Parametrised Simplicial Volume and S^1-Actions

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Introduction

The simplicial volume is a homotopy invariant of oriented closed connected manifolds introduced by Gromov [24]. Given an oriented closed connected manifold M the *simplicial volume of M* is defined by

$$\|M\| := \inf\bigl\{|c|_1 \bigm| c \text{ is an } \mathbb{R}\text{-fundamental cycle of } M\bigr\},$$

where $|\cdot|_1$ is the ℓ^1-norm on the singular chain complex $C_*(M;\mathbb{R})$ with real coefficients that is given by

$$\Bigl|\sum_{j=1}^{k} a_j \cdot \sigma_j\Bigr|_1 := \sum_{j=1}^{k} |a_j|$$

on singular chains (in reduced form).

There are interesting connections between simplicial volume and Riemannian geometry: For Riemannian manifolds the ratio of the simplicial volume to the Riemannian volume only depends on the isometry type of the Riemannian universal covering of the manifold [24, Section 2.3][44, p. 6.9][6, 17, 34, 33]. For example, this ratio for hyperbolic oriented closed connected n-manifolds is known to be $1/v_n$, where v_n is the maximal volume of an ideal geodesic n-simplex in hyperbolic n-space [24, Section 2.2][44, Theorem 6.2]. This directly implies that two homotopy equivalent closed hyperbolic manifolds have the same volume, which is the key for Gromov's alternative proof of Mostow rigidity [40]. Recall that Mostow rigidity states that every homotopy equivalence between hyperbolic oriented closed connected n-manifolds with $n \geq 3$ is homotopic to an isometry.

Another interesting connection between simplicial volume and Riemannian geometry is the following: Let M be a smooth oriented closed connected manifold M. Then, the *minimal volume of M* is given by

$$\operatorname{minvol}(M) := \inf_g \operatorname{vol}(M, g),$$

where g runs through all complete Riemannian metrics on M such that all sectional curvatures lie in the closed interval between -1 and 1 and where $\operatorname{vol}(M, g)$ denotes the Riemannian volume of M with respect to g. Gromov proved that the simplicial volume yields a lower bound for the minimal volume [24, p. 12][2, Théorème D]; more precisely for all $n \in \mathbb{N}$ there exists a constant $\text{const}_n \in \mathbb{R}_{\geq 0}$ with

$$\|M\| \leq \text{const}_n \cdot \operatorname{minvol}(M)$$

for all smooth oriented closed connected n-manifolds M. This inequality also holds (with different constants) if one replaces the simplicial volume by the L^2-Betti-numbers: For all $n \in \mathbb{N}$ there is a constant $\mathrm{const}_n \in \mathbb{R}_{\geq 0}$ such that

$$\sum_{i=0}^{n} b_i^{(2)}(M) \leq \mathrm{const}_n \cdot \mathrm{minvol}(M)$$

holds for all smooth oriented closed connected n-manifolds M. This follows from a statement of Gromov [23, Section 5.33] which has been proven in detail by Sauer [42]. For the definition and basic properties of L^2-Betti numbers we refer to the literature [38].

Whether also simplicial volume yields an upper bound for the L^2-Betti numbers is an open question which was asked by Gromov [22, p. 232].

Question 1 (Gromov's question)**.** *Let M be an aspherical oriented closed connected manifold. Does then vanishing of the simplicial volume of M imply that all L^2-Betti numbers of M are zero? In particular, do we have*

$$\|M\| = 0 \implies \chi(M) = 0\,?$$

In general, the question is wide open but there are some cases for which the answer is known to be affirmative, for example for smooth manifolds with vanishing minimal volume. One method to try to answer Gromov's question to the positive is to introduce integral approximations of the simplicial volume.

Integral Approximations of the Simplicial Volume

One integral approximation of the simplicial volume is the *stable integral simplicial volume* that is given by

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf\bigl\{ \|\overline{M}\|_{\mathbb{Z}}/d \bigm| \overline{M} \longrightarrow M \text{ is a } d\text{-sheeted covering} \bigr\}$$

for an oriented closed connected manifold M. Here, $\|\overline{M}\|_{\mathbb{Z}}$ denotes the integral simplicial volume that is defined analogously to the simplicial volume but with integral coefficients instead of real coefficients (see Section 1.1). Stable integral simplicial volume yields an upper bound for the L^2-Betti numbers because integral simplicial volume bounds the L^2-Betti numbers from above [43, Corollary 5.6] and L^2-Betti numbers are multiplicative with respect to coverings. In order to answer Gromov's question to the positive it therefore helps to investigate the connection between simplicial volume and stable integral simplicial volume.

In general, manifolds do not satisfy *integral approximation*, i.e. $\|M\|_{\mathbb{Z}}^{\infty} = \|M\|$, for example $\|S^2\|_{\mathbb{Z}}^{\infty} = \|S^2\|_{\mathbb{Z}} = 2 > 0 = \|S^2\|$, but it is reasonable to ask whether aspherical manifolds M with residually finite fundamental group satisfy integral approximation. Surfaces satisfy integral approximation [37, Example 6.2], but in this generality the question was answered to the negative: Francaviglia, Frigerio and Martelli [16, Theorem 5.2] showed that there is a gap between the simplicial volume and the stable integral simplicial volume of hyperbolic oriented closed connected manifolds of dimension ≥ 4. However, if we additionally assume $\|M\| = 0$, the question for having integral approximation is again open. We will see examples for manifolds with vanishing simplicial volume that satisfy integral approximation later.

Another example for an integral approximation of simplicial volume is the *integral foliated simplicial volume* (written as $\big| M \big|$ for a manifold M) that combines the rigidity of integral coefficients with the flexibility of $\mathbb{Z}$-valued functions on standard probability spaces that are equipped with a measure preserving action of the fundamental group of the manifold (see Section 1.2.2 for the definition). The integral foliated simplicial volume fits into the sandwich [37]

$$\|M\| \leq \big| M \big| \leq \|M\|_{\mathbb{Z}}^{\infty}$$

and it also yields an upper bound for the L^2-Betti numbers, i.e., for all $n \in \mathbb{N}$ there exists a constant const_n such that

$$\sum_{i=0}^{n} b_i^{(2)}(M) \leq \mathrm{const}_n \cdot \big| M \big|$$

holds for all oriented closed connected n-manifolds M. The latter was already suggested by Gromov [23, Remark (e) on p. 307] and verified by Schmidt [43]. Schmidt also asked whether $\|M\| = \big| M \big|$ holds for all aspherical oriented closed connected manifolds M [43, p. 65] and an affirmative answer would imply an affirmative answer to Gromov's question. Löh and Pagliantini proved the equality for hyperbolic oriented closed connected 3-manifolds [37, Theorem 1.1] and Frigerio, Löh, Pagliantini and Sauer improved this result to the statement that hyperbolic 3-manifolds satisfy integral approximation but they also adapted the arguments of Francaviglia, Frigerio and Martelli in the case of stable integral simplicial volume to show that there is also a gap between the simplicial volume and the integral foliated simplicial volume of hyperbolic oriented closed connected manifolds of dimension ≥ 4 [19, Theorem 1.8]. Again, if we additionally assume $\|M\| = 0$, Schmidt's question is open.

A Refined Version of Gromov's Question

By Schmidt's bound of the L^2-Betti numbers in terms of integral foliated simplicial volume we know that an affirmative answer to the following refined version of Gromov's question would imply an affirmative answer to Gromov's question.

Question 2 (Gromov's question refined [19, Question 1.12])**.** *Let M be an aspherical oriented closed connected manifold. Does then vanishing of the simplicial volume of M imply vanishing of the integral foliated simplicial volume $\big| M \big|$ of M? I.e., do we have*

$$\|M\| = 0 \implies \big| M \big| = 0\,?$$

An affirmative answer to the refined version of Gromov's question is known for the following examples of oriented closed connected manifolds:

- Seifert 3-manifolds with infinite fundamental group [37, Proposition 8.1],
- aspherical manifolds with amenable fundamental group [19, Theorem 1.9],
- aspherical Riemannian manifolds with vanishing minimal volume [3, Corollary 1.9],
- smooth manifolds with non-trivial smooth S^1-action such that all orbits are π_1-injective [12] (see Theorem 3),
- smooth manifolds that are the total space of a smooth S^1-bundle over a smooth manifold where all fibres are π_1-injective in the total space [13, Proposition 4.2] (see Theorem 5) and
- generalised graph manifolds in the sense of the work of Friedl, Löh and the author [13] (see Theorem 3.3.6).

Actually, all the cited results are stronger than stated above and even imply integral approximation (i.e., also $\|M\|_{\mathbb{Z}}^{\infty} = 0$) for (aspherical) manifolds with residually finite fundamental group.

Furthermore, the answer to Question 2 would be affirmative if the integral foliated simplicial volume was functorial on aspherical oriented closed connected manifolds (see Theorem 2.2.2).

Integral Foliated Simplicial Volume and S^1-Actions

In this work, we focus on the vanishing of the integral foliated simplicial volume of smooth manifolds with non-trivial S^1-action and smooth manifolds that are the total space of a smooth S^1-bundle. First, let M be a smooth oriented closed connected manifold with a non-trivial smooth S^1-action. Then, it is a classical result independently proved by Gromov [24, p. 41] and Yano [47] that the simplicial volume of M vanishes. The main purpose of this work is to show that the answer to the refined version of Gromov's question is affirmative for smooth manifolds with non-trivial S^1-action, i.e., that also the integral foliated simplicial volume of M vanishes. More precisely, we prove the following statement.

Theorem 3 (integral foliated simplicial volume and S^1-actions [12])**.** *Let M be a smooth oriented compact connected manifold (possibly with boundary) with a non-trivial smooth S^1-action. We assume that all orbits are π_1-injective in M. Then* $\left| M, \partial M \right| = 0$.

This directly implies the following corollary for aspherical manifolds using a result on the structure of non-trivial S^1-actions on aspherical closed manifolds of Lück [38, Corollary 1.43]. Note that this corollary also follows from Braun's estimate of the integral foliated simplicial volume in terms of the minimal volume [3, Corollary 5.6].

Corollary 4 ([12])**.** *Let M be an aspherical oriented closed connected smooth manifold with a non-trivial smooth S^1-action. Then $\big| M \big| = 0$.*

The idea of the proof of Theorem 3 is to adapt Yano's proof [47] in the classical setting such that it fits to the parametrised setting and also works for compact manifolds with boundary.

Since the proof idea of Theorem 3 turns out to also work for smooth manifolds that are the total space of a smooth S^1-bundle, we get the following result.

Theorem 5 (integral foliated simplicial volume of S^1-bundles [13, Prop. 4.2])**.** *Let M be a smooth oriented compact connected manifold that is the total space of a smooth S^1-bundle $p\colon M \longrightarrow B$ over a smooth oriented compact connected manifold B. We assume that all fibres are π_1-injective in M. Then we have $\big| M, \partial M \big| = 0$.*

Actually, we will prove stronger versions of the stated results (i.e., Theorem 3, Corollary 4 and Theorem 5): In the situation of these results we even have that if $\pi_1(M)$ is residually finite then also $\|M, \partial M\|_{\mathbb{Z}}^{\infty} = 0$ holds (see Corollary 4.4.1).

Organisation of this Work

In Chapter 1, we recall the definition of simplicial volume and integral foliated simplicial volume. More precisely, we introduce normed chain complexes and define the induced ℓ^1-semi-norm in homology in Section 1.1, which leads to various simplicial volumes. In Section 1.2, we define normed chain complexes with twisted coefficients and eventually introduce the integral foliated simplicial volume. Furthermore, we discuss the local criterion for parametrised (relative) fundamental cycles. Finally, we recall the definition of singular homology with local coefficients in Section 1.3. This yields an alternative description of integral foliated simplicial volume, which we will use in the proofs of Theorem 3 and Theorem 5.

Chapter 2 is all about functoriality of some invariants related to the simplicial volume and whether and how functoriality affects the relation to simplicial volume. More precisely, in Section 2.1, we show that there exist finite functorial semi-norms on singular homology with real coefficients that are exotic in the sense that they are not carried by the ℓ^1-semi-norm. Afterwards, in Section 2.2, we prove that if the integral foliated simplicial volume was functorial on aspherical manifolds, then it would carry the simplicial volume on aspherical manifolds, i.e., the answer to Question 2 would be positive.

Then, in Chapter 3, we begin by recalling the uniform boundary condition (UBC) and classical results by Matsumoto and Morita [39] in Section 3.1. In Section 3.2, we repeat the proof of Löh and the author [14] that the parametrised chain complex of tori with respect to an essentially free parameter space satisfies parametrised UBC, which we need for the proofs of Theorem 3 and Theorem 5. At the end of this chapter, we state more results of Löh and the author [14] and exhibit glueing formulas for parametrised simplicial volumes as an application of the uniform boundary condition (see Section 3.3).

Finally, in Chapter 4 we prove Theorem 3 and Theorem 5. The proofs can be decomposed into two parts: The deconstruction step and the filling step. We give a unified proof for the filling step in both cases in Section 4.1. Afterwards, we discuss the deconstruction step for the case of S^1-actions in Section 4.2 and the deconstruction step for the case of S^1-bundles in Section 4.3. In the end, we give some applications of Theorem 3 and Theorem 5 in Section 4.4.

In Appendix A we recall basic definitions and results on smooth compact Lie group actions on smooth compact connected manifolds. In particular we summarise the properties of smooth S^1-actions that we need in the proof of Theorem 3.

Then, in Appendix B we define hollowings and manifolds with corners in detail, that are the main technical tools needed for the deconstruction step in both of the the proofs of Theorem 3 and Theorem 5.

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Contents

1. Simplicial Volumes

Given a triangulable oriented closed connected manifold, one way of measuring its "complexity" is by the minimal number of simplices needed to triangulate the manifold. In the late 70's, Gromov came up with the idea to relaxate this to "triangulations" by singular simplices and to count the arising simplices with weights, more precisely to sum the absolute values of the weights of the simplices in a singular chain with real coefficients that represents the real fundamental class of the manifold in homology. Taking the infimum of this number for all real fundamental cycles of a manifold gives rise to a homotopy invariant called *simplicial volume* (also *Gromov invariant* or *Gromov norm*) [24].

Replacing the reals $\mathbb{R}$ by other coefficients like the integers $\mathbb{Z}$ leads to other versions of simplicial volume like *integral simplicial volume*, which are introduced in Section 1.1.

Also twisted coefficients can be used to define variants of the simplicial volume. This leads to the definition of integral foliated simplicial volume, which is explained in Section 1.2.

In Section 1.3, we explain how to define simplicial volumes via local coefficients and we will obtain an alternative description of integral foliated simplicial volume.

1.1. Normed Chain Complexes and Simplicial Volumes

In this section, we recall normed chain complexes and induced semi-norms on homology. We will directly work with the relative singular chain complex to obtain the definition of the relative simplicial volume. We follow the work of Löh and the author [14, Section 2].

Definition 1.1.1 ((semi-)normed abelian group)**.** A *normed abelian group* is an abelian group A together with a *norm on* A, i.e., a map $|\cdot|\colon A \longrightarrow \mathbb{R}_{\geq 0}$ with the following properties:

1. We have $|x| = 0$ if and only if $x = 0$ for all $x \in A$.
2. For all $x, y \in A$, we have $|x + y| \leq |x| + |y|$.
3. For all $x \in A$, we have $|-x| = |x|$.

If the map $|\cdot|$ does not satisfy the first condition but the condition $|0| = 0$ and 2. and 3., we call $|\cdot|$ a *semi-norm on* A and A a *semi-normed abelian group*.

A homomorphism $f\colon A \longrightarrow B$ between (semi-)normed abelian groups $(A, |\cdot|_A)$ and $(B, |\cdot|_B)$ is called *bounded*, if there exists a constant $C \in \mathbb{R}_{\geq 0}$ such that for all $x \in A$ we have $|f(x)|_B \leq C \cdot |x|_A$.

Definition 1.1.2 ((semi-)normed chain complex)**.** A *(semi-)normed chain complex (of abelian groups)* is a chain complex in the category of (semi-)normed abelian groups together with bounded group homomorphisms.

Remark 1.1.3. A (semi-)normed chain complex is nothing but a chain complex in the category of abelian groups, where we endow every chain group with a (semi-)norm such that all boundary maps are bounded.

Example 1.1.4 (ℓ^1-norm on the singular chain complex). Let A be an abelian group with norm $|\cdot|_A$. For a topological space X, the singular chain complex $C_*(X;A)$ becomes a normed chain complex via the *ℓ^1-norm* that is given by

$$\begin{aligned} |\cdot|_{1,A}\colon C_n(X;A) &\longrightarrow \mathbb{R}_{\geq 0} \\ \sum_{i=1}^{k} a_i \cdot \sigma_i &\longmapsto \sum_{i=0}^{k} |a_i|_A \end{aligned}$$

for all $n \in \mathbb{N}$. Here, we assume that $\sum_{i=1}^{k} a_i \cdot \sigma_i$ is in *reduced form*, i.e., $\sigma_i \neq \sigma_j$ if $i \neq j$. Note that chain maps that are induced by continuous maps are norm non-increasing.

If $Y \subset X$ is a subspace, the quotient norm

$$\begin{aligned} C_n(X,Y;A) = C_n(X;A)/C_n(Y;A) &\longrightarrow \mathbb{R}_{\geq 0} \\ c + C_n(Y;A) &\longmapsto \inf\bigl\{|c-d|_{1,A} \bigm| d \in C_n(Y;A) \subset C_n(X;A)\bigr\} \end{aligned}$$

on $C_*(X,Y;A)$ induced by the ℓ^1-norm is also denoted by $|\cdot|_{1,A}$.

Definition 1.1.5 (induced semi-norm on homology)**.** Let (C_*, ∂_*) be a normed chain complex. Then the *induced semi-norm in homology* is given by

$$\begin{aligned} H_n(C_*, \partial_*) &\longrightarrow \mathbb{R}_{\geq 0} \\ \alpha &\longmapsto \inf\bigl\{|c| \bigm| c \in C_n, \partial c = 0, [c] = \alpha\bigr\} \end{aligned}$$

for all $n \in \mathbb{N}$.

Example 1.1.6 (ℓ^1-semi-norm on singular homology). Let A be a normed abelian group. For a pair of topological spaces (X,Y), the ℓ^1-norm on the singular chain complex $C_*(X,Y;A)$ induces a semi-norm on singular homology $H_*(X,Y;A)$, the *ℓ^1-semi-norm*, which we denote by $\|\cdot\|_{1,A}$.

Definition 1.1.7 (simplicial volumes)**.** Let M be an oriented compact connected manifold. Let A be a normed abelian group together with a group homomorphism $i\colon \mathbb{Z} \longrightarrow A$. Then the *$A$-simplicial volume of M* is defined by

$$\|M, \partial M\|_A := \|[M, \partial M]_A\|_{1,A} \,,$$

where $[M, \partial M]_A$ denotes the image of $[M, \partial M]$ under the change of coefficients map $H_n(M, \partial M; \mathbb{Z}) \longrightarrow H_n(M, \partial M; A)$ induced by i. If $\partial M = \emptyset$, we write

$$\|M\|_A := \|M, \partial M\|_A.$$

The definition of A-simplicial volumes in general depends on the group homomorphism i. For $A = \mathbb{R}$ and i the inclusion $\mathbb{Z} \subset \mathbb{R}$ the definition of the A-simplicial volume coincides with the definition of the simplicial volume in the introduction. We also call $\mathbb{Z}$-simplicial volume (with i the identity) *integral simplicial volume*.

1.2. Simplicial Volume and Twisted Coefficients

In this section, we introduce simplicial volumes that are defined via singular homology with twisted coefficients. We mainly follow the work of Löh and the author [14, Section 2.1].

In Section 1.2.1, we recall the definition of singular homology with twisted coefficients and the corresponding twisted ℓ^1-norm and we explain how we get simplicial volumes out of this. Using special twisted simplicial volumes we introduce integral foliated simplicial volume in Section 1.2.2. Finally, we prove a local criterion in Section 1.2.3 which characterises parametrised fundamental cycles.

1.2.1. Singular Homology with Twisted Coefficients and Simplicial Volumes

We now recall the definition of singular homology with twisted coefficients together with the twisted ℓ^1-norm and obtain twisted simplicial volumes in the end. Actually, we repeat the procedure of Section 1.1 with the difference that we replace the abelian group A by a $\mathbb{Z}\Gamma$-module for a countable group Γ. Here, $\mathbb{Z}\Gamma$ denotes the group ring of Γ over $\mathbb{Z}$.

Definition 1.2.1 (normed $\mathbb{Z}\Gamma$-module)**.** Let Γ be a group. A *normed right-$\mathbb{Z}\Gamma$-module* is a right-$\mathbb{Z}\Gamma$-module A together with a semi-norm on A (as an abelian group) that is Γ-*invariant*, i.e., $|a \cdot \gamma| = |a|$ for all $a \in A$ and all $\gamma \in \Gamma$ (or equivalently, the homomorphism $\cdot\gamma\colon A \longrightarrow A$ is norm non-increasing for all $\gamma \in \Gamma$).

Definition 1.2.2 (singular homology with twisted coefficients)**.** We say that a topological space M *admits a universal covering*, if it is connected, locally path-connected and semi-locally simply connected. Let M be a topological space that admits a universal covering $q\colon \widetilde{M} \longrightarrow M$ and let $\Gamma := \pi_1(M)$. Let A be a normed right-$\mathbb{Z}\Gamma$-module. Then

$$C_*(M; A) := A \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z})$$

together with the boundary maps $\mathrm{id}_A \otimes \partial_*$ becomes a semi-normed chain complex of abelian groups via the ℓ^1-semi-norm induced by the semi-norm on A, i.e., the semi-norm

$$\begin{aligned} |\cdot|_{1,A} \colon C_n(M;A) &\longrightarrow \mathbb{R}_{\geq 0} \\ \sum_{i=1}^{k} a_i \otimes \sigma_i &\longmapsto \sum_{i=1}^{k} |a_i|_A \,. \end{aligned}$$

Here, we assume that $\sum_{i=1}^{k} a_i \otimes \sigma_i$ is in *reduced form*, i.e., $q \circ \sigma_i \neq q \circ \sigma_j$ if $i \neq j$ and we consider the left-action of Γ on $C_n(\widetilde{M};\mathbb{Z})$ induced by the deck transformation action of Γ on $\widetilde{M}$.

Let $U \subset M$ be a subspace. Then we define

$$C_*^M(U;A) := A \otimes_{\mathbb{Z}\Gamma} C_*\bigl(q^{-1}(U);\mathbb{Z}\bigr)$$

and the relative chain complex

$$C_*(M,U;A) := C_*(M;A)/C_*^M(U;A) \cong A \otimes_{\mathbb{Z}\Gamma} C_*\bigl(\widetilde{M}, q^{-1}(U);\mathbb{Z}\bigr).$$

Here, the last isomorphism follows from compatibility of tensor products with quotients and the definition of the boundary operators. Then, $C_*(M,U;A)$ together with the induced boundary maps becomes a semi-normed chain complex of abelian groups via the quotient norm of $|\cdot|_{1,A}$. Let

$$H_*(M,U;A) := H_*\bigl(C_*(M,U;A)\bigr);$$

we write $\|\cdot\|_{1,A}$ for the induced semi-norm on $H_*(M,U;A)$.

Remark 1.2.3. Let A be a right-$\mathbb{Z}\Gamma$-module with trivial Γ-action. Then for all topological spaces M that admit a universal covering $q\colon \widetilde{M} \longrightarrow M$ there are mutually inverse homomorphisms

$$\begin{aligned} A \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z}) &\longrightarrow A \otimes_{\mathbb{Z}} C_n(M;\mathbb{Z}) \\ a \otimes \widetilde{\sigma} &\longmapsto a \otimes q \circ \widetilde{\sigma} \end{aligned} \qquad \begin{aligned} A \otimes_{\mathbb{Z}} C_n(M;\mathbb{Z}) &\longrightarrow A \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M};\mathbb{Z}) \\ b \otimes \tau &\longmapsto b \otimes \widetilde{\tau}, \end{aligned}$$

where $\widetilde{\tau}$ is a q-lift of τ. In the following, we will identify these two expressions and write $C_n(M;A)$ for both of them. We act analogously in the relative case.

Definition 1.2.4 (simplicial volume with twisted coefficients)**.** Let M be an oriented compact connected n-manifold with fundamental group Γ. Let A be a normed right-$\mathbb{Z}\Gamma$-module together with a $\mathbb{Z}\Gamma$-linear map $i\colon \mathbb{Z} \longrightarrow A$ (where we consider the trivial Γ-action on $\mathbb{Z}$). Then the *relative A-simplicial volume of M* is given by

$$\|M,\partial M\|_A := \|[M,\partial M]_A\|_{1,A},$$

where $[M,\partial M]_A$ denotes the image of $[M,\partial M]$ under the change of coefficients map $H_n(M,\partial M;\mathbb{Z}) \longrightarrow H_n(M,\partial M;A)$ induced by i. If $\partial M = \emptyset$, we write

$$\|M\|_A := \|M,\partial M\|_A.$$

1.2.2. Integral Foliated Simplicial Volume

We are now prepared to define parametrised simplicial volumes as a special case of simplicial volumes with twisted coefficients. This leads to the definition of integral foliated simplicial volume.

We first introduce the twisted modules that we work with and fix some notation, which we will use in the following.

Definition 1.2.5 (standard Γ-space, essentially free)**.** Let Γ be a countable group. A *standard* Γ *space*

$$\alpha = \Gamma \curvearrowright (X, \mu)$$

is a standard Borel probability space (X, μ) (e.g., the unit interval with the Lebesgue measure) together with a measurable left-Γ-action which is measure preserving.

A standard Γ-space is said to be *essentially free* if the Γ-action is free almost everywhere.

For the definition of standard Borel probability spaces we refer to the literature [29].

Let Γ be a countable group and let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-space. We write $L^\infty(\alpha, \mathbb{Z}) := L^\infty(X, \mu, \mathbb{Z})$ for the quotient of the abelian group of all bounded measurable maps $X \longrightarrow \mathbb{Z}$ by the subgroup of all bounded measurable maps $X \longrightarrow \mathbb{Z}$ that are zero almost everywhere. Here, we equip $\mathbb{Z}$ with the discrete σ-algebra.

Definition 1.2.6. Let M be a topological space that admits a universal covering and such that $\Gamma := \pi_1(M)$ is countable. Let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-probability space. Then $L^\infty(\alpha, \mathbb{Z})$ becomes a normed right-$\mathbb{Z}\Gamma$-module via the right-Γ-action that is given by

$$(f \cdot \gamma)(x) := f(\gamma \cdot x)$$

for all $\gamma \in \Gamma$ and all $x \in X$ and the Γ-invariant semi-norm

$$\begin{aligned} L^\infty(\alpha, \mathbb{Z}) &\longrightarrow \mathbb{R}_{\geq 0} \\ f &\longmapsto \int_X |f| \, d\mu. \end{aligned}$$

We write $A := L^\infty(\alpha, \mathbb{Z})$. The Γ-invariant semi-norm induces an ℓ^1-semi-norm on the so-called *α-parametrised chain complex* $C_*(M; \alpha) := C_*(M; A)$ and a semi-norm $|\cdot|_{1,\alpha} := |\cdot|_{1,L^\infty(\alpha,\mathbb{Z})}$ in homology $H_*(M; \alpha) := H_*(M; A)$ as it is explained in Definition 1.2.2 (and analogously for the relative versions). We have a canonical inclusion of $\mathbb{Z}$ into $L^\infty(\alpha, \mathbb{Z})$ as the constant functions.

Definition 1.2.7 ((relative) parametrised fundamental cycle)**.** Let M be an oriented compact connected n-manifold with fundamental group Γ. Let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-space. Then we also write $[M, \partial M]^\alpha := [M, \partial M]_{L^\infty(\alpha,\mathbb{Z})}$ for the so-called *relative α-parametrised fundamental class of M* (that is the image of $[M, \partial M]$ under the change of coefficients map $H_n(M, \partial M; \mathbb{Z}) \longrightarrow H_n(M, \partial M; \alpha)$ induced by the inclusion of $\mathbb{Z}$ into $L^\infty(\alpha, \mathbb{Z})$). If $\partial M = \emptyset$, we also write $[M]^\alpha := [M, \partial M]^\alpha$. A cycle in $C_n(M, \partial M; \alpha)$ representing $[M, \partial M]^\alpha$ is called a *relative α-parametrised fundamental cycle of M*.

Definition 1.2.8 (relative integral foliated simplicial volume)**.** Let M be an oriented compact connected n-manifold with fundamental group Γ. Then the *relative integral foliated simplicial volume of M* is given by

$$\left| M, \partial M \right| := \inf_\alpha \left| M, \partial M \right|^\alpha,$$

where α runs through all standard Γ-spaces and

$$\left| M, \partial M \right|^\alpha := \| [M, \partial M]^\alpha \|_{1,\alpha}$$

is also called the *relative α-parametrised simplicial volume of M*.

If $\partial M = \emptyset$, we also write

$$\left| M \right|^\alpha := \left| M, \partial M \right|^\alpha \quad \text{and} \quad \left| M \right| := \left| M, \partial M \right| .$$

1.2.3. The Local Criterion for Parametrised Fundamental Cycles

In this section, we prove that for a parametrised cycle being a parametrised fundamental cycle is a local property by introducing a local criterion, which characterises parametrised fundamental cycles (see Proposition 1.2.15). We follow the work of the author [12, Section 3.3].

Let M be an oriented compact connected n-manifold and let $U \subset M$ be an embedded open n-ball in M such that the closure $\overline{U}$ lies in the interior M° of M. In the classical setting of $\mathbb{Z}$-coefficients we have isomorphisms

$$H_n(M, \partial M; \mathbb{Z}) \xrightarrow{\cong} H_n(M, M \setminus U; \mathbb{Z}) \xleftarrow{\cong} H_n(\overline{U}, \partial \overline{U}; \mathbb{Z})$$

that are induced by the according inclusions. By definition, the images of $[\overline{U}, \partial \overline{U}]_\mathbb{Z}$ and $[M, \partial M]_\mathbb{Z}$ coincide in $H_n(M, M \setminus U; \mathbb{Z})$. This gives us a local criterion for relative fundamental cycles of M in the following sense: Let $c \in C_n(M, \partial M; \mathbb{Z})$ be a cycle. Then c is a relative fundamental cycle of M if and only if c represents the image of $[\overline{U}, \partial \overline{U}]_\mathbb{Z}$ in $H_n(M, M \setminus U)$.

In practice, this criterion can be used for example as follows: We start with a relative fundamental cycle c of M. If we perform some changes to c that do not affect the simplices of c that have non-trivial intersection with U in a way that we obtain a relative cycle $c' \in C_n(M, \partial M; \mathbb{Z})$, then we know that $c' = c$ holds in $C_n(M, M \setminus U; \mathbb{Z})$. Therefore, by the local criterion, also c' is a relative fundamental cycle of M.

Keeping this in mind, we will now prove a local criterion for parametrised relative fundamental cycles.

Definition 1.2.9 ((co)invariants)**.** Let Γ be a group and let A be a right-$\mathbb{Z}\Gamma$-module. Then the Γ-*invariants* and Γ-*coinvariants* are defined by

$$A^\Gamma := \{x \in A \mid \forall_{g\in\Gamma}\ x \cdot g = x\} \quad \text{and} \quad A_\Gamma := A/B,$$

where B is the subgroup of A generated by $\{a \cdot g - a \mid g \in \Gamma, a \in A\}$. Note that the inherited Γ-actions on A^Γ and A_Γ are trivial.

Lemma 1.2.10. *Let Γ be a countable group. Let $\alpha = \Gamma \curvearrowright (X, \mu)$ be a standard Γ-space. We write $A := L^\infty(\alpha, \mathbb{Z})$. Then, the canonical homomorphism $A^\Gamma \longrightarrow A_\Gamma$ from the Γ-invariants to the Γ-coinvariants is injective.*

Proof. Let $f \in A^\Gamma$ be non-trivial. Then there exists a Γ-invariant measurable subset $Y \subset X$ with positive measure such that

$$f|_Y \geq 1 \quad \text{or} \quad f|_Y \leq -1.$$

Integration over the Γ-invariant set Y yields a map

$$\int_Y \cdot \, d\mu \colon A^\Gamma \longrightarrow \mathbb{R},$$

which factors through the canonical map $A^\Gamma \longrightarrow A_\Gamma$ since μ is Γ-invariant. Now, assume for a contradiction that $[f] = 0$ in A_Γ. Then

$$0 = \left| \int_Y f \, d\mu \right| \geq \mu(Y) > 0,$$

which is impossible. □

Lemma 1.2.11. *Let M be an oriented compact connected manifold with fundamental group Γ. Let A be a right $\mathbb{Z}\Gamma$-module. Then we have*

$$H^0(M; A) \cong A^\Gamma$$

as $\mathbb{Z}$-modules.

Proof. Recall the definition of the cochain complex with twisted coefficients:

$$C^*(M; A) := \operatorname{Hom}_{\mathbb{Z}\Gamma}\big(C_*(\widetilde{M}; \mathbb{Z}), A\big);$$

here, we consider A as a left $\mathbb{Z}\Gamma$-module via the involution on Γ, i.e.,

$$\gamma \cdot a := a \cdot \gamma^{-1}.$$

The coboundary operator is given by $\delta^*(f) := f \circ \partial_*$ for all $f \in C^*(M; A)$. Then we have $H^0(M; A) = \ker \delta^0$. We define a map $\varphi \colon A^\Gamma \longrightarrow H^0(M; A)$ as follows: For all $a \in A^\Gamma$ we define

$$\begin{aligned} \varphi(a) \colon C_0(\widetilde{M}; \mathbb{Z}) &\longrightarrow A \\ \widetilde{M} \ni x &\longmapsto a. \end{aligned}$$

Then, $\varphi(a)$ is indeed a $\mathbb{Z}\Gamma$-homomorphism since a is Γ-invariant and $\varphi(a) \in \ker \delta^0$, because for all $\eta \in \operatorname{map}(\Delta^1, \widetilde{M})$ we have

$$\delta(\varphi(a))(\eta) = \varphi(a)(\partial\eta) = \varphi(a)(\eta(e_1) - \eta(e_0)) = a - a = 0.$$

Here, e_i denotes the i-th vertex of the standard n-simplex in $\mathbb{R}^{n+1}$.

Since φ is obviously injective, it is left to show that φ is also surjective. Let $f \in \ker \delta^0$. Then f is constant on the points of $\widetilde{M}$ because $\widetilde{M}$ is path-connected: Let $x, y \in \widetilde{M}$. Then there exists a path η from x to y and we have

$$0 = \delta(f)(\eta) = f \circ \partial\eta = f(y) - f(x).$$

Moreover, the value of f on points in $\widetilde{M}$ is Γ-invariant, since

$$f(x) = f(\gamma^{-1} \cdot x) = \gamma^{-1} \cdot f(x) = f(x) \cdot \gamma$$

for all $x \in \widetilde{M}$ and all $\gamma \in \Gamma$. Therefore, φ is surjective and the claim follows. □

Lemma 1.2.12. *Let M be an oriented compact connected manifold and let Γ be its fundamental group. Let A be a right $\mathbb{Z}\Gamma$-module. Let F be the composition*

$$A^\Gamma \xrightarrow{\varphi} H^0(M; A) \xrightarrow{PLD} H_n(M, \partial M; A) \xrightarrow{j} H_n(M, \partial M; A_\Gamma) \xrightarrow{\psi} A_\Gamma,$$

where φ is the isomorphism defined in the proof of Lemma 1.2.11, PLD is the Poincaré-Lefschetz-Duality isomorphism with twisted coefficients, j is the change of coefficients map induced by the projection $A \longrightarrow A_\Gamma$ and ψ is the classical isomorphism that comes from Poincaré-Duality (with untwisted coefficients).

Then, F equals the canonical projection $A^\Gamma \longrightarrow A_\Gamma$, i.e., the upper right square in the diagram Figure 1.1 commutes.

Proof. Recall the definition of the cap-product:

$$\begin{aligned} \cdot \cap \cdot : C^0(M; A) \times C_n(M; \mathbb{Z}) &\longrightarrow C_n(M; A) \\ (\varphi(a), \sigma\colon \Delta^n \longrightarrow M) &\longmapsto \varphi(a)({}_0\lfloor\widetilde{\sigma}) \otimes \widetilde{\sigma}\rfloor_n = a \otimes \widetilde{\sigma}, \end{aligned}$$

where $\widetilde{\sigma}$ is a lift of σ to the universal covering of M and $\widetilde{\sigma}\rfloor_n := \widetilde{\sigma}[0, \dots, n] = \widetilde{\sigma}$ and ${}_0\lfloor\widetilde{\sigma} := \sigma[n-0, \dots, n] = \sigma[n]$; here we write

$$\begin{aligned} [j_1, \dots, j_k]\colon \Delta^k &\longrightarrow \Delta^n \\ (t_0, \dots t_k) &\longrightarrow \sum_{i=0}^{k} t_i \cdot e_{j_i} \end{aligned}$$

for all $k \leq n$ and $j_1, \dots, j_k \in \{0, \dots, k\}$. Let $c \in C_n(M; \mathbb{Z})$ be a relative fundamental cycle of M. Then the above composition of maps is given by

$$a \longmapsto \varphi(a) \longmapsto [\varphi(a) \cap c] = [a \otimes \widetilde{c}] \longmapsto \big[[a] \otimes c\big] \longmapsto [a]$$

for all $a \in A^\Gamma$, where $\widetilde{c}$ is a lift of c. Since we work with Γ-invariants above, the maps do not depend on the choice of the lifts. □

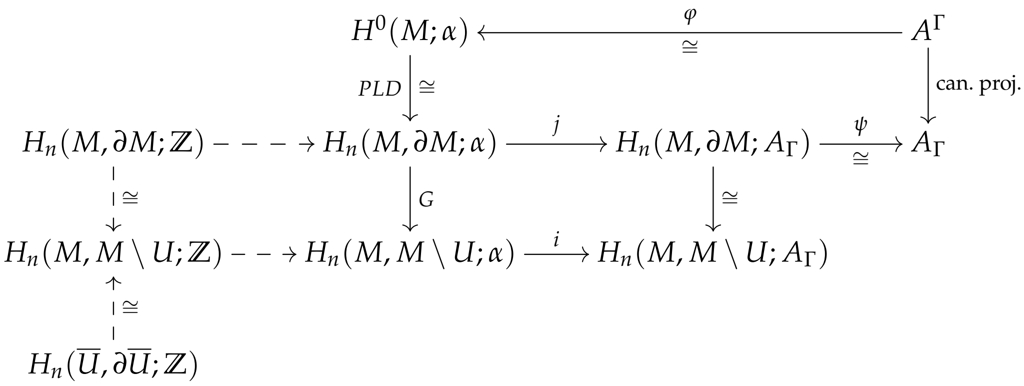

Figure 1.1.: Proving the local criterion for parametrised fundamental classes

Corollary 1.2.13. *Let M be an oriented compact connected n-manifold and let $\Gamma := \pi_1(M)$. Let $U \subset M$ be an embedded open n-ball in M such that the closure $\overline{U}$ lies in the interior M° of M. Let $\alpha = \Gamma \curvearrowright (X,\mu)$ be a standard Γ-space and let $q\colon \widetilde{M} \longrightarrow M$ denote the universal covering of M.*

Then, the map

$$G\colon H_n(M,\partial M;\alpha) \longrightarrow H_n(M,M\setminus U;\alpha)$$

induced by the inclusion $(M,\partial M) \subset (M,M\setminus U)$ is injective.

Proof. We write $A := L^\infty(\alpha,\mathbb{Z})$. We consider the composition F of homomorphisms

$$H_n(M,\partial M;\alpha) \xrightarrow{G} H_n(M,M\setminus U;\alpha) \xrightarrow{i} H_n(M,M\setminus U;A_\Gamma),$$

where i is the change of coefficients homomorphism induced by the canonical projection $A \longrightarrow A_\Gamma$. Analogously to the classical local criterion, the homomorphism $H_n(M,\partial M;A_\Gamma) \longrightarrow H_n(M,M\setminus U;A_\Gamma)$ that is induced by the inclusion is an isomorphism. Then F is (up to isomorphism) the canonical projection $A^\Gamma \longrightarrow A_\Gamma$ by Lemma 1.2.12 and the commutativity of the right part of Figure 1.1. Therefore, it follows from Lemma 1.2.10 that F is injective and thus, also G is injective. □

Definition 1.2.14 (local parametrised fundamental cycles)**.** Let M be an oriented compact connected n-manifold and let $\Gamma := \pi_1(M)$. Let $U \subset M$ be an embedded open n-ball in M such that $\overline{U} \subset M^\circ$. Let $\alpha = \Gamma \curvearrowright (X,\mu)$ be a standard Γ-space.

A relative cycle $c \in C_n(M,\partial M;\alpha)$ is called a *U-local relative α-parametrised fundamental cycle of M* if c represents the image of the class $[\overline{U},\partial\overline{U}]_\mathbb{Z}$ in $H_n(M,M\setminus U;\alpha)$.

Proposition 1.2.15 (local criterion for parametrised fundamental cycles)**.** *Let M be an oriented compact connected n-manifold and let $\Gamma := \pi_1(M)$. Let $U \subset M$ be an embedded open n-ball in M such that $\overline{U} \subset M^\circ$. Let $\alpha = \Gamma \curvearrowright (X,\mu)$ be a standard Γ-space and let $c \in C_n(M,\partial M;\alpha)$ be a relative cycle. Then the following are equivalent:*

- *The relative cycle c is a relative α-parametrised fundamental cycle of M.*
- *The relative cycle c is a U-local relative α-parametrised fundamental cycle of M.*

Proof. This directly follows from Corollary 1.2.13 and the fact that the images of the classes $[M, \partial M]_{\mathbb{Z}}$ and $[\overline{U}, \partial\overline{U}]_{\mathbb{Z}}$ in $H_n(M, M \setminus U; \mathbb{Z})$ coincide (see Figure 1.1). □

1.3. Simplicial Volume and Local Coefficients

In this section, we recall singular homology with local coefficients. We will obtain an alternative way for computing the integral foliated simplicial volume via local coefficients. We will need this characterisation later: In Chapter 4, we will mainly work with local coefficients, because then we can avoid dealing with the universal covering and we can work with non-connected spaces. We mainly follow the work of Friedl, Löh, and the author [13] in this section.

More precisely, we recall the definition of singular homology with local coefficients in Section 1.3.1. Then, in Section 1.3.2, we compare singular homology with twisted coefficients to singular homology with local coefficients. In Section 1.3.3, we give an alternative definition of integral foliated simplicial volume via local coefficients and finally, in Section 1.3.4, we state the local criterion from Section 1.2.3 in terms of local coefficients.

1.3.1. Singular Homology with Local Coefficients

In this section, we recall the definition of singular homology with local coefficients and the corresponding ℓ^1-semi-norm.

Definition 1.3.1 (normed local system)**.** Let X be a topological space. A *normed local (coefficient) system on* X is a functor $L\colon \pi(X) \longrightarrow \mathrm{Mod}^{\mathrm{sn}}_{\mathbb{Z}}$.

Here, $\pi(X)$ denotes the *fundamental groupoid* of X, i.e., the category

- whose set of objects is X,
- for all $x, y \in X$, the set of morphisms from x to y is given by all homotopy classes $[\gamma]_*$ of paths $\gamma\colon x \to y$ from x to y with respect to homotopies relative to the start and endpoints of the paths
- and the composition is given by the concatenation $\cdot * \cdot$ of paths (representing the corresponding homotopy classes).

Moreover, $\mathrm{Mod}^{\mathrm{sn}}_{\mathbb{Z}}$ is the category of all semi-normed $\mathbb{Z}$-modules together with semi-norm non-increasing $\mathbb{Z}$-linear maps.

Example 1.3.2. Let X be a path-connected topological space, let $x_0 \in X$. Then we have the following correspondence between normed right-$\pi_1(X, x_0)$-modules and normed local coefficient systems on X.

Let A be a normed right-$\pi_1(X, x_0)$-module. Then we define the corresponding normed local system L_A on X as follows: For all $x \in X$, we choose a path γ_x from x_0 to x, where we set $\gamma_{x_0} := \mathrm{const}_{x_0}$. Then we set $L_A(x) := A$ for all $x \in X$ and

$$\begin{aligned} L_A([\gamma\colon x \to y]_*)\colon A &\longrightarrow A \\ a &\longmapsto a \cdot [\gamma_x * \gamma * \gamma_y^{-1}]_*. \end{aligned}$$

Here, $\cdot^{-1}$ indicates that we reverse the corresponding path.

Conversely, if $L\colon \pi(X) \longrightarrow \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{sn}}$ is a normed local system on X then the normed $\mathbb{Z}$-module $L(x_0)$ becomes a normed right-$\pi_1(X, x_0)$-module via

$$x \cdot [\gamma]_* := L([\gamma]_*)(x)$$

for all $x \in L(x_0)$ and all $[\gamma]_* \in \pi_1(X, x_0)$. Indeed, we have

$$\begin{aligned} (x \cdot [\gamma]_*) \cdot [\eta]_* &= L([\eta]_*)\big(L([\gamma]_*)(x)\big) \\ &= L([\eta]_*) \circ L([\gamma]_*)(x) \\ &= L([\gamma * \eta]_*)(x) = x \cdot ([\gamma]_* \cdot [\eta]_*). \end{aligned}$$

We have $L_A(x_0) = A$ as normed right-$\pi_1(X, x_0)$-modules and $L_{L(x_0)} \cong L$ as normed local coefficient systems on X. The first statement is immediate, for the second we consider the following mutually inverse natural transformations

$$T\colon L \Longrightarrow L_{L(x_0)} \qquad\qquad S\colon L_{L(x_0)} \Longrightarrow L$$

given by (norm non-increasing) mutually inverse homomorphisms

$$T(x) := L([\gamma_x^{-1}]_*)\colon L(x) \longrightarrow L(x_0) \qquad S(x) := L([\gamma_x]_*)\colon L(x_0) \longrightarrow L(x)$$

for all $x \in X$.

Definition 1.3.3 (singular homology with local coefficients)**.** Let X be a topological space and let L be a normed local system on X. Then the *normed singular chain complex of X with local coefficients in L* is given by

$$C_*(X; L) := \bigoplus_{\sigma \in \mathrm{map}(\Delta^*, X)} L\big(\sigma(e_0)\big) \cdot \sigma$$

with boundary maps

$$\begin{aligned} C_n(X; L) &\longrightarrow C_{n-1}(X; L) \\ a \cdot \sigma &\longmapsto L\big(\sigma[0,1]\big)(a) \cdot \partial^0 \sigma + \sum_{i=1}^{n} (-1)^i \cdot a \cdot \partial^i \sigma \end{aligned}$$

and the ℓ^1-semi-norm $|\cdot|_{1,L}$ induced by L. Here, $\partial^i \sigma := \sigma[0, \ldots, i-1, i+1, \ldots, n]$ denotes the i-th face of σ for all $i \in \{0, \ldots, n\}$. If Y is a subspace of X, we write

$$C_*^X(Y; L) := \big\{c \in C_*(X; L) \bigm| \mathrm{supp}(c) \subset Y\big\}$$

and the *normed singular chain complex of X relative to Y with local coefficients in L* is defined by

$$C_*(X,Y;L) := C_*(X;L)/C_*^X(Y;L)$$

together with the quotient norm of $|\cdot|_{1,L}$ (which we again denote by $|\cdot|_{1,L}$). Here, for a chain $c = \sum_{i=1}^k a_i \cdot \sigma_i \in C_n(X;L)$ we define the *support of c* by

$$\operatorname{supp}(c) := \bigcup_{i=1}^{k} \operatorname{im}(\sigma_i)$$

Moreover, we define the *restriction of c to Y* by

$$c|_Y := \sum_{j \in J} a_j \cdot \sigma_j \in C_n(Y; L|_Y) \subset C_n(X;L),$$

where $J \subset \{1,\dots,k\}$ is the set of all indices j with $\operatorname{im}(\sigma_j) \subset Y$ and $L|_Y$ is the restriction of L along the inclusion $Y \subset X$, i.e.,

$$L|_Y := L \circ \big(\pi(Y) \longrightarrow \pi(X)\big).$$

Then, *singular homology of X relative to Y with local coefficients in L* is given by

$$H_*(X,Y;L) := H_*\big(C_*(X,Y;L)\big).$$

We denote the induced semi-norm on $H_*(X,Y;L)$ by $\|\cdot\|_{1,L}$. If $Y = \emptyset$ then we write $H_*(X;L) := H_*(X,Y;L)$.

1.3.2. Local vs. Twisted Coefficients

Now, we compare singular homology with local coefficients to singular homology with twisted coefficients while taking the norms into account.

Proposition 1.3.4 (local vs. twisted coefficients)**.** *Let M be a path-connected topological space that admits a universal covering $q\colon \widetilde{M} \longrightarrow M$. Let $x_0 \in M$, let $N \subset M$ be a subspace and let A be a normed right-$\pi_1(M,x_0)$-module. Let D be a set-theoretic fundamental domain of the Deck transformation action of $\pi_1(M,x_0)$ on $\widetilde{M}$.*

Then we have mutually inverse norm non-increasing chain maps

$$\begin{array}{ll} f_*\colon C_*(M;A) \longrightarrow C_*(M;L_A) & g_*\colon C_*(M;L_A) \longrightarrow C_*(M;A), \\ \quad a \otimes \sigma \longmapsto a \cdot q \circ \sigma & \quad a \cdot \tau \longmapsto a \otimes \widetilde{\tau} \\ \text{where } \sigma(e_0) \in D & \quad \text{where } \widetilde{\tau} \text{ is the lift of } \tau \\ & \quad \text{with } \widetilde{\tau}(e_0) \in D \end{array}$$

which induce well-defined isometric chain isomorphisms

$$C_*(M,N;A) \cong C_*(M,N;L_A)$$

on the quotients. Hence, these chain maps induce an isometric (with respect to the induced semi-norms) isomorphism in homology, i.e.,

$$H_n(M;A) \cong H_n(M;L_A) \quad \textit{and} \quad H_n(M,N;A) \cong H_n(M,N;L_A)$$

for all $n \in \mathbb{N}$.

Proof. We only prove that f_* is a chain map. We require that the family $(\gamma_x)_{x\in M}$ of paths that is part of the definition of L_A satisfies the following: For all $x \in M$, every q-lift of γ_x has its start and endpoint in the same translate of the fundamental domain D. More precisely, we define the paths γ_x as follows: Let $\widetilde{x}_0$ be the q-lift of x_0 in D. For all $\widetilde{x} \in D$, let $\widetilde{\gamma}_{\widetilde{x}}$ be a path from $\widetilde{x}_0$ to $\widetilde{x}$ in $\widetilde{M}$, where we set $\widetilde{\gamma}_{\widetilde{x}_0} := \mathrm{const}_{x_0}$. Then, for all $x \in M$, we set $\gamma_x := q \circ \widetilde{\gamma}_{\widetilde{x}}$, where $\widetilde{x}$ is the q-lift of x in D.

Let $n \in \mathbb{N}$. Let $a \in A$ and let $\widetilde{\sigma} \in C_n(\widetilde{M};\mathbb{Z})$ with $\widetilde{\sigma}(e_0) \in D$. We write $\sigma := q \circ \widetilde{\sigma}$. Then we have

$$\begin{aligned}\partial \circ f_n(a \otimes \widetilde{\sigma}) &= \partial(a \cdot \sigma)\\ &= L\big(\sigma[0,1]\big)(a) \cdot \partial^0\sigma + \sum_{i=1}^{n}(-1)^i \cdot a \cdot \partial^i\sigma\\ &= \big(a \cdot \big[\gamma_{\sigma(e_0)} * \sigma[0,1] * \gamma^{-1}_{\sigma(e_1)}\big]\big) \cdot \partial^0\sigma + \sum_{i=1}^{n}(-1)^i \cdot a \cdot \partial^i\sigma.\end{aligned}$$

Now, let $g \in \pi_1(M,x_0)$ be the unique element with $\widetilde{\sigma}(e_1) \in g \cdot D$. Then we have

$$\begin{aligned}f_{n-1} \circ \partial(a \otimes \widetilde{\sigma}) &= f_{n-1}(a \otimes \partial\widetilde{\sigma})\\ &= f_{n-1}(a \cdot g \otimes g^{-1} \cdot \partial^0\widetilde{\sigma}) + \sum_{i=1}^{n} f_{n-1}\big(a \otimes (-1)^i \cdot \partial^i\widetilde{\sigma}\big)\\ &= (a \cdot g) \cdot \partial^0\sigma + \sum_{i=1}^{n}(-1)^i \cdot a \cdot \partial^i\sigma.\end{aligned}$$

Therefore, it remains to prove

$$g = \big[\gamma_{\sigma(e_0)} * \sigma[0,1] * \gamma^{-1}_{\sigma(e_1)}\big] =: h.$$

It suffices to show $h \cdot \widetilde{x}_0 \in g \cdot D$. The path $\eta := \widetilde{\gamma}_{\widetilde{\sigma}(e_0)} * \widetilde{\sigma}[0,1] * (g \cdot \widetilde{\gamma}_{g^{-1}\cdot\widetilde{\sigma}(e_1)})^{-1}$ is the q-lift of $\gamma_{\sigma(e_0)} * \sigma[0,1] * \gamma^{-1}_{\sigma(e_1)}$ with $\eta(0) = \widetilde{x}_0$. Then, by definition, we have $h \cdot \widetilde{x}_0 = \eta(1)$. Moreover, we have that $\eta(1)$ lies in the same translate of D as $\widetilde{\sigma}[0,1](1) = \widetilde{\sigma}(e_1)$, i.e., $\eta(1) \in g \cdot D$. Hence, $h \cdot \widetilde{x}_0 \in g \cdot D$ holds. It follows that $g = h$ and therefore, f_* is a chain map. □

Remark 1.3.5. Let M be a path-connected topological space that admits a universal covering $q\colon \widetilde{M} \longrightarrow M$. Let $x_0 \in M$ and let L be a normed local system on M. For all $x \in M$, let γ_x be a path from x_0 to x, where we set $\gamma_{x_0} := \mathrm{const}_{x_0}$. Then the natural isomorphism $T\colon L \Longrightarrow L_{L(x_0)}$ defined as in Example 1.3.2 induces an isometric chain isomorphism

$$\begin{aligned} C_n(M;L) &\longrightarrow C_n(M;L_{L(x_0)}) \\ a\cdot\sigma &\longmapsto L([\gamma_{\sigma(e_0)}^{-1}]_*)(a)\cdot\sigma \end{aligned}$$

and by Proposition 1.3.4 we therefore have an isometric isomorphism

$$H_n(M;L) \cong H_n(M;L(x_0))$$

for all $n \in \mathbb{N}$. Analogously, also the relative version of this statement holds.

1.3.3. Integral Foliated Simplicial Volume via Local Coefficients

In this section, we explain how to define integral foliated simplicial volume using local coefficients.

Definition 1.3.6 (standard G-space, essentially free)**.** Let G be a groupoid. A *standard G-space* is a functor $\alpha\colon G \longrightarrow \mathrm{SBP}$, where SBP is the category of all standard Borel probability spaces and probability measure preserving transformations.

Note that for all objects x in G, the standard Borel probability space $\alpha(x)$ becomes a standard $\mathrm{Aut}_G(x)$-space via the induced action $\mathrm{Aut}_G(x) \curvearrowright \alpha(x)$, where $\mathrm{Aut}_G(x)$ denotes the group of automorphisms of x in G.

A standard G-space $\alpha\colon G \longrightarrow \mathrm{SBP}$ is *essentially free*, if for every object $x \in G$ the the standard $\mathrm{Aut}_G(x)$-space $\alpha(x)$ is essentially free.

Definition 1.3.7 (associated normed local coefficient system to a standard G-space)**.** Let G be a groupoid and let α be a standard G-space. Then the *associated normed local (coefficient) system $L^\infty(\alpha,\mathbb{Z})$ to α on G* is given by

$$L^\infty(\alpha,\mathbb{Z})(x) := L^\infty(\alpha(x),\mathbb{Z})$$

for all objects x of G and

$$\begin{aligned} L^\infty(\alpha,\mathbb{Z})(g)\colon L^\infty(\alpha(x),\mathbb{Z}) &\longrightarrow L^\infty(\alpha(y),\mathbb{Z}) \\ f &\longmapsto f\circ\alpha(g^{-1}) \end{aligned}$$

for all morphisms $g\colon x \longrightarrow y$ of G.

Definition 1.3.8 (parametrised (local) relative fundamental class)**.** Let $n \in \mathbb{N}$, let M be an oriented compact (not necessarily connected) n-manifold and let α be a standard G-space. Then the *relative α-parametrised fundamental class $[M,\partial M]^\alpha$ of M* is defined to be the image of the integral relative fundamental class $[M,\partial M]_\mathbb{Z}$ under the change of coefficients map

$$H_n(M,\partial M;\mathbb{Z}) \longrightarrow H_n\big(M,\partial M;L^\infty(\alpha,\mathbb{Z})\big) =: H_n(M,\partial M;\alpha)$$

that is induced by the "inclusion" of $\mathbb{Z}$ (as constant local system) into $L^\infty(\alpha,\mathbb{Z})$.

A *relative α-parametrised fundamental cycle of M* is a cycle in $C_n(M,\partial M;\alpha)$ that represents $[M,\partial M]^\alpha$.

Definition 1.3.9 (relative integral foliated simplicial volume with local coefficients)**.** Let $n \in \mathbb{N}$, let M be a oriented compact (not necessarily connected) n-manifold and let α be a standard $\pi(M)$-space. Then the *α-parametrised relative simplicial volume of M* is defined by

$$\left| M, \partial M \right|^{\alpha} := \|[M,\partial M]^{\alpha}\|_{1,\alpha} ,$$

where we write $\|\cdot\|_{1,\alpha} := \|\cdot\|_{1,L^{\infty}(\alpha,\mathbb{Z})}$. The *relative integral foliated simplicial volume* $\left| M, \partial M \right|$ *of M with local coefficients* is given by the infimum over all parametrised relative simplicial volumes of M.

Proposition 1.3.10 (comparison with the twisted version of integral foliated simplicial volume)**.** *Let $n \in \mathbb{N}$, let M be an oriented compact connected n-manifold and let $x_0 \in M$. Let α be a standard $\pi(M)$-space. Then the isometric isomorphism*

$$H_n(M, \partial M; \alpha) \longrightarrow H_n\big(M, \partial M; \alpha(x_0)\big)$$

defined as in Remark 1.3.5 sends fundamental class to fundamental class. Hence, we have

$$\left| M, \partial M \right|^{\alpha} = \left| M, \partial M \right|^{\alpha(x_0)},$$

i.e., the integral foliated simplicial volume with local coefficients coincides with the one with twisted coefficients.

Proof. Let i denote the isometric chain isomorphism

$$i \colon C_n(M, \partial M; \alpha) \longrightarrow C_n\big(M, \partial M; \alpha(x_0)\big)$$

defined as in Remark 1.3.5. The calculation

$$\begin{aligned} i(\mathrm{const}_1 \cdot \sigma) &= \big(L^{\infty}(\alpha,\mathbb{Z})([\gamma_{\sigma(e_0)}^{-1}]_*)\big)(\mathrm{const}_1) \otimes \widetilde{\sigma} \\ &= \mathrm{const}_1 \circ \alpha([\gamma_{\sigma(e_0)}]_*) \otimes \widetilde{\sigma} = \mathrm{const}_1 \otimes \widetilde{\sigma} \end{aligned}$$

(with the notation from Proposition 1.3.4 and Remark 1.3.5) shows that the image of $[M,\partial M]_{\mathbb{Z}}$ in $H_n(M,\partial M;\alpha)$ is sent to the image of $[M,\partial M]_{\mathbb{Z}}$ in $H_n\big(M,\partial M;\alpha(x_0)\big)$ under $H_n(i)$ and the claim follows. □

1.3.4. The Local Criterion for Parametrised Fundamental Cycles with Local Coefficients

We now reformulate the local criterion from Section 1.2.3 for parametrised fundamental cycles with local coefficients.

Proposition 1.3.11 (local criterion for parametrised fundamental cycles with local coefficients)**.** *Let M be an oriented compact connected n-manifold. Let $U \subset M$ be an embedded open n-ball in M such that $\overline{U} \subset M^{\circ}$. Let α be a standard $\pi(M)$-space. Let $c \in C_n(M, \partial M; \alpha)$ be a relative cycle. Then the following are equivalent:*

1. *The cycle c is a relative α-parametrised fundamental cycle of M.*
2. *The cycle c is a U-*local relative α-parametrised fundamental cycle of *M, i.e., c represents the image of $[\overline{U}, \partial\overline{U}]_{\mathbb{Z}}$ in $H_n(M, M \setminus U; \alpha)$.*

Proof. Let $x_0 \in M$. Then the statement is a direct consequence of Proposition 1.2.15 and the fact that the isometric isomorphisms

$$H_n(M, \partial M; \alpha) \cong H_n\big(M, \partial M; \alpha(x_0)\big) \quad \text{and} \quad H_n(M, M \setminus U; \alpha) \cong H_n\big(M, M \setminus U; \alpha(x_0)\big)$$

defined as in Remark 1.3.5 send (local) fundamental classes to (local) fundamental classes (see Proposition 1.3.10 for the first isomorphism; the second can be shown analogously). □

2. Exotic Finite Functorial Semi-Norms, URC-Manifolds and Gromov's Question

This chapter splits into two parts, which are connected by the following definition.

Definition 2.0.1. Let A be a class and let $f, g\colon A \longrightarrow \mathbb{R}_{\geq 0}$ be maps. Then we say that f *carries* g if

$$\forall_{a\in A} \quad f(a) = 0 \implies g(a) = 0.$$

In Section 2.1, we investigate functorial semi-norms on $H_*(\,\cdot\,;\mathbb{R})$ and recall a result of Löh and the author on the existence of such semi-norms that are finite and not carried by the ℓ^1-norm.

Then, in Section 2.2, we give a sufficient condition to prove that the answer to Gromov's question (Question 1) is affirmative: We show that if the integral foliated simplicial volume is functorial on aspherical manifolds then the integral foliated simplicial volume is carried by the simplicial volume on aspherical manifolds.

2.1. Exotic Finite Functorial Semi-Norms

Let $n \in \mathbb{N}$. A *(finite) functorial semi-norm on* $H_n(\,\cdot\,;\mathbb{R})$ is a lift of $H_n(\,\cdot\,;\mathbb{R})$ (as a functor $\mathrm{Top} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$ from topological spaces to real vector spaces) to a functor $\mathrm{Top} \longrightarrow \mathrm{Vect}_{\mathbb{R}}^{\mathrm{sn}}$, where $\mathrm{Vect}_{\mathbb{R}}^{\mathrm{sn}}$ is the category of semi-normed real vector spaces with norm non-increasing linear maps. More concretely, a functorial semi-norm is a choice of a (finite) semi-norm on the vector space $H_n(X;\mathbb{R})$ for all topological spaces X such that all maps in homology that are induced by continuous maps are norm non-increasing. We say that a functorial semi-norm A *carries* another functorial semi-norm B if the semi-norm on $H_n(X;\mathbb{R})$ coming from A carries the one coming from B for all topological spaces X.

Crowley and Löh showed that there are (non-finite) *exotic* functorial semi-norms on singular homology [9, Theorem 1.2], i.e., functorial semi-norms that are not carried by the ℓ^1-semi-norm. Furthermore, they ask whether the ℓ^1-semi-norm carries every finite functorial semi norm on singular homology [9, Question 5.8]. Löh and the author answered that question to the negative by proving the following theorem.

Theorem 2.1.1 (exotic finite functorial semi-norms [15, Theorem 1.2])**.** *Let $n \in \{3\} \cup \mathbb{N}_{\geq 5}$. Then there exists an exotic finite functorial semi-norm on $H_n(\,\cdot\,;\mathbb{R})$.*

Before we give the idea of the proof we recall the following definition.

Definition 2.1.2 ((strongly) inflexible)**.** Let $n \in \mathbb{N}$. Let M be an oriented closed connected n-manifold. For all oriented closed connected n-manifolds N we define

$$D(N, M) := \{\deg f \mid f \in \operatorname{map}(N, M)\}.$$

Recall that the *degree of* f is the integer $\deg f \in \mathbb{Z}$ such that

$$H_n(f; \mathbb{Z})[N] = \deg f \cdot [M]$$

holds. Then, we define the following notions:

- We call M *inflexible* if $D(M, M)$ is finite, i.e., $D(M, M) \subset \{-1, 0, 1\}$.
- If M is not inflexible, we call M *flexible*.
- We call M *strongly inflexible* if $D(N, M)$ is finite for all oriented closed connected n-manifolds N.
- If M is not strongly inflexible, we call M *weakly flexible*.

Note that weakly flexible manifolds have vanishing simplicial volume. In particular hyperbolic manifolds are strongly inflexible. Moreover, products of strongly inflexible manifolds are strongly inflexible [15, Proposition 3.2].

The idea of the proof of Theorem 2.1.1 now is the following:

1. We prove that there exists an (aspherical) oriented closed connected n-manifold M that is strongly inflexible and has vanishing simplicial volume:
   - There exist (aspherical) oriented closed connected 3-manifolds that are strongly inflexible and that have vanishing simplicial volume [15, Example 3.3]. Let M_1 be such a manifold. If $n = 3$ we are done.
   - If $n \geq 5$ then let M_2 be a hyperbolic oriented closed connected $(n-3)$-manifold. Then the product $M := M_1 \times M_2$ is an aspherical oriented closed connected n-manifold, which is strongly inflexible and has vanishing simplicial volume.
2. There exists a finite functorial semi-norm $|\cdot|_M$ on $H_n(\,\cdot\,; \mathbb{R})$ with

   $$|[N]_{\mathbb{R}}|_M = \sup\{|d| \mid d \in D(N, M)\}$$

   for all oriented closed connected n-manifolds N [9, Section 7.1]. In particular $|[M]_{\mathbb{R}}|_M = 1$ and therefore, $|\cdot|_M$ is *not* carried by the ℓ^1-semi norm.

2.2. Gromov's Question and Functoriality of the Integral Foliated Simplicial Volume

The following question is a reformulation of the refined version of Gromov's question (Question 2).

Question 2.2.1. Does the simplicial volume carry the integral foliated simplicial volume on the class of aspherical oriented closed connected manifolds?

If one drops the condition "aspherical", the answer to this question is obviously "no", because for example

$$\|S^2\| = 0 < 2 = \|S^2\|_{\mathbb{Z}} = \|S^2\|_{\mathbb{Z}}^{\infty} = \bigl| S^2 \bigr| .$$

Since there exists a map of degree 1 from the 2-torus T^2 to the 2-sphere S^2 and because $\bigl| T^2 \bigr| = 0$, we know that the integral foliated simplicial volume cannot come from a (finite) functorial semi norm. Still, one could hope that the integral foliated simplicial volume is functorial on aspherical manifolds, because in this case no counterexamples are known. In this section, we prove the following result.

Theorem 2.2.2 (functoriality and Gromov's question)**.** *Let $n \in \mathbb{N}$. We assume that the integral foliated simplicial volume was known to be* functorial on aspherical n-manifolds *in the following sense: For all aspherical oriented closed connected n-manifolds M, N and all continuous maps $f \colon M \longrightarrow N$ we have*

$$|\deg f| \cdot \bigl| N \bigr| \leq \bigl| M \bigr| .$$

Under this assumption, the answer to the refined version of Gromov's question is affirmative in dimension n, i.e., for all aspherical oriented closed connected n-manifolds we have

$$\|M\| = 0 \quad \Longrightarrow \quad \bigl| M \bigr| = 0.$$

2.2.1. Gaifullin's URC-Manifolds

In this section, we recall the definition of URC-manifolds by Gaifullin [20].

Definition 2.2.3. Let $n \in \mathbb{N}$. A URC-*n-manifold* is an oriented closed connected n-manifold M satisfying the *universal realisation of cycles* property: For all topological spaces X and all homology classes $\alpha \in H_n(X;\mathbb{Z})$ there exists a URC-*triple for α*, i.e., a triple (p,k,f), where $p\colon \overline{M} \longrightarrow M$ is a finite-sheeted covering, $k \in \mathbb{Z} \setminus \{0\}$ and $f\colon \overline{M} \longrightarrow X$ is a continuous map with

$$H_n(f;\mathbb{Z})[\overline{M}] = k \cdot \alpha.$$

Using manifolds introduced by Tomei [45] with fundamental groups, which were calculated by Davis [10], Gaifullin proved the following result.

Theorem 2.2.4 (asphercial URC-manifolds [21])**.** *For all $n \in \mathbb{N}$ there exists a URC-n-manifold that is aspherical.*

Having a URC-manifold we can define associated finite functorial semi-norms on $H_*(\,\cdot\,;\mathbb{Z})$.

Definition 2.2.5 (URC-norm)**.** Let M be a URC-n-manifold. Then we define a (finite) functorial semi-norm on $H_n(\,\cdot\,;\mathbb{Z})$ by

$$|\alpha|_M^{\mathrm{URC}} := \inf \frac{|\deg p|}{|k|}$$

for all topological spaces X and all $\alpha \in H_n(X;\mathbb{Z})$, where the infimum goes over all URC-triples (p,k,f) for α.

The following result connects the URC norms to the ℓ^1-norm. Gaifullin proved this by reduction to Tomei manifolds and using the structure of these manifolds.

Theorem 2.2.6 (URC-norm vs. ℓ^1-norm [20, Theorem 6.1])**.** *Let M be a URC-n-manifold. Then there exists $C_M \in \mathbb{R}_{>0}$ such that for all topological spaces X and all $\alpha \in H_n(X;\mathbb{Z})$ we have*

$$|\alpha|_M^{\mathrm{URC}} \leq C_M \cdot \|\alpha\|_{1,\mathbb{R}}.$$

In particular, the associated semi-norm to a URC-manifold is carried by the ℓ^1-semi-norm.

2.2.2. Proof of Theorem 2.2.2

Now we are prepared to prove Theorem 2.2.2 using the existence of aspherical URC-manifolds (Theorem 2.2.4) and that the URC-norm is carried by the ℓ^1-norm (Theorem 2.2.6).

Recall that we assume that the integral foliated simplicial volume is functorial on aspherical n-manifolds. This is not known to be true! By Theorem 2.2.4, there exists an aspherical URC-n-manifold M. Let N be an aspherical oriented closed connected n-manifold with $\|N\| = 0$. We want to show that $\bigl| N \bigr| = 0$. To this end, let $(p\colon \overline{M} \longrightarrow M, k, f\colon \overline{M} \longrightarrow N)$ be a URC-triple for $[N]$. Since, by assumption, the integral foliated simplicial volume is functorial on aspherical n-manifolds and because the integral foliated simplicial volume is multiplicative under finite coverings [37, Theorem 4.22] we have

$$|k| \cdot \bigl| N \bigr| = |\deg f| \cdot \bigl| N \bigr| \leq \bigl| \overline{M} \bigr| = |\deg p| \cdot \bigl| M \bigr| .$$

Taking the infimum over all URC-triples and applying Theorem 2.2.6 we obtain

$$\bigl| N \bigr| \leq \bigl|[N]\bigr|_M^{\mathrm{URC}} \cdot \bigl| M \bigr| \leq C_M \cdot \|N\| \cdot \bigl| M \bigr| = 0.$$

Hence, we have $\bigl| N \bigr| = 0$ as desired. □

3. The Uniform Boundary Condition

A normed chain complex satisfies the uniform boundary condition (or UBC), if every null-homologous cycle can be filled by a chain efficiently, i.e., the ratio of the norm of the chain by the norm of the null-homologous cycle is uniformly bounded (see Definition 3.1.1). For the singular chain complex with real coefficients together with the ℓ^1-norm there is a full characterisation for satisfying the uniform boundary condition in terms of bounded cohomology by Matsumoto and Morita [39] (see Section 3.1). For the proofs of Theorem 3 and Theorem 5 we need that the parametrised chain complex of S^1 with respect to an essentially free standard $\pi_1(S^1)$-space together with the parametrised ℓ^1-norm satisfies the uniform boundary condition (see Theorem 3.2.1). Therefore, we will state and for the sake of completeness also prove a result of Löh and the author on the parametrised uniform boundary condition of tori [14] (see Section 3.2). In Section 3.3, we summarize more results of Löh and the author and we give an additional application of UBC (apart from Theorem 3 and Theorem 5), namely glueing results for (parametrised) simplicial volumes.

3.1. The Uniform Boundary Condition for the Singular Chain Complex

We begin by defining the uniform boundary condition for normed chain complexes and stating a classical result by Matsumoto and Morita [39].

Definition 3.1.1 (UBC)**.** Let R be a ring (with unit). Let C_* be a normed chain complex of R-modules. Let $n \in \mathbb{N}$. Then, we say that C_* satisfies the *uniform boundary condition in degree n* (or n-UBC) if there exists $K_n \in \mathbb{R}_{>0}$ such that for every null-homologous cycle $c \in C_n$ there exists $b \in C_{n+1}$ with $\partial b = c$ and $|b| \leq K_n \cdot |c|$. Here, K_n is called a UBC-*constant of C_* in degree n*.

We say that C_* satisfies the *uniform boundary condition* (or UBC) if C_* satisfies the uniform boundary condition in every degree.

First, we observe that the uniform boundary condition is preserved under bounded chain homotopies.

Proposition 3.1.2 (UBC and chain homotopy equivalences)**.** *Let R be a ring and let $f_*\colon C_* \longrightarrow D_*$ and $g_*\colon D_* \longrightarrow C_*$ be chain maps between normed chain complexes of R-modules such that there exists a chain homotopy h^C_* from id_C to $g \circ f$.*

*If the R-linear maps f_n, g_{n+1} and h^C_n are all bounded for some $n \in \mathbb{N}$ then the following holds: If D_* satisfies n-*UBC*, then so does C_*.*

The proof is straight-forward and can be found in the work of the author [12, Proposition 3.15].

This implies that for the singular or parametrised chain complex of a space, satisfying UBC only depends on the homotopy type of the space.

We will now state a classical result about UBC of Matsumoto and Morita [39].

Theorem 3.1.3 (UBC and singular homology)**.** *Let M be a topological space and let $n \in \mathbb{N}$. Then the following conditions are equivalent:*

- *The normed chain complex $C_*(M;\mathbb{R})$ satisfies n-*UBC*.*
- *The comparison map $H_b^{n+1}(M;\mathbb{R}) \longrightarrow H^{n+1}(M;\mathbb{R})$ is injective.*

In particular, $C_*(M;\mathbb{R})$ satisfies UBC *if $\pi_1(M)$ is amenable.*

Here, $H_b^*(M;\mathbb{R})$ is the so-called *bounded cohomology of M (with coefficients in $\mathbb{R}$)*; it is the cohomology of the subcomplex $C_b^*(M;\mathbb{R}) \subset C^*(M;\mathbb{R})$ of the singular cochain complex of M with real coefficients consisting of all linear maps that are bounded with respect to the ℓ^∞-norm on $C^*(M;\mathbb{R})$.

The proof of Theorem 3.1.3 is based on elementary homological algebra and functional analysis. We will not discuss the details here because for dealing with parametrised UBC we (have to) use techniques, which are very different. The last statement in Theorem 3.1.3 directly follows from the fact that bounded cohomology of topological spaces with amenable fundamental group is trivial [5, 24].

3.2. The Uniform Boundary Condition and the Parametrised Chain Complex of Tori

In this section, we investigate the uniform boundary condition for parametrised chain complexes of tori with the ℓ^1-norm. As mentioned before we cannot use the arguments of Matsumoto and Morita in the classical case because the parametrised chain complex is built up from the singular chain complex with $\mathbb{Z}$-coefficients and it seems that there is no adequate definition of bounded cohomology in this case. Instead, we focus on tori and apply a geometric Følner argument on the chain level. The following is an extract of an article by Löh and the author [14].

Theorem 3.2.1 (parametrised UBC for tori [14, Theorem 1.3])**.** *Let $n \in \mathbb{N}$ and let $M := (S^1)^n$ be the n-torus. Let $\Gamma := \pi_1(M) \cong \mathbb{Z}^d$ be its fundamental group. Let $\alpha = \Gamma \curvearrowright (X,\mu)$ be an essentially free standard Γ-space. Then, $C_*(M;\alpha)$ satisfies* UBC *(with* UBC*-constants any $K_n \in \mathbb{R}_{>1}$).*

We prove this theorem in Section 3.2.2. The proof is based on the filling lemma and the lifting lemma (see Section 3.2.1) as well as the Rokhlin lemma for free abelian groups [8, 28].

Observe that Theorem 3.2.1 also holds for local coefficients by Proposition 1.3.4. Using local coefficients, we can also deal with spaces with multiple connected components in the following sense.

Remark 3.2.2 (local parametrised UBC). Let M be a topological space with finitely many components $M_1, \dots, M_k$ for some $k \in \mathbb{N}_{\geq 1}$. Let α be a standard $\pi(M)$-space. Let α_i be the composition $\pi(M_i) \longrightarrow \pi(M) \xrightarrow{\alpha} \mathrm{SBP}$ for all $i \in \{1, \dots, k\}$. Let $n \in \mathbb{N}$. Then we have the following: If $C_*(M_i; \alpha_i)$ satisfies n-UBC for all $i \in \{1, \dots, k\}$ then so does $C_*(M; \alpha)$.

3.2.1. The Filling Lemma and the Lifting Lemma

In this section, we prepare for the proof of Theorem 3.2.1 by presenting the filling lemma and the lifting lemma.

Lemma 3.2.3 (filling lemma [14, Lemma 4.1]). *Let M be a contractible topological space, let A be a normed $\mathbb{Z}$-module, and let $n \in \mathbb{N}$. For every $c \in C_n(M; A)$ there exists a chain $c' \in C_n(M; A)$ with*

$$\partial c' = \partial c \quad \text{and} \quad |c'|_1 \leq |\partial c|_1.$$

This lemma can be proved via a simple coning-off argument. For the details we refer to the literature [19, Lemma 6.3][14, Lemma 4.1]. Observe that this lemma also says that contractible spaces satisfy (parametrised) UBC.

Lemma 3.2.4 (lifting lemma [14, Lemma 4.2]). *Let M be a topological space that admits a universal covering $\widetilde{M}$, let $\Gamma := \pi_1(M)$ and let $n \in \mathbb{N}$. Let A be a normed right-$\mathbb{Z}\Gamma$-module, and let $\widetilde{a} \in A \otimes_{\mathbb{Z}} C_n(\widetilde{M}; \mathbb{Z})$ be a lift of $0 \in A \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M}; \mathbb{Z})$, i.e., $\widetilde{a}$ is mapped to zero under the canonical projection*

$$A \otimes_{\mathbb{Z}} C_n(\widetilde{M}; \mathbb{Z}) \longrightarrow A \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M}; \mathbb{Z}).$$

Then there is a constant $C \in \mathbb{R}_{>0}$ and a finite set $S \subset \Gamma$ such that the following holds: For every finite subset $F \subset \Gamma$ we have

$$|F \cdot \widetilde{a}|_1 \leq C \cdot |\partial_S F|,$$

where $|\cdot|_1$ is the ℓ^1-norm on $A \otimes_{\mathbb{Z}} C_n(\widetilde{M}; \mathbb{Z})$ induced by the norm on A and $|\cdot|$ denotes the cardinality of sets.

In the situation of Lemma 3.2.4, the *S-boundary of F in Γ* is given by

$$\partial_S F := \{\gamma \in F \mid \exists_{s \in S}\ \gamma \cdot s \notin F\}.$$

Moreover, we set $F \cdot \widetilde{a} := \sum_{\gamma \in F} \gamma \cdot \widetilde{a}$, where Γ acts on $A \otimes_{\mathbb{Z}} C_n(\widetilde{M}; \mathbb{Z})$ by

$$\gamma \cdot (x \otimes \sigma) := x \cdot \gamma^{-1} \otimes \gamma \cdot \sigma$$

for all $\gamma \in \Gamma$, $x \in A$ and $\sigma \in \mathrm{map}(\Delta^n, \widetilde{M})$.

We add a proof of the lifting lemma, which is slightly different from the proof in our original paper and possibly more accessible. It is based on unpublished notes of Löh.

Proof. We write $\widetilde{a} = \sum_{j=1}^{k} f_{\tau_j} \otimes \tau_j$ in reduced form in $A \otimes_{\mathbb{Z}} C_n(\widetilde{M};\mathbb{Z})$, i.e., $\tau_i \neq \tau_j$ if $i \neq j$. Let $q\colon \widetilde{M} \longrightarrow M$ denote the universal covering and let

$$K := \{q \circ \tau_j \mid j \in \{1,\dots,k\}\}.$$

For all $\tau \in K$ we choose a lift $\widetilde{\tau} \in C_n(\widetilde{M};\mathbb{Z})$ that occurs in $\widetilde{a}$ and we set

$$S(\tau) := \{\gamma \in \Gamma \mid \exists_{j\in\{1,\dots,k\}}\ \tau_j = \gamma \cdot \widetilde{\tau}\}.$$

Let $S := \bigcup_{\tau\in K}\big(S(\tau) \cup S(\tau)^{-1}\big) \subset \Gamma$. Then S is a finite and symmetric subset of Γ. Now, we can write

$$\widetilde{a} = \sum_{\tau\in K}\sum_{s\in S(\tau)} f_{s\cdot\widetilde{\tau}} \otimes s \cdot \widetilde{\tau} = \sum_{\tau\in K}\sum_{s\in S} f_{s\cdot\widetilde{\tau}} \otimes s \cdot \widetilde{\tau}$$

in $A \otimes_{\mathbb{Z}} C_n(\widetilde{M};\mathbb{Z})$, where we set $f_{s\cdot\widetilde{\tau}} := 0$ for all $s \in S \setminus S(\tau)$. We have

$$\sum_{s\in S} f_{s\cdot\widetilde{\tau}} \cdot s = 0$$

for all $\tau \in K$ because $\widetilde{a}$ is a lift of zero.

Let $F \subset \Gamma$ be a finite subset. The goal is to estimate the ℓ^1-norm of

$$F \cdot \widetilde{a} = \sum_{\gamma\in F} \gamma \cdot \widetilde{a} = \sum_{\gamma\in F}\sum_{\tau\in K}\sum_{s\in S} f_{s\cdot\widetilde{\tau}} \cdot \gamma^{-1} \otimes \gamma \cdot s \cdot \widetilde{\tau}.$$

In $A \otimes_{\mathbb{Z}} C_n(\widetilde{M};\mathbb{Z})$ we have

$$\begin{aligned}
F \cdot \widetilde{a} &= \sum_{s\in S}\sum_{\gamma\in F}\sum_{\tau\in K} f_{s\cdot\widetilde{\tau}} \cdot \gamma^{-1} \otimes \gamma \cdot s \cdot \widetilde{\tau} \\
&= \sum_{s\in S}\sum_{\eta\in F\cdot s}\sum_{\tau\in K} f_{s\cdot\widetilde{\tau}} \cdot s \cdot \eta^{-1} \otimes \eta \cdot \widetilde{\tau} \\
&= R_F + \sum_{s\in S}\sum_{\eta\in F}\sum_{\tau\in K} f_{s\cdot\widetilde{\tau}} \cdot s \cdot \eta^{-1} \otimes \eta \cdot \widetilde{\tau} \\
&= R_F + \sum_{\eta\in F}\sum_{\tau\in K}\Big(\sum_{s\in S} f_{s\cdot\widetilde{\tau}} \cdot s\Big) \cdot \eta^{-1} \otimes \eta \cdot \widetilde{\tau} \\
&= R_F,
\end{aligned}$$

where

$$\begin{aligned}
R_F := &\sum_{s\in S}\sum_{\eta\in F\cdot s\setminus F}\sum_{\tau\in K} f_{s\cdot\widetilde{\tau}} \cdot s \cdot \eta^{-1} \otimes \eta \cdot \widetilde{\tau} \\
&- \sum_{s\in S}\sum_{\eta\in F\setminus F\cdot s}\sum_{\tau\in K} f_{s\cdot\widetilde{\tau}} \cdot s \cdot \eta^{-1} \otimes \eta \cdot \widetilde{\tau}.
\end{aligned}$$

Now, we estimate the ℓ^1-norm of the remainder R_F. We define

$$m := |K| \cdot \max\{|f_{\gamma\cdot\widetilde{\tau}}|_A \mid \gamma \in S, \tau \in K\}.$$

Then, we have

$$\begin{aligned}|R_F|_1 &\leq \sum_{s\in S} |F\cdot s \setminus F| \cdot m + \sum_{s\in S} |F \setminus F\cdot s|\cdot m \\ &= \sum_{s\in S} |F\cdot s \setminus F| \cdot m + \sum_{s\in S} |F\cdot s^{-1} \setminus F|\cdot m \\ &\leq 2\cdot |S|\cdot |\partial_S F|\cdot m.\end{aligned}$$

Hence, the result follows for

$$C := 2\cdot |S| \cdot m. \qquad \square$$

3.2.2. Proof of Theorem 3.2.1

Let $c \in C_n(M;\alpha)$ be a null-homologous cycle, i.e., there exists $b \in C_{n+1}(M;\alpha)$ with $\partial b = c$. We will use b to find a more efficient filling of c. Let

$$p\colon L^\infty(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C_*(\widetilde{M};\mathbb{Z}) \longrightarrow L^\infty(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z}) = C_*(M;\alpha)$$

be the canonical projection.

The idea of the proof is as follows.

- *Lifting step:* First, we choose p-lifts $\widetilde{c}$ and $\widetilde{b}$ of c and b with controlled ℓ^1-norm. We consider the lift $\widetilde{a} := \partial\widetilde{b} - \widetilde{c}$ of zero and apply the lifting lemma (Lemma 3.2.4) to a "Følner set" F, i.e., we consider the translates of $\widetilde{a}$ by F and estimate the ℓ^1-norm of this chain. We now have a bound for the ℓ^1-norm of $F\cdot\widetilde{b}$ in terms of the ℓ^1-norm of c and the size of the "boundary" of F.

- *Quotient step:* We continue by applying the Rokhlin lemma for free abelian groups [8, 28] to the transformations of X given by multiplication with elements in F. That gives us a decomposition of X into a set A and F-translates of this set. This allows us to decompose the chain $F\cdot\widetilde{b}$ according to the decomposition of X. It turns out that it is enough to consider only one of the pieces $\widetilde{b}_0$ of $F\cdot\widetilde{b}$ such that the ℓ^1-norm of $\partial\widetilde{b}_0$ is bounded by the quotient $|\partial(F\cdot\widetilde{b})|_1/|F|$.

- *Filling step:* We apply the filling lemma (Lemma 3.2.3) and obtain an efficient filling $\widetilde{b}_0'$ of $\widetilde{b}_0$. By construction, we have $\partial b_0' = c$ up to a correction term of small norm. This gives us the desired efficient filling of c.

For the sake of completeness we will now discuss the proof in more detail. The following is taken from the work of Löh and the author [14].

Lifting step: Let $\widetilde{c}$ be a p-lift of c with $|\widetilde{c}|_1 \leq |c|_1$ (e.g., by lifting c simplex by simplex) and let $\widetilde{b}$ be a p-lift of b. We now consider

$$\widetilde{a} := \partial\widetilde{b} - \widetilde{c} \in L^\infty(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(\widetilde{M};\mathbb{Z}).$$

Then $\widetilde{a}$ is a p-lift of $0 \in C_n(M;\alpha)$. By the lifting lemma (Lemma 3.2.4), there exist a constant $C \in \mathbb{R}_{>0}$ and a finite subset $S \subset \Gamma$ such that the following holds: for all finite sets $F \subset \Gamma$ we have

$$|F \cdot \widetilde{a}|_1 \leq C \cdot |\partial_S F|.$$

For all $k \in \mathbb{N}$ we define

$$F_k := \{0, \dots, k\}^d \subset \Gamma$$

via an isomorphism $\Gamma \cong \mathbb{Z}^d$. Then $(F_k^{-1})_{k \in \mathbb{N}}$ is a Følner sequence for Γ with respect to S in the sense that

$$\lim_{k \to \infty} \frac{|\partial_S(F_k^{-1})|}{|F_k|} = 0.$$

Then, for all $k \in \mathbb{N}$ we have

$$|\partial(F_k^{-1} \cdot \widetilde{b})|_1 = |F_k^{-1} \cdot \widetilde{a} + F_k^{-1} \cdot \widetilde{c}|_1 \leq C \cdot |\partial_S F_k^{-1}| + |F_k| \cdot |\widetilde{c}|_1.$$

Quotient step: Let $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}_{>0}$. By the Rokhlin lemma for free abelian groups [8, 28], there exists a measurable subset $A_k \subset X$ such that the translates $(\gamma \cdot A_k)_{\gamma \in F_k}$ are pairwise disjoint and the complement

$$B_k := X \setminus F_k \cdot A_k$$

has measure less than ε.

Since $L^\infty(X;\mathbb{Z})$ is a $L^\infty(X;\mathbb{Z})$-$\mathbb{Z}$-bimodule, it follows that

$$L^\infty(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(\widetilde{M};\mathbb{Z})$$

is a left-$L^\infty(X;\mathbb{Z})$-module. Therefore, we can define

$$\widetilde{b}_{k,\gamma} := \chi_{\gamma \cdot A_k} \cdot (F_k^{-1} \cdot \widetilde{b}) \in L^\infty(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(\widetilde{M};\mathbb{Z})$$

for all $\gamma \in F_k$. Then we have

$$\begin{aligned} F_k^{-1} \cdot \widetilde{b} = \chi_X \cdot (F_k^{-1} \cdot \widetilde{b}) &= \chi_{B_k} \cdot (F_k^{-1} \cdot \widetilde{b}) + \sum_{\gamma \in F_k} \chi_{\gamma \cdot A_k} \cdot (F_k^{-1} \cdot \widetilde{b}) \\ &= \chi_{B_k} \cdot (F_k^{-1} \cdot \widetilde{b}) + \sum_{\gamma \in F_k} \widetilde{b}_{k,\gamma}. \end{aligned}$$

Because the chains $(\partial \widetilde{b}_{k,\gamma})_{\gamma \in F_k}$ have pairwise disjoint support, we obtain

$$\Big| \sum_{\gamma \in F_k} \partial \widetilde{b}_{k,\gamma} \Big|_1 = \sum_{\gamma \in F_k} |\partial \widetilde{b}_{k,\gamma}|_1$$

and by the box principle it follows that there exists $\gamma_0 \in F_k$ with

$$|\partial \widetilde{b}_{k,\gamma_0}|_1 \leq \frac{1}{|F_k|} \cdot \big(|F_k^{-1} \cdot \partial \widetilde{b}|_1 + |\chi_{B_k} \cdot (F_k^{-1} \cdot \partial \widetilde{b})|_1 \big).$$

We write $\widetilde{b} = \sum_{j=1}^{m} a_j \otimes \sigma_j \in L^\infty(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C_{n+1}(\widetilde{M};\mathbb{Z})$ in reduced form and set

$$|\widetilde{b}|_{1,\infty} := \sum_{j=1}^{m} |a_j|_\infty.$$

Then

$$\begin{aligned}|\partial \widetilde{b}_{k,\gamma_0}|_1 &\leq \frac{1}{|F_k|} \cdot \big(|F_k^{-1} \cdot \partial \widetilde{b}|_1 + \mu(B_k) \cdot |F_k| \cdot (n+2) \cdot |\widetilde{b}|_{1,\infty}\big) \\ &\leq C \cdot \frac{|\partial_S(F_k^{-1})|}{|F_k|} + |c|_1 + \varepsilon \cdot (n+2) \cdot |\widetilde{b}|_{1,\infty}.\end{aligned}$$

Filling step: By Lemma 4.1.4, there exists a parametrised chain

$$\widetilde{b}'_{k,\gamma_0} \in L^\infty(X;\mathbb{Z}) \otimes_{\mathbb{Z}} C_{n+1}(\widetilde{M};\mathbb{Z})$$

with $\partial \widetilde{b}'_{k,\gamma_0} = \partial \widetilde{b}_{k,\gamma_0}$ and

$$|\widetilde{b}'_{k,\gamma_0}|_1 \leq |\partial \widetilde{b}_{k,\gamma_0}|_1 \leq C \cdot \frac{|\partial_S(F_k^{-1})|}{|F_k|} + |c|_1 + \varepsilon \cdot (n+2) \cdot |\widetilde{b}|_{1,\infty}.$$

Let

$$b_{k,\gamma_0} := p(\widetilde{b}_{k,\gamma_0}) \in C_{n+1}(M;\alpha) \quad \text{and} \quad b'_{k,\gamma_0} := p(\widetilde{b}'_{k,\gamma_0}) \in C_{n+1}(M;\alpha).$$

Because Γ is abelian and $\chi_{\gamma \cdot A_k} = \chi_{A_k} \cdot \gamma^{-1}$ holds for all $\gamma \in \Gamma$, we obtain in the chain module $C_n(M;\alpha)$ that

$$\begin{aligned}\partial b'_{k,\gamma_0} &= \partial b_{k,\gamma_0} \\ &= \sum_{j=1}^{m} \sum_{\gamma \in F_k} \chi_{\gamma_0 \cdot A_k} \cdot (a_j \cdot \gamma) \otimes \gamma^{-1} \cdot \partial \sigma_j \\ &= \sum_{j=1}^{m} \sum_{\gamma \in F_k} \chi_{A_k} \cdot (a_j \cdot \gamma \cdot \gamma_0) \otimes \gamma_0^{-1} \cdot \gamma^{-1} \cdot \partial \sigma_j \\ &= \sum_{j=1}^{m} \sum_{\gamma \in F_k} \chi_{A_k} \cdot (a_j \cdot \gamma_0 \cdot \gamma) \otimes \gamma^{-1} \cdot \gamma_0^{-1} \cdot \partial \sigma_j \\ &= \sum_{j=1}^{m} \sum_{\gamma \in F_k} \chi_{\gamma \cdot A_k} \cdot (a_j \cdot \gamma_0) \otimes \gamma_0^{-1} \cdot \partial \sigma_j,\end{aligned}$$

which almost looks like c. We define the correction term

$$r_k := \sum_{j=1}^{m} \chi_{\gamma_0 \cdot B_k} \cdot a_j \otimes \sigma_j = \sum_{j=1}^{m} \chi_{B_k} \cdot (a_j \cdot \gamma_0) \otimes \gamma_0^{-1} \cdot \sigma_j$$

and observe that the following holds in $C_n(M;\alpha)$:

$$\partial(b'_{k,\gamma_0} + r_k) = \sum_{j=1}^{m} \chi_X \cdot (a_j \cdot \gamma_0) \otimes \gamma_0^{-1} \cdot \partial\sigma_j = \sum_{j=1}^{m} a_j \otimes \partial\sigma_j = \partial b = c.$$

Finally, we have

$$\begin{aligned} |b'_{k,\gamma_0} + r_k|_1 &\le |b'_{k,\gamma_0}|_1 + |r_k|_1 \\ &\le C \cdot \frac{|\partial_S(F_k^{-1})|}{|F_k|} + |c|_1 + \varepsilon \cdot (n+2) \cdot |\widetilde{b}|_{1,\infty} + \varepsilon \cdot |\widetilde{b}|_{1,\infty} \\ &\le C \cdot \frac{|\partial_S(F_k^{-1})|}{|F_k|} + |c|_1 + \varepsilon \cdot (n+3) \cdot |\widetilde{b}|_{1,\infty}. \end{aligned}$$

Because $(F_k^{-1})_{k\in\mathbb{N}}$ is a Følner sequence for Γ with respect to S, for $k \to \infty$ and $\varepsilon \to 0$ the chains $b'_{k,\gamma_0} + r_k$ are efficient fillings of c. □

3.3. More Results on the Uniform Boundary Condition and Applications

In this section, we state more results by Löh and the author [14] on the uniform boundary condition without repeating the proofs. Moreover, we recall an important application of the parametrised uniform boundary condition namely glueing results for the integral foliated simplicial volume.

3.3.1. More Results on the Uniform Boundary Condition

For the sake of completeness, we state further results on stable and parametrised UBC of Löh and the author [14].

Theorem 3.3.1 (UBC for the stable integral ℓ^1-norm [14, Theorem 1.2]). *Let M be an aspherical topological space with countable amenable residually finite fundamental group and let $n \in \mathbb{N}$. Then there is a constant $K \in \mathbb{R}_{>0}$ such that: If $c \in C_n(M;\mathbb{Z})$ is a null-homologous cycle, then there is a sequence $(b_k)_{k\in\mathbb{N}}$ of chains and a sequence $(M_k)_{k\in\mathbb{N}}$ of covering spaces of M with the following properties:*

- *For each $k \in \mathbb{N}$, there is a regular finite-sheeted covering $p_k\colon M_k \longrightarrow M$ of M, and the covering degrees d_k satisfy $\lim_{k\to\infty} d_k = \infty$.*
- *For each $k \in \mathbb{N}$ we have $b_k \in C_{n+1}(M_k;\mathbb{Z})$ and*

$$\partial b_k = c_k \quad \text{and} \quad |b_k|_1 \le d_k \cdot K \cdot |c|_1,$$

where $c_k \in C_n(M_k;\mathbb{Z})$ denotes the full p_k-lift of c, i.e., if $c = \sum_{j=1}^{m} a_j \cdot \sigma_j$ then

$$c_k = \sum_{j=1}^{m} a_j \cdot \Bigl(\sum_{\substack{\tau \in \operatorname{map}(\Delta^n, M_k) \\ p_k \circ \tau = \sigma_j}} \tau \Bigr).$$

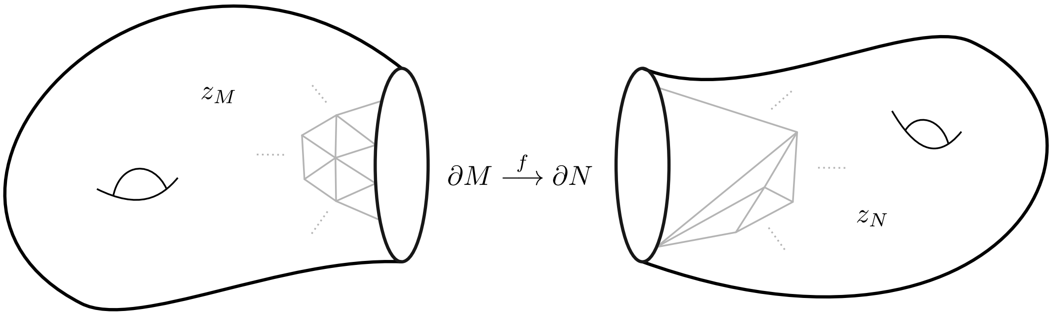

Figure 3.1.: A glueing of two manifolds along their "common" boundary; the sum of two relative fundamental cycles will not be a cycle in general

It seems to be hard to show parametrised UBC for a general aspherical topological space with amenable fundamental group. However, it is possible to fill integral null-homologous cycles by parametrised chains in an efficient way.

Theorem 3.3.2 (mixed UBC for the twisted ℓ^1-norm [14, Theorem 1.4]). *Let M be an aspherical topological space with amenable fundamental group and let $n \in \mathbb{N}$. Let $\alpha = \Gamma \curvearrowright (X, \mu)$ be an (essentially) free standard Γ-space. Then there is a constant $K \in \mathbb{R}_{>0}$ such that: For every cycle $c \in C_n(M;\mathbb{Z}) \subset C_n(M;\alpha)$ that is null-homologous in $C_n(M;\mathbb{Z})$ there exists a parametrised chain $b \in C_{n+1}(M;\alpha)$ with*

$$\partial b = c \quad \textit{and} \quad |b|_1 \leq K \cdot |c|_1.$$

3.3.2. Application: Glueing Results for Parametrised Simplicial Volumes

We first explain the idea how UBC can help with proving glueing results [7, Remark 6.2]. Let M and N be oriented compact connected n-manifolds with connected boundary and with an orientation reversing homeomorphism $f\colon \partial M \cong \partial N$. Then we consider the glued manifold $M \cup_f N$. Let z_M be a relative real fundamental cycle of M and let z_N be a relative real fundamental cycle of N. See Figure 3.1 for an illustration of this situation. Then, $\partial z_M + \partial z_N$ is a null-homologous cycle in $C_{n-1}(\partial M;\mathbb{R})$ (here and in the following we identify ∂M and ∂N via f). To show that

$$\|M \cup_f N\| \leq \|M, \partial M\| + \|N, \partial N\|$$

it suffices to verify the following:

- We can choose z_M and z_N such that $\partial z_M + \partial z_N$ has small ℓ^1-norm and
- that $\partial z_M + \partial z_N$ can be filled efficiently; this for example holds if $C_*(\partial M;\mathbb{R})$ satisfies the uniform boundary condition in degree $n-1$.

These conditions are for example fulfilled if ∂M has amenable fundamental group because then we can find a relative fundamental cycle z_M of M such that ∂z_M has arbitrarily small norm (and analogously for N) by the Equivalence Theorem of Gromov [24, p. 57][7][18, Section 4.5]. Moreover, $C_{n-1}(\partial M;\mathbb{R})$ satisfies the uniform boundary condition by Theorem 3.1.3. Recall that one can even prove the equality

$$\|M \cup_f N\| = \|M, \partial M\| + \|N, \partial N\|$$

in this case [24, 31, 7].

However, if we work with parametrised fundamental cycles, things get more involved. For example there is no version of the Equivalence Theorem known in this case. Still, we can formulate the following result.

Theorem 3.3.3 (parametrised simplicial volumes and glueings)**.** *Let M and N be oriented compact connected n-manifolds such that there exists an orientation reversing homeomorphism $f\colon M' \longrightarrow N'$, where M' and N' are connected components of ∂M and ∂N respectively with the inherited orientations. Then we consider the oriented compact connected manifold $V := M \cup_f N$. Let i_M, i_N and $i_{M'}$ be the corresponding inclusions into V. Let α be a standard $\pi(V)$-space and let α_M, α_N and $\alpha_{M'}$ be the restrictions of α along i_M, i_N and $i_{M'}$ respectively. We require that $C_*(M';\alpha_{M'})$ satisfies the uniform boundary condition in degree $n-1$ and that for all $\varepsilon \in \mathbb{R}_{>0}$ there exist α_M- and α_N-parametrised fundamental cycles z_M and z_N of M and N, respectively, with*

$$|z_M|_{1,\alpha_M} + \big|(\partial z_M)|_{M'}\big|_{1,\alpha_{M'}} < \big| M, \partial M \big|^{\alpha_M} + \varepsilon$$

and

$$|z_N|_{1,\alpha_N} + \big|(\partial z_N)|_{M'}\big|_{1,\alpha_{M'}} < \big| N, \partial N \big|^{\alpha_N} + \varepsilon.$$

Then we have

$$\big| V, \partial V \big|^{\alpha} \leq \big| M, \partial M \big|^{\alpha_M} + \big| N, \partial N \big|^{\alpha_N}.$$

Proof. Let $K \in \mathbb{R}_{>0}$ be a UBC-constant for $C_*(M';\alpha_{M'})$ in degree $n-1$ and let $\varepsilon \in \mathbb{R}_{>0}$. Let z_M and z_N be α_M- and α_N-parametrised fundamental cycles of M and N respectively with

$$|z_M|_{1,\alpha_M} + \big|(\partial z_M)|_{M'}\big|_{1,\alpha_{M'}} < \big| M, \partial M \big|^{\alpha_M} + \varepsilon$$

and

$$|z_N|_{1,\alpha_N} + \big|(\partial z_N)|_{M'}\big|_{1,\alpha_{M'}} < \big| N, \partial N \big|^{\alpha_N} + \varepsilon.$$

Then, the chain

$$\widetilde{z} := (\partial z_M)|_{M'} + (\partial z_N)|_{M'} \in C_{n-1}(M';\alpha_{M'})$$

is a null-homologous cycle because both $(\partial z_M)|_{M'}$ and $(\partial z_N)|_{M'}$ are $\alpha_{M'}$-parametrised fundamental cycles of M' (with opposite orientations). Because $C_*(M';\alpha_{M'})$ satisfies the uniform boundary condition in degree $n-1$ with UBC-constant K, there exists a chain $b \in C_n(M';\alpha_{M'})$ with

$$\partial b = \widetilde{z} \quad \text{and} \quad |b|_{1,\alpha_{M'}} \leq K \cdot |\widetilde{z}|_{1,\alpha_{M'}} < K \cdot 2 \cdot \varepsilon.$$

Then we have that $z_M + z_N - b$ is an α-parametrised fundamental cycle of V (this can be verified easily using the local criterion; see Proposition 1.3.11) with ℓ^1-norm less than

$$|M, \partial M|^{\alpha_M} + |N, \partial N|^{\alpha_N} + 2 \cdot \varepsilon + K \cdot 2 \cdot \varepsilon.$$

Since we can choose ε arbitrarily small, the result follows. □

If one drops the condition on the existence of parametrised fundamental cycles with boundary of small norm, the bounds get worse and will depend on the UBC-constant in general. Still we can formulate the following result for glueings along tori.

Theorem 3.3.4. *Let M and N be oriented compact connected n-manifolds such that there exists an orientation reversing homeomorphism $f\colon T \longrightarrow T'$, where T and T' are connected components of ∂M and ∂N respectively with the inherited orientations that are homeomorphic to the torus $T^{n-1} = (S^1)^{n-1}$. Then we consider the oriented compact connected manifold $V := M \cup_f N$. Let i_M, i_N and i_T be the corresponding inclusions into V. Let α be an essentially free standard $\pi(V)$-space with the property that the restriction α_T along i_T is essentially free. Let α_M and α_N be the restrictions of α along i_M and i_N respectively. Then we have*

$$|V, \partial V|^{\alpha} \leq (n+3) \cdot \big(|M, \partial M|^{\alpha_M} + |N, \partial N|^{\alpha_N} \big).$$

Proof. Let $K \in \mathbb{R}_{>0}$ be a UBC-constant for $C_{n-1}(T; \alpha_T)$; this exists by Theorem 3.2.1 and can be chosen such that $K \leq 1 + 1/(n+1)$. Let $\varepsilon \in \mathbb{R}_{>0}$. Let z_M and z_N be α_M- and α_N-parametrised fundamental cycles of M and N respectively with

$$|z_M|_{1,\alpha_M} \leq |M, \partial M|^{\alpha_M} + \varepsilon \quad \text{and} \quad |z_N|_{1,\alpha_N} \leq |N, \partial N|^{\alpha_N} + \varepsilon.$$

Then, the chain

$$\widetilde{z} := (\partial z_M)|_T + (\partial z_N)|_T \in C_{n-1}(T; \alpha_T)$$

is a null-homologous cycle. Because $C_{n-1}(T; \alpha_T)$ satisfies the uniform boundary condition, there exists a chain $b \in C_n(T; \alpha_T)$ with $\partial b = \widetilde{z}$ and

$$|b|_{1,\alpha_T} \leq K \cdot |\widetilde{z}|_{1,\alpha_T} < K \cdot (n+1) \cdot \big(|M, \partial M|^{\alpha_M} + \varepsilon + |N, \partial N|^{\alpha_N} + \varepsilon \big).$$

Then we have that $z_M + z_N - b$ is an α-parametrised fundamental cycle of V (this can be verified easily using the local criterion; see Proposition 1.3.11). Because $K \leq 1 + 1/(n+1)$ we have

$$\begin{aligned} |V, \partial V|^{\alpha} &\leq |z_M + z_N - b|_{1,\alpha} \\ &< \big(1 + K \cdot (n+1)\big) \cdot \big(|M, \partial M|^{\alpha_M} + \varepsilon + |N, \partial N|^{\alpha_N} + \varepsilon \big) \\ &\leq (n+3) \cdot \big(|M, \partial M|^{\alpha_M} + |N, \partial N|^{\alpha_N} + 2 \cdot \varepsilon \big). \end{aligned}$$

Since we can choose ε arbitrarily small it follows that

$$|V, \partial V|^{\alpha} < (n+3) \cdot \big(|M, \partial M|^{\alpha_M} + |N, \partial N|^{\alpha_N} \big),$$

as desired. □

The following is a direct consequence of the above theorem.

Corollary 3.3.5 (integral foliated simplicial volume and glueings along tori [13, Proposition 4.4])**.** *If we are in the situation of Theorem 3.3.4 and we additionally require that*

$$\left| M, \partial M \right|^{\alpha_M} = 0 \quad \textit{and} \quad \left| N, \partial N \right|^{\alpha_N} = 0$$

holds, then we have $\left| V, \partial V \right|^{\alpha} = 0$ *and in particular,* $\left| V, \partial V \right| = 0$.

Corollary 3.3.5 and Theorem 5 are important ingredients in the proof of vanishing of integral foliated simplicial volume of generalised graph manifolds in the sense of Friedl, Löh and the author [13] (these have toroidal boundary and in particular include all 3-dimensional graph manifolds that are not finitely covered by S^3).

Theorem 3.3.6 (integral foliated simplicial volume and graph manifolds [13, Theorem 1.7])**.** *Let M be an oriented compact connected generalised graph manifold. Let α be an essentially free standard $\pi(M)$-space. Then* $\left| M, \partial M \right|^{\alpha} = 0$.

This implies vanishing of the stable integral simplicial volume of oriented compact connected generalised graph manifolds with residually finite fundamental group [19, Theorem 2.6].

4. Integral Foliated Simplicial Volume and S^1-Actions

In this chapter we will show vanishing of the integral foliated simplicial volume of

- smooth oriented compact connected manifolds with a non-trivial smooth S^1-action such that all orbits are π_1-injective (see Theorem 3) and
- smooth oriented compact connected manifolds that are the total space of a smooth S^1-bundle over a smooth oriented compact connected manifold with π_1-injective fibres (see Theorem 5).

The proofs are very similar, and based on Yano's proof of vanishing of simplicial volume of smooth oriented closed connected manifolds with a smooth non-trivial S^1-action [47].

Yano's proof consists of two steps, which can be sketched as follows: Let $n \in \mathbb{N}$ and let M be a smooth oriented compact connected n-manifold with a non-trivial smooth S^1-action.

1. **Deconstruction step:** In this step, we construct a trivial S^1-bundle M' with an equivariant map to M.
   - Recall that there is a decomposition of M into invariant submanifolds with constant isotropy group such that distinct submanifolds have distinct isotropy groups; then each of these submanifolds has the structure of a smooth fiber bundle over a manifold (with fiber S^1 or a point).
   - We can choose a triangulation of the orbit space $\overline{M}$ that respects the decomposition of M from above, i.e., such that open simplices (or more precisely their preimages in M) have constant isotropy group.
   - We may assume that the S^1-action is effective. Then, we know that the dimension of the complement of the submanifold corresponding to the trivial isotropy group has dimension less than $n-2$.
   - Now we remove invariant tubular neighbourhoods of all the submanifolds corresponding to non-trivial orbit types. We do this inductively by equivariant hollowings in M at the pullbacks of the skeleta of $\overline{M}$ of dimension less than $n-3$. This gives us a combinatorial structure on the resulting manifold M' as a manifold with corners.
   - Since the induced S^1-action on M' is free by construction, the projection $M' \longrightarrow \overline{M}'$ to the orbit space is a principal S^1-bundle. Moreover, the

space $\overline{M}'$ has the homotopy type of a 1-complex. Therefore, $M' \longrightarrow \overline{M}'$ is a trivial S^1-bundle.

2. **Filling step:** In this step, we construct a fundamental cycle of M of small norm.

   - Since trivial S^1-bundles have vanishing simplicial volume, we can choose a fundamental cycle of M' of small norm.

   - Using the combinatorial structure of M' and an efficient filling strategy for null-homologous cycles in contractible spaces or in spaces that are homotopy equivalent to S^1 we can make the pushforward of this cycle a fundamental cycle of M of small norm again.

We have to adapt Yano's proof in two ways. First, we have to deal with compact manifolds with boundary and second, we need a setup with twisted/local coefficients and a filling strategy for null-homologous parametrised cycles in S^1 (or the point). We fortunately already have such a filling strategy in form of the uniform boundary condition (see Chapter 3).

This chapter is built up as follows: In Section 4.1, we set up the notation to deal with twisted/local coefficients and give a unified proof for the filling step for S^1-actions and S^1-bundles. Then, in Section 4.2 and Section 4.3, we focus on the deconstruction step for S^1-actions and S^1-bundles, respectively, in such a way that the unified filling step can be applied in both cases. Finally, in Section 4.4, we give some applications of Theorem 3 and Theorem 5.

4.1. The Abstract Filling Step

In this section, we set up the notation that is needed to prove the filling step in the parametrised world and then prove the filling step in a way that it can be applied in both cases, the case of S^1-actions and the case of S^1-bundles. Here we need the notion of an *admissible sequence of hollowings*, which is explained in Appendix B in more detail; concrete examples are the sequences of hollowings appearing in the deconstruction step for S^1-actions (see Section 4.2) or S^1-bundles (see Section 4.3).

A hollowing in a manifold M at a submanifold N can be thought to be the following: We remove a tubular neighbourhood of N from M and glue a collar to the resulting boundary. We call the obtained manifold M' and define a map $p\colon M' \longrightarrow M$ by projecting the collar to the tubular neighbourhood of N such that the boundary of the collar is mapped to N. This map is called *hollowing in M at N* with *trace N* and *hollow wall $p^{-1}(N)$*. For a subspace $L \subset M$ the *pullback p^*L of L along p* is the closure of $p^{-1}(L \setminus N)$. For the precise definitions we refer to Appendix B.

4.1.1. The Setup for the Filling Step

Let $n \in \mathbb{N}$. Let M be a smooth oriented compact connected n-manifold. Let $d \in \mathbb{N}$ and let

$$M_d \xrightarrow{p_{d-1}} M_{d-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{p_1} M_1 \xrightarrow{p_0} M_0 = M$$

be an admissible sequence of hollowings p_i in manifolds with corners M_i at submanifolds $X_i \subset M_i$ with hollow walls $N_i = p_i^{-1}(X_i) \subset M_{i+1}$ (see Definition B.3.1).

This vaguely means that every component Y of each X_i is either

- transverse to the boundary ∂M_i or
- is a subset of the "old part" of ∂M_i that did not arise from the previous hollowings and which is transverse to the "new part" of ∂M_i.

For more details about and references for manifolds with corners and hollowings at submanifolds see Appendix B. In particular, we explain there how to hollow at submanifolds that fulfil the second of the above conditions, i.e., that are submanifolds at the boundary, since this construction is non-standard.

We need some notation regarding the geometric construction: Let

$$\widetilde{N}_{-1} := p_{d,0}^* \partial M \subset M_d \quad \text{and} \quad \widetilde{N}_{d-1} := N_{d-1} \subset M_d$$

and for all $i \in \{0, \ldots d-2\}$ let

$$\widetilde{N_i} := p_{d,i+1}^* N_i \subset M_d,$$

where

$$p_{j,i}^* L := p_{j-1}^* \ldots p_i^* L \subset M_j$$

for all $i, j \in \{0, \ldots, d\}$ with $i < j$ and all subspaces $L \subset M_i$. For all $i, j \in \{0, \ldots, d\}$ with $i < j$ we define

$$p_{j,i} := p_i \circ \cdots \circ p_{j-1} \colon M_j \longrightarrow M_i$$

and we set $p_{i,i} := \mathrm{id}_{M_i}$. We define

$$\widetilde{N}_{j_1,\ldots,j_k} := \bigcap_{i \in \{j_1,\ldots,j_k\}} \widetilde{N}_i \subset M_d$$

and

$$X_{j_1,\ldots,j_k} := p_{d,j_1}(\widetilde{N}_{j_1,\ldots,j_k}) \subset M_{j_1}.$$

for all $k \in \{2, \ldots, d+1\}$ and pairwise distinct $j_1, \ldots, j_k \in \{-1, \ldots, d-1\}$. We set

$$\widetilde{N}_{j_1,\ldots,j_k} = \emptyset \quad \text{and} \quad X_{j_1,\ldots,j_k} = \emptyset$$

if the $j_1, \ldots, j_k$ are not pairwise distinct. See Table 4.1 for an overview of the relative positions of the defined objects so far.

$M_d \xrightarrow{p_{d-1}} \dots$	$\xrightarrow{p_{j_1+1}} M_{j_1+1}$	$\xrightarrow{p_{j_1}} M_{j_1}$	$\xrightarrow{p_{j_1-1}} \dots$	$\xrightarrow{p_0} M_0 = M$
$\widetilde{N}_{-1}$ $\dots$	-	-	$\dots$	∂M
$\widetilde{N}_{j_1}$ $\dots$	N_{j_1}	X_{j_1}	$\dots$	-
$\cup$		$\cup$		
$\widetilde{N}_{j_1,\dots,j_k}$ $\dots$	-	$X_{j_1,\dots,j_k}$	$\dots$	-

Table 4.1.: Relative position of the defined objects

Moreover, we need notation to work with parametrised chains: Let α be an essentially free standard $\pi(M)$-space. We define a sequence

$$C_*(M_d;\alpha_d) \xrightarrow{P_{d-1}} \cdots \xrightarrow{P_0} C_*(M_0;\alpha_0)$$

of chain maps: Let $\alpha_0 = \alpha$ and for all $j \in \{1,\dots,d\}$ let α_j be the standard $\pi(M_j)$-space given by

$$\alpha_j := \alpha \circ \pi(p_{j,0})\colon \pi(M_j) \longrightarrow \text{SBP}\,.$$

Now, we can define chain maps

$$\begin{aligned} P_j\colon C_*(M_{j+1};\alpha_{j+1}) &\longrightarrow C_*(M_j;\alpha_j) \\ f\cdot\sigma &\longmapsto f\cdot p_j\circ\sigma \end{aligned}$$

for all $j \in \{0,\dots,d-1\}$. Notice that the chain maps P_j are *not* relative chain maps regarding the boundaries of M_{j+1} and M_j. Obviously, we have $||P_j|| \le 1$ with respect to the parametrised ℓ^1-norm and we define

$$P_{j,i} := P_i \circ \cdots \circ P_{j-1}\colon C_*(M_j;\alpha_j) \longrightarrow C_*(M_i;\alpha_i)$$

for $i < j$ and we set $P_{i,i} := \text{id}$. Let $k \in \{1,\dots,d+1\}$ and $j_1,\dots,j_k \in \{-1,\dots,d-1\}$ be pairwise distinct. Let $\iota_{j_1,\dots,j_k}\colon \pi(X_{j_1,\dots,j_k}) \longrightarrow \pi(M_{j_1})$ be the functor induced by the inclusion $X_{j_1,\dots,j_k} \subset M_{j_1}$. Then we have an inclusion

$$C_*(X_{j_1,\dots,j_k};\alpha'_{j_1,\dots,j_k}) \subset C_*(M_{j_1};\alpha_{j_1}),$$

where $\alpha'_{j_1,\dots,j_k} := \alpha_{j_1} \circ \iota_{j_1,\dots,j_k}$.

For all chains $z \in C_n(M_d;\alpha_d)$ we define the following: For all $j \in \{-1,\dots,d-1\}$ let

$$z_j := (\partial z)|_{\widetilde{N}_j}.$$

Recall that for a chain $c = \sum_{j=1}^k a_j \cdot \sigma_j \in C_n(M;\alpha)$ and a subspace $L \subset M$ we define the restriction of c to L by

$$c|_L := \sum_{\substack{j\in\{1,\dots,k\} \\ \text{im}(\sigma_j)\subset L}} a_j \cdot \sigma_j \in C_n(M;\alpha).$$

Then, we inductively define

$$z_{j_1,\dots,j_k} := (\partial z_{j_1,\dots,j_{k-1}})|_{\widetilde{N}_{j_1,\dots,j_k}}$$

for all $k \in \{2,\dots,d+1\}$ and all $j_1,\dots,j_k \in \{-1,\dots,d-1\}$ that are pairwise distinct. See Table 4.2 for an overview of all defined objects.

We require that the following statements hold:

1. There exists an embedded n-ball $U \subset M^\circ$ in the interior of M that is not affected by any of the hollowings, i.e., with $U \cap p_{d,0}(\partial M_d) = \emptyset$.

2. We have that

$$\angle^{(k)} M_d = \bigcup_{j_1,\dots,j_k \in \{-1,\dots,d-1\}} \widetilde{N}_{j_1,\dots,j_k} \quad \text{and} \quad \partial \widetilde{N}_{j_1,\dots,j_k} = \bigcup_{j=-1}^{d-1} \widetilde{N}_{j_1,\dots,j_k,j}$$

   are decompositions into subpolyhedra for all $k \in \{2,\dots,d+1\}$ and $j_1,\dots,j_k \in \{-1,\dots,d-1\}$. Moreover, we assume $d \geq n-1$ and $\widetilde{N}_{j_1,\dots,j_k} = \emptyset$ if either
   - $k = n-1$ and $j_1,\dots,j_k \in \{0,\dots,d-1\}$ with $\{j_1,\dots,j_k\} \neq \{0,\dots,n-2\}$
   - or $k \in \{n,\dots,2n-2\}$ and $j_1,\dots,j_k \in \{-1,\dots,2n-4\}$.

3. For all $k \in \{1,\dots,d\}$ and $j_1,\dots,j_k \in \{0,\dots,d-1\}$ with $j_1 > \cdots > j_k$ the space $X_{j_1,\dots,j_k}$ satisfies parametrised UBC, i.e., the normed chain complex $C_*(X_{j_1,\dots,j_k}; \alpha'_{j_1,\dots,j_k})$ satisfies UBC. We also assume that $X_{j_1,\dots,j_k,-1}$ is the union of the connected components Y in $X_{j_1,\dots,j_k}$ with

$$Y \subset p_{n-2,j_1}(\widetilde{N}_{-1}).$$

   Moreover, we have that $H_\ell(X_{j_1,\dots,j_k}; \alpha'_{j_1,\dots,j_k}) = 0$ holds for all $\ell \geq 2$.

4. We have $\left| M_d, \partial M_d \right|^{\alpha_d} = 0$. More precisely, for every $\varepsilon \in \mathbb{R}_{>0}$ there exists a relative α_d-parametrised fundamental cycle z of M_d that respects the structure of M_d as a manifold with corners, i.e., every open k-face of the support of each simplex in ∂z (in reduced form) is contained in exactly one of the $\angle^{(l)} M_d \setminus \angle^{(l+1)} M_d$ with $l \in \{1,\dots,n-k\}$. Furthermore, we assume that

$$P_{d,n-2}(z_{n-2,\dots,0})$$

   is a null-homologous cycle in $C_1(X_{n-2,\dots,0}; \alpha'_{n-2,\dots,0})$.

4.1.2. The Filling Step

We are now prepared to do the filling step if we are in the situation of Section 4.1.1. More precisely, given a nice relative parametrised fundamental cycle z of M_d with small ℓ^1-norm we repair $P_{d,0}(z)$ efficiently to obtain a relative parametrised fundamental cycle of M with small norm via an inductive filling technique.

object	information	in words
p_j	$M_{j+1} \longrightarrow M_j$	hollowing in M_j at X_j
$p_{j,i}$	$M_j \longrightarrow M_i$	composition of the corresponding hollowings
$p_j^* L$	$\subset M_{j+1}$	pullback of L along p_j
$p_{j,i}^* L$	$\subset M_j$	pullback of L along $p_{j,i}$
X_j	$\subset M_j$	trace of p_j
N_j	$\subset M_{j+1}$	hollow wall of p_j
$\widetilde{N}_{-1}$	$\subset M_d$	pullback of the boundary of M along $p_{d,0}$
$\widetilde{N}_j$	$\subset M_d$	pullback of N_j along $p_{d,j+1}$
$\widetilde{N}_{j_1,\ldots,j_k}$	$\subset \widetilde{N}_{j_1}$	intersection of $\widetilde{N}_{j_1}, \ldots, \widetilde{N}_{j_k}$
$X_{j_1,\ldots,j_k}$	$\subset X_{j_1}$	pushforward of $\widetilde{N}_{j_1,\ldots,j_k}$ along p_{d,j_1}
α	standard $\pi(M)$-space	the standard $\pi(M)$-space we start with
α_j	standard $\pi(M_j)$-space	restriction of α along $\pi(p_{j,0})$
P_j	$C_*(M_{j+1}; \alpha_{j+1}) \longrightarrow C_*(M_j; \alpha_j)$	chain map induced by p_j
$P_{j,i}$	$C_*(M_j; \alpha_j) \longrightarrow C_*(M_i; \alpha_i)$	composition of the corresponding chain maps
$\alpha'_{j_1,\ldots,j_k}$	standard $\pi(X_{j_1,\ldots,j_k})$-space	restriction of α_{j_1} along $\pi(X_{j_1,\ldots,j_k}) \subset \pi(M_{j_1})$
z	$\in C_n(M_d; \alpha_d)$	the parametrised chain we start with
z_j	$\in C_{n-1}(M_d; \alpha_d)$	restriction of ∂z to $\widetilde{N}_j$
$z_{j_1,\ldots,j_k}$	$\in C_{n-k}(M_d; \alpha_d)$	restriction of $\partial z_{j_1,\ldots,j_{k-1}}$ to $\widetilde{N}_{j_1,\ldots,j_k}$

Table 4.2.: Overview of all defined objects

Theorem 4.1.1 (integral foliated simplicial volume and hollowings)**.** *If we are in the situation described in Section 4.1.1 then*

$$\left| M, \partial M \right|^{\alpha} = 0.$$

Proof of Theorem 4.1.1. Assume that we are in the situation described in Section 4.1.1. Let $\varepsilon \in \mathbb{R}_{>0}$. Then, by Assumption 4 in Section 4.1.1 there exists a (representative of a) relative α_d-parametrised fundamental cycle $z \in C_n(M_d; \alpha_d)$ with $|z|_1 \leq \varepsilon$ that respects the structure of ∂M_d as a manifold with corners.

We need the following lemmas:

Lemma 4.1.2. *We have*

$$\partial z = \sum_{j=-1}^{d-1} z_j \quad \textit{and} \quad \partial z_{j_1,\ldots,j_k} = \sum_{j=-1}^{d-1} z_{j_1,\ldots,j_k,j}$$

for all $k \in \{1, \ldots, d\}$ and all $j_1, \ldots, j_k \in \{0, \ldots, d-1\}$ that are pairwise distinct.

Proof. This immediately follows from the fact that z respects the structure of ∂M_d as a manifold with corners and from Assumption 2 in Section 4.1.1. □

Note that by Lemma 4.1.2 and Assumption 2 in Section 4.1.1, it follows that we have $z_{j_1,\ldots,j_k} = 0$ if $k \geq n$ or ($k = n-1$ and $\{j_1, \ldots, j_k\} \in \{0, \ldots, d-1\}$ with $\{j_1, \ldots, j_k\} \neq \{0, \ldots, n-2\}$).

Lemma 4.1.3. *Let $k \in \{1, \ldots, d+1\}$ and let τ be a permutation of $\{1, \ldots, k\}$. Then we have*

$$z_{j_1,\ldots,j_k} = \operatorname{sign}(\tau) \cdot z_{j_{\tau(1)},\ldots,j_{\tau(k)}}$$

for all $j_1, \ldots, j_k \in \{-1, \ldots, d-1\}$ that are pairwise distinct.

Proof. We may assume that τ is a transposition. Furthermore, it suffices to consider the case of swapping the last two indices, i.e., to show that

$$z_{j_1,\ldots,j_k} = -z_{j_1,\ldots,j_{k-2},j_k,j_{k-1}},$$

because: Let $\tau = (\ell\ m)$ with $\ell < m$. Then, we know that

$$z_{j_1,\ldots,j_{\ell-1},j_\ell,j_m} = -z_{j_1,\ldots,j_{\ell-1},j_m,j_\ell}$$

by swapping the last two indices. It follows that

$$z_{j_1,\ldots,j_{\ell-1},j_\ell,j_m,j_{\ell+1}} = -z_{j_1,\ldots,j_{\ell-1},j_m,j_\ell,j_{\ell+1}}$$

holds. Now, by swapping the last two indices, we have

$$z_{j_1,\ldots,j_{\ell-1},j_\ell,j_{\ell+1},j_m} = -z_{j_1,\ldots,j_{\ell-1},j_m,j_{\ell+1},j_\ell}.$$

Continuing this pattern, we obtain

$$z_{j_1,\ldots,j_m} = -z_{j_1,\ldots,j_{\ell-1},j_m,j_{\ell+1},\ldots,j_{m-1},j_\ell}.$$

Then it follows that

$$z_{j_1,\ldots,j_k} = -z_{j_1,\ldots,j_{\ell-1},j_m,j_{\ell+1},\ldots,j_{m-1},j_\ell,j_{m+1},\ldots,j_k}$$

holds. Therefore, we only have to prove the case of swapping the last two indices.

By Lemma 4.1.2 we have

$$0 = \partial\partial z_{j_1,\ldots,j_{k-2}} = \sum_{j=-1}^{d-1} \partial z_{j_1,\ldots,j_{k-2},j} = \sum_{j=-1}^{d-1}\sum_{i=-1}^{d-1} z_{j_1,\ldots,j_{k-2},j,i}$$

Since z respects the structure of M_d as a manifold with corners by assumption, it follows that the only term that can cancel $z_{j_1,\ldots,j_k}$ out is a term that has the same indices, namely $z_{j_1,\ldots,j_{k-2},j_k,j_{k-1}}$ and therefore,

$$z_{j_1,\ldots,j_k} = -z_{j_1,\ldots,j_{k-2},j_k,j_{k-1}}$$

holds. □

Lemma 4.1.4. *For all $k \in \{1,\ldots,d\}$ and all $j_1,\ldots,j_k \in \{0,\ldots,d-1\}$ there exist families of chains*

$$w_{j_1,\ldots,j_k} \in C_{n-k+1}(X_{j_1,\ldots,j_k}; \alpha'_{j_1,\ldots,j_k})$$

and

$$w_{j_1,\ldots,j_k,-1} \in C_{n-k}(X_{j_1,\ldots,j_k,-1}; \alpha'_{j_1,\ldots,j_k,-1})$$

satisfying the following conditions:

1. *The chains $w_{j_1,\ldots,j_k}$ and $w_{j_1,\ldots,j_k,-1}$ are alternating with respect to permutations of the indices $j_1,\ldots,j_k$.*
2. *We have*
$$\partial w_{j_1,\ldots,j_k} = P_{d,j_1}(z_{j_1,\ldots,j_k}) - \sum_{j=-1}^{d-1} w_{j_1,\ldots,j_k,j}$$
and
$$\partial w_{j_1,\ldots,j_k,-1} = P_{d,j_1}(z_{j_1,\ldots,j_k,-1}) + \sum_{j=0}^{d-1} w_{j_1,\ldots,j_k,j,-1}.$$
3. *Let C be a ℓ-UBC-constant for all $X_{j_1,\ldots,j_k}$ for all degrees $\ell \in \{1,\ldots,n-1\}$ (which exists by Assumption 3 in Section 4.1.1). Then we have*
$$|w_{j_1,\ldots,j_k}|_1 \leq C \cdot B^{n-k+1} \cdot (n+1)! \cdot |z|_1$$
where $B := 1 + C \cdot d$.

Proof. We prove the lemma by downward induction on k. If $k \geq n$ we set $w_{j_1,\dots,j_k} := 0$. Let $k = n-1$. If $\{j_1,\dots,j_k\} \neq \{0,\dots,n-2\}$, we set $w_{j_1,\dots,j_k} := 0$. By Lemma 4.1.2 and Assumption 2 in Section 4.1.1 we have

$$\partial z_{n-2,\dots,0} = \sum_{j=-1}^{d-1} z_{n-2,\dots,0,j} = 0$$

Therefore, $\widetilde{z}_{n-2,\dots,0} := P_{d,n-2}(z_{n-2,\dots,0})$ is a cycle in $C_1(X_{n-2,\dots,0};\alpha'_{n-2,\dots,0})$, which is null-homologous by Assumption 4 in Section 4.1.1. Because $X_{n-2,\dots,0}$ satisfies parametrised UBC by Assumption 3 in Section 4.1.1, there exists a chain $w_{n-2,\dots,0}$ with

$$\partial w_{n-2,\dots,0} = P_{d,n-2}(z_{n-2,\dots,0})$$

and

$$|w_{n-2,\dots,0}|_1 \leq C \cdot |z_{n-2,\dots,0}|_1 \leq C \cdot (n+1)! \cdot |z|_1,$$

where the term $(n+1)!$ comes from the operator norm of the boundary maps. For each permutation τ of $\{0,\dots,n-2\}$ we set

$$w_{\tau(n-2),\dots,\tau(0),} := \operatorname{sign}(\tau) \cdot P_{d,\tau(n-2)}(w_{n-2,\dots,0}).$$

The induction step consists of two parts. Let $k \in \{1,\dots,n-2\}$ such that $w_{j_1,\dots,j_k,j}$ is defined for all $j_1,\dots,j_k \in \{0,\dots,d-1\}$ and all $j \in \{0,\dots,d-1\}$. We first construct $w_{j_1,\dots,j_k,-1}$ and then $w_{j_1,\dots,j_k}$ for all $j_1,\dots,j_k \in \{0,\dots,d-1\}$. To this end, let $j_1,\dots,j_k \in \{0,\dots,d-1\}$ with $j_1 > j_2 > \dots > j_k$.

Step 1: We want to define $w_{j_1,\dots,j_k,-1}$. We consider

$$\hat{z}_{j_1,\dots,j_k} := P_{d,j_1}(z_{j_1,\dots,j_k}) - \sum_{j=0}^{d-1} w_{j_1,\dots,j_k,j} \in C_{n-k}(X_{j_1,\dots,j_k};\alpha'_{j_1,\dots,j_k})$$

and use Lemma 4.1.2 and the formula in statement 2 of Lemma 4.1.4 for all $\partial w_{j_1,\dots,j_k,j}$ with $j \geq 0$ to verify that

$$\begin{aligned}
\partial \hat{z}_{j_1,\dots,j_k} &= P_{d,j_1}(\partial z_{j_1,\dots,j_k}) - \sum_{j=0}^{d-1} \partial w_{j_1,\dots,j_k,j} \\
&= \sum_{j=-1}^{d-1} P_{d,j_1}(z_{j_1,\dots,j_k,j}) - \sum_{j=0}^{d-1} P_{d,j_1}(z_{j_1,\dots,j_k,j}) + \sum_{j=0}^{d-1}\sum_{i=-1}^{d-1} w_{j_1,\dots,j_k,j,i} \\
&= P_{d,j_1}(z_{j_1,\dots,j_k,-1}) + \sum_{j=0}^{d-1} w_{j_1,\dots,j_k,j,-1},
\end{aligned}$$

where for the last equality we observe that

$$\sum_{j=0}^{d-1}\sum_{i=0}^{d-1} w_{j_1,\dots,j_k,j,i} = 0$$

since the $w_{j_1,\dots,j_k}$ are alternating with respect to permutations of the indices $j_1, \dots, j_k$. We have that

$$\partial \hat{z}_{j_1,\dots,j_k} \in C_{n-k-1}(X_{j_1,\dots,j_k,-1}; \alpha'_{j_1,\dots,j_k,-1})$$

is a cycle, which is null-homologous because $X_{j_1,\dots,j_k,-1}$ is a union of connected components in $X_{j_1,\dots,j_k}$ and we can set

$$w_{j_1,\dots,j_k,-1} := (\hat{z}_{j_1,\dots,j_k})|_{X_{j_1,\dots,j_k,-1}} \in C_{n-k}(X_{j_1,\dots,j_k,-1}; \alpha'_{j_1,\dots,j_k,-1}).$$

For all $i_1, \dots, i_k \in \{0, \dots, d-1\}$, we define $w_{i_1,\dots,i_k,-1} := 0$ if $i_1, \dots, i_k$ are not pairwise distinct and otherwise we define

$$w_{i_1,\dots,i_k,-1} := \operatorname{sign}(\tau) \cdot P_{i_{\tau(1)},i_1}(w_{i_{\tau(1)},\dots,i_{\tau(k)},-1}),$$

where τ is the permutation on $\{1, \dots, k\}$ with $i_{\tau(1)} > \dots > i_{\tau(k)}$.

Step 2: Now, we want to define $w_{j_1,\dots,j_k}$. We observe that

$$\widetilde{z}_{j_1,\dots,j_k} := P_{d,j_1}(z_{j_1,\dots,j_k}) - \sum_{j=-1}^{d-1} w_{j_1,\dots,j_k,j} = \hat{z}_{j_1,\dots,j_k} - w_{j_1,\dots,j_k,-1}$$

is a cycle in $C_{n-k}(X_{j_1,\dots,j_k}; \alpha'_{j_1,\dots,j_k})$. Furthermore, the cycle $\widetilde{z}_{j_1,\dots,j_k}$ is null-homologous because by assumption we have

$$H_\ell(X_{j_1,\dots,j_k}; \alpha'_{j_1,\dots,j_k}) \cong 0$$

for all $\ell \in \mathbb{N}_{\geq 2}$. We can apply the parametrised uniform boundary condition for $X_{j_1,\dots,j_k}$. Then, there exists a chain $w_{j_1,\dots,j_k} \in C_{n-k+1}(X_{j_1,\dots,j_k}; \alpha'_{j_1,\dots,j_k})$ with

$$\partial w_{j_1,\dots,j_k} = \widetilde{z}_{j_1,\dots,j_k} = P_{d,j_1}(z_{j_1,\dots,j_k}) - \sum_{j=-1}^{d-1} w_{j_1,\dots,j_k,j}$$

and

$$\begin{aligned}
|w_{j_1,\dots,j_k}|_1 &\leq C \cdot |\widetilde{z}_{j_1,\dots,j_k}|_1 \\
&\leq C \cdot |\hat{z}_{j_1,\dots,j_k}|_1 \\
&\leq C \cdot (n+1)! \cdot |z|_1 + C \cdot d \cdot C \cdot B^{n-k-1+1} \cdot (n+1)! \cdot |z|_1 \\
&\leq (1 + C \cdot d) \cdot B^{n-k} \cdot C \cdot (n+1)! \cdot |z|_1 \\
&= C \cdot B^{n-k+1} \cdot (n+1)! \cdot |z|_1,
\end{aligned}$$

where the second inequality follows from the definition of $w_{j_1,\dots,j_k,-1}$ and the last inequality follows from $B \geq 1$. For all $i_1, \dots, i_k \in \{0, \dots, d-1\}$, we define $w_{i_1,\dots,i_k} :=$ 0 if $i_1, \dots, i_k$ are not pairwise distinct and otherwise we define

$$w_{i_1,\dots,i_k} := \operatorname{sign}(\tau) \cdot P_{i_{\tau(1)},i_1}(w_{i_{\tau(1)},\dots,i_{\tau(k)}}),$$

where τ is the unique permutation on $\{1, \dots, k\}$ with $i_{\tau(1)} > \dots > i_{\tau(k)}$.

By induction the lemma follows. □

$$\begin{array}{ccccc} C_n(U,\partial U;\mathbb{Z}) & \longrightarrow & C_n(M_d, M_d \setminus U; \alpha_d) & \longleftarrow & C_n(M_d, \partial M_d; \alpha_d) \\ \| & & \downarrow & & \\ C_n(U,\partial U;\mathbb{Z}) & \longrightarrow & C_n(M, M \setminus U; \alpha) & \longleftarrow & C_n(M, \partial M; \alpha) \end{array}$$

Figure 4.1.: Proving that z' is a (local) fundamental cycle

To finish the proof of Theorem 4.1.1 we set

$$z' := P_{d,0}(z) - \sum_{j=0}^{d-1} P_{j,0}(w_j).$$

Then, z' is a relative cycle with

$$|z'|_1 \leq C \cdot B^n \cdot (n+1)! \cdot \varepsilon.$$

It is left so show, that z' is a relative α-parametrised fundamental cycle of M. Let $U \subset M^\circ$ be an embedded n-ball in the interior of M with $U \cap p_{d,0}(\partial M_d) = \emptyset$, i.e., U is not affected by any of the hollowings (this exists by Assumption 1). By the local criterion for parametrised fundamental cycles (see Proposition 1.3.11) it is left to show that z' represents the image of $[U, \partial U]_{\mathbb{Z}}$ in $H_n(M, M \setminus U; \alpha)$. By construction and the choice of U we have $P_{d,0}(z) = z'$ in $C_n(M, M \setminus U; \alpha)$. Since z is a parametrised relative fundamental cycle of M_d, we have that z represents the image of $[U, \partial U]_{\mathbb{Z}}$ in $H_n(M_d, M_d \setminus U; \alpha_d)$ (note that this also holds if M_d is not connected). By the commutativity of the diagram Figure 4.1 it follows that also $P_{d,0}(z)$ represents the image of $[U, \partial U]_{\mathbb{Z}}$ in $H_n(M, M \setminus U; \alpha)$. Hence, z' is a relative α-parametrised fundamental cycle of M.

Since ε can be chosen arbitrarily small, we have $\left| M, \partial M \right|^{\alpha} = 0$. □

4.2. The Deconstruction Step for the Case of S^1-Actions

Now, we focus on manifolds with non-trivial S^1-action. Since we already worked out the filling step in the previous section, we will now mainly discuss the deconstruction step and then we have to verify that we can apply the filling technique from Section 4.1. The goal is to prove the following theorem.

Theorem 4.2.1 (parametrised simplicial volume and S^1-actions [12, Theorem 1.1])**.** *Let M be a smooth oriented compact connected manifold that admits a non-trivial smooth S^1-action. Let α be a standard $\pi(M)$-space that restricts to essentially free standard spaces on every orbit. Then, $\left| M, \partial M \right|^{\alpha} = 0$.*

In particular, if the inclusion of every orbit $S^1 \cdot x$ into M is π_1-injective then we have $\left| M, \partial M \right|^{\alpha} = 0$ for all essentially free standard $\pi(M)$-spaces α.

Proof. We begin by investigating the structure on M that comes with the non-trivial S^1-action. The details and references can be found in Appendix A. Without loss of generality, we may assume that the S^1-action is *effective*, i.e., there exists no $g \in S^1 \setminus \{e\}$ with $g \cdot x = x$ for all $x \in M$. From that, it follows that the *principal orbit type* is (1), i.e., the set of points in M with isotropy group $\{1\} \subset S^1$ is dense in M. Let F be the fixed point set, for all $r \in \mathbb{N}_{\geq 2}$, let L_r be the set of points whose isotropy groups contain $\{0, 1/r, \ldots, (r-1)/r\} \subset \mathbb{R}/\mathbb{Z} \cong S^1$ and let $L := \bigcup_{r \in \mathbb{N}_{\geq 2}} L_r$. Then F, L and each L_r are smooth invariant submanifolds of M and for sufficiently large r the set $L_r \setminus F$ is empty. Furthermore, there exists a triangulation of the orbit space $\overline{M}$ such that open simplices have constant isotropy group and $\overline{F}$ and all $\overline{L}_r$ are subcomplexes. We have

$$\dim F \leq n-2 \quad \text{and} \quad \dim L \setminus F \leq n-2$$

and therefore for the orbit spaces we have

$$\dim \overline{F} \leq n-2 \quad \text{and} \quad \dim \overline{L} \setminus \overline{F} \leq n-3.$$

Now, we consider the sequence of hollowings

$$M_{2n-3} \xrightarrow{p_{2n-4}} M_{2n-4} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{p_1} M_1 \xrightarrow{p_0} M_0 = M$$

specified in the following: For all $j \in \{0, \ldots, 2n-4\}$ the map p_j is a hollowing of M_j at X_j, where

- for all $j \in \{0, \ldots, n-2\}$ we define

$$X_j := p_{j-1}^* \cdots p_0^* q^{-1}\big(\overline{M}^{(j)} \cap \overline{F}\big)$$

- and for all $j \in \{n-1, \ldots, 2n-4\}$ we set

$$X_j := p_{j-1}^* \cdots p_0^* q^{-1}\big(\overline{M}^{(j-n+1)}\big).$$

Here $q \colon M \longrightarrow \overline{M}$ is the orbit map. This is an admissible sequence of equivariant hollowings (see Definition B.3.1). Furthermore, note that we never hollow at an $(n-2)$-simplex in the triangulation of $\overline{M}$ that is part of $\partial\overline{M}$ because the dimension of the fixed point set of ∂M is bounded by $n-3$ (see Appendix B).

In the following we use the notation from Section 4.1.1. In order to apply Theorem 4.1.1 we need to show that the assumptions listed in Section 4.1.1 are satisfied.

Assumption 1 holds obviously, since we can set U to be an open ball in the preimage of an open $(n-1)$-simplex in the triangulation of $\overline{M}$.

The first part of Assumption 2, i.e., that we have decompositions

$$\angle^{(k)} M_{2n-3} = \bigcup \widetilde{N}_{j_1,\ldots,j_k} \quad \text{and} \quad \partial \widetilde{N}_{j_1,\ldots,j_k} = \bigcup_{j=-1}^{2n-4} \widetilde{N}_{j_1,\ldots,j_k,j}$$

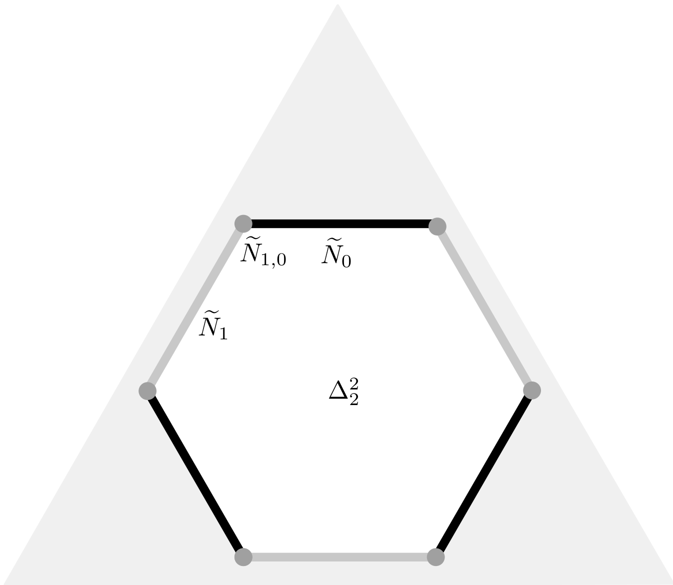

Figure 4.2.: Decomposition of the boundary of Δ^2 with the 0-skeleton and the 1-skeleton removed (also compare to Figure 4.3)

into subpolyhedra, also follows from the construction (see Figure 4.2 for an illustration of such a decomposition). We will discuss the second part of Assumption 2 later (see Lemma 4.2.6).

We first want to investigate the structure of the $X_{j_1,\dots,j_k}$ and of M_{2n-3}.

Lemma 4.2.2. *Let $k \in \{1, \dots, 2n-3\}$ and let $j_1, \dots, j_k \in \{0, \dots, 2n-4\}$ with $j_1 > \dots > j_k$. Then each connected component of $\overline{X}_{j_1,\dots,j_k}$ is contractible.*

Proof. Let $\ell \in \mathbb{N}$. We consider the sequence of hollowings

$$\Delta^\ell_\ell \longrightarrow \Delta^\ell_{\ell-1} \longrightarrow \cdots \longrightarrow \Delta^\ell_1 \longrightarrow \Delta^\ell_0 = \Delta^\ell$$

of the standard ℓ-simplex with the standard triangulation, where we hollow at the pullbacks of the skeleta (see Figure 4.3 for an illustration of this sequence for $\ell = 2$). By construction, each connected component of $\overline{X}_{j_1}$ is homeomorphic to Δ^ℓ_ℓ and thus is contractible, where $\ell = j_1$ if $j_1 \in \{0, \dots, n-2\}$ and $\ell = j_1 - n + 1$ if $j_1 \in \{n-1, \dots, 2n-4\}$.

Under this identification, each connected component of $\overline{X}_{j_1,\dots,j_k}$ for $k \geq 2$ is one of the closures of the connected components of

$$\angle^{(k-1)}\Delta^\ell_\ell \setminus \angle^{(k)}\Delta^\ell_\ell.$$

Indeed, two distinct components A, B of $\angle^{(k-1)}\Delta^\ell_\ell \setminus \angle^{(k)}\Delta^\ell_\ell$ with non-trivial intersection $\overline{A} \cap \overline{B}$ of the closures are mapped to faces of Δ that do *not* have the same dimension, but all components of $X_{j_1,\dots,j_k}$ are mapped to ℓ-dimensional faces of Δ^ℓ,

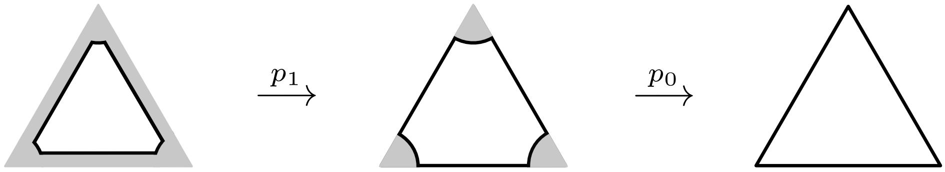

Figure 4.3.: Sequence of hollowings of Δ^2

where $\ell = j_k$ if $j_k \in \{0, \ldots, n-2\}$ and $\ell = j_k - n + 1$ if $j_k \in \{n-1, \ldots, 2n-4\}$. From the above, it follows that $\overline{X}_{j_1,\ldots,j_k}$ is homeomorphic to $\Delta^{\ell-k+1}_{\ell-k+1}$ and thus is contractible. $\square$

Lemma 4.2.3. *Let $k \in \{1, \ldots, 2n-3\}$ and let $j_1, \ldots, j_k \in \{0, \ldots, 2n-4\}$ with $j_1 > \cdots > j_k$. Then we have*

$$X_{j_1,\ldots,j_k} \cong \begin{cases} \overline{X}_{j_1,\ldots,j_k} & \text{if } 0 \leq j_1 \leq n-2 \\ \overline{X}_{j_1,\ldots,j_k} \times S^1 & \text{if } n-1 \leq j_1 \leq 2n-4. \end{cases}$$

Proof. If $j_1 \in \{0, \ldots, n-2\}$, we have by construction

$$X_{j_1,\ldots,j_k} \subset X_{j_1} \subset p^*_{j_1-1} \ldots p^*_0 F.$$

That means that S^1 acts trivially on $X_{j_1,\ldots,j_k}$ and therefore

$$X_{j_1,\ldots,j_k} \cong \overline{X}_{j_1,\ldots,j_k}.$$

If $j_1 \in \{n-1, \ldots 2n-4\}$ and Y is a connected component of $X_{j_1,\ldots,j_k}$ then either the restriction of the S^1-action on M_{j_1} to Y is free or the orbit type of all points in Y is $(H) = \big(\{0, 1/r, 2/r, \ldots, (r-1)/r\}\big)$ for some $r \in \mathbb{R}_{\geq 2}$ (since we already removed the fixed points and because open simplices in $\overline{M}$ have constant orbit type). In the second case we consider the free action of $S^1 \cong S^1/H$ on Y. Thus, $X_{j_1,\ldots,j_k}$ is the total space of a principal S^1-bundle over the contractible space $\overline{X}_{j_1,\ldots,j_k}$ and therefore a trivial S^1-bundle. $\square$

Lemma 4.2.4. *We have $M_{2n-3} \cong \overline{M}_{2n-3} \times S^1$ as manifolds with corners.*

Proof. As discussed above, we have $\dim \overline{F} \leq n-2$ and $\dim \overline{L} \setminus \overline{F} \leq n-3$. Therefore, the S^1-action on M_{2n-3} is free by construction. The space

$$\overline{M}' := \overline{M} \setminus \big(\overline{M}^{(n-3)} \cup \overline{F}\big)$$

and thus also $\overline{M}_{2n-3}$ has the homotopy type of a 1-complex. More precisely, we consider the graph G that is dual to the simplicial complex K, which we have chosen to triangulate $\overline{M}$, given by

- one vertex for each $(n-1)$-simplex and one for each $(n-2)$-simplex that is not part of $\overline{F}$ and
- an edge between two vertices if and only if one vertex corresponds to an $(n-1)$-simplex and the other corresponds to an $(n-2)$-face of that simplex.

Note, that there exist no simplices of dimension n or higher and that all $(n-2)$-simplices are a face of an $(n-1)$-simplex in our case. The graph G can be identified with a subcomplex of the barycentric subdivision of K such that the geometric realization of this subcomplex lies in $\overline{M}'$. One can now show that this embedded graph is a deformation retract of $\overline{M}'$ for example by retracting every $(n-1)$-simplex to a 1-dimensional star (that is the part of the embedded graph lying in that simplex) in a way that these retractions fit together on the connecting $(n-2)$-simplices.

Since any principal S^1-bundle over a 1-complex is trivial, we have that M_{2n-3} and $\overline{M}_{2n-3} \times S^1$ are homeomorphic via a homeomorphism f such that the diagram

$$\begin{array}{ccc} M_{2n-3} & \xrightarrow{f} & \overline{M}_{2n-3} \times S^1 \\ {\scriptstyle q_{2n-3}}\downarrow & & \downarrow{\scriptstyle p_{\overline{M}_{2n-3}}} \\ \overline{M}_{2n-3} & = & \overline{M}_{2n-3} \end{array}$$

commutes. That the homeomorphism f respects the structures of manifolds with corners follows from the commutativity of above diagram together with the fact that the corners $\angle^{(k)} M_{2n-3}$ are invariant by construction. Here and in the following, we consider the structure as a manifold with corners on $\overline{M}_{2n-3}$ (with the orientation such that the homeomorphism f is orientation-preserving) that arises from the "hollowings" $\overline{p}_k$ in the orbit spaces, i.e., such that

$$\angle^{(k)} \overline{M}_{2n-3} = q_{2n-3}(\angle^{(k)} M_{2n-3}),$$

and the canonical structure as a manifold with corners on $\overline{M}_{2n-3} \times S^1$, i.e., such that

$$\angle^{(k)} (\overline{M}_{2n-3} \times S^1) = (\angle^{(k)} \overline{M}_{2n-3}) \times S^1. \qquad \square$$

Now we can prove that the second part of Assumption 2 in Section 4.1.1 holds.

Lemma 4.2.5. *Let $k \in \{1, \dots, 2n-3\}$ and let $j_1, \dots, j_k \in \{0, \dots, 2n-4\}$ with $j_1 > \dots > j_k$. If $j_1 > n-2$ and $k > j_1 - n + 2$, then we have $\widetilde{N}_{j_1,\dots,j_k} = \emptyset$.*

Proof. If there are $i, i' \in \{1, \dots, 2n-3\}$ with $i' = i + n - 1$ then $\overline{N}_{i',i} = \overline{N}_{i'} \cap \overline{N}_i = \emptyset$ because: Let Y, Y' be connected components of $\overline{N}_i$ and $\overline{N}_{i'}$ respectively. Then there are i-simplices Δ, Δ' in the triangulation of $\overline{M}$ such that Y is the pullback of $\overline{p}_i^{-1}(\Delta_i)$ and Y' is the pullback of $p_{i'}^{-1}(\Delta_{i'})$, where $\Delta_i = \overline{p}_{i,0}^{*}\Delta$ denotes the pullback. Now, $Y \cap Y' = \emptyset$ follows from $\Delta_i \cap \Delta'_{i'} = \emptyset$. This holds obviously: Either Δ and Δ' are already disjoint, or they have a common face of dimension less than i and then the pullbacks Δ_i and $\Delta'_{i'}$ are disjoint.

If there is an $i \in \{1, \dots, 2n-3\}$ with $j_1 - n + 2 \leq i \leq n-2$ then $\overline{N}_{j_1,i} = \emptyset$, because: Let Y, Y' be connected components of $\overline{N}_{j_1}$ and $\overline{N}_i$ respectively. Then there is a $(j_1 - n + 1)$-simplex Δ and an i-simplex Δ' in the triangulation of $\overline{M}$ such that Y is the pullback of $p_{j_1}^{-1}(\Delta_{j_1})$ and Y' is the pullback of $p_i^{-1}(\Delta_i)$. Now, we are in one of the following three cases:

1. Δ and Δ' are disjoint,
2. $\Delta \cap \Delta'$ is a common face of dimension less than $j_1 - n + 1$, or
3. Δ is a face of Δ'.

In the first two cases $Y \cap Y' = \emptyset$ holds obviously. The third case is not possible because Δ' is a simplex in the simplicial complex $\overline{F}$, i.e., the orbit space of the fixed points, but Δ is not.

The result follows by elementary combinatorics: By the second paragraph of the proof, we may assume that for all $i \in \{1, \dots, k\}$ we either have

$$0 \leq j_i \leq j_1 - n + 1 \quad \text{or} \quad n - 1 \leq j_i \leq j_1.$$

Since $k > j_1 - n + 2$ there exist $i, i' \in \{1, \dots, k\}$ with $j_{i'} = j_i + n - 1$. Therefore,

$$\widetilde{N}_{j_1,\dots,j_k} \subset \widetilde{N}_{i,i'} = \emptyset$$

by the first paragraph of the proof. □

Lemma 4.2.6. *We have $\widetilde{N}_{j_1,\dots,j_k} = \emptyset$ if either*

- *$k = n - 1$ and $j_1, \dots, j_k \in \{0, \dots, 2n-4\}$ with $\{j_1, \dots, j_k\} \neq \{0, \dots, n-2\}$ or*
- *$k \in \{n, \dots, 2n-2\}$ and $j_1, \dots, j_k \in \{-1, \dots, 2n-4\}$.*

Proof. The first part follows from Lemma 4.2.5, because we may assume that $j_1 > j_2 > \cdots > j_k$ holds and then we have

$$k = n - 1 > n - 2 = 2n - 4 - n + 2 \geq j_1 - n + 2.$$

The second part directly follows from Lemma 4.2.4 because

$$\angle^{(n)}(\overline{M}_{2n-3} \times S^1) = \emptyset. \qquad \square$$

The following two lemmas show that Assumption 3 in Section 4.1.1 is satisfied. Note that we have

$$H_\ell(X_{j_1,\dots,j_k}; \alpha'_{j_1,\dots,j_k}) = 0$$

for all $\ell \in \mathbb{N}_{\geq 2}$ by Lemma 4.2.3 and Lemma 4.2.2.

Lemma 4.2.7. *For all $k \in \{1, \dots, 2n-3\}$ and $j_1, \dots, j_k \in \{0, \dots, 2n-4\}$ with $j_1 > \cdots > j_k$, the standard $\pi(X_{j_1,\dots,j_k})$-space $\alpha'_{j_1,\dots,j_k}$ is essentially free and $C_*(X_{j_1,\dots,j_k}; \alpha'_{j_1,\dots,j_k})$ satisfies UBC in every degree.*

Proof. By Lemma 4.2.2 and Lemma 4.2.3 we have that

$$X_{j_1,\dots,j_k} \cong \overline{X}_{j_1,\dots,j_k} \times S^1 \quad \text{or} \quad X_{j_1,\dots,j_k} \cong \overline{X}_{j_1,\dots,j_k}$$

and $\overline{X}_{j_1,\dots,j_k}$ is contractible. Hence, the composition $X_{j_1,\dots,j_k} \subset M_{j_1} \xrightarrow{p_{j_1,0}} M$ is homotopic to the inclusion of finitely many disjoint S^1-orbits and therefore the restriction $\alpha'_{j_1,\dots,j_k}$ of $\alpha_{j_1,\dots,j_k}$ is essentially free by the assumptions of Theorem 4.2.1. The normed chain complex $C_*(X_{j_1,\dots,j_k}; \alpha'_{j_1,\dots,j_k})$ satisfies parametrised UBC because we know parametrised UBC for the point and for S^1 (see Theorem 3.2.1). □

For the case with boundary we need in addition the following observation:

Lemma 4.2.8. *For all $k \in \{1, \dots, 2n-3\}$ and $j_1, \dots, j_k \in \{0, \dots, 2n-4\}$ with $j_1 > \dots > j_k$, we have that $X_{j_1,\dots,j_k,-1}$ is the union of those connected components Y of $X_{j_1,\dots,j_k}$ with*

$$Y \subset p_{2n-3,j_1}(\widetilde{N}_{-1}).$$

Proof. Because $X_{j_1,\dots,j_k,-1} \subset p_{2n-3,j_1}(\widetilde{N}_{-1})$ and $X_{j_1,\dots,j_k,-1} \subset X_{j_1,\dots,j_k}$, it suffices to prove that all connected components Y of $X_{j_1,\dots,j_k}$ with $Y \subset p_{2n-3,j_1}(\widetilde{N}_{-1})$ are contained in $X_{j_1,\dots,j_k,-1}$.

Let $j \in \{0, \dots, 2n-4\}$. First, we show the statement for $X_{j,-1} \subset X_j$. Let $Y \subset \overline{X}_j$ be a connected component. As in the proof of Lemma 4.2.4 we observe that Y is the pullback $\overline{p}^*_{\ell-1} \dots \overline{p}^*_0 \Delta$ of an ℓ-simplex Δ in the triangulation of $\overline{M}$, where $\ell = j$ if $j \in \{0, \dots, n-2\}$ and $\ell = j - n + 1$ if $j \in \{n-1, \dots, 2n-4\}$. There are two possibilities:

1. We have that $\Delta \subset \partial\overline{M}$, or
2. we have that $\Delta \cap \partial\overline{M}$ is empty or a simplex of dimension less than ℓ.

Assume that we are in the first case. Let Δ' be a $(n-2)$-simplex with $\Delta' \subset \partial\overline{M}$ and such that Δ is a face of Δ'. We already observed, that we never hollow at pullbacks of $(n-2)$-simplices at the boundary. It follows that

$$Y \subset \overline{p}_{2n-3,j}(\overline{p}^*_{2n-3,0}\Delta') \subset \overline{p}_{2n-3,j}(\overline{N}_{-1}).$$

Therefore, we have

$$\begin{aligned} Y \subset \overline{X}_j \cap \overline{p}_{2n-3,j}(\overline{N}_{-1}) &= \overline{p}_{2n-3,j}(\overline{N}_j) \cap \overline{p}_{2n-3,j}(\overline{N}_{-1}) \\ &\subset \overline{p}_{2n-3,j}(\overline{N}_{j,-1}) = \overline{X}_{j,-1}. \end{aligned}$$

Here, the last inclusion follows from

$$\overline{p}_{n-2,j}(\overline{N}_j \setminus \overline{N}_{-1}) \cap \overline{p}_{n-2,j}(\overline{N}_{-1} \setminus \overline{N}_j) = \emptyset.$$

This is a consequence of

$$\overline{p}_{n-2,j}(\overline{N}_{-1} \setminus \overline{N}_j) \subset \overline{p}_{n-2,j}(\overline{N}_{-1}) \setminus \overline{p}_{n-2,j}(\overline{N}_j),$$

which holds by construction (see Lemma B.3.2; here we identify a connected component of $\overline{N}_{-1}$ with a hollowed $(n-2)$-simplex and a component of $\overline{N}_{j,-1}$ with the pullback of a hollowed j-simplex).

In the second case, we have that $Y \cap \overline{p}_{2n-3,j}(\overline{N}_{-1}) = \emptyset$ holds and therefore, we have $Y \cap \overline{X}_{j,-1} = \emptyset$.

For the case of more than one index it is enough to show that

$$\overline{X}_{j_1,\dots,j_k,-1} = \overline{X}_{j_1,\dots,j_k} \cap \overline{X}_{j_1,-1}$$

holds. This follows similarly as before from

$$\overline{p}_{2n-3,j_1}(\overline{N}_{j_1,\dots,j_k} \setminus \overline{N}_{j_1,-1}) \cap \overline{p}_{2n-3,j_1}(\overline{N}_{j_1,-1} \setminus \overline{N}_{j_1,\dots,j_k}) = \emptyset.$$

This again is a consequence of

$$\overline{p}_{2n-3,j_1}(\overline{N}_{j_1,-1} \setminus \overline{N}_{j_1,\dots,j_k}) \subset \overline{p}_{2n-3,j_1}(\overline{N}_{j_1,-1}) \setminus \overline{p}_{2n-3,j_1}(\overline{N}_{j_1,\dots,j_k}),$$

which holds by construction (see Lemma B.3.2; here we identify a connected component of $\overline{N}_{j_1,-1}$ with the pullback of the preimage of a hollowed ℓ-simplex in a hollowed $(n-2)$-simplex where $\ell = j_1$ if $j_1 \in \{0,\dots,n-2\}$ and $\ell = j_1 - n + 1$ if $j_1 \in \{n-1,\dots,2n-4\}$ and a component of $\overline{N}_{j_1,\dots,j_k,-1}$ with the pullback of the preimage of a hollowed $(\ell - k + 1)$-simplex; note that if $k > \ell + 1$ then $\overline{N}_{j_1,\dots,j_k} = \emptyset$ by Lemma 4.2.5). □

Now, we construct a relative parametrised fundamental cycle of M_{2n-3} which satisfies Assumption 4 in Section 4.1.1. Let α be a standard $\pi(M)$-space that restricts to essentially free standard spaces on every orbit. By Lemma 4.2.4 we have $M_{2n-3} \cong \overline{M}_{2n-3} \times S^1$ as manifolds with corners. Let $\overline{K}$ be a triangulation of $\overline{M}_{2n-3}$ that respects the structure of $\overline{M}_{2n-3}$ as a manifold with corners. Since $\overline{M}_{2n-3}$ is an oriented compact connected manifold we can construct a relative fundamental cycle $\overline{z} \in C_{n-1}(\overline{M}_{2n-3};\mathbb{Z})$ out of the triangulation $\overline{K}$. Let $\varepsilon \in \mathbb{R}_{>0}$. Let $\iota\colon \{x\} \times S^1 \longrightarrow M_{2n-3}$ be the inclusion of an orbit of the trivial S^1-bundle M_{2n-3} (for some $x \in \overline{M}_{2n-3}$). By assumption on α the restriction $\alpha' := \alpha_{2n-3} \circ \pi(\iota)$ is an essentially free $\pi(S^1)$-space and therefore, there exists a parametrised fundamental cycle $c_{S^1} \in C_1(S^1;\alpha')$ of small norm [43, Lemma 10.8] such that

$$z := \overline{z} \times c_{S^1} \in C_n(M_{2n-3};\alpha_{2n-3})$$

has ℓ^1-norm less than ε. For the definition of a cross product map in the case of parametrised chains with twisted coefficients we refer to the work of Schmidt [43, Definition 5.32]. Via Proposition 1.3.4 we also obtain a cross product for parametrised chains with local coefficients.

By Lemma 4.2.6 we have $z_{j_1,\dots,j_k} = 0$ if $k > n-1$ or

$$k = n-1 \quad \text{and} \quad \{j_1,\dots,j_k\} \neq \{0,\dots,n-2\}.$$

The chain $P_{2n-3,n-2}(z_{n-2,\dots,0}) \in C_1(X_{n-2,\dots,0};\alpha'_{n-2,\dots,0})$ is a cycle by Lemma 4.1.2 and Lemma 4.2.6 which is null-homologous because every component of $X_{n-2,\dots,0}$ is contractible by Lemma 4.2.2 and Lemma 4.2.3.

By Theorem 4.1.1, we have $\left| M, \partial M \right|^{\alpha} = 0$. □

4.3. The Deconstruction Step for S^1-Bundles

In this section, we will discuss the deconstruction step for proving vanishing of parametrised simplicial volume of smooth manifolds that are the total space of a smooth S^1-bundle. Afterwards we prove that the filling step, which is explained in Section 4.1, can be applied also in the case of S^1-bundles. We aim to prove the following theorem.

Theorem 4.3.1 (parametrised simplicial volume and S^1-bundles [13, Prop. 4.2]). *Let M be a smooth oriented compact connected n-manifold that is the total space of a smooth S^1-bundle $q\colon M \longrightarrow B$ over a compact smooth $(n-1)$-manifold B. Then, for all standard $\pi(M)$-spaces α that restrict to essentially free standard spaces on every fibre, we have $\left| M, \partial M \right|^{\alpha} = 0$.*

In particular, if all fibres are π_1-injective in M then we have $\left| M, \partial M \right|^{\alpha} = 0$ for all essentially free standard $\pi(M)$-spaces α.

Proof. We choose a triangulation of B; then the bundle p is trivial over every simplex in this triangulation. We consider the admissible sequence

$$M_{n-1} \xrightarrow{p_{n-2}} M_{n-2} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{p_1} M_1 \xrightarrow{p_0} M_0 = M,$$

of hollowings p_j in M_j at the pullback X_j of the preimage of the j-skeleton of B (see Definition B.3.1), i.e.,

$$X_j := p_{j-1}^* \dots p_0^* p^{-1}(B^{(j)})$$

for all $j \in \{0, \dots n-3\}$ and

$$X_{n-2} := p_{n-3}^* \dots p_0^* p^{-1}(\overline{B^{(n-2)} \setminus \partial B}).$$

In the last step we only hollow at pullbacks of preimages of $(n-2)$-faces of B that are not part of the boundary. The resulting space would be the same (up to isomorphism of manifolds with corners) if we were hollowing at all faces, but we would lose the information about which simplices were part of the boundary. We can perform the hollowings in a way that the tubular neighbourhoods that we remove are smooth subbundles. Then we have that all the M_i are S^1-bundles and the maps p_i are smooth bundle maps. We write $q_j\colon M_j \longrightarrow B_j$ for the according bundles and $\overline{p}_{j-1}\colon B_j \longrightarrow B_{j-1}$ for the according maps for all $j \in \{1, \dots, n-1\}$.

Now, let α be a $\pi(M)$-space whose restrictions to the fibres are essentially free. In the following we use the notation from Section 4.1.1. Before we prove that the assumptions listed in Section 4.1.1 are satisfied, we investigate the structure of M_{n-1} and the $X_{j_1,\dots,j_k}$.

For every $(n-1)$-simplex Δ in the set $B^{[n-1]}$ of all $(n-1)$-simplices in the triangulation of B we consider the induced sequence of hollowings

$$\Delta_{n-1} \longrightarrow \cdots \longrightarrow \Delta_0 = \Delta$$

at the skeleta of Δ. Then

$$M_{n-1} \cong \coprod_{\Delta \in B^{[n-1]}} \Delta_{n-1} \times S^1$$

as manifolds with corners (with an appropriate orientation of the Δ_{n-1}).

Now, we study the structure of the $X_{j_1,\dots,j_k}$. Let $k \in \{1,\dots,n-1\}$ and $j_1,\dots,j_k \in \{0,\dots,n-2\}$ with $j_1 > j_2 > \cdots > j_k$. Analogously to Lemma 4.2.3 in the case of S^1-bundles, we can prove that $X_{j_1,\dots,j_k} \cong \overline{X}_{j_1,\dots,j_k} \times S^1$ and $\overline{X}_{j_1,\dots,j_k}$ is contractible. Here, $\overline{X}_{j_1,\dots,j_k} := q_{j_1}(X_{j_1,\dots,j_k}) \subset B_{j_1}$. Similarly as in Lemma 4.2.8 we can prove that $X_{j_1,\dots,j_k,-1}$ is a union of connected components of $X_{j_1,\dots,j_k}$.

Assumption 1 in Section 4.1.1 is satisfied by construction: we can choose an open n-ball V in the interior of some $\Delta_{n-1} \times S^1 \subset M_{n-1}$ such that $p_{n-1,0}|_V \colon V \longrightarrow p_{n-1,0}(V) = V$ is the identity. Then let U be an open n-ball in V with $\overline{U} \subset V$.

The first part of Assumption 2 holds by construction and we will not give the details. The second part can be proved analogously to Lemma 4.2.6 in the case of S^1-actions.

Assumption 3 in Section 4.1.1 holds in this case because for all $k \in \{1,\dots,n-1\}$ and all $j_1,\dots,j_k \in \{0,\dots,n-2\}$ with $j_1 > j_2 > \cdots > j_k$ we know

- that $X_{j_1,\dots,j_k}$ is homotopy equivalent to S^1 and
- the restriction $\alpha'_{j_1,\dots,j_k}$ of α are essentially free by assumption since they are restrictions to fibres.

Hence, $C_*(X_{j_1,\dots,j_k};\alpha'_{j_1,\dots,j_k})$ satisfies UBC by Theorem 3.2.1.

Now, we deal with Assumption 4 in Section 4.1.1. Let $\varepsilon \in \mathbb{R}_{>0}$. We will now construct a relative α_{n-1}-parametrised fundamental cycle of M_{n-1} with small norm that fulfils Assumption 4 in Section 4.1.1. For each $\Delta \in B^{[n-1]}$, we choose a triangulation of Δ_{n-1} that respects the structure of Δ_{n-1} as a manifold with corners. Let $z_\Delta \in C_n(\Delta_{n-1};\mathbb{Z})$ be a relative fundamental cycle obtained from the triangulation of Δ_{n-1} and the orientation inherited from the orientation of M_{n-1} (for example by taking the barycentric subdivision [37, Remark 3.6]). Since the restriction of α to every fibre is essentially free, we have that the restriction α_Δ of α_{n-2} to $\Delta_{n-1} \times S^1$ is essentially free. Therefore, there exits an α_Δ-parametrised fundamental cycle $c_\Delta^{S^1} \in C_1(S^1;\alpha_\Delta)$ of S^1 with $|c_\Delta^{S^1}|_1 < \varepsilon$ [43, Proposition 5.30] (note that Schmidt's proof also works for essentially free standard $\pi(S^1)$-spaces). We define

$$z := \sum_{\Delta \in B^{[n-1]}} z_\Delta \times c_\Delta^{S^1} \in C_n(M_{n-1};\alpha_{n-1}).$$

Then, z is a relative α_{n-1}-parametrised fundamental cycle of M_{n-1} with

$$|z|_1 \leq n \cdot \left|B^{[n-1]}\right| \cdot A \cdot \varepsilon,$$

where $A := \max\bigl\{|z_\Delta|_1 \bigm| \Delta \in B^{[n-1]}\bigr\}$.

Since $z_{n-2,\dots,0}$ is a cycle by Lemma 4.1.2 and Lemma 4.2.6 and P_{n-2} is a chain map, it is left to show that $P_{n-2}(z_{n-2,\dots,0})$ is null-homologous in $C_1(X_{n-2,\dots,0};\alpha'_{n-2,\dots,0})$. Observe that the hollowing map $\overline{p}_{n-2}\colon B_{n-1} \longrightarrow B_{n-2}$ is given by identifying some of the hollowed $(n-2)$-faces of B_{n-1} (namely the components of $\overline{N}_{n-2}$) in pairs. The space $\widetilde{N}_{n-2,\dots,0}$ then is a collection of circles in $\partial\widetilde{N}_{n-2}$ that are pairwise identified under p_{n-2}. By construction of the cycle z, we have that

- $z_{n-2,\dots,0}$ is an $\alpha'_{n-2,\dots,0}$-parametrised fundamental cycle of a disjoint union of several copies of S^1, one for every circle in $\widetilde{N}_{n-2,\dots,0}$ and
- the restrictions of $z_{n-2,\dots,0}$ to two different circles in $\widetilde{N}_{n-2,\dots,0}$ that are identified under p_{n-2} have different signs.

Therefore, $P_{n-2}(z_{n-2,\dots,0})$ is a null-homologous cycle in $C_1(X_{n-2,\dots,0};\alpha'_{n-2,\dots,0})$.

By Theorem 4.1.1, we have $\left| M, \partial M \right|^{\alpha} = 0$. □

4.4. Applications

In this section, we list some applications of Theorem 4.2.1 and Theorem 4.3.1 to gradient invariants. These are mainly taken from the work of the author [12].

We have a new approximation result for simplicial volume and hence vanishing results for the Betti number gradient, the torsion homology gradient and the rank gradient and cost of (fundamental groups of) aspherical smooth oriented compact connected manifolds that admit a non-trivial smooth S^1-action or are the total space of a smooth S^1-bundle over a compact smooth manifold with π_1-injective fibres.

First, we recall some definitions: The *(relative) stable integral simplicial volume* of an oriented compact connected manifold M with fundamental group Γ is given by

$$\|M,\partial M\|_{\mathbb{Z}}^{\infty} := \inf\bigl\{ \|\overline{M},\partial\overline{M}\|_{\mathbb{Z}}/d \bigm| \overline{M} \longrightarrow M \text{ is a } d\text{-sheeted covering}\bigr\}.$$

A *residual chain* in a finitely generated group Γ is a descending sequence $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ of normal finite index subgroups of Γ whose intersection is trivial.

Corollary 4.4.1 (stable integral simplicial volume and S^1-actions/-bundles)**.** *Let M be a smooth oriented compact connected manifold with residually finite fundamental group Γ that*

- *admits a non-trivial smooth S^1-action such that all orbits are π_1-injective, or*
- *is the total space of a smooth S^1-bundle over a smooth compact connected manifold with π_1-injective fibres.*

Then, we have

$$\|M,\partial M\|_{\mathbb{Z}}^{\infty} = 0.$$

More generally: If $(\Gamma_i)_{i\in\mathbb{N}}$ is a Farber chain [1] (e.g., a residual chain) in Γ, then

$$\inf_{i\in\mathbb{N}} \frac{\|M_i, \partial M_i\|_{\mathbb{Z}}}{[\Gamma : \Gamma_i]} = 0,$$

where $M_i \longrightarrow M$ denotes the covering of M associated to Γ_i for all $i \in \mathbb{N}$.

Proof. A Farber chain in a finitely generated group Γ yields a standard Γ-space that is essentially free [1, Section 3]. Now, the result follows from (a relative version of) the relation between integral foliated simplicial volume and stable integral simplicial volume [19, Theorem 2.6] and Theorem 4.2.1 or Theorem 4.3.1. □

Let Γ be a finitely generated group. Then the *rank gradient* of Γ (with respect to a Farber chain $(\Gamma_i)_{i\in\mathbb{N}}$ in Γ) is defined as

$$\operatorname{rg}(\Gamma) := \inf_{\Lambda\in F(\Gamma)} \frac{d(\Lambda)-1}{[\Gamma:\Lambda]} \quad \text{and} \quad \operatorname{rg}\big(\Gamma, (\Gamma_i)_{i\in\mathbb{N}}\big) := \inf_{i\in\mathbb{N}} \frac{d(\Gamma_i)-1}{[\Gamma:\Gamma_i]},$$

where $F(\Gamma)$ is the set of finite index subgroups of Γ and $d(\Gamma)$ denotes the *rank of* Γ, i.e., the minimal cardinality of a generating set of Γ.

The *cost* of a finitely generated group is a dynamical version of the rank gradient (as integral foliated simplicial volume is a dynamical version of simplicial volume); for the definition we refer to the literature [30].

Furthermore, let tors A denote the torsion of a finitely generated abelian group A and let $\operatorname{rk}_R B$ denote the R-dimension of the free part of a finitely generated R-module B, where R is a principal ideal domain.

Corollary 4.4.2 (gradient invariants and S^1-actions/-bundles)**.** *Let M be a smooth oriented closed connected n-manifold with residually finite fundamental group Γ that*

- *admits a non-trivial smooth S^1-action such that all orbits are π_1-injective, or*
- *is the total space of a smooth S^1-bundle over a smooth compact connected manifold with π_1-injective fibres.*

Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a Farber chain in Γ and let $M_i \longrightarrow M$ denote the covering of M associated to Γ_i for all $i \in \mathbb{N}$. Then

1. *for all $k \in \mathbb{N}$ and for every principal ideal domain R, we have*

$$\limsup_{i\to\infty} \frac{\operatorname{rk}_R H_k(M_i;R)}{[\Gamma:\Gamma_i]} \le \operatorname{const}_n \cdot \inf_{i\in\mathbb{N}} \frac{\|M_i\|_{\mathbb{Z}}}{[\Gamma:\Gamma_i]} = 0$$

$$\limsup_{i\to\infty} \frac{\log|\operatorname{tors} H_k(M_i;\mathbb{Z})|}{[\Gamma:\Gamma_i]} \le \inf_{i\in\mathbb{N}} \frac{\|M_i\|_{\mathbb{Z}}}{[\Gamma:\Gamma_i]} = 0,$$

2. *we have*

$$\operatorname{rg}\big(\Gamma, (\Gamma_i)_{i\in\mathbb{N}}\big) \le \inf_{i\in\mathbb{N}} \frac{\|M_i\|_{\mathbb{Z}}}{[\Gamma:\Gamma_i]} = 0;$$

in particular, $\operatorname{rg}(\Gamma) = 0$ *and*

3. *it follows that* $\operatorname{cost}(\Gamma) = 1$.

Proof. The first part of the corollary follows from a result on homology bounds by Frigerio, Löh, Pagliantini, and Sauer [19, Theorem 1.6] and Corollary 4.4.1.

The second part follows from the result by Löh that stable integral simplicial volume yields an upper bound for the rank gradient [36, Theorem 1.1] and Corollary 4.4.1.

The third part follows from the dynamical version of the latter, namely the cost estimate by Löh [35], which is given by

$$\operatorname{cost}(\Gamma) - 1 \leq |M| .$$

Since $\operatorname{cost}(\Gamma) \geq 1$, we have $\operatorname{cost}(\Gamma) = 1$ by Theorem 3 or Theorem 5. □

A. A Survey on Smooth Lie Group Actions

In this appendix, we recall the basic notions of smooth Lie group actions on smooth manifolds in order to understand the deconstruction step for smooth manifolds with non-trivial smooth S^1-action in Section 4.2 better. In particular, we investigate unions of orbits of a given orbit type, e.g., the fixed point set, which turn out to be manifolds; we are interested in the dimension of these manifolds. For the details we refer to the literature [4, Chapter VI][27].

In Section A.1, we recall some basic notions and classical results (without proofs) about compact smooth Lie group actions on smooth oriented compact connected manifolds. Then, in Section A.2, we work out the case of smooth S^1-actions in more detail. We conclude this chapter with Section A.3, where we investigate equivariant triangulations of orbit spaces.

A.1. Basic Notions and Classical Results

In this section, we recall some of the very basics about compact smooth Lie group actions on smooth oriented compact connected manifolds. Afterwards, we state some classical results that we need. In the end, we briefly discuss the connection between free Lie group actions and principal fiber bundles.

Throughout this section, let G be a compact Lie group.

Definition A.1.1 (smooth G-action, G-manifold)**.** Let M be a smooth compact connected manifold. A group action of G on M is called *smooth* if the map $G \times M \longrightarrow M$ given by the group action is smooth.

In this case we also call M a *G-manifold*.

Definition A.1.2 (orbit, isotropy groups, fixed points, orbit type)**.** Let M be a G-manifold and let $x \in M$. Then we define

- the *orbit of x* by
$$G \cdot x := \{g \cdot x \mid g \in G\} \subset M,$$
- the *isotropy group (or stabilizer) of x* by
$$G_x := \{g \in G \mid g \cdot x = x\} \subset G,$$
- the *set of fixed points* by
$$M^G := \{y \in M \mid \forall_{g \in G}\, g \cdot y = y\} \subset M,$$

- the *orbit type of* $G \cdot x$ by

$$(G_x) := \{G_{g\cdot x} \mid g \in G\},$$

- and the *orbit space* by

$$M/G := \{G \cdot y \mid y \in M\}$$

  together with the quotient topology.

Remark A.1.3 (orbit types are conjugacy classes). Let M be a G-manifold and let $x \in M$ and $g \in G$. Then we have

$$G_{g\cdot x} = g \cdot G_x \cdot g^{-1}$$

and therefore the orbit type (G_x) is the conjugacy class of the isotropy group G_x in G. To explain the name "orbit type" we consider the induced map $G/G_x \longrightarrow M$. This is an equivariant smooth embedding with image $G \cdot x$. In particular, G/G_x and $G \cdot x$ are equivariantly diffeomorphic. Let $y \in M$. Then G/G_x and G/G_y are equivariantly diffeomorphic if and only if G_x and G_y are in the same conjugacy class. Putting it all together we have that the orbits $G \cdot x$ and $G \cdot y$ are isomorphic (i.e., equivariantly diffeomorphic) G-manifolds if and only if they have the same orbit type.

Now, we state some basic facts about G-manifolds and orbit types. For the proofs we refer to the work of Jänich [27].

Theorem A.1.4. *Let M be a G-manifold. Then there are only finitely many orbit types of M.*

Theorem A.1.5. *Let M be a G-manifold and let (H) be an orbit type. Then the union $M_{(H)}$ of all orbits of type (H) is a smooth submanifold of M.*

We define an ordering on the orbit types of a G-manifold M as follows: If (H_1) and (H_2) are two orbit types in M, then we write $(H_1) \leq (H_2)$ if and only if there are representatives H_1' and H_2' with $H_2' \subset H_1'$.

Theorem A.1.6 (principal orbit type)**.** *Let M be a G-manifold. Then there exists a maximal orbit type (H) among all orbit types of M. The union $M_{(H)}$ of all orbits of type (H) is open and dense in M.*

We call (H) the principal orbit type *of M. A* principal orbit *is an orbit of principal orbit type.*

Definition A.1.7 (singular orbit, exceptional orbit)**.** Let M be a G-manifold. An orbit is called

- *singular* if the dimension of the orbit is less than the dimension of a principal orbit and

- *exceptional* if the dimension equals the dimension of a principal orbit but the orbit is not principal.

An orbit type is called *singular* or *exceptional* if it is the orbit type of a singular or exceptional orbit, respectively.

Theorem A.1.8 (dimension of the union of orbits of a singular orbit type)**.** *Let M be a G-manifold of dimension n. Let s be the maximal dimension of orbits in M. Let (U) be a singular orbit type. Then we have*

$$\dim M_{(U)} \leq n - s + \dim G/U - 1.$$

Theorem A.1.9 (dimension of the union of all exceptional orbits [4, Chapter III.3.10 and 3.11])**.** *Let M be a G-manifold of dimension n. Let E be the union of all exceptional orbits. We assume that all exceptional orbits are orientable. Then we have* $\dim E \leq n - 2$.

For several conclusions in Chapter 4 we need a connection between G-manifolds and G-principal bundles.

Remark A.1.10 (principal G-bundles and free G-actions). If M is a G-manifold with a free G-action, then the orbit map $M \longrightarrow M/G$ is a principal G-bundle [4, Theorem II.5.8].

A.2. Example: Smooth S^1-Actions

As we are particularly interested in S^1-actions, we will now apply the results from the previous section to smooth S^1-actions on smooth manifolds. To do so, let M be a compact connected smooth n-manifold with a smooth S^1-action. We first focus on isotropy groups and orbit types: It is easy to see that all possible isotropy groups are of the form $\{1\}$, S^1, or

$$T_r := \left\{0, \frac{1}{r}, \frac{2}{r}, \ldots, \frac{r-1}{r}\right\}$$

for an $r \in \mathbb{N}_{\geq 2}$, where we view S^1 as $[0,1]/(0 \sim 1)$. Since S^1 is abelian, the orbit types correspond to the isotropy groups.

Let F be the set of fixed points in M and let L_r be the union of orbits in M of orbit type T_r or smaller, i.e., the set of points whose isotropy groups contain T_r. Let L denote the union of all L_r. By Theorem A.1.5 we obtain that F, L and each L_r are submanifolds of M. Since S^1 is compact, Theorem A.1.4 implies that we have only finitely many orbit types and therefore, for large r we have $L_r \setminus F = \emptyset$.

By Theorem A.1.6 we obtain the following cases for the principal orbit type:

- If S^1 is the principal orbit type, then F is dense in M and therefore, the S^1-action is trivial.

- If T_r is the principal orbit type, then $L_r = M$ and therefore, the S^1-action is not effective. Recall that effective means that there exists no $g \in S^1$ with $g \cdot x = x$ for all $x \in M$. Note that in this case we can consider the action of $S^1/T_r \cong S^1$ on M, which is effective.
- The S^1-action is effective if and only if $\{1\}$ is the principal orbit type. If $\{1\}$ is the principal orbit type, the fixed points are the singular orbits, which implies $\dim F \leq n-2$ by Theorem A.1.8. The orbits of type T_r with $r \in \mathbb{N}_{\geq 2}$ are the exceptional orbits; they are homeomorphic to S^1, hence orientable, and therefore, $\dim L \setminus F \leq n-2$ by Theorem A.1.9.

A.3. Triangulations of Orbit Spaces

A crucial ingredient for the proof of vanishing of the integral foliated simplicial volume of smooth manifolds with non-trivial smooth S^1-action with π_1-injective orbits (see Section 4.2) is the existence of an appropriate triangulation of the orbit space.

Theorem A.3.1 (equivariant triangulations)**.** *Let M be a G-manifold. Then there exists a triangulation on M/G such that all points in an open simplex have the same orbit type (and open faces of a simplex have smaller or equal orbit type than the type of the open simplex).*

This was first proved in the case without boundary by Verona [46]. Extending Verona's result, Illman proved this also for manifolds with boundary using smooth equivariant collars and doublings [25, Section 7]. Since the reference that Illman gives for equivariant collars probably contains a typo we refer for this to the work of Bredon [4, Theorem V.1.5], who uses the Covering Homotopy Theorem of Palais [41, Theorem 2.4.1].

B. Manifolds with Corners and Hollowings

An important part in the deconstruction steps of the proofs of vanishing integral foliated simplicial volume in the case of S^1-actions or S^1-bundles presented in Section 4.2 and Section 4.3 is to hollow in a manifold at a submanifold. We first introduce the notion of manifolds with corners in Section B.1, which allows us to store combinatorial data given by a sequence of hollowings and which also arise naturally in the process of a hollowing. In Section B.2, we explain hollowings at submanifolds that are transverse to the boundary. Finally, in Section B.3, we explain how to hollow equivariantly also at submanifolds that are part of the boundary in some cases.

B.1. Manifolds with Corners

We now introduce manifolds with corners. A manifold locally looks like $\mathbb{R}^n$ and a manifold with boundary locally looks like a half space in $\mathbb{R}^n$. Continuing this pattern, manifolds with corners locally look like $\mathbb{R}^{n-k} \times [0, \infty)^k$. We follow the introduction to manifolds with corners by Douady [11].

We first define the building blocks for manifolds with corners.

Definition B.1.1 (sectors)**.** Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$. A *sector of dimension n with index k* is a set of the form

$$A = \{x \in \mathbb{R}^n \mid x_{i_1} \geq 0, \dots, x_{i_k} \geq 0\}$$

with pairwise distinct $i_1, \dots, i_k \in \{1, \dots, n\}$. In this case, for a point $x \in A$ the *index of x in A* is defined by

$$\operatorname{ind}_A(x) := \big|\{i \in \{i_1, \dots, i_k\} \mid x_i = 0\}\big|.$$

Observe that the index of A equals $\operatorname{ind}_A(0)$. Furthermore, note that $\mathbb{R}^n$ is a sector of dimension n with index 0.

It is not hard to see that every sector is homeomorphic to either a half space in $\mathbb{R}^n$ or to $\mathbb{R}^n$ itself for some $n \in \mathbb{N}$. In particular, a manifold with corners will be topologically just a manifold with (possibly empty) boundary. To make a difference, we will require that the transition maps are smooth. To do so we define smooth maps on sectors.

Definition B.1.2 (smooth maps on sectors)**.** Let $n, k, m \in \mathbb{N}$ with $n \geq k$ and let A be a sector of dimension n with index k. Let $U \subset A$ be open and let $V \subset \mathbb{R}^n$. A map $\varphi\colon U \longrightarrow V$ is called *smooth* if for all $x \in U$ there exists an open neighbourhood $U_x \subset \mathbb{R}^n$ of x and a smooth map $\varphi_x\colon U_x \longrightarrow V$ that coincides with φ on $U_x \cap U$.

Note that the above definition of a smooth map on an open set U in a sector of dimension n is equivalent to the existence of a smooth extension to an open subset of $\mathbb{R}^n$ that contains U [32, Lemma 2.27]. Moreover, observe that if A and A' are smoothly diffeomorphic sectors then A and A' have the same dimension and index.

Definition B.1.3 (atlas with corners)**.** Let $n \in \mathbb{N}$ and let M be a topological space. An *atlas with corners for M of dimension n* is a family $(U_i, V_i, \varphi_i)_{i \in I}$ where $(V_i)_{i \in I}$ is an open cover of M, each U_i is an open subset in a sector A_i of dimension n and $\varphi_i\colon U_i \longrightarrow V_i$ are homeomorphisms (called *charts*) such that all *transition maps are smooth*, i.e., for all $i, j \in I$ we have that

$$(\varphi_j^{-1} \circ \varphi_i)|_{U_i \cap U_j}$$

is a smooth map (in the sense of Definition B.1.2) that is a smooth diffeomorphism onto its image.

We define an equivalence relation on atlases with corners for M as follows: Two atlases for M are *equivalent* if and only if their union is an atlas with corners for M.

Finally, we are prepared to define manifolds with corners.

Definition B.1.4 (manifold with corners)**.** Let $n \in \mathbb{N}$. A *manifold with corners of dimension n* is a Hausdorff second countable topological space M together with an equivalence class of an atlas with corners for M of dimension n.

Let M be a manifold with corners and let $x \in M$. The *index of x in M* is given by

$$\operatorname{ind}_M(x) := \operatorname{ind}_{A_i}\big(\varphi_i^{-1}(x)\big),$$

where $\varphi_i\colon U_i \longrightarrow V_i$ is a chart with $x \in V_i$ and A_i is the sector of dimension n where U_i lives. Note that this definition does not depend on the choice of the chart and that we can always find a chart in a maximal atlas such that $\varphi_i(0) = x$.

For all $k \in \mathbb{N}$ we define

$$\angle^{(k)} M := \big\{ x \in M \bigm| \operatorname{ind}_M(x) \geq k \big\}.$$

Let M, N be manifolds with corners. A map $f\colon M \longrightarrow N$ is called *smooth* if for all $x \in M$, there exist charts $\varphi^M\colon U^M \longrightarrow V^M$ of M with $x \in V^M$ and $\varphi^N\colon U^N \longrightarrow V^N$ of M with $f(x) \in V^N$ such that $\varphi^N \circ f \circ (\varphi^M)^{-1}$ is smooth (in the sense of Definition B.1.2).

Since our goal is to define hollowings in manifolds with corners at submanifolds, we need to explain what submanifolds are in this case. As a preparation we define subsectors.

Definition B.1.5 (subsector)**.** Let $n, k, p, \ell, j \in \mathbb{N}$ with $n \geq k, p \geq \ell$. Let

$$A = \{x \in \mathbb{R}^n \mid x_{i_1} \geq 0, \ldots, x_{i_k} \geq 0\}$$

be a sector of dimension n with index k. A *subsector of A (of codimension p, coindex ℓ in A and complementary index j)* is a subset $A' \subset A$ given by replacing ℓ of the inequalities defining A by the corresponding equalities and adding $p - l$ new equalities of the form $x_r = 0$ and j inequalities of the form $x_s \geq 0$ with new and distinct indices. More precisely, there are sets $L \subset \{i_1, \ldots, i_k\}$ with ℓ elements, $P \subset \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ with $p - \ell$ elements and $J \subset \{1, \ldots, n\} \setminus (\{i_1, \ldots, i_k\} \cup P)$ with j elements such that

$$A' = \{x \in A \mid \forall_{i \in L}\, x_i = 0, \forall_{i \in P}\, x_i = 0, \forall_{i \in J}\, x_i \geq 0\}.$$

Note that A' can be viewed as a sector of dimension $n - p$ with index $k - \ell + j$. A *relative face of A' in A* is a subset of A' obtained by replacing at least one of the j inequalities $x_i \geq 0$ with $i \in J$ by the corresponding equality. We call A' a subsector *without relative faces* if $J = \emptyset$.

Example B.1.6 (sector, subsector). We discuss now some examples of sectors and subsectors (see Figure B.1). The set

$$A_1 := \{x \in \mathbb{R}^2 \mid x_2 \geq 0\}$$

is a sector of dimension 2 with index 1.

Then the set

$$A_2 := \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\}$$

is a subsector of A_1 (of codimension 1, coindex 0 in A_1 and complementary index 0) without relative faces. Viewed in $\mathbb{R}$ it is a sector of dimension 1 with index 1.

The set

$$A_3 := \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$$

is a subsector of A_1 (of codimension 0, coindex 0 in A_1 and complementary index 1) with relative face A_2. It is a sector of dimension 2 with index 2.

Subsectors can also be parts of the boundary of a sector: The set

$$A_4 := \{x \in \mathbb{R}^2 \mid x_2 = 0\}$$

is a subsector of A_1 (of codimension 1, coindex 1 in A_1 and complementary index 0) without relative faces. Viewed in $\mathbb{R}$ it is a sector of dimension 1 with index 0.

Now, we can define submanifolds with corners.

Definition B.1.7 (submanifold with corners)**.** Let M be a manifold with corners. A closed subspace $N \subset M$ is called *submanifold (with corners)* if for all $x \in N$ there exists a chart $\varphi\colon U \longrightarrow V$ in M, where $U \subset A$, and a subsector $A' \subset A$ such that

$$\varphi(0) = x \quad \text{and} \quad \varphi^{-1}(V \cap N) = U \cap A'.$$

In this case, N is *without relative boundary* if all of the subsectors $A' \subset A$ that are part the above charts are without relative faces.

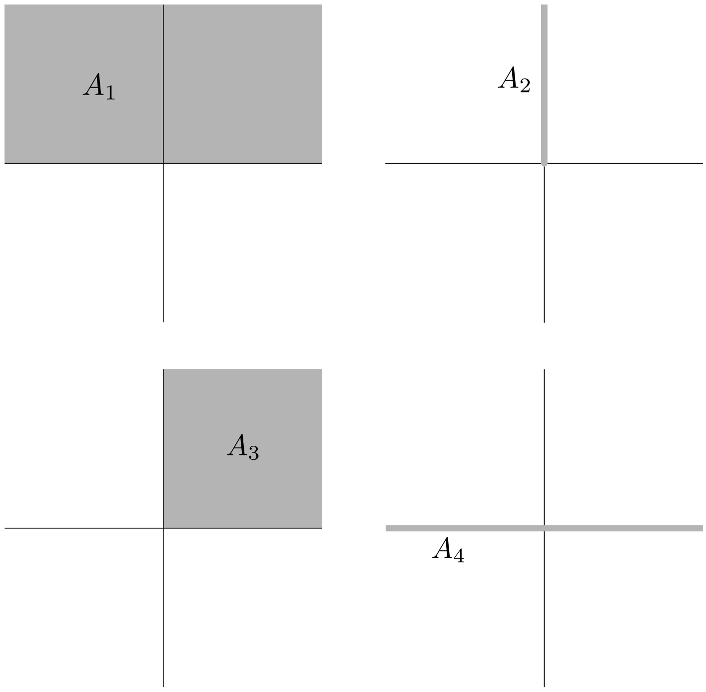

Figure B.1.: Sectors and subsectors from Example B.1.6

For our goal to define hollowings we have to introduce tubular neighbourhoods in manifolds with corners.

Let $n \in \mathbb{N}$. Let M be a manifold with corners of dimension n. We define the tangent bundle $TM \longrightarrow M$ as follows: Let $x \in M$. Then, T_xM is defined to be the dual of the real vector space m/m^2, where m is the maximal ideal in the local ring $\mathcal{O}_x$ of germs of smooth $\mathbb{R}$-valued functions on M at x consisting of all germs f with $f(x) = 0$. Let TM be as a set the disjoint union of all of the T_xM and let $TM \longrightarrow M$ be the map that sends $v \in T_xM$ to x. For the topology and smooth structure of the tangent bundle, we refer to the literature [32, Lemma 4.1].

Let $x \in M$ and let $\varphi\colon U \longrightarrow V$ be a chart with $\varphi(0) = x$. Then we have an isomorphism $\varphi'(0)$ from $\mathbb{R}^n$ to T_xM. Let A_xM be the image of the sector A (that contains U) in T_xM under $\varphi'(0)$. This does not depend on the choice of the chart. Let N be a submanifold of M. Then, for all $x \in N$ we write

$$T_x(M : N) := T_xM/T_xN$$

and we define the *transverse sector* $A_x(M : N)$ *of* N *in* M to be the image of A_xM in $T_x(M : N)$.

Definition B.1.8 (tubular neighbourhood)**.** Let M be a manifold with corners. Let N be a submanifold of M without relative boundary. A *tubular neighbourhood of N in M* is a triple (T, μ, ψ) consisting of

1. a family $(\mu_x)_{x\in N}$ of scalar products on $T_x(M:N)$ that "smoothly depend" on x, i.e., they are obtained from a smooth section of the smooth vector bundle $\mathcal{T}^2(M:N) \longrightarrow N$ of covariant 2-tensors on $T(M:N)$

2. and a diffeomorphism ψ from $B := B(M:N;\mu)$ to a submanifold T of M, where for all $x \in N$ we define $B_x := B_x(M:N;\mu)$ to be the trace of $A_x(M:N)$ of the unit ball in $T_x(M:N)$ with respect to μ_x, i.e.,

$$B_x(M:N;\mu) := A_x(M:N) \cap B_1^{\mu_x}(0) \subset T_x(M:N)$$

such that the following holds:

- The restriction of ψ to the zero section of B (which we identify with N) is the inclusion $N \subset M$.
- The map
$$\varphi\colon A(B:N) \longrightarrow A(T:N)$$
that is induced by ψ on the fibres is the identity on $A(M:N)$.

Theorem B.1.9 (tubular neighbourhoods of submanifolds without relative boundary [11, Théorème 1])**.** *Let M be a manifold with corners and let N be a submanifold of M without relative boundary. Then there exists a tubular neighbourhood of N in M.*

B.2. Hollowings at Submanifolds

In this section, we will define hollowings. Afterwards, we shortly explain equivariant hollowings in G-manifolds for some compact Lie group G and finally, fiber preserving hollowings in smooth fiber bundles. We follow the exposition of Yano [47, Section 2].

Definition B.2.1 (hollowings)**.** Let $n \in \mathbb{N}$. Let M be an n-manifold with corners and let N be a compact submanifold with corners of M that is transverse to each $\angle^{(k)}M \setminus \angle^{(k+1)}M$ and such that $\angle^{(k)}N = N \cap \angle^{(k)}M$. Then there exists a tubular neighbourhood $\nu(N)$ of N in M by Theorem B.1.9 and it has the structure of a disc bundle over N (this follows from the transversality conditions). Let $\nu_s(N)$ be the total space of the associated sphere bundle to $\nu(N)$. The polar coordinate map $S^n \times [0,1] \longrightarrow D^n$ given by $(x,r) \longmapsto r\cdot x$ induces a bundle map $\psi\colon \nu_s(N) \times [0,1] \longrightarrow \nu(N)$ (over id_N) via local trivialisations such that $\psi|_{\nu_s(N)\times\{1\}} = \mathrm{id}_{\nu_s(N)}$ and $\psi|_{\nu_s(N)\times\{0\}}$ is the projection to $N \subset \nu(N)$. Let

$$M' := \overline{M \setminus \nu(N)} \cup_{\psi|_{\nu_s(N)\times\{1\}}} \big(\nu_s(N) \times [0,1]\big).$$

Then we have a map

$$p := \big(\mathrm{id}_{\overline{M\setminus\nu(N)}} \cup_{\psi|_{\nu_s(N)\times\{1\}}} \psi\big)\colon M' \longrightarrow M$$

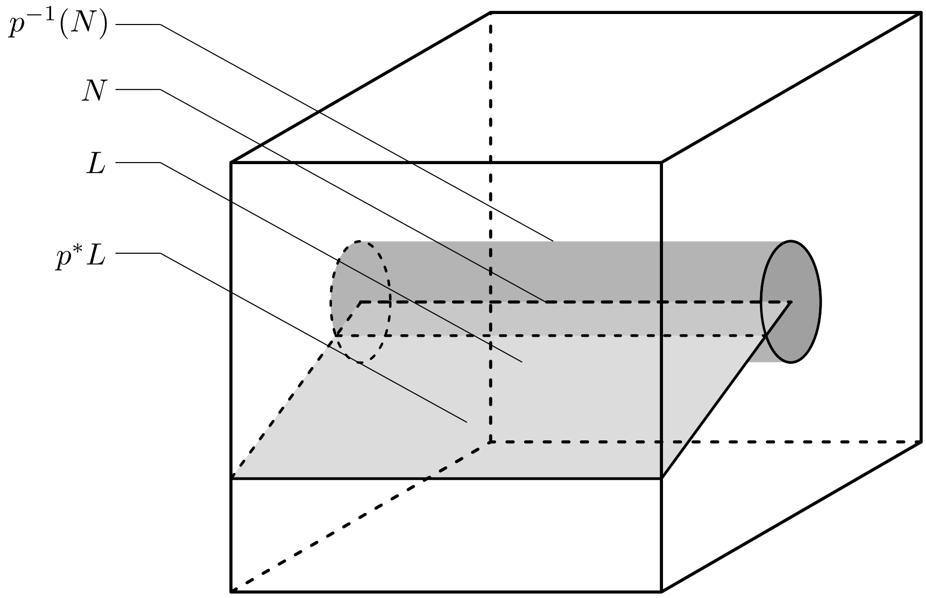

Figure B.2.: A hollowing in a cube at an interval schematically, where we view the hollowed cube as a submanifold of the filled cube

and there exists a structure of an n-manifold with corners on M' such that p is a smooth map. We call p a *hollowing in M at N* and the submanifolds $N \subset M$ and $p^{-1}(N) = \nu_s(N) \times \{0\} \subset M'$ are called *trace* and *hollow wall* of the hollowing respectively. For a subspace $L \subset M$ we write

$$p^*L := \overline{p^{-1}(L \setminus N)}$$

for the pullback of L along p.

Note that for all $x \in \big(\nu_s(N) \times \{0\}\big) \cap p^*\partial M$ we have

$$\mathrm{ind}_M(x) = \mathrm{ind}_N\big(p(x)\big) + 1.$$

See Figure B.2 for an illustration of a hollowing.

Remark B.2.2 (equivariant/fiber preserving hollowings). Let $n \in \mathbb{N}$ and let G be a compact Lie group. Let M be a compact n-manifold with corners with a smooth G-action, i.e., a smooth map $G \times M \longrightarrow M$, where we equip $G \times M$ with the structure as a manifold with corners given by the product of the structures of G and M as manifold with corners. Let N be a G-invariant submanifold with corners of M that fulfils the transversality conditions in Definition B.2.1. Then there exists a tubular neighbourhood $\nu(N)$ of N by Theorem B.1.9. This can be chosen to be G-invariant [27, p. 4]. We define a G-action on the corresponding sphere bundle $\nu_s(N) \times [0,1]$ by taking the unique extension of the pullback action of G

on $\nu_s(N) \times (0,1]$ via ψ that is given by

$$g \cdot x := \psi^{-1}\big(g \cdot \psi(x)\big)$$

for all $g \in G$ and $x \in \nu_s(N) \times (0,1]$. After we proceed as in Definition B.2.1 we obtain an n-manifold with corners M' which has a canonical smooth G-action that makes p equivariant. We call $p\colon M' \longrightarrow M$ an *equivariant hollowing in M at N*.

Analogously, we define *fibre preserving hollowings* in a total space of a G-bundle at a submanifold that is a G-subbundle, i.e., a union of fibres. Here, we need to remove a tubular neighbourhood that is a G-subbundle itself. This can be chosen to be the preimage of a tubular neighbourhood of the base space of the subbundle we are hollowing at.

A similar version of hollowings and how to get invariant tubular neighbourhoods in manifolds with corners with a smooth Lie group action is briefly explained in work of Jänich [26, Section 1.3].

B.3. Equivariant Hollowings at the Boundary

In this section, we explain what an admissible sequence of hollowings is. For the definition of hollowings in Section B.2 we did not really need to restrict to submanifolds that are transverse to the boundary but only to submanifolds without relative faces in order to apply Theorem B.1.9. But since we need to hollow equivariantly and want to follow the work of Illman [25] we restrict to hollowings as defined in Definition B.2.1.

Let $n \in \mathbb{N}$ and let G be a compact Lie group. Let M be a smooth compact connected n-manifold with a smooth G-action. As described by Illman in more detail [25], we can construct the double DM of M such that it has the structure of a smooth oriented closed connected n-manifold (which restricts to the smooth structure of M on the two copies of M in DM) with a smooth $G \times \mathbb{Z}/2$-action (where the $\mathbb{Z}/2$-action is given by interchanging the two copies of M in DM; this commutes with the G-action) and such that ∂M is a $(G \times \mathbb{Z}/2)$-invariant submanifold of DM.

In order to avoid speaking about the double of a manifold with corners, we want to deal with a whole sequence of hollowings at once.

Definition B.3.1 (admissible sequence of hollowings)**.** Let $d \in \mathbb{N}$ and let G be a compact Lie group. Let M be a smooth compact connected manifold with a smooth G-action and let $X_i' \subset M$ be invariant subspaces for all $i \in \{0, \dots, d-1\}$. We consider M as a manifold with corners with $\angle^{(1)}(M) = \partial M$ and $\angle^{(2)}(M) = \emptyset$. Then, the pair $\big(M, (X_i')_i\big)$ is called *admissible* if the following holds:

- The space X_0' is a closed (possibly non-connected) submanifold of M. We write $(DM)_0 := DM$, $M_0 := M$ and $DX_0 := DX_0' \subset (DM)_0$ for the corresponding submanifold. Let $Dp_0\colon (DM)_1 \longrightarrow (DM)_0$ be an equivariant hollowing in $(DM)_0$ at DX_0. Then, we write $M_1 := (DM)_1/(\mathbb{Z}/2)$, $p_0 := \overline{Dp_0}$, $DX_1 := Dp_0^* DX_1'$ and $X_1 := DX_1/(\mathbb{Z}/2)$.

- Let $i \in \{1, \dots, d-1\}$. Let inductively $(DM)_j$, M_j, DX_j, X_j, equivariant hollowings $Dp_{j-1}\colon (DM)_j \longrightarrow (DM)_{j-1}$ and maps $p_{j-1}\colon M_j \longrightarrow M_{j-1}$ be already defined for all $1 \leq j \leq i$. Then DX_i satisfies the conditions in Definition B.2.1, i.e., DX_i is a compact submanifold of $(DM)_i$ which is transverse to each $\angle^{(k)}(DM)_i \setminus \angle^{(k+1)}(DM)_i$ and that we have
$$\angle^{(k)} DX_i = DX_i \cap \angle^{(k)}(DM)_i$$
for all appropriate k. Let
$$Dp_i\colon (DM)_{i+1} \longrightarrow (DM)_i$$
be an equivariant hollowing in $(DM)_i$ at DX_i. We write
$$M_{i+1} := (DM)_{i+1}/(\mathbb{Z}/2), \quad p_i := \overline{Dp}_i\colon M_{i+1} \longrightarrow M_i,$$
and if $i < d-1$ we write
$$DX_{i+1} := Dp_i^* \dots Dp_0^* DX'_{i+1} \quad \text{and} \quad X_{i+1} := DX_{i+1}/(\mathbb{Z}/2).$$
We also call p_i a *hollowing in M_i at X_i* and use the same terms and notation as in Definition B.2.1 like *hollow wall*, *trace* and *pullback* $p_i^* L$ for $L \subset M_i$.

The associated sequence
$$M_d \xrightarrow{p_{d-1}} \cdots \xrightarrow{p_1} M_1 \xrightarrow{p_0} M_0 = M$$
to an admissible pair is called an *admissible sequence of hollowings (in M_i at X_i)*.

See Figure B.3 for an illustration of a hollowing at the boundary using the double. We prove the following lemma about faces in a hollowed standard simplex.

Lemma B.3.2. *Let $\ell, k \in \mathbb{N}$ with $k \leq \ell$. Let*
$$\Delta^\ell_\ell \xrightarrow{p_{\ell-1}} \Delta^\ell_{\ell-1} \xrightarrow{p_{\ell-2}} \cdots \xrightarrow{p_0} \Delta^\ell_0 = \Delta^\ell$$
together with the sets $X'_i := (\Delta^\ell)^{(i)}$, i.e., the i-skeleton of Δ^ℓ (with respect to the canonical triangulation of Δ^ℓ), be an admissible sequence of hollowings. Then, each p_i is a hollowing in Δ^ℓ_i at $X_i := p_{i,0}^* X'_i$.

Let Δ be a k-face of Δ^ℓ. Then we have
$$p_{\ell,k}\big(\Delta^\ell_\ell \setminus p^*_{\ell,k+1} p_k^{-1}(\Delta_k)\big) \subset \Delta^\ell_k \setminus \Delta_k.$$
Moreover, if Δ' is a j-face of Δ^ℓ with $\Delta \subset \Delta' \subset \Delta^\ell$ then
$$p_{\ell,k}\big(p^*_{\ell,j+1} p_j^{-1}(\Delta'_j) \setminus p^*_{\ell,k+1} p_k^{-1}(\Delta_k)\big) \subset \Delta'_k \setminus \Delta_k.$$

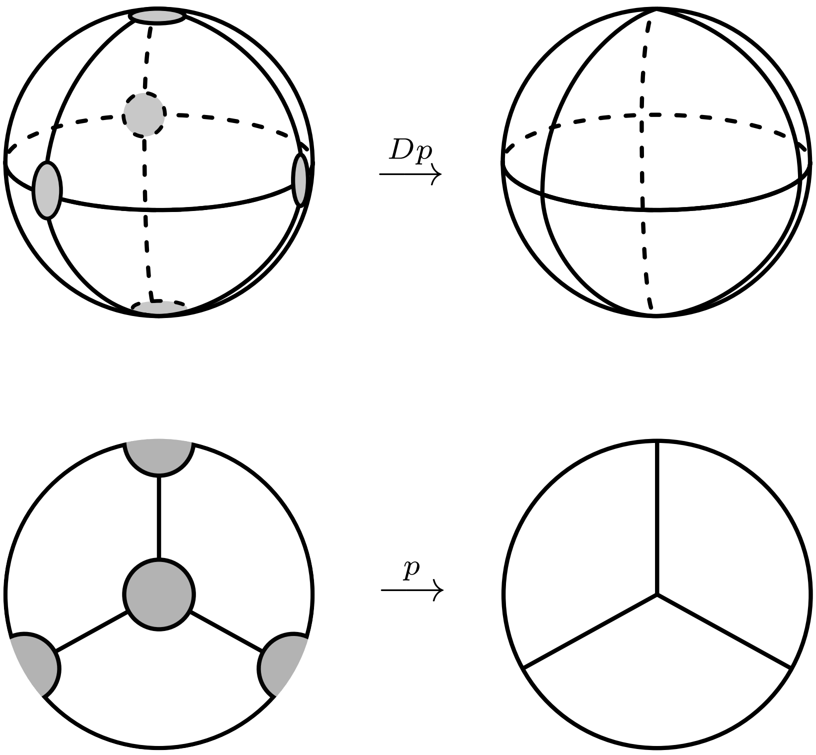

Figure B.3.: A hollowing in a disc at the 0-skeleton of the given triangulation which gives rise to a manifold with corners

Proof. We prove the following by induction: For all $d \in \{k, \dots, \ell\}$ we have

$$p_{d,k}^{-1}(\Delta_k) = p_{d,k+1}^{*} p_k^{-1}(\Delta_k).$$

This obviously holds for $d = k$ and $d = k + 1$. Let now $d \in \{k, \dots, \ell\}$ and assume that the claim holds for d. Then it also holds for $d + 1$: We set

$$A := p_{d,k}^{-1}(\Delta_k) = p_{d,k+1}^{*} p_k^{-1}(\Delta_k)$$

It suffices to prove that

$$p_{d+1}^{-1}(A) = p_{d+1}^{*} A.$$

We observe that

$$p_{d+1}^{-1}(A) = p_{d+1}^{-1}(\overline{A}) = p_{d+1}^{-1}(\overline{A \setminus X_{d+1}}) = \overline{p_{d+1}^{-1}(A \setminus X_{d+1})} = p_{d+1}^{*} A.$$

Here we use that p_{d+1} is a continuous closed map (as a continuous map between compact sets in $\mathbb{R}^{\ell+1}$) and that $A = \overline{A} = \overline{A \setminus X_{d+1}}$ holds (for dimension reasons and because of the transversality conditions). □

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