Subharmonic functions and real-valued differential forms on non-archimedean curves

DISSERTATION

ZUR ERLANGUNG DES DOKTORGRADES
DER NATURWISSENSCHAFTEN (DR. RER. NAT.)
DER FAKULTÄT FÜR MATHEMATIK
DER UNIVERSITÄT REGENSBURG

vorgelegt von

Veronika Wanner

aus

Straubing

im Jahr 2019

Promotionsgesuch eingereicht am:	08.04.2019
Die Arbeit wurde angeleitet von:	Prof. Dr. Walter Gubler

Prüfungsausschuss:

Vorsitzender:	Prof. Dr. Harald Garcke
1. Gutachter:	Prof. Dr. Walter Gubler
2. Gutachter:	Prof. Dr. José Ignacio Burgos Gil
Weiterer Prüfer:	Prof. Dr. Clara Löh
Ersatzprüfer:	Prof. Dr. Klaus Künnemann

Abstract

We show that the approach by Chambert–Loir and Ducros of defining plurisubharmonic functions on Berkovich spaces via real-valued differential forms is an extension of Thuillier's very well developed theory on non-archimedean curves. More precisely, we prove that a continuous function on the Berkovich analytification of a smooth proper algebraic curve over a non-archimedean field is plurisubharmonic in the sense of Chambert–Loir and Ducros if and only if it is subharmonic in the sense of Thuillier. Hence Thuillier's theory enables us to verify some of the characteristic properties of plurisubharmonic functions for this new approach by Chambert–Loir and Ducros. For example, it follows directly that for continuous functions being plurisubharmonic is stable under pullback with respect to morphisms of curves. Moreover, we deduce an analogue of the monotone regularization theorem on the Berkovich analytifications of $\mathbb{P}^1$ and Mumford curves.

Furthermore, we study the tropical Dolbeault cohomology for the Berkovich analytifications of $\mathbb{P}^1$ and Mumford curves. We show that it satisfies Poincaré duality and behaves analogously to the cohomology of curves over the complex numbers. We also give a complete calculation of the dimension of the cohomology on a basis of the topology.

Another part of this thesis is a generalization of the Energy Minimization Principle to the analytification of a general smooth proper curve over a non-archimedean field. This was known before only for the Berkovich analytification of $\mathbb{P}^1$ by work of Baker and Rumely. As an application, we generalize an equidistribution result on elliptic curves due to Baker and Petsche.

Contents

CHAPTER 1

Introduction

Potential theory is a very old area of mathematics and originates in mathematical physics, in particular in electrostatic and gravitational problems. A fundamental part of it is the study of subharmonic functions. Potential theory has been extended to non-archimedean analytic geometry in the one-dimensional case. This is done by Baker and Rumely in [**BR10**] for the Berkovich analytification of the projective line and by Thuillier in [**Thu05**] for general analytic curves. Naturally, one is interested in developing potential theory also in higher dimensions. Ideas and concepts from classical pluripotential theory have been already introduced into the theory of non-archimedean analytic spaces by several authors such as Zhang [**Zha93**], Boucksom, Favre, and Jonsson [**BFJ15, BFJ16**], Chambert-Loir and Ducros [**CD12**], and Gubler and Künnemann [**GK17, GK15**]. For example, the approach by Chambert-Loir and Ducros in [**CD12**] is based on their theory of real-valued differential forms and currents on Berkovich spaces. Plurisubharmonicity of the function is characterized by positivity of a corresponding current. Their definition is therefore analogous to the one in classical complex analysis. Just like Thuillier's notion in the one-dimensional case, their notion is locally analytic and works without any hypotheses on the characteristic. Furthermore, they introduced the Monge–Ampère measure for plurisubharmonic functions that are locally approximable by smooth plurisubharmonic functions. This is a partial analogue of the complex Bedford–Taylor theory. One would desire an analogue of this whole theory, and also a monotone regularization theorem in this setting would be worthwhile. Moreover, we do not know if Chambert-Loir and Ducros' notion of plurisubharmonicity is stable under pullback. In contrast, we already know that subharmonic functions on analytic curves in the sense of Thuillier satisfy all of the above mentioned properties (cf. [**Thu05**, §3.2]). However, his theory only works in dimension one.

The main purpose of this thesis is to compare the notion of plurisubharmonic functions by Chambert-Loir and Ducros on curves with Thuillier's theory and use the comparison to find out whether the above mentioned characteristic properties hold in this one-dimensional case. The main result of this thesis is that for continuous functions these notions coincide (see Theorem 1).

For the most of this thesis, we work in the situation where X is a smooth proper algebraic curve over an algebraically closed, complete, non-archimedean, non-trivially valued field K. We denote by X^{an} the Berkovich analytification of X. We call a function *subharmonic* if it is subharmonic in the sense of Thuillier, and we call a function *plurisubharmonic* (shortly *psh*) if it is plurisubharmonic in the sense of Chambert-Loir and Ducros. In both approaches there are characterizations of continuous (pluri)subharmonic functions that follow a similar pattern, which we briefly explain. One defines a class of smooth functions and a Laplacian for them. The Laplacian can be extended canonically to the dual space. As every continuous function defines an element in the dual space, we have a corresponding Laplacian. A continuous functions is then called (pluri)subharmonic if the corresponding Laplacian is positive. Note that the classes of smooth functions are different in both approaches, and so it is not clear whether the notions of subharmonicity coincide.

In Thuillier's theory, smooth functions are roughly speaking pullbacks of piecewise affine functions along retraction maps of skeleta. We call them *lisse* and write $A^0(W)$ for the class of lisse functions on an open subset W of X^{an}. The corresponding Laplacian of a lisse function $g \in A^0(W)$ is a discretely supported measure on W and it is denoted by $dd^c g$. Then a continuous function f on W is *subharmonic* if and only if $\int f\ dd^c g \geq 0$ for all non-negative $g \in A^0(W)$ with compact support in W.

Chambert-Loir and Ducros' potential theory is based on their double complex of real-valued differential forms $(\mathcal{A}_X^{p,q}, d', d'')$ on X^{an} with the differential operators d' and d''. This should be thought of as an analogue of the sheaf of (p,q)-differential forms with differential operators ∂ and $\overline{\partial}$ on a complex manifold. Their class of smooth functions on an open subset W of X^{an} is defined as $\mathcal{C}^\infty(W) := \mathcal{A}_X^{0,0}(W)$. We call these functions *smooth*. They locally look like pullbacks of smooth functions on $\mathbb{R}^r$ along tropicalization maps. The corresponding Laplacian of a smooth function $g \in \mathcal{C}^\infty(W)$ is given by $d'd''g$ which defines a Radon measure on W. Then a continuous function f on W is *psh* if and only if $\int f\ d'd''g \geq 0$ for all non-negative $g \in \mathcal{C}^\infty(W)$ with compact support in W.

Although the classes of smooth functions and Laplacians in both approaches are obviously different, we can prove the following result in Corollary 5.2.12:

THEOREM 1 (Comparison Theorem). Let $f\colon W \to \mathbb{R}$ be a continuous function on an open subset W of X^{an}. Then f is subharmonic if and only if it is psh.

The Comparison Theorem and the resulting corollaries were proven by the author in [**Wan18**]. As already mentioned, the theory of subharmonic functions by Thuillier is very well developed and most of the analogous statements to complex geometry are proven in his thesis. From our Comparison Theorem (Theorem 1) we can deduce some of the characteristic properties also for psh functions. For example, we prove in Corollary 5.3.1 stability under pullback:

COROLLARY 2. Let X, X' be smooth proper curves over K and let $\varphi\colon W' \to W$ be a morphism of K-analytic spaces for open subsets $W \subset X^{\mathrm{an}}$ and $W' \subset (X')^{\mathrm{an}}$. If a continuous function $f\colon W \to \mathbb{R}$ is psh on W, then $\varphi^* f$ is psh on $\varphi^{-1}(W)$.

One of the main characteristic properties of complex subharmonic functions is regularization by smooth subharmonic functions. Thuillier proved a non-archimedean analogue of the regularization theorem in the case of curves in his thesis. Since the two classes of smooth functions do not coincide, a regularization theorem in the setting of Chambert-Loir and Ducros does not follow immediately. However, we prove in Corollary 5.3.4 the following regularization result using Theorem 1 and Thuillier's regularization theorem:

COROLLARY 3. Let X be $\mathbb{P}^1$ or a Mumford curve over K and let $f\colon W \to \mathbb{R}$ be a continuous psh function on an open subset W of X^{an}. Then f is locally the uniform limit of a decreasing sequence of smooth psh functions.

Here, a Mumford curve is a smooth proper algebraic curve of genus $g > 0$ such that the special fiber of a semistable formal model has only rational irreducible components (cf. Section 4.2.3 for some general properties of these curves).

The second main part of this thesis is the study of the tropical Dolbeault cohomology $\mathrm{H}^{p,q}(V) := \mathrm{H}^q(\mathcal{A}_X^{p,\bullet}(V), d'')$ for a fixed p and an open subset V of X^{an}. This whole part is joint work with Philipp Jell and was published in [**JW18**]. We let $\mathcal{A}_{X,c}^{p,\bullet}(V)$ be the sections of $\mathcal{A}_X^{p,\bullet}(V)$ with compact support in V and we write $\mathrm{H}_c^{p,q}(V) := \mathrm{H}^q(\mathcal{A}_{X,c}^{p,\bullet}(V), d'')$. The main result is Poincaré duality for certain open subsets of X^{an} (see Corollary 4.2.47):

THEOREM 4 (Poincaré Duality). Let X be a smooth curve over K and $V \subset X^{\mathrm{an}}$ an open subset such that all points of type II in V have genus 0. Then

$$\mathrm{PD}\colon\ \mathrm{H}^{p,q}(V) \to \mathrm{H}_c^{1-p,1-q}(V)^*,\ [\alpha] \mapsto ([\beta] \mapsto \int_V \alpha \wedge \beta)$$

is an isomorphism for all p, q.

In particular, the map PD is an isomorphism for every open subset V of X^{an} if X is $\mathbb{P}^1$ or is a Mumford curve. Normally, we only work over non-trivially valued fields. However, we show that the statement is still true for any open subset V of $\mathbb{P}^{1,\mathrm{an}}$ if K is trivially valued. For the definition of the genus of a type II point we also refer to Section 4.2.3.

Theorem 4 allows us to give a complete calculation of the dimensions $h^{p,q}(X^{\mathrm{an}})$ for $\mathbb{P}^1$ and Mumford curves (cf. Theorem 4.2.50). Note that $h^{p,q}(X^{\mathrm{an}}) = 0$ for every algebraic curve X if $p > 1$ or $q > 1$. We indeed find as in the complex case the following dimensions.

THEOREM 5. Let X either be $\mathbb{P}^1$ or a Mumford curve over K. We denote by g the genus of X and let $p, q \in \{0, 1\}$. Then we have

$$h^{p,q}(X^{\mathrm{an}}) = \begin{cases} 1 \text{ if } p = q, \\ g \text{ else.} \end{cases}$$

Again, we also prove this theorem for $X = \mathbb{P}^1$ if K is trivially valued.

A basis of the topology of X^{an} is given by so called absolutely simple open subsets (see Definition 4.2.51). Theorem 4 enables us to calculate $h^{p,q}$ and $h_c^{p,q}$ for absolutely simple open subsets which do not contain type II points of positive genus (cf. Theorem 4.2.54). By the boundary of such an absolutely simple open subset we mean the topological boundary in X^{an}. In particular, we show in Corollary 2.3.31 that this boundary is finite.

THEOREM 6. Let X be a smooth proper curve over K and $p, q \in \{0, 1\}$. For every absolutely simple open subset V of X^{an} containing no type II points of positive genus and $k := \#\partial V$, we have

$$h^{p,q}(V) = \begin{cases} 1 & \text{if } (p,q) = (0,0) \\ k-1 & \text{if } (p,q) = (1,0) \\ 0 & \text{if } q \neq 0 \end{cases} \quad \text{and} \quad h_c^{p,q}(V) = \begin{cases} 1 & \text{if } (p,q) = (1,1) \\ k-1 & \text{if } (p,q) = (0,1) \\ 0 & \text{if } q \neq 1. \end{cases}$$

For any smooth algebraic curve X of genus g, the space X^{an} contains at most g points of type II with positive genus [**BPR13**, Remark 4.18]. Thus this theorem describes the cohomology locally at all but finitely many points. Furthermore, if X is $\mathbb{P}^1$ or a Mumford curve, then X^{an} contains no type II points of positive genus (Theorem 4.2.29 and Proposition 4.2.26). Thus Theorem 6 describes the cohomology on a basis of the topology.

There is also a link between the above mentioned cohomology results and the comparison of the notions of harmonic functions on X^{an}. If X is a Mumford curve, we can use Theorem 4 and Theorem 6 to show in Section 4.2.7 (independently from Theorem 1) that a continuous function $h\colon W \to \mathbb{R}$ on an open subset W of X^{an} is harmonic in the sense of Thuillier if and only if h and $-h$ are psh. This link is also joint work with Philipp Jell. Note that this is of course also a direct consequence of Theorem 1, but the proof of Theorem 1 is much harder.

Baker and Rumely developed in [**BR10**] a potential theory on $\mathbb{P}^{1,\mathrm{an}}$ independently from Thuillier. However, one can show that Thuillier's definition of subharmonic functions extends the one by Baker and Rumely (see Proposition 3.1.10). On the other hand, they show more potential theory results in the case of $X = \mathbb{P}^1$. The third part of this thesis is the generalization of some results from [**BR10**] to all smooth proper curves following Matt Baker's suggestions. The main result of this part is a generalization of the Energy Minimization Principle. To obtain such a principle, we first need to show the existence of an Arakelov–Green's function g_μ on $X^{\mathrm{an}} \times X^{\mathrm{an}}$ for a given probability measure μ on X^{an} with continuous potentials analogous to the complex geometrical setting. For the definition of having continuous potentials we refer to Definition 3.2.33. Complex Arakelov–Green's functions are characterized by a special list of properties. We show in Corollary 3.2.42 the existence and uniqueness of a function g_μ satisfying the following analogous list of properties. This results extends the theory from [**BR10**, §8.10] to all smooth proper curves X:

THEOREM 7. For a probability measure μ on X^{an} with continuous potentials, there exists a unique symmetric function $g_\mu\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty, \infty]$ such that the following holds.

i) (Semicontinuity) The function g_μ is finite and continuous off the diagonal and strongly lower semi-continuous on the diagonal in the sense that
$$g_\mu(x_0, x_0) = \liminf_{(x,y)\to(x_0,x_0),x\neq y} g_\mu(x, y).$$
ii) (Differential equation) For each fixed $y \in X^{\mathrm{an}}$ the function $g_\mu(\cdot, y)$ is an element of $D^0(X^{\mathrm{an}})$ and
$$dd^c g_\mu(\cdot, y) = \mu - \delta_y.$$
iii) (Normalization)
$$\int\int g_\mu(x, y)\ d\mu(x)d\mu(y) = 0.$$

The function g_μ is called the *Arakelov–Green's function* corresponding to μ. With the help of g_μ, we can define the μ-energy integral of an arbitrary probability measure ν on X^{an} as
$$I_\mu(\nu) := \int\int g_\mu(x, y)\ d\nu(y)d\nu(x).$$
We formulate and prove in Theorem 3.2.44 the following Energy Minimization Principle analogous to the one in complex potential theory and [**BR10**, §8.10]:

THEOREM 8 (Energy Minimization Principle). Let μ be a probability measure on X^{an} with continuous potentials. Then

i) $I_\mu(\nu) \geq 0$ for each probability measure ν on X^{an}, and
ii) $I_\mu(\nu) = 0$ if and only if $\nu = \mu$.

As a direct application of the Energy Minimization Principle, we can give a generalization and a different proof of the non-archimedean local discrepancy result from [**BP05**] for an elliptic curve E over K. Note that in [**BP05**] everything was worked out for K coming from a number field. For our general K, we define the *local discrepancy* of a subset $Z_n \subset E(K)$ consisting of n distinct points as
$$D(Z_n) := \frac{1}{n^2}\left(\sum_{P\neq Q\in Z_n} g_\mu(P, Q) + \frac{n}{12}\log^+|j_E|\right),$$

where μ is the canonical measure and j_E is the j-invariant of E (see Subsection 3.2.6 for definitions). Note that this definition is consistent with the definition of local discrepancy from [**BP05**] and [**Pet09**]. We show in Corollary 3.2.58 the following generalization of [**BP05**, Corollary 5.6] using the Energy Minimization Principle:

Corollary 9. For each $n \in \mathbb{N}$, let $Z_n \subset E(K)$ be a set consisting of n distinct points and let δ_n be the probability measure on E^{an} that is equidistributed on Z_n. If $\lim_{n\to\infty} D(Z_n) = 0$, then δ_n converges weakly to μ on E^{an}.

We now outline the organization of this thesis. In Chapter 2, we recall the Berkovich analytification X^{an} of an algebraic variety X over K (see Section 2.1) and two very important tools to study X^{an}, tropicalizations and skeleta. In Section 2.2, we give a short overview of the concept of tropicalization. This provides the basis of the definition of the superforms $\mathcal{A}_X^{p,q}$ on X^{an} in Chapter 4.

In Section 2.3, we consider skeleta of non-archimedean curves. These are deformation retracts of X^{an} and have the structure of metrized graphs. Thuillier's theory of lisse and subharmonic functions is based on them, and so they play a key role in Chapter 3. There is also a basis of open neighborhoods in X^{an} that can be characterized by skeleta, which are called *simple open* subsets (cf. Definition 2.3.29). We study these special sets in Subsection 2.3.3.

In Chapter 3, we present the fundamental principles of potential theory developed by Thuillier. First, we recall in Subsection 3.1.1 his definition of subharmonic functions and their basic properties. We also give a proof that his definition is a generalization of the one by Baker and Rumely in [**BR10**]. In Subsection 3.1.2, we explain the class A^0 of Thuillier's smooth functions, which we call *lisse*, and the construction of their Laplacian dd^c. The Laplacian of a lisse function, which is a measure with discrete support, is positive if and only if the function is subharmonic. One can extend the Laplacian to the dual space $D^0(W) := A_c^0(W)^*$ of lisse functions with compact support in W. An upper semi-continuous function $f\colon W \to [-\infty, \infty)$ is then subharmonic if and only if $f \in D^0(W)$ with $\langle dd^c f, g\rangle := \langle f, dd^c g\rangle \geq 0$ for every non-negative $g \in A_c^0(W)$. Inspired by [**BR10**, Corollary 8.35], we prove in Subsection 3.1.3 a Domination Theorem. Given two subharmonic functions f and g on an open subset W of X^{an} with $dd^c f \geq dd^c g$ and satisfying some boundary condition, then we have $f \leq g$ on W (cf. Theorem 3.1.36).

In Section 3.2, we generalize another result from [**BR10**] to smooth proper curves, the Energy Minimization Principle (see Theorem 8). Analogously to [**BR10**], we introduce the potential kernel, capacity, potential functions and Arakelov–Green's functions. Some of these concepts also appear in [**Thu05**] and we compare them with our constructions. In Subsection 3.2.5, we prove analogues of the complex Maria's theorem and Frostman's theorem to deduce the Energy Minimization Principle from them as Baker and Rumely did in [**BR10**].

In Chapter 4, we first introduce the theory of real-valued differential forms $\mathcal{A}_X^{p,q}$ on the Berkovich analytification X^{an} of an algebraic variety X. These forms were defined by Chambert-Loir and Ducros in their preprint [**CD12**]. Here, we follow the equivalent algebraic approach by Gubler from [**Gub16**]. One obtains a double complex of sheaves $(\mathcal{A}_X^{\bullet,\bullet}, d', d'')$, which should be thought of as an analogue of the sheaf of (p,q)-differential forms with differential operators ∂ and $\overline{\partial}$ on a complex manifold.

In Section 4.2, we study the cohomology groups $\mathrm{H}_c^{p,q}(V) := \mathrm{H}^q(\mathcal{A}_{X,c}^{p,\bullet}(V), d'')$ in the one-dimensional case. The content of this section can be seen as an application of the tropical statements of [**JSS19**, Chapter 4] to the Berkovich analytic setting. To do so, we work with Jell's approach of using $\mathbb{A}$-tropical charts to define $\mathcal{A}_X^{p,q}$. Note that this approach also works in the trivially valued case and is equivalent to the one by Gubler and Chambert-Loir and Ducros in the non-trivially valued case. We first show in Subsection 4.2.5 that Poincaré duality is a local question and every open subset of $\mathbb{A}^{1,\mathrm{an}}$ satisfies it. Then Poincaré duality follows for all open subsets of X^{an} not containing type II points of positive genus (cf. Theorem 4) since these are locally isomorphic to open subsets of $\mathbb{A}^{1,\mathrm{an}}$. In Subsection 4.2.6, we apply Poincaré duality to prove Theorem 5 and Theorem 6. At the end of this section, we introduce pluriharmonic functions using differential forms. We deduce from the forgoing results, more precisely Theorem 4 and Theorem 6, that the different notions of harmonicity coincide for $\mathbb{P}^1$ and for Mumford curves.

In Chapter 5, we first introduce plurisubharmonic (shortly psh) functions in all dimensions following [**CD12**]. Note that a function h is pluriharmonic as defined in Subsection 4.2.7 if and only if h and $-h$ are plurisubharmonic. We outline some facts about these psh functions from [**CD12**]. As mentioned at the beginning, this theory is in the early stages of development and we do not know yet if all characteristic properties of plurisubharmonic are satisfied. At the end of this thesis, we can prove two of them in the case of a (Mumford) curve.

In Section 5.2, we compare subharmonic functions with plurisubharmonic functions, and so automatically harmonic functions with pluriharmonic functions, for an arbitrary smooth proper curve X over K. We prove the Comparison Theorem (Theorem 1) in several steps. The main ingredients are model functions, which are very well accessible in both theories. On the one hand they are lisse and on the other hand they can be written as the difference of uniform limits of smooth psh functions.

As an application of the fact that both notions of subharmonicity coincide, we obtain stability under pullback for psh functions and a regularization theorem in the setting of Chambert-Loir and Ducros in Section 5.3.

We now explain how this thesis interacts with the papers [**JW18**] and [**Wan18**]. Section 4.2 is precisely [**JW18**] with some minor changes in notations and referring to Section 2.1 for the definitions regarding polyhedral complexes. Differential forms are now defined in every dimension in Section 4.1 following Gubler's approach from [**Gub16**]. When we introduce Jell's new approach of $\mathbb{A}$-tropical charts in Subsection 4.2.1, we partially refer to Section 4.1.

Chapter 5 follows closely [**Wan18**], where the introductory part to Thuillier's theory from [**Wan18**] is moved to Section 3.1.

Terminology. In this thesis, let K be an algebraically closed field that is complete with respect to a non-archimedean absolute value $|\cdot|$. We write $\mathrm{val}(K^\times) := \log|K^\times|$ and $R := \{x \in K \mid |x| \leq 1\}$ with maximal ideal $\mathfrak{m} \subset R$ and residue field $\widetilde{K} := R/\mathfrak{m}$. By suitably normalizing the absolute value $|\cdot|$, we may assume that $\mathbb{Z}$, and so $\mathbb{Q}$, is contained in $\mathrm{val}(K^\times)$.

In $A \subset B$, the subset A may be equal to B, and the complement of A in B is always denoted by $B \backslash A$. All rings and algebras are commutative with 1. If A is a ring, then $A^\times$ denotes the multiplicative group of units. A variety over K is an irreducible separated reduced scheme of finite type over K and a curve is a 1-dimensional variety over K.

Acknowledgements. There are many people whom I would like to thank for helping me doing my research and writing this thesis. First of all, I am very grateful to my advisor Walter Gubler for suggesting this topic to me, for many inspiring and helping discussions and his general guidance during my PhD.

I would like to thank Philipp Jell for explaining his thesis to me as a starting point of my own research and his ideas and contributions to the joint paper [**JW18**]. Moreover, many thanks to him for being always so helpful and open for questions.

I would also like to express my gratitude to Matt Baker for inviting me to a two month research stay at Georgie Institute of Technology, many fruitful discussions during this time and especially for suggesting the generalization of parts of his book to general non-archimedean curves to me.

Furthermore, I am also very grateful to Johann Haas for his openness towards general questions and discussions.

There are a lot more people I would like to thank namely for interesting discussions and asking inspiring questions regarding this thesis as Ana María Botero, Antoine Chambert-Loir, Thomas Fenzl, Robin de Jong, Mattias Jonsson, Klaus Künnemann, Julia Menzel, Rosemarie Martienssen, Joe Rabinoff, Helene Sigloch, Matt Stevenson, Martino Stoffel and Christian Vilsmeier.

Moreover, I am thankful to the referees of the papers [**JW18**] and [**Wan18**] for their very precise report and helpful suggestions.

Finally, I thank the collaborative research center 'SFB 1085 Higher Invariants' for its financial support as well as the great activities, talks and conferences enabled by it.

CHAPTER 2

Preliminaries

2.1. Berkovich spaces

Let K be an algebraically closed field that is complete with respect to a non-archimedean absolute value $|\cdot|$. We denote by R its valuation ring and by $\widetilde{K}$ its residue field. For an algebraic variety X over K, we always denote by X^{an} the Berkovich analytification of X. We briefly recall the construction of this analytification.

Definition 2.1.1. For an open affine subset $U = \mathrm{Spec}(A)$ of the algebraic variety X, the analytification U^{an} is the set of all multiplicative seminorms on A extending the given absolute value $|\cdot|$ on K. We endow the set U^{an} with the coarsest topology such that $U^{\mathrm{an}} \to \mathbb{R}$, $p \mapsto p(a)$ is continuous for every element $a \in A$. By gluing, we get a topological space X^{an}, which is connected, locally compact and Hausdorff. We call the space X^{an} the *(Berkovich) analytification* of X which is a K-analytic space in the sense of [**Ber90**, §3.1].

For a morphism $\phi\colon X \to Y$ of algebraic varieties over K we have a canonical map $\phi^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ defined by pulling back seminorms on suitable affine open subsets.

Remark 2.1.2. In the case of a curve, the points of X^{an} can be classified in four different types following [**Ber90**, §1.4], [**Thu05**, §2.1] and [**BPR16**, §3.5]: For a point $x \in X^{\mathrm{an}}$, let $\mathscr{H}(x)$ be the completed residue field at x and $\widetilde{\mathscr{H}}(x)$ be its residue field. Then $\widetilde{\mathscr{H}}(x)$ has transcendence degree $s(x) \leq 1$ over $\widetilde{K}$, and the abelian group $|\mathscr{H}(x)^{\times}|/|K^{\times}|$ has rank $t(x) \leq 1$. Moreover, the integers $s(x)$ and $t(x)$ satisfy the *Abhyankar inequality* [**Vaq00**, Theorem 9.2]

$$s(x) + t(x) \leq 1.$$

We say that x is of *type II* if $s(x) = 1$ and of *type III* if $t(x) = 1$. If $s(x) = t(x) = 0$, then x is called of *type I* when $\mathscr{H}(x) = K$ and of *type IV* otherwise.

The points of type I can be identified with the rational points $X(K)$. By $I(X^{\mathrm{an}})$ we denote the subset of points of type II or III, and by $\mathbb{H}(X^{\mathrm{an}})$ the subset of points of type II, III and IV, i.e. $\mathbb{H}(X^{\mathrm{an}}) = X^{\mathrm{an}} \backslash X(K)$. For any subset S of X^{an}, we write $I(S) := S \cap I(X^{\mathrm{an}})$ and $\mathbb{H}(S) := S \cap \mathbb{H}(X^{\mathrm{an}})$. The set of type II points (and hence also $I(X^{\mathrm{an}})$) is dense in X^{an}. If K is non-trivially valued, then $X(K)$ is dense in X^{an} as well.

Example 2.1.3. The Berkovich analytification $\mathbb{A}^{r,\mathrm{an}}$ of the affine space $\mathbb{A}^r$ over K is the set of multiplicative seminorms on $K[T_1, \ldots, T_r]$ that extend the absolute value $|\cdot|$ on K, endowed with the coarsest topology such that for every $f \in K[T_1, \ldots, T_r]$ the map $\mathbb{A}^{r,\mathrm{an}} \to \mathbb{R}$, $p \mapsto p(f)$ is continuous. A basis of open subsets of $\mathbb{A}^{r,\mathrm{an}}$ is thus given by sets of the form

$$\{p \in \mathbb{A}^{r,\mathrm{an}} \mid a_i < p(f_i) < b_i\}$$

for $a_i, b_i \in \mathbb{R} \cup \{\infty\}$ and $f_i \in K[T_1, \ldots, T_r]$. We call sets of this form *standard open subsets* of $\mathbb{A}^{r,\mathrm{an}}$.

In the case of the affine line $\mathbb{A}^1$, its analytification $\mathbb{A}^{1,\mathrm{an}}$ is very well understood and we can describe its points explicitly: A point of $\mathbb{A}^{1,\mathrm{an}}$ that is given by a seminorm $p\colon K[T] \to \mathbb{R}_{\geq 0}$ is

i) of type I if p is of the form $f \mapsto f(a)$ for some $a \in K$,
ii) of type II (resp. type III) if p is of the form $f \mapsto \sup_{c\in D(a,r)} |f(c)|$ for some $a \in K$ and $r \in |K^\times|$ (resp. $r \notin |K^\times|$), where $D(a,r) := \{c \in K \mid |a-c| \leq r\}$, and
iii) of type IV if p is given by $f \mapsto \lim_{i\to\infty} \sup_{c\in D(a_i,r_i)} |f(c)|$ for a decreasing nested sequence $(D(a_i,r_i))$ of closed balls with empty intersection.

As a set, the Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}$ can be obtained from $\mathbb{A}^{1,\mathrm{an}}$ by adding a type I point denoted by ∞.

We end this subsection by studying some properties of measures on the analytification X^{an} of a smooth proper algebraic curve X. Here, we use the nice properties of the topological space X^{an}.

Definition 2.1.4. Let Y be a locally compact Hausdorff space and let μ be a positive Borel measure on Y. If E is a Borel subset of Y, then μ is called *outer regular on E* if

$$\mu(E) = \inf\{\mu(U) \mid E \subset U,\ U \text{ open}\},$$

and *inner regular on E* if

$$\mu(E) = \sup\{\mu(E') \mid E' \subset E,\ E' \text{ compact}\}.$$

We say that μ is a *Radon measure* if it is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open subsets.

More generally, a signed Borel measure is said to be a *signed Radon measure* if and only if its positive and negative parts are both Radon measures.

Proposition 2.1.5. *Let X be a smooth proper algebraic curve. Then every finite signed Borel measure on X^{an} is a signed Radon measure.*

Proof. Let μ be a finite signed Borel measure on X^{an}. By the definition of a signed Radon measure, we may assume μ to be positive. As X^{an} is a compact Hausdorff space, it remains by [**Roy88**, §13.2 Proposition 14] to show that

$$\mu(V) = \sup\{\mu(E) \mid E \text{ compact in } X^{\mathrm{an}} \text{ with } E \subset V\} =: M$$

for every open subset V of X^{an}. The open subset V is paracompact by [**Ber90**, Theorem 4.2.1 & 4.3.2], and so V is the countable union of compact subsets E_n of X^{an} by [**CD12**, (2.1.5)]. As the finite union of compact subsets is compact, we may assume $E_n \subset E_{n+1}$. Continuity from below for measures implies

$$\mu(V) = \mu\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) = \lim_{n\to\infty} \mu(E_n) \leq M.$$

Obviously, $\mu(V) \geq \mu(E)$ for every compact subset E of V as μ is positive. Hence, we also have $\mu(V) \geq M$ implying our assertion. □

Corollary 2.1.6. *Let X be a smooth proper algebraic curve. Then every net $(\nu_\alpha)_\alpha$ of probability measures ν_α on X^{an} has a subnet that converges weakly to a probability measure ν on X^{an}.*

Proof. Since every probability measure is a Radon measure by Proposition 2.1.5, the assertion follows by the Prohorov's theorem for nets (see for example [**BR10**, Theorem A.11]). □

2.2. Tropicalization

In this section, we briefly recall the concept of tropicalization. Its relation to polyhedral complexes shown by Bieri and Groves is the fundamental tool to define real-valued differential forms on X^{an} following Chambert-Loir and Ducros (resp. Gubler) in Chapter 4.

Again we consider an algebraic variety X over an algebraically closed field K endowed with a complete non-archimedean absolute value $|\cdot|$.

Definition 2.2.1.
i) A *polyhedron* in $\mathbb{R}^r$ is the intersection of finitely many half-spaces $H_i := \{w \in \mathbb{R}^r | \langle u_i, w\rangle \leq c_i\}$ with $u_i \in \mathbb{R}^{r*}$ and $c_i \in \mathbb{R}$. A *face* of a polyhedron σ is a polyhedron that is obtained by turning some of the defining inequalities of σ into equalities. Let Γ be a subgroup of $\mathbb{R}$, then a polyhedron σ is called Γ-*rational* if we may choose u_i and c_i such that all u_i have integer coefficients and all $c_i \in \Gamma$.
ii) A *polyhedral complex* $\mathscr{C}$ in $\mathbb{R}^r$ is a finite set of polyhedra in $\mathbb{R}^r$ satisfying the following two properties:
   (a) If τ is a closed face of a polyhedron $\sigma \in \mathscr{C}$, then $\tau \in \mathscr{C}$.
   (b) If $\sigma, \tau \in \mathscr{C}$, then $\sigma \cap \tau$ is a closed face of both.

   The polyhedral complex is called Γ-*rational* for a subgroup Γ of $\mathbb{R}$ if every polyhedron $\sigma \in \mathscr{C}$ is Γ-rational.

Definition 2.2.2. Let $\mathscr{C}$ be a polyhedral complex in $\mathbb{R}^r$.
i) We say that $\mathscr{C}$ is of *dimension* d if the maximal dimension of its polyhedra is d. A polyhedral complex $\mathscr{C}$ is called *pure dimensional of dimension* d if every maximal polyhedron in $\mathscr{C}$ has dimension d.
ii) The *support* $|\mathscr{C}|$ of $\mathscr{C}$ is the union of all polyhedra in $\mathscr{C}$. Let Y be a subset of $\mathbb{R}^r$ with $Y = |\mathscr{C}|$. Then $\mathscr{C}$ is called a *polyhedral structure* on Y.
iii) For $\sigma \in \mathscr{C}$, we denote by $\mathrm{relint}(\sigma)$ the relative interior of σ, by $\mathbb{A}(\sigma)$ the affine space that is spanned by σ and by $\mathbb{L}(\sigma)$ the corresponding linear subspace of $\mathbb{R}^r$.

Definition 2.2.3. A *weighted polyhedral complex* $\mathscr{C}$ is a pure d-dimensional polyhedral complex such that every face $\sigma \in \mathscr{C}$ of dimension d is equipped with a weight $m_\sigma \in \mathbb{Z}$.

Let $\mathscr{C}$ be a weighted Γ-rational polyhedral complex of pure dimension d. Then for every polyhedron $\sigma \in \mathscr{C}$ there is a canonical lattice of full rank $\mathbb{Z}(\sigma) \subset \mathbb{L}(\sigma)$. For a d-dimensional polyhedron and a $(d-1)$-dimensional face τ of σ, we denote by $\nu_{\tau,\sigma}$ a representative of the unique generator in $\mathbb{Z}(\sigma)/\mathbb{Z}(\tau)$ that is pointing outwards of σ in the direction of τ. $\mathscr{C}$ is called *balanced* if every $(d-1)$-dimensional polyhedron τ of $\mathscr{C}$ fulfills the following *balancing condition*

$$\sum_{\sigma \supset \tau} m_\sigma \nu_{\tau,\sigma} \in \mathbb{Z}(\tau),$$

where σ runs over all d-dimensional polyhedra of $\mathscr{C}$ containing τ.

A weighted Γ-rational polyhedral complex is called a *tropical cycle* if it is balanced.

Definition 2.2.4. We set $\mathbb{G}_m^r := \operatorname{Spec} K[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$. Recall that its Berkovich analytification $\mathbb{G}_m^{r,\mathrm{an}}$ is the set of all multiplicative seminorms on $K[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$ extending the given absolute value $|\cdot|$ on K endowed with the coarsest topology such that $\mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{R},\ p \mapsto p(f)$ is continuous for every $f \in K[T_1^{\pm 1}, \dots, T_r^{\pm 1}]$. We define

$$\mathrm{trop}\colon \mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{R}^r,\ p \mapsto (\log(p(T_1)), \dots, \log(p(T_r))).$$

Note that trop is a proper continuous map of topological spaces.

Let U be an open subset of the algebraic variety X over K and let $\varphi\colon U \to \mathbb{G}_m^r$ be a closed embedding, then we set $\varphi_{\mathrm{trop}} := \mathrm{trop} \circ \varphi^{\mathrm{an}}$ and $\mathrm{Trop}_\varphi(U) := \varphi_{\mathrm{trop}}(U^{\mathrm{an}})$.

REMARK 2.2.5. Note that we differ in sign from e.g. [**Gub16, Gub13, BPR13, Wan18**] here because of the following extension used in Subsection 4.2: For a closed subvariety Z of $\mathbb{A}^r = \mathrm{Spec}(K[T_1, \dots, T_r])$, we define $\mathrm{Trop}(Z)$ to be the image of Z^{an} under the (extended) tropicalization map

$$\begin{aligned} \mathrm{trop}\colon\ \mathbb{A}^{r,\mathrm{an}} &\to \mathbb{T}^r, \\ p &\mapsto (\log(p(T_1)), \dots, \log(p(T_r))). \end{aligned}$$

Here $\mathbb{T} := [-\infty, \infty)$ and we endow it with the topology of a half-open interval, and then $\mathbb{T}^r$ is equipped with the product topology. Clearly, $\mathbb{G}_m^{r,\mathrm{an}} = \mathrm{trop}^{-1}(\mathbb{R}^r)$.

THEOREM 2.2.6 (Bieri–Groves). *For every open subset U of X and every closed embedding $\varphi\colon U \to \mathbb{G}_m^r$, the set* $\mathrm{Trop}_\varphi(U)$ *is the support of an* $\mathrm{val}(K^\times)$*-rational polyhedral complex of pure dimension $n = \dim(X)$.*

PROOF. See [**BG84**, Theorem A]. □

REMARK 2.2.7. There is a canonical way to associate positive weights to $\mathrm{Trop}_\varphi(U)$ (see [**Gub13**, 13.10]). It is shown in [**Gub13**, Theorem 13.11] that $\mathrm{Trop}_\varphi(U)$ is the support of a tropical cycle with respect to these weights.

2.3. Skeleta

Skeleta of the analytification X^{an} of an algebraic variety X are deformation retracts of X^{an} and they help us study the Berkovich analytification. In this section, we restrict ourselves to the curve case and recall the concept of skeleta of X^{an} for a smooth curve X over a non-trivially valued field K following [**Thu05**] and [**BPR13**]. Thuillier's potential theory is based on skeleta as we will see in Chapter 3.

2.3.1. Strictly analytic domains and strictly semistable formal models. In this subsection, we recall the class of strictly analytic domains, which includes all strictly affinoid domains and X^{an} itself. Strictly analytic domains are accessible to study as they always have a strictly semistable model. These models lead to skeleta as explained in the next subsection. To define strictly analytic domains, we need the notion of strictly affinoid domains. We give a brief recall and we refer to [**Ber90**] or [**Tem11**] for more details.

DEFINITION 2.3.1. For every $r \in \mathbb{R}^n_{>0}$, we set

$$K\{r^{-1}T\} := \{f = \sum_{\nu=0}^{\infty} a_\nu T^\nu \mid a_\nu \in K \text{ and } |a_\nu| r^\nu \to 0 \text{ as } \nu \to \infty\},$$

where $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ and we use the notations $|\nu| = \nu_1 + \ldots + \nu_n$, $T^\nu = T_1^{\nu_1} \cdots T_n^{\nu_n}$ and $r^\nu = (r_1^\nu, \dots, r_n^\nu)$. This is a commutative K-Banach algebra with respect to the multiplicative norm $\|f\| := \max_\nu |a_\nu| r^\nu$.

If $r = 1 \in \mathbb{R}^n_{>0}$, the K-Banach algebra $K\{r^{-1}T\}$ is called the *Tate algebra in n variables*, and it is usually denoted by $K\langle T_1, \dots, T_n\rangle$.

DEFINITION 2.3.2. An *affinoid algebra* is a Banach K-algebra A with an admissible epimorphism $K\{r^{-1}T\} \to A$, i.e. A is isomorphic to a quotient $K\{r^{-1}T\}/I$ for some ideal I and its norm is equivalent to the residue norm of $K\{r^{-1}T\}/I$. If $r \in |K^\times|^n$, we say that A is a *strictly affinoid algebra*.

For an affinoid algebra A, we define the *Berkovich spectrum* $\mathcal{M}(A)$ of A as the set of all bounded multiplicative seminorms on A extending the given absolute value $|\cdot|$ on K. As for the analytification of X, we endow $\mathcal{M}(A)$ with the coarsest topology such that the maps $p \to p(f)$ are continuous for all $f \in A$ and we obtain a nonempty compact topological space.

DEFINITION 2.3.3. A closed subset V of X^{an} is called an *affinoid domain* in X^{an} if there exists a morphism $\phi\colon \mathcal{M}(A_V) \to X^{\mathrm{an}}$ for an affinoid algebra A_V whose image coincides with V and such that any morphism $\mathcal{M}(B) \to X^{\mathrm{an}}$ for an affinoid algebra B with image in V factors through ϕ.

If V is an affinoid domain such that A_V is a strictly affinoid algebra, then V is called a *strictly affinoid domain*. We identify a (strictly) affinoid domain V with its Berkovich spectrum $\mathcal{M}(A_V)$ as one can show that they are homeomorphic.

DEFINITION 2.3.4. A *(strictly) analytic domain* in X^{an} is a subset of X^{an} having a locally finite covering of (strictly) affinoid domains.

For example X^{an} is a strictly analytic domain.

REMARK 2.3.5. There are different notions of boundary for a subset W of X^{an}. If nothing is stated otherwise, we always talk about the Berkovich boundary of W, which is the topological boundary in X^{an}. For example, the limit boundary of W is defined as the set of limit points of sequences in W which are not contained in W. If W is open in the topology of X^{an}, its limit boundary coincides with the Berkovich boundary by [**Poi13**, Corollaire 5.5].

For an affinoid domain $V = \mathcal{M}(A_V)$, one can also define the Shilov boundary as the smallest subset $\Gamma(V) \subset V$ such that every function $p \mapsto p(f)$ for $f \in A_V$ attains its maximum at a point in $\Gamma(V)$. Then these three notions coincide and the boundary is always a finite set of points of type II or III in X^{an} [**Thu05**, Proposition 2.1.12]. If the affinoid domain is strictly affinoid, all boundary points are of type II.

DEFINITION 2.3.6. A *closed (resp. open) ball of radius* $|a|$ in $\mathbb{A}^{1,\mathrm{an}}$ is a set of the form $\mathrm{trop}^{-1}([-\infty, \log|a|])$ (resp. $\mathrm{trop}^{-1}([-\infty, \log|a|)))$ for $a \in K^\times$.

A *closed (resp. open) annulus of inner radius* $|a|$ *and outer radius* $|b|$ in $\mathbb{A}^{1,\mathrm{an}}$ is a set of the form $\mathrm{trop}^{-1}([\log|a|, \log|b|])$ (resp. $\mathrm{trop}^{-1}((\log|a|, \log|b|)))$ for $a, b \in K^\times$ with $|a| \leq |b|$ (resp. $|a| < |b|$).

DEFINITION 2.3.7. A subset of X^{an} is called *closed ball (resp. closed annulus, resp. open ball, resp. open annulus) in* X^{an} if it is isomorphic to a closed ball (resp. closed annulus, resp. open ball, resp. open annulus) as K-analytic spaces.

If A is an open annulus in X^{an} isomorphic to $\mathrm{trop}^{-1}((\log|a|, \log|b|))$ for some $a, b \in K^\times$ with $|a| < |b|$, then $\log|b| - \log|a|$ is called the *modulus of* A, which is well-defined by [**BPR13**, Corollary 2.6].

REMARK 2.3.8. Balls and annuli are objects that we understand very well. For example, they are uniquely path-connected. Moreover, we can say something about its boundary. Let B be an open or a closed ball in X^{an} isomorphic to $\mathrm{trop}^{-1}([-\infty, \log|a|))$ resp. isomorphic to $\mathrm{trop}^{-1}([-\infty, \log|a|])$ via an isomorphism φ for some $a \in K^\times$. Then the boundary of B is a single point in X^{an} of type II by [**ABBR15**, Lemma 3.3]. If B is a closed ball, then B is a strictly affinoid domain in X^{an} isomorphic to $\mathcal{M}(K\{r^{-1}T\})$ for $r = |a|$.

If A is an open or a closed annulus in X^{an} isomorphic to $\mathrm{trop}^{-1}((\log|a|, \log|b|))$ resp. isomorphic to $\mathrm{trop}^{-1}([\log|a|, \log|b|])$ via an isomorphism φ for some $a, b \in K^\times$ with $|a| \leq |b|$, then the boundary of A is a set of at most two type II points in X^{an} by [**ABBR15**, Lemma 3.6]. If A is a closed annulus, then A is a strictly affinoid domain in X^{an} isomorphic to $\mathcal{M}(K\{r^{-1}T\}/(T_1T_2 - 1))$ for $r = (|a|, |b|^{-1})$.

Definition 2.3.9. An R-algebra $\mathcal{A}$ is called *admissible* if it is R-flat and isomorphic to a quotient $R\langle T_1, \ldots, T_n\rangle/I$ for $R\langle T_1, \ldots, T_n\rangle := K\langle T_1, \ldots, T_n\rangle \cap R[\![T_1, \ldots, T_n]\!]$ and an ideal I. Then $\mathcal{A} \otimes_R K$ is an affinoid algebra.

A formal scheme $\mathcal{X}$ over R is called *admissible* if there is a locally finite covering of open subsets isomorphic to formal affine schemes $\mathrm{Spf}(\mathcal{A})$ for admissible R-algebras $\mathcal{A}$. For the definition of formal schemes we refer to [**Bos14**, §7.2].

The *generic fiber* $\mathcal{X}_\eta$ of an admissible formal scheme $\mathcal{X}$ is the analytic space locally defined by the Berkovich spectrum of the affinoid algebra $A := \mathcal{A} \otimes_R K$. Moreover, we define the *special fiber* $\mathcal{X}_s$ of $\mathcal{X}$ as the $\widetilde{K}$-scheme locally given by $\mathrm{Spec}(A/\widetilde{K}A)$, i.e. $\mathcal{X}_s$ is a scheme of locally finite type over $\widetilde{K}$ with the same topological space as $\mathcal{X}$ and structure sheaf $\mathcal{O}_{\mathcal{X}} \otimes_R \widetilde{K}$.

One can define a *reduction map* $\mathrm{red}\colon \mathcal{X}_\eta \to \mathcal{X}_s$ assigning to each seminorm p in a neighborhood $\mathcal{M}(A)$ the prime ideal $\{a \in \mathcal{A} \mid p(a \otimes 1) < 1\}/\widetilde{K}A$. This map is surjective and anti-continuous. If $\mathcal{X}_s$ is reduced, then red coincides with the reduction map in [**Ber90**, §2.4] and $\mathrm{red}^{-1}(\zeta)$ is a single point of $\mathcal{X}_\eta$ for every generic point ζ of an irreducible component of $\mathcal{X}_s$ [**Ber90**, Proposition 2.4.4].

Definition 2.3.10. A *semistable R-curve* is an integral admissible formal R-curve $\mathcal{X}$ whose special fiber $\mathcal{X}_s$ has only ordinary double points as singularities. In particular, $\mathcal{X}_s$ is reduced. A semistable R-curve is called *strictly semistable* if in addition the irreducible components of the special fiber are smooth.

Definition 2.3.11. Let Y be a compact strictly analytic domain in X^{an}, then a *(strictly) semistable formal model* of Y is a (strictly) semistable proper formal R-curve $\mathcal{Y}$ together with an isomorphism $\varphi\colon Y \to \mathcal{Y}_\eta$.

From now on we identify $\mathcal{Y}_\eta$ with Y via φ.

Theorem 2.3.12. *Let Y be a compact strictly analytic domain in X^{an}, then there exists a strictly semistable model of Y.*

Proof. See [**Thu05**, Théorème 2.3.8], which is based on the semistable reduction theorem from [**BL85**, Ch.7]. □

2.3.2. Skeleta via strictly semistable formal models. In this subsection, we explain how strictly semistable formal models lead to skeleta and give their most important properties.

Definition 2.3.13. Let $\mathcal{Y}$ be a strictly semistable formal model of a compact strictly analytic domain Y. We define

$$S_0(\mathcal{Y}) := \{\mathrm{red}^{-1}(\zeta) \mid \zeta \text{ generic point of an irreducible component of } \mathcal{Y}_s\},$$

which is a finite set of type II points in Y [**Ber90**, Proposition 2.4.4].

Proposition 2.3.14. *Let $\mathcal{Y}$ be a strictly semistable formal model of a compact strictly analytic domain Y. Then Y can be written as the disjoint union*

$$Y = S_0(\mathcal{Y}) \cup \bigcup_{j=1,\ldots,m} A_j \cup \bigcup_{i \in I} B_i \tag{2.3.1}$$

for finitely many open annuli A_j and infinitely many open balls B_i in X^{an}.

PROOF. The set $S_0(\mathcal{Y})$ is by definition the preimage of all generic points in $\mathcal{Y}_s$, which is a finite set of points (cf. [**Ber90**, Proposition 2.4.4]). Thuillier showed in [**Thu05**, Lemme 2.1.13] that the connected components of $Y\backslash S_0(\mathcal{Y})$ are the fibers of the closed points in $\mathcal{Y}_s$ under the reduction map $\mathrm{red}\colon Y \to \mathcal{Y}_s$. The closed points of $\mathcal{Y}_s$ consist of infinitely many smooth closed points and of finitely many ordinary double points as $\mathcal{Y}$ is strictly semistable. By a result of Bosch and Lüthkebohmert [**BL85**, Propositions 2.2 and 2.3], the preimage of a smooth closed point is an open ball and the preimage of an ordinary double point is an open annulus. We therefore get the decomposition of Y stated in (2.3.1). □

DEFINITION 2.3.15. Let Y be a compact strictly analytic domain in X^{an} and $\mathcal{Y}$ be a strictly semistable model of Y. Then we define $S(\mathcal{Y})$ to be the set of all points in Y that do not have an affinoid neighborhood that is isomorphic to a closed ball in Y and is disjoint from $S_0(\mathcal{Y})$.

We define the canonical *retraction map* $\tau_{\mathcal{Y}}\colon Y \to S(\mathcal{Y})$ as follows. If $x \in S(\mathcal{Y})$, we set $\tau_{\mathcal{Y}}(x) := x$. If $x \in Y\backslash S(\mathcal{Y})$, then the connected component B_x of $Y\backslash S(\mathcal{Y})$ that contains x is an open ball with unique boundary point ζ_x in $S(\mathcal{Y})$ (cf. [**BPR13**, Proposition 2.4 & Lemma 2.12]). We set $\tau_\Gamma(x) := \zeta_x$. Note that this retraction map is indeed continuous.

We call $S(\mathcal{Y})$ together with its retraction map $\tau_{\mathcal{Y}}\colon Y \to S(\mathcal{Y})$ the *skeleton of Y corresponding to $\mathcal{Y}$*.

DEFINITION 2.3.16. Let Y be a compact strictly analytic domain in X^{an}. A pair (Γ, Γ_0) consisting of a subset Γ of Y and a finite subset Γ_0 of points of type II contained in Y is called a *skeleton of Y* if there is a strictly semistable formal model $\mathcal{Y}$ of Y such that $\Gamma = S(\mathcal{Y})$ and $\Gamma_0 = S_0(\mathcal{Y})$. We then use the notation $\tau_\Gamma := \tau_{\mathcal{Y}}$.

REMARK 2.3.17. A skeleton (Γ, Γ_0) can be thought of as a pair of a finite metric graph Γ and a vertex set Γ_0 of Γ. We explain in Remark 2.3.23 the metric graph structure more precisely. Often, we are just interested in Γ without a specific vertex set Γ_0. We therefore just say a skeleton Γ without mentioning Γ_0.

Note that there always exist skeleta of all compact strictly analytic domains in X^{an} since all of them do have strictly semistable formal models by Theorem 2.3.12. In particular, the space X^{an} itself has a model if X is proper.

Baker, Payne and Rabinoff also give an explanation of the theory of skeleta of X^{an} in [**BPR13**]. They define skeleta with the help of (strongly) semistable vertex sets. These are finite subsets of type II points such that X^{an} can be decomposed as in (2.3.1). In the case of $Y = X^{\mathrm{an}}$ is compact, both definitions agree. Note that the analytification X^{an} of our considered smooth curve X is compact if and only if X is proper.

PROPOSITION 2.3.18. *Let X be proper. A pair (Γ, Γ_0) is a skeleton of X^{an} as in Definition 2.3.16 if and only if Γ_0 is a strongly semistable vertex set and Γ is the induced skeleton as defined in* [**BPR13**, Definition 3.3].

PROOF. Follows by [**BPR13**, Theorem 4.11] and [**BPR13**, Lemma 3.4 (4)]. □

REMARK 2.3.19. The analytification $\mathbb{P}^{1,\mathrm{an}}$ of the projective line is uniquely path-connected [**BR10**, Lemma 2.10]. Hence every finite subset Γ_0 of type II points induces a skeleton $\Gamma = \bigcup_{\zeta,\zeta' \in \Gamma_0} [\zeta, \zeta']$ by [**BPR13**, Definition 3.3] and Proposition 2.3.18. Here $[\zeta, \zeta']$ denotes the unique path between two points $\zeta, \zeta' \in \mathbb{P}^{1,\mathrm{an}}$. In particular, the skeleton of every connected strictly affinoid domain Y in $\mathbb{P}^{1,\mathrm{an}}$ is automatically a skeleton of $\mathbb{P}^{1,\mathrm{an}}$.

In the following, we will see some important properties of skeleta.

PROPOSITION 2.3.20. *Let Γ be a skeleton of a compact strictly analytic domain Y in X^{an}, then the following are true:*

i) If Y is connected, then Γ is connected.
ii) Γ is a compact subset of points of type II and III.

PROOF. First of all, recall that the retraction map τ_Γ is continuous. Thus assertion i) and compactness in ii) follow directly as $\Gamma = \tau_\Gamma(Y)$ and Y is connected and compact.

A point $x \in Y$ of type I or IV is contained in an open ball or in an open annulus by (2.3.1) as Γ_0 is a set of points of type II. Hence we may assume x to be a point in $\mathbb{P}^{1,\mathrm{an}}$. Then [**BR10**, Proposition 1.6] implies that x has a basis of open neighborhoods isomorphic to open balls, and so x cannot be contained in Γ. □

REMARK 2.3.21. Let X be proper, and so X^{an} is compact. Since the analytic space X^{an} is connected, every skeleton Γ of X^{an} is connected as well by Proposition 2.3.20. Note that the space X^{an} is in fact path-connected. In this thesis a *path* from x to y is a continuous injective map $\gamma\colon [a,b] \to X^{\mathrm{an}}$ with $\gamma(a) = x$ and $\gamma(b) = y$. If there is a unique path between two points $x, y \in X^{\mathrm{an}}$, we write $[x,y]$ for this path. Often we also use the notations $(x,y) := [x,y]\backslash\{x,y\}$, $(x,y] := [x,y]\backslash\{x\}$ and $[x,y) := [x,y]\backslash\{y\}$.

In the subsequent proposition, we will see that we can enlarge a skeleton as a graph by blowing up the corresponding model. The definition of an admissible blowing up can be found in [**Bos14**, §8.2] or [**Tem11**, §5.3.2].

PROPOSITION 2.3.22. *Let Y be a compact strictly analytic domain and $\Gamma = S(\mathcal{Y})$ a skeleton corresponding to a strictly semistable formal model $\mathcal{Y}$ of Y.*

i) For a finite subset S of type II and III points in Y, there is an admissible blowing up $\mathcal{Y}'$ of $\mathcal{Y}$ such that $\mathcal{Y}'$ defines a strictly semistable model of Y, the skeleton $S(\mathcal{Y}')$ of Y contains $S(\mathcal{Y})$ as a finite metric subgraph and $S \subset S(\mathcal{Y}')$.
ii) For a finite subset S of type II points in $S(\mathcal{Y})$, there is an admissible blowing up $\mathcal{Y}'$ of $\mathcal{Y}$ such that $\mathcal{Y}'$ defines a strictly semistable model of Y, the skeleton $S(\mathcal{Y}')$ of Y contains $S(\mathcal{Y})$ as a finite metric subgraph with $S_0(\mathcal{Y}) \cup S = S_0(\mathcal{Y}')$, i.e. $S(\mathcal{Y}')$ and $S(\mathcal{Y})$ are equal as sets.

PROOF. See [**Thu05**, Lemme 2.2.22]. □

In the following, we sketch the metric graph structure of a skeleton.

REMARK 2.3.23. Let Y be a connected compact strictly analytic domain and (Γ, Γ_0) a skeleton of Y. Then Γ is connected by Proposition 2.3.20. We can show that Γ has the structure of a metric graph in the following way: Let A be an open annulus in the decomposition (2.3.1) and let m be its modulus. Then $\partial A \subset \Gamma_0$ and by defining $S(A)$ as the subset of points in A not having an affinoid neighborhood isomorphic to a closed ball, we can identify $S(A)$ with an open interval I_A of length m connecting the boundary points ∂A [**BPR13**, Proposition 2.4]. This identification is up to isometries of $\mathbb{R}$ of the form $r \mapsto \pm r + a$ for some $a \in |K^\times|$.

Using the decomposition (2.3.1) from Proposition 2.3.14 and the fact that Γ is connected, we get that

$$\Gamma = \Gamma_0 \cup \bigcup_{j=1,\dots,m} S(A_j) = \bigcup_{j=1,\dots,m} \left(\partial A_j \cup S(A_j)\right).$$

Then Γ has the structure of a finite metric graph with vertices $\Gamma_0 = \bigcup_{j=1,\dots,m} \partial A_j$, open edges $S(A_j)$ and the shortest-path metric. Note that it is the incidence graph of the irreducible components of $\mathcal{Y}_s$ if $\mathcal{Y}$ is a strictly semistable formal model of Y such that $\Gamma = S(\mathcal{Y})$ and $\Gamma_0 = S_0(\mathcal{Y})$. One should mention that our graph Γ does not have any loop edges (i.e. ∂A consists of two elements for every open annulus in the decomposition (2.3.1)) as we required our semistable formal to be strictly semistable.

Definition 2.3.24. We call the shortest-path metric on a skeleton Γ of a connected compact strictly analytic domain from Remark 2.3.23 the *skeletal metric*.

With the help of the skeletal metric on every skeleton, one can define a metric ρ on $\mathbb{H}(X^{\mathrm{an}})$ (cf. [**BPR13**, §5]), which is again called the *skeletal metric*.

As the skeleton of a connected compact strictly analytic domain Y has the structure of a finite metric graph, we may study piecewise affine functions on skeleta.

Definition 2.3.25. Let Γ be a skeleton of a connected compact strictly analytic domain Y.

i) A *piecewise affine function* on Γ is a continuous function $F\colon \Gamma \to \mathbb{R}$ such that $F|_e \circ \alpha_e$ is piecewise affine for every edge e of Γ, where α_e is an identification of e with a real closed interval.
ii) We define the *outgoing slope* of a piecewise affine function F on Γ at a point $x \in \Gamma$ along a tangent direction v_e at x corresponding to an adjacent edge e as
$$d_{v_e}F(x) := \lim_{\varepsilon \to 0} (F|_e \circ \alpha_e)'(\alpha_e^{-1}(x) + \varepsilon).$$
One obtains a finite measure on X^{an} by putting
$$dd^c F := \sum_{x \in \Gamma} \left(\sum_{v_e} d_{v_e}F(x) \right) \delta_x,$$
where e is running over all edges in Γ at x. Since F is piecewise affine, $\sum_{v_e} d_{v_e}F(x) \neq 0$ for only finitely many points in Γ.
iii) Let S be a finite subset of Γ, then we say that a piecewise affine function F on Γ is *harmonic* on $\Gamma \backslash S$ if $dd^c F(x) = 0$ for all $x \in \Gamma \backslash S$. We define $H(\Gamma, S)$ as the vector space of harmonic functions on $\Gamma \backslash S$.

2.3.3. Simple open subsets. Baker, Payne and Rabinoff proved in [**BPR13**] that there is a basis of open neighborhoods of type II points that consists of preimages of special open subsets of skeleta. Open balls, open annuli and this kind of open subsets are called *simple*. We introduce them here following [**BPR13**] and study their boundaries.

Definition 2.3.26. Let Γ be a skeleton of a compact strictly analytic domain Y of X^{an}. Then a subset Ω of Γ is a *star-shaped open subset* of Γ if Ω is a simply-connected open subset of Γ and there is a point $x_0 \in \Omega$ such that $\Omega \backslash \{x_0\}$ is a disjoint union of open intervals. We call x_0 the *center* of Ω.

Theorem 2.3.27. *Let $x_0 \in X^{\mathrm{an}}$. There is a fundamental system of open neighborhoods $\{V_\alpha\}$ of x_0 of the following form:*

i) If x_0 is of type I or type IV, then the V_α are open balls.
ii) If x_0 is of type III, then the V_α are open annuli with x_0 contained in $S(V_\alpha)$ (cf. Remark 2.3.23).
iii) If x_0 is of type II, then $V_\alpha = \tau_\Gamma^{-1}(\Omega_\alpha)$ for a skeleton Γ of X^{an} and a star-shaped open subset Ω_α of Γ. Hence each $V_\alpha \backslash \{x_0\}$ is a disjoint union of open balls and open annuli.

Proof. See [**BPR13**, Corollary 4.27]. $\square$

REMARK 2.3.28. Theorem 2.3.27 implies directly that X^{an} is locally path-connected.

DEFINITION 2.3.29. An open subset of the described form in Theorem 2.3.27 is called *simple open*.

LEMMA 2.3.30. *Let Γ be a skeleton of X^{an} and Ω an open subset of Γ. Then the set $W := \tau_\Gamma^{-1}(\Omega)$ is open with $\partial W = \partial\Omega$, where $\partial\Omega$ is the limit boundary of Ω inside the compact set Γ.*

Note that Γ satisfies the first countability axiom as it is a finite graph.

PROOF. Clearly, W is an open subset of X^{an} as τ_Γ is continuous.

Since $\Omega \subset W$, we have that $\partial\Omega$ is contained in $\overline{W}$. If there is a point $x \in \partial\Omega \cap W$, then $x = \tau_\Gamma(x) \in \Omega$, where we use that τ_Γ is the identity on the skeleton. This contradicts $x \in \partial\Omega$, and so $\partial\Omega \subset \partial W$.

Suppose now that $x \in \partial W$. Then by the definition of W, we have $\tau_\Gamma(x) \in \Gamma\backslash\Omega$. First, we show that $\tau_\Gamma(x) = x$. Assume that x lies in $X^{\mathrm{an}}\backslash\Gamma$ and let B be the connected component of $X^{\mathrm{an}}\backslash\Gamma$ containing x. Then $\tau_\Gamma(B) = \tau_\Gamma(x) \in \Gamma\backslash\Omega$ by the definition of the retraction map. Hence B is an open neighborhood of x in $X^{\mathrm{an}}\backslash W$ contradicting $x \in \partial W$. Thus, $x = \tau_\Gamma(x) \in \Gamma\backslash\Omega$. Now, let (x_n) be a sequence in W converging to x. Then $(\tau_\Gamma(x_n))$ defines a sequence in Ω converging to $\tau_\Gamma(x) = x$ as τ_Γ is continuous. Consequently, $x \in \partial\Omega$ because we know that $x = \tau_\Gamma(x) \notin \Omega$. □

COROLLARY 2.3.31. *The boundary ∂W of a simple open subset W of X^{an} is finite.*

PROOF. If W is an open ball or an open annulus, it has at most two boundary points. Otherwise, this is a direct consequence of Lemma 2.3.30 and the fact that $\partial\Omega$ (for Ω an star-shaped open subset of a skeleton Γ of X^{an}) is finite. The latter is the case because Ω is a connected open subset of a finite graph. □

COROLLARY 2.3.32. *Let Γ be a skeleton of X^{an} and let Ω either be a star-shaped open subset of Γ or $\Omega = \{x_0\}$ for a type II point $x_0 \in \Gamma$. If the closure $\overline{\Omega}$ of Ω in Γ is a proper simply-connected subset such that all points in the corresponding boundary $\partial\Omega$ are of type II, then $Y := \tau_\Gamma^{-1}(\overline{\Omega})$ is a strictly affinoid domain with $\partial Y = \partial\Omega$.*

PROOF. First, we show that $\partial Y = \partial\Omega$. For every $x \in \Gamma$, the set $\tau_\Gamma^{-1}(x)$ is closed as τ_Γ is continuous. The set $\tau_\Gamma^{-1}(x)\backslash\{x\}$ is the disjoint union of infinitely many open balls having x as unique boundary point, and so it is open. Hence $\partial\tau_\Gamma^{-1}(x) = \{x\}$ because $\Gamma \neq \{x\}$. In particular, this shows $\partial Y = \partial\Omega$ if $\Omega = \{x_0\}$. We therefore consider $Y = \tau_\Gamma^{-1}(\overline{\Omega})$ for a star-shaped open subset Ω of Γ. We start with $\partial\Omega \subset \partial Y$. Since Ω is a subset of the closed set Y, we obviously have $\partial\Omega \subset Y$. Assume there is a point $x \in \partial\Omega \cap Y^\circ$ for the relative interior Y° of Y. Note that $\partial\Omega = \partial\overline{\Omega}$ since $\overline{\Omega}$ is a proper simply-connected star-shaped open subset of a finite graph. Then there is a sequence of points (x_n) in $\Gamma\backslash\overline{\Omega}$ converging to x. Due to $Y = \tau_\Gamma^{-1}(\overline{\Omega})$, the sequence is contained in $X^{\mathrm{an}}\backslash Y$ contradicting $x \in Y^\circ$. Hence $\partial\Omega \subset \partial Y$.

To see $\partial Y \subset \partial\Omega$, note that $\partial\Omega$ is a finite set of type II points and $\partial\tau_\Gamma^{-1}(x) = \{x\}$ for each $x \in \partial\Omega$. It follows from Lemma 2.3.30 that

$$\partial Y = \partial(\tau_\Gamma^{-1}(\partial\Omega \cup \Omega)) \subset \partial(\tau_\Gamma^{-1}(\partial\Omega)) \cup \partial(\tau_\Gamma^{-1}(\Omega)) = \partial\Omega.$$

Consequently, $\partial Y = \partial\Omega$.

It remains to prove that Y is a strictly affinoid domain. Note that $Y \neq X^{\mathrm{an}}$ as $\overline{\Omega} \subsetneq \Gamma$. By Proposition 2.3.22, we may assume that x_0 respectively the endpoints of Ω, which are all of type II, are vertices of Γ. Deleting all open annuli in the decomposition of

$X^{\mathrm{an}} \backslash \Gamma_0$ disjoint from Ω leads to a strictly affinoid domain (cf. [**BPR13**, Lemma 4.12]). As Y is a connected component of this strictly affinoid domain, it is one itself. $\square$

DEFINITION 2.3.33. If Y is a strictly affinoid domain of the form described in Corollary 2.3.32, then we call Y a *strictly simple domain*.

CHAPTER 3

Potential theory on non-archimedean curves via skeleta

In this chapter, we always consider a smooth proper curve X over a non-trivially valued K.

3.1. Subharmonic functions according to Thuillier

Thuillier developed in his thesis [**Thu05**] a potential theory on non-archimedean curves, which is based on skeleta. In this section, we introduce his subharmonic functions and his class of smooth (here called *lisse*) functions with their corresponding Laplacian. We present their typical and (for us most) important properties from [**Thu05**] referring mostly to [**Thu05**] for their proofs. Note that there is also an independent very well-developed potential theory on $\mathbb{P}^{1,\mathrm{an}}$ by Baker and Rumely [**BR10**]. In their book, Baker and Rumely proved some statements that are not written down in Thuillier's thesis. Matt Baker suggested to the author some of these results for generalization and we give proofs of them in this section and also in Section 3.2. Additionally, we show that Thuillier's definition of subharmonic functions is indeed a generalization of the one by Baker and Rumely.

3.1.1. Definition and basic properties. We start by defining harmonic functions and using them to introduce subharmonic functions as in the complex case. Next to presenting and citing some direct properties from [**Thu05**], we compare in this subsection Thuillier's (sub)harmonic functions to (sub)harmonic functions on $\mathbb{P}^{1,\mathrm{an}}$ defined by Baker and Rumely in [**BR10**].

Definition 3.1.1. Let Y be a strictly affinoid domain in X^{an}. Then there exists a strictly semistable formal model $\mathcal{Y}$ of Y (see Theorem 2.3.12), and we define the *harmonic functions on* Y as

$$H(Y) := \tau_{\mathcal{Y}}^{*}(H(S(\mathcal{Y}), \partial Y)).$$

Note that the definition is independent of $\mathcal{Y}$ [**Thu05**, Proposition 2.3.3].

Definition 3.1.2. A continuous function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} is called *harmonic on* W if for every strictly affinoid domain Y in W we have $f|_Y \in H(Y)$. Harmonic functions form a subsheaf of the sheaf $\mathcal{C}^0$ of real-valued continuous function on X^{an} (cf. [**Thu05**, Corollaire 2.3.15]), which we denote by $\mathcal{H}_X$. Note that we have $H(Y) = \mathcal{H}_X(Y \backslash \partial Y) \cap \mathcal{C}^0(Y)$ for every strictly affinoid domain Y (see [**Thu05**, Corollaire 2.3.14]).

Lemma 3.1.3. *Let $X = \mathbb{P}^1$ and let W be an open subset of $\mathbb{P}^{1,\mathrm{an}}$. Then this notion of harmonic functions on W (following Thuillier) coincides with the definition of a harmonic function by Baker and Rumely in* [**BR10**, §7].

Proof. Consider a function $h\colon W \to \mathbb{R}$. At first, assume that h is harmonic in the sense of [**BR10**] and let Y be a strictly affinoid domain in W. Then there is a strictly semistable model $\mathcal{Y}$ of Y and its corresponding skeleton $S(\mathcal{Y})$. Note that $S(\mathcal{Y})$ defines a finite subgraph in the sense of [**BR10**] and the retraction maps are defined in the same way. By [**BR10**, Proposition 7.12], we have $h = H \circ \tau_{\mathcal{Y}}$ on Y with $H := h|_{S(\mathcal{Y})}$. As h is harmonic on $\tau_{\mathcal{Y}}^{-1}(S(\mathcal{Y})^\circ)$ as defined in [**BR10**], [**BR10**, Proposition 3.11 & Example 5.18] imply that H is a piecewise affine function on $S(\mathcal{Y})$, whose outgoing slopes at every point in $S(\mathcal{Y})^\circ = S(\mathcal{Y})\backslash\partial Y$ sum up to zero. Hence $dd^cH = 0$ on $S(\mathcal{Y})\backslash\partial Y$, and so $h \in H(Y)$. This proves that h is harmonic on W.

For the other direction, we assume that h is harmonic as defined in Definition 3.1.2, and show that h is also harmonic in [**BR10**]. We have to find for every point $x_0 \in W$ an open connected neighborhood V in W such that h is strongly harmonic on V in the sense of [**BR10**]. Let Y be a strictly simple domain Y in W with $x_0 \in Y^\circ$ (cf. Theorem 2.3.27 and Corollary 2.3.32) and let V be the connected component of Y° containing x_0. As Y is a strictly affinoid domain in W, we have $h \in H(Y)$ because we assumed h to be harmonic on W. Thus $h = H \circ \tau_{\mathcal{Y}}$ for some strictly semistable model $\mathcal{Y}$ of Y and a piecewise affine function H on $S(\mathcal{Y})$ such that the outgoing slopes at every point in $S(\mathcal{Y})^\circ = S(\mathcal{Y})\backslash\partial Y$ sum up to zero. By [**BR10**, Example 5.18 & Definition 7.1], h is strongly harmonic on V in the sense of [**BR10**]. This shows that h is also harmonic in [**BR10**]. □

Before introducing subharmonic functions with the help of harmonic functions we recall upper respectively lower semi-continuity.

REMARK 3.1.4. A function $f\colon W \to [-\infty, \infty)$ on an open subset W of a topological space is *upper semi-continuous* (shortly *usc*) in a point x_0 of W if

$$\limsup_{x\to x_0} f(x) \leq f(x_0),$$

where the limit superior in this context is defined as

$$\limsup_{x\to x_0} f(x) := \sup_{U\in\mathcal{U}(x_0)} \inf_{x\in U\backslash\{x_0\}} f(x),$$

where $\mathcal{U}(x_0)$ is any basis of open neighborhoods of x_0. We say that f is *upper semi-continuous* (shortly *usc*) *on* W if it is usc in all points of W.

Analogously, a function $f\colon W \to (-\infty, \infty]$ on an open subset W of a topological space is *lower semi-continuous* (shortly *lsc*) in a point x_0 of W if

$$\liminf_{x\to x_0} f(x) \geq f(x_0),$$

where the limit inferior in this context is defined as

$$\liminf_{x\to x_0} f(x) := \inf_{U\in\mathcal{U}(x_0)} \sup_{x\in U\backslash\{x_0\}} f(x),$$

where $\mathcal{U}(x_0)$ is any basis of open neighborhoods of x_0. We say that f is *lower semi-continuous* (shortly *lsc*) *on* W if it is lsc in all points of W.

DEFINITION 3.1.5. Let W be an open subset of X^{an}. Then $f\colon W \to [-\infty, \infty)$ is called *subharmonic* if f is upper semi-continuous, $f \not\equiv -\infty$ on every connected component of W and for every strictly affinoid domain Y in W and every harmonic function h on Y, we have

$$(f|_{\partial Y} \leq h|_{\partial Y}) \Rightarrow (f|_Y \leq h).$$

REMARK 3.1.6. A function $h\colon W \to \mathbb{R}$ on an open subset W of X^{an} is harmonic if and only if h and $-h$ are both subharmonic.

We follow [**Thu05**, §3] and give some direct properties of subharmonic functions from there.

LEMMA 3.1.7. *Let W be an open subset of X^{an} and let $f, g\colon W \to [-\infty, \infty)$ be two subharmonic functions on W. Then*

i) $\alpha f + \beta g$ *is subharmonic on* W *for all* $\alpha, \beta \in \mathbb{R}_{\geq 0}$, *and*
ii) $\max(f, g)$ *is subharmonic on* W.

PROOF. At first note that for all $\alpha, \beta \in \mathbb{R}_{\geq 0}$, the functions $\alpha f + \beta g$ and $\max(f, g)$ are still usc on W. Moreover, they are not identically $-\infty$ on a connected component of W if f and g are not. Thus let Y be a strictly affinoid domain in W and let $h \in H(Y)$. We start with ii). If $\max(f, g) \leq h$ on ∂Y, we clearly have $f \leq h$ and $g \leq h$ on ∂Y, and so also on Y as f and g are subharmonic. Consequently, $\max(f, g) \leq h$ on Y as well.

For i), we assume $\alpha f + \beta g \leq h$ on ∂Y. By the Dirichlet problem for strictly affinoid domains [**Thu05**, Corollaire 3.1.21], there is $h_g \in H(Y)$ such that we have $h_g(y) = \beta g(y)$ whenever $y \in \partial Y$ with $g(y) \neq -\infty$ and $h_g(y) + \alpha f(y) \leq h(y)$ whenever $y \in \partial Y$ with $g(y) = -\infty$. By construction, $g \leq \beta^{-1} h_g$ on ∂Y, and so $\beta g \leq h_g$ on Y as g is subharmonic and $\beta^{-1} h_g \in H(Y)$. Moreover, for every $y \in \partial Y$ with $g(y) \neq -\infty$ we have

$$\alpha f(y) = \alpha f(y) + \beta g(y) - \beta g(y) \leq h(y) - \beta g(y) = h(y) - h_g(y).$$

As $\alpha f(y) \leq h(y) - h_g(y)$ whenever $y \in \partial Y$ with $g(y) = -\infty$, we have $\alpha f \leq h - h_g$ on ∂Y. Due to $\alpha^{-1}(h - h_g) \in H(Y)$, we get $\alpha f \leq h - h_g$ on Y since f is subharmonic. Because of $\beta g \leq h_g$ on Y, this leads to

$$\alpha f + \beta g \leq h - h_g + h_g = h$$

on Y. □

We have the following Maximum Principle for subharmonic functions.

PROPOSITION 3.1.8. *Let W be an open subset of X^{an}. Then a subharmonic function $f\colon W \to [-\infty, \infty)$ admits a local maximum in a point x_0 of W if and only if it is locally constant at x_0.*

PROOF. See [**Thu05**, Proposition 3.1.11]. □

PROPOSITION 3.1.9. *Let W be an open subset of X^{an} and let $f\colon W \to [-\infty, \infty)$ be an upper semi-continuous function such that $f \not\equiv -\infty$ on every connected component of W. Then the following are equivalent:*

i) f *is subharmonic on* W.
ii) For every open subset W' of W and every harmonic function h on W', the function $f|_{W'} - h$ satisfies the Maximum Principle from Proposition 3.1.8.

PROOF. See [**Thu05**, Proposition 3.1.11]. □

With the help of this characterization, we can show that Thuillier's definition of subharmonic functions on X^{an} is an extension of those by Baker and Rumely in [**BR10**]. Note that we already know that the notions of harmonic functions agree on open subsets of $\mathbb{P}^{1,\mathrm{an}}$ (see for example Lemma 3.1.3).

PROPOSITION 3.1.10. *Let $X = \mathbb{P}^1$ and let W be an open subset of $\mathbb{P}^{1,\mathrm{an}}$. Then our notion of subharmonic functions on W (following Thuillier) coincides with the definition by Baker and Rumely in* [**BR10**, §8].

PROOF. At first, let $f\colon W \to [-\infty, \infty)$ be a subharmonic function in the sense of Thuillier. Then by [**BR10**, Theorem 8.19], f is subharmonic in [**BR10**] if

(a) for every open subset V of W that is a connected component of $\mathbb{P}^{1,\mathrm{an}}\backslash\{x_0\}$ for $x_0 \in I(\mathbb{P}^{1,\mathrm{an}})$, or V is of the form $V = \tau_\Gamma^{-1}(\Sigma^0)$ for a skeleton Γ of $\mathbb{P}^{1,\mathrm{an}}$ and a connected finite subgraph Σ of Γ, and

(b) for every continuous function h on $\overline{V} = \partial\Sigma \cup V$ (see Proposition 2.3.30) with $h|_V$ harmonic,

we have $f \leq h$ on ∂V implies $f \leq h$ on $\overline{V}$. Note that h is then also harmonic on V in the sense of Thuillier (see Lemma 3.1.3), but $\overline{V}$ is not necessarily a strictly affinoid domain. Since $f - h$ is upper semi-continuous on the connected compact subset $\overline{V}$, it achieves a maximum on $\overline{V}$. By Proposition 3.1.9, $f - h$ satisfies the Maximum Principle on V, and so $f - h$ achieves its maximum on ∂V. Thus $f \leq h$ on $\overline{V}$ if $f \leq h$ on ∂V. The considered function f is therefore subharmonic in the sense of [**BR10**].

Now, let $f \colon W \to [-\infty, \infty)$ be a subharmonic function in the sense of [**BR10**]. We have to consider a strictly affinoid domain Y in W and $h \in H(Y)$ with $f \leq h$ on ∂Y. We may assume Y to be connected. Then there is a strictly semistable model $\mathcal{Y}$ of Y such that $Y = \tau_{\mathcal{Y}}^{-1}(S(\mathcal{Y}))$ for the skeleton $S(\mathcal{Y})$ corresponding to $\mathcal{Y}$. Note that the finite metric graph $S(\mathcal{Y})$ defines also a skeleton of $\mathbb{P}^{1,\mathrm{an}}$ (cf. Remark 2.3.19). Then $h \in H(Y)$ is continuous on Y and harmonic on $Y^\circ = \tau_{\mathcal{Y}}^{-1}(S(\mathcal{Y})^\circ) \cup \left(\tau_{\mathcal{Y}}^{-1}(\partial Y)\backslash\partial Y\right)$ in the sense of [**BR10**] (cf. Definition 3.1.2 and Lemma 3.1.3). The set $\tau_{\mathcal{Y}}^{-1}(\partial Y)\backslash\partial Y$ is the disjoint union of infinitely many open balls, i.e. connected components of $\mathbb{P}^{1,\mathrm{an}}\backslash\{y\}$ for $y \in \partial Y \subset I(\mathbb{P}^{1,\mathrm{an}})$. Hence the described characterization of subharmonic functions in [**BR10**] from above, tells us that $f \leq h$ on Y. Hence f is subharmonic in the sense of Thuillier. □

Moreover, it can be deduced from the local characterization in Proposition 3.1.9 that subharmonic functions form a sheaf on X^{an}.

Proposition 3.1.11. *The subharmonic functions form a sheaf on X^{an}.*

Proof. See [**Thu05**, Corollaire 3.1.13]. □

Corollary 3.1.12. *Let $f \colon W \to [-\infty, \infty)$ be a subharmonic function on an open subset W of X^{an}. Then f is finitely valued on $I(W)$.*

Proof. Assume there is a point $x_0 \in I(W)$ with $f(x_0) = -\infty$. Let Y be a strictly affinoid domain in W with x_0 in its interior. By Theorem 2.3.12 and Proposition 2.3.22, there is a strictly semistable model $\mathcal{Y}$ such that x_0 is contained in the skeleton $S(\mathcal{Y})$. Let B_0 be a connected component of $\tau_{\mathcal{Y}}^{-1}(x_0)\backslash\{x_0\}$, which is an open ball contained in W with unique boundary point x_0. As f is upper semi-continuous on the compact set $\overline{B_0}$, the function f attains a maximum on $\overline{B_0}$. This maximum has to be attained in x_0 by Proposition 3.1.8 as $\overline{B_0}$ is connected. Hence f is identically $-\infty$ on $\overline{B_0}$. By Proposition 3.1.11, f is subharmonic on the open ball B_0 contradicting $f \equiv -\infty$ on B_0. □

Proposition 3.1.13. *Let X, X' be smooth proper curves over K and let $\varphi \colon W' \to W$ be a morphism of K-analytic spaces for open subsets $W \subset X^{\mathrm{an}}$ and $W' \subset (X')^{\mathrm{an}}$. If $f \colon W \to \mathbb{R}$ is a subharmonic function on W, then $\varphi^* f$ is a subharmonic function on $\varphi^{-1}(W)$.*

Proof. See [**Thu05**, Proposition 3.1.14]. □

There are further stability properties about subharmonic functions that are not written down in [**Thu05**], but were proven for $X = \mathbb{P}^1$ in [**BR10**]. As they are also interesting in the general case we give proofs of them. In the subsequent proposition, we see that

[**BR10**, Lemma 8.28] also holds for an arbitrary smooth proper curve X. Later on, we will see more statements of this kind.

PROPOSITION 3.1.14. *Let $f\colon W \to [-\infty, \infty)$ be a subharmonic function on an open subset W of X^{an} and let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a convex and non-decreasing function, then $\varphi \circ f$ is subharmonic on W (where $\varphi(-\infty) := \lim_{t\to-\infty} \varphi(t)$).*

PROOF. The assertion can be directly deduced from the definition of subharmonic functions using some general ingredients for convex functions. The proof is therefore analogous to the one for $X = \mathbb{P}^1$ in [**BR10**, Lemma 8.28]. It is lined out for the convenience of the reader. Since φ is required to be convex and non-decreasing, the composition of φ with the upper semi-continuous function f is still upper semi-continuous. Moreover, we have the following description of φ

$$\varphi(t) = \sup_{(a,b)\in A} a \cdot t + b$$

for the set $A := \{(a, b) \in \mathbb{R}_{\geq 0} \times \mathbb{R} | a \cdot t + b \leq \varphi(t) \text{ for all } t \in \mathbb{R}\}$. Thus

$$(\varphi \circ f)(x) = \sup_{(a,b)\in A} a \cdot f(x) + b \tag{3.1.1}$$

on W. Let Y be a strictly affinoid domain in W and $h \in H(Y)$ such that $\varphi \circ f \leq h$ on ∂Y. Due to (3.1.1), we have $a \cdot f + b \leq h$ on ∂Y for all $(a, b) \in A$. The functions $a \cdot f + b$ are subharmonic by Lemma 3.1.7 since f is, and so the harmonic function h dominates $a \cdot f + b$ on all of Y for every $(a, b) \in A$. Consequently, $\varphi \circ f \leq h$ on Y by (3.1.1). Thus $\varphi \circ f$ is subharmonic on W. □

3.1.2. Subharmonic functions and the Laplacian. In complex potential theory, there is a further very important way of characterizing subharmonic functions via smooth functions and a Laplacian operator. In the following, we introduce Thuillier's smooth functions that are defined with the help of skeleta (called *lisse* here) and their corresponding Laplacian operator dd^c. Afterwards, we will see that subharmonic functions can be characterized as in the complex case, as currents with positive Laplacian.

It should be noted that we use Thuillier's characterization in [**Thu05**, Proposition 3.2.4] for the definition of lisse functions.

DEFINITION 3.1.15. Let $W \subset X^{\mathrm{an}}$ be open. A continuous function $f\colon W \to \mathbb{R}$ is called *lisse* if for every strictly analytic domain $Y \subset W$ there exists a strictly semistable formal model $\mathcal{Y}$ of Y such that

$$f|_Y = F \circ \tau_{\mathcal{Y}}$$

for a piecewise affine function F on $S(\mathcal{Y})$. We denote by $A^0(W)$ the vector space of lisse functions on W, and by $A^0_c(W)$ the subspace of lisse functions on W with compact support in W.

DEFINITION 3.1.16. We write $A^1(W)$ for the set of real measures on W with discrete support in $I(W)$, and use $A^1_c(W)$ for those with compact support in W. Then for every lisse function $f \in A^0(W)$, there is a unique real measure $dd^c f$ in $A^1(W)$ such that

$$dd^c f = dd^c F$$

on Y° whenever $f = F \circ \tau_{\mathcal{Y}}$ on a compact strictly analytic domain Y with formal model $\mathcal{Y}$ (see [**Thu05**, Théorème 3.2.10]). We call this linear operator $dd^c\colon A^0(W) \to A^1(W)$ the *Laplacian*. Note that $A^0_c(W)$ is mapped to $A^1_c(W)$ under dd^c [**Thu05**, Corollaire 3.2.11].

The Laplacian of a lisse functions tells us whether it is (sub)harmonic or not.

PROPOSITION 3.1.17. *A function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} is harmonic on W if and only if $f \in A^0(W)$ and $dd^c f = 0$.*

PROOF. See [**Thu05**, Corollaire 3.2.11]. □

PROPOSITION 3.1.18. *A lisse function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} is subharmonic if and only if $dd^c f \geq 0$.*

PROOF. See [**Thu05**, Proposition 3.4.4]. □

As in the complex case, there is a regularization theorem with respect to this class of smooth functions.

PROPOSITION 3.1.19. *Let W be an open subset of X^{an} and let f be a subharmonic function on W. For every relatively compact open subset W' of W, there is a decreasing net $\langle f_\alpha \rangle$ of lisse subharmonic functions on W' converging pointwise to f.*

PROOF. See [**Thu05**, Théorème 3.4.2]. □

We continue with the existence of some special lisse functions, which will be important in later sections.

PROPOSITION 3.1.20. *For any two points $x, y \in I(X^{\mathrm{an}})$ there is a unique lisse function $g_{x,y} \in A^0(X^{\mathrm{an}})$ such that*

i) $dd^c g_{x,y} = \delta_x - \delta_y$, and
ii) $g_{x,y}(x) = 0$.

PROOF. See [**Thu05**, Proposition 3.3.7]. □

PROPOSITION 3.1.21. *Let Y be a connected affinoid domain in X^{an} and $x \in I(Y)\backslash\partial Y$. Then there exists a unique lisse function $g_x^Y \in A_c^0(X^{\mathrm{an}})$ such that the following are true:*

i) g_x^Y is strictly positive on the connected component V of $Y\backslash\partial Y$ containing x and it is equal to zero on $X^{\mathrm{an}}\backslash V$.
ii) $dd^c g_x^Y$ is supported on $\partial Y \cup \{x\}$ with $dd^c g_x^Y = -\delta_x$ in a neighborhood of x.
iii) For every harmonic function h on Y we have

$$h(x) = \int_{\partial Y} h \; dd^c g_x^Y. \tag{3.1.2}$$

PROOF. See [**Thu05**, Proposition 3.3.7 & Corollaire 3.3.9]. □

REMARK 3.1.22. Let Y be a connected affinoid domain in X^{an}, let $x \in I(Y)\backslash\partial Y$, and let g_x^Y be the unique lisse function on X^{an} from Proposition 3.1.21. As g_x^Y is strictly-positive on the connected component V of $Y\backslash\partial Y$ containing x and zero outside of it, we have $\int_{\{y_i\}} dd^c g_x^Y > 0$ for every boundary point $y_i \in \partial Y$. Considering the constant harmonic function $h \equiv 1$ on Y, Proposition 3.1.21 iii) implies directly that the sum of all outgoing slopes at all boundary points (there are only finitely many) have to sum up to 1,

$$1 = h(x) = \sum_{y_i \in \partial Y} \int_{\{y_i\}} dd^c g_x^Y.$$

Moreover, if f is a subharmonic function on an open neighborhood of Y, then f is finitely valued on ∂Y (see Corollary 3.1.12). Hence there is a continuous function h

on Y with $h \equiv f$ on ∂Y that is harmonic on Y° (by the Dirichlet problem [**Thu05**, Corollaire 3.1.21]). Proposition 3.1.21 iii) directly implies

$$f(x) \leq \left(\sum_{y_i \in \partial Y} \int_{\{y_i\}} dd^c g_x^Y \right) f(y_i).$$

Corollary 3.1.23. *Let $W \subset X^{\mathrm{an}}$ be open. A continuous function $f\colon W \to \mathbb{R}$ is subharmonic if and only if for every connected strictly affinoid domain $Y \subset W$ and every point $x \in Y \backslash \partial Y$ of type II or III, we have*

$$\int_W f \; dd^c g_x^Y \geq 0.$$

Proof. Consider a strictly affinoid domain Y in W and a harmonic function h on Y with $f \leq h$ on ∂Y. As explained in [**Thu05**, Remarque 3.1.10], we may assume $f = h$ on ∂Y, and without loss of generality Y is connected. For every $x \in Y \backslash \partial Y$ of type II or III, Equation (3.1.2) implies that

$$\int_W f \; dd^c g_x^Y = \int_{\partial Y} f \; dd^c g_x^Y - f(x) = \int_{\partial Y} h \; dd^c g_x^Y - f(x) = h(x) - f(x).$$

Hence we have $\int_W f \; dd^c g_x^Y \geq 0$ for every type II or III point in Y if and only if $f \leq h$ on $I(Y)$. Since f and h are continuous and the type II and III points are dense, the last is equivalent to $0 \leq h - f$ on Y. □

With the help of these special lisse functions, we can prove a similar stability statement as in [**BR10**, Proposition 8.27] for $X = \mathbb{P}^1$. Note that we require f to be upper semi-continuous in contrast to [**BR10**].

Proposition 3.1.24. *Let μ be a positive measure on a measure space T, and let W be a connected open subset of X^{an}. Suppose that $F\colon W \times T \to [-\infty, \infty)$ is a measurable function such that*

i) for each $t \in T$, the function $F_t := F(\cdot, t)$ is subharmonic on W, and
ii) the function

$$f := \int_T F(\cdot, t) \; d\mu(t)$$

is upper semi-continuous on W.

Then f is either subharmonic on W or $f \equiv -\infty$ on W.

Proof. Assume that $f \not\equiv -\infty$, and let Y be a strictly affinoid domain in W with boundary $\partial Y = \{y_1, \ldots, y_n\}$ and let h be a harmonic function on Y with $f(y_i) \leq h(y_i)$ for every $i \in \{1, \ldots, n\}$. Consider at first a point $x \in I(Y) \backslash \partial Y$ and the corresponding function $g_x^Y \in A_c^0(W)$ from Proposition 3.1.21. Set $\lambda_i(x) := \int_{\{y_i\}} dd^c g_x^Y$. Then Remark 3.1.22 tells us $\lambda_i(x) \in [0, 1]$, $\sum_{i=1,\ldots,n} \lambda_i(x) = 1$ and

$$F(x,t) = F_t(x) \leq \sum_{i=1}^n \lambda_i(x) F_t(y_i) = \sum_{i=1}^n \lambda_i(x) F(y_i, t)$$

for every $t \in T$ because each F_t is subharmonic on W. Then

$$f(x) = \int_T F(x,t) \; d\mu(t) \leq \sum_{i=1}^n \lambda_i(x) \int_T F(y_i, t) \; d\mu(t) = \sum_{i=1}^n \lambda_i(x) f(y_i).$$

As all $0 \leq \lambda_i(x)$ and $f \leq h$ on ∂Y, we get

$$f(x) \leq \sum_{i=1}^{n} \lambda_i(x) f(y_i) \leq \sum_{i=1}^{n} \lambda_i(x) h(y_i) = h(x),$$

where the last equation follows by Proposition 3.1.21 iii).

If $x \in Y \backslash \partial Y$ is a point of type I or IV, then x has an open ball V as a neighborhood in $Y \backslash \partial Y$ with a unique boundary point y of type II. By the Maximum Principle (Proposition 3.1.8), we have $h(x) = h(y)$. Moreover, we know that $F_t(x) \leq F_t(y)$ for every $t \in T$ since $F_t(x)$ is also subharmonic on the open ball V. Hence $f(x) \leq f(y)$. On $I(Y)$ we have already seen that $f \leq h$. As $y \in I(Y)$, we obtain

$$f(x) \leq f(y) \leq h(y) = h(x).$$

Thus $f \leq h$ on all of Y. □

REMARK 3.1.25. Let W be an open subset of X^{an} and S a finite subset of $I(W)$. Then we set

$$\begin{aligned} A_c^0(W)_S &:= A_c^0(W) \cap \mathcal{H}_X(W \backslash S) \\ A_c^1(W)_S &:= \{\mu \in A^1(W) \mid \operatorname{supp}(\mu) \subset S\} = \bigoplus_{x \in S} \mathbb{R}\delta_x. \end{aligned}$$

These finite dimensional vector spaces are equipped with the canonical topology. Since $A_c^0(W)$ (resp. $A_c^1(W)$) can be written as the direct limit over these vector spaces $A_c^0(W)_S$ (resp. $A_c^1(W)_S$), we can endow $A_c^0(W)$ (resp. $A_c^1(W)$) with the induced topology (see [**Thu05**, §3.2.2]).

DEFINITION 3.1.26. Let W be an open subset of X^{an}. We denote by $D^0(W)$ (resp. $D^1(W)$) the topological dual of $A_c^1(W)$ (resp. $A_c^0(W)$).

Note that they are as sets equal to the algebraic duals.

PROPOSITION 3.1.27. *The map*

$$D^0(W) \to \operatorname{Hom}(I(W), \mathbb{R}),\ T \mapsto (x \mapsto \langle T, \delta_x \rangle)$$

is an isomorphism of topological vector spaces, where $\operatorname{Hom}(I(W), \mathbb{R})$ *is a topological vector space with respect to pointwise convergence.*

PROOF. See [**Thu05**, Proposition 3.3.3]. □

In the following, we always use this identification.

REMARK 3.1.28. The Laplacian $dd^c \colon A_c^0(W) \to A_c^1(W)$ on an open subset $W \subset X^{\text{an}}$ leads naturally by duality to an $\mathbb{R}$-linear operator

$$dd^c \colon D^0(W) \to D^1(W),\ T \mapsto (g \mapsto \langle dd^c T, g \rangle := \langle T, dd^c g \rangle)$$

such that

$$\begin{tikzcd} A^0(W) \arrow[r, "dd^c"] \arrow[d] & A^1(W) \arrow[d] \\ D^0(W) \arrow[r, "dd^c"] & D^1(W) \end{tikzcd}$$

commutes. It follows from the definition that this map is continuous (see [**Thu05**, Proposition 3.3.4]).

We close this subsection, by showing that non-lisse subharmonic functions also can be characterized by the Laplacian dd^c.

DEFINITION 3.1.29. We say that a current $T \in D^1(W)$ on an open subset W of X^{an} is *positive* if $\langle T, g\rangle \geq 0$ for every non-negative lisse function $g \in A_c^0(W)$.

PROPOSITION 3.1.30. *An upper semi-continuous function $f\colon W \to [-\infty,\infty)$ on an open subset W of X^{an} is subharmonic if and only if $f \in D^0(W)$ and $dd^c f \geq 0$.*

PROOF. Thuillier showed the assertion in [**Thu05**, Théorème 3.4.12], but we outline parts of the proof here as well. First, assume that f is subharmonic. By Corollary 3.1.12, the subharmonic function f is finitely valued on $I(W)$, and so f defines a current in $D^0(W)$ by Proposition 3.1.27. To show that $dd^c f \geq 0$, we consider a non-negative $g \in A_c^0(W)$. Since X^{an} is a locally compact Hausdorff space, we can find for the set $\mathrm{supp}(dd^c g) \subset \mathrm{supp}(g)$ a relatively compact open neighborhood W' in W. Then the regularization theorem in Proposition 3.1.19 tells us that f is the pointwise limit of a decreasing net $\langle f_\alpha\rangle$ of lisse subharmonic functions on W'. We get

$$\langle dd^c f, g\rangle = \int_W f\ dd^c g = \int_{W'} f\ dd^c g = \lim_\alpha \int_{W'} f_\alpha\ dd^c g = \lim_\alpha \int_{W'} g\ dd^c f_\alpha,$$

where the last identity follows by [**Thu05**, Proposition 3.2.12]. Each f_α is lisse and subharmonic. Hence each $dd^c f_\alpha$ is a positive measures by Proposition 3.1.18. Finally,

$$\langle dd^c f, g\rangle = \lim_\alpha \int_{W'} g\ dd^c f_\alpha \geq 0$$

since $g \geq 0$.

For the other direction, we assume $dd^c f$ to be a positive current. Then in particular, $\langle dd^c f, g_x^Y\rangle \geq 0$ for every connected affinoid domain Y in W and every $x \in I(Y)\backslash \partial Y$, where g_x^Y is the lisse function from Proposition 3.1.21. It is shown in [**Thu05**, Lemma 3.4.1] that this gives the claim that f is subharmonic. □

COROLLARY 3.1.31. *A continuous function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} is harmonic if and only if $f \in D^0(W)$ with $dd^c f = 0$.*

PROOF. Applying Proposition 3.1.30 for f and $-f$ gives the claim (cf. Remark 3.1.6). □

REMARK 3.1.32. Let $f\colon W \to [-\infty,\infty)$ be a subharmonic function on an open subset W of X^{an}. Let $[x_0, y_0]$ be an interval (i.e. a segment of an edge) in a skeleton Γ of X^{an} such that $\tau_\Gamma^{-1}((x_0,y_0)) \subset W$. We will deduce from Proposition 3.1.30 that f is convex restricted to the relative interior of $I = [x_0, y_0]$.

Let $x, y \in I^\circ = (x_0, y_0)$, let $\lambda \in [0,1]$, and let z be the point in I° that corresponds to $(1-\lambda)x + \lambda y$ via some identification of I with an interval in $\mathbb{R}$. There is a unique (continuous) piecewise affine function Ψ on Ω that is zero outside of $[x,y]$ and $dd^c\Psi$ is supported on $\{x,y,z\}$ with $\int_{\{x\}} dd^c\Psi = (1-\lambda)$ and $\int_{\{y\}} dd^c\Psi = \lambda$. Then

$$\int_{\{z\}} dd^c\Psi = -\lambda - (1-\lambda) = -1,$$

and $\psi := \Psi \circ \tau_\Gamma$ defines a non-negative function in $A_c^0(W)$ with

$$dd^c\psi = (1-\lambda)\delta_x + \lambda\delta_y - \delta_z.$$

Note that we have $\mathrm{supp}(\psi) \subset \tau_\Gamma^{-1}((x_0, y_0)) \subset W$. As f is subharmonic on W, we get by Proposition 3.1.30

$$0 \leq \int f \; dd^c\psi = (1-\lambda)f(x) + \lambda f(y) - f(z).$$

Thus it is convex on I°.

3.1.3. The Domination Theorem. In the following, we prove a generalization of [**BR10**, Corollary 8.35], which gives conditions regarding the Laplacian when $f \leq g$ for two subharmonic functions f and g.

LEMMA 3.1.33. *Let W be an open subset of X^{an} and let f be a subharmonic function on W. Then for every path $[z, y]$ in W that is contained in an open ball or in an open annulus of X^{an}, we have*

$$f(y) = \lim_{x \to y, x \in [z,y)} f(x).$$

PROOF. Every open ball or open annulus can be identified with an open ball respectively an open annulus in $\mathbb{P}^{1,\mathrm{an}}$. Hence Proposition 3.1.10 and [**BR10**, Proposition 8.11] imply the claim. □

LEMMA 3.1.34. *Let W be an open subset of X^{an}, let Y be an affinoid domain in W and let $f, g \in D^0(W)$ be two currents on W with $\langle dd^c f, \varphi \rangle \geq \langle dd^c g, \varphi \rangle$ for every non-negative $\varphi \in A_c^0(W)$ with $\mathrm{supp}(\varphi) \subset Y$. If f and g are subharmonic on the relative interior $V := Y^\circ$ of Y, then*

$$f \leq \max_{\partial Y}(f - g) + g$$

on Y.

PROOF. First, we explain why the assertion $f \leq \max_{\partial Y}(f-g) + g$ is well-defined. Since $f, g \in D^0(W)$ and $\partial Y \subset I(W)$, both are finitely valued on ∂Y (see Proposition 3.1.27), and so $\max_{\partial Y}(f - g)$ is well-defined. Moreover, the assertion holds trivially on ∂Y because

$$f(y) = f(y) - g(y) + g(y) \leq \max_{\partial Y}(f-g) + g(y)$$

for every $y \in \partial Y$.

Next we deal with $y \in I(Y)\backslash \partial Y$. We may assume Y to be connected and can consider the non-negative lisse function $g_y^Y \in A_c^0(X^{\mathrm{an}})$ from Proposition 3.1.21. Then $g_y^Y = 0$ on $X^{\mathrm{an}}\backslash V$ and whose Laplacian $dd^c g_y^Y$ is supported on $\partial Y \cup \{y\}$ with $dd^c g_y^Y = -\delta_y$ in a neighborhood of y. Due to our requirement, we get

$$0 \leq \langle dd^c f - dd^c g, g_y^Y \rangle = \int_Y (f-g) \; dd^c g_y^Y.$$

Let $\partial Y = \{y_1, \ldots, y_n\}$. Since $\int_{\{y_i\}} dd^c g_y^Y \in (0,1]$ and $\sum_{y_i \in \partial Y} \left(\int_{\{y_i\}} dd^c g_y^Y \right) = 1$ by Remark 3.1.22, we get

$$\begin{aligned} 0 \leq \int_{X^{\mathrm{an}}} (f-g) \; dd^c g_y^Y &= \sum_{y_i \in \partial Y} \left(\int_{\{y_i\}} dd^c g_y^Y \right) (f-g)(y_i) - (f-g)(y) \\ &\leq \max_{y_i \in \partial Y} (f-g)(y_i) - (f-g)(y) \\ &= \max_{y_i \in \partial Y} (f-g)(y_i) + g(y) - f(y). \end{aligned}$$

Thus the assertion is true for all points in $I(Y)$.

Next, consider $y \in Y \backslash I(Y)$. Then $y \in V$ as all boundary points of Y are of type II or III. Since $y \notin I(Y)$, we can find a neighborhood of y in V that is an open ball by Theorem 2.3.27. As type II and III points are dense, we can find a path $[z, y]$ that lies in this open ball and $[z, y)$ is contained in $I(Y)$. By Lemma 3.1.33, we have

$$f(y) = \lim_{x \to y, x \in [z,y)} f(x) \text{ and } g(y) = \lim_{x \to y, x \in [z,y)} g(x).$$

Above, we have proved the desired inequality for all points in $I(Y)$, and hence these equations imply

$$f(y) = \lim_{x \to y, x \in [z,y)} f(x) \leq \max_{y_i \in \partial Y} (f-g)(y_i) + \lim_{x \to y, x \in [z,y)} g(x) = \max_{y_i \in \partial Y} (f-g)(y_i) + g(y).$$

□

COROLLARY 3.1.35. *Let W be an open subset of X^{an}, let Y be an affinoid domain in W and let $f, g \in D^0(W)$ be two currents on W with $\langle dd^c f, \varphi \rangle = \langle dd^c g, \varphi \rangle$ for every $\varphi \in A_c^0(W)$ with $\operatorname{supp}(\varphi) \subset Y$. If f and g are subharmonic on the relative interior $V := Y^\circ$ of Y and $f = g$ on ∂Y, then $f = g$ on Y.*

PROOF. Follows directly by symmetry and the lemma above. □

THEOREM 3.1.36. *Let W be a proper open subset of X^{an}. Suppose f and g are two subharmonic functions on W such that*

i) for each $z \in \partial W$ we have

$$\limsup_{x \to z, x \in W'} (f(x) - g(x)) \leq 0,$$

where W' is defined as W without points with $f(x) = g(x) = -\infty$, and

ii) $dd^c f \geq dd^c g$ on W.

Then $f \leq g$ on W.

PROOF. We write $h := f - g$ on W' and note that we already know that our assertion is true on $W \backslash W'$. Recall from Corollary 3.1.12 and Remark 2.3.5 that subharmonic functions are finite on $I(W)$ and the boundary of an affinoid domain is a finite set of points in $I(W)$.

The proof follows as in [**BR10**, Proposition 8.35] using Lemma 3.1.34. For every point y in W not satisfying $f(y) = g(y) = -\infty$, let $(Y_\alpha)_\alpha$ be the directed system of strictly affinoid domains contained in W and containing y. Note that the union of two strictly affinoid domains Y_1, Y_2 in X^{an} with $Y_1 \cup Y_2 \neq X^{\text{an}}$ is again a strictly affinoid domain in X^{an} by [**Thu05**, Corollaire 2.1.17]. For every Y_α, we can choose by Lemma 3.1.34 a point $x_\alpha \in \partial Y_\alpha$ such that $h(y) \leq h(x_\alpha)$. Then $\langle x_\alpha \rangle_\alpha$ defines a net in W. As $\overline{W}$ is compact, there is a subnet $\langle z_\alpha \rangle_\alpha$ in W converging to a point $z \in \overline{W}$. Due to $W = \bigcup_\alpha Y_\alpha$ and $z_\alpha \in \partial Y_\alpha$, the point z has to ly in the boundary ∂W. Because the z_α are chosen such that $h(y) \leq h(z_\alpha)$ and $\langle z_\alpha \rangle_\alpha$ is a net in W converging to z, we obtain

$$h(y) \leq \limsup_{\alpha} h(z_\alpha) \leq \limsup_{x \to z, x \in W'} h(x) \leq 0,$$

where the last inequality is true due to requirement i). □

3.2. The Energy Minimization Principle

In this section, we prove a non-archimedean Energy Minimization Principle for a smooth proper curve X over K. For $X = \mathbb{P}^1$ this was worked out by Baker and Rumely in [**BR10**]. We generalize there results and proofs for our arbitrary smooth proper X.

The Energy Minimization Principle is a result regarding Arakelov–Green's functions. Hence our first assignment is to extend Baker and Rumely's definition of an Arakelov–Green's functions on X^{an}. To do so, we first need to develop a potential kernel on X^{an}.

3.2.1. The potential kernel. On the way to define Arakelov–Green's functions and prove an Energy Minimization Principle, we have to introduce a lot of other things first. Our most fundamental tool is the potential kernel that is a function $g_\zeta(\cdot, y)$ for fixed ζ and y that inverts the Laplacian in the sense that $dd^c g_\zeta(\cdot, y) = \delta_\zeta - \delta_y$. A function with this property was already seen in Proposition 3.1.20 for $\zeta, y \in I(X^{\mathrm{an}})$. We start this subsection with a definition of a potential kernel for an arbitrary $y \in X^{\mathrm{an}}$ and $\zeta \in I(X^{\mathrm{an}})$, which is an extension of the function from Proposition 3.1.20 and also extends the one in [**BR10**] for $X = \mathbb{P}^1$. At the end of this subsection, we also give a definition for $\zeta \notin I(X^{\mathrm{an}})$.

As in [**BR10**, §3.3], we first define a potential kernel on a metric graph.

Definition 3.2.1. Let Γ be a metric graph. For two fixed points $\zeta, y \in \Gamma$, let $g_\zeta(\cdot, y)_\Gamma \colon \Gamma \to \mathbb{R}_{\geq 0}$ be the unique piecewise affine function on Γ such that

i) $dd^c g_\zeta(\cdot, y)_\Gamma = \delta_\zeta - \delta_y$, and
ii) $g_\zeta(\zeta, y)_\Gamma = 0$.

We call $g_\zeta(x, y)_\Gamma$ the *potential kernel* on Γ.

Lemma 3.2.2. *Let Γ be a metric graph, then the potential kernel $g_\zeta(x, y)_\Gamma$ on Γ is non-negative, bounded, symmetric in x and y, and jointly continuous in x, y, ζ. For every $\zeta' \in \Gamma$, we have*

$$g_\zeta(x, y)_\Gamma = g_{\zeta'}(x, y)_\Gamma - g_{\zeta'}(x, \zeta)_\Gamma - g_{\zeta'}(y, \zeta)_\Gamma + g_{\zeta'}(\zeta, \zeta)_\Gamma.$$

Proof. Follows by [**BR10**, Proposition 3.3]. □

Since every skeleton of X^{an} has the structure of a metric graph, we can define a potential kernel on every skeleton. Using the skeletal metric $\rho\colon \mathbb{H}(X^{\mathrm{an}}) \times \mathbb{H}(X^{\mathrm{an}}) \to \mathbb{R}_{\geq 0}$ from Definition 2.3.24, we can extend the potential kernel to all of X^{an}.

Remark 3.2.3. Let V be a uniquely path-connected subset of X^{an} and let ζ be a point in V. For two points $x, y \in V$, we denote by $w_\zeta(x, y)$ the unique point in V where the paths $[x, \zeta]$ and $[y, \zeta]$ first meet. For example, for a skeleton Γ of X^{an} and a point $x_0 \in \Gamma$, the subset $\tau_\Gamma^{-1}(x_0)$ is uniquely path-connected. We therefore can define for two points $x, y \in \tau_\Gamma^{-1}(x_0)$ the point $w_\Gamma(x, y) := w_{x_0}(x, y)$.

Definition 3.2.4. Let $\zeta \in I(X^{\mathrm{an}})$. We define the corresponding *potential kernel* $g_\zeta\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty, \infty]$ by

$$g_\zeta(x, y) := \begin{cases} \infty & \text{if } (x, y) \in \mathrm{Diag}(X(K)), \\ g_\zeta(\tau_\Gamma(x), \tau_\Gamma(y))_\Gamma & \text{if } \tau_\Gamma(x) \neq \tau_\Gamma(y), \\ g_\zeta(\tau_\Gamma(y), \tau_\Gamma(x))_\Gamma + \rho((w_\Gamma(x, y), \tau_\Gamma(y)) & \text{else} \end{cases}$$

for a skeleton Γ of X^{an} containing ζ and the skeletal metric $\rho\colon \mathbb{H}(X^{\mathrm{an}}) \times \mathbb{H}(X^{\mathrm{an}}) \to \mathbb{R}_{\geq 0}$.

Proposition 3.2.5. *The function g_ζ is well-defined for every $\zeta \in I(X^{\mathrm{an}})$.*

Proof. We have to show that g_ζ is independent of the skeleton Γ. Thus we consider $(x, y) \notin \mathrm{Diag}(X(K))$. Let Γ_1 and Γ_2 be two skeleta containing ζ, and we may assume that $\Gamma_1 \subset \Gamma_2$. Since Γ_1 is already a skeleton of X^{an}, Γ_2 arises by just adding additional edges and vertices to Γ_1 without getting new loops or cycles. Working inductively, we may assume that Γ_2 equals to the graph Γ_1 and one new edge e attached to a vertex z in Γ_1.

Note that for every $w \in \Gamma_1$, due to uniqueness of the potential kernel and because its Laplacian is supported on $\{\zeta, w\}$, we have

$$g_\zeta(\cdot, w)_{\Gamma_1} \equiv g_\zeta(\cdot, w)_{\Gamma_2} \text{ on } \Gamma_1, \text{ and } g_\zeta(\cdot, w)_{\Gamma_2} \equiv g_\zeta(z, w)_{\Gamma_2} \text{ on } e. \tag{3.2.1}$$

Hence $g_\zeta(v, w)_{\Gamma_1} = g_\zeta(v, w)_{\Gamma_2}$ for every pair $(v, w) \in \Gamma_1 \times \Gamma_1$.

First, we consider (x, y) with $\tau_{\Gamma_1}(x) \neq \tau_{\Gamma_1}(y)$. Then automatically $\tau_{\Gamma_2}(x) \neq \tau_{\Gamma_2}(y)$ and $\tau_{\Gamma_1}(x) = \tau_{\Gamma_2}(x)$ or $\tau_{\Gamma_1}(y) = \tau_{\Gamma_2}(y)$. Without loss of generality $\tau_{\Gamma_1}(y) = \tau_{\Gamma_2}(y)$. As argued in (3.2.1) and using symmetry, we get

$$\begin{aligned} g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(y))_{\Gamma_1} &= g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(y))_{\Gamma_2} \\ &= g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_1}(y))_{\Gamma_2} \\ &= g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_2}(y))_{\Gamma_2}. \end{aligned}$$

Note for the second equation that either $\tau_{\Gamma_1}(x) = \tau_{\Gamma_2}(x)$ or $\tau_{\Gamma_1}(x) = z$ and $\tau_{\Gamma_2}(x) \in e$.

Now consider the case $\tau_{\Gamma_1}(x) = \tau_{\Gamma_1}(y)$, and we set $w := w_{\Gamma_1}(x, y)$ (cf. Remark 3.2.3). Then (3.2.1) implies

$$g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(y))_{\Gamma_1} = g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(y))_{\Gamma_2} = g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_1}(y))_{\Gamma_2}. \tag{3.2.2}$$

Note again that that either $\tau_{\Gamma_1}(x) = \tau_{\Gamma_2}(x)$ or $\tau_{\Gamma_1}(x) = z$ and $\tau_{\Gamma_2}(x) \in e$. In the case $\tau_{\Gamma_2}(x) = \tau_{\Gamma_2}(y) = \tau_{\Gamma_1}(y) = \tau_{\Gamma_1}(x)$, then the line above implies the claim.

If $\tau_{\Gamma_2}(x) = \tau_{\Gamma_2}(y) \neq \tau_{\Gamma_1}(y) = \tau_{\Gamma_1}(x)$, then $\tau_{\Gamma_1}(y) = \tau_{\Gamma_1}(x) = z$ and we have

$$\rho(w, \tau_{\Gamma_1}(x)) = \rho(w, \tau_{\Gamma_2}(x)) + \rho(\tau_{\Gamma_2}(x), \tau_{\Gamma_1}(x)).$$

Identity (3.2.2) implies

$$\begin{aligned} g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(x))_{\Gamma_1} &= g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_1}(x))_{\Gamma_2} \\ &= g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_2}(x))_{\Gamma_2} \\ &= g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_2}(x))_{\Gamma_2} - \rho(\tau_{\Gamma_1}(x), \tau_{\Gamma_2}(x)), \end{aligned}$$

where we use for the last equation that $g_\zeta(\cdot, \tau_{\Gamma_2}(x))_{\Gamma_2}$ restricted to the path $[z, \tau_{\Gamma_2}(x)]$ is affine with slope 1. Adding these two equations up, we get

$$g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(x))_{\Gamma_1} + \rho(w, \tau_{\Gamma_1}(x)) = g_\zeta(\tau_{\Gamma_2}(x), \tau_{\Gamma_2}(x))_{\Gamma_2} + \rho(w, \tau_{\Gamma_2}(x))$$

as we desired.

If $\tau_{\Gamma_2}(x) \neq \tau_{\Gamma_2}(y)$, then $z = \tau_{\Gamma_1}(x) = \tau_{\Gamma_1}(y)$ and $w = \tau_{\Gamma_2}(x)$ or $w = \tau_{\Gamma_2}(y)$. Without loss of generality, $w = \tau_{\Gamma_2}(y)$. The potential kernel $g_\zeta(\cdot, \tau_{\Gamma_2}(x))_{\Gamma_2}$ restricted to the path $[z, \tau_{\Gamma_2}(x)]$ (which contains $w = \tau_{\Gamma_2}(y)$) is affine with slope 1. Hence (3.2.2) and symmetry yield

$$g_\zeta(\tau_{\Gamma_1}(x), \tau_{\Gamma_1}(y))_{\Gamma_1} = g_\zeta(\tau_{\Gamma_1}(y), \tau_{\Gamma_2}(x))_{\Gamma_2} = g_\zeta(\tau_{\Gamma_2}(y), \tau_{\Gamma_2}(x))_{\Gamma_2} - \rho(\tau_{\Gamma_1}(y), w).$$

Consequently, $g_\zeta(x, y)$ is well-defined. □

Proposition 3.2.6. *If $X = \mathbb{P}^1$, the function g_ζ coincides with the potential kernel j_ζ from* [**BR10**, §4.2] *for every $\zeta \in I(X^{\mathrm{an}})$.*

Proof. We compare the definition of the potential kernel g_ζ to the characterization of j_ζ in [**BR10**, (4.10)]. For $(x, y) \notin \mathrm{Diag}(X^{\mathrm{an}} \backslash I(X^{\mathrm{an}}))$ we can find a skeleton Γ that contains ζ and $w_\Gamma(x, y) \in \Gamma$. Note that $X^{\mathrm{an}} = \mathbb{P}^{1,\mathrm{an}}$ is uniquely path-connected and Γ is a finite subgraph in the sense of [**BR10**]. Thus the assertion follows directly outside of $\mathrm{Diag}(X^{\mathrm{an}} \backslash I(X^{\mathrm{an}}))$.

So it remains to consider points of the form (x,x), where x is a point of type I or IV. If x is of type I, then $g_\zeta(x,x) = \infty = j_\zeta(x,x)$ and there is nothing to show. Thus let x be of type IV. Let Γ be a skeleton containing ζ. Then Γ is a tree, and adding the path $[x, \tau_\Gamma(x)]$ to Γ yields to a finite subgraph Σ in the sense of [**BR10**] and Σ has the structure of a metric graph. Thus we can consider its potential kernel $g_\zeta(\cdot,\cdot)_\Sigma$ on $\Sigma \times \Sigma$. In the case $X = \mathbb{P}^1$, one has

$$j_\xi(y,z) = g_\xi(y,z)_{\Gamma'} = \rho(\xi, w_\zeta(y,z))$$

for every finite subgraph Γ' of $\mathbb{P}^{1,\mathrm{an}}$ and $y,z,\xi \in \Gamma'$ (cf. [**BR10**, §4.2]). Hence

$$\begin{aligned} j_\zeta(x,x) = \rho(\zeta,x) &= \rho(\zeta,\tau_\Gamma(x)) + \rho(\tau_\Gamma(x),x) \\ &= g_\zeta(\tau_\Gamma(x),\tau_\Gamma(x))_\Gamma + \rho(\tau_\Gamma(x),x) \\ &= g_\zeta(x,x). \end{aligned}$$

Thus g_ζ and j_ζ coincide. □

LEMMA 3.2.7. *Fix $\zeta \in I(X^{\mathrm{an}})$. As a function of two variables $g_\zeta(x,y)$ satisfies the following properties:*

i) It is non-negative and $g_\zeta(\zeta,y) = 0$.
ii) $g_\zeta(x,y) = g_\zeta(y,x)$.
iii) For every $\zeta' \in I(X^{\mathrm{an}})$, we have

$$g_\zeta(x,y) = g_{\zeta'}(x,y) - g_{\zeta'}(x,\zeta) - g_{\zeta'}(y,\zeta) + g_{\zeta'}(\zeta,\zeta). \tag{3.2.3}$$

iv) It is finitely valued and continuous off the diagonal and it is lsc on $X^{\mathrm{an}} \times X^{\mathrm{an}}$ (where we understand $X^{\mathrm{an}} \times X^{\mathrm{an}}$ set theoretically and endowed with the product topology).

PROOF. All properties follow by construction and the properties of the potential kernel on a metric graph from Lemma 3.2.2. Note for the third assertion that we choose a skeleton such that $\zeta, \zeta' \in \Gamma$.

We explain iv) a little bit more in detail. By the definition of the potential kernel $g_\zeta(x,y)$, it is finite off the diagonal. To show continuity off the diagonal, we consider $(x_0,y_0) \in X^{\mathrm{an}} \times X^{\mathrm{an}}$ with $x_0 \neq y_0$. We choose a skeleton Γ such that ζ is contained in it and x_0, y_0 are retracted to different points in Γ. Then (x_0,y_0) has a basis of open neighborhoods that are of the form $V_{x_0} \times V_{y_0}$ for disjoint simple open neighborhoods V_{x_0} of x_0 and V_{y_0} of y_0, i.e. points in V_{x_0} cannot be retracted to the same point as points in V_{y_0}. Hence $g_\zeta(x,y) = g_\zeta(\tau_\Gamma(x),\tau_\Gamma(y))_\Gamma$ on $V_{x_0} \times V_{y_0}$. Continuity of the retraction map and the potential kernel on a metric graph imply continuity of $g_\zeta(x,y)$ off the diagonal.

Next, we show that g_ζ is lsc on the diagonal. Consider a point (x_0,x_0) in the diagonal and let Γ be any skeleton that contains ζ. If $x_0 \in X^{\mathrm{an}}(K)$, we have $g_\zeta(x_0,x_0) = \infty$. Hence we have to show that $g_\zeta(x,y)$ tends to ∞ as (x,y) tends to (x_0,x_0). Since x_0 is of type I, we can find an open neighborhood V_{x_0} of x_0 contained in the connected component of $X^{\mathrm{an}} \backslash \Gamma$ that contains x_0. On the open neighborhood $V := V_{x_0} \times V_{x_0}$ of (x_0,x_0), the function $g_\zeta(x,y)$ is either equal to ∞ or to $g_\zeta(\tau_\Gamma(x_0),\tau_\Gamma(x_0))_\Gamma + \rho(\tau_\Gamma(x_0), w_\Gamma(x,y))$. Since the latter also tends to ∞ as (x,y) tends to (x_0,x_0), the function g_ζ is lsc in (x_0,x_0).

If x_0 is of type IV, then $g_\zeta(x_0,x_0) = g_\zeta(\tau_\Gamma(x_0),\tau_\Gamma(x_0))_\Gamma + \rho(x_0,\tau_\Gamma(x_0))$. We can find V_{x_0} as above, and we either have $g_\zeta(x,y) = \infty > g_\zeta(x_0,x_0)$ or

$$g_\zeta(x_0,x_0) - g_\zeta(x,y) = \rho(x_0,\tau_\Gamma(x_0)) - \rho(w_\Gamma(x,y),\tau_\Gamma(x_0))$$

on $V_{x_0} \times V_{x_0}$. It is easy to see that we can find for every $\varepsilon > 0$ an open neighborhood such that the right hand side is smaller or equal than ε. Thus g_ζ is lsc in (x_0,x_0).

Now we consider $x_0 \in I(X^{\mathrm{an}})$ and we may assume that $x_0 \in \Gamma$. Hence we have $g_\zeta(x_0, x_0) = g_\zeta(x_0, x_0)_\Gamma$. Since the potential kernel $g_\zeta(\cdot, \cdot)_\Gamma$ is continuous on $\Gamma \times \Gamma$, there is for every $\varepsilon > 0$ an open neighborhood $\Omega = \Omega_1 \times \Omega_2$ of (x_0, x_0) in $\Gamma \times \Gamma$ such that $g_\zeta(z_1, z_2)_\Gamma \geq g_\zeta(x_0, x_0)_\Gamma - \varepsilon$ for every $(z_1, z_2) \in \Omega$. The skeletal metric is non-negative, so

$$g_\zeta(x, y) \geq g_\zeta(\tau_\Gamma(x), \tau_\Gamma(y))_\Gamma \geq g_\zeta(x_0, x_0)_\Gamma - \varepsilon$$

for every $(x, y) \in \tau_\Gamma^{-1}(\Omega_1) \times \tau_\Gamma^{-1}(\Omega_2)$, which is an open neighborhood of (x_0, x_0) in $X^{\mathrm{an}} \times X^{\mathrm{an}}$. Hence g_ζ is lsc on the diagonal, and so on $X^{\mathrm{an}} \times X^{\mathrm{an}}$. □

PROPOSITION 3.2.8. *For fixed points $\zeta \in I(X^{\mathrm{an}})$ and $y \in X^{\mathrm{an}}$, we consider the function $G_{\zeta,y} := g_\zeta(\cdot, y) \colon X^{\mathrm{an}} \to (-\infty, \infty]$. Then $G_{\zeta,y}$ defines a current in $D^0(X^{\mathrm{an}})$ with*

$$dd^c G_{\zeta,y} = \delta_\zeta - \delta_y.$$

Moreover, the following hold:

- *i) If y is of type II or III, then $G_{\zeta,y} \in A^0(X^{\mathrm{an}})$ and coincides with $g_{y,\zeta}$ from Proposition 3.1.20.*
- *ii) If y is of type IV, the function $G_{\zeta,y}$ is finitely valued and continuous on X^{an}.*
- *iii) If y is of type I, then $G_{\zeta,y}$ is finitely valued on $X^{\mathrm{an}} \backslash \{y\}$ and continuous on X^{an} when we endow $(-\infty, \infty]$ with the topology of a half-open interval.*

Hence $G_{\zeta,y}$ is subharmonic on $X^{\mathrm{an}} \backslash \{y\}$ for every fixed $y \in X^{\mathrm{an}}$.

PROOF. First, note that by construction $G_{\zeta,y}(x) = \infty$ if and only if $x = y \in X(K)$. Thus the restriction of $G_{\zeta,y}$ to $I(X^{\mathrm{an}})$ is always finite, and so $G_{\zeta,y}$ defines a current in $D^0(X^{\mathrm{an}})$. Here, one should have in mind that the topological vector space $D^0(X^{\mathrm{an}})$ is isomorphic to the vector space $\mathrm{Hom}(I(X^{\mathrm{an}}), \mathbb{R})$ endowed with the pointwise convergence (see Proposition 3.1.27). We always use this identification.

Let Γ always be a skeleton that contains ζ. To calculate the Laplacian, we first consider a point $y \in I(X^{\mathrm{an}})$. We may extend Γ such that $y \in \Gamma$. Then $G_{\zeta,y} = g_\zeta(\cdot, y)_\Gamma \circ \tau_\Gamma$ on X^{an} since

$$\rho(w_\Gamma(x, y), y) = \rho(\tau_\Gamma(x), \tau_\Gamma(y)) = 0$$

if $\tau_\Gamma(x) = \tau_\Gamma(y) = y$. Since the potential kernel $g_\zeta(\cdot, y)_\Gamma$ is the unique piecewise affine function on the metric graph Γ such that $dd^c g_\zeta(\cdot, y)_\Gamma = \delta_\zeta - \delta_y$ and $g_\zeta(\zeta, y)_\Gamma = 0$, we have $G_{\zeta,y} = g_\zeta(\cdot, y) \in A^0(X^{\mathrm{an}})$ and $dd^c G_{\zeta,y} = \delta_\zeta - \delta_y$ on X^{an} by the construction of the Laplacian. In particular, the function $G_{\zeta,y}$ is continuous on X^{an}. Uniqueness in Proposition 3.1.20 implies that $G_{\zeta,y}$ coincides with $g_{y,\zeta}$.

Now consider an arbitrary $y \in X^{\mathrm{an}} \backslash I(X^{\mathrm{an}})$ and let $(y_n)_{n \in \mathbb{N}}$ be a sequence of points $y_n \in I(X^{\mathrm{an}})$ converging to y. Then $g_\zeta(\cdot, y_n)$ converges to $g_\zeta(\cdot, y)$ in the topological vector space $\mathrm{Hom}(I(X^{\mathrm{an}}), \mathbb{R}) \simeq D^0(X^{\mathrm{an}})$, i.e. for every fixed point $x \in I(X^{\mathrm{an}})$ we have $g_\zeta(x, y_n) = g_\zeta(y_n, x)$ converges to $g_\zeta(x, y) = g_\zeta(y, x)$ for $n \to \infty$ since $g_\zeta(\cdot, x)$ is lisse, and so continuous. The differential operator $dd^c \colon D^0(X^{\mathrm{an}}) \to D^1(X^{\mathrm{an}})$ is continuous by [**Thu05**, Proposition 3.3.4], and hence $dd^c g_\zeta(\cdot, y) = \delta_\zeta - \delta_y$.

If y is a point of type I or IV, the connected component U of $X^{\mathrm{an}} \backslash \Gamma$ containing y is an open ball. For a type IV point y, we have

$$\begin{aligned} G_{\zeta,y}(x) &= g_\zeta(\tau_\Gamma(y), \tau_\Gamma(y))_\Gamma + \rho(w_\Gamma(x, y), \tau_\Gamma(x)) \\ &= g_\zeta(\tau_\Gamma(y), \tau_\Gamma(y))_\Gamma + \rho(w_{\tau_\Gamma(y)}(x, y), \tau_\Gamma(x)) \end{aligned}$$

for every $x \in U$. Note that $\tau_\Gamma(x) = \tau_\Gamma(y)$ and $\zeta \in \Gamma$. If y is of type I, we have this identity on $U \backslash \{y\}$. Since the path distance metric ρ is continuous on U, it follows that

$G_{\zeta,y}$ is continuous on U in both cases with $\lim_{x\to y} G_{\zeta,y}(x) = G_{\zeta,y}(y) = \infty$ if y is of type I.

In particular, $G_{\zeta,y}$ is upper semi-continuous on $X^{\mathrm{an}}\backslash\{y\}$ with $dd^c G_{\zeta,y} = \delta_\zeta - \delta_y$ for every fixed $y \in X^{\mathrm{an}}$. Hence $G_{\zeta,y}$ is subharmonic on $X^{\mathrm{an}}\backslash\{y\}$ by Proposition 3.1.30. □

To introduce a capacity theory and define potential functions on X^{an} in the following subsections, we define a potential kernel $g_\zeta(x,y)$ for every point $\zeta \in X^{\mathrm{an}}$ (cf. [**BR10**, §4.4]). In [**BR10**], this is done with the help of the Gauss point. In our case, we have to fix a base point for the definition.

Definition 3.2.9. Fix $\zeta_0 \in I(X^{\mathrm{an}})$. We define $g_{\zeta_0}\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \times X^{\mathrm{an}} \to [-\infty,\infty]$ as $g_{\zeta_0}(\zeta,x,y) = \infty$ if $x = y = \zeta \in X(K)$ and else as

$$g_{\zeta_0}(\zeta,x,y) := g_{\zeta_0}(x,y) - g_{\zeta_0}(x,\zeta) - g_{\zeta_0}(y,\zeta).$$

Corollary 3.2.10. *For fixed points $\zeta_0 \in I(X^{\mathrm{an}})$ and $\zeta, y \in X^{\mathrm{an}}$, the potential kernel $g_{\zeta_0}(\zeta,\cdot,y)$ defines a current in $D^0(X^{\mathrm{an}})$ with*

$$dd^c g_{\zeta_0}(\zeta,\cdot,y) = \delta_\zeta - \delta_y,$$

and it extends $g_\zeta(x,y)$ in the following way

$$g_\zeta(\zeta,x,y) = g_\zeta(x,y)$$

if $\zeta \in I(X^{\mathrm{an}})$.

Proof. Since all terms of $g_{\zeta_0}(\zeta,\cdot,y)$ are currents, it is a current itself. Proposition 3.2.8 implies

$$\begin{aligned} dd^c g_{\zeta_0}(\zeta,\cdot,y) &= dd^c g_{\zeta_0}(\cdot,y) - dd^c g_{\zeta_0}(\cdot,\zeta) - dd^c g_{\zeta_0}(y,\zeta) \\ &= \delta_{\zeta_0} - \delta_y - \delta_{\zeta_0} + \delta_\zeta + 0 \\ &= \delta_\zeta - \delta_y. \end{aligned}$$

Moreover, if $\zeta \in I(X^{\mathrm{an}})$, then

$$g_\zeta(\zeta,x,y) = g_\zeta(x,y) - g_\zeta(x,\zeta) - g_\zeta(y,\zeta) = g_\zeta(x,y)$$

for every $x \in X^{\mathrm{an}}$ since $g_\zeta(x,\zeta) = g_\zeta(y,\zeta) = 0$ by Lemma 3.2.7. □

3.2.2. Capacity theory. The main goal of Section 3.2 is to prove an analogue of the Energy Minimization Principle. On this way, we need to prove some partial results as for example Frostman's theorem. One of the tools of showing a Frostman's theorem is capacity. There is already a notion of relative capacity (relative to some special surrounding subset) by Thuillier for analytic curves in [**Thu05**, §3.6.1]. Since we need something more general (relative to an arbitrary point outside the given set), we introduce capacity analogously as in [**BR10**, §6.1], show all needed properties and compare our notion with Thuillier's (see Proposition 3.2.19).

Definition 3.2.11. Let $\zeta_0 \in I(X^{\mathrm{an}})$ be a fixed base point. Then for a point $\zeta \in X^{\mathrm{an}}$ and for a probability measure ν on X^{an} with $\mathrm{supp}(\nu) \subset X^{\mathrm{an}}\backslash\{\zeta\}$, we define the *energy integral* as

$$I_{\zeta_0,\zeta}(\nu) := \int\int g_{\zeta_0}(\zeta,x,y)\ d\nu(x)d\nu(y).$$

Recall from Definition that 3.2.9 the extended potential kernel $g_{\zeta_0}(\zeta,\cdot,\cdot)$, which is lower semi-continuous on $X^{\mathrm{an}}\backslash\{\zeta\} \times X^{\mathrm{an}}\backslash\{\zeta\}$ by Lemma 3.2.7 and Proposition 3.2.8. Hence the Lebesgue integral with respect to ν is well-defined.

With the help of the energy integral, we can introduce the *capacity* of a proper E of X^{an} with respect to $\zeta \in X^{\mathrm{an}} \backslash E$ as

$$\gamma_{\zeta_0,\zeta}(E) := e^{-\inf_\nu I_{\zeta_0,\zeta}(\nu)}$$

where ν varies over all probability measures supported on E. We say that E *has positive capacity* if there is a $\zeta_0 \in I(X^{\mathrm{an}})$ and a point $\zeta \in X^{\mathrm{an}} \backslash E$ such that $\gamma_{\zeta_0,\zeta}(E) > 0$, i.e. there exists a probability measure ν supported on E with $I_{\zeta_0,\zeta}(\nu) < \infty$. Otherwise, we say that E *has capacity zero.*

REMARK 3.2.12. It follows from the definition of the capacity of E with respect to $\zeta \in X^{\mathrm{an}} \backslash E$ that

$$\gamma_{\zeta_0,\zeta}(E) = \sup_{E' \subset E,\ E' \text{ compact}} \gamma_{\zeta_0,\zeta}(E').$$

LEMMA 3.2.13. *Having positive capacity is independent of the choice of the chosen base point ζ_0.*

PROOF. Consider $\zeta_0, \zeta_0' \in I(X^{\mathrm{an}})$, a proper subset E of X^{an}, a point $\zeta \in X^{\mathrm{an}} \backslash E$ and a probability measure ν supported on E. We show that $I_{\zeta_0,\zeta}(\nu)$ is finite if and only if $I_{\zeta_0',\zeta}(\nu)$ is finite. Using Definition 3.2.9 and Lemma 3.2.7, we obtain for every $x, y \in E$ (note that $\zeta \notin E$)

$$\begin{aligned} g_{\zeta_0}(\zeta, x, y) &= g_{\zeta_0}(x,y) - g_{\zeta_0}(x,\zeta) - g_{\zeta_0}(y,\zeta) \\ &= g_{\zeta_0'}(x,y) - g_{\zeta_0'}(x,\zeta_0) - g_{\zeta_0'}(y,\zeta_0) + g_{\zeta_0'}(\zeta_0,\zeta_0) \\ &\quad - (g_{\zeta_0'}(x,\zeta) - g_{\zeta_0'}(x,\zeta_0) - g_{\zeta_0'}(\zeta,\zeta_0) + g_{\zeta_0'}(\zeta_0,\zeta_0)) \\ &\quad - (g_{\zeta_0'}(y,\zeta) - g_{\zeta_0'}(y,\zeta_0) - g_{\zeta_0'}(\zeta,\zeta_0) + g_{\zeta_0'}(\zeta_0,\zeta_0)) \\ &= g_{\zeta_0'}(x,y) - g_{\zeta_0'}(x,\zeta) - g_{\zeta_0'}(y,\zeta) + 2g_{\zeta_0'}(\zeta,\zeta_0) - g_{\zeta_0'}(\zeta_0,\zeta_0) \\ &= g_{\zeta_0'}(\zeta,x,y) + 2g_{\zeta_0'}(\zeta,\zeta_0) - g_{\zeta_0'}(\zeta_0,\zeta_0), \end{aligned}$$

where the last two terms are finite for all $\zeta \in X^{\mathrm{an}} \backslash E$. Considering the energy integrals, we get

$$\begin{aligned} I_{\zeta_0,\zeta}(\nu) &= \int\int g_{\zeta_0}(\zeta,x,y)\ d\nu(x)d\nu(y) \\ &= \int\int g_{\zeta_0'}(\zeta,x,y)\ d\nu(x)d\nu(y) + 2g_{\zeta_0'}(\zeta,\zeta_0) - g_{\zeta_0'}(\zeta_0,\zeta_0) \\ &= I_{\zeta_0',\zeta}(\nu) + 2g_{\zeta_0'}(\zeta,\zeta_0) - g_{\zeta_0'}(\zeta_0,\zeta_0). \end{aligned}$$

Hence they differ by a finite constant. □

For the rest of the subsection, we therefore just fix a base point $\zeta_0 \in I(X^{\mathrm{an}})$.

REMARK 3.2.14. Let E be a proper subset of X^{an} and let ν be a probability measure supported on E. Then for every $\zeta \in X^{\mathrm{an}} \backslash E$

$$\begin{aligned} I_{\zeta_0,\zeta}(\nu) &= \int\int g_{\zeta_0}(\zeta,x,y)\ d\nu(x)d\nu(y) \\ &= \int\int g_{\zeta_0}(x,y)\ d\nu(x)d\nu(y) - 2\int g_{\zeta_0}(x,\zeta)\ d\nu(x), \end{aligned}$$

where the last term of the right hand side is finite since $g_{\zeta_0}(\cdot,\zeta)$ is continuous on the compact set $\mathrm{supp}(\nu)$ by Proposition 3.2.8. Thus $I_{\zeta_0,\zeta}(\nu)$ is finite if and only if $I_{\zeta_0,\xi}(\nu)$ is finite for every point $\xi \in X^{\mathrm{an}} \backslash E$.

LEMMA 3.2.15. *If E is a proper subset of X^{an} containing a point of $\mathbb{H}(X^{\mathrm{an}})$, then E has positive capacity.*

PROOF. Choose a point $\zeta \in X^{\mathrm{an}} \backslash E$ and assume there is a point $z \in \mathbb{H}(X^{\mathrm{an}}) \cap E$. Then the Dirac measure $\nu := \delta_z$ is a probability measure supported on E and

$$I_{\zeta_0,\zeta}(\nu) = \int\int g_{\zeta_0}(\zeta, x, y)\ d\nu(x)d\nu(y) = g_{\zeta_0}(z,z) - 2g_{\zeta_0}(\zeta, z) < \infty$$

due to $z \in \mathbb{H}(X^{\mathrm{an}})$. □

Note that $I_{\zeta_0,\zeta_0}(\nu)$ is also well-defined for a probability measure ν supported on X^{an} with $\zeta_0 \in \mathrm{supp}(\nu)$ as

$$I_{\zeta_0,\zeta_0}(\nu) = \int\int g_{\zeta_0}(\zeta_0, x, y)\ d\nu(x)d\nu(y) = \int\int g_{\zeta_0}(x, y)\ d\nu(x)d\nu(y)$$

by Corollary 3.2.10 and g_{ζ_0} is lsc on $X^{\mathrm{an}} \times X^{\mathrm{an}}$ by Lemma 3.2.7.

LEMMA 3.2.16. *Let ζ be a point in X^{an}, let E be a subset of $X^{\mathrm{an}}\backslash\{\zeta\}$ that has capacity zero and let ν be a probability measure on X^{an}. If*

i) $\mathrm{supp}(\nu) \subset X^{\mathrm{an}}\backslash\{\zeta\}$ *with $I_{\zeta_0,\zeta}(\nu) < \infty$ for some base point $\zeta_0 \in I(X^{\mathrm{an}})$, or*

ii) $\zeta \in I(X^{\mathrm{an}})$ with $I_{\zeta,\zeta}(\nu) < \infty$,

then $\nu(E) = 0$.

PROOF. The proof is analogous to [**BR10**, Lemma 6.16]. Note that $g_{\zeta_0}(\cdot, \zeta)$ is continuous on the compact set $\mathrm{supp}(\nu)$ and $g_{\zeta_0}(x, y)$ as a function of two variables is lsc on $\mathrm{supp}(\nu) \times \mathrm{supp}(\nu)$ (Proposition 3.2.8 and Lemma 3.2.7). Hence the extended potential kernel $g_{\zeta_0}(\zeta, x, y) = g_{\zeta_0}(x, y) - g_{\zeta_0}(x, \zeta) - g_{\zeta_0}(y, \zeta)$ is bounded from below on $\mathrm{supp}(\nu) \times \mathrm{supp}(\nu)$ by a constant if i) is satisfied. If $\zeta \in I(X^{\mathrm{an}})$, then the function $g_\zeta(\zeta, x, y) = g_\zeta(x, y)$ (cf. Corollary 3.2.10) is lsc on $X^{\mathrm{an}} \times X^{\mathrm{an}}$ by Lemma 3.2.7, and so also bounded from below on $\mathrm{supp}(\nu)$. In both cases let C be this constant. If $\nu(E) > 0$, then there is a compact subset e of E such that $\nu(e) > 0$. Consider the probability measure $\omega := (1/\nu(e)) \cdot \nu|_e$ on e. Then

$$\begin{aligned}
I_{\zeta_0,\zeta}(\omega) &= \int\int g_{\zeta_0}(\zeta, x, y)\ d\omega(x)d\omega(y) \\
&= \int\int (g_{\zeta_0}(\zeta, x, y) - C)\ d\omega(x)d\omega(y) + \int\int C\ d\omega(x)d\omega(y) \\
&\leq \frac{1}{\nu(e)^2} \cdot \int\int (g_{\zeta_0}(\zeta, x, y) - C)\ d\nu(x)d\nu(y) + C \\
&= \frac{1}{\nu(e)^2} \cdot \int\int g_{\zeta_0}(\zeta, x, y)\ d\nu(x)d\nu(y) - \frac{1}{\nu(e)^2} \cdot \int\int C\ d\nu(x)d\nu(y) + C \\
&= \frac{1}{\nu(e)^2} \cdot I_{\zeta_0,\zeta}(\nu) - \frac{\nu(E)^2}{\nu(e)^2} \cdot C + C < \infty
\end{aligned}$$

contradicting that E has capacity zero. Note that in case ii) we have $\zeta_0 = \zeta$ in the calculation. □

COROLLARY 3.2.17. *Let ζ be a point in X^{an} and let E_n be a countable collection of Borel sets in $X^{\mathrm{an}}\backslash\{\zeta\}$ such that E_n has capacity zero for every $n \in \mathbb{N}$. Then the set $E := \bigcup_{n\in\mathbb{N}} E_n$ has capacity zero.*

PROOF. Assume E has positive capacity, i.e. there is a $\zeta \in X^{\mathrm{an}} \backslash E$ and a probability measure ν supported on E such that $I_{\zeta_0,\zeta}(\nu) < \infty$. The set E is measurable since all E_n are, and $\sum_{n\in\mathbb{N}} \nu(E_n) \geq \nu(E) = 1$. Thus there has to be an E_n such that $\nu(E_n) > 0$ contradicting Lemma 3.2.16. $\square$

REMARK 3.2.18. Thuillier introduced in [**Thu05**, §3.6.1] relative capacity in an open subset Ω of X^{an} with a non-empty boundary $\partial\Omega \subset I(X^{\mathrm{an}})$. The capacity of a compact subset E of Ω is then defined as

$$C(E,\Omega)^{-1} := \left(\inf_{\nu} \int_E \int_E -g_x(y)\ d\nu(x) d\nu(y)\right) \in [0,\infty]$$

where ν runs over all probability measures supported on E. Here $g_x \colon \Omega \to [-\infty, 0)$ for $x \in \Omega$ is the unique subharmonic function on Ω such that

i) $dd^c g_x = \delta_x$, and
ii) $\lim_{y\in\Omega,\ y\to\zeta} g_x(y) = 0$

for every $\zeta \in \partial\Omega$ (see [**Thu05**, Lemma 3.4.14]). This notion of relative capacity can be extended canonically to all subsets of Ω by

$$C(E,\Omega) := \sup_{E'\subset E \text{ compact}} C(E',\Omega).$$

In the next proposition, we compare the relative capacity to our notion of capacity in a special situation. We use this comparison and Thuillier's theory to prove that every polar set has capacity zero at the end of this subsection (see Corollary 3.2.21).

PROPOSITION 3.2.19. *Let E be subset of X^{an} such that $E \subset \Omega$ for an open subset Ω with $\partial\Omega = \{\zeta\} \subset I(X^{\mathrm{an}})$. Then E has positive capacity if and only if $C(E,\Omega) > 0$.*

PROOF. We may assume E to be compact (cf. Remark 3.2.12). Consider for the given $\zeta \in I(X^{\mathrm{an}})$ and for a point $x \in \Omega$ the function $-G_{\zeta,x} = -g_\zeta(\cdot, x) = -g_\zeta(x,\cdot)$ from Proposition 3.2.8. We have seen that $-G_{\zeta,x} \colon X^{\mathrm{an}} \to [-\infty,\infty)$ is continuous on X^{an} with $-G_{\zeta,x}(y) = -\infty$ if and only if $y = x \in X(K)$,

$$dd^c(-G_{\zeta,x}) = \delta_x - \delta_\zeta = \delta_x$$

on Ω and

$$\lim_{y\in\Omega,\ y\to\zeta} -G_{\zeta,x}(y) = -G_{\zeta,x}(\zeta) = -g_\zeta(x,\zeta) = -g_\zeta(\zeta,x) = 0.$$

Thus $-G_{\zeta,x}$ is subharmonic on Ω (see Proposition 3.1.30), and it satisfies the characterizing properties of g_x on Ω for every $x \in \Omega$. Hence the functions g_x and $-G_{\zeta,x}$ coincide on Ω. Since having positive capacity is independent of the base point (Lemma 3.2.13), we can choose ζ. Thus $g_\zeta(\zeta,x,y) = g_\zeta(x,y) = G_{\zeta,x}(y)$ by Corollary 3.2.10. Plugging in implies

$$I_{\zeta,\zeta}(\nu) = \int\int G_{\zeta,x}(y)\ d\nu(x)d\nu(y) = \int\int -g_x(y)\ d\nu(x)d\nu(y)$$

for every probability measure ν supported on E. Hence E has positive capacity if and only if $C(E,\Omega) > 0$. $\square$

DEFINITION 3.2.20. Let W be an open subset of X^{an} and let $f \colon W \to [-\infty,\infty)$ be a subharmonic function on W. Then we define its *polar set* to be

$$\mathcal{P}_W(f) := \{x \in W \mid f(x) = -\infty\}.$$

Baker and Rumely proved in [**BR10**, Corollary 8.40] that polar sets have capacity zero. There is also a version for general curves in [**Thu05**, Théorème 3.6.11], which we translate into our setting.

COROLLARY 3.2.21. *Let $f\colon W \to [-\infty, \infty)$ be a subharmonic function on an open connected subset W of X^{an}. Then $\mathcal{P}_W(f)$ has capacity zero.*

PROOF. By Remark 3.2.12, it remains to show that every compact subset of $\mathcal{P}_W(f)$ has capacity zero, so we consider $E \subset \mathcal{P}_W(f)$ compact. [**Thu05**, Proposition 3.4.10] tells us that $\mathcal{P}_W(f)$, and so E consists only of points of type I. Thus every point x in E has an open ball B_x as a neighborhood in W (cf. Theorem 2.3.27), and so finitely many of these open balls $B_{x_1}, \ldots, B_{x_m}$ cover E. We show that each set $E \cap B_{x_i}$ has capacity zero and use Corollary 3.2.17. As f is subharmonic on W, the function f is also subharmonic on every B_{x_i}. Our function f is usc, and so each $E \cap B_{x_i} = \bigcap_{n\in\mathbb{N}} (f|_{B_{x_i}})^{-1}([-\infty, -n))$ is a Borel set. Then [**Thu05**, Théorème 3.6.9 & 3.6.11] implies $C(E \cap B_{x_i}, B_{x_i}) = 0$ for every $i = 1, \ldots, n$ (note that B_{x_i} has only one boundary point and this point is of type II). Using Proposition 3.2.19, the set $E \cap B_{x_i}$ has capacity zero for every $i = 1, \ldots, m$. Due to $E = \bigcup_{i=1,\ldots,m} (E \cap B_{x_i})$, Corollary 3.2.17 imply that E itself has capacity zero. ☐

3.2.3. The potential function. With the help of the potential kernel from Subsection 3.2.1, one can introduce potential functions on X^{an} attached to a finite signed Borel measure. Baker and Rumely defined these functions on the Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}$ in [**BR10**, §6.3]. For the generalization to X^{an}, we have to fix a type II or III point ζ_0 serving as a base point as the Gauss point does for $\mathbb{P}^{1,\mathrm{an}}$. We define potential functions with respect to this base point and use them to define Arakelov–Green's functions in Section 3.2.4. Later in Lemma 3.2.38, we see that the definition of the Arakelov–Green's functions is independent of this choice.

DEFINITION 3.2.22. Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be any finite signed Borel measure on X^{an}. For every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$, we define the corresponding *potential function* as

$$u_{\zeta_0,\nu}(x,\zeta) := \int_{X^{\mathrm{an}}} g_{\zeta_0}(\zeta, x, y)\ d\nu(y)$$

for every $x \in X^{\mathrm{an}}$. Here $g_{\zeta_0}(\zeta, x, y)$ is the potential kernel defined in Definition 3.2.9.

LEMMA 3.2.23. *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be any finite signed Borel measure on X^{an}. For every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$, the function $u_{\zeta_0,\nu}(\cdot,\zeta)$ is well-defined on X^{an} with values in $\mathbb{R} \cup \{\pm\infty\}$ and we can write*

$$u_{\zeta_0,\nu}(\cdot,\zeta) = \int g_{\zeta_0}(\cdot, y)\ d\nu(y) - \nu(X^{\mathrm{an}}) g_{\zeta_0}(\cdot, \zeta) + C_{\zeta_0,\zeta} \tag{3.2.4}$$

on X^{an} for a finite constant $C_{\zeta_0,\zeta}$.

PROOF. By the definition of the potential kernel $g_{\zeta_0}(\zeta, x, y)$, we get for every $x \in X^{\mathrm{an}}$

$$\begin{aligned} u_{\zeta_0,\nu}(x,\zeta) &= \int g_{\zeta_0}(x, y)\ d\nu(y) - \int g_{\zeta_0}(x, \zeta)\ d\nu(y) - \int g_{\zeta_0}(y, \zeta)\ d\nu(y) \\ &= \int g_{\zeta_0}(x, y)\ d\nu(y) - \nu(X^{\mathrm{an}}) g_{\zeta_0}(x, \zeta) - \int g_{\zeta_0}(y, \zeta)\ d\nu(y). \end{aligned}$$

Since $g_{\zeta_0}(\cdot, \zeta)$ is continuous on the compact subset $\mathrm{supp}(\nu)$ if $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$ (cf. Proposition 3.2.8), the last term is always a finite constant, and so we get the description in (3.2.4) with $C_{\zeta_0,\zeta} := -\int g_{\zeta_0}(y, \zeta)\ d\nu(y)$.

To prove that $u_{\zeta_0,\nu}(\cdot,\zeta)$ is well-defined, we have to show that $\infty - \infty$ or $-\infty + \infty$ cannot occur.

If $\zeta \in I(X^{\mathrm{an}})$, then $g_{\zeta_0}(x, \zeta)$ is finite for every $x \in X^{\mathrm{an}}$, and so $u_{\zeta_0,\nu}(\cdot,\zeta)$ is well-defined.

Next, we consider $\zeta \notin \mathrm{supp}(\nu)$. For every $x \neq \zeta$, we know that $g_{\zeta_0}(x,\zeta)$ is finite as well, and so $\infty - \infty$ or $-\infty + \infty$ cannot occur. It remains to show that the function is well-defined in $x = \zeta \notin \mathrm{supp}(\nu)$. Since $g_{\zeta_0}(x,\cdot)$ is continuous on the compact subset $\mathrm{supp}(\nu)$ as $x \notin \mathrm{supp}(\nu)$, the first term $\int g_{\zeta_0}(x,y)\ d\nu(y)$ is finite, and so $u_{\zeta_0,\nu}(x,\zeta)$ is well-defined in $x = \zeta$. □

REMARK 3.2.24. For another chosen base point $\zeta_0' \in I(X^{\mathrm{an}})$, we have

$$g_{\zeta_0}(\zeta,x,y) = g_{\zeta_0'}(\zeta,x,y) + 2g_{\zeta_0'}(\zeta,\zeta_0) - g_{\zeta_0'}(\zeta_0,\zeta_0)$$

for all $\zeta, x, y \in X^{\mathrm{an}}$ by Lemma 3.2.7 (as in the proof of Lemma 3.2.13). For every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$ and $x \in X^{\mathrm{an}}$, we get

$$\begin{aligned}
u_{\zeta_0,\nu}(x,\zeta) &= \int \Big(g_{\zeta_0'}(\zeta,x,y) + 2g_{\zeta_0'}(\zeta,\zeta_0) - g_{\zeta_0'}(\zeta_0,\zeta_0) \Big)\ d\nu(y) \\
&= \int g_{\zeta_0'}(\zeta,x,y)\ d\nu(y) + 2\nu(X^{\mathrm{an}}) g_{\zeta_0'}(\zeta,\zeta_0) - \nu(X^{\mathrm{an}}) g_{\zeta_0'}(\zeta_0,\zeta_0) \\
&= u_{\zeta_0',\nu}(x,\zeta) + 2\nu(X^{\mathrm{an}}) g_{\zeta_0'}(\zeta,\zeta_0) - \nu(X^{\mathrm{an}}) g_{\zeta_0'}(\zeta_0,\zeta_0)
\end{aligned}$$

i.e. the corresponding potential function differ by a constant depending on ζ_0', ζ_0 and ζ.

LEMMA 3.2.25. *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be any finite signed Borel measure on X^{an}. For every skeleton Γ of X^{an} or every path $\Gamma = [z,\omega] \subset \mathbb{H}(X^{\mathrm{an}})$, and for every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$, the restriction of $u_{\zeta_0,\nu}(\cdot,\zeta)$ to Γ is finite and continuous.*

PROOF. First, we consider a skeleton Γ of X^{an}. We may assume $\zeta_0 \in \Gamma$ by Remark 3.2.24. Note that the potential kernel satisfies by construction a retraction formula as in [**BR10**, Proposition 4.5], i.e.

$$g_{\zeta_0}(x,y) = g_{\zeta_0}(x,\tau_\Gamma(y))_\Gamma = g_{\zeta_0}(x,\tau_\Gamma(y)) \tag{3.2.5}$$

for every $x \in \Gamma$ and $y \in X^{\mathrm{an}}$. Furthermore, recall the description of $u_{\zeta_0,\nu}(\cdot,\zeta)$ in (3.2.4).

Then for every $x \in \Gamma$

$$\begin{aligned}
u_{\zeta_0,\nu}(x,\zeta) &= \int_{X^{\mathrm{an}}} g_{\zeta_0}(x,y)\ d\nu(y) - \nu(X^{\mathrm{an}}) g_{\zeta_0}(x,\zeta) + C_{\zeta_0,\zeta} \\
&= \int_{X^{\mathrm{an}}} g_{\zeta_0}(x,\tau_\Gamma(y))_\Gamma\ d\nu(y) - \nu(X^{\mathrm{an}}) g_{\zeta_0}(x,\tau_\Gamma(\zeta))_\Gamma + C_{\zeta_0,\zeta} \\
&= \int_{\Gamma} g_{\zeta_0}(x,t)_\Gamma\ d((\tau_\Gamma)_*\nu)(t) - \nu(X^{\mathrm{an}}) g_{\zeta_0}(x,\tau_\Gamma(\zeta))_\Gamma + C_{\zeta_0,\zeta}.
\end{aligned}$$

The first term is finite and continuous by Lemma 3.2.2 and the second one is as well by Lemma 3.2.8. Hence $u_{\zeta_0,\nu}(\cdot,\zeta)$ is finite and continuous on Γ.

In the following, we consider a path $\Sigma := [z,\omega]$. Recall that $\mathbb{H}(X^{\mathrm{an}})$ is the set of points of type II, III and IV, and every point of type IV has only one tangent direction in X^{an} [**BPR13**, Lemma 5.12]. We already know that $u_{\zeta_0,\nu}(\cdot,\zeta)$ restricted to every skeleton is finite and continuous. Moreover, every path $[z,\omega]$ for $z,\omega \in I(X^{\mathrm{an}})$ lies in some skeleton. Thus it remains to consider paths of the form $[z,\tau_\Gamma(z)]$ for a type IV point z and an arbitrary large skeleton Γ of X^{an}. From now on let Σ be the considered path $[z,\omega]$ with $\omega := \tau_\Gamma(z)$.

Let ζ_0 be some base point in $I(X^{\mathrm{an}}) \cap \Gamma$, which we may choose that way by Remark 3.2.24. Again, we consider each term of

$$u_{\zeta_0,\nu}(x,\zeta) = \int_{X^{\mathrm{an}}} g_{\zeta_0}(x,y)\ d\nu(y) - \nu(X^{\mathrm{an}}) g_{\zeta_0}(x,\zeta) + C_{\zeta_0,\zeta}$$

for $x \in \Sigma$ separately. The second term is finite and continuous in x by Proposition 3.2.8 (note that $\Sigma \cap X(K) = \emptyset$).

It remains to consider the first term. Let V be the connected component of $X^{\mathrm{an}} \backslash \Gamma$ containing z, which is an open ball with unique boundary point $\omega = \tau_\Gamma(z)$. We can consider the canonical retraction map $\tau_\Sigma \colon \overline{V} \to [z, \omega]$, where a point $x \in \overline{V}$ is retracted to $w_\Gamma(x,z)$ (cf. Remark 3.2.3). Note that for $x \in \Sigma$, we have

$$g_{\zeta_0}(x,y) = \begin{cases} g_{\zeta_0}(\omega, \tau_\Gamma(y))_\Gamma & \text{if } y \notin V, \\ g_{\zeta_0}(\omega, \tau_\Gamma(y))_\Gamma + \rho(w_\Gamma(x,y), \omega) & \text{if } y \in V = \tau_\Sigma^{-1}([z,\omega)). \end{cases}$$

Hence for $x \in \Sigma$ the following is true

$$\begin{aligned} \int_{X^{\mathrm{an}}} g_{\zeta_0}(x,y)\, d\nu(y) &= \int_{X^{\mathrm{an}}} g_{\zeta_0}(\omega, \tau_\Gamma(y))_\Gamma \, d\nu(y) + \int_{\tau_\Sigma^{-1}((\omega,z])} \rho(w_\Gamma(x,y),\omega)\, d\nu(y) \\ &= \int_{X^{\mathrm{an}}} g_{\zeta_0}(\omega, \tau_\Gamma(y))_\Gamma \, d\nu(y) + \int_{\tau_\Sigma^{-1}((\omega,z])} \rho(w_\Gamma(x,\tau_\Sigma(y)),\omega)\, d\nu(y) \\ &= \int_{X^{\mathrm{an}}} g_{\zeta_0}(\omega, \tau_\Gamma(y))_\Gamma \, d\nu(y) + \int_{\Sigma} \rho(w_\Gamma(x,t),\omega)\, d((\tau_\Sigma)_*\nu)(t) \\ &= \int_{X^{\mathrm{an}}} g_{\zeta_0}(\omega, \tau_\Gamma(y))_\Gamma \, d\nu(y) + \int_{\Sigma} g_\omega(x,t)_\Sigma \, d((\tau_\Sigma)_*\nu)(t). \end{aligned}$$

Note that our path $\Sigma = [z,\omega] \subset \mathbb{H}(X^{\mathrm{an}})$ is a metric graph, and so we can consider the potential kernel $g_\omega(x,t)_\Sigma$ on Σ from Definition 3.2.1. For the last identity we used $\rho(w_\Gamma(x,t),\omega) = \rho(w_\omega(x,t),\omega) = g_\omega(x,t)_\Sigma$, which follows by Proposition 3.2.6 and [**BR10**, §4.2 p. 77]. Then Lemma 3.2.2 tells us again that the second term is finite and continuous. As $g_{\zeta_0}(\omega, \tau_\Gamma(\cdot))_\Gamma = g_{\zeta_0}(\omega, \cdot)$ (see (3.2.5)) is finitely valued and continuous on the compact set $\mathrm{supp}(\nu)$ by Proposition 3.2.8, the first one is a finite constant, and hence the claim follows. □

PROPOSITION 3.2.26. *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be a positive Radon measure on X^{an}. Then for every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$ the following are true:*

i) If $\zeta \notin X(K)$, then $u_{\zeta_0,\nu}(\cdot,\zeta)$ is finitely valued and continuous on $X^{\mathrm{an}} \backslash \mathrm{supp}(\nu)$ and it is lsc on X^{an}.

ii) If $\zeta \in X(K)$, then $u_{\zeta_0,\nu}(\cdot,\zeta)$ is continuous on $X^{\mathrm{an}} \backslash (\mathrm{supp}(\nu) \cup \{\zeta\})$ with $u_{\zeta_0,\nu}(x,\zeta) = \infty$ if and only if $x = \zeta$, and it is lsc on $X^{\mathrm{an}} \backslash \{\zeta\}$.

iii) For each $z \in X^{\mathrm{an}}$ and each path $[z,\omega]$, we have

$$\liminf_{t\to z} u_{\zeta_0,\nu}(t,\zeta) = \liminf_{\substack{t\to z,\\ t\in I(X^{\mathrm{an}})}} u_{\zeta_0,\nu}(t,\zeta) = \lim_{\substack{t\to z,\\ t\in[\omega,z)}} u_{\zeta_0,\nu}(t,\zeta) = u_{\zeta_0,\nu}(z,\zeta). \tag{3.2.6}$$

Note that every probability measure on X^{an} is a positive Radon measure by Proposition 2.1.5.

PROOF. Recall from (3.2.4) that we can write

$$u_{\zeta_0,\nu}(\cdot,\zeta) = \int g_{\zeta_0}(\cdot,y)\, d\nu(y) - \nu(X^{\mathrm{an}}) g_{\zeta_0}(\cdot,\zeta) + C_{\zeta_0,\zeta}.$$

Since $g_{\zeta_0}(\cdot,\zeta)$ is finitely valued and continuous on X^{an} if $\zeta \notin X(K)$ and $g_{\zeta_0}(\cdot,\zeta)$ is finitely valued and continuous on $X^{\mathrm{an}} \backslash \{\zeta\}$ if $\zeta \in X(K)$ by Proposition 3.2.8, it remains to show the assertions i) and ii) for the function $f(x) := \int g_{\zeta_0}(x,y)\, d\nu(y)$ on X^{an}. As g_{ζ_0} is finitely valued and continuous off the diagonal by Lemma 3.2.7 and $\mathrm{supp}(\nu)$ is a compact subset, it follows that f is finitely valued and continuous on $X^{\mathrm{an}} \backslash \mathrm{supp}(\nu)$. For the lower semi-continuity of f we use techniques from the proof of [**BR10**, Proposition 6.12]. By Lemma 3.2.7, g_{ζ_0} is lower semi-continuous on the compact space $X^{\mathrm{an}} \times X^{\mathrm{an}}$, and so it is bounded from below by a constant M. Using [**BR10**, Proposition A.3], we get the identity

$$f(x) = \sup \left\{ \int_{X^{\mathrm{an}}} g(x,y)\ d\nu(y) \mid g \in \mathcal{C}^0(X^{\mathrm{an}} \times X^{\mathrm{an}}),\ M \leq g \leq g_{\zeta_0} \right\}$$

on X^{an}. Due to the compactness of X^{an}, the integral function $x \mapsto \int_{X^{\mathrm{an}}} g(x,y)\ d\nu(y)$ is continuous on X^{an} for every $g \in \mathcal{C}^0(X^{\mathrm{an}} \times X^{\mathrm{an}})$. Then [**BR10**, Lemma A.2] tells us that f has to be lower semi-continuous on X^{an}.

Thus it remains to prove identity (3.2.6). First, we show the last equation

$$\lim_{t \to z, t \in [\omega, z)} u_{\zeta_0,\nu}(t,\zeta) = u_{\zeta_0,\nu}(z,\zeta).$$

If $z \notin X(K)$, then by shrinking our path we may assume $[z,\omega] \subset \mathbb{H}(X^{\mathrm{an}})$, and so the restriction of $u_{\zeta_0,\nu}(\cdot,\zeta)$ to $[z,\omega]$ is continuous by Lemma 3.2.25 and the equation is true. If $z \in X(K)$, we may assume that $[z,\omega]$ lies in a connected component of $X^{\mathrm{an}} \backslash \Gamma$ for a skeleton Γ of X^{an} with $\zeta_0 \in \Gamma$. Then $\tau_\Gamma(t) = \tau_\Gamma(z)$ for every $t \in (z,\omega]$, and so for every $y \in X^{\mathrm{an}}$ and $t \in (z,\omega]$

$$g_{\zeta_0}(t,y) = \begin{cases} g_{\zeta_0}(\tau_\Gamma(z), \tau_\Gamma(y))_\Gamma & \text{if } \tau_\Gamma(z) \neq \tau_\Gamma(y), \\ g_{\zeta_0}(\tau_\Gamma(z), \tau_\Gamma(y))_\Gamma + \rho(w_\Gamma(t,y), \tau_\Gamma(z)) & \text{if } \tau_\Gamma(z) = \tau_\Gamma(y). \end{cases}$$

Since $\rho(w_\Gamma(t,y), \tau_\Gamma(z))$ increases monotonically as t tends to z along $(z,\omega]$ for every $y \in X^{\mathrm{an}}$, the Monotone Convergence Theorem implies as in the proof of [**BR10**, Proposition 6.12] that the integral function $\int g_{\zeta_0}(t,y)\ d\nu(y)$ converges to $\int g_{\zeta_0}(z,y)\ d\nu(y)$ as t tends to z along $(z,\omega]$. Furthermore, $g_{\zeta_0}(t,\zeta)$ converges to $g_{\zeta_0}(z,\zeta)$ as t tends to z along $(z,\omega]$ by Proposition 3.2.8. At most one of the terms $\int g_{\zeta_0}(z,y)\ d\nu(y)$ and $g_{\zeta_0}(z,\zeta)$ is infinite (due to $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$), so the description stated at the beginning of the proof (or see (3.2.4)) implies

$$\lim_{t \to z, t \in [\omega, z)} u_{\zeta_0,\nu}(t,\zeta) = u_{\zeta_0,\nu}(z,\zeta).$$

Now, we deduce the rest of (3.2.6) from that. When $\zeta = z \in X(K)$, we have

$$\liminf_{t \to z} u_{\zeta_0,\nu}(t,\zeta) \leq \liminf_{\substack{t \to z, \\ t \in I(X^{\mathrm{an}})}} u_{\zeta_0,\nu}(t,\zeta) \leq \lim_{\substack{t \to z, \\ t \in [\omega, z)}} u_{\zeta_0,\nu}(t,\zeta) = u_{\zeta_0,\nu}(z,\zeta) = -\infty,$$

and so clearly (3.2.6) is true. When $\zeta \notin X(K)$ or $\zeta \neq z$, then $u_{\zeta_0,\nu}(\cdot,\zeta)$ is lsc at z by i) and ii), and so we get

$$u_{\zeta_0,\nu}(z,\zeta) \leq \liminf_{t \to z} u_{\zeta_0,\nu}(t,\zeta) \leq \liminf_{\substack{t \to z, \\ t \in I(X^{\mathrm{an}})}} u_{\zeta_0,\nu}(t,\zeta) \leq \lim_{\substack{t \to z, \\ t \in [\omega, z)}} u_{\zeta_0,\nu}(t,\zeta) = u_{\zeta_0,\nu}(z,\zeta).$$

Hence we also have equality. □

PROPOSITION 3.2.27. *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$ and let ν be any finite signed Borel measure on X^{an}. Then for every $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$, the potential function $u_{\zeta_0,\nu}(\cdot,\zeta)$ defines a current in $D^0(X^{\mathrm{an}})$ with*

$$dd^c u_{\zeta_0,\nu}(\cdot,\zeta) = \nu(X^{\mathrm{an}})\delta_\zeta - \nu.$$

PROOF. A function on X^{an} defines a current in $D^0(X^{\mathrm{an}})$ if and only if its restriction to $I(X^{\mathrm{an}})$ is finite (cf. Proposition 3.1.27). Recall from (3.2.4) that for every $x \in X^{\mathrm{an}}$

$$u_{\zeta_0,\nu}(\cdot,\zeta) = \int g_{\zeta_0}(\cdot,y)\ d\nu(y) - \nu(X^{\mathrm{an}})g_{\zeta_0}(\cdot,\zeta) + C_{\zeta_0,\zeta}.$$

If we fix $x \in I(X^{\mathrm{an}})$, the function $g_{\zeta_0}(x,\cdot) = g_{\zeta_0}(\cdot,x)$ (symmetry follows by Lemma 3.2.7) is a finitely valued continuous function on X^{an} by Proposition 3.2.8 i). Hence all terms define currents in $D^0(X^{\mathrm{an}})$, and so does $u_{\zeta_0,\nu}(\cdot,\zeta)$. For the first term we also use that $\mathrm{supp}(\nu)$ is compact. Furthermore, we know by Proposition 3.2.8 that for any fixed y we have $dd^c g_{\zeta_0}(\cdot,y) = \delta_{\zeta_0} - \delta_y$. Due to the calculation

$$\begin{aligned}\langle dd^c\left(\int g_{\zeta_0}(\cdot,y)\ d\nu(y)\right),\varphi\rangle &= \int \langle dd^c g_{\zeta_0}(\cdot,y),\varphi\rangle\ d\nu(y)\\ &= \int\left(\int \varphi\ d(\delta_{\zeta_0}-\delta_y)(x)\right)d\nu(y)\\ &= \int \varphi\ d(\nu(X^{\mathrm{an}})\delta_{\zeta_0}-\nu)(y)\end{aligned}$$

for every $\varphi \in A_c^0(X^{\mathrm{an}})$, we obtain

$$dd^c\left(\int g_{\zeta_0}(\cdot,y)\ d\nu(y)\right) = \nu(X^{\mathrm{an}})\delta_{\zeta_0} - \nu.$$

Hence

$$dd^c u_{\zeta_0,\nu}(\cdot,\zeta) = \nu(X^{\mathrm{an}})\delta_{\zeta_0} - \nu - \nu(X^{\mathrm{an}})(\delta_{\zeta_0}-\delta_\zeta) = \nu(X^{\mathrm{an}})\delta_\zeta - \nu.$$

□

In potential theory there are some important statements involving potential functions as for instance the Riesz Decomposition Theorem, which was shown by Baker and Rumely for $X = \mathbb{P}^1$ in [**BR10**, Theorem 8.38]. Using the results from above and some statements from [**Thu05**], we can generalize the Riesz Decomposition Theorem (see Theorem 3.2.29) to our arbitrary smooth proper curve X, and deduce Corollary 3.2.30 from it as Baker and Rumely did in [**BR10**, Proposition 8.42].

REMARK 3.2.28. Having a subharmonic function $f\colon W \to [-\infty,\infty)$, then $dd^c f$ can be identified with a positive Radon measure on W by [**Thu05**, Théorème 3.4.12]. For every strictly affinoid domain Y in W, the restriction $(dd^c f)|_Y$ is still a positive Radon measure by Proposition 2.1.5. We consider in the following $u_{\zeta_0,\nu}(\cdot,\zeta)$ for $\nu := (dd^c f)|_Y$, for a fixed base point $\zeta_0 \in I(X^{\mathrm{an}})$ and for $\zeta \in I(X^{\mathrm{an}})$ or $\zeta \notin \mathrm{supp}(\nu)$.

Recall from Definition 2.3.33 that a strictly simple domain is a strictly affinoid domain of the form $\tau_\Gamma^{-1}(\overline{\Omega})$ for a skeleton Γ of X^{an} and a suitable star-shaped open subset Ω of Γ or $\Omega = \{x_0\}$ for a type II point $x_0 \in \Gamma$.

THEOREM 3.2.29 (Riesz Decomposition Theorem). *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$, let W be an open subset of X^{an}, and let f be a subharmonic function on W. Let Y be a strictly simple domain in W and fix $\zeta \in X^{\mathrm{an}}\backslash Y$. Then for $\nu := (dd^c f)|_Y$ there is a continuous function h on Y that is harmonic on $V := Y^\circ$ and*

$$f = h - u_{\zeta_0,\nu}(\cdot,\zeta)$$

on Y.

PROOF. Recall from Remark 2.3.5 that ∂Y is a finite set of points of type II. The subharmonic function f can therefore only attain finite values on ∂Y by Corollary 3.1.12. By Proposition 3.2.27 and Proposition 3.1.27, the potential function $u_{\zeta_0,\nu}(\cdot,\zeta)$ is also finitely valued on ∂Y. Thus there exists a continuous function $h\colon Y\to\mathbb{R}$ that is harmonic on V and coincides with $f+u_{\zeta_0,\nu}(\cdot,\zeta)$ on ∂Y by [**Thu05**, Corollaire 3.1.21]. Let Γ be a skeleton of X^{an} such that the strictly simple domain Y is given by $\tau_\Gamma^{-1}(\overline{\Omega})$ for a suitable star-shaped open subset Ω of Γ or $\Omega=\{x_0\}$ for some type II point $x_0\in\Gamma$. Then $h=\Phi\circ\tau_\Gamma$ on Y for a piecewise affine function Φ on $\overline{\Omega}$. We choose an open neighborhood Ω' of $\overline{\Omega}$ in Γ such that Ω' is still simply-connected ($\overline{\Omega}$ is required to be simply-connected by Definition 2.3.33). Then we can extend Φ to Ω' by constant values, and the extension $h=\Phi\circ\tau_\Gamma$ defines a lisse function on $\tau_\Gamma^{-1}(\Omega')$, which is harmonic on the open subset V. We may choose Ω' small enough such that $\zeta\notin\tau_\Gamma^{-1}(\Omega')$. Set $W':=\tau_\Gamma^{-1}(\Omega')\cap W$, then h is lisse on W', f is subharmonic on W' (and so $f\in D^0(W')$ cf. Proposition 3.1.30), and $Y\subset W'$.

The strategy of proving $f=h-u_{\zeta_0,\nu}(\cdot,\zeta)$ on Y is to show that $g:=h-u_{\zeta_0,\nu}(\cdot,\zeta)$

(a) is a current in $D^0(W')$ with $\langle dd^cg,\varphi\rangle=\langle dd^cf,\varphi\rangle$ for every $\varphi\in A_c^0(W')$ with $\mathrm{supp}(\varphi)\subset Y$,
(b) g is subharmonic on V, and
(c) $f=g$ on ∂Y.

Then Corollary 3.1.35 implies the claim. By construction, we know that (c) is satisfied. Hence it remains to show (a) and (b). Proposition 3.2.27 tells us that the potential function $u_{\zeta_0,\nu}(\cdot,\zeta)$ defines a current in $D^0(W')$ with

$$dd^cu_{\zeta_0,\nu}(\cdot,\zeta)=\nu(X^{\mathrm{an}})\delta_\zeta-\nu=-\nu$$

on W' (note $\zeta\notin W'$). Hence g defines a current in $D^0(W')$ with $dd^cg=dd^ch+\nu$ on W'. In particular, it defines a current in $D^0(V)$ with

$$dd^cg=dd^ch+\nu=\nu\geq 0$$

on V as h is harmonic on V, and so g is subharmonic on V by Proposition 3.1.30. Thus (b) is true as well. To verify (a), let $\varphi\in A_c^0(W')$ with $\mathrm{supp}(\varphi)\subset Y$. Then $\varphi\equiv 0$ outside of $V=Y^\circ$, and so

$$\langle dd^ch,\varphi\rangle=\int_{W'}\varphi\;dd^ch=\int_{\partial Y}\varphi\;dd^ch=0$$

as dd^ch is supported on ∂Y by construction and $\varphi\equiv 0$ on ∂Y. Consequently,

$$\langle dd^cg,\varphi\rangle=\langle dd^c(-u_{\zeta_0,\nu}(\cdot,\zeta)),\varphi\rangle=\int_{W'}\varphi\;d\nu=\int_Y\varphi\;d\nu=\int_{W'}\varphi\;dd^cf=\langle dd^cf,\varphi\rangle$$

since $\varphi\equiv 0$ outside of $V=Y^\circ$. This proves (a). Corollary 3.1.35 implies $g=f$ on Y, and the assertion follows. □

COROLLARY 3.2.30. *Let ζ_0 be a chosen base point in $I(X^{\mathrm{an}})$, let W be an open subset of X^{an}, and let f be a continuous subharmonic function on W. Let Y be a strictly simple domain in W. Fix $\zeta\in X^{\mathrm{an}}\backslash Y$, and set $\nu:=(dd^cf)|_Y$. Then the potential function $u_{\zeta_0,\nu}(\cdot,\zeta)$ is continuous on X^{an}.*

PROOF. Recall from Definition 2.3.33 that Y is of the form $\tau_\Gamma^{-1}(\overline{\Omega})$ for a skeleton Γ of X^{an} and for a star-shaped open subset Ω of Γ or for $\Omega=\{x_0\}$ for a type II point $x_0\in\Gamma$. By Corollary 2.3.32, we have $\partial Y=\partial\Omega$, which is a finite set of type II points.

It is already known that $u_{\zeta_0,\nu}(\cdot,\zeta)$ is continuous on $X^{\mathrm{an}}\backslash Y$ by Proposition 3.2.26 as $\mathrm{supp}(\nu)\subset Y$. By Theorem 3.2.29, there is a continuous function h on Y that is harmonic on $V = Y^\circ$ such that $u_{\zeta_0,\nu}(\cdot,\zeta) = h - f$ on Y. In particular, $u_{\zeta_0,\nu}(\cdot,\zeta)$ is continuous in every point of V. Hence it remains to show continuity in ∂Y. We therefore consider a point $y \in \partial Y$. At first, note that the description $u_{\zeta_0,\nu}(\cdot,\zeta) = h - f$ on Y with f and h continuous on Y already implies

$$\lim_{x \to y, x \in V} u_{\zeta_0,\nu}(x,\zeta) = u_{\zeta_0,\nu}(y,\zeta). \tag{3.2.7}$$

Since every boundary point of Y is of type II, there is a simple open neighborhood U of y in $W \backslash \{\zeta\}$ (cf. Theorem 2.3.27), i.e. $U = \tau_{\Gamma'}^{-1}(\Omega')$ for a skeleton Γ' of X^{an} and an open subset $\Omega' = \bigcup_{i=1,\dots,n}[y,a_i)$ of Γ'. We may assume $\Gamma \subset \Gamma'$. We have to show that

$$\lim_{x \to y, x \in \widetilde{U}} u_{\zeta_0,\nu}(x,\zeta) = u_{\zeta_0,\nu}(y,\zeta)$$

for every connected component $\widetilde{U}$ of $U \backslash \{y\}$. A connected component of $U \backslash \{y\}$ is either an open ball or an open annulus $\tau_{\Gamma'}^{-1}((y,a_i))$ for some $i = 1,\dots,n$.

Recall that we required $Y = \tau_\Gamma^{-1}(x_0)$ for some type II point $x_0 \in \Gamma$ or $Y = \tau_\Gamma^{-1}(\overline{\Omega})$ for an star-shaped open subset Ω of the skeleton $\Gamma \subset \Gamma'$ of X^{an}. Due to the form of Y, either $\widetilde{U}$ is

(a) contained in one of the connected components of $Y \backslash \partial Y$ that is an open ball, or
(b) an open annulus and has non-empty intersection with Ω, or
(c) an open annulus and is disjoint from Y.

Note that there can be at most one $\widetilde{U} = \tau_{\Gamma'}^{-1}((y,a_i))$ fulfilling the second case as y is x_0 in the case of $Y = \tau_\Gamma^{-1}(x_0)$ or y is an endpoint of Ω otherwise. We may shrink (y,a_i), and so the simple open neighborhood U of y, such that $\widetilde{U}$ is contained in V as well. Thus we have already dealt with (a) and (b) in (3.2.7).

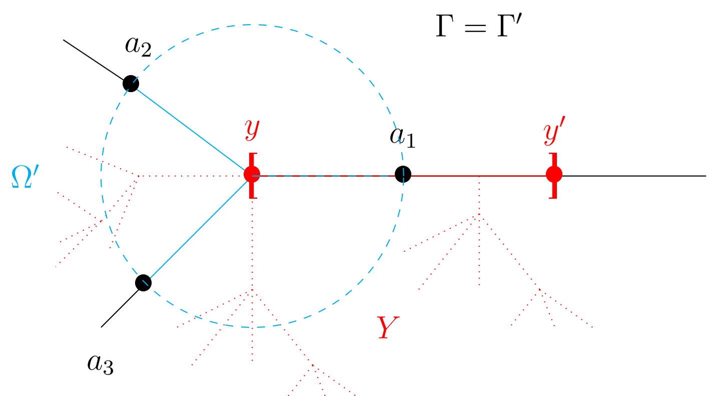

FIGURE 1. An example how the situation could look like if Ω is an open interval (y,y') in the skeleton $\Gamma = \Gamma'$.

It remains to show (3.2.7) for (c), i.e. $\widetilde{U} = \tau_{\Gamma'}^{-1}((y,a_i))$ disjoint from Y. Then our function $u_{\zeta_0,\nu}(\cdot,\zeta)$ is continuous on $\widetilde{U} \subset X^{\mathrm{an}} \backslash Y$ and

$$(dd^c u_{\zeta_0,\nu}(\cdot,\zeta))|_{\widetilde{U}} = (\delta_\zeta - \nu)|_{\widetilde{U}} = 0$$

by Proposition 3.2.27, where $\mathrm{supp}(\nu) \subset Y$ and $\zeta \notin U$. Consequently, $u_{\zeta_0,\nu}(\cdot,\zeta)$ is harmonic on $\widetilde{U}$ by Corollary 3.1.31. The Maximum Principle (cf. Proposition 3.1.8) tells us that $u_{\zeta_0,\nu}(\cdot,\zeta)$ is constant on every preimage $\tau_{\Gamma'}^{-1}(x)$ of a point $x \in (y,a_i)$ as every connected component of $\tau_{\Gamma'}^{-1}(x)\backslash\{x\}$ is an open ball with unique boundary point x. Consequently, we have

$$u_{\zeta_0,\nu}(\cdot,\zeta) = u_{\zeta_0,\nu}(\tau_{\Gamma'}(\cdot),\zeta)$$

on $\widetilde{U} = \tau_{\Gamma'}^{-1}((y,a_i))$. Using that $u_{\zeta_0,\nu}(\cdot,\zeta)$ is continuous on $[y,a_i]$ by Lemma 3.2.25, we obtain

$$\lim_{x\to y, x\in\widetilde{U}} u_{\zeta_0,\nu}(x,\zeta) = \lim_{x\to y, x\in(y,a_i]} u_{\zeta_0,\nu}(x,\zeta) = u_{\zeta_0,\nu}(y,\zeta).$$

Hence $\lim_{x\to y, x\in\widetilde{U}} u_{\zeta_0,\nu}(x,\zeta) = u_{\zeta_0,\nu}(y,\zeta)$ holds for every connected component $\widetilde{U}$ of $U\backslash\{y\}$, and so the assertion follows. □

3.2.4. The Arakelov–Green's function. Baker and Rumely developed a theory of Arakelov–Green's functions on $\mathbb{P}^{1,\mathrm{an}}$ in [**BR10**, §8.10]. This class of functions arise naturally in the study of dynamics and can be seen as a generalization of the potential kernel from Subsection 3.2.1. Arakelov–Green's functions are characterized by a list of properties which can be found in Definition 3.2.31. We generalize Baker and Rumely's definition of an Arakelov–Green's function from $\mathbb{P}^{1,\mathrm{an}}$ to X^{an}, and show that the characteristic properties are still satisfied by using results about potential functions from the previous Subsection 3.2.3.

Definition 3.2.31. A symmetric function g on $X^{\mathrm{an}}\times X^{\mathrm{an}}$ that satisfies the following list of properties for a probability measure μ on X^{an} is called a *normalized Arakelov–Green's function* on X^{an}.

i) (Semicontinuity) The function g is finite and continuous off the diagonal and strongly lower semi-continuous on the diagonal in the sense that
$$g(x_0,x_0) = \liminf_{(x,y)\to(x_0,x_0), x\neq y} g(x,y).$$
ii) (Differential equation) For each fixed $y\in X^{\mathrm{an}}$ the function $g(\cdot,y)$ is an element of $D^0(X^{\mathrm{an}})$ and
$$dd^c g(\cdot,y) = \mu - \delta_y.$$
iii) (Normalization)
$$\int\int g(x,y)\ d\mu(x)d\mu(y) = 0.$$

The list of properties is an analog of the one in the complex case and can for example also be found in [**BR06**, §3.5 (B1)-(B3)].

Remark 3.2.32. As in the complex case, the list of properties in Definition 3.2.31 for a probability measure μ on X^{an} determines a normalized Arakelov–Green's function on X^{an} uniquely. If $\widetilde{g}$ is another symmetric function on $X^{\mathrm{an}}\times X^{\mathrm{an}}$ satisfying i)-iii), then for a fixed $y\in X^{\mathrm{an}}$

$$g(\cdot,y) - \widetilde{g}(\cdot,y) = h_y$$

on $I(X^{\mathrm{an}})$ for a harmonic function h_y on X^{an} by property ii) and [**Thu05**, Lemme 3.3.12]. This harmonic function h_y has to be constant on X^{an} by the Maximum Principle (Proposition 3.1.8). Since $I(X^{\mathrm{an}})$ is dense in X^{an}, the identity holds on all of X^{an} by property i).

Thanks to the symmetry of g and $\widetilde{g}$, the constant function h_y is independent of y. The last property iii), implies that this constant has to be zero, i.e. $g=\widetilde{g}$ on $X^{\mathrm{an}}\times X^{\mathrm{an}}$.

In the following, we define functions satisfying these properties using techniques from [**BR10**].

DEFINITION 3.2.33. A probability measure μ on X^{an} *has continuous potentials* if each $\zeta \in I(X^{\mathrm{an}})$ defines a continuous function

$$X^{\mathrm{an}} \to \mathbb{R},\ x \mapsto \int_{X^{\mathrm{an}}} g_\zeta(x,y)\ d\mu(y).$$

These functions are bounded as X^{an} is compact.

REMARK 3.2.34. Let μ be a probability measure on X^{an}. If there exists a point $\zeta_0 \in I(X^{\mathrm{an}})$ such that $X^{\mathrm{an}} \to \mathbb{R},\ x \mapsto \int_{X^{\mathrm{an}}} g_{\zeta_0}(x,y)\ d\mu(y)$ defines a continuous function, then μ has continuous potentials:

By Lemma 3.2.7, the following identity holds for every $\zeta \in I(X^{\mathrm{an}})$

$$\begin{aligned}\int g_\zeta(\cdot,y)\ d\mu(y) &= \int \left(g_{\zeta_0}(\cdot,y) - g_{\zeta_0}(\cdot,\zeta) - g_{\zeta_0}(y,\zeta) + g_{\zeta_0}(\zeta,\zeta)\right)\ d\mu(y)\\ &= \int g_{\zeta_0}(\cdot,y)\ d\mu(y) - g_{\zeta_0}(\cdot,\zeta) - \int g_{\zeta_0}(y,\zeta)\ d\mu(y) + g_{\zeta_0}(\zeta,\zeta).\end{aligned}$$

Since $g_{\zeta_0}(\cdot,\zeta)$ is continuous and bounded on X^{an} by Proposition 3.2.8, the function $\int g_\zeta(\cdot,y)$ differs from $\int g_{\zeta_0}(\cdot,y)$ by a finitely valued continuous function on X^{an}.

EXAMPLE 3.2.35. Let μ be a probability measure supported on a skeleton Γ of X^{an} (e.g. $\mu = \delta_z$ for some $z \in I(X^{\mathrm{an}})$), then μ has continuous potentials:

As explained in the last remark, it remains to consider a point $\zeta \in \Gamma \subset I(X^{\mathrm{an}})$. Then $g_\zeta(x,y) = g_\zeta(\tau_\Gamma(x),y)_\Gamma$ for every $x \in X^{\mathrm{an}}$ and $y \in \mathrm{supp}(\mu) \subset \Gamma$. Since $g_\zeta(\cdot,\cdot)_\Gamma$ is continuous and bounded on $\Gamma \times \Gamma$ by Lemma 3.2.2,

$$x \mapsto \int_{X^{\mathrm{an}}} g_\zeta(x,y)\ d\mu(y) = \int_\Gamma g_\zeta(\tau_\Gamma(x),y)_\Gamma\ d\mu(y)$$

defines a finitely valued continuous function on X^{an}.

DEFINITION 3.2.36. For every probability measure μ on X^{an} with continuous potentials and a fixed base point $\zeta_0 \in I(X^{\mathrm{an}})$, we define $g_{\zeta_0,\mu}\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty,\infty]$ by

$$g_{\zeta_0,\mu}(x,y) := g_{\zeta_0}(x,y) - \int_{X^{\mathrm{an}}} g_{\zeta_0}(x,\zeta)\ d\mu(\zeta) - \int_{X^{\mathrm{an}}} g_{\zeta_0}(y,\zeta)\ d\mu(\zeta) + C_{\zeta_0},$$

where C_{ζ_0} is a constant chosen such that

$$\int\int g_{\zeta_0,\mu}(x,y)\ d\mu(x)d\mu(y) = 0.$$

REMARK 3.2.37. Recall that $g_{\zeta_0}(\zeta_0,x,y) = g_{\zeta_0}(x,y)$ (see Definition 3.2.9) by Corollary 3.2.10, and so the potential function from Subsection 3.2.3 can be written as

$$u_{\zeta_0,\mu}(\cdot,\zeta_0) = \int g_{\zeta_0}(\zeta_0,\cdot,\zeta)\ d\mu(\zeta) = \int g_{\zeta_0}(\cdot,\zeta)\ d\mu(\zeta).$$

Hence we have the description

$$g_{\zeta_0,\mu}(x,y) = g_{\zeta_0}(x,y) - u_{\zeta_0,\mu}(x,\zeta_0) - u_{\zeta_0,\mu}(y,\zeta_0) + C_{\zeta_0} \tag{3.2.8}$$

on $X^{\mathrm{an}} \times X^{\mathrm{an}}$.

In the following lemma, we see that this function is independent of the chosen base point, and hence we just write g_μ.

LEMMA 3.2.38. *For every probability measure μ on X^{an} with continuous potentials, the function $g_{\zeta_0,\mu}$ is independent of the chosen base point ζ_0.*

PROOF. First, we determine C_{ζ_0}

$$\begin{aligned}0 = \int\int g_{\zeta_0,\mu}(x,y)\ d\mu(x)d\mu(y) &= \int\int g_{\zeta_0}(x,y)\ d\mu(x)d\mu(y) - \int\int g_{\zeta_0}(x,\zeta)\ d\mu(\zeta)d\mu(x) \\ &\quad - \int\int g_{\zeta_0}(y,\zeta)\ d\mu(\zeta)d\mu(y) + C_{\zeta_0} \\ &= -\int\int g_{\zeta_0}(x,y)\ d\mu(x)d\mu(y) + C_{\zeta_0}.\end{aligned}$$

Hence $C_{\zeta_0} = \int\int g_{\zeta_0}(x,y)\ d\mu(x)d\mu(y)$. Now let $\zeta_0' \in I(X^{\mathrm{an}})$. Applying Lemma 3.2.7 to C_{ζ_0}, we get

$$\begin{aligned}C_{\zeta_0} &= \int\int g_{\zeta_0}(x,y)\ d\mu(x)d\mu(y) \\ &= \int\int \Big(g_{\zeta_0'}(x,y) - g_{\zeta_0'}(x,\zeta_0) - g_{\zeta_0'}(y,\zeta_0) + g_{\zeta_0'}(\zeta_0,\zeta_0)\Big)\ d\mu(x)d\mu(y) \\ &= C_{\zeta_0'} - 2\int g_{\zeta_0'}(x,\zeta_0)\ d\mu(x) + g_{\zeta_0'}(\zeta_0,\zeta_0),\end{aligned}$$

where $-2\int g_{\zeta_0'}(\zeta,\zeta_0)\ d\mu(\zeta) + g_{\zeta_0'}(\zeta_0,\zeta_0)$ is a finite constant as μ has continuous potentials.

Using Lemma 3.2.7 also for the other terms of $g_{\zeta_0,\mu}$, i.e. for $g_{\zeta_0}(x,y)$, $g_{\zeta_0}(x,\zeta)$ and $g_{\zeta_0}(y,\zeta)$, and plugging in the identity from above, we get

$$\begin{aligned}g_{\zeta_0,\mu}(x,y) &= g_{\zeta_0}(x,y) - \int g_{\zeta_0}(x,\zeta)\ d\mu(\zeta) - \int g_{\zeta_0}(y,\zeta)\ d\mu(\zeta) + C_{\zeta_0} \\ &= g_{\zeta_0'}(x,y) - \int g_{\zeta_0'}(x,\zeta)\ d\mu(\zeta) - \int g_{\zeta_0'}(y,\zeta)\ d\mu(\zeta) + 2\int g_{\zeta_0'}(\zeta,\zeta_0)\ d\mu(\zeta) \\ &\quad - g_{\zeta_0'}(\zeta_0,\zeta_0) + C_{\zeta_0} \\ &= g_{\zeta_0'}(x,y) - \int g_{\zeta_0'}(x,\zeta)\ d\mu(\zeta) - \int g_{\zeta_0'}(y,\zeta)\ d\mu(\zeta) + 2\int g_{\zeta_0'}(\zeta,\zeta_0)\ d\mu(\zeta) \\ &\quad - g_{\zeta_0'}(\zeta_0,\zeta_0) + C_{\zeta_0'} - 2\int g_{\zeta_0'}(x,\zeta_0)\ d\mu(x) + g_{\zeta_0'}(\zeta_0,\zeta_0) \\ &= g_{\zeta_0',\mu}(x,y).\end{aligned}$$

□

PROPOSITION 3.2.39. *Let μ be a probability measure on X^{an} with continuous potentials. Then as a function of two variables $g_\mu\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty,\infty]$ is symmetric, finite and continuous off the diagonal, and strongly lower semi-continuous on the diagonal in the sense that*

$$g_\mu(x_0,x_0) = \liminf_{(x,y)\to(x_0,x_0),x\neq y} g_\mu(x,y),$$

where we understand $X^{\mathrm{an}} \times X^{\mathrm{an}}$ set theoretically and endowed with the product topology.

PROOF. As

$$g_\mu(x,y) = g_{\zeta_0}(x,y) - \int g_{\zeta_0}(x,\zeta)\ d\mu(\zeta) - \int g_{\zeta_0}(y,\zeta)\ d\mu(\zeta) + C_{\zeta_0}$$

for some base point $\zeta_0 \in I(X^{\mathrm{an}})$ and as we required μ to has continuous potentials, Lemma 3.2.7 implies that $g_\mu\colon X^{\mathrm{an}} \times X^{\mathrm{an}} \to (-\infty,\infty]$ is symmetric, finite and continuous off the diagonal and lsc on $X^{\mathrm{an}} \times X^{\mathrm{an}}$. Thus we only need to prove

$$g_\mu(x_0,x_0) \geq \liminf_{(x,y)\to(x_0,x_0),x\neq y} g_\mu(x,y) = \sup_{U\in\mathcal{U}((x_0,x_0))} \inf_{(x,y)\in U\setminus(x_0,x_0)} g_\mu(x,y). \tag{3.2.9}$$

Here $\mathcal{U}((x_0, x_0))$ is any basis of open neighborhoods of (x_0, x_0) in $X^{\mathrm{an}} \times X^{\mathrm{an}}$ endowed with the product topology.

In the following, let Γ be any skeleton of X^{an} with $\zeta_0 \in \Gamma$. If x_0 is of type I, we have $g_\mu(x_0, x_0) = g_{\zeta_0}(x_0, x_0) = \infty$ by the definition of the potential kernel, and so (3.2.9) is obviously true.

If x_0 is of type II or III, we may choose $\zeta_0 = x_0$ by Lemma 3.2.38, and so

$$g_\mu(x_0, x_0) = g_{x_0}(x_0, x_0) - \int g_{x_0}(x_0, \zeta)\ d\mu(\zeta) - \int g_{x_0}(x_0, \zeta)\ d\mu(\zeta) + C_{\zeta_0} = C_{\zeta_0}$$

as $g_{x_0}(x_0, \zeta) = 0$ for every $\zeta \in X^{\mathrm{an}}$ by Lemma 3.2.7. On the other hand, every U in $\mathcal{U}((x_0, x_0))$ contains an element of the form (x_0, y) with $y \in X^{\mathrm{an}} \backslash \{x_0\}$, and

$$\begin{aligned} g_\mu(x_0, y) &= g_{x_0}(x_0, y) - \int g_{x_0}(x_0, \zeta)\ d\mu(\zeta) - \int g_{x_0}(y, \zeta)\ d\mu(\zeta) + C_{\zeta_0} \\ &= - \int g_{x_0}(y, \zeta)\ d\mu(\zeta) + C_{\zeta_0} \leq C_{\zeta_0} \end{aligned}$$

since μ and $g_{x_0}(y, \cdot)$ are non-negative (see Lemma 3.2.7 i)). Thus (3.2.9) has to be true.

For the rest of the proof let x_0 be of type IV. There is a basis of open neighborhoods of x_0 that is contained in the connected component V of $X^{\mathrm{an}} \backslash \Gamma$ that contains x_0 (cf. Theorem 2.3.27). Consider the corresponding basis of open neighborhoods $\mathcal{U}((x_0, x_0))$ of (x_0, x_0) in $X^{\mathrm{an}} \times X^{\mathrm{an}}$ endowed with the product topology. In every $U \in \mathcal{U}((x_0, x_0))$ we consider tuples of the form (x_0, y) where y lies in the interior of the unique path $[x_0, \tau_\Gamma(x_0)]$ (such tuples always exist). Then $\tau_\Gamma(y) = \tau_\Gamma(x_0)$ and $w_\Gamma(x_0, y) = y$ (recall its definition from Remark 3.2.3), and so

$$\begin{aligned} g_{\zeta_0}(x_0, x_0) - g_{\zeta_0}(x_0, y) &= g_{\zeta_0}(\tau_\Gamma(x_0), \tau_\Gamma(x_0))_\Gamma + \rho(w_\Gamma(x_0, x_0), \tau_\Gamma(x_0)) \\ &\quad - (g_{\zeta_0}(\tau_\Gamma(x_0), \tau_\Gamma(y))_\Gamma + \rho(w_\Gamma(x_0, y), \tau_\Gamma(y))) \\ &= \rho(w_\Gamma(x_0, x_0), \tau_\Gamma(x_0)) - \rho(w_\Gamma(x_0, y), \tau_\Gamma(y)) \\ &= \rho(x_0, \tau_\Gamma(x_0)) - \rho(y, \tau_\Gamma(x_0)) \\ &= \rho(x_0, y). \end{aligned}$$

Consequently, we get

$$\begin{aligned} g_\mu(x_0, x_0) - g_\mu(x_0, y) &= g_{\zeta_0}(x_0, x_0) - 2 \int g_{\zeta_0}(x_0, \zeta)\ d\mu(\zeta) + C_{\zeta_0} \\ &\quad - (g_{\zeta_0}(x_0, y) - \int g_{\zeta_0}(x_0, \zeta)\ d\mu(\zeta) - \int g_{\zeta_0}(y, \zeta)\ d\mu(\zeta) + C_{\zeta_0}) \\ &= g_{\zeta_0}(x_0, x_0) - g_{\zeta_0}(x_0, y) - \int g_{\zeta_0}(x_0, \zeta)\ d\mu(\zeta) + \int g_{\zeta_0}(y, \zeta)\ d\mu(\zeta) \\ &= \rho(x_0, y) + \int g_{\zeta_0}(y, \zeta) - g_{\zeta_0}(x_0, \zeta)\ d\mu(\zeta). \qquad (3.2.10) \end{aligned}$$

To prove (3.2.9), we need to show that (3.2.10) is non-negative.

Recall that y lies in the interior of the unique path $[x_0, \tau_\Gamma(x_0)]$. We denote by V_0 the connected component of $V \backslash \{y\}$ that contains x_0 (note that V_0 is an open ball as x_0 is of type IV and V is an open ball). We will see that $g_{\zeta_0}(y, \zeta) - g_{\zeta_0}(x_0, \zeta)$ in (3.2.10) is zero for every $\zeta \in X^{\mathrm{an}} \backslash V_0$. Recall that V is the connected component of $X^{\mathrm{an}} \backslash \Gamma$ that contains x_0. Hence V is an open ball with $\partial V = \{\tau_\Gamma(x_0)\}$ and $V_0 \subset V$. Furthermore, one should have in mind that $\tau_\Gamma(y) = \tau_\Gamma(x_0)$ as $y \in [x_0, \tau_\Gamma(x_0)]$.

If $\zeta \in X^{\mathrm{an}}\backslash V$, then by the definition of the potential kernel

$$g_{\zeta_0}(y,\zeta) - g_{\zeta_0}(x_0,\zeta) = g_{\zeta_0}(\tau_\Gamma(y),\tau_\Gamma(\zeta))_\Gamma - g_{\zeta_0}(\tau_\Gamma(x_0),\tau_\Gamma(\zeta))_\Gamma = 0.$$

If $\zeta \in V\backslash V_0$, then $\tau_\Gamma(\zeta) = \tau_\Gamma(x_0) = \tau_\Gamma(y)$ and $w_\Gamma(x_0,\zeta) = w_\Gamma(y,\zeta)$, and hence

$$\begin{aligned} g_{\zeta_0}(y,\zeta) - g_{\zeta_0}(x_0,\zeta) &= g_{\zeta_0}(\tau_\Gamma(y),\tau_\Gamma(\zeta))_\Gamma + \rho(w_\Gamma(y,\zeta),\tau_\Gamma(y)) \\ &\quad - (g_{\zeta_0}(\tau_\Gamma(x_0),\tau_\Gamma(\zeta))_\Gamma + \rho(w_\Gamma(x_0,\zeta),\tau_\Gamma(x_0))) \\ &= 0. \end{aligned}$$

Thus $g_{\zeta_0}(y,\zeta) - g_{\zeta_0}(x_0,\zeta) = 0$ for every $\zeta \in X^{\mathrm{an}}\backslash V_0$.

For every $\zeta \in V_0$ we have $\tau_\Gamma(\zeta) = \tau_\Gamma(x_0) = \tau_\Gamma(y)$, $w_\Gamma(y,\zeta) = y$, and $w_\Gamma(x_0,\zeta) \in [x_0,y]$. Hence

$$\begin{aligned} g_{\zeta_0}(y,\zeta) - g_{\zeta_0}(x_0,\zeta) &= g_{\zeta_0}(\tau_\Gamma(y),\tau_\Gamma(\zeta))_\Gamma + \rho(w_\Gamma(y,\zeta),\tau_\Gamma(y)) \\ &\quad - (g_{\zeta_0}(\tau_\Gamma(x_0),\tau_\Gamma(\zeta))_\Gamma + \rho(w_\Gamma(x_0,\zeta),\tau_\Gamma(x_0))) \\ &= \rho(w_\Gamma(y,\zeta),\tau_\Gamma(y)) - \rho(w_\Gamma(x_0,\zeta),\tau_\Gamma(x_0)) \\ &= \rho(y,\tau_\Gamma(x_0)) - \rho(w_\Gamma(x_0,\zeta),\tau_\Gamma(x_0)) \\ &= -\rho(w_\Gamma(x_0,\zeta),y) \end{aligned}$$

for every $\zeta \in V_0$. Plugging everything in (3.2.10), we get

$$\begin{aligned} g_\mu(x_0,x_0) - g_\mu(x_0,y) &= \rho(x_0,y) + \int_{V_0} -\rho(w_\Gamma(x_0,\zeta),y)\ d\mu(\zeta) \\ &\geq \int_{V_0} \rho(x_0,y) - \rho(w_\Gamma(x_0,\zeta),y)\ d\mu(\zeta) \\ &\geq 0 \end{aligned}$$

as $\rho(x_0,y) \geq \rho(w_\Gamma(x_0,\zeta),y)$ on V_0 and μ is a non-negative measure.

Consequently, (3.2.9) has to be also true for x_0 of type IV. $\square$

Proposition 3.2.40. *For every probability measure μ on X^{an} with continuous potentials and for every fixed $y \in X^{\mathrm{an}}$, the function $G_{\mu,y} := g_\mu(\cdot,y)\colon X^{\mathrm{an}} \to (-\infty,\infty]$ defines a current in $D^0(X^{\mathrm{an}})$ and satisfies*

$$dd^c G_{\mu,y} = \mu - \delta_y.$$

Moreover, $G_{\mu,y}$ is continuous on X^{an} with $G_{\mu,y}(x) = \infty$ if and only if $x = y \in X(K)$. In particular, $G_{\mu,y}$ is subharmonic on $X^{\mathrm{an}}\backslash\{y\}$.

Proof. By the definition of the Arakelov–Green's function, we have

$$G_{\mu,y}(x) = g_{\zeta_0}(x,y) - \int g_{\zeta_0}(x,\zeta)\ d\mu(\zeta) - \int g_{\zeta_0}(y,\zeta)\ d\mu(\zeta) + C_{\zeta_0}$$

for every $x \in X^{\mathrm{an}}$. Due to Proposition 3.2.8, the first term is continuous on X^{an} and attains values in $\mathbb{R}\cup\{\infty\}$ with $g_{\zeta_0}(x,y) = \infty$ if and only if $x = y \in X(K)$. In particular, the first term is finitely valued on $I(X^{\mathrm{an}})$. Since μ has continuous potentials, the other two terms are finitely valued and continuous on X^{an}. Hence $G_{\mu,y}\colon X^{\mathrm{an}} \to (-\infty,\infty]$ is continuous on X^{an} with $G_{\mu,y}(x) = \infty$ if and only if $x = y \in X(K)$. In particular, $G_{\mu,y}$ is finitely valued on $I(X^{\mathrm{an}})$, and so defines a current in $D^0(X^{\mathrm{an}})$ by Proposition 3.1.27. It remains to calculate the Laplacian of $G_{\mu,y}$. By (3.2.8), we have

$$G_{\mu,y} = g_{\zeta_0}(\cdot,y) - u_{\zeta_0,\mu}(\cdot,\zeta_0) - u_{\zeta_0,\mu}(y,\zeta_0) + C_{\zeta_0}.$$

Proposition 3.2.8 and Proposition 3.2.27 imply

$$\begin{aligned} dd^c G_{\mu,y} &= dd^c G_{\zeta_0,y} - dd^c u_{\zeta_0,\mu}(\cdot,\zeta_0) \\ &= \delta_{\zeta_0} - \delta_y - (\delta_{\zeta_0} - \mu) \\ &= \mu - \delta_y. \end{aligned}$$

Then $G_{\mu,y}$ is subharmonic on $X^{\mathrm{an}}\backslash\{y\}$ by Proposition 3.1.30. □

REMARK 3.2.41. For a probability measure μ on X^{an} and for a point $y \in I(X^{\mathrm{an}})$, Thuillier constructs in his thesis [**Thu05**, §3.4.3] a unique function $g_{y,\mu}\colon X^{\mathrm{an}} \to [-\infty,\infty)$ such that $dd^c g_{y,\mu} = \mu - \delta_y$, $g_{y,\mu}(y) = 0$ and its restriction to $X^{\mathrm{an}}\backslash\{y\}$ is subharmonic. His construction uses [**Thu05**, Théorème 3.3.13 & 3.4.12]. If μ has continuous potentials, then $g_{y,\mu}$ and $G_{\mu,y}$ define two currents in $D^0(X^{\mathrm{an}})$ (cf. Proposition 3.1.30) having the same Laplacian $\mu - \delta_y$. [**Thu05**, Lemma 3.3.12] implies that $g_{y,\mu}$ and $G_{\mu,y}$ differ only by a harmonic function on X^{an}, which has to be constant by the Maximum Principle.

Using the previous propositions, g_μ is indeed a normalized Arakelov–Green's function as defined in Definition 3.2.31 for every probability measure μ on X^{an} with continuous potentials.

COROLLARY 3.2.42. *Let μ be a probability measure on X^{an} with continuous potentials. Then the function g_μ is a normalized Arakelov–Green's function on X^{an}.*

PROOF. We need to know that all properties of the list in Definition 3.2.31 hold. Property i) and symmetry are true due to Proposition 3.2.39, ii) was shown in Proposition 3.2.40, and iii) follows by construction. □

3.2.5. The Energy Minimization Principle. The Energy Minimization Principle is a very important theorem in dynamics and has many applications. The goal is to translate this principle into our non-archimedean setting. For $X = \mathbb{P}^1$ this was already done in [**BR10**, §8.10], and Matt Baker suggested to generalize their definition of Arakelov–Green's functions and their result to the author. In the following section, we give a proof of the Energy Minimization Principle for a smooth proper curve X over our non-archimedean field K using the techniques from [**BR10**, §8.10].

DEFINITION 3.2.43. Let μ be a probability measure on X^{an} with continuous potentials. Then for every probability measure ν on X^{an}, we define the corresponding *μ-energy integral* as

$$I_\mu(\nu) := \int\int g_\mu(x,y)\ d\nu(y)d\nu(x).$$

Note that the integral is well-defined since g_μ is lower semi-continuous on $X^{\mathrm{an}} \times X^{\mathrm{an}}$ by Proposition 3.2.39, and hence Borel measurable.

THEOREM 3.2.44 (Energy Minimization Principle). *Let μ be a probability measure on X^{an} with continuous potentials. Then*

i) $I_\mu(\nu) \geq 0$ for each probability measure ν on X^{an}, and
ii) $I_\mu(\nu) = 0$ if and only if $\nu = \mu$.

We show the principle in several steps. At first, we prove analogues of Maria's theorem (Theorem 3.2.50) and Frostman's theorem (Theorem 3.2.53). In Maria's theorem we study the boundedness of the generalized potential function that is defined in the subsequent definition.

DEFINITION 3.2.45. Let μ be a probability measure on X^{an} with continuous potentials. Then for every probability measure ν on X^{an}, we define the corresponding *generalized potential function* by

$$u_\nu(\cdot,\mu) := \int g_\mu(\cdot,y)\ d\nu(y).$$

LEMMA 3.2.46. *Let μ be a probability measure with continuous potentials and let ν be an arbitrary probability measure on X^{an}. Then for every $\zeta_0 \in I(X^{\mathrm{an}})$ we can write*

$$u_\nu(\cdot,\mu) = u_{\zeta_0,\nu}(\cdot,\zeta_0) - u_{\zeta_0,\mu}(\cdot,\zeta_0) + C \tag{3.2.11}$$

on X^{an} for a finite constant C.

PROOF. Let ζ_0 be a point in $I(X^{\mathrm{an}})$. Then by Corollary 3.2.10

$$u_{\zeta_0,\nu}(\cdot,\zeta_0) = \int g_{\zeta_0}(\zeta_0,\cdot,\zeta)\ d\nu(\zeta) = \int g_{\zeta_0}(\cdot,\zeta)\ d\nu(\zeta).$$

The same identity is true for μ, i.e. $u_{\zeta_0,\mu}(\cdot,\zeta_0) = \int g_{\zeta_0}(\cdot,\zeta)\ d\mu(\zeta)$, which is a finitely valued continuous function on X^{an} as μ has continuous potentials. Thus we can write using the definition of the Arakelov–Green's function (Definition 3.2.36)

$$\begin{aligned} u_\nu(x,\mu) &= \int g_\mu(x,y)\ d\nu(y) \\ &= \int g_{\zeta_0}(x,y)\ d\nu(y) - \int g_{\zeta_0}(x,\zeta)\ d\mu(\zeta) - \int\int g_{\zeta_0}(y,\zeta)\ d\mu(\zeta)d\nu(y) + C_{\zeta_0} \\ &= u_{\zeta_0,\nu}(x,\zeta_0) - u_{\zeta_0,\mu}(x,\zeta_0) - \int u_{\zeta_0,\mu}(y,\zeta_0)\ d\nu(y) + C_{\zeta_0} \end{aligned}$$

for every $x \in X^{\mathrm{an}}$. Since $u_{\zeta_0,\mu}(\cdot,\zeta_0)$ is bounded and continuous on X^{an}, we get

$$u_\nu(\cdot,\mu) = u_{\zeta_0,\nu}(\cdot,\zeta_0) - u_{\zeta_0,\mu}(\cdot,\zeta_0) + C$$

on X^{an} for a finite constant C. □

PROPOSITION 3.2.47. *Let μ be a probability measure with continuous potentials and let ν be an arbitrary probability measure on X^{an}. Then $u_\nu(\cdot,\mu)\colon X^{\mathrm{an}} \to (-\infty,\infty]$ is continuous on $X^{\mathrm{an}}\setminus \operatorname{supp}(\nu)$ and lsc on X^{an}. Moreover, the restriction of $u_\nu(\cdot,\mu)$ to every skeleton Γ of X^{an} and to every path $[y,z]$ is finite and continuous.*

PROOF. Let ζ_0 be some point in $I(X^{\mathrm{an}})$, then

$$u_\nu(\cdot,\mu) = u_{\zeta_0,\nu}(\cdot,\zeta_0) - u_{\zeta_0,\mu}(\cdot,\zeta_0) + C$$

on X^{an} for a finite constant C by Lemma 3.2.46. Since μ has continuous potentials, $u_{\zeta_0,\mu}(\cdot,\zeta_0)$ is a finitely valued continuous function on X^{an}. Thus it remains to prove the continuity assertions for $u_{\zeta_0,\nu}(\cdot,\zeta_0)$. But these were all already shown in Lemma 3.2.25 and Proposition 3.2.26. □

PROPOSITION 3.2.48. *Let μ be a probability measure with continuous potentials and let ν be an arbitrary probability measure on X^{an}. Then $u_\nu(\cdot,\mu)$ defines a current in $D^0(X^{\mathrm{an}})$ with*

$$dd^c u_\nu(\cdot,\mu) = \mu - \nu.$$

In particular, $u_\nu(\cdot,\mu)$ is subharmonic on $X^{\mathrm{an}}\setminus \operatorname{supp}(\nu)$.

PROOF. Let ζ_0 be a point in $I(X^{\mathrm{an}})$, then

$$u_\nu(\cdot,\mu) = u_{\zeta_0,\nu}(\cdot,\zeta_0) - u_{\zeta_0,\mu}(\cdot,\zeta_0) + C$$

on X^{an} for a finite constant C by Lemma 3.2.46. By Proposition 3.2.27, $u_{\zeta_0,\nu}(\cdot,\zeta_0)$ and $u_{\zeta_0,\mu}(\cdot,\zeta_0)$ belong to $D^0(X^{\mathrm{an}})$ with

$$dd^c u_{\zeta_0,\nu}(\cdot,\zeta_0) = \delta_{\zeta_0} - \nu,$$
$$dd^c u_{\zeta_0,\mu}(\cdot,\zeta_0) = \delta_{\zeta_0} - \mu.$$

Hence $u_\nu(\cdot,\mu)$ belongs to $D^0(X^{\mathrm{an}})$ as well with

$$dd^c u_\nu(\cdot,\mu) = dd^c u_{\zeta_0,\nu}(\cdot,\zeta_0) - dd^c u_{\zeta_0,\mu}(\cdot,\zeta_0) = \mu - \nu.$$

By Proposition 3.2.47 and Proposition 3.1.30, the generalized potential function $u_\nu(\cdot,\mu)$ is therefore subharmonic on $X^{\mathrm{an}}\backslash\operatorname{supp}(\nu)$. □

The key tool of the proof of Maria's theorem in [**BR10**] is [**BR10**, Proposition 8.16], which we can translate to our situation in the following form.

LEMMA 3.2.49. *Let W be an open ball or an open annulus in X^{an} and let f be a subharmonic function on a connected open subset V of W with $\overline{V} \subset W$. For every $x \in \mathbb{H}(V)$, there is a path Λ from x to a boundary point $y \in \partial V$ such that f is non-decreasing along Λ.*

PROOF. Since $\overline{V}$ is contained in an open ball or in an open annulus, we can view it is a subset of $\mathbb{P}^{1,\mathrm{an}}$. Then [**BR10**, Proposition 8.16] and Proposition 3.1.10 yield the claim. □

With the help of Proposition 3.2.47 and Lemma 3.2.49, we can prove Maria's theorem.

THEOREM 3.2.50 (Maria). *Let μ be a probability measure on X^{an} with continuous potentials and let ν be an arbitrary probability measure on X^{an}. If there is a constant $M < \infty$ such that $u_\nu(\cdot,\mu) \leq M$ on $\operatorname{supp}(\nu)$, then $u_\nu(\cdot,\mu) \leq M$ on X^{an}.*

PROOF. Let V be a connected component of $X^{\mathrm{an}}\backslash\operatorname{supp}(\nu)$ and assume there is a point $x_0 \in V$ such that $u_\nu(x_0,\mu) > M$. Note that V is path-connected since X^{an} is locally path-connected. If B is an open ball in X^{an}, then between two points $x, y \in B$ there is only one path in X^{an} by the structure of X^{an}. Thus $V \cap B$ is uniquely path-connected for every open ball B in X^{an}. We have seen in Proposition 3.2.47 that the generalized potential function $u_\nu(\cdot,\mu)$ is continuous on $V \subset X^{\mathrm{an}}\backslash\operatorname{supp}(\nu)$. Hence we may assume x_0 to be contained in the dense subset $I(V)$ of V, and so we can choose a skeleton Γ of X^{an} containing x_0 by Proposition 2.3.22.

Let $(Y_\alpha)_\alpha$ be the directed system of connected strictly affinoid domains contained in V and containing x_0. Note that the union of two connected strictly affinoid domains Y_1, Y_2 in X^{an} both containing x_0 with $Y_1 \cup Y_2 \neq X^{\mathrm{an}}$ is again a connected strictly affinoid domain in X^{an} by [**Thu05**, Corollaire 2.1.17]. Then $u_\nu(\cdot,\mu)$ is continuous on Y_α and subharmonic on the relative interior Y_α° by Proposition 3.2.47 and Proposition 3.2.48. Hence $u_\nu(\cdot,\mu)$ attains a maximum on Y_α in a point $z_\alpha \in \partial Y_\alpha$ (see Maximum Principle 3.1.8), i.e.

$$u_\nu(z_\alpha,\mu) = \max_{x\in Y_\alpha} u_\nu(x,\mu) \geq \max_{x \in Y_\alpha^\circ} u_\nu(x,\mu) \geq u_\nu(x_0,\mu) > M \tag{3.2.12}$$

for every α. Then $\langle z_\alpha\rangle_\alpha$ defines a net of type II points in V. As $\overline{V}$ is compact, we may assume by passing to a subnet that $\langle z_\alpha\rangle_\alpha$ converges to a point $z \in \overline{V}$. Due to $V = \bigcup_\alpha Y_\alpha$ and $z_\alpha \in \partial Y_\alpha$, the point z has to ly in $\partial V \subset \operatorname{supp}(\nu)$. In the following, we use this net to get a contradiction to $u_\nu(\cdot,\mu) \leq M$ on ∂V. Recall that Γ is a skeleton of X^{an} containing x_0.

If $z \in \partial V \backslash \Gamma$, there exists an open ball B_z in $X^{\mathrm{an}} \backslash \Gamma$ containing z. We can find B_z such that $\overline{B_z} = B_z \cup \{\zeta_z\} \subset X^{\mathrm{an}} \backslash \Gamma$. We may assume $\langle z_\alpha \rangle_\alpha$ to ly in B_z. Then every path from a z_α to x_0, or more generally to the skeleton, goes by construction through ζ_z. Hence for every α the path $[z_\alpha, \zeta_z]$ lies inside Y_α as z_α and x_0 do, and so $u_\nu(z_\alpha, \mu) \geq u_\nu(\cdot, \mu)$ on $[z_\alpha, \zeta_z]$ by (3.2.12). Assume we have equality for every α, then

$$u_\nu(\cdot, \mu) \equiv u_\nu(z_\alpha, \mu) \geq u_\nu(x_0, \mu)$$

on $(z, \zeta_z]$ since we can write $(z, \zeta_z] \subset \bigcup_\alpha [z_\alpha, \zeta_z]$ as z_α converges to z. Proposition 3.2.47 implies

$$u_\nu(z, \mu) = \lim_{x \in [\zeta_z, z),\ x \to z} u_\nu(x, \mu) = u_\nu(\zeta_z, \mu) \geq u_\nu(x_0, \mu) > M$$

contradicting $u_\nu(\cdot, \mu) \leq M$ on $\mathrm{supp}(\nu)$. Consequently, we may assume that there is a z_α and a point $y_\alpha \in (z_\alpha, \zeta_z]$ such that $u_\nu(z_\alpha, \mu) > u_\nu(y_\alpha, \mu)$. Our function $u_\nu(\cdot, \mu)$ is subharmonic on the connected open subset $V \cap B_z$ and $z_\alpha \in I(V \cap B_z)$, and so there exists a path Λ from z_α to a boundary point of $V \cap B_z$ by Lemma 3.2.49 such that $u_\nu(\cdot, \mu)$ is non-decreasing along Λ. The boundary points of $V \cap B_z$ consist of points in ∂V and ζ_z. Since we have already seen that there is a point $y_\alpha \in (z_\alpha, \zeta_z]$ such that $u_\nu(z_\alpha, \mu) > u_\nu(y_\alpha, \mu)$, Λ cannot be the path $[z_\alpha, \zeta_z]$. Hence Λ is a path to a boundary point $z' \in \partial V$ and we get the contradiction

$$u_\nu(z', \mu) = \lim_{x \in \Lambda^\circ,\ x \to z'} u_\nu(x, \mu) \geq u_\nu(z_\alpha, \mu) > u_\nu(x_0, \mu) > M,$$

where $u_\nu(\cdot, \mu)$ restricted to Λ is continuous by Proposition 3.2.47.

If $z \in \partial V \cap \Gamma$, we show that $\langle \tau_\Gamma(z_\alpha) \rangle_\alpha$ defines a net in $V \cap \Gamma$ converging to z with $u_\nu(\tau_\Gamma(z_\alpha), \mu) \geq u_\nu(x_0, \mu) > M$ for every α. Then we use again Proposition 3.2.47. Since τ_Γ is continuous, the net $\langle \tau_\Gamma(z_\alpha) \rangle_\alpha$ converges to $\tau_\Gamma(z) = z$. Clearly, $\langle \tau_\Gamma(z_\alpha) \rangle_\alpha$ lies in Γ. The open set V is path-connected, and so there exists a path between z_α and x_0 in V. By the construction of the retraction map and due to $x_0 \in \Gamma$, $\tau_\Gamma(z_\alpha)$ lies inside this path, and hence it lies in V. We continue with $u_\nu(\tau_\Gamma(z_\alpha), \mu) \geq u_\nu(x_0, \mu)$ for every z_α. Assume that $z_\alpha \neq \tau_\Gamma(z_\alpha)$ because otherwise we are done by (3.2.12). Denote by B_α the connected component of $X^{\mathrm{an}} \backslash \Gamma$ containing z_α, and choose a sequence of type II points $\zeta_n \in [z_\alpha, \tau_\Gamma(z_\alpha)]^\circ$ converging to $\tau_\Gamma(z_\alpha)$. Note that there is only one path from z_α to $\tau_\Gamma(z_\alpha)$ in X^{an}, and this path lies in V because $z_\alpha, \tau_\Gamma(z_\alpha) \in V$ and V is path-connected. Thus each ζ_n lies in V as well. Let $B_{\alpha,n}$ be the open ball containing z_α and having ζ_n as unique boundary point. Since $u_\nu(\cdot, \mu)$ is subharmonic on $V \cap B_{\alpha,n}$ for every $n \in \mathbb{N}$, there is a path Λ_n from z_α to a boundary point z'_n in $\partial(V \cap B_{\alpha,n}) \subset \partial V \cup \{\zeta_n\}$ such that $u_\nu(\cdot, \mu)$ is non-decreasing along Λ_n by Lemma 3.2.49. If there exists an $n \in \mathbb{N}$ with $z'_n \in \partial V$, then Proposition 3.2.47 and (3.2.12) imply

$$u_\nu(z'_n, \mu) = \lim_{x \in \Lambda_n^\circ,\ x \to z'_n} u_\nu(x, \mu) \geq u_\nu(z_\alpha, \mu) \geq u_\nu(x_0, \mu) > M$$

contradicting $u_\nu(\cdot, \mu) \leq M$ on $\mathrm{supp}(\nu)$. Hence $\Lambda_n = [z_\alpha, \zeta_n]$ for all $n \in \mathbb{N}$. Recall that $(\zeta_n)_n$ is a sequence in V converging to $\tau_\Gamma(z_\alpha) \in V$. Since $u_\nu(\cdot, \mu)$ is continuous on V and $u_\nu(\cdot, \mu)$ is non-decreasing along $\Lambda_n = [z_\alpha, \zeta_n]$, Proposition 3.2.47 yields

$$u_\nu(\tau_\Gamma(z_\alpha), \mu) = \lim_{n \to \infty} u_\nu(\zeta_n, \mu) \geq u_\nu(z_\alpha, \mu). \tag{3.2.13}$$

Altogether, we have a net $(\tau_\Gamma(z_\alpha))_\alpha$ in $V \cap \Gamma$ converging to z such that

$$u_\nu(\tau_\Gamma(z_\alpha), \mu) \geq u_\nu(x_0, \mu) > M$$

for every α. Proposition 3.2.47 tells us that $u_\nu(\cdot,\mu)$ restricted to Γ is continuous, and hence using (3.2.12) and (3.2.13) we get

$$u_\nu(z,\mu) = \lim_{\alpha} u_\nu(\tau_\Gamma(z_\alpha),\mu) \geq u_\nu(x_0,\mu) > M$$

contradicting $u_\nu(\cdot,\mu) \leq M$ on $\mathrm{supp}(\nu)$.

Hence there cannot exist a point x_0 in V with $u_\nu(x_0,\mu) > M$. □

DEFINITION 3.2.51. Let μ be a probability measure with continuous potentials, then we define the *μ-Robin constant* as

$$V(\mu) := \inf_{\nu} I_\mu(\nu),$$

where ν is running over all probability measures supported on X^{an}.

LEMMA 3.2.52. *We have $V(\mu) \in \mathbb{R}_{\leq 0}$ and there exists a probability measure ω on X^{an} such that $I_\mu(\omega) = V(\mu)$.*

PROOF. First, we explain why $V(\mu)$ is a non-positive real number. The normalized Arakelov–Green's function g_μ is bounded from below as a lsc function on the compact space $X^{\mathrm{an}} \times X^{\mathrm{an}}$ by Proposition 3.2.39, and hence we have

$$V(\mu) = \int\int g_\mu(x,y)\ d\nu(x)d\nu(y) > -\infty.$$

On the other hand,

$$V(\mu) \leq I_\mu(\mu) = \int\int g_\mu(x,y)\ d\mu(x)d\mu(y) = 0$$

by the normalization of g_μ. Thus $V(\mu) \in \mathbb{R}_{\leq 0}$.

We show the second part of the assertion applying the same argument used to prove the existence of an equilibrium measure in [**BR10**, Proposition 6.6]. Let ω_i be a sequence of probability measures such that $\lim_{i\to\infty} I_\mu(\omega_i) = V(\mu)$. By Corollary 2.1.6, we can pass to a subsequence converging weakly to a probability measure ω on X^{an}. Due to $I_\mu(\omega) \geq V(\mu)$ by the definition of the Robin constant, it remains to show the inequality $I_\mu(\omega) \leq V(\mu)$. By Proposition 3.2.39, the normalized Arakelov–Green's function g_μ is lsc on the compact space $X^{\mathrm{an}} \times X^{\mathrm{an}}$, and so it is bounded from below by some constant $M \in \mathbb{R}$. Proposition 2.1.5 tells us that ω is a Radon measure, and so [**BR10**, Proposition A.3] yields the following description

$$I_\mu(\omega) = \int\int g_\mu(x,y)\ d\omega(x)d\omega(y) = \sup_{\substack{g\in\mathcal{C}^0(X^{\mathrm{an}}\times X^{\mathrm{an}}),\\ M\leq g\leq g_\mu}} \int\int g(x,y)\ d\omega(x)d\omega(y),$$

for the space $\mathcal{C}^0(X^{\mathrm{an}} \times X^{\mathrm{an}})$ of real-valued continuous functions on $X^{\mathrm{an}} \times X^{\mathrm{an}}$. For every $g \in \mathcal{C}^0(X^{\mathrm{an}} \times X^{\mathrm{an}})$ satisfying $M \leq g \leq g_\mu$, we have

$$\begin{aligned}\int\int g(x,y)\ d\omega(x)d\omega(y) &= \lim_{i\to\infty} \int\int g(x,y)\ d\omega_i(x)d\omega_i(y)\\ &\leq \lim_{i\to\infty} \int\int g_\mu(x,y)\ d\omega_i(x)d\omega_i(y)\\ &= \lim_{i\to\infty} I_\mu(\omega_i) = V(\mu),\end{aligned}$$

where the first identity is proven for example in [**BR10**, Lemma 6.5] and the inequality holds as every ω_i is positive. Hence $I_\mu(\omega) \leq V(\mu)$. □

THEOREM 3.2.53 (Frostman). *Let μ be a probability measure on X^{an} with continuous potentials and let ω be a probability measure on X^{an} such that $I_\mu(\omega) = V(\mu)$. Then we have*

$$u_\omega(\cdot, \mu) \equiv V(\mu)$$

on X^{an}.

PROOF. The strategy is as in the proof of [**BR10**, Proposition 8.55] with using analogous capacity results from Subsection 3.2.2.

1. Step: Show that $E := \{x \in X^{\mathrm{an}} \mid u_\omega(x, \mu) < V(\mu)\} \subset X(K)$.

By Lemma 3.2.15, it remains to show that E is a proper subset of X^{an} of capacity zero. Assume that $E \subset \mathrm{supp}(\omega)$, then we get the contradiction

$$V(\mu) = I_\mu(\omega) = \int\int g_\mu(x, y)\ d\omega(y)d\omega(x) = \int u_\omega(x, \mu)\ d\omega(x) < V(\mu).$$

Thus there has to be a point $\xi \in \mathrm{supp}(\omega)\backslash E$, and so E is indeed a proper subset of X^{an}. To show that it has capacity zero, we consider

$$E_n := \{x \in X^{\mathrm{an}} \mid u_\omega(x, \mu) \leq V(\mu) - 1/n\}$$

for every $n \in \mathbb{N}_{\geq 1}$. Clearly, $\xi \notin E_n$ for every $n \in \mathbb{N}_{\geq 1}$. Since $u_\omega(\cdot, \mu)$ is lsc on X^{an} by Proposition 3.2.47, each E_n is closed and so compact as a closed subset of a compact space. If every E_n has capacity zero, then $E = \bigcup_{n \in \mathbb{N}_{\geq 1}} E_n$ has capacity zero as well by Corollary 3.2.17.

We therefore assume that there is an E_n with positive capacity, i.e. there exist a probability measure ν supported on E_n, a base point $\zeta_0 \in I(X^{\mathrm{an}})$ and $\zeta \in X^{\mathrm{an}}\backslash E_n$ such that $I_{\zeta_0,\zeta}(\nu) < \infty$. Since E_n is closed and $I(X^{\mathrm{an}})$ is a dense subset of X^{an}, we may choose $\zeta_0 = \zeta \in I(X^{\mathrm{an}})\backslash E_n$ by Remark 3.2.14. Then

$$I_{\zeta_0,\zeta_0}(\nu) = \int\int g_{\zeta_0}(\zeta_0, x, y)\ d\nu(x)d\nu(y) = \int\int g_{\zeta_0}(x, y)\ d\nu(x)d\nu(y) < \infty, \tag{3.2.14}$$

where we used $g_{\zeta_0}(\zeta_0, x, y) = g_{\zeta_0}(x, y)$ from Corollary 3.2.10. We can write by the definition of the Arakelov–Green's function g_μ

$$\begin{aligned}
I_\mu(\nu) &= \int\int g_\mu(x, y)\ d\nu(y)d\nu(x) \\
&= \int\int g_{\zeta_0}(x, y)\ d\nu(y)d\nu(x) - \int\int g_{\zeta_0}(x, \zeta)\ d\mu(\zeta)d\nu(x) \\
&\quad - \int\int g_{\zeta_0}(y, \zeta)\ d\mu(\zeta)d\nu(y) + C_{\zeta_0} \\
&= \int\int g_{\zeta_0}(x, y)\ d\nu(x)d\nu(y) - 2\int\int g_{\zeta_0}(x, \zeta)\ d\mu(\zeta)d\nu(x) + C_{\zeta_0} \\
&= I_{\zeta_0,\zeta_0}(\nu) - 2\int\int g_{\zeta_0}(x, \zeta)\ d\mu(\zeta)d\nu(x) + C_{\zeta_0}.
\end{aligned}$$

Since μ has continuous potentials, the term $2\int\int g_{\zeta_0}(x, \zeta)\ d\mu(\zeta)d\nu(x)$ is finite. Hence $I_{\zeta_0,\zeta_0}(\nu) < \infty$ implies $I_\mu(\nu) < \infty$.

Recall that ξ is a point in $\mathrm{supp}(\omega)\backslash E_n$ and $u_\omega(\xi, \mu) \geq V(\mu)$. Since $u_\omega(\cdot, \mu)$ is lsc on X^{an} by Proposition 3.2.47, we can find an open neighborhood U of ξ such that

$u_\omega(\cdot,\mu) > V(\mu) - 1/(2n)$ on $\overline{U}$. Then $\overline{U} \cap E_n = \emptyset$ and $M := \omega(\overline{U}) > 0$ using that ω is a positive measure and $\xi \in \overline{U} \cap \operatorname{supp}(\omega)$. We define the following measure on X^{an}

$$\sigma := \begin{cases} M \cdot \nu & \text{on } E_n, \\ -\omega & \text{on } \overline{U}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\sigma(X^{\mathrm{an}}) = M \cdot \nu(E_n) - \omega(\overline{U}) = 0$ as ν is a probability measure supported on E_n. Moreover, we can consider

$$\begin{aligned} I_\mu(\sigma) &:= \int\int g_\mu(x,y)\ d\sigma(x)d\sigma(y) \\ &= M^2 \cdot \int_{E_n}\int_{E_n} g_\mu(x,y)\ d\nu(x)d\nu(y) - 2M \cdot \int_{E_n}\int_{\overline{U}} g_\mu(x,y)\ d\nu(x)d\omega(y) \\ &\quad + \int_{\overline{U}}\int_{\overline{U}} g_\mu(x,y)\ d\omega(x)d\omega(y). \end{aligned}$$

We will explain why $I_\mu(\sigma)$ is finite. Note that g_μ is lsc on the compact space $X^{\mathrm{an}} \times X^{\mathrm{an}}$ (cf. Proposition 3.2.39), and so bounded from below. The first term is equal to $M^2 \cdot I_\mu(\nu)$, and we have already seen that $I_\mu(\nu) < \infty$. Since g_μ is bounded from below and ν is a positive measure, the first term is finite. The second term is finite because $\overline{U}$ and E_n are compact disjoint sets and g_μ is continuous off the diagonal (see Proposition 3.2.39). The third term has to be finite as well as g_μ is bounded from below, ω is a positive measure, and we have

$$\int\int g_\mu(x,y)\ d\omega(x)d\omega(y) = I_\mu(\omega) = V(\mu) \in \mathbb{R}$$

by Lemma 3.2.52. Consequently, $I_\mu(\sigma)$ is finite.

For every $t \in [0,1]$, we define the probability measure $\omega_t := \omega + t\sigma$ on X^{an}. Then

$$\begin{aligned} I_\mu(\omega_t) - I_\mu(\omega) &= \int\int g_\mu(x,y)\ d\omega_t(x)d\omega_t(y) - \int\int g_\mu(x,y)\ d\omega(x)d\omega(y) \\ &= \int\int g_\mu(x,y)\ d\omega(x)d\omega(y) + 2\int\int g_\mu(x,y)\ d\omega(x)d(t\sigma)(y) \\ &\quad + \int\int g_\mu(x,y)\ d(t\sigma)(x)d(t\sigma)(y) - \int\int g_\mu(x,y)\ d\omega(x)d\omega(y) \\ &= 2t \cdot \int u_\omega(y,\mu)\ d\sigma(y) + t^2 \cdot I_\mu(\sigma). \end{aligned}$$

Inserting the definition of the measure σ, we obtain

$$I_\mu(\omega_t) - I_\mu(\omega) = 2t \cdot \left(M \cdot \int_{E_n} u_\omega(y,\mu)\ d\nu(y) - \int_{\overline{U}} u_\omega(y,\mu)\ d\omega(y) \right) + t^2 \cdot I_\mu(\sigma).$$

Since $u_\omega(\cdot,\mu) \leq V(\mu) - 1/n$ on E_n and $\operatorname{supp}(\nu) \subset E_n$, $u_\omega(\cdot,\mu) > V(\mu) - 1/(2n)$ on $\overline{U}$ and $M = \omega(\overline{U}) > 0$, we get

$$\begin{aligned} I_\mu(\omega_t) - I_\mu(\omega) &\leq 2t \cdot (M \cdot (V(\mu) - 1/n) - M \cdot (V(\mu) - 1/(2n))) + t^2 \cdot I_\mu(\sigma) \\ &= (-M/n) \cdot t + t^2 \cdot I_\mu(\sigma). \end{aligned}$$

The right hand side is negative for sufficiently small $t > 0$ as $I_\mu(\sigma)$ is finite, and so this contradicts $I_\mu(\omega) = V(\mu)$. Hence each E_n has capacity zero, and so does E. By Lemma 3.2.15, we get $E \cap \mathbb{H}(X^{\mathrm{an}}) = \emptyset$.

2. Step: Show that $\omega(E) = 0$.

Pick a base point $\zeta_0 \in I(X^{\mathrm{an}})$. We have seen in Step 1 that $E \subset X(K)$, so ζ_0 cannot by contained in E. Because of $I_{\zeta_0,\zeta_0}(\omega) = \int\int g_{\zeta_0}(x,y)\ d\omega(x)d\omega(y)$ by Corollary 3.2.10, we have

$$\begin{aligned} I_\mu(\omega) &= \int\int g_\mu(x,y)\ d\omega(y)d\omega(x) \\ &= \int\int g_{\zeta_0}(x,y)\ d\omega(y)d\omega(x) - \int\int g_{\zeta_0}(x,\zeta)\ d\mu(\zeta)d\omega(x) \\ &\quad - \int\int g_{\zeta_0}(y,\zeta)\ d\mu(\zeta)d\omega(y) + C_{\zeta_0} \\ &= I_{\zeta_0,\zeta_0}(\omega) - 2\int\int g_{\zeta_0}(x,\zeta)\ d\mu(\zeta)d\omega(x) + C_{\zeta_0}, \end{aligned}$$

where the double integral is finite since μ has continuous potentials. As $I_\mu(\omega) = V(\mu)$ is finite by Lemma 3.2.52, it follows directly from the calculation that $I_{\zeta_0,\zeta_0}(\omega) < \infty$. Moreover, we have seen in the proof of Step 1 that E has capacity zero and we also know that $\zeta_0 \notin E$. Lemma 3.2.16 yields $\omega(E) = 0$.

3. Step: Show that $u_\omega(\cdot,\mu) \leq V(\mu)$ on X^{an}.

Using Maria's theorem 3.2.50, it remains to prove $u_\omega(\cdot,\mu) \leq V(\mu)$ on $\mathrm{supp}(\omega)$. Assume there is a point $z \in \mathrm{supp}(\omega)$ such that $u_\omega(z,\mu) > V(\mu)$. Choose $\varepsilon > 0$ such that $u_\omega(z,\mu) > V(\mu) + \varepsilon$. Since $u_\omega(\cdot,\mu)$ is lsc on X^{an} by Proposition 3.2.47, there is an open neighborhood U_z of z with $u_\omega(\cdot,\mu) > V(\mu) + \varepsilon$ on U_z. Then $\omega(U_z) > 0$ as $z \in \mathrm{supp}(\omega)$. By the construction of E, we have $u_\omega(\cdot,\mu) < V(\mu)$ on E. Hence E and U_z are disjoint and we get the following decomposition of $V(\mu) = I_\mu(\omega)$

$$\begin{aligned} V(\mu) &= \int_{X^{\mathrm{an}}} u_\omega(x,\mu)\ d\omega(x) \\ &= \int_{U_z} u_\omega(x,\mu)\ d\omega(x) + \int_{X^{\mathrm{an}}\setminus(U_z\cup E)} u_\omega(x,\mu)\ d\omega(x). \end{aligned}$$

Note that we also use that the integral of $u_\omega(\cdot,\mu)$ over E has to be zero as $\omega(E) = 0$ by Step 2. For the first term we know that $u_\omega(\cdot,\mu) > V(\mu) + \varepsilon$ on U_z and $\omega(U_z) > 0$. Thus

$$\int_{X^{\mathrm{an}}} u_\omega(x,\mu)\ d\omega(x) \geq \omega(U_z)\cdot(V(\mu)+\varepsilon). \tag{3.2.15}$$

We have $u_\omega(\cdot,\mu) \geq V(\mu)$ on $X^{\mathrm{an}}\setminus E$ by the definition of E, and so

$$\int_{X^{\mathrm{an}}\setminus(U_z\cup E)} u_\omega(x,\mu)\ d\omega(x) \geq (1-\omega(U_z)-\omega(E))\cdot V(\mu). \tag{3.2.16}$$

Putting (3.2.15), (3.2.16) and $\omega(E) = 0$ together, we get the contradiction

$$\begin{aligned} V(\mu) &\geq \omega(U_z)\cdot(V(\mu)+\varepsilon) + (1-\omega(U_z)-\omega(E))\cdot V(\mu) \\ &= \omega(U_z)\cdot(V(\mu)+\varepsilon) + (1-\omega(U_z))\cdot V(\mu) \\ &= V(\mu) + \omega(U_z)\varepsilon > V(\mu). \end{aligned}$$

Hence $u_\omega(\cdot,\mu) \leq V(\mu)$ on $\mathrm{supp}(\omega)$. Maria's theorem 3.2.50, implies that $u_\omega(\cdot,\mu) \leq V(\mu)$ on X^{an}. This shows the third step.

By the first step we know that $u_\omega(\cdot,\mu) \geq V(\mu)$ on $X^{\mathrm{an}} \backslash X(K)$. For every point $y \in X(K)$, we can find a path $[z,y]$ from a point $z \in I(X^{\mathrm{an}})$ to y such that $[z,y)$ is contained in $I(X^{\mathrm{an}}) \subset X^{\mathrm{an}} \backslash X(K)$. Then Proposition 3.2.47 implies

$$u_\omega(y,\mu) = \lim_{x \in [z,y)} u_\omega(x,\mu) \geq V(\mu).$$

Hence $E = \{x \in X^{\mathrm{an}} \mid u_\omega(x,\mu) < V(\mu)\}$ is empty, and so $u_\omega(\cdot,\mu) \geq V(\mu)$ on X^{an}. Step 3 implies $u_\omega(\cdot,\mu) \equiv V(\mu)$ on X^{an}. □

Proof of Theorem 3.2.44. Let ω be a probability measure on X^{an} that minimizes the energy integral, i.e. $I_\mu(\omega) = V(\mu)$. Such a measure always exists by Lemma 3.2.52. By Frostman's theorem 3.2.53, $u_\omega(\cdot,\mu)$ is constant on X^{an}, and hence

$$0 = dd^c u_\omega(\cdot,\mu) = \mu - \omega$$

by Proposition 3.2.47. Thus ω minimizes the energy integral if and only if $\omega = \mu$. Since $I_\mu(\mu) = \int\int g_\mu(x,y)\, d\mu(y) d\mu(x) = 0$ by the normalization of the Arakelov–Green's function g_μ, it follows that $I_\mu(\nu) \geq 0$ for every probability measure ν on X^{an}. □

Corollary 3.2.54. *Let $\zeta \in I(X^{\mathrm{an}})$ and μ be a probability measure on X^{an} with continuous potentials. Then $g_\mu(\zeta,\zeta) \geq 0$, and $g_\mu(\zeta,\zeta) = 0$ if and only if $\mu = \delta_\zeta$.*

Proof. Since

$$g_\mu(\zeta,\zeta) = \int\int g_\mu(x,y)\, d\delta_\zeta(x) d\delta_\zeta(y) = I_\mu(\delta_\zeta),$$

the Energy Minimization Principle (Theorem 3.2.44) gives the assertion immediately. □

3.2.6. Local discrepancy. Let E be an elliptic curve over K with j-invariant j_E. In this subsection, we give a different proof of the local discrepancy result from [**BP05**, Corollary 5.6] using our Energy Minimization Principle (Theorem 3.2.44).

Remark 3.2.55. In the following, let Γ be the minimal skeleton of E^{an}. Then Γ is a single point ζ_0 when E has good reduction and Γ corresponds to the circle $\mathbb{R}/\mathbb{Z}$ when it has multiplicative reduction. One has a canonical probability measure μ supported on Γ, where

i) μ is the dirac measure in ζ_0 if E has good reduction, and
ii) μ is the uniform probability measure (i.e. Haar measure) supported on the circle $\Gamma \simeq \mathbb{R}/\mathbb{Z}$ if E has multiplicative reduction.

Then μ has in particular continuous potentials by Example 3.2.35. Hence we can consider its corresponding Arakelov–Green's function g_μ on $E^{\mathrm{an}} \times E^{\mathrm{an}}$.

Definition 3.2.56. Let $Z = \{P_1,\dots,P_N\}$ be a set of N distinct points in $E(K)$. Then the *local discrepancy* of Z is defined as

$$D(Z) := \frac{1}{N^2}\left(\sum_{i\neq j} g_\mu(P_i,P_j) + \frac{N}{12}\log^+|j_E|\right).$$

Remark 3.2.57. Baker and Petsche defined the local discrepancy in [**BP05**, §3.4] and [**Pet09**, §2.2] of a set $Z = \{P_1,\dots,P_N\}$ of N distinct points in $E(K)$ as

$$\frac{1}{N^2}\left(\sum_{i\neq j} \lambda(P_i - P_j) + \frac{N}{12}\log^+|j_E|_v\right)$$

for the Néron function $\lambda\colon E(K)\backslash\{O\} \to \mathbb{R}$ (cf. [**Sil94**, §VI.1]).

Note that our definition is consistent with theirs. As it is also mentioned in [**BP05**, Remark 5.3], the Néron function can be extend to an Arakelov–Green's function corresponding to the canonical measure μ on E^{an}. By the uniqueness of the Arakelov–Green's function (see Remark 3.2.32), we have $g_\mu(P,Q) = \lambda(P-Q)$ for $P \neq Q \in E(K)$.

Baker and Petsche showed in [**BP05**, Corollary 5.6] the following result for the local discrepancy when $K = \mathbb{C}_v$. Here, v is a non-archimedean place of a number field k and $\mathbb{C}_v$ is the completion of the algebraic closure of the completion of k with respect to v. We can prove this statement for our general K using our characterization of the local discrepancy and the Energy Minimization Principle (Theorem 3.2.44).

Corollary 3.2.58. *For each $n \in \mathbb{N}$, let $Z_n \subset E(K)$ be a set consisting of n distinct points and let δ_n be the probability measure on E^{an} that is equidistributed on Z_n. If $\lim_{n\to\infty} D(Z_n) = 0$, then δ_n converges weakly to μ on E^{an}.*

Proof. By passing to a subsequence we may assume that δ_n converges weakly to a probability measure ν on E^{an} (see Corollary 2.1.6). The strategy is to show that the μ-energy integral $I_\mu(\nu)$ is zero, and then $\mu = \nu$ follows by the Energy Minimization Principle 3.2.44. We have also seen in the Energy Minimization Principle that $I_\mu(\nu) \geq 0$. Thus it remains to show $I_\mu(\nu) \leq 0$. Due to the definition of the μ-energy integral and [**BR10**, Lemma 7.54], the following inequality holds

$$\begin{aligned}
I_\mu(\nu) &= \int\int_{E^{\mathrm{an}}\times E^{\mathrm{an}}} g_\mu(x,y)\ d\nu(x)d\nu(y) \\
&\leq \liminf_{n\to\infty} \int\int_{(E^{\mathrm{an}}\times E^{\mathrm{an}})\backslash\Delta} g_\mu(x,y)\ d\delta_n(x)d\delta_n(y) \\
&= \liminf_{n\to\infty} \frac{1}{n^2} \sum_{P\neq Q\in Z_n} g_\mu(P,Q),
\end{aligned}$$

where $\Delta := \mathrm{Diag}(E^{\mathrm{an}})$. Since $D(Z_n) = \frac{1}{n^2}\sum_{P\neq Q\in Z_n} g_\mu(P,Q) + \frac{1}{12n}\log^+|j_E|$ converges to zero, and $\frac{1}{12n}\log^+|j_E|$ does as well, we have

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{P\neq Q\in Z_n} g_\mu(P,Q) = 0.$$

Hence $I_\mu(\nu) \leq 0$. The Energy Minimization Principle yields $\mu = \nu$. $\square$

CHAPTER 4

Real-valued differential forms and tropical Dolbeault cohomology

4.1. Differential forms on Berkovich spaces

In this section, we assume K to be non-trivially valued and let X be an n-dimensional algebraic variety over K. Chambert-Loir and Ducros introduced real-valued differential forms on Berkovich analytic spaces in their fundamental preprint [**CD12**] by using tropicalizations. In the algebraic situation, i.e. the analytification X^{an} of an algebraic variety, there is a slightly different approach by Gubler which is equivalent to the one by Chambert-Loir and Ducros. We only work in the algebraic case and we follow [**Gub16**] for the definition of these forms $\mathcal{A}_X^{p,q}$ on X^{an}. There is also a natural way to define currents, which are linear functionals on $\mathcal{A}_{X,c}^{p,q}(W)$ for an open subset W of X^{an} satisfying a continuity property. Here, $\mathcal{A}_{X,c}^{p,q}(W)$ denotes the sections of $\mathcal{A}_X^{p,q}(W)$ with compact support. Since this continuity property does not act a part in the following, we refer to [**CD12**, §4.2] or [**Gub16**, §6] for it. Via these differential forms Chambert-Loir and Ducros introduced a potential theory in every dimension, which we introduce in Section 5.

Before we start introducing these forms, we outline the idea of their definition. Lagerberg defined (p,q)-superforms on open subsets of $\mathbb{R}^r$ in [**Lag12**]. This theory of superforms leads to superforms on polyhedral complexes developed in [**CD12**]. The tropicalization of an open affine subset via a closed embedding into $\mathbb{G}_m^r$ has the structure of a polyhedral complex by the theorem of Bieri–Groves. By pulling back superforms on polyhedral complexes, we obtain real-valued differential forms on the analytification X^{an}.

Definition 4.1.1. i) For an open subset $U \subset \mathbb{R}^r$ denote by $\mathcal{A}^p(U)$ the space of smooth real differential forms of degree p. The space of *superforms of bidegree* (p,q) on U is defined as

$$\mathcal{A}^{p,q}(U) := \mathcal{A}^p(U) \otimes_{\mathcal{C}^\infty(U)} \mathcal{A}^q(U) = \mathcal{A}^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} = \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^q(U).$$

ii) There are *differential operators*

$$\begin{aligned} d' \colon \ \mathcal{A}^{p,q}(U) &= \mathcal{A}^p(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} \to \mathcal{A}^{p+1}(U) \otimes_{\mathbb{R}} \Lambda^q \mathbb{R}^{r*} = \mathcal{A}^{p+1,q}(U) \\ d'' \colon \ \mathcal{A}^{p,q}(U) &= \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^q(U) \to \Lambda^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^{q+1}(U) = \mathcal{A}^{p,q+1}(U) \end{aligned}$$

that are given by $D \otimes (-1)^q \operatorname{id}$ and $(-1)^p \operatorname{id} \otimes D$, where D is the usual exterior derivative.

iii) There is a *wedge product*

$$\begin{aligned} \wedge : \mathcal{A}^{p,q}(U) \times \mathcal{A}^{p',q'}(U) &\to \mathcal{A}^{p+p',q+q'}(U), \\ (\alpha \otimes \psi, \beta \otimes \nu) &\mapsto (-1)^{p'q} \alpha \wedge \beta \otimes \psi \wedge \nu, \end{aligned}$$

that is, up to sign, induced by the usual wedge product.

iv) We have the following *canonical involution* $J\colon \mathcal{A}^{p,q}(U) \to \mathcal{A}^{q,p}(U)$ given by

$$\alpha = \sum_{I,J} \alpha_{IJ} d'x_I \wedge d''x_J \mapsto (-1)^{pq} \sum_{I,J} \alpha_{IJ} d'x_J \wedge d''x_I,$$

where $d'x_I \wedge d''x_J := (dx_{i_1} \otimes \ldots \otimes dx_{i_p}) \otimes_{\mathbb{R}} (dx_{j_1} \otimes \ldots \otimes dx_{j_q})$ for $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$.

For all p, q the functor $U \mapsto \mathcal{A}^{p,q}(U)$ defines a sheaf on $\mathbb{R}^r$ and we have $\mathcal{A}^{p,q} = 0$ if $\max(p,q) > r$.

Recall polyhedral complexes from Section 2.2. With the help of (p,q)-forms on open subsets of $\mathbb{R}^r$, one can define also forms on open subsets of the support of a polyhedral complex as follows.

Definition 4.1.2. Let $\mathscr{C}$ be a polyhedral complex and let Ω be an open subset of $|\mathscr{C}|$. Then a *superform* $\alpha \in \mathcal{A}^{p,q}(\Omega)$ *of bidegree* (p,q) on Ω is given by a superform $\alpha' \in \mathcal{A}^{p,q}(V)$ where V is an open subset of $\mathbb{R}^r$ with $V \cap |\mathscr{C}| = \Omega$. We can view α' as a map $V \times (\mathbb{R}^r)^p \times (\mathbb{R}^r)^q \to \mathbb{R}$. Two forms $\alpha' \in \mathcal{A}^{p,q}(V)$ and $\alpha'' \in \mathcal{A}^{p,q}(W)$ with $V \cap |\mathscr{C}| = W \cap |\mathscr{C}| = \Omega$ define the same form in $\mathcal{A}^{p,q}(\Omega)$ if for each $\sigma \in \mathscr{C}$ the restrictions of α' and α'' to $(\Omega \cap \sigma) \times \mathbb{L}(\sigma)^p \times \mathbb{L}(\sigma)^q$ agree. If $\alpha \in \mathcal{A}^{p,q}(\Omega)$ is given by $\alpha' \in \mathcal{A}^{p,q}(V)$, we write

$$\alpha'|_\Omega = \alpha.$$

Remark 4.1.3. Let $F\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ be an affine map. If $\mathscr{C}'$ is a polyhedral complex of $\mathbb{R}^{r'}$ and $\mathscr{C}$ a polyhedral complex of $\mathbb{R}^r$ with $F(|\mathscr{C}'|) \subset |\mathscr{C}|$, then the pullback $F^*\colon \mathcal{A}^{p,q}(|\mathscr{C}|) \to \mathcal{A}^{p,q}(|\mathscr{C}'|)$ is well-defined and compatible with the differential operators d' and d''. In particular, we have the operators d' and d'' on $\mathcal{A}^{p,q}(|\mathscr{C}|)$ given by the restriction of the corresponding operators on $\mathcal{A}^{p,q}(\mathbb{R}^r)$.

Due to theorem of Bieri–Groves' (see Theorem 2.2.6), we can use these forms on polyhedral complexes for differential forms on X^{an}. For the construction, we first need to introduce canonical embeddings of very affine subsets and tropical charts.

Definition 4.1.4. Let U, U' be open affine subsets of X and let $\varphi\colon U \to \mathbb{G}_m^r$ and $\varphi'\colon U' \to \mathbb{G}_m^{r'}$ be closed embeddings. We say that φ' *refines* φ if $U' \subset U$ and there is an affine homomorphism (i.e. group homomorphism composed with a multiplicative translation) $\psi\colon \mathbb{G}_m^{r'} \to \mathbb{G}_m^r$ of multiplicative tori such that $\varphi = \psi \circ \varphi'$. This homomorphism induces an integral affine map $\mathrm{Trop}(\psi)\colon \mathbb{R}^{r'} \to \mathbb{R}^r$ such that $\varphi_{\mathrm{trop}} = \mathrm{Trop}(\psi) \circ \varphi'_{\mathrm{trop}}$.

We call an open affine subset U of X *very affine* if $\mathcal{O}(U)$ is generated as a K-algebra by $\mathcal{O}(U)^\times$. If U is very affine, then there exists a *canonical closed embedding* (up to multiplicative translation) $\varphi_U\colon U \to \mathbb{G}_m^r$ which refines all other closed embeddings $\varphi\colon U \to \mathbb{G}_m^{r'}$ (see [**Gub16**, 4.12]). For the canonical embedding φ_U, we use the notations $\mathrm{trop}_U := (\varphi_U)_{\mathrm{trop}} = \mathrm{trop} \circ \varphi_U^{\mathrm{an}}$ and $\mathrm{Trop}(U) := \mathrm{trop}_U(U^{\mathrm{an}})$.

Definition 4.1.5. Let W be an open subset of X^{an}. A *tropical chart* (V, φ_U) of W consists of the canonical closed embedding $\varphi_U\colon U \to \mathbb{G}_m^r$ of a very affine open subset U of X and an open subset V of W that is of the form $V = \mathrm{trop}_U^{-1}(\Omega)$ for an open subset Ω of $\mathrm{Trop}(U)$.

We say that $(V', \varphi_{U'})$ is a *tropical subchart* of (V, φ_U) if $V' \subset V$ and $U' \subset U$. In this situation, $\varphi_{U'}$ refines φ_U.

Remark 4.1.6. The theorem of Bieri–Groves' (see Theorem 2.2.6) tells us that $\mathrm{Trop}(U)$ is the support of a one-dimensional polyhedral complex, and so allows us to consider forms $\mathcal{A}^{p,q}_{\mathrm{Trop}(U)}(\mathrm{trop}_U(V))$ for a tropical chart (V, φ_U) of X^{an}. Let $(V', \varphi_{U'})$ be another tropical chart of X^{an}, then $(V \cap V', \varphi_{U\cap U'})$ is a tropical subchart of both by [**Gub16**, Proposition 4.16] with $\varphi_{U\cap U'} = \varphi_U \times \varphi_{U'}$. We get a canonical homomorphism $\psi_{U,U\cap U'} \colon \mathbb{G}_m^{r+r'} \to \mathbb{G}_m^r$ of the underlying tori with

$$\varphi_U = \psi_{U,U\cap U'} \circ \varphi_{U\cap U'}$$

on $U \cap U'$ and an associated affine map $\mathrm{Trop}(\psi_{U,U\cap U'}) \colon \mathbb{R}^{r+r'} \to \mathbb{R}^r$ such that

$$\mathrm{trop}_U = \mathrm{Trop}(\psi_{U,U\cap U'}) \circ \mathrm{trop}_{U\cap U'}$$

and the tropical variety $\mathrm{Trop}(U \cap U')$ is mapped onto $\mathrm{Trop}(U)$ (see [**Gub16**, 5.1]). We define the *restriction* of $\alpha \in \mathcal{A}^{p,q}_{\mathrm{Trop}(U)}(\mathrm{trop}_U(V))$ to $\mathrm{trop}_{U\cap U'}(V \cap V')$ as

$$\mathrm{Trop}(\psi_{U,U\cap U'})^*\alpha \in \mathcal{A}^{p,q}_{\mathrm{Trop}(U\cap U')}(\mathrm{trop}_{U\cap U'}(V \cap V'))$$

and write $\alpha|_{V\cap V'}$.

Definition 4.1.7. Let X be an algebraic variety over K and let W be an open subset of X^{an}. An element of $\mathcal{A}^{p,q}_X(W)$ is given by a family $(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}$ such that:

i) For all $i \in I$ the pair (V_i, φ_{U_i}) is a tropical chart and $W = \bigcup_{i\in I} V_i$.
ii) For all $i \in I$ we have $\alpha_i \in \mathcal{A}^{p,q}_{\mathrm{Trop}(U_i)}(\mathrm{trop}_{U_i}(V_i))$.
iii) For all $i, j \in I$ the restrictions $\alpha_i|_{V_i\cap V_j} = \alpha_j|_{V_i\cap V_j}$ agree.

Another such family $(V_j, \varphi_j, \alpha_j)_{j\in J}$ defines the same form on V if $(V_i, \varphi_i, \alpha_i)_{i\in I\cup J}$ still defines a form on V.

Remark 4.1.8. Let $V' \subset V$ and $\alpha \in \mathcal{A}^{p,q}_X(W)$ given by $(V_i, \varphi_i, \alpha_i)_{i\in I}$. The subset V' can be covered by tropical subcharts (W_{ij}, φ_{ij}) of (V_i, φ_i) [**Gub16**, Proposition 4.16]. We define $\alpha|_{V'}$ to be given by the family $(W_{ij}, \varphi_{ij}, \mathrm{Trop}(\psi_{i,ij})^*\alpha_i)_{ij}$. Note that $\alpha|_{V'}$ is independent of all choices.

Then $W \mapsto \mathcal{A}^{p,q}_X(W)$ defines a sheaf on X^{an}, which we denote by $\mathcal{A}^{p,q}_X$ and we write $\mathcal{A}^{p,q}_{X,c}(W)$ for the sections with compact support in W. If the space of definition is clear, we often just use the notations $\mathcal{A}^{p,q}$ and $\mathcal{A}^{p,q}_c$. By Theorem 2.2.6, we have $\mathcal{A}^{p,q}_X = 0$ if $\max(p, q) > n = \dim(X)$.

The differentials d' and d'' and the wedge product carry over. Moreover, for every open subset W of X^{an} there is a non-trivial integration map $\int \colon \mathcal{A}^{n,n}_c(W) \to \mathbb{R}$ which is compatible with pullback and satisfies Stokes' Theorem (see [**Gub16**, Theorem 5.17]), i.e. $\int_W d''\alpha = 0$ for all $\alpha \in \mathcal{A}^{n,n-1}_c(W)$ and $\int_W d'\beta = 0$ for all $\beta \in \mathcal{A}^{n-1,n}_c(W)$.

Remark 4.1.9. The approach by Gubler to define differential forms via canonical charts, which was presented here, is equivalent to the one by Chambert-Loir and Ducros in [**CD12**] as is shown in [**Gub16**, Proposition 7.2 & Proposition 7.11].

Note that smooth differential forms of bidegree $(0, 0)$ are well-defined continuous functions.

Definition 4.1.10. Let W be an open subset of X^{an}. A function $f \colon W \to \mathbb{R}$ is called *smooth* if $f \in \mathcal{A}^{0,0}(W)$. We write $\mathcal{C}^\infty(W) := \mathcal{A}^{0,0}(W)$ and $\mathcal{C}^\infty_c(W) := \mathcal{A}^{0,0}_c(W)$.

Proposition 4.1.11. *Let W be an open subset of X^{an} and let $\alpha \in \mathcal{A}^{n,n}(W)$. Then there is a unique signed Radon measure μ_α on W such that $\int_W f\ d\mu_\alpha = \int_W f \wedge \alpha$ for every $f \in \mathcal{C}^\infty_c(W)$. If α has compact support on W, so has μ_α and $|\mu_\alpha|(W) < \infty$.*

Proof. See [**Gub16**, Proposition 6.8]. ☐

4.2. Poincaré duality for non-archimedean Mumford curves

Let K be an algebraically closed field endowed with a complete, non-archimedean absolute value $|\cdot|$ and let X be an algebraic curve over K. In this section, we do not require $|\cdot|$ to be non-trivial (if not stated otherwise). The formalism of Subsection 4.1 does not work in the trivially valued case (cf. [**Jel16a**, Example 3.3.1]), but the one introduced in Subsection 4.2.1 does.

The contents of this section are joint work with Philipp Jell and were published in [**JW18**].

4.2.1. Differential forms using $\mathbb{A}$-tropical charts. In Section 4.1, we have seen the definition of the sheaf $\mathcal{A}_X^{p,q}$ of smooth differential forms on X^{an}. The key tool to define these forms were tropical charts. Jell developed in his thesis [**Jel16a**, §3.2.2] a further approach using so called $\mathbb{A}$-tropical charts, which also works in the trivially valued case. In the non-trivially valued case, his approach leads to the same forms as the approach introduced in Section 4.1. We will present this way of defining smooth differential forms on X^{an} for curves, so let X be from now on an algebraic curve over K.

In Subsection 2.2, we gave definitions regarding polyhedral complexes in any dimension. To introduce the $\mathbb{A}$-tropical chart approach, we need some further definitions in this setting. As we are are only interested in curves, we will restrict these definitions to dimension one. The defined objects are named as in [**JW18**] and for higher dimensions we refer to [**Jel16a**].

Definition 4.2.1. A polyhedral complex $\mathscr{C}$ of dimension one in $\mathbb{R}^r$ is a finite set of closed intervals, half-lines, lines and points in $\mathbb{R}^r$ such that all endpoints are in $\mathscr{C}$ and the intersection of two elements is empty or a point in $\mathscr{C}$. We call the closed intervals, half-lines and lines *edges* of $\mathscr{C}$ and the points are called *vertices* of $\mathscr{C}$. We only consider non-trivial one-dimensional polyhedral complexes, i.e. complexes which do not only consist of vertices. A *polyhedral $\mathbb{R}$-rational curve Y in $\mathbb{T}^r$* is the topological closure of $|\mathscr{C}|$ for a one-dimensional $\mathbb{R}$-rational polyhedral complex $\mathscr{C}$. A *polyhedral structure $\mathscr{C}$* on Y is a polyhedral structure $\mathscr{C}'$ of $Y \cap \mathbb{R}^r$ plus the vertices at infinity (i.e. points in $Y \setminus \mathbb{R}^r$). Then $\mathscr{C}$ is called *weighted* (resp. *balanced*) if $\mathscr{C}'$ is weighted (resp. balanced). We call Y *weighted* if it has a weighted polyhedral structure $\mathscr{C}$.

A *tropical curve Y* is a $\mathbb{R}$-rational polyhedral curve with a balanced weighted polyhedral structure $\mathscr{C}$, up to weight preserving subdivision of $\mathscr{C}$. Using terms from Subsection 2.2, a tropical curve is the topological closure of the support of some tropical cycle of pure dimension one.

Up to now, we always considered closed embeddings $\varphi\colon U \to \mathbb{G}_m^r$ of open subsets U of X. The approach presented in this section, works instead with closed embeddings into $\mathbb{A}^r$.

Definition 4.2.2. Set $\mathbb{T} := [-\infty, \infty)$ and equip it with the topology of a half-open interval. Then $\mathbb{T}^r$ is equipped with the product topology.

For a closed subvariety Z of $\mathbb{A}^r = \mathrm{Spec}(K[T_1, \dots, T_r])$, we define $\mathrm{Trop}(Z)$ to be the image of Z^{an} under the (extended) tropicalization map

$$\begin{aligned} \mathrm{trop}\colon\ \mathbb{A}^{r,\mathrm{an}} &\to \mathbb{T}^r, \\ p &\mapsto (\log(p(T_1)), \dots, \log(p(T_r))). \end{aligned}$$

Let U be an open subset of X and $\varphi\colon U \to \mathbb{A}^r$ be a closed embedding such that one has $\varphi(U) \cap \mathbb{G}_m^r \neq \emptyset$. We say that φ *is given by* $f_1, \dots, f_r \in \mathcal{O}_X(U)$ if the corresponding K-algebra homomorphism $\varphi^\sharp\colon K[T_1, \dots, T_r] \to \mathcal{O}_X(U)$ maps T_i to f_i. Then the function

$\varphi_{\mathrm{trop}} := \mathrm{trop} \circ \varphi^{\mathrm{an}} \colon U^{\mathrm{an}} \to \mathbb{T}^r$ is given by $p \mapsto (\log(p(f_1)), \ldots, \log(p(f_r)))$. The map φ_{trop} is proper in the sense of topological spaces. We write $\mathrm{Trop}_\varphi(U) := \varphi_{\mathrm{trop}}(U^{\mathrm{an}})$.

Recall from Subsection 2.2 that for every open subset U of X and for every closed embedding $\varphi \colon U \to \mathbb{G}_m^r$, the tropicalization $\mathrm{Trop}_\varphi(U)$ is the support of a tropical cycle with the canonical weights from [**Gub13**, 13.10]. We see in the subsequent theorem an analog for closed embeddings $\varphi \colon U \to \mathbb{A}^r$ with $\varphi(U) \cap \mathbb{G}_m^r \neq \emptyset$.

THEOREM 4.2.3. *The space* $\mathrm{Trop}_\varphi(U)$ *from Definition 4.2.2 with the canonical weights from* [**Gub13**, 13.10] *is a tropical curve.*

PROOF. Follows directly from Theorem 2.2.6 combined with Remark 2.2.7. □

In Section 4.1, we introduced smooth differential forms on the support of a polyhedral complex. We can extend this definition to their closures, i.e. in our case to polyhedral $\mathbb{R}$-rational curves in $\mathbb{T}^r$.

DEFINITION 4.2.4. Let Y be an $\mathbb{R}$-rational polyhedral curve in $\mathbb{T}^r$ and Ω an open subset of Y. A form α of bidegree (p,q) on Ω is given by a form $\alpha' \in \mathcal{A}^{p,q}(\Omega')$ for $\Omega' := \Omega \cap \mathbb{R}^r$ (cf. Definition 4.1.2), satisfying the following *boundary conditions*: For each $x \in \Omega \setminus \Omega'$, there exists a neighborhood Ω_x of x in Ω such that

i) if $(p,q) = (0,0)$, then $\alpha'|_{\Omega_x \cap \mathbb{R}^r}$ is constant (note here that α' is indeed a function on Ω' in that case),
ii) otherwise, we require $\alpha'|_{\Omega_x \cap \mathbb{R}^r} = 0$.

We denote the space of such (p,q)-forms on Ω by $\mathcal{A}_Y^{p,q}(\Omega)$ and by $\mathcal{A}_{Y,c}^{p,q}(\Omega)$ the space of forms with compact support.

The presheaf $\Omega \mapsto \mathcal{A}_Y^{p,q}(\Omega)$ is indeed a sheaf on Y. We define the differentials and the wedge product by applying the respective operation to the forms on Ω'.

REMARK 4.2.5. Let Y be a tropical curve and let Ω be an open subset of Y. There is also a canonical non-trivial integration map $\int \colon \mathcal{A}_{Y,c}^{1,1}(\Omega) \to \mathbb{R}$ which satisfies $\int d''\alpha = 0$ for all $\alpha \in \mathcal{A}_{Y,c}^{1,0}(\Omega)$ [**JSS19**, Definition 4.5 & Theorem 4.9]. Note here that open subsets of tropical curves are tropical spaces in the sense of [**JSS19**, Definition 4.8].

DEFINITION 4.2.6. Let $U' \subset \mathbb{T}^{r'}$ and $U \subset \mathbb{T}^r$ be open subsets. An *extended linear resp. affine map* $F \colon U' \to U$ is the continuous extension (which may not always exist (!)) of a linear resp. affine map $F \colon U' \cap \mathbb{R}^{r'} \to U \cap \mathbb{R}^r$.

REMARK 4.2.7. There is a well-defined, functorial pullback of superforms along extended affine maps F which commutes with d'' and $\wedge$. If F maps a polyhedral curve Y_1 to a polyhedral curve Y_2, then this induces a pullback $F^* \colon \mathcal{A}_{Y_2}^{p,q} \to F_* \mathcal{A}_{Y_1}^{p,q}$.

THEOREM 4.2.8 (Poincaré lemma). *Let Y be a tropical curve and let Ω be a connected open subset of Y, which, for some polyhedral structure $\mathscr{C}$ on Y, contains at most one vertex. Let $\alpha \in \mathcal{A}_Y^{p,q}(\Omega)$ such that $q > 0$ and $d''\alpha = 0$. Then there exists $\beta \in \mathcal{A}_Y^{p,q-1}(\Omega)$ such that $d''\beta = \alpha$.*

PROOF. If Ω contains a vertex z which is not at infinity, then it is polyhedrally star-shaped with center z in the sense of [**Jel16b**, Definition 2.2.11]. If Ω contains no vertex, then it is just an open line segment and polyhedrally star shaped with respect to any of its points. Thus if Ω contains no vertex at infinity, the result follows from [**Jel16b**, Theorem 2.16]. If the vertex is at infinity, Ω is a half-open line with vertex at infinity, thus basic open in the sense of [**JSS19**, Definition 3.7] and the statement follows from [**JSS19**, Theorem 3.22 & Proposition 3.11]. □

DEFINITION 4.2.9. Let $\varphi\colon U \to \mathbb{A}^r$ be a closed embedding. Then another embedding $\varphi'\colon U' \to \mathbb{A}^{r'}$ is called a *refinement* of φ if $U' \subset U$ and there exists a torus equivariant map $\psi\colon \mathbb{A}^{r'} \to \mathbb{A}^r$ such that $\varphi|_{U'} = \psi \circ \varphi'$. If that is the case, we obtain an extended linear map $\operatorname{Trop}(\psi)\colon \operatorname{Trop}_{\varphi'}(U') \to \operatorname{Trop}_\varphi(U)$ which satisfies $\varphi_{\mathrm{trop}} = \operatorname{Trop}(\psi) \circ \varphi'_{\mathrm{trop}}$ on $(U')^{\mathrm{an}}$. Since we have $\operatorname{Trop}_{\varphi'}(U') = \varphi'_{\mathrm{trop}}((U')^{\mathrm{an}})$, the map $\operatorname{Trop}(\psi)$ is independent of ψ. We call this map the *transition map* between φ and φ'.

Note that if for $i = 1, \dots, n$ we have closed embeddings $\varphi_i\colon U_i \to \mathbb{A}^{r_i}$, then the product map $\varphi_1 \times \dots \times \varphi_n\colon \bigcap_{i=1}^n U_i \to \prod_{i=1}^n \mathbb{A}^{r_i}$ is a common refinement of all the φ_i.

DEFINITION 4.2.10. An $\mathbb{A}$-*tropical chart* is given by a pair (V, φ), where $V \subset X^{\mathrm{an}}$ is an open subset and $\varphi\colon U \to \mathbb{A}^r$ is a closed embedding of an affine open subset U of X such that $V = \varphi_{\mathrm{trop}}^{-1}(\Omega)$ for an open subset $\Omega \subset \operatorname{Trop}_\varphi(U)$.

Another $\mathbb{A}$-tropical chart (V', φ') is called an $\mathbb{A}$-*tropical subchart* of (V, φ) if φ' is a refinement of φ and $V' \subset V$.

Note that if (V, φ) is a tropical chart and $\varphi'\colon U' \to \mathbb{A}^{r'}$ is a refinement of φ for an affine open subset U' of X such that $V \subset U'^{\mathrm{an}}$, then (V, φ') is an $\mathbb{A}$-tropical chart, thus in particular a subchart of (V, φ). To see this, let $\operatorname{Trop}(\psi)$ be the transition map between φ and φ' and Ω such that $V = \varphi_{\mathrm{trop}}^{-1}(\Omega)$. Then $V = \varphi'^{-1}_{\mathrm{trop}}(\operatorname{Trop}(\psi)^{-1}(\Omega))$.

$\mathbb{A}$-tropical charts form a basis of the topology of X^{an} [**Jel16a**, Lemma 3.2.35 & Lemma 3.3.2]. We will show this for $X = \mathbb{A}^1$ in Lemma 4.2.41.

Using these $\mathbb{A}$-tropical charts, we can define forms analogously as in Section 4.1.

DEFINITION 4.2.11. Let X be an algebraic curve and let V be an open subset of X^{an}. An element of $\mathcal{A}^{p,q}_{X,\mathbb{A}}(V)$ is given by a family $(V_i, \varphi_i, \alpha_i)_{i \in I}$ such that:

i) For all $i \in I$ the pair (V_i, φ_i) is an $\mathbb{A}$-tropical chart and $\bigcup_{i \in I} V_i = V$.
ii) For all $i \in I$ we have $\alpha_i \in \mathcal{A}^{p,q}_{\operatorname{Trop}_{\varphi_i}(U_i)}(\varphi_{i,\mathrm{trop}}(V_i))$.
iii) For all $i, j \in I$ the forms α_i and α_j agree when pulled back to $V_i \cap V_j$ via the corresponding transition maps.

Another such family $(V_j, \varphi_j, \alpha_j)_{j \in J}$ defines the same form on V if $(V_i, \varphi_i, \alpha_i)_{i \in I \cup J}$ still defines a form on V.

Let $V' \subset V$ and let $\alpha \in \mathcal{A}^{p,q}_{X,\mathbb{A}}(V)$ be given by $(V_i, \varphi_i, \alpha_i)_{i \in I}$. The subset V' can be covered by $\mathbb{A}$-tropical subcharts (W_{ij}, φ_{ij}) of (V_i, φ_i) [**Jel16a**, Lemma 3.2.35]. We define $\alpha|_{V'}$ to be given by the family $(W_{ij}, \varphi_{ij}, \operatorname{Trop}(\psi_{i,ij})^* \alpha_i)_{ij}$. Note that $\alpha|_{V'}$ is independent of all choices.

Then $V \mapsto \mathcal{A}^{p,q}_{X,\mathbb{A}}(V)$ defines a sheaf on X^{an}, which we denote by $\mathcal{A}^{p,q}_{X,\mathbb{A}}$. By Theorem 4.2.3, we have $\mathcal{A}^{p,q}_{X,\mathbb{A}} = 0$ if $\max(p, q) > 1$.

The differentials d', d'' and the wedge product carry over.

DEFINITION 4.2.12. For an $\mathbb{A}$-tropical chart (V, φ), we call the map

$$\mathcal{A}^{p,q}_{X,\mathbb{A}}(\varphi_{\mathrm{trop}}(V)) \to \mathcal{A}^{p,q}_{X,\mathbb{A}}(V)$$

the *pullback along* φ_{trop}.

PROPOSITION 4.2.13. *Let (V, φ) be an $\mathbb{A}$-tropical chart. Then the pullback along φ is injective. Furthermore, if a form $\alpha \in \mathcal{A}^{p,q}_{X,\mathbb{A}}(V)$ is given by (V, φ, α'), then we have $\varphi_{\mathrm{trop}}(\operatorname{supp}(\alpha)) = \operatorname{supp}(\alpha')$.*

PROOF. The first statement is shown in [**Jel16a**, Lemma 3.2.42]. The second statement then follows from the first as in [**CD12**, Corollaire 3.2.3]. □

For $V \subset X^{\mathrm{an}}$ open, there is a non-trivial integration map $\int : \mathcal{A}^{1,1}_{X,\mathbb{A},c}(V) \to \mathbb{R}$ which is compatible with pullback and satisfies $\int d''\alpha = 0$ for all $\alpha \in \mathcal{A}^{1,0}_{X,\mathbb{A},c}(V)$. For an explicit description we refer to [**Jel16a**, Definition 3.2.58].

PROPOSITION 4.2.14. *Let K be non-trivially valued and let X be an algebraic curve over K. Then for the sheaf $\mathcal{A}^{p,q}_X$ on X^{an} from Section 4.1, we have an isomorphism of sheaves*

$$\mathcal{A}^{p,q}_X \simeq \mathcal{A}^{p,q}_{X,\mathbb{A}}$$

for all $p, q \in \{0,1\}$ such that integration is compatible with this isomorphism.

PROOF. See [**Jel16a**, Theorem 3.2.41& Lemma 3.2.58]. □

REMARK 4.2.15. Note that $\mathcal{A}^{p,q}_{X,\mathbb{A}}$ is defined also for higher dimensional varieties in [**Jel16a**, §3.2.2] and we also have the isomorphism in every dimension.

Due to this isomorphism which commutes with all considered constructions, we will not distinguish the sheaves anymore and also write $\mathcal{A}^{p,q}_X$ for $\mathcal{A}^{p,q}_{X,\mathbb{A}}$, or just $\mathcal{A}^{p,q}$ if the space of definition is clear. Note that we do just work with $\mathbb{A}$-tropical charts instead of tropical charts in Section 4.2.

4.2.2. Tropical Dolbeault cohomology. In this subsection, we consider the induced cohomology of the complex $(\mathcal{A}^{p,\bullet}, d'')$, the so called *tropical Dolbeault cohomology*. We also give here some preliminary results for the proofs later on.

DEFINITION 4.2.16. Let X be an algebraic resp. an $\mathbb{R}$-rational polyhedral curve and V an open subset of X^{an} resp. X. Then we denote the presheaf $V \mapsto \mathcal{A}^{p,q}_{X,c}(V)^*$ by $\mathcal{A}^{p,q}_{X,c}{}^*$. Note that we do not put any topology on $\mathcal{A}^{p,q}_X$, thus the dual is always meant in the sense of linear algebra. We write $\mathrm{H}^{p,q}(V) := \mathrm{H}^q(\mathcal{A}^{p,\bullet}(V), d'')$ and $\mathrm{H}^{p,q}_c(V) := \mathrm{H}^q(\mathcal{A}^{p,\bullet}_c(V), d'')$ and denote by $h^{p,q}(V)$ resp. $h^{p,q}_c(V)$ the respective $\mathbb{R}$-dimensions. We denote by d''^* the dual of the differential operator and use the notations $\mathcal{L}^p_Y := \ker(d'' \colon \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X)$ and $\mathcal{G}^p_Y := \ker(d''^* \colon \mathcal{A}^{p,1}_{X,c}{}^* \to \mathcal{A}^{p,0}_{Y,c}{}^*)$.

We denote by $h^q_{\mathrm{sing}}(V)$ resp. $h^q_{c,\mathrm{sing}}(V)$ the $\mathbb{R}$-dimension of the singular cohomology $\mathrm{H}^q_{\mathrm{sing}}(V, \mathbb{R})$ resp. $\mathrm{H}^q_{c,\mathrm{sing}}(V, \mathbb{R})$.

THEOREM 4.2.17. *Let X be an algebraic resp. a tropical curve and let V be an open subset of X^{an} resp. X. Then $\mathcal{L}^p_V \to (\mathcal{A}^{p,\bullet}_V, d'')$ is an acyclic resolution of $\mathcal{L}^p$. In particular, we have $\mathrm{H}^q(V, \mathcal{L}^p) = \mathrm{H}^q((\mathcal{A}^{p,\bullet}(V), d''))$ and $\mathrm{H}^q_c(V, \mathcal{L}^p) = \mathrm{H}^q((\mathcal{A}^{p,\bullet}_c(V), d''))$. Here H^q_c denotes sheaf cohomology with compact support. We also have $\mathcal{L}^0_V = \underline{\mathbb{R}}$, where $\underline{\mathbb{R}}$ denotes the constant sheaf with stalks $\mathbb{R}$. As a consequence, we obtain $h^{0,q}(V) = h^q_{\mathrm{sing}}(V)$ and $h^{0,q}_c(V) = h^q_{c,\mathrm{sing}}(V)$.*

PROOF. The sheaves $\mathcal{A}^{p,q}_X$ are fine for a tropical curve X [**JSS19**, Lemma 2.15]. As a consequence, the sheaves $\mathcal{A}^{p,q}_X$ are also fine for an algebraic curve X [**Jel16a**, Lemma 3.2.17, Proposition 3.2.46 & Lemma 3.3.6]. That $(\mathcal{A}^{p,\bullet}_V, d'')$ is exact in positive degree follows from Theorem 4.2.8 for the tropical case and from [**Jel16a**, Theorem 3.4.3] for the algebraic case. That $\mathcal{L}^0_V = \underline{\mathbb{R}}$ is shown in [**Jel16a**, Lemma 3.4.5]. The rest now follows from standard sheaf theory since V is Hausdorff and locally compact. The fact that $\mathcal{A}^{p,q}_V$ are acyclic for both the functor of global section respectively global sections with compact support follows from [**Wel80**, Chapter II, Proposition 3.5 & Theorem 3.11] and [**Ive86**, III, Theorem 2.7] and identification with singular cohomology comes from [**Bre97**, Chapter III, Theorem 1.1]. □

LEMMA 4.2.18. *Let X be an algebraic resp. an $\mathbb{R}$-rational polyhedral curve. Then the presheaf ${\mathcal{A}^{p,q}_{X,c}}^*$ is a sheaf and flasque. Furthermore,*

$$\mathrm{H}^q(\mathcal{A}^{1-p,1-\bullet *}_{X,c}(V), d''^*) = \mathrm{H}^{1-p,1-q}_c(V)^* \tag{4.2.1}$$

for every open subset V of X^{an} resp. X.

PROOF. It is a sheaf because $\mathcal{A}^{p,q}_X$ admits partitions of unity and it is flasque because for all $W \subset V$ the map $\mathcal{A}^{p,q}_{X,c}(W) \to \mathcal{A}^{p,q}_{X,c}(V)$ is injective. The second assertion is true since dualizing is an exact functor. □

DEFINITION 4.2.19. Let X be an algebraic resp. a tropical curve and let V be an open subset of X^{an} resp. X. We define

$$\mathrm{PD}\colon \mathcal{A}^{p,q}(V) \to \mathcal{A}^{1-p,1-q}_c(V)^*,\ \alpha \mapsto (\beta \mapsto \varepsilon \int \alpha \wedge \beta)$$

where $\varepsilon = 1$ if $p = q = 0$ and $\varepsilon = -1$ else. The morphism PD defined above induces a morphism of complexes $\mathrm{PD}\colon \mathcal{A}^{p,\bullet}(V) \to \mathcal{A}^{1-p,1-\bullet}_c(V)^*$ where the complex $\mathcal{A}^{p,\bullet}(V)$ is equipped with d'' and $\mathcal{A}^{1-p,1-\bullet}_c(V)^*$ with its dual map d''^*.

Hence we get a morphism $\mathrm{PD}\colon \mathrm{H}^{p,q}(V) \to \mathrm{H}^{1-p,1-q}_c(V)^*$ by (4.2.1). We say that V *has* PD if the map on cohomology is an isomorphism for all (p, q).

LEMMA 4.2.20. *The map* PD *defined above induces a monomorphism $\mathcal{L}^p \to \mathcal{G}^{1-p}$.*

PROOF. That PD maps $\mathcal{L}^p$ to $\mathcal{G}^{1-p}$ follows because PD is a morphism of complexes. To show that this map is injective it is sufficient to show that $\mathrm{PD}\colon \mathcal{A}^{p,0} \to \mathcal{A}^{1-p,1*}_c$ is injective, i.e. that for all $\alpha \in \mathcal{A}^{p,0}(V) \setminus \{0\}$ there exists $\beta \in \mathcal{A}^{1-p,1}_c(V)$ such that $\int_V \alpha \wedge \beta \neq 0$. In the tropical situation, there exists an open subset Ω which is contained in an edge σ and a coordinate x on σ such that $\alpha|_\Omega = \pm f$ resp. $\alpha|_\Omega = f d'x$ for $f > 0$. Then letting g be a bump function with compact support in Ω and $\beta = \pm g d'x \otimes d''x$ resp. $\beta = g d''x$ suffices.

In the algebraic situation, let α be given by $(V_i, \varphi_i, \alpha_i)$. Choose i such that $\alpha_i \neq 0$ and $\beta_i \in \mathcal{A}^{1-p,1}_c(\varphi_{i,\mathrm{trop}}(V_i))$ with $\int_{\varphi_{i,\mathrm{trop}}(V_i)} \alpha_i \wedge \beta_i \neq 0$. By definition of integration, we have $\int_V \alpha \wedge \varphi^*_{i,\mathrm{trop}}(\beta_i) = \int_{\varphi_{i,\mathrm{trop}}(V_i)} \alpha_i \wedge \beta_i$ which proves the claim. □

DEFINITION 4.2.21. Let z be a vertex of a tropical curve Y and let $\sigma_0, \ldots, \sigma_k$ be the edges which contain z. We define $\mathrm{val}(z) := k + 1$. Furthermore, we define $\dim(z) := \dim(\mathbb{L}(\sigma_0) + \ldots + \mathbb{L}(\sigma_k))$ if z is not at infinity and $\dim(z) := 0$ if z is at infinity.

A tropical curve Y is called *smooth* if all weights are 1 and for every vertex z we have $\mathrm{val}(z) = \dim(z) + 1$.

THEOREM 4.2.22. *Open subsets of smooth tropical curves have* PD.

PROOF. Let Y be a smooth tropical curve and let Ω be an open subset. Let $\mathscr{C}$ be a polyhedral structure on Y. Since tropical manifolds in the sense of [**JSS19**, Definition 4.15] have PD by [**JSS19**, Theorem 4.33], it is enough to show that any point $z \in \Omega$ has a neighborhood which is either isomorphic to an open subset of $\mathbb{T}$ or to an open subset of a Bergmann fan $B(M)$ of a matroid M. If z is not a vertex, it has a neighborhood which is isomorphic to an open interval, thus to an open subset of $\mathbb{T}$. If z is a vertex at infinity, it has a neighborhood isomorphic to $[-\infty, b) \subset \mathbb{T}$ for some $b \in \mathbb{R}$. Now let z be a vertex which is not at infinity and let k be as in Definition 4.2.21. Let $U_{2,k}$ be the uniform matroid of rank 2 on k Elements, i.e. the base set is $\{1, \ldots, k\}$ and the rank function is $A \mapsto \max\{\#A, 2\}$. Following the construction of the Bergman fan in [**Sha13**, §2.4] we find that $B(U_{2,k})$ is the fan whose rays are spanned by $-e_1, \ldots, -e_k$

and $\sum_{i=1}^k e_i$ where e_i denotes the i-th unit vector in $\mathbb{R}^k$. Now, after translation of z to the origin, $\nu_{z,\sigma_i} \mapsto e_i$ for $i = 1, \dots, k$ provides a linear isomorphism of a neighborhood of z with an open neighborhood of the origin in $B(U_{2,k})$. Note here that $-\sigma_{z,\sigma_0} \mapsto \sum e_i$ by the balancing condition and that ν_{z,σ_i} for $i = 1, \dots, k$ are linearly independent since $\dim(z) = \mathrm{val}(z) + 1$. □

CONSTRUCTION 4.2.23. We now describe the operation of tropical modification (for a more detailed introduction see [**BIMS15**]). Let $Y \subset \mathbb{T}^r$ be a tropical curve and $P \colon Y \to \mathbb{R}$ a continuous, piecewise affine function with integer slopes. The graph $\Gamma_Y(P)$ of P is a polyhedral $\mathbb{R}$-rational curve in $\mathbb{T}^{r+1}$. Choosing a polyhedral structure $\mathscr{C}$ on Y such that P is affine on every edge and defining the weight $m_{\Gamma_\sigma(P)} := m_\sigma$ for every $\sigma \in \mathscr{C}$ makes $\Gamma_Y(P)$ into a weighted $\mathbb{R}$-rational polyhedral curve. It is however not balanced because P is only piecewise affine. Let $z \in Y$ be a point where P is not affine. If we add a line $\sigma_z := [(z, P(z)), (z, -\infty)]$, then there is a unique weight m_{σ_z} to make $\Gamma(P) \cup \sigma_z$ balanced at z. If we do this for every such $z \in Y$, we obtain a tropical curve Y'. The projection $\pi \colon \mathbb{T}^{r+1} \to \mathbb{T}^r$ restricts to a map $\delta \colon Y' \to Y$ and we call δ a *tropical modification.*

Note that δ is a proper map in the sense of topological spaces.

PROPOSITION 4.2.24. *Let $\delta \colon Y' \to Y$ be a tropical modification. Let V be an open subset of Y and set $V' := \delta^{-1}(V)$. Then*

$$\delta^* \colon \mathrm{H}^{p,q}(V) \to \mathrm{H}^{p,q}(V') \qquad \textit{and} \qquad \delta^* \colon \mathrm{H}^{p,q}_c(V) \to \mathrm{H}^{p,q}_c(V')$$

are isomorphisms which are compatible with the Poincaré duality map.

PROOF. The proof uses identification of $\mathrm{H}^{p,q}$ with tropical cohomology [**JSS19**, Theorem 3.22]. It is shown in [**Sha15**, Theorem 4.13] that $P(V)$ is a strong deformation retract in V' and that this retraction and the homotopy respect the polyhedral structure of V resp. V'. Thus the usual prism operator argument for singular cohomology shows that the tropical cohomology groups of V and V' agree and thus [**JSS19**, Theorem 3.22 & Proposition 3.24] show that δ^* induces an isomorphism on $\mathrm{H}^{p,q}$ resp. $\mathrm{H}^{p,q}_c$.

That this is compatible with the PD map is just saying that integration of a $(1,1)$-form with compact support commutes with pullback along δ. This follows from the tropical projection formula [**Gub16**, Proposition 3.10]. □

4.2.3. Non-archimedean Mumford curves. In this subsection, we recall the definition of Mumford curves and give a further characterization of them which is needed in Sections 4.2.5 and 4.2.6.

DEFINITION 4.2.25. We say that an analytic space Y is *locally isomorphic to* $\mathbb{P}^{1,\mathrm{an}}$ if there is a cover of Y by open subsets which are isomorphic to open subsets of $\mathbb{P}^{1,\mathrm{an}}$ in the sense of analytic spaces.

Let X be a smooth algebraic curve over K and $x \in X^{\mathrm{an}}$. We denote by $\mathscr{H}(x)$ the completed residue field at x and by $\widetilde{\mathscr{H}(x)}$ its residue field. The point x is of type II if $\widetilde{\mathscr{H}(x)}$ is of transcendence degree 1 over the residue field $\widetilde{K}$ of K. If this is the case, the *genus* of x is defined as the genus of the smooth projective $\widetilde{K}$-curve with function field $\widetilde{\mathscr{H}(x)}$.

Note that there are only finitely many points of type II of positive genus in X^{an} [**BPR13**, Remark 4.18].

Proposition 4.2.26. *Let X be a smooth curve and let $V \subset X^{\mathrm{an}}$ be an open subset. Then V is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$ if and only if it does not contain any point of type II with positive genus.*

Proof. By [**BR10**, Proposition 2.3], every type II point of $\mathbb{P}^{1,\mathrm{an}}$ has genus zero. Hence if V is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$, all its type II points have genus 0.

Now, we assume that every type II point in V has genus zero. Then [**Ber07**, Proposition 2.2.1] says that we have an open covering of V by open balls, open annuli and sets which are isomorphic to $\mathbb{P}^{1,\mathrm{an}}$ without the disjoint union of a finite number of closed balls. Note that we have used that the definition of the genus of a point in [**Ber07**] is equal to the one in Definition 4.2.25 [**Ber07**, p. 31]. Hence V is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$. □

Definition 4.2.27. A smooth proper curve X of genus $g \geq 1$ is called a *Mumford curve* if there is a semistable formal model $\mathscr{X}$ of X^{an} such that all irreducible components of the special fiber $\mathscr{X}_s$ are rational (cf. [**Ber90**, Theorem 4.4.1]).

Remark 4.2.28. In [**Ber90**] (and also in [**JW18**]), one uses algebraic models of X instead of formal models of X^{an} for the definition of Mumford curves. However, this does not make any difference because having an algebraic model of X (resp. a formal model of X^{an}) there is always a formal model of X^{an} (resp. an algebraic model of X) with the same special fiber (cf. [**BPR13**, Remark 4.2]).

Moreover, note that there exist no Mumford curves over a trivially valued field K since otherwise $R = K$ and so the only semistable algebraic model of X is X itself, which cannot be rational due to $g \geq 1$.

Theorem 4.2.29. *Let X be a smooth proper curve over K of genus g. Then the following properties are equivalent:*

i) X is a Mumford curve or is isomorphic to $\mathbb{P}^1$.
ii) X^{an} is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$.
iii) $h^{0,1}(X^{\mathrm{an}}) = g$.

Proof. If $g = 0$, then X is isomorphic to $\mathbb{P}^1$ and all three properties are satisfied. Indeed, the third one is true since $h^{0,1}(\mathbb{P}^{1,\mathrm{an}}) = h^1_{\mathrm{sing}}(\mathbb{P}^{1,\mathrm{an}})$ by Theorem 4.2.17 and $\mathbb{P}^{1,\mathrm{an}}$ is contractible.

If $g \geq 1$, property i) implies ii) by [**Ber90**, Theorem 4.4.1]. Note here that for any analytic space Y, the topological universal cover $\pi \colon Z \to Y$ of Y is given by the analytic structure which makes π into a local isomorphism. Thus X^{an} is locally isomorphic to its universal cover.

On the other hand, if ii) is satisfied, we know from Proposition 4.2.26 that every type II point in X^{an} has genus zero. Let $\mathscr{X}$ be any semistable model. Then every irreducible component of the special fiber corresponds to a type II point $x \in X^{\mathrm{an}}$, and we denote this component by C_x. The curve C_x is birationally equivalent to the smooth proper $\widetilde{K}$-curve with function field $\widetilde{\mathscr{H}}(x)$ by [**Ber90**, Proposition 2.4.4]. We know that the latter curve is of genus zero, and so C_x is as well. Thus every irreducible component C_x is rational.

Since the skeleton of a Berkovich space is a deformation retract, iii) is equivalent to the skeleton of X^{an} having first Betti number equal to g by Theorem 4.2.17. Thus we know from [**Ber90**, Theorem 4.6.1], that we have $h^{0,1}(X^{\mathrm{an}}) = g$ if and only if X is a Mumford curve or a principal homogeneous space over a Tate elliptic curve. The first sentence in the proof of [**Ber90**, Lemma 4.6.2] shows that if K is algebraically closed, the only principal homogeneous space over any Tate elliptic curve is the curve itself.

Since Tate elliptic curves are indeed Mumford curves, we have equivalence of i) and iii) also for $g \geq 1$. Thus all properties are equivalent. □

4.2.4. Cohomology of open subsets of the Berkovich affine line. The goal of this subsection is to get a better description of the cohomology of a basis of open subsets of $\mathbb{A}^{1,\mathrm{an}}$ (cf. Theorem 4.2.43). We use this description to prove Poincaré duality for a special class of open subsets of the analytification X^{an} of a smooth algebraic curve X in Subsection 4.2.5.

First, we consider a special class of embeddings of the affine line into affine spaces, which are called linear embeddings. We show that their tropicalizations are smooth tropical curves and that their refinements induce tropical modifications (Theorem 4.2.37 and Theorem 4.2.36).

Definition 4.2.30. A closed embedding $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^r$ is called a *linear embedding* if φ is given by linear polynomials $(x - a_i)_{i \in [r]}$, where $[r] := \{1, \ldots, r\}$.

Another linear embedding $\varphi'\colon \mathbb{A}^1 \to \mathbb{A}^{r'}$ is called a *linear refinement* if $\varphi = \pi \circ \varphi'$, where π is the projection to a set of coordinates.

Lemma 4.2.31. *Let $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^r$ be a linear embedding. Then all weights on* $\mathrm{Trop}_\varphi(\mathbb{A}^1)$ *are equal to 1.*

Proof. Let φ be given by $x - a_1, \ldots, x - a_r$ and let σ be an edge of $\mathrm{Trop}_\varphi(\mathbb{A}^1)$. Then there exists some coordinate i such that the restriction of the i-th coordinate function to σ is not constant. Obviously, φ is a refinement of the linear $\mathbb{A}$-tropical chart $\varphi'\colon \mathbb{A}^1 \to \mathbb{A}^1$ which is given by $x - a_i$. The weights on $\mathrm{Trop}_{\varphi'}(\mathbb{A}^1)$ are obviously all 1 and thus so is the one on σ by the Sturmfels–Tevelev multiplicity formula and the construction of the pushforward of tropical cycles [**Gub16**, 4.10 & Proposition 4.11]. Note here, that by the choice of i, the edge σ is not contracted to a point when projecting from $\mathrm{Trop}_\varphi(\mathbb{A}^1)$ to $\mathrm{Trop}_{\varphi'}(\mathbb{A}^1)$. □

Lemma 4.2.32. *Let R' be a commutative ring with* 1 *and let $p\colon R' \to \mathbb{R}_{\geq 0}$ be a non-archimedean multiplicative seminorm on R'. Furthermore, let $x, a_i, a_j, b \in R'$ such that $p(x - a_i) \leq p(x - a_j)$ and $p(x - a_j) \neq p(b - a_j)$. Then we have*

$$\max(p(x - a_i), p(b - a_i)) = \max(p(x - a_j), p(b - a_j)).$$

Proof. If the maximum on the left hand side is attained uniquely, then both sides equal $p(x - b)$ by the ultrametric triangle inequality, and so the lemma holds.

If not, we have

$$\begin{aligned} p(x - b) &\leq \max(p(x - a_i), p(b - a_i)) = p(x - a_i) \\ &\leq p(x - a_j) \leq \max(p(x - a_j), p(b - a_j)) = p(x - b), \end{aligned}$$

and so equality holds as well. □

We fix a closed embedding $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^r$ which is given by r linear polynomials $x - a_1, \ldots, x - a_r$. For $b \in K$ we want to understand the behavior of the refinement $\varphi'\colon \mathbb{A}^1 \to \mathbb{A}^{r+1}$ which is given by $x - a_1, \ldots, x - a_r, x - b$. We will for the moment assume that $b \neq a_j$ for all $j \in [r]$.

Definition 4.2.33. We define a map $P := P_{a_1, \ldots, a_r, b}\colon \mathrm{Trop}_\varphi(\mathbb{A}^1) \to \mathbb{R}$ in the following way: For $z \in \mathrm{Trop}_\varphi(\mathbb{A}^1)$ choose $i \in [r]$ such that $z_i \leq z_j$ for all $j \in [r]$ and define $P(z) := \max(z_i, \log|b - a_i|)$.

The next proposition shows the basic properties of P.

Proposition 4.2.34. *In the situation above, we have:*

i) $P(z)$ *is well-defined, independent of the choice of* i.

ii) For $j \in [r]$ *we have*

$$P(z) = \max(z_j, \log|b - a_j|)$$

if the maximum on the right hand side is attained uniquely.

Proof. The second statement follows from Lemma 4.2.32 by choosing $R' = K[x]$ and letting p be a point in $\mathbb{A}^{1,\mathrm{an}}$ which maps to z. The first follows from the second and the fact that for two minima z_i, z_j the equality $\max(z_i, \log|b-a_i|) = \max(z_j, \log|b-a_j|)$ is trivial if $z_i = \log|b - a_i| = z_j = \log|b - a_j|$. □

Lemma 4.2.35. *The function* P *is continuous. It is affine at every point except* $(\log|a_i - b|)_{i\in[r]}$. *In particular,* P *is piecewise affine.*

Proof. Let $\mathscr{C}$ be a polyhedral structure on $\mathrm{Trop}_\varphi(\mathbb{A}^1)$ such that for each $\sigma \in \mathscr{C}$ there exists $i_\sigma \in [r]$ such that for all $z \in \sigma$ we have $z_{i_\sigma} \le z_j$ for all $j \in [r]$. Then $P|_\sigma$ is continuous by definition, thus P is continuous.

If $z \neq (\log|a_i - b|)_{i\in[r]}$, then there exists $j \in [r]$ such that $\max(z_j, \log|b - a_j|)$ is attained uniquely. Since this maximum is then attained uniquely for all z' in a neighborhood of z, Proposition 4.2.34 ii) shows that P is either constant or the projection to the j-th coordinate on this neighborhood, so in particular affine. □

Theorem 4.2.36. *Let* $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^r$ *be a linear embedding and let* $\varphi'\colon \mathbb{A}^1 \to \mathbb{A}^{r+1}$ *be a linear refinement. We consider the commutative diagram*

$$\begin{tikzcd} & \mathbb{A}^{r+1} \arrow[d, "\pi"] \\ \mathbb{A}^1 \arrow[ur, "\varphi'"] \arrow[r, "\varphi"] & \mathbb{A}^r \end{tikzcd}$$

and the map $\mathrm{Trop}(\pi)\colon \mathrm{Trop}_{\varphi'}(\mathbb{A}^1) \to \mathrm{Trop}_\varphi(\mathbb{A}^1)$ *induced by the projection* $\mathbb{T}^{r+1} \to \mathbb{T}^r$. *Then* $\mathrm{Trop}(\pi)$ *is a tropical modification.*

Proof. Let φ be given by $(x - a_1), \ldots, (x - a_r)$ and φ' additionally by $(x - b)$. If there exists $j \in [r]$ such that $b = a_j$, then $z \mapsto (z, z_j)$ is an extended linear map which is an inverse of $\mathrm{Trop}(\pi)$. In particular, $\mathrm{Trop}(\pi)$ is an isomorphism and we are done. Thus we may assume that $b \neq a_j$ for all $j \in [r]$.

In the following, we write Y (resp. Y') for $\mathrm{Trop}_\varphi(\mathbb{A}^1)$ (resp. $\mathrm{Trop}_{\varphi'}(\mathbb{A}^1)$), use P as defined in Definition 4.2.33 and use the notation z_s for the point $(\log|b - a_i|)_{i\in[r]} \in \mathbb{R}^r$. We want to show that Y' is the completion of the graph $\Gamma_Y(P) \subset Y \times \mathbb{R} \subset \mathbb{T}^{r+1}$ to a tropical curve as explained in Construction 4.2.23.

We show that if $z \in Y \setminus \{z_s\}$, the unique preimage of z under $\mathrm{Trop}(\pi)\colon Y' \to Y$ is the point $(z, P(z))$ and that the preimage of z_s is the line $[(z_s, -\infty), (z_s, P(z_s))]$. We then conclude that P is indeed not linear in z_s and that the line $[(z_s, -\infty), (z_s, P(z_s))]$ is precisely needed to rebalance the graph $\Gamma_Y(P)$, which proves the claim.

At first, let $z \in Y\backslash\{z_s\}$ and consider $p \in \mathbb{A}^{1,\mathrm{an}}$ such that $\varphi_{\mathrm{trop}}(p) = z$. Then the ultrametric triangle inequality implies $P(z) = \log(p(x - b))$, which precisely means $\varphi'_{\mathrm{trop}}(p) = (z, P(z))$, and that this is the unique preimage of z under $\mathrm{Trop}(\pi)$.

Next, we consider the preimage of the remaining point $z_s \in Y$. Let $\eta(b, t) \in \mathbb{A}^{1,\mathrm{an}}$ be given by the multiplicative seminorm $f \mapsto \sup_{c\in D(b,t)} |f(c)|$ with $t \ge 0$. Using the ultrametric triangle inequality, we get $\varphi_{\mathrm{trop}}(\eta(b,t)) = z_s$ and $\varphi'_{\mathrm{trop}}(\eta(b,t)) = (z_s, \log(t))$, for $-\infty \leq \log(t) \leq P(z_s)$. This shows

$$[(z_s, -\infty), (z_s, P(z_s))] \subset \mathrm{Trop}(\pi)^{-1}(\{z_s\}). \tag{4.2.2}$$

For the other inclusion, observe that by the ultrametric triangle inequality, we have $p(x-b) \leq \max(p(x-a_i), |a_i - b|)$ for all $i \in [r]$, which shows $p(x-b) \leq P(z_s)$ for all $p \in \varphi_{\mathrm{trop}}^{-1}(z_s)$. Thus the $(r+1)$-th coordinate of any point in the fiber of z_s is bounded above by $P(z_s)$. This shows that we have equality in (4.2.2).

All together, we obtain

$$Y' = \Gamma_Y(P) \cup [(z_s, -\infty), (z_s, P(z_s))].$$

By Lemma 4.2.35, P is affine everywhere except possibly at z_s. Furthermore, we know by Theorem 4.2.3 and Lemma 4.2.31 that Y' is a tropical curve with all weights equal to 1. Thus P can not be affine in z_s because otherwise Y' would not satisfy the balancing condition. Consequently, the line $[(z_s, -\infty), (z_s, P(z_s))]$ is precisely needed to rebalance the graph $\Gamma_Y(P)$ as explained in Construction 4.2.23. □

Theorem 4.2.37 is a special instance of the fact that tropicalizations of linear subspaces are tropical manifolds, which was to our knowledge first observed by Speyer [**Spe08**]. We give a self-contained proof in our case using Theorem 4.2.36.

THEOREM 4.2.37. *Let $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^r$ be a linear embedding. Then the tropical curve* $\mathrm{Trop}_\varphi(\mathbb{A}^1)$ *is smooth.*

PROOF. We do induction on r, with $r = 1$ being obvious since $\mathbb{T}$ is smooth. For the induction step let $\varphi'\colon \mathbb{A}^1 \to \mathbb{A}^{r+1}$ be given by $(x-a_1), \ldots, (x-a_r), (x-b)$ and we need to show that $\mathrm{Trop}_{\varphi'}(\mathbb{A}^1)$ is smooth. Note that we already know that all weights of $\mathrm{Trop}_{\varphi'}(\mathbb{A}^1)$ are equal to 1 by Lemma 4.2.31. For the other required properties in Definition 4.2.21, we consider $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^r$ which is given by $(x-a_1), \ldots, (x-a_r)$.

We have seen in Theorem 4.2.36 that $\mathrm{Trop}(\pi)\colon \mathrm{Trop}_{\varphi'}(\mathbb{A}^1) \to \mathrm{Trop}_\varphi(\mathbb{A}^1)$ is a tropical modification and the vertices of $\mathrm{Trop}_{\varphi'}(\mathbb{A}^1)$ are precisely the preimages of the ones of $\mathrm{Trop}_\varphi(\mathbb{A}^1)$, plus $(z_s, -\infty)$ and $(z_s, P(z_s))$, where z_s denotes the point $(\log|b-a_i|)_{i\in[r]}$ and P the function from Definition 4.2.33.

By induction hypothesis, we know that $\mathrm{Trop}_\varphi(\mathbb{A}^1)$ is smooth. For a vertex z of $\mathrm{Trop}_{\varphi'}(\mathbb{A}^1)$ which is neither $(z_s, -\infty)$ nor $(z_s, P(z_s))$, we have invariance of valence $\mathrm{val}(z) = \mathrm{val}(\mathrm{Trop}(\pi)(z))$ and dimension $\dim(z) = \dim(\mathrm{Trop}(\pi)(z))$. Thus z is a smooth point since $\mathrm{Trop}(\pi)(z)$ is.

We examine now the situation at $z = (z_s, P(z_s))$. Let $\sigma_1, \ldots, \sigma_k$ be the edges adjacent to z_s. Denote by σ_i^P the image of σ under the map $y \mapsto (y, P(y))$. The edges adjacent to z are then given by $\sigma_1^P, \ldots, \sigma_k^P, \{z_s\} \times [P(z_s), -\infty]$ and thus $\mathrm{val}(z) = \mathrm{val}(z_s) + 1$. We have $\dim(z_s) = \dim\langle \nu_{z_s,\sigma_1}, \ldots, \nu_{z_s,\sigma_k}\rangle$ and $\nu_{z,\sigma_i^P} = (\nu_{z_s,\sigma_i}, c_i)$ for some $c_i \in \mathbb{R}$. Then

$$\begin{aligned}\dim(z) &= \dim\langle \nu_{z,\sigma_1^P}, \ldots, \nu_{z,\sigma_k^P}, (0,\ldots,0,1)\rangle \\ &= \dim\langle (\nu_{z,\sigma_1}, 0), \ldots, (\nu_{z,\sigma_k}, 0), (0,\ldots,0,1)\rangle = \dim(z_s) + 1.\end{aligned}$$

and consequently $\mathrm{val}(z) = \dim(z) + 1$.

The point $(z_s, -\infty)$ lies at infinity, thus $\dim(z) = 0$, and has only the adjacent edge $[(z_s, -\infty), (z_s, P(z_s))]$, thus $\mathrm{val}(z) = 1$. Altogether, we have $\mathrm{val}(z) = \dim(z) + 1$ for all vertices of $\mathrm{Trop}_{\varphi'}(\mathbb{A}^1)$, which precisely means that $\mathrm{Trop}_{\varphi'}(\mathbb{A}^1)$ is smooth, completing the induction. □

COROLLARY 4.2.38. *Let $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^r$ be a linear embedding. Then every open subset Ω of $\operatorname{Trop}_\varphi(\mathbb{A}^1)$ has* PD.

PROOF. The assertion follows directly by Theorem 4.2.37 and Theorem 4.2.22. □

In the following, we introduce linear $\mathbb{A}$-tropical charts and show that for any standard open subset V of $\mathbb{A}^{1,\mathrm{an}}$ it suffices to consider one linear $\mathbb{A}$-tropical chart (V,φ) to determine the cohomology with compact support of V (cf. Theorem 4.2.43).

DEFINITION 4.2.39. An $\mathbb{A}$-tropical chart (V,φ) is called a *linear $\mathbb{A}$-tropical chart* if the map φ is a linear embedding.

An $\mathbb{A}$-tropical subchart (V',φ') is called a *linear $\mathbb{A}$-tropical subchart* if φ' is a linear refinement of φ.

The next proposition shows that, when defining forms on $\mathbb{A}^{1,\mathrm{an}}$, we may restrict our attention to linear $\mathbb{A}$-tropical charts.

PROPOSITION 4.2.40. *Let V be an open subset of $\mathbb{A}^{1,\mathrm{an}}$ and let (V,φ) be an $\mathbb{A}$-tropical chart. Then there exists a linear $\mathbb{A}$-tropical chart (V,φ') such that for all $\alpha \in \mathcal{A}^{p,q}(\varphi_{\mathrm{trop}}(V))$ there exists $\alpha' \in \mathcal{A}^{p,q}(\varphi'_{\mathrm{trop}}(V))$ such that*

$$(V,\varphi,\alpha) = (V,\varphi',\alpha') \in \mathcal{A}^{p,q}(V).$$

PROOF. Let $U \subset \mathbb{A}^1$ be the domain of φ and let f_i, g_i be such that $\varphi\colon U \to \mathbb{A}^r$ is given by f_i/g_i. We write $f_i = c_i \prod_{j=1}^{s_i}(x-a_{ij})$ and $g_i = d_i \prod_{k=1}^{t_i}(x-b_{ik})$. We denote by $\varphi'\colon \mathbb{A}^1 \to \prod_{i=1}^r \left(\mathbb{A}^{s_i}\times\mathbb{A}^{t_i}\right)$ the closed embedding given by $(x-a_{ij})$ and $(x-b_{ik})$ for $i=1,\ldots,r$, $j=1,\ldots,s_i$ and $k=1,\ldots,t_i$. Note that $\varphi'(U)$ is contained in $\prod_{i=1}^r \left(\mathbb{A}^{s_i}\times\mathbb{G}_m^{t_i}\right)$ since the g_i do not vanish on U. The maps

$$\eta_i\colon\ \mathbb{A}^{s_i}\times\mathbb{G}_m^{t_i} \to \mathbb{A}^1$$

which are given by $x \mapsto \frac{c_i\prod_{j=1}^{s_i}T_{ij}}{d_i\prod_{k=1}^{t_i}S_{ik}}$ induce a map

$$\eta\colon\ \prod_{i=1}^r \left(\mathbb{A}^{s_i}\times\mathbb{G}_m^{t_i}\right) \to \mathbb{A}^r\,.$$

Restricting our attention to the respective images of U, one can easily check on coordinate rings that the diagram

$$\begin{tikzcd}
 & (\varphi\times\varphi')(U) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\varphi(U) & & \varphi'(U) \arrow[ll, "\eta"] \\
 & U \arrow[ul, "\varphi"] \arrow[ur, "\varphi'"'] \arrow[uu, "\varphi\times\varphi'"'] &
\end{tikzcd}$$

commutes, where $\varphi\times\varphi'\colon U \to \mathbb{A}^r \times \prod_{i=1}^r \left(\mathbb{A}^{s_i}\times\mathbb{A}^{t_i}\right)$.

Since η is a torus equivariant map composed with a multiplicative translation, it induces an extended affine map $\operatorname{Trop}(\eta)$ on the tropicalizations, which is given in the following way: We denote a point in $\prod_{i=1}^r \left(\mathbb{T}^{s_i}\times\mathbb{T}^{t_i}\right)$ by the tuple $(y_1,z_1,\ldots,y_r,z_r)$ where $y_i = (y_{i,1},\ldots,y_{i,s_i}) \in \mathbb{T}^{s_i}$ and $z_i = (z_{i,1},\ldots,z_{i,t_i}) \in \mathbb{T}^{t_i}$. Then for each i we have

$$\begin{aligned}
\operatorname{Trop}(\eta_i)\colon \mathbb{T}^{s_i}\times\mathbb{R}^{t_i} &\to \mathbb{T} \\
(y_{i,1},\ldots,y_{i,s_i},z_{i,1},\ldots,z_{i,t_i}) &\mapsto \sum y_{i,j} - \sum z_{i,k} + \log(c_i/d_i).
\end{aligned}$$

and

$$\begin{aligned}\operatorname{Trop}(\eta)\colon \prod_{i=1}^{r}\left(\mathbb{T}^{s_i}\times\mathbb{R}^{t_i}\right) &\to \mathbb{T}^r\\ (y_1,z_1,\dots,y_r,z_r) &\mapsto (\operatorname{Trop}(\eta_i)(y_i,z_i))_{i\in[r]}.\end{aligned}$$

We obtain the following commutative diagram of tropicalizations:

$$\begin{tikzcd} & \operatorname{Trop}_{\varphi\times\varphi'}(U) \arrow[dl, "\operatorname{Trop}(\pi_1)"'] \arrow[dr, "\operatorname{Trop}(\pi_2)"] & \\ \operatorname{Trop}_{\varphi}(U) & & \operatorname{Trop}_{\varphi'}(U) \arrow[ll, "\operatorname{Trop}(\eta)"] \end{tikzcd}$$

For $\alpha\in\mathcal{A}^{p,q}(\varphi_{\mathrm{trop}}(V))$ we define $\alpha' := \operatorname{Trop}(\eta)^*\alpha\in\mathcal{A}^{p,q}(\varphi'_{\mathrm{trop}}(V))$. Note that we have $V=\varphi'^{-1}_{\mathrm{trop}}(\operatorname{Trop}(\eta)^{-1}(\varphi_{\mathrm{trop}}(V)))$, thus (V,φ') is indeed a tropical chart. Now the discussion after Definition 4.2.10 shows that $(V,\varphi\times\varphi')$ is a common subchart of (V,φ) and (V,φ'). The commutativity of the last diagram shows that $\operatorname{Trop}(\pi_1)^*\alpha=\operatorname{Trop}(\pi_2)^*\alpha'$ which precisely means $(V,\varphi,\alpha)=(V,\varphi',\alpha')\in\mathcal{A}^{p,q}(V)$. $\square$

LEMMA 4.2.41. *Let V be a standard open subset of $\mathbb{A}^{1,\mathrm{an}}$ (see Example 2.1.3). Then V admits a linear $\mathbb{A}$-tropical chart (V,φ).*

PROOF. By definition

$$V=\{p\in\mathbb{A}^{1,\mathrm{an}}\mid b_i<p(f_i)<c_i, i=1,\dots,r\}$$

for polynomials $f_1,\dots,f_r\in K[x]$ and elements $b_i\in\mathbb{R}$ and $c_i\in\mathbb{R}_{>0}$. We take $g_1,\dots,g_s$ such that $f_1,\dots,f_r,g_1,\dots,g_s$ generate $K[x]$ as a K-algebra and denote by φ the corresponding closed embedding. Then V is precisely the preimage under $\varphi_{\mathrm{trop}}\colon\ \mathbb{A}^{1,\mathrm{an}}\to\mathbb{T}^{r+s}$ of the product of intervals of the form $[-\infty,\log(c_i))$, $(\log(b_i),\log(c_i))$ and $[-\infty,\infty)$ in $\mathbb{T}^{r+s}$. Thus (V,φ) is an $\mathbb{A}$-tropical chart, and so the claim follows by Proposition 4.2.40. $\square$

LEMMA 4.2.42. *Let V be an open subset of $\mathbb{A}^{1,\mathrm{an}}$ which admits an $\mathbb{A}$-tropical chart (V,φ). For every $\alpha\in\mathcal{A}_c^{p,q}(V)$ there exists an $\mathbb{A}$-tropical chart (V,Φ) with Φ defined on all of $\mathbb{A}^1$ such that α is the pullback of a form $\alpha'\in\mathcal{A}_c^{p,q}(\Phi_{\mathrm{trop}}(V))$.*

PROOF. Let $\alpha\in\mathcal{A}_c^{p,q}(V)$ be given by a family $(V_i,\varphi_i,\alpha_i)_{i\in I}$. We fix a finite subset I' of I such that $\operatorname{supp}(\alpha)\subset V' := \bigcup_{i\in I'}V_i$. Since linear $\mathbb{A}$-tropical charts in particular are defined on all of $\mathbb{A}^1$, we may assume by Proposition 4.2.40 that each of the φ_i and φ is defined on all of $\mathbb{A}^1$. Denote $\Phi:=\varphi\times\prod_{i\in I'}\varphi_i$, which is a refinement of all φ_i with $i\in I'$ and φ. Thus (V,Φ) and (V_i,Φ) for $i\in I'$ are $\mathbb{A}$-tropical charts by the discussion after Definition 4.2.10 and consequently (V',Φ) is. We denote by α'_i the pullback of α_i to $\Phi_{\mathrm{trop}}(V_i)$. Then $\alpha'_i|_{\Phi_{\mathrm{trop}}(V_j)}-\alpha'_j|_{\Phi_{\mathrm{trop}}(V_i)}=0$ since $\Phi^*_{\mathrm{trop}}(\alpha'_i|_{\Phi_{\mathrm{trop}}(V_j)}-\alpha'_j|_{\Phi_{\mathrm{trop}}(V_i)})=\alpha|_{V_i\cap V_j}-\alpha|_{V_j\cap V_i}=0$ and Φ^*_{trop} is injective by Proposition 4.2.13. Thus the forms $(\alpha'_i)_{i\in I'}$ glue to a form $\alpha'\in\mathcal{A}^{p,q}(\Phi_{\mathrm{trop}}(V'))$ which pulls back to $\alpha|_{V'}$. By Proposition 4.2.13, we have $\operatorname{supp}(\alpha')=\Phi_{\mathrm{trop}}(\operatorname{supp}(\alpha|_{V'}))$, thus it is compact. Then extending α' by zero to a form on $\Phi_{\mathrm{trop}}(V)$ shows that α can be defined by one triple (V,Φ,α'). $\square$

THEOREM 4.2.43. *Let V be a standard open subset of $\mathbb{A}^{1,\mathrm{an}}$. Then*

$$\mathcal{A}_c^{p,q}(V)=\varinjlim\mathcal{A}_c^{p,q}(\varphi_{\mathrm{trop}}(V)) \text{ and} \tag{4.2.3}$$

$$\mathrm{H}_c^{p,q}(V)=\varinjlim\mathrm{H}_c^{p,q}(\varphi_{\mathrm{trop}}(V)), \tag{4.2.4}$$

where the limits run over the linear $\mathbb{A}$-tropical charts (V,φ).

Furthermore, for a linear $\mathbb{A}$-tropical chart (V,φ) we have that

$$\mathrm{H}_c^{p,q}(\varphi_{\mathrm{trop}}(V)) \to \mathrm{H}_c^{p,q}(V) \tag{4.2.5}$$

is an isomorphism.

Proof. For any $\mathbb{A}$-tropical chart (V,φ), the pullback along the proper map φ_{trop} induces a well-defined morphism $\mathcal{A}_c^{p,q}(\varphi_{\mathrm{trop}}(V)) \to \mathcal{A}_c^{p,q}(V)$. By Definition 4.2.11, this map is compatible with pullback between charts. Thus the universal property of the direct limit leads to a morphism $\Psi\colon \varinjlim \mathcal{A}_c^{p,q}(\varphi_{\mathrm{trop}}(V)) \to \mathcal{A}_c^{p,q}(V)$, where the limit runs over all linear $\mathbb{A}$-tropical charts of V. By Lemma 4.2.41, there exists a $\mathbb{A}$-tropical chart for V. Furthermore, by Lemma 4.2.42, every $\alpha \in \mathcal{A}_c^{p,q}(V)$ can be defined by one chart (V,φ) which we may assume to be a linear $\mathbb{A}$-tropical chart by Proposition 4.2.40. Hence Ψ is surjective. Since the pullback along φ_{trop} is injective by Proposition 4.2.13 and $\alpha \in \mathcal{A}_c^{p,q}(V)$ can be defined by only one chart, the morphism Ψ is injective as well. This shows (4.2.3). Equation (4.2.4) follows because direct limits commute with cohomology.

By Theorem 4.2.36, in (4.2.4) all transition maps are pullbacks along compositions of tropical modifications, thus isomorphisms by Proposition 4.2.24. This shows (4.2.5). □

4.2.5. Poincaré duality. The goal of this subsection is to prove Poincaré duality for a class of open subsets of X^{an} for a smooth algebraic curve X. To do this, we will first prove a lemma which lets us deduce Poincaré duality from local considerations. Poincaré duality is our key statement to prove Theorem 4.2.50 and Theorem 4.2.54, where we calculate the dimension of the cohomology.

Remark 4.2.44. From the definitions of $\mathcal{A}^{p,q}$ given in this thesis (cf. Definition 4.1.7 and Definition 4.2.11), it is not clear that these definitions (and the associated d'', $\wedge$ and integration) are functorial along analytic morphisms which do not come from algebraic ones. When K is non-trivially valued, both approaches coincide, and they lead to the same forms (and the associated d'', $\wedge$ and integration) as defined by Chambert–Loir and Ducros in [**CD12**] as is shown in [**Gub16**, Proposition 7.2 & Proposition 7.11]. Since their approach is purely analytic, our constructions are in particular functorial along analytic morphisms. This implies that PD is also functorial as well since this is just a combination of the wedge product and integration.

Recall the definitions of $\mathcal{L}^p$ and $\mathcal{G}^p$ from Definition 4.2.16. Note that $\mathcal{L}^p(V) = \mathrm{H}^{p,0}(V)$ and $\mathcal{G}^p(V) = \mathrm{H}_c^{1-p,1}(V)^*$ for an open subset $V \subset X^{\mathrm{an}}$. We will use many times that $\mathrm{H}^q(\mathcal{A}_c^{1-p,1-\bullet *}(V), d''^*) = \mathrm{H}_c^{1-p,1-q}(V)^*$ (cf. Lemma 4.2.18). We start with the following general observation, which allows us to prove PD using local considerations.

Lemma 4.2.45. *Let X be an algebraic curve and let $V \subset X^{\mathrm{an}}$ be an open subset. Assume that* PD: $\mathcal{L}_V^p \to \mathcal{G}_V^{1-p}$ *is an isomorphism of sheaves on V and that further the complex $(\mathcal{A}_{V,c}^{1-p,1-\bullet *}, d''^*)$ is exact in positive degree. Then V has* PD.

Proof. In the given situation, PD: $\mathcal{A}_V^{p,\bullet} \to \mathcal{A}_{V,c}^{1-p,1-\bullet *}$ is a quasi-isomorphism of complexes of sheaves on V. This is the case because the complexes are both exact in positive degree (by assumption resp. Theorem 4.2.17) and the map restricts to an isomorphism on the zeroth cohomology (also by assumption). Thus the left diagram is a commutative diagram of acylic resolutions (note that $\mathcal{A}_{V,c}^{1-p,1-\bullet *}$ is flasque by Lemma 4.2.18), and $\mathrm{PD}\colon \mathcal{L}_V^p \to \mathcal{G}_V^{1-p}$ is an isomorphism.

$$\begin{tikzcd}
\mathcal{L}_V^p \arrow[r] \arrow[d, "\mathrm{PD}"] & (\mathcal{A}_V^{p,\bullet}, d'') \arrow[d, "\mathrm{PD}"] \\
\mathcal{G}_V^{1-p} \arrow[r] & (\mathcal{A}_{V,c}^{1-p,1-\bullet *}, d''^*)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{H}^q(V, \mathcal{L}_V^p) \arrow[r, equal] \arrow[d, "\mathrm{PD}"] & \mathrm{H}^q(\mathcal{A}^{p,\bullet}(V), d'') \arrow[d, "\mathrm{PD}"] \\
\mathrm{H}^q(V, \mathcal{G}_V^{1-p}) \arrow[r, equal] & \mathrm{H}^q(\mathcal{A}_c^{1-p,1-\bullet}(V)^*, d''^*)
\end{tikzcd}$$

We thus obtain the diagram on the right since we can use the acyclic resolutions on the left to calculate the sheaf cohomology of $\mathcal{L}_V^p$ resp. $\mathcal{G}_V^{1-p}$. The result follows due to $\mathrm{H}^q(\mathcal{A}^{p,\bullet}(V), d'') = \mathrm{H}^{p,q}(V)$ and $\mathrm{H}^q(\mathcal{A}_c^{1-p,1-\bullet}(V)^*, d''^*) = \mathrm{H}_c^{1-p,1-q}(V)^*$. $\square$

THEOREM 4.2.46. *Let V be an open subset of $\mathbb{P}^{1,\mathrm{an}}$. Then* $\mathrm{PD}\colon \mathcal{L}_V^p \to \mathcal{G}_V^{1-p}$ *is an isomorphism of sheaves on V and the complex $(\mathcal{A}_{V,c}^{1-p,1-\bullet *}, d''^*)$ is exact in positive degree. In particular, V has PD.*

PROOF. That $\mathrm{PD}\colon \mathcal{L}_V^p \to \mathcal{G}_V^{1-p}$ is an isomorphism of sheaves on V and that the complex $(\mathcal{A}_{V,c}^{1-p,1-\bullet *}, d''^*)$ is exact in positive degree are local conditions, thus we may assume $V \subset \mathbb{A}^{1,\mathrm{an}}$. We can cover V by linear $\mathbb{A}$-tropical charts (W, φ_W) contained in V by Lemma 4.2.41, and we can choose the W sufficiently small such that $\Omega_W := \varphi_{W,\mathrm{trop}}(W)$, which is an open subset of the tropical curve $\mathrm{Trop}_{\varphi_W}(\mathbb{A}^1)$, is connected and has at most one vertex for some polyhedral structure $\mathscr{C}$ on $\mathrm{Trop}_{\varphi_W}(\mathbb{A}^1)$. By Corollary 4.2.38, Ω_W has PD. Thus $\mathrm{H}_c^{1-p,1-q}(\Omega_W)$ vanishes if and only if $\mathrm{H}^{p,q}(\Omega_W)$ vanishes. Since Ω_W has at most one vertex, by the Poincaré Lemma 4.2.8 these groups vanish for $q > 0$. Now by Theorem 4.2.43, we have an isomorphism $\mathrm{H}_c^{1-p,1-q}(W) \simeq \mathrm{H}_c^{1-p,1-q}(\Omega_W)$, which shows that $\mathrm{H}_c^{1-p,1-q}(W)^*$ vanishes for $q > 0$. This proves exactness of $\mathcal{A}_{V,c}^{1-p,1-\bullet *}$ in positive degree.

We further have the following maps

$$\mathcal{L}^p(\Omega_W) \hookrightarrow \mathcal{L}^p(W) \hookrightarrow \mathcal{G}^{1-p}(W) \simeq \mathcal{G}^{1-p}(\Omega_W) \simeq \mathcal{L}^p(\Omega_W).$$

Here, the first map is the pullback along $\varphi_{W,\mathrm{trop}}$, which is injective by Proposition 4.2.13. The second map is the PD map on W, which is injective by Lemma 4.2.20. The third map is the dual of the pullback of $\varphi_{W,\mathrm{trop}}$ in cohomology with compact support, which is an isomorphism by Theorem 4.2.43. The fourth map is the inverse of the PD map on Ω_W, which is an isomorphism by Corollary 4.2.38. Since PD commutes with pullbacks, the composition is indeed the identity. In particular, $\mathrm{PD}\colon \mathcal{L}^p \simeq \mathcal{G}^{1-p}$ is an isomorphism.

That V has PD now follows from Lemma 4.2.45. $\square$

In the next corollary, we need K to be non-trivially valued to ensure functoriality along analytic maps.

COROLLARY 4.2.47. *Assume that K is non-trivially valued. Let X be a smooth curve and let $V \subset X^{\mathrm{an}}$ be an open subset such that all points of type II in V have genus* 0. *Then V has* PD. *In particular, if X is a Mumford curve, every open subset of X^{an} has* PD.

PROOF. We claim that $\mathrm{PD}\colon \mathcal{L}_V^p \to \mathcal{G}_V^{1-p}$ is an isomorphism of sheaves on V and that the complex $(\mathcal{A}_{V,c}^{1-p,1-\bullet *}, d''^*)$ is exact in positive degree.

Because this is a purely local question and V is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$ by Proposition 4.2.26, we may assume that V is isomorphic to an open subset of $\mathbb{P}^{1,\mathrm{an}}$. Since $\mathcal{A}^{\bullet,\bullet}$, d'' and PD are functorial along analytic maps by Remark 4.2.44, we may thus assume $V \subset \mathbb{P}^{1,\mathrm{an}}$. Now the claim follows from Theorem 4.2.46. $\square$

That V has PD now follows from Lemma 4.2.45. For a Mumford curve X, the analytification X^{an} contains no type II points of positive genus by Theorem 4.2.29 and Proposition 4.2.26. Thus the statement for open subsets of Mumford curves follows. □

REMARK 4.2.48. Philipp Jell gave later on in [**Jel17**] an if and only if condition when the analytification X^{an} of a smooth proper curve X over K satisfies PD [**Jel17**, Theorem A]. In particular, he showed that X^{an} satisfies PD if the residue field of K is the algebraic closure of a finite field. With the help of this equivalence, we now know that there are indeed smooth proper curves such that X^{an} does not satisfy PD. He also proved some local results, where he crucially used the local results from [**JW18**].

COROLLARY 4.2.49. *Let V be a standard open subset of $\mathbb{A}^{1,\mathrm{an}}$ and let (V,φ) be a linear $\mathbb{A}$-tropical chart. Then we have $\mathrm{H}^{p,q}(\varphi_{\mathrm{trop}}(V)) \simeq \mathrm{H}^{p,q}(V)$.*

PROOF. Since the wedge product and the integration map are compatible with pullback along φ_{trop}, we find the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^{p,q}(\varphi_{\mathrm{trop}}(V)) \arrow[r, "\mathrm{PD}"] \arrow[d] & \mathrm{H}_c^{1-p,1-q}(\varphi_{\mathrm{trop}}(V))^* \\
\mathrm{H}^{p,q}(V) \arrow[r, "\mathrm{PD}"] & \mathrm{H}_c^{1-p,1-q}(V)^*. \arrow[u]
\end{tikzcd}$$

Both PD maps are isomorphisms by Corollary 4.2.38 and Theorem 4.2.46 and the right vertical map is an isomorphism by Theorem 4.2.43. Thus the left vertical map is one as well. □

4.2.6. Cohomology of non-archimedean Mumford curves. In this subsection, we give a calculation of $h^{p,q}(X^{\mathrm{an}})$ for $\mathbb{P}^1$ and Mumford curves (cf. Theorem 4.2.50). For these curves we can further determine $h^{p,q}$ and $h_c^{p,q}$ on a basis of the topology of X^{an}. Note that the Poincaré lemma proved in [**Jel16b**] does not give a basis of open subsets V such that $\mathrm{H}^{p,q}(V) = 0$ for $q > 0$. We will show that we obtain such a basis for open subsets of $\mathbb{P}^1$ and Mumford curves.

The key statement in this calculation is Poincaré duality for certain open subsets V of X^{an} for a smooth algebraic curve X from Subsection 4.2.5.

THEOREM 4.2.50. *Let either X be $\mathbb{P}^1$ or a Mumford curve over a non-trivially valued K. We denote by g the genus of X and let $p, q \in \{0,1\}$. Then we have*

$$h^{p,q}(X^{\mathrm{an}}) = \begin{cases} 1 & \textit{if } p = q, \\ g & \textit{else.} \end{cases}$$

PROOF. We have $h^{0,0}(X^{\mathrm{an}}) = 1$ by Theorem 4.2.17 and $h^{0,1}(X^{\mathrm{an}}) = g$ by Theorem 4.2.29. Thus $h^{1,1}(X^{\mathrm{an}}) = 1$ and $h^{1,0}(X^{\mathrm{an}}) = g$ follow from Theorem 4.2.46 if $X = \mathbb{P}^1$, and from Corollary 4.2.47 if X is a Mumford curve over a non-trivially valued field. □

We now use the results on the structure of non-archimedean curves from [**BPR13**, Section 3 & 4], which were mostly outlined in Section 2.3. Since these results are only worked out for a non-trivially valued K, we assume from now on again that K is non-trivially valued.

Recall from Definition 2.3.29 that a simple open subset is either an open ball, an open annulus or is of the form $\tau_\Gamma^{-1}(\Omega)$ for a skeleton Γ of X^{an} and for a star-shaped open subset Ω of Γ. In the following, we consider a special subclass of simple open subsets.

DEFINITION 4.2.51. A simple open subset V is called *absolutely simple* if its closure in X^{an} is simply connected in the case of an open ball or an open annulus or if the closure of Ω in Γ is simply connected for $V = \tau_\Gamma^{-1}(\Omega)$.

Note that in [**JW18**] these open subsets are called strictly simple instead of absolutely simple. We changed the naming as in this thesis strictly in the context of a set always means that the boundary points are all of type II.

PROPOSITION 4.2.52. *Absolutely simple open subsets form a basis of the topology of X^{an}.*

PROOF. For simple open subsets, this follows from Theorem 2.3.27. Thus it is enough to show that we can cover simple subsets by absolutely simple ones. If V is an open ball or annulus, then we can cover V by open balls and annuli whose closures are contained in V, thus their closures are simply connected again. If $V = \tau_\Gamma^{-1}(\Omega)$ for a skeleton Γ of X^{an} and for $\Omega \subset \Gamma$ open and star-shaped, then we can cover Ω by open subsets Ω_i whose closure in Γ is contained in Ω. Thus the closure of Ω_i is simply connected, and so every $V_i := \tau_\Gamma^{-1}(\Omega_i)$ is absolutely simple. □

Recall that we have seen in Corollary 2.3.31 that the boundary of a simple open subset is finite.

LEMMA 4.2.53. *Let X be a smooth proper curve and let $V \subset X^{\mathrm{an}}$ be an absolutely simple open subset such that $\#\partial V = k$. Then V is properly homotopy equivalent to the one point union of k copies of half-open intervals, glued at the closed ends.*

PROOF. At first, we consider the case where V is an open ball of radius r or an open annulus with radii $r_1 < r_2$. Note that if V is a ball, then $k = 1$. If V is an annulus, then $k = 2$ since the closure of V is simply connected. In both situations, we may thus assume that $X = \mathbb{P}^1$. For any $R > 0$ we denote by ζ_R the point in $\mathbb{P}^{1,\mathrm{an}}$ which is given by the seminorm $f \mapsto \sup_{c \in D(0,R)} |f(c)|$ and by $[\zeta_R, \zeta_{R'}]$ the unique path between two such points in the uniquely path-connected space $\mathbb{P}^{1,\mathrm{an}}$ [**BR10**, Lemma 2.10].

If V is an open ball of radius r, by change of coordinates we may assume that V is the connected component of $\mathbb{P}^{1,\mathrm{an}} \backslash \{\zeta_r\}$ containing 0. Then $\Gamma := [\zeta_{R_1}, \zeta_{R_2}]$ with $R_1, R_2 \in |K^\times|$ and $R_1 < r < R_2$ defines a skeleton of $\mathbb{P}^{1,\mathrm{an}}$ with $\Gamma_0 = \{\zeta_{R_1}, \zeta_{R_2}\}$. We find $V = \tau_\Gamma^{-1}([\zeta_{R_1}, \zeta_r))$. Since τ_Γ is a homotopy equivalence between compact spaces, it induces a proper homotopy equivalence $V \to [\zeta_{R_1}, \zeta_r)$.

If V is an annulus with radii $r_1 < r_2$, we take $R_1, R_2 \in |K^\times|$ with $R_1 < r_1$ and $r_2 < R_2$. Then $\Gamma = [\zeta_{R_1}, \zeta_{R_2}]$ defines again a skeleton of $\mathbb{P}^{1,\mathrm{an}}$ with $\Gamma_0 = \{\zeta_{R_1}, \zeta_{R_2}\}$, and now $V = \tau_\Gamma^{-1}((\zeta_{r_1}, \zeta_{r_2}))$. Again, $(\zeta_{r_1}, \zeta_{r_2})$ is properly homotopy equivalent to V.

Now let $V = \tau_\Gamma^{-1}(\Omega)$ for a simply connected open subset Ω of Γ for some skeleton Γ of X^{an}. Again, Ω is properly homotopy equivalent to V. Since $\overline{\Omega}$ is simply connected, Ω is the interior of a simply connected finite graph contained in Γ, thus properly homotopy equivalent to the one point union of $\#\partial\Omega$ copies of half-open intervals, glued at the closed ends. The result now follows from $\partial V = \partial \Omega$ (see Lemma 2.3.30). □

Note that the following theorem applies to all absolutely simple open subsets V of X^{an} if X is a Mumford curve or $\mathbb{P}^1$ by Theorem 4.2.29 and Proposition 4.2.26.

THEOREM 4.2.54. *Let X be a smooth proper curve over K and let $p, q \in \{0, 1\}$. Let V be an absolutely simple open subset of X^{an} such that all type II points in V have* *genus* 0 *and denote by* $k := \#\partial V$ *the finite number of boundary points. Then we have*

$$h^{p,q}(V) = \begin{cases} 1 & \text{if } (p,q) = (0,0) \\ k-1 & \text{if } (p,q) = (1,0) \\ 0 & \text{if } q \neq 0 \end{cases} \quad \text{and} \quad h_c^{p,q}(V) = \begin{cases} 1 & \text{if } (p,q) = (1,1) \\ k-1 & \text{if } (p,q) = (0,1) \\ 0 & \text{if } q \neq 1. \end{cases}$$

Proof. First, note that V has PD by Corollary 4.2.47. Thus it is sufficient to calculate $h^{0,q}(V)$ and $h_c^{0,q}(V)$. By identification with singular cohomology (cf. Theorem 4.2.17), we only have to calculate $h^q_{\text{sing}}(V)$ and $h^q_{c,\text{sing}}(V)$, which are invariant under proper homotopy equivalences. Thus by Lemma 4.2.53, we have to calculate $h^q_{\text{sing}}(Y)$ and $h^q_{c,\text{sing}}(Y)$ for Y a one point union of k intervals. Since Y is connected and contractible, we have $h^0_{\text{sing}}(Y) = 1$ and $h^1_{\text{sing}}(Y) = 0$. Calculating its cohomology with compact support is an exercise in algebraic topology, which we lay out for the convenience of the reader. We have

$$\mathrm{H}^q_{c,\text{sing}}(Y) = \varinjlim_{E \subset Y \text{ compact}} \mathrm{H}^q_{\text{sing}}(Y, Y \setminus E)$$

by [**Hat02**, p. 244]. Each compact subset E' of Y is contained in a connected compact subset E where E intersects all intervals from which Y is glued. Thus we may restrict our attention to those E. For every pair $E_1 \subset E_2$ of such subsets, $(Y, Y \setminus E_2) \hookrightarrow (Y, Y \setminus E_1)$ is a homotopy equivalence, thus all transition maps in the limit are isomorphisms. We have $\mathrm{H}^0_{\text{sing}}(Y) = \mathbb{R}$, $\mathrm{H}^0_{\text{sing}}(Y \setminus E) = \mathbb{R}^k$ and for both Y and $Y \setminus E$ higher cohomology groups vanish. Now using the long exact sequence of pairs gives the desired result. □

4.2.7. Comparison of two notions of harmonicity via Poincaré duality. In this subsection, we require again that $|\cdot|$ is non-trivial and we let X be an algebraic variety over K of dimension n. We introduce a notion of harmonicity via differential forms and link it to Thuillier's definition from Section 3.1 using Poincaré duality and our cohomology results from the previous subsections. This link between Thuillier's potential theory and Chambert-Loir and Ducros' differential forms is also joint work with Philipp Jell, but it is not part of [**JW18**].

Remark 4.2.55. Let X be an algebraic variety of dimension n over K. Consider a continuous function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an}. Then f defines an element in $\mathcal{A}_c^{n,n}(W)^*$ by setting

$$[f](\alpha) := \int_W f \, d\mu_\alpha$$

for every $\alpha \in \mathcal{A}_c^{n,n}(W)$, where μ_α is the measure from Proposition 4.1.11 corresponding to α. Note that the defined map $\mathcal{C}^0(W) \to \mathcal{A}_c^{n,n}(W)^*$, $f \mapsto [f]$ is injective by [**CD12**, Proposition 5.4.9].

If X is a curve and $f \in \mathcal{C}^\infty(W) := \mathcal{A}^{0,0}(W)$, then we have $[f] = \mathrm{PD}(f)$ for the map PD defined in Definition 4.2.19.

Definition 4.2.56. Let X be an algebraic variety of dimension n over K and let W be an open subset of X^{an}. A continuous function $h\colon W \to \mathbb{R}$ is called *pluriharmonic* if $[h] \in \ker(d'^* d''^*)$ for the dual differential operators d'^* and d''^*, i.e.

$$[h](d'd''\alpha) = \int_W h \, d\mu_{(d'd''\alpha)} = 0$$

for every $\alpha \in \mathcal{A}_c^{n-1,n-1}(W)$.

By construction, these functions define a sheaf on X^{an}, which we denote by $\mathcal{H}_X^{CD}$.

REMARK 4.2.57. For every $h \in \mathcal{C}^\infty(W)$, one has $d'd''h = 0$ if and only if $d'^*d''^*[h] = 0$ (follows by using the theorem of Stokes [**Gub16**, Theorem 5.17]).

When introducing plurisubharmonic functions following [**CD12**] in Chapter 5, we will see that h is pluriharmonic if and only if h and $-h$ are plurisubharmonic (cf. Remark 5.1.4).

In Section 3.1, we introduced the sheaf $\mathcal{H}_X$ of harmonic functions on the analytification of a smooth proper curve defined by Thuillier. It arises the question whether the sheaves $\mathcal{H}_X^{CD}$ and $\mathcal{H}_X$ coincide. With the tools from the previous subsections, we can answer this question easily if X is the projective line or a Mumford curve. The general case is harder to prove and we postpone it to Chapter 5.

For the rest of the subsection, let X be a smooth proper algebraic curve over K.

REMARK 4.2.58. First, one should mention that Thuillier proved that $\mathcal{H}_X(W)$ is a subspace of $C^\infty(W)$ for an open subset W of X^{an} if

i) $\widetilde{K}$ is algebraic over a finite field, or
ii) W is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$.

This is shown in [**Thu05**, Théorème 2.3.21] and we have in fact in these situations that

$$\mathcal{H}_X(W) = \ker(d'd'') \subset \mathcal{C}^\infty(W) \tag{4.2.6}$$

(see for example [**Wan16**, Corollary 5.3.21]).

THEOREM 4.2.59. *Let X be a smooth proper curve and let W be an open subset of X^{an} that does not contain any type II point of positive genus. Then every pluriharmonic function on W is smooth. In particular,*

$$\mathcal{H}_X^{CD}(V) = \mathcal{H}_X(V)$$

for every open subset V of W.

REMARK 4.2.60. Up to now, we always considered for an open subset V of X^{an} the cohomology groups $\mathrm{H}^{p,q}(V) := \mathrm{H}^q(\mathcal{A}^{p,\bullet}(V), d'')$ and $\mathrm{H}_c^{p,q}(V) := \mathrm{H}^q(\mathcal{A}_c^{p,\bullet}(V), d'')$. In the proof of the theorem we also need to work with $\mathrm{H}^q(\mathcal{A}^{\bullet,q}(V), d')$ and $\mathrm{H}^q(\mathcal{A}_c^{\bullet,q}(V), d')$, thus we use the notations $\mathrm{H}^{p,q}_{d'',(c)}(V)$ and $\mathrm{H}^{p,q}_{d',(c)}(V)$ for the corresponding cohomology groups. Note that the canonical involution J induces an isomorphism $\mathrm{H}^{p,q}_{d'}(V) \simeq \mathrm{H}^{q,p}_{d''}(V)$ for all p, q [**Jel16a**, Lemma 3.4.2] and we have all cohomology results which were shown in the previous subsections also for $\mathrm{H}^{p,q}_{d',(c)}$.

PROOF. First, we prove that a pluriharmonic function $h\colon W \to \mathbb{R}$ is smooth. Since smooth functions form a sheaf and every point $x \in W$ has an absolutely simple open neighborhood in W (cf. Proposition 4.2.52), we may assume W to be absolutely simple.

By definition and Lemma 4.2.18, we have

$$d''^*[h] \in \ker(d'^*\colon \mathcal{A}_c^{1,0}(W)^* \to \mathcal{A}_c^{0,0}(W)^*) = \mathrm{H}^{1,0}_{d',c}(W)^*.$$

Since $\mathrm{PD}\colon \mathrm{H}^{0,1}_{d'}(W) \to \mathrm{H}^{1,0}_{d',c}(W)^*$ is an isomorphism by Corollary 4.2.47, there is a form $\alpha \in \ker(d') \subset \mathcal{A}^{0,1}(W)$ such that $\mathrm{PD}(\alpha) = d''^*[h]$. Our open subset is absolutely simple, so Theorem 4.2.54 yields $\mathrm{H}^{0,1}_{d''}(W) = 0$. Hence there is a $g \in \mathcal{C}^\infty(W) = \mathcal{A}^{0,0}(W)$ such that $d''g = \alpha$. Since PD is a morphism of complexes, we have $\mathrm{PD}(d''g) = d''^*\,\mathrm{PD}(g)$. Altogether,

$$d''^*[h] = \mathrm{PD}(\alpha) = \mathrm{PD}(d''g) = d''^*\,\mathrm{PD}(g) = d''^*[g]$$

in $\mathcal{A}_c^{1,0}(W)^*$. Thus $[h-g] \in \ker(d''^*) = \mathrm{H}^{1,1}_{d'',c}(W)^*$ (cf. Lemma 4.2.18). Using Poincaré duality again, there is an $f \in \mathrm{H}^{0,0}_{d''}(W) = \ker(d''\colon \mathcal{A}^{0,0}(W) \to \mathcal{A}^{0,1}(W))$ such that

$$[h-g] = \mathrm{PD}(f) = [f].$$

By [**CD12**, Proposition 5.4.9], we have $h - g = f$ on W. Hence h is smooth on W as $f, g \in \mathcal{C}^\infty(W)$.

Next, we prove $(\mathcal{H}_X^{CD})|_W = (\mathcal{H}_X)|_W$. Since W is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$ by Proposition 4.2.26, we know that

$$(\mathcal{H}_X)|_W = \ker(d'd'')|_W \subset \mathcal{C}^\infty|_W$$

as explained in Remark 4.2.58. On the other hand, we have

$$\ker(d'd'') = (\mathcal{H}_X^{CD}) \cap \mathcal{C}^\infty$$

(cf. Remark 4.2.57). We have seen in the first part of the proof that $(\mathcal{H}_X^{CD})|_W \subset \mathcal{C}^\infty|_W$. Thus

$$(\mathcal{H}_X)|_W = \ker(d'd'')|_W = (\mathcal{H}_X^{CD})|_W.$$

□

COROLLARY 4.2.61. *Let X be $\mathbb{P}^1$ or a Mumford curve over K. Then $\mathcal{H}_X^{CD} = \mathcal{H}_X$.*

PROOF. Follows directly by Theorem 4.2.59 using Theorem 4.2.29 and Proposition 4.2.26. □

COROLLARY 4.2.62. *Let X be a smooth proper curve and let $h\colon W \to \mathbb{R}$ be a pluriharmonic function on an open subset W of X^{an}. Then h is lisse on W (cf. Definition 3.1.15).*

PROOF. Note that a function f is lisse on an open subset V of X^{an} if and only if V has a locally finite covering by affinoid domains $(Y_i)_{i\in I}$ such that $f \in \mathcal{H}_X(Y_i\backslash\partial Y_i)\cap\mathcal{C}^0(Y_i)$ for every $i \in I$ by [**Thu05**, Proposition 3.2.4]. Let $G(W)$ be the set of type II points of positive genus in W. This set is finite by [**BPR13**, Remark 4.18], and so $W' := W\backslash G(W)$ is an open subset of X^{an}. Then $h\colon W' \to \mathbb{R}$ is harmonic on W' by Theorem 4.2.59, and hence lisse on W' by Proposition 3.1.17. Thus there is a locally finite covering of W' by affinoid domains $(Y_i)_{i\in I}$ such that $h \in \mathcal{H}_X(Y_i\backslash\partial Y_i) \cap \mathcal{C}^0(Y_i)$ for every $i \in I$. We write $G(W) = \{y_1, \ldots, y_m\}$, then we can choose affinoid domains $\widetilde{Y}_1, \ldots, \widetilde{Y}_m$ in W such that $y_j \in \partial\widetilde{Y}_j$ and $G(W) \cap (\widetilde{Y}_j\backslash\partial\widetilde{Y}_j) = \emptyset$ (use for instance Proposition 4.2.52 and Corollary 2.3.32 for the construction). Since h is continuous on W and harmonic on $W' = W\backslash G(W)$, we have $h \in \mathcal{H}_X(\widetilde{Y}_j\backslash\partial\widetilde{Y}_j) \cap \mathcal{C}^0(\widetilde{Y}_j)$ for every $j = 1, \ldots, m$. Then $(Y_i)_{i\in I}\cup(\widetilde{Y}_j)_{j=1,\ldots,m}$ is a required locally finite covering of W, and thus h is lisse on W. □

REMARK 4.2.63. In the next chapter, we also give a definition of plurisubharmonic functions following [**CD12**]. We compare these plurisubharmonic functions to Thuillier's subharmonic functions in the curve case in Section 5.2. In particular, we also get the comparison for pluriharmonic functions in the case of a general smooth proper curve.

CHAPTER 5

Potential theory via differential forms

In this whole chapter, let K be non-trivially valued. The results in this chapter were published in [**Wan18**].

5.1. Plurisubharmonicity via real-valued differential forms

With the help of the (p,q)-forms $\mathcal{A}^{p,q}$ defined in Subsection 4.1, Chambert-Loir and Ducros introduced a notion of plurisubharmonic functions on the analytification X^{an} of an n-dimensional algebraic variety X over K. We introduce their approach and also their developed Bedford–Taylor theory in this section.

DEFINITION 5.1.1. Let U be an open subset of $\mathbb{R}^r$. A superform $\alpha \in \mathcal{A}^{p,p}(U)$ is called *strongly positive* if there exist finitely many superforms $\alpha_{j,s}$ of type $(0,1)$ and non-negative smooth functions f_s on U such that

$$\alpha = \sum_s f_s \alpha_{1,s} \wedge J(\alpha_{1,s}) \wedge \ldots \wedge \alpha_{p,s} \wedge J(\alpha_{p,s}).$$

Let $\mathscr{C}$ be a polyhedral complex and let Ω be an open subset of $|\mathscr{C}|$. A superform $\alpha \in \mathcal{A}^{p,p}(\Omega)$ is called *strongly positive* if there is a polyhedral decomposition of $\mathscr{C}$ such that the restriction of α to $\mathrm{relint}(\sigma) \cap \Omega$ is strongly positive for every polyhedron $\sigma \in \mathscr{C}$.

For an open subset W of X^{an}, a smooth form $\omega \in \mathcal{A}^{p,p}(W)$ is called *strongly positive* if for every point x in W there is a tropical chart (V, φ_U) with $x \in V$ such that $\omega = \alpha \circ \mathrm{trop}_U$ on V for a strongly positive form $\alpha \in \mathcal{A}^{p,p}_{\mathrm{Trop}(U)}(\mathrm{trop}_U(V))$.

Note that for forms of type $(0,0)$, $(1,1)$, $(n-1,n-1)$ and (n,n) the notion of strongly positivity defined here coincides with the other positivity notions from [**CD12**, §5.1]. Thus we just say that a smooth form ω in $\mathcal{A}^{p,p}(W)$ is *positive* if it is of one of these types and it is strongly positive.

A smooth function f is (strongly) positive as a form if and only if $f \geq 0$.

DEFINITION 5.1.2. Let $W \subset X^{\mathrm{an}}$ be open. We call a function $f\colon W \to [-\infty, \infty]$ *locally integrable* if f is integrable with respect to every measure μ_α associated to a form $\alpha \in \mathcal{A}^{n,n}_c(W)$ (cf. Proposition 4.1.11). We write $\int_W f \wedge \alpha := \int_W f \, d\mu_\alpha$.

Then for every locally integrable (e.g. continuous) function $f\colon W \to [-\infty, \infty]$ one can define a current in the sense of [**CD12**, §4.2] and [**Gub16**, §6] by

$$d'd''[f]\colon \mathcal{A}^{n-1,n-1}_c(W) \to \mathbb{R}, \ \alpha \mapsto \int_W f \wedge d'd''\alpha$$

(see [**Gub16**, 6.9]).

Note that $d'd''[f] = -d'^{*}d''^{*}[f]$ for the dual operators d'^{*} and d''^{*} and the element $[f] \in \mathcal{A}^{n-1,n-1}(W)^{*}$ that we defined for a continuous function $f\colon W \to \mathbb{R}$ in Remark 4.2.55.

Definition 5.1.3. A locally integrable function $f\colon W \to [-\infty, \infty)$ is called *plurisubharmonic* (shortly *psh*) if f is upper semi-continuous and $d'd''[f]$ is a positive current, i.e. $d'd''[f](\omega) \geq 0$ for all positive forms $\omega \in \mathcal{A}_c^{n-1,n-1}(W)$.

Note that we do not require that a psh function has to be continuous, contrary to [**CD12**, Définition 5.5.1].

Remark 5.1.4. One can show that a continuous function $h\colon W \to \mathbb{R}$ on an open subset W of X^{an} is pluriharmonic as defined in Subsection 4.2.7 if and only if h and $-h$ are psh: Recall that h is pluriharmonic if and only if $\int_W h \wedge d'd''\alpha = 0$ for every $\alpha \in \mathcal{A}_c^{n-1,n-1}(W)$. Hence pluriharmonic implies directly that h and $-h$ are psh. Next, assume that h and $-h$ are psh and consider $\alpha \in \mathcal{A}_c^{n-1,n-1}(W)$. Then by [**CD12**, Lemme 5.3.3] there are positive forms $\alpha^+, \alpha^- \in \mathcal{A}_c^{n-1,n-1}(W)$ such that $\alpha = \alpha^+ - \alpha^-$ on W. Let $\eta \in \mathcal{C}_c^\infty(W)$ with $\eta \equiv 1$ on $\mathrm{supp}(\alpha)$, which can be found by [**CD12**, Corollaire 3.3.4]. Then have the following identity on W

$$\alpha = \eta\alpha = \eta\alpha^+ - \eta\alpha^-,$$

where $\eta\alpha^+$ and $\eta\alpha^-$ are positive forms in $\mathcal{A}_c^{n-1,n-1}(W)$. Thus $\int_W h \wedge d'd''(\eta\alpha^\pm) \geq 0$ and $-\int_W h \wedge d'd''(\eta\alpha^\pm) = \int_W -h \wedge d'd''(\eta\alpha^\pm) \geq 0$, as h and $-h$ are psh (cf. Definition 5.1.3). Therefore, $\int_W h \wedge d'd''(\eta\alpha^\pm) = 0$. This implies

$$\int_W h \wedge d'd''\alpha = \int_W h \wedge d'd''(\eta\alpha^+ - \eta\alpha^-) = \int_W h \wedge d'd''(\eta\alpha^+) - \int_W h \wedge d'd''(\eta\alpha^-) = 0.$$

Thus h is pluriharmonic.

In Section 4.2.7, we started to link these pluriharmonic functions with Thuillier's sheaf $\mathcal{H}_X$ of harmonic functions from Section 3.1. In Section 5.2, we show that every continuous function is plurisubharmonic if and only it is subharmonic (in the sense of Section 3.1).

The notion of plurisubharmonic functions introduced by Chambert-Loir and Ducros works for every dimension of X and the variety is not required to be smooth or proper. However, the theory of these psh functions is in the early stages of development and only a few properties are already known. We line out parts of these results to the reader, which were originally proven by Chambert-Loir and Ducros in [**CD12**, §5 & §6].

Proposition 5.1.5. *Let W be a paracompact open subset of X^{an} and let $(V_i)_{i\in I}$ be an open covering of W. Then there are smooth non-negative functions $(\eta_j)_{j\in J}$ with compact support on W such that:*

i) The family $(\mathrm{supp}(\eta_j))_{j\in J}$ is locally finite on W.
ii) We have $\sum_{j\in J} \eta_j \equiv 1$ on W.
iii) For every $j \in J$, there is a $i(j) \in I$ such that $\mathrm{supp}(\eta_j) \subset V_{i(j)}$.

We call $(\eta_j)_{j\in J}$ a partition of unity *subordinated to the open covering $(V_i)_{i\in I}$.*

Proof. See [**Gub16**, Proposition 5.10]. □

Remark 5.1.6. Note that every open subset W of X^{an} is paracompact if X is a curve [**Ber90**, Theorem 4.2.1 & 4.3.2].

Proposition 5.1.7. *The psh functions form a sheaf on X^{an}.*

Proof. Let W be an open subset of X^{an} and let $(W_i)_{i\in I}$ be an open covering of W. Consider a function $f\colon W \to [-\infty, \infty)$. If f is psh on W, the restrictions $f|_{W_i}$ are obviously psh for every $i \in I$.

Assume that $f|_{W_i}$ is psh on W_i for every $i \in I$ and show that f is then psh on W. Clearly, f is usc if every restriction is. Consider $\omega \in \mathcal{A}_c^{n,n}(W)$ and let V be a paracompact open neighborhood of $\mathrm{supp}(\omega)$ in W, which we can find by [**CD12**, 2.1.5 & Lemme 2.1.6]. Then $\omega \in \mathcal{A}_c^{n,n}(V)$, the family $(V_i := W_i \cap V)_{i\in I}$ defines an open covering of V and the restrictions $f|_{V_i}$ are psh for every $i \in I$. Let $(\eta_j)_{j\in J}$ be a partition of unity subordinated to the covering $(V_i)_{i\in I}$ (see Proposition 5.1.5). Then for every $j \in J$, we have $\eta_j\omega \in \mathcal{A}_c^{n,n}(V_{i(j)})$. Since $f|_{V_{i(j)}}$ is psh, the integral $\int_{V_{i(j)}} f \wedge \eta_j\omega$ has to be finite for every $j \in J$. Furthermore, we can write $\omega = \sum_{j\in J} \eta_j\omega$ on V, where this sum has to be finite as ω has compact support on V. Hence

$$\int_W f \wedge \omega = \int_V f \wedge \omega = \sum_{j\in J} \int_{V_{i(j)}} f \wedge \eta_j\omega$$

has to be finite as well, i.e. f is locally integrable.

Now, consider a positive form $\omega \in \mathcal{A}_c^{n-1,n-1}(W)$. Again, we work over the paracompact open neighborhood V of $\mathrm{supp}(\omega)$ in W and consider ω as a form in $\mathcal{A}_c^{n-1,n-1}(V)$. As above, we have a partition of unity $(\eta_j)_{j\in J}$ subordinated to the covering $(V_i)_{i\in I}$, and $\omega = \sum_{j\in J} \eta_j\omega$, where the sum is finite since ω has compact support. Then $\eta_j\omega \in \mathcal{A}_c^{n-1,n-1}(V_{i(j)})$ and $\eta_j\omega$ is a positive form for every $j \in J$. Thus

$$d'd''[f|_V](\omega) = d'd''[f](\sum\nolimits_j \eta_j\omega) = \sum\nolimits_j d'd''[f](\eta_j\omega) \geq 0$$

by linearity and the fact that $f|_{V_{i(j)}}$ is psh for every $j \in J$. Since we have the identity $d'd''[f](\omega) = d'd''[f|_V](\omega)$, the function f is psh on W. □

Next, we translate a very useful characterization of smooth psh functions from [**CD12**, Lemme 5.5.3] to our setting.

Proposition 5.1.8. *Let W be an open subset of X^{an}. A smooth function $f\colon W \to \mathbb{R}$ is psh if and only if for every $x \in W$ there is a tropical chart $(V, \varphi_U\colon U \to \mathbb{G}_m^r)$ of W with $x \in V$ such that $f = \psi \circ \mathrm{trop}_U$ on V for a smooth function $\psi\colon \mathbb{R}^r \to \mathbb{R}$ whose restriction $\psi|_\sigma$ to every polyhedron σ in $\mathbb{R}^r$ with $\sigma \subset \mathrm{trop}_U(V)$ is convex.*

Proof. First, assume that f is psh on W, i.e. $d'd''[f]$ defines a positive current on W. Since f is smooth, we can find for every $x \in W$ a tropical chart $(V, \varphi_U\colon U \to \mathbb{G}_m^r)$ in W with $x \in V$ such that $f = \psi \circ \mathrm{trop}_U$ on $V = \mathrm{trop}_U^{-1}(\Omega)$ for a smooth function $\psi\colon \mathbb{R}^r \to \mathbb{R}$ and an open subset Ω of $\mathrm{Trop}(U)$. We choose a compact neighborhood B of $\mathrm{trop}_U(x)$ in Ω. Then the preimage $Y := \mathrm{trop}_U^{-1}(B)$ under the proper map trop_U is a compact analytic domain. The restriction of the current $d'd''[f]$ to the compact analytic domain Y is still positive. Applying [**CD12**, Lemme 5.5.3] to $f = \psi \circ \mathrm{trop}_U\colon Y \to \mathbb{R}$, we know that for every polyhedron Δ in $\mathbb{R}^r$ with $\Delta \subset B$ the restriction $\psi|_\Delta$ is convex. Now, let Ω' be an open neighborhood of $\mathrm{trop}_U(x)$ in B and consider the open neighborhood $V' := \mathrm{trop}_U^{-1}(\Omega')$ of x in $\mathrm{trop}_U^{-1}(B)$. Then the pair (V', φ_U) is a tropical chart in W that contains x and $f|_{V'} = \psi \circ \mathrm{trop}_U$, where ψ is smooth. Consider a polyhedron σ in $\mathbb{R}^r$ with $\sigma \subset \Omega' = \mathrm{trop}_U(V')$. Then $\sigma \subset B$, and so $\psi|_\sigma$ is convex.

Next, we assume that there is for every $x \in W$ a tropical chart (V, φ_U) of W with $x \in V$ such that $f = \psi \circ \mathrm{trop}_U$ on V for a function ψ satisfying i) and ii). Our goal is to show that f is psh in a neighborhood of x. As above, we choose a compact neighborhood B of $\mathrm{trop}_U(x)$ in $\mathrm{trop}_U(V)$ and set $Y := \mathrm{trop}_U^{-1}(B)$. Since every polyhedron Δ in $\mathbb{R}^r$ with $\Delta \subset B$ is contained in $\mathrm{trop}_U(V)$, the restriction $\psi|_\Delta$ is convex by ii). Again [**CD12**, Lemme 5.5.3] tells us that f is psh on the compact analytic domain Y. Choosing (V', φ_U) as above, we obtain a tropical chart of W with $x \in V'$. Due to $V' \subset Y$, the function f is psh on the open neighborhood V' of x. Psh functions form a sheaf by Proposition 5.1.7, and so the claim follows. $\square$

Chambert-Loir and Ducros transferred parts of the complex Bedford–Taylor theory to their theory of psh functions. They defined a Monge–Ampère measure for those functions which are locally the difference of two limits of smooth psh functions.

DEFINITION 5.1.9. A function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} is called *locally psh-approximable* if every point of W has a neighborhood V in W such that f is the uniform limit of smooth psh functions f_i on V. A function f is *locally approximable* if it is locally the difference of two locally psh-approximable functions. Furthermore, we say that f is *globally psh-approximable* (resp. *globally approximable*) on W if f is a uniform limit (resp. the difference of two uniform limits) of smooth psh functions on W.

Let $f\colon W \to \mathbb{R}$ be a locally psh-approximable function, then there exists a unique positive Radon measure $\mathrm{MA}(f)$ on W such that for every open subset $V \subset W$, $g \in \mathcal{C}_c^\infty(V)$ and smooth psh functions $f_i \in \mathcal{C}^\infty(V)$ converging uniformly to $f|_V$, we have

$$\int_V g \; d\,\mathrm{MA}(f) = \lim_{i\to\infty} \int_V g \wedge (d'd'' f_i)^n,$$

where $n = \dim(X)$ and $(d'd''f_i)^n$ is defined as the n-th wedge product of $d'd''f_i$. We call $\mathrm{MA}(f)$ the *Monge–Ampère measure* of f.

For a locally approximable function f that is given locally by $f_V^+ - f_V^-$ for locally psh-approximable functions f_V^+ and f_V^- on V, we define the Monge–Ampère measure $\mathrm{MA}(f)$ to be the measure obtained by gluing $\mathrm{MA}(f_V^+) - \mathrm{MA}(f_V^-)$. Note that the definition is independent of the decompositions. For details see [**CD12**, Corollaire 5.6.5 & 5.6.6] and [**CD12**, Définition 5.6.7].

REMARK 5.1.10. It would be desirable to have a Monge–Ampère measure for all continuous psh functions as in the complex Bedford–Taylor theory. If we would have an analogous regularization theorem as in the complex case, i.e. every continuous psh function is locally psh-approximable, we would get this immediately, but up to now, this is an unsolved problem. At the end of this thesis, we will see that we have a regularization theorem in this setting if X is for example a non-archimedean Mumford curve.

Furthermore, stability of psh functions under pullback is in general an open question. Let X and X' be algebraic varieties over K, let $\varphi\colon W' \to W$ be a morphism of analytic spaces for open subsets $W' \subset (X')^{\mathrm{an}}$ and $W \subset X^{\mathrm{an}}$, and let $f\colon W \to [-\infty, \infty)$ be a psh function on W. If f is smooth, it follows directly from the definition of smooth functions by Chambert-Loir and Ducros [**CD12**, 3.1.3] and [**Gub16**, Proposition 7.2] that $\varphi^* f$ is smooth on $\varphi^{-1}(W)$. Furthermore, [**Gub16**, Proposition 7.2] and Proposition 5.1.8 imply that $\varphi^* f$ is also psh on $\varphi^{-1}(W)$. If f is not smooth, it is not clear whether $\varphi^* f$ is psh or not. At the end of the thesis, we will answer this question positively for a continuous psh function f on an open analytic subset of the analytification of a smooth proper algebraic curve.

Model functions form a subclass of locally approximable functions and their Monge–Ampère measures are very well understood. Hence they are an important tool for the proofs of the main theorems in Section 5.2. In the following, we introduce model functions and state the most important result about their Monge–Ampère measures.

DEFINITION 5.1.11. Let X be a proper and normal variety over K and let L be a line bundle on X. A *continuous metric* $\| \ \|$ on L^{an} associates to every section $s \in \Gamma(U, L)$ on a Zariski open subset U of X a continuous function $\|s\|\colon U^{\mathrm{an}} \to [0, \infty)$ such that

$\|f \cdot s\| = |f| \cdot \|s\|$ holds for every $f \in \mathcal{O}_X(U)$ and $\|s(x)\| = 0$ if and only if s vanishes in x.

We call a continuous metric $\| \ \|$ on L^{an} *smooth* (resp. *psh*) if for every open subset U of X and for every invertible section s of L on U the function $-\log\|s\|$ is smooth (resp. psh) on U^{an}.

A continuous metric $\| \ \|$ on L^{an} is *psh-approximable* if there is a sequence of smooth psh metrics $\| \ \|_k$ on L^{an} such that $\sup_{x\in X^{\mathrm{an}}} |\log(\|s_x(x)\|/\|s_x(x)\|_k)|$ converges to zero for any local section s_x of L that does not vanish in x. Clearly, this is independent of the choice of s_x.

In Section 2.3, we have seen (strictly) semistable formal models of a proper algebraic curve X (see Definition 2.3.11). In the definition of model functions, these (strictly) semistable formal models of our algebraic variety are needed and for the definition in the higher dimensional case we refer to [**GH17**, Appendix B].

DEFINITION 5.1.12. Let X be a proper variety over K and let L be a line bundle on X.

A *semistable formal model* of (X, L) is a pair $(\mathcal{X}, \mathcal{L})$ consisting of a semistable formal model $\mathcal{X}$ of X^{an} and a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}|_{X^{\mathrm{an}}} \simeq L^{\mathrm{an}}$. Note that we always may assume that $\mathcal{X}$ is strictly semistable by the semistable reduction Theorem [**BL85**, Ch. 7] since K is algebraically closed.

Let $(\mathcal{X}, \mathcal{L})$ be a semistable formal model of $(X, L^{\otimes m})$ for $m \in \mathbb{N}_{>0}$. Then one can define a continuous metric on L^{an} in the following way: If $\mathcal{U}$ is a formal trivialization of $\mathcal{L}$ and s is a section of L^{an} on $\mathcal{U}_\eta$ such that $s^{\otimes m}$ corresponds to $\lambda \in \mathcal{O}_{X^{\mathrm{an}}}(\mathcal{U}_\eta)$ with respect to this trivialization, then

$$-\log\|s(x)\|_{\mathcal{L}} := -\frac{1}{m}\log|\lambda(x)|$$

for all $x \in \mathcal{U}_\eta$. This definition is independent of all choices and shows immediately that the defined metric is continuous. Metrics of this form are called $\mathbb{Q}$-*formal metrics*, and they are called *formal metrics* if $m = 1$.

Let $\mathcal{O}_X$ be the trivial line bundle on X. A function $f\colon X^{\mathrm{an}} \to \mathbb{R}$ of the form $f = -\log\|1\|_{\mathcal{L}}$ for a formal metric $\| \ \|_{\mathcal{L}}$ associated to a semistable formal model of $(X, \mathcal{O}_X)$ is called *model function.*

We have the following statement for model functions, which is a direct consequence of a result of Chambert-Loir and Ducros in [**CD12**, §6.3].

THEOREM 5.1.13. *Let X be a projective variety over K and let $f = -\log\|1\|_{\mathcal{L}}$ be a model function on X^{an} for a semistable formal model $(\mathcal{X}, \mathcal{L})$ of $(X, \mathcal{O}_X)$.*

i) The function f is locally approximable on X^{an}, and so the Monge–Ampère measure $\mathrm{MA}(f)$ *exists.*
ii) We have the following identity of measures

$$\mathrm{MA}(f) = \sum_Y \deg_{\mathcal{L}}(Y)\delta_{\zeta_Y},$$

where Y runs over all irreducible components of the special fiber $\mathcal{X}_s$ and ζ_Y is the unique point in X^{an} mapped to the generic point of Y under the reduction map (see [**Ber90**, Proposition 2.4.4]*).*

PROOF. By [**CD12**, Corollaire 6.3.5], we know that there are line bundles L_1, L_2 on X with formal models $\mathcal{L}_1$, $\mathcal{L}_2$ such that $\mathcal{O}_X = L_1 \otimes L_2^{-1}, \mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$, the corresponding metrics $\| \ \|_{\mathcal{L}_1}$ and $\| \ \|_{\mathcal{L}_2}$ are psh-approximable and

$$-\log \| \cdot \|_{\mathcal{L}} = -\log \| \cdot \|_{\mathcal{L}_1} + \log \| \cdot \|_{\mathcal{L}_2}$$

on X^{an}. Let $\| \ \|_{i,k}$ be a sequence of smooth psh metrics converging uniformly to $\| \cdot \|_{\mathcal{L}_i}$ on X^{an}. For every point x in X^{an}, let U be an open subset of X with $x \in U^{\mathrm{an}}$ and s an invertible section of $L_1 = L_2$ on U. Then $-\log \|s\|_{\mathcal{L}_{i,k}}$ is a smooth psh function on U^{an} for $i = 1, 2$, and for every $k \in \mathbb{N}$

$$-\log \|1\|_{\mathcal{L}} = -\log \|s\|_{\mathcal{L}_1} + \log \|s\|_{\mathcal{L}_2} = \lim_{k\to\infty} -\log \|s\|_{1,k} + \lim_{k\to\infty} \log \|s\|_{2,k}.$$

Hence the function $f = -\log \|1\|_{\mathcal{L}}$ is locally approximable.

The second assertion is shown in [**CD12**, §6.9]. □

5.2. Comparison Theorem

In this section, we always consider a smooth proper curve X over K. We compared in Section 4.2.7 harmonic functions (see Definition 3.1.2) with pluriharmonic functions (see Definition 4.2.56 or Remark 5.1.4) on X^{an} when X is the projective line or a Mumford curve. It arises the question whether they do also coincide in the general case and whether the corresponding notions of subharmonic functions do coincide as well. In this section, we answer both questions positively for continuous functions.

5.2.1. Preliminaries. In this subsection, we give some preliminary results in the case of a curve most of them regarding the Monge–Ampère measure. The main statement is that for a model function f on X^{an} the Monge–Ampère measure $\mathrm{MA}(f)$ is equal to the measure $dd^c f$. The statement can be deduced from Theorem 5.1.13 using the slope formula in [**KRZ16**]. It is the main tool for the proofs in Subsections 5.2.2 and 5.2.3. Moreover, we can conclude directly using our results from Section 4.2.7 that harmonic functions are the same as pluriharmonic functions (cf. Corollary 5.2.6).

In this subsection, we always consider a smooth proper curve X over K.

LEMMA 5.2.1. *Let W be an open subset of X^{an} and let $f\colon W \to \mathbb{R}$.*

i) If f is locally approximable, we have

$$d'd''[f](g) = \int_W g \; d\,\mathrm{MA}(f)$$

for every $g \in \mathcal{C}_c^\infty(W)$.

ii) If f is locally psh-approximable, then f is psh.

PROOF. At first, note that every locally (psh-)approximable function is continuous, and so locally integrable. We start with assertion i) and assume that f is locally approximable. We therefore can cover W by open subsets V_i on that f is the difference of uniform limits

$$f = \lim_{k\to\infty} f_{ik}^+ - \lim_{k\to\infty} f_{ik}^-$$

of smooth psh functions f_{ik}^+ and f_{ik}^- on V_i. Choose a partition of unity $(\eta_j)_{j\in J}$ subordinated to the covering $(V_i)_{i\in I}$ (see Proposition 5.1.5). We write for simplicity V_j instead of $V_{i(j)}$ and f_{jk} instead of $f_{i(j)k}$. Then for every $j \in J$, we have $\eta_j g \in \mathcal{C}_c^\infty(V_j)$. Furthermore, $g = \sum_{j\in J} \eta_j g$ on W. Since g has compact support on W, the sum has to be finite.

Hence

$$\begin{aligned} d'd''[f](g) &= \sum_{j\in J} \int_{V_j} f \wedge d'd''(\eta_j g) \\ &= \sum_{j\in J} \left(\lim_{k\to\infty} \int_{V_j} f_{jk}^{+} \wedge d'd''(\eta_j g) - \lim_{k\to\infty} \int_{V_j} f_{jk}^{-} \wedge d'd''(\eta_j g) \right). \end{aligned}$$

Using the theorem of Stokes [**Gub16**, Theorem 5.17] twice, we get

$$\int_{V_j} f_{jk}^{\pm} \wedge d'd''(\eta_j g) = \int_{V_j} \eta_j g \wedge d'd'' f_{jk}^{\pm},$$

and so we finally obtain

$$\begin{aligned} d'd''[f](g) &= \sum_{j\in J} \left(\lim_{k\to\infty} \int_{V_j} \eta_j g \wedge d'd'' f_{jk}^{+} - \lim_{k\to\infty} \int_{V_j} \eta_j g \wedge d'd'' f_{jk}^{-} \right) \\ &= \sum_{j\in J} \left(\int_{V_j} \eta_j g \; d\,\mathrm{MA}(f^{+}) - \int_{V_j} \eta_j g \; d\,\mathrm{MA}(f^{-}) \right) \\ &= \sum_{j\in J} \int_{V_j} \eta_j g \; d\,\mathrm{MA}(f) = \int_W g \; d\,\mathrm{MA}(f). \end{aligned}$$

For assertion ii), we assume that f is locally psh-approximable, i.e. we can cover W by open subsets $(V_i)_{i\in I}$ such that f is the uniform limit $f = \lim_{k\to\infty} f_{ik}$ of smooth psh functions f_{ik} on V_i. As above let $(\eta_j)_{j\in J}$ be a partition of unity subordinated to $(V_i)_{i\in I}$. For every non-negative function $g \in \mathcal{C}_c^\infty(W)$, the smooth function $\eta_j g$ is also non-negative as $\eta_j \geq 0$ and has compact support on V_j. From the calculations above, we get

$$d'd''[f](g) = \sum_{j\in J} \lim_{k\to\infty} \int_{V_j} \eta_j g \wedge d'd'' f_{jk} \geq 0.$$

This proves that f is psh on W. □

LEMMA 5.2.2. *Let W be an open subset of X^{an} and let $f\colon W \to \mathbb{R}$ be a globally psh-approximable function, i.e. f is the uniform limit of smooth psh functions f_i on W, and $g \in \mathcal{C}_c^0(W)$, then*

$$\int_W g \; d\,\mathrm{MA}(f) = \lim_{i\to\infty} \int_W g \wedge d'd'' f_i.$$

PROOF. By [**CD12**, Proposition 3.3.5], there are smooth functions $g_k \in \mathcal{C}_c^\infty(W)$ converging uniformly to g. Then

$$\begin{aligned} \int_W g \; d\,\mathrm{MA}(f) &= \lim_{k\to\infty} \int_W g_k \; d\,\mathrm{MA}(f) \\ &= \lim_{k\to\infty} \lim_{i\to\infty} \int_W g_k \wedge d'd'' f_i \\ &= \lim_{i\to\infty} \lim_{k\to\infty} \int_W g_k \wedge d'd'' f_i \\ &= \lim_{i\to\infty} \int_W g \wedge d'd'' f_i. \end{aligned}$$

Note that we may change the order of the limits since $\int_W g_k \wedge d'd'' f_i$ converges uniformly to $\int_W g \wedge d'd'' f_i$ in $i \in \mathbb{N}$. $\square$

One of the main ingredients of the proof of the comparison of the two different notions of subharmonicity is the following theorem. It can be deduced directly from the slope formula for line bundles by Katz, Rabinoff, and Zureick-Brown in [**KRZ16**] using Theorem 5.1.13.

THEOREM 5.2.3. *Let $f = -\log \|1\|_{\mathcal{L}}$ be a model function on X^{an} for a semistable formal model $(\mathcal{X}, \mathcal{L})$ of $(X, \mathcal{O}_X)$. The restriction F of f to $S(\mathcal{X})$ is a piecewise affine function and $f = F \circ \tau_{\mathcal{X}}$ on X^{an}. Furthermore, we have*

$$\mathrm{MA}(f) = dd^c F = dd^c f.$$

PROOF. This follows directly from [**KRZ16**, Theorem 2.6] using Theorem 5.1.13. $\square$

This result can be used to link the measure $dd^c f$ and the current $d'd''[f]$ for every lisse function f on an open subset of X^{an}.

REMARK 5.2.4. Every signed Radon measure μ on an open subset W of X^{an} defines the following current

$$[\mu] \colon \mathcal{C}_c^\infty(W) \to \mathbb{R}, g \mapsto \int_W g \, d\mu$$

(see [**Gub16**, Example 6.3]). Consider a smooth function $f \in \mathcal{C}^\infty(W)$, then we have seen in Proposition 4.1.11 that the smooth form $d'd''f$ corresponds to a signed Radon measure which we also denote by $d'd''f$. Using the theorem of Stokes [**Gub16**, Theorem 5.17], we get $d'd''[f] = [d'd''f]$. Recall that a function is called psh if and only if this current is positive.

Analogously, for every lisse function $f \colon W \to \mathbb{R}$ we get a current $[dd^c f]$ for the corresponding measure $dd^c f$ from Definition 3.1.16 which is positive if and only if f is subharmonic (cf. Proposition 3.1.18).

COROLLARY 5.2.5. *Let W be an open subset of X^{an} and let $f \colon W \to \mathbb{R}$ be a lisse function. For every type II point $x \in W$, there is an open neighborhood V of x in W on which the currents $d'd''[f]$ and $[dd^c f]$ agree.*

PROOF. Consider a point $x \in W$ and let Y be a strictly affinoid domain containing x in its interior. Since f is lisse, there is a strictly semistable formal model $\mathcal{Y}$ of Y with corresponding skeleton $S(\mathcal{Y})$ and $f = F \circ \tau_{\mathcal{Y}}$ on Y for a piecewise affine function F on $S(\mathcal{Y})$.

If $x \notin S(\mathcal{Y})$, then f is constant on an open neighborhood V of x, and so

$$d'd''[f] = [d'd''f] = [0] = [dd^c f]$$

on V.

If $x \in S(\mathcal{Y})$, we may assume by Proposition 2.3.22 that x is a vertex in $S(\mathcal{Y})$. Let $e_1 = [x, y_1], \ldots, e_r = [x, y_r]$ be the edges in $S(\mathcal{Y})$ emanating from x, let $v_1, \ldots, v_r$ be the corresponding tangent directions and set $\lambda_i := d_{v_i} F(x)$. By blowing up $\mathcal{Y}$, we may assume that $y_i \neq y_j$ for all $i, j \in \{1, \ldots, r\}$, the restrictions $F|_{e_i}$ are affine, $d(x, y_i) \in \mathbb{Q}$ and that we divide the edge e_i by an additional vertex y_i' with $d(x, y_i) = 2d(x, y_i')$ (cf. Proposition 2.3.22). Denote this blowing up by $\mathcal{Y}'$, and the tangent direction corresponding to $[y_i', y_i]$ by v_i'. Define the piecewise affine functions F_i on the metric subgraph $\Gamma := \bigcup_{i=1,\ldots,r} [x, y_i]$ of $S(\mathcal{Y}')$ by the following data

$$F_i(x) = 0, \ d_{v_i}(F_i)(x) = \mathrm{sgn}(\lambda_i)\delta_{ij} \ \text{ and } \ d_{v_i'}(F_i)(y_i') = -d_{v_i}(F_i)(x).$$

Set $f_i = F_i \circ \tau_{\mathcal{Y}'}$ on $Y' := \tau_{\mathcal{Y}'}^{-1}(\Gamma)$. Note that Y' is a strictly affinoid domain in W with $\partial Y' = \partial\Gamma$ by Corollary 2.3.32. By construction, $f_i = 0$ on $\partial Y'$ for every $i \in \{1, \dots, r\}$. Hence we can extend f_i to X^{an} by setting $f_i = 0$ on $X^{\mathrm{an}} \backslash Y'$. We therefore have a G-covering of X^{an} on which f_i is piecewise linear, and so f_i is a model function on X^{an} [**GK17**, Proposition 8.11]. Set $\Gamma' := \bigcup_{i=1,\dots,r}[x, y_i']$ and $V := \tau_{\mathcal{Y}'}^{-1}((\Gamma')^\circ)$, which is an open neighborhood of x in W. By the definition of f_i on V, we have on $V \subset Y$

$$f = F \circ \tau_{\mathcal{Y}} = F \circ \tau_{\mathcal{Y}'} = (\sum_{i=1}^{r} |\lambda_i| F_i) \circ \tau_{\mathcal{Y}'} + F(x) = \sum_{i=1}^{r} |\lambda_i| \cdot f_i + F(x) \tag{5.2.1}$$

and

$$dd^c f = dd^c F = \sum_{i=1}^{r} \lambda_i \delta_x = \sum_{i=1}^{r} |\lambda_i| (dd^c(f_i)). \tag{5.2.2}$$

Since the functions f_i are model functions, we know that they are locally approximable on V and $\mathrm{MA}(f_i) = dd^c(f_i)$ by Theorem 5.1.13 and Theorem 5.2.3. Let $0 \leq g \in \mathcal{C}_c^\infty(V)$. Then for every $i \in \{1, \dots, r\}$ we have by Lemma 5.2.1 and $\mathrm{MA}(f_i) = dd^c(f_i)$ that

$$d'd''[f_i](g) = \int_V g \, d\,\mathrm{MA}(f_i) = \int_V g \; dd^c(f_i) = [dd^c f_i](g).$$

Linearity and the Equations (5.2.1) and (5.2.2) imply $d'd''[f](g) = [dd^c f](g)$. □

Recall the definitions of harmonic and pluriharmonic functions from Definition 3.1.2 resp. Definition 4.2.56. Harmonic functions can be characterized as those functions such that f and $-f$ are subharmonic, and pluriharmonic functions as those such that f and $-f$ are psh. Combining Corollary 5.2.5, Theorem 4.2.59 and Corollary 4.2.62, we get directly that these classes of functions coincide.

COROLLARY 5.2.6. *Let W be an open subset of X^{an}. Then a continuous function on W is pluriharmonic if and only if it is harmonic.*

PROOF. Let $h \colon W \to \mathbb{R}$ be continuous. Note that pluriharmonic resp. harmonic functions form a sheaf $\mathcal{H}_X^{CD}$ resp. $\mathcal{H}_X$ on X^{an}, and so we may show the assertion locally. Furthermore, h is lisse in both cases by Corollary 4.2.62 resp. Proposition 3.1.17. We consider $x_0 \in W$. Note that there are only finitely many points of type II that are of positive genus [**BPR13**, Remark 4.18]. If x_0 is of type I, III or IV, we therefore can find an open neighborhood V of x_0 in W that does not contain any of these positive genus points. Then $\mathcal{H}_X^{CD}(V) = \mathcal{H}_X(V)$ holds by Theorem 4.2.59, and so $h|_V$ is pluriharmonic if and only if it is harmonic.

If x_0 is a point of type II in W, Corollary 5.2.5 tells us that there is an open neighborhood V of x_0 in W such that the currents $d'd''[h]$ and $[dd^c h]$ coincide on V. If h is harmonic, the measure $dd^c h$ is zero on V, and so $d'd''[h](g) = [dd^c h](g) = 0$ for every $g \in \mathcal{C}_c^\infty(V)$. Hence h is pluriharmonic on V.

Next, we assume h to be pluriharmonic on V. As h is lisse on V, its Laplacian $dd^c h$ is a discretely supported measure on V. By shrinking V, we assume $\mathrm{supp}(dd^c h)|_V \subset \{x_0\}$. Assume $\mathrm{supp}(dd^c h)|_V = \{x_0\}$. Then we choose a smooth function $g \in \mathcal{C}_c^\infty(V)$ with $g(x_0) \neq 0$. Since the currents $d'd''[h]$ and $[dd^c h]$ coincide on V, we have

$$g(x_0) \cdot \left(\int_{\{x_0\}} dd^c h \right) = \int_V g \; dd^c h = [dd^c h](g) = d'd''[h](g) = 0.$$

Thus $\int_{\{x_0\}} dd^c h = 0$, and so the measure $dd^c h$ on V has to be zero. Proposition 3.1.17 implies that h is harmonic on V. □

5.2.2. Thuillier's subharmonic functions are also subharmonic in the sense of Chambert-Loir and Ducros. In this subsection, we show that every subharmonic function on an open subset of X^{an} is psh. First, we prove this for every lisse function and then we use the fact that a subharmonic function is the limit of lisse subharmonic functions from Proposition 3.1.19 to prove the general claim.

PROPOSITION 5.2.7. *Let W be an open subset of X^{an}. Then every lisse subharmonic function on W is psh.*

PROOF. Let $f\colon W \to \mathbb{R}$ be a lisse subharmonic function on W. Note that being psh, i.e. $d'd''[f] \geq 0$, is a local property by Proposition 5.1.7. We therefore consider a point $x \in W$ and we choose a strictly affinoid domain Y in W that contains x in its interior. Since f is lisse, there is a strictly semistable model $\mathcal{Y}$ of Y with corresponding skeleton $S(\mathcal{Y})$ and a piecewise affine function F on $S(\mathcal{Y})$ such that $f = F \circ \tau_{\mathcal{Y}}$ on Y. In particular, f is constant on $\tau_{\mathcal{Y}}^{-1}(y)$ for every $y \in S(\mathcal{Y})$.

If x is of type I or IV, the point x is contained in $Y \backslash S(\mathcal{Y})$ and so the lisse function f is constant on an open neighborhood V of x in W. Hence $d'd''[f] = [d'd''f] = 0$ on V.

If x is of type II, we have seen in Proposition 5.2.5 that there is an open neighborhood V of x in W such that $d'd''[f] = [dd^cf]$ on V. Since f is subharmonic, the measure dd^cf on W is positive by Proposition 3.1.18, and so we have $d'd''[f] = [dd^cf] \geq 0$ on V.

It remains to consider the case when x is of type III. If x is not contained in $S(\mathcal{Y})$, then there is again an open neighborhood V of x in W on which f is constant, and so $d'd''[f] = [d'd''f] = 0$ on V.

If $x \in S(\mathcal{Y})$, then there is an edge e in $S(\mathcal{Y})$ such that x lies in the interior of e. The closed annulus $A := \tau_{\mathcal{Y}}^{-1}(e)$ is isomorphic to a closed annulus $A' = \mathrm{trop}^{-1}([\mathrm{val}(a), \mathrm{val}(b)])$ in $\mathbb{G}_m^{1,\mathrm{an}}$ for some $a, b \in K^\times$ with $|a| < |b|$ and $\mathrm{trop} = \log|T|$. Thus we can identify e with the real interval $[\mathrm{val}(a), \mathrm{val}(b)]$ via $\mathrm{trop} \circ \Phi$ for a fixed isomorphism $\Phi\colon A \xrightarrow{\sim} A'$. Since e is isometric to $[\mathrm{val}(a), \mathrm{val}(b)]$, we can define a function

$$\psi\colon [\mathrm{val}(a), \mathrm{val}(b)] \to \mathbb{R},\ z \mapsto F((\mathrm{trop} \circ \Phi)^{-1}(z)).$$

Then ψ extends to a piecewise affine function on $\mathrm{trop}(\mathbb{G}_m^{1,\mathrm{an}}) = \mathbb{R}$, and its restriction to the connected components of $\mathbb{R} \backslash \{\mathrm{trop}(\Phi(x))\}$ is affine with outgoing slopes at $\mathrm{trop}(\Phi(x))$ equal to the ones of F at x on e. We required that f is subharmonic, so the sum of the outgoing slopes at x is greater than or equal to zero by Proposition 3.1.18. Hence ψ is convex on $\mathbb{R}$, and so we can find smooth convex functions ψ_i on $\mathbb{R}$ converging uniformly to ψ. Then $\psi_i \circ \mathrm{trop}$ are smooth psh functions on $\mathrm{trop}^{-1}(\mathrm{val}(a), \mathrm{val}(b)) \subset \mathbb{G}_m^{1,\mathrm{an}}$ by Proposition 5.1.8. By Remark 5.1.10, the pullbacks $f_i := \Phi^*(\psi_i \circ \mathrm{trop})$ are smooth psh function on $V' := \tau_{\mathcal{Y}}^{-1}(e^\circ)$ converging uniformly to f on V'. The set V' is an open neighborhood of x in W. Note that by [**BPR13**, Lemma 2.13 & 3.8], the map $\mathrm{trop} = \log|T|$ factors through τ_A and $\tau_A = \tau_{\mathcal{Y}}$ on A. Hence the function f is itself psh on V' by Lemma 5.2.1, i.e. $d'd''[f] \geq 0$ on V'. □

PROPOSITION 5.2.8. *Let W be an open subset of X^{an}. Then every subharmonic function on W is locally integrable.*

PROOF. Let $f\colon W \to [-\infty, \infty)$ be subharmonic. We have to show that $\int_W f \wedge \omega$ is finite for every $\omega \in \mathcal{A}_c^{1,1}(W)$. By [**Gub16**, Proposition 5.13], we may assume that the $(1,1)$-form ω is of the form $\omega = \mathrm{trop}_U^* \omega_{\mathrm{trop}}$ for a tropical chart (V, φ_U) of W and a form $\omega_{\mathrm{trop}} \in \mathcal{A}_c^{1,1}(\mathrm{trop}_U(V))$. The closed embedding φ_U is given by $\gamma_1, \dots, \gamma_r \in \mathcal{O}(U)^\times$, and we denote by H the set of zeros and poles of $\gamma_1, \dots, \gamma_r$ on X. We choose a strictly semistable model $\mathcal{X}$ of X^{an} such that the type I points of H lie in distinct connected components of $X^{\mathrm{an}} \backslash S_0(\mathcal{X})$ (see [**BPR13**, Theorem 4.11 & 5.2]), where $S_0(\mathcal{X})$ denotes the vertices of the skeleton $S(\mathcal{X})$ corresponding to $\mathcal{X}$. By [**BPR13**, Lemma 2.13 & 3.8], we have the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "\mathrm{trop}_U"] \arrow[d, "\tau_{\mathcal{X}}"'] & \Omega \\ S(\mathcal{X}) \arrow[ur, "\mathrm{trop}_U"'] & \end{tikzcd}$$

where $\tau_{\mathcal{X}} \colon X^{\mathrm{an}} \to S(\mathcal{X})$ is the retraction map corresponding to $\mathcal{X}$ and $\Omega := \mathrm{trop}_U(V)$. The retraction map $\tau_{\mathcal{X}}$ is defined in such a way that every connected component of $X^{\mathrm{an}} \backslash S(\mathcal{X})$ is retracted to a single point in $S(\mathcal{X})$. Due to this fact and the commutativity of the diagram, the form $\omega = \mathrm{trop}_U^* \omega_{\mathrm{trop}}$ is supported on $S(\mathcal{X})$. Since the restriction of the subharmonic function f to $S(\mathcal{X})$ is continuous by [**Thu05**, Proposition 3.4.6], we get that

$$\int_W f \wedge \omega = \int_W (f \circ \tau_{\mathcal{X}}) \wedge \omega$$

is finite. Hence f is locally integrable. □

Theorem 5.2.9. *Let W be an open subset of X^{an}. Then every subharmonic function on W is psh.*

Proof. Let $f \colon W \to [-\infty, \infty)$ be a subharmonic. We already know by Proposition 5.2.8 that f is locally integrable. Thus it remains to show that the current $d'd''[f]$ is non-negative, which is a local property by Proposition 5.1.7. Since X^{an} is a locally compact Hausdorff space, we can find for every $x \in W$ a relatively compact neighborhood W' of x in W. By Proposition 3.1.19, there is a decreasing net $\langle f_\alpha \rangle$ of lisse subharmonic functions converging pointwise to f on W'. Consider a non-negative function $g \in \mathcal{C}_c^\infty(W')$. Then there are smooth forms $\omega^+, \omega^- \in \mathcal{A}^{1,1}(X^{\mathrm{an}})$ such that $d'd''g = \omega^+ - \omega^-$ and the corresponding signed Radon measures from Proposition 4.1.11 are positive [**CD12**, Lemme 5.3.3]. By [**CD12**, Corollaire 3.3.4], we can find a smooth non-negative function $\eta \in \mathcal{C}_c^\infty(W')$ such that $\eta \equiv 1$ on $\mathrm{supp}(g)$. Hence

$$d'd''g = \eta d'd''g = \eta\omega^+ - \eta\omega^-$$

on W'. The smooth $(1,1)$-forms $\eta\omega^\pm$ are contained in $\mathcal{A}_c^{1,1}(W')$ and the corresponding Radon measures are still non-negative and have compact support by Proposition 4.1.11. Thus $\int_{W'} f \wedge \eta\omega^\pm$ is finite (see Proposition 5.2.8), and we have

$$\int_{W'} f \wedge \eta\omega^\pm = \lim_\alpha \int_{W'} f_\alpha \wedge \eta\omega^\pm$$

by [**Fol99**, Proposition 7.12]. Together, we get

$$\begin{aligned} d'd''[f](g) &= \int_{W'} f \wedge \eta d'd''g = \int_{W'} f \wedge \eta\omega^+ - \int_{W'} f \wedge \eta\omega^- \\ &= \lim_\alpha \Big(\int_{W'} f_\alpha \wedge \eta\omega^+ - \int_{W'} f_\alpha \wedge \eta\omega^-\Big) = \lim_\alpha \int_{W'} f_\alpha \wedge d'd''g. \end{aligned}$$

By Proposition 5.2.7, we have $\int_{W'} f_\alpha \wedge d'd''g \geq 0$ for every f_α, and so $d'd''[f](g) \geq 0$. □

5.2.3. Continuous subharmonic functions in the sense of Chambert-Loir and Ducros are subharmonic in the sense of Thuillier. In this subsection, we prove that every continuous psh function is subharmonic in the sense of Thuillier. The key tool of the proof is that for a model function g the Monge–Ampère measure is equal to its Laplacian $dd^c g$ (see Theorem 5.2.3).

LEMMA 5.2.10. *Let W be an open subset of X^{an} and let $f\colon W \to \mathbb{R}$ be a continuous function. If f is not subharmonic, then there is a strictly semistable formal model $(\mathcal{X}, \mathcal{L})$ of $(X, \mathcal{O}_X)$ such that the model function $g := -\log \|1\|_{\mathcal{L}}$ satisfies the following properties:*

i) $\operatorname{supp}(g)$ *is a connected strictly affinoid domain $Y \subset W$.*

ii) g is non-negative on Y.

iii)

$$\int_W f dd^c g < 0.$$

PROOF. By Corollary 3.1.23, there is a connected strictly affinoid domain Y and a type II or III point x in $Y\backslash\partial Y$ with $\int_W f dd^c g_x^Y < 0$. Note that g_x^Y is a non-negative lisse function on X^{an} with $\operatorname{supp}(g_x^Y) = Y$. Since g_x^Y is lisse, we can find a strictly semistable formal model $\mathcal{Y}$ of Y and a piecewise affine function G_x^Y on the corresponding skeleton $S(\mathcal{Y})$ such that $g_x^Y = G_x^Y \circ \tau_{\mathcal{Y}}$ on Y. By blowing up $\mathcal{Y}$, we may assume that each edge of $S(\mathcal{Y})$ has at most one endpoint that lies in the boundary ∂Y (see Proposition 2.3.22). We will explain in steps, why there is a strictly semistable formal model $(\mathcal{X}, \mathcal{L})$ such that the corresponding model function $g = -\log \|1\|_{\mathcal{L}}$ on X^{an} satisfies $g \geq 0$ on $Y\backslash\partial Y$, $g = 0$ elsewhere, and $\int_W f dd^c g < 0$.

0. Step: Strategy of the proof.

We construct a $\mathrm{val}(K^\times)$-rational function (see definition below) G on $S(\mathcal{Y})$ such that

$$G|_{S(\mathcal{Y})\backslash\partial Y} \geq 0,\ G|_{\partial Y} = 0 \text{ and } \int_{S(\mathcal{Y})} f dd^c G < 0. \tag{5.2.3}$$

Then $g := G \circ \tau_{\mathcal{Y}}$ is piecewise $\mathbb{Q}$-linear on Y in the sense of [**GK15**]. Setting $g \equiv 0$ on $X^{\mathrm{an}}\backslash Y$, we get a continuous non-negative function g on X^{an} and a G-covering of X^{an} on which g is piecewise $\mathbb{Q}$-linear. Thus there is a semistable formal model $(\mathcal{X}, \mathcal{L})$ such that $g = -\frac{1}{m}\log \|1\|_{\mathcal{L}}$ on X^{an} by [**GK17**, Proposition 8.11& 8.13]. We always may assume $\mathcal{X}$ to be strictly semistable by the semistable reduction Theorem [**BL85**, Ch. 7] since K is algebraically closed. Then the formal model $(\mathcal{X}, \mathcal{L}^{\otimes m})$ of $(X, \mathcal{O}_X)$ gives the claim.

Before we start with the construction, note that a $\mathrm{val}(K^\times)$-rational function is a piecewise affine function on $S(\mathcal{Y})$ (we refine the vertex set such that G is affine on every edge) such that the following properties are satisfied

(a) $dd^c G$ is only supported on points of type II.
(b) G has values in $\mathrm{val}(K^\times)$ at every vertex of $S(\mathcal{Y})$.
(c) G has rational slopes.

Our piecewise affine function G_x^Y on $S(\mathcal{Y})$ satisfies (5.2.3), but it is not (necessarily) $\mathrm{val}(K^\times)$-rational. We therefore modify G_x^Y in the following three steps.

1. Step: Replace G_x^Y by a piecewise affine function G' such that (5.2.3) is still satisfied and (a) additionally holds.

If x is of type II, the support of $dd^c G_x^Y$ consists only of type II points. If x is of type III, we use that the points of type II are dense in X^{an}. Let $e = [y_1, y_2]$ be the edge of $S(\mathcal{Y})$

having x in its interior. Let x_n be a sequence of type II points in e converging to x with respect to the skeletal metric. Consider the piecewise affine functions G_n on $S(\mathcal{Y})$ that are given by the affine function on $[y_1, x_n]$ (resp. on $[x_n, y_2]$) connecting the points $G_x^Y(y_1)$ and $G_x^Y(x_n)$ (resp. $G_x^Y(x_n)$ and $G_x^Y(y_2)$) and $G_n \equiv G_x^Y$ on $S(\mathcal{Y})\backslash e$. It is easy to see that the slopes converge to the ones of G_x^Y, i.e. $dd^cG_n \to dd^cG_x^Y$ for $n \to \infty$. Furthermore, f is continuous, so we can find n big enough such that $|f(x_n)dd^cG_n(x_n) - f(x)dd^cG_x^Y(x)|$ is so small that we still have $\int_{S(\mathcal{Y})} f dd^c G_n < 0$ (cf. [**Thu05**, Proposition 3.3.4]). Set $G' := G_n$ for such an n.

2. Step: Replace G' by a piecewise affine function G'' such that (5.2.3) and (a) are still satisfied and (b) additionally holds.

Due to normalizing the absolute value $|\cdot|$, we assume that $\mathbb{Q}$ is contained, and so dense, in $\mathrm{val}(K^\times)$. Let z be a vertex of $S(\mathcal{Y})$. If $G'(z)$ is not in $\mathbb{Q}$, let $(a_n)_n$ be a sequence of rational points converging to $G'(z)$. Then the slopes of the piecewise affine functions G'_n on $S(\mathcal{Y})$ resulting by replacing $G'(z)$ by a_n converge to the slopes of G'. Thus we can choose an $n \in \mathbb{N}$ such that $G'' := G'_n$ still satisfies (5.2.3) and takes only values in $\mathbb{Q}$ at every vertex of $S(\mathcal{Y})$. We choose these values such that $G'' \geq 0$ on $S(\mathcal{Y})$.

3. Step: Replace G'' by a piecewise affine function G such that (5.2.3), (a) and (b) are still satisfied and (c) additionally holds.

Now, consider an edge $e = [y_1, y_2]$ of $S(\mathcal{Y})$. Denote by $d(y_1, y_2)$ the distance between these two points with respect to the skeletal metric. If $d(y_1, y_2) \in \mathbb{Q}$, we are done. If not, we can find points y'_1 and y'_2 of type II in e with distance $d(y'_1, y'_2) \in \mathbb{Q}$ arbitrary close to $d(y_1, y_2)$. These points are chosen so that we can decompose the edge e into $[y_1, y'_1] \cup [y'_1, y'_2] \cup [y'_2, y_2]$.

Let G be the piecewise affine function on e defined by the data $G \equiv G''(y_1)$ on $[y_1, y'_1]$, $G \equiv G''(y_2)$ on $[y'_2, y_2]$ and $G|_{[y'_1, y'_2]}$ is affine (see Figure 2). Since $G(y'_i) = G''(y_i) \in \mathbb{Q}$, the constructed function G on e has rational slopes. Choose y'_i with $d(y'_1, y'_2)$ close enough to $d(y_1, y_2)$ such that (5.2.3) is still satisfied. Again this is possible since f is continuous.

We do this for all edges except the ones containing the boundary points ∂Y, where we just move the other vertex. Then the function G on $S(\mathcal{Y})$ has slopes in $\mathbb{Q}$ and is consequently the required function.

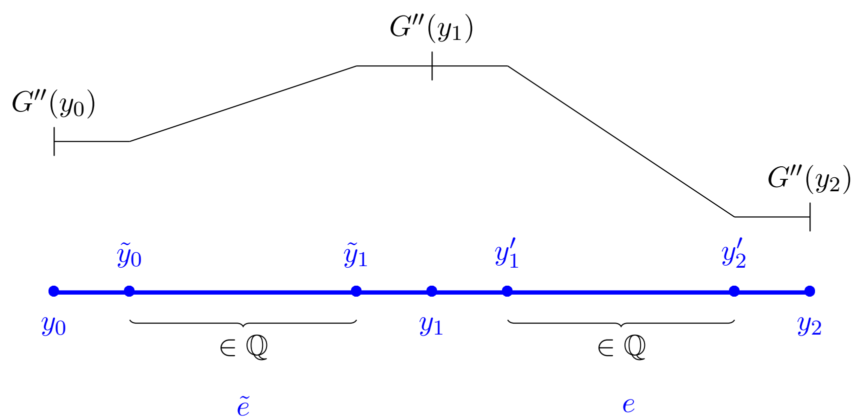

FIGURE 2. Representing the construction of G in Step 3.

$\square$

THEOREM 5.2.11. *Let W be an open subset of X^{an} and let $f\colon W \to \mathbb{R}$ be a continuous function. If f is psh, then f is subharmonic.*

PROOF. The proof is by contradiction. We assume that there is a strictly semistable formal model $(\mathcal{X}, \mathcal{L})$ of $(X, \mathcal{O}_X)$ such that $\int_W f dd^c g < 0$ for $g := -\log \|1\|_{\mathcal{L}}$ as in Lemma 5.2.10.

0. Step: Strategy of the proof.

We know that $g = G \circ \tau_{\mathcal{X}}$ on X^{an} for the piecewise affine function $G = g|_{S(\mathcal{X})}$ on $S(\mathcal{X})$, the Monge–Ampère measure exists and $dd^c g = dd^c G = \mathrm{MA}(g)$ by Theorem 5.2.3.

Assume that we have the following situation: We can write $g = g^+ - g^-$ on an open subset V of W that contains the connected strictly affinoid domain $Y = \mathrm{supp}(g)$ such that

(a) $g^{\pm}$ is the uniform limit of smooth psh functions $g_k^{\pm}$ on V, i.e. g is globally approximable on V,
(b) $g_k := g_k^+ - g_k^- \geq 0$ on V, and
(c) g_k has compact support on V.

Moreover, we want to have a connected open subset V' of V with $Y \subset V'$, $\overline{V'} \subset V$ and $g_k \in \mathcal{C}_c^{\infty}(V')$ and a continuous map η on V such that $\eta \equiv 1$ on V' and η has compact support in V.

In this situation, we have $\eta f \in \mathcal{C}_c^0(V)$ and $f = \eta f$ on V'. Moreover, by the definition of the Monge–Ampère measure and Theorem 5.2.3, we have the identity of measures

$$dd^c g = \mathrm{MA}(g) = \mathrm{MA}(g^+) - \mathrm{MA}(g^-)$$

whose support is finite and contained in $Y \subset V' \subset V$. Due to our assumption, this implies

$$0 > \int_W f \, d\,\mathrm{MA}(g) = \int_V \eta f \, d\,\mathrm{MA}(g) = \int_V \eta f \, d\,\mathrm{MA}(g^+) - \int_V \eta f \, d\,\mathrm{MA}(g^-).$$

Applying Lemma 5.2.2 and (a) to the right hand side and using $g_k \in \mathcal{C}_c^{\infty}(V')$, we get

$$\begin{aligned} 0 > \int_W f \, d\,\mathrm{MA}(g) &= \lim_{k\to\infty} \int_V \eta f \wedge d'd'' g_k^+ - \lim_{k\to\infty} \int_V \eta f \wedge d'd'' g_k^- \\ &= \lim_{k\to\infty} \int_V \eta f \wedge d'd'' g_k \\ &= \lim_{k\to\infty} \int_{V'} f \wedge d'd'' g_k \\ &= \lim_{k\to\infty} d'd''[f|_{V'}](g_k). \end{aligned}$$

We know that $d'd''[f]$ is positive on W, and so it is on V'. Thus

$$0 > \int_W f \, d\,\mathrm{MA}(g) = \lim_{k\to\infty} d'd''[f|_{V'}](g_k) \geq 0,$$

and so we have a contradiction. Hence f has to be subharmonic.

We explain in several steps how to construct V, V', η and the functions g_k^+, g_k^- such that one has the described situation.

1. Step: Show that the function g is globally approximable on W.

The curve X is projective, so we may assume $\mathcal{X}$ to be projective as well and we therefore can find very ample line bundles $\mathcal{L}_1, \mathcal{L}_2$ such that $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$. Thus we can write

$$g = -\log \|1\|_{\mathcal{L}} = -\log \|s_1\|_{\mathcal{L}_1} + \log \|s_2\|_{\mathcal{L}_2}$$

on W for global sections s_1, s_2 that coincide on the generic fiber. Since we may work on every open subset of W containing the compact subset Y by Proposition 5.1.7, we may assume that s_1 and s_2 do not vanish on W. Due to the very ampleness, we can find closed embeddings $\varphi_i \colon \mathcal{X} \to \mathbb{P}^{n_i}_R$ such that $\mathcal{L}_i \simeq \varphi_i^* \mathcal{O}_{\mathbb{P}^{n_i}_R}(1)$ and $s_i = \varphi_i^* x_{j_i}$ for $i = 1, 2$. Here, let $x_0, \ldots, x_{n_i}$ be the coordinates of $\mathbb{P}^{n_i}_R$ and $j_i \in \{0, \ldots, n_i\}$. Without loss of generality, we assume $x_{j_1} = x_{n_1}$ and $x_{j_2} = x_{n_2}$. These closed embeddings induce closed embeddings $\varphi_i \colon X \to \mathbb{P}^{n_i}_K$. Then

$$\begin{aligned} g &= -\log \|s_1\|_{\mathcal{L}_1} + \log \|s_2\|_{\mathcal{L}_2} \\ &= (\varphi_1^{\mathrm{an}})^*(-\log |x_{n_1}| + \max_{j \in \{0, \ldots, n_1\}} \log |x_j|) - (\varphi_2^{\mathrm{an}})^*(-\log |x_{n_2}| + \max_{j \in \{0, \ldots, n_2\}} \log |x_j|) \end{aligned}$$

on W. We approximate $\phi_i := -\log |x_{n_i}| + \max_{j \in \{0, \ldots, n_i\}} \log |x_j|$ for $i = 1, 2$ by smooth convex functions on $\{\eta \in \mathbb{P}^{n_i, \mathrm{an}}_K \mid |x_{n_i}|_\eta \neq 0\}$ as in [**CD12**, Proposition 6.3.2].

For every $k \in \mathbb{N}_{>0}$ and $n \in \mathbb{N}$, there is a smooth, convex function $M_{n, \frac{1}{k}}$ on $\mathbb{R}^{n+1}$ that is non-decreasing in every variable and has the following properties:

i) $\max(t_0, \ldots, t_n) \leq M_{n, \frac{1}{k}}(t_0, \ldots, t_n) \leq \max(t_0, \ldots, t_n) + \frac{1}{k}$.
ii) If $t_l + \frac{2}{k} \leq \max_{j \neq l} t_j$, then $M_{n, \frac{1}{k}}(t_0, \ldots, t_n) = M_{n-1, \frac{1}{k}}(t_0, \ldots, \widehat{t_l}, \ldots, t_n)$.
iii) For all $t \in \mathbb{R}$, we have $M_{n, \frac{1}{k}}(t_0 + t, \ldots, t_n + t) = M_{n, \frac{1}{k}}(t_0, \ldots, t_n) + t$.

Define for $k \in \mathbb{N}_{>0}$ and $i \in \{1, 2\}$ the following function

$$\phi_{i,k} := -\log |x_{n_i}| + M_{n_i, \frac{1}{k}}(\log |x_0|, \ldots, \log |x_{n_i}|)$$

on $\mathbb{P}^{n_i, \mathrm{an}}_K$. For every point ζ in $\{\eta \in \mathbb{P}^{n_i, \mathrm{an}}_K \mid |x_{n_i}|_\eta \neq 0\}$ there is an open neighborhood of ζ such that $\phi_{i,k}$ is smooth and psh on this neighborhood. Both are local properties and s_i does not vanish on W, so $(\varphi_i^{\mathrm{an}})^* \phi_{i,k}$ is a smooth psh function on W (see Remark 5.1.10) converging uniformly to $-\log \|s_i\|_{\mathcal{L}_i}$ on W. We use the notations $g^+ := -\log \|s_1\|_{\mathcal{L}_1}$ and $g^- := -\log \|s_2\|_{\mathcal{L}_2}$. Furthermore, we set $g_k^+ := (\varphi_1^{\mathrm{an}})^* \phi_{1,k}$ and $g_k^- := (\varphi_2^{\mathrm{an}})^* \phi_{2,k}$. Note that we have by construction

$$g^\pm \leq g_k^\pm \leq g^\pm + \frac{1}{k}.$$

These functions do not necessarily satisfy (b) and (c), so we need to modify $g_k^\pm$.

2. Step: Construct a suitable V and study the behavior of $g_k^\pm$ outside of Y.

At the end of Step 2, one can find an illustration of the construction in Figure 3. The boundary of the strictly affinoid domain Y is a finite set of points of type II. By blowing up our model $\mathcal{X}$, we may assume that the points ∂Y are vertices in $S(\mathcal{X})$ (see Proposition 2.3.22). Note that we always may assume that an admissible blowing up is strictly semistable again by the semistable reduction Theorem [**BL85**, Ch. 7]. Consider a point $y \in \partial Y$. Since y is of type II, Theorem 2.3.27 tells us that there is a strictly semistable formal model $\mathcal{X}_y$ of X and a star-shaped open neighborhood Ω_y of y in $S(\mathcal{X}_y)$ such that $\tau_{\mathcal{X}_y}^{-1}(\Omega_y)$ is an open neighborhood of y in W. Here, a star-shaped open neighborhood Ω_y of y in $S(\mathcal{X}_y)$ is a simply-connected open neighborhood of y in $S(\mathcal{X}_y)$ such that the intersection of Ω_y with any edge e in $S(\mathcal{X}_y)$ emanating from y is a half-open interval $I_{y,e} = [y, x_e)$ with endpoints y and x_e. We may choose the endpoints x_e also of type II. By blowing up $\mathcal{X}$ and modifying Ω_y, we may assume that we can find this star-shaped open neighborhood Ω_y in $S(\mathcal{X})$. We explain how to do this. The model $\mathcal{X}$ has to be blown up such that every vertex of $S(\mathcal{X}_y)$ is a vertex in the new skeleton $S(\mathcal{X})$, i.e. $S_0(\mathcal{X}_y) \subset S_0(\mathcal{X})$. Then we can modify Ω_y in the following way. Consider an edge e of the new skeleton $S(\mathcal{X})$. Then the interior of e is either contained in an edge $\tilde{e}$ of $S(\mathcal{X}_y)$ or lies in a connected component of $X^{\mathrm{an}} \backslash S(\mathcal{X}_y)$ isomorphic to an open ball. In the first case, we shrink the interval such that $I_{y,e}$ is a half-open interval in e. Note that $\tau_{\mathcal{X}}^{-1}(I_{y,e}) = \tau_{\mathcal{X}_y}^{-1}(I_{y,e})$, and so it is still contained in W. In the second case, we just add a new half-open interval $I_{y,e}$ to Ω_y. Then $\tau_{\mathcal{X}}^{-1}(I_{y,e}) \subset \tau_{\mathcal{X}_y}^{-1}(y) \subset W$. We do this blowing ups and modifications for all boundary points. Moreover, we always choose Ω_y such that $\overline{\Omega_y} \subset W$. Before we can construct V, we have to blow up $\mathcal{X}$ one more time. We find this admissible formal blowing up $\mathcal{X}'$ of $\mathcal{X}$ such that $Y = \mathcal{Y}_\eta$ for a formal open subset $\mathcal{Y}$ of $\mathcal{X}'$, which is possible by [**BL85**, Lemma 4.4]. Then we have $Y = \tau_{\mathcal{X}'}^{-1}(S(\mathcal{X}') \cap Y)$. As described above, we modify Ω_y such that it is a star-shaped open neighborhood of y in $S(\mathcal{X}')$. Altogether, we have a strictly semistable formal model $\mathcal{X}'$ of X such that every boundary point y of Y has an open neighborhood $\tau_{\mathcal{X}'}^{-1}(\Omega_y)$ in W for a star-shaped open neighborhood Ω_y of y in $S(\mathcal{X}')$ and $Y = \tau_{\mathcal{X}'}^{-1}(S(\mathcal{X}') \cap Y)$.

For every point $z \in (S(\mathcal{X}') \cap Y) \backslash \partial Y$, we choose a star-shaped open neighborhood Ω_z of z in $S(\mathcal{X}')$ with $\Omega_z \subset Y \backslash \partial Y$. Then $\tau_{\mathcal{X}'}^{-1}(\Omega_z)$ is automatically contained in W due to $Y = \tau_{\mathcal{X}'}^{-1}(S(\mathcal{X}') \cap Y)$ and $Y \subset W$.

We have constructed for every point in $S(\mathcal{X}') \cap Y$ a star-shaped open neighborhood of it in $S(\mathcal{X}')$, Ω_y for $y \in \partial Y$ and Ω_z for $z \notin \partial Y$, and so these open subsets clearly cover our compact subset $S(\mathcal{X}') \cap Y$. Thus there is a finite subset Y_0 of $S(\mathcal{X}') \cap Y$ such that

$$Y = \tau_{\mathcal{X}'}^{-1}(S(\mathcal{X}') \cap Y) \subset \bigcup_{z \in Y_0} \tau_{\mathcal{X}'}^{-1}(\Omega_z).$$

By construction, the set of boundary points ∂Y is contained in Y_0. Furthermore, we choose Y_0 minimal, i.e. removing one open subset $\tau_{\mathcal{X}'}^{-1}(\Omega_z)$ from the covering would no longer cover Y. Set

$$V := \bigcup_{z \in Y_0} \tau_{\mathcal{X}'}^{-1}(\Omega_z),$$

then V is an open subset of W containing $Y = \mathrm{supp}(g)$.

Let y be a point in ∂Y, let e be an edge emanating from y in $S(\mathcal{X}')$ not contained in Y and let $I_{y,e}$ be the corresponding half-open interval in the star-shaped open neighborhood Ω_y. Note that $g_{|e} = 0$. We may shrink the half-open interval $I_{y,e}$ in e, and so Ω_y and V, such that

$$\begin{aligned} g^+ &= (\varphi_1^{\mathrm{an}})^*(-\log|x_{n_1}| + \max_{j \in \{0,\dots,n_1\}} \log|x_j|) = (\varphi_1^{\mathrm{an}})^*(-\log|x_{n_1}| + \log|x_{l_1}|) \\ g^- &= (\varphi_2^{\mathrm{an}})^*(-\log|x_{n_2}| + \max_{j \in \{0,\dots,n_2\}} \log|x_j|) = (\varphi_2^{\mathrm{an}})^*(-\log|x_{n_2}| + \log|x_{l_2}|)) \end{aligned}$$

on $I_{y,e}$ for some $l_i \in \{0, \dots, n_i\}$. Define the map $N_{i,\max} \colon I_{y,e} \to \mathbb{N}$ as follows

$$N_{i,\max}(\eta) := |\{j \in \{0, \dots, n_i\} | (\varphi_i^{\mathrm{an}})^*(\log|x_j|) = (\varphi_i^{\mathrm{an}})^*(\log|x_{l_i}|)\}|.$$

Shrink $I_{y,e}$ again such that $(\varphi_i^{\mathrm{an}})^*(\log|x_j|)$ is affine on $I_{y,e}$ for every $j \in \{0,\dots,n_i\}$. Consequently, the function $N_{i,\max}$ is constant on $I_{y,e}\backslash\{y\}$ because we have on $I_{y,e}$ $(\varphi_i^{\mathrm{an}})^*(\log|x_{l_i}|) = \max(\varphi_i^{\mathrm{an}})^*(\log|x_j|)$. Hence we write $N_{i,\max}$ for this constant value.

Define

$$\begin{aligned}\Omega_{i,k} :=& \{\eta \in I_{y,e} \mid (\varphi_i^{\mathrm{an}})^*(\log|x_j|)(\eta) + \frac{2}{k} < (\varphi_i^{\mathrm{an}})^*(\log|x_{l_i}|)(\eta) \\ &\text{if } (\varphi_i^{\mathrm{an}})^*(\log|x_j|)|_{I_{y,e}} \neq (\varphi_i^{\mathrm{an}})^*(\log|x_{l_i}|)|_{I_{y,e}},\ j \in \{0,\dots,n_i\}\}\end{aligned}$$

Let $N' \in \mathbb{N}$ such that $\Omega_k := (\Omega_{1,k} \cap \Omega_{2,k})\backslash\{y\}$ is a non-empty connected open subset of $I_{y,e}$ for every $k \geq N'$. We have $\Omega_k \subset \Omega_{k+1}$, and we work in the following with $\Omega_{y,e} := \Omega_{N'}$. Then we get for every $k \geq N'$ using properties i)-iii) that

$$\begin{aligned} g_k^+ &= (\varphi_1^{\mathrm{an}})^*(-\log|x_{n_1}|) + M_{n_1,\frac{1}{k}}((\varphi_1^{\mathrm{an}})^*(\log|x_0|),\dots,(\varphi_1^{\mathrm{an}})^*(\log|x_{n_1}|)) \\ &= (\varphi_1^{\mathrm{an}})^*(-\log|x_{n_1}|) + M_{N_{1,\max}-1,\frac{1}{k}}((\varphi_1^{\mathrm{an}})^*(\log|x_{l_1}|),\dots,(\varphi_1^{\mathrm{an}})^*(\log|x_{l_1}|)) \\ &= (\varphi_1^{\mathrm{an}})^*(-\log|x_{n_1}|) + (\varphi_1^{\mathrm{an}})^*(\log|x_{l_1}|) + M_{N_{1,\max}-1,\frac{1}{k}}(0,\dots,0) \\ &= g^+ + M_{N_{1,\max}-1,\frac{1}{k}}(0,\dots,0) \end{aligned} \tag{5.2.4}$$

on $\Omega_{y,e}$. Set $C_k^+ := M_{N_{1,\max}-1,\frac{1}{k}}(0,\dots,0) \in [0,\frac{1}{k}]$. Analogously,

$$g_k^- = g^- + M_{N_{2,\max}-1,\frac{1}{k}}(0,\dots,0) \tag{5.2.5}$$

on $\Omega_{y,e}$. Set $C_k^- := M_{N_{2,\max}-1,\frac{1}{k}}(0,\dots,0) \in [0,\frac{1}{k}]$. Due to $g = g^+ - g^- = 0$ on $\Omega_{y,e}$, we have

$$g_k^+ - g_k^- = M_{N_{1,\max}-1,\frac{1}{k}}(0,\dots,0) - M_{N_{2,\max}-1,\frac{1}{k}}(0,\dots,0)$$

on $\Omega_{y,e}$.

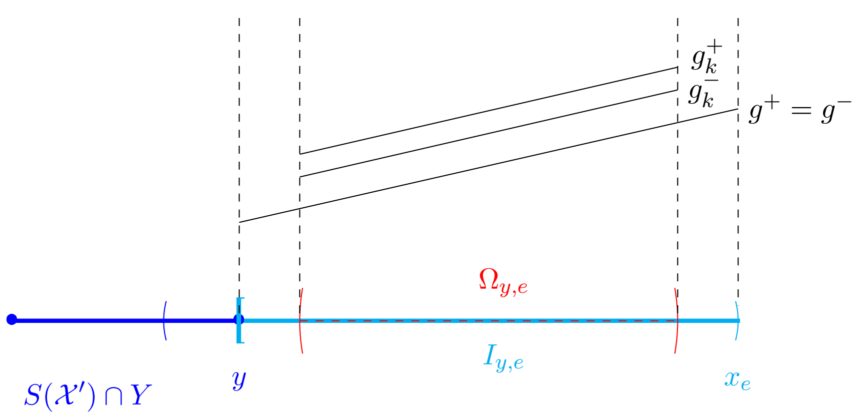

FIGURE 3. Representing the behavior of the functions $g_k^\pm$ on the skeleton outside of Y close to a boundary point y of Y.

3. Step: Modify the constructed functions $g_k^\pm$ on the open subset $V \subset W$ such that (b) and (c) are satisfied.

We start with (b). To ensure that the difference $g_k^+ - g_k^- = (\varphi_1^{\mathrm{an}})^*\phi_{1,k} - (\varphi_2^{\mathrm{an}})^*\phi_{2,k}$ is non-negative on V, we work with $g_k^+ := (\varphi_1^{\mathrm{an}})^*\phi_{1,k} + \frac{1}{k}$ on V instead of $(\varphi_1^{\mathrm{an}})^*\phi_{1,k}$. This is still a smooth psh function on V converging uniformly to the function $g^+ = -\log \|s_1\|_{\mathcal{L}_1}$, and

$$\begin{aligned}(\varphi_1^{\mathrm{an}})^*\phi_{1,k} + \frac{1}{k} - (\varphi_2^{\mathrm{an}})^*\phi_{2,k} &\geq (\varphi_1^{\mathrm{an}})^*\phi_{1,k} + \frac{1}{k} - g^+ + g^- - (\varphi_2^{\mathrm{an}})^*\phi_{2,k} \\ &\geq 0 + \frac{1}{k} - \frac{1}{k} = 0\end{aligned}$$

on V. Note that we used $g = g^+ - g^- \geq 0$ on V and property i) of $M_{n_i,k}$.

Next, we deal with (c), i.e. we modify g_k^+ and g_k^- such that $g_k := g_k^+ - g_k^-$ has compact support on V. As in Step 2, let y be a boundary point of Y, let Ω_y be a star-shaped open neighborhood of y and let $I_{y,e} = [y, x_e)$ be a half-open interval of Ω_y contained in an edge e of $S(\mathcal{X}')$ emanating from y and pointing outwards of Y. Recall that we have constructed in the second step the open subset $\Omega_{y,e}$ of $I_{y,e}$. Here, we start with the construction of an affine function on the closed annulus $A := \tau_{\mathcal{X}'}^{-1}([y, x_e])$ whose graph intersects the graphs of all our functions $g_k^{\pm}$ on $\Omega_{y,e}$ for k big enough. The annulus A is isomorphic to a closed annulus $A' := S(a,b) = \mathrm{trop}^{-1}([\mathrm{val}(a), \mathrm{val}(b)])$ in $\mathbb{G}_m^{1,\mathrm{an}}$ for some $a, b \in K^\times$ with $|a| < |b|$ and $\mathrm{trop} = \log|T|$, where it is meant that a seminorm p is mapped to $\log(p(T))$. Let $\Phi_{y,e} \colon A \xrightarrow{\sim} A'$ be an isomorphism. For simplicity, we may assume that $\mathrm{val}(a) = 0$. Then we can identify $I_{y,e} = [y, x_e)$ with the real half-open interval $[0, \mathrm{val}(b))$ via $\mathrm{trop}_{y,e} := \log|T| \circ \Phi_{y,e}$. Choose points $\zeta_{y,e}, \zeta'_{y,e} \in \Omega_{y,e} \subset e$, and $m \in \mathbb{N}$, $c \in \mathbb{R}$ such that the function

$$\psi_{y,e} \colon I_{y,e} \to \mathbb{R},\ \zeta \mapsto m \cdot \mathrm{trop}_{y,e}(\zeta) + c$$

satisfies

$$\psi_{y,e}(\zeta_{y,e}) = ((\varphi_1^{\mathrm{an}})^*(-\log|x_{n_1}|) + (\varphi_1^{\mathrm{an}})^*(-\log|x_{l_1}|))(\zeta_{y,e}) = g^+(\zeta_{y,e}),$$

and $\psi_{y,e}(\zeta'_{y,e}) = g_{N'}^+(\zeta'_{y,e})$. Recall that N' was fixed in Step 2 to define $\Omega_{y,e}$. Since $g_k^{\pm}$ converges uniformly to $g^+ = g^-$ on $\Omega_{y,e}$, there is an $N'' \geq N'$ such that

$$\sup_{x \in \Omega_{y,e}} (g_k^{\pm} - g^{\pm}) = \sup_{x \in \Omega_{y,e}} |g_k^{\pm} - g^{\pm}| \leq C_{N'}^+$$

for every $k \geq N''$. By (5.2.4), we have $g_{N'}^+ - g^{\pm} = C_{N'}^+ + 1/N'$, and hence

$$g_k^{\pm} - g_{N'}^+ = g_k^{\pm} - g^{\pm} + g^{\pm} - g_{N'}^+ < 0$$

on $\Omega_{y,e}$ for every $k \geq N''$. Thus for every $k \geq N''$ there is a point ζ_k^+ in $\Omega_{y,e}$ such that $\psi_{y,e}(\zeta_k^+) = g_k^+(\zeta_k^+)$. Due to $g^+ = g^- \leq g_k^- \leq g_k^+$ by (5.2.5) and construction, there is also for every $k \geq N''$ a point ζ_k^- in $\Omega_{y,e}$ such that $\psi_{y,e}(\zeta_k^-) = g_k^-(\zeta_k^-)$. Recall that $g^- = g^+$ and g_k^+ and g_k^- with $k \geq N'$ are affine on $\Omega_{y,e}$.

We choose ε such that

$$\begin{aligned}\Omega_{y,e,<} &:= \{\zeta \in \Omega_{y,e} \mid \psi_{y,e}(\zeta) + 2\varepsilon < g^+(\zeta)\} \\ \Omega_{y,e,>} &:= \{\zeta \in \Omega_{y,e} \mid \psi_{y,e}(\zeta) > g^+(\zeta) + 2\varepsilon\}\end{aligned}$$

are non-empty open subsets of $\Omega_{y,e}$, and we set $\Gamma_{y,e} := \Omega_{y,e} \backslash (\Omega_{y,e,<} \cup \Omega_{y,e,>})$.

In the following, we smoothen the piecewise affine functions $\max(g_k^{\pm}, \psi_{y,e})$ in a proper way. One can construct a smooth symmetric convex 1-Lipschitz continuous function $\theta_\varepsilon \colon \mathbb{R} \to (0, \infty)$ such that $\theta_\varepsilon(a) = |a|$ if $|a| \geq \varepsilon$. We set

$$m_\varepsilon(a,b) := \frac{a + b + \theta_\varepsilon(a-b)}{2}. \tag{5.2.6}$$

Then the smooth function $m_\varepsilon \colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the following properties:

i) m_ε is convex.
ii) $\max(a,b) \le m_\varepsilon(a,b) \le \max(a,b) + \frac{\varepsilon}{2}$.
iii) $m_\varepsilon(a,b) = \max(a,b)$ whenever $|a-b| \ge \varepsilon$.
iv) m_ε is increasing in every variable.

We define the functions

$$\widetilde{g}_k^+ := m_\varepsilon(g_k^+, \psi_{y,e})$$
$$\widetilde{g}_k^- := m_\varepsilon(g_k^-, \psi_{y,e})$$

on $\Omega_{y,e}$. Then $\widetilde{g}_k^+$ (resp. $\widetilde{g}_k^-$) coincides with g_k^+ (resp. with g_k^-) on $\Omega_{y,e,<}$ for every $k \ge N''$ since $g^+ = g^- \le g_k^- \le g_k^+$ on $I_{y,e}$ by (5.2.5) and by construction. The functions g_k^+ converge uniformly to $g^- = g^+$, so we can choose $N_{y,e} \ge N''$ such that for all $k \ge N_{y,e}$, we have $g^+ + \varepsilon \ge g_k^+ \ge g_k^-$ on $\Omega_{y,e}$. Then $\widetilde{g}_k^+$ and $\widetilde{g}_k^-$ coincide with $\psi_{y,e}$ on $\Omega_{y,e,>}$ for all $k \ge N_{y,e}$. Thus $\widetilde{g}_k^+ - \widetilde{g}_k^- = 0$ on $\Omega_{y,e,>}$ for every $k \ge N_{y,e}$. We do this for every $y \in \partial Y$ and for every edge e in $S(\mathcal{X}')$ emanating from y and pointing outwards of Y.

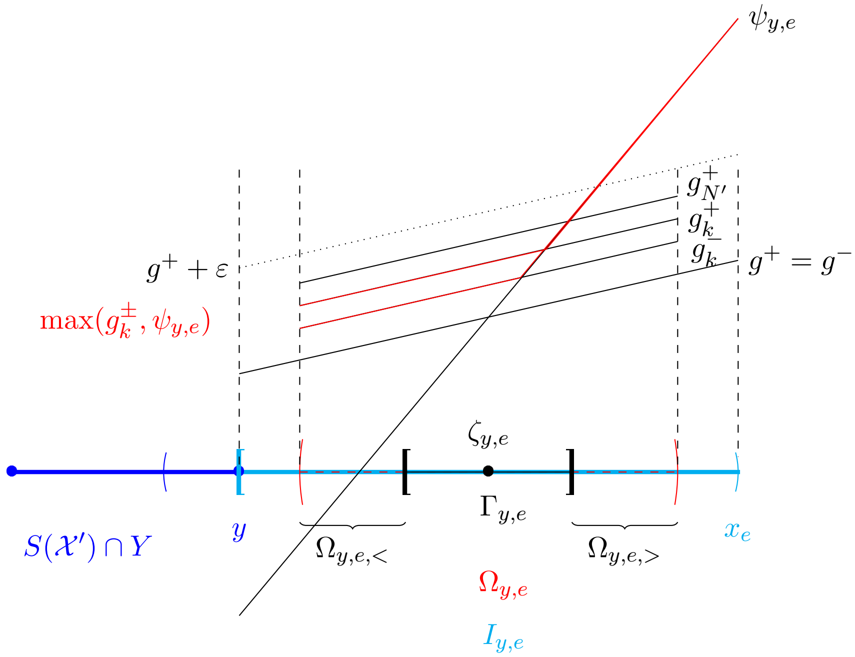

FIGURE 4. Representing the strategy of the modification of $g_k\pm$ outside of Y.

Recall that we work on the open subset $V = \bigcup_{z \in Y_0} \tau_{\mathcal{X}'}^{-1}(\Omega_z)$ of W containing Y, where Ω_z is a star-shaped open neighborhood of z in $S(\mathcal{X}')$ and $I_{z,e} = \Omega_z \cap e$ for every edge e in $S(\mathcal{X}')$ emanating from z. We have $\partial Y \subset Y_0$ and the sets $\Omega_y \backslash Y$ with $y \in \partial Y$ are disjoint of all other star-shaped open neighborhoods Ω_z. We write S_y for the set of edges e in $S(\mathcal{X}')$ emanating from y and pointing outwards of Y. Then for every $y \in \partial Y$ and $e \in S_y$, we have constructed the open subset $\Omega_{y,e}$ of $I_{y,e} = [y, x_e) \subset \Omega_y$.

Let V_Y be the connected component of $V \backslash (\bigcup_{y \in \partial Y, e \in S_y} \Gamma_{y,e})$ containing Y and let V_e be the connected component of $V \backslash \Gamma_{y,e}$ containing x_e. Note that by the construction of V all connected components V_e are pairwise disjoint. We can extend $\widetilde{g}_k^\pm$ to a continuous function on V by

$$\widetilde{g}_k^{\pm} := \begin{cases} m_\varepsilon(g_k^{\pm}, \psi_{y,e} \circ \tau_{\mathcal{X}'}) & \text{on } \tau_{\mathcal{X}'}^{-1}(\Omega_{y,e}), \\ g_k^{\pm} & \text{on } V_Y, \\ \psi_{y,e} \circ \tau_{\mathcal{X}'} & \text{on } V_e. \end{cases}$$

Then the functions $\widetilde{g}_k^+ - \widetilde{g}_k^-$ are non-negative and continuous on V with compact support for every $k \geq \max_{y,e} N_{y,e}$. Recall that by property iv), m_ε is increasing in every variable. From now on we only consider $k \in \mathbb{N}$ with $k \geq \max_{y \in \partial Y, e \in S_y} N_{y,e}$.

4. Step: Show that the modified functions $\widetilde{g}_k^+$ (resp. $\widetilde{g}_k^-$) converge uniformly to a function $\widetilde{g}^+$ (resp. $\widetilde{g}^-$) such that $g = \widetilde{g}^+ - \widetilde{g}^-$ on V.

Define the following functions on V

$$\widetilde{g}^{\pm} := \begin{cases} m_\varepsilon(g^{\pm}, \psi_{y,e} \circ \tau_{\mathcal{X}'}) & \text{on } \tau_{\mathcal{X}'}^{-1}(\Omega_{y,e}), \\ g^{\pm} & \text{on } V_Y, \\ \psi_{y,e} \circ \tau_{\mathcal{X}'} & \text{on } V_e. \end{cases}$$

By construction, these functions are well-defined and continuous. Since $g = g^+ - g^-$ on V with $g = g^+ - g^- = 0$ on $V \backslash Y$, we clearly have $g = \widetilde{g}^+ - \widetilde{g}^-$ on V.

Next, we show that $\widetilde{g}_k^+$ (resp. $\widetilde{g}_k^-$) converge uniformly to $\widetilde{g}^+$ (resp. to $\widetilde{g}^-$) on V. We know that g_k^+ (resp. g_k^-) converge uniformly to g^+ (resp. to g^-) on V, thus $\widetilde{g}_k^+$ (resp. $\widetilde{g}_k^-$) converge uniformly to g^+ (resp. to g^-) on V_Y. Since $\widetilde{g}_k^{\pm} = \psi_{y,e} \circ \tau_{\mathcal{X}'} = \widetilde{g}^{\pm}$ on V_e, it remains to consider the open subset $\tau_{\mathcal{X}'}^{-1}(\Omega_{y,e})$. For every $x \in \tau_{\mathcal{X}'}^{-1}(\Omega_{y,e})$, we have

$$\begin{aligned} |\widetilde{g}_k^{\pm}(x) - \widetilde{g}^{\pm}(x)| &= \left| m_\varepsilon(g_k^{\pm}(x), \psi_{y,e}(\tau_{\mathcal{X}'}(x)) - m_\varepsilon(g^{\pm}(x), \psi_{y,e}(\tau_{\mathcal{X}'}(x))) \right| \\ &= \left| \frac{g_k^{\pm}(x) - g^{\pm}(x) + \theta_\varepsilon(g_k^{\pm}(x) - \psi_{y,e}(\tau_{\mathcal{X}'}(x))) - \theta_\varepsilon(g^{\pm}(x) - \psi_{y,e}(\tau_{\mathcal{X}'}(x)))}{2} \right| \\ &\leq \left| \frac{g_k^{\pm}(x) - g^{\pm}(x)}{2} \right| + \left| \frac{g_k^{\pm}(x) - g^{\pm}(x)}{2} \right| \\ &\leq |g_k^{\pm}(x) - g^{\pm}(x)| \end{aligned}$$

where we used that θ_ε is 1-Lipschitz continuous to get the last inequality. Due to the uniform convergence of $g_k^{\pm}$ to $g^{\pm}$ on V, which contains $\tau_{\mathcal{X}'}^{-1}(\Omega_{y,e})$, we get

$$\lim_{k \to \infty} \sup_{x \in \tau_{\mathcal{X}'}^{-1}(\Omega_{y,e})} |\widetilde{g}_k^{\pm}(x) - \widetilde{g}^{\pm}(x)| = 0.$$

5. Step: Show that the modified functions $\widetilde{g}_k^+$ and $\widetilde{g}_k^-$ are smooth and psh on V.

Note that both properties are local. We already know that $g_k^{\pm} \in \mathcal{C}^\infty(V_Y)$, so it remains to find for every point x in $\tau_{\mathcal{X}'}^{-1}(\Omega_{y,e}) \cup V_e$ an open neighborhood V_x in V such that $\widetilde{g}_k^{\pm}$ is smooth and psh on V_x.

We start with a point $x \in V_e$. Choose an open neighborhood V_x of x in the open subset $V_e \subset V$. Then $\widetilde{g}_k^{\pm}$ is given by $\psi_{y,e} \circ \tau_{\mathcal{X}'}$ on V_x. For every $\zeta \in V_x \subset A = \tau_{\mathcal{X}'}^{-1}([y, x_e])$, we have

$$\widetilde{g}_k^{\pm}(\zeta) = (\psi_{y,e} \circ \tau_{\mathcal{X}'})(\zeta) = (\psi_{y,e} \circ \tau_A)(\zeta) = \Phi_{y,e}^*(m \cdot \log |T| + c)(\zeta)$$

on V_x by [**BPR13**, Lemma 2.13 & 3.8]. We first show

$$m \cdot \log|T| + c \in \ker(d'd''\colon \mathcal{C}^\infty(\mathbb{G}_m^{1,\mathrm{an}}) \to \mathcal{A}^{1,1}(\mathbb{G}_m^{1,\mathrm{an}})).$$

Consider the tropical chart $(V, \varphi_U) = (\mathbb{G}_m^{1,\mathrm{an}}, \mathrm{id})$ of $\mathbb{G}_m^{1,\mathrm{an}}$. Then $\mathrm{trop}_U = \log|T|$, and so $m \cdot \log|T| + c$ can be written as the triple $(\mathbb{G}_m^{1,\mathrm{an}}, \mathrm{id}, \lambda)$, where $\lambda\colon \mathbb{R} \to \mathbb{R}$ is the affine function $t \mapsto mt{+}c$. Thus $m{\cdot}\log|T|{+}c$ is a smooth function on $\mathbb{G}_m^{1,\mathrm{an}}$ (cf. Definition 4.1.7). The $(1,1)$-form $d'd''(m \cdot \log|T| + c)$ is given by the triple $(\mathbb{G}_m^{1,\mathrm{an}}, \mathrm{id}, d'd''\lambda)$. Since λ is affine, the form $d'd''\lambda$ is zero, and so is $d'd''(m \cdot \log|T| + c)$. Consequently, $m \cdot \log|T| + c$ lies in $\ker(d'd''\colon \mathcal{C}^\infty(\mathbb{G}_m^{1,\mathrm{an}}) \to \mathcal{A}^{1,1}(\mathbb{G}_m^{1,\mathrm{an}}))$. This implies that $\widetilde{g}_k^{\pm}|_{V_x} = \Phi_{y,e}^*(m{\cdot}\log|T|{+}c)$ is a smooth psh function on V_x (see Remark 5.1.10).

Now, consider $x \in \tau_{\mathcal{X}'}^{-1}(\Omega_{y,e})$. We have just seen that $\psi_{y,e} \circ \tau_{\mathcal{X}'}$ is a smooth psh function on $\tau_{\mathcal{X}'}^{-1}((y, x_e))$. Using Proposition 5.1.8, there is a chart (V_x, φ_{U_x}) with $x \in V_x \subset \tau_{\mathcal{X}'}^{-1}((y, x_e))$ such that $\psi_{y,e} \circ \tau_{\mathcal{X}'} = \phi \circ \mathrm{trop}_{U_x}$ on V_x for a smooth function ϕ on $\mathbb{R}^r$ that is convex restricted to every polyhedron contained in $\mathrm{trop}_{U_x}(V_x)$. On the other hand, we know that the function $g_k^{\pm}$ is smooth and psh on $\tau_{\mathcal{X}'}^{-1}((y, x_e))$ as well. Hence there is also a chart $(V'_x, \varphi_{U'_x})$ with $x \in V'_x \subset \tau_{\mathcal{X}'}^{-1}((y, x_e))$ such that $g_k^{\pm} = \phi' \circ \mathrm{trop}_{U'_x}$ on V'_x for a smooth function ϕ' on $\mathbb{R}^{r'}$ that is convex restricted to every polyhedron contained in $\mathrm{trop}_{U'_x}(V'_x)$. Working on the intersection $(V_x \cap V'_x, \varphi_{U_x} \times \varphi_{U'_x})$, which is a subchart of both [**Gub16**, Proposition 4.16], we get

$$\psi_{y,e} \circ \tau_{\mathcal{X}'} = (\phi \circ \mathrm{Trop}(\pi)) \circ \mathrm{trop}_{U_x \cap U'_x}$$
$$g_k^{\pm} = (\phi' \circ \mathrm{Trop}(\pi')) \circ \mathrm{trop}_{U_x \cap U'_x}$$

for the corresponding transition functions π, π' satisfying $\varphi_{U_x} = \pi \circ (\varphi_{U_x} \times \varphi_{U'_x})$ and $\varphi_{U'_x} = \pi' \circ (\varphi_{U_x} \times \varphi_{U'_x})$. Since $\mathrm{Trop}(\pi)$ and $\mathrm{Trop}(\pi')$ are integral affine functions on $\mathbb{R}^{r+r'}$, the composition $\phi \circ \mathrm{Trop}(\pi)$ (resp. $\phi' \circ \mathrm{Trop}(\pi')$) is still a smooth function on $\mathbb{R}^{r+r'}$ with a convex restriction to every polyhedron. Thus $m_\varepsilon(\phi \circ \mathrm{Trop}(\pi), \phi' \circ \mathrm{Trop}(\pi'))$ is a smooth function on $\mathbb{R}^{r+r'}$, and the properties i) and iv) of m_ε imply that the restriction to every polyhedron is convex since the restriction of $\phi \circ \mathrm{Trop}(\pi)$ and $\phi' \circ \mathrm{Trop}(\pi')$ are. We have

$$\widetilde{g}_k^{\pm} = m_\varepsilon(g_k^{\pm}, \psi_{y,e} \circ \tau_{\mathcal{X}'}) = m_\varepsilon(\phi \circ \mathrm{Trop}(\pi), \phi' \circ \mathrm{Trop}(\pi')) \circ \mathrm{trop}_{U_x \cap U'_x}$$

on $V_x \cap V'_x$ for every $x \in \tau_{\mathcal{X}'}^{-1}(\Omega_{y,e})$, so $\widetilde{g}_k^{\pm}$ is a smooth psh function on $\tau_{\mathcal{X}'}^{-1}(\Omega_{y,e})$ by Proposition 5.1.8, which proves Step 5.

Altogether, $\widetilde{g}_k^{\pm}$ are smooth psh function on V. Setting

$$\widetilde{g}_k := \widetilde{g}_k^{+} - \widetilde{g}_k^{-},$$

the functions $\widetilde{g}_k$ satisfy by construction all the required properties in Step 0.

6. Step: Construction of V' and η.

By the construction of V and $\widetilde{g}_k$, we can construct easily the required set V', i.e. an open subset V' of V containing Y such that $\overline{V'} \subset V$ and $\widetilde{g}_k \in \mathcal{C}_c^\infty(V')$. Furthermore, let V'' be an open neighborhood of $\overline{V'}$ in V with $\overline{V''} \subset V$. The topological space X^{an} is a compact Hausdorff space. Urysohn's Lemma states the existence of a continuous function $\eta\colon X^{\mathrm{an}} \to [0,1]$ with $\eta \equiv 1$ on $\overline{V'}$ and $\eta \equiv 0$ on $X^{\mathrm{an}} \backslash V''$. Thus η has compact support in V, i.e. it is the required function in Step 0.

Thus we have constructed everything as it was described in Step 0 proving the theorem. $\square$

COROLLARY 5.2.12. *A continuous function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} is subharmonic if and only it is psh.*

PROOF. Follows directly from Theorem 5.2.9 and Theorem 5.2.11. □

5.3. Stability under pullback and a regularization theorem

Due to the equivalence in Corollary 5.2.12, we know that a continuous psh function has all the nice properties that were shown for subharmonic functions in [**Thu05**]. More precisely, we now know that the property psh for continuous functions is stable under pullback with respect to morphisms of curves. Furthermore, we show that there is a monotone regularization theorem in the setting of Chambert-Loir and Ducros under certain conditions, e.g. if X is $\mathbb{P}^1$ or a Mumford curve.

COROLLARY 5.3.1. *Let X, X' be smooth proper algebraic curves over K and let $\varphi\colon W' \to W$ be a morphism of K-analytic spaces for open subsets $W \subset X^{\mathrm{an}}$ and $W' \subset (X')^{\mathrm{an}}$. If a continuous function $f\colon W \to \mathbb{R}$ is psh on W, then $\varphi^* f$ is psh on $\varphi^{-1}(W)$.*

PROOF. By Corollary 5.2.12, the function f is subharmonic on W, and so $\varphi^* f$ is subharmonic on $\varphi^{-1}(W)$ by Proposition 3.1.13. Using again Corollary 5.2.12, the pullback $\varphi^* f$ is psh on $\varphi^{-1}(W)$. □

To obtain a monotone regularization theorem in the setting of Chambert-Loir and Ducros for certain X, e.g. for the projective line or a Mumford curve, we use the monotone regularization theorem in Thuillier's setting. Hence we first need to show that every point has an open neighborhood such that every lisse subharmonic, and so psh, function is globally psh-approximable on it, i.e. it is the uniform limit of smooth psh functions. Recall the definition of the sheaf $\mathcal{C}^\infty$ of smooth functions on X^{an} from Definition 4.1.10. The key tool of this step is to use that for certain X, the sheaf of harmonic functions $\mathcal{H}_X$ is a subsheaf of $\mathcal{C}^\infty$ (see Remark 4.2.58). Note that in general, every continuous function f with $d'd''[f] = 0$ is harmonic (Corollary 5.2.6).

PROPOSITION 5.3.2. *Let X be a smooth proper algebraic curve such that the sheaf $\mathcal{H}_X$ of harmonic functions on X^{an} is a subsheaf of the sheaf $\mathcal{C}^\infty$ of smooth functions on X^{an}. Then every lisse function $f\colon Y \to \mathbb{R}$ on a strictly affinoid domain Y of X^{an} which is subharmonic on the relative interior $W := Y \backslash \partial Y$ of Y is globally psh-approximable on W. More precisely, there is a monotone decreasing sequence of continuous functions f_k on Y that are smooth and psh on W, and converge uniformly to f on Y.*

PROOF. Since f is lisse, we can find a strictly semistable model $\mathcal{Y}$ such that $f = F \circ \tau_{\mathcal{Y}}$ on Y for a piecewise affine function F on the skeleton $S(\mathcal{Y})$. We construct continuous functions on Y converging uniformly to f that are smooth and psh on W using techniques as in the proof of Theorem 5.2.11.

Let S be the set of points in $S(\mathcal{Y}) \backslash \partial Y$ that are contained in the support of the discretely supported measure $dd^c F$. Consider in the following a point x in S. Then $dd^c F > 0$ in an open neighborhood of x because f is subharmonic. The considered point x is either of type II or III. If x is of type II, we may assume x to be a vertex of $S(\mathcal{Y})$ by Proposition 2.3.22. We denote by $e_{x,1}, \dots, e_{x,n}$ the adjacent edges in $S(\mathcal{Y})$ and by x_i the second endpoint of $e_{x,i}$. If x is of type III, the point x is contained in the interior of an edge e_x with endpoints x_1 and x_2 and we denote by $e_{x,1}$ and $e_{x,2}$ the segments $[x_1, x]$ and $[x, x_2]$ of e_x. By blowing up $\mathcal{Y}$, we may assume that no x_i belongs to S and that F restricted to every $e_{x,i}$ is affine.

In both situations, we can find a piecewise affine function G_x on the metric subgraph $\Gamma_x := \bigcup_{i=1,\dots,n} e_{x,i}$ of $S(\mathcal{Y})$ such that

(a) $G_x(x) = F(x)$,
(b) $G_x < F$ on $\Gamma_x \backslash \{x\}$,
(c) $dd^c G_x = 0$ in a neighborhood of x, and
(d) $(G_x)|_{e_{x,i}}$ is affine for every $i = 1, \dots, n$.

Choose $\varepsilon_{x,i} > 0$ with $F(x_i) - G_x(x_i) > 2\varepsilon_{x,i}$. Then there exists a point $y_i \in (x, x_i)$ such that $F(y_i) = G_x(y_i) + \varepsilon_{x,i}$, $F < G_x + \varepsilon_{x,i}$ on $[x, y_i)$ and $F > G_x + \varepsilon_{x,i}$ on $(y_i, x_i]$. Since S has only finitely many points and corresponding adjacent edges, we can set $\varepsilon_0 := \min_{x\in S,i} \varepsilon_{x,i}$. Then for every $x \in S$, every $e_{x,i} = [x, x_i]$, and every $0 < \varepsilon \leq \varepsilon_0$ the inequalities

$$G_x(x) + \varepsilon - F(x) > \varepsilon/2, \qquad F(x_i) - (G_x(x_i) + \varepsilon) > \varepsilon/2 \tag{5.3.1}$$

hold.

For every $x \in S$, the set $V_x := \tau_{\mathcal{Y}}^{-1}(\Gamma_x^\circ)$ is an open neighborhood of x in W. By construction, these sets are pairwise disjoint. We define for every $0 < \varepsilon \leq \varepsilon_0$ the following function on Y

$$f_\varepsilon := \begin{cases} m_{\frac{\varepsilon}{2}}(G_x \circ \tau_{\mathcal{Y}} + \varepsilon, f) & \text{on } V_x \text{ for } x \in S, \\ f & \text{on } Y \backslash \bigcup_{x\in S} V_x, \end{cases} \tag{5.3.2}$$

where m_ε is the smooth maximum defined in (5.2.6) (see proof of Theorem 5.2.11). By (5.3.1) and property iii) of $m_{\varepsilon/2}$ in (5.2.6), the function f_ε coincides with $G_x \circ \tau_{\mathcal{Y}} + \varepsilon$ in an open neighborhood of x and with f in an open neighborhood of x_i. Here, x_i is the other vertex for an adjacent $e_{x,i} = [x, x_i]$. Thus f_ε is continuous on Y.

We will later use functions of this form to construct our desired sequence, but first we show that f_ε is smooth and psh on W. By construction, there is an open neighborhood W' of $W \backslash \bigcup_{x\in S} V_x$ such that f_ε coincides with f and $f = F \circ \tau_{\mathcal{Y}}$ is harmonic on W'. Since we required that $\mathcal{H}_X$ is a subsheaf of $\mathcal{C}^\infty$ and every harmonic function is psh by Proposition 3.1.18 and Theorem 5.2.11, the function f_ε is a smooth psh function on W'.

On the other hand, for every $x \in S$ the constructed function f_ε coincides with the harmonic function $G_x \circ \tau_{\mathcal{Y}} + \varepsilon$ on an open neighborhood of x, and so it is locally smooth and psh at x as well. It remains to consider f_ε on $\tau_{\mathcal{Y}}^{-1}((x, x_i))$ for every $x \in S$ and every adjacent $e_{x,i} = [x, x_i]$. Since f and $G_x \circ \tau_{\mathcal{Y}} + \varepsilon$ are harmonic, and so smooth and psh on $\tau_{\mathcal{Y}}^{-1}((x, x_i))$, one can show as in Step 5 in the proof of Theorem 5.2.11 that $f_\varepsilon = m_{\frac{\varepsilon}{2}}(G_x \circ \tau_{\mathcal{Y}} + \varepsilon, f)$ is still smooth and psh on $\tau_{\mathcal{Y}}^{-1}((x, x_i))$. Altogether, f_ε is a smooth psh function on W.

With the help of the function f_ε defined in (5.3.2), we construct now a monotonically decreasing sequence $(f_k)_{k\in\mathbb{N}}$ of smooth psh functions converging uniformly to f on Y. For every $k \in \mathbb{N}$, we define $\varepsilon_k > 0$ recursively starting with ε_0 from above, and set $f_k := f_{\varepsilon_k}$. To do so, we need to consider the subsets

$$\Omega_k := \bigcup_{x\in S} \left\{ y \in V_x \mid |G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_k - f(y)| < \frac{\varepsilon_k}{2} \right\}$$

on which f_k does not necessarily coincide with $\max(G_x \circ \tau_{\mathcal{Y}} + \varepsilon_k, f)$ for some $x \in S$. For a given ε_k, we choose ε_{k+1} such that $0 < \varepsilon_{k+1} < \varepsilon_k/3$. Then $\Omega_k \cap \Omega_{k+1} = \emptyset$ for every $k \in \mathbb{N}$ and $\varepsilon_k \to 0$ for $k \to \infty$.

We show that the sequence $(f_k)_k$ converges pointwise to f and $f_{k+1} \leq f_k$ on Y. If $y \in Y \backslash \bigcup_{x\in S} V_x$, then all f_k coincide with f, and so both assertions are trivial. We therefore assume that $y \in V_x$ for some $x \in S$. In the case of $y \in \tau_{\mathcal{Y}}^{-1}(x)$, we have

$$f_k(y) = G_x(x) + \varepsilon_k = F(x) + \varepsilon_k = f(y) + \varepsilon_k,$$

and so $f_k(y) \geq f_{k+1}(y)$ and $f_k(y)$ converges to $f(y)$ for $k \to \infty$. If $y \in V_x \backslash \{\tau_{\mathcal{Y}}^{-1}(x)\}$, we can find ε_N small enough such that $f(y) - (G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_N) > \varepsilon_N/2$. Hence for every $k \geq N$ we have $f(y) - (G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_k) > \varepsilon_k/2$, and so

$$f_k(y) = \max(G_x \circ \tau_{\mathcal{Y}} + \varepsilon_k, f) = f(y).$$

Thus $f_k(y)$ converges to $f(y)$. Next, we consider an arbitrary $k \in \mathbb{N}$ and we show $f_k(y) \geq f_{k+1}(y)$. If $y \notin \Omega_k \cup \Omega_{k+1}$, then

$$f_{k+1}(y) = \max(G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_{k+1}, f(y)) \leq \max(G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_k, f(y)) = f_k(y)$$

since $\varepsilon_k > \varepsilon_{k+1}$. If $y \in \Omega_k$, by the choice of ε_{k+1}, we have $y \notin \Omega_{k+1}$. Thus

$$f_{k+1}(y) = \max(G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_{k+1}, f(y)) \leq \max(G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_k, f(y)) \leq f_k(y),$$

where the last inequality is true by property (b) following (5.2.6). Finally, let $y \in \Omega_{k+1}$, and so $y \notin \Omega_k$. Then $G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_k \geq f(y)$ as $\varepsilon_{k+1} < \varepsilon_k/3$, and so

$$f_k(y) = \max(G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_k, f(y)) = G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_k \geq f(y) + \varepsilon_k/2$$

as $y \notin \Omega_k$. By property (b) following (5.2.6), $\varepsilon_{k+1} < \varepsilon_k/3$ and the last inequality, we get

$$\begin{aligned} f_{k+1}(y) &\leq \max(G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_{k+1}, f(y)) + \varepsilon_{k+1}/4 \\ &\leq \max(G_x(\tau_{\mathcal{Y}}(y)) + \varepsilon_k, f(y) + \varepsilon_k/2) \\ &\leq f_k(y). \end{aligned}$$

Thus the sequence $(f_k)_k$ of continuous functions converges pointwise to the continuous function f on Y and $f_{k+1} \leq f_k$. Since Y is compact, the sequence $(f_k)_k$ converges uniformly by Dini's theorem. We have already seen above that each f_k is smooth and psh on W. $\square$

Before we use this proposition to prove a monotone regularization theorem in the setting of Chambert-Loir and Ducros for Mumford curves, we show that for these curves the sheaf of harmonic functions $\mathcal{H}_X$ is a subsheaf of $\mathcal{C}^\infty$. Recall the definition of Mumford curves from Definition 4.2.27.

LEMMA 5.3.3. *Let X be a smooth proper curve over K. If $\widetilde{K}$ is algebraic over a finite field or X is the projective line or a Mumford curve, then $\mathcal{H}_X$ is a subsheaf of $\mathcal{C}^\infty$.*

PROOF. At first, note that if $\widetilde{K}$ is algebraic over a finite field or X^{an} is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$ (see Definition 4.2.25), the sheaf $\mathcal{H}_X$ is a subsheaf of $\mathcal{C}^\infty$ on X^{an} as explained in Remark 4.2.58. By Proposition 4.2.29, X^{an} is locally isomorphic to $\mathbb{P}^{1,\mathrm{an}}$ if X is the projective line or a Mumford curve. Hence the assertion follows. $\square$

COROLLARY 5.3.4. *Let X be a smooth proper algebraic curve over K. If $\widetilde{K}$ is algebraic over a finite field or X is the projective line or a Mumford curve, then every continuous psh function $f\colon W \to \mathbb{R}$ on an open subset W of X^{an} is locally psh-approximable. More precisely, the sequence of smooth psh functions can be chosen monotonically decreasing.*

PROOF. In the given situation, we may apply Proposition 5.3.2 by Lemma 5.3.3.

To prove the corollary, we have to show that every point x in W has an open neighborhood in W such that f is a uniform limit of smooth psh functions. The continuous psh function f is subharmonic in the sense of Thuillier by Theorem 5.2.11, and we therefore can use Thuillier's monotone regularization theorem (see Proposition 3.1.19). We can find for every $x \in W$ a relatively compact neighborhood W' of x in W and a decreasing net $\langle f_\alpha \rangle$ of lisse subharmonic functions on W' converging pointwise to f. Let Y be a strictly affinoid domain in W' having x in its interior Y°. Then the decreasing net $\langle f_\alpha \rangle$ converges uniformly to the continuous function f on the compact set Y by Dini's theorem. Thus one can construct inductively a decreasing sequence of lisse subharmonic functions on W' converging uniformly to f on Y and we write $(f_k)_{k\in\mathbb{N}}$ for this sequence.

We have seen in Proposition 5.3.2 that each f_k is the uniform limit of a decreasing sequence of smooth psh functions on Y°. Hence we can choose a decreasing sequence of smooth psh functions on Y° converging uniformly to f. □

Remark 5.3.5. Note that there are curves such that the sheaf $\mathcal{H}_X$ of harmonic functions is not a subsheaf of the sheaf $\mathcal{C}^\infty$ of smooth functions on X^{an}. A counter example of such a curve can be constructed by the proof of [**Thu05**, Théorème 2.3.21] (see for example [**Wan16**, Corollary 5.3.23]). For those curves we do not know whether every psh function is locally psh-approachable.

Bibliography

[ABBR15] Omid Amini, Matthew Baker, Erwan Brugallé, and Joseph Rabinoff. Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta. *Res. Math. Sci.*, 2:Art. 7, 67, 2015.

[Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[Ber07] Vladimir G. Berkovich. *Integration of one-forms on p-adic analytic spaces*, volume 162 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2007.

[BFJ15] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Solution to a non-Archimedean Monge-Ampère equation. *J. Amer. Math. Soc.*, 28(3):617–667, 2015.

[BFJ16] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Singular semipositive metrics in non-Archimedean geometry. *J. Algebraic Geom.*, 25(1):77–139, 2016.

[BG84] Robert Bieri and J. R. J. Groves. The geometry of the set of characters induced by valuations. *J. Reine Angew. Math.*, 347:168–195, 1984.

[BIMS15] Erwan Brugallé, Ilia Itenberg, Grigory Mikhalkin, and Kristin Shaw. Brief introduction to tropical geometry. In *Gökova Geometry-Topology conference*, 2015.

[BL85] Siegfried Bosch and Werner Lütkebohmert. Stable reduction and uniformization of abelian varieties. I. *Math. Ann.*, 270(3):349–379, 1985.

[Bos14] Siegfried Bosch. *Lectures on formal and rigid geometry*, volume 2105 of *Lecture Notes in Mathematics*. Springer, Cham, 2014.

[BP05] Matthew Baker and Clayton Petsche. Global discrepancy and small points on elliptic curves. *Int. Math. Res. Not.*, (61):3791–3834, 2005.

[BPR13] Matthew Baker, Sam Payne, and Joseph Rabinoff. On the structure of non-Archimedean analytic curves. In *Tropical and non-Archimedean geometry*, volume 605 of *Contemp. Math.*, pages 93–121. Amer. Math. Soc., Providence, RI, 2013.

[BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *Algebr. Geom.*, 3(1):63–105, 2016.

[BR06] Matthew H. Baker and Robert Rumely. Equidistribution of small points, rational dynamics, and potential theory. *Ann. Inst. Fourier (Grenoble)*, 56(3):625–688, 2006.

[BR10] Matthew Baker and Robert Rumely. *Potential theory and dynamics on the Berkovich projective line*, volume 159 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[CD12] Antoine Chambert–Loir and Antoine Ducros. Formes différentielles réelles et courants sur les espaces de Berkovich. 2012. `http://arxiv.org/abs/1204.6277`.

[Fol99] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[GH17] Walter Gubler and Julius Hertel. Local heights of toric varieties over non-archimedean fields. In *Actes de la Conférence "Non-Archimedean Analytic Geometry: Theory and Practice"*, volume 2017/1 of *Publ. Math. Besançon Algèbre Théorie Nr.*, pages 5–77. Presses Univ. Franche-Comté, Besançon, 2017.

[GK15] Walter Gubler and Klaus Künnemann. Positivity properties of metrics and delta-forms. 2015. `https://arxiv.org/abs/1509.09079`, to appear in *J. Reine Angew. Math.*

[GK17] Walter Gubler and Klaus Künnemann. A tropical approach to nonarchimedean Arakelov geometry. *Algebra Number Theory*, 11(1):77–180, 2017.

[Gub13] Walter Gubler. A guide to tropicalizations. In *Algebraic and combinatorial aspects of tropical geometry*, volume 589 of *Contemp. Math.*, pages 125–189. Amer. Math. Soc., Providence, RI, 2013.

[Gub16] Walter Gubler. Forms and currents on the analytification of an algebraic variety (after Chambert-Loir and Ducros). In Matthew Baker and Sam Payne, editors, *Nonarchimedean and Tropical Geometry*, Simons Symposia, pages 1–30, Switzerland, 2016. Springer.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[Ive86] Birger Iversen. *Cohomology of sheaves*. Universitext. Springer-Verlag, Berlin, 1986.

[Jel16a] Philipp Jell. Differential forms on Berkovich analytic spaces and their cohomology. 2016. PhD Thesis, availible at `http://epub.uni-regensburg.de/34788/1/ThesisJell.pdf`.

[Jel16b] Philipp Jell. A Poincaré lemma for real-valued differential forms on Berkovich spaces. *Math. Z.*, 282(3-4):1149–1167, 2016.

[Jel17] Philipp Jell. Tropical Hodge numbers of non-archimedean curves. 2017. `https://arxiv.org/abs/1706.05895`, to appear in Israel J. Math.

[JSS19] Philipp Jell, Kristin Shaw, and Jascha Smacka. Superforms, tropical cohomology and Poincaré duality. *Advances in Geometry*, 19:101–130, 2019.

[JW18] Philipp Jell and Veronika Wanner. Poincaré duality for the tropical Dolbeault cohomology of non-archimedean Mumford curves. *J. Number Theory*, 187:344–371, 2018.

[KRZ16] Eric Katz, Joseph Rabinoff, and David Zureick-Brown. Uniform bounds for the number of rational points on curves of small Mordell-Weil rank. *Duke Math. J.*, 165(16):3189–3240, 2016.

[Lag12] Aron Lagerberg. Super currents and tropical geometry. *Math. Z.*, 270(3-4):1011–1050, 2012.

[Pet09] Clayton Petsche. Nonarchimedean equidistribution on elliptic curves with global applications. *Pacific J. Math.*, 242(2):345–375, 2009.

[Poi13] Jérôme Poineau. Les espaces de Berkovich sont angéliques. *Bull. Soc. Math. France*, 141(2):267–297, 2013.

[Roy88] H. L. Royden. *Real analysis*. Macmillan Publishing Company, New York, third edition, 1988.

[Sha13] Kristin Shaw. A tropical intersection product in matroidal fans. *SIAM J. Discrete Math.*, 27(1):459–491, 2013.

[Sha15] Kristin Shaw. Tropical surfaces. 2015. `http://arxiv.org/abs/1506.07407`.

[Sil94] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[Spe08] David E. Speyer. Tropical linear spaces. *SIAM J. Discrete Math.*, 22(4):1527–1558, 2008.

[Tem11] Michael Temkin. Introduction to berkovich analytic spaces. `https://arxiv.org/abs/1010.2235v2`, 2011.

[Thu05] Amaury Thuillier. Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov. 2005.

[Vaq00] Michel Vaquié. Valuations. In *Resolution of singularities (Obergurgl, 1997)*, volume 181 of *Progr. Math.*, pages 539–590. Birkhäuser, Basel, 2000.

[Wan16] Veronika Wanner. Harmonic functions on the Berkovich projective line. 2016. Master Thesis, availible at `https://epub.uni-regensburg.de/35758/1/MA.pdf`.

[Wan18] Veronika Wanner. Comparison of two notions of subharmonicity on non-archimedean curves. *Mathematische Zeitschrift*, 2018. https://doi.org/10.1007/s00209-018-2205-z.

[Wel80] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1980.

[Zha93] Shouwu Zhang. Admissible pairing on a curve. *Invent. Math.*, 112(1):171–193, 1993.