Partial Fermionization: Spectral Universality in 1D Repulsive Bose Gases

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Because of the vast growth of the many-body level density with excitation energy, its smoothed form is of central relevance for spectral and thermodynamic properties of interacting quantum systems. We compute the cumulative of this level density for confined one-dimensional continuous systems with repulsive short-range interactions. We show that the crossover from an ideal Bose gas to the strongly correlated, fermionized gas, i.e., partial fermionization, exhibits universal behavior: Systems with very few and up to many particles share the same underlying spectral features. In our derivation we supplement quantum cluster expansions with short-time dynamical information. Our nonperturbative analytical results provide predictions for excitation spectra that enable access to finite-temperature thermodynamics in large parameter ranges.

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The huge progress in cold atom physics has enabled precision experiments which allow us to confine, control, and study ensembles of atoms with particle numbers ranging from very few [1–5] to nearly macroscopically many [6–9]. The high control over parameters, trapping to low dimensions, and tunability of interactions has lead to a synergetic understanding of highly correlated many-body (MB) systems [10], in many cases based on theories of one-dimensional integrable models [11,12] and correspondingly tailored experiments [13,14]. However, in situations deviating from integrability (see, e.g., Refs. [15–19]) the theoretical treatment of systems with an intermediate number \( N \) of interacting identical particles is particularly hard, especially when the observed spectral, thermodynamic, or dynamical properties involve highly excited multiparticle states.

The conceptual challenges are numerous: First, systems with fixed \( N \) require a canonical treatment, in particular when approaching the few-body regime, where grand canonical approaches often fail [20]. Second, due to strong inter-particle correlations that can experimentally be pushed up to the limit of fermionization in Bose gases [3,7,8,21,22], and especially for small \( N \), mean-field approaches or more generally \( 1/N \) expansions get problematic. Elaborate MB techniques allow for calculating ground and low excited states of such interacting multi-particle systems with high precision (see, e.g., Refs. [23,24]). However, these methods reach their limits when increasing \( N \) or the degree of excitation since this implies vastly growing Hilbert space dimensions.

This goes along with a close to exponential increase of the MB density of states (DOS) with excitation energy for continuous \( N \)-particle systems, even in the 1D case. The universal Bethe law [25,26] and variants [27,28] for sufficiently low-lying excitations in large-\( N \) fermionic systems represent a famous example in nuclear physics. There, the effect of (residual) interactions is merely a broadening of the otherwise highly degenerate noninteracting MB spectrum [29,30], while for small to intermediate \( N \) interactions have nontrivial effects and the Bethe law generally fails [31,32].

Nonetheless, the spacing between MB levels as well as the associated fluctuations tend to zero such that individual highly excited MB levels are usually no longer resolvable. Hence the (locally) energy-averaged, smooth MB DOS \( \rho^{(N)}(E) \) gains particular relevance [33]. In particular it plays the central role for computing thermodynamic equilibrium properties at finite temperature. Beyond that, \( \rho^{(N)}(E) \) is a key ingredient to nonequilibrium quantum work statistics that has drawn much attention lately [36–39], not least due to a recently revealed connection to information scrambling [40].

This calls for developing genuinely interacting MB techniques specifically devised to directly compute the smooth DOS, thereby circumventing the intricate [41–44] or simply impossible calculation of individual (highly) excited MB levels which requires additional information that is afterwards smoothed out anyway.

Similar to the single particle case [45–48], a smooth MB DOS corresponds to and requires dynamical information from MB quantum propagation on finite time scales only. Invoking such short-time information in a quantum cluster expansion (QCE) [49–51] implies, as we will show, that interaction effects in the smooth DOS arise
FIG. 1. Many-body level counting function for six interacting bosons in a harmonic trap (spacing $\hbar\omega$) for different contact interaction strengths $\alpha$. Numerically exact results for $N(E)$ (staircases) exhibit characteristic shifts $\Delta_\alpha$ in $E$ towards the limit $\alpha \to \infty$ of full fermionization. These shifts carry universal features and are quantitatively explained by our theory (solid lines) based on Eqs. (8) and (11). Dotted lines denote analytical QCE-based approximations [Eq. (5)] invoking the limiting cases of weak and strong $\alpha$, see main text.

nonperturbatively from universal cluster kernels dressed with terms depending on the confinement potential, required to be a homogeneous function. Specifically, we consider not directly the MB DOS $\rho^{(N)}(E)$ but the (smooth) MB level counting function $N(E) = \int_{-\infty}^{\infty} dE \rho^{(N)}(E)$, depicted in Fig. 1 for a harmonically trapped Bose gas. $N(E)$ exhibits interaction-dependent characteristic horizontal shifts $\Delta_\alpha$ indicating what we call partial fermionization. We will analytically show that these shifts, and thereby $N$ and $\rho^{(N)}$, follow with high accuracy $N$-independent universal laws, i.e., broad classes of interacting bosonic systems ranging from very few to many particles possess equal spectral features. Remarkably, these robust features are reminiscent of the spectral shifts in the famous solvable Calogero-Sutherland models [52,53] which admit an interpretation in terms of fractional exclusion statistics [54–56].

We first outline the main steps of our QCE for the canonical partition function providing the basis for our further (asymptotic) analysis to derive our main result, a universal law for partial fermionization.

Canonical partition function.—The MB DOS $\rho^{(N)}_\pm(E)$ of a system of $N$ identical quantum particles ("±" denoting bosons and fermions) is related to the canonical partition function $Z^{(N)}(\beta)$ through the inverse Laplace transform $\rho^{(N)}_\pm(E) = L^{-1}_\beta[Z^{(N)}_\pm(\beta)](E)$ with $\beta = 1/(k_B T)$. Furthermore, $Z^{(N)}_\pm(\beta) = \text{Tr}_B K^{(N)}(t = -i\hbar\beta)$ is the trace over the propagator $K^{(N)}$ for $N$ distinguishable particles in the properly (anti-)symmetrized basis.

For $N$ noninteracting particles of mass $m$, each with coordinates $q$, confined by a homogeneous potential $U(q) = w^m U(q/w)$, it can be expressed in closed form [31,57],

$$Z^{(N)}_{0,\pm}(\beta) = \sum_{l=1}^{N} \sum_{\alpha=1}^{\infty} \left( \frac{V_{\text{eff}}}{\lambda\hbar^2} \right)^l,$$

with universal constants $z^{(N,\alpha)}_{\pm,l}$, physical dimension $D$, and effective dimension $d = D(1 + (2/\mu))$. Setting $\hbar^2/(2m) = 1$, the thermal wavelength is $\lambda_T = \sqrt{4\pi\beta}$ and the effective volume is $V_{\text{eff}} = (4\pi)^{D/2} \int d^D q \exp[-U(q)]$. The case without external potential is included as $\mu \to \infty$, then $d = D$ and $V_{\text{eff}}$ equals the physical volume $V_D$.

Quantum cluster expansion.—The noninteracting part $K^{(N)}_0$ of the propagator factorizes into single-particle (SP) propagators, see Fig. 2(a). A contribution to $Z^{(N)}_{0,\pm}$ corresponding to a permutation $P$ is a product of cluster terms, resembling the decomposition of $P$ into cycles [62]. Using the semigroup property of the SP propagator and identifying $q_{n+1} = q_1 = q$, each cycle involving a subset of $n$ particles [see Fig. 2(b)] yields the amplitude $A_n(t) = \int d^D q K^{(1)}_0(q; q; nt)$. In line with our major assumption of short-time propagation we can use [31] $K^{(1)}_0(q, q; t) := \exp[-(i/\hbar)U(q)]/K_{\text{free}}(q, q; t)$ where $K_{\text{free}}$ stands for unconfined propagation. The full contribution to $Z^{(N)}_{0,\pm}$ of a permutation is then $A_{\mathcal{P}}(-i\hbar\beta) = \prod_{K \in \mathcal{P}} \alpha_{\mathcal{P}(K)} (-i\hbar\beta)$, in terms of the multiset $\mathcal{P} = \{n_1, n_2, \ldots, n_{|\mathcal{P}|}\}$ of cycle lengths, see Fig. 2(c). Further evaluation of these amplitudes eventually yields the explicit result Eq. (1) [31,57].

The implementation of interaction effects begins with a cluster expansion [49–51] of $K^{(N)}$ to first order in the interaction by decomposing the full two-body propagator $K^{(2)} = K^{(2)}_0 + \Delta K^{(2)}_\alpha$ into $K^{(2)}_0$ and nonperturbative interaction contributions $\Delta K^{(2)}_\alpha$ where $\alpha$ is an energy associated with the coupling strength [63]. To calculate interaction effects we choose all pairs $\{k, l\}$ of particles and replace $K^{(1)}_0(q_{P(k)}, q_k; t) K^{(1)}_0(q_{P(l)}, q_l; t)$ in $A_{\mathcal{P}}$ by the interaction term $\Delta K^{(2)}_\alpha((q_{P(k)}, q_{P(l)}), (q_k, q_l); t)$, see Fig. 2(d).
interaction can link two particles involved in either the
same [see Fig. 2(e)] or in two different cycles [see Fig. 2(f)]
of $P$, referred to as intra- and inter-cycle contributions
$A_{n_1,n_2}^{\text{inter}}$, where $n_1, n_2$ denotes the distribution of the $n = n_1 + n_2$ particles. Evaluation of the diagram classes in
Figs. 2(e) and 2(f) yields

$$Z_{a,\pm} = Z_{0,\pm} + \sum_{n=2}^{N} (-1)^n Z_{0,\pm}^{N-n} \sum_{n_1=1}^{n-1} A_{n_1,n-1,\pm}$$

(2)

with amplitudes of the form

$$A_{n_1,n_2}^{\pm} = \frac{1}{2} \left[ A_{n_1,n_2}^{\text{inter}} \pm A_{n_1,n_2}^{\text{inter}} \right] = \frac{V_{\text{eff}}}{{\mathcal{R}^d_n}} a_{n_1,n_2}^{\pm}(\beta \alpha),$$

(3)

defining the interaction kernels $a_{n_1,n_2}^{\pm}(\beta \alpha)$, see below.

The philosophy behind cluster expansions implies that
the form Eq. (3) of amplitudes is generic for arbitrary short-
nation kernels, i.e., to first order, effective system size
systems [10,65,66] we consider Hamiltonians

$$\hat{H} = \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial q_i^2} + U(q_i) \right) + \sqrt{8\alpha} \sum_{i<j} \delta(q_i - q_j)$$

(6)

of $N$ interacting bosons with coordinates $q_i$ in 1D. One obtains [57] explicit analytical expressions for the kernels
$a_{n_1,n_2}^{\pm}(\beta \alpha)$ in Eq. (3). Closed explicit expressions for the
$g_{\pm,j}^{(N,d)}(E/\alpha)$ in Eq. (5) follow for the prominent 1D cases
of $U(q) = 0$ ($d = 1$), harmonic confinement ($d = 2$), and
linear potential wells ($d = 3$) [57].

Before addressing representative cases we note that the
QCE [Eqs. (4) and (5)], evaluated to first order, although
devised for weak interaction, can also be applied to the
complementary regime of strong coupling [57] by means of
fermionization [67,68] due to an exact duality [69] of
strongly coupled bosons and weakly coupled spinless
fermions.

**Harmonic confinement.**—We first consider $U(q_i) = \pm (\hbar \omega)^2 q_i^2/4$, for which $V_{\text{eff}} = 4\pi/(\hbar \omega)$, and compare in
Fig. 1 analytical QCE results (dotted lines) for $N_a(E)$ with
extensive numerical calculations (staircases) based on exact
diagonalization and hence restricted to roughly the first 40
excited MB levels for $N = 6$. The first-order QCE, imple-
mented as weak- and dual strong-coupling expansions,
indeed is valid in the respective regimes. However,
for intermediate couplings (here $\alpha \approx 2\hbar \omega$) it degrades.
Moreover, such deviations grow with increasing $N$ calling
for an improved method that adequately treats intermediate
couplings.

**Partial fermionization.**—Interactions predominantly
cause characteristic shifts of $N_a(E)$ towards larger energies
(see visible in Fig. 1). Presuming knowledge of the non-
interacting spectra, the shifts $\Delta_e$ of individual levels contain
all information about the interacting spectra. We adopt this
formulation of the problem to develop a method that
directly addresses these shifts on average. Our approach
further enables asymptotic considerations that strongly
simplify the MB problem and highlight the universality
behind partial fermionization.

For the interaction-induced energy shift at fixed $N$,

$$\Delta_n \equiv \langle E^{(n)}(\alpha) - E^{(n)}(0) \rangle_a \equiv \langle E^{(n)}(\alpha) \rangle_a - E_0,$$

(7)

averaged over a bunch of individual MB levels $E^{(n)}$ we propose, in a first-order implementation, the ansatz

$$\Delta_n \approx \chi^{(N,d)}(E/\alpha) \Delta^{(N,d)}_\infty(E_0, V_{\text{eff}}),$$

(8)

where $E = E_0 + \Delta_n$ is the shifted energy, separating the
$V_{\text{eff}}$ dependence from an $\alpha$-dependent function $\chi^{(N,d)}$, in
view of the notable structure of $N_a$ within QCE [Eq. (5)]
and corroborated by a general consistency argument
[57,64]. $\Delta_\infty$ denotes the full “horizontal” shift (see Fig. 1) between fermionized and noninteracting bosonic
levels for fixed $N \equiv \langle n \rangle_a = N_a(E_0)$. We find [57]

$$\Delta^{(N,d)}_\infty \approx \text{const} \times V_{\text{eff}}^{(2/d-1)/N}.$$
The $\alpha$-dependent factor $\chi \in [0,1]$ in Eq. (8) continuously interpolates between the free Bose gas $\chi \to 0$ and the fully fermionized gas $\chi \to 1$, quantifying partial fermionization. Most notably, the central function $\chi^{(N,d)}(E/\alpha)$ in Eq. (8) is uniquely obtained from QCE [Eq. (5)] by matching

$$N_\alpha(E) = N = N_0(E_0) = N_0(E - \Delta_\alpha) \quad (10)$$

in the regime $E^{1/2}V_{\text{eff}} \gg 1$ of weak quantum degeneracy, where the first-order QCE becomes increasingly accurate. For the lhs of Eq. (10) we apply QCE [Eq. (5)], while for the rhs we use the result Eq. (1) for $\alpha = 0$, and implement the shift $\Delta_\alpha$, Eq. (8), as an expansion around $E$ in the small parameter $\Delta_{\text{sc}}^{(N,d)} = E = \mathcal{O}(E^{-d/2}V_{\text{eff}}^{-1})$. Matching the next-to-leading order $\mathcal{O}(V_{\text{eff}}^{N-\epsilon})$ in Eq. (10) fixes $\chi^{(N,d)}(E/\alpha) \propto -g^{(N,d)}_{[0,N-1]}(E/\alpha)$ [57], which, remarkably, is fully determined by two-body clusters for which the first-order QCE is exact.

A solution for $N_\alpha(E)$ is achieved by determining the partial fermionization for a given initial noninteracting energy $E_0$, reducing the problem to finding, in view of Eq. (8), the root of

$$x = \chi^{(N,d)}(E/\alpha) = \chi^{(N,d)}[(E_0 + x\Delta_{\text{sc}})/\alpha]. \quad (11)$$

This implicitly defines $x = \chi$ as a function of $E_0$, $\alpha$, and $N$. The method efficiently emulates the effect of higher-order clusters in terms of the smallest ones, giving excellent predictions (see solid curves in Fig. 1). While the presented lowest-order version involves only two-body clusters it can be pushed to second (and higher) order in a controlled way [57,64] by extending the ansatz Eq. (8) to admit corrections to $\chi$ in powers of $(E_{\text{sc}}^{(N,d)})^{-d/2}$ [see Eq. (13) below]. Those incorporate three-body (and larger) clusters that correct for multiparticle collision effects in the deeply quantum degenerate regime.

**Asymptotics and universality.**—An asymptotic analysis [57] of $\chi$ and $\Delta_{\text{sc}}$ for large $N$ further reveals the existence of a specific finite limit, i.e., in first order,

$$x \approx \lim_{N \to \infty} \chi^{(N,d)}(N\tilde{E}) = 1 - e^{d/(2\tilde{E})} \text{erfc} \sqrt{d/(2\tilde{E})}, \quad (12)$$

where $\tilde{E} = [E_0 + x\Delta_{\text{sc}}^{(N,d)}]/\alpha^{\text{sc}}$ and, with $n_{\text{eff}} = N/V_{\text{eff}}$,

$$E_0^{\text{sc}} = \frac{E_0}{N}, \quad \alpha^{\text{sc}} = \alpha^{\text{sc}}/N, \quad \tilde{E} = \frac{2\pi h^2}{m}n_{\text{eff}}^{2/d}, \quad (13)$$

implying a universal law for the partial fermionization,

$$\chi^{N \to \infty} \approx \chi(d, E_0^{\text{sc}}, \alpha^{\text{sc}}). \quad (14)$$

For systems with different $N \gg 1$ it predicts that $\chi$, and hence $N_\alpha$, depend in the same peculiar way on $\alpha$ and the energy per particle $E_0/N$ of the corresponding noninteracting system, both appropriately scaled in terms of the energy unit $\tilde{E} = (2\pi h^2/m)n_{\text{eff}}^{2/d}$, establishing a key feature of the observed universality: It relates high excitations in large-$N$ systems to low-lying excitations in corresponding systems with smaller, but still considered large, $N$. Explicit approximants for $\chi(d, E_0^{\text{sc}}, \alpha^{\text{sc}})$ can be found by iteration [57].

In Figs. 3(a) and 3(b) we compare these predictions with numerically obtained data based on MB levels of Eq. (6) for the paradigmatic Lieb-Liniger model [70–73] $[U(q)] = 0$ on a ring with length $L$, i.e., $n_{\text{eff}} = N/V_{\text{eff}} = N/L$. We find that the universality is fulfilled with remarkable accuracy for the whole range of interactions and particle numbers, most notably even down to $N = 2$. Moreover, spectral fluctuations, not included in our analytical approach, are strongly suppressed for growing $N$, implying approximate analytical predictability of individual excited MB energies for arbitrary parameters.

Inaccuracies at very low energies and couplings are cured by extending the energy shifting from first order (dashed line), based on two-body processes, to second order (solid line) involving three-cluster diagrams which, again, can be calculated analytically [64,74]. Our approach amounts to a description of the entire smooth spectrum in terms of only two- or three-body processes which nonperturbatively interpolates between $\alpha = 0$ and $\alpha \to \infty$.

Figure 3(c) shows results for harmonic confinement, for which $\Delta_{\text{sc}} = 1/2$, representing a generic nonintegrable $N$-particle system, see also Fig. 1. The full lines display the analytical solutions for finite $N = 3, 4, 6, 8$, converging to the universal large-$N$ limit Eq. (14). The universal prediction (inset) shows, besides fermionization $\chi \approx 1$ (roof) and the perturbative regime $\chi \approx 2\alpha^{\text{sc}}/(\pi E_0^{\text{sc}})$ (right flank), a nonperturbative quantum regime for $E_0^{\text{sc}} \ll (\alpha^{\text{sc}})^{1/3}$, $\alpha^{\text{sc}} \ll 1$ where $\chi \approx 2(\alpha^{\text{sc}}/\pi)^{1/3}$ becomes independent of $E_0^{\text{sc}}$ [57]. This peculiarity connects our findings to the solvable Calogero-Sutherland models with $\sim 1/r^2$ interaction [52,53,75] where spectra are exact bosonic-to-fermionic interpolations that we identify as a specific realization of $\chi$ fulfilling universality, in this case level by level and constant in $E_0^{\text{sc}}$. Here we find a generalization (including nonintegrable systems) where $\chi$ is allowed to vary over the (smoothed) spectrum in a way characteristic for the particular type of interaction. This illustrates that due to the generality of the QCE approach [Eq. (5)], the shifting procedure [Eqs. (8)–(11)], and the subsequent asymptotic analysis, universality [Eq. (14)] of $\chi$ is not restricted to contact interaction [Eq. (6)]. We stress that it also applies to higher-order implementations.

We close with a few remarks. (i) Our method provides predictions for regions of excitation spectra and particle numbers that are barely accessible via full numerical calculations. (ii) Universality [Eq. (14)] of $\chi$ directly implies, through Eq. (10), universal features for $N_\alpha(E) = N_0(E - \chi_{\Delta_{\text{sc}}})$ and for the MB DOS $\rho^{(N)}(E) = \rho_0^{(N)}(E - \chi_{\Delta_{\text{sc}}})[1 - (d/dE)(\chi_{\Delta_{\text{sc}}})]$, both represented in
terms of their noninteracting limits at shifted energy. (iii) Corresponding expressions for the microcanonical and canonical partition functions and thereby thermodynamic quantities follow right away. For example, Eq. (14) implies that the microcanonical temperature $T$ can be determined as well by the scaled variables [Eq. (13)].

Thus, in the thermodynamic limit $N$, $V_{\text{eff}} \to \infty$ with $n_{\text{eff}}$ fixed, partial fermionization $\chi(T, \alpha, n_{\text{eff}})$ is an intensive quantity. (iv) Experimentally accessible [76] local pair correlations provide a direct probe of $\chi$ through a simple exact relation [57]. (v) Equation (5) also holds true for fermions indicating that our approach can be generalized to fermions. (vi) Another application concerns MB scattering through interacting media due to a fundamental relation between the smooth DOS and the average dwell time [77,78] that is, in the single-particle case, robust against disorder implying universality [79].

To conclude we have shown that the consistent use of short-time dynamical information in the description of short-range-interacting systems enables a separation of interaction and confinement effects implying universal features of smoothed MB spectra and related thermodynamic properties. Here we have made this explicit [Eq. (12)] and have confirmed it for two models of 1D contact-interacting bosons. On top, the way universality [Eq. (14)] is derived is not restricted to 1D systems and does not depend on details of the interactions apart from being sufficiently short-ranged and excluding strongly attractive forces if supporting MB bound states beyond small cluster formation [80]. Hence we envisage applications and benchmarks in higher dimensions and for other types of interaction, e.g., by utilizing Beth-Uhlenbeck type formulas [81,82].

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[33] See, e.g., Refs. [34] and [35] for recent calculations for the Sachdev-Ye-Kitaev model and quantum spin-1/2 Ising model with tilted field.
[57] See Supplemental Material at http://link.aps.org supplemental/10.1103/PhysRevLett.122.240601, Appendix A, for the derivation of Eq. (1), Appendix B, for the derivation and explicit calculation of interaction kernels, Appendix C, for the derivation and explicit calculation of Eq. (5), Appendix D, for the QCE as strong coupling expansion, Appendix E, for energy shifting, partial fermionization, and the consistancy argument supporting Eq. (8), Appendix F, for large-N asymptotics of the χ function, the full shift Δ(ν) of, and universality of partial fermionization, Appendix G, for the identification of the three characteristic regimes of partial fermionization and explicit approximants, Appendix H, for the relation between local pair correlations and partial fermionization, which includes Refs. [58–61].