HOCHSCHILD AND CYCLIC HOMOLOGY FOR BORNOLOGICAL COARSE SPACES

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Die Arbeit wurde angeleitet von: Prof. Dr. Ulrich Bunke

Prüfungsausschuss:	Vorsitzender:	Prof. Dr. Helmut Abels
	1. Gutachter:	Prof. Dr. Ulrich Bunke
	2. Gutachter:	Prof. Dr. Denis-Charles Cisinski
	weiterer Prüferin:	Prof. Dr. Clara Löh

Summary

The main goal of the thesis is to construct equivariant coarse versions of the classical Hochschild and cyclic homologies of algebras. These are lax symmetric monoidal functors from the category of equivariant bornological coarse spaces to the cocomplete stable ∞-category of chain complexes and are called *equivariant coarse Hochschild and cyclic homology*. If k is a field, the evaluation at the one point bornological coarse space induces equivalences with the classical Hochschild and cyclic homologies of k. In the equivariant setting, the G-equivariant coarse Hochschild (cyclic) homology of (a canonical bornological coarse space associated to) the group G agrees with the classical Hochschild (cyclic) homology of the associated group algebra $k[G]$.

The second aim of the thesis is the construction of natural transformations from equivariant coarse algebraic K-homology to equivariant coarse Hochschild and cyclic homology, and of natural transformations from equivariant coarse Hochschild and cyclic homology to equivariant coarse ordinary homology. This is achieved by using trace-like maps and gives a natural transformation from equivariant coarse algebraic K-homology to equivariant coarse ordinary homology.

We conclude the dissertation with two additional investigations: we give a comparison result between the forget-control map for equivariant coarse Hochschild homology and the associated generalized assembly map, and we show a Segal-type localization theorem for equivariant Hochschild and cyclic homology.

Contents

Introduction

Coarse geometry is the study of metric spaces from a large-scale point of view, meaning that one wants to capture the behavior at infinity of metric spaces, discrete groups or more general spaces. The general principle, in coarse geometry, is that two spaces are considered equivalent if they "look the same from great distances". This viewpoint has lead Roe to the abstract notion of *coarse spaces*, objects that encode the large-scale geometry and properties of metric spaces [Roe93, Roe96, Roe03].

In order to apply homological methods in coarse geometry and perform index theory on non-compact manifolds, a large-scale analogue of ordinary (co-)homology has also been introduced by Roe and has been called *coarse (co-)homology.* This coarse cohomology theory, together with a coarse version of topological K-theory, provided new invariants and have applications in index theory, homotopy theory, K-theory of C^*-algebras and, maybe most importantly, in studying assembly map conjectures [Roe93, Mit01, BEKW18]. Among the applications of coarse geometry to other mathematical fields there is, *e.g.,* Mostow's proof of his rigidity theorem [Mos73]. For further references and a survey on coarse geometry we refer to [Roe03].

A new axiomatic and homotopic approach to coarse geometry and coarse algebraic topology has been recently developed by Bunke and Engel [BE16], and then generalized to the equivariant framework by Bunke, Engel, Kasprowski and Winges [BEKW17]. In their set-up, the main objects are called *bornological coarse spaces*: these are triples $(X, \mathcal{C}, \mathcal{B})$ consisting of a set X, equipped with a bornology $\mathcal{B}$ (describing the bounded subsets of X) and a coarse structure $\mathcal{C}$ (a family of neighborhoods of the diagonal in $X \times X$ encoding the large-scale properties of X), with bornology and coarse structure being compatible in a suitable way [BE16, Def. 2.1]. Every metric space is a bornological coarse space in a canonical way, and, more generally, a coarse space, as defined by Roe, is a coarse space in this framework.

Let G be a group. Among different invariants of (G-equivariant) bornological coarse spaces, *i.e.,* bornological coarse spaces with a G-action by automorphisms, there are *(equivariant) coarse homology theories*, *i.e.,* functors

$$E\colon G\mathbf{BornCoarse} \to \mathbf{C}$$

from the category of G-bornological coarse spaces $G\mathbf{BornCoarse}$ to a cocomplete stable ∞-category $\mathbf{C}$ (*e.g.*, the ∞-categories of chain complexes or spectra), satisfying some additional axioms: coarse invariance, flasqueness, coarse excision and u-continuity [BEKW17, Def. 3.10]. Examples of coarse homology theories arise as coarsifications of locally finite homology theories [BE16]. Among others, there are coarse versions of ordinary homology and of topological K-theory [BE16], of equivariant algebraic K-homology and of topological Hochschild homology [BEKW17, BC17], and of Waldhausen's A-theory [BKW18].

This new homotopic approach to coarse geometry has brought new applications and insight (for example, the dualizing spectrum of a group is now known to be a coarse invariant [BE17b]), and it is now developing as a new field of research [BC17, BE17a, BE17b, BE17c, Bun18, BKW18].

Hochschild and cyclic homology for bornological coarse spaces

Classically, Hochschild and cyclic homology are homological invariants of algebras [Lod98], and have been extended to dg-algebras, schemes and, more generally, additive categories and exact categories [McC94, Kel99]. The goal of this thesis is to construct (equivariant) Hochschild and cyclic homology theories

$$\mathcal{X}\mathrm{HH}_k^G\colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty \quad \text{and} \quad \mathcal{X}\mathrm{HC}_k^G\colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$$

(as described below) for bornological coarse spaces within the homotopic framework developed by Bunke and Engel [BE16, BEKW17].

Our main motivation comes from equivariant coarse algebraic K-homology [BEKW17, BC17], as we now explain. Various versions of algebraic K-theory come equipped with trace maps (*e.g.,* the Dennis trace map from algebraic K-theory of rings to Hochschild homology, or the refined version, the cyclotomic trace, from the algebraic K-theory spectrum to the topological cyclic homology spectrum) and these trace maps have been of fundamental importance in the understanding of algebraic K-theory; see, for example, [DGM13]. A version of algebraic K-theory for bornological coarse spaces has recently also been defined [BEKW17]; hence, inspired by the classical case, our aim is to define coarse versions of Hochschild and cyclic homology and to define trace maps from equivariant coarse algebraic K-homology to equivariant coarse Hochschild and cyclic homology.

We now present the main result of the thesis. Let k be field and let G be a group. The symbols HH and HC refer to Hochschild and cyclic homology; we denote by C_*^{HH} and C_*^{HC} the chain complexes computing Hochschild homology and cyclic homology (of k-algebras) respectively. We deonte by $\{*\}$ the one point bornological coarse space; the G-equivariant bornological coarse space $G_{\mathrm{can,min}}$ denotes a canonical equivariant bornological coarse space associated to the group G (see Example 1.1.22).

The main result of the dissertation is the following theorem (see Definition 3.4.6, Theorem 3.4.7, Proposition 4.1.3 and Proposition 4.1.4):

Theorem. *There are functors*

$$\mathcal{X}\mathrm{HH}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty \quad \textit{and} \quad \mathcal{X}\mathrm{HC}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$$

from the category of G-equivariant bornological coarse spaces to the cocomplete stable ∞-category $\mathbf{Ch}_\infty$ of chain complexes, satisfying the following properties:

(i) $\mathcal{X}\mathrm{HH}_k^G$ *and* $\mathcal{X}\mathrm{HC}_k^G$ *are G-equivariant coarse homology theories;*

(ii) there are equivalences of chain complexes

$$\mathcal{X}\mathrm{HH}_k^G(*) \simeq C_*^{\mathrm{HH}}(k) \quad \textit{and} \quad \mathcal{X}\mathrm{HC}_k^G(*) \simeq C_*^{\mathrm{HC}}(k)$$

between the evaluations of $\mathcal{X}\mathrm{HH}_k^G$ *and* $\mathcal{X}\mathrm{HC}_k^G$ *at the one point bornological coarse space $\{*\}$, endowed with the trivial G-action, and the chain complexes computing Hochschild and cyclic homology of k;*

(iii) there are equivalences

$$\mathcal{X}\mathrm{HH}_k^G(G_{\mathrm{can,min}}) \simeq C_*^{\mathrm{HH}}(k[G];k) \quad \textit{and} \quad \mathcal{X}\mathrm{HC}_k^G(G_{\mathrm{can,min}}) \simeq C_*^{\mathrm{HC}}(k[G];k)$$

of chain complexes between the evaluations at the G-bornological coarse space $G_{\mathrm{can,min}}$ and the chain complexes computing Hochschild and cyclic homology of the k-algebra $k[G]$.

The functors $\mathcal{X}\mathrm{HH}_k^G$ and $\mathcal{X}\mathrm{HC}_k^G$ are called *equivariant coarse Hochschild homology* and *equivariant coarse cyclic homology* respectively. The names are justified by the use of a suitable Hochschild homology (and cyclic homology) functor in their construction, and by the comparison results: the evaluation at the one point space and at the (canonical G-bornological coarse space associated to the) group G agree with the classical Hochschild and cyclic homologies of the base field k and of the group algebra $k[G]$.

Our construction of the functors $\mathcal{X}\mathrm{HH}_k^G$ and $\mathcal{X}\mathrm{HC}_k^G$ uses a cyclic homology theory for additive exact categories that satisfies certain additive and localizing properties in the sense of Tabuada [Tab07]; see also [CT11]. This is *Keller's cone construction* for exact categories [Kel99]

$$C \colon \mathbf{Ex} \to \mathbf{Mix}, \tag{0.0.1}$$

which is a functor from the category $\mathbf{Ex}$ of small exact categories to Kassel's category $\mathbf{Mix}$ of mixed complexes [Kas87]. As shown by Kassel, Hochschild and cyclic homologies can be defined in terms of mixed complexes, consistently with the classical definitions for k-algebras [Lod98]. Hochschild and cyclic homologies of the Keller's mixed complex $C(\mathcal{A})$, associated to a small exact category $\mathcal{A}$, will be called *Keller's Hochschild and cyclic homology of $\mathcal{A}$*.

To every G-bornological coarse space X, we can associate a suitable k-linear category; this is the k-linear category $V_k^G(X)$ of G-equivariant X-controlled (finite dimensional) k-vector spaces ([BEKW17], or Definition 2.1.2). In fact, this correspondence is functorial in X, and describes a functor

$$V_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Cat}_k$$

from the category of G-bornological coarse spaces to the category $\mathbf{Cat}_k$ of small k-linear categories [BEKW17]. The k-linear category $V_k^G(X)$ is then equipped with the exact structure given by the short split exact sequences. We define equivariant coarse Hochschild (and cyclic) homology $\mathcal{X}\mathrm{HH}_k^G$ and $\mathcal{X}\mathrm{HC}_k^G$, as Keller's Hochschild and cyclic homology of the k-linear category $V_k^G(X)$ equipped with the exact structure given by the short split exact sequences. In Theorem 3.4.2, we show that these compositions of functors satisfy the axioms describing a coarse homology theory.

Keller's functor C satisfies several properties, like an additivity and a localization property [Kel99]. We review them in Section 3.2. The localization property [Kel99, Thm. 1.5], in particular, is fundamental to us in order to prove coarse excision (Theorem 3.3.8), one of the axioms defining a coarse homology theory. A simpler definition of Hochschild and cyclic homologies for exact k-linear categories is due to McCarthy: his definition uses the *additive cyclic nerve* $\mathrm{CN}(\mathcal{A})$ of the k-linear category $\mathcal{A}$ [McC94]. However, the homology theories defined by McCarthy do not (a priori) satisfy the localization property [Kel99, Example 1.8, Example 1.9]: a cyclic homology functor satisfying localization has to take also negative values, and McCarthy's Hochschild and cyclic homologies are only positively graded. However, when restricted to additive categories with the split exact structure, Keller's and McCarthy's definitions of Hochschild and cyclic homologies are equivalent (see also Lemma 3.4.4 and Remark 3.4.5 for a discussion about it). This has led us to define two intermediate coarse homology theories

$$\mathcal{X}\mathrm{Mix}_k^G \colon G\mathbf{BornCoarse} \xrightarrow{V_k^G} \mathbf{Cat}_k \xrightarrow{\mathrm{Mix}} \mathbf{Mix} \xrightarrow{loc} \mathbf{Mix}_\infty$$

of Definition 3.4.1 and

$$\widetilde{\mathcal{X}C}_k^G \colon G\mathbf{BornCoarse} \xrightarrow{V_k^G} \mathbf{Cat}_k \longrightarrow \mathbf{Ex}_k \xrightarrow{C} \mathbf{Mix} \xrightarrow{loc} \mathbf{Mix}_\infty$$

of Definition 3.3.1; here, the category $\mathbf{Mix}_\infty$ is Kassel's ∞-category of mixed complexes, the functor C is Keller's cone construction (0.0.1) and the functor Mix is the functor that associates a canonical mixed complex to the additive cyclic nerve of a k-linear category. Finally, the functor $\mathbf{Cat}_k \to \mathbf{Ex}_k$ sends a k-linear category to the same k-linear category equipped with the split exact structure.

The functors $\mathcal{X}\mathrm{Mix}_k^G$ and $\widetilde{\mathcal{X}C}_k^G$ provide equivalent coarse homology theories. Coarse

Hochschild and cyclic homology are then defined by post-composition of Hochschild and cyclic homology for mixed complexes with either $\mathcal{X}\mathrm{Mix}_k^G$ or $\widetilde{\mathcal{X}C}_k^G$; in this thesis, we decide to use the functor $\mathcal{X}\mathrm{Mix}_k^G$ (see Section 3.4), but we prove most of the theorems by using the functor $\widetilde{\mathcal{X}C}_k^G$ (see Remark 3.3.3).

Trace-like maps

The construction of equivariant coarse Hochschild homology follows the same ideas of [BEKW17], where the equivariant coarse algebraic K-homology $\mathcal{X}\mathbf{A}K^G$ (with values in an additive category $\mathbf{A}$ with strict G-action) has been defined as the non-connective K-theory of the additive category $V_{\mathbf{A}}^G(X)$ of G-equivariant X-controlled $\mathbf{A}$-objects (see Definition 2.1.2). When $\mathbf{A}$ is the category of finitely dimensional k-vector spaces, we denote by $\mathrm{K}\mathcal{X}_k^G$ the corresponding G-equivariant coarse K-homology functor. By using an extension to exact categories of the classical Dennis trace map we get the following (see Proposition 4.4.1):

Theorem. *There are natural transformations of equivariant coarse homology theories*

$$\mathrm{K}\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G \quad \textit{and} \quad \mathrm{K}\mathcal{X}_k^G \to \mathcal{X}\mathrm{HC}_k^G$$

induced by the classical trace maps from algebraic K-theory to Hochschild and cyclic homology.

Equivariant coarse ordinary homology

$$\mathcal{X}C^G\colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$$

is defined in terms of equivariant locally finite controlled maps $X^{n+1} \to k$ (see Definition 1.5.1), where X is a G-bornological coarse space and k is the field of coefficients. We review its definition in Section 1.5.

The second main result of the thesis is the construction of a natural transformation from equivariant coarse Hochschild homology to equivariant coarse ordinary homology. We prove the following result (see Theorem 4.3.8 and Proposition 4.3.9):

Theorem. *There is a natural transformation*

$$\Phi_{\mathcal{X}\mathrm{HH}_k^G}\colon \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}C^G.$$

of G-equivariant $\mathbf{Ch}_\infty$-valued coarse homology theories, which induces an equivalence of chain complexes when evaluated at the one point space $\{*\}$.

The transformation $\Phi_{\mathcal{X}\mathrm{HH}_k^G}\colon \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}C^G$ is defined by using trace-like maps and extends to coarse cyclic homology as well. The idea, developed in Section 4.3, is the following.

We start by considering the additive cyclic nerve of the k-linear category $V_k^G(X)$ of G-equivariant X-controlled finite dimensional k-vector spaces. The n-th component of the additive cyclic nerve of a k-linear category $\mathcal{C}$ is

$$\mathrm{CN}_n(\mathcal{C}) = \bigoplus_{(C_0,\dots,C_n)} \left(\bigotimes_{i=0}^{n} \mathrm{Hom}_{\mathcal{C}}(C_{i+1}, C_i) \right)$$

where the index i runs cyclically in the ordered set $(0,\dots,n)$ and the sum ranges over all the tuples $(C_0,\dots,C_n)$ of objects of $\mathcal{C}$. When $\mathcal{C}$ is the k-linear category $V_k^G(X)$, we let $A_0 \otimes A_1 \otimes \cdots \otimes A_n$, with $A_i \colon M_{i+1} \to M_i$, be an element in the n-th component of the additive cyclic nerve $\mathrm{CN}_n(V_k^G(X)))$ and $(x_0,\dots,x_n)$ be a point of X^{n+1}. There are well-defined evaluations $M_i(x_j)$ of the object $M_i \in V_k^G(X)$ at the point $x_j \in X$ and there are induced linear operators $A_i^{x_i,x_{i+1}} \colon M_i(x_i) \to M_{i+1}(x_{i+1})$ between finite dimensional k-vector spaces. We consider the following composition:

$$M_0(x_n) \xrightarrow{A_n^{x_n,x_{n-1}}} M_n(x_{n-1}) \xrightarrow{A_{n-1}^{x_{n-1},x_{n-2}}} \dots \xrightarrow{A_1^{x_1,x_0}} M_1(x_0) \xrightarrow{A_0^{x_0,x_n}} M_0(x_n), \qquad (0.0.2)$$

that is an endomorphism of $M_0(x_n)$ (a finite dimensional k-vector space). To every tuple $(A_0,\dots,A_n)$ and every $(n+1)$-tuple $(x_0,\dots,x_n)$ of points of X, we can associate an element of the field k, that is the trace of the composition (0.0.2). We show that this describes an equivariant coarse n-chain of X and extends linearly to a chain map defined on the additive cyclic nerve $\mathrm{CN}(V_k^G(X)))$. It extends to the canonical mixed complex associated to $\mathrm{CN}(V_k^G(X))$, and this extension yields a natural transformation $\Phi_{\mathcal{X}\mathrm{HH}_k^G} \colon \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}C^G$ (see Theorem 4.3.8). The computation at the one point space induces an equivalence of chain complexes and this implies that the transformation is not trivial (as the equivariant coarse ordinary homology of the point is non-zero).

By composition of these two natural transformations, we get a natural transformation

$$K\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}\mathrm{H}_k^G$$

from equivariant coarse algebraic K-homology to (the spectra-valued) equivariant coarse ordinary homology. We believe that the study of this transformation may be useful for the understanding and detection of coarse K-theory classes.

Assembly maps and localization results

One of the main applications of coarse algebraic topology is in the study of assembly map conjectures, like the Farrell-Jones and the Baum-Connes assembly map conjectures. If $\mathcal{F}$ is a family of subgroups of a group G (meaning that it is a collection of subgroups of G closed under conjugation and under taking subgroups), and if $G\mathbf{Orb}$ denotes the orbit category (*i.e.,* the category of transitive G-sets and G-equivariant maps), then we can consider the full subcategory $G\mathbf{Orb}_{\mathcal{F}}$ of $G\mathbf{Orb}$ given by the G-sets whose stabilizers belong to $\mathcal{F}$. The assembly map for an equivariant homology theory (assumed to be defined on $G\mathbf{Orb}$), with respect to the family of subgroups $\mathcal{F}$, is the map induced (after taking colimits) in homology by the inclusion $G\mathbf{Orb}_{\mathcal{F}} \to G\mathbf{Orb}$; roughly, it approximates the starting functor with its restrictions in $G\mathbf{Orb}_{\mathcal{F}}$ [Lue18]. The Farrell-Jones conjecture, for example, asserts that the assembly map for algebraic K-theory with respect to the family of virtually cyclic subgroups is an isomorphism, and this conjecture has many connections to other outstanding conjectures, like the Borel conjecture, the Kaplansky conjecture or the Novikov conjecture. We refer to [BLR07, RV17, Lue18] for recent surveys on the subject.

If $E\colon G\mathbf{BornCoarse} \to \mathbf{C}$ is a G-equivariant coarse homology theory, then the composition

$$E \circ \mathcal{O}^{\infty}_{\mathrm{hlg}}\colon G\mathbf{Top} \to \mathbf{C}$$

with a suitable cone functor $\mathcal{O}^{\infty}_{\mathrm{hlg}}$ [BEKW17], describes a G-equivariant homology theory. Furthermore, one can associate to E a fiber sequence of functors, called a *cone sequence* [BEKW17, Def. 11.9] whose boundary map is called the *forget-control* map. In the coarse approach to the assembly map conjectures, one studies the forget-control map of a coarse homology theory E, trying to infer properties of the associated G-homology theory $E \circ \mathcal{O}^{\infty}_{\mathrm{hlg}}$. Under some mild assumptions, the assembly map for this G-homology theory is equivalent to the forget control map for E [BEKW17, BEKW18]; see [BKW18] for a recent application of these methods to the split-injectivity of the assembly map in Waldhausen A-theory.

We apply a comparison result (between the forget-control map and the assembly map [BEKW17, Thm. 11.16]) to equivariant coarse Hochschild and cyclic homology as well. Unfortunately, we do not know (and we do not expect) that coarse Hochschild and cyclic homologies are additive coarse homology theories, hence we cannot apply directly the more general results on the split-injectivity of the associated assembly map [BEKW18, Thm. 1.11]. However, we show (see Proposition 4.2.9) that, when restricted to the family **Fin** of finite subgroups of G, the forget-control map for G-equivariant coarse Hochschild (cyclic) homology agrees with the assembly map for the associated G-equivariant homology theory $\mathbf{HH}^G_k := \mathcal{X}\mathrm{HH}^G_k \circ \mathcal{O}^{\infty}_{\mathrm{hlg}}$:

Theorem. *The forget-control map for $\mathcal{X}\mathrm{HH}^G_k$ is equivalent to the assembly map for the G-homology theory $\mathbf{HH}^G_k$.*

Our last application is inspired by Segal's Localization Theorem [Seg68a], that we briefly recall.

Atiyah gives a definition of G-equivariant K-theory K^*_G of a G-space X in terms of G-vector bundles [Ati64, Seg68a]. From this definition, the evaluation at the one point space $\{*\}$ induces an isomorphism of rings between $K^0_G(*)$ and the representation ring (called *character ring* therein) $R(G)$ of G; hence the G-equivariant K-theory $K_G^*(X)$ of a G-space X has the structure of an $R(G)$-algebra over the representation ring because every space has a natural map onto a point.

Segal's Localization Theorem [Seg68a, Prop. 4.1] says that, after localization at a certain prime ideal $\mathfrak{p}$ of $R(G)$ (that is associated to a conjugacy class γ of G), the inclusion $X^\gamma \to X$ of the subspace of γ-fixed points X^γ in X induces (for locally compact G-spaces) an isomorphism

$$K_G^*(X)_{\mathfrak{p}} \to K_G^*(X^\gamma)_{\mathfrak{p}}$$

in equivariant K-theory.

Inspired by Segal's Localization Theorem, general localization results for equivariant coarse homology theories, with the goal to develop new tools for studying equivariant coarse algebraic K-theory, have been shown [BCb]. In this thesis, we apply these general coarse localization results to (the lax-symmetric monoidal refinements of) equivariant coarse Hochschild and cyclic homology, obtaining the following localization theorem for the associated G-equivariant Hochschild homology (see Corollary 4.5.7):

Theorem. *Let G be a finite group and γ be a conjugacy class of G. Let W be a finite G CW-complex and let W^γ be the sub-complex of γ-fixed points. Then, the inclusion $W^\gamma \to W$ induces an equivalence*

$$\mathbf{HH}^G_{(\gamma)}(W^\gamma) \to \mathbf{HH}^G_{(\gamma)}(W)$$

of chain complexes.

Organization of the thesis

The thesis is structured as follows.

In *Chapter 1* we recall and study the category $G\mathbf{BornCoarse}$ of G-bornological coarse spaces and its invariants, the G-equivariant coarse homology theories. This is done in Section 1.1 and Section 1.2, where we also introduce the necessary notation and terminology for the subsequent sections and chapters. In Section 1.3, we define the forget-control map for a G-equivariant coarse homology theory. We introduce the notions of G-uniform bornological coarse spaces and of the cone functor; the cone functor in particular is important in the study of assembly map conjectures because it provides a bridge between G-homology theories and G-equivariant coarse homology theories. We then describe the general comparison result between the forget-control maps and the usual assembly maps [BEKW17]. In Section 1.4, we describe the general coarse localization theorems [BCb]. The last section, Section 1.5, is devoted to a first example of coarse homology theories: this is coarse ordinary homology.

In *Chapter 2*, we study in detail the symmetric monoidal category $V^G_{\mathbf{A}}(X)$ of G-equivariant X-controlled $\mathbf{A}$-objects, where X is a G-bornological coarse space and $\mathbf{A}$ denotes an additive category with strict G-action. In Section 2.1, we analyze this category and the associated functor $V^G_{\mathbf{A}}\colon G\mathbf{BornCoarse} \to \mathbf{Add}$ from G-bornological coarse spaces to the category of small additive categories. Furthermore, we study the behavior of this functor under coarse equivalences, colimits and flasque spaces [BEKW17]. In Section 2.2, we discuss the behavior of the functor $V^G_{\mathbf{A}}$ under products and we see that it admits a lax symmetric monoidal refinement [BCa]. In the last section, Section 2.3, we recall the definition of equivariant coarse algebraic K-homology, which is defined as the non-connective K-theory of the category $V^G_{\mathbf{A}}(X)$ [BEKW17].

In *Chapter 3*, we construct coarse versions of the classical Hochschild and cyclic homologies of algebras as functors

$$\mathcal{X}\mathrm{HH}^G_k\colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty \quad \text{and} \quad \mathcal{X}\mathrm{HC}^G_k\colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$$

from the category of G-bornological coarse spaces to the cocomplete stable ∞-category of chain complexes. In order to define these functors, we use Keller's cyclic homology [Kel99] of additive exact categories, which is recalled in Section 3.1 and Section 3.2. Section 3.3 and Section 3.4 are the core of the thesis, where we prove that, in complete analogy to the construction of coarse algebraic K-homology of [BEKW17], the composition of the functor V^G_k and Keller's cone functor provide a coarse homology theory. In Section 3.5, we prove that equivariant coarse Hochschild and cyclic homologies refine to lax symmetric monoidal functors.

In *Chapter 4*, we describe some features and properties of the coarse homology theories defined in Chapter 3. In Section 4.1, we see that coarse Hochschild and cyclic homologies do extend the classical Hochschild and cyclic homologies for algebras: we prove that the evaluation at the one point space is equivalent to the usual Hochschild homology of the base field, and that the evaluation at (a canonical G-bornological coarse space associated to) G is equivalent to the classical Hochschild homology of the group algebra $k[G]$. In Section 4.2, we study the forget-control map for coarse Hochschild and cyclic homology, by using the general results reviewed in Section 1.3 and we see that this forget-controlling is equivalent to the assembly map for the G-equivariant homology theory associated to coarse Hochschild and cyclic homology. In Section 4.3 and Section 4.4, we construct the trace-like natural transformations from coarse algebraic K-homology to coarse Hochschild homology, and from coarse Hochschild homology to coarse ordinary homology. In the last section, Section 4.5, we apply the general localization theorems for equivariant coarse homology theories, reviewed in Section 1.4, to the case of coarse Hochschild and cyclic homology.

Finally, in the Appendix, we mainly recollect some useful facts concerning differential graded categories.

Conventions

In this thesis we freely employ the language of ∞-categories. More precisely, we model ∞-categories as quasi-categories [Cis, Lur09, Lur14]. Our main reference on dg-categories is Keller's ICM [Kel06].

The results of this work are stated for small categories. However, the categories of bornological coarse spaces, the category of dg-categories and dg-modules, etc., are not small. There are classical ways to overcome size problems; we assume the Grothendieck theory of nested universes.

In the thesis, G will usually denote a group, k a field and the symbol $\otimes$ denotes tensor products over k. We assume that 0 belongs to the natural numbers and that $\{*\}$ is the one point space.

Wandrers Nachtlied

Über allen Gipfeln
Ist Ruh;
In allen Wipfeln
Spürest du
Kaum einen Hauch;
Die Vögelein schweigen im Walde.
Warte nur, balde
Ruhest du auch.

Goethe

Chapter 1

Bornological coarse spaces and coarse homology theories

A *bornological coarse space* is a set endowed with a bornology and a coarse structure (see Definition 1.1.11). When a group G acts on such a space by automorphisms, we have the notion of a G-equivariant bornological coarse space.

Following our main references [BE16, BEKW17], in this chapter we define the category $G\mathbf{BornCoarse}$ of G-equivariant bornological coarse spaces and we recall the definition of equivariant coarse homology theories. These are functors

$$E\colon G\mathbf{BornCoarse} \to \mathbf{C}$$

from the category of G-bornological coarse spaces to some cocomplete stable ∞-category $\mathbf{C}$ satisfying additional axioms, as reviewed in Definition 1.2.1. We then recollect some general properties and results about equivariant coarse homology theories, focusing in particular on forget-control and assembly maps [BEKW17, BEKW18] and on Segal-type localization theorems [Seg68a].

The chapter is organized as follows. In Section 1.1, we give the definitions of bornological coarse spaces and G-bornological coarse spaces, introducing notations and terminology for the subsequent sections and chapters. In Section 1.2 we study the (equivariant) coarse homology theories and we introduce the notions of strongness and continuity, necessary for studying coarse assembly map conjectures.

In Section 1.3 we focus on forget-control and coarse assembly maps for equivariant coarse homology theories. Following [BEKW17], we give a comparison result between these different maps, showing that in some special cases they are equivalent. We introduce the necessary background and the core definition of *cone functor* $\mathcal{O}^\infty$. In Section 1.4, following [BCb], we focus on general localization theorems for coarse homology theories.

We conclude the chapter with a very first example of coarse homology theories: equivariant coarse ordinary homology.

1.1 G-Equivariant bornological coarse spaces

In this section we give the fundamental definition of G-bornological coarse spaces (see Definition 1.1.21). The main source of examples is given by metric spaces, for which the bornology is the family of bounded sets and the coarse structure is generated by the family of controlled sets (see Example 1.1.12).

We denote by (G)**BornCoarse** the category of $(G$-$)$bornological coarse spaces and (G-equivariant) proper controlled maps (see Definition 1.1.3 and Definition 1.1.6). After reviewing some basic definitions, namely Definition 1.1.24–1.1.27, we describe the canonical symmetric monoidal structure on G**BornCoarse**.

Let X be a set and let $\mathcal{P}(X)$ denote its power set.

Definition 1.1.1. [BE16, Def 2.1] A *bornology* on a set X is a subset $\mathcal{B} \subseteq \mathcal{P}(X)$ such that:

- $\mathcal{B}$ is closed under taking subsets;
- $\mathcal{B}$ is closed under taking finite unions;
- the set X is covered by elements of $\mathcal{B}$: $X = \cup_{B\in\mathcal{B}} B$.

A *bornological space* is a pair $(X, \mathcal{B})$ where $\mathcal{B}$ is a bornology on the set X.

The elements B in $\mathcal{B}$ are called $\mathcal{B}$*-bounded sets*, or simply *bounded sets* when the given bornology is clear from the context. Intersection of bornologies on a set X is again a bornology on X; if A is a subset of X, the bornology *generated* by A is the minimal bornology on X containing A. This bornology is denoted by $\mathcal{B}\langle A\rangle$.

Notation 1.1.2. In order to emphasize the dependence of the bornology $\mathcal{B}$ on the set X, the bornology $\mathcal{B}$ on X will also be denoted by $\mathcal{B}(X)$.

Let $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ be two bornological spaces and let $f\colon X \to X'$ be a map between the underlying sets.

Definition 1.1.3. Let $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ be two bornological spaces and let $f\colon X \to X'$ be a map between the underlying sets. The map f is called *proper* if the pre-image of every $\mathcal{B}'$-bounded set is $\mathcal{B}$-bounded.

Let X be a set. Let U and U' be subsets of $X \times X$; we consider the following operations:

Inversion: $U^{-1} := \{(x, y) \in X \times X \mid (y, x) \in U\}$;

Composition: $U \circ U' := \{(x, y) \in X \times X \mid \exists z \in X : (x, z) \in U \text{ and } (z, y) \in U'\}$.

We denote by $\Delta_X \coloneqq \{(x,x) \in X \times X \mid x \in X\}$ the diagonal of X in the product $X \times X$.

Definition 1.1.4. [BE16, Def. 2.3] A *coarse structure* on a set X is a subset $\mathcal{C} \subseteq \mathcal{P}(X\times X)$ that contains the diagonal Δ_X and is closed under taking subsets, finite unions, inverses, and compositions.

The pair $(X, \mathcal{C})$ given by a set X and a coarse structure $\mathcal{C}$ on X is called a *coarse space*; elements of $\mathcal{C}$ are called *entourages*, or *$\mathcal{C}$-controlled sets*.

Notation 1.1.5. In order to emphasize the dependence on X, the coarse structure $\mathcal{C}$ on X will also be denoted by $\mathcal{C}(X)$.

Definition 1.1.6. Let $(X, \mathcal{C})$ and $(X', \mathcal{C}')$ be two coarse spaces and let $f\colon X \to X'$ be a map between the underlying sets. The map $f\colon (X, \mathcal{C}) \to (X', \mathcal{C}')$ is called *controlled* (or *coarse*) if for every entourage U of $\mathcal{C}$ the set $(f \times f)(U)$ belongs to $\mathcal{C}'$.

Intersection of coarse structures on X is again a coarse structure on X. When a subset F of $\mathcal{P}(X \times X)$ is given, the coarse structure generated by F is the minimal coarse structure on X that contains F. It is denoted by $\mathcal{C}\langle F\rangle$.

Example 1.1.7. (i) Let X be a set. The minimal bornology $\mathcal{B}_{\min}$ on X consists of all the finite subsets of X; the maximal bornology $\mathcal{B}_{\max}$ on X is $\mathcal{P}(X)$. The minimal coarse structure $\mathcal{C}_{\min}$ is the set $\mathcal{C}_{\min} = \mathcal{P}(\Delta_X)$ generated by the diagonal Δ_X, and the maximal coarse structure $\mathcal{C}_{\max}$ is $\mathcal{P}(X \times X)$.

(ii) Let (X, d) be a metric space. There is a canonical bornology and a canonical coarse structure associated to (X, d), denoted by $\mathcal{B}_d$ and $\mathcal{C}_d$ respectively. The family

$$\mathcal{B}_d \coloneqq \mathcal{B}\langle\{B(x,r) \mid x \in X,\ r \geq 0\}\rangle$$

is the bornology generated by the d-bounded balls $B(x, r)$ of (X, d). We observe that the bornology $\mathcal{B}_d$ on X contains all the d-bounded subsets of X. The coarse structure $\mathcal{C}_d$ is generated by the family of subsets of $X \times X$

$$U_r \coloneqq \{(x,y) \mid d(x,y) \leq r)\} \tag{1.1.1}$$

for every $r \geq 0$.

By the previous example we can see that the definition of a bornology on a set X generalizes the notion of *bounded sets* of a metric space (X, d) and that the definition of a coarse structure generalizes the notion, for two points of X, of *being uniformly close.*

Let U be a subset of $X \times X$ and let B be a subset of X; the *U-thickening of B* is the subset of X defined as follows:

$$U[B] \coloneqq \{x \in X \mid \exists b \in B,\ (x,b) \in U\} \tag{1.1.2}$$

If U is an entourage of a coarse space $(X, \mathcal{C})$, the U-thickening is also called a *controlled thickening*.

Example 1.1.8. Let (X, d) be a metric space. If $p \in X$ is a point of X and $U_r \in \mathcal{C}_d$ is a controlled set as in (1.1.1), then the U_r-thickening of p is the ball of radius r. More generally, the U_r-thickening of a ball of radius s is contained in the ball of radius $r + s$.

Remark 1.1.9. Let X be a set endowed with a bornology and a coarse structure. For every entourage U and V in $\mathcal{C}(X)$ and bounded set B in $\mathcal{B}(X)$ we have an inclusion $U[V[B]] \subseteq (U \circ V)[B]$ of controlled thickenings.

Recall the definition of proper (Definition 1.1.3) and controlled maps (Defnition 1.1.6).

Definition 1.1.10. A bornology $\mathcal{B}$ and a coarse structure $\mathcal{C}$ on a set X are *compatible* if every controlled thickening of every bounded set is bounded.

Definition 1.1.11. [BE16, Definition 2.6 & 2.10] A *bornological coarse space* is a triple $(X, \mathcal{C}, \mathcal{B})$ given by a set X, a bornology $\mathcal{B}$ and a coarse structure $\mathcal{C}$ on X such that $\mathcal{B}$ and $\mathcal{C}$ are compatible. We denote by **BornCoarse** the category of bornological coarse spaces and proper controlled maps.

When the bornology $\mathcal{B}$ and the coarse structure $\mathcal{C}$ on a set X are clear from the context, we will omit them and we will refer to a bornological coarse space just by its underlying set. Proper and controlled maps between bornological coarse spaces are also called *morphisms* of bornological coarse spaces.

Example 1.1.12. Examples of bornological coarse spaces arise from metric spaces (X, d), by equipping the underlying set X with the associated bornology $\mathcal{B}_d$ and coarse structure $\mathcal{C}_d$. By Example 1.1.8, $\mathcal{C}_d$ and $\mathcal{B}_d$ are compatible and the space $X_d \coloneqq (X, \mathcal{C}_d, \mathcal{B}_d)$ is a bornological coarse space.

Example 1.1.13. Let S be a set. We denote by $S_{\min,\max}$ the bornological coarse space $(S, \mathcal{C}_{\min}, \mathcal{B}_{\max})$ whose underlying set is S, equipped with the minimal coarse structure and the maximal bornology.

Example 1.1.14. Let $(X', \mathcal{C}', \mathcal{B}')$ be a bornological coarse space and let X be a set. A map $f\colon X \to X'$ of sets induces a bornological coarse structure on X. In fact, let

$$f^*\mathcal{C}' \coloneqq \mathcal{C}\langle\{(f \times f)^{-1}(U') \mid U' \in \mathcal{C}'\}\rangle$$

and

$$f^*\mathcal{B}' \coloneqq \mathcal{B}\langle\{f^{-1}(B') \mid B' \in \mathcal{B}'\}\rangle$$

be the coarse structure and bornology on X generated by the preimages. Then, the triple $(X, f^*\mathcal{C}, f^*\mathcal{B})$ is a bornological coarse space and $f\colon (X, f^*\mathcal{C}, f^*\mathcal{B}) \to (X', \mathcal{C}', \mathcal{B}')$ is a morphism of bornological coarse spaces. When Z is a subset of X' and f is the inclusion of Z in X', we refer to this construction as *inclusion* of bornological coarse spaces.

If X and Y are metric spaces, two maps $f, f'\colon X \to Y$ are close to each other if $d_Y(f(x), f'(x))$ is bounded, uniformly in X. In the context of bornological coarse spaces this leads to the following definition:

Definition 1.1.15. Let $f_0, f_1\colon (X, \mathcal{C}, \mathcal{B}) \to (X', \mathcal{C}', \mathcal{B}')$ be morphisms in **BornCoarse**. We say that f_0 and f_1 are *close* to each other if the image of the diagonal $(f_0, f_1)(\Delta_X)$ is an entourage of X'.

Observe that being close to each other is an equivalence relation on the set of morphisms between X and X'.

Definition 1.1.16. [BE16, Def. 3.14] A morphism $f\colon (X, \mathcal{C}, \mathcal{B}) \to (X', \mathcal{C}', \mathcal{B}')$ of bornological coarse spaces is an *equivalence* if there exists an inverse $g\colon (X', \mathcal{C}', \mathcal{B}') \to (X, \mathcal{C}, \mathcal{B})$ such that the compositions $g \circ f$ and $f \circ g$ are close to the identity maps. In this case, the spaces X and X' are called *coarsely equivalent*.

Example 1.1.17. A quasi-isometry between metric spaces induces a coarse equivalence between the associated bornological coarse spaces.

Definition 1.1.18. [BE16, Def. 3.21] A bornological coarse space X is called *flasque* if it admits a morphism $f\colon X \to X$ such that:

(i) f is close to the identity map;

(ii) for every entourage U of X, the union $\bigcup_{k\in\mathbb{N}}(f^k \times f^k)(U)$ is an entourage;

(iii) for every bounded set B of X there exists a natural number k such that $f^k(X) \cap B = \emptyset$.

The definition says that the morphism f is equivalent to the identity, is uniformly controlled, together with all its powers, and that each bounded set has eventually no intersections with the image of f^n. An example of such a morphism on the bornological coarse space $\mathbb{N}$ (*i.e.,* the bornological coarse space associated to the set of natural numbers endowed with the standard metric) is the function $\mathbb{N} \to \mathbb{N}$ sending the natural number n to $n+1$. On the other hand, the bornological coarse space associated to the integers $\mathbb{Z}$ is not flasque.

Definition 1.1.19. [BE16, Def. 3.2 & 3.5] Let $(X, \mathcal{C}, \mathcal{B})$ be a bornological coarse space.

1. A *big family* $\mathcal{Y}$ on X is a filtered family $(Y_i)_{i\in I}$ of subsets of X satisfying the following:
$$\forall\, i \in I, \quad \forall\, U \in \mathcal{C}, \quad \exists j \in I \text{ such that } U[Y_i] \subseteq Y_j.$$

2. A pair $(Z, \mathcal{Y})$ consisting of a subset Z of X and of a big family $\mathcal{Y} = (Y_i)_{i\in I}$ on X is called a *complementary pair* if there exists an index $i \in I$ for which $Z \cup Y_i = X$.

Example 1.1.20. Let $X = \mathbb{R}$, endowed with bornology and coarse structure induced by the standard euclidean metric on the real numbers. Let $Z = (-\infty, 0]$, and $Y_n = [-n, \infty)$, for $n \in \mathbb{N}$. Then, the pair $(Z, \mathcal{Y})$, with $\mathcal{Y} = (Y_n)_{n\in\mathbb{N}}$, is a complementary pair on $\mathbb{R}$.

Let G be a group acting by automorphisms on a bornological coarse space X. The G-action on X induces a diagonal G-action on $X \times X$, hence a G-action on the power set $\mathcal{P}(X \times X)$ of X and on the set of entourages of X. Let $\mathcal{C}^G$ be the partially ordered subset of $\mathcal{C}$ consisting of the set-wise G-fixed entourages.

Definition 1.1.21. [BEKW17, Definition 2.1] A *G-bornological coarse space* is a bornological coarse space $(X, \mathcal{C}, \mathcal{B})$ equipped with a G-action by automorphisms such that the set of invariant entourages $\mathcal{C}^G$ is cofinal in $\mathcal{C}$.

A *morphism* of G-bornological coarse spaces is a morphism of bornological coarse spaces that is also G-equivariant. We denote by G**BornCoarse** the category of G-bornological coarse spaces and G-equivariant, proper controlled maps.

For a subset U of $X \times X$ we set

$$GU \coloneqq \bigcup_{g\in G} (g \times g)(U).$$

The cofinality assumption of $\mathcal{C}^G$ in $\mathcal{C}$ is equivalent to requiring that, for every entourage U of $\mathcal{C}$, the set GU belongs to $\mathcal{C}$.

Example 1.1.22. We provide some examples of G-bornological coarse spaces:

(i) If G acts on a metric space (X, d) by isometries, then $(X, \mathcal{C}_d, \mathcal{B}_d)$ is a G-bornological coarse space.

(ii) Let G be a group, $\mathcal{B}_{\min}$ be the minimal bornology on its underlying set, and let $\mathcal{C}_{\text{can}} \coloneqq \langle\{G(B \times B) \mid B \in \mathcal{B}_{\min}\}\rangle$ be the coarse structure on G generated by the G-orbits. The space $G_{\text{can,min}} \coloneqq (G, \mathcal{C}_{\text{can}}, \mathcal{B}_{\min})$ is a G-bornological coarse space.

(iii) Let G be a countable group equipped with a proper left invariant metric d; then, $(G, \mathcal{C}_d, \mathcal{B}_d)$ is the same as $(G, \mathcal{C}_{\text{can}}, \mathcal{B}_{\min})$ and different choices of d induce the same G-bornological coarse space.

(iv) Let X be a G-bornological coarse space and let Z be a G-invariant subset of X. As in Example 1.1.14, we define the induced coarse structure and bornology on Z by restriction: $\mathcal{C}_Z \coloneqq \{(Z \times Z) \cap U \mid U \in \mathcal{C}\}$ and $\mathcal{B}_Z \coloneqq \{Z \cap B \mid B \in \mathcal{B}\}$. Then, $Z_X \coloneqq (Z, \mathcal{C}_Z, \mathcal{B}_Z)$ is a G-bornological coarse space and the inclusion $Z \hookrightarrow X$ is a morphism of G-bornological coarse spaces.

(v) Let X be a G-bornological coarse space and let U be a G-invariant entourage of X. The induced coarse structure $\mathcal{C}_U$ is the coarse structure on X generated by U. This is compatible with the bornology $\mathcal{B}$ and $X_U \coloneqq (X, \mathcal{C}_U, \mathcal{B})$ is a G-bornological coarse space. Moreover, the identity map $X \to X_U$ is a morphism of G-bornological coarse spaces and, if $U \subseteq U'$, there is a natural induced morphism $X_U \to X_{U'}$.

Let X be a G-set. In general, a bornology $\mathcal{B}$ on X is not closed under G-action. However, we can *complete* the bornology $\mathcal{B}$, obtaining a new G-bornological space with this property:

Definition 1.1.23. [BEKW17, Def. 2.12] Let $(X, \mathcal{C}, \mathcal{B})$ be a G-bornological coarse space. The *G-completion* of $(X, \mathcal{C}, \mathcal{B})$ is the G-bornological coarse space $B_G X := (X, \mathcal{C}, \mathcal{B}_G)$, where $\mathcal{B}_G$ is the bornology generated by all the sets GB, with B in $\mathcal{B}$.

A bornological coarse space $X \in \mathbf{BornCoarse}$ is a G-bornological coarse space with trivial G-action and the category $\mathbf{BornCoarse}$ of bornological coarse spaces is a full subcategory of $G\mathbf{BornCoarse}$. The definitions 1.1.15, 1.1.16, 1.1.18 and 1.1.19, have natural extensions to G-bornological coarse spaces:

Definition 1.1.24. [BEKW17, Def. 3.1] Two morphisms between G-bornological coarse spaces are *close* to each other if they are close to each other as morphisms of bornological coarse spaces (see Definition 1.1.15). A morphism f of G-bornological coarse spaces is an equivalence if it admits an inverse g such that the two compositions are close to the identities.

Definition 1.1.25. [BEKW17, Def. 3.8] A G-bornological coarse space X is called *flasque* if it admits a morphism $f \colon X \to X$ such that:

(i) f is close to the identity map;

(ii) for every entourage U of X, the union $\bigcup_{k \in \mathbb{N}} (f^k \times f^k)(U)$ is again an entourage of X;

(iii) for every bounded set B of X there exists k such that $f^k(X) \cap GB = \emptyset$.

Definition 1.1.26. [BEKW17, Def. 3.5 & 3.7] Let X be a G-bornological coarse space.

(i) An *equivariant big family* on X is a big family (see Definition 1.1.19) consisting of G-invariant subsets.

(ii) An *equivariant complementary pair* $(Z, \mathcal{Y})$ on X is a complementary pair (see Definition 1.1.19) where Z is a G-invariant subset of X and $\mathcal{Y}$ is an equivariant big family.

Example 1.1.27. Let X be a G-bornological coarse space and A a G-invariant subset of X. Then, the family

$$\{A\} := (U[A])_{U \in \mathcal{C}^G}$$

is an equivariant big family on X (generated by A).

The categories **BornCoarse** and G**BornCoarse** have natural symmetric monoidal structures (see Definition A.3.1). We conclude the section with a description of them (see [BE16, Example 2.30] and [BEKW17, Example 2.17]).

Let G**Set** be the category of G-sets and G-equivariant maps. Let

$$\mathcal{F} : G\mathbf{BornCoarse} \to G\mathbf{Set}$$

be the forgetful faithful functor, which associates to every G-bornological coarse space X its underlying G-set. The category G**Set** has the symmetric monoidal structure given by the cartesian product. The symmetric monoidal structure on G**BornCoarse** is then defined by pulling back the symmetric monoidal structure on G**Set**, as we now describe.

Let X and X' be G-bornological coarse spaces. Then, their tensor product

$$X \otimes_{G\mathbf{BornCoarse}} X'$$

is the G-bornological coarse space defined as follows:

1. The underlying G-set of $X \otimes_{G\mathbf{BornCoarse}} X'$ is the cartesian product $X \times X'$ of the underlying G-sets, with the diagonal action.
2. The bornology on $X \times X'$ is generated by all the subsets $B \times B'$, with B and B' varying among all the bounded sets of X and X'.
3. The coarse structure on $X \times X'$ is generated by the entourages $U \times U'$ with U in $\mathcal{C}(X)$ and U' in $\mathcal{C}(X')$.

If $f : X \to Y$ and $f' : X' \to Y'$ are morphisms of G-bornological coarse spaces, then their tensor product

$$f \otimes f' : X \otimes_{G\mathbf{BornCoarse}} Y \to X' \otimes_{G\mathbf{BornCoarse}} Y'$$

is the equivariant map between the underlying G-sets $(x, y) \mapsto (f(x), f(y))$.

This describes a bifunctor

$$- \otimes_{G\mathbf{BornCoarse}} - : G\mathbf{BornCoarse} \times G\mathbf{BornCoarse} \to G\mathbf{BornCoarse}, \qquad (1.1.3)$$

which agrees with the cartesian symmetric monoidal structure of G**Set** on the underlying G-sets. The tensor unit $1_{G\mathbf{BornCoarse}}$ (Definition A.3.1.2) is given by the one-point space $*$.

The functor $\mathcal{F}$ preserves the unit and the tensor product strictly, *i.e.,* the morphisms 1 and 2 in Definition A.3.4 are identities. The associator, unit and symmetry constraints are imported from those in G**Set** and are implemented by morphisms of G-bornological coarse spaces; the relations of Definition A.3.1 are then satisfied and this finishes the description of the symmetric monoidal structure on the category G**BornCoarse**.

Remark 1.1.28. The one-point space $*$ is a commutative algebra object in $G\mathbf{BornCoarse}$.

Notation 1.1.29. We will use the shorter notation $X \otimes X'$ for the tensor product $X \otimes_{G\mathbf{BornCoarse}} X'$.

1.2 Equivariant coarse homology theories

The goal of this section is to recall the definition of equivariant coarse homology theories [BEKW17, Def. 3.10]. A coarse homology theory is a functor

$$E\colon G\mathbf{BornCoarse} \to \mathbf{C}$$

from the category of G-bornological coarse spaces to a cocomplete stable ∞-category $\mathbf{C}$ (*e.g.,* the ∞-category of chain complexes $\mathbf{Ch}_\infty$ or the ∞-category of spectra $\mathbf{Sp}$), satisfying additional axioms: coarse invariance, flasqueness, coarse excision and u-continuity. These are reviewed in Definition 1.2.1.

In order to study general properties of equivariant coarse homology theories, suitable categories $G\mathbf{Spc}\mathcal{X}$ of motivic coarse spaces and $G\mathbf{Sp}\mathcal{X}$ of motivic coarse spectra have been constructed [BE16, BEKW17]; in particular, the category $G\mathbf{Sp}\mathcal{X}$ is the target of a universal coarse homology theory

$$\mathrm{Yo}^s_G\colon G\mathbf{BornCoarse} \to G\mathbf{Sp}\mathcal{X}$$

with the property that every equivariant coarse homology theory $E\colon G\mathbf{BornCoarse} \to \mathbf{C}$ has an essentially unique factorization over Yo^s_G. After reviewing these facts,we see that equivariant coarse homology theories can be described as colimit-preserving functors on the category of motivic coarse spaces and we conclude the section with the definitions of a strong and of a continuous equivariant coarse homology theory; we will use these notions in Section 1.3 in relation with forget-control and assembly maps.

In the following, let $\mathbf{C}$ be a cocomplete stable ∞-category, let $E\colon G\mathbf{BornCoarse} \to \mathbf{C}$ be a functor and let X be a G-bornological coarse space.

If $\mathcal{Y} = (Y_i)_{i\in I}$ is a filtered family of G-invariant subsets of X, we define the value of the functor E at the family $\mathcal{Y}$ as the colimit in $\mathbf{C}$

$$E(\mathcal{Y}) \coloneqq \mathrm{colim}_{i\in I} E(Y_i). \tag{1.2.1}$$

Here, the subsets Y_i of X are endowed with the G-bornological coarse structure induced from X (by Example 1.1.22 (iv)). Observe that there is a natural morphism $E(\mathcal{Y}) \to E(X)$ induced by all the inclusions $Y_i \to X$.

Let Z be a G-invariant subset of X, with the induced structures. If $\mathcal{Y}$ is an equivariant big family on X (see Definition 1.1.26), then the intersection $Z \cap \mathcal{Y} \coloneqq (Z \cap Y_i)_{i\in I}$ is an equivariant big family on Z and $E(Z \cap \mathcal{Y})$ denotes the value of E at this filtered family; moreover, we observe that there are natural morphisms $E(Z \cap \mathcal{Y}) \to E(Z)$ and $E(Z \cap \mathcal{Y}) \to E(\mathcal{Y})$.

Let $\{0,1\}_{\max,\max}$ be the G-bornological coarse space whose underlying set is $\{0,1\}$, endowed with the maximal coarse structure and the maximal bornology; the G-action is the trivial one. Recall that $\mathcal{C}^G$ denotes the family of G-invariant entourages of X. Recall Definition 1.1.25 of a flasque G-bornological coarse space and Definition 1.1.26 of equivariant complementary pair. We recall that the space X_U, for U a G-invariant entourage, is the G-bornological coarse space with coarse structure induced by U of Example 1.1.22 (v). We can now give the fundamental definition of an equivariant coarse homology theory:

Definition 1.2.1. [BEKW17, Definition 3.10] Let G be a group and let $G\mathbf{BornCoarse}$ be the category of G-bornological coarse spaces. Let $\mathbf{C}$ be a cocomplete stable ∞-category. A *G-equivariant* $\mathbf{C}$*-valued coarse homology theory* is a functor

$$E\colon G\mathbf{BornCoarse} \longrightarrow \mathbf{C}$$

with the following properties:

i. **Coarse invariance:** for all $X \in G\mathbf{BornCoarse}$, the projection

$$\{0,1\}_{\max,\max} \otimes X \to X$$

is sent by E to an equivalence of $\mathbf{C}$;

ii. **Flasqueness:** if X is a flasque G-bornological coarse space, then $E(X) \simeq 0$;

iii. **Coarse excision:** $E(\emptyset) \simeq 0$, and for every equivariant complementary pair $(Z, \mathcal{Y})$ on X, the diagram

$$\begin{array}{ccc} E(Z \cap \mathcal{Y}) & \longrightarrow & E(Z) \\ \downarrow & & \downarrow \\ E(\mathcal{Y}) & \longrightarrow & E(X) \end{array}$$

is a push-out square;

iv. **u-continuity:** for every G-bornological coarse space $(X, \mathcal{C}, \mathcal{B})$, the canonical morphisms $X_U \to X$ induce an equivalence $E(X) \simeq \operatorname{colim}_{U \in \mathcal{C}^G} E(X_U)$.

We recall that the product $\otimes$ of G-bornological coarse spaces in Definition 1.2.1 is the symmetric monoidal product (1.1.3).

Remark 1.2.2. [BEKW17, Rem. 3.11] Coarse invariance in Definition 1.2.1 is equivalent to asking that the functor E sends equivalences of G-bornological coarse spaces (see Definition 1.1.24) to equivalences of $\mathbf{C}$.

Examples of (equivariant) coarse homology theories are: coarse ordinary homology and coarse topological K-theory [BE16], coarse algebraic K-homology and coarse topological Hochschild homology [BEKW17, BC17], coarse A-theory [BKW18], coarse Hochschild and cyclic homology (see Theorem 3.4.7). We will recall the definitions of coarse ordinary homology in Section 1.5 and of coarse algebraic K-theory in Section 2.3.

Equivariant $\mathbf{C}$-valued coarse homology theories can be equivalently seen as colimit-preserving functors $G\mathbf{Sp}\mathcal{X} \to \mathbf{C}$, where $G\mathbf{Sp}\mathcal{X}$ is a suitable category of motivic coarse spectra, as we now explain. In [BE16, Section 3.4] and [BEKW17, Section 4.1], ∞-categories $\mathbf{Spc}\mathcal{X}$ and $G\mathbf{Spc}\mathcal{X}$ of motivic coarse spaces and of G-equivariant motivic coarse spaces respectively, together with the corresponding stable versions $\mathbf{Sp}\mathcal{X}$ and $G\mathbf{Sp}\mathcal{X}$, have been constructed. This is implemented by first completing the category of bornological coarse spaces (by embedding it in the category of spaces-valued presheaves) and then by localizing (in the realm of ∞-categories) the obtained ∞-category at various sets of morphisms: these encode the properties of descent, coarse equivalences, vanishing on flasque G-bornological coarse spaces, and u-continuity. The ∞-categories of motivic coarse spectra $\mathbf{Sp}\mathcal{X}$ and $G\mathbf{Sp}\mathcal{X}$ are defined as the stabilizations of the ∞-categories $\mathbf{Spc}\mathcal{X}$ and $G\mathbf{Spc}\mathcal{X}$ of motivic coarse spaces [BE16, Sec. 4].

Every bornological coarse space X represents a motivic coarse space $\mathrm{Yo}(X)$ in $\mathbf{Spc}\mathcal{X}$. In fact, this is described by *Yoneda functors*

$$\mathrm{Yo} : \mathbf{BornCoarse} \to \mathbf{Spc}\mathcal{X} \tag{1.2.2}$$

and

$$\mathrm{Yo}_G : G\mathbf{BornCoarse} \to G\mathbf{Spc}\mathcal{X}, \tag{1.2.3}$$

sending a bornological coarse space (or a G-bornological coarse space) to its (equivariant) coarse motivic space. The stable versions of these functors are denoted by

$$\mathrm{Yo}^s : \mathbf{BornCoarse} \to \mathbf{Sp}\mathcal{X} \tag{1.2.4}$$

and

$$\mathrm{Yo}^s_G : G\mathbf{BornCoarse} \to G\mathbf{Sp}\mathcal{X}. \tag{1.2.5}$$

The category $G\mathbf{Sp}\mathcal{X}$ of motivic coarse spectra is a stable cocomplete ∞-category and the functor $\mathrm{Yo}^s_G \colon G\mathbf{BornCoarse} \to G\mathbf{Sp}\mathcal{X}$ is the *universal G-equivariant coarse homology theory* [BEKW17, Def. 4.9]: every equivariant coarse homology theory factorizes in an essentially unique way over it.

Let $\mathbf{C}$ be a cocomplete stable ∞-category and let $\mathbf{CoarseHomologyTheories}_{\mathbf{C}}$ be the full subcategory of $\mathbf{Fun}(\mathbf{BornCoarse}, \mathbf{C})$ consisting of all functors satisfying coarse invariance, excision, continuity and vanishing on flasque spaces (as in Definition 1.2.1). Let $\mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$ be the category of colimit-preserving functors from $\mathbf{Spc}\mathcal{X}$ to $\mathbf{C}$.

Remark 1.2.3. [BE16, Cor. 3.34] For every cocomplete ∞-category $\mathbf{C}$, we have an equivalence between the ∞-categories $\mathbf{CoarseHomologyTheories}_{\mathbf{C}}$ and $\mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$.

Analogously, pre-composition with the functor Yo^s_G, yields an equivalence between the ∞-category of G-equivariant coarse homology theories and the category of colimit-preserving functors $G\mathbf{Sp}\mathcal{X} \to \mathbf{C}$ [BEKW17, Cor. 4.10].

Notation 1.2.4. If X is in $\mathbf{BornCoarse}$ (or $G\mathbf{BornCoarse}$) and E is a colimit-preserving functor in $\mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$ (or in $\mathbf{Fun}^{\mathrm{colim}}(G\mathbf{Spc}\mathcal{X}, \mathbf{C})$), then we write $E(X)$ instead of $E(\mathrm{Yo}(X))$ (or $E(\mathrm{Yo}_G(X))$).

The categories $\mathbf{BornCoarse}$ and $G\mathbf{BornCoarse}$ have symmetric monoidal structures denoted by $\otimes$ (1.1.3). In the same way, the categories $\mathbf{Spc}\mathcal{X}$ and $G\mathbf{Spc}\mathcal{X}$ have symmetric monoidal structures, also denoted by $\otimes$, which are essentially uniquely determined by the requirement that the functors Yo and Yo_G refine to symmetric monoidal functors. For X in $G\mathbf{Spc}\mathcal{X}$ and Y in $G\mathbf{BornCoarse}$ we will often write $Y \otimes X$ instead of $\mathrm{Yo}_G(Y) \otimes X$.

Definition 1.2.5. Let $E\colon G\mathbf{BornCoarse} \to \mathbf{C}$ be a functor and let X be a G-bornological coarse space. The twist E_X of E by X is the functor

$$E_X \colon G\mathbf{BornCoarse} \to \mathbf{C}$$

defined by $E_X(Y) := E(X \otimes Y)$.

Remark 1.2.6. Analogously, we can twist a coarse homology theory E by a coarse motivic space $\mathrm{Yo}_G(X)$ in $G\mathbf{Spc}\mathcal{X}$: $E_X(Y) := E(\mathrm{Yo}_G(X) \otimes Y)$.

If E is a G-equivariant coarse homology theory, it is natural to ask whether the twisted version is also a coarse homology theory:

Lemma 1.2.7. [BEKW17, Lemma 4.17] *If E is a G-equivariant coarse homology theory and X is a G-bornological coarse space, then the twist E_X is a G-equivariant coarse homology theory.*

We conclude the section with the definitions of *strongness* and *continuity* for a (equivariant) coarse homology theory. These properties are important is the study of coarse assembly maps. We first need a more general definition of flasque spaces (recall Definition 1.1.25):

Definition 1.2.8. [BEKW17] A G-bornological coarse space X is called *weakly flasque* if it admits a morphism $f\colon X \to X$ such that:

1. $\mathrm{Yo}^s_G(f) \simeq \mathrm{id}_{\mathrm{Yo}^s_G}(X)$;
2. for every entourage U of X, the union $\bigcup_{k\in\mathbb{N}}(f^k \times f^k)(U)$ is again an entourage of X;

3. for every bounded set B of X there exists k such that $f^k(X) \cap GB = \emptyset$.

Definition 1.2.9. [BEKW17, Def. 4.19] Let $E\colon G\mathbf{BornCoarse} \to \mathbf{C}$ be a G-equivariant coarse homology theory. Then, E is *strong* if $E(X) \simeq 0$ for all weakly flasque G-bornological coarse spaces X.

Remark 1.2.10. [BEKW17, Lemma 11.25] Let $E\colon G\mathbf{BornCoarse} \to \mathbf{C}$ be a G-equivariant coarse homology theory. Let X be a G-bornological coarse space. If E is strong, then also the twist E_X by X is strong equivariant coarse homology theory.

We now proceed with the definition of a continuous coarse homology theory. Let X be a G-bornological coarse space.

Definition 1.2.11. [BEKW17, Def. 5.1] Let F be a subset of X. Then, F is called *locally finite* if $B \cap F$ is finite for every bounded set $B \in \mathcal{B}(X)$.

Definition 1.2.12. [BEKW17, Def. 5.6] Let $\mathcal{Y} = (Y_i)_{i \in I}$ be a filtered family of invariant subsets of X. The family $\mathcal{Y}$ is called a *trapping exhaustion* if, for every locally finite G-invariant subset F of X, there exists an index $i \in I$ for which F is contained in Y_i.

Let $E\colon G\mathbf{BornCoarse} \to \mathbf{C}$ be a G-equivariant coarse homology theory. Recall that, for a given filtered family $\mathcal{Y} = (Y_i)_i$, we define $E(\mathcal{Y})$ as the colimit of the $E(Y_i)$ (1.2.1).

Definition 1.2.13. [BEKW17, Def. 5.15] A G-equivariant coarse homology theory E is *continuous* if for every trapping exhaustion $\mathcal{Y}$ of a G-bornological coarse space X, the morphism

$$E(\mathcal{Y}) \to E(X)$$

is an equivalence.

1.3 Forget-control and assembly maps

One of the main applications of coarse geometry is in the study of (coarse) assembly map conjectures. We refer to [BLR07, RV17, Lue18] for recent surveys on assembly map conjectures and controlled methods. The coarse geometry approach to these conjectures consists in analyzing the so-called *forget-control* map and then, by comparison, in getting information about the assembly maps, as we shall see.

In this section, we recall the definitions of forget-control map [BEKW17, Def. 11.10] and of coarse assembly map [BEKW17, Def. 10.24] in the framework of [BEKW17]. We first need to introduce the category $G\mathbf{UBC}$ of G-uniform bornological coarse spaces (see Definition 1.3.3). Then, we define the so-called cone functors

$$\mathcal{O}, \mathcal{O}^\infty \colon G\mathbf{UBC} \to G\mathbf{BornCoarse}$$

that are functors from the category of G-uniform bornological coarse spaces to the category of G-bornological coarse spaces. From these functors, we get a fiber sequence

$$F^0 \to F \to F^\infty \to \Sigma F^0$$

of functors $G\mathbf{BornCoarse} \to G\mathbf{Sp}\mathcal{X}$ (1.3.2). The forget-control map is then defined as the connecting morphism associated to this fiber sequence.

After giving the definition of coarse assembly map (Definition 1.3.19), that is related to the classical assembly map [BEKW17, BEKW18], we conclude the section with a comparison result [BEKW17, Thm. 11.16]: under some mild assumptions, the coarse assembly map and the forget-control map provide equivalent morphisms when restricted to the family of subgroups (see Definition 1.3.16) **Fin** of finite subgroups.

We start by recalling the definition of G-uniform bornological coarse space.

Definition 1.3.1. Let X be a G-set. A *G-uniform structure* on X is a non-empty subset $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ with the following properties:

- every element of $\mathcal{U}$ contains the diagonal Δ_X of X;
- $\mathcal{U}$ is closed under inverses, compositions, finite intersections and supersets;
- for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;
- for every U in $\mathcal{U}$ the set $\bigcap_{g \in G} gU$ belongs to $\mathcal{U}$.

A *G-uniform space* $(X,\mathcal{U})$ is a pair given by a G-set X and a G-uniform structure $\mathcal{U}$ on X. If $(X,\mathcal{U})$ and $(X',\mathcal{U}')$ are G-uniform spaces, a G-equivariant map $f\colon X \to X'$ is *uniform* if $(f \times f)^{-1}(U') \in \mathcal{U}$ for every $U' \in \mathcal{U}'$.

Definition 1.3.2. Let X be a G-set endowed with a coarse structure $\mathcal{C}$ and a uniform structure $\mathcal{U}$. The structures $\mathcal{U}$ and $\mathcal{C}$ are *compatible* if $\mathcal{C}^G \cap \mathcal{U}^G$ is not empty.

We can define a category $G\mathbf{UBC}$ of G-uniform bornological coarse spaces:

Definition 1.3.3. [BEKW17, Def. 9.9] A *G-uniform bornological coarse space* is a G-bornological coarse space with a compatible G-uniform structure. Morphisms of G-uniform bornological coarse spaces are morphisms of G-bornological coarse spaces that are also uniform.

Remark 1.3.4. The category $G\mathbf{UBC}$ is symmetric monoidal. If X and Y are G-uniform bornological coarse spaces, the product $X \otimes Y$ is defined to be the G-bornological coarse space $X \otimes Y$ endowed with the (compatible) G-uniform structure generated by all the products $U_X \times U_Y$ of invariant uniform entourages of X and Y.

Example 1.3.5. Let (X, d) be a metric space with G-action. The uniform structure $\mathcal{U}_d$ on X associated to the metric d is generated by the sets

$$U_r = \{(x, y) \in X \times X \mid d(x, y) \leq r\}$$

for all r in $(0, \infty)$. This is compatible with the coarse structure $\mathcal{C}_d$ of Example 1.1.22 (i). We denote by X_{du} the G-uniform bornological coarse space obtained in this way.

Let X be a G-uniform bornological coarse space and let $\mathcal{Y} = (Y_i)_{i\in I}$ be a G-equivariant filtered family on X. Consider an order-preserving function $\varphi\colon I \to \mathcal{P}(X \times X)^G$, where $\mathcal{P}(X \times X)^G$ is the family of G-invariant subsets of $X \times X$ ordered by the opposite of the inclusion relation. We say that φ is $\mathcal{U}^G$*-admissible* [BEKW17, Def. 9.14] if for every $U \in \mathcal{U}^G$ there exists i in I such that $\varphi(i) \subseteq U$.

For a $\mathcal{U}^G$-admissible function $\varphi\colon I \to \mathcal{P}(X \times X)^G$, define the G-invariant entourages

$$U_\varphi := \{(x, y) \in X \times X \mid (\forall i \in I \mid (x, y) \in Y_i \times Y_i \text{ or } (x, y) \in \varphi(i))\}.$$

Assume also that $\mathcal{Y} = (Y_i)_{i\in I}$ is a G-equivariant big family on X (see Definition 1.1.26); then, the *hybrid coarse structure* $\mathcal{C}_h$ is the coarse structure generated by the entourages $U \cap U_\varphi$ for all the G-invariant entourages U in $\mathcal{C}^G$ and all the $\mathcal{U}^G$-admissible functions $\varphi\colon I \to \mathcal{P}(X \times X)^G$ [BEKW17, Def. 9.15].

Definition 1.3.6. [BEKW17, Def. 9.16] Let X be a G-uniform bornological coarse space. The *hybrid space* X_h is the G-bornological coarse space $X_h := (X, \mathcal{C}_h, \mathcal{B})$ with the hybrid coarse structure and the same bornology.

In order to define the forget-control map, we first need to define the cone functor $\mathcal{O}\colon G\mathbf{UBC} \to G\mathbf{BornCoarse}$ [BEKW17, Sec. 9.4] from the category of G-uniform bornological coarse spaces to the category of G-bornological coarse spaces. We now explain how to construct it.

One first considers the metric space $[0, \infty)$ endowed with the trivial G-action. By Example 1.3.5, we get the G-uniform bornological coarse space $[0, \infty)_{du}$. If X is a G-uniform bornological coarse space, the product $[0, \infty)_{du} \otimes X$ is still a G-uniform bornological coarse space by Remark 1.3.4. Let $\mathcal{Y}$ be the G-equivariant big family $\mathcal{Y}(X) := ([0, n] \times X)_{n\in\mathbb{N}}$ on $[0, \infty)_{du} \otimes X$.

Definition 1.3.7. [BEKW17, Def. 9.24] The *cone functor*

$$\mathcal{O}\colon G\mathbf{UBC} \to G\mathbf{BornCoarse}$$

sends a G-uniform bornological coarse space X to the hybrid G-bornological coarse space $([0, \infty)_{du} \otimes X)_h$. If $f\colon X \to X'$ is a morphism of G-uniform bornological coarse spaces, then $\mathcal{O}(f)\colon \mathcal{O}(X) \to \mathcal{O}(X')$ is the morphism of G-bornological coarse spaces induced by $\mathrm{id}_{[0,\infty)} \times f$.

For further details on the functoriality of the cone functor $\mathcal{O}$, we refer to [BEKW17, Sec. 9.4].

Let $\mathbb{R}$ be the set of real numbers endowed with the euclidean metric; hence, $\mathbb{R}$ is a G-uniform bornological coarse space (with trivial G-action). For a G-uniform bornological coarse space X, the product $\mathbb{R} \otimes X$ is again a G-uniform bornological coarse space; let $\mathcal{Y}$ be the equivariant big family $\mathcal{Y} := ((-\infty, n] \times X)_{n \in \mathbb{N}}$ on it. As above, these data describe a functor:

Definition 1.3.8. [BEKW18, Def. 4.9] We denote by

$$\mathcal{O}^\infty_{\mathrm{geom}} \colon G\mathbf{UBC} \to G\mathbf{BornCoarse}$$

the functor that sends a G-uniform bornological coarse space to the hybrid G-bornological coarse space $\mathcal{O}^\infty_{\mathrm{geom}}(X) := (\mathbb{R} \otimes X)_h$.

Let $\mathrm{Yo}^s_G : G\mathbf{BornCoarse} \to G\mathbf{Sp}\mathcal{X}$ be the stable Yoneda functor (1.2.5) and let $G\mathbf{Sp}\mathcal{X}$ be the category of motivic coarse spectra.

Definition 1.3.9. [BEKW18, Def. 4.10] We denote by $\mathcal{O}^\infty$ the composition

$$\mathcal{O}^\infty := \mathrm{Yo}^s_G \circ \mathcal{O}^\infty_{\mathrm{geom}} \colon G\mathbf{UBC} \to G\mathbf{Sp}\mathcal{X}.$$

This composition is called *cone-at-infinity functor*.

Let $\mathcal{F} \colon G\mathbf{UBC} \to G\mathbf{BornCoarse}$ be the forgetful functor (which forgets the uniform structure). By [BEKW17, Cor. 9.30] (see also [BEKW18, Sec. 4]), we have a fibre sequence of functors

$$\cdots \to \mathrm{Yo}^s_G \circ \mathcal{F} \to \mathrm{Yo}^s_G \circ \mathcal{O} \to \mathcal{O}^\infty \to \Sigma \,\mathrm{Yo}^s_G \circ \mathcal{F} \to \dots \tag{1.3.1}$$

called the *cone sequence*. The first map of the sequence is induced by the inclusion of X in the product $[0, \infty) \times X$, the second by the inclusion of $\mathcal{O}(X)$ in $\mathcal{O}^\infty_{\mathrm{geom}}(X)$, and the third is described as follows: the spaces $\mathcal{O}^\infty_{\mathrm{geom}}(X)$ and $\mathbb{R} \otimes \mathcal{F}(X)$ have the same underlying sets and the identity map induces a morphism $\mathrm{Yo}^s_G(\mathcal{O}^\infty_{\mathrm{geom}}(X)) \to \mathrm{Yo}^s_G(\mathbb{R} \otimes \mathcal{F}(X))$. The boundary map $\mathcal{O}^\infty \to \Sigma\,\mathrm{Yo}^s_G \circ \mathcal{F}$ is then obtained after applying the equivalence $\mathrm{Yo}^s_G(\mathbb{R} \otimes \mathcal{F}(X)) \simeq \Sigma\,\mathrm{Yo}^s_G(\mathcal{F}(X))$ (that is obtained by using excision with respect to the complementary pair of Example 1.1.20).

Definition 1.3.10. Let X be a bornological coarse space and U an entourage of X. Let μ be a probability measure on the measurable space $(X, \mathcal{P}(X))$. Then, μ is called *finite U-bounded* if there is a finite subset of X such that $F \times F$ is contained in U and $\mu(F) = 1$. The *support* $\mathrm{supp}(\mu)$ of μ is the smallest subset of X of μ-measure 1.

Definition 1.3.11. [BEKW17, Def. 11.1] Let X be a bornological coarse space and U an entourage of X. Then, we denote by $P_U(X)$ the set of finite U-bounded probability measures on X.

The set $P_U(X)$ is actually a G-simplicial complex [BEKW17, Sec. 11.1]. On the set $P_U(X)$, we consider the bornology generated by the sets $P_U(B)$ for all the bounded sets B of X and the coarse and uniform structures induced by the path metric. Then, $P_U(X)$, with these structures, is a G-uniform bornological coarse space [BEKW17, Def. 11.5] that we denote by $P_U(X)_u$. When equipped with only the bornology and the coarse structure (hence after forgetting the uniform structure), it is denoted by $P_U(X)_d$.

The construction of $P_U(X)$ can be made functorial (in pairs (X, U) of bornological coarse spaces and fixed entourage) as described in [BEKW18, Sec. 4].

Definition 1.3.12. [BEKW17, Def. 11.9] Let X be a G-bornological coarse space, U a G-invariant entourage and let $P_U(X)$ be the set of Definition 1.3.11 equipped with the bornology and the uniform and coarse structure described above. Then, we define:

$$\begin{aligned} F(X) &:= \operatorname{colim}_{U \in \mathcal{C}^G(X)} \mathrm{Yo}^s_G(P_U(X)_u) \\ F^\infty(X) &:= \operatorname{colim}_{U \in \mathcal{C}^G(X)} \mathcal{O}^\infty(P_U(X)_u) \\ F^0(X) &:= \operatorname{colim}_{U \in \mathcal{C}^G(X)} \mathrm{Yo}^s_G(P_U(X)_d). \end{aligned}$$

These refine to functors $F^0, F, F^\infty\colon G\mathbf{BornCoarse} \to G\mathbf{Sp}\mathcal{X}$ [BEKW17, Rmk. 11.11]. Composition of the functor P with the fibre sequence (1.3.1) induces a fibre sequence of functors $G\mathbf{BornCoarse} \to G\mathbf{Sp}\mathcal{X}$

$$F^0 \to F \to F^\infty \to \Sigma F^0 \tag{1.3.2}$$

(see [BEKW17, Def. 11.9] or also [BEKW18, Def. 4.15]). We are particularly interested in the boundary map $F^\infty \to \Sigma F^0$ induced by the cone boundary:

Definition 1.3.13. [BEKW17, Def. 11.10] Let X be a G-bornological coarse space. Then, the map

$$\beta_X\colon F^\infty(X) \to \Sigma F^0(X)$$

in the cone sequence (1.3.2) is called the *forget-control* map.

Remark 1.3.14. The functors F^0, F^∞ are not coarse homology theories, hence β is not a transformation of coarse homology theories; however, if E is a strong equivariant coarse homology theory, then the induced forget-control map

$$\beta\colon E \circ F^\infty \to \Sigma E \circ F^0$$

is a transformation of equivariant coarse homology theories [BEKW17, Cor. 11.26].

We now introduce the necessaryterminology for defining the coarse assembly map.

For a given group G, let $G\mathbf{Orb}$ be the category of transitive G-sets and G-equivariant maps. Let $G\mathbf{Top}$ be the category of topological spaces (compactly generated weakly Hausdorff spaces) endowed with a continuous G-action.

The cone functor $\mathcal{O}^\infty$ of Definition 1.3.9 induces a new functor [BEKW17, Def. 10.10]

$$\mathcal{O}^\infty_{\mathrm{hlg}}\colon G\mathbf{Top} \to G\mathbf{Sp}\mathcal{X} \tag{1.3.3}$$

that is an equivariant homology theory [BEKW17, Prop. 10.11]. By Elmendorf's Theorem, equivariant homology theories with values is a cocomplete stable ∞-category $\mathbf{C}$ are the same as functors $G\mathbf{Orb} \to \mathbf{C}$ (see [Blu17, Thm. 1.3.8] for a recent reformulation of the classical Elmendorf's Theorem [Elm83]). The equivariant homology theory $\mathcal{O}^\infty_{\mathrm{hlg}}$ is essentially uniquely characterized by the natural equivalence

$$\mathcal{O}^\infty_{\mathrm{hlg}}(S) \simeq \Sigma\,\mathrm{Yo}^s_G(S_{min,max}) \tag{1.3.4}$$

for S in $G\mathbf{Orb}$, where $S_{\mathrm{min,max}}$ denotes the G-bornological coarse space S with the minimal coarse structure and the maximal bornology (see [BEKW17, Prop. 9.35] and the text after [BEKW18, Rmk. 8.17]). We refer to [BEKW18, Def. 8.16] for a construction of the functor $\mathcal{O}^\infty_{\mathrm{hlg}}$.

Let $\mathbf{C}$ be a stable cocomplete ∞-category and let E be a G-equivariant $\mathbf{C}$-valued coarse homology theory, seen as a colimit preserving functor on $G\mathbf{Sp}\mathcal{X}$.

Remark 1.3.15. [BEKW17, Prop. 9.35] The composition $E \circ \mathcal{O}^\infty_{\mathrm{hlg}}\colon G\mathbf{Top} \to \mathbf{C}$ is a G-equivariant homology theory. By the equivalence (1.3.4), if E_L is the twist of E by a G-bornological coarse space L, then we get the equivalences

$$E_L \circ \mathcal{O}^\infty_{hlg}(S) \simeq \Sigma E_L(\mathrm{Yo}^s_G(S_{min,max}))$$

for S in $G\mathbf{Orb}$.

Definition 1.3.16. Let $\mathcal{F}$ be a non-empty set of subgroups of G. The set $\mathcal{F}$ is called a *family of subgroups* if it is closed under conjugation and under taking subgroups.

Example 1.3.17. Let G be a group. The trivial family $\{*\}$, the set $\mathbf{Fin}$ of all finite subgroups of G and the set $\mathbf{All}$ of all subgroups of G are families of subgroups.

Let G be a discrete group. We recall that a classifying space for a family of subgroups $\mathcal{F}$ of G is a G-CW-complex $E_{\mathcal{F}}G$ (unique up to G-homotopy) with the property that, for every H in $\mathcal{F}$, the set of H-fixed points of $E_{\mathcal{F}}G$ is contractible, and is empty otherwise.

Definition 1.3.18. We denote by $E_{\mathcal{F}}G^{cw}$ the choice of a model of the classifying space for the family of subgroups $\mathcal{F}$ of G.

The projection $E_{\mathcal{F}}G^{cw} \to \{*\}$ to the one point space is a morphism in $G\mathbf{Top}$. By applying an equivariant homology theory F we get a map $F(E_{\mathcal{F}}G^{cw}) \to F(\{*\})$, that is the *assembly map* for F and the family of subgroups $\mathcal{F}$. When F is the homology theory $\mathcal{O}^\infty_{\mathrm{hlg}}\colon G\mathbf{Top} \to G\mathbf{Sp}\mathcal{X}$, we get an assembly map that is called the *universal assembly map* [BEKW17, Def. 10.21].

For a G-topological space X, the projection $X \to \{*\}$ to the one point space induces a morphism $\alpha_X \colon \mathcal{O}^\infty_{\mathrm{hlg}}(X) \to \mathcal{O}^\infty_{\mathrm{hlg}}(*)$.

Definition 1.3.19. [BEKW17, Def. 10.24] Let Q be a G-bornological coarse space such that the G-action on the underlying G-set is free and let X be a G-topological space. Then, the morphism

$$\alpha_{X,Q} \coloneqq \alpha_X \otimes \mathrm{Yo}^s(Q) \colon \mathcal{O}^\infty_{\mathrm{hlg}}(X) \otimes \mathrm{Yo}^s(Q) \to \Sigma\, \mathrm{Yo}^s_G(Q)$$

is called the *motivic assembly map* with twist Q.

We refer to [BEKW17, Sec. 10.3] for further details. The main example for us is given by $Q = G_{\mathrm{can,min}}$ of Example 1.1.22(ii) and for X by the choice of a model of $E_{\mathbf{Fin}}G$. In fact, with these assumptions, we get the assembly map for the equivariant homology theory $E \circ \mathcal{O}^\infty_{\mathrm{hlg}}$.

We also consider twists $\beta_{X,Q}$ of the forget-control map β_X (Definition 1.3.13).

We conclude the section with a comparison result between the coarse assembly map $\alpha_{E_{\mathbf{Fin}}G, G_{\mathrm{can,min}}}$ and the forget-control map $\beta_{G_{\mathrm{can,min}}, G_{\mathrm{max,max}}}$:

Theorem 1.3.20. [BEKW17, Thm. 11.16] *Let $E \colon G\mathbf{BornCoarse} \to \mathbf{C}$ be a continuous G-equivariant coarse homology theory. Then, the coarse assembly map $\alpha_{E_{\mathbf{Fin}}G, G_{\mathrm{can,min}}}$ and the forget-control map $\beta_{G_{\mathrm{can,min}}, G_{\mathrm{max,max}}}$ induce equivalent morphisms $E(\alpha_{E_{\mathbf{Fin}}G, G_{\mathrm{can,min}}})$ and $E(\beta_{G_{\mathrm{can,min}}, G_{\mathrm{max,max}}})$.*

New developments and applications to split-injectivity results of the assembly maps can be found in [BEKW18].

1.4 Localization theorems for coarse homology theories

Let G be a discrete group and denote by $R(G)$ its representation ring [Seg68b]. Let γ be a conjugacy class of G and denote by $\mathfrak{p}$ the (prime) ideal of $R(G)$ consisting of representations in $R(G)$ with trace 0. Segal proves a localization theorem [Seg68a, Prop. 4.1] saying that, after localization at the prime ideal $\mathfrak{p}$, the map

$$K^*_G(X) \to K^*_G(X^\gamma)$$

in equivariant K-theory induced from the inclusion of the γ-fixed points X^γ in X (for a locally compact G-space X) is an isomorphism.

Inspired by Segal's localization theorem, with the aim to develop new tools for studying equivariant coarse algebraic K-homology, a general localization result for equivariant coarse homology theories has been proved [BCb]. In this section we recall and state the principal results of [BCb], in particular the coarse localization theorems Theorem 1.4.15 and Theorem 1.4.21. We will apply these theorems in Section 4.5, where a localization theorem for G-equivariant coarse Hochschild homology and cyclic homology is given.

Definition 1.4.1. [Mac71, Exercise IX.6.3] Let $\mathbf{I}$ be an ordinary category. The *twisted arrow category* $\mathbf{Tw}(\mathbf{I})$ is the category defined as follows:

1. objects: the objects of $\mathbf{Tw}(\mathbf{I})$ are arrows $i \to j$ in $\mathbf{I}$.
2. morphisms: a morphism $(i \to j) \to (i' \to j')$ is a commuting diagram

$$\begin{tikzcd} i \arrow[r] & j \arrow[d] \\ i' \arrow[u] \arrow[r] & j' \end{tikzcd}$$

with the natural compositions.

Observe that the twisted arrow category of $\mathbf{I}$ comes equipped with a functor

$$\pi : \mathbf{Tw}(\mathbf{I}) \to \mathbf{I}^{op} \times \mathbf{I} \ , \quad (i \to j) \mapsto (i, j) \ .$$

Definition 1.4.2. [GHN17, Def. 2.5] Let $\mathbf{I}$ be a category and let $\mathbf{C}$ be a cocomplete ∞-category. For a functor $F : \mathbf{I}^{op} \times \mathbf{I} \to \mathbf{C}$, we define its *coend*

$$\int^{\mathbf{I}} F := \operatorname{colim}_{\mathbf{Tw}(\mathbf{I})} F \circ \pi.$$

as the colimit over the twisted arrow category $\mathbf{Tw}(\mathbf{I})$ of $F \circ \pi$.

Let G be a discrete group, $G\mathbf{Orb}$ the category of transitive G-sets and G-equivariant maps and let $\mathbf{C}$ be a stable cocomplete ∞-category. Recall that, by Remark 1.2.3, a (non-equivariant) coarse homology theory is the same as a functor of $\mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$.

Consider a functor

$$E \colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$$

with values in non-equivariant coarse homology theories and let X be in

$$\mathbf{Fun}(G\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{X}).$$

Then, we can consider the induced functor

$$E \circ X \colon G\mathbf{Orb} \times G\mathbf{Orb}^{\mathrm{op}} \to \mathbf{C} \tag{1.4.1}$$

defined by sending a pair (S, T) to the object $E(S)(X(T))$ of $\mathbf{C}$. We can now apply the coend formula of Definition 1.4.2:

Definition 1.4.3. [BCb, Def. 3.13] As above, let $E\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$ be a functor with values in a cocomplete stable ∞-category $\mathbf{C}$. We define the functor E^G associated to E by

$$E^G(X) \coloneqq \int^{G\mathbf{Orb}} E \circ X.$$

where the index category is $G\mathbf{Orb}$ and X is a functor $X\colon G\mathbf{Orb}^{\mathrm{op}} \to \mathbf{Spc}\mathcal{X}$.

A transformation of functors $X \to X'$ induces a transformation of the pairing $E \circ X \to E \circ X'$ (that uses the functoriality of $E(S)$ as well), hence a transformation between the coend formulas.

Remark 1.4.4. We now briefly explain the motivation behind Definition 1.4.3. Let $\mathbf{C}$ be a cocomplete stable ∞-category. In equivariant homotopy theory, as a consequence of Elmendorf's theorem [Elm83], a $\mathbf{C}$-valued equivariant homology theory is essentially uniquely determined by a functor $H\colon G\mathbf{Orb} \to \mathbf{C}$. There is an equivalence of ∞-categories between $\mathbf{C}$ and the colimit-preserving functors from the ∞-category of spaces $\mathbf{Spc}$ to $\mathbf{C}$, and by using this equivalence, we can interpret H as a functor

$$H\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}, \mathbf{C}).$$

If X is G-space, there is a well-defined pairing $H \circ X$ defined by sending the pair $(S,T) \in G\mathbf{Orb} \times G\mathbf{Orb}^{\mathrm{op}}$ to the object $H(S)(X(T)) \in \mathbf{C}$ and its evaluation can be described in terms of a coend formula [BCb, Rmk. 3.11]. This interpretation can be used in order to get abstract localization results [BCb]. In the case of equivariant coarse homology theories and of G-bornological coarse spaces, the ∞-category $\mathbf{Spc}$ of spaces is replaced by the ∞-category $\mathbf{Spc}\mathcal{X}$ of coarse motivic spaces, which leads to the Definition 1.4.3.

The functor E^G is a coarse homology theory when pre-composed with a suitable orbit functor, as we now explain.

Let G be a group and H a subgroup of G and denote by $W_G(H) \coloneqq N_G(H)/H$ the associated Weyl group. If X is a G-bornological coarse space, we consider the subset X^H of H-fixed points, which has an induced $W_G(H)$-action. In order to endow X^H with a compatible coarse structure and bornology, we embed it in the G-completion $B_G X$ of X (see Definition 1.1.23); the set X^H, endowed with the bornology and coarse structure induced by the inclusion $X^H \to B_G X$ (see Example 1.1.14), is a $W_G(H)$-bornological coarse space. A morphism $f\colon X \to Y$ of G-bornological coarse spaces restricts to a morphism $f^H\colon X^H \to Y^H$ of $W_G(H)$-bornological coarse spaces. Then, we get an H-fixed points functor

$$(-)^H\colon G\mathbf{BornCoarse} \to W_G(H)\mathbf{BornCoarse}$$

sending X to the H-fixed points X^H of X. The composition with the Yoneda functor $\mathrm{Yo}_{W_G(H)}\colon W_G(H)\mathbf{BornCoarse} \to W_G(H)\mathbf{Spc}\mathcal{X}$ (1.2.3) describes a new functor

$$G\mathbf{BornCoarse} \to W_G(H)\mathbf{Spc}\mathcal{X},$$

that can be refined to a *motivic orbit functor* [BCb, Lemma 3.6 & Def. 3.8]

$$\tilde{Y}\colon G\mathbf{BornCoarse} \to \mathbf{Fun}(G\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{X}) \tag{1.4.2}$$

(roughly) defined by sending a G-bornological coarse space X to the functor which sends the G -set G/H to the motivic coarse space $\mathrm{Yo}(X^H)$ of the bornological coarse space of H-fixed points X^H. In this way, we take the dependence on the subgroup H into account properly; we refer to [BCb, Sec. 3.3] for further details on the definition of the motivic orbit functor.

Let E^G be the functor constructed in Definition 1.4.3. Then, we get a G-equivariant coarse homology theory:

Lemma 1.4.5. [BCb, Cor. 3.15] *Let $E\colon G\mathbf{Orb} \to \mathbf{Fun}^{colim}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$ be a functor with values in a stable cocomplete ∞-category $\mathbf{C}$. Then, the functor*

$$E^G \circ \tilde{Y}\colon G\mathbf{BornCoarse} \to \mathbf{C}$$

is a G-equivariant coarse homology theory.

We do not know whether all the $\mathbf{C}$-valued equivariant coarse homology theories are of the form E^G for some functor $E\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$ [BCb, Rmk. 3.16]. Such functors will be called of Bredon-style:

Definition 1.4.6. [BCb, Def. 3.17] Let $E\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$ as above. A G-equivariant coarse homology theory of the form E^G is called a *Bredon-style equivariant coarse homology theory.*

Bredon-style equivariant coarse homology theories are important to us because of the coarse localization theorems Theorem 1.4.15 and Theorem 1.4.21; in fact, these theorems are stated for Bredon-style equivariant coarse homology theories. However, every equivariant coarse homology theory E can be approximated by a Bredon-style equivariant coarse homology theory E^{Bredon} as follows.

Remark 1.4.7. Let $\mathbf{C}$ be a stable ∞-category and let $E \in \mathbf{Fun}^{\mathrm{colim}}(G\mathbf{Spc}\mathcal{X}.\mathbf{C})$ be a functor corresponding to an equivariant coarse homology theory $E \circ \mathrm{Yo}_G$. Consider the functor

$$\underline{E}\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$$

defined as $\underline{E}(S)(X) \coloneqq E(S_{\min,\max} \otimes X)$. We can then apply the coend formula to $\underline{E}$ and $X \in \mathbf{Fun}(G\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{X})$ as in Definition 1.4.3 obtaining a new functor $\underline{E}^G$.

We call $Y\colon G\mathbf{Spc}\mathcal{X} \to \mathbf{Fun}(G\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{X})$ the factorization of the motivic orbit functor (1.4.2) through the Yoneda functor Yo_G.

Definition 1.4.8. [BCb, Def. 3.29] The *Bredon-style equivariant coarse homology theory* E^{Bredon} associated to a functor $E\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$ is the composition

$$E^{Bredon} := \underline{E}^G \circ Y \colon G\mathbf{Spc}\mathcal{X} \to \mathbf{C}.$$

Consider the following categories:

Definition 1.4.9. [BCb, Def. 3.34] Let $\mathbf{Sp}\mathcal{X}_{bd}$ denote the full subcategory of $\mathbf{Sp}\mathcal{X}$ generated under colimits by the objects $\mathrm{Yo}^s(X)$, with X a bounded bornological coarse space.

Definition 1.4.10. [BCb, Def. 3.46] Let

$$G\mathbf{Sp}\mathcal{X}\langle G\mathbf{Orb} \otimes \mathbf{Sp}\mathcal{X}_{bd}\rangle$$

be the full subcategory of $G\mathbf{Sp}\mathcal{X}$ generated under colimits by the motives $S_{min,max} \otimes X$ for S in $G\mathbf{Orb}$ and X in $\mathbf{Sp}\mathcal{X}_{bd}$.

Let $E\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$ be a functor and E^{Bredon} be its associated Beredon-style coarse homology theory. There is a comparison map

$$E^{Bredon} \to E \tag{1.4.3}$$

[BCb, Sec. 3.5]. One may ask under which conditions on the functor E (and on the bornological coarse space X) the comparison map (1.4.3) provides an equivalence. By [BCb, Cor. 3.43], this is true if the group G is finite and X belongs to the category $G\mathbf{Sp}\mathcal{X}\langle G\mathbf{Orb} \otimes \mathbf{Sp}\mathcal{X}\rangle$, or, by [BCb, Cor. 3.47], if E is *hyperexcisive* [BCb, Def. 3.32] and X belongs to the category $G\mathbf{Sp}\mathcal{X}\langle G\mathbf{Orb} \otimes \mathbf{Sp}\mathcal{X}_{bd}\rangle$.

We do not give the definition of hyperexcisive coarse homology theory here; however, if E is a coarse homology theory and $E_{G_{\mathrm{can,min}}}$ denotes its twist by the G-bornological coarse space $G_{\mathrm{can,min}}$, then the following is true:

Remark 1.4.11. [BCb, Lemma 3.37] The twist $E_{G_{\mathrm{can,min}}}$ of a continuous equivariant coarse homology theory E is an hyperexcisive coarse homology theory.

Let G be a discrete group and let $\mathcal{F}$ be a set of subgroups of G that is invariant under conjugation. It determines the full subcategory $G_{\mathcal{F}}\mathbf{Orb}$ of $G\mathbf{Orb}$ of transitive G-sets with stabilizers in $\mathcal{F}$. Moreover, the inclusion of $G_{\mathcal{F}}\mathbf{Orb}$ in $G\mathbf{Orb}$ induces adjunctions:

$$\mathrm{Ind}_{\mathcal{F}} : \mathbf{Fun}(G_{\mathcal{F}}\mathbf{Orb}^{op}, \mathbf{D}) \leftrightarrows \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathbf{D}) : \mathrm{Res}_{\mathcal{F}}$$

and

$$\mathrm{Res}_{\mathcal{F}} : \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathbf{D}) \leftrightarrows \mathbf{Fun}(G_{\mathcal{F}}\mathbf{Orb}^{op}, \mathbf{D}) : \mathrm{Coind}_{\mathcal{F}}$$

for every presentable ∞-category $\mathbf{D}$ [Lur09, Corollary 5.5.2.9].

Definition 1.4.12. Let $\mathcal{F}$ be a conjugation invariant set of subgroups of G. Then, $\mathcal{F}^{\perp} := \mathbf{All} \setminus \mathcal{F}$ denotes the complement of $\mathcal{F}$ in the set $\mathbf{All}$ of all subgroups of G.

Definition 1.4.13. Let X be in $\mathbf{Fun}(G\mathbf{Orb}^{\mathrm{op}}, \mathbf{D})$, where $\mathbf{D}$ is a presentable ∞-category. We define

$$X^{\mathcal{F}} := \mathrm{Ind}_{\mathcal{F}^{\perp}} \mathrm{Res}_{\mathcal{F}^{\perp}} X \tag{1.4.4}$$

and we denote by $X^{\mathcal{F}} \to X$ the morphism induced by the counit of the adjunction (1.4).

Definition 1.4.14. [BCb, Def. 2.6] Let $E\colon G\mathbf{Orb} \to \mathbf{C}$ be a functor, with $\mathbf{C}$ a stable ∞-category, and let $\mathcal{F}$ be a conjugation invariant set of subgroups of G. We say that the functor E *vanishes on* $\mathcal{F}$ if $E(S) \simeq 0$ for all S in $G_{\mathcal{F}}\mathbf{Orb}$.

We can now formulate the Coarse Abstract Localization Theorem I. Let $\mathcal{F}$ be a family of subgroups of G and let X be an object of $\mathbf{Fun}(G\mathbf{Orb}^{op}, \mathbf{Spc}\mathcal{X})$. Let $E\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$ be a functor with $\mathbf{C}$ a stable cocomplete ∞-category.

Theorem 1.4.15 (Coarse Abstract Localization Theorem I)**.** [BCb, Thm. 4.1] *If E vanishes on $\mathcal{F}$, then the map $X^{\mathcal{F}} \to X$ induces an equivalence*

$$E^G(X^{\mathcal{F}}) \to E^G(X)$$

where E^G is the functor associated to E in Definition 1.4.3.

For a second version of the localization theorem, we first need to fix a conjugacy class γ of G and to restrict to a class of *nice* bornological coarse spaces, as we shall see.

Definition 1.4.16. [BCb, Def. 4.2] Let X be a G-bornological coarse space and γ be a conjugacy class of the group G. Define the G-bornological coarse space of γ-fixed points X^{γ} as the G-invariant subset of X

$$X^{\gamma} := \bigcup_{g \in \gamma} X^g$$

with the induced bornology and coarse structures.

A G-invariant subset A of X generates a big family $\{A\}$ (see Example 1.1.27). If F is a functor from $G\mathbf{BornCoarse}$ to a cocomplete category, then we let $F(\{A\})$ be the colimit

$$F(\{A\}) := \mathrm{colim}_{A' \in \{A\}} F(A').$$

Recall the Yoneda functor $\mathrm{Yo}_G : G\mathbf{BornCoarse} \to G\mathbf{Spc}\mathcal{X}$ (1.2.3).

Definition 1.4.17. [BCb, Def. 4.3] We say that a G-invariant subset A of X is a *nice* subset if the natural morphism

$$\mathrm{Yo}_G(A) \to \mathrm{Yo}_G(\{A\})$$

is an equivalence.

Recall the construction of the functor $\mathcal{O}^{\infty}_{\mathrm{geom}}\colon G\mathbf{UBC} \to G\mathbf{BornCoarse}$ of Definition 1.3.8.

Example 1.4.18. [BCb] Let W be a finite G-simplicial complex and γ a conjugacy class of G. Then, W is a uniform G-bornological coarse space with the uniform structure induced from the path metric. The subset W^{γ} of γ-fixed points is a uniform G-bornological coarse space. The cones $\mathcal{O}^{\infty}_{\mathrm{geom}}(W)$ and $\mathcal{O}^{\infty}_{\mathrm{geom}}(W^{\gamma})$ are G-bornological coarse spaces. By [BCb, Lemma 3.4], $\mathcal{O}^{\infty}_{\mathrm{geom}}(W^{\gamma})$ is equivalent to $\mathcal{O}^{\infty}_{\mathrm{geom}}(W)^{\gamma}$; moreover, $\mathcal{O}^{\infty}_{\mathrm{geom}}(W)^{\gamma}$ is a nice subset of $\mathcal{O}^{\infty}_{\mathrm{geom}}(W)$.

Definition 1.4.19. Let γ be a conjugacy class of the group G. Let $\mathcal{F}(\gamma)$ be the set of subgroups of G

$$\mathcal{F}(\gamma) \coloneqq \{H < G \mid H \cap \gamma = \emptyset\}$$

that do not intersect the conjugacy class γ.

Remark 1.4.20. The set $\mathcal{F}(\gamma)$ is a family of subgroups.

We can finally state the Coarse Abstract Localization Theorem II.

Consider a functor $E\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{C})$, with $\mathbf{C}$ a stable cocomplete ∞-category. Let X be a G-bornological coarse space and γ a conjugacy class of G. Let $\tilde{Y}$ be the motivic orbit functor (1.4.2).

Theorem 1.4.21 (Coarse Abstract Localization Theorem II)**.** [BCb, Thm. 4.4] *Assume that:*

1. *X^{γ} is a nice subset of X;*
2. *E vanishes on $\mathcal{F}(\gamma)$;*

then, the map

$$E^{G}(\tilde{Y}(X^{\gamma})) \to E^{G}(\tilde{Y}(X))$$

induced by the inclusion $X^{\gamma} \to X$ is an equivalence.

Let E be a $\mathbf{C}$-valued equivariant coarse homology theory seen as a functor in $\mathbf{Fun}^{\mathrm{colim}}(G\mathbf{Spc}\mathcal{X}, \mathbf{C})$ and $\underline{E}$ the functor of Remark 1.4.7. We have the following corollary of the coarse localization theorem:

Corollary 1.4.22. [BCb, Cor. 4.9] *Let X be a G-bornological coarse space and γ a conjugacy class of G. Let X^{γ} be a nice subset of X and assume that $\underline{E}$ vanishes on $\mathcal{F}(\gamma)$. Then, the morphism*

$$E^{Bredon}(\mathrm{Yo}_G(X^{\gamma})) \to E^{Bredon}(\mathrm{Yo}_G(X)$$

induced by the inclusion is an equivalence.

1.5 Equivariant coarse ordinary homology

A first example of equivariant coarse homology theories is given by the coarse version of ordinary homology
$$\mathcal{X}\mathrm{H}^G\colon G\mathbf{BornCoarse} \to \mathbf{Sp},$$
that is a spectra-valued equivariant coarse homology theory [BE16, BEKW17]. The goal of this section is to recall its construction.

Let X be a bornological coarse space, $n \in \mathbb{N}$ a natural number, B a bounded set of X, and $x = (x_0, \dots, x_n)$ a point of X^{n+1}. We say that x *meets* B if there exists an index $i \in \{0, \dots, n\}$ such that x_i belongs to B. If U is an entourage of X, we say that x is U*-controlled* if, for each i and j in $\{0, \dots, n\}$, the pair (x_i, x_j) belongs to U.

An n*-chain* c on X is a function $c\colon X^{n+1} \to \mathbb{Z}$; its *support* $\mathrm{supp}(c)$ is defined as the set of points for which the function c is non-zero:

$$\mathrm{supp}(c) = \{x \in X^{n+1} \mid c(x) \neq 0\}. \tag{1.5.1}$$

We say that an n-chain c is U*-controlled* if every point x of $\mathrm{supp}(c)$ is U-controlled. The chain c is *locally finite* if, for every bounded set B, the set of points in $\mathrm{supp}(c)$ which meet B is finite. An n-chain $c\colon X^{n+1} \to \mathbb{Z}$ is *controlled* if it is locally finite and U-controlled for some entourage U of X.

Definition 1.5.1. Let X be a bornological coarse space and $n \in \mathbb{N}$ a natural number. Then, $\mathcal{X}C_n(X)$ denotes the free abelian group generated by the locally finite controlled n-chains on X.

We will also represent n-chains as formal sums

$$\sum_{x \in X^{n+1}} c(x)x,$$

that are locally finite and U-controlled. The boundary map $\partial\colon \mathcal{X}C_n(X) \to \mathcal{X}C_{n-1}(X)$ is defined as the alternating sum $\partial \coloneqq \sum_i (-1)^i \partial_i$ of the face maps:

$$\partial_i(x_0, \dots, x_n) \coloneqq (x_0, \dots, \hat{x}_i, \dots, x_n).$$

As the n-chains of $\mathcal{X}C_n(X)$ are locally finite and controlled, then ∂_i extends linearly to a map $\partial_i\colon \mathcal{X}C_n(X) \to \mathcal{X}C_{n-1}(X)$.

Lemma 1.5.2. [BE16, Sec. 6.3] *Let X be a bornological coarse space. The graded abelian group $\mathcal{X}C_*(X)$, endowed with the boundary operator ∂ extended linearly to $\mathcal{X}C_*(X)$, is a chain complex.*

When X is a G-bornological coarse space, we let $\mathcal{X}C_n^G(X)$ be the subgroup of $\mathcal{X}C_n(X)$ given by the locally finite controlled n-chains that are also G-invariant. The boundary operator restricts to $\mathcal{X}C_*^G(X)$, and $(\mathcal{X}C_*^G(X), \partial)$ is a subcomplex of $(\mathcal{X}C_*(X), \partial)$.

It is easy to see that the chain complexes $\mathcal{X}C_*(X)$ and $\mathcal{X}C_*^G(X)$ are functorial in X; namely, if $f \colon X \to Y$ is a morphism of G-bornological coarse spaces, then we consider the map on the products $X^n \to Y^n$ sending $(x_0, \ldots, x_n)$ to $(f(x_0), \ldots, f(x_n))$. This extends linearly to a map

$$\mathcal{X}C^G(f) \colon \mathcal{X}C_n^G(X) \to \mathcal{X}C_n^G(Y) \tag{1.5.2}$$

that is well-defined because the map f is proper and controlled and because the chains are locally finite. In fact, the map $\mathcal{X}C^G(f)$ involves sums over the pre-images by f, that are finite. Moreover, $\mathcal{X}C^G(f)$ sends controlled n-chains to controlled n-chains because f is controlled.

We have described a functor

$$\mathcal{X}C^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch} \tag{1.5.3}$$

with values in the category $\mathbf{Ch}$ of chain complexes over the integers $\mathbb{Z}$.

The ∞-category $\mathbf{Ch}_\infty$ of chain complexes is defined as the localization (in the realm of ∞-categories) of the nerve of the category $\mathbf{Ch}$ at the class W of quasi-isomorphism of chain complexes:

$$\mathbf{Ch}_\infty := \mathrm{N}(\mathbf{Ch})[W^{-1}]. \tag{1.5.4}$$

as described in [Lur14, Sec. 1.3.4].

There is a standard way to go from chain complexes to spectra that uses the equivalence between the ∞-category of chain complexes $\mathbf{Ch}_\infty$ and $H\mathbb{Z} - \mathbf{Mod}$ of $H\mathbb{Z}$-modules [Shi07, Thm. 1.1]. This is the Eilenberg-MacLane correspondence

$$\mathcal{EM} \colon \mathbf{Ch} \xrightarrow{loc} \mathbf{Ch}_\infty \xrightarrow{\simeq} H\mathbb{Z} - \mathbf{Mod} \to \mathbf{Sp} \tag{1.5.5}$$

between the ∞-category of chain complexes and the ∞-category of spectra, where *loc* is the localization functor (see [BE16, Sec. 6.3]); among the properties of the functor $\mathcal{EM}$, it is remarkable to say that it sends chain homotopic maps to equivalences, short exact sequences of chain complexes to spectra and it preserves filtered colimits. By post-composing the functor $\mathcal{X}C^G$ with the functor $\mathcal{EM}$, we get a coarse version of equivariant coarse ordinary homology:

$$\mathcal{X}\mathrm{H}^G := \mathcal{EM} \circ \mathcal{X}C^G \colon G\mathbf{BornCoarse} \to \mathbf{Sp}. \tag{1.5.6}$$

Theorem 1.5.3 ([BE16, BEKW17]). *The functor $\mathcal{X}\mathrm{H}^G$ is a G-equivariant* $\mathbf{Sp}$*-valued coarse homology theory.*

Notation 1.5.4. When we want to specify the ring of coefficients, we denote by $\mathcal{X}\mathrm{H}_R^G(X)$

the G-equivariant coarse ordinary homology with R coefficients.

Example 1.5.5. Assume that $X = \{*\}$ is the one point space. Then, the chain complex $\mathcal{X}C_*(X)$ has one free generator in each dimension, and the boundary maps are either the null map or the identity, depending on the degree. The coarse homology groups are 0 in positive and negative degree and $\mathbb{Z}$ otherwise. Hence, $\mathcal{X}\mathrm{H}(*) \simeq \mathrm{H}\mathbb{Z}$.

Example 1.5.6. If $X \in G\mathbf{BornCoarse}$ has a trivial G-action, then $\mathcal{X}\mathrm{H}^G(X) \simeq \mathcal{X}\mathrm{H}(X)$.

Example 1.5.7 (Proposition 7.5 [BEKW17])**.** Let G be a group, and consider the G-bornological coarse space $G_{can,mn}$ endowed with the canonical coarse structure and the minimal bornology as described in Example 1.1.22 (ii); then the homology of the chain complex $\mathcal{X}C^G(G_{\mathrm{can,min}})$ is isomorphic to the group homology of G:

$$\mathrm{H}_*(\mathcal{X}C^G(G_{\mathrm{can,min}})) \cong \mathrm{H}_*(G;\mathbb{Z}).$$

Chapter 2

The symmetric monoidal category of controlled objects

Let X be a G-bornological coarse space (Definition 1.1.21) and let $\mathbf{A}$ be an additive category with a strict G-action (Definition A.1.3). From these data we can define a category of objects that are "parametrized" on X: this is the category $V_{\mathbf{A}}^G(X)$ of G-equivariant X-controlled $\mathbf{A}$-objects [BEKW17, Def. 8.3]. The definition of this category is of fundamental importance for studying properties of bornological coarse spaces; for example, it is a flasque category provided the G-bornological coarse space X is flasque (Lemma 2.1.15), and coarse equivalences induce equivalent categories (Lemma 2.1.14).

The correspondence $X \mapsto V_{\mathbf{A}}^G(X)$ is functorial in the G-bornological coarse space X and yields a functor

$$V_{\mathbf{A}}^G \colon G\mathbf{BornCoarse} \to \mathbf{Add}$$

from the category of G-bornological coarse spaces to the category of small additive categories $\mathbf{Add}$. This functor, when considered with values in the ∞-category $\mathbf{Add}_\infty$ of additive categories (see Definition A.3.7), is lax symmetric monoidal [BCa, Thm. 3.26] and this fact allows us to refine many equivariant coarse homology theories to lax symmetric monoidal functors. An example of equivariant coarse homology theory is G-equivariant coarse K-homology $K\mathbf{A}\mathcal{X}^G$ [BEKW17, Def. 8.8], defined as the (non-connective) K-theory of the additive category $V_{\mathbf{A}}^G(X)$ of G-equivariant X-controlled $\mathbf{A}$-objects:

$$K\mathbf{A}\mathcal{X}^G := K \circ V_{\mathbf{A}}^G \colon G\mathbf{BornCoarse} \to \mathbf{Sp}.$$

In Section 2.1, we recall the definition of the category $V_{\mathbf{A}}^G(X)$, we construct the functor $V_{\mathbf{A}}^G \colon G\mathbf{BornCoarse} \to \mathbf{Add}$ and we review its main properties. In Section 2.2, we we see how to refine the functor $V_{\mathbf{A}}^G$ to a lax symmetric monoidal one. In the last section we recall the definition of equivariant coarse algebraic K-homology as defined in [BEKW17].

2.1 The category of controlled objects

The goal of this section is to recall the definition of the additive category $V_{\mathbf{A}}^G(X)$ of G-equivariant X-controlled $\mathbf{A}$-objects [BEKW17, Def. 8.3] and of the functor

$$V_{\mathbf{A}}^G\colon G\mathbf{BornCoarse} \to \mathbf{Add}.$$

sending a G-bornological coarse space to the category $V_{\mathbf{A}}^G(X)$. We will show that close morphisms of bornological coarse spaces (see Definition 1.1.24) induce naturally isomorphic functors between the associated categories of controlled objects (see Lemma 2.1.14). Moreover, we will show that the category $V_{\mathbf{A}}^G(X)$ is a flasque category (see Definition A.1.4) provided the G-bornological coarse space X is also flasque (see Lemma 2.1.15).

Our main reference is [BEKW17, Sec. 8.2] where the main results of this section are given.

Let G be a group and let X be a G-bornological coarse space.

Remark 2.1.1. The bornology $\mathcal{B}(X)$ on X defines a poset with the partial order induced by subset inclusion; hence, $\mathcal{B}(X)$ can be seen as a category.

Let $\mathbf{A}$ be an additive category with strict G-action (Definition A.1.3). For every element g in G and every functor $F\colon \mathcal{B}(X) \to \mathbf{A}$, let $gF\colon \mathcal{B}(X) \to \mathbf{A}$ denote the functor sending a bounded set B in $\mathcal{B}(X)$ to the $\mathbf{A}$-object $g(F(g^{-1}(B)))$ (and defined on morphisms $B \subseteq B'$ as the induced morphism of $\mathbf{A}$ $(gF)(B \subseteq B')\colon gF(g^{-1}(B)) \to gF(g^{-1}(B')))$. If $\eta\colon F \to F'$ is a natural transformation between two functors $F, F'\colon \mathcal{B}(X) \to \mathbf{A}$, we denote by $g\eta\colon gF \to gF'$ the induced natural transformation between gF and gF'.

Definition 2.1.2. [BEKW17, Def. 8.3] Let G be a group, X a G-bornological coarse space and $\mathbf{A}$ an additive category with strict G-action. A *G-equivariant X-controlled* $\mathbf{A}$-*object* is a pair (M, ρ) consisting of a functor $M\colon \mathcal{B}(X) \to \mathbf{A}$ and a family $\rho = (\rho(g))_{g\in G}$ of natural isomorphisms $\rho(g)\colon M \to gM$, satisfying the following conditions:

1. $M(\emptyset) \cong 0$;
2. for all B, B' in $\mathcal{B}(X)$, the commutative diagram
$$\begin{array}{ccc} M(B\cap B') & \longrightarrow & M(B) \\ \downarrow & & \downarrow \\ M(B') & \longrightarrow & M(B\cup B') \end{array}$$
is a push-out;
3. for all B in $\mathcal{B}(X)$ there exists a finite subset F of B such that the inclusion induces an isomorphism $M(F) \xrightarrow{\cong} M(B)$;

4. for all elements g, g' in G we have the relation $\rho(gg') = g\rho(g') \circ \rho(g)$, where $g\rho(g')$ is the natural transformation from gM to $gg'M$ induced by $\rho(g')$.

Notation 2.1.3. If (M, ρ) is an X-controlled $\mathbf{A}$-object and x is an element of X, we will often write $M(x)$ istead of $M(\{x\})$ for the value of the functor M at the bounded set $\{x\}$ of X.

The following lemma is a consequence of the above definition:

Lemma 2.1.4. [BEKW17, Lemma 8.4] *In the same hypothesis of Definition 2.1.2, let (M, ρ) be a G-equivariant X-controlled $\mathbf{A}$-object. Then,*

(i) the canonical morphism $\sum_{x\in F}(M(\{x\} \hookrightarrow F))\colon \bigoplus_{x\in F} M(\{x\}) \to M(F)$ is an isomorphism for every finite subset F of X;

(ii) for all finite subsets $F \subseteq F'$ of X the following diagram is commutative:

$$\begin{tikzcd}
\bigoplus_{x\in F} M(\{x\}) \arrow[r, "\sum_{x\in F}(M(\{x\}\subseteq F))"] \arrow[d] & M(F) \arrow[d] \\
\bigoplus_{x\in F'} M(\{x\}) \arrow[r, "\sum_{x\in F'}(M(\{x\}\subseteq F'))"] & M(F')
\end{tikzcd}$$

(iii) for each bounded set B of X, there exists a unique minimal finite subset $F_B \subseteq B$ for which the induced morphism $M(F_B) \to M(B)$ is an isomorphism. Moreover, for every subset B' of X with $F_B \subseteq B' \subseteq B$, the morphisms $M(F_B) \to M(B')$ and $M(B') \to M(B)$ are isomorphisms.

Proof. The items (i) and (ii) are clear from Definition 2.1.2.

We prove (iii). Let B be a bounded set of X and suppose that F and F' are two finite subsets of B such that the morphisms $M(F) \to M(B)$ and $M(F') \to M(B)$ induced by the inclusions $F \subseteq B$ and $F' \subseteq B$ are isomorphisms. The square

$$\begin{tikzcd}
M(F \cap F') \arrow[r] \arrow[d] & M(F) \arrow[d, "\cong"] \\
M(F') \arrow[r, "\cong"] & M(B)
\end{tikzcd}$$

is a push-out square and all the maps are inclusions of direct summands. This implies that the morphism $M(F \cap F') \to M(B)$ is an isomorphism; hence, there exists a minimal finite subset F_B of B for which the morphism $M(F_B) \to M(B)$ is an isomorphism.

In a similar way one proves that, for every subset B' of X such that $F_B \subseteq B' \subseteq B$, the morphism $M(F_B) \to M(B')$ is an isomorphism, hence $M(B') \to M(B)$ is an isomorphism as well. $\square$

Definition 2.1.5. Let (M,ρ) be an equivariant X-controlled $\mathbf{A}$-object. The function σ which sends a bounded set B in $\mathcal{B}(X)$ to the minimal finite subset F_B of Lemma 2.1.4 is called the *support function* of (M,ρ).

Let X be a G-bornological coarse space. Recall that for every bounded set B of X and every entourage U we can consider the U-thickening $U[B]$ (1.1.2), which is again bounded due to the compatibility of the bornology and coarse structure on X (see Definition 1.1.10). As U preserves the inclusions of bounded sets, this describes a functor

$$U[-]\colon \mathcal{B}(X)\to\mathcal{B}(X),$$

where $\mathcal{B}(X)$ is the family of bounded sets of X seen as a category (see Remark 2.1.1). Observe that, if U is a G-invariant entourage of X, then $U[gB]=g(U[B])$ for every g in G.

Definition 2.1.6. [BEKW17, Def. 8.6] Let X be a G-bornological coarse space, let (M,ρ) and (M',ρ') be G-equivariant X-controlled $\mathbf{A}$-objects and let $U\in\mathcal{C}^G(X)$ be a G-invariant entourage of X. A *G-equivariant U-controlled morphism* $\varphi\colon(M,\rho)\to(M',\rho')$ is a natural transformation

$$\varphi\colon M(-)\to M'\circ U[-]$$

such that $\rho'(g)\circ\varphi=(g\varphi)\circ\rho(g)$ for all g in G.

The set of G-equivariant U-controlled morphisms $\varphi\colon(M,\rho)\to(M',\rho')$ is denoted by $\mathrm{Mor}_U((M,\rho),(M',\rho'))$.

Remark 2.1.7. The set $\mathrm{Mor}_U((M,\rho),(M',\rho'))$ is an abelian group with operation induced by $\mathbf{A}$.

Let X be a G-bornological coarse space and let U and U' be G-invariant entourages of X with $U\subseteq U'$. For every bounded set B of X, the inclusion $U\subseteq U'$ induces an inclusion $U[B]\subseteq U'[B]$; this yields a natural transformation of functors $M'\circ U[-]\to M'\circ U'[-]$, hence a map

$$\mathrm{Mor}_U((M,\rho),(M',\rho'))\to\mathrm{Mor}_{U'}((M,\rho),(M',\rho'))$$

by post-composition.

By using these structure maps we define the abelian group of *G-equivariant controlled morphisms* from (M,ρ) to (M',ρ') as the colimit

$$\mathrm{Hom}_{V^G_{\mathbf{A}}(X)}((M,\rho),(M',\rho')) := \mathrm{colim}_{U\in\mathcal{C}^G}\,\mathrm{Mor}_U((M,\rho),(M',\rho')).$$

We now describe the composition. Consider two morphisms in $\mathrm{Hom}_{V^G_{\mathbf{A}}(X)}((M,\rho),(M',\rho'))$ and $\mathrm{Hom}_{V^G_{\mathbf{A}}(X)}((M',\rho'),(M'',\rho''))$ respectively and let $\varphi\colon M\to M'\circ U[-]$ and $\varphi'\colon M'\to M''\circ U'[-]$ represent them. By using Remark 1.1.9, we see that φ and φ' induce the following composition of functors

$$M \to M' \circ U[-] \to M'' \circ (U \circ U')[-]$$

where $U \circ U'$ denotes the composition of entourages. The composition of the two morphisms is defined as the morphism of $\mathrm{Hom}_{V^G_{\mathbf{A}}(X)}((M, \rho), (M'', \rho''))$ represented by the composition $M \to M' \circ U[-] \to M'' \circ (U \circ U')[-]$.

Definition 2.1.8. Let X be a G-bornological coarse space and let $\mathbf{A}$ be an additive category with strict G-action. The category $V^G_{\mathbf{A}}(X)$ is the category of G-equivariant X-controlled $\mathbf{A}$-objects and G-equivariant controlled morphisms.

Notation 2.1.9. When $\mathbf{A}$ is the category of finitely generated free R-modules, with R a commutative ring, then we denote by $V^G_R(X)$ the associated category of G-equivariant X-controlled (finitely generated free) R-modules.

Recall that an additive category is a category enriched on abelian groups, with a zero object and all finite biproducts (Definition A.1.1).

Lemma 2.1.10. [BEKW17, Lemma 8.7] *The category $V^G_{\mathbf{A}}(X)$ of G-equivariant X-controlled $\mathbf{A}$-objects is an additive category.*

Proof. Let X be a G-bornological coarse space with bornology $\mathcal{B}$. Let (M, ρ) and (M', ρ') be two equivariant controlled objects in $V^G_{\mathbf{A}}(X)$. The category $\mathbf{A}$ is an additive category and it induces a direct sum operation in the functor category $\mathbf{Fun}(\mathcal{B}, \mathbf{A})$. Define the functor $M \oplus M' \colon \mathcal{B} \to \mathbf{A}$ as the direct sum in $\mathbf{Fun}(\mathcal{B}, \mathbf{A})$. In the same way, we define $\rho \oplus \rho'(g) \coloneqq \rho(g) \oplus \rho'(g)$ for every g in G.

The pair $(M \oplus M', \rho \oplus \rho')$ satisfies the conditions of Definition 2.1.2 and is a G-equivariant X-controlled $\mathbf{A}$-object. There are isomorphisms of G-sets

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{B},\mathbf{A})}(M \oplus M', C \circ U[-]) \cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{B},\mathbf{A})}(M, C \circ U[-]) \times \mathrm{Hom}_{\mathbf{Fun}(\mathcal{B},\mathbf{A})}(M', C \circ U[-])$$

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{B},\mathbf{A})}(C, M \oplus M' \circ U[-]) \cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{B},\mathbf{A})}(C, M \circ U[-]) \times \mathrm{Hom}_{\mathbf{Fun}(\mathcal{B},\mathbf{A})}(C, M' \circ U[-])$$

for every G-invariant entourage U of X and G-equivariant X-controlled $\mathbf{A}$-object (C, η). This implies that $(M \oplus M', \rho \oplus \rho')$ is also a biproduct in the category $V^G_{\mathbf{A}}(X)$.

The biproduct of morphisms of $V^G_{\mathbf{A}}(X)$ is defined analogously and is induced by the direct sum operation of morphisms in the functor category. □

Remark 2.1.11. If the category $\mathbf{A}$ is a k-linear category with strict G-action, then the category of equivariant X-controlled $\mathbf{A}$-objects $V^G_{\mathbf{A}}(X)$ is a k-linear category. For example, the category $V^G_k(X)$ of G-equivariant X-controlled finite dimensional k-vector spaces is a k-linear category.

We now describe the functoriality of $V^G_{\mathbf{A}}(X)$ with respect to X.

Let $f\colon (X, \mathcal{C}, \mathcal{B}) \to (X', \mathcal{C}', \mathcal{B}')$ be a morphism of G-bornological coarse spaces. If (M, ρ) is a G-equivariant X-controlled $\mathbf{A}$-object, we consider the functor

$$f_*M\colon \mathcal{B}' \to \mathbf{A}$$

defined by $f_*M(B') := M(f^{-1}(B'))$ for every bounded set B' in $\mathcal{B}'$ and defined on morphisms in the canonical way; observe that the functor is well-defined as f is proper. For every g in G, the family of transformation $f_*\rho = ((f_*\rho)(g))_{g\in G}$ is given by the natural isomorphisms $(f_*\rho)(g)\colon f_*M \to g(f_*M)$ with

$$((f_*\rho)(g))(B') := \rho(g)(f^{-1}(B')).$$

The pair $f_*(M, \rho) := (f_*M, f_*\rho)$ defined in this way is a G-equivariant X'-controlled $\mathbf{A}$-object [BEKW17, Sec. 8.2]. Assume also that U is an invariant entourage of X and that $\varphi\colon (M, \rho) \to (M', \rho')$ is an equivariant U-controlled morphism (Definition 2.1.6). Then, the set $V := (f \times f)(U)$ is a G-invariant entourage of X' as f is controlled; moreover, the relation $U[f^{-1}(B')] \subseteq f^{-1}(V[B'])$ is true for all bounded subsets B' of X'. Therefore, we obtain an induced V-controlled morphism on X':

$$f_*\varphi := \left(f_*M(B') \xrightarrow{\varphi_{f^{-1}(B')}} M(U[f^{-1}(B')]) \to f_*M(V[B']) \right)_{B'\in\mathcal{B}'}. \tag{2.1.1}$$

We have just described a functor

$$f_* := V^G_{\mathbf{A}}(f)\colon V^G_{\mathbf{A}}(X) \to V^G_{\mathbf{A}}(X') \tag{2.1.2}$$

between the additive categories of equivariant controlled objects; observe that f_* is also exact.

We denote by

$$V^G_{\mathbf{A}}\colon G\mathbf{BornCoarse} \to \mathbf{Add}. \tag{2.1.3}$$

the functor from the category of G-bornological coarse spaces to the category of small additive categories obtained in this way.

Remark 2.1.12. If $\mathbf{A}$ is a k-linear category, then the functor $V^G_{\mathbf{A}}\colon G\mathbf{BornCoarse} \to \mathbf{Add}$ refines to a functor $V^G_{\mathbf{A}}\colon G\mathbf{BornCoarse} \to \mathbf{Cat}_k$ from the category of G-bornological coarse spaces to the category of small k-linear categories.

We conclude the section we some further properties of the functor $V^G_{\mathbf{A}}$.

Remark 2.1.13. Let $(X, \mathcal{C}, \mathcal{B})$ be a G-bornological coarse space and $U \in \mathcal{C}^G$ be a G-invariant entourage of X. Then, $X_U := (X, \mathcal{C}_U, \mathcal{B})$ is a G-bornological coarse space by restriction (see Example 1.1.22 (v)). The natural map $X_U \to X$ induces an additive functor $\Phi_U\colon V^G_{\mathbf{A}}(X_U) \to V^G_{\mathbf{A}}(X)$ which is the identity on objects as the definition of equivariant X controlled $\mathbf{A}$-objects does not depend on the coarse structure. Moreover, the category $V_{\mathbf{A}}^G(X_U)$ can be seen as a subcategory of $V_{\mathbf{A}}^G(X)$. On the other hand, every controlled morphism in $V_{\mathbf{A}}^G(X)$ is U-controlled for some entourage U in $\mathcal{C}^G$. Therefore, the category $V_{\mathbf{A}}^G(X)$ is the filtered colimit

$$V_{\mathbf{A}}^G(X) \cong \operatorname{colim}_{U \in \mathcal{C}^G} V_{\mathbf{A}}^G(X_U)$$

indexed on the poset of G-invariant entourages of X.

Recall the definition of closeness for morphisms of G-bornological coarse spaces Definition 1.1.24.

Lemma 2.1.14. [BEKW17, Lemma 8.11] *Let $f, g \colon X \to X'$ be two morphisms of G-bornological coarse spaces. If f and g are close to each other, then they induce naturally isomorphic functors $f_* \cong g_* \colon V_{\mathbf{A}}^G(X) \to V_{\mathbf{A}}^G(X')$.*

Proof. As f and g are close to each other, there exists an entourage U' of X' such that $(f,g)(\Delta_X) \subseteq U'$. We can assume that U' is also symmetric (*i.e.,* $U' = (U')^{-1}$). Then, if B' is a bounded set of X', then $f^{-1}(B') \subseteq g^{-1}(U'[B'])$ and $g^{-1}(B') \subseteq f^{-1}(U'[B'])$.

Let (M, ρ) be a G-equivariant X-controlled $\mathbf{A}$-object. The morphisms

$$M(f^{-1}(B')) \to M(g^{-1}(U'[B']))$$

and

$$M(g^{-1}(B')) \to M(f^{-1}(U'[B']))$$

induced by the inclusions define natural transformations $\eta \colon f_*M \to g_*M$ and $\eta' \colon g_*M \to f_*M$. The composition $\eta' \circ \eta$ is the identity $\eta' \circ \eta = \mathrm{id}_{f_*M}$; in fact, it is given by the transformation

$$\big(M(f^{-1}(B') \subseteq f^{-1}((U')^2[B'])) \colon f_*M \to f_*M \circ (U')^2[-]\big)_{B'}$$

where B' runs among the bounded sets of X'. We conclude that η is a natural isomorphism with inverse of the same form. □

Recall the notion of flasque category Definition A.1.4.

Lemma 2.1.15. [BEKW17, Lemma 8.13] *If X is a flasque G-bornological coarse space, then the category $V_{\mathbf{A}}^G(X)$ of G-equivariant X-controlled $\mathbf{A}$-objects is a flasque category.*

Proof. The space X is flasque, hence there exists a morphism $f \colon X \to X$ as required in Definition 1.1.25. Consider the following functor:

$$S \coloneqq \bigoplus_{n \in \mathbb{N}} (f^n)_* \colon V_{\mathbf{A}}^G(X) \to V_{\mathbf{A}}^G(X).$$

For every equivariant controlled object (M,ρ), the direct sum $\bigoplus_n (f^n)_* M$ is well-defined because for every bounded set B there exists an index n for which $(f^n)^{-1}(B) = \emptyset$. The same is true for the family ρ. For a U-controlled morphism $\varphi\colon (M,\rho) \to (M',\rho')$ let $V := \bigcup_n (f\times f)^n(U)$; this is an entourage of X by the assumptions on f of Definition 1.1.25. Hence, the map $\bigoplus_n (f^n)_*\varphi$ is V-controlled and S describes in fact an endofunctor.

In order to conclude, observe that, as f is close to the identity $\mathrm{id}_{V^G_{\mathbf{A}}(X)}$, then the functor $f_* \circ S$ is naturally isomorphic to S. Hence, $\mathrm{id}_{V^G_{\mathbf{A}}(X)} \oplus S \cong \mathrm{id}_{V^G_{\mathbf{A}}(X)} \oplus (f_* \circ S) \cong S$. $\square$

2.2 The symmetric monoidal refinement of $V^G_{\mathbf{A}}$

The goal of this section is to explain how to refine the functor

$$V^G_{\mathbf{A}}\colon G\mathbf{BornCoarse} \to \mathbf{Add}$$

of (2.1.3) to a lax symmetric monoidal functor (see Definition A.3.7). This is [BCa, Thm. 3.26] and allow us to refine several equivariant coarse homology theories to lax symmetric monoidal functors; for the sake of completeness, we report the entire construction. In order to describe this lax symmetric monoidal refinement, we will also describe the Grothendieck construction associated to the functor $V^G_{\mathbf{A}}$.

Let $\mathbf{A}$ be a small additive category with a strict G-action (Definition A.1.3).

Assumption 2.2.1. *For the rest of the section we assume that the additive category* $\mathbf{A}$ *has a symmetric monoidal structure and that the strict action of* G *on* $\mathbf{A}$ *has a refinement to an action by symmetric monoidal functors.*

Let $\mathbf{C}$ be a small category and let $F\colon \mathbf{C} \to \mathbf{Cat}$ be a functor from $\mathbf{C}$ to the category of small categories $\mathbf{Cat}$. We recall that the Grothendieck construction for F provides a category $\mathcal{F}$ and a projection (a cocartesian fibration) $\pi_F\colon \mathcal{F} \to \mathbf{C}$. We spell this out in the case in which $\mathbf{C}$ is the category of G-bornological coarse spaces and F is the functor $V^G_{\mathbf{A}}\colon G\mathbf{BornCoarse} \to \mathbf{Add}$ (2.1.3) viewed as a functor from $G\mathbf{BornCoarse}$ to the category of small categories $\mathbf{Cat}$:

Definition 2.2.2. The *Grothendieck construction* for the functor $V^G_{\mathbf{A}}\colon G\mathbf{BornCoarse} \to \mathbf{Cat}$ is the category $\mathcal{V}^G_{\mathbf{A}}$ defined as follows:

1. The objects of $\mathcal{V}^G_{\mathbf{A}}$ are pairs $(X,(M,\rho))$ where X is a G-bornological coarse space in $G\mathbf{BornCoarse}$ and (M,ρ) is a G-equivariant X-controlled $\mathbf{A}$-object in $V^G_{\mathbf{A}}(X)$.
2. A morphism $(f,\varphi) : (X,(M,\rho)) \to (X',(M',\rho'))$ in $\mathcal{V}^G_{\mathbf{A}}$ consists of a morphism $f : X \to X'$ in $G\mathbf{BornCoarse}$ and of a morphism $\varphi : f_*(M,\rho) \to (M',\rho')$ in $V^G_{\mathbf{A}}(X')$.

3. The composition of morphisms is given by

$$(f', \varphi') \circ (f, \varphi) := (f' \circ f, \varphi' \circ f'_*(\varphi)) \ .$$

The projection

$$\pi_{V_{\mathbf{A}}^G} : \mathcal{V}_{\mathbf{A}}^G \to G\mathbf{BornCoarse} \ , \quad (X, (M, \rho)) \mapsto X$$

is the functor that forgets the second component.

Let $\mathbf{C}$ be a small category and let $F \colon \mathbf{C} \to \mathbf{Cat}$ be a functor to the category of small categories $\mathbf{Cat}$. Let $\pi_F \colon \mathcal{F} \to \mathbf{C}$ be the associated projection (via the Grothendieck construction). Assume that the categories $\mathbf{C}$ and $\mathcal{F}$ have symmetric monoidal structures and that the projection π_F preserves the tensor product strictly:

$$\pi_F((X, A) \otimes_{\mathcal{F}} (Y, B)) = X \otimes_{\mathbf{C}} X'.$$

Then, the tensor product $(X, A) \otimes_{\mathcal{F}} (Y, B)$ can be written as $(X \otimes_{\mathbf{C}} X', A \boxtimes_{X,X'} B)$ for some object $A \boxtimes_{X,X'} B$ of $F(X \otimes_{\mathbf{C}} X')$. For every pair of objects X and X' of $\mathbf{C}$, we get a bifunctor

$$\boxtimes_{X,X'} \colon F(X) \times F(X') \to F(X \otimes_{\mathbf{C}} X')$$

and for every pair of morphisms $f \colon X \to X'$ and $g \colon Y \to Y'$, an induced morphism

$$F(f \otimes_{\mathbf{C}} g)(A \boxtimes_{X,Y} B) \to F(f)(A) \boxtimes_{X',Y'} F(g)(B) \tag{2.2.1}$$

where A is in $F(X)$ and B in $F(Y)$ (see [BCa, Sec. 2.2] for further details).

Assume now that the functor F is actually a functor $F \colon \mathbf{C} \to \mathbf{Add}$ from a symmetric monoidal category $\mathbf{C}$ to the category of small additive categories $\mathbf{Add}$ endowed with the symmetric monoidal structure of Definition A.3.2. Denote by $\mathbf{Add}_\infty^\otimes$ the resulting symmetric monoidal ∞-category of additive categories (see Definition A.3.7 and the text below). Assume that also the Grothendieck construction $\mathcal{F}$ of F has a symmetric monoidal structure. Then, with some mild assumptions, the functor F can be refined to a symmetric monoidal functor in the sense of Definition A.3.7:

Theorem 2.2.3. [BCa, Thm. 2.3] *Let $\mathbf{C}$ be a symmetric monoidal category and let $F \colon \mathbf{C} \to \mathbf{Add}$ be a functor to the category of small additive categories $\mathbf{Add}$. Assume that:*

1. *the Grothendieck construction $\mathcal{F}$ for the functor F has a symmetric monoidal structure;*
2. *the projection $\pi_F \colon \mathcal{F} \to \mathbf{C}$ preserves the tensor product, the tensor unit as well as the associator, unit, and symmetry transformations.*

3. *The functors* $\boxtimes_{X,X'}$ *are bi-additive for all* X, X' *in* $\mathbf{C}$.
4. *For objects* (X, A) *and* (Y, B) *in* $\mathcal{F}$ *and morphisms* $f : X \to X'$ *and* $g : Y \to Y'$ *in* $\mathbf{C}$, *the morphism* (2.2.1)

$$F(f \otimes_{\mathbf{C}} g)(A \boxtimes_{X,Y} B) \to F(f)(A) \boxtimes_{X',Y'} F(g)(B) \ .$$

is an isomorphism.

Then, the data provide a lax symmetric monoidal refinement (Definition A.3.7)

$$F^{\otimes} : \mathrm{N}(\mathbf{C}^{\otimes}) \to \mathbf{Add}_{\infty}^{\otimes}$$

of the functor F.

Before explaining how to refine the functor $V_{\mathbf{A}}^{G} \colon G\mathbf{BornCoarse} \to \mathbf{Add}$ to a lax symmetric monoidal functor by using the theorem above, we need a more explicit description of the morphisms in the category $V_{\mathbf{A}}^{G}(X)$ [BCa, Sec. 3.3].

We use the following convention:

Convention 2.2.4. *Let* $\mathbf{A}$ *be an additive category. If* $(A_i)_{i\in I}$ *is a family of objects of* $\mathbf{A}$ *with at most finitely many non-zero members, then we use the symbol* $\bigoplus_{i\in I} A_i$ *in order to denote a choice of an object of* $\mathbf{A}$ *that comes together with a family of morphisms* $(A_j \to \bigoplus_{i\in I} A_i)_{j\in I}$ *and represents the coproduct of the family.*

Since in an additive category finite coproducts and products coincide, for every j *in* I *we furthermore have a canonical projection*

$$\bigoplus_{i\in I} A_i \to A_j$$

such that the diagram

$$\begin{tikzcd} A_j \arrow[r] \arrow[rr, bend left, "\mathrm{id}_{A_j}"] & \bigoplus_{i\in I} A_i \arrow[r] & A_j \end{tikzcd}$$

commutes.

If $(A'_{i'})_{i'\in I'}$ *is a second family of this type and* $(\varphi_{i,i'} : A'_{i'} \to A_i)_{(i',i)\in I'\times I}$ *is a family of morphisms in* $\mathbf{A}$, *then we have a unique morphism* $\oplus\varphi_{i,i'}$ *such that the squares*

$$\begin{tikzcd} A'_{i'} \arrow[r, "\varphi_{i,i'}"] \arrow[d] & A_i \arrow[d] \\ \bigoplus_{i'\in I'} A'_{i'} \arrow[r, "\oplus\varphi_{i,i'}"] & \bigoplus_{i\in I} A_i \end{tikzcd}$$

commute for every i' in I' and i in I (where the vertical morphisms are the inclusions of summands).

Let X be a G-bornological coarse space and let (M, ρ) be an equivariant X-controlled **A**-object. Let B be a bounded set of X and let x be a point in B. The inclusion $\{x\} \to B$ induces a morphism $M(\{x\}) \to M(B)$ of **A**. The conditions of Definition 2.1.2 imply that $M(\{x\}) = 0$ for all but finitely many points of B and that the canonical morphism (induced by the universal property of the coproduct in **A**)

$$\bigoplus_{x \in B} M(\{x\}) \xrightarrow{\cong} M(B) \tag{2.2.2}$$

is an isomorphism.

Let U be an invariant entourage of X and let $\varphi : (M, \rho) \to (M', \rho')$ be an equivariant U-controlled morphism, *i.e.,* a natural transformation of functors $\varphi : M \to M' \circ U[-]$ satisfying an equivariance condition (see Definition 2.1.6). For every point x of X, the transformation φ induces a morphism

$$M(\{x\}) \xrightarrow{\varphi_{\{x\}}} M'(U[\{x\}]) \overset{(2.2.2)}{\cong} \bigoplus_{x' \in U[\{x\}]} M(\{x'\}) \ . \tag{2.2.3}$$

We let

$$\varphi_{x',x} : M(\{x\}) \to M'(\{x'\}) \tag{2.2.4}$$

denote the composition of (2.2.3) with the projection onto the summand corresponding to x'. For every pair of points x, x' of X we get a morphism $\varphi_{x',x}$, hence a family of morphisms $(\varphi_{x',x})_{x',x \in X}$ in **A**. In a similar way, for g in G, the transformation $\rho(g) : M \to gM$ gives rise to a family of morphisms

$$\big(\rho(g)_x : M(\{x\}) \to gM(\{g^{-1}x\})\big)_{x \in X} \ . \tag{2.2.5}$$

By construction the family $(\varphi_{x',x})_{x',x \in X}$ satisfies the following conditions.

1. For all x, x' in X the condition $\varphi_{x',x} \neq 0$ implies that $(x', x) \in U$.
2. We have $\rho'(g)_{x'} \circ \varphi_{x',x} = (g\varphi)_{g^{-1}x', g^{-1}x} \circ \rho(g)_x$ for all x, x' in X and g in G.

Lemma 2.2.5. [BCa, Lemma 3.14] *Let X be a G-bornological coarse space and let U be a G-invariant entourage of X. The, we have a bijection between the equivariant U-controlled morphisms $\varphi : (M, \rho) \to (M', \rho')$ and the families $(\varphi_{x',x})_{x',x \in X}$ of morphisms as in* (2.2.4) *satisfying the conditions 1 and 2.*

Let $f : X_0 \to X_1$ be a morphism of G-bornological coarse spaces and (M_i, ρ_i) be objects of $\mathcal{V}_{\mathbf{A}}^G(X_i)$ for $i = 0, 1$. Then, a morphism

$$\varphi : f_*(M_0, \rho_0) \to (M_1, \rho_0) \tag{2.2.6}$$

induces a matrix

$$\left(\varphi^f_{x_1,x_0} : M_0(\{x_0\}) \to M_1(\{x_1\})\right)_{x_0\in X_0, x_1\in X_1} . \tag{2.2.7}$$

In fact, we observe that

$$(f_*M_0)(\{x_1'\}) = M_0(f^{-1}(\{x_1'\})) \overset{(2.2.2)}{\cong} \bigoplus_{x_0\in f^{-1}(\{x_1'\})} M_0(\{x_0\})$$

so that, for x_1, x_1' in X', the induced morphism $\varphi_{x_1,x_1'} : (f_*M_0)(\{x_1'\}) \to M_1(x_1)$ is written as sums $\oplus_{x_0\in f^{-1}(\{x_1'\})}\varphi^f_{x_0,x_1} : M_0(f^{-1}(\{x_1'\})) \cong \bigoplus_{x_0\in f^{-1}(\{x_1'\})} M_0(\{x_0\}) \to M_1(\{x_1\})$ and

$$\left(\varphi_{x_1,x_1'} := \oplus_{x_0\in f^{-1}(\{x_1'\})}\varphi^f_{x_0,x_1} : M_0(f^{-1}(\{x_1'\})) \cong \bigoplus_{x_0\in f^{-1}(\{x_1'\})} M_0(\{x_0\}) \to M_1(\{x_1\})\right)_{x_1',x_1\in X_1}$$

is the matrix representing φ according to Lemma 2.2.5. The following, is a consequence of Lemma 2.2.5:

Corollary 2.2.6. [BCa, Cor. 3.15] *A matrix* (2.2.7) *represents a morphism* (2.2.6) *iff the following conditions are satisfied:*

1. *There exists an entourage U_1 in $\mathcal{C}(X_1)$ such that for every x_0 in X_0 and x_1 in X_1 the condition $\varphi^f_{x_1,x_0} \neq 0$ implies that $(x_1, f(x_0)) \in U_1$.*
2. *For every g in G we have the equality*

$$\rho_1(g)_{x_1} \circ \varphi^f_{x_1,x_0} = (g\varphi^f)_{g^{-1}x_1,g^{-1}x_0} \circ \rho(g)_{x_0} .$$

With this explicit description of the morphisms of $V^G_{\mathbf{A}}(X)$, we can now define the symmetric monoidal structure on the Grothendieck construction $\mathcal{V}^G_{\mathbf{A}}$ for the functor $V^G_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Cat}$ (se Definition 2.2.2). First, we describe the bifunctor

$$- \otimes_{\mathcal{V}^G_{\mathbf{A}}} - : \mathcal{V}^G_{\mathbf{A}} \times \mathcal{V}^G_{\mathbf{A}} \to \mathcal{V}^G_{\mathbf{A}} \tag{2.2.8}$$

on objects.

Recall that the category $G\mathbf{BornCoarse}$ of G-bornological coarse spaces has a symmetric monoidal structure as reviewed in Section 1.1. Let $(X, (M, \rho))$ and $(X', (M', \rho'))$ be objects in $\mathcal{V}^G_{\mathbf{A}}$ and define the functor

$$M \boxtimes M' : \mathcal{B}(X \otimes X') \to \mathbf{A} \tag{2.2.9}$$

(where $\mathcal{B}(X \otimes X')$ denotes the (category associated to the) bornology of the product of G-bornological coarse spaces $X \otimes X'$) as follows:

1. For every B in $\mathcal{B}(X \otimes X')$ we set (see Convention 2.2.4)
$$(M \boxtimes M')(B) := \bigoplus_{(x,x') \in B} M(\{x\}) \otimes_{\mathbf{A}} M'(\{x'\}) \ . \tag{2.2.10}$$
Note that the sum has only finitely many non-zero summands because of Definition 2.1.2.

2. If B' is in $\mathcal{B}(X \otimes X')$ such that $B' \subseteq B$, then the morphism
$$(M \boxtimes M')(B' \subseteq B) \colon (M \boxtimes M')(B') \to (M \boxtimes M')(B)$$
is given by the canonical map
$$\bigoplus_{(x,x') \in B'} M(\{x\}) \otimes_{\mathcal{A}} M'(\{x'\}) \to \bigoplus_{(x,x') \in B} M(\{x\}) \otimes_{\mathcal{A}} M'(\{x'\})$$
as described in Convention 2.2.4.

By using our Convention 2.2.4 and the universal property of the direct sum, one easily checks that this describes a functor satisfying the first three conditions of Definition 2.1.2.

The family $\rho \boxtimes \rho'$ is analogously defined as the sum $\bigoplus_{(x,x') \in B} \rho(g)_x \otimes \rho'(g)_{x'}$:
$$(\rho \boxtimes \rho')(g)_B \colon \bigoplus_{(x,x') \in B} M(\{x\}) \otimes_{\mathcal{A}} M'(\{x'\}) \to \bigoplus_{(x,x') \in B} gM(\{g^{-1}x\}) \otimes_{\mathcal{A}} gM'(\{g^{-1}x'\})$$
where we use the notation (2.2.5). For every object $B \in \mathcal{B}(X \otimes X')$ the map $(\rho \boxtimes \rho')(g)_B$ provides an isomorphism as $\rho(g)_x$ and $\rho'(g)_{x'}$ are isomorphisms for every $(x, x') \in B$. By using our Convention 2.2.4 and the universal property of the direct sum, it is easy to see that the assignment is functorial in B and this provides a family of natural isomorphisms.

Remark 2.2.7. The pair $(M \boxtimes M', \rho \boxtimes \rho')$ satisfies the remaining condition of Definition 2.1.2 and therefore belongs to $V_{\mathbf{A}}^G(X \otimes X')$.

We now describe the bifunctor $\otimes_{\mathcal{V}_{\mathbf{A}}^G}$ on morphisms. Let $(f, \varphi) : (X_0, (M_0, \rho_0)) \to (X_1, (M_1, \rho_1))$ be a morphism in $\mathcal{V}_{\mathbf{A}}^G$ (see Definition 2.2.2 (2)). Then, we define the morphism
$$(g, \psi) := (f, \varphi) \otimes_{\mathcal{V}_{\mathbf{A}}^G} (X', (M', \rho')) : (X_0 \otimes X', (M_0 \boxtimes M', \rho_0 \boxtimes \rho')) \to (X_1 \otimes X', (M_1 \boxtimes M', \rho_1 \boxtimes \rho'))$$
as follows.

1. We set $g := f \otimes \mathrm{id}_{X'} : X_0 \otimes X' \to X_1 \otimes X'$ by using the tensor product in G**BornCoarse**.

2. In order to describe the morphism

$$\psi : (f \otimes \mathrm{id}_{X'})_*(M_0 \boxtimes M', \rho_0 \boxtimes \rho') \to (M_1 \boxtimes M', \rho_1 \boxtimes \rho')$$

we use Corollary 2.2.6, hence we have to describe the matrix

$$(\psi^{f\otimes \mathrm{id}_{X'}}_{(x_1,y'),(x_0,x')})_{(x_0,x')\in X_0\times X',(x_1,y')\in X_1\times X'} .$$

By definition of $M_0 \boxtimes M'$, for x_0 in X_0 and x' in X', we have

$$(M_0 \boxtimes M')((x_0, x')) \cong M(\{x_0\}) \otimes_{\mathcal{A}} M'(\{x'\})$$

so that we can set

$$\psi^{f\otimes \mathrm{id}_{X'}}_{(x_1,y'),(x_0,x')} := \varphi^f_{x_1,x_0} \otimes_{\mathcal{A}} (\mathrm{id}_{(M',\rho')})_{y',x'} : M_0(\{x_0\}) \otimes_{\mathcal{A}} M'(\{x'\}) \to M_1(\{x_1\}) \otimes_{\mathcal{A}} M'(\{y'\}) .$$

This matrix satisfies the conditions listed in Corollary 2.2.6 and represents the desired morphism.

In a similar way, we define $(X, (M, \rho)) \otimes (f', \varphi')$ for a morphism $(f', \varphi') : (X'_0, (M'_0, \rho'_0)) \to (X'_1, (M'_1, \rho_1))$. Moreover, this description is compatible with compositions.

Definition 2.2.8. [BCa, Def. 3.18 & 3.19] Let $\mathcal{V}^G_{\mathbf{A}}$ be the Grothendieck construction of Definition 2.2.2. The bifunctor $\otimes_{\mathcal{V}^G_{\mathbf{A}}}$ (2.2.8) is defined on objects $(X, (M, \rho)), (X', (M', \rho'))$ of $\mathcal{V}^G_{\mathbf{A}}$ by

$$(X, (M, \rho)) \otimes_{\mathcal{V}^G_{\mathbf{A}}} (X', (M', \rho')) := (X \otimes X', (M \boxtimes M', \rho \boxtimes \rho'))$$

where $\otimes$ denotes the tensor product of G**BornCoarse** and on morphisms by the preceding description.

We refer to [BCa] for the descriptions of the tensor unit $1_{\mathcal{V}^G_{\mathbf{A}}}$, of the associative constrain $\alpha^{\mathcal{V}^G_{\mathbf{A}}}$, of the unit constraint $\eta^{\mathcal{V}^G_{\mathbf{A}}}$, the symmetry constraint $\sigma^{\mathcal{V}^G_{\mathbf{A}}}$ (see Definition A.3.1), as they are defined by the same reasoning.

Proposition 2.2.9. [BCa, Prop. 3.23] *Let $\mathcal{V}^G_{\mathbf{A}}$ be the Grothendieck construction for the functor $V^G_{\mathbf{A}}$: G**BornCoarse** $\to$ **Cat** and let $\pi : \mathcal{V}^G_{\mathbf{A}} \to G$**BornCoarse** be the associated projection. Then:*

1. *the functor $- \otimes_{\mathcal{V}^G_{\mathbf{A}}} -$ and the object $1_{\mathcal{V}^G_{\mathbf{A}}}$ together with the natural isomorphisms $\alpha^{\mathcal{V}^G_{\mathbf{A}}}$, $\eta^{\mathcal{V}^G_{\mathbf{A}}}$ and $\sigma^{\mathcal{V}^G_{\mathbf{A}}}$ define a symmetric monoidal structure on $\mathcal{V}^G_{\mathbf{A}}$;*
2. *the functor $\pi : \mathcal{V}^G_{\mathbf{A}} \to G$**BornCoarse** preserves the tensor product strictly and the tensor unit as well as the associator, unit, and symmetry transformations.*

The categories $\mathcal{V}^G_{\mathbf{A}}$ and G**BornCoarse** have symmetric monoidal structures such that the projection π preserves the tensor product strictly. This implies that, for all

G-bornological coarse spaces X and X', we obtain a bifunctor

$$\boxtimes_{X,X'} : V^G_{\mathbf{A}}(X) \times V^G_{\mathbf{A}}(X') \to V^G_{\mathbf{A}}(X \otimes X') \tag{2.2.11}$$

that preserves also the linear structure:

Proposition 2.2.10. [BCa, Prop. 3.24] *Let X, X' be G-bornological coarse spaces. Then, the functor*

$$\boxtimes_{X,X'} : V^G_{\mathbf{A}}(X) \times V^G_{\mathbf{A}}(X') \to V^G_{\mathbf{A}}(X \otimes X')$$

is additive in both variables.

Proof. Let (M_i, ρ_i) be in $V^G_{\mathbf{A}}(X)$ for $i = 0, 1$ and (M', ρ) be in $V^G_{\mathbf{A}}(X')$. In view of the symmetry it suffices to show that the canonical morphism

$$(M_0 \boxtimes_{X,X'} M') \oplus (M_1 \boxtimes_{X,X'} M') \to (M_0 \oplus M_1) \boxtimes_{X,X'} M'$$

is an isomorphism. It suffices to show that

$$\left[(M_0 \boxtimes_{X,X'} M') \oplus (M_1 \boxtimes_{X,X'} M')\right] (\{(x, x')\}) \to \left[(M_0 \oplus M_1) \boxtimes_{X,X'} M'\right] (\{(x, x')\})$$

is an isomorphism for every point (x, x') in $X \times X'$. By inserting the definitions we see that this morphism is the same as

$$(M_0(\{x\}) \otimes_{\mathbf{A}} M'(\{x'\})) \oplus (M_1(\{x\}) \otimes_{\mathbf{A}} M'(\{x'\})) \to (M_0(\{x\}) \oplus M_1(\{x\})) \otimes_{\mathbf{A}} M'(\{x'\}) .$$

But this last morphism is an isomorphism since the tensor product in $\mathbf{A}$ is additive in the first argument. $\square$

Remark 2.2.11. If the category $\mathbf{A}$ is also k-linear, then bifunctor $\boxtimes_{X,X'}$ preserves the k-linear structure as well.

We can now apply Theorem 2.2.3 to the functor $V^G_{\mathbf{A}} \colon G\mathbf{BornCoarse} \to \mathbf{Cat}$. In fact, the Grothendieck construction $\mathcal{V}^G_{\mathbf{A}}$ for the functor $V^G_{\mathbf{A}}$, seen as a functor to the category of small categories, has a symmetric monoidal structure by Proposition 2.2.9; the projection $\pi : \mathcal{V}^G_{\mathbf{A}} \to G\mathbf{BornCoarse}$ preserves the tensor product strictly and the tensor unit as well as the associator, unit, and symmetry transformations. By Proposition 2.2.10, $\boxtimes_{X,X'} : V^G_{\mathbf{A}}(X) \times V^G_{\mathbf{A}}(X') \to V^G_{\mathbf{A}}(X \otimes X')$ is bi-additive for all $X, X' \in G\mathbf{BornCoarse}$. By [BCa, Lemma 3.25], Theorem 2.2.3 (2.2.1) is also satisfied and the data provide a lax symmetric monoidal refinement as in Definition A.3.7:

Theorem 2.2.12. [BCa, Thm. 3.26] *If $\mathbf{A}$ is a symmetric monoidal additive category with a strict action of G by symmetric monoidal functors, then the functor $V^G_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Add}$ admits a refinement to a symmetric monoidal functor.*

Remark 2.2.13. If $\mathbf{A}$ is a symmetric monoidal k-linear category with a strict action of G by symmetric monoidal functors, then the functor $V^G_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Cat}_k$ takes values in small additive categories that are in addition k-linear. Hence, by Theorem 2.2.12 and [Lur14, Prop. 2.2.1.1], the functor $V^G_{\mathbf{A}}$ admits a refinement to a symmetric monoidal functor.

2.3 Equivariant coarse algebraic K-homology

In this section we recall the definition of equivariant coarse algebraic K-homology as defined in [BEKW17]. This is a coarse homology theory

$$K\mathbf{A}\mathcal{X}^G : G\mathbf{BornCoarse} \to \mathbf{Sp}.$$

with values in the ∞-category $\mathbf{Sp}$ of spectra and is defined as the (non-connective) K-theory of the category $V^G_{\mathbf{A}}(X)$ of G-equivariant X-controlled $\mathbf{A}$-objects of Definition 2.1.2. We remark that the equivariant coarse K-homology has been studied in several works [BE16, BEKW17, BC17, BEKW18]. In this section we briefly give its definition and we state the most important properties for us, referring to those papers for further details.

Let $\mathbf{Add}$ be the category of small additive categories and exact functors (see Definition A.1.1). Consider the non-connective version of K-theory of additive categories

$$K : \mathbf{Add} \to \mathbf{Sp};$$

a construction (for more general exact categories) of this functor has been given by Schlichting [Sch06]. We list some of its properties:

- if R is a ring, then the K-theory of the additive category of finitely generated free R-modules is equivalent to the non-connective K-theory of the ring R;
- it sends isomorphic exact functors to equivalences of spectra;
- it commutes with filtered colimits of categories;
- it satisfies additivity and is a localizing invariant (see Section 3.2 for a description of these properties in the case of Hochschild homology);
- it sends flasque categories to zero;
- it is a lax symmetric monoidal functor.

Recall Definition A.1.3 of an additive category with strict G-action.

Definition 2.3.1. [BEKW17, Def. 8.8] Let G be a group and let $\mathbf{A}$ be an additive category with strict G-action. The G-equivariant coarse algebraic K-homology associated to $\mathbf{A}$ is the K-theory of the additive category of $\mathbf{A}$-controlled objects:

$$K\mathbf{A}\mathcal{X}^G := K \circ V^G_{\mathbf{A}} \colon G\mathbf{BornCoarse} \to \mathbf{Sp}.$$

Notation 2.3.2. When $\mathbf{A}$ is the category of finitely generated free R-modules, we denote by $K\mathcal{X}^G_R$ the associated G-equivariant K-homology.

The above properties of the K-theory functor are used in order to prove that the composition $K\mathbf{A}\mathcal{X}^G$ provides a coarse homology theory:

Theorem 2.3.3. [BEKW17, Thm. 8.9] *Let G be a group and let $\mathbf{A}$ be an additive category with strict G-action. Then, the functor $K\mathbf{A}\mathcal{X}^G$ is a G-equivariant $\mathbf{Sp}$-valued coarse homology theory.*

By Theorem 2.2.12, the functor $V^G_{\mathbf{A}} \colon G\mathbf{BornCoarse} \to \mathbf{Add}$ admits a lax symmetric monoidal refinement. As a consequence, coarse equivariant K-homology is a lax symmetric monoidal functor as well:

Theorem 2.3.4. [BCa, Thm. 3.27] *Let G be a group and let $\mathbf{A}$ be an additive category with strict G-action. The functor $K\mathbf{A}\mathcal{X}^G$ admits a lax symmetric monoidal refinement.*

For a proof of this theorem we refer to [BCa, Sec. 3.5].

In Farrell-Jones type questions, the twist (see Definition 1.2.5) of coarse algebraic K-homology by the motivic coarse space $\mathrm{Yo}^s_G(G_{\mathrm{can,min}})$ associated to a group G (see Example 1.1.22 (ii)) is of fundamental importance. In order to state this computation, we use the following category:

Definition 2.3.5. [BR07, Def. 2.1] Let $\mathbf{A}$ be an additive category with strict G-action. Let X be a G-set. The additive category $\mathbf{A} *_G X$ is described as follows:

Objects: an object of $\mathbf{A} *_G X$ is a family $A = (A_x)_{x\in X}$ of objects of $\mathbf{A}$ such that the set $\{x \in X \mid A_x \neq 0\}$ is finite;

Morphisms: a morphism $\varphi\colon A \to B$ between A and B is a family of morphisms $\varphi = (\varphi_{x,g})_{(x,g)\in X\times G}$ where $\varphi_{x,g} \colon A_x \to g(B_{g^{-1}x})$ is a morphism in $\mathbf{A}$ and the set of pairs (x, g) for which $\varphi_{x,g} \neq 0$ is finite.

The composition is defined as the convolution product and the addition component-wise.

We recall that every G-set X, endowed with the minimal coarse structure and the maximal bornology, belongs to $G\mathbf{BornCoarse}$ (Example 1.1.13).

Proposition 2.3.6. [BEKW17, Prop. 8.24] *For every G-set X we have an equivalence*

$$V^G_{\mathbf{A}}(X_{min,max} \otimes G_{\mathrm{can,min}}) \simeq \mathbf{A} *_G X.$$

We refer to [BEKW17, Prop. 8.24] for further details and the proof. As a consequence of this proposition, we have:

Remark 2.3.7. Let G be a group and H a subgroup of G. Let G/H be endowed with the minimal coarse structure and the maximal bornology; then the previous proposition yields the following computation:

$$K\mathbf{A}\mathcal{X}^G((G/H)_{\mathrm{min,max}} \otimes G_{\mathrm{can,min}}) \simeq K(\mathbf{A} *_G (G/H))$$

When $\mathbf{A}$ is the additive category of finitely generated free R-modules, R a ring, the category $\mathbf{A} *_G (G/H)$ is equivalent to the category of finitely generated free $R[H]$-modules.

Chapter 3

A coarse version of Hochschild and cyclic homology

The goal of this chapter is to define equivariant coarse versions

$$\mathcal{X}\mathrm{HH}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$$

and

$$\mathcal{X}\mathrm{HC}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$$

of the classical Hochschild and cyclic homology for k-algebras [Lod98, Sec. 1.1 & 2.1], where k is a field.

Our definition of equivariant coarse Hochschild and cyclic homology is analogous to the definition of equivariant coarse algebraic K-homology [BEKW17, BC17] recalled in Definition 2.3.1. In fact, we first consider the functor $V_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Cat}_k$ (see Remark 2.1.12) sending a G-bornological coarse space X to the k-linear category $V_k^G(X)$ of G-equivariant X-controlled finite dimensional k-vector spaces (see Definition 2.1.2 and Notation 2.1.9). Then, we define equivariant coarse Hochschild (and cyclic) homology as Keller's Hochschild (and cyclic) homology [Kel99] of the k-linear category $V_k^G(X)$ (see Definition 3.4.6).

The chapter is structured as follows. In Section 3.2 we recall Keller's definition of Hochschild and cyclic homology for k-linear exact categories; as this definition uses Kassel's approach to cyclic homology via mixed complexes [Kas87], we start the chapter in Section 3.1 by reviewing the main properties of the (∞-)category of mixed complexes. In Section 3.3, we prove our main result Theorem 3.4.2, where we show that the composition of Keller's Hochschild homology with the functor V_k^G yields a G-equivariant coarse homology theory. In Section 3.4, we define coarse Hochschild and cyclic homology by using the results of Section 3.3. We conclude the chapter with Section 3.5, where, by using the results of Section 2.2, we refine equivariant coarse Hochschild $\mathcal{X}\mathrm{HH}_k^G$ and equivariant coarse cyclic homology $\mathcal{X}\mathrm{HC}_k^G$ to lax symmetric monoidal functors.

We will freely use the language of additive and exact categories, and of differential graded categories, as reviewed in the Appendix.

In the following, as set in our conventions, we will always assume that k is a field and tensor products $\otimes$ are tensor products over k. However, most of the proofs of this chapter are also true for a general commutative ring. We choose to work with coefficients in a field k instead of a ring because several properties of these coarse homology theories (as discussed in Chapter 4) depend on the choice of a base field (see, *e.g.,* the construction of the natural transformation to equivariant coarse ordinary homology of Section 4.3).

3.1 The category of mixed complexes

In this section we describe the (cocomplete stable ∞-)category of (unbounded) mixed complexes; we mainly follow Kassel's viewpoint [Kas87].

We start with Kassel's definition of mixed complexes:

Definition 3.1.1. [Kas87, §1] A *mixed complex* (C, b, B) is a triple consisting of a $\mathbb{Z}$-graded k-module $C = \{C_p\}_{p\in\mathbb{Z}}$ together with differentials b and B

$$b = (b_p \colon C_p \to C_{p-1})_{p\in\mathbb{Z}} \quad \text{and} \quad B = (B_p \colon C_p \to C_{p+1})_{p\in\mathbb{Z}}$$

of degree -1 and 1 respectively, satisfying the following identities:

$$b^2 = 0, \quad B^2 = 0, \quad bB + Bb = 0.$$

Morphisms $f\colon (C, b, B) \to (C', b', B')$ of mixed complexes are given by sequences $f = (f_p \colon C_p \to C'_p)_{p\in\mathbb{Z}}$ of maps commuting with both the differentials b and B. The category of mixed complexes and morphisms of mixed complexes is denoted by **Mix**.

When the differentials are clear from the context, we refer to a mixed complex (C, b, B) by C.

In the category of mixed complexes there is a notion of *shifted mixed complexes* and of *cone* of a morphism of mixed complexes:

- the shifted mixed complex $(C[1], b_{C[1]}, B_{C[1]})$ of a mixed complex (C, b, B) is defined by $(C[1])_p := C_{p-1}$, for all p, with differentials $b_{C[1]} := -b_C$ and $B_{C[1]} := -B_C$.
- Let $f\colon (C, b, B) \to (C', b', B')$ be a morphism of mixed complexes. The cone of f is defined as the mixed complex

$$\operatorname{cone}(f) := \left(C' \oplus C[1], \begin{bmatrix} b_{C'} & f \\ 0 & -b_C \end{bmatrix}, \begin{bmatrix} B_{C'} & 0 \\ 0 & -B_C \end{bmatrix} \right)$$

where $C[1]$ is the shifted mixed complex of C.

Recall the definitions of dg-algebras and dg-modules over a dg-algebra, Definition A.2.1. Let Λ be the dg-algebra over the field k

$$\Lambda := \cdots \to 0 \to k\epsilon \xrightarrow{0} k \to 0 \to \cdots \tag{3.1.1}$$

generated by an indeterminate ϵ of degree 1, with $\epsilon^2 = 0$ and differential (of degree -1) $d(\epsilon) = 0$. Then, mixed complexes are nothing but dg-modules over the dg-algebra Λ:

Remark 3.1.2. [Kas87] The category **Mix** of mixed complexes is equivalent (in fact, isomorphic) to the category of left dg Λ-modules, which we denote by Λ-**Mod**, as we now explain. A mixed complex (C, b, B) yields a differential graded left Λ-module whose underlying differential graded module is (C, b) and where the multiplication $\epsilon \cdot c$ is defined by the differential B, *i.e.,* $\epsilon \cdot c := B(c)$. Vice versa, left Λ-dg-modules (M, d) correspond to mixed complexes (C, b, B) by defining $C_p := M_p$, $b(c) := d(c)$ and $B(c) := \epsilon \cdot c$ for every $c \in C_p$. The identity $B^2 = 0$ corresponds to the relation $\epsilon^2 = 0$, and by using the equality $d(\epsilon) = 0$ one gets the identity $bB+Bb = 0$. Observe that also morphisms of mixed complexes correspond one-to-one to morphisms of dg-Λ-modules (see Definition A.2.1 and the text thereafter) via this correspondence. We denote by $L\colon \mathbf{Mix} \to \Lambda\text{-}\mathbf{Mod}$ the induced functor sending a mixed complex to the associated Λ-dg-module and by $R\colon \Lambda\text{-}\mathbf{Mod} \to \mathbf{Mix}$ its inverse sending a Λ-dg-module to its associated mixed complex.

A mixed complex (C, b, B) functorially determines a double chain complex $\mathcal{B}C$ [Lod98, § 2.5.10] by means of the differentials b and B:

$$\mathcal{B}C := \Big(\dots \xleftarrow{0} (C, b) \xleftarrow{B} (C[-1], b_{C[-1]}) \xleftarrow{B} \dots \xleftarrow{B} (C[-n], b_{C[-n]}) \xleftarrow{B} \dots\Big); \tag{3.1.2}$$

here, the chain complex (C, b) is placed in bi-degree $(0, *)$, *i.e.,* $\mathcal{B}C_{(0,*)} = (C_*, b)$, and the chain complex $(C[-n], b_{C[-n]})$, placed in bi-degree $(n, *)$, is the chain complex (C, b) shifted by $-n$, hence $\mathcal{B}C_{(p,q)} = C_{q-p}$ for $p \geq 0$ and $\mathcal{B}C_{(p,q)} = 0$ for $p < 0$. The total chain complex $\mathrm{Tot}(\mathcal{B}C)$, functorially associated to the double chain complex $\mathcal{B}C$, is the chain complex defined in degree n by $\mathrm{Tot}_n(\mathcal{B}C) = \bigoplus_{i\geq 0} C_{n-2i}$ with differential d acting as follows:

$$d(c_n, c_{n-2}, \dots) := (bc_n + Bc_{n-2}, \dots).$$

Let **Ch** be the category of chain complexes over k. Consider the forgetful functor

$$\text{forget}\colon \mathbf{Mix} \to \mathbf{Ch} \tag{3.1.3}$$

sending a mixed complex (M, b, B) to its underlying chain complex (M, b), and the functor

$$\mathrm{Tot}(\mathcal{B}-)\colon \mathbf{Mix} \to \mathbf{Ch} \tag{3.1.4}$$

just described above. We observe that the chain complex $\mathrm{Tot}(\mathcal{B}C)$ associated to the mixed complex (C, b, B) represents the derived tensor product $C \otimes^{\mathbb{L}}_{\Lambda} k$.

Hochschild homology (and all its cyclic variants: cyclic, negative and periodic) can be defined as functors from the category of mixed complexes:

Definition 3.1.3. [Kas87, Sec. 1] Let (C, b, B) be a mixed complex. The *Hochschild homology* $\mathrm{HH}_*(C)$ of (C, b, B) is the homology of the underlying chain complex (C, b). Its *cyclic homology* $\mathrm{HC}_*(C)$ is the homology of the associated chain complex $\mathrm{Tot}(\mathcal{B}C)$.

Let M be a cyclic module (see Definition A.4.3) and let d_i and s_i denote the i-th face and i-th degeneracy maps of M respectively. Let t_{n+1} be the cyclic operator in degree n and let $b\colon M_n \to M_{n-1}$ be the alternating sum

$$b \coloneqq \sum_{i=0}^{n} (-1)^i d_i \tag{3.1.5}$$

of face maps . Let $N \coloneqq \sum_{i=0}^{n} t_{n+1}^i$ be the sum of the powers of the cyclic operator t_{n+1} and define the cochain map $B\colon M_n \to M_{n+1}$ as the composition

$$B \coloneqq (-1)^{n+1}(1 - t_{n+1})sN. \tag{3.1.6}$$

Here s denotes the extra degeneracy $s = (-1)^{n+1} t_{n+1} s_n \colon M_n \to M_{n+1}$.

Remark 3.1.4. Let M be a cyclic module. Let b be the differential $b \coloneqq \sum_{i=0}^{n}(-1)^i d_i$ (3.1.5) and B the differential $B \coloneqq (-1)^{n+1}(1 - t_{n+1})sN$ (3.1.6). Then, (M, b, B) is a mixed complex. Morphisms of cyclic modules commute with face and degeneracy maps, and with the cyclic operators as well; hence yield morphisms of mixed complexes. This describes a functor from the category of cyclic modules to the category of mixed complexes.

Example 3.1.5. If A is a k-algebra (that is associative and unital), then by Example A.4.5, we get a cyclic module $Z_*(A)$. The classical Hochschild homology $\mathrm{HH}(A)$ of A is defined as the homology (of the underlying chain complex) of the cyclic module $Z_*(A)$. If $C(A)$ is the mixed complex associated to the cyclic module $Z_*(A)$ as described in Remark 3.1.4, then the Hochschild homology of the mixed complex $C(A)$ of Definition 3.1.3 coincides with the Hochschild homology $\mathrm{HH}(A)$. Furthermore, the cyclic homology of the mixed complex $C(A)$ is the classical cyclic homology of the k-algebra A [Kas87, Prop. 1.3].

The category of dg-modules over k admits a combinatorial model structure (the projective model structure), whose weak equivalences are the objects-wise quasi-isomorphisms of dg-modules (see Definition A.2.10 and Remark A.2.11). In the language of mixed complexes, the definition of quasi-isomorphism of dg-modules translate as follows:

Definition 3.1.6. A morphism $(C, b, B) \to (C', b', B')$ of mixed complexes is a *quasi-isomorphism* if the underlying b-complexes are quasi-isomorphic via the induced chain map $(C, b) \to (C', b')$.

Remark 3.1.7. Consider the functors $L\colon \mathbf{Mix} \to \Lambda\text{-}\mathbf{Mod}$ sending a mixed complex to the associated Λ-dg-module and $R\colon \Lambda\text{-}\mathbf{Mod} \to \mathbf{Mix}$ sending a Λ-dg-module to its associated mixed complex, as defined in Remark 3.1.2. Then, quasi-isomorphisms of mixed complexes correspond by the functor L to quasi-isomorphisms of Λ-dg-modules and vice versa, *i.e.,* the functors L and R preserve quasi-isomorphisms.

Remark 3.1.8. As observed by Keller, the correspondence $M \mapsto (M, b, B)$ from cyclic modules to mixed complexes, using the differentials b (3.1.5) and B (3.1.6), is not functorial when one considers morphisms of cyclic modules that do not commute with degeneracies (as opposed to morphisms of cyclic modules that commute with both faces and degeneracies) [Kel98, Sec. 2.1]. However, Keller defines a functor from the category of precyclic modules (*i.e.,* cyclic modules without degeneracies) to the category of mixed complexes; for the sake of completeness we report also this approach [Kel98, Sec. 2.1]. Let C be a precyclic module, let d_i denote the i-th face map and set $b' := \sum_{i=0}^{n-1} d_i$ and $b := \sum_{i=0}^{n} d_i$. Let t be the cyclic operator and let N be the sum of its powers. Keller defines the mixed complex $\tilde{M}$ as follows:

$$\tilde{M} := \left(C \oplus C[1], \begin{bmatrix} b & (1-t) \\ 0 & -b' \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix} \right)$$

This correspondence yields a functor from precyclic modules to mixed complexes. Moreover, there is a canonical morphism (induced by $[1, (1-t)s]$) between this mixed complex and the mixed complex constructed in Remark 3.1.4 by using the differentials (3.1.5) and (3.1.6), and this canonical morphism of mixed complexes is a quasi-isomorphism of mixed complexes [Kel98, Sec. 2.1].

To every small dg-category $\mathcal{A}$, one can functorially associate a cyclic module, called the *additive cyclic nerve* $\mathrm{CN}(\mathcal{A})$ as described in Definition A.4.7; hence, a mixed complex by Remark 3.1.4. The composition of these two functors describes then a functor from the category of small dg-categories to the category of mixed complexes, that we call Mix:

Definition 3.1.9. [Kel99, Def. 1.3] Let

$$\mathrm{Mix}\colon \mathbf{dgcat}_k \to \mathbf{Mix}$$

be the functor from the category of small dg-categories over k to the category of mixed complexes defined as composition of the additive cyclic nerve of Definition A.4.7 and of the functor of Remark 3.1.4.

We now introduce the ∞-category $\mathbf{Mix}_\infty$ of mixed complexes. We recall that, if $\mathbf{C}$ is an ordinary category and W denotes a collection of morphisms of $\mathbf{C}$, then $\mathrm{N}(\mathbf{C})[W^{-1}]$ is the ∞-category obtained by the nerve $\mathrm{N}(\mathbf{C})$ of $\mathbf{C}$ by inverting the set of morphisms W [Cis, Def. 7.1.2 & Prop. 7.1.3], [Lur14, Def. 1.3.4.1].

The ∞-category $\mathbf{Mix}_\infty$ of mixed complexes is then defined as the localization of the nerve of the category $\mathbf{Mix}$ at the class W_{mix} of quasi-isomorphisms of mixed complexes

(Definition 3.1.6):

$$\mathbf{Mix}_\infty \coloneqq \mathrm{N}(\mathbf{Mix})[W_{mix}^{-1}]. \tag{3.1.7}$$

Analogously, the ∞-category Λ-$\mathbf{Mod}_\infty$ is defined as the localization of the category Λ-$\mathbf{Mod}$ of dg-Λ-modules at the class W of quasi-isomorphisms of dg-Λ-modules (Definition A.2.10):

$$\Lambda\text{-}\mathbf{Mod}_\infty \coloneqq \mathrm{N}(\Lambda\text{-}\mathbf{Mod})[W^{-1}]. \tag{3.1.8}$$

Proposition 3.1.10. *The ∞-category Λ-$\mathbf{Mod}_\infty$ is a cocomplete stable ∞-category.*

Proof. The category Λ-$\mathbf{Mod}$ is a (pre-triangulated) dg-category. By applying the dg-nerve functor N_{dg} [Lur14, Constr. 1.3.1.6] we obtain an ∞-category $\mathrm{N}_{\mathrm{dg}}(\Lambda\text{-}\mathbf{Mod})$ [Lur14, Prop. 1.3.1.10]. The dg-nerve functor sends pre-triangulated dg-categories to stable ∞-categories [Fao17, Thm. 4.3.1], [Lur14, Prop. 1.3.1.10]. The ∞-category $\mathrm{N}_{\mathrm{dg}}(\Lambda\text{-}\mathbf{Mod})$ is a stable ∞-category and its homotopy category can be identified (as triangulated category) with the derived category $\mathcal{D}(\Lambda)$ associated to the dg-algebra Λ (see Definition A.2.14).

The category Λ-$\mathbf{Mod}$ is equipped with a combinatorial simplicial model structure by Remark A.2.11. By [Lur14, Prop. 1.3.1.17] and by the fact that the simplicial nerve of the simplicial category associated to Λ-$\mathbf{Mod}$ is equivalent to the localization $\mathrm{N}(\Lambda\text{-}\mathbf{Mod})[W^{-1}]$ (by [Lur14, Rem. 1.3.4.16 & Thm. 1.3.4.20] where we also use that the model category Λ-$\mathbf{Mod}$ is combinatorial, hence admits functorial factorizations), the two constructions $\mathrm{N}(\Lambda\text{-}\mathbf{Mod})[W^{-1}]$ and $\mathrm{N}_{\mathrm{dg}}(\Lambda\text{-}\mathbf{Mod})$ present equivalent ∞-categories. Hence, the ∞-category Λ-$\mathbf{Mod}_\infty$ is a stable ∞-category.

The ∞-category Λ-$\mathbf{Mod}_\infty$ is also cocomplete by [Lur14, Prop. 1.3.4.22] because the model category Λ-$\mathbf{Mod}$ is combinatorial. □

Proposition 3.1.11. *The ∞-category $\mathbf{Mix}_\infty$ is a cocomplete stable ∞-category.*

Proof. The categories $\mathbf{Mix}$ and Λ-$\mathbf{Mod}$ are isomorphic by Remark 3.1.2 and the functor $L\colon \mathbf{Mix} \to \Lambda\text{-}\mathbf{Mod}$ and its inverse $R\colon \Lambda\text{-}\mathbf{Mod} \to \mathbf{Mix}$ preserve quasi-isomorphisms by Remark 3.1.7. This yields an equivalence of ∞-categories

$$\mathrm{N}(\mathbf{Mix})[W_{mix}^{-1}] \to \mathrm{N}(\Lambda\text{-}\mathbf{Mod})[W^{-1}].$$

The statement is then a consequence of this equivalence and of Proposition 3.1.10. □

We observe that one can prove that the ∞-category $\mathbf{Mix}_\infty$ is a cocomplete stable ∞-category by following the same arguments of [Fao17, Thm. 4.3.1]. In fact, the category $\mathbf{Mix}$ is cocomplete, has a 0-object, and the cone of a morphism of mixed complexes and the shift mixed complexes can be used for proving that $\mathbf{Mix}_\infty$ is also stable.

Remark 3.1.12. The homotopy category of the stable ∞-category $\mathbf{Mix}_\infty$ is canonically equivalent to the derived category $\mathcal{D}(\Lambda)$ of the dg-algebra Λ (by Proposition 3.1.10 and Proposition 3.1.11).

3.2 Keller's cyclic homology of exact categories

In this section we recall Keller's construction of Hochschild and cyclic homology for exact k-linear categories [Kel99]. In fact, Keller constructs a functor

$$C\colon \mathbf{Ex}_k \to \mathbf{Mix}$$

from the category $\mathbf{Ex}_k$ of small exact k-linear categories to the category $\mathbf{Mix}$ of Kassel's mixed complexes. The functor C is also called the *cone functor* because its definition uses the mapping cone for morphisms of mixed complexes. Hochschild and cyclic homology are homological invariants of mixed complexes, as explained in Definition 3.1.3. Hence, Keller's Hochschild (or cyclic) homology of a k-linear exact category $\mathcal{A}$ is the Hochschild (or cyclic) homology of the mixed complex $C(\mathcal{A})$ associated to $\mathcal{A}$ by the functor C.

Among the various properties of Keller's Hochschild homology, we have the following [Kel99]:

Agreement: the Hochschild homology of the exact category of finitely generated projective modules over a (unital) algebra agrees with the Hochschild homology of the algebra.

Invariance: Hochschild homology is preserved by exact functors inducing equivalences in the bounded derived categories.

Localization: suitable sequences of exact categories (roughly, inducing short exact sequences of bounded derived categories) are sent to triangles of the triangulated category $\mathcal{D}(\Lambda)$ in a sense to be made precise.

Additivity: Hochschild homology is additive.

Trace maps: there are trace maps linking the K-theory of an exact category to its Hochschild homology.

We review and explain these properties below in Theorem 3.2.5 and Theorem 3.2.7. The relation with K-theory is discussed in Proposition 4.4.1. The most important property for us is the localization property: it allows us to deduce Mayer-Vietoris type conclusions (see Theorem 3.3.8) in the same way as it has been already done for other localizing invariants [BEKW17, BC17].

We start with the definition of the functor $C\colon \mathbf{Ex}_k \to \mathbf{Mix}$.

Let $\mathcal{E}$ be a small k-linear exact category (see Definition A.1.5). The category $\mathrm{Ch}^b(\mathcal{E})$ of bounded chain complexes in $\mathcal{E}$ and the sub-category $\mathrm{Acy}^b(\mathcal{E})$ of bounded acyclic chain complexes in $\mathcal{E}$ have the structure of dg-categories, as described in Example A.2.6. We denote these dg-categories by $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E})$ and $\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{E})$ respectively.

For a small dg-category $\mathcal{A}$, let $\mathrm{Mix}(\mathcal{A})$ be the mixed complex associated to $\mathcal{A}$ as described in Definition 3.1.9.

Definition 3.2.1. [Kel99, Sec. 1.4] Let $\mathcal{E}$ be an exact k-linear category. The mixed complex $C(\mathcal{E})$ associated to $\mathcal{E}$ is the cone

$$C(\mathcal{E}) \coloneqq \operatorname{cone}(\operatorname{Mix}(\operatorname{Acy}^b_{\mathrm{dg}}(\mathcal{E})) \to \operatorname{Mix}(\operatorname{Ch}^b_{\mathrm{dg}}(\mathcal{E}))) \tag{3.2.1}$$

of the morphism of mixed complexes $\operatorname{Mix}(\operatorname{Acy}^b_{\mathrm{dg}}(\mathcal{E})) \to \operatorname{Mix}(\operatorname{Ch}^b_{\mathrm{dg}}(\mathcal{E}))$ induced by the inclusion $\operatorname{Acy}^b_{\mathrm{dg}}(\mathcal{E}) \to \operatorname{Ch}^b_{\mathrm{dg}}(\mathcal{E})$ of dg-categories.

Remark 3.2.2. [Kel99, Sec. 1.4] The cone construction C of Definition 3.2.1 is functorial with respect to (k-linear) exact functors between (k-linear) exact categories.

The cone construction $C\colon \mathbf{Ex}_k \to \mathbf{Mix}$ is a functor from the category $\mathbf{Ex}_k$ of small exact k-linear categories to the category $\mathbf{Mix}$ of mixed complexes. We can describe the functor as follows.

For a category $\mathbf{C}$, we denote by $\mathbf{C}^{\Delta^1}$ the arrow category $\mathbf{Fun}(\{0 \to 1\}, \mathbf{C})$. Then, the cone functor of Definition 3.2.1 can be written as the composition:

$$C\colon \mathbf{Ex}_k \longrightarrow \mathbf{dgcat}_k^{\Delta^1} \xrightarrow{\mathrm{Mix}} \mathbf{Mix}^{\Delta^1} \xrightarrow{\mathrm{cone}} \mathbf{Mix} \tag{3.2.2}$$

where the first functor associates to a k-linear exact category $\mathcal{E}$ the morphism of dg-categories $\operatorname{Acy}^b_{\mathrm{dg}}(\mathcal{E}) \to \operatorname{Ch}^b_{\mathrm{dg}}(\mathcal{E})$ in $\mathbf{dgcat}^{\Delta^1}$ (and on morphisms it is defined in the canonical way), the second functor uses Definition 3.1.9 and the third functor is the cone in the category $\mathbf{Mix}$.

Definition 3.2.3. Let $\mathcal{E}$ be a small k-linear exact category. Keller's Hochschild (cyclic) homology of $\mathcal{E}$ is defined as the Hochschild (cyclic) homology of the mixed complex $C(\mathcal{E})$ associated to $\mathcal{E}$ by the functor C of Definition 3.1.3.

Before describing the properties of the functor $C\colon \mathbf{Ex}_k \to \mathbf{Mix}$, we need some more terminology:

Definition 3.2.4. [Kel99, Sec. 1.5] Let $\mathcal{A}, \mathcal{B}$ be additive categories and let $\mathcal{T}, \mathcal{T}'$ and $\mathcal{T}''$ be triangulated categories.

(i) A *factor-dense* subcategory $\mathcal{A}'$ of $\mathcal{A}$ is a full subcategory such that each object of $\mathcal{A}$ is a direct factor of a finite direct sum of objects of $\mathcal{A}'$.

(ii) An *equivalence up to factors* is an additive functor $\mathcal{A} \to \mathcal{B}$ that induces an equivalence onto a factor-dense subcategory of $\mathcal{B}$.

(iii) A sequence
$$\mathcal{T}' \to \mathcal{T} \to \mathcal{T}''$$
of triangulated categories is *exact up to factors* if the composition is zero, the functor $\mathcal{T}' \to \mathcal{T}$ is fully faithful and the induced functor $\mathcal{T}/\mathcal{T}' \to \mathcal{T}''$ from the Verdier quotient to $\mathcal{T}''$ is an equivalence up to factors.

Let $\mathcal{E}$ be a small exact k-linear category and let $\mathcal{D}^b(\mathcal{E})$ be the derived category of $\mathcal{E}$ (see Definition A.1.10). We recall that, by Example 3.1.5, one can associate a mixed complex $C(A)$ to every k-algebra A. Recall also that $\mathcal{D}(\Lambda)$ denotes the derived category of the dg-algebra Λ.

Theorem 3.2.5. [Kel99, Theorem 1.5] *Let k be a commutative ring.*

1. *If A is a k-algebra, there is a natural isomorphism*

$$C(A) \to C(\mathrm{proj}A)$$

in $\mathcal{D}(\Lambda)$, where $\mathrm{proj}A$ *is the exact category of finitely generated projective modules.*

2. *If $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor between exact categories that induces an equivalence up to factors $\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B})$, then F induces an isomorphism*

$$C(\mathcal{A}) \to C(\mathcal{B})$$

in $\mathcal{D}(\Lambda)$.

3. *If $F\colon \mathcal{A}' \to \mathcal{A}$ and $G\colon \mathcal{A} \to \mathcal{A}''$ are exact functors between exact categories such that the sequence*

$$\mathcal{D}^b(\mathcal{A}') \to \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A}'')$$

is exact up to factors, then there is a canonical morphism $\partial(F,G)$ such that the sequence

$$C(\mathcal{A}') \longrightarrow C(\mathcal{A}) \longrightarrow C(\mathcal{A}'') \xrightarrow{\partial(F,G)} C(\mathcal{A}')[1]$$

is a triangle in $\mathcal{D}(\Lambda)$.

Remark 3.2.6. After application of the functor C, the inclusion of an exact k-linear category in its idempotent completion induces an isomorphism in $\mathcal{D}(\Lambda)$.

If $\mathcal{E}$ is a small k-linear exact category, let $\mathrm{Ex}\,\mathcal{E}$ denote the category of admissible short exact sequences of $\mathcal{E}$ (which is again exact, provided that short exact sequences are defined component-wise). Consider the following exact functors:

$$I\colon \mathcal{E} \to \mathrm{Ex}\,\mathcal{E} \quad A \mapsto (A \xrightarrow{1} A \to 0)$$

$$R\colon \mathrm{Ex}\,\mathcal{E} \to \mathcal{E} \quad (A \to B \to C) \mapsto A$$

$$P\colon \mathrm{Ex}\,\mathcal{E} \to \mathcal{E} \quad (A \to B \to C) \mapsto C$$

$$S\colon \mathcal{E} \to \mathrm{Ex}\,\mathcal{E} \quad C \mapsto (0 \to C \xrightarrow{1} C)$$

The functors I and P are left adjoint to the functors R and S, respectively. Keller proves that Hochschild and cyclic homology are additive invariants:

Theorem 3.2.7 (Theorem 1.12 [Kel99])**.** *Let $\mathcal{E}$ be a small k-linear exact category. Then, the functors P and R induce an isomorphism*

$$C(\mathrm{Ex}\,\mathcal{E}) \xrightarrow{\sim} C(\mathcal{E}) \oplus C(\mathcal{E})$$

in $\mathcal{D}(\Lambda)$.

We now interpret Keller's results in the context of ∞-categories.

Let $\mathbf{dgcat}_k$ be the category of small dg-categories over k. The category $\mathbf{dgcat}_k$ is endowed with the Morita model structure of Theorem A.2.16; we denote by $\mathbf{dgcat}_{k,\infty}$ the underlying ∞-category:

$$\mathbf{dgcat}_{k,\infty} \coloneqq \mathrm{N}(\mathbf{dgcat}_k)[W_{\mathrm{Morita}}^{-1}] \tag{3.2.3}$$

where W_{Morita} is the class of Morita equivalences between dg-categories.

To every dg-category we can associate a cyclic module, *i.e.,* its additive cyclic nerve of Definition A.4.7, hence a mixed complex by Remark 3.1.4. Let

$$\mathrm{Mix}\colon \mathbf{dgcat}_k \to \mathbf{Mix}$$

be the functor of Definition 3.1.9. By Theorem 3.2.5 (2), the functor Mix sends Morita equivalences of dg-categories to quasi-isomorphisms of mixed complexes. A morphism in an ∞-category is an equivalence if its image in the homotopy category is an isomorphism, hence, the functor Mix descends to a functor

$$\begin{tikzcd} \mathbf{dgcat}_k \arrow[r, "\mathrm{Mix}"] \arrow[d, "\mathrm{loc}"] & \mathbf{Mix} \arrow[d, "\mathrm{loc}"] \\ \mathbf{dgcat}_{k,\infty} \arrow[r, "\mathrm{Mix}_\infty"] & \mathbf{Mix}_\infty \end{tikzcd} \tag{3.2.4}$$

on the localizations. Observe that the functor Mix preserves filtered colimits and that quasi-isomorphisms of mixed complexes commute with filtered colimits of mixed complexes. Hence, $\mathrm{loc} \circ \mathrm{Mix}$ preserves filtered colimits because the localization preserves filtered colimits as well.

Let $\mathbf{Ex}_k$ be the category of small exact k-linear categories. Let W_{ex} be the class of exact functors between exact k-linear categories inducing equivalences up to factors between the associated bounded derived categories (Definition 3.2.4). Let $\mathrm{N}(\mathbf{Ex}_k)[W_{ex}^{-1}]$ be the ∞-category associated to $\mathbf{Ex}_k$ by inverting the morphisms of W_{ex} [Lur14, Def. 1.3.4.1]. By Theorem 3.2.5 (2), the cone functor $C\colon \mathbf{Ex}_k \to \mathbf{Mix}$ (3.2.2) sends morphisms in W_{ex} to quasi-isomorphisms of mixed complexes, hence it descends to a functor between the associated ∞-categories

$$\begin{tikzcd} \mathbf{Ex}_k \arrow[r, "C"] \arrow[d, "\mathrm{loc}"] & \mathbf{Mix} \arrow[d, "\mathrm{loc}"] \\ \mathrm{N}(\mathbf{Ex}_k)[W_{ex}^{-1}] \arrow[r] & \mathbf{Mix}_\infty \end{tikzcd}$$

The functor C commutes with filtered colimits (see also Proposition 3.3.7).

The ∞-category $\mathbf{Mix}_\infty$ is stable by Proposition 3.1.11. Cofiber sequences of $\mathbf{Mix}_\infty$ [Lur14, Def. 1.1.1.6] are detected in its homotopy category, *i.e.,* in $\mathcal{D}(\Lambda)$. By Theorem 3.2.5 (3), the composition $\mathrm{loc} \circ C$ (and, analogously, composition with the functor Mix) sends sequences of exact categories inducing equivalences up to factors (between the associated bounded derived categories) to cofiber sequences of the stable ∞-category $\mathbf{Mix}_\infty$.

3.3 The equivariant coarse homology theory $\widetilde{\mathcal{X}C}^G_k$

The goal of this section is to construct an equivariant $\mathbf{Mix}_\infty$-valued coarse homology theory (see Definition 1.2.1)

$$\widetilde{\mathcal{X}C}^G_k : G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$$

from the category of G-bornological coarse spaces to the cocomplete stable ∞-category of mixed complexes (Proposition 3.1.11) that uses Keller's functor $C\colon \mathbf{Ex}_k \to \mathbf{Mix}$ of Definition 3.2.1. In fact, the functor $\widetilde{\mathcal{X}C}^G_k$ is defined (see Definition 3.3.1) as Keller's cone C of the k-linear category $V_k^G(X)$ of G-equivariant X-controlled finite dimensional k-vector spaces (Definition 2.1.2) equipped with the exact structure given by the short split exact sequences.

The main result of the section is Theorem 3.4.2, where we prove that the functor $\widetilde{\mathcal{X}C}^G_k : G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$ is a G-equivariant coarse homology theory; this is achieved by showing that the functor $\widetilde{\mathcal{X}C}^G_k$ satisfies:

(i) coarse invariance, see Proposition 3.3.4;

(ii) vanishing on flasque spaces, see Proposition 3.3.6;

(iii) u-continuity, see Proposition 3.3.7;

(iv) coarse excision, see Theorem 3.3.8;

i.e., the axioms describing an equivariant coarse homology theory (see Definition 1.2.1).

We start with the definition of the functor $\widetilde{\mathcal{X}C}^G_k : G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$.

By Remark 2.1.11, the category $V_k^G(X)$ of G-equivariant X-controlled finite dimensional k-vector spaces is a k-linear category. It is also exact with the exact structure given by the short split exact sequences by Example A.1.6. Hence, the functor V_k^G (2.1.3) can be seen as a functor

$$V_k^G\colon G\mathbf{BornCoarse} \to \mathbf{Ex}_k$$

from the category of G-bornological coarse spaces to the category of small exact k-linear categories. By abuse of notation, with denote the functor V_k^G from $G\mathbf{BornCoarse}$ to $\mathbf{Add}$ or to $\mathbf{Ex}_k$ with the same symbol.

Let loc be the localization functor loc: $\mathbf{Mix} \to \mathrm{N}(\mathbf{Mix}) \to \mathrm{N}(\mathbf{Mix}[W_{mix}^{-1}]) = \mathbf{Mix}_\infty$ [Cis, Def. 7.1.2], [Lur14, Def. 1.3.4.1]. Let $C\colon \mathbf{Ex}_k \to \mathbf{Mix}$ denote Keller's cone construction of Definition 3.2.1.

Definition 3.3.1. We denote by $\widetilde{\mathcal{X}C}_k^G$ the following composition of functors

$$\widetilde{\mathcal{X}C}_k^G : G\mathbf{BornCoarse} \xrightarrow{V_k^G} \mathbf{Ex}_k \xrightarrow{C} \mathbf{Mix} \xrightarrow{\text{loc}} \mathbf{Mix}_\infty$$

from the category of bornological coarse spaces with G-action (Definition 1.1.21) to the ∞-category of mixed complexes (Definition 3.1.1 and (3.1.7)).

Theorem 3.3.2. *The functor*

$$\widetilde{\mathcal{X}C}_k^G : G\mathbf{BornCoarse} \longrightarrow \mathbf{Mix}_\infty$$

is a G-equivariant $\mathbf{Mix}_\infty$-valued coarse homology theory.

The proof of Theorem 3.3.2 follows the ideas of [BEKW17, BC17] and it is a combination of the following Proposition 3.3.4, Proposition 3.3.6, Proposition 3.3.7 and Theorem 3.3.8.

Let split: $\mathbf{Cat}_k \to \mathbf{Ex}_k$ denote the functor that sends a k-linear category to the same category equipped with the split exact structure.

Remark 3.3.3. We have decided to provide a complete proof for the composition

$$\widetilde{\mathcal{X}C}_k^G : G\mathbf{BornCoarse} \xrightarrow{V_k^G} \mathbf{Cat}_k \xrightarrow{\text{split}} \mathbf{Ex}_k \xrightarrow{C} \mathbf{Mix} \xrightarrow{\text{loc}} \mathbf{Mix}_\infty,$$

(that uses the functor $C\colon \mathbf{Ex}_k \to \mathbf{Mix}$ of Definition 3.2.1 instead of the functor Mix of Definition 3.4.1), because the proof of Theorem 3.3.2 works in the more general context in which the k-linear category $V_k^G(X)$ is equipped with other (non-split) exact structures.

We proceed with the proof of the axioms of Definition 1.2.1.

Proposition 3.3.4. *The functor* $\widetilde{\mathcal{X}C}_k^G : G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$ (3.3.1) *satisfies coarse invariance.*

Proof. If $f\colon X \to Y$ is a coarse equivalence of G-bornological coarse spaces, then it induces a natural equivalence $f_*\colon V_k^G(X) \to V_k^G(Y)$ by Lemma 2.1.14. Keller's cone functor 3.2.1 sends equivalences of k-linear exact categories to equivalences of mixed complexes by Theorem 3.2.5 (2). Hence, the functor f_* induces the equivalence

$$\widetilde{\mathcal{X}C}^G_k(X) \xrightarrow{\sim} \widetilde{\mathcal{X}C}^G_k(Y)$$

in $\mathbf{Mix}_\infty$. The functor $\widetilde{\mathcal{X}C}^G_k$ satisfies then coarse invariance by Remark 1.2.2. □

We recall that the homotopy category of $\mathbf{Mix}_\infty$ is the derived category $\mathcal{D}(\Lambda)$. The following is [Sch11, Theorem 2.3.11], reinterpreted in our setting:

Theorem 3.3.5. *Let $\mathcal{E}$ and $\mathcal{E}'$ be exact categories and let $F' \to F \to F''$ be a sequence of exact functors $\mathcal{E} \to \mathcal{E}'$ such that $F'(A) \to F(A) \to F''(A)$ is a short exact sequence for all objects $A \in \mathcal{E}$. Then, the functors F and $F' \oplus F''$ induce an equivalence*

$$C(F) \simeq C(F' \oplus F'')\colon C(\mathcal{E}) \to C(\mathcal{E}')$$

in $\mathbf{Mix}_\infty$.

Proof. Recall the exact functors I, P, R, S defined in Theorem 3.2.7; the induced functors D^bI and D^bP are left adjoint to D^bR and D^bS, with unit and counit the isomorphisms $\mathrm{id} \to D^bR \circ D^bI$ and $D^bP \circ D^bS \to \mathrm{id}$.

The sequence of functors $F' \to F \to F''$ induces an exact functor $F_\bullet\colon \mathcal{E} \to \mathrm{Ex}\,\mathcal{E}'$ sending an object A to the sequence $F'(A) \to F(A) \to F''(A)$. By Theorem 3.2.7, the composition of the functors $I \oplus S$ and (R, P)

$$G\colon \mathrm{Ex}\,\mathcal{E}' \xrightarrow{(R,P)} \mathcal{E}' \times \mathcal{E}' \xrightarrow{I \oplus S} \mathrm{Ex}\,\mathcal{E}'$$

being mutual inverses in $\mathcal{D}(\Lambda)$, induces a map $C(G)\colon C(\mathrm{Ex}\,\mathcal{E}') \to C(\mathrm{Ex}\,\mathcal{E}')$ that is isomorphic to the identity in $\mathcal{D}(\Lambda)$.

Let $M\colon \mathrm{Ex}\,\mathcal{E}' \to \mathcal{E}'$ be the functor sending a short exact sequence $A \to B \to C$ to the object B. Then, the functors F and $F' \oplus F''$ can be written as the following compositions

$$\mathcal{E} \xrightarrow{F_\bullet} \mathrm{Ex}\,\mathcal{E}' \xrightarrow{M} \mathcal{E}' \quad \text{and} \quad \mathcal{E} \xrightarrow{F_\bullet} \mathrm{Ex}\,\mathcal{E}' \xrightarrow{G} \mathrm{Ex}\,\mathcal{E}' \xrightarrow{M} \mathcal{E}'$$

and they induce isomorphic maps $C(\mathcal{E}) \to C(\mathcal{E}')$ in $\mathcal{D}(\Lambda)$. □

Recall the definition of a flasque category Definition A.1.4. As a corollary of Schlichting's theorem, we get that the functor $\widetilde{\mathcal{X}C}^G_k$ vanishes on flasque spaces:

Proposition 3.3.6. *The functor* $\widetilde{\mathcal{X}C}^G_k\colon G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$ (3.3.1) *vanishes on flasque spaces.*

Proof. Let X be a flasque space. By Lemma 2.1.15, the category $V_k^G(X)$ is a flasque category, hence there exists an endofunctor $S\colon V_k^G(X) \to V_k^G(X)$ such that $\mathrm{id}_{V_k^G(X)} \oplus S \cong S$. By Theorem 3.3.5, $C(\mathrm{id}) \oplus C(S)$ and $C(\mathrm{id} \oplus S) \cong C(S)$ induce equivalent morphisms in $\mathbf{Mix}_\infty$. This means that the morphism

$$\widetilde{\mathcal{X}C}_k^G(\mathrm{id})\colon \widetilde{\mathcal{X}C}_k^G(X) \to \widetilde{\mathcal{X}C}_k^G(X)$$

is equivalent to the 0-morphism, hence that $\widetilde{\mathcal{X}C}_k^G(X)$ is equivalent to the object 0 in $\mathbf{Mix}_\infty$, concluding the proof. □

Proposition 3.3.7. *The functor* $\widetilde{\mathcal{X}C}_k^G : G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$ (3.3.1) *is u-continuous.*

Proof. Let X be a G-bornological coarse space, and let $\mathcal{C}^G$ be the poset of G-invariant controlled sets. By Remark 2.1.13, there is an equivalence $V_k^G(X) \simeq \mathrm{colim}_{U \in \mathcal{C}^G} V_k^G(X_U)$ of k-linear exact categories.

The canonical maps

$$\mathrm{colim}_U \mathrm{Ch}^b_{\mathrm{dg}}(V_k^G(X_U)) \to \mathrm{Ch}^b_{\mathrm{dg}}(V_k^G(X))$$

and

$$\mathrm{colim}_U \mathrm{Acy}^b_{\mathrm{dg}}(V_k^G(X_U)) \to \mathrm{Acy}^b_{\mathrm{dg}}(V_k^G(X))$$

are equivalences of k-linear exact categories. The functor C sends equivalences to equivalences of mixed complexes, hence we get the equivalence

$$\widetilde{\mathcal{X}C}_k^G(X) \simeq \mathrm{colim}_{U \in \mathcal{C}^G} \widetilde{\mathcal{X}C}_k^G(X_U)$$

in $\mathbf{Mix}_\infty$. This proves that the functor $\widetilde{\mathcal{X}C}_k^G$ is u-continuous. □

Theorem 3.3.8. *The functor* $\widetilde{\mathcal{X}C}_k^G : G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$ (3.3.1) *satisfies coarse excision.*

The proof of the theorem follows the main ideas of [CP97, BE16, BEKW17]; we first need some more terminology.

Definition 3.3.9. [Kas15] A full additive subcategory $\mathcal{A}$ of an additive category $\mathcal{U}$ is a *Karoubi-filtration* if every diagram

$$X \to Y \to Z$$

in $\mathcal{U}$, with $X, Z \in \mathcal{A}$, admits an extension

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "\cong"] & Z \\ U \arrow[r, "i"] & U \oplus U^\perp \arrow[r, "p"] & U \arrow[u] \end{tikzcd}$$

with $U \in \mathcal{A}$.

By [Kas15, Lemma 5.6], this definition is equivalent to the classical one [Kar70, CP97]. If $\mathcal{A}$ is a Karoubi-filtration of $\mathcal{U}$, we can construct a quotient category $\mathcal{U}/\mathcal{A}$. Its objects are the objects of $\mathcal{U}$, and the morphisms sets are defined as follows:

$$\mathrm{Hom}_{\mathcal{U}/\mathcal{A}}(U,V) \coloneqq \mathrm{Hom}_{\mathcal{U}}(U,V)/\sim$$

where the relation identifies pairs of maps $U \to V$ whose difference factorizes through an object of $\mathcal{A}$.

Let X be a G-bornological coarse space and let $\mathcal{Y} = (Y_i)_{i\in I}$ be an equivariant big family on X (see Definition 1.1.26). The bornological coarse space Y_i is a subspace of X with the induced bornology and coarse structure. The inclusion $Y_i \hookrightarrow X$ induces a functor $V_k^G(Y_i) \to V_k^G(X)$ which is injective on objects. The categories $V_k^G(Y_i)$ and $V_k^G(\mathcal{Y}) \coloneqq \mathrm{colim}_{i\in I} V_k^G(Y_i)$ are full subcategories of $V_k^G(X)$.

Lemma 3.3.10. [BEKW17, Lemma 8.14] *Let $\mathcal{Y}$ be an equivariant big family on the G-bornological coarse space X. Then, the full additive subcategory $V_k^G(\mathcal{Y})$ of $V_k^G(X)$ is a Karoubi filtration.*

Let X be a G-bornological coarse space, and $(Z,\mathcal{Y})$ be an equivariant complementary pair. Consider the functor

$$a\colon V_k^G(Z)/V_k^G(Z\cap\mathcal{Y}) \to V_k^G(X)/V_k^G(\mathcal{Y}) \tag{3.3.1}$$

induced by the inclusion of Z in X; on objects, it coincides with $i_*\colon V_k^G(Z) \to V_k^G(X)$, but on morphisms it sends an equivalence class $[A]$ of A in the equivalence $[i_*(A)]$ of $i_*(A)$.

Lemma 3.3.11. [BEKW17, Prop. 8.15] *The functor a* (3.3.1) *is an equivalence of categories.*

Remark 3.3.12. By [Sch04, Ex. 1.8, Prop. 2.6], if $\mathcal{A} \subseteq \mathcal{U}$ is a Karoubi filtration with $\mathcal{A}$ idempotent complete, the sequence of exact categories $\mathcal{A} \to \mathcal{U} \to \mathcal{U}/\mathcal{A}$ induces a sequence of triangulated categories

$$D^b(\mathcal{A}) \to D^b(\mathcal{U}) \to D^b(\mathcal{U}/\mathcal{A})$$

that is exact up to factors. When $\mathcal{A}$ is not idempotent complete, this is achieved by first applying [Sch00, Lemma 3.8] to the Karoubi filtration $\mathcal{A} \subseteq U$ and then again [Sch04, Prop. 2.6].

We use this remark and the previous lemmas to prove Theorem 3.3.8:

Proof of Theorem 3.3.8. Let X be a G-bornological coarse space, and let $(Z, \mathcal{Y})$ be an equivariant complementary pair on X. By Lemma 3.3.10, $V_k^G(Z \cap \mathcal{Y}) \subseteq V_k^G(Z)$ and $V_k^G(\mathcal{Y}) \subseteq V_k^G(X)$ are Karoubi filtrations. Consider the following sequences of k-linear exact categories:

$$V_k^G(Z \cap \mathcal{Y}) \to V_k^G(Z) \to V_k^G(Z)/V_k^G(Z \cap \mathcal{Y})$$

and

$$V_k^G(\mathcal{Y}) \to V_k^G(X) \to V_k^G(X)/V_k^G(X \cap \mathcal{Y}).$$

By Remark 3.3.12, the induced sequences of bounded derived categories are exact up to factors, hence the assumptions of Theorem 3.2.5 are satisfied and we get cofiber sequences of mixed complexes:

$$C(V_k^G(Z \cap \mathcal{Y})) \to C(V_k^G(Z)) \to C(V_k^G(Z)/V_k^G(Z \cap \mathcal{Y})) \to C(V_k^G(Z \cap \mathcal{Y}))[1]$$

and

$$C(V_k^G(\mathcal{Y})) \to C(V_k^G(X)) \to C(V_k^G(X)/V_k^G(X \cap \mathcal{Y})) \to C(V_k^G(\mathcal{Y}))[1]$$

The inclusion $Z \hookrightarrow X$ induces a commutative diagram

$$\begin{tikzcd}
C(V_k^G(Z \cap \mathcal{Y})) \arrow[r] \arrow[d] & C(V_k^G(Z)) \arrow[r] \arrow[d] & C(V_k^G(Z)/V_k^G(Z \cap \mathcal{Y})) \arrow[d, dashed, "a_*"] \\
C(V_k^G(\mathcal{Y})) \arrow[r] & C(V_k^G(X)) \arrow[r] & C(V_k^G(X)/V_k^G(X \cap \mathcal{Y}))
\end{tikzcd}$$

where a_* is the map induced by $a \colon V_k^G(Z)/V_k^G(Z \cap \mathcal{Y}) \to V_k^G(X)/V_k^G(\mathcal{Y})$ (3.3.1). By Lemma 3.3.11, the functor a_* yields an equivalence of categories, hence the left square is a co-Cartesian square in $\mathbf{Mix}_\infty$.

In order to conclude the proof, we recall that $\mathcal{X}C_k^G(\mathcal{Y})$ is the filtered colimit $\mathcal{X}C_k^G(\mathcal{Y}) = \operatorname{colim}_i \mathcal{X}C_k^G(Y_i)$ (1.2.1) and that

$$V_k^G(\mathcal{Y}) \coloneqq \operatorname{colim}_{i \in I} V_k^G(Y_i).$$

The functor C commutes with filtered colimits of k-linear categories, hence we have the equivalence $C(V_k^G(\mathcal{Y})) = C(\operatorname{colim}_i V_k^G(Y_i)) \simeq \operatorname{colim}_i C(V_k^G(Y_i))$ and the same holds for $Z \cap \mathcal{Y}$. By using these identifications, we obtain the co-Cartesian square in $\mathbf{Mix}_\infty$

$$\begin{tikzcd}
\widetilde{\mathcal{X}C}_k^G(Z \cap \mathcal{Y}) \arrow[r] \arrow[d] & \widetilde{\mathcal{X}C}_k^G(Z) \arrow[d] \\
\widetilde{\mathcal{X}C}_k^G(\mathcal{Y}) \arrow[r] & \widetilde{\mathcal{X}C}_k^G(X)
\end{tikzcd}$$

meaning that $\widetilde{\mathcal{X}C}_k^G$ satisfies coarse excision. $\square$

Proof of Theorem 3.3.2. By Proposition 3.1.11, the ∞-category $\mathbf{Mix}_\infty$ is stable and cocomplete. By Proposition 3.3.4, Proposition 3.3.6, Proposition 3.3.7 and Theorem 3.3.8, the functor $\widetilde{\mathcal{X}C}^G_k$ of Definition 3.3.1 is a G-equivariant coarse homology theory. $\square$

3.4 Coarse Hochschild and coarse cyclic homology

In this section, we define Hochschild homology

$$\mathcal{X}\mathrm{HH}^G_k \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$$

and cyclic homology

$$\mathcal{X}\mathrm{HC}^G_k \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$$

for G-bornological coarse spaces (see Definition 3.4.6). These are equivariant coarse homology theories with values in the cocomplete stable ∞-category $\mathbf{Ch}_\infty$ of chain complexes.

By Theorem 3.3.2, Keller's cone functor $C\colon \mathbf{Ex}_k \to \mathbf{Mix}$ yields an equivariant coarse homology theory $\widetilde{\mathcal{X}C}^G_k \colon G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$ with values in the cocomplete stable ∞-category $\mathbf{Mix}_\infty$ of mixed complexes. In this section, namely in Lemma 3.4.4, we see that the functor C and the functor $\mathrm{Mix}\colon \mathbf{dgcat}_k \to \mathbf{Mix}$ of Definition 3.1.9 are naturally equivalent when restricted to k-linear categories (equipped with the exact structure given by split short exact sequences). Hence, also the functor Mix yields a G-equivariant coarse homology theory that we call $\mathcal{X}\mathrm{Mix}^G_k$ (see Definition 3.4.1).

Then, we define the functors $\mathcal{X}\mathrm{HH}^G_k$ and $\mathcal{X}\mathrm{HC}^G_k$ as the Hochschild and cyclic homology, respectively, of the mixed complex $\mathrm{Mix}(V^G_k(X))$ associated to the k-linear category $V^G_k(X)$ of G-equivariant X-controlled finite dimensional k-vector spaces (see Definition 2.1.2).

We start with the definition of the functor $\mathcal{X}\mathrm{Mix}^G_k \colon G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$.

Let $\mathbf{Cat}_k$ be the category of small k-linear categories, let $V^G_k \colon G\mathbf{BornCoarse} \to \mathbf{Cat}_k$ be the functor of Remark 2.1.12, let $\mathrm{Mix}\colon \mathbf{dgcat}_k \to \mathbf{Mix}$ be the functor of Definition 3.1.9, let $\iota\colon \mathbf{Cat}_k \to \mathbf{dgcat}_k$ be the functor that associates to a k-linear category the dg-category with the same objects and morphisms complexes concentrated in degree zero (see Example A.2.5) and let loc be the localization functor $\mathrm{loc}\colon \mathbf{Mix} \to \mathrm{N}(\mathbf{Mix}) \to \mathrm{N}(\mathbf{Mix}[W^{-1}_{mix}]) = \mathbf{Mix}_\infty$.

Definition 3.4.1. Let k be a field. We denote by $\mathcal{X}\mathrm{Mix}^G_k$ the following composition of functors

$$\mathcal{X}\mathrm{Mix}^G_k \colon G\mathbf{BornCoarse} \xrightarrow{V^G_k} \mathbf{Cat}_k \xrightarrow{\iota} \mathbf{dgcat}_k \xrightarrow{\mathrm{Mix}} \mathbf{Mix} \xrightarrow{loc} \mathbf{Mix}_\infty$$

from the category of G-bornological coarse spaces (Definition 1.1.21) to the ∞-category of mixed complexes (Definition 3.1.1).

Theorem 3.4.2. *The functor*

$$\mathcal{X}\mathrm{Mix}_k^G \colon G\mathbf{BornCoarse} \longrightarrow \mathbf{Mix}_\infty$$

is a G-equivariant $\mathbf{Mix}_\infty$-valued coarse homology theory.

Proof. The theorem is a consequence of Theorem 3.3.2 and the next Lemma 3.4.4. □

Let $\mathcal{A}$ be a small additive category, equipped with the exact structure given by the split short exact sequences (Example A.1.6). Consider the category of bounded chain complexes in $\mathcal{A}$ and recall that a bounded chain complex is acyclic in degree n if the boundary operator factors through an object Z_{n+1}

$$\begin{tikzcd} K_{n+1} \arrow[rr, "d_K^{n+1}"] \arrow[dr, "p_{n+1}"'] & & K_n \\ & Z_{n+1} \arrow[ur, "i_{n+1}"'] & \end{tikzcd}$$

with p_{n+1} a deflation and i_{n+1} an inflation (see Definition A.1.8).

For an additive category $\mathcal{A}$, let $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})$ be the associated dg-category of bounded chain complexes and $\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{A})$ be the sub-dg-category of bounded acyclic chain complexes; let $\mathrm{Mix}\colon \mathbf{dgcat}_k \to \mathbf{Mix}$ be the functor of Definition 3.1.9.

Lemma 3.4.3. *Let $\mathcal{A}$ be an additive category. Then, the mixed complex $\mathrm{Mix}(\mathrm{Acy}^b_{dg}(\mathcal{A}))$ associated to the dg-category of bounded acyclic chain complexes in $\mathcal{A}$ is quasi-isomorphic to the mixed complex 0.*

Proof. The exact structure on $\mathcal{A}$ is given by the short split exact sequences. Let $K = (K_p)_{p\in\mathbb{Z}}$ be a bounded acyclic chain complex in $\mathrm{Ch}^b(\mathcal{A})$. Let H be another bounded chain complex in $\mathrm{Ch}^b(\mathcal{A})$ and consider the chain complex $\mathrm{Hom}_{\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})}(H, K)$. First, we want to see that this chain complex is acyclic.

Let g be a morphism in $\mathrm{Hom}_{\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})}(H, K)$; without loss of generality, we can assume that the degree of g is zero, hence $g = (g_p\colon H_p \to K_p)_{p\in\mathbb{Z}}$. We want to construct an element $f = (f_p)_{p\in\mathbb{Z}}$ of $\mathrm{Hom}_{\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})}(H, K)$ of degree 1 whose differential is g. In order to construct $f_n\colon H_n \to K_{n+1}$ we use the split exact structure of $\mathcal{A}$. In fact, by acyclicity, we have the diagram:

$$\begin{tikzcd} K_{n+1} \arrow[rr, "d_K^{n+1}"] \arrow[dr, "p_{n+1}"'] & & K_n \arrow[rr, "d_K^n"] \arrow[dr, "p_n"'] & & K_{n-1} \\ & Z_{n+1} \arrow[ur, "i_{n+1}"'] & & Z_n \arrow[ur, "i_n"'] & \end{tikzcd}$$

where the i's are inflations and the p's deflations of $\mathcal{A}$. The split exact structure of $\mathcal{A}$ provides splittings of both the i's and p's, say $j_{n+1}\colon K_n \to Z_{n+1}$ and $q_n\colon Z_n \to K_n$. Hence, a lift of $g_n\colon H_n \to K_n$ is constructed as $f_n \coloneqq q_{n+1} \circ j_{n+1} \circ g_n$, and this for every $n \in \mathbb{Z}$. This shows the existence of a map $f = (f_p)_{p\in\mathbb{Z}}$ of degree 1. By using again the splitting at K_n, we see that the differential $d(f) = d_K \circ f - f \circ d_H$ is the starting map g. This is enough to prove that the chain complex $\mathrm{Hom}_{\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})}(H, K)$ is acyclic.

The additive cyclic nerve associated to the dg-category $\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{A})$ is given in degree n by direct sums of complexes

$$\mathrm{Hom}(K_1, K_0) \otimes_k \mathrm{Hom}(K_2, K_1) \otimes_k \cdots \otimes_k \mathrm{Hom}(K_0, K_n)$$

where every bounded chain complex K_i over $\mathcal{A}$ is acyclic, hence the chain complexes $\mathrm{Hom}(K_i, K_{i-1})$ are acyclic. The tensor product of acyclic complexes is again acyclic, hence $\mathrm{CN}_n(\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{A}))$ is acyclic for each n. The whole complex $\mathrm{CN}_*(\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{A}))$ is now acyclic because it is a double chain complex with acyclic rows. Therefore, the associated mixed complex $\mathrm{Mix}(\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{A}))$ is quasi-isomorphic to the zero mixed complex. □

Consider the following functors:

(a) $\overline{C}\colon \mathbf{Cat}_k \xrightarrow{\text{split}} \mathbf{Ex}_k \xrightarrow{C} \mathbf{Mix} \xrightarrow{\text{loc}} \mathbf{Mix}_\infty$ where the first functor sends a k-linear category to the exact k-linear category with split exact structure and C is Keller's cone functor (3.2.2).

(b) $\overline{\mathrm{Mix}}\colon \mathbf{Cat}_k \xrightarrow{\iota} \mathbf{dgcat}_k \xrightarrow{\mathrm{Mix}} \mathbf{Mix} \xrightarrow{\text{loc}} \mathbf{Mix}_\infty$ where the first functor associates to a k-linear category the dg-category with the same objects and morphisms complexes concentrated in degree zero (Example A.2.5) and Mix is the functor of Definition 3.1.9.

(c) $\overline{\mathrm{Mix} \circ \mathrm{Ch}^b_{\mathrm{dg}}}\colon \mathbf{Cat}_k \xrightarrow{\iota} \mathbf{dgcat}_k \xrightarrow{\mathrm{Ch}^b_{\mathrm{dg}}} \mathbf{dgcat}_k \xrightarrow{\mathrm{Mix}} \mathbf{Mix} \xrightarrow{\text{loc}} \mathbf{Mix}_\infty$ where the functor $\mathrm{Ch}^b_{\mathrm{dg}}$ sends a k-linear category $\mathcal{A}$ to the dg-category $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})$.

Lemma 3.4.4. *Let $\overline{C}$, $\overline{\mathrm{Mix}}$ and $\overline{\mathrm{Mix} \circ \mathrm{Ch}^b_{dg}}$ be the functors defined in* (a),(b) *and* (c)*, respectively. Then, there are natural isomorphisms:*

(i) $\overline{\mathrm{Mix}} \xrightarrow{\cong} \overline{\mathrm{Mix} \circ \mathrm{Ch}^b_{dg}}$*;*

(ii) $\overline{\mathrm{Mix}} \xrightarrow{\cong} \overline{C}$*.*

Proof. (i) Let $\mathcal{A}$ be a k-linear category, seen as dg-category by the functor $\iota\colon \mathbf{Cat}_k \to \mathbf{dgcat}_k$. Let id be the identity functor of $\mathbf{dgcat}_k$ and let $\mathrm{Ch}^b_{\mathrm{dg}}\colon \mathbf{dgcat}_k \to \mathbf{dgcat}_k$ be the functor sending a dg-category $\mathcal{A}$ to the dg-category $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})$ of bounded chain complexes in $\mathcal{A}$. Then, the transformation $\overline{\mathrm{Mix}} \to \overline{\mathrm{Mix} \circ \mathrm{Ch}^b_{\mathrm{dg}}}$ is induced by the transformation $\eta\colon \mathrm{id} \to \mathrm{Ch}^b_{\mathrm{dg}}$ defined by

$$\eta_{\mathcal{A}}\colon \mathrm{id}(\mathcal{A}) \to \mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A}),$$

the canonical inclusion of $\mathcal{A}$ in $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})$.

The inclusion of $\mathcal{A}$ in the dg-category of bounded chain complexes represents the inclusion in the pretriangulated hull, as remarked in Example A.2.13, and the inclusion in the pretriangulated hull is a Morita equivalence by Example A.2.15. Therefore, the transformation η induces an equivalence $\mathrm{Mix}(\mathcal{A}) \simeq \mathrm{Mix}(\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A}))$ of mixed complexes by Theorem 3.2.5 (2). This is enough to prove that the transformation η induces a natural isomorphism.

(ii) Observe that the functor $\mathbf{Cat}_k \xrightarrow{\iota} \mathbf{dgcat}_k \xrightarrow{\mathrm{Mix}} \mathbf{Mix}$ is equivalent to the functor $\mathbf{Cat}_k \xrightarrow{\iota'} \mathbf{dgcat}_k^{\Delta^1} \to \mathbf{Mix}^{\Delta^1} \xrightarrow{\mathrm{cone}} \mathbf{Mix}$ where a k-linear category $\mathcal{A}$ in $\mathbf{Cat}_k$ is sent by the functor ι' to the morphism $0 \to \mathcal{A}$ of $\mathbf{dgcat}_k$; the other functors are the same functors as in (3.2.2).

Let $j\colon \mathbf{dgcat}_k \to \mathbf{dgcat}_k^{\Delta^1}$ be the functor sending a dg-category $\mathcal{A}$ to the morphism $\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})$ (defined on morphisms of $\mathbf{dgcat}_k$ in the natural way) and let $\rho\colon \iota' \to j$ be the transformation defined as:

$$\rho_{\mathcal{A}}\colon \iota'(\mathcal{A}) \to j(\mathcal{A}), \quad (0 \to \mathcal{A}) \mapsto (\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{A}) \to \mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})).$$

Then, ρ induces a transformation $\overline{\mathrm{Mix}} \to \overline{C}$. By Lemma 3.4.3, the mixed complex $\mathrm{Mix}(\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{A}))$ is equivalent to 0 and by the equivalence (i), we have $\mathrm{Mix}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Mix}(\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A}))$. After applying the cone,we get the equivalence

$$\mathrm{cone}(\mathrm{Mix}(0) \to \mathrm{Mix}(\mathcal{A})) \xrightarrow{\simeq} \mathrm{cone}(\mathrm{Mix}(\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{A})) \to \mathrm{Mix}(\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A}))),$$

i.e., $\overline{\mathrm{Mix}}(\mathcal{A}) \xrightarrow{\simeq} \overline{C}(\mathcal{A})$.

□

Remark 3.4.5. There are two different cyclic homology theories for k-linear exact categories: McCarthy's cyclic homology [McC94] and Keller's cyclic homology [Kel99]. McCarthy's cyclic homology of a k-linear category $\mathcal{A}$ equipped with the exact structure of split short exact sequences is defined as the cyclic homology of the additive cyclic nerve associated to $\mathcal{A}$. Keller's cyclic homology of an exact category is defined in terms of the functor C (3.2.2). The above lemma says that McCarthy's Hochschild homology (and cyclic homology) in terms of the additive cyclic nerve [McC94] and Keller's Hochschild homology (and cyclic homology) in terms of the cone construction in the category of mixed complexes are equivalent functors when restricted to the category of small k-linear categories (equipped with the split exact structure). This fact is probably known to the experts, but for convenience of the reader we have provided a proof of it.

Let $\mathbf{Ch}$ be the category of chain complexes (over k). Recall that there are functors forget: $\mathbf{Mix} \to \mathbf{Ch}$ (3.1.3), sending a mixed complex to the underlying chain complex, and $\mathrm{Tot}(\mathcal{B}-)\colon \mathbf{Mix} \to \mathbf{Ch}$ (3.1.4), sending a mixed complex to the total complex of the associated bicomplex. These functors send quasi-isomorphisms of mixed complexes to quasi-isomorphisms of chain complexes. Hence, they induce the functors

$$\mathrm{forget}_\infty\colon \mathbf{Mix}_\infty \to \mathbf{Ch}_\infty \tag{3.4.1}$$

and

$$\mathrm{Tot}(\mathcal{B}-)_\infty\colon \mathbf{Mix}_\infty \to \mathbf{Ch}_\infty \tag{3.4.2}$$

where $\mathbf{Ch}_\infty$ is the ∞-category of chain complexes.

Definition 3.4.6. Let k be a field, G a group and $\mathbf{Ch}_\infty$ the ∞-category of chain complexes (over k). The *G-equivariant coarse Hochschild homology* $\mathcal{X}\mathrm{HH}_k^G$ (with k-coefficients) is the G-equivariant $\mathbf{Ch}_\infty$-valued coarse homology theory defined as the following composition:

$$\mathcal{X}\mathrm{HH}_k^G\colon G\mathbf{BornCoarse} \xrightarrow{\mathcal{X}\mathrm{Mix}_k^G} \mathbf{Mix}_\infty \xrightarrow{\text{forget}} \mathbf{Ch}_\infty.$$

where $\mathcal{X}\mathrm{Mix}_k^G$ is the equivariant coarse homology theory of Definition 3.4.1. The composition

$$\mathcal{X}\mathrm{HC}_k^G\colon G\mathbf{BornCoarse} \xrightarrow{\mathcal{X}\mathrm{Mix}_k^G} \mathbf{Mix}_\infty \xrightarrow{\mathrm{Tot}(\mathcal{B}-)} \mathbf{Ch}_\infty.$$

is called *G-equivariant coarse cyclic homology.*

Theorem 3.4.7. *The functors*

$$\mathcal{X}\mathrm{HH}_k^G\colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty \quad \textit{and} \quad \mathcal{X}\mathrm{HC}_k^G\colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$$

of Definition 3.4.6 are G-equivariant coarse homology theories.

Proof. By Theorem 3.4.2, the functor $\mathcal{X}\mathrm{Mix}_k^G\colon G\mathbf{BornCoarse} \to \mathbf{Mix}_\infty$ is an equivariant coarse homology theory and satisfies coarse invariance, coarse excision, u-continuity, and vanishing on flasques. The functors $\mathrm{forget}_\infty\colon \mathbf{Mix}_\infty \to \mathbf{Ch}_\infty$ (3.4.1) and the functor $\mathrm{Tot}(\mathcal{B}-)_\infty\colon \mathbf{Mix}_\infty \to \mathbf{Ch}_\infty$ (3.4.2) commute with filtered colimits, and send cofiber sequences to cofiber sequences. The two compositions with $\mathcal{X}\mathrm{Mix}_k^G$ satisfy coarse invariance, coarse excision, u-continuity, and vanishing on flasques, and the functors $\mathcal{X}\mathrm{HH}_k^G$ and $\mathcal{X}\mathrm{HC}_k^G$ are equivariant coarse homology theories as well. □

Question 3.4.8. The analogue functors providing negative and periodic cyclic homologies are not u-continuous and not excisive, as direct products do not commute with colimits. Hence, we are not able to construct coarse versions of negative and periodic cyclic homology with the same methods. It remains open whether a different definition of coarse versions of negative and periodic cyclic homologies can be given.

By applying coarse excision, we can compute the value of coarse Hochschild (and cyclic) homology on coarse spaces finitely generated by one point (endowed with trivial G-action).

Example 3.4.9. By Example 1.1.20, we get:

$$\mathcal{X}\mathrm{HH}(\mathbb{R}^n) \cong \Sigma^n\, \mathcal{X}\mathrm{HH}(*)$$

where $\mathbb{R}^n$ is endowed with the euclidean metric.

3.5 A symmetric monoidal refinement of $\mathcal{X}\mathrm{HH}_k^G$

The goal of this section is to refine the functors $\mathcal{X}\mathrm{HH}_k^G$ and $\mathcal{X}\mathrm{HC}_k^G$ of Definition 3.4.6 to lax symmetric monoidal functors.

This is achieved by providing lax symmetric monoidal refinements of the functors V_k^G, ι and loc $\circ$Mix in the definition

$$\mathcal{X}\mathrm{Mix}_k^G\colon G\mathbf{BornCoarse} \xrightarrow{V_k^G} \mathbf{Cat}_k \xrightarrow{\iota} \mathbf{dgcat}_k \xrightarrow{\mathrm{Mix}} \mathbf{Mix} \xrightarrow{\mathrm{loc}} \mathbf{Mix}_\infty$$

of $\mathcal{X}\mathrm{Mix}_k^G$.

We start with the functor Mix: $\mathbf{dgcat}_k \to \mathbf{Mix}$; recall that Mix induces a functor $\mathrm{Mix}_\infty\colon \mathbf{dgcat}_{k,\infty} \to \mathbf{Mix}_\infty$ (3.2.4). By [Kas87, Thm. 2.4], the functor Mix_∞ has a lax symmetric monoidal refinement (see also [CT12]):

Theorem 3.5.1. *The functor* $\mathrm{Mix}_\infty\colon \mathbf{dgcat}_{k,\infty} \to \mathbf{Mix}_\infty$ *has a a lax symmetric monoidal refinement.*

The lax symmetric monoidal structure on the functor $\mathrm{Mix}_\infty\colon \mathbf{dgcat}_{k,\infty} \to \mathbf{Mix}_\infty$ (3.2.4) induced by [Kas87, Thm. 2.4] uses shuffle products, that for convenience of the reader we now recall, before giving the proof of the theorem.

Remark 3.5.2. The category of mixed complexes has a natural symmetric monoidal structure induced by the canonical symmetric monoidal structure of (the underlying) chain complexes: if (C, b, B) and (C', b', B') are two mixed complexes, then their tensor product is the mixed complex

$$(C \otimes C', b \otimes \mathrm{id} + \mathrm{id} \otimes b, B \otimes \mathrm{id} + \mathrm{id} \otimes B)$$

where all the tensor products are taken over k. The unit object is k.

Let M and N two simplicial modules with associated chain complexes M_* and N_*. The product $M \times N$ is the simplicial module $(M \times N)_n \coloneqq M_n \otimes N_n$ with face and degeneracies maps $d_i(m \otimes n) \coloneqq d_i^M(m) \otimes d_i^N(n)$ and $s_j(m \otimes n) \coloneqq s_j^M(m) \otimes s_j^N(n)$, where

d_i^M and s_j^M are the face and degeneracy maps of M and d_i^N and s_j^N are the face and degeneracy maps of N.

Let p, q be two non-negative integer numbers; a (p, q)-*shuffle* (μ, ν) is a partition of the set of natural numbers $\{0, \dots, p+q-1\}$ into two disjoint subsets

$$\mu_1 < \mu_2 < \cdots < \mu_p \quad \text{and} \quad \nu_1 < \nu_2 < \cdots < \nu_q.$$

The tuple $(\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)$ is the result of a permutation of the ordered set $(0, \dots, p+q-1)$. Let $\operatorname{sgn}(\mu, \nu)$ be the sign of this permutation.

For simplicial modules M and N, define a homomorphism (*a shuffle map*) [Lod98, Sec. 1.6.8]

$$\mathrm{sh}\colon M_* \otimes N_* \to (M \times N)_*$$

in the product $M \times N$ by extending linearly

$$\mathrm{sh}(m \otimes n) = \sum_{(\mu,\nu)} \operatorname{sgn}(\mu, \nu)(s_{\nu_q} \dots s_{\nu_1}(m) \otimes s_{\mu_p} \dots s_{\mu_1}(n)) \tag{3.5.1}$$

where m belongs to M_p, n to N_q and (μ, ν) runs over all the (p, q)-shuffles. The Eilenberg-Zilber Theorem says that the shuffle map sh is a quasi-isomorphism.

Proof of Theorem 3.5.1. The category $\mathbf{dgcat}_k$ has a natural symmetric monoidal structure induced by the object-wise tensor product of Definition A.3.3. By Remark A.3.6, we get a symmetric monoidal ∞-category $\mathrm{N}(\mathbf{dgcat}_k^{\otimes}) \to \mathrm{N}(\mathbf{Fin}_*)$ whose underlying ∞-category is $\mathrm{N}(\mathbf{dgcat}_k)$. As k is a field, this tensor product preserves the Morita equivalences of the Morita model structure on $\mathbf{dgcat}_k$. Hence, we can consider the symmetric monoidal localization [Hin16, Prop. 3.2.2]

$$\mathbf{dgcat}_{k,\infty}^{\otimes} := \mathrm{N}(\mathbf{dgcat}_k^{\otimes})[W_{\mathrm{Morita}}^{\otimes,-1}] \to \mathrm{N}(\mathbf{Fin}_*)$$

whose underlying ∞-category is $\mathbf{dgcat}_{k,\infty} = \mathrm{N}(\mathbf{dgcat}_k)[W_{\mathrm{Morita}}^{-1}]$ (3.2.3).

By [Hin16, Prop. 3.2.2], we also get a symmetric monoidal ∞-category

$$\mathbf{Mix}_{\infty}^{\otimes} := \mathrm{N}(\mathbf{Mix}^{\otimes})[W_{mix}^{\otimes,-1}] \to \mathrm{N}(\mathbf{Fin}_*)$$

because the tensor product of mixed complexes preserves equivalences. Its underlying ∞-category is $\mathbf{Mix}_{\infty} = \mathrm{N}(\mathbf{Mix})[W_{mix}^{-1}]$ (3.1.7).

We observe that one can define the shuffle map (3.5.1) in the category of mixed complexes as well, but the resulting map sh does not commute with the operator B, hence it is not a morphism of mixed complexes. However, thanks to the work of Kassel [Kas87], we know that the shuffle map can be extended to a map commuting with B up to higher homotopies. Therefore, the functor Mix has a lax symmetric monoidal refinement (by [Kas87, Thm. 2.4])

$$\mathrm{Mix}_{\infty}^{\otimes}\colon \mathbf{dgcat}_{k,\infty}^{\otimes} \to \mathbf{Mix}_{\infty}^{\otimes}$$

induced by the shuffle map. □

As a consequence of Kassel's theorem, we also have lax symmetric monoidal refinements of equivariant coarse Hochschild and cyclic homology:

Proposition 3.5.3. *Coarse Hochschild homology* $\mathcal{X}\mathrm{HH}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$ *and coarse cyclic homology* $\mathcal{X}\mathrm{HC}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$ *admit lax symmetric monoidal refinements:*

$$\mathcal{X}\mathrm{HH}_k^{G,\otimes} \colon \mathrm{N}(G\mathbf{BornCoarse}^{\otimes}) \to \mathbf{Ch}_\infty^{\otimes}$$

and

$$\mathcal{X}\mathrm{HC}_k^{G,\otimes} \colon \mathrm{N}(G\mathbf{BornCoarse}^{\otimes}) \to \mathbf{Ch}_\infty^{\otimes}.$$

where $\mathbf{Ch}_\infty^{\otimes}$ *is the* ∞*-category of chain complexes with the induced symmetric monoidal structure.*

Proof. Consider the functor

$$\mathcal{X}\mathrm{Mix}_k^G \colon G\mathbf{BornCoarse} \xrightarrow{V_k^G} \mathbf{Cat}_k \xrightarrow{\iota} \mathbf{dgcat}_k \xrightarrow{\mathrm{Mix}} \mathbf{Mix} \xrightarrow{\mathrm{loc}} \mathbf{Mix}_\infty$$

By Remark 2.2.13, the functor $V_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Cat}_k$ admits a lax symmetric monoidal refinement. The inclusion $\iota \colon \mathbf{Cat}_k \to \mathbf{dgcat}_k$ is evidently symmetric monoidal, due to Definition A.3.2, and we get a functor $\iota_\infty^{\otimes} \colon \mathbf{Cat}_{k,\infty}^{\otimes} \to \mathbf{dgcat}_{k,\infty}^{\otimes}$ of symmetric monoidal ∞-categories. By Theorem 3.5.1, the functor Mix has a lax symmetric monoidal refinement $\mathrm{Mix}_\infty^{\otimes} \colon \mathbf{dgcat}_{k,\infty}^{\otimes} \to \mathbf{Mix}_\infty^{\otimes}$, hence so does $\mathcal{X}\mathrm{Mix}_k^G$ (3.4.1):

$$\mathcal{X}\mathrm{Mix}_k^{G,\otimes} \colon \mathrm{N}(G\mathbf{BornCoarse}^{\otimes}) \to \mathbf{Mix}_\infty^{\otimes}.$$

Finally, as the functors forget (3.4.1) and $\mathrm{Tot}(\mathcal{B}-)$ (3.4.2) are lax symmetric monoidal, also coarse Hochschild and cyclic homology admit lax symmetric monoidal refinements. □

Chapter 4

A transformation to coarse ordinary homology and further properties

In this chapter we study some additional properties of the functors $\mathcal{X}\mathrm{HH}_k^G$ and $\mathcal{X}\mathrm{HC}_k^G$, equivariant coarse Hochschild and cyclic homology, of Definition 3.4.6. The main goal is to define a natural transformation from equivariant coarse algebraic K-homology $\mathcal{X}K_k^G$ (see Definition 2.3.1) to equivariant coarse ordinary homology $\mathcal{X}\mathrm{H}_k^G$ (1.5.6), factoring through coarse Hochschild homology $\mathcal{X}\mathrm{HH}_k^G$.

The chapter is organized as follows. In the first section, Section 4.1, we give some comparison results: we compare G-equivariant coarse Hochschild and cyclic homology with the classical Hochschild and cyclic homology of algebras. In Section 4.2 we analyze the forget-control map for coarse Hochschild homology and we show an equivalence between the forget-control map for equivariant coarse Hochschild homology and the associated generalized assembly map.

In the third section, Section 4.3, we construct a natural transformation $\Phi_{\mathcal{X}\mathrm{HH}_k^G}$ from equivariant coarse Hochschild homology to equivariant coarse ordinary homology, which is the main result of the chapter. In Section 4.4, we see that the classical Dennis trace map extends to a transformation from G-equivariant coarse algebraic K-homology to equivariant coarse Hochschild homology. By composition of this natural transformation with the transformation $\Phi_{\mathcal{X}\mathrm{HH}_k^G}\colon \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}\mathrm{H}^G$, we get a transformation

$$K\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}\mathrm{H}^G$$

of spectra-valued coarse homology theories from equivariant coarse algebraic K-homology to equivariant coarse ordinary homology. We conclude the chapter, in Section 4.5, with a Segal-type localization result for coarse Hochschild homology, in the spirit of Section 1.4.

4.1 Comparison results

In this section, we compare equivariant coarse Hochschild homology with the classical version of Hochschild homology for algebras. We show that the evaluation at the one point (bornological coarse) space, endowed with the trivial action, agrees with the Hochschild homology of k (see Proposition 4.1.3) and that the evaluation at the group G, seen as a G-bornological coarse space, is equivalent to the Hochschild homology of the group algebra $k[G]$ (see Proposition 4.1.4).

Let A be a k-algebra, where k is a field. By Example A.4.5, the cyclic nerve $Z_*(A)$ of A is a cyclic module. By Remark 3.1.4, we get a mixed complex that we have denoted by $C(A)$ (consistently with Keller's functor C of Definition 3.2.1).

Notation 4.1.1. Let A be a k-algebra. We denote by

$$C_*^{\mathrm{HH}}(A;k) \quad \text{and} \quad C_*^{\mathrm{HC}}(A;k)$$

the chain complexes computing the Hochschild and cyclic homology of the mixed complex $C(A)$ (with k coefficients) as defined in Definition 3.1.3.

Notation 4.1.2. We denote by C_∞ the composition $C_\infty\colon \mathbf{Ex}_k \xrightarrow{C} \mathbf{Mix} \xrightarrow{\mathrm{loc}} \mathbf{Mix}_\infty$, where C is the cone construction of Definition 3.2.1 and loc the localization functor to the ∞-category of mixed complexes (3.1.7).

Observe that the homologies of $C_*^{\mathrm{HH}}(A;k)$ and $C_*^{\mathrm{HC}}(A;k)$ compute the Hochschild and cyclic homology of the k-algebra A (see Example 3.1.5). We shall omit the field of coefficients k, when clear from the context.

Let $\mathcal{X}\mathrm{HH}_k$ and $\mathcal{X}\mathrm{HC}_k$ be the non-equivariant coarse Hochschild and coarse cyclic homology defined in Definition 3.4.6. Let $\{*\}$ be the one point bornological coarse space, endowed with a trivial G-action.

Proposition 4.1.3. *There are equivalences of chain complexes*

$$\mathcal{X}\mathrm{HH}_k(*) \simeq C_*^{\mathrm{HH}}(k;k) \quad \text{and} \quad \mathcal{X}\mathrm{HC}_k(*) \simeq C_*^{\mathrm{HC}}(k;k)$$

between the coarse Hochschild (cyclic) homology of the point and the classical Hochschild (cyclic) homology of k.

Proof. Let A be a k-algebra. By Theorem 3.2.5 (1), the mixed complex $C(A)$ associated to A is equivalent to the mixed complex associated to the exact category of finitely generated projective A-modules. When X is a point endowed with a trivial G-action and k is a field, the k-linear category $V_k(X)$ is isomorphic to the category $\mathbf{Vect}_k^{f.d.}$ of finite dimensional k-vector spaces; hence we get the equivalence of mixed complexes

$$C(V_k(\{*\})) = C\big(\mathbf{Vect}_k^{f.d.}\big) \simeq C(k),$$

i.e., $\widetilde{\mathcal{X}C}_k^G(*) \simeq C_\infty(k)$; this proves the statement for both coarse Hochschild and coarse cyclic homology by Lemma 3.4.4 and the fact that both the forgetful functor and the functor $\mathrm{Tot}(\mathcal{B}-)$ preserve equivalences. □

Let G be a group. By Example 1.1.22, there is a canonical G-bornological coarse space $G_{\mathrm{can,min}} = (G, \mathcal{C}_{\mathrm{can}}, \mathcal{B}_{\mathrm{min}})$ associated to it.

Proposition 4.1.4. *There are equivalences of chain complexes:*

$$\mathcal{X}\mathrm{HH}_k^G(G_{\mathrm{can,min}}) \simeq C_*^{\mathrm{HH}}(k[G];k)$$

and

$$\mathcal{X}\mathrm{HC}_k^G(G_{\mathrm{can,min}}) \simeq C_*^{\mathrm{HC}}(k[G];k)$$

between the G-equivariant coarse Hochschild and cyclic homologies of $G_{\mathrm{can,min}}$ and the classical Hochschild and cyclic homologies of the group algebra $k[G]$.

Proof. The category $V_k^G(G_{\mathrm{can,min}})$ of G-equivariant $G_{\mathrm{can,min}}$-controlled finite dimensional k-vector spaces is equivalent to the category $\mathbf{Mod}^{\mathrm{fg,free}}(k[G])$ of finitely generated free $k[G]$-modules [BEKW17, Proposition 8.24] (see also Example 2.3.7). By Remark 3.2.6, Keller's mixed complex $C(\mathbf{Mod}^{\mathrm{fg,free}}(k[G]))$ of the category of finitely generated free $k[G]$-modules is equivalent to the mixed complex associated to the category $\mathbf{Mod}^{\mathrm{fg,proj}}(k[G])$ of finitely generated projective modules. Therefore, the result follows from the chain of equivalences of mixed complexes

$$C(V_k^G(G)) \simeq C(\mathbf{Mod}^{\mathrm{fg,free}}(k[G])) \simeq C(\mathbf{Mod}^{\mathrm{fg,proj}}(k[G])) \simeq C(k[G]),$$

where the last equivalence is given by Theorem 3.2.5 (1), by using the same reasoning of Proposition 4.1.3. □

Let X be a G-set and let $X_{\mathrm{min,max}}$ denote the G-bornological coarse space with minimal coarse structure and maximal bornology. We conclude the section with the calculations on G-bornological coarse spaces of the form $X_{\mathrm{min,max}} \otimes G_{\mathrm{can,min}}$:

Lemma 4.1.5. *Let X be a G-set. Then, we have an equivalence of mixed complexes*

$$\widetilde{\mathcal{X}C}_k^G(X_{\mathrm{min,max}} \otimes G_{\mathrm{can,min}}) \simeq C_\infty\left(\mathbf{Vect}_k^{f.d.} *_G X\right)$$

where $\mathbf{Vect}_k^{f.d.} *_G X$ is the category of Definition 2.3.5.

Proof. The result follows from Proposition 2.3.6. □

Recall that, when $\mathbf{A}$ is the additive category of finitely generated free R-modules, where R is a ring, then the category $\mathbf{A} *_G (G/H)$ is equivalent to the category of finitely generated free $R[H]$-modules. As in Remark 2.3.7, we also have:

Remark 4.1.6. Let G be a group, H a subgroup of G and endow the set G/H with the minimal coarse structure and the maximal bornology; then, by Lemma 4.1.5, we get an equivalence of chain complexes:

$$\mathcal{X}\mathrm{HH}_k^G((G/H)_{\mathrm{min,max}} \otimes G_{\mathrm{can,min}}) \simeq C_*^{\mathrm{HH}}(k[H]; k);$$

the same holds for equivariant coarse cyclic homology.

4.2 The forget-control map for coarse Hochschild homology

In Section 1.3, we have discussed how to get equivariant homology theories from equivariant coarse homology theories and we have defined the forget-control and the assembly maps. In this section, we give a comparison result between the forget-control maps for equivariant coarse Hochschild and cyclic homology and the associated assembly maps (see Proposition 4.2.9). In order to apply the results of Section 1.3, we first need to prove that coarse Hochschild and cyclic homologies are strong (Lemma 4.2.1) and continuous (Lemma 4.2.3) equivariant coarse homology theories.

Recall the definitions of strongness (Definition 1.2.9) for a coarse homology theory. In the following proofs, we use the equivariant coarse homology theory $\widetilde{\mathcal{X}C}_k^G$ (3.3.1) (see Remark 3.3.3). As in Notation 4.1.2, we denote by C_∞ the composition $\mathrm{loc} \circ C$.

Lemma 4.2.1. *The functor* $\mathcal{X}\mathrm{HH}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$ *is strong.*

Proof. Let X be a weakly flasque space as defined in Definition 1.2.8. Let $f \colon X \to X$ be a morphism that implements the weak flasqueness of X. Observe that the functor

$$S \coloneqq \bigoplus_{n \in \mathbb{N}} (f^n)_* \colon V_k^G(X) \to V_k^G(X).$$

defined in Lemma 2.1.15 is still well-defined. By Definition 1.2.8, we get the equivalence $\mathrm{id}_{\widetilde{\mathcal{X}C}_k^G(X)} \simeq \widetilde{\mathcal{X}C}_k^G(f)$.

The functor S is naturally isomorphic to the functor $\mathrm{id}_{V_k^G(X)} \oplus f_* \circ S$. By Theorem 3.3.5, we get the chain of equivalences of mixed complexes

$$C_\infty(\mathrm{id}_{V_k^G(X)} \oplus f_* \circ S) \simeq C_\infty(\mathrm{id}_{V_k^G(X)}) \oplus C_\infty(f_* \circ S) \simeq \mathrm{id}_{\widetilde{\mathcal{X}C}_k^G(X)} \oplus \widetilde{\mathcal{X}C}_k^G(f) \circ C_\infty(S)$$

hence,

$$C_\infty(S) \simeq \mathrm{id}_{\mathcal{X}C_k^G(X)} + C_\infty(S);$$

this shows that the morphism $\mathrm{id}_{\mathcal{X}C_k^G(X)}$ is equivalent to zero. By Lemma 3.4.4, also the chain complex $\mathcal{X}\mathrm{HH}_k^G(X)$ is equivalent to 0. □

As a corollary of the previous lemma, we get:

Corollary 4.2.2. *The functor* $\mathcal{X}\mathrm{HC}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$ *is strong.*

Proof. Equivariant coarse cyclic homology $\mathcal{X}\mathrm{HC}_k^G$ has been defined (Definition 3.4.6) as the composition of the functors $\mathcal{X}\mathrm{Mix}_k^G$ and $\mathrm{Tot}(\mathcal{B}-)$; this last functor preserves equivalences. Hence, the result follows from Lemma 4.2.1. □

Recall the definition of continuity (Definition 1.2.13) for a coarse homology theory.

Lemma 4.2.3. *The functor* $\mathcal{X}\mathrm{HH}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$ *is continuous.*

Proof. Let $\mathcal{Y}$ be a trapping exhaustion of X. Let (M, ρ) be a G-equivariant X-controlled finite dimensional k-vector space of $V_k^G(X)$ and denote by F the subset

$$F = \{x \in X \mid M(\{x\}) \neq 0\}$$

of X. The set F is locally finite: if B is a bounded set of X, then $F \cap B = \sigma(B)$, where σ is the support function of (M, ρ) (Definition 2.1.5). Therefore, the controlled module (M, ρ) belongs to the subcategory $V_k^G(F)$ of $V_k^G(X)$.

This proves that the category $V_k^G(X)$ is the filtered colimit of subcategories $V_k^G(F)$ indexed on locally finite subsets of X:

$$V_k^G(X) = \mathrm{colim}_{F \subseteq X} V_k^G(F).$$

As the mixed complex functor and the forgetful functor commute with filtered colimits, we get the equivalence $\mathcal{X}\mathrm{HH}_k^G(\mathcal{Y}) \simeq \mathcal{X}\mathrm{HH}_k^G(X)$. This shows that equivariant coarse Hochschild homology is continuous. □

As the functor $\mathrm{Tot}(\mathcal{B}-)$ commutes with filtered colimits as well, the same proof of Lemma 4.2.3 shows the following:

Corollary 4.2.4. *The functor* $\mathcal{X}\mathrm{HC}_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$ *is continuous.*

We now proceed with the comparison result of the forget-control map for coarse Hochschild homology and the associated assembly map. Recall, from Section 1.3, the definitions of the cone $\mathcal{O}_{hlg}^\infty$ (1.3.3), of the forget-control map β (Definition 1.3.13) and of the coarse assembly map α (Definition 1.3.19). By [BEKW17, Theorem 11.16], the forget-control map for a G-equivariant coarse homology theory E can be compared with the classical assembly map for the associated G-equivariant homology theory $E \circ \mathcal{O}_{hlg}^\infty \colon G\mathbf{Top} \to \mathbf{C}$ (see Remark 1.3.15).

By applying the Eilenberg-MacLane correspondence (1.5.5), we can assume that the equivariant coarse homology theories $\mathcal{X}\mathrm{HH}_k^G$ and $\mathcal{X}\mathrm{HC}_k^G$ are equivariant spectra-valued coarse homology theories.

Definition 4.2.5. Let $\mathbf{HH}_k^G \coloneqq \mathcal{X}\mathrm{HH}_k^G \circ \mathcal{O}_{\mathrm{hlg}}^\infty$ be the G-equivariant homology theory associated to equivariant coarse Hochschild homology (with k-coefficients).

Let $i\colon G\mathbf{Orb} \to G\mathbf{BornCoarse}$ be the functor that associates to a transitive G-set S in $G\mathbf{Orb}$ the G-bornological coarse space $S_{\min,\max}$ with the minimal coarse structure and maximal bornology. Recall that a coarse homology theory can be twisted by a bornological coarse space (see Definition 1.2.5).

Remark 4.2.6. By Lemma 4.1.5, if $\mathcal{X}\mathrm{HH}^G_{k,G_{\mathrm{can},\min}}$ is the twist of the coarse homology theory $\mathcal{X}\mathrm{HH}^G_k$ by the G-bornological coarse space $G_{\mathrm{can},\min}$, then we get

$$\mathbf{HH}^G_k \simeq \mathcal{X}\mathrm{HH}^G_{k,G_{\mathrm{can},\min}} \circ i.$$

Let $\mathbf{Fin}$ be the family of finite subgroups of G.

Proposition 4.2.7. *The forget-control map $\beta_{G_{\mathrm{can},\min},G_{\max,\max}}$ for $\mathcal{X}\mathrm{HH}^G_k$ is equivalent to the assembly map $\alpha_{E_{\mathbf{Fin}}G,G_{\mathrm{can},\min}}$ for the G-homology theory $\mathbf{HH}^G_k$.*

Proof. The functors $\mathcal{X}\mathrm{HH}^G_k$ and $\mathcal{X}\mathrm{HC}^G_k$ are strong by Lemma 4.2.1, and continuous by Lemma 4.2.3. Then, the result follows by Theorem 1.3.20. □

Remark 4.2.8. The assembly map $\alpha_{E_{\mathbf{Fin}}G,G_{\mathrm{can},\min}}$ for the G-homology theory $\mathbf{HH}^G_k$ (hence, the forget-control map $\beta_{G_{\mathrm{can},\min},G_{\max,\max}}$ for $\mathcal{X}\mathrm{HH}^G_k$) is split injective by [LR06, Thm 1.7].

By using the same reasoning, we get the same result for equivariant coarse cyclic homology:

Proposition 4.2.9. *The forget-control map $\beta_{G_{\mathrm{can},\min},G_{\max,\max}}$ for $\mathcal{X}\mathrm{HC}^G_k$ is equivalent to the assembly map $\alpha_{E_{\mathbf{Fin}}G,G_{\mathrm{can},\min}}$ for the G-homology theory $\mathbf{HC}^G_k := \mathcal{X}\mathrm{HC}^G_k \circ \mathcal{O}^\infty_{hlg}$.*

4.3 A transformation to coarse ordinary homology

Let $\mathcal{X}\mathrm{HH}^G_k, \mathcal{X}C^G\colon G\mathbf{BornCoarse} \to \mathbf{Ch}_\infty$ denote the G-equivariant coarse Hochschild homology (see Definition 3.4.6) and the equivariant (chain complexes-valued) ordinary homology (see Section 1.5) with coefficients in a field k, respectively. The goal of this section is to define a natural transformation

$$\Phi_{\mathcal{X}\mathrm{HH}^G_k}\colon\ \mathcal{X}\mathrm{HH}^G_k \longrightarrow \mathcal{X}C^G$$

from equivariant coarse Hochschild homology to the chain valued equivariant coarse ordinary homology $\mathcal{X}C^G$, and a natural transformation $\Phi_{\mathcal{X}\mathrm{HC}^G_k}$ from equivariant coarse cyclic homology.

The idea of our construction uses the fact that, if $A_0 \otimes A_1 \otimes \cdots \otimes A_n$ belongs to the additive cyclic nerve $\mathrm{CN}_n(V^G_k(X)))$ associated to the category of controlled objects $V^G_k(X)$ (see Definition 2.1.2 and Definition A.4.7), then, after evaluation at $(n+1)$-tuples $(x_0, \ldots, x_n)$ of points of X, there are well-defined linear operators $(A_0 \circ \cdots \circ A_n)|(x_0, \ldots, x_n)$

(see Notation 4.3.2) which are endomorphisms of finite dimensional k-vector spaces; we associate to a such linear operator its trace, *i.e.,* an element of k. This will give well-defined chains in coarse ordinary homology and a natural transformation $\Phi_{\mathcal{X}\mathrm{HH}_k^G}$ (see Theorem 4.3.8) of equivariant coarse homology theories.

Recall that $V_k^G(X)$ denotes the k-linear category of G-equivariant X-controlled (finite dimensional) k-vector spaces (Definition 2.1.8). Recall that the objects of $V_k^G(X)$ are pairs (M,ρ) where M is a functor on the category associated to the poset of bounded sets of X.

The construction of the transformation $\Phi_{\mathcal{X}\mathrm{HH}_k^G}$ proceeds as follows:

1. For every G-bornological coarse space X, we consider its associated k-linear category $V_k^G(X)$ of controlled objects, hence the associated additive cyclic nerve $\mathrm{CN}(V_k^G(X))$ (see Example A.4.6);

2. for every tensor element $A_0 \otimes \ldots \otimes A_n$ in the additive cyclic nerve of $V_k^G(X)$ and every $n+1$ points $x_0, \ldots, x_n$ of X, we define a trace-like map, which gives an element of k (see Notation 4.3.2);

3. by letting $x_0, \ldots, x_n$ vary, this yields a G-equivariant locally finite controlled chain on X, *i.e.,* an element of $\mathcal{X}C_n^G(X)$ (see Definition 4.3.3 and Lemma 4.3.4);

4. by letting $A_0 \otimes \ldots \otimes A_n$ vary we get a map $\varphi\colon \mathrm{CN}_*(V_k^G(X)) \to \mathcal{X}C_*^G(X)$ that is a chain map with respect to the differential $d = \sum d_i$ of $\mathrm{CN}(V_k^G(X))$ (see Proposition 4.3.6);

5. the additive cyclic nerve $\mathrm{CN}(V_k^G(X))$ yields a mixed complex with the differentials b and B (3.1.6) and the chain map φ extends to a map of mixed complexes $\tilde{\varphi}$ (see Lemma 4.3.7);

6. the map $\tilde{\varphi}$ yields a natural transformation of equivariant coarse homology theories $\Phi_{\mathcal{X}\mathrm{HH}_k^G}\colon \mathcal{X}\mathrm{HH}_k^G \longrightarrow \mathcal{X}C^G$ (see Theorem 4.3.8).

We now proceed with the construction of the natural transformation $\Phi_{\mathcal{X}\mathrm{HH}_k^G}\colon \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}C^G$.

Let $n \geq 0$ be a natural number. The n-th component of the additive cyclic nerve of $V_k^G(X)$ (see Definition A.4.7 or Example A.4.6) is :

$$\mathrm{CN}_n(V_k^G(X)) = \bigoplus_{((M_0,\rho_0),\ldots,(M_n,\rho_n))} \left(\bigotimes_{i=0}^{n} \mathrm{Hom}((M_{i+1},\rho_{i+1}),(M_i,\rho_i)) \right)$$

where the index i runs cyclically in the set $\{0,\ldots,n\}$ and the sum ranges over all the tuples $((M_0,\rho_0),\ldots,(M_n,\rho_n))$ of objects of $V_k^G(X)$.

Remark 4.3.1. For every controlled morphism $A_i\colon (M_{i+1},\rho_{i+1}) \to (M_i,\rho_i)$ (see Definition 2.1.6) in $\mathrm{Hom}((M_{i+1},\rho_{i+1}),(M_i,\rho_i))$ and for every pair of points x and y of X, there is a well-defined linear map

$$A_i^{x,y}\colon M_{i+1}(x) \to M_i(y)$$

(see also (2.2.4), where the notation is $A_{i,y,x}$).

In order to simplify the exposition of later proofs, we use the following notation:

Notation 4.3.2. Let $A_0 \otimes \ldots \otimes A_n$ be an element of $\bigotimes_{i=0}^n \mathrm{Hom}((M_{i+1},\rho_{i+1}),(M_i,\rho_i))$ with $A_i\colon (M_{i+1},\rho_{i+1}) \to (M_i,\rho_i)$ and $((M_0,\rho_0),\ldots,(M_n,\rho_n))$ a tuple of objects of $V_k^G(X)$. Let $(x_0,\ldots,x_n)$ be a point of X^{n+1}. The symbol

$$(A_0 \circ \cdots \circ A_n)|(x_0,\ldots,x_n)$$

denotes the linear operator $(A_0 \circ \cdots \circ A_n)|(x_0,\ldots,x_n)\colon M_0(x_n) \to M_0(x_n)$ defined as the composition

$$\begin{aligned}&(A_0 \circ \ldots \circ A_n)|(x_0,\ldots,x_n) :=\\ &\qquad M_0(x_n) \xrightarrow{A_n^{x_n,x_{n-1}}} M_n(x_{n-1}) \xrightarrow{A_{n-1}^{x_{n-1},x_{n-2}}} \ldots \xrightarrow{A_1^{x_1,x_0}} M_1(x_0) \xrightarrow{A_0^{x_0,x_n}} M_0(x_n)\end{aligned}$$

of the induced operators $A_i^{x_i,x_{i+1}}\colon M_i(x_i) \to M_{i+1}(x_{i+1})$. It is an endomorphism of $M_0(x_n)$, which is a finite dimensional k-vector space.

Let X be a G-bornological coarse space and let $\mathcal{X}C_n(X)$ be the k-linear vector space generated by the locally finite controlled n-chains on X (see Definition 1.5.1).

Definition 4.3.3. Let X be a G-bornological coarse space and let n be a natural number. We let $\varphi_n\colon \mathrm{CN}_n(V_k^G(X)) \to \mathcal{X}C_n(X)$ be the map defined on elementary tensors as

$$\begin{aligned}&\varphi_n\colon A_0 \otimes \ldots \otimes A_n \longmapsto\\ &\qquad \sum_{(x_0,\ldots,x_n)\in X^{n+1}} \mathrm{tr}\left((A_0 \circ \cdots \circ A_n)|(x_0,\ldots,x_n)\colon M_0(x_n) \to M_0(x_n)\right)\cdot(x_0,\ldots,x_n)\end{aligned}$$

where we use Notation 4.3.2 and the symbol tr denotes the trace map.

The map φ_n is extended to $\mathrm{CN}_n(V_k^G(X))$ by linearity.

Lemma 4.3.4. *Let X be a G-bornological coarse space and let $A_0 \otimes \ldots \otimes A_n$ be a tensor element of $\mathrm{CN}_n(V_k^G(X))$ with $A_i\colon (M_{i+1},\rho_{i+1}) \to (M_i,\rho_i)$. Then, the n-chain $\varphi_n(A_0 \otimes \ldots \otimes A_n)$, given by the sum*

$$\sum_{(x_0,\ldots,x_n)\in X^{n+1}} \mathrm{tr}((A_0 \circ \cdots \circ A_n)|(x_0,\ldots,x_n))\cdot(x_0,\ldots,x_n),$$

of Definition 4.3.3, is locally finite and controlled.

Proof. In order to prove that $\varphi_n(A_0 \otimes \ldots \otimes A_n)$ is locally finite and controlled we show that its support (1.5.1) is locally finite and that there exists an entourage U of X such that every $x = (x_0, \ldots, x_n)$ in $\mathrm{supp}(\varphi_n(A_0 \otimes \ldots \otimes A_n))$ is U-controlled.

We first observe that the operators $A_i\colon (M_{i+1}, \rho_{i+1}) \to (M_i, \rho_i)$ are U_i-controlled for some entourage U_i of X. By Definition 2.1.6, A_i is given by a natural transformation of functors $M_{i+1} \to M_i \circ U_i[-]$ satisfying an equivariance condition. For every point x in X, by using the Convention 2.2.4, A_i restricts to a morphism

$$M_{i+1}(x) \to M_i(U_i[x]) \cong \bigoplus_{x' \in U_i[x]} M_i(x')$$

where the direct sum has only finitely many non-zero summands.

Let K be a bounded set of X. The set of points $x_n \in K$ for which $M_0(x_n)$ is non-zero is finite (as a consequence of Definition 2.1.8). For such a fixed x_n, there are only finitely many points $x_{n-1} \in U_n[K]$ such that the corresponding map $A_n^{x_n, x_{n-1}}\colon M_0(x_n) \to M_n(x_{n-1})$ is non-zero. The set $U_n[K]$ is a bounded set of X, the morphism $A_i\colon M_{i+1} \to M_i$ is U_i-controlled and we can repeat the same argument for each A_i. This implies that the n-chain is locally finite because, for the given bounded set K, we have found only finitely tuples $(x_0, \ldots, x_n)$ in the support of $\varphi_n(A_0 \otimes \ldots \otimes A_n)$ that meet K.

The chain is also U-controlled, where U is the entourage $U \coloneqq U_0 \circ \cdots \circ U_n$ of X. $\square$

Remark 4.3.5. Let X be a G-bornological coarse space. Let (M, ρ) be a G-equivariant X-controlled finite dimensional k-vector space and let g be an element of the group G. Then, $\rho(g)$ (Definition 2.1.2) is a natural isomorphism between the functors M and gM. The morphisms in the category $V_k^G(X)$ satisfy a G-equivariant condition (see Definition 2.1.6). These observations imply that the following diagram is commutative

$$\begin{tikzcd}
M_0(gx_n) \arrow[r, "A_n^{gx_n, gx_{n-1}}"] \arrow[d, "\cong"] & M_n(gx_{n-1}) \arrow[r] \arrow[d, "\cong"] & \ldots \arrow[r, "A_0^{gx_0, gx_n}"] \arrow[d, dotted, "\cong"] & M_0(gx_n) \arrow[d, "\cong"] \\
gM_0(x_n) \arrow[r, "gA_n^{x_n, x_{n-1}}"] & gM_n(x_{n-1}) \arrow[r] & \ldots \arrow[r, "gA_0^{x_0, x_n}"] & gM_0(x_n)
\end{tikzcd}$$

for $A_0 \otimes \ldots \otimes A_n$ in $\mathrm{CN}_n(V_k^G(X))$ with $A_i\colon (M_{i+1}, \rho_{i+1}) \to (M_i, \rho_i)$, where the isomorphisms are induced by $\rho_i(g)$. Hence, the image of φ_n is a G-invariant locally finite controlled n-chain on X.

Let $\partial_i\colon \mathcal{X}C_n^G(X) \to \mathcal{X}C_{n-1}^G(X)$ be the i-th differential of the chain complex $\mathcal{X}C^G(X)$ and let $d_i\colon \mathrm{CN}_n(V_k^G(X)) \to \mathrm{CN}_{n-1}(V_k^G(X))$ be the i-th face map of $\mathrm{CN}(V_k^G(X))$ (see Example A.4.6). Consider the chain complex underlying the additive cyclic nerve $\mathrm{CN}(V_k^G(X))$.

Proposition 4.3.6. *Let X be a G-bornological coarse space. The maps $\varphi_n\colon \mathrm{CN}_n(V_k^G(X)) \to \mathcal{X}C_n^G(X)$ of Definition 4.3.3 extend to a chain map*

$$\varphi\colon (\mathrm{CN}(V_k^G(X)), d) \to (\mathcal{X}C^G(X), \partial).$$

Proof. We prove that the following square is commutative:

$$\begin{tikzcd} \mathrm{CN}_n(V_k^G(X)) \arrow[r, "\varphi_n"] \arrow[d, "d_i"] & \mathcal{X}C_n^G(X) \arrow[d, "\partial_i"'] \\ \mathrm{CN}_{n-1}(V_k^G(X)) \arrow[r, "\varphi_{n-1}"'] & \mathcal{X}C_{n-1}^G(X) \end{tikzcd}$$

Consider the case $i \neq n$ and let $A_0 \otimes \ldots \otimes A_n$ be an elementary tensor.

By Definition A.4.7 and by using Notation 4.3.2, we get:

$$\begin{aligned}
&\varphi_{n-1}(d_i(A_0 \otimes \ldots \otimes A_n)) = \\
&= \sum_{(x'_0,\ldots,\hat{x}'_i,\ldots,x'_n)} \operatorname{tr}(A_0 \circ \cdots \circ (A_i \circ A_{i+1}) \circ \cdots \circ A_n | (x'_0, \ldots, \hat{x}'_i, \ldots, x'_n)) \cdot (x'_0, \ldots, \hat{x}'_i, \ldots, x'_n) \\
&= \sum_{(x'_0,\ldots,\hat{x}'_i,\ldots,x'_n)} \left(\sum_{x'_i \in X} \operatorname{tr}(A_0 \circ \cdots \circ A_n | (x'_0, \ldots, x'_i, \ldots, x'_n)) \right) \cdot (x'_0, \ldots, \hat{x}'_i, \ldots, x'_n)
\end{aligned}$$

where we used that the trace is additive and that the morphism $A_i \circ A_{i+1}$ factors through all the points x'_i of X (that give contribution zero up to finitely many).

On the other hand, we also get

$$\begin{aligned}
&\partial_i(\varphi_n(A_0 \otimes \ldots \otimes A_n)) = \\
&= \partial_i \left(\sum_{(x'_0,\ldots,x'_i,\ldots,x'_n)} \operatorname{tr}(A_0 \circ \cdots \circ A_n | (x'_0, \ldots, x'_i, \ldots, x'_n)) \right) \cdot (x'_0, \ldots, x'_i, \ldots, x'_n) \\
&= \sum_{(x'_0,\ldots,\hat{x}'_i,\ldots,x'_n)} \operatorname{tr}(A_0 \circ \cdots \circ A_n | (x'_0, \ldots, x'_i, \ldots, x'_n)) \cdot (x'_0, \ldots, \hat{x}'_i, \ldots, x'_n)
\end{aligned}$$

and, by gathering together the preimages, we can write the sum as follows:

$$\begin{aligned}
&\partial_i(\varphi_n(A_0 \otimes \ldots \otimes A_n)) = \\
&= \sum_{(x'_0,\ldots,\hat{x}'_i,\ldots,x'_n)} \left(\sum_{x'_i \in X} \operatorname{tr}(A_0 \circ \cdots \circ A_n | (x'_0, \ldots, x'_i, \ldots, x'_n)) \right) \cdot (x'_0, \ldots, \hat{x}'_i, \ldots, x'_n)
\end{aligned}$$

Therefore, for the i-face maps d_i and ∂_i, with $i \neq n$, the above diagram is commutative.

For $i = n$, we recall that $d_n(A_0 \otimes \cdots \otimes A_n) = (A_n \circ A_0 \otimes A_1 \otimes \cdots \otimes A_{n-1})$. Hence, the above diagram commutes for $i = n$ as well because the trace map is invariant under cyclic permutations. □

Let (M, b) be a cyclic module. By Remark 3.1.4, we get a mixed complex whose differential b is the differential $d = \sum_{i=0}^{n} (-1)^i d_i$ of above. We recall that the operator B (3.1.6) is defined by $B \coloneqq (-1)^{n+1}(1 - t_{n+1})sN$, where s denotes the extra degeneracy

$s = (-1)^{n+1} t_{n+1} s_n$. The chain complex $\mathcal{X}C^G(X)$ is a mixed complex with the differential $B = 0$.

Lemma 4.3.7. *Let X be a G-bornological coarse space. The chain map $\varphi\colon \mathrm{CN}(V_k^G(X)) \to \mathcal{X}C^G(X)$ of Definition 4.3.3 extends to a map*

$$\tilde{\varphi}\colon \mathrm{Mix}(V_k^G(X)) \to \mathcal{X}C^G(X)$$

that is a morphism of mixed complexes.

Proof. Let $A_0 \otimes \ldots \otimes A_n$ be an element of $\mathrm{CN}_n(V_k^G(X))$. Then, $N(A_0 \otimes \ldots \otimes A_n) = \sum_{i=0}^n A_i \otimes \ldots \otimes A_{i+n}$. We have:

$$sN(A_0 \otimes \ldots \otimes A_n) = (-1)^{n+1} \sum_{i=0}^n 1 \otimes A_i \otimes \ldots \otimes A_{i+n}$$

and then

$$\begin{aligned} B(A_0 \otimes \ldots \otimes A_n) &= \sum_{i=0}^n 1 \otimes A_i \otimes \ldots \otimes A_{i+n} - \sum_{i=0}^n A_{i+n} \otimes 1 \otimes A_i \otimes \ldots \otimes A_{i+n-1} \\ &= \sum_{i=0}^n \left(1 \otimes A_i \otimes \ldots \otimes A_{i+n} - A_{i+n} \otimes 1 \otimes A_i \otimes \ldots \otimes A_{i+n-1}\right). \end{aligned}$$

By applying the map φ_n, we see that all the terms of $B(A_0 \otimes \ldots \otimes A_n)$ cancel pairwise, because the trace is invariant under cyclic permutations. This is enough to show that the map φ extends to $\mathrm{Mix}(V_k^G(X))$. □

We can finally construct the natural transformation $\Phi_{\mathcal{X}\mathrm{HH}_k^G}\colon \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}C^G$:

Theorem 4.3.8. *There are natural transformations*

$$\Phi_{\mathcal{X}\mathrm{HH}_k^G}\colon \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}C^G.$$

and

$$\Phi_{\mathcal{X}\mathrm{HC}_k^G}\colon \mathcal{X}\mathrm{HC}_k^G \to \bigoplus_{n\in\mathbb{N}} \mathcal{X}C^G$$

of G-equivariant $\mathbf{Ch}_\infty$-valued coarse homology theories.

Proof. The map $\varphi\colon (\mathrm{CN}(V_k^G(X)), d) \to (\mathcal{X}C^G(X), \partial)$ of Definition 4.3.3 is a chain map by Proposition 4.3.6. Let $f\colon X \to Y$ be a morphism of G-equivariant bornological coarse spaces. Consider the induced chain map $\mathcal{X}C^G(f)\colon \mathcal{X}C^G(X) \to \mathcal{X}C^G(Y)$ (1.5.2) and the induced functor $f_* = V_k^G(f)\colon V_k^G(X) \to V_k^G(X')$ (2.1.2). By functoriality of the additive cyclic nerve, f_* induces a morphism $\mathrm{CN}(f_*)\colon \mathrm{CN}_*(V_k^G(X)) \to \mathrm{CN}_*(V_k^G(Y))$ of cyclic modules (hence, a chain map between the underlying chain complexes as well).

The diagram

$$\begin{tikzcd} \mathrm{CN}_n(V_k^G(X)) \arrow[r, "\varphi_n"] \arrow[d, "\mathrm{CN}(f_*)"] & \mathcal{X}C_n^G(X) \arrow[d, "\mathcal{X}C^G(f)"] \\ \mathrm{CN}_n(V_k^G(Y)) \arrow[r, "\varphi_n"] & \mathcal{X}C_n^G(Y) \end{tikzcd}$$

is commutative, as we now explain. If $A_0 \otimes \ldots \otimes A_n$ is a tensor element of the additive cyclic nerve associated to $V_k^G(X)$, then $\mathrm{CN}(f_*)(A_0 \otimes \ldots \otimes A_n)$ is the elementary tensor $f_*A_0 \otimes \ldots \otimes f_*A_n$ and belongs to the tensor product $\bigotimes_{i=0}^n \mathrm{Hom}_{V_k^G(Y)}(f_*(M_{i+1}, \rho_{i+1}), f_*(M_i, \rho_i))$. After application of the map φ_n we get:

$$\varphi_n(f_*A_0 \otimes \ldots \otimes f_*A_n) = \sum_{(y_0,\ldots,y_n)} \mathrm{tr}\left((f_*A_0 \circ \cdots \circ f_*A_n)|(y_0,\ldots,y_n)\right) \cdot (y_0,\ldots,y_n).$$

By definition of f_*A_i (2.1.1), the composition $(f_*A_0 \circ \cdots \circ f_*A_n)|(y_0,\ldots,y_n)$ decomposes as a sum over the fibers $(x_0,\ldots,x_n) \in X^n$ of f at $(y_0,\ldots,y_n)$, *i.e.*,

$$\varphi_n(f_*A_0 \otimes \ldots \otimes f_*A_n) = \sum_{(y_0,\ldots,y_n)} \left(\sum_{(x_0,\ldots,x_n)} \mathrm{tr}\left((A_0 \circ \cdots \circ A_n)|(x_0,\ldots,x_n)\right) \right) \cdot (y_0,\ldots,y_n)$$

where the second sum runs over the fibers of $(y_0,\ldots,y_n)$. But this last term is in fact $\mathcal{X}C^G(f)(\varphi_n(A_0 \otimes \ldots \otimes A_n))$, proving that the diagram of above is commutative.

The map φ extends to the associated mixed complexes by Lemma 4.3.7 and this extension preserves the commutative diagram (of associated mixed complexes). After localization and application of the forgetful functor (recall the definition of equivariant coarse Hochschild homology in terms of $\mathcal{X}\mathrm{Mix}_k^G$, Definition 3.4.6), the map φ yields a natural transformation of equivariant coarse homology theories

$$\Phi_{\mathcal{X}\mathrm{HH}_k^G} \colon \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}C^G.$$

To every mixed complex C, we associate the chain complex $\mathrm{Tot}(\mathcal{B}C)$ (3.1.2) defined by $\mathrm{Tot}_n(\mathcal{B}C) = \bigoplus_{i\geq 0} C_{n-2i}$ with differential $d(c_n, c_{n-2}, \ldots) = (bc_n + Bc_{n-2}, \ldots)$. By Lemma 4.3.7, we conclude that the map φ extends to a chain map on the total complex as well, and to a natural transformation of coarse homology theories

$$\Phi_{\mathcal{X}\mathrm{HC}_k^G} \colon \mathcal{X}\mathrm{HC}_k^G \to \bigoplus_{n\in\mathbb{N}} \mathcal{X}C^G$$

where the sum is indexed on the natural numbers because the (mixed complex associated to) the additive cyclic nerve of $V_k^G(X)$ is positively graded. $\square$

The following result implies that the transformation $\Phi_{\mathcal{X}\mathrm{HH}_k} \colon \mathcal{X}\mathrm{HH}_k \to \mathcal{X}C$ is non-zero.

Proposition 4.3.9. *If X is the one point space $\{*\}$, then the transformation*

$$\Phi_{\mathcal{X}\mathrm{HH}_k}\colon \mathcal{X}\mathrm{HH}_k(*) \to \mathcal{X}C(*)$$

induces an equivalence of chain complexes.

Proof. Let $c\colon \{*\}^{n+1} \to k$ be an n-chain in $\mathcal{X}C_n(*)$; we identify this chain with the element $c \in k$ that is its image. Let $\iota_n\colon \mathcal{X}C_n(*) \to \mathrm{CN}_n(V_k(*)))$ be the map sending c to the element $(\cdot c) \otimes (\cdot 1_k) \otimes \ldots \otimes (\cdot 1_k)$. This extends to a chain map that gives a section of the trace map, *i.e.*, $\varphi \circ \iota = \mathrm{id}$.

As coarse Hochschild homology and coarse ordinary homology of the point are both isomorphic to the Hochschild homology of the ground field k (by Example 1.5.5 and Proposition 4.1.3), we get equivalences of chain complexes

$$\mathcal{X}\mathrm{HH}_k(*) \simeq C_*^{\mathrm{HH}}(k) \simeq \mathcal{X}C_k(*).$$

By using these equivalences and the section $\varphi \circ \iota = \mathrm{id}$, we obtain that, when X is the one point space, the transformation $\Phi_{\mathcal{X}\mathrm{HH}_k}$ induces an equivalence of chain complexes $\mathcal{X}\mathrm{HH}_k(*) \to \mathcal{X}C(*)$. □

4.4 A transformation from coarse algebraic K-homology

In this section, we construct natural transformations $K\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G$ and $K\mathcal{X}_k^G \to \mathcal{X}\mathrm{HC}_k^G$ from equivariant coarse algebraic K-homology (see Definition 2.3.1) to equivariant coarse Hochschild and cyclic homology (Definition 3.4.6). In order to construct these natural transformations we use the trace maps constructed by McCarthy [McC94, Sec. 4.4] and Keller [Kel99, Theorem 1.13]. As equivariant coarse algebraic K-homology takes values in spectra, by applying the Eilenberg-MacLane correspondence $\mathcal{EM}$ (1.5.5), we assume in this section that also equivariant coarse Hochschild and cyclic homology take values in the ∞-category **Sp** of spectra.

Let $K\mathcal{X}_k^G$ be the equivariant coarse algebraic K-homology (Definition 2.3.1). Let $\mathrm{HH}^{\mathrm{McC}}$ and $\mathrm{HH}^{\mathrm{Kel}}$ denote McCarthy's [McC94] and Keller's [Kel99] Hochschild homology, respectively. Classically, there are trace maps from algebraic K-theory to (topological) Hochschild homology and its cyclic variants. For example, the Dennis trace map is a transformation from the algebraic K-theory of rings to Hochschild homology. McCarthy has extended the Dennis trace map to a transformation from the algebraic K-theory (of exact categories) to (McCarthy's) Hochschild homology of exact categories [McC94, Sec. 4.4]. We use this extension to construct a transformation from equivariant coarse algebraic K-homology to equivariant coarse Hochschild homology:

Proposition 4.4.1. *There are natural transformations*

$$\mathrm{K}\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G \quad \text{and} \quad \mathrm{K}\mathcal{X}_k^G \to \mathcal{X}\mathrm{HC}_k^G$$

induced by the Dennis trace maps from algebraic K-theory to Hochschild homology.

Proof. When k is a field, Keller constructs a natural transformation $\mathrm{HH}^{\mathrm{McC}} \to \mathrm{HH}^{\mathrm{Kel}}$ between McCarthy's Hochschild homology of exact categories and Keller's Hochschild homology [Kel99, Theorem 1.13]. Trace maps between algebraic K-theory of exact categories and Keller's cyclic homology

$$K \to \mathrm{HH}^{\mathrm{McC}} \to \mathrm{HH}^{\mathrm{Kel}}\,. \tag{4.4.1}$$

are then constructed by composing the transformation $\mathrm{HH}^{\mathrm{McC}} \to \mathrm{HH}^{\mathrm{Kel}}$ with the trace maps $K \to \mathrm{HH}^{\mathrm{McC}}$ defined by McCarthy [McC94, Sec. 4.4].

Let $V_k^G \colon G\mathbf{BornCoarse} \to \mathbf{Add}$ be the functor (2.1.3), seen as a functor to the category of small exact categories (by equipping them with the split exact structure). Then, composition of the transformation $K \to \mathrm{HH}^{\mathrm{Kel}}$ (4.4.1) with the functor V_k^G, induces a transformation $K\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G$ of equivariant coarse homology theories, as equivariant coarse algebraic K-homology $K\mathcal{X}_k^G$ and equivariant coarse Hochschild homology $\mathcal{X}\mathrm{HH}_k^G$ are defined as the algebraic K-theory and Keller's Hochschild homology, respectively, of the k-linear category of controlled objects $V_k^G(X)$. In the same way, we have a transformation to equivariant coarse cyclic homology. □

When restricted to k-linear categories with split exact structure, McCarthy's and Keller's cyclic homologies provide equivalent complexes (in the sense of Lemma 3.4.4 and Remark 3.4.5), hence equivalent spectra after applying the Eilenberg-MacLane correspondence $\mathcal{EM}$ (1.5.5). In particular, when X is the G-bornological coarse space $G_{\mathrm{can,min}}$, the induced map

$$K\mathcal{X}_k^G(G_{\mathrm{can,min}}) \to \mathcal{X}\mathrm{HH}_k^G(G_{\mathrm{can,min}})$$

is the classical Dennis trace map $K(k[G]) \to \mathrm{HH}(k[G])$ by McCarthy's agreement result [McC94, Sec. 4.5], by Proposition 2.3.6 and Remark 2.3.7, and by Proposition 4.1.4.

A symmetric monoidal natural transformation is a natural transformation between lax symmetric monoidal functors that respects the symmetric monoidal structures.

Corollary 4.4.2. *The transformation* $\mathrm{K}\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G$ *is symmetric monoidal.*

Proof. Both functors $K\mathcal{X}_k^G$ and $\mathcal{X}\mathrm{HH}_k^G$ admit lax symmetric monoidal refinements by Theorem 2.3.4 and Proposition 3.5.3. By [McC94, Prop. 4.4.3 & Sec. 4.5], the trace map $K \to \mathrm{HH}^{\mathrm{McC}}$ is lax symmetric monoidal, and the transformation $\mathrm{K}\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G$ respects the symmetric monoidal structures and is symmetric monoidal. □

Consider the transformations $K\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G$ and $\mathcal{EM} \circ \Phi_{\mathcal{X}\mathrm{HH}_k^G}\colon \mathcal{X}\mathrm{HH}_k^G \to \mathcal{X}\mathrm{H}_k^G$ of spectra-valued equivariant coarse homology theories. By combining Remark 2.3.7, Proposition 4.1.4, Example 1.5.7, Proposition 4.4.1 and Theorem 4.3.8, we get the following evaluations:

$$\begin{tikzcd}
K\mathcal{X}_k^G(G_{\mathrm{can,min}}) \arrow[r] \arrow[d, "\simeq"] & \mathcal{X}\mathrm{HH}_k^G(G_{\mathrm{can,min}}) \arrow[r, "\mathcal{EM}\circ\Phi_{\mathcal{X}\mathrm{HH}_k^G}"] \arrow[d, "\simeq"] & \mathcal{X}\mathrm{H}_k^G(G_{\mathrm{can,min}}) \arrow[d, "\simeq"] \\
K(k[G]) \arrow[r] & \mathrm{HH}(k[G]) \arrow[r] & H(G;k)
\end{tikzcd}$$

where the map $K(k[G]) \to \mathrm{HH}(k[G])$ is the Dennis trace map and the induced map $\mathrm{HH}(k[G]) \to H(G;k)$ can be described as follows:

1. the k-algebra $k[G]$ can be seen as a category with a single object. The inclusion of this category in the category of finitely generated free $k[G]$-modules induces an equivalence of cyclic sets between the associated additive cyclic nerves.

2. By Proposition 2.3.6, the category of finitely generated free $k[G]$-modules is equivalent to the category $V_k^G(G_{\mathrm{can,min}})$ and this equivalence of additive categories induces an equivalence between the associated additive cyclic nerves; then, we get an equivalence of cyclic sets $\mathrm{CN}(k[G]) \to \mathrm{CN}(V_k^G(G_{\mathrm{can,min}}))$.

3. We apply the map φ of Definition 4.3.3 to the additive cyclic nerve associated to the category $V_k^G(G_{\mathrm{can,min}})$;

4. by [BEKW17, Prop. 7.5] there is an equivalence $\psi\colon \mathcal{X}C^G(G_{\mathrm{can,min}}) \to C_*(G;k)$ of chain complexes, where $\mathcal{X}C^G(G_{\mathrm{can,min}})$ is the equivariant ($\mathbf{Ch}_\infty$-valued) coarse ordinary homology of X and $C_*(G;k)$ is the chain complex computing the homology of G with k-coefficients.

5. All the maps extend to morphisms of mixed complexes, and their composition (after localization and application of the Eilenberg-MacLane correspondence $\mathcal{EM}$) yields the map $\mathrm{HH}(k[G]) \to H(G;k)$ in the diagram.

We believe that further studies of the natural transformation

$$K\mathcal{X}_k^G \to \mathcal{X}\mathrm{HH}_k^G \xrightarrow{\mathcal{EM}\circ\Phi_{\mathcal{X}\mathrm{HH}_k^G}} \mathcal{X}\mathrm{H}_k^G$$

from equivariant coarse algebraic K-homology to (the spectra-valued) coarse ordinary homology, may be useful for a better understanding of equivariant coarse algebraic K-homology and for detecting coarse K-theory classes.

4.5 A Segal-type localization theorem for coarse Hochschild homology

In this section we apply the abstract localization results of Section 1.4 to coarse Hochschild and cyclic homology; we also assume that the field k is the field of complex numbers $k = \mathbb{C}$. Hence, the additive category with strict G-action of Definition 2.1.2 is the category $\mathbf{Vect}_{\mathbb{C}}^{\mathrm{f.d.}}$ of finite dimensional $\mathbb{C}$-vector spaces. For simplicity of notation, we will omit the subscript $\mathbb{C}$ in the symbol $\mathcal{X}\mathrm{HH}_{\mathbb{C}}^{G}$ denoting equivariant coarse Hochschild homology.

In this section G is always supposed to be a finite group.

By Remark 1.1.28, the one point space $* \in G\mathbf{BornCoarse}$ is a commutative algebra object in the category $G\mathbf{BornCoarse}$. By Proposition 3.5.3, equivariant coarse Hochschild homology is a lax symmetric monoidal functor, hence the evaluation at the one point space

$$R_{\mathbb{C}} \coloneqq \mathcal{X}\mathrm{HH}^{G}(*)$$

is a commutative algebra object in the ∞-category of chain complexes and the functor $\mathcal{X}\mathrm{HH}^{G}$ refines to a $\mathbf{Mod}(R_{\mathbb{C}})$-valued equivariant coarse homology theory (where $\mathbf{Mod}(R_{\mathbb{C}})$ is the cocomplete stable ∞-category of modules over $R_{\mathbb{C}}$).

The functor $\mathcal{X}\mathrm{HH}^{G}$, being a coarse homology theory, factors through the category of motivic coarse spaces over the Yoneda functor $\mathrm{Yo}_G \colon G\mathbf{BornCoarse} \to G\mathbf{Spc}\mathcal{X}$ (1.2.3). We denote with the same symbol $\mathcal{X}\mathrm{HH}^{G}$ the factorization $\mathcal{X}\mathrm{HH}^{G} \colon G\mathbf{Spc}\mathcal{X} \to \mathbf{Mod}(R_{\mathbb{C}})$, which is still lax symmetric monoidal (Yo_G being lax symmetric monoidal as described in Section 1.2).

Remark 4.5.1. The twist of $\mathcal{X}\mathrm{HH}^{G}$ by a G-bornological coarse space T, as in Definition 1.2.5, gives an equivariant $\mathbf{Mod}(R_{\mathbb{C}})$-valued coarse homology theory, *i.e.,* an element $\mathcal{X}\mathrm{HH}_{T}^{G}$ of $\mathbf{Fun}^{\mathrm{colim}}(G\mathbf{Spc}\mathcal{X}.\mathbf{C})$.

Remark 4.5.2. Equivariant coarse Hochschild homology is defined as the Hochschild homology of the category $V_{\mathbb{C}}^{G}(X)$. When X is a point, the category $V_{\mathbb{C}}^{G}(*)$ is equivalent to the category of finitely generated free $\mathbb{C}[G]$-modules that are of finite dimension over $\mathbb{C}$. Hence, by Theorem 3.2.5, Hochschild homology of this category is equivalent to the Hochschild homology of $\mathbb{C}[G]$. An easy computation shows that the Hochschild homology of the group algebra $\mathbb{C}[G]$ is given by the representation ring $R(G)$ of G and it is concentrated in degree 0. It then follows that $H_*(R_{\mathbb{C}})$ is isomorphic to the representation ring $R(G)$.

Let $\mathfrak{p}$ be a prime ideal of $R(G)$. Then, we can form the lax symmetric monoidal localization functor (a Bousfield localization)

$$(-)_{\mathfrak{p}} \colon \mathbf{Mod}(R_{\mathbb{C}}) \to L_{\mathfrak{p}}\mathbf{Mod}(R_{\mathbb{C}})$$

generated by the set of morphisms $(f\colon R_{\mathbb{C}} \to R_{\mathbb{C}})_{f \in R_{\mathbb{C}} \setminus \mathfrak{p}}$ and a lax symmetric monoidal functor

$$\mathbf{Mod}(R_{\mathbb{C}}) \to \mathbf{Mod}(R_{\mathbb{C}})$$

sending an $R_{\mathbb{C}}$-module M to the localization $M_{\mathfrak{p}}$ (see [BCa, Sec. 2.3] or [BCL18]). For every G-bornological coarse space X, the twisted equivariant coarse Hochschild homology $\mathcal{X}\mathrm{HH}^G_T(X)$ is an $R_{\mathbb{C}}$-module, and we denote by $\mathcal{X}\mathrm{HH}^G_{T,\mathfrak{p}}(X)$ its localization at the prime ideal $\mathfrak{p}$ of $R(G)$ (by using the isomorphism between $R(G)$ and the homology of $R_{\mathbb{C}}$).

We can apply the construction of Remark 1.4.7 to the functor $\mathcal{X}\mathrm{HH}^G_{T,\mathfrak{p}}$, getting the functor

$$\underline{\mathcal{X}\mathrm{HH}^G_{T,\mathfrak{p}}}\colon G\mathbf{Orb} \to \mathbf{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{X}, \mathbf{Mod}(R_{\mathbb{C}})). \tag{4.5.1}$$

We now proceed with proving the coarse localization theorem for coarse Hochschild homology.

Recall that a family of subgroups is invariant under conjugation and taking subgroups (Definition 1.3.16). Let γ be a conjugacy class of G. Then, the family $\mathcal{F}(\gamma) = \{H < G \mid H \cap \gamma = \emptyset\}$ is a family of subgroups (see Definition 1.4.19 and Lemma 1.4.20).

Definition 4.5.3. Let γ be a conjugacy class of the group G. The ideal of $R(G)$ given by the representations ρ with $\mathrm{tr}(\rho(g)) = 0$ for all $g \in \gamma$ is denoted by (γ).

Remark 4.5.4. We observe that the ideal (γ) of Definition 4.5.3 is a prime ideal of the representation ring $R(G)$.

Let H be a subgroup in $\mathcal{F}(\gamma)$. Then Segal [Seg68b] has shown that there exists an element η in the representation ring $R(G)$ of G such that:

1. $\eta_{|H} = 0$ and
2. $\mathrm{tr}(\eta(g)) \neq 0$ for all g in γ.

Hence, if H is any subgroup of G, we can form the localization $R(H)_{(\gamma)}$ of the $R(G)$-module $R(H)$ at the prime ideal (γ).

Let $\mathbf{C}$ be a stable ∞-category. Recall (Definition 1.4.14) that a functor $E\colon G\mathbf{Orb} \to \mathbf{C}$ vanishes on the family of subgroups $\mathcal{F}(\gamma)$ of G if $E(S)$ is equivalent to 0 for all S in $G_{\mathcal{F}(\gamma)}\mathbf{Orb}$.

Lemma 4.5.5. *Let $\underline{\mathcal{X}\mathrm{HH}^G_{T,(\gamma)}}$ be the functor* (4.5.1) *localized at the prime ideal* (γ) *associated to the conjugacy class γ of G. Then, the functor $\underline{\mathcal{X}\mathrm{HH}^G_{T,(\gamma)}}$ vanishes on the family $\mathcal{F}(\gamma)$ of Definition 1.4.19.*

Proof. We want to show that

$$\mathcal{X}\mathrm{HH}^G_{(\gamma)}((G/K)_{\text{min, max}} \otimes X) \simeq 0$$

for every K in $\mathcal{F}(\gamma)$ and every X in $G\mathbf{BornCoarse}$. We first need to explicitly understand the $R(G)$-module structure of $\mathcal{X}\mathrm{HH}^G(X)$ for a G-bornological coarse space. We proceed as in [BCb, Prop. 4.11 & 4.15].

The module structure is induced by the morphism of $\mathbb{C}$-linear categories

$$V^G_{\mathbb{C}}(*) \otimes V^G_{\mathbb{C}}(X) \to V^G_{\mathbb{C}}(X)$$

that sends $((V,\pi),(H,\rho))$ to $(V \otimes H, \pi \otimes \rho)$. Here (V,π) is a representation of G on a finitely dimensional $\mathbb{C}$-vector spaces, and (H,ρ) is an equivariant X-controlled $\mathbf{Vect}^{\mathrm{f.d.}}_{\mathbb{C}}$-object. This construction extends to morphisms in the natural way and we get an exact functor between additive categories

$$(V,\pi) \otimes (-) : V^G_{\mathbb{C}}((G/K)_{min,max} \otimes X) \to V^G_{\mathbb{C}}((G/K)_{min,max} \otimes X)\ .$$

We assume now that $(V^{\pm},\pi^{\pm})$ are two representations of G such that there exists an isomorphism $W : (V^+,\pi^+)_{|K} \to (V^-,\pi^-)_{|K}$. Then there is a controlled isomorphism [BCb, Prop. 4.15]

$$\tilde{W}_{(H,\rho)} : (V^+ \otimes H, \pi^+ \otimes \rho) \to (V^- \otimes H, \pi^- \otimes \rho)$$

providing a natural isomorphism of functors

$$\begin{tikzcd} V^G_{\mathbb{C}}((G/K)_{min,max} \otimes X) \arrow[rr, bend left, "(V^+,\pi^+)\otimes(-)"] \arrow[rr, bend right, "(V^-,\pi^-)\otimes(-)"'] \arrow[rr, phantom, "\Downarrow \tilde{W}"] & & V^G_{\mathbb{C}}((G/K)_{min,max} \otimes X) \end{tikzcd}\ .$$

The existence of the isomorphism $\tilde{W}$ implies that the multiplication by the classes $[V^+,\pi^+]$ and $[V^-,\pi^-]$ in $R(G) \cong H_*(R_{\mathbb{C}})$ on the homology of $\mathcal{X}\mathrm{HH}^G((G/K)_{min,max} \otimes X)$ coincide. Hence, if η is an element of $R(G)$ which restricts to 0 in K, then η acts trivially on the homology of $\mathcal{X}\mathrm{HH}^G((G/K)_{min,max} \otimes X)$.

Let now H be a subgroup of G in $\mathcal{F}(\gamma)$. Let $\eta \in R(G)$ be the representation constructed by Segal satisfying (1) and (2). By (2), the multiplication by η acts as an isomorphism on the homology of $\mathcal{X}\mathrm{HH}^G_{(\gamma)}((G/H)_{min,max} \otimes X)$, but by (1) it vanishes. This implies that the homology of $\mathcal{X}\mathrm{HH}^G_{(\gamma)}((G/H)_{min,max} \otimes X)$ is zero, hence

$$\mathcal{X}\mathrm{HH}^G_{(\gamma)}((G/H)_{min,max} \otimes X) \simeq 0.$$

The lemma is proved for any T by setting $X = T \otimes Y$ in the above argument. $\square$

Let X be a G-bornological coarse space and γ a conjugacy class of G. In Definition 1.4.10, we have defined a category $G\mathbf{Sp}\mathcal{X}\langle G\mathbf{Orb} \otimes \mathbf{Sp}\mathcal{X}_{bd}\rangle$; we consider here also the category $G\mathbf{Sp}\mathcal{X}\langle G\mathbf{Orb} \otimes \mathbf{Sp}\mathcal{X}\rangle$ defined analogously [BCb, Def. 3.43]. Recall Definition 1.4.16 of X^γ and Definition 1.4.17 of a nice coarse space. Let T be an object in $G\mathbf{Sp}\mathcal{X}$. The following is the coarse localization theorem for coarse Hochschild homology.

Theorem 4.5.6. *Let G be a finite group and let γ be a conjugacy class of G. Let X be a G-bornological coarse space and assume that the G-bornological coarse space of γ-fixed points X^γ is a nice subset of X. Assume one of:*

1. *G is finite and both X^γ and X belong to $G\mathbf{Sp}\mathcal{X}\langle G\mathbf{Orb} \otimes \mathbf{Sp}\mathcal{X}\rangle$;*
2. *$T = G_{\mathrm{can,min}}$ and both the G-bornological coarse spaces X^γ and X belong to $G\mathbf{Sp}\mathcal{X}\langle G\mathbf{Orb} \otimes \mathbf{Sp}\mathcal{X}_{bd}\rangle$.*

Then, the inclusion $X^\gamma \to X$ induces an equivalence

$$\mathcal{X}\mathrm{HH}^G_{T,(\gamma)}(X^\gamma) \to \mathcal{X}\mathrm{HH}^G_{T,(\gamma)}(X).$$

The same theorem holds if one replaces $\mathcal{X}\mathrm{HH}^G$ with coarse cyclic homology $\mathcal{X}\mathrm{HC}^G$ with $\mathbb{C}$-coefficients.

Proof. Assume (1); then, by [BCb, Cor. 3.43], we can replace the functor $\mathcal{X}\mathrm{HH}^G_{T,(\gamma)}$ with its Bredon-style approximation. In the case (2), we use [BCb, Cor. 3.47], Lemma 4.2.3 and Example 1.4.11.

By Lemma 4.5.5, the functor $\underline{\mathcal{X}\mathrm{HH}^G_{T,(\gamma)}}$ vanishes on $\mathcal{F}(\gamma)$. By Corollary 1.4.22, the morphism

$$\mathcal{X}\mathrm{HH}^G_{T,(\gamma)}(X^\gamma) \to \mathcal{X}\mathrm{HH}^G_{T,(\gamma)}(X).$$

is then an equivalence. $\square$

The localization theorem for coarse Hochschild homology implies a localization theorem for the associated equivariant Hochschild homology $\mathbf{HH}^G \coloneqq \mathcal{X}\mathrm{HH}^G_{k,G_{\mathrm{can,min}}} \circ \mathcal{O}^\infty_{\mathrm{hlg}}$:

Corollary 4.5.7. *Let G be a finite group and let γ be a conjugacy class of G. Let W be a finite G CW-complex and let W^γ be the sub-complex of γ-fixed points. Then, the inclusion $W^\gamma \to W$ induces an equivalence*

$$\mathbf{HH}^G_{(\gamma)}(W^\gamma) \to \mathbf{HH}^G_{(\gamma)}(W)$$

of chain complexes.

The proof is the same as [BCb, Cor. 4.14 & Rem. 4.20], where the same result has been obtained in the case of equivariant coarse topological and equivariant coarse algebraic K-homology. The result translates in the case of equivariant coarse cyclic homology as well.

Appendix A

In this appendix we collect some information about additive categories, dg-categories, symmetric monoidal categories and cyclic objects. In this way we introduce the notation and language that we need in this thesis, and specially in Chapter 3 in order to define coarse Hochschild and cyclic homology.

In this appendix we let k be a commutative ring; we write $\otimes$ for the tensor product over k when clear from the context.

A.1 Additive categories

In this section we recollect definitions and properties of additive categories, k-linear categories and exact categories.

We start with the definition of an additive category.

Definition A.1.1. [Mac71, Section VIII.2] An *additive category* is a category enriched over (the category of) abelian groups, that has a zero object and that admits a biproduct for each pair of objects.

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. The functor F is called *additive* if it preserves finite biproducts; hence it sends the 0-object to the 0-object and for every two objects A, B in $\mathcal{A}$ there is a canonical isomorphism $F(A \oplus B) \cong F(A) \oplus F(B)$, where $\oplus$ denotes the biproduct of $\mathcal{A}$. If, in addition, F preserves also finite limits and colimits, then it is called *exact*.

We denote by **Add** the category of small additive categories and exact functors.

Definition A.1.2. A *k-category* is a category enriched on k-modules; a *k-linear category* is a k-category that is also additive.

An example of k-linear categories is given by k-algebras: a k-algebra is a k-linear category with a single object. Morphisms of k-linear categories are additive functors that preserve the k-linear structure as well. We denote by $\mathbf{Cat}_k$ the category of small k-linear categories and exact functors.

Let $\mathcal{A}$ be a small additive category and let G be a group.

Definition A.1.3. The additive category $\mathcal{A}$ is an *additive category with strict G-action* (on the right) if:

1. for every $g \in G$ we have an additive isomorphism $F_g \colon \mathcal{A} \to \mathcal{A}$;

2. if e is the unit element of G, then $F_e = \mathrm{id}_{\mathcal{A}}$;

3. for every g, h in G we have $F_h \circ F_g = F_{gh}$ satisfying strictly associative relations.

The equalities in the definition express the strictness of the G-action. In general, they are replaced by natural isomorphisms subject to coherence associative relations. Observe that, if BG denotes the category with a single object and endomorphisms the given group G, then an additive category with strict G-action can be equivalently described as a functor $BG \to \mathbf{Add}$ from the category BG to the category of small additive categories.

Let $\mathcal{A}$ be an additive category and denote by $\oplus$ its biproduct.

Definition A.1.4. An additive category $\mathcal{A}$ is called *flasque* if it admits an endofunctor $S \colon \mathcal{A} \to \mathcal{A}$ and a natural isomorphism $\mathrm{id}_{\mathcal{A}} \oplus S \cong S$.

Following Keller, we now recall the definition of an exact category [Kel96, Sec. 4].

Let $\mathcal{A}$ be an additive category. A pair (i, p) of composable morphisms

$$A \xrightarrow{i} B \xrightarrow{p} C$$

in $\mathcal{A}$ is called *exact* if i is a kernel for p and p is a cokernel for i. We use Keller's terminology: exact pairs (i, p), $A \xrightarrow{i} B \xrightarrow{p} C$, are called *conflations*, the morphism i is called an *inflation* and the morphism p a *deflation*.

Definition A.1.5. An *exact category* is an additive category equipped with a class of exact pairs satisfying the following axioms:

1. The identity morphism of the zero object is a deflation.

2. A composition of two deflations is a deflation.

3. A composition of two inflations is an inflation.

4. The pull-back of a deflation exists and is a deflation.

5. The push-out of an inflation exists and is an inflation.

These set of axioms are equivalent to the Quillen's axioms [Qui73], as proved by Keller [Kel90]. If $\mathcal{A}$ and $\mathcal{B}$ are exact categories, an additive functor $F \colon \mathcal{A} \to \mathcal{B}$ is *exact* if it sends conflations to conflations.

Example A.1.6. Let $\mathcal{A}$ be an additive category. The family of all split short exact sequences is an exact structure on $\mathcal{A}$. We call this exact structure on $\mathcal{A}$ the *split exact structure.*

Example A.1.7. Let $\mathcal{A}$ be an additive category and let $\mathrm{Ch}(\mathcal{A})$ be the category of chain complexes in $\mathcal{A}$, whose objects are chain complexes in $\mathcal{A}$ and morphisms are chain maps. The category $\mathrm{Ch}(\mathcal{A})$, equipped with the class of exact pairs (i, p) for which (i_n, p_n) is a split short exact sequence in $\mathcal{A}$ for all $n \in \mathbb{Z}$, is an exact category.

We now give the definition of acyclic chain complexes:

Definition A.1.8. [Kel90] Let $\mathcal{E}$ be an exact category. A chain complex $K \in \mathrm{Ch}(\mathcal{E})$ is called *acyclic in degree* n if the boundary operator d_K^{n+1} factors as follows:

$$\begin{tikzcd} K_{n+1} \arrow[rr, "d_K^{n+1}"] \arrow[dr, "p_{n+1}"'] & & K_n \\ & Z_{n+1} \arrow[ur, "i_{n+1}"'] & \end{tikzcd}$$

where p_{n+1} is a deflation, and i_{n+1} is an inflation. The complex K is called *acyclic* if it is acyclic in degree n for every n.

We denote by $\mathrm{Ch}^b(\mathcal{E})$ the category of bounded chain complexes over $\mathcal{E}$ and by $\mathrm{Acy}^b(\mathcal{E})$ the category of bounded acyclic chain complexes over $\mathcal{E}$.

Remark A.1.9. Let $\mathcal{E}$ be an exact category and let K be an acyclic chain complex. This means that we have conflations

$$Z_{n+1} \xrightarrow{i_{n+1}} K_n \xrightarrow{p_n} Z_n$$

for every $n \in \mathbb{N}$.

Let $\mathcal{E}$ be a small exact k-linear category. Following [Kel96, Sec. 11], we now explain how to define the bounded derived category $\mathcal{D}^b(\mathcal{E})$. Let $\mathcal{K}^b(\mathcal{E})$ be the homotopy category of bounded chain complexes. The objects are the bounded chain complexes in $\mathcal{E}$ and the morphisms are given by the chain maps up to chain homotopy. The category $\mathcal{K}^b(\mathcal{E})$ is a triangulated category. Let $\mathcal{K}^b_{acy}(\mathcal{E})$ be the sub-category of acyclic chain complexes, and observe that this is closed under taking cones and shifts in $\mathcal{K}^b(\mathcal{E})$. Hence, it is a full triangulated sub-category of $\mathcal{K}^b(\mathcal{E})$. The bounded derived category $\mathcal{D}^b(\mathcal{E})$ is then defined as the Verdier quotient

$$\mathcal{D}^b(\mathcal{E}) \coloneqq \mathcal{K}^b(\mathcal{E})/\mathcal{K}^b_{acy}(\mathcal{E}).$$

Definition A.1.10. The category $\mathcal{D}(\mathcal{E})$ ($\mathcal{D}^b(\mathcal{E})$) is called the *(bounded) derived category* of $\mathcal{E}$.

We refer to [Kel96, Kel06] for more details.

A.2 Differential graded categories

In this section we recall the main notions concerning differential graded categories. We give the definitions of dg-algebras, dg-modules and dg-categories. We also review the notions of (strngly) pretriangulated categories and of Morita equivalences. Our main references are [Kel06, Toë11].

We start with the definition of a dg-algebra:

Definition A.2.1. A differential graded k-algebra A (shortly, a dg-algebra), is a $\mathbb{Z}$-graded k-algebra $A = \bigoplus_{p\in\mathbb{Z}} A^p$ endowed with a differential d either of degree -1 (chain complex convention) or of degree 1 (cochain complex convection) that satisfies the Leibniz rule

$$d(ab) = d(a)b + (-1)^p a d(b)$$

for all $a \in A^p$ and $p \in \mathbb{Z}$, and all $b \in A$.

A left dg-module M over a dg-algebra A is a left graded module $M = \oplus_{p\in\mathbb{Z}} M^p$ endowed with a differential d (of the same degree as the differential of A) such that $d(ma) = d(m)a + (-1)^p m d(a)$ for every $m \in M^p$ and $a \in A$. A morphism of dg-modules is a homogeneous morphism of degree 0 of the underlying graded modules commuting with the differentials.

The category of dg-modules over the dg-algebra A and morphisms of dg-modules is denoted by A-**Mod**.

A dg-category over a ring k is a category enriched on (the category of) chain complexes of k-modules. We spell it out:

Definition A.2.2. A *small differential graded category* $\mathcal{A}$ (shortly, a dg-category) consists of the following data:

- a small set of objects $\mathrm{obj}(\mathcal{A})$ (denoted $\mathcal{A}$ as well);
- for each pair of objects A, B of $\mathcal{A}$, a chain complex of k-modules $\mathrm{Hom}_{\mathcal{A}}(A, B)$;
- for each triple of objects A, B, C in $\mathcal{A}$ compositions

  $$\mathrm{Hom}_{\mathcal{A}}(A, B) \otimes \mathrm{Hom}_{\mathcal{A}}(B, C) \to \mathrm{Hom}_{\mathcal{A}}(A, C)$$

  satisfying the associativity relations;
- for each object A of $\mathcal{A}$, a morphism $k \to \mathrm{Hom}_{\mathcal{A}}(A, A)$ satisfying the usual unit condition with respect to the compositions.

Example A.2.3. Every dg-algebra is a dg-category with a single object.

Definition A.2.4. Let $\mathcal{A}$ and $\mathcal{A}'$ be small dg-categories. A *dg-functor* $F\colon \mathcal{A} \to \mathcal{A}'$ consists of a map $F\colon \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{A}')$ of sets and, for each pair of objects A, B in $\mathcal{A}$, of a morphism $F(A,B)\colon \mathrm{Hom}_{\mathcal{A}}(A,B) \to \mathrm{Hom}_{\mathcal{A}'}(F(A),F(B))$ of chain complexes satisfying the usual unit and composition conditions.

We denote by $\mathbf{dgcat}_k$ the category of small dg-categories (over k) and dg-functors.

Example A.2.5. An additive category $\mathcal{A}$ is a dg-category: for every object A in $\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(A,A)$ is a chain complex concentrated in degree 0. This gives a functor

$$\iota\colon \mathbf{Add} \to \mathbf{dgcat} \tag{A.1}$$

that sends an additive category to the corresponding dg-category.

Example A.2.6. Let $\mathcal{E}$ be a small k-linear exact category. The category $\mathrm{Ch}(\mathcal{E})$ of chain complexes over $\mathcal{E}$ has the structure of a dg-category: if K and K' are two chain complexes, then $\mathrm{Hom}_n(K,K')$ consists of the homogeneous morphisms f of degree n of graded objects: $(f\colon K_p \to K'_{p+n})_p$. The differential $d_n\colon \mathrm{Hom}_n(K,K') \to \mathrm{Hom}_{n-1}(K,K')$ is defined by the Leibniz rule

$$d(f) \coloneqq d_{K'} \circ f - (-1)^n f \circ d_K.$$

We denote by $\mathrm{Ch}_{dg}(\mathcal{E})$ this dg-structure of the category of chain complexes over $\mathcal{E}$. Analogously, the category of bounded chain complexes over $\mathcal{E}$ is a dg category that we denote by $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{E})$.

The category $\mathrm{Acy}^b(\mathcal{E})$ of bounded acyclic chain complexes (see Definition A.1.8) is a sub-category of the category of bounded chain complexes; we denote by $\mathrm{Acy}^b_{\mathrm{dg}}(\mathcal{E})$ the category of bounded acyclic chain complexes with the induced dg-structure.

Example A.2.7. If $\mathcal{A}$ is a dg-category, then the dg-category $\mathcal{A}^{\mathrm{op}}$ defined as the category with the same objects as $\mathcal{A}$ and morphisms $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(A,B) \coloneqq \mathrm{Hom}_{\mathcal{A}}(B,A)$, is a dg-category.

We have defined dg-modules over a dg-algebra; we now generalize the definition of dg-modules to dg-categories.

Definition A.2.8. Let $\mathcal{A}$ be a dg-category. A left (right) *dg-module* over $\mathcal{A}$ is a dg-functor

$$\mathcal{A}^{\mathrm{op}} \to \mathrm{Ch}_{dg}(k)$$

where $\mathrm{Ch}_{dg}(k)$ is the dg-category of chain complexes over the dg-algebra k.

There is a natural category of dg-modules over a dg-category $\mathcal{A}$, whose objects are the dg-modules over $\mathcal{A}$ and whose morphisms are the natural transformations of dg-functors $F\colon M \to N$ such that $F(A)\colon M(A) \to N(A)$ is a morphism of complexes for every object A of $\mathcal{A}$.

Remark A.2.9. When $\mathcal{A}$ is a dg-category with one object (*i.e.,* a dg-algebra), the category of modules over $\mathcal{A}$ just described is equivalent to the category of dg-modules (over a dg-algebra) defined in the text after Definition A.2.1.

Let $\mathcal{A}$ be a dg-category and let M, N be two dg-modules over $\mathcal{A}$. Then, the evaluations of M and N at an object $A \in \mathcal{A}$ are chain complexes.

Definition A.2.10. A morphism of dg-modules $M \to N$ is a *quasi-isomorphism* if it induces a quasi-isomorphism of chain complexes $M(A) \to N(A)$ for every object A in $\mathcal{A}$.

Remark A.2.11. The category of dg-modules (over a dg-algebra or a dg-category) admits two Quillen model structures where the weak equivalences are the object-wise quasi-isomorphisms of dg-modules; these are the injective and the projective model structure induced from the injective and projective model structure on chain complexes, respectively. We remark that the category of dg-modules over a dg-algebra, equipped with the projective model structure (hence the fibrations are the object-wise epimorphisms), is a combinatorial model category; see, for example [Coh, Rem. 2.14].

In analogy with chain complexes, we can define shifts and cones of objects in a dg-category, as we now explain.

Let $\mathcal{A}$ be a dg-category and let A be an object of $\mathcal{A}$. The *n-translation* of A, for n an integer in $\mathbb{Z}$, is an object $A[n]$ of $\mathcal{A}$ representing the dg-functor

$$B \mapsto \mathrm{Hom}_{\mathcal{A}}(B, A)[n]$$

where $\mathrm{Hom}_{\mathcal{A}}(B, A)[n]$ is the n-shifted chain complex.

If A and B are objects of a dg-category $\mathcal{A}$, and $\varphi\colon A \to A'$ is a morphism of $\mathcal{A}$, then its cone $\mathrm{cone}(\varphi)$ is the object of $\mathcal{A}$ that represents the dg-functor

$$B \mapsto \mathrm{cone}(\mathrm{Hom}_{\mathcal{A}}(B, A) \xrightarrow{\mathrm{Hom}(B,\varphi)} \mathrm{Hom}_{\mathcal{A}}(B, A'))$$

where the cone between the Hom-complexes here denotes the usual mapping cone of chain complexes.

Definition A.2.12. [BV08, Dri04] The dg-category $\mathcal{A}$ is *strongly pretriangulated* if it admits a zero object 0 and the translations $A[n]$ and cones $\mathrm{cone}(\varphi)$ exist for every object $A \in \mathcal{A}$, $n \in \mathbb{Z}$, and morphism φ.

For every dg-category $\mathcal{A}$ there exist a dg-category $\mathcal{A}^{pretr}$ called the *pretriangulated hull* of $\mathcal{A}$ and a universal dg-functor $\mathcal{A} \to \mathcal{A}^{pretr}$, *i.e.,* a dg-functor inducing an equivalence $\mathrm{Hom}_{\mathbf{dgcat}}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathbf{dgcat}}(\mathcal{A}^{pretr}, \mathcal{B})$ for every strong pretriangulated dg-category $\mathcal{B}$ [Kel06, Sec. 4.5] and [BK91].

We observe here that a weaker notion, namely of a pretriangulated category, is usually considered (see *e.g.,* [Coh, Kel06]). However, the pretriangulated hull $\mathcal{A}^{pretr}$ of $\mathcal{A}$ is always strongly pretriangulated [BK91, Dri04].

Example A.2.13. [BV08, BK91] If $\mathcal{A}$ is an additive category, seen as a dg-category by $\iota(\mathcal{A})$ (A.1), then its pretriangulated hull $\mathcal{A}^{pretr}$ is the dg-category $\mathrm{Ch}^b_{\mathrm{dg}}(\mathcal{A})$ of bounded chain complexes in $\mathcal{A}$.

If $\mathcal{A}$ is a dg-category, we can define an associated derived category:

Definition A.2.14. [Kel06, Sec. 3.2] Let $\mathcal{A}$ be a dg-category. The *derived category* $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is the localization of the category of dg-modules over $\mathcal{A}$ at the class of quasi-isomorphisms.

The objects of $\mathcal{D}(\mathcal{A})$ are the dg-modules over $\mathcal{A}$ and the morphisms are obtained from morphisms of dg-modules by inverting the quasi-isomorphisms. It is a triangulated category with shift functor induced by the 1-translation and triangles coming from short exact sequences of complexes.

Let $\mathcal{A}$ and $\mathcal{B}$ be two small dg-categories. A dg-functor $F\colon \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if it induces an equivalence of derived categories. For a precise definition of Morita equivalences we refer to [Kel06, Sec. 3.8], or [Coh, Def. 2.29]. The most important example to us is the following:

Example A.2.15. [Kel06, Sec. 4.6] Let $\mathcal{A}$ be a dg-category. Then, the canonical morphism $\mathcal{A} \to \mathcal{A}^{pretr}$ in the pretriangulated hull is a Morita equivalence.

Theorem A.2.16. [Tab05, Thm. 5.1] *The category* $\mathbf{dgcat}_k$ *of small dg-categories over* k *admits the structure of a combinatorial model category whose weak equivalences are the Morita equivalences.*

For a description of fibrations and cofibrations we refer to [Tab05, Thm. 5.1], or [Kel06, Thm. 4.1].

The localization [Lur14, Def. 1.3.4.1] of the category of small dg-categories **dgcat** at the class of Morita equivalences is the ∞-category of (idempotent-complete) dg-categories:

$$\mathbf{dgcat}_\infty \coloneqq \mathrm{N}(\mathbf{dgcat})[W^{-1}_{\mathrm{Morita}}]$$

where W_{Morita} denotes the class of Morita equivalences of small dg-categories.

Remark A.2.17. The Morita model structure on **dgcat** presents the ∞-category of stable ∞-categories [Coh, Fao17]. The bridge between the two theories is the dg-nerve functor N_{dg} that sends a dg-category to a stable ∞-categories [Lur14, Sec. 1.3.1].

We refer to Robalo's thesis [Rob14, Sec. 6.1 & 6.2] for a comparison between the theory of dg-categories and the theory of stable ∞-categories.

A.3 Symmetric monoidal structures

In this section we recall the definitions of a symmetric monoidal category and (lax) symmetric monoidal functors.

A symmetric monoidal structure on a category $\mathcal{C}$ consists of a functor $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a tensor unit $1_{\mathcal{C}}$, and the associator, symmetry and unit-transformations that must satisfy some compatibility relations. We review these relations in the following definition:

Definition A.3.1. [Mac71, Sec. VII.1.] A *symmetric monoidal structure* on a category $\mathcal{C}$ is given by the following data:

1. a bifunctor $(- \otimes_{\mathcal{C}} -) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
2. an object $1_{\mathcal{C}}$ (the *tensor unit*),
3. a natural isomorphism (the *associativity constraint*)
$$\alpha^{\mathcal{C}} : (- \otimes_{\mathcal{C}} -) \circ ((- \otimes_{\mathcal{C}} -) \times \mathrm{id}_{\mathcal{C}}) \to (- \otimes_{\mathcal{C}} -) \circ (\mathrm{id}_{\mathcal{C}} \times (- \otimes_{\mathcal{C}} -)) \ ,$$
4. a natural isomorphism $\eta^{\mathcal{C}} : 1_{\mathcal{C}} \otimes_{\mathcal{C}} - \to \mathrm{id}_{\mathcal{C}}$ (the *unit constraint*),
5. a natural isomorphism (the *symmetry*) $\sigma^{\mathcal{C}} : (- \otimes_{\mathcal{C}} -) \circ T \to (- \otimes_{\mathcal{C}} -)$, where $T : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the flip functor.

This data has to satisfy the following relations:

1. the pentagon relation;
$$\begin{tikzcd}
((C \otimes_{\mathbf{C}} C')\otimes_{\mathbf{C}})C'' \otimes_{\mathbf{C}} C''' \arrow[rr, "\alpha_{C\otimes_{\mathbf{C}}C',C'',C'''}"] \arrow[d, "\alpha_{C,C',C''}\times \mathrm{id}_{C'''}"] & & (C \otimes_{\mathbf{C}} C') \otimes_{\mathbf{C}} (C'' \otimes_{\mathbf{C}} C''') \arrow[d, "\alpha_{C,C',C''\otimes_{\mathbf{C}}C'''}"] \\
(C \otimes_{\mathbf{C}} (C' \otimes_{\mathbf{C}} C'')) \otimes_{\mathbf{C}} C''' \arrow[dr, "\alpha_{C,C'\otimes_{\mathbf{C}}C'',C'''}"] & & C \otimes_{\mathbf{C}} (C' \otimes_{\mathbf{C}} (C'' \otimes_{\mathbf{C}} C''')) \\
& C \otimes_{\mathbf{C}} ((C' \otimes_{\mathbf{C}} C'') \otimes_{\mathbf{C}} C''') \arrow[ur, "\mathrm{id}_C\otimes\alpha_{C',C'',C'''}"] &
\end{tikzcd}$$
2. the triangle relation;
$$\begin{tikzcd}
(C \otimes 1_{\mathbf{C}}) \otimes_{\mathbf{C}} C' \arrow[rr, "\alpha_{C,1_{\mathbf{C}},C'}"] \arrow[dr, "\eta_C\times\mathrm{id}_C"'] & & C \otimes_{\mathbf{C}} (1_{\mathbf{C}} \otimes C') \arrow[dl, "\mathrm{id}_{1_{\mathbf{C}}}\times(\eta_{C'}\circ\sigma_{1_{\mathbf{C}},C'})"] \\
& C \otimes_{\mathbf{C}} C' &
\end{tikzcd} \ ,$$
3. the inverse relation; $\sigma_{C,C'} \circ \sigma_{C',C} = \mathrm{id}_{C'\otimes C}$,

4. the associativity coherence.

$$\begin{tikzcd}
(C \otimes_{\mathbf{C}} C') \otimes_{\mathbf{C}} C'' \arrow[r, "\alpha_{C,C',C''}"] \arrow[d, "\sigma_{C,C'} \times \mathrm{id}_{C''}"'] & C \otimes_{\mathbf{C}} (C' \otimes_{\mathbf{C}} C'') \arrow[r, "\sigma_{C,C' \otimes_{\mathbf{C}} C''}"] & (C' \otimes_{\mathbf{C}} C'') \otimes_{\mathbf{C}} C \arrow[d, "\alpha_{C',C'',C}"] \\
(C' \otimes_{\mathbf{C}} C) \otimes_{\mathbf{C}} C'' \arrow[r, "\alpha_{C',C,C''}"'] & C' \otimes_{\mathbf{C}} (C \otimes_{\mathbf{C}} C'') \arrow[r, "\mathrm{id}_{C'} \times \sigma_{C'',C}"'] & C' \otimes_{\mathbf{C}} (C'' \otimes_{\mathbf{C}} C)
\end{tikzcd} .$$

A *symmetric monoidal category* is a category with a symmetric monoidal structure.

The category $\mathbf{Cat}_k$ of small k-linear categories has a symmetric monoidal structure:

Definition A.3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be small k-linear categories. The *tensor product* $\mathcal{A} \otimes_{\mathbf{Cat}_k} \mathcal{B}$ is the k-linear category whose objects are pairs (A, B) of objects with A in $\mathcal{A}$ and B in $\mathcal{B}$ and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{A} \otimes_{\mathbf{Cat}_k} \mathcal{B}}((A_0, B_0), (A_1, B_1)) \coloneqq \mathrm{Hom}_{\mathcal{A}}(A_0, A_1) \otimes_k \mathrm{Hom}_{\mathcal{B}}(B_0, B_1).$$

Analogously, when k is a field, the category of small dg-categories has a natural symmetric monoidal structure:

Definition A.3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be small dg-categories over k. The *tensor product* $\mathcal{A} \otimes_{\mathbf{dgcat}_k} \mathcal{B}$ is the dg-category whose objects are pairs (A, B) of objects with A in $\mathcal{A}$ and B in $\mathcal{B}$ and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{A} \otimes_{\mathbf{dgcat}_k} \mathcal{B}}((A_0, B_0), (A_1, B_1) \coloneqq \mathrm{Hom}_{\mathcal{A}}(A_0, A_1) \otimes_k \mathrm{Hom}_{\mathcal{B}}(B_0, B_1),$$

where $\otimes_k$ denotes the tensor product of chain complexes over k.

When k is not a field, the tensor product just described does not respect, in general, the Morita model structure on the category $\mathbf{dgcat}_k$ of Theorem A.2.16 because it is not compatible with Morita equivalences. However, if k is a field, a dg-category over k is also flat, and the tensor product induces a symmetric monoidal structure on $\mathbf{dgcat}_k$

$$- \otimes_k - \colon \mathbf{dgcat}_k \times \mathbf{dgcat}_k \to \mathbf{dgcat}_k \tag{A.1}$$

that preserves the Morita equivalences. We denote by $\mathbf{dgcat}_k^{\otimes}$ the symmetric monoidal category obtained in this way.

Definition A.3.4. [Mac71, Sec. VII.1.] Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between symmetric monoidal categories. A *symmetric monoidal structure* on F is given by the following data:

1. an isomorphism $\epsilon^F : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$,
2. a natural isomorphism $\mu^F : (- \otimes_{\mathcal{D}} -) \circ (F \times F) \to F \circ (- \otimes_{\mathcal{C}} -)$.

This data has to satisfy the following relations:

1. associativity;

2. unitality;

3. symmetry.

We refer to [Mac71, Sec. VII.1.] for further details.

Remark A.3.5. If we weaken the assumption and only require that that ϵ^F and μ^F are natural transformations, then we get the definition of a *lax symmetric monoidal functor*.

Remark A.3.6. Let $\mathcal{C}$ be a symmetric monoidal category. Then, it gives rise to a category $\mathcal{C}^{\otimes}$ [Lur14, Constr. 2.0.0.1] and to a symmetric monoidal ∞-category

$$\mathrm{N}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathbf{Fin}_*)$$

[Lur14, Ex. 2.1.2.21] with the property that the underlying ∞-category of $\mathrm{N}(\mathcal{C}^{\otimes})$ is the nerve $\mathrm{N}(\mathcal{C})$.

In Section 2.2, we need to refine functors with target the category of small additive categories **Add**. Let $\mathcal{C}$ be an ordinary category. A functor $F : \mathcal{C} \to \mathbf{Add}$ gives rise to a functor between ∞-categories $F_\infty : \mathrm{N}(\mathcal{C}) \to \mathbf{Add}_\infty$ in the natural way, *i.e.,* as the composition

$$F_\infty : \mathrm{N}(\mathcal{C}) \overset{\mathrm{N}(F)}{\to} \mathrm{N}(\mathbf{Add}) \to \mathrm{N}(\mathbf{Add})[W_{\mathbf{Add}}^{-1}] = \mathbf{Add}_\infty$$

where $W_{\mathbf{Add}}$ is the class of equivalences of additive categories.

Recall that a map of ∞-operads [Lur14, Def. 2.1.2.7] can be thought of as a (lax) symmetric monoidal functor [Lur14, Def. 2.1.3.7] between the underlying categories.

Definition A.3.7. Let $F : \mathcal{C} \to \mathbf{Add}$ be a functor to the category of small additive categories. A *lax symmetric monoidal refinement* of F is a morphism

$$F^{\otimes} : \mathrm{N}(\mathcal{C}^{\otimes}) \to \mathbf{Add}_\infty^{\otimes}$$

of ∞-operads that induces a functor equivalent to F_∞ between the underlying ∞-categories.

A.4 Cyclic objects and the additive cyclic nerve

Following [Con83, Lod98, NS17], we recall in this section the definition of cyclic objects in a category $\mathcal{C}$; then, we recall the definition of additive cyclic nerve of a dg-category.

A cyclic object X is a simplicial object with extra structure: in addition to face and degeneracy maps, there is an action of the cyclic group of order $i + 1$ acting on the objects X_i, for every i. There are different equivalent definitions of cyclic objects; we define them as controvariant functors on Connes' cyclic category, as we now explain.

We start with the definition of the paracyclic category Λ_∞ [NS17, Appendix B].

The *paracyclic category* Λ_∞ is a full subcategory of the category $\mathbb{Z}\mathbf{PoSet}$ of partially ordered sets equipped with a $\mathbb{Z}$-action and non-decreasing equivariant maps. The objects of Λ_∞ are all the objects of $\mathbb{Z}\mathbf{PoSet}$ that are isomorphic to

$$\frac{1}{n}\mathbb{Z} := \left\{ \frac{k}{n} \mid k \in \mathbb{Z} \right\}$$

for $n \geq 1$, equipped with its natural ordering and the $\mathbb{Z}$ action by addition. We use the notation $[n]_{\Lambda_\infty} := \frac{1}{n}\mathbb{Z}$.

There is an action of $\mathbb{Z}$ on the morphism sets of Λ_∞: if τ is a generator of $\mathbb{Z}$ and $f\colon [n]_{\Lambda_\infty} \to [m]_{\Lambda_\infty}$ is a morphism of Λ_∞, then $\tau(f)\colon [n]_{\Lambda_\infty} \to [m]_{\Lambda_\infty}$ is defined as $\tau(f) = f + 1$.

Definition A.4.1. The *cyclic category* Λ is the full subcategory of Λ_∞ with the same objects as Λ_∞; the morphisms between two objects of Λ_∞ are given by the morphisms of Λ_∞ with the $\mathbb{Z}$-action by τ divided out.

It is a standard fact that the category Λ is self-dual [NS17, Appendix B]. If Δ is the category of totally ordered non-empty finite sets, there is a functor $\Delta \to \Lambda$ sending $[n] = \{0, \dots, n\} \in \Delta$ to $[n+1]_\Lambda$ inducing faces and degeneracy maps. The cyclic operator in Λ descends from the $\mathbb{Z}$ action by the generator τ described above. Then, the previous definition of Λ is equivalent to the following more explicit one:

Definition A.4.2. [Lod98, Def. 6.1.1] The *cyclic category* Λ has objects $[n]$, for every $n \in \mathbb{N}$, and morphisms generated by:

- *faces* $\delta_i\colon [n-1] \to [n]$ for $i \leq n$;
- *degeneracies* $\sigma_j\colon [n+1] \to [n]$ for $j \leq n$;
- *cyclic operators* $\tau_n\colon [n] \to [n]$;

satisfying the following relations:

(i) $\delta_j \delta_i = \delta_i \delta_{j-1}$ for $i < j$;

$\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$ for $i \leq j$;

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{for } i < j \\ \mathrm{id}_{[n]} \sigma_{j-1} & \text{for } i = j, i = j+1 \\ \delta_{i-1} \sigma_j & \text{for } i > j+1 \end{cases}$$

(ii) $\tau_n \delta_i = \delta_{i-1} \tau_{n-1}$ for $1 \leq i \leq n$ and $\tau_n \delta_0 = \delta_n$;

$\tau_n\sigma_i = \sigma_{i-1}\tau_{n+1}$ for $1 \leq i \leq n$ and $\tau_n\sigma_0 = \sigma_n\tau_{n+1}^2$;

(iii) $\tau_n^{n+1} = \mathrm{id}_{[n]}$.

Definition A.4.3. Let $\mathcal{C}$ be a category. A *cyclic object* in $\mathcal{C}$ is a functor

$$X\colon \Lambda^{\mathrm{op}} \to \mathcal{C}.$$

Morphisms of cyclic objects are given by natural transformations of functors.

Notation A.4.4. We denote by X_n the value of X at $[n]$.

There is a category of cyclic objects in $\mathcal{C}$ and morphisms of cyclic objects. When $\mathcal{C}$ is the category k-**Mod** of k-modules over a ring k, then a cyclic object in k-**Mod** is called a *cyclic module.* Hence, observe that a cyclic module C is a simplicial k-module endowed for all n with an action of the cyclic group of order $n+1$ on C_n. Morphism of cyclic modules are morphisms of simplicial k-modules that commute with the cyclic structure.

Example A.4.5. Let k be a commutative ring with identity 1 and let A be a k-algebra. Then, for each $n \in \mathbb{N}$ define

$$Z_n(A) \coloneqq A^{\otimes_k n+1}.$$

The following degeneracies s_i, face maps d_i and cyclic maps t

$$\begin{aligned}
d_i(a_0,\ldots,a_n) &= \begin{cases} (a_0,a_1,\ldots,a_ia_{i+1},\ldots,a_n) & \text{if } 0 \leq i \leq n-1 \\ (a_na_0,a_1,\ldots,a_n) & \text{if } i = n \end{cases} \\
s_i(a_0,\ldots,a_n) &= \quad (a_0,a_1,\ldots,a_i,1,a_{i+1},\ldots,a_n) \quad \text{if } 0 \leq i \leq n \\
t_{n+1}(a_0,\ldots,a_n) &= \quad (a_n,a_0,\ldots,a_{n-1})
\end{aligned}$$

make $Z_*(A)$ a cyclic module [Goo85, Sec II.1].

Example A.4.6. [McC94, Def. 2.1.1] Let $\mathcal{A}$ be an additive category and k-**Mod** be the category of k-modules, k being a ring. The previous construction generalizes to additive categories, describing a cyclic module, called the *additive cyclic nerve*,

$$\mathrm{CN}_*(\mathcal{A})\colon \Lambda^{\mathrm{op}} \to k\text{-}\mathbf{Mod}$$

that in degree n is given by:

$$\mathrm{CN}_n(\mathcal{A}) \coloneqq \bigoplus \mathrm{Hom}_{\mathcal{A}}(A_1,A_0) \otimes \mathrm{Hom}_{\mathcal{A}}(A_2,A_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(A_0,A_n)$$

where the sum runs over all the objects $(A_0,A_1,\ldots,A_n)$ in $\mathcal{A}^{n+1}$ and tensor products $\otimes$ are tensor products over k. We give a description of the face maps, degeneracies and cyclic maps:

$$d_i(f_0 \otimes \cdots \otimes f_n) = \begin{cases} f_0 \otimes f_1 \otimes \cdots \otimes f_i \circ f_{i+1} \otimes \cdots \otimes f_n & \text{if } 0 \leq i \leq n-1 \\ f_n \circ f_0 \otimes f_1 \otimes \cdots \otimes f_n & \text{if } i = n \end{cases}$$
$$s_i(f_0 \otimes \cdots \otimes f_n) = \begin{cases} f_0 \otimes f_1 \otimes \cdots \otimes f_i \otimes \mathrm{id}_{A_{i+1}} \otimes f_{i+1} \otimes \cdots \otimes f_n & \text{if } 0 \leq i \leq n-1 \\ f_0 \otimes f_1 \otimes \cdots \otimes f_n \otimes \mathrm{id}_{A_0} & \text{if } i = n \end{cases}$$
$$t(f_0 \otimes \cdots \otimes f_n) = \quad (f_n \otimes f_0 \otimes \cdots \otimes f_{n-1})$$

The additive cyclic nerve is a covariant functor from the category of small k-linear categories to the category of cyclic k-modules.

By [McC94, Ex. 2.2.1], when applied to a k-algebra A seen as a k-linear category with a single object, the additive cyclic nerve $\mathrm{CN}_*(A)$ is the cyclic module Z_*A of Example A.4.5.

We can do the same for dg-categories. For reference, we spell this out:

Definition A.4.7. The *additive cyclic nerve* of a differential graded k-linear category $\mathcal{C}$ is the cyclic k-module defined for each n by

$$\mathrm{CN}_n(\mathcal{C}) \coloneqq \bigoplus \mathrm{Hom}_{\mathcal{C}}(C_1, C_0) \otimes \mathrm{Hom}_{\mathcal{C}}(C_2, C_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(C_0, C_n)$$

where the sum runs over all the objects $(C_0, C_1, \ldots, C_n)$ in $\mathcal{C}^{n+1}$. The face and degeneracy maps, and the cyclic action, are defined as follows:

$$d_i(f_0 \otimes \cdots \otimes f_n) = \begin{cases} f_0 \otimes f_1 \otimes \cdots \otimes f_i \circ f_{i+1} \otimes \cdots \otimes f_n & \text{if } 0 \leq i \leq n-1 \\ (-1)^{n+\sigma} f_n \circ f_0 \otimes f_1 \otimes \cdots \otimes f_n & \text{if } i = n \end{cases}$$
$$s_i(f_0 \otimes \cdots \otimes f_n) = \begin{cases} f_0 \otimes f_1 \otimes \cdots \otimes f_i \otimes \mathrm{id}_{C_{i+1}} \otimes f_{i+1} \otimes \cdots \otimes f_n & \text{if } 0 \leq i \leq n-1 \\ f_0 \otimes f_1 \otimes \cdots \otimes f_n \otimes \mathrm{id}_{C_0} & \text{if } i = n \end{cases}$$
$$t(f_0 \otimes \cdots \otimes f_n) = \quad (-1)^{n+\sigma}(f_n \otimes f_0 \otimes \cdots \otimes f_{n-1})$$

where $\sigma = (\deg f_n)(\deg f_{n-1} + \cdots + \deg f_0)$.

The differential b is defined as $b = (-1)^n d_n + \sum_{i=0}^{n-1} d_i$.

Observe that the definition of additive cyclic nerve for dg-categories generalizes both Example A.4.5 and Example A.4.6. In particular, when applied to an additive category, seen as a dg-category by means of the functor $\iota\colon \mathbf{Cat}_k \to \mathbf{dgcat}_k$ (A.1), we get (up to sign) the additive cyclic nerve for additive categories of Example A.4.6 (because the maps f_i of the above definition have only degree 0).

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