On continuous K-theory and cohomology of rigid spaces

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Zusammenfassung

In dieser Dissertation wird ein Zusammenhang zwischen stetiger K-Theorie auf der einen und der Kohomologie rigider Räume auf der anderen Seite etabliert.

Für einen starr-analytischen Raum endlicher Dimension über einem vollständigen diskret bewerteten Körper verschwindet dessen stetige K-Theorie stets in den Graden unterhalb des Negativen der Dimension. Ebenfalls verschwinden stets die Kohomologiegruppen in den Graden oberhalb der Dimension.

Bezüglich dieser Schranken besagt das Hauptresultat vorliegender Abhandlung, dass es einen Isomorphismus zwischen der niedrigsten möglicherweise nicht verschwindenden stetigen K-Gruppe und der höchsten möglicherweise nicht verschwindenden Kohomologiegruppe mit Koeffizienten in den ganzen Zahlen gibt.

Eine entscheidende Rolle im Beweis des Hauptresultates spielt ein Vergleich von Kohomologiegruppen eines Zariski-Riemann-Raumes bezüglich verschiedener Topologien; und zwar der RH-Topologie, die einen Bezug zur K-Theorie hat, sowie der Zariski-Topologie, die besagten Kohomologiegruppen zugrunde liegt.

Summary

This thesis establishes a connection between continuous K-theory on the one hand and cohomology of rigid spaces on the other hand.

Given an rigid analytic space over a complete discretely valued field, its continuous K-groups vanish in degrees below the negative of the dimension. Likewise, the cohomology groups vanish in degrees above the dimension.

The main result of this thesis provides the existence of an isomorphism between the lowest possibly non-vanishing continuous K-group and the highest possibly non-vanishing cohomology group with integral coefficients.

A key role in the proof is played by a comparison between cohomology groups of a Zariski-Riemann space with respect to different topologies; namely, the rh-topology which is related to K-theory as well as the Zariski topology whereon the cohomology groups in question rely.

Résumé

Dans cette thèse on établit une relation entre la K-théorie continue d'une part et la cohomologie des espaces rigides d'autre part.

En général, pour un espace rigide analytique au-dessus d'un corps complet relativement à une valuation discrète, les K-groupes continus s'annulent en degré strictement inférieur à l'opposé de sa dimension. Dans le même ordre d'idée, les groupes de cohomologie s'annulent en degré strictement supérieur à la dimension.

Se referent à ce borne, le résultat principal de cet traité énonce un isomorphisme entre la K-théorie continue en degré minimal éventuellement non nulle et le groupe de cohomologie à coefficients entiers en degré maximal éventuellement non nulle.

Un rôle capital dans la démonstration est joué par une comparaison des groupes de cohomologie d'un espace Zariski-Riemann relativement à diverses topologies, à savoir la topologie rh qui est relié à la K-théorie ainsi que la topologie de Zariski qui est à la base des groupes de cohomologie en question.

Sintesi

In questa tesi viene stabilita una connessione tra la K-teoria continua da una parte e la coomologia di spazi rigidi dall'altra.

In generale, per un spazio rigido analitico su un campo completo relativamente a una valuazione discreta, i K-gruppi continui si annullano in grado minore meno la dimensione. Similmente, i gruppi di coomologia si annullano in gradi al di sopra della dimensione.

Rapportantesi a queste limitazioni, il risultato principale della presente trattazione enuncia un isomorfismo tra la K-teoria continua al grado più basso possibilmente non evanescente ed il gruppo di coomologia con coefficienti interi al grado più alto possibilmente non evanescente.

Un ruolo decisivo nella dimostrazione è giocato da un confronto dei gruppi di coomologia di uno spazio di Zariski-Riemann, relativamente a topologie diverse; e cioè la topologia rh, che è in relazione alla K-teoria, come anche la topologia di Zariski, che è alla base dei gruppi di coomologia anzidetti.

Für meine Brüder

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Introduction

The negative algebraic K-theory of a scheme is related to its singularities. If X is a regular scheme, then $\mathrm{K}_{-i}(X)$ vanishes for $i > 0$. For an arbitrary noetherian scheme X of dimension d we know that

(i) $\mathrm{K}_{-i}(X) = 0$ for $i > d$,

(ii) $\mathrm{K}_{-d}(X) \cong \mathrm{K}_{-d}(\mathbf{A}^n_X)$ for $n \geq 1$, and

(iii) $\mathrm{K}_{-d}(X) \cong \mathrm{H}^d_{\mathrm{cdh}}(X;\mathbf{Z})$.

The cdh-cohomology group $\mathrm{H}^d_{\mathrm{cdh}}(X,\mathbf{Z})$ in question describes (in nice cases) the combinatorics of the irreducible components of X. The affine case of (i) was a question of Weibel [Wei80, 2.9] who proved (i) and (ii) for $d \leq 2$ [Wei01, 2.3, 2.5, 4.4]. For varieties in characteristic zero (i)-(iii) were proven by Cortiñas-Haesemeyer-Schlichting-Weibel [CHSW08] and for any characteristic by Kerz-Strunk-Tamme [KST18b]. As an example for the lowest possibly non-vanishing group $\mathrm{K}_{-d}(X)$, the cusp $C = \{y^2 = x^3\}$ over a field has $\mathrm{K}_{-1}(C) = 0$ whereas the node $N = \{y^2 = x^3 + x^2\}$ over a field (of characteristic not 2) has $\mathrm{K}_{-1}(N) = \mathbf{Z}$; more generally, for a nice curve the rank is the number of loops [Wei01, 2.3]. The main result of the present thesis is an analogous statement of (iii) for *continuous K-theory* of rigid analytic spaces in the sense of Morrow [Mor16].

There is a long history of versions of K-theory for topological rings that take the topology into account. For instance, the higher algebraic K-groups of a ring A can be defined via the classifying space $\mathrm{BGL}(A)$ of the general linear group $\mathrm{GL}(A)$. If A happens to be a Banach algebra over the complex numbers, it also makes sense to consider $\mathrm{GL}(A)$ as a topological group and to define *topological K-theory* $\mathrm{K}^{\mathrm{top}}(A)$ analoguously in terms of the classifying space $\mathrm{BGL}^{\mathrm{top}}(A)$. This yields a better behaved K-theory for complex Banach algebras which satisfies homotopy invariance and excision (which does not hold true in general for algebraic K-theory). Unfortunately, a similar approach for *nonarchimedean* algebras does not behave well since the nonarchimedean topology is totally disconnected. Karoubi-Villamayor [KV71] and Calvo [Cal85] generalised topological K-theory to arbitrary Banach algebras (either nonarchimedean or complex) in terms of the ring of power series converging on a unit disc. A different approach is to study **continuous K-theory** which is the limit

$$\mathrm{K}^{\mathrm{cont}}(R) = \lim_n \mathrm{K}(R/I^n)$$

where R is an I-adic ring with respect to some ideal $I \subset R$ (e.g. $\mathbf{Z}_p$ with the p-adic topology or $\mathbf{F}_p[\![t]\!]$ with the t-adic topology). Such "continuous" objects have been studied amply in the literature – cf. Wagoner [Wag76a, Wag76b], Dundas [Dun98], Geisser-Hesselholt [GH06a, GH06b], or Beilinson [Bei14] – and they were related by Bloch-Esnault-Kerz to the Hodge conjecture for abelian varieties [BEK14a] and the p-adic variational Hodge conjecture [BEK14b]. Morrow [Mor16] suggested an extension of continuous K-theory to rings A admitting an open subring A_0 which is I-adic with respect to some ideal I of A_0 (e.g. $\mathbf{Q}_p = \mathbf{Z}_p[p^{-1}]$ or $\mathbf{F}_p((t)) = \mathbf{F}_p[\![t]\!][t^{-1}]$).[1] This notion was recently studied by Kerz-Saito-Tamme [KST18a] and they showed that it coincides in non-positive degrees with the groups studied by Karoubi-Villamayor and Calvo. For an affinoid algebra A over a discretely valued field, Kerz proved the corresponding analytical statements to (i) and (ii); that is replacing algebraic K-theory by continuous K-theory and the polynomial ring by the ring of power series converging on a unit disc [Ker18]. Continuous K-theory extends to a sheaf on rigid k-spaces for any discretely valued field k. Analoguously to the isomorphism (iii) above, the main result of this thesis describes the edge degree of continuous K-theory of a rigid k-space.

Theorem A (Theorem 3.2.14)**.** *Let X be a quasi-compact and quasi-separated rigid k-space of dimension d over a discretely valued field k. Assume that $d \geq 2$ or that there exists a formal model of X which is algebraic (e.g. X is affinoid or projective). Then there is an isomorphism*

$$\mathrm{K}_{-d}^{\mathrm{cont}}(X) \cong \mathrm{H}^d(X;\mathbf{Z})$$

where the right-hand side is sheaf cohomology with respect to the admissible topology on the category of rigid k-varieties.

There are several approaches to nonarchimedean analytic geometry. Our proof uses rigid analytic spaces in the sense of Tate [Tat71] and adic spaces introduced by Huber [Hub94]. Another approach is the one of Berkovich spaces [Ber90] for which there is also a version of our main result as conjectured in the affinoid case by Kerz [Ker18, Conj. 14].

Corollary B (Corollary 3.1.2)**.** *Let X be a quasi-compact and quasi-separated rigid analytic space of dimension d over a discretely valued field. Assume that $d \geq 2$ or that there exists a formal model of X which is algebraic (e.g. X is affinoid or projective). Then there is an isomorphism*

$$\mathrm{K}_{-d}^{\mathrm{cont}}(X) \cong \mathrm{H}^d(X^{\mathrm{berk}};\mathbf{Z})$$

where X^{berk} is the Berkovich space associated with X.

If X is smooth over k or the completion of a k-scheme of finite type, then there is an isomorphism

$$\mathrm{H}^d(X^{\mathrm{berk}};\mathbf{Z}) \cong \mathrm{H}^d_{\mathrm{sing}}(X^{\mathrm{berk}};\mathbf{Z})$$

[1] Actually, Morrow does only talk about affinoid algebras.

with singular cohomology by results of Berkovich [Ber99] and Hrushovski-Loeser [HL16]. The identification of Corollary B is very helpful since it is hard to actually compute K-groups whereas the cohomology of Berkovich spaces is amenable for computations. For instance, the group $\mathrm{H}^d(X^{\mathrm{berk}};\mathbf{Z})$ is finitely generated since X^{berk} has the homotopy type of a finite CW-complex; such a finiteness statement is usually unknown for K-theory.

An important tool within the proof of Theorem A is the **Zariski-Riemann space** $\langle X\rangle_U$ which we will associate, more generally, with every quasi-compact and quasi-separated scheme X with open subscheme U. The Zariski-Riemann space $\langle X\rangle_U$ is given by the limit of all U-modifications of X in the category of locally ringed spaces (Definition 2.1.1). In our case of interest where A is an affinoid algebra and A° its open subring of power-bounded elements, then we will set $X = \mathrm{Spec}(A^\circ)$ and $U = \mathrm{Spec}(A)$. We shall relate its Zariski cohomology to the cohomology with respect to the so-called **rh-topology**, i.e. the minimal topology generated by the Zariski topology and abtract blow-up squares (Definition 2.2.2). To every topology τ on the category of schemes (e.g. Zar, Nis, rh, cdh), there is a corresponding appropriate site $\mathrm{Sch}_\tau(\langle X\rangle_U)$ for the Zariski-Riemann space (Definition 2.3.10). We show the following which is later used in the proof of Theorem A.

Theorem C (Theorem 2.3.16)**.** *For every constant abelian rh-sheaf F on $\mathrm{Sch}(\langle X\rangle_U)$ the canonical map*

$$\mathrm{H}^*_{\mathrm{Zar}}(\langle X\rangle_U \smallsetminus U;F) \longrightarrow \mathrm{H}^*_{\mathrm{rh}}(\langle X\rangle_U \smallsetminus U;F)$$

is an isomorphism. In particular,

$$\operatorname*{colim}_{X'\in\mathrm{Mdf}(X,U)} \mathrm{H}^*_{\mathrm{Zar}}(X' \smallsetminus U;F) = \operatorname*{colim}_{X'\in\mathrm{Mdf}(X,U)} \mathrm{H}^*_{\mathrm{rh}}(X' \smallsetminus U;F).$$

where $\mathrm{Mdf}(X,U)$ is the category of all U-modifications of X and $X' \smallsetminus U$ is equipped with the reduced scheme structure. The same statement also holds if one replaces 'Zar' by 'Nis' and 'rh' by 'cdh'.

Let A be a Tate ring with ring of definition A_0 and pseudo-uniformiser π (Definition 1.3.3), e.g. A an affinoid k-algebra, $A_0 = A^\circ$, and $\pi \in k$ such that $|\pi| < 1$. For this we show the following statement which is independent of Theorem C and not relevant for Theorem A.

Theorem D (Theorem 2.4.10)**.** *We have an equivalence of spectra*

$$\mathrm{K}\big(\langle A_0\rangle_A \text{ on } \pi\big) \longrightarrow \mathrm{K}\big(\mathrm{Coh}(\langle A_0\rangle_A/\pi)\big).$$

We also show an rh-version of a cdh-result of Kerz-Strunk-Tamme [KST18b, 6.3]. This is not a new proof but the observation that the analogous proof goes through. The statement will enter in the proof of Theorem A.

Theorem E (Theorem 2.2.14). *Let X be a finite dimensional noetherian scheme. Then the canonical maps of rh-sheaves with values in spectra on* Sch_X

$$\mathrm{L_{rh}\,K_{\geq 0}} \longrightarrow \mathrm{L_{rh}\,K} \longrightarrow \mathrm{KH}$$

are equivalences.

Sketch of the proof of the main result

We shall briefly sketch the proof of Theorem A in the affinoid case (Theorem 3.1.1). For every affinoid algebra A and every model $X' \to \mathrm{Spec}(A^\circ)$ over the subring A° of power-bounded elements with pseudo-uniformiser π there exists a fibre sequence [KST18a, 5.8]

$$\mathrm{K}(X' \text{ on } \pi) \longrightarrow \mathrm{K}^{\mathrm{cont}}(X') \longrightarrow \mathrm{K}^{\mathrm{cont}}(A).$$

For $n < 0$ and $\alpha \in \mathrm{K}_n(X' \text{ on } \pi)$ there exists by Raynaud-Gruson's *platification par éclatement* an admissible blow-up $X'' \to X'$ such that the pullback of α vanishes in $\mathrm{K}_n(X'' \text{ on } \pi)$ [Ker18, 7]. In the colimit over all models this yields that $\mathrm{K}_n^{\mathrm{cont}}(A) \cong \mathrm{K}_n^{\mathrm{cont}}(\langle A_0 \rangle_A)$. For $d = \dim(A)$ we have $\mathrm{K}_d^{\mathrm{cont}}(\langle A_0 \rangle_A) \cong \mathrm{K}_d(\langle A_0 \rangle_A/\pi)$ and the latter is isomorphic to $\mathrm{H}^d_{\mathrm{rh}}(\langle A_0 \rangle_A/\pi; \mathbf{Z})$ via a descent spectral sequence argument (Theorem 2.2.18). Using Theorem C (Theorem 2.3.16) we can pass to Zariski cohomology. Now the result follows from identifying $\langle A^\circ \rangle_A$ with the adic spectrum $\mathrm{Spa}(A, A^\circ)$ (Theorem 2.5.7).

Leitfaden

In chapter 1 we present the definition of continuous K-theory (Definition 1.3.5) after some recollections on algebraic K-theory and on pro-objects. Afterwards, in section 1.5, a short proof of the main result under the additional assumption of regularity and resolution of singularities will be explained (Theorem 1.5.2). This proof relies on the existence of a nice regular model and uses the framework of Berkovich skeleta which will be sketched prior to the proof in section 1.4.

In chapter 2 we study Zariski-Riemann spaces. They play the role of a substitute for nice regular models so that we do not have to assume resolution of singularities. First, we introduce schematic Zariski-Riemann spaces (§2.1). Then we make a détour to the rh-topology and give a proof of Theorem E (Theorem 2.2.14) which will be an ingredient for the main result's proof. The heart of this thesis is section 2.3 where we compare the cohomology of Zariski-Riemann spaces culminating in the proof of Theorem C (Theorem 2.3.16). The key step for this is that any rh-cover – after some admissible pullback – can be refined by a closed cover (Proposition 2.3.9). Afterwards, we treat briefly the K-theory of Zariski-Riemann spaces (§2.4) including a proof of Theorem D (Theorem 2.4.10). Finally, we will identify admissible Zariski-Riemann spaces with formal Zariski-Riemann spaces and the latter ones with adic spaces (Theorem 2.5.7).

Chapter 3 contains the proof of the main result Theorem A. In first instance, we proof the result for affinoid algebras (Theorem 3.1.1). Afterwards we conjecture the global case (Conjecture 3.2.1) and prove it in dimension at least 2 or in the algebraic case (Theorem 3.2.14) by reduction.

Appendix A is dedicated to the exhibition of the theory of sheaves of spaces and spectra for topologies of cd-structures. This merely rephrases Voevodsky's work [Voe10a, Voe10b] in a hopefully more accessible account and with modern language. Being of homotopical nature, it is convenient to express this content using the language of ∞-categories. More precisely, we will refer to Lurie's *Higher Topos Theory* [Lur09].

Finally, in Appendix B we state some facts on limits of locally ringed spaces and on topologies which are used in the main text.

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General remarks

- In definitions we write "iff" as an abbreviation for the phrase "if and only if". In statements however, we desist from this practice in order to avoid possible misunderstandings.
- We assume a model $\mathbb{V}$ for ZFC-set theory satisfying the large cardinal axiom. There exists a universe $\mathbb{U} \in \mathbb{V}$ [Bou72] satisfying itself the axioms of ZFC [Wil69]. The term "small set" refers to a set in $\mathbb{U}$ whereas the term "large set" refers to a set in $\mathbb{V}$ which is not in $\mathbb{U}$. Denote by Set the category of small sets.
- The term "(discrete) category" describes a 1-category, i.e. a (possibly large) set of objects together with, for any two objects, a (possibly large) set of morphisms between them and composition maps satisfying the usual conditions. A category is said to be "small" iff all sets in question are small sets.
- The term "∞-category" refers to the notion of an $(\infty,1)$-category introduced by Boardman-Vogt [BV73] (as "weak Kan complexes") and further studied by Joyal [Joy08] (as "quasi-categories") and Lurie [Lur09]. Every discrete category can be seen as an ∞-category via the nerve functor which is usually omitted in the notation.
- Discrete categories are denoted by upright letters whereas genuine ∞-categories are denoted by **bold letters**.
- The term "space" refers to an object of the ∞-category **Spc** of spaces [Lur09, 1.2.16.1]. whereas the term "topological space" refers to an object of the discrete category Top of topological spaces.
- In an ∞-category, commutativity is not a property, but a structure. A "commutative diagram" implies the choice of a homotopy.
- Given a scheme X we denote by Sch_X the category of separated schemes of finite type over X. If X is noetherian, then every scheme in Sch_X is noetherian as well.

1. Continuous K-theory

1.1. Algebraic K-theory

Algebraic K-theory emerged from Grothendieck's work on a generalisation of the Riemann-Roch theorem [BGI71, pp. 20-77] where he defined what is nowadays called the *Grothendieck group* $\mathrm{K}_0(X)$ of a scheme X. Bass defined a group $\mathrm{K}_1(R)$ for a ring R [Bas64, §12] and proved the *Fundamental Theorem*, saying that the sequence

$$0 \to K_1(R) \xrightarrow{\Delta} \mathrm{K}_1(R[t]) \oplus \mathrm{K}_1(R[t^{-1}]) \xrightarrow{\pm} \mathrm{K}_1(R[t,t^{-1}]) \to \mathrm{K}_0(R) \to 0$$

is exact [Bas68, VII (7.4)]. Motivated by this he also introduced negative algebraic K-groups $\mathrm{K}_{-n}(R)$ for $n \geq 1$ iteratively defined as the cokernel of the map

$$\mathrm{K}_{-n+1}(R[t]) \oplus \mathrm{K}_{-n+1}(R[t^{-1}]) \xrightarrow{\pm} \mathrm{K}_{-n+1}(R[t,t^{-1}]).$$

Higher algebraic theory was invented by Quillen [Qui73] who constructed a K-theory space $\mathrm{K}^{\mathrm{Q}}(\mathcal{A})$ for an exact category $\mathcal{A}$. For a scheme X, the homotopy groups of the space $\mathrm{K}^{\mathrm{Q}}(X) := \mathrm{K}^{\mathrm{Q}}(\mathrm{Vec}(X))$ in degree 0 and 1 coincide with the groups $\mathrm{K}_0(R)$ and $K_1(R)$. Another construction of a K-theory space was given by Waldhausen [Wal85]. Based on the latter one, Thomason-Trobaugh [TT90, §3] constructed their connective K-theory spectrum $\mathrm{K}_{\geq 0}(X)$ for a scheme X in terms of the derived category $\mathrm{Perf}(X)$ of perfect complexes [TT90, 2.2.10]. If X has an ample family of line bundles, then $\mathrm{K}_{\geq 0}(X)$ is equivalent to $\mathrm{K}^{Q}(X)$ [TT90, 3.10]. Every scheme which is quasi-projective over an affine scheme has an ample family of line bundles [TT90, 2.1.2]. Mimicking Bass' definition, Thomason-Trobaugh delooped the spectrum $\mathrm{K}_{\geq 0}(X)$ to a nonconnective K-theory spectrum $\mathrm{K}(X)$ whose negative homotopy groups coincide in the affine case with Bass' negative K-groups. Schlichting [Sch06] defined an axiomatic framework for the negative K-theory of exact categories. In the spirit of the constructions of Waldhausen and Schlichting, Blumberg-Gepner-Tabuada defined nonconnective K-theory as a functor

$$\mathrm{K}\colon \mathbf{Cat}_\infty^{\mathrm{ex}} \longrightarrow \mathbf{Sp}$$

from the ∞-category of small stable ∞-categories with exact functors to the ∞-category of spectra [BGT13, §7.1, §9.1]. Furthermore, they gave a universal characterisation of this functor as the universal localising invariant with values in spectra [BGT13, 9.8].

Definition 1.1.1. Let X be a scheme. We denote by $\mathrm{K}(X)$ the nonconnective K-theory spectrum $\mathrm{K}(\mathbf{Perf}(X))$ à la Blumberg-Gepner-Tabuada associated with the ∞-category $\mathbf{Perf}(X)$ of perfect complexes on X. For a ring A, we write $\mathrm{K}(A)$ denoting $\mathrm{K}(\mathrm{Spec}(A))$. For $i \in \mathbf{Z}$ we denote by $\mathrm{K}_i(X)$ and $\mathrm{K}_i(A)$ the i-th homotopy group of $\mathrm{K}(X)$ and $\mathrm{K}(A)$, respectively.

Remark 1.1.2. For a scheme X, the homotopy category $\mathrm{Ho}(\mathbf{Perf}(X))$ is equivalent to the derived category of perfect complexes $\mathrm{Perf}(X)$ and the K-theory spectrum $\mathrm{K}(X)$ is equivalent to the one constructed by Thomason-Trobaugh. Every scheme in this treatise is quasi-projective over an affine scheme, hence admits an ample family of line bundles. Thus K-theory can be computed in terms of the category $\mathrm{Vec}(X)$ of vector bundles (i.e. locally free $\mathcal{O}_X$-modules). In view of Bass' Fundamental Theorem, for $n \geq 1$ the group $\mathrm{K}_{-n}(X)$ is a quotient of $\mathrm{K}_0(X \times \mathbf{G}_m^n)$ wherein elements coming from $\mathrm{K}_0(X \times \mathbf{A}^n)$ vanish.

For a first account of algebraic K-theory we refer the reader to Schlichting's introductory article [Sch11]. A very detailed presentation is Weibel's *K-book* [Wei13]; we cite general facts from this source. Beyond that, we only need the following lemma for our concerns.

Lemma 1.1.3. *Let Y be a noetherian scheme of finite dimension d. Then for $n \geq d$ we have*

$$\mathrm{K}_{-n}(Y) \cong \mathrm{K}_{-n}(Y_{\mathrm{red}}).$$

Proof. Let X be a noetherian scheme of finite dimension d. We consider the Zariski-descent spectral sequence [TT90, Thm. 10.3]

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}(X; \mathcal{K}_q) \Rightarrow \mathrm{K}_{q-p}(X)$$

where $\mathcal{K}_q$ is the Zariski sheafification of the presheaf $U \mapsto \mathrm{K}_q(U)$ on Sch_X.

Claim. For $q \leq 0$ and for all $p \geq 0$ we have $\mathrm{H}^p_{\mathrm{Zar}}(X; \mathcal{K}_q) \cong \mathrm{H}^p_{\mathrm{Zar}}(X_{\mathrm{red}}; \mathcal{K}_q)$.
Let $q \leq 0$. The K-groups K_q are nil-invariant, i.e. $\mathrm{K}_q(A) \cong \mathrm{K}_q(A_{\mathrm{red}})$ where $A_{\mathrm{red}} = A/\mathrm{nil}(A)$ where $\mathrm{nil}(A)$ is the nil-radical of A. By [TT90, Rem. 10.4], we have $(\mathcal{K}_q)_x \cong \mathrm{K}_q(\mathcal{O}_{X,x})$. Now let $\iota\colon X_{\mathrm{red}} \to X$ be the inclusion of the reduced subscheme. The underlying space map is a homeomorphism and the map $\iota^{-1}\mathcal{K}_q \to \mathcal{K}_q$ of sheaves on X_{red} is an isomorphism since for $x = \iota(x) \in X_{\mathrm{red}} = X$ the equality

$$(\iota^{-1}\mathcal{K}_q)_x = (\mathcal{K}_q)_{\iota(x)} = \mathrm{K}_q(\mathcal{O}_{X,\iota(x)}) = \mathrm{K}_q((\mathcal{O}_{X,\iota(x)})_{\mathrm{red}}) = \mathrm{K}_q(\mathcal{O}_{X_{\mathrm{red}},x}) = (\mathcal{K}_q)_x$$

holds. This shows the claim. Hence the whole $(q \leq 0)$-half of the spectral sequence does only depend on the reduced structure. Being interested in $\mathrm{K}_{-n}(X)$ for $n \geq d$, all contributing cohomology groups $\mathrm{H}^p_{\mathrm{Zar}}(X, \mathcal{K}_q)$ in question only depend on the reduced structure due to the claim. This shows the lemma. □

Remark 1.1.4. Cracking a nut with a sledgehammer, Lemma 1.1.3 follows also instantly from the vanishing of negative K-theory for noetherian schemes below the dimension [KST18b, Thm B] and the identification $\mathrm{K}_{-d}(X) \cong \mathrm{H}^d_{\mathrm{cdh}}(X;\mathbf{Z})$ [KST18b, Cor. D].

1.2. Pro-objects

In this section, we briefly recall the notion of pro-objects and, in particular, of pro-spectra. The content of this section is taken from Kerz-Saito-Tamme [KST18a, §2] where the interested reader may find proofs or more detailled references.

Given an ∞-category $\mathcal{C}$ which is assumed to be accessible [Lur09, § 5.4] and to admit limits, one can built its **pro-category**

$$\mathbf{Pro}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex,acc}}(\mathcal{C},\mathbf{Spc})^{\mathrm{op}}$$

where $\mathrm{Fun}^{\mathrm{lex,acc}}(\mathcal{C},\mathbf{Spc})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathbf{Spc})$ consisting of functors which are accessible (i.e. preserve κ-small colimits for some regular cardinal number κ) and left-exact (i.e. commute with finite limits). The category $\mathbf{Pro}(\mathcal{C})$ has finite limits and, if $\mathcal{C}$ has, also finite colimits which both can be computed level-wise. If $\mathcal{C}$ is stable, then also $\mathbf{Pro}(\mathcal{C})$ is.

As a matter of fact, a pro-object in $\mathcal{C}$ can be represented by a functor $X\colon I \to \mathcal{C}$ where I is a small cofiltered ∞-category. In this case, we write "lim"$_{i\in I} X_i$ for the corresponding object in $\mathbf{Pro}(\mathcal{C})$. In our situations, the index category I will always be the poset of natural numbers $\mathbf{N}$.

Our main example of interest is the category $\mathbf{Pro}(\mathbf{Sp})$ of **pro-spectra** whereas we are interested in another notion of equivalence. For this purpose, let $\iota\colon \mathbf{Sp}^+ \hookrightarrow \mathbf{Sp}$ be the inclusion of the full stable subcategory spanned by bounded above spectra (i.e. whose higher homotopy groups eventually vanish). The induced inclusion $\mathbf{Pro}(\iota)\colon \mathbf{Pro}(\mathbf{Sp}^+) \hookrightarrow \mathbf{Pro}(\mathbf{Sp})$ is a localising subcategory whose fully faithful left-adjoint will be denoted by ι^*.

A map $X \to Y$ of pro-spectra is said to be a **weak equivalence** iff the induced map $\iota^* X \to \iota^* Y$ is an equivalence in $\mathbf{Pro}(\mathbf{Sp}^+)$. This nomenclature is justified by the fact that the map $X \to Y$ is a weak equivalence if and only if some truncation is an equivalence and the induced map on pro-homotopy groups are pro-isomorphisms. Similarly, one defines the notions of **weak fibre sequence** and **weak pullback**.

1.3. Continuous K-theory

Reminder 1.3.1. Let A_0 be be a ring and let I be an ideal of A_0. Then the ideals $(I^n)_{n\geq 0}$ form a basis of neighbourhoods of zero in the so-called I**-adic topology**. An **adic ring** is a topolocical ring A_0 such that its topology coincides with the I-adic topology for some ideal I of A_0. We say that I is an **ideal of definition**. Note that adic rings have usually more than one ideal of definition. If the ideal I is finitely generated, the completion $\hat{A}_0$ is naturally isomorphic to the limit $\lim_{n\geq 1} A_0/I^n$.

Definition 1.3.2. Let A_0 be a complete I-adic ring for some ideal I of A_0. The **continuous K-theory** of A_0 is defined as the pro-spectrum

$$\mathrm{K}^{\mathrm{cont}}(A_0) = \text{“}\lim_{n\geq 1}\text{”}\,\mathrm{K}(A_0/I^n)$$

where K is nonconnective algebraic K-theory (Definition 1.1.1). This is independent of the choice of the ideal of definition.

Definition 1.3.3. A topological ring A is called a **Tate ring** if there exists an open subring $A_0 \subset A$ which is a complete π-adic ring (i.e. it is complete with respect to the (π)-adic topology) for some $\pi \in A_0$ such that $A = A_0[\pi^{-1}]$. We call such a subring A_0 a **ring of definition** of A and such an element π a **pseudo-uniformiser**. A **Tate pair** (A, A_0) is a Tate ring together with the choice of a ring of definition and a **Tate triple** (A, A_0, π) is a Tate pair together with the choice of a pseudo-uniformiser.[1]

Example 1.3.4. Every affinoid algebra is a Tate ring. Let k be a nonarchimedean field which is complete with respect to an absolute value $|-|$. Then one defines the **Tate algebra** (in n variables) as

$$k\langle t_1,\dots,t_n\rangle := \Big\{ \sum_{\nu\in\mathbf{N}^n} c_\nu t^\nu \in k[\![t_1,\dots,t_n]\!] \;\Big|\; |c_\nu| \to 0 \text{ as } |\nu| \to \infty \Big\}.$$

These are precisely those power series converging on the unit ball

$$\mathbf{B}^n(\bar{k}) := \{(x_1,\dots,x_n) \in \overline{k}^n \mid \forall i : |x_i| \leq 1\}$$

in an algebraic closure $\overline{k}$ of k. Equipped with the **maximum norm**[2]

$$\Big|\sum_{\nu\in\mathbf{N}^n} c_\nu t^\nu\Big| := \max_\nu |c_\nu|$$

the Tate algebra $k\langle t_1,\dots,t_n\rangle$ is a complete k-algebra. The Tate algebra is noetherian, factorial, jacobson, and of finite Krull dimension n.

An **affinoid k-algebra** A is a k-algebra admitting an epimorphism of k-algebras $\alpha: k\langle t_1,\dots,t_n\rangle \to A$ for some $n \in \mathbf{N}$. One can equip A with the **residue norm**

$$|a|_\alpha := \inf\Big\{|f| \;\Big|\; f \in k\langle t_1,\dots,t_n\rangle \wedge \alpha(f) = a\Big\}$$

for $a \in A$. Affinoid algebras are noetherian, jacobson, and satisfy Noether normalisation (i.e. there exists a finite monomorphism $k\langle t_1,\dots,t_d\rangle \hookrightarrow A$ where d is the Krull dimesnion of A). Affinoid algebras are the buidling blocks of rigid spaces ananogously to algebras of finite type in algebraic geometry.

[1]One should not confuse our notion of a Tate pair with the notion of an *affinoid Tate ring* (A, A^+), i.e. a Tate ring A together with an open subring A^+ of the power-bounded elements of A which is integrally closed in A. The latter one is used in the context of adic spaces.

[2]Bosch calls it the *Gauß norm* [Bos14, §2.2].

The following notion was suggested by Morrow [Mor16] and studied by Kerz-Saito-Tamme [KST18a].

Definition 1.3.5. Let (A, A_0, π) be a Tate triple. We define the **continuous K-theory** $\mathrm{K}^{\mathrm{cont}}(A)$ of A as the pushout

$$\begin{array}{ccc} \mathrm{K}(A_0) & \longrightarrow & \mathrm{K}(A) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(A_0) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A) \end{array}$$

in the ∞-category $\mathbf{Pro}(\mathbf{Sp})$ of pro-spectra.

Remark 1.3.6. In the situation of Definition 1.3.5 we obtain a fibre sequence

$$\mathrm{K}(A_0 \text{ on } \pi) \longrightarrow \mathrm{K}^{\mathrm{cont}}(A_0) \longrightarrow \mathrm{K}^{\mathrm{cont}}(A).$$

If $A = A_0'[\lambda^{-1}]$ for another complete λ-adic ring A_0', one obtains a weakly equivalent pro-spectrum, i.e. there is a zig-zag of maps inducing pro-isomorphisms on pro-homotopy groups [KST18a, Prop. 5.4].

For regular rings, algebraic K-theory vanishes in negative degrees. For continuous K-theory this may be not the case since it sees the reduction type of a regular ring of definition.

Example 1.3.7. Let (A, A_0) be a Tate pair. There is the Mayer-Vietoris exact sequence

$$\ldots \to \mathrm{K}_{-1}(A_0) \to \mathrm{K}^{\mathrm{cont}}_{-1}(A_0) \oplus \mathrm{K}_{-1}(A) \to \mathrm{K}^{\mathrm{cont}}_{-1}(A) \to \mathrm{K}_{-2}(A_0) \to \ldots.$$

If both A and A_0 are regular, it follows that $\mathrm{K}^{\mathrm{cont}}_{-1}(A) \cong \mathrm{K}_{-1}(A_0)$. If A_0 is a π-adic ring, then $\mathrm{K}^{\mathrm{cont}}_{-1}(A_0) = \mathrm{K}_{-1}(A_0/\pi)$ due to nilinvariance of negative algebraic K-theory. Now let k be a discretely valued field and let $\pi \in k^\circ$ be a uniformiser.

(i) If $A \cong k\langle x, y\rangle/(x^3 - y^2 + \pi)$, we can choose $A_0 := k^\circ\langle x, y\rangle/(x^3 - y^2 + \pi)$ so that both A and A_0 are regular. The reduction $A_0/\pi =\cong \tilde{k}\langle x, y\rangle/(x^3 - y^2)$ is the "cusp" over $\tilde{k}$. Thus $\mathrm{K}^{\mathrm{cont}}_{-1}(A) = \mathrm{K}(A_0/\pi) = 0$ [Wei01, 2.4].

(ii) If $A \cong k\langle x, y\rangle/(x^3 + x^2 - y^2 + \pi)$, we can choose $A_0 := k^\circ\langle x, y\rangle/(x^3 + x^2 - y^2 + \pi)$ so that both A and A_0 are regular. The reduction $A_0/\pi =\cong \tilde{k}\langle x, y\rangle/(x^3 + x^2 - y^2)$ is the "node" over $\tilde{k}$. If $\mathrm{char}(k) \neq 2$, then $\mathrm{K}^{\mathrm{cont}}_{-1}(A) = \mathrm{K}(A_0/\pi) = \mathbf{Z}$ does not vanish [Wei01, 2.4].

We state some properties of continuous K-theory.

Proposition 1.3.8 (Kerz-Saito-Tamme)**.** *Let (A, A_0, π) be a Tate triple.*

(i) The canonical map $\mathrm{K}_0(A) \to \mathrm{K}^{\mathrm{cont}}_0(A)$ is an isomorphism.

(ii) $\mathrm{K}_1^{\mathrm{cont}}(A) \cong \text{“}\lim_n\text{”} \mathrm{K}_1(A)/(1+\pi^n A_0)$.

(iii) *Continuous K-theory satifies an analytic version of Bass Fundamental Theorem; more precisely, for $i \in \mathbf{Z}$ there is an exact sequence*

$$0 \to \mathrm{K}_i^{\mathrm{cont}}(A) \to \mathrm{K}_i^{\mathrm{cont}}(A\langle t\rangle) \oplus \mathrm{K}_i^{\mathrm{cont}}(A\langle t^{-1}\rangle) \to \mathrm{K}_i^{\mathrm{cont}}(A\langle t,t^{-1}\rangle) \to \mathrm{K}_{i-1}^{\mathrm{cont}}(A) \to 0.$$

(iv) *Continuous K-theory coincides in negative degrees with the groups defined by Karoubi-Villamayor [KV71, 7.7][3] and Calvo [Cal85, 3.2].*

There are not always rings of definition which behave nice enough so that we will have to deal with other models which may not be affine. Hence we define similarly to Definition 1.3.5 the following.

Definition 1.3.9. Let X be a scheme over a π-adic ring A_0. Its continuous K-theory is

$$\mathrm{K}^{\mathrm{cont}}(X) := \text{“}\lim_{n\geq 1}\text{”} K(X/\pi^n)$$

where $X/\pi^n := X \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(A_0/\pi^n)$.

Proposition 1.3.10 (Kerz-Saito-Tamme [KST18a, 5.8])**.** *Let (A, A_0, π) be a Tate triple such that A_0 is noetherian and let $X \to \mathrm{Spec}(A_0)$ be an admissible blow-up, i.e. a proper morphism which is an isomorphism over* $\mathrm{Spec}(A)$. *Then there exists a weak fibre sequence*

$$\mathrm{K}(X \text{ on } \pi) \longrightarrow \mathrm{K}^{\mathrm{cont}}(X) \longrightarrow \mathrm{K}^{\mathrm{cont}}(A)$$

of pro-spectra.

For a more detailed account of continuous K-theory we refer the reader to the recent preprint by Kerz-Saito-Tamme [KST18a, §6].

1.4. Skeleta of Berkovich spaces

There are several models for nonarchimedean analytic geometry. In the approach of Berkovich [Ber90] the building blocks are the Berkovich spectra $\mathrm{Spb}(A)$ of affinoid algebras over a complete nonarchimedean base field. As a set, $\mathrm{Spb}(A)$ is the set of all multiplicative semi-norms which are bounded with respect to a nonarchimedean norm on an affinoid algebra over a complete nonarchimedean base field k. As a topological space, $\mathrm{Spb}(A)$ is compact and hausdorff. These spectra can be glued together to what Berkovich called **k-analytic spaces** and what nowadays is also called **Berkovich spaces**. An overview of Berkovich spaces is given in Ducros' Bourbaki article [Duc07] and more detailed account is given by Temkin's chapter [Tem15] of the book *Berkovich Spaces and Applications* [DFN15].

[3]Unfortunately, Karoubi-Villamayor call these groups “positive”.

In nice cases, the homotopy type of these spaces can be described in terms of their **Berkovich skeleta** which are of combinatorial nature and hence amenable to computations. We will briefly state the facts we we need for our puroses. Following a survey article of Nicaise [Nic14], we restrict our attention to Berkovich spaces over complete discretely valued fields which arise as the analytification of k-schemes.

Notation. In section 1.4, let k be a complete discretely valued field with valuation ring k°, uniformiser π, and residue field $\tilde{k}$.

Definition 1.4.1. Let X be a scheme. A **strict normal crossing divisor** (or **snc-divisor**) of X is a closed subscheme D such that [Sta19, Tag 0CBN]

(i) D is reduced,

(ii) every irreducible component D_i of D is an effective Cartier divisor, and

(iii) for every subset $J \subset I$, the intersection $\bigcap_{i \in J} D_i$ is a regular closed subscheme of X of pure codimension $\#J$.

Given a closed subscheme D of X, we define its **dual complex** $\Delta(D)$ to be the simplicial complex whose n-simplices are the irreducible components of all intersections $\bigcap_{i \in J} D_i$ where J runs over all subsets of J having cardinality $n+1$.

Example 1.4.2. (i) Consider the node $N = \mathrm{Spec}\big(\tilde{k}[x,y]/(x^3+x^2-y^2)\big)$ over some base field $\tilde{k}$. It is a closed subscheme of $\mathbf{A}^2_{\tilde{k}}$, reduced, irreducible, an effective Cartier divisor, and of codimension 1. But it is not regular, hence not a strict normal crossing divisor. It has a singular point at the origin $(0,0)$ (i.e. the prime ideal (X,Y)) and it is regular elsewhere. The intersection complex $\Delta(N)$ is just one point.

(ii) The preimage N' of N under the blow-up $\mathrm{Bl}_{(0,0)}(\mathbf{A}^2_{\tilde{k}}) \to \mathbf{A}^2_{\tilde{k}}$ has two irreducible components. On the one hand the exceptional divisor, i.e. the preimage of the singular point which is a projective line $\mathbf{P}^1_{\tilde{k}}$ corresponding to the tangent directions at the origin. On the other hand the strict transform, i.e. the closure of the preimage of the regular locus which is an affine line $\mathbf{A}^1_{\tilde{k}}$ intersecting the exceptional divisor at two different points. Considered as a closed subscheme of $\mathrm{Bl}_{(0,0)}(\mathbf{A}^2_{\tilde{k}})$, the scheme N' is a strict normal crossing divisor. The intersection complex $\Delta(N')$ has two 0-simplices and two 1-simplices both connecting the two 0-simplices. Hence $|\Delta(N')|$ has the homotopy type of a circle.

(iii) The blow-up N'' of N' at one of the two intersection points adds another irreducible component $\mathbf{P}^1_{\tilde{k}}$ which intersects each of the two components of C' at precisely one point. It is still a strict normal crossing divisor of $\mathrm{Bl}_{(0,0)}(\mathbf{A}^2_{\tilde{k}})$. The intersection complex $\Delta(N'')$ has three 0-simplices and three 1-simplices connecting each two 0-simplices. Hence $|\Delta(N'')|$ is a triangle and still has the homotopy type of a circle.

Definition 1.4.3. Let X_k be a connected and smooth k-variety. A k°**-model** of X_k is a flat separated k°-scheme of finite type X together with an isomorphism $X \times_{k^\circ} k \xrightarrow{\cong} X_k$. An **snc-model** of X_k is a k°-model X which is regular and whose special fibre $X/\pi := X \times_{k^\circ} k^\circ/\pi$ is an snc divisor of X.

Remark 1.4.4. There exists an **analytification functor** which associates to every connected and smooth k-variety X_k a Berkovich space X_k^{berk}. As a topological space, X_k^{berk} is hausdorff. It is compact if and only if X_k is a proper k-variety. Every model X of X_k yields a formal scheme $\hat{X}$ which is the formal completion of X along the closed subscheme X/π. To $\hat{X}$, one can associate a compact Berkovich space $\hat{X}_\eta$ which is called its *generic fibre*. In general, $\hat{X}_\eta$ is a closed subspace of X_k^{berk}.

Example 1.4.5. If $X_k = \mathrm{Spec}(A)$ for a k-algebra A of finite type, then any open and bounded subring A_0 gives rise to a model $X = \mathrm{Spec}(A_0)$ and a formal scheme $\hat{X} = \mathrm{Spf}(A_0)$. Its generic fibre is the Berkovich spectrum $\mathrm{Spb}(\hat{A})$ which is a subspace of X^{berk}.

Theorem 1.4.6 (Berkovich, Thuillier, cf. [Nic14, 2.4.6,2.4.9])**.** *Let X_k be a connected and smooth k-variety. Assume that there exists an snc-model X of X_k. Then there exists a continuous map*

$$\Phi\colon |\Delta(X/\pi)| \longrightarrow \hat{X}_\eta,$$

whose image $\mathrm{Sk}(X)$ *we call the* ***Berkovich skeleton*** *of X, such that*

(i) *the induced map* $|\Delta(X/\pi)| \to \mathrm{Sk}(X)$ *is a homeomorphism and*

(ii) $\mathrm{Sk}(X)$ *is a strong deformation retract of* $\hat{X}_\eta$.

1.5. Main result for regular algebras assuming resolution of singularities

In this section, we give a proof of the main result for regular affinoid algebras under the assumption of resolution of singularities. Logically, this section is redundant as we will later give another proof for a more general statement. However, this section may serve as a heuristic for what comes later.

Notation. In section 1.5, let k be a complete discretely valued field with valuation ring k°, uniformiser π, and residue field $\tilde{k}$.

Definition 1.5.1. A *regular* affinoid k-algebra A is said to have **admissible resolution of singularities** if there exists a noetherian ring of definition A_0 such that there exists an *admissible snc-model* over A_0 (i.e. a proper morphism $X \to \mathrm{Spec}(A_0)$ with X being regular which is an isomorphism over $\mathrm{Spec}(A)$ and whose special fibre X/π is a snc-divisor).[4]

[4]This is condition $(\dagger)_A$ in [KST18a, 3.3].

Theorem 1.5.2. *Let A be a regular affinoid k-algebra of dimension d. Assume that A satisfies admissible resolution of singularities and assume that the residue field $\tilde{k} := k^\circ/\pi$ is perfect and admits resolution of singularities. Then*

$$\mathrm{K}^{\mathrm{cont}}_{-d}(A) \cong \mathrm{H}^d(\mathrm{Spb}(A);\mathbf{Z})$$

where $\mathrm{Spb}(A)$ *is the Berkovich space associated with A.*

The proof makes use of the cdh-topology which will be introduced later in section 2.2.

Proof. Let $X \to \mathrm{Spec}(A_0)$ as in Defintion 1.5.1. Kerz-Saito-Tamme [KST18a, 5.7] showed that there is a fibre sequence

$$\mathrm{K}(X \text{ on } \pi) \longrightarrow \mathrm{K}^{\mathrm{cont}}(X) \longrightarrow \mathrm{K}^{\mathrm{cont}}(A)$$

where K is non-connective algebraic K-theory and $\mathrm{K}^{\mathrm{cont}}(X) = \text{“}\lim\text{”}_{n\geq 1} \mathrm{K}(X/\pi^n)$ is continuous K-theory of the model X. We have a commutative diagram where the lines are fibre sequences.

$$\begin{array}{ccccc}
\mathrm{K}(X \text{ on } \pi) & \longrightarrow & \mathrm{K}(X) & \longrightarrow & \mathrm{K}(X_A) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{G}(X/\pi) & \longrightarrow & \mathrm{G}(X) & \longrightarrow & \mathrm{G}(X_A)
\end{array}$$

As X is regular and hence also $X_A := X \times_{A_0} A \cong \mathrm{Spec}(A)$, the two right vertical arrows are equivalences, hence also the left one. This implies that we have for $n \geq 1$ that

$$\mathrm{K}^{\mathrm{cont}}_{-n}(X) \cong \mathrm{K}^{\mathrm{cont}}_{-n}(A).$$

We have $d = \dim(A) = \dim(X/\pi)$. By definition and Lemma 1.1.3 we have

$$\mathrm{K}^{\mathrm{cont}}_{-d}(X) = \text{“}\lim_{n\geq 1}\text{”}\, \mathrm{K}_{-d}(X/\pi^n) = \mathrm{K}_{-d}(X/\pi).$$

By [KST18b, Cor. D], we have $\mathrm{K}_{-d}(X/\pi) \cong \mathrm{H}^d_{\mathrm{cdh}}(X/\pi;\mathbf{Z})$.

Let $(D_i)_{i\in I}$ be the irreducible components of X/π so that the map $\sqcup_{i\in I} D_i \to X/\pi$ is a cdh-cover. In order to compute the cdh-cohomology of X/π in terms of this cover, we use some facts about Nisnevich sheaves with transfers.

The constant presheaf $\mathbf{Z}$ is a presheaf with transfers [MVW06, Ex. 2.2] and it is homotopy invariant. If F is a presheaf with transfers, its sheafification with respect to the Nisnevich topology admits transfers [MVW06, Thm. 13.1]. Thus the constant sheaf $\mathbf{Z}$ is a homotopy invariant Nisnevich sheaf with transfers. Let Y be a connected and smooth $\tilde{k}$-scheme. Then we have (using that resolution of singularities holds over $\tilde{k}$) that

$$\mathrm{H}^n_{\mathrm{cdh}}(Y;\mathbf{Z}) \cong \mathrm{H}^n_{\mathrm{Nis}}(Y;\mathbf{Z}) \cong \mathrm{H}^n_{\mathrm{Zar}}(Y;\mathbf{Z}) \cong \begin{cases} \mathbf{Z} & (n=0) \\ 0 & (n \geq 1) \end{cases}$$

where the first isomorphism is a result by Suslin-Voevodsky [SV00, Cor. 5.12.3] (see also [MVW06, Prop. 13.27]), the second isomorphism is a result by Mazza-Voevodsky-Weibel [MVW06, Prop. 13.9] and the third isomorphism follows since the sheaf $\mathbf{Z}$ is flasque on connected schemes, hence it is Zariski acyclic.

For $n \geq 0$, let E_n be the disjoint union of the irreducible components of all intersections $\bigcap_{i\in J} D_i$ with $J \subset I$ such that $\#J = n+1$. Since X/π is a snc-divisor, all D_n are cdh-acyclic. Hence the exact sequence of cdh-sheaves

$$\mathbf{Z} \longrightarrow \mathbf{Z}_{E_0} \longrightarrow \mathbf{Z}_{E_1} \longrightarrow \mathbf{Z}_{E_2} \longrightarrow \dots$$

on X/π is a cdh-acyclic resolution where $\mathbf{Z}_{E_i} = (E_i \to X/\pi)_*\mathbf{Z}$. Since cdh-sheaves are additive for disjoint unions, $\mathrm{H}^i_{\mathrm{cdh}}(X/\pi;\mathbf{Z})$ identifies with the simplicial cohomology group $\mathrm{H}^i(\Delta(X/\pi);\mathbf{Z})$. Now the claim follows from Theorem 1.4.6. □

2. Zariski-Riemann spaces

Under the assumption of resolution of singularities, one can obtain for a non-regular scheme X a regular scheme $\tilde{X}$ which admits a proper, birational morphism $\tilde{X} \to X$. For many purposes $\tilde{X}$ behaves similarly to X. For instance, in the proof of Theorem 1.5.2 we used a regular model X of an affinoid algebra A to work with the fibre sequence

$$\mathrm{K}(X \text{ on } \pi) \longrightarrow \mathrm{K}^{\mathrm{cont}}(X) \longrightarrow \mathrm{K}^{\mathrm{cont}}(A)$$

instead of working with the analoguous fibre sequence for a subring of definition A_0 of A from Definition 1.3.5. In characteristic zero, resolution of singularities for all dimension was established by Hironaka [Hir64]. More precisely, he proved resolution for integral schemes of finite type over a local quasi-excellent ring of residue characteristic zero. The most general result nowadays is due to Temkin [Tem08] who proved resolution for integral schemes of finite type over any quasi-excellent scheme of characteristic zero. For a detailed account we refer the reader to Kollar [Kol07]. Unfortunately, resolution of singularities is not available at the moment in positive characteristic.

A good workaround for this inconvenience is to work with a Zariski-Riemann type space which is defined as the inverse limit of all models, taken in the category of locally ringed spaces. This is not a scheme anymore, but behaves in the world of K-theory almost as good as a regular model does. For instance $\mathrm{K}_n^{\mathrm{cont}}(A) \cong \mathrm{K}_n^{\mathrm{cont}}(\langle A_0 \rangle_A)$ for negative n where $\langle A_0 \rangle_A$ is a Zariski-Riemann space associated with A (Definition 2.1.15).

The notion of Zariski-Riemann spaces goes back to Zariski [Zar44] who called them "Riemann manifolds" and was further studied by Temkin [Tem11]. Recently, Kerz-Strunk-Tamme [KST18b] used them to prove that homotopy algebraic K-theory [Wei89] is the cdh-sheafification of algebraic K-theory. Also Huber-Kelly [HK18] used them related to the cdh-topology. It seems that Zariski-Riemann spaces are a promising tool for working with the cdh-topology.

The key part of this thesis is section 2.3 where we establish a comparison of rh-cohomology and Zariski cohomology for Zariski-Riemann spaces (Theorem 2.3.16). Furthermore, we will see that Zariski-Riemann spaces for formal schemes are closely related to adic spaces (Theorem 2.5.7).

2.1. Schematic Zariski-Riemann spaces

In this section, we will deal with Zariski-Riemann spaces which arise from schemes. We will see that they can be equipped with a corresponding topology for every topology on schemes (Definition 2.3.11). For their special fibres, we will compare the rh-cohomology groups with the Zariski cohomology groups (Theorem 2.3.16).

Notation. In section 2.1, let X be a *reduced* quasi-compact and quasi-separated scheme and let U be a quasi-compact open subscheme of X.

Definition 2.1.1. A **U-modification** of X is a projective morphism $X' \to X$ of schemes which is an isomorphism over U. Denote by $\mathrm{Mdf}(X,U)$ the category of U-modifications of X with morphisms over X. We define the **U-admissible Zariski-Riemann space** of X to be the limit

$$\langle X \rangle_U = \lim_{X' \in \mathrm{Mdf}(X,U)} X'$$

in the category of locally ringed spaces; it exists due to Proposition B.1.9.

Remark 2.1.2. The reason why we demand the morphisms in Definition 2.1.1 to be projective is that it implies both proper and quasi-projective. The first one is needed in order to obtain abstract blow-up squares and therefore rh-covers. The second one enables us to work with vector bundles instead of perfect complexes to compute K-theory.

The following is just a special case of Proposition B.1.9.

Lemma 2.1.3. *The underlying topological space of $\langle X \rangle_U$ is coherent and sober and for any $X' \in \mathrm{Mdf}(X,U)$ the projection $\langle X \rangle_U \to X'$ is quasi-compact.*

The notion of a U-admissible modification is quite general. However, one can restrict to more concrete notion, namely U-admissible blow-ups.

Definition 2.1.4. A **U-admissible blow-up** is a blow-up $\mathrm{Bl}_Z(X) \to X$ whith centre $Z \subseteq X \setminus U$. Denote by $\mathrm{Bl}(X,U)$ the category of U-admissible blow-ups with morphisms over X.

Proposition 2.1.5. *The inclusion $\mathrm{Bl}(X,U) \hookrightarrow \mathrm{Mdf}(X,U)$ is cofinal. In particular, the canonical morphism*

$$\langle X \rangle_U = \lim_{X' \in \mathrm{Mdf}(X,U)} X' \longrightarrow \lim_{X' \in \mathrm{Bl}_{(X,U)}} X'.$$

is an isomorphism of locally ringed spaces.

Proof. Since a blow-up is projective and an isomorphism outside its centre $\mathrm{Bl}(X,U)$ lies in $\mathrm{Mdf}(X,U)$. On the other hand, every U-modification is dominated by a U-admissible blow-up [Tem08, Lem. 2.1.5]. Hence the inclusion is cofinal and the limits agree.[1] □

[1] Cf. the proof of Lemma 2.2.12.

Lemma 2.1.6. *The full subcategory* $\mathrm{Mdf}^{\mathrm{red}}(X,U)$ *spanned by reduced schemes is cofinal in* $\mathrm{Mdf}(X,U)$.

The following results depend on Raynaud-Gruson's *platification par éclatement* [RG71, 5.2.2]. These results and their proofs are modified versions of results of Kerz-Strunk-Tamme who considered birational and projective schemes over X instead of U-modifications, cf. Lemma 6.5 and Proof of Proposition 6.4 in [KST18b].

Definition 2.1.7. Let $(Y,\mathcal{O}_Y)$ be a locally ringed space and let $n \geq 0$. An $\mathcal{O}_Y$-module F is said to have **Tor-dimension** $\leq$ **n** iff there exists an exact sequence

$$0 \to E_n \to \ldots \to E_1 \to E_0 \to F \to 0$$

where $E_n,\ldots,E_1,E_0$ are locally free $\mathcal{O}_Y$-modules. Denote by $\mathrm{Mod}^{\leq n}(Y)$ and $\mathrm{Coh}^{\leq n}(Y)$ the full subcategories of $\mathrm{Mod}(Y)$ resp. $\mathrm{Coh}(Y)$ spanned by $\mathcal{O}_Y$-modules of Tor-dimension $\leq n$.

Definition 2.1.8. For $X' \in \mathrm{Mdf}(X,U)$ and $Z' = X' \setminus U$, denote by $\mathrm{Coh}^{\leq 1}_{Z'}(X')$ the intersection of $\mathrm{Coh}_{Z'}(X')$ and $\mathrm{Coh}^{\leq 1}(X')$. Analogously, define $\mathrm{Coh}^{\leq 1}_{\tilde{Z}}(\langle X\rangle_U)$.

Lemma 2.1.9 (cf. [KST18b, 6.5 (i)]). *If* U *is dense in* X*, then for every* U*-modification* $p: X' \to X$ *the pullback functor*

$$p^*: \mathrm{Mod}^{\mathrm{fp},\leq 1}(X) \longrightarrow \mathrm{Mod}^{\mathrm{fp},\leq 1}(X')$$

is exact.

Proof. If F is an $\mathcal{O}_X$-module of Tor-dimension ≤ 1, there exists an exact sequence $0 \to E_1 \overset{\varphi}{\to} E_0 \to F \to 0$ where E_1 and E_0 are flat (i.e. locally free) $\mathcal{O}_X$-modules. Then the pulled back sequence

$$0 \longrightarrow p^*E_1 \overset{p^*\varphi}{\longrightarrow} p^*E_0 \longrightarrow p^*F \longrightarrow 0$$

is exact at p^*E_0 and p^*F. We claim that the map $p^*\varphi$ is injective. The $\mathcal{O}_{X'}$-modules p^*E_1 and p^*E_0 are locally free, say of rank n and m, repectively. Let η be a generic point of an irreducible component of X'. Since U is dense, the map $(p^*\varphi)_\eta: \mathcal{O}^n_{X',\eta} \to \mathcal{O}^m_{X',\eta}$ is injective since it identifies with the injective map $\varphi_{p(\eta)}: (E_1)_{p(\eta)} \hookrightarrow (E_0)_{p(\eta)}$. For every specialisation x of η, the stalk $\mathcal{O}_{X',x}$ embeds into $\mathcal{O}_{X',\eta}$ [GW10, 3.29], hence the induced map $(p^*\varphi)_x: \mathcal{O}^n_{X',x} \to \mathcal{O}^m_{X',x}$ is injective. Thus $p^*\varphi$ is injective at every point of X', hence injective. Now the exactness of p^* follows from the nine lemma. □

Lemma 2.1.10. *Let* X *be a noetherian scheme admitting an ample family of line bundles (e.g.* X *quasi-projective over an affine scheme) and assume that* U *is dense in* X*. Then the inclusion*

$$\mathrm{Mod}^{\mathrm{fp},\leq 1}_{\tilde{Z}}(\langle X\rangle_U) \longrightarrow \mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X\rangle_U)$$

is an equivalence of categories.

Proof. It suffices to show that the functor in question is essentially surjective. Let $F \in \mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X\rangle_U)$. By Proposition B.1.9 (iv), there exists an $X' \in \mathrm{Mdf}(X,U)$ and $F' \in \mathrm{Mod}^{\mathrm{fp}}_{Z'}(X')$ such that $F \cong p^*F'$. Since X has an ample family of line bundles and the map $X' \to X$ is quasi-projective, also X' has an ample family of line bundles [TT90, 2.1.2.(h)]. Hence there exists an exact sequence $E'_1 \overset{\varphi}{\to} E'_0 \to F' \to 0$ where E'_1, E'_0 are locally free $\mathcal{O}_{X'}$-modules. By our assumptions,

$$\mathrm{im}(\varphi)|_U = \ker\big(E'_0 \to \mathrm{coker}(\varphi)\big)|_U = \ker\big(E'_0 \to F'\big)|_U = E'_0|_U$$

is flat. By *platification par éclatement* [RG71, 5.2.2], there exists a U-admissible blow-up $q{:}X'' \to X'$ such that the strict transform $q^{\mathrm{st}}\mathrm{im}(\varphi)$ is flat, i.e. locally free. Furthermore, $q^{\mathrm{st}}\mathrm{im}(\varphi) = \mathrm{im}(q^*\varphi)$ (see Remark 2.1.11). Hence we obtain an exact sequence

$$0 \longrightarrow \mathrm{im}(q^*\varphi) \longrightarrow q^*E'_0 \longrightarrow q^*F' \longrightarrow 0,$$

hence $q^*F' \in \mathrm{Coh}^{\le 1}(X'')$. By Lemma 2.1.6, we may assume that all schemes in question are reduced. By a result of Kerz-Strunk-Tamme, the pullback along projective and birational morphisms with reduced domain is exact on the full subcategory of modules of Tor-dimension ≤ 1 by Lemma 2.1.9. Hence $F \in \mathrm{Mod}^{\mathrm{fp},\le 1}_{\tilde{Z}}(\langle X\rangle_U)$ which was to be shown. □

Remark 2.1.11. In the situation of the proof of Lemma 2.1.10 we used that $\mathrm{im}(q^*\varphi)$ is the strict transform of $\mathrm{im}(\varphi)$ along the U-admissible blow-up $q{:}X'' \to X'$. By definition, $q^{\mathrm{st}}\mathrm{im}(\varphi)$ is the quotient of $q^*\mathrm{im}(\varphi)$ by its submodule of sections whose support is contained in the centre of the blow-up q [Sta19, Tag 080D]. Since the surjective map $q^*\mathrm{im}(\varphi) \to q^{\mathrm{st}}\mathrm{im}(\varphi)$ factors over the map $\mathrm{im}(q^*\varphi) \subset p^*E'_0$, the following commutative diagram has exact rows and exact columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \ker(\sigma) & \longrightarrow & \mathrm{im}(q^*\varphi) & \overset{\sigma}{\longrightarrow} & q^{\mathrm{st}}\mathrm{im}(\varphi) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\tau) & \longrightarrow & q^*E'_0 & \overset{\tau}{\longrightarrow} & q^{\mathrm{st}}E'_0 & \longrightarrow & 0
\end{array}$$

Since E'_0 is a vector bundle, $q^*E'_0 = q^{\mathrm{st}}E_0$ [Sta19, Tag 080F] which implies the claim.

If a locally ringed space is cohesive, i.e. its structure sheaf is coherent (Definition B.1.3), then a module is coherent if and only if it is finitely presented (Lemma B.1.5). Unfortunately, we do not know whether or not this is also true for the Zariski-Riemann space $\langle X\rangle_U$. But passing to the complement $\langle X\rangle_u \setminus U$ we have the following.

Proposition 2.1.12. *Let $\tilde{Z}$ be the complement of U in $\langle X\rangle_U$. An $\mathcal{O}_{\langle X\rangle_U}$-module with support in $\tilde{Z}$ is coherent if and only if it is finitely presented. In particular, the canonical functor*

$$\operatorname*{colim}_{X'\in \mathrm{Mdf}(X,U)} \mathrm{Coh}_{Z'}(X') \longrightarrow \mathrm{Coh}_{\tilde{Z}}(\langle X\rangle_U)$$

is an equivalence of categories where $Z' = X' \setminus U$ and the colimit is taken in the 2-category of categories.

Proof. We have an equivalence $\mathrm{colim}_{X'} \mathrm{Mod}^{\mathrm{fp}}(X') \to \mathrm{Mod}^{\mathrm{fp}}(\langle X\rangle_U)$ according to Proposition B.1.9. The colimit is compatible with the restriction to the subcategory of modules with support. Since the schemes in question are cohesive locally ringed spaces, we get an equivalence $\mathrm{colim}_{X'} \mathrm{Coh}_{Z'}(X') \to \mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X\rangle_U)$. It remains to show that every finitely presented $\mathcal{O}_{\langle X\rangle_U}$-module with support in $\tilde{Z}$ is coherent.

Let F be an $\mathcal{O}_{\langle X\rangle_U}$-module of finite presentation with support on $\tilde{Z}$. By definition, F is of finite type. Let V be an open subset of $\langle X\rangle_U$ and let $\varphi\colon \mathcal{O}_V^n \to F|_U$ be a morphism. We need to show that $\ker(\varphi)$ is of finite type. Since $\langle X\rangle_U$ is coherent, V is quasi-compact. By passing iteratively to another U-modification, by using Proposition B.1.8 (i) and Proposition B.1.9 (iv) and (v), there exists an $X' \in \mathrm{Mdf}(X,U)$ with canonical projection $p_{X'}\colon \langle X\rangle_U \to X'$ such that

- $F = (p_{X'})^* F_X$ for some $F_X \in \mathrm{Mod}^{\mathrm{fp}}_{Z'}(X')$ (Proposition B.1.9 (iv)),
- F_X has Tor-dimension ≤ 1 (Lemma 2.1.10),
- $V = (p_{X'})^{-1}(V')$ for some open subset V' of X' (Proposition B.1.8),
- φ is induced by a morphism $\varphi'\colon \mathcal{O}^n_{V'} \to F_X|_{V'}$ (Proposition B.1.9 (v)), and
- $\mathrm{coker}(\varphi)$ has Tor-dimension ≤ 1 (Lemma 2.1.10).

Since $\ker(\varphi')$ is of finite type we may assume that there exists a surjection $\mathcal{O}_{V'} \twoheadrightarrow \ker(\varphi')$ (otherwise we have to shrink V'). By Lemma 2.1.9, $(p_X)^*\big(\ker(\varphi')\big) = \ker(\varphi)$ is of finite type. □

Theorem 2.1.13. *Let $\tilde{Z}$ be the complement of U in $\langle X\rangle_U$. Assume that U is dense in X. Then the canonical functors*

$$\begin{array}{ccc} \mathrm{Coh}^{\leq 1}_{\tilde{Z}}(\langle X\rangle_U) & \longrightarrow & \mathrm{Coh}_{\tilde{Z}}(\langle X\rangle_U) \\ \downarrow & & \downarrow \\ \mathrm{Mod}^{\mathrm{fp},\leq 1}_{\tilde{Z}}(\langle X\rangle_U) & \longrightarrow & \mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X\rangle_U) \end{array}$$

are equivalences. In particular, the canonical functor

$$\operatorname*{colim}_{X'\in \mathrm{Mdf}(X,U)} \mathrm{Coh}_{Z'}(X') \longrightarrow \mathrm{Coh}_{\tilde{Z}}(\langle X\rangle_U)$$

is an equivalence of categories where $Z' = X' \setminus U$.

Proof. The lower horizontal funtor is an equivalence by Lemma 2.1.10 and the right vertical functor is an equivalence by Lemma 2.1.12. The other two functors are equivalence since the square is a pullback square. The equivalence from Proposition B.1.9 restricts to an equivalence

$$\operatorname*{colim}_{X' \in \mathrm{Mdf}(X,U)} \mathrm{Mod}^{\mathrm{fp}}_{Z'}(X') \longrightarrow \mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X \rangle_U)$$

which yields the desired statement. □

Proposition 2.1.14. *Let X be a noetherian scheme and let $U \subset X$ be an open subscheme. Denote by $\tilde{Z}$ the closed complement of U in $\langle X \rangle_U$. Then $\mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X \rangle_U)$ is a Serre subcategory of $\mathrm{Mod}^{\mathrm{fp}}(\langle X \rangle_U)$ and the inclusion $j\colon U \hookrightarrow \langle X \rangle_U$ induces an equivalence of categories*

$$j^*\colon \mathrm{Mod}^{\mathrm{fp}}(\langle X \rangle_U)/\mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X \rangle_U) \xrightarrow{\cong} \mathrm{Mod}^{\mathrm{fp}}(U).$$

In particular, the sequence of abelian categories

$$\mathrm{Coh}_{\tilde{Z}}(\langle X \rangle_U) \longrightarrow \mathrm{Mod}^{\mathrm{fp}}(\langle X \rangle_U) \longrightarrow \mathrm{Coh}(U) \tag{2.1}$$

is exact.

Proof. Since j^* is exact, the subcategory $\mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X \rangle_U) \subset \mathrm{Mod}^{\mathrm{fp}}(\langle X \rangle_U)$ is closed under subobjects, quotients, and extensions, hence a Serre subcategory. By definition, the restriction $j^*\colon \mathrm{Mod}^{\mathrm{fp}}(\langle X \rangle_U) \to \mathrm{Mod}^{\mathrm{fp}}(U)$ factors through the quotient of $\mathrm{Mod}^{\mathrm{fp}}(\langle X \rangle_U)$ by $\mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X \rangle_U)$. We will show that the induced functor is essentially surjective and fully faithful.

Note that noetherian schemes are cohesive locally ringed spaces, i.e. their structure sheaf is coherent (Definition B.1.3). Hence a module is coherent if and only if it is finitely presented (Lemma B.1.5).

For essential surjectiveness let $F \in \mathrm{Mod}^{\mathrm{fp}}(U) = \mathrm{Coh}(U)$. The inclusion $j_X\colon U \hookrightarrow X$ induces an equivalence of categories

$$(j_X)^*\colon \mathrm{Mod}^{\mathrm{fp}}(X)/\mathrm{Mod}^{\mathrm{fp}}_Z(X) = \mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \longrightarrow \mathrm{Coh}(U) = \mathrm{Mod}^{\mathrm{fp}}(U)$$

where $Z := X \setminus U$ [Sch11, 2.3.7][2]. Thus there exists an $F_X \in \mathrm{Mod}^{\mathrm{fp}}(X)$ such that $(j_X)^* F_X \cong F$. Since j_X factors as $p_X \circ j$, it follows that

$$F \cong j_X^* F_X \cong (p_X \circ j)^* F_X \cong j^*(p_X^* F_X),$$

i.e. F comes from a finitely presented module $p_X^* F_X \in \mathrm{Mod}^{\mathrm{fp}}(\langle X \rangle_U)$.

For fullness, let $\varphi\colon j^*F \to j^*G$ be a morphism of $\mathcal{O}_U$-modules where F and G are in $\mathrm{Mod}^{\mathrm{fp}}(\langle X \rangle_U)$. The quotient of $\mathrm{Mod}^{\mathrm{fp}}(\langle X \rangle_U)$ by $\mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X \rangle_U)$ is the localisation of

[2] Originally, this is due to Gabriel [Gab62].

$\mathrm{Mod}^{\mathrm{fp}}(\langle X\rangle_U)$ along those morphisms which are sent to isomorphisms by j^*. Consider the pullback diagram

$$\begin{array}{ccc} H & \longrightarrow & G \\ \alpha\downarrow & & \downarrow\beta \\ F \longrightarrow j_*j^*F & \xrightarrow{j_*(\varphi)} & j_*j^*G \end{array} \qquad (\Delta)$$

in $\mathrm{Mod}^{\mathrm{fp}}(\langle X\rangle_U)$. Since j^* is exact, the square $j^*(\Delta)$ is also a pullback; hence $j^*(\alpha)$ is an isomorphism (as $j^*(\beta)$ is one). Thus the span $F \leftarrow H \to G$ represents a morphism ψ in the quotient category such that $\varphi = j^*(\varphi)$, hence j^* is full. Faithfulness follows from the construction.

The last assertion follows from the equality $\mathrm{Coh}(U) = \mathrm{Mod}^{\mathrm{fp}}(U)$ and the equivalence

$$\mathrm{Coh}_{\tilde{Z}}(\langle X\rangle_U) \cong \operatorname*{colim}_{X'\in\mathrm{Mdf}(X,U)} \mathrm{Coh}_{Z'}(X') \cong \operatorname*{colim}_{X'\in\mathrm{Mdf}(X,U)} \mathrm{Mod}^{\mathrm{fp}}_{Z'}(X') \cong \mathrm{Mod}^{\mathrm{fp}}_{\tilde{Z}}(\langle X\rangle_U)$$

where the first equivalence is Lemma 2.1.12, the second one follows as all X' are cohesive locally ringed spaces, and the last one holds true for any cofiltered limit of quasi-compact and quasi-separated schemes (Proposition B.1.9(iv)). □

Zariski-Riemann spaces of complete Tate rings

This subsection is merely fixing notation for the application of schematic Zariski-Riemann spaces to the context of Tate rings.

Notation. In subsection 2.1, let (A, A_0, π) be a Tate triple (Definition 1.3.3).

Definition 2.1.15. Setting $X = \mathrm{Spec}(A_0)$ and $U = \mathrm{Spec}(A)$, we are in the situation of Definition 2.1.1. For simplicity, we denote

$$\mathrm{Adm}(A_0) := \mathrm{Mdf}(\mathrm{Spec}(A), \mathrm{Spec}(A_0))$$

and call its objects **admissible blow-ups**. Furthermore, we call the locally ringed space

$$\langle A_0\rangle_A := \langle \mathrm{Spec}(A_0)\rangle_{\mathrm{Spec}(A)} = \lim_{X\in\mathrm{Adm}(A_0)} X$$

the **admissible Zariski-Riemann space** associated to the pair (A, A_0).

Remark 2.1.16. The Zariski-Riemann space $\langle A_0\rangle_A$ depends on the choice of the the ring of definition A_0. However, if B_0 is another ring of definition, then also the intersection $C_0 := A_0 \cap B_0$ is. Hence we get a span

$$\mathrm{Spec}(A_0) \longrightarrow \mathrm{Spec}(C_0) \longleftarrow \mathrm{Spec}(B_0)$$

which is compatible with the inclusions of $\mathrm{Spec}(A)$ into these. Hence every admissible blow-up $X \to \mathrm{Spec}(C_0)$ induces by pulling back an admissible blow-up

$X_{A_0} \to \operatorname{Spec}(A_0)$ and a morphism $X \to X_{A_0}$. Precomposed with the canonical projections we obtain a map $\langle A_0 \rangle_A \to X$. Hence the universal property yields a morphism $\langle A_0 \rangle_A \to \langle C_0 \rangle_A$. The same way, we get a morphism $\langle B_0 \rangle_A \to \langle C_0 \rangle_A$. One checks that the category of all Zariski-Riemann spaces associated with A is filtered.

2.2. Détour: some remarks on the rh-topology

Abstract blow-up squares and the rh-topology

In this subsection, we examine the rh-topology introduced by Goodwillie-Lichtenbaum [GL01, 1.2]. We use a different definition in terms of abstract blow-up squares and show that both definitions agree (Corollary 2.2.8).

Notation. Every scheme in this section is noetherian of finite dimension. Under these circumstances, a birational morphism is an isomorphism over a dense open subset of the target [Sta19, Tag 01RN].

Definition 2.2.1. An **abstract blow-up square** is a cartesian diagram of schemes

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array} \qquad \text{(abs)}$$

where $Z \to X$ is a closed immersion, $\tilde{X} \to X$ is proper, and the induced morphism $\tilde{X} \setminus E \to X \setminus Z$ is an isomorphism.

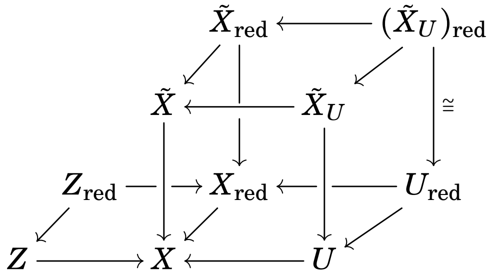
Definition 2.2.2. Let S be a noetherian scheme. The **rh-topology** on Sch_S is the topology generated by Zariski squares and abstract blow-up squares (cf. Definition B.2.10) as well as the empty cover of the empty scheme. The **cdh-topology** (completely decomposed h-topology) is the topology generated by Nisnevich squares and abstract blow-up squares as well as the empty cover of the empty scheme.

Remark 2.2.3. Our definition of an abstract blow-up square coincides with the one given in [KST18b]. Other authors demand instead the weaker condition that the induced morphism $(\tilde{X} \setminus E)_{\mathrm{red}} \to (X \setminus Z)_{\mathrm{red}}$ on the associated reduced schemes is an isomorphism, e.g. [MVW06, Def. 12.21]. Indeed, both notions turn out to yield the same topology. To see this, first note that the map $X_{\mathrm{red}} \to X$ is a rh-cover since

$$\begin{array}{ccc} \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \downarrow \\ X_{\mathrm{red}} & \longrightarrow & X \end{array}$$

is an abstract blow-up square in the sense of Definition 2.2.1 since $X_{\mathrm{red}} \to X$ is a closed immersion, $\varnothing \to X$ is proper, and the induced map on the complements $\varnothing \to \varnothing$

is an isomorphism. Now we consider the following situation as indicated in the diagram



where $U := X \setminus Z$ and $\tilde{X}_U := \tilde{X} \times_X U$. The morphism $Z \sqcup \tilde{X} \to X$ is a cover in the sense of [MVW06] but not a priori in the sense of Definition 2.2.2. However, it can be refined by the composition $Z_{\mathrm{red}} \sqcup \tilde{X}_{\mathrm{red}} \to X_{\mathrm{red}} \to X$ in which both maps are rh-covers.

Definition 2.2.4. A morphism of schemes $\tilde{X} \to X$ satisfies the **Nisnevich lifting property** iff every point $x \in X$ has a preimage $\tilde{x} \in \tilde{X}$ such that the induced morphism $\kappa(x) \to \kappa(\tilde{x})$ on residue fields is an isomorphism.

Lemma 2.2.5. *Let $p: \tilde{X} \to X$ be a proper map satisfying the Nisnevich lifting property and assume X to be reduced. Then there exists a closed subscheme X' of $\tilde{X}$ such that the restricted map $p|_{X'}: X' \to X$ is birational.*

Proof. Let η be a generic point of X. By assumption, there exists a point $\tilde{\eta}$ of $\tilde{X}$ mapping to η. Since p is a closed map, we have $p\left(\overline{\{\tilde{\eta}\}}\right) \supset \overline{p(\{\tilde{\eta}\})} = \overline{\{\eta\}}$ and hence equality holds. Thus the restriction $\overline{\{\tilde{\eta}\}} \to \overline{\{\eta\}}$ is a morphism between reduced and irreducible schemes inducing an isomorphism on the stalks of the generic points, hence it is birational. Thus setting X' to be the (finite) union of all $\overline{\{\tilde{\eta}\}}$ for all generic points η of X does the job. □

Lemma 2.2.6 ([Voe10b, 2.18])**.** *A proper map is an rh-cover if and only if it satisfies the Nisnevich lifting property.*

Proof. It follows immediately from the definitions that every rh-cover satisfies the Nisnevich lifting property. For the converse let $p: \tilde{X} \to X$ be a proper map satisfying the Nisnevich lifting property. We may assume that X is reduced and irreducible and proceed by induction on the dimension.

If $\dim(X) = 0$, then $X = \mathrm{Spec}(k)$ for some field k. Due to the Nisnevich lifting property the map p admits a section, hence is an rh-cover.

Now let $\dim(X) > 0$. By Lemma 2.2.5, there exists a closed subscheme X' of $\tilde{X}$ such that the restriction $X' \to X$ is birational and hence an isomorphism over a dense open subset U of X. Set $Z := X \setminus U$. By construction, the map $X' \sqcup Z \to X$ is an rh-cover. Consider the diagram

$$\begin{array}{ccc} (X' \times_X \tilde{X}) \sqcup (Z \times_X \tilde{X}) & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ X' \sqcup Z & \longrightarrow & X. \end{array}$$

The first component of the left vertical map is an rh-cover since it splits. The second component can be assumed an rh-cover by induction hypothesis. Thus the map $\tilde{X} \to X$ can be refined by an rh-cover, hence it is an rh-cover. □

Definition 2.2.7. A **proper rh-cover** is a proper map which is also an rh-cover, i.e. a proper map satisfying the Nisnevich lifting property. By Lemma 2.2.5, every proper rh-cover of a reduced scheme has a refinement by a proper birational rh-cover.

Corollary 2.2.8. *The rh-topology equals the topology which is generated by Zariski covers and by proper rh-covers.*

Definition 2.2.9. We say that a set-valued presheaf F satisifes **rh-excision** iff for every abstract blow-up square (abs) as in Definition 2.2.1 the induces square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Z) \\ \downarrow & & \downarrow \\ F(\tilde{X}) & \longrightarrow & F(E) \end{array}$$

is a pullback square.

Proposition 2.2.10. *A Zariski sheaf is an rh-sheaf if and only if it satisfies rh-excision.*

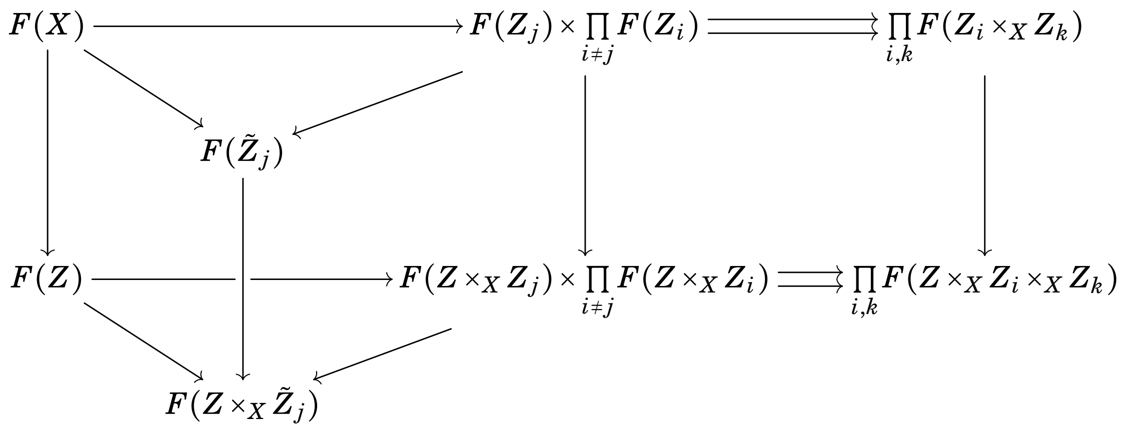
The proof is adapted from the corresponding statement for the Nisnevich topology, cf. [MVW06, 12.7].

Proof. For every Zariski sheaf F and for every abstract blow-up square (abs) as in Definition 2.2.1, the diagram $F(X) \to F(\tilde{X}) \times F(Z) \rightrightarrows F(E)$ is an equaliser. Hence F is rh-excisive.

For the converse, let F be an excisive Zariski sheaf and let $\{Z_i \to X\}_{i \in I}$ be an rh-cover. We may assume that X is reduced and irreducible. By Lemma 2.2.5, there exists an index $j \in I$ and a closed subscheme $\tilde{Z}_j$ of Z_j such that the restrictied map $\tilde{Z}_j \to X$ is birational, hence an isomorphism over an open subset U of X. Let $Z := X \setminus U$ be the closed complement. By assumption, the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(\tilde{Z}_j) \\ \downarrow & & \downarrow \\ F(Z) & \longrightarrow & F(Z \times_X \tilde{Z}_j) \end{array}$$

is a pullback square. By noetherian induction, we can assume that the lower horizontal line in the diagram



is an equaliser. Then a diagram chase shows that also the upper horizontal line is an equaliser. Thus F is an rh-sheaf. □

Definition 2.2.11. The **hrh-topology** (honest rh-topology) (resp. the **hcdh-topology**) is the topology generated by honest blow-up squares

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Bl}_Z(X) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

and Zariski squares (resp. Nisnevich squares) as well as the empty cover of the empty scheme.

Lemma 2.2.12. *Let S be a noetherian scheme. On Sch_S, the hrh-topology equals the rh-topology and the hcdh-topology equals the cdh-topology.*

Proof. Every hrh-cover is an rh-cover. We have to show conversely that every rh-cover can be refined by an hrh-cover. Let $X \in \mathrm{Sch}_S$. It suffices to show that a cover coming from an abstract blow-up square over X can be refined by an hrh-cover. As $\mathrm{Bl}_{X_{\mathrm{red}}}(X) = \varnothing$, the map $X_{\mathrm{red}} \to X$ is an hrh-cover. Hence we can assume that X is reduced since pullbacks of abstract blow-up squares are abstract blow-up squares again. Let $X = X_1 \cup \ldots \cup X_n$ be the decomposition into irreducible components. For a closed subscheme Z of X one has

$$\mathrm{Bl}_Z(X) = \mathrm{Bl}_Z(X_1) \cup \ldots \cup \mathrm{Bl}_Z(X_n).$$

If $Z = X_n$, then $\mathrm{Bl}_{X_n}(X_n) = \varnothing$ and $\mathrm{Bl}_{X_n}(X_i)$ is irreducible for $i \in \{1, \ldots, n-1\}$ [GW10, Cor. 13.97]. By iteratively blowing-up along the irreducible components, we can hence reduce to the case where X is irreducible. Let

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

be an abstract blow-up square. As X is irreducible, the complement $U := X \smallsetminus Z$ is schematically dense in X. As $p: \tilde{X} \to X$ is proper and birational, also $\tilde{X}$ is irreducible and $p^{-1}(U)$ is schematically dense in $\tilde{X}$. Thus p is a U-modification and a result of Temkin tells us that there exists a U-admissible blow-up factoring over $\tilde{X}$ [Tem08, Lem. 2.1.5].[3] That means that there exists a closed subscheme Z' of X which lies in $X \smallsetminus U = Z$ such that $\mathrm{Bl}_{Z'}(X) \to X$ factors over $\tilde{X} \to X$. Thus the rh-cover $\{Z \to X, \tilde{X} \to X\}$ can be refined by the hrh-cover $\{Z' \to X, \mathrm{Bl}_{Z'}(X) \to X\}$ which was to be shown. The second part has the same proof. □

rh-versions of cdh-results

The cdh-topology relates to the Nisnevich topology in the same way as the rh-topology relates to the Zariksi topology. Thus a lot of results in the literature concerning the cdh-topology are also valid for the rh-topology. Possible occurences of the Nisnevich topology may be substituted by the Zariski topology. In this spirit, we will show that one can decompose rh-covers into proper rh-covers and Zariski covers (Proposition 2.2.13) and that the rh-sheafification of algebraic K-theory is homotopy K-theory (Theorem 2.2.14). These statements will be used to prove the main result.

Notation. Let S be a scheme and let $\mathrm{Sh}_{\mathrm{Ab}}(\mathrm{Sch}_S^{\mathrm{rh}})$ be the category of rh-sheaves on Sch_S with values in abelian groups. Its inclusion into the category $\mathrm{PSh}_{\mathrm{Ab}}(\mathrm{Sch}_S)$ of presheaves on Sch_S with values in abelian groups admits an exact left adjoint a_{rh}. Similarly, the inclusion $\mathbf{Sh}_{\mathbf{Sp}}(\mathrm{Sch}_S^{\mathrm{rh}}) \hookrightarrow \mathbf{PSh}_{\mathbf{Sp}}(\mathrm{Sch}_S)$ of rh-sheaves on Sch_S with values in the ∞-category of spectra $\mathbf{Sp}$ admits an exact left adjoint L_{rh}.

Proposition 2.2.13 (cf. [SV00, 5.9]). *Let S be a noetherian base scheme and let $X \in \mathrm{Sch}_S$. Every rh-cover of X admits a refinement of the form $U \xrightarrow{f} \tilde{X} \xrightarrow{p} X$ where f is a Zariski cover and p is a proper rh-cover.*

The analogous result for the cdh-topology is due to Suslin-Voevodsky [SV00, 5.9]. Their proof applies here almost verbatim by exchanging the terms “cdh” by “rh”, “Nisnevich” by “Zariksi”, and “étale morphism” by “open immersion”. Nevertheless, we present here an adapted version of the proof given in [MVW06, 12.27,12.28].

Proof. The inclusion of the underlying reduced subscheme and the disjoint union of the irreducible components both are proper rh-covers. Thus one can reduce to the case where X is integral.

[3] Even though most parts of Temkin's article [Tem08] deal with characteristic zero, this is not the case for the mentioned result.

First, assume that we have given a cover $T \to U \to X$ where $T \to U$ is a proper rh-cover and $U \to X$ is a Zariski cover. Let $(U_i)_i$ be the irreducible components of U and set $T_i := T \times_U U_i$. By refining the cover we may assume that the proper maps $T_i \to U_i$ are also birational. By construction, the morphisms $T_i \to X$ are flat over some (necessarily dense) open subsets Y_i of X. Applying platification par éclatement [RG71, 5.2.2] to the $\mathcal{O}_X$-module $\mathcal{O}_{T_i}$ yields Y_i-admissible blow-ups $X'_i \to X$ such that the strict transforms T'_i of T_i are flat over X'_i. Let $X = \mathrm{Bl}_Z(X) \to X$ be a blow-up dominating the $X'_i \to X$ and identify the T'_i with their pullbacks along the respective maps $X' \to X'_i$. Setting $U'_i := U_i \times_X X'$ we are in the situation described in the following commutative diagram.

$$\begin{array}{ccccccc} T'_i & \longrightarrow & T_i \times_X X' & \longrightarrow & U'_i & \longrightarrow & X' \\ & & \downarrow & & \downarrow & & \downarrow \\ & & T_i & \longrightarrow & U_i & \longrightarrow & X \end{array}$$

Now the compositions $T'_i \to U'_i \to X'$ are flat and $U'_i \to X'$ are open immersions, hence unramified. Thus the maps $T'_i \to U'_i$ are also flat. On the other hand, they are also proper and birational, hence isomorphisms (since the U'_i are irreducible by construction). It follows that $T \times_X X' \to X'$ admits a refinement $U' := \sqcup_i U'_i \cong \sqcup_i T'_i \to X'$ which is a Zariski cover.

The centre Z of the blow-up $X' \to X$ has dimension strictly smaller than the dimension of X. Since the zero-dimesnional case is clear, we can assume by induction that the induced cover $T \times_X Z \to U \times_X Z \to Z$ admits a refinement $V' \to Z' \to Z$ where $V' \to Z'$ is a Zariski cover and $Z' \to Z$ is a proper rh-cover. Thus the initial cover $T \to X$ can be refined by the cover

$$V := V' \sqcup U' \longrightarrow \tilde{X} := Z' \sqcup X' \to X$$

where $V \to \tilde{X}$ is a Zariski cover and $\tilde{X} \to X$ is a proper h-cover.

A general rh-cover has a refinement of the form

$$X_n \to X_{n-1} \to \dots X_1 \to X$$

where every map $X_i \to X_{i-1}$ is either a Zariski cover or a proper rh-cover. By the argument above, one can by iterative refinements bring all Zariski covers to the left so that we obtain our desired refinement. □

The following theorem and its proof are just rh-variants of the corresponding statement for the cdh-topology by Kerz-Strunk-Tamme [KST18b, 6.3]. The theorem goes back to Haesemeyer [Hae04]. Another recent proof for the cdh-topology which also works for the rh-topology was recently given by Kelly-Morrow [KM18, 3.4].

Theorem 2.2.14. *Let S be a finite-dimensional noetherian scheme. Then the canoni-*

cal maps of rh-sheaves with values in spectra on Sch_S

$$\mathrm{L}_{\mathrm{rh}}\mathrm{K}_{\geq 0} \longrightarrow \mathrm{L}_{\mathrm{rh}}\mathrm{K} \longrightarrow \mathrm{KH}$$

are equivalences.

In the proof of the theorem we will make use of the following lemma.

Lemma 2.2.15. *Let S be a finite dimensional noetherian scheme and let F be a presheaf of abelian groups on* Sch_S. *Assume that*

(i) for every reduced affine scheme X and every element $\alpha \in F(X)$ there exists a proper birational morphism $X' \to X$ such that $F(X) \to F(X')$ maps α to zero and that

(ii) $F(Z) = 0$ if $\dim(Z) = 0$ and Z is reduced.

Then $\mathrm{a}_{\mathrm{rh}}F = 0$.

Proof. It suffices to show that for every affine scheme X the map

$$F(X) \longrightarrow \mathrm{a}_{\mathrm{rh}}F(X)$$

vanishes as this implies that the rh-stalks are zero. Consider the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & \mathrm{a}_{\mathrm{rh}}F(X) \\ \downarrow & & \downarrow \\ F(X_{\mathrm{red}}) & \longrightarrow & \mathrm{a}_{\mathrm{rh}}F(X_{\mathrm{red}}). \end{array}$$

As $X_{\mathrm{red}} \to X$ is an rh-cover, the right vertical map is an isomorphism. Thus we can assume that X is reduced. For any $\alpha \in F(X)$ there exists, by condition (i), a proper birational morphism $f\colon X' \to X$ such that α maps to zero in $F(X')$. Let U be an open dense subscheme of X over which f is an isomorphism and set $Z := (X \smallsetminus U)_{\mathrm{red}}$. Then $\dim(Z) < \dim(X)$ as U is dense. By condition (ii) we can argue by induction that α maps to zero in $\mathrm{a}_{\mathrm{rh}}F(Z)$. Since $X' \sqcup Z \to X$ is an rh-cover by construction, α vanishes on some rh-cover of Y. Thus the map $F(X) \to \mathrm{a}_{\mathrm{rh}}F(X)$ maps every element to zero which finishes the proof. □

Example 2.2.16. (i) For $i < 0$, the functor $F = \mathrm{K}_i$ satisfies the conditions of Lemma 2.2.15. Zero-dimensional reduced schemes are regular and hence their negative K-theory vanishes, and condition (i) was proven by Kerz-Strunk [KS17, Prop. 5].

(ii) Another example is the functor $F = N\mathrm{K}_i$ for $i \in \mathbf{Z}$ which is defined by $N\mathrm{K}_i(X) := \mathrm{K}_i(\mathbf{A}^1_X)/\mathrm{K}_i(X)$. The K-theory of regular schemes is homotopy invariant, and condition (i) was proven by Kerz-Strunk-Tamme [KST18b, Prop. 6.4].

Proof of Theorem 2.2.14. In the light of Corollary A.2.14 we can test the desired equivalences on stalks. Since spheres are compact, taking homotopy groups commutes with filtered colimits and we can check on the sheaves of homotopy groups of the stalks whether the maps are equivalences. Thus the first equivalence follows directly by applying Lemma 2.2.15 and Example 2.2.16 (i) and since the connective cover has isomorphic non-negative homotopy groups.

For the second equivalence we assume for a moment the existence of a weakly convergent spectral sequence

$$E^1_{p,q} = \mathrm{a}_{\mathrm{rh}} N^p \mathrm{K}_q \Rightarrow \mathrm{a}_{\mathrm{rh}} \mathrm{KH}_{p+q}$$

in $\mathrm{Sh}_{\mathrm{Ab}}(\mathrm{Sch}^{\mathrm{rh}}_S)$. It suffices to show that $\mathrm{a}_{\mathrm{rh}} N^p \mathrm{K}_q = 0$ for $p \geq 1$ which follows from Lemma 2.2.15 and Example 2.2.16 (ii). Thus the proof is finished by the following lemma. □

Lemma 2.2.17. *There is a weakly convergent spectral sequence*

$$E^1_{p,q} = \mathrm{a}_{\mathrm{rh}} N^p \mathrm{K}_q \Rightarrow \mathrm{a}_{\mathrm{rh}} \mathrm{KH}_{p+q}$$

of rh-sheaves of abelian groups on Sch_X.

Proof. For every ring R there is a weakly convergent spectral sequence [Wei13, IV.12.3]

$$E^1_{p,q} = N^p \mathrm{K}_q(R) \Rightarrow \mathrm{KH}_{p+q}(R).$$

This yields a spectral sequence $E^1_{p,q} = N^p \mathrm{K}_q$ on the associated presheaves of abelian groups on Sch_X and hence a spectral sequence $E^1_{p,q} = \mathrm{a}_{\mathrm{rh}} N^p \mathrm{K}_q$ of the associated rh-sheafifications. We have to check that the latter one converges to $\mathrm{a}_{\mathrm{rh}} \mathrm{KH}_{p+q}$. This can be tested on rh-stalks which are filtered colimits of the weakly convergent spectral sequence above. As filtered colimits commute with colimits and finite limits, a filtered colimit of weakly convergent spectral sequence yields a weakly convergent spectral sequence. Hence we are done. □

Theorem 2.2.18. *Let X be a d-dimensional noetherian scheme. Then there exists a canonical isomorphism*

$$\mathrm{K}_{-d}(X) \cong \mathrm{H}^d_{\mathrm{rh}}(X;\mathbf{Z}).$$

Proof. As KH is an rh-sheaf, the Zariski descent spectral sequence appears as

$$E^{p,q}_2 = \mathrm{H}^p_{\mathrm{rh}}(X, \mathrm{a}_{\mathrm{rh}}(\mathrm{K}_{-q})) \Rightarrow \mathrm{KH}_{-p-q}(X).$$

We know the following:

- $E^{p,q}_2 = 0$ for $p > d$ as the rh-cohomological dimension is bounded by the dimension (Corollary A.2.8).

- $\mathrm{a}_{\mathrm{rh}}(\mathrm{K}_{\geq 0})_0 = \mathbf{Z}$ since $(\mathrm{K}_{\geq 0})_0(R) = \mathrm{K}_0(R) = \mathbf{Z}$ for any local ring R.

- $K_{-d}(X) \cong KH_{-d}(X)$ by K_{-d}-regularity and the vanishing of K_{-i} for $i > \dim(X)$ [KST18b, Thm. B] together with the spectral sequence relating K-theory and KH-theory [Wei13, IV.12.3].

- $a_{rh} K_{-q} = 0$ for $q > 0$ by Lemma 2.2.15 and Example 2.2.16.

This implies that in the 2-page only the term $E_2^{d,0} = H^d_{rh}(X; \mathbf{Z})$ contributes on the line $-p-q = -d$ and that already $E_2^{d,0} = E_\infty^{d,0}$ since all differentials of all $E_i^{d,0}$ for $i \geq 2$ come from or go to zero. Hence the theorem follows. □

2.3. Cohomology of Zariski-Riemann spaces

This section is the heart of the present thesis providing the key ingredient for the proof of our main result; namely, a comparison of Zariski cohomology and rh-cohomology for Zariski-Riemann spaces (Theorem 2.3.16). This will be done in two steps passing through the biZariski topology.

Definition 2.3.1. Let S be a noetherian scheme. The **biZariski topology** is the topology generated by Zariski covers as well as by **closed covers**, i.e. covers of the form $\{Z_i \to X\}_i$ where $X \in \mathrm{Sch}_S$ and the Z_i are *finitely many* jointly surjective closed subschemes of X. This yields a site $\mathrm{Sch}_S^{\mathrm{biZar}}$.

Lemma 2.3.2. *The points on the biZariski site are precisely the spectra of integral local rings.*

Proof. This follows from the fact that local rings are points for the Zariski topology and integral rings are points for the closed topology [GK15]. □

Lemma 2.3.3. *Let X be a noetherian scheme. The cover of X by its irreducible components is a subcover of every closed cover.*

Proof. Let $(X_i)_i$ be the irreducible components of X with generic points $\eta_i \in X_i$. Let $X = \bigcup_\alpha Z_\alpha$ be a closed cover. For every i there exists an α such that $\eta_i \in Z_\alpha$, hence $X_i = \overline{\{\eta_i\}} \subseteq \overline{Z}_\alpha = Z_\alpha$. By maximality of the irreducible components we have equality and are done. □

Lemma 2.3.4. *Let S be a noetherian scheme. Every constant Zariski sheaf on* Sch_S *is already a bi-Zariski sheaf.*

Proof. Let A be an abelian group. Via the Yoneda embedding we consider A as a presheaf and we denote by $\underline{A}$ its associated Zariski sheaf. Let $\iota: Z \to X$ be a closed immersion in Sch_S and denote by $\underline{A}_Z$ and $\underline{A}_X$ the restrictions of $\underline{A}$ to $\mathbf{Sh}(Z)$ respectively $\mathbf{Sh}(X)$. There is a morphism $\underline{A}_X \to \iota_* \underline{A}_Z$ of sheaves on X which is given by the restriction

$$\underline{A}_X(U) = \underline{A}(U) \to \underline{A}(U \cap Z) = \underline{A}_Z(U \cap Z) = \iota_* \underline{A}_Z(U).$$

for $U \subset X$ open. Hence we get an adjoint map $\iota^{-1}\underline{A}_X \to \underline{A}_Z$. This map is an isomorphism; indeed, for $x \in Z$ we have $(\iota^{-1}\underline{A}_X)_x \cong (\underline{A}_X)_x \cong (\underline{A}_Z)_x$.

We have to show that $\underline{A}$ satisfies the sheaf axiom for closed covers. Let $\{Z_i \to X\}_i$ be one. We claim that the sequence

$$0 \to \underline{A}(X) \to \prod_i \underline{A}(Z_i) \to \prod_{i,j} \underline{A}(Z_i \times_X Z_j)$$

is exact. Indeed, by the above considerations we get

$$\underline{A}(Z_i) = \underline{A}_{Z_i}(Z_i) = (Z_i \hookrightarrow X)^{-1}\underline{A}_X(Z_i) = \operatorname*{colim}_{Z_i \subset U} \underline{A}(U)$$

where the colimit runs over all open subsets $U \subset X$ containing Z_i. This implies that an element $(f_i)_i$ of $\prod_i \underline{A}(Z_i)$ is represented by an element $(f_i')_i$ in $\prod_i \underline{A}(U_i)$ for suitable open subschemes U_i containing Z_i. In particular, $\{U_i \to X\}_i$ is a Zariski cover of X. Now we can use the sheaf condition for this cover. Here there is a small subtlety. If f_i and f_j agree on $Z_i \cap Z_j$, then f_i' and f_j' do not necessarily have to agree on $U_i \cap U_j$, but they agree in some open subset $V_{i,j}$ which satisfies $U_i \cap U_j \supset V_{i,j} \supset Z_i \cap Z_j$. Then one has to replace every U_i by the smaller set $\bigcap_{j \neq i}\big(V_{i,j} \cup (U_i \smallsetminus Z_j)\big)$ which does the job. □

Alternative proof. Let A be an abelian group. For an open subset U of $X \in \mathrm{Sch}_S$, the sections over U are precisely the locally constant functions $f: U \to A$. By Lemma 2.3.3, it suffices to check the sheaf condition for the cover of U by its irreducible components $(U_i)_i$. We only have to show the glueing property. If $f_i: U_i \to A$ are locally constant functions which agree on all intersections, then they glue to a function $f: U \to A$. We have to show that f is locally constant. If $x \in U$, for every i such that $x \in U_i$ there exists an open neighbourhood V_i of x in U such that f becomes constant when restricted to $U_i \cap V_i$. Hence f becomes also constant when restricted to the intersection of all these V_i. Thus f is locally constant. □

Remark 2.3.5. For an arbitrary sheaf F on $\mathrm{Sch}_S^{\mathrm{Zar}}$ and a closed immersion $\iota: Z \to X$ it is not true that the canonical map $\iota^{-1}F_X \to F_Z$ is an isomorphism. For instance, let $F = \mathcal{O}$ be the structure sheaf on $\mathrm{Sch}_{\mathbf{Z}}$, $X = \mathrm{Spec}(\mathbf{Z})$ and $Z = \mathrm{Spec}(\mathbf{Z}/2)$. Then the morphism $\iota^{-1}\mathcal{O}_{\mathrm{Spec}(\mathbf{Z})} \to \mathcal{O}_{\mathrm{Spec}(\mathbf{Z}/2)}$ yields on global sections the map $\mathbf{Z}_{(2)} \to \mathbf{Z}/2$.

Lemma 2.3.6. *Let X be a scheme and let τ and σ be two topologies on Sch_X. Let F be a τ-sheaf on Sch_X. Then:*

(i) Let $(S_i \to U)_i$ be a σ-cover of some $U \in \mathrm{Sch}_X$. Then F satisfies the sheaf condition with respect to the σ-cover $(S_i)_i$ if there exists a τ-cover $(T_\alpha \to U)_\alpha$ such that for all α, β the sheaf condition with respect to the σ-covers $(T_\alpha \times_U S_i \to T_\alpha)_i$ and $(T_\alpha \times_U T_\beta \times_U S_i \to T_\alpha \times_U T_\beta)_i$ are satisfied.

(ii) The restriction $F|_U$ is a σ-sheaf on Sch_U if and only if for any τ-cover $(T_\alpha \to U)_\alpha$ and for every α the restriction $F|_{T_\alpha}$ is a σ-sheaf on Sch_{T_α}.

More concisely, a τ-sheaf is a σ-sheaf if and only if it is τ-locally a σ-sheaf.

Proof. For (i) let $U \in \mathrm{Sch}_X$ and let $(S_i)_i$ be a σ-cover of U. If the second and the third row in the commutative diagram with exact columns

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(U) & \longrightarrow & \prod_i F(S_i) & \longrightarrow & \prod_{i,j} F(S_{i,j}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_\alpha F(T_\alpha \times U) & \longrightarrow & \prod_\alpha \prod_i F(T_\alpha \times S_i) & \longrightarrow & \prod_\alpha \prod_{i,j} F(T_\alpha \times S_{i,j}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_{\alpha,\beta} F(T_{\alpha,\beta} \times U) & \longrightarrow & \prod_{\alpha,\beta} \prod_i F(T_{\alpha,\beta} \times S_i) & \longrightarrow & \prod_{\alpha,\beta} \prod_{i,j} F(T_{\alpha,\beta} \times S_{i,j}) \end{array}$$

are exact, then also the first row (where $T_{\alpha,\beta} := T_\alpha \times T_\beta$ and $S_{i,j} := S_i \times S_j$). This shows the claim. The statement (ii) follows similarly. □

Lemma 2.3.7. *Let S be a noetherian base scheme and let $X \in \mathrm{Sch}_S$. For any constant sheaf A on $\mathrm{Sch}_S^{\mathrm{Zar}}$, we have $\mathrm{H}_{\mathrm{Zar}}(X,A) \cong \mathrm{H}_{\mathrm{biZar}}(X,A)$.*

Proof. Let $u\colon \mathrm{Sch}_S^{\mathrm{biZar}} \to \mathrm{Sch}_S^{\mathrm{Zar}}$ be the change of topology morphism of sites. Using the Leray spectral sequence

$$\mathrm{H}^p_{\mathrm{Zar}}(X, R^q u_* A) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{biZar}}(X,A)$$

it is enough to show that the higher images $R^q u_* A$ vanish for $q > 0$. We know that $R^q u_* A$ is the Zariski sheaf associated to the presheaf

$$\mathrm{Sch}_S^{\mathrm{Zar}} \ni U \mapsto \mathrm{H}^q_{\mathrm{biZar}}(U;A)$$

and that its stalks are given by $\mathrm{H}^q_{\mathrm{biZar}}(X;A)$ for X a local scheme (i.e. the spectrum of a local ring). As the biZariski sheafification of $R^q u_* A$ is zero amd using Lemma 2.3.3, we see that $\mathrm{H}^q_{\mathrm{biZar}}(X;A) = 0$ for every irreducible local scheme X. For a general local scheme X we can reduce to the case where X is covered by two irreducible components Z_1 and Z_2.

First, let $q = 1$. We have an exact Mayer-Vietoris sequence

$$0 \to A(X) \to A(Z_1) \times A(Z_2) \overset{\alpha}{\to} A(Z_1 \cap Z_2) \overset{\partial}{\to} \qquad (\Delta)$$
$$\overset{\partial}{\to} \mathrm{H}^1_{\mathrm{biZar}}(X;A) \to \mathrm{H}^1_{\mathrm{biZar}}(Z_1;A) \times \mathrm{H}^1_{\mathrm{biZar}}(Z_2;A) \to \mathrm{H}^1_{\mathrm{biZar}}(Z_1 \cap Z_2;A).$$

Since local schemes are connected, the map α is surjective, hence $\partial = 0$ and the second line remains exact with a zero added on the left. Thus $\mathrm{H}^1_{\mathrm{biZar}}(X;A) = 0$ for any local scheme, hence $R^1 u_* A$ vanishes.

For $q > 1$ we proceed by induction. Let

$$0 \to A \to I \to G \to 0$$

be an exact sequence of biZariski sheaves such that I is injective. This yields a commutative diagram with exact rows and columns

$$\begin{array}{ccccc} I(Z_1)\times I(Z_2) & \longrightarrow & I(Z_1\cap Z_2) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ G(Z_1)\times G(Z_2) & \xrightarrow{\beta} & G(Z_1\cap Z_2) & & \\ & & \downarrow & & \\ & & \mathrm{H}^1_{\mathrm{biZar}}(Z_1\cap Z_2;A). & & \end{array}$$

Being a closed subscheme of a local scheme, $Z_1 \cap Z_2$ is also a local scheme. By the case $q = 1$, the group $\mathrm{H}^1_{\mathrm{biZar}}(Z_1 \cap Z_2;A)$ vanishes. Hence the map β is surjective. Using the analogous Mayer-Vietoris sequence (Δ) above for G instead of A, we can conclude that $R^2u_*A \cong R^1u_*G = 0$. Going on, we get the desired vanishing of $R^q u_* A$ for every $q > 0$. □

Notation. For the rest of this section, let X be a quasi-compact and quasi-separated scheme and let U be a quasi-compact dense open subscheme of X. We denote by Z the closed complement equipped with the reduced scheme structure.

Definition 2.3.8. For any morphism $p: X' \to X$ we get an analogous decomposition

$$X'_Z \longrightarrow X' \longleftarrow X'_U$$

where $X'_Z := X' \times_X Z$ and $X'_U := X' \times_X U$. By abuse of notation, we call X_Z the **special fibre** of X' and X_U the **generic fibre** of X'. An **(abstract) admissible blow-up** of X' is a proper map $X'' \to X'$ inducing an isomorphism $X''_U \xrightarrow{\cong} X'_U$ over X. In particular, one obtains an abstract blow-up square

$$\begin{array}{ccc} X''_Z & \longrightarrow & X'' \\ \downarrow & & \downarrow \\ X'_Z & \longrightarrow & X'. \end{array}$$

At the end of this section, we will see that the Zariski cohomology and the rh-cohomology on the Zariski-Riemann space coincide (Theorem 2.3.16). The following proposition will be used in the proof to reduce from the rh-topology to the biZariski topology.

Proposition 2.3.9. *Let X be noetherian and let $X' \in \mathrm{Sch}_X$. Then for every proper rh-cover of the special fibre X'_Z there exists an admissible blow-up $X'' \to X'$ such that the induced rh-cover of X''_Z can be refined by a closed cover.*

Proof. We may assume that X' is reduced. Every proper rh-cover can be refined by a birational proper rh-cover (Lemma 2.2.5). Thus a cover yields a blow-up square which can be refined by an honest blow-up square

$$\begin{array}{ccc} E' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ V' & \longrightarrow & X'_Z \end{array}$$

i.e. an abstract blow-up square where $Y' = \mathrm{Bl}_{V'}(X'_Z)$ (Lemma 2.2.12). We consider the honest blow-up square

$$\begin{array}{ccc} V'' & \longrightarrow & X'' := \mathrm{Bl}_{V'}(X') \\ \downarrow & & \downarrow \\ V' & \longrightarrow & X'_Z \end{array}$$

which is an admissible blow-up as $V' \subseteq X'_Z$ and decomposes into two cartesian squares

$$\begin{array}{ccccc} V'' & \longrightarrow & \tilde{X}''_Z & \longrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ V' & \longrightarrow & X'_Z & \longrightarrow & X' \end{array}$$

where all the horizontal maps are closed immersions. By functoriality of blow-ups, we obtain a commutative diagram

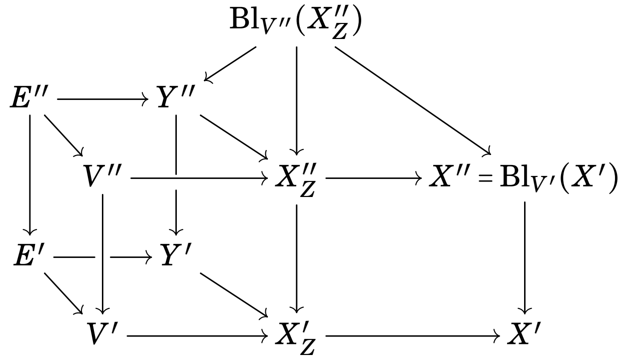
$$\begin{array}{ccc} \mathrm{Bl}_{V''}(X''_Z) & \longrightarrow & \mathrm{Bl}_{V''}(X'') = X'' \\ \downarrow & & \downarrow \\ X''_Z & \longrightarrow & X'' \end{array}$$

wherein both horizontal maps are closed immersions and the right vertical map is an isomorphism by the universal property of the blow-up. Thus $\mathrm{Bl}_{V''}(X''_Z) \to X''_Z$ is a closed immersion [GW10, Rem. 9.11]. Functoriality of blow-ups yields a commutative square

$$\begin{array}{ccc} \mathrm{Bl}_{V''}(X''_Z) & \longrightarrow & \mathrm{Bl}_{V'}(X'_Z) = Y' \\ \downarrow & & \downarrow \\ X''_Z & \longrightarrow & X'_Z. \end{array}$$

By the universal property of the pullback, there exists a unique map $\mathrm{Bl}_{V''}(X''_Z) \to$

$Y'' := Y' \times_{X'_Z} X''_Z$ such that following diagram commutes.



To sum up, we have shown that the pullback of the proper rh-cover $V' \sqcup Y' \to X'_Z$ along $X''_Z \to X'_Z$ can be refined by the closed cover $V'' \sqcup \mathrm{Bl}_{V''}(X''_Z) \to X''_Z$ which was to be shown. □

Given a topology on (some appropriate subcategory of) the category of schemes, we want to have a corresponding topology on Zariski-Riemann spaces. For this purpose, we will work with an appropriate site.

Notation. Let τ be a topology on the category Sch_X. One obtains compatible topologies on the slice categories $\mathrm{Sch}_{X'} = (\mathrm{Sch}_X)_{/X'}$ for all U-modifications $X' \in \mathrm{Mdf}(X,U)$.

Definition 2.3.10. Consider the category

$$\mathrm{Sch}(\langle X\rangle_U) := \operatorname*{colim}_{X' \in \mathrm{Mdf}(X,U)} \mathrm{Sch}_{X'}.$$

More precisely, the set of objects is the set of morphisms of schemes $Y' \to X'$ for some $X' \in \mathrm{Mdf}(X,U)$. The set of morphisms between two objects $Y' \to X'$ and $Y'' \to X''$ is given by

$$\operatorname*{colim}_{\tilde{X}} \mathrm{Hom}_{\mathrm{Sch}}(Y' \times_{X'} \tilde{X}, Y'' \times_{X''} \tilde{X})$$

where $\tilde{X}$ runs over all modifications $\tilde{X} \in \mathrm{Mdf}(X,U)$ dominating both X' and X''. Analogously, define the category

$$\mathrm{Sch}(\langle X\rangle_U \smallsetminus U) := \operatorname*{colim}_{X' \in \mathrm{Mdf}(X,U)} \mathrm{Sch}_{X' \smallsetminus U}$$

where the $X' \smallsetminus U$ are equipped with the reduced scheme structure.

Definition 2.3.11. Let $Y' \to X'$ be an object of $\mathrm{Sch}(\langle X\rangle_U)$. We declare a sieve R on Y' to be a **τ-covering sieve** of $Y' \to X'$ iff there exists a U-modification $p: X'' \to X'$ such that the pullback sieve p^*R lies in $\tau(Y' \times_{X'} X'')$. Analogously we define τ-covering sieves in $\mathrm{Sch}(\langle X\rangle_U \smallsetminus U)$.

Lemma 2.3.12. *The collection of τ-covering sieves in Definition 2.3.11 defines a topology on the categories* $\mathrm{Sch}(\langle X\rangle_U)$ *and* $\mathrm{Sch}(\langle X\rangle_U \smallsetminus U)$ *which we will denote with the same symbol τ.*

Remark 2.3.13. In practice, for working with the site $(\mathrm{Sch}(\langle X\rangle_U),\tau)$ it is enough to consider τ-covers in the category Sch_X and identifying them with their pullbacks along U-modifications.

Caveat 2.3.14. The category $\mathrm{Sch}(\langle X\rangle_U)$ is not a slice category, i.e. a scheme Y together with a morphism of locally ringed spaces $Y \to \langle X\rangle_U$ does not necissarily yield an object of $\mathrm{Sch}(\langle X\rangle_U)$. Such objects were studied e.g. by Hakim [Hak72]. In contrast, an object of $\mathrm{Sch}(\langle X\rangle_U)$ is given by a scheme morphism $Y \to X'$ for some $X' \in \mathrm{Mdf}(X,U)$ and it is isomorphic to its pullbacks along admissible blow-ups.

In the proof of the main theorem we will need the following statement which follows from the construction of our site.

Proposition 2.3.15. *Let F be a constant sheaf of abelian groups on* $\mathrm{Sch}(\langle X\rangle_U)$. *Then the canonical morphism*

$$\operatorname*{colim}_{X'\in\mathrm{Mdf}(X,U)} \mathrm{H}^*_\tau(X';F) \longrightarrow \mathrm{H}^*_\tau(\langle X\rangle_U;F)$$

is an isomorphism. Analogously, if F is a constant sheaf of abelian groups on $\mathrm{Sch}(\langle X\rangle_U \smallsetminus U)$*, then the canonical morphism*

$$\operatorname*{colim}_{X'\in\mathrm{Mdf}(X,U)} \mathrm{H}^*_\tau(X'_Z;F) \longrightarrow \mathrm{H}^*_\tau(\langle X\rangle_U \smallsetminus U;F)$$

is an isomorphism.

Proof. This is a special case of [Sta19, Tag 09YP] where the statement is given for any compatible system of abelian sheaves. □

Theorem 2.3.16. *For any constant sheaf F on* $\mathrm{Sch}^{\mathrm{qc}}_{\mathrm{rh}}(\langle X\rangle_U)$*, the canonical map*

$$\mathrm{H}^*_{\mathrm{Zar}}(\langle X\rangle_U \smallsetminus U;F) \longrightarrow \mathrm{H}^*_{\mathrm{rh}}(\langle X\rangle_U \smallsetminus U;F)$$

is an isomorphism.

Proof. By construction, any rh-cover of $\langle X\rangle_U \smallsetminus U$ is represented by an rh-cover of X'_Z for some $X' \in \mathrm{Adm}(A_0)$. By Proposition 2.2.13, we find a refinement $V' \xrightarrow{q} Y' \xrightarrow{p} X'_Z$ where p is a proper rh-cover and q is a Zariski cover. The cover $Y' \to X'_Z$ is given by an abstract blow-up square

$$\begin{array}{ccc} E' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ V' & \longrightarrow & X'_Z. \end{array}$$

This is the situation of Proposition 2.3.9. Thus there exists an admissible blow-up $X'' \to X'$ and a refinement $V'' \sqcup \mathrm{Bl}_{V''}(X''_Z) \to X''_Z$ of the pulled back cover which consists of two closed immersions. Hence we have refined our given cover of $\langle X\rangle_U \smallsetminus U$ by a composition of a Zariski cover and a closed cover which yields a bi-Zariski cover. This implies that $\mathrm{H}^*_{\mathrm{rh}}(\langle X\rangle_U \smallsetminus; F)$ equals $\mathrm{H}_{\mathrm{biZar}}(\langle X\rangle_U \smallsetminus; F)$. Now the assertion follows from Lemma 2.3.7. □

Corollary 2.3.17. *For any constant sheaf, we have*

$$\operatorname*{colim}_{X'\in \mathrm{Mdf}(X,U)} \mathrm{H}^*_{\mathrm{Zar}}(X'_Z; F) = \operatorname*{colim}_{X'\in \mathrm{Mdf}(X,U)} \mathrm{H}^*_{\mathrm{rh}}(X'_Z; F).$$

Proof. This is a formal consequence by the construction of the topology on the category $\mathrm{Sch}^{\mathrm{qc}}(\langle X\rangle_U)$ since the cohomology of a limit site is the colimit of the cohomologies [Sta19, Tag 09YP]. □

2.4. K-theory of Zariski-Riemann spaces

For a regular noetherian scheme, the canonical map from $\mathrm{K}(X) \to G(X)$ from K-theory to G-theory is an equivalence [Wei13, V.3.4]. A similar result for Zariski-Riemann spaces was shown by Kerz-Strunk-Tamme.

Theorem 2.4.1 ([KST18b, Proof of Proposition 6.4])**.** *Let X be a reduced scheme which is quasi-projective over a noetherian ring, let $\mathcal{J}$ be the cofiltered category of reduced schemes which are projective and birational over X and set $\langle X\rangle = \lim_{X'\in\mathcal{J}} X'$ its limit in the category of locally ringed spaces. Then*

$$\mathrm{K}(\mathrm{Vec}(\langle X\rangle)) \longrightarrow \mathrm{K}(\mathrm{Coh}(\langle X\rangle))$$

is an equivalence of spectra.

This indicates that – from the K-theoretic point of view – Zariski-Riemann spaces behave as if they were regular.

Notation. In section 2.4, let (A, A_0, π) be a Tate triple (Definition 1.3.3) and let $\mathrm{Adm}(A_0)$ be the category of admissible blow-ups (Definition 2.1.15).

Remark 2.4.2. The closed immersion $\mathrm{Spec}(A_0/\pi) \hookrightarrow \mathrm{Spec}(A_0)$ induces for every admissible blow-up $X \in \mathrm{Adm}(A_0)$ a closed immersion $X/\pi \hookrightarrow X$. In the limit, they yield a closed map

$$\langle A_0\rangle_A/\pi \hookrightarrow \langle A_0\rangle_A.$$

whose complement is $\mathrm{Spec}(A)$.

Definition 2.4.3. Denote by $\mathrm{Coh}_\pi(X)$ and $\mathrm{Coh}_\pi(\langle A_0\rangle_A)$ the full subcategories of $\mathrm{Coh}(X)$ respectively of $\mathrm{Coh}(\langle A_0\rangle_A)$ spanned by those modules which are π-torsion, i.e. killed by some power of π.

Lemma 2.4.4. *Let* $Z := \langle A_0\rangle_A/\pi$. *Then* $\mathrm{Coh}_\pi(\langle A_0\rangle_A) = \mathrm{Coh}_Z(\langle A_0\rangle_A)$. *Similarly, for every* $X \in \mathrm{Adm}(A_0)$ *and* $Y := X \setminus \mathrm{Spec}(A)$ *one has* $\mathrm{Coh}_\pi(X) = \mathrm{Coh}_Y(X)$.

Lemma 2.4.5. *The inclusion* $\mathrm{Coh}(\langle A_0\rangle_A/\pi) \to \mathrm{Coh}_\pi(\langle A_0\rangle_A)$ *induces an equivalence*

$$\mathrm{K}(\mathrm{Coh}(\langle A_0\rangle_A/\pi)) \xrightarrow{\simeq} \mathrm{K}(\mathrm{Coh}_\pi(\langle A_0\rangle_A)).$$

Proof. We want to apply the Devissage Theorem [Wei13, V.4.1]. Let $M \in \mathrm{Coh}_\pi(\langle A_0\rangle_A)$, i.e. there exist an $n \geq 0$ such that $\pi^n M = 0$. Thus M has a finite filtration

$$0 = \pi^n M \subset \pi^{n-1} M \subset \ldots \subset \pi M \subset M$$

where every subquotient $\pi^i M/\pi^{i-1}M$ lies in $\mathrm{Coh}(\langle A_0\rangle_A/\pi)$. □

Proposition 2.4.6. *There is a fibre sequence*

$$\mathrm{K}\big(\mathrm{Coh}(\langle A_0\rangle_A/\pi)\big) \longrightarrow \mathrm{K}\big(\mathrm{Mod}^{\mathrm{fp}}(\langle A_0\rangle_A)\big) \longrightarrow \mathrm{K}(\mathrm{Coh}(A)).$$

Proof. By Propostion 2.1.14, there is an exact sequence

$$\mathrm{Coh}_{\langle A_0\rangle_A/\pi}(\langle A_0\rangle_A) \longrightarrow \mathrm{Mod}^{\mathrm{fp}}(\langle A_0\rangle_A) \longrightarrow \mathrm{Coh}(A)$$

of abelian categories. By the localisation theorem [Wei13, V.5.1] we get an induced fibre sequence

$$\mathrm{K}\big(\mathrm{Coh}_{\langle A_0\rangle_A/\pi}(\langle A_0\rangle_A)\big) \longrightarrow \mathrm{K}\big(\mathrm{Mod}^{\mathrm{fp}}(\langle A_0\rangle_A)\big) \longrightarrow \mathrm{K}\big(\mathrm{Coh}(A)\big).$$

Then the claim follows from Lemma 2.4.4 and Lemma 2.4.5. □

Definition 2.4.7. We define the **K-theory of the Zariski-Riemann space** as

$$\mathrm{K}(\langle A_0\rangle_A) = \operatorname*{colim}_{X \in \mathrm{Adm}(A_0)} \mathrm{K}(X)$$

an subsequently the **K-theory with support** as

$$\mathrm{K}(\langle A_0\rangle_A \text{ on } \pi) := \mathrm{fib}\big(\mathrm{K}(\langle A_0\rangle_A) \to \mathrm{K}(A)\big)$$

Remark 2.4.8. Since filtered colimits commute with finite limits and with K-theory, one has

$$\begin{aligned} \mathrm{K}(\langle A_0\rangle_A \text{ on } \pi) &= \mathrm{fib}\big(\mathrm{K}(\langle A_0\rangle_A)) \longrightarrow \mathrm{K}(A)\big) \\ &\simeq \mathrm{fib}\big(\operatorname*{colim}_{X \in \mathrm{Adm}(A_0)} \mathrm{K}(X) \longrightarrow \operatorname*{colim}_{X \in \mathrm{Adm}(A_0)} \mathrm{K}(A)\big) \\ &\simeq \operatorname*{colim}_{X \in \mathrm{Adm}(A_0)} \mathrm{fib}\big(\mathrm{K}(X) \longrightarrow \mathrm{K}(A)\big) \\ &\simeq \operatorname*{colim}_{X \in \mathrm{Adm}(A_0)} \mathrm{K}(X \text{ on } \pi). \end{aligned}$$

Theorem 2.4.9 (Kerz). *For $i < 0$ one has* $\mathrm{K}_i(\langle A_0\rangle_A \text{ on } \pi) = 0$.

Proof. Since $\mathrm{K}_i(\langle A_0\rangle_A \text{ on } \pi) \simeq \operatorname{colim}_{X\in \mathrm{Adm}(A_0)} \mathrm{K}(X \text{ on } \pi)$ by design, this follows as the right-hand side vanishes due to a result of Kerz [Ker18, Prop. 7]. □

Theorem 2.4.10. *We have an equivalence of spectra*

$$\mathrm{K}\big(\langle A_0\rangle_A \text{ on } \pi\big) \longrightarrow \mathrm{K}\big(\mathrm{Coh}(\langle A_0\rangle_A/\pi)\big).$$

Proof. Let $\mathrm{Coh}_\pi^{\leq 1}(\langle A_0\rangle_A)$ be the full subcategory of those $M \in \mathrm{Coh}_\pi(\langle A_0\rangle_A)$ such that there exists a resolution

$$0 \to P_1 \to P_0 \to M \to 0$$

where P_1, P_0 are locally-free $\mathcal{O}_{\langle A_0\rangle_A}$-modules. Then

$$\begin{aligned}\mathrm{K}(\langle A_0\rangle_A \text{ on } \pi) &\simeq \operatorname*{colim}_{X\in\mathrm{Adm}(A_0)} \mathrm{K}(X \text{ on } \pi)\\ &\simeq \operatorname*{colim}_{X\in\mathrm{Adm}(A_0)} \mathrm{K}\big(\mathrm{Coh}_\pi^{\leq 1}(X)\big)\\ &\simeq \mathrm{K}\big((\mathrm{Coh}_\pi^{\leq 1}(\langle A_0\rangle_A)\big).\end{aligned}$$

by [TT90, 5.7 (e)] as every scheme in $\mathrm{Adm}(A_0)$ has an ample family of line bundles since they are quasi-projective over an affine scheme [TT90, 2.1.2 (h)]. Now the claim follows from Theorem 2.1.13. □

Example 2.4.11. Let k be a discretely valued field with valuation ring k° and residue field $\tilde{k}$. Consider the scheme $X_0 := \mathrm{Spec}(k^\circ\langle t\rangle)$, its special fibre $X_0/\pi = \mathrm{Spec}(\tilde{k}\langle t\rangle)$ and the open complement $U := \mathrm{Spec}(k\langle t\rangle)$. Let $x_0\colon \mathrm{Spec}(\tilde{k}) \to X_0/\pi$ be the zero section. Then we define the blow-up $X_1 := \mathrm{Bl}_{\{x_0\}}(X_0)$. Its special fibre X_1/π is an affine line over $\tilde{k}$ with an $\mathbf{P}^1_{\tilde{k}}$ attached to the origin. We get a commutative diagram

$$\begin{array}{ccccc} E_1 & \longrightarrow & X_1/\pi & \longrightarrow & X_1 = \mathrm{Bl}_{\{x_0\}}(X_0)\\ \downarrow & & \downarrow \quad \nwarrow & \mathrm{Bl}_{\{x_0\}}(X_0/\pi) \nearrow & \downarrow\\ & & \swarrow{\scriptstyle \cong} & & \\ \{x_0\} & \longrightarrow & X_0/\pi & \longrightarrow & X_0 \end{array}$$

where the two squares are cartesian, all horizontal maps are closed immersions, the blow-up $\mathrm{Bl}_{\{x_0\}}(X_0/\pi) \to X_0/\pi$ is an isomorphism, and the map $\mathrm{Bl}_{\{x_0\}}(X_0/\pi) \to X_1/\pi$ is also closed immersion. Hence there is a closed immersion $X_0/\pi \hookrightarrow X_1/\pi$ which splits the canonical projection. Now choose a closed point $x_1\colon \mathrm{Spec}(\tilde{k}) \to E_1 \setminus X_0/\pi$ and define $X_2 := \mathrm{Bl}_{\{x_1\}}(X_1)$ and iterate this construction. Thus we obtain a stricly increasing chain of closed immersions

$$X_0/\pi \hookrightarrow X_1/\pi \hookrightarrow \ldots \hookrightarrow X_n/\pi \hookrightarrow \ldots\ldots$$

and we can consider X_i/π as a closed subscheme of X_n/π for $i < n$. In each step, X_{n+1}/π is obtained from X_n/π by attaching a $\mathbf{P}^1_{\tilde{k}}$ to at a closed point not contained in X_{n-1}/π. Passing to the Zariski-Riemann space $\langle X_0\rangle_U$, its special fibre $\langle X_0\rangle_U/\pi$ admits a strictly decreasing chain of closed subschemes

$$\langle X_0\rangle_U/\pi = p_0^*(X_0/\pi) \supsetneq p_1^*(X_0/\pi) \supsetneq \ldots \supsetneq p_n^*(X_0/\pi) \supsetneq \ldots\ldots$$

where $p_n\colon \langle X_0\rangle_U/\pi \to X_n/\pi$ denotes the canonical projection. Thus we obatin a strictly increasing chain of ideals

$$0 = \mathcal{I}_0 \subsetneq \mathcal{I}_1 \subsetneq \ldots \subsetneq \mathcal{I}_n \subsetneq \ldots\ldots$$

in $\mathcal{O}_{\langle X_0\rangle_U/\pi}$ where $\mathcal{I}_n$ denotes the ideal sheaf corresponding to $p_n^*(X_0/\pi)$. By construction, they all have support in $\langle X_0\rangle_U/\pi$. Thus the category $\mathrm{Coh}(\langle X_0\rangle_U/\pi)$ is a non-noetherian abelian category whose negative K-theory vanishes due to Theorem 2.4.9 and Theorem 2.4.10.

Remark 2.4.12. Example 2.4.11 gives further evidence for a conjecture of Schlichting whereby the negative K-theory of any abelian category vanishes. Schlichting himself proved the vanishing in degree -1 and subsequently the conjucture in the noetherian case [Sch06, 9.1, 9,3, 9.7]. Further progress in this direction was achieved by Antieau-Gepner-Heller [AGH19].

2.5. Formal Zariski-Riemann spaces and adic spaces

In this section, we will deal with Zariski-Riemann spaces which arise from formal schemes. They are isomorphic to certain adic spaces (Theorem 2.5.7). This identification is used in the proof of the main theorem (Theorem 3.1.1) to obtain the adic spectrum $\mathrm{Spa}(A, A^\circ)$ in the statement.

Formal Zariski-Riemann spaces

Notation. In the remainer of section 2.5, let R be a ring of one of the following types:[4]

(V) R is an adic valuation ring ring with finitely generated ideal of definition.

(N) R is a noetherian adic ring with ideal of definition I such that R does not have I-torsion.

Definition 2.5.1. Let $\mathcal{X}$ be a formal scheme. Let $\mathrm{Adm}(\mathcal{X})$ be the poset of all coherent open ideals of $\mathcal{O}_\mathcal{X}$. For $\mathcal{A} \in \mathrm{Adm}(\mathcal{X})$ denote by $\mathcal{X}_\mathcal{A}$ the admissible formal blow-up of

[4] Cf. [Bos14, §7.3].

$\mathcal{X}$ at $\mathcal{A}$. The **Zariksi-Riemann space** associated with $\mathcal{X}$ is defined as the limit

$$\langle \mathcal{X} \rangle := \lim_{\mathcal{A} \in \mathrm{Adm}(\mathcal{X})} \mathcal{X}_{\mathcal{A}}$$

in the category of locally ringed spaces.

Remark 2.5.2. Let R be a valuation ring with finitely generated ideal of definition I. For any scheme X over $\mathrm{Spec}(R)$ one can associate a formal scheme $\hat{X} := \lim_n X/I^n$ over $\mathrm{Spf}(R)$. Setting $U := X \setminus X/I$, for every U-admissible blow-up $X' \to X$ the induced morphism of $\hat{X}' \to \hat{X}$ is an admissible formal blow-up [Abb10, 3.1.3]. An admissible formal blow-up $\mathcal{X}' \to \hat{X}$ will be called **algebraic** whenever it is induced from a U-admissible blow up of X.

Lemma 2.5.3. *Let X be an R-scheme. Every admissible formal blow-up of $\hat{X}$ is algebraic.*

Proof. Let I be an ideal of definition, set $X/I := X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/I)$, and let $\mathcal{I}$ be the ideal sheaf of $\mathcal{O}_X$ defining X/I. Let $\mathcal{X}' \to \hat{X}$ be an admissible formal blow-up defined by an open ideal $\mathcal{A}$ of $\mathcal{O}_{\hat{X}}$. In particular, there exists an $n \in \mathbf{N}$ such that $\mathcal{I}^n \mathcal{O}_{\hat{X}} \subset \mathcal{A}$. Let $Z_n := X/I^n$ be the closed subscheme of X defined by $\mathcal{I}^n$. This yields a surjective map $\varphi = i^{\#}: \mathcal{O}_X \to i_* \mathcal{O}_{Z_n}$ of sheaves on X where $i: Z_n \to X$ denotes the inclusion. Let $\tilde{\mathcal{A}} := \varphi^{-1}\big(\mathcal{A}/(\mathcal{I}^n \mathcal{O}_{\hat{X}})\big)$. By construction, $i^{-1}\tilde{\mathcal{A}} = \mathcal{A}$ since both have the same pullback to $Z_k = (Z, \mathcal{O}_X/\mathcal{I}^k) = (Z, \mathcal{O}_{\hat{X}}/\mathcal{I}^k)$. Thus $\mathcal{X} = \hat{X}_{\tilde{\mathcal{A}}}$. □

Remark 2.5.4. In particular, this implies that any admissible formal blow-up of an affine formal scheme is algebraic. Indeed, an affine formal scheme $\mathrm{Spf}(A_0)$ is isomorphic to the formal completion of $\mathrm{Spec}(A_0)$. The analogous statement is also true for quasi-affine formal schemes, i.e. open formal subschemes of affine formal schemes. For this purpose we need a previous lemma.

Lemma 2.5.5. *The family* $\big(\mathrm{Spf}(A_0\langle f^{-1}\rangle\big)_{f \in A_0}$ *is a basis of the topology of* $\mathrm{Spf}(A_0)$.

Proof. The family $\big((\mathrm{Spec}(A_0[f^{-1}])\big)_{f \in A_0}$ forms a basis of the topology of $\mathrm{Spec}(A_0)$. Topologically, $\mathrm{Spf}(A_0)$ is a closed subspace of $\mathrm{Spec}(A_0)$. Thus the induced family $\big(\mathrm{Spec}(A_0[f^{-1}] \cap \mathrm{Spf}(A_0)\big)_{f \in A_0}$ is a basis of the topology of $\mathrm{Spf}(A_0)$. As topological spaces, $\mathrm{Spf}(A_0\langle f^{-1}\rangle) = \mathrm{Spec}(A_0[f^{-1}]) \cap \mathrm{Spf}(A_0)$. Hence we are done. □

Lemma 2.5.6. *Every admissible formal blow-up of a quasi-affine formal scheme is algebraic.*

Proof. Let $j: \mathcal{U} \hookrightarrow \mathcal{X} = \mathrm{Spf}(A_0)$ be the inclusion of an open formal subscheme. and let $\mathcal{U}' \to \mathcal{U}$ be an admissible formal blow-up defined by a coherent open ideal $\mathcal{A}_U \subseteq \mathcal{O}_{\mathcal{U}}$. Then there exists a coherent open ideal $\mathcal{A} \subseteq \mathcal{O}_{\mathcal{X}}$ such that $\mathcal{A}|_U \cong \mathcal{A}_U$ and $\mathcal{A}|_V \cong \mathcal{O}_V$ whenever $V \cap U = \varnothing$ [Bos14, §8, Prop. 13]. In particular, $\mathcal{U}' \to \mathcal{U}$ extends to an admissible formal blow-up $\mathcal{X}' \to \mathcal{X}$. By Lemma 2.5.3, this blow-up comes from an an admissible blow-up $p: X' \to X = \mathrm{Spec}(A_0)$. By Lemma 2.5.5, we can write $\mathcal{U} = \bigcup_{i=1}^n \mathcal{U}_i$

with $\mathcal{U}_i = \mathrm{Spf}(A_0\langle f_i^{-1}\rangle)$ for suitable $f_1,\dots,f_n \in A_0$. Setting $U_i := \mathrm{Spec}(A_0[f_i^{-1}])$ and $U_i' := p^{-1}(U_i)$ and $U' := \bigcup_{i=1}^n U_i'$ the union in X', then we obtain that

$$\mathcal{U}' = \bigcup_{i=1}^{n} \hat{U}_i = \hat{U}'$$

which finishes the proof. □

Formal Zariski-Riemann spaces and adic spaces

In this section, we relate the Zariski-Riemann space associated with a formal scheme with the adic space associated with its generic fibre.

Notation. In the remainer of this section, let k be a complete nonarchimedean field, i.e. a topological field whose topology is induced by a nonarchimedean norm.

Theorem 2.5.7 ([Sch12, 2.22])**.** *Let X^{ad} be a quasi-compact and quasi-separated adic space locally of finite type over k. Then there exists a formal model $\mathcal{X}$ of X^{ad} and there is a homeomorphism $X^{\mathrm{ad}} \xrightarrow{\cong} \langle \mathcal{X} \rangle$ which extends to an isomorphism*

$$(X, \mathcal{O}_X^+) \xrightarrow{\cong} \lim_{\mathcal{X}' \in \mathrm{Adm}(\mathcal{X})} (\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$$

of locally ringed spaces.[5]

Since there seems to be no proof in the literature, we present an argument here.

Proof. The category of quasi-separated adic spaces locally of finite type over k is equivalent to the category of quasi-separated rigid analytic k-varieties [Hub94, 4.5(iv)] and this equivalence restricts to the full subcategories of quasi-compact objects on both sides. Hence there exists a quasi-compact and quasi-separated rigid analytic k-variety X^{rig} corresponding to the adic space X. Furthermore, there is a functorial map $\varrho: X^{\mathrm{ad}} \to X^{\mathrm{rig}}$ of ringed sites [Hub94, 4.3].

According to Raynaud, the generic fibre functor $(-)_\eta: \mathrm{FSch}_{k^\circ} \to \mathrm{Rig}_k$ induces an equivalence from the localisation of the category of admissible formal schemes over k° by the admissible formal blow-ups to the category of quasi-compact and quasi-separated rigid analytic k-varieties. Hence there exists a quasi-compact and quasi-separated formal scheme $\mathcal{X}$ over k° whose generic fibre is X^{rig}. This shows the existence of the formal model. The rest follows from Proposition 2.5.9 below. □

Definition 2.5.8. Given a formal scheme $\mathcal{X}$ over k°, one can assign to it a rigid k-space $\mathcal{X}_\eta$ using Raynaud's generic fibre functor. Then we call the corresponding adic space the **adic generic fibre** of $\mathcal{X}$.

[5] In fact, this is also true if we replace quasi-compact by the weaker notion of quasi-paracompactness.

Proposition 2.5.9. *Let $\mathcal{X}$ be a quasi-compact and quasi-separated formal scheme over k°, let X^{rig} be Raynaud's generic fibre of $\mathcal{X}$, and let X^{ad} be its adic generic fibre. Then there exist functorial maps $X^{\mathrm{ad}} \overset{\varrho}{\to} X^{\mathrm{rig}} \overset{\mathrm{sp}}{\to} \langle \mathcal{X} \rangle$ which induce equivalences of sites*

$$\mathrm{Sh}(X^{\mathrm{ad}}) \xrightarrow{\varrho_*} \mathrm{Sh}(X^{\mathrm{rig}}) \xrightarrow{\mathrm{sp}_*} \mathrm{Sh}(\langle \mathcal{X} \rangle)$$

In particular, the map $X^{\mathrm{ad}} \to \langle \mathcal{X} \rangle$ is a homeomorphism.

Proof. The map ϱ was constructed by Huber [Hub94, 4.3] who also showed that ϱ_* is an equivalence [Hub94, 4.5(i)]. The second map is described in Bosch's lecture notes [Bos14, p. 222] and the second equivalence, for abelian sheaves, is sketched in [Bos14, §9, Prop. 4] but the proof works for arbitrary set-valued sheaves as well. The last statement follows from the fact that the functor sending sober topological spaces to its sheaf topoi is fully faithful. □

Remark 2.5.10. (i) In the situation of Proposition 2.5.9, one could also show that the functor sp_* is an equivalence as follows. First one has to show that it factors as

$$\mathrm{Sh}(\mathcal{X}_\eta) \longrightarrow \operatorname*{colim}_{\mathcal{X}' \in \mathrm{Adm}(\mathcal{X})} \mathrm{Sh}(\mathcal{X}') \longrightarrow \mathrm{Sh}(\langle \mathcal{X} \rangle).$$

Then the first functor is an equivalence due to Abbes [Abb10, 4.5.12] and the second one follows from the construction of $\langle \mathcal{X} \rangle$ as a cofiltered limit of *sober* spaces [FK18, ch. 0, 2.7.16].

(ii) So far, we did not address the statement that the homeomorphism $X^{\mathrm{ad}} \overset{\simeq}{\to} \langle \mathcal{X} \rangle$ extends to an isomorphism of locally ringed spaces. This follows since locally ringed spaces whose underlying topological spaces are sober can be detected by their ringed sheaf topoi. Unfortunately, no reference was found. But that is not relevant for us since we will not need it later and explaining it would lead us too far away.

3. Main result

Notation. In this chapter, let k be a complete discretely valued field with valuation ring k° and uniformiser π. This implies that the ring k° is noetherian.

3.1. The affinoid case

Theorem 3.1.1. *Let A be an affinoid k-algebra of dimension d. Then there is an isomorphism*

$$\mathrm{K}_{-d}^{\mathrm{cont}}(A) \xrightarrow{\cong} \mathrm{H}^d\big(\mathrm{Spa}(A,A^\circ);\mathbf{Z}\big)$$

where $\mathrm{Spa}(A,A^\circ)$ *is the adic spectrum of A with respect to its subring A° of power-bounded elements and the right-hand side is sheaf cohomology.*

Before proving the result, we first deduce an immediate consequence.

Corollary 3.1.2. *Let A be an affinoid k-algebra of dimension d. Then there is an isomorphism*

$$\mathrm{K}_{-d}^{\mathrm{cont}}(A) \xrightarrow{\cong} \mathrm{H}^d\big(\mathrm{Spb}(A);\mathbf{Z}\big)$$

where $\mathrm{Spb}(A)$ *is the Berkovich spectrum of A and the right-hand side is sheaf cohomology.*

Proof. The category of so-called overconvergent sheaves on an adic spectrum is equivalent to the category of sheaves on the Berkovich spectrum [vdPS95, 5, Thm. 6]. The locally constant sheaf $\mathbf{Z}$ is overconvergent, hence the first part follows from Theorem 3.1.1. □

Remark 3.1.3. If A is smooth over k or the completion of a k-algebra of finite type, then there is an isomorphism [Bre97, III.1.1]

$$\mathrm{H}^d(\mathrm{Spb}(A);\mathbf{Z}) \cong \mathrm{H}^d_{\mathrm{sing}}(\mathrm{Spb}(A);\mathbf{Z})$$

with singular cohomology since the Berkovich spectrum $\mathrm{Spb}(A)$ is locally contractible. For smooth Berkovich spaces this is a result of Berkovich [Ber99, 9.1] and for completions of k-algebras of finite type this was proven by Hrushovski-Loeser [HL16].

Proof of Theorem 3.1.1. We may assume that A is reduced as the statement is nil-invariant. Let A° be the subring of A consisting of power-bounded elements of A. Then the pair (A,A°) is a Tate pair [BGR84, §6.2.4, Thm. 1] and A° is noetherian

[BGR84, §6.4.3, Prop. 3 (i)]. For any $X \in \mathrm{Adm}(A^\circ)$ one has $X_A = \mathrm{Spec}(A)$ and thus by [KST18a, 5.7] there is a fibre sequence

$$\mathrm{K}(X \text{ on } \pi) \longrightarrow \mathrm{K}^{\mathrm{cont}}(X) \longrightarrow \mathrm{K}^{\mathrm{cont}}(A).$$

Passing to the colimit over all admissible models we obtain a fibre sequence

$$\operatorname*{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{K}(X \text{ on } \pi) \longrightarrow \operatorname*{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{K}^{\mathrm{cont}}(X) \longrightarrow \mathrm{K}^{\mathrm{cont}}(A).$$

For $i < 0$ one has $\mathrm{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{K}_i(X \text{ on } \pi) = 0$ [Ker18, Prop. 7] and hence

$$\mathrm{K}_i^{\mathrm{cont}}(A) \cong \operatorname*{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{K}_i^{\mathrm{cont}}(X).$$

Lemma 1.1.3 and Theorem 2.2.18 yield

$$\begin{aligned} \mathrm{K}_{-d}^{\mathrm{cont}}(\langle A^\circ \rangle_A) &\cong \operatorname*{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{K}_{-d}(X/\pi) \\ &\cong \operatorname*{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{K}_{-d}(X/\pi) \\ &\cong \operatorname*{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{H}^d_{\mathrm{rh}}(X/\pi; \mathbf{Z}). \end{aligned}$$

where the last isomorphism uses that $d = \dim(X/\pi)$ if $X \in \mathrm{Adm}(A^\circ)$ is reduced. Corollary 2.3.17 says

$$\operatorname*{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{H}_{\mathrm{rh}}(X/\pi; \mathbf{Z}) \cong \operatorname*{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{H}_{\mathrm{Zar}}(X/\pi; \mathbf{Z}).$$

The Zariski cohomology is just ordinary sheaf cohomology. The latter one commutes with colimits of coherent and sober spaces with quasi-compact transition maps [FK18, ch. 0, 4.4.1]. Since the admissible Zariski-Riemann space is such a colimit we obtain

$$\operatorname*{colim}_{X \in \mathrm{Adm}(A^\circ)} \mathrm{H}_{\mathrm{Zar}}(X/\pi; \mathbf{Z}) \cong \mathrm{H}_{\mathrm{Sh}}(\langle A^\circ \rangle_A/\pi; \mathbf{Z}).$$

By Proposition 2.5.9 we finally get that

$$\mathrm{H}_{\mathrm{Sh}}\big(\langle A^\circ \rangle_A/\pi; \mathbf{Z}\big) \cong \mathrm{H}_{\mathrm{Sh}}\big(\mathrm{Spa}(A, A^\circ); \mathbf{Z}\big).$$

□

3.2. From local to global

In this section, we will present Morrow's extension of continuous K-theory to a global functor $\mathrm{K}^{\mathrm{cont}}$ from rigid k-spaces to pro-spectra. We conjecture that an analogous version of our main result (Theorem 3.1.1) for rigid spaces is true. We prove this conjecture in the algebraic case (e.g. affinoid or projective) and in dimension at least two (Theorem 3.2.14). This section is rather sketchy and a full development of the formalism which will be based on adic spaces needs to be examined in future work.

For a moment let us take for granted that continuous K-theory extends to a functor on rigid k-spaces (Corollary 3.2.4). For the general theory on rigid k-spaces we refer the reader to Bosch's lecture notes [Bos14, pt. I].

Conjecture 3.2.1. *Let X be a quasi-compact and quasi-separated rigid k-space of dimension d. Then there is an isomorphism*

$$\mathrm{K}^{\mathrm{cont}}_{-d}(X) \cong \mathrm{H}^d(X;\mathbf{Z})$$

of pro-abelian groups. In particular, the pro-abelian group $\mathrm{K}^{\mathrm{cont}}_{-d}(X)$ is constant.

Now let us see that continuous K-theory, as defined for algebras in Definition 1.3.5, satisfies descent and hence defines a sheaf for the admissible topology. The result and its proof are due to Morrow. For an affinoid k-algebra A denote by $\mathrm{Spm}(A)$ its associated affinoid k-space [Bos14, §3.2].[1]

Lemma 3.2.2 ([Mor16, 3.4])**.** *Let $\mathrm{Spm}(A)$ be an affinoid k-variety which is assumed to be covered by two open subspaces $\mathrm{Spm}(A^1)$ and $\mathrm{Spm}(A^2)$. We set*

$$A^3 := A^1 \hat{\otimes}_A A^2 := k \otimes_{k_0} (A^1_0 \otimes_{A_0} A^2_0)^\wedge$$

where A_0, A^1_0, A^2_0 are respective subrings of definition of A, A^1, A^2 and $(A^1_0 \otimes_{A_0} A^2_0)^\wedge$ denotes the π-adic completion. Then the diagram

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{cont}}(A) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A^1) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(A^2) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A^3) \end{array}$$

is weakly cartesian in $\mathbf{Pro}(\mathbf{Sp})$, i.e. cartesian in $\mathbf{Pro}(\mathbf{Sp}^+)$.

Proof. We find an admissible blow-up $X \to \mathrm{Spec}(A_0)$ where the open cover can be realised, i.e. there exist opens $\mathrm{Spec}(B^1)$ and $\mathrm{Spec}(B^2)$ in X such that, setting $B^3 := B^1 \otimes_A B^2$, one has $A^i_0 = \hat{B^i}$ for $i \in \{1,2,3\}$. The diagram of fibre sequences

$$\begin{array}{ccccc} \text{“lim”}_n \mathrm{K}(B^i \text{ on } \pi^n) & \longrightarrow & \mathrm{K}(B^i) & \longrightarrow & \text{“lim”}_n \mathrm{K}(B^i/\pi^n) \\ \downarrow & & \downarrow & & \downarrow \\ \text{“lim”}_n \mathrm{K}(A^i_0 \text{ on } \pi^n) & \longrightarrow & \mathrm{K}(A^i_0) & \longrightarrow & \text{“lim”}_n \mathrm{K}(A^i_0/\pi^n) \end{array}$$

[1] Bosch uses the notation $\mathrm{Sp}(A)$.

has equivalent cofibres since $B^i/\pi^n \cong A_0^i/\pi^n$ for every $n \geq 1$. Hence its left-hand square is cartesian. The diagram of fibre sequences

$$\begin{array}{ccccc} \mathrm{K}(B^i \text{ on } \pi) & \longrightarrow & \mathrm{K}(B^i) & \longrightarrow & \mathrm{K}(B^i[\pi^{-1}]) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}(A_0^i \text{ on } \pi) & \longrightarrow & \mathrm{K}(A_0^i) & \longrightarrow & \mathrm{K}(A_0^i[\pi^{-1}]) = \mathrm{K}(A^i) \end{array}$$

has equivalent fibres since the maps $B^i \to A_0^i$ are analytic isomorphisms, cf. [Vor79, 1.5] and [Wei80, 1.1], thus its right-hand square is cartesian. Together this yields that the diagram

$$\begin{array}{ccc} \text{“lim”}_n \mathrm{K}(B^i \text{ on } \pi^n) & \longrightarrow & \mathrm{K}(B^i[\pi^{-1}]) \\ \downarrow & & \downarrow \\ \text{“lim”}_n \mathrm{K}(A_0^i \text{ on } \pi^n) & \longrightarrow & \mathrm{K}(A^i) \end{array}$$

is cartesian and hence the sequence

$$\underset{n}{\text{“lim”}} \mathrm{K}(B^i \text{ on } \pi^n) \longrightarrow \mathrm{K}(B^i[\pi^{-1}]) \longrightarrow \mathrm{K}^{\mathrm{cont}}(A^i)$$

is a fibre sequence. The two diagrams

$$\begin{array}{ccc} \mathrm{K}(X) & \longrightarrow & \mathrm{K}(B^1) \\ \downarrow & & \downarrow \\ \mathrm{K}(B^2) & \longrightarrow & \mathrm{K}(B^3) \end{array} \qquad \begin{array}{ccc} \text{“lim”}_n \mathrm{K}(X/\pi^n) & \longrightarrow & \text{“lim”}_n \mathrm{K}(B^1/\pi^n) \\ \downarrow & & \downarrow \\ \text{“lim”}_n \mathrm{K}(B^2/\pi^n) & \longrightarrow & \text{“lim”}_n \mathrm{K}(B^3/\pi^n) \end{array}$$

are cartesian by Zariski descent, hence the diagram of respective fibres

$$\begin{array}{ccc} \text{“lim”}_n \mathrm{K}(X \text{ on } \pi) & \longrightarrow & \text{“lim”}_n \mathrm{K}(B^1 \text{ on } \pi^n) \\ \downarrow & & \downarrow \\ \text{“lim”}_n \mathrm{K}(B^2 \text{ on } \pi^n) & \longrightarrow & \text{“lim”}_n \mathrm{K}(B^3 \text{ on } \pi^n) \end{array}$$

is cartesian as well. Again by Zariski descent, the diagram

$$\begin{array}{ccc} \mathrm{K}(A) & \longrightarrow & \mathrm{K}(B^1[\pi^{-1}]) \\ \downarrow & & \downarrow \\ \mathrm{K}(B^2[\pi^{-1}]) & \longrightarrow & \mathrm{K}(B^3[\pi^{-1}]) \end{array}$$

is cartesian. By pro-cdh-descent for the upper left corner and by the fibre sequence above for other corners, the diagram

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{cont}}(A) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A^1) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(A^2) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A^3) \end{array}$$

is the square of respective cofibres of the two last squares and hence cartesian. □

The following result is taken from unpublished work of Kerz-Saito-Tamme and permits us to extend continuous K-theory to the category of rigid k-spaces.

Proposition 3.2.3 (Kerz-Saito-Tamme)**.** *The inclusion* $\iota\colon \mathrm{Rig}_k^{\mathrm{aff}} \hookrightarrow \mathrm{Rig}_k$ *of affinoid k-spaces into rigid k-spaces induces an equivalence*

$$\iota^* : \mathbf{Sh}(\mathrm{Rig}_k) \xrightarrow{\simeq} \mathbf{Sh}(\mathrm{Rig}_k^{\mathrm{aff}}).$$

Moreover, for every ∞-category $\mathcal{D}$ which admits small limits, the canonical map

$$\iota^* : \mathbf{Sh}_{\mathcal{D}}(\mathrm{Rig}_k) \xrightarrow{\simeq} \mathbf{Sh}_{\mathcal{D}}(\mathrm{Rig}_k^{\mathrm{aff}}).$$

is an equivalence.

Proof. The functor ι^* has a right adjoint ι_*; explicitly, for $F \in \mathbf{PSh}(\mathrm{Rig}^{\mathrm{aff}})$ and $X \in \mathrm{Rig}$ we have

$$(\iota_* F)(X) \simeq \lim_{V \in (\mathrm{Rig}_k^{\mathrm{aff}})^{\mathrm{op}}_{/X}} F(V).$$

The counit $\iota^*\iota_* F \to F$ is an equivalence since $(\iota_* F)(X) = F(X)$ if X is affinoid. Since every rigid k-space has an affinoid cover, every covering sieve in Rig can be refined by a covering sieve that is generated by an affinoid cover. Thus ι_* preserves sheaves and the unit $F \to \iota_*\iota^* F$ is an equivalence. The second statement follows from the equivalence $\mathbf{Sh}_{\mathcal{D}}(\mathbf{Sh}(\mathcal{C})) \simeq \mathbf{Sh}_{\mathcal{D}}(\mathcal{C})$ for any site $\mathcal{C}$ [Lur18, 1.3.1.7]. □

Corollary 3.2.4 ([Mor16, 3.5])**.** *There exists a unique sheaf* $\mathrm{K}^{\mathrm{cont}}$ *on the category* Rig_k *(equipped with the admissible topology) that has values in* $\mathbf{Pro}(\mathbf{Sp}^+)$ *and satisfies* $\mathrm{K}^{\mathrm{cont}}(\mathrm{Spm}(A)) \simeq \mathrm{K}^{\mathrm{cont}}(A)$ *for every affinoid k-algebra A.*

Corollary 3.2.5. *The functor*

$$\mathrm{K}^{\mathrm{cont}}((\_)_\eta) : (\mathrm{FSch}_{\mathrm{Zar}})^{\mathrm{op}} \to \mathrm{Rig}^{\mathrm{op}} \to \mathbf{Pro}(\mathbf{Sp}^+), \quad \mathcal{X} \mapsto \mathcal{X}_\eta \mapsto \mathrm{K}^{\mathrm{cont}}(\mathcal{X}_\eta)$$

is a sheaf.

Proof. This follows from the fact that Zariski covers of formal schemes induce on generic fibres admissible covers of rigid spaces. □

Lemma 3.2.6. *Let* $\mathrm{Spf}(A_0)$ *be an affine formal scheme over* k° *which is assumed to be covered by two open formal subschemes* $\mathrm{Spf}(A_0^1)$ *and* $\mathrm{Spf}(A_0^2)$*. Setting* $A_0^3 := (A_0^1 \otimes_{A_0} A_0^2)^\wedge$ *we get a cartesian square*

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{cont}}(A_0) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A_0^1) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(A_0^2) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A_0^3) \end{array}$$

in the category $\mathbf{Pro}(\mathbf{Sp})$

Proof. For every $n \geq 1$, the special fibre $\mathrm{Spec}(A_0/\pi^n)$ is covered by $\mathrm{Spec}(A_0^1/\pi^n)$ and $\mathrm{Spec}(A_0^2/\pi^n)$ with intersection $\mathrm{Spec}(A_0^3/\pi^n)$. Applying algebraic K-theory one obtains cartesian squares by Zariski descent. Now the claim follows as finite limits in the pro-category can be computed levelwise. □

Corollary 3.2.7. *There exists a unique sheaf* $\mathrm{K}^{\mathrm{cont}}$ *on* $\mathrm{FSch}_{\mathrm{Zar}}$ *that has values in* $\mathbf{Pro}(\mathbf{Sp}^+)$ *and satisfies* $\mathrm{K}^{\mathrm{cont}}(\mathrm{Spf}(A_0)) \simeq \mathrm{K}^{\mathrm{cont}}(A_0)$ *for every* π*-adic ring* A_0*.*

Definition 3.2.8. For an affine formal scheme $\mathrm{Spf}(A_0)$ with associated generic fibre $\mathrm{Spm}(A)$ where $A = A_0 \otimes_{k^\circ} k$, there is by definition a map $\mathrm{K}^{\mathrm{cont}}(A_0) \to \mathrm{K}^{\mathrm{cont}}(A)$. This map can be seen as a natural transformation $\mathrm{FSch}^{\mathrm{aff}} \to \mathbf{Pro}(\mathbf{Sp}^+)$ which extends to a natural transformation

$$\mathrm{K}^{\mathrm{cont}}(\_) \to \mathrm{K}^{\mathrm{cont}}((\_)_\eta) : \mathrm{FSch}^{\mathrm{op}} \longrightarrow \mathbf{Pro}(\mathbf{Sp}^+).$$

For a formal scheme $\mathcal{X}$ we define

$$\mathrm{K}^{\mathrm{cont}}(\mathcal{X} \text{ on } \pi) := \mathrm{fib}\big(\mathrm{K}^{\mathrm{cont}}(\mathcal{X}) \to \mathrm{K}^{\mathrm{cont}}(\mathcal{X}_\eta)\big)$$

where $\mathcal{X}_\eta$ is the associated generic fibre. By construction and by Corollary 3.2.7 and Corollary 3.2.5 the induced functor

$$\mathrm{K}^{\mathrm{cont}}(\_ \text{ on } \pi) : (\mathrm{FSch}_{\mathrm{Zar}})^{\mathrm{op}} \to \mathbf{Pro}(\mathbf{Sp}^+)$$

is a sheaf.

Lemma 3.2.9. *Let* X *be a quasi-compact and quasi-separated* k°*-scheme. Then there is a canonical equivalence*

$$\mathrm{K}(X \text{ on } \pi) \xrightarrow{\simeq} \mathrm{K}^{\mathrm{cont}}(\hat{X} \text{ on } \pi).$$

Proof. If $X = \mathrm{Spec}(A_0)$ is affine we have by Definition 1.3.5 a pushout

$$\begin{array}{ccc} \mathrm{K}(A_0) & \longrightarrow & \mathrm{K}(A_0[\pi^{-1}]) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(A_0) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A_0[\pi^{-1}]). \end{array}$$

Since the category **Sp** is stable, this also holds for **Pro**(**Sp**). Thus the square is also a pullback and we have an equivalence $\mathrm{K}(A_0 \text{ on } \pi) \simeq \mathrm{K}^{\mathrm{cont}}(A_0 \text{ on } \pi)$ of the horizontal fibres. For general X choose a finite affine cover $(U_i)_i$ which yields a commutative diagram

$$\begin{array}{ccc} \mathrm{K}(X \text{ on } \pi) & \longrightarrow & \lim_{\Delta} \mathrm{K}(\check{U}_\bullet \text{ on } \pi) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(\hat{X} \text{ on } \pi) & \longrightarrow & \lim_{\Delta} \mathrm{K}^{\mathrm{cont}}(\check{\hat{U}}_\bullet \text{ on } \pi) \end{array}$$

where $\check{U}_\bullet$ and $\check{\hat{U}}_\bullet$ are the Čech nerves of the cover $(U_i)_i$ of X respectively the induced cover $(\hat{U}_i)_i$ of $\hat{X}$. Thus the horizontal maps are equivalences. By the affine case, the right vertical map is an equivalence, hence also the left vertical map as desired. □

Corollary 3.2.10. *Let X be a quasi-compact and quasi-separated k°-scheme. Then the square*

$$\begin{array}{ccc} \mathrm{K}(X) & \longrightarrow & \mathrm{K}(X_k) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(\hat{X}) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(\hat{X}_\eta) \end{array}$$

is cartesian in **Pro**($\mathbf{Sp}^+$) *where* $X_k := X \times_{\mathrm{Spec}(k^\circ)} \mathrm{Spec}(k)$.

Proof. There is a commutative diagram of fibre sequences

$$\begin{array}{ccccc} \mathrm{K}(X \text{ on } \pi) & \longrightarrow & \mathrm{K}(X) & \longrightarrow & \mathrm{K}(X_k) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(\hat{X} \text{ on } \pi) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(\hat{X}) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(\hat{X}_\eta) \end{array}$$

where the left vertical map is an isomorphism due to Lemma 3.2.9. □

Corollary 3.2.11. *Let $\mathcal{X}$ be an algebraic formal scheme. For $n \geq 1$ have*

$$\operatorname*{colim}_{\mathcal{X}'} \mathrm{K}^{\mathrm{cont}}_{-n}(\mathcal{X}' \text{ on } \pi) = 0$$

where $\mathcal{X}'$ runs over all admissible formal blow-ups of $\mathcal{X}$.

Proof. Since every admissible formal blow-up of an algebraic formal scheme is algebraic (Remark 2.5.4), due to Lemma 3.2.9, and by Theorem 2.4.9 we have

$$\operatorname*{colim}_{\mathcal{X}'} \mathrm{K}^{\mathrm{cont}}_{-n}(\mathcal{X}' \text{ on } \pi) \cong \operatorname*{colim}_{X'} \mathrm{K}^{\mathrm{cont}}_{-n}(\hat{X}' \text{ on } \pi) \cong \operatorname*{colim}_{X'} \mathrm{K}_{-n}(X' \text{ on } \pi) = 0$$

where the latter two colimits are indexed by all X_k-admissible blow-ups of X where X is a quasi-compact and quasi-separated k°-scheme such that $\mathcal{X} \cong \hat{X}$ and where $X_k := X \times_{\mathrm{Spec}(k^\circ)} \mathrm{Spec}(k)$. □

Lemma 3.2.12. *Let $\mathcal{X}$ be a formal scheme. For $n \geq 2$ have*

$$\operatorname*{colim}_{\mathcal{X}'} \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{X}' \text{ on } \pi) = 0$$

where $\mathcal{X}'$ runs over all admissible formal blow-ups of $\mathcal{X}$.

Proof. Let $\alpha \in \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{X} \text{ on } \pi)$. We choose a finite affine cover $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{X}$ where $I = \{1,\ldots,k\}$. By the affine case, we find for every $i \in I$ an admissible formal blow-up $\mathcal{U}_i' \to \mathcal{U}_i$ such that the map $\mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{U}_i) \to \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{U}_i')$ sends $\alpha|_{\mathcal{U}_i}$ to zero. There exists an admissible formal blow-up $\mathcal{X}' \to \mathcal{X}$ locally dominating these local blow-ups, i.e. for every $i \in I$ the pullback $\mathcal{X}' \times_{\mathcal{X}} \mathcal{U}_i \to \mathcal{U}_i$ factors over $\mathcal{U}' \to \mathcal{U}$ [Bos14, 8.2, Prop. 14]. We may assume that $\mathcal{X}' \times_{\mathcal{X}} \mathcal{U}_i = \mathcal{U}_i'$. Setting $\mathcal{V} := \mathcal{U}_2 \cup \ldots \mathcal{U}_k$ one obtains a commutative diagram

$$\begin{array}{ccccc}
\mathrm{K}_{-n+1}^{\mathrm{cont}}(\mathcal{U}_1 \cap \mathcal{V} \text{ on } \pi) & \longrightarrow & \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{X} \text{ on } \pi) & \longrightarrow & \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{U}_1 \text{ on } \pi) \oplus \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{V} \text{ on } \pi) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{K}_{-n+1}^{\mathrm{cont}}(\mathcal{U}_1' \cap \mathcal{V}' \text{ on } \pi) & \longrightarrow & \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{X}' \text{ on } \pi) & \longrightarrow & \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{U}_1' \text{ on } \pi) \oplus \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{V}' \text{ on } \pi)
\end{array}$$

of Mayer-Vietoris sequences. By the affine case and by induction on the cardinality of the affine cover, α maps to zero in $\mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{U}_1') \oplus \mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{V}')$. Hence its image in $\mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{X}' \text{ on } \pi)$ comes from an element α' in $\mathrm{K}_{-n+1}^{\mathrm{cont}}(\mathcal{U}_1' \cap \mathcal{V}')$. As an admissible formal blow-up of the quasi-affine formal scheme $\mathcal{U}_1 \cap \mathcal{V}$, the formal scheme $\mathcal{U}_1' \cap \mathcal{V}'$ is algebraic according to Lemma 2.5.6. Hence there exists an admissible formal blow-up of $\mathcal{U}_1' \cap \mathcal{V}'$ where α' vanishes. As above, this can be dominated by an admissible formal blow-up $\mathcal{X}'' \to \mathcal{X}'$ so that the image of α in $\mathrm{K}_{-n}^{\mathrm{cont}}(\mathcal{X}'' \text{ on } \pi)$ vanishes. □

Next we do another similar reduction to the affinoid case.

Lemma 3.2.13. *For every formal scheme $\mathcal{X}$ and every constant rh-sheaf F the canonical map*

$$\operatorname*{colim}_{\mathcal{X}'} \mathrm{H}_{\mathrm{Zar}}^{*}(\mathcal{X}'/\pi; F) \longrightarrow \operatorname*{colim}_{\mathcal{X}'} \mathrm{H}_{\mathrm{rh}}^{*}(\mathcal{X}'/\pi; F)$$

is an isomorphism.

Proof. This is similar to the proof of Proposition 3.2.12. By a Mayer-Vietoris argument and by induction on the number of affine formal schemes needed to cover $\mathcal{X}$, we can reduce to one degree less. Fortunately, this also works in degree 0 due to the sheaf condition. □

We now prove Conjecture 3.2.1 in almost all cases.

Theorem 3.2.14. *Let X be a quasi-compact and quasi-separated rigid k-variety of dimension d. Assume that $d \geq 2$ or that there exists a formal model which is algebraic (e.g. X is affinoid or projective). Then there is an isomorphism*

$$\mathrm{K}_{-d}^{\mathrm{cont}}(X) \cong \mathrm{H}^{d}(X; \mathbf{Z})$$

where the right-hand side is sheaf cohomology with respect to the admissible topology on the category of rigid k-varieties.

Proof. Let $\mathcal{X}$ be a formal model of X. By Definition 3.2.8 there is a fibre sequence

$$\mathrm{K}^{\mathrm{cont}}(\mathcal{X} \text{ on } \pi) \longrightarrow \mathrm{K}^{\mathrm{cont}}(\mathcal{X}) \longrightarrow \mathrm{K}^{\mathrm{cont}}(X).$$

If $\operatorname{colim}_{\mathcal{X}'} \mathrm{K}^{\mathrm{cont}}_{-n}(\mathcal{X}' \text{ on } \pi) = 0$ for $n \in \{d-1, d\}$, then the induced map

$$\operatorname*{colim}_{\mathcal{X}'} \mathrm{K}^{\mathrm{cont}}_{-d}(\mathcal{X}') \longrightarrow \mathrm{K}^{\mathrm{cont}}_{-d}(X)$$

is an isomorphism; this is the case if $\mathcal{X}$ is algebraic (Corollary 3.2.11) or if $\dim(X) \geq 2$ (Lemma 3.2.12). By Lemma 1.1.3, Theorem 2.2.18, and we conclude

$$\mathrm{K}^{\mathrm{cont}}_{-d}(X) \cong \operatorname*{colim}_{\mathcal{X}'} \mathrm{K}^{\mathrm{cont}}_{-d}(\mathcal{X}) \cong \operatorname*{colim}_{\mathcal{X}'} \mathrm{K}_{-d}(\mathcal{X}/\pi) \cong \operatorname*{colim}_{\mathcal{X}'} \mathrm{H}_{\mathrm{rh}}(\mathcal{X}/\pi; \mathbf{Z}).$$

By Lemma 3.2.13 we have that

$$\operatorname*{colim}_{\mathcal{X}'} \mathrm{H}^{*}_{\mathrm{rh}}(\mathcal{X}/\pi; \mathbf{Z}) \cong \operatorname*{colim}_{\mathcal{X}'} \mathrm{H}^{*}_{\mathrm{Zar}}(\mathcal{X}'/\pi; \mathbf{Z}).$$

Since every formal scheme is homeomorphic to its special fibre, the latter one identifies with $\operatorname{colim}_{\mathcal{X}'} \mathrm{H}^d(\mathcal{X}'; \mathbf{Z})$ as sheaf cohomology only depends on the topology. By Proposition B.1.9 and Theorem 2.5.7 we conclude that

$$\operatorname*{colim}_{\mathcal{X}'} \mathrm{H}^d(\mathcal{X}'; \mathbf{Z}) \cong \mathrm{H}^d(\langle \mathcal{X} \rangle; \mathbf{Z})) \cong \mathrm{H}^d(X^{\mathrm{ad}}; \mathbf{Z}) \cong \mathrm{H}^d(X; \mathbf{Z})$$

where $\langle \mathcal{X} \rangle = \lim_{\mathcal{X}'} \mathcal{X}'$ is the formal Zariski-Riemann space associated with $\mathcal{X}$ and X^{ad} is the adic space associated to X. □

Remark 3.2.15. The cases of Conjecture 3.2.1 which are not covered by Theorem 3.2.14 are curves which are not algebraic. In particular, they must not be affine nor projective nor even smooth proper[2].

As the constant sheaf $\mathbf{Z}$ is overconvergent we infer the following.

Corollary 3.2.16. *Let X be a quasi-compact and quasi-separated rigid analytic space of dimension d over a discretely valued field. Assume that $d \geq 2$ or that there exists a formal model of X which is algebraic (e.g. X is affinoid or projective). Then there is an isomorphism*

$$\mathrm{K}^{\mathrm{cont}}_{-d}(X) \cong \mathrm{H}^d(X^{\mathrm{berk}}; \mathbf{Z})$$

where X^{berk} is the Berkovich space associated with X.

Remark 3.2.17. If X^{berk} is smooth over k or the completion of a k-scheme of finite type, then there is an isomorphism [Bre97, III.1.1]

$$\mathrm{H}^d(X^{\mathrm{berk}}; \mathbf{Z}) \cong \mathrm{H}^d_{\mathrm{sing}}(X^{\mathrm{berk}}; \mathbf{Z})$$

[2] Cf. [Lüt16, 1.8.1].

with singular cohomology since the Berkovich space X^{berk} is locally contractible. For smooth Berkovich spaces this is a result of Berkovich [Ber99, 9.1] and for completions of k-schemes of finite type this was proven by Hrushovski-Loeser [HL16].

A. Sheaves for cd-topologies

A.1. Sheaves of spectra

In this section, we will eventually deal with sheaves of spectra on sites whose underlying category is (an appropriate subcategory of) the category of schemes equipped with a topology which is induced by a cd-structure. This is used in the main body when dealing with K-theory spectra and their sheafifications in the rh-topology (Proof of Theorem 2.2.14). Being of homotopical nature, it is convenient to express this chapter's content within the language of ∞-categories. However, some results we are using from the literature are expressed in the language of model categories. For this reason, a more detailed account on this subject is presented. Some content of this section is based on a lecture series held by Florain Strunk and Georg Tamme at the Homotopy Summer in June 2018 in Berlin as well as on a course of Florian Strunk in the winter term 2018/19 at Universität Regensburg. First, let us deal with sheaves of spaces.

Notation. In section A.1, let $\mathcal{C}$ be a small ∞-category equipped with a topology τ (B.2.4). Denote by $\mathbf{PSh}$ the category $\mathbf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$ with values in the ∞-category of spaces $\mathbf{Spc}$. For an object X of $\mathcal{C}$ denote by $y(X)$ its associated representable presheaf.

Sheaves

Definition A.1.1. A presheaf $F \in \mathbf{PSh}$ is said to be a τ**-sheaf** iff the map

$$\mathrm{Map}(y(X), F) \longrightarrow \mathrm{Map}(U, F)$$

is an equivalence for every τ-covering sieve $U \hookrightarrow y(X)$. Let $\mathbf{Sh}_\tau$ be the full subcategory of $\mathbf{PSh}$ spanned by τ-sheaves.

Proposition A.1.2. *There is an adjunction*

$$\mathrm{L}_\tau : \mathbf{PSh} \rightleftarrows \mathbf{Sh}_\tau : \mathrm{incl}$$

and the category $\mathbf{Sh}_\tau$ *is an* ∞*-topos.*

Proof. The τ-sheaves are precisely the τ-local objects [Lur09, 5.5.4.1]. Since the category $\mathbf{PSh}$ is presentable, the inclusion of τ-local objects has a left adjoint L_τ and $\mathbf{Sh}_\tau$ is presentable [Lur09, 5.5.4.15]. The functor L_τ is a topological localisation [Lur09, 6.2.1.4], hence $\mathbf{Sh}_\tau$ is an ∞-topos [Lur09, 6.2.2.7]. □

Definition A.1.3. We call the left-adjoint L_τ in Proposition A.1.2 the ***τ*-sheafification**. If there is no confusion in sight, we omit the τ in the notation and just write L.

In many cases of interest, in order to check whether a given presheaf is a τ-sheaf it suffices to check the sheaf condition on certain types of covers which are more concrete.

Proposition A.1.4 ([Hoy14, C.2])**.** *Assume that the topology τ is induced by a pretopology τ_p (B.2.11). Then a presheaf F is a τ-sheaf if and only if for every τ_p-cover $\{U_i \to X\}_i$ the map*

$$\mathrm{Map}(X,F) \longrightarrow \mathrm{Map}(|\check{C}(U_\bullet)|,F)$$

is an equivalence where $|\check{C}(U_\bullet)|$ is the Čech nerve of the cover.

Points and stalkwise equivalences

We introduce the notion of a point which is inspired by the corresponding notion for pretopologies for schemes which has been introduced by Goodwillie-Lichtenbaum [GL01, § 2] and further studied by Gabber-Kelly [GK15].

Definition A.1.5. A ***τ*-point** of $\mathcal{C}$ is a pair (X,p) consisting of an object X of $\mathcal{C}$ and a morphism $p\colon P \to X$ in $\mathcal{C}$ that is an element of every τ-covering sieve of X. We shall also say that p is a τ-point of X. Denote by $\mathrm{Pts}_\tau(X)$ the set of all τ-points of X.

Example A.1.6. Let $\mathcal{C} = \mathrm{Top}$ be the category of topological spaces equipped with the topology τ generated by covers of open subsets. Given a topological space X, every point $x \in X$ in the classical sense defines a τ-point $\{x\} \to X$. If X is hausdorff, then every τ-point is of this form. On the other hand, consider the Sierpinski space S, i.e. the two-point space with one closed point s and one generic point η. Then S has precisely three τ-points, namely $\{s\} \to S$, $\{\eta\} \to S$, and $\mathrm{id}_S\colon S \to S$.

Remark A.1.7. A τ-point p of an object X of $\mathcal{C}$ yields an ∞-topos-theoretic point of the ∞-topos $\mathbf{Sh}_\tau$, i.e. a geometric morphism $p_*\colon \mathbf{Spc} \to \mathbf{Sh}_\tau$, i.e. a functor which admits a left exact left adjoint p^* [Lur09, 6.3.1.1]. The functor p^* is the stalk functor we introduce in Definition A.1.12. By construction, p^* commutes with small colimits and hence admits a right adjoint p_* [Lur09, 5.5.2.9]. Since filtered colimits commute with finte limits the stalk functor is left exact on τ-sheaves.

Definition A.1.8. A ***τ*-neighbourhood** of a τ-point (X,p) is a pair $(U \xrightarrow{f} X, q)$ consisting of a morphsim $f\colon U \to X$ in $\mathcal{C}$ that is contained in some covering sieve of X and τ-point $q\colon P \to U$ such that $p = f \circ q$. Denote by $\mathrm{Nbh}_\tau(X,p)$ the set of all τ-neighbourhods of (X,p). Define a partial ordering on $\mathrm{Nbh}_\tau(X,p)$ as follows: given two τ-neighbourhoods (U,q) and (U',q') of (X,p), then declare $(U,q) \le (U',q')$ iff there exists a morphism $i\colon U \to U'$ over X such that $q' = i \circ q$.

Example A.1.9. For the category of topological spaces with the global topology τ, a neighbourhood of a neighbourhood of $x \in X$ is a τ-neighbourhood of the τ-point $\{x\} \to X$. If $S = \{s, \eta\}$ is the Sierpinski space, then $(\mathrm{id}_S, \mathrm{id}_S)$ is the only τ-neighbourhood of the point id_S.

Lemma A.1.10. *The fibre product of two τ-neighbourhoods of a given τ-point (X,p) is canonically a τ-neighbourhood of (X,p). In particular, the set $\mathrm{Nbh}_\tau(X,p)$ of all τ-neighbourhoods of (X,p) is cofiltered.*

Proof. Given $(U \xrightarrow{f} X, q), (U' \xrightarrow{f'} X, q') \in \mathrm{Nbh}_\tau(X,p)$, we have $f \circ q = p = f' \circ q'$ by definition. Hence there exists a unique map $\tilde{q}: P \to \tilde{U} := U \times_X U'$ rendering $\tilde{U}$ a τ-neighbourhood of (X,p) which is smaller than both initial neighbourhoods. □

Remark A.1.11. Assume that the topology τ is induced by a pretopology wherein every cover is a family of morphisms satisfying a certain property, e.g. the Zariski, Nisnevich, or étale topology for schemes. Then there is also an easier notion of a neighbourhood of a point: define a neighbourhood to be a morphism of that certain property along which the point lifts. This notion is not the same as the one of Definition A.1.5 since it yields less neighbourhoods. However, both notions are essentially equivalent: since every covering sieve can be refined by a sieve coming from a covering family, these neighbourhoods are cofinal in our ones. Hence they will yield equivalent stalk functors.

Definition A.1.12. Let $(X,p) \in \mathrm{Pts}_\tau(X)$ be a τ-point of X. Define its **stalk functor** as

$$\begin{aligned} (-)_p : \mathbf{PSh} &\longrightarrow \mathbf{Spc} \\ F &\mapsto F_p := \operatorname*{colim}_{(U,q) \in \mathrm{Nbh}(X,p)} F(U) \end{aligned}$$

and call F_p the τ-**stalk** of F at the point p of X. A map of presheaves $f: E \to F$ is called a τ-**stalkwise equivalence** iff for every $X \in \mathcal{C}$ and for every τ-point $(X,p) \in \mathrm{Pts}_\tau(X)$ the induced map $f_p: E_p \to F_p$ on stalks is an equivalence of spaces.

Example A.1.13. Consider the category of topological spaces equipped with the global topology τ. Let X be a topological space and let $x \in X$ be a classical point. A set-valued presheaf $F' \in \mathrm{PSh}(\mathrm{Top})$ defines a space-valued presheaf F via the canonical map $\mathrm{N}(\mathrm{Top}) \to \mathbf{Spc} \simeq \mathrm{N}(\mathrm{Top})[W^{-1}]$. Then the classical stalk F'_x is isomorphic to $\pi_0(F_x)$.

Caveat A.1.14. For discrete sheaves, i.e. sheaves with values in sets, one can detect isomorphisms on stalks. Unfortunately, the analogous statement for sheaves of spaces is *not* true in general, i.e. not every stalkwise equivalence is an equivalence, cf. [MV99, 1.30]. However, such a statement is true if the cohomological dimension of the site in question is finite and if the site has enough points; cf. Theorem A.1.32.

Lemma A.1.15. *Let F be a presheaf on $\mathcal{C}$. Then the map $F \to \mathrm{L}F$ is a stalkwise equivalence.*

Proof. We take for granted that the functor $\mathrm{L}F$ can be described explicitly as follows: for an object X of $\mathcal{C}$ we have

$$\mathrm{L}F(X) \simeq \operatorname*{colim}_{R \in \tau(X)^{\mathrm{op}}} \lim_{(V \to X) \in R^{\mathrm{op}}} F(V).$$

Taking now the colimit over all neighbourhoods of a given point p of X we obtain

$$(\mathrm{L}F)_p \simeq \operatorname*{colim}_{(U,q) \in \mathrm{Nbh}_\tau(X,p)} \operatorname*{colim}_{R \in \tau(U)^{\mathrm{op}}} \lim_{(V \to U) \in R^{\mathrm{op}}} F(V).$$

Every $V \to U$ along which the point p does not lift will not contribute to the limit any more when passing to a later stage in the first colimit, hence $(\mathrm{L}F)_p \simeq \mathrm{L}_p$ by cofinality. □

Truncated (pre)sheaves

Definition A.1.16. Let $n \geq -1$. A space X is called ***n*-truncated** iff $\pi_i(X,x) = 0$ for every $i > n$ and every base point $x \in X$. Denote by $\tau_{\leq n}\mathbf{Spc}$ the full subcategory of $\mathbf{Spc}$ spanned by n-truncated spaces.

Lemma A.1.17. *The inclusion $\tau_{\leq n}\mathbf{Spc} \hookrightarrow \mathbf{Spc}$ preserves limits.*

Proof. It suffices to show that the inclusion preserves products and pullbacks, cf. [Lur09, 4.4.2.7] for the dual statement. For products this follows since $\pi_i(X,x) = [(S^i,*),(X,x)]_\bullet$. For pullbacks this follows from the long exact sequence of homotopy groups. □

Definition A.1.18. An object F in an ∞-category $\mathcal{D}$ is called ***n*-truncated** iff the space $\mathrm{Map}(E,F)$ is n-truncated for every object E of $\mathcal{D}$. Denote by $\tau_{\leq n}\mathcal{D}$ the full subcategory of n-truncated objects in $\mathcal{D}$.

Proposition A.1.19 ([Lur09, 5.5.6.5, 5.5.6.18])**.** *Let $\mathcal{D}$ be an ∞-category and let $n \geq 1$.*

(i) The full subcategory $\tau_{\leq n}\mathcal{D} \subset \mathcal{D}$ is closed under limits which exist in $\mathcal{D}$.

(ii) If $\mathcal{D}$ is presentable, there exists an adjunction $\tau_{\leq n} : \mathcal{D} \rightleftarrows \tau_{\leq n}\mathcal{D} : \mathrm{incl}$.

Definition A.1.20. Assume that $\mathcal{D}$ is presentable. We call the functor $\tau_{\leq n}$ in Proposition A.1.19 the ***n*-truncation functor**. For any F in $\mathcal{D}$ call both the object $\tau_{\leq n}F$ and the unit map $F \to \tau_{\leq n}F$ the ***n*-truncation** of F.

Now we turn back to presheaves and sheaves on a site $(\mathcal{C},\tau)$. To avoid confusion, we write $\tau_{\leq n}\colon \mathbf{PSh} \to \tau_{\leq n}\mathbf{PSh}$ for the truncation of presheaves and $\mathrm{L}_{\leq n}\colon \mathbf{Sh}_\tau \to \tau_{\leq n}\mathbf{Sh}_\tau$ for the truncation of sheaves.

Lemma A.1.21. *Let $n \geq -1$.*

*(i) A presheaf F lies in $\tau_{\leq n}$**PSh** if and only if $F(X)$ is n-truncated for every object X of $\mathcal{C}$.*

(ii) The n-truncation of a sheaf is the sheafification of the n-truncated presheaf, i.e. $\mathrm{L}_{\leq n} \simeq \mathrm{L} \circ \tau_{\leq n} \circ \iota$ where $\iota\colon \mathbf{Sh}_\tau \hookrightarrow \mathbf{PSh}$ is the inclusion.

Proof. (i) If $F \in \tau_{\leq n}\mathbf{PSh}$, then $F(X) = \mathrm{Map}_{\mathbf{PSh}}(y(X), F)$ is n-truncated. Conversely, we can write $\mathrm{Map}_{\mathbf{PSh}}(E,F) \simeq \lim_X \mathrm{Map}_{\mathbf{PSh}}(y(X),F) = \lim_X F(X)$ as a limit and appeal to Lemma A.1.17. (ii) follows formally. □

Definition A.1.22. Let $n \geq 0$, $F \in \mathbf{PSh}$, $X \in \mathcal{C}$, and $x \in F(X)$. We call

$$\pi_i^X(F,x)\colon (\mathcal{C}_{/X})^{\mathrm{op}} \longrightarrow \mathrm{Set}$$
$$U \mapsto \pi_i(F(U), x|_U).$$

the **homotopy presheaf** of F on X with base point x.

Lemma A.1.23. *If $F \in \tau_n\mathbf{Sh}$ is an n-truncated sheaf, then $\pi_n^X(F,x)$ is a sheaf for every $X \in \mathrm{Sch}_S$ and every $x \in F(X)$.*

Proof. Let $R \in \tau(U)$ be a τ-covering sieve of $U \in \mathcal{C}_{/X}$. Then $F(U) \simeq \lim_{(V \to U) \in R^{\mathrm{op}}} F(V)$ lies in $\tau_{\leq n}\mathbf{Spc}$ by Lemma A.1.17. The forgetful functor $\mathbf{Spc}_\bullet \to \mathbf{Spc}$ from the category of pointed spaces to spaces is a right adjoint and hence preserves limits. As a limit of pointed spaces receives a canonical base point, the forgetful functor even creates limits. Thus $(F(U), x|_U) \simeq \lim_{(V \to U) \in R^{\mathrm{op}}}(F(V), x|_V)$ is a limit in $\tau_{\leq n}\mathbf{Spc}_\bullet$. Analogously to the proof of Lemma A.1.17 we see via the long exact sequence of homotopy groups that the functor $\pi_n\colon \mathbf{Spc}_\bullet \to \mathrm{Set}_*$ preserves pullbacks, hence limits. In particular, $\pi_n(F(U), x|_U) \cong \lim_{(V \to U) \in R^{\mathrm{op}}} \pi_i(F(V), x|_V)$, i.e. the sheaf condition is satisfied. □

Lemma A.1.24. *Let $F \in \tau_n\mathbf{Sh}_\tau$ be an n-truncated sheaf. If $\pi_n^X(F,x) \simeq *$ for every $X \in \mathcal{C}$ and every $x \in F(X)$, then F is $(n-1)$-truncated.*

Proof. This follows from [Lur09, 6.5.1.6, 6.5.1.7]. □

Corollary A.1.25. *Let $E, F \in \tau_n\mathbf{Sh}_\tau$ be n-truncated sheaves. If a map $E \to F$ is a stalkwise equivalence, then it is an equivalence.*

Postnikov completeness and hypercompleteness

Definition A.1.26. Let $\mathbf{N}_\infty$ be the poset $\mathbf{N} \cup \{\infty\}$ where the element ∞ is bigger than any other element. A **Postnikov tower** in an ∞-category $\mathcal{D}$ is a functor $F\colon \mathbf{N}_\infty^{\mathrm{op}} \to \mathcal{D}$, i.e. a diagram

$$F_\infty \to \ldots \to F_2 \to F_1 \to F_0$$

in $\mathcal{D}$, such that for every $n \in \mathbf{N}$ the map $F_\infty \to F_n$ is an n-truncation of F_∞, i.e. it is equivalent to the unit map $F_\infty \to \tau_{\leq n} F_\infty$. A Postnikov tower $F_\bullet$ is said to be **convergent** iff the canonical map $F_\infty \to \lim_n F_n$ is an equivalence. We call $\mathcal{D}$ **Postnikov complete** iff every Postnikov tower in it is convergent.

Example A.1.27. (i) The ∞-category **Spc** of spaces is Postnikov complete.

(ii) The ∞-topos $\mathbf{Sh}_\tau(\mathcal{C})$ of τ-sheaves need not be Postnikov complete. For an example consider Morel-Voevodsky [MV99, 1.30].

A kind of orthogonal notion to truncation is the concept of connectedness. This leads to Whitehead's lemma which is not true in every ∞-topos but in *hypercomplete* ones.

Definition A.1.28. Let $\mathcal{X}$ be an ∞-topos and $n \in \mathbf{N}_\infty$. A morphism $f : X \to Y$ in $\mathcal{X}$ is called **n-connective** iff it is an effective epimorphism [Lur09, 6.2.3.5] and $\pi_i(f) = 0$ for all $0 \leq i < n$. That means, that f induces an isomorphism on homotopy groups in degrees smaller than n. An object Z of $\mathcal{X}$ is said to be **hypercomplete** iff it is local with respect to ∞-connected morphisms, i.e. for every ∞-connected morphism $X \to Y$ the induced map $\mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is an equivalence. Denote by $\mathcal{X}^\wedge$ the full subcategory spanned by hypercomplete objects. We say that $\mathcal{X}$ is hypercomplete iff $\mathcal{X} = \mathcal{X}^\wedge$. Note that, plainly by definition, Whitehead's lemma holds in a hypercomplete ∞-topos.

Proposition A.1.29. *(i) There is a left exact localisation*

$$(-)^\wedge : \mathcal{X} \rightleftarrows \mathcal{X}^\wedge : \mathrm{incl}$$

and $\mathcal{X}^\wedge$ is a hypercomplete ∞-topos.

(ii) A Postnikov complete ∞-topos is hypercomplete.

Proof. (i) is [Lur09, p. 669]. For (ii) let $\mathcal{X}$ be a Postnikov complete ∞-topos. We have to show that every object $X \in \mathcal{X}$ is hypercomplete. Since $\mathcal{X}$ is presentable, X can be extended to a Postnikov tower [Lur09, 7.2.1.11]

$$X = X_\infty \to \ldots \to X_1 \to X_0.$$

By construction, X_n is n-truncated for any $n \in \mathbf{N}$, hence hypercomplete [Lur09, 6.5.2.9]. Thus the limit $\lim_n X_n$ is hypercomplete. Since $\mathcal{X}$ is Posnikov complete, $X \simeq \lim_n X_n$ is hypercomplete. □

Equivalences on stalks

Definition A.1.30. A map of presheaves $f : E \to F$ is said to be a **τ-local equivalence** iff the sheafified map $\mathrm{L}f : \mathrm{L}E \to \mathrm{L}F$ is an equivalence.

Lemma A.1.31. *Every τ-local equivalence is a τ-stalkwise equivalence.*

Proof. Let $E \to F$ be a τ-local equivalence, i.e. the map $\mathrm{L}E \to \mathrm{L}F$ is an equivalence. Thus, for every point (X,p), there exists a commutative diagram

$$\begin{array}{ccc} E_p & \longrightarrow & F_p \\ \downarrow & & \downarrow \\ (\mathrm{L}E)_p & \longrightarrow & (\mathrm{L}F)_p \end{array}$$

where the lower vertical map is an equivalence. By Lemma A.1.15 also the two vertical maps are equivalences, hence the upper horizontal one as desired. □

Theorem A.1.32. *Assume that the ∞-topos $\mathbf{Sh}_\tau$ is Postnikov complete. Then a map of presheaves is a τ-local equivalence if and only if it is a τ-stalkwise equivalence.*

Proof. Let $f\colon E \to F$ be a stalkwise equivalence. There exists a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ \mathrm{L}E & \longrightarrow & \mathrm{L}F. \end{array}$$

The two vertical maps are both local equivalences and stalkwise equivalences by Lemma A.1.15. Hence the map $\mathrm{L}E \to \mathrm{L}F$ is a stalkwise equivalence and we may assume that E and F are sheaves. The truncation functor $\mathrm{L}_{\leq n} = \mathrm{L} \circ \tau_{\leq n}$ is left adjoint to the inclusion $\tau_{\leq n}\mathbf{Sh} \hookrightarrow \mathbf{Sh}$. Truncation for presheaves (i.e. objectwise truncation) and sheafification both preserve stalkwise equivalences. Thus the map $\mathrm{L}_{\leq n}E \to \mathrm{L}_{\leq n}F$ is a stalkwise equivalence, hence an equivalence by Corollary A.1.25. Since $\mathbf{Sh}_\tau$ is Postnikov complete by assumption, the vertival maps in the commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ \lim_n \mathrm{L}_{\leq n}E & \longrightarrow & \lim_n \mathrm{L}_{\leq n}F \end{array}$$

are equivalences. The lower horizontal map is an equivalence since it is a limit of equivalences. Thus $E \to F$ is an equivalence. □

Model structures on simplicial presheaves

In this subsection, we present different model structures on the category of simplicial presheaves on a small discrete site. Namely, the injective model structure, the Joyal-Jardine model structure, and the Čech-model structure. Their underlying ∞-categories are the ∞-topoi of presheaves, hypercomplete sheaves, resp. all sheaves (Proposition A.1.40). This will be needed in order to show that the ∞-topos of sheaves for certain cd-topologies is hypercomplete (Theorem A.2.10).

Notation. In this subsection, let $\mathcal{C}$ be a small discrete category. Denote by $\mathrm{sPSh}(\mathcal{C})$ the category of simplicial presheaves on $\mathcal{C}$, i.e. the category of simplicial objects in the category of set-valued presheaves on $\mathcal{C}$. Equivalentely, $\mathrm{sPSh}(\mathcal{C})$ may be described as the category of presheaves on $\mathcal{C}$ with values in simplicial sets.

Proposition A.1.33 (cf. [Lur09, A.3.3.2]). *There is a simplicial proper model structure on the category* $\mathrm{sPSh}(\mathcal{C})$ *where*

- *the weak equivalences are the objectwise weak equivalences,*
- *the cofibrations are the monomorphisms, and*
- *the fibrations are the maps which have the right lifting property with respect to all trivial cofibrations.*

This model structure is called the ***injective model structure*** *and its weak equivalences are called the* ***global weak equivalences***.

In classical homotopy theory of CW complexes, every weak homotopy equivalence is a homotopy equivalence. In this spirit, Joyal [Joy84] and Jardine [Jar87] both defined (different, but equivalent) model structures on the category of simplicial presheaves wherein the weak equivalences can be characterised as maps inducing isomorphisms on homotopy sheaves (Proposition A.1.37). Those are sheaves of sets and hence one can test on stalks whether a map is an isomorphism (if the site has enough points in the 1-categorical sense). The fibrant objects of these model structures have been characterised by Dugger-Hollander-Isaksen [DHI04] as those simplicial presehaves which satisfy so-called *hyperdescent*, i.e. they satisfy a sheaf condition with respect to so-called *hypercovers*. As we see later, the common underlying ∞-category of these model structures are the *hypercomplete* space-valued sheaves on the nerve of $\mathcal{C}$ (Proposition A.1.40).

Proposition A.1.34. *There is a left proper, combinatorial, simplicial model structure on the category* $\mathrm{sPSh}(\mathcal{C})$ *where*

- *the weak equivalences are the local trivial Kan fibrations, i.e. those maps $E \to F$ such that for every object X of $\mathcal{C}$ and every commutative diagram*

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & E(X) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & E(X) \end{array}$$

  there exists a covering $\{X_i \to X\}_i$ such that for every i there exists a lift for the induced diagram as indicated below.

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & E(X_i) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & F(X_i) \end{array}$$

- *the cofibrations are the monomorophisms, i.e. the cofibrations in the injective model structure, and*
- *the fibrations are the maps which have the right lifting property with respect to all trivial cofibrations.*

We call this model structure the ***Joyal-Jardine model structure*** *and the weak equivalences are called the* ***local weak equivalences****.*

Caveat A.1.35. In the literature, the Joyal-Jardine model structure is often called "local model structure" and local weak equivalences are then called "local equivalences", cf. [Voe10a, p. 1391] or [Lur09, 6.5.2.1]. However, our terminology is helpful to distinguish from the τ-local model structure and τ-local equivalences.

As indicated above, the local weak equivalences can be desribed as maps which are locally homotopy equivalences. To make this precise, we introduce homotopy sheaves.

Definition A.1.36. Let F be a simplicial presheaf on $\mathcal{C}$. For $n \geq 0$, an object X of $\mathcal{C}$, and a base point $x \in F(X)$, define the **homotopy sheaf** $\pi_n^X(F,x)$ as the sheafification of the presheaf $U \mapsto \pi_n(F(U), x|_U)$ on $\mathcal{C}_{/X}$. A morphism $E \to F$ of simplicial presheaves on $\mathcal{C}$ is called a **local homotopy equivalence** iff for every $n \geq$, every object X of $\mathcal{C}$, and every base point $x \in E(X)$ the induced map

$$\pi_n^X(E,x) \longrightarrow \pi_n^X(F,f(x))$$

is an isomorphism of set-valued sheaves on $\mathcal{C}$; we wrote $f(x)$ to denote the base point $f(X)(x) \in F(X)$.

Proposition A.1.37 ([Jar87, 1.12])**.** *A morphism between simplicial presheaves which are fibrant with respect to the Joyal-Jardine model structure is a local weak equivalence if and only if it is a local homotopy equivalence.*

Since the fibrant objects in the Joyal-Jardine model structure are the presheaves which satisfy hyperdescent, it is natural to ask about a model structure wherein the fibrant objects are those satisfying descent, i.e. the usual sheaf condition. This yields the Čech-model structure which is defined as the τ-model structure for τ being the class of τ-covering sieves. Such a model structure was studied by Dugger-Hollander-Isaksen [DHI04, App. A].

Remark A.1.38 (Bousfield localisation)**.** The injective model struture on $\mathrm{sPSh}(\mathcal{C})$ is combinatorial [Lur09, A.2.8.2]. This ensures the existence of Bousfield localisations [Lur09, A.3.7].

Definition A.1.39 (Dugger-Hollander-Isaksen)**.** For an object X in $\mathcal{C}$ denote by $h_X\colon C^{\mathrm{op}} \to \mathrm{sSet}$ the associated simplicial presheaf. Let τ be a topology on $\mathcal{C}$. A simplicial presheaf F on $\mathcal{C}$ is called **τ-local** iff for every X in $\mathcal{C}$ and every τ-covering $U \to h_X$ the induced map

$$\mathrm{Map}(h_X, F) \longrightarrow \mathrm{Map}(U, F)$$

on simplicial mapping spaces is a weak equivalence of simplicial sets (cf. Definition A.1.1 for the corresponding notion of space valued-sheaves). A morphism $E \to E'$ of simplicial presheaves is called a **τ-local equivalence** iff for every τ-local object F the induced map

$$\mathrm{Map}(E', F) \longrightarrow \mathrm{Map}(E, F)$$

between the deriveed mapping spaces is a weak equivalence of simplicial sets. The **τ-local model structure** on $\mathrm{sPSh}(\mathcal{C})$ is the Bousfield localisation [Lur09, A.3.7.3] of $\mathrm{sPSh}(\mathcal{C})$ equipped with the injective model structure with respect to the τ-equivalences.

Proposition A.1.40. *Let A, B, and C be the category* $\mathrm{sPSh}(\mathcal{C})$ *equipped with the injective model structure, the Čech-model structure, and the Joyal Jardine model structure, respectively. Let A°, B°, and C° be the full subcategories spanned by the fibrant-cofibrant objects with respect to the respective model structure. Denote by $\mathcal{A} = \mathrm{N}(A^\circ)$, $\mathcal{B} = \mathrm{N}(B^\circ)$, and $\mathcal{C} = \mathrm{N}(C^\circ)$, respectively, their underlying ∞-categories* [Lur09, § A.2]. *Then:*

(a) $\mathcal{A}$ is equivalent to the ∞-topos $\mathbf{PSh}(\mathrm{N}(\mathcal{C}))$ of space-valued presheaves.

(b) $\mathcal{B}$ is equivalent to the ∞-topos $\mathbf{Sh}_\tau(\mathrm{N}(\mathcal{C}))$ of space-valued τ-sheaves.

(c) $\mathcal{C}$ is equivalent to the ∞-topos $\mathbf{Sh}_\tau(\mathrm{N}(\mathcal{C}))^\wedge$ of hypercomplete space-valued τ-sheaves.

Proof. (a) follows from [Lur09, 5.1.1.1], cf. p. 666 ibid. (b) follows from (a) since the fibrant-cofibrant objects in the Čech-model structure are precisely the presheaves satisfying the sheaf condition, cf. [Lur09, A.3.7.8]. (c) is [Lur09, 6.5.2.14]. □

A.2. Homotopy theory for cd-structures

Inspired by the work of Brown-Gersten [BG73], Voevodsky [Voe10a] studied topologies which are generated by *cd-structures*. On the one hand, the Brown-Gersten model structure is more explicit than the Joyal-Jardine model structure. On the other hand, it is only defined for simplcial presheaves on a finite-dimensional noetherian topological space. Voevodsky generalised the approach of Brown-Gersten to sites whose topology is induced by a cd-structure. In such topologies, one can check the sheaf condition on *elementary covers*. Under some additional assumtions the ∞-topos of space-valued sheaves will be hypercomplete (Theorem A.2.10).

Notation. In section A.2, let $\mathcal{C}$ be a small discrete category.

cd-structures and their topologies

Definition A.2.1. Assume that $\mathcal{C}$ has a strict initial object, i.e. every map $X \to \varnothing$ is an isomorphism. A **cd-structure** on $\mathcal{C}$ is a collection P of so-called **distinguished squares**

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p_Q \\ A & \xrightarrow{j_Q} & X \end{array} \tag{Q}$$

satisfiying

(CD1) P is closed under isomorphisms of squares.

A cd-structure is called **complete** iff it satisfies

(CD2) Pullbacks of squares in P exist and are in P.

A cd-structure is called **regular** iff it satisfies

(CD3) All squares in P are pullback squares.

(CD4) The map $j_Q: A \to X$ is a monomorphism.

(CD5) For any square (Q) in P the square

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B \times_A B & \longrightarrow & Y \times_X Y \end{array}$$

is also in P.

The phrase "let $\mathcal{C}$ be a category with a cd-structure" shall implicitly include that the category $\mathcal{C}$ has a strict initial object.

Caveat A.2.2. In Voevoedsky's original definition of a cd-structure [Voe10a, 2.1], the initial object is not assumed to be strict. Also the notions of being complete [Voe10a, 2.3] and regular [Voe10a, 2.10] are defined slightly different. However, if the category $\mathcal{C}$ has a strict initial object, then our notions imply Voevodsky's ones [Voe10a, 2.5,2.11].

Definition A.2.3 (Voevodsky)**.** Let S be a scheme. We consider the following collections of squares on the category Sch_S which can easily seen to be cd-structures.

(i) The **additive cd-structure** has distinguished squares

$$\begin{array}{ccc} \varnothing & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \sqcup Y \end{array}$$

(ii) The **Zariski cd-structure** has distinguished squares (Q) as in Definition A.2.1 which are pullback squares and where j_Q and p_Q are jointly surjective open immersions.

(iii) The **closed cd-structure** has distinguished squares (Q) as in Definition A.2.1 which are pullback squares and where j_Q and p_Q are jointly surjective closed immersions.

(iv) The **biZariski cd-structure** is the union of the Zariski cd-structure and the closed cd-structure.

(v) The **Nisnevich cd-structure** has distinguished squares (Q) as in Definition A.2.1 which are pullback squares and where j_Q is an open immersion, p_Q is étale, and the induced morphism $(Y \smallsetminus B)_{\mathrm{red}} \to (X \smallsetminus A)_{\mathrm{red}}$ is an isomorphism.

(vi) The **closed Nisnevich cd-structure** is the union of the closed cd-structure and the Nisnevich cd-structure.

(vii) The **abstract blow-up cd-structure** has as distinguished squares the *abstract blow-up squares*, i.e. squares (Q) as in Definition A.2.1 which are pullback squares and where j_Q is a closed immersion, p_Q is proper, and the induced morphism $(Y \smallsetminus B) \to (X \smallsetminus A)$ is an isomorphism.

(viii) The **rh-cd-structure** is the union of the Zariski cd-structure and the blow-up cd-structure.

(ix) The **cdh-cd-structure**[1] is the union of the Nisnevich cd-structure and the abstract blow-up cd-structure.

These cd-structures are called the **standard cd-structures**.

Remark A.2.4. For an arbitrary category with initial object, Voevodsky defined the notion of a *density structure* on a category with initial object [Voe10a, 2.20]. A density structure allows the notion of a dimension and it can bound a cd-structure. This leads to the notion of a **bounded cd-structure** [Voe10a, 2.22] which will not be defined here.

Theorem A.2.5 (Voevodsky)**.** *Let S be a noetherian scheme. Then all the standard cd-structures on* Sch_S *are complete, regular, and bounded. Moreover, when restricted to the full subcategory category* Sm_S *of smooth schemes, the Zariski cd-structure and the Nisnevich cd-structure are complete, bounded, and regular.*

Proof. The entire statement is [Voe10b, 2.2]. In more detail: The empty scheme is a strict initial object in Sch_S. Completeness follows since all the properties of morphisms of schemes which are used in the definitions are stable under pullbacks. Regularity can be checked on the generating cd-structures (i.e. additive, Zariski,

[1]This name is somewhat unfortunate as it is a pleonasm.

closed, Nisnevich, and abstract blow-up), see [Voe10b, 2.14]. All the generating cd-structure are equipped with the standard density structure [Voe10b, p. 1401] which is linked to the Krull dimension. Boundedness is proven geometrically and the most tricky part [Voe10b, 2.4-2.13]. □

Definition A.2.6. Let $\mathcal{C}$ be equipped with a cd-structure P. Its **associated topology** τ_P is the topology which is generated (Definition B.2.10) by the families $\{A \overset{j_Q}{\to} X, Y \overset{p_Q}{\to} X\}$ for every square $Q \in P$ as in Definition A.2.1.

Theorem A.2.7 (Voevodsky [Voe10b, 2.27])**.** *Let P be a complete and regular cd-structure on $\mathcal{C}$ which is bounded by a density structure D. Then for every object $X \in \mathcal{C}$ and every τ_P-sheaf F of abelian groups on $\mathcal{C}/X$ we have*

$$\mathrm{H}^i_{\tau_P}(X;F) = 0$$

for $i > \dim_D(X)$ where $\dim_D$ is the dimension of the density structure.

Corollary A.2.8. *Let X be a noetherian scheme of finite dimension d and let P be one of the standard cd-structures of Definition A.2.3 on the category Sch_X. Then for every τ_P-sheaf F of abelian groups on Sch_X we have*

$$\mathrm{H}^i_{\tau_P}(X;F) = 0$$

for $i > d$. In particular, this holds for the rh-topology.

Sheaves for topologies of cd-structures

Definition A.2.9. Let $\mathcal{C}$ be equipped with a cd-structure P. A presheaf F on $\mathcal{C}$ is called P**-excisive** iff for every square $Q \in P$ as in Definition A.2.1 the induced square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(A) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(B) \end{array}$$

in $\mathbf{PSh}(\mathcal{C})$ is cartesian.

Theorem A.2.10 (Voevodsky, Asok-Hoyois-Wendt)**.** *Assume that $\mathcal{C}$ is equipped with a cd-structure P (Definition A.2.1).*

(i) If P is complete and regular, then a space-valued presheaf on $\mathcal{C}$ is P-excisive if and only if it is a τ_P-sheaf.

(ii) If P is complete, regular, and bounded, then a space-valued presheaf on $\mathcal{C}$ is P-excisive if and only if it is a hypercomplete τ_p-sheaf.

In particular, if P is complete, regular, and bounded, then $\mathbf{Sh}_\tau(\mathcal{C})$ is equivalent to $\mathbf{Sh}_\tau(\mathcal{C})^\wedge$, hence hypercomplete.

Proof. Voevodsky proved that, if P is assumed to be complete and bounded, a morphism between P-excisive simplicial presehaves is a local homotopy equivalence if and only if it is a global equivalence [Voe10a, 3.5]. Hence a fibrant replacement $F \to \mathrm{R}F$ is a global equivalence if F is P-excisive. Thus (ii) follows from Proposition A.1.40 since the hypercomplete τ_P-sheaves are precisely the fibrant objects in the Joyal-Jardine model structure. The claim of (i) follows from the corresponding statement for simplicial presheaves which was proven by Asok-Hoyois-Wendt [AHW17, 3.2.5]. □

Proposition A.2.11. *Let S be a noetherian scheme of finite dimension d and let $\mathrm{Sch}_S^{\le d}$ be the category of S-schemes of dimension $\le d$. Let P be one of the standard cd-structures on Sch_S (Definition A.2.3).*

(i) The cd-structure P restricts to a cd-structure on $\mathrm{Sch}_S^{\le d}$.

(ii) The ∞-topos $\mathbf{Sh}(\mathrm{Sch}_S^{\le d})$ is locally of cohomological dimension $\le d$.

(iii) The ∞-topos $\mathbf{Sh}(\mathrm{Sch}_S^{\le d})$ is hypercomplete.

(iv) The ∞-topos $\mathbf{Sh}(\mathrm{Sch}_S^{\le d})$ is locally of homotopy dimension $\le \max\{2, d\}$.

(v) The ∞-topos $\mathbf{Sh}(\mathrm{Sch}_S^{\le d})$ is Postnikov complete.

Proof. (i) Consider a distinguished square

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle p_Q} \\ A & \xrightarrow[j_Q]{} & X \end{array} \tag{Q}$$

in the cd-structure P on Sch_S. If $\dim(X) \le d$, then A, B, and Y all have dimension less than or equal to d since the dimension does not decrease from the source to the target of a morphism of schemes which is an immersion (or étale resp. proper). Hence P restricts to the full subcategory $\mathrm{Sch}_S^{\le d}$.

(ii) For sheaves of abelian groups, the categorical cohomology groups of the ∞-topos $\mathbf{Sh}(\mathrm{Sch}_S^{\le d})$ identify with the classical cohomology groups on the discrete topos $\mathrm{Sh}(\mathrm{Sch}_S^{\le d})$ of set-valued sheaves [Lur09, 7.2.2.17]. Hence the cohomological dimension is bounded by d according to Theorem A.2.7.

(iii) By Theorem A.2.5 one can apply Theorem A.2.10.

(iv) The notion of having locally finite homotopy dimension is defined in Lurie [Lur09, 7.2.1.1]. This follows from (ii) and (iii) since, if $n \ge 2$, an ∞-topos is locally of homotopy dimension $\le n$ if and only if it is both hypercomplete and locally of finite cohomological dimension $\le n$.

(v) This follows from (iv) by [Lur09, 7.2.1.10] where the terminology is different, cf. [Lur18, A.7.2.2]. □

Corollary A.2.12. *In the situation of Proposition A.2.11, the ∞-topos $\mathbf{Sh}_\tau(\mathrm{Sch}_S)$ is Postnikov complete. In particular, it is hypercomplete.*

Proof. Let $F_\bullet : \mathbf{N}_\infty \to \mathbf{Sh}_\tau(\mathrm{Sch}_S)$ be a Postnikov tower and let $(X \xrightarrow{f} S) \in \mathrm{Sch}_S$. Since our schemes are assumed to be finitely generated, X is of finite dimension, say d. Restricting level-wise along $\mathrm{Sch}_X^{\leq d} \hookrightarrow \mathrm{Sch}_X \xrightarrow{f_*} \mathrm{Sch}_S$, we obtain a Postnikov tower in the ∞-topos $\mathbf{Sh}(\mathrm{Sch}_X^{\leq d})$ which is Postnikov complete by Proposition A.2.11. Hence the map $F_\infty(X) \to \lim_n F(X)$ is an equivalence for every object $X \in \mathrm{Sch}_S$. Thus the map $F_\infty \to \lim_n F_n$ is an equivalence of presheaves. □

Corollary A.2.13. *Let S be a finite-dimensional noetherian scheme and let Sch_X be equipped with one of the standard cd-structures (Definition A.2.3). Then the t-structure on $\mathbf{Sh}_{\mathbf{Sp}}(\mathrm{Sch}_S^{\leq d})$ is left complete, i.e. the canonical map into the limit of the truncations is an equivalence.*

Proof. Since the ∞-category $\mathcal{C}$ is admits small limits, the ∞-category $\mathbf{Sh}_{\mathbf{Sp}}(\mathrm{Sch}_S^{\leq d})$ is equivalent to the ∞-category $\mathbf{Sh}_{\mathbf{Sp}}(\mathbf{Sh}(\mathrm{Sch}^{\leq d}))$ of sheaves of spectra on the ∞-topos $\mathbf{Sh}(\mathrm{Sch}_S^{\leq d})$ [Lur18, 1.3.1.7]. Hence the ∞-topos $\mathbf{Sh}_{\mathbf{Sp}}(\mathrm{Sch}_S^{\leq d})$ of connective objects is Postnikov complete. We can write every object F as the colimit $\mathrm{colim}_{n\in\mathbf{N}} F_{\geq -n}$ of objects which are (up to a shift) connective. This shows the claim. □

Corollary A.2.14. *Let S be a finite-dimensional noetherian scheme and let Sch_X be equipped with one of the standard cd-structures (Definition A.2.3). A map of τ_P-sheaves on Sch_S with values in spaces or spectra is an equivalence if and only if it is a τ_P-stalkwise equivalence (Definition A.1.12)*

Proof. By Proposition A.2.11 (v) we can apply Theorem A.1.32 and get the claim for spaces. For spectra, one proceeds similarly to the proof of Corollary A.2.13. □

B. Some background

B.1. Limits of locally ringed spaces

In this section, we collect some facts about locally ringed spaces and filtered limits of those. This is based on the exposition by Fujiwara-Kato [FK18, ch. 0, §4.2].

Definition B.1.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space and let F be an $\mathcal{O}_X$-module.

(i) F is of **finite type** iff for every $x \in X$ there exists an open neighbourhood U of X and an epimorphism $\mathcal{O}_U^n \twoheadrightarrow F|_U$ for some $n \geq 0$.

(ii) F is **finitely presented** iff for any $x \in X$ there exists an open neighbourhood U of x and a short exact sequence

$$\mathcal{O}_U^m \longrightarrow \mathcal{O}_U^n \longrightarrow F|_U \longrightarrow 0$$

for some $m, n \geq 0$.

(iii) F is **coherent** iff both of the following conditions are satisfied:

a) F is of finite type and

b) for any open $U \subseteq X$ and any morphism $\varphi: \mathcal{O}_U^n \to F|_U$ with $n \geq 0$, the kernel of φ is of finite type.

(iv) F is **locally free (of finite rank)** iff for every $x \in X$ there exists an open neighbourhood U of x and an isomorphism $\mathcal{O}_U^n \xrightarrow{\cong} F|_U$ for some $n \geq 0$.

We obtain a chain of full subcategories

$$\mathrm{Mod}(X) \supset \mathrm{Mod}^{\mathrm{ft}}(X) \supset \mathrm{Mod}^{\mathrm{fp}}(X) \supset \mathrm{Coh}(X) \supset \mathrm{Vec}(X)$$

which are spanned by all $\mathcal{O}_X$-modules, $\mathcal{O}_X$-modules of finite type, finitely presented $\mathcal{O}_X$-modules, coherent $\mathcal{O}_X$-modules, and locally free $\mathcal{O}_X$-modules, respectively.

Lemma B.1.2 ([Sta19, Tag 01BY])**.** *The category* $\mathrm{Coh}(X)$ *is a weak Serre subcategory of* $\mathrm{Mod}(X)$ *and hence an abelian category.*[1] *The category* $\mathrm{Vec}(X)$ *is an exact subcategory of* $\mathrm{Coh}(X)$.

[1] This is the terminology used in the Stacks Project [Sta19, Tag 02MN]. Fujiwara-Kato call it a *thick* subcategory [FK18, ch. 0, §C.5].

Definition B.1.3. We say that a locally ringed space $(X, \mathcal{O}_X)$ is **cohesive** iff its structure sheaf $\mathcal{O}_X$ is coherent.

Example B.1.4. For a locally noetherian scheme X, an $\mathcal{O}_X$-module is coherent if and only if it is finitely presented [Sta19, Tag 01XZ]. Thus a locally noetherian scheme is a cohesive locally ringed space.

Lemma B.1.5 ([FK18, ch. 0, 4.1.8, 4.1.9]). *(i) If $(X, \mathcal{O}_X)$ is cohesive, then an $\mathcal{O}_X$-module F is coherent if and only if it is finitely presented.*

(ii) If $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces and $(X, \mathcal{O}_X)$ is cohesive, then $f^*: \mathrm{Mod}(Y) \to \mathrm{Mod}(X)$ restricts to a functor $f^*: \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$.

Definition B.1.6. A topological space is said to be **coherent** iff it is quasi-compact, quasi-separated, and admits an open basis of quasi-compact subsets. A topological space is called **sober** iff it is a T_0-space and any irreducible closed subset has a (unique) generic point.

Example B.1.7. The underlying topological space of a quasi-compact and quasi-separated scheme is coherent and sober.

Proposition B.1.8 ([FK18, ch. 0, 2.2.9, 2.2.10]). *Let $(X_i, (p_{ij})_{j \in I})_{i \in I}$ be a filtered system of topological spaces. Denote by X its limit and by $p_i: X \to X_i$ the projection maps.*

(i) Assume that the topologies of the X_i are generated by quasi-compact open subsets and that the transition maps p_{ij} are quasi-compact. Then every quasi-compact open subset $U \subset X$ is the preimage of a quasi-compact open subset $U_i \subset X_i$ for some $i \in I$.

(ii) Assume that all the X_i are coherent and sober and that the transition maps p_{ij} are quasi-compact. Then X is coherent and sober and the p_i are quasi-compact.

Proposition B.1.9. *Let $(X_i, \mathcal{O}_{X_i}, (p_{ij})_{j \in I})_{i \in I}$ be a filtered system of locally ringed spaces. Then its limit $(X, \mathcal{O}_X)$ in the category of locally ringed spaces exists. Let $p_i: X \to X_i$ be the canonical projections.*

(i) The underlying topological space X is the limit of $(X_i)_{i \in I}$ in Top.

(ii) $\mathcal{O}_X = \mathrm{colim}_{i \in I}\, p_i^{-1} \mathcal{O}_{X_i}$ in $\mathrm{Mod}(X)$.

(iii) For every $x \in X$ have $\mathcal{O}_{X,x} = \mathrm{colim}_{i \in I}\, p_i^{-1} \mathcal{O}_{X_i, p_i(x)}$ in Ring.

Assume additionally that every X_i is coherent and sober and that all transitions maps are quasi-compact.

(iv) *The canonical functor*

$$\operatorname*{colim}_{i\in I} \mathrm{Mod}^{\mathrm{fp}}(X_i) \longrightarrow \mathrm{Mod}^{\mathrm{fp}}(X)$$

is an equivalence in the 2-category of categories. In particular, for any finitely presented $\mathcal{O}_X$-module F there exists an $i \in I$ and a finitely presented $\mathcal{O}_{X_i}$-module F_i such that $F \cong p_i^* F_i$.

(v) *For any morphism $\varphi: F \to G$ between finitely presented $\mathcal{O}_X$-modules there exists an $i \in I$ and a morphism $\varphi_i: F_i \to G_i$ between finitely presented $\mathcal{O}_{X_i}$-modules such that $\varphi \cong p_i^* \varphi_i$. Additionally, if φ is an isomorphism or an epimorphism, then one can choose φ_i to be an isomorphism or an epimorphism, respectively.*

(vi) *For every $i \in I$ let F_i be an $\mathcal{O}_{X_i}$-module and for every $i \le j$ in I let $\varphi_{ij}: p_{ij}^* F_i \to F_j$ be a morphism of $\mathcal{O}_{X_j}$-modules such that $\varphi_{ik} = \varphi_{jk} \circ p_{jk}^* \varphi_{ij}$ whenever $i \le j \le k$ in I. Denote by F the $\mathcal{O}_X$-module $\operatorname{colim}_i p_i^* F_i$. Then the canonical map*

$$\operatorname*{colim}_{i\in I} \mathrm{H}^*(X_i, F_i) \longrightarrow \mathrm{H}^*(X, F)$$

is an isomorphism of abelian groups.

Proof. The existence, (i), (ii), and (iii) are [FK18, ch. 0, 4.1.10] and (iv) and (v) are [FK18, ch. 0, 4.2.1–4.2.3]. Finally, (vi) is [FK18, ch. 0, 4.4.1]. □

Definition B.1.10. Let $(X, \mathcal{O}_X)$ be a locally ringed space, Z a closed subset of X, and $j: (X, \mathcal{O}_X) \hookrightarrow (U, \mathcal{O}_U)$ the inclusion of the open complement. An $\mathcal{O}_X$-module F has **support in** Z iff $j^* F = F|_U$ vanishes. Denote with a "Z" in the lower index the full subcategory of those $\mathcal{O}_X$-modules which have support in Z, e.g. $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$.

B.2. Topologies and pretopologies

The content of this section is based on the exposition by Vistoli [Vis05]. Another recommendable source is, as in many cases, the Stacks Project [Sta19, Tag 00UZ]. An exposition for ∞-categories is given by Lurie [Lur09, §6.2.2]. However, a topology on an ∞-category is just a topology on the underlying homotopy category [Lur09, 6.2.2.3].

Notation. In section B.2, let $\mathcal{C}$ be a small discrete category. For an object U in $\mathcal{C}$ denote by $y(U): \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ its associated representable set-valued presheaf.

Topologies

One advantage of the notion of topologies in comparison to pretopologies (which are defined in terms of families of morphism with fixed target) is that one can intersect and unify them. This is helpful since some important topologies for schemes are not induced by pretopologies since their covers are not "of a single type".

Definition B.2.1. Let $U \in \mathcal{C}$. A **sieve** on U is a subfunctor $S \subset y(U)$. In other words, a sieve S is for every obejct T of $\mathcal{C}$ a subset $S(T) \subset \mathrm{Hom}_{\mathcal{C}}(T,U)$ with the following property: for every $T \to U$ in $S(U)$ and any morphism $T' \to T$ in $\mathcal{C}$ the composition $T' \to T \to U$ lies also in $S(U)$.

Example B.2.2. Let $\mathcal{U} := \{f_i : U_i \to U\}_i$ be a family of morphisms with fixed target in $\mathcal{C}$. Then one obtains a sieve $y(\mathcal{U})$ on U which is the union of the images of the maps $y(U_i) \to h_U$. That means that

$$y(\mathcal{U})(T) = \{g : T \to U \mid \exists g_i : T \to U_i : g = f_i \circ g_i\}$$

for every $T \in \mathcal{C}$. The sieve $y(\mathcal{U})$ is the unique smallest sieve on U such that every map $U_i \to U$ lies in $y(\mathcal{U})$ [Sta19, Tag 00Z0].

Example B.2.3. Let $V \to U$ be a morphism in $\mathcal{C}$ and S be a sieve on U. Then there exists a **pullback sieve** $S \times_U V$ which is given by

$$S \times_U V(T) := \{T \to V \mid (T \to V \to U) \in S(U)\}$$

for $T \in \mathcal{C}$.

Definition B.2.4. A **topology** τ on $\mathcal{C}$ is, for every object U of $\mathcal{C}$, a subset $\tau(U)$ of the set of sieves on U satisfying for every object V of $\mathcal{C}$ and every morphism $V \to U$ in $\mathcal{C}$ the following conditions:

(i) the representable $y(U)$ lies in $\tau(U)$,

(ii) for every $S \in \tau(U)$ the pullbakc sieve $S \times_U V$ is in $\tau(V)$, and

(iii) for every $S \in \tau(U)$ and every sieve R on U such that for all $U' \to U$ in $S(U)$ have $R \times_U U' \in \tau(U')$, it follows that R is in $\tau(U)$.

We call the objects of $\tau(U)$ the **τ-covering sieves** on U.

Let τ and σ be two topologies on $\mathcal{C}$. We say that σ is **finer** than τ, or equivalentely, that τ is **coarser** than σ, iff $\tau(U) \subset \sigma(U)$ for every object $U \in \mathcal{C}$.

Remark B.2.5. A topology on $\mathcal{C}$ is just a topology on its homotopy category [Lur09, 6.2.2.3].

Example B.2.6. There is always the **discrete topology** wherein every sieve is a covering sieve. It is finer than any other topology. On the other hand, the **indiscrete topology** has for every object U of $\mathcal{C}$ the single covering sieve $y(U)$. it is coarser than any other topology.

Example B.2.7. The **canonical topology** can on $\mathcal{C}$ is the finest topology such that every representable presheaf is a sheaf. A topology coarser than the canonical topology is called **subcanonical**.

Lemma B.2.8. *Let $(\tau_i)_{i \in I}$ be a familiy of topologies on $\mathcal{C}$.*

(i) *The set* $\left\{\bigcap_{i\in I}\tau_i(U)\;\middle|\;U\in\mathcal{C}\right\}$ *is a topology on* $\mathcal{C}$.

(ii) *There is a coarsest topology finer than all* τ_i.

Proof. The intersections $\bigcap_{i\in I}\tau_i(U)$ satifiy the condidions of Definition B.2.4 since all $\tau_i(U)$ satisfy them by assumption. For (ii) take the intersection of all topologies which are finer than all the τ_i at once. This intersection is non-empty since there is at least the discrete topology (cf. Example B.2.6). □

Lemma B.2.9. *Let* $(\sigma_i)_{i\in I}$ *be a family of sets of covering sieves for every object in* $\mathcal{C}$. *Then there exists a unique coarest topology wherein every sieve which is contained in some* σ_i *is a covering sieve.*

Proof. For every $i\in I$, there is a unique coarsest topology τ_i wherein every sieve of σ_i is a covering sieve. Namely, this is the intersection of all topologies which contain σ_i. Hence the desired topology is the coarest topology finer than all τ_i. □

Definition B.2.10. In the situation of Lemma B.2.9 we call the obtained topology the **topology generated by** the given sieves.

Pretopologies

In some topologies, it is sufficient to consider ony simple covers of a certain type.

Definition B.2.11. A **pretopology** on $\mathcal{C}$ is given by a set $\mathrm{Cov}(\mathcal{C})$ of coverings, i.e. a set of families of morphism with fixed target $(U_i\to U)_i$ satisfying the following conditions:

(i) for every isomorphism $V\to U$ have $\{V\to U\}\in\mathrm{Cov}(\mathcal{C})$,

(ii) if $\{U_i\to U\}_i\in\mathrm{Cov}(\mathcal{C})$ and if $\{U_{ij}\to U_i\}_j\in\mathrm{Cov}(\mathcal{C})$ for every i, then $\{U_{ij}\to U\}_{i,j}\in\mathrm{Cov}(\mathcal{C})$, and

(iii) if $\{U_i\to U\}_i\in\mathrm{Cov}(\mathcal{C})$ and if $V\to U$ is a morphism, then the fibre products $U_i\times_U V$ exist and $\{U_i\times_U V\to V\}_i\in\mathrm{Cov}(\mathcal{C})$.

Every pretopology defines a topology.

Proposition B.2.12. *Let* $\mathrm{Cov}(\mathcal{C})$ *be a pretopology on* $\mathcal{C}$. *Then there is a topology* τ *defined as follows: a sieve* S *on an object* U *is an element of* $\tau(U)$ *iff there exists a cover* $\mathcal{U}=\{U_i\to U\}_i$ *such that* $y(\mathcal{U})\subset S$ *(cf. B.2.2).*

Definition B.2.13. In the situation of Proposition B.2.12, call τ the **induced topology** of the pretopology $\mathrm{Cov}(\mathcal{C})$. We say that two pre-topologies $\mathrm{Cov}(\mathcal{C})$ and $\mathrm{Cov}'(\mathcal{C})$ on $\mathcal{C}$ are **equivalent** iff their induced pretopologies coincide.

Lemma B.2.14. *Let* $\mathrm{Cov}(\mathcal{C})$ *and* $\mathrm{Cov}'(\mathcal{C})$ *be two pretopologies on* $\mathcal{C}$ *with induced topologies* τ *resp.* τ'. *Then* τ' *is finer than* τ *if and only if every* τ*-cover can be refined by a* τ'*-cover.*

Corollary B.2.15. *Assume for every object U of $\mathcal{C}$ a set of families of morphisms with fixed target U. Then there is a unique coarest topology such that all given families are covers.*

Proof. Every familiy of morphisms with fixed target defines a sieve (B.2.2), hence one can apply Lemma B.2.9. □

Example B.2.16. Let S be a base scheme. On the category Sch_S we have the following chain of refinements of topologies:

$$\mathrm{Zar} \subset \mathrm{Nis} \subset \text{ét} \subset \mathrm{fppf} \subset \mathrm{fpqc} \subset \mathrm{can}$$

It is a theorem of Grothendieck that the fpqc-topology is subcanonical [Sta19, Tag 023Q]. For a detailed account of various topologies on schemes consider Gabber-Kelly [GK15].

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