RHO INVARIANTS FOR MANIFOLDS WITH BOUNDARY AND LOW-DIMENSIONAL TOPOLOGY

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Summary

This thesis is dedicated to the definition of a 3-manifold invariant and to a systematic study of its properties. The invariant is built on the work of Kirk and Lesch, who introduced a generalization of the Atiyah-Patodi-Singer rho invariant to manifolds with boundary. Their invariant is not purely topological, in that it depends on the choice of a Riemannian metric on the boundary (up to pseudo-isotopy). The starting point of this thesis is the observation that, on a torus, the choice of a *framing*, i.e. of a basis for its first homology group, is enough to define such a metric. This elementary fact leads to the definition of our main invariant: a real number $\rho_\alpha(X, \mathcal{F})$ associated to a compact, oriented 3-manifold X whose boundary is a union of tori with a specified framing $\mathcal{F}$, and a representation $\alpha\colon \pi_1(X) \to U(n)$. In particular, we obtain in this way a new invariant for links in S^3.

One of the main techniques to study the new invariant is the use of gluing formulas. This is made effective thanks to an enhancement of the formulas of Kirk and Lesch, which stands as one of our main results. A special emphasis is put on the computation of the rho invariant of the solid torus $D^2 \times S^1$ and of the thick torus $[0,1] \times T^2$ for all possible framings and representations, as they appear, via the gluing formulas, as correction terms respectively to Dehn fillings and to changes of framing. The problem turns out to be a complicated one, and it remains open in general. Interestingly, it leads to the definition of two families of functions whose behavior is still partially mysterious, which have relations with some classical functions in analytic number theory. The thesis is concluded with applications to knot theory. An invariant $\varrho_\alpha(L)$ of a link L in S^3 with a representation $\alpha\colon \pi_1(S^3 \setminus L) \to U(n)$ arises as the rho invariant of the link exterior, with framing given by the usual meridian and longitudes of L. We show that $\varrho_\alpha(L)$ is a generalization of the rho invariants of the closed manifolds obtained by Dehn surgeries on L. For abelian α, this leads to a comparison with signatures of links, providing for them an alternative definition. The invariant $\varrho_\alpha(L)$ is tested with success to be an effective tool in simplifying some proofs of classical results, showing potential for further discoveries.

Contents

Introduction

Atiyah-Patodi-Singer rho invariants

Given a closed, oriented, odd-dimensional manifold M with a representation $\alpha\colon \pi_1(M) \to U(n)$, the Atiyah-Patodi-Singer rho invariant $\rho_\alpha(M)$ is a real number with the following property: if there exists a compact oriented manifold W whose boundary is M and such that α extends to $\pi_1(W)$, then $\rho_\alpha(M)$ is an integer and it satisfies the equality

$$\rho_\alpha(M) = n\,\sigma(W) - \sigma_\alpha(W), \tag{1}$$

where $\sigma(W)$ and $\sigma_\alpha(W)$ are the traditional and twisted signature of W. In this sense, the rho invariant can be seen as an odd-dimensional counterpart to the signature, and it shares many features with it. However, its direct computation is often very difficult. In fact, the invariant is defined by fixing a Riemannian metric on M and taking the difference

$$\rho_\alpha(M) := \eta_\alpha(M) - n\,\eta(M), \tag{2}$$

where the numbers $\eta(M)$ and $\eta_\alpha(M)$ are spectral invariants of some first-order differential operators on N (namely of the *odd signature operator*, whose square is the Hodge Laplacian), and of its twisted version. These eta invariants appear as correction terms in the celebrated signature theorem for manifolds with boundary of Atiyah, Patodi and Singer [2], of which (1) is a direct consequence [3]. Eta invariants depend indeed on the Riemannian metric. However, the rho invariant is independent of this choice, as it follows by applying the signature theorem to the product $[0,1]\times M$ with two different metrics at the two ends.

Rho invariants and knot theory

The fact that the right-hand term of (1) only depends on the boundary M and on the restriction of α to $\pi_1(M)$ was known before the Atiyah-Patodi-Singer signature theorem was proved. This lead people to use variations of

(1) as *definitions* of invariants of odd-dimensional manifolds. Such disguised forms of rho invariants appear in classical papers in knot theory such as those of Casson and Gordon [10, 11], Litherland [33] and Gilmer [25], often in relationship to the Levine-Tristram signature function $\sigma_K\colon S^1 \to \mathbb{Z}$ of a knot K in S^3. A well known result, whose origin goes back to a computation of Viro, states in fact that, if M_K is the closed manifold obtained by 0-framed surgery on K and $\alpha\colon \pi_1(M_K) \to U(1)$ is the representation sending the meridian of K to $\omega \in U(1)$, then

$$\rho_\alpha(M_K) = -\sigma_K(\omega).$$

This is indeed proved using (1), since an appropriate 4-manifold W with $\partial W = M_K$ can be constructed in this case. Atiyah-Patodi-Singer rho invariants of the 0-framed surgery manifold of a knot or link, associated to higher-dimensional, non-abelian representations, were used in knot theory by Levine [31, 32] and Friedl [22, 23] as obstructions to concordance.

Cut-and-paste formulas and Wall's non-additivity

In order to work with rho invariants, it is often useful to have the machinery of *cut-and-paste*. Namely, if we have three closed manifolds that decompose along a codimension-1 submanifold Σ as $X_1 \cup_\Sigma X_2$, $X_1 \cup_\Sigma X_0$ and $-X_0 \cup_\Sigma X_2$, and α is a representation on $\pi_1(X_1 \cup X_2 \cup X_3)$, we want to compute the correction term C in the formula

$$\rho_\alpha(X_1 \cup_\Sigma X_2) = \rho_\alpha(X_1 \cup_\Sigma X_0) + \rho_\alpha(-X_0 \cup_\Sigma X_2) + C. \tag{3}$$

Now, if $X_1 \cup_\Sigma X_0$ and $-X_0 \cup_\Sigma X_2$ bound manifolds W_1 and W_2 such that the representation extends, then a theorem of Wall about *non-additivity* of the signature, together with (1), tells us how to compute the correction term. Namely, in that case we have

$$C = \tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}) - n\,\tau(V_{X_0}, V_{X_1}, V_{X_2}), \tag{4}$$

where τ is the *Maslov triple index* in the twisted and untwisted cohomology of Σ in degree $k = \dim \Sigma/2$, and the variables are the *canonical Lagrangians* of the three manifolds with boundary, i.e. the subspaces

$$V_{X_i} := \operatorname{im}(H^k(X_i;\mathbb{C}) \to H^k(\partial X_i;\mathbb{C}) \tag{5}$$

and their twisted equivalents. We show that the correction term C of (3) is always given by (4), no matter whether the manifolds W_1 and W_2 exist, and that Wall's non-additivity theorem is enough to prove it (see Proposition 2.1.7).

Eta and rho invariants for manifolds with boundary

In the nineties, several authors studied versions of eta invariants for manifolds with boundary and gluing formulas for them. In most cases, such formulas were only stated up to integer contributions. An exception to this trend is a paper of Bunke [7], where the integer contribution is described in terms of indices of some non-canonical linear operators. However, the correction term remains in practice quite mysterious.

Eventually, Kirk and Lesch [29] proved a gluing formula for the eta invariant that identifies the correction term more explicitly, making it computable in the case of the odd signature operator. Suppose that X is a compact oriented manifold, with a Riemanniam metric having product form near the boundary, and let $\alpha\colon \pi_1(X) \to U(n)$ be a representation. In order to get a well-defined eta invariant, boundary conditions have to be fixed, and these can be can be specified by the choice of a Lagrangian subspace $L \subseteq H^*(\partial X; \mathbb{C}^n_\alpha)$. This leads to the invariants $\eta_\alpha(X, L)$ considered by Kirk and Lesch. A natural choice for L is the extension to all degrees of the canonical Lagrangian V^α_X defined in (5), but more flexibility is often useful. If Y is a Riemannian manifold with the same boundary Σ as X but oriented the opposite way, we can form a closed manifold $X \cup_\Sigma Y$ by gluing them along their boundary. Kirk and Lesch prove the formula

$$\eta_\alpha(X \cup_\Sigma Y) = \eta_\alpha(X, V^\alpha_X) + \eta_\alpha(Y, V^\alpha_Y) + m(V^\alpha_X, V^\alpha_Y), \tag{6}$$

where m is a real number depending on the Riemannian metric and on the relative position of V^α_X and V^α_Y. In order to get an even more treatable correction term, we introduce a slightly modified version of their eta invariant, denoted by $\overline{\eta}_\alpha(X, L)$, that still coincides with the Atiyah-Patodi-Singer invariant whenever X is a closed manifold. Using their result, we show that, for each choice of L, we have

$$\eta_\alpha(X \cup_\Sigma Y) = \overline{\eta}_\alpha(X, L) + \overline{\eta}_\alpha(Y, L) + \tau(L, V^\alpha_X, V^\alpha_Y) \tag{7}$$

(see Proposition 2.3.4.) The advantage of (7) is that the correction term is now an integer, it is independent of the metric and it computable by linear algebra. Using either (6) or (7), it is easy to prove a cut-and-paste formula for the eta invariant of closed manifolds. We complete an argument of Kirk and Lesch in relating this to the cut-and-paste formulas discussed above, by reducing the correction term to the middle degree (see Section 2.3.4).

The topological significance of the eta invariants for manifolds with boundary is that, as it happened for closed manifolds, taking a relative version of them reduces the dependence on the Riemannian metric. The result proved by Kirk and Lesch [29, 28] is that, given Lagrangian subspaces $L_1 \subseteq H^*(\partial X; \mathbb{C}^n_\alpha)$ and $L_2 \subseteq H^*(\partial X; \mathbb{C}^n)$, the difference

$$\rho_\alpha(X, g, L_1, L_2) := \eta_\alpha(X, L_1) - \eta_\varepsilon(X, L_2), \tag{8}$$

where ε is the trivial n-dimensional representation, is independent of the metric in the interior of X, and it depends on the metric g on ∂X only up to (pseudo-)isotopy. This last observation will be crucial in order to define a topological invariant later on.

The gluing formula for cobordisms

The gluing formulas (6) and (7) only allow to express the Atiyah-Patodi-Singer eta invariant of a *closed* manifold in terms of its constituent parts. If the only goal of rho invariants for manifolds with boundary is to use them as building blocks to compute Atiyah-Patodi-Singer rho invariants, this might be enough. As our ultimate goal is to use them as topological invariants on their own, a more general formula is needed.

For this purpose, suppose that X is an odd-dimensional manifold whose boundary components are partitioned as $\partial X = -\Sigma' \sqcup \Sigma$. We see then X as a cobordism from Σ to Σ'. Let $\mathcal{L}ag(H)$ denote the set of Lagrangians of a complex symplectic space H. Employing a formalism of Turaev [45], this leads X to induce a *Lagrangian action*

$$V_X \colon \mathcal{L}ag(H^*(\Sigma'; \mathbb{C})) \to \mathcal{L}ag(H^*(\Sigma; \mathbb{C})),$$

that behaves well under the stacking of cobordisms. Heuristically, if L' is an element of the first set, $V_X(L')$ is the canonical Lagrangian of a fictional manifold X' obtained from X by capping the boundary piece Σ' with a manifold whose canonical Lagrangian is glued to L'. In particular, if Σ' is empty (so that $L' = 0$), the result is the canonical Lagrangian V_X itself.

If Y is a manifold with boundary $\partial Y = -\Sigma \sqcup \Sigma''$, we can glue X and Y along Σ, obtaining a manifold Z with boundary $\partial Z = -\Sigma' \sqcup \Sigma''$. Suppose now that $\alpha \colon \pi_1(Z) \to U(n)$ is a representation, and set

$$H = H^*(\Sigma, \mathbb{C}^n_\alpha), \quad H' = H^*(\Sigma', \mathbb{C}^n_\alpha), \quad H'' = H^*(\Sigma'', \mathbb{C}^n_\alpha).$$

Under a mild topological assumption, the formalism of Lagrangians actions can be extended to the twisted setting, leading to maps $V^\alpha_X \colon \mathcal{L}ag(H') \to \mathcal{L}ag(H)$ and $V^\alpha_{Y^t} \colon \mathcal{L}ag(H'') \to \mathcal{L}ag(H)$. Provide Z with a Riemannian metric which is of product form near Σ, Σ' and Σ''. We have then the following result, which is the content of Theorem 2.4.3.

Theorem 1. *Let $L \in \mathcal{L}ag(H)$, $L' \in \mathcal{L}ag(H')$ and $L'' \in \mathcal{L}ag(H'')$ be arbitrary Lagrangian subspaces. Then, we have*

$$\overline{\eta}_\alpha(Z, L' \oplus L'') = \overline{\eta}_\alpha(X, L' \oplus L) + \overline{\eta}_\alpha(Y, L \oplus L'') + \tau(L, V^\alpha_X(L'), V^\alpha_{Y^t}(L'')).$$

Theorem 1 gives a satisfying generalization of (7). Its proof depends on the fact that the triple Maslov index interacts well with the Lagrangian actions, as it was already observed by Turaev [45].

Rho invariants of 3-manifolds with toroidal boundary

We define a *framing* on a 2-torus T to be an ordered basis (μ, λ) for its first homology group $H_1(T;\mathbb{Z})$, and extend this definition to disjoint union of tori in the obvious way. The element μ is called the *meridian* of the framing, and the element λ is called its *longitude*. We say that a Riemannian metric g on T is *compatible* with (μ, λ) if there exists an isometry φ from (T, g) to the standard flat torus $S^1 \times S^1$ such that $(\varphi_*(\mu), \varphi_*(\lambda))$ is the canonical basis of $H_1(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}^2$. The observation that is crucial for what follows is that, up to isotopy, there is exactly one metric on T which is compatible with (μ, λ).

Suppose that X is a compact, oriented 3-manifold whose boundary is a union of tori, and let $\mathcal{F}$ be a framing on ∂X. We call the pair $(X, \mathcal{F})$ a 3-*manifold with framed toroidal boundary*. Given such a pair, let $g_\mathcal{F}$ be a Riemannian metric on ∂X which is compatible with $\mathcal{F}$. Then, given a representation $\alpha\colon \pi_1(X) \to U(n)$, we define the rho invariant of $(X, \mathcal{F})$ as the real number

$$\rho_\alpha(X, \mathcal{F}) := \rho(X, g_\mathcal{F}, L^\alpha_\mathcal{F}, L^\epsilon_\mathcal{F}),$$

where $L^\alpha_\mathcal{F} \subseteq H^1(\partial X; \mathbb{C}^n_\alpha)$ and $L^\epsilon_\mathcal{F} \subseteq H^1(\partial X, \mathbb{C})$ are some Lagrangian subspaces that are naturally associated to the longitudes of $\mathcal{F}$. The observations made so far, together with the properties of the rho invariant (8) of Kirk and Lesch, lead to the following result, which is the starting point of the subsequent work (see Theorem 3.2.3 for a more precise statement)

Theorem 2. *The number $\rho_\alpha(X, \mathcal{F})$ is well defined, it is invariant under orientation-preserving diffeomorphisms, and it changes sign if the orientation of X is reversed.*

Once that an appropriate Riemannian metric on X is fixed, the rho invariant $\rho_\alpha(X, \mathcal{F})$ can be described in terms of the modified eta invariants $\overline{\eta}$ as

$$\rho_\alpha(X, \mathcal{F}) = \overline{\eta}_\alpha(X, M^\alpha_\mathcal{F}) - n\,\overline{\eta}(X, M_\mathcal{F}), \tag{9}$$

where $M_{\mathcal{F}}^{\alpha}$ and $M_{\mathcal{F}}$ are now Lagrangians depending on the meridians of $\mathcal{F}$ (see Proposition 3.2.6). Using Theorem 1, it is then possible to obtain nice gluing formulas for the rho invarant of 3-manifolds with framed toroidal boundary. Before doing so, it is convenient to introduce some more Lagrangian subspaces, which correspond to the restriction to middle degree of the Poincaré duals of some of the Lagrangians discussed so far. Namely, we set

$$\mathcal{V}_X := \ker(H_1(\partial X;\mathbb{C}) \to H_1(X;\mathbb{C})) \subseteq H_1(\partial X;\mathbb{C})$$

and, if the framing $\mathcal{F}$ has meridians $\mu_1, \ldots, \mu_k$,

$$\mathcal{M}_{\mathcal{F}} := \mathrm{Span}_{\mathbb{C}}\{\mu_1, \ldots, \mu_k\} \subseteq H_1(\partial X;\mathbb{C}).$$

We present now a simplified version of the main gluing formula (see Theorem 3.2.10), corresponding to the case where the two manifolds are glued along all of their boundary, and the twisted homology of this manifold vanishes (which is true in most applications).

Theorem 3. *Let $M = X \cup_{\Sigma} Y$ be a closed, oriented manifold which is the union of two 3-manifolds X, Y over a disjoint union of tori Σ. Let $\mathcal{F}$ be any framing on Σ, and let $\alpha\colon \pi_1(X) \to U(n)$ be a representation such that $H_*(\Sigma;\mathbb{C}^n_{\alpha}) = 0$. Then, we have*

$$\rho_{\alpha}(M) = \rho_{\alpha}(X,\mathcal{F}) + \rho_{\alpha}(Y,\mathcal{F}) - n\,\tau(\mathcal{M}_{\mathcal{F}}, \mathcal{V}_X, \mathcal{V}_Y). \tag{10}$$

It is normally easy to find generators for the Lagrangians $\mathcal{V}_X$, and $\mathcal{V}_Y$, and this makes the correction term computable explicitly in most situations. Suppose now for simplicity that X has single boundary component T, framed by (μ, λ) (everything extends easily to the several component case). Then, given coprime integers p and q, Theorem 3 can be used to compare the rho invariant of $(X,\mathcal{F})$ with the Atiyah-Patodi-Singer rho invariant of the closed manifolds obtained by *Dehn filling* on X, i.e. by gluing a solid torus $D^2 \times S^1$ to X in such a way that the homology class of $\partial D^2 \times \{1\}$ corresponds to $p\mu + q\lambda \in H_1(T;\mathbb{Z})$. In fact, the rho invariant of $(X,\mathcal{F})$ contains potentially more information than the Atiyah-Patodi-Singer rho invariants of its Dehn fillings, as not every representation $\alpha\colon \pi_1(X) \to U(n)$ can be extended to the fundamental group of some Dehn filling of X.

Rho invariants of solid tori and lens spaces

In order to get an explicit formula for the difference between $\rho_{\alpha}(X,\mathcal{F})$ and the rho invariant of the p/q-Dehn filling along one of its boundary components, we have to be able to compute the rho invariant of the solid torus

$D^2 \times S^1$ with the induced framing on its boundary. This framing depends on the slope p/q of the filling. In fact, the number $p/q \in \mathbb{Q} \cup \{\infty\}$ can be used to classify all framings (within a specified "orientation") on $\partial D^2 \times S^1$ up to orientation-preserving self-diffeomorphism of $D^2 \times S^1$. We get in this way a family of framings $\mathcal{F}_r$ on $\partial D^2 \times S^1$, for $r \in \mathbb{Q} \cup \{\infty\}$.

Using the symmetries of the solid torus, it is easy to show that, for all representations $\alpha\colon \pi_1(D^2 \times S^1) \to U(n)$, we have

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}_0) = \rho_\alpha(D^2 \times S^1, \mathcal{F}_\infty) = 0. \tag{11}$$

In general, however the rho invariant of $(D^2 \times S^1, \mathcal{F}_r)$ is non-trivial, and it is surprisingly hard to compute. In order to study this problem systematically, we introduce the following notation. For each $r \in \mathbb{Q} \cup \{\infty\}$, we define a 1-periodic function $S_r\colon \mathbb{R} \to \mathbb{R}$ by setting

$$S_r(t) := \rho_{\alpha_t}(D^2 \times S^1, \mathcal{F}_r), \tag{12}$$

where $\alpha_t\colon \pi_1(D^2 \times S^1) \to U(1)$ is the representation sending the class of $\{0\} \times S^1$ to $e^{2\pi i t} \in U(1)$. We show that the functions S_r are continuous outside of $\mathbb{Z}$ (where their value is 0), and prove that they satisfy many interesting relations (see Proposition 3.3.31). One of these is the reciprocity formula (with the convention $\operatorname{sgn}(\infty) := 0$)

$$S_r(t) + S_{1/r}(t) = -\operatorname{sgn}(r) \quad \text{for all } t \in \mathbb{R} \setminus \mathbb{Z}.$$

By gluing together two copies of the solid torus along some framing of their boundaries, we obtain a 3-dimensional lens space $L(p,q)$, where the integers p and q depend on the framings. Atiyah-Patodi-Singer eta and rho invariants were computed since early on [3], and expressed in many different fashions. Let $\alpha\colon \pi_1(L(p,q)) = \mathbb{Z}/p \to U(1)$ be the representation sending the natural generator to a p^{th} root of unity $e^{2\pi i y}$. Then, starting from a description of Casson and Gordon [10] of $\rho_\alpha(L(p,q))$ in terms of a count of lattice points inside of a triangle, we find, for $p > 0$, that (see (3.9))

$$\rho_\alpha(L(p,q)) = -4(s_{0,y}(q,p) - s(q,p)), \tag{13}$$

$s(a,c)$ is a classical *Dedekind sum*, and $s_{x,y}(a,c)$, for $x, y \in \mathbb{R}$, is a generalized Dedekind sum due to Rademacher. Lens spaces have a natural Riemannian metric, coming from their description as quotients of S^3. Combining (13) with a computation of Atiyah, Patodi and Singer for the untwisted eta invariant of $L(p,q)$ with respect to this metric, we reach the following description for the twisted eta invariant (with respect to the same metric).

Theorem 4. *Let p, q coprime integers with $p \geq 0$, and let $\alpha\colon \pi_1(L(p,q)) = \mathbb{Z}/p \to U(1)$ be the representation sending 1 to $e^{2\pi i y}$ (for some $y \in \frac{1}{p}\mathbb{Z}$). Then, we have*

$$\eta(L(p,q), e^{2\pi i y}) = -4 s_{0,y}(q,p).$$

In fact, for computations, the right-hand term of (13) can be replaced by a fairly simple expression (see Corollary 3.3.25). Thanks to this knowledge about the rho invariants of lens spaces, using the gluing formula (10) we can compute $S_r(t)$ for many values of t (see Corollary 3.3.32). With the help of a computer program, it is now easy to visualize the known values of $S_r(t)$ for any reasonable choice of $r = p/q$. In Appendix A.3, we added some images of this kind. The above method, however, will always only give the answer for a *discrete* subset of $\mathbb{R}$. The problem of computing $S_r(t)$ for all values of r and t stays open.

A similar treatment can be given for the (related) problem of computing the rho invariant of a *thick torus* $[0,1] \times T^2$, with two different framings on the two boundary components. By the gluing formula, this serves the goal of being able to express explicitly the difference $\rho_\alpha(X, \mathcal{F}) - \rho_\alpha(X, \mathcal{F}')$ for two different framings $\mathcal{F}, \mathcal{F}'$ on the (toroidal) boundary of a same 3-manifold X. The rho invariant of $[0,1] \times T^2$ leads to the definition of a function

$$\Theta\colon \operatorname{SL}(2,\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R},$$

which is related to the *Rademacher function* $\Phi\colon \operatorname{SL}(2,\mathbb{Z}) \to \frac{1}{3}\mathbb{Z}$. The computation of $\Theta(A, v)$ for some choices of $A \in \operatorname{SL}(2,\mathbb{Z})$ and $v \in \mathbb{R}^2$ is possible thanks to a result of Bohn [5], who computed the Atiyah-Patodi-Singer rho invariants of torus bundles T_A over the circle.

An invariant for links

Let L be an oriented link in S^3, and let X_L denote its *exterior*, i.e. the complement of an open tubular neighborhood of L. Then, X_L is a compact oriented 3-manifold whose boundary is a disjoint union of tori. The usual meridians and longitudes of L determine a basis for $H_1(\partial X_L;\mathbb{Z})$, and thus framing on ∂X_L. Given a representation $\alpha\colon \pi_1(X_L) \to U(n)$, we can now define a link invariant, called the *rho invariant of* L, as

$$\varrho_\alpha(L) := \rho_\alpha(X_L, \mathcal{F}_L). \tag{14}$$

We introduce the following notation, which will turn useful in a moment. If the components of L are $L_1, \dots, L_k$, let $\Lambda_L \in \mathbb{Z}^{k\times k}$ denote matrix whose (i,j)-coefficient is given by the linking number $\Lambda_{ij} := \operatorname{lk}(L_i, L_j)$, with the convention that $\Lambda_{ii} = 0$. As Λ_L is a symmetric matrix, its signature $\operatorname{sign}\Lambda_L \in \mathbb{Z}$ is well defined.

Performing 0-*framed Dehn surgery* on L, i.e. filling every boundary component of X_L with a solid torus $D^2 \times S^1$ in such a way that the classes $\partial D^2 \times \{1\}$ correspond to the longitudes of the link, we obtain a closed manifold that is denoted by M_L. As we have anticipated, the Atiyah-Patodi-Singer rho invariant of M_L is a well studied invariant [31, 32, 22, 23]. In particular, it is interesting to compare $\varrho_\alpha(L)$ to $\rho_\alpha(M_L)$. Using the gluing formula and (11), we show that the following simple relation holds between the two invariants (see Theorem 4.1.9).

Theorem 5. *Let L be a link and let $\alpha\colon \pi_1(M_L) \to U(n)$ be a representation such that $H_*(\partial X_L; \mathbb{C}^n_\alpha) = 0$. Then, we have*

$$\rho_\alpha(M_L) = \varrho_\alpha(L) + n \operatorname{sign}\Lambda_L.$$

In particular, if $L = K$ is a knot, we have $\rho_\alpha(M_K) = \varrho_\alpha(K)$.

Observe that $\varrho_\alpha(L)$ is defined for every representation $\alpha\colon \pi_1(X_L) \to U(n)$, and not all of them extend to $\pi_1(M_L)$. In particular, by Theorem 5, the invariant $\varrho_\alpha(L)$ is a strict extension of $\rho_\alpha(M_L)$.

Very often, more general surgeries than the 0-framed one are considered in knot theory. By allowing the flexibility of a framing different from the standard one in the definition (14), this leads to a useful generalization of Theorem 5 (see Theorem 4.1.16).

The thesis is concluded by a thorough treatment of the case of one-dimensional representations $\psi\colon \pi_1(X_L) \to U(1)$. If L has k components $L_1, \dots, L_k$, the set of such representations are in a natural correspondence with $\mathbb{T}^k := (S^1)^k$. One of the driving goals of this project was a comparison between the rho invariant and the *multivariable signature* of L, which is a function

$$\sigma'_L\colon (\mathbb{T} \setminus \{1\})^k \to \mathbb{Z}$$

defined by Cimasoni and Florens as a generalization of the Levine-Tristram signature function $\sigma_L\colon \mathbb{T} \to \mathbb{Z}$. We managed to prove the following result, which implies that $\varrho_\psi(L)$ (for $\psi\colon \pi_1(X_L) \to U(1)$) and the function σ'_L contain the same amount of topological information about the link (see Theorem 4.2.23 for the complete statement).

Theorem 6. *Let $L = L_1 \cup \dots \cup L_k$ be an oriented link in S^3, let $(\omega_1 \dots, \omega_k) \in (\mathbb{T} \setminus \{1\})^k$ and let $\psi\colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the representation sending the meridian of L_i to ω_i for $i = 1, \dots, k$. Then, we have*

$$\varrho_\psi(L) = -\sigma_L(\omega_1, \dots, \omega_k) + C(\Lambda_L, \omega_1, \dots, \omega_k),$$

where the real number $C(\Lambda_L, \omega_1, \ldots, \omega_k)$ *only depends on the linking matrix* Λ_L *and and on the* k*-tuple* $(\omega_1, \ldots, \omega_k)$.

On the way to the proof of Theorem 6, we reformulate the definition of the Levine-Tristram and Cimasoni-Florens signature in terms of Atiyah-Patodi-Singer rho invariants of closed manifolds (see Theorem 4.2.7, Proposition 4.2.21). Using the invariant $\varrho_\psi(L)$ and its framed version, together with gluing formulas, this allows to simplify many classical proofs in knot theory. As an example, we give new short proofs of two classical results. The first is a theorem of Litherland expressing the Levine-Tristram signature of a satellite knot in terms of the signatures of its companion and orbit (see Theorem 4.2.10). The second is a theorem of Casson and Gordon giving a computation for the Atiyah-Patodi-Singer rho invariant of a 3-manifold (with respect to some representations) in terms of the Levine-Tristram signature of a link (see Theorem 4.2.11).

Organization of the work

In Chapter 1, we review several concepts that are needed in the rest of the thesis. In particular, we recall the basics about complex symplectic spaces and the Maslov triple index of Lagrangians, we give several different views on twisted homology, we review twisted signatures of manifolds and the non-additivity theorem of Wall, and we conclude with a section on the formalism of cobordisms and Lagrangian relations.

Chapter 2 is dedicated to the general theory of rho invariants for closed manifolds and for manifolds with boundary. Apart from recalling the main results about them, we prove a cut-and-paste formula for closed manifolds and our main gluing formula for cobordisms.

In Chapter 3, we define the main invariant of our interest, namely the rho invariant of a 3-manifold with framed toroidal boundary. We study its general properties and rewrite the gluing formulas in this context. We also focus on explicit computations for the rho invariant of the solid torus and of the thick torus.

In Chapter 4, we use the rho invariant of the previous chapter to define an invariant of links, and we compare it to several previously known invariants in knot theory.

In the Appendix, we recall the construction and some basic results about 3-dimensional lens spaces and some basics about Dedekind sums and of the Rademacher Φ function. Finally, we include a series of images representing some of our results about the rho invariant of the solid torus.

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Chapter 1

Preliminaries

1.1 Hermitian and skew-Hermitian forms

In this section, we fix conventions and recall some basic properties about the linear algebra of sesquilinear forms on finite-dimensional complex vector spaces. In Section 1.1.1 we discuss the basics on reflexive sesquilinear forms. In Section 1.1.2, we restrict our attention on Hermitian forms and their signature. In Section 1.1.3, we focus instead on skew-Hermitian forms, leading to the concept of a complex symplectic space. In Section 1.1.4, we introduce the so-called Hermitian symplectic spaces, where a symplectic form arises from a Hermitian structure together with an automorphism of order four.

1.1.1 Reflexive sesquilinear forms

A *sesquilinear form* φ on a complex vector space H is a map

$$\varphi\colon H \times H \to \mathbb{C}$$

that is linear in the first variable and antilinear in the second variable. Given a sesquilinear form φ_1 on a space H_1 and a sesquilinear form φ_2 on the space H_2, we define the sesquilinear form $\varphi_1 \oplus \varphi_2$ on $H_1 \oplus H_2$ by

$$(\varphi_1 \oplus \varphi_2)((v_1, v_2), (w_1, w_2)) := \varphi_1(v_1, w_1) + \varphi_2(v_2, w_2).$$

If the space H has finite dimension n and $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis, the matrix $A \in M_n(\mathbb{C})$ representing a sesquilinear form φ with respect to $\mathcal{B}$ is defined by $A_{i,j} = \varphi(b_i, b_j)$. If $\psi\colon V \times V \to \mathbb{R}$ is a bilinear form on the real vector space V, we get a sesquilinear form $\psi_{\mathbb{C}}$ on the complex vector space $V \otimes \mathbb{C}$, called the *complexification* of ψ, by

$$\psi_{\mathbb{C}}(v \otimes x, w \otimes y) := x\overline{y}\, \psi(v, w).$$

Definition 1.1.1. A sesquilinear form φ on a complex vector space H is

- *Hermitian* if, for all $a, b \in H$, $\varphi(b, a) = \overline{\varphi(a, b)}$;
- *skew-Hermitian* if, for all $a, b \in H$, $\varphi(b, a) = -\overline{\varphi(a, b)}$;
- *reflexive* if, for all $a, b \in H$, $\varphi(a, b) = 0 \iff \varphi(b, a) = 0$.

We see immediately that all Hermitian and skew-Hermitian sesquilinear forms are reflexive. We shall normally just speak of "Hermitian forms" and "skew-Hermitian forms" when referring to these, omitting the adjective "sesquilinear". A reflexive sesquilinear form φ on H has a well-defined *radical*, which is the subspace of H given by

$$\mathrm{rad}(\varphi) := \{a \in H \mid \varphi(a, b) = 0 \,\forall\, b \in H\} = \{a \in H \mid \varphi(b, a) = 0 \,\forall\, b \in H\}$$

(for a general sesquilinear form, the two descriptions above need not coincide, and they define two different spaces called the left and right radical). We say that φ is *non-degenerate* if $\mathrm{rad}(\varphi) = 0$. Otherwise, we say that φ is *degenerate*.

Given a complex vector space H with a reflexive sesquilinear form φ, we define the *orthogonal complement* (with respect to φ) of a subspace $V \subseteq H$ as the subspace

$$V^{\varphi} := \{a \in H \mid \varphi(a, v) = 0 \,\forall\, v \in V\}.$$

Once again, because of reflexivity, it does not matter whether we write a as the first or second variable in the above definition. Moreover, we see immediately that $\mathrm{rad}(\varphi) = H^{\varphi}$ and that, for all subspaces V, we have $V \subseteq (V^{\varphi})^{\varphi}$.

Definition 1.1.2. Let H be a complex vector space with a reflexive sesquilinear form φ. A subspace $V \subseteq H$ is called, with respect to φ:

- *isotropic* , if $V \subseteq V^{\varphi}$;
- *maximal isotropic*, if $V \subseteq V^{\varphi}$ and there is no isotropic subspace that properly contains V;
- *Lagrangian* (or "a Lagrangian"), if $V = V^{\varphi}$.

It is immediate to see that every Lagrangian subspace is maximal isotropic, but the converse in general is not true. Restricting ourselves to non-degenerate sesquilinear forms, we get the following result.

Proposition 1.1.3. *Let H be a finite-dimensional complex vector space and let φ be a non-degenerate reflexive sesquilinear form on H. Then, for all subspace $V \subseteq H$, we have*

$$\dim V + \dim V^{\varphi} = \dim H.$$

As a consequence, $(V^{\varphi})^{\varphi} = V$.

Proof. Let φ be the given sesquilinear form, and consider the linear map $\varphi^*\colon H \to \overline{H}^*$ from H to the dual space of the complex conjugate of H, defined by $\varphi^*(a)(\bar{b}) := \varphi(a,b)$. As φ is non-degenerate, the map φ^* is injective, and hence it is an isomorphism as $\dim \overline{H}^* = \dim H$. We consider now the map $f\colon H \to \overline{V}^*$ given as the composition

$$H \xrightarrow{\varphi^*} \overline{H}^* \xrightarrow{p} \overline{V}^*,$$

where p denotes the restriction to $\overline{V}$. Clearly p is surjective and hence f is also surjective. Moreover, the null-space of f coincides by definition with the orthogonal complement V^{φ} of V. From the rank-nullity theorem, we get

$$\dim H = \dim \overline{V}^* + \dim V^{\varphi}.$$

The first statement follows as $\dim \overline{V}^* = \dim V$. By a double application of the formula (to V^{φ} and to V), we get then

$$\dim (V^{\varphi})^{\varphi} = \dim H - \dim V^{\varphi} = \dim V,$$

and the equality $(V^{\varphi})^{\varphi} = V$ follows as V is cointained in $(V^{\varphi})^{\varphi}$. □

An immediate consequence of Proposition 1.1.3 is that, under the hypotheses of the lemma, if H admits a Lagrangian subspace V, then the dimension of H is even and V is a half-dimensional subspace. Moreover, we have the following.

Corollary 1.1.4. *Let H be a finite dimensional complex vector space with a non-degenerate reflexive sesquilinear form, and let $V \subseteq H$ be an isotropic subspace such that $\dim V \geq \dim H/2$. Then, $\dim H$ is even and V is Lagrangian.*

1.1.2 The signature of a Hermitian form

Let H be a complex vector space, and let φ be a Hermitian form H. From the equation $\varphi(h,h) = \overline{\varphi(h,h)}$, we see that the value $\varphi(h,h)$ is a real number for all $h \in H$. We say that a nonzero vector $h \in H$ is *positive* (with respect to φ) if $\varphi(h,h) > 0$, *negative* if $\varphi(h,h) < 0$ and *isotropic* if $\varphi(h,h) = 0$. We say that a subspace $V \subseteq H$ is *positive* if all of its non-zero elements are positive, and *negative* if all of its non-zero elements are negative. We suppose from now on that H is finite dimensional. Then, we can define integers

$$n_+(\varphi) := \max\{\dim V_+ \mid V_+ \text{ is a positive subspace}\},$$
$$n_-(\varphi) := \max\{\dim V_- \mid V_- \text{ is a negative subspace}\}.$$

Lemma 1.1.5. *If φ is non-degenerate, there exists a positive subspace V_+ and a negative subspace V_- that are mutually orthogonal and satisfy*

$$H = V_+ \oplus V_-.$$

Moreover, for each such a decomposition, V_+ and V_- have maximal dimension among subspaces of their sign. In particular, we have

$$\dim H = n_+(\varphi) + n_-(\varphi).$$

Proof. For brevity, we only sketch the proof. Start by choosing any negative subspace V_- of maximal dimension, and set $V_+ := V_-^{\varphi}$. By maximality of V_-, there are no negative vectors in V_+, and hence $V_+ \cap V_- = 0$. By Proposition 1.1.3, we have $\dim H = \dim V_- + \dim V_+$, and hence there is a direct sum decomposition

$$H = V_+ \oplus V_-.$$

The fact that V_+ is positive and not just "semi-positive" follows from the fact that φ is non-degenerate. It has maximal dimension among positive subspaces, because every supsbace of higher dimension intersects V_- non-trivially and hence it contains at least a negative vector. In particular, $n_+(\varphi) = \dim V_+$ and $n_-(\varphi) = \dim V_-$, and the second statement follows. □

Definition 1.1.6. Let H be a complex vector space of finite dimension, and let φ be a Hermitian form on H. The *signature* of φ is the integer

$$\operatorname{sign}(\varphi) := n_+(\varphi) - n_-(\varphi).$$

The following properties of the signature are immediate to be verified.

Proposition 1.1.7. *Let H, H' finite-dimensional vector space with Hermitian forms φ, φ' respectively. Then:*

(i) $\operatorname{sign}(-\varphi) = -\operatorname{sign}(\varphi)$*;*

(ii) $\operatorname{sign}(\varphi \oplus \varphi') = \operatorname{sign}(\varphi) + \operatorname{sign}(\varphi')$*;*

(iii) if $f\colon H \to H'$ is a surjective map such that $\varphi'(f(a), f(b)) = \varphi(a, b)$ for all $a, b \in H$, then $\operatorname{sign}\varphi' = \operatorname{sign}\varphi$.

Remark 1.1.8. In particular, we see from (iii) that we can always obtain a non-degenerate Hermitian form φ' with the same signature of φ by looking at the well-defined Hermitian form

$$\begin{aligned}\varphi'\colon V/\operatorname{rad}(\varphi) \times V/\operatorname{rad}(\varphi) &\to \mathbb{C}\\ ([a],[b]) &\mapsto \varphi(a,b).\end{aligned}$$

A Hermitian form φ on a complex vector space H is called *metabolic* if there is a subspace of H that is Lagrangian with respect to φ. The following result is often useful.

Proposition 1.1.9. *Let φ be a metabolic Hermitian form on a complex vector space of finite dimension. Then* $\operatorname{sign}(\varphi) = 0$.

Proof. Let V be a Lagrangian for φ, and let V_+, V_- a positive and a negative subspace of maximal dimension. As all vectors $v \in V$ satisfy $\varphi(v, v) = 0$, the Lagrangian V intersects trivially with both V_+ andd V_-. We have then

$$\dim H \geq \dim V_+ + \dim V, \quad \dim H \geq \dim V_- + \dim V. \tag{1.1}$$

Suppose now for the moment that φ is non-degenerate. Then, from Proposition 1.1.3 and Lemma 1.1.5 we have

$$\dim H = 2\dim V, \quad \dim H = \dim V_+ + \dim V_-. \tag{1.2}$$

Comparing (1.1) and (1.2), we see that

$$\dim V_+ = \dim V_- = \dim H/2.$$

It follows that $\operatorname{sign}(\varphi) = \dim V_+ - \dim V_-$ is 0. If φ has a non-trivial radical, we consider the non-degenerate Hermitian form φ' on $H/\operatorname{rad}(\varphi)$, that has the same signature of φ. Then the subspace $V/\operatorname{rad}(\varphi)$ is a Lagrangian for φ', and hence we can conclude by the above argument. ☐

Remark 1.1.10. In case of a degenerate Hermitian form, with our definition, it is not enough to find a half-dimensional isotropic subspace to deem it metabolic. For example, the space $\mathbb{C}^2$ with the Hermitian form represented in the standard basis by the matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ has a 1-dimensional isotropic subspace but it is not metabolic, and it has signature 1 (see Proposition 1.1.11). However, if the space H is decomposed as the direct sum of two isotropic subspaces V_1 and V_2, then the Hermitian form is metabolic. This can be seen by applying Corollary 1.1.4 to the non-degenerate form φ' on $H/\operatorname{rad}\varphi$ and observing that metabolicity of φ' implies metabolicity of φ.

Proposition 1.1.11. *Let H be a complex vector space of finite dimension, and let φ a Hermitian form on H. Let A be the matrix associated to φ with respect to some basis of H. Let $e_+(A)$ be the number of positive eigenvalues of A, and let $e_-(A)$ be the number of negative eigenvalues. Then*

$$\operatorname{sign}(\varphi) = e_+(A) - e_-(A).$$

Proof. Let $\mathcal{B}$ be the basis of H with respect to which φ is represented by the matrix A. By the spectral theorem for Hermitian matrices, we can find a unitary matrix U such that UAU^* is diagonal. The matrix $D := UAU^*$ is the matrix representing φ with respect to the basis $\mathcal{B}'$ obtained from $\mathcal{B}$ by seeing U as change of basis matrix. Moreover, the elements on the diagonal of D are exactly the eigenvalues of A. The positive eigenvalues correspond to the elements of $\mathcal{B}'$ that are positive with respect to φ, and the negative eigenvalues correspond to the elements of $\mathcal{B}'$ that are negative with respect to φ.

Let V_+ be the span of the positive elements of $\mathcal{B}'$, and let V_- be the span of the negative ones, so that $e_+(A) = \dim V_+$ and $e_-(A) = \dim V_-$. We can conclude if we show that V_+ have maximal dimension among subspaces of their sign. If φ is non-degenerate, this is a consequence of Lemma 1.1.5, as H gets decomposed as an orthogonal sum $V_+ \oplus V_-$ in this case. Otherwise, we take the associate non-degenerate form on $H/\operatorname{rad}(\varphi)$, and observe that the dimension of V_+ and V_- in the quotient is not affected. □

1.1.3 Complex symplectic spaces

A *complex symplectic space* is a pair (H, ω), where H is a complex vector space of finite dimension and ω is a non-degenerate skew-Hermitian form on H, called the *symplectic form*. The symplectic form ω will often be implicit, and we shall call H itself a complex symplectic space. We define the *opposite* of a complex symplectic space (H, ω) as the complex symplectic space $(H, -\omega)$. When the symplectic form is implicit we shall denote the opposite of H by H^-. Given two complex symplectic spaces (H_1, ω_1) and (H_2, ω_2), we define their *direct sum* as the complex symplectic space $(H_1 \oplus H_2, \omega_1 \oplus \omega_2)$. A linear isomorphism $f\colon H_1 \to H_2$ is called *symplectic* if $\omega_2(f(v), f(w)) = \omega_1(v, w)$ for all $v, w \in H_1$. If a symplectic isomorphism $f\colon H_1 \to H_2$ exists, we say that the symplectic spaces H_1 and H_2 are isomorphic.

The main difference with the theory of real symplectic spaces is that a complex symplectic space need not be even-dimensional, and a maximal isotropic subspace need not be a Lagrangian. This is illustrated by the following example.

Example 1.1.12. Consider the vector space $\mathbb{C}$ together with the skew-Hermitian form ω given by $\omega(z, w) = iz\overline{w}$. Then, $(\mathbb{C}, \omega)$ is a complex symplectic space. The only isotropic subspace is the trivial subspace 0, hence it is maximal isotropic. However, its orthogonal is the whole space $\mathbb{C}$ and hence it is not a Lagrangian.

In order to get a theory resembling more the theory of real symplectic spaces, we have to add an additional assumption. It is immediate to check that, as the symplectic form ω is skew-Hermitian, the sesquilinear form $i\omega$ is Hermitian. This leads us to the following definition.

Definition 1.1.13. A complex symplectic space (H, ω) is called *balanced* if the Hermitian form $i\omega$ has signature 0.

From Lemma 1.1.5, we see that the signature of a non-degenerate Hermitian form has the same parity of the dimension of the space. As a consequence, balanced complex symplectic spaces are are always even-dimensional. Observe now that, if (V, ψ) is a real symplectic space (i.e. V is a real vector space and ψ is a non-degenerate skew-symmetric form), then $(V \otimes \mathbb{C}, \psi_{\mathbb{C}})$ is a complex symplectic space, called the *complexification* of (V, ψ). The following result characterizes balanced complex symplectic spaces.

Proposition 1.1.14. *Let H be a complex symplectic space. Then, the following conditions are equivalent:*

(i) H is balanced;

(ii) every maximal isotropic subspace of H is Lagrangian;

(iii) there exists a Lagrangian subspace of H;

(iv) H is isomorphic to the complexification of a real symplectic space.

Proof. Let ω be the symplectic form. We show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$, and then that (i) is equivalent to (iv). Suppose hence that H is balanced, and let L be a maximal isotropic subspace. We need to prove that $L = L^\omega$. Thanks to Lemma 1.1.5 we can write $H = V_+ \oplus V_-$, with V_+ a positive subspace and V_- a negative subspace for the Hermitian form $i\omega$, orthogonal one to each other. As H is balanced, there is a natural number n such that

$$\dim H = 2n, \qquad \dim V_+ = \dim V_- = n.$$

Thanks to Lemma 1.1.3 we also have $\dim L + \dim L^\omega = 2n$. As $L \subseteq L^\omega$ by assumption, it is enough to show that $\dim L = \dim L^\omega = n$. Suppose by contradiction that $\dim L^\omega > n$. Then, there are non-zero vectors

$$v_+ \in L^\omega \cap V_+, \qquad v_- \in L^\omega \cap V_-,$$

that we can choose in such a way that $i\,\omega(v_+, v_+) = 1$ and $i\,\omega(v_-, v_-) = -1$. Clearly v_+ and v_- do not belong to L, as L is isotropic for $\imath\,\omega$. Then at least one between $v_+ + v_-$ and $v_+ - v_-$ does not belong to L. Call this vector v. Then v is isotropic, as $\omega(v, v) = \omega(v_+, v_+) + \omega(v_-, v_-) = -i + i = 0$, and it is orthogonal to L. Hence, the subspace $L \oplus \mathbb{C}v$ is also isotropic, contradicting the maximality of L.

We have thus proved that (i) implies (ii). It is immediate that (ii) implies (iii), as 0 is an isotropic subspace and it is clearly contained in some maximal isotropic subspace, that turns out to be a Lagrangian. From Proposition 1.1.9, it follows that (iii) implies (i), as a Lagrangian for ω is also a Lagrangian for $i\,\omega$.

Let us prove the equivalence between (i) and (iv). We start by supposing that H is isomorphic to the complexification of a real symplectic space. It follows that there is a basis $\{v_1, \dots v_n, w_1 \dots w_n\}$ of H such that $\omega(v_j, v_k) = \omega(w_j, w_k) = 0$ and $\omega(v_j, w_k) = -\delta_{jk}$ for all i, j. Then, the subspace generated by $v_1, \dots, v_n$ is a Lagrangian for the Hermitian form $i\omega$, so that the signature of $i\omega$ is 0 by Proposition 1.1.9 and hence H is balanced. Suppose now that H is balanced. We can diagonalize the Hermitian form $i\omega$ and find a basis $\{v_1, \dots v_n, w_1, \cdots, w_n\}$ of H such that, for all i, j,

$$i\,\omega(v_j, w_k) = 0, \quad i\,\omega(v_j, v_k) = \delta_{jk}, \quad i\,\omega(w_j, w_k) = -\delta_{jk},$$

the v_j's and the w_j's being in equal number because $\operatorname{sign}(i\omega) = 0$. We define now a linear map $f\colon \mathbb{C}^{2n} \to H$ by setting, for $j = 1, \dots, n$,

$$f(e_j) := v_j + iw_j, \quad f(e_{n+j}) := iv_j + w_j,$$

where $\{e_i, \dots, e_{2n}\}$ is the standard basis of $\mathbb{C}^{2n}$. Then f is a symplectic isomorphism if we provide $\mathbb{C}^{2n}$ with the symplectic form associated to the matrix $A = \left(\begin{smallmatrix} 0 & 2\,\mathrm{Id} \\ -2\,\mathrm{Id} & 0 \end{smallmatrix}\right)$. Since the A has real coefficients, the complex symplectic space $(\mathbb{C}^{2n}, A)$ is clearly isomorphic to the complexification of the real symplectic space $(\mathbb{R}^{2n}, A)$. □

1.1.4 Hermitian symplectic spaces

Following Kirk and Lesch [28], we define a *Hermitian symplectic space* as a triple $(H, \langle \cdot, \cdot \rangle, \gamma)$, where H is a finite-dimensional complex vector space, $\langle \cdot, \cdot \rangle$ is an inner product on H (i.e. a positive-definite Hermitian form), and $\gamma\colon H \mapsto H$ is a unitary operator such that $\gamma^2 = -\,\mathrm{id}$, called the *symplectic operator*. A Hermitian symplectic space $(H, \langle \cdot, \cdot \rangle, \gamma)$ has a natural underlying structure of a complex symplectic space, with the symplectic form ω given by

$$\omega(a, b) \ = \langle a, \gamma(b) \rangle.$$

The sesquilinear form ω is non-degenerate as γ is an isomorphism, and it is skew-Hermitian as

$$\omega(b, a) = \langle b, \gamma(a) \rangle = \langle \gamma(b), \gamma^2(a) \rangle = -\langle \gamma(b), a \rangle = -\overline{\langle a, \gamma(b) \rangle} = -\overline{\omega(a, b)}.$$

Given a Hermitian symplectic space $(H, \langle \cdot, \cdot \rangle, \gamma)$, we define its *opposite* as the Hermitian symplectic space $(H, \langle \cdot, \cdot \rangle, -\gamma)$. Whenever the symplectic operator γ is assumed implicitly, we use the notation H^- to refer to this space. Given two Hermitian symplectic spaces $(H_1, \langle \cdot, \cdot \rangle_1, \gamma_1), (H_2, \langle \cdot, \cdot \rangle_2, \gamma_2)$, their *direct sum* is defined as the Hermitian symplectic space $(H_1 \oplus H_2, \langle \cdot, \cdot \rangle_1 \oplus \langle \cdot, \cdot \rangle_2, \gamma_1 \oplus \gamma_2)$. A linear isomorphism $f\colon H_1 \to H_2$ is called an *isomorphism of Hermitian symplectic spaces* if it is unitary with respect to the inner products and it satisfies $f \circ \gamma_1 = \gamma_2 \circ f$. It is immediate to check that all of these notions correspond to those given in Section 1.1.3 if we restrict our attention to the underlying complex symplectic spaces.

Let $(H, \langle \cdot, \cdot \rangle, \gamma)$ be a Hermitian symplectic space. As γ is unitary and it satisfies the equation $\gamma^2 = -\,\mathrm{id}$, it is diagonalizable with eigenvalues $\pm i$. Let $E_+(\gamma)$ denote its i-eigenspace and $E_-(\gamma)$ denote its $(-i)$-eigenspace. We have then an orthogonal decomposition $H = E_+ \oplus E_-$. Properties about the symplectic structure of a Hermitian symplectic space can be easily reformulated in terms of the symplectic operator γ and orthogonality with respect to the Hermitian product.

Lemma 1.1.15. *Let $(H, \langle \cdot, \cdot \rangle, \gamma)$ be a Hermitian symplectic space. Then* *(i) a subspace V is isotropic* $\iff$ $\gamma(V) \subseteq V^\perp$*;*

(ii) a subspace V is Lagrangian $\iff$ $\gamma(V) = V^\perp$*;*

(iii) H is balanced $\iff$ $\dim E_+(\gamma) = \dim E_-(\gamma)$.

Proof. The first two points are immediate. For (iii), fix an orthonormal basis of H. Let A be the matrix representing the Hermitian form $i\omega$ (defined by $i\,\omega(a,b,) = i\langle a, \gamma(b)\rangle$) and let B the matrix representing γ. We have then $B = i\overline{A}$, so that $A^2 = 1$, and every occurrence of the eigenvalue 1 of A corresponds to an occurrence of the eigenvalue i of B, and the same for -1 with $-i$. By Proposition 1.1.11, the signature of $i\omega$ coincide hence with the difference between the dimensions of $E_+(\gamma)$ and $E_-(\gamma)$. □

1.2 The Maslov triple index

In this section, we discuss the Maslov triple index, which is a well known function associating an integer to a triple of Lagrangians of a symplectic space. In Section 1.2.1, we review the definition of this function. In Section 1.2.2, we give a simple formula for the function in the 2-dimensional case. In Section 1.2.3, we discuss a function defined for pairs of Lagrangians in a Hermitian symplectic space that is related to the Maslov triple index.

1.2.1 Definition and first properties

Let (H,ω) be a balanced complex symplectic space, and let $\mathcal{L}ag(H)$ denote the set of its Lagrangian subspaces. Our goal is to define a function

$$\tau_H \colon \mathcal{L}ag(H) \times \mathcal{L}ag(H) \times \mathcal{L}ag(H) \to \mathbb{Z}.$$

Given three Lagrangian subspaces $L_1, L_2, L_3 \in \mathcal{L}ag(H)$, we first introduce a sesquilinear form by

$$\begin{aligned} \psi_{L_1L_2L_3} \colon (L_1 + L_2) \cap L_3 \times (L_1 + L_2) \cap L_3 &\to \mathbb{C} \\ (a,b) &\mapsto \omega(a_1, b), \end{aligned}$$

where a_1 is any element in L_1 such that $a - a_1 \in L_2$.

Lemma 1.2.1. *The sesquilinear form $\psi_{L_1L_2L_3}$ is well defined and Hermitian.*

Proof. Given a different element $a_1' \in L_1$ such that $a - a_1'$ lies in L_2, the difference $a_1' - a_1$ lies in $L_1 \cap L_2$. Thus, for all $b \in L_1 + L_2$, we have

$$\omega(a_1', b) - \omega(a_1, b) = \omega(a_1' - a_1, b) = 0,$$

and hence $\psi_{L_1 L_2 L_3}(a, b)$ does not depend on this choice. We prove now that the form is Hermitian. Chosen $b_1 \in L_1$ such that $b - b_1$ lies in L_2 and using the fact all three subspaces are isotropic, we get

$$\begin{aligned}\psi_{L_1 L_2 L_3}(b, a) &= \omega(b_1, a) = \omega(b, a) - \omega(b - b_1, a) = 0 - \omega(b - b_1, a) = \\ &= -\omega(b - b_1, a - a_1) - \omega(b - b_1, a_1) = 0 - \omega(b - b_1, a_1) = \\ &= -\omega(b, a_1) + \omega(b_1, a_1) = -\omega(b, a_1) + 0 = \overline{\omega(a_1, b)} = \\ &= \overline{\psi_{L_1 L_2 L_3}(a, b)}.\end{aligned}$$

□

We are now ready to define the function of our interest.

Definition 1.2.2. The *Maslov triple index* [1] of (L_1, L_2, L_3) is the integer

$$\tau_H(U, V, W) := \operatorname{sign} \psi_{L_1 L_2 L_3}.$$

The Maslov index satisfies the following properties.

Proposition 1.2.3. (Properties of the Maslov triple index)

(i) *Let* $L_1, L_2, L_3 \in \mathcal{L}ag(H)$, *and let* α *be a permutation of the set* $\{1, 2, 3\}$. *Then*

$$\tau_H(L_{\alpha(1)}, L_{\alpha(2)}, L_{\alpha(3)}) = \operatorname{sgn}(\alpha)\, \tau_H(L_1, L_2, L_3).$$

In particular, $\tau_H(L_1, L_2, L_3) = 0$ *if two of the Lagrangians coincide.*

(ii) *Let* $L_1, L_2, L_3 \in \mathcal{L}ag(H)$ *and* $L_1', L_2', L_3' \in \mathcal{L}ag(H')$. *Then*

$$\tau_{H \oplus H'}(L_1 \oplus L_1', L_2 \oplus L_2', L_3 \oplus L_3') = \tau_H(L_1, L_2, L_3) + \tau_{H'}(L_1', L_2', L_3').$$

(iii) *Let* $g\colon H \to H'$ *be a symplectic isomorphism, and* $L_1, L_2, L_3 \in \mathcal{L}ag(H)$. *Then*

$$\tau_{H'}(g(L_1), g(L_2), g(L_3)) = \tau_H(L_1, L_2, L_3).$$

[1] Also known as the *Hörmander-Kashiwara index.*

(iv) *On the complex symplectic space $\mathbb{C}^2$ with the symplectic form represented by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we have,*

$$\tau_{\mathbb{C}^2}(\mathbb{C}(1,0), \mathbb{C}(1,1), \mathbb{C}(0,1)) = 1.$$

(v) *Let $L_1, L_2, L_3 \in \mathcal{L}ag(H)$. Then $\tau_{H^-}(L_1, L_2, L_3) = -\tau_H(L_1, L_2, L_3)$.*

(vi) *Let $L_1, L_2, L_3, L_4 \in \mathcal{L}ag(H)$. Then*

$$\tau_H(L_1, L_2, L_3) - \tau_H(L_1, L_2, L_4) + \tau_H(L_1, L_3, L_4) - \tau_H(L_2, L_3, L_4) = 0.$$

Proof. It is enough to prove (i) for two elements that generate the group S_3. The case of $\alpha = (12)$ is easy, as the subspace of H where the Hermitian forms $\psi_{L_1L_2L_3}$ and $\psi_{L_2L_1L_3}$ are defined is the same. Given $a, b \in (L_1 + L_2) \cap L_3$, with $a = a_1 + a_2$ for elements $a_1 \in L_1$ and $a_2 \in L_2$, we see that

$$\omega(a_2, b) = \omega(a, b) - \omega(a_1, b) = -\omega(a_1, b)$$

(because L_3 is Lagrangian), and hence we have $\psi_{L_2L_1L_3} = -\psi_{L_1L_2L_3}$ and thus $\tau_H(L_2, L_1, L_3) = -\tau_H(L_1, L_2, L_3)$ as desired. Let us now prove the statement for $\alpha = (123)$. Consider the map

$$f\colon \frac{(L_1 + L_2) \cap L_3}{L_1 \cap L_3} \to \frac{(L_3 + L_1) \cap L_2}{L_1 \cap L_2}$$

sending the class of $a = a_1 + a_2 \in (L_1 + L_2) \cap L_3$ (with $a_1 \in L_1, a_2 \in L_2$) to the class of a_2. It easy to show that f is well-defined and it is an isomorphism. As $L_1 \cap L_3$ is contained in $\operatorname{rad} \psi_{L_1L_2L_3}$ and $L_1 \cap L_2$ is contained in $\operatorname{rad} \psi_{L3L_1L_2}$, these forms descend to forms $\psi'_{L_1L_2L_3}$ and $\psi'_{L3L_1L_2}$ on the quotient spaces, having the same signatures. Moreover, for elements $a = a_1 + a_2$ and $b = b_1 + b_2$ of $(L_1 + L_2) \cap L_3$, we have

$$\psi_{L3L_1L_2}(a_2, b_2) = \omega(a, b_2) = \omega(a_1, b) = \psi_{L_1L_2L_3}(a, b),$$

so that

$$\psi'_{L3L_1L_2}(f(a + L_1 \cap L_3), f(b + L_1 \cap L_3)) = \psi'_{L_1L_2L_3}(a + L_1 \cap L_3, b + L_1 \cap L_3).$$

As a consequence, we have $\operatorname{sign}(\psi'_{L3L_1L_2}) = \operatorname{sign}(\psi'_{L1L_2L_3})$, and it follows that $\tau_H(L_3, L_1, L_2) = \tau_H(L_1, L_2, L_3)$ as desired.

Properties (ii), (iii) and (iv) follow immediately from the properties of the signature (see Proposition 1.1.7). For proving (iv), we observe that the relevant Hermitian form $\psi\colon \mathbb{C}(0,1)\times\mathbb{C}(0,1)\to\mathbb{C}$ is defined on the basis element $(0,1)$ as

$$\psi((0,1),(0,1)) = \omega((-1,0),(0,1)) = 1.$$

In particular, ψ is positive definite and it has hence signature 1. For the proof of (vi), see Remark 1.2.16 or the book of Turaev [45, Chapter IV, 3.6]. □

The Maslov triple index was studied in depth by Cappell, Lee and Miller. Working with real symplectic spaces, they proved in particular that the first four properties above characterize the Maslov triple index [9, Theorem 8.1]. Their proof can be adapted without any formal change to the framework of balanced complex symplectic spaces, yielding the following.

Theorem 1.2.4 (Cappell-Lee-Miller)**.** *If a family of functions*

$$\widetilde{\tau}_H\colon \mathcal{L}ag(H)\times\mathcal{L}ag(H)\times\mathcal{L}ag(H)\to\mathbb{R}$$

satisfies the properties (i), (ii), (iii) and (iv) of Proposition 1.2.3, then it coincides with the Maslov triple index.

Warning 1.2.5. Given our definition of the Maslov index in the complex case, it is possible to get a real version of it by

$$\tau_H^{\mathbb{R}}(L_1,L_2,L_3) := \tau_{H\otimes\mathbb{C}}(L_1\otimes\mathbb{C}, L_2\otimes\mathbb{C}, L_3\otimes\mathbb{C}).$$

This function corresponds to the triple index of Wall [47], but it differs from the function τ of Cappell, Lee and Miller by a sign. In fact, it takes the value -1 on their standard triple considered in Property IV [9, p. 163], as they take $\mathbb{R}^2$ with the symplectic structure corresponding to the matrix $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$, which is opposite to our convention.

We conclude the section with the following result, that will turn out useful later on.

Lemma 1.2.6. *Let H be a complex symplectic space and let L_1, L_2 be two Lagrangian subspaces with $L_1\cap L_2 = 0$. Let U,V,W three Lagrangian subspaces that decompose as $U = U_1+U_2$, $V = V_1+V_2$, $W = W_1+W_2$, with $U_i,V_i,W_i\subseteq L_i$ for $i=1,2$. Then $\tau_H(U,V,W)=0$.*

Proof. It is enough to prove the result in the special case $W = L_1$. The full statement follows then from the properties of the Maslov triple index

(Proposition 1.2.3 (i) and (vi)). By definition, $\tau_H(U,V,L_1)$ is the signature of the Hermitian form

$$\begin{aligned}\psi\colon (U+V)\cap L_1 \times (U+V)\cap L_1 &\to \mathbb{C}\\ (u+v,x') &\mapsto \omega(u,y) \qquad \text{if } u\in U,\ v\in V.\end{aligned}$$

Since both U and V split with respect to the direct sum decomposition L_1+L_2, we have $(U+V)\cap L_1 = U_1+V_1$. In particular, in the definition of ψ, we can take the elements u and v to belong to U_1 and V_1 respectively. It is then clear that $\omega(u,y)=0$, as both u and y belong to L_1, which is a Lagrangian subspace. Hence, ψ is the trivial Hermitian form and its signature is 0. □

1.2.2 Computations in the symplectic plane

Suppose now that (H,ω) is a balanced complex symplectic space of dimension 2.

Definition 1.2.7. An ordered basis (μ,λ) of H is called *symplectic* if the form ω is represented in this basis by the matrix $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, i.e. if

$$\omega(\mu,\mu)=\omega(\lambda,\lambda)=0, \quad \omega(\mu,\lambda)=-1.$$

Let (μ,λ) be a symplectic basis of H. Given a 1-dimensional subspace L of H, generated by a vector $a\mu+b\lambda$, we define the *slope* of L with respect to (μ,λ) as the number

$$\operatorname{slope}(L) := a/b \in \mathbb{C}\cup\{\infty\}.$$

The slope is indeed well defined, and it gives a bijection between the set of lines in H and $\mathbb{C}\cup\{\infty\}$. It is immediate to check that a vector $a\mu+b\lambda$ is isotropic if and only if $a\bar{b}$ is real. As a consequence, the Lagrangian subspaces of H are exactly the lines with slope in $\mathbb{R}\cup\{\infty\}$. For such subspaces, we can always find a generator $a\mu+b\lambda$ with $a,b\in\mathbb{R}$. Thanks to Proposition 1.2.3 (iii), the Maslov index of three Lagrangian subspaces of H only depends on their slopes, i.e. there is a function

$$\tau\colon (\mathbb{R}\cup\{\infty\})\times(\mathbb{R}\cup\{\infty\})\times(\mathbb{R}\cup\{\infty\}) \to \{-1,0,1\}$$

such that

$$\tau_H(L_1,L_2,L_3)=\tau(s_1,s_2,s_3)$$

whenever L_1,L_2,L_3 are three Lagrangian subspace of a balanced 2-dimensional complex symplectic space, with $\operatorname{slope}(L_i)=s_i$ for $i=1,2,3$ (see Degtyarev, Florens and Lecuona [20, Section 2] for more details). The function τ is then computed in the following way [20, Corollary 2.2].

Proposition 1.2.8. *A complete description of the function τ is:*

$$\tau(s_1, s_2, s_3) = \begin{cases} \operatorname{sgn}((s_2 - s_1)(s_3 - s_2)(s_1 - s_3)) & \text{if } s_1, s_2, s_3 \neq \infty \\ \operatorname{sgn}(s_2 - s_3) & \text{if } s_1 = \infty,\ s_2, s_3 \neq \infty \\ \operatorname{sgn}(s_3 - s_1) & \text{if } s_2 = \infty,\ s_1, s_3 \neq \infty \\ \operatorname{sgn}(s_1 - s_2) & \text{if } s_3 = \infty,\ s_1, s_2 \neq \infty \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.2.9. Proposition 1.2.8 allows us to compute the Maslov index explicitly for all triples of Lagrangian subspaces in a 2-dimensional H once we know their slopes. In practice, a convenient method for quick computations is to draw the subspaces as real lines L_1, L_2, L_3 in $\operatorname{Span}_{\mathbb{R}}\{\mu, \lambda\}$ and counting:

- $+1$ if the three lines are distinct and, rotating L_1 counterclockwise, L_2 is met before L_3;
- -1 if the three lines are distinct and, rotating L_1 counterclockwise, L_3 is met before L_2;
- 0 if at least two lines coincide.

1.2.3 The function m of pairs of Lagrangians

Suppose now that (H, γ) is a balanced Hermitian symplectic space. Using this additional structure, we introduce a function

$$m_H \colon \mathcal{L}ag(H) \times \mathcal{L}ag(H) \to \mathbb{R},$$

that was studied extensively by Bunke [7] and then by Kirk and Lesch [29, 28] in relation to eta invariants for manifolds with boundary, after appearing in a formula of Lesch and Wojciechowski [30].

As we have seen, there is an orthogonal decomposition $H = E_+ \oplus E_-$, where $E_\pm$ denotes the $\pm i$-eigenspace of γ, and $\dim E_+ = \dim E_-$ as the space is balanced. The next result shows that the Lagrangians subspaces of H are in a natural bijection with the set $U(E_+, E_-)$ of isometries $E_+ \to E_-$.

Lemma 1.2.10. *There is a bijection $L \colon U(E_+, E_-) \to \mathcal{L}ag(H)$ given by*

$$L(A) := \{x + Ax \mid x \in E_+\}.$$

Proof. First of all, we show that $L(A)$ is a Lagrangian subspace, i.e. that $\gamma(L(A)) = L(A)^{\perp}$. The inclusion $\gamma(L(A)) \subseteq L(A)^{\perp}$ follows from the fact that, for all $x, y \in E_+$, we have

$$\langle x + Ax, \gamma(y + Ay)\rangle = \langle x + Ax, iy - iAy\rangle = \langle x, iy\rangle - \langle Ax, Aiy\rangle = 0.$$

As $L(A)$ is half-dimensional, thus, it is Lagrangian. As consequence, the map L is well defined, and clearly it is injective.

To show surjectivity, we observe that every Lagrangian subspace $V \in \mathcal{L}ag(H)$ has trivial intersection with both E_+ and E_-, and hence there is a unique linear map $A\colon E_+ \to E_-$ such that $V = \{x + Ax \,|\, x \in E_+\}$. In order to conclude the proof, it remains to be shown that A is unitary, and this follows from the fact that, given $x, y \in E_+$, we can write

$$\langle Ax, Ay\rangle = i\langle x + Ax, iy - iAy\rangle + \langle x, y\rangle = i\langle x + Ax, \gamma(y + Ay)\rangle + \langle x, y\rangle$$

and the first summand vanishes as V is Lagrangian. □

Let $\Phi\colon \mathcal{L}ag(H) \to U(E_+, E_-)$ denote the inverse of L, so that every Lagrangian subspace V gets written as

$$V = \{x + \Phi(V)(x) \,|\, x \in E_+\}.$$

Moreover, given two Lagranians $V, W \in \mathcal{L}ag(H)$, we set

$$\Phi_{V,W} := \Phi(V)\Phi(W)^{-1} \in U(E_-).$$

Definition 1.2.11. We define now the function of our interest as

$$m_H(V, W) := -\frac{1}{\pi} \sum_{\substack{e^{i\lambda} \in \mathrm{Spec}(-\Phi_{V,W}) \\ \lambda \in (-\pi, \pi)}} \lambda.$$

Remark 1.2.12. In the definition of $m_H(V, W)$, the sum ranges along all eigenvalues of $-\Phi_{V,W}$ that are different from -1. The -1-eigenspace, corresponding to the intersection $V \cap W$, is expressly not counted.

The following properties of the function m are proved immediately using the definition.

Proposition 1.2.13. *Let H, H' Hermitian symplectic spaces. Then, for Lagrangian subspaces $V, W \in \mathcal{L}ag(H)$ and $V', W' \in \mathcal{L}ag(H')$, we have:*

(i) $m_H(V, W) = -m_H(W, V)$ *(hence,* $m_H(V, V) = 0$*);*

(ii) $m_{H\oplus H'}(V\oplus V', W\oplus W') = m_H(V,W) + m_{H'}(V',W')$*;*

(iii) If $g\colon H \to H'$ *is an isometry such that* $g\gamma_1 = \gamma_2 g$*, then* $m_{H'}(g(V), g(W)) = m_H(V,W)$*;*

(iv) $m_{H^-}(V,W) = -m_H(V,W)$*;*

Remark 1.2.14. Combining (i) and (iii), for $m = m_H$ we get the useful formulas

$$m(\gamma(V), \gamma(W)) = m(V,W), \quad m(\gamma(V), W) = m(V, \gamma(W)), \quad m(V, \gamma(V)) = 0.$$

Since every Hermitian symplectic space is also a complex symplectic space, given three Lagrangian subspaces V, W, Z it is possible to define their Maslov index $\tau_H(V,W,Z)$, which is an integer and does not depend on the Hermitian structure. The following result relates the function m_H to the Maslov triple index τ_H. We give a proof which is due to Kirk and Lesch [28, Section 8.3],

Proposition 1.2.15. *Let* U*,* V *and* W *any three Lagrangian subspaces of a balanced Hermitian symplectic space* H*. Then,*

$$m_H(U,V) + m_H(V,W) + m_H(W,U) = \tau_H(U,V,W).$$

Proof. For any complex symplectic space H, pick a Hermitian symplectic structure and define $\widetilde{\tau}_H(U,V,W) := m_H(U,V) + m_H(V,W) + m_H(W,U)$. As proved by Kirk and Lesch, $\widetilde{\tau}_H$ satisfies the first four properties of Proposition 1.2.3 [29, Proposition 8.19] [2]. The result follows then by Theorem 1.2.4. □

Remark 1.2.16. Proposition 1.2.15 gives an easy way to prove the cocycle property of the Maslov index (Proposition 1.2.3 (vi)). In fact, once a Hermitian symplectic structure on H is fixed, we can write (for $\tau = \tau_H$ and $m = m_H$)

$$\begin{aligned}
&\tau(L_1,L_2,L_3) - \tau(L_1,L_2,L_4) = \\
&= m(L_1,L_2) + m(L_2,L_3) + m(L_3,L_1) - m(L_1,L_2) - m(L_2,L_4) - m(L_4,L_1) = \\
&= m(L_2,L_3) + m(L_3,L_1) - m(L_2,L_4) - m(L_4,L_1) = \\
&= -m(L_1,L_3) - m(L_3,L_4) - m(L_4,L_1) + m(L_2,L_3) + m(L_3,L_4) + m(L_4,L_2) = \\
&= -\tau(L_1,L_3,L_4) + \tau(L_2,L_3,L_4).
\end{aligned}$$

[2] Thanks to (iii), $\widetilde{\tau}_H$ is independent of the specific Hermitian structure inducing the given symplectic one, and hence for (iv) it is enough to check the result for $\mathbb{C}^2$ with the standard Hermitian product and the operator γ corresponding to the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, as they do.

1.3 Homology and cohomology with local coefficients

In this section, we discuss various definitions of twisted homology and cohomology. In the Section 1.3.1, we take the point of view of local coefficient systems, which will be the preferred one in most abstract developments. In Section 1.3.2, we see how under certain circumstances it is possible to view a local coefficient system on a topological space as a representation of the fundamental group of the space, and we use this fact to define twisted homology and cohomology more explicitly. In Section 1.3.3, we quickly discuss the de Rham version of this theory, available for smooth manifolds.

1.3.1 Local coefficient systems and homology theory

Let X and F be topological spaces. By a *fiber bundle* on X with *typical fiber* F, we mean a topological space E together with a surjective map $\pi\colon E \to X$ such that there exists an open cover $\{U_j\}$ of X provided with homeomorphisms φ_j fitting into commutative diagrams

$$\begin{array}{ccc} \pi^{-1}(U_j) & \xrightarrow{\varphi_j} & U_j \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & U_j. & \end{array}$$

The family $\{(U_\alpha, \varphi_\alpha)\}$ is called a *local trivialization* for the fiber bundle. For all $x \in X$, the space $\xi_x := \pi^{-1}(x)$ is called the *fiber over* x. It is homeomorphic to the typical fiber, but not in a canonical way. For every open set U_α belonging to a local trivialization and such that $x \in U_\alpha$, we have a homeomorphism $\varphi_\alpha^x\colon \xi_x \to F$ obtained by restricting φ_x to ξ_x.

Notation 1.3.1. Let $\mathbb{C}^n_{\mathrm{dis}}$ denote the vector space $\mathbb{C}^n$ equipped with the discrete topology.

Definition 1.3.2. A *local coefficient system* of complex vector spaces on X of dimension n is a fiber bundle $\xi \to X$ with typical fiber $\mathbb{C}^n_{\mathrm{dis}}$ and the structure of a complex vector space on every fiber, which admits a local trivialization $\{(U_\alpha, \varphi_\alpha)\}$ such that every homeomorphism $\varphi_\alpha^x\colon \xi_x \to \mathbb{C}^n_{\mathrm{dis}}$ is a linear isomorphism.

Notation 1.3.3. We let $\mathcal{L}_n(X)$ denote the class of all local coefficient systems of complex vector spaces of dimension n over X.

Example 1.3.4. The product $\tau_n = X \times \mathbb{C}^n_{\mathrm{dis}}$ together with the natural projection $X \times \mathbb{C}^n_{\mathrm{dis}} \to X$ is a local coefficient system of complex vector spaces on X, called the *trivial local coefficient system of dimension* n.

A local coefficient system of complex vector spaces over X can be used for defining graded complex vector spaces $H_*(X;\mathbb{C}^n_\xi)$, $H^*(X;\mathbb{C}^n_\xi)$, forming the *homology* and *cohomology with (local) coefficients in* ξ [26, Chapter 3.H] [18, Chapter 5]. If ξ is the product bundle $X \times \mathbb{C}^n$ (with the discrete topology on $\mathbb{C}^n$), (co)homology with local coefficients in L is naturally isomorphic to the ordinary (co)homology with values in $\mathbb{C}^n$.

Homology and cohomology with local coefficients preserve most of the formal properties of the ordinary homology and cohomology. Given a continuous map $f\colon X' \to X$, the pull-back bundle $\xi' := f^*\xi$ is a local coefficient system of complex vector spaces over Y, and there are induced maps

$$f_*\colon H_*(X';\mathbb{C}^n_{\xi'}) \to H_*(X;\mathbb{C}^n_\xi), \quad f^*\colon H^*(X;\mathbb{C}^n_\xi) \to H^*(X';\mathbb{C}^n_{\xi'})$$

that behave functorially with respect to composition. Relative homology and cohomology can also be defined, and they fit in long exact sequences generalizing the one for ordinary homology and cohomology. If M is a compact oriented manifold of dimension m, there are Poincaré duality isomorphisms

$$H^k(M,\partial M;\mathbb{C}^n_\alpha) \xrightarrow{\sim} H_{m-k}(M;\mathbb{C}^n_\alpha)$$
$$H^k(M;\mathbb{C}^n_\alpha) \xrightarrow{\sim} H_{m-k}(M,\partial M;\mathbb{C}^n_\alpha)$$

arising from cap products generalizing those on homology with traditional coefficients.

We define now a local coefficient system $\xi\colon \mathcal{L}_n(X)$ to be *Hermitian* if it is equipped with a scalar product on the fibers that varies continuously.

Notation 1.3.5. The class of Hermitian coefficient system will be denoted by $\mathcal{U}_n(X)$.

If ξ is Hermitian, the scalar product induces a bundle map $\xi \otimes \overline{\xi} \to \varepsilon$, where $\overline{\xi}$ denotes the local cofficient system obtained from ξ by taking the opposite complex structure on all fibers, and ε is the trivial 1-dimensional local coefficient system on X. As a consequence, given subspaces $A, B \subseteq X$ that form an excisive pair, there is a bilinear cup product

$$\smile_\xi\colon H^j(X,A;\mathbb{C}^n_\xi) \times H^l(X,B;\mathbb{C}^n_{\overline{\xi}}) \to H^{j+l}(X,A\cup B;\mathbb{C}).$$

Using the Hermitian metric on the fibers, we can also define an evaluation map

$$\langle\cdot,\cdot\rangle_\xi\colon H^j(X,A;\mathbb{C}^n_{\overline{\xi}}) \times H_j(X,A;\mathbb{C}^n_\xi) \to \mathbb{C}$$

that gives a natural isomorphism $H^j(X,A;\mathbb{C}^n_\xi) \cong \mathrm{Hom}(H_j(X,A;\mathbb{C}^n_\xi),\mathbb{C})$. These facts are well known to the experts. Some details about them, albeit treated in higher generality and from the point of view of Section 1.3.2, can be found in the thesis of Conway [15, Chapter 5].

1.3.2 Representations of the fundamental group

In practice, local coefficient systems arise most of the times in the following way. Suppose that X is path connected, locally path connected and locally simply connected (we assume the latter two conditions from now on), and that a base point $x_0 \in X$ is fixed. Suppose that

$$\alpha\colon \pi_1(X,x_0) \to GL(V)$$

is a representation of the fundamental group $\pi := \pi(X,x_0)$ of X into a complex vector space V. Then, we can can construct a local coefficient system ξ^α associated to α by seeing the universal cover $\widetilde{X}$ as a principal π-bundle and taking the associated fiber bundle

$$\xi^\alpha := \widetilde{X} \times_\pi V.$$

The above construction can be reversed. Starting from a local coefficient system $\xi \in \mathcal{L}(X,n)$, we observe that the projection $\xi \to X$ is a covering space. In particular, given an element $a \in \pi_1(X,x_0)$, that we see as a loop $a\colon [0,1] \to X$, for each element v of ξ_{x_0} there is a unique lift $\widetilde{a}\colon [0,1] \to \xi$ of a such that $\widetilde{a}(0) = v$. We can hence define a map

$$\alpha_\xi\colon \pi(X,x_0) \to \mathrm{GL}(\xi_{x_0})$$

, by $\alpha_\xi(a)(v) := \widetilde{a}(1)$, which can be easily verified to be a representation into the fiber over x_0. The correspondences $\alpha \mapsto \xi^\alpha$ and $\xi \to \alpha_\xi$ are inverses one of each other in the sense that there are canonical isomorphisms between the representations α_{ξ^α} and α, and between the fiber bundles ξ^{α_ξ} and ξ.

Remark 1.3.6. Because of the dependence on base points, working with representations of the fundamental group can be cumbersome in some cases (for example for dealing with induced maps and with non-connected spaces), making implicit choices and some abuse of notation often necessary in order to keep the proofs readable. This is the reason why in the theory we prefer to adopt the point of view of local coefficient systems instead. However, whenever explicit descriptions of our local coefficients systems are needed in the applications, we shall normally see them as representations, taking advantage on the correspondence described above.

Representations of the fundamental group lead to the following alternative view on homology and cohomology with local coefficients. Consider the natural left action of the fundamental group $\pi_1(X, x_0)$ on the universal cover $\widetilde{X}$ of X. We transform this left action into a right action by replacing the action of an element by that of its inverse. The resulting action induces the structure of right $\mathbb{Z}[\pi]$-modules on the singular complex $C_*(\widetilde{X})$. Given a representation $\alpha\colon \pi_1(X, x_0) \to \mathrm{GL}(V)$ into a complex vector space of dimension n, we have then natural isomorphisms [18, Theorem 5.8 and 5.9]

$$H_k(X; \mathbb{C}^n_{\xi^\alpha}) \cong H_k(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} V)$$

$$H^k(X; \mathbb{C}^n_{\xi^\alpha}) \cong H^k(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X})^t, V)),$$

where the t denotes the fact that we are turning the right $\mathbb{Z}[\pi]$-module structure back into a left one in this instance.

Remark 1.3.7. If the vector space V has a scalar product, and the representation $\alpha\colon \pi_1(X) \to U(V)$ is unitary, then the associated local coefficient system ξ^α has a well-defined induced Hermitian structure, i.e. $\xi^\alpha \in \mathcal{U}_n(X)$. Conversely, if $\xi \in \mathcal{U}_n(X)$ is a Hermitian local coefficient system, then the associated representation on one of the fibers is unitary, i.e. we can write $\alpha_\xi\colon \pi_1(X, x_0) \to U(\xi_{x_0})$.

An elementary property of twisted cohomology is that $H^0(X, \mathbb{C}^n_\alpha)$ is isomorphic to the subspace of $\mathbb{C}^n$ on which α acts trivially [18, Proposition 5.14]. We apply this easy fact in the next computation, that will be useful later on.

Lemma 1.3.8. *Let $X = S^1$ or $X = T^2$, and let $\alpha\colon \pi_1(X) \to U(n)$ be a representation that has no trivial subrepresentation. Then, $H_*(X; \mathbb{C}^n_\alpha) = 0$. In particular, homology vanushes for all non-trivial $\alpha \in \mathcal{U}_1(X)$.*

Proof. Let us prove the result for the circle first. From the above description of H^0, we see that $H^0(S^1; \mathbb{C}^n_\alpha) = 0$. Then, by the Poincaré duality and universal coefficient isomorphisms, we have

$$\dim H_0(S^1; \mathbb{C}^n_\alpha) = \dim H^1(S^1; \mathbb{C}^n_\alpha) = \dim H_1(S^1; \mathbb{C}^n_\alpha) = \dim H^0(S^1; \mathbb{C}^n_\alpha) = 0,$$

and the conclusion follows. In the case of the solid torus, with the same argument we find

$$H_0(T^2; \mathbb{C}^n_\alpha) = H_2(T^2; \mathbb{C}^n_\alpha) = 0$$

As the Euler charachteristic of T^2 is 0, it follow that $H_1(T^2, \mathbb{C}^n_\alpha)$ also has to vanish. □

1.3.3 Flat connections and twisted de Rham cohomology

We sketch here one more point of view of twisted cohomology (for details, see the lecture notes of Bunke [8, Chapter 4]). Let M be a manifold. Given a (smooth) complex vector bundle $E \to M$, we define the space of *differential k-forms with values in E* as

$$\Omega^k(X, E) := \Gamma\left(\bigwedge^k T^*M \otimes E\right).$$

If E is equipped with a *flat connection*, i.e. with vanishing curvature form, we call it a *flat vector bundle*. Associated to a flat vector bundle E, there is a twisted de Rham complex

$$0 \to \Omega^0(X, E) \xrightarrow{d^0} \Omega^1(X, E) \xrightarrow{d^1} \Omega^2(X, E) \xrightarrow{d^2} \cdots,$$

whose cohomology $H^k(M; E) := \ker d^k / \operatorname{im} d^{k-1}$ is called the *twisted de Rham cohomology* of X (with values in E).

Suppose now that $\xi \in \mathcal{L}_n(X)$ is a local coefficient system. Then, there is an associated vector bundle $\pi\colon E_\xi \to X$. To define it we set E_ξ to be equal to ξ as a set, we take π to coincide with the projection $\xi \to X$, and we keep the vector space structure on the fibers. We pick then any local trivialization $(U_\alpha, \varphi_\alpha)$ for ξ, and we equip E_ξ with the only topology that makes all the compositions

$$\pi^{-1}(U_\alpha) \xrightarrow{\varphi_\alpha} U_\alpha \times \mathbb{C}^n_{\text{dis}} \to U_\alpha \times \mathbb{C}^n$$

into diffeomorphisms. Such a topology exists uniquely, it makes E_ξ a vector bundle and it is independent of the choice of the trivialization [43, pp. 12-14]. A local trivialization for E_ξ obtained in this way is such that the transition functions

$$\varphi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C})$$

are locally constant. As a consequence, we can define a flat connection on E_α by pulling back, for each chart U_α, the trivial connection on $U_\alpha \times \mathbb{C}^n$, and patching these together. There is then a *twisted de Rham isomorphism*

$$H^k(M; E_\xi) \cong H^k(M; \mathbb{C}^n_\xi)$$

(for an explanation of this fact, based on sheaf theory, see Ramanan [41, 5.14]).

1.4 Twisted intersection forms and signatures

In this section, we review twisted intersection forms and signatures of manifolds. In Section 1.4.1, we recall the definition of the intersection form with local coefficients. In Section 1.4.2, we see how the pairing in middle degree can be used to define twisted signatures and, for closed manifolds, the structure of a complex symplectic space on twisted cohomology. In Section 1.4.3, we discuss a theorem of Wall about the behavior of the signature under gluing of manifolds along part of their boundary.

1.4.1 The intersection form with local coefficients

Suppose now that M is a compact oriented manifold of dimension m with a Hermitian local coefficient system $\xi \in \mathcal{U}_n(M)$. The conventional intersection pairing on M can be generalized in the following way.

Definition 1.4.1. The *intersection form of M with coefficients in ξ* is the sesquilinear form on $H^*(M, \partial M; \mathbb{C}^n_\xi)$ coming from the pairings

$$I^\xi_M \colon H^j(M, \partial M; \mathbb{C}^n_\xi) \times H^{m-j}(M, \partial M; \mathbb{C}^n_\xi) \to \mathbb{C}$$

defined for $j = 0, \dots, m$ by

$$I^\xi_M(a, b) := \langle a \smile_\xi \overline{b}, [M] \rangle,$$

where the bar denotes the obvious antilinear bijection from $H^*(M, \partial M; \mathbb{C}^n_\xi)$ to $H^*(M, \partial M; \mathbb{C}^n_{\overline{\xi}})$.

For $a \in H^j(M, \partial M; \mathbb{C}^n_\xi)$ is a j-form and $b \in H^{m-j}(M, \partial M; \mathbb{C}^n_\xi)$, from the basic properties of the cup product we get the formula

$$I^\xi_M(b, a) = (-1)^{j(m-j)} \overline{I^\xi_M(a, b)}. \tag{1.3}$$

Thanks to (1.3) the sesquilinear form I^ξ_M on $H^*(M, \partial M; \mathbb{C}^n_\xi)$ is reflexive. The intersection pairing can be described alternatively as follows.

Lemma 1.4.2. *Let $p \colon M \to (M, \partial M)$ be the natural inclusion. Then*

$$I^\xi_M(a, b) = \langle \overline{b}, [M] \frown p^*(a) \rangle_\xi.$$

Proof. It is enough to check that the equality $\langle \psi, \sigma \frown \varphi \rangle_\xi = \langle \varphi \smile_\xi \psi, \sigma \rangle$, generalizing a classical formula relating the cup and cup product, holds in this context. ☐

As a consequence, we have the following description of the radical of the interesection form.

Proposition 1.4.3. *The radical of I_M^ξ coincides with the kernel of the natural map $H^*(M, \partial M; \mathbb{C}^n_\xi) \to H^*(M; \mathbb{C}^n_\xi))$. In particular, if M is closed, then I_M^ξ is non-degenerate.*

Proof. In Section 1.3.1, we have seen that cap product with the fundamental class gives rise to an isomorphism by Poincaré duality, and that the evaluation pairing between homology and cohomology is non-degenerate. The statement now follows immediately from Lemma 1.4.2. □

Given a compact, oriented manifold with boundary X with a local coefficient system $\xi \in \mathcal{U}(X)$, we are now going to define a natural Lagrangian subspace of $H^*(\partial X; \mathbb{C}^n_\xi)$ that will be central in this work.

Convention 1.4.4. For the orientation induced by a manifold on its boundary, we follow the usual "outward normal first" convention. A consequence of this convention is that the boundary of $(-\varepsilon, 0] \times M$ is oriented the same way as M, while the boundary of $[0, \varepsilon) \times M$ is identified with $-M$. For this reason we shall prefer to write collars of the boundary as $(-\varepsilon, 0] \times \partial X$.

Warning 1.4.5. Our orientation convention is opposite to the one used by Kirk and Lesch, who consider collars $[0, \varepsilon) \times \partial X$ [28, Section 2].

Define

$$V_X^\xi = \operatorname{im}(H^*(X; \mathbb{C}^n_\xi) \to H^*(\partial X; \mathbb{C}^n_\xi)). \tag{1.4}$$

Proposition 1.4.6. *The subspace $V_X^\xi \subseteq H^*(\partial X; \mathbb{C}^n_\xi)$ is Lagrangian with respect to the intersection form $I_{\partial X}^\xi$.*

Proof. Let $i \colon \partial X \to X$ denote the inclusion, and consider the induced maps

$$i_* \colon H_*(\partial X; \mathbb{C}^n_\xi) \to H_*(X; \mathbb{C}^n_\xi), \quad i^* \colon H^*(X; \mathbb{C}^n_\xi) \to H^*(\partial X; \mathbb{C}^n_\xi),$$

so that, by definition, $V_X^\xi = \operatorname{im} i^*$. Thanks to Lemma 1.4.2, the orthogonal complement of V_X^ξ is made of all the elements a such that

$$\langle \bar{b}, \mathrm{PD}(a) \rangle_\xi = 0, \quad \forall\, b \in V_X^\xi,$$

where $PD(a) = [\partial X] \cap a$ is the Poincaré duality isomorphism on ∂X. By definition of V_X^ξ, we can rewrite this property as

$$\langle \overline{i^* b}, \mathrm{PD}(a) \rangle_\xi = 0, \quad \forall\, b \in H^*(X; \mathbb{C}^n_\xi).$$

Using the fact that i_* and i^* are dual one of each other with respect to the evaluation map, we rewrite this one more time as

$$\langle \bar{b}, i_* \operatorname{PD}(a) \rangle_\xi = 0, \quad \forall\, b \in H^*(X; \mathbb{C}^n_\xi).$$

As the evaluation pairing is non-degenerate, this happens if and only if $i_* \operatorname{PD}(a) = 0$. In order to conclude, we need to show that $\ker(i_* \circ \operatorname{PD}) = \operatorname{im} i^*$. This is clear from the following commutative diagram, where the first row is exact, and the vertical maps are isomorphism by Poincaré duality.

$$\begin{array}{ccccc} H^*(X; \mathbb{C}^n_\xi) & \xrightarrow{i^*} & H^*(\partial X; \mathbb{C}^n_\xi) & \longrightarrow & H^*(X, \partial X; \mathbb{C}^n_\xi) \\ & & \wr \big\downarrow \operatorname{PD} & & \wr \big\downarrow \operatorname{PD} \\ & & H_*(\partial X; \mathbb{C}^n_\xi) & \xrightarrow{i_*} & H_*(X; \mathbb{C}^n_\xi). \end{array}$$

□

Definition 1.4.7. We refer to V^ξ_X defined by (1.4) as the *canonical Lagrangian* associated to X and ξ.

1.4.2 The pairing in middle degree and signatures

We suppose now that M is a compact oriented manifold of even dimension $2k$ with a local coefficient system $\xi \in \mathcal{U}_n(M)$. By (1.3), the intersection form in middle-degree

$$I^\xi_M \colon H^k(M, \partial M; \mathbb{C}^n_\xi) \times H^k(M, \partial M; \mathbb{C}^n_\xi) \to \mathbb{C}$$

is Hermitian if k is even and skew-Hermitian if k is odd. This observation leads to the following definition.

Definition 1.4.8. Let M be a compact oriented manifold of dimension $2k$ with a local coefficient system $\xi \in \mathcal{U}_n(M)$. We define the *signature of M with coefficients in ξ* as the integer

$$\sigma_\xi(M) := \begin{cases} \operatorname{sign}(I^\xi_M), & \text{if } k \text{ is even}, \\ \operatorname{sign}(i\, I^\xi_M), & \text{if } k \text{ is odd}, \end{cases}$$

where I^ξ_M denotes the intersection pairing in degree k.

If M is an oriented manifold, let $-M$ denote the same manifold with the opposite orientation. The signature has the following immediate properties.

Proposition 1.4.9 (Properties of the signature)**.**

(i) *If $f\colon M' \to M$ is an orientation-preserving diffeomorphism and α is an element of $\mathcal{U}_n(M)$, we have $\sigma_{f^*\alpha}(M') = \sigma_\alpha(M)$.*

(ii) *On the other hand, we have $\sigma_\alpha(-M) = -\sigma_\alpha(M)$.*

(iii) *If $\alpha, \beta \in \mathcal{U}_n(M)$ are isomorphic, we have $\sigma_\alpha(M) = \sigma_\beta(M)$.*

(iv) *For $\alpha \in \mathcal{U}_n(M)$ and $\beta \in \mathcal{U}_m(M)$, we have $\sigma_{\alpha\oplus\beta}(M) = \sigma_\alpha(M) + \sigma_\beta(M)$.*

(v) *If $\varepsilon \in \mathcal{U}_n(M)$ is trivial, then $\sigma_\varepsilon(M) = n\,\sigma(M)$.*

(vi) *For $\alpha \in \mathcal{U}_n(M \sqcup M')$, we have $\sigma_\alpha(M \sqcup M') = \sigma_\alpha(M) + \sigma_\alpha(N')$.*

Corollary 1.4.10. *Let N be a closed, oriented odd-dimensional manifold. Then, for all $\xi \in \mathcal{U}_n([0,1] \times N)$, we have*

$$\sigma_\xi([0,1] \times N) = 0.$$

Proof. Consider the orientation reversing diffeomorphism $f\colon [0,1] \times N \to [0,1] \times N$ defined by $f(t,x) = (1-t, x)$, that we see as orientation-preserving diffeomorphism $[0,1] \times N \to -[0,1] \times N$. As f is homotopic to the identity, $f^*\xi$ must be isomorphic to ξ. We have hence

$$\sigma_\xi([0,1] \times N) \overset{(iii)}{=} \sigma_{f^*\xi}([0,1] \times N) \overset{(i)}{=} \sigma_\xi(-[0,1] \times N) \overset{(ii)}{=} -\sigma_\xi([0,1] \times N),$$

and the conclusion follows. □

Remark 1.4.11. If $\varepsilon \in \mathcal{U}_1(M)$ is the trivial one-dimensional local coefficient system, then $\sigma_\tau(M)$ is the classical (or *untwisted*) signature $\sigma(M)$. This is an interesting invariant only if k is even, i.e. if the dimension of the manifold is a multiple of 4. In fact, if k is odd, the intersection form

$$I_M\colon H^k(M, \partial M; \mathbb{C}) \otimes H^k(M, \partial M; \mathbb{C}) \to \mathbb{C}$$

is the complexification of the corresponding skew-symmetric form on cohomology with real coefficients, and consequently the complex symplectic space $H^k(M, \partial M; \mathbb{C}) / \operatorname{rad}(I_M)$ is balanced by Proposition 1.1.14. This means, in other words, that $\sigma(M) = \operatorname{sign}(i\, I_M) = 0$ for all manifolds of dimension of the form $4l + 2$. The same is not true for twisted signatures.

Remark 1.4.12. In principle, we could make the whole pairing

$$I_M^\xi \colon H^*(M, \partial M; \mathbb{C}^n_\xi) \times H^*(M, \partial M; \mathbb{C}^n_\xi) \to \mathbb{C}$$

into a Hermitian form, by multiplying it by some power of the complex number i on the various degrees. For the purpose of extracting the signature, however, this would not add any extra information. In fact, the space

$$H := \bigoplus_{j \neq k} H^j(M; \mathbb{C}^n_\xi)$$

admits two complementary isotropic subspaces, given by the cohomology in degrees smaller and bigger than k respectively. Thus, the signature of this modified intersection pairing is 0 outside of middle degree by 1.1.10. For the same reason, the signature of the intersection form is 0 on odd-dimensional manifolds, and thus it does not yield an interesting invariant in that case.

Signatures with local coefficients can be substantially different invariants from the classical signature only in the case of manifolds with non-empty boundary. In fact, for closed manifolds, the following classical result is true.

Theorem 1.4.13 (Atiyah-Singer, Hirzebruch)**.** *Let M be a closed, oriented manifold of even dimension, and let $\alpha \in \mathcal{U}_n(M)$ be a local coefficient system. Then*

$$\sigma_\alpha(M) = n\,\sigma(M).$$

Proof. The formula follows from a comparison between the ordinary Hirzebruch signature theorem with its analogue for signatures with local coefficients [4, Theorem 4.7]. □

If the closed even-dimensional manifold M bounds a compact oriented manifold X such that the local coefficient system ξ extends to X, the Lagrangian subspace V_X^ξ defined in Section 1.4.1 is also Lagrangian when restricted to middle degree. By Proposition 1.1.9, we see that $\sigma_\xi(M) = 0$ in this case. This is in particular true for the untwisted signature, as it is well known. In this case, the (trivial) coefficient system can always be extended. Thanks to Theorem 1.4.13, we can hence improve the above observation to the following result.

Corollary 1.4.14. *Let M be a closed, oriented manifold of even dimension, and suppose that there is a compact oriented manifold X such that $\partial X = M$. Then, for all local coefficient system $\xi \in \mathcal{U}_n(M)$, we have $\sigma_\xi(M) = 0$.*

On closed manifolds, we shall often consider a skew-Hermitian form instead of a Hermitian one, namely the sesquilinear form on $H^k(M;\mathbb{C}^n_\xi)$ defined by

$$\omega_M^\xi := \begin{cases} I_M^\xi, & \text{if } k \text{ is odd,} \\ i\, I_M^\xi, & \text{if } k \text{ is even.} \end{cases}$$

This skew-Hermitian form ω_M^ξ is non-degenerate for closed manifolds thanks to 1.4.3, which makes the pair $(H^k(M;\mathbb{C}^n_\xi), \omega_M^\xi)$ a complex symplectic space. As a consequence of Theorem 1.4.13, we have the following result.

Corollary 1.4.15. *The complex symplectic space $(H^k(M;\mathbb{C}^n_\xi), \omega_M^\xi)$ is balanced if and only if the signature of M is 0.*

Proof. By definition, the complex symplectic space of the statement is balanced if and only if the signature of $i\,\omega_M^\xi$ is 0. It is immediate to see that this form coincides up to sign with the Hermitian form used to the define the signature with coefficients in ξ. In particular, thanks to Theorem 1.4.13, we have

$$\operatorname{sign}(i\omega_M^\xi) = \pm\sigma_\xi(M) = \pm n\sigma(M),$$

and the result follows. □

Combining the results and observations of this section, we see in particular that the complex symplectic space of our interest is always balanced for manifolds of dimension of the form $4l+2$, and it is balanced on manifolds of dimension $4l$ at least when they bound a compact oriented manifold.

1.4.3 Wall's non-additivity of the signature

Let M be a compact oriented manifold. Suppose that there is a closed properly embedded submanifold $X \subseteq M$ of codimension 1 that splits M into two manifolds with boundary M_1 and M_2. We use in this case the notation

$$M = M_1 \cup_X M_2.$$

Warning 1.4.16. Notice that X is a boundary component of both M_1 and M_2, but the orientation that it inherits from being a boundary component of M_1 is different from the one obtained from being a boundary component of M_2.

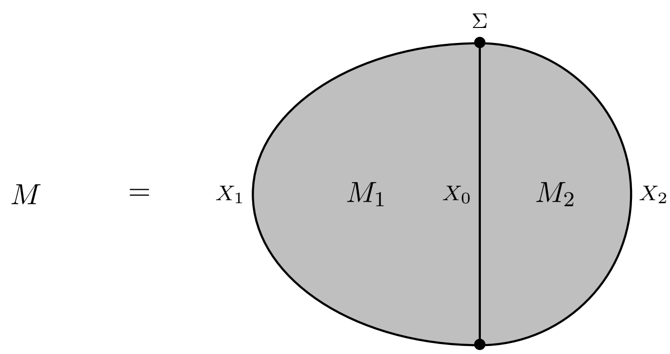
Suppose now that the dimension of $M = M_1 \cup_X M_2$ is an even number $2k$ (for the moment, we can further suppose that k is even). A classical result, first observed by Novikov and subsequently referred to as *Novikov additivity*, states that in this case we have

$$\sigma(M) = \sigma(M_1) + \sigma(M_2).$$

The proof of this fact follows from a careful decomposition of $H^k(M, \partial M; \mathbb{R})$ with respect to the Mayer-Vietoris sequence for the decomposition of M as the union of M_1 and M_2 [21, Theorem 27.5], and it generalizes without complications to signatures with local coefficients, once a local coefficient system $\xi \in \mathcal{U}_n(M)$ is given.

We shall now revise a more general result due to Wall. Suppose that M is a compact oriented manifold of dimension $2k$ which is split as $M = M_1 \cup_{X_0} M_2$ along a properly embedded submanifold X_0 of codimension 1, which is now allowed to have boundary Σ (note that M_1 and M_2 are just topological manifolds unless some way to "smooth the corners" is specified, but neither the definition of the signature nor the next result need the hypothesis of smoothness). Let $X_1 := \partial M_1 \setminus \mathrm{int}(X_0)$ and $X_2 := \partial M_2 \setminus \mathrm{int}(X_0)$. At the level of unoriented manifolds, we have then $\partial X_1 = \partial X_1 = \partial X_2 = \Sigma$ and

$$\partial M_1 = X_1 \cup_\Sigma X_0, \quad \partial M_2 = X_0 \cup_\Sigma X_2, \quad \partial M = X_1 \cup_\Sigma X_2. \tag{1.5}$$



We pick on X_1 the orientation coming from being a codimension 0 submanifold of ∂M_1, and we give Σ the orientation coming from being the boundary of X_1. Then, we have the following result, originally proved by Wall for the untwisted signature [47] (see also the paper of Py [38, (3.2)] for a more detailed proof, where it can be easily checked that the result extends to signatures with local coefficients).

Theorem 1.4.17 (Wall)**.** *Let $\xi \in \mathcal{U}_n(X)$ be a local coefficient system, and set $H := H^{k-1}(\Sigma; \mathbb{C}^n_\xi)$. Then*

$$\sigma_\xi(M) = \sigma_\xi(M_1) + \sigma_\xi(M_2) - \tau_H(V^\xi_{X_0}, V^\xi_{X_1}, V^\xi_{X_2}).$$

Remark 1.4.18. The correction term changes sign if the order of $V^{\xi}_{X_1}$ and $V^{\xi}_{X_2}$ is exchanged, which might be surprising at first sight, as the two manifolds seem to play perfectly symmetric roles. In fact, the orientation that we are giving Σ would be opposite if we saw it as the oriented boundary of $X_2 \subseteq \partial M_2$. If we want to truly exchange the roles of M_1 and M_2, we should also reverse the orientation of Σ. We see then that in this case the correction term is

$$-\tau_{H^-}(V^{\xi}_{X_0}, V^{\xi}_{X_2}, V^{\xi}_{X_1}) = \tau_H(V^{\xi}_{X_0}, V^{\xi}_{X_2}, V^{\xi}_{X_1}) = -\tau_H(V^{\xi}_{X_0}, V^{\xi}_{X_1}, V^{\xi}_{X_2}),$$

i.e. it coincides with that of Theorem 1.4.17.

Remark 1.4.19. Very often, in the applications, it is not the manifold M to be given from the start, but instead we have two manifolds M_1, M_2 such that

$$\partial M_1 = X_1 \cup_{\Sigma_1} Y_1, \quad M_2 = Y_2 \cup X_2,$$

and an orientation-reversing homeomorphism $f\colon Y_1 \to Y_2$. We can then define

$$M := M_1 \cup_f M_2 = M_1 \sqcup M_2/\sim,$$

where $\sim$ is the equivalence relation that identifies $x \in Y_1$ with $f(x) \in Y_2$. Then, M is an oriented manifold and we have natural inclusions of M_1 and M_2 into M as oriented submanifolds. Moreover, the submanifold $X_0 := Y_1/\sim = Y_2/\sim$ splits M as $M = M_1 \cup_{X_0} M_2$. We orient Σ_1 and Σ_2 as

$$\Sigma_1 = \partial X_1 = -\partial Y_1, \quad \Sigma_2 = \partial X_2 = -\partial Y_2,$$

and observe that f restricts to an orientation-reversing homeomorphism $f\colon \Sigma_1 \to \Sigma_2$. If we have local coefficient systems $\xi_1 \in \mathcal{U}(M_1)$ and $\xi_2 \in \mathcal{U}_n(M_2)$, in order to apply Theorem 1.4.17 we need them to glue well together into a local coefficient system $\xi \in \mathcal{U}_n(M)$, and this is the case if $f^*(\xi_2|_{Y_2}) = \xi_1|_{Y_1}$. Set $H_1 := H^k(\Sigma_1; \mathbb{C}^n_{\xi_1})$. Then, by Theorem 1.4.17, we have

$$\sigma_\xi(M) = \sigma_{\xi_1}(M_1) + \sigma_{\xi_2}(M_2) - \tau_{H_1}(V^{\xi_1}_{Y_1}, V^{\xi_1}_{X_1}, f^*(V^{\xi_2}_{X_2}).$$

1.5 Lagrangian subspaces and cobordisms

In this section, we discuss how Lagrangian subspaces of the twisted cohomology of a manifold propagate through cobordisms. In Section 1.5.1, we review the algebraic setting about Lagrangian relations and Lagrangian actions. In Section 1.5.2, we see how these concepts arise in topology. In Section 1.5.3, we describe the interplay between Lagrangian actions and the triple Maslov index.

1.5.1 Lagrangian relations and actions

Given a Hermitian symplectic space (H, ω), let $\mathcal{L}ag(H)$ denote the set of Lagrangian subspaces of H.

Warning 1.5.1. A vector subspace $V \subseteq H$ is Lagrangian for (H, ω) if and only if it is Lagrangian for $(H, -\omega)$. In this sense, we have $\mathcal{L}ag(H) = \mathcal{L}ag(H^-)$, and we shall usually not change the notation for a subspace when it is seen as a Lagrangian subspace of one symplectic space or the other.

A Lagrangian subspace $L \in \mathcal{L}ag(H_1^- \oplus H_2)$ is sometimes called a *Lagrangian relation* from H_1 to H_2, and denoted by $L\colon H_1 \Rightarrow H_2$. The terminology and notation are justified by the following construction: given two Lagrangian relations $L_1\colon H_1 \Rightarrow H_2$ and $L_2\colon H_2 \Rightarrow H_3$, we define their *composition* as the subspace

$$L_2L_1 := \{(h_1, h_3) \in H_1^- \oplus H_3 \mid \exists\, h_2 \in H_2, \text{ such that } (h_1, h_2) \in V, (h_2, h_3) \in W\}.$$

It is not hard to prove that L_2L_1 is a Lagrangian subspace of $H_1^- \oplus H_3$ (i.e. it is a Lagrangian relation $H_1 \Rightarrow H_3$), and that this composition is associative [13, Lemma 2.5 and Theorem 2.7]. We define for each complex symplectic space H the *diagonal relation* $\Delta_H\colon H \Rightarrow H$ as the Lagrangian subspace

$$\Delta_H := \{(h, -h) \in H^- \oplus H\} \in \mathcal{L}ag(H^- \oplus H),$$

and verify that $\Delta_H L = L$ and $L\Delta_H = L$ whenever this composition makes sense. In other words, the class of complex symplectic spaces with Lagrangian relations as morphisms form a category, where the identities are the diagonal relations. A Lagrangian relation $L\colon H_1 \Rightarrow H_2$ induces a map of sets

$$L_*\colon \mathcal{L}ag(H_1) \to \mathcal{L}ag(H_2),$$

called the *Lagrangian action* of L and defined by

$$L_*(U) := p_2((U \oplus H_2) \cap L),$$

where p_2 denotes the projection on the second term. It is not hard to see that this subspace is indeed Lagrangian [45, Chapter IV, 3.4]. The diagonal relation induces the identity, and we have the covariant relation $(L_2L_1)_* = (L_2)_* \circ (L_1)_*$.

Given a Lagrangian relation $L\colon H_1 \Rightarrow H_2$, we obtain a Lagrangian relation $L^t\colon H_2 \Rightarrow H_1$ by considering the subspace

$$L^t = \{(h_2, h_1) \in H_2^- \oplus H_1 \,|\, (h_1, h_2) \in L\} \in \mathcal{L}ag(H_2^- \oplus H_1).$$

Hence, we also get an induced map

$$L^t_*\colon \mathcal{L}ag(H_2) \to \mathcal{L}ag(H_1),$$

that can be defined more explicitly as

$$L^t_*(V) = p_1((H_1 \oplus V) \cap L).$$

Given two Lagrangian relations $L_1\colon H_1 \Rightarrow H_2$ and $L_2\colon H_2 \Rightarrow H_3$, we have an equality $(L_2L_1)^t = L_1^t L_2^t\colon H_3 \Rightarrow H_1$, and hence $(L_2L_1)^t_* = (L_1^t)_*(L_2^t)_*$ holds at the level of induced maps (as the correspondence is contravariant, Turaev uses the notation L^* for the induced map L^t_*).

Example 1.5.2. Suppose that $L\colon H_1 \Rightarrow H_2$ is of the form $L = L_1 \oplus L_2$, with $L_1 \in \mathcal{L}ag(H_1)$ and $L_2 \in \mathcal{L}ag(H_2)$. Then, for all $U_1 \in \mathcal{L}ag(H_1)$ and $U_2 \in \mathcal{L}ag(H_2)$ we have

$$L_*(U_1) = L_2, \quad L^t_*(U_2) = L_1.$$

Remark 1.5.3. If we see Lagrangian subspaces of a space H as Lagrangian relations $0 \Rightarrow H$, the action of a Lagrangian relation $L\colon H_1 \Rightarrow H_2$ can be interpreted as particular case of the composition, in the sense that

$$L_*(U) = LU.$$

Remark 1.5.4. It is immediate to see that, if $f_1\colon H_1 \to K_1$ and $f_2\colon H_2 \to K_2$ are symplectic isomorphisms, if $L\colon H_1 \Rightarrow H_2$ is a Lagrangian relation and if $U \subseteq \mathcal{L}ag(H_1)$ is a Lagrangian subspace, then we have

$$f_2(L_*(U)) = ((f_1 \oplus f_2)(L))_*(f_1(U)).$$

1.5.2 Propagation through cobordisms

The motivation for the definitions of Section 1.5.1 comes from the following topological setting. Given two closed, oriented manifolds Σ, Σ' of dimension $2k$, we say that a $(2k+1)$-dimensional oriented manifold X is a *cobordism* from Σ to Σ' if ∂X is identified to $-\partial\Sigma \sqcup \Sigma'$.

If a cobordism X from Σ to Σ' comes with a local coefficient system $\alpha \in \mathcal{U}_n(X)$, we can see the canonical Lagrangian subspace $V_X^\alpha \subseteq H^*(\partial X, \mathbb{C}^n_\alpha)$ as a Lagrangian relation

$$V_X^\alpha\colon H^*(\Sigma, \mathbb{C}^n_\alpha) \Rightarrow H^*(\Sigma', \mathbb{C}^n_\alpha).$$

We have then a Lagrangian action

$$(V_X^\alpha)_*\colon \mathcal{L}ag(H^*(\Sigma, \mathbb{C}^n_\alpha)) \to \mathcal{L}ag(H^*(\Sigma', \mathbb{C}^n_\alpha))$$

(this corresponds to the map L_X^α of Kirk and Lesch [28, Section 4]).

Example 1.5.5. A product $[0,1]\times\Sigma$ is a cobordism from Σ to itself, called the *trivial cobordism*, as $\partial([0,1]\times\Sigma) = -(\{0\}\times\Sigma)\sqcup(\{1\}\times\Sigma)$ (see Convention 1.4.4). If α is a local coefficient system of product form, then V_X^α is the diagonal relation $\Delta\colon H^*(\Sigma,\mathbb{C}^n_\alpha) \Rightarrow H^*(\Sigma,\mathbb{C}^n_\alpha)$, and $(V_X^\alpha)_*$ is the identity on $\mathcal{L}ag(H^*(\Sigma,\mathbb{C}^n_\alpha))$.

Observe that, if X is a cobordism from Σ to Σ', the same underlying manifold also yields a cobordism from $-\Sigma'$ to $-\Sigma$. By reversing the orientation, we get a cobordism from Σ' to Σ, that we denote by X^t. In terms of Lagrangian relations, it is immediate to see that $V^\alpha_{X^t} = (V^\alpha_X)^t$. We have thus an induced map

$$(V^\alpha_{X^t})_* = (V^\alpha_X)^t_*\colon \mathcal{L}ag(H^*(\Sigma',\mathbb{C}^n_\alpha)) \to \mathcal{L}ag(H^*(\Sigma,\mathbb{C}^n_\alpha)).$$

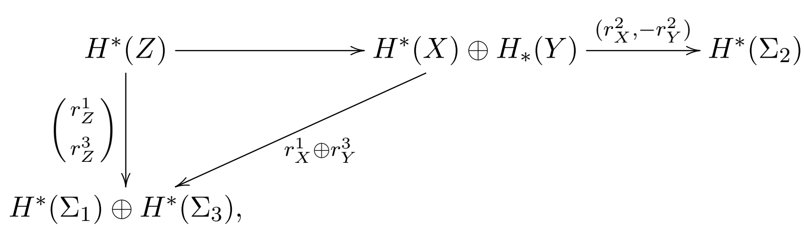
If X is a cobordism from Σ_1 to Σ_2 and X is a cobordism from Σ_2 to Σ_3, we can glue them together along Σ_2 and get a cobordism $YX := X\cup_{\Sigma_2} Y$ from Σ_1 to Σ_3. The following result says that the canonical Lagrangian relation of YX is then the composition of the canonical Lagrangian relations of X and Y. This implies that, if $\Sigma_1=\emptyset$, the Lagrangian action of V^α_Z transforms V^α_X into V^α_{YX} (in the words of Kirk and Lesch, "the bordism propagates the distinguished Lagrangian" [28, Theorem 4.1]).

Proposition 1.5.6. *Let X be a cobordism from Σ_1 to Σ_2 and Y be a cobordism from Σ_2 to Σ_3. Let $\alpha\in\mathcal{U}_n(YX)$ be a local coefficient system. Then*

$$V^\alpha_{YX} = V^\alpha_Y V^\alpha_X$$

In particular, if $\Sigma_1=\emptyset$, we have $V^\alpha_{YX} = (V^\alpha_Y)_*(V^\alpha_X)$.

Proof. Let $Z := YX$. For $i=1,2,3$ and W being either X, Y or Z, we denote the restrictions maps by $r^i_W\colon H^*(W)\to H^*(\Sigma_i)$. Here and in the rest of the proof, coefficients in $\mathbb{C}^n_\alpha$ are assumed. We consider then the commutative diagram



where the first row is exact by Mayer-Vietoris. We have then

$$V^\alpha_Z = \operatorname{im}\begin{pmatrix} r^1_Z \\ r^3_Z \end{pmatrix} = (r^1_X\oplus r^3_Y)(\ker(r^2_X,-r^2_Y)),$$

or, put differently,

$$(a_1, a_3) \in V_Z^\alpha \iff \exists (x, y) \text{ s.t. } r_X^1(x) = a_1,\, r_Y^3(y) = a_3,\, r_X^2(x) = r_Y^2(y).$$

Call (P) the property on the right side of the equivalence. If (P) is satisfied, by taking $a_2 = r_X^2(x) = r_Y^2(y)$, we see that $(a_1, a_2) = \begin{pmatrix} r_X^1 \\ r_X^2 \end{pmatrix}(x)$ and $(a_2, a_3) = \begin{pmatrix} r_Y^2 \\ r_Y^3 \end{pmatrix}(y)$, and hence that $(a_1, a_2) \in V_X^\alpha$ and $(a_2, a_3) \in V_Y^\alpha$. Conversely, if there is an element $a_2 \in H^*(\Sigma_2)$ such that $(a_1, a_2) \in V_X^\alpha$ and $(a_2, a_3) \in V_Y^\alpha$, we can find x and y such that (P) is satisfied. We have hence shown that

$$V_Z^\alpha = \{(a_1, a_3) \in V_Z^\alpha \,|\, a_2 \text{ s.t. } (a_1, a_2) \in V_X^\alpha \text{ and } (a_2, a_3) \in V_Y^\alpha\},$$

and this is $V_Y^\alpha V_X^\alpha$. by definition.

If $\Sigma_1 = \emptyset$, V_X^α is a Lagrangian subspace of $H^*(\Sigma_2, \mathbb{C}_\alpha^n)$ and V_{YX}^α is a Lagrangian subspace of $H^*(\Sigma_3, \mathbb{C}_\alpha^n)$. By Remark 1.5.3, in this case we can reinterpret the composition as $V_Y^\alpha V_X^\alpha = (V_Y^\alpha)_*(V_X^\alpha)$, and the desired result follows. □

Remark 1.5.7. We can give now a heuristic interpretation of the Lagrangian action associated to a cobordism in the following way. Let Y be a cobordism from Σ to Σ'', and consider the Lagrangian action

$$(V_Y^\alpha)_* \colon \mathcal{L}ag(H^*(\Sigma, \mathbb{C}_\alpha^n)) \to \mathcal{L}ag(H^*(\Sigma', \mathbb{C}_\alpha^n)).$$

Then, if $L \in \mathcal{L}ag(H^*(\Sigma, \mathbb{C}_\alpha^n))$ is any Lagrangian, we can interpret its image $(V_Y^\alpha)_*(L)$ as the canonical Lagrangian $V_{Y'}^\alpha$ of a fictional manifold Y' obtained from Y by capping Σ with a manifold X to which α extends and whose canonical Lagrangian is identified to L under the gluing. Of course, such a manifold in general does not exist.

Notation 1.5.8. For not overloading the notation of some formulas, we shall sometimes drop the lower star from the induced map of the canonical Lagrangian relation, and just write V_X^α instead of $(V_X^\alpha)_*$. Because of Remark 1.5.3, the notation shall not create any ambiguity.

Remark 1.5.9. Cobordisms of dimension $(2k+1)$ up to the right notion of equivalence can be seen as the morphisms of a category whose elements are $2k$-dimensional closed, oriented manifolds. Composition is given by gluing, and the identity morphisms are the trivial cobordisms $[0, 1] \times \Sigma$. This notion can be extended to the context of cobordisms with a local coefficient system, which can be glued when the local systems on the two pieces of boundary that we are attaching agree. Then, the correspondence

$$(\Sigma, \alpha) \mapsto H^*(\Sigma, \mathbb{C}^n_\alpha), \quad (X, \alpha) \mapsto V^\alpha_X$$

becomes a covariant functor to the category of complex symplectic spaces with Lagrangian relations as morphism. As discussed in the first part of this section, the induced map L^α_X is the image of a further covariant functor to the category of sets.

1.5.3 Lagrangian actions and the Maslov index

We discuss now some results relating the Maslov index with the Lagrangian relations introduced in Section 1.5.1. First observe that, by Proposition 1.2.3 (ii) and (v), we have

$$\tau_{H_1^- \oplus H_2}(U_1 \oplus U_2, V_1 \oplus V_2, W_1 \oplus W_2) = -\tau_{H_1}(U_1, V_1, W_1) + \tau_{H_2}(U_2, V_2, W_2).$$

If the Lagrangian $W_1 \oplus W_2$ is replaced by any Lagrangian relation $W \colon H_1 \Rightarrow H_2$, this formula generalizes in the following way.

Proposition 1.5.10. *Let $W \colon H_1 \Rightarrow H_2$ be a Lagrangian relation, and let $U_1, V_1 \in \mathcal{L}ag(H_1)$ and $U_2, V_2 \in \mathcal{L}ag(H_2)$ Lagrangian subspaces. Then,*

$$\tau_{H_1^- \oplus H_2}(U_1 \oplus U_2, V_1 \oplus V_2, W) = -\tau_{H_1}(U_1, V_1, W^t_*(U_2)) + \tau_{H_2}(U_2, V_2, W_*(V_1)).$$

Proof. The argument is substantially contained in the book of Turaev [45, Lemma 3.7], but for the convenience of the reader we write it here in full detail. We first prove the result in the special case where $U_1 = V_1$. Namely, we prove

$$\tau_{H_1^- \oplus H_2}(V_1 \oplus U_2, V_1 \oplus V_2, W) = \tau_{H_2}(U_2, V_2, W_*(V_1)). \tag{1.6}$$

Let ω_1 be the symplectic form on H_1, ω_2 the symplectic form on H_2 and $\omega = (-\omega_1) \oplus \omega_2$ the symplectic form on $H_1^- \oplus H_2$. The Maslov index in the left side of the equation is by definition the signature of a Hermitian form on $((V_1 \oplus U_2) + (V_1 \oplus V_2)) \cap W = (V_1 \oplus (U_2 + V_2)) \cap W$, namely of

$$\begin{aligned} \beta \colon (V_1 \oplus (U_2 + V_2)) \cap W \times (V_1 \oplus (U_2 + V_2)) \cap W \to \mathbb{C} \\ ((x_1, u_2 + v_2), (x_1', y_2)) \mapsto \omega((0, u_2), (x_1', y_2)) = \omega_2(u_2, y_2). \end{aligned}$$

On the other hand, the Maslov index on the right-hand of (1.6) is the signature of a Hermitian form on $(U_2+V_2)\cap W_*(V_1)$, namely of

$$\beta'\colon (U_2+V_2)\cap W_*(V_1)\times (U_2+V_2)\cap W_*(V_1)\to \mathbb{C}$$
$$((u_2+v_2),(y_2))\mapsto \omega_2(u_2,y_2).$$

Using the definition of $W_*(V_1)$, we have

$$(U_2+V_2)\cap W_*(V_1)=(U_2+V_2)\cap p_2((V_1\oplus H_2)\cap W)=p_2((V_1\oplus(U_2+V_2))\cap W),$$

and clearly p_2 brings the form β to the form β'. Hence, their signatures coincide, and (1.6) is established. In the same way we prove the equivalent statement for the special case $U_2=V_2$, i.e.

$$\tau_{H_1^-\oplus H_2}(U_1\oplus U_2,V_1\oplus U_2,W)=-\tau_{H_1}(U_1,V_1,W_*^t(U_2)). \tag{1.7}$$

We can now prove the general formula. Using the properties of the Maslov index (Proposition 1.2.3 (i) and (vi)), we write, for $\tau:=\tau_{H_1^-\oplus H_2}$

$$\tau(U_1\oplus U_2,V_1\oplus V_2,W)=\tau(V_1\oplus U_2,V_1\oplus V_2,W)+$$
$$+\tau(U_1\oplus U_2,V_1\oplus U_2,W)+\tau(U_1\oplus U_2,V_1\oplus V_2,V_1\oplus U_2).$$

The last term of the sum vanishes, as it can be seen by Proposition 1.2.3 (ii) and (i). By applying (1.6) to the first term and (1.7) to the second term, we get to the desired formula. □

Specializing Proposition 1.5.10 to the diagonal relation yields the following useful formula.

Corollary 1.5.11. *Let H be a complex symplectic space and let A,B,C,D four Lagrangian subspaces. Then*

$$\tau_{H^-\oplus H}(A\oplus B,C\oplus D,\Delta_H)=\tau_H(A,B,C)-\tau_H(B,C,D).$$

Remark 1.5.12. Applying Proposition 1.5.10 to both sides of the identity

$$\tau_{H_1^-\oplus H_2}(U_1\oplus U_2,V_1\oplus V_2,W)=-\tau_{H_1^-\oplus H_2}(V_1\oplus V_2,U_1\oplus U_2,W),$$

and rearranging the terms, we get

$$\tau_{H_1}(U_1,V_1,W^*(U_2))-\tau_{H_1}(U_1,V_1,W_*^t(V_2))+\tau_{H_2}(W_*(U_1),U_2,V_2)$$
$$-\tau_{H_2}(W_*(V_1),U_2,V_2)=0.$$

This is a generalization of the cocycle property of the Maslov index (Proposition 1.2.3 (vi)), which corresponds to the case $H_1=H_2$ and $W=\Delta_{H_1}$.

Using Proposition 1.2.15 and Corollary 1.5.11, we prove now easily the following result about the function m.

Corollary 1.5.13. *Let H be a Hermitian symplectic space, and let $A, B \in \mathcal{L}ag(H)$ two Lagrangian subspaces. Then*

$$m_{H^- \oplus H}(\Delta_H, A \oplus B) = m_H(A, B).$$

Proof. We first prove the statement in the special case $A = B$, i.e. we show that $m_{H^- \oplus H}(\Delta_H, A \oplus A) = 0$. Consider the map $g\colon H^- \oplus H \to H \oplus H^-$ that exchanges the two coordinates. Then g is an isometry which transforms the symplectic operator of the first space into the symplectic operator of the second space. Moreover, g preserves the Lagrangian subspaces Δ_H and $A \oplus A$. Let m stand for $m_{H^- \oplus H}$. By Proposition 1.2.13 (3. and 4.), we get

$$-m(\Delta_H, A \oplus A) = m(\Delta_H, A \oplus A),$$

whence $m(\Delta_H, A \oplus A) = 0$ as expected. We now prove the result in the general case. By Proposition 1.2.15, we have

$$\tau(A \oplus A, A \oplus B, \Delta_H) = m(A \oplus A, A \oplus B) + m(A \oplus B, \Delta_H) + m(\Delta_H, A \oplus A).$$

By Corollary 1.5.11, the left hand side is equal to $\tau_H(A, A, A) - \tau_H(A, A, B) = 0$. On the right hand side, the first summand is equal to $-m(A, A) + m(A, B) = m(A, B)$, and the last summand vanish by the first part of the proof. Hence, the last formula can be rewritten as

$$0 = m_H(A, B) + m(A \oplus B, \Delta_H),$$

which is equivalent to the one in the statement by the antisymmetry of m. ∎

Chapter 2

Eta and rho invariants for manifolds with boundary

2.1 The Atiyah-Patodi-Singer rho invariant

This section is dedicated to the eta and rho invariants of closed manifolds. In Section 2.1.1, we briefly recall the definition of the eta and invariant and its role in the Atiyah-Patodi-Singer signature theorem. In Section 2.1.2, we define the Atiyah-Patodi-Singer rho invariant and highlight its relationship to signatures. In Section 2.1.3, we prove a cut-and-paste formula for the rho invariant.

2.1.1 The signature theorem for manifolds with boundary

Let N be a closed, oriented, Riemannian manifold of dimension $2k-1$, with a local coefficient system $\alpha \in \mathcal{U}_n(N)$ (if N is connected, by the results of Section 1.3.2 we can see α as a representation $\pi_1(N) \to U(n)$). Let $E_\alpha \to N$ be the associated flat vector bundle (see Section 1.3.3), and consider the subspace

$$\Omega^{\mathrm{ev}}(N, E_\alpha) := \bigoplus_{q=0}^{k-1} \Omega^{2q}(N, E_\alpha)$$

of twisted differential forms of even degree. Let D_N^α be the *twisted odd signature operator*, i.e. the first-order differential operator on $\Omega^{\mathrm{ev}}(N, E_\alpha)$ defined by

$$D_N^\alpha\, \phi := (-1)^{q+1} i^k (\star d - d\star)\phi, \qquad \text{for } \phi \in \Omega^{2q}(N, E_\alpha).$$

The operator D_N^α can be extended to a self-adjoint elliptic operator with discrete spectrum and, by the results of Atiyah, Patodi and Singer [2], it has a well defined *eta invariant*

$$\eta_\alpha(N) := \eta(D_N^\alpha) \in \mathbb{R},$$

that is defined as the value at 0 of a meromorphic extension of the eta function

$$\eta(s) = \sum_{\substack{\lambda \in \mathrm{Spec}(D_N^\alpha) \\ \lambda \neq 0}} \operatorname{sgn} \lambda \, |\lambda|^{-s}.$$

We say that a compact Riemannian manifold M has metric of *product form* near the boundary, if there exists a neighborhood of ∂M that is isometric to $(-\varepsilon, 0] \times \partial M$ with the product metric. The main result about the eta invariant of the twisted signature operator is the following [3, Theorem 2.2].

Theorem 2.1.1 (Atiyah-Patodi-Singer)**.** *Let M be a compact, oriented manifold with $\partial M = N$, equipped with Riemannian metric of product form near N, and let $\alpha \in \mathcal{U}_n(M)$ be a local coefficient system. Then*

$$\sigma_\alpha(M) = n \int_M L(p) - \eta_\alpha(N),$$

where $L(p)$ is the Hirzebruch L-polynomial in the Pontryagin forms of M.

Note that both summands on the right-hand term depend on the Riemannian metric. As the left-hand term is a topological invariant of M, we see that the integral of the L-polynomial only depends on the metric on the boundary N (this fact can be proved more easily by applying the signature theorem to the closed double of M). We shall not dwell upon the geometrical significance of this summand, as it will disappear soon in the paper.

The Atiyah-Patodi-Singer eta invariant shares many properties with the signature with local coefficients. In the following, we assume the manifolds to be closed, oriented and with a metric. The following result follows easily from the definition (compare with Proposition 1.4.9).

Proposition 2.1.2 (Properties of the APS eta invariant)**.**

(i) If $f\colon N' \to N$ is an orientation-preserving isometry and $\alpha \in \mathcal{U}_n(N)$, then $\eta_{f^*\alpha}(N') = \eta_\alpha(N)$.

(ii) On the other hand, we have $\eta_\alpha(-N) = -\eta_\alpha(N)$.

(iii) If $\alpha, \beta \in \mathcal{U}_n(N)$ are isomorphic, we have $\eta_\alpha(N) = \eta_\beta(N)$.

(iv) For $\alpha \in \mathcal{U}_n(N)$ and $\beta \in \mathcal{U}_m(N)$, we have $\eta_{\alpha\oplus\beta}(N) = \eta_\alpha(N) + \eta_\beta(N)$.

(v) If $\varepsilon \in \mathcal{U}_n(N)$ is trivial, then $\eta_\varepsilon(N) = n\,\eta(N)$.

(vi) For $\alpha \in \mathcal{U}_n(N \sqcup N')$, we have $\eta_\alpha(N \sqcup N') = \eta_\alpha(N) + \eta_\alpha(N')$.

2.1.2 Rho invariants and signatures

We are now going to define the rho invariant, which is a relative version of the eta invariant. Note that, for the untwisted odd signature operator D_N on $\Omega^{\mathrm{ev}}(N, \mathbb{C})$, we set $\eta(N) := \eta(D_N)$.

Definition 2.1.3. Let N be a closed, oriented manifold of odd dimension, and let $\alpha \in \mathcal{U}_n(N)$ be a local coefficient system. The *Atiyah-Patodi-Singer rho invariant* of N associated to α is the real number

$$\rho_\alpha(N) := \eta_\alpha(N) - n\,\eta(N),$$

where the eta invariants are computed for an arbitrary Riemannian metric on N.

We shall see in a moment that the difference $\eta_\alpha(N) - n\,\eta(N)$ is independent of the Riemannian metric, so that $\rho_\alpha(N)$ is well defined. From the properties of the eta invariant, it is clear that $\rho_\alpha(N)$ also only depends on the isomorphism class of the local coefficient system and that it is additive under direct sum of local coefficient systems and disjoint union of manifolds. Moreover, $\rho_\tau(N)$ is 0 for trivial local coefficient systems, and it satisfies

$$\rho_\alpha(-N) = -\rho_\alpha(N). \tag{2.1}$$

The main theorem about the rho invariant is the following [3, Theorem 2.4]. As the proof is simple (once Theorem 2.1.1 is settled), we repeat it here.

Theorem 2.1.4 (Atiyah-Patodi-Singer)**.**

(i) $\rho_\alpha(N)$ is independent of the Riemannian metric on N.

(ii) If M is a compact, oriented manifold with $\partial M = N$ and α extends to M, then

$$\rho_\alpha(N) = n\,\sigma(M) - \sigma_\alpha(M).$$

Proof. We first prove that (ii) is satisfied for any Riemannian metric on N. This is an immediate conseguence of Theorem 2.1.1, for any Riemannian metric on M that extends the metric on N and has product form near it. It is enough to apply the theorem tin the twisted and untwisted case, and observe that the L-polynomial summand is the same in the two cases. To prove (i), let N' be a copy of N with a different Riemanniann metric. Consider the manifold $M = [0, 1] \times N$, and extend α as a product. Pick Riemannian metrics on ∂M in such a way that, up to an orientation-preserving isometry, we have $\partial M = -N \sqcup N'$. Then, by (i) and the basic properties of the rho invariant we obtain

$$\rho_\alpha(N') - \rho_\alpha(N) = n\,\sigma([0, 1] \times N) - \sigma_\alpha([0, 1] \times N),$$

and the conclusion follows as all signatures of $[0, 1] \times N$ are 0 by Corollary 1.4.10. □

We state one more result about the rho invariant that will turn useful later on.

Proposition 2.1.5. *Let M and N be closed, oriented manifolds, of dimension respectively $2m$ and $2k - 1$. Let $\alpha \in \mathcal{U}_n(M)$ and $\beta \in \mathcal{U}_r(N)$ be local coefficient systems. Then*

$$\rho_{\alpha\times\beta}(M \times N) = (-1)^{mk} n\,\sigma(M)\rho_\beta(N).$$

In particular, if m is odd, we have $\rho_{\alpha\times\beta}(M \times N) = 0$.

Proof. As explained by Neumann [36, Theorem 1.2 (v)], it follows from a direct computation about the eta invariants that

$$\rho_{\alpha\times\beta}(M \times N) = (-1)^{mk}\sigma_\alpha(M)\rho_\beta(N)$$

(note that the "tensor product representation" on $\pi_1(M \times N)$ corresponds to the cartesian product of local coefficient systems). The statement follows then from Theorem 1.4.13. □

We will only need the following consequence of Proposition 2.1.5 (compare with the similar results [23, Proposition 7.1] [17, Lemma 4.2]).

Corollary 2.1.6. *Let F be a closed, oriented surface, and let $\psi \in \mathcal{U}_1(F \times S^1)$ be a local coefficient system. Then $\rho_\psi(F' \times S^1) = 0$.*

Proof. Without loss of generality, we can suppose that F is connected. We see in this case ψ as a representation $\psi\colon \pi_1(F \times S^1) \to U(1)$. As $U(1)$ is abelian, ψ factors through

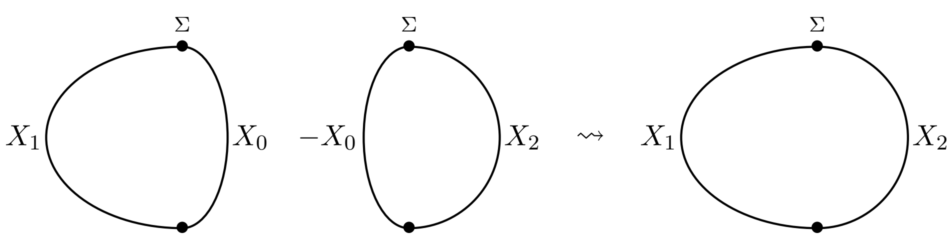
$$\psi'\colon H_1(F' \times S^1; \mathbb{Z}) \cong H_1(F'; \mathbb{Z}) \oplus H_1(S^1; \mathbb{Z}) \to U(1).$$

Then, there are $\alpha'\colon H_1(F'; \mathbb{Z}) \to U(1)$ and $\beta'\colon H_1(S^1) \to U(1)$ such that ψ' is equivalent to $\alpha' \otimes \beta'$. Then ψ is equivalent to the direct product of the corresponding representations $\alpha\colon \pi_1(F) \to U(1)$ and $\beta\colon \pi_1(S^1) \to U(1)$, and the result follows from Proposition 2.1.5 . □

2.1.3 A cut-and-paste formula

Suppose we have a closed, oriented $(2k-1)$-dimensional manifold that is split by a codimension one closed manifold Σ, yielding a decomposition $X_1 \cup_\Sigma X_2$. In many concrete situations, it is possible to find a manifold X_0 with $\partial X_0 = -\Sigma$ such that $X_1 \cup_\Sigma X_0$ and $-X_0 \cup_\Sigma X_2$ are "simpler" than $X_1 \cup_\Sigma X_2$. The operation of obtaining the first manifold from the latter two is often called *cut-and-paste*. Schematically, we have hence

$$X_1 \cup_\Sigma X_0 \sqcup -X_0 \cup_\Sigma X_2 \rightsquigarrow X_1 \cup_\Sigma X_2.$$



It is then useful to be able to compute invariants of $X_1 \cup_\Sigma X_2$ in terms of invariants of the other two manifolds.

Using Wall's non-additivity theorem and the Atiyah-Patodi-Singer theorem, we prove such a cut-and-paste formula for the rho invariant. In order to upgrade this formula to one valid for the eta invariant, a delicate treatment of the differential geometry near the boundary is needed (see Remark 2.1.8). As we shall see in Section 2.3.4, a cut-and-paste formula for the eta invariant can be proved directly by means of gluing formulas for the corresponding invariants of manifolds with boundary. The relationship between gluing and cut-and-past formulas was already investigated by Bunke [7, 2.5] and Kirk and Lesch [29, Section 8.3]. The following result is closely related to their treatments, and it overlaps partially with their results.

Proposition 2.1.7. *Let X_1, X_2 and X_0 be compact, oriented manifolds of dimension $2k-1$ with $\partial X_1 = \Sigma = -\partial X_0 = -\partial X_2$. Then, for every $\alpha \in \mathcal{U}_n(X_1 \cup X_2 \cup X_0)$, we have*

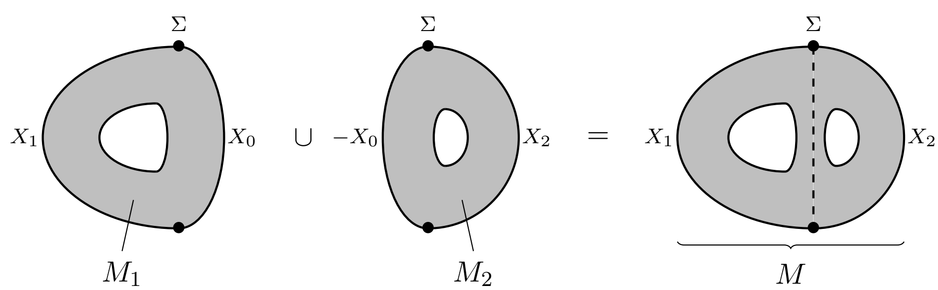
$$\rho_\alpha(X_1 \cup_\Sigma X_2) = \rho_\alpha(X_1 \cup_\Sigma X_0) + \rho_\alpha(-X_0 \cup_\Sigma X_2) + \\ + \tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}) - n\,\tau(V_{X_0}, V_{X_1}, V_{X_2}),$$

where the first Maslov triple index is performed on $H^{k-1}(\Sigma; \mathbb{C}^n_\alpha)$, and the second on $H^{k-1}(\Sigma; \mathbb{C})$.

Proof. Consider the oriented manifolds

$$M_1 := [0,1] \times (X_1 \cup_\Sigma X_0), \quad M_2 := [0,1] \times (-X_0 \cup_\Sigma X_2),$$

and extend the local coefficient system α on them as a product. We glue then M_1 with M_2 along $\{1\} \times X_0$, obtaining a topological oriented manifold M to which α extends.



The boundary of M can be described as

$$\partial M = (-(X_1 \cup_\Sigma X_0) \sqcup -(-X_0 \cup_\Sigma X_2)) \sqcup (X_1 \cup_\Sigma X_2), \tag{2.2}$$

and we can equip M of a smooth structure such that (2.2) is satisfied in the smooth sense [44, 15.10.3]. We compute now the terms in the equation

$$\rho_\alpha(\partial M) = n\,\sigma(M) - \sigma_\alpha(M). \tag{2.3}$$

given by Theorem2.1.4. By (2.2), the left-hand term is given by

$$\rho_\alpha(\partial M) = -\rho_\alpha(X_1 \cup_\Sigma X_0) - \rho_\alpha(-X_0 \cup_\Sigma X_2) + \rho_\alpha(X_1 \cup_\Sigma X_2). \tag{2.4}$$

By Wall's non-additivity (Theorem 1.4.17), together with Corollary 1.4.10 (which ensures that the signatures of M_1 and M_2 vanish) we can compute the twisted and untwisted signature of M as

$$\sigma(M) = 0 + 0 - \tau(V_{X_0}, V_{X_1}, V_{X_2}), \quad \sigma_\alpha(M) = 0 + 0 - \tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}). \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.3), we obtain the desired formula. □

Remark 2.1.8. If X_0, X_1 and X_2 are equipped with Riemannian metrics that coincide on Σ and have product form near it, the three closed manifold obtained by their gluing inherit well-defined Riemannian metrics. We can then ask ourselves whether the following formula is true:

$$\eta_\alpha(X_1 \cup_\Sigma X_2) = \eta_\alpha(X_1 \cup_\Sigma X_0) + \eta_\alpha(-X_0 \cup_\Sigma X_2) + \tau(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}). \quad (2.6)$$

Such a formula, if proved for both α and the trivial local coefficient system, would imply Proposition 2.1.7 immediately by definition of the rho invariant. As we have already mentioned, we shall prove such a formula by a different means in Section 2.3.4 (see Proposition 2.3.9). However, we discuss here an approach based on the proof of Proposition 2.1.7.

In such proof, we can equip M_1 and M_2 with Riemannian metrics that extend the given metrics on their boundary, and have product form near them. Applying the Atiyah-Patodi-Singer signature theorem (Theorem2.1.1) to M_1 and M_2, we see then that

$$\int_{M_1} L(p) = \int_{M_2} L(p) = 0,$$

as their signatures vanish by Corollary 1.4.10, and the eta invariant of their boundaries is 0 as they are made up of two copies of the same manifold with opposite orientations. Bunke [7, pp. 414-415] suggests that, by choosing an appropriate smoothing of the corner, in gluings of this kind it is possible to equip the union $M = M_1 \cup_{X_0} M_2$ with a metric of product form near the boundary and such that

$$\int_M L(p) = \int_{M_1} L(p) + \int_{M_2} L(p) = 0.$$

Theorem 2.1.1 applied to M tells us hence that

$$\sigma_\alpha(M) = \eta_\alpha(X_1 \cup_\Sigma X_0) + \eta_\alpha(-X_0 \cup_\Sigma X_2) - \eta_\alpha(X_1 \cup_\Sigma X_2),$$

and the proof of (2.6) is then concluded by Wall's non-additivity applied to M.

Remark 2.1.9. Bunke [7, Lemma 2.12] uses the aforementioned argument about smoothing the corner while keeping control of the integral of the L-form to give a proof of (2.6) for the untwisted eta invariant. As we have said, in Section 2.3.4 we shall prove (2.6) in full generality without need of this delicate operation.

If N_1 and N_2 are closed, oriented manifolds of the same dimension m, we can form the connected sum $N_1 \# N_2$. If $m \geq 3$, we have an isomorphism $\pi_1(N_1 \# N_2) \cong \pi_1(N_1) \star \pi_1(N_2)$. As a consequence, if $\alpha_1 \in \mathcal{U}_n(N_1)$ and $\alpha_2 \in \mathcal{U}_n(N_2)$ are local coefficient systems, there is a local coefficient system $\alpha_1 \star \alpha_2 \in \mathcal{U}_n(N_1 \# N_2)$ that restricts, up to isomorphism, to α_1 and α_2 on the two sides (this can be seen by passing to the associate representations). An immediate consequence of Proposition 2.1.7 is the following.

Corollary 2.1.10. $\rho_{\alpha_1 * \alpha_2}(N_1 \# N_2) = \rho_{\alpha_1}(N_1) + \rho_{\alpha_2}(N_2)$.

Proof. It is enough apply the cut-and-paste formula with Σ a sphere and X_0 a disk. Then, the twisted cohomology of Σ coincides with the ordinary one (as Σ is simply connected), and both Maslov triple indices vanish as the cohomology of Σ is trivial in middle degree. □

2.2 Eta and rho invariants of Kirk and Lesch

In this section, we review the main properties of eta and rho invariants for manifolds with boundary, as they were defined and studied by Kirk and Lesch. In Section 2.2.1, we set some notational convention. In Section 2.2.2, we sketch the definition and recall some properties of the eta invariants for manifolds with boundary. In Section 2.2.3, we discuss the corresponding rho invariants and their dependence on the metric.

2.2.1 The Hermitian symplectic structure

Let Σ be a closed, oriented, Riemannian manifold of dimension $2k$. Let $\alpha \in \mathcal{U}_n(\Sigma)$ be a local coefficient system, and let E_α denote the associated flat vector bundle. Then, the space of twisted differential forms $\Omega^*(\Sigma; E_\alpha)$ inherits a Hermitian product and a unitary operator $\star$ such that $\star^2 = (-1)^p$ on $\Omega^p(\Sigma; \mathbb{C}^n_\alpha)$. Following Kirk and Lesch [28, Section 2], we define

$$\gamma := (-1)^{kp}(-1)^{\frac{p(p+1)}{2}} i^{k+1} \star \quad \text{on} \quad \Omega^p(\Sigma; \mathbb{C}^n_\alpha). \tag{2.7}$$

This renormalization of the star operator satisfies $\gamma^2 = -1$.

Warning 2.2.1. The expression defining γ coincides only up to a minus sign with the one used by Kirk and Lesch. This difference is there in order to compensate the different convention in orienting Σ when it is the boundary of a $(2k+1)$-dimensional oriented manifold X (see Warning 1.4.5).

Restrict now to the finite-dimensional subspace $\mathcal{H}^*(\Sigma; E_\alpha)$ of harmonic forms with respect to the twisted de Rham operator D_α. The Hodge-de Rham theorem gives an identification

$$H^p(\Sigma; \mathbb{C}^n_\alpha) \cong \mathcal{H}^p(\Sigma; E_\alpha),$$

that we use to push the Hermitian product and the operator γ to complex vector space $H^*(\Sigma; \mathbb{C}^n_\alpha)$. This makes the pair $(H^*(\Sigma; \mathbb{C}^n_\alpha), \gamma)$ a Hermitian symplectic space (see Section 1.1.4).

Associated to the Hermitian symplectic structure, there is a symplectic form ω on $H^*(\Sigma; \mathbb{C}^n_\alpha)$ that is defined by

$$\omega(x, y) := \langle x, \gamma y \rangle.$$

It can be checked that ω is independent of the Riemannian metric, as it can be described in terms of the intersection form I^α_Σ as

$$\omega(x, y) = (-1)^{pk+k}(-1)^{\frac{p(p+1)}{2}} i^{k+1} I^\alpha_\Sigma(x, y) \qquad \text{if } x \text{ has degree } p.$$

Note that, on degree k, the form ω coincides up to sign with the symplectic form ω^α_Σ defined in Section 1.4.2.

Remark 2.2.2. In general, the complex symplectic space $(H^*(\Sigma; \mathbb{C}^n_\alpha), \omega)$ is not balanced (see Definition 1.1.13). From Remark 1.4.12 and Theorem 1.4.13, however, we see that

$$\operatorname{sign}(i\omega) = \pm \operatorname{sign}(i(I^\alpha_\Sigma)_k) = \pm\sigma_\alpha(\Sigma) = \pm n\, \sigma(\Sigma),$$

so that $H^*(\Sigma; \mathbb{C}^n_\alpha) = 0$ whenever the ordinary signature of Σ vanishes (compare with Corollary 1.4.15). In particular, it is balanced whenever k is odd or Σ bounds a compact, oriented manifold of dimension $2k + 1$.

2.2.2 The eta invariant of the odd signature operator

Let X be a compact, oriented manifold of dimension $2k + 1$, with boundary $\partial X = \Sigma$, provided with a Riemannian metric of product form on a collar $(-\varepsilon, 0] \times \Sigma$ of the boundary of X. Given a local coefficient system $\alpha \in \mathcal{U}_n(X)$, we consider the twisted signature operator D^α_X on $\Omega^{\mathrm{ev}}(X; E_\alpha)$. In order for it to have a well-defined eta invariant, we need to impose boundary conditions. Following Kirk and Lesch, near the boundary we have an identification

$$\Omega^{\mathrm{ev}}((-\varepsilon, 0] \times \Sigma; E_\alpha) \cong C^\infty((-\varepsilon, 0], \Omega^*(\Sigma; E_\alpha))$$

under which the signature operator is rewritten locally as

$$D_X^\alpha = \gamma\left(\tfrac{\partial}{\partial t} - A_b^\alpha\right),$$

where γ is the bundle isomorphism defined in (2.7), t is the coordinate on $(-\epsilon, 0]$ and the boundary operator A_b^α is a square root of the twisted Hodge Laplacian on $\Omega^*(\Sigma, E_\alpha)$. For details, see Kirk and Lesch [29, Section 8.1] (the difference of sign in the local form of D_X^α is due to our different conventions on orientations). The operator A_b^α has a discrete spectrum, giving an orthogonal decomposition

$$L^2\Omega^*(\Sigma; E_\alpha)) = F^- \oplus \ker A_b^\alpha \oplus F^+,$$

and the null-space $\ker A_b^\alpha$ corresponds to the space of twisted harmonic differential forms $\mathcal{H}^*(\Sigma, \mathbb{C}^n_\alpha)$. Given a Lagrangian subspace V of $H^*(\Sigma, \mathbb{C}^n_\alpha)$, we use the Hodge–de Rham theorem to identify it to a subspace of the harmonic forms $\mathcal{H}^*(\Sigma, \mathbb{C}^n_\alpha) = \ker A_b^\alpha$. Let $D_{X,V}^\alpha$ be the odd signature operator with the Atiyah-Patodi-Singer boundary conditions given by the projection on $V \oplus F^+$. Then, $D_{X,V}^\alpha$ has a well-defined eta invariant $\eta(D_{X,V}^\alpha)$ (see for example Kirk Lesch [29, Theorem 3.1]). We set

$$\eta_\alpha(X, V) := \eta(D_{X,V}^\alpha).$$

As we have seen in Section 2.2.1, associated to X there is a canonical Lagrangian $V_X^\alpha \subseteq H^*(\Sigma, \mathbb{C}^n_\alpha)$ defined as the image of the restriction map $H^*(\Sigma, \mathbb{C}^n_\alpha) \to H^*(\Sigma, \mathbb{C}^n_\alpha)$.

We recall here some of the basic properties of the eta invariants of Kirk and Lesch. The manifolds are understood to be odd dimensional, compact and oriented, with a metric of product form near the boundary. We start from the following list of results extending the corresponding ones for the Atiyah-Patodi-Singer eta invariants (see Proposition 2.1.2). They are mostly implicit in the work of Kirk and Lesch, but they can all be proved easily from the definition.

Proposition 2.2.3 (Properties of the Kirk-Lesch eta invariant)**.**

(i) Let $f\colon Y \to X$ be an orientation preserving isometry. Then, we have $\eta_{f^*\alpha}(Y, f^*(V)) = \eta_\alpha(X, V)$.

(ii) On the other hand, we have $\eta_\alpha(-X, V) = -\eta_\alpha(X, V)$.

(iii) If $\alpha, \beta \in \mathcal{U}_n(X)$ are isomorphic and $\varphi\colon H^*(X; \mathbb{C}^n_\alpha) \to H^*(X; \mathbb{C}^n_\beta)$ is the induced isomorphism, we have $\eta_\alpha(X, V) = \eta_\beta(X, \varphi(V))$.

(iv) $\eta_{\alpha\oplus\beta}(X, V \oplus W) = \eta_\alpha(X, V) + \eta_\beta(X, W)$.

(v) *If* $\varepsilon \in \mathcal{U}_n(X)$ *is trivial and* V *is a Lagrangian subspace of* $H^*(\Sigma; \mathbb{C})$, *we have* $\eta_\varepsilon(X, V^n) = n\,\eta(X, V)$.

(vi) $\eta_\alpha(X \sqcup X', V \oplus V') = \eta_\alpha(X, V) + \eta_\alpha(X', V')$.

The following consequence of the first two properties of Proposition 2.2.3 is a very useful computational tool.

Corollary 2.2.4. *If* X *admits an orientation-reversing isometry* $f\colon X \to X$ *such that* $f^*\alpha = \alpha$ *and* $f^*V = V$, *then* $\eta_\alpha(X, V) = 0$.

Proof. The same map f can be seen as an orientation preserving isometry from X to $-X$. We have hence

$$\eta_\alpha(X, V) = \eta_{f^*\alpha}(X, f^*(V)) \overset{(i)}{=} \eta_\alpha(-X, V) \overset{(ii)}{=} -\eta_\alpha(X, V),$$

and the statement follows immediately. □

The main two results of Kirk and Lesch about the eta invariant for manifolds with boundary can be summarized in the following way.

Theorem 2.2.5 (Kirk-Lesch)**.** *Let* X *be a compact, oriented Riemannian manifold of odd dimension with product metric near* $\Sigma = \partial X$. *Let* $\alpha \in \mathcal{U}(X)$ *be a local coefficient system, and set* $H = H^*(\partial X; \mathbb{C}^n_\alpha)$. *Then:*

(i) *for every Lagrangian subspace* $L \in \mathcal{L}ag(H)$, *we have*

$$\eta_\alpha(X, L) - \eta_\alpha(X, V^\alpha_X) = m_H(V^\alpha_X, \gamma(L));$$

(ii) *if* Y *is a compact, oriented Riemannian manifold with* $\partial Y = -\Sigma$ *and product metric near the boundary, and* α *extends on* $X \cup_\Sigma Y$, *then*

$$\eta_\alpha(X \cup Y) = \eta_\alpha(X, V^\alpha_X) + \eta_\alpha(Y, V^\alpha_Y) + m_H(V^\alpha_X, V^\alpha_Y).$$

Proof. (i) is the statement of [28, Theorem 3.2 (i)], and (ii) is the content of [29, p. 618 (8.32)] (see also [28, p. 632 (2.5)]). □

Remark 2.2.6. Although we adopt a different convention in orienting the boundary, the formulas coincide with those of Kirk and Lesch because the operator γ and hence the structure of Hermitian symplectic space is also reversed (see Warning 1.4.5 and 2.2.1).

Remark 2.2.7. In (ii), the correction term is not symmetric in X and Y, as $m_H(V_Y^\alpha, V_X^\alpha) = -m_H(V_X^\alpha, V_Y^\alpha)$. The reason is that the orientation of Σ coincides by assumption with the one induced from being the boundary of X, while $\partial Y = -\Sigma$ instead. If we want to exchange the role of X and Y, hence, we need to consider Σ with the opposite orientation, and the minus sign above is compensated by the fact that $m_{H^-} = -m_H$.

2.2.3 The rho invariant for manifolds with boundary

As in the case of closed manifolds, we can define a rho invariant as the difference between the twisted and untwisted eta invariant, and hope to get rid of the dependence on the metric. Unfortunately this is only true for the metric in the interior, while the invariant will still depend on the metric on the boundary (as well as on the boundary conditions). Following Kirk and Lesch, we give the following definition.

Definition 2.2.8. Let X be an odd dimensional compact, oriented manifold with boundary. Let g be a Riemannian metric on $\Sigma = \partial X$. Let $\alpha \in \mathcal{U}_n(X)$ be a local coefficient system, and let $V \subseteq H^*(X, \mathbb{C}^n_\alpha)$ and $W \subseteq H^*(X, \mathbb{C}^n)$ be two Lagrangian subspaces. We set

$$\rho_\alpha(X, g, V, W) := \eta_\alpha(X, V) - \eta_\varepsilon(X, W),$$

where the eta invariants are defined using any Riemannian metric X of product form near Σ extending the metric g.

Notation 2.2.9. For the rho invariant corresponding to the canonical Lagrangians of X, we use the shortened notation

$$\rho_\alpha(X, g) := \rho_\alpha(X, g, V_X^\alpha, V_X^\epsilon) = \eta_\alpha(X, V_X^\alpha) - n\,\eta(X, V_X).$$

A priori, $\rho_\alpha(X, g, V, W)$ depends on the chosen Riemannian metric on X that extends the metric g of the boundary. In fact, it does not, as the following result of Kirk and Lesch shows. They prove the result using a gluing formula for the so-called "reduced eta-invariants" [29, Lemma 8.16], that we have not introduced. As the argument is simple, we repeat the proof using the gluing formula of Theorem 2.2.5 instead.

Theorem 2.2.10 (Kirk-Lesch). *$\rho_\alpha(X, g, V, W)$ is independent of the metric in the interior of X.*

Proof. Let X, X' denote the same manifold with two different Riemannian metrics of product form near the boundary Σ, and suppose that the two Riemannian metrics agree on Σ. We need to show that

$$\eta_\alpha(X, V) - \eta_\varepsilon(X, W) = \eta_\alpha(X', V) - \eta_\varepsilon(X', W) \tag{2.8}$$

Consider the closed Riemannian manifold $M := X \cup (-X')$, and pull back α to M by means of the natural retraction $M \to X$. As a manifold, M is just the double of X, and we have the identifications $V^\alpha_{X'} = V^\alpha_X$ and $V^\epsilon_{X'} = V^\epsilon_X$. By Theorem 2.2.5 (i) and Proposition 2.2.3 (ii), we get hence

$$\eta_\alpha(M) = \eta_\alpha(X, V^\alpha_X) - \eta_\alpha(X', V^\alpha_{X'}), \quad \text{and} \quad \eta_\varepsilon(M) = \eta_\varepsilon(X, V^\epsilon_X) - \eta_\varepsilon(X', V^\epsilon_{X'}).$$

As $\eta_\varepsilon(M)$ is just n times the eta invariant $\eta(M)$ associated to the untwisted odd signature operator, the difference $\eta_\alpha(M) - \eta_\varepsilon(M)$ is the Atiyah-Patodi-Singer rho invariant of M, and it is thus independent of the metric by Theorem 2.1.4. It is actually zero, as M admits an orientation-reversing self-diffeomorphism f such that $f^*\alpha = \alpha$, constructed by sending the element $x \in X$ to the corresponding $x \in -X'$. By rearranging the terms, it follows that

$$\eta_\alpha(X, V^\alpha_X) - \eta_\varepsilon(X, V^\alpha_X) = \eta_\alpha(X', V^\alpha_{X'}) - \eta_\varepsilon(X', V^\alpha_{X'}).$$

We have thus proved (2.8) for the canonical boundary conditions. The statement can be extended to arbitrary boundary conditions V, W by Theorem 2.2.5 (i), whose correction term does not depend on the Riemannian metric on the interior. □

The results Proposition 2.2.3 can be readily restated in terms of rho invariants. We shall only write explicitly the first two properties.

Proposition 2.2.11. *Let $f\colon X \to Y$ be an orientation preserving diffeomorphism. Then, for every local coefficient system $\alpha \in \mathcal{U}_n(Y)$ and Lagrangian subspaces $V \subseteq H^*(\partial Y; \mathbb{C}^n_\alpha)$ and $W \subseteq H^*(\partial Y; \mathbb{C}^n_\varepsilon)$, we have*

$$\rho_{f^*\alpha}(X, f^*g, f^*V, f^*W) = \rho_\alpha(Y, g, V, W).$$

On the other hand, we have $\rho_\alpha(-X, g, V, W) = -\rho_\alpha(X, g, V, W)$.

Theorem 2.2.5 also gives for free formulas for the rho invariant. For example, the gluing formula for $M = X \cup_\Sigma Y$ gets written as

$$\rho_\alpha(M) = \rho_\alpha(X, g) + \rho_\alpha(Y, g) + m(V^\alpha_X, V^\alpha_Y) - n\, m(V_X, V_Y).$$

There is one more result that will be essential in the applications. Recall that a *pseudo-isotopy* between two diffeomorphisms $f, g : \Sigma \to \Sigma$ is a diffeomorphism $F : [0,1] \times \Sigma \to [0,1] \times \Sigma$ such that

$$F(0,x) = (0, f(x)), \quad F(1,x) = (1, g(x)) \quad \forall x \in \Sigma.$$

It is clear that isotopic diffeomorphisms are always pseudo-isotopic, and that pseudo-isotopic diffeomorphisms are smoothly homotopic.

Definition 2.2.12. Two Riemannian metrics g, g' on Σ are said to be *pseudo-isotopic* if there exists a self-diffeomorphism f on Σ which is pseudo-isotopic to the identity and such that $g' = f^*(g)$.

Kirk and Lesch prove the following result for the canonical boundary conditions [28, Corollary 5.2]. We repeat here their proof, and check that it actually applies to arbitrary boundary conditions.

Proposition 2.2.13. *$\rho_\alpha(X, g, V, W)$ only depends on g up to pseudo-isotopy.*

Proof. Let g, g' be two pseudo-isotopic Riemannian metrics on ∂X, and let $f\colon \partial X \to \partial X$ be a diffeomorphism that is pseudo-isotopic to the identity such that $g' = f^*(g)$. Let $F : [0,1] \times \partial X \to [0,1] \times \partial X$ be a pseudo-isotopy, so that $F(0,\cdot) = \mathrm{id}, F(1,\cdot) = f$. Now, we can see $(0,1] \times \partial X$ as a collar of ∂X in X, and extend F to a diffeomorphism $F\colon X \to X$ by defining it to be the identity outside of $(0,1] \times \partial X$. By definition, F is homotopic to the identity on X, and hence we have a natural isomorphism $F^*\alpha \cong \alpha$, up to which there is an identification $F^*V = V$, and of course $F^*W = W$ as F^* is the identity on untwisted cohomology. As F restricts to f on the boundary, moreover, we have $F^*g = g'$. From Proposition 2.2.11, it follows now, as desired, that

$$\rho_\alpha(X, g', V, W) = \rho_\alpha(X, g, V, W).$$

□

2.3 Eta invariants and the Maslov index

In this section, we develop some gluing formulas whose correction term is represented by a Maslov triple index. In Section 2.3.1, we define a slightly modified eta invariant that behaves better for this purpose. In Section 2.3.2, we prove the main result about the decomposition of a closed manifold into two manifolds with boundary. In Section 2.3.3, we show that in many cases the Maslov index is only non trivial in middle cohomological degree. In Section 2.3.4, we improve the cut-and-paste formula for the Atiyah-Patodi-Singer rho invariant discussed in Section 2.1.3.

2.3.1 A little change of the definition

We start by introducing the following notation.

Notation 2.3.1. Let X be a compact, oriented Riemannian manifold of odd dimension, with metric of product form near the boundary. Let $\alpha \in \mathcal{U}_n(X)$ be a local coefficient system, and let $V \subseteq H^*(\partial X; \mathbb{C}^n_\Sigma)$ be a Lagrangian subspace. We set

$$\overline{\eta}_\alpha(X, V) := \eta_\alpha(X, \gamma(V)).$$

Let us compute explicitly the difference between $\eta_\alpha(X)$ and $\overline{\eta}_\alpha(X)$. For this purpose, we first prove a more general result that will turn useful many times.

Lemma 2.3.2. *Suppose that $H^*(\partial X, \mathbb{C}^n_\alpha)$ splits as an orthogonal sum of balanced Hermitian symplectic subspaces $H_1 \oplus H_2$, and let $V_1 \in \mathcal{L}ag(H_1)$ and $V_2 \in \mathcal{L}ag(H_2)$ be two Lagrangian subspaces. Then*

$$\eta_\alpha(X, V_1 \oplus \gamma(V_2)) - \eta_\alpha(X, V_1 \oplus V_2) = \tau(\gamma(V_1) \oplus V_2, \gamma(V_1) \oplus \gamma(V_2), V^\alpha_X).$$

Proof. By adding and subtracting $\eta_\alpha(X, V^\alpha_X)$, a double application of Theorem 2.2.5 (i) gives us

$$\eta_\alpha(X, V_1 \oplus \gamma(V_2)) - \eta_\alpha(X, V_1 \oplus V_2) = m(V^\alpha_X, \gamma(V_1) \oplus V_2) - m(V^\alpha_X, \gamma(V_1) \oplus \gamma(V_2)).$$

We add $0 = m(\gamma(V_1) \oplus V_2, \gamma(V_1) \oplus \gamma(V_2))$ and, by skew-symmetry of m, we rewrite the right-hand term as

$$m(V^\alpha_X, \gamma(V_1) \oplus V_2) + m(\gamma(V_1) \oplus \gamma(V_2), V^\alpha_X) + m(\gamma(V_1) \oplus V_2, \gamma(V_1) \oplus \gamma(V_2)).$$

The first statement follows then from Proposition 1.2.15. □

Proposition 2.3.3. $\overline{\eta}_\alpha(X, V) = \eta_\alpha(X, V) + \tau(V, \gamma(V), V^\alpha_X)$.

Proof. It is enough to apply Lemma 2.3.2 to the case where H_1 and V_1 are trivial, so that $V_2 = V$. □

Given an odd dimensional manifold X with a local coefficient system $\alpha \in \mathcal{U}_n(X)$, a Riemannian metric g on $\Sigma = \partial X$ and Lagrangian subspaces $V \subseteq H^*(\Sigma, \mathbb{C}^n_\alpha)$, $W \subseteq H^*(\Sigma, \mathbb{C}^n_\varepsilon)$, we can define

$$\overline{\rho}_\alpha(X, g, V, W) := \rho_\alpha(X, g, \gamma(V), \gamma(W)).$$

This means that $\overline{\rho}_\alpha(X, g, V, W) = \overline{\eta}_\alpha(X, V) - \overline{\eta}_\varepsilon(X, W)$ for some choice of Riemannian metric on X of cylindrical form near Σ extending g. In particular, $\overline{\rho}_\alpha(X, g, V, W)$ satisfies gluing and change of boundary conditions formulas which are the relative version of those listed above, the correction term being a difference of Maslov indices that do not depend on the Riemannian metric. We shall not write these formulas explicitly, but refer instead to the gluing formulas about the eta invariants whenever needed.

2.3.2 Gluing along the boundary

Let M be a closed Riemannian manifold that splits as $M = X \cup_\Sigma Y$, with $\Sigma = \partial X$, such that the metric has product form around Σ. Let $\alpha \in \mathcal{U}_n(M)$ be a local coefficient system. Set $H = H_*(\Sigma, \mathbb{C}^n_\alpha)$. The driving reason for the definition of $\overline{\eta}$ is the following result.

Proposition 2.3.4. *Let $L \subseteq H$ be any Lagrangian subspace. Then*

$$\eta_\alpha(M) = \overline{\eta}_\alpha(X, L) + \overline{\eta}_\alpha(Y, L) + \tau_H(L, V^\alpha_X, V^\alpha_Y).$$

Proof. By Proposition 2.2.3 (iv), we have

$$\eta_\alpha(M) = \eta_\alpha(X, V^\alpha_X) + \eta_\alpha(Y, V^\alpha_Y) + m(V^\alpha_X, V^\alpha_Y). \tag{2.9}$$

Changing the boundary conditions on X and Y with (iii) of the same proposition, and using then the definition of $\overline{\eta}$ and the basic properties of m (see Proposition 1.2.13), we get

$$\eta_\alpha(X, V^\alpha_X) = \eta_\alpha(X, \gamma(L)) - m(V^\alpha_X, L) = \overline{\eta}_\alpha(X, L) + m_H(L, V^\alpha_X),$$

$$\eta_\alpha(Y, V^\alpha_Y) = \eta_\alpha(Y, \gamma(L)) - m_{H^-}(V^\alpha_Y, L) = \overline{\eta}_\alpha(Y, V) + m_H(V^\alpha_X, L).$$

Substituting these two equations into (2.9) and applying Proposition 1.2.15, we get the desired result. □

Remark 2.3.5. Using the invariants η instead of $\overline{\eta}$, the main formula of Proposition 2.3.4 reads as

$$\eta_\alpha(M) = \eta_\alpha(X, V) + \eta_\alpha(Y, V) + \tau_H(\gamma(L), V^\alpha_X, V^\alpha_Y).$$

In particular, even though the correction term is expressed in terms of a Maslov index, it still depends on the Riemannian metric, because the operator γ does. Instead, with the introduction of $\overline{\eta}$, we obtain an invariant whose correction term under gluing is independent of the Riemannian metric.

2.3.3 The Maslov index of graded Lagrangians

We call a Lagrangian subspace $V \subseteq H^*(\Sigma; \mathbb{C}^n_\alpha)$ *graded* if it splits as

$$V = \bigoplus_{i=0}^{2k} V^i, \quad V^i \subseteq H^i(\Sigma; \mathbb{C}^n_\alpha) \quad \forall i.$$

Example 2.3.6. Canonical Lagrangians V^α_X are graded and their Lagrangian actions (associated to cobordisms) transform graded Lagrangians into graded Lagrangians.

The following result allows to reduce the computation of the Maslov index of graded Lagrangians to middle degree.

Proposition 2.3.7. *Let Σ be any $2k$-dimensional closed, oriented manifold with a local coefficient system $\alpha \in \mathcal{U}_n(\Sigma)$, and let $U, V, W \subseteq H^*(\Sigma; \mathbb{C}^n_\alpha)$ be three graded Lagrangians. Then*

$$\tau(U, V, W) = \tau(U^k, V^k, W^k).$$

Proof. The complex symplectic space $H^*(\Sigma; \mathbb{C}^n_\alpha)$ splits as a direct sum of two complex symplectic subspaces in the following way:

$$H^*(\Sigma; \mathbb{C}^n_\alpha) = H^k(\Sigma; \mathbb{C}^n_\alpha) \oplus H^{\neq k}(\Sigma; \mathbb{C}^n_\alpha),$$

where $H^{\neq k}(\Sigma; \mathbb{C}^n_\alpha) := \bigoplus_{i \neq k} H^i(\Sigma; \mathbb{C}^n_\alpha)$. Since the three Lagrangian subspaces split accordingly as $U = U^k \oplus U^{\neq k}$, $V = V^k \oplus V^{\neq k}$ and $W = W^k \oplus W^{\neq k}$, in view of Proposition 1.2.3 (ii), we have

$$\tau(U, V, W) = \tau(U^k, V^k, W^k) + \tau(U^{\neq k}, V^{\neq k}, W^{\neq k}).$$

In order to prove the desired result, it is enough to show that the second summand is zero. The spaces $H^{<k}(\Sigma; \mathbb{C}^n_\alpha)$ and $H^{>k}(\Sigma; \mathbb{C}^n_\alpha)$ are two Lagrangian subspaces of $H^{\neq k}(\Sigma; \mathbb{C}^n_\alpha)$, and once again, the Lagrangians $U^{\neq k}$, $V^{\neq k}$ and $W^{\neq k}$ split accordingly. Thanks to Lemma 1.2.6, we have

$$\tau(U^{\neq k}, V^{\neq k}, W^{\neq k}) = 0,$$

and the proof is completed. □

2.3.4 Cut-and-paste revisited

We use now the gluing formulas, together with the above "reduction to middle degree", in order to prove a cut-and-paste formula for the Atiyah-Patodi-Singer eta invariants, that implies its version for rho invariants Proposition 2.1.7 as an immediate corollary. As in Section 2.1.3, suppose we have splittings of Riemannian manifolds $X_1 \cup_\Sigma X_2$, $X_1 \cup_\Sigma X_0$ and $-X_0 \cup_\Sigma X_2$, such that the metrics have product form around Σ (look at the referred section for more details). We have then the following result. Let $2k+1$ be the dimension of these manifolds, so that Σ has dimension $2k$.

Warning 2.3.8. In the next result, the same notation $V^\alpha_{X_i}$ is used for both the full canonical Lagrangian in $H^*(\Sigma;\mathbb{C}^n_\alpha)$ and for its restriction to $H^k(\Sigma;\mathbb{C}^n_\alpha)$. The space will be specified in the text.

Proposition 2.3.9. *Let $\alpha \in \mathcal{U}_n(X_1 \cup X_2 \cup X_0)$ be a local coefficient system, and set $H := H^k(\Sigma;\mathbb{C}^n_\alpha)$. Then, we have*

$$\eta_\alpha(X_1 \cup_\Sigma X_2) = \eta_\alpha(X_1 \cup_\Sigma X_0) + \eta_\alpha(-X_0 \cup_\Sigma X_2) + \tau_H(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}).$$

Proof. We apply Proposition 2.3.4 to the three gluings, in all cases with $L = V^\alpha_{X_0}$, obtaining thus

$$\begin{aligned}
\eta_\alpha(X_1 \cup_\Sigma X_2) &= \overline{\eta}_\alpha(X_1, V^\alpha_{X_0}) + \overline{\eta}_\alpha(X_1, V^\alpha_{X_0}) + \tau_{H'}(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}),\\
\eta_\alpha(X_1 \cup_\Sigma X_0) &= \overline{\eta}_\alpha(X_1, V^\alpha_{X_0}) + \overline{\eta}_\alpha(X_0, V^\alpha_{X_0}) + \tau_{H'}(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_0}),\\
\eta_\alpha(-X_0 \cup_\Sigma X_2) &= -\overline{\eta}_\alpha(X_0, V^\alpha_{X_0}) + \overline{\eta}_\alpha(X_1, V^\alpha_{X_0}) + \tau_{H'}(V^\alpha_{X_0}, V^\alpha_{X_0}, V^\alpha_{X_2}),
\end{aligned}$$

where H' is the symplectic space $H^*(\Sigma;\mathbb{C}^n_\alpha)$. Note that, in the second and third formula, the Maslov triple index vanishes, as two of the variables coincide. Comparing the three formulas, we obtain

$$\eta_\alpha(X_1 \cup_\Sigma X_2) = \eta_\alpha(X_1 \cup_\Sigma X_0) + \eta_\alpha(-X_0 \cup_\Sigma X_2) + \tau_{H'}(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}). \quad (2.10)$$

In order to complete the proof, we need to show that

$$\tau_{H'}(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}) = \tau_H(V^\alpha_{X_0}, V^\alpha_{X_1}, V^\alpha_{X_2}),$$

i.e. that the Maslov triple index computed on the full symplectic space $H^*(\Sigma;\mathbb{C}^n_\alpha)$ coincides with the one computed on its subspace $H^k(\Sigma;\mathbb{C}^n_\alpha)$. As the three Lagrangian subspaces are all graded, this follows from Proposition 2.3.7. □

Remark 2.3.10. The first part of the proof, up to (2.10), can be also carried out with the original formulas of Theorem 2.2.5, together with Proposition 1.2.15 in order to rewrite the correction term as a Maslov triple index. This was probably noticed by Kirk and Lesch, who performed the same computation in their discussion about Wall's non-additivity for the signature [29, Section 8.3]. We made this statement more explicit, and completed the argument by showing that the contribution of the Maslov index is indeed only non-trivial in middle degree.

2.4 Eta invariants and cobordisms

The gluing formulas discussed so far only apply when the two manifolds are glued along their whole boundary. In this section, using the formalism about Lagrangians actions, we generalize such formulas to the gluing of cobordisms, obtaining the main theorem of this chapter. In Section 2.4.1, we start our discussion by studying the eta invariant of a trivial cobordism, while in Section 2.4.2, we prove the main formula.

2.4.1 Eta invariants of cylinders

If Σ is an even dimensional Riemannian manifold with a local coefficient system $\alpha \in \mathcal{U}_n(\Sigma)$, for all $r > 0$ we can consider the Riemannian product

$$X_r := [0, r] \times \Sigma.$$

The local coefficient system clearly extends as a product over X_r, and we can ask ourselves what is the value of the eta invariant of this manifold with a given boundary condition. Taking care of the orientations, the boundary of X_r can be described as $\partial X_r = (-\{0\} \times \Sigma) \sqcup (\{r\} \times \Sigma)$, so that

$$H^*(\partial X_r, \mathbb{C}^n_\alpha) = H^*(\Sigma, \mathbb{C}^n_\alpha)^- \oplus H^*(\Sigma, \mathbb{C}^n_\alpha),$$

where, as usual, the minus sign on top denotes the reversal of the symplectic structure. Let H denote the Hermitian symplectic space $H^*(\Sigma, \mathbb{C}^n_\alpha)$.

Proposition 2.4.1. *For all Lagrangian subspace $L \subseteq H^- \oplus H$, we have*

$$\overline{\eta}_\alpha(X_r, L) = m_{H^- \oplus H}(\Delta_H, L).$$

In particular, the eta invariant of X_r does not depend on the length r.

Proof. The canonical Lagrangian $V^{\alpha}_{X_r}$ is the diagonal $\Delta_H \subseteq H^- \oplus H$, and it is preserved by the orientation-reversing isometry $f\colon X_r \to X_r$ defined by $f(t,x) := (r-t,x)$. The local coefficient system is also preserved , in the sense that $f^*\alpha = \alpha$. Hence, by Corollary 2.2.4, we have $\eta_\alpha(X_r, V^{\alpha}_{X_r}) = 0$. Using the definition of $\overline{\eta}$ and Proposition 2.2.3 (iii), we get

$$\overline{\eta}_\alpha(X_r, L) = \eta_\alpha(X_r, \gamma(L)) = \eta_\alpha(X_r, V^{\alpha}_{X_r}) + m_{H^-\oplus H}(V^{\alpha}_{X_r}, L) = 0 + m_{H^-\oplus H}(\Delta_H, L).$$

□

Suppose now that the signature of Σ is 0, so that the Hermitian symplectic space $H^*(\Sigma, \mathbb{C}^n_\alpha)$ is balanced. Then, we can consider Lagrangian subspaces of $H^*(\partial X_r, \mathbb{C}^n_\alpha)$ of the form $L_1 \oplus L_2$, with $L_1 \in \mathcal{L}ag(H^-)$ and $L_2 \in \mathcal{L}ag(H)$. Identifying as usual the sets $\mathcal{L}ag(H^-)$ and $\mathcal{L}ag(H)$, we get the following result, that can be compared with the original formula of Lesch and Wojcechowski for eta invariants of differential operators on cyclinders [30, Theorem 2.1].

Corollary 2.4.2. *Let* $L_1, L_2 \in \mathcal{L}ag(H)$. *Then* $\overline{\eta}_\alpha(X_r, L_1 \oplus L_2) = m_H(L_1, L_2)$.

Proof. By Proposition 2.4.1, we have

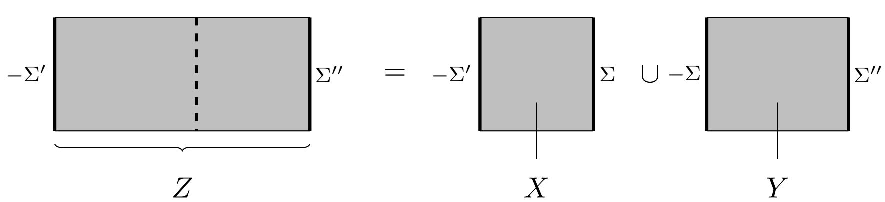
$$\overline{\eta}_\alpha(X_r, L_1 \oplus L_2) = m_{H^-\oplus H}(\Delta_H, L_1 \oplus L_2),$$

and the result follows immediately from Corollary 1.5.13. □

2.4.2 The main gluing formula

Proposition 2.3.4 has a limitation, in that it only applies to the case where two manifolds are attached along their full boundary, giving rise to a closed manifold. We give now a generalization that allows X and Y to have extra boundary components.

Let Z be a compact, oriented manifold that splits as $Z = X \cup_\Sigma Y$, with $\partial X = -\Sigma' \sqcup \Sigma$ and $\partial Y = -\Sigma \sqcup \Sigma''$, as it is schematically represented by the next picture.



Then, we can see X as a cobordism from Σ' to Σ and Y as a cobordism from Σ to Σ''. In this way, as a composition of cobordisms we have $Z = YX$. Let now $\alpha \in \mathcal{U}_n(Z)$ be a local coefficient system, and set

$$H = H_*(\Sigma, \mathbb{C}^n_\alpha), \quad H' = H_*(\Sigma, \mathbb{C}^n_\alpha), \quad H = H_*(\Sigma'', \mathbb{C}^n_\alpha).$$

Then, as in Section 1.5.1, we have Lagrangian actions

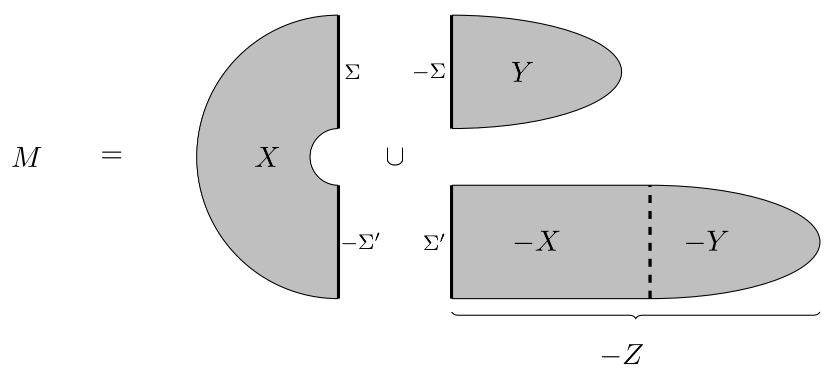
$$(V^\alpha_X)_* \colon \mathcal{L}ag(H') \to \mathcal{L}ag(H), \quad (V^\alpha_{Y^t})_* \colon \mathcal{L}ag(H'') \to \mathcal{L}ag(H)$$

(we will often omit the lower $*$ from the notation). Suppose that $Z = YX$ is equipped with a Riemannian metric, which has product form near $\partial Z = -\Sigma' \sqcup \Sigma''$ and around the gluing hypersurface Σ. We have then the following result, generalizing Proposition 2.3.4.

Theorem 2.4.3. *Let $Z = X \cup_\Sigma Y$ as above, and let $L \subseteq H$, $L' \subseteq H'$ and $L'' \subseteq H''$ be arbitrary Lagrangian subspaces. Then*

$$\overline{\eta}_\alpha(Z, L' \oplus L'') = \overline{\eta}_\alpha(X, L' \oplus L) + \overline{\eta}_\alpha(Y, L \oplus L'') + \tau_H(L, V^\alpha_X(L'), V^\alpha_{Y^t}(L'')).$$

Proof. We first prove the result in the case where $\Sigma'' = \emptyset$, so that only X has boundary components outiside of the gluing area. In this case, consider the Riemannian double $M := (-Z) \cup_{\Sigma'} Z$, and pull back the local coefficient system α to M using the natural retraction $M \to Z$. We decompose now M as $M = X \cup_{(-\Sigma') \sqcup \Sigma} ((-Z) \sqcup Y)$.



Applying Proposition 2.3.4 to the above decomposition, we get

$$\eta_\alpha(M) = \overline{\eta}_\alpha(X, L' \oplus L) + \overline{\eta}_\alpha((-Z) \sqcup Y, L' \oplus) + \tau(L' \oplus L, V^\alpha_X, V^\alpha_Z \oplus V^\alpha_Y).$$

For symmetry reasons, we have $\eta_\alpha(M) = 0$. Moreover, for the disjoint union $Y \sqcup (-Z)$ we have

$$\overline{\eta}_\alpha((-Z) \sqcup Y, L' \oplus L) = \overline{\eta}_\alpha(Y, L) - \overline{\eta}_\alpha(Z, L').$$

Rearranging the terms in the equation and applying an even permutation to the variables of the Maslov index, we obtain

$$\overline{\eta}_\alpha(Z, L') = \overline{\eta}_\alpha(X, L \oplus L') + \overline{\eta}_\alpha(Y, L) + \tau(V_Z^\alpha \oplus V_Y^\alpha, L' \oplus L, V_X^\alpha).$$

The complex symplectic space on which we are computing the Maslov index is $H^*(\Sigma'; \mathbb{C}^n_\alpha)^- \oplus H^*(\Sigma; \mathbb{C}^n_\alpha)$. Seeing V_X^α as a Lagrangian relation from $H^*(\Sigma'; \mathbb{C}^n_\alpha)$ to $H^*(\Sigma; \mathbb{C}^n_\alpha)$, we can apply Proposition 1.5.10 and rewrite the correction term as

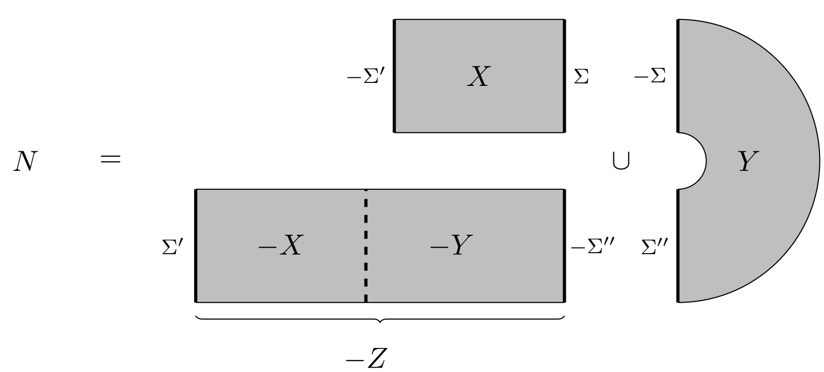
$$\tau(V_Z^\alpha \oplus V_Y^\alpha, L' \oplus L, V_X^\alpha) = -\tau(V_Z^\alpha, L', V_{X^t}^\alpha(V_Y^\alpha)) + \tau(V_Y^\alpha, L, V_X^\alpha(L'))$$

(following Notation 1.5.8, we dropped the stars out of the induced map). In the first summand of the right side of the equation, two of the three variables coincide, as $V_{X^t}^\alpha(V_Y^\alpha) = V_Z^\alpha$ by Proposition 1.5.6. Hence, the first summand vanishes. As for the second summand, we apply an even permutation and get

$$\overline{\eta}_\alpha(Z, L') = \overline{\eta}_\alpha(X, L \oplus L') + \overline{\eta}_\alpha(Y, L) + \tau(L, V_X^\alpha(L'), V_Y^\alpha), \tag{2.11}$$

which is the desired result in the case $\Sigma'' = \emptyset$.

Let us now prove the general case. We form in this case the Riemannian manifold $N := (-Z) \cup_{\Sigma''} Z$, which has boundary $\partial N = (-\Sigma') \sqcup \Sigma'$. We extend the local coefficient system α on N by symmetry as we did before. We consider now the splitting $N = (X \sqcup (-Z)) \cup_{\Sigma \sqcup (-\Sigma'')} Y$.



As Y gets glued along all of its boundary components, we are in the setting of the first part of the proof. Applying (2.11) to this decomposition of N, we obtain

$$\overline{\eta}_\alpha(N, L' \oplus L') = \overline{\eta}_\alpha(X \sqcup (-Z), (L \oplus L') \oplus (L' \oplus L'')) + \overline{\eta}_\alpha(Y, L \oplus L'') + \\ + \tau(L \oplus L'', V_X^\alpha(L') \oplus V_Z^\alpha(L'), V_Y^\alpha).$$

Similarly to what happened in the first part of the proof, we have $\overline{\eta}_\alpha(N, L' \oplus L')=0$. Writing the eta invariant of $X \sqcup (-Z)$ as a sum of the eta invariants of X and $-Y$ and rearranging the terms, we get

$$\overline{\eta}_\alpha(Z, L' \oplus L'') = \overline{\eta}_\alpha(X, L \oplus L') + \overline{\eta}_\alpha(Y, L \oplus L'') + \tau(L \oplus L'', V_X^\alpha(L') \oplus V_Z^\alpha(L'), V_Y^\alpha).$$

The complex symplectic space on which we are computing the Maslov index is $H^*(\Sigma; \mathbb{C}^n_\alpha) \oplus H^*(\Sigma''; \mathbb{C}^n_\alpha)^-$. Proposition 1.5.10 now tells us

$$\tau(L \oplus L'', V_X^\alpha(L') \oplus V_Z^\alpha(L'), V_Y^\alpha) = \\ = \tau(L, V_X^\alpha(L'), V_{Y^t}^\alpha(L'')) - \tau(L'', V_Z^\alpha(L'), V_Y^\alpha(V_X^\alpha(L'))).$$

(we had to put the minus sign on the second summand instead that on the first, because in our splitting the opposite structure is on the right-hand summand). But the second summand vanishes, because $V_Y^\alpha(V_X^\alpha(L')) = V_Z^\alpha(L')$ thanks to Proposition 1.5.10 and covariance of the induced map. Hence, we are left with the desired formula. □

Remark 2.4.4. In the situation of $\Sigma'' = \emptyset$, which corresponds to the first step in our proof, Kirk and Lesch prove by analytical means the gluing formula [28, Theorem 4.1]

$$\eta_\alpha(Z, V_Z^\alpha) = \eta_\alpha(X, V_Z^\alpha \oplus \gamma(V_Y^\alpha)) + \eta_\alpha(Y, V_Y^\alpha). \tag{2.12}$$

Let us check that (2.12) can be retrieved using Theorem 2.4.3. By choosing $L = \gamma(V_Y^\alpha)$ and $L' = \gamma(V_Z^\alpha)$ and passing from $\overline{\eta}$ to η, our theorem gives

$$\eta_\alpha(Z, V_Z^\alpha) = \eta_\alpha(X, V_Z^\alpha \oplus V_Y^\alpha) + \eta_\alpha(Y, V_Y^\alpha) + \tau(\gamma(V_Y^\alpha), L_X^\alpha(\gamma(V_Z^\alpha)), V_Y^\alpha). \tag{2.13}$$

We want to show that (2.12) and (2.13) are equivalent. Applying Lemma 2.3.2 to the splitting $H^*(\partial X; \mathbb{C}^n_\alpha) = \overline{H^*(\Sigma'; \mathbb{C}^n_\alpha)} \oplus H^*(\Sigma; \mathbb{C}^n_\alpha)$, we get

$$\eta_\alpha(X, V_Z^\alpha \oplus \gamma(V_Y^\alpha)) = \eta_\alpha(X, V_Z^\alpha \oplus V_Y^\alpha) + \tau(\gamma(V_Z^\alpha) \oplus V_Y^\alpha, \gamma(V_Z^\alpha) \oplus \gamma(V_Y^\alpha), V_X^\alpha),$$

and, thanks to Proposition 1.5.10, the correction term can be rewritten as

$$\tau(\gamma(V_Z^\alpha) \oplus V_Y^\alpha, \gamma(V_Z^\alpha) \oplus \gamma(V_Y^\alpha), V_X^\alpha) = \\ = -\tau(\gamma(V_Z^\alpha), \gamma(V_Z^\alpha), L_{X^t}^\alpha(V_Y^\alpha)) + \tau(V_Y^\alpha, \gamma(V_Y^\alpha), L_X^\alpha(\gamma(V_Z^\alpha))) = \\ = 0 + \tau(\gamma(V_Y^\alpha), L_X^\alpha(\gamma(V_Z^\alpha)), V_Y^\alpha),$$

which is what we wanted to show.

Remark 2.4.5. Suppose that X is a cobordism from Σ_0 to Σ_1, Y is a cobordism from Σ_1 to Σ_2 and Z is a cobordism from Σ_2 to Σ_3. In computing the eta invariant of $N = X \cup_{\Sigma_1} Y \cup_{\Sigma_2} Z$, we can either first glue X with Y and then glue the resulting manifold with Z, or first glue Y with Z, and then glue X with the resulting manifold, or even glue $X \sqcup Z$ with Y in a single step. One can check that the correction term coincides at the end in three cases. Let $H_i := H^*(\Sigma_i; \mathbb{C}^n_\alpha)$, and fix Lagrangian subspaces $L_i \subseteq H_i$ for $i = 0, 1, 2, 3$.

(i) By first gluing X with Y and then $X \cup_{\Sigma_1} Y$ with Z, from Theorem 2.4.3 we get a correction term

$$C_1 = \tau_{H_1}(L_1, V^\alpha_X(L_0), V^\alpha_{Y^t}(L_2)) + \tau_{H_2}(L_2, V^\alpha_{X\cup Y}(L_0), V^\alpha_{Z^t}(L_3)).$$

(ii) By first gluing Y with Z, and then X with $Y \cup_{\Sigma_2} Z$, we get

$$C_2 = \tau_{H_1}(L_1, V^\alpha_X(L_0), V^\alpha_{(Y\cup Z)^t}(L_3)) + \tau_{H_2}(L_2, V^\alpha_Y(L_1), V^\alpha_{Z^t}(L_3)).$$

(iii) By gluing $X \sqcup Z$ with Y, we get

$$C_3 = \tau_{H_1 \oplus H_2^-}(L_1 \oplus L_2, V^\alpha_X(L_0) \oplus V_{Z^t}(L_2), V^\alpha_Y).$$

Using Proposition 1.5.6 and Proposition 1.5.10, it is not hard to show that $C_1 = C_2 = C_3$.

Chapter 3

Rho invariants of 3-manifolds with toroidal boundary

3.1 Framed tori and 3-manifolds

In this section we introduce the main objects of our study and prove some basic results about them. In Section 3.1.1, we define framed tori and 3-manifolds with framed toroidal boundary and give some examples. In Section 3.1.2, we study the twisted cohomology of a framed torus, and some important Lagrangians subspaces. In Section 3.1.3, we prove the important observation that a framing on a torus can be used to define (up to isotopy) Riemannian metric on the torus.

3.1.1 Basic definitions

We call a *torus* any surface which is diffeomorphic to $S^1 \times S^1$.

Definition 3.1.1. A *framing* on a torus T is an ordered basis of $H_1(T; \mathbb{Z})$. If $\mathcal{F} = (\mu, \lambda)$ is a framing on T, we call μ the *meridian* of $\mathcal{F}$, and λ the *longitude* of $\mathcal{F}$. We call the triple (T, μ, λ) a *framed torus*.

A compact, oriented 3-manifold whose boundary is a disjoint union of tori with a specified framing will be called a 3-*manifold with framed toroidal boundary*. It is convenient to see 3-manifolds with framed toroidal boundary as couples $(X, \mathcal{F})$, where X is the manifold and $\mathcal{F}$ (the *framing*) is the data of a framing for each boundary component. Framings on the boundary tori of 3-manifolds arise naturally in many topological contexts. Here are two examples.

Example 3.1.2. Let F be a compact oriented surface. Then $F \times S^1$ is a compact, oriented 3-manifolds with toroidal boundary. We give a framing to each boundary component $C \times S^1$ ($C \subseteq \partial F$) by choosing $\mu = [C]$ as the meridian and $\lambda = [S^1]$ as the longitude. We call this framing the *product framing* on $F \times S^1$ and use the notation $\mathcal{F}_F^{\times}$ or simply $\mathcal{F}^{\times}$.

Example 3.1.3. Let L be an oriented link in S^3. The link exterior $X_L := S^3 \backslash N(L)$ is an oriented 3-manifold with toroidal boundary, and a framing $\mathcal{F}_L$ on ∂X_L can be given by the standard definitions of meridians and longitudes of L (see Section 4.1.1). If no additional assumption is given, the longitude of a component K of L is the one being characterized by being homologically trivial in $S^3 \setminus K$. More generally, if L is a *framed* link, the longitudes of ∂X_L are prescribed by the framing on L.

If T is a torus with a framing $\mathcal{F} = (\mu, \lambda)$ and $f : T \to T'$ is a diffeomorphism, there is an induced framing $f_*(\mathcal{F}) := (f_*(\mu), f_*(\lambda))$. The definition extends readily to the case of a diffeomorphism between 3-manifolds with toroidal boundary, one of the two with a framing.

Definition 3.1.4. Two framings on a disjoint union of tori are said to *coincide up to signs* if its is possible to get from one to the other by reversing the sign of some number of meridians and longitudes.

Once a framing $\mathcal{F}_0$ on a torus T is fixed, the set of all framings on T is in a natural bijection with $\mathrm{GL}(2, \mathbb{Z})$, where we can associate to any framing $\mathcal{F}$ the change of basis matrix from $\mathcal{F}$ to $\mathcal{F}_0$. If a framed torus T is given an orientation, there is a well defined intersection pairing $H_1(T, \mathbb{Z}) \times H_1(T, \mathbb{Z}) \to \mathbb{Z}$, and $\mu \cdot \lambda$ is either 1 or -1.

Definition 3.1.5. A framing (μ, λ) on an oriented torus is said to be *standardly oriented* if $\mu \cdot \lambda = -1$. It is said to be *non-standardly oriented* if $\mu \cdot \lambda = 1$.

The terminology comes from the fact that, if (μ, λ) is a standardly oriented framing, then its image in the homology with complex coefficients is a symplectic basis of $H_1(T; \mathbb{C})$ in the sense of Definition 1.2.7. As we shall see more in detail, if in Example 3.1.3 we choose the meridian of a link component K in such a way that $\mathrm{lk}(\mu, K) = 1$, and the longitude λ is oriented coherently with the orientation of K, then the framing is standardly oriented. On the other hand, the product framing of Example 3.1.2 is naturally non-standardly oriented.

3.1.2 Symplectic structure and Lagrangians

Let Σ be a closed, oriented surface with a local coefficient system $\alpha \in \mathcal{U}_n(\Sigma)$. The construction of Section 2.2.1 gives the structure of a complex symplectic vector space to the cohomology with twisted coefficients $H^*(\Sigma; \mathbb{C}^n_\alpha)$. Thanks to Proposition 2.3.7), we shall mostly care about middle degree $H^1(\Sigma; \mathbb{C}^n_\alpha)$, where we see that the symplectic form ω coincides with the intersection form I^α_Σ (see Section 2.2.1). In our applications, Σ is a union of tori with a framing, which is given in terms of (integer) homology classes. For these reason, it is often more natural to work with homology instead of cohomology. We consider thus the Poincaré duality isomorphism

$$\mathrm{PD}_\Sigma \colon H_1(\Sigma; \mathbb{C}^n_\alpha) \xrightarrow{\sim} H^1(\Sigma; \mathbb{C}^n_\alpha),$$

and use it to induce the structure of a complex symplectic space on $H_1(\Sigma; \mathbb{C}^n_\alpha)$. If Σ is the boundary of a compact, oriented 3-manifold X with a local coefficient system $\alpha \in \mathcal{U}_n(X)$, we define now

$$\mathcal{V}^\alpha_X := \ker(H_1(\Sigma; \mathbb{C}^n_\alpha) \to H_1(X; \mathbb{C}^n_\alpha)).$$

Notation 3.1.6. In the case of the trivial one-dimensional local system, leading to the usual cohomology with complex coefficients, we let the above subspace be denoted simply by $\mathcal{V}_X$.

Let $(V^\alpha_X)^1 := (V^\alpha_X) \cap H^1(\Sigma; \mathbb{C}^n_\alpha)$ denote the topological Lagrangian restricted to middle degree.

Lemma 3.1.7. *The space $\mathcal{V}^\alpha_X$ is a Lagrangian subspace of $H_1(\Sigma; \mathbb{C}^n_\alpha)$, and*

$$\mathrm{PD}_\Sigma(\mathcal{V}^\alpha_X) = (V^\alpha_X)^1.$$

If the boundary components of X is decomposed as a disjoint union $\partial X = -Sigma_1 \sqcup \Sigma_2$ and X is seen as a cobordism from Σ_1 to Σ_2, we can use the formalism of Section 1.5.1 and see the above subspace as a Lagrangian relation $\mathcal{V}^\alpha_X \colon H_1(\Sigma_1; \mathbb{C}^n_\alpha) \Rightarrow H_1(\Sigma_2; \mathbb{C}^n_\alpha)$. We have then the following result, where the data of an appropriate local coefficient system α is implicit.

Proposition 3.1.8. *Let Y be a cobordism from a surface Σ_1 to a surface Σ_2, and let $\mathcal{L} \in H_1(\Sigma_1; \mathbb{C}^n_\alpha)$ be a Lagrangian subspace. Then, we have*

$$\mathrm{PD}_{\Sigma_2}((\mathcal{V}^\alpha_Y)_*(\mathcal{L})) = (V^\alpha_Y)^1_*(\mathrm{PD}_{\Sigma_1}(\mathcal{L})).$$

In particular, if X is a compact, oriented 3-manifold with $\partial X = \Sigma$, we have

$$(\mathcal{V}^\alpha_Y)_*(\mathcal{V}^\alpha_X) = \mathcal{V}^\alpha_{YX}.$$

Proof. Consider the symplectic isomoprhism $\mathrm{PD}_{\partial Y}\colon H_1(\partial Y;\mathbb{C}^n_\alpha) \to H^1(\partial Y;\mathbb{C}^n_\alpha)$, that thanks to Lemma 3.1.7 is such that

$$\mathrm{PD}_{\partial Y}(\mathcal{V}^\alpha_Y) = (V^\alpha_Y)^1.$$

As $\partial Y = -\Sigma_1 \sqcup \Sigma_2$, this ismorphism can be described as

$$\mathrm{PD}_{\partial Y} = \mathrm{PD}_{-\Sigma_1} \oplus \mathrm{PD}_{\Sigma_2},$$

and we can thus write the right-hand term of the equation of the first statement as $(\mathrm{PD}_{-\Sigma_1} \oplus \mathrm{PD}_{\Sigma_2}(\mathcal{V}^\alpha_Y))_*(\mathrm{PD}_{\Sigma_1}(\mathcal{L}))$. The statement follows then from Remark 1.5.4, as $\mathrm{PD}_{-\Sigma_1} = -\mathrm{PD}_{\Sigma_1}$. The second statement is a consequence of the first, as we can now write

$$\mathrm{PD}_{\Sigma_2}((\mathcal{V}^\alpha_Y)_*(\mathcal{V}^\alpha_X)) = (V^\alpha_Y)^1_*(\mathrm{PD}_{\Sigma_1}(\mathcal{V}^\alpha_X)),$$

and then, thanks to Lemma 3.1.7 and Proposition 1.5.6, we have

$$(V^\alpha_Y)^1_*(\mathrm{PD}_{\Sigma_1}(\mathcal{V}^\alpha_X)) = (V^\alpha_Y)^1_*((V^\alpha_X)^1) = (V^\alpha_{YX})^1 = \mathrm{PD}_{\Sigma_2}(\mathcal{V}^\alpha_{YX}).$$

□

Remark 3.1.9. The composition of Lagrangian relations as it is defined in Section 1.5.1 is better suited for cohomology than for homology. In fact, the formula $\mathcal{V}^\alpha_{YX} = \mathcal{V}^\alpha_Y\mathcal{V}^\alpha_X$, if Z is the composition of two cobordisms X and Y, fails in this case. Moreover, the Lagrangian $\mathcal{V}^\alpha_C$ associated to a product cobordism $[0,1] \times \Sigma$ is not the diagonal relation, but the "anti-diagonal" instead, and thus it is not the neutral element under composition. [1] Notice in any case that, even though $V^\alpha_{YX} = V^\alpha_Y V^\alpha_X$ is false at the level of Lagrangian relations, the equality

$$(\mathcal{V}^\alpha_Y)_* \circ (\mathcal{V}^\alpha_X)_* = (\mathcal{V}^\alpha_{YX})_*$$

holds true for the maps induced on the sets of Lagrangian subspaces.

Let T be an oriented torus with a local coefficient system $\alpha \in \mathcal{U}_n(T)$, that we shall think of as a representation $\alpha\colon \pi_1(T) \to U(n)$. We give a

[1] The reason of this difference is that the sign of the Poincaré duality isomorphism depends on the orientation of the surface, and thus it is opposite when viewing the same attaching surface from the two cobordisms that are being glued. In principle, the definition of the composition can be slightly modified in order to suit better in this context. We shall not do it, as Proposition 3.1.8 is enough for our goals, and we do not need to ever talk about composition of Lagrangian relations.

way to associate to any element $\nu \in H_*(T;\mathbb{Z})$ a Lagrangian subspace $\mathcal{W}_\nu^\alpha$ of $H_1(T;\mathbb{C}^n_\alpha)$. Represent the class $\nu \in H_*(T;\mathbb{Z})$ as a curve $f\colon S^1 \to T$, and let $\alpha' = f^*\alpha$ the induced local coefficient system on S^1. Then, the map $f_*\colon H_*(S^1;\mathbb{C}^n_{\alpha'}) \to H_*(T;\mathbb{C}^n_\alpha)$ only depends on the initial element ν.

$$\mathcal{W}_\nu^\alpha := \operatorname{im} f_* \subseteq H_*(T;\mathbb{C}^n_\alpha). \tag{3.1}$$

Proposition 3.1.10. *The space $\mathcal{W}_\nu^\alpha$ constructed above is a Lagrangian subspace of $H_1(T;\mathbb{C}^n_\alpha)$.*

Proof. We decompose the representation space $\mathbb{C}^n_\alpha$ as $A \oplus B$, where A is the maximal subspace where the representation acts trivially, and B is a complementary subspace. By Lemma 1.3.8, we have $H_*(T;B) = 0$, and thus there is a natural isomorphism

$$\varphi\colon H_1(T;\mathbb{C}^n_\alpha) \xrightarrow{\sim} H_1(T;A) \oplus H_1(T;B) = H_1(T;A).$$

The isomorphism φ clearly respects the symplectic structure. The induced representation α' on S^1 gets decomposed accordingly, and we get a commutative diagram

$$\begin{array}{ccc} H_1(S^1;\mathbb{C}^n_\alpha) & \xrightarrow{\ f_*\ } & H_1(T;\mathbb{C}^n_\alpha) \\ \downarrow \wr & & \varphi \downarrow \wr \\ H_1(S^1,A) \oplus H_*(S^1,B) & \longrightarrow & H_1(T,A) \oplus 0, \end{array}$$

where the lower horizontal map is also induced by f. In particular $\mathcal{W}_\nu^\alpha$, which is by definition the image of f_*, is isomorphic through φ to

$$\operatorname{im}(H_1(S^1,A) \to H_1(T,A)) = \operatorname{Span}_{\mathbb{C}}\{a_1 \otimes \nu, \dots, a_r \otimes \nu\},$$

where $\{a_1, \dots, a_r\}$ is any basis of A. This is clearly a Lagrangian subspace, because it is a half-dimensional subspace of $H_1(T,A)$ on which the intersection form vanishes. Hence $\mathcal{W}_\nu^\alpha \subseteq H_1(T;\mathbb{C}^n_\alpha)$ is Lagrangian as well. ☐

Example 3.1.11. Let α be a 1-dimensional unitary representation. If α is nontrivial, the whole symplectic space $H_1(T;\mathbb{C}_\alpha)$ vanishes thanks to Lemma 1.3.8. If α is the trivial one-dimensional representation, we have $H_1(T;\mathbb{C}_\alpha) = H_1(T;\mathbb{C})$, and we can describe our Lagrangian explicitly as

$$\mathcal{W}_c^\alpha = \operatorname{Span}_{\mathbb{C}}\{c\}.$$

Let now $\Sigma = T_1 \cup \cdots \cup T_r$ be a collection of framed oriented tori, with framing $\mathcal{F}$ given by a meridian μ_i and longitude λ_i for each torus T_i. Let $\alpha \in \mathcal{U}_n(\Sigma)$ be a local coefficient system. We define Lagrangian subspaces of $H_1(\Sigma; \mathbb{C}^n_\alpha) = \bigoplus_{i=1}^r H_1(T_i; \mathbb{C}^n_\alpha)$ by

$$\mathcal{M}^\alpha_{\mathcal{F}} := \bigoplus_{i=1}^r \mathcal{W}^\alpha_{\mu_i}, \quad \mathcal{L}^\alpha_{\mathcal{F}} := \bigoplus_{i=1}^r \mathcal{W}^\alpha_{\lambda_i}.$$

We call $\mathcal{M}^\alpha_{\mathcal{F}}$ the Lagrangian *generated by the meridians* of $\mathcal{F}$, and $\mathcal{L}^\alpha_{\mathcal{F}}$ the Lagrangian *generated by the longitudes* of $\mathcal{F}$. If α is the trivial one-dimensional system, we remove it from the notation. In view of Example 3.1.11, we have in this case $\mathcal{M}_{\mathcal{F}} = \mathrm{Span}_{\mathbb{C}}\{\mu_1, \ldots, \mu_r\}$ and $\mathcal{L}_{\mathcal{F}} = \mathrm{Span}_{\mathbb{C}}\{\lambda_1, \ldots, \lambda_r\}$. We define corresponding Lagrangian subspaces in cohomology as

$$M^\alpha_{\mathcal{F}} := \mathrm{PD}(H_0(\Sigma; \mathbb{C}^n_\alpha) \oplus \mathcal{M}^\alpha_{\mathcal{F}}) = \mathrm{PD}(\mathcal{M}^\alpha_{\mathcal{F}}) \oplus H^2(\Sigma; \mathbb{C}^n_\alpha),$$
$$L^\alpha_{\mathcal{F}} := \mathrm{PD}(H_0(\Sigma; \mathbb{C}^n_\alpha) \oplus \mathcal{L}^\alpha_{\mathcal{F}}) = \mathrm{PD}(\mathcal{L}^\alpha_{\mathcal{F}}) \oplus H^2(\Sigma; \mathbb{C}^n_\alpha).$$

Clearly, both Lagrangians are graded.

3.1.3 Compatible Riemannian metrics

Let T be a framed torus with meridian μ and longitude λ. We say that a Riemannian metric g on T is *compatible with the framing* if there is an isometry φ from T to the standard flat torus such that $\varphi_*(\mu)$ and $\varphi_*(\lambda)$ are the two canonical generators of $H_1(S^1 \times S^1) \cong \mathbb{Z}^2$. The following result says that up to isotopy there is exactly one such a metric on T (we recall that two Riemannian metric on T are called isotopic if there is a self-diffeomorphism of T isotopic to the identity that is an isometry between the two metrics).

Proposition 3.1.12. *Let T be a framed torus. Then there is at least one Riemannian metric compatible with the framing. Moreover, any two such metrics are isotopic.*

Proof. In order to construct a metric on T which is compatible with the framing it is enough to choose a diffeomorphism $\varphi\colon T \to S^1 \times S^1$ such that $\varphi_*(\mu)$ and $\varphi_*(\lambda)$ are the two canonical generators of $H_1(S^1 \times S^1) \cong \mathbb{Z}^2$, and equip T with the pull-back metric $g := \varphi^* h$ of the standard flat metric h.

Let now g_1 and g_2 be two Riemannian metrics on T that are compatible with the framing. Then, we have $g_1 = \varphi_1^* h$ and $g_2 = \varphi_2^* h$ for two diffeomorphisms $\varphi_1, \varphi_2\colon T \to S^1 \times S^1$ such that, for $i = 1, 2$, $(\varphi_i)_*(\mu)$ and $(\varphi_i)_*(\lambda)$ are the two canonical generators of $H_1(S^1 \times S^1)$. It is a well-know fact that two self-diffeomorphisms of the torus are isotopic if and only if they induce the same map on $\pi_1 = H_1$ (see Rolfsen [42, Theorem 4 of Section 2D]). As a consequence, φ_1 and φ_2 (and hence g_1 and g_2) are isotopic. □

Suppose now that $\alpha \in \mathcal{U}_n(T)$ is a local coefficient system. As we have seen in Section 2.2.1, a Riemannian metric on T allows us to identify $H^*(T;\mathbb{C}^n_\alpha)$ with the space of twisted harmonic forms $\mathcal{H}^*(T;\mathbb{C}^n_\alpha)$ via the Hodge–de Rham theorem. These identification give $H^*(T;\mathbb{C}^n_\alpha)$ an inner product and a unitary operator $\star$ on $H^*(T;\mathbb{C}^n_\alpha)$. By defining

$$\gamma = \begin{cases} -\star, \text{ on } H^0 \text{ and } H^1, \\ \star \text{ on } H^2, \end{cases} ,$$

we obtain a unitary operator on $H^*(T;\mathbb{C}^n_\alpha)$ with the property that

$$\gamma^2 = -\operatorname{id}, \quad \langle x, \gamma(y)\rangle = \omega(x,y).$$

This makes $H^*(T;\mathbb{C}^n_\alpha)$ a Hermitian symplectic space. We shall now have a closer look at Hermitian symplectic structure on a framed torus with a compatible metric. We start with the untwisted case.

Lemma 3.1.13. *Let (T,μ,λ) be a framed oriented torus, and let g be a compatible Riemannian metric. Then, $\{\mathrm{PD}(\mu),\mathrm{PD}(\lambda)\}$ is an orthonormal basis of $H^1(T;\mathbb{C})$, and $\gamma(\mathrm{PD}(\mu)) = \pm\,\mathrm{PD}(\lambda)$.*

Proof. Suppose that the framing is standardly oriented, i.e. that $\mu\cdot\lambda = -1$. The proof can be easily adapted to the case where $\mu\cdot\lambda = 1$, and result will have its sign reversed. Consider the canonical 1-forms on $S^1\times S^1$ whose pull-backs to $\mathbb{R}^2$ under the exponential map are dx and dx respectively. Using an isometry $\varphi\colon T\to S^1\times S^1$ sending λ and μ respectively to the first and second canonical generator of $H_1(S^1\times S^1;\mathbb{Z})$, we pull back these forms further to 1-forms $d\theta_\lambda$, $d\theta_\mu$ on T which are orthogonal, harmonic, of norm 2π, and such that

$$\int_\mu d\theta_\mu = 2\pi, \quad \int_\lambda d\theta_\mu = 0, \quad \int_\lambda d\theta_\lambda = 2\pi, \quad \int_\mu d\theta_\lambda = 0. \tag{3.2}$$

Moreover, as $\lambda\cdot\mu = 1$, the diffeomorphism φ is orientation-preserving, and hence we have $\star d\theta_\lambda = d\theta_\mu$.

In order to understand how the star operator acts on Poincaré duals of homology classes, we need to see to what elements in $H_1(T;\mathbb{C})$ the differential forms $d\theta_\lambda$ and $d\theta_\mu$ correspond. We will show that, up to the Hodge–de Rham isomorphism, we have

$$d\theta_\lambda = -2\pi\,\mathrm{PD}(\mu), \quad d\theta_\mu = 2\pi\,\mathrm{PD}(\lambda). \tag{3.3}$$

It follows from the properties of $d\theta_\lambda$ and $d\theta_\mu$ that $\{\mathrm{PD}(\lambda), \mathrm{PD}(\mu)\}$ is an orthonormal basis of $H^1(T;\mathbb{C})$. Moreover, the relation $\star d\theta_\lambda = d\theta_\mu$ gets rewritten as $\star(-2\pi\,\mathrm{PD}(\mu)) = 2\pi\,\mathrm{PD}(\lambda)$, from which we deduce $\star\,\mathrm{PD}(\mu) = -\,\mathrm{PD}(\lambda)$. It follows from the definition of γ that $\gamma(\mathrm{PD}(\mu)) = \mathrm{PD}(\lambda)$, so that the proof is complete.

Let us hence prove (3.3). Using the fact that the Poincaré duality map is inverse to capping with the fundamental class, and the standard formula $\langle \alpha \cup \beta, c\rangle = \langle \alpha, \beta \frown c\rangle$, we compute

$$\begin{aligned}
\langle \mathrm{PD}(\mu), \lambda\rangle &= \langle \mathrm{PD}(\mu), \mathrm{PD}(\lambda) \frown [T]\rangle = \langle \mathrm{PD}(\mu) \cup \mathrm{PD}(\lambda), [T]\rangle = \mu \cdot \lambda = -1,\\
\langle \mathrm{PD}(\mu), \mu\rangle &= \langle \mathrm{PD}(\mu), \mathrm{PD}(\mu) \frown [T]\rangle = \langle \mathrm{PD}(\mu) \cup \mathrm{PD}(\mu), [T]\rangle = \mu \cdot \mu = 0,\\
\langle \mathrm{PD}(\lambda), \mu\rangle &= \langle \mathrm{PD}(\lambda), \mathrm{PD}(\mu) \frown [T]\rangle = \langle \mathrm{PD}(\lambda) \cup \mathrm{PD}(\mu), [T]\rangle = \lambda \cdot \mu = 1,\\
\langle \mathrm{PD}(\lambda), \lambda\rangle &= \langle \mathrm{PD}(\lambda), \mathrm{PD}(\lambda) \frown [T]\rangle = \langle \mathrm{PD}(\lambda) \cup \mathrm{PD}(\lambda), [T]\rangle = \lambda \cdot \lambda = 0.
\end{aligned}$$

By the universal coefficient theorem, the pairing with homology identifies cohomology elements in an unequivocal way. Comparing these formulas with those of (3.2), we see that $d\theta_\lambda$ corresponds to $\mathrm{PD}(\mu)$ up to a factor -2π, and $d\theta_\mu$ corresponds to $\mathrm{PD}(\lambda)$ up to a factor 2π, so that (3.3) is verified. □

As a consequence of Lemma 3.1.13, we get the following result about the Lagrangians associated to a framing.

Proposition 3.1.14. *Let Σ be a collection of tori with a framing $\mathcal{F}$ and a compatible Riemannian metric. Then, for all $\alpha \in \mathcal{U}_n(\Sigma)$, we have $\gamma(\mathrm{PD}(\mathcal{M}^\alpha_\mathcal{F})) = \mathrm{PD}(\mathcal{L}^\alpha_\mathcal{F})$.*

Proof. By working, on each component separately, it is enough to prove the result for a single framed torus $(T, \mathcal{F})$, with $\mathcal{F} = (\mu, \lambda)$. Arguing like in the proof of Proposition 3.1.10, we have natural isomorphisms

$$H^1(T, \mathbb{C}^n_\alpha) \cong A \otimes H^1(T;\mathbb{C}), \quad H_1(T, \mathbb{C}^n_\alpha) \cong A \otimes H_1(T;\mathbb{C}),$$

where A is the maximal subspace of $\mathbb{C}^n$ where α acts trivially. The Poincaré duality isomorphism decomposes accordingly, and the symplectic operator γ corresponds to $-\,\mathrm{id} \otimes \star$, where $\star$ denotes now the operator on $H_1(T;\mathbb{C})$. Since $\mathcal{M}_\mathcal{F} = A \otimes \mathrm{Span}_\mathbb{C}(\mu)$ and $\mathcal{L}_\mathcal{F} = A \otimes \mathrm{Span}_\mathbb{C}(\lambda)$, the conclusion follows now from Lemma 3.1.13. □

3.2 The main invariant

In this section, we introduce the rho invariant of a 3-manifold with framed toroidal boundary. In Section 3.2.1, we define the invariant and prove some basic properties. In Section 3.2.2 we study how the invariant changes under exchanging the role of meridians and longitudes. In Section 3.2.3, we see how the general gluing formulas proved in Chapter 2 get translated in this framework.

3.2.1 Definition and first properties

We are now ready to define the main invariant of our study.

Definition 3.2.1. Let $(X, \mathcal{F})$ be a 3-manifold with framed toroidal boundary and a local coefficient system $\alpha \in \mathcal{U}_n(X)$. The *rho invariant* of $(X, \mathcal{F})$ associated to α is the real number

$$\rho_\alpha(X, \mathcal{F}) := \rho_\alpha(X, g_{\mathcal{F}}, L^\alpha_{\mathcal{F}}, L^\epsilon_{\mathcal{F}}),$$

where ϵ is the trivial n-dimensional local coefficient system on X and $g_{\mathcal{F}}$ is any Riemannian metric on ∂X that is compatible with $\mathcal{F}$.

Remark 3.2.2. In other words, $\rho_\alpha(X, \mathcal{F})$ is defined in the following way: we choose any Riemannian metric on X of product form near ∂X that extends $g_{\mathcal{F}}$, and we set

$$\rho_\alpha(X, \mathcal{F}) = \eta_\alpha(X, L^\alpha_{\mathcal{F}}) - n\,\eta(X, L_{\mathcal{F}}).$$

Theorem 3.2.3. *The invariant $\rho_\alpha(X, \mathcal{F})$ is well defined, and it is functorial in the following way: if $f\colon X \to X'$ is an orientation-preserving diffeomorphism, $\mathcal{F}' := f_*(\mathcal{F})$ and $\alpha' := (f^{-1})^*(\alpha)$, we have*

$$\rho_\alpha(X, \mathcal{F}) = \rho_{\alpha'}(X', \mathcal{F}').$$

On the other hand, we have $\rho_\alpha(-X, \mathcal{F}) = -\rho_\alpha(X, \mathcal{F})$.

Proof. Thanks to Proposition 3.1.12, all metrics on ∂X that are compatible with $\mathcal{F}$ are isotopic. Then, $\rho_\alpha(X, \mathcal{F})$ is well defined by Proposition 2.2.13. Let us prove the functoriality property. As $\mathcal{F}' = f_*(\mathcal{F})$, it is clear that the metric $g_{\mathcal{F}'}$ on $\partial X'$ such that $g_{\mathcal{F}} = f^* g'_{\mathcal{F}}$ is compatible with $\mathcal{F}'$. Moreover, since $\mathcal{L}^{\alpha'}_{\mathcal{F}'} = f_*(\mathcal{L}^\alpha_{\mathcal{F}})$, we have

$$\begin{aligned} L^\alpha_{\mathcal{F}} &= \mathrm{PD}_X(\mathcal{L}^\alpha_{\mathcal{F}}) \oplus H^2(\partial X; \mathbb{C}^n_\alpha) = f^*\,\mathrm{PD}_{X'}\, f_*(\mathcal{L}^\alpha_{\mathcal{F}}) \oplus H^2(\partial X; \mathbb{C}^n_\alpha) = \\ &= f^*(\mathrm{PD}_{X'}(\mathcal{L}^{\alpha'}_{\mathcal{F}'}) \oplus H^2(\partial X; \mathbb{C}^n_\alpha)) = f^*(L^{\alpha'}_{\mathcal{F}'}), \end{aligned}$$

and in the same way we get $L^\varepsilon_{\mathcal{F}} = f^*(L^\varepsilon_{\mathcal{F}'})$. By definition, we get thus

$$\rho_\alpha(X, \mathcal{F}) = \rho_\alpha(X, g_{\mathcal{F}}, L^\alpha_{\mathcal{F}}, L^\varepsilon_{\mathcal{F}}) = \rho_{f^*\alpha'}(Y, f^* g_{\mathcal{F}'}, f^*(L^{\alpha'}_{\mathcal{F}'}), f^*(L^\varepsilon_{\mathcal{F}'})).$$

while

$$\rho_{\alpha'}(X',\mathcal{F}') = \rho_{\alpha'}(X', g_{\mathcal{F}'}, L^{\alpha'}_{\mathcal{F}'}, L^{\varepsilon}_{\mathcal{F}'}).$$

The conclusion follows by applying Proposition 2.2.11. □

Remark 3.2.4. In fact, the metric $g_{\mathcal{F}}$ and and the Lagrangians $L^{\alpha}_{\mathcal{F}}$ and $L^{\varepsilon}_{\mathcal{F}}$ only depend on the framing up to the sign of the meridians and longitudes. In particular, $\rho_\alpha(X,\mathcal{F}) = \rho_\alpha(X,\mathcal{G})$ if $\mathcal{F}$ and $\mathcal{G}$ coincide up to sign. As a consequence, in the setting of Theorem 3.2.3, we have $\rho_\alpha(X,\mathcal{F}) = \rho_{\alpha'}(X',\mathcal{F}')$ even if $\mathcal{F}'$ coincides with $f_*(\mathcal{F})$ just up to signs.

Remark 3.2.5. Thanks to Theorem 3.2.3 and Remark 3.2.4, if $f\colon X \to X'$ is orientation *reversing* and $\mathcal{F}'$ coincides with $f_*(\mathcal{F})$ up to signs, we have

$$\rho_\alpha(X,\mathcal{F}) = -\rho_{\alpha'}(X',\mathcal{F}').$$

The flexibility about signs is here very important for proving vanishing results, as an orientation-reversing self-diffeomorphism of X cannot satisfy $f_*(\mathcal{F}) = \mathcal{F}$.

By choosing appropriate orientation-reversing self-diffeomorphisms, this last remark can be readily used to prove that the rho invariant of a solid torus $D^2 \times S^1$ and of a cylinder $I \times S^1 \times S^1$, provided with the product framing discussed in Example 3.1.2, vanish for all local coefficient system. We postpone however the details to the following sections, where the rho invariants of these manifolds will be studied for more general framings.

In Section 2.3 we introduced a slight modification of the eta invariants, which fits somehow better in the gluing formulas. Namely, we set $\overline{\eta}_\beta(X,V) = \eta_\beta(X,V)$. The next result describes the freshly defined rho invariant of a 3-manifold with framed toroidal boundary in terms of these modified eta invariants.

Proposition 3.2.6. *Let $(X,\mathcal{F})$ be a 3-manifold with framed toroidal boundary, and let $\alpha \in \mathcal{U}_n(X)$ be a local coefficient system. Then, for every Riemannian metric on X that restricts to $g_{\mathcal{F}}$ on the boundary and has product form near it, we have*

$$\rho_\alpha(X,\mathcal{F}) = \overline{\eta}_\alpha(X, M^\alpha_{\mathcal{F}}) - n\,\overline{\eta}(X, M_{\mathcal{F}}).$$

Proof. We need to show that $\overline{\eta}_\beta(X, M^\alpha_{\mathcal{F}}) = \eta_\beta(X, L^\beta_{\mathcal{F}})$ for $\beta = \alpha, \epsilon$. By definition, we have $\overline{\eta}_\beta(X, M^\alpha_{\mathcal{F}}) = \eta_\beta(X, \gamma(M^\alpha_{\mathcal{F}}))$. Let H^k be a shorthand for $H^k(\Sigma;\mathbb{C}^n_\beta)$. Thanks to Proposition 3.1.14, $\gamma(M^\beta_{\mathcal{F}})$ coincides with $L^\beta_{\mathcal{F}}$ in degree 1, so that. we have the decompositions

$$L^\beta_{\mathcal{F}} = U \oplus H^2, \quad \gamma(M^\beta_{\mathcal{F}}) = U \oplus H^0,$$

where $U = \mathrm{PD}(\mathcal{L}^\beta_{\mathcal{F}})$. Since $H^0 = \gamma(H^2)$, we can apply Lemma 2.3.2, that gives

$$\eta_\beta(X, \gamma(M^\beta_{\mathcal{F}})) - \eta_\beta(X, L^\beta_{\mathcal{F}}) = \tau(\gamma(U) \oplus H^2, \gamma(U) \oplus H^0, V^\beta_X).$$

Because all Lagrangians involved are graded, the Maslov index can be computed on the degree-one summands (Proposition 2.3.7), where it clearly vanishes as two of them coincide. Hence, $\eta_\beta(X, \gamma(M^\beta_{\mathcal{F}})) = \eta_\beta(X, L^\beta_{\mathcal{F}})$ as desired. □

3.2.2 The reverse framing

Given a framing $\mathcal{F}$, let $\mathcal{F}^\star$ denote the framing whose meridians are the longitudes of $\mathcal{F}$, and whose longitudes are the meridians of $\mathcal{F}$. We call $\mathcal{F}^\star$ the *reverse framing* of $\mathcal{F}$. It is immediate to see that the Lagrangians depending on the two framings are related by

$$M^\alpha_{\mathcal{F}^\star} = L^\alpha_{\mathcal{F}}, \quad L^\alpha_{\mathcal{F}^\star} = M^\alpha_{\mathcal{F}}.$$

The rho invariants of $(X, \mathcal{F})$ and $(X, \mathcal{F}^\star)$ differ by an integer, which can be computed using Maslov indices.

Notation 3.2.7. In order to make the notation a bit lighter, in this chapter we shall denote every triple Maslov index just by τ, omitting the name of the complex symplectic space. It shall hopefully not create any confusion, as it is normally clear from the variables. Whenever there is some ambiguity related to whether we are using some symplectic structure or its opposite, we shall specify the space in the text.

Proposition 3.2.8. *Let $(X, \mathcal{F})$ a 3-manifold with framed toroidal boundary and let $\alpha \in \mathcal{U}_n(Y)$ be a local coefficient esystem. Then,*

$$\rho_\alpha(X, \mathcal{F}^\star) - \rho_\alpha(X, \mathcal{F}) = \tau(\mathcal{L}^\alpha_{\mathcal{F}}, \mathcal{M}^\alpha_{\mathcal{F}}, \mathcal{V}^\alpha_X) - n\,\tau(\mathcal{L}_{\mathcal{F}}, \mathcal{M}_{\mathcal{F}}, \mathcal{V}_X).$$

Proof. Using Proposition 3.2.6, we have

$$\rho_\alpha(X, \mathcal{F}^\star) = \overline{\eta}_\alpha(X, M^\alpha_{\mathcal{F}^\star}) - \overline{\eta}_\varepsilon(X, M^\varepsilon_{\mathcal{F}^\star}) = \overline{\eta}_\alpha(X, L^\alpha_{\mathcal{F}}) - \overline{\eta}_\varepsilon(X, L^\varepsilon_{\mathcal{F}}).$$

On the other hand, by definition we have $\rho_\alpha(X,\mathcal{F}) = \eta_\alpha(X, L^\alpha_{\mathcal{F}}) - \eta_\varepsilon(X, L^\varepsilon_{\mathcal{F}})$. By Lemma 2.3.2, we compute hence

$$\rho_\alpha(X,\mathcal{F}^\star) - \rho_\alpha(X,\mathcal{F}) = \tau(L^\alpha_{\mathcal{F}}, M^\alpha_{\mathcal{F}}, V^\alpha_X) - \tau(L^\varepsilon_{\mathcal{F}}, M^\varepsilon_{\mathcal{F}}, V^\varepsilon_X).$$

Since all the Lagrangians involved are graded, we can apply Proposition 2.3.7 and restrict the Maslov index to the degree 1 summands. On degree 1, we have $L^\alpha_{\mathcal{F}} = \mathrm{PD}(\mathcal{L}^\alpha_{\mathcal{F}})$, $M^\alpha_{\mathcal{F}} = \mathrm{PD}(\mathcal{M}^\alpha_{\mathcal{F}})$ and $V^\alpha_X = \mathrm{PD}(\mathcal{V}^\alpha_X)$, and hence we can compute the Maslov index in homology as

$$\tau(L^\alpha_{\mathcal{F}}, M^\alpha_{\mathcal{F}}, V^\alpha_X) = \tau(\mathcal{L}^\alpha_{\mathcal{F}}, \mathcal{M}^\alpha_{\mathcal{F}}, \mathcal{V}^\alpha_X).$$

The same observation holds for the trivial local system. The proof is concluded by observing that $\tau(\mathcal{L}^\varepsilon_{\mathcal{F}}, \mathcal{M}^\varepsilon_{\mathcal{F}}, \mathcal{V}^\varepsilon_X) = n\,\tau(\mathcal{M}_{\mathcal{F}}, \mathcal{L}_{\mathcal{F}}, \mathcal{V}_X)$. This follows from the fact that there is a natural symplectic isomorphism

$$H_1(\Sigma; \mathbb{C}^n_\varepsilon) \xrightarrow{\sim} (H_1(\Sigma;\mathbb{C}))^n,$$

which transforms our initial Lagrangians to direct sums of n identical copies of the ones appearing in the final formula. □

3.2.3 Gluing formulas

Suppose that a 3-manifold Z with framed toroidal boundary is split along some disjoint union of (framed) tori Σ as a union $X \cup_\Sigma Y$. For comparing the rho invariant of Z with those of X and Y, we shall employ the gluing formulas of Section 2.3. It is hence convenient to see these manifolds as cobordisms and adopt the formalism of Section 1.5.1. We shall then specialize the formula to the case where the result is a closed manifold, for which such a formalism is not needed.

We can see an oriented 3-manifold X with boundary $\partial X = -\Sigma_1 \cup \Sigma_2$ as a cobordism from Σ_1 to Σ_2. Given a local coefficient system $\alpha \in \mathcal{U}_n(X)$, the canonical Lagrangian $\mathcal{V}^\alpha_X$ will be seen as a Lagrangian relation

$$\mathcal{V}^\alpha_X \colon H_1(\Sigma_1, \mathbb{C}^n_\alpha) \Rightarrow H_1(\Sigma_2, \mathbb{C}^n_\alpha).$$

If Σ_1 is a disjoint union of tori with a framing $\mathcal{F}_1$ and Σ_2 is a disjoint union of tori with a framing $\mathcal{F}_2$, we shall denote the framing on ∂X coinciding with $\mathcal{F}_1$ on $-\Sigma_1$ and with $\mathcal{F}_2$ on Σ_2 as $\mathcal{F}_1 \cup \mathcal{F}_2$. The framing $\mathcal{F}_1$ determines a Lagrangian subspace $\mathcal{M}^\alpha_{\mathcal{F}_1} \subseteq H_1(\Sigma_1, \mathbb{C}^n_\alpha)$ and the framing $\mathcal{F}_2$ determines a Lagrangian subspace $\mathcal{M}^\alpha_{\mathcal{F}_2} \subseteq H_1(\Sigma_2, \mathbb{C}^n_\alpha)$.

We obtain a new Lagrangian subspace of $H_1(\Sigma_2, \mathbb{C}^n_\alpha)$ by the action of the cobordism on $\mathcal{M}^\alpha_{\mathcal{F}_1}$.

Notation 3.2.9. We set $\mathcal{V}^\alpha_{X,\mathcal{F}_1} := (\mathcal{V}^\alpha_X)_*(\mathcal{M}^\alpha_{\mathcal{F}_1})$.

As we will see in Lemma 3.3.20, $\mathcal{V}^\alpha_{X,\mathcal{F}_1}$ has to be thought of (at least heuristically) as the canonical Lagrangian associated to the manifold obtained by gluing solid tori to X on Σ_1 by capping the meridians of $\mathcal{F}_1$ with a disk (compare with Remark 1.5.7). The relative position of $\mathcal{M}^\alpha_{\mathcal{F}_2}$ and $\mathcal{V}^\alpha_{X,\mathcal{F}_1}$ will be relevant in the next result, that is the main gluing formula for rho invariants of 3-manifolds with framed toroidal boundary.

Theorem 3.2.10. *Let $Z = X \cup_\Sigma Y$ an oriented 3-manifold, with $\partial X = -\Sigma' \sqcup \Sigma$ and $\partial Y = -\Sigma \sqcup \Sigma''$. Suppose that Σ, Σ' and Σ'' are all disjoint unions of tori, with framings $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ respectively. Let $\alpha \in \mathcal{U}_n(Z)$ be a local coefficient system. Then*

$$\begin{aligned}\rho_\alpha(Z, \mathcal{F}' \cup \mathcal{F}'') = \rho_\alpha(X, \mathcal{F}' \cup \mathcal{F}) + \rho_\alpha(Y, \mathcal{F} \cup \mathcal{F}'') +& \\ + \tau(\mathcal{M}^\alpha_{\mathcal{F}}, \mathcal{V}^\alpha_{X,\mathcal{F}'}, \mathcal{V}^\alpha_{Y^t,\mathcal{F}''}) - n\,\tau(\mathcal{M}_{\mathcal{F}}, \mathcal{V}_{X,\mathcal{F}'}, \mathcal{V}_{Y^t,\mathcal{F}''}),&\end{aligned} \tag{3.4}$$

where the two Maslov triple indices are taken respectively on $H_1(\Sigma; \mathbb{C}^n_\alpha)$ and $H_1(\Sigma; \mathbb{C})$.

Proof. By Proposition 3.2.6, we have

$$\begin{aligned}\rho_\alpha(Z, \mathcal{F}' \cup \mathcal{F}'') &= \overline{\eta}_\alpha(Z, M^\alpha_{\mathcal{F}'} \oplus M^\alpha_{\mathcal{F}''}) - \overline{\eta}_\alpha(Z, M^\varepsilon_{\mathcal{F}'} \oplus M^\varepsilon_{\mathcal{F}''}),\\ \rho_\alpha(X, \mathcal{F}' \cup \mathcal{F}) &= \overline{\eta}_\alpha(X, M^\alpha_{\mathcal{F}'} \oplus M^\alpha_{\mathcal{F}}) - \overline{\eta}_\alpha(X, M^\varepsilon_{\mathcal{F}'} \oplus M^\varepsilon_{\mathcal{F}}),\\ \rho_\alpha(Y, \mathcal{F} \cup \mathcal{F}'') &= \overline{\eta}_\alpha(Y, M^\alpha_{\mathcal{F}} \oplus M^\alpha_{\mathcal{F}''}) - \overline{\eta}_\alpha(Y, M^\varepsilon_{\mathcal{F}} \oplus M^\varepsilon_{\mathcal{F}''}).\end{aligned}$$

Thanks to Theorem 2.4.3, we have

$$\rho_\alpha(Z, \mathcal{F}' \cup \mathcal{F}'') = \rho_\alpha(X, \mathcal{F}' \cup \mathcal{F}) + \rho_\alpha(Y, \mathcal{F} \cup \mathcal{F}'') + C,$$

where

$$C = \tau(M^\alpha_{\mathcal{F}}, V^\alpha_X(M^\alpha_{\mathcal{F}'}), V^\alpha_{Y^t}(M^\alpha_{\mathcal{F}''})) - \tau(M^\varepsilon_{\mathcal{F}}, V^\varepsilon_X(M^\varepsilon_{\mathcal{F}'}), V^\varepsilon_{Y^t}(M^\varepsilon_{\mathcal{F}''})).$$

The above Lagrangian subspaces are all graded, and thus the Maslov triple index is 0 outside of degree 1. Thanks to Proposition 3.1.8 and the definition of the Lagrangians, on degree 1 we have, for $\beta = \alpha, \tau$ and $\mathrm{PD} = \mathrm{PD}_\Sigma$,

$$(M^\beta_{\mathcal{F}})^1 = \mathrm{PD}(\mathcal{M}^\beta_{\mathcal{F}}), \quad V^\beta_X(M^\beta_{\mathcal{F}'})^1 = \mathrm{PD}(\mathcal{V}^\beta_{X,\mathcal{F}'}), \quad V^\beta_{Y^t}(M^\beta_{\mathcal{F}''})^1 = \mathrm{PD}(\mathcal{V}^\beta_{Y^t,\mathcal{F}''}).$$

The proof is then concluded by using the same argument as in the proof of Proposition 3.2.8. ∎

In many applications, the correction term gets simplified. in the applications, for example, we shall normally restrict to local coefficient systems α such that $H_*(\partial X; \mathbb{C}^n_\alpha) = 0$ because of Lemmma 1.3.8, and hence the Maslov triple index in twisted homology disappears. Another common situation is the one where the manifold $X \cup_\Sigma Y$ is closed, and the cobordism formalism becomes unneeded. We make this explicit in the following.

Corollary 3.2.11. *Let $M = X \cup_\Sigma Y$ be a closed, oriented manifold which is the union of two 3-manifolds X, Y over a disjoint union of tori Σ. Let $\mathcal{F}$ be any framing on Σ and $\alpha \in \mathcal{U}_n(M)$ be a local coefficient system. Then*

$$\rho_\alpha(M) = \rho_\alpha(X, \mathcal{F}) + \rho_\alpha(Y, \mathcal{F}) + \tau(\mathcal{M}^\alpha_\mathcal{F}, \mathcal{V}^\alpha_X, \mathcal{V}^\alpha_Y) - n\,\tau(\mathcal{M}_\mathcal{F}, \mathcal{V}_X, \mathcal{V}_Y).$$

Proof. We apply Theorem 3.2.10 with $Z = M$. As $\Sigma' = \Sigma'' = \emptyset$, we have

$$\mathcal{V}^\alpha_{X,\mathcal{F}'} = (\mathcal{V}^\alpha_X)_*(0) = \mathcal{V}^\alpha_X, \quad \mathcal{V}^\alpha_{Y^t,\mathcal{F}''} = (\mathcal{V}^\alpha_{Y^t})_*(0) = \mathcal{V}^\alpha_{Y^t},$$

and similarly for the trivial local system. □

3.3 Solid tori and Dehn fillings

The simplest 3-manifold with non-empty toroidal boundary is the solid torus $D^2 \times S^1$. The computation of $\rho_\alpha(D^2 \times S^1, \mathcal{F})$ for every framing $\mathcal{F}$ and local coefficient system α turns out to be an interesting problem, which stays open in general. In Section 3.3.1, we study the different framings on $\partial(D^2 \times S^1)$. In Section 3.3.2, we prove some basic results that allow us to compute the invariant for the product framing and for its reverse framing. In Section 3.3.3, we study in detail the behavior of the rho invariant of a 3-manifold when a solid torus is glued to one of its toroidal boundary components. In Section 3.3.4, we recall a classical formula for the Atiyah-Patodi-Singer rho invariants of 3-dimensional lens spaces, and rewrite it in many different ways. Using this computation and the gluing formulas, the computation of $\rho_\alpha(D^2 \times S^1, \mathcal{F})$ is then approached again in Section 3.3.5.

3.3.1 Framings on the solid torus

The simplest 3-manifold with non-empty toroidal boundary is the solid torus $D^2 \times S^1$. The first integral homology group of its boundary is freely generated by the classes $[\partial D^2]$ and $[S^1]$. Classically, $[\partial D^2]$ is called the meridian of the solid torus, and $[S^1]$ is called the longitude (with respect to the specific product structure at hand, which we take as fixed). The classical denomination coincide with the one arising from the product framing $\mathcal{F}^\times$ defined in Example 3.1.2. However different framings can be considered. As usual, the set of *all* framings on $\partial(D^2 \times S^1)$ is in a natural bijection with $\mathrm{GL}(2,\mathbb{Z})$ (once a preferred framing is fixed). If we are only interested in framings up to orientation-preserving diffeomorphisms of the solid torus, though, the classification becomes simpler.

Definition 3.3.1. Given torus T with a framing $\mathcal{F} = (\mu, \lambda)$, define the *slope with respect to* $\mathcal{F}$ of an element $\gamma = a\mu + b\lambda \in H_1(T;\mathbb{Z})$ as the number $a/b \in \mathbb{Q} \cup \{\infty\}$. We define the *gradient* of $\mathcal{F}$ as the slope of the class $[\partial D^2]$ with respect to $\mathcal{F}$.

Convention 3.3.2. We set $1/\infty = 0$ and $-\infty = \infty$, so that the operations of taking the opposite and the reciprocal of an element of $\mathbb{Q} \cup \{\infty\}$ are well defined. Moreover, we extend the sign function to $\mathbb{Q}\cup\{\infty\}$ (or even $\mathbb{R}\cup\{\infty\}$) as

$$\mathrm{sgn}(r) := \begin{cases} 1, & \text{if } r \neq \infty,\ r > 0 \\ -1, & \text{if } r \neq \infty,\ r < 0 \\ 0, & \text{if } r \in \{0, \infty\}. \end{cases}$$

If $\mathcal{F}$ is standardly oriented, passing to complex coefficients, the slope of γ with respect to a framing $\mathcal{F} = (\mu, \lambda)$ coincides with the slope of the Lagrangian subspace $\mathrm{Span}_{\mathbb{C}}\{\gamma\}$ in $H_1(T;\mathbb{C})$ with respect to the symplectic basis (μ, λ), as it was introduced in Section 1.2.1. However, we shall now focus on framings on the boundary solid torus that are *non-standardly* oriented. There is a double reason for this choice: firstly, the most natural framing, i.e. the product framing, is non-standardly oriented; secondly, very often we shall use solid tori to perform Dehn fillings on 3-manifolds with framed toroidal boundary (see Section 3.3.3), and the induced framing on the boundary of the solid torus has in this case the opposite orientation.

The following result is a rephrasing of an elementary classical result.

Lemma 3.3.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two non-standardly oriented framings on $\partial(D^2 \times S^1)$. Then, there exists an orientation-preserving diffeomorphism $f\colon D^2 \times S^1 \to D^2 \times S^1$ such that $\mathcal{G} = f_*(\mathcal{F})$ if and only if $\mathcal{F}$ and $\mathcal{G}$ have the same gradient.*

As a consequence of Lemma 3.3.3, non-standardly oriented framings up to diffeomorphism are classified by $\mathbb{Q}\cup\{\infty\}$. In fact, for every $r \in \mathbb{Q}\cup\{\infty\}$ we can choose coprime integers p, q such that $r = p/q$ and associate to r the class of the framing $\mathcal{F}_r = (\mu, \lambda)$ with $\mu = b[\partial D^2] - q[S^1]$ and $\lambda = -a[\partial D^2] + p[S^1]$, for some integers a, b such that $bp - aq = 1$. It is immediate to check that this choice of $\mathcal{F}_r$ gives a non-standardly oriented framing with $[\partial D^2] = p\mu + q\lambda$, and hence with gradient r.

Example 3.3.4. The framings $\mathcal{F}_\infty$ of gradient ∞ are those with meridian $\mu = \pm[\partial D^2]$ and longitude $\lambda = k[\partial D^2] \pm [S^1]$, with $k \in \mathbb{Z}$ and the same sign in the two equations. In particular, the product framing $\mathcal{F}^\times$ is in this class. The other framings in this class can be seen as the product framings corresponding to different choices of a product structure on the solid torus.

Warning 3.3.5. Given $r \in \mathbb{Q} \cup \{\infty\}$, a non-standardly oriented framing $\mathcal{F}_r$ of gradient r is only determined up to orientation-preserving diffeomorphism of $D^2 \times S^1$. Nevertheless, we shall often speak of $\mathcal{F}_r$ without specifying explicitly which representative we are considering. As we are about to prove, the rho invariant $\rho_\alpha(D^2 \times S^1, \mathcal{F}_r)$ does not depend on this choice.

Remark 3.3.6. An analogous classification holds for standardly oriented framings. As we have already said, we shall focus here on non-standardly oriented framings. This is enough for the purpose of computing rho invariants, as any standardly oriented framing can be turned into a non-standardly oriented one by reversing (precisely) one between the meridian and the longitude, without changing the result. Notice though that this process also changes the sign of the gradient. In other words, a standardly oriented framing on $\partial(D^2 \times S^1)$ of gradient r coincides up to sign with a non-standardly oriented framing of gradient $-r$. In order to avoid confusion, we shall reserve the notation $\mathcal{F}_r$ to non-standardly oriented framings.

3.3.2 First explicit computations

We shall think of local coefficient systems $\alpha \in \mathcal{U}_n(D^2 \times S^1)$ as representations $\alpha\colon \pi_1(D^2 \times S^1) \cong \mathbb{Z} \to U(n)$ Let t be the generator of $\pi_1(D^2 \times S^1) \cong \mathbb{Z}$ corresponding to the fundamental class $[S^1] \in H_1(S^1; \mathbb{Z})$. A representation $\alpha\colon \pi_1(D^2 \times S^1) \to U(n)$ is determined by the matrix $\alpha(t) \in U(n)$. Unitary matrices are diagonalizable, and the equivalence class of a representation α is determined by its eigenvalues of $\alpha(t)$. It follows that, for all α, the complex conjugate representation $\overline{\alpha}$ is equivalent to the one sending t to $\alpha(t)^{-1}$.

We shall now prove a series of results about the rho invariants of $D^2 \times S^1$ by taking advantage of the symmetries of this manifold.

Proposition 3.3.7. *For all framings $\mathcal{F}$ on $\partial(D^2 \times S^1)$ and for all representations $\alpha\colon \pi_1(D^2 \times S^1) \to U(n)$, we have*

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}) = \rho_{\overline{\alpha}}(D^2 \times S^1, \mathcal{F}).$$

Proof. Let φ be a reflection of D^2, and ψ be a reflection of S^1. Then the map $f := \varphi \times \psi \colon D^2 \times S^1 \to D^2 \times S^1$ is an orientation preserving diffeomorphism. The restriction of f to the boundary induces minus the identity on $H_1(\partial(D^2 \times S^1); \mathbb{Z})$, and hence $f_*(\mathcal{F})$ coincides with $\mathcal{F}$ up to signs. The induced map f_* on $\pi_1(D^2 \times S^1) \to U(n)$ sends the generator t to t^{-1}, and hence, by the discussion above, the induced representation $f^*\alpha$ is equivalent to $\overline{\alpha}$. The result follows now from Proposition 3.2.8. □

Corollary 3.3.8. *The number $\rho_\alpha(D^2 \times S^1, \mathcal{F}_r)$ does not depend on the choice of $\mathcal{F}_r$ inside the class of non-standardly oriented framings of slope r.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be two non-standardly oriented framings of gradient r. By Lemma 3.3.3, there is an orientation-preserving self-diffeomorphism of $D^2 \times S^1$ such that $f_*(\mathcal{F}) = \mathcal{G}$. The induced map f_* on $\pi_1(D^2 \times S^1) \to U(n)$ is either the identity or the involution $t \mapsto t^{-1}$. In the first case, Proposition 3.2.8 tells us immediately that $\rho_\alpha(D^2 \times S^1, \mathcal{F}) = \rho_\alpha(D^2 \times S^1, \mathcal{G})$. In the second case, it tells us that $\rho_{\overline{\alpha}}(D^2 \times S^1, \mathcal{F}) = \rho_\alpha(D^2 \times S^1, \mathcal{G})$, and the same conclusion follows now from Proposition 3.3.7. □

Proposition 3.3.9. *For all $r \in \mathbb{Q} \cup \{\infty\}$ and for all $\alpha \colon \pi_1(D^2 \times S^1) \to U(n)$, we have*

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}_{-r}) = -\rho_\alpha(D^2 \times S^1, \mathcal{F}_r).$$

Proof. Let $\mathcal{F}_r = (\mu, \lambda)$ be a non-standardly oriented framing of gradient r, i.e. such that $[\partial D^2] = p\mu + q\lambda$ with $p/q = r$. Let φ be again a reflection of D^2. We consider now the map $g := \varphi \times \mathrm{id} \colon D^2 \times S^1 \to D^2 \times S^1$, which is an orientation reversing diffeomorphism. The induced framing $g_*(\mathcal{F}_r) = (\mu', \lambda')$ is by definition such that $g_*([\partial D^2]) = p\mu' + q\lambda'$, i.e. $[\partial D^2] = -p\mu' - q\lambda'$. However, it is standardly oriented, because the restriction of g to $\partial(D^2 \times S^1)$ is also orientation-reversing. By turning the sign of one between μ' and λ', we get a non-standardly oriented framing of gradient $-p/q = -r$. In particular, $g_*(\mathcal{F}_r)$ coincides up to sign with $\mathcal{F}_{-r}$. Moreover, we have $g^*\alpha = \alpha$ for all choices of α, as g is homotopic to the identity on $D^2 \times S^1$. The conclusion follows once again from Proposition 3.2.8.

□

Corollary 3.3.10. *For every choice of $\alpha \colon \pi_1(S^1 \times D^2) \to U(n)$ we have*

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}_0) = \rho_\alpha(D^2 \times S^1, \mathcal{F}_\infty) = 0.$$

Proof. The gradients $r = 0$ and $r = \infty$ satisfy $r = -r$. Applying Proposition 3.3.9 to these values of r, we get

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}_r) = -\rho_\alpha(D^2 \times S^1, \mathcal{F}_r),$$

which implies that the rho invariant vanishes in the two cases. □

For the next result, it is convenient to restrict the attention to one-dimensional representations.

Proposition 3.3.11. *For all $r \in \mathbb{Q} \cup \{\infty\}$ and non-trivial $\alpha\colon \pi_1(S^1 \times D^2) \to U(1)$, we have*

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}_{1/r}) = -\rho_\alpha(D^2 \times S^1, \mathcal{F}_r) - \operatorname{sgn}(r).$$

Proof. Let $\mathcal{F}_{-r} = (\mu, \lambda)$ be a non-standardly oriented framing of gradient $-r$, so that $[\partial D] = p\mu + q\lambda$ with $p/q = -r$. Then, the framing (μ', λ') defined by $\mu' = \lambda$, $\lambda' = -\mu$ is non-standardly oriented and has gradient $-1/r$, so that we can legitimately call it $\mathcal{F}_{1/r}$. Of course $\mathcal{F}_{1/r}$ coincides up to sign with the reverse framing $\mathcal{F}^\star_{-r} = (\lambda, \mu)$ (which is standardly oriented). In particular, we have

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}_{1/r}) = \rho_\alpha(D^2 \times S^1, \mathcal{F}^\star_{-r}).$$

We apply now Proposition 3.2.8. Since the representation is one-dimensional and non-trivial, the twisted homology of $\partial(D^2 \times S^1)$ is 0 and the first Maslov index vanishes. We remain hence with

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}_{1/r}) = \rho_\alpha(D^2 \times S^1, \mathcal{F}_{-r}) - \tau(\mathcal{L}_{\mathcal{F}_{-r}}, \mathcal{M}_{\mathcal{F}_{-r}}, \mathcal{V}_{D^2 \times S^1}).$$

By definition we have $\mathcal{L}_{\mathcal{F}_{-r}} = \mathbb{C}\lambda$ and $\mathcal{M}_{\mathcal{F}_{-r}} = \mathbb{C}\mu$. Moreover, it is immediate to see that $\mathcal{V}_{D^2 \times S^1} = \ker(H_1(\partial(D^2 \times S^1); \mathbb{C}) \to H_1(D^2 \times S^1; \mathbb{C}))$ is generated by $[\partial D] = p\mu + q\lambda$. We have hence

$$\tau(\mathcal{L}_{\mathcal{F}_{-r}}, \mathcal{M}_{\mathcal{F}_{-r}}, \mathcal{V}_{D^2 \times S^1}) = \tau(0, \infty, -p/q).$$

In the last equality, we are using the notation of Section 1.2.1, and writing slopes with respect to the symplectic basis $(\mu, -\lambda)$. By Proposition 1.2.8, together with Convention 3.3.2, this equals $-\operatorname{sgn}(p/q) = \operatorname{sgn}(r)$. □

Corollary 3.3.12. *For all non-trivial $\alpha\colon \pi_1(S^1 \times D^2) \to U(1)$, we have*

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}_1) = -\frac{1}{2}.$$

Proof. Applying Proposition 3.3.11 with $r = 1$, we get the equation

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}_1) = -\rho_\alpha(D^2 \times S^1, \mathcal{F}_1) - 1,$$

which yields immediately the desired result. □

We conclude with the following result, showing that the canonical Lagrangian $\mathcal{V}^\alpha_{D^2\times S^1}$ corresponds to the Lagrangian $\mathcal{M}^\alpha_{\mathcal{F}_\infty}$ generated by the meridian of the framing $\mathcal{F}_\infty = \mathcal{F}^\times$.

Lemma 3.3.13. *Let $\alpha\colon \pi_1(D^2 \times S^1) \to U(n)$ be any representation. Then,*

$$\mathcal{V}^\alpha_{D^2\times S^1} = \mathcal{M}^\alpha_{\mathcal{F}_\infty}.$$

Proof. The two spaces we want to compare are described as

$$\mathcal{M}^\alpha_{\mathcal{F}_\infty} = \operatorname{im}(H_1(\partial D^2 \times 1; \mathbb{C}^n_\alpha) \to H_1(\partial D^2 \times S^1; \mathbb{C}^n_\alpha)),$$

$$\mathcal{V}^\alpha_{D^2\times S^1} = \ker(H_1(\partial D^2 \times S^1; \mathbb{C}^n_\alpha) \to H_1(D^2 \times S^1; \mathbb{C}^n_\alpha)).$$

In particular, proving $\mathcal{M}^\alpha_{\mathcal{F}_\infty} = \mathcal{V}^\alpha_{D^2\times S^1}$ is the same as showing that the sequence

$$H_1(1 \times \partial D^2; \mathbb{C}^n_\alpha) \to H_1(S^1 \times \partial D^2; \mathbb{C}^n_\alpha) \to H_1(S^1 \times D^2; \mathbb{C}^n_\alpha)$$

induced by the inclusions is exact. It is clear that the composition of the two maps is trivial, because it factors through $H_1(1 \times D^2, \mathbb{C}^n_\alpha) = 0$. As a consequence, the inclusion $\mathcal{M}^\alpha_{\mathcal{F}_\infty} \subseteq \mathcal{V}^\alpha_{D^2\times S^1}$ is satisfied, and the two spaces have to coincide as they are both half-dimensional. □

3.3.3 Dehn fillings

Let X be an oriented 3-manifold and let T be a connected component of ∂X which is provided with a framing $\mathcal{F} = (\mu, \lambda)$. Let r be an element of $\mathbb{Q} \cup \{\infty\}$. Then there are coprime integers p, q such that $p/q = r$ (with the convention that $\pm 1/0 = \infty$), which are determined by r up to simultaneous change of sign. We obtain a new oriented manifold $D_{p/q}(X, \mathcal{F})$ by gluing a solid torus $V = D^2 \times S^1$ to X through an orientation-reversing diffeomorphism $f\colon \partial V \to T$ such that the induced map $f_*\colon H_1(\partial V; \mathbb{Z}) \to H_1(T, \mathbb{Z})$ sends the meridian $[\partial D^2]$ of V to the element $p\mu + q\lambda$.

Definition 3.3.14. The manifold $D_{p/q}(X, \mathcal{F})$ is said to be obtained through a *p/q-framed Dehn filling of X along $\mathcal{F}$.*

The manifold $D_{p/q}(X, \mathcal{F})$ is only well-defined up to orientation-preserving diffeomorphism. The well-posedness in this sense of the above definition depends on the following well-known elementary facts:

a) a diffeomorphism f as the one required above always exists;

b) for two such diffeomorphisms $f_1, f_2 \colon \partial V \to T$, there is an orientation-preserving diffeomorphism $g \colon V \to V$ such that $f_2 = f_1 \circ f_1|_{\partial V}$.

Remark 3.3.15. The r-framed Dehn filling along T identifies T with the boundary $\partial(D^2 \times S^1)$ of the solid torus. Under this identification, the framing $\mathcal{F}$ on T induces a framing on $\partial(D^2 \times S^1)$. If $\mathcal{F}$ is standardly oriented, then the induced framing on $\partial(D^2 \times S^1)$ is a non-standardly oriented framing $\mathcal{F}_r$ of gradient r.

Suppose that X is connected and that $x_0 \in T$ is fixed, so that we can see local coefficient systems $\alpha \in \mathcal{U}_n(X)$ as representations $\alpha \colon \pi_1(X, x_0) \to U(n)$. If $i \colon T \to X$ is the inclusion map, we have an induced map

$$i_* \colon H_1(T; \mathbb{Z}) \cong \pi_1(T; x_0) \to \pi_1(X, x_0),$$

and by Seifert-van Kampen's theorem we have a natural isomorphism

$$\pi_1(D_{p/q}(X, \mathcal{F}), x_0) = \pi_1(X, x_0)/\langle i_*(p\mu + q\lambda)\rangle.$$

As a consequence, a representation $\alpha \colon \pi_1(X) \to U(n)$ can be extended to $\pi_1(D_{p/q}(X, \mathcal{F}))$ if and only if it is trivial on $i_*(p\mu + q\lambda)$. In that case, the extension is unique, and we shall normally keep calling it α. Notice in particular that α extends to $D_\infty(X, \mathcal{F})$ if and only if it is trivial on the meridian, and to $D_0(X, \mathcal{F})$ if and only if it is trivial on the longitude.

Suppose now that $(X, \mathcal{G})$ is a 3-manifold with framed toroidal boundary. If T is a boundary component of X, we can write $\mathcal{G} = \mathcal{F} \cup \mathcal{F}'$, where $\mathcal{F}$ is the framing on T and $\mathcal{F}'$ is the framing on the remaining components. Then, a Dehn filling $D_r(X, \mathcal{F})$ is again a 3-manifold with framed toroidal boundary, namely with framing $\mathcal{F}'$. We can now apply the gluing formulas of Section 3.2.3 to the case of Dehn fillings. For $r \in \mathbb{Q} \cup \{\infty\}$, it is useful to consider the Lagrangian subspace generated by a curve of slope r, namely

$$\mathcal{W}^\alpha_{\mathcal{F},r} := \mathcal{W}^\alpha_{p\mu+q\lambda},$$

for $\mathcal{F} = (\mu, \lambda)$ and p, q coprime integers such that $p/q = r$ (see Section 3.1.2 (3.1) for the definition of $\mathcal{W}^\alpha_c$). From the definition, we see immediately that

$$\mathcal{W}^\alpha_{\mathcal{F},\infty} = \mathcal{M}^\alpha_{\mathcal{F}}, \quad \mathcal{W}^\alpha_{\mathcal{F},0} = \mathcal{L}^\alpha_{\mathcal{F}}$$

and, for the trivial 1-dimensional representation,

$$\mathcal{W}_{\mathcal{F},r} = \mathrm{Span}_{\mathbb{C}}(p\mu + q\lambda).$$

We have then the following result.

Proposition 3.3.16. *Suppose that $\mathcal{F}$ is standardly oriented. If $\alpha \in \mathcal{U}_n(X)$ extends to $D_r(X, \mathcal{F})$, we have*

$$\rho_\alpha(D_r(X, \mathcal{F}), \mathcal{F}') = \rho_\alpha(X, \mathcal{F} \cup \mathcal{F}') + \rho_\alpha(D^2 \times S^1, \mathcal{F}_r) + C,$$

where $C = \tau(\mathcal{M}^\alpha_{\mathcal{F}}, \mathcal{V}^\alpha_{X,\mathcal{F}'}, \mathcal{W}^\alpha_{\mathcal{F},r}) - n\,\tau(\mathcal{M}_{\mathcal{F}}, \mathcal{V}_{X,\mathcal{F}'}, \mathcal{W}_{\mathcal{F},r})$.

Proof. We use the gluing formula of Theorem 3.2.10 to get

$$\rho_\alpha(D_r(X, \mathcal{F}), \alpha, \mathcal{F}') = \rho_\alpha(X, \mathcal{F} \cup \mathcal{F}') + \rho_\alpha(D^2 \times S^1, \mathcal{F}) + C,$$

with $C = \tau(\mathcal{M}^\alpha_{\mathcal{F}}, \mathcal{V}^\alpha_{X,\mathcal{F}'}, \mathcal{V}^\alpha_{D^2 \times S^1}) - n\,\tau(\mathcal{M}_{\mathcal{F}}, \mathcal{V}_{X,\mathcal{F}'}, \mathcal{V}_{D^2 \times S^1})$. Thanks to Remark 3.3.15, the framing $\mathcal{F}$ on the boundary of the solid torus is a non-standardly oriented framing of gradient r. In particular, we have

$$\rho_\alpha(D^2 \times S^1, \mathcal{F}) = \rho_\alpha(D^2 \times S^1, \mathcal{F}_r).$$

The proof is hence completed if we show that, for a general α, $\mathcal{V}^\alpha_{D^2 \times S^1}$ gets identified by the gluing with $\mathcal{W}^\alpha_{\mathcal{F},r}$. By Lemma 3.3.13, we have $\mathcal{V}^\alpha_{D^2 \times S^1} = \mathcal{M}^\alpha_{\mathcal{F}_\infty}$. In other words, it is the Lagrangian generated by the meridian of $\mathcal{F}_\infty$, i.e. by $[D^2] \in H_1(\partial(D^2 \times S^1); \mathbb{Z})$. The gluing sends $[D^2]$ to a curve $p\mu + q\lambda \in H_1(T; \mathbb{Z})$ of slope r, and hence it identifies $\mathcal{V}^\alpha_{D^2 \times S^1}$ with $\mathcal{W}^\alpha_{\mathcal{F},r}$ as desired. ☐

Proposition 3.3.16 gives a relationship between the rho invariant of the manifold obtained by a Dehn filling and the one of the original manifold. In order to make this useful in the applications, it would be good to know the value of $\rho_\alpha(D^2 \times S^1, \mathcal{F}_r)$ for as many choices of α and r as possible. The problem is addressed in Section 3.3.5. The correction term C is often easy to compute, as we see the following remark.

Remark 3.3.17. Suppose that the restriction of α to T has no trivial summand. Then, by Lemma 1.3.8, we have $H_*(T; \mathbb{C}^n_\alpha) = 0$. In particular, the term $\tau(\mathcal{M}^\alpha_{\mathcal{F}}, \mathcal{V}^\alpha_{X,\mathcal{F}'}, \mathcal{W}^\alpha_{\mathcal{F},r})$ vanishes, and the correction term is just given by

$$C = -n\,\tau(\mathcal{M}_{\mathcal{F}}, \mathcal{V}_{X,\mathcal{F}'}, \mathcal{W}_{\mathcal{F},r}).$$

This can be computed easily as soon as we are able to identify the Lagrangian $\mathcal{V}_{X,\mathcal{F}'} \subseteq H_1(T;\mathbb{C})$. In fact, if a generator of this Lagrangian has slope s with respect to $\mathcal{F}$, by Proposition 1.2.8 we have

$$\tau(\mathcal{M}_{\mathcal{F}}, \mathcal{V}_{X,\mathcal{F}'}, \mathcal{W}_{\mathcal{F},r}) = \tau(\infty, s, r) = \operatorname{sgn}(s-r),$$

where $\operatorname{sgn}(s-r)$ has to be interpreted as 0 as soon as $r=\infty$ or $s=\infty$. Thus, under the assumption that α has no trivial summand on T, we have

$$\rho_\alpha(D_r(X,\mathcal{F})) = \rho_\alpha(X,\mathcal{F}) + \rho_\alpha(D^2\times S^1,\mathcal{F}_r) + n\operatorname{sgn}(r-s). \tag{3.5}$$

Without any assumption on the representation (apart from the extension), moreover, the correction term vanishes in the case of ∞-framed Dehn filling, as expressed by the following result.

Corollary 3.3.18. *In the setting of Proposition 3.3.16, suppose that α extends to $D_\infty(X,\mathcal{F}_T)$. We have then*

$$\rho_\alpha(D_\infty(X,\mathcal{F}),\mathcal{F}') = \rho_\alpha(X,\mathcal{F}\cup\mathcal{F}').$$

Proof. The result follows immediately from Proposition 3.3.16, once it has been observed that $\mathcal{W}^\alpha_{\mathcal{F},\infty} = \mathcal{M}^\alpha_{\mathcal{F}}$ and $\mathcal{W}_{\mathcal{F},\infty} = \mathcal{M}_{\mathcal{F}}$. ☐

Remark 3.3.19. Sometimes, we are interested in doing a Dehn filling on *all* boundary components of $(X,\mathcal{G})$, in order to obtain a closed manifold. We introduce notation for one specific case that we will need in the applications, namely that of the 0-framed filling along all boundary components of X along the framing $\mathcal{G}$. We let $D_0(X,\mathcal{G})$ denote the closed manifold obtained in this way (this agrees with the usual notation if X only has one boundary component). Suppose now that $\alpha \in \mathcal{U}_n(X_L)$ extends to $D_0(X,\mathcal{G})$. As a 0-framed filling is the same thing as an ∞-filling on the reverse framing $\mathcal{F}^*$, and as the ∞-filling (whenever it is allowed) does not change the rho invariant, we have thus $\rho_\alpha(D_0(X,\mathcal{G})) = \rho_\alpha(X,\mathcal{G}^*)$ and thus, by Proposition 3.2.8,

$$\rho_\alpha(D_0(X,\mathcal{G})) = \rho_\alpha(X,\mathcal{G}) + \tau(\mathcal{L}^\alpha_{\mathcal{G}}, \mathcal{M}^\alpha_{\mathcal{G}}, \mathcal{V}^\alpha_X) - n\,\tau(\mathcal{L}_{\mathcal{G}}, \mathcal{M}_{\mathcal{G}}, \mathcal{V}_X).$$

The correction term in the gluing formulas seen so far appears to be more complicated when there are non-glued boundary components, because the Lagrangians of the form $\mathcal{V}^\alpha_{X,\mathcal{F}}$ do not carry an immediate geometrical meaning and it might be hard to describe them explicitly using the definition. We conclude the section with a more geometrical description for them in terms of Dehn fillings, which is available for some representations.

Let X be a 3-manifold with framed toroidal boundary, with ∂X partitioned into two groups of components Σ and Σ', with framings $\mathcal{F}$ and $\mathcal{F}'$ respectively. Let $\alpha \in \mathcal{U}_n(X)$ be a local coefficient system.

Lemma 3.3.20. *Let X be a compact, oriented 3-manifold, and let $\Sigma \subseteq \partial X$ be a disjoint union of tori with framing $\mathcal{F}$. If α extends to the manifold X' obtained by an ∞-framed surgery along $\mathcal{F}$ on all components of Σ, then*

$$\mathcal{V}^{\alpha}_{X,\mathcal{F}} = \mathcal{V}^{\alpha}_{X'}.$$

Proof. The manifold X' is obtained by gluing to X a disjoint union Y of solid tori. By definition, ∞-framed Dehn filling identifies the meridians of the solid tori to the meridians of the framing. In particular, $\mathcal{M}^{\alpha}_{\mathcal{F}}$ is identified under the gluing with the corresponding Lagrangian $\mathcal{M}^{\alpha}_{\mathcal{F}_\infty}$ for the product framing of Y. By Lemma 3.3.13, we have hence $\mathcal{M}^{\alpha}_{\mathcal{F}} = \mathcal{V}^{\alpha}_{Y}$, and the result follows now from Proposition 1.5.6 about the propagation under bordisms of the canonical Lagrangians. □

Remark 3.3.21. The above description of $\mathcal{V}^{\alpha}_{X,\mathcal{F}}$ always works if α is a trivial local coefficient system. In general, α might not extend to $D_\infty(X, \mathcal{F})$, and in those cases the right term is not defined. We can think of $\mathcal{V}^{\alpha}_{X,\mathcal{F}}$ as a formal replacement for it. In many cases, we shall restrict our attention to 1-dimentional representations that restrict nontrivially to each boundary component. In those cases, twisted homology of the boundary vanishes, and the only correction term to be computed is the one for the trivial representation, which can be done with the help of Lemma 3.3.20.

3.3.4 Atiyah-Patodi-Singer invariants of lens spaces

Before going on with the computation for solid tori, it is useful to recall and reformulate some classical results about eta and rho invariants of 3-dimensional solid tori. For coprime integers p, q, consider the 3-dimensional lens spaces $L(p, q)$ (see Appendix A.1 for the conventions). A Hermitian local coefficient system on $L(p, q)$ can be seen as a unitary representation $\alpha\colon \pi_1(L(p, q)) = \mathbb{Z}/p \to U(n)$. As every such α can be written as a direct sum of 1-dimensional representations, we shall focus on representations $\alpha\colon \mathbb{Z}/p \to U(1)$. The case $p = 0$ is not interesting. In fact, in that case we have defined $L(0, \pm 1)$ to be $S^2 \times S^1$, and its rho invariant is trivial because it admits an orientation-reversing self-diffeomorphism that is trivial on the fundamental group.

We suppose from now on that p is different from 0. We observe in this case that the representations $\alpha\colon \mathbb{Z}/p \to U(1)$ are in a natural bijection with the set of p^{th} roots of unity: to each such root ω, we associate the representation α_ω sending 1 to ω.

Notation 3.3.22. Given a p^{th} root of unity ω, we write

$$\rho(L(p,q),\omega) := \rho_{\alpha_\omega}(L(p,q)).$$

Remark 3.3.23. Thanks to the equalities and diffeomorphisms of Proposition A.1.1, for $r \equiv q \mod p$ and $s \equiv q^{-1} \mod p$, we have

$$\begin{aligned}\rho(L(p,q),\omega) &= \rho(L(p,q),\omega^{-1}) = -\rho(L(-p,q),\omega) = -\rho(L(p,-q),\omega)\\ &= \rho(L(p,r),\omega) = \rho(L(p,s),\omega^s).\end{aligned}$$

Formulas for the rho invariants of lens spaces were given since the original paper of Atiyah, Patodi and Singer [3, Proposition 2.12], using the G-signature theorem. The result can be expressed in many equivalent ways. We choose as a starting point a description of Casson and Gordon in terms of lattice points in a triangle. For $(x,y) \in \mathbb{R}^2$, let $\Delta(x,y)$ be the triangle with vertices $(0,0)$, $(x,0)$ and (x,y). For such a triangle, we consider the number $\operatorname{int}(\Delta(x,y))$ given by counting:

- $+1$ for every point of $\mathbb{Z}^2$ that lies in the interior of $\Delta(x,y)$;
- $+1/2$ for every point of $\mathbb{Z}^2$ that lies in the interior of its edges;
- $+1/4$ for every point of $\mathbb{Z}^2 \setminus \{(0,0)\}$ that coincides with one of the vertices.

Then, the following formula is verified.

Theorem 3.3.24 (Casson-Gordon)**.** *Let p,q be two positive coprime integers, and let $\zeta = e^{2\pi i/p}$. Then, for $k \in \{1,2,\dots,p-1\}$, we have*

$$\rho(L(p,q),\zeta^{kq}) = 4\left(\operatorname{int}\Delta\left(k,\frac{kq}{p}\right) - \operatorname{area}\Delta\left(k,\frac{kq}{p}\right)\right).$$

Proof. Set $z := e^{2\pi i n/p}$, with $n = \gcd(p,k)$, and set $r := k/n$. Then, we have $\zeta^{kq} = z^{rq}$. The representation sending 1 to ω^q has as its image the set of m^{th} roots of unity, with m such that $p = mn$. In this setting, Casson and Gordon [10, pp.187-188] gave the formula [2]

$$\rho(L(p,q),z^{rq}) = -4\left(\operatorname{area}\Delta\left(nr,\frac{rq}{m}\right) - \operatorname{int}\Delta\left(nr,\frac{rq}{m}\right)\right),$$

that can be immediately rewritten as in the statement. □

[2] Thanks to the Atiyah-Patodi-Singer theorem, their invariant σ coincides indeed with the rho invariant up to a minus sign.

We rewrite now the formula of Casson and Gordon in a way that make-explicit computations more feasible. In the following let $(\!(\cdot)\!) : \mathbb{R} \to \left(-\frac{1}{2}, \frac{1}{2}\right)$ be the periodic sawtooth function defined by

$$(\!(x)\!) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases}$$

Corollary 3.3.25. *Let p, q be two coprime integers with $p \neq 0$, and let $\zeta = e^{2\pi i/p}$. Then, for $k \in \{1, 2, \dots, |p| - 1\}$, we have*

$$\rho(L(p,q), \zeta^{kq}) = -4 \sum_{j=1}^{k-1} \left(\!\!\left(\frac{qj}{p}\right)\!\!\right) - 2 \left(\!\!\left(\frac{qk}{p}\right)\!\!\right).$$

Proof. We first suppose $p, q > 0$, in order to apply Theorem 3.3.24. It is clear that $4 \operatorname{area} \Delta \left(k, \frac{kq}{p}\right) = \frac{2q}{p} k^2$. Moreover, we can count the lattice points inside the triangle by following vertical lines $\{(x, y) \mid x = j\}$, for $j = 1, \dots, k$, and then summing over j. We obtain

$$\begin{aligned} 4 \operatorname{int} \Delta \left(k, \frac{kq}{p}\right) &= 4 \sum_{j=1}^{k-1} \left(\frac{1}{2} + \left\lfloor \frac{jq}{p} \right\rfloor\right) + 4 \left(\frac{1}{4} + \frac{1}{2} \left\lfloor \frac{kq}{p} \right\rfloor\right) = \\ &= 2k - 1 + 4 \sum_{j=1}^{k-1} \left\lfloor \frac{jq}{p} \right\rfloor + 2 \left\lfloor \frac{kq}{p} \right\rfloor, \end{aligned}$$

and it follows thus from Theorem 3.3.24 that

$$\rho(L(p,q), \zeta^{kq}) = -\frac{2q}{p} k^2 + 2k - 1 + 4 \sum_{j=1}^{k-1} \left\lfloor \frac{jq}{p} \right\rfloor + 2 \left\lfloor \frac{kq}{p} \right\rfloor. \tag{3.6}$$

This is exactly -4 times the final expression of Remark A.2.5, and thus it can be rewritten as it appears in the statement. As the two sides of the identity behave in the same way when either p or q is changed sign, the result keeps holding for non-positive choices of p and q. □

Remark 3.3.26. Even though it is less elegant than the formula in the statement, the intermediate step (3.6) can be more useful for quick computations. For example, in the case $q = 1$ it leads immediately to

$$\rho(L(p,1), \zeta^k) = -\frac{2}{p} k^2 + 2k - 1 = \frac{2k(p-k)}{p} - 1.$$

Alternatively, in terms of $t = k/p \in \frac{1}{p}\mathbb{Z}$, this can be written as

$$\rho(L(p,1), e^{2\pi i t}) = 2p \cdot t(1-t) - 1.$$

The lens space $L(p,q)$ has a natural Riemannian metric inherited from the standard metric of S^3. It is possible to get a sharper result than Theorem 3.3.24, identifying precisely the contribution of the twisted and untwisted eta invariant with respect to this metric. Namely, in the original series of paper about eta invariants, Atiyah, Patodi and Singer proved that

$$\eta(L(p,q)) = -\frac{1}{p}\sum_{j=1}^{p-1} \cot\frac{\pi k}{p}\cot\frac{\pi kq}{p}. \tag{3.7}$$

Up to a multiplicative constant, this is a classical expression for the value of the *Dedekind sum* $s(q,p)$. We show that, more generally, the twisted eta invariant $\eta(L(p,q),\omega) := \eta_\omega(L(p,q))$ can be expressed in terms of the *Dedekind-Rademacher sum*

$$s_{x,y}(a,c) := \sum_{j=0}^{|c|-1}\left(\left(\frac{a(j+x)}{c}+y\right)\right)\left(\left(\frac{j+x}{c}\right)\right),$$

defined for coprime integers a,c with $c\neq 0$ and real numbers x,y (note that $s_{x,y}(a,c)$ is 1-periodic in both x and y). This is a true generalization of the classical Dedekind sums, as we have $s_{0,0}(a,c) = s(a,c)$. See Appendix A.2 for some basic results about the classical and generalized Dedekind sums.

Theorem 3.3.27. *Let p,q coprime integers with $p\geq 0$, and let $y\in\frac{1}{p}\mathbb{Z}$. Then, we have*

$$\eta(L(p,q),e^{2\pi iy}) = -4s_{0,y}(q,p).$$

Proof. If y is an integer, we are looking at the untwisted eta invariant. By the 1-periodicity of the Dedekind-Rademacher sums, we have $s_{0,y}(q,p) = s_{0,0}(q,p) = s(q,p)$. So, the result to be proved in this case is

$$\eta(L(p,q)) = -4s(q,p). \tag{3.8}$$

This follows from (3.7) via the cotangent formula for the Dedekind sums (A.4), as it was observed by Atiyah [1, p. 356].

In the general case, find $k,n\in\mathbb{Z}$ such that $y = kq/p+n$. Then, by Corollary 3.3.25 we have

$$\rho(L(p,q),e^{2\pi iy}) = -4\left(\sum_{j=1}^{k-1}\left(\left(\frac{qj}{p}\right)\right)+\frac{1}{2}\left(\left(\frac{qk}{p}\right)\right)\right).$$

Thanks to Lemma A.2.4, we can rewrite the above expression as

$$\rho(L(p,q),e^{2\pi iy}) = -4\left(s_{0,y}(q,p)-s(q,p)\right). \tag{3.9}$$

On the other hand, we have by definition

$$\rho(L(p,q), e^{2\pi i y}) = \eta(L(p,q), e^{2\pi i y}) - \eta(L(p,q)),$$

and the conclusion follows then from (3.8). □

3.3.5 More computations on the solid torus

In Section 3.3.1, we classified all positively-oriented framings on the boundary of a solid torus up to orientation-preserving diffeomorphism. Namely, we saw that they are in bijection with $\mathbb{Q} \cup \{\infty\}$ through their gradient (see Definition 3.3.1 and Lemma 3.3.3). Moreover, we saw that the number $\rho_\alpha(D^2 \times S^1, \mathcal{F}_r)$ does not depend on the choice of the specific framing $\mathcal{F}_r$ among the positively-oriented framings of gradient r, and established some properties and first computations. With the help of the result of Section 3.3.4, together with our gluing formulas, we wish now to compute the rho invariants of solid tori for more values of α and r. As we will see, the invariant has (maybe unexpectedly) a very complicated behavior.

First of all, we observe that every representation $\alpha\colon \pi_1(D^2 \times S^1) \to U(n)$ can be written as the direct sum of 1-dimensional representations. As rho invariants are additive with respect to direct sums of representations, hence, it is enough to compute $\rho_\alpha(D^2 \times S^1, \mathcal{F}_r)$ for 1-dimensional representations. Every 1-dimensional representation, in turn, can be represented by an element in $U(1)$, corresponding to the image of the class $[S^1]$. Pre-composing with the exponential map, this leads to a family of 1-periodic functions $S_r\colon \mathbb{R} \to \mathbb{R}$ given, for fixed $r \in \mathbb{Q} \cup \{\infty\}$, by

$$S_r(t) := \rho_{\alpha_t}(D^2 \times S^1, \mathcal{F}_r),$$

where α_t is the representation sending $[S^1]$ to $e^{2\pi i t}$. Of course we have $S_r(0) = 0$ for all r, as 0 corresponds to the trivial representation. More generally, by periodicity, $S_r(n) = 0$ for all integers n, and we can focus on the study of the function on $\mathbb{R} \setminus \mathbb{Z}$ or even just on $(0,1)$. We prove now that S_r is continuous outside of $\mathbb{Z}$, and recall the properties about it that we proved in Section 3.3.1.

Proposition 3.3.28. *For all $r \in \mathbb{Q} \cup \{\infty\}$, $S_r\colon \mathbb{R} \setminus \mathbb{Z} \to \mathbb{R}$ is a continuous function, satisfying, for all $t \in \mathbb{R} \setminus \mathbb{Z}$,*

(i) $S_r(t+1) = S_r(t)$.

(ii) $S_r(-t) = S_r(t)$.

(iii) $S_{-r}(t) = -S_r(t)$*. As a consequence,* $S_0(t) = S_\infty(t) = 0$*.*

(iv) $S_r(t) + S_{1/r}(t) = -\operatorname{sgn}(r)$*. As a consequence,* $S_1(t) = -\frac{1}{2}$*.*

Proof. Property (i) is the 1-periodicity of S_p, that we have already discussed. Properties (ii), (iii) and (iv) follow respectively from Proposition 3.3.7, Proposition 3.3.9 and Proposition 3.3.11.

Now, set $Y := D^2 \times S^1$. For proving that S_r is continuous on $\mathbb{R} \setminus \mathbb{Z}$, we write

$$S_r(t) = \eta_{\alpha_t}(D^2 \times S^1, L_{\mathcal{F}_r}^{\alpha_t}) - \eta(D^2 \times S^1, L_{\mathcal{F}_r}),$$

where the eta invariants are computed with respect to any Riemannian metric on Y whose restriction to ∂Y is compatible with $\mathcal{F}_r$. Hence, S_r is continuous exactly where the function $t \mapsto \eta_{\alpha_t}(Y, L_{\mathcal{F}_r}^{\alpha_t})$ is continuous. For every non-trivial representation $\alpha\colon \pi_1(Y) \to U(1)$, by Lemma 1.3.8 we have

$$H^*(Y; \mathbb{C}_\alpha) = 0, \quad H^*(\partial Y; \mathbb{C}_\alpha) = 0. \tag{3.10}$$

It follows that, for non-trivial α, we have $L_{\mathcal{F}_r}^{\alpha} = V_Y^{\alpha} = 0$. For $t \in \mathbb{R}$, let D_t denote the odd signature operator on Y associated to the representation α_t and the canonical boundary conditions $V_Y^{\alpha_t}$ (see Section 2.2.2). For $t \notin \mathbb{Z}$ we have then, by definition,

$$\eta_{\alpha_t}(Y, L_{\mathcal{F}_r}^{\alpha_t}) = \eta_{\alpha_t}(Y, 0) = \eta(D_t).$$

Now, fix $s \in \mathbb{R} \setminus \mathbb{Z}$. By the results of Kirk and Lesch [28, (7.6)], for h small enough we have

$$\eta(D_{s+h}) - \eta(D_s) = 2\,\mathrm{SF}(h) - (\dim\ker D_{s+h} - \dim\ker D_s) + I(h), \tag{3.11}$$

where $\mathrm{SF}(h) \in \mathbb{Z}$ denotes the *spectral flow* of D_t between s and $s+h$, and $I(h) = \int_s^{s+h} \frac{d\eta(D_t)}{dt} dt$ is a smooth function with respect to h with the property that $I(0) = 0$. From (3.10) and the general observation that there is an isomorphism [29, Lemma 8.6]

$$\ker D_{Y, V_Y^\alpha}^{\alpha} \cong \operatorname{im}(H^{\mathrm{even}}(Y, \partial Y; \mathbb{C}_\alpha) \to H^{\mathrm{even}}(Y; \mathbb{C}_\alpha)),$$

we see immediately that, for $t \notin \mathbb{Z}$, we have $\ker(D_t) = 0$. From these facts, it also follows that $\mathrm{SF}(h)$ is zero. As a consequence, (3.11) can be rewritten as

$$\eta(D_{s+h}) - \eta(D_s) = I(h),$$

and it follows that $t \mapsto \eta(D_t)$ is continuous at the point s. □

Thanks to properties (ii) and (iii), we can focus on studying the function S_r for $r \geq 1$, and derive it then for all other values of r. For example, with the information at hand we can also see that $S_{-1}(t) = \frac{1}{2}$ for all $t \notin \mathbb{Z}$.

We use the above result, together with the gluing formulas for Dehn fillings, in order to relate the functions S_r with the rho invariants of lens spaces. It is convenient to set up the following notation.

Notation 3.3.29. Given coprime integers p and q, with $p \neq 0$, for $k \in \mathbb{Z}$ we set

$$\ell(p, q, k) := \rho(L(p, q), e^{2\pi i k q/p}).$$

Remark 3.3.30. The value of $\ell(p, q, k)$ is clearly p-periodic on k. According to the result of Section 3.3.4, for $p > 0$ and $k \in \{1, \dots p-1\}$, we have equalities

$$\begin{aligned}\ell(p, q, k) &= -\frac{2q}{p}k^2 + 2k - 1 + 4\sum_{j=1}^{k-1} \left\lfloor \frac{jq}{p} \right\rfloor + 2\left\lfloor \frac{kq}{p} \right\rfloor = \\ &= -4\sum_{j=1}^{k-1} \left(\left(\frac{qj}{p} \right) \right) - 2\left(\left(\frac{qk}{p} \right) \right) \\ &= -4\left(s_{0,kq/p}(q, p) - s(q, p) \right).\end{aligned}$$

Proposition 3.3.31. *Let (a, c) and (p, q) be two pairs of coprime integers, and let b, d integers such that $ad - bc = 1$. Then, for $t = \frac{k}{pc+qa} \in \frac{1}{pc+qa}\mathbb{Z}$, $t \notin \mathbb{Z}$, we have*

$$S_{p/q}(t) + S_{a/c}((pd + qb)t) + \operatorname{sgn}\left(\tfrac{p}{q} + \tfrac{a}{c}\right) = \ell(pc + qa, pd + qb, k).$$

Proof. Let V, W be two copies of the solid torus $D^2 \times S^1$, and denote their usual meridians and longitudes by

$$\mu_V, \lambda_V \in H_1(\partial V; \mathbb{Z}), \quad \mu_W, \lambda_W \in H_1(\partial W; \mathbb{Z})$$

(i.e. μ_V and μ_W correspond to the class $[\partial D^2]$, while λ_V and λ_W correspond to $[S^1]$). Moreover, let $\mathcal{F} = (\mu, \lambda)$ be the standardly oriented framing on ∂V defined by the relations

$$\begin{cases}\mu_V = -p\mu + q\lambda \\ \lambda_V = -r\mu + s\lambda,\end{cases} \qquad \text{with } \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{3.12}$$

Thanks to Proposition 3.3.16 (see also Remark 3.3.17), for every representation $\alpha\colon \pi_1(V) \to U(1)$ that extends to the filling, we have

$$\rho_\alpha(D_{a/c}(V,\mathcal{F})) = \rho_\alpha(V,\mathcal{F}) + \rho_\alpha(W,\mathcal{F}_{a/c}) + \operatorname{sgn}\left(\frac{p}{q} + \frac{a}{c}\right). \tag{3.13}$$

By definition, the framing $\mathcal{F}$ coincides up to sign with a non-standardly oriented framing $\mathcal{F}_{p/q}$ of gradient p/q. Hence, $\rho_\alpha(V,\mathcal{F}) = \rho_\alpha(V,\mathcal{F}_{p/q})$.

Now, the manifold $D_{a/c}(V,\mathcal{F})$ can be built as the union of V and W along a gluing diffeomorphism $\varphi\colon \partial W \to \partial V$ such that

$$\begin{cases} \varphi_*(\mu_W) = a\mu + c\lambda \\ \varphi_*(\lambda_W) = b\mu + d\lambda \end{cases}$$

(such a map is indeed orientation-reversing, as the framing (μ_W, λ_W) is non-standardly oriented). Its inverse map $\psi\colon \partial V \to \partial W$, thus, is such that

$$\begin{cases} \psi_*(\mu) = d\mu_W - c\lambda_W \\ \psi_*(\lambda) = -b\mu_W + a\lambda_W. \end{cases} \tag{3.14}$$

Putting together (3.12) and (3.14), we see thus that

$$\begin{cases} \psi_*(\mu_V) = -(pd+qb)\mu_W + (pc+qa)\lambda_W \\ \psi_*(\lambda_V) = -(rd+sb)\mu_W + (rc+sa)\lambda_W. \end{cases} \tag{3.15}$$

From the first equation of (3.15), it follows that there is an orientation-preserving diffeomorphism

$$f\colon D_{a/c}(V,\mathcal{F}) \xrightarrow{\sim} L(pc+qa, pd+qb)$$

such that the induced map on the fundamental groups sends the class of λ_W to $[1] \in \mathbb{Z}/(pc+qa)$ (see Proposition A.1.2 and Remark A.1.3)

For $t = \frac{k}{pc+qa}$, we consider now the $(pc+qa)^{\text{th}}$ root of unity $\omega = e^{2\pi i t}$, and we let $\alpha\colon \pi_1(V) \to U(1)$ be the representation sending λ_V to ω. From the inverse relations of (3.15), we see that such a representation extends to the filling, and that the class corresponding to λ_W is sent to $\omega^{pd+qb} = e^{2\pi i(pd+qb)t}$. It follows that, with respect to this representation, we have

$$\rho_\alpha(V,\mathcal{F}) = \rho_\alpha(V,\mathcal{F}_{p/q}) = S_{p/q}(t), \quad \rho_\alpha(W,\mathcal{F}_{a/c}) = S_{a/c}((pd+qb)t)$$

and

$$\rho_\alpha(D_{a/c}(V,\mathcal{F})) = \rho(L(pc+qa,pd+qb),\omega^{pd+qb}) = \ell(pc+qa,pd+qb,k).$$

Plugging these values into (3.13), we find the desired formula. □

Proposition 3.3.31 allows us to compute the value of $S_{p/q}(t)$ for many values of $t \in (0,1)$.

Corollary 3.3.32. *Let p, q be two positive coprime integers, with $p > q$. Then, we have*

(i) $S_{p/q}(k/q) = \ell(q,p,k)$ *for all* $k \in \mathbb{Z}$*;*

(ii) $S_{p/q}(k/p) = -\ell(p,q,k) - 1$ *for all* $k \in \mathbb{Z}$, $k \not\equiv 0 \mod p$*;*

(iii) $S_{p/q}(k/(p+q)) = -\ell(p+q,q,k) - 1/2$ *for all* $k \in \mathbb{Z}$, $k \not\equiv 0 \mod p+q$*;*

(iv) $S_{p/q}(k/(p-q)) = -\ell(p-q,q,k) - 3/2$ *for all* $k \in \mathbb{Z}$, $k \not\equiv 0 \mod p-q$.

In Appendix A.3 we added graphs, generated with the software Mathematica, representing all the values that can be computed for $S_{p/q}(t)$ for $p = 31$ and all $1 \leq q \leq 30$, by the use of Corollary 3.3.32. This gives at least an idea of the shape of the graph of these functions on the whole interval $(0,1)$.

Remark 3.3.33. As the formula for the rho invariant of $L(p,q)$ is particularly simple in the case $q = 1$ (see Remark 3.3.26), there are very nice expressions for the value $S_p(t)$ ($p \in \mathbb{N}$) at some values of t. Namely, for positive p, we find from the formulas (ii), (iii) and (iv) of Corollary 3.3.32 that

$$S_p(t) = \begin{cases} -2pt(1-t), & \text{if } t \in \frac{1}{p}\mathbb{Z} \cap (0,1), \\ -2(p+1)t(1-t) + \frac{1}{2}, & \text{if } t \in \frac{1}{p+1}\mathbb{Z} \cap (0,1), \\ -2(p-1)t(1-t) - \frac{1}{2}, & \text{if } t \in \frac{1}{p-1}\mathbb{Z} \cap (0,1). \end{cases} \tag{3.16}$$

In fact, the expression $S_p(t) = -2pt(1-t)$ stays true for every $t \in \frac{1}{p}\mathbb{Z} \cap (0,1)$. This can be seen by applying Proposition 3.3.31 with $a = p$, $c = q = 1$, $b = -1$, $d = 0$, leading to

$$S_p(t) + S_p(-t) + 1 = \ell(2p, -1, k).$$

This can be rewritten as

$$2S_p(t) = -\ell(2p, 1, k) - 1,$$

which implies the said expression. The formulas of (3.16) show that, on some discrete subset of $(0,1)$, the points of the graph of $S_p(t)$ distribute themselves along three different parabolas, all having the same vertex at $t = 1/2$, where

$$S_p(\tfrac{1}{2}) = -\tfrac{p}{2}.$$

See for example the first graph of Appendix A.3, corresponding to the case $p = 31$.

Even though using Corollary 3.3.32 we are able to identify the value of $S_{p/q}(t)$ for t a discrete subset of $(0, 1)$, that grows bigger for bigger p and q, the function is continuous and a complete relation with already known functions remains so far mysterious. The fact that $S_{p/q}(t)$ coincides with $-4(s_{0,pt}(p, q) - s(p, q))$ for $t \in \frac{1}{q}\mathbb{Z}$ and that both functions are well-defined 1-periodic functions on $\mathbb{R}$ might lead to the expectation that the functions coincide for all values. In fact, however, this conjecture is quickly disproved by direct computation, and the function on the right is not even continuous in t.

The family of functions S_r seems to contain a quite high amount of number-theoretical information, and it satisfies a reciprocity formula that is even simpler than the one satisfied by Dedekind sums, namely

$$S_{p/q}(t) + S_{q/p}(t) = -1$$

for positive p and q. For these reasons, we conclude the section with the following (loosely formulated) open problem.

Problem 3.3.34. *What is the value of $S_r(t) = \rho_{\alpha_t}(D^2 \times S^1, \mathcal{F}_r)$ for generic $r \in \mathbb{Q} \cup \{\infty\}$ and $t \in \mathbb{R}$? What is the true relation of this family of functions with analytic number theory and with other mathematical objects?*

3.4 Additional topics and problems

In this section, we give a precise formulation and partial solutions to two more important problems regarding the rho invariant of 3-manifolds with framed toroidal boundary. In Section 3.4.1, we attack the problem of studying a formula for a general change of framing on the boundary of a manifold. In Section 3.4.2, we study the rho invariant of a manifold of the form $F \times S^1$, where F is a compact oriented surface.

3.4.1 Change of framing formulas and thick tori

Given two different framings $\mathcal{F}$ and $\mathcal{F}'$ on the (toroidal) boundary of a compact, oriented 3-manifold X with a local coefficient system $\alpha \in \mathcal{U}_n(X)$, we can ask ourselves how $\rho_\alpha(X, \mathcal{F})$ and $\rho_\alpha(X, \mathcal{F}')$ are related. The dependence of Kirk-Lesch rho invariants on the boundary conditions is well understood thanks to Theorem 2.2.5 (i). However, a framing prescribes not only the boundary conditions, but also the Riemannian metric on the boundary, which makes the problem much harder. The case of the conjugate framing is easier to analyze, as it induces the same Riemannian metric as the original framing. For this specific change of framing, we have computed in Proposition 3.2.8 that

$$\rho_\alpha(X, \mathcal{F}^\star) - \rho_\alpha(X, \mathcal{F}) = \tau(\mathcal{M}^\alpha_{\mathcal{F}}, \mathcal{L}^\alpha_{\mathcal{F}}, \mathcal{V}^\alpha_X) - n\, \tau(\mathcal{M}_{\mathcal{F}}, \mathcal{L}_{\mathcal{F}}, \mathcal{V}_X). \tag{3.17}$$

Suppose now for simplicity that X has a single boundary component T, and let $\mathcal{F}$ and $\mathcal{F}'$ be any two framings on T. Then, up to diffeomorphism, we can see the manifold $(X, \mathcal{F}')$ as obtained from gluing to $(X, \mathcal{F})$ a cylinder $[0,1] \times T$, with framing $\mathcal{F}$ on $-\{0\} \times T$ and $\mathcal{F}'$ on $\{1\} \times T$. We set from now on $I := [0,1]$. We obtain thus the following result.

Proposition 3.4.1. *The difference* $\rho_\alpha(X, \mathcal{F}') - \rho_\alpha(X, \mathcal{F})$ *is given by*

$$\rho_\alpha(I \times T, \mathcal{F} \cup \mathcal{F}') + \tau(\mathcal{M}^\alpha_{\mathcal{F}}, \mathcal{V}^\alpha_X, \mathcal{M}^\alpha_{\mathcal{F}'}) - n\tau(\mathcal{M}_{\mathcal{F}}, \mathcal{V}_X, \mathcal{M}_{\mathcal{F}'}).$$

Proof. It is a direct application of the gluing formula of Theorem 3.2.10, together with the observation that the cylinder $I \times T$ acts as the identity in the propagation of Lagrangian subspaces in the sense of Section 1.5.2. □

It is thus clear that, in order to get an explicit change-of-framing formula, it is needed to compute the rho invariant of a *thick torus* $I \times T$ with two different framings at the two ends. We shall see local coefficient systems $\alpha \in \mathcal{U}_n(I \times T)$ as representations $\alpha\colon \pi_1(I \times T) \to U(n)$. As in the case of solid tori, it is enough to consider 1-dimensional representations, as every other representation splits as a direct sum of them.

In order to study the rho invariants of thick tori, we introduce the following notation. Given an oriented torus T with two standardly oriented framings $\mathcal{F} = (\mu, \lambda)$ and $\mathcal{F}' = (\mu', \lambda')$, there are integers a, b, c, d such that

$$\begin{cases} \mu = a\mu' + c\lambda' \\ \lambda = b\mu' + d\lambda', \end{cases} \qquad \text{with } A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{3.18}$$

We write in this case $\mathcal{F}' = A\mathcal{F}$. In this way, $SL(2,\mathbb{Z})$ acts freely and transitively on the left on the sets of all standardly oriented framings on T.

Remark 3.4.2. Note that, if $\mathcal{F}$ is standardly oriented, then the conjugate framing $\mathcal{F}^*$ is non-standardly oriented. Thus, it does not fall in the previous description. However, by reversing either the meridian or the longitude, we obtain a standardly oriented framing that coincides with $\mathcal{F}^*$ up to sign. The two framings obtained in this way correspond to the matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Observe now that there is a natural isomorphism $\pi_1(T) = H_1(T; \mathbb{Z})$, and a framing $\mathcal{F}$ on T prescribes two generators μ, λ, for this abelian group. We set the following notation.

Notation 3.4.3. Given two framings $\mathcal{F}, \mathcal{F}'$ on a torus T, let $\mathcal{F} \cup \mathcal{F}'$ denote the framing on $\partial(I \times T)$ given by $\mathcal{F}$ on $-\{0\} \times T$ and $\mathcal{F}'$ on $\{1\} \times T$.

We define now a function $\Theta\colon \mathrm{SL}(2, \mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}$

$$\Theta(A, (x, y)) := \rho_\alpha(I \times T, \mathcal{F} \cup A\mathcal{F}),$$

where $(T, \mathcal{F})$ is any standardly oriented torus, $A\mathcal{F} = (\mu', \lambda)$, and $\alpha\colon \pi_1(I \times T) \to U(1)$ is the representation defined by $\alpha(\mu') = e^{2\pi i x}$ and $\alpha(\lambda') = e^{2\pi i y}$. From the diffeomorphism properties of rho invariants (Theorem 3.2.3), it is immediate to see that Θ is well defined function, which is 1-periodic in both of its real variables.

Set $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$. We state now some of the basic properties of Θ.

Proposition 3.4.4. *Let $v \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, and let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), B = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$. Then*

(i) $\Theta(\mathrm{Id}, v) = \Theta(S, v) = 0$;

(ii) $\Theta(-A, v) = \Theta(A, v)$;

(iii) $\Theta(A, v) = -\Theta(A^{-1}, A^t v)$;

(iv) $\Theta(AB, v) = \Theta(A, B^t v) + \Theta(B, v) - \mathrm{sgn}(cc'c'')$*, where* $AB =: \left(\begin{smallmatrix} a'' & b'' \\ c'' & d'' \end{smallmatrix}\right)$.

Moreover, for all $A \in \mathrm{SL}(2, \mathbb{Z})$, the function $\Theta(A, \cdot)$ is continuous in $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

Proof. (ii) is trivial, as the framings corresponding to A and $-A$ coincide up to sign. We prove then (iii). Using the reflection on I, for every representation α we have by Theorem 3.2.3

$$\rho_\alpha(I \times T, \mathcal{F} \cup A\mathcal{F}) = -\rho_\alpha(I \times T, A\mathcal{F} \cup \mathcal{F}). \tag{3.19}$$

Now, let $\mathcal{F} = (\mu, \lambda)$, $A\mathcal{F} = (\mu', \lambda')$ and $v = (x, y)$. Then, it follows from (3.18) that

$$\begin{cases} \alpha(\mu') = e^{2\pi i x}, \\ \alpha(\lambda') = e^{2\pi i y}, \end{cases} \implies \begin{cases} \alpha(\mu) = e^{2\pi i (ax+cy)}, \\ \alpha(\lambda) = e^{2\pi i (bx+dy)}. \end{cases} \tag{3.20}$$

This implies that (3.19) gets rewritten as $\Theta(A, v) = -\Theta(A^{-1}, A^t v)$, as desired. We prove now (i). The equality $\Theta(\mathrm{Id}, v) = 0$ follows immediately from (iii). For proving the other equality, observe that, by definition, $\Theta(S, v) = \rho_\alpha(I \times T, \mathcal{F} \cup \mathcal{F}^*)$. Using (3.17), we see (after an easy Maslov index computation) that

$$\Theta(S, v) = \rho_\alpha(I \times T, \mathcal{F}^* \cup \mathcal{F}) = \Theta(S^{-1}, S^t v).$$

Then, (iii) also implies that $\Theta(S, v) = 0$, and the proof of (i) is complete. For proving (iv), we write $\Theta(AB, v) = \rho_\alpha([0, 2] \times T, \mathcal{F} \cup AB\mathcal{F})$. We decompose the thick torus above as

$$[0, 2] \times T = ([0, 1] \times T) \cup_{\{1\} \times T} ([1, 2] \times T),$$

and provide the torus $\{1\} \times T$ with the framing $B\mathcal{F}$. Then, by Theorem 3.2.10, we have

$$\begin{aligned} \Theta(AB, v) =& \rho_\alpha([0, 1] \times T, \mathcal{F} \cup B\mathcal{F}) + \rho_\alpha([1, 2] \times T, B\mathcal{F} \cup AB\mathcal{F}) + \\ & - \tau(\mathcal{M}_{B\mathcal{F}}, \mathcal{M}_{\mathcal{F}}, \mathcal{M}_{AB\mathcal{F}}). \end{aligned}$$

From the same observation as the one used for proving (iii), we have

$$\rho_\alpha([0, 1] \times T, \mathcal{F} \cup B\mathcal{F}) = \Theta(A, B^t v), \quad \rho_\alpha([1, 2] \times T, B\mathcal{F} \cup AB\mathcal{F}) = \Theta(B, v),$$

and it only remains to identify the Maslov index. Set $\mathcal{G} := B\mathcal{F}$ and write now $\mathcal{G} = (\mu, \lambda)$. Then, we have

$$\tau(\mathcal{M}_{B\mathcal{F}}, \mathcal{M}_{\mathcal{F}}, \mathcal{M}_{AB\mathcal{F}}) = \tau(\mathcal{M}_{\mathcal{G}}, \mathcal{M}_{B^{-1}\mathcal{G}}, \mathcal{M}_{A\mathcal{G}}).$$

From the descriptions

$$\mathcal{M}_{\mathcal{G}} = \mathrm{Span}_{\mathbb{C}}(\mu), \quad \mathcal{M}_{B^{-1}\mathcal{G}} = \mathrm{Span}_{\mathbb{C}}(a'\mu + c'\lambda), \quad \mathcal{M}_{A\mathcal{G}} = \mathrm{Span}_{\mathbb{C}}(d\mu - c\lambda),$$

we can hence compute, using Proposition 1.2.8,

$$\tau(\mathcal{M}_{\mathcal{G}}, \mathcal{M}_{B^{-1}\mathcal{G}}, \mathcal{M}_{A\mathcal{G}}) = \tau\left(\infty, \frac{a'}{c'}, -\frac{d}{c}\right) = \mathrm{sgn}\left(\frac{a'}{c'} + \frac{d}{c}\right) = \mathrm{sgn}\left(\frac{ca' + dc'}{cc'}\right).$$

This is exactly the desired result, as we have the identity $ca' + dc' = c''$, and $\mathrm{sgn}(\infty)$ is 0 by definition.

The proof of continuity is the same as the one for the analogous result about the function S_r (see Proposition 3.3.31). □

The function Θ is at least as complicated as the family of functions S_r (for $r \in \mathbb{Q} \cup \{\infty\}$) defined in Section 3.3.5. In fact, we have the following result.

Proposition 3.4.5. *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. Then, for all $\omega \in U(1)$, we have*

$$\Theta(A,(0,t)) = -S_{d/c}(t).$$

Proof. Consider the standard torus $T = S^1 \times S^1$ with the standardly oriented framing $\mathcal{F}_\times = (-e_1, e_2)$, where

$$e_1 := -[\{1\} \times S^1], \quad e_2 := -[S^1] \times \{1\}.$$

By definition, then, we have

$$\Theta(A,1,\omega) = \rho_\beta(I \times T, (A^{-1}\mathcal{F}_\times) \cup \mathcal{F}_\times),$$

where β is the representation sending $-e_1$ to 1 and e_2 to $e^{2\pi i t}$. Observe that, as $\mathcal{F}$ and $A^{-1}\mathcal{F}$ are standardly oriented on T, $A^{-1}\mathcal{F}$ is non-standardly oriented on the boundary component $-\{0\} \times T$, while $\mathcal{F}$ is standardly oriented on $\{1\} \times T$. We can perform an ∞-framed filling on $\{1\} \times T = \{1\} \times S^1 \times S^1$, by capping the first S^1-factor with a disk, without changing the rho invariant thanks to Corollary 3.3.18. The resulting manifold is a solid torus $D^2 \times S^1$, with framing $\mathcal{F} = (\mu, \lambda)$ corresponding to $A^{-1}\mathcal{F}_\times$. Recalling that $\mathcal{F}_\times = (-e_1, e_2)$ and that in the gluing we have identified $[\partial D^2]$ with $-e_2$, by (3.18) we have

$$[\partial D^2] = d\mu - c\lambda.$$

This means that $\mathcal{F}$ is a non-standardly oriented framing of slope $-d/c$. As a consequence, we have

$$\Theta(A,(0,t)) = \rho_\beta(D^2 \times S^1, \mathcal{F}_{-d/c}) = S_{-d/c}(t) = -S_{d/c}(t).$$

□

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ denote from now on a generic element of $\mathrm{SL}(2,\mathbb{Z})$. Consider now the 3-manifold with framed toroidal boundary $(I \times T, \mathcal{F} \cup A\mathcal{F})$, with $\mathcal{F} = (\mu, \lambda)$ and $A\mathcal{F} = (\mu', \lambda')$. We can form a closed 3 manifold by gluing $-\{0\} \times T$ with $\{1\} \times T$ under the identifications given by $\mu = \mu'$ and $\lambda = \lambda'$. Because of our orientation conventions for the framing, the manifold obtained in this way is the torus bundle $-T_A$, i.e. the opposite of the $s^1 \times S^1$-bundle over S^1 with monodromy A. A representation $\alpha \colon \pi_1(I \times T) \to U(1)$, prescribed as usual by $v \in \mathbb{R}^2$, then, extends to

$-T_A$ if and only if $(\mathrm{Id} - A^t)v \in \mathbb{Z}^2$. In that case, we can compare $\Theta(A, v)$ with the rho invariant of T_A. In order to do so, we consider the function $\nu\colon \mathrm{SL}(2, \mathbb{Z}) \to \mathbb{Z}$ defined as

$$\nu(A) := \begin{cases} \operatorname{sgn}(b), & \text{if } A = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), \\ \operatorname{sgn}(c(a + d - 2)), & \text{otherwise.} \end{cases}$$

(see Appendix A.2.2).

Proposition 3.4.6. *Let $A \in \mathrm{SL}(2, \mathbb{Z})$, and let $v \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ be an element with the property that $(\mathrm{Id} - A^t)v \in \mathbb{Z}^2$. Let $\alpha\colon \pi_1(T_A) \to U(1)$ be the representation induced by v. Then, we have*

$$\Theta(A, v) = -\rho_\alpha(T_A) + \nu(A).$$

Proof. As usual, let $\mathcal{F} = (\mu, \lambda)$ and $\mathcal{F}' = (\mu', \lambda')$. Instead of obtaining T_A from $I \times T$ by gluing its two ends together, we can obtain the same result by gluing $I \times T$ together with another framed thick torus $(I \times T', \mathcal{F} \cup \mathcal{F})$ (where T' is a copy of T) under diffeomorphisms

$$f_0\colon \{0\} \times T' \to \{1\} \times T, \quad f_1\colon \{1\} \times T' \to \{0\} \times T,$$

where f_0 gives the identifications of (3.17), while f_1 is just the identity on T. From Corollary 3.2.11, we have hence

$$\rho_\alpha(-T_A) = \rho_\alpha(I \times T, \mathcal{F} \cup A\mathcal{F}) + \rho_\alpha(I \times T, \mathcal{F} \cup \mathcal{F}) - \tau(\mathcal{M}_{\mathcal{F} \oplus A\mathcal{F}}, \mathcal{V}_{I \times T}, \mathcal{V}_{I \times T'}).$$

Now, it is clear that $\rho_\alpha(-T_A) = -\rho_\alpha(T_A)$, that $\rho_\alpha(I \times T, \mathcal{F} \cup A\mathcal{F}) = \Theta(A, v)$ and that $\rho_\alpha(I \times T, \mathcal{F} \cup \mathcal{F}) = 0$. Thus, in order to complete the proof, we need only show that $\tau := \tau(\mathcal{M}_{\mathcal{F} \oplus A\mathcal{F}}, \mathcal{V}_{I \times T}, \mathcal{V}_{I \times T'})$ coincides with $\nu(A)$. We have the descriptions

$$\begin{cases} \mathcal{M}_{\mathcal{F} \oplus A\mathcal{F}} = \operatorname{Span}_{\mathbb{C}}(\mu, \mu'), \\ \mathcal{V}_{I \times T} = \operatorname{Span}_{\mathbb{C}}(\mu - a\mu' - c\lambda', \lambda - b\mu' - d\lambda'), \\ \mathcal{V}_{I \times T'} = \operatorname{Span}_{\mathbb{C}}(\mu - \mu', \lambda - \lambda'). \end{cases}$$

The space $W := (\mathcal{M}_{\mathcal{F} \oplus A\mathcal{F}} \oplus \mathcal{V}_{I \times T}) \cap \mathcal{V}_{I \times T'}$, then, surely contains $\mu - \mu'$ as a generator, but it does not contribute to the Maslov index because it also lies in $\mathcal{M}_{\mathcal{F} \oplus A\mathcal{F}}$. As consequence, the τ is 0 if $\lambda - \lambda'$ does not belong to W, while, if it does, we have

$$\tau = \operatorname{sgn} \Psi(\lambda - \lambda', \lambda - \lambda'),$$

where Ψ is the bilinear form associated to the triple of Lagrangians (see Section 1.2.1).

Let us check the case $A = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ first. In that case, we have

$$\mathcal{V}_{I\times T} = \mathrm{Span}_{\mathbb{C}}(\mu - \mu', \lambda - \lambda' - b\mu'),$$

and we can write

$$\lambda - \lambda' = (b\mu') + (\lambda - \lambda' - b\mu'),$$

where the first summand belongs to $\mathcal{M}_{\mathcal{F}\oplus A\mathcal{F}}$, and the second to $\mathcal{V}_{I\times T}$. In particular, $\lambda - \lambda' \in W$, and we have by definition

$$\Psi(\lambda - \lambda', \lambda - \lambda') = (b\mu') \cdot (\lambda - \lambda' - b\mu') = -b\mu' \cdot \lambda' = b,$$

as $A\mathcal{F}$ is standardly oriented on $\{1\} \times T$. As a consequence, we have $\tau = \mathrm{sgn}(b) = \nu(A)$ as desired.

Suppose now that $c = 0$. Then $\lambda - \lambda'$ does not belong to W unless we have $d = 1$, which forces A to be of the form already considered. Excluding the matrices of that form, thus, we have $\tau = 0$ as expected.

We can now prove the general formula for $c \neq 0$. If this is the case, in fact, we can write

$$\lambda - \lambda' = \left(\tfrac{1-d}{c}\mu - \tfrac{1-a}{c}\mu'\right) + \left(\tfrac{1-d}{c}(\mu - a\mu' - c\lambda') + (\lambda - b\mu' - d\lambda')\right),$$

(to verify the equality, the fact that $\det A = 1$ has to be used), where the first summand belongs to $\mathcal{M}_{\mathcal{F}\oplus A\mathcal{F}}$, and the second to $\mathcal{V}_{I\times T}$. As a consequence, $\lambda - \lambda'$ belongs to W, and we can compute

$$\begin{aligned}\Psi(\lambda - \lambda', \lambda - \lambda') &= \left(\tfrac{1-d}{c}\mu - \tfrac{1-a}{c}\mu'\right) \cdot \left(\tfrac{1-d}{c}(\mu - a\mu' - c\lambda') + (\lambda - b\mu' - d\lambda')\right) = \\ &= \left(\tfrac{1-d}{c}\mu - \tfrac{1-a}{c}\mu'\right) \cdot \left(\lambda - \lambda'\right) = -\tfrac{1-d}{c} - \tfrac{1-a}{c} = \tfrac{a+d-2}{c},\end{aligned}$$

from which it follows that $\tau = \mathrm{sgn}(c(a + d - 2)$ as desired. □

We introduce now a classical function $\Phi\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{Q}$ called the Rademacher function (see Appendix A.2.2), defined by

$$\Phi(A) := \begin{cases} \frac{b}{3d}, & \text{if } c = 0, \\ \frac{a+d}{3c} - 4\,\mathrm{sgn}(c)s(a,c), & \text{otherwise,} \end{cases}$$

where, again, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The rho invariant of T_A is computed by Bohn by subdividing the problem in three cases, depending on whether A is *elliptic*, *parabolic* or *hyperbolic*, i.e. on whether we have, respectively $\mathrm{tr}\, A^2 - 4 < 0$,

$\operatorname{tr} A^2 - 4 = 0$ or $\operatorname{tr} A^2 - 4 > 0$. For briefness, we recall here Bohn's result only in the case of hyperbolic matrices (which is the "generic" case). In order to do so, we set

$$P_2(x) := \{x\}^2 - \{x\} - \frac{1}{6}, \quad \text{where } \{x\} = x - \lfloor x \rfloor$$

and, for A *hyperbolic* (i.e. such that $\operatorname{tr}(A)^2 > 4$, which implies $c \neq 0$)

$$\Phi_{x,y}(A) := \frac{2(a+d)}{c} P_2(x) - 4 \operatorname{sgn}(c) s_{x,y}(a,c)$$

(observe that, for $(x,y) \in \mathbb{Z}^2$ and A hyperbolic, we have $\Phi_{x,y}(A) = \Phi(A)$). Now, an element $v = (x,y) \in \mathbb{R}^2$ satisfying $(\operatorname{Id} - A^t)v \in \mathbb{Z}^2$ induces a one-dimensional representation α of

$$H_1(T_A;\mathbb{Z}) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}t / \langle e_1 - ae_1 - ce_2, e_2 - be_1 - de_2 \rangle$$

by the rules $\alpha(e_1) := e^{2\pi i x}$, $\alpha(e_2) := e^{2\pi i y}$, $\alpha(t) := 1$. Then, Bohn's result [5, Theorem 4.4.20, (4.67)] can be expressed in a compact way as follows.

Theorem 3.4.7 (Bohn)**.** *Let $A \in \mathrm{SL}(2,\mathbb{Z})$ be hyperbolic, and let $v \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ be an element with the property that $(\operatorname{Id} - A^t)v \in \mathbb{Z}^2$. Let $\alpha\colon \pi_1(T_A) \to U(1)$ be the representation induced by v. Then, we have*

$$\rho_\alpha(T_A) = \Phi_v(A) - \Phi(A) + \nu(A). \tag{3.21}$$

Combining Theorem 3.4.7 with Proposition 3.4.6, we obtain thus the following result.

Corollary 3.4.8. *Let $A \in \mathrm{SL}(2,\mathbb{Z})$ hyperbolic, and let $v \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ be such that $(\operatorname{Id} - A^t)v \in \mathbb{Z}^2$. Then, we have*

$$\Theta(A,v) = \Phi(A) - \Phi_v(A).$$

Remark 3.4.9. We conclude with following long list of only partially developed observations.

(1) Theorem 3.4.7 is proved by picking an appropriate Riemannian metric on T_A and using it to compute the twisted and untwisted eta invariant separately.

(2) The study of the untwisted eta invariant $\eta(T_A)$ goes back to Atiyah [1]. In fact, an *adiabatic limit* (i.e. the limit of $\eta(T_A)$ under a rescaling of the metric by $\varepsilon \mapsto 0$) is needed in order to get a value that is independent from the specific metric chosen, but we shall ignore this technical issue here.

(3) Atiyah proved that the coboundary of $-\eta(T_A) \in \mathbb{Q}$ is the *signature cocycle* [1, (5.12) Proposition]. In particular, from the work of Kirby and Melvin [27, Theorem 6.1], it follows that, for every $A \in \mathrm{SL}(2, \mathbb{Z})$, we have

$$\eta(T_A) = \Phi(A) - \nu(A). \tag{3.22}$$

(see Appendix A.2.2 for a brief discussion about the signature cocycle and the result of Kirby and Melvin).[3]

(4) By the definition of rho invariants, we have now $\rho_\alpha(T_A) = \eta_\alpha(T_A) - \eta(T_A)$. For hyperbolic A, Bohn computes that $\eta_\alpha(T_A) = \Phi_v(A)$, and together with (3.22) (which Bohn reproves by a different method) this gives Theorem 3.4.7.

(5) The invariant $\rho_\alpha(T_A)$ is computed by Bohn not only for hyperbolic elements, but also for elliptic [5, Theorem 4.4.4] and parabolic [5, Theorem 4.4.8] ones. Using his results, thus, it is possible to calculate $\eta_\alpha(T_A)$ in the three different cases. In view of the results about the untwisted eta invariant, it would be now satisfactory to find a single nice expression extending Φ_v to a function $\Phi_v \colon \mathrm{SL}(2, \mathbb{Z}) \to \mathbb{R}$ that satisfies $\Phi_v(A) = \eta_\alpha(T_A)$ for elliptic, parabolic and hyperbolic elements at the same time (whenever the hypothesis $(\mathrm{Id} - A^t)v \in \mathbb{Z}^2$ is satisfied).

(6) The same kind of reasonings can be applied to the eta invariants of a thick torus $I \times T$ with an appropriate metric depending on A (this can be obtained by cutting $-T_A$ open along a fiber). Letting $(I \times T)_A$ denote such Riemannian manifold, and observing that the Maslov index computed in the proof of Proposition 3.4.6 comes from the untwisted eta invariant only, we see that (up to adiabatic limits) we have

$$\eta((I \times T)_A, \mathcal{F} \cup A\mathcal{F}) = -\eta(T_A) - \nu(A) = -\Phi(A). \tag{3.23}$$

(7) In view of formula (A.6) of Appendix A.2.2, the equality (3.23) explains the correction term $-\operatorname{sgn}(cc'c'')$ of Proposition 3.4.4 (iv). Put differently, (3.23) can be used to give a new proof, via gluing formulas, of the fact that

$$\delta\Phi(A, B) = \operatorname{sgn}(c_A c_B c_{AB}),$$

[3] A word of warning here is due. Atiyah also gave explicit expressions for $\eta(T_A)$ in the three cases of elliptic, parabolic and hyperbolic elements, but his results seems to differ from (3.22) by a global sign. Bohn computed $\eta(T_A)$ for elliptic [5, Corollary 4.4.5] and hyperbolic elements [5, Theorem 4.4.13] by different methods, and his result coincides with the formulas of Atiyah in the former case, and with (3.22) in the latter.

where c_A denotes the coefficient c in the usual description $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of an element $A \in \mathrm{SL}(2, \mathbb{Z})$.

(8) Because of (3.23), the twisted eta invariant of $(I \times T)_A$ corresponds to the function

$$\widetilde{\Theta}(A, v) := \Theta(A, v) + \Phi(A).$$

This function, in turn, satisfies the cocycle condition

$$\widetilde{\Theta}(AB, v) = \widetilde{\Theta}(A, B^t v) + \widetilde{\Theta}(B, v). \tag{3.24}$$

The open problem of determining the value of $\Theta(A, v)$ for all A and v can be now attacked by trying to solve the same problem for $\widetilde{\Theta}(A, v)$, which satisfies the nicer property (3.24). By Corollary 3.4.8, we already know that if A is hyperbolic and $v \in \mathbb{R}^2$ satisfies $(\mathrm{Id} - A^t)v \in \mathbb{Z}^2$, we have $\widetilde{\Theta}(A, v) = -\Phi_v(A)$.

3.4.2 Rho invariants of products

Let F be a compact, oriented surface, and consider the manifold $F \times S^1$. As discussed in Example 3.1.2, there is a natural framing $\mathcal{F}_F^\times$ (often just denoted by $\mathcal{F}^\times$) on $\partial F \times S^1$, called the *product framing*, which is associated to the product structure. It is defined in such a way that the meridians are the classes of the boundary curves of F, and the longitudes are the classes of $[S^1]$. A natural question is: how to compute $\rho_\alpha(F \times S^1, \mathcal{F}^\times)$ for a Hermitian local coefficient system $\alpha \in \mathcal{U}(F \times S^1)$? We shall try to face this problem for 1-dimensional α's.

We suppose now that F is connected, and adopt the following convention. As usual, we shall think of an element $\alpha \in \mathcal{U}_1(F \times S^1)$ as a representation $\alpha\colon \pi_1(F \times S^1) \to U(1)$. As $U(1)$ is an abelian group, α factors through the abelianization $\mathrm{ab}\colon \pi_1(F \times S^1) \to H_1(F \times S^1; \mathbb{Z})$, and hence it is determined by the unique representation $\psi\colon H_1(X; \mathbb{Z}) \to U(1)$ such that $\psi \circ \mathrm{ab} = \alpha$.

Convention 3.4.10. By a little abuse of notation, we shall see $\alpha \in \mathcal{U}_1(F \times S^1)$ itself as a representation of $\alpha\colon H_1(F \times S^1; \mathbb{Z}) \to U(1)$.

Given a representation $\alpha\colon H_1(F \times S^1; \mathbb{Z}) \to U(1)$, we have an induced representation $\partial\alpha\colon H_1(\partial F \times S^1; \mathbb{Z}) \to U(1)$, obtained by pulling back through the inclusion of the boundary. The following result says that $\rho_\alpha(F \times S^1, \mathcal{F}^\times)$ is determined by $\partial F \times S^1$ together with the given product structure and the induced representation $\partial\alpha$.

Proposition 3.4.11. *Let F_1 and F_2 be two compact, oriented surfaces, with representations $\alpha_i\colon H_1(F \times S^1;\mathbb{Z}) \to U(1)$ for $i = 1, 2$. Suppose that $f\colon \partial F_1 \times S^1 \to \partial F_2 \times S^1$ is a diffeomorphism between their boundaries such that $f_*(\mathcal{F}^{\times}_{F_1}) = \mathcal{F}^{\times}_{F_2}$ and $\partial\alpha_1 = f^*(\partial\alpha_2)$. Then, we have*

$$\rho_{\alpha_1}(F_2 \times S^1, \mathcal{F}^{\times}_1) = \rho_{\alpha_2}(F_2 \times S^1, \mathcal{F}^{\times}_2).$$

Proof. From the fact that the representations coincide on the longitudes, we have $\alpha_1([S^1]) = \alpha_2([S^1]$. If this number $\zeta \in U(1)$ is the trivial element, then both rho invariants are 0, as a reflection on the S^1-factor produces appropriate orientation-reversing self-diffeomorphisms of $F_1 \times S^1$ and $F_2 \times S^1$. We can thus suppose $\zeta \neq 1$, which implies that the representations are non-trivial on all boundary components of the 3-manifolds, so that twisted homology of the framed tori vanishes. Now, form the closed oriented surface $F := F_1 \cup (-F_2)$. Then, the representations α_1 and α_2 glue well to a representation α on F. By Corollary 3.2.11, we get

$$\rho_\alpha(F \times S^1) = \rho_{\alpha_1}(F_2 \times S^1, \mathcal{F}^{\times}_1) - \rho_{\alpha_2}(F_2 \times S^1, \mathcal{F}^{\times}_2),$$

as the twisted Maslov index vanishes because of the assumption $\zeta \neq 1$, and the untwisted one vanishes because $\mathcal{V}_{F_1 \times S^1} = \mathcal{V}_{F_2 \times S^1}$. The conclusion follows from the fact that $\rho_\alpha(F \times S^1) = 0$, which is a consequence of Corollary 2.1.6. □

Suppose now that F has k boundary components $C_1, \dots C_k$, and let

$$\mu_i := [C_i] \subseteq H_1(C_k \times S^1;\mathbb{Z}), \quad \lambda_i := [S^1] \subseteq H_1(C_k \times S^1;\mathbb{Z}),$$

for $i = 1, \dots, k$, denote the meridians and longitudes of the product framing $\mathcal{F}^{\times}$. Then, a representation $\alpha\colon H_1(\partial F \times S^1;\mathbb{Z}) \to U(1)$ is determined by the values of $\alpha(\mu_i), \alpha(\lambda_i) \in U(1)$ for $i = 1, \dots, k$. Moreover, if α is the restriction to the boundary of a representation $H_1(F \times S^1;\mathbb{Z}) \to U(1)$ (that we shall also call α), we have

$$\prod_{i=1}^{k} \alpha(\mu_i) = 1, \quad \alpha(\lambda_1) = \cdots = \alpha(\lambda_k).$$

Set $\mathbb{T} := U(1)$. Thanks to Proposition 3.4.11, then, $\rho_\alpha(F \times S^1, \mathcal{F}^{\times})$ is determined by the k-tuple

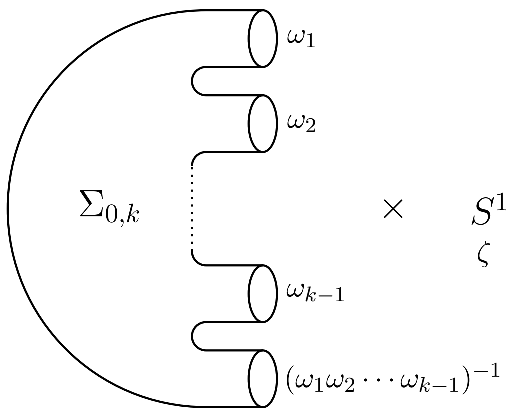
$$(\alpha(\mu_1), \dots, \alpha(\mu_{k-1}), \beta(\lambda_1)) \in \mathbb{T}^k.$$

This observation leads to the definition of a function

$$\begin{aligned}\Gamma_{k-1}\colon \mathbb{T}^{k-1} \times \mathbb{T} &\to \mathbb{R}\\ (\omega_1, \dots, \omega_{k-1}; \zeta) &\mapsto \rho_\beta(\Sigma_{0,k} \times S^1, \mathcal{F}^\times),\end{aligned}$$

where $\Sigma_{0,k}$ denotes the k-punctured sphere, and $\beta\colon H_1(\Sigma_{0,k} \times S^1; \mathbb{Z}) \to U(1)$ is the representation determined by the equations

$$\begin{cases}\beta(\mu_i) = \omega_i & \text{for } i = 1, \dots, k,\\ \beta([S^1]) = \zeta.\end{cases}$$



By the above discussion, if F is any compact, oriented surface and $\alpha\colon H_1(F \times S^1; \mathbb{Z}) \to U(1)$ is a representation, we have

$$\rho_\alpha(F \times S^1, \mathcal{F}^\times) = \Gamma_{k-1}(\alpha(\mu_1), \dots, \alpha(\mu_{k-1}); \alpha(\lambda_1)),$$

and we have reduced our problem to the one of describing the function Γ_k for $k \geq 0$.

Remark 3.4.12. A quick computation using Proposition 3.2.8 shows that if we use the reverse of the product framing (i.e. with meridians corresponding to $[S^1]$ and longitudes corresponding to the boundary class), then nothing changes. More precisely, we have

$$\rho_\alpha(F \times S^1, (\mathcal{F}^\times)^*) = \rho_\alpha(F \times S^1, \mathcal{F}^\times) = \Gamma_{k-1}(\alpha(\mu_1), \dots, \alpha(\mu_{k-1}); \alpha(\lambda_1)).$$

This observation shall be useful in knot theory constructions, where the S^1-factor is more naturally glued to the meridian of a link.

The function has the following elementary properties.

Lemma 3.4.13. *Let $k \in \mathbb{N}$ and let $(\omega_1, \ldots, \omega_k; \zeta) \in \mathbb{T}^k \times \mathbb{T}$. Then:*

(i) $\Gamma_k(\omega_1, \ldots, \omega_k; \zeta)$ *is invariant under any permutation of* $\omega_1, \ldots, \omega_k$*;*

(ii) $\Gamma_k(\omega_1, \ldots, \omega_k; \zeta) = \Gamma_k(\omega_1, \ldots, \omega_{k-1}, (\omega_1 \cdots \omega_k)^{-1}; \zeta)$*;*

(iii) $\Gamma_k(\omega_1^{-1}, \ldots, \omega_k^{-1}; \zeta) = -\Gamma_k(\omega_1, \ldots, \omega_k; \zeta)$*;*

(iv) $\Gamma_k(\omega_1, \ldots, \omega_k; \zeta^{-1}) = -\Gamma_k(\omega_1, \ldots, \omega_k; \zeta)$*;*

(v) $\Gamma_{k+1}(\omega_1, \ldots, \omega_k, 1; \zeta) = \Gamma_k(\omega_1, \ldots, \omega_k; \zeta)$*;*

(vi) Γ_0 *and* Γ_1 *are* 0 *at every point;*

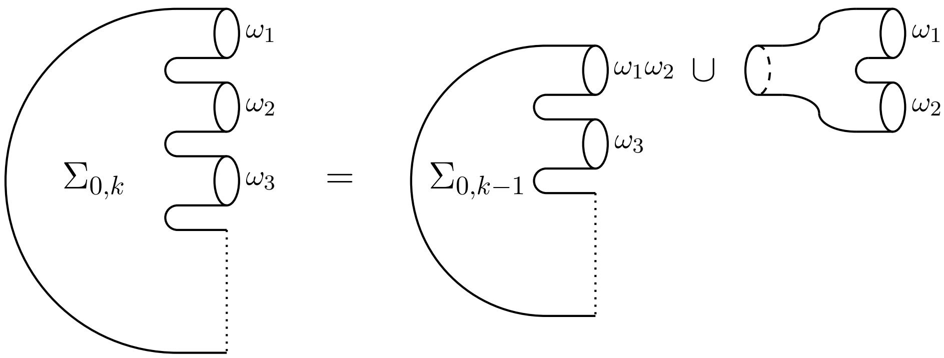
(vii) if $k \geq 2$*, we have*

$$\Gamma_k(\omega_1, \ldots, \omega_k; \zeta) = \Gamma_2(\omega_1, \omega_2; \zeta) + \Gamma_{k-1}(\omega_1\omega_2, \omega_3, \ldots, \omega_k; \zeta).$$

Proof. (i) and (ii) follow immediately from the definition of Γ_k. (iii) and (iv) are obtained by applying Theorem 3.2.3 to an appropriate orientation-reversing self-diffeomorphism of $\Sigma_{0,k+1} \times S^1$. (v) is a consequence of Corollary 3.3.18, as we can do an ∞-framed fillling to the boundary component with coefficient 1, which accounts to capping the boundary with a disk. (vi) follows from (ii) and (iii) (or equivalently from the observation that Γ_0 and Γ_1 represent rho invariants of the solid torus and of the cylinder $I \times T$ with the product framing, which were proven to be trivial with the same argument). For proving (vii), we shall employ a gluing formula. If $\zeta = 1$, all terms are 0 as consequence of (iv), and the result is trivially satisfied. We suppose hence $\zeta \neq 1$. We decompose the k-punctured sphere as

$$\Sigma_{0,k} = \Sigma_{0,k-1} \cup_C \Sigma_{0,3}$$

(where C is a boundary component of $\Sigma_{0,k-1}$) as shown in the following picture.



Let $\mathcal{F} = (\mu, \lambda)$ the restriction of the product framing to $C \times S^1 \subseteq \partial\Sigma_{0,k} \times S^1$, and let $\mathcal{F}', \mathcal{F}''$ denote the product framing on the remaining component of respectively $\partial\Sigma_{0,k-1} \times S^1$ and $\Sigma_{0,3} \times S^1$ (after the identification of one of the boundary components of $\Sigma_{0,3}$ with C given by the gluing). By taking the product with S^1, by Theorem 3.2.10 we get

$$\Gamma_k(\omega_1, \dots, \omega_k; \zeta) = \Gamma_{k-1}(\omega_1\omega_2, \omega_3, \dots, \omega_k; \zeta) + \Gamma_2(\omega_1, \omega_2; \zeta) + N,$$

where the correction term N is given by the Maslov triple index

$$N = -\tau(\mathcal{M}_{\mathcal{F}}, \mathcal{V}_{\Sigma_{0,k-1},\mathcal{F}'}, \mathcal{V}_{\Sigma_{0,3},\mathcal{F}''})$$

(note that the Maslov index in twisted homology is 0 because it follows from the assumption that $\zeta \neq 1$ that the twisted homology vanishes on the gluing torus). From Lemma 3.3.20, it follows that the second and third Lagrangian subspaces correspond to the canonical Lagrangians of manifolds obtained from $\Sigma_{0,k-1} \times S^1$ and $\Sigma_{0,3} \times S^1$ by capping the punctured spheres with disks along all the components outside of the gluing, which gives solid tori. This means that

$$\mathcal{M}_{\mathcal{F}} = \mathcal{V}_{\Sigma_{0,k-1},\mathcal{F}'} = \mathcal{V}_{\Sigma_{0,3},\mathcal{F}''} = \mathrm{Span}_{\mathbb{C}}(\mu).$$

As a consequence we have $N = 0$, and the result is proved. □

By a repeated application of Lemma 3.4.13 we can always reduce our computation of Γ_k to the computation of Γ_2, which represents the rho invariant of $P \times S^1$, where P is a pair of pants. The next result is to be taken as an example, and it shows that this function is non-trivial.

Proposition 3.4.14. *Let $\zeta = e^{2\pi i/3}$. Then, we have $\Gamma_2(\zeta, \zeta; \zeta) = -\frac{1}{6}$.*

Proof. Let C_1, C_2, C_3 be the three boundary components of a pair of pants $P = \Sigma_{0,3}$. By definition, we have,

$$\Gamma_2(\zeta, \zeta; \zeta) = \rho_\alpha(P \times S^1, \mathcal{F}^\times), \tag{3.25}$$

where α is the representation satisfying

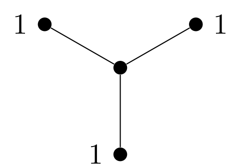
$$\alpha(\mu_i) = \alpha(\lambda_i) = \zeta \quad \text{for } i = 1, 2, 3$$

on the meridians and longitudes of $\mathcal{F}^\times$. We perform now a Dehn filling on every component of $\partial P \times S^1$ by orientation-reversing diffeomorphisms $\varphi_i \colon \partial D^2 \times S^1 \to C_i \times S^1$ giving the identifications

$$\begin{cases} [\partial D^2] = \mu_i - \lambda_i, \\ [S^1] = \mu_i, \end{cases} \tag{3.26}$$

that is, we perform a -1-framed filling with respect to $\mathcal{F}^\times$ on every component. Let N be the closed manifold obtained in this way. In the language of plumbing calculus [37], the manifold obtained by this filling can be represented and simplified as

$N \quad \cong$



$\cong \quad -3\,\bullet \quad \cong \quad L(3,1).$

Clearly, the representation α extends to the filling $L(3,1)$. It is irrelevant whether it sends the standard generator of $\pi_1(L(3,1)) = \mathbb{Z}/3$ to ζ or to $\zeta^{-1} = \zeta^2$, as in both cases the formula of Remark 3.3.26 gives

$$\rho_\alpha(N) = \rho(L(3,1), \zeta^{\pm 1}) = \frac{1}{3}. \tag{3.27}$$

In order to find $\Gamma_2(\zeta, \zeta; \zeta)$, we use now a gluing formula. As $\mathcal{F}^\times$ is non-standardly oriented on $\partial P \times S^1$ and as we prefer to compute the correction term for the three glued pieces at once, we shall not apply Proposition 3.3.16, but the more general gluing formula of Corollary 3.2.11. Namely, we find

$$\rho_\alpha(N) = \rho_\alpha(P \times S^1, \mathcal{F}^\times) + \rho_\alpha(Y, \mathcal{F}^\times) - \tau(\mathcal{M}_{\mathcal{F}^\times}, \mathcal{V}_{P \times S^1}, \mathcal{V}_Y), \tag{3.28}$$

where Y is the disjoint union of the three solid tori of the filling. We still have to identify the last two summands of (3.28). First, we observe that, on each of the solid tori that we have glued, because of (3.26), the framing $\mathcal{F}^\times$ induces a standardly oriented framing of gradient -1. Up to sign, in turn, this coincides with a non-standardly oriented framing $\mathcal{F}_1$ of gradient 1 (see Remark 3.3.6). As a consequence, thanks to Corollary 3.3.12, we have

$$\rho_\alpha(Y, \mathcal{F}^\times) = 3\,\rho_\alpha(D^2 \times S^1, \mathcal{F}_1) = -\frac{3}{2}. \tag{3.29}$$

For computing the triple Maslov index, we first identify the three Lagrangians as $\mathcal{M}_{\mathcal{F}^\times} = \mathrm{Span}_\mathbb{C}\{\mu_1, \mu_2, \mu_3\}$, $\mathcal{V}_{P\times S^1} = \mathrm{Span}_\mathbb{C}\{\mu_1+\mu_2+\mu_3, \lambda_1-\lambda_2, \lambda_1-\lambda_3\}$ and $\mathcal{V}_Y = \mathrm{Span}_\mathbb{C}\{\mu_1-\lambda_1, \mu_2-\lambda_2, \mu_3-\lambda_3\}$. Then, recalling that $\mu_i \cdot \lambda_i = 1$, it is not hard to compute directly with the definition of the triple index that

$$\tau(\mathcal{M}_{\mathcal{F}^\times}, \mathcal{V}_{P\times S^1}, \mathcal{V}_Y) = -2. \tag{3.30}$$

Substituting (3.25), (3.27), (3.29) and (3.30) into (3.28), we find

$$\frac{1}{3} = \Gamma_2(\zeta, \zeta; \zeta) - \frac{3}{2} + 2,$$

and the conclusion follows. □

Remark 3.4.15. It follows from Lemma 3.4.13 (i), (iii) and (vii) that, for all $(\omega_1, \omega_2; \zeta) \in \mathbb{T}^2 \times \mathbb{T}$, we have

$$\Gamma_2(\omega_1, \omega_2; \zeta) + \Gamma_2(\omega_1\omega_2, \omega_3; \zeta) = \Gamma_2(\omega_1, \omega_2\omega_3; \zeta) + \Gamma_2(\omega_2, \omega_3; \zeta).$$

This means that, for fixed $\zeta \in \mathbb{T}$, Γ_2 is a 2-*cocycle* on $\mathbb{T}$. By adapting the proof of continuity in Proposition 3.3.28, moreover, one can prove that Γ_2 is continuous if restricted to the subset of points such that ω_1, ω_2, $\omega_1\omega_2$ and ζ are all different from 1.

Set $\mathbb{T}_* := \mathbb{T} \setminus \{1\}$. Remark 3.4.15 leads to the following conjecture.

Conjecture 3.4.16. *There is a function $\gamma\colon \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, continuous on $\mathbb{T}_* \times \mathbb{T}_*$, such that $\Gamma_2 = \partial\gamma$ with respect to the first two variables, i.e. such that*

$$\Gamma_2(\omega_1, \omega_2; \zeta) = \gamma(\omega_1, \zeta) + \gamma(\omega_2, \zeta) - \gamma(\omega_1\omega_2, \zeta). \tag{3.31}$$

If Conjecture 3.4.16 is true, then for every surface F with k boundary components we have

$$\rho_\alpha(F \times S^1, \mathcal{F}^\times) = \sum_{i=1}^{k} \gamma(\omega_i, \zeta),$$

where the ω_i's are values of α on the meridians of $\mathcal{F}^\times$, and $\zeta = \alpha([S^1])$. Thus, the problem of computing rho invariants of products would reduce itself to that of computing the function γ.

Remark 3.4.17. If we restrict our attention to the finite subgroup $\mathbb{T}_N \subseteq \mathbb{T}$ of the N^{th} roots of unity, then Conjecture 3.4.16 has positive answer. Namely, from the fact that $H^i(\mathbb{Z}/N;\mathbb{Q}) = 0$ for $i > 0$, it follows that, for all $\zeta \in \mathbb{T}$, there exists a unique function $\gamma_N(\cdot,\zeta)\colon \mathbb{T}_N \to \mathbb{Q}$ satisfying (3.31) for all $(\omega_1,\omega_2) \in \mathbb{T}^2$. The questions, then, are

(a) whether these functions for $N \in \mathbb{N}$ patch together well into determining a continuous function on $\mathbb{T}_*$;

(b) if (a) is satisfied, whether the resulting function varies continuously with respect to ζ.

Chapter 4

Rho invariants and knot theory

4.1 The rho invariant of a link

In this section, we use the rho invariant of 3-manifolds with framed toroidal boundary in order to define a new invariant for links. In Section 4.1.1, we fix some conventions about the exterior of a link. On Section 4.1.2, we define the rho invariant of a link and prove some basic properties about it. In Section 4.1.3 we give a mild generalization of this invariant that is useful in the applications. In Section 4.1.4, we prove a general formula for the rho invariant of a satellite knot.

4.1.1 Topological setting

Let L be an oriented link in S^3 (from now on, just a *link*). By removing from S^3 the interior of a closed tubular neighbourhood $N(L)$, we get the *link exterior*

$$X_L := S^3 \setminus \operatorname{int}(N(L)).$$

The link exterior is a compact, oriented 3-manifold, whose boundary is a union of tori: to each link component $K \subseteq L$, there corresponds a boundary component $T_K = -\partial(N(K))$ (this is the orientation coming from being part of the boundary of X_L, and it is the one we shall always consider). Every component $T_K \subseteq \partial X_L$ has a well defined framing $\mathcal{F}_K = (\mu_K, \lambda_K)$ in the sense of Definition 3.1.1, given by the following description:

- the *meridian* is the only element $\mu_K \in H_1(T_K;\mathbb{Z})$ whose image in $H_1(N(K);\mathbb{Z})$ is 0 and such that $\operatorname{lk}(\mu_K, K) = 1$;

- the *longitude* is the only element $\lambda_K \in H_1(T_K; \mathbb{Z})$ whose image in $H_1(N(K); \mathbb{Z})$ is homologous to K and such that $\mathrm{lk}(\lambda_K, K) = 0$.

The above pair of elements forms a basis of $H_1(T_K; \mathbb{Z})$, and hence $\mathcal{F}_K$ is indeed a framing. As it is easy to check, the framing $\mathcal{F}_K$ is standardly oriented (see Definition 3.1.5), as we have

$$\mu_K \cdot \lambda_K = -1.$$

Definition 4.1.1. The framing on ∂X_L given by the collection of the framings $\mathcal{F}_{L_i}$ defined above is called the *standard framing* associated to L, and it is denoted by $\mathcal{F}_L$.

It is now clear that the pair $(X_L, \mathcal{F}_L)$ is a 3-manifold with framed toroidal boundary.

Definition 4.1.2. The *linking matrix* of L is the symmetric matrix $\Lambda_L = (\Lambda_{ij})_{i,j} \in \mathbb{Z}^{k \times k}$ defined by

$$\Lambda_{ij} = \begin{cases} \mathrm{lk}(L_i, L_j), & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Notation 4.1.3. Given a link $L = L_1 \cup \cdots \cup L_k$, if there is no danger of confusion, we shall freely use the shorthands

$$T_i = T_{L_i}, \quad \mu_i = \mu_{L_i}, \quad \lambda_i = \lambda_{L_i}.$$

Similarly, for a knot K, we shall often just use T, μ and λ for the boundary torus, the meridian and the longitude.

For every link component L_i of L, we can consider the natural map $H_1(T_i; \mathbb{Z}) \to H_1(X_L; \mathbb{Z})$. We shall keep calling meridian and longitude the images of μ_i and λ_i in $H_1(X_L; \mathbb{Z})$, and keep the same notation for them. The following is an elementary well-known result.

Lemma 4.1.4. *The abelian group $H_1(X_L; \mathbb{Z})$ is freely generated by the meridians $\mu_1, \ldots, \mu_k$, and every longitude λ_i satisfies*

$$\lambda_i = \sum_{j=1}^{k} \Lambda_{ij}\, \mu_j \in H_1(X_L; \mathbb{Z}).$$

As an immediate consequence of Lemma 4.1.4, the canonical Lagrangian $\mathcal{V}_{X_L} = \ker(H_1(\partial X_L;\mathbb{C}) \to H_1(X_L;\mathbb{C}))$ can be described explicitly as

$$\mathcal{V}_{X_L} = \mathrm{Span}_{\mathbb{C}} \left\{ \lambda_i - \sum_{j=1}^{k} \Lambda_{ij}\mu_j \,\middle|\, i = 1, \ldots, k \right\} \subseteq H_1(\partial X_L;\mathbb{C}). \tag{4.1}$$

In Chapter 3, we used extensively two more Lagrangians, depending on the framing. In this context, they are the subspaces of $H_1(\partial X_L;\mathbb{C})$ given by

$$\mathcal{M}_{\mathcal{F}_L} = \mathrm{Span}_{\mathbb{C}}\{\mu_1, \ldots, \mu_k\}, \quad \mathcal{L}_{\mathcal{F}_L} = \mathrm{Span}_{C}\{\lambda_1, \ldots, \lambda_k\}.$$

Recalling that the signature of a real symmetric matrix is defined as the difference between the numbers of its positive and negative eigenvalues, the Maslov triple index of the three Lagrangians $\mathcal{M}_{\mathcal{F}_L}$, $\mathcal{L}_{\mathcal{F}_L}$ and $\mathcal{V}_{X_L}$ can be computed in the following way.

Lemma 4.1.5. $\tau(\mathcal{M}_{\mathcal{F}_L}, \mathcal{L}_{\mathcal{F}_L}, \mathcal{V}_{X_L}) = \mathrm{sign}\,\Lambda_L$.

Proof. The two Lagrangians $\mathcal{M}_{\mathcal{F}_L}$ and $\mathcal{L}_{\mathcal{F}_f}$ are transverse, so that we can write every element $v \in H_1(X_L;\mathbb{C})$ in a unique way as $v = v' + v''$, with $v' \in \mathcal{M}_{\mathcal{F}_L}$ and $v' \in \mathcal{L}_{\mathcal{F}_f}$. By definition of the Maslov triple index, $\tau(\mathcal{M}_{\mathcal{F}_L}, \mathcal{L}_{\mathcal{F}_f}, \mathcal{V}_{X_L})$ is thus the signature of the Hermitian form

$$\begin{aligned} \varphi \colon \mathcal{V}_{X_L} \times \mathcal{V}_{X_L} &\to \mathbb{C} \\ (v, w) &\mapsto v' \cdot w''. \end{aligned}$$

For $i = 1, \ldots, k$, let $v_i := \lambda_i - \sum_j \Lambda_{ij}\mu_j$, so that $\mathcal{B} = (v_1, \ldots, v_k)$ is a basis of $\mathcal{V}_{X_L}$ by (4.1). As $\mu_i \cdot \lambda_j = -\delta_{ij}$, we have then

$$\varphi(v_i, v_l) = \left(-\sum_{j=1}^{k} \Lambda_{ij}\mu_j \right) \cdot \lambda_l = \Lambda_{il}.$$

This means that the form φ is represented in the basis $\mathcal{B}$ by the matrix Λ_L, and hence $\mathrm{sign}\,\psi = \mathrm{sign}\,\Lambda_L$ by Proposition 1.1.11. □

4.1.2 Definition and first properties

We are now ready to define the main invariant of this chapter.

Definition 4.1.6. Let $L \subseteq S^3$ be a link, and let $\alpha \in \mathcal{U}_n(X_L)$ be a local coefficient system. We define the *rho invariant of* L associated to α as the real number

$$\varrho_\alpha(L) := \rho_\alpha(X_L, \mathcal{F}_L).$$

As usual, once a base point $x_0 \in X_L$ has been fixed, up to isomorphism we can see the local coefficient system $\alpha \in \mathcal{U}_n(X_L)$ as a representation $\alpha\colon \pi_1(X_L) \to U(n)$. For each boundary component T_i of X_L, the restriction of α to T_i is well defined as a local coefficient system. Notice however that the induced maps $\pi_1(T_i) \to \pi_1(X_L)$ only make sense once a base point x_i for T_i and a path from x_i to x_0 are fixed. Hence for working in the language of representations of the fundamental groups, we need to make these choices in order to get induced representations $\alpha_i\colon \pi_1(T_i) \to U(n)$ (note that we shall normally call α_i just α).

Warning 4.1.7. We take these choices for given, and do not make them explicit in the notation.

Observe in any case that two different choices give equivalent representations $\pi_1(T_i) \to U(n)$. In particular, the eigenvalues of the image of an element of $\pi_1(T_i)$ are independent of the choices. In order to get simpler formulas, we shall often make an additional assumption.

Definition 4.1.8. We say that $\alpha \in \mathcal{U}_n(X_L)$ is non-degenerate if, for every boundary component T_i of X_L, the restriction of α to T_i (as a local coefficient system) admits no trivial summand.

In terms of representations, this means that the restriction $\alpha\colon \pi_1(T_i) \to U(n)$ admits no trivial subrepresentation. Because of lemma 1.3.8, we have the equivalence

$$\alpha \text{ is non-degenerate } \iff H_*(\partial X_L; \mathbb{C}^n_\alpha) = 0.$$

In particular, if this is the case, then all triple Maslov indices on $H_*(\partial X_L; \mathbb{C}^n_\alpha)$ vanish.

We compare now the rho invariant of L with a classical invariant. Let M_L denote the manifold obtained by 0-*framed surgery* on the link L. This means that M_L is the result of a 0-framed filling on every boundary component of X_L along the framing $\mathcal{F}_L$. In the notation of Remark 3.3.19, we have thus

$$M_L = D_0(X_L, \mathcal{F}_L).$$

If the local coefficient system extends to M_L, we can consider the Atiyah-Patodi-Singer rho invariant of M_L. In terms of representations, this is the case if and only if $\alpha(\lambda_i) = \mathrm{Id}$ for all longitude λ_i. The invariant $\rho_\alpha(M_L)$ was studied among others by Levine [31, 32] and Friedl [22, 23]. The next result shows that, whenever $\rho_\alpha(M_L)$ is defined, it coincides with our invariant

$\rho_\alpha(M_L)$ up to an integer correction term that only depends on the linking numbers of L. In other words, the invariant $\varrho_\alpha(L)$ is for all practical purposes an extension of the Atiyah-Patodi-Singer rho invariant of the manifold obtained by 0-framed surgery.

Theorem 4.1.9. *Let L be a link and let $\alpha \in \mathcal{U}_n(X_L)$ be a non-degenerate local coefficient system that extends to M_L. Then*

$$\rho_\alpha(M_L) = \varrho_\alpha(L) + n \operatorname{sign} \Lambda_L.$$

In particular, if $L = K$ is a knot, we have

$$\rho_\alpha(M_K) = \varrho_\alpha(K).$$

Proof. By the discussion in Remark 3.3.19, we have

$$\rho_\alpha(M_L) = \rho_\alpha(X_L, \mathcal{F}_L) + \tau(\mathcal{L}^\alpha_{\mathcal{F}_L}, \mathcal{M}^\alpha_{\mathcal{F}_L}, \mathcal{V}^\alpha_{X_L}) - n\tau(\mathcal{L}_{\mathcal{F}_L}, \mathcal{M}_{\mathcal{F}_L}, \mathcal{V}_{X_L}).$$

Now, $\rho_\alpha(X_L, \mathcal{F}_L) = \varrho_\alpha(L)$ by definition, the first Maslov index is 0 because the local coefficient system is non-degenerate, while

$$-n\,\tau(\mathcal{L}_{\mathcal{F}_L}, \mathcal{M}_{\mathcal{F}_L}, \mathcal{V}_{X_L}) = n\,\tau(\mathcal{M}_{\mathcal{F}_L}, \mathcal{L}_{\mathcal{F}_L}, \mathcal{V}_{X_L}) = n \operatorname{sign}(\Lambda_L)$$

thanks to Lemma 4.1.5. □

Remark 4.1.10. The Atiyah-Patodi-Singer rho invariant $\rho_\alpha(M_L)$ of the manifold obtained by 0-framed surgery on a knot or link L was used By Levine [31, 32] and by Friedl as *concordance* invariant. Theorem 4.1.9 gives a new interpretation for this invariant. Moreover, while $\rho_\alpha(M_L)$ is only defined for representations that extend to the fundamental group of M_L, the invariant $\varrho_\alpha(L)$ exists for every representation $\alpha\colon \pi_1(X_L) \to U(n)$. As a consequence, its employment might lead to generalizations of the results of Levine and of Friedl about concordance of links.

4.1.3 Framed links and surgery descriptions

Sometimes it is convenient to have a more flexible invariant, allowing framings different from $\mathcal{F}_L$. We start with recalling the following classical definition.

Definition 4.1.11. A *framing* on a link $L \subseteq S^3$ is a tuple of integers $f \in \mathbb{Z}^{\pi_0(L)}$, each associated to a component of L. The pair (L, f) is called a *framed link*.

If the link is described as $L = L_1 \cup \cdots \cup L_k$, we shall write framings on L as k-tuples $f = (f_1, \ldots, f_k) \in \mathbb{Z}^k$. A framing $f = (f_1, \ldots, f_k)$ on a link $L = L_1 \cup \cdots \cup L_k$ determines a framing on ∂X_L that may differ from the standard framing $\mathcal{F}_L$. For each component $T_i = -\partial N(L_i)$, we set

$$\mathcal{F}'_i = (\mu'_i, \lambda'_i), \quad \text{with} \quad \begin{cases} \mu'_i := \mu_i, \\ \lambda'_i := \lambda_i + f_i\,\mu_i, \end{cases}$$

and call $\mathcal{F}_f$ the framing on ∂X_L obtained by taking the union of these.

Definition 4.1.12. Let (L, f) be a framed link. Given a local coefficient system $\alpha \in \mathcal{U}_n(X_L)$, we define the *rho invariant of* (L, f) associated to α as the real number

$$\varrho_\alpha(L, f) := \rho_\alpha(X_L, \mathcal{F}_f).$$

Remark 4.1.13. For the 0-framing $f = (0, \ldots, 0)$, the invariant $\varrho_\alpha(L, f)$ coincides with $\varrho_\alpha(L)$. For a general framing f, the difference between $\varrho_\alpha(L, f)$ and $\varrho_\alpha(L)$ can be easily written in terms of the function Θ of Section 3.4.1. in particular, an explicit computation seems hard in general.

Given a framed link (L, f), it is also convenient to introduce a modification of the linking matrix as

$$\Lambda_f := \Lambda_L + \operatorname{diag}(f_1, \ldots, f_k).$$

We shall call Λ_f the *framed linking matrix* of (L, f). Of course, if f is the 0-framing, we have $\Lambda_f := \Lambda_L$. We have then the following generalization of Lemma 4.1.5.

Lemma 4.1.14. $\tau(\mathcal{M}_{\mathcal{F}_L}, \mathcal{L}_{\mathcal{F}_f}, \mathcal{V}_{X_L}) = \operatorname{sign} \Lambda_f$.

Proof. Let $\lambda'_i = \lambda_i + f_i \mu_i$ as above, so that $\mathcal{L}_{\mathcal{F}_f} = \operatorname{Span}_C\{\lambda'_1, \ldots, \lambda'_k\}$. We set moreover

$$a_{ij} := \begin{cases} \Lambda_{ij}, & \text{if } i \neq j, \\ f_i, & \text{if } i = j, \end{cases}$$

so that $\Lambda_f = (a_{ij})_{i,j}$. Then, the basis elements of $\mathcal{V}_{X_L}$ given by (4.1) can be written as $v_i = \lambda'_i - \sum_{j=1}^k a_{ij}\mu_j$. Once again, the meridians and the longitudes generate transversal Lagrangians, and they satisfy $\mu_i \cdot \lambda_j = -\delta_{ij}$. The rest of the proof follows exactly like the one of Lemma 4.1.5, but this time the matrix representation of the relevant Hermitian form is $\Lambda_f = (a_{ij})$. □

Given a framed link (L, f), we can consider the closed manifold $M_L(f)$ obtained by the *Dehn surgery on* (L, f), i.e. by performing, on each component T_i of the boundary of X_L, an f_i-framed Dehn filling with respect to the standard framing $\mathcal{F}_L$. Equivalently, $M_L(f)$ can be described as the result of a 0-framing filling along the modified framing $\mathcal{F}_f$, so that

$$M_L(f) = D_0(X_L, \mathcal{F}_f)$$

in the notation of Remark 3.3.19. By a famous theorem of Lickorish and Wallace, every closed, oriented 3-manifold can be obtained as the result of Dehn surgery along a framed link.

Definition 4.1.15. Given a framed link L with a local coefficient system $\alpha \in \mathcal{U}_n(X_L)$, we say that a framing f on L is *compatible* with α if α extends to $M_L(f)$.

If we see the local coefficient system as a representation $\alpha\colon \pi_1(X_L) \to U(n)$, then

$$f \text{ is compatible with } \alpha \iff \alpha(\lambda_i)\alpha(\mu_i)^{f_i} = 1 \ \forall i.$$

Note that, in general, given a local coefficient system α, there might be no framing that is compatible with it. We have the following generalization of Theorem 4.1.9.

Theorem 4.1.16. *Let $L = L_1 \cup \cdots \cup L_k$ be a link, $\alpha \in \mathcal{U}_n(X_L)$ a non-degenerate local coefficient system and f be a compatible framing on L. Then, we have*

$$\rho_\alpha(M_L(f)) = \varrho_\alpha(L, f) + n \operatorname{sign} \Lambda_f. \tag{4.2}$$

More generally, if in addition g is any framing on L, we have

$$\rho_\alpha(M_L(f)) = \varrho_\alpha(L, g) + n \operatorname{sign} \Lambda_f + \sum_{i=1}^{k} \sum_{j=1}^{n} S_{f_i - g_i}(t_{ij}), \tag{4.3}$$

where S is the family of functions defined in Section 3.3.5, and the real numbers t_{ij} are such that, for $j = 1, \ldots, k$, the matrix $\alpha(\mu_j) \in U(n)$ has eigenvalues $e^{2\pi i t_{j1}}, \ldots, e^{2\pi i t_{jn}}$.

Proof. For proving (4.2), it is enough to reproduce the proof of Theorem 4.1.9, using Lemma 4.1.14 instead of Lemma 4.1.5. In order to prove (4.3), instead, we apply the gluing formula of Corollary 3.2.11 directly. We glue a union $Y = D_1 \times S^1 \sqcup \cdots \sqcup D_k \times S^1$ of solid tori to X_L according to the framing f, but equip the boundary tori with the framing $\mathcal{F}_g$. We obtain then

$$\rho_\alpha(M_L(f)) = \rho_\alpha(X_L, \mathcal{F}_g) + \sum_{i=1}^{k} \rho_{\alpha_i}(D_i \times S^1, \mathcal{F}^i) - n\,\tau(\mathcal{M}_{\mathcal{F}_g}, \mathcal{V}_{X_L}, \mathcal{V}_Y),$$

where α_i denotes the restriction of α to $D_i \times S^1$, and $\mathcal{F}^i$ denotes the framing induced by $\mathcal{F}_g$ on $\partial D_i \times S^1$, after the identification with T_i given by the gluing. Now, $\mathcal{M}_{\mathcal{F}_g} = \mathcal{M}_{\mathcal{F}_L}$, while $\mathcal{V}_Y$ is generated by the meridians of the solid tori, which are identified with $\lambda_i + f_i\mu_i$, so that $\mathcal{V}_Y = \mathcal{L}_{\mathcal{F}_f}$. By Lemma 4.1.14, we get thus

$$-n\,\tau(\mathcal{M}_{\mathcal{F}_g}, \mathcal{V}_{X_L}, \mathcal{V}_Y) = n\,\tau(\mathcal{M}_{\mathcal{F}_L}, \mathcal{L}_{\mathcal{F}_f}, \mathcal{V}_{X_L}) = n \operatorname{sign} \Lambda_f.$$

The framings on the solid tori are $\mathcal{F}^i = (\mu_i, \lambda_i')$,with $\lambda_i' = \lambda_i + g_i\mu_i$. As the meridian $[\partial D_i]$ is glued according to the framing f, we have

$$[\partial D_i] = \lambda_i + f_i\mu_i = (f_i - g_i)\mu_i + \lambda_i'.$$

This means that the framing $\mathcal{F}^i$ on $\partial D_i \times S^1$ is a non-standardly oriented framing of gradient $f_i - g_i$ (see Definition 3.3.1), and we can use the notation $\mathcal{F}^i = \mathcal{F}_{f_i - g_i}$. Now, the class $[S_1] \in H_1(D_i \times S^1; \mathbb{Z})$, instead, is identified with μ_i, and thus it is sent to $\alpha_i(\mu_i) \in U(n)$ by the representation. Up to equivalence, α_i can be decomposed as $\alpha_i = \alpha_{i1} \oplus \cdots \oplus \alpha_{in}$, with $\alpha_{ij}\colon \pi_1(D_i \times S^1) \to U(1)$ sending the standard generator to ω_{ij}. We see that

$$\rho_{\alpha_i}(D_i \times S^1, \mathcal{F}^i) = \sum_{j=1}^{n} \rho_{\alpha_i j}(D_i \times S^1, \mathcal{F}_{f_i - g_i}) = \sum_{j=1}^{n} S_{f_i - g_i}(t_{ij})$$

by definition of the function $S_{f_i - g_i}$, and the proof is concluded. □

4.1.4 The satellite construction

We start by recalling the satellite construction for knots. Let $C \subseteq S^3$ be a knot, and let $K \cup A \subseteq S^3$ be a 2-component link, such that A is unknotted. The exterior X_A of A is a solid torus, whose meridian is the longitude λ_A of A. The component K can then be seen as a knot in X_A, i.e. as a *pattern*. Choose a diffeomorphism $\varphi\colon \partial X_A \to \partial X_C$ be a diffeomorphism such that

$$\varphi_*(\lambda_A) = \mu_C, \quad \varphi_*(\mu_A) = \lambda_C. \tag{4.4}$$

Then we have an orientation-preserving diffeomorphism $\Phi\colon X_C \cup_\varphi X_A \to S^3$, and the isotopy class of $\Phi(K) \subseteq S^3$ only depends on the knot C and on the link $K \cup A$. The knot

$$S(C, K, A) := \Phi(K)$$

is called a *satellite knot* with *companion* C, *orbit* K and *axis* A. The integer $\mathrm{lk}(K, A)$ is called the *winding number* of the satellite construction.

Let S be a shorthand for $S(C, K, A)$. By construction, we have $X_S = \Phi(X_C \cup_{T_A} X_{K\cup A})$, in such a way that the restriction to T_K acts by $\Phi_*(\mathcal{F}_K) = \mathcal{F}_S$. because of this, we identify the two manifolds completely, by writing

$$X_S = X_C \cup_{T_A} X_{K\cup A}, \quad \mathcal{F}_S = \mathcal{F}_K.$$

We have the following fairly general result.

Proposition 4.1.17. *Let $S := S(C, K, A)$ be a satellite knot, and let $\alpha \in \mathcal{U}_n(X_S)$ be a local coefficient system whose restriction to X_C is either trivial or non-degenerate. Then, we have*

$$\varrho_\alpha(S) = \varrho_\alpha(C) + \varrho_\alpha(K \cup A).$$

Proof. By (4.4), under the gluing $X_S = X_C \cup_{T_A} X_{K\cup A}$ we have the identification $\mathcal{F}_C^* = \mathcal{F}_A$ (where $\mathcal{F}_C^* = (\lambda_C, \mu_C)$ is the reverse framing of $\mathcal{F}_K$). Suppose that α is non-degenerate on X_C. Then, the twisted homology of T_A vanishes. By Theorem 3.2.10, we have

$$\rho_\alpha(X_S, \mathcal{F}_S) = \rho_\alpha(X_C, \mathcal{F}_C^*) + \rho_\alpha(X_{K\cup A}, \mathcal{F}_{K\cup A}) - n\,\tau(\mathcal{M}_{\mathcal{F}_C^*}, \mathcal{V}_{X_C}, \mathcal{V}_{X_{K\cup A},\mathcal{F}_K}). \tag{4.5}$$

Now, by definition we have

$$\rho_\alpha(X_{K\cup A}, \mathcal{F}_{K\cup A}) = \varrho_\alpha(K \cup A),$$

and by Proposition 3.2.8 we compute

$$\rho_\alpha(X_C, \mathcal{F}_C^*) = \rho_\alpha(X_C, \mathcal{F}_C) - \tau(\mathcal{L}_{\mathcal{F}_C}, \mathcal{M}_{\mathcal{F}_C}, \mathcal{V}_{X_C}) = \varrho_\alpha(C), \tag{4.6}$$

where the Maslov index vanishes because $\mathcal{V}_{X_C} = \mathcal{L}_{\mathcal{F}_C} = \mathrm{Span}_\mathbb{C}(\lambda_C)$. Thus, in order to conclude it is enough to show that the Maslov index in (4.5) also vanishes. Once again, this is immediate from the equality $\mathcal{M}_{\mathcal{F}_C^*} = \mathcal{L}_{\mathcal{F}_C} = \mathcal{V}_C$.

We have thus proved the result in the case when α is non-degenerate on X_C. The other case that we have to consider is the one where α is trivial on X_C (in which case, the desired formula reduces to $\varrho_\alpha(S) = \varrho_\alpha(K \cup A)$).

The only difference in the proof is that in (4.5) and (4.6) there is a priori an extra correction term, given by the Maslov index in the twisted cohomology. However, from the fact that α is trivial on X_S, we have

$$\mathcal{M}^{\alpha}_{\mathcal{F}^*_C} = \mathcal{L}^{\alpha}_{\mathcal{F}_C} = \mathcal{V}^{\alpha}_C$$

as for the corresponding untwisted Lagrangians. As a consequence, both Maslov indices in twisted homology are also 0. □

4.2 The abelian case

In this section, we focus on the rho invariant of a link associated to representations into $U(1)$. In Section 4.2.1, we recall the classical definition of the Levine-Tristram signature function of a link and a characterization in terms of signatures of manifolds. In Section 4.2.2, we express this function in two ways: as the Atiyah-Patodi-Singer rho invariant of some simple 3-manifold associated to the link, and in terms of the rho invariant of the link itself as defined above. Using this description, in Section 4.2.3 we give new easy proofs to two classical results about links. In Section 4.2.4, we compare the rho invariant of a knot with the Atiyah-Patodi-Singer rho invariant of the closed manifolds obtained by Dehn surgery on it. In Section 4.2.5, we discuss the multivariable signatures of Cimasoni and Florens, and a characterization of these in terms of twisted signatures of manifolds. In Section 4.2.6, we express the multivariable signature function as the Atiyah-Patodi-Singer rho invariant of some 3-manifold associated to the link (more complicated than the one used for the Levine-Tristram signature). We conclude with a comparison of the multivariable signatures with the rho invariant of the link itself.

Convention 4.2.1. Let X be a topological space. As $U(1)$ is an abelian group, every representation $\alpha\colon \pi_1(X) \to U(1)$ factors through the abelianization $\mathrm{ab}\colon \pi_1(X) \to H_1(X;\mathbb{Z})$, and hence α is determined by the unique representation $\psi\colon H_1(X;\mathbb{Z}) \to U(1)$ such that $\psi \circ \mathrm{ab} = \alpha$. For this reason, we shall draw out the abelianization from the picture altogether, and simply speak of the representation $\psi\colon H_1(X;\mathbb{Z}) \to U(1)$. Whenever an invariant requires a representation on $\pi_1(X)$ to be defined, it will be meant that we are considering the composition $\alpha = \psi \circ \mathrm{ab}$

4.2.1 The Levine-Tristram signature function

Let L be an oriented link in S^3. We start by recalling the definition of the Levine-Tristram signature function of L. First of all, consider a *Seifert* *surface* S for L, i.e. an embedded connected oriented surface $F \subseteq S^3$ such that $\partial F = L$. Associated to F there is a bilinear form V_F on $H_1(F;\mathbb{Z})$, called the *Seifert pairing* and defined essentially by $V_F(a,b) = \operatorname{lk}(a, b^+)$, where the superscript $+$ means that we are pushing a representative of the class b in the positive direction of a tubular neighborhood $[-1,1] \times F \subseteq S^3$ of F. By taking its complexification, we can also see V_F and V_F^t as sesquilinear forms on $H_1(F;\mathbb{C})$. We identify now $U(1)$ with $S^1 \subseteq \mathbb{C}$. For $\omega \in U(1)$, we define now the form

$$\Phi_F(\omega) := (1-\omega)V_F + (1-\overline{\omega})V_F^t.$$

It can be checked that $\Phi_F(\omega)$ is Hermitian, and that the signature of $\Phi_F(\omega)$ only depends on the link L and not on the choice of the Seifert surface F. We have thus the following definition.

Definition 4.2.2. The *Levine-Tristram signature* of an oriented link L is the function $\sigma_L \colon U(1) \to \mathbb{Z}$ defined by

$$\sigma_L(\omega) := \operatorname{sign}(\Phi_F(\omega)),$$

where F is any Seifert surface for L.

The Levine-Tristram signature has the following well-known 4-dimensional description (a detailed account of this discussion, albeit for knots, can be found inside the proof of Lemma 5.4 of Cochran, Orr and Teichner [14]; the generalization to links is harmless). See S^3 as the boundary of the 4-ball D^4, and consider the 4-manifold with boundary W_F obtained by taking the exterior of a push-in of F in D^4. Then, the group $H_1(W_F;\mathbb{Z})$ is free with one generator (the *meridian* of F), corresponding to the image of any of the meridians of L. We can define the representation $\psi \colon H_1(W_F;\mathbb{Z}) \to U(1)$ sending this generator to ω. One computes then that $H_2(W_F;\mathbb{C}_\psi)$ is isomorphic to $H_1(W_F;\mathbb{Z})$, and that up to this identification and Poincaré duality, the intersection form $I^\psi_{W_F}$ is given by

$$I^\psi_{W_F} = (1-\omega)V_F^t + (1-\overline{\omega})V_F.$$

In particular, $I^\psi_{W_F}$ has the same signature as $\Phi_F(\omega)$, as matrix representations for these Hermitian forms with respect to the same basis are transpose one of the other, and hence they have the same eigenvalues. It follows then, by definition of the Levine-Tristram signature and of the signature with local coefficients of a 4-manifold, that

$$\operatorname{sign}_L(\omega) = \sigma_\psi(W_F). \tag{4.7}$$

Remark 4.2.3. With an argument based on Wall's non-additivity theorem, it is possible to show that $\operatorname{sign}_L(\omega) = \sigma_\psi(W_S)$ whenever W_S is the exterior of a properly embedded connected oriented surface $S \subseteq D^4$ such that $\partial S = L$, even if it is not the push-in of a Seifert $F \subseteq S^3$.

This last observation gives an alternative definition for the Levine-Tristram signature that is purely 4-dimensional. However, both approaches depend on the non-canonical choice of either a Seifert surface $F \subseteq S^3$ or of such a "bounding surface" $S \subseteq D^4$. In the next section, we will give two descriptions of $\operatorname{sign}_L(\omega)$ that, instead, are independent of any choice, and that imply the claim of Remark 4.2.3 as a corollary. The first is a description of $\operatorname{sign}_L(\omega)$ as the Atiyah-Patodi-Singer rho invariant of a closed 3-manifold N_L obtained naturally from L. The second is a description of $\operatorname{sign}_L(\omega)$ in terms of the rho invariant of L with some framing (or, in other words, as the rho invariant of the manifolds with boundary X_L with an appropriate framing on ∂X_L).

4.2.2 Rho invariants and Levine-Tristram signatures

We define the following framing on an oriented link L.

Definition 4.2.4. The *Seifert framing* on an oriented link $L = L_1 \cup \cdots \cup L_k$ is the framing $g_L = (g_1, \cdots, g_n)$ defined by

$$g_i := -\sum_{j=1}^{k} \Lambda_{ij}.$$

Remark 4.2.5. The longitudes $\lambda_i' = \lambda_i + g_i\mu_i$ associated to the Seifert framing correspond to the intersections of a Seifert surface with the boundary tori T_i.

Consider the surjection $\epsilon\colon H_1(X_L;\mathbb{Z}) \to \mathbb{Z}$ sending all meridians to 1. We shall consider here representations $\psi\colon H_1(X_L;\mathbb{Z}) \to U(1)$ that factor through ϵ, i.e that send all meridians to the same value $\omega \in U(1)$. Of course these representations are in a natural bijection with $U(1)$ itself.

Lemma 4.2.6. *The Seifert framing is compatible with every representation* $\psi\colon H_1(X_L;\mathbb{Z}) \to U(1)$ *that factors though* $\epsilon\colon H_1(X_L;\mathbb{Z}) \to \mathbb{Z}$.

Proof. It is enough to prove that all longitudes $\lambda_i' = \lambda_i + g_i\mu_i$ are sent to 0 by ϵ. Clearly ϵ factors through the abelianization $H_1(X_L;\mathbb{Z})$, and by

Lemma 4.1.4 we see that

$$\epsilon(\lambda_i) = \sum_{j=1}^{k} \Lambda_{ij}\epsilon(\mu_j) = \sum_{j=1}^{k} \Lambda_{ij} = -\epsilon(g_i\mu_i),$$

and hence $\epsilon(\lambda'_i) = 0$ for all i. □

We construct a closed 3-manifold Y_L in the following way: we take a k-punctured sphere $\Sigma_{0,k} := S^3 \setminus (D_1 \sqcup \cdots \sqcup D_k)$, and define a diffeomorphism

$$\varphi\colon \partial(-\Sigma_{0,k} \times S^1) = (\partial D_1 \sqcup \cdots \sqcup \partial D_k) \times S^1 \to \partial X_L \tag{4.8}$$

as the union of the orientation-reversing diffeomorphisms $\partial D_i \times S^1 \to T_i$ (determined up to isotopy) that give the identifications

$$\begin{cases} -[\partial D_i] = \lambda'_i = \lambda_i + g_i\mu_i, \\ [S^1] = \mu_i. \end{cases}$$

We define then the closed, oriented 3-manifold

$$Y_L := X_L \cup_\varphi (-\Sigma_{0,k} \times S^1). \tag{4.9}$$

The representation $\psi\colon H_1(X_L;\mathbb{Z}) \to U(1)$ sending all the meridians to a same value ω extends in a unique way to a representation $\psi\colon H_1(Y_L;\mathbb{Z}) \to U(1)$, whose restriction to $\Sigma_{0,k} \times S^1$ is the map

$$H_1(\Sigma_{0,k} \times S^1;\mathbb{Z}) \cong H_1(\Sigma_{0,k}) \oplus H_1(S^1) \to U(1)$$

that is 0 on $H_1(\Sigma_{0,k})$ and that sends $[S^1]$ to ω. We are now ready to state our result.

Theorem 4.2.7. *Let L be an oriented link and $\psi\colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the representation sending every meridian to the same $\omega \in U(1)$. Let g be the Seifert framing on L. Then*

$$\varrho_\psi(L, g_L) = \rho_\psi(Y_L) = -\sigma_L(\omega). \tag{4.10}$$

In particular, if K is a knot, we have

$$\varrho_\psi(K) = \rho_\psi(M_K) = -\sigma_K(\omega). \tag{4.11}$$

Proof. For a knot K, the Seifert framing is the 0-framing, and the 3-manifold $\Sigma_{0,1} \times S^1$ glued to X_L to obtain Y_L is just a solid torus. In particular, Y_K is the 0-framed surgery M_K, and the second statement is just a specialization of the first.

We prove the second equality of (4.10) first, by a cut-and-paste argument. As the statement is trivial for $\omega = 1$ (where it reads as $0 = 0$), we shall suppose that $\omega \in U(1) \setminus \{1\}$. By (4.7), the Levine-Tristram signature can be written as

$$\sigma_K(\omega) = \sigma_\psi(W_F),$$

where W_F is the exterior of a pushed-in Seifert surface F, and ψ is the representation sending the meridian of F to ω (the notation coincides on purpose with that of the representation in the statement, as this map restricts to the desired representation on X_L). Thanks to 1.4.3, we see easily that the untwisted intersection form is trivial on W_F. Hence, by Theorem 2.1.4 we have

$$\rho_\psi(\partial W_F) = \sigma(W_F) - \sigma_\psi(W_F) = -\sigma_K(\omega). \tag{4.12}$$

By choosing an appropriate trivialization for the tubular neighborhood of the pushed-in surface in the 4-ball, we see that the boundary of ∂W_F can be written, up to diffeomorphism, as

$$\partial W_F = X_L \cup_{\varphi'} (-F \times S^1).$$

In the above formula, the gluing diffeomorphism $\varphi' \colon \partial(-F \times S^1) \to \partial X_L$ behaves exactly like the diffeomorpphism φ of (4.8), in that it sends the homology classes of the boundary components of F to the longitudes $\lambda_i' = \lambda_i + g_i\mu_i$ of the Seifert framing, and the classes coming from the circle to the meridians of L. We have thus

$$(F \times S^1) \cup_{\varphi^{-1}\varphi'} (-\Sigma_{0,k} \times S^1) = F' \times S^1,$$

where F' is the closed surface by gluing $-F$ with $\Sigma_{0,k}$ along their boundary components. By Proposition 2.1.7, we obtain now

$$\rho_\psi(Y_L) = \rho_\psi(\partial W_F) + \rho_\psi(F' \times S^1) - \tau(\mathcal{V}_{F \times S^1}, \mathcal{V}_{X_L}, \mathcal{V}_{\Sigma_{0,k} \times S^1}),$$

where the Maslov triple index in twisted cohomology does not appear as $H_*(\partial X_L; \mathbb{C}_\psi) = 0$. After the identification given by the gluing, we easily see that $\mathcal{V}_{\Sigma_{0,k} \times S^1} = \mathcal{V}_{F \times S^1}$, and hence the untwisted Maslov triple index also vanishes. Thanks to Corollary 2.1.6, we also have $\rho_\psi(F' \times S^1) = 0$. We are hence left with

$$\rho_\psi(Y_L) = \rho_\psi(\partial W_F),$$

that together with (4.12) gives us $\rho_\psi(Y_L) = -\sigma_L(\omega)$ as desired.

We prove now the equality $\varrho_\psi(L, g_L) = \rho_\psi(Y_L)$. Thanks to Corollary 3.2.11, for any framing $\mathcal{F}$ on ∂X_L, we have

$$\rho_\psi(Y_L) = \rho_\psi(X_L, \mathcal{F}) - \rho_\psi(\Sigma_{0,k} \times S^1, \mathcal{F}) - \tau(\mathcal{M}_\mathcal{F}, \mathcal{V}_{X_L}, \mathcal{V}_{\Sigma_{0,k}\times S^1}).$$

We choose $\mathcal{F} = \mathcal{F}_{g_L}$ so that $\rho_\psi(X_L, \mathcal{F}) = \varrho_\psi(L, g_L)$ by definition. Moreover, by definition of the gluing, $\mathcal{F}$ coincides on $\partial\Sigma_{0,k} \times S^1$ with the reverse of the product framing defined in Example 3.1.2. As the representation ψ on $H_1(\Sigma_{0,k}\times S^1; \mathbb{Z})$ is trivial on the surface summand, the term $\rho_\psi(\Sigma_{0,k}\times S^1, \mathcal{F})$ vanishes by Remark 3.4.12 and Lemma 3.4.13 (iii). We are thus left with

$$\rho_\psi(Y_L) = \varrho_\psi(L, g_L) - \tau(\mathcal{M}_\mathcal{F}, \mathcal{V}_{X_L}, \mathcal{V}_{\Sigma_{0,k}\times S^1}),$$

and we need to show that the Maslov triple index vanishes. If we set $\Lambda_{g_L} = (a_{ij})$, as in the proof of Lemma 4.1.14, we have the explicit descriptions

$$\begin{cases} \mathcal{M}_\mathcal{F} = \operatorname{Span}_\mathbb{C}\{\mu_i, \dots, \mu_k\}, \\ \mathcal{V}_{X_L} = \operatorname{Span}_\mathbb{C}\left\{ \lambda'_i - \sum_{j=1}^k a_{ij}\mu_j \;\middle|\; i = 1, \dots, k\right\} \\ \mathcal{V}_{\Sigma_{0,k}\times S^1} = \operatorname{Span}_\mathbb{C}\left\{\mu_1 - \mu_k, \dots, \mu_{k-1} - \mu_k, \sum_{i=1}^k \lambda'_i\right\}. \end{cases}$$

Up to sign, the Maslov index we want to compute is the signature of the Hermitian form

$$\Psi \colon (\mathcal{M}_\mathcal{F} + \mathcal{V}_{\Sigma_{0,k}\times S^1}) \cap \mathcal{V}_{X_L} \times (\mathcal{M}_\mathcal{F} + \mathcal{V}_{\Sigma_{0,k}\times S^1}) \cap \mathcal{V}_{X_L} \to \mathbb{C}$$

defined by $\Psi(a, b) = a' \cdot b$, where $a = a' + a''$ is any decomposition of a such that $a' \in \mathcal{M}_\mathcal{F}$ and $a'' \in \mathcal{V}_{\Sigma_{0,k}\times S^1}$. By the explicit descriptions of the Lagrangians, we see that the space where the form is defined is one dimensional. Namely, we have

$$(\mathcal{M}_\mathcal{F} + \mathcal{V}_{\Sigma_{0,k}\times S^1}) \cap \mathcal{V}_{X_L} = \operatorname{Span}_\mathbb{C}\left(\sum_{i=1}^k \lambda'_i - \sum_{i=1}^k\sum_{j=1}^k a_{ij}\mu_j\right).$$

From the definition of the Seifert framing, we see however that

$$\sum_{i=1}^k\sum_{j=1}^k a_{ij}\mu_j = \sum_{j=1}^k\left(\sum_{i=1}^k a_{ij}\right)\mu_j = \sum_{j=1}^k\left(g_j + \sum_{i=1}^k \Lambda_{ij}\right)\mu_j = 0,$$

so that the generator is actually just $v := \sum_i \lambda'_i$. We compute now easily that

$$\Psi(v, v) = 0 \cdot v = 0,$$

so that $\operatorname{sign}(\Psi) = 0$ and the Maslov triple index vanishes as desired. □

As the Seifert framing g_L is compatible with every $\psi\colon H_1(X_L;\mathbb{Z}) \to U(1)$ that sends all meridians to the same ω, such representations also extend to the closed 3-manifold $M_L(g_L)$ obtained by surgery along g_L. The Atiyah-Patodi-Singer rho invariant of $M_L(g_L)$ is related to the Levine-Tristram signature of L in the following way.

Corollary 4.2.8. $\rho_\psi(M_L(g_L)) = -\sigma_L(\omega) + \operatorname{sign}\Lambda_{g_L}$.

Proof. By Theorem 4.1.16, we have $\rho_\psi(M_L(g_L)) = \varrho_\psi(L, g_L) + \operatorname{sign}\Lambda_{g_L}$, which is equivalent to the desired formula because of (4.10). □

Remark 4.2.9. If L has vanishing pairwise linking numbers, then g_L is the 0-framing, and we have the equalities

$$\varrho_\psi(L) = \rho_\psi(Y_L) = \rho_\psi(M_L) = -\sigma_L(\omega).$$

4.2.3 A short proof of two classical results

With the help of rho invariants and gluing formulas, it is now possible to give an easy proof of a classical result of Litherland [33, Theorem 2] about the Levine-Tristram signature of a satellite knot.

Theorem 4.2.10 (Litherland)**.** *Let $S := S(C, K, A)$ be a satellite knot with winding number n, and let $\omega \in U(1)$. Then, we have*

$$\sigma_S(\omega) = \sigma_C(\omega^n) + \sigma_K(\omega).$$

Proof. Let $\psi\colon H_1(X_S) \to U(1)$ defined by $\psi(\mu_S) = \omega$. By Proposition 4.1.17, we have

$$\varrho_\alpha(S) = \varrho_\alpha(C) + \varrho_\alpha(K \cup A),$$

and we only need to write the above rho invariants as Levine-Tristram signatures of knots. Now, we can easily see that, under the identification $X_S = X_C \cup X_{K\cup A}$, the restriction of ψ to $X_{K\cup A}$ is the representation sending μ_A to 1 and μ_k to ω, while its restriction to X_C sends μ_C to ω^n. From Theorem 4.2.7 and the above discussion, it follows immediately that

$$\varrho_\psi(S) = -\sigma_S(\omega), \quad \varrho_\psi(C) = -\sigma_C(\omega^n).$$

As the restriction of ψ to $X_{K\cup A}$ is trivial on μ_A, we can fill the link component A with a solid torus in a trivial way. The representation will extend, and the resulting manifold is (up to an orientation-preservig diffeomorphism that preserves the framings) just X_K. By Corollary 3.3.18, and the definitions, hence, we get

$$\varrho_\psi(K \cup A) = \rho_\psi(X_K, \mathcal{F}_K) = \varrho_\psi(K) = -\sigma_K(\omega).$$

Putting together all the equations, we obtain the desired formula. □

We reprove now a result of Casson and Gordon [11, Lemma 3.1] about the rho invariant of a 3-manifold M with a representation $\pi_1(M) \to U(1)$ that factors through a finite cyclic group, once a presentation of M as the surgery along a framed link is known (see also the later article of Gilmer [25, Theorem 3.6], who proved independently a slightly more general formula).

Theorem 4.2.11 (Casson-Gordon)**.** *Let $\psi\colon \pi_1(X_L) \to U(1)$ be the reprentation sending every meridian to $\omega = e^{2\pi i r/p}$, with $p \in \mathbb{N}$ and $r \in \{1, \cdots, p-1\}$. Then, for all compatible framing f, we have*

$$\rho_\psi(M_L(f)) = -\sigma_L(\omega) - \frac{2(\sum_{i,j} a_{ij})r(p-r)}{p^2} + \operatorname{sign}\Lambda_f,$$

where $(a_{ij}) = \Lambda_f$ is the framed linking matrix of (L, f).

Proof. As usual, let $g_L = (g_1, \ldots, g_k)$ denote the Seifert framing on L. It is immediate to check that, in the given context, a framing $f = (f_1, \cdots, f_k)$ is compatible if and only if, for all i,

$$f_i = g_i + n_i, \qquad \text{for some } n_i \in \frac{p}{\gcd(p, r)}\mathbb{Z}.$$

By Theorem 4.1.16, together with (4.10), we obtain

$$\rho_\psi(M_L(f)) = -\sigma_L(\omega) + \sum_{i=1}^{k} S_{n_i}(r/p) + \operatorname{sign}\Lambda_f. \tag{4.13}$$

Thanks to Remark 3.3.33 we can compute

$$S_{n_i}(r/p) = -\frac{2n_i r}{p}\left(1 - \frac{r}{p}\right) = -\frac{2n_i r(p-r)}{p^2}.$$

Plugging these values in (4.13), we obtain

$$\rho_\psi(M_L(f)) = -\sigma_L(\omega) - \frac{2(\sum_i n_i)r(p-r)}{p^2} + \operatorname{sign}\Lambda_f. \tag{4.14}$$

Note that, by definition of the framing g_L, we can now write

$$\sum_i n_i = \sum_i (f_i - g_i) = \sum_i f_i + \sum_{i,j} \Lambda_{ij} = \sum_{i,j} a_{ij},$$

so that (4.14) is equivalent to the formula in the statement. □

Remark 4.2.12. As it is observed by Casson and Gordon, every closed, oriented 3-manifold N with a representation $\psi\colon \pi_1(N) \to U(1)$ that factors through $\mathbb{Z}/p$ can be written as the result of the surgery on a link L with a representation sending every meridian to $\omega = e^{2\pi i r/p}$ along a compatible framing. Hence, (4.14) gives a general formula for computing rho invariants of 3-manifolds with such representations, once a surgery description is known.

4.2.4 Dehn surgery and signatures of knots

Let K be a knot, and let p, q be coprime integers. We let $M_K(p/q)$ denote the closed 3-manifolds obtained by p/q-Dehn surgery on K . In the notation of Dehn fillings, this is described as

$$M_K(p/q) = D_{p/q}(X_K, \mathcal{F}_K).$$

A local coefficient system $\psi \in \mathcal{U}(1)$, seen as a representation $\psi\colon H_1(X_K) \to U(1)$, extends to $M_K(p/q)$ if and only if $\psi(\mu)^p = 1$, i.e. if and only if $\omega := \psi(\mu)$ is a p^{th} root of unity.

The next result shows that the Atiyah-Patodi-Singer rho invariant of $M_K(p/q)$ can be nicely expressed in terms of the Levine-Tristram signature and of the rho invariant of an appropriate lens space. We were not able to trace this result in the literature.

Proposition 4.2.13. *Let K be a knot, let ω be a p^{th} root of unity, and let $\psi\colon H_1(X_K;\mathbb{Z}) \to U(1)$ be the representation defined by $\psi(\mu) = \omega$. Then, we have*

$$\rho_\psi(M_K(p/q)) = -\sigma_K(\omega) - \rho(L(p,q), \omega).$$

Proof. Using formula (3.5) of Remark 3.3.17, we can write

$$\rho_\psi(M_K(p/q)) = \rho_\psi(X_K, \mathcal{F}_K) + \rho_\psi(D^2 \times S^1, \mathcal{F}_{p/q}) + \operatorname{sgn}(p/q). \tag{4.15}$$

Now, by definition of $\varrho_\psi(K)$, together with Theorem 4.2.7, we have

$$\rho_\psi(X_K, \mathcal{F}_K) = \varrho_\psi(K) = -\sigma_K(\omega). \tag{4.16}$$

Under the gluing, the class $[S^1] \in H_1(D^2 \times S^1)$ is identified with $b\mu + d\lambda$, where b, d are integers such that $pd - qb = 1$. It follows that we have

$$\psi([S^1]) = \psi(\mu)^b = \omega^b.$$

Let $k \in \mathbb{Z}$ such that $\omega = e^{2\pi i k/p}$. Then, by definition of the function $S_{p/q}$, together with Corollary 3.3.32 (ii) and the definition of ℓ, we have

$$\rho_\psi(D^2 \times S^1, \mathcal{F}_{p/q}) = S_{p/q}(bk/p) = -\ell(p, q, bk) - \operatorname{sgn}(p/q).$$

As bq is just 1 mod p, we have by definition

$$\ell(p, q, bk) = \rho(L(p, q), e^{2\pi i bkq/p}) = \rho(L(p, q), \omega),$$

so that

$$\rho_\psi(D^2 \times S^1, \mathcal{F}_{p/q}) = -\rho(L(p, q), \omega) - \operatorname{sgn}(p/q). \tag{4.17}$$

Plugging (4.16) and (4.17) into (4.15), we obtain the desired result. □

Using some "exceptional surgeries", we get now the following description of the Levine-Tristram signature of a torus knot in terms of rho invariants of lens spaces. Given the explicit expressions of Remark 3.3.30, it would be interesting to compare this result to the computations of Borodzik and Oleszkiewicz [6].

Corollary 4.2.14. *Let r, s be positve coprime integers, and let $T(r, s)$ denote the (r, s)-torus knot. Let $\zeta := e^{2\pi i (rs-1)}$. Then, for all $0 \leq k \leq rs - 2$, we have*

$$\sigma_{T(r,s)}(\zeta^k) = -\ell(rs - 1, s^2, kr) - \ell(rs - 1, 1, k).$$

Proof. The $(rs-1)$-Dehn surgery on $T(r, s)$ gives a manifold which is orientation-preserving diffeomorphic to the lens space $L(rs-1, s^2)$ [35, Proposition 3.2]. Keeping track of the induced map on the fundamental group under this diffeomorphisms, by Proposition 4.2.13 (applied with $\omega = \zeta^k$) we obtain

$$\rho_\psi(L(rs - 1, s^2), \zeta^{krs^2}) = -\sigma_{T(r,s)}(\zeta^k) - \rho(L(rs - 1, 1), \zeta^k),$$

which immediately gives the desired equality. □

4.2.5 Cimasoni-Florens signatures

Using a generalizition of the concept of Seifert surfaces (called *C-complexes*), Cimasoni and Florens [12] defined a multivariable version of the Levine-Tristram signature of a link. In their work, a link L is considered with the extra structure of a *coloring*, i.e. a partition of its components into n non-empty subsets, indexed by $\{1, \ldots, n\}$, and the signature is a function

$$\sigma_L \colon \mathbb{T}_*^n \to \mathbb{Z},$$

where $\mathbb{T}_* := \mathbb{T} \setminus \{1\} = U(1) \setminus \{1\}$. At the two extremes there are the case where $n = 1$ and all components are grouped together, where σ_L coincides with the Levine-Tristram signature function, and the case where n corresponds to the number of components of L, and every components is grouped on its own. As the resulting function contains in this latter case the highest amount of information and all other functions can be easily recovered from it, for simplicity we shall not talk of colored links, and we shall always consider the function corresponding to this maximal coloring. Given a link L witk k components, in order to distinguish this function from the Levine-Tristram signature function, we shall use the notation

$$\sigma'_L \colon \mathbb{T}_*^k \to \mathbb{Z}$$

for the multivariable signature with maximal coloring.

Remark 4.2.15. By a result of Cimasoni and Florens [12, Proposition 2.5], the Levine-Tristram signature can be recovered from the multivariable signature by the formula

$$\sigma_L(\omega) = \sigma'_L(\omega, \ldots, \omega) - \sum_{i<j} \Lambda_{ij},$$

where Λ_{ij} denotes as usual the linking number $\operatorname{lk}(L_i, L_j)$.

Instead of giving the original definition of the multivariable signatures, we shall employ a 4-dimensional characterization, due to Conway, Nagel and the author [17, Proposition 3.5]. We need the following definition.

Definition 4.2.16. A *bounding surface* for a link $L = L_1 \cup \cdots \cup L_k \subseteq S^3 = \partial D^4$ is a union $F = F_1 \cup \cdots \cup F_k$ of properly embedded, locally flat, compact, oriented surfaces $F_i \subset D^4$ with $\partial F_i = L_i$ and which only intersect each other transversally in double points.

Given a bounding surface $F = F_1 \cup \cdots \cup F_k \subseteq D^4$ for a link L, we can take a small tubular neighbourhood $N(F_i)$ of each surface F_i and define the *exterior* of F in D^4 as the 4-manifold with boundary

$$W_F := D^4 \setminus (N(F_1) \cup \cdots \cup N(F_k)).$$

It is easy to show that $H_1(W_F; \mathbb{Z})$ is freely generated by the images of the meridians $\mu_1, \ldots, \mu_k$ of the link L. We have then the following result [17, Proposition 3.5].

Proposition 4.2.17. *Let L be a link in S^3 and let F be a bounding surface for L. Given a k-tuple $(\omega_1, \ldots, \omega_k) \in \mathbb{T}^k_*$, let $\psi\colon H_1(W_F; \mathbb{Z}) \to U(1)$ be the representation sending μ_i to ω_i, for $i = 1, \ldots, k$. Then, we have*

$$\sigma'_L(\omega_1, \ldots, \omega_k) = \sigma_\psi(W_F).$$

Remark 4.2.18. The proof of Proposition 4.2.17 is made of two steps. First, one proves that the value of $\sigma_\psi(W_F)$ is independent of the choice of the bounding surface. This is originally due to Viro [46, Theorem 2.A] (see also Degtyarev, Florens and Lecuona [19, Proposition 3.9]). As a second step, one shows the result for the specific case of F being a *pushed-in* C-complex. This is based on a computation of Conway, Friedl and the author [16, Theorem 1.2].

Remark 4.2.19. Multivariable signatures are good concordance and cobordism invariants. Let $\mathbb{T}^k_!$ denote the dense subset of $\mathbb{T}^k_*$ made of k-tuples $(\omega_1, \ldots, \omega_k)$ such that there is no Laurent polynomial $p \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ which satisfies $p(\omega_1, \ldots, \omega_k) = 0$ and $p(1, \ldots, 1) = \pm 1$. Then, by a result of Conway, Nagel and the author [17, Theorem 1.5], the invariant $\sigma_L(\omega_1, \cdots, \omega_k)$ is invariant under *0.5-solvable cobordism* of links for all for $(\omega_1, \ldots, \omega_k) \in \mathbb{T}^k_!$.

4.2.6 Rho invariants of links and multivariable signatures

In this section, we shall give analogues of the results of Section 4.2.2 about the Levine-Tristram signature. First, given a link L, we shall construct a closed manifold Y'_L which is only determined by L and whose Atiyah-Patodi-Singer rho invariant coincides with the multivariable signature of L. Then, we shall compare the multivariable signature of L with the rho invariant $\varrho_\psi(L)$ defined in Section 4.1.2.

Let $L = L_1, \cdots, L_k$ be a k-component link. In order to define the closed manifold Y'_L of our need, we first build a 3-manifold with boundary P_L in the following way.

1. Take k copies $D_1, \cdots, D_k$ of the 2-disk D^2.
2. For each ordered pair (i, j), remove $p_{ij} := |\Lambda_{ij}|$ disjoint disks $D_{ij}^1, \ldots D_{ij}^{p_{ij}}$ from D_i.
3. For each i, let $\widetilde{D}_i$ denote the surface obtained from D_i by removing $\sum_{j \neq i} p_{ij}$ small disks in the way described above.
4. For each triple (i, j, m) with $i < j$ and $1 \leq m \leq p_{ij}$, let

$$\varphi_{ij}^m \colon \partial D_{ij}^m \times S^1 \to \partial D_{ji}^m \times S^1$$

be the orientation-reversing diffeomorphism given by

$$\varphi_{ij}^m(x, y) = \begin{cases} (y, x), & \text{if } \operatorname{sgn} \Lambda_{ij} > 0, \\ (y^{-1}, x^{-1}), & \text{if } \operatorname{sgn} \Lambda_{ij} < 0, \end{cases}$$

where $\varepsilon = \operatorname{sign} \Lambda_{ij}$ and we are implicitly using an identification of ∂D_{ij}^m and ∂D_{ji}^m with S^1.
5. Set

$$P_L := \left(\bigsqcup_{i=1}^{k} -\widetilde{D}_i \times S^1 \right) / \sim,$$

where $\sim$ is the equivalence relation given by the gluings with respect to the diffeomorphisms φ_{ij}^m for all triples (i, j, k).

In other words, P_L is obtained by plumbing along the graph having k disks $-D_1, \cdots, -D_k$ as vertices, and edges corresponding to the linking numbers between the corresponding link components [17, 4.2]. By construction, P_L is an oriented 3 manifolds with k boundary components $\Sigma_1, \cdots, \Sigma_k$, given by $\Sigma_i = -\partial D_i \times S^1$. On the other hand, the boundary of the link exterior X_L is a union of tori $T_1, \cdots, T_k$, where T_i is the boundary of a tubular neighborhood of L_i, and has a standardly oriented framing with meridian μ_i and longitude λ_i. We set now

$$Y_L' := X_L \cup_{\varphi} P_L,$$

where the gluing map φ is given by the collection of orientation-reversing diffeomorphisms $\Sigma_i \to T_i \subseteq \partial X_L$ determined up to isotopy by the identifications

$$\begin{cases} [\partial D_i] = \lambda_i, \\ [S^1] = \mu_i. \end{cases} \tag{4.18}$$

We use the identifications given by (4.18) to define a framing of ∂P_L which thus, by construction, coincides with $\mathcal{F}_L$ under the gluing that gives rise to Y'_L. We describe next the canonical Lagrangian $\mathcal{V}_{P_L} \subseteq H_1(\partial P_L;\mathbb{Z})$ in terms of the meridians μ_i and longitudes λ_i.

Lemma 4.2.20. *The canonical Lagrangian of P_L is given by*

$$\mathcal{V}_{P_L} = \operatorname{Span}_{\mathbb{C}} \left\{ \lambda_i - \sum_{j=1}^{k} \Lambda_{ij}\mu_j \,\middle|\, i = 1, \dots, k \right\}.$$

In particular, $\mathcal{V}_{P_L}$ coincides with $\mathcal{V}_{X_L}$ under the gluing $X_L \cup_\varphi P_L$.

Proof. The first statement is an easy Mayer-Vietoris calculation [17, Lemma 4.7]. The second statement follows immediately from (4.1). □

An immediate consequence of Lemma 4.2.20 is that the images of the meridians μ_i are linearly independent in $H_1(P_L;\mathbb{Z})$ and in $H_1(Y'_L;\mathbb{Z})$. In particular, given a k-tuple $(\omega_1 \dots, \omega_k) \in \mathbb{T}^k_*$, we can define a representation

$$\psi \colon H_1(Y'_L;\mathbb{Z}) \to U(1)$$

by sending every meridian μ_i to ω_i, and the complement of the subgroup generated by the meridians to 1. We have then the following result.

Proposition 4.2.21. *Let $L = L_1 \cup \cdots \cup L_k$ be a link in S^3, let $(\omega_1 \dots, \omega_k) \in \mathbb{T}^k_*$ and let $\psi \colon H_1(Y'_L;\mathbb{Z}) \to U(1)$ be the representation corresponding to $(\omega_1 \dots, \omega_k)$ as described above. Then, we have*

$$\rho_\psi(Y'_L) = -\sigma'_L(\omega_1, \dots, \omega_k).$$

Proof. We shall only give the sketch of a proof. More details for most of the claims are found in the references. See S^3 as the boundary of D^4, and pick a bounding surface $F = F_1 \cup \cdots \cup F_k \subseteq D^4$ for L. The boundary of its exterior W_F, then, is given by $X_L \cup_\partial M_F$, where M_F is a 3-manifold obtained by plumbing along the graph whose vertices are the surfaces $-F_1, \dots, -F_k$ and whose edges correspond to the intersection points between them, with the sign of the intersection as decoration [17, Example 4.12]. Of course, the representation ψ extends to W_F. By the cut-and-paste formula for the rho invariant (Proposition 2.1.7), we find now

$$\rho_\psi(Y'_L) = \rho_\psi(\partial W_F) + \rho_\psi(-M_F \cup_\partial P_L) - \tau(\mathcal{V}_{M_F}, \mathcal{V}_{X_L}, \mathcal{V}_{P_L}) \tag{4.19}$$

(as usual we compute Maslov indices in homology instead of cohomology, and the twisted one is 0 because all ω_i's are non-trivial by assumption).

From Lemma 4.2.20, we see that $\tau(\mathcal{V}_{M_F}, \mathcal{V}_{X_L}, \mathcal{V}_{P_L})$ is also 0. The term $\rho_\psi(-M_F \cup_\partial P_L)$ is also 0 thanks to a computation of Conway, Nagel and the author [17, Proposition 4.10], because $-M_F \cup P_L$ can be seen as the plumbing of a *balanced* graph with vertices the closed surfaces $F_i \cup_\partial (-D_i)$ (otherwise, the same result can be proved by using gluing formulas and Proposition 3.4.11). Using the Atiyah-Patodi-Singer signature theorem, (4.19) can be now rewritten as

$$\rho_\psi(Y'_L) = \rho_\psi(\partial W_F) = \operatorname{sign}(W_F) - \operatorname{sign}_\psi(W_F).$$

As the untwisted signature of W_F is 0 [17, Proposition 3.3], the result follows now from Proposition 4.2.17. □

Remark 4.2.22. The original definition of the multivariable signature [12] depends on the choice of a C-complex in S^3. Proposition 4.2.17 can be used to give an alternative definition of the multivariable signature as the twisted signature of an appropriate 4-manifold [17, 19]. However, this 4-manifold depends on the non-canonical choice of a bounding surface for the link. Proposition 4.2.21, in turn, gives a description of the multivariable signature as the rho invariant of a closed 3-manifold which is unequivocally determined by the link, providing a possible more intrinsic definition.

We conclude with a comparison between rho invariants for links (in the abelian case) and multivariable signatures.

Theorem 4.2.23. *Let $L = L_1 \cup \cdots \cup L_k$ be a link in S^3, let $(\omega_1 \ldots, \omega_k) \in \mathbb{T}^k_*$ and let $\psi \colon H_1(X_L;\mathbb{Z}) \to U(1)$ be the representation sending μ_i to ω_i for $i = 1, \ldots, k$. Then, we have*

$$\varrho_\psi(L) = -\sigma_L(\omega_1, \ldots, \omega_k) - \rho_\psi(P_L, \mathcal{F}_L).$$

In particular, the difference between $\varrho_\psi(L)$ and $-\sigma_L(\omega_1, \ldots, \omega_k)$ only depends on the linking matrix Λ_L and and on the k-tuple $(\omega_1, \ldots, \omega_k)$.

Proof. Clearly, we can extend ψ to the whole of Y'_L in the usual way. From the description $Y'_L = X_L \cup_\varphi (-P_L)$, using the gluing formula of Corollary 3.2.11 we obtain now

$$\rho_\psi(Y'_L) = \rho_\psi(X_L, \mathcal{F}_L) + \rho_\psi(P_L, \mathcal{F}_L) - \tau(\mathcal{M}_{\mathcal{F}_L}, \mathcal{V}_{X_L}, \mathcal{V}_{P_L}).$$

The first summand on the right-hand side is exactly $\varrho_\psi(L)$, and the Maslov triple index vanishes as we have seen that $\mathcal{V}_{X_L} = \mathcal{V}_{P_L}$. From Proposition 4.2.21, we get thus

$$\varrho_\psi(L) = -\sigma_L(\omega_1 \ldots, \omega_k) - \rho_\psi(P_L, \mathcal{F}_L)$$

as desired. □

Remark 4.2.24. The manifold P_L whose rho invariant corresponds to the correction term $C(\Lambda_L, \omega_1, \ldots, \omega_k)$ of Theorem 4.2.23 is built by gluing products $-\widetilde{D}_i \times S^1$, where $\widetilde{D}_i$ is a punctured disk. With the help of gluing formulas, we can thus express the result in terms of the rho invariants of these manifolds, corresponding to the function Γ of Section 3.4.2. A lengthy Maslov index calculation, namely, leads to the formula

$$\varrho_\psi(L) = -\sigma_L(\omega_1, \ldots, \omega_k) - \operatorname{sign} \Lambda_L + \sum_{i=1}^{k} \Gamma_{p_i}(\boldsymbol{v}_i),$$

where $p_i := \sum_{j \neq i} |\Lambda_{ij}|$, and $\boldsymbol{v}_i \in \mathbb{T}_*^{p_i}$ is a tuple made of $|\Lambda_{ij}|$ repetitions of $\omega_j^{\operatorname{sgn} \Lambda_{ij}}$ for each $j \neq i$. In particular, if we were able to compute the rho invariants of products $F \times S^1$ effectively, we would have a more explicit formula for the difference between the invariant $\varrho_\psi(L)$ and (minus) the multivariable signature.

Remark 4.2.25. As a consequence of Theorem 4.2.23, we see that the link invariant $\varrho_\psi(L)$ is as good as a concordance invariant as the multivariable signatures. Even though it might be hard to compute explicitly, its topological description is nicer than that of multivariable signatures (it is just the rho invariant of the link exterior), and can be effectively used at their place in proofs requiring constructions that start from the link exterior.

Appendix

A.1 3-dimensional lens spaces

Let $p, q \in \mathbb{Z}$ be two coprime integers. We define the 3-dimensional *lens space* $L(p, q)$ as follows.

- For $p > 0$, we set $L(p, q) = S^3/\mathbb{Z}/p$ where we see S^3 as standarly embedded in $\mathbb{C}^2$ and the quotient is with respect to the action

$$\begin{aligned} \mathbb{Z}/p \times S^3 &\to S^3 \\ ([k], (z, w)) &\mapsto (e^{2\pi i k/p} z, e^{2\pi i k q/p} w). \end{aligned}$$

  Notice that $L(p, q)$ inherits from S^3 the structure of a closed, oriented 3-manifold.

- For $p < 0$, we set $L(p, q) := -L(-p, q)$.

- For $p = 0$, we set $L(0, 1) = L(0, -1) := S^2 \times S^1$.

In all three cases, there is a natural isomorphism $\pi_1(L(p, q)) \cong \mathbb{Z}/p$. We prove now the following basic results.

Proposition A.1.1 (Relationship between different lens spaces)**.**

(i) $L(-p, q) = -L(p, q)$ *for* $p \neq 0$*;*

(ii) $L(p, q) = L(p, r)$ *if* $r \equiv q \mod p$*;*

(iii) there is an orientation-reversing diffeomorphism $f\colon L(p, -q) \to L(p, q)$ *such that* $f_*\colon \mathbb{Z}/p \to \mathbb{Z}/p$ *satisfies* $f_*([1]) = [1]$*;*

(iv) there is an orientation-preserving diffeomorphism $g\colon L(p, q) \to L(p, q)$ *such that* $g_*\colon \mathbb{Z}/p \to \mathbb{Z}/p$ *satisfies* $f_*([1]) = [-1]$*;*

(v) if $s \equiv q^{-1} \mod p$*, there is an orientation-preserving diffeomorphism* $h\colon L(p,s) \to L(p,q)$ *such that* $h_*\colon \mathbb{Z}/p \to \mathbb{Z}/p$ *satisfies* $h_*([1]) = [s]$.

Proof. Point (i) follows from the definition of $L(a,b)$ for negative a. Because of the same definition, it is enough to prove the other points for $p > 0$ and for the special case $p = 0$. We start with $p > 0$. Then, point (ii) follows immediately from the definition. For the other three points, we define the diffeomorphisms explicitly as

$$f([(z,w)]) = [(z,\overline{w})], \quad g([(z,w)]) = [(\overline{z},\overline{w})], \quad h([(z,w)]) = [(w,z)],$$

and easily verify that the induced map in the fundamental group are those listed in the statement. For $p = 0$, (ii) is trivial and (v) is either trivial or coincides with (iv). For (iii) and (iv), we have to provide diffeomorphisms $f,g\colon S^2 \times S^1 \to S^2 \times S^1$ with f orientation-reversing and trivial on the fundamental group, and g orientation preserving and non-trivial on the fundamental group. We can thus define $f = \tau \times \mathrm{id}_{S^1}$ and $g = \tau \times \sigma$ with τ a reflection of S^2 and σ a reflection of S^1. □

There is another classical description of lens spaces in terms of genus 1 Heegaard splittings. Given a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\det A = -1$, we denote by H_A the closed 3-manifold obtained by gluing two copies V_1, V_2 of the standard solid torus $D^2 \times S^1 \subseteq \mathbb{C}^2$ through the orientation-reversing diffeomorphism

$$\begin{aligned} f_A\colon \partial V_2 = S^1 \times S^1 &\to \partial V_1 = S^1 \times S^1 \\ (z,w) &\mapsto (z^a w^c, z^b w^d). \end{aligned}$$

By Seifert–Van Kampen, the fundamental group of H_A can be described in terms of the positive generators $x \in \pi_1(V_1)$ and $y \in \pi_1(V_2)$ as

$$\pi_1(H_A) = \langle x, y \mid x^c = 1, y = x^d \rangle \cong \langle x \mid x^c \rangle.$$

We have the following result.

Proposition A.1.2. *Let* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be a matrix with* $\det A = -1$*. Then, there is an orientation-preserving diffeomorphism*

$$\varphi_A\colon H_A \xrightarrow{\sim} L(c, -a)$$

such that $(\varphi_A)_*\colon \langle x \mid x^c \rangle = \pi_1(H_A) \to \pi_1(L(c,-a)) = \mathbb{Z}/c$ *is the map sending* x *to* $[1]$.

Proof. For positive c, the diffeomorphism is constructed explicitly in the lecture notes of Friedl [24], once the inverse of the matrix is taken, which corresponds to using the inverse of the gluing diffeomorphism. The observation about the fundamental group is immediate from the explicit description of the diffeomorphism.

For negative c, consider the matrix $A' := \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$. Since we defined in this case $L(c,-a)$ to be $-L(-c,-a)$, it is enough to find an orientation-reversing diffeomorphism $g\colon H_A \to -H_{A'}$ which behaves correctly on the fundamental group, and then define φ_A as the composition

$$H_A \xrightarrow{g} -H_{A'} \xrightarrow{\varphi_{A'}} -L(-c,-a) = L(c,-a).$$

By definition, $H_{A'}$ is built by gluing two copies V_1', V_2' of $D^2 \times S^1$ through the map

$$\begin{aligned} f_{A'}\colon \partial V_2 = S^1 \times S^1 &\to \partial V_1 = S^1 \times S^1 \\ (z,w) &\mapsto (z^a w^{-c}, z^{-b} w^d). \end{aligned}$$

We define the map $g\colon V_1 \cup_{f_A} V_2 = H_A \to H_{A'} = V_1' \cup_{f_A'} V_2'$ as the union of the maps

$$\begin{aligned} g_1\colon V_1 &\to V_1' & g_2\colon V_2 &\to V_2' \\ (z,w) &\mapsto (z^{-1}, w) & (z,w) &\mapsto (z^{-1}, w). \end{aligned}$$

It is immediate to check that the two maps respect the gluing, so that g is well defined. Moreover, if $x' \in \pi_1(V_1')$ is the positive generator, so that we have a description $\pi_1(H_{A'}) = \langle x' \mid (x')^{-c} \rangle$ analogous to the one of $\pi_1(H_A)$, we have the induced map $g_*(x) = (x')$, and hence $(\varphi_A)_* = (\varphi_{A'})_* \circ g_*$ sends $x \in \pi_1(H_A)$ to $[1] \in \mathbb{Z}/p$ as desired.

It remains to consider the case $c = 0$, which implies $a = \pm 1$. We defined in these cases $L(0,1) = L(0,-1) = S^2 \times S^1$. So, we have to find an orientation-preserving diffeomorphism $\varphi_A\colon H_A \xrightarrow{\sim} S^2 \times S^1$ such that

$$(\varphi_A)_*\colon \langle x \rangle = \pi_1(H_A) \to \pi_1(S^2 \times S^1) = \mathbb{Z}$$

sends x to 1. The fact that H_A is diffeomorphic to $S^2 \times S^1$ is well-known (see Rolfsen [42, Chapter 9.B]). By composing the chosen diffeomorphism with an appropriate self-diffeomorphism of $S^2 \times S^1$, we obtain easily the map φ_A with the desired properties. □

Remark A.1.3. In practice, in order to build the lens space $L(p,q)$ out of two solid tori V_1, V_2, with meridians μ_1, μ_2 and longitudes λ_1, λ_2 respectively, it is enough to choose an orientation-reversing diffeomorphism $f\colon \partial V_2 \to \partial V_1$ with

$$f_*(\mu_2) = -q\mu_1 + p\lambda_1.$$

The element $[1] \in \mathbb{Z}/p = \pi_1(L(p,q))$ concides then with the image of the generator of $\pi_1(V_1)$. With respect to the bases (μ_1, λ_1) of $H_1(\partial V_1;\mathbb{Z})$ and (μ_2, λ_2) of $H_1(\partial V_2;\mathbb{Z})$, the diffeomorphism f is represented in homology by a matrix

$$A = \begin{pmatrix} -q & a \\ p & b \end{pmatrix}, \qquad \det A = -1.$$

The values $a, b \in \mathbb{Z}$ are not unequivocally determined by p and q, but they do not play a role in the determination of the diffeomorphism class of the resulting manifold. However, it is sometimes important to keep track of them, as they appear if we try to look at the fundamental group of $\pi_1(L(p,q))$ from the point of view of V_2. In fact, the generator of $\pi_1(V_2)$ corresponds to the element $[b] \in \mathbb{Z}/p$.

A.2 Dedekind sums and the Rademacher function

The topic of this section is a very classical one, but we shall not try to outline the history of the results or to be exhaustive in the treatment. Instead, we shall just outline the definitions and results that we need in relationship to our goal, which is centered in the computation of the eta and rho invariants of some particular manifolds.

A.2.1 Classical Dedekind sums

For coprime integers a, c such that $c \neq 0$, the *Dedekind sum* of the pair (a, c) is the rational number

$$s(a,c) := \sum_{j=1}^{|c|-1} \left(\!\!\left(\frac{aj}{c}\right)\!\!\right)\left(\!\!\left(\frac{j}{c}\right)\!\!\right), \tag{A.1}$$

where $(\!(\cdot)\!) : \mathbb{R} \to \left(-\frac{1}{2}, \frac{1}{2}\right)$ be the periodic sawtooth function defined by

$$(\!(x)\!) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases}$$

From the oddness of the function $(\!(\cdot)\!)$, it is immediate to see that

$$s(-a,c) = -s(a,c), \quad s(a,-c) = s(a,c).$$

From the 1-periodicity of $(\!(\cdot)\!)$, we see that $s(a,c)$ only depends on a up to equivalence modulo c. For the same reason, we can see the sum in the definition of $s(a,c)$ as running through elements modulo c, and then an easy change of variable leads to the equality

$$s(a,c) = s(a',c), \quad \text{if} \quad aa' \equiv 1 \mod c. \tag{A.2}$$

Suppose now that a and c are positive coprime integers. One of the most important relations satisfied by the Dedekind sums, admitting, several different proofs, is the *reciprocity formula* [40, Chapter 2]

$$s(a,c) + s(c,a) = \frac{1}{12}\left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a}\right) - \frac{1}{4}. \tag{A.3}$$

Another useful property is the *cotangent formula* [40, (26)]

$$s(a,c) = \frac{1}{4c}\sum_{j=1}^{c-1} \cot\frac{\pi k}{c}\cot\frac{\pi a k}{c}. \tag{A.4}$$

A.2.2 The Rademacher function

In this section, we shall consider various functions defined on $\mathrm{SL}(2,\mathbb{Z})$ and on $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$. Given a function $f\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{R}$ we shall consider its *coboundary* $\delta f\colon \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{R}$, given by

$$\delta f(A,B) := f(A) + f(B) - f(AB).$$

Given an element $A \in \mathrm{SL}(2,\mathbb{Z})$, we define its coefficients by

$$A = \begin{pmatrix} a_A & b_A \\ c_A & d_A \end{pmatrix}.$$

When a single matrix is being considered, we shall normally omit its name from the notation of the coefficients.

We start with the definition of the main function of our interest. The *Rademacher function* $\Phi\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{Q}$ is defined as

$$\Phi(A) := \begin{cases} \frac{b}{3d}, & \text{if } c = 0, \\ \frac{a+d}{3c} - 4\,\mathrm{sgn}(c)s(a,c), & \text{otherwise.} \end{cases} \tag{A.5}$$

Remark A.2.1. In fact, the original function considered by Rademacher is 3Φ, and it takes integer values as a consequence of the reciprocity formula (A.3) for the Dedekind sums [39, p. 50].

Remark A.2.2. It is immediate to check that $\Phi(-A) = \Phi(A)$, so that Φ actually gives rise to a well-defined function $\mathrm{PSL}(2,\mathbb{Z}) \to \mathbb{Z}$.

Rademacher was able to give a simple formula for the coboundary of the function Φ. Namely, he proved that

$$\delta\Phi(A,B) = \mathrm{sgn}(c_A c_B c_{AB}). \tag{A.6}$$

An analytic proof of this fact based on the transformation properties of the Dedekind eta function can be find in the book of Rademacher and Grosswald [40, (62)]. In the same text, a reference to the original (purely arithmetic but lengthy) proof is also given.

Remark A.2.3. Both the Rademacher function $\Phi\colon \mathrm{PSL}(2,\mathbb{Z}) \to \mathbb{Z}$ and the function $\varepsilon\colon \mathrm{PSL}(2,\mathbb{Z}) \times \mathrm{PSL}(2,\mathbb{Z}) \to \mathbb{Z}$ defined by

$$\varepsilon(A,B) = \mathrm{sgn}(c_A c_B c_{AB}) \tag{A.7}$$

have geometric interpretations in terms of invariants of triangles in the hyperbolic plane, as observed by Kirby and Melvin [27, Section 1]. They call the latter (with the opposite sign normalization) the *area cocycle*, and show by geometric means that $\delta\Phi = \varepsilon$.

Following the work of Kirby and Melvin, we define one more function $\nu\colon \mathrm{SL}(2,\mathbb{Z}) \to \mathbb{Z}$ as

$$\nu(A) := \begin{cases} \mathrm{sgn}(b), & \text{if } A = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), \\ \mathrm{sgn}(c(a+d-2)), & \text{otherwise.} \end{cases}$$

The authors prove [27, Theorem 6.1] that the coboundary of $-\Phi + \nu$ is the famous *signature cocycle*

$$\sigma\colon \mathrm{SL}(2,Z) \times \mathrm{SL}(2,Z) \to \mathbb{Z},$$

of Meyer, originally defined as the twisted signature of a torus bundle over a pair of pants [34]. Together with (A.6), this gives the simple expression

$$\sigma(A,B) = \nu(A) + \nu(B) - \nu(AB) - \mathrm{sgn}(c_A c_B c_{AB}).$$

A.2.3 Dedekind-Rademacher sums

For coprime integers a, c such that $c \neq 0$, and real numbers $x, y \in \mathbb{R}$, we define the *Dedekind-Rademacher sum*

$$s_{x,y}(a,c) := \sum_{j=0}^{|c|-1} \left(\left(\frac{a(j+x)}{c} + y\right)\right)\left(\left(\frac{j+x}{c}\right)\right). \tag{A.8}$$

It is immediate to see that, for fixed a and c, $s_{x,y}(a,c)$ is 1-periodic in both x and y, so that it can be seen as a function of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. Clearly, moreover, we have

$$s_{0,0}(a,c) = s(a,c).$$

Hence, this is a true generalization of the classical Dedekind sums. The function $s_{x,y}(a,c)$ was first defined in this generality by Rademacher [39]. We took our notation from [5, Appendix C.2.2].

We shall now focus on the special case $x = 0$, establishing some formulas that will be useful in the applications. By definition, we have

$$s_{0,y}(a,c) = \sum_{j=0}^{|c|-1} \left(\left(\frac{aj}{c} + y\right)\right)\left(\left(\frac{j}{c}\right)\right).$$

First of all, using the oddness and the periodicity of the sawtooth function, we observe that

$$s_{0,y}(a,c) = -s_{0,y}(-a,c) = s_{0,y}(a,-c) = s_{0,-y}(a,c)$$

and

$$s_{0,y}(a',c) = s_{0,y}(a,c) \quad \text{if} \quad a' \equiv a \mod c.$$

If cy is an integer, we obtain more relations. For example, for $d \in \mathbb{Z}$ such that $ad \equiv 1 \mod c$, and $y = \frac{n}{c} \in \frac{1}{c}\mathbb{Z}$, we have

$$s_{0,ay}(a,c) = s_{0,y}(d,c),$$

as it can be seen with the substitution $k = a(j+n)$ in the expression

$$s_{0,ay}(a,c) = \sum_{j \bmod c} \left(\left(\frac{a(j+n)}{c}\right)\right)\left(\left(\frac{j}{c}\right)\right).$$

Moreover, we have the following result.

Lemma A.2.4. *Let a, c coprime integers with $c > 0$, and let $y = k/c$ with $k \in \{1, \dots, c-1\}$. Then*

$$s_{0,ay}(a,c) - s(a,c) = \sum_{j=1}^{k-1} \left(\left(\frac{aj}{c}\right)\right) + \frac{1}{2}\left(\left(\frac{ak}{c}\right)\right). \tag{A.9}$$

Proof. By definition, we have

$$s_{0,ay}(a,c) = \sum_{i \mod c} \left(\left(\frac{a(i+k)}{c}\right)\right)\left(\left(\frac{i}{c}\right)\right),$$

and with the substitution $j = i + k$ we obtain

$$s_{0,ay}(a,c) = \sum_{j \bmod c} \left(\left(\frac{aj}{c}\right)\right)\left(\left(\frac{j-k}{c}\right)\right) = \sum_{j=1}^{c-1} \left(\left(\frac{aj}{c}\right)\right)\left(\left(\frac{j-k}{c}\right)\right).$$

Now, it is immediate to verify that

$$\left(\left(\frac{j-k}{c}\right)\right) = \begin{cases} \left(\left(\frac{j}{c}\right)\right) + 1 - \frac{k}{c}, & \text{if } 1 \le j \le k-1, \\ 0, & \text{if } j = k, \\ \left(\left(\frac{j}{c}\right)\right) - \frac{k}{c}, & \text{if } k+1 \le j \le c-1. \end{cases}$$

As a consequence, we can write

$$\begin{aligned} s_{0,ay}(a,c) - s(a,c) &= \sum_{j=1}^{c-1} \left(\left(\frac{aj}{c}\right)\right)\left[\left(\left(\frac{j-k}{c}\right)\right) - \left(\left(\frac{j}{c}\right)\right)\right] = \\ &= \sum_{j=1}^{k-1} \left(\left(\frac{aj}{c}\right)\right) - \frac{k}{c}\sum_{j=1}^{c-1} \left(\left(\frac{aj}{c}\right)\right) + \left[\frac{k}{c} - \left(\left(\frac{k}{c}\right)\right)\right]\left(\left(\frac{ak}{c}\right)\right). \end{aligned}$$

The second summand in the last expression is 0, as $\left(\left(\frac{aj}{c}\right)\right) = -\left(\left(\frac{a(c-j)}{c}\right)\right)$, and thus all terms in the sum get canceled. By observing that

$$\frac{k}{c} - \left(\left(\frac{k}{c}\right)\right) = \frac{k}{c} - \left(\frac{k}{c} - \frac{1}{2}\right) = \frac{1}{2},$$

we obtain the desired result. □

Remark A.2.5. Expanding (A.9), we get

$$\sum_{j=1}^{k-1}\left(\left(\frac{aj}{c}\right)\right)+\frac{1}{2}\left(\left(\frac{ak}{c}\right)\right)=\sum_{j=1}^{k-1}\left(\frac{aj}{c}-\left\lfloor\frac{aj}{c}\right\rfloor-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{ak}{c}-\left\lfloor\frac{ak}{c}\right\rfloor-\frac{1}{2}\right)=$$

$$=\frac{ak(k-1)}{2c}-\sum_{j=1}^{k-1}\left\lfloor\frac{aj}{c}\right\rfloor-\frac{k-1}{2}+\frac{ak}{2c}-\frac{1}{2}\left\lfloor\frac{ak}{c}\right\rfloor-\frac{1}{4}=$$

$$=\frac{ak^2}{2c}-\frac{k}{2}+\frac{1}{4}-\sum_{j=1}^{k-1}\left\lfloor\frac{aj}{c}\right\rfloor-\frac{1}{2}\left\lfloor\frac{ak}{c}\right\rfloor.$$

Note that here the assumption $c > 0$ is not needed. This computation will turn useful in the study of rho invariants of lens spaces (see Corollary 3.3.25).

A.3 Some software-generated images

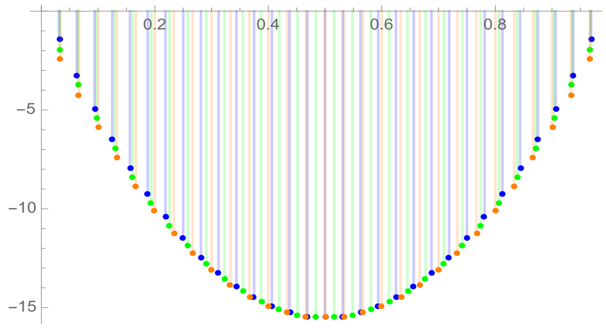
In Section 3.3.5, we introduced a family of 1-periodic functions $S_r : \mathbb{R} \to \mathbb{R}$, for $r \in \mathbb{Q} \cup \{\infty\}$, so that S_r corresponds to the rho invariant of the solid torus $D^2 \times S^1$ with a non-standardly oriented framing $\mathcal{F}_r$ of gradient r, for all possible representations $\alpha : \pi_1(D^2 \times S^1) \to U(1)$.

Because of the relations between different functions in this family, we have observed that it is enough to know S_r for $r > 1$ in order to recover all other functions. One way to proceed is to fix $p \in \mathbb{N} \setminus \{0, 1\}$, and consider the rational numbers $r = p/q$ for all integers q that are coprime with p and satisfy $1 \leq q \leq p - 1$. In this way, for each p we get a finite number of functions, and by letting p vary in $\mathbb{N} \setminus \{0, 1\}$ we recover all of them.

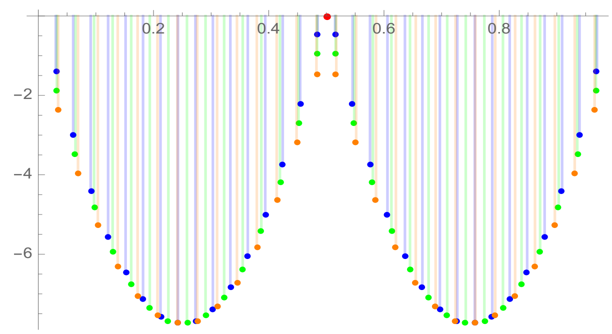
The general features of the finite sequence of functions corresponding to a choice of p seem not to vary by changing p. We illustrate next the sequence of 30 functions corresponding to the choice $p = 31$. For all $r = 31/q$ with $1 \leq q \leq 30$, we append the graph, generated with the software Mathematica, representing all the values of $S_r(t)$ for $t \in (0, 1)$ that we are able to compute, thanks to Corollary 3.3.32, as rho invariants of lens spaces. The color scheme is the following (see Notation 3.3.29):

- red dots represent $\ell(q, 31, k)$ for $1 \leq k \leq q - 1$;
- green dots represent $-\ell(31, q, k) - 1$ for $1 \leq k \leq 30$;
- blue dots represent $-\ell(31 + q, q, k) - 1/2$ for $1 \leq k \leq 30 + q$;
- orange dots represent $-\ell(p - q, q, k) - 3/2$ for $1 \leq k \leq 30 - q$.

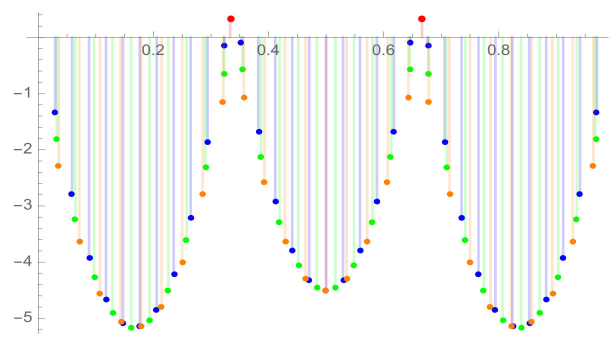
S_{31}



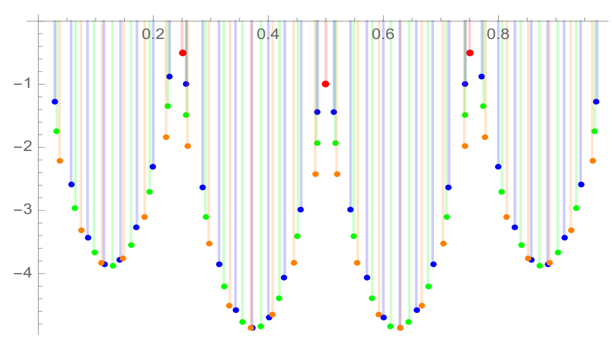
$S_{31/2}$



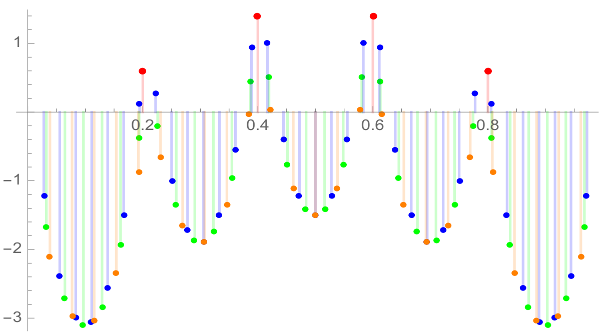
$S_{31/3}$



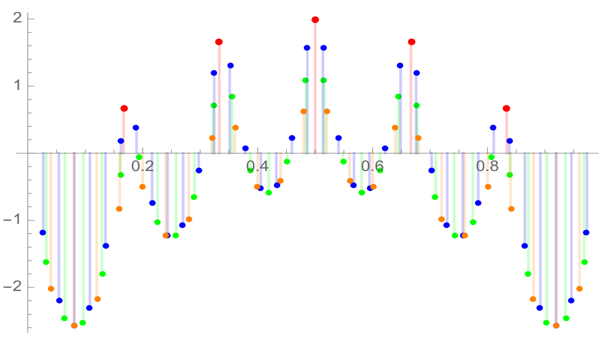
$S_{31/4}$



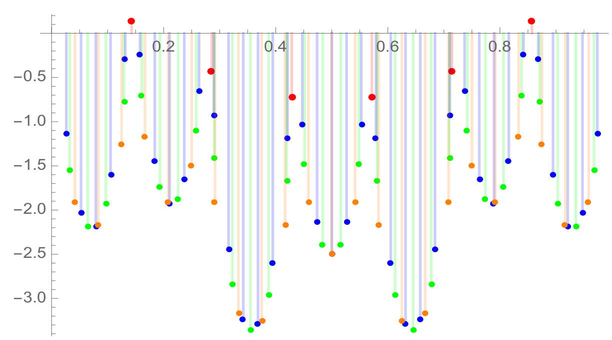
$S_{31/5}$



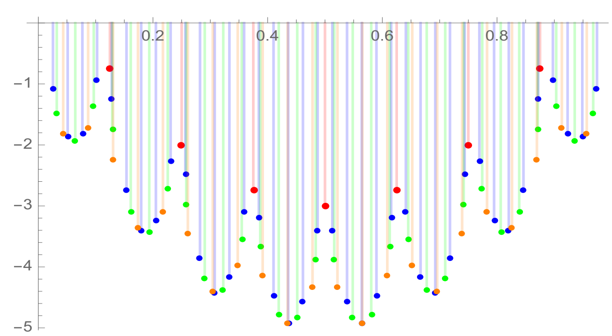
$S_{31/6}$



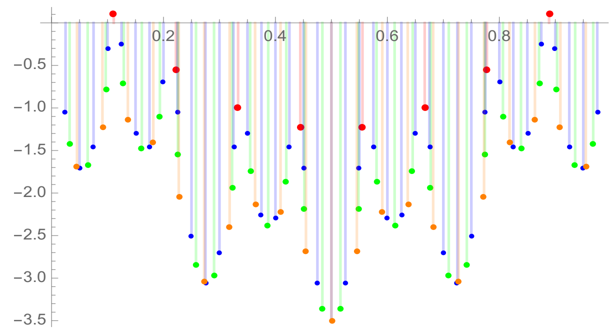
$S_{31/7}$



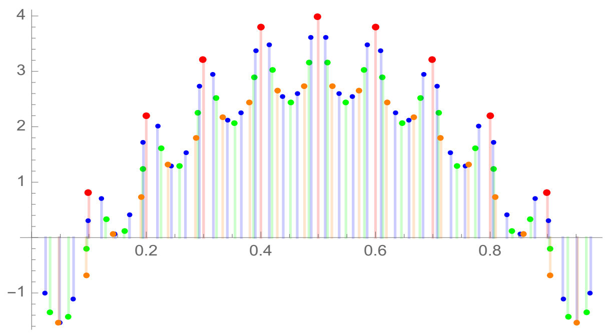
$S_{31/8}$



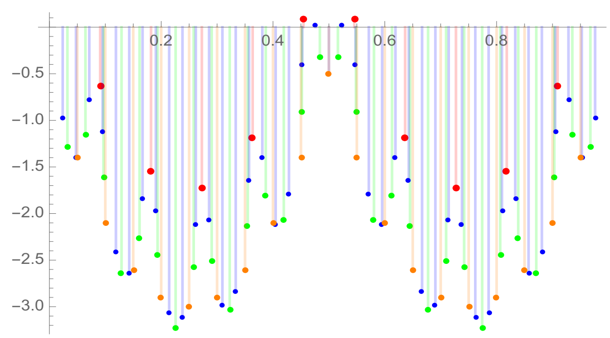
$S_{31/9}$



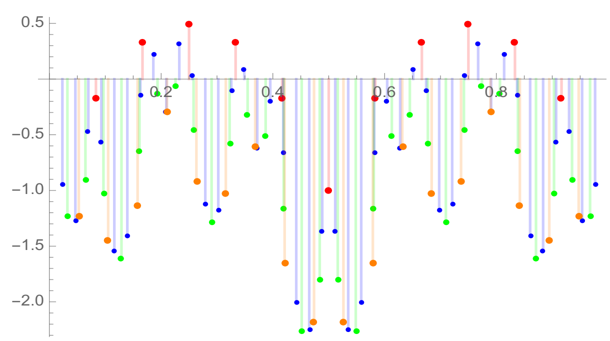
$S_{31/10}$



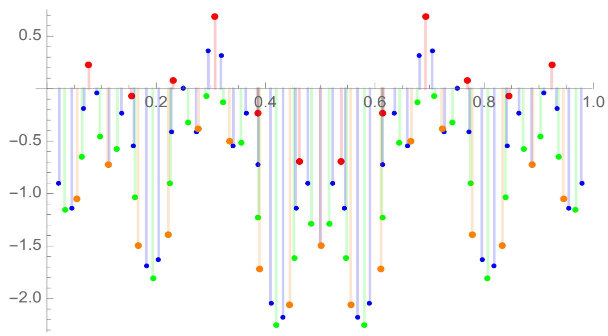
$S_{31/11}$



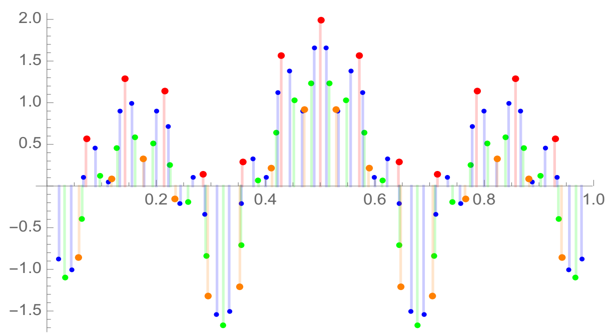
$S_{31/12}$



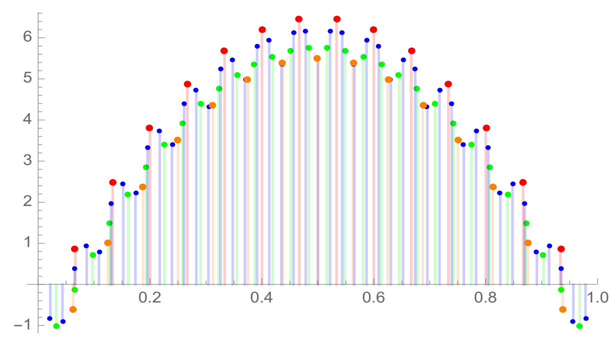
$S_{31/13}$



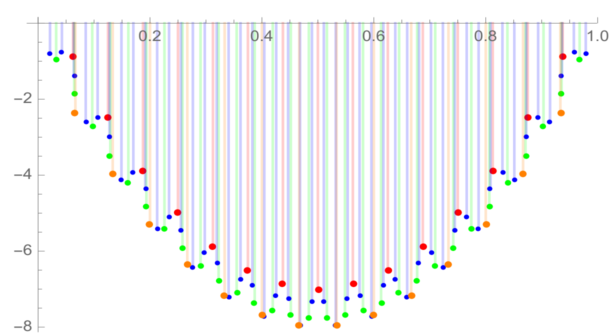
$S_{31/14}$



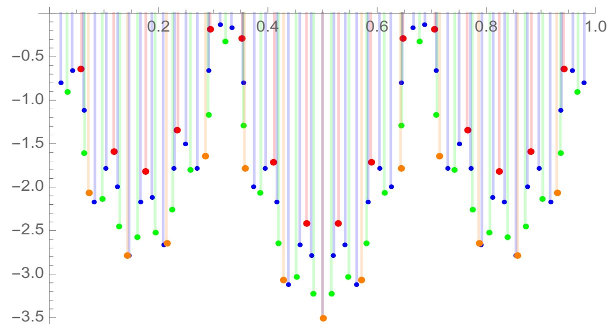
$S_{31/15}$



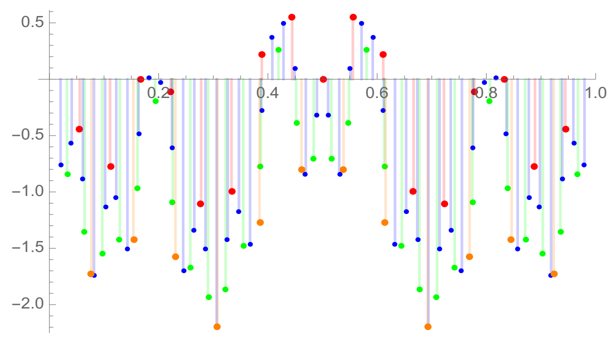
$S_{31/16}$



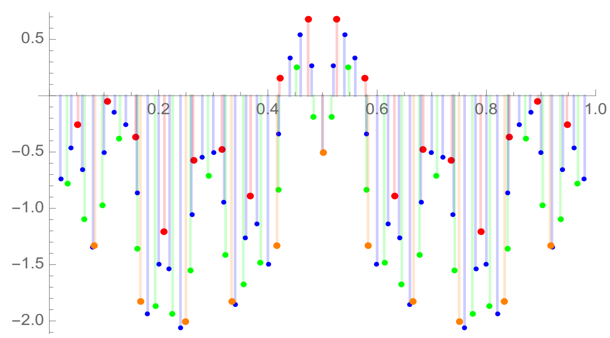
$S_{31/17}$



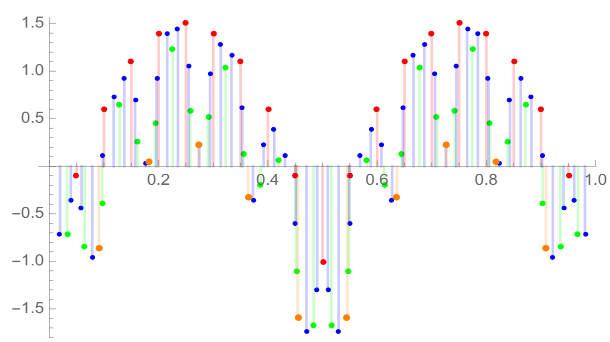
$S_{31/18}$



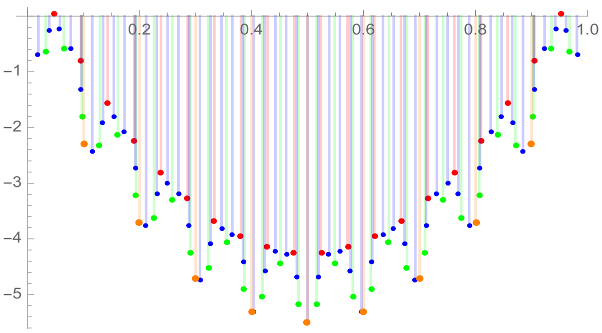
$S_{31/19}$



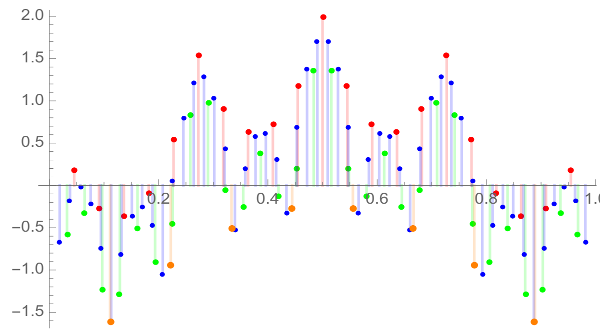
$S_{31/20}$



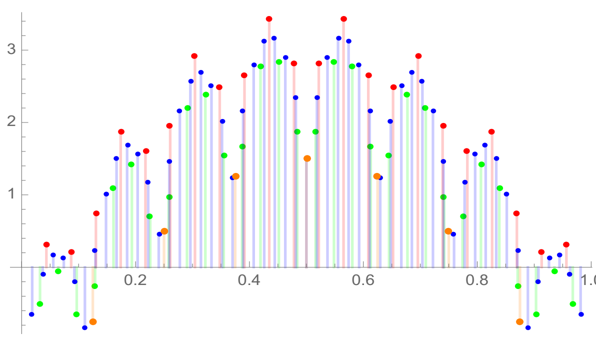
$S_{31/21}$



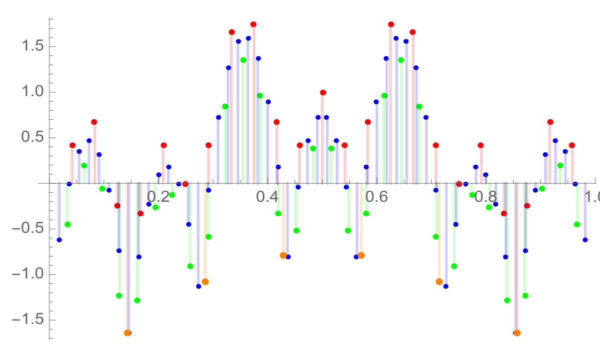
$S_{31/22}$



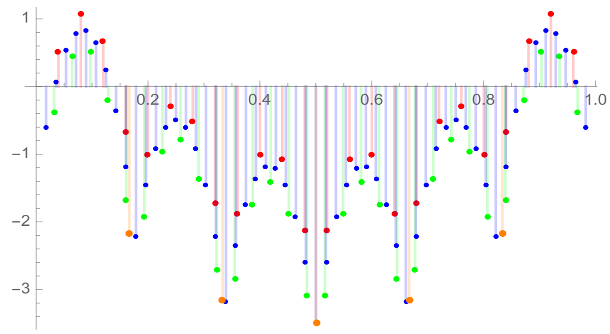
$S_{31/23}$



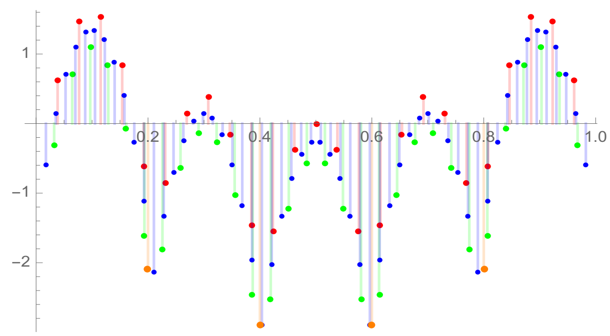
$S_{31/24}$



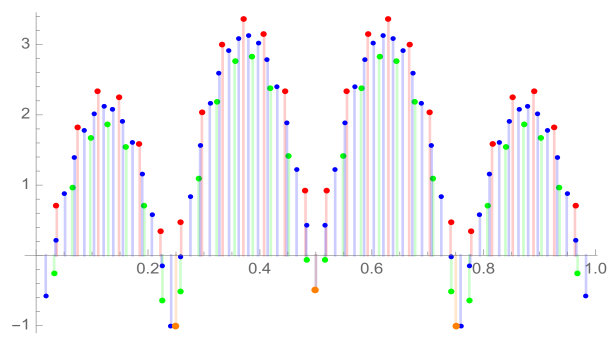
$S_{31/25}$



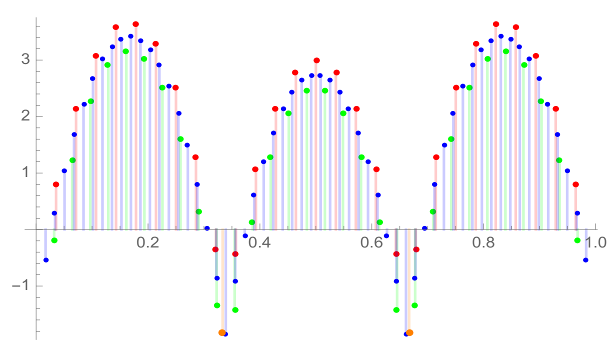
$S_{31/26}$



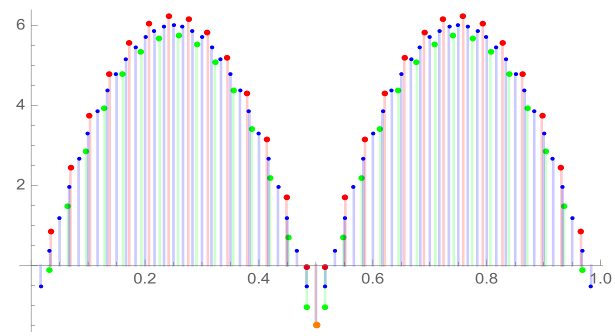
$S_{31/27}$



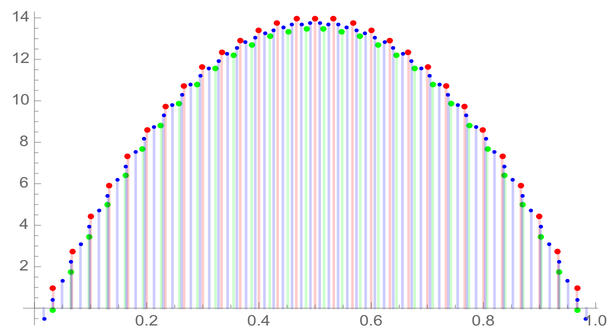
$S_{31/28}$



$S_{31/29}$



$S_{31/30}$



Bibliography

[1] M. Atiyah. The logarithm of the Dedekind η-function. *Math. Ann.*, 278(1-4):335–380, 1987.

[2] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[3] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. II. *Math. Proc. Cambridge Philos. Soc.*, 78(3):405–432, 1975.

[4] N. Berline, E. Getzler, and TITLE = Heat kernels and Dirac operators SERIES = Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] VOLUME = 298 PUBLISHER = Springer-Verlag, Berlin YEAR = 1992 PAGES = viii+369 ISBN = 3-540-53340-0 MRCLASS = 58G10 (58G11) MR-NUMBER = 1215720 MRREVIEWER = Alejandro Uribe DOI = 10.1007/978-3-642-58088-8 URL = http://dx.doi.org/10.1007/978-3-642-58088-8 Vergne, M.

[5] M. Bohn. *On Rho invariants of fiber bundles*. PhD thesis, University of Bonn, 2009. http://hss.ulb.uni-bonn.de/2009/1830/1830.htm.

[6] M. Borodzik and K. Oleszkiewicz. On the signatures of torus knots. *Bulletin of the Polish Academy of Sciences Mathematics*, 58, 02 2010.

[7] U. Bunke. On the gluing problem for the η-invariant. *J. Differential Geom.*, 41(2):397–448, 1995.

[8] U. Bunke. Algebraic Topology 1. 2015. Lecture notes, https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/Bunke/topos.pdf.

[9] S. E. Cappell, R. Lee, and E. Y. Miller. On the Maslov index. *Comm. Pure Appl. Math.*, 47(2):121–186, 1994.

[10] A. J. Casson and C. McA. Gordon. Cobordism of classical knots. In *À la recherche de la topologie perdue*, volume 62 of *Progr. Math.*, pages 181–199. Birkhäuser Boston, Boston, MA, 1986.

[11] A. J. Casson and C.McA. Gordon. On slice knots in dimension three. *Proc. Sympos. Pure Math.*, 32:339–53, 1978.

[12] D. Cimasoni and V. Florens. Generalized Seifert surfaces and signatures of colored links. *Trans. Amer. Math. Soc.*, 360(3):1223–1264 (electronic), 2008.

[13] D. Cimasoni and V. Turaev. A Lagrangian representation of tangles. *Topology*, 7:747 – 767, 2005.

[14] T. D. Cochran, K. E. Orr, and P. Teichner. Structure in the classical knot concordance group. *Comment. Math. Helv.*, 79(1):105–123, 2004.

[15] A. Conway. *Invariants of colored links and generalizations of the Burau representation*. PhD thesis, University of Geneva, 2017. `http://www.unige.ch/math/folks/conway/ThesisAnthonyConway2.pdf`.

[16] A. Conway, S. Friedl, and E. Toffoli. The Blanchfield pairing of colored links. *Indiana Univ. Math. J.*, 67:2151–2180, 2018.

[17] A. Conway, M. Nagel, and E. Toffoli. Multivariable signatures, genus bounds and 0.5-solvable cobordisms. 2017. Preprint `https://arxiv.org/abs/1703.07540`, to appear in Mich. Math. J.

[18] J. F. Davis and P. Kirk. *Lecture notes in algebraic topology*, volume 35 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[19] A. Degtyarev, V. Florens, and A. G. Lecuona. The signature of a splice. *Int. Math. Res. Not.*, 2016.

[20] A. Degtyarev, V. Florens, and A. G. Lecuona. Slopes and signatures of links. 2018. Preprint `https://arxiv.org/abs/1802.01836`.

[21] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov. *Modern Geometry-Methods and Applications. Part III. Introduction to Homology Theory.* Graduate Texts in Mathematics, vol. 124. Springer Verlag, 1990.

[22] S. Friedl. Eta invariants as sliceness obstructions and their relation to casson–gordon invariants. *Algebr. Geom. Topol.*, 4(2):893–934, 2004.

[23] S. Friedl. Link concordance, boundary link concordance and eta-invariants. *Math. Proc. Cambridge Philos. Soc.*, 138(3):437–460, 2005.

[24] S. Friedl. Algebraic Topology I+II+III+III.5. 2018. Lecture notes, https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/friedl/papers/2018_algebraic-topology-iv-september-27.pdf.

[25] P. M. Gilmer. Configuration of surfaces in 4-manifolds. *Trans. Amer. Math. Soc.*, 264(2):333–380, 1981.

[26] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[27] R. Kirby and P. Melvin. Dedekind sums, μ-invariants and the signature cocycle. *Math. Ann.*, 299(2):231–267, 1994.

[28] P. Kirk and M. Lesch. On the rho invariant for manifolds with boundary. *Algebr. Geom. Topol.*, 3:623–675, 2003.

[29] P. Kirk and M. Lesch. The η-invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary. *Forum Math.*, 16(4):553–629, 2004.

[30] M. Lesch and K. P. Wojciechowski. *Illinois J. Math.*, (1):30–46, 03.

[31] J. P. Levine. Link invariants via the eta invariant. *Comment. Math. Helv.*, 69(1):82–119, 1994.

[32] J. P. Levine. Concordance of boundary links. *J. Knot Theory Ramifications*, 16(9):1111–1120, 2007.

[33] R. A. Litherland. Signatures of iterated torus knots. In *Topology of low-dimensional manifolds*, volume 62, pages 71–84. Springer-Verlag, 1979.

[34] W. Meyer. Die Signatur von Flächenbündeln. *Math. Ann.*, 201:239–264, 1973.

[35] L. Moser. Elementary surgery along a torus knot. *Pacific J. Math.*, 38(3):737–745, 1971.

[36] W. D. Neumann. Signature related invariants of manifolds. I. Monodromy and γ-invariants. *Topology*, 18(2):147–172, 1979.

[37] W. D. Neumann. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *Trans. Amer. Math. Soc.*, 268(2):299–344, 1981.

[38] P. Py. Indice de Maslov et théorème de Novikov-Wall. *Bol. Soc. Mat. Mexicana*, 11(2):303–331, 2005.

[39] H. Rademacher. Some remarks on certain generalized Dedekind sums. *Acta Arith.*, 9:97–105, 1964.

[40] H. Rademacher and E. Grosswald. *Dedekind sums*. The Mathematical Association of America, Washington, D.C., 1972. The Carus Mathematical Monographs, No. 16.

[41] S. Ramanan. *Global calculus*, volume 65 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[42] D. Rolfsen. *Knots and links*. Publish or Perish, Inc., Berkeley, Calif., 1976. Mathematics Lecture Series, No. 7.

[43] N. Steenrod. *The Topology of Fibre Bundles*. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951.

[44] T. tom Dieck. *Algebraic Topology*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[45] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[46] O. Viro. Twisted acyclicity of a circle and signatures of a link. *J. Knot Theory Ramifications*, 18(6):729–755, 2009.

[47] C. T. C. Wall. Non-additivity of the signature. *Invent. Math.*, 7:269–274, 1969.