p)), \qquad F_{s,s} = \mathrm{Id}_M .$$

By uniqueness of solutions and smooth dependence on initial data for each $t \in [T_1, T_2]$ the map $F_{t,s}$ is a smooth bijection. That it is indeed a diffeomorphism follows from the following consideration analogous to the use of the Wronski determinant in the theory of ordinary differential equations. We apply the rule for differentiation of determinants, that reads for $A(t)$ an invertible, time dependent matrix

$$\frac{\mathrm{d}}{\mathrm{d}t} \det(A(t)) = \det(A) \operatorname{tr}(A^{-1} \partial_t A(t)).$$

Thus, using $F_{s,s} = \mathrm{Id}_M$ and therefore $\det(\mathrm{d}F_{s,s}) = 1$ we can calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\det(\mathrm{d}F_{t,s}) &= \mathrm{tr}\left(\mathrm{d}F_{t,s}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}F_{t,s}\right)\det(\mathrm{d}F_{t,s})\\ &= \mathrm{tr}\left(\mathrm{d}F_{t,s}^{-1}\mathrm{d}\big(\frac{\mathrm{d}}{\mathrm{d}t}F_{t,s}\big)\right)\det(\mathrm{d}F_{t,s})\\ &= \mathrm{tr}\left(\mathrm{d}F_{t,s}^{-1}\mathrm{d}\big(Y(F_{t,s},t)\big)\right)\det(\mathrm{d}F_{t,s})\\ &= \mathrm{tr}\left(\mathrm{d}F_{t,s}^{-1}\big(\mathrm{d}Y(F_{t,s},t)\mathrm{d}F_{t,s}\big)\right)\det(\mathrm{d}F_{t,s})\\ &= \mathrm{tr}\left(\mathrm{d}Y(F_{t,s},t)\right)\det(\mathrm{d}F_{t,s})\end{aligned}$$

and we conclude that

$$\det(\mathrm{d}F_{t,s}) = \det(\mathrm{Id})\exp\left(\int_s^t \mathrm{tr}\left(\mathrm{d}Y(F_{\tau,s},\tau)\right)\mathrm{d}\tau\right) > 0.$$

Moreover, we see that

$$\begin{aligned}\partial_t(\varphi_t(F_{t,s})) =&\Psi_t(F_{t,s}) + \mathrm{d}\varphi_t(F_{t,s})\partial_t F_{t,s}\\ =&\Psi_t(F_{t,s}) - \mathrm{d}\varphi_t(F_{t,s})[\mathrm{d}\varphi_t(F_{t,s})]^{-1}\Psi_t^k(F_{t,s})X_{k,t}(F_{t,s})\\ =&\Psi_t^N(F_{t,s})\nu_t(F_{t,s}),\end{aligned}$$

that is $\varphi_t \circ F_{t,s}$ is normal along $\varphi_t(M)$. □

Definition 1.8
For $T_1, T_2 \in \mathbb{R}$ and a smooth family of smooth immersions $(\varphi_t)_{t\in[T_1,T_2]}$, we call

$$\mathcal{T} := \bigcup_{(p,t)\in M\times[T_1,T_2]} \{p\}\times\{\varphi_t(p)\}\times\{t\}$$

the *trajectory* of φ_t. We say a function is a *material* function if its domain is $M \times [T_1, T_2]$ and *spatial* if its domain is $\mathcal{T}$.

Observe that with this definition the notion of a spatial function is well defined even if φ_t is only a family of immersions rather than of embeddings.

For $t \in \mathbb{R}$ fixed, consider points $x \in \mathbb{R}^{d+1}$ and $p \in M$ such that $x = \varphi_t(p)$. For a smooth function $f : \mathbb{R}^{d+1} \times [T_1, T_2] \to \mathbb{R}$ and a vector $v \in \mathbb{R}^{d+1}$ we denote the directional derivative by $\partial_v f(x,t) := \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} f(x+sv,t)$ and the tangential gradient by $\nabla_{g_t} f(x,t) := g_t^{ij}(p)\partial_{X_{i,t}} f(x,t)X_{j,t}$. This yields a splitting $\mathrm{D}f(x,t) = \nabla_{g_t} f(x,t) + \partial_{\nu_t} f(x,t)\nu\otimes\nu$.

We have by the chain rule the identity

$$\partial_t\left[f(p,\varphi_t(p),t)\right] = \partial_t f(p,\varphi_t(p),t) + \partial_{\nu_t} f(p,\varphi_t(p),t)\Psi_t^N(p) + \nabla_{g_t} f(p,\varphi_t(p),t)\cdot(\Psi_t^k X_{k,t}).$$

For a spatial function f that is defined on the trajectory $\mathcal{T}$ of φ_t it is not clear what the expression $\partial_{\nu_t} f(p,\varphi_t(p),t)\Psi_t^N(p)$ should mean. To that account, a useful notion of material derivatives for quantities on trajectories is provided by Prüss and Simonett [77, §2.5.3].

Definition 1.9
Let φ_t be a family of immersions with trajectory $\mathcal{T}$ and let $\Psi_t := \partial_t\varphi_t$ denote its time derivative. Let $F_{t,s} : M \to M$ be a smooth two parameter family of diffeomorphisms of M such that $F_{t,t} = \mathrm{Id}_M$ and $\eta_{t,s} = \varphi_t \circ F_{t,s}$ is normal along $\varphi_t(M)$, whose existence is guaranteed by Lemma 1.7. For a spatial function $g : \mathcal{T} \to \mathbb{R}$ we call

$$\left.\frac{\mathrm{D}}{\mathrm{D}t}g\right|_{(p,\varphi(p),t)} = \partial_t\left(g(p,\varphi_t(p),t)\right)$$

the Lagrange (or material) derivative of g with respect to φ_t and

$$\frac{\mathrm{D}_n}{\mathrm{D}t} g\bigg|_{(p,\varphi_t(p),t)} = \partial_s|_{s=0} \left(g(F_{t+s,t}(p), \eta_{t+s,t}(p), t+s)\right)$$

the normal derivative of g along φ_t.

To find the relation between the Lagrange and the normal derivative of a function on the trajectory we cast the calculation Prüss and Simonett [77, §2.5.3] in a lemma.

Lemma 1.10
For $T_1, T_2 \in \mathbb{R}$ let $(\varphi_t)_{t\in[T_1,T_2]}$ be a family of immersions, $\mathcal{T}$ its trajectory, and $f : \mathcal{T} \to \mathbb{R}$. Then,

$$\frac{\mathrm{D}}{\mathrm{D}t} f = \frac{\mathrm{D}_n}{\mathrm{D}t} f + \langle \nabla_{\varphi_t(M)} f, \Psi_t^k X_{k,t}\rangle.$$

Proof. First of all, we observe that since φ_t are immersions, around any $(p,t) \in M \times [T_1, T_2]$ there is a neighborhood U, where the map $(p,t) \mapsto (\varphi_t(p), t)$ is a bijection. To prove the assertion for all $\tau \in \mathcal{T}$, we may restrict our consideration to such a neighborhood. Therefore, we may w.l.o.g. consider the situation exactly as discussed in [77, §2.5.3], where

$$\mathcal{T} = \bigcup_{(p,t)\in M\times[T_1,T_2]} \{\varphi_t(p)\} \times \{t\},$$

and assume that f only depends on $\varphi_t(p) \in \mathbb{R}^{d+1}$ and t.

Following the argument in [77, §2.5.3], we start by choosing an extension $\tilde{f}$ of f to an open neighborhood of $\mathcal{T}$ in $\mathbb{R}^{d+1} \times [T_1, T_2]$.Using chain rule we see that in $x = \varphi_t(p)$, we have

$$\frac{\mathrm{D}}{\mathrm{D}t} f(x,t) = \partial_t \left(\tilde{f}(\varphi_t(p), t)\right) = \partial_t \tilde{f}(x,t) + \langle \nabla_{\mathbb{R}^{d+1}} \tilde{f}(x,t), \Psi_t(p)\rangle$$

and

$$\frac{\mathrm{D}_n}{\mathrm{D}t} f(x,t) = \partial_s|_{s=0} \left(\tilde{f}(\eta_{t,t+s}(p), t+s)\right) = \partial_t \tilde{f}(x,t) + \Psi_t^N \langle \nabla_{\mathbb{R}^{d+1}} \tilde{f}(x,t), \nu_t(p)\rangle.$$

Now splitting $\nabla_{\mathbb{R}^{d+1}} \tilde{f} = \nabla_{\varphi_t} \tilde{f} + \nu_t \partial_{\nu_t} \tilde{f}$ and $\partial_t \varphi_t = \Psi_t^N \nu_t + \Psi_t^k X_{k,t}$, we obtain the claimed identity independent of the choice of the extension. □

This lemma shows that in order to compute a material derivative of spatial quantities, it is enough to compute the normal derivative and the surface gradient.

In order to study integral functionals on manifolds it is important to have a formula for $\frac{\mathrm{d}}{\mathrm{d}t} \int_M f(\varphi, t) \mathrm{d}\mu_{\varphi_t(M)}$, when $f : \mathcal{T} \to \mathbb{R}$. To find such formula, we need to calculate the evolution of some geometric quantities associated to φ. We follow [77, §2.5.4] and [68, Sec. 1.2]. In the following we drop the arguments of functions.

Lemma 1.11
Let $\varphi_t : M \to \mathbb{R}^{d+1}$ be a family of immersions with trajectory $\mathcal{T}$ and let $\Psi_t := \partial_t \varphi_t$ denote its time derivative. Let $\frac{\mathrm{D}}{\mathrm{D}t}$ denote the material time derivative along φ_t. For the canonical tangent vectors $X_{i,t} = \partial_i \varphi_t$ we find

$$\partial_t X_{i,t} = \partial_i \Psi_t.$$

The metric changes according to

$$\partial_t g_{ij,t} = 2\Psi_t^N h_{ij,t} + g_{kj,t} \partial_i \Psi_t^k + g_{ik,t} \partial_j \Psi_t^k.$$

For the matrix g_t^{ij} representing inverse of the matrix $g_{ij,t}$ we find

$$\partial_t g_t^{ij} = -g_t^{i\ell} g_t^{jk} \partial_t g_{\ell k,t}.$$

Let $f : \mathcal{T} \to \mathbb{R}$ be a spatial quantity, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M f \mathrm{d}\mu_{\varphi_t} = \int_M \frac{\mathrm{D}_n}{\mathrm{D}t} f - f \Psi_t^N H_t \mathrm{d}\mu_{\varphi_t},$$

where H_t is the mean curvature associated to φ_t That is, the time derivative of the integral only depends on the normal derivative of f and the normal velocity of φ_t.

Proof. To keep the calculations as simple as possible, it is helpful to do them in the following order and to use normal coordinates, i.e. we assume that in one point in space and time we have $\langle \partial_i X_j, X_k \rangle = \Gamma_{ij}^k = 0$. Such coordinates can be constructed using the exponential map $\exp : T_p M \to M$. The first identity follows straight from the definitions,

$$\frac{\mathrm{D}}{\mathrm{D}t} X_{i,t} = \partial_i \Psi_t.$$

For further calculations we split Ψ_t in a normal and a tangential part,

$$\Psi_t = \Psi_t^N \nu_t + \Psi_t^k X_{k,t}.$$

With this definition, we compute in normal coordinates and with help of the Weingarten equation

$$\begin{aligned} \partial_t g_{ij,t} =& \partial_t \langle X_{i,t}, X_{j,t} \rangle \\ =& \langle \partial_i \Psi_t, X_{j,t} \rangle + \langle X_{i,t}, \partial_j \Psi_t \rangle \\ =& \langle (\partial_i \Psi_t^N) \nu_t + \Psi_t^N \partial_i \nu_t + (\partial_i \Psi_t^k) X_{k,t} + \Psi_t^k \partial_i X_{k,t}, X_{j,t} \rangle \\ &+ \langle X_{i,t}, (\partial_j \Psi_t^N) \nu_t + \Psi_t^N \partial_j \nu_t + (\partial_j \Psi_t^k) X_{k,t} + \Psi_t^k \partial_j X_{k,t} \rangle \\ =& -2 \Psi_t^N h_{ij,t} + g_{kj,t} \partial_i \Psi_t^k + g_{ik,t} \partial_j \Psi_t^k. \end{aligned}$$

Differentiating

$$\delta_k^i = g_t^{ij} g_{jk,t}$$

we find that

$$g_{jk,t} \partial_t g_t^{ij} = -g^{ij} \partial_t g_{jk,t}$$

and thus, multiplying with $g_t^{k\ell}$, summing up and renaming indices, we find

$$\partial_t g_t^{ij} = -g_t^{i\ell} g_t^{jk} \partial_t g_{\ell k,t}.$$

To find the derivative of the volume element, we apply the rule for differentiation of determinants, that reads for $A(t)$ an invertible, time dependent matrix

$$\frac{\mathrm{d}}{\mathrm{d}t} \det(A(t)) = \det(A) \operatorname{tr}(A^{-1} \partial_t A(t)).$$

Therefore, for $g_t = \det(g_{ij,t})$ we find

$$\begin{aligned} \partial_t \sqrt{g_t} =& \frac{1}{2\sqrt{g_t}} g_t \delta_i^k g_t^{ij} \partial_t g_{jk,t} \\ =& \frac{1}{2} \sqrt{g_t} \cdot \delta_i^k g_t^{ij} (-2\Psi_t^N h_{jk,t} + g_{\ell k,t} \partial_j \Psi_t^\ell + g_{j\ell,t} \partial_k \Psi_t^\ell \\ =& -\Psi_t^N H + \operatorname{div}_{\varphi_t(M)} (\Psi_t^k X_k). \end{aligned}$$

We find using Gauß' theorem

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M f\mathrm{d}\mu_{\varphi_t} &= \int_M \partial_t(f\sqrt{g_t})\mathrm{d}x\\ &= \int_M \frac{\mathrm{D}_n}{\mathrm{D}t}f + \langle\nabla_{\varphi_t(M)}f, \Psi_t^k X_{k,t}\rangle - f(\Psi_t^N H_t - \mathrm{div}_{\varphi_t(M)}(\Psi_t^k X_{k,t}))\mathrm{d}\mu_{\varphi_t}\\ &= \int_M \frac{\mathrm{D}_n}{\mathrm{D}t}f - f\Psi_t^N H_t \mathrm{d}\mu_{\varphi_t}.\end{aligned}$$

We observe, that in this formula the right hand side does in particular not depend on the tangential part of Ψ_t. □

For curves, i.e. $M = \mathbb{S}^1$, the formulas above are, of course, still true, but can be simplified. Moreover, we adapt the formulas to the notation explained in Section 1.3,so that it matches with that of other authors (e.g. [21, 28, 91]) considering geometric evolution equations for curves. The family of immersions is in the curve case denoted by γ, the metric tensor is simply $g_{11} = |\dot{\gamma}|^2$ and there is only one curvature quantity (up to orientation) that we denote by κ. We consider a normal family $\gamma_t = \gamma(\cdot, t)$ with trajectory $\mathcal{T}$ and $\partial_t\gamma = V\nu$. The following formulas appear almost verbatim in the literature (e.g. [28, (2.3)-(2.8)]). One finds

$$\begin{aligned}\partial_t\partial_s &= \partial_s\partial_t + V\kappa\partial_s,\\ \partial_t \mathrm{d}s &= -V\kappa \mathrm{d}s,\\ \partial_t\tau &= (\partial_s V)\nu,\\ \partial_t\nu &= -(\partial_s V)\tau.\end{aligned}$$

1.6 Multiplication and Composition in Sobolev Spaces

Dealing with non-linear partial differential equations one has to consider products of Sobolev functions and their composition with classically differentiable functions. These topics are of course closely related and we start with the first, the latter being an application then. The results that are stated in the following can be found (in more general versions) in the Chapters 4 and 5 of the book by Runst and Sickel [79]. We state a simple version of their Theorems 4.8.2/1 and 4.8.2/2.

Theorem 1.12 (Multiplication in Sobolev spaces)
For $m \in \mathbb{N}_0$ and (M, g) a smooth, compact d-dimensional Riemannian manifold with empty or C^1 boundary, take $1 \le p, q, r < \infty$ such that $r \le \min(p, q)$ and

$$2m - \frac{d}{p} - \frac{d}{q} > m - \frac{d}{r}.$$

Then, there exists $C > 0$ such that for all $f \in W^{m,p}(M)$, $g \in W^{m,q}(M)$ the product fg is in $W^{m,r}$ with the estimate

$$\|fg\|_{W^{m,r}(M)} \le C\|f\|_{W^{m,p}(M)}\|g\|_{W^{m,q}(M)}.$$

Remark 1.13
1. For $p = q$ and $m > \frac{d}{p}$ we have $W^{m,p}(M) \hookrightarrow C^0(M)$ and $2m - \frac{d}{p} - \frac{d}{p} > m - \frac{d}{p}$. Thus, $W^{m,p}(M)$ is closed under multiplication. Banach spaces with such a multiplication property are called Banach Algebras.
2. With the help of more advanced techniques one can prove that the theorem is also true for Sobolev spaces of fractional order (e.g. [79, Theorem 4.8.2/1, 4.8.2/2]).

3. There is the improved estimate

$$\|fg\|_{W^{m,p}(M)} \leq C\big(\|f\|_{W^{m,p}(M)}\|g\|_{L^\infty(M)} + \|f\|_{L^\infty(M)}\|g\|_{W^{m,p}(M)}\big). \tag{1.2}$$

See [79, Theorem 4.6.4/2].

In view of Taylor series expansion one could expect that the Sobolev spaces which are closed under multiplication are also closed under composition with sufficiently smooth functions. Precisely, the following theorem [79, Theorem 5.5/2] holds.

Theorem 1.14
For $m, N \in \mathbb{N}$ and (M, g) a smooth compact Riemannian manifold of dimension $d \in \mathbb{N}$ with empty or C^1 boundary, take $1 \leq p < \infty$ such that

$$m - \frac{d}{p} > 0.$$

Then, for all $f \in C^m(\mathbb{R}^N)$ we have $f(u) \in W^{m,p}(M)$. If additionally $f(0) = 0$ holds, then for $R > 0$ there is a constant $C > 0$ such that for all $u \in W^{m,p}(M)^N$ with $\|u\|_\infty \leq R$ it holds

$$\|f(u)\|_{W^{m,p}(M)} \leq C(\|f\|_{C^m(B(0,R))})\|u\|_{W^{m,p}(M)}(1 + R^{m-1}).$$

For $f \in C^{m+1}(\mathbb{R}^N)$, the induced composition operator is locally Lipschitz continuous, i.e. for all $R > 0$ there is $L > 0$ such that for all $u, v \in B(0, R) \subset W^{m,p}(M)^N$

$$\|f(u) - f(v)\|_{W^{m,p}(M)} \leq L\|u - v\|_{W^{m,p}(M)^N}.$$

The Lipschitz constant L only depends on R and $\|f\|_{C^{m+1}(B(0,R))}$.

Applying this theorem to first variation of a composition operator shows that the operator is Fréchet differentiable [79, Sec. 5.5.3].

Corollary 1.15
For $d, m, N \in \mathbb{N}\, k \in \mathbb{N}_0$ and (M, g) a smooth compact d-dimensional Riemannian manifold with empty or C^1 boundary, take $1 \leq p < \infty$ such that

$$m - \frac{d}{p} > 0.$$

Then, for all $f \in C^{m+k+1}(\mathbb{R}^N)$ the map

$$F : W^{m,p}(M, \mathbb{R}^N) \to W^{m,p}(M), \qquad u \mapsto f(u)$$

is of class $C^k\big(W^{m,p}(M, \mathbb{R}^N), W^{m,p}(M)\big)$ and

$$D^k F : W^{m,p}(M, \mathbb{R}^N) \to L_k\big((W^{m,p}(M, \mathbb{R}^N)^k, W^{m,p}(M)\big)$$

is locally Lipschitz continuous, where L_k denotes the space of k-linear maps .

If f is analytic, so is F.

For spaces with continuous embedding to L^∞ the non-autonomuous case follows immediately. Theorem 1 in [79, Sec. 5.5.4] is even more general. We note the following.

Corollary 1.16
For $d, m, N \in \mathbb{N}$, $k \in \mathbb{N}_0$ and (M, g) a smooth compact d-dimensional Riemannian manifold with empty or C^1 boundary, let $1 \leq p < \infty$ such that

$$m - \frac{d}{p} > 0.$$

Then, for all $f \in C^{m+k+1}(M \times \mathbb{R}^N)$ the map

$$F : W^{m,p}(M, \mathbb{R}^N) \to W^{m,p}(M), \qquad u \mapsto f(\cdot, u(\cdot))$$

is of class $C^k\big(W^{m,p}(M, \mathbb{R}^N), W^{m,p}(M)\big)$ and

$$D^k F : W^{m,p}(M, \mathbb{R}^N) \to L\big((W^{m,p}(M, \mathbb{R}^N)^k, W^{m,p}(M)\big)$$

is locally Lipschitz continuous.

If f is analytic, so is F.

We have the estimate

$$\begin{aligned} \|F(u)\|_{W^{m,p}(M)} \leq C(\|f\|_{C^m(M \times B(0, \|u\|_{L^\infty(M)}))}) \|u\|_{W^{m,p}(M)} (1 + \|u\|^{m-1}_{L^\infty(M)}) \\ + C\|f(\cdot, 0)\|_{W^{m,p}(M)}. \end{aligned} \tag{1.3}$$

The foregoing results tell us that the spaces $W^{m,p}(M)$ for $m > d/p$ inherit the properties of C^0. The statement of the next theorem is, that if $m > d/p + 1$, the spaces $W^{m,p}$ also have some properties of C^1. Let M be a closed oriented manifold of dimension d, N a C^∞-manifold, and s an integer. We define the space $H^s(M, N)$ of Sobolev maps between manifolds and for $s > d/2 + 1$ the space of diffeomorphisms of M of Sobolev regularity s, denoted by $D^s(M) \subset H^s(M, M) \subset C^1(M, M)$ as in [52, Chap. 1]. Then, the following theorem holds.

Theorem 1.17 (Theorem 1.2 from [52])
Let M be a closed oriented manifold of dimension d, N a C^∞-manifold, and s an integer satisfying $s > d/2 + 1$. Then for any $r \in \mathbb{N}_0$,

$$i) \quad \mu : H^{s+r}(M, N) \times D^s(M) \to H^s(M, N), \; (f, \varphi) \mapsto f \circ \varphi$$

and

$$ii) \quad \mathrm{inv} : D^{s+r}(M) \to D^s(M), \; \varphi \mapsto \varphi^{-1}$$

are both C^r-maps.

We use this theorem to prove an approximation and reparametrization result for hypersurfaces. For the approximation lemma we follow Prüss and Simonett [77, Sec. 2.3] and extend their result on approximation of embedded hypersurfaces to immersed hypersurfaces.

Lemma 1.18
Let (M, g) be an orientable smooth closed Riemannian manifold of dimension $d \in \mathbb{N}$ and for $\alpha \in (0, 1)$ let $\varphi \in C^{1,\alpha}(M)$ be an immersion. Then, φ can be approximated in the sense that for every $\varepsilon > 0$ there exists a smooth immersion φ^*, a function $f^* \in C^1(M)$ and a reparametrization $\Psi \in C^1(M, M)$ such that

$$\varphi \circ \Psi = \varphi^* + f^* \nu^*$$

with $\|f^*\|_{C^1(M)} < \varepsilon$. *That is, we can write* φ *as a graph over* φ^*. *We call* φ^* *a* reference immersion *and* $\varphi^*(M)$*a* reference surface *for* φ *and* f^* *the corresponding height function.*

Moreover, if $s \in \mathbb{N}$, $s > d/2 + 1$ *and* $\varphi \in H^s(M, \mathbb{R}^{d+1})$, *then* $\Psi \in H^s(M, M)$, $\varphi \circ \Psi \in H^s(M, \mathbb{R}^{d+1})$ *and* $f^* \in H^s(M)$.

Proof. **Step 1: Approximation in $C^1(M)$**
Choosing a smooth partition of unity and arguing in local charts we follow e.g. the book by Evans [32, Sec. 5.3, App. C.4] to find an approximating sequence $\varphi_n \in C^\infty(M)$ by convolution such that

$$\varphi_n \to \varphi \text{ in } C^1(M).$$

We argue as follows. Let $(U_i, x_i)_{i\in I}$ be a finite atlas for M with the following property: There is $r > 0$ such that for all $i \in I$ and $p \in U_i$ with $d(x_i(p), \partial\{x_i(U_i)\}) < r$ there exists $j \in I$ such that $p \in U_i \cap U_j$. Let ξ_i be a subordinate partition of unity with the property that $\xi_i(p) = 1$ if $d(x_i(p), \partial\{x_i(U_i)\}) > r/2$ and $\xi_i(p) = 0$ if $d(x_i(p), \partial\{x_i(U_i)\}) < r/4$. Moreover, let ζ_i be another partition of unity with the property that $\zeta_i(p) = 1$ if $d(x_i(p), \partial\{x_i(U_i)\}) > r/8$ and $\zeta_i(p) = 0$ if $d(x_i(p), \partial\{x_i(U_i)\}) < r/16$. That such specific collection of charts and subordinate partitions of unity exists is guaranteed by the existence of compact exhaustions for manifolds.

Let $\eta : \mathbb{R}^d \to \mathbb{R}$ denote a symmetric convolution kernel such that $\operatorname{supp}\eta = B(0, 1)$ and $\int_{\mathbb{R}^d} \eta \mathrm{d}x = 1$. For $n \in \mathbb{N}, n > \frac{64}{r}$ we set $\eta_n(x) = n^d\eta(nx)$ and

$$\varphi_n = \sum_{i\in I} \xi_i(\cdot) \cdot \big(\eta_n * (\zeta_i \cdot \varphi \circ x_i)\big) \circ x_i^{-1}.$$

With this configuration, we need not consider the cut-off functions in the subsequent calculations.

Step 2: Parametrising one hypersurface over another
In general, let $\varphi_1 \in C^1(M, \mathbb{R}^{d+1})$ and $\varphi_2 \in C^2(M, \mathbb{R}^{d+1})$ be two immersions with associated unit normals ν_i, let A_2 be the second fundamental form associated to φ_2 and set

$$A_{2,\max} = \max_{x\in M} |A_2(x)|$$

and $\varepsilon = \frac{1}{2(A_{2,\max}+1)}$. Following Prüss and Simonett [77, Section 2.3/3, pp. 67,68] we note that $\|\varphi_1 - \varphi_2\|_{C^0(M)} \le \varepsilon$ together with the condition $\|\langle\nu_1, \nu_2\rangle\|_{C^0(M)} > 1/2$ implies that there exists $\Psi \in C^1(M, M)$ and $f \in C^1(M)$ such that for all $p \in M$ we have

$$\varphi_1 \circ \Psi(p) = \varphi_2(p) + f(p)\nu_2(p)$$

with $f(p) = d(\varphi_2(M), \varphi_1 \circ \Psi(p))$.

To use this argument to find a reference surface φ^* for a C^1 immersion φ, we have to control the norms of the second fundamental forms of the approximating sequence.

Step 3: Estimates on the second fundamental form
The following argument is inspired by that of Conti, De Lellis and Székelyhidi [18, Proposition 1.6]. We consider the situation only in a single chart. Thus, let $U \subset M$ be an open set, $x : U \to \mathbb{R}^d$ one chart as defined above and let η be asymmetricstandard convolution kernel. Since $\varphi \in C^{1,\alpha}(U)$ we have from Taylor expansion that for $y_1, y_2 \in x(U)$ it holds

$$\varphi(x^{-1}(y_1)) - \varphi(x^{-1}(y_2)) = \mathrm{D}(\varphi \circ x^{-1})(y_1)(y_1 - y_2) + r(y_1)(y_1 - y_2)$$

with $\sup_{y\in x(U)} |r(y)(h)| \le C|h|^{1+\alpha}$. Therefore, for $p \in U$ and $\varphi_n = (\eta_n * (\varphi \circ x^{-1})) \circ x$ we have

$$\begin{aligned}|\varphi_n(p) - \varphi(p)| &= \left| \int_{B(x(p),1/n)} \eta_n(x(p)-y)\big(\varphi(p) - \varphi(x^{-1}(y))\big)\mathrm{d}y \right| \\ &= \left| \int_{B(x(p),1/n)} \eta_n(x(p)-y)\big(\mathrm{D}(\varphi\circ x^{-1})(x(p))(x(p)-y) + r(x(p))(x(p)-y)\big)\mathrm{d}y \right|\end{aligned}$$

and using the symmetry we conclude

$$\int_{B(x(p),1/n)} \eta_n(x(p)-y)\mathrm{D}(\varphi\circ x^{-1})(x(p))(x(p)-y)\mathrm{d}y = 0.$$

Therefore, the asymptotic of the remainder r implies $|\varphi_n(p) - \varphi(p)| \le Cn^{-1-\alpha}$, establishing a rate for the convergence of φ_n in C^0. Conversely, considering $\partial_{ij}\varphi_n$ we find

$$\begin{aligned}|\partial_{ij}\varphi_n(p)| &= |\partial_j\eta_n * \partial_i\varphi| \\ &= \left| n^d \int_{B(x(p),1/n)} n\partial_j\eta(n(x(p)-y))\partial_i\varphi(y)\mathrm{d}y \right| \\ &\le Cn.\end{aligned}$$

Together these estimates imply that $|A_{\varphi_n}| \le Cn$ and $|\varphi_n - \varphi| \le Cn^{-1-\alpha}$. Inserting these estimates in the C^0 condition from step 2, we see that it is fulfilled for $n > (4C)^{1/\alpha}$. By convergence of φ_n to φ in C^1, for n large enough also the condition on the unit normals is fulfilled. Then, we can parametrize φ over φ_n .

Step 4: The height function
Together, Step 1 and Step 2 imply the existence of diffeomorphisms Ψ_n and height functions f_n such that $\varphi = \varphi_n \circ \Psi_n + (f_n\nu_n) \circ \Psi_n$. We can derive the claimed C^1-estimate for n large enough with help of some explicit calculation, again following [77, Section 2.3/1, pp. 65,66].

The construction in Step 1 yields the following. For n large enough, such that the map $\Lambda_n : M \times (-C/(n^{1+\alpha}), C/(n^{1+\alpha})) \to \mathbb{R}^{d+1}$, $(x,r) \mapsto \varphi_n(x) + r\nu_n$ is smoothly invertible on its image due to the curvature estimate, we denote the inverse map by (Π_n, d_n). Thus,

$$\varphi = \varphi_n \circ \Pi_n \circ \varphi + (d_n \circ \varphi)(\nu_n \circ \Pi_n \circ \varphi)$$

and $f_n = d_n \circ \varphi$. Hence, we have to show that $\|f_n\|_{C^1} \to 0$. Since $|d_n \circ \varphi| \le |\varphi - \varphi_n| \le Cn^{-1-\alpha}$, we see that $f_n \to 0$ in C^0. We prove in the following that $\mathrm{d}f_n \to 0$.

For $y \in \Lambda_n\big(M, (-1/(n^{1+\alpha}C), 1/(n^{1+\alpha}C))\big)$ the derivative of d_n is normal to $\varphi_n(\Pi_n(y))$ and given by $\mathrm{D}d_n(y) = \nu_n(\Pi_n(y))$. Additionally, we have that $y - \varphi_n(\Pi_n(y)) = d_n(y)\nu_n(\Pi_n(y))$. Differentiating this identity and justifying the invertibility of the first factor in the following we have

$$\mathrm{D}\Pi_n(y) = \big[\mathrm{d}\varphi_n(\Pi_n(y)) - d_n(y)\mathrm{d}\nu_n(\Pi_n(y))\big]^{-1}\big(I_{\mathbb{R}^{d+1}} - \nu_n(\Pi_n(y)) \otimes \nu_n(\Pi_n(y))\big).$$

Here, $\big(I_{\mathbb{R}^{d+1}} - \nu_n(\Pi_n(y)) \otimes \nu_n(\Pi_n(y))\big)$ is the projection onto $\mathrm{d}\varphi_n(\Pi_n(y))(T_{\Pi_n(y)}M)$, where $\mathrm{d}\varphi_n(\Pi_n(y))$ is invertible. Since $|\mathrm{d}\nu_n| = |A_{\varphi_n}| \le Cn$ and $|d_n| \le Cn^{-1-\alpha}$ we conclude that $|d_n \circ \varphi \mathrm{d}\nu_n(\Pi_n \circ \varphi)| \le Cn^{-\alpha}$ and thus that also $\mathrm{d}\varphi_n(\Pi_n(y)) - d_n(y)\mathrm{d}\nu_n(\Pi_n(y))$ is invertible for n large enough and $|\mathrm{D}\Pi_n|$ is bounded.

Moreover, we have

$$\partial_i f_n = \nu_n \circ \Pi_n \circ \varphi \cdot \partial_i\varphi.$$

In a chart $x : U \to V \subset \mathbb{R}^d$ around $p \in M$ we analyze the expression $\Pi_n \circ \varphi$ further. By the formula for $\mathrm{D}\Pi_n$ we find for $y \in V$ that

$$\begin{aligned}&|\Pi_n(\varphi(x^{-1}(y))) - y|\\ \le\ &|\Pi_n(\varphi_n(x^{-1}(y))) + D\Pi_n(\varphi_n(x^{-1}(y)))(\varphi(x^{-1}(y)) - \varphi_n(x^{-1}(y))) - y|\\ \le\ &|D\Pi_n(\varphi_n(x^{-1}(y)))||\varphi(x^{-1}(y)) - \varphi_n(x^{-1}(y))| \le Cn^{-1-\alpha}.\end{aligned}$$

With the C^α regularity of ν and $\nu_n - \nu$ we invoke again Taylor's Theorem and furnish the remainder terms r with an index indicating the function that is approximated. We infer

$$\begin{aligned}&\nu_n \circ \Pi_n \circ \varphi \cdot \partial_i\varphi\\ =\ &\nu \circ \Pi_n \circ \varphi \cdot \partial_i\varphi + (\nu_n - \nu) \circ (\Pi_n \circ \varphi) \cdot \partial_i\varphi\\ =\ &\nu \cdot \partial_i\varphi + r_\nu(\Pi_n \circ \varphi - \mathrm{Id}) \cdot \partial_i\varphi + (\nu_n - \nu) \cdot \partial_i\varphi + r_{(\nu_n-\nu)}(\Pi_n \circ \varphi - \mathrm{Id}) \cdot \partial_i\varphi.\end{aligned}$$

This converges to 0 since $\nu \cdot \partial_i\varphi = 0$ and the asymptotics for the remainder terms. Thus we have

$$\varphi_n \circ \Pi_n \circ \varphi + f_n\nu_n \circ \Pi_n \circ \varphi = \varphi.$$

By the chain rule, we observe that $\Pi_n \circ \varphi : M \to M$ converges to Id_M in $C^1(M, M)$. Therefore also the inverse map $(\Pi_n \circ \varphi)^{-1}$ converges and $f_n \circ (\Pi_n \circ \varphi)^{-1}$ converges to 0 in C^1. Therefore, for n large enough we set $\varphi^* = \varphi_n$, $\nu^* = \nu_n$, $\Psi = (\Pi_n \circ \varphi)^{-1}$ and $f^* = f_n \circ \Psi$ and the claim

$$\varphi \circ \Psi = \varphi^* + f^*\nu^*$$

follows.

Step 5: Additional Sobolev regularity
Now assume $\varphi \in H^s(M, \mathbb{R}^{d+1})$, then the Sobolev regularity of the compositions is preserved by Theorem 1.17. □

Remark 1.19
In Lemma 1.18 it would be pleasant to have an estimate that guarantees the smallness or at least boundedness of the height functions also in the Sobolev norms. Unfortunately, to the author's knowledge, such estimate is not available in the literature.

A possible strategy of proof might consist in following the steps of the proof of Lemma 1.20, applying the Banach space version of the implicit function theorem. However, it is not clear to the author, how the non-linear operators can be estimated to obtain convergence without imposing strong additional conditions on the regularity of the approximated immersion.

However, the next lemma shows that, when we are close enough to a fixed smooth immersion, we can also control the Sobolev norm of the corresponding height function. The result is a generalization of a lemma of Dall'Acqua, Pozzi, Spener [22, Lemma 4.1].

Lemma 1.20
Let (M, g) be an orientable smooth closed Riemannian manifold of dimension $d \in \mathbb{N}$ and let $\varphi^* \in C^\infty(M, \mathbb{R}^{d+1})$ be a smooth immersion with normal ν^* and fix $k \in \mathbb{N}$, $k > d/2 + 1$. Then for given $\sigma > 0$ there exists $\tilde\sigma > 0$ such that for all $\varphi \in H^k(M, \mathbb{R}^{d+1})$ with $\|\varphi - \varphi^*\|_{H^k(M,\mathbb{R}^{d+1})} \le \tilde\sigma$ there exists a diffeomorphism $\Psi \in H^k(M, M)$ and a function $f \in H^k(M)$ with $\|f\|_{H^k(M)} \le \sigma$ such that

$$\varphi \circ \Psi = \varphi^* + f\nu^*.$$

Proof. We pursue a strategy similar to that of Dall'Acqua, Pozzi, Spener [22, Lemma 4.1].

In the following we consider M together with the pullback metric of the scalar product on $\mathbb{R}^{d+1}$ via φ^* as a Riemannian manifold.Let π_M denote the canonical projection $TM \to M$ and let the space of vector fields on M with Sobolev regularity H^s be denoted by

$$\Gamma^s(TM) = \{X \in H^s(M, TM) | \; \pi_M(X) = \mathrm{Id}_M\}.$$

Let $U \subset \Gamma^s(TM)$ and $V \subset H^s(M)$ denote open neighborhoods of the respective element 0. We make use of the exponential maps for $p \in M$ denoted by $\exp_p : T_pM \to M$ to turn vector fields into variations of diffeomorphisms. For $\mathrm{Id}_M \in H^s(M, M)$ and $X \in \Gamma^s(TM)$ we define $\mathrm{Id}_M + X$ by $(\mathrm{Id}_M + X)(p) = \exp_p(X(p))$. We consider the map

$$F : U \times V \to H^s(M, \mathbb{R}^{d+1}), \; (X, f) \mapsto \varphi^* \circ (\mathrm{Id}_M + X) + f\nu^* \circ (\mathrm{Id}_M + X).$$

Since φ^* is smooth, so is the exponential map, therefore yielding a smooth map of Sobolev spaces. Thus, also the map F is smooth. In particular we use that by the definition of $\exp_p$ we have $\mathrm{d}\exp_p(0) = \mathrm{Id}_{T_pM}$, to find $F'(0, 0) = (\mathrm{d}\varphi^*, \nu^*)$ which is a continuous bijection $\Gamma^s(TM) \times H^s(M) \to H^s(M, \mathbb{R}^{d+1})$ and thus F is smoothly invertible in a neighborhood of φ^* (see [95, Theorem 4.f, Corollary 4.37]). That is, for a sequence $\varphi_k \in H^s(M, \mathbb{R}^{d+1})$ that converges to φ^* we conclude that there exist sequences $f_k \in H^s(M)$ and $X_k \in \Gamma^s(M)$, $f_k, \; X_k \to 0$ such that $\varphi_k = \varphi^* \circ (\mathrm{Id}_M + X_k) + f_k\nu^* \circ (\mathrm{Id}_M + X_k)$. Since by Theorem 1.17 inversion is a continuous operation on $H^s(M, M)$ and composition is continuous as a map $H^s(M) \times H^s(M, M) \to H^s(M)$, we infer that $f_k \circ (\mathrm{Id}_M + X_k)^{-1} \to 0$. Therefore, for$\sigma > 0$given, there exists $\tilde{\sigma} > 0$such that for all $\varphi \in H^s(M, \mathbb{R}^{d+1})$ with $\|\varphi - \varphi^*\|_{H^s(M,\mathbb{R}^{d+1})} \leq \tilde{\sigma}$ there is a diffeomorphism $\Psi \in H^s(M, M)$ and $f \in H^s(M)$ with $\|f\|_{H^s(M)} \leq \sigma$ such that

$$\varphi \circ \Psi = \varphi^* + f\nu^*.$$

□

1.7 Spaces for Parabolic Problems

For the treatment of parabolic problems, we need to apply the above results to Lebesgue-Bochner spaces. In this section M is a smooth d-dimensional closed manifold, $T_0 \geq T > 0$ are real numbers and $p \in \mathbb{N}, k \in \mathbb{Z}$ are integers. We study the properties of spaces

$$H^1(0, T; H^{k-2p}(M)) \cap L^2(0, T; H^k(M)).$$

Moreover, we fix $T_0 \in (0, \infty)$ and it will be our main concern that the following estimates on $[0, T) \subset [0, T_0)$ are uniformin $T < T_0$. We state some well known theorems.

Theorem 1.21 (Intermediate derivative theorem)
For all $s \in (0, 1)$ there is a continuous embedding

$$H^1(0, T; H^{k-2p}(M)) \cap L^2(0, T; H^k(M)) \hookrightarrow H^s(0, T; H^{k-2ps}(M)) \; .$$

Proof. The proof can be found for example in the books by Lions and Magenes [64, Theorem 2.3] or a paper of Denk, Saal and Seiler [24, Lemma 4.3]. □

Combining the intermediate derivative theorem with the usual embedding theorems for Sobolev spaces (cf. Theorem 1.3) we obtain useful embedding results.

Corollary 1.22
For $p \in \mathbb{N}, k \in \mathbb{Z}$, for all $r \in [1, \infty)$ with

$$\frac{1}{r} \geq \frac{2p + d - 2k}{4p + 2d}$$

there is an embedding

$$H^1(0, T; H^{k-2p}(M)) \cap L^2(0, T; H^k(M)) \hookrightarrow L^r(0, T; L^r(M))$$

and for all $\alpha \in (0, 1)$ with

$$\alpha \leq \frac{2k - 2p - d}{4p + 2}$$

there is an embedding

$$H^1(0, T; H^{k-2p}(M)) \cap L^2(0, T; H^k(M)) \hookrightarrow C^\alpha(0, T; C^\alpha(M)).$$

Proof. As stated in Theorem 1.21, for all $s \in (0, 1)$ there is an embedding

$$H^1(0, T; H^{k-2p}(M)) \cap L^2(0, T; H^k(M)) \hookrightarrow H^s(0, T; H^{k-2ps}(M)) \ .$$

To prove the embedding to the Lebesgue space we recall that the embeddings as stated in Theorem 1.3 yield

$$H^s(0, T; H^{k-2ps}(M)) \to L^q(0, T; L^r(M)),$$

for $q, r \in [1, \infty)$ satisfying $1/q \geq \frac{1-2s}{2} =: \frac{1}{q^*}$ and $1/r \geq \frac{d+4ps-2k}{2d} =: \frac{1}{r^*}$. For $s = \frac{k}{2p+d}$ we have

$$\begin{aligned} \frac{1}{r^*} &= \frac{d + 4ps - 2k}{2d} = \frac{d + 4p\frac{k}{2p+d} - 2k}{2d} \\ &= \frac{2p + d - 2k}{4p + 2d} = \frac{1 - \frac{2k}{2p+d}}{2} = \frac{1 - 2s}{2} = \frac{1}{q^*}. \end{aligned}$$

Moreover, the embedding results into Hölder spaces yield

$$H^s(0, T; H^{k-2ps}(M)) \to C^\alpha(0, T; C^\beta(M))$$

for $\alpha, \beta \in (0, 1)$ satisfying $\alpha < \frac{2s-1}{2} =: \alpha^*$ and $\beta < \frac{2k-4ps-d}{2} =: \beta^*$. Thus, for $s = \frac{2k-d+1}{2p+d}$ we have

$$\begin{aligned} 2\alpha^* &= \frac{4k - 2d + 2}{4p + 2} - 1 = \frac{4k - 4p - 2d}{4p + 2} \\ &= \frac{8kp + 4k - 8kp + 4pd - 4p - 4pd - 2d}{4p + 2} = 2k - 4p\frac{2k - d + 1}{4p + 2} - d = 2\beta^*. \end{aligned}$$

□

Even though there is no embedding $H^{1/2}(0, T) \to C^0(0, T)$, we have the following trace theorem as stated by Amann [2, III.4.10.2] or again Denk, Saal, Seiler [24, Lemma 4.4]).

Theorem 1.23 (Trace spaces)
For $p \in \mathbb{N}$, $k \in \mathbb{Z}$ there is a continuous embedding

$$H^1(0, T; H^{k-2p}(M)) \cap L^2(0, T; H^k(M)) \hookrightarrow BUC(0, T; H^{k-p}(M)).$$

This yields continuity of the map

$$\gamma : H^1(0, T; H^{k-2p}(M)) \cap L^2(0, T; H^k(M)) \to H^{k-p}(M), \qquad f \mapsto f(0).$$

This map is surjective and has a continuous right-inverse. That is, for $f_0 \in H^k(M)$ we can find a continuation $f \in L^2(0,T_0;H^k) \cap H^1(0,T_0;H^{k-2p})$ of f_0 such that $f(0) = f_0$ and

$$\|f\|_{L^2(0,T_0;H^k)\cap H^1(0,T_0;H^{k-2p})} \leq C\|f_0\|_{H^{k-p}},$$

In Corollary 1.25 we discuss how the dependence of the constants on T can be controlled.

As mentioned above, it will be important for the analysis of non-linear operators later on, to control the constants for these embeddings.

Definition 1.24
On $L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p})$ we define the norm $\|.\|_{X_T}$ by

$$\|f\|_{X_T} = \|f\|_{L^2(0,T;H^k)\cap H^1(0,T;H^{k-2p})} + \|f(0)\|_{H^{k-p}}.$$

This norm has the following useful property.

Corollary 1.25
The norm $\|.\|_{X_T}$ on $L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p})$ is equivalent to the usual norm and for this norm the constants $C = C(T)$ in the embeddings from Theorems 1.21 and 1.23 are uniformly bounded for all $T < T_0$.

Proof. The norm equivalence is implied by the trace theorem as stated in Theorem 1.23.

On $X_T^0 := \{x \in L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p}) | \gamma(x) = 0\}$ we define the extension operator $E : X_T^0 \to X_{T_0}^0$ by

$$(Eg)(t) = \begin{cases} g(t) \text{ for } t \leq T, \\ g(T-t) \text{ for } 2T \geq t > T, \\ 0 \text{ for } t > 2T. \end{cases}$$

For this operator we have for $u \in X_T^0$ the estimate

$$\|Eu\|_{L^2(0,T_0;H^k)\cap H^1(0,T_0;H^{k-2p})} \leq 2\|u\|_{L^2(0,T;H^k)\cap H^1(0,T;H^{k-2p})}.$$

Let $f \in L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p})$ be a function such that $f(0) = f_0 \in H^{k-p}$. Then we denote by $f_{T_0} \in L^2(0,T_0;H^k) \cap H^1(0,T_0;H^{k-2p})$ a continuation of this f_0 as in Theorem 1.23 and obtain for all $s \in (0,1)$ that

$$\|f\|_{H^s(0,T;H^{k-2ps})} \leq C\big(\|f - f_{T_0}\|_{H^s(0,T;H^{k-2ps})} + \|f_{T_0}\|_{H^s(0,T;H^{k-2ps})}\big).$$

Using the reflection E explained above we find

$$\begin{aligned}\|f\|_{H^s(0,T;H^{k-2ps})} \leq& \|E(f - f_{T_0})\|_{H^s(0,T_0;H^{k-2ps})} + \|f_{T_0}\|_{H^s(0,T_0;H^{k-2ps})} \\ \leq& C(T_0)\|E(f - f_{T_0})\|_{L^2(0,T_0;H^k)\cap H^1(0,T_0;H^{k-2p})} + \|f_{T_0}\|_{H^s(0,T_0;H^{k-2ps})} \\ \leq& C(T_0)(\|f\|_{L^2(0,T;H^k)\cap H^1(0,T;H^{k-2p})} + \|f(0)\|_{H^{k-p}})\end{aligned}$$

and

$$\begin{aligned}\|f\|_{C^{0,s-\frac{1}{2}}(0,T;H^{k-2ps})} \leq& \|E(f - f_{T_0})\|_{C^{0,s-\frac{1}{2}}(0,T;H^{k-2ps})} + \|f_{T_0}\|_{C^{0,s-\frac{1}{2}}(0,T;H^{k-2ps})} \\ \leq& C(T_0)\|E(f - f_{T_0})\|_{L^2(0,T_0;H^k)\cap H^1(0,T_0;H^{k-2p})} + \|f_{T_0}\|_{C^{0,s-\frac{1}{2}}(0,T;H^{k-2ps})} \\ \leq& C(T_0)\|f\|_{L^2(0,T;H^k)\cap H^1(0,T;H^{k-2p})} + \|f(0)\|_{H^{k-p}}),\end{aligned}$$

with the constant $C = C(T_0)$ independent of $T < T_0$. ☐

In the next lemma we use the intermediate derivative theorem in the following way. Philosophically, if we consider some norm of a function not taking into account all the function's (space) regularity, we can trade the remaining (space) regularity for some time regularity. This yields then, that for small times the deviation from the initial data will be small.

Lemma 1.26
For $\ell \in \mathbb{N}$ satisfying $\ell \leq 2p-1$, consider $f \in L^2(0,T;H^k(M)) \cap H^1(0,T;H^{k-2p}(M))$. We fix $\tau \in (0,\frac{1}{2})$ and set

$$q = \begin{cases} \frac{2d}{d+4p\tau+2\ell-2k} & \text{if } d+4p\tau+2\ell-2k>0 \\ q=\infty & \text{else.} \end{cases}$$

Then, for all $1<r<r^*=\frac{2}{1-2\tau}$ there is $s>0$ such that

$$\|\nabla^\ell f\|_{L^r(0,T;L^q(M))} \leq CT^s\|f\|_{X_T}$$

and there is $\tilde{s}$ such that

$$\|\nabla^\ell f\|_{L^2(0,T;H^{k-2p}(M))} \leq CT^s\|f\|_{X_T}$$

If we have additionally that $k-p-\ell>d/2$, then

$$\|\nabla^\ell f\|_{BUC(0,T,C^0(M))} \leq CT^s\big(\|f\|_{X_T}\big) + C\|f(0)\|_{H^{k-p}(M)}. \tag{1.4}$$

Proof. In the following, we do not denote the spatial domain M in the function spaces.
For $f \in L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p})$ as demanded in the statement of the theorem with initial data

$$f(0,\cdot) \in H^{k-p},$$

let $f_{0,T} \in L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p})$ be a continuation of $f(0)$ with the property

$$\|f_{0,T}\|_{L^2(0,T;H^k)\cap H^1(0,T;H^{k-2p})} \leq C\|f_0\|_{H^{k-p}},$$

whose existence is guaranteed by the trace theorem as stated in Theorem 1.23. To generate a factor T^α with $\alpha>0$, we use Hölders inequality in time, Sobolev embeddings (see Theorem 1.3) in space and time, and Theorem 1.21. We calculate for $1<r<r^*=\frac{2}{1-2\tau}$

$$\begin{aligned}
\|\nabla^\ell f\|_{L^r(0,T;L^q)} \leq & T^{\frac{1}{r}-\frac{1}{r^*}}\|\nabla^\ell f\|_{L^{r^*}(0,T;L^q)} \\
\leq & CT^{\frac{1}{r}-\frac{1}{r^*}}\|\nabla^\ell f\|_{H^\tau(0,T;H^{k-2p\tau})} \\
\leq & CT^{\frac{1}{r}-\frac{1}{r^*}}\big(\|\nabla^\ell(f-f_{0,T})\|_{H^\tau(0,T;H^{k-2p\tau})} + \|\nabla^\ell f_{0,T}\|_{H^\tau(0,T;H^{k-2p\tau})}\big) \\
\leq & CT^{\frac{1}{r}-\frac{1}{r^*}}\big(\|\nabla^\ell(f-f_{0,T})\|_{X_T} + \|f(0)\|_{H^{k-p}}\big) \\
\leq & CT^{\frac{1}{r}-\frac{1}{r^*}}\big(\|f\|_{X_T} + \|f(0)\|_{H^{k-p}}\big)
\end{aligned}$$

The other estimate is similar. We find with Theorem 1.21

$$\begin{aligned}
& \|\nabla^\ell f_T\|_{BUC(0,T,C^0)} \\
\leq & T^\alpha\|(f_T-f_{0,T})\|_{C^{0,\alpha}([0,T],C^\ell)} + \|\nabla^\ell f_{0,T}\|_{BUC(0,T,C^0)} \\
\leq & CT^\alpha\|f_T-f_{0,T}\|_{X_T} + \|\nabla^\ell f_{0,T}\|_{BUC(0,T,C^0)} \\
\leq & CT^\alpha(\|f_T\|_{X_T} + \|f_{0,T}\|_{L^2(0,T;H^k)\cap H^1(0,T;H^{k-2p})}) + \|\nabla^\ell f(0)\|_{C^0} \\
\leq & CT^\alpha\|f_T\|_{X_T} + C\|f(0)\|_{H^{k-p}}.
\end{aligned}$$

Here, any $\alpha \in (0,d/(4r))$ is possible. □

We combine the previous results to study the mapping properties of composition operators in the parabolic setting.

Lemma 1.27
For $f \in C^{k+2}(M \times [0,T) \times \mathbb{R})$ and $p, k \in \mathbb{N}$ with $k > d/2 + p$ the map

$$\begin{aligned} F : L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p}) &\to L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p}), \\ u &\mapsto f(\cdot,\cdot,u) \end{aligned}$$

is well defined, of class C^k, and the k-th Fréchet derivative

$$\begin{aligned} D^k F : &L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p}) \to \\ &L_k\big((L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p}))^k, L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p})\big) \end{aligned}$$

is locally Lipschitz, where L_k denotes the space of k-linear maps.

Proof. For fixed t we infer by Theorem 1.14 that $u \mapsto f(\cdot,t,u)$ is locally Lipschitz continuous in H^k with Lipschitz constant $L(t,R)$.Thus, for $u_1, u_2 \in B(0,R) \subset L^2(0,T;H^k) \cap H^1(0,T;H^{k-2p})$ we have

$$\|F(u_1) - F(u_2)\|_{L^2(0,T;H^k)\cap H^1(0,T;H^{k-2p})} \le \sup_{t\in[0,T)} L(t,R) \|u_1 - u_2\|_{L^2(0,T;H^k)\cap H^1(0,T;H^{k-2p})}.$$

Here, $\sup_{t\in[0,T)} L(t,R)$ is finite, since for all $t \in [0,T]$ the Lipschitz constant only depends on R and $\|f\|_{C^{k+2}(B(0,R))}$ as stated in Corollary 1.15 in combination with the estimate (1.3). Applying this to the derivatives of F yields the desired assertion. □

2

Curves and Vector Fields—an Anisotropic Approach

2.1 Vector Field in the Background

A firstapproach to study the interplay between evolving immersions and vector fields might consist in adaption of the framework of Wheeler [91] to the mean curvature flow. Rather than a spontaneous curvature, we interpret the ambient vector field as an additional weight in the arc length measure.

Suppose $M : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth vector field and $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is a regular smooth curve. Let s be the arc length parameter of γ, τ and ν be the unit tangent and normal, respectively. By κ we denote the curvature of γ. We consider the energy

$$E_M(\gamma) = \int_\gamma 1 + \langle M(\gamma(s)), \tau(s)\rangle^2 \mathrm{d}s.$$

Some elementary calculations show that this energy cannotbe interpreted as the length of the curve in the plane furnished with a suitable Riemannian metric. It can be interpreted however as an inhomogeneous and anisotropic surface energy. A brief discussion of the historical background can be found in the introduction of this work.

Now we calculate the first variation of this specific energy. Let $\phi : \mathbb{S}^1 \to \mathbb{R}$ be a smooth function. As this energy is invariant under reparametrization, we consider a family of normal variations $\gamma_\varepsilon = \gamma + \varepsilon\phi\nu$ to determine the L^2 gradient flow of the energy. To compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} E_M(\gamma_\varepsilon)$ the following formulas are very useful. For a quantity f_ε depending on ε we write f' to mean $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0} f_\varepsilon$. We find for $\varepsilon = 0$ the relations

$$\begin{aligned}
\partial_\varepsilon \partial_s &= \partial_s \partial_\varepsilon + \phi\kappa\partial_s \\
\mathrm{d}s' &= -\phi\kappa \mathrm{d}s \\
\tau' &= (\partial_s \phi)\nu \\
\nu' &= -(\partial_s \phi)\tau \\
M' &= \phi \mathrm{D}M\nu.
\end{aligned}$$

Thus,

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E_M(\gamma_\varepsilon) &= \int_\gamma 2\langle M,\tau\rangle\left(\langle M',\tau\rangle+\langle M,\tau'\rangle\right)\mathrm{d}s + \int_\gamma \left(1+\langle M(\gamma(s)),\tau(s)\rangle^2\right)\mathrm{d}s' \\ &= \int_\gamma 2\langle M,\tau\rangle\left(\langle \phi \mathrm{D}M\nu,\tau\rangle+\langle M,\partial_s\phi\nu\rangle\right)\mathrm{d}s - \int_\gamma \left(1+\langle M(\gamma(s)),\tau(s)\rangle^2\right)\phi\kappa\mathrm{d}s.\end{aligned}$$

To get rid of the term involving derivatives of ϕ we integrate by parts. Since the curve is closed, we do not pick up any boundary terms and find

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E_M(\gamma_\varepsilon) = &-\int_\gamma 2\phi\partial_s\left(\langle M,\tau\rangle\langle M,\nu\rangle\right)\mathrm{d}s + \int_\gamma 2\phi\langle \mathrm{D}M\nu,\tau\rangle\langle M,\tau\rangle\mathrm{d}s \\ &-\int_\gamma \left(1+\langle M(\gamma(s)),\tau(s)\rangle^2\right)\phi\kappa\mathrm{d}s.\end{aligned}$$

We interpret this as

$$(\nabla_{L^2}E_M,\phi)_{L^2}$$

and thus the normal velocity of the corresponding gradient flow is given by

$$\begin{aligned}V =& -\langle\nabla_{L^2}E_M,\nu\rangle \\ =&(1+\langle M,\tau\rangle^2)\kappa + 2\partial_s\left(\langle M,\tau\rangle\langle M,\nu\rangle\right) - 2\langle \mathrm{D}M\nu,\tau\rangle\langle M,\tau\rangle \\ =&(1+\langle M,\tau\rangle^2)\kappa - 2\langle \mathrm{D}M\nu,\tau\rangle\langle M,\tau\rangle \\ &+2\left(\langle \mathrm{D}M\tau,\tau\rangle\langle M,\nu\rangle + \kappa\langle M,\nu\rangle\langle M,\nu\rangle + \langle M,\tau\rangle\langle \mathrm{D}M\tau,\nu\rangle - \kappa\langle M,\tau\rangle\langle M,\tau\rangle\right) \\ =&(1-|M|^2+3\langle M,\nu\rangle^2)\kappa + 2\left(\langle \mathrm{D}M\tau,\tau\rangle\langle M,\nu\rangle + \langle \mathrm{D}M\tau,\nu\rangle\langle M,\tau\rangle - \langle \mathrm{D}M\nu,\tau\rangle\langle M,\tau\rangle\right).\end{aligned} \tag{2.1}$$

We used the fact that ν and τ form an orthonormal basis of $\mathbb{R}^2$ and thus $|M|^2 = \langle M,\nu\rangle^2+\langle M,\tau\rangle^2$. We observe that this flow is parabolic under the assumption that $|M|<1$ globally.

This equation fits in the following general framework developed by Angenent, Oaks and Zhu generalising the results of Gage, Hamilton and Grayson.

In the early 1990s Angenent published two articles [3, 4] in which he gave a short-time existence result for a very general class of anisotropic geometric evolution problems. For closed curves $\gamma: S^1\to M$ in a 2-dimensional manifold M he considered flows of the form

$$\partial_t\gamma = V(\tau,\kappa)\nu,$$

where τ is the unit tangent to γ, κ is its curvature and V is a suitable map from the set of all possible tangents—the sphere bundle $S^1(M)$–and $\mathbb{R}$. Imposing growth and smoothness conditions on V he was able to prove well-posedness of the evolution equation for curves with merely p-integrable curvature, for a suitable p [3, Theorem A]. He also studied the formation of singularities and succeeded to characterize the possible behavior with the additional assumption of symmetry $V(-\tau,-\kappa) = -V(\tau,\kappa)$ [4, Section 6]. This condition means that the evolution does not depend on the orientation of the curve and also allows to further enlarge the class of admissible initial curves, so that it contains the class of possible (reduced) limit curves in the case of singularities [4, Theorem C]. This allows to define the notion of a weak solution that can pass through singularities, resembling the result of the surgery procedure of Huisken and Sinestrari [51] for the mean curvature flow of two-convex hypersurfaces. Oaks [73] was able to characterize the formation of singularities even further. When a singularity occurs, the curve either shrinks to a point or it loses a self-intersection. This implies, that embedded curves can only develop singularities, when they shrink to a point.

At the same time Gage [37] and Gage, Li [39] considered for curves $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ and a smooth, strictly positive and symmetric function $\alpha : \mathbb{S}^1 \to \mathbb{R}$ the flow

$$\partial_t \gamma = V(\tau, \kappa) = \alpha(\tau)\kappa\nu \tag{2.2}$$

motivated by the curve shortening problem in Minkowski geometry. The main theorem in this paper of Gage is the natural extension of his and Hamilton's [36, 38] work on the curve shortening problem in euclidean geometry to Minkowski geometry. The gradient flow of the Minkowski length functional is an evolution of the form (2.2) and an initially convex curve will shrink to a point with its shape converging in Hausdorff distance to the solution of the isoperimetric problem in the Minkowski geometry. Proving the existence of self-similar solutions Gage and Li [39, Corollary 0.2] were able to show that every evolution of the form (2.2) is the curve shortening flow of a uniquely determined Minkowski length functional.

Grayson's theorem [42] for the mean curvature flow states, that any embedded curve will become convex under the curve shortening flow before a singularity can occur. That this is also true for curve shortening in Minkowski geometry was proven by Zhu [96] building up on the work of Oaks [73]. Indeed, his result is even more general. For a 2-dimensional manifold M maps $\Phi, \Psi : S^1(M) \to \mathbb{R}$ obeying three conditions

(H1): Φ, $S^1(M) \to \mathbb{R}$ are smooth and bounded.

(H2): There exists $\lambda > 0$ such that $\lambda \leq \Phi \leq \lambda^{-1}$.

(H3): For all $T \in S^1(M)$ it holds $\Phi(T) = \Phi(-T)$ and $\Psi(T) = -\Psi(-T)$

he considered the flow

$$\partial_t \gamma = (\Psi(\tau, \gamma)\kappa + \Phi(\tau, \gamma))\nu.$$

He shows [96, Section 4] that if a singularity occurs in finite time, then γ shrinks to a point $p^* \in M$, the asymptotic shape is given by the Minkowski isoperimetrix of the Minkowski length functional associated to $\Phi(\cdot, p^*)$ and the rescaled curves converge in C^∞.

In our case, condition (H1) is fullfilled whenever M is smooth. Condition (H2) corresponds to the parabolicity of the system and is ensured by the prerequisite that $|M| < 1$. Condition (H3) is fullfilled since a change of sign in τ also implies the same change in $\nu = R\tau$. Therefore, the asymptotic behavior of the gradient flow (2.1) is determined by the result of Zhu.

2.2 Vector Field on the Curve

As a next attempt in studying a system that couples the motion of a curve to a vector field we consider a vector field along the curve, analogous to the ideas of Bartels, Dolzmann, Nochetto and Raisch [7]. A possible Energy in this context is given by

$$E(\gamma, M) = L(\gamma) + \int_\gamma \langle M(s), \tau(s)\rangle^2 \mathrm{d}s + \frac{1}{2}\int_\gamma |\partial_s M(s)|^2 \mathrm{d}s.$$

In computing the variation of this energy we have to make a choice for the change of M under a normal variation of γ. The easiest choice which we will adopt in the following is $M' = 0$. That is, M is constant under normal variations of γ. We set $\gamma_\varepsilon = \gamma + \varepsilon\phi\nu$ and $M_\varepsilon = M + \varepsilon\eta$, where

$\eta : \mathbb{S}^1 \to \mathbb{R}^2$ is an arbitrary variation of M. We obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} E(\gamma_\varepsilon, M) &= -\int_\gamma 2\phi\partial_s\left(\langle M,\tau\rangle\langle M,\nu\rangle\right)\mathrm{d}s + \int |\partial_s M|^2\phi\kappa\mathrm{d}s \\
&\quad -\int_\gamma \left(1 + \langle M(s),\tau(s)\rangle^2 + \frac{1}{2}|\partial_s M(s)|^2\right)\phi\kappa\mathrm{d}s \\
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} E(\gamma, M_\varepsilon) &= \int_\gamma (-M_{ss} + 2\langle M,\tau\rangle\tau)\cdot\eta\mathrm{d}s.
\end{aligned}$$

Thus, for the gradient flow we deduce analogous to the derivation in (2.1) the normal velocity $V = \langle\gamma_t, \nu\rangle$ as

$$V = \kappa(1 - |M|^2 + 3\langle M,\nu\rangle^2 - \frac{1}{2}|\partial_s M|^2) + 2\left(\langle\partial_s M,\tau\rangle\langle M,\nu\rangle + \langle\partial_s M,\nu\rangle\langle M,\tau\rangle\right)$$

and

$$\partial_t M = \partial_{ss}M - 2\langle M,\tau\rangle\tau \tag{2.3}$$

as the evolution equation for M. Here, the term $-\frac{1}{2}|\partial_s M|^2$ has the wrong sign that might cause the equation to be backward parabolic in some parts of the curve. Since this is very undesirable, we will start with the assumption, that at least for the initial condition we have

$$(1 - |M|^2 + 3\langle M,\nu\rangle^2 - \frac{1}{2}|\partial_s M|^2) > 0.$$

Analysis of Some Special Cases

When we start the flow with initial condition $M(0) \equiv 0$, we see that the flow coincides with the curve shortening flow. If we start with a round circle of radius R and for the vector field the initial configuration is a constant multiple of the unit normal $M(0) = S\nu$, the solution will preserve the symmetry due to uniqueness and we can reduce the problem to a coupled system of ordinary differential equations. To achieve this reduction, we observe that in the case of such spherical symmetry, we have

$$\begin{aligned}
\kappa &= 1/R, & \partial_s M &= -S\kappa\tau = -(S/R)\tau, \\
\partial_{ss}M &= (-S/R^2)\nu, & |M| &= S, \\
\langle M,\nu\rangle &= S, & \langle M,\tau\rangle &= 0, \\
V &= -\dot{R}.
\end{aligned}$$

We conclude

$$\begin{aligned}
\dot{R} &= -\frac{1}{R}(1 - S^2 + 3S^2 - \frac{S^2}{2R^2}) - 2\frac{S^2}{R} = -\frac{1}{R}(1 - S^2\frac{1}{2R^2}), \\
\dot{S} &= -\frac{S}{R^2}, \\
S(0) &= S_0 > 0, \\
R(0) &= R_0 > 0.
\end{aligned} \tag{2.4}$$

In the following, we want to study the behavior of this coupled system of ordinary differential equations.

Proposition 2.1
For all $S_0 \in \mathbb{R}\backslash\{0\}$ and $R_0 > 0$ the initial value problem (2.4) *has a unique solution* $(R(t), S(t))$ *and there is a $T_{\max} > 0$ such that for all $t \in [0, T_{max})$ we have $R(t) > 0$ and $S(t) \neq 0$ and*

$$\lim_{t \to T_{max}} R(t) = \lim_{t \to T_{max}} S(t) = 0.$$

Proof. First, we infer by the Picard-Lindelöf theorem, that there is a unique solution, that exists until either R or S diverges to ∞ or R becomes 0. W.l.o.g. we assume that $S_0 > 0$. If $S_0 = 0$ the system reduces to $\dot{R} = -\frac{1}{R}$ for which our claim holds. If $S_0 < 0$ we consider the equations for R and $-S$.

Step 1: S and R are bounded from above.
We observe, that $\dot{S} < 0$ for all $S, R > 0$ and thus S is strictly decreasing. Also R is decreasing, whenever $R^2 > \frac{S^2}{2}$. Therefore, R is bounded from above by $R_{\max} := \max\{R_0, \frac{S_0^2}{2}\}$.

Step 2: S decreases at least exponentially.
We observe that

$$\dot{S} = -\frac{S}{R^2} < -\frac{S}{2{R_{\max}}^2}.$$

Thus, due to a comparison principle for ordinary differential equations [90, §9], $S(t)$ is bounded from above by the solution of the ordinary differential equation

$$\dot{u} = -\frac{u}{2R_{\max}^2}, \quad u(0) = S_0$$

which is solved by

$$u(t) = S_0 e^{-\frac{1}{2R_{\max}^2}t},$$

which becomes arbitrarily small as $t \to \infty$.

Step 3: R and S can only vanish simultanuously.
By inspection of the ordinary differential equation (2.4) R is increasing, if $R^2 < \frac{S^2}{2}$. This implies, that R can only go to zero, when S does. Next we show, that S is bounded from below by a positive constant, whenever R is. This follows from comparing S to the solution of

$$\dot{u} = -\frac{2u}{R_{\min}^2}, \quad u(0) = S_0$$

which is solved by

$$u(t) = S_0 e^{-\frac{2}{R_{\min}^2}t}.$$

Step 4: The interval of existence is finite.
We want to gain insights on the ratio of S and R. We can assume without loss of generality that

$$R^2 > \frac{S^2}{2}. \tag{2.5}$$

If this condition does not hold from the beginning, the differential equation implies that R is non-decreasing while S is decreasing exponentially. Thus, at a certain point t_0, we will have equality in (2.5). At this point $\dot{R}(t_0) = 0$ and $\dot{S}(t_0) < 0$. Comparing the difference quotients at t_0, for all $h > 0$ small enough we will have

$$R^2(t_0 + h) > \frac{S^2(t_0 + h)}{2}.$$

If (2.5) is true for $t_1 \in [0, T_{\max})$ it remains true for all $t \in [t_1, T_{\max})$. To see this, let

$$t_c = \inf\{t > t_1,\, R^2(t) \leq S^2(t)/2\}.$$

By the differential equation we see that $\dot{R}(t_c) \geq 0$ and $\dot{S}(t_c) < 0$. Again considering difference quotients implies that for all $h > 0$ small enough $R^2(t_c - h) < S^2(t_c - h)/2$. But this is a contradiction to the minimality of t_c.

To conclude finite extinction time, we need a suitable upper bound for R. For $\alpha \in \mathbb{R}$, we derive the equation

$$\begin{aligned}(RS^\alpha)^\cdot =& \dot{R}S^\alpha + \alpha RS^{\alpha-1}\dot{S}\\ =& -\frac{S^\alpha}{R}(1 - \frac{S^2}{2R^2}) - \alpha RS^{\alpha-1}\frac{S}{R^2}\\ =& -\frac{S^\alpha}{R}(1 - \frac{S^2}{2R^2}) - \alpha\frac{S^\alpha}{R}\\ =& -\frac{S^\alpha}{R}(1 + \alpha - \frac{S^2}{2R^2}).\end{aligned}$$

We observe, that this term is negative, when

$$1 + \alpha - \frac{S^2}{2R^2} > 0 \quad \Leftrightarrow \quad R^2 > \frac{S^2}{2 + 2\alpha}. \tag{2.6}$$

We consider the special case $\alpha = -3/4$. Our aim is to obtain an estimate of the form

$$RS^{-3/4} < C$$

and then to conclude

$$R^2 < CS^{3/2}.$$

We see that if $RS^{-3/4}$ is non-decreasing, then from (2.6) we have $R^2 \leq 2S^2$ and thus $R^2S^{-3/2} \leq 2S^{1/2} \leq 2S_0^{1/2}$. On the other hand, if $RS^{-3/4}$ is decreasing, then it is bounded from above by the initial value or its value at the last stationary point, where it started to decrease. But in this point we have $R^2 = 2S^2 \leq 2S(0)^2$. This implies for all $t \in [0, T_{\max})$ that $R^2(t) < C(R_0, S_0)S^{3/2}(t)$ for a suitable constant $C(R_0, S_0) > 0$. We put this in the equation for $\dot{S}$ to see

$$\dot{S} = -\frac{S}{R^2} < \frac{S}{CS^{3/2}} = -\frac{1}{CS^{1/2}}.$$

Therefore, comparing S to the solution of

$$\dot{u} = -\frac{1}{Cu^{1/2}}, \quad u(0) = S_0$$

which has on $[0, CS_0^{3/2})$ the solution

$$u(t) = \left(S_0^{3/2} - t/C\right)^{2/3},$$

we see that the maximal time of existence is finite.

All steps together prove the proposition. □

Numerical experiments reproduce the behaviour predicted by the above proposition. S and R tend to zero in finite time and if we start with $R_0 < \frac{S_0}{\sqrt{2}}$ we see that R reaches its maximum, when $R^2 = \frac{S^2}{2}$. For the plots in figures 2.1 and 2.2 Matlab's ode45 routine was used.

Evolution of Important Quantities

Our first objective is to determine whether the equation remains parabolic, if the initial data is suitable. We would like to apply a minimum principle to see that

$$1 - |M|^2 - \frac{1}{2}|M_s|^2 + 3\langle M, \nu\rangle^2.$$

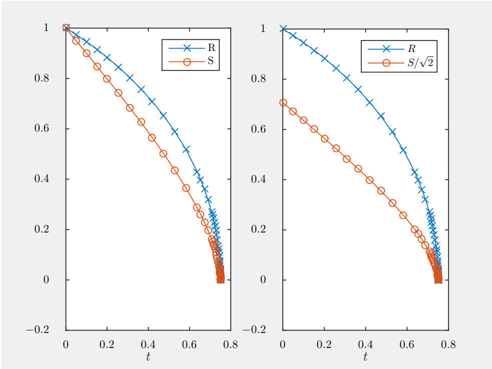

Figure 2.1: Plots for $S_0 = R_0 = 1$.

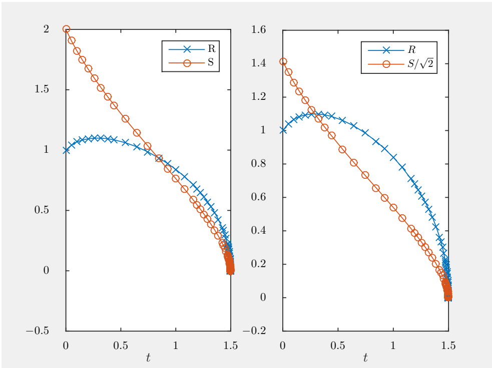

Figure 2.2: Plots for $S_0 = 2$, $R_0 = 1$.

has a fixed sign. The formula for interchange of arc length and time derivatives yields

$$\begin{aligned}\partial_t \partial_s M =& \partial_s \partial_t M + \kappa V \partial_s M \\ =& \partial^2_{ss}(\partial_s M) - 2(\langle \partial_s M, \tau\rangle \tau + \kappa \langle M, \nu\rangle \tau + \kappa\langle M, \tau\rangle \nu) + \kappa V \partial_s M.\end{aligned}$$

Moreover,

$$\begin{aligned}\partial_t \langle M, \nu\rangle =& \langle M_t, \nu\rangle + \langle M, \nu_t\rangle \\ =& \langle M_{ss}, \nu\rangle - \partial_s V \langle M, \tau\rangle \\ \partial_s \langle M, \nu\rangle =& \langle M_s, \nu\rangle - \kappa\langle \nu, \tau\rangle\end{aligned}$$

For a smooth vector valued function $u(s)$ iterated product rule yields

$$\begin{aligned}\partial_s |u|^2 =& 2\langle u, \partial_s u\rangle \\ \partial^2_{ss}|u|^2 =& 2\langle u, \partial^2_{ss} u\rangle + 2|\partial_s u|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\partial_t |M|^2 =& 2\langle M_{ss}, M\rangle - 4\langle M, \tau\rangle^2, \\ \partial_{ss}|M|^2 =& 2\langle M_{ss}, M\rangle + 2|M_s|^2, \\ \partial_t |M_s|^2 =& 2\langle \partial^2_{ss} M_s - 2(\langle M_s, \tau\rangle\tau + \kappa\langle M, \nu\rangle \tau + \kappa\langle M, \tau\rangle \nu) + \kappa V M_s, M_s\rangle, \\ \partial_{ss}|M_s|^2 =& 2\langle \partial^2_{ss} M_s, M_s\rangle + 2|M_{ss}|^2, \\ \partial_t \langle M, \nu\rangle^2 =& 2\langle M, \nu\rangle(\langle M_{ss}, \nu\rangle - \partial_s V\langle M, \tau\rangle), \\ \partial_{ss}\langle M, \nu\rangle^2 =& 2\langle M, \nu\rangle(\langle M_{ss}, \nu\rangle - 2\kappa\langle M_s, \tau\rangle - \kappa^2\langle M, \nu\rangle - \kappa_s\langle M, \tau\rangle) + 2(\partial_s\langle M, \nu\rangle)^2.\end{aligned}$$

We use these identities to obtain an equation for

$$|M|^2 + \frac{1}{2}|M_s| - 3\langle M, \nu\rangle^2$$

to show that this quantity cannot attain the value 1, what would imply loss of parabolicity. We calculate

$$\begin{aligned}\partial_t\langle M, \nu\rangle^2 = \partial_{ss}\langle M, \nu\rangle^2 - 2\partial_s V\langle M, \nu\rangle\langle M, \tau\rangle + 4\kappa\langle M, \nu\rangle\langle M_s, \tau\rangle \\ +2\kappa^2\langle M, \nu\rangle^2 + 2\kappa_s\langle M, \nu\rangle\langle M, \tau\rangle - 2(\partial_s\langle M, \nu\rangle)^2\end{aligned}$$

and hence

$$\begin{aligned}&\partial_t(|M|^2 + \frac{1}{2}|M_s|^2 - 3\langle M, \nu\rangle^2) \\ =\ &\partial_{ss}(|M|^2 + \frac{1}{2}|M_s|^2 - 3\langle M, \nu\rangle^2) \\ &-4\langle M, \tau\rangle^2 - 2|M_s|^2 - |M_{ss}|^2 - 6(\partial_s\langle M, \nu\rangle)^2 - 2\langle M_s, \tau\rangle^2 \\ &-2\kappa\langle M, \nu\rangle\langle \tau, M_s\rangle - 2\kappa\langle M, \tau\rangle\langle \nu, M_s\rangle + \kappa V|M_s|^2 + 6\kappa^2\langle M, \nu\rangle^2 \\ &-6\partial_s V\langle M, \nu\rangle\langle M, \tau\rangle + 12\kappa\langle M_s, \tau\rangle\langle M, \nu\rangle + 6\kappa_s\langle M, \tau\rangle\langle M, \nu\rangle.\end{aligned}$$

Examining the signs of the involved terms it seems rather difficult rearrange them in a manner that allows the application of a minimum principle. We present some numerical experiments in the appendix of this work.

3

A Helfrich-Type Model for Biomembranes

In his seminal work, Helfrich [47] discussed different types of elastic energies for models of bilipid membranes. In the following we discuss the analytic properties of an energy for biomembranes-taking into account the bending energy as well as a term stemming from a vector field on the surface representing the orientation of the bilipid molecules the membrane consists of. This model was introduced by Bartels, Dolzmann, Nochetto and Raisch [7] in 2011. It is a generalized form of a Helfrich energy, that penalizes the deviation of the surface's mean curvature from a spontaneous curvature. There have been different approaches on how to choose this spontaneous curvature. In some cases it is considered constant [21, 28] or it is given by a vector field on the surrounding space [91]. The approach of Bartels, Dolzmann, Nochetto and Raisch tries to take the molecular structure of bio membranes into account explaining the spontaneous curvature by the above mentioned vector field on the moving surface. The model is discussed in detail in the following section.

Let M be a smooth, orientable manifold of dimension $d \in \mathbb{N}$, then we recall for $k \in \mathbb{N}$, $k \geq 2$ the notation from Definition 1.5 introducing $C^1_{\text{imm}}(M, \mathbb{R}^{d+1})$ and $H^k_{\text{imm}}(M, \mathbb{R}^{d+1})$.

Let $\varphi : M \to \mathbb{R}^{d+1}$ be a smooth immersion. We can pull back the metric of $\mathbb{R}^{d+1}$ and the unit normal ν via φ, to have a Riemannian manifold (M, g). For a map $n : M \to \mathbb{R}^{d+1}$ we can calculate the gradient, divergence and Laplacian as explained in Section 1.2.

For physical constants $\lambda > 0$ and $\delta \geq 0$ we are interested in the analytic properties of the energy functional $E : H^2_{\text{imm}}(M, \mathbb{R}^{d+1}) \times H^1(M, \mathbb{R}^{d+1}) \to \mathbb{R}$ given by

$$E(\varphi, n) = \frac{1}{2} \int_M (\operatorname{div}_\varphi \nu_\varphi - \delta \operatorname{div}_\varphi n)^2 \, \mathrm{d}\mu_\varphi + \frac{\lambda}{2} \int_M |\nabla_\varphi n|^2 \, \mathrm{d}\mu_\varphi. \tag{3.1}$$

Moreover, from a physical point of view one may want to impose additional constraints. Let ω_{d+1} denote the volume of $B(0,1) \subset \mathbb{R}^{d+1}$. For V_0, $A_0 \in \mathbb{R}$ with $A_0 \geq n V_0^{\frac{d}{d+1}} \omega_{d+1}$ we may consider the constraints

$$\begin{aligned}
&\text{Fixed signed volume enclosed by } \varphi(M): && V(\varphi) = \int_M \varphi \cdot \nu_\varphi \, \mathrm{d}\mu_\varphi = V_0, \\
&\text{Fixed surface area of } \varphi(M): && A(\varphi) = \int_M 1 \, \mathrm{d}\mu_\varphi = A_0, \\
&\text{Length constraint for } n: && \forall p \in M : \quad \|n(p)\|_{\mathbb{R}^{d+1}} = 1
\end{aligned} \tag{3.2}$$

In the remainder of this chapter, we will discuss analytic properties of this energy functional. We apply the direct method of calculus of variations and derive the Euler-Lagrange equations to study existence and regularity of minimizers in the case of curves, i.e. $d = 1$. Furthermore, from the Euler-Lagrange equations, we obtain an L^2-gradient flow equation as a model for the motion of a vesicle.

3.1 Scaling Properties

We want to address briefly the scaling properties of the energy introduced above. As the energy is a combination of the Willmore energy with the Dirichlet energy it inherits their scaling behavior.

Let M be a d-dimensional smooth manifold and $\varphi : M \to \mathbb{R}^{d+1}$ an immersion. For $0 < \lambda \in \mathbb{R}$, also $\lambda\varphi$ is an immersion and we can compare the induced metrics and mean curvature. Since

$$g_{ij,\varphi} = \langle \partial_i \varphi, \partial_j \varphi \rangle_{\mathbb{R}^{d+1}}$$

we see that

$$g_{ij,\lambda\varphi} = \lambda^2 g_{ij,\varphi}.$$

As the tangent vectors $X_{i,\lambda} = \partial_i(\lambda\varphi) = \lambda\partial_i\varphi$ are only scaled by the factor λ, it holds for all $i = 1, \dots, d$ and $\lambda > 0$ that $\nu \perp X_{i,\lambda}$ and thus the unit normal is invariant under scaling of the immersion. Thus,

$$h_{ij,\lambda\varphi} = \langle \partial_{ij}(\lambda\varphi), \nu_\lambda \rangle = \lambda h_{ij,\varphi}$$

and

$$H_{\lambda\varphi} = g^{ij}_{\lambda\varphi} h_{ij,\lambda\varphi} = \lambda^{-1} H_\varphi.$$

For the surface measure we find

$$\mathrm{d}\mu_{\lambda\varphi} = \sqrt{\det(g_{ij,\lambda\varphi})} = \lambda^d \,\mathrm{d}\mu_\varphi.$$

Now let $f : M \to \mathbb{R}$ be a C^1 map, then

$$\nabla_{\lambda\varphi} f = g^{ij}_{\lambda\varphi} \partial_i f X_{j,\lambda\varphi} = \lambda^{-1} \nabla_\varphi f$$

and for a not necessarily tangential vector field $v : M \to \mathbb{R}^{d+1}$ we have

$$\mathrm{div}_{\lambda\varphi}\, v = g^{ij}_{\lambda\varphi} \langle \partial_i v, X_{j,\lambda\varphi} \rangle = \lambda^{-1}\, \mathrm{div}_\varphi\, v.$$

Putting these formulas together we observe that the behavior of the energy (3.1) under a scaling of the immersion depends on the dimension d of the manifold M. We find

$$E(\lambda\varphi, n) = \lambda^{d-2} E(\varphi, n).$$

That is, in the case $d = 2$ of surfaces the energy is invariant under scaling of the immersion, whereas in the case of curves, that is $d = 1$, dilation with $\lambda > 1$ reduces the energy. In higher dimensions, shrinking will decrease the energy.

3.2 An Adaption of the Energy for Curves

In view of the foregoing discussion, in the case $M = \mathbb{S}^1$ we add a penalization term for the length of the curve. That is, for a regular curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ and a vector field $n : \mathbb{S}^1 \to \mathbb{R}^2$ we set

$$E_\alpha(\gamma, n) = E(\gamma, n) + \alpha L(\gamma). \tag{3.3}$$

Without loss of generality, we can restrict our analysis to the case $\alpha = 1$, since for $t > 0$ we have

$$E_\alpha(\alpha^t \gamma, n) = \alpha^{-t} E(\gamma, n) + \alpha^{1+t} L(\gamma)$$

and thus with $t = -\frac{1}{2}$ we find

$$E_1(\gamma, n) = \alpha^{-1/2} E_\alpha(\alpha^{-1/2}\gamma, n)$$

after rearranging the terms. In the following, we will simply write E for curves and surfaces to designate the penalized and non-penalized energy, respectively.

3.2.1 Existence of minimizers

With a bound on the length of curves along any minimizing sequence we can deduce the existence of minimizers.

Theorem 3.1
For $\delta \geq 0$, $\lambda > 0$ and $\mathcal{U} = H^2_{\text{imm}}(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2)$ the energy $E : \mathcal{U} \to \mathbb{R}$ given by

$$E(\gamma, n) = \int_\gamma \frac{1}{2}(\kappa + \delta \operatorname{div}(n))^2 + \frac{\lambda}{2} |\partial_s n|^2 + 1 \, \mathrm{d}s$$

has a smooth global minimizer. There also exists a global minimizer if we impose any combination of the constraints in (3.2).

Moreover, there is a constant $C = C(\delta, \lambda) > 0$ such that for all $(\gamma, n) \in \mathcal{U}$ it holds

$$\frac{C}{E(\gamma, n)} \leq L(\gamma) \leq E(\gamma, n). \tag{3.4}$$

Proof. We start with the study of the uncronstrained problem.Using the invariance of the energy under translation and reparametrization we take steps along the usual concept of the direct method, proving existence of a minimizer by showing weak convergence of a minimizing sequence in $H^2(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2)$. But since the open set $H^2_{\text{imm}}(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2)$ is neither convex nor closed, we have to show that the weak limit lies in $\mathcal{U}$. Smoothness of the minimizer is then guaranteed by Theorem 3.2.

We start by observing that by Young's inequality with $\varepsilon = \frac{1}{2} + \frac{\lambda}{4\delta^2}$ for all $(\gamma, n) \in \mathcal{U}$ we have

$$\begin{aligned}\int_\gamma (\kappa + \delta \operatorname{div}(n))^2 + \lambda |\partial_s n|^2 \, \mathrm{d}s &= \int_\gamma \kappa^2 + 2\delta\kappa \operatorname{div}(n) + \delta^2 \operatorname{div}(n)^2 + \lambda |\partial_s n|^2 \, \mathrm{d}s \\ &\geq \int_\gamma \frac{\lambda}{2\delta^2 + \lambda} \kappa^2 + \frac{\lambda}{2} |\partial_s n|^2 \, \mathrm{d}s,\end{aligned}$$

when we use that we can estimate $\| \operatorname{div}(n) \|_{L^2} \leq \| \partial_s n \|_{L^2}$. Moreover, from Poincaré's inequality for $\partial_s \gamma$ (see [28, (2.18)]) we have

$$4\pi^2 \leq L(\gamma) \int_\gamma \kappa^2 \, \mathrm{d}s.$$

The exact value of the Poincaré constant can be seen by considering Fourier series expansions. Together, these elementary estimates provide a non-trivial, global lower bound for E:

$$\begin{aligned}E(\gamma, n) &= \frac{1}{2} \int_\gamma (\kappa + \delta \operatorname{div}(n))^2 + \lambda |\partial_s n|^2 \, \mathrm{d}s + L(\gamma) \\ &\geq \frac{1}{2} \frac{\lambda}{2\delta^2 + \lambda} \int_\gamma \kappa^2 \, \mathrm{d}s + L(\gamma) \\ &\geq \frac{2\pi^2 \lambda}{2\delta^2 + \lambda} L^{-1}(\gamma) + L(\gamma).\end{aligned}$$

From this, we obtain for all $(\gamma, n) \in \mathcal{U}$ that

$$E \geq 2\sqrt{\frac{2\pi^2\lambda}{2\delta^2+\lambda}} \qquad \text{and} \qquad L^{-1}(\gamma) \leq \frac{2\delta^2+\lambda}{2\pi^2\lambda}E(\gamma, n).$$

Let γ_i, n_i, $i \in \mathbb{N}$ be a minimizing sequence. Since E is invariant under translations of γ and n and under reparametrization of γ we can assume without loss of generality for all $i \in \mathbb{N}$ that γ_i is parametrized proportional to arc length and that

$$\int_{\gamma_i} \gamma_i \,\mathrm{d}s = 0 \qquad \text{and} \qquad \int_{\gamma_i} n_i \,\mathrm{d}s = 0. \tag{3.5}$$

Therefore, along the minimizing sequence the energy reduces to

$$E(\gamma_i, n_i) = L^{-1}(\gamma)\frac{1}{2}\int_0^1 \left|\partial_x^2\gamma_i + \delta\langle\partial_x n_i, \partial_x\gamma_i\rangle R\partial_x\gamma_i\right|^2 + \lambda|\partial_x n_i|^2 \,\mathrm{d}x + L(\gamma).$$

We introduce R as the rotation with angle $\pi/2$ mapping τ to ν. Thus,

$$\|\partial_x n_i\|_{L^2}^2 \leq \frac{2}{\lambda}L(\gamma)E(\gamma_i, n_i)$$

and

$$\|\partial_x^2\gamma_i\|_{L^2}^2 \leq 2L(\gamma)E(\gamma_i, n_i).$$

Hence, by Poincaré's inequality the minimizing sequence is bounded in the reflexive space $H^2(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2)$ and therefore has a weakly convergent subsequence that we also refer to as $(\gamma_i, n_i) \rightharpoonup (\gamma^*, n^*)$ in $H^2(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2)$. Since $H^2 \hookrightarrow C^1$ compactly we have that $L(\gamma_i) \to L(\gamma^*) > 0$ and that γ^* is parametrized proportional to arc length and thus a regular curve.Therefore we conclude that $(\gamma^*, n^*) \in \mathcal{U}$.

Since norms are weakly sequentially lower semi-continuous and $L(\gamma_i)$ are positive and converging to a positive quantity, the energy is also lower semi-continuous and thus

$$E(\gamma^*, n^*) \leq \lim_{i\to\infty} E(\gamma_i, n_i),$$

i.e. (γ^*, n^*) is a global minimizer.

Considering suitably scaled ellipses of different eccentricity shows that the set of admissible curves is non-empty for any combination of the constraints (3.2).Choosing a minimizing sequence that fulfills the imposed constraints, it will have a weak limit by the arguments above. Observe that the condition (3.5) of vanishing mean value for the components of n is only used to infer boundedness of the minimizing sequence in L^2. This condition is obsolete when $\|n\|_\infty = 1$ is guaranteed already by the constraint.

Since the constraining maps $L(\gamma)$, $A(\gamma)$ and $|n|$ are continuous on C^1 and C^0 respectively, the constraints will be preserved under weak convergence due to the compact embedding of $H^2(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2) \hookrightarrow C^1(\mathbb{S}^1, \mathbb{R}^2) \times C^0(\mathbb{S}^1, \mathbb{R}^2)$. □

3.2.2 Regularity of stationary points

Once we have established the existence of minimizers, we use the Euler-Lagrange equation to prove their regularity. Indeed, the following result holds true for all stationary points of the energy.

Theorem 3.2
If a regular closed plane curve $\gamma \in H^2_{\mathrm{imm}}(\mathbb{S}^1, \mathbb{R}^2)$ together with a vector field $n \in H^1(\mathbb{S}^1, \mathbb{R}^2)$

is a stationary point of the energy E from (3.3) *with constants $\delta \geq 0$, $\lambda > 0$ and with any combination of the constraints* (3.2)*, then there is a reparametrization $\Psi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $(\gamma, n) \circ \Psi$ is smooth.*

Since we want to prove regularity of a stationary point with an equality constraint, we need a 'Lagrange multiplier' theorem for Banach space valued constraints. We cite the book by Deimling.

Theorem 3.3 (Theorem 26.1 from [23])
Consider real Banach spaces X, Y and $B_r(x_0) \subset X$, $E : B_r(x_0) \to \mathbb{R}$ and a continuously differentiable map $F : B_r(x_0) \to Y$, with $F(x_0) = 0$ and $R(F'(x_0))$ closed. Suppose also that

$$E(x_0) = \min\{E(x) | x \in B_r(x_0) \text{ and } F(x) = 0\}.$$

Then there exist 'Lagrange multipliers' $\lambda \in \mathbb{R}$ and $\varphi \in Y^*$, not all zero, such that

$$\lambda E'(x_0) + (F'(x_0))^* \varphi = 0.$$

If $R(F'(x_0)) = Y$, then $\lambda \neq 0$.

With this tool, we can prove the regularity theorem.

Proof of Theorem 3.2. We adapt the proof by Dall'Acqua and Pluda [20] to our setting.

We observe that for a pair $(\gamma, n) \in H^2_{\text{imm}} \times H^1$ we know that the curvature is square integrable, i.e. $\kappa \in L^2(\mathbb{S}^1)$, and we can reparametrize the curve γ by arc length through a map $\Psi : \mathbb{S}^1 \to \mathbb{S}^1$. Moreover, we see that the set $H^2_{\text{imm}} \times H^1$ is open in $H^2 \times H^1$ since the embedding $H^2 \hookrightarrow C^1$ is continuous.

Step 1: We determine the Euler-Lagrange equation.
In a first step, we show that E is C^1 as a map from $H^2 \times H^1 \to \mathbb{R}$ in an open neighbourhood of any point with finite energy and we identify the Fréchet derivative E' of E, that is a map in $C(H^2 \times H^1, (H^2 \times H^1)')$. For $(\gamma, n) \in H^2_{\text{imm}} \times H^1$, with γ parametrized by arc length, and an arbitrary pair of functions $(\varphi, \eta) \in H^2 \times H^1$ we consider the first variation

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} E(\gamma + \varepsilon\varphi, n + \varepsilon\eta).$$

For curves, we have a very explicit formula for the above expression. Let γ_ε be given by

$$\gamma_\varepsilon = \gamma + \varepsilon\varphi,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} |\dot{\gamma}_\varepsilon| = \frac{\langle \dot{\varphi}, \dot{\gamma} \rangle}{|\dot{\gamma}|} = \langle \partial_s \varphi, \partial_s \gamma \rangle |\dot{\gamma}|$$

and thus, using the identity for the second arc length derivative (1.1) we obtain for the curvature vector $\overrightarrow{\kappa} := \partial_s^2 \gamma$ the identity

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \overrightarrow{\kappa}_\varepsilon &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \frac{\ddot{\gamma}_\varepsilon}{|\dot{\gamma}_\varepsilon|^2} - \frac{\dot{\gamma}_\varepsilon \langle \dot{\gamma}_\varepsilon, \ddot{\gamma}_\varepsilon \rangle}{|\dot{\gamma}_\varepsilon|^4} \\
&= \frac{\ddot{\varphi}}{|\dot{\gamma}|^2} + (-2) \frac{\ddot{\gamma}}{|\dot{\gamma}|^3} \langle \partial_s \varphi, \partial_s \gamma \rangle |\dot{\gamma}| \\
&\quad - \left[\frac{\dot{\varphi} \langle \dot{\gamma}, \ddot{\gamma} \rangle}{|\dot{\gamma}|^4} + \frac{\dot{\gamma} \langle \dot{\varphi}, \ddot{\gamma} \rangle}{|\dot{\gamma}|^4} + \frac{\dot{\gamma} \langle \dot{\gamma}, \ddot{\varphi} \rangle}{|\dot{\gamma}|^4} + (-4) \frac{\dot{\gamma} \langle \dot{\gamma}, \ddot{\gamma} \rangle}{|\dot{\gamma}|^5} \langle \partial_s \varphi, \partial_s \gamma \rangle |\dot{\gamma}| \right] \\
&= \partial_{ss} \varphi - 2 \overrightarrow{\kappa} \langle \partial_s \varphi, \partial_s \gamma \rangle - \dot{\gamma} \left[\frac{\langle \dot{\varphi}, \ddot{\gamma} \rangle}{|\dot{\gamma}|^4} + \frac{\langle \dot{\gamma}, \ddot{\varphi} \rangle}{|\dot{\gamma}|^4} - 6 \frac{\langle \dot{\gamma}, \ddot{\gamma} \rangle}{|\dot{\gamma}|^4} \langle \partial_s \varphi, \partial_s \gamma \rangle \right].
\end{aligned}$$

With the modeling assumption (cf. [7, p. 20]) $n_\varepsilon \equiv n$, yielding $\partial_\varepsilon n = 0$, and denoting by R the rotation with angle $\pi/2$ mapping τ to ν, we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \operatorname{div}_\varepsilon(n)\nu_\varepsilon &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \frac{1}{|\dot\gamma_\varepsilon|^3}\langle \dot n, \dot\gamma_\varepsilon\rangle R\dot\gamma_\varepsilon \\
&= -3\frac{1}{|\dot\gamma|^3}\langle \partial_s\varphi, \partial_s\gamma\rangle\langle \dot n, \dot\gamma\rangle R\dot\gamma + \frac{1}{|\dot\gamma|^3}\langle \dot n, \dot\varphi\rangle R\dot\gamma + \frac{1}{|\dot\gamma|^3}\langle \dot n, \dot\gamma\rangle R\dot\varphi \\
&= -3\langle \partial_s\varphi, \partial_s\gamma\rangle\langle \partial_s n, \partial_s\gamma\rangle\nu + \langle \partial_s n, \partial_s\varphi\rangle\nu + \langle \partial_s n, \partial_s\gamma\rangle R\partial_s\varphi.
\end{aligned}$$

We use dominated convergence, the above identities, and the relation $\langle \nu, \dot\gamma\rangle = 0$, to compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} E(\gamma_\varepsilon, n) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \int_{\gamma_\varepsilon} \frac{1}{2}|\partial_{s_\varepsilon}^2\gamma_\varepsilon + \delta \operatorname{div}_\varepsilon(n)\nu_\varepsilon|^2 + \frac{\lambda}{2}|\partial_{s_\varepsilon} n|^2 + 1 \ \mathrm{d}s_\varepsilon \\
= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} &\frac{1}{2}\int_0^1 \left| \frac{\ddot\gamma_\varepsilon}{|\dot\gamma_\varepsilon|^2} - \frac{\dot\gamma_\varepsilon\langle\dot\gamma_\varepsilon, \ddot\gamma_\varepsilon\rangle}{|\dot\gamma_\varepsilon|^4} + \left(\delta\frac{1}{|\dot\gamma_\varepsilon|^3}\langle \dot n, \dot\gamma_\varepsilon\rangle\right) R\dot\gamma_\varepsilon\right|^2 |\dot\gamma_\varepsilon|\,\mathrm{d}x \\
&+ \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \frac{\lambda}{2}\int_0^1 |\frac{1}{|\dot\gamma_\varepsilon|}\dot n|^2 |\dot\gamma_\varepsilon|\,\mathrm{d}x + \int_0^1 |\dot\gamma_\varepsilon|\,\mathrm{d}x \\
= \int_\gamma &\big\langle(\kappa + \delta \operatorname{div}(n))\nu, \partial_s^2\varphi - 2\kappa\langle\partial_s\varphi, \partial_s\gamma\rangle\nu\big\rangle\,\mathrm{d}s \\
&+ \int_\gamma \big\langle(\kappa + \delta \operatorname{div}(n))\nu, -3\delta\langle\partial_s n, \tau\rangle\langle\partial_s\varphi, \tau\rangle\nu + \delta\langle\partial_s n, \partial_s\varphi\rangle\nu + \delta\langle\partial_s n, \tau\rangle R\partial_s\varphi\big\rangle\,\mathrm{d}s \\
&+ \frac{1}{2}\int_\gamma (\kappa + \delta \operatorname{div}(n))^2\langle\partial_s\varphi, \partial_s\gamma\rangle\,\mathrm{d}s + \frac{\lambda}{2}\int_\gamma -|\partial_s n|^2\langle\partial_s\varphi, \partial_s\gamma\rangle\,\mathrm{d}s + \int_\gamma \langle\partial_s\varphi, \partial_s\gamma\rangle\,\mathrm{d}s.
\end{aligned}$$

The variation with respect to the vector field is easier to compute. For $\eta \in H^1(S^1, \mathbb{R}^2)$ and $n_\varepsilon = n + \varepsilon\eta$ we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} E(\gamma, n_\varepsilon) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \int_\gamma \frac{1}{2}|\partial_s^2\gamma + \delta \operatorname{div}(n_\varepsilon)\nu|^2 + \frac{\lambda}{2}|\partial_s n_\varepsilon|^2 + 1 \ \mathrm{d}s \\
= \int_\gamma &\langle\partial_s^2\gamma + \delta \operatorname{div}(n)\nu, \delta \operatorname{div}(\eta)\nu\rangle + \langle\partial_s n, \partial_s\eta\rangle\,\mathrm{d}s.
\end{aligned}$$

That the first variation is indeed the Fréchet derivative follows from the continuous embedding $H^2 \hookrightarrow C^1$ in one space dimension and Hölder's inequality. By inspection of the L^2 products given above, we see that for all $(\gamma, n) \in H^2_{\mathrm{imm}} \times H^1$ and $(\varphi, \eta) \in H^2 \times H^1$ the map

$$E'(\gamma, n)(\varphi, \eta) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} E(\gamma + \varepsilon\varphi, n + \varepsilon\eta) \leq C(\gamma, n)\|(\varphi, \eta)\|_{H^2\times H^1}$$

is a continuous linear operator and that E' depends continuously on γ and n.

For the maps

$$\begin{aligned}
A &: H^2(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2) \to \mathbb{R}, \quad (\gamma, n) \mapsto \int_\gamma 1\,\mathrm{d}s, \\
V &: H^2(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2) \to \mathbb{R}, \quad (\gamma, n) \mapsto \int_\gamma \langle\gamma, \nu\rangle\,\mathrm{d}s, \\
G &: H^2(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2) \to H^1(\mathbb{S}^1), \quad (\gamma, n) \mapsto \|n\|^2_{\mathbb{R}^2} - 1,
\end{aligned}$$

realizing the constraints—observe that G maps to H^1 as H^1 is closed under multiplication in dimension one—we obtain the derivatives

$$\begin{aligned}
A'(\gamma, n)[\varphi, \eta] &= \int_\gamma \langle\partial_s\varphi, \partial_s\gamma\rangle\,\mathrm{d}s, \\
V'(\gamma, n)[\varphi, \eta] &= \int_\gamma \langle\varphi, \nu\rangle + \langle\gamma, R\partial_s\varphi\rangle + \langle\gamma, \nu\rangle\langle\partial_s\varphi, \partial_s\gamma\rangle\,\mathrm{d}s, \\
G'(\gamma, n)[\varphi, \eta] &= 2\langle n, \eta\rangle_{\mathbb{R}^2}.
\end{aligned}$$

Summarizing the above considerations we obtain the Euler-Lagrange equation with Lagrange multipliers as a necessary condition that a minimizer must fulfill according to Theorem 3.3. Observe, that A' and V' are linearly independent if and only if γ^* is not a round circle. In this case, γ^* is already smooth and the regularity of n^* follows by the usual theory for the Laplace equation. Thus, we can assume in the following that the differential of the constraining map is surjective.That is, in the following we consider $(\gamma^*, n^*) \in H^2_{\text{imm}}(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2)$ for which there exist $a, b \in \mathbb{R}$ such that for all $\varphi \in H^2(\mathbb{S}^1, \mathbb{R}^2)$ we have

$$\big[\frac{\partial}{\partial\gamma}E(\gamma^*, n^*) + a\frac{\partial}{\partial\gamma}A(\gamma^*, n^*) + b\frac{\partial}{\partial\gamma}V(\gamma^*, n^*)\big]\varphi = 0.$$

Here we mean by $\frac{\partial}{\partial\gamma}$ the Frechét derivative with respect to the first component. We observe that with the exception of the first term in $\frac{\partial}{\partial\gamma}E$, only the first derivative of φ is involved. We rearrange the terms to obtain

$$\begin{aligned}
&\int_{\gamma^*} \big\langle (\kappa^* + \delta \operatorname{div}(n^*))\nu^*, \partial_s^2\varphi \big\rangle \, \mathrm{d}s \\
= &\int_{\gamma^*} \Big(\big((\kappa^* + \delta \operatorname{div}(n^*)) \big(2\kappa^* + 3\delta \operatorname{div}(n^*) \big) - \frac{1}{2}\big((\kappa^* + \delta \operatorname{div}(n^*))^2 - \lambda |\partial_s n^*|^2 \big) + 1 \Big) \langle \partial_s\varphi, \partial_s\gamma^* \rangle \\
&-\big((\kappa^* + \delta \operatorname{div}(n^*) \big) \big((\delta \langle \partial_s n^*, \partial_s\varphi \rangle + \delta \operatorname{div}(n^*) \langle \nu^*, R\partial_s\varphi \rangle \big) \, \mathrm{d}s \\
&+a \int_{\gamma^*} \langle \partial_s\varphi, \partial_s\gamma^* \rangle \, \mathrm{d}s + b \int_{\gamma^*} \langle \varphi, \nu^* \rangle + \langle \gamma^*, R\partial_s\varphi \rangle + \langle \gamma^*, \nu^* \rangle \langle \partial_s\varphi, \partial_s\gamma^* \rangle \, \mathrm{d}s =: F(\varphi).
\end{aligned} \tag{3.6}$$

Step 2: The quantity $\kappa^* + \delta \operatorname{div}(n^*)$ is essentially bounded.
Since we know that κ^*, $\operatorname{div}(n^*) \in L^2(\mathbb{S}^1)$ and $\tau, \nu \in L^\infty(\mathbb{S}^1)$, we can estimate the right hand side with

$$|F(\varphi)| \leq C\|\partial_s\varphi\|_\infty.$$

Thus,

$$|F(\varphi)| \leq C\|\varphi\|_{W^{1,\infty}(\mathbb{S}^1)} \leq C\|\varphi\|_{W^{2,1}(\mathbb{S}^1)}$$

by the special embedding $W^{2,1}(\mathbb{S}^1) \hookrightarrow W^{1,\infty}(\mathbb{S}^1)$ that is only true in one space dimension. This estimate shows that for a stationary point (γ^*, n^*) also

$$\int_{\gamma^*} (\kappa^* + \delta \operatorname{div}(n^*)) \langle \nu^*, \partial_{ss}\varphi \rangle \, \mathrm{d}s \leq C\|\varphi\|_{W^{2,1}(\mathbb{S}^1)}.$$

Still following the lines of [20, Proof of Proposition 4.1] let $\sigma \in C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ be a smooth map and observe that we may assume that γ^* is parametrized by arc length, i.e. $|\partial_x\gamma^*| = L(\gamma^*) > 0$ is constant. We set

$$\tilde{\varphi}(x) = \int_0^x |\partial_x\gamma^*(y)|^2 \int_0^y \sigma(t) \, \mathrm{d}t \, \mathrm{d}y.$$

For $\alpha(\sigma) := \partial_s\tilde{\varphi}(1) - 2\tilde{\varphi}(1)$, $\beta(\sigma) := \tilde{\varphi}(1) - \partial_s\tilde{\varphi}(1)$ we set

$$\varphi(x) = \tilde{\varphi}(x) + \alpha x + \beta x^2.$$

Consequently,

$$\varphi(0) = \varphi(1) = 0, \quad \partial_s\varphi(0) = \partial_s\varphi(1) = 0 \quad \text{and} \quad \partial_s^2\varphi = \sigma + 2\frac{\beta}{|\partial_x\gamma^*|^2}$$

and thus $\varphi \in W^{2,2}(\mathbb{S}^1, \mathbb{R}^2)$, with

$$|\alpha(\sigma)|, |\beta(\sigma)|, \|\partial_s\varphi\|_\infty \leq C\|\sigma\|_{L^1(\mathbb{S}^1)}.$$

Plugging this special test function into equation (3.6) we see that

$$\int_{\gamma^*} \big(\kappa^* + \delta \operatorname{div}(n^*)\big)\langle \nu^*, \sigma\rangle \,\mathrm{d}s \le C\|\sigma\|_{L^1}.$$

Therefore, for $i = 1, 2$ both maps $\sigma \mapsto \int_{\gamma^*} \big(\kappa^* + \delta \operatorname{div}(n^*)\big)\nu_i^* \sigma_i \,\mathrm{d}s$ can be extended to map $\xi_i \in \big(L^1(\mathbb{S}^1)\big)^*$ that is for all $u \in L^1(\mathbb{S}^1)$ represented by a function $z_i \in L^\infty(\mathbb{S}^1)$ via $\xi_i(u) = \int_{\gamma^*} z_i u \,\mathrm{d}s$. For all $v \in C^\infty(\mathbb{S}^1)$ however, we know that $\int_{\gamma^*} \big((\kappa^* + \delta \operatorname{div}(n^*))\nu_i^* - z_i\big) v \,\mathrm{d}s = 0$ and thus $(\kappa^* + \delta \operatorname{div}(n^*))\nu_i^* = z_i$ in $L^2(\mathbb{S}^1)$. From this and $\nu^* \in L^\infty$ we conclude that $\kappa^* + \delta \operatorname{div}(n^*) \in L^\infty(\mathbb{S}^1)$.

Step 3: We analyze the Lagrange multiplier for $\|n^*\|_{\mathbb{R}^2} \equiv 1$.
We turn to the equation for the variation with respect to n. Here, the Lagrange multiplier is an element $\varphi \in H^1(\mathbb{S}^1, \mathbb{R})'$. In a point (γ^*, n^*) where the constraint $G(\gamma^*, n^*) = 0$ is satisfied, the derivative

$$G'(\gamma^*, n^*) : H^2 \times H^1 \to H^1, \quad (\varphi, u) \mapsto 2\langle n^*, u\rangle_{\mathbb{R}^2}$$

is onto, since we can explicitly construct a right inverse as follows. When $\|n^*\|_{\mathbb{R}^2} \equiv 1$, we define a map $H : H^1(\mathbb{S}^1, \mathbb{R}) \to H^2(\mathbb{S}^1, \mathbb{R}^2) \times H^1(\mathbb{S}^1, \mathbb{R}^2)$ by

$$w \mapsto (0, \frac{1}{2} w n^*)$$

that is well defined, linear and continuous since H^1 is closed under multiplication for a one-dimensional domain. As $\langle n^*, n^*\rangle_{\mathbb{R}^2} \equiv 1$ we conclude that

$$G'(\gamma^*, n^*) \circ H = \mathrm{Id}_{H^1(\mathbb{S}^1)}\,.$$

To avoid confusion, in the following we will use the symbol † to denote an adjoint operator. The Banach space version of the Lagrange multiplier theorem as stated in Theorem 3.3 now yields the existence of $\varphi \in H^1(\mathbb{S}^1, \mathbb{R})^*$ such that

$$\frac{\partial E}{\partial n}(\gamma^*, n^*) + (G')^\dagger(\gamma^*, n^*)\varphi = 0$$

Multiplying the equation by the dual operator $H^\dagger$ we find

$$\varphi = -H^\dagger \frac{\partial E}{\partial n}(\gamma^*, n^*).$$

For an element $u \in H^1(\mathbb{S}^1, \mathbb{R})$ we use $\varphi(u) = -H^\dagger \frac{\partial E}{\partial n}(\gamma^*, n^*)(u) = \frac{\partial E}{\partial n}(\gamma^*, n^*)(Hu)$ and obtain

$$\begin{aligned}\frac{\partial E}{\partial n}(\gamma^*, n^*)(Hu) &= \int_{\gamma^*} \langle \partial_s n^*, \partial_s (n^* u)\rangle + (\kappa^* + \delta \operatorname{div}(n^*)) \operatorname{div}(n^* u) \,\mathrm{d}s \\ &= \int_{\gamma^*} u\langle \partial_s n^*, \partial_s n^*\rangle + (\kappa^* + \delta \operatorname{div}(n^*))(u \operatorname{div}(n^*) + \langle n, \nabla u\rangle) \,\mathrm{d}s \\ &\le C(\|u\|_\infty + \|u\|_{W^{1,1}(\mathbb{S}^1)}).\end{aligned}$$

Here we used that from $\langle n^*, n^*\rangle_{\mathbb{R}^2} \equiv 1$ we infer $\langle n^*, \partial_s n^*\rangle_{\mathbb{R}^2} = 0$ and that we already know $\kappa^* + \delta \operatorname{div}(n^*) \in L^\infty(\mathbb{S}^1)$.

Thus, for (γ^*, n^*) we have for all $\eta \in H^1(\mathbb{S}^1, \mathbb{R}^2)$ that

$$\begin{aligned}0 &= \int_{\gamma^*} \langle \partial_s n^*, \partial_s \eta\rangle + (\kappa^* + \delta \operatorname{div}(n^*)) \operatorname{div}(\eta) \,\mathrm{d}s - \varphi(G'\eta) \\ &= \int_{\gamma^*} \langle \partial_s n^*, \partial_s \eta\rangle + (\kappa^* + \delta \operatorname{div}(n^*)) \operatorname{div}(\eta) \,\mathrm{d}s \\ &\quad - \int_{\gamma^*} \langle n^*, \eta\rangle \langle \partial_s n^*, \partial_s n^*\rangle + (\kappa^* + \delta \operatorname{div}(n^*))(\langle n^*, \eta\rangle \operatorname{div}(n^*) + \langle n^*, \nabla \langle n^*, \eta\rangle\rangle) \,\mathrm{d}s.\end{aligned}$$

Using again $(\kappa^* + \delta \operatorname{div}(n^*)) \in L^\infty(\mathbb{S}^1)$ and that $W^{1,1}(\mathbb{S}^1)$ is a Banach algebra, we estimate

$$\begin{aligned}
&\int_{\gamma^*} \langle \partial_s n^*, \partial_s \eta \rangle \, \mathrm{d}s \\
&= \int_{\gamma^*} (\kappa^* + \delta \operatorname{div}(n^*)) \operatorname{div}(\eta) \, \mathrm{d}s \\
&- \int_{\gamma^*} \langle n^*, \eta \rangle \langle \partial_s n^*, \partial_s n^* \rangle + (\kappa^* + \delta \operatorname{div}(n^*))(\langle n^*, \eta \rangle \operatorname{div}(n^*) + \langle n^*, \nabla \langle n^*, \eta \rangle \rangle) \, \mathrm{d}s \\
&\leq C \|\eta\|_{W^{1,1}(\mathbb{S}^1)}.
\end{aligned} \tag{3.7}$$

For $\sigma \in C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ we plug the test function

$$\eta(x) = \int_0^x |\gamma(t)| \sigma(t) \, \mathrm{d}t - \Big(\int_0^1 |\gamma(t)| \sigma(t) \, \mathrm{d}t \Big) x$$

into (3.7) and conclude again by duality that $\partial_s n^* \in L^\infty(\mathbb{S}^1)$ and hence also $\kappa^* \in L^\infty(\mathbb{S}^1)$.

Step 4: We establish regularity for distributional derivatives.
Examining F further we can exploit the additional regularity of κ^* and n^* to estimate

$$|F(\varphi)| \leq C \|\varphi\|_{W^{1,1}(\mathbb{S}^1)}.$$

Integrating by parts in the left hand sides of (3.6) and (3.7), we observe that the distributional derivatives of $\kappa^* + \delta \operatorname{div}(n^*)$ and $\partial_s n^*$ are indeed in L^∞, which yields $(\kappa^* + \delta \operatorname{div}(n^*)) \in W^{1,\infty}$. Integrating by parts in the equation for n^*, we obtain $n^* \in W^{2,\infty}$ and hence $\kappa^* \in W^{1,\infty}$. Interpreting (3.6) and (3.7) as weak formulations of elliptic partial differential equations with L^2 right hand sides, we can use L^2 regularity theory for elliptic equations [83, Ch. 5, Theorem 11.1]for a bootstrap procedure and inductively improve the regularity of κ^* and n^*. From this we conclude $(\gamma^*, n^*) \in C^\infty(\mathbb{S}^1)$. □

Remark 3.4
One is now tempted to conjecture what the minimizing shape might look like. Looking at the calculations in Section 3.5 one could assume that the symmetric configuration of a circle together with a multiple of the unit normal is optimal in the unconstrained setting. But as soon as constraints come into play, the situation becomes rather involved and numerical experiments show a variety of different and potentially rather wild, at least numerically stable, configurations.

3.3 The L^2-Gradient Flow

We return to the general setting of immersions φ of a d-dimensional manifold M to $\mathbb{R}^{d+1}$ and vector fields n on M. We obtain a geometric evolution equation to model the motion of a membrane that is not an equilibrium point of the energy $E(\varphi, n)$ from (3.1) as a gradient flow. To that account we take the first variation of the energy under normal perturbations of φ and arbitrary perturbations of n. Then, we can apply the theory from Section 1.5. Following the calculations of Bartels, Dolzmann, Nochetto and Raisch in [7] and integrating by parts we obtain a gradient flow dynamic for φ and n.

Proposition 3.5
For a smooth immersion $\varphi : M \to \mathbb{R}^{d+1}$ and a smooth vector field $n : M \to \mathbb{R}^{d+1}$ the first

variation of the energy (3.1) *for smooth functions* $\Psi : M \to \mathbb{R}$*and* $\eta : M \to \mathbb{R}^{d+1}$ *is given by*

$$\frac{\delta E}{\delta \varphi}(\varphi, n)(\Psi \nu_\varphi) = \langle H_\varphi + \delta \operatorname{div}_\varphi n, \Delta_\varphi \Psi + \Psi |\nabla_\varphi \nu_\varphi|^2 - \delta \Psi \nabla_\varphi n : \nabla_\varphi \nu + \delta(\nu_\varphi^T \nabla_\varphi n) \cdot \nabla_\varphi \Psi \rangle_{L^2}$$
$$- \frac{1}{2} \langle (H_\varphi + \delta \operatorname{div}_\varphi n)^2 + \lambda |\nabla_\varphi n|^2, H_\varphi \Psi \rangle_{L^2} - \lambda \langle (\nabla_\varphi n)^T : \left[\nabla_\varphi \nu_\varphi (\nabla_\varphi n)^T\right], \Psi \rangle_{L^2}$$

and

$$\frac{\delta E}{\delta n}(\varphi, n)(\eta) = \delta \langle H_\varphi + \delta \operatorname{div}_\varphi n, \operatorname{div}_\varphi \eta \rangle_{L^2} + \lambda \langle \nabla_\varphi n, \nabla_\varphi \eta \rangle_{L^2}.$$

The gradient flow equation for a time dependent family of immersions $\varphi : M \times [0, T) \to \mathbb{R}^{d+1}$ *and vector fields* $n : M \times [0, T) \to \mathbb{R}^{d+1}$ *with respect to the* L^2 *scalar product is given by*

$$\begin{aligned} \partial_t \varphi =& \Big[- \Delta_\varphi (H_\varphi + \delta \operatorname{div}_\varphi n) - (H_\varphi + \delta \operatorname{div}_\varphi n) |\nabla_\varphi \nu_\varphi|^2 + \delta (H_\varphi + \delta \operatorname{div}_\varphi n) \nabla_\varphi n : \nabla_\varphi \nu_\varphi \\ &+ \delta \operatorname{div}_\varphi ((H_\varphi + \delta \operatorname{div}_\varphi n) \nu_\varphi^T \nabla_\varphi n) + \lambda (\nabla_\varphi n)^T : \left[\nabla_\varphi \nu_\varphi (\nabla_\varphi n)^T\right] \\ &+ \frac{1}{2} (H_\varphi + \delta \operatorname{div}_\varphi n)^2 H_\varphi + \frac{\lambda}{2} |\nabla_\varphi n|^2 H_\varphi \Big] \, \nu_\varphi, \\ \partial_t n =& \lambda \Delta_\varphi n + \delta \nabla_\varphi (H_\varphi + \delta \operatorname{div}_\varphi n) + \delta (H_\varphi + \delta \operatorname{div}_\varphi n) H_\varphi \nu_\varphi. \end{aligned} \tag{3.8}$$

Proof. We consider the family of immersions given by $\varphi_\varepsilon = \varphi + \varepsilon \Psi \nu_0$ which is normal at $\varepsilon = 0$. In view of Lemma 1.11 the time derivative of a surface integral of a spatial function does not depend on the tangential part of the time derivative of the family of immersions. Therefore, it is sufficient to consider only a normal variation of φ. We furnish all geometric quantities that depend on ε with the respective index and write f' to mean $\partial_\varepsilon|_{\varepsilon=0} f_\varepsilon$ for such a quantity f_ε. Lemma 1.11 already provides some formulas, we continue the calculation following e.g. Nochetto and Dǒgan [26]. Differentiating

$$1 = \langle \nu_\varepsilon, \nu_\varepsilon \rangle \quad \text{and} \quad 0 = \langle \nu_\varepsilon, X_{j,\varepsilon} \rangle$$

we see that $\partial_\varepsilon \nu_\varepsilon$ is tangential. Thus,

$$\langle \nu', X_j \rangle = -\langle \nu, X_j' \rangle = -\langle \nu, \partial_j (\Psi \nu) \rangle = -\langle \nu, \partial_j \Psi \nu \rangle = -\partial_j \Psi.$$

For a tangential vector $v \in T_p M$ we have the representation $v = g^{ij} \langle v, X_i \rangle X_j$. We find

$$\nu' = -g^{ij} \partial_i \Psi X_j = -\nabla \Psi.$$

We compute the evolution of the mean curvature

$$\begin{aligned} H' =& \, \partial_\varepsilon|_{\varepsilon=0} \operatorname{div}_{\varphi_\varepsilon}(\nu_\varepsilon) = - \, \partial_\varepsilon|_{\varepsilon=0} \, (g_\varepsilon^{ij} (\partial_i \nu_\varepsilon) \cdot X_{j,\varepsilon}) \\ =& 2 \Psi h^{ij} h_{ij} + \operatorname{div}_\varphi (\nabla \Psi) - (g^{ij} \partial_i \nu \cdot ((\partial_j \Psi) \nu + \Psi \partial_j \nu) \\ =& \Delta_\varphi \Psi + \Psi |A_\varphi|^2. \end{aligned}$$

Here, A_φ is the second fundamental form of φ. The only terms that require additional effort are $\partial_\varepsilon|_{\varepsilon=0} \operatorname{div}_{\varphi_\varepsilon}(n)$ and $\partial_\varepsilon|_{\varepsilon=0} |\nabla_{\varphi_\varepsilon} n|^2$. We follow [7] and recall, that for a map $v : M \to \mathbb{R}^{d+1}$ the expression $\nabla_\varphi v$ is short for the matrix $\sum_\alpha e_\alpha \otimes \nabla_\varphi v^\alpha$. The rows of this matrix are the tangential gradients of the components of v. Thus, for two such vector fields v, w, we find the following identities. For the matrix scalar product we have

$$\nabla_\varphi v : \nabla_\varphi w = g^{ij} \langle \partial_i v, \partial_j w \rangle.$$

For a function $f : M \to \mathbb{R}$ we have

$$\nabla_\varphi v \nabla_\varphi f = \sum_\alpha e_\alpha (g_\varphi^{ij} \partial_i v^\alpha \partial_j f)$$

and for matrix multiplication

$$\nabla_\varphi v(\nabla_\varphi w)^T = \sum_{\alpha,\beta} e_\alpha \otimes e_\beta (g_\varphi^{ij}\partial_i v^\alpha \partial_j w^\beta) = \sum_\beta \nabla_\varphi v \nabla_\varphi w^\beta.$$

From here, we suppress the index φ. Observe that it was a modeling assumption in [7, p. 20] to set $n_\varepsilon \equiv n$ and thus have $n' = 0$, which we will adopt in the following. Using these formulas and the Weingarten relation $\nabla\nu X_i = \partial_i \nu = -h_{ik}g^{k\ell}X_\ell$ we calculate

$$\begin{aligned}\partial_\varepsilon|_{\varepsilon=0} \operatorname{div}_\varepsilon(n_\varepsilon) = \partial_\varepsilon|_{\varepsilon=0} &\left(g_\varepsilon^{ij}\langle\partial_i n_\varepsilon, X_{j,\varepsilon}\rangle\right) \\ =&2\Psi h_{k\ell}g^{ki}g^{\ell j}\langle\partial_i n, X_j\rangle + g^{ij}\langle\partial_i n, \nu\partial_j\Psi + \Psi h_j^k X_k\rangle \\ =&-2\Psi\nabla\nu : \nabla n + \nu^T\nabla n\cdot\nabla\Psi + \Psi\nabla\nu : \nabla n\end{aligned}$$

and

$$\begin{aligned}\partial_\varepsilon|_{\varepsilon=0}|\nabla_\varepsilon n_\varepsilon|^2 =& \partial_\varepsilon|_{\varepsilon=0}\sum_\alpha g_\varepsilon^{ij}\partial_i n_\varepsilon^\alpha \partial_j n_\varepsilon^\alpha \\ =&\sum_\alpha 2\Psi h_{k\ell}g^{ki}g^{\ell j}\partial_i n^\alpha\partial_j n^\alpha \\ =&-2\Psi\sum_\alpha \nabla n^\alpha\cdot\nabla\nu\nabla n^\alpha \\ =&-2\Psi(\nabla n)^T : \left[\nabla\nu(\nabla n)^T\right].\end{aligned}$$

For sufficiently regular φ and n repeatedly integrating by parts yields an L^2-gradient-like structure for this first variation that constitutes the gradient flow equation (3.8). □

The proof of well-posedness of this flow is given in Chapter 4, where we follow Huisken and Polden [50, Chap. 7] and Mantegazza [68, Chap. 1, App. A], including the cases of area and volume constraints as well as additional lower order components in the energy.

We will construct a solution, that solves equation (3.8) only up to tangential motion away from time 0. This can be corrected by reparametrizing M by solving an ordinary differential equation as in the proof of Proposition 1.3.4 in Mantegazza's book [68]. In our case, however, we have to take into account what happens to n under reparametrization of M.

Lemma 3.6 (Reparametrization of solutions)
For $T > 0$, let $\varphi : M \times [0,T) \to \mathbb{R}^{d+1}$ and $n : M \times [0,T) \to \mathbb{R}^{d+1}$ be an immersion and a vector field. Given a tangential vector field $X : M \times [0,T) \to \mathbb{R}^{d+1}$ with $X(p,t) \in \mathrm{d}\varphi(p,t)[T_pM]$, i.e. $X \in \Gamma(\varphi^*TM)$, if (φ, n) is a smooth solution of the system

$$\begin{aligned}\partial_t\varphi =& \Big[-\Delta_\varphi(H_\varphi + \delta\operatorname{div}_\varphi n) - (H_\varphi + \delta\operatorname{div}_\varphi n)|\nabla_\varphi\nu|^2 + \delta(H_\varphi + \delta\operatorname{div}_\varphi n)\nabla_\varphi n : \nabla_\varphi\nu_\varphi \\ &+ \delta\operatorname{div}_\varphi((H_\varphi + \delta\operatorname{div}_\varphi n)\nu_\varphi^T\nabla_\varphi n) + \lambda(\nabla_\varphi n)^T : \left[\nabla_\varphi\nu_\varphi(\nabla_\varphi n)^T\right] \\ &+ \frac{1}{2}(H_\varphi + \delta\operatorname{div}_\varphi n)^2 H_\varphi + \frac{\lambda}{2}|\nabla_\varphi n|^2 H_\varphi\Big]\nu_\varphi + X, \\ \partial_t n =& \lambda\Delta_\varphi n + \delta\nabla_\varphi(H_\varphi + \delta\operatorname{div}_\varphi n) - \delta(H_\varphi + \delta\operatorname{div}_\varphi n)H_\varphi\nu_\varphi + \nabla_\varphi n X,\end{aligned}$$

then there is a family of diffeomorphisms $\Psi : M \times [0,T) \to M$ such that

$$(\tilde\varphi(p,t), \tilde n(p,t)) := (\varphi(\Psi(p,t),t), n(\Psi(p,t),t))$$

solves the original equation (3.8).

Proof. Let $\Psi : M \times [0,T) \to M$ be the family of diffeomorphisms constructed in Lemma 1.7 and set

$$(\tilde{\varphi}(p,t), \tilde{n}(p,t)) := (\varphi(\Psi(p,t),t), n(\Psi(p,t),t)),$$

then $\mathrm{d}\varphi(\Psi(p,t),t)\partial_t\Psi(p,t) = -X(p,t)$ and by the chain rule

$$\begin{aligned}\partial_t\tilde{\varphi}(p,t) =& \partial_t\varphi(\Psi(p,t),t) + \mathrm{d}\varphi(\Psi(p,t),t)\partial_t\Psi(p,t),\\ \partial_t\tilde{n} =& \partial_t n(\Psi(p,t),t) + \nabla_\varphi n(\Psi(p,t),t)\mathrm{d}\varphi(\Psi(p,t),t)\partial_t\Psi(p,t).\end{aligned}$$

That is, $(\tilde{\varphi}(p,t), \tilde{n}(p,t))$ is a solution of the original equation (3.8). □

In Chapter 5 we will mainly consider the evolution of geometric quantities during the flow for curves. Therefore, we state the evolution equation for curves explicitly.

Remark 3.7
Let $\gamma : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a family of planar curves with tangent τ, unit normal ν and curvature κ and let $n : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a time dependent vector field. Then, we can also slightly simplify the evolution equation using the formulas from Section1.3 yielding

$$\nabla n : \nabla\nu = (\partial_s n\tau^T) : (\partial_s\nu\tau^T) = -\kappa \operatorname{div} n$$

and

$$\nabla n^T : \nabla\nu(\nabla n)^T = -\kappa|\partial_s n|^2.$$

Thus, the equation for curves reads

$$\begin{aligned}\langle\partial_t\gamma,\nu\rangle =& -\partial_s^2(\kappa+\delta\operatorname{div} n) - \kappa^2(\kappa+\delta\operatorname{div}(n))\\ &-\delta(\kappa+\delta\operatorname{div} n)\kappa\operatorname{div} n + \delta\operatorname{div}\big((\kappa+\delta\operatorname{div}(n))\nu^T\nabla n\big)\\ &+\frac{1}{2}\kappa(\kappa+\delta\operatorname{div}(n))^2 - \frac{\lambda}{2}\kappa|\nabla n|^2 + \kappa\\ \partial_t n =& \lambda\partial_s^2 n + \delta\nabla(\kappa+\delta\operatorname{div} n) + \delta(\kappa+\delta\operatorname{div} n)\kappa\nu + \langle\partial_t\gamma,\tau\rangle\nabla n\tau.\end{aligned} \tag{3.9}$$

3.4 The Projected L^2-Gradient Flow

In this section, we compute the projection of the full L^2-gradient of E on the tangent space of the set fulfilling some of the constraints stated in (3.2). This projected gradient flow will still decrease the energy but obey the imposed constraints during the evolution. We remark the following about gradient flows with constraints: For $k, m \in \mathbb{N}$ and given an energy functional $\mathcal{E} : \mathbb{R}^k \to \mathbb{R}$ and a smooth function $g : \mathbb{R}^k \to \mathbb{R}^m$, $m < k$, we can consider the gradient flow of $\mathcal{E}$ subject to the constraint of g being constant. As long as the derivative of g has full rank, meaning that the gradients of the component functions $g_i, i = 1, \dots, m$ are linearly independent, for $c \in \mathbb{R}^m$ the level set $g = c$ is a smooth submanifold of $\mathbb{R}^k$ of codimension m or empty.Let $N_g(x)$ denote the linear subspace spanned by the gradients of $g_i(x)$. This space is then orthogonal to the tangent space of the level set $g = c$ at the point x. Then we consider the flow

$$\dot{x}(t) = -\operatorname{grad}\mathcal{E}(x(t)) + P_{N_g(x(t))}(\operatorname{grad}\mathcal{E}(x(t))) = -P_{T_g(x(t))}(\operatorname{grad}\mathcal{E}(x(t))),$$

where $P_{N_g(x(t))}$ and $P_{T_g(x(t))}$ are the orthogonal projections on $N_g(x(t))$ and $T_g(x(t))$, respectively. Since the gradient of a function on a submanifold can be obtained by projecting the full gradient of an extension of the function to the surrounding space, it is reasonable to consider this flow as the gradient flow of $\mathcal{E}$ with respect to the constraint $g(x(t)) = g(x(0))$.Using that orthogonal projections are idempotent and self-adjoint we obtain from

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(x(t)) = -\langle \operatorname{grad}\mathcal{E}(x(t)), P_{T_g(x(t))}(\operatorname{grad}\mathcal{E}(x(t)))\rangle \leq 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}g_i(x(t)) = \langle \operatorname{grad} g_i(x(t)), -\operatorname{grad}\mathcal{E}(x(t)) + P_{N_g(x(t))}(\operatorname{grad}\mathcal{E}(x(t)))\rangle = 0$$

that the energy still decreases during the flow and the condition $g = c$ is preserved during the flow. In our setting, g has three components: One is the area functional, the second is the enclosed volume and the third is the Hilbert space valued length condition for the vector field. Projecting onto the normal spaces yields corresponding correction terms.

Hence, if one considers a geometric flow for an immersion $\varphi : M \times [0, T) \to \mathbb{R}^{d+1}$ with normal velocity $v(\varphi)$, that is $\langle \partial_t\varphi, \nu_\varphi\rangle = v(\varphi)$, an area and volume preserving version is obtained by choosing the new normal velocity

$$v_{\text{constrained}}(\varphi) = v(\varphi) - \int_M v(\varphi)\,\mathrm{d}\mu_\varphi \frac{1}{\displaystyle\int_M 1\,\mathrm{d}\mu_\varphi} - \int_M v(\varphi)(H_\varphi - \overline{H}_\varphi)\,\mathrm{d}\mu_\varphi \frac{(H_\varphi - \overline{H}_\varphi)}{\displaystyle\int_M (H_\varphi - \overline{H}_\varphi)^2\,\mathrm{d}\mu_\varphi}. \tag{3.10}$$

Here, $\overline{H}_\varphi$ denotes the mean value of the mean curvature on the surface

$$\overline{H}_\varphi = \frac{1}{\displaystyle\int_M 1\,\mathrm{d}\mu_\varphi}\int_M H_\varphi\,\mathrm{d}\mu_\varphi.$$

We calculated the derivative of the length constraint earlier in the proof of Theorem 3.2. Also in the theory of the harmonic map heat flow it is important to have an evolution of a map with image in a fixed submanifold [63]. In our case the length is rather easy to control. Let $m : M \times [0, T) \to \mathbb{R}^{d+1}$ be a time dependent family of vector fields, then $|m|^2 \equiv 1$ implies $\partial_t|m|^2 = 0$ and $\langle \partial_t m, m\rangle = 0$. Thus, when the evolution for m is given by $\partial_t m = F(m)$ then a length preserving evolution is given by $\partial_t m = F(m) - \langle F(m), \frac{m}{|m|}\rangle$.

3.5 Analysis of Some Special Cases

To get a better feeling for the behavior of this flow, we start by considering situations of high symmetry.

Example 3.8
For $R, S \in \mathbb{R}$, $R > 0$, consider the initial configuration $M = \mathbb{S}^1$, $\varphi_0 : \mathbb{S}^1 \to \mathbb{R}^2$, $p \mapsto Rp$, $n_0 : \mathbb{S}^1 \to \mathbb{R}^2$, $p \mapsto S\nu(p)$. Inspecting the evolution law, we can explicitly determine an immersion φ and a vector field n that are solutions of equation (3.9). By existence and uniqueness of solutions (that we prove in the next chapter, cf. Theorem 4.24), the flow will preserve the symmetry of this initial configurationand we get a solution $\varphi_t(p) = R(t)p$, $n_t = S(t)\nu$. In this case we have curves with constant curvature $\kappa(p, t) = 1/R(t)$ and $\operatorname{div}(n(t)) = -S(t)/R(t)$. The evolution equation reduces to a system of ordinary differential equations for S and R that reads

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}R &= +\frac{1-\delta S}{R^3} - \delta\frac{(1-\delta S)S}{R^3} - \frac{1}{2}\frac{(1-\delta S)^2}{R^3} + \frac{\lambda}{2}\frac{S^2}{R^3} - \frac{1}{R}, \\
&= \frac{\frac{1}{2}(1-\delta S)^2 + \frac{\lambda}{2}S^2 - R^2}{R^3} \\
\frac{\mathrm{d}}{\mathrm{d}t}S &= -\lambda\frac{S}{R^2} + \delta\frac{1-\delta S}{R^2} = -\frac{\lambda S - \delta(1-\delta S)}{R^2}.
\end{aligned}$$

This system has the stationary point $S = \frac{\delta}{\lambda+\delta^2}$, $R^2 = \frac{\lambda}{2(\lambda+\delta^2)}$, which is linearly stable, as the Jakobi matrix of the corresponding map $\mathbb{R}^2 \to \mathbb{R}^2$ in the critical point is

$$\begin{pmatrix} -\frac{1}{4R^2} & 0 \\ 0 & -\frac{\lambda+\delta^2}{R^2} \end{pmatrix}.$$

We can impose the length constraint $\|n\|_{\mathbb{R}^2} = 1$, that is here $S = 1$. In this case, we do no longer need an equation for S and the equation is then

$$\frac{\mathrm{d}}{\mathrm{d}t} R = \frac{\frac{1}{2}\big((1-\delta)^2 + \lambda\big) - R^2}{R^3}$$

which still has the stationary point $R^2 = \frac{1}{2}\big((1-\delta)^2 + \lambda\big)$. Moreover, we can fix the enclosed area or the length of the curve, both resulting in a fixed value for R.

We can also apply similar considerations in the case of 2 dimensions.

Example 3.9
For 2-dimensional surfaces, we take the energy without penalization of surface area or enclosed volume. In this case a little more care is needed when calculating the matrix valued gradients.

For the symmetric configuration

$$\varphi = R\,\mathrm{Id}_{\mathbb{S}^2} \qquad \text{and} \qquad n = S\nu$$

we find

$$H = \frac{2}{R}, \qquad \nabla\nu(p) = -\frac{1}{R}\,\mathrm{Id}_{T_p\mathbb{S}^2}, \qquad \text{and} \qquad \Delta\nu = \frac{2}{R^2}\nu.$$

From this we see that here n and $\partial_t n$ are co-linear and thus we can derive an ordinary differential equation for S and R. Since the energy in two dimensions is invariant under scaling of the surface, the radius R is stationary and we get an equation only for S reading

$$\frac{\mathrm{d}}{\mathrm{d}t} S = \frac{1}{R^2}\big(4\delta - (4\delta^2 + 2\lambda)S\big).$$

The solution of this equation can be written down explicitly. We find for the initial condition $S(0) = S_0$ the solution

$$S(t) = \big(S_0 - \frac{4\delta}{4\delta^2 + 2\lambda}\big)e^{-\frac{4\delta^2+2\lambda}{R^2}t} + \frac{4\delta}{4\delta^2 + 2\lambda}$$

From this we infer that for n being a constant length multiple of the unit normal, every round sphere is stationary.

One might hope that these stationary solutions will turn out to be at least local minimizers of the original energy. We will discuss general stability questions in Chapter 5. The following calculation shows that, at least in the presence of constraints,depending on the parameter δ, the symmetric configuration for curves discussed in Example 3.8 above is not stable.

Example 3.10
We fix the parameter δ of the energy (3.1) as $\delta > 1$ and consider the stationary point from Example 3.8 with fixed length $|n| = 1$.That is, for

$$R = \sqrt{\frac{1}{2}\big((1-\delta)^2 + \lambda\big)}$$

we consider $\varphi : \mathbb{S}^1 \to \mathbb{R}^2 : p \mapsto Rp$. We identify $\mathbb{S}^1$ with the interval $[0, 2\pi]$, then the inner unit normal ν at

$$\varphi(\theta) = R \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

for $\theta \in [0, 2\pi]$ is given by

$$\nu(\theta) = \begin{pmatrix} -\cos(\theta) \\ -\sin(\theta) \end{pmatrix}.$$

We can calculate the energy of this configuration, we obtain

$$E(\varphi, \nu) = \int \big(\kappa + \delta \operatorname{div}(\nu)\big)^2 \,\mathrm{d}s + \frac{\lambda}{2} \int |\nabla_s \nu|^2 \,\mathrm{d}s + L(\varphi) = 2\pi R\big(\frac{(1-\delta)^2}{R^2} + \frac{\lambda}{2R^2} + 1\big).$$

For $\varepsilon > 0$ we consider the unit vector field

$$n_\varepsilon(\theta) = \begin{pmatrix} -\cos(\theta + \varepsilon) \\ -\sin(\theta + \varepsilon) \end{pmatrix}$$

which forms a fixed small angle ε with ν. The tangential divergence of n_ε is given by

$$\begin{aligned} \operatorname{div}_s(n_\varepsilon) =& \partial_s n_\varepsilon \cdot \tau \\ =& \frac{1}{R} \begin{pmatrix} \sin(\theta + \varepsilon) \\ -\cos(\theta + \varepsilon) \end{pmatrix} \cdot \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \\ =& -\frac{1}{R}(\sin(\theta + \varepsilon)\sin(\theta) + \cos(\theta + \varepsilon)\cos(\theta)) = -\frac{\cos(\varepsilon)}{R}. \end{aligned}$$

The norm of the tangential gradient is still

$$|\nabla_s n_\varepsilon|^2 = |\partial_s n_\varepsilon|^2 = \frac{1}{R^2}$$

and thus

$$\begin{aligned} E(\varphi, n_\varepsilon) &= \int \big(\kappa + \delta \operatorname{div}(n_\varepsilon)\big)^2 \,\mathrm{d}s + \frac{\lambda}{2} \int |\nabla_s n_\varepsilon|^2 \,\mathrm{d}s + L(\varphi) \\ &= 2\pi R\big(\frac{(1 - \delta\cos(\varepsilon))^2}{R^2} + \frac{\lambda}{2R^2} + 1\big). \end{aligned}$$

Since $\delta > 1$ and the cosine has a strict local maximum in 0, we see that for $0 < |\varepsilon| \leq \arccos(\frac{1}{\delta})$ the energy $E(\varphi, n_\varepsilon)$ is strictly decreasing in ε. This intuition is complemented by the following calculation.

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(1 - \delta\cos(\varepsilon))^2 \Big|_{\varepsilon=0} = 0$$

and

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}(1 - \delta\cos(\varepsilon))^2 \Big|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} 2\delta \sin(\varepsilon)(1 - \delta\cos(\varepsilon)) \Big|_{\varepsilon=0} \\ &= 2\delta\cos(\varepsilon)(1 - \delta\cos(\varepsilon)) + 2\delta^2 \sin^2(\varepsilon)\big|_{\varepsilon=0} \\ &= 2\delta(1 - \delta) < 0. \end{aligned}$$

This consideration shows that for $\delta > 1$ the stationary configuration (φ, ν) is not a local minimizer subject to the constraint $|n| = 1$.

Our next observation is that the circle of radius $\sqrt{\frac{1}{2}\big((1-\delta)^2 + \lambda\big)}$ together with the vector field $n_{\arccos(\frac{1}{\delta})}$ is not stationary either. Since the curvature part of the energy is zero now, the radius of the circle given by the immersion φ will also change. The optimal radius is now given by the minimum $R = \sqrt{\lambda/2}$ of

$$E(\varphi, n_\varepsilon) = 0 + \frac{\lambda}{2}\frac{2\pi}{R} + 2\pi R.$$

4

Short-Time Existence for the Generalized Helfrich Flow

Our main objective in this chapter is the proof of local well-posedness of the geometric evolution problem (3.8). To this end, there is a variety of possible strategies with different advantages. A detailed exposition can be found in Mantegazza's book [68, Sec. 1.5]. Hamilton uses anintegrability criterium [45], DeTurck [25] considered a flow with a special tangent component. Our approach relies on Huisken's and Polden's idea [50, Sec. 7] of using Fermi coordinates and height functions as it generalizes best to higher order equations. Then, one can apply the standard approach to quasilinear parabolic equations, which is to linearize and then solve an equivalent fixed-point problem. Then, it is determined by the specific structure of the problem, what part of the proof is the most challenging. Often, it is the boundary values or making sense of the problem for very weak initial regularity. In our case the main issue will be the non-trivial coupling of equations of different order yielding a non-homogeneous symbol of the differential operator. We will tackle this problem with energy methods.

4.1 The Evolution Equation for the Height Function

Given $d \in \mathbb{N}$, a d-dimensional orientable compact smooth manifold M without boundary, an immersion $\varphi : M \to \mathbb{R}^{d+1}$, and a vector field $n : M \to \mathbb{R}^{d+1}$, by inspection of the evolution law (3.8), we can already interpret this equation as a partial differential equation for φ and n. Due to the reparametrization invariance of the geometric quantities, this system is degenerate in tangential directions. This problem and one way to deal with it was described by Huisken and Polden [50, Section 3]. For a smooth initial immersion $\varphi_0 : M \to \mathbb{R}^{d+1}$, they observe that a smoothly evolving hypersurface given by a family of immersions $\varphi : M \times [0, T) \to \mathbb{R}^{d+1}$ can be represented for small $T > 0$ by a smooth scalar function $f : M \times [0, T) \to \mathbb{R}$ as

$$\varphi(p, t) = \varphi_0(p) + f(p, t)\nu_0(p). \tag{4.1}$$

In this section, we adapt this strategy to our setting and reformulate the geometric evolution problem (3.8) from the previous chapter as equations in the variables f and n. The next lemmagathers important identities from Mantegazza's book and the work by Huisken and Polden.

Lemma 4.1

Given $d \in \mathbb{N}$, a d-dimensional orientable compact manifold M without boundary and a smooth immersion $\varphi^* : M \to \mathbb{R}^{d+1}$, let A^* denote the second fundamental form of the immersed surface and $A^*_{\max} = \max_{p \in M} |A^*(p)|$. For a smooth scalar function $f : M \to (-\frac{1}{2A^*_{\max}}, \frac{1}{2A^*_{\max}})$ the map $\varphi_f, p \mapsto \varphi^*(p) + f(p)\nu^*(p)$ is an immersion and one can express the metric and curvature of the corresponding surface in terms of the same on the surface associated to φ^* and derivatives $f_i = \nabla^*_i f$ of f. We use the star $*$ to indicate that a geometric object only depends on the immersion φ^*. We find

$$X_i = X^*_i + f_i \nu^* - f h^{k,*}_i X^*_k,$$

for the tangent vectors and

$$\nu = \frac{\nu^* - f_i g^{ij} X_j}{|\nu^* - f_i g^{ij} X_j|}$$

for the unit normal. The perturbed metric is given by

$$g_{ij} = g^*_{ij} - 2f h^*_{ij} + f^2 h^*_{ik} g^{kl,*} h^*_{lj} + f_i f_j,$$

and with a smooth tensor p_{ij} we get

$$h_{ij} = \langle \nu, f_{ij} \nu^* \rangle + p_{ij}(p, f, \nabla^* f).$$

This yields

$$H = \langle \nu, \nu^* \rangle g^{ij} f_{ij} + r(p, f, \nabla^* f)$$

for the curvature, where r is a suitable smooth function. Moreover, for another function $u : M \to \mathbb{R}$ we find with a smooth remainder term r' the identity

$$\Delta_f u = g^{ij} u_{ij} + r'(p, u, \nabla^* u, f, \nabla^* f, \nabla^{*2} f)$$

for the Laplace-Beltrami operator on the evolving surface.

If $n : M \to \mathbb{R}^{d+1}$ is a non-tangential vector field, then

$$\operatorname{div}_f n = g^{ij} \langle \partial_i n, X_j \rangle$$

depends linearly on n with coefficients depending on f and $\nabla^* f$.

Proof. First off, we observe that it can be seen from Lemma 1.18, that φ_f is an immersion. The order of the formulas is chosen in a way that they follow from each other. A very detailed discussion can be found in the work by Huisken and Polden [50, Sec. 7.5] and in Mantegazza's book [68, Section 1.5]. □

We cast the above considerations in the following proposition that relates the geometric evolution problem to a quasilinear partial differential equation problem.

Proposition 4.2

For natural numbers $d, r \in \mathbb{N}$, $r > d/2 + 3$ let $\varphi_0 : M \to \mathbb{R}^{d+1}$ be an immersion of Sobolev regularity $H^r(M, \mathbb{R}^{d+1})$ and $n_0 : M \to \mathbb{R}^{d+1}$ be a vector field of Sobolev regularity $H^r(M, \mathbb{R}^{d+1})$. Then, there exists a smooth immersion $\varphi^* : M \to \mathbb{R}^{d+1}$ and $f_0 \in H^r(M)$ such that $\varphi_0 = \varphi^* + f_0 \nu^*$. To solve the geometric initial value problem (3.8) *is equivalent to finding $T > 0$*

and a solution (f, n) on $M \times [0, T)$ of the quasilinear evolution equation that is given in local coordinates by

$$\begin{aligned}
\partial_t f &= -g^{ij} g^{k\ell} \nabla_i^* \nabla_j^* \nabla_k^* \nabla_\ell^* f - \delta \frac{g^{ij} \nabla_i^* \nabla_j^* \operatorname{div}_f n}{\langle \nu_f, \nu^* \rangle} \\
&\quad + b_1(p, t, f, \nabla^* f, \nabla^{*2} f, \nabla^{*3} f, n, \nabla^* n, \nabla^{*2} n), \\
\partial_t n &= \lambda g^{ij} \nabla_i^* \nabla_j^* n + \delta g^{k\ell} \nabla_k^* (g^{ij} \nabla_i^* \nabla_j^* f + \delta \operatorname{div}_f n) X_\ell \\
&\quad + b_2(p, t, f, \nabla^* f, \nabla^{*2} f, n, \nabla^* n) + g^{ij} \nabla_i^* n \otimes X_j \nu^* \partial_t f, \\
f(\cdot, 0) &= f_0, \\
n(\cdot, 0) &= n_0.
\end{aligned} \tag{4.2}$$

Here b_1 and b_2 are the functions arising when we write the lower order geometric expressions in terms of f and n rather than in terms of the evolving geometric quantities. They are smooth in their arguments at least in a neighborhood of the initial data.

Proof. The existence of φ^* and f_0 is guaranteed by Lemma 1.18 and in view of Lemma 3.6, it is enough to find a solution of the system

$$\begin{aligned}
\langle \partial_t \varphi, \nu_\varphi \rangle &= -\Delta_\varphi (H_\varphi + \delta \operatorname{div}_\varphi n) - (H_\varphi + \delta \operatorname{div}_\varphi n) |\nabla_\varphi \nu_\varphi|^2 \\
&\quad + \delta (H_\varphi + \delta \operatorname{div}_\varphi n) \nabla_\varphi n : \nabla_\varphi \nu_\varphi + \delta \operatorname{div}_\varphi ((H_\varphi + \delta \operatorname{div}_\varphi n) \nu_\varphi^T \nabla_\varphi n) \\
&\quad + \lambda (\nabla_\varphi n)^T : \left[\nabla_\varphi \nu_\varphi (\nabla_\varphi n)^T \right] + \frac{1}{2} (H_\varphi + \delta \operatorname{div}_\varphi n)^2 H_\varphi + \frac{\lambda}{2} |\nabla_\varphi n|^2 H_\varphi, \\
\partial_t n &= \lambda \Delta_\varphi n + \delta \nabla_\varphi (H_\varphi + \delta \operatorname{div}_\varphi n) - \delta (H_\varphi + \delta \operatorname{div}_\varphi n) H_\varphi \nu_\varphi + \nabla_\varphi n \partial_t \varphi
\end{aligned}$$

and then to reparametrize M in order to obtain a solution of the original equation (3.8). Since we aim to construct a strong solution in the sense that it is given by a continuous family of C^2 surfaces, we can assume that for the first short period of its existence it is given as in equation (4.1) as a graph over the smooth reference surface associated to φ^*. We calculate for $\varphi = \varphi^* + f\nu^*$

$$\langle \partial_t \varphi, \nu_\varphi \rangle = \langle \nu^*, \nu_\varphi \rangle \partial_t f.$$

Then the assertion follows from the identities in Lemma 4.1. □

4.2 The Linearized Problem

In order to solve the quasilinear problem, we need an optimal existence and regularity result for the linearized problem. We consider the highest order terms of the equation (4.2) with frozen coefficients for $f = 0$ and thus representing the geometric quantities induced by φ^*. In this section, we work on the manifold M with fixed metric g_{ij}^* and all geometric quantities and differential operators such as div, grad and Δ are meant to be those induced by φ^*. The linear problem with frozen coefficients which we want to solve for $T \in (0, \infty]$ reads

$$\begin{aligned}
\partial_t f + \Delta^2 f + \delta \Delta \operatorname{div} n &= x \quad \text{on } M \times (0, T), \\
\partial_t n - \lambda \Delta n - \delta \nabla (\Delta f + \delta \operatorname{div} n) &= y \quad \text{on } M \times (0, T), \\
f(\cdot, 0) &= f_0 \quad \text{on } M, \\
n(\cdot, 0) &= n_0 \quad \text{on } M.
\end{aligned} \tag{4.3}$$

4.2.1 Weak solutions

Following the strategy of Huisken and Polden [50, Sec. 7.2], wefind a weak solution with help of a version of the Lax-Milgram lemma due to Friedman and then show a priori estimates. We use estimates for difference quotients to get higher differentiability and prove that the weak solution is a strong solution. These notions will be made precise along with the introduction of function spaces suitable for the treatment of this system later on. This method will yield a solution that exists for all positive times. To solve the non-linear problem, we will have to work on finite time intervals. So we will have to adapt our results later.

Definition 4.3
For $f, g \in C_c^\infty(M \times [0,\infty))$, $m, n \in C_c^\infty(M \times [0,\infty))^{d+1}$, $a > 0$ we define

$$(f,g)_{LL_a(M)} = \int_0^\infty e^{-2at} \int_M fg \,\mathrm{d}\mu \,\mathrm{d}t,$$

$$(m,n)_{LL_a(M,\mathbb{R}^{d+1})} = \int_0^\infty e^{-2at} \int_M m \cdot n \,\mathrm{d}\mu \,\mathrm{d}t.$$

and for $s \in \mathbb{N}$ we set

$$\langle f,g\rangle_{LH_a^s(M)} = \int_0^\infty e^{-2at} \langle f(\cdot,t), g(\cdot,t)\rangle_{H^s(M)} \,\mathrm{d}t \ ,$$

$$\langle m,n\rangle_{LH_a^s(M,\mathbb{R}^{d+1})} = \int_0^\infty e^{-2at} \langle m(\cdot,t), n(\cdot,t)\rangle_{H^s(M,\mathbb{R}^{d+1})} \,\mathrm{d}t \ ,$$

and

$$\begin{aligned}\langle (f,m),(g,n)\rangle_{W_a(0,\infty,M)} = &\langle \partial_t f, \partial_t g\rangle_{LL_a(M)} + \langle \partial_t m, \partial_t n\rangle_{LL_a(M,\mathbb{R}^{d+1})} \\ &+ \langle m,n\rangle_{LH_a^1(M,\mathbb{R}^{d+1})} + \langle f,g\rangle_{LL_a(M)} \\ &+ \langle \Delta f + \delta \operatorname{div}(m), \Delta g + \delta \operatorname{div}(n)\rangle_{LL_a(M)}.\end{aligned}$$

Now $LL_a(M)$, $LL_a(M,\mathbb{R}^{d+1})$, $LH_a^s(M,\mathbb{R})$, and $LH_a^s(M,\mathbb{R}^{d+1})$ are the closures of $C_c^\infty(M \times [0,\infty))$ and $C_c^\infty(M \times [0,\infty))^{d+1}$, respectively, in the norm induced by the inner products given above. That is, we set

$$\begin{aligned}LH_a^s(M,\mathbb{R}) &:= \overline{C_c^\infty(M \times [0,\infty))}^{\|\cdot\|_{LH_a^s(M,\mathbb{R})}}, \\ LH_a^s(M,\mathbb{R}^{d+1}) &:= \overline{C_c^\infty(M \times [0,\infty), \mathbb{R}^{d+1})}^{\|\cdot\|_{LH_a^s(M,\mathbb{R}^{d+1})}}\end{aligned}$$

and similarly

$$\begin{aligned}W_a(0,\infty,M) &:= \overline{C_c^\infty(M \times [0,\infty)) \times C_c^\infty(M \times [0,\infty))^{d+1}}^{\|\cdot\|_{W_a(0,\infty,M)}}, \\ W_{a,0}(0,\infty,M) &:= \overline{C_c^\infty(M \times (0,\infty)) \times C_c^\infty(M \times (0,\infty))^{d+1}}^{\|\cdot\|_{W_a(0,\infty,M)}}.\end{aligned}$$

By a little abuse of notation, we will from now on often suppress the target space (and sometimes also the domain) in the notation if they are clear from context. So we write $\|n\|_{LH_a^s(M)}$ or $\langle n,m\rangle_{LH_a^s}$ even for vector valued functions.

Remark 4.4
Considering these spaces involving time weights one might wonder whether the regularity of an element of such space can be characterized, when it is considered on a finite time interval.

Indeed, when $f \in LH_a^k$ then $f|_{[0,T)} \in L^2(0,T;H^k)$, since the weight is uniformly bounded from above and below, as for all $t \in [0,T)$ we have $e^{-2aT} \le e^{-2at} \le 1$.

To illustrate the advantage of an exponential time weight, consider the heat equation

$$\partial_t u = \Delta u \quad \text{on } M \times [0,\infty).$$

The solution of this equation for the initial condition $u(\cdot,0) = 1$ on M is given by $u = 1$ on $M \times [0,\infty)$, which is perfectly smooth, but not an element of $L^2(0,\infty;H^2(M))$. However, it is an element in $LH_a^2(M)$ for any $a > 0$. We will see that the exponential weighting enables us to treat problems on $M \times [0,\infty)$, where the solution grows exponentially in time, in a proper functional analytic setting and to derive a priori estimates as in the case of finite time intervals.

For functions in the space $W_a(0,\infty,M)$ we can infer some more properties.

Lemma 4.5
If a pair (f,n) is in $W_a(0,\infty,M)$, we may conclude $f \in LH_a^2(M)$, $n \in LH_a^1(M)$, and $\partial_t f, \partial_t n \in LL_a(M)$. Moreover, for $T > 0$ fixed, it holds that $f,n \in C([0,T],L^2(M))$ and we have the estimate

$$\begin{aligned}\|f\|_{LH_a^2(M)} + \|\partial_t f\|_{LL_a(M)} + \|n\|_{LH_a^1(M)} + \|\partial_t n\|_{LL_a(M)} \\ +\|f(\cdot,0)\|_{L^2(M)} + \|n(\cdot,0)\|_{L^2(M)} \le C\|(f,n)\|_{W_a(0,\infty,M)}\end{aligned}$$

with a constant $C = C(M,T)$ independent of f and n.

Proof. Partially, this is directly seen from the definition, only the assertion $f \in LH_a^2(M)$ and the estimate require additional effort. We find that the reverse triangle inequality yields as $n \in LH_a^1(M)$ the estimate

$$\|\Delta f\|_a + \delta\|\operatorname{div} n\|_a \le \max\{1,2\delta\}(\|n\|_{LH_a^1(M)} + \|\Delta f + \delta \operatorname{div}(n)\|_{LL_a(M)}.$$

From elliptic regularity theory (see e.g. the book by Taylor [83, Ch. 5, Theorem 11.1]) we infer that whenever

$$\Delta f \in L^2(M),$$

then

$$\|f\|_{H^2(M)} \le C\|\Delta f\|_{L^2(M)} + C\|f\|_{L^2(M)}. \tag{4.4}$$

Since $\Delta f \in L^2(M)$ and $f \in L^2(M)$ for almost every $t \in [0,\infty)$, we may conclude that $f \in H^2(M)$ for almost every t. Integration in t then shows

$$\|f\|_{LH_a^2} \le C\big(\|\Delta f\|_{LL_a(M)} + \|f\|_{LL_a(M)}\big).$$

In combination with the trace theorem 1.23 we have the asserted estimate. □

We introduce the following notion of weak solutions to equation (4.3).

Definition 4.6
Consider $x \in LL_a,\ y \in (LH_a^1)^{d+1}$ and $f_0 \in C^\infty(M),\ n_0 \in C^\infty(M)^{d+1}$. A pair of functions

$$(f,n) \in W_a(0,\infty,M)$$

is called a weak solution of (4.3) if

$$(f - f_0, n - n_0) \in W_{a,0}(0,\infty,M)$$

and for any

$$(\varphi, \eta) \in C_c^\infty(M \times (0,\infty)) \times C_c^\infty(M \times (0,\infty))^{d+1}$$

we have

$$\begin{aligned}&\langle \partial_t f, \varphi \rangle_{LL_a(M)} + \langle \partial_t n, \eta \rangle_{LL_a(M)} + \lambda \langle \nabla n, \nabla \eta \rangle_{LL_a(M)} \\ &\quad + \langle \Delta f + \delta \operatorname{div}(n), \Delta \varphi + \delta \operatorname{div}(\eta) \rangle_{LL_a(M)} = \langle x, \varphi \rangle_{LL_a(M)} + \langle y, \eta \rangle_{LL_a(M)}.\end{aligned}$$

Lemma 4.7
If $(f, n) \in W_a(0, \infty, M)$ is a weak solution of equation (4.3) *in the sense of Definition 4.6, then for all $(\varphi, \eta) \in LW_a^2(M) \times LW_a^1(M)$ the identity*

$$\begin{aligned}&\langle \partial_t f, \varphi \rangle_{LL_a(M)} + \langle \partial_t n, \eta \rangle_{LL_a(M)} + \lambda \langle \nabla n, \nabla \eta \rangle_{LL_a(M)} \\ &\quad + \langle \Delta f + \delta \operatorname{div}(n), \Delta \varphi + \delta \operatorname{div}(\eta) \rangle_{LL_a(M)} = \langle x, \varphi \rangle_{LL_a(M)} + \langle y, \eta \rangle_{LL_a(M)}.\end{aligned}$$

holds.

Proof. For $(f, n) \in W_a(0, \infty, M)$ the map

$$(\varphi, \eta) \mapsto \begin{pmatrix} \langle \partial_t f, \varphi \rangle_{LL_a(M)} + \langle \partial_t n, \eta \rangle_{LL_a(M)} + \lambda \langle \nabla n, \nabla \eta \rangle_{LL_a(M)} \\ + \langle \Delta f + \delta \operatorname{div}(n), \Delta \varphi + \delta \operatorname{div}(\eta) \rangle_{LL_a(M)} - \langle x, \varphi \rangle_{LL_a(M)} - \langle y, \eta \rangle_{LL_a(M)} \end{pmatrix}$$

is continuous in the $LW_a^2(M) \times LW_a^1(M)$ topology. Therefore, the assertion holds by density of $C_c^\infty(M \times (0,\infty)) \times C_c^\infty(M \times (0,\infty))^{d+1}$ in $LW_a^2(M) \times LW_a^1(M)$. ☐

Before we prove the existence of weak solutions we can a priori derive the follwing estimate.

Lemma 4.8
Let $(f, n) \in W_a(0, \infty, M)$ be a weak solution of equation (4.3) *in the sense of Definition 4.6 for $x \in LL_a$, $y \in LH_a^1$ and smooth initial values f_0, n_0. Then, we have the a priori estimate*

$$\|f\|_{LH_a^2} + \|n\|_{LH_a^1} \leq C(\|(x,y)\|_{LL_a(M)} + \|f_0\|_{L^2(M)} + \|n_0\|_{L^2(M)}). \tag{4.5}$$

Proof. In view of Lemma 4.7 we may use test functions with non-vanishing initial data in the weak formulation. Using the solution as a test function yields

$$\begin{aligned}&\langle \partial_t f, f \rangle_{LL_a(M)} + \langle \partial_t n, n \rangle_{LL_a(M)} + \langle \nabla_M n, \nabla_M n \rangle_{LL_a(M)} \\ &\quad + \langle \Delta_M f + \delta \operatorname{div}(n), \Delta_M f + \delta \operatorname{div}(n) \rangle_{LL_a(M)} = \langle x, f \rangle_{LL_a(M)} + \langle y, n \rangle_{LL_a(M)} .\end{aligned}$$

Since the weak solution has also one weak time derivative we can use integration by parts in time, yielding

$$\langle \partial_t f, f \rangle_{LL_a(M)} = \int_0^\infty e^{-2at} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} \int_M f^2 \, \mathrm{d}\mu \, \mathrm{dt} = a \int_0^\infty e^{-2at} \int_M f^2 \, \mathrm{d}\mu \, \mathrm{dt} - \frac{1}{2} \int_M f(\cdot, 0)^2 \, \mathrm{d}\mu.$$

The second boundary term

$$\lim_{T \to \infty} e^{-2at} \int_M f^2(\cdot, T) \, \mathrm{d}\mu$$

exists and vanishes, since $e^{-at} f \in H^1(0, \infty; L^2(M))$. We will use this very often in subsequent calculations. With Young's inequality we infer

$$\|f\|_a + \|n\|_{LH_a^1} + \|\Delta f + \delta \operatorname{div}(n)\|_a \leq C\big(\|(x,y)\|_a + \|f_0\|_{L^2(M)} + \|n_0\|_{L^2(M)}\big).$$

Using Lemma 4.5 we may conclude $\|\nabla_M^2 f\|_a \leq C \|\Delta_M f\|_a \leq C(\|(x,y)\| + \|f_0\| + \|n_0\|)$, as the norm of the second derivative is controlled by the norm of the Laplacian. ☐

Proposition 4.9
Let $x \in L^2(0,T;L^2(M))$, $y \in L^2(0,T;H^1(M))^{d+1}$ be two functions and consider smooth initial data $f_0 \in C^\infty(M)$ and $n_0 \in C^\infty(M)^{d+1}$. Then, for all $a > 0$ equation (4.3) has a unique weak solution in the sense of Definition 4.6.

To find a weak solution of our parabolic problem, we follow Polden [76, Sec. 2.2] and Friedman's version of Lax-Milgram [34, Chapter 10, Theorem 16] is one main ingredient for the proof.

Lemma 4.10
Let H be a Hilbert space and Φ a (not necessarily complete) space with a scalar product continuously embedded in H. Moreover, let $P : H \times \Phi \to \mathbb{R}$ be a bilinear form such that

i) the linear map $h \mapsto P(h,\varphi)$ is continuous for every fixed $\varphi \in \Phi$,

ii) $P|_{\Phi\times\Phi}$ is coercive. That is, there exists $C > 0$ such that $P(\varphi,\varphi) \geq C\|\varphi\|_H^2$, for every $\varphi \in \Phi$.

Then, for every $K \in \Phi^*$ there exists $u \in H$ such that for every $\varphi \in \Phi$ it holds that

$$P(u,\varphi) = K(\varphi).$$

With this, we are in a position to establish existence of weak solutions to equation (4.3).

Proof of Proposition 4.9. First, assume $f_0 = 0$ and $n_0 = 0$. Furthermore, we observe that finding a solution to equation (4.3) is equivalent to finding $(\tilde{f},\tilde{n}) \in W_a^0(0,\infty,M)$ solving

$$\partial_t(\tilde{f},\tilde{n}) - (L - \mathrm{Id})(\tilde{f},\tilde{n}) = (x,y)e^{-t}. \tag{4.6}$$

This spectral shift will give coercivity of the bilinear form used later. For simplicity we write f and n instead of $\tilde{f}$ and $\tilde{n}$ in the sequel. The solution to the original equation can then be obtained by multiplication with e^t. If $(e^{-t}f, e^{-t}n) \in W_a(0,\infty,M)$, then $(f,n) \in W_{a+1}(0,\infty,M)$. To fit in the setting of Lemma 4.10 we set

$$\begin{aligned} H &= (W_{a,0}(0,\infty,M), \langle\cdot,\cdot\rangle_{W_{a,0}(0,\infty,M)}) \\ \Phi &= ((C_c^\infty(M\times(0,\infty)) \times C_c^\infty(M\times(0,\infty))^{d+1}), \langle\cdot,\cdot\rangle_{W_{a,0}(0,\infty,M)}) \end{aligned}$$

and we define the bilinear map

$$P : W_{a,0}(0,\infty,M) \times (C_c^\infty(M\times(0,\infty)) \times C_c^\infty(M\times(0,\infty))^{d+1}) = H\times\Phi \to \mathbb{R}$$

and the functional

$$K : C_c^\infty(M\times(0,\infty)) \times C_c^\infty(M\times(0,\infty))^{d+1} = \Phi \to \mathbb{R}$$

by

$$\begin{aligned} P((f,n),(\varphi,\eta)) = {} & \langle\partial_t f, \partial_t\varphi\rangle_{LL_a(M)} + \langle\partial_t n, \partial_t\eta\rangle_{LL_a(M)} + \lambda\langle\nabla n, \nabla\partial_t\eta\rangle_{LL_a(M)} + \langle n, \partial_t\eta\rangle_{LL_a(M)} \\ & + \langle f, \partial_t\varphi\rangle_{LL_a(M)} + \langle\Delta f + \delta \operatorname{div} n, \Delta\partial_t\varphi + \delta\operatorname{div}(\partial_t\eta)\rangle_{LL_a(M)} \end{aligned}$$

and

$$K((\varphi,\eta)) = \langle x, \partial_t\varphi\rangle_{LL_a(M)} + \langle y, \partial_t\eta\rangle_{LL_a(M)} .$$

Basically, K and P emerge from testing equation (4.6) with time derivatives of test functions. The terms $\langle n, \partial_t \eta\rangle_{LL_a(M)}$ and $\langle f, \partial_t \varphi\rangle_{LL_a(M)}$ arise due to the spectral shift. Integration by parts in time shows that P is coercive for any positive a and continuous in the first argument for any fixed $(\varphi, \eta) \in C_c^\infty(M \times (0,\infty)) \times C_c^\infty(M \times (0,\infty))^{d+1}$. We use that for φ we find

$$\int_0^\infty e^{-2at} \int_M \varphi \partial_t \varphi \,\mathrm{d}\mu \,\mathrm{d}t = \int_0^\infty e^{-2at} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_M \varphi^2 \,\mathrm{d}\mu \,\mathrm{d}t$$
$$= a \int_0^\infty e^{-2at} \int_M \varphi^2 \,\mathrm{d}\mu \,\mathrm{d}t,$$

since φ has compact support in $(0, \infty)$. We see

$$\begin{aligned} P((\varphi,\eta),(\varphi,\eta)) &= \langle \partial_t\varphi, \partial_t\varphi\rangle_{LL_a(M)} + \langle \partial_t\eta, \partial_t\eta\rangle_{LL_a(M)} + \lambda\langle \nabla\eta, \nabla\partial_t\eta\rangle_{LL_a(M)} + \langle \eta, \partial_t\eta\rangle_{LL_a(M)} \\ &\quad + \langle \varphi, \partial_t\varphi\rangle_{LL_a(M)} + \langle \Delta\varphi + \delta \operatorname{div}\eta, \Delta\partial_t\varphi + \delta \operatorname{div}(\partial_t\eta)\rangle_{LL_a(M)} \\ &= \|(\partial_t\varphi, \partial_t\eta)\|^2_{LL_a(M)} + a\|\eta\|^2_{LL_a(M)} + a\lambda\|\nabla\eta\|^2_{LL_a(M)} \\ &\quad + a\|\varphi\|^2_{LL_a(M)} + \|\Delta\varphi + \delta \operatorname{div}\eta\|^2_{LL_a(M)} \\ &\geq \min\{a\lambda, a, 1\}\|(\varphi,\eta)\|^2_{W_{a,0}(0,\infty,M)} \end{aligned}$$

and by Cauchy Schwarz inequality

$$P((f,n),(\varphi,\eta)) \leq C(\varphi,\eta)\|(f,n)\|_{W_{a,0}(0,\infty,M)}$$

the map $(f,n) \to P((f,n),(\varphi,\eta))$ is continuous for every fixed (φ,η) as required in Lemma 4.10. Hence, by Lemma 4.10 we may conclude that there exists $(f,n) \in W_{a,0}(0,\infty,M)$ such that

$$P((f,n),(\varphi,\eta)) = K((\varphi,\eta))$$

for all $(\varphi,\eta) \in C_c^\infty(M \times (0,\infty)) \times C_c^\infty(M \times (0,\infty))^{d+1}$. That is, (f,n) satisfies the weak formulation of equation (4.6) when testing with time derivatives. To see that (f,n) is indeed a weak solution we employ the exponential weighting. For a pair of test functions $(\varphi,\eta) \in C_c^\infty(M \times (0,\infty)) \times C_c^\infty(M \times (0,\infty))^{d+1}$, which we extend by zero to the whole real line, we set for $0 < S \in \mathbb{R}$

$$(\tilde{\varphi}(x,t), \tilde{\eta}(x,t)) = (\varphi(x,t), \eta(x,t)) - (\varphi(x,t-S), \eta(x,t-S)),$$

which has compact support and zero mean value in time. Thus, it has a compactly supported anti-derivative and is therefore the time derivative of a test function. Hence,

$$P((\tilde{f},\tilde{n}),(\tilde{\varphi},\tilde{\eta})) = K((\tilde{\varphi},\tilde{\eta})).$$

But since $P((f,n),(\varphi(\cdot,\cdot-S),\eta(\cdot,\cdot-S)))$ and $K((\varphi(\cdot,\cdot-S),\eta(\cdot,\cdot-S)))$ tend to zero as we send $S \to \infty$, we have

$$\begin{aligned} &\langle \partial_t f, \varphi\rangle_{LL_a(M)} + \langle \partial_t n, \eta\rangle_{LL_a(M)} + \langle \nabla_M n, \nabla_M \eta\rangle_{LL_a(M)} \\ &\qquad + \langle \Delta_M f + \delta \operatorname{div}(n), \Delta_M\varphi + \delta \operatorname{div}(\eta)\rangle_{LL_a(M)} = \langle x, \varphi\rangle_{LL_a(M)} + \langle y, \eta\rangle_{LL_a(M)}. \end{aligned}$$

That is, (f,n) is a weak solution of equation (4.6). Thus, we find a weak solution of (4.3) by multiplication with e^t.

To find a solution when the initial data (f_0, n_0) do not vanish, we introduce the operator L by

$$L(f,n) = \begin{pmatrix} -\Delta^2 f - \delta\Delta \operatorname{div} n \\ +\Delta n + \delta\nabla(\Delta f + \delta \operatorname{div} n) \end{pmatrix}$$

and consider the problem

$$\begin{aligned}\partial_t(\tilde{f},\tilde{n}) - L(\tilde{f},\tilde{n}) &= (x,y) + L(f_0,n_0),\\ \tilde{f}(0) &= 0,\\ \tilde{n}(0) &= 0.\end{aligned}$$

Observe, that up to now we only consider smooth initial data, on which we can apply any differential operators and which is integrable in time with respect to our weighted measure. Therefore $(x, y) + L(f_0, n_0)$ is an admissible right-hand side. The arguments above imply that a weak solution $(\tilde{f}, \tilde{n})$ of this new problem exists and that $(f, n) = (\tilde{f} + f_0, \tilde{n} + n_0)$ then solves the original equation with smooth non-zero initial data.

Uniqueness of the solution follows from the a priori estimate in Lemma 4.8 □

4.2.2 Regularity

To show that such weak solution is indeed a strong solution and to derive estimates for the higher order derivatives we use difference quotients. This technique is well established in the literature, e.g. in Evans' book [32, Sec 5.8.2, Sec 6.3]. In the context of geometric evolution equations it was used by Polden in his thesis [76, Sec 2.3] and in the overview article with Huisken [50, Sec 7.3]. The main strategy is to exploit the coercivity of the elliptic operator to obtain a Gårding type inequality and Caccioppoli estimates. Firstly, we define suitable function spaces and borrow the notation of Huisken and Polden.

Definition 4.11
For $f, g \in C_c^\infty(M \times [0,\infty))$, $m, n \in C_c^\infty(M \times [0,\infty))^{d+1}$ we define

$$\langle (f,m),(g,n)\rangle_{P_a^k(0,\infty,M)} = \langle \partial_t f, \partial_t g\rangle_{LH_a^k} + \langle f,g\rangle_{LH_a^{4+k}} + \langle \partial_t m, \partial_t n\rangle_{LH_a^{1+k}} + \langle m,n\rangle_{LH_a^{3+k}}$$

and the space $P_a^k(0,\infty,M)$ as the closure of $C_c^\infty(M \times [0,\infty)) \times C_c^\infty(M \times [0,\infty))^{d+1}$ in the induced norm. Instead of P_a^0 we just write P_a.

Lemma 4.12
Let $(f, n) \in W_a(0, \infty, M)$ be a weak solution of equation (4.3) in the sense of Definition 4.6 for $x \in LL_a$, $y \in LH_a^1$ and smooth initial values f_0, n_0. Then,

$$(f,n) \in P_a(0,\infty,M)$$

and moreover we have the a priori estimate

$$\|(f,n)\|_{P_a(0,\infty,M)} \le C(\|x\|_{LL_a(M)} + \|y\|_{LH_a^1} + \|f_0\|_{H^2} + \|n_0\|_{H^2})$$

The key elements of the proof are localization and the use of difference quotients. Therefore, we make a short comment here on how to use difference quotients on manifolds and give two important formulas. We choose a system of charts $(U_i, x_i)_i$ for M and let $u : M \to \mathbb{R}$ be a smooth function with compact support in only a single set U_j. Then the difference quotient in a direction $v \in \mathbb{S}^{d-1}$ is defined for $|h| < \frac{1}{2}\operatorname{dist}(\partial \operatorname{supp}(u \circ x^{-1}), \partial U_j)$ as

$$D_h^v u(p) = \frac{u(x_j^{-1}(x_j(p) + vh)) - u(p)}{h}.$$

In the following, by abuse of notation we mean $u \circ x_j^{-1}$ whenever we write just u. Let ϕ be a second smooth function with its support contained also in U_j. Then, we have the following chain rule

$$D_h^v(u \cdot \phi) = D_h^v u \phi(\cdot + hv) + u D_h^v \phi$$

that implies a rule for integration by parts for difference quotients that reads

$$\begin{aligned}\int_M u D_h^v \phi \,\mathrm{d}\mu &= \int_{x_j(U_j)} u D_h^v \phi \sqrt{g}\,\mathrm{d}x \\ &= -\int_{x_j(U_j)} D_{-h}^v u \phi \sqrt{g}\,\mathrm{d}x - \int_{x_j(U_j)} u\phi(\cdot - hv) D_{-h}^v \sqrt{g}\,\mathrm{d}x.\end{aligned}$$

In the following we denote the signed measure $D_h^v \sqrt{g}\,\mathrm{d}x$ by $\mathrm{d}D_h\mu$. Now we can prove the lemma.

Proof of Lemma 4.12. Let $\xi \in C_c^\infty(V)$ be a smooth cut-off function compactly supported in an open set $V \subset M$ and $\xi \equiv 1$ in $U \subset\subset V$ where V is contained in a single chart. Then we take difference quotients in space. Since the estimates have to hold for an arbitrary $v \in S^{d-1}$ we do not denote v explicitly in the following. By virtue of Lemma 4.7,a pair of functions $(D_{-h}(\xi^2 D_h f), D_{-h}(\xi^2 D_h n))$ may be put in the weak formulation of the equation

$$\begin{aligned}&\langle x, D_{-h}(\xi^2 D_h f)\rangle_{LL_a} + \langle y, D_{-h}(\xi^2 D_h n)\rangle_{LL_a} \\ &= \langle \partial_t f, D_{-h}(\xi^2 D_h f)\rangle_{LL_a} + \langle \partial_t n, D_{-h}(\xi^2 D_h n)\rangle_{LL_a} + \lambda\langle \nabla_M n, \nabla_M D_{-h}(\xi^2 D_h n)\rangle_{LL_a} \\ &\quad + \langle \Delta_M f + \delta\,\mathrm{div}(n), \Delta_M D_{-h}(\xi^2 D_h f) + \delta\,\mathrm{div}(D_{-h}(\xi^2 D_h n)\rangle_{LL_a}.\end{aligned} \tag{4.7}$$

We integrate by parts in time first and use the formula for integration by parts for difference quotients in space to find

$$\begin{aligned}&\langle \partial_t f, D_{-h}(\xi^2 D_h f)\rangle_{LL_a} \\ &= -\int_0^\infty e^{-2at}\int_M (\xi D_h \partial_t f)(\xi D_h f)\,\mathrm{d}\mu\,\mathrm{d}t - \int_0^\infty e^{-2at}\int_M \xi^2 \partial_t f(\cdot + vh) D_h f\,\mathrm{d}D_h\mu\,\mathrm{d}t \\ &= -a\int_0^\infty e^{-2at}\int_M (\xi D_h f)^2\,\mathrm{d}\mu\,\mathrm{d}t + \frac{1}{2}\int_M (\xi D_h f(\cdot,0))^2\,\mathrm{d}\mu \\ &\quad -\int_0^\infty e^{-2at}\int_M \xi^2 \partial_t f(\cdot + vh) D_h f\,\mathrm{d}D_h\mu\,\mathrm{d}t.\end{aligned}$$

The difference quotient on the volume element can be estimated since the metric on M is smooth. Using Young's inequality with ε on the left-hand terms in (4.7) and the terms involving the time derivatives we get

$$\begin{aligned}&\|\xi D_h \nabla_M n\|_{LL_a}^2 + \|\xi D_h(\Delta_M f + \delta\,\mathrm{div}_M n)\|_{LL_a}^2 \\ &\leq \frac{C}{\varepsilon}(\|f\|_{LH_a^2}^2 + \|n\|_{LH_a^1}^2 + \|f_0\|_{H^1(M)}^2 + \|n_0\|_{H^1(M)}^2 + \|x\|_{LL_a}^2 + \|y\|_{LH_a^1}^2) \\ &\quad + \varepsilon(\|\xi\partial_t f\|_{LL_a}^2 + \|\xi\partial_t n\|_{LL_a}^2)\end{aligned}$$

for a positive constant C depending on the manifold (M, g), but not on f, n or h. Terms, where derivatives or difference quotients fall on ξ or $\sqrt{g}$ are of lower order and can be treated by interpolation. For the terms involving n we do the exact same estimates as for those in f with the only difference that we use integration by parts to shift one difference quotient on y.

We use this estimate and the estimate for the Laplace operator (4.4) to see that

$$\begin{aligned}\|\xi D_h f\|_{LH_a^2} \leq &\frac{C}{\varepsilon}(\|f\|_{LH_a^2} + \|n\|_{LH_a^1} + \|f_0\|_{H^1} + \|n_0\|_{H^1} + \|x\|_{LL_a} + \|y\|_{LH_a^1}) \\ &+ \varepsilon(\|\xi\partial_t f\|_{LL_a} + \|\xi\partial_t n\|_{LL_a}).\end{aligned}$$

Using the a priori estimate (4.5) for $\|f\|_{LH^2_a} + \|n\|_{LH^1_a}$ we find

$$\|f\|_{LH^3_a} + \|n\|_{LH^2_a} \leq \frac{C}{\varepsilon}(\|f_0\|_{H^1} + \|n_0\|_{H^1} + \|x\|_{LL_a} + \|y\|_{LH^1_a}) + \varepsilon(\|\xi\partial_t f\|_{LL_a} + \|\xi\partial_t n\|_{LL_a}).$$

We observe that this means, we can consider the second equation of the system in an LL_a sense. That is,

$$\partial_t n = \Delta n + \delta\nabla(\Delta f + \delta \operatorname{div} n) + y \quad \text{in } LL_a.$$

The same computations for all mixed difference quotients $D^2_{-h}(\xi^2 D^2_h f)$ and $D^2_{-h}(\xi^2 D^2_h n)$ yield

$$\begin{aligned}&\|\xi D_h f\|_{LH^3_a} + \|\xi D_h n\|_{LH^2_a}\\ &\leq \frac{C}{\varepsilon}(\|f_0\|_{H^2} + \|n_0\|_{H^2} + \|x\|_{LL_a(M)} + \|y\|_{LH^1_a}) + \varepsilon(\|\xi\partial_t f\|_{LL_a} + \|\xi D_h\partial_t n\|_{LL_a}).\end{aligned}$$

That is, we have to control the difference quotient of the time derivative of n. But

$$\begin{aligned}\langle \xi D_h\partial_t n, \xi D_h\partial_t n\rangle_{LL_a} &= \langle \xi D_h(\Delta n + \delta\nabla(\Delta f + \delta \operatorname{div} n)), \xi D_h(\Delta n + \delta\nabla(\Delta f + \delta \operatorname{div} n))\rangle_{LL_a}\\ &\leq C(\|\xi D_h f\|^2_{LH^3_a} + \|\xi D_h n\|^2_{LH^2_a}).\end{aligned}$$

Thus, we can absorb the term $C\varepsilon(\|\xi D_h f\|_{LH^3_a} + \|\xi D_h n\|_{LH^2_a})$. This allows us to conclude that also the first equation of the system holds in a strong LH_a sense. As a next step, we need estimates for $\|\partial_t f\|_{LL_a}$ and $\|\partial_t n\|_{LH^1_a}$. This can now be obtained by using the equation. We find

$$\begin{aligned}&\langle \xi\partial_t f, \xi\partial_t f\rangle_{LL_a} + \langle \xi D_h\partial_t n, \xi D_h\partial_t n\rangle_{LL_a}\\ &= \langle \xi\Delta(\Delta f + \delta \operatorname{div} n) + \xi x, \xi\Delta(\Delta f + \delta \operatorname{div} n) + \xi x\rangle_{LL_a}\\ &\quad + \langle \xi D_h(\lambda\Delta n + \delta\nabla(\Delta f + \delta \operatorname{div} n)) + \xi y, \xi D_h(\lambda\Delta n + \delta\nabla(\Delta f + \delta \operatorname{div} n)) + \xi y\rangle_{LL_a}\\ &\leq \frac{C}{\varepsilon}(\|f_0\|^2_{H^2} + \|n_0\|^2_{H^2} + \|x\|^2_a + \|y\|^2_{LH^1_a}) + \varepsilon(\|\xi\partial_t f\|^2_{LL_a} + \|\xi D_h\partial_t n\|^2_{LL_a}),\end{aligned}$$

which yields the final local estimate

$$\|\xi(f,n)\|_{P_a(0,\infty,M)} \leq \frac{C}{\varepsilon}(\|f_0\|_{H^2} + \|n_0\|_{H^2} + \|x\|_{LL_a(M)} + \|y\|_{LH^1_a}).$$

To get the global estimates from the local ones we only use that we may cover M by finitely many charts. ☐

By approximating the initial data in the trace space, we can now prove existence of strong solutions.

Theorem 4.13
For $k \geq 2$, $(f_0, n_0) \in H^k \times (H^k)^{d+1}$, $x \in LH^{k-2}_a$, and $y \in (LH^{k-1}_a)^{d+1}$ equation (4.3) has a solution $(f,n) \in P^k_a$ with an a priori estimate

$$\|(f,n)\|_{P^k_a} \leq C(\|f_0\|_{H^{k-2}} + \|n_0\|_{H^{k-1}} + \|x\|_{LH^{k-2}_a} + \|y\|_{LH^{k-1}_a}),$$

Proof. Assume for $k \in \mathbb{N}$ that

$$\begin{gathered}(f_0, n_0) \in H^{k+2}(M) \times (H^{k+2})^{d+1}\\ (x,y) \in LH^k_a \times (LH^{k+1}_a)^{d+1},\end{gathered}$$

then we use

$$(D^{k+2}_{-h}\xi^{k+2}D^{k+2}_h f, D^{k+2}_{-h}\xi^{k+2}D^{k+2}_h n)$$

as a test function. We do the same calculations as above in (4.7) but we have to exploit the regularity of x and y by putting k or $k+1$, respectively, difference quotients on them. Approximating f_0, n_0 by smooth functions together with the a priori estimates completes the proof, since a H^k Cauchy sequence of initial values gives a P_a^k Cauchy sequence of solutions. □

In order to prove a short time existence result for the full equation we have to adapt our result to finite time intervals and spaces without weighting in time.

Definition 4.14
Let $k \geq 2$ be a real number, then define

$$\begin{aligned} X_T^1 =& H^1(0,T;H^{k-2}) \cap L^2(0,T,H^{k+2}) \\ X_T^2 =& (H^1(0,T;H^{k-1}) \cap L^2(0,T,H^{k+1}))^{d+1} \end{aligned}$$

with norms

$$\begin{aligned} \|f\|_{X_T^1} =& \|\partial_t f\|_{L^2(0,T,H^{k-2})} + \|f\|_{L^2(0,T,H^{k+2})} + \|f_0\|_{H^k}, \\ \|n\|_{X_T^2} =& \|\partial_t n\|_{L^2(0,T,H^{k-1})} + \|n\|_{L^2(0,T,H^{k+1})} + \|n_0\|_{H^k}, \end{aligned}$$

and set

$$X_T = X_T^1 \times X_T^2$$

with $\|\cdot\|_{X_T} = \|\cdot\|_{X_T^1 \times X_T^2}$. Moreover, we consider

$$\begin{aligned} Y_T^1 &= L^2(0,T,H^{k-2}) \\ Y_T^2 &= (L^2(0,T,H^{k-1}))^{d+1} \\ Y_T &= Y_T^1 \times Y_T^2, \end{aligned}$$

with the usual product norm. Finally, we set

$$X_\gamma = H^k \times (H^k)^{d+1},$$

which is the right space for the trace according to Theorem 1.23, and

$$X_{T,\text{imm}} = X_{T,\text{imm}}^1 \times X_T^2,$$

where

$$X_{T,\text{imm}}^1 = H^1(0,T;H_{\text{imm}}^{k-2}(M,\mathbb{R}^{d+1})) \cap L^2(0,T;H_{\text{imm}}^{k+2}(M,\mathbb{R}^{d+1})).$$

as the space for solutions of the full flow equation.

In this setting we deduce the following result.

Theorem 4.15
For any $T > 0$, $k \in \mathbb{R}$, $k \geq 2$, $(f_0, n_0) \in H^k \times (H^k)^{d+1}$, and $x, y \in Y_T$ equation (4.3) has a solution $(f,n) \in X_T$ with an a priori estimate

$$\|(f,n)\|_{X_T} \leq C(\|(f_0,n_0)\|_{X_\gamma} + \|(x,y)\|_{Y_T}),$$

where the constant C remains bounded, when T tends to zero.

Proof. Firstly, we consider the case, where k is an integer number. Since Y_T and the LH_a spaces do not imply any time regularity for their elements, we can interpret x and y as LH_a^{k-2} and LH_a^{k-1} functions, just by setting them to zero on (T, ∞). Then we use Theorem 4.13 to get a solution (f, n) in P_a^k. But

$$\|(f,n)\|_{X_T} \leq e^{2aT} \|(f,n)\|_{P_a^k} + \|(f_0, n_0)\|_{X_\gamma}$$

and thus

$$\|(f,n)\|_{X_T} \leq C(\|(f_0, n_0)\|_{X_\gamma} + \|(x,y)\|_{Y_T}).$$

It remains to show that C does not blow up for $T \to 0$. This can be seen as follows. Fix an arbitrary $T_0 > 0$ and take $T < T_0$. Now $Y_T \subset Y_{T_0}$ in the sense that we can again extend functions by zero on (T, T_0). Moreover if $(f, n) \in X_T$ and $(\tilde{f}, \tilde{n}) \in X_{T_0}$ such that $(f, n) = (\tilde{f}, \tilde{n})$ on $[0, T)$, then

$$\|(f,n)\|_{X_T} \leq \|(\tilde{f}, \tilde{n})\|_{X_{T_0}}.$$

If $((\tilde{x}, \tilde{y}) \in Y_{T_0}$ is the extension of (x, y) to $[0, T_0)$ by zero, then we have

$$\|(x,y)\|_{Y_T} = \|(x,y)\|_{Y_{T_0}}.$$

Now let (f, n) be the solution of equation (4.3) on $[0, T)$ for right hand side (x, y) and $(\tilde{f}, \tilde{n})$ be the solution on $[0, T_0)$ for right hand side $(\tilde{x}, \tilde{y})$. The functions (f, n) and $(\tilde{f}, \tilde{n})$ coincide on $[0, T)$ and we have

$$\|(f,n)\|_{X_T} \leq \|(\tilde{f}, \tilde{n})\|_{X_{T_0}} \leq C(T_0)(\|((\tilde{x}, \tilde{y})\| + \|(f_0, n_0)\|_{X_\gamma}) = C(T_0)(\|(x,y)\| + \|(f_0, n_0)\|_{X_\gamma})$$

as claimed.

If $k \in \mathbb{R}$ is not an integer, we can now employ interpolation. The maximal regularity result above implies that the stationary operator that we denote in the following by A generates an analytic semi group, which is equivalent to sectoriality (see Theorem 4.4.4 together with Remark 4.1.3 and Chapter 6, Section 1.4 in the book of Prüss and Simonett [77]). Consider Hilbert spaces $X_1 \subset X$ and an unbounded operator $A : X \to X$ with domain $D(A) \subset X$. Now let $\Sigma_\theta = \{z \in \mathbb{C}\backslash\{0\} \big| |\arg z| < \theta\}$ denote a sector of the complex plain. If the set

$$\{A(\lambda + A)^{-1} : \lambda \in \Sigma_\theta\}$$

is bounded in $L(X, X)$ and

$$\{A(\lambda + A)^{-1} : \lambda \in \Sigma_\theta\}$$

is bounded in $L(X_1, X_1)$, too, then

$$\{A(\lambda + A)^{-1} : \lambda \in \Sigma_\theta\}$$

is also bounded in $L(X_\alpha, X_\alpha)$ for any X_α which is an interpolation space of X and X_1. Hence, A has maximal regularity in X_α. □

4.3 The Full Equation

In this section we consider the full equation which is quasilinear as a system. But under further inspection there turns out to be a difference between f and n. The coefficients in front of the highest order term of f and n only depend on the derivatives of f. So somehow the equation is only semi-linear in n. We define the operators

$$
\begin{aligned}
Q_1(f,n) :&= \partial_t f + g^{ij} g^{k\ell} \nabla_i^* \nabla_j^* \nabla_k^* \nabla_\ell^* f + \frac{\delta g^{ij} \nabla_i^* \nabla_j^* \operatorname{div}_f n}{\langle \nu(p,t,\nabla^* f), \nu_0(p) \rangle} \\
&+ b_1(p,t,f,\nabla^* f, \nabla^{*2} f, \nabla^{*3} f, n, \nabla^* n, \nabla^{*2} n) \\
Q_2(f,n) :&= \partial_t n - g^{ij} \nabla_i^* \nabla_j^* n - \delta g^{k\ell} \nabla_k^* (g^{ij} \nabla_i^* \nabla_j^* f + \delta \operatorname{div}_f n) X_\ell + g^{ij} \nabla_i^* m \otimes X_j \nu^* \partial_t f \\
&+ b_2(p,t,f,\nabla^* f, \nabla^{*2} f, n, \nabla^* n) \\
Q(f,n) :&= \begin{pmatrix} Q_1(f,n) \\ Q_2(f,n) \end{pmatrix}
\end{aligned}
\tag{4.8}
$$

with b_1 and b_2 smooth functions as in (4.2). Moreover, we set

$$
L(f,n) := \begin{pmatrix} \partial_t f + \Delta^2 f + \delta \Delta \operatorname{div} n \\ \partial_t n - \Delta n - \delta \nabla (\Delta f + \delta \operatorname{div} n) \end{pmatrix}. \tag{4.9}
$$

Remark 4.16
The definition of X_T in Definition 4.14 depends on the number k, which we do not explicitly denote. The following arguments are valid for all $k > \frac{d}{2} + 3$.

The key ingredient of the short time existence proof is the following result.

Lemma 4.17
For $\varepsilon, R > 0$, the map

$$
X_T \to Y_T : \quad F(f,n) := Q(f,n) - L(f,n)
$$

is Lipschitz on a closed subset

$$
U_\varepsilon = \overline{B_{X_T}(0,R)} \cap \{ \|(f,n) \in X_T | \ \|f(0,\cdot)\|_{C^1} \leq \varepsilon \}
$$

in X_T with Lipschitz constant

$$
C_L(T,R,\varepsilon) \to c\varepsilon \tag{4.10}
$$

for $T \to 0$.

Proof. For each summand in F, we have to estimate the Y_T^1 and Y_T^2 norms of differences. In the following we suppress the $*$ at the differential operator ∇.

We start with the analysis of b_1 and b_2. We observe that since $k > d/2 + 3$, we have $\nabla^3 f \in BUC(0,T;H^{k-3})$ and $\nabla^2 n \in BUC(0,T;H^{k-2})$. Thus, we apply Lemma 1.27 to establish Lipschitz continuity in a larger space and Lemma 1.26 to obtain the assertedscaling of the Lipschitz constant and thus the claimed estimate.

For the term involving the highest order derivatives, we make use of the additional linear terms introduced by the operator L.We observe that the coefficients in front of the highest order terms only depend on f and ∇f, but not on higher derivatives. Therefore, the following calculation applies to all the highest order terms. Thus, the following generic computation can be adapted to all the highest order terms

$$
\begin{aligned}
&g^{ij}(\nabla f) g^{k\ell}(\nabla f) \nabla^4_{ijk\ell} f - \Delta^2 f, && g^{ij}(\nabla f) g^{k\ell} \langle \nabla^3_{ijk} n, X_\ell(\nabla f) \rangle - \delta \operatorname{div}(n), \\
&g^{ij}(\nabla f) \nabla^2_{ij} n - \Delta n, && g^{ij}(\nabla f) g^{k\ell}(\nabla f) \nabla^3_{ik\ell} f X_j(\nabla f) - \nabla(\Delta f), \\
&g^{ij}(\nabla f) g^{k\ell}(\nabla f) \langle \nabla^2_{ik} n, X_\ell(\nabla f) \rangle X_j(\nabla f) - \nabla \operatorname{div}(n).
\end{aligned}
$$

For a smooth coefficient function (or tensor) a, we have

$$\begin{aligned}&\|a(\nabla f_1)\nabla^4 f_1 - a(\nabla f_2)\nabla^4 f_2 - a(0)\nabla^4(f_1 - f_2)\|_{Y_T^1}\\ &\leq C\|[a(\nabla f_1) - a(0)](\nabla^4 f_1 - \nabla^4 f_2)\|_{Y_T^1} + \|[a(\nabla f_1) - a(\nabla f_2)]\nabla^4 f_2\|_{Y_T^1}.\end{aligned}$$

Then, we use the Leibnitz rule for one such summand and obtain

$$\begin{aligned}&\|[a(\nabla f_1) - a(0)](\nabla^4 f_1 - \nabla^4 f_2)\|_{Y_T^1}\\ &\leq C \sum_{\ell \leq r, r \leq k-2} \|\nabla^{r-\ell}[a(\nabla f_1) - a(0)]\nabla^{4+\ell}(f_1 - f_2)\|_{L^2(0,T;L^2)}.\end{aligned}$$

We only do the estimates for $r = k - 2$, since then the cases with fewer derivatives follow immediately. Now we want to see in what spaces the factors can be estimated. For every $t \in [0, T]$, we have by the trace theorem as stated in Theorem 1.23, Corollary 1.15 and our assumption $k > d/2 + 3$ that

$$a(\nabla f_i) \in B(0, LR) \subset H^1(0, T; H^{k-2-1}(M)) \cap L^2(0, T; H^{k+2-1}(M)).$$

Moreover, by Corollary 1.22 we have for $s \in \mathbb{Z}$ embedding

$$H^1(0, T; H^{s-4}(M)) \cap L^2(0, T; H^k(M)) \to L^p(0, T; L^p)$$

for all $p \in [1, \infty)$ satisfying

$$1/p \geq \frac{4 + d - 2s}{2d + 8}.$$

The embedding is continuous for every $p \in [1, \infty)$ if $4 + d - 2s \leq 0$. For $4 + d - 2s < 0$, we even have

$$H^1(0, T; H^{s-4}(M)) \cap L^2(0, T; H^s(M)) \to C^{0,\alpha}([0, T]; C^{0,\alpha})$$

for suitable $\alpha > 0$. Thus, for $k > d/2 + 3$, we see that

$$\begin{aligned}&\nabla^{k-\ell-2}a(\nabla f_i) \in H^1(0, T; H^{\ell-1}(M)) \cap L^2(0, T; H^{\ell+3}(M)) \to L^p(0, T; L^p)\\ &\text{for all } 1/p \geq 1/p^* := \max\{0, \frac{d - 2\ell - 2}{2d + 8}\}\end{aligned}$$

and

$$\begin{aligned}&\nabla^{\ell+4} f_i \in H^1(0, T; H^{k-6-\ell}(M)) \cap L^2(0, T; H^{k-2-\ell}(M)) \to L^q(0, T; L^q)\\ &\text{for all } 1/q \geq 1/q^* := \max\{0, \frac{d + 8 + 2\ell - 2k}{2d + 8}\}.\end{aligned}$$

There are four different cases for $\frac{1}{p^*} + \frac{1}{q^*}$.

1.) If $\frac{1}{p^*} > 0$ and $\frac{1}{q^*} > 0$ we can simply add them to discover

$$\frac{1}{p^*} + \frac{1}{q^*} = \frac{d - 2\ell - 2}{2d + 8} + \frac{d + 6 + 2\ell - 2k}{2d + 8} = \frac{2d + 4 - 2k}{2d + 8}$$

and

$$\frac{2d + 4 - 2k}{2d + 8} < \frac{1}{2} \Leftrightarrow k \geq d/2,$$

which is true since $k > d/2 + 3$.

2.) If $\frac{1}{p^*} = 0$ we have

$$\frac{1}{q^*} = \frac{d + 6 + 2\ell - 2k}{2d + 8} \leq \frac{1}{2} \Leftrightarrow k > \ell + 2,$$

which is true when $\ell < r = k - 2$.

3.) If $\frac{1}{q^*} = 0$ we have see that

$$\frac{1}{p^*} = \frac{d-2\ell-2}{2d+8} < \frac{1}{2} \Leftrightarrow 2\ell > -6,$$

which is always true since ℓ is non-negative.

4.) Finally, if $\frac{1}{q^*} = \frac{1}{p^*} = 0$ there is nothing to show.

These considerations justify the following computation, whenever we can choose p and q such that $1/p + 1/q = 1/2$and either $p < p^*$ or $q < q^*$, leaving space to create a factor T^α by using Hölder's inequality or directly by Lemma 1.26. Then, we find

$$\begin{aligned}
&\|\nabla^{k-2-\ell}[a(\nabla f_1) - a(\nabla f_0)]\nabla^{4+\ell}(f_1 - f_2)\|_{L^2(0,T;L^2)} \\
&\le \|\nabla^{k-2-\ell}[a(\nabla f_1) - a(\nabla f_0)]\|_{L^p(0,T;L^p)}\|\nabla^{4+\ell}(f_1 - f_2)\|_{L^q(0,T;L^q)} \\
&\le CT^\alpha\|\nabla^{k-2-\ell}[a(\nabla f_1) - a(\nabla f_0)]\|_{L^{p^*}(0,T;L^{p^*})}\|\nabla^{4+\ell}(f_1 - f_2)\|_{L^{q^*}(0,T;L^{q^*})} \\
&\le CT^\alpha\|f_1 - f_2\|_{X_T^1}\|f_1 - f_2\|_{X_T^1} \\
&\le CRT^\alpha\|f_1 - f_2\|_{X_T^1}
\end{aligned}$$

The remaining case is $\ell = k-2$ which yields $q^* = 2$ and forces us to set $p = \infty$. Then, we need to employ equation (1.4) from Lemma 1.26 and use that then $d - 2k - 2 < 0$ and thus $a(\nabla f_i) \in C^\alpha(0,T;C^\alpha(M))$ for $\alpha > 0$. From this originates the scaling invariant part of the Lipschitz constant depending on ε.The term $g^{ij}\nabla_i^* m \otimes X_j \nu^* \partial_t f$ is treated also by exactly this argument together with the initial smallness of $\langle X_j, \nu^*\rangle$.

Then the calculation above in the setting of Y_T^2 yields the analogous result and we obtain the Lipschitz continuity of F, with small constant for small T. ☐

Proposition 4.18
For $d \in \mathbb{N}$ let M be a d-dimensional, smooth, orientable closed manifold. Let $k > d/2 + 3$ be a natural number and let X_γ, X_T be as in Definition 4.14 depending on k. Let c denote the constant from equation (4.10) *and fix* $0 < \varepsilon < \frac{1}{8c}$.

Then, for all $(f_0, n_0) \in H^k(M, \mathbb{R}^{d+1})$ such that $\|f_0\|_{C^1} < \varepsilon$, there is a $T > 0$, such that equation (4.2) *has a unique solution $(f, n) \in X_T$, which depends continuously on the initial data.*

Proof. With the help of the above lemmata, we can follow the strategy of [77][ch. 5.1].

Step 1: Existence
We observe the following. The coefficients are only smooth for small values of the arguments and contain for example fractions where we might end up dividing by zero, for some large values of f or its derivatives. But this problem can be overcome. We can assume without loss of generality that the coefficients are globally smooth in their arguments. This can be seen as follows. We modify the coefficients to be globally smooth and agree with the original coefficients on a neighborhood of zero. Then we solve the equation. Now by Sobolev embeddings derivatives of f are continuous in space and time up to $\nabla^3 f$. Hence, at least for a short period of time they stay in the area where the modified and the original coefficients are the same. So we restrict our solution to this interval and have a solution to the original problem.

Looking at our equation, we see that finding a solution is equivalent to finding a solution of a fixed point problem. Recall Q and L as given by equations (4.8) and (4.9), respectively. We introduce the operator $L^{-1} : Y_T \times X_\gamma \to X_T$, which maps a combination of right-hand side and initial data to the unique solution of problem (4.3). This operator is continuous by Theorem

4.15. Then finding a fixed point of the map

$$L^{-1}(Q(\cdot) - L\cdot, f_0, n_0) : X_T \to X_T$$

is equivalent to finding a solution of equation (4.2).

We want to apply the contraction mapping principle. Therefore, we have to show that $L^{-1}(Q(\cdot) - L\cdot, f_0, n_0)$ is a contraction on a suitable closed subset of X_T and maps this set to itself. We take $R > \|(f_0, n_0)\|_{H^k}$ such that the Ball $B(0, R) \subset X_T$ contains functions with the given initial data.

We have seen in Theorem 4.15 and Lemma 4.17 that $Q - L$ is a Lipschitz map on subsets

$$U_\varepsilon = B(0, R) \cap \{\|(f, n) \in X_T| \ \|f(0, \cdot)\|_{H^k} \leq \varepsilon\} \subset X_T \to Y_T$$

and the Lipschitz constant tends to $c\varepsilon$ if T and is sufficiently small.

We start with $T_0 > 0$ and take an arbitrary element x in X_{T_0} with the right initial data and take $y = L^{-1}(Q(x) - Lx)$. Now choose $R > 2\max\{\|x\|_{X_{T_0}}, \|y\|_{X_{T_0}}\}$. For any time $T < T_0$ we get an element $x \in X_T$ by restricting x to the shorter time intervall. The same is true for y. Moreover, it will be still true that $y = L^{-1}(Q(x) - Lx)$ and we have

$$\|x\|_{X_T} \leq \|x\|_{X_{T_0}}, \ \|y\|_{X_T} \leq \|y\|_{X_{T_0}}.$$

In addition, for all $z \in U_\varepsilon$ we have

$$\|z - x\|_{X_T} < 2R.$$

Now we choose T so that the Lipschitz constant C_L of $L^{-1}(Q(z) - Lz, f_0, n_0)$ is smaller than 1/4. This is possible since $\varepsilon < 1/8$ and C_L is of the form $C(R)T^\alpha + \varepsilon$. Hence, we have

$$\|L^{-1}(Q(z) - Lz, f_0, n_0) - y\|_{X_T} < R/2.$$

Since $L^{-1}(\cdot, f_0, n_0)$ does not change the time trace at $t = 0$, this implies that for T small enough $L^{-1}(Q(\cdot) - L\cdot, f_0, n_0)$ maps U_ε to itself. Thus, by the Banach fixed point theorem, the map $L^{-1}(F)$ has a fixed point in U_ε and this means we have a solution to our equation.

Step 2: Uniqueness Suppose the solution found in step 1 is not unique. Then, we choose R big enough, that both solutions lie in some U_ε.

For T small enough $L^{-1}(F)$ is a contraction also on this set. Thus, there is a unique fixed point. But both solutions give a fixed point of $L^{-1}(F)$ in U_ε, which is a contradiction.

Step 3: Continuous dependence on the initial data Suppose $(f_1, n_1) \in X_\gamma$ and $(f_2, n_2) \in X_\gamma$ are suitable initial data, that is $\|f_1\|_{C^1}, \|f_2\|_{C^1} \leq \varepsilon$. With U_ε as above, we calculate for $x_1, x_2 \in U_\varepsilon$

$$\begin{aligned} &\|L^{-1}(Q(x_1) - Lx_1, (f_1, n_1)) - L^{-1}(Q(x_2) - Lx_2, (f_2, n_2))\|_{X_T} \\ &\leq C(\|(Q - L)(x_1) - (Q - L)(x_2)\|_{Y_T} + \|(f_1 - f_2, n_1 - f_2)\|_{X_\gamma}) \\ &\leq \frac{1}{2}\|x_1 - x_2\|_{X_T} + C\|(f_1 - f_2, n_1 - f_2)\|_{X_\gamma}. \end{aligned} \tag{4.11}$$

Here we used again, that $Q - L$ is Lipschitz on U_ε with small Lipschitz constant. Suppose for $i = 1, 2$ the functions x_i are solutions of (4.2) for initial data (f_i, n_i), then

$$x_i = L^{-1}(Q(x_i) - Lx_i, (f_i, n_i)).$$

Thus, by (4.11) we conclude

$$\|x_1 - x_2\|_{X_T} \leq C\|(f_1 - f_2, n_1 - n_2)\|_{X_\gamma}.$$

□

Theorem 4.19
For $d \in \mathbb{N}$ let M be a d-dimensional, smooth, orientable closed manifold. Let $k > d/2 + 3$ be a natural number and let $X_{T,\mathrm{imm}}$ be as in Definition 4.14 depending on k. Let an immersion $\varphi_0 \in H^k_{\mathrm{imm}}(M, \mathbb{R}^{d+1})$ and a vector field $n_0 \in H^k(M, \mathbb{R}^{d+1})$ be given.

Then, there is a $T > 0$, such that the gradient flow equation (3.8) of the energy (3.1) has a unique solution $(\varphi, n) \in X_{T,\mathrm{imm}}$, which depends continuously on the initial data.

Proof. As a first step, we write φ_0 as the graph of a function $f_0 \in H^k(M)$ over a smooth immersion φ^*. We have seen in Proposition 4.2 that this leads to a quasilinear system of equations for functions f, n given by (4.2). Following Lemma 1.18 the C^1-norm of the initial data f_0 can be made arbitrarily small by taking better approximations of φ. Let c denote the constant from equation (4.10), then we fix $0 < \varepsilon < \frac{1}{8c}$ and arrange that $\|f_0\|_{C^1} < \varepsilon$.

Existence, uniqueness and continuous dependence on the initial datum of a solution for the equation (4.2) is guaranteed by Proposition 4.18. A solution (φ, n) of can then be obtained by setting $\tilde{\varphi} = \varphi_0 + f\nu_0$ and reparametrization by Lemma 3.6. □

4.4 Some Non-Local Constraints

Since in the model the image of the immersion φ represents the shape of a membrane, it is reasonable, to consider some additional constraints as given in (3.2). In Section 3.4 it was discussed what correction terms are necessary to obtain a flow equation whose solution preserves the constrained quantities and still decreases the energy.Our aim is now to show that for the flow (3.8) the correction terms for volume and area preservation are of lower order. We therefore have to calculate them in terms of n and the height function f. It will turn out to be useful to have at hand a rather general estimate on the difference of integral terms in Bochner spaces.

Lemma 4.20
For $k, p \in \mathbb{N}$ satisfying $k > n/2$ and $p \leq k$ let $a \in C^{k+2}(\mathbb{R} \times [0, T])$ be a real valued function. Then the map

$$
\begin{aligned}
A : L^2(0,T; H^{k+p}(M)) \cap H^1(0,T; H^{k-p}(M)) &\to L^2(0,T) \\
f &\mapsto \int_M a(f,t)\,\mathrm{d}\mu
\end{aligned}
$$

is well defined, of class C^k and the k-th Fréchet derivative $D^k A$ is locally Lipschitz continuous.

Proof. First off, we recall that Lemma 1.27 states that $f \mapsto a(f,t)$ is a well-defined map from $L^2(0,T; H^{k+p}(M)) \cap H^1(0,T; H^{k-p}(M))$ to itself and differentiable. For two functions $f_1, f_2 \in L^2(0,T; H^{k+p}(M)) \cap H^1(0,T; H^{k-p}(M))$ Hölder's inequality gives

$$
\int_0^T |\int_M a(f_1,t) - a(f_2,t)\,\mathrm{d}\mu|^2\,\mathrm{d}t \leq |M| \big\| a(f_1,t) - a(f_2,t) \big\|^2_{L^2(0,T;L^2(M))},
$$

but this norm can be estimated by Lemma 1.26 in combination with Theorem 1.14 □

With this basic observation, we can analyze the correction terms for volume and area constraints and obtain local well-posedness of the constrained flow.

Theorem 4.21
For $d \in \mathbb{N}$ let M be a d-dimensional, smooth, orientable, closed manifold. Let $k > d/2+3$ be a natural number and let $X_{T,\mathrm{imm}}$ be as in Definition 4.14 depending on k. Let $\varphi_0 \in H^k_{\mathrm{imm}}(M, \mathbb{R}^{d+1})$ be an immersion satisfying $A(\varphi_0) = A_0$ and $Vol(\varphi_0) = V_0$ for A_0, V_0 as in (3.2) and let $n_0 \in H^k(M, \mathbb{R}^{d+1})$ be a vector field.

Then, there is a $T > 0$, such that the area and volume preserving gradient flow equation (3.10) of the energy (3.1) has a unique solution in $X_{T,\mathrm{imm}}$, which depends continuously on the initial data. The result remains valid, if we only impose one of the two constraints.

Proof. First off, we observe that when volume and area preservation are enforced and $\varphi_0(M)$ is a round sphere, φ will be stationary due to isoperimetric restrictions. The result then follows from short-time existence for the harmonic map heat flow. Thus, in the following, when we enforce area and volume preservation, we can assume that $\varphi(M)$ is not a round sphere and the mean value free mean curvature is not identically 0. Thus, we can safely divide its L^2-norm.

To prove this result, we show that the proof of Theorem 4.19 can be generalized. We deduce that the correction that is necessary to preserve area and/or volume is of lower order, in the sense that it can be absorbed in the b_i terms in equation (4.2), when we formulate the problem in terms of the height function. First off, we remark that for (M, g) with finite volume, we have an embedding $\mathbb{R} \to H^k(M)$, since the constant functions are integrable.

We have seen in Lemma 4.1 that if the evolving immersion φ is given as $\varphi = \varphi^* + f\nu^*$, then the normal velocity $v = \langle \partial_t \varphi, \nu \rangle$ in terms of the height function f is

$$v = \partial_t f \langle \nu^*, \nu \rangle.$$

Hence, v in terms of f and n is given by the right-hand side of equation (4.2) times $\langle \nu^*, \nu(t) \rangle$. We set

$$\begin{aligned} R(f,n) = g^{ij} g^{k\ell} \nabla^*_i \nabla^*_j \nabla^*_k \nabla^*_\ell f + \frac{\delta g^{ij} \nabla^*_i \nabla^*_j \operatorname{div}_f n}{\langle \nu, \nu^* \rangle} \\ + b_1(f, \nabla^* f, \nabla^{*2} f, \nabla^{*3} f, n, \nabla^* n, \nabla^{*2} n), \end{aligned}$$

hence,

$$v = \langle \nu^*, \nu \rangle R(f,n)$$

then we end up with the equation

$$\partial_t f = R(f,n) - \frac{\displaystyle\int_M v \,\mathrm{d}\mu}{\langle \nu^*, \nu \rangle \displaystyle\int_M \mathrm{d}\mu} - \frac{\displaystyle\int_M v(H - \overline{H}) \,\mathrm{d}\mu}{\langle \nu^*, \nu \rangle \displaystyle\int_M (H - \overline{H})^2 \,\mathrm{d}\mu} (H - \overline{H}).$$

Observe that also $\mathrm{d}\mu$ has a dependence on f and also v and H depend on f and n. Since we see from equation (3.8) that the main part of v is a divergence, we find

$$\begin{aligned} \int_M v \,\mathrm{d}\mu &= \int_M -\Delta_\varphi (H_\varphi + \delta \operatorname{div}_\varphi n) + b_1(\nu, \nabla_\varphi \nu, H, \nabla_\varphi H, n, \nabla_\varphi n, \nabla^2_\varphi n) \,\mathrm{d}\mu \\ &= \int_M b_1(\nu, \nabla_\varphi \nu, H, \nabla_\varphi H, n, \nabla_\varphi n, \nabla^2_\varphi n) \,\mathrm{d}\mu. \end{aligned}$$

To establish Lipschitz continuity of the correction terms we have to estimate their difference. Since the volume element depends also on the integrands it is useful to perform a change of variables and obtain all integrals with respect to $\mathrm{d}\mu^*$. Thus, for a function ϕ on M we find that

$$\int_M \phi \,\mathrm{d}\mu = \int_M \phi J \,\mathrm{d}\mu^*,$$

where J is in local coordinates given as

$$J = \frac{\sqrt{g}}{\sqrt{g^*}}.$$

Hence, for $R, \varepsilon > 0$ setting again $U_\varepsilon = \overline{B_{X_T}(0,R)} \cap \{\|(f,n) \in X_T| \ \|f(0,\cdot)\|_{C^1} \leq \varepsilon\}$ we have to compute for two pairs of functions $(f_1,n_1),(f_2,n_2) \in U_\varepsilon$ the difference

$$\frac{\int_M B_1 \,\mathrm{d}\mu}{\langle \nu^*, \nu_1\rangle \int_M 1 \,\mathrm{d}\mu} - \frac{\int_M B_2 \,\mathrm{d}\mu}{\langle \nu^*, \nu_2\rangle \int_M 1 \,\mathrm{d}\mu}.$$

Here $B_i = b_1(p,t,f_i,\nabla^* f_i, \nabla^{*2} f_i, \nabla^{*3} f_i, n_i, \nabla^* n_i, \nabla^{*2} n_i)$ for $i = 1,2$ and b_1 from equation (3.8) which is smooth in its arguments. Moreover, let $J_i = J(f_i)$ denote the change of volume elements. Then,

$$\begin{aligned}
&\frac{\int_M B_1 \,\mathrm{d}\mu}{\langle \nu^*, \nu_1\rangle \int_M 1 \,\mathrm{d}\mu} - \frac{\int_M B_2 \,\mathrm{d}\mu}{\langle \nu^*, \nu_2\rangle \int_M 1 \,\mathrm{d}\mu} \\
&= \frac{\int_M B_1 J_1 \,\mathrm{d}\mu^*}{\langle \nu^*, \nu_1\rangle \int_M J_1 \,\mathrm{d}\mu^*} - \frac{\int_M B_2 J_2 \,\mathrm{d}\mu^*}{\langle \nu^*, \nu_2\rangle \int_M J_2 \,\mathrm{d}\mu^*} \\
&= \frac{\int_M B_1 J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^* \langle \nu^*, \nu_2\rangle - \int_M B_2 J_2 \,\mathrm{d}\mu^* \int_M J_1 \,\mathrm{d}\mu^* \langle \nu^*, \nu_1\rangle}{\langle \nu^*, \nu_1\rangle \langle \nu^*, \nu_2\rangle \int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*} \\
&= \frac{\int_M B_1 J_1 \,\mathrm{d}\mu^* \Big[\langle \nu^*, \nu_2\rangle \int_M J_2 \,\mathrm{d}\mu^* - \langle \nu^*, \nu_1\rangle \int_M J_1 \,\mathrm{d}\mu^*\Big]}{\langle \nu^*, \nu_1\rangle \langle \nu^*, \nu_2\rangle \int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*} \\
&\quad - \frac{\langle \nu^*, \nu_1\rangle \int_M J_1 \,\mathrm{d}\mu^* \Big[\int_M B_2 J_2 \,\mathrm{d}\mu^* - \int_M B_1 J_1 \,\mathrm{d}\mu^*\Big]}{\langle \nu^*, \nu_1\rangle \langle \nu^*, \nu_2\rangle \int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*}
\end{aligned}$$

We consider these two terms separately. After calculating

$$\begin{aligned}
&\frac{\int_M B_1 J_1 \,\mathrm{d}\mu^* \Big[\langle \nu^*, \nu_2\rangle \int_M J_2 \,\mathrm{d}\mu^* - \langle \nu^*, \nu_1\rangle \int_M J_1 \,\mathrm{d}\mu^*\Big]}{\langle \nu^*, \nu_1\rangle \langle \nu^*, \nu_2\rangle \int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*} \\
&= \int_M B_1 J_1 \,\mathrm{d}\mu^* \frac{\langle \nu^*, \nu_2\rangle \big(\int_M J_2 - J_1 \,\mathrm{d}\mu^*\big) + \int_M J_1 \,\mathrm{d}\mu^* \big(\langle \nu^*, \nu_2\rangle - \langle \nu^*, \nu_1\rangle\big)}{\langle \nu^*, \nu_1\rangle \langle \nu^*, \nu_2\rangle \int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*} \\
&= \int_M B_1 J_1 \,\mathrm{d}\mu^* \frac{\int_M J_2 - J_1 \,\mathrm{d}\mu^*}{\langle \nu^*, \nu_1\rangle \int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*} + \int_M B_1 J_1 \,\mathrm{d}\mu^* \frac{\langle \nu^*, \nu_2\rangle - \langle \nu^*, \nu_1\rangle}{\langle \nu^*, \nu_1\rangle \langle \nu^*, \nu_2\rangle \int_M J_2 \,\mathrm{d}\mu^*},
\end{aligned}$$

we set

$$I := \int_M B_1 J_1 \,\mathrm{d}\mu^* \frac{\displaystyle\int_M J_2 - J_1 \,\mathrm{d}\mu^*}{\langle \nu^*, \nu_1 \rangle \displaystyle\int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*},$$

$$II := \int_M B_1 J_1 \,\mathrm{d}\mu^* \frac{\langle \nu^*, \nu_2 \rangle - \langle \nu^*, \nu_1 \rangle}{\langle \nu^*, \nu_1 \rangle \langle \nu^*, \nu_2 \rangle \displaystyle\int_M J_2 \,\mathrm{d}\mu^*}$$

and

$$\begin{aligned} III &:= \frac{\langle \nu^*, \nu_1 \rangle \displaystyle\int_M J_1 \,\mathrm{d}\mu^* \Big[\int_M B_2 J_2 \,\mathrm{d}\mu^* - \int_M B_1 J_1 \,\mathrm{d}\mu^* \Big]}{\langle \nu^*, \nu_1 \rangle \langle \nu^*, \nu_2 \rangle \displaystyle\int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*} \\ &= \frac{\Big(\displaystyle\int_M B_2 J_2 \,\mathrm{d}\mu^* - \int_M B_1 J_1 \,\mathrm{d}\mu^* \Big)}{\langle \nu^*, \nu_2 \rangle \displaystyle\int_M J_2 \,\mathrm{d}\mu^*}. \end{aligned}$$

We have to calculate $\|I\|_{Y_T^1}, \|II\|_{Y_T^1}$, and $\|III\|_{Y_T^1}$. We observe that since $J(0) = 1$ and by continuity $J > 1/2$ in a neighborhood of 0 and as $\|\nabla f_i\|_{BUC(0,T,C^0)} \leq \varepsilon + C(R)T^\alpha$, the terms $\langle \nu^*, \nu_i \rangle$ and $\frac{1}{\int J_i \,\mathrm{d}\mu^*}$ are bounded from below by $1/4$ for ε and T sufficiently small. Thus,

$$\begin{aligned} \|I\|_{Y_T^1} &= \Big\| \Big\| \int_M B_1 J_1 \,\mathrm{d}\mu^* \frac{\displaystyle\int_M J_2 - J_1 \,\mathrm{d}\mu^*}{\langle \nu^*, \nu_1 \rangle \displaystyle\int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*} \Big\|_{H^{k-2}} \Big\|_{L^2(0,T)} \\ &= \Big\| \Big\| \frac{1}{\langle \nu^*, \nu_1 \rangle} \Big\|_{H^{k-2}} \int_M B_1 J_1 \,\mathrm{d}\mu^* \frac{\displaystyle\int_M J_2 - J_1 \,\mathrm{d}\mu^*}{\displaystyle\int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*} \Big\|_{L^2(0,T)} \\ &= \Big\| \frac{1}{\langle \nu^*, \nu_1 \rangle} \Big\|_{BUC(0,T,H^{k-2})} \Big\| \Big| \int_M J_2 - J_1 \,\mathrm{d}\mu^* \frac{\displaystyle\int_M B_1 J_1 \,\mathrm{d}\mu^*}{\displaystyle\int_M J_1 \,\mathrm{d}\mu^* \int_M J_2 \,\mathrm{d}\mu^*} \Big| \Big\|_{L^2(0,T)} \\ &\leq C(R, \varepsilon) \|J_1 - J_2\|_{L^2(0,T,L^2)} \\ &\leq C(R, \varepsilon) T^\alpha \|(f_1 - f_2, n_1 - n_2)\|_{X_T} \end{aligned}$$

by Lemma 1.26 and 4.20.

Analogously, we can find estimates for II and III, leaving us with an estimate for the Lipschitz constant of the first non-local term reading

$$\left\| \frac{\displaystyle\int_M B_1 \,\mathrm{d}\mu}{\displaystyle\int_M 1 \,\mathrm{d}\mu} - \frac{\displaystyle\int_M B_2 \,\mathrm{d}\mu}{\displaystyle\int_M 1 \,\mathrm{d}\mu} \right\|_{Y_T} \leq C(R, \varepsilon) T^\alpha \|(f_1 - f_2, n_1 - n_2)\|_{X_T}.$$

The calculation for the area preserving correction is a little more involved, since H depends on the evolution of the surface. Moreover, by using Gauß' theorem the highest order term does not simply vanish but produces a third order term involving the gradient of the curvature. That is, with the gradient vector field $Z = Z(f, n) = \mathrm{grad}_f(H_f + \delta \,\mathrm{div}_f(n))$ and all lower order terms collected in $B = B(f,n)$ we integrate by parts to find

$$\begin{aligned}
&\frac{\int_M v(H-\overline{H})\,\mathrm{d}\mu}{\langle\nu^*,\nu\rangle\int_M (H-\overline{H})^2\,\mathrm{d}\mu}(H-\overline{H})\\
&=\frac{\int_M \langle Z,\operatorname{grad}_f H\rangle + B(H-\overline{H})\,\mathrm{d}\mu}{\langle\nu^*,\nu\rangle\int_M (H-\overline{H})^2\,\mathrm{d}\mu}(H-\overline{H})\\
&=\frac{\int_M \big(\langle Z,\operatorname{grad}_f H\rangle + B(H-\overline{H})\big)J\,\mathrm{d}\mu^*}{\langle\nu^*,\nu\rangle\int_M (H-\overline{H})^2 J\,\mathrm{d}\mu^*}(H-\overline{H}),
\end{aligned}$$

since $\overline{H}$ is constant in space. We take $(f_1,n_1),(f_2,n_2)\in U_\varepsilon$ again and calculate, denoting the dependence of a quantity on f_i and n_i only by an index.

$$\begin{aligned}
&\frac{\int_M \big(\langle Z_1,\operatorname{grad}_{f_1} H_1\rangle + B_1(H_1-\overline{H}_1)\big)J_1\,\mathrm{d}\mu^*}{\langle\nu^*,\nu_1\rangle\int_M (H_1-\overline{H}_1)^2 J_1\,\mathrm{d}\mu^*}(H_1-\overline{H}_1)\\
&-\frac{\int_M \big(\langle Z_2,\operatorname{grad}_{f_2} H_2\rangle + B_2(H_2-\overline{H}_2)\big)J_2\,\mathrm{d}\mu^*}{\langle\nu^*,\nu_2\rangle\int_M (H_2-\overline{H}_2)^2 J_2\,\mathrm{d}\mu^*}(H_2-\overline{H}_2)\\
=&\frac{\int_M \big(\langle Z_1,\operatorname{grad}_{f_1} H_1\rangle + B_1(H_1-\overline{H}_1)\big)J_1\,\mathrm{d}\mu^*}{\langle\nu^*,\nu_1\rangle\int_M (H_1-\overline{H}_1)^2 J_1\,\mathrm{d}\mu^*}(H_1-\overline{H}_1-(H_2-\overline{H}_2))\\
&+\Big(\frac{\int_M \big(\langle Z_1,\operatorname{grad}_{f_1} H_1\rangle + B_1(H_1-\overline{H}_1)\big)J_1\,\mathrm{d}\mu^*}{\langle\nu^*,\nu_1\rangle\int_M (H_1-\overline{H}_1)^2 J_1\,\mathrm{d}\mu^*}\\
&-\frac{\int_M \big(\langle Z_2,\operatorname{grad}_{f_2} H_2\rangle + B_2(H_2-\overline{H}_2)\big)J_2\,\mathrm{d}\mu^*}{\langle\nu^*,\nu_2\rangle\int_M (H_2-\overline{H}_2)^2 J_2\,\mathrm{d}\mu^*}\Big)(H_2-\overline{H}_2).
\end{aligned}$$

We write the second term as

$$\begin{aligned}
&\Big(\frac{\int_M \big(\langle Z_1,\operatorname{grad}_{f_1} H_1\rangle + B_1(H_1-\overline{H}_1)\big)J_1\,\mathrm{d}\mu^*}{\langle\nu^*,\nu_1\rangle\int_M (H_1-\overline{H}_1)^2 J_1\,\mathrm{d}\mu^*}\\
&-\frac{\int_M \big(\langle Z_2,\operatorname{grad}_{f_2} H_2\rangle + B_2(H_2-\overline{H}_2)\big)J_2\,\mathrm{d}\mu^*}{\langle\nu^*,\nu_2\rangle\int_M (H_2-\overline{H}_2)^2 J_2\,\mathrm{d}\mu^*}\Big)(H_2-\overline{H}_2)\\
=&\Big(\int_M \big(\langle Z_1,\operatorname{grad}_{f_1} H_1\rangle + B_1(H_1-\overline{H}_1)\big)J_1\,\mathrm{d}\mu^*\langle\nu^*,\nu_2\rangle\int_M (H_2-\overline{H}_2)^2 J_2\,\mathrm{d}\mu^*\\
&-\int_M \big(\langle Z_2,\operatorname{grad}_{f_2}\rangle + B_2(H_2-\overline{H}_2)\big)J_2\,\mathrm{d}\mu^*\langle\nu^*,\nu_1\rangle\int_M (H_1-\overline{H}_1)^2 J_1\,\mathrm{d}\mu^*\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{H_2 - \overline{H}_2}{\langle \nu^*, \nu_1 \rangle \int_M (H_1 - \overline{H}_1)^2 J_1 \,\mathrm{d}\mu^* \langle \nu^*, \nu_2 \rangle \int_M (H_2 - \overline{H}_2)^2 J_2 \,\mathrm{d}\mu^*} \\
= &\Big(\int_M \big(\langle Z_1, \operatorname{grad}_{f_1} H_1 \rangle + B_1 (H_1 - \overline{H}_1) \big) J_1 \,\mathrm{d}\mu^* \\
&\times \Big[\langle \nu^*, \nu_2 \rangle \int_M (H_2 - \overline{H}_2)^2 J_2 \,\mathrm{d}\mu^* - \langle \nu^*, \nu_1 \rangle \int_M (H_1 - \overline{H}_1)^2 J_1 \,\mathrm{d}\mu^* \Big] \\
&+ \langle \nu^*, \nu_1 \rangle \int_M (H_1 - \overline{H}_1)^2 J_1 \,\mathrm{d}\mu^* \\
&\times \big(\int_M (\langle Z_2, \operatorname{grad}_{f_2} H_2 \rangle + B_2 (H_2 - \overline{H}_2)) J_2 \,\mathrm{d}\mu^* \\
&- \int_M (\langle Z_1, \operatorname{grad}_{f_1} H_1 \rangle + B_1 (H_1 - \overline{H}_1)) J_1 \,\mathrm{d}\mu^* \big) \Big) \\
&\times \frac{H_2 - \overline{H}_2}{\langle \nu^*, \nu_1 \rangle \int_M (H_1 - \overline{H}_1)^2 J_1 \,\mathrm{d}\mu^* \langle \nu^*, \nu_2 \rangle \int_M (H_2 - \overline{H}_2)^2 J_2 \,\mathrm{d}\mu^*}
\end{aligned}
$$

Thus, we have split the term

$$
\frac{\int_M v (H - \overline{H}) \,\mathrm{d}\mu}{\langle \nu^*, \nu \rangle \int_M (H - \overline{H})^2 \,\mathrm{d}\mu} (H - \overline{H})
$$

into several parts, each of which we can estimate from above by the Lemmas 1.26 and 4.20. This shows that the correction terms arising from area and volume preservation are Lipschitz continuous as operators on the solution space. That is, they do not affect the strategy to prove short-time existence. □

4.5 The Unit-Length Constraint for n

In the following we will consider the case, where the length of n is fixed and equal to one. The necessary correction term was already discussed in Section 3.4. We denote the identity on $\mathbb{R}^{d+1}$ by E_{d+1}, then the adjusted equation reads

$$
\begin{aligned}
\partial_t n = \big(E_{d+1} - n \otimes n \big) \Big[&\lambda g^{ij} \nabla_i^* \nabla_j^* n + \delta g^{k\ell} \nabla_k^* (g^{ij} \nabla_i^* \nabla_j^* f + \delta \operatorname{div}_f n) X_\ell \\
&+ b_2(p, t, f, \nabla^* f, \nabla^{*2} f, n, \nabla^* n) + g^{ij} \nabla_i^* n \otimes X_j \nu^* \partial_t f \Big].
\end{aligned} \tag{4.12}
$$

That this correction does the trick and guaranties that n is a map to the unit sphere can be seen by differentiating $\|n\|^2$ in time. This gives $\frac{\mathrm{d}}{\mathrm{d}t} \|n\|^2 = 2 \langle \partial_t n, n \rangle$.Now, since $\partial_t n$ is by construction orthogonal to n, we see that the length of n is constant. Hence, n maps to the unit sphere, if the initial data maps to the unit sphere.

This however, changes the coefficients of the highest order terms. This kind of correction is also needed if one studies the harmonic map heat flow. The crucial observation is, that the normal component of the Laplace Beltrami operator is of lower order (see e.g. [63, Ch. 1]). This means we may interchange $E_{d+1} - n \otimes n$ and $g^{ij} \nabla_{ij}^2 n$ only producing a lower order term which is covered by a modification of b_2. Nevertheless, we have to adjust the linear system,as there is still another change in the highest order terms.

If we want to solve the flow equation with initial datum $n_0 \in H^k(M, \mathbb{S}^d)$, we approximate n_0 by $a \in C^\infty(M, \mathbb{S}^d)$ in H^k. Since $k > d/2$, this is possible by standard approximation results [8, Sec. 1]. Therefore, we have $n_0 = a + \tilde{n}_0$

We linearize the whole system (4.2) with the equationfor n replaced by (4.12) around a. The new linear problem that we obtain is therefore in $\tilde{n}$ from which n is reconstructed by $n = a + \tilde{n}$. We find the system

$$\begin{aligned} \partial_t f + \Delta(\Delta f + \delta \operatorname{div} \tilde{n}) &= x \quad \text{on } M \times (0,T), \\ \partial_t \tilde{n} - \lambda \Delta \tilde{n} - (E_{d+1} - a \otimes a)\, \delta \nabla(\Delta f + \delta \operatorname{div} \tilde{n}) &= y \quad \text{on } M \times (0,T), \\ f(\cdot, 0) &= f_0 \quad \text{on } M, \\ \tilde{n}(\cdot, 0) &= \tilde{n}_0 \quad \text{on } M. \end{aligned} \tag{4.13}$$

In the following we write n instead of $\tilde{n}$

We want to use the method of continuity to solve this equation. The following propositionand a proof can e.g. be found in the textbook by Gilbarg and Trudinger [41, Theorem 5.2]

Proposition 4.22 (Method of continuity)
Let X and Y be Banach spaces and $\{L_t\}_{t\in[0,1]}$ a continuous family of operators from X to Y, such that there is a uniform a priori estimate

$$\|x\|_X \leq C \|L_t x\|_Y$$

for all $t \in [0,1]$. Then L_1 is surjective if and only if L_0 is surjective.

With help of this result, we prove maximal regularity for the new linear operator.

Theorem 4.23
For $\big((x,y),(f_0,n_0)\big) \in Y_T \times X_\gamma$ and $\delta < 2\sqrt{\lambda}$ the equation (4.13) *has a unique solution in X_T and*

$$\|(f,n)\|_{X_T} \leq C(\|(x,y)\|_{Y_T} + \|(f_0,n_0)\|_{X_\gamma}).$$

Proof. We apply the method of continuity to the family of operators

$$L_\theta(f,n) = \left(\begin{pmatrix} \partial_t f + \Delta^2 f + \delta \Delta \operatorname{div} n \\ \partial_t n - \lambda \Delta n - (E_{d+1} - \theta a \otimes a) \delta \nabla(\Delta f + \delta \operatorname{div} n) \end{pmatrix}, \begin{pmatrix} f(0) \\ n(0) \end{pmatrix} \right).$$

Hence, we have to show an a priori estimate. Suppose (f,n) are smooth and the functions ζ form a partition of unity subordinate to a covering of M with charts. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multiindex of order $2 \leq \ell \leq k$. Again, k is related to the regularity of functions in X_T as in Definition 4.14. Then, we take the $L^2(M)$ scalar product of $L_\theta(f,n)$ with

$$(-1)^{|\alpha|} D^\alpha\big(\zeta^{2|\alpha|+2} D^\alpha(f,n)\big).$$

In the following calculation, where derivatives went onto the cut-off functions, we denote the resulting lower order terms only by "+L.O.T."These terms may be ignored since they can be bounded by some interpolation inequality.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta^{|\alpha|+1}D^\alpha f\|^2_{L^2(M)} + \frac{\mathrm{d}}{\mathrm{d}t}\|\zeta^{|\alpha|+1}D^\alpha n\|^2_{L^2(M)} + (\lambda - \frac{\theta\delta\sqrt{\lambda}}{2})\|\zeta^{|\alpha|+1}\nabla D^\alpha n\|^2_{L^2(M)}\\
&+(1-\frac{\theta\delta}{2\sqrt{\lambda}})\|\zeta^{|\alpha|+1}\Delta D^\alpha f + \zeta^{|\alpha|+1}\delta\operatorname{div}(D^\alpha n)\|^2_{L^2(M)}\\
\leq\ &\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta^{|\alpha|+1}D^\alpha f\|^2_{L^2(M)} + \frac{\mathrm{d}}{\mathrm{d}t}\|\zeta^{|\alpha|+1}D^\alpha n\|^2_{L^2(M)}\\
&+\|\zeta^{|\alpha|+1}\Delta D^\alpha f + \delta\zeta^{|\alpha|+1}\operatorname{div}(D^\alpha n)\|^2_{L^2(M)} + \lambda\|\zeta^{|\alpha|+1}\nabla D^\alpha n\|^2_{L^2(M)}\\
&-\theta\delta\langle\zeta^{|\alpha|+1}\Delta D^\alpha f + \zeta^{|\alpha|+1}\delta\operatorname{div}(D^\alpha n), \zeta^{|\alpha|+1}\operatorname{div}(a\otimes aD^\alpha n)\rangle_{L^2(M)} + \text{L.O.T.}\\
=\ &\langle(-1)^{|\alpha|}D^\alpha[\zeta^{2|\alpha|+2}D^\alpha f], \partial_t f + \Delta^2 f + \delta\Delta\operatorname{div} n\rangle_{L^2(M)}\\
&+\langle(-1)^{|\alpha|}D^\alpha[\zeta^{2|\alpha|+2}D^\alpha n], \partial_t n - \lambda\Delta n - (E_{d+1}-\theta a\otimes a)\delta\nabla(\Delta f + \delta\operatorname{div} n)\rangle_{L^2(M)} + \text{L.O.T.}
\end{aligned}
\tag{4.14}
$$

We integrate by parts and use Schwarz'and Young's inequality to obtain

$$
\begin{aligned}
&\langle x, D^\alpha\zeta D^\alpha f\rangle_{L^2(M)} + \langle y, D^\alpha\zeta D^\alpha n\rangle_{L^2(M)}\\
\leq\ &C_\varepsilon\|D^{\alpha-\beta}x\|^2_{L^2(M)} + \varepsilon\|\zeta D^\beta D^\alpha f\|^2_{L^2(M)} + C_\varepsilon\|D^{\alpha-\gamma}y\|^2_{L^2(M)} + \varepsilon\|D^\gamma\zeta D^\alpha n\|^2_{L^2(M)},
\end{aligned}
$$

with $\beta, \gamma \leq \alpha$ and $|\beta| = 2$ and $|\gamma| = 1$.We note, that we have to move one more derivative in the term involving f, due to its higher differentiability. Setting $x = \partial_t f + \Delta^2 f + \delta\Delta\operatorname{div} n$ and $y = \partial_t n - \lambda\Delta n - (E_{d+1} - \theta a\otimes a)\delta\nabla(\Delta f + \delta\operatorname{div} n)$ we discover by integrating (4.14) in time

$$
\begin{aligned}
&\|\zeta^{|\alpha|+1]}f\|_{L^2(0,T;H^{\ell+2}(M))} + \|\zeta^{|\alpha|+1]}n\|_{L^2(0,T;H^{\ell+1}(M))}\\
\leq\ &C\big[\sup_{t\in[0,T]}\|\zeta^{|\alpha|+1}D^\alpha f\|^2_{L^2(M)} + \sup_{t\in[0,T]}\|\zeta^{|\alpha|+1}D^\alpha n\|^2_{L^2(M)}\\
&+\|\zeta^{|\alpha|+1}\Delta D^\alpha f + \delta\zeta^{|\alpha|+1}\operatorname{div}(D^\alpha n)\|^2_{L^2(0,T;L^2(M))} + \|\zeta^{|\alpha|+1}\nabla D^\alpha n\|^2_{L^2(0,T;L^2(M))}\big]\\
\leq\ &C(\|D^{\alpha-\beta}(\partial_t f + \Delta^2 f + \delta\Delta\operatorname{div} n)\|^2_{L^2(0,T;L^2(M))} + \text{L.O.T.}\\
&+\|D^{\alpha-\gamma}(\partial_t n - \lambda\Delta n - (E_{d+1} - \theta a\otimes a)\delta\nabla(\Delta f + \delta\operatorname{div} n))\|^2_{L^2(0,T;L^2(M))}\\
&+\|\zeta^{|\alpha|+1}D^\alpha f(0)\|^2_{L^2(M)} + \|\zeta^{|\alpha|+1}D^\alpha n(0)\|^2_{L^2(M)}) + \text{L.O.T.}
\end{aligned}
$$

This can be done for all ℓ from 2 up to k. Next, we replace in

$$(-1)^{|\alpha|}D^\alpha\big(\zeta^{2|\alpha|+2}D^\alpha(f,n)\big)$$

four space derivatives of f and two of n each by one time derivative, that is we test with

$$(-1)^{|\alpha|}D^\alpha\big(\zeta^{2|\alpha|+2}D^{\alpha-\eta_1}\partial_t f, \zeta^{2|\alpha|+2}D^{\alpha-\eta_2}\partial_t n\big)$$

for $\eta_1, \eta_2 < \alpha$ and $|\eta_1| = 4$, $|\eta_2| = 2$. This provides us with analogous estimates for

$$\|\zeta^{|\alpha|+1}f\|_{H^1(0,T(H^{\ell-2}(M))} + \|\zeta^{|\alpha|+1}n\|_{H^1(0,T(H^{\ell-1}(M))}.$$

Finally, we can sum up the results from all charts to get

$$\|(f,n)\|_{X_T} \leq C\|L_\theta(f,n)\|_{Y_T\times X_\gamma}$$

as desired. Now the claim follows directly from the application of Proposition 4.22. □

That means, we have an optimal result for this linear problem, hence we are now able to treat the quasilinear problem.

Theorem 4.24
For $d \in \mathbb{N}$ let M be a d-dimensional, smooth, orientable, closed manifold. Let $k > d/2 + 3$ be a natural number and let $X_{T,\mathrm{imm}}$ be as in Definition 4.14 depending on k. Let $\varphi_0 \in H^k(M, \mathbb{R}^{d+1})$ be an immersion satisfying $A(\varphi_0) = A_0$ and $Vol(\varphi_0) = V_0$ for A_0, V_0 as in (3.2) and let $n_0 \in H^k(M, \mathbb{R}^{d+1})$ be a vector field with $\|n_0\|_{R^{d+1}} \equiv 1$.

Then, there is a $T > 0$, such that the gradient flow equation of the energy (3.1), with constants $\delta > 0, \lambda > 0$satisfying $2\sqrt{\lambda} > \delta$, preserving any combination of the constraints (3.2) has a unique solution $(\varphi, n) \in X_{T,\mathrm{imm}}$, which depends continuously on the initial data. If the unit-length constraint is neglected, also the condition on n_0, δ, and λ are obsolete.

Proof. Again, we observe that when volume and area preservation are enforced and $\varphi_0(M)$ is a round sphere, φ will be stationary due to isoperimetric restrictions. The result then follows from short-time existence for the harmonic map heat flow. Thus, in the following, when we enforce area and volume preservation, we can assume that $\varphi(M)$ is not a round sphere.

We have shown in the foregoing section and Theorem 4.23, how the modified equation is linearized and how to solve the linearized problem. Thus the proof of Theorem 4.19 is still valid. □

4.6 The Parameter Trick and Implications of Maximal Regularity

In this last section of this chapter, we will use the so called parameter trick and exploit some general results about maximal L^p-regularity to obtain smooth solutions and parabolic regularization. We introduce time weighted spaces. They are relatively common in this context and we follow the definitions given in the book of Prüss and Simonett [77, ch. 3.2], that gives an overview on state of the art techniques for parabolic problems in the Sobolev setting.

Definition 4.25
For a Banach space Y, $1 < p < \infty$, and $1/p < \mu \leq 1$, we define

$$L^p_\mu(0,T;Y) := \{u : (0,T) \to Y, t^{1-\mu}u(t) \in L^p(0,T;Y)\}$$

with the norm

$$\|u\|_{L^p_\mu(0,T;Y)} = \big(\int_0^T |t^{1-\mu}u(t)|^p_Y \,\mathrm{d}t\big)^{1/p}.$$

Accordingly, we set

$$H^{1,p}_\mu(0,T;Y) := \{u \in L^p_\mu(0,T;Y) \cap H^{1,1}(0,T;Y) | \dot{u} \in L^p_\mu(0,T;Y)\}$$

and equip it with the norm

$$\|u\|_{H^{1,p}_\mu(0,T;Y)} = \big(\|u\|^p_{L^p_\mu(0,T;Y)} + \|\dot{u}\|^p_{L^p_\mu(0,T;Y)}\big)^{1/p}.$$

With these norms, the above spaces turn into Banach spaces. We denote the Hilbert spaces$H^{k,2}_\mu$ by H^k_μ, since we only consider the case $p = 2$ in this work.

As for the unweighted spaces, the problem of the right trace space is very important. We state the following result, again from [77, Theorem 3.4.8].

Theorem 4.26
There is a continuous and surjective trace map

$$\begin{aligned} L^p_\mu(0,T;H^{k+p}) \cap H^1_\mu(0,T;H^{k-p}) &\to H^{k-p+2p(\mu-\frac{1}{2})}, \\ u &\mapsto u(0). \end{aligned}$$

Proof. To identify the objects in the original formulation in Theorem 3.4.8 of [77], it is very helpful to study their Example 3.4.9. Especially the operator A appearing in their Theorem 3.4.8 can in our case be set to $(1+\Delta)^p$ and $\alpha = 1$. □

We remark, that for positive times, the space regularity is the same as in the unweighted case since we can divide by the weighting function. Moreover, there is a correspondence between results for weighted and unweighted spaces. We state the following theorem [77, Theorem 3.5.4].

Theorem 4.27
Let X be a Banach space, $p \in (1,\infty)$, and $1/p < \mu \leq 1$. Then anoperator L has maximal L^p-regularity if and only if it has maximal L^p_μ regularity.

In our case $p = 2$, $X = H^{k-2} \times (H^{k-1})^{d+1}$ and L is the operator

$$L(f,n) = \begin{pmatrix} \Delta^2 f + \delta\Delta \operatorname{div} n, \\ -\Delta n - (E_{d+1} - a \otimes a)\delta\nabla(\Delta f + \delta \operatorname{div} n) \end{pmatrix}.$$

In the foregoing sections, we have shown, that the operator has the property of maximal regularity in L^2. We adapt the definition of our spaces to the weighted setting.

Definition 4.28
For $1/2 < \mu \leq 1$ and $k \geq 2 - 4(1-\mu)$ we define the weighted spaces $X_{T,\mu}$ and $Y_{T,\mu}$ exactly analougously to the unweighted case. The natural trace space is now denoted by

$$X_{\gamma,\mu} = H^{k+4(\mu-1)} \times (H^{k+2(\mu-1)})^{d+1}$$

in consistence with Theorem 4.26. Moreover, we introduce $X_{T,\mu,\mathrm{imm}}$ in anology to Definition 4.14.

With this, we can adapt the short-time existence result to the new setting.

Theorem 4.29
For $d \in \mathbb{N}$ let M be a d-dimensional smooth closed orientable manifold. Let $k > d/2+3$ be a natural number and choose $\mu \in (1/2,1)$ such that $k - 4(1-\mu) > d/2+3$. Let $\varphi_0 \in H^k(M,\mathbb{R}^{d+1})$ be an immersion and $n_0 \in H^k(M,\mathbb{R}^{d+1})$ a vector field with $\|n_0\|_{\mathbb{R}^{d+1}} \equiv 1$.

Then, there is a $T > 0$, such that the area and volume preserving gradient flow equation of the energy (3.1) with $\delta < 2\sqrt{\lambda}$ has a unique solution $(\varphi, n) \in X_{T,\mu,\mathrm{imm}}$ and $n : M \times [0,T) \to \mathbb{S}^d$. The result remains valid, if we impose no or only one or two of the three constraints. If the unit length constraint is abolished, the condition on the initial datum and δ, λ is obsolete. Moreover, for positive times the surface and the vector field are smooth in space and time.

Proof. Again, we observe that when volume and area preservation are enforced and $\varphi_0(M)$ is a round sphere, φ will be stationary due to isoperimetric restrictions. The result then follows from short-time existence for the harmonic map heat flow. Thus, in the following, when we enforce area and volume preservation, we can assume that $\varphi(M)$ is not a round sphere.

We consider again equation (4.2).The strategy is as in the unweighted setting. We first linearize the equation and then use the contraction mapping principle to find a solution for the non-linear problem.

Firstly, the mapping properties of the linearized operator are determined by Theorem 4.27.

Secondly, we have to prove results analogous to Lemma 1.26 and Lemma 1.27 in the weighted setting. Our choices of k and μ ensure that $\nabla^3 f$ and $\nabla^2 n$ are still C^0 and even a little bit more regular. The weighted version of Lemma 1.26 is again proven by a direct calculation making use of the trace theorem for weighted spaces as stated in this work in 4.26, the intermediate derivative theorem for weighted parabolic spaces as proven by Meyries and Schnaubelt [70, Theorem 4.2] and repeated use of Hölder's inequality as in the proof of the unweighted version.

In order to control the non-linearities, one can either determine them explicitly and estimate them one-by-one as demonstrated e.g. by Prüss and Simonett [77, Sec. 9.1 and 9.5]. Alternatively, one can prove a weighted version of Lemma 1.27 yielding a rather general result, potentially leaving space for further optimization in terms of necessary regularity of the initial datum. The proof is as in the unweighted case, since the crucial estimate is done at every point in time, making use of the trace space and Theorem 1.14.

Thus, the usual technique for quasilinear equations as described in [77, chap. 5 and 9] applies, that is the proof of Theorem 4.24 generalizes.

To show the claimed regularization property, we argue as follows. For positive times, we can simply divide by the weighting and the trace space is given by $X_{T,1}$. So at time $t_1 > 0$ we use the short-time existence theorem for k increased by $1/2$ and μ decreased by $1/4$ and obtain improved regularity for $t > t_1$. At $t_2 > t_1$ we increase k again. Inductively, we see that for any time $t > 0$ the solution is in $(H^k)^{d+2}$ for all $k \in \mathbb{N}$ and thus is smooth in space. As we can choose t_i as small as we please, the solution is regular for any positive time.

For the regularity in time, we employ Angenent's parameter trick. To explain this trick, we use the following notation. We set $u = (f, n) \in X_{T,\mu}$, $A(u)u = Q(f, n) - \partial_t(f, n)$ and $F(u) = Q(u) - Lu$ from equations (4.8) and (4.9). The result is formulated by Prüss and Simonett [77, theorem 5.2.1], we only sketch the proof. The crucial idea is to consider for $\varepsilon > 0$ the family of functions

$$u_\lambda(t) := u(\lambda t), \quad \lambda \in (1-\varepsilon, 1+\varepsilon), \quad t \in J_\varepsilon := [0, T/(1+\varepsilon)).$$

Since $\partial_t u_\lambda(t) = \lambda \partial_t u(\lambda t)$ we have

$$\partial_t u_\lambda + \lambda A(u_\lambda)u = \lambda F(u).$$

Now we consider the map

$$H : (1-\varepsilon, 1+\varepsilon) \times B_{X_{\gamma,\mu}}(u_0, r_0) \times X_{T_\varepsilon,\mu} \to Y_{T_\varepsilon,\mu} \times X_{\gamma,\mu}$$

defined by

$$H(\lambda, v, w) = (\partial_t w + \lambda A(w)w - \lambda F(w), w(0) - v).$$

If u^* is a solution of equation (4.2) for initial data u_0, then $H(1, u_0, u^*) = 0$. By the considerations in Corollary 1.15, H is C^ℓ for all $\ell \in \mathbb{N}$ and we calculate the Fréchet derivatives of H. We see, that H satisfies the prerequisites of the implicit function theorem. It turns out, that the parametrization of the level set of 0 just corresponds to $u_\lambda(t, v)$ for $u(t, v)$ being a solution of equation (4.2) for initial data v. Since this parametrization is C^ℓ, also

$$(\lambda, v) \mapsto u_\lambda(\cdot, v)$$

is smooth. But smoothness in λ implies smoothness in t. □

Weremark that there is also a way to recover parabolic regularization avoiding the use of time-weighted spaces and the parameter trick. Since the short-time existence result Theorem 4.24 holdson a whole range of spaces, we can argue as follows. When the initial datum is of class H^k the solution (f, n) will be in an intersection of spaces $L^2(0, T; H^{k+2} \times H^{k+1}) \cap H^1(0, T; H^{k-2} \times H^{k-1})$. From this we infer that for $\varepsilon > 0$ arbirary, at almost every time $t \in [0, \varepsilon)$ the function $(f, n)(t)$ has a representative in $H^{k+2} \times H^{k+1}$ but without control of the norm. to this representative we apply the short-time existence result with initial H^{k+1} regularity. We see inductively that the solution is smooth in space for positive time since ε was arbitrary. Now differentiating the equation in time or using the parameter trick yields also regularity for the time derivatives.

The advantage of the time-weighted setting is that it also provides an estimate on the stronger norm. This will be in particular important for the application of the Łojasiewicz-Simon inequality in Theorem 5.32. However, such estimate can also be derived by integral estimates. To make this work self-contained in this point, this is done in the next section.

4.7 Some Useful A Priori Estimates

To complement the short-time existence result, we derive a priori estimates that quantify the regularization of the solution that we observed in the foregoing section. To simplify notation, we use the symbol # to denote metric contraction of tensors in some indices.

Theorem 4.30
Let M be a smooth, closed manifold of dimension $d \in \mathbb{N}$ without boundary.For $T > 0$, $k \in \mathbb{N}$ with $k \geq d/2 + 3$ let $(f, n) \in C^\infty(0, T, C^\infty(M))$ be a smooth solution of the system (4.2) *and set $R = \|(f, n)\|_{X_T}$. Then for $j \in \mathbb{N}$ we have*

$$\int_M |\nabla^{k+j} f(p,t)|^2 + |\nabla^{k+j} n(p,t)|^2 \,\mathrm{d}\mu \leq \frac{C(R,j)}{t^j}.$$

Proof. First we introduce $A^i \in C^\infty(TM^*, (TM)^{d+1})$, the tensor corresponding to coefficients of the divergence of a vector field n as

$$A^i(\nabla f)\nabla_i n := \sum_{\alpha=1}^{d+1} A^{i;\alpha}(\nabla f)\nabla_i n^\alpha = g^{im}(\nabla f)\langle \nabla_i n, X_m(\nabla f)\rangle$$

and $G^i(\nabla f)$ corresponding to the gradient for a function u as

$$G^i(\nabla f)\nabla_i u = g^{im}(\nabla f)\nabla_i u X_m(\nabla f).$$

We recall from (4.2) that

$$\partial_t f = -g^{im}(\nabla f)g^{k\ell}(\nabla f)\nabla_i\nabla_m\nabla_k\nabla_\ell f - \frac{g^{im}(\nabla f)\nabla_i\nabla_m \delta A^i(\nabla f)\nabla_i n)}{\langle \nu(\nabla f), \nu^*\rangle} + b_1(\nabla^3 f, \nabla^2 n)$$

and

$$\partial_t n = \lambda g^{im}\nabla_i\nabla_m n + \delta G^k \nabla_k(g^{im}\nabla_i\nabla_m f + \delta A^i \nabla_i n) + b_2(\nabla^2 f, \nabla n)$$

For sake of shortness and readability, we suppress from here on the dependence of the tensors A^i, G^i and g^{im} and the normal ν on ∇f. Now we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla^r f|^2 + |\nabla^r n|^2 \,\mathrm{d}\mu$$

for $r \geq k$. In a first step we simply use the equation and integrate by parts. Moreover, we collect the terms where no derivatives go on the coefficient functions. Those terms come with a minus sign after intergrating by part once or twice, respectively, due to the parabolicity of the equation. We introduce tensors T_i, $i = 1, \dots, 4$ where we collect all terms emerging, when a derivative falls on a coefficient function. The estimate reads then

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla^r f|^2 + |\nabla^r n|^2 \,\mathrm{d}\mu \\
&= \int \langle \nabla^r f, \nabla^r\big(- g^{im} g^{k\ell} \nabla_i \nabla_m \nabla_k \nabla_\ell f - \frac{g^{im}\nabla_i\nabla_m \delta A^i \nabla_i n)}{\langle \nu, \nu^* \rangle} + b_1(\nabla^3 f, \nabla^2 n)\big)\rangle \,\mathrm{d}\mu \\
&\quad + \int \langle \nabla^r n, \nabla^r\big(\lambda g^{im}\nabla_i\nabla_m n + \delta G^k \nabla_k (g^{im}\nabla_i\nabla_m f + \delta A^i \nabla_i n) + b_2(\nabla^2 f, \nabla n)\big)\rangle \,\mathrm{d}\mu \\
&\leq - \int |g^{im}\nabla_i\nabla_m\nabla^r f + \delta A^i \nabla_i \nabla^r n|^2 + \lambda g^{im}\nabla_i \nabla^r n \nabla_m \nabla^r n \,\mathrm{d}\mu \\
&\quad + \int \langle \Delta^2 \nabla^{r-2} f, \nabla^{r-3}\big(\nabla T_1(\nabla f)\nabla^2 f \# \nabla^4 f + \nabla T_2(\nabla f)\nabla^2 f \# \nabla^3 n\big)\rangle \,\mathrm{d}\mu \\
&\quad + \int \langle \Delta \nabla^{r-1} n, \nabla^{r-2}\big(\nabla T_3(\nabla f)\nabla^2 f \# \nabla^2 n + \nabla T_4(\nabla f)\nabla^2 f \# \nabla^3 f\big)\rangle \,\mathrm{d}\mu \\
&\quad + \int \langle g^{im}\nabla_i\nabla_m \nabla^r f, \nabla^r\big((\delta A^i \nabla_i n)(\frac{1}{\langle \nu, \nu^* \rangle} - 1)\big)\rangle \,\mathrm{d}\mu \\
&\quad + \int \langle \Delta^2 \nabla^{r-2} f, \nabla^{r-2} b_1(\nabla^3 f, \nabla^2 n))\rangle \,\mathrm{d}\mu + \int \langle \Delta \nabla^{r-1} n, \nabla^{r-1} b_2(\nabla^2 f, \nabla n)\rangle \,\mathrm{d}\mu.
\end{aligned}
$$

We use the parabolicity of the coefficients, Hölder's and Young's inequality to find

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla^r f|^2 + |\nabla^r n|^2 \,\mathrm{d}\mu \\
&\leq - \int \frac{\lambda}{4\delta^2 + 2\lambda} |\nabla^{r+2} f|^2 + \frac{1}{4}|\nabla^{r+1} n|^2 \,\mathrm{d}\mu + c\varepsilon \int |\nabla^{r+2} f|^2 + |\nabla^{r+1} n|^2 \,\mathrm{d}\mu \\
&\quad + \frac{1}{4\varepsilon}\int |\nabla^{r-3}\big(\nabla T_1(\nabla f)\nabla^2 f \# \nabla^4 f\big)|^2 + |\nabla^{r-3}\big(\nabla T_2(\nabla f)\nabla^2 f \# \nabla^3 n\big)|^2 \\
&\qquad\qquad + |\nabla^{r-2}\big(\nabla T_3(\nabla f)\nabla^2 f \# \nabla^2 n\big)|^2 + |\nabla^{r-2}\big(\nabla T_4(\nabla f)\nabla^2 f \# \nabla^3 f\big)|^2 \,\mathrm{d}\mu \\
&\quad + \frac{1}{4\varepsilon}\int |\nabla^{r-2} b_1(\nabla^3 f, \nabla^2 n)|^2 + |\nabla^{r-1} b_2(\nabla^2 f, \nabla n)|^2 + |\nabla^r\big((\delta A^i \nabla_i n)(\frac{1}{\langle \nu, \nu^* \rangle} - 1)\big)|^2 \,\mathrm{d}\mu.
\end{aligned}
$$

At this point we employ the theory of multiplication and Nemitsky operators in Sobolev spaces as developed by Runst and Sickel [79] and stated in Section 1.6 of this work. Estimates (1.2) and (1.3) imply

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla^r f|^2 + |\nabla^r n|^2 \,\mathrm{d}\mu \\
&\leq -\frac{1}{2}\int \frac{\lambda}{4\delta^2 + 2\lambda} |\nabla^{r+2} f|^2 + \frac{1}{4}|\nabla^{r+1} n|^2 \,\mathrm{d}\mu + C(R)(\|f\|^2_{H^r} + \|n\|^2_{H^r}) \qquad (4.15) \\
&\leq -\frac{1}{2}\int \frac{\lambda}{4\delta^2 + 2\lambda} |\nabla^{r+2} f|^2 + \frac{1}{4}|\nabla^{r+1} n|^2 \,\mathrm{d}\mu + C(R)\int_M |\nabla^r f|^2 + |\nabla^r n|^2 \,\mathrm{d}\mu + C(R)R^2
\end{aligned}
$$

From this we can immediately deduce Caccioppoli inequalities by integration in time. For $t \in (0, T)$ they read

$$
\begin{aligned}
&\int_M |\nabla^r f|^2 + |\nabla^r n|^2 \,\mathrm{d}\mu(t) + \frac{1}{2}\int_0^t \int \frac{\lambda}{4\delta^2 + 2\lambda}|\nabla^{r+2} f|^2 + \frac{1}{4}|\nabla^{r+1} n|^2 \,\mathrm{d}\mu \,\mathrm{d}s \\
&\leq \int_M |\nabla^r f|^2 + |\nabla^r n|^2 \,\mathrm{d}\mu(0) + C(R)\int_0^t \int_M |\nabla^r f|^2 + |\nabla^r n|^2 \,\mathrm{d}\mu \,\mathrm{d}s + \int_0^t C(R)R^2 \,\mathrm{d}s.
\end{aligned}
$$

Applying Gronwall's Theorem, these inequalities imply in particular that a smooth solution will remain smooth as long as the H^k-norm is bounded. Until now, we have shown that there is no regularity loss in parabolic equations. Theorem 4.29 even yields an improvement of regularity, but not yet an estimate. For parabolic equations it is a useful strategy to use a sequence of cut-off functions $\chi_j : [0,T] \to [0,1]$ in time, to weaken the restriction on regularity due to the initial data. In the context of geometric evolution equations such strategy is carried out e.g. by Kuwert and Schätzle [55, Theorem 3.5]. There are different choices for the cut-off functions. The important properties are

$$\chi_0 \equiv 1, \quad \chi_j(T) = 1 \text{ and } \quad \dot{\chi}_j \leq \frac{c(j)}{T}\chi_{j-1}.$$

Kuwert and Schätzle give a piecewise defined sequences. For $m \in \mathbb{N}$ and $j \leq m$ they set

$$\chi_j(t) = \begin{cases} 0 & \text{for } t \leq (j-1)\frac{T}{m} \\ \frac{m}{T}(t-(j-1)\frac{T}{m}) & \text{in between} \\ 1 & \text{for } t \geq j\frac{T}{m}. \end{cases}$$

But also $\chi_j(t) = \frac{t^j}{T^j}$ is possible. Following their argument we set

$$e_j(t) := \chi_j(t)\int_M |\nabla^{k+j} f(t)|^2 + |\nabla^{k+j} n(t)|^2 \,\mathrm{d}\mu$$

and use (4.15) to calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} e_j &= -c\chi_j(t)\int |\nabla^{k+j+2} f|^2 + |\nabla^{k+j+1} n|^2 \,\mathrm{d}\mu \\ &\quad + \dot{\chi}_j(t)\int_M |\nabla^{k+j} f(t)|^2 + |\nabla^{k+j} n(t)|^2 \,\mathrm{d}\mu + \big(C(R)e_j(t) + 1\big). \end{aligned}$$

We integrate both sides in time to see

$$\begin{aligned} & e_j(T) + c\int_0^T \chi_j \int_M |\nabla^{k+j+2} f(t)|^2 + |\nabla^{k+j+1} n(t)|^2 \,\mathrm{d}\mu\,\mathrm{d}t \\ \leq & \int_0^T \dot{\chi}_j(t)\int_M |\nabla^{k+j} f(t)|^2 + |\nabla^{k+j} n(t)|^2 \,\mathrm{d}\mu\,\mathrm{d}t + \int_0^T C(R)e_j(t)\,\mathrm{d}t + \int_0^T C(R)\,\mathrm{d}t \\ \leq & \int_0^T C(R)\big(1 + e_j(t)\big)\,\mathrm{d}t + \frac{c}{T}\int_0^T \chi_{j-1}(t)\int_M |\nabla^{k+j} f(t)|^2 + |\nabla^{k+j} n(t)|^2 \,\mathrm{d}\mu\,\mathrm{d}t. \end{aligned}$$

We show by induction that

$$\begin{aligned} & e_j(T) + c\int_0^T \chi_j \int_M |\nabla^{k+j+1} f(t)|^2 + |\nabla^{k+j+1} n(t)|^2 \,\mathrm{d}\mu\,\mathrm{d}t \\ \leq & \frac{C(R,j)}{T^j}, \end{aligned}$$

the case $j = 0$ being (4.15). At this point, it is useful to observe the following implication of Gronwall's lemma. Let $f, g : [0,T] \to \mathbb{R}$ denote continuous functions with $g > 0$ and $0 < a,\ 0 < b \in \mathbb{R}$ positiv constants. Moreover, let f and g fulfill the inequality

$$f(t) + g(t) \leq a + \int_0^t bf(t)\,\mathrm{d}t.$$

Since $g > 0$ we conclude that

$$f(t) \leq a + \int_0^t ab\exp(b(t-s)) \leq a + aC(T)bt \leq aC(b,T).$$

Putting this in the original inequality, we see

$$f(t) + g(t) \leq a + \int bC \,\mathrm{d}t \leq aC(b, T).$$

We emphasize that the estimate depends linearly on the constant a. We apply this to

$$\begin{aligned} &e_j(T) + c\int_0^T \chi_j \int_M |\nabla^{k+j+2} f(t)|^2 + |\nabla^{k+j+1} n(t)|^2 \,\mathrm{d}\mu\,\mathrm{d}t \\ \leq &\int_0^T C(R)\big(1 + e_j(t)\big)\,\mathrm{d}t + \frac{c}{T}\int_0^T \chi_{j-1}(t)\int_M |\nabla^{k+j} f(t)|^2 + |\nabla^{k+j} n(t)|^2 \,\mathrm{d}\mu\,\mathrm{d}t \end{aligned}$$

to conclude

$$\begin{aligned} &e_j(T) + c\int_0^T \chi_j \int_M |\nabla^{k+j+2} f(t)|^2 + |\nabla^{k+j+1} n(t)|^2 \,\mathrm{d}\mu\,\mathrm{d}t \\ &\leq C(R)\big(1 + \frac{c}{T}\int_0^T \chi_{j-1}(t)\int_M |\nabla^{k+j} f(t)|^2 + |\nabla^{k+j} n(t)|^2 \,\mathrm{d}\mu\,\mathrm{d}t\big) \\ &\leq \frac{C(R, j)}{T^j} \end{aligned}$$

by the induction hypothesis. By the Gagliardo-Nirenberg interpolation inequality we conclude that

$$\begin{aligned} &\int_M |\nabla^{k+j+1} f(t)|^2 \,\mathrm{d}\mu \leq C\Big(\int_M |\nabla^{k+j+2} f(t)|^2 \,\mathrm{d}\mu + \int_M |\nabla^k f(t)|^2 \,\mathrm{d}\mu\Big) \\ \leq &\, C\Big(\int_M |\nabla^{k+j+2} f(t)|^2 \,\mathrm{d}\mu + C(R)\Big) \end{aligned}$$

and thus

$$\int_0^T \chi_j \int_M |\nabla^{k+j+1} f(t)|^2 \,\mathrm{d}\mu\,\mathrm{d}t \leq \int_0^T \chi_j C\Big(\int_M |\nabla^{k+j+2} f(t)|^2 \,\mathrm{d}\mu + C(R)\Big)\,\mathrm{d}\mu\,\mathrm{d}t \leq \frac{C(R, j)}{T^j}.$$

Since all estimates are invariant under decrease of T, we can simply rename it into t and this concludes the proof. □

Remark 4.31

Since the proof of Theorem 4.30 relies essentially on the coercivity of the main part and the remainder terms are estimated with a very general approach, we assume in the following that the assertion of Thereom 4.30 also holds in the presence of constraints, when the prerequesites for the respective short-time existence result are satisfied.

5

Long-Time Behavior for Solutions of the Gradient-Flow Equation

In this chapter, we analyze the coupled Helfrich Flow in greater detail. For the flow of curves as explained in Section 3.2, we give a global geometric quantity that has to become unbounded in the case of a singularity. Moreover, we discuss stability of local minimizers in arbitrary dimension.

5.1 A Criterion Granting Global Existence for the Flow of Curves

Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth immersion yielding a plane closed curve. We parametrize γ by arc length and consider a vector field $n : \mathbb{S}^1 \to \mathbb{R}^2$. We recall that in the curve case we examine for $\delta, \lambda \in \mathbb{R}_{>0}$ the energy

$$E(\gamma, n) = \frac{1}{2} \int_\gamma (\kappa + \delta \operatorname{div} n)^2 \, \mathrm{d}s + \frac{\lambda}{2} \int_\gamma |\nabla_s n|^2 \, \mathrm{d}s + \int_\gamma 1 \, \mathrm{d}s. \tag{5.1}$$

In the following, we want to find conditions under which the L^2-gradient flow of this energy exists globally. To this end, we will use integral estimates and Gagliardo-Nirenberg-type interpolation inequalities. We recall from (3.4) that the energy (5.1) suffices to control the length of the curve from below and above during the flow. So constants may depend on positive or negative powers of L still yielding uniform in time estimates.

In the following we assume that $(\gamma, n) \in C^\infty(\mathbb{S}^1 \times [0, T), \mathbb{R}^2 \times \mathbb{R}^2)$ is a smooth solution of equation (3.8).Using the dependent variable $z := \kappa + \delta \operatorname{div} n$, we recall from Remark 3.7 that the normal velocity V of the flow is given for curves by (3.9) reading

$$\begin{aligned}
V = & -\partial_s^2(\kappa + \delta \operatorname{div} n) - \kappa^2(\kappa + \delta \operatorname{div}(n)) \\
& +\delta\kappa \nabla n : \nabla \nu + \delta \operatorname{div}(\kappa \nu^T \nabla n) + \delta^2 \operatorname{div}(n)(\nabla n : \nabla \nu) + \delta^2 \operatorname{div}(\operatorname{div}(n)\nu^T \nabla n) \\
& + \frac{1}{2}\kappa(\kappa + \delta \operatorname{div}(n))^2 + \lambda \nabla n^T : (\nabla \nu (\nabla n)^T) + \frac{\lambda}{2}\kappa |\nabla n|^2 + \kappa \\
= & -\partial_s^2 z - \kappa^2 z + \delta \nabla n : \nabla \nu z + \delta \operatorname{div}(z \nu^T \nabla n) + \frac{1}{2}\kappa z^2 \\
& + \lambda \nabla n^T : (\nabla \nu (\nabla n)^T) + \frac{\lambda}{2}\kappa |\nabla n|^2 + \kappa.
\end{aligned}$$

We also have an equation for n which reads

$$n_t = \lambda \partial_s^2 n + \delta \nabla z + \delta z \kappa \nu, \tag{5.2}$$

since $\gamma_t = V\nu$.As a first step we analyze the scaling properties of these equations.

Lemma 5.1
Let $(\gamma(x,t), n(x,t)) : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2 \times \mathbb{R}^2$ be a smooth solution of the evolution given by (3.8)*, then for $0 < \alpha$ the rescaled functions $(\alpha^{-1}\gamma(x, \alpha^4 t), n(x, \alpha^4 t))$ are a smooth solution on $\mathbb{S}^1 \times [0, T\alpha^{-4})$ for the system of equations*

$$\begin{aligned}
\langle \gamma_t, \nu \rangle = & -\partial_s^2 z - \kappa^2 z + \delta \nabla n : \nabla \nu z + \delta \operatorname{div}(z \nu^T \nabla n) + \frac{1}{2}\kappa z^2 \\
& + \lambda \nabla n^T : \nabla \nu (\nabla n) + \frac{\lambda}{2}\kappa |\nabla n|^2 + \alpha^{-2}\kappa \\
n_t = & \, \alpha^2 \Big(\lambda \partial_s^2 n + \delta \nabla z + \delta z \kappa \nu \Big)
\end{aligned} \tag{5.3}$$

Proof. We note that for a function $f : \mathbb{S}^1 \times [0,T) \to \mathbb{R}$ we have

$$\partial_{s,\alpha} f = \frac{1}{|\partial_x \gamma_\alpha|} \partial_x f = \alpha \partial_s f.$$

and recall $\operatorname{div}(\cdot) = \langle \partial_s \cdot, \tau \rangle$ and $\nabla(\cdot) = \partial_s(\cdot) \otimes \tau$.The assertion follows from chain rule and the considerations in Section 3.1. □

For the motion of curves, we obtain evolution equations for κ and $\partial_s n$ with help of the formulas

$$\partial_t \kappa = \partial_s^2 V + V\kappa^2, \tag{5.4}$$

$$\partial_t \partial_s f = \partial_s \partial_t f + \kappa V \partial_s f, \tag{5.5}$$

where $f : \mathbb{S}^1 \times [0,T) \to \mathbb{R}$is meant to be an arbitrary smooth function. They follow from differentiating $\sqrt{\langle \partial_x \gamma, \partial_x \gamma \rangle}$ and are well established in the literature (e.g. [28, Lemma 2.1]). In the remainder of this section, we find parabolic evolution equations for κ and n and their derivatives. After a suitable testing procedure, we will gather the highest order terms on one side leaving the lower order but non-linear terms on the other side. We will estimate those terms by Gagliardo-Nirenberg-type inequalities. Absorbing the highest order terms, we inductively get bounds on all the derivatives of κ and n.

Definition 5.2
For $\sigma, \mu \in \mathbb{N}$, $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ a smooth regular curve and a function $f : \mathbb{S}^1 \to \mathbb{R}$, we denote by

$P^\mu_\sigma(f)$ any linear combination of products of derivatives of f, such that each product has σ factors and μ derivatives distributed over the single factors. For tensors $\phi_1, \dots, \phi_k$, $k \in \mathbb{N}$ we mean by

$$P^\mu_\sigma(\phi_1, \dots \phi_k)$$

any linear combination of products

$$\prod_i P^{\mu_i}_{\sigma_i}(\phi_i) \text{ with } \sum_i \mu_i = \mu, \sum_i \sigma_i = \sigma.$$

Moreover,we will use the notation

$$P^\mu(\phi_1 \dots \phi_k)$$

to mean any linear combination of terms

$$\partial_s^{\mu_1}\phi_1 * \dots * \partial_s^{\mu_k}\phi_k \text{ with } \sum_i \mu_i = \mu$$

denoting by $*$ any metric contraction. However, such expression may still be vector valued. Lastly, we use scaling invariant norms for functions on curves. We set

$$\|f\|_{k,p} = \sum_{i=0}^{k} L(\gamma)^{i-\frac{1}{p}} \left(\int_\gamma |\partial_s^i f|^p \, \mathrm{d}s \right)^{1/p}$$

and

$$\|f\|_p = L(\gamma)^{-\frac{1}{p}} \left(\int_\gamma |f|^p \, \mathrm{d}s \right)^{1/p}.$$

For these norms, too, the following Gagliardo-Nirenberg-type interpolation inequalities hold as stated in [28, Lemma 2.4] and [21, Appendix C].

Theorem 5.3 (Interpolation inequalities)
Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth regular curve and $f : \mathbb{S}^1 \to \mathbb{R}$ a smooth function. Then for any $k \in \mathbb{N}$, $p \geq 2$ and $0 \leq j < k$ we have

$$\|f\|_{j,2} \leq C\|f\|_p^{1-\beta}\|f\|_{k,2}^{\beta} \tag{5.6}$$

for $0 < j < k$, $\beta = (j + \frac{1}{2} - \frac{1}{p})/k$ and $C = C(k)$.

Proof. See [21, Appendix C]. □

For the integral estimates it will be of importance that we can deduce an estimate for κ from estimates for z and n.

Lemma 5.4
Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth curve with curvature κ and $n : \mathbb{S}^1 \to \mathbb{R}^2$ a smooth vector field. We have for suitable $\gamma_1 > 1$ that

$$\begin{aligned} \|\operatorname{div}(n)\|_{k,2} &\leq C(\|\partial_s n\|_{k,2} + \|\partial_s n\|_\infty \|\tau\|_{k,2}), \\ \|\tau\|_{k,2} &\leq C\big(\|\kappa\|_{k-1,2} + \|\kappa\|_\infty + \|\kappa\|_\infty^{\gamma_1} + 1\big) \end{aligned}$$

and for $\delta > 0$ we find $\gamma_2 > 1$ such that

$$\|\kappa\|_{k,2} \leq C\big(\|z\|_{k,2} + \|\partial_s n\|_{k,2} + \|z\|_\infty + \|\partial_s n\|_\infty + \|z\|_\infty^{\gamma_2} + \|\partial_s n\|_\infty^{\gamma_2}\big)$$

for a constant C depending only on k and $L(\gamma)$.

Proof. Since $\operatorname{div}(n) = \langle \partial_s n, \tau\rangle$, it follows by equation (1.2) that

$$\|\operatorname{div}(n)\|_{k,2} = \|\langle \partial_s n, \tau\rangle\|_{k,2} \le C(\|\tau\|_\infty \|\partial_s n\|_{k,2} + \|\partial_s n\|_\infty \|\tau\|_{k,2})$$

and $\|\tau\|_\infty = 1$. Moreover, we make repeated use of estimate (1.2) and the interpolation inequality (5.6)to see that

$$\begin{aligned}\|\tau\|_{k,2} &= \|\kappa\nu\|_{k-1,2} + 1 \le C(\|\kappa\|_\infty \|\nu\|_{k-1,2} + \|\kappa\|_{k-1,2}) + 1\\ &\le C(\|\kappa\|_\infty(\|\kappa\tau\|_{k-2,2} + 1) + \|\kappa\|_{k-1,2}) + 1\\ &\le C(\|\kappa\|_\infty^2 \|\tau\|_{k-2,2} + \|\kappa\|_\infty + \|\kappa\|_\infty \|\kappa\|_{k-2,2} + \|\kappa\|_{k-1,2}) + 1\\ &\le C(\|\kappa\|_\infty^2 \|\tau\|_{k,2}^{\beta_1} + \|\kappa\|_\infty + \|\kappa\|_\infty^{2-\beta_2} \|\kappa\|_{k-1,2}^{\beta_2} + \|\kappa\|_{k-1,2}) + 1\\ &\le \varepsilon \|\tau\|_{k,2} + C\|\kappa\|_{k-1,2} + C\frac{1}{\varepsilon}(\|\kappa\|_\infty + \|\kappa\|_\infty^{\gamma_1} + 1)\end{aligned}$$

for suitable exponents $\gamma_1 > 0$ and $\beta_1, \beta_2 \in (0, 1)$. We use this and interpolation to calculate

$$\begin{aligned}\|\kappa\|_{k,2} &= \|\kappa + \delta \operatorname{div}(n) - \delta \operatorname{div}(n)\|_{k,2} \le \|z\|_{k,2} + \delta \|\operatorname{div}(n)\|_{k,2}\\ &\le \|z\|_{k,2} + C\delta \|\partial_s n\|_\infty \|\tau\|_{k,2} + C\|\partial_s n\|_{k,2}\\ &\le \|z\|_{k,2} + C\delta\|\partial_s n\|_\infty C\big(\|\kappa\|_{k-1,2} + \|\kappa\|_\infty + \|\kappa\|_\infty^{\gamma_1} + 1\big) + C\|\partial_s n\|_{k,2}\\ &\le \|z\|_{k,2} + C\delta\|\partial_s n\|_\infty C\big(\varepsilon\|\kappa\|_{k,2} + \frac{C}{\varepsilon}(\|\kappa\|_\infty + \|\kappa\|_\infty^{\gamma_1} + 1)\big) + C\|\partial_s n\|_{k,2}.\end{aligned}$$

Since the assertion is trivial when n is constant, setting $\varepsilon = \frac{1}{4\min\{1, \|\partial_s n\|_\infty\}}$ we conclude

$$\|\kappa\|_{k,2} \le C(\|z\|_{k,2} + \|\partial_s n\|_{k,2} + \|z\|_\infty + \|\partial_s n\|_\infty + \|z\|_\infty^{\gamma_2} + \|\partial_s n\|_\infty^{\gamma_2})$$

for a suitable exponent γ_2. □

Remark 5.5
We recall that for $n \in \mathbb{N}$ and positive real numbers $a_1, \dots, a_n$ iteration of Young's inequality yields for $p_1, \dots, p_n \in (1, \infty)$ with $\sum \frac{1}{p_i} = 1$ that

$$\prod a_i \le \sum \frac{a_i^{p_i}}{p_i}.$$

The following interpolation inequalities will be important tools to derive integral estimates.

Proposition 5.6 (Proposition 2.5 from [28])
Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth regular curve and $f : \mathbb{S}^1 \to \mathbb{R}$ a smooth map. Then for $k \in \mathbb{N}$ and $r \in \mathbb{R}$, $r \ge 1$ and any term $P_\sigma^\mu(f)$ with $\sigma \ge 1$ which contains only derivatives of f of order at most $k-1$, we have

$$\int_\gamma |P_\sigma^\mu(f)|^r \,\mathrm{d}s \le CL^{1-r\mu-r\sigma} \|f\|_2^{r\sigma-\beta} \|f\|_{k,2}^\beta,$$

where $\beta = (r\mu + r\sigma/2 - 1)/k$ and $C > 0$ is a positive constant. Moreover, if $r\mu + r\sigma/2 < 2k + 1$, then $\beta < 2$ and we have for all $\varepsilon > 0$ and a constant $c > 0$ the estimate

$$\int_\gamma |P_\sigma^\mu(f)|^r \,\mathrm{d}s \le \varepsilon \int_\gamma |\partial_s^k f|^2 \,\mathrm{d}s + c\varepsilon^{-\frac{\beta}{2-\beta}} \left(\int_\gamma f^2 \,\mathrm{d}s\right)^{\frac{r\sigma-\beta}{2-\beta}} + c\left(\int_\gamma f^2 \,\mathrm{d}s\right)^{r\mu+r\sigma-1}.$$

Proof. The proof of Dziuk, Kuwert and Schätzle uses Hölder's and Young's inequality and the Gagliardo-Nirenberg inequality (5.6) on $\mathbb{S}^1$.The original assertion of Proposition 2.5 from [28] covers only the case $r = 1$. But for $r > 1$ the proof generalizes by using the Hölder's inequality with exponents $r\sigma$ instead of σ and taking the r-th power of the whole estimate. □

From this result, we derive an estimate for products of up to three different functions.

Proposition 5.7 (Interpolation for mixed products)
Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth curve and n, κ, z be smooth maps as specified above. Let $k, \sigma_n, \sigma_z, \sigma_\kappa, \mu_n, \mu_z, \mu_\kappa \in \mathbb{N}$ be natural numbers, such that the expression condensed in the expression $P^{\mu_n}_{\sigma_n}(\partial_s^2 n)P^{\mu_\kappa}_{\sigma_\kappa}(\partial_s\kappa)P^{\mu_z}_{\sigma_z}(\partial_s z)$ does only contain derivatives of n and z of order at most $k-1$ and derivatives of κ of order at most $k-2$. We set $k_z = k$ and $k_n = k_\kappa = k-1$. Then, we find for p_n, p_κ and p_z with $\frac{1}{p_n} + \frac{1}{p_\kappa} + \frac{1}{p_z} = 1$ and $\vartheta_\kappa = \frac{\mu_\kappa + \sigma_\kappa/2 - 1}{k_\kappa}$, $\vartheta_n = \frac{\mu_n + \sigma_n/2}{k_n}$, $\vartheta_z = \frac{\mu_z + \sigma_z/2}{k_z}$ and a constant $C(L) > 0$ an estimate

$$\int_\gamma |P^{\mu_n}_{\sigma_n}(\partial_s^2 n)||P^{\mu_z}_{\sigma_z}(\partial_s z)||P^{\mu_\kappa}_{\sigma_\kappa}(\partial_s\kappa)|\,\mathrm{d}s$$
$$\leq C(L)\|\partial_s n\|_{1,2}^{\sigma_n - \vartheta_n/p_n}\|\partial_s n\|_{k_n,2}^{\vartheta_n/p_n}\|z\|_{1,2}^{\sigma_z-\vartheta_z/p_z}\|z\|_{k_z,2}^{\vartheta_z/p_z}\|\kappa\|_{1,2}^{\sigma_\kappa-\vartheta_\kappa/p_\kappa}\|\kappa\|_{k_\kappa,2}^{\vartheta_\kappa/p_\kappa}.$$

If in addition $\sum_i \frac{\vartheta_i}{p_i} < 2$, there exists $\beta > 0$ such that for any $\varepsilon > 0$ we have an estimate

$$\int_\gamma |P^{\mu_n}_{\sigma_n}(\partial_s^2 n)||P^{\mu_z}_{\sigma_z}(\partial_s z)||P^{\mu_\kappa}_{\sigma_\kappa}(\partial_s\kappa)|\,\mathrm{d}s$$
$$\leq \varepsilon\int_\gamma |\partial_s^k n|^2 + |\partial_s^k z|^2\,\mathrm{d}s + C(L)\left(\|\partial_s n\|_{1,2} + \|z\|_{1,2}\right) + C(\varepsilon, L)\left(\|\partial_s n\|_{1,2}^\beta + \|z\|_{1,2}^\beta\right).$$

Proof. We employ Hölder's inequality with p_z, p_n and p_κ with $\frac{1}{p_z} + \frac{1}{p_n} + \frac{1}{p_\kappa} = 1$ and the interpolation estimate Proposition 5.6 to find

$$\int_\gamma |P^{\mu_n}_{\sigma_n}(\partial_s^2 n)||P^{\mu_z}_{\sigma_z}(\partial_s z)||P^{\mu_\kappa}_{\sigma_\kappa}(\partial_s\kappa)|\,\mathrm{d}s$$
$$\leq \left(\int_\gamma |P^{\mu_n}_{\sigma_n}(\partial_s^2 n)|^{p_n}\,\mathrm{d}s\right)^{\frac{1}{p_n}}\left(\int_\gamma |P^{\mu_z}_{\sigma_z}(\partial_s z)|^{p_z}\,\mathrm{d}s\right)^{\frac{1}{p_n}}\left(\int_\gamma |P^{\mu_\kappa}_{\sigma_\kappa}(\partial_s\kappa)|^{p_\kappa}\,\mathrm{d}s\right)^{\frac{1}{p_\kappa}}$$
$$\leq C(L)\|\partial_s n\|_{1,2}^{\sigma_n - \vartheta_n/p_n}\|\partial_s n\|_{k_n,2}^{\vartheta_n/p_n}\|z\|_{1,2}^{\sigma_z-\vartheta_z/p_z}\|z\|_{k_z,2}^{\vartheta_z/p_z}\|\kappa\|_{1,2}^{\sigma_\kappa-\vartheta_\kappa/p_\kappa}\|\kappa\|_{k_\kappa,2}^{\vartheta_\kappa/p_\kappa}$$

with $\vartheta_i = \frac{p_i\mu_i + \frac{1}{2}p_i\sigma_i - 1}{k_i - 1}$.

Concerning the second part of the assertion, when $\vartheta := \sum_i \frac{\vartheta_i}{p_i} < 2$, we can use Young's inequality to obtain the integral estimate. We also use that for a function $f : \mathbb{S}^1 \to \mathbb{R}$ we have $\|f\|_{k,2} \leq c(k)(\|f\|_2 + L^k\|\partial_s^k f\|_2)$. We calculate

$$C(L)\|\partial_s n\|_{1,2}^{\sigma_n - \vartheta_n/p_n}\|\partial_s n\|_{k_n,2}^{\vartheta_n/p_n}\|z\|_{1,2}^{\sigma_z-\vartheta_z/p_z}\|z\|_{k_z,2}^{\vartheta_z/p_z}\|\kappa\|_{1,2}^{\sigma_\kappa-\vartheta_\kappa/p_\kappa}\|\kappa\|_{k_\kappa,2}^{\vartheta_\kappa/p_\kappa}$$
$$\leq C(L)\Big(\|\partial_s n\|_{1,2}^{\frac{\sigma_n p_n\vartheta}{\vartheta_n}-\vartheta}\|\partial_s n\|_{k_n,2}^{\vartheta} + \|z\|_{1,2}^{\frac{\sigma_z p_z\vartheta}{\vartheta_z}-\vartheta}\|z\|_{k_z,2}^{\vartheta} + \|\kappa\|_{1,2}^{\frac{\sigma_\kappa p_\kappa\vartheta}{\vartheta_\kappa}-\vartheta}\|\kappa\|_{k_\kappa,2}^{\vartheta}\Big)$$
$$\leq C(L)\Big(\|\partial_s n\|_{1,2}^{\frac{\sigma_n p_n\vartheta}{\vartheta_n}} + C(\varepsilon, L)\|\partial_s n\|_{1,2}^{\left(\frac{\sigma_n p_n\vartheta}{\vartheta_n}-\vartheta\right)\frac{2}{2-\vartheta}} + \varepsilon C(L)^{-1}\|\partial_s^{k_n+1}n\|_2^2$$
$$+\|z\|_{1,2}^{\frac{\sigma_z p_z\vartheta}{\vartheta_z}} + C(\varepsilon, L)\|z\|_{1,2}^{\left(\frac{\sigma_z p_z\vartheta}{\vartheta_z}-\vartheta\right)\frac{2}{2-\vartheta}} + \varepsilon C(L)^{-1}\|\partial_s^{k_z}z\|_2^2$$
$$+\|\kappa\|_{1,2}^{\frac{\sigma_\kappa p_\kappa\vartheta}{\vartheta_\kappa}} + C(\varepsilon, L)\|\kappa\|_{1,2}^{\left(\frac{\sigma_\kappa p_\kappa\vartheta}{\vartheta_\kappa}-\vartheta\right)\frac{2}{2-\vartheta}} + \varepsilon C(L)^{-1}\|\partial_s^{k_\kappa}\kappa\|_2^2\Big)$$
$$\leq \varepsilon\left(\int_\gamma |\partial_s^{k_n+1}n|^2\,\mathrm{d}s + \int_\gamma |\partial_s^{k_z}z|^2\,\mathrm{d}s + \int_\gamma |\partial_s^{k_\kappa}\kappa|^2\,\mathrm{d}s\right)$$
$$+C(L)\left(\|\partial_s n\|_{1,2} + \|z\|_{1,2} + \|\kappa\|_{1,2}\right) + C(\varepsilon, L)\left(\|\partial_s n\|_{1,2}^\beta + \|z\|_{1,2}^\beta + \|\kappa\|_{1,2}^\beta\right).$$

Using Lemma 5.4 we can eliminate κ from the right hand side and the assertion follows. □

Remark 5.8
We keep the notation from Proposition 5.7. The observations in this remark give useful conditions on μ_i and σ_i that imply $\sum \vartheta_i/p_i < 2$.

If we choose $\frac{1}{p_n} = \frac{1}{p_\kappa} = 1/2$ and $p_z = \infty$ and μ_i, σ_i fulfill

$$\frac{\mu_n + \frac{1}{2}\sigma_n + \mu_\kappa + \frac{1}{2}\sigma_\kappa - 1}{k_n - 1} + \frac{\mu_z + \frac{1}{2}\sigma_z}{k_z - 1} < 2$$

then by slightly increasing p_n and p_κ we obtain a finite value for p_z and the inequality is still valid.

Moreover, if $\sigma_z \geq 1$ then

$$\sum \vartheta_i/p_i < \frac{\mu_n + \frac{1}{2}\sigma_n + \mu_\kappa + \frac{1}{2}\sigma_\kappa + \mu_z + \frac{1}{2}\sigma_z - 1}{k_n - 1}$$

as $k_n = k_z - 1$.

Lastly, for a smooth function $f : \mathbb{S}^1 \to \mathbb{R}$, if we want to estimate $\int_\gamma P^\mu_\sigma(f)\,\mathrm{d}s$ interpolating between $\|f\|_{1,2}$ and $\|f\|_{k,2}$ the following observation is useful. Depending on the number of terms on that the μ derivatives are distributed, there is a number $r \in \mathbb{N}$, $2 \leq r \leq \sigma$ such that $P^\mu_\sigma(f) = f^{\sigma - r} P^{\mu - r}_r(\partial_s f)$ and thus

$$\int_\gamma P^\mu_\sigma(f)\,\mathrm{d}s \leq \|f\|^{\sigma-r}_\infty \int_\gamma |P^{\mu-r}_r(\partial_s f)|\,\mathrm{d}s \leq \|f\|^r_\infty \|f\|^{1-\beta}_{1,2} \|f\|^\beta_{k,2} \leq \|f\|^{1-\beta+r}_{1,2} \|f\|^\beta_{k,2}$$

with $\beta = \frac{\mu - r + r/2 - 1}{k-1} \leq \frac{\mu-2}{k-1}$.

We calculate the evolution equations of the geometric quantities under the scaled evolution (5.3).

Lemma 5.9
For $T > 0$, a family of curves $\gamma : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ that evolves according to the law $\partial_t \gamma = V\nu$ for smooth $V : \mathbb{S}^1 \times [0,T) \to \mathbb{R}$, and for a time dependent vector field $n : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ evolving according to the second equation in the scaled system (5.3) *we have for $m \in \mathbb{N}_0$ the evolution equations*

$$\partial_t \partial^m_s \kappa = \partial^{m+2}_s V + P^m(\kappa^2 V)$$

and

$$\partial_t \partial^m_s n = \alpha^2 \Big(\lambda \partial^{m+2}_s n + \delta \partial^m_s (\tau \partial_s z + z\kappa\nu)\Big) + P^{m-1}(\kappa \partial_s n V).$$

Moreover,

$$\begin{aligned} \partial_t \operatorname{div}(n) &= \partial_t(\partial_s n \cdot \tau) = \operatorname{div}(\partial_t n) + \kappa V \partial_s n \cdot \tau + \partial_s V \partial_s n \cdot \nu, \\ \partial_t \partial^m_s \operatorname{div}(n) &= \partial^m_s(\operatorname{div}(\partial_t n) + \partial_s V \partial_s n \cdot \nu) + P^m(\kappa V \partial_s n \cdot \tau) \end{aligned}$$

and for $z = \kappa + \delta \operatorname{div}(n)$ we have

$$\partial_t \partial^m_s z = \partial^{m+2}_s V + P^m(\kappa^2 V) + \delta\big(\partial^m_s(\operatorname{div}(\partial_t n) + \partial_s V \partial_s n \cdot \nu) + P^m(\kappa V \partial_s n \cdot \tau)\big).$$

For the normal velocity given by (5.3) *a representation in the generic P notation is given by*

$$\begin{aligned} V &= -\partial_s^2 z - \kappa^2 z + \delta \nabla n : \nabla \nu z + \delta \operatorname{div}(z \nu^T \nabla n) + \frac{1}{2}\kappa z^2 \\ &+\lambda \nabla n^T : \nabla \nu (\nabla n) + \frac{\lambda}{2}\kappa |\nabla n|^2 + \alpha^{-2}\kappa \\ &= -\partial_s^2 z + P_3^0(\kappa, z, \partial_s n) + P_3^1(\nu, z, \partial_s n) + \alpha^{-2}\kappa. \end{aligned} \tag{5.7}$$

Proof. By formulas (5.4) and (5.5) the assertions follow by induction. □

For the flow the occurrence of a singularity, means that the norm of some derivative of the curve γ or of the vector field n goes to infinity. The parabolicity of the equation allows usto deduce Caccioppoli inequalities for the higher derivatives of z and n. The next theorem introduces a quantity that necessarily must blow up, when a singularity develops.

Theorem 5.10
Let $\gamma, n : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2 \times \mathbb{R}^2$ *be a smooth solution of the scaled flow equation* (5.3) *with* $\alpha > 0$*. Then there exists* $\varepsilon_1 > 0$ *and* $C(L,\alpha) > 0$ *with the following significance. If for* $t \in [0,T)$ *and* $z = \kappa + \delta \operatorname{div}(n)$ *we have*

$$\begin{aligned} F(\gamma(t), n(t)) :&= L^{-1}[\gamma(t)] \int_{\gamma(t)} |z|^2 \,\mathrm{d}s + L[\gamma(t)] \int_{\gamma(t)} |\partial_s z|^2 \,\mathrm{d}s \\ &+\alpha^{-2} L^{-3}[\gamma(t)] \int_{\gamma(t)} |\partial_s n|^2 \,\mathrm{d}s + \alpha^{-2} L^{-1}[\gamma(t)] \int_{\gamma(t)} |\partial_s^2 n|^2 \,\mathrm{d}s < \varepsilon_1, \end{aligned}$$

then $T - t > C(L,\alpha)$ *and for all* $m \in \mathbb{N}$, $m \geq 1$ *there exists* $r > 0$ *such that the estimate*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma(t)} |\nabla^m z|^2 + \alpha^{-2} |\nabla^{m+1} n|^2 \,\mathrm{d}s + \int_{\gamma(t)} |\nabla^{m+2} z|^2 + \lambda |\nabla^{m+2} n|^2 \,\mathrm{d}s \\ &\leq C(L) \left(\|\partial_s n(t)\|_{1,2} + \|z(t)\|_{1,2} \right) + C(L) \left(\|\partial_s n(t)\|_{1,2}^r + \|z(t)\|_{1,2}^r \right) \end{aligned}$$

holds.

Proof. **Step 1:** We show the claimed a-priori estimates under the assumption that F is small. We set $V_\ell = V + \partial_s^2 z$ and using the generic P-notation from Definition 5.2 and the formulas from Lemma 5.9 we integrate by parts in the highest order terms to find

$$\begin{aligned} &\alpha^{-2} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int |\partial_s^{m+1} n|^2 \,\mathrm{d}s + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int (\partial_s^m z)^2 \,\mathrm{d}s + \lambda \int |\partial_s^{m+2} n|^2 \,\mathrm{d}s + \int (\partial_s^{m+2} z)^2 \\ &= \int \langle \partial_s^{m+1} n, \Big((\delta \partial_s^{m+1} (\tau \partial_s z + z \kappa \nu) + \alpha^{-2} P^m(\kappa \partial_s n V) \Big) \rangle - \frac{\alpha^{-2}}{2} |\partial_s^{m+1} n|^2 \kappa V \,\mathrm{d}s \\ &+ \int \partial_s^m z (\partial_s^{m+2} V_\ell + P^m(\kappa^2 V)) \,\mathrm{d}s - \frac{1}{2} \int (\partial_s^m z)^2 \kappa V \,\mathrm{d}s \\ &+\delta \int \partial_s^m z (\partial_s^m (\alpha^2 \operatorname{div}(\partial_s^2 n + \nabla z) + \partial_s V \partial_s n \cdot \nu) + P^m(\kappa V \partial_s n \cdot \tau)) \,\mathrm{d}s. \end{aligned} \tag{5.8}$$

Using the representation of V in the generic P-notation from (5.7) we find integrating by parts if necessary

$$\begin{aligned} &\int \langle \partial_s^{m+1} n, (\alpha^{-2} P^m(\kappa \partial_s n V)) \rangle \,\mathrm{d}s \\ &= \alpha^{-2} \int P_4^{2m+2}(\kappa, \partial_s n, z) + P_6^{2m}(\kappa, \partial_s n, z) \\ &\quad +P_6^{2m+1}(\nu, \kappa, \partial_s n, z) + \alpha^{-2} P_4^{2m}(\partial_s n, \kappa) \,\mathrm{d}s, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned}&\int \langle \partial_s^{m+1} n, (\delta \partial_s^{m+1}(\tau \partial_s z + z\kappa\nu)\rangle \,\mathrm{d}s \\ &= \int P_3^{2m+2}(n,\tau,z) + P_4^{2m+2}(n,\nu,\kappa,z)\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\alpha^{-2}\int |\partial_s^{m+1} n|^2 \kappa V \,\mathrm{d}s \\ &= \alpha^{-2}\int P_4^{2m+2}(\kappa,\partial_s n,z) + P_6^{2m}(\kappa,\partial_s n,z) \\ &\qquad + P_6^{2m+1}(\nu,\kappa,\partial_s n,z) + \alpha^{-2} P_4^{2m}(\kappa,\partial_s n)\,\mathrm{d}s\end{aligned} \tag{5.10}$$

for the terms arising from the time derivative of n. Furthermore, we obtain

$$\begin{aligned}&\int \partial_s^m z(\partial_s^{m+2} V_\ell + P^m(\kappa^2 V))\,\mathrm{d}s \\ &\leq \int P_4^{2m+2}(\kappa,\partial_s n,z) + P_4^{2m+3}(\nu,\kappa,\partial_s n,z) + \alpha^{-2}|\partial_s^{m+1} z||\partial_s^{m+1}\kappa| \\ &\quad + P_4^{2m+2}(\kappa,z) + P_6^{2m+1}(\nu,\kappa,\partial_s n,z) + P_6^{2m}(\kappa,\partial_s n,z) + \alpha^{-2}P_4^{2m}(\kappa,z)\,\mathrm{d}s,\end{aligned} \tag{5.11}$$

from the time derivative of κ as a part of z,

$$\begin{aligned}&\int (\partial_s^m z)^2 \kappa V \,\mathrm{d}s \\ &= \int P_6^{2m}(\kappa,\partial_s n,z) + P_4^{2m+1}(\nu,\kappa,\partial_s n,z) + P_4^{2m+2}(\kappa,z) + \alpha^{-2}P_4^{2m}(\kappa,z)\,\mathrm{d}s,\end{aligned} \tag{5.12}$$

from the time derivative of $\mathrm{d}s$,

$$\begin{aligned}&\int \partial_s^m z(\partial_s^m(\alpha^2 \operatorname{div}(\partial_s^2 n + \nabla z) + \partial_s V \partial_s n\cdot\nu)\,\mathrm{d}s \\ &= \int \alpha^2 P_3^{2m+2}(\nu,\partial_s n,z) + \alpha^2 P_4^{2m+2}(\nu,z) + P_4^{2m+3}(\nu,\partial_s n,z) \\ &\quad + P_6^{2m+1}(\nu,\kappa,\partial_s n,z) + P_6^{2m+2}(\nu,\kappa,\partial_s n,z) + \alpha^{-2}P_4^{2m+1}(\nu,\kappa,\partial_s n,z)\,\mathrm{d}s,\end{aligned} \tag{5.13}$$

from the time derivative of $\operatorname{div} n$ as a part of z and lastly

$$\begin{aligned}&\int \partial_s^m z P^m(\kappa V \partial_s n\cdot\tau)\,\mathrm{d}s \\ &= \int P_5^{2m+2}(\nu,\kappa,\partial_s n,z) + P_7^{2m}(\nu,\kappa,\partial_s n,z) \\ &\quad + P_6^{2m+1}(\nu,\kappa,\partial_s n,z) + \alpha^{-2}P_5^{2m}(\nu,\kappa,\partial_s n,z)\,\mathrm{d}s.\end{aligned} \tag{5.14}$$

We apply the conditions from Proposition 5.7 and Remark 5.8. First we exploit that we are looking for estimates between highest and first derivatives of $\kappa, \partial_s n$ and z and then we observe in what terms z appears in a non-trivial way. Thus, we find interpolation estimates for all terms of the form P_ν^μ with $\mu \leq 2m+2$ and ν arbitrary, as long as we can arrange integrating by parts, that not more than $m+1$ derivative falls on z or n. But there are a few terms where this is inevitable and two, where $\mu = 2m+3$. We analyse these terms one-by-one in the following. In

(5.9), when all m derivatives fall on z, there is a term to that we apply Hölder's and Young's inequality, to obtain

$$\begin{aligned}&\int \langle \partial_s^{m+1} n, \kappa \partial_s n \partial_s^{m+2} z\rangle \,\mathrm{d}s\\ &\le \varepsilon \int |\partial_s^{m+2} z|^2 \,\mathrm{d}s + \frac{1}{2\varepsilon}\int |\partial_s^{m+1} n \kappa \partial_s n|^2 \,\mathrm{d}s\\ &= \varepsilon \int |\partial_s^{m+2} z|^2 \,\mathrm{d}s + \frac{1}{2\varepsilon}\int P_6^{2m}(\kappa, \partial_s n)\,\mathrm{d}s.\end{aligned}$$

The second term can now be estimated by Proposition 5.7 again and Young's inequality with ε^2. Also in (5.9) there is the term from the coupling, where $m+2$ derivatives fall on n. We use Hölder and Young again to find

$$\begin{aligned}&\int \partial_s^{m+1} n * \kappa * \partial_s n * \nu * z * \partial_s^{m+2} n \,\mathrm{d}s\\ &\le \varepsilon \int |\partial_s^{m+2} n|^2 \,\mathrm{d}s + \frac{1}{2\varepsilon}\int |\partial_s^{m+1} n \kappa z \partial_s n|^2 \,\mathrm{d}s\\ &= \varepsilon \int |\partial_s^{m+2} n|^2 \,\mathrm{d}s + \frac{\|\kappa\|_\infty^2 \|z\|_\infty^2 \|\partial_s n\|_\infty^2}{2\varepsilon}\int |\partial_s^{m+1} n|^2 \,\mathrm{d}s,\end{aligned}$$

denoting contraction again by $*$. This can be estimated directly by interpolation and Young's inequality.

The next critical term is $P_4^{2m+3}(\nu, \kappa, \partial_s n, z)$ from (5.11) which originates from

$$\int \partial_s^m z \partial_s^{m+2} \operatorname{div}(\nu z \partial_s n)\,\mathrm{d}s.$$

We integrate by parts to have a term $\partial_s^{m+2} z$. If not all the remaining derivatives fall on n we use Hölder's and Young's inequality and Proposition 5.7 exactly as before. Thus, the most critical case is, when all derivatives fall on n. Here we have to use the smallness assumption on the quantity F that controls in particular the L^∞-norm of z. We estimate

$$\begin{aligned}\int \partial_s^{m+2} z \nu^T z \partial_s^{m+2} n \,\mathrm{d}s &\le \|z\|_\infty \left(\frac{1}{2}\int |\partial_s^{m+2} z|^2 \,\mathrm{d}s + \frac{1}{2}\int |\partial_s^{m+2} n|^2 \,\mathrm{d}s\right)\\ &\le C\sqrt{\varepsilon_1}\left(\frac{1}{2}\int |\partial_s^{m+2} z|^2 \,\mathrm{d}s + \frac{1}{2}\int |\partial_s^{m+2} n|^2 \,\mathrm{d}s\right).\end{aligned}$$

The term

$$\int \partial_s^m z \partial_s^m \operatorname{div}(\partial_s^2 n)\,\mathrm{d}s$$

from (5.13) can be estimated as those from (5.9) with Hölder, Young and interpolation after intergration by parts. This applies as well to the P_4^{2m+3} and P_6^{2m+2} terms. Putting everything together we obtain for suitable $r > 1$ the estimates

$$\begin{aligned}&\alpha^{-2}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\partial_s^{m+1} n)^2 \,\mathrm{d}s + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\partial_s^m z)^2 \,\mathrm{d}s + \lambda \int (\partial_s^{m+2} n)^2 \,\mathrm{d}s + \int (\partial_s^{m+2} z)^2\\ &\le (N\varepsilon + C\varepsilon_1)\int \lambda |\partial_s^{m+2} n|^2 + |\partial_s^{m+2} z|^2 \,\mathrm{d}s + C(L)\left(\|\partial_s n\|_{1,2} + \|z\|_{1,2}\right)\\ &\quad + C(\varepsilon, L)\left(\|\partial_s n\|_{1,2}^r + \|z\|_{1,2}^r\right).\end{aligned}$$

Here, N is the number of terms estimated by interpolation, each contributing one ε. Choosing $\varepsilon < \frac{1}{4N}$ and $\varepsilon_1 < \frac{1}{4C}$ small enough, we can absorb these highest order terms and the claimed estimate is proven. Thus, as long as F is small, the second derivative of n and the first derivative of z are sufficient to control the flow. Moreover, $\int |\partial_s z|^2 + |\partial_s^2 n|^2$ is finite as long F is finite. Thus, by short-time existence and the argument of Dziuk, Kuwert and Schätzle [28] completing the proof of their Theorem 3.1, we can continue the flow as long as $F < \varepsilon_1$.

Step 2: We show a suitable upper bound for the time derivative of F. That is, when F is small initially, it remains so for a controlled period of time. We use the standard Sobolev embedding and Hölder's inequality to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L[\gamma] &= \int -\kappa V \,\mathrm{d}s = \int \kappa\partial_s^2 z + P_4^0(\kappa,\partial_s n, z) + P_4^1(\nu,\kappa,\partial_s n, z) - \kappa^2 \,\mathrm{d}s \\ &\leq L^{-1}\|\kappa\|_{1,2}\|z\|_{1,2} + L\|\kappa\|_{1,2}^3\|z\|_{1,2} + L\|\kappa\|_{1,2}^2\|z\|_{1,2}\|\partial_s n\|_{1,2} \\ &\quad + L^2\|\kappa\|_{1,2}\|z\|_{1,2}^2\|\partial_s n\|_{1,2}^2 + L\|\kappa\|_{1,2}^2\|z\|_{1,2}^2 + L\|\kappa\|_{1,2}^2\|\partial_s n\|_{1,2}^2.\end{aligned}$$

Therefore, the evolution of L is controlled by lower order quantities. Since it is not exactly covered by the estimates for the higher space derivatives we calculate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |z|^2 \,\mathrm{d}s = \int & z(\partial_s^2 V + \kappa^2 V + \alpha^2 \operatorname{div}(\partial_s^2 n + \delta\nabla z) + \kappa V \partial_s n\cdot\tau \\ & + \partial_s V\partial_s n\cdot\nu) - \kappa V |z|^2 \,\mathrm{d}s\end{aligned}$$

and

$$\alpha^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_s n|^2 \,\mathrm{d}s = \int \langle \partial_s n, \Big(\lambda\partial_s^3 n + \delta\partial_s(\tau\partial_s z + z\kappa\nu)\Big) + 2\alpha^{-2}\kappa\partial_s n V\rangle \,\mathrm{d}s.$$

We split these terms as above. We have

$$\begin{aligned}\int \langle \partial_s n, (\alpha^{-2}\kappa\partial_s n V)\rangle \,\mathrm{d}s = \alpha^{-2}\int & P_4^2(\kappa,\partial_s n, z) + P_6^0(\kappa,\partial_s n, z) \\ & + P_6^1(\nu,\kappa,\partial_s n, z) + \alpha^{-2}P_4^0(\partial_s n,\kappa)\,\mathrm{d}s,\end{aligned} \tag{5.15}$$

$$\begin{aligned}\alpha^{-2}\int (\partial_s n)^2\kappa V \,\mathrm{d}s = \alpha^{-2}\int & P_4^2(\kappa,\partial_s n, z) + P_6^0(\kappa,\partial_s n, z) \\ & + P_6^1(\nu,\kappa,\partial_s n, z) + \alpha^{-2}P_4^0(\kappa,\partial_s n)\,\mathrm{d}s,\end{aligned} \tag{5.16}$$

$$\begin{aligned}\int z(\partial_s^2 V_\ell + \kappa^2 V)\,\mathrm{d}s \leq \int & P_4^2(\kappa,\partial_s n, z) + P_4^3(\nu,\kappa,\partial_s n, z) + \alpha^{-2}|\partial_s^1 z||\partial_s^{m+1}\kappa| \\ & + P_4^2(\kappa, z) + P_6^1(\nu,\kappa,\partial_s n, z) + P_6^0(\kappa,\partial_s n, z) + \alpha^{-2}P_4^0(\kappa, z)\,\mathrm{d}s,\end{aligned} \tag{5.17}$$

$$\int z^2\kappa V \,\mathrm{d}s = \int P_6^0(\kappa,\partial_s n, z) + P_4^1(\nu,\kappa,\partial_s n, z) + P_4^2(\kappa, z) + \alpha^{-2}P_4^0(\kappa, z)\,\mathrm{d}s, \tag{5.18}$$

$$\begin{aligned}&\int z(\alpha^2 \operatorname{div}(\partial_s^2 n + \nabla z) + \partial_s V\partial_s n\cdot\nu)\,\mathrm{d}s \\ &= \int \alpha^2 P_3^2(\nu,\partial_s n, z) + \alpha^2 P_4^2(\nu, z) + P_4^3(\nu,\partial_s n, z) \\ &\quad + P_6^1(\nu,\kappa,\partial_s n, z) + P_6^2(\nu,\kappa,\partial_s n, z) + \alpha^{-2}P_4^1(\nu,\kappa,\partial_s n, z)\,\mathrm{d}s,\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}\int z\kappa V\partial_s n\cdot\tau \,\mathrm{d}s = \int & P_5^2(\nu,\kappa,\partial_s n, z) + P_7^0(\nu,\kappa,\partial_s n, z) \\ & + P_6^1(\nu,\kappa,\partial_s n, z) + \alpha^{-2}P_5^0(\nu,\kappa,\partial_s n, z)\end{aligned} \tag{5.20}$$

Inspecting these expressions term by term we find that some of them can simply be bounded by products of L, $\|z\|_{1,2}$ and $\|\partial_s n\|_{1,2}$. For the remaining terms we employ again Proposition 5.7. When we apply Young's inequality, we take care to generate the right power of L in front of the term we would like to absorb.

To calculate $\frac{\mathrm{d}}{\mathrm{d}t}F(\gamma, n)$ we use the estimates (5.9)-(5.13) and (5.15)-(5.20) to find again an exponent $r > 0$ such that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}F(\gamma, n) = & -3\alpha^{-2}L^{-4}\frac{\mathrm{d}L}{\mathrm{d}t}\int |\partial_s n|^2 \,\mathrm{d}s + \alpha^{-2}L^{-3}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_s n|^2 \,\mathrm{d}s \\
& -L^{-2}\frac{\mathrm{d}L}{\mathrm{d}t}\int |z|^2 \,\mathrm{d}s + L^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\int |z|^2 \,\mathrm{d}s \\
& -\alpha^{-2}L^{-2}\frac{\mathrm{d}L}{\mathrm{d}t}\int |\partial_s^2 n|^2 \,\mathrm{d}s + \alpha^{-2}L^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_s^2 n|^2 \,\mathrm{d}s \\
& +\frac{\mathrm{d}L}{\mathrm{d}t}\int |\partial_s z|^2 \,\mathrm{d}s + L\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_s z|^2 \,\mathrm{d}s \\
\leq & -L\int |\partial_s^2 z|^2 \,\mathrm{d}s - L^{-1}\int |\partial_s^3 n|^2 \,\mathrm{d}s \\
& +C(L,\alpha,\varepsilon)\left(\|\partial_s n\|_{1,2} + \|z\|_{1,2} + \|\partial_s n\|_{1,2}^r + \|z\|_{1,2}^r\right) \\
\leq & -L\int |\partial_s^2 z|^2 \,\mathrm{d}s - L^{-1}\int |\partial_s^3 n|^2 \,\mathrm{d}s + C(L,\alpha,\varepsilon)\left(F(\gamma, n) + F(\gamma, n)^r\right).
\end{aligned}$$

For $t \in (0, T)$ assume $F(\gamma(t), n_0) < \varepsilon_1/2 < 1$. We conclude for $s > 0$ that

$$\begin{aligned}
F(\gamma(t+s), n(t+s)) &\leq F(\gamma(s), n(s)) + \int_0^s \frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t), n(t))\,\mathrm{d}t \\
&\leq F(\gamma_0, n_0) + C(L,\alpha)s\varepsilon_1.
\end{aligned}$$

Thus, $F(\gamma(t+s), n(t+s)) < \varepsilon_1$ for $s < \frac{C(L,\alpha)}{2}$. Therefore we may conclude that $T - t > \frac{C(L,\alpha)}{2}$ and this completes the proof. ☐

Corollary 5.11
For $T > 0$ let $(\gamma, n) : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2 \times \mathbb{R}^2$ be a smooth solution of equation (3.8)*, that cannot be smoothly extended beyond T. Then, for $z = \kappa + \delta\,\mathrm{div}(n)$ one has*

$$\lim_{t\to T}\int_{\gamma(t)} |\partial_s z|^2 + |\partial_s^2 n|^2 \,\mathrm{d}s = \infty.$$

Proof. We set

$$\begin{aligned}
F_{\max} = \sup_{t\in[0,T)} L^{-3}[\gamma_0]\int |\partial_s n|^2 \,\mathrm{d}s + L^{-1}[\gamma_0]\int |z|^2 \,\mathrm{d}s \\
+ L[\gamma_0]\int |\partial_s z|^2 \,\mathrm{d}s + L^{-1}[\gamma_0]\int |\partial_s^2 n|^2 \,\mathrm{d}s
\end{aligned}$$

and $\alpha = \varepsilon_1/F_{\max}$ and consider $(\gamma_\alpha, n_\alpha) = (\alpha^{-1}\gamma(x, \alpha^4 t), n(x, \alpha^4 t))$ that is a solution of equation (5.3) on $\mathbb{S}^1 \times [0, T\alpha^{-4})$. Due to the scaling properties of F we know that $\sup_{t\in[0,T)} F(\gamma_\alpha, n_\alpha) < \varepsilon_1$. Thus by Theorem 5.10 we conclude that we can find a smooth extension of $(\gamma_\alpha, n_\alpha)$ to $[0, T_1\alpha^{-4})$ with $T_1 > T$. Scaling back, we have a smooth solution $(\gamma, n) : \mathbb{S}^1 \times [0, T_1) \to \mathbb{R}^2 \times \mathbb{R}^2$ in contradiction to the maximality of T. ☐

Remark 5.12
In the proof of Theorem 5.10 we only make use of smallness of z in an L^∞-sense. Observing

this, one might conjecture that this is already enough to extend the flow. An L^∞-bound for z is however a still surprisingly strong prerequisite as for curves it is often sufficient to control the L^2 or even any L^p-norm of the curvature for $p > 1$ to continue the flow [3, 14, 28, 38]. For a class of second order flows, Angenent [4] was even able to extend the flow as long the curves are locally Lipschitz.

Since we cannot a priori rule out the occurrence of singularities, a next step in the analysis of this flow could consist in the construction of a blow-up limit in this case. To that account, it would be necessary to show that a strong enough quantity with suitable rescaling behavior concentrates in any singular point. Parabolic rescaling then often allows the construction of a limit object and reveals more insights or even allows to extend the flow. This strategy is subject to recent research and lies in the heart of many famous theorems of geometric Analysis [44, 51, 55]. In the case of the mean curvature flow, Mantegazza [68] gives a good introduction to the analysis of singularities, including Huisken's monotonicity formula.

In our case any non-trivial blow-up would already be a contradiction to finite energy of the limit curve. However, it proved to be difficult to find the right quantity and rescaling due to the different scaling regimes of the coupled equations.

5.2 A Łojasiewicz-Simon Inequality in the Unconstrained Case

In this and the following sections we derive a Łojasiewicz-Simon inequality in the framework of Feehan and Maridakis. We use it together with the short-time existence result and the smoothness of stationary points to obtain global existence for solutions that start close to a local minimizer and convergence to a local minimizer for global solutions. Our approach is inspired by works of different authors that applied a Łojasiwicz-Simon inequality in the context of Willmore or Helfrich flow.

As already noted in the introduction of this work, Chill, Fasangova and Schätzle used this abstract framework to prove that Willmore blow-ups are never compact [16]. For elastic curves in $\mathbb{R}^d$ subject to different boundary conditions Lin [62] proved global existence of solutions and convergence up to translation of a subsequence to a minimizer. Here, Dall'Acqua, Pozzi and Spener [22] were able to obtain smooth convergence of the whole flow by means of Chill's result on the Łojasiewicz-Simon inequality. For a Helfrich-type model for 2-dimensional surfaces in $\mathbb{R}^3$, Lengeler [60] proved stability of local minimizers and global existence for solutions starting close by.

In the following we will make repeated use of the concepts of analytic maps between Banach spaces and Fredholm operators. For the convenience of the reader, we state the important definitions and properties. The following definition is taken from the book of Zeidler.

Definition 5.13 (Analytic maps between Banach spaces [95, § 8.2])
For real Banach spaces X and Y let F denote a map $F : D \to Y$, $D \subset X$ open. For $k \in \mathbb{N}$ consider a continuous, symmetric, and k-linear operator T. We introduce the norm (cf. [95, Definition 4.15])

$$\|T\| = \sup_{\|x_1\|_X, \dots, \|x_k\|_X \le 1} \|T(x_1, \dots, x_k)\|_Y$$

for such multilinear operators and for $x \in X$ we write Tx^k to denote $T(x, \dots, x)$. Note that we use these notations also for "0-linear" operators—that are simply elements in Y—to improve readability.

Now, F is called *analytic* in $x_0 \in X$ if and only if there exists a collection $(T_k)_{k\in\mathbb{N}_0}$, where T_k is a k-linear operator, and $r > 0$ such that for all x with $\|x - x_0\|_X < r$ the series

$$\sum_{k=0}^{\infty} \|T_k\| \|x - x_0\|_X^k$$

converges and

$$F(x) = \sum_{k=0}^{\infty} T_k(x - x_0)^k.$$

Moreover, F is called analytic in D, if and only if F is analytic in every $x \in D$.

Concerning Fredholm operators, we recall the following definition and theorem that can be found in Zeidler's book as well.

Definition 5.14 (Fredholm operator [95, Definition 8.13])
Let X and Y be real Banach spaces. A linear operator $A : X \to Y$ is called a *Fredholm operator* if A is continuous and $\dim(\ker A)$ and $\operatorname{codim}(R(A))$ are both finite. The number $\operatorname{ind}(A) = \dim(\ker A) - \operatorname{codim}(R(A))$ is called *index* of A.

We will make repeated use of the fact that the Fredholm property and the index are invariant under compact perturbations.

Theorem 5.15 ([95, Theorem 8.14(3)])
Let X and Y be real Banach spaces. For a Fredholm operator $A \in L(X, Y)$ and a compact operator $C \in L(X, Y)$ it is true that

$$\operatorname{ind}(A) = \operatorname{ind}(A + C).$$

To prove a Łojasiewicz-Simon inequality we apply the following result by Feehan and Maridakis [33, Theorem 3]. First, we need the concept of a gradient map (cf. [33, Definition 1.5] and the reference therein).

Definition 5.16 (Gradient map)
Let $K, \tilde{K}$ be Banach spaces such that there are continuous embeddings$K \subset \tilde{K} \subset K^*$. In the following we consider K to be a subset of $\tilde{K}$ and $\tilde{K}$ to be a subset of K^* via these embeddings without writing them down explicitly.Let U denote an open subset of K. A continuous map, $\mathcal{M} : U \to \tilde{K}$ is called a *gradient map* if there exists a C^1-function $E : U \to \mathbb{R}$, such that for all $x \in U$ and for all $v \in K$ it holds that

$$E'(x)v = \langle \mathcal{M}(x), v\rangle_{K^*\times K},$$

where $\langle \cdot, \cdot\rangle_{K^*\times K}$ is the canonical bilinear form on $K^* \times K$.The real-valued function E is called a potential for the gradient map $\mathcal{M}$.

Moreover, we note the following. On a Banach space E we call a bilinear form $b : E \times E \to \mathbb{R}$ *definite* if $b(x, x) \neq 0$ for all $x \in E \setminus \{0\}$. For a continuous map $\iota : E \to E^*$ we say that ι is *definite*,if the bilinear form $(x, y) \mapsto \langle \iota(y), x\rangle_{E^*\times E}$ is definite. As Feehan and Maridakis note, this is e.g. the case when there is a continuous embedding $J : E \to H$ of E into a

Hilbert space H. Indeed, the adjoint operator J^* then embeds E^* into H and we have for all $x, y \in E$ that$\langle \iota(y), x\rangle_{E^*\times E} = \langle J(y), J(x)\rangle_H$ and thus $\langle \iota(x), x\rangle_{E^*\times E} = 0$ implies $x = 0$. See also [12, Remark 3, p.136] for details.

Theorem 5.17 (Theorem 3 from [33])
Let K and $\tilde{K}$ be Banach spaces with continuous embeddings $K \subset \tilde{K} \subset K^*$ and such that the embedding $K \subset K^*$ is definite. Let $U \subset K$ be an open subset, $E : U \to \mathbb{R}$ be an analytic function, and $x_\infty \in U$ be a critical point of E, that is $E'(x_\infty) = 0$. Assume that there are Banach spaces G and $\tilde{G}$ and continuous embeddings of Banach spaces $K \subset G \subset \tilde{G}$ and $K \subset \tilde{K} \subset \tilde{G}$, such that the compositions $K \subset G \subset \tilde{G}$ and $\tilde{K} \subset \tilde{G} \subset K^*$induce the same embedding $K \subset \tilde{G}$. Let $\mathcal{M} : U \to \tilde{K}$ be an analytic gradient map for E in the sense of Definition 5.16. Suppose that for each $x \in U$ the bounded, linear operator,

$$\mathcal{M}'(x) : K \to \tilde{K}$$

has an extension

$$\mathcal{M}_1(x) : G \to \tilde{G}$$

such that the map $U \ni x \mapsto \mathcal{M}_1(x) \in L(G, \tilde{G})$ is continuous. If $\mathcal{M}'(x_\infty) : K \to \tilde{K}$ and $\mathcal{M}_1(x_\infty) : G \to \tilde{G}$ are Fredholm operators with index zero, then there are constants $c \in (0, \infty)$, $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$, with the following significance. If $x \in U$ obeys $\|x - x_\infty\|_K < \sigma$, then

$$\|\mathcal{M}(x)\|_{\tilde{G}} \geq c|E(x) - E(x_\infty)|^\theta.$$

Remark 5.18 (Embeddings and identifications)
We will apply this theorem in the case where all the spaces are L^2-based Sobolev spaces. For $k, \ell \in \mathbb{N}_0$, $k \geq \ell$ we embed $H^k \to H^\ell$ and $H^{-\ell} \to H^{-k}$ by inclusion and $H^k \to H^{-\ell}$ via $H^k \to L^2 \cong (L^2)^* \to H^{-\ell}$. Therefore, we will also use the term L^2-gradient map.

For $d \in \mathbb{N}$ let M be a smooth, orientable, closed d-dimensional manifold. We will analyze the stability of local minimizers of the energy $E : H^2_{\mathrm{imm}}(M, \mathbb{R}^{d+1}) \times H^1(M, \mathbb{R}^{d+1}) \to \mathbb{R}$ as introduced in Chapter 3 with help of a Łojasiewicz-Simon inequality. To do so, we have to study its smoothness properties. It is continuous with respect to the $H^2_{\mathrm{imm}} \times H^1$ topology, but we will also identify prerequisites under which it has two proper Fréchet derivatives and when it is even analytic. Moreover, we will address the Fredholm property of its second derivative.

As in the proof of short-time existence, the geometric invariance of the energy causes a degeneracy of the corresponding differential operators. The solution is again to parametrize the evolving surfaces over a fixed reference surface. With this strategy also constraints can be incorporated, as done for a problem related to the Helfrich flow by Lengeler [60] and nicely presented for the harmonic map heat flow by Feehan and Maridakis [33, Section 3]. Fix $(\varphi, \eta) \in C^\infty_{\mathrm{imm}}(M, \mathbb{R}^{d+1}) \times C^\infty(M, \mathbb{R}^{d+1})$ and let ν denote a unit normal associated to φ, then there is an open neighborhood U of $(0, 0)$ in $(H^2(M) \cap C^1(M)) \times H^1(M, \mathbb{R}^{d+1})$ such that for $(f, n) \in U$ the map $\varphi + f\nu$ is still an immersion. We define $F : U \to \mathbb{R}$ by

$$F(f, n) := E(\varphi + f\nu, \eta + n). \tag{5.21}$$

In the following we will examine the properties of F, when restricted to spaces of higher regularity. It is especially important that F has a gradient map $\mathcal{M}$ and F and $\mathcal{M}$ are analytic on Sobolev spaces of enough regularity. A corresponding result can therefore be found in every work employing a Łojasiewicz-Simon inequality, see e.g. [16, Lemma 3.2], [22, Lemma 3.4], [60, Lemma 1.1].

Lemma 5.19
For $d \in \mathbb{N}$ let M be a smooth, orientable, closed d-dimensional manifold and $\varphi : M \to \mathbb{R}^{d+1}$ a smooth immersion. For $d = 1$ we set $k = 3$, else we set $k > d/2 + 3$. Set $K = H^k(M) \times H^{k-1}(M, \mathbb{R}^{d+1})$, $\tilde{K} = H^{k-4}(M) \times H^{k-3}(M, \mathbb{R}^{d+1})$ with properties as asserted in Theorem 5.17and let $U \subset K$ be an open neighborhood of $(0, 0)$ such that for $(f, n) \in U$ the map $\varphi + f\nu$ is still an immersion. Then, the map F from equation (5.21) has a gradient map $\mathcal{M} : U \to \tilde{K}$ in the sense of Definition 5.16. Moreover, the map

$$F : U \to \mathbb{R} \quad \textit{and the gradient map} \quad \mathcal{M} : U \to \tilde{K}$$

are real analytic.

Proof. First, we recall from Proposition 3.5 and Lemma 1.11 that for all

$$\psi, m \in H^k_{\mathrm{imm}}(M, \mu_\varphi; \mathbb{R}^{d+1}) \times H^{k-1}(M, \mu_\varphi; \mathbb{R}^{d+1})$$

the map

$$\mathrm{d}E : H^k(M, \mu_\varphi; \mathbb{R}^{d+1}) \times H^{k-1}(M, \mu_\varphi; \mathbb{R}^{d+1}) \to K^*$$

is given by

$$\begin{aligned}
\mathrm{d}_\psi E(\psi, m) = \Big[&- \Delta_\psi (H_\psi + \delta \operatorname{div}_\psi m) - (H_\psi + \delta \operatorname{div}_\psi m)|\nabla_\psi \nu_\psi|^2 \\
&+ \delta (H_\psi + \delta \operatorname{div}_\psi m) \nabla_\psi m : \nabla_\psi \nu_\psi + \delta \operatorname{div}_\psi ((H_\psi + \delta \operatorname{div}_\psi m) \nu_\psi^T \nabla_\psi m) \\
&+ \lambda (\nabla_\psi m)^T : \left[\nabla_\psi \nu (\nabla_\psi m)^T\right] + \frac{1}{2}(H_\psi + \delta \operatorname{div}_\psi m)^2 H_\psi + \frac{\lambda}{2}|\nabla_\psi m|^2 H_\psi \Big] \nu_\psi, \\
\mathrm{d}_m E(\psi, m) = {}& \lambda \Delta_\psi m + \delta \nabla_\psi (H_\psi + \delta \operatorname{div}_\psi m) - \delta (H_\psi + \delta \operatorname{div}_\psi m) H_\psi \nu,
\end{aligned}$$

when we set for $(\xi, \mu) \in K$

$$\langle \mathrm{d}E(\psi, m), (\xi, \mu) \rangle_{K^* \times K} = \langle \iota_1 (\mathrm{d}E_\psi (\psi, m)), \xi \rangle_{(H^k)^* \times H^k} + \langle \iota_2 (\mathrm{d}E_m (\psi, m)), \mu \rangle_{(H^{k-1})^* \times H^{k-1}},$$

where ι is the inclusion $\tilde{K} \to K^*$. We infer that the first variation of F around (f, n) is given by

$$\mathrm{d}F(f, n)[(g, m)] = \mathrm{d}E(\varphi + \nu_\varphi f, \eta + n)[(\nu_\varphi g, m)].$$

The sought map $\mathcal{M}$ now has to satisfy for all $(f, n), (g, m) \in U$ the relation

$$\langle \mathcal{M}(f, n), (g, m) \rangle_{K^* \times K} = \mathrm{d}F(f, n)[(g, m)].$$

Observe that in all prior descriptions of the expression $\mathrm{d}E(\varphi + \nu_\varphi f, \eta + n)[(\nu_\varphi g, m)]$ we integrated with respect to the measure $\mu_{\varphi + f\nu_\varphi}$ that is in local coordinates given by the density $\sqrt{g_{\varphi + f\nu_\varphi}}$. But here, it is important that we use the $L^2(M, \mu_\varphi)$ scalar product to represent $\mathrm{d}F(f, n)$ by $\mathcal{M}$. Thus, we have to introduce the correction term

$$J_f = \frac{\sqrt{g_{\varphi + f\nu_\varphi}}}{\sqrt{g_\varphi}}.$$

In Lemma 4.1 we discussed, how to write the geometric quantities in terms of derivatives of f and n. For readability we furnish the quantities depending on φ only with a $*$-symbol and replace the index $\varphi + f\nu_\varphi$ simply by the index f. Applying the formulas from Lemma 4.1 again, we find

$$\mathcal{M}(f, n) = \begin{pmatrix} J_f \Big(- \langle \nu_f, \nu_\varphi \rangle g_f^{ij} g_f^{k\ell} \nabla^{*4}_{ijk\ell} f - \delta g_f^{ij} \nabla^{*2}_{ij} \operatorname{div}_f n \\ + b_1(p, t, f, \nabla^* f, \nabla^{*2} f, \nabla^{*3} f, n, \nabla^* n, \nabla^{*2} n) \Big) \\ \\ J_f \Big(\lambda g_f^{ij} \nabla^{*2}_{ij} n + \delta g_f^{k\ell} \nabla^*_k (g^{ij} \nabla^{*2}_{ij} f + \delta \operatorname{div}_f n) X_\ell \\ + b_2(p, t, f, \nabla^* f, \nabla^{*2} f, n, \nabla^* n) \Big) \end{pmatrix} \tag{5.22}$$

with suitable functions b_1, b_2 as in equation (4.2) To complete the proof we have to show analyticity of two kinds of maps. Firstly, multiplication and secondly composition. Fortunately, we can treat both issues by citations from literature. We proceed along the lines of the proofs of [16, Lemma 3.2], [22, Lemma 3.4], [60, Lemma 1.1]. That is, we apply repeatedly Theorem 4 of [79, Sec. 5.5.3] to obtain analyticityof the composition operators and the results from [79, Sec. 4.4.3 and 4.4.4] (that are stated as Theorems 1.14 and 1.12 in this work) to determine the regularity of products. In the curve case $d = 1$, the highest order term involves the distribution $\partial_x^4 f$. It can be treated by [79, Sec. 4.4.3], since the other factors are multipliers for the space of respective test functions. □

Lemma 5.20
For $d \in \mathbb{N}$ let M be a d-dimensional closed orientable manifold. For $d = 1$ we set $k = 3$, else we set $k > d/2 + 3$. Set $K = H^k(M) \times H^{k-1}(M, \mathbb{R}^{d+1})$, $\tilde{K} = H^{k-4}(M) \times H^{k-3}(M, \mathbb{R}^{d+1})$, $G = H^3(M) \times H^2(M, \mathbb{R}^{d+1})$ and $\tilde{G} = H^{-1}(M) \times L^2(M, \mathbb{R}^{d+1})$. For $(\varphi, \eta) \in C^\infty_{\text{imm}}(M, \mathbb{R}^{d+1}) \times C^\infty(M, \mathbb{R}^{d+1})$ as in equation (5.21) let ν be a unit normal associated to φ and let $U \subset K$ be a neighbourhood of $(0, 0)$ such that for all $(f, n) \in U$ the maps $(\varphi, \eta) + (f\nu, n)$ are still immersions. For F as in equation (5.21), let $\mathcal{M} \in C^\omega(U, \tilde{K})$ be the gradient map from Lemma 5.19. Then for each $(f, n) \in U$ the bounded, linear operator,

$$\mathcal{M}'(f, n) : K \to \tilde{K}$$

has an extension

$$\mathcal{M}_1 : G \to \tilde{G}$$

such that the map for $U \ni (f, n) \mapsto \mathcal{M}_1(f, n) \in L(G, \tilde{G})$ is continuous and $\mathcal{M}'(0)$ and $\mathcal{M}_1(0)$ are both Fredholm of index 0.

Proof. The proof consists of two major steps. First we compute the second variation of F, then we discuss the regularity of coefficients of the associated differential operator.

We start with the calculation of the second derivative. With $A_{\varphi+tf\nu}$ we denote the second fundamental form of the perturbed surface and by $\nabla_{\varphi+tf\nu}$ the covariant derivative with respect to the varying metric.

Lemma 5.19 gives an expression for $\mathcal{M}$ and $\mathcal{M}' \in C^\omega(U, L(K, \tilde{K}))$ and $\mathcal{M}'$ is a differential operator with coefficients of Sobolev regularity depending smoothly on f and n, that are in particular continuous. Thus, for all $(f, n) \in U$ the operator $\mathcal{M}'(f, n) \in L(K, \tilde{K})$ extends to an operator $\mathcal{M}_1(f, n) \in L(G, \tilde{G})$ and the map $U \ni (f, n) \mapsto \mathcal{M}_1(f, n)$ is continuous.

The exact form of the differential operator $\mathcal{M}'(0)$ is now given by

$$\mathcal{M}'(0)(f, n) = \begin{pmatrix} -\Delta_\varphi(\Delta_\varphi f + \delta \operatorname{div}_\varphi n) \\ +a_1 \nabla^3_\varphi f + a_2 \nabla^2_\varphi f + a_3 \nabla_\varphi f + a_4 f + a_5 \nabla^2_\varphi n + a_6 \nabla_\varphi n + a_7 n \\ \\ \Delta_\varphi n + \delta \nabla_\varphi(\Delta_\varphi f + \delta \operatorname{div}_\varphi n) \\ +b_1 \nabla^2_\varphi f + b_2 \nabla_\varphi f + b_3 \nabla_\varphi n + b_4 n \end{pmatrix}$$

with smooth coefficient tensors $a_i = a_i(\varphi, \eta)$, $i = 1, \ldots, 7$ and $b_j = b_j(\varphi, \eta)$, $j = 1, \ldots 4$ that are contracted with the derivatives of f and n. Due to the regularity of φ and η we see that $\mathcal{M}'(0) \in L(K, \bar{K})$ can be extended to a map $\mathcal{M}_1 \in L(G, \bar{G})$. Since the Fredholm property and index are stable under compact perturbations it is sufficient to consider the main parts of $\mathcal{M}'(0)$ and $\mathcal{M}_1(0)$ with a spectral shift if necessary. As in the proof of short-time existence in Chapter 4 we invoke the theory of the Laplace equation on manifolds [83, Chap. 5]. Moreover we split the system by the introduction of a new dependent variable $\zeta = \Delta f + \delta \operatorname{div}(\eta)$. This was useful for a numerical approximation scheme [7] and also in Chapter 4. That is, we consider for $(x, y) \in \bar{G}$ the system

$$L \begin{pmatrix} \zeta \\ f \\ n \end{pmatrix} = \begin{pmatrix} -\Delta\zeta + \zeta \\ -\Delta f + f - \zeta + \delta \operatorname{div}(\eta) \\ -\Delta n + n + \delta \nabla \zeta \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}.$$

This system has a unique solution $(f, n) \in G$ with a priori estimates

$$\|(f,n)\|_G \leq C\big(\|(x,y)\|_{\bar{G}}\big).$$

Thus, the operator L is an isomorphism of Hilbert spaces and therefore Fredholm of index 0. Since the Fredholm index is invariant under compact perturbations, we obtain that also $\mathcal{M}_1$ is Fredholm of index 0 as it is a compact perturbation of L. Application of higher regularity theory for the Laplacian shows that with the same argument $\mathcal{M}'(0) \in L(K, \bar{K})$ is Fredholm of index 0, too. ☐

In Lemma 5.19 and Lemma 5.20 we have checked all assumptions of Theorem 5.17 and can therefore infer the validity of a Łojasiewicz inequality for the energy F. We do this by application of the result of Feehan and Maridakis stated in Theorem 5.17.

Theorem 5.21 (Łojasiewicz inequality for F.)
For $d \in \mathbb{N}$ let M be a smooth, orientable, closed d-dimensional manifold. For $d = 1$ we set $k = 3$, else we set $k > d/2 + 3$ and let $(\varphi^*, n^*) \in C^\infty(M, \mathbb{R}^{d+1}) \times C^\infty(M, \mathbb{R}^{d+1})$ be a critical point of the energy functional $E : H^k_{\mathrm{imm}}(M, \mathbb{R}^{d+1}) \times H^{k-1}(M, \mathbb{R}^{d+1}) \to \mathbb{R}$ as introduced in Chapter 3. Let ν^* denote a unit normal associated to φ^* and let U be a neighborhood of $(0,0)$ in $H^k(M) \times H^{k-1}(M)$ such that for $(f,n) \in U$ the map $(\varphi, \eta) + (f\nu, n)$ is still an immersion. Then, for $F : H^k(M) \times H^{k-1}(M) \to \mathbb{R}$ given by $F(f,n) = E(\varphi + f\nu, \eta + n)$ with gradient map $\mathcal{M}$ given by equation (5.22), there exists $\theta \in [\frac{1}{2}, 1)$,$C > 0$ and $\sigma > 0$ such that for every $(f,n) \in H^k(M) \times H^{k-1}(M, \mathbb{R}^{d+1})$ with $\|(f,n)\|_{H^k(M) \times H^{k-1}(M, \mathbb{R}^{d+1})} \leq \sigma$ the inequality

$$\|F(f,n) - F(0)\|^\theta \leq C\|\mathcal{M}(f,n)\|_{H^{-1}(M) \times L^2(M, \mathbb{R}^{d+1})}$$

is satisfied.

Proof. The proof follows from Theorem 5.17 in this work. The conditions there are fulfilled by Lemma 5.19 and Lemma 5.20 above. ☐

To prove a Łojasiewicz inequality for E we need to find a normal reparametrization for every regular immersion that is H^k close to a stationary point. In the following, we identify the differential $\mathrm{d}E$ with the gradient map of E.

Theorem 5.22 (Łojasiewicz inequality for E.)
For $d \in \mathbb{N}$ let M be a smooth, orientable, closed d-dimensional manifold. For $d = 1$ we set $k = 3$, else we set $k > d/2 + 3$ and let $(\varphi^*, n^*) \in C^\infty(M, \mathbb{R}^{d+1}) \times C^\infty(M, \mathbb{R}^{d+1})$ be a critical point of the energy functional $E : H^k_{\mathrm{imm}}(M, \mathbb{R}^{d+1}) \times H^{k-1}(M, \mathbb{R}^{d+1}) \to \mathbb{R}$ as introduced in Chapter 3. Then, there exists $\theta \in [\frac{1}{2}, 1)$,$C > 0$ and $\sigma > 0$ such that for every $(\varphi, n) \in H^k(M, \mathbb{R}^{d+1}) \times H^{k-1}(M, \mathbb{R}^{d+1})$ with $\|(\varphi, n)\|_{H^k(M, \mathbb{R}^{d+1}) \times H^{k-1}(M, \mathbb{R}^{d+1})} \leq \sigma$ the inequality

$$\|E(\varphi, n) - E(\varphi^*, n^*)\|^\theta \leq C\|dE(\varphi, n)\|_{H^{-1}(M, \mathbb{R}^{d+1}) \times L^2(M, \mathbb{R}^{d+1})} \tag{5.23}$$

is satisfied.

Proof. We apply the same argument as [60, Theorem 1.4], [16, Theorem 3.1], [22, Lemma 4.1]. We write φ as a graph over φ^* after reparametrization of M by $\Phi : M \to M$ with $\varphi = \varphi^* \circ \Psi + f\nu^* \circ \Psi$. This is possible with Lemma 1.18 and yields $\|f\|_{H^k(M)} \leq C\|\varphi - \varphi^*\|_{H^k(M,\mathbb{R}^{d+1})}$. Now we see by the parametrization invariance of E and the equivalence of the metrics g_{φ^*} and g_φ that

$$\begin{aligned}\|E(\varphi, n) - E(\varphi^*, n^*)\|^\theta &\leq C\|F(f,n) - F(0)\|^\theta \\ &\leq C\|F'(f,n)\|_{H^{-1}(M)\times L^2(M,\mathbb{R}^{d+1})} \\ &= C\|E'(\varphi,n)\|_{H^{-1}(M,\mathbb{R}^{d+1})\times L^2(M,\mathbb{R}^{d+1},\mu_{\varphi^*})} \\ &\leq C\|E'(\varphi,n)\|_{H^{-1}(M,\mathbb{R}^{d+1})\times L^2(M,\mathbb{R}^{d+1},\mu_{\varphi})}\end{aligned}$$

and this concludes the proof. □

Due to the scaling properties of E we will not find stationary points in dimension $d > 2$ without any further restrictions. But when we demand that the immersion must include a certain area or have a fixed volume, we cannot decrease the energy by scaling.

5.3 A Łojasiewicz-Simon Inequality in the Presence of Constraints

It will be our aim to establish a Łojasiewicz-Simon inequality also for constrained functionals. For the Helfrich flow coupled to a fluid dynamical system such result was developed by Lengeler [60], Simons original work [80] and also a part of the work by Feehan and Maridakis [33, Chapter 3] derivea gradient inequality for the harmonic map heat flow. Here, the image of the flow has to stay on the embedded target manifold. This condition can also be treated as a constraint in our sense. We will adapt the arguments of these authors to our coupled problem.

Definition 5.23
For Banach spaces K, C, let $E : K \to \mathbb{R}$ and $\Theta : K \to C$ be C^1 maps. Let $x_S \in K$ be a point in K such that $\Theta(x_S) = 0, \Theta'(x_S)$ is surjective and $\ker \Theta'(x_S)$ is a complemented subspace of K with projection π. We call x_S a stationary point with respect to the constraint $\Theta = 0$, if for all $y \in \ker \Theta'(x_S)$ it holds that

$$\langle E'(x_S), y\rangle_{K^*\times K} = 0$$

which is equivalent to $\pi^* E'(x_S) = 0$.

A condition that allows us to draw conclusions about the existence of a gradient map in the constrained case is the following.

Definition 5.24 (Admissible constraints)
In the situation of Theorem 5.17 let C denote another Banach space let $U \subset K$ be an open neighborhood of 0. We call a constraint map $\Theta : U \to C$ with $\Theta(0) = 0$ admissible for the functional E, if and only if the following conditions hold.

1. Θ is analytic.
2. $\Theta'(0)$ is surjective and $X := \ker \Theta'(0)$ is a closed complemented subspace of K with projection π.
3. The adjoint projection $\pi^* : K^* \to K^*$ leaves the spaces $\tilde{K}$ and $\tilde{G}$ invariant.

4. There exists an open (in the trace topology of X) neighborhood $V \subset X$ with $0 \in V$and an analytic map $\gamma : V \to K$ such that for all $x \in V$ it holds that $\Theta(x + \gamma(x)) = 0$ and additionally $\gamma(0) = 0$ and $\mathrm{d}\gamma(0) = 0$

5. For all $x \in V$ the adjoint map $\big(\mathrm{d}\gamma(x)\big)^* : K^* \to X^*$ leaves $\tilde{K}$ and $\tilde{G}$ invariant.

Proposition 5.25
In the situation of Theorem 5.17 let $U \subset K$ be an openneighborhood of 0. Given another Banach space C and $\Theta : U \to C$ a constraint function that is admissible for F, we set $X = \ker \Theta'(0)$, $V = U \cap X$ and define the functional $\tilde{F} : V \to \mathbb{R}$ by $\tilde{F}(x) = F(x + \gamma(x))$. With $\pi : K \to X$ we denote the projection on X that exists by the admissibility of Θ. Then, $\tilde{F}$ admits a gradient map $\tilde{\mathcal{M}} : U \to \tilde{X}$ with $\tilde{X} = \pi^* \tilde{K}$ that is given by $\pi^*(\mathcal{M}(x + \gamma(x)) + \mathrm{d}\gamma(x)^* \mathcal{M}(x + \gamma(x)))$. For $G_X = \pi G$ and $\tilde{G}_X = \pi^* \tilde{G}$, for each $x \in V$ the operator $\tilde{\mathcal{M}}'(x) \in L(X, \tilde{X})$ has an extension $\tilde{\mathcal{M}}_1(v) \in L(G_X, \tilde{G}_X)$ and the map $v \mapsto \tilde{\mathcal{M}}_1(v)$ is continuous. Moreover, if $\tilde{\mathcal{M}}(0) = 0$ the operator $\tilde{\mathcal{M}}'(0) : X \to \tilde{X}$ is given by $\pi^*\big(\mathcal{M}'(0) + \mathrm{d}(\mathrm{d}\gamma^*)\mathcal{M}(0)\big)$.

Proof. We consider the functional $\tilde{F} : \ker \Theta' \to \mathbb{R}$ given by $\tilde{F}(x) = F(x + \gamma(x))$ and calculate by chain rule $\mathrm{d}\tilde{F} = \mathrm{d}F(\mathrm{Id} + \gamma) \circ (\mathrm{Id} + \mathrm{d}\gamma)$. We know that there is a function $\mathcal{M} : K \to \tilde{K}$ fulfilling for all $v, h \in K$ that

$$\mathrm{d}F(v)h = \langle \mathcal{M}(v), h \rangle_{K^* \times K}.$$

Therefore, we conclude that for all $x, h \in X \subset K$ we have

$$\begin{aligned} \mathrm{d}\tilde{F}(x)h &= \mathrm{d}F(x + \gamma(x))(h + \mathrm{d}\gamma(x)h) \\ &= \langle \mathcal{M}(x + \gamma(x)), (h + \mathrm{d}\gamma(x)h) \rangle_{K^* \times K} \\ &= \langle \mathcal{M}(x + \gamma(x)) + \mathrm{d}\gamma(x)^* \mathcal{M}(x + \gamma(x)), \pi h \rangle_{K^* \times K} \\ &= \langle \pi^*(\mathcal{M}(x + \gamma(x)) + \mathrm{d}\gamma(x)^* \mathcal{M}(x + \gamma(x))), h \rangle_{X^* \times X} \end{aligned}$$

From the assumption of admissibility we conclude that

$$\pi^*(\mathcal{M}(x + \gamma(x)) + \mathrm{d}\gamma(x)^* \mathcal{M}(x + \gamma(x))) \in \tilde{K}$$

and set $\tilde{X} = \pi^* \tilde{K}$. This shows that $\tilde{\mathcal{M}} = \pi^*(\mathcal{M}(x + \gamma(x)) + \mathrm{d}\gamma(x)^* \mathcal{M}(x + \gamma(x)))$ is the desired analytic gradient map of $\tilde{F}$. In the next step we consider the map $\tilde{\mathcal{M}}'(0) : X \to \tilde{X}$. Using that $\mathrm{d}\gamma(0) = 0$ we see that

$$\tilde{\mathcal{M}}'(0) = \pi^*\big(\mathcal{M}'(0) + \mathrm{d}(\mathrm{d}\gamma^*)\mathcal{M}(0)\big). \tag{5.24}$$

□

The next lemma shows that real-valued constraint functions that admit an L^2-gradient of sufficient regularity are admissible for the functional F from (5.21).

Lemma 5.26 (Real valued constraints)
For $d \in \mathbb{N}$ let M be a d-dimensional closed orientable manifold. For $d = 1$ we set $k = 3$, else we set $k > d/2 + 3$. Set $K = H^k(M) \times H^{k-1}(M, \mathbb{R}^{d+1})$, $\tilde{K} = H^{k-4}(M) \times H^{k-3}(M, \mathbb{R}^{d+1})$, $G = H^3(M) \times H^2(M, \mathbb{R}^{d+1})$ and $\tilde{G} = H^{-1}(M) \times L^2(M, \mathbb{R}^{d+1})$. For a smooth pair of functions $(\varphi, \eta) \in C^\infty_{\mathrm{imm}}(M, \mathbb{R}^{d+1}) \times C^\infty(M, \mathbb{R}^{d+1})$ let ν be a unit normal associated to φ and let let $U \subset K$ be an open neighborhood of $(0, 0)$ such that for $(f, n) \in U$ the map $\varphi + f\nu$ is still an immersion. For F as in equation (5.21), let $\mathcal{M} \in C^\omega(U, \tilde{K})$ be the gradient map from Lemma 5.19. For $\ell \in \mathbb{N}$ let $\Theta : K \to \mathbb{R}^\ell$, given through analytic maps $(f, n) \mapsto \Theta_i(f)$, be an analytic map that has gradient maps $\mathcal{M}_{\Theta_i} : K \to \tilde{K}$, $i = 1, \ldots, \ell$, that fulfill for all $\psi \in K$ the relations

$$\langle \mathcal{M}_{\Theta_i}(f), \psi \rangle_{K^* \times K} = \langle \Theta_i'(f), \psi \rangle_{K^* \times K}.$$

Suppose $\Theta(0) = 0$*,* $\mathcal{M}_{\Theta_i}(0) \in C^\infty(M)$ *and are linearly independent. Then,* Θ *is admissible for the functional* F *from equation* (5.21)*.*

Proof. We have to check the different assumptions of admissibility. We set $X = \ker \Theta'(0)$ and $Y = \operatorname{span}\{\mathcal{M}_{\Theta_i}(0)\}$ which are complementary subspaces of K. The linear independence of the gradient vectors guarantees the surjectivity of $\Theta'(0)$. Since we assumed that the gradient map at 0 is represented by smooth functions $\mathcal{M}_{\Theta_i}(0)$, there is a smooth L^2-orthonormal basis of Y and we can use the L^2-orthogonal projection onto X and this projection is regularity preserving. Therefore, its adjoint leaves $\tilde{K}$ and $\tilde{G}$ invariant. Moreover, there exists $U \subset X$ a neighborhood of 0 and analytic functions $\gamma_i : U \to \mathbb{R}$ such that for all $f \in U$ we have $\Theta(f + \sum_i \gamma_i(f)\mathcal{M}_{\Theta_i}(0)) = 0$ by the implicit function theorem [95, Theorem 4.B]. That the assumptions of this theorem are satisfied is easier to see, when we consider $\tilde{\Theta} : U \times \mathbb{R}^\ell \to \mathbb{R}^\ell : (f, s) \mapsto \Theta(f + \sum_i s_i \mathcal{M}_{\Theta_i}(0))$. Surely, $\tilde{\Theta}$ is differentiable, as it is analytic, and the partial Fréchet derivative $\mathrm{D}_s\tilde{\Theta}(0, 0)$ is an isomorphism of $\mathbb{R}^\ell$since the vectors $\mathcal{M}_{\Theta_i}(0)$ are linearly independent. For $f \in U$ and $g \in X$ we calculate

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Theta_i(f + tg + \sum_j \gamma_j(f + tg)\mathcal{M}_{\Theta_j}(0)) \\
&= \langle \Theta_i'\big(f + \sum_j \gamma_j(f)\mathcal{M}_{\Theta_j}(0)\big), g + \sum_j \langle \gamma_j'(f), g\rangle_{K^*\times K} \mathcal{M}_{\Theta_j}(0)\rangle_{K^*\times K} \\
&= \langle \mathcal{M}_{\Theta_i}\big(f + \sum_j \gamma_j(f)\big), g + \sum_j \langle \gamma_j'(f), g\rangle_{K^*\times K} \mathcal{M}_{\Theta_j}(0)\rangle_{K^*\times K},
\end{aligned}$$

and thus, since $\gamma_j'(f)g$ are real numbers, we find

$$\begin{aligned}
&\sum_j \langle \gamma_j'(f), g\rangle_{K^*\times K} \big\langle \mathcal{M}_{\Theta_i}\big(f + \sum_k \gamma_k(f) M_{\Theta_k}(0)\big),\ \mathcal{M}_{\Theta_j}(0)\big\rangle_{K^*\times K} \\
&= -\langle \mathcal{M}_{\Theta_i}\big(f + \sum_j \gamma_j(f) M_{\Theta_j}(0)\big),\ g\rangle_{K^*\times K}.
\end{aligned}$$

From the continuity of the vectors $\mathcal{M}_{\Theta_i}$ we infer that the matrices $m(f) \in \mathbb{R}^{\ell\times\ell}$ with

$$m_{ij}(f) = \langle \mathcal{M}_{\Theta_i}\big(f + \sum_k \gamma_k(f)\mathcal{M}_{\Theta_k}(0)\big), \mathcal{M}_{\Theta_j}(0)\rangle_{K^*\times K}$$

are invertible in a neighborhood of 0. We denote the entries of the inverse matrices by $m^{ij}(f)$ and we can compute

$$\gamma_i'(f)g = -\sum_j m^{ij}(f)\langle \mathcal{M}_{\Theta_j}(f + \sum_k \gamma_k(f)\mathcal{M}_{\Theta_k}(0)), g\rangle_{K^*\times K}.$$

Since $X = \ker(\Theta'(0))$ we have $\gamma_i'(0) = 0$. Defining $\gamma : X \to \tilde{K}$ by $\gamma(f) = \gamma_i(f)\mathcal{M}_{\Theta_i}(0)$ we have $\mathrm{d}\gamma(f)g = \sum_i \gamma_i'(f)g\mathcal{M}_{\Theta_i}(0)$. Thus, for $g \in X$, $h \in H^{k-4}(M)$ we have

$$\begin{aligned}
\langle \mathrm{d}\gamma(f)g, h\rangle_{K^*\times K} &= -\sum_i \sum_j m^{ij}(f)\langle \mathcal{M}_{\Theta_j}\big(f + \sum_k \gamma_k(f)\mathcal{M}_{\Theta_k}(0)\big), g\rangle_{K^*\times K} \langle \mathcal{M}_{\Theta_i}(0), h\rangle_{K^*\times K} \\
&= -\langle g, -\sum_i \sum_j \mathcal{M}_{\Theta_j}\big(f + \sum_k \gamma_k(f)\mathcal{M}_{\Theta_k}(0)\big) m^{ji}(f)\langle \mathcal{M}_{\Theta_i}(0), h\rangle_{K^*\times K}\rangle_{K^*\times K} \\
&= \langle g, \mathrm{d}\gamma(f)^* h\rangle_{K^*\times K},
\end{aligned} \tag{5.25}$$

showing that $\mathrm{d}\gamma(f)^* : H^k(M)^* \to H^k(M)^*$ indeed maps $\tilde{K}$ and $\tilde{G}$ to $K \subset \tilde{K} \subset \tilde{G}$ and thus leaves them invariant. ∎

Remark 5.27
In the proof of Lemma 5.26 we did not use the fact that Θ does not depend on the second variable n. Indeed, it seems possible to adapt the proof to the general situation of Theorem 5.17. A subtle point is then that we used that $\operatorname{span}\{\mathcal{M}_{\Theta_i}(0)\} \subset K$, which is an additional prerequisite. Moreover, in the Banach space case, we cannot use an orthogonal projection.

Lemma 5.28 (The unit length constraint)
For $d \in \mathbb{N}$ let M be a d-dimensional closed orientable manifold. For $d = 1$ we set $k = 3$, else we set $k > d/2 + 3$. We set $K = H^k(M) \times H^{k-1}(M, \mathbb{R}^{d+1})$, $\tilde{K} = H^{k-4}(M) \times H^{k-3}(M, \mathbb{R}^{d+1})$, $G = H^3(M) \times H^2(M, \mathbb{R}^{d+1})$ and $\tilde{G} = H^{-1}(M) \times L^2(M, \mathbb{R}^{d+1})$. Moreover, we set $C = H^{k-1}(M)$. For $(\varphi, \eta) \in C^\infty_{\mathrm{imm}}(M, \mathbb{R}^{d+1}) \times C^\infty(M, \mathbb{R}^{d+1})$ with $|\eta|_{\mathbb{R}^{d+1}} = 1$, let ν be a unit normal associated to φ and let $U \subset K$ be a neighborhood of $(0, 0)$ such that for all $(f, n) \in U$ the maps $(\varphi, \eta) + (f\nu, n)$ are still immersions. For F as in equation (5.21), let $\mathcal{M} \in C^\omega(U, \tilde{K})$ be the gradient map from Lemma 5.19. Then, the constraint map $\Theta : K \to C$, $(f, n) \mapsto |\eta + n|^2 - 1$ is admissible for F.

Proof. We see that for $X := \{n \in K | \langle n, \eta\rangle_{\mathbb{R}^{d+1}} = 0\}$ the kernel of $\Theta'(0)$ is given by $\ker \Theta'(0) = H^k(M) \times X$ this space is complemented by the space $Y = \{0\} \times (\eta \cdot H^{k-1}(M, \mathbb{R}^{d+1}))$ and we consider the projection $\pi : (f, n) \mapsto (0, n - \eta\langle n, \eta\rangle_{\mathbb{R}^{d+1}})$ that is self-adjoint with respect to the L^2 scalar product and maps $\tilde{K}$ to itself, due to the regularity of η. We observe that

$$|n + \frac{(1 - |\eta + n|)n + \eta}{|\eta + n|}| = 1$$

and thus we choose $\gamma : X \to K$ as

$$\gamma(n) = \frac{n + \eta}{|\eta + n|} - n$$

with derivative $\mathrm{d}\gamma$, that is for $s \in X$ given by

$$\mathrm{d}\gamma(n)s = \frac{s}{|\eta + n|} + \frac{\eta + n}{|\eta + n|^3}\langle \eta + n, s\rangle - s, \tag{5.26}$$

vanishing at $n = 0$. For a function $r \in H^{k-2}(M, \mathbb{R}^{d+1})$ we see that

$$\langle \mathrm{d}\gamma(n)s, r\rangle_{L^2(M,\mathbb{R}^{d+1})} = \langle s, \mathrm{d}\gamma(n)r\rangle_{L^2(M,\mathbb{R}^{d+1})},$$

that is $\mathrm{d}\gamma(n)^*$ maps $\tilde{K}$ to itself. ☐

To establish a Łojasiewicz inequality in the presence of constraints it is necessary to prove the Fredholm property of the respective derivative of the gradient map that we denoted by $\tilde{\mathcal{M}}'(0)$ in the preceding computations.

Proposition 5.29
In the situation of Lemma 5.20 with $|\eta| = 1$ on M, for $C = \mathbb{R} \times \mathbb{R} \times H^{k-1}(M)$, $A_0 = A(\varphi)$ and $V_0 = V(\varphi)$, we set

$$\Theta : K \to C, (f, n) \mapsto (A(\varphi + f\nu_\varphi) - A_0, V(\varphi + f\varphi_\varphi) - V_0, |\eta + n|_{\mathbb{R}^{d+1}} - 1),$$

where A and V are the area and volume functionals as in (3.2) and we assume additionally that φ is not a round sphere.

Then, Θ is an admissible constraint and with $X = \ker \Theta'(0) \subset K$, $\pi_X : K \to K$ the L^2-orthogonal projection onto X as introduced in Lemma 5.26 and Lemma 5.28.Moreover, for

$\tilde{X} = \pi_X \tilde{K}$, $G_X = \pi G$, and $\tilde{G}_X = \pi_X \tilde{G}$, for each $x \in X$ the operator $\tilde{\mathcal{M}}'(x) : X \to \tilde{X}$ has an extension $\tilde{\mathcal{M}}_1(x) \in L(G_X, \tilde{G}_X)$ and $\tilde{\mathcal{M}}'(0)$ and $\tilde{\mathcal{M}}_1(0)$ are Fredholm of index 0.

Proof. We recall that the gradient maps of $\tilde{A} : (f, n) \mapsto (A(\varphi + f\nu_\varphi)$ and $\tilde{V} : (f, n) \mapsto V(\varphi + f\nu_\varphi)$ in a neighborhood of 0. The full gradients of A and V are given by $-H\nu$ and $-\nu$, respectively. In this parametrized setting, we obtain by the same reasoning as in Lemma5.19 the analytic gradient maps

$$\mathcal{M}_{\tilde{A}}(f, n) = -J_f \langle \nu_f, \nu_\varphi \rangle H_f, \quad \mathcal{M}_{\tilde{V}}(f, n) = -J_f \langle \nu_f, \nu_\varphi \rangle.$$

Moreover they take values in the right spaces since H_f involves only two and J_f, ν_f only one derivative.

Therefore, it follows from Lemma 5.26 and Lemma 5.28 that Θ is admissible and we can characterize

$$\ker \Theta'(0) = \left\{ (f, n) \in K \,\middle|\, \int_M f \,\mathrm{d}\mu_\varphi = \int_M f H_\varphi \,\mathrm{d}\mu_\varphi = 0 \text{ and } \langle n, \eta \rangle_{\mathbb{R}^{d+1}} \equiv 0 \right\}.$$

With $X = \ker \Theta'(0)$ and

$$\bar{H}_\varphi = \frac{1}{\int_M 1 \,\mathrm{d}\mu_\varphi} \int_M H_\varphi \,\mathrm{d}\mu_\varphi, \quad \bar{f} = \frac{1}{\int_M 1 \,\mathrm{d}\mu_\varphi} \int_M f \,\mathrm{d}\mu_\varphi$$

the orthogonal L^2-projection onto X is given by

$$\pi_X(f, n) = \big(f - \bar{f} - \frac{(H_\varphi - \bar{H}_\varphi)}{\int_M (H_\varphi - \bar{H}_\varphi \,\mathrm{d}\mu_\varphi} \int_M f(H_\varphi - \bar{H}_\varphi) \,\mathrm{d}\mu_\varphi,\ n - \eta \langle n, \eta \rangle_{\mathbb{R}^{d+1}} \big)$$

Together with the considerations in the proof of Lemma 5.20 it follows that for all $x \in X$ the operator $\tilde{\mathcal{M}}'(x)$ is a differential operator $X \to \tilde{X}$ that has an extension $\tilde{\mathcal{M}}_1(x) : G_X \to \tilde{G}_X$ depending continuously on x.

In the following we prove the Fredholm property of $\tilde{\mathcal{M}}(0)$ and $\tilde{\mathcal{M}}_1(0)$. We follow the ideas of Chill, Fasangova and Schätzle [16, Lemma 3.3] and recall from equation (5.24) that these operators are given by

$$\pi_X^* \big(\mathcal{M}'(0) + \mathrm{d}(\mathrm{d}\gamma^*) \mathcal{M}(0) \big)$$

By inspection of the representations of $\mathrm{d}\gamma^*$ in equation (5.25) and equation (5.26), we observe that the operator $\pi_X^* \circ \mathrm{d}(\mathrm{d}\gamma^*)\mathcal{M}(0)$ maps X and G_X to themselves and therefore it is compact as an operator $X \to \tilde{X}$ and $G_X \to \tilde{G}_X$. Since compact perturbations leave the Fredholm index invariant (cf. Theorem 5.15), we only have to show that the operator $\pi_X^* \circ \mathcal{M}'(0)$ is Fredholm of index 0. We introduce π_X^1 and π_X^2 to denote the two components of π_X Thus, we have to consider for

$$L(f, n) = \begin{pmatrix} \pi_X^1 \Delta(\Delta f + \delta \operatorname{div} n) + f, \\ \pi_X^2 \left(-\Delta n - \delta \nabla(\Delta f + \delta \operatorname{div} n) \right) + n \end{pmatrix}$$

the differential equation

$$L(f, n) = (x, y) \tag{5.27}$$

for $(x, y) \in \tilde{X}$ and $(x, y) \in \tilde{G}_X$. To that account, we introduce a notion of weak solution and find such by the Lax-Milgram theorem. Afterwards, we employ regularity theory for the Laplacian. We introduce the space

$$Y = \pi_X \big(H^2(M) \times H^1(M, \mathbb{R}^{d+1}) \big),$$

the bilinear operator

$$\begin{aligned} B : Y \times Y \to& \mathbb{R} \\ B[(f, n), (g, m)] =& \int (\Delta f + \delta \operatorname{div} n)(\Delta g + \delta \operatorname{div} m) + \lambda \langle \nabla n, \nabla m \rangle + fg + \langle n, m \rangle \,\mathrm{d}\mu \end{aligned}$$

and for $(x, y) \in Y^* \subset (H^2)^* \times (H^1)^*$ the functional

$$K : Y \to \mathbb{R}, \quad K(f, n) = \langle x, f\rangle_{H^{2*}\times H^2} + \langle y, n\rangle_{H^{1*}\times H^1}.$$

We call (f, n) a weak solution of equation (5.27) if for all $(\psi, \rho) \in Y$ it holds that

$$B[(f, n), (\psi, \rho)] = K(\psi, \rho).$$

The functional K and the operator B are continuous on Y and B satisfies the coercivity condition

$$B[(f, n), (f, n)] \geq C\|(f, n)\|_Y^2.$$

Thus, by the Lax-Milgram theorem, for all $(x, y) \in Y^*$ there exists a unique weak solution $(f, n) \in Y$ and $\|(f, n)\|_Y \leq C\|(x, y)\|_{Y^*}$.

To apply regularity theory for the Laplacian [83, Ch. 5, Theorem 11.1], we have to calculate the commutator of π_X and Δ. For $u \in \pi_X^1 C^\infty(M)$ we calculate

$$\begin{aligned}\pi_X^1 \Delta u =& \Delta u - \frac{1}{|M|}\int \Delta u \,\mathrm{d}\mu - \frac{H-\bar{H}}{\|H-\bar{H}\|_{L^2}^2}\int (H-\bar{H})\Delta u \,\mathrm{d}\mu \\ =& \Delta u - \frac{H-\bar{H}}{\|H-\bar{H}\|_{L^2}^2}\int u\Delta(H-\bar{H})\,\mathrm{d}\mu\end{aligned}$$

using Gauß' theorem. Extending the Laplacian to an operator $H^1 \to (H^1)^*$ by density, yielding

$$\langle \Delta f, g\rangle_{(H^1)^*\times H^1} = \langle \nabla f, \nabla g\rangle_{L^2},$$

we see that for all $f \in H^1(M)$ we have $\pi_X^1\Delta f = \Delta f - \frac{H-\bar{H}}{\|H-\bar{H}\|_{L^2}^2}\int f\Delta(H-\bar{H})\,\mathrm{d}\mu$.But the map $f \mapsto \frac{\bar{H}}{\|\bar{H}\|_{L^2}}\int f\Delta\bar{H}\,\mathrm{d}\mu$ is compact on H^1 since its image is one-dimensional. Moreover, the image is contained in $C^\infty(M)$ since $H - \bar{H}$ is smooth. Also for $\pi_X^2\Delta$ the projection induces only a lower order perturbation. To see this, we take $n \in \pi_X^2 H^1(M, \mathbb{R}^{d+1})$ and then by definition we have $\langle \eta, n\rangle_{\mathbb{R}^{d+1}} = 0$ on M. Using this we calculate

$$0 = \eta\Delta\langle\eta, n\rangle = \eta\sum_{i=1}^d \Delta(\eta_i n_i) = \eta\sum_{i=1}^d n_i\Delta\eta_i + \eta_i\Delta n_i + 2\langle\nabla\eta_i, \nabla n_i\rangle$$

thus

$$\eta\langle\eta, \Delta n\rangle = -\eta\langle\Delta\eta, n\rangle - 2\eta\sum_{i=1}^d\langle\nabla n_i, \nabla\eta_i\rangle$$

which only involves first derivatives of n and is therefore for all $k \in \mathbb{N}$ an operator $H^k \to H^{k-1}$. For all $k \in \mathbb{N}\cup\{0\}$, we denote this perturbations by

$$p_1(f) = \frac{\bar{H}}{\|\bar{H}\|_{L^2}}\int f\Delta\bar{H}\,\mathrm{d}\mu$$

and

$$p_2(n) = \eta\langle\Delta\eta, n\rangle + 2\eta\sum_{i=1}^d\langle\nabla n_i, \nabla\eta_i\rangle$$

as maps $\pi_X^1(H^k(M)) \to \bar{H}\cdot\mathbb{R}$ and $\pi_X^2(H^{k+1}(M, \mathbb{R}^{d+1})) \to \times H^k(M, \mathbb{R}^{d+1})$. Therefore, if

$$L(f, n) = (x, y)$$

for $(x, y) \in \pi_X(H^1 * \times L^2)$, we proceed in the same way as in the parabolic case in Section 4.2.2, Improving the regularity of $\Delta f + \delta \operatorname{div} n$, then that of n and lastly that of f. First, we examine the equation

$$\Delta(\Delta f + \delta \operatorname{div}(n)) = x - f + p_1(\Delta f + \delta \operatorname{div})$$

and conclude that $\Delta f + \delta \operatorname{div} n \in H^1(M)$ and

$$\|\Delta f + \delta \operatorname{div} n\|_{H^1} \leq C(\|x - f\|_{H^{1*}} + \|\Delta f + \delta \operatorname{div} n\|_{L^2}) \leq C\|(x, y)\|_{H^{1*} \times L^2}$$

by elliptic regularity theory and the a priori estimate for the weak solution. Secondly, we look at the second equation to find

$$-\Delta n + n = y - p_2(n) + \delta \pi_X^2 \nabla(\Delta f + \delta \operatorname{div} n)$$

and infer $n \in H^2$ with $\|n\|_{H^2} \leq C\|(x, y)\|_{H^{1*} \times L^2}$. Using this, we can improve the regularity of f to $f \in H^3$ and finally have the estimate

$$\|(f, n)\|_{G_X} \leq C\|(x, y)\|_{\tilde{G}_X}$$

For more regular x and y, we can repeat this bootstrapping procedure. This shows that the operator L is an isomorphism of Hilbert spaces and thus the original operators $\tilde{\mathcal{M}}'(0)$ and $\tilde{\mathcal{M}}_1$, that differ from L only by compact perturbations, are Fredholm of index 0. ☐

Proposition 5.30 (Łojasiewicz-Simon inequality for $\tilde{F}$)
In the situation of Lemma 5.29 assume that $\tilde{\mathcal{M}}(0) = 0$. Then, the map $\tilde{F} : U \to \mathbb{R}$ given by $\tilde{F}(x) = F(x + \gamma(x))$ satisfies a Łojasiewicz-Simon inequality around 0. That is there exists $\theta \in [\frac{1}{2}, 1)$,$C > 0$ and $\sigma > 0$ such that for every $(f, n) \in X$ with $\|(f, n)\|_X \leq \sigma$ the inequality

$$\|\tilde{F}(f, n) - \tilde{F}(0)\|^\theta \leq C\|\tilde{\mathcal{M}}(f, n)\|_{\tilde{G}_X}$$

is satisfied.

Proof. The assertion follows again from the result of Feehan and Maridakis stated in Theorem 5.17 in this work applied to the spaces $X \subset \tilde{X}$, $G_X \subset \tilde{G}_X$ that satisfy also $X \subset G_X$ and $\tilde{X} \subset \tilde{G}_X$, since $K, \tilde{K}, G, \tilde{G}$ from Lemma 5.20 have this property. These spaces are closed complemented subspacesof $K, \tilde{K}, G, \tilde{G}$ and turn into Banach spaces when we equip them with the trace topology. The further conditions in Theorem 5.17 are then fulfilled by Lemma 5.19, Proposition 5.25 and Lemma 5.29 above. ☐

To prove a Łojasiewicz inequality for E subject to constraints we argue exactly as in the unconstrained case. For $d \in \mathbb{N}$ let M be a smooth, orientable, closed d-dimensional manifold. For $d = 1$ we set $k = 3$, else we set $k > d/2 + 3$. Let A and V denote the area and volume functionals as in Definition (3.2). For $(\varphi^*, n^*) \in C^\infty(M, \mathbb{R}^{d+1}) \times C^\infty(M, \mathbb{R}^{d+1})$ such that $|n^*| \equiv 1$ and $\varphi^*(M)$ is not a round sphere let U be an open neighborhood of (φ^*, n^*) in $H^k_{\text{imm}} \times H^{k-1}$. Moreover, we denote for $\varepsilon > 0$ by

$$\Gamma = \left\{(\varphi, n) \in B((\varphi^*, n^*), \varepsilon) \subset H^k_{\text{imm}} \times H^{k-1} \middle| \, A(\varphi) = A(\varphi^*), V(\varphi) = V(\varphi^*), |n| \equiv 1\right\}$$

the set of immersions and vector fields in a neighborhood of (φ^*, n^*) fulfilling the constraints. We equip $\Gamma \subset H^k_{\text{imm}}(M, \mathbb{R}^{d+1}) \times H^{k-1}(M, \mathbb{R}^{d+1})$ with the trace topology and set $\tilde{E} = E|_\Gamma$. We observe that the map $\Theta : U \to \mathbb{R} \times \mathbb{R} \times H^k$ given by

$$\Theta(\varphi, n) = \big(A(\varphi) - A(\varphi^*),\ V(\varphi) - V(\varphi^*),\ |n| - 1\big)$$

has the Frechét derivative

$$\Theta'(\varphi,n)[(\psi,\eta)] = \langle H_\varphi \nu_\varphi + \nu_\varphi, \psi\rangle_{L^2} + \langle n,\eta\rangle_{\mathbb{R}^{d+1}}$$

which is surjective. The kernel $\ker\Theta'(\varphi,n)$ is given by

$$\ker\Theta'(\varphi,n) = \big\{(\psi,\eta)\in H^k\times H^{k-1} \big|\ \int \langle\psi, H_\varphi\nu_\varphi + \nu_\varphi\rangle \,\mathrm{d}\mu_\varphi = 0,\ \langle n,\eta\rangle = 0\big\}$$

which is the image of the projection $\pi_{(\varphi,n)} : H^k\times H^{k-1}\to H^k\times H^{k-1}$ given by

$$\pi_{(\varphi,n)}(\psi,\eta) = \Big(\psi - \frac{\nu_\varphi}{|M|}\int\langle\psi,\nu_\varphi\rangle\,\mathrm{d}\mu_\varphi - \frac{\bar H_\varphi\nu_\varphi}{\int|\bar H_\varphi|^2}\int\langle\psi,\bar H_\varphi\nu_\varphi\rangle\,\mathrm{d}\mu_\varphi,\ \eta - n\langle\eta,n\rangle\Big)$$

for $\bar H = H - \frac{1}{|M|}\int H\,\mathrm{d}\mu_\varphi$. Thus, according to Definition 5.23 the differential of $\tilde E$ in (φ,n) is given by

$$\mathrm{d}\tilde E(\varphi,n) = \pi_{(\varphi,n)}E'(\varphi,n).$$

With this notation, we can state the following theorem.

Theorem 5.31 (Łojasiewicz-Simon inequality for $\tilde E$)
For $d\in\mathbb{N}$ let M be a smooth, orientable, closed d-dimensional manifold. For $d=1$ we set $k=3$, else we set $k>d/2+3$. For A and V the area and volume functionals as in Definition (3.2)*. let $(\varphi^*,n^*)\in C^\infty(M,\mathbb{R}^{d+1})\times C^\infty(M,\mathbb{R}^{d+1})$ be a critical point with respect to the constraints*

$$\Theta(\varphi,n) = \big(A(\varphi)-A(\varphi^*),\ V(\varphi)-V(\varphi^*),\ |n|-1\big) = 0$$

of the energy functional $E : H^k_{\mathrm{imm}}(M,\mathbb{R}^{d+1})\times H^{k-1}(M,\mathbb{R}^{d+1})\to\mathbb{R}$ as introduced in Chapter 3. Moreover, assume that $\varphi^*(M)$ is not a round sphere. Then, there exists $\theta\in[\frac12,1)$,$C>0$ and $\sigma>0$ such that for every $(\varphi,n)\in H^k(M,\mathbb{R}^{d+1})\times H^{k-1}(M,\mathbb{R}^{d+1})$ with $\Theta(\varphi,n)=0$ and $\|(\varphi,n)-(\varphi^*,n^*)\|_{H^k(M,\mathbb{R}^{d+1})\times H^{k-1}(M,\mathbb{R}^{d+1})}\le\sigma$ the inequality

$$|E(\varphi,n)-E(\varphi^*,n^*)|^\theta \le C\|\mathrm{d}\tilde E(\varphi,n)\|_{H^{-1}(M,\mathbb{R}^{d+1})\times L^2(M,\mathbb{R}^{d+1})} \tag{5.28}$$

is satisfied.

Proof. We write φ as a graph over φ^* after reparametrization of M by $\Psi : M\to M$ with $\varphi = \varphi^*\circ\Psi + f\nu^*\circ\Psi$. This is possible by Lemma 1.18 which yields the existence of such f together with an estimate $\|f\|_{H^k(M)}\le C\|\varphi-\varphi^*\|_{H^k(M,\mathbb{R}^{d+1})}$. Moreover, observe that the projection π onto X from Lemma 5.29 is the inverse of the map $x\mapsto x+\gamma(x)$ from the admissibility of Θ for F in Lemma 5.29. We write again $\tilde F = F(\mathrm{Id}+\gamma)$. Now we see by Proposition 5.30 and the parametrization invariance of E that

$$\begin{aligned}|E(\varphi,n)-E(\varphi^*,n^*)|^\theta &= |\tilde F(\pi(f,n))-\tilde F(0)|^\theta\\ &\le C\|\tilde{\mathcal{M}}'(\pi(f,n))\|_{H^{-1}(M)\times L^2(M,\mathbb{R}^{d+1})}\\ &= C\|\mathrm{d}\tilde E(\varphi,n)\|_{H^{-1}(M,\mathbb{R}^{d+1},\mu_{\varphi^*})\times L^2(M,\mathbb{R}^{d+1},\mu_{\varphi^*})}\\ &\le C\|\mathrm{d}\tilde E(\varphi,n)\|_{H^{-1}(M,\mathbb{R}^{d+1},\mu_\varphi)\times L^2(M,\mathbb{R}^{d+1},\mu_\varphi)},\end{aligned}$$

and this concludes the proof. □

5.4 Stability of Minimizers

In this section we apply the Łojasiewiczinequalities from the previous section to show stability of minimizers.

Theorem 5.32
For $d \in \mathbb{N}$, let M be an orientable d-dimensional smooth, closed manifold. For $d = 1$ we set $k = 3$, else let $k > d/2 + 3$ be an integer, and let $(\varphi^*, n^*) \in C^\infty(M, \mathbb{R}^{d+1}) \times C^\infty(M, \mathbb{R}^{d+1})$ be a smooth local minimizer of $E : H^k(M, \mathbb{R}^{d+1}) \times H^{k-1}(M, \mathbb{R}^{d+1}) \to \mathbb{R}$ as introduced in Chapter 3, including the length penalization in the case $d = 1$,with respect to any combination of constraints (3.2)*. When we impose the unit-length constraint on n for the physical constants $\lambda > 0, \delta > 0$in E it has to hold $\delta < 2\sqrt{\lambda}$.*

Then there exists $\varepsilon > 0$ such that for all initial data $(\varphi_0, n_0) \in H^k(M, \mathbb{R}^{d+1}) \times H^k(M, \mathbb{R}^{d+1})$ with

$$\|(\varphi_0 - \varphi^*, n_0 - n^*)\|_{H^k(M,\mathbb{R}^{d+1}) \times H^{k-1}(M,\mathbb{R}^{d+1})} < \varepsilon$$

the gradient flow (3.8) *has a global solution $(\varphi, n) : M \times [0, \infty) \to \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$, smooth away from time* 0*, that converges smoothly to a possibly different local minimizer $(\tilde{\varphi}, \tilde{n})$ as $t \to \infty$ and $E(\varphi^*, n^*) = E(\tilde{\varphi}, \tilde{n})$.*

This is the analogue for the energy E from Chapter 3 of the results of Chill, Fasangova, and Schätzle [16, Lemma 4.1], Dall'Acqua, Pozzi and Spener [22, Theorem 1.2] and Lengeler [60, Theorem 2.2]. Thus, the method of proof is very similar. The use of the Łojasiewiczinequality was already proposed by Łojasiewicz himself [67] for ordinary differential equations in $\mathbb{R}^n$, we have to adapt the argument to the infinite dimensional setting.

Proof of Theorem 5.32. First we observe, that in the case where volume and area are fixed and the initial immersion is a round sphere, it will remain so and not move at all. Then the convergence in n follows from the analogous result for the harmonic map flow [33, Section 3]. Therefore, we can assume in the following that φ^* is not a round sphere, when the volume and area are fixed. In the following calculations we write E to mean either E or $\tilde{E}$ since the argument is identical in both cases.

Choose $\sigma > 0$ such that an immersion $\varphi \in H^k$ can be written as a graph over φ^* whenever $\|\varphi - \varphi^*\|_{H^k} \leq \sigma$. Let $(\varphi, n) : M \times [0, T^*) \to \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ be a smooth solution of the flow equation (3.8) with

$$T^* = \max_{t \in \mathbb{R}} \{\forall s \leq t | \ \|\big((\varphi, n) - (\varphi^*, n^*)\big)(s)\|_{H^k(M,\mathbb{R}^{d+1}) \times H^k(M,\mathbb{R}^{d+1})} \leq \sigma\}.$$

That such a solution exists, can be seen from short-time existence Theorem 4.19. As long as the solution is bounded in the phase space H^k, it can be continued. In the following, we prove $T^* = \infty$.

Observe that the smoothness of the solution justifies the following calculation. In particular it allows us to use a chain rule to determine the time derivative of $E(\varphi, n)$ and grants the necessary regularity of the L^2 gradient map denoted by ∇E given by equation (5.2).Using the gradient flow property and the Łojasiewicz-Simon inequalities (5.23) and (5.28) we find

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}|E(\varphi, n) - E(\varphi^*, n^*)|^\theta &= -\theta|E(\varphi, n) - E(\varphi^*, n^*)|^{\theta-1}\langle \nabla E(\varphi, n), (\partial_t\varphi, \partial_t n)\rangle_{L^2} \\ &= \theta|E(\varphi, n) - E(\varphi^*, n^*)|^{\theta-1}\|\nabla E(\varphi, n)\|_{L^2}\|(\partial_t\varphi, \partial_t n)\|_{L^2} \\ &\geq \frac{\theta}{C}\|(\partial_t\varphi, \partial_t n)\|_{L^2}. \end{aligned} \tag{5.29}$$

Hence, we can estimate

$$\begin{aligned}
&\|(\varphi(t),n(t))-(\varphi^*,n^*)\|_{L^2}\\
&=\|(\varphi(0),n(0))+\int_0^t(\partial_t\varphi(s),\partial_t n(s))\mathrm{d}s-(\varphi^*,n^*)\|_{L^2}\\
&\le\|(\varphi(0),n(0))-(\varphi^*,n^*)\|_{L^2}+\int_0^t\|(\partial_t\varphi(s),\partial_t n(s))\|_{L^2}\mathrm{d}s\\
&\le\|(\varphi(0),n(0))-(\varphi^*,n^*)\|_{L^2}-C\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}|E(\varphi(s),n(s))-E(\varphi^*,n^*)|^\theta\mathrm{d}s\\
&\le\|(\varphi(0),n(0))-(\varphi^*,n^*)\|_{L^2}\\
&\quad-C\big(|E(\varphi(t),n(t))-E(\varphi^*,n^*)|^\theta-|E(\varphi(0),n(0))-E(\varphi^*,n^*)|^\theta\big)\\
&\le\|(\varphi(0),n(0))-(\varphi^*,n^*)\|_{L^2}+C|E(\varphi(0),n(0))-E(\varphi^*,n^*)|^\theta\\
&\le C\|(\varphi(0),n(0))-(\varphi^*,n^*)\|^\theta_{C^2\times C^1}.
\end{aligned}$$

This means that the solution will remain inside a small enough L^2-ball around a local minimizer. This, however, does not exclude a blow up of some higher derivative. At this point, we employ the short-time existence result from Theorem 4.19 and the parabolic smoothing effect explained in Theorem 4.30 to show that the solution will also remain in a small H^k-ball around (φ^*,n^*), in particular with radius small than $\sigma/2$.

Now, we use Lemma 1.20 to reformulate the problem again in terms of a height function f as for the proof of short-time existence. Observe that then the stationary immersion corresponds to $f=0$.For $\tilde\sigma$ sufficiently small, there exists a reparametrization Ψ of M and a function $f\in H^k(M)$ with $\|f\|_{H^k(M)}\le\sigma$ such that

$$\varphi\circ\Psi=\varphi^*+f\nu^*.$$

From Theorem 4.30 (observe Remark 4.31 in the presence of constraints, where we need $\delta<2\sqrt{\lambda}$)we know that as long as f is small in H^k, it will also be bounded in stronger norms. By interpolation, we obtain the following estimate for the H^k norm,

$$\begin{aligned}
&\|(f,n)\|_{H^k}\le C\|(f,n)\|^\beta_{L^2}\|(f,n)\|^{1-\beta}_{H^{k+1}}\\
&\le\|(f,n)\|^{1-\beta}_{H^{k+1}}\|(f(0),n(0))-(0,n^*)\|^{\beta\theta}_{C^2\times C^1}\le C(\sigma,k)\varepsilon^{\beta\theta}\le\frac{\sigma}{2}
\end{aligned}$$

on $[T^*/2,T^*)$ for ε small enough. Therefore, $T^*=\infty$ and the flow exists globally.

Now we prove convergence to a local minimizer. From (5.29) we infer

$$(\partial_t f,\partial_t n)\in L^1\big(0,\infty;L^2(M)\times L^2(M,\mathbb{R}^{d+1})\big)$$

and thus $(f,n)\to(\tilde f,\tilde n)$ in L^2. Theorem 4.30 gives boundedness in all $H^k(M)\times H^k(M,\mathbb{R}^{d+1})$. By weak compactness and compact embeddings we conclude for any $k\in\mathbb{N}$ convergence in $H^k(M)\times H^k(M,\mathbb{R}^{d+1})$ and hence smooth convergence. Moreover it holds then, that $E(\varphi^*,n^*)=E(\varphi^*+\tilde f\nu^*,n^*+\tilde n)$ by application of the Lojasiewicz-Simon inequality (cf. the proof of Lemma 4.1 in [16, pp. 359-362]). □

Remark 5.33

In the results of this section, we assumed throughout the smoothness of stationary points. In the curve case, this is justified by Theorem 3.1. In space dimension $d=2$ there are results [78, 81] concerning the regularity of minimizers of the Willmore functional. But it is by no means clear, if and how such proof could be adapted. In higher space dimensions the situation becomes even more involved.

Employing elliptic and parabolic estimates for partial differential equations with non-smooth coefficients, one might be able to weaken the assumption of smoothness along the lines of [33, Section 3]. The minimal regularity needed for the proof would then probably be comparable to the minimal regularity of initial data for the short-time existence result in Theorem 4.29.

A

Numerical Experiments

In this chapter we discuss numerical computations that approximate the behavior of curves and vector fields under geometric evolution equations. Of course, Bartels, Dolzmann, Nochetto and Raisch [7] present a finite element method for the flow of two dimensional surfaces and it is by no means the aspiration of this chapter to improve their techniques, we rather aim for a flexible method to visualize different curvature flows, relying on Matlab's built-in solvers for ordinary differential equations. Observe that the functionality of Matlab and Octave concerning ordinary differential equations is slightly different.

We approximate all derivatives by simple difference quotients, to obtain an approximation of the normal velocity. Then, we follow the idea of Elliottand Fritz [30] and add tangential motion according to the harmonic map heat flow, to avoid mesh degeneracy. This is fairly simple for curves, since the reference manifold is $\mathbb{S}^1$ which is intrinsically flat.

For the presented simulations the parameters λ and δ in the energy (3.1) were chosen as follows. While λ was always set to 1 playing with different values of δ and different combinations of the constraints from(3.2) we explore the behavior of the flow. Observe that the depicted curves are all scaled to the same size.

In Figure A.1, we do not impose constraints and start with a randomly perturbed circle. The energy decreases very rapidly and a round shape is approached.

In Figures A.2 and A.3, we start with ellipses. Without constraints a circular shape is approached. When volume and area are constrained a dumbbell-like shape is attained.

In Figure A.4 also the length of the vector field is restricted and we start with a randomly perturbed initial configuration.

In Figure A.5 we start with a perturbed curve. The unperturbed curve is parametrized by $\gamma(t) = (4\sin(t), \cos(3t) - \cos(t))$.

A particularly interesting evolution is presented in Figure A.6. For $\delta = 3$ with all constraints imposed and initial random perturbation of the symmetric configuration it takes comparably long until the symmetry is broken entirely.

We include the code for the curve diffusion flow of closed curves. The essential steps can be seen already here and the reader might adapt it.

```
main.m
% We start by generating an initial curve
J=128;
h=1/J;
x = 0:h:1-h;
c = [sin(2*pi*x),(cos(6*pi*x) - cos(2*pi*x))/4];
SIZE = size(c)/2;
N = int16(numel(c)/2);

% It is useful to provide events to the ODE routine,
% e.g. when the curve vanishes or a singularity occurs.
% Moreover, we fix a maximal time-step size.
options = odeset('Events', @CurvatureBlowUp, 'MaxStep', 1e-2);

% The evolution law is prescribed in cdf.m
EvoFun = @(t,X) cdf(X.',h);
T_max = 1;

% ode15s is a special routine for stiff problems that uses an implicit method,
% observe that the initial value has to be a column vector.
[t, y] = ode15s(EvoFun, [0 T_max], c.', options);

% The following loop draws the evolving curve
%after a certain number of time steps
f = figure;
set(f, 'Units', 'normalized', 'Position', [0.2, 0.1, 0.7, 0.7]);
axis equal;
for ITER = 1:ceil(numel(t)/30):numel(t)
plot(y(ITER,1:N), y(ITER,(N+1):2*N),'-x');
xlim([-1.5 1.5]);
axis equal
pause(0.2);
end;
```

```
cdf.m
% This function implements the actual CDF with tangential correction
function [ Vec ] = cdf(c , h)
% An index must be an integer
N = int16(numel(c)/2);

% the elements in c are rearranged,
% so that they correspond to x and y-coordinates
c = [c(1:N); c((N+1):2*N)];

% the derivative of c is approximated by difference quotients
c_t = ([c(:,2:N),c(:,1)] - [c(:,N),c(:,1:N-1)])/(2*h);

% We calculate the necessary geometric quantities
v = sqrt(c_t(1,:).^2 + c_t(2,:).^2);
v2 = v.^2;
tang = [c_t(1,:)./v;c_t(2,:)./v];
norm = [-tang(2,:);  +tang(1,:)];
c_tt = ([c(:,2:N),c(:,1)] + [c(:,N),c(:,1:N-1)] - 2*c)/h/h;
H = (c_t(1,:).*c_tt(2,:) - c_tt(1,:).*c_t(2,:))...
./((c_t(1,:).^2 + c_t(2,:).^2).^(3/2));
H_t = ([H(:,2:N),H(:,1)] - [H(:,N),H(:,1:N-1)])/(2*h);
H_tt = ([H(:,2:N),H(:,1)] + [H(:,N),H(:,1:N-1)] - 2*H)/h/h;
H_ss =  H_tt./(v2) - H_t.*((c_t(1,:).*c_tt(1,:)...
+ c_t(2,:).*c_tt(2,:))./(v2.^2));

% The normalspeed for the cdf is given by
normalspeed = -H_ss;

% We add the tangential correction c_tt - v2*H*norm.
Vec = [((normalspeed - v2.*H).*norm(1,:) + c_tt(1,:));...
((normalspeed - v2.*H).*norm(2,:)+ c_tt(2,:))];
Vec = reshape(Vec.',[2*N,1]);
```

```
CurvatureBlowUp.m
% This event tells ode15s to stop, based on a blow-up criterion for the CDF
function [CurvEn, isterminal, direction] = CurvatureBlowUp(t,y)
SIZE = size(y);
c = reshape(y.', [SIZE(1)/2, 2]).';
SIZE = size(c);
N = int16(numel(c)/2);
c_t = ([c(:,2:N),c(:,1)] - [c(:,N),c(:,1:N-1)])/2;
v = sqrt(c_t(1,:).^2 + c_t(2,:).^2);
v2 = v.^2;
tang = [c_t(1,:)./v;c_t(2,:)./v];
norm = [-tang(2,:);  +tang(1,:)];
c_tt = ([c(:,2:N),c(:,1)] + [c(:,N),c(:,1:N-1)] - 2*c);
H = (c_t(1,:).*c_tt(2,:) - c_tt(1,:).*c_t(2,:))...
./((c_t(1,:).^2 + c_t(2,:).^2).^(3/2));
dE = H.^2.*v;
CurvEn = 1e2 - sum(dE);
isterminal = 1;
direction = 0;
```

The pictures in Figure A.7 have been generated with the above code. With the necessary adaptions also the curve shortening flow can be implemented as depicted in Figure A.8. Observe that due to the use of *events* the ODE-routine stops automatically, when a singularity occurs. The code for the implementation of the generalized Helfrich flow is essentially the same, with the necessary changes due to the additional variable n.

A simulation of the flow according to equation (2.3) is depicted in Figure A.9. From the initial dumbbell shape it becomes circular before it shrinks to a point.

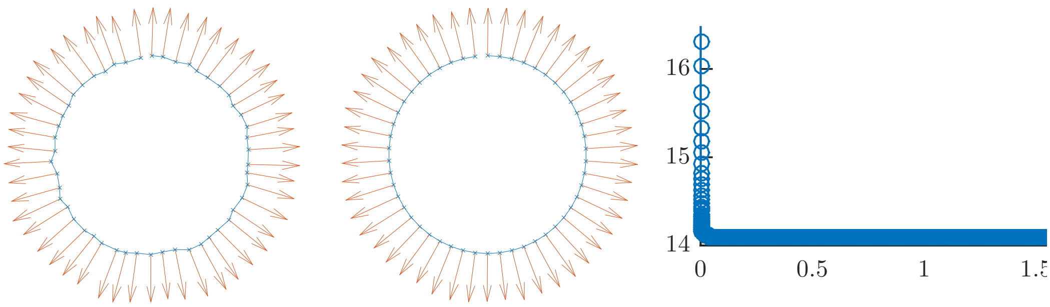

Figure A.1: The generalized Helfrich flow with volume and unit-length constraint. An initial random perturbation is smoothed out corresponding to an initially rapid decrease of the energy.

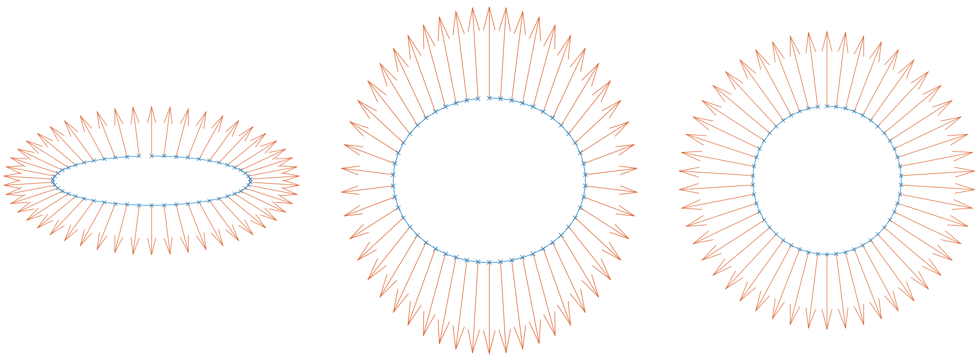

Figure A.2: Without constraints the generalized Helfrich flow evolves an ellipse into a circle and the vector field to the normal.

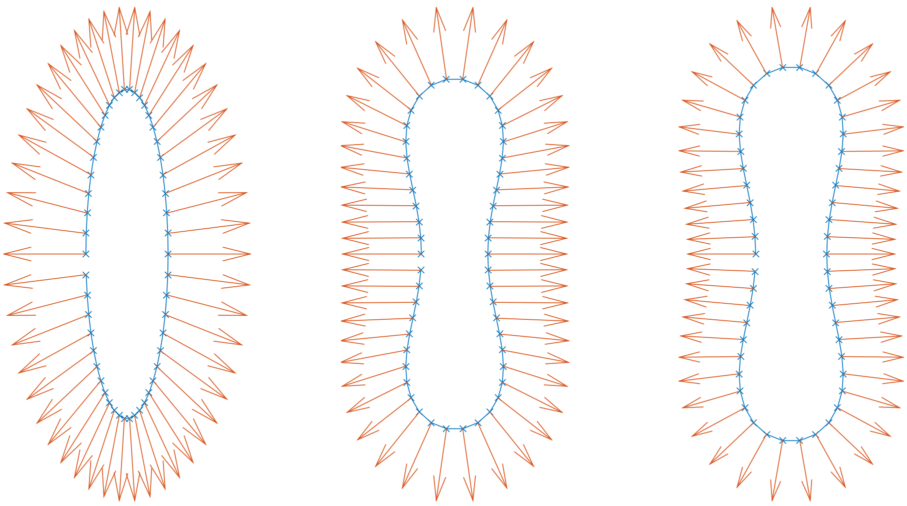

Figure A.3: With volume and area constraint the ellipse evolves into a dumbbell. The vector field changes most in the part of the curve where the curvature is large.

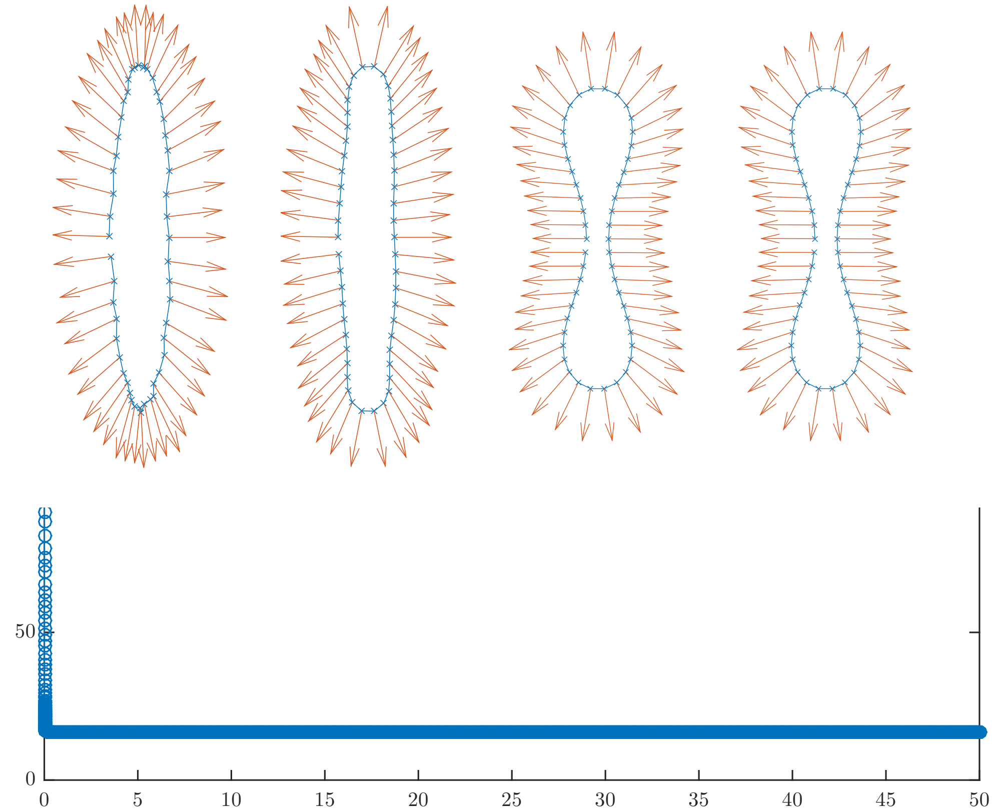

Figure A.4: The generalized Helfrich flow preserving length and enclosed volume of the curve and with fixed length of the vector field. The energy decreases very fast in the beginning smoothing out the initial random perturbation.

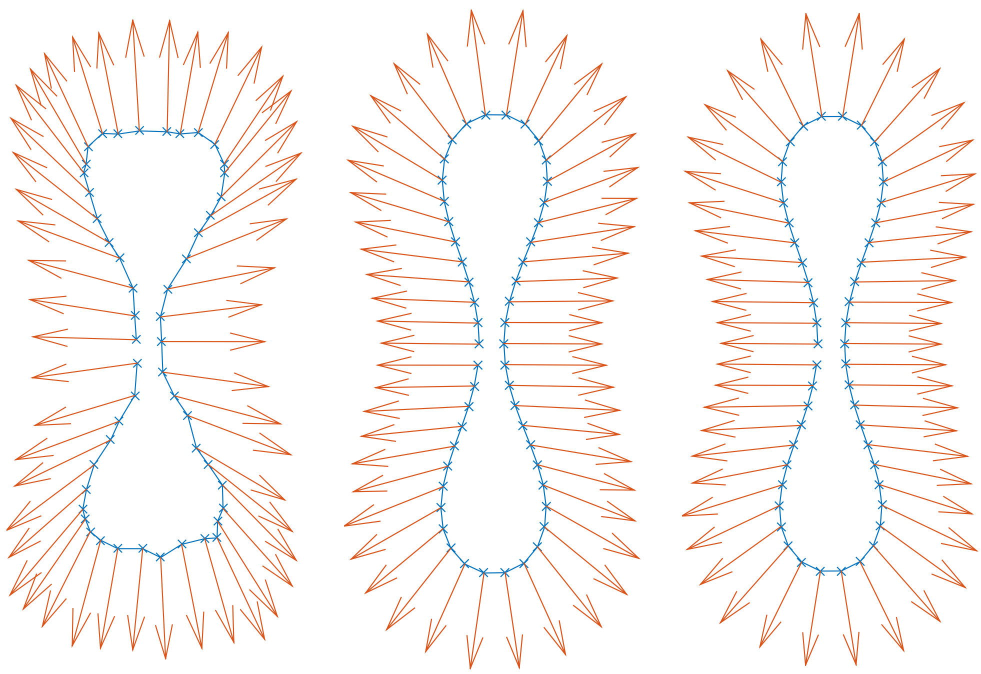

Figure A.5: The generalized Helfrich flow for $\delta = 1$ subject to length, area and unit-length constraint. The initial perturbation is smoothed out and the symmetry is preserved.

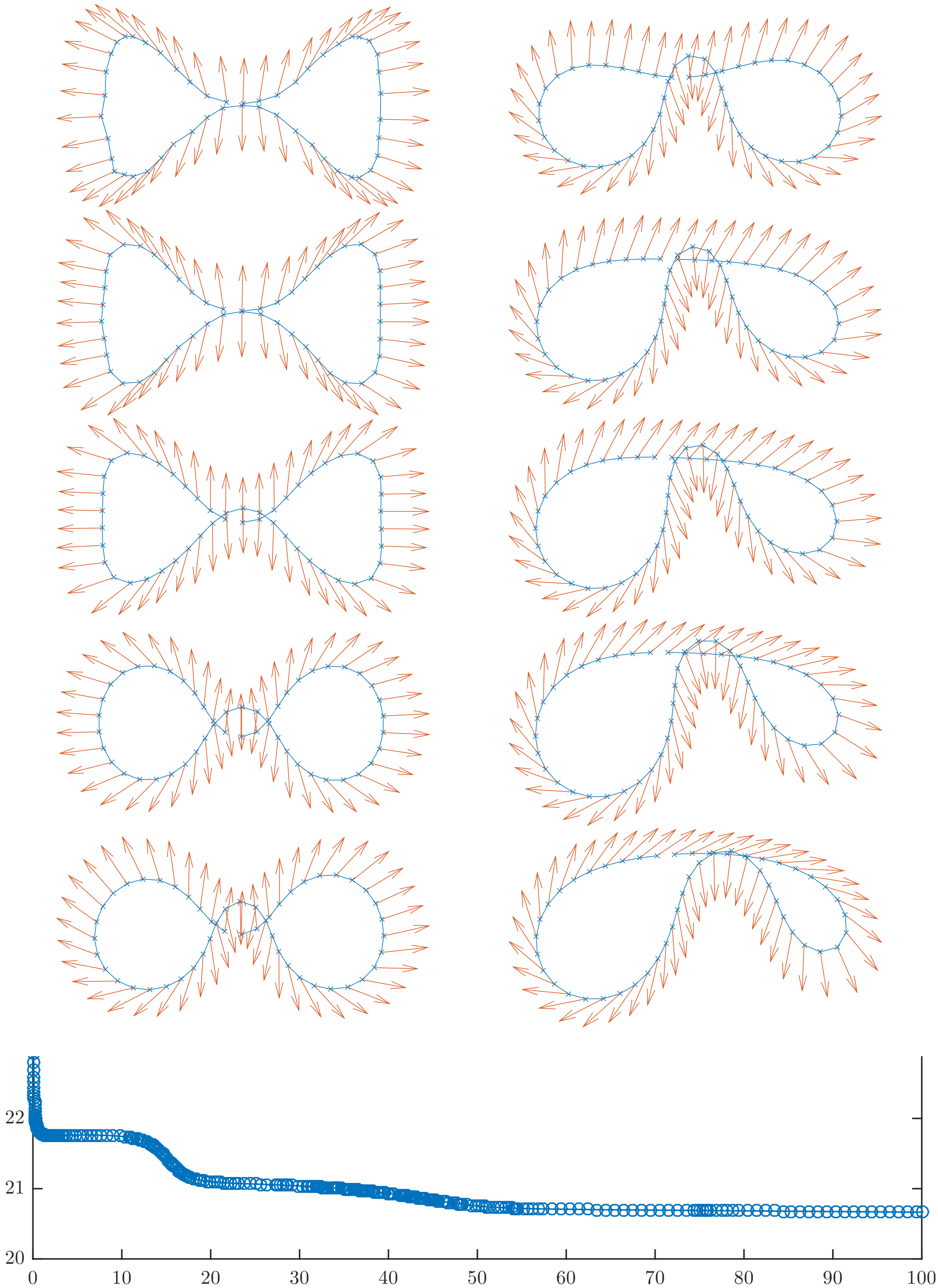

Figure A.6: The generalized Helfrich flow for $\delta = 3$ subject to length, area and unit-length constraint. The initial perturbation is smoothed out almost immediately, but it takes long until an apparently stable configuration is reached.

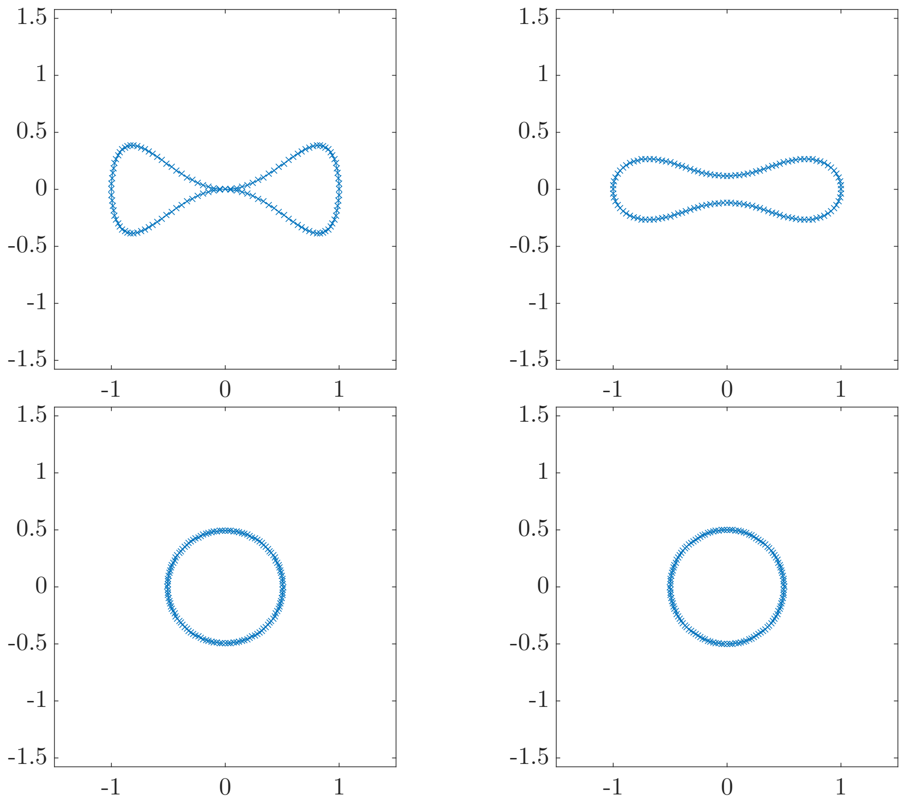

Figure A.7: The curve diffusion flow reduces the length but preserves the enclosed volume.

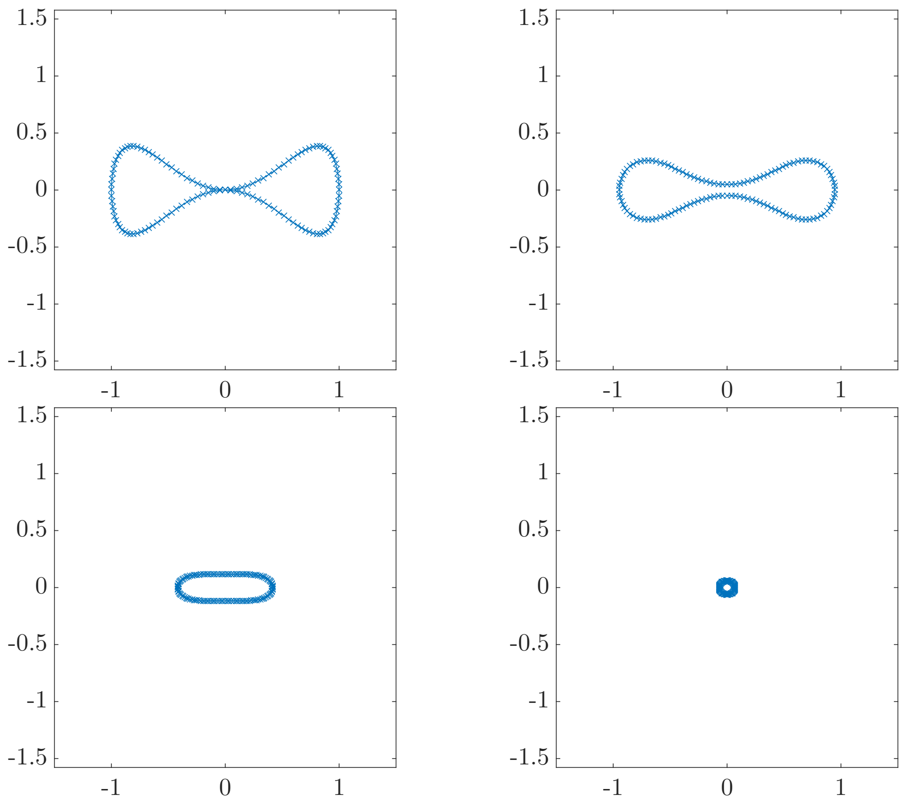

Figure A.8: The curve shortening flow is the L^2-gradient flow of the length.

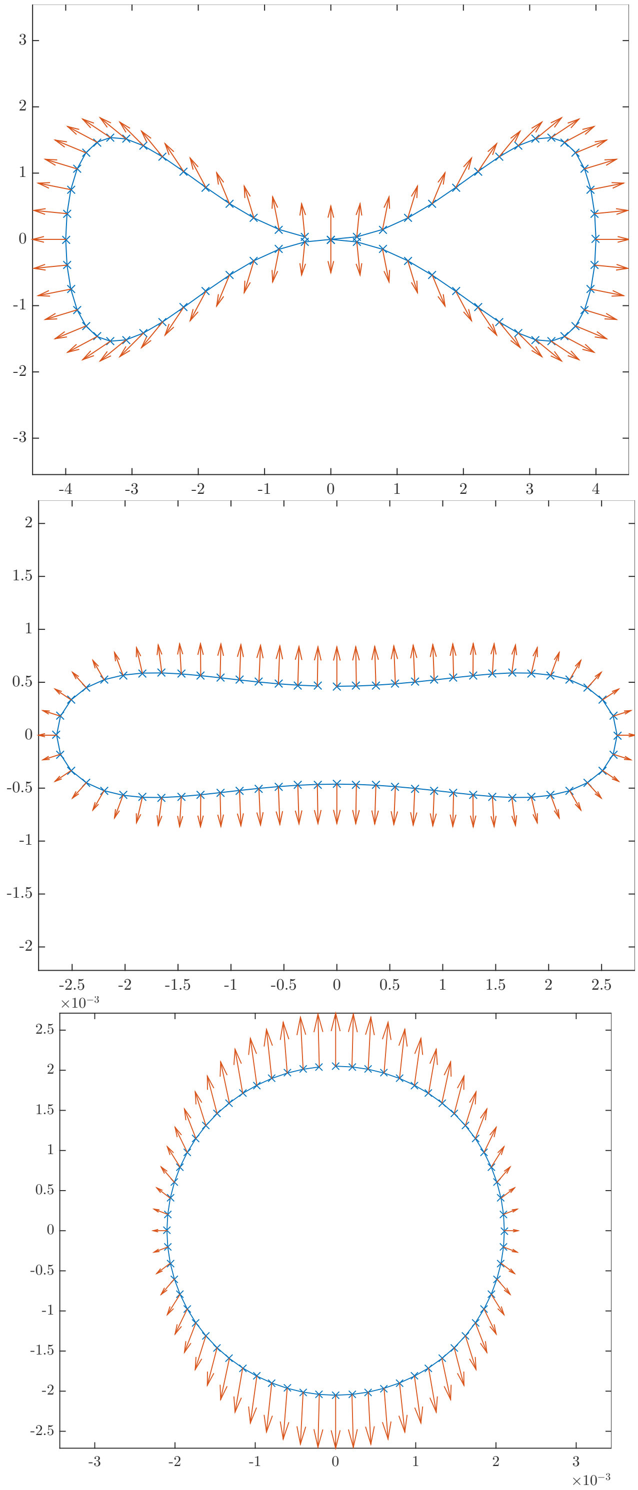

Figure A.9: The flow according to equation (2.3). Observe that the last picture is scaled by a factor 10^3.

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