

Distribution of entanglement and correlations in all finite dimensions

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The physics of a many-particle system is determined by the correlations in its quantum state. Therefore, analyzing these correlations is the foremost task of many-body physics. Any ‘a priori’ constraint for the properties of the global vs. the local states—the so-called marginals—would help in order to narrow down the wealth of possible solutions for a given many-body problem, however, little is known about such constraints. We derive an equality for correlation-related quantities of any multipartite quantum system composed of finite-dimensional local parties. This relation defines a necessary condition for the compatibility of the marginal properties with those of the joint state. While the equality holds both for pure and mixed states, the pure-state version containing only entanglement measures represents a fully general monogamy relation for entanglement. These findings have interesting implications in terms of conservation laws for correlations, and also with respect to topology.

Correlations between different parts of many-particle quantum systems are manifestly different from their classical counterparts. The hallmark of the latter is that, if the global state is completely known, so are the local states, that is, the individual states of each particle. For quantum states this may be different: Even if the global state is known with certainty, there may be no information whatsoever regarding the local states such as, for example, in maximally entangled states of two d -dimensional systems [1, 2].

Often, different degrees of departure from the classical behavior are distinguished, namely entanglement, steerability, and nonlocality [3]. While all these correlation types are based on en-

tanglement, nonlocal correlations are regarded as the strongest. This is because all nonlocal states are entangled, but only those for which there are no local models may violate a Bell inequality [4, 5]. Already such basic considerations reveal that there must exist mathematical tools to characterize quantitative aspects of correlations in a multipartite quantum state. Entanglement measures [6, 7] constitute one example of such tools, but there are many others [8, 9] and to date it is not clear how to exhaustively characterize quantum correlations, not even for finite-dimensional systems.

Such a characterization can be achieved via mathematical constraints for different correlation functions and/or quantifiers. A Bell inequality is precisely one such constraint: It relates the existence of a local model to the maximum absolute value of a certain linear combination of correlation functions. Another example are so-called monogamy inequalities for entanglement [10, 11, 12, 13, 14] or nonlocality [16, 17] which establish constraints between the correlations that may coexist in different subsystems of a multipartite quantum system.

It is remarkable that there exist *exact* relations, that is, *equalities* between certain entanglement measures in qubit systems. Coffman et al. [10] discovered the now famous relation for pure states ψ_3 of three qubits (denoted by A , B , and C)

$$\tau_3(\psi_3) = \tau_A - C_{AB}^2 - C_{AC}^2 \quad , \quad (1)$$

where τ_A denotes the linear entropy of the reduced state $\rho_A = \text{Tr}_{BC} |\psi_3\rangle\langle\psi_3|$, C_{AB} is the concurrence of the two-qubit reduced state $\rho_{AB} = \text{Tr}_C |\psi_3\rangle\langle\psi_3|$ (analogously for C_{AC}), and $\tau_3(\psi_3)$ is the absolute value of a polynomial local SL (LSL) invariant that depends on the state coefficients of ψ_3 . It turned out only recently [18] that similar relations exist for any N -qubit system, the

simplest being

$$2|H(\psi_N)|^2 = \sum_{k=1}^N (-1)^{k+1} \sum_{j_1 < \dots < j_k} \tau_{A_{j_1} \dots A_{j_k}}, \quad (2)$$

where now $\tau_{A_{j_1} \dots A_{j_k}}$ is the linear entropy of the reduced state of qubits A_{j_1}, \dots, A_{j_k} and $H(\psi_N) = \langle \psi_N^* | \sigma_2^{\otimes N} | \psi_N \rangle$ is another polynomial LSL invariant [20, 7] (here $|\psi_N^*\rangle$ is the complex conjugate of $|\psi_N\rangle$ and σ_2 is the second Pauli matrix). Strikingly, all of the quantities in Eqs. (1), (2) are measures for entanglement, either of the individual party (the linear entropies and the concurrences) or in the global state (the polynomial LSL invariants). These relations impose rigid constraints between the properties of the global and the reduced states, and are therefore intimately connected with the *quantum marginal problem* [19]. However, since their validity appears to be restricted to qubit systems, it is not clear whether they bear any relevance to a general physical setting.

This is the motivation for the present work: We show that the analogue of Eq. (2) holds for *any* multipartite system of finite-dimensional constituents. In passing, by this relation a local unitary invariant of the pure state is introduced, which we term *distributed concurrence*, because it describes the distribution of bipartite entanglement in the pure state and is a natural generalization both of the bipartite concurrence and of $|H(\psi_N)|$ in Eq. (2).

1 Universal state inversion

1.1 Definition

The key ingredient in our investigation is the operator obtained by applying the so-called *universal state inversion* [21, 22], see also Refs. [23, 24, 25, 26]. Consider a d -dimensional Hilbert space \mathcal{H} and $\mathcal{B}(\mathcal{H})$, the set of bounded positive semidefinite operators on \mathcal{H} . Horodecki et al. [21] defined the operator

$$S_d(\mathcal{O}) = \text{Tr}(\mathcal{O}) \mathbb{1} - \mathcal{O} \quad , \quad \mathcal{O} \in \mathcal{B}(\mathcal{H}) \quad (3)$$

which was used by Rungta et al. [22] to write the concurrence of a state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ as

$$C(\psi) = \sqrt{\langle \psi | S_{d_A} \otimes S_{d_B} (|\psi\rangle\langle\psi|) | \psi \rangle} \quad (4)$$

$$= \sqrt{2 - \text{Tr}_A \rho_A^2 - \text{Tr}_B \rho_B^2} \quad , \quad (5)$$

(where $d_j = \dim \mathcal{H}_j$, $j = A, B$; $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$, $\rho_B = \text{Tr}_A |\psi\rangle\langle\psi|$ are the reduced states for the parties). They termed the operator S_d the *universal state inverter*. We will show below that, for non-Hermitian operators \mathcal{O} , the appropriate definition of the single-system state inverter is $\tilde{\mathcal{O}} = \text{Tr}(\mathcal{O}^\dagger) \mathbb{1} - \mathcal{O}^\dagger$ (see Section 1.4).

It is straightforward to generalize universal state inversion toward density matrices ρ of N -partite systems, i.e., $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N)$, $X \in \{1, \dots, N\}$

$$\tilde{\rho} = \mathcal{S}_{d_1} \mathcal{S}_{d_2} \dots \mathcal{S}_{d_n}(\rho) \quad (6)$$

$$= \left[\prod_{X=1}^N (\text{Tr}_X(\cdot) \otimes \mathbb{1}_X - \mathbb{1}) \right] \rho \quad , \quad (7)$$

where $\mathbb{1}_X$ denotes the identity on a single subsystem X , as opposed to $\mathbb{1}$, the identity on the full system. We use the tilde notation for the inverted state $\tilde{\rho}$ in order to make reference to Wootters' notation in Ref. [27], Eq. (5), because that definition for two qubits is naturally generalized by Eq. (7), as we will see below. Note that if the dimension of any party is larger than 2, the trace of $\tilde{\rho}$ is larger than 1. Nonetheless we will call $\tilde{\rho}$ the 'tilde state' if no ambiguity is possible.

The operator $\tilde{\rho}$ appears quite frequently in algebraic considerations regarding properties of quantum states, in particular, entanglement. Yet it is not obvious to attribute to it a physical significance beyond the interpretation of Eq. (3) given in Ref. [22]: It takes a pure state to the maximally mixed state in the subspace orthogonal to the pure state, multiplied by the dimension of that subspace. The latter statement can be re-expressed in the following way: Geometrically, the single-system inverter maps a pure state of a d -dimensional system to the barycenter of the hypersurface opposite to it, that is, the convex hull of those pure states which, together with the original state, form an orthonormal basis. The result is then scaled with a prefactor $(d-1)$, cf. Fig. 1.

This geometrical interpretation can be extended to the multipartite case. To this end, consider a state of the computational basis $|j_1 \dots j_N\rangle \in \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$. Then we have

$$|j_1 \dots j_N\rangle \widetilde{\langle j_1 \dots j_N|} = \bigotimes_{k=1}^N (\mathbb{1}_k - |j_k\rangle\langle j_k|) \quad , \quad (8)$$

which corresponds to the barycenter of the hypersurface formed by all those states of the multipartite computational basis whose k -th tensor

factor is different from $|j_k\rangle$. This state is multiplied by a factor $\prod_{k=1}^N (d_k - 1)$ that keeps track of the total dimension of that hypersurface.

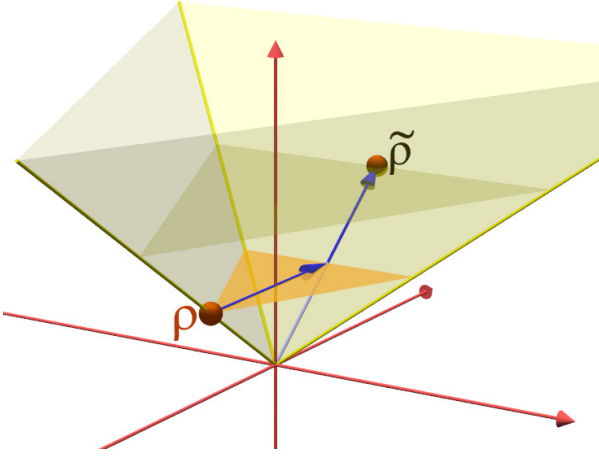


Figure 1: Illustration of the action of the state inverter on a pure state ρ in dimension $d = 3$. The orange triangle is a section through the qutrit Bloch body containing the pure state ρ at one corner (orange sphere). The other two corners of the triangle are pure states of which $\tilde{\rho}$ is a linear combination. The vertical axis denotes the multiples of the identity matrix determining the trace. All density matrices have trace 1, thus the Bloch body is located at a constant height. The axis crosses the Bloch body in the completely mixed state. The state inverter maps the pure state to the state lying furthest on the opposite site of the completely mixed state, as shown by the blue arrow crossing the vertical axis. This state is an equal mixture of the other two pure states in the triangle. At the same time, it scales the vector by the factor $(d - 1) = 2$, as indicated by the second blue arrow. This scaling maps the Bloch body to a copy at height $(d - 1)$, shown as semitransparent yellow triangle. The resulting operator $\tilde{\rho}$ is again marked by a small orange sphere.

1.2 Representation in terms of reduced states

There are several noteworthy ways to explicitly write the state $\tilde{\rho}$. The first and obvious possibility is to expand the operator product in Eq. (7) which leads to an alternating sum over all bipartitions A [23]

$$\tilde{\rho} = \sum_A (-1)^{|A|} (\text{Tr}_{\bar{A}} \rho) \otimes \mathbb{1}_{\bar{A}} . \quad (9)$$

Here A denotes a subset of $\{1, \dots, N\}$, $|A|$ is the number of elements in A , \bar{A} is the complement of the subset A , and $\rho_A = \text{Tr}_{\bar{A}} \rho$ is the reduced density matrix of the parties in the subset A .

1.3 Relation to the Bloch representation of the original state

A second important way to write $\tilde{\rho}$ is its Bloch representation. In order to obtain it we first consider the Bloch representation of ρ itself. An N -qubit state ρ can be written as

$$\rho = \frac{1}{d_1 \cdots d_N} \sum_{j_1 \dots j_N} r_{j_1 \dots j_N} \mathbf{g}_{j_1} \otimes \cdots \otimes \mathbf{g}_{j_N} \quad (10)$$

$$= \frac{1}{d_{\text{tot}}} (\mathbb{1} + P_1 + P_2 + \dots + P_N) . \quad (11)$$

Here, $\{\mathbf{g}_{j_k}\}$ are sets of $d_k^2 - 1$ traceless generators of $\text{SU}(d_k)$, $\mathbf{g}_{j_k} = \mathbb{1}_k$ for $j_k = 0$, $r_{j_1 \dots j_N} \in \mathbb{C}$. Moreover, $d_{\text{tot}} = d_1 \cdots d_N$ is the total dimension of the Hilbert space. We use the symbol P_q for the sum of all those terms in Eq. (10) that contain the same number q of nontrivial generators.

The Bloch representation of $\tilde{\rho}$ is particularly nice if all subsystem dimensions are equal, i.e., $d_1 = d_2 = \dots = d_N \equiv d$ and $d_{\text{tot}} = d^N$. In order to avoid cumbersome notation we restrict ourselves to this case. Then it is easy to see by directly applying the product form of the state inversion operator Eq. (7) to each term in the Bloch representation Eq. (11) separately that

$$\begin{aligned} \tilde{\rho} = \frac{1}{d^N} & \left[(d-1)^N \mathbb{1} - (d-1)^{N-1} P_1 + \right. \\ & + (d-1)^{N-2} P_2 - \dots + \\ & \left. + (-1)^{N-1} (d-1) P_{N-1} + (-1)^N P_N \right] . \end{aligned} \quad (12)$$

It is obvious from Eq. (12) that for $d > 2$ the universal state inverter is not a trace-preserving map. Further, it shows that only for $d = 2$ the tilde state of a pure state $\Pi_{N\text{qubit}}$ is pure as well, because in that case we have for the purity of the tilde state

$$\begin{aligned} (\text{Tr } \tilde{\Pi}_{N\text{qubit}})^2 &= \text{Tr } \tilde{\Pi}_{N\text{qubit}}^2 \\ &= \frac{1}{2^N} \sum_{j_1 \dots j_N} |r_{j_1 \dots j_N}|^2 \\ &= 1 . \end{aligned}$$

1.4 Explicit generator representation

Finally, we want to find a representation of $\tilde{\rho}$ in terms of generators of the unitary group. To

this end, we use the generalized Gell-Mann matrices [28] in d dimensions

$$\mathbf{x}_{kl}^{(d)} = \sqrt{\frac{d}{2}}(|k\rangle\langle l| + |l\rangle\langle k|) \quad (13a)$$

$$\mathbf{y}_{kl}^{(d)} = \sqrt{\frac{d}{2}}(-i|k\rangle\langle l| + i|l\rangle\langle k|) \quad (13b)$$

$$\mathbf{z}_l^{(d)} = \sqrt{\frac{d}{l(l+1)}} \left(-l|l\rangle\langle l| + \sum_{j=0}^{l-1} |j\rangle\langle j| \right) \quad (13c)$$

where $0 \leq k < l < d$. The normalization is chosen such that

$$\begin{aligned} \text{Tr}(\mathbf{x}_{kl}\mathbf{x}_{mn}) &= \text{Tr}(\mathbf{y}_{kl}\mathbf{y}_{mn}) = d \delta_{km}\delta_{ln} \quad , \\ \text{Tr}(\mathbf{z}_l\mathbf{z}_n) &= d \delta_{ln} \quad . \end{aligned} \quad (14)$$

Note that, up to a factor, these are the usual d -dimensional Gell-Mann matrices.

Just as above in Eq. (10) it is useful to include the identity matrix, as this gives a complete basis for the Hermitian matrices. Sometimes we will use a uniform labeling with only one index and therefore introduce also the alternative form

$$\mathbf{h}_0^{(d)} = \mathbb{1} \quad (15a)$$

$$\mathbf{h}_{l^2+2k}^{(d)} = \mathbf{x}_{kl} \quad (15b)$$

$$\mathbf{h}_{l^2+2k+1}^{(d)} = \mathbf{y}_{kl} \quad (15c)$$

$$\mathbf{h}_{l^2+2l}^{(d)} = \mathbf{z}_l \quad (15d)$$

where the index corresponds to the usual numbering of the Gell-Mann matrices. Relations (14) then simplify to

$$\text{Tr}(\mathbf{h}_k\mathbf{h}_l) = d \delta_{kl} \quad . \quad (16)$$

For the following considerations we need two formulae for the single-system density matrix $\rho \in \mathcal{B}(\mathcal{H})$ with $\dim \mathcal{H} = d$ (a proof is given in Appendix A)

$$(\text{Tr } \rho)\mathbb{1} = \frac{1}{d} \sum_{k=0}^{d^2-1} \mathbf{h}_k \rho \mathbf{h}_k \quad (17)$$

$$\rho^T = \frac{1}{d} \sum_{k=0}^{d^2-1} \mathbf{h}_k^T \rho \mathbf{h}_k \quad . \quad (18)$$

By applying these relations to Eq. (3) we find for the tilde state of a single system

$$\begin{aligned} \tilde{\rho} &= (\text{Tr } \rho)\mathbb{1} - \rho \\ &= \left[(\text{Tr } \rho)\mathbb{1} - \rho^T \right]^* \\ &= \frac{1}{d} \sum_{k=0}^{d^2-1} \left[\mathbf{h}_k \rho \mathbf{h}_k - \mathbf{h}_k^T \rho \mathbf{h}_k \right]^* \quad . \end{aligned} \quad (19)$$

To proceed from here, we note that the generators \mathbf{y}_{kl} change their sign on transposition, while all other generators remain unchanged under that operation. Therefore the sum simplifies to

$$\begin{aligned} \tilde{\rho} &= \frac{2}{d} \sum_{k=0}^{d-2} \sum_{l=k+1}^{d-1} (\mathbf{y}_{kl} \rho \mathbf{y}_{kl})^* \\ &= \frac{2}{d} \sum_{k=0}^{d-2} \sum_{l=k+1}^{d-1} \mathbf{y}_{kl} \rho^* \mathbf{y}_{kl} \quad . \end{aligned} \quad (20)$$

It is straightforward to extend the result (20) to states of multipartite systems $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N)$. For this purpose, we introduce the simplified notation

$$\begin{aligned} \mathbf{k} &= (k_1, \dots, k_N) \\ \mathbf{y}_{kl} &= \mathbf{y}_{k_1 l_1} \otimes \mathbf{y}_{k_2 l_2} \otimes \cdots \otimes \mathbf{y}_{k_N l_N} \\ \sum_{k < l} &\equiv \sum_{k_1=0}^{d_1-1} \sum_{l_1=k_1+1}^{d_1-1} \cdots \sum_{k_N=0}^{d_N-1} \sum_{l_N=k_N+1}^{d_N-1} \end{aligned}$$

and can then write

$$\tilde{\rho} = \frac{2^N}{d_{\text{tot}}} \sum_{k < l} \mathbf{y}_{kl} \rho^* \mathbf{y}_{kl} \quad . \quad (21)$$

This is a remarkable result, not only because of its simplicity. It makes explicit that universal state inversion is an antilinear operation. In fact, $\tilde{\rho}$ in Eq. (21) is in all respects the natural generalization of Wootters' tilde state. We note also that for pure states $\rho_\psi = |\psi\rangle\langle\psi| \equiv \Pi_\psi$ the tilde state $\tilde{\Pi}_\psi$ in general is mixed, only for $d_k = 2$ the generator sum contains a single term, and $\tilde{\Pi}_\psi$ becomes pure as well.

Finally, it is immediately clear from Eq. (21) that the tilde state $\tilde{\rho}$ is positive: With ρ , also ρ^* is positive and therefore $\rho^* = \sqrt{\rho^*} \cdot \sqrt{\rho^*}$, so that

$$\begin{aligned} \tilde{\rho} &= \frac{2^N}{d_{\text{tot}}} \sum_{k < l} \mathbf{y}_{kl} \sqrt{\rho^*} \sqrt{\rho^*} \mathbf{y}_{kl} \\ &= \frac{2^N}{d_{\text{tot}}} \sum_{k < l} \left(\sqrt{\rho^*} \mathbf{y}_{kl} \right)^\dagger \sqrt{\rho^*} \mathbf{y}_{kl} \quad . \end{aligned} \quad (22)$$

That is, $\tilde{\rho}$ is a sum of positive operators and therefore also positive. An important consequence of Eq. (22) is

$$\text{Tr}(\rho\tilde{\rho}) = \text{Tr} \left[\left(\sqrt{\tilde{\rho}} \sqrt{\rho} \right)^\dagger \left(\sqrt{\tilde{\rho}} \sqrt{\rho} \right) \right] = 0 \quad . \quad (23)$$

2 Distributed concurrence

2.1 New concurrence-type invariant

By using the notation $|\psi\rangle\langle\psi| \equiv \Pi_\psi$ we can express the concurrence Eq. (5) for the pure state of a bipartite system as $C(\psi) = \sqrt{\text{Tr}(\Pi_\psi \tilde{\Pi}_\psi)}$. It is straightforward to extend this definition to pure states of multipartite systems, $\psi \in \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$. We try

$$\begin{aligned} C_D(\psi)^2 &= \text{Tr}(\Pi_\psi \tilde{\Pi}_\psi) \\ &= \frac{2^N}{d_{\text{tot}}} \sum_{k < l}^{d-1} \text{Tr}(\Pi_\psi y_{kl} \Pi_\psi^* y_{kl}) \\ &= (-1)^N \frac{2^N}{d_{\text{tot}}} \sum_{k < l}^{d-1} |\langle \psi^* | y_{kl} | \psi \rangle|^2 . \end{aligned}$$

Here we see that for odd N , the quantity $C_D(\psi)$ vanishes identically, for any local dimension, in analogy with the behavior of the N -qubit LSL invariant $H(\phi_N) = \langle \phi_N^* | \sigma_2^{\otimes N} | \phi_N \rangle$ (where $\phi_N \in \mathbb{C}_2^{\otimes N}$). This is because $\langle \phi_N^* | y_{kl} | \phi_N \rangle = \langle \phi_N | y_{kl}^* | \phi_N^* \rangle^* = \langle \phi_N^* | (-y_{kl}) | \phi_N \rangle \equiv 0$ for an odd number N of factors in the tensor product y_{kl} .

We can define now the *distributed concurrence*

$$C_D(\psi) = \sqrt{\text{Tr}(\Pi_\psi \tilde{\Pi}_\psi)} \quad (24)$$

$$= \sqrt{\frac{2^N}{d_{\text{tot}}} \sum_{k < l}^{d-1} |\langle \psi^* | y_{kl} | \psi \rangle|^2} . \quad (25)$$

Since the universal state inverter Eq. (6) commutes with local unitary operations [22], the distributed concurrence is a local unitary invariant. We will show below that $C_D(\psi)$ in certain cases is also an entanglement monotone.

Again, the quantity $C_D(\psi)$ generalizes the known concurrence-type quantities in the most natural way: For N -qubit states ϕ_N the LSL invariant $|H(\phi_N)|$ is obtained, in particular for $N = 2$ we get Wootters' concurrence $|\langle \phi_2^* | \sigma_2 \otimes \sigma_2 | \phi_2 \rangle|$. On the other hand, for two parties in dimensions $d_1 = d_2 > 2$ we get the standard $d \times d$ concurrence, Eq. (5) (cf. also Ref. [29, 30, 31]).

Concluding this section, we note that the operator $\Pi_\psi \tilde{\Pi}_\psi \Pi_\psi$ is proportional to Π_ψ because of the projector property of Π_ψ . Then, by virtue of Eq. (24), we have

$$\Pi_\psi \tilde{\Pi}_\psi \Pi_\psi = C_D(\psi)^2 \Pi_\psi . \quad (26)$$

2.2 Distributed concurrence is an entanglement monotone for local dimensions smaller than four

In order to investigate the monotone property of the distributed concurrence we need to study whether or not, for pure states, $C_D(\psi)$ can increase under the action of separable Kraus operators $O_j = A_1^{(j)} \otimes \dots \otimes A_N^{(j)}$ (where $\sum_j O_j^\dagger O_j = \mathbb{1}$), which represent stochastic local operations and classical communication (SLOCC) [32, 33]. If C_D is an entanglement monotone for pure states, it can be extended to the mixed states, as usual, through the convex roof [34].

As the distributed concurrence acts symmetrically on the parties it suffices to consider actions on the first party only, $\mathcal{A}_j = A_j \otimes \mathbb{1}_2 \otimes \dots \otimes \mathbb{1}_N$. We may assume that the number of parties N is even and that the reduced density matrix of the first party has rank $r_1 > 1$ because otherwise the monotone property is trivially valid. Further, since all generalized measurements may be composed from two-outcome measurements, we will consider Kraus operators with $A_1^\dagger A_1 + A_2^\dagger A_2 = \mathbb{1}$. Then, for $\psi \in \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$ we need to show

$$\begin{aligned} C_D(\psi) &= p_1 C_D\left(\frac{\mathcal{A}_1 \psi}{\sqrt{p_1}}\right) + p_2 C_D\left(\frac{\mathcal{A}_2 \psi}{\sqrt{p_2}}\right) \\ &= C_D(\mathcal{A}_1 \psi) + C_D(\mathcal{A}_2 \psi) \end{aligned} \quad (27)$$

with $p_j = \text{Tr}(\mathcal{A}_j \Pi_\psi \mathcal{A}_j^\dagger)$. In the second line we have used the property $C_D(\alpha \psi) = |\alpha|^2 C_D(\psi)$.

We now rewrite Eq (27) by using the decomposition $\Pi_\psi = \sum_{jk} |e_j f_j\rangle \langle e_k f_k|$, where $\{|e_j\rangle\}$ are orthonormal vectors that span the Hilbert space of the first party and are chosen such that they diagonalize the operator $\mathcal{A}_1^\dagger \mathcal{A}_1$. The vectors $\{|f_j\rangle\}$ span the Hilbert space of the last $(N-1)$ parties in ψ and are not necessarily orthogonal or normalized, but they obey $\sum_j \langle f_j | f_j \rangle = 1$. We obtain

$$\begin{aligned} \text{tr}(\Pi_\psi \tilde{\Pi}_\psi) &= 2 \sum_{jk} F_{kjjk} \\ \text{tr}(\mathcal{A}_1 \Pi_\psi \mathcal{A}_1^\dagger) (\widetilde{\mathcal{A}_1 \Pi_\psi \mathcal{A}_1^\dagger}) &= 2 \sum_{kl} F_{kjjk} D_j D_k \end{aligned}$$

where we have used the definitions

$$\begin{aligned} D_k \delta_{jk} &\equiv \langle e_k | \mathcal{A}_1^\dagger \mathcal{A}_1 | e_j \rangle , \quad 0 \leq D_k \leq 1 , \\ F_{klmj} &\equiv \langle f_k | \left(\widetilde{|f_l\rangle \langle f_m|} \right) | f_j \rangle . \end{aligned}$$

Note that $F_{klmj} = F_{lkjm} = -F_{lkmj}$, which follows with the explicit inversion formula (21). Finally, $p_1 = \text{Tr}(\mathcal{A}_1 \Pi_\psi \mathcal{A}_1^\dagger) = \sum_j D_j \langle f_j | f_j \rangle$ and $p_2 = \text{Tr}(\mathcal{A}_2 \Pi_\psi \mathcal{A}_2^\dagger) = \sum_j (1 - D_j) \langle f_j | f_j \rangle$.

With all this, we can transform Eq. (27) into the equivalent inequality

$$\begin{aligned} \sqrt{\sum_{jk} F_{jkkj}} &= \sqrt{\sum_{jk} F_{jkkj} D_j D_k} + \\ &+ \sqrt{\sum_{jk} F_{jkkj} (1 - D_j)(1 - D_k)}. \end{aligned} \quad (28)$$

Now it is straightforward to prove the monotone property for the well-known cases $d = 2$, N arbitrary, and $N = 2$, d arbitrary, as we show in Appendix B. For the general case, we continue transforming Eq. (28) by subtracting the second term on the right, squaring and simplifying the result, to obtain

$$\left(\sum_{jk} w_{jk} D_j \right)^2 = \sum_{jk} w_{jk} D_j D_k, \quad (29)$$

with the weights $w_{jk} \equiv F_{jkkj} / (\sum_{lm} F_{lmm})$. Note that, if the denominator here were zero, the monotone property would be trivially satisfied. It is possible to construct examples of $\{D_k\}$ and $\{F_{lmm}\}$ such that inequality (29) is violated for $\dim \mathcal{H}_1 = 4$ (see Appendix C). Therefore, C_D is *not* an entanglement monotone if a party has local dimension $d_j = 4$ and the party number $N = 4$. Similarly, such sets can be found to show that C_D^2 is not a monotone for $d_j = 3$, $N = 4$.

Remarkably, however, Eq. (29) does hold for $\dim \mathcal{H}_1 = 3$, that is, C_D is an entanglement monotone for any number of parties if all local dimensions are not larger than three. This behavior resembles that of LSL invariants for which it is known that the monotone property depends on the homogeneity degree [35] and, in fact, also on the local dimension.

For the proof of Eq. (29) in the case $\dim \mathcal{H}_1 = 3$ we collect only the D_j^2 terms on the left and multiply both sides by 4. Moreover, we use $2(w_{01} + w_{02} + w_{12}) = 1$ and obtain

$$\begin{aligned} (1 - 2w_{12})^2 D_0^2 + (1 - 2w_{02})^2 D_1^2 + (1 - 2w_{01})^2 D_2^2 &= D_0 \sqrt{2(2w_{01}) - (1 - 2w_{12})(1 - 2w_{02})} D_1 + \\ &+ D_0 \sqrt{2(2w_{02}) - (1 - 2w_{12})(1 - 2w_{01})} D_2 + \\ &+ D_1 \sqrt{2(2w_{12}) - (1 - 2w_{01})(1 - 2w_{02})} D_2 + \\ &+ D_1 \sqrt{2(2w_{01}) - (1 - 2w_{12})(1 - 2w_{02})} D_0 + \\ &+ D_2 \sqrt{2(2w_{12}) - (1 - 2w_{02})(1 - 2w_{01})} D_1 + \\ &+ D_2 \sqrt{2(2w_{02}) - (1 - 2w_{12})(1 - 2w_{01})} D_0. \end{aligned} \quad (30)$$

The peculiar way of writing Eq. (30) indicates how one can see that this is precisely a Cauchy-Schwarz inequality, and therefore is valid. This concludes the proof that Eq. (29) holds for $\dim \mathcal{H}_1 = 3$.

To summarize this section (see also Fig. 2), we have found that both C_D and C_D^2 are entanglement monotones only in the well-known cases and, trivially, for odd number of parties N . For local dimensions $d_j \leq 3$ and even party number N , C_D is an entanglement monotone, but not C_D^2 .

For local dimensions $d_j = 4$ neither C_D nor C_D^2 is an entanglement monotone for any even $N > 2$.

3 Distribution of correlations and monogamy of entanglement

In Ref. [18], the monogamy relation Eq. (2) for N qubits was derived. The strategy there was to write down a local SL invariant for an N -qubit state ρ_N via combining ρ_N and $\tilde{\rho}_N$, and, by ex-

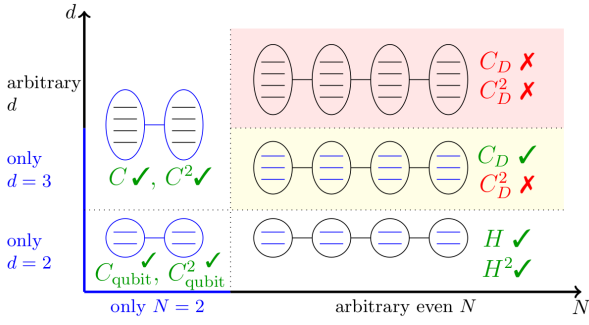


Figure 2: The local unitary invariant C_D , cf. Eq. (24), termed distributed concurrence, naturally generalizes the well-known cases of bipartite concurrence $d = 2$, $N = 2$ [22, 29] and H invariant for N -qubit system [20]. While in those cases both C_D and C_D^2 are monotones, the general case $d > 2$, $N > 2$ is different: C_D (but not C_D^2) is an entanglement monotone only if all local dimensions $d \leq 3$. If there is one subsystem of dimension at least four, neither C_D nor C_D^2 are entanglement monotones.

panding that expression in terms of Bloch components to obtain an identity which connects the invariant (for the global state) with combinations of local quantities (linear entropies). Now we ask whether one can obtain a new result by generalizing Eq. (2) in the frame of the present formalism. The answer is affirmative, as we will show.

We consider a state ρ of a finite-dimensional multipartite quantum system $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N)$. Clearly, $\text{Tr}(\rho\tilde{\rho})$ is a local unitary invariant. We expand this expression as a sum over all bipartite splits $A|\bar{A}$ by applying Eq. (9)

$$\begin{aligned} \text{Tr}(\rho\tilde{\rho}) &= 1 + \sum_{|A|>0} (-1)^{|A|} \text{Tr} \left[\rho \text{Tr}_{\bar{A}} \rho \otimes \mathbf{1}_{\bar{A}} \right] \\ &= 1 + \sum_{|A|>0} (-1)^{|A|} \text{Tr}_A \left[\text{Tr}_{\bar{A}} \rho \right]^2 \\ &= \frac{1}{2} \sum_{|A|>0} (-1)^{|A|+1} 2 \left(1 - \text{Tr} \rho_A^2 \right) \end{aligned}$$

By using the definition of the linear entropy on the partition A , $\tau_A = 2(1 - \text{Tr} \rho_A^2)$ we obtain

$$0 \leq 2 \text{Tr}(\rho\tilde{\rho}) = \sum_{|A|>0} (-1)^{|A|+1} \tau_A . \quad (31)$$

This is an equality for the distribution of correlations and holds both for pure and for mixed states of any finite-dimensional multipartite system. The general validity of Eq. (31) is quite remarkable: In Ref. [18] it was derived by using a qubit-specific formalism, therefore its full generality could not be expected.

It is in order here to recall the quantum marginal problem [19], which poses the question whether or not a given set of marginal states is compatible with a (possibly unique) global state. The equality (31) represents a simple necessary (but not sufficient) condition for a favorable answer to that question.

It is interesting here to consider the special case of mixed states ρ_{ABC} of *three-parties* A, B, and C. From Eq. (31) and $\text{Tr}(\rho\tilde{\rho}) = 0$ we find

$$\tau_A + \tau_B + \tau_C + \tau_{ABC} = \tau_{AB} + \tau_{AC} + \tau_{BC} . \quad (32)$$

This reminds of the *strong subadditivity inequality* for the von-Neumann entropy S [37], which reads $S_{ABC} + S_B \geq S_{AB} + S_{BC}$. However, inequality (32) is symmetrized between the parties and has the opposite direction.

We emphasize that the τ_A in Eq. (31) are simply the linear entropies of the reduced states ρ_A , so they are *not* entanglement measures. However, if we consider *pure* states $|\psi\rangle\langle\psi| \equiv \Pi_\psi$, we note $\tau_A(\psi) = 0$ for $|A| = N$, and the equality reads

$$\begin{aligned} 0 \leq 2C_D^2(\psi) &= \sum_{|A|>0} (-1)^{|A|+1} \tau_A(\psi) \\ &= \sum_{N>|A|>0} (-1)^{|A|+1} C_{A|\bar{A}}^2(\psi) . \end{aligned} \quad (33)$$

Here we denote by $C_{A|\bar{A}}(\psi)$ the bipartite concurrence on the split $A|\bar{A}$ according to Eq. (5). We see that Eq. (33) represents a true monogamy relation, because all the terms on the right-hand side are entanglement measures. If the local dimensions are not larger than three, also C_D is an entanglement measure, so that we even have a monogamy equality. Thus, Eq. (33) is a constraint for the distribution of entanglement across all bipartitions of an arbitrary multipartite pure state of finite dimensionality. For odd N the equality is trivial, because the purities on complementary partitions A and \bar{A} enter it with a different sign, confirming that $C_D(\psi) = 0$ in this case. We note that aspects of the results discussed here and summarized in Eqs. (31), (33) were obtained in Ref. [36].

It is an interesting feature of Eqs. (33), (32) that they describe all local parties in a symmetric manner, whereas the Coffman-Kundu-Wootters identity Eq. (1) or the strong subadditivity inequality single out one party. Our way of deriving the general monogamy relation (33) seems to

suggest that the symmetric form has a peculiar role, just because the tilde state $\tilde{\rho}$ is symmetric in the local parties as well.

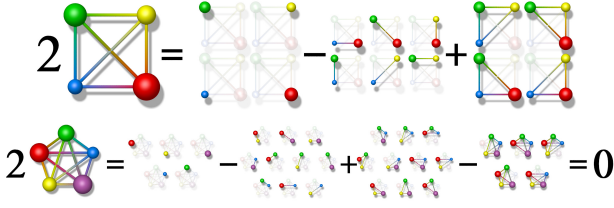


Figure 3: Illustration of the N -qudit monogamy relation Eq. (33). The left-hand side symbolizes C_D^2 while, on the right-hand side, there are the representations of the linear entropies of all possible bipartite splits in the set of N qudits.

Top panel: Four qudits. Complementary partitions A and \bar{A} enter with the same sign.

Bottom panel: Five qudits. For odd N , the monogamy relation is trivial because the linear entropy of a partition A equals that of its complementary partition \bar{A} , but contributes with opposite sign.

4 Other interesting aspects

4.1 Conservation laws for correlations

In the previous section we have emphasized that in Eq. (31) all terms are local unitary invariants. The left-hand side is locally invariant on each single party, whereas the terms on the right-hand side are locally invariant on their respective bipartition. We explain now that Eq. (31) gives rise to nontrivial unitary invariants on an entire subset of the parties.

To this end, let us consider an example of a four-party state ρ_{1234} where each index denotes the respective party. The reduced state, e.g., of parties $\{13\}$ is $\rho_{13} = \text{Tr}_{24} \rho_{1234}$. Correspondingly, the linear entropy of parties $\{13\}$ is $\tau_{13} = 2(1 - \text{Tr}_{13} \rho_{13}^2)$. Then we have

$$\begin{aligned} \text{Tr} \rho_{1234} \tilde{\rho}_{1234} &= \tau_1 + \tau_2 + \tau_3 + \tau_4 - \tau_{12} - \dots - \\ &\quad - \tau_{34} + \tau_{123} + \tau_{124} + \tau_{134} + \tau_{234} - \\ &\quad - \tau_{1234} . \end{aligned} \quad (34)$$

Assume now that the subsystem of parties $\{13\}$ is subject to unitary evolution $U_{13}(t) = \mathbb{1}_{24} \otimes \exp(-iH_{13}t)$ generated by a (time-independent) Hamiltonian H_{13} . The full state ρ_{1234} be mixed, e.g., the thermal state at some temperature. Let us collect all the terms in Eq. (34) in which both

of the parties 1 and 3, or none of them, are traced out:

$$\begin{aligned} -\tau_2 - \tau_4 + \tau_{13} + \tau_{24} - \tau_{123} - \tau_{134} + \tau_{1234} &= \\ = -\text{Tr} \rho_{1234} \tilde{\rho}_{1234} + \tau_1 + \tau_3 - \tau_{12} - \tau_{14} - \\ - \tau_{23} - \tau_{34} + \tau_{124} + \tau_{234} \end{aligned} \quad (35)$$

The left-hand side of Eq. (35) is invariant under the evolution $U_{13}(t)$ while each of the terms on the right will vary in a nontrivial way, as the time evolution generates or degrades correlations of the corresponding parties with the other subsystems. This means Eq. (35) represents a *conservation law for correlations*. Analogous conservation laws can be found for Hamiltonian subsystem dynamics of any finite-dimensional multipartite state.

4.2 Relation between distributed concurrence and topology

A striking property of Eqs. (31) and (33) is their apparent connection with topology. Just the form of those equations hints at the so-called Euler characteristic [38], a number characterizing the structure of a topological space. Yet in the present case the nature of the topological entity, described in particular by Eq. (33) is not really clear. Another problem is that Eq. (33) in general is not a relation between integers, which makes its topological interpretation questionable.

However, there are more indications in the properties of C_D that there indeed exists a relation to topology. First, it is known that a closed topological space of odd dimension has Euler characteristic 0. This coincides with the property $C_D \equiv 0$ for systems with odd number of parties N . Further, the Euler characteristic of a product manifold equals the product of the Euler characteristics of the factors. Again, it is easy to see from the explicit definition of C_D , Eq. (25), that the distributed concurrence has the same property. Finally, the Euler characteristic bears a close relation with the inclusion-exclusion principle. The derivation of Eq. (33) for qubits in Ref. [18] via the Bloch representation is an explicit application of this principle.

We can only speculate what the topological nature of Eq. (33) is. A tempting interpretation is the following. We regard the N parties as vertices of a standard $(N-1)$ -simplex. In a simplex, each

vertex is connected to each other vertex. This appears to correspond to the distribution of entanglement in a genuinely entangled [33] pure state: A given party is entangled with any other party in such a pure state. In a strict sense there is no distinction, such as ‘only pairwise entanglement’, as is possible for mixed states. The boundary of an $(N - 1)$ -simplex is topologically equivalent to a sphere S^{N-2} and therefore has Euler characteristic $1 + (-1)^{N-2}$ which corresponds to the behavior of C_D (vanishing C_D for odd N). Finally, if the pure state can be written as a product of exactly two factors (with party numbers k and $N - k$), the topology would be that of two disjoint simplices, one with k and the other with $N - k$ vertices.

5 Conclusions

We have analyzed universal state inversion applied to multipartite quantum states with finite-dimensional parties. This map takes pure states to the opposite hypersurface of the state space and therefore algebraically encodes extremal properties of quantum states. We made explicit that this map is antilinear. In the framework of our formalism we were able to derive a universally valid equality, Eq. (31), that relates a local unitary invariant depending on the global state to the linear entropies of all possible reduced states. Remarkably, if we apply this equality to pure states we obtain the first known monogamy relation for bipartite entanglement in a multipartite state, Eq. (33), which is valid for any number of parties and any finite local dimension. These findings appear to have a close relation with the topological properties of multipartite quantum systems, as we have argued in Sec. 4.2. Moreover, our correlation equalities give rise to conservation laws for correlations, which are valid for any type of unitary subsystem dynamics.

The local unitary invariant mentioned above, the distributed concurrence, describes the quantitative distribution of bipartite entanglement in a multipartite pure state. It naturally generalizes the concurrence in the well-known cases, that is, for bipartite systems of arbitrary dimension and for many-qubit systems. Thus, it provides both the mathematical and conceptual link between these hitherto somewhat unrelated quantities, see Eqs (2),(5). Intriguingly, for systems

of even party number $N = 4$ the distributed concurrence is an entanglement monotone only below a ‘critical’ local dimension $d \leq 3$. Note, however, that for its role as a ‘generator’ of the monogamy inequality the monotone property of the distributed concurrence is not essential.

Our work shows that, by means of algebraic methods, it is possible to express the mathematical constraints to quantum states, in particular their positivity, in terms of physically relevant quantities and principles. Importantly, our results establish necessary relations between the global and the local states which may be considered as a partial answer in the context of the quantum-marginal problem. Clearly, our result highlights only one specific aspect of correlations, and the question is whether also other mathematical constraints allow for a similar formulation with direct access to their physical meaning.

Acknowledgments

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A Proof of useful generator formulae

The interesting relations (17), (18) are closely linked to the well-known completeness relation for the set of generators $\{h_k\}$ (see, e.g., [28]). It is instructive to derive all these relations in the same framework, i.e., by consequently using the terminology of quantum information.

An important ingredient is the SWAP operator. Consider two states $|\psi\rangle, |\phi\rangle \in \mathcal{H}$. Then we define

$$\text{SWAP} \cdot (|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle \quad . \quad (36)$$

With the computational basis $\{|j\rangle\}$ of \mathcal{H} it is straightforward to show that $\text{SWAP} =$

$\sum_{jk} |jk\rangle\langle kj|$. Let us denote by $\text{Tr}_{[1]}$ the partial trace over the first tensor factor in $\mathcal{H} \otimes \mathcal{H}$, and correspondingly by $\text{Tr}_{[2]}$ the partial trace over the second factor. Then we find by explicit calculation in the computational basis that for two operators $A, B \in \mathcal{B}(\mathcal{H})$

$$\text{Tr}_{[1]}(\text{SWAP} \cdot (A \otimes B)) = A \cdot B \quad (37)$$

$$\text{Tr}_{[2]}(\text{SWAP} \cdot (A \otimes B)) = B \cdot A \quad (38)$$

and

$$\text{Tr}_{[1]}((A \otimes B) \cdot \text{SWAP}) = B \cdot A \quad (39)$$

$$\text{Tr}_{[2]}((A \otimes B) \cdot \text{SWAP}) = A \cdot B \quad (40)$$

Important special cases of these relations are obtained by setting, e.g., $B = \mathbf{1}$,

$$\text{Tr}_{[1]}(\text{SWAP} \cdot (A \otimes \mathbf{1})) = A \quad (41)$$

Now consider a set $\{\mathbf{h}_j\}$, $j = 1 \dots (d^2 - 1)$ of traceless Hermitian generators of $\text{SU}(d)$, $\text{Tr}(\mathbf{h}_j \mathbf{h}_k) = d\delta_{jk}$, $\mathbf{h}_0 \equiv \mathbf{1}$, so that we can expand

$$A = \frac{1}{d} \sum_{j=0}^{d^2-1} a_j \mathbf{h}_j = \frac{1}{d} \sum_{j=0}^{d^2-1} \text{Tr}(\mathbf{h}_j A) \mathbf{h}_j \quad (42)$$

$$= \text{Tr}_{[1]} \left[\left(\frac{1}{d} \sum_{j=0}^{d^2-1} \mathbf{h}_j \otimes \mathbf{h}_j \right) \cdot (A \otimes \mathbf{1}) \right] \quad (43)$$

Because A is an arbitrary operator here, comparison of Eqs. (41), (43) yields the completeness relation for the generators \mathbf{h}_j

$$\text{SWAP} = \frac{1}{d} \sum_{j=0}^{d^2-1} \mathbf{h}_j \otimes \mathbf{h}_j \quad (44)$$

With these preliminary considerations we can turn to the proof of the operator relations in Sec. 1.4. Inserting Eq. (44) into the identity

$$\text{SWAP} \cdot (A \otimes \mathbf{1}) \cdot \text{SWAP} = \mathbf{1} \otimes A$$

and tracing over the second party gives

$$\begin{aligned} \text{Tr}(A) \mathbf{1} &= \frac{1}{d^2} \sum_{j,k=0}^{d^2-1} (\mathbf{h}_j A \mathbf{h}_k) \text{Tr}(\mathbf{h}_j \mathbf{h}_k) \\ &= \frac{1}{d} \sum_{j=0}^{d^2-1} \mathbf{h}_j A \mathbf{h}_j \quad , \end{aligned} \quad (45)$$

which proves Eq. (17).

From the explicit representation of SWAP in the computational basis it is obvious that it is

related to the maximally entangled state $|\Phi^+\rangle \equiv \frac{1}{\sqrt{d}} \sum_j |jj\rangle$ by virtue of partial transposition, e.g., on the first party, $T_{[1]}$,

$$\begin{aligned} \text{SWAP}^{T_{[1]}} &= \left(\sum_{jk} |jk\rangle\langle kj| \right)^{T_{[1]}} \\ &= \sum_{jk} |kk\rangle\langle jj| \\ &= d |\Phi^+\rangle\langle\Phi^+| \quad . \end{aligned} \quad (46)$$

We conclude from Eqs. (44) and (46) that

$$\text{SWAP}^{T_{[1]}} = \frac{1}{d} \sum_{j=0}^{d^2-1} \mathbf{h}_j^T \otimes \mathbf{h}_j \quad (47)$$

In analogy with Eqs. (37)–(40) we obtain now (by explicit calculation in the computational basis)

$$\text{Tr}_{[1]}(d |\Phi^+\rangle\langle\Phi^+| \cdot (A \otimes B)) = A^T \cdot B \quad (48)$$

$$\text{Tr}_{[2]}(d |\Phi^+\rangle\langle\Phi^+| \cdot (A \otimes B)) = B^T \cdot A \quad (49)$$

$$\text{Tr}_{[1]}((A \otimes B) \cdot d |\Phi^+\rangle\langle\Phi^+|) = B \cdot A^T \quad (50)$$

$$\text{Tr}_{[2]}((A \otimes B) \cdot d |\Phi^+\rangle\langle\Phi^+|) = A \cdot B^T \quad (51)$$

By using Eq. (48) with $B = \mathbf{1}$, the permutation invariance of $|\Phi^+\rangle$, and Eq. (47) we find

$$\begin{aligned} A^T &= \text{Tr}_{[1]}(d |\Phi^+\rangle\langle\Phi^+| \cdot (A \otimes \mathbf{1})) \\ &= \text{Tr}_{[1]}(\text{SWAP} \cdot d |\Phi^+\rangle\langle\Phi^+| \cdot (A \otimes \mathbf{1})) \\ &= \frac{1}{d} \sum_{j=0}^{d^2-1} \text{Tr}_{[1]}(\text{SWAP} \cdot [(\mathbf{h}_j^T A) \otimes \mathbf{h}_j]) \end{aligned}$$

and, finally, with Eq. (37)

$$A^T = \frac{1}{d} \sum_{j=0}^{d^2-1} \mathbf{h}_j^T A \mathbf{h}_j \quad , \quad (52)$$

which proves Eq. (18).

B Proof of monotone property in the known cases

For the sake of completeness, we present the proof for the monotone property of both C_D and C_D^2 in the known cases in the frame of the formalism in Sec. 2.2.

$d = 2$, N arbitrary: Here, there are only two indices $j \in \{0, 1\}$ and only one

value of $F_{0110} = F_{1001}$, so that the inequality (28) becomes $\sqrt{2F_{0110}} = \sqrt{2F_{0110}D_0D_1} + \sqrt{2F_{0110}(1-D_0)(1-D_1)}$ from which we have

$$1 = \sqrt{D_0D_1} + \sqrt{(1-D_0)(1-D_1)}$$

and further

$$(1 - \sqrt{D_0D_1})^2 = (1 - D_0)(1 - D_1)$$

which is equivalent to

$$D_0 + D_1 = 2\sqrt{D_0D_1}$$

and always fulfilled because of the relation between arithmetic and geometric mean.

The monotone condition Eq. (27) for the square of C_D reads

$$\begin{aligned} C_D^2(\psi) &= p_1 C_D^2\left(\frac{\mathcal{A}_1\psi}{\sqrt{p_1}}\right) + p_2 C_D^2\left(\frac{\mathcal{A}_2\psi}{\sqrt{p_2}}\right) \\ &= \frac{C_D^2(\mathcal{A}_1\psi)}{p_1} + \frac{C_D^2(\mathcal{A}_2\psi)}{p_2} . \end{aligned} \quad (53)$$

In the proof for C_D^2 we use the probability $p_1 = \langle f_0|f_0\rangle D_0 + \langle f_1|f_1\rangle D_1$ (recall that $\sum_j \langle f_j|f_j\rangle = 1$), and we have to show that

$$1 = \frac{D_0D_1}{p_1} + \frac{(1-D_0)(1-D_1)}{1-p_1} .$$

This inequality is equivalent to

$$0 = (p_1 - D_0)(p_1 - D_1)$$

which is always fulfilled as is easily checked with the explicit formula for p_1 .

$N = 2$, d arbitrary: In this case we have

$$\begin{aligned} F_{jkkj} &= \langle f_j | \left(\widetilde{|f_k\rangle\langle f_k|} \right) | f_j \rangle \\ &= \langle f_j | \left(\text{Tr}(|f_k\rangle\langle f_k|) \mathbf{1} - |f_k\rangle\langle f_k| \right) | f_j \rangle \\ &= \langle f_j | f_j \rangle \langle f_k | f_k \rangle - |\langle f_j | f_k \rangle|^2 , \end{aligned}$$

because the subnormalized vectors $\{|f_j\rangle\}$ belong to a single party. and inequality (28) takes the form

$$\begin{aligned} \sqrt{1 - \sum_{jk} |\langle f_j | f_k \rangle|^2} &= \sqrt{p_1^2 - \sum_{jk} |\langle f_j | f_k \rangle|^2 D_j D_k} + \\ &+ \sqrt{(1-p_1)^2 - \sum_{jk} |\langle f_j | f_k \rangle|^2 (1-D_j)(1-D_k)} . \end{aligned}$$

The right-hand side (rhs) of this relation can

be upper-bounded by using the concavity of the square root. Subsequent squaring yields

$$\begin{aligned} (\text{rhs})^2 &\leq 1 - \frac{\sum_{jk} |\langle f_j | f_k \rangle|^2 D_j D_k}{p_1} - \frac{\sum_{jk} |\langle f_j | f_k \rangle|^2 (1-D_j)(1-D_k)}{1-p_1} \\ &\leq 1 - \sum_{jk} |\langle f_j | f_k \rangle|^2 \frac{p_1(1-p_1) + (p_1 - D_j)(p_1 - D_k)}{p_1(1-p_1)} . \end{aligned}$$

Hence, in order to prove the monotone property it remains to show that

$$0 \leq \sum_{jk} |\langle f_j | f_k \rangle|^2 (p_1 - D_j)(p_1 - D_k) .$$

This can be seen by noting that

$$\begin{aligned} \sum_{jk} |\langle f_j | f_k \rangle|^2 (p_1 - D_j)(p_1 - D_k) &= \text{Tr}(W^\dagger W) \\ &= 0 \end{aligned}$$

with the Hermitian operator

$$W = \sum_j |f_j\rangle\langle f_j| (p_1 - D_j) ,$$

which concludes the proof. We note that application of the concavity of the square root above implies that also C_D^2 is a monotone in this case.

C Examples for states violating the monotone property

As we have mentioned in Sec. 2.2, it is possible to find examples violating the monotone conditions, Eqs. (28), (29), which implies that C_D cannot be an entanglement monotone for the local dimension in question, and all higher dimensions as well.

A numerical strategy for finding such counterexamples for $d = 4$ is the following. We can restrict our attention to channels which are diagonal in the Schmidt basis of the first party. Then, the decomposition $|\psi\rangle = \sum_j |e_j f_j\rangle$ from Sec. 2.2 becomes the Schmidt decomposition $|\psi\rangle = \sum_j \sqrt{\lambda_j} |e_j \bar{f}_j\rangle$ with orthonormal $\{|\bar{f}_j\rangle\}$ and $\sum_j \lambda_j = 1$. A strategy to look for parameters to violate Eq. (29) is to construct a matrix $F_{kjjk} \equiv \lambda_j \lambda_k \bar{F}_{kjjk}$ (where $\bar{F}_{kjjk} \equiv \langle \bar{f}_k | \left(\widetilde{|\bar{f}_j\rangle\langle \bar{f}_j|} \right) | \bar{f}_k \rangle$) with sufficiently extremal entries, that is, with values close to 0 and others of order 1.

A matrix \bar{F}_{kjjk} with such extremal entries can be found as follows. First, we generate a state ψ_1 with random entries and apply the state inverter to the projector Π_{ψ_1} . Since we consider an odd number of parties, the state ψ_1 is an eigenstate of $\tilde{\Pi}_{\psi_1}$ with eigenvalue 0. In general, the other eigenvalues of $\tilde{\Pi}_{\psi_1}$ are distributed between 0 and numbers of order 1, so that we can choose three of those other eigenstates of $\tilde{\Pi}_{\psi_1}$ with sufficiently different eigenvalues. Once we have computed the matrix \bar{F}_{kjjk} we can perform a random search for Schmidt coefficients $\{\lambda_j\}$ and diagonal entries of the channel $\{D_j\}$ and readily find numerical examples that violate the monotone condition for C_D or C_D^2 . The same procedure can be applied for $d = 3$ in order to find a counterexample for the monotone condition of C_D^2 .

To conclude this part, we give an explicit analytical counterexample for a four-party state

(with local dimensions 4, 2, 2, 2) that violates the monotone condition both for C_D and C_D^2 . It reads

$$\begin{aligned} |\psi_{4222}\rangle = \frac{1}{\sqrt{8}} (&|0000\rangle + |0011\rangle + |1100\rangle + \\ &+ |1111\rangle + |2000\rangle - |2011\rangle - \\ &- |3100\rangle + |3111\rangle) . \end{aligned}$$

The local channel acting on the first party is given by the projectors

$$\begin{aligned} \mathcal{A}_1 &= \text{diag}(1, 1, 0, 0) \\ \mathcal{A}_2 &= \text{diag}(0, 0, 1, 1) . \end{aligned}$$

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