EVOLUTION OF INTERFACES IN TWO-PHASE PROBLEMS WITH NINETY DEGREE CONTACT ANGLE

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Abstract

In this thesis we are concerned with the analysis of contact angle problems for the free boundary in two-phase flows. In particular, we consider the Mullins-Sekerka problem with a ninety degree angle condition at the points where the free interface meets the boundary. Here, the domain in two or three space dimensions is smooth and bounded. The main result is existence and uniqueness of local strong solutions. We hereby develop a comprehensive $L_p - L_q$-maximal regularity theory for the linear problem.

Furthermore, we are interested in qualitative properties of solutions. We show global existence and nonlinear stability for flat interfaces in a cylindrical geometry in two or three space dimensions, and provide a complete analysis of the linearized stability properties of stationary solutions for general geometries in two dimensions.

Moreover we consider a sharp interface model given by the two-phase Navier-Stokes equations with surface tension coupled to the Mullins-Sekerka problem. Again we prove the existence and uniqueness of local-in-time strong solutions. Furthermore, global-in-time existence and stability for solutions starting close to equilibria is obtained.

We then introduce a thermodynamically consistent model for the two-phase Navier-Stokes/Mullins-Sekerka equations with gravity and prove the presence of a so called Rayleigh-Taylor instability.

Zusammenfassung

Diese Arbeit behandelt die Analysis von Kontaktwinkelproblemen in Zwei-Phasen Strömungen. Insbesondere betrachten wir das Mullins-Sekerka Problem mit einem Kontaktwinkel von neunzig Grad am Rand des Gebietes. Das Gebiet hat zwei oder drei Raumdimensionen, glatten Rand und ist beschränkt. Das Hauptresultat hier ist Existenz und Eindeutigkeit starker Lösungen. Wir entwickeln hierbei eine umfassende $L_p - L_q$-Theorie und zeigen maximale Regularität des linearen Problems.

Ebenfalls werden Resultate über qualitatives Verhalten von Lösungen bewiesen. Wir zeigen globale Existenz und nichtlineare Stabilität für flache Grenzschichten in einer zylindrischen Geometrie. Ebenfalls geben wir eine umfassende lineare Stabilitätsanalyse für allgemeine Gebiete in zwei Raumdimensionen.

Danach betrachten wir ein Zwei-Phasen Navier-Stokes/Mullins-Sekerka Modell und zeigen Existenz und Eindeutigkeit von starken Lösungen für kurze Zeiten. Ebenfalls werden Stabilitätsresultate für Lösungen, die nahe an Equilibria starten, bewiesen.

Fernerhin leiten wir ein thermodynamisch konsistentes Zwei-Phasen Navier-Stokes/ Mullins-Sekerka Modell mit Gravitationskraft her und zeigen Rayleigh-Taylor-Instabilität.

Für Mama und Papa.

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CHAPTER 1

Introduction

1.1. Moving interfaces, contact angles, and maximal L_p-regularity

The evolution of interfaces is a phenomenon arising in mathematics, physics, and other natural sciences. The most prominent example might be the geometric evolution law *Mean Curvature Flow* for a family of closed hypersurfaces. This problem is an example for a curvature driven geometric evolution equation. It states that the normal velocity of the interface is given by its mean curvature, up to a sign depending on the sign convention used. Here, the quantities which are responsible for the flow are given explicitly on the interface: the normal velocity and the mean curvature of the hypersurface representing the interface. There are also problems which do not satisfy this local property. A problem which has been extensively studied is the so called *Stefan problem*. It it used for instance to describe the melting process of an ice cube surrounded by water. In this two-phase problem one has to solve an equation away from the interface, the solution of which then determines the normal velocity of the interface. These two-phase problems are typically harder to solve due to their nonlocal structure.

Interfaces appear also very naturally in the context of fluid dynamics. If one considers a situation where two or more fluids are involved, the question on how these fluids interact with each other and how their dynamical behaviour evolves in time arises immediately. In a setting where the fluids are assumed to be immiscible, there is no mixing zone and a sharp interface appears to separate the domains occupied by the respective fluids. A prominent example to model this type of situation with two different fluids are the *two-phase Navier-Stokes equations with surface tension.* However, there are also models which allow for partial mixing by introducing a diffuse interface layer in which the mixing takes place. This is also very reasonable, but the sharp interface models are accepted as an idealized model.

In applications, one often has to deal with interfaces not being closed, but with interfaces touching a part of the so called boundary: think of two fluids in a bounded container. One can have the situation where a droplet of oil is completely surrounded by water, but also two films of oil and water having a common interface which touches the boundary of the container. These problems are called *contact angle problems*. Compared to the evolution of closed interfaces, these problems are analytically harder to solve since one has to take extra care for the boundary condition which models the contact angle. These contact angles in general can range from a static ninety degree angle, to angles different from ninety degrees, up to dynamic contact angles depending on time. However, already the simplest case of ninety degree contact angles can cause major difficulties in the analysis of the respective problems. From a physical viewpoint, this condition may be seen as somewhat of a idealization of the real behaviour observed in nature. In view of the vast applications, these contact angle problems are of high interest. We refer e.g. to DeSimone, Grunewald, Otto [**15**], Guo, Tice [**28**], Knüpfer, Masmoudi [**37**], Otto [**50**], Pukhnachev, Solonnikov [**59**], and Wilke [**64**].

There are different possibilities to describe the free interface and its evolution in time. One way is to pick a fixed reference frame and measure the distance of the free boundary with respect to this reference geometry. This way one can think of reducing the geometric problem for the free hypersurface to some, in general highly nonlinear, evolution equation for the distance function. This ansatz is preferrably used when the distance of the free interface to the fixed configuration is small. One can then linearize the equations at the reference geometry and show *maximal regularity* for the linearized problem. For a given operator, this is usually a nontrivial task. Maximal regularity roughly speaking means that there is an isomorphism between solution and data space of the linearized operator. In case where the equations can be reduced to a single abstract evolution equation in some Banach space, the concept of maximal L_p-regularity has proven to be a very useful tool to obtain solvability of the equation in a strong sense and deduce further results, e.g. stability or instability properties. The literature on maximal L_p-regularity is vast. We therefore only want to refer to the recent book by Prüss, Simonett [**57**] and the references therein for further discussion. In [**57**] it is displayed in a great way that the abstract theory of maximal L_p-regularity proves flexible and hence can be applied to a wide array of problems arising in the local description of free boundaries and moving interfaces.

1.2. The Mullins-Sekerka problem

In this section we describe and collect results on the classical *Mullins-Sekerka problem*. We follow the survey article [**24**].

The classical Mullins-Sekerka problem for closed interfaces has been widely studied. It describes the evolution of the spatial distribution of two phases in time. The two phases occupy two regions, separated by a sharp interface. The motion is driven by the reduction of interfacial area and limited by diffusion. The area of the two phases is conserved in time. The Mullins-Sekerka evolution law can be derived from conservation laws and the principles of thermodynamics, cf. [**17**], [**30**], but also in the context of gradient flows, cf. [**23**], [**24**], [**49**].

The classical model is the following. Let $\Omega \subset \mathbb{R}^n$ denote a bounded domain in $\mathbb{R}^n$, where $n \in \mathbb{N}$ is the dimension. For a time-dependent family of closed and compactly embedded interfaces $\Gamma(t)$ separating the two disjoint phases $\Omega^+(t)$ and $\Omega^-(t)$, the problem, given an initial surface Γ_0, reads as

$$\begin{aligned} -\Delta\mu &= 0, && \text{in } \Omega^\pm(t), \\ V &= -[\![\nabla\mu]\!] \cdot \nu, && \text{on } \Gamma(t), \\ \mu &= H, && \text{on } \Gamma(t), \end{aligned}$$

supplemented by Neumann boundary conditions for μ on $\partial\Omega$. Hereby, V denotes the normal velocity and H the mean curvature of $\Gamma(t)$ with respect to the unit normal ν of $\Gamma(t)$ pointing inside $\Omega^+(t)$. By $[\![\cdot]\!]$ we denote the jump of a given quantity across the interface in direction of ν. It now readily follows that the enclosed volume by

$\Gamma(t)$ is constant in time, whereas the area of $\Gamma(t)$ is non-increasing,

$$\frac{d}{dt}\mathrm{Area}(\Gamma(t)) = -\int_\Omega |\nabla\mu|^2 dx \leq 0,$$

cf. Proposition 2 in [**24**]. In comparison to the classical Mean Curvature Flow, the Mullins-Sekerka problem is nonlocal in space. The hypersurface $\Gamma(t)$ determines the domains in which we have to solve the Laplace equation, $\Delta\mu = 0$ in $\Omega\backslash\Gamma(t)$. At the same time, the solution μ to the Laplace equation determines the normal velocity of the evolving interface. In this way, the interface at time t is also an unknown and has to be determined as part of the problem as well.

In order to formulate the problem in a suitable setting, one can transform the equations defined on the time dependent domains $\Omega\backslash\Gamma(t)$ to a fixed reference configuration. This ansatz goes back to Hanzawa [**32**]. Note that this transformation naturally depends on the solution $\Gamma(t)$. By this transformation, the equations have to be transformed to the fixed reference geometry and a highly nonlinear problem arises. Using this approach, local existence of unique strong solutions was obtained by Chen, Hong, Yi [**13**], and Escher, Simonett [**21**]. In [**13**], the authors work in classical Hölder spaces, whereas in [**21**] height functions in so-called little Hölder spaces are considered. Both works [**13**] and [**21**] show that for positive times $t > 0$ the solution is smooth in the classical sense. In [**5**], Alikakos et al. consider the case of ninety degree contact in two space dimensions in the case where the initial interface is assumed to be smooth and close to a part of a circle. Their arguments rely on an harmonic extension of the curvature and finding explicit formulas in complex variables.

In general, classical or strong solutions to the Mullins-Sekerka problem only exist for short times as topological changes and singularities may occur. For long-time existence results one can turn to weak formulations, since this solution concept can handle different scenarios where classical solutions break down. We refer to Luckhaus [**41**], Luckhaus, Sturzenhecker [**42**], and Röger [**60**]. As a part of this thesis we will consider the Mullins-Sekerka problem when the interface forms a contact angle to the boundary of the domain.

1.3. The two-phase Navier-Stokes/Mullins-Sekerka equations

In this section we introduce the system given by the Mullins-Sekerka equations coupled to the *two-phase Navier-Stokes equations with surface tension*. We follow the introductory chapter of [**4**].

Consider the flow of two incompressible, viscous fluids inside a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. We assume that the fluids are of Newtonian type, do not mix, and possess a common sharp interface $\Gamma(t)$. This interface separates the two regions occupied by the fluids. In the model without boundary contact which we present in this introduction, the inner phase is compactly embedded in the domain, that is, has a positive distance to the boundary of the fixed domain for all times during the evolution. The viscosities of the two fluids are constant but may be different from each other. In particular, we model the case where surface tension at the common interface is present. For simplicity in this introduction, we set the density equal to one in both fluids.

Let v be the velocity and p the pressure. The stress tensor is given by $T(v,p) = \mu^{\pm}(Dv + Dv^{\top}) - pI$ in the bulk regions $\Omega\backslash\Gamma(t)$. The model reads as

$$
\begin{aligned}
\partial_t v + v\cdot\nabla v - \operatorname{div} T(v,p) &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\operatorname{div} v &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\Delta\mu &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
-\nu\cdot[\![T(v,p)]\!] &= \sigma H\nu, && \text{on } \Gamma(t),\\
[\![v]\!] &= 0, && \text{on } \Gamma(t),\\
V &= v\cdot\nu - [\![\nabla\mu]\!]\cdot\nu, && \text{on } \Gamma(t),\\
\mu &= \sigma H, && \text{on } \Gamma(t),
\end{aligned}
$$

subject to boundary and initial conditions. Here, $\sigma > 0$ is a constant representing the surface tension.

One motivation to consider this problem is that it appears as a sharp interface limit of the Navier-Stokes/Cahn-Hilliard model

$$
\begin{aligned}
\partial_t v + v\cdot\nabla v - \operatorname{div}(\mu(c)Dv) + \nabla p &= -\epsilon\operatorname{div}(\nabla c\otimes\nabla c), && \text{in } \Omega\times\mathbb{R}_+,\\
\operatorname{div} v &= 0, && \text{in } \Omega\times\mathbb{R}_+,\\
\partial_t c + v\cdot\nabla c &= \Delta\mu, && \text{in } \Omega\times\mathbb{R}_+,\\
\mu &= \epsilon^{-1}f'(c) - \epsilon\Delta c, && \text{in } \Omega\times\mathbb{R}_+,
\end{aligned}
$$

equipped with boundary and initial conditions, cf. [**4**]. Here, c is the concentration of one of the fluids, and f is a suitable double well potential. The small order parameter $\epsilon > 0$ is related to the interfacial thickness, where a partial mixing of the fluids is allowed. We refer to Halperin [**31**], Gurtin, Polignone, Viñals [**29**], and Abels, Garcke, Grün [**2**] for further discussion.

For local existence of strong solutions and stability results in the case of closed interfaces we refer to [**4**]. In this thesis we consider the case where the free boundary meets the solid wall of the container at a constant contact angle. For a similar coupled problem between Navier-Stokes and Mean Curvature Flow we refer to Liu, Sato, Tonegawa [**40**].

1.4. Rayleigh-Taylor Instability

Let us consider a fixed container filled with two incompressible, immiscible, and viscous fluids. Again we assume that these two fluids are separated by a common and sharp interface. Let us additionally consider the case where the two fluids possess different densities. As an external force we consider gravity acting on the fluids.

When the heavier fluid is on top, one expects the upper fluid to sink down in the lower phase. This effect is well known as *Rayleigh-Taylor-Instability.* This appears when the difference of the densities is large enough compared to the surface tension between these fluids. If the lighter fluid is on top and gravity acts in the downwards direction, we expect a different behaviour, namely that the lighter fluid stays on top.

As before, we can model the flow of the two fluids by the two-phase Navier-Stokes equations with surface tension, now additionally with gravity acting as an external force. Rayleigh-Taylor-Instability for this model was first rigorously proven in an L_p-setting by Prüss and Simonett [**54**], in the case of a two-phase full space problem. In this case the problem reads as

$$\begin{aligned}
\rho^{\pm}(\partial_t v + (v\cdot\nabla)v) - \mu^{\pm}\Delta v + \nabla p &= -\rho^{\pm}\mathrm{g}e_n, && \text{in } \Omega\backslash\Gamma(t),\\
\operatorname{div} v &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
[\![v]\!] &= 0, && \text{on } \Gamma(t),\\
-\nu\cdot[\![T(v,p)]\!] &= \sigma H\nu, && \text{on } \Gamma(t),\\
V &= v\cdot\nu, && \text{on } \Gamma(t),
\end{aligned}$$

subject to initial conditions. Hereby $\rho^{\pm} > 0$ denote the two constant densities in the fluids and $\mathrm{g} > 0$ is the gravitational acceleration constant. In particular we point out that the gravitational force $-\rho^{\pm}\mathrm{g}e_n$ enters the momentum balance of the Navier-Stokes equation.

For a result on Rayleigh-Taylor-Instability for the two-phase Navier-Stokes equations in a cylindrical domain we refer to Wilke [**64**]. As a part of this thesis we will show linearized Rayleigh-Taylor-Instability for the two-phase Navier-Stokes/Mullins-Sekerka equations in cylindrical domains, where $\Gamma(t)$ forms a ninety degree angle at the boundary.

1.5. Preliminaries and Function Spaces

1.5.1. Notation. Let $\mathbb{N}$ denote the set of all natural numbers $1,2,3,\dots$ and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. For two Banach spaces X and Y, we denote by $\mathcal{B}(X,Y)$ the set of all bounded, linear operators from X to Y. Given some Hilbert space H, we denote the scalar product in H by $(.,.)_H$. For a given domain $\Omega\subset\mathbb{R}^n$, the set of continuous functions on Ω is denoted by $C^0(\Omega)$. For $k\in\mathbb{N}$, the set of k-times continuously differentiable functions is denoted by $C^k(\Omega)$ and for $\alpha>0, \alpha\notin\mathbb{N}$, we denote by $C^\alpha(\Omega)$ the Hölder space with exponent α. $BUC(\Omega)$ is the space of all functions which are bounded on Ω and uniformly continuous. The classical Lebesgue spaces are denoted by $L_p(\Omega)$, $1\le p\le\infty$, and the L_p-Sobolev spaces are denoted by $W^k_p(\Omega)$. Sometimes, for $k=1, p=2$, we also write $H^1(\Omega)$ only. By $C^\infty(\Omega)$ we mean the set of smooth functions and $C^\infty_0(\Omega)$ are smooth functions with compact support in Ω. In function spaces or inequalities with fractional exponents we simply write e.g. $1-1/2q$ instead of $1-1/(2q)$ from time to time. This should make complex expressions more readable while the meaning still should be very clear from the context. The constant $C>0$ usually is a generic constant and may change from line to line. We also employ classical Vinogradov notation, that is, $A\lesssim B$ means there is a constant $C>0$ independent of A and B, such that $A\le CB$.

1.5.2. Interpolation theory. Let X, Y be two Banach spaces. We say that (X,Y) is an interpolation couple, if both X and Y are continuously embedded into another topological Hausdorff vector space Z.

Let (X,Y) be an interpolation couple and denote by $(\cdot,\cdot)_{\theta,p}$ the real interpolation functor with respect to (θ,p) for $0<\theta<1, 1\le p\le\infty$, cf. Lunardi [**44**]. Then $(X,Y)_{\theta,p}$ is called the real interpolation space of X and Y with respect to (θ,p). It is well known that

$$(X,Y)_{\theta,p} = (Y,X)_{1-\theta,p}, \quad 0<\theta<1, 1\le p\le\infty.$$

Furthermore, for $0 < \theta < 1, 1 \leq p \leq q \leq \infty$,

$$X \cap Y \subset (X,Y)_{\theta,p} \subset (X,Y)_{\theta,q} \subset X + Y.$$

If now $Y \subset X$, for $0 < \theta_1 < \theta_2 < 1$ we have that

$$(X,Y)_{\theta_2,\infty} \subset (X,Y)_{\theta_1,1}.$$

Therefore $(X,Y)_{\theta_2,p} \subset (X,Y)_{\theta_1,q}$ for all $0 < \theta_1 < \theta_2 < 1, 1 \leq p,q \leq \infty$. An important estimate regarding interpolation theory is the following, cf. Corollary 1.7 in [**44**].

LEMMA 1.1. *Let (X,Y) be an interpolation couple. For $0 < \theta < 1$ and $1 \leq p \leq \infty$ there is a constant $C = C(\theta,p) > 0$, such that*

$$|z|_{(X,Y)_{\theta,p}} \leq C|z|_X^{1-\theta}|z|_Y^{\theta},$$

for all $z \in X \cap Y$. Hereby, $|\cdot|_{(X,Y)_{\theta,p}}$ denotes the norm of $(X,Y)_{\theta,p}$, cf. Definition 1.2 in [44].

Let us also note the following well known result, cf. Theorem 1.6 in [**44**].

LEMMA 1.2. *Let (X_1,Y_1) and (X_2,Y_2) be interpolation couples. If now $T \in \mathcal{B}(X_1,X_2) \cap \mathcal{B}(Y_1,Y_2)$, then $T \in \mathcal{B}((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})$, for any $\theta \in (0,1), p \in [1,\infty]$. In particular,*

$$|T|_{\mathcal{B}((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})} \leq |T|_{\mathcal{B}(X_1,X_2)}^{1-\theta}|T|_{\mathcal{B}(Y_1,Y_2)}^{\theta}.$$

For futher discussion we refer to Bergh, Löfström [**11**], and Lunardi [**44**].

1.5.3. Bessel-Potential and Besov spaces. As usual, we will denote the classical L_p-Sobolev spaces on $\mathbb{R}^n$ by $W_p^k(\mathbb{R}^n)$, where k is a natural number and $1 \leq p \leq \infty$. The well known Bessel-potential spaces will be denoted by $H_p^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$. We will now introduce the so-called (nonhomogeneous) Besov spaces, as is done in [**1**].

Let (φ_j), $j \in \mathbb{N}_0$, be a dyadic partition of unity on $\mathbb{R}^n$. This is a partition of unity (φ_j), $j \in \mathbb{N}_0$, on $\mathbb{R}^n$ with $\varphi_j \in C_0^\infty(\mathbb{R}^n)$ such that each support satisfies

$$\operatorname{supp}\varphi_j \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \quad j \geq 0,$$

and the support of φ_0 is contained in the closed ball $\overline{B_2(0)}$. This can be realized as follows. Choose a smooth bump function $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_0 = 1$ for $|\xi| \leq 1$ and $\varphi_0 = 0$ for $|\xi| \geq 2$. Then let $\varphi_j, j \in \mathbb{N}$, be defined by means of $\varphi_j(\xi) := \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)$.

For a smooth function $f \in C_0^\infty(\mathbb{R}^n)$ and $j \in \mathbb{N}_0$, define the dyadic block Δ_j of f by means of $\Delta_j f := \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$. It is then well known that the dyadic blocks can be extended to the space of tempered distributions $\mathcal{S}'$. We can then define the Besov space as follows.

DEFINITION 1.3. Let $s \in \mathbb{R}$ and $1 \leq p,q \leq \infty$. Then the Besov space $B_{pq}^s(\mathbb{R}^n)$ is defined as

$$B_{pq}^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : |f|_{B_{pq}^s(\mathbb{R}^n)} < \infty\},$$

where, if $q < \infty$,

$$|f|_{B_{pq}^s(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq}|\Delta_j f|_{L_p(\mathbb{R}^n)}^q\right)^{1/q},$$

and else

$$|f|_{B^s_{p\infty}(\mathbb{R}^n)} := \sup_{j\in\mathbb{N}_0} 2^{js}|\Delta_j f|_{L_p(\mathbb{R}^n)}.$$

It is well known that by Plancherel's theorem $B^s_{22}(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$, and the simple relations

$$B^s_{pq_1}(\mathbb{R}^n) \hookrightarrow B^s_{pq_2}(\mathbb{R}^n), \quad B^{s+\varepsilon}_{p\infty}(\mathbb{R}^n) \hookrightarrow B^s_{p1}(\mathbb{R}^n),$$

for all $s \in \mathbb{R}, \varepsilon > 0, 1 \le p \le \infty, 1 \le q_1 \le q_2 \le \infty$, hold true, cf. Lemma 6.5 in [**1**]. There is also a well known Sobolev-type embedding theorem for Besov spaces, cf. Theorem 6.15 in [**1**].

LEMMA 1.4. *Let $s, s_1 \in \mathbb{R}$ with $s \le s_1$, $1 \le q_1 \le q \le \infty$, and $1 \le p_1 \le p \le \infty$, such that $s - n/p \le s_1 - n/p_1$. Then $B^{s_1}_{p_1q_1}(\mathbb{R}^n) \hookrightarrow B^s_{pq}(\mathbb{R}^n)$.*

The following lemma is very well known and can easily be shown by using paraproduct estimates, see Corollary 2.86 in [**9**].

LEMMA 1.5. *For any $s > 0, 1 < p, q < \infty$,*

$$|vw|_{B^s_{pq}(\mathbb{R}^n)} \lesssim |v|_{B^s_{pq}(\mathbb{R}^n)}|w|_{L_\infty(\mathbb{R}^n)} + |v|_{L_\infty(\mathbb{R}^n)}|w|_{B^s_{pq}(\mathbb{R}^n)} \tag{1.1}$$

for all $v, w \in B^s_{pq}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$. In particular, the space $B^s_{pq}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ is a Banach algebra under pointwise multiplication.

The above lemma also holds true for smooth, bounded domains $\Omega \subset \mathbb{R}^n$. To see this consider the extension operator from $B^s_{pq}(\Omega)$ to $B^s_{pq}(\mathbb{R}^n)$. By a careful inspection, e.g. formula (7) in Section 2.9.2 in [**62**], we see that the extension operator is also bounded as a mapping $[L_\infty(\Omega) \to L_\infty(\mathbb{R}^n)]$. This way, we can first extend to $\mathbb{R}^n$ and then use Lemma 1.5 to obtain also the case of bounded, smooth domains $\Omega \subset \mathbb{R}^n$.

1.5.4. Triebel-Lizorkin spaces. Let $(\varphi_j)_j$ be the dyadic decomposition and Δ_j the dyadic blocks as before. We can then define the Triebel-Lizorkin spaces as follows.

DEFINITION 1.6. Let $s \in \mathbb{R}$ and $1 \le p \le \infty$, $1 \le q < \infty$. Then the Triebel-Lizorkin space $F^s_{pq}(\mathbb{R}^n)$ is defined as

$$F^s_{pq}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : |f|_{F^s_{pq}(\mathbb{R}^n)} < \infty\},$$

where

$$|f|_{F^s_{pq}(\mathbb{R}^n)} := \left| (\sum_{j=0}^{\infty} 2^{jsq}|\Delta_j f(.)|^q)^{1/q} \right|_{L_p(\mathbb{R}^n)}.$$

Note that with the usual modification one can also define F^s_{pq} for $q = \infty$, cf. [**62**]. For further discussion we refer to [**11**], [**1**], [**62**], [**57**].

1.5.5. Banach space valued function spaces. In this thesis we also need the above function spaces in a Banach-valued version. Let X be a Banach space. It is well known that one can define the so called vector-valued or Banach space-valued Lebesgue spaces $L_p(\Omega; X)$, cf. Appendix A.4 in [1]. Let $J = (0,T)$, $T > 0$, or $J = (0,\infty) = \mathbb{R}_+$. The Banach space-valued versions of the other function spaces for functions defined on J with values in X are denoted by $W_p^k(J;X)$, $H_p^s(J;X)$, $W_p^s(J;X)$, $B_{pr}^s(J;X)$, $F_{pr}^s(J;X)$, respectively. For precise definitions we refer to, e.g. [48].

1.5.6. Sectorial operators and maximal regularity. We first define the notion of sectorial operators as in Definition 3.1.1 in [57].

Definition 1.7. Let X be a complex Banach space and A be a closed linear operator on X. Then A is called *sectorial*, if both domain and range of A are dense in X, the resolvent set of A contains $(-\infty, 0)$, and there is some $C > 0$ such that $|t(t+A)^{-1}|_{\mathcal{B}(X)} \leq C$ for all $t > 0$.

The concept of $\mathcal{R}$-bounded families of operators is an important tool in modern analysis. We refer to Definition 4.1.1 in [57].

Definition 1.8. Let X and Y be Banach spaces and $\mathcal{T} \subset \mathcal{B}(X,Y)$. We say that $\mathcal{T}$ is $\mathcal{R}$-bounded, if there is some $C > 0$ and $p \in [1,\infty)$, such that for each $N \in \mathbb{N}, \{T_j : j = 1, ..., N\} \subset \mathcal{T}$, $\{x_j : j = 1, ..., N\} \subset X$, and for all independent, symmetric, ±1-valued random variables ε_j on a probability space $(\Omega, \mathcal{A}, \mu)$ the inequality

$$\left| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right|_{L^p(\Omega;Y)} \leq C \left| \sum_{j=1}^{N} \varepsilon_j x_j \right|_{L^p(\Omega;X)} \tag{1.2}$$

is valid. The smallest $C > 0$ such that (1.2) holds is called $\mathcal{R}$-bound of $\mathcal{T}$. We denote it by $\mathcal{R}(\mathcal{T})$.

We can now define $\mathcal{R}$-sectoriality of an operator as is done in Definition 4.4.1 in [57].

Definition 1.9. Let X be a Banach space and A a sectorial operator on X. Then A is $\mathcal{R}$-sectorial, if $\mathcal{R}_A(0) := \mathcal{R}\{t(t+A)^{-1} : t > 0\}$ is finite. We can then define the $\mathcal{R}$-angle of A by means of $\varphi_A^R := \inf\{\theta \in (0,\pi) : \mathcal{R}_A(\pi - \theta) < \infty\}$. Here, $\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda + A)^{-1} : |\arg\lambda| \leq \theta\}$.

We now define the important class of operators which admit a bounded $\mathcal{H}^\infty$-calculus as in Definition 3.3.12 in [57]. For the well known Dunford functional calculus and an extension of which we refer to Sections 3.1.4 and 3.3.2 in [57]. Let $0 < \varphi \leq \pi$ and $\Sigma_\varphi := \{z \in \mathbb{C} : |\arg z| < \varphi\}$. Let $H(\Sigma_\varphi)$ be the set of all holomorphic functions $f : \Sigma_\varphi \to \mathbb{C}$ and $H^\infty(\Sigma_\varphi)$ the subset of all bounded functions of $H(\Sigma_\varphi)$. The norm in $H^\infty(\Sigma_\varphi)$ is given by

$$|f|_{H^\infty(\Sigma_\varphi)} := \sup\{|f(z)| : z \in \Sigma_\varphi\}.$$

Furthermore let

$$H_0(\Sigma_\varphi) := \bigcup_{\alpha,\beta<0} H_{\alpha,\beta}(\Sigma_\varphi),$$

where $H_{\alpha,\beta}(\Sigma_\varphi) := \{f \in H(\Sigma_\varphi) : |f|_{\alpha,\beta}^{\varphi} < \infty\}$, and $|f|_{\alpha,\beta}^{\varphi} := \sup\{|z^\alpha f(z)| : |z| \leq 1\} + \sup\{|z^{-\beta} f(z)| : |z| \geq 1\}$.

Definition 1.10. Let X be a Banach space and A a sectorial operator on X. Then A admits a bounded $\mathcal{H}^\infty$-calculus, if there is some $\varphi > \varphi_A$ and a constant $K_\varphi < \infty$, such that

$$|f(A)|_{\mathcal{B}(X)} \leq K_\varphi |f|_{H^\infty(\Sigma_\varphi)} \tag{1.3}$$

for all $f \in H_0(\Sigma_\varphi)$. The class of operators admitting a bounded $\mathcal{H}^\infty$-calculus on X will be denoted by $\mathcal{H}^\infty(X)$. The $\mathcal{H}^\infty$-angle of A is defined by the infimum of all $\varphi > \varphi_A$, such that (1.3) is valid, that is, $\varphi_A^\infty := \inf\{\varphi > \varphi_A : (1.3) \text{ holds}\}$.

Let us recall the property of maximal L_p-regularity, cf. Definition 3.5.1 in [**57**].

Definition 1.11. Let X be a Banach space, $J = (0,T), 0 < T < \infty$ or $J = \mathbb{R}_+$ and A a closed, densely defined operator on X with domain $D(A) \subset X$. Then the operator A has maximal L_p-regularity on J, if and only if for every $f \in L_p(J;X)$ there is a unique $u \in W_p^1(J;X) \cap L_p(J;D(A))$ solving

$$\frac{d}{dt}u(t) + Au(t) = f(t), \quad t \in J, \qquad u|_{t=0} = 0,$$

in an almost-everywhere sense in $L_p(J;X)$.

There is a wide class of results on operators having maximal regularity, we refer to Sections 3.5 and 4 in [**57**] for further discussion. For results on $\mathcal{R}$-boundedness and interpolation we refer to [**36**].

There is a strong connection between maximal L_p-regularity and the so-called semigroup theory. For an introduction to semigroups we refer to Engel, Nagel [**19**], [**20**], Pazy [**51**], and Prüss, Simonett [**57**].

1.6. Overview over this thesis

This thesis deals with the analysis of free boundary problems for two-phase flows with boundary contact, in particular the Mullins-Sekerka problem and the two-phase Navier-Stokes/Mullins-Sekerka equations with surface tension. In these problems we impose the condition of a constant ninety degree angle contact between the free interface and the boundary of the domain. This condition is justified as an idealization. One major tool of tackling these problems is the theory of maximal L_p-regularity. Besides well-posedness and maximal regularity of these systems we are also interested in long-time behaviour, that is, stability and instability results. In particular we will present a first result on the so called Rayleigh-Taylor instability for the coupled system of two-phase Navier-Stokes equations and Mullins-Sekerka flow.

1.6.1. Mullins-Sekerka with ninety degree angle contact. The first part of this thesis deals with the analysis of the Mullins-Sekerka equations in a bounded, smooth domain in space dimension $n \in \{2,3\}$ with a ninety degree contact angle condition for the free interface. Since there were no results available involving a contact line between interface and boundary of the domain we will start at the very beginning and give a careful analysis of the underlying linear model problems.

Several technical difficulties appear, such as the curved boundary of the domain, as well as the necessity to treat the linear problem in an $L_p - L_q$-theory with $p \neq q$. Hereby, $q < 2$ is needed for reflection techniques across the boundary and $p > 6$ for the Hanzawa transform to be a C^1-diffeomorphism. First showing maximal $L_p - L_q$-regularity for the model problems we are able to show well-posedness for the nonlinear problem for cylindrical and general, curved domains by a localization procedure and a fixed point argument.

We then consider nonlinear stability of solutions and global-in-time existence in cylindrical domains. In this simpler geometry the ninety degree angle boundary condition, which is in general highly nonlinear, reduces to a linear condition involving only the gradient of the height function. Hence it allows for an application of the generalized principle of linearized stability of Prüss, Simonett, and Zacher, cf. [**58**].

In the spirit of [**25**] we give a linearized stability analysis of stationary solutions of the Mullins-Sekerka flow with ninety degree angle contact in two dimensions. Here the domain and the stationary solution may both be curved. The relevant quantities deciding on linear stability or instability are the length of the stationary solution, its curvature, as well as the curvature of the boundary of the domain. We then obtain different results on linear stability for the trivial solution depending on the values of these quantities.

These results are also published in the preprints [**3**], [**27**].

1.6.2. Two-phase Navier-Stokes/Mullins-Sekerka with ninety degree angle contact. We can also couple the Mullins-Sekerka problem with the ninety degree angle contact condition to the two-phase Navier-Stokes equations with surface tension. This model describes the flow of two immiscible, incompressible fluids inside a bounded domain. The model without boundary contact has already been studied by Abels and Wilke [**4**].

We first show local existence and uniqueness of strong solutions. Then we investigate the long-time behaviour of solutions starting close to certain equilibria. We show that for two constant but maybe different densities these solutions exist globally in time, are stable, and converge to an equilibrium solution at an exponential rate. Since the evolution equation for the height function is now however non-local in time, the generalized principle of [**58**] cannot be applied directly.

1.6.3. Rayleigh-Taylor instability for Navier-Stokes/Mullins-Sekerka with ninety degree contact and gravity. In this part we extend the Navier-Stokes/Mullins-Sekerka system with ninety degree contact in the sense that we allow for gravity to act on the fluids. This is interesting since the fluids may have different densities. We then formulate a model which is thermodynamically consistent and well-posed. The main result then is to show the presence of linearized Rayleigh-Taylor instability whenever the heavier fluid lies on top. Rayleigh-Taylor instability for two-phase Navier-Stokes equations with surface tension was already adressed by Prüss, Simonett [**55**], and Wilke [**64**].

CHAPTER 2

The Mullins-Sekerka equations with ninety degree angle boundary contact: well-posedness

2.1. Introduction

In this chapter we study the Mullins-Sekerka problem inside a bounded domain $\Omega \subset \mathbb{R}^n, n = 2, 3$, where the interface separating the two materials meets the boundary of Ω at a constant ninety degree angle. This leads to a free boundary problem involving a contact angle problem as well. The domain Ω can here be either smooth and bounded, or of cylindrical type. The latter means that there is some smooth, bounded domain $G \subset \mathbb{R}^{n-1}$, such that $\Omega = G \times (a, b)$, for some $a < b$. Precise assumptions will be made later.

Let us introduce the model. We assume that the domain Ω can be decomposed as $\Omega = \Omega^+(t) \dot\cup \mathring{\Gamma}(t) \dot\cup \Omega^-(t)$, where $\mathring{\Gamma}(t)$ denotes the interior of $\Gamma(t)$, an $(n-1)$-dimensional submanifold with boundary. We interpret $\Gamma(t)$ to be the interface separating the two phases, $\Omega^\pm(t)$, which will be assumed to be connected. The boundary of $\Gamma(t)$ will be denoted by $\partial\Gamma(t)$. Furthermore we assume $\Gamma(t)$ to be orientable, the unit normal vector field on $\Gamma(t)$ pointing from $\Omega^-(t)$ to $\Omega^+(t)$ will be denoted by $n_{\Gamma(t)}$.

The precise model we study reads as

$$
\begin{aligned}
V_{\Gamma(t)} &= -[\![n_{\Gamma(t)} \cdot \nabla\mu]\!], && \text{on } \Gamma(t),\\
\mu|_{\Gamma(t)} &= H_{\Gamma(t)}, && \text{on } \Gamma(t),\\
\Delta\mu &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
n_{\partial\Omega} \cdot \nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega,\\
\mathring{\Gamma}(t) &\subset \Omega, &&\\
\partial\Gamma(t) &\subset \partial\Omega, &&\\
\angle(\Gamma(t), \partial\Omega) &= \pi/2, && \text{on } \partial\Gamma(t),
\end{aligned}
\tag{2.1}
$$

subject to the initial condition

$$\Gamma|_{t=0} = \Gamma_0.$$

Here, $V_{\Gamma(t)}$ denotes the normal velocity and $H_{\Gamma(t)}$ the mean curvature of the free interface $\Gamma(t)$, which is given by the sum of the principal curvatures. By $[\![\cdot]\!]$ we denote the jump of a quantity across $\Gamma(t)$ in direction of $n_{\Gamma(t)}$, that is,

$$[\![f]\!](x) := \lim_{\varepsilon\to 0+} [f(x + \varepsilon n_{\Gamma(t)}) - f(x - \varepsilon n_{\Gamma(t)})], \quad x \in \Gamma(t).$$

Equation $(2.1)_7$ prescribes the angle at which the interface $\Gamma(t)$ has contact with the fixed boundary $\partial\Omega$, which will be a constant ninety degree angle during the evolution. We can alternatively write $(2.1)_7$ as the condition that the normals are perpendicular on the boundary of the interface,

$$n_{\Gamma(t)} \cdot n_{\partial\Omega} = 0, \quad \text{on } \partial\Gamma(t). \tag{2.2}$$

We also pose, motivated by (2.2), a compatibility condition at time $t = 0$,

$$\angle(\Gamma_0, \partial\Omega) = \pi/2 \quad \text{on } \partial\Gamma_0.$$

Let us first state some simple properties of this evolution. The volume of each of the two phases is conserved in time,

$$\frac{d}{dt}|\Omega^{\pm}(t)| = 0, \quad t \in \mathbb{R}_+. \tag{2.3}$$

Here, $\Omega^{\pm}(t)$ denote the two different phases separated by the sharp interface $\Gamma(t)$, $\Omega = \Omega^+(t) \cup \mathring{\Gamma}(t) \cup \Omega^-(t)$. Then (2.3) stems from

$$\begin{aligned}\frac{d}{dt}|\Omega^+(t)| = \int_{\Gamma(t)} V_{\Gamma(t)} d\mathcal{H}^{n-1} &= -\int_{\Gamma(t)} [\![n_{\Gamma(t)} \cdot \nabla\mu]\!] d\mathcal{H}^{n-1} \\ &= \int_{\Omega^+(t)} \Delta\mu dx = 0,\end{aligned}$$

cf. Theorem 5.4 in [**17**]. However, the energy given by the surface area of the free interface $\Gamma(t)$ satisfies

$$\frac{d}{dt}|\Gamma(t)| \le 0, \quad t \in \mathbb{R}_+.$$

Indeed, an integration by parts and the ninety degree contact angle condition readily give

$$\begin{aligned}\frac{d}{dt}|\Gamma(t)| = -\int_{\Gamma(t)} H_{\Gamma(t)} V_{\Gamma(t)} d\mathcal{H}^{n-1} &= \int_{\Gamma(t)} \mu|_{\Gamma(t)} [\![n_{\Gamma(t)} \cdot \nabla\mu]\!] d\mathcal{H}^{n-1} \\ &= -\int_{\Omega} |\nabla\mu|^2 dx \le 0,\end{aligned}$$

cf. Theorem 2.32 in [**10**]. We are concerned with existence of strong solutions to the Mullins-Sekerka problem (2.1). To this end we will later pick some reference surface Σ inside the domain Ω, also intersecting the boundary at a constant ninety degree angle, and write the moving interface as a graph over Σ by a height function h, depending on $x \in \Sigma$ and time $t \ge 0$. Pulling back the equations then to the time-independent domain $\Omega\backslash\Sigma$ we reduce the problem to a nonlinear evolution equation for h. The corresponding linearization for the spatial differential operator for h then turns out to be a nonlocal pseudo-differential operator of order three, cf. [**22**]. We also refer to the introduction of Escher, Simonett [**22**] for further properties of the Mullins-Sekerka problem.

In the following, we will be interested in height functions h with regularity

$$h \in W^1_p(0, T; W^{1-1/q}_q(\Sigma)) \cap L_p(0, T; W^{4-1/q}_q(\Sigma)),$$

where p and q are different in general. We will choose $q < 2$ and p finite but large, to ensure that the real interpolation space

$$X_\gamma := (W_q^{1-1/q}(\Sigma), W_q^{4-1/q}(\Sigma))_{1-1/p,p} = B_{qp}^{4-1/q-3/p}(\Sigma) \tag{2.4}$$

continuously embeds into $C^2(\Sigma)$, cf. Amann [7]. This is needed to ensure that the transformation mapping the problem to a fixed reference configuration is a C^1-diffeomorphism. By an ansatz where $p = q < 2$, this is not achievable, since $h(t), t \in [0, T]$, will not be in $C^2(\Sigma)$ in general. We need however the restriction $q < 2$ to avoid additional compatibility conditions for the elliptic problem and certain reflection techniques, cf. also Section 2.5.1 and Section 2.3. However, the fact that $p \neq q$ in the solution space requires a maximal regularity result of type $L_p - L_q$ of the underlying linearized problem.

Let us give an overview of this chapter. We will first show local well-posedness for the Mullins-Sekerka problem with ninety degree angle boundary contact. We will hereby describe the motion of the moving interface by a height function over a fixed reference surface and thus rewrite the free boundary problem of the moving interface as a nonlinear problem for the height function parametrizing the interface. Using the theory of maximal regularity together with a linearization of the equations and a localization argument we will prove well-posedness of the full nonlinear problem via the contraction mapping principle. Here one difficulty will lie in choosing the right space for the Neumann trace of the height function and showing maximal $L_p - L_q$-regularity for the linear problem. Section 2.5 is devoted to the analysis of the underlying linear problem, where an extensive analysis is made on the half-space model problems. This is needed since these model problems at the contact line are not well-understood until now. The main result is then maximal $L_p - L_q$ regularity for the linear problem. We then use a fixed point argument to show that the nonlinear problem is also well-posed.

2.2. The Neumann trace of the height function

In this section we characterize the optimal trace space for the Neumann trace of the height function h and show that it is a Banach algebra with respect to pointwise multiplication.

THEOREM 2.1. *Let $n = 2, 3$, $0 < T \leq \infty, 6 < p < \infty$ and $q \in (5/3, 2) \cap (2p/(p+1), 2p)$ and let Σ be the flat interface $\mathbb{R}^n_+ \cap \{x_1 = 0\}$. Let again $X_0 := W_q^{1-1/q}(\Sigma)$ and $X_1 := W_q^{4-1/q}(\Sigma)$. Then*

$$\begin{aligned} &{}_0W^1_p(0,T; X_0) \cap L_p(0,T; X_1) \ni h \mapsto \\ &\qquad \mapsto \nabla h|_{\partial\Sigma} \in {}_0F_{pq}^{1-2/(3q)}(0,T; L_q(\partial\Sigma)) \cap L_p(0,T; B_{qq}^{3-2/q}(\partial\Sigma)) \end{aligned} \tag{2.5}$$

is bounded, linear, and has a continuous right inverse E, such that $\partial_n Eg|_{\partial\Sigma} = g$ for all $g \in {}_0F_{pq}^{1-2/(3q)}(0,T; L_q(\partial\Sigma)) \cap L_p(0,T; B_{qq}^{3-2/q}(\partial\Sigma))$. Here, the subscript denotes vanishing traces at $t = 0$, e.g. ${}_0W^1_p(0,T; X_0) := \{f \in W^1_p(0,T; X_0) : f(0) = 0\}$.

Furthermore, there exists some constant $C > 0$ independent of the length of the time interval T, such that

$$|\nabla h|_{\partial\Sigma}|_{F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))} \leq C|h|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)},$$

for all $h \in {}_0W_p^1(0,T;X_0) \cap L_p(0,T;X_1)$ *and*

$$|Eg|_{W_p^1(0,T;X_0)\cap L_p(0,T;X_1)} \leq C|g|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-\frac{2}{q}}(\partial\Sigma))},$$

for all $g \in {}_0F_{pq}^{1-2/(3q)}(0,T;L^q(\partial\Sigma)) \cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))$.

REMARK 2.2. The time trace at $t=0$ in ${}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))$ is well defined since $1-2/3q > 1/p$ is ensured, cf. [**48**].

PROOF. We may use Propositions 5.37 and 5.39 in [**35**] to get an embedding

$${}_0W_p^1(0,T;X_0) \cap L_p(0,T;X_1) \hookrightarrow {}_0F_{pq}^{1-1/(3q)}(0,T;W_q^1(\Sigma)), \tag{2.6}$$

where the embedding constant is independent of T. This can be seen as follows. Since we restrict ourselves to functions with vanishing trace at $t=0$, we may extend the function to the half line $\mathbb{R}_+$ by reflection. We then apply the result of [**35**] and restrict the extensions back to the finite time interval $(0,T)$. This way, we first obtain by Proposition 5.37 in [**35**] that

$${}_0W_p^1(0,T;X_0) \cap L_p(0,T;X_1) \hookrightarrow {}_0H_p^1(0,T;B_{qq}^{1-1/q}(\Sigma)) \cap H_p^{1/2}(0,T;B_{qq}^{5/2-1/q}(\Sigma)),$$

hence interpolating according to Proposition 5.39 in [**35**] gives

$${}_0H_p^1(0,T;X_0) \cap H_p^{1/2}(0,T;B_{qq}^{5/2-1/q}(\Sigma)) \hookrightarrow F_{pq}^{1-1/(3q)}(0,T;H_q^1(\Sigma)).$$

Hence (2.6) yields that for any $h \in {}_0W_p^1(0,T;X_0) \cap L_p(0,T;X_1)$,

$$\nabla h \in {}_0F_{pq}^{1-1/(3q)}(0,T;L_q(\Sigma)) \cap L_p(0,T;B_{qq}^{3-1/q}(\Sigma)).$$

Concerning the traces of ∇h on the boundary $\partial\Sigma$, we use Proposition 5.23 in [**35**] to write this intersection space on the right hand side as an anisotropic Triebel-Lizorkin space $F_{\vec{p},q}^{s,\vec{a}}$ and use the trace theory developed in [**35**] for these particular spaces. For a definition of $F_{\vec{p},q}^{s,\vec{a}}$ we refer to Definition 5.15 in [**35**]. By Proposition 5.23 in [**35**],

$$F_{pq}^{1-1/(3q)}(0,T;L_q(\Sigma)) \cap L_p(0,T;B_{qq}^{3-1/q}(\Sigma)) \equiv F_{\vec{p},q}^{s,\vec{a}}((0,T)\times\Sigma),$$

where

$$s=1, \quad \vec{a} = \left(\frac{1}{l},...,\frac{1}{l},\frac{1}{t}\right), \quad \vec{p} = (q,...,q,p), \quad t = 1-\frac{1}{3q}, \quad l = 3-\frac{1}{q},$$

and we take $n-1$ copies of $1/l$ and q, respectively. Now, for taking traces in these anisotropic Triebel-Lizorkin spaces we refer to [**34**]. With the notation used there in equations (2.1) and (2.11) we use Corollary 2.7 in [**34**] to get that the trace operator onto the boundary $\partial\Sigma$,

$$\mathrm{tr}_{\partial\Sigma}: F_{\vec{p},q}^{s,\vec{a}}((0,T)\times\Sigma) \to F_{\vec{p''},q}^{s-\frac{1}{ql},\vec{a''}}((0,T)\times\partial\Sigma),$$

is bounded. Here $\vec{a''}$ and $\vec{p''}$ are used as introduced in the beginning of Section 2.1 in [**34**]. In our particular case,

$$\vec{a''} = \left(\frac{1}{l},...,\frac{1}{l},\frac{1}{t}\right), \quad \vec{p''} = (q,...,q,p),$$

taking now $n-2$ copies of $1/l$ and q, respectively. We note at this point that by the order of integration with respect to the different exponents in $\vec{p}$ as explained in equation (3.1) in [**34**], we have to take traces in "x_1-direction" in the notation of [**34**] and not in "x_n-direction" and therefore have to use Corollary 2.7 instead of Corollary 2.8 in [**34**].

Again using Proposition 5.23 in [**35**],

$$F^{s-\frac{1}{ql},\vec{a''}}_{\vec{p''},q}((0,T)\times\partial\Sigma) = F^{(s-\frac{1}{ql})t}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{(s-\frac{1}{ql})l}_{qq}(\partial\Sigma)).$$

Clearly,

$$\left(s-\frac{1}{ql}\right)t = \left(1-\frac{1}{3q-1}\right)\left(1-\frac{1}{3q}\right) = 1-\frac{2}{3q},$$

as well as

$$\left(s-\frac{1}{ql}\right)l = 3\left(s-\frac{1}{ql}\right)t = 3-\frac{2}{q}.$$

Hence

$$F^{s-\frac{1}{ql},\vec{a''}}_{\vec{p''},q}((0,T)\times\partial\Sigma) = F^{1-2/(3q)}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma)).$$

Concludingly we have shown so far that the mapping $[h\mapsto \mathrm{tr}_{\partial\Sigma}\nabla h]$ between the spaces in (2.5) is bounded.

It remains to construct a continuous right inverse. We use again Corollary 2.7 in [**34**] for $j=1$ to get a bounded right inverse E of

$$F^{s,\vec{a}}_{\vec{p},q}((0,T)\times\Sigma)\to F^{s-a_1-\frac{a_1}{p_1},\vec{a''}}_{\vec{p''},p_1}((0,T)\times\partial\Sigma),\quad h\mapsto \mathrm{tr}_{\partial\Sigma}\nabla h. \tag{2.7}$$

Note that by restriction we then get an inverse also on the spaces with vanishing time trace. Now, let $\tilde{s} = s-a_1-a_1/p_1$. If we choose $\tilde{s}, t$ and l to satisfy

$$\tilde{s}t = 1-\frac{2}{3q},\quad \tilde{s}l = 3-\frac{2}{q},$$

for instance by choice of

$$\tilde{s}=1,\quad t = 1-\frac{2}{3q},\quad l = 3-\frac{2}{q},$$

we obtain

$$s = 1+\frac{1}{l}+\frac{1}{lq} = \frac{4q-1}{3q-2},$$

whence, by characterization of anisotropic Triebel-Lizorkin spaces, (2.7) reads as

$$\begin{aligned}{}_0F^{\frac{4}{3}-\frac{1}{3q}}_{pq}(0,T;L^q(\Sigma))\cap L^p(0,T;B^{4-\frac{1}{q}}_{qq}(\Sigma))\ni h\mapsto\\ \mathrm{tr}\,|_{\partial\Sigma}\nabla h\in {}_0F^{1-\frac{2}{3q}}_{pq}(0,T;L^q(\partial\Sigma))\cap L^p(0,T;B^{3-\frac{2}{q}}_{qq}(\partial\Sigma)).\end{aligned}$$

So in other words for given Neumann data

$$b\in {}_0F^{1-\frac{2}{3q}}_{pq}(0,T;L^q(\partial\Sigma))\cap L^p(0,T;B^{3-\frac{2}{q}}_{qq}(\partial\Sigma)),$$

there exists some $h = E(b) \in {}_0F_{pq}^{\frac{4}{3}-\frac{1}{3q}}(0,T;L^q(\Sigma)) \cap L^p(0,T;B_{qq}^{4-\frac{1}{q}}(\Sigma))$, such that $\nabla h|_{\partial\Sigma} = b$ and

$$\|h\|_{F_{pq}^{\frac{4}{3}-\frac{1}{3q}}(0,T;L^q(\Sigma))\cap L^p(0,T;B_{qq}^{4-\frac{1}{q}}(\Sigma))} \lesssim \|b\|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))\cap L^p(0,T;B_{qq}^{3-\frac{2}{q}}(\partial\Sigma))}.$$

An application of Proposition 5.38 in [35] with

$$\sigma = \frac{1}{\frac{4}{3}-\frac{1}{3q}} = \frac{3q}{4q-1} \in (0,1)$$

gives ${}_0F_{pq}^{\frac{4}{3}-\frac{1}{3q}}(0,T;L^q(\Sigma)) \cap L^p(0,T;B_{qq}^{4-\frac{1}{q}}(\Sigma)) \hookrightarrow {}_0W_p^1(0,T;B_{qq}^{1-\frac{1}{q}}(\Sigma))$. Concludingly,

$$\|E(b)\|_{{}_0W_p^1(0,T;X_0)\cap L^p(0,T;X_1)} \lesssim \|b\|_{{}_0F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))\cap L^p(0,T;B_{qq}^{3-\frac{2}{q}}(\partial\Sigma))},$$

and the theorem is shown. We again point out that the constant is only independent of T since we restrict ourselves to functions having vanishing trace at $t = 0$. □

The boundedness of the Neumann trace operator can easily be generalized to the case of a curved interface by a standard argument involving a partition of unity and a localization argument.

THEOREM 2.3. *Let $\Omega \subset \mathbb{R}^n, n = 2,3$, bounded and smooth and Σ a smooth interface of dimension $n-1$ in the sense that Σ is a submanifold with interior inside Ω meeting the boundary at a ninety degree angle. Then the Neumann trace $\mathrm{tr}_{\partial\Sigma} \nabla_\Sigma$ is bounded as a mapping from ${}_0W_p^1(0,T;X_0)\cap L_p(0,T;X_1)$ to ${}_0F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))$.*

The next result states that the Neumann trace space is a Banach algebra under pointwise multiplication. We need this for contraction estimates in the nonlinear problem later.

THEOREM 2.4. *Let $n = 2,3$, $0 < T \le \infty, 6 < p < \infty$ and $q \in (3/2,2) \cap (2p/(p+1),2p)$. Then the Neumann trace space with vanishing time trace at $t = 0$ above is a Banach algebra, that is, the product estimate*

$$\begin{aligned} &\|fg\|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))\cap L^p(0,T;B_{qq}^{3-\frac{2}{q}}(\partial\Sigma))} \lesssim \\ &\quad \lesssim \|f\|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))\cap L^p(0,T;B_{qq}^{3-\frac{2}{q}}(\partial\Sigma))} \|g\|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))\cap L^p(0,T;B_{qq}^{3-\frac{2}{q}}(\partial\Sigma))} \end{aligned} \tag{2.8}$$

holds for all $f,g \in {}_0F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma)) \cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))$. In particular, the constant in (2.8) is independent of the length of the time interval.

PROOF. We first show that

$${}_0F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma)) \cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma)) \hookrightarrow L_\infty(0,T;L_\infty(\partial\Sigma)). \tag{2.9}$$

Using Proposition 5.38 in [35] and a standard reflection argument in time and localization argument in space, we obtain a continuous embedding

$$\begin{aligned} &{}_0F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma)) \cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma)) \\ &\qquad \hookrightarrow {}_0H_p^{(1-\frac{2}{3q})\theta}(0,T;B_{qq}^{(3-2/q)(1-\theta)}(\partial\Sigma)) \end{aligned}$$

for any $\theta \in (0,1)$, where the embedding constant is independent of T since the functions have vanishing trace at $t = 0$. Note that if θ is so small such that the space on the right hand side does not have a well defined time trace at $t = 0$, we simply replace it with

$$H_p^{(1-\frac{2}{3q})\theta}(0,T; B_{qq}^{(3-2/q)(1-\theta)}(\partial\Sigma)).$$

Now, since $n = 2$ or 3, the boundary $\partial\Sigma$ has at most Hausdorff dimension 1, whence the latter space on the right hand side surely embeds into $L_\infty(0,T; L_\infty(\partial\Sigma))$, if

$$(1 - 2/(3q))\,\theta - 1/p > 0, \quad (3 - 2/q)\,(1-\theta) - 1/q > 0.$$

These both equations are equivalent to finding some $\theta \in (0,1)$ such that

$$\frac{1}{p}\frac{3q}{3q-2} < \theta < 1 - \frac{1}{3q-2}.$$

Simple calculations show that for any $q \in (3/2, 2)$,

$$1 - \frac{1}{3q-2} > \frac{3}{5}, \quad \frac{3q}{3q-2} < \frac{9}{5},$$

whence $p \geq 3$ ensures $\theta = 3/5$ is a solid choice. Therefore we know for sure that the Neumann trace space embeds continuously into $L_\infty(0,T; L_\infty(\partial\Sigma))$. Using paraproduct estimates, cf. Lemma 1.5,

$$\begin{aligned}
&|fg|_{L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))} \leq \\
&\leq \Big||f(t)|_{L_\infty}|g(t)|_{B_{qq}^{3-2/q}}\Big|_{L_p(0,T)} + \Big||f(t)|_{B_{qq}^{3-2/q}}|g(t)|_{L_\infty}\Big|_{L_p(0,T)} \\
&\leq |f|_{L_\infty(L_\infty)}|g|_{L_p(B_{qq}^{3-2/q})} + |f|_{L_p(B_{qq}^{3-2/q})}|g|_{L_\infty(L_\infty)}.
\end{aligned}$$

From Proposition 5.7 in [**48**] we get

$$\begin{aligned}
|fg|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L_q(\partial\Sigma))} &\lesssim |f|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L_q(\partial\Sigma))}|g|_{L_\infty(0,T;L_\infty(\partial\Sigma))} \\
&\quad + |f|_{L_\infty(0,T;L_\infty(\partial\Sigma))}|g|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L_q(\partial\Sigma))}.
\end{aligned}$$

These two estimates and (2.9) finish the proof. □

Remark 2.5. Note that the above proof even shows that the Neumann trace space embeds into $C([0,T] \times \partial\Sigma)$.

We now give a product estimate in the Triebel-Lizorkin part of the norm if one of the functions is time-independent.

Lemma 2.6. *Let $1 < p, q < \infty$, $f \in {}_0F_{pq}^{1-\frac{2}{3q}}(0,T; L^q(\partial\Sigma))$ and $g \in L^\infty(\partial\Sigma)$ independent of the time variable. Then $gf \in {}_0F_{pq}^{1-\frac{2}{3q}}(0,T; L^q(\partial\Sigma))$ and*

$$\|gf\|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))} \leq C\|g\|_{L^\infty(\partial\Sigma)}\|f\|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))}$$

for some constant $C > 0$ independent of f, g, and T.

Proof. This estimate is a consequence of Proposition 5.4 in [**48**]. Indeed, we choose $\sigma = 1, s = 1 - 2/(3q)$ and $X = Y = L_q(\partial\Sigma)$ and directly get the claimed estimate. It also easily follows from Theorem 2.7 below. □

The following theorem gives a characterization of the Triebel-Lizorkin norm on the half line $\mathbb{R}_+$ via differences. It will be used to gain estimates with regards to the trace space norm of h.

THEOREM 2.7. *Let $s \in (0,1)$. Define $\Delta_h f(t) := f(t+h) - f(t)$ for all $t, h > 0$. Furthermore, let*

$$[f]_{F^s_{pq}(0,\infty;L^q(\partial\Sigma))} := \left\| \left(\int_0^\infty t^{-(s+1)q} \left(\int_{|h|\le z} \|\Delta_h f(.)\|_{L^q(\partial\Sigma)} dh \right)^q \frac{dz}{z} \right)^{1/q} \right\|_{L^p(0,\infty)}. \tag{2.10}$$

Then

$$\|.\|_{F^s_{pq}(\mathbb{R}_+;L^q(\partial\Sigma))} \simeq \|.\|_{L^p(\mathbb{R}_+;L^q(\partial\Sigma))} + [.]_{F^s_{pq}(\mathbb{R}_+;L^q(\partial\Sigma))}$$

are equivalent up to a constant, where the expression on the left hand side is the $F^s_{pq}(\mathbb{R}_+; L^q(\partial\Sigma))$ norm.

PROOF. Since $0 < s < 1$, we may choose $m = 1$ in Proposition 2.3 in [**47**] and get the result. $\square$

REMARK 2.8. For every $q \in (5/3, 2)$, we obtain that $1 - 2/(3q) \in (0,1)$. Hence the result above is applicable to the trace space of the Neumann trace of h in our setting.

LEMMA 2.9. *Let $G : \mathbb{R} \to \mathbb{R}$ be smooth, $0 < T' < \infty$, $0 < T < T'$, and $R_0 > r_0 > 0$. Then there is a constant $C = C(R_0) > 0$ independent of $T > 0$ and $r_0 > 0$, such that*

$$\begin{aligned} &|G(f) - G(g)|_{F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} \\ &\qquad \le C(R_0)|f-g|_{F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))}, \end{aligned}$$

for all functions

$$f, g \in {}_0W^1_p(0,T;X_0) \cap L_p(0,T;X_1) \cap \{u : |u|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)} \le r_0\}. \tag{2.11}$$

Here, the subscript zero denotes vanishing traces at $t = 0$.

PROOF. First note that $[G(f) - G(g)]|_{t=0} = 0$ for all functions f, g in (2.11). Also, $f, g \in C^0([0,T] \times \partial\Sigma)$. Hence we can deduce the following pointwise estimate. Pick some $(x,t) \in \partial\Sigma \times [0,T]$. Since G is smooth,

$$G(f(x,t)) - G(g(x,t)) = \int_0^1 G'(g(x,t) + \tau(f(x,t) - g(x,t)))d\tau(g(x,t) - f(x,t)).$$

Since $G'(0)$ is not necessarily zero, we rewrite this equality as

$$\begin{aligned} &G(f(x,t)) - G(g(x,t)) \\ &\qquad = \int_0^1 G'(g(x,t) + \tau(f(x,t) - g(x,t))) - G'(0)d\tau(g(x,t) - f(x,t)) \\ &\qquad + G'(0)(g(x,t) - f(x,t)). \end{aligned}$$

Note that this holds for any $(x,t) \in \partial\Sigma \times [0,T]$. We can now apply triangle inequality and the product estimate of Theorem 2.4, since $\int_0^1 G'(g+\tau(f-g)) - G'(0) d\tau$ has vanishing trace at $t=0$. This entails

$$\begin{aligned}
&|G(f)-G(g)|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} \\
&\leq \int_0^1 |G'(g+\tau(f-g))-G'(0)|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} d\tau \times \\
&\qquad\qquad \times |f-g|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} \\
&\quad + |G'(0)||f-g|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))}.
\end{aligned}$$

Note that for any $\tau \in [0,1]$, $g+\tau(f-g)$ belongs to (2.11). We now show that the map $[u \mapsto \tilde{G}(u) - \tilde{G}(0)]$ satisfies the following: for any given smooth $\tilde{G} : \mathbb{R} \to \mathbb{R}$, we show that for any $R_0 > 0$, there is a constant $C(R_0) > 0$ independent of u, such that

$$|\tilde{G}(u)-\tilde{G}(0)|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} \leq C(R_0), \tag{2.12}$$

provided only that u belongs to (2.11), in particular, $|u|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)} \leq R_0$.

Let us start with the following result on the trace operator. The trace onto the boundary as a mapping

$$\begin{aligned}
\mathrm{tr}_{\partial\Sigma} : {}_0W^1_p(0,T;X_0) &\cap C^0([0,T];C^2(\Sigma)) \\
&\to {}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma)) \cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma)),
\end{aligned} \tag{2.13}$$

is bounded with constant independent of T. Hence

$$\begin{aligned}
&|\tilde{G}(u)-\tilde{G}(0)|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} \\
&\quad \leq C|\tilde{G}(u)-\tilde{G}(0)|_{{}_0W^1_p(0,T;X_0)\cap C^0([0,T];C^2(\Sigma))}.
\end{aligned}$$

Let us show (2.13) later.

We now take a function u in (2.11) with $|u|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)} \leq R_0$. By the embedding ${}_0W^1_p(0,T;X_0) \cap L_p(0,T;X_1) \hookrightarrow C^0([0,T];X_\gamma) \hookrightarrow C^0([0,T];C^2(\Sigma))$, we see that

$$|u|_{C^0([0,T];C^2(\Sigma))} \leq C_1 R_0,$$

where C_1 is independent of R_0 and T. Now,

$$\begin{aligned}
&|\tilde{G}(u)-\tilde{G}(0)|_{C^0([0,T];C^2(\Sigma))} \\
&\quad \leq |\tilde{G}(u)-\tilde{G}(0)|_{C^0([0,T];C^0(\Sigma))} + |\tilde{G}'(u)\nabla u|_{C^0([0,T];C^0(\Sigma))} \\
&\quad + |\tilde{G}''(u)\nabla u \nabla u|_{C^0([0,T];C^0(\Sigma))} + |\tilde{G}'(u)\nabla^2 u|_{C^0([0,T];C^0(\Sigma))} \\
&\quad \leq C(R_0),
\end{aligned}$$

since $\tilde{G}$ is smooth and $|u|_{C^0([0,T]\times\Sigma)} \leq C_1 R_0$. Moreover,

$$\begin{aligned}
&|\tilde{G}(u)-\tilde{G}(0)|_{W^1_p(0,T;X_0)} \\
&\quad \leq |\tilde{G}(u)-\tilde{G}(0)|_{L_p(0,T;X_0)} + |\tilde{G}'(u)\partial_t u|_{L_p(0,T;X_0)} \\
&\quad \leq |\tilde{G}(u)-\tilde{G}(0)|_{C^0([0,T];C^1(\Sigma))} + |\tilde{G}'(u)|_{C^0([0,T];C^1(\Sigma))}|\partial_t u|_{L_p(0,T;X_0)} \\
&\quad \leq C(R_0).
\end{aligned}$$

Altogether, this shows (2.12). It remains to prove the embedding (2.13). Let us first show that the trace is bounded as a mapping

$$[u \mapsto u|_{\partial\Sigma}], \quad {}_0W^1_p(0,T;X_0) \cap C^0([0,T];C^2(\Sigma)) \to L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma)). \tag{2.14}$$

Clearly, ${}_0W^1_p(0,T;X_0) \cap C^0([0,T];C^2(\Sigma)) \hookrightarrow L^p(0,T;H^2_{\tilde q}(\Sigma))$, for any $1 < \tilde q < \infty$. The trace operator $[u \mapsto u|_{\partial\Sigma}]$ is now bounded as a mapping $[H^2_{\tilde q}(\Sigma) \to B^{2-1/\tilde q}_{\tilde q\tilde q}(\partial\Sigma)]$, by classical results. Given $q < 2$, we may choose $\tilde q = \tilde q(q) < \infty$ large enough to satsify $2 - 1/\tilde q > 3 - 2/q$. Then, since $\tilde q \geq q$, we have $B^{2-1/\tilde q}_{\tilde q\tilde q}(\partial\Sigma) \hookrightarrow B^{3-2/q}_{qq}(\partial\Sigma)$. This proves (2.14).

It remains to show boundedness of

$$[u \mapsto u|_{\partial\Sigma}], \quad {}_0W^1_p(0,T;X_0) \cap C^0([0,T];C^2(\Sigma)) \to {}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma)).$$

Clearly, ${}_0W^1_p(0,T;X_0) \cap C^0([0,T];C^2(\Sigma)) \hookrightarrow {}_0W^1_p(0,T;X_0) \cap L_p(0,T;W^{2-1/q}_q(\Sigma))$. Interpolating according to Proposition 5.39 in [**35**],

$${}_0W^1_p(0,T;X_0) \cap L_p(0,T;W^{2-1/q}_q(\Sigma)) \hookrightarrow {}_0F^{1-2/3q}_{pq}(0,T;H^{1-1/3q}_q(\Sigma)).$$

Since $1 - 1/3q - 1/q > 0$, $[u \mapsto u|_{\partial\Sigma}]$ is bounded from $H^{1-1/3q}_q(\Sigma)$ to $L_q(\partial\Sigma)$. The proof of (2.13) is complete. ☐

Based on this proof we need to show a more involved estimate for functions with time trace different from zero.

LEMMA 2.10. *Let $G : \mathbb{R} \to \mathbb{R}$ be smooth, $0 < T' < \infty$, $0 < T < T'$, and $R_0 > R > 0$. Assume*

$$f, g \in {}_0W^1_p(0,T;X_0) \cap L_p(0,T;X_1) \cap \{u : |u|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)} \leq R\},$$

and $\tilde f := f + h_*$, $\tilde g := g + h_*$, where $h_* \in \mathbb{E}(T)$ is a given function satisfying $|h_*|_{\mathbb{E}(T)} + |h_*|_{C^0([0,T];C^2(\Sigma))} \leq 1$. Then

$$\begin{aligned} &|G(\tilde f) - G(\tilde g)|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} \\ &\qquad \leq C(R_0)|\tilde f - \tilde g|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))}, \end{aligned}$$

for a constant $C = C(R_0) > 0$ independent of f, g, h_*, R, and $T > 0$.

PROOF. Note that $\tilde f(t=0) = \tilde g(t=0) = h_*(t=0)$. Let $h_0 := h_*(t=0)$. We can show that pointwise

$$\begin{aligned} G(\tilde f(x,t)) - G(\tilde g(x,t)) &= G'(h_0(x))(\tilde g(x,t) - \tilde f(x,t)) \\ &\quad + \int_0^1 G'(\tilde g(x,t) + \tau(\tilde f(x,t) - \tilde g(x,t))) - G'(h_0(x)) d\tau (\tilde g(x,t) - \tilde f(x,t)). \end{aligned}$$

Now, $\int_0^1 G'(\tilde g + \tau(\tilde f - \tilde g)) - G'(h_0) d\tau$ has vanishing trace at $t = 0$, whence we may apply the same product estimate argument as before. Note that we can also use the boundedness of the embedding (2.13) if we apply it to $G'(\tilde g + \tau(\tilde f - \tilde g)) - G'(h_0)$. Since $\tilde f = f + h_*$ and f itself has vanishing trace,

$$|\tilde f|_{\mathbb{E}(T)} \leq |f|_{\mathbb{E}(T)} + |h_*|_{\mathbb{E}(T)} \leq R_0 + 1,$$

$$|\tilde f|_{C^0([0,T];C^2(\Sigma))} \leq |f|_{C^0([0,T];C^2(\Sigma))} + |h_*|_{C^0([0,T];C^2(\Sigma))} \leq C_1 R_0 + 1.$$

Here, C_1 is as before independent of T since the embedding constant in

$$_0W_p^1(0,T;X_0)\cap L_p(0,T;X_1)\hookrightarrow C^0([0,T];X_\gamma)$$

is independent of $T\in(0,T')$. Therefore mimicking the proof of Lemma 2.9 gives the desired estimate. $\square$

2.3. Reflection operators

We denote the upper half space of $\mathbb{R}^n$ by $\mathbb{R}^n_+ := \{x\in\mathbb{R}^n : x_n>0\}$. We will denote by R the even reflection of a function defined on $\mathbb{R}^n_+$ across the boundary $\partial\mathbb{R}^n_+$ in x_n-direction, that is, we define R as an extension operator via $Ru(x_1,...,x_n) := u(x_1,...,-x_n)$ for all $x_n<0$. Note that R admits a bounded operator $R: L_q(\mathbb{R}^n_+)\to L_q(\mathbb{R}^n)$. The following theorems state that even more is true.

Theorem 2.11. *Let $1<q<\infty$. The even reflection in x_n-direction R induces a bounded linear operator from $W_q^{1+\alpha}(\mathbb{R}^n_+)$ to $W_q^{1+\alpha}(\mathbb{R}^n)$, whenever $0\le\alpha<1/q$.*

Proof. It is straightforward to verify that for a given $u\in W_q^{1+\alpha}(\mathbb{R}^n_+)$,

$$\partial_j Ru(x_1,...,x_n)=\partial_j u(x_1,...,-x_n),\quad j=1,...,n-1,\ x_n<0,$$

and $\partial_n Ru(x_1,...,x_n)=-\partial_n u(x_1,...,-x_n)$. Hence $R: W_q^1(\mathbb{R}^n_+)\to W_q^1(\mathbb{R}^n)$ is a bounded operator. To show the claim for the mapping $[W_q^{1+\alpha}(\mathbb{R}^n_+)\to W_q^{1+\alpha}(\mathbb{R}^n)]$, it remains to show that the odd reflection of $Du\in W_q^\alpha(\mathbb{R}^n_+)$, that is, say TDu, is again $W_q^\alpha(\mathbb{R}^n)$ and that the corresponding bounds hold true.

We first note that $TDu(x_1,...,x_n)=e_0Du(x_1,...,x_n)-e_0Du(x_1,...,-x_n)$, where e_0 denotes the extension by zero to the lower half plane. Let $W^1_{q,0}(\mathbb{R}^n_+) := \{u\in W_q^1(\mathbb{R}^n_+) : u=0 \text{ on } \partial\mathbb{R}^n_+\}$. Note that by real interpolation method,

$$W_q^\alpha(\mathbb{R}^n_+)=\left(L_q(\mathbb{R}^n_+),W^1_{q,0}(\mathbb{R}^n_+)\right)_{\alpha,q},\quad W_q^\alpha(\mathbb{R}^n)=\left(L_q(\mathbb{R}^n),W^1_q(\mathbb{R}^n)\right)_{\alpha,q},$$

since $0<\alpha<1/q$, cf. [**62**]. Now, both zero extension operators

$$e_0: L_q(\mathbb{R}^n_+)\to L_q(\mathbb{R}^n),\quad e_0: W^1_{q,0}(\mathbb{R}^n_+)\to W^1_q(\mathbb{R}^n),$$

are bounded and linear. From Theorem 1.1.6 in [**44**] we obtain that e_0 is therefore also a bounded, linear operator between the corresponding interpolation spaces, hence the theorem is proven. $\square$

Note that the above proof makes essential use of the fact that the derivative of $u\in W_q^{1+\alpha}(\mathbb{R}^n_+)$ has no trace on $\partial\mathbb{R}^n_+$ for $\alpha<1/q$. If one has a trace it needs to be zero to reflect appropriately, which is the statement of the next theorem.

Theorem 2.12. *Let q and R be as above. Then R induces a bounded linear operator*

$$W_q^{1+\beta}(\mathbb{R}^n_+)\cap\{u:\partial_{x_n}u|_{x_n=0}=0\}\to W_q^{1+\beta}(\mathbb{R}^n)$$

for all $\beta\in(1/q,1)$.

Proof. We modify the proof of Theorem 2.11. We already have that R induces bounded linear operators

$$L^q(\mathbb{R}^n_+)\to L^q(\mathbb{R}^n),\quad W_q^1(\mathbb{R}^n_+)\to W_q^1(\mathbb{R}^n).$$

Given a function $u \in W_q^{1+\beta}(\mathbb{R}_+^n) \cap \{u : \partial_{x_n} u|_{x_n=0} = 0\}$, it now remains to show that the odd reflection, again say T, of the derivative $Du \in W_q^\beta(\mathbb{R}_+^n)$ is now $W_q^\beta(\mathbb{R}^n)$. This follows now again as in the proof of Theorem 2.11, since

$$T\partial_{x_n} u(x_1, ..., x_n) = e_0 \partial_{x_n} u(x_1, ..., x_n) - e_0 \partial_{x_n} u(x_1, ..., -x_n),$$

and the fact that the zero extension operator

$$e_0 : W_q^1(\mathbb{R}_+^n) \cap \{f : f|_{x_n=0} = 0\} \to W_q^1(\mathbb{R}^n)$$

is bounded. Using $W_q^1(\mathbb{R}_+^n) \cap \{f : f|_{x_n=0} = 0\} = W_{q,0}^1(\mathbb{R}_+^n)$ and the interpolation argument in Theorem 2.11 completes the proof. □

We can also reflect the initial data. The result reads as follows.

THEOREM 2.13. *The even reflection R induces a bounded linear operator*

$$W_q^{3+\alpha}(\mathbb{R}_+^n) \cap \{u : \partial_{x_n} u|_{x_n=0} = 0\} \to W_q^{3+\alpha}(\mathbb{R}^n)$$

for all $\alpha \in (0, 1/q)$, $q \in (3/2, 2)$. In particular, R also induces a bounded linear operator

$$B_{qp}^{4-1/q-3/p}(\mathbb{R}_+^n) \cap \{u : \partial_{x_n} u|_{x_n=0} = 0\} \to B_{qp}^{4-1/q-3/p}(\mathbb{R}^n)$$

for all $q \in (3/2, 2)$ and $p > 3/(2 - 3/q)$.

PROOF. The second statement follows from the first one for $\alpha = 1 - 1/q - 3/p < 1/q$ since $q < 2$. The first claim is shown as in the proof of Theorem 2.11, using additionally that $\partial_{x_n} \partial_{x_n} Ru = R\partial_{x_n} \partial_{x_n} u$. □

2.4. Transformation to a fixed Reference Surface

In this section we transform the problem (2.1) to a fixed reference configuration. To this end we construct a suitable transformation, taking into account the possibly curved boundary of $\partial\Omega$, by locally introducing curvilinear coordinates. The idea of these transforms goes back to Hanzawa [**32**], curvilinear coordinates go back to the work of Vogel [**63**].

Let $\Sigma \subset \Omega$ be a smooth reference surface and $\partial\Omega$ smooth at least in a neighbourhood of $\partial\Sigma$. Furthermore, let $\angle(\Sigma, \partial\Omega) = \pi/2$ on $\partial\Sigma$. From Proposition 3.1 in [**63**] we get the existence of so called curvilinear coordinates at least in a small neighbourhood of Σ, that is, there is some possibly small $a > 0$ depending on the curvature of Σ and $\partial\Omega$, such that

$$X : \Sigma \times (-a, a) \to \mathbb{R}^n, \quad (p, w) \mapsto X(p, w),$$

is a smooth diffeomorphism onto its image and $X(.,.)$ is a curvilinear coordinate system. One feature of these coordinates is that points on the boundary $\partial\Sigma$ only get transported along the boundary of the domain, $X(p, w) \in \partial\Omega$ for all $p \in \partial\Sigma, w \in (-a, a)$. We need to make use of these coordinates since the boundary $\partial\Omega$ may be curved. Therefore a transport only in normal direction of n_Σ is not sufficient here.

The curvilinear coordinates X in [**63**] are of form

$$X(p, w) = p + wn_\Sigma(p) + \tau(p, w)\vec{T}(p), \quad p \in \Sigma, r \in (-a, a), \tag{2.15}$$

where $\tau\vec{T}$ is responsible for the tangential correction. More precisely, n_Σ denotes the unit normal vector field on Σ with fixed orientation and $\vec{T}$ is a smooth vector field defined on the closure of Σ with the following properties: it is tangent to Σ, normal to $\partial\Sigma$, of unit length on $\partial\Sigma$, and vanishing outside a neighbourhood of $\partial\Sigma$. In particular, $\vec{T}$ is bounded. Furthermore, $\tau = \tau(p, w)$ is a smooth scalar function such that $X(p, w)$ lies on $\partial\Omega$ whenever $p \in \partial\Sigma$. It satisfies $\tau(p, 0) = 0$ for all $p \in \Sigma$. In particular then, $X(p, 0) = p$ for all $p \in \Sigma$. For more details we refer to [**63**].

2.4.1. Hanzawa transform. With the help of these coordinates we may parametrize the free interface as follows. We assume that at time $t \geq 0$, the free interface is given as a graph over the reference surface Σ, that is, there is some $h : \Sigma \times [0, T] \to (-a, a)$, such that

$$\Gamma(t) = \Gamma_h(t) := \{X(p, h(p, t)) : p \in \Sigma\}, \quad t \in [0, T], \tag{2.16}$$

for small $T > 0$, at least. With the help of this coordinate system we may construct a Hanzawa-type transform as follows.

Let $\chi \in C_0^\infty(\mathbb{R})$ be a fixed bump function satisfying $\chi(s) = 1$ for $|s| \leq 1/3$, $\chi(s) = 0$ for $|s| \geq 2/3$, and $|\chi'(s)| \leq 4$ for all $s \in \mathbb{R}$. Let $\Sigma_a := X(\Sigma \times (-a, a))$. Then for a given height function $h : \Sigma \to (-a, a)$ describing an interface Γ_h we define

$$\Theta_h(x) := \begin{cases} x, & x \notin \Sigma_a, \\ (X \circ F_h \circ X^{-1})(x), & x \in \Sigma_a, \end{cases} \tag{2.17}$$

where

$$F_h(p, w) := (p, w - \chi((w - h(p))/a)h(p)), \quad p \in \Sigma, w \in (-a, a).$$

Recall that by properties of the curvilinear coordinate system we have that $\Sigma = \{x \in \mathbb{R}^n : x = X(p, 0), p \in \Sigma\}$. Let X_γ be as in (2.4) and define

$$\mathcal{U} := \{h \in X_\gamma : |h|_{L_\infty(\Sigma)} < a/5\}.$$

Then we have the following result.

Theorem 2.14. *For fixed $h \in \mathcal{U}$, the transformation $\Theta_h : \Omega \to \Omega$ is a C^1-diffeomorphism satisfying $\Theta_h(\Gamma_h) = \Sigma$.*

Proof. Let $h \in \mathcal{U}$. We begin with the second identity. Pick $x \in \Gamma_h$. Then, in curved coordinates, $x = X(p, h(p))$ for some uniquely determined $p \in \Sigma$. Therefore,

$$\begin{aligned} \Theta_h(x) &= X \circ F_h \circ X^{-1}(X(p, h(p))) \\ &= X \circ F_h(p, h(p)) \\ &= X(p, h(p) - \chi(0)h(p)) \\ &= X(p, 0), \end{aligned}$$

since the coordinates X are a diffeomorphism on Σ_a and $\chi(0) = 1$. Therefore $\Theta_h(x) \in \Sigma$, since $X(p, 0) \in \Sigma$. To show surjectivity, let $x \in \Sigma$. Then $x = X(x, 0)$ and $y := X(x, h(x))$ satisfies $\Theta_h(y) = x$.

Now we show next that $\Theta_h : \Omega \to \Omega$ is bijective. Clearly, $\Theta_h : \Omega\backslash\Sigma_a \to \Omega\backslash\Sigma_a$ is bijective since it is simply the identity map there. Now, because $X : \Sigma \times [-a, a] \to \Sigma_a$ is bijective, we only have to show that $F_h : \Sigma \times [-a, a] \to \Sigma \times [-a, a]$ is bijective.

We first show injectivity of F_h. Let $F_h(p_1,w_1) = F_h(p_2,w_2)$ for some $p_1,p_2 \in \Sigma, w_1,w_2 \in [-a,a]$. From the definition of F_h we directly get $p_1 = p_2$ and therefore

$$w_1 - \chi\left(\frac{w_1 - h(p_1)}{a}\right) h(p_1) = w_2 - \chi\left(\frac{w_2 - h(p_1)}{a}\right) h(p_1).$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$r \mapsto f(r) := r - \chi\left(\frac{r - h(p_1)}{a}\right) h(p_1).$$

By showing $\frac{d}{dr} f(r) > 0$ for all $r \in [-a,a]$, we readily get $w_1 = w_2$. This however easily follows from

$$\frac{d}{dr} f(r) = 1 - \chi'\left(\frac{r - h(p_1)}{a}\right) \frac{1}{a} h(p_1),$$

combined with the estimate

$$\left|\chi'\left(\frac{r - h(p_1)}{a}\right) \frac{1}{a} h(p_1)\right| \leq \|\chi'\|_\infty \frac{1}{a} \|h\|_\infty \leq \frac{4}{5}, \tag{2.18}$$

since $|\chi'| \leq 4$ and $h \in \mathcal{U}$. Surjectivity of $F_h : \Sigma \times [-a,a] \to \Sigma \times [-a,a]$ easily follows since F_h is continuous and

$$F_h(p,a) = (p,a), \quad F_h(p,-a) = (p,-a), \tag{2.19}$$

for every $p \in \Sigma$. Hereby (2.19) follows from

$$(F_h(p,a))_2 = a - \chi\left(\frac{a - h(p)}{a}\right) h(p) = a,$$

since $|a - h(p)|/a = |1 - h(p)/a| \geq 2/3$. Note that we used $|h(p)/a| \leq 1/5$, since $h \in \mathcal{U}$. Concludingly, we have shown that $\Theta_h : \Omega \to \Omega$ is bijective.

To show that Θ_h and its inverse are in fact C^1-mappings, we need to know that Θ_h is C^1 and $D\Theta_h$ is regular in every point. Then the claim follows by the inverse function theorem. Clearly, by chain rule,

$$D\Theta_h|_x = DX|_{F_h \circ X^{-1}(x)} DF_h|_{X^{-1}(x)} DX^{-1}|_x.$$

We can easily compute

$$DF_h(p,w) = \begin{pmatrix} \mathrm{id}_{T_p\Sigma} & 0 \\ \chi'\left(\frac{w-h(p)}{a}\right) \frac{1}{a} h(p) \partial_p h(p) - \chi\left(\frac{w-h(p)}{a}\right) \partial_p h(p) & 1 - \chi'\left(\frac{w-h(p)}{a}\right) \frac{1}{a} h(p) \end{pmatrix},$$

hence by (2.18) we readily see that DF_h is invertible in every point (p,w). The proof is complete. □

2.4.2. Transformed mean curvature operator. The following lemma gives a decomposition of the transformed curvature operator $K(h) := H_{\Gamma_h} \circ \Theta_h$ for $h \in \mathcal{U}$. The result and proof are an adpation of the work in Lemma 2.1 in [4] and Lemma 3.1 in [22]. However, since we have boundary contact we need a new proof which takes into account the curved boundary of the domain.

LEMMA 2.15. *Let $n = 2, 3$, $q \in (3/2, 2), 3/(2-3/q) < p < \infty$, and $\mathcal{U} \subset X_\gamma$ as before. Then there are functions*

$$P \in C^1(\mathcal{U}, \mathcal{B}(W_q^{4-1/q}(\Sigma), W_q^{2-1/q}(\Sigma))), \quad Q \in C^1(\mathcal{U}, W_q^{2-1/q}(\Sigma)),$$

such that

$$K(h) = P(h)h + Q(h), \quad \text{for all } h \in \mathcal{U} \cap W_q^{4-1/q}(\Sigma).$$

Moreover,

$$P(0) = -\Delta_\Sigma,$$

where Δ_Σ denotes the Laplace-Beltrami operator with respect to the surface Σ.

REMARK 2.16. Note that the orthogonality relations (3.2) in [**22**] do not hold if we take X to be curvilinear coordinates, since in X we not only have a variation in normal direction but also in tangential directions. Therefore we have to modify the proofs in [**4**], [**22**].

PROOF. We recall, cf. (2.15), that the curvilinear coordinates X are of form

$$X(s, r) = s + rn_\Sigma(s) + \tau(s, r)\vec{T}(s), \quad s \in \Sigma, r \in (-a, a), \tag{2.20}$$

where n_Σ, τ, and $\vec{T}$ are as before. In particular we recall that $\tau(s, 0) = 0$ for all $s \in \Sigma$. Moreover, since Σ and $\partial\Omega$ form a ninety degree contact angle, we have that

$$\partial_r \tau(s, 0) = 0, \quad s \in \partial\Sigma. \tag{2.21}$$

Hence we may choose τ in [**63**] to satisfy (2.21) for all $s \in \Sigma$. Let us give a proof of (2.21).

The ninety degree contact angle condition means that for every $s \in \partial\Sigma$, the tangent vector which is given by

$$n_\Sigma(s) + \partial_r\tau(s, 0)\vec{T}(s)$$

is a multiple of $n_\Sigma(s)$. Since $n_\Sigma(s)$ and $\vec{T}(s)$ are orthogonal and $\vec{T}$ is of unit length on $\partial\Sigma$, in particular $\vec{T}(s)$ is not zero, (2.21) follows. Then, in the proof of [**63**], we simply choose the extension of τ to satisfy (2.21) for all $s \in \Sigma$.

We will now derive a formula for the transformed mean curvature operator $K(h)$ in local coordinates. We follow the arguments of [**22**].

The surface $\Gamma_h(t)$ is the zero level set of the function

$$\varphi_h(x, t) := (X^{-1})_2(x) - h((X^{-1})_1(x), t), \quad x \in \Sigma_a, t \in \mathbb{R}_+,$$

whence we define

$$\Phi_h(s, r) := \varphi_h(X(s, r), t) = r - h(s, t), \quad s \in \Sigma, r \in (-a, a).$$

We obtain that since $X : \Sigma \times (-a, a) \to \mathbb{R}^n$ is a smooth diffeomorphism onto its image, it induces a Riemannian metric g_X on $\Sigma \times (-a, a)$. We denote the induced differential operators gradient, Laplace-Beltrami and the Hessian with respect to $(\Sigma \times (-a, a), g_X)$ by ∇_X, Δ_X and hess_X. As in equation (3.1) in [**22**] we find that

$$K(h)|_s = \frac{1}{\|\nabla_X \Phi_h\|_X} \left(\Delta_X \Phi_h - \frac{[\text{hess}_X \Phi_h](\nabla_X \Phi_h, \nabla_X \Phi_h)}{\|\nabla_X \Phi_h\|_X^2} \right) |_{(s,h(s))},$$

for all $s \in \Sigma$, where $\|\nabla_X \Phi_h\|_X := (g_X(\nabla_X \Phi_h, \nabla_X \Phi_h))^{1/2}$. Note at this point that since X induces also a variation in tangential directions, the orthogonality relations (3.2) in [22] do not hold in general. However,

$$\partial_n X = n_\Sigma + \partial_r \tau \vec{T},$$

hence we get in local coordinates that

$$(\partial_j X|\partial_n X) = (\partial_j X|n_\Sigma) + \partial_r \tau (\partial_j X|\vec{T}), \quad j \in \{1, ..., n-1\},$$

and

$$\begin{aligned}(\partial_n X|\partial_n X) &= (n_\Sigma|n_\Sigma) + 2\partial_r \tau (n_\Sigma|\vec{T}) + (\partial_r \tau)^2 (\vec{T}|\vec{T}) \\ &= 1 + (\partial_r \tau)^2 (\vec{T}|\vec{T}).\end{aligned}$$

In particular we see that on the surface Σ the relations (3.2) in [22] still hold, but not away from Σ in general.

Let us now make the following observation. Denote the first fundamental form with respect to X by $(w_{ij})_{i,j=1,\dots,n}$, that is,

$$w_{ij} := (\partial_i X|\partial_j X), \quad 1 \le i, j \le n, \tag{2.22}$$

where we surpress the dependence of (s, r). Its inverse is given by $(w^{ij})_{i,j=1,\dots,n}$ as usual. Recall that (w_{ij}) and therefore its inverse are symmetric. Note at this point that in contrary to [22], $w_{jn} = \delta_{jn}$ does not hold in general. In particular, also $w^{nn} \neq 1$. However, we are able to show that w^{nn} is not "too far away" from 1. Indeed, note that the matrix $(w_{ij})_{1\le i,j\le n}$ at a point $X(s, r)$ as is given by (2.22) converges to

$$\begin{pmatrix} \left(w_{ij}^\Sigma|_s\right)_{1\le i,j\le n-1} & 0 \\ 0 & 1 \end{pmatrix}$$

in any matrix norm as $(r, |\partial_r \tau|_{L_\infty}) \to 0$. Hereby (w_{ij}^Σ) denotes the first fundamental form of Σ. This is an easy consequence, since $(\partial_j X|\partial_n X) \to 0$ and $(\partial_n X|\partial_n X) \to 1$ as $(r, |\partial_r \tau|_{L_\infty}) \to 0$. Note that all coefficients are smooth and depend smoothly on their parameters. Hence, since then inversion is smooth by the implicit function theorem, we have convergence

$$(w^{ij})_{1\le i,j\le n} \to \begin{pmatrix} \left(w_\Sigma^{ij}\right)^{-1}_{1\le i,j\le n-1} & 0 \\ 0 & 1 \end{pmatrix},$$

as $(r, |\partial_r \tau|_{L_\infty}) \to 0$. Therefore there exists some possibly small $\delta > 0$, such that if $|r| + |\partial_r \tau|_\infty \le \delta$,

$$w^{nn}|_{(s,r')} \ge \frac{1}{2} > 0 \tag{2.23}$$

for all $|r'| \le |r|, s \in \Sigma$. Note that since $\partial\Omega$ is smooth and Σ intersects $\partial\Omega$ at a ninety degree angle,

$$\tau(s, 0) = 0, \quad \partial_r \tau(s, 0) = 0,$$

for all $s \in \Sigma$, see the considerations for (2.21). Hence we can choose, say, $|r| + \|\tau\|_{L^\infty(\Sigma, C^1((-a,a)))} \le a + \|\tau\|_{L^\infty(\Sigma, C^1((-a,a)))}$, to be arbitrarily small, simply by choosing $a > 0$ small enough. Therefore we can assume without loss of generality that estimate (2.23) holds true.

We now derive the formula for the mean curvature operator K in local coordinates. We will use well-known representation formulas for ∇_X, Δ_X, and hess_X in local coordinates, cf. [22]. By definition,

$$\nabla_X \Phi_h = \sum_{i,j=1,\dots,n} w^{ij} \partial_i \Phi_h \partial_j X.$$

Recall that since g_X is a Riemannian metric,

$$g_X(\partial_i X, \partial_j X) = w_{ij}, \quad i,j = 1,\dots,n.$$

Therefore,

$$\begin{aligned} g_X(\nabla_X \Phi_h, \nabla_X \Phi_h) &= g_X\left(\sum_{i,j} w_X^{ij} \partial_i \Phi_h \partial_j X, \sum_{l,m} w_X^{lm} \partial_l \Phi_h \partial_m X\right) \\ &= \sum_{i,j,l,m} w_X^{ij} w_X^{lm} \partial_i \Phi_h \partial_l \Phi_h g_X(\partial_j X, \partial_m X) \\ &= \sum_{i,j,l,m} w_X^{ij} w_{jm}^X w_X^{lm} \partial_i \Phi_h \partial_l \Phi_h \\ &= \sum_{i,l} w_X^{il} \partial_i \Phi_h \partial_l \Phi_h, \end{aligned}$$

where we used that $\sum_j w_X^{ij} w_{jm}^X = \delta_{im}$. Clearly,

$$\partial_j \Phi_h = -\partial_j h, \quad j = 1,\dots,n-1, \qquad \partial_n \Phi_h = 1.$$

Hence,

$$\partial_j \partial_k \Phi_h = -\partial_j \partial_k h, \quad j,k = 1,\dots,n-1, \qquad \partial_j \partial_n \Phi_h = 0, \quad j = 1,\dots,n. \tag{2.24}$$

Therefore,

$$\ell_X(h) := \|\nabla_X \Phi_h\|_X = \sqrt{\sum_{i,l=1}^{n-1} w_X^{li} \partial_i h \partial_l h - 2\sum_{l=1}^{n-1} w^{ln} \partial_l h + w_X^{nn}},$$

where we remind ourselves of (2.23). Recall the well known formulas

$$\Delta_X \Phi_h = \sum_{j,k=1}^{n} w_X^{jk} \partial_i \partial_k \Phi_h - \sum_{j,k,l=1}^{n} w_X^{jk} \Gamma_{jk,X}^l \partial_l \Phi_h,$$

and

$$\begin{aligned} &\mathrm{hess}_X \Phi_h(\nabla_X \Phi_h, \nabla_X \Phi_h) = \\ &\qquad = \sum_{i,j=1}^{n} \left(\partial_i \partial_j \Phi_h - \sum_{k=1}^{n} \Gamma_{ij,X}^k \partial_k \Phi_h\right) [\mathrm{dx}^i \otimes \mathrm{dx}^j] \left(\nabla_X \Phi_h, \nabla_X \Phi_h\right), \end{aligned}$$

where $\Gamma_{jk,X}^i$ as usually denote the Christoffel symbols with respect to g_X. We also have that

$$\Gamma_{jk,X}^i = \sum_{m=1}^{n} w_X^{mi} (\partial_j \partial_k X | \partial_m X).$$

Again writing the expressions in local coordinates,

$$
\begin{aligned}
&[\mathrm{dx}^i \otimes \mathrm{dx}^j]\,(\nabla_X \Phi_h, \nabla_X \Phi_h) = \\
&\qquad = [\mathrm{dx}^i \otimes \mathrm{dx}^j] \left(\sum_{p,q=1}^{n} w_X^{pq} \partial_q \Phi_h \partial_p X, \sum_{r,s=1}^{n} w_X^{rs} \partial_s \Phi_h \partial_r X \right) \\
&\qquad = \sum_{p,q,r,s=1}^{n} w_X^{pq} w_X^{rs} \partial_q \Phi_h \partial_s \Phi_h [\mathrm{dx}^i \otimes \mathrm{dx}^j]\,(\partial_p X, \partial_r X) \\
&\qquad = \sum_{p,q,r,s=1}^{n} w_X^{pq} w_X^{rs} \partial_q \Phi_h \partial_s \Phi_h \delta_{ip} \delta_{jr} \\
&\qquad = \sum_{q,s=1}^{n} w_X^{iq} w_X^{js} \partial_q \Phi_h \partial_s \Phi_h,
\end{aligned}
$$

where we used that $[\mathrm{dx}^i \otimes \mathrm{dx}^j](\partial_p X, \partial_r X) = \delta_{ip}\delta_{jr}$. Hence,

$$
\begin{aligned}
&\mathrm{hess}_X \Phi_h(\nabla_X \Phi_h, \nabla_X \Phi_h) = \\
&\quad = \sum_{i,j=1}^{n} (\partial_i \partial_j \Phi_h - \sum_{k=1}^{n} \Gamma_{ij,X}^k \partial_k \Phi_h)[\mathrm{dx}^i \otimes \mathrm{dx}^j](\nabla_X \Phi_h, \nabla_X \Phi_h) \\
&\quad = \sum_{i,j,q,s=1}^{n} w_X^{iq} w_X^{js} \partial_q \Phi_h \partial_s \Phi_h \partial_i \partial_j \Phi_h \\
&\quad - \sum_{i,j,k,q,s=1}^{n} \Gamma_{ij,X}^k w_X^{iq} w_X^{js} \partial_k \Phi_h \partial_q \Phi_h \partial_s \Phi_h.
\end{aligned}
$$

Therefore

$$
\ell_X(h)^2 \Delta_X \Phi_h - \mathrm{hess}_X\, \Phi_h(\nabla_X \Phi_h, \nabla_X \Phi_h) = I + II + III,
$$

where

$$
\begin{aligned}
I &:= \ell_X(h)^2 \left(\sum_{j,k=1}^{n} w_X^{jk} \partial_j \partial_k \Phi_h - \sum_{j,k,l=1}^{n} \Gamma_{jk,X}^l w_X^{jk} \partial_l \Phi_h \right), \\
II &:= - \sum_{i,j,q,s=1}^{n} w_X^{iq} w_X^{js} \partial_i \partial_j \Phi_h \partial_q \Phi_h \partial_s \Phi_h, \\
III &:= \sum_{i,j,k,q,s=1}^{n} \Gamma_{ij,X}^k w_X^{iq} w_X^{js} \partial_k \Phi_h \partial_q \Phi_h \partial_s \Phi_h,
\end{aligned}
$$

as well as $K(h) = \frac{1}{\ell_X(h)^3}(I + II + III)$. A straightforward calculation using (2.24) gives

$$I = -\ell_X(h)^2 \sum_{j,k=1}^{n-1} w_X^{jk} \partial_j \partial_k h + \ell_X(h)^2 \sum_{l=1}^{n-1} \sum_{j,k=1}^{n} w_X^{jk} \Gamma_{jk,X}^l \partial_l h \\ - \ell_X(h)^2 \sum_{j,k=1}^{n} w_X^{jk} \Gamma_{jk,X}^n,$$

as well as

$$II = \sum_{s,q,i,j=1}^{n-1} w_X^{iq} w_X^{js} \partial_q h \partial_s h \partial_i \partial_j h - \sum_{s,i,j=1}^{n-1} w_X^{in} w_X^{js} \partial_s h \partial_i \partial_j h \\ - \sum_{q,i,j=1}^{n-1} w_X^{iq} w_X^{jn} \partial_q h \partial_i \partial_j h + \sum_{i,j=1}^{n-1} w_X^{in} w_X^{jn} \partial_i \partial_j h.$$

A careful splitting of the sums entails

$$III = -\sum_{s,q,k=1}^{n-1} \sum_{i,j=1}^{n} \Gamma_{ij}^k w^{iq} w^{js} \partial_k h \partial_q h \partial_s h + \sum_{s,q=1}^{n-1} \sum_{i,j=1}^{n} \Gamma_{ij,X}^n w^{iq} w^{js} \partial_q h \partial_s h \\ + \sum_{s,k=1}^{n-1} \sum_{i,j=1}^{n} \Gamma_{ij}^k w^{in} w^{js} \partial_k \partial_s h - \sum_{s=1}^{n-1} \sum_{i,j=1}^{n} \Gamma_{ij}^n w^{in} w^{js} \partial_s h \\ + \sum_{q,k=1}^{n-1} \sum_{i,j=1}^{n} \Gamma_{ij}^k w^{iq} w^{jn} \partial_k h \partial_q h - \sum_{q=1}^{n-1} \sum_{i,j=1}^{n} \Gamma_{ij}^n w^{iq} w^{jn} \partial_q h \\ - \sum_{k=1}^{n-1} \sum_{i,j=1}^{n} \Gamma_{ij}^k w^{in} w^{jn} \partial_k h + \sum_{i,j=1}^{n} \Gamma_{ij}^n w^{in} w^{jn}.$$

We sort now the terms by appearance of derivatives in h. We find, in local coordinates,

$$K(h)|_s = \left(\sum_{j,k=1}^{n-1} a_{jk}(h) \partial_j \partial_k h + \sum_{j=1}^{n-1} a_j(h) \partial_j h + a(h) \right) |_{(s,h(s))},$$

where

$$a_{jk}(h) = \frac{1}{\ell_X(h)^3} \bigg(- \ell_X(h)^2 w^{jk} + w^{jn} w^{kn} - \sum_{l=1} g^{jl} g^{kn} \partial_l h \\ - \sum_{l=1}^{n-1} g^{jn} g^{kl} \partial_l h + \sum_{l,m=1}^{n-1} g^{jm} g^{kl} \partial_l h \partial_m h \bigg),$$

as well as

$$\begin{aligned} a_j(h) = \frac{1}{\ell_X(h)^3}\Bigg(&\ell_X(h)^2 \sum_{l,k=1}^{n} \Gamma^j_{lk} w^{lk} - \sum_{q,k=1}^{n-1}\sum_{i,l=1}^{n} \Gamma^k_{il} w^{iq} w^{lj} \partial_k h \partial_q h \\ &+ \sum_{q=1}^{n-1}\sum_{i,l=1}^{n} \Gamma^n_{il} w^{iq} w^{lj} \partial_q h + \sum_{k=1}^{n-1}\sum_{i,l=1}^{n} \Gamma^k_{il} w^{in} w^{lj} \partial_k h - \sum_{i,l=1}^{n} \Gamma^n_{il} w^{in} w^{lj} \\ &+ \sum_{k=1}^{n-1}\sum_{i,l=1}^{n} \Gamma^k_{il} w^{ij} w^{ln} \partial_k h - \sum_{i,l=1}^{n} \Gamma^n_{il} w^{ij} w^{ln} - \sum_{i,l=1}^{n} \Gamma^j_{il} w^{in} w^{ln}\Bigg), \end{aligned}$$

and

$$a(h) = -\frac{1}{\ell_X(h)} \sum_{j,k=1}^{n} \Gamma^n_{jk} w^{jk} + \frac{1}{\ell_X(h)^3} \sum_{i,j=1}^{n} \Gamma^n_{ij} w^{in} w^{jn}.$$

Let

$$\begin{aligned} P(h)|_s &= \left(\sum_{j,k=1}^{n-1} a_{jk}(h) \partial_j \partial_k + \sum_{j=1}^{n-1} a_j(h) \partial_j \right) |_{(s,h(s))}, \\ Q(h)|_s &= a(h)|_{(s,h(s))}, \end{aligned} \tag{2.25}$$

in local coordinates. We will show that $K(h) = P(h)h + Q(h)$ is the desired decomposition of K. Let us recall that

$$w^X_{jk}|_{(s,h(s))} = \frac{\partial X}{\partial j} \cdot \frac{\partial X}{\partial k}|_{(s,h(s))}, \quad \Gamma^i_{jk,X}|_{(s,h(s))} = \sum_{m=1}^{n} w^{mi}_X \partial_j \frac{\partial X}{\partial k} \cdot \frac{\partial X}{\partial m}|_{(s,h(s))}.$$

Note that since Σ is smooth, also $\frac{\partial X}{\partial j}$, w^X_{ij} and w^{ij}_X are smooth. Therefore the evaluation at the point $(s, h(s))$ in the above formulas is a composition of a smooth function with h.

We briefly recall useful results on composition operators, as found in Section 2.1 in [**4**] and [**61**]. For a smooth function $G \in C^\infty(\mathbb{R})$ with $G(0) = 0$, we have that

$$G(f) \in B^s_{pq}(\mathbb{R}^n) \tag{2.26}$$

for any function $f \in B^s_{pq}(\mathbb{R}^n)$, provided that $s - n/p > 0, 1 \le p, q \le \infty$. This in particular implies that $f^{-1} \in B^s_{pq}(A)$ for all $f \in B^s_{pq}(A)$ such that $|f|_\infty \ge c_0 > 0$ if A is a bounded Lipschitz domain. Note that $s > n/p$ ensures $B^s_{pq}(A) \hookrightarrow L^\infty(A)$. Moreover, if the previous conditions are satisfied, the mapping $[f \mapsto G(f)]$ is bounded as a map from $B^s_{pq}(\mathbb{R}^n)$ to $B^s_{pq}(\mathbb{R}^n)$. Furthermore, c.f. Section 2.1 in [**4**],

$$G(.) \in C^1(B^s_{p,q}(\mathbb{R}^n)) \tag{2.27}$$

for any $G \in C^\infty(\mathbb{R})$ such that $G(0) = 0$, provided $s > n/p$. Again we mention as in Section 2.1 in [**4**] that these results carry over directly if one replaces $\mathbb{R}^n$ by a finite dimensional smooth compact manifold. Then $G(0) = 0$ is no longer required. For further discussion we refer to Section 2.1 in [**4**].

Recall that $\mathcal{U} := \{h \in X_\gamma : \|h\|_{L^\infty} < a/5\} \subset X_\gamma$ and $X_\gamma = B^{4-1/q-3/p}_{qp}(\Sigma)$. Now, for $h \in \mathcal{U}$, by (2.26),

$$w^X_{jk}(h) \in X_\gamma.$$

This follows from the fact that

$$s := 4 - \frac{1}{q} - \frac{3}{p} > \frac{n}{q}, \quad \text{for } n = 2, 3.$$

Note that for $n = 3$ (which is sufficent to consider) this is equivalent to $p > \frac{3}{4-4/q}$, which is easily satisfied since even $p > \frac{3}{3-4/q}$ by assumption. Arguing as in [4], $\det[(w_{jk})_{jk}] \geq c_0 > 0$, hence also

$$w_X^{jk}(h) \in X_\gamma.$$

As the regularity index of Besov spaces behaves well under differentiation,

$$\partial_j h \in B_{qp}^{3-1/q-3/p}(\Sigma), \quad j \in 1, ..., n-1,$$

whence

$$w^{jk}\partial_j h \partial_k h \in B_{qp}^{3-1/q-3/p}(\Sigma). \tag{2.28}$$

Note at this point that we used the fact that since $p > \frac{3}{3-4/q}$,

$$3 - 1/q - 3/p > \frac{n}{q}, \quad n = 2, 3,$$

therefore the space $B_{qp}^{3-1/q-3/p}(\Sigma)$ is a Banach algebra under pointwise multiplication and we have a product estimate. More generally speaking, for $s > n/q$, $B_{q,q_0}^s \hookrightarrow L^\infty$ and

$$\|fg\|_{B_{q,q_0}^s} \lesssim \|f\|_{B_{q,q_0}^s} \|g\|_{B_{q,q_0}^s},$$

for all $f, g \in B_{q,q_0}^s$, see Lemma 1.5. Note that this lemma also holds true on $B_{q,q_0}^s(\Sigma)$ since the extension operator $[B_{q,q_0}^s(\Sigma) \to B_{q,q_0}^s(\mathbb{R}^{n-1})]$ is also bounded from $[L_\infty(\Sigma) \to L_\infty(\mathbb{R}^{n-1})]$, cf. [62]. By (2.28), the composition result (2.26), $h \in \mathcal{U}$, and $w_X^{nn} \geq \frac{1}{2}$,

$$\ell_X(h) \in B_{qp}^{3-1/q-3/p}(\Sigma).$$

As in [4], we proceed this way to get

$$a_{jk}(h), a_j(h), a(h) \in B_{qp}^{3-1/q-3/p}(\Sigma),$$

for all $h \in \mathcal{U}$. By the product estimate

$$\|fg\|_{W_q^{2-1/q}} \lesssim \|f\|_{B_{q,p}^{3-1/q-3/p}} \|g\|_{W_q^{2-1/q}}$$

for all $f \in B_{q,p}^{3-1/q-3/p}(\mathbb{R}^n), g \in W_q^{2-1/q}(\mathbb{R}^n)$, we readily obtain

$$P \in C^1(\mathcal{U}, \mathcal{B}(W_q^{4-1/q}(\Sigma), W_q^{2-1/q}(\Sigma))), \quad Q \in C^1(\mathcal{U}, W_q^{2-1/q}(\Sigma)).$$

Indeed, the operators are compositions of C^1-mappings, cf. (2.27). Moreover we have the estimates

$$\|Q(h)\|_{W_q^{2-1/q}} \lesssim \|a(h)\|_{B_{q,p}^{3-1/q-3/p}} \|1\|_{W_q^{2-1/q}} \lesssim \|a(h)\|_{B_{q,p}^{3-1/q-3/p}},$$

for all $h \in \mathcal{U}$, as well as

$$\begin{aligned}\|P(h)\tilde{h}\|_{W_q^{2-1/q}} &\lesssim \sum_{j,k=1}^{n-1} \|a_{jk}(h)\partial_j\partial_k\tilde{h}\|_{W_q^{2-1/q}} + \sum_{j=1}^{n-1} \|a_j(h)\partial_j\tilde{h}\|_{W_q^{2-1/q}} \\ &\lesssim \sum_{j,k=1}^{n-1} \|a_{jk}(h)\|_{B_{q,p}^{3-1/q-3/p}} \|\partial_j\partial_k\tilde{h}\|_{W_q^{2-1/q}} + \\ &\quad + \sum_{j=1}^{n-1} \|a_j(h)\|_{B_{q,p}^{3-1/q-3/p}} \|\partial_j\tilde{h}\|_{W_q^{2-1/q}} \\ &\lesssim \left(\sum_{j,k=1}^{n-1} \|a_{jk}(h)\|_{X'_{p,q}}\| + \sum_{j=1}^{n-1} \|a_j(h)\|_{X'_{p,q}} \right) \|\tilde{h}\|_{W_q^{4-1/q}},\end{aligned}$$

for all $h \in \mathcal{U}, \tilde{h} \in W_q^{4-1/q}(\Sigma)$.

Now we investigate the operator $P(0)$. From (2.25), in local coordinates,

$$P(0)|_s = \left(\sum_{j,k=1}^{n-1} a_{jk}(0)\partial_j\partial_k + \sum_{j=1}^{n-1} a_j(0)\partial_j \right) |_{(s,0)}.$$

Firstly,

$$\partial_n X|_{(s,0)} = \partial_r X(s,0) = \nu_\Sigma(s), \quad s \in \Sigma,$$

as well as

$$\partial_j X|_{(s,0)} = \partial_j^\Sigma|_s, \quad j = 1, ..., n-1, s \in \Sigma.$$

Hence,

$$w_{jn}^X|_{(s,0)} = \delta_{jn}, \quad j = 1, ..., n,$$

where δ_{jn} denotes the Kronecker delta. Inversion formula for block type matrices implies

$$w_X^{jn}|_{(s,0)} = \delta_{jn}, \quad j = 1, ..., n.$$

In particular,

$$\ell_X(0) = \sqrt{w_X^{nn}|_{(s,0)}} = 1.$$

From the general formulas for a_{jk} and a_j we obtain that

$$a_{jk}(0) = -w_\Sigma^{jk},$$

as well as

$$\begin{aligned}a_j(0) = &\sum_{l,k=1}^{n} w_\Sigma^{lk}\Gamma_{lk,\Sigma}^j - \sum_{i,l=1}^{n} \Gamma_{il,\Sigma}^n w_\Sigma^{in} w_\Sigma^{lj} \\ &- \sum_{i,l=1}^{n} \Gamma_{il,\Sigma}^n w_\Sigma^{ij} w_\Sigma^{ln} - \sum_{i,l=1}^{n} \Gamma_{il,\Sigma}^j w_\Sigma^{in} w_\Sigma^{ln},\end{aligned}$$

for all $j, k = 1, ...n-1$. Now, using $w_\Sigma^{mn} = \delta_{mn}$ for all $m = 1, ..., n$,

$$a_j(0) = \sum_{l,k=1}^{n-1} w_\Sigma^{lk}\Gamma_{lk,\Sigma}^j + \Gamma_{nn,\Sigma}^j + \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3,$$

where

$$\mathcal{D}_1 := -\sum_{i,l=1}^{n} \Gamma^n_{il,\Sigma} w^{in}_\Sigma w^{lj}_\Sigma, \quad \mathcal{D}_2 := -\sum_{i,l=1}^{n} \Gamma^n_{il,\Sigma} w^{ij}_\Sigma w^{ln}_\Sigma, \quad \mathcal{D}_3 := -\sum_{i,l=1}^{n} \Gamma^j_{il,\Sigma} w^{in}_\Sigma w^{ln}_\Sigma.$$

Now, again using $w^{mn}_\Sigma = \delta_{mn}$ for all $m = 1, ..., n$ and that the Christoffel symbols are symmetric in the lower two indices, we get

$$\mathcal{D}_1 = \mathcal{D}_2 = -\sum_{l=1}^{n} \Gamma^n_{nl,\Sigma} w^{lj}_\Sigma, \quad \mathcal{D}_3 = \Gamma^j_{nn,\Sigma}.$$

Since $1 \leq j \leq n-1$, we again use $w^{nj}_\Sigma = 0$ to deduce

$$\mathcal{D}_1 = \mathcal{D}_2 = -\sum_{l=1}^{n-1} \Gamma^n_{nl,\Sigma} w^{lj}_\Sigma.$$

Now by characterization of the Christoffel symbols via the first fundamental form,

$$\Gamma^n_{in,\Sigma} = \frac{1}{2} \sum_{k=1}^{n} w^{nk}_\Sigma \left(\frac{\partial w_{nk}}{\partial x_i} + \frac{\partial w_{ik}}{\partial x_n} - \frac{\partial w_{in}}{\partial x_k} \right), \quad i = 1, ..., n.$$

Using once more $w^{nk}_\Sigma = \delta_{nk}$,

$$\Gamma^n_{in,\Sigma} = \frac{1}{2} \left(\frac{\partial w_{nn}}{\partial x_i} + \frac{\partial w_{in}}{\partial x_n} - \frac{\partial w_{in}}{\partial x_n} \right) = \frac{1}{2} \frac{\partial w_{nn}}{\partial x_i}, \quad i = 1, ..., n.$$

Now we claim that

$$\frac{\partial w^\Sigma_{nn}}{\partial x_i} = 0, \quad i = 1, ..., n-1.$$

Note that this would imply that $\mathcal{D}_1 = \mathcal{D}_2 = 0$. So let $1 \leq i \leq n-1$. Then

$$\partial_{s_i} w^\Sigma_{nn}(s,r) = \left(2\partial_{s_j} (\partial_r t(s,r) \vec{T}(s)), \partial_r t(s,r) \vec{T}(s) \right), \quad s \in \Sigma, r \in (-\varepsilon, \epsilon),$$

for $\varepsilon > 0$ sufficiently small, whenceforth,

$$\partial_{s_i} w^\Sigma_{nn}(s,r) = 0.$$

Hence

$$a_j(0) = \sum_{l,k=1}^{n-1} w^{lk}_\Sigma \Gamma^j_{lk,\Sigma}.$$

Since

$$\Delta_\Sigma = \sum_{j,k=1}^{n-1} w^{jk}_\Sigma \partial_j \partial_k - \sum_{j=1}^{n-1} \sum_{l,k=1}^{n-1} w^{lk}_\Sigma \Gamma^j_{lk,\Sigma} \partial_j$$

in local coordinates, we have $P(0) = -\Delta_\Sigma$. The proof is complete. $\square$

2.4.3. Transformation to a fixed domain. We assume that the free interface is given as a graph of a function h, $\Gamma(t) = \Gamma_h(t) := \{X(p, h(p,t)) : p \in \Sigma\}$, $t \geq 0$, cf. (2.16). Using the Hanzawa-type transformation $\Theta_h^t := \Theta_{h(t)} : \Omega \to \Omega$, cf. (2.17), we transform system (2.1) to a fixed time-independent reference configuration.

Introduce a new variable η by

$$\eta\left(\Theta_h^t(x), t\right) = \mu(x,t), \quad x \in \Omega, t \in \mathbb{R}_+.$$

Chain rule entails

$$\nabla\mu(x,t) = [D\Theta_h^t]^T(x)\nabla\eta(\Theta_h^t(x), t).$$

We also have

$$\Delta\mu(x,t) = \sum_{j,l=1}^n \alpha_{j,l}^h(x)\partial_j\partial_l\eta(\Theta_h^t(x),t) + \sum_{l=1}^n \alpha_l^h(x)\partial_l\eta(\Theta_h^t(x),t), \tag{2.29}$$

where the coefficients are given by

$$\alpha_{j,l}^h(x) = \sum_{k=1}^n \partial_k(\Theta_h^t)_j(x)\partial_k(\Theta_h^t)_l(x), \quad \alpha_l^h(x) = \sum_{k=1}^n \partial_k\partial_k(\Theta_h^t)_l(x).$$

Let us economize notation. Define transformed differential operators

$$\nabla_h := (D\Theta_h^t)^T\nabla, \quad \operatorname{div}_h u := \operatorname{Tr}(\nabla_h u), \quad \Delta_h := \operatorname{div}_h \nabla_h.$$

Is is then easy to check that $\nabla\mu(x,t) = \nabla_h\eta(\Theta_h^t(x),t)$ and $\Delta\mu(x,t) = \Delta_h\eta(\Theta_h^t(x),t)$. Let us introduce

$$n_{\partial\Omega}^h := n_{\partial\Omega} \circ \Theta_h^t.$$

Since the Hanzawa transformation maps points from $\Sigma = \{p = X(p,0) : p \in \Sigma\}$ to points on $\Gamma_h = \{p = X(p, h(p,t)) : p \in \Sigma\}$ and vice versa, also the normals change since the boundary points are moving.

Let us deduce a formula for the normal velocity of the free interface $\Gamma_h(t) = \Gamma_{h(t)}$ in terms of the height function h. By Hanzawa transformation, $\Theta_h^t(\Gamma_{h(t)}) = \Sigma$, in other words, the inverse mapping

$$\Xi_h^t := (\Theta_h^t)^{-1} : \Sigma \to \Gamma_{h(t)}$$

is a parametrization of $\Gamma_{h(t)}$ over the fixed reference surface Σ. By equation (2.79) in [57], the normal velocity satisfies

$$V_{\Gamma_h}(t,p) = \left(\partial_t\Xi_h^t(x)|\nu_{\Gamma_h}(t,p)\right), \quad p = \Xi_h^t(x), x \in \Sigma.$$

On the reference surface, we have that

$$\Xi_h^t|_\Sigma = X \circ (F_h^t)^{-1} \circ X^{-1}|_\Sigma,$$

where X denotes the curvilinear coordinates in a neighbourhood of Σ,

$$X(s,r) = s + r\nu_\Sigma(s) + \tau(r,s)\vec{T}_\Sigma(s), \quad s \in \Sigma, r \in (-a,a),$$

for $a > 0$ small enough. Then, picking a point $s \in \Sigma$, we have that $X(s,0) = s$, whence

$$\Xi_h^t(s) = X \circ (F_h^t)^{-1} \circ X^{-1}(s) = X \circ (F_h^t)^{-1}(s,0) = X(s, \chi(-h(s,t)/a)h(s,t)).$$

In particular, if we choose h small enough, say $|h(t)|_{L_\infty(\Sigma)} < a/3$, we may use that $\chi(x) = 1$ for all $|x| < 1/3$ to conclude that

$$\Xi_h^t(s) = X(s, h(s,t)), \quad s \in \Sigma, \quad |h(t)|_{L_\infty(\Sigma)} < a/3.$$

Hence

$$\partial_t \Xi_h^t(s) = \partial_r X(s, h(s,t))\partial_t h(s,t), \quad s \in \Sigma.$$

At this point note that, cf. (2.20),

$$\partial_r X(s,r) = \nu_\Sigma(s) + \partial_r \tau(s,r)\vec{T}(s), \quad s \in \Sigma, r \in (-a,a).$$

Hence

$$V_{\Gamma_h} = \big(\partial_t \Xi_h^t | \nu_{\Gamma_h}\big) = \partial_t h(\nu_\Sigma + \partial_r \tau(h)\vec{T} | \nu_{\Gamma_h}).$$

Let

$$a(h) := (\nu_\Sigma + \partial_r \tau(h)\vec{T} | \nu_{\Gamma_h}). \tag{2.30}$$

Observe $a(0) = 1$. Employing Hanzawa transform then yields a nonlinear system on a fixed reference configuration,

$$\begin{aligned}
a(h)\partial_t h &= -[\![n_{\Gamma_h} \cdot \nabla_h \eta]\!], && \text{on } [\Sigma \times \{t > 0\}],\\
\eta|_\Sigma &= K(h), && \text{on } [\Sigma \times \{t > 0\}],\\
\Delta_h \eta &= 0, && \text{in } [\Omega \backslash \Sigma \times \{t > 0\}],\\
n_{\partial\Omega}^h \cdot \nabla_h \eta|_{\partial\Omega} &= 0, && \text{on } [\partial\Omega \backslash \partial\Sigma \times \{t > 0\}],\\
n_{\partial\Omega}^h \cdot n_{\Gamma_h} &= 0, && \text{on } [\partial\Sigma \times \{t > 0\}],\\
h|_{t=0} &= h_0, && \text{at } t = 0.
\end{aligned} \tag{2.31}$$

Here $K(h)$ is the transformed mean curvature operator, $n_{\partial\Omega}^h := n_{\partial\Omega} \circ \Theta_h^t$, and h_0 a suitable description of the initial configuration such that $\Omega^\pm(t=0) = \Omega_0^\pm$. Note that we have the compatibility condition $n_{\partial\Omega} \cdot n_{\Gamma_{h_0}} = 0$ at time $t = 0$.

2.4.4. Differentiability properties of transformed operators. We now prove differentiability of the transformed differential operators

$$\nabla_h := (D\Theta_h^t)^T \nabla, \quad \operatorname{div}_h u := \operatorname{Tr}(\nabla_h u), \quad \Delta_h := \operatorname{div}_h \nabla_h,$$

and the transformed normals $n_{\partial\Omega}^h$, n_{Γ_h}.

LEMMA 2.17. *Let $n = 2,3$, $q \in (3/2, 2)$, $p > 3/(3 - 4/q)$ and $\mathcal{U} \subset X_\gamma$ as before. Then*

$$\begin{gathered}
[h \mapsto \Delta_h] \in C^1(\mathcal{U}; \mathcal{B}(W_q^2(\Omega\backslash\Sigma); L_q(\Omega))),\\
[h \mapsto \nabla_h] \in C^1(\mathcal{U}; \mathcal{B}(W_q^k(\Omega\backslash\Sigma); W_q^{k-1}(\Omega\backslash\Sigma))), \quad k = 1,2,\\
[h \mapsto n_\Sigma^h],\ [h \mapsto n_{\partial\Omega}^h] \in C^1(\mathcal{U}; C^1(\Sigma)).
\end{gathered}$$

PROOF. The proof follows the lines of Section 4 in [4], since the trace space satisfies $X_\gamma \hookrightarrow C^2(\Sigma)$ by choice of p and q. Indeed, going back to equation (2.29), we see that the transformed Laplace operator can be written as

$$\Delta_h = \sum_{j,l=1}^n a_{j,l}^h \partial_j \partial_l + \sum_{l=1}^n a_l^h \partial_l,$$

where

$$a^h_{j,l}(x) = \sum_{k=1}^{n} \partial_k(\Theta^t_h)_j((\Theta^t_h)^{-1}(x))\partial_k(\Theta^t_h)_l((\Theta^t_h)^{-1}(x)),$$
$$a^h_l(x) = \sum_{k=1}^{n} \partial_k\partial_k(\Theta^t_h)_l((\Theta^t_h)^{-1}(x)).$$

The coefficients depend on h itself and up to two derivatives of h,

$$a^h_{j,k} = a_{j,k}(x, h, \nabla h, \nabla^2 h), \quad a^h_j = a_j(x, h, \nabla h, \nabla^2 h),$$

and the dependence of $a_{j,k}$ and a_j of $(x, h, \nabla h, \nabla^2 h)$ is smooth. This together with $X_\gamma \hookrightarrow C^2(\Sigma)$ allows us to follow the lines of [4]. For the differentiability properties of the transformed normals we refer to (2.27) and [61]. Note that the derivative satisfies $\nabla h \in C^1(\Sigma)$ for $h \in X_\gamma$ and the transformed normals depend smoothly on $(x, h, \nabla h)$ as well. $\square$

2.5. Linearization and model problems

The main result of this chapter is maximal regularity in $L_p - L_q$ for the principal linearization. We start with maximal regularity for the model problems and then apply a localization procedure.

2.5.1. The shifted model problem on the half space. Let $n = 2, 3$. In this section we will be concerned with the principal linearization on the whole upper half space $\mathbb{R}^n_+$ with a flat interface $\Sigma := \{x \in \mathbb{R}^n_+ : x_1 = 0\}$. More precisely, we will consider

$$\begin{aligned} \partial_t h + \omega^3 h + [\![n_\Sigma \cdot \nabla\mu]\!] &= g_1, && \text{on } \Sigma,\\ \mu|_\Sigma + \Delta_{x'} h &= g_2, && \text{on } \Sigma,\\ \omega^2\mu - \Delta\mu &= g_3, && \text{on } \mathbb{R}^n_+\backslash\Sigma,\\ e_n \cdot \nabla\mu|_{\partial\mathbb{R}^n_+} &= g_4, && \text{on } \partial\mathbb{R}^n_+,\\ e_n \cdot \nabla_{x'} h|_{\partial\Sigma} &= g_5, && \text{on } \partial\Sigma,\\ h|_{t=0} &= h_0, && \text{on } \Sigma, \end{aligned} \tag{2.32}$$

where $x' = (x_2, ..., x_n)$. Here $\omega > 0$ is a fixed shift parameter we need to introduce to get maximal regularity results on the unbounded time-space domain $\mathbb{R}_+ \times \mathbb{R}^n_+$.

Let us discuss the optimal regularity classes for the data. We search for a solution h of this evolution equation in the space

$$W^1_p(\mathbb{R}_+; W^{1-1/q}_q(\Sigma)) \cap L_p(\mathbb{R}_+; W^{4-1/q}_q(\Sigma)),$$

where p and q are specified below. In particular, $\mu \in L_p(\mathbb{R}_+; W^2_q(\mathbb{R}^n_+\backslash\Sigma))$. Let

$$X_0 := W^{1-1/q}_q(\Sigma), \quad X_1 := W^{4-1/q}_q(\Sigma),$$

and the real interpolation space

$$X_\gamma := (X_0, X_1)_{1-1/p,p} = B^{4-1/q-3/p}_{qp}(\Sigma).$$

By simple trace theory, cf. [62], we may deduce the necessary conditions

$$g_1 \in L_p(\mathbb{R}_+; X_0), \quad g_2 \in L_p(\mathbb{R}_+; W_q^{2-1/q}(\Sigma)), \tag{2.33}$$
$$g_3 \in L_p(\mathbb{R}_+; L_q(\mathbb{R}^n_+)), \quad g_4 \in L_p(\mathbb{R}_+; W_q^{1-1/q}(\partial\mathbb{R}^n_+)), \quad h_0 \in X_\gamma.$$

It is now a delicate matter to find the optimal regularity condition for g_5, which turns out to be

$$g_5 \in F_{pq}^{1-2/(3q)}(\mathbb{R}_+; L_q(\partial\Sigma)) \cap L_p(\mathbb{R}_+; W_q^{3-2/q}(\partial\Sigma)), \tag{2.34}$$

cf. Theorem 2.1. Note that g_5 has a time trace at $t = 0$, whenever $1-2/(3q)-1/p > 0$. Hence there is a compatibility condition inside the system whenever this inequality is satisfied, namely

$$g_5|_{t=0} = e_n \cdot \nabla_{x'} h_0|_{\partial\Sigma} = \partial_n h_0|_{\partial\Sigma}, \quad \text{on } \partial\Sigma. \tag{2.35}$$

Note that there is no compatibility condition stemming from $(2.32)_2$ and $(2.32)_4$ on $\partial\Sigma$, whenever $q < 2$. The following theorem now states that these conditions are also sufficient. Note that the assumptions in Theorem 2.18 imply that $q < 2$ and $1 - 2/(3q) - 1/p > 0$ hold.

THEOREM 2.18. *Let $p \in (6,\infty)$, $q \in (3/2, 2) \cap (2p/(p+1), 2p)$, and $\omega > 0$. Then (2.32) has maximal $L_p - L_q$-regularity on $\mathbb{R}_+$. More precisely, for every $(g_1, g_2, g_3, g_4, g_5, h_0)$ satisfying the regularity conditions (2.33),(2.34), and the compatibility condition (2.35), there is a unique solution*

$$(h,\mu) \in (W_p^1(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; X_1)) \times L_p(\mathbb{R}_+; W_q^2(\mathbb{R}^n_+\backslash\Sigma))$$

of the shifted half space problem (2.32).

Furthermore,

$$|h|_{W_p^1(\mathbb{R}_+;X_0)\cap L_p(\mathbb{R}_+;X_1)} + |\mu|_{L_p(\mathbb{R}_+;W_q^2(\mathbb{R}^n_+\backslash\Sigma))}$$

is bounded by

$$|g_1|_{L_p(\mathbb{R}_+;X_0)} + |g_2|_{L_p(\mathbb{R}_+;W_q^{2-1/q}(\Sigma))} + |g_3|_{L_p(\mathbb{R}_+;L_q(\mathbb{R}^n_+))} +$$
$$|g_4|_{L_p(\mathbb{R}_+;W_q^{1-1/q}(\partial\mathbb{R}^n_+))} + |g_5|_{F_{pq}^{1-2/(3q)}(\mathbb{R}_+;L_q(\partial\Sigma))\cap L_p(\mathbb{R}_+;W_q^{3-2/q}(\partial\Sigma))} + |h_0|_{X_\gamma},$$

up to a constant $C = C(\omega) > 0$, which may depend on $\omega > 0$.

PROOF. We first reduce to a trivial initial value by extending h_0 to $\tilde{\Sigma} = \{0\} \times \mathbb{R}^{n-1}$ using standard extension results of [62] and solving an $L_p - L_q$ auxiliary problem on $\mathbb{R}^{n-1}$ using results of Section 4 in [58] to find some $h_S \in W_p^1(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; X_1)$ such that $h_S|_{t=0} = h_0$, cf. problem (2.39). Then define $\tilde{g}_5 := g_5 - \partial_n h_S|_{\partial\Sigma}$. Clearly,

$$\tilde{g}_5|_{t=0} = g_5|_{t=0} - \partial_n h_0|_{\partial\Sigma} = 0, \quad \text{on } \partial\Sigma,$$

by the compatibility condition (2.35). This allows us to use Theorem 2.1 to find some $\tilde{h} \in {}_0W_p^1(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; X_1)$ such that

$$\partial_n \tilde{h}|_{\partial\Sigma} = \tilde{g}_5, \quad \text{on } \partial\Sigma.$$

By simple trace theory, cf. [62], we may find $\mu_4 \in L_p(\mathbb{R}_+; W^2_q(\mathbb{R}^n_+\backslash\Sigma))$ such that $\partial_n\mu_4|_{\partial\mathbb{R}^n_+} = g_4$ on $\partial\mathbb{R}^n_+$. Let $\tilde{\Sigma} := R\Sigma := \{x \in \mathbb{R}^n : x_1 = 0\}$. We then solve the elliptic auxiliary problem

$$\begin{aligned} \omega^2\tilde{\mu} - \Delta\tilde{\mu} &= Rg_3 - R(\omega^2 - \Delta)\mu_4, && \text{on } \mathbb{R}^n\backslash\tilde{\Sigma}, \\ \tilde{\mu}|_{\tilde{\Sigma}} &= -R\Delta_{x'}\tilde{h} - R\Delta_{x'}h_S + Rg_2 - R\mu_4|_{\tilde{\Sigma}}, && \text{on } \tilde{\Sigma}, \end{aligned} \tag{2.36}$$

by a unique $\tilde{\mu} \in L_p(\mathbb{R}_+; W^2_q(\mathbb{R}^n\backslash\tilde{\Sigma}))$, cf. [6]. Note at this point that we used that

$$-R\Delta_{x'}\tilde{h} - R\Delta_{x'}h_S + Rg_2 - R\mu_4|_{\tilde{\Sigma}} \in L_p(\mathbb{R}_+; W^{2-1/q}_q(\tilde{\Sigma})),$$

by Theorem 2.11 since $q < 2$. By construction $\tilde{\mu}$ is even in x_n-direction since both the data in (2.36) are. Hence

$$\partial_n\tilde{\mu} = 0, \quad \text{on } \partial\mathbb{R}^n_+.$$

We have reduced the problem to the case where $(g_2, g_3, g_4, g_5, h_0) = 0$, that is, we are left to solve

$$\begin{aligned} \partial_t h + \omega^3 h + [\![n_\Sigma \cdot \nabla\mu]\!] &= g_1, && \text{on } \Sigma, \\ \mu|_\Sigma + \Delta_{x'}h &= 0, && \text{on } \Sigma, \\ \omega^2\mu - \Delta\mu &= 0, && \text{on } \mathbb{R}^n_+\backslash\Sigma, \\ e_n \cdot \nabla\mu|_{\partial\mathbb{R}^n_+} &= 0, && \text{on } \partial\mathbb{R}^n_+, \\ e_n \cdot \nabla_{x'}h|_{\partial\Sigma} &= 0, && \text{on } \partial\Sigma, \\ h|_{t=0} &= 0, && \text{on } \Sigma, \end{aligned} \tag{2.37}$$

for possibly modified g_1 not to be relabeled in an $L_p - L_q$-setting. We reflect the problem once more across the boundary $\partial\mathbb{R}^n_+$ using the even reflection in x_n-direction R. We obtain a full space problem with a flat interface and that the conditions $(2.37)_4$ and $(2.37)_5$ are fulfilled automatically. We obtain the problem

$$\begin{aligned} \partial_t h + \omega^3 h + [\![n_\Sigma \cdot \nabla\mu]\!] &= Rg_1, && \text{on } \tilde{\Sigma}, \\ \mu|_{\tilde{\Sigma}} + \Delta_{x'}h &= 0, && \text{on } \tilde{\Sigma}, \\ \omega^2\mu - \Delta\mu &= 0, && \text{on } \mathbb{R}^n\backslash\tilde{\Sigma}, \\ h|_{t=0} &= 0, && \text{on } \tilde{\Sigma}, \end{aligned} \tag{2.38}$$

where $Rg_1 \in L_p(\mathbb{R}_+; W^{1-1/q}_q(\tilde{\Sigma}))$. Let us denote by $S(h)$ the unique solution of the elliptic problem $(2.38)_{2,3}$. We can then write the system as an abstract evolution equation as follows. Define $\mathcal{A}h := [\![n_\Sigma \cdot \nabla S(h)]\!] + \omega^3 h$ and its realization in $W^{1-1/q}_q(\tilde{\Sigma})$ by $A : D(A) \to W^{1-1/q}_q(\tilde{\Sigma})$ with domain $D(A) := W^{4-1/q}_q(\tilde{\Sigma})$. Then we can modify the results of [58] to obtain that the operator A has the property of maximal L_q-regularity on the whole half line $\mathbb{R}_+$. A general principle of maximal regularity going back to Dore [16] and Bourgain [12] now gives that A has also maximal L_p-regularity on $\mathbb{R}_+$, since $1 < p < \infty$, cf. [57]. We give the full details below. Having this at hand we can solve the initial value problem

$$\begin{aligned} \frac{d}{dt}h(t) + Ah(t) &= \tilde{f}(t), \quad t \in \mathbb{R}_+, \\ h(0) &= \tilde{h}_0, \end{aligned} \tag{2.39}$$

for any $\tilde{f} \in L_p(\mathbb{R}_+; W_q^{1-1/q}(\tilde{\Sigma}))$ and $\tilde{h}_0 \in B_{qp}^{4-1/q-3/p}(\tilde{\Sigma})$ by a unique function $h \in W_p^1(\mathbb{R}_+; W_q^{1-1/q}(\tilde{\Sigma})) \cap L_p(\mathbb{R}_+; W_q^{4-1/q}(\tilde{\Sigma}))$. By choosing

$$f := R[\![n_\Sigma \cdot \nabla(\tilde{\mu} + \mu_4)]\!] - R\partial_t(\tilde{h} + h_S) + Rg_1, \quad \tilde{h}_0 := 0,$$

we obtain a unique solution $(h, S(h))$ of the problem (2.38) in the proper $L_p - L_q$-regularity classes on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. The estimate easily follows and the proof is complete.

Let us give the details on how we obtain maximal L_q-regularity for A on $\mathbb{R}_+$. We take Fourier transform with respect to $(x_2, ..., x_n) \in \mathbb{R}^{n-1}$ to obtain a system

$$\begin{aligned} \omega^3 \hat{h} + \partial_t \hat{h} + [\![\partial_1 \hat{\pi}]\!] &= \hat{f}, && \xi \in \mathbb{R}^{n-1}, \\ \omega^2 \hat{\pi} + |\xi|^2 \hat{\pi} - \partial_1^2 \hat{\pi} &= 0, && (x_1, \xi) \in \dot{\mathbb{R}} \times \mathbb{R}^{n-1}, \\ \hat{\pi}|_{x_1=0} + |\xi|^2 \hat{h} &= 0, && \xi \in \mathbb{R}^{n-1}, \\ \hat{h}|_{t=0} &= 0, && \xi \in \mathbb{R}^{n-1}, \end{aligned}$$

where $\hat{\pi} = \hat{\pi}(t, x_1, \xi)$, $\hat{h} = \hat{h}(t, \xi)$, and $\hat{f} = \hat{f}(t, \xi)$ denote the Fourier transforms of π, h, and f with respect to the last $n-1$ variables $(x_2, ..., x_n) \in \mathbb{R}^{n-1}$. We can now solve the second order differential equation for $\hat{\pi}$ to the result

$$\hat{\pi}(x_1, \xi) = -|\xi|^2 \hat{h}(\xi) \exp\left(-\sqrt{\omega^2 + |\xi|^2}|x_1|\right), \quad x_1 \in \mathbb{R}, \xi \in \mathbb{R}^{n-1}.$$

We can easily compute the jump to be

$$[\![\partial_1 \hat{\pi}]\!] = 2|\xi|^2 \sqrt{\omega^2 + |\xi|^2} \hat{h},$$

whence we obtain a modified version of the evolution equation in [**58**], namely

$$\begin{aligned} (\partial_t + \omega^3)\hat{h} + \left(2|\xi|^2 \sqrt{\omega^2 + |\xi|^2}\right) \hat{h} = \hat{f}, \quad t \in \mathbb{R}_+, \\ \hat{h}(t = 0) = 0. \end{aligned}$$

Let now B_1 be the negative Laplacian on $L_q(\mathbb{R}^{n-1})$ with domain $W_q^2(\mathbb{R}^{n-1})$. It is now well known that B_1 admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus on $L_q(\mathbb{R}^{n-1})$ with corresponding $\mathcal{RH}^\infty$-angle zero, $\varphi_{B_1}^{\mathcal{RH}^\infty} = 0$, cf. the proof of Proposition 8.3.1 in [**57**]. Let furthermore B_2 be the operator given by $(\omega^2 - \Delta)^{1/2}$ on $L_q(\mathbb{R}^{n-1})$ with natural domain $W_q^1(\mathbb{R}^{n-1})$. Then by Example 4.5.16(i) in [**57**] we know that B_2 is invertible, admits a bounded $\mathcal{H}^\infty$-calculus on $L_q(\mathbb{R}^{n-1})$, and the $\mathcal{H}^\infty$-angle is zero, $\varphi_{B_2}^\infty = 0$. We now apply Corollary 4.5.12(iii) in [**57**] to get that $P := 2B_1 B_2$ is a closed, sectorial operator which itself admits a bounded $\mathcal{H}^\infty$-calculus on $L_q(\mathbb{R}^{n-1})$ as well and that the $\mathcal{H}^\infty$-angle of P is zero. The fact that B_1 and B_2 commute stems from the fact that these are given as Fourier multiplication operators.

We now show that P admits a bounded $\mathcal{H}^\infty$-calculus also on $W_q^s(\mathbb{R}^{n-1})$ for all $0 < s < 1$, in particular for $s = 1 - 1/q$. To this end we show the claim for $s = 1$ and use real interpolation method. We will use the fact that $(I - \Delta)^{1/2}$ is a bounded isomorphism from $W_q^1(\mathbb{R}^{n-1})$ to $L_q(\mathbb{R}^{n-1})$ with inverse $(I - \Delta)^{-1/2}$.

Let $\varphi > 0$ and $\Sigma_\varphi := \{z \in \mathbb{C} : |\arg z| < \varphi\}$. Since P admits a bounded $\mathcal{H}^\infty$-calculus on $L_q(\mathbb{R}^{n-1})$, there is a constant $K_\varphi > 0$, such that

$$|h(P)|_{\mathcal{B}(L_q(\mathbb{R}^{n-1});L_q(\mathbb{R}^{n-1}))} \le K_\varphi |h|_{H^\infty(\Sigma_\varphi)}$$

for all $h \in H_0(\Sigma_\varphi)$. Let $u \in C_0^\infty(\mathbb{R}^{n-1})$. Taking Fourier transform just as in the proof of Theorem 6.1.8 in [**57**] gives

$$\mathcal{F}[h(P)u](\xi) = h(\mathcal{P}(\xi))\mathcal{F}u(\xi),$$

where $\mathcal{P}(\xi) = 2|\xi|^2\sqrt{\omega^2 + |\xi|^2}$ is the corresponding symbol of P. Whence we have the representation formula

$$h(P)u = \mathcal{F}^{-1}[h(\mathcal{P}(\xi))\mathcal{F}u]$$

for all $u \in C_0^\infty(\mathbb{R}^{n-1})$, in other words, the symbol of $h(P)$ is in fact $h(\mathcal{P}(\xi))$. Since $h(P)$ and the shift operators $(I - \Delta)^{\pm 1/2}$ commute we easily see that P admits a bounded $\mathcal{H}^\infty$-calculus on $W_q^1(\mathbb{R}^{n-1})$. Indeed,

$$h(P)u = h(P)(I-\Delta)^{-1/2}(I-\Delta)^{1/2}u = (I-\Delta)^{-1/2}h(P)(I-\Delta)^{1/2}u,$$

for all $u \in C_0^\infty(\mathbb{R}^{n-1})$, which implies

$$\begin{aligned}
|h(P)u|_{W_q^1(\mathbb{R}^{n-1})} &= |(I-\Delta)^{-1/2}h(P)(I-\Delta)^{1/2}u|_{W_q^1(\mathbb{R}^{n-1})} \\
&\lesssim |h(P)(I-\Delta)^{1/2}u|_{L_q(\mathbb{R}^{n-1})} \\
&\lesssim |h(P)|_{\mathcal{B}(L_q(\mathbb{R}^{n-1});L_q(\mathbb{R}^{n-1}))}|(I-\Delta)^{1/2}u|_{L_q(\mathbb{R}^{n-1})} \\
&\lesssim |h(P)|_{\mathcal{B}(L_q(\mathbb{R}^{n-1});L_q(\mathbb{R}^{n-1}))}|u|_{W_q^1(\mathbb{R}^{n-1})},
\end{aligned}$$

for all $u \in C_0^\infty(\mathbb{R}^{n-1}), h \in H_0(\Sigma_\varphi)$. By density this also holds true for all $u \in W_q^1(\mathbb{R}^{n-1})$, whence P also admits a bounded $\mathcal{H}^\infty$-calculus on $W_q^1(\mathbb{R}^{n-1})$ and the corresponding constant K'_φ is bounded by a multiple of K_φ. Now by real interpolation method, $W_q^s(\mathbb{R}^{n-1}) = (L_q(\mathbb{R}^{n-1}), W_q^1(\mathbb{R}^{n-1}))_{s,q}$, $s \in (0,1)$, whence

$$|h(P)|_{\mathcal{B}(W_q^s(\mathbb{R}^{n-1});W_q^s(\mathbb{R}^{n-1}))} \lesssim |h(P)|^s_{\mathcal{B}(L_q(\mathbb{R}^{n-1});L_q(\mathbb{R}^{n-1}))}|h(P)|^{1-s}_{\mathcal{B}(W_q^1(\mathbb{R}^{n-1});W_q^1(\mathbb{R}^{n-1}))}$$

for all $h \in H_0(\Sigma_\varphi)$, which implies

$$|h(P)|_{\mathcal{B}(W_q^s(\mathbb{R}^{n-1});W_q^s(\mathbb{R}^{n-1}))} \lesssim |h(P)|_{\mathcal{B}(L_q(\mathbb{R}^{n-1});L_q(\mathbb{R}^{n-1}))} \lesssim K_\varphi |h|_{H^\infty(\Sigma_\varphi)}$$

for all $h \in H_0(\Sigma_\varphi)$. In other words, P also admits a bounded $\mathcal{H}^\infty$-calculus on $W_q^s(\mathbb{R}^{n-1})$ for all $s \in (0,1)$.

The canonical extension to $L_p(\mathbb{R}_+; W_q^s(\mathbb{R}^{n-1}))$, which we will also denote by P, then admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(\mathbb{R}_+; W_q^s(\mathbb{R}^{n-1}))$ for all $0 < s < 1$ with angle zero.

We now apply a version of Dore-Venni theorem, cf. [**52**]. To this end let B be the operator on $L_p(\mathbb{R}_+; W_q^{1-1/q}(\mathbb{R}^{n-1}))$ defined by $B = \frac{d}{dt} + \omega^3$ with domain

$$D(B) = {}_0W_p^1(\mathbb{R}_+; W_q^{1-1/q}(\mathbb{R}^{n-1})).$$

Then B is sectorial and admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(\mathbb{R}_+; W_q^s(\mathbb{R}^{n-1}))$ of angle $\pi/2$, cf. [**57**]. Furthermore, $B : D(B) \to L_p(\mathbb{R}_+; W_q^s(\mathbb{R}^{n-1}))$ is invertible. Let as above P be the operator on $L_p(\mathbb{R}_+; W_q^{1-1/q}(\mathbb{R}^{n-1}))$ with domain

$$D(P) = L_p(\mathbb{R}_+; W_q^{4-1/q}(\mathbb{R}^{n-1})),$$

given by its symbol $2|\xi|^2(\omega^2 + |\xi|^2)^{1/2}$. By the Dore-Venni theorem, the sum $B + P$ with domain $D(B + P) = D(B) \cap D(P)$ is closed, sectorial, and invertible. In other words, the evolution equation $Bu + Pu = f$ posesses for every $f \in L_p(\mathbb{R}_+; W_q^{1-1/q}(\mathbb{R}^{n-1}))$ a unique solution $u \in D(B) \cap D(P)$, hence the proof of maximal regularity is complete. $\square$

2.5.2. Dependence of the maximal regularity constant on the shift parameter. Note that at this point it is a-priori not clear how the maximal regularity constant depends on the shift parameter $\omega > 0$. However, we will need a good understanding of this dependence later on when we want to solve the bent halfspace problems.

We will now introduce suitable ω-dependent norms in both data and solution space and show that the maximal regularity constant is then independent of ω with respect to these norms.

To this end we will proceed with a scaling argument. Fix $\omega > 0$ and let (h, μ) be the solution on $\mathbb{R}_+ \times \mathbb{R}^n_+$ of the ω-shifted half space problem (2.32). Define new functions

$$\tilde{h}(x,t) := \omega^2 h(x/\omega, t/\omega^3), \quad \tilde{\mu}(x,t) := \mu(x/\omega, t/\omega^3), \quad x \in \mathbb{R}^n_+, t \in \mathbb{R}_+.$$

It is then easy to check that $(\tilde{h}, \tilde{\mu})$ solves

$$\begin{aligned} \partial_t \tilde{h} + \tilde{h} + [\![n_\Sigma \cdot \nabla \tilde{\mu}]\!] &= \tilde{g}_1, && \text{on } \Sigma, \\ \tilde{\mu}|_\Sigma + \Delta_{x'} \tilde{h} &= \tilde{g}_2, && \text{on } \Sigma, \\ \tilde{\mu} - \Delta \tilde{\mu} &= \tilde{g}_3, && \text{on } \mathbb{R}^n_+ \backslash \Sigma, \\ e_n \cdot \nabla \tilde{\mu}|_{\partial \mathbb{R}^n_+} &= \tilde{g}_4, && \text{on } \partial \mathbb{R}^n_+, \\ e_n \cdot \nabla_{x'} \tilde{h}|_{\partial \Sigma} &= \tilde{g}_5, && \text{on } \partial \Sigma, \\ \tilde{h}|_{t=0} &= \tilde{h}_0, && \text{on } \Sigma, \end{aligned} \tag{2.40}$$

where

$$\begin{aligned} \tilde{g}_1(x,t) &:= \omega^{-1} g_1(x/\omega, t/\omega^3), \quad \tilde{g}_2(x,t) := g_2(x/\omega, t/\omega^3), \\ \tilde{g}_3(x,t) &:= \omega^{-2} g_3(x/\omega, t/\omega^3), \quad \tilde{g}_4(x,t) := \omega^{-1} g_4(x/\omega, t/\omega^3), \\ \tilde{g}_5(x,t) &:= \omega g_5(x/\omega, t/\omega^3), \quad \tilde{h}_0(x) := \omega^2 h_0(x/\omega), \quad x \in \mathbb{R}^n_+, t \in \mathbb{R}_+. \end{aligned}$$

Since the operator on the left hand side of (2.40) is now independent of ω, we get by the previous theorem that there is some constant $M > 0$ independent of ω, such that

$$|\tilde{h}|_{W^1_p(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; X_1)} + |\tilde{\mu}|_{L_p(\mathbb{R}_+; W^2_q(\mathbb{R}^n_+ \backslash \Sigma))} \tag{2.41}$$

is bounded by

$$\begin{aligned} M \big(&|\tilde{g}_1|_{L_p(\mathbb{R}_+; X_0)} + |\tilde{g}_2|_{L_p(\mathbb{R}_+; W_q^{2-1/q}(\Sigma))} + |\tilde{g}_3|_{L_p(\mathbb{R}_+; L_q(\mathbb{R}^n_+))} + \\ &+ |\tilde{g}_4|_{L_p(\mathbb{R}_+; W_q^{1-1/q}(\partial \mathbb{R}^n_+))} + |\tilde{g}_5|_{F^{1-2/(3q)}_{pq}(\mathbb{R}_+; L_q(\partial \Sigma)) \cap L_p(\mathbb{R}_+; W_q^{3-2/q}(\partial \Sigma))} + |\tilde{h}_0|_{X_\gamma} \big). \end{aligned}$$

Clearly, the ω-dependence is now hidden in the norms. We now have to analyze how the norms transform in ω. Since we need to transform norms of h and μ, we proceed in a general setting as follows. For $\alpha \in \{0, 2\}$, $Z \in \{\mathbb{R}^n_+, \partial\mathbb{R}^n_+, \Sigma, \partial\Sigma\}$, we consider a function $u = u(x,t)$ defined on $Z \times \mathbb{R}_+$ and consider a transformation via

$$\tilde{u}(x,t) := \omega^\alpha u(x/\omega, t/\omega^3), \quad x \in Z, t \in \mathbb{R}^+.$$

Note that all spaces in Z are invariant under this scaling. For $\beta \in \{0,1\}$ and $\gamma \in \mathbb{N}_0^n$, we calculate

$$\partial_t^\beta D_x^\gamma \tilde{u}(x,t) = \omega^{\alpha-3\beta-|\gamma|}\partial_t^\beta D_x^\gamma u(x/\omega, t/\omega^3), \quad x \in Z, t \in \mathbb{R}^+.$$

Hereby D_x denotes the derivative with respect to $x \in Z$. We need this distinction to keep track on how the norm depends on ω with respect to the dimension of Z. We calculate

$$\begin{aligned}
\|\partial_t^\beta D^\gamma \tilde{u}\|_{L^p(0,\infty;L^q(Z))} &= \left(\int_0^\infty \|\partial_t^\beta D^\gamma \tilde{u}(t)\|^p_{L^q(Z)} dt\right)^{1/p} \\
&= \left(\int_0^\infty \left(\int_Z |\partial_t^\beta D^\gamma \tilde{u}(x,t)|^q dx\right)^{p/q} dt\right)^{1/p} \\
&= \omega^{\alpha-3\beta-|\gamma|}\left(\int_0^\infty \left(\int_Z \left|\partial_t^\beta D^\gamma u\left(\frac{x}{\omega}, \frac{t}{\omega^3}\right)\right|^q dx\right)^{p/q} dt\right)^{1/p} \\
&= \omega^{\alpha-3\beta-|\gamma|}\left(\int_0^\infty \left(\int_Z \left|\partial_t^\beta D^\gamma u\left(x, \frac{t}{\omega^3}\right)\right|^q dx\omega^{\dim Z}\right)^{p/q} dt\right)^{1/p} \\
&= \omega^{\alpha-3\beta-|\gamma|+\frac{\dim Z}{q}}\left(\int_0^\infty \left(\int_Z \left|\partial_t^\beta D^\gamma u\left(x, t\right)\right|^q dx\right)^{p/q} dt\omega^3\right)^{1/p} \\
&= \omega^{\alpha-3\beta-|\gamma|+\frac{\dim Z}{q}+\frac{3}{p}}\|\partial_t^\beta D^\gamma u\|_{L^p(0,\infty;L^q(Z))},
\end{aligned}$$

where we used chain rule and transformation formula. For the Sobolev-Slobodeckij seminorms we have, for $\sigma \in (0,1)$,

$$\begin{aligned}
&[\partial_t^\beta D^\gamma \tilde{u}(t)]_{W_q^\sigma(Z)} \\
&= \left(\int_Z\int_Z \frac{|\partial_t^\beta D^\gamma \tilde{u}(x,t) - \partial_t^\beta D^\gamma \tilde{u}(y,t)|^q}{|x-y|^{\dim Z+\sigma q}} dxdy\right)^{1/q} \\
&= \omega^{\alpha-3\beta-|\gamma|}\left(\int_Z\int_Z \frac{|\partial_t^\beta D^\gamma u(\frac{x}{\omega}, \frac{t}{\omega^3}) - \partial_t^\beta D^\gamma u(\frac{y}{\omega}, \frac{t}{\omega^3})|^q}{|x-y|^{\dim Z+\sigma q}} dxdy\right)^{1/q} \\
&= \omega^{\alpha-3\beta-|\gamma|}\left(\int_Z\int_Z \frac{|\partial_t^\beta D^\gamma u(x, \frac{t}{\omega^3}) - \partial_t^\beta D^\gamma u(y, \frac{t}{\omega^3})|^q}{|\omega x-\omega y|^{\dim Z+\sigma q}} dxdy\omega^{2\dim Z}\right)^{1/q} \\
&= \omega^{\alpha-3\beta-|\gamma|+\frac{\dim Z}{q}-\sigma}[\partial_t^\beta D^\gamma u(\frac{t}{\omega^3})]_{W_q^\sigma(Z)}.
\end{aligned}$$

Integration in time gives

$$\begin{aligned}\|[\partial_t^\beta D^\gamma \tilde{u}(t)]_{W_q^\sigma(Z)}\|_{L^p(0,\infty)} &= \omega^{\alpha-3\beta-|\gamma|+\frac{\dim Z}{q}-\sigma}\|[\partial_t^\beta D^\gamma u(\frac{t}{\omega^3})]_{W_q^\sigma(Z)}\|_{L^p(0,\infty)}\\ &= \omega^{\alpha-3\beta-|\gamma|+\frac{\dim Z}{q}-\sigma+\frac{3}{p}}\|[\partial_t^\beta D^\gamma u(t)]_{W_q^\sigma(Z)}\|_{L^p(0,\infty)},\end{aligned}$$

again by transformation formula. Now, we have

$$\begin{aligned}&\|\tilde{h}\|_{W_p^1(0,\infty;X_0)\cap L^p(0,\infty;X_1)} + \|\tilde{\mu}\|_{L^p(0;\infty;W_q^2(\mathbb{R}_+^n\setminus\Sigma))} \simeq\\ &\quad \|\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))} + \|D\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))} + \|D^2\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))}+\\ &\quad + \|D^3\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))} + \|\partial_t\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))} + \|[\partial_t\tilde{h}]_{X_0}\|_{L^p(0,\infty)}+\\ &\quad + \|[D^3\tilde{h}]_{X_0}\|_{L^p(0,\infty)} + \|\tilde{\mu}\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))} + \|D\tilde{\mu}\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))}+\\ &\quad + \|D^2\tilde{\mu}\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))}.\end{aligned}$$

Using the above results now for $\tilde{h}$ and $\tilde{\mu}$, we readily get

$$\begin{aligned}\|\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))} &= \omega^{2+\frac{n-1}{q}+\frac{3}{p}}\|h\|_{L^p(0,\infty;L^q(\Sigma))},\\ \|D\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))} &= \omega^{1+\frac{n-1}{q}+\frac{3}{p}}\|Dh\|_{L^p(0,\infty;L^q(\Sigma))},\\ \|D^2\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))} &= \omega^{\frac{n-1}{q}+\frac{3}{p}}\|D^2h\|_{L^p(0,\infty;L^q(\Sigma))},\\ \|D^3\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))} &= \omega^{-1+\frac{n-1}{q}+\frac{3}{p}}\|D^3h\|_{L^p(0,\infty;L^q(\Sigma))},\\ \|\partial_t\tilde{h}\|_{L^p(0,\infty;L^q(\Sigma))} &= \omega^{-1+\frac{n-1}{q}+\frac{3}{p}}\|\partial_t h\|_{L^p(0,\infty;L^q(\Sigma))},\\ \|[\partial_t\tilde{h}]_{X_0}\|_{L^p(0,\infty)} &= \omega^{-2+\frac{n-1}{q}+\frac{1}{q}+\frac{3}{p}}\|[\partial_t\tilde{h}]_{X_0}\|_{L^p(0,\infty)},\\ \|[D^3\tilde{h}]_{X_0}\|_{L^p(0,\infty)} &= \omega^{-2+\frac{n-1}{q}+\frac{1}{q}+\frac{3}{p}}\|[D^3\tilde{h}]_{X_0}\|_{L^p(0,\infty)},\\ \|\tilde{\mu}\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))} &= \omega^{\frac{n}{q}+\frac{3}{p}}\|\mu\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))},\\ \|D\tilde{\mu}\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))} &= \omega^{-1+\frac{n}{q}+\frac{3}{p}}\|D\mu\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))},\\ \|D^2\tilde{\mu}\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))} &= \omega^{-2+\frac{n}{q}+\frac{3}{p}}\|D^2\mu\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))}.\end{aligned}$$

For the terms on the right hand side we have

$$\|\tilde{g}_1\|_{L^p(0,\infty;X_0)} = \omega^{-1+\frac{n-1}{q}+\frac{3}{p}}\|g_1\|_{L^p(0,\infty;L^q(\Sigma))} + \omega^{-2+\frac{n-1}{q}+\frac{1}{q}+\frac{3}{p}}\|[g_1]_{X_0}\|_{L^p(0,\infty)},$$

as well as

$$\begin{aligned}\|\tilde{g}_2\|_{L^p(0,\infty;W_q^{2-1/q}(\Sigma))} &= \omega^{\frac{n-1}{q}+\frac{3}{p}}\|g_2\|_{L^p(0,\infty;L^q(\Sigma)}\\ &\quad + \omega^{-1+\frac{n-1}{q}+\frac{3}{p}}\|Dg_2\|_{L^p(0,\infty;L^q(\Sigma)}\\ &\quad + \omega^{-2+\frac{1}{q}+\frac{n-1}{q}+\frac{3}{p}}\|[Dg_2]_{X_0}\|_{L^p(0,\infty)},\\ \|\tilde{g}_3\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma))} &= \omega^{-2+\frac{n}{q}+\frac{3}{p}}\|g_3\|_{L^p(0,\infty;L^q(\mathbb{R}_+^n\setminus\Sigma)},\\ \|\tilde{g}_4\|_{L^p(0,\infty;W_q^{1-1/q}(\partial\mathbb{R}_+^n))} &= \omega^{-1+\frac{n-1}{q}+\frac{3}{p}}\|g_4\|_{L^p(0,\infty;L^q(\partial\mathbb{R}_+^n))}\\ &\quad + \omega^{-2+\frac{1}{q}+\frac{n-1}{q}+\frac{3}{p}}\|[g_4]_{X_0}\|_{L^p(0,\infty)}.\end{aligned}$$

For the Triebel-Lizorkin norm on the boundary $\partial\Sigma$ we proceed as follows. Note that since $q<2$, $3-2/q<2$. Moreover, $1-2/(3q)<1$. Hence

$$\begin{aligned}\|\tilde g_5\|_{F_{pq}^{1-\frac{2}{3q}}(0,\infty;L^q(\partial\Sigma))\cap L^p(0,\infty;B_{qq}^{3-\frac{2}{q}}(\partial\Sigma))} &\simeq \|\tilde g_5\|_{L^p(0,T;L^q(\partial\Sigma))}+\|D\tilde g_5\|_{L^p(0,T;L^q(\partial\Sigma))}\\ &+\|[D\tilde g_5]_{W_q^{2-2/q}(\partial\Sigma)}\|_{L^p(0,T)}+[\tilde g_5]_{F_{pq}^{1-\frac{2}{3q}}(0,\infty;L^q(\partial\Sigma))},\end{aligned}$$

where the Triebel-seminorm $[.]_{F_{pq}^{1-\frac{2}{3q}}(0,\infty;L^q(\partial\Sigma))}$ is defined as in (2.10). Now,

$$\begin{aligned}\|\tilde g_5\|_{L^p(0,\infty;L^q(\partial\Sigma))} &= \omega^{1+\frac{n-2}{q}+\frac{3}{p}}\|g_5\|_{L^p(0,\infty;L^q(\partial\Sigma))},\\ \|D\tilde g_5\|_{L^p(0,\infty;L^q(\partial\Sigma))} &= \omega^{\frac{n-2}{q}+\frac{3}{p}}\|Dg_5\|_{L^p(0,\infty;L^q(\partial\Sigma))},\\ \|[D\tilde g_5]_{W_q^{2-2/q}(\partial\Sigma)}\|_{L^p(0,\infty)} &= \omega^{-2+\frac{n}{q}+\frac{3}{p}}\|[Dg_5]_{W_q^{2-2/q}(\partial\Sigma)}\|_{L^p(0,\infty)},\\ [\tilde g_5]_{F_{pq}^{1-\frac{2}{3q}}(0,\infty;L^q(\partial\Sigma))} &= \omega^{-2+\frac{n}{q}+\frac{3}{p}}[g_5]_{F_{pq}^{1-\frac{2}{3q}}(0,\infty;L^q(\partial\Sigma))}.\end{aligned}\tag{2.42}$$

Here, (2.42) follows from the following observations. From (2.10) we get a characterization of the seminorm via differences, namely

$$[\tilde g_5]_{F_{pq}^s(0,\infty;L^q(\partial\Sigma))}=\left\|\left(\int_0^\infty z^{-(s+1)q}\left(\int_{|h|\le z}\|\Delta_h\tilde g_5(.)\|_{L^q(\partial\Sigma)}dh\right)^q\frac{dz}{z}\right)^{1/q}\right\|_{L^p(0,\infty)},$$

where $s=1-\frac{2}{3q}$. Now,

$$\begin{aligned}\|\Delta_h\tilde g_5(t)\|_{L^q(\partial\Sigma)} &= \left(\int_{\partial\Sigma}|\tilde g_5(x,t+h)-\tilde g_5(x,t)|^q dx\right)^{1/q}\\ &= \omega\left(\int_{\partial\Sigma}|g_5(\frac{1}{\omega}x,\frac{1}{\omega^3}(t+h))-g_5(\frac{1}{\omega}x,\frac{1}{\omega^3}t)|^q dx\right)^{1/q}\\ &= \omega^{1+\frac{\dim\partial\Sigma}{q}}\left(\int_{\partial\Sigma}|g_5(x,\frac{1}{\omega^3}t+\frac{1}{\omega^3}h)-g_5(x,\frac{1}{\omega^3}t)|^q dx\right)^{1/q}\\ &= \omega^{1+\frac{n-2}{q}}\|\Delta_{\frac{1}{\omega^3}h}g_5\left(\frac{1}{\omega^3}t\right)\|_{L^q(\partial\Sigma)}.\end{aligned}$$

Furthermore,

$$\int_{|h|\le z}\|\Delta_{\frac{1}{\omega^3}h}g_5\left(\frac{1}{\omega^3}t\right)\|_{L^q(\partial\Sigma)}dh=\omega^3\int_{|h|\le\frac{z}{\omega^3}}\|\Delta_h g_5\left(\frac{1}{\omega^3}t\right)\|_{L^q(\partial\Sigma)}dh.$$

By transformation formula,

$$\begin{aligned}
&\int_0^\infty z^{-(s+1)q-1}\left(\int_{|h|\le z/\omega^3}\|\Delta_h g_5\left(\frac{1}{\omega^3}t\right)\|_{L^q(\partial\Sigma)}dh\right)^q dz\\
&=\omega^3\int_0^\infty(\omega^3 z)^{-(s+1)q-1}\left(\int_{|h|\le z}\|\Delta_h g_5\left(\frac{1}{\omega^3}t\right)\|_{L^q(\partial\Sigma)}dh\right)^q dz\\
&=\omega^{-3(s+1)q}\int_0^\infty z^{-(s+1)q-1}\left(\int_{|h|\le z}\|\Delta_h g_5\left(\frac{1}{\omega^3}t\right)\|_{L^q(\partial\Sigma)}dh\right)^q dz.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&[\tilde g_5]_{F^s_{pq}(0,\infty;L^q(\partial\Sigma))}\\
&=\left\|\left(\int_0^\infty z^{-(s+1)q}\left(\int_{|h|\le z}\|\Delta_h\tilde g_5(.)\|_{L^q(\partial\Sigma)}dh\right)^q\frac{dz}{z}\right)^{1/q}\right\|_{L^p(0,\infty)}\\
&=\omega^{1+\frac{n-2}{q}}\left\|\left(\int_0^\infty z^{-(s+1)q}\left(\int_{|h|\le z}\|\Delta_{h/\omega^3}g_5(\frac{\cdot}{\omega^3})\|_{L^q(\partial\Sigma)}dh\right)^q\frac{dz}{z}\right)^{1/q}\right\|_{L^p(0,\infty)}\\
&=\omega^{4+\frac{n-2}{q}}\left\|\left(\int_0^\infty z^{-(s+1)q}\left(\int_{|h|\le z/\omega^3}\|\Delta_h g_5(\frac{\cdot}{\omega^3})\|_{L^q(\partial\Sigma)}dh\right)^q\frac{dz}{z}\right)^{1/q}\right\|_{L^p(0,\infty)}\\
&=\omega^{4+\frac{n-2}{q}-3(s+1)}\left\|\left(\int_0^\infty z^{-(s+1)q}\left(\int_{|h|\le z}\|\Delta_h g_5(\frac{\cdot}{\omega^3})\|_{L^q(\partial\Sigma)}dh\right)^q\frac{dz}{z}\right)^{1/q}\right\|_{L^p(0,\infty)}\\
&=\omega^{1+(n-2)/q+3/p-3s}\left\|\left(\int_0^\infty z^{-(s+1)q}\left(\int_{|h|\le z}\|\Delta_h g_5(.)\|_{L^q(\partial\Sigma)}dh\right)^q\frac{dz}{z}\right)^{1/q}\right\|_{L^p(0,\infty)}\\
&=\omega^{-2+n/q+3/p}[g_5]_{F^s_{pq}(0,\infty;L^q(\partial\Sigma))},
\end{aligned}$$

which gives (2.42).

Now, $X_\gamma = B^{4-1/q-3/p}_{qp}(\Sigma) = (W^{1-1/q}_q(\Sigma), W^{4-1/q}_q(\Sigma))_{1-1/p,p}$ via the real interpolation method. Recall that $\tilde h_0 = \omega^2(h_0\circ F_\omega)$, where $F_\omega(x) = x/\omega$. Define $T_\omega : W^k_p(\Sigma)\to W^k_p(\Sigma)$, $k\in\mathbb{N}$, by $T_\omega h_0 := \omega^2(h_0\circ F_\omega)$. Then T_ω is a bounded, linear operator for every $k\in\mathbb{N}$ by chain rule, hence by real interpolation method also on every $W^s_p(\Sigma), s>0$. Since X_γ is naturally an interpolation space, there is a constant $C = C(p) > 0$, such that

$$|T_\omega|_{\mathcal{B}(X_\gamma;X_\gamma)} \le C|T_\omega|^{1-p}_{\mathcal{B}(W^{1-1/q}_q(\Sigma);W^{1-1/q}_q(\Sigma))}|T_\omega|^p_{\mathcal{B}(W^{4-1/q}_q(\Sigma);W^{4-1/q}_q(\Sigma))}.$$

Hence

$$|T_\omega h_0|_{X_\gamma} = |\tilde h_0|_{X_\gamma} \le C(p,q,\omega)|h_0|_{X_\gamma}. \tag{2.43}$$

Define $K(\omega) = K(p,q,\omega) := C(p,q,\omega)$ to be the constant from (2.43). We then obtain from (2.41) that

$$\begin{aligned}
&\omega^{4-1/q}|h|_{L_p(\mathbb{R}_+;L_q(\Sigma))} + \omega^{3-1/q}|Dh|_{L_p(\mathbb{R}_+;L_q(\Sigma))} + \omega^{2-1/q}|D^2h|_{L_p(\mathbb{R}_+;L_q(\Sigma))} +\\
&+ \omega^{1-1/q}|D^3h|_{L_p(\mathbb{R}_+;L_q(\Sigma))} + |[\partial_t h]_{X_0}|_{L_p(\mathbb{R}_+)} + |[D^3h]_{X_0}|_{L_p(\mathbb{R}_+)} +\\
&+ \omega^2|\mu|_{L^p(\mathbb{R}_+;L_q(\mathbb{R}^n_+\setminus\Sigma))} + \omega|D\mu|_{L_p(\mathbb{R}_+;L_q(\mathbb{R}^n_+\setminus\Sigma))} + |D^2\mu|_{L_p(\mathbb{R}_+;L_q(\mathbb{R}^n_+\setminus\Sigma))} \leq\\
&\leq M\Big(\omega^{1-1/q}|g_1|_{L_p(\mathbb{R}_+;L_q(\Sigma))} + |[g_1]_{X_0}|_{L_p(\mathbb{R}_+)} + \omega^{2-1/q}|g_2|_{L_p(\mathbb{R}_+;L^q(\Sigma))} +\\
&+ \omega^{1-1/q}|Dg_2|_{L_p(\mathbb{R}_+;L_q(\Sigma))} + |[Dg_2]_{X_0}|_{L_p(\mathbb{R}_+)} + |g_3|_{L_p(\mathbb{R}_+;L_q(\mathbb{R}^n_+\setminus\Sigma))} +\\
&+ \omega^{1-1/q}|g_4|_{L_p(\mathbb{R}_+;L_q(\partial\mathbb{R}^n_+))} + |[g_4]_{W_q^{1-1/q}(\partial\mathbb{R}^n_+)}|_{L_p(\mathbb{R}_+)} +\\
&+ \omega^{3-2/q}|g_5|_{L_p(\mathbb{R}_+,L_q(\partial\Sigma))} + \omega^{2-2/q}|Dg_5|_{L_p(\mathbb{R}_+,L_q(\partial\Sigma))}\\
&+ |[Dg_5]_{W_q^{2-2/q}(\partial\Sigma)}|_{L_p(\mathbb{R}_+)} + [g_5]_{F_{pq}^{1-\frac{2}{3q}}(\mathbb{R}_+;L_q(\partial\Sigma))} + K(\omega)|h_0|_{X_\gamma}.\Big),
\end{aligned}$$

We now define norms as follows. Let

$$\begin{aligned}
|h|_{E,1,\omega} &:= \omega^{4-1/q}|h|_{L_p(\mathbb{R}_+;L^q(\Sigma))} + \omega^{3-1/q}|Dh|_{L_p(\mathbb{R}_+;L_q(\Sigma))}\\
&\quad + \omega^{2-1/q}|D^2h|_{L_p(\mathbb{R}_+;L_q(\Sigma))} + \omega^{1-1/q}|D^3h|_{L_p(\mathbb{R}_+;L_q(\Sigma))}\\
&\quad + |[\partial_t h]_{X_0}|_{L_p(\mathbb{R}_+)} + |[D^3h]_{X_0}|_{L_p(\mathbb{R}_+)},\\
|\mu|_{E,2,\omega} &:= \omega^2|\mu|_{L^p(\mathbb{R}_+;L_q(\mathbb{R}^n_+\setminus\Sigma))} + \omega|D\mu|_{L_p(\mathbb{R}_+;L_q(\mathbb{R}^n_+\setminus\Sigma))} + |D^2\mu|_{L_p(\mathbb{R}_+;L_q(\mathbb{R}^n_+\setminus\Sigma))},\\
|g_1|_{F,1,\omega} &:= \omega^{1-1/q}|g_1|_{L_p(\mathbb{R}_+;L_q(\Sigma))} + |[g_1]_{X_0}|_{L_p(\mathbb{R}_+)},\\
|g_2|_{F,2,\omega} &:= \omega^{2-1/q}|g_2|_{L_p(\mathbb{R}_+;L^q(\Sigma))} + \omega^{1-1/q}|Dg_2|_{L_p(\mathbb{R}_+;L_q(\Sigma))} + |[Dg_2]_{X_0}|_{L_p(\mathbb{R}_+)},\\
|g_3|_{F,3,\omega} &:= |g_3|_{L_p(\mathbb{R}_+;L_q(\mathbb{R}^n_+\setminus\Sigma))},\\
|g_4|_{F,4,\omega} &:= \omega^{1-1/q}|g_4|_{L_p(\mathbb{R}_+;L_q(\partial\mathbb{R}^n_+))} + |[g_4]_{W_q^{1-1/q}(\partial\mathbb{R}^n_+)}|_{L_p(\mathbb{R}_+)},\\
|g_5|_{F,5,\omega} &:= \omega^{3-2/q}|g_5|_{L_p(\mathbb{R}_+,L_q(\partial\Sigma))} + \omega^{2-2/q}|Dg_5|_{L_p(\mathbb{R}_+,L_q(\partial\Sigma))}\\
&\quad + |[Dg_5]_{W_q^{2-2/q}(\partial\Sigma)}|_{L_p(\mathbb{R}_+)} + [g_5]_{F_{pq}^{1-\frac{2}{3q}}(\mathbb{R}_+;L_q(\partial\Sigma))},
\end{aligned}$$

and $|h_0|_{F,6,\omega} := K(\omega)|h_0|_{X_\gamma}$. This way we obtain that $|h|_{E,1,\omega} + |\mu|_{E,2,\omega}$ is bounded by

$$M(|g_1|_{F,1,\omega} + |g_2|_{F,2,\omega} + |g_3|_{F,3,\omega} + |g_4|_{F,4,\omega} + |g_5|_{F,5,\omega} + |h_0|_{F,6,\omega}),$$

where we point out that $M > 0$ is independent of $\omega > 0$. Note that this estimate also holds true on bounded intervals $J = (0,T) \subset \mathbb{R}_+$, as can be seen as follows. First again reduce to trivial initial data as in the proof of Theorem 2.18. Then we can simply extend the data (g_1, g_2, g_3, g_4) to the half line $\mathbb{R}_+$ by zero. Regarding g_5 we note that after the reduction procedure, $g_5|_{t=0} = 0$, whence we may use Section 3.4.3 in [**62**] and Corollary 5.12 in [**35**] to extend g_5 to a function on the half line $\mathbb{R}_+$. Then on J the same estimate holds true if we replace M by $2M$.

2.5.3. Bent half space problems. In this section we consider the shifted model problem (2.32) on a bent half space $\mathbb{R}^n_\gamma := \{x \in \mathbb{R}^n : x_n > \gamma(x_1, ..., x_{n-1})\}$, where $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ is a sufficiently smooth function with sufficiently small norm in $C^1(\mathbb{R}^{n-1})$. Since also the reference surface may be curved, we consider a slightly bent interface $\Sigma_\beta := \{x \in \overline{\mathbb{R}^n_\gamma} : x_1 = \beta(x_2, ..., x_n)\}$. Again, $\beta : \mathbb{R}^{n-1} \to \mathbb{R}$ is suitably smooth and the $C^1(\mathbb{R}^{n-1})$-norm is sufficiently small. The bent half space problem reads as

$$\begin{aligned}
\partial_t h + \omega^3 h + [\![n_{\Sigma_\beta} \cdot \nabla\mu]\!] &= g_1, && \text{on } \Sigma_\beta,\\
\mu|_{\Sigma_\beta} + \Delta_{\Sigma_\beta} h &= g_2, && \text{on } \Sigma_\beta,\\
\omega^2\mu - \Delta_x\mu &= g_3, && \text{on } \mathbb{R}^n_\gamma \backslash \Sigma_\beta,\\
n_\gamma \cdot \nabla\mu|_{\partial\mathbb{R}^n_\gamma} &= g_4, && \text{on } \partial\mathbb{R}^n_\gamma,\\
n_\gamma \cdot \nabla_{\Sigma_\beta} h|_{\partial\Sigma_\beta} &= g_5, && \text{on } \partial\Sigma_\beta,\\
h|_{t=0} &= h_0, && \text{on } \Sigma_\beta,
\end{aligned}$$

where n_γ denotes the outer unit normal of $\mathbb{R}^n_\gamma$. The smallness assumption on $|\beta|_{C^1} + |\gamma|_{C^1}$ implies that the bent domain and interface are only a small perturbation of the half space and the flat interface. We will now solve this problem on the bent half space by transforming it back to the regular half space.

LEMMA 2.19. *Let $k \in \mathbb{N}$ and $\beta, \gamma \in C^k(\mathbb{R}^{n-1})$. Then there is some $F \in C^k(\mathbb{R}^n; \mathbb{R}^n)$, such that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^k-diffeomorphism and such that additionally the restriction $F|_{\mathbb{R}^n_\gamma} : \mathbb{R}^n_\gamma \to \mathbb{R}^n_+$ is a C^k-diffeomorphism as well. Furthermore, F maps Σ_β to the flat interface $\overline{\mathbb{R}^n_+} \cap \{x_1 = 0\}$. We also have that $|I - DF|_{C^l(\mathbb{R}^n)} \lesssim |\beta|_{C^{l+1}(\mathbb{R}^{n-1})} + |\gamma|_{C^{l+1}(\mathbb{R}^{n-1})}$, for all $l = 0, ..., k-1$.*

PROOF. To economize notation, let $n = 3$. We first transform in x_3-direction via $\Phi_1 : \mathbb{R}^3 \to \mathbb{R}^3, x \mapsto (x_1, x_2, x_3 - \gamma(x_1, x_2))$. It is then easy to see that the surface $\Phi_1(\Sigma_\beta)$ is given by the set

$$\Phi_1(\Sigma_\beta) = \{(\beta(x_2, x_3), x_2, x_3 - \gamma(\beta(x_2, x_3), x_2)) : x_2 \in \mathbb{R}\} \cap \mathbb{R}^3_+.$$

Note that this is equivalent to

$$\Phi_1(\Sigma_\beta) = \{(\beta(x_2, x_3), H(x_2, x_3)) : (x_2, x_3) \in \mathbb{R}^2\} \cap \mathbb{R}^3_+,$$

where

$$H : \mathbb{R}^2 \to \mathbb{R}^2, (x_2, x_3) \mapsto (x_2, x_3 - \gamma(\beta(x_2, x_3), x_2)).$$

Now note that whenever $|(\beta, \gamma)|_{C^1}$ is sufficiently small, $|H - \mathrm{id}_{\mathbb{R}^2}|_{C^1}$ is small. Then $|\det DH| \geq 1/2$ on $\mathbb{R}^2$ and $H : \mathbb{R}^2 \to \mathbb{R}^2$ is globally invertible. Hence the surface $\Phi_1(\Sigma_\beta)$ can be parametrized by $\beta \circ H^{-1}$,

$$\Phi_1(\Sigma_\beta) = \{(\beta(H^{-1}(x_2, x_3)), x_2, x_3) : (x_2, x_3) \in \mathbb{R}^2\} \cap \mathbb{R}^3_+.$$

Note that by the inverse function theorem, H^{-1} is $C^1(\mathbb{R}^2, \mathbb{R}^2)$. Then we transform via $\Phi_2 : \mathbb{R}^3 \to \mathbb{R}^3, x \mapsto (x_1 - \beta \circ H^{-1}(x_2, x_3), x_2, x_3)$. We easily check that $F := \Phi_2\Phi_1$ satisfies the desired properties. $\square$

To pull back the equations to the regular upper half space $\mathbb{R}^n_+ \backslash \Sigma$, we define now new functions G_1, G_2, G_3, G_4, G_5 and G_6 via

$$g_j(x, t) = G_j(t, F(x)),\ j = 1, ..., 5, \quad h_0(x) = G_6(F(x)), \quad x \in \mathbb{R}^n_\gamma, t \in \mathbb{R}_+,$$

cf. [33]. We also introduce $(\bar{h},\bar{\mu}) := (h,\mu)\circ F^{-1}$. This way, the functions $(\bar{h},\bar{\mu})$ are defined on the regular upper half space. Then the problem for (h,μ) is equivalent to the upper half space problem for $(\bar{h},\bar{\mu})$ reading as

$$\begin{aligned}
\partial_t\bar{h} + \omega^3\bar{h} + [\![n_\Sigma\cdot\nabla\bar{\mu}]\!] &= \mathcal{B}_1(\bar{\mu})\circ F^{-1} + G_1, && \text{on } \Sigma,\\
\bar{\mu}|_\Sigma + \Delta_\Sigma\bar{h} &= \mathcal{B}_2(\bar{h})\circ F^{-1} + G_2, && \text{on } \Sigma,\\
\omega^2\bar{\mu} - \Delta_x\bar{\mu} &= \mathcal{B}_3(\bar{\mu})\circ F^{-1} + G_3, && \text{on } \mathbb{R}^n_+\backslash\Sigma,\\
e_n\cdot\nabla\bar{\mu}|_{\partial\mathbb{R}^n} &= \mathcal{B}_4(\bar{\mu})\circ F^{-1} + G_4, && \text{on } \partial\mathbb{R}^n_+,\\
e_n\cdot\nabla_\Sigma\bar{h}|_{\partial\Sigma} &= \mathcal{B}_5(\bar{h})\circ F^{-1} + G_5, && \text{on } \partial\Sigma,\\
\bar{h}|_{t=0} &= G_6, && \text{on } \Sigma,
\end{aligned}$$

where the perturbation operators are given by

$$\begin{aligned}
\mathcal{B}_1(\bar{\mu}) &= [\![n_{\Sigma_\beta}\cdot\nabla(\bar{\mu}\circ F)]\!] - [\![(n_\Sigma\circ F)\cdot(\nabla\bar{\mu}\circ F)]\!],\\
\mathcal{B}_2(\bar{h}) &= \Delta_{\Sigma_\beta}(\bar{h}\circ F) - \Delta_\Sigma\bar{h}\circ F,\\
\mathcal{B}_3(\bar{\mu}) &= \Delta_x(\bar{\mu}\circ F) - \Delta\bar{\mu}\circ F,\\
\mathcal{B}_4(\bar{\mu}) &= e_n\cdot(\nabla\bar{\mu}\circ F) - n_\gamma\cdot\nabla(\bar{\mu}\circ F),\\
\mathcal{B}_5(\bar{h}) &= e_n\cdot(\nabla_\Sigma\bar{h}\circ F) - n_\gamma\cdot\nabla_{\Sigma_\beta}(\bar{h}\circ F).
\end{aligned}$$

Define $\mathcal{B} := (\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,0)$. We will now show that the operator norm of $\mathcal{B}$ is as small as we like in terms of the ω-dependent norms by choosing $\omega > 0$ large enough and the time interval and $|\beta|_{C^1} + |\gamma|_{C^1}$ small enough. By a reduction argument, we can again reduce to the case where $G_6 = 0$.

Let

$$ {}_0\mathbb{E}(T) := [{}_0W^1_p(0,T;X_0)\cap L_p(0,T;X_1)]\times L_p(0,T;W^2_q(\mathbb{R}^n_+\backslash\Sigma), \tag{2.44}$$

and

$$\begin{aligned}
{}_0\mathbb{F}(T) := L_p(0,T;X_0)\times L_p(0,T;W^{2-1/q}_q(\Sigma))\times L_p(0,T;L_q(\mathbb{R}^n_+))\times \\
\times L_p(0,T;W^{1-1/q}_q(\partial\mathbb{R}^n_+))\times \\
\times [{}_0F^{1-2/(3q)}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;W^{3-2/q}_q(\partial\Sigma))]\times X_\gamma.
\end{aligned} \tag{2.45}$$

We equip ${}_0\mathbb{E}(T)$ and ${}_0\mathbb{F}(T)$ with the ω-weighted norms of Section 2.5.2. Then we can show the following estimate.

Lemma 2.20. *There is some small $\alpha = \alpha(p) > 0$, such that*

$$|\mathcal{B}|_{\mathcal{B}({}_0\mathbb{E}(T);{}_0\mathbb{F}(T))} \le C(\beta,\gamma)(\omega^{-1/q}+\omega^{-1}) + \varepsilon C(\omega,\beta,\gamma,F) + T^\alpha C(\omega,\beta,\gamma) \tag{2.46}$$

for some constants $C(\beta,\gamma), C(\omega,\beta,\gamma,F), C(\omega,\beta,\gamma) > 0$, whenever $|\beta|_{C^1}+|\gamma|_{C^1}\le\varepsilon$. Note that by first choosing $\omega > 0$ sufficiently large and then $\varepsilon > 0$ and $T > 0$ sufficiently small, the right hand side gets as small as we like.

Hereby, the norm in $\mathcal{B}({}_0\mathbb{E}(T);{}_0\mathbb{F}(T))$ in (2.46) is taken with respect to the ω-weighted norms of Section 2.5.2.

Let us postpone the proof of the estimate to a later point. Having this estimate at hand, we can show maximal regularity for the bent half space problem by a Neumann series argument.

THEOREM 2.21. *Let β, γ be smooth curves. Then there exists some possibly large $\omega_0 > 0$, some small $T > 0$, and some small $\varepsilon > 0$, such that if $\omega \geq \omega_0$, $|\beta|_{C^1} + |\gamma|_{C^1} \leq \varepsilon$, the bent half space problem has maximal $L_p - L_q$-regularity. To be more precise, this means that if we replace Σ by Σ_β and $\mathbb{R}^n_+$ by $\mathbb{R}^n_\gamma$ in (2.44) and (2.45), there is for every $(g_1, g_2, g_3, g_4, g_5, 0) \in \mathbb{F}(T)$ a unique solution $(h, \mu) \in {}_0\mathbb{E}(T)$. Furthermore, $|h|_{E,1,\omega} + |\mu|_{E,2,\omega}$ is bounded by*

$$2M(|g_1|_{F,1,\omega} + |g_2|_{F,2,\omega} + |g_3|_{F,3,\omega} + |g_4|_{F,4,\omega} + |g_5|_{F,5,\omega}),$$

where $M > 0$ is as in (2.41) and in particular independent of ω.

PROOF. Denote by L_ω be the linear operator defined by the left hand side of (2.40). Let $M > 0$ be as in (2.41). By (2.46) there exist some $\omega_0 > 0$, $\varepsilon > 0$, and $T_0 > 0$, such that

$$|\mathcal{B}|_{\mathcal{B}(\mathbb{E}(T_0);\mathbb{F}(T_0))} \leq \frac{1}{2M}.$$

Then a Neumann series argument shows that $L_\omega + \mathcal{B} = L_\omega(I + L_\omega^{-1}\mathcal{B})$ is invertible between the spaces equipped with the (ω-weighted) norms, since

$$|L_\omega^{-1}\mathcal{B}\ |_{\mathcal{B}(\mathbb{E}(T_0);\mathbb{E}(T_0))} \leq |L_\omega^{-1}\ |_{\mathcal{B}(\mathbb{F}(T_0);\mathbb{E}(T_0))}|\mathcal{B}\ |_{\mathcal{B}(\mathbb{E}(T_0);\mathbb{F}(T_0))} \leq M\frac{1}{2M} = \frac{1}{2}.$$

Furthermore,

$$|(L_\omega + \mathcal{B})^{-1}|_{\mathcal{B}(\mathbb{E}(T_0);\mathbb{E}(T_0))} \leq |L_\omega^{-1}\ |_{\mathcal{B}(\mathbb{F}(T_0);\mathbb{E}(T_0))} \sum_{k=0}^{\infty} |L_\omega^{-1}\mathcal{B}\ |^k_{\mathcal{B}(\mathbb{E}(T_0);\mathbb{E}(T_0))} \leq 2M,$$

where the right hand side is independent of ω. This way, we can draw back the bent half space problem to $\mathbb{R}^n_+\backslash\Sigma$ and the theorem is proven. $\square$

It remains to show Lemma 2.20.

PROOF OF LEMMA 2.20. We first need to write each $\mathcal{B}_i$ in such a way that we can give suitable estimates. For convenience we drop the bars. Now, for $x \in \mathbb{R}^n_\gamma$,

$$\begin{aligned}&[\![n_\beta \cdot \nabla(\mu \circ F)]\!] - [\![n_\Sigma(F) \cdot \nabla\mu(F)]\!]\\ &\quad = [\![(n_\beta - n_\Sigma \circ F) \cdot \nabla\mu \circ F]\!] + [\![n_\beta \cdot (DF^T - I)\nabla\mu(F)]\!].\end{aligned}$$

For the Laplacian on $\mathbb{R}^n$, a straightforward calculation using chain rule entails

$$\Delta(\mu \circ F) = \sum_{j,k,l=1}^{n} (\partial_j\partial_l\mu \circ F)\partial_k F_j \partial_k F_l + \sum_{k,l=1}^{n} (\partial_l\mu \circ F)\partial_k\partial_k F_l.$$

Therefore,

$$\Delta(\mu \circ F) - \Delta\mu \circ F = \sum_{j,l=1}^{n} (\partial_j\partial_l\mu \circ F)[\sum_{k=1}^{n} \partial_k F_j \partial_k F_l - \delta_{jl}] + \sum_{k,l=1}^{n} (\partial_l\mu \circ F)\partial_k\partial_k F_l.$$

Let us briefly recall the definition of surface gradient and Laplace-Beltrami operator, see Section 2.1 in [**57**]. For a sufficiently regular scalar valued function $f : \Sigma_\beta \to \mathbb{R}$, we define the surface gradient in local coordinates via

$$\nabla_{\Sigma_\beta} f = (\nabla_{\Sigma_\beta} f)_i, \quad (\nabla_{\Sigma_\beta} f)_i = \sum_{j=1}^{n-1} g^{ij}\partial_j f,$$

where $g = (g_{ij})_{ij}$ denotes the first fundamental form and $(g^{ij})_{ij}$ its inverse with respect to the surface Σ_β. For a given sufficiently smooth vector field, say, $v : \Sigma_\beta \to \mathbb{R}^{n-1}$, we define the surface divergence in local coordinates by

$$\operatorname{div}_{\Sigma_\beta} v = \sum_{i=1}^{n-1} \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} v_i \right).$$

Now, we can define the Laplace-Beltrami operator in local coordinates,

$$\Delta_{\Sigma_\beta} := \operatorname{div}_{\Sigma_\beta} \nabla_{\Sigma_\beta}.$$

Hence

$$\Delta_{\Sigma_\beta} = \sum_{i,j=2}^{n-1} g^{ij} \partial_i \partial_j + \frac{1}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} g^{ij} \right) \partial_j,$$

in local coordinates. Let us now deduce a formula for g_{ij} and g^{ij} in terms of β. Since Σ_β is given by a parametrization,

$$\mathbb{R}^{n-1} \to \mathbb{R}^n, \ (x_2, ..., x_n) \mapsto (\beta(x_2, ..., x_n), x_2, ..., x_n),$$

we have

$$g_{ij} = \delta_{ij} + \partial_i \beta \partial_j \beta, \quad 2 \leq i, j \leq n.$$

Therefore,

$$g = (g_{ij})_{ij} = I_{n-1} + (\partial_i \beta \partial_j \beta)_{ij},$$

where I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix. Now, a straightforward application of chain rule gives

$$\partial_j (h \circ F) = \sum_{l=2}^{n} (\partial_l h \circ F) \partial_j F_l,$$

as well as

$$\partial_i \partial_j (h \circ F) = \sum_{k,l=2}^{n} (\partial_l \partial_k h \circ F) \partial_i F_k \partial_j F_l + \sum_{l=2}^{n} (\partial_l h \circ F) \partial_i \partial_j F_l.$$

Again by chain rule,

$$\partial_i \left(\sqrt{\det g} g^{ij} \right) = \frac{1}{2} \frac{1}{\sqrt{\det g}} \partial_i (\det g) g^{ij} + \sqrt{\det g} \partial_i g^{ij}.$$

Altogether,

$$\Delta_{\Sigma_\beta} (h \circ F) = \sum_{i,j,k,l=2}^{n} g^{ij} (\partial_k \partial_l h \circ F) \partial_i F_k \partial_j F_l + \sum_{l=2}^{n} \partial_l (h \circ F) T_l,$$

where

$$T_l = \sum_{i,j=2}^{n} g^{ij} \partial_i \partial_j F_l + \left(\frac{1}{2} \frac{\partial_i (\det g)}{\sqrt{\det g}} g^{ij} + \partial_i g^{ij} \right) \partial_j F_l.$$

Altogether,

$$\Delta_{\Sigma_\beta} (h \circ F) - \Delta_\Sigma h \circ F = \sum_{k,l=2}^{n} (\partial_k \partial_l h \circ F) S_{k,l} + \sum_{l=2}^{n} (\partial_l h \circ F) T_l,$$

where

$$S_{k,l} = \left(\sum_{i,j=2}^{n} g^{ij} \partial_i F_k \partial_j F_l \right) - \delta_{kl}.$$

Let us discuss the boundary equations. We clearly have

$$\begin{aligned} e_n \cdot & (\nabla \mu \circ F) - n_\gamma \cdot (DF^T (\nabla \mu \circ F)) \\ & = (e_n - n_\gamma) \cdot (\nabla \mu \circ F) + n_\gamma \cdot ((DF^T - I)(\nabla \mu \circ F)). \end{aligned}$$

Furthermore,

$$\nabla_{\Sigma_\beta} (h \circ F) = \sum_{j=1}^{n-1} g^{ij} \partial_j (h \circ F) = \sum_{j,l=1}^{n-1} g^{ij} (\partial_l h \circ F) \partial_j F_l.$$

Regarding the boundary condition for h we note that

$$\begin{aligned} e_n \cdot & (\nabla_\Sigma h \circ F) - n_\gamma \cdot \nabla_{\Sigma_\beta} (h \circ F) \\ & = (e_n - n_\gamma) \cdot (\nabla_\Sigma h \circ F) + n_\gamma \cdot (\nabla_\Sigma h \circ F - \nabla_{\Sigma_\beta} (h \circ F)), \end{aligned}$$

By comparing components,

$$[\nabla_{\Sigma_\beta} (h \circ F) - \nabla_\Sigma h \circ F]_i = \sum_{j=1}^{n-1} g^{ij} \sum_{l=1}^{n-1} (\partial_l h \circ F) \partial_j F_l - \partial_i h \circ F \tag{2.47}$$

for every $i = 1, ..., n-1$. Now,

$$\partial_i h \circ F = \sum_{j,l=1}^{n-1} \delta_{ij} \delta_{jl} (\partial_l h \circ F).$$

Hence (2.47) is equivalent to

$$\sum_{j=1}^{n-1} \sum_{l=1}^{n-1} \left\{ g^{ij} \partial_j F_l - \delta_{ij} \delta_{jl} \right\} (\partial_l h \circ F).$$

Conclusively, in short notation,

$$\begin{aligned} \mathcal{B}_1(\mu) &= [\![(n_\beta - n_\Sigma) \cdot \nabla \mu]\!] + [\![n_\beta \cdot ((DF^T - I) \nabla \mu)]\!], \\ \mathcal{B}_2(h) &= \sum_{k,l=2}^{n} \partial_k \partial_l h S_{k,l} + \sum_{l=2}^{n} \partial_l h T_l, \\ \mathcal{B}_3(\mu) &= \sum_{l,j=1}^{n} \partial_j \partial_l \mu \left(\sum_{k=1}^{n} \partial_k F_j \partial_k F_l - \delta_{jl} \right) + \sum_{l=1}^{n} \partial_l \mu \left(\sum_{k=1}^{n} \partial_k \partial_k F_l \right), \\ \mathcal{B}_4(\mu) &= (e_n - n_\gamma) \cdot \nabla \mu + n_\gamma \cdot ((DF^T - I) \nabla \mu), \\ \mathcal{B}_5(h) &= (e_n - n_\gamma) \cdot \nabla_\Sigma h + n_\gamma \cdot (\nabla_\Sigma h \circ F - \nabla_{\Sigma_\beta} (h \circ F)). \end{aligned}$$

We estimate each $\mathcal{B}_i$ seperately. We start with $\mathcal{B}_1$. Firstly,

$$\begin{aligned} |\mathcal{B}_1(\mu)|_{\omega,T} &= \omega^{1-1/q} \|\mathcal{B}_1(\mu)\|_{L^p(0,T;L^q(\Sigma))} + \|[\mathcal{B}_1(\mu)]_{X_0}\|_{L^p(0,T)} \\ &\le 2\omega^{1-1/q} \|\mathcal{B}_1(\mu)\|_{L^p(0,T;X_0)}. \end{aligned}$$

Extend the normal vector fields n_β, n_Σ defined on Σ to functions $\tilde{n}_\beta, \tilde{n}_\Sigma$ on $\mathbb{R}^n_+\backslash\Sigma$ by a bounded extension operator, cf. [62]. Then

$$\begin{aligned}\|\mathcal{B}_1(\mu)\|_{L^p(0,T;X_0)} \leq{}& |(\tilde{n}_\beta - \tilde{n}_\Sigma)\cdot\nabla\mu|_{L_p(0,T;W^1_q(\mathbb{R}^n_+\backslash\Sigma))} \\ &+ |\tilde{n}_\beta\cdot((DF^T - I)\nabla\mu)|_{L_p(0,T;W^1_q(\mathbb{R}^n_+\backslash\Sigma))}.\end{aligned}$$

Hence

$$\begin{aligned}|\mathcal{B}_1(\mu)|_{\omega,T} \lesssim{}& \omega^{1-1/q}|(\tilde{n}_\beta - \tilde{n}_\Sigma)\cdot\nabla\mu|_{L_p(0,T;L_q(\mathbb{R}^n_+\backslash\Sigma))}+ \\ &+\omega^{1-1/q}|\nabla(\tilde{n}_\beta - \tilde{n}_\Sigma)\cdot\nabla\mu|_{L_p(0,T;L_q(\mathbb{R}^n_+\backslash\Sigma))}+ \\ &+\omega^{1-1/q}|(\tilde{n}_\beta - \tilde{n}_\Sigma)\cdot\nabla^2\mu|_{L_p(0,T;L_q(\mathbb{R}^n_+\backslash\Sigma))}+ \\ &+\omega^{1-1/q}|\tilde{n}_\beta\cdot((DF^T - I)\nabla\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+\backslash\Sigma))}+ \\ &+\omega^{1-1/q}|\nabla\tilde{n}_\beta\cdot((DF^T - I)\nabla\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+\backslash\Sigma))}+ \\ &+\omega^{1-1/q}|\tilde{n}_\beta\cdot(\nabla(DF^T - I)\nabla\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+\backslash\Sigma))}+ \\ &+\omega^{1-1/q}|\tilde{n}_\beta\cdot((DF^T - I)\nabla^2\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+\backslash\Sigma))}.\end{aligned}$$

The right hand side can be controlled by

$$\begin{aligned}&\omega^{1-1/q}|\nabla\mu|_{L_p(L_q(\mathbb{R}^n_+\backslash\Sigma))}\left(|\tilde{n}_\beta - \tilde{n}_\Sigma|_{C^1} + |\tilde{n}_\beta|_{C^1}\right)\left(1 + |DF - I|_{L^\infty} + |D^2F|_{L^\infty}\right) \\ &+\omega^{1-1/q}|\nabla^2\mu|_{L_p(L_q(\mathbb{R}^n_+\backslash\Sigma))}\left(|\tilde{n}_\beta - \tilde{n}_\Sigma|_{L^\infty} + |\tilde{n}_\beta|_{L^\infty}|DF - I|_{L^\infty}\right).\end{aligned}$$

It easily follows that

$$\begin{aligned}|\mathcal{B}_1(\mu)|_{\omega,T} \lesssim{}& \omega^{-1/q}\left(1 + |\tilde{\gamma}|_{C^2} + |\gamma|_{C^2}\right)|\mu|_{1,\omega,T} \\ &+ C(\omega)|\mu|_{1,\omega,T}\left(|\beta|_{C^1} + |\gamma|_{C^1}\right)\left(1 + |\beta|_{C^1}\right).\end{aligned}$$

Now we concern $\mathcal{B}_2$. Firstly,

$$|\mathcal{B}_2(h)|_{\omega,T} \lesssim \lambda^{2-1/q}|\mathcal{B}_2(h)|_{L^p(0,T;W^{2-1/q}_q(\Sigma)}.$$

Furthermore,

$$\begin{aligned}|\mathcal{B}_2(h)|_{L^p(0,T;W^{2-1/q}_q(\Sigma))} \leq{}& \sum_{k,l}|\partial_k\partial_l hS_{kl}|_{L^p(0,T;W^{2-1/q}_q(\Sigma))} \\ &+ \sum_l |\partial_l hT_l|_{L^p(0,T;W^{2-1/q}_q(\Sigma))}.\end{aligned}$$

Recall that for all $s > 0, 1 \leq p_1, r_1 \leq \infty$,

$$\|fg\|_{B^s_{p_1,r_1}} \lesssim \|f\|_{B^s_{p_1,r_1}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{B^s_{p_1,r_1}}$$

for all $f, g \in B^s_{p_1 r_1} \cap L^\infty$, see Lemma 1.5. Note that S_{kl} is independent of time. Hence

$$\begin{aligned}\|\partial_k\partial_l hS_{kl}\|_{L^p(0,T;W^{2-1/q}_q(\Sigma))} \lesssim{}& \|\partial_k\partial_l h\|_{L^p(0,T;W^{2-1/q}_q(\Sigma))}\|S_{kl}\|_{L^\infty(\Sigma)} \\ &+ \|S_{kl}\|_{W^{2-1/q}_q(\Sigma)}\|\partial_k\partial_l h\|_{L^p(0,T;L^\infty(\Sigma))}.\end{aligned}$$

Let us analyse the terms S_{kl}. We can write

$$S_{kl} = \begin{cases} (g^{kk}-1)\partial_k F_k \partial_k F_k + (\partial_k F_k - 1)(\partial_k F_k + 1) + \\ \quad + \sum_{j\neq k} g^{jj}\partial_j F_k \partial_j F_k + \sum_{j\neq j'} g^{jj'}\partial_j F_k \partial_{j'} F_k, & k = l, \\ \sum_{i,j}(g^{ij}-\delta_{ij})\partial_i F_k \partial_j F_l + \sum_j \partial_j F_k \partial_j F_l, & k \neq l. \end{cases}$$

If $k = l$, we get

$$\begin{aligned} \|S_{kk}\|_{L^\infty} \leq \|g^{kk}-1\|_{L^\infty}\|DF\|^2_{L^\infty} + \|DF - I\|_{L^\infty}\|DF + I\|_{L^\infty} + \\ + \sum_{j\neq k}\|g^{jj}\|_{L^\infty}\|\partial_j F_k\|^2_{L^\infty} + \sum_{j'\neq j}\|g^{j'j}\|_{L^\infty}\|\partial_{j'} F_k\|_{L^\infty}\|\partial_j F_k\|_{L^\infty}. \end{aligned} \tag{2.48}$$

Clearly,

$$\|g_{ij} - \delta_{ij}\|_{L^\infty} + \|g^{ij} - \delta_{ij}\|_{L^\infty} \lesssim \|\beta\|_{C^1}$$

for all $1 \leq i, j \leq n-1$. In the first sum in (2.48), $j \neq k$, hence

$$\|\partial_j F_k\|_{L^\infty} \leq \|DF - I\|_{L^\infty}.$$

In the second sum in (2.48), either j or j' is not equal to k, hence

$$\|\partial_{j'} F_k\|_{L^\infty}\|\partial_j F_k\|_{L^\infty} \leq \|DF - I\|_{L^\infty}\|DF\|_{L^\infty}.$$

If now $k \neq l$,

$$\|S_{kl}\|_{L^\infty} \leq \sum_{i,j}\|g^{ij} - \delta_{ij}\|_{L^\infty}\|DF\|^2_{L^\infty} + \sum_j \|\partial_j F_k\|_{L^\infty}\|\partial_j F_l\|_{L^\infty}.$$

In the second sum j can not be k and l at the same time, hence

$$\|S_{kl}\|_{L^\infty} \lesssim \left(1 + \|DF\|_{L^\infty} + \|DF\|^2_{L^\infty}\right)\|\beta\|_{C^1}.$$

Since the first fundamental form g and DF depend only on at most one derivative of β and γ,

$$\|S_{kl}\|_{W^{2-1/q}_q} \lesssim \|S_{kl}\|_{C^2} \lesssim \|\beta\|_{C^3} + \|\gamma\|_{C^3}.$$

By Hölder inequality,

$$|\partial_k\partial_l h|_{L^p(0,T;L^\infty(\Sigma))} \leq T^{1/p}|h|_{L^\infty(0,T;X_\gamma)} \lesssim T^{1/p}|h|_{L^p(0,T;X_1)\cap W^1_p(0,T;X_0)},$$

since $h \in \mathbb{E}(T)$. By paraproduct estimates, cf. Lemma 1.5,

$$|\partial_l h(t) T_l|_{W^{2-1/q}_q(\Sigma)} \lesssim |\partial_l h(t)|_{W^{2-1/q}_q(\Sigma)}|T_l|_{L^\infty(\Sigma)} + |\partial_l h(t)|_{L^\infty(\Sigma)}|T_l|_{W^{2-1/q}_q(\Sigma)},$$

for almost every $t \in (0,T)$. Furthermore,

$$\|T_l\|_{L^\infty(\Sigma)} + \|T_l\|_{W^{2-1/q}_q} \lesssim \|T_l\|_{C^2} \lesssim \|\gamma\|_{C^4} + \|\tilde{\gamma}\|_{C^4},$$

as well as

$$|\partial_l h(t)|_{L^\infty(\Sigma)} + |\partial_l h(t)|_{W^{2-1/q}_q(\Sigma)} \lesssim |h(t)|_{W^{3-1/q}_q(\Sigma)},$$

for almost every $t \in (0,T)$, since $W^{3-1/q}_q(\Sigma) \hookrightarrow W^1_\infty(\Sigma)$. By Lemma 2.23,

$$|h|_{L^p(0,T;W^{3-1/q}_q(\Sigma))} \lesssim T^{1/(3p)}|h|_{L^p(0;T;X_1)\cap W^1_p(0,T;X_0)}.$$

Hence

$$|\partial_l h T_l|_{L_p(0,T;W^{2-1/q}_q(\Sigma))} \lesssim T^{\frac{1}{3p}}|h|_{L_p(0;T;X_1)\cap W^1_p(0,T;X_0)}\left(\|\beta\|_{C^4} + \|\gamma\|_{C^4}\right).$$

Altogether,

$$\begin{aligned}
&\|\mathcal{B}_2(h)\|_{L^p(0,T;W_q^{2-1/q}(\Sigma))} \\
&\quad\lesssim \left(1+\|DF\|_{L^\infty}+\|DF\|_{L^\infty}^2\right)\|\beta\|_{C^1}\|h\|_{L_p(0,T;X_1)\cap W_p^1(0,T;X_0)} \\
&\quad+ T^{1/p}\left(\|\beta\|_{C^3}+\|\gamma\|_{C^3}\right)\|h\|_{L^p(0,T;X_1)\cap W_p^1(0,T;X_0)} \\
&\quad+ T^{\frac{1}{3p}}\left(\|\beta\|_{C^4}+\|\gamma\|_{C^4}\right)\|h\|_{L^p(0,T;X_1)\cap W_p^1(0,T;X_0)}.
\end{aligned}$$

Hence

$$\begin{aligned}
|\mathcal{B}_2(h)|_{\lambda,T} \lesssim C(\lambda)&\left(1+\|D\Phi\|_{L^\infty}+\|D\Phi\|_{L^\infty}^2\right)\|\tilde\gamma\|_{C^1}|h|_{1,\lambda,T} \\
&+C(\lambda)T^{1/p}\left(\|\gamma\|_{C^3}+\|\tilde\gamma\|_{C^3}\right)|h|_{1,\lambda,T} \\
&+C(\lambda)T^{\frac{1}{3p}}\left(\|\gamma\|_{C^4}+\|\tilde\gamma\|_{C^4}\right)|h|_{1,\lambda,T}.
\end{aligned}$$

Now let us consider $\mathcal{B}_3$. We write

$$\mathcal{B}_3(\mu)=\sum_{l,j}R_{jl}\partial_l\partial_j\mu+\sum_l\tilde R_l\partial_l\mu, \tag{2.49}$$

where

$$R_{j,l}:=\begin{cases}\partial_j\Phi_j\partial_j\Phi_j-1+\sum_{k\neq j}\partial_k\Phi_j\partial_k\Phi_j, & j=l,\\ \sum_k\partial_k\Phi_j\partial_k\Phi_l, & j\neq l,\end{cases}\qquad \tilde R_l:=\sum_k\partial_k\partial_k\Phi_l. \tag{2.50}$$

Now if $j=l$, we have $R_{j,l}=(\partial_j\Phi_j+1)(\partial_j\Phi_j-1)+\sum_{k\neq j}\partial_k\Phi_j\partial_k\Phi_j$. Clearly for $k\neq j$, also $\partial_k\Phi_j=(D\Phi-I)_{jk}$. Hence if $j=l$,

$$\begin{aligned}
\|R_{j,j}\|_{L^\infty}&\lesssim\|DF+I\|_{L^\infty}\|DF-I\|_{L^\infty}+\|DF\|_{L^\infty}\|DF-I\|_{L^\infty}\\
&\lesssim\left(1+|\beta|_{C^1}+|\gamma|_{C^1}\right)\left(|\beta|_{C^1}+|\gamma|_{C^1}\right).
\end{aligned}$$

The argument if $j\neq l$ is completely the same since j and l can not be k at the same time. Clearly,

$$\|\tilde R_l\|_{L^\infty}\lesssim 1+\|\gamma\|_{C^2}+\|\tilde\gamma\|_{C^2}.$$

Now, by taking norms in (2.49),

$$\|\mathcal{B}_3(\mu)\|_{L^p(L^q)}\le\sum_{l,j}\|\partial_{x_l}\partial_{x_j}\mu\|_{L^p(L^q)}\|R_{j,l}\|_{L^\infty}+\sum_l\|\partial_{x_l}\mu\|_{L^p(L^q)}\|\tilde R_l\|_{L^\infty}.$$

Now this yields

$$|\mathcal{B}_3(\mu)|_{\lambda,T}=\|\mathcal{B}_3(\mu)\|_{L^p(L^q)}\lesssim|\mu|_{1,\lambda,T}\sum_{j,l}\|R_{j,l}\|_{L^\infty}+\frac{1}{\lambda}|\mu|_{1,\lambda,T}\sum_l\|\tilde R_l\|_{L^\infty},$$

whence

$$\begin{aligned}
|\mathcal{B}_3(\mu)|_{\lambda,T}\lesssim|\mu|_{1,\lambda,T}&\left(1+\|\gamma\|_{C^1}+\|\tilde\gamma\|_{C^1}\right)\left(\|\gamma\|_{C^1}+\|\tilde\gamma\|_{C^1}\right)\\
&+\frac{1}{\lambda}|\mu|_{1,\lambda,T}\left(1+\|\gamma\|_{C^2}+\|\tilde\gamma\|_{C^2}\right).
\end{aligned}$$

Let us be concerned with $\mathcal{B}_4$. Clearly,

$$|\mathcal{B}_4(\mu)|_{\lambda,T}\lesssim\lambda^{1-1/q}\|\mathcal{B}_4(\mu)\|_{L^p(0,T;W_q^{1-1/q}(\Sigma))}.$$

We use a similar strategy as before. We extend n_γ to $\mathbb{R}^n_+$ using a bounded extension operator, cf. [**62**]. The extension is denoted by $\tilde{n}_\gamma$. We obtain

$$\begin{aligned}|\mathcal{B}_4(\mu)|_{L^p(0,T;W_q^{1-1/q}(\partial\mathbb{R}^n_+))} \le\ & |(\tilde{n}_\gamma - e_n)\cdot(DF^T\nabla\mu)|_{L_p(0,T;W^1_q(\mathbb{R}^n_+))}\\ &+ |e_n\cdot((DF^T-I)\nabla\mu)|_{L_p(0,T;W^1_q(\mathbb{R}^n_+))}.\end{aligned}$$

The right hand side is bounded by

$$\begin{aligned}&|(\tilde{n}_\gamma - e_n)\cdot(DF^T\nabla\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+))}\\ &\quad+ |\nabla(\tilde{n}_\gamma - e_n)\cdot(DF^T\nabla\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+))}\\ &\quad+ |(\tilde{n}_\gamma - e_n)\cdot(\nabla DF^T\nabla\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+))}\\ &\quad+ |(\tilde{n}_\gamma - e_n)\cdot(DF^T\nabla^2\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+))}\\ &\quad+ |e_n\cdot((DF^T-I)\nabla\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+))}\\ &\quad+ |e_n\cdot(\nabla(DF^T-I)\nabla\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+))}\\ &\quad+ |e_n\cdot((DF^T-I)\nabla^2\mu)|_{L_p(0,T;L_q(\mathbb{R}^n_+))}.\end{aligned}$$

Hence

$$\begin{aligned}|\mathcal{B}_4(\mu)|_{\omega,T} \lesssim\ & \omega^{1-1/q}\|\nabla\mu\|_{L^p(L^q)}\left(1+\|\tilde{n}_\gamma\|_{C^1}\right)\left(1+\|DF\|_{L^\infty}+\|D^2F\|_{L^\infty}\right)\\ &+ \left(\|DF-I\|_{L^\infty}+\|\tilde{n}_\gamma - e_n\|_{L^\infty}\right)\|\nabla^2\mu\|_{L^p(L^q)}.\end{aligned}$$

This entails

$$\begin{aligned}|\mathcal{B}_4(\mu)|_{\omega,T} \lesssim\ & \lambda^{-1/q}\left(1+\|\gamma\|_{C^2}\right)\left(1+\|DF\|_{L^\infty}+\|D^2F\|_{L^\infty}\right)|\mu|_{1,\lambda,T}\\ &+ \left(\|\beta\|_{C^1}+\|\gamma\|_{C^1}\right)|\mu|_{1,\lambda,T}.\end{aligned}$$

Let us give the estimates for $\mathcal{B}_5$. Recall

$$\mathcal{B}_5(h) = (e_n - n_\gamma)\cdot\nabla_\Sigma h + \sum_i (n_\gamma)_i \sum_{j,l}(g^{ij}\partial_j F_l - \delta_{ij}\delta_{jl})\partial_l h.$$

Note that the normals, the first fundamental form, and the transformation F are time-independent. Therefore, by Lemma 2.6,

$$\begin{aligned}|\mathcal{B}_5(h)|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))} \lesssim\ & \Big(|n_\gamma - e_n|_{L_\infty(\partial\Sigma)} + |n_\gamma|_{L_\infty(\partial\Sigma)}|g^{ij}-\delta_{ij}|_{L_\infty(\partial\Sigma)}|DF|_{L_\infty}\\ &+ |n_\gamma|_{L_\infty(\partial\Sigma)}|g^{ij}|_{L_\infty}|DF-I|_{L_\infty}\Big)|\nabla_\Sigma h|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))}.\end{aligned}$$

Up to a constant, the right hand side is bounded by

$$(1+|\beta|_{C^1}+|\gamma|_{C^1})(|\beta|_{C^1}+|\gamma|_{C^1})|h|_{W^1_p(0,T;X_0)\cap L^p(0,T;X_1)}.$$

It remains to estimate $|\mathcal{B}_5(h)|_{L^p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))}$. We have, for almost every $t \in (0,T)$,

$$\begin{aligned}|\mathcal{B}_5(h)(t)|_{B^{3-2/q}_{qq}(\partial\Sigma)} \le{}& |(n_\gamma - e_n)\cdot\nabla_\Sigma h|_{B^{3-2/q}_{qq}(\partial\Sigma)}\\ &+ \sum_{i,j,l} |(n_\gamma)_i(g^{ij}-\delta_{ij})\partial_j F_l \partial_l h|_{B^{3-2/q}_{qq}(\partial\Sigma)}\\ &+ \sum_{i,j,l} |(n_\gamma)_i\delta_{ij}(\partial_j F_l - \delta_{jl})\partial_l h|_{B^{3-2/q}_{qq}(\partial\Sigma)}.\end{aligned}$$

By paraproduct estimates (1.1),

$$\begin{aligned}|\mathcal{B}_5(h)(t)|_{B^{3-2/q}_{qq}(\partial\Sigma)} \lesssim{}& |n_\gamma - e_n|_{B^{3-2/q}_{qq}(\partial\Sigma)}|\nabla_\Sigma h|_{L_\infty(\partial\Sigma)}\\ &+ |n_\gamma - e_n|_{L_\infty(\partial\Sigma)}|\nabla_\Sigma h|_{B^{3-2/q}_{qq}(\partial\Sigma)}\\ &+ \sum_{i,j,l} |n_i^\gamma(g^{ij}-\delta_{ij})\partial_j F_l|_{B^{3-2/q}_{qq}(\partial\Sigma)}|\partial_l h|_{L_\infty(\partial\Sigma)}\\ &+ \sum_{i,j,l} |n_i^\gamma(g^{ij}-\delta_{ij})\partial_j F_l|_{L_\infty(\partial\Sigma)}|\partial_l h|_{B^{3-2/q}_{qq}(\partial\Sigma)}\\ &+ \sum_{i,j,l} |n_i^\gamma\delta_{ij}(\partial_j F_l-\delta_{jl})|_{B^{3-2/q}_{qq}(\partial\Sigma)}|\partial_l h|_{L_\infty(\partial\Sigma)}\\ &+ \sum_{i,j,l} |n_i^\gamma\delta_{ij}(\partial_j F_l-\delta_{jl})|_{L_\infty(\partial\Sigma)}|\partial_l h|_{B^{3-2/q}_{qq}(\partial\Sigma)},\end{aligned}$$

for almost every $t \in (0,T)$. By Theorem 2.1,

$$|\nabla h|_{L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} \lesssim |h|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)}.$$

Hölder inequality entails

$$|\nabla h|_{L_p(0,T;L_\infty(\partial\Sigma))} \le T^{1/p}|\nabla h|_{L_\infty(0,T;L_\infty(\partial\Sigma))} \lesssim T^{1/p}|h|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)},$$

since $h \in \mathbb{E}(T)$. Furthermore,

$$\begin{gathered}|g^{ij}-\delta_{ij}|_{L_\infty} + |DF - I|_{L_\infty} \lesssim |\beta|_{C^1} + |\gamma|_{C^1},\\ |n_\gamma - e_n|_{B^{3-2/q}_{qq}} + |g^{ij}-\delta_{ij}|_{B^{3-2/q}_{qq}} + |DF|_{B^{3-2/q}_{qq}} \lesssim 1 + |\beta|_{C^3} + |\gamma|_{C^3}.\end{gathered}$$

Hence

$$\begin{aligned}&|\mathcal{B}_5(h)|_{L_p(0;T;B^{3-2/q}_{qq}(\partial\Sigma))}\\ &\qquad\lesssim (1+|\beta|_{C^3}+|\gamma|_{C^3})(|\beta|_{C^1}+|\gamma|_{C^1}+T^{1/p})|h|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)}.\end{aligned}$$

whence readily altogether

$$\begin{aligned}\|\mathcal{B}_5(h)\|_{F^{1-\frac{2}{3q}}_{pq}(0,T;L^q(\partial\Sigma))\cap L^p(0;T;B^{3-2/q}_{qq}(\partial\Sigma))} &\lesssim (1+\|\gamma\|_{C^3}+\|\tilde\gamma\|_{C^3})\times\\ &\times(\|\gamma\|_{C^1}+\|\tilde\gamma\|_{C^1}+T^{1/p})\|h\|_{W^1_p(0,T;X_0)\cap L^p(0,T;X_1)}.\end{aligned}$$

Hence, since the norms are equivalent up to a constant $C(\lambda) > 0$,

$$|\mathcal{B}_5(h)|_{\lambda,T} \le C(\lambda)(\|\gamma\|_{C^1}+\|\tilde\gamma\|_{C^1}+T^{1/p})|h|_{1,\lambda,T}.$$

The proof is complete. $\square$

2.5.4. Localization. Let us now be concerned with the shifted problem in a bounded smooth domain $\Omega \subset \mathbb{R}^n$, where Σ is a perpendicular smooth surface inside. More precisely, the system reads as

$$\begin{aligned} \partial_t h + \omega^3 h + [\![n_\Sigma \cdot \nabla\mu]\!] &= g_1, && \text{on } \Sigma,\\ \mu|_\Sigma + \Delta_\Sigma h &= g_2, && \text{on } \Sigma,\\ \omega^2\mu - \Delta\mu &= g_3, && \text{on } \Omega\backslash\Sigma,\\ n_{\partial\Omega} \cdot \nabla\mu|_{\partial\Omega} &= g_4, && \text{on } \partial\Omega,\\ n_{\partial\Sigma} \cdot \nabla_\Sigma h|_{\partial\Sigma} &= g_5, && \text{on } \partial\Sigma,\\ h|_{t=0} &= h_0, && \text{on } \Sigma, \end{aligned} \tag{2.51}$$

where $\omega \geq \omega_0$ and $\omega_0 > 0$ is as in Theorem 2.21. Let again $X_0 := W_q^{1-1/q}(\Sigma)$, $X_1 := W_q^{4-1/q}(\Sigma)$, $X_\gamma := B_{qp}^{4-1/q-3/p}(\Sigma)$,

$$\mathbb{E}(T) := [W_p^1(0,T;X_0) \cap L_p(0,T;X_1)] \times L_p(0,T;W_q^2(\Omega\backslash\Sigma)),$$

and

$$\begin{aligned} \mathbb{F}(T) := L_p(0,T;X_0) \times L_p(0,T;W_q^{2-1/q}(\Sigma)) \times L_p(0,T;L_q(\Omega))\times \\ \times L_p(0,T;W_q^{1-1/q}(\partial\Omega))\times \\ \times [F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma)) \cap L_p(0,T;W_q^{3-2/q}(\partial\Sigma))] \times X_\gamma. \end{aligned}$$

The main result reads as follows.

THEOREM 2.22. *Let $n = 2,3$, $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain, $\omega \geq \omega_0$, $6 \leq p < \infty$, $q \in (3/2,2) \cap (2p/(p+1),2p)$ and Σ be a smooth surface inside intersecting $\partial\Omega$ at a constant ninety degree angle.*

Then there is some $T > 0$, such that for every $(g_1,g_2,g_3,g_4,g_5,h_0) \in \mathbb{F}(T)$ satisfying (2.35) there is a unique solution $(h,\mu) \in \mathbb{E}(T)$ of (2.51).

PROOF. We can reduce the system to the case where $(g_2,g_3,g_4,h_0) = 0$ by solving auxiliary problems first, cf. the proof of Theorem 2.18 and Theorem A.7. We are then left to solve

$$\begin{aligned} \partial_t h + \omega^3 h + [\![n_\Sigma \cdot \nabla\mu]\!] &= g_1, && \text{on } \Sigma,\\ \mu|_\Sigma + \Delta_\Sigma h &= 0, && \text{on } \Sigma,\\ \omega^2\mu - \Delta\mu &= 0, && \text{on } \Omega\backslash\Sigma,\\ n_{\partial\Omega} \cdot \nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega,\\ n_{\partial\Sigma} \cdot \nabla_\Sigma h|_{\partial\Sigma} &= g_5, && \text{on } \partial\Sigma,\\ h|_{t=0} &= 0, && \text{on } \Sigma, \end{aligned} \tag{2.52}$$

for possibly modified right hand sides which we do not relabel.

We will now show existence and uniqueness of the solution of this system via the localization method, cf. [**57**]. To this end let $(\varphi_j)_{j=0,\dots,N} \subseteq C_0^\infty(\mathbb{R}^n)$ be a smooth partition of unity with respect to Ω and the open sets $(U_j)_{j=0,\dots,N} \subseteq \mathbb{R}^n$, that is, the support of φ_j is contained in U_j for each $j = 0,\dots,N$ and $\Omega \subseteq \bigcup_{j=0,\dots,N} U_j$. Furthermore, let $(\psi_j)_{j=0,\dots,N} \subseteq C_0^\infty(\mathbb{R}^n)$ be smooth functions with compact support in U_j such that $\psi_j \equiv 1$ on supp φ_j for every $0 \leq j \leq N$.

Now, by choosing N finite but sufficiently large and, corresponding to that, the open sets U_j sufficiently small, we can assume that, up to a rotation, for each $j = 0, ..., N$ there exist smooth curves γ_j, β_j such that

$$U_j \cap \Omega = \mathbb{R}^n_{\gamma_j} \cap \Omega, \quad U_j \cap \Sigma = \mathbb{R}^n_{\gamma_j} \cap \Sigma_{\beta_j}.$$

Furthermore, again by a smallness argument, we can choose γ_j and β_j such that the C^1-norm is as small as we like.

We now assume for a moment that we have a solution (h, μ) of (2.52) to derive an explicit representation formula. We therefore multiply every equation with φ_j and get corresponding equations for the localized functions $(h^j, \mu^j) := \varphi_j(h, \mu)$. By doing so, we obtain

$$\begin{aligned}
\omega^3 h^j + \partial_t h^j + [\![n_\Sigma \cdot \nabla \mu^j]\!] &= \varphi_j g_1 - \mu|_{\Sigma_j} \nabla \varphi_j \cdot n_\Sigma, && \text{on } \Sigma_j,\\
\mu^j + \Delta_\Sigma h^j &= (\Delta_\Sigma \varphi_j) h + 2 \textstyle\sum_{lm} g^{lm} \partial_l \varphi_j \partial_m h, && \text{on } \Sigma_j,\\
\omega^2 \mu^j - \Delta \mu^j &= (\Delta \varphi_j)\mu + 2 \nabla \varphi_j \cdot \nabla \mu, && \text{on } \Omega_j \backslash \Sigma_j,\\
n_{\partial\Omega} \cdot \nabla \mu^j|_{\partial\Omega_j} &= n_{\partial\Omega} \cdot \nabla \varphi_j \mu|_{\partial\Omega_j}, && \text{on } \partial\Omega_j,\\
n_{\partial\Sigma} \cdot \nabla_\Sigma h^j|_{\partial\Sigma_j} &= \varphi_j g_5 + n_{\partial\Sigma} \cdot \nabla_\Sigma \varphi_j h|_{\partial\Sigma_j}, && \text{on } \partial\Sigma_j,\\
h^j|_{t=0} &= 0, && \text{on } \Sigma_j,
\end{aligned}$$

where $\Sigma_j := \Sigma_{\beta_j}$, $\Omega_j := \mathbb{R}^n_{\gamma_j}$, (g_{lm}) is the first fundamental form of Σ_j with respect to the surface Σ and (g^{lm}) its inverse. This way, we obtain a finite number of bent half space problems. Denote by $L^j = L^j_\omega$ the linear operator on the right hand side of the above system. Moreover let $G^j := (\varphi_j g_1, 0, 0, 0, \varphi_j g_5, 0)$, and the perturbation operator R^j be such that the right hand side equals $G^j + R^j(h, \mu)$. We can write the system of localized equations as

$$L^j(h^j, \mu^j) = G^j + R^j(h, \mu), \quad j = 0, ..., N.$$

Since each L^j is invertible, this is equivalent to

$$(h^j, \mu^j) = (L^j)^{-1}[G^j + R^j(h, \mu)], \quad j = 0, ..., N.$$

Since $(\varphi_j)_{j=0,...,N}$ is a partition of unity and $\psi_j = 1$ on the support of φ_j,

$$(h, \mu) = \sum_{j=0}^N (h^j, \mu^j) = \sum_{j=0}^N \psi_j (h^j, \mu^j).$$

This way, we may derive the representation formula

$$(h, \mu) = \sum_{j=0}^N \psi_j (L^j)^{-1} G^j + \sum_{j=0}^N \psi_j (L^j)^{-1} R^j(h, \mu). \tag{2.53}$$

Since now $R := \sum_{j=0}^N \psi_j (L^j)^{-1} R^j$ is of lower order, we can show that if $T > 0$ is small enough,

$$|R|_{\mathcal{B}(\mathbb{E}(T);\mathbb{E}(T))} \leq 1/2. \tag{2.54}$$

Hence a Neumann series argument then yields that $I - R$ is invertible if $T > 0$ is small enough, hence we can rewrite (2.53) as

$$(h, \mu) = (I - R)^{-1} \sum_{j=0}^{N} \psi_j (L^j)^{-1} G^j. \tag{2.55}$$

We will show (2.54) below.

Let L be the linear operator from the left hand side of (2.52). We obtain from (2.55) that L is injective, has closed range and a left inverse. It remains to show that $L : \mathbb{E}(T) \to \mathbb{F}(T)$ has a right inverse. To this end let $z \in \mathbb{F}(T)$ be arbitrary. Define

$$S := (I - R)^{-1} \sum_{j=0}^{N} \psi_j (L^j)^{-1} \varphi_j. \tag{2.56}$$

Let $u := Sz$. Then

$$u = (I - R)^{-1} \sum_{j=0}^{N} \psi_j (L^j)^{-1} \varphi_j z.$$

Applying $I - R$ to both sides of (2.56) yields

$$u - Ru = \sum_{j=0}^{N} \psi_j (L^j)^{-1} \varphi_j z.$$

Therefore

$$u = \sum_{j=0}^{N} \psi_j (L^j)^{-1} R^j u + \sum_{j=0}^{N} \psi_j (L^j)^{-1} \varphi_j z. \tag{2.57}$$

Applying L to both sides of (2.57) and using that $L = L^j$ on U_j,

$$\begin{aligned}
Lu &= \sum_{j=0}^{N} L\psi_j (L^j)^{-1} R^j u + \sum_{j=0}^{N} L\psi_j (L^j)^{-1} \varphi_j z \\
&= \sum_{j=0}^{N} L^j\psi_j (L^j)^{-1} R^j u + \sum_{j=0}^{N} L^j\psi_j (L^j)^{-1} \varphi_j z \\
&= \sum_{j=0}^{N} L^j\psi_j (L^j)^{-1} R^j u + \sum_{j=0}^{N} [L^j, \psi_j] (L^j)^{-1} \varphi_j z + \sum_{j=0}^{N} \psi_j L^j (L^j)^{-1} \varphi_j z \\
&= \sum_{j=0}^{N} L^j\psi_j (L^j)^{-1} R^j u + \sum_{j=0}^{N} [L^j, \psi_j] (L^j)^{-1} \varphi_j z + \sum_{j=0}^{N} \psi_j \varphi_j z \\
&= \sum_{j=0}^{N} L^j\psi_j (L^j)^{-1} R^j u + \sum_{j=0}^{N} [L^j, \psi_j] (L^j)^{-1} \varphi_j z + \sum_{j=0}^{N} \varphi_j z \\
&= \sum_{j=0}^{N} L^j\psi_j (L^j)^{-1} R^j u + \sum_{j=0}^{N} [L^j, \psi_j] (L^j)^{-1} \varphi_j z + z.
\end{aligned}$$

Since $u = Sz$,

$$LSz = z + \sum_{j=0}^{N} L^j \psi_j (L^j)^{-1} R^j Sz + \sum_{j=0}^{N} [L^j, \psi_j](L^j)^{-1}\varphi_j z, \quad z \in \mathbb{F}(T).$$

Let $S^R := \sum_{j=0}^{N} L^j \psi_j (L^j)^{-1} R^j S + \sum_{j=0}^{N}[L^j, \psi_j](L^j)^{-1}\varphi_j$. We can show, using again a Neumann series argument involving the fact that the commutator is lower order, that $I + S^R$ is invertible if $T > 0$ is small enough. Indeed, below we will show that

$$|S^R|_{\mathcal{B}(\mathbb{F}(T);\mathbb{F}(T))} \leq 1/2, \tag{2.58}$$

if $T > 0$ is sufficiently small. The right inverse of L is therefore given by $S(I+S^R)^{-1}$. This then concludes the proof.

Let us show (2.54). To show that R^j gets small when $T > 0$ is small enough, we need to extract more time regularity for μ. For almost every time t, $\mu(t)$ is also a weak solution of

$$\begin{aligned} \omega^2\mu(t) - \Delta\mu(t) &= 0, && \text{in } \Omega\backslash\Sigma, \\ \mu(t)|_\Sigma &= \Delta_\Sigma h(t), && \text{on } \Sigma, \\ n_{\partial\Omega}\cdot\nabla\mu(t) &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Hence, cf. (A.6) in the Appendix,

$$|\mu(t)|_{W^1_q(\Omega\backslash\Sigma)} \leq C(\omega)|\Delta_\Sigma h(t)|_{W_q^{1-1/q}(\Sigma)},$$

for almost every $t \in (0,T)$. Using Hölder inequality, we can extract more time regularity for μ:

Lemma 2.23. *Let $T > 0$. Under the assumptions on p and q from above, there exists some $r > p$, such that $h \in L_r(0,T;W_q^{3-1/q}(\Sigma))$, whenever $h \in \mathbb{E}(T)$. In particular, we can choose $r = 3p/2$. Furthermore,*

$$|h|_{L_p(0,T;W_q^{3-1/q}(\Sigma))} \leq T^{1/(3p)}|h|_{L_{3p/2}(0,T;W_q^{3-1/q}(\Sigma))},$$

as well as

$$|h|_{L_{3p/2}(0,T;W_q^{3-1/q}(\Sigma))} \leq C|h|_{\mathbb{E}(T)},$$

for some constant $C > 0$ independent of h, for all $h \in \mathbb{E}(T)$. When restricting to functions with vanishing trace at $t = 0$, the constant C is independent of T.

Proof. By real interpolation method,

$$W_q^{3-1/q}(\Sigma) = (W_q^{4-1/q}(\Sigma), W_q^{1-1/q}(\Sigma))_{\theta,q},$$

for $\theta = 1/3$. Choosing $r = 3p/2$, we obtain $r(1-\theta) = 2r/3 = p$. Hence

$$\begin{aligned} \|h\|^r_{L^r(0,T;W_q^{3-1/q})} &= \int_0^T \|h(t)\|^r_{W_q^{3-1/q}}dt \\ &\lesssim \int_0^T \|h(t)\|^{r\theta}_{W_q^{1-1/q}}\|h(t)\|^{r(1-\theta)}_{W_q^{4-1/q}}dt \\ &\lesssim \|h\|^{r\theta}_{L^\infty(0,T;W_q^{1-1/q})}\|h\|^p_{L^p(0,T;W_q^{4-1/q})}. \end{aligned}$$

Hence

$$\|h\|_{L^r(0,T;W_q^{3-1/q})} \lesssim \|h\|^{\theta}_{L^\infty(0,T;W_q^{1-1/q})}\|h\|^{1-\theta}_{L^p(0,T;W_q^{4-1/q})},$$

since $p/r = 1-\theta$. In particular,

$$\|h\|_{L^r(0,T;W_q^{3-1/q})} \lesssim \|h\|^{\theta}_{W_p^1(0,T;W_q^{1-1/q})}\|h\|^{1-\theta}_{L^p(0,T;W_q^{4-1/q})} \lesssim \|h\|_{\mathbb{E}_{1,T}}.$$

This shows $h \in L^r(0,T;W_q^{1-1/q}(\Sigma))$ for $r = 3p/2$ and the second estimate. The first one then easily follows from Hölder inequality. Indeed,

$$\|h\|^p_{L^p(0,T;W_q^{3-1/q})} \leq \left(\int_0^T \|h(t)\|^{p\frac{r}{p}}_{W_q^{3-1/q}}dt\right)^{\frac{p}{r}}\left(\int_0^T 1dt\right)^{\frac{r-p}{r}}.$$

Hence

$$\|h\|_{L^p(0,T;W_q^{3-1/q})} \leq T^{\frac{r-p}{rp}}\|h\|_{L^r(0,T;W_q^{3-1/q})},$$

so using $\frac{r-p}{rp} = \frac{1}{3p}$ completes the proof of the lemma. □

Having this at hand, we can give the estimates for R^j. Clearly,

$$|R|_{\mathcal{B}(\mathbb{E}(T);\mathbb{E}(T))} \leq C \sup_{j=0,\dots,N}|R^j|_{\mathcal{B}(\mathbb{E}(T);\mathbb{F}(T))}.$$

We estimate each component in the corresponding norm. By Lemma 2.23,

$$\begin{aligned}|\nabla\varphi_j\cdot n_\Sigma[\![\mu]\!]|_{L_p(0,T;W_q^{1-1/q}(\Sigma))} &\lesssim |\nabla\varphi_j\cdot n_\Sigma|_{C^1(\Sigma)}\|[\![\mu]\!]\|_{L_p(0,T;W_q^{1-1/q}(\Sigma))}\\ &\lesssim |\nabla\varphi_j\cdot n_\Sigma|_{C^1(\Sigma)}|\mu|_{L_p(0,T;W_q^1(\Omega\setminus\Sigma))}\\ &\lesssim |\nabla\varphi_j\cdot n_\Sigma|_{C^1(\Sigma)}T^{1/(3p)}|h|_{\mathbb{E}(T)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&|(\Delta_\Sigma\varphi_j)h + 2\textstyle\sum_{kk'} g^{kk'}\partial_k\varphi_j\partial_{k'}h|_{L_p(0,T;W_q^{2-1/q}(\Sigma))}\\ &\qquad\lesssim \left(|\Delta_\Sigma\varphi_j|_{C^2} + |g|_{C^2}|\varphi_j|_{C^3}\right)|h|_{L_p(0,T;W_q^{3-1/q}(\Sigma))}\\ &\qquad\lesssim \left(|\varphi_j|_{C^4} + |g|_{C^2}|\varphi_j|_{C^3}\right)T^{1/(3p)}|h|_{\mathbb{E}(T)}.\end{aligned}$$

Also,

$$\begin{aligned}|\Delta\varphi_j\mu + 2\nabla\varphi_j\cdot\nabla\mu|_{L_p(L_q)} &\leq \left(|\Delta\varphi_j|_{L^\infty} + |\nabla\varphi_j|_{L^\infty}\right)|\mu|_{L_p(W_q^1)}\\ &\lesssim |\varphi_j|_{C^2}T^{1/(3p)}|h|_{\mathbb{E}(T)},\end{aligned}$$

as well as

$$\begin{aligned}|n_{\partial\Omega}\cdot\nabla\varphi_j\mu|_{\partial\Omega}|_{L_p(0,T;W_q^{1-1/q}(\partial\Omega))} &\lesssim |n_{\partial\Omega}\cdot\nabla\varphi_j|_{C^1}|\mu|_{\partial\Omega}|_{L^p(0,T;W_q^{1-1/q}(\partial\Omega))}\\ &\lesssim |n_{\partial\Omega}\cdot\nabla\varphi_j|_{C^1}|\mu|_{L^p(0,T;W_q^1(\Omega\setminus\Sigma))}\\ &\lesssim |\varphi_j|_{C^2}T^{1/(3p)}|h|_{\mathbb{E}(T)}.\end{aligned}$$

By paraproduct estimates of Lemma 1.5,

$$\begin{aligned}|n_{\partial\Omega}\cdot\nabla_\Sigma\varphi_j h|_{\partial\Sigma}|_{L^p(0,T;W_q^{3-2/q}(\partial\Sigma))} &\lesssim |n_{\partial\Omega}\cdot\nabla_\Sigma\varphi_j|_{C^2}|h|_{L_p(0,T;W_q^{3-1/q}(\Sigma))}\\ &\lesssim |\varphi_j|_{C^3}T^{1/(3p)}|h|_{\mathbb{E}(T)}.\end{aligned}$$

Since the partition of unity is time-independent, Lemma 2.6 renders

$$\|n_{\partial\Omega}\cdot\nabla_\Sigma\varphi_j h|_{\partial\Sigma}\|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))}\lesssim\|n_{\partial\Omega}\cdot\nabla\varphi_j\|_{L^\infty}\|h|_{\partial\Sigma}\|_{F_{pq}^{1-\frac{2}{3q}}(0,T;L^q(\partial\Sigma))}.$$

Lemma 2.24 below entails there is some $\epsilon>0$ such that

$$|h|_{\partial\Sigma}|_{F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma))}\lesssim T^\epsilon|h|_{\mathbb{E}(T)},$$

since $h\in\mathbb{E}(T)$ with vanishing time trace. Concludingly, we have shown (2.54).

Let us now show (2.58). Recall that

$$S^R:=\sum_{j=0}^N L^j\psi_j(L^j)^{-1}R^jS+\sum_{j=0}^N[L^j,\psi_j](L^j)^{-1}\varphi_j,$$

$S=(I-R)^{-1}\sum_{j=0}^N\psi_j(L^j)^{-1}\varphi_j$, and $R=\sum_{j=0}^N\psi_j(L^j)^{-1}R^j$. By maximal regularity of L^j,

$$\begin{aligned}|L^j\psi_j(L^j)^{-1}R^jSz|_{\mathbb{F}(T)}&\lesssim|\psi_j(L^j)^{-1}R^jSz|_{\mathbb{E}(T)}\\&\lesssim|(L^j)^{-1}R^jSz|_{\mathbb{E}(T)}\\&\lesssim|R^jSz|_{\mathbb{F}(T)}\\&\lesssim T^\epsilon|Sz|_{\mathbb{E}(T)},\end{aligned}$$

up to a harmless constant also depending on ω_0. Using that $I-R:\mathbb{E}(T)\to\mathbb{E}(T)$ has a bounded inverse,

$$|Sz|_{\mathbb{E}(T)}\lesssim\sum_{j=0}^N|(L^j)^{-1}\varphi_jz|_{\mathbb{E}(T)}\lesssim\sum_{j=0}^N|\varphi_jz|_{\mathbb{E}(T)}\lesssim|z|_{\mathbb{E}(T)}.$$

We have therefore shown that

$$|L^j\psi_j(L^j)^{-1}R^jS|_{\mathcal{B}({}_0\mathbb{E}(T);{}_0\mathbb{F}(T))}\to0,$$

as $T\to0$. Now, for the second sum in S^R, we first comment on the commutator. Since $[L^j,\psi_j]v=L^j(\psi_jv)-\psi_jL^jv$, we easily see that the commutator is of lower order and actually takes the form of R^j with ψ_j replacing φ_j. More precisely, for $v=(v_1,v_2)$,

$$[L^j,\psi_j]v=\begin{pmatrix}-\nabla\psi_j\cdot n_\Sigma[v_2]\\-(\Delta_\Sigma\psi_j)v_1-2\sum_{lm}g^{lm}\partial_l\psi_j\partial_mv_1\\(\Delta\psi_j)v_2+2\nabla\psi_j\cdot\nabla v_2\\n_{\partial\Omega}\cdot\nabla\psi_jv_2|_{\partial\Omega}\\n_{\partial\Omega}\cdot\nabla_\Sigma\psi_jv_1|_{\partial\Sigma}\\0\end{pmatrix}.$$

Hence, by the same arguments as before, the commutator satisfies

$$|[L^j,\psi_j]|_{\mathcal{B}({}_0\mathbb{E}(T);{}_0\mathbb{F}(T))}\to0,$$

as $T\to0$. This entails

$$|[L^j,\psi_j](L^j)^{-1}\varphi_j|_{\mathcal{B}({}_0\mathbb{F}(T);{}_0\mathbb{F}(T))}\to0,$$

as $T\to0$. This shows that $|S^R|_{\mathcal{B}({}_0\mathbb{F}(T);{}_0\mathbb{F}(T))}\to0$ as $T\to0$, hence the proof is complete.

□

LEMMA 2.24. *Let* $T' \in (0,\infty)$, $T \in (0,T')$, $p \in (6,\infty)$, *and* $q \in (5/3,2) \cap (2p/(p+1),2p)$. *Then there is* $\epsilon > 0$, *such that*

$$|\nabla u|_{\partial\Sigma}|_{L_\infty(0,T;L_\infty(\partial\Sigma))} \lesssim T^\epsilon |u|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)},$$

as well as

$$|u|_{\partial\Sigma}|_{F^{1-2/(3q)}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;W^{3-2/q}_q(\partial\Sigma))} \lesssim T^\epsilon |u|_{W^1_p(0,T;X_0)\cap L^p(0,T;X_1)},$$

both for all $u \in {}_0W^1_p(0,T;X_0) \cap L_p(0,T;X_1)$. *The constants in* $\lesssim$ *are independent of the length of the time interval* $(0,T)$.

PROOF. Let $p \in (6,\infty)$ and $q \in (5/3,2) \cap (2p/(p+1),2p)$. Choose some $p_1 \in (6,p)$, such that $q \in (5/3,2) \cap (2p_1/(p_1+1),2)$ and

$$1/p_1 - 1/p < 1/3q. \tag{2.59}$$

Note that by choosing $p_1 < p$ close enough to p this is possible due to the assumption on q. Since $p_1 < p$, Hölder inequality gives

$$|u|_{W^1_{p_1}(0,T;X_0)\cap L_{p_1}(0,T;X_1)} \le T^{(p-p_1)/(pp_1)} |u|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)},$$

for all $u \in W^1_p(0,T;X_0) \cap L_p(0,T;X_1)$. From Theorem 2.1 we obtain that the Neumann trace is bounded as a mapping

$$\begin{aligned} \mathrm{tr}_{\partial\Sigma} \nabla_\Sigma : {}_0W^1_{p_1}(0,T;X_0) &\cap L_{p_1}(0,T;X_1) \\ &\to {}_0F^{1-2/3q}_{p_1 q}(0,T;L_q(\partial\Sigma)) \cap L_{p_1}(0,T;B^{3-2/q}_{qq}(\partial\Sigma)). \end{aligned}$$

Since the functions have time trace zero at $t=0$, the operator norm is independent of $T>0$. It remains to show ${}_0F^{1-2/3q}_{p_1 q}(0,T;L_q(\partial\Sigma)) \cap L_{p_1}(0,T;B^{3-2/q}_{qq}(\partial\Sigma)) \hookrightarrow L_\infty(0,T;L_\infty(\partial\Sigma))$, with an embedding constant independent of $T>0$. This can be seen by using Proposition 5.38 in [**35**]. Choose $\sigma \in (0,1)$ such that $\sigma > \frac{1}{p_1}\frac{3q}{3q-2}$, $\sigma < 1 - \frac{1}{3q-2}$. Note that such a choice is possible. Then,

$$\begin{aligned} {}_0F^{1-2/3q}_{p_1 q}(0,T;L_q(\partial\Sigma)) &\cap L_{p_1}(0,T;B^{3-2/q}_{qq}(\partial\Sigma)) \\ &\hookrightarrow {}_0H^{\sigma(1-2/3q)}_{p_1}(0,T;B^{(1-\sigma)(3-2/q)}_{qq}(\partial\Sigma)), \end{aligned}$$

where we note that $\sigma(1-2/3q) > 1/p_1 > 1/p$, hence the trace at $t=0$ is well-defined in the space on the right hand side. For this σ, we obtain

$${}_0H^{\sigma(1-2/3q)}_{p_1}(0,T;B^{(1-\sigma)(3-2/q)}_{qq}(\partial\Sigma)) \hookrightarrow L_\infty(0,T;L_\infty(\partial\Sigma))$$

and the proof of the first statement is complete.

Regarding the second inequality, by Lemma 2.23,

$$|u|_{\partial\Sigma}|_{L_p(0,T;W^{3-2/q}_q(\partial\Sigma))} \lesssim |u|_{L_p(0,T;W^{3-1/q}_q(\Sigma))} \lesssim T^{1/(3p)} |u|_{W^1_p(0,T;X_0)\cap L_p(0,T;X_1)}.$$

The constant is independent of T since u has vanishing trace at $t=0$. By (2.6),

$${}_0W^1_{p_1}(0,T;X_0) \cap L_{p_1}(0,T;X_1) \hookrightarrow {}_0F^{1-1/(3q)}_{p_1 q}(0,T;W^1_q(\Sigma)).$$

Choose now $p_1 < p$, such that $1/(3q) > 1/p_1 - 1/p$. Then by (2.59) and Theorem 1.2 in [**46**], ${}_0F^{1-1/(3q)}_{p_1 q}(0,T;W^1_q(\Sigma)) \hookrightarrow {}_0F^{1-2/(3q)}_{pq}(0,T;W^1_q(\Sigma))$. The embedding constant is independent of T. Hence

$$\begin{aligned}|u|_{\partial\Sigma}|_{{}_0F^{1-2/(3q)}_{pq}(0,T;L_q(\partial\Sigma))} &\lesssim |u|_{{}_0F^{1-1/(3q)}_{p_1q}(0,T;W^1_q(\Sigma))}\\ &\lesssim |u|_{{}_0W^1_{p_1}(0,T;X_0)\cap L_{p_1}(0,T;X_1)}\\ &\lesssim T^{(p-p_1)/(pp_1)}|u|_{{}_0W^1_p(0,T;X_0)\cap L_p(0,T;X_1)}.\end{aligned}$$

The proof is complete. □

2.5.5. The non-shifted linear problem on bounded domains. In this section we are concerned with problem (2.51) for $\omega = 0$. The main result is the following.

THEOREM 2.25. *Let $n = 2,3$, $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain, $p \in (6,\infty)$, $q \in (3/2,2) \cap (2p/(p+1), 2p)$, and Σ be a smooth submanifold with boundary $\partial\Sigma$ such that $\mathring{\Sigma}$ is inside Ω and Σ meets $\partial\Omega$ at a constant ninety degree angle.*

Then there is some $T > 0$, such that for every $(g_1, g_2, g_3, g_4, g_5, h_0) \in \mathbb{F}(T)$ satisfying the compatibility condition (2.35) there is a unique solution $(h,\mu) \in \mathbb{E}(T)$ of (2.51) for $\omega = 0$. Furthermore, the solution map is continuous between these spaces.

PROOF. As in the previous section we may reduce to the case $(g_2, g_3, g_4, h_0) = 0$. It is also clear that the ω^3-shift in equation (2.51) can easily be resolved to the case $\omega = 0$ by an exponential shift in solution and data. We are therefore left to solve

$$\begin{aligned}\partial_t h + [\![n_\Sigma \cdot \nabla T_0 \Delta_\Sigma h]\!] &= g_1, && \text{on } \Sigma,\\ n_{\partial\Sigma}\cdot\nabla_\Sigma h|_{\partial\Sigma} &= g_5, && \text{on } \partial\Sigma,\\ h|_{t=0} &= 0, && \text{on } \Sigma,\end{aligned} \tag{2.60}$$

where $T_0 g$ is defined as the unique solution of the two-phase elliptic problem

$$\begin{aligned}-\Delta u &= 0, && \text{in } \Omega\backslash\Sigma,\\ u|_\Sigma &= g, && \text{on } \Sigma,\\ n_{\partial\Omega}\cdot\nabla u|_{\partial\Omega} &= 0, && \text{on } \partial\Omega,\end{aligned}$$

cf. Appendix A. Also from Appendix A we obtain that

$$T_0\Delta_\Sigma h = T_\eta \Delta_\Sigma h + \eta(\eta-\Delta)^{-1}T_0\Delta_\Sigma h,$$

for all $\eta \geq \eta_0$. This implies that problem (2.60) is equivalent to

$$\begin{aligned}\partial_t h + [\![n_\Sigma \cdot \nabla T_\eta \Delta_\Sigma h]\!] &= g_1 + \mathcal{B}_\eta(h), && \text{on } \Sigma,\\ n_{\partial\Sigma}\cdot\nabla_\Sigma h|_{\partial\Sigma} &= g_5, && \text{on } \partial\Sigma,\\ h|_{t=0} &= 0, && \text{on } \Sigma,\end{aligned} \tag{2.62}$$

provided $\eta \geq \eta_0$, where

$$\mathcal{B}_\eta(h) = \eta[\![n_\Sigma \cdot \nabla(\eta-\Delta)^{-1}T_0\Delta_\Sigma h]\!].$$

Now choose large enough η to render the left hand side of (2.62) to be an invertible operator. We now fix this $\eta > 0$ and show that the perturbation operator $\mathcal{B}_\eta$ satisfies

$$|\mathcal{B}_\eta(h)|_{L_p(0,T;X_0)} \leq C(\eta)T^{1/(3p)}|h|_{\mathbb{E}(T)}, \quad h \in \mathbb{E}(T), \tag{2.63}$$

from which it follows that $|\mathcal{B}_\eta|_{\mathcal{B}({}_0\mathbb{E}(T);L_p(0,T;X_0))} \to 0$ as $T \to 0$. Choosing $T > 0$ sufficiently small, a standard Neumann series argument completes the proof.

Let us show (2.63). For almost every time $t \in (0,T)$,

$$\begin{aligned}\|\mathcal{B}_\eta(h)(t)\|_{X_0} &\le C\eta\|\nabla(\eta-\Delta)^{-1}T_0\Delta_\Sigma h(t)\|_{W^1_q(\Omega\setminus\Sigma)}\\ &\le C\eta\|(\eta-\Delta)^{-1}T_0\Delta_\Sigma h(t)\|_{W^2_q(\Omega\setminus\Sigma)}\\ &\le C(\eta)\|T_0\Delta_\Sigma h(t)\|_{L_q(\Omega)}\\ &\le C(\eta)\|\Delta_\Sigma h(t)\|_{W^{1-1/q}_q(\Sigma)}\\ &\le C(\eta)\|h(t)\|_{W^{3-1/q}_q(\Sigma)},\end{aligned}$$

cf. (A.6) in the Appendix. Integration in time gives

$$\|\mathcal{B}_\eta(h)\|_{L^p(0,T;X_0)} \le C(\eta)\|h\|_{L^p(0,T;W^{3-1/q}_q(\Sigma))}.$$

Therefore Hölder inequality in time, see also the proof of Lemma 2.23, gives the claimed estimate and the proof is complete. □

Two remarks are in order.

REMARK 2.26. We remark that, see Theorem III.4.10.2 in [7],

$$\mathbb{E}(T) = W^1_p(0,T;W^{1-1/q}_q(\Sigma)) \cap L_p(0,T;W^{4-1/q}_q(\Sigma)) \hookrightarrow C([0,T];X_\gamma).$$

Under the assumptions on p and q above, $\mathbb{E}(T) \hookrightarrow C([0,T];C^2(\Sigma))$. Indeed, there exists some $\varepsilon > 0$ such that

$$X_\gamma = B^{4-1/q-3/p}_{qp}(\Sigma) \hookrightarrow B^{2+\varepsilon}_{\infty\infty}(\Sigma) \hookrightarrow C^{2+\varepsilon}(\Sigma) \hookrightarrow C^2(\Sigma),$$

by classical Besov embedding, cf. [62].

REMARK 2.27. Note that the maximal regularity constant, or in other words the operator norm of the solution map, in general depends on $T > 0$. Regarding contraction estimates in the nonlinear problem it is now an important feature that the constant stays bounded as $T \to 0$, if we consider the problem with initial value zero, $h_0 = 0$. This is due to the fact that whenever $h_0 = 0$, we can first extend the data to the half line by reflection and then solve the problem on a larger time interval. This way, the operator norm of the solution map between spaces with vanishing traces at $t = 0$, say $[{}_0\mathbb{F}(T) \to {}_0\mathbb{E}(T)]$, stays bounded as $T \to 0$.

2.6. Nonlinear well-posedness

In this section we will show local well-posedness for the full nonlinear (transformed) system (2.31). We will use the maximal $L_p - L_q$ regularity result for the underlying linear problem and a contraction argument via Banach's fixed point principle, cf. e.g. Zeidler [65]. Let again $X_0 := W^{1-1/q}_q(\Sigma)$, $X_1 := W^{4-1/q}_q(\Sigma)$, $X_\gamma := B^{4-1/q-3/p}_{qp}(\Sigma)$,

$$\mathbb{E}(T) := [W^1_p(0,T;X_0) \cap L_p(0,T;X_1)] \times L_p(0,T;W^2_q(\Omega\setminus\Sigma)),$$

and

$$\begin{aligned}\mathbb{F}(T) := L_p(0,T;X_0) &\times L_p(0,T;W_q^{2-1/q}(\Sigma)) \times L_p(0,T;L_q(\Omega))\times\\ &\times L_p(0,T;W_q^{1-1/q}(\partial\Omega))\times\\ &\times [F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma)) \cap L_p(0,T;W_q^{3-2/q}(\partial\Sigma))] \times X_\gamma,\end{aligned}$$

as well as the spaces with vanishing traces at $t=0$,

$$\begin{aligned}{}_0\mathbb{E}(T) &:= \mathbb{E}(T) \cap \{(h,\mu) \in \mathbb{E}(T) : h|_{t=0} = 0\},\\ {}_0\mathbb{F}(T) &:= \mathbb{F}(T) \cap \{(g_1,g_2,g_3,g_4,g_5,g_6) \in \mathbb{F}(T) : g_5|_{t=0} = 0\}.\end{aligned}$$

2.6.1. Main result. The main result reads as follows.

THEOREM 2.28. *Let $p \in (6,\infty)$, $q \in (5/3,2) \cap (2p/(p+1),2p)$, and $h_0 \in X_\gamma$. Then there is some $\tau_0 > 0$, such that for every $0 < \tau < \tau_0$, there is some $\delta = \delta(\tau) > 0$, such that (2.31) has a unique strong solution on $(0,\tau)$, that is, there are*

$$h \in W_p^1(0,\tau;X_0) \cap L_p(0,\tau;X_1), \quad \mu \in L_p(0,\tau;W_q^2(\Omega\backslash\Sigma)),$$

solving (2.31) on $(0,\tau)$, whenever h_0 satisfies the initial compatibility condition

$$n_{\partial\Omega}^{h_0} \cdot n_\Gamma^{h_0} = 0, \quad \text{on } \partial\Sigma,$$

and the smallness condition $|h_0|_{X_\gamma} \le \delta(\tau)$.

PROOF. Define the linear operator $L : \mathbb{E}(T) \to \mathbb{F}(T)$ by

$$L(h,\mu) = \begin{pmatrix} \partial_t h + [\![n_\Sigma \cdot \nabla\mu]\!] \\ \mu|_\Sigma - P(0)h \\ \Delta\mu \\ n_{\partial\Omega} \cdot \nabla\mu|_{\partial\Omega} \\ n_{\partial\Sigma} \cdot \nabla_\Sigma h|_{\partial\Sigma} \\ h|_{t=0} \end{pmatrix}.$$

We now reduce to trivial initial data as follows, cf. [**39**], [**64**]. We can not directly solve the linear problem with right hand side $(0,0,0,0,0,h_0)$, since h_0 does not satisfy $n_{\partial\Sigma} \cdot \nabla_\Sigma h_0 = 0$ on $\partial\Sigma$. However, by a standard extension argument and solving an auxiliary problem, cf. [**58**], we may find $\tilde{h} \in W_p^1(\mathbb{R}_+;X_0) \cap L_p(\mathbb{R}_+;X_1)$, such that $\tilde{h}(t=0) = h_0$. We may then solve

$$Lz_* = (0,0,0,0,b^*,h_0), \tag{2.64}$$

by some $z_* = (h_*,\mu_*) \in \mathbb{E}(T)$, where $b^* := n_{\partial\Sigma} \cdot \nabla_\Sigma \tilde{h}|_{\partial\Sigma} - n_{\partial\Omega}^{\tilde{h}} \cdot n_\Gamma^{\tilde{h}}$. Note that the necessary compatibility condition for (2.64) is satisfied,

$$n_{\partial\Sigma} \cdot \nabla_\Sigma h_0|_{\partial\Sigma} = b^*(t=0),$$

cf. the analysis of the linear problem in Section 2.5. Then the problem (2.31) is equivalent to finding some $z = (h,\mu) \in {}_0\mathbb{E}(T)$ solving

$$L(z) = N(z+z_*) - Lz_* =: \tilde{N}(z), \quad \text{in } {}_0\mathbb{F}(T),$$

where the nonlinear part is given by

$$N(z+z_*) := \begin{pmatrix} [\![n_\Sigma \cdot \nabla(\mu+\mu_*)]\!] - \frac{1}{a(h+h_*)}[\![n_\Sigma^{h+h_*} \cdot \nabla_{h+h_*}(\mu+\mu_*)]\!] \\ K(h+h_*) - P(0)(h+h_*) \\ (\Delta - \Delta_{h+h_*})(\mu+\mu_*) \\ n_{\partial\Omega} \cdot \nabla(\mu+\mu_*)|_{\partial\Omega} - n_{\partial\Omega}^{h+h_*} \cdot \nabla_{h+h_*}(\mu+\mu_*)|_{\partial\Omega} \\ n_{\partial\Sigma} \cdot \nabla_\Sigma(h+h_*)|_{\partial\Sigma} - n_{\partial\Sigma}^{h+h_*} \cdot n_\Sigma^{h+h_*} \\ h_0 \end{pmatrix}.$$

We may now define $\mathsf{K} : {}_0\mathbb{E}(T) \to {}_0\mathbb{E}(T)$ by $[z \mapsto L^{-1}\tilde{N}(z) = L^{-1}(N(z+z_*) - Lz_*)]$. By restricting to functions with vanishing trace at time zero, we get that the operator norm $|L^{-1}|_{\mathcal{B}({}_0\mathbb{F}(T);{}_0\mathbb{E}(T))}$ stays bounded as $T \to 0$.

LEMMA 2.29. *Let $T > 0$, $\delta > 0$, $r_0 > 0$, and $|h_0|_{X_\gamma} \le \delta$. The mapping $N : \mathbb{E}(T) \to \mathbb{F}(T)$ is well-defined and bounded. Furthermore, N allows for contraction estimates in a neighbourhood of zero, that is,*

$$\begin{aligned} &|N(z_1+z_*) - N(z_2+z_*)|_{{}_0\mathbb{F}(T)} \\ &\qquad \le C(r_0)(T^{1/p} + r + C(T)\delta)|z_1 - z_2|_{{}_0\mathbb{E}(T)}, \end{aligned} \tag{2.65}$$

for all $z_1, z_2 \in \mathsf{B}(r;0) \subset {}_0\mathbb{E}(T)$, if $0 < r \le r_0$ and $T = T(r) > 0$, $\delta = \delta(T) > 0$ are sufficiently small. Here, $\mathsf{B}(r;0)$ denotes the closed ball around 0 with radius $r > 0$.

Let now $\delta > 0$, such that $|h_0|_{X_\gamma} \le \delta$. Note at this point that $|z_*|_{\mathbb{E}(T)} \le C(T)|h_0|_{X_\gamma} \le C(T)\delta$. By choosing $r > 0$, $T = T(r) > 0$ and $\delta = \delta(T) > 0$ sufficiently small, we ensure K to be a $1/2$-contraction on $\mathsf{B}(r,0) \subset {}_0\mathbb{E}(T)$. Indeed, by Lemma 2.29,

$$\begin{aligned} |\mathsf{K}(z_1) - \mathsf{K}(z_2)|_{{}_0\mathbb{E}(T)} &= |L^{-1}[\tilde{N}(z_1) - \tilde{N}(z_2)]|_{{}_0\mathbb{E}(T)} \\ &\le |L^{-1}|_{\mathcal{B}({}_0\mathbb{F}(T);{}_0\mathbb{E}(T))}|N(z_1+z_*) - N(z_2+z_*)|_{{}_0\mathbb{F}(T)} \\ &\le C(T^{1/p} + |z_1|_{{}_0\mathbb{E}(T)} + |z_2|_{{}_0\mathbb{E}(T)} + |z_*|_{\mathbb{E}(T)})|z_1 - z_2|_{{}_0\mathbb{E}(T)} \\ &\le C(T^{1/p} + 2r + C(T)\delta)|z_1 - z_2|_{{}_0\mathbb{E}(T)}, \end{aligned}$$

if $z_1, z_2 \in \mathsf{B}(r,0) \subset {}_0\mathbb{E}(T)$, and $r > 0$, $T = T(r) > 0$ sufficiently small. By choosing $r > 0$, $T = T(r) > 0$ and $\delta = \delta(T) > 0$ even smaller, we see that K is a $1/2$-contraction on $\mathsf{B}(r,0) \subset {}_0\mathbb{E}(T)$.

Let us note that

$$\tilde{N}(0) = N(z_*) - Lz_*, \quad \mathsf{K}(0) = L^{-1}\tilde{N}(0).$$

Furthermore, $\tilde{N}(0) \in {}_0\mathbb{F}(T)$, whence

$$|\mathsf{K}(0)|_{{}_0\mathbb{E}(T)} \le |L^{-1}|_{\mathcal{B}({}_0\mathbb{F}(T);{}_0\mathbb{E}(T))}|\tilde{N}(0)|_{{}_0\mathbb{F}(T)}.$$

Now we note that $\tilde{N}(0)$ has quadratic growth in $z_* = (h_*, \mu_*)$ at zero, except for the term $Q(h_*)$ in $\tilde{N}(0)_2$. By Lemma 2.15,

$$\begin{aligned} &|Q(h_*)|_{L_p(0,T;W_q^{2-1/q}(\Sigma))} \\ &\qquad \le T^{1/p}|Q(h_*) - Q(0)|_{L_\infty(0,T;W_q^{2-1/q}(\Sigma))} + T^{1/p}|Q(0)|_{L_\infty(0,T;W_q^{2-1/q}(\Sigma))} \\ &\qquad \le C(T)|z_*|_{\mathbb{E}(T)} + CT^{1/p}, \end{aligned}$$

if $|h_*|_{\mathbb{E}(T)}$ is small. Moreover,

$$|z_*|_{\mathbb{E}(T)} \leq |L^{-1}|_{\mathcal{B}(\mathbb{F}(T);\mathbb{E}(T))}|h_0|_{X_\gamma} \leq C(T)\delta.$$

Hence, if $z_1 \in \mathsf{B}(r,0)$, $r > 0$ sufficiently small,

$$\begin{aligned}|\mathsf{K}(z_1)|_{{}_0\mathbb{E}(T)} &\leq |\mathsf{K}(z_1) - K(0)|_{{}_0\mathbb{E}(T)} + |K(0)|_{{}_0\mathbb{E}(T)} \\ &\leq |z_1|_{{}_0\mathbb{E}(T)}/2 + |\mathsf{K}(0)|_{{}_0\mathbb{E}(T)} \\ &\leq |z_1|_{{}_0\mathbb{E}(T)}/2 + C|\tilde{N}(0)|_{{}_0\mathbb{F}(T)} \\ &\leq |z_1|_{{}_0\mathbb{E}(T)}/2 + C(|z_*|^2_{\mathbb{E}(T)} + |Q(h_*)|_{L_p(0,T;W_q^{2-1/q}(\Sigma))}) \\ &\leq |z_1|_{{}_0\mathbb{E}(T)}/2 + C(C(T)^2\delta^2 + C(T)\delta + T^{1/p}).\end{aligned}$$

This entails that K maps $\mathsf{B}(r,0) \subset {}_0\mathbb{E}(T)$ to $\mathsf{B}(r,0) \subset {}_0\mathbb{E}(T)$ again, provided first $r > 0, T = T(r) > 0$, and then $\delta = \delta(T) > 0$ are chosen small enough. Note that we can further decrease $r > 0$, $T > 0$, and $\delta > 0$ and K still is self-map and contraction on $\mathsf{B}(r,0) \subset {}_0\mathbb{E}(T)$. Hence the Banach fixed point principle yields the existence of a unique fixed point $\bar{z} \in \mathsf{B}(r,0) \subset {}_0\mathbb{E}(T)$.

We now show that the solution is unique in the sense that if there is another solution, it exists on $(0,T)$ and coincides there with the above fixed point. Assume therefore that there is a second solution given as $z_* + z_2$ for $z_2 \in {}_0\mathbb{E}(T_2)$, with norm maybe bigger than $r > 0$ and time of existence T_2 different from T. Note that this is no contradiction to the uniqueness argument stemming from Banach's fixed point theorem above. We now show that z_2 exists at least on $(0,T)$ and coincides with $\bar{z}$ on $(0,T)$. Assume that

$$|z_2|_{{}_0\mathbb{E}(T_2)} = R_2, \tag{2.66}$$

for some $T_2 > 0$, $R_2 < \infty$. Since the norm in (2.66) is an integral norm,

$$|z_2|_{{}_0\mathbb{E}(T_2)} \to 0, \quad \text{as } T_2 \to 0.$$

Hence there exists $\tilde{T}_2 = \tilde{T}_2(z_2) > 0$, such that

$$|z_2|_{{}_0\mathbb{E}(\tilde{T}_2)} \leq r.$$

Performing the above fixed point argument on the closed ball $\mathsf{B}(r,0)$ in the space ${}_0\mathbb{E}(\min(T,\tilde{T}_2))$ we see that by uniqueness $\bar{z}$ and z_2 coincide on $(0,\min(T,\tilde{T}_2))$. Let

$$T_* := \sup\{t \geq 0 : \bar{z}(s) = z_2(s), 0 \leq s \leq t\}.$$

By the above arguments, $T_* > 0$. Assume that $T_* < T$, otherwise there is nothing to prove. We now want to solve the full nonlinear problem with initial value $(z_*+\bar{z})(T_*)$. Clearly, this belongs to the interpolation space X_γ and satisfies the relevant nonlinear compatibility condition since $z_* + \bar{z}$ solves the nonlinear problem on $(0,T_*)$. It remains to verify the smallness condition. We have

$$\begin{aligned}|(z_* + \bar{z})(T_*)|_{X_\gamma} &\leq |z_*|_{L_\infty(0,T;X_\gamma)} + |\bar{z}|_{L_\infty(0,T;X_\gamma)} \\ &\leq C(T)|h_0|_{X_\gamma} + C'|\bar{z}|_{{}_0\mathbb{E}(T)},\end{aligned} \tag{2.67}$$

where C' is explicitly independent of T since $\bar{z}$ has vanishing time trace. By decreasing $|h_0|_{X_\gamma}$ and the radius of the ball in the fixed point argument where the fixed point is unique, we can achieve that the right hand side of (2.67) is again bounded by $\delta > 0$. Hence we may restart the flow of the nonlinear problem with initial value $(z_* + \bar{z})(T_*)$ at time T_* to conclude that there is a unique solution on the time interval $(0, T_* + \epsilon_*)$, for some $\epsilon_* > 0$. This contradicts the definition of T_*, whence the claim is shown. $\square$

Proof of Lemma 2.29. To economize notation, $z_1 = (h_1, \mu_1)$, $z_2 = (h_2, \mu_2)$. We want to estimate $N(z_1+z_*) - N(z_2+z_*)$. We write down every component seperately. Recall that for functions with vanishing time trace at $t = 0$, the embedding constant in ${}_0\mathbb{E}(T) \hookrightarrow L_\infty(0, T; X_\gamma)$ is independent of T.

Estimates for N_1. We have

$$\begin{aligned} &N_1(z_1 + z_*) - N_1(z_2 + z_*) \\ &\quad = [\![n_\Sigma \cdot \nabla(\mu_1 + \mu_*)]\!] - \frac{1}{a(h_1 + h_*)} [\![n_\Sigma^{h_1+h_*} \cdot \nabla_{h_1+h_*}(\mu_1 + \mu_*)]\!] \\ &\quad - [\![n_\Sigma \cdot \nabla(\mu_2 + \mu_*)]\!] + \frac{1}{a(h_2 + h_*)} [\![n_\Sigma^{h_2+h_*} \cdot \nabla_{h_2+h_*}(\mu_2 + \mu_*)]\!]. \end{aligned} \tag{2.68}$$

Recall, cf. (2.30), that a depends smoothly on h and $a(0) = 1$. If h is small, a^{-1} is well-defined and also depends smoothly on h. Moreover $n_\Sigma^h = n_\Sigma$ and $\nabla_h = \nabla$ for $h = 0$. Moreover we recall the properties of n_Σ^h and ∇_h stated in Lemma 2.17. Using the product estimate

$$\begin{aligned} &|a^{-1}(g_1) [\![n_\Sigma^{g_2} \cdot \nabla_{g_3} \tilde{\mu}]\!]|_{L_p(0,T;X_0)} \\ &\quad \le |a^{-1}(g_1)|_{L_\infty(0,T;C^1(\Sigma))} |n_\Sigma^{g_2}|_{L_\infty(0,T;C^1(\Sigma))} |\nabla_{g_3} \tilde{\mu}|_{L_p(0,T;W_q^1(\Omega^+))}, \end{aligned}$$

for $g_1, g_2, g_3 \in [\mathbb{E}(T)]_1$, $\tilde{\mu} \in [\mathbb{E}(T)]_2$, gives estimates of form (2.65) for N_1 using the structure of N_1 in (2.68).

Estimates for N_2. Recall that by Lemma 2.15,

$$\begin{aligned} N_2(z_1 + z_*) - N_2(z_2 + z_*) &= [P(h_1 + h_*) - P(0)](h_1 - h_2) \\ &\quad + [P(h_1 + h_*) - P(h_2 + h_*)](h_2 + h_*) \\ &\quad + Q(h_1 + h_*) - Q(h_2 + h_*). \end{aligned}$$

By Lemma 2.15,

$$|Q(h_1(t) + h_*(t)) - Q(h_2(t) + h_*(t))|_{W_q^{2-1/q}(\Sigma)} \lesssim |h_1(t) - h_2(t)|_{X_\gamma},$$

for almost every $t \in (0, T)$. Integration in time gives

$$|Q(h_1 + h_*) - Q(h_2 + h_*)|_{L_p(0,T;W_q^{2-1/q}(\Sigma))} \lesssim T^{1/p} |h_1 - h_2|_{L_\infty(0,T;X_\gamma)}.$$

Also for almost every $t \in (0, T)$,

$$\begin{aligned} &|[P(h_1(t) + h_*(t)) - P(0)](h_1(t) - h_2(t))|_{W_q^{2-1/q}(\Sigma)} \\ &\quad \lesssim |P(h_1(t) + h_*(t)) - P(0)|_{\mathcal{B}(W_q^{4-1/q}(\Sigma); W_q^{2-1/q}(\Sigma))} |h_1(t) - h_2(t)|_{W_q^{4-1/q}(\Sigma)}. \end{aligned}$$

Inferring differentiability of P from Lemma 2.15,

$$\begin{aligned} &|[P(h_1 + h_*) - P(0)](h_1 - h_2)|_{L_p(0,T;W_q^{2-1/q}(\Sigma))} \\ &\quad \lesssim |h_1 + h_*|_{L_\infty(0,T;X_\gamma)} |h_1 - h_2|_{L_p(0,T;W_q^{4-1/q}(\Sigma))}. \end{aligned}$$

The other term is estimated in the same way.

Estimates for N_3. We have

$$\begin{aligned}&N_3(z_1+z_*)-N_3(z_2+z_*)\\&\qquad=(\Delta-\Delta_{h_1+h_*})(\mu_1-\mu_2)+(\Delta_{h_2+h_*}-\Delta_{h_1+h_*})(\mu_2+\mu_*).\end{aligned}$$

Hence

$$\begin{aligned}&|N_3(z_1+z_*)-N_3(z_2+z_*)|_{L_p(0,T;L_q(\Omega))}\\&\quad\lesssim(|h_1|_{L_\infty(0,T;X_\gamma)}+|h_*|_{L_\infty(0,T;X_\gamma)})|\mu_1-\mu_2|_{L_p(0,T;W^2_q(\Omega\setminus\Sigma))}\\&\quad+|h_1-h_2|_{L_\infty(0,T;X_\gamma)}\big(|\mu_2|_{L_p(0,T;W^2_q(\Omega\setminus\Sigma))}+|\mu_*|_{L_p(0,T;W^2_q(\Omega\setminus\Sigma))}\big),\end{aligned}$$

which gives estimates for N_3.

Estimates for N_4. Note

$$\begin{aligned}N_4(z_1+z_*)-N_4(z_2+z_*)&=n^{h_2+h_*}_{\partial\Omega}\cdot(\nabla_{h_2+h_*}-\nabla)(\mu_2-\mu_1)\\&+n^{h_2+h_*}_{\partial\Omega}\cdot(\nabla_{h_2+h_*}-\nabla_{h_1+h_*})(\mu_1+\mu_*)\\&+(n^{h_2+h_*}_{\partial\Omega}-n^{h_1+h_*}_{\partial\Omega})\cdot(\nabla_{h_1+h_*}-\nabla)(\mu_1+\mu_*)\\&+(n^{h_2+h_*}_{\partial\Omega}-n_{\partial\Omega})\cdot\nabla(\mu_2-\mu_1)\\&+(n^{h_2+h_*}_{\partial\Omega}-n^{h_1+h_*}_{\partial\Omega})\cdot\nabla(\mu_1+\mu_*).\end{aligned}$$

Lemma 2.17 together with the product estimate

$$|fg|_{W^{1-1/q}_q(\partial\Omega)}\lesssim|f|_{C^1(\partial\Omega)}|g|_{W^{1-1/q}_q(\partial\Omega)},\tag{2.69}$$

for all $f\in C^1(\partial\Omega)$, $g\in W^{1-1/q}_q(\partial\Omega)$, give estimates for N_4. For a proof of (2.69) we refer to [**62**].

Estimates for N_5. The estimates for N_5 are more involved. Firstly,

$$\begin{aligned}&N_5(z_1+z_*)-N_5(z_2+z_*)\\&\qquad=n_{\partial\Sigma}\cdot\nabla_\Sigma(h_1-h_2)-[n^{h_1+h_*}_{\partial\Sigma}\cdot n^{h_1+h_*}_{\Sigma}-n^{h_2+h_*}_{\partial\Sigma}\cdot n^{h_2+h_*}_{\Sigma}].\end{aligned}\tag{2.70}$$

Linking now the surface gradient to the normal of the interfaces is now a complicated matter due to the curved boundary of the domain. For better readability we dedicate the next subsection to the proof of contraction estimates for (2.70). ☐

Remark 2.30. We point out that the proof of Theorem 2.28 also gives well-posedness of (2.31) in the case where $\Omega=G\times(L_1,L_2)$ is a bounded, cylindrical container in $\mathbb{R}^n$, $n=2,3$. Hereby $G\subset\mathbb{R}^{n-1}$ is a smooth, bounded domain. In this case there is another model problem in the localization procedure for the linear problem stemming from when the top and bottom of the container $G\times\{L_1,L_2\}$ intersect the walls $\partial G\times(L_1,L_2)$. This elliptic problem, although being a problem on a domain with corners, admits full regularity for the solution, cf. the Appendix in Section A.2.

2.6.2. Contraction estimates for the $90°$-angle condition. To start with, we recall the basic situation of a cylindrical domain $\Omega = G \times (L_1, L_2)$, where the free interface is given as a graph over the flat reference surface $G \times \{0\} \subset \mathbb{R}^n$. Here, the surface is given as the set $\{(x_1, ..., x_{n-1}, f(x_1, ..., x_{n-1})) : (x_1, ..., x_{n-1}) \in G\}$, for some sufficiently regular function f. If $n = 2$, the normal direction is given as a rotation of the tangent vector $(1, f'(x_1))$, namely $(-f'(x_1), 1)$. If $n = 3$, the normal direction is given as the cross product of the two tangent vectors $(1, 0, \partial_1 f(x_1, x_2))$ and $(0, 1, \partial_2 f(x_1, x_2))$, which is well known to be $(-\nabla f(x_1, x_2), 1)$. In this simple geometry, the connection between the gradient of the height function, here f, and the normal to the graph of f is obvious. The important ingredient here is the fact that there is no need for tangential correction and curvilinear coordinates. In the situation of a smooth, curved domain with curved reference surface we now establish a similar correspondence.

So let us consider the case where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded, smooth domain. Recall that for a suitable height function h, the free surface is given as $\Gamma_{h(t)} = \{x \in \mathbb{R}^n : x = X(s, h(s,t)) : s \in \Sigma\}$, where X denotes the curvilinear coordinate system in a neighbourhood of Σ, cf. [**63**]. The boundary condition then reads as

$$n_{\partial\Omega}(\bar{q}) \cdot n_{\Gamma_{h(t)}}(\bar{q}) = 0, \quad \bar{q} \in \partial\Gamma_{h(t)}. \tag{2.71}$$

Note that we can replace the unit normal $n_{\Gamma_{h(t)}}$ with any normal direction vector of $\Gamma_{h(t)}$ in (2.71). Let us also remark that the linearization of the boundary condition (2.71) with respect to the unit normal vector was already calculated in [**14**]. In our case it will later turn out to be convenient to replace the unit normal vector with a suitable normal direction vector. To then account for this change and be able to deduce contraction estimates for (2.70), we will modify the curvilinear coordinates X in the following way.

Consider a smooth function $f : \overline{\Sigma} \to \mathbb{R}$, with the property that either $0 < c_0 \le f(p) \le C_0$, or $-C_0 \le f(p) \le -c_0 < 0$, for all $p \in \overline{\Sigma}$ and some constants $c_0, C_0 > 0$. Then, $\tilde{X}(p, w) := p + wf(p)n_\Sigma(p) + \tilde{t}(w,p)\vec{T}(p)$ is also a curvilinear coordinate system in the sense of [**63**], with the properties $\tilde{X}(p, 0) = p$, for all $p \in \Sigma$, and $\partial_w \tilde{X}(p, 0) = f(p)\nu_\Sigma(p)$. Compared to the case where $f \equiv 1$, the correction function $\tilde{t}(w,p)$ changes, the ninety degree angle condition of Σ and $\partial\Omega$ however ensures that $\partial_w \tilde{t}(p, 0) = 0$, for any $p \in \partial\Sigma$. Basically, this approach amounts to the construction of curvilinear coordinates in Proposition 3.2 in [**63**] by pointwise solving a differential equation in normal direction to the interface. For readability, we refer to $\tilde{X}$ again as X.

For readability, we consider the involved case where $n = 3$. Then Σ is a two-dimensional surface, which may be parametrized locally by $\varphi : U \subset \mathbb{R}^2 \to \Sigma$. At a point on the surface Σ, the two tangent vectors are given by $\tau_j(p) := \tau_j^\Sigma(p) := \partial_j\varphi(s)$, $s = \varphi^{-1}(p)$, $p \in \Sigma$, $j = 1, 2$. Note that the parametrization of the free interface $\Gamma_{h(t)}$ then is given as $[s \mapsto X(\varphi(s), h(\varphi(s), t))]$. Differentiating with respect to s_j gives a tangential vector to $\Gamma_{h(t)}$,

$$[D_p X(\varphi(s), h(\varphi(s), t)D_s\varphi(s)]e_j + \partial_w X(\varphi(s), h(\varphi(s), t))\nabla h(\varphi(s), t) \cdot \partial_j\varphi(s).$$

For readability, we surpress the dependence of p and define $W_j(h) := [D_pX(h)D_s\varphi]e_j$, $j = 1, 2$, as well as

$$\tau_j^h := W_j(h) + \partial_w X(h)\partial_{\bar{\tau}_j} h, \quad j = 1, 2.$$

The normal direction at the free interface in terms of h is therefore given by

$$\tau_1^h \times \tau_2^h, \tag{2.72}$$

the normal direction to the reference surface Σ by $\tau_1 \times \tau_2$. We now want to remark that

$$W_j(0) = \tau_j^\Sigma, \quad j = 1, 2.$$

This follows from the fact that D_pX is the identity on $T_p\Sigma$, which in turn stems from differentiating the initial condition $X(p, 0) = p$ with respect to p. Concludingly, the boundary equation (2.71) is equivalent to

$$n_{\partial\Omega}^h(\bar{q}) \cdot (\tau_1^h \times \tau_2^h)(\bar{q}) = 0, \quad \bar{q} \in \partial\Sigma. \tag{2.73}$$

We can therefore replace the unit normals in N_5 with the corresponding normal direction vector of (2.73). We now rewrite the surface gradient in a suitable way. Firstly,

$$\nabla_\Sigma h = \sum_{i,j} g^{ij}(\partial_{\tau_j} h)\tau_i,$$

where (g_{ij}) is the first fundamental form of Σ,

$$g_{ij} = \begin{pmatrix} \tau_1 \cdot \tau_1 & \tau_1 \cdot \tau_2 \\ \tau_1 \cdot \tau_2 & \tau_2 \cdot \tau_2 \end{pmatrix},$$

and (g^{ij}) its inverse,

$$g^{ij} = \frac{1}{\det g} \begin{pmatrix} \tau_2 \cdot \tau_2 & -\tau_1 \cdot \tau_2 \\ -\tau_1 \cdot \tau_2 & \tau_1 \cdot \tau_1 \end{pmatrix}.$$

Using the well known vector identities

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c, \quad (a \times b) \times c = (a \cdot c)b - (b \cdot c)a, \quad a, b, c \in \mathbb{R}^3,$$

we readily obtain

$$(\tau_1 \times \tau_2) \times \tau_2 = (\tau_1 \cdot \tau_2)\tau_2 - (\tau_2 \cdot \tau_2)\tau_1, \quad \tau_1 \times (\tau_1 \times \tau_2) = (\tau_1 \cdot \tau_2)\tau_1 - (\tau_1 \cdot \tau_1)\tau_2.$$

In particular,

$$\begin{aligned} \nabla_\Sigma h &= (g^{11}\tau_1 + g^{12}\tau_2)\partial_{\tau_1} h + (g^{12}\tau_1 + g^{22}\tau_2)\partial_{\tau_2} h \\ &= (\det g)^{-1}[(\tau_2 \cdot \tau_2)\tau_1 + (-\tau_1 \cdot \tau_2)\tau_2]\partial_{\tau_1} h \\ &+ (\det g)^{-1}[(-\tau_1 \cdot \tau_2)\tau_1 + (\tau_1 \cdot \tau_1)\tau_2]\partial_{\tau_2} h \\ &= (\det g)^{-1}[-(\tau_1 \times \tau_2) \times \tau_2]\partial_{\tau_1} h \\ &+ (\det g)^{-1}[-\tau_1 \times (\tau_1 \times \tau_2)]\partial_{\tau_2} h. \end{aligned}$$

Choose the curvilinear coordinates X in such a way that

$$\partial_w X(p, 0) = -\frac{(\tau_1 \times \tau_2)(p)}{\det g(p)}, \quad p \in \Sigma.$$

Note that this is possible since $(\tau_1 \times \tau_2)(p)$ is a multiple of $n_\Sigma(p)$, for every $p \in \Sigma$. In short notation, we may rewrite the surface gradient as

$$\nabla_\Sigma h = [\partial_w X(0) \times \tau_2]\partial_1 h + [\tau_1 \times \partial_w X(0)]\partial_2 h. \tag{2.74}$$

As shown before,

$$\tau_1^h \times \tau_2^h = [W_1(h) \times W_2(h)] + [\partial_w X(h) \times W_2(h)]\partial_1 h + [W_1(h) \times \partial_w X(h)]\partial_2 h. \tag{2.75}$$

Going back to (2.70), where the unit normals at the free surface are replaced by the direction vectors of (2.72),

$$\begin{aligned} &N_5(z_1 + z_*) - N_5(z_2 + z_*) \\ &= n_{\partial\Sigma} \cdot \nabla_\Sigma (h_1 - h_2) - [n_{\partial\Sigma}^{h_1+h_*} \cdot (\tau_1^{h_1+h_*} \times \tau_2^{h_1+h_*}) - n_{\partial\Sigma}^{h_2+h_*} \cdot (\tau_1^{h_2+h_*} \times \tau_2^{h_2+h_*})]. \end{aligned} \tag{2.76}$$

We now have to give estimates for the right hand side of (2.76). For readability, $\tilde{h}_j := h_j + h_*$, $j = 1, 2$. By (2.75),

$$\begin{aligned} n_{\partial\Sigma}^{\tilde{h}_j} \cdot (\tau_1^{\tilde{h}_j} \times \tau_2^{\tilde{h}_j}) = n_{\partial\Sigma}^{\tilde{h}_j} &\cdot [W_1(\tilde{h}_j) \times W_2(\tilde{h}_j)] \\ &+ n_{\partial\Sigma}^{\tilde{h}_j} \cdot [\partial_w X(\tilde{h}_j) \times W_2(\tilde{h}_j)]\partial_1 \tilde{h}_j \\ &+ n_{\partial\Sigma}^{\tilde{h}_j} \cdot [W_1(\tilde{h}_j) \times \partial_w X(\tilde{h}_j)]\partial_2 \tilde{h}_j. \end{aligned}$$

Hence

$$\begin{aligned} N_5(z_1 + z_*) - N_5(z_2 + z_*) = n_{\partial\Sigma} &\cdot \nabla_\Sigma (\tilde{h}_1 - \tilde{h}_2) \\ &+ n_{\partial\Sigma}^{\tilde{h}_2} \cdot [W_1(\tilde{h}_2) \times W_2(\tilde{h}_2)] \\ &- n_{\partial\Sigma}^{\tilde{h}_1} \cdot [W_1(\tilde{h}_1) \times W_2(\tilde{h}_1)] \\ &+ n_{\partial\Sigma}^{\tilde{h}_2} \cdot [\partial_w X(\tilde{h}_2) \times W_2(\tilde{h}_2)]\partial_1 \tilde{h}_2 \\ &- n_{\partial\Sigma}^{\tilde{h}_1} \cdot [\partial_w X(\tilde{h}_1) \times W_2(\tilde{h}_1)]\partial_1 \tilde{h}_1 \\ &+ n_{\partial\Sigma}^{\tilde{h}_2} \cdot [W_1(\tilde{h}_2) \times \partial_w X(\tilde{h}_2)]\partial_2 \tilde{h}_2 \\ &- n_{\partial\Sigma}^{\tilde{h}_1} \cdot [W_1(\tilde{h}_1) \times \partial_w X(\tilde{h}_1)]\partial_2 \tilde{h}_1. \end{aligned}$$

By (2.74),

$$\begin{aligned} N_5(z_1 + z_*) - N_5(z_2 + z_*) = n_{\partial\Sigma}^{\tilde{h}_2} &\cdot [W_1(\tilde{h}_2) \times W_2(\tilde{h}_2)] \\ &- n_{\partial\Sigma}^{\tilde{h}_1} \cdot [W_1(\tilde{h}_1) \times W_2(\tilde{h}_1)] \\ &+ n_{\partial\Sigma}^{\tilde{h}_2} \cdot [\partial_w X(\tilde{h}_2) \times W_2(\tilde{h}_2)]\partial_1 \tilde{h}_2 \\ &- n_{\partial\Sigma}^{\tilde{h}_1} \cdot [\partial_w X(\tilde{h}_1) \times W_2(\tilde{h}_1)]\partial_1 \tilde{h}_1 \\ &+ n_{\partial\Sigma} \cdot [\partial_w X(0) \times \tau_2]\partial_1 (\tilde{h}_1 - \tilde{h}_2) \\ &+ n_{\partial\Sigma}^{\tilde{h}_2} \cdot [W_1(\tilde{h}_2) \times \partial_w X(\tilde{h}_2)]\partial_2 \tilde{h}_2 \\ &- n_{\partial\Sigma}^{\tilde{h}_1} \cdot [W_1(\tilde{h}_1) \times \partial_w X(\tilde{h}_1)]\partial_2 \tilde{h}_1 \\ &+ n_{\partial\Sigma} \cdot [\tau_1 \times \partial_w X(0)]\partial_2 (\tilde{h}_1 - \tilde{h}_2). \end{aligned}$$

Define

$$
\begin{aligned}
N_{5,1} &:= n_{\partial\Sigma}^{\tilde h_2}\cdot[W_1(\tilde h_2)\times W_2(\tilde h_2)] - n_{\partial\Sigma}^{\tilde h_1}\cdot[W_1(\tilde h_1)\times W_2(\tilde h_1)],\\
N_{5,2} &:= n_{\partial\Sigma}^{\tilde h_2}\cdot[\partial_w X(\tilde h_2)\times W_2(\tilde h_2)]\partial_1\tilde h_2 - n_{\partial\Sigma}^{\tilde h_1}\cdot[\partial_w X(\tilde h_1)\times W_2(\tilde h_1)]\partial_1\tilde h_1\\
&\quad + n_{\partial\Sigma}\cdot[\partial_w X(0)\times\tau_2]\partial_1(\tilde h_1-\tilde h_2),\\
N_{5,3} &:= n_{\partial\Sigma}^{\tilde h_2}\cdot[W_1(\tilde h_2)\times\partial_w X(\tilde h_2)]\partial_2\tilde h_2 - n_{\partial\Sigma}^{\tilde h_1}\cdot[W_1(\tilde h_1)\times\partial_w X(\tilde h_1)]\partial_2\tilde h_1\\
&\quad + n_{\partial\Sigma}\cdot[\tau_1\times\partial_w X(0)]\partial_2(\tilde h_1-\tilde h_2).
\end{aligned}
$$

We estimate each term seperately. Remark that since $h_1,h_2\in{}_0\mathbb{E}(T)$, $\tilde h_1(t=0)=\tilde h_2(t=0)=h_*(t=0)=h_0$. Therefore we need to be careful when applying the product estimate of Theorem 2.4 or the contraction estimate of Lemma 2.10. By Lemma 2.10,

$$
\begin{aligned}
&|N_{5,1}|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\\
&\qquad\le C(r)|\tilde h_2-\tilde h_1|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\\
&\qquad\le C(r)|h_2-h_1|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}.
\end{aligned}
$$

By Lemma 2.24, there is some $\epsilon>0$ independent of r and T, such that

$$
|N_{5,1}|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\le T^\epsilon C(r)|h_2-h_1|_{{}_0\mathbb{E}(T)}.
$$

Let us now estimate $N_{5,2}$. Define $\tilde F(\tilde u):=n_{\partial\Sigma}^{\tilde u}\cdot[\partial_w X(\tilde u)\times W_2(\tilde u)]$. We rewrite

$$
N_{5,2}=(\tilde F(\tilde h_2)-\tilde F(\tilde h_1))\partial_1\tilde h_2+(\tilde F(0)-\tilde F(\tilde h_1))\partial_1(\tilde h_1-\tilde h_2).
$$

Recall that $\tilde h_j(t=0)=h_0$. We rewrite the first term as

$$
(\tilde F(\tilde h_2)-\tilde F(\tilde h_1))\partial_1\tilde h_2=(\tilde F(\tilde h_2)-\tilde F(\tilde h_1))\partial_1 h_2+(\tilde F(\tilde h_2)-\tilde F(\tilde h_1))\partial_1 h_*.
$$

Hence

$$
\begin{aligned}
&|(\tilde F(\tilde h_2)-\tilde F(\tilde h_1))\partial_1\tilde h_2|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\\
&\quad\le|(\tilde F(\tilde h_2)-\tilde F(\tilde h_1))\partial_1 h_2|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\\
&\quad+|(\tilde F(\tilde h_2)-\tilde F(\tilde h_1))\partial_1 h_*|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\\
&\quad\le C|(\tilde F(\tilde h_2)-\tilde F(\tilde h_1))|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\times\\
&\qquad\times|\partial_1 h_2|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\\
&\quad+C(T)|(\tilde F(\tilde h_2)-\tilde F(\tilde h_1))|_{{}_0F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\times\\
&\qquad\times|\partial_1 h_*|_{F_{pq}^{1-2/3q}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B_{qq}^{3-2/q}(\partial\Sigma))}\\
&\quad\le CC(r_0)|h_2-h_1|_{{}_0\mathbb{E}(T)}|h_2|_{{}_0\mathbb{E}(T)}+C(T)C(r_0)|h_2-h_1|_{{}_0\mathbb{E}(T)}|h_*|_{\mathbb{E}(T)}\\
&\quad\le CC(r_0)|h_2-h_1|_{{}_0\mathbb{E}(T)}|h_2|_{{}_0\mathbb{E}(T)}+C(T)C(r_0)|h_2-h_1|_{{}_0\mathbb{E}(T)}|h_0|_{X_\gamma}\\
&\quad\le CC(r_0)r|h_2-h_1|_{{}_0\mathbb{E}(T)}+C(T)C(r_0)|h_2-h_1|_{{}_0\mathbb{E}(T)}\delta.
\end{aligned}
$$

Similarly we rewrite

$$(\tilde{F}(0)-\tilde{F}(\tilde{h}_1))\partial_1(\tilde{h}_1-\tilde{h}_2) = (\tilde{F}(h_*)-\tilde{F}(\tilde{h}_1))\partial_1(\tilde{h}_1-\tilde{h}_2)+(\tilde{F}(0)-\tilde{F}(h_*))\partial_1(\tilde{h}_1-\tilde{h}_2).$$

Hence

$$\begin{aligned}
&|(\tilde{F}(0)-\tilde{F}(\tilde{h}_1))\partial_1(\tilde{h}_1-\tilde{h}_2)|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} \\
&\leq CC(r_0)|\tilde{h}_1-h_*|_{{}_0\mathbb{E}(T)}|h_1-h_2|_{{}_0\mathbb{E}(T)} + CC(T)|h_*|_{{}_0\mathbb{E}(T)}|h_1-h_2|_{{}_0\mathbb{E}(T)} \\
&\leq CC(r_0)|h_1|_{{}_0\mathbb{E}(T)}|h_1-h_2|_{{}_0\mathbb{E}(T)} + CC(T)|h_0|_{X_\gamma}|h_1-h_2|_{{}_0\mathbb{E}(T)} \\
&\leq CC(r_0)r|h_1-h_2|_{{}_0\mathbb{E}(T)} + CC(T)\delta|h_1-h_2|_{{}_0\mathbb{E}(T)}.
\end{aligned}$$

Altogether,

$$|N_{5,2}|_{{}_0F^{1-2/3q}_{pq}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;B^{3-2/q}_{qq}(\partial\Sigma))} \leq C(r_0)(r+C(T)\delta)|h_2-h_1|_{{}_0\mathbb{E}(T)}.$$

By choosing $r > 0$ small, then $T = T(r) > 0$, and then $\delta = \delta(T) > 0$ small, $C(r_0)(r + C(T)\delta)$ gets arbitrarily small. Since $N_{5,3}$ obeys the same structure as $N_{5,2}$, we also obtain estimates for $N_{5,3}$. The proof of contraction estimates for N_5 is complete.

CHAPTER 3

The Mullins-Sekerka equations with ninety degree angle boundary contact: qualitative behaviour

3.1. Introduction

In this chapter we study the long-time behaviour of solutions to the Mullins-Sekerka problem with ninety degree contact angle which start close to equilibria.

This chapter consists of two major results. In the first part we consider a cylindrical domain in two or three space dimensions and show a result on nonlinear stability. We will prove that solutions starting close to certain equilibria exist globally in time, are stable, and converge to an equilibrium solution at an exponential rate. The simpler geometry of a bounded cylinder reduces the nonlinear boundary angle condition to a linear one. In a way, it allows to recast the problem as one single abstract evolution equation for the height function and for an application of the generalized principle of linearized stability. In the second part we consider a general, bounded, smooth domain in two space dimensions and give a complete linearized stability analysis around stationary solutions. Here, both the domain at the contact points and the stationary solution may be curved. We will show that the relevant quantities deciding on stability or instability are the length of the curve of the stationary solution, its constant curvature, and the curvature of the domain at the two contact points. These results are published in the articles [**3**], [**27**].

Let us recall the Mullins-Sekerka problem with ninety degree contact angle,

$$
\begin{aligned}
V_{\Gamma(t)} &= -[\![n_{\Gamma(t)} \cdot \nabla\mu]\!], && \text{on } \Gamma(t),\\
[\![\mu]\!] = 0, \quad \mu|_{\Gamma(t)} &= H_{\Gamma(t)}, && \text{on } \Gamma(t),\\
\Delta\mu &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
n_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega,\\
\mathring{\Gamma}(t) \subset \Omega, \quad \partial\Gamma(t) &\subset \partial\Omega, &&\\
\angle(\Gamma(t),\partial\Omega) &= \pi/2, && \text{on } \partial\Gamma(t),\\
\Gamma|_{t=0} &= \Gamma_0.
\end{aligned}
\tag{3.1}
$$

Again we recall that equation (3.1)$_6$ fixes the contact angle to be constant ninety degrees.

3.2. Nonlinear stability and convergence to equilibria in cylindrical domains

This section is devoted to the long-time behaviour of solutions to (3.1) in a cylindrical geometry, starting close to equilibria. We will characterize the set of equilibria, study the spectrum of the linearization of the transformed Mullins-Sekerka equations around an equilibrium and apply the generalized principle of linearized stability to show that solutions starting sufficiently close to certain equilibria converge to an equilibrium solution at an exponential rate in X_γ.

We note that the potential μ can always be reconstructed by $\Gamma(t)$ by solving the elliptic two-phase problem

$$\begin{aligned} \mu|_{\Gamma(t)} &= H_{\Gamma(t)}, && \text{on } \Gamma(t), \\ \Delta\mu &= 0, && \text{in } \Omega\backslash\Gamma(t), \\ n_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Whence we may concentrate on the set of equilibria for $\Gamma(t)$.

It can now easily be shown that for a stationary solution Γ of (3.1) with $V_\Gamma = 0$ the corresponding chemical potential μ is constant, since then μ and $\nabla\mu$ have no jump across the interface Γ and $\mu \in W^2_q(\Omega)$ solves a homogeneous Neumann problem on Ω. By $(3.2)_1$, the mean curvature H_Γ is constant. The set of equilibria for the flow $\Gamma(t)$ is therefore given by

$$\mathcal{E} = \{\Gamma : H_\Gamma = \text{const.}\}.$$

Let us now consider the case where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded cylinder, that is, $\Omega := \Sigma \times (L_1, L_2)$, where $-\infty < L_1 < 0 < L_2 < \infty$ and $\Sigma \subset \mathbb{R}^{n-1} \times \{0\}$ is a bounded domain and $\partial\Sigma$ is smooth.

Note that flat interfaces are equilibria. Arcs of circles intersecting $\partial\Omega$ at a ninety degree angle also belong to $\mathcal{E}$, since then $(3.1)_6$ is also satisfied.

If we now additionally assume that the contact points between Γ and $\partial\Omega$ are only on the walls of the cylinder and Γ is given as a graph over Σ, we may even deduce that $H_\Gamma = 0$, that is, Γ is a flat interface described by a constant height function over the reference surface. This follows from two facts. Note that the normal to the boundary of the domain $n_{\partial\Omega}$ at the walls of the cylinder is independent of h. Furthermore, the last entry of $n_{\partial\Omega}$ is zero. Hence the boundary equation can be recast as

$$n_{\partial\Omega}\cdot(-\nabla h, 0) = 0, \quad \text{on } \partial\Sigma. \tag{3.3}$$

Also, we have the well-known formula

$$H_\Gamma = \operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) \tag{3.4}$$

for the mean curvature of a graph. This together with the boundary condition (3.3) on $\partial\Sigma$ renders $H_\Gamma = 0$. Indeed, assume that $\Gamma = \Gamma_h$ is a graph of h over Σ. We may assume that h has mean value zero. Otherwise we consider $h - \frac{1}{|\Sigma|}\int_\Sigma h dx$. This shift leaves the mean curvature of the interface and the boundary condition for the height function invariant. We already know H_Γ is constant, but may be different from zero. An integration by parts entails

$$0 = H_\Gamma \int_\Sigma h dx = \int_\Sigma h H_\Gamma dx = -\int_\Sigma \frac{|\nabla h|^2}{\sqrt{1+|\nabla h|^2}} dx. \tag{3.5}$$

The boundary integral vanishes due to (3.3) and renders ∇h to be zero in Σ, hence h is constant. This implies $H_\Gamma = 0$.

We will now study the problem for the height function in an L_p-setting. We rewrite the geometric problem (3.1) as an abstract evolution equation for the height function h, cf. [4], [58], [22]. As seen before in Section 2.4, by means of Hanzawa transform, the full system reads as

$$\begin{aligned} \partial_t h &= -(\nu_\Sigma|\nu_{\Gamma_h})^{-1}[\![n_\Sigma^h \cdot \nabla_h \mu]\!], && \text{on } \Sigma, \\ \mu|_\Sigma &= H_\Gamma(h), && \text{on } \Sigma, \\ \Delta_h \mu &= 0, && \text{in } \Omega\backslash\Sigma, \\ n_{\partial\Omega} \cdot \nabla_h \mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega, \\ n_{\partial\Sigma} \cdot \nabla_\Sigma h|_{\partial\Sigma} &= 0, && \text{on } \partial\Sigma, \\ h|_{t=0} &= h_0, && \text{on } \Sigma. \end{aligned} \tag{3.6}$$

Let us note that due to working in a cylinder, the nonlinear angle condition reduces to a linear one, condition $(3.6)_5$. Define now $B(h)g := (\nu_\Sigma|\nu_{\Gamma_h})^{-1}[\![n_\Sigma^h \cdot \nabla_h g]\!]$ and $S(h)g$ as the unique solution of the elliptic problem

$$\begin{aligned} \mu|_\Sigma &= g, && \text{on } \Sigma, \\ \Delta_h \mu &= 0, && \text{in } \Omega\backslash\Sigma, \\ n_{\partial\Omega} \cdot \nabla_h \mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Recalling Lemma 2.15, we may rewrite (3.6) as an abstract evolution equation,

$$\begin{aligned} \frac{d}{dt}h(t) + A(h(t))h(t) &= F(h(t)), \quad t \in \mathbb{R}_+, \\ h(0) &= h_0, \end{aligned} \tag{3.7}$$

where $A(h)g := B(h)S(h)P(h)g$, equipped with domain

$$D(A(h)) := W_q^{4-1/q}(\Sigma) \cap \{g : n_{\partial\Sigma} \cdot \nabla_\Sigma g = 0 \text{ on } \partial\Sigma\},$$

and $F(g) := -B(g)S(g)Q(g)$. We now want to study (3.7) in an L_p-setting. Define

$$X_0 := W_q^{1-1/q}(\Sigma), \quad X_1 := W_q^{4-1/q}(\Sigma), \quad X_\gamma := (X_0, X_1)_{1-1/p,p}.$$

We now interpret problem (3.7) as an evolution equation in $L_p(\mathbb{R}_+; X_0)$, fitting in the setting of Prüss, Simonett, and Zacher [58]. Regarding the linearization we have the following result.

LEMMA 3.1. *Let $p \in (6, \infty)$, $q \in (5/3, 2) \cap (2p/(p+1), 2p)$. Then the following statements are true.*

(1) *The derivative of H_Γ at $h = 0$ is given by $[h \mapsto \Delta_\Sigma h]$.*
(2) *There is an open neighbourhood of zero $V \subset X_\gamma$, such that*

$$(A, F) \in C^1(V; \mathcal{B}(X_1; X_0) \times X_0).$$

(3) *The linearization of A at zero is given by $A_0 = A(0)$, where $A_0 : D(A_0) \to X_0$, $A_0 h = -[\![n_\Sigma \cdot T\Delta_\Sigma h]\!]$, with domain*

$$D(A_0) = X_1 \cap \{h : n_{\partial\Sigma} \cdot \nabla_\Sigma h|_{\partial\Sigma} = 0 \text{ on } \partial\Sigma\}.$$

Here, $T : W_q^{2-1/q}(\Sigma) \to W_q^2(\Omega\backslash\Sigma), g \mapsto \chi$, is the solution operator for the elliptic two-phase problem

$$\begin{aligned} \Delta\chi &= 0, && \text{in } \Omega\backslash\Sigma,\\ \chi|_\Sigma &= g, && \text{on } \Sigma, \\ n_{\partial\Omega}\cdot\nabla\chi|_{\partial\Omega} &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{3.8}$$

(4) *In a neighbourhood of zero in X_γ, the set of equilibria, that is, the solutions to $A(h)h = F(h)$, is given by $\mathcal{E} = \{h = const.\}$.*
(5) *A_0 has maximal L_p-regularity.*
(6) *The kernel of A_0 are the constant functions, $N(A_0) = \{h = const.\}$.*
(7) *$N(A_0) = N(A_0^2)$.*

Proof. (1) This stems from linearizing (3.4) at $h = 0$, cf. also [**14**].

(2) Again by Lemma 2.15, there is a small neighbourhood of zero $V \subset X_\gamma$, such that $P \in C^1(V; \mathcal{B}(X_1; W_q^{2-1/q}(\Sigma))$ and $Q \in C^1(V; W_q^{2-1/q}(\Sigma))$. Following the lines of [4] using Lemma 2.17, we now show that

$$S \in C^1(V; \mathcal{B}(W_q^{2-1/q}(\Sigma); W_q^2(\Omega\backslash\Sigma))).$$

By Lemma 2.17, $[h \mapsto \Delta_h] \in C^1(V; \mathcal{B}(W_q^2(\Omega\backslash\Sigma); L_q(\Omega)))$, as well as $[h \mapsto \nabla_h] \in C^1(V; \mathcal{B}(W_q^2(\Omega\backslash\Sigma); W_q^1(\Omega\backslash\Sigma)))$. Hence by trace theory,

$$[h \mapsto \nabla_h|_{\partial\Omega}] \in C^1(V; \mathcal{B}(W_q^2(\Omega\backslash\Sigma); W_q^{1-1/q}(\partial\Omega))),$$

since $q < 2$. Again by Lemma 2.17, $[h \mapsto n^h_{\partial\Omega}] \in C^1(V; C^1(\partial\Omega))$, therefore the product estimate of (2.69) gives

$$[h \mapsto n^h_{\partial\Omega}\cdot\nabla_h|_{\partial\Omega}] \in C^1(V; \mathcal{B}(W_q^2(\Omega\backslash\Sigma); W_q^{1-1/q}(\partial\Omega))).$$

Hence $[h \mapsto (\Delta_h, \mathrm{tr}_\Sigma, n^h_{\partial\Omega}\cdot\nabla_h|_{\partial\Omega})]$ belongs to

$$C^1(V; \mathcal{B}(W_q^2(\Omega\backslash\Sigma); L_q(\Omega) \times W_q^{2-1/q}(\Sigma) \times W_q^{1-1/q}(\partial\Omega))).$$

Since inversion is smooth and $S(h)g = (\Delta_h, \mathrm{tr}_\Sigma, n^h_{\partial\Omega}\cdot\nabla_h|_{\partial\Omega})^{-1}(0, g, 0)$,

$$S \in C^1(V; \mathcal{B}(W_q^{2-1/q}(\Sigma); W_q^2(\Omega\backslash\Sigma))).$$

Regarding B we may write $B(h) = \sum_{j=1}^{n-1} b_j^h \,\mathrm{tr}_\Sigma\, \partial_j$, where the coefficients $b_j^h = b_j(x, h, \nabla h)$ depend smoothly on $(x, h, \nabla h)$. This yields $b_j(\cdot, h, \nabla h) \in C^1(\Sigma)$, since the derivatives of h belong to $B_{qp}^{3-1/q-3/p}(\Sigma) \hookrightarrow C^1(\Sigma)$. Hence,

$$[h \mapsto b_j^h \,\mathrm{tr}_\Sigma\, \partial_j] \in C^1(V; \mathcal{B}(W_q^2(\Omega\backslash\Sigma); W_q^{1-1/q}(\Sigma))),$$

as well as

$$B \in C^1(V; \mathcal{B}(W_q^2(\Omega\backslash\Sigma); W_q^{1-1/q}(\Sigma))).$$

This shows that $(A, F) \in C^1(V; \mathcal{B}(X_1; X_0) \times X_0)$.

(3) This stems from the fact that $A_0 = A(0)$ and Lemma 2.15.

(4) Let $h \in D(A)$ satisfy $A(h)h = F(h)$. It then follows that $B(h)S(h)H_\Gamma(h) = 0$ on Σ, that is,

$$[\![n^h_\Sigma \cdot \nabla_h [S(h)H_\Gamma(h)]]\!] = 0, \quad \text{on } \Sigma.$$

Note that $S(h)H_\Gamma(h)$ is the solution of an h-perturbed elliptic problem with homogeneous Neumann boundary conditions. Therefore $S(h)H_\Gamma(h)$ has to be constant, if

$|h|_{X_\gamma}$ is small enough by a perturbation argument. Since $S(h)H_\Gamma(h)$ equals $H_\Gamma(h)$ on Σ, also $H_\Gamma(h)$ is constant. We then obtain that the mean curvature H_Γ of the interface given as a graph of h over Σ is constant. As in (3.5) we may even deduce using formula (3.4) that $H_\Gamma = 0$ and h is constant.

(5) This stems from Theorem 2.25.

(6) Clearly, every constant function is an element of $N(A_0)$. For the converse, let $h \in D(A_0)$, such that $A_0 h = 0$. Hence $\chi := T\Delta_\Sigma h$ is constant, where T is the solution operator of (3.8). Therefore $\Delta_\Sigma h$ is constant. Since $h \in D(A_0)$, an integration by parts shows $\Delta_\Sigma h = 0$ on Σ. Again since $h \in D(A_0)$, h has to be constant.

(7) We only need to show $N(A_0^2) \subset N(A_0)$. Pick some $h \in N(A_0^2)$. Then $A_0 h \in D(A_0) \cap N(A_0)$. Hence $A_0 h$ is constant. Also, $A_0 h$ is in the range of A_0. Let us show that every element in the range of A_0 has mean value zero. Having this at hand it follows that $A_0 h = 0$, hence $h \in N(A_0)$.

Let $g \in R(A_0)$. Pick some $\bar{h} \in D(A_0)$, such that $A_0\bar{h} = g$. Then

$$\int_\Sigma g dx = \int_\Sigma A_0 \bar{h} dx = -\int_\Sigma [\![n_\Sigma \cdot \nabla \chi]\!] dx,$$

where $\chi := T\Delta_\Sigma \bar{h}$ and T is the solution operator stemming from (3.8). Then

$$\int_\Sigma [\![n_\Sigma \cdot \nabla \chi]\!] dx = 0,$$

since $\Delta\chi = 0$ in $\Omega\backslash\Sigma$ and the Neumann boundary condition for χ. Hence g is mean value free. The proof is complete. $\square$

The following theorem enables us to apply the generalized principle of linearized stability of Prüss, Simonett, and Zacher [**58**] to the evolution equation (3.7).

THEOREM 3.2. *Let $p \in (6, \infty)$, $q \in (5/3, 2) \cap (2p/(p+1), 2p)$. Then the trivial equilibrium $h_* = 0$ is normally stable.*

More precisely:

(1) *Near $h_* = 0$, the set of equilibria $\mathcal{E}$ is a C^1-manifold in X_1 of dimension one.*
(2) *The tangent space of $\mathcal{E}$ at $h_* = 0$ is given by the kernel of the linearization, $T_0\mathcal{E} = N(A_0)$.*
(3) *Zero is a semi-simple eigenvalue of A_0, i.e. $X_0 = N(A_0) \oplus R(A_0)$.*
(4) *The spectrum $\sigma(A_0)$ satisfies $\sigma(A_0)\backslash\{0\} \subset \mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$.*

PROOF. (1) Around h_*, the set of equilibria only consists of constant functions, hence is a one-dimensional linear subspace of X_1.

(2) This stems from Lemma 3.1.

(3) Since $D(A_0)$ compactly embeds into $W_q^{1-1/q}(\Sigma)$, the operator A_0 has a compact resolvent and the spectrum $\sigma(A_0)$ only consists of eigenvalues, cf. [**19**]. Furthermore, every spectral value in $\sigma(A_0)$ is a pole of finite algebraic multiplicity. By using $N(A_0) = N(A_0^2)$ and Proposition A.2.2 and Remark A.2.4 in [**43**] we may conclude that the range of A_0 is closed in X_0 and that there is a spectral decomposition $X_0 = N(A_0) \oplus R(A_0)$. Hence $\lambda = 0$ is semi-simple.

(4) Pick $\lambda \in \sigma(A_0)$ with corresponding eigenfunction $h \in D(A_0)$, in other words

$$\lambda h - A_0 h = 0, \quad \text{in } X_0. \tag{3.9}$$

By definition of A_0, $A_0 h = -[\![n_\Sigma \cdot T\Delta_\Sigma h]\!]$. Testing this equality in $L_2(\Sigma)$ with $\Delta_\Sigma h$ yields

$$0 = |\nabla \chi|^2_{L_2(\Omega)} + (A_0 h | \Delta_\Sigma h)_{L_2(\Sigma)},$$

where $\chi := T\Delta_\Sigma h$. Note that since $q > 3/2$, $W^{1-1/q}_q(\Sigma) \hookrightarrow L_2(\Sigma)$. Testing the resolvent equation (3.9) now with $\Delta_\Sigma h$ in $L_2(\Sigma)$ finally yields

$$\lambda |\nabla h|^2_{L_2(\Sigma)} = |\nabla \chi|^2_{L_2(\Omega)}.$$

This shows that λ is real and $\lambda \geq 0$. In particular, $\sigma(A_0)\backslash\{0\} \subset (0, \infty)$. Hence h_* is normally stable. $\square$

The following theorem is the main result on stability of solutions.

THEOREM 3.3. *The trivial equilibrium $h_* = 0$ is stable in X_γ, and there is some $\delta > 0$ such that the evolution equation*

$$\frac{d}{dt} h(t) + A(h(t))h(t) = 0, \; t > 0, \quad h(0) = h_0,$$

with initial value $h_0 \in X_\gamma$ satisfying $|h_0 - h_*|_{X_\gamma} \leq \delta$ has a unique global in-time solution on $\mathbb{R}_+$,

$$h \in W^1_p(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; D(A_0)),$$

which converges at an exponential rate in X_γ to some $h_\infty \in \mathcal{E}$ as $t \to +\infty$.

PROOF. It is an application of the generalized principle of linearized stability of Prüss, Simonett, and Zacher [**58**] to the evolution equation (3.7). $\square$

THEOREM 3.4 (Geometrical version). *Suppose that the initial surface Γ_0 is given as a graph, $\Gamma_0 = \{(x, h_0(x)) : x \in \Sigma\}$ for some function $h_0 \in X_\gamma$. Then, for each $\varepsilon > 0$ there is some $\delta(\varepsilon) > 0$, such that if the initial value $h_0 \in X_\gamma$ satisfies $|h_0|_{X_\gamma} \leq \delta(\varepsilon)$, there exists a global-in-time strong solution h on $\mathbb{R}_+$ of the evolution equation, $h \in L_p(\mathbb{R}_+; D(A_0)) \cap W^1_p(\mathbb{R}_+; X_0)$. The solution satisfies $|h(t)|_{X_\gamma} \leq \varepsilon$ for all $t \geq 0$.*

Moreover, there is some constant h_∞, such that $\Gamma_h \to \Gamma_{h_\infty}$ in the sense of $h(t) \to h_\infty$ in X_γ. The convergence is at an exponential rate.

Note that by the following theorem we can characterize the limit. It is a priori not clear to which equilibrium the solution converges to by the generalized principle of linearized stability.

THEOREM 3.5. *The limit h_∞ from above has the same mean value as h_0, in other words, $h_\infty = \frac{1}{|\Sigma|}\int_\Sigma h_0 dx$.*

PROOF. The theorem is a consequence of the fact that the Mullins-Sekerka system conserves the measure of the domains separated by the interface in time. Hence the solution h from Theorem 3.3 satisfies

$$\frac{d}{dt}\int_\Sigma h(t,x)dx = 0, \quad t > 0.$$

In particular,

$$\int_\Sigma h(t,x)dx = \int_\Sigma h_0(x)dx, \quad t > 0.$$

Since $h(t) \to h_\infty$ as $t \to \infty$ in $X_\gamma \hookrightarrow L_1(\Sigma)$, we get the result. $\square$

3.3. Linearized stability analysis in curved domains

In this chapter we study the two-phase Mullins-Sekerka problem inside a bounded, smooth domain in two space dimensions with boundary contact.

Let us precisely state the model. We consider a fixed, smooth, and bounded domain in two space dimensions $\Omega \subset \mathbb{R}^2$. As before, we assume that the domain can be decomposed as $\Omega = \Omega^+(t)\dot{\cup}\mathring{\Gamma}(t)\dot{\cup}\Omega^-(t)$, where $\mathring{\Gamma}(t)$ denotes the interior of $\Gamma(t)$, here a smooth one-dimensional curve with two boundary points $\partial\Gamma(t)$ on $\partial\Omega$.

Again we consider the Mullins-Sekerka problem with ninety degree contact angle,

$$\begin{aligned} V_\Gamma &= -[\![n_\Gamma \cdot \nabla\mu]\!], && \text{on } \Gamma(t), \\ \mu|_\Gamma &= H_\Gamma, && \text{on } \Gamma(t), \\ \Delta\mu &= 0, && \text{in } \Omega\backslash\Gamma(t), \\ n_{\partial\Omega} \cdot \nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\partial\Gamma(t), \\ n_\Gamma \cdot n_{\partial\Omega} &= 0, && \text{on } \partial\Gamma(t), \\ \Gamma(0) &= \Gamma_0. \end{aligned} \tag{3.10}$$

Hereby, V_Γ denotes again the normal velocity and H_Γ the (mean) curvature of $\Gamma(t)$ with respect to n_Γ, where we use the sign convention that H is negative for convex spheres. In particular, the sphere of radius $R > 0$ and center $x_0 \in \mathbb{R}^2$ has negative curvature $-1/R$.

Using curvilinear coordinates and a Hanzawa type transformation, cf. Section 2.4, we can pull back the equations to the fixed reference configuration $\Omega\backslash\Sigma$ by means of the Hanzawa transform Θ_h, cf. also [**3**], [**64**], [**57**], [**22**]. Recall that Σ may be any smooth curve intersecting the boundary perpendicularly at the two contact points. For convenience, we recall that the transformed system reads as

$$\begin{aligned} \partial_t h &= -a^{-1}(h)[\![n_{\Gamma_h} \cdot \nabla_h \eta]\!], && \text{on } \Sigma, \\ \eta|_\Sigma &= K(h), && \text{on } \Sigma, \\ \Delta_h \eta &= 0, && \text{in } \Omega\backslash\Sigma, \\ n^h_{\partial\Omega} \cdot \nabla_h \eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega, \\ n^h_{\partial\Omega} \cdot n_{\Gamma_h} &= 0, && \text{on } \partial\Sigma, \\ h|_{t=0} &= h_0, && \text{on } \Sigma. \end{aligned}$$

Hereby, $K(h)$ is the transformed (mean) curvature operator, cf. Section 2.4, $n^h_{\partial\Omega} := n_{\partial\Omega} \circ \Theta^t_h$, and the transformed differential operators are given by

$$\nabla_h := (D\Theta^t_h)^\top \nabla, \quad \operatorname{div}_h := \operatorname{Tr} \nabla_h, \quad \Delta_h := \operatorname{div}_h \nabla_h.$$

Furthermore, h_0 is a suitable description of the initial configuration at time $t = 0$ and $a^{-1}(h)(t)$ depends only on $h(t)$. Also $a^{-1}(0) = 1$ and $a^{-1}(h)$ depends smoothly on h.

3.3.1. The linearized problem. Let Σ_* be a stationary solution to the Mullins-Sekerka problem with boundary contact (3.10). In particular, the (mean) curvature of Σ_* is constant and Σ_* is a flat surface or part of a circle intersecting $\partial\Omega$ perpendicularly. We now consider the full linearization of (3.10) at any stationary solution Σ_*.

Referring to [14], given an equilibrium solution Σ_*, the linearization of the transformed mean curvature operator at $h = 0$ is given by

$$K'(0) = \Delta_{\Sigma_*} + |\kappa_*|^2,$$

where κ_* is the constant curvature of Σ_*. Furthermore, the linearization of the nonlinear ninety degree angle condition at the boundary at $h = 0$ is given by

$$\nabla_{\Sigma_*} h \cdot n_{\partial\Sigma_*} = -S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})h, \quad \text{on } \partial\Sigma_*,$$

where $S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})$ is the second fundamental form of $\partial\Omega$ with respect to the outer unit normal $n_{\partial\Omega}$, cf. [14]. In particular, we have the formula

$$S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})h = -(n_{\Sigma_*} \cdot \partial_{n_{\Sigma_*}} n_{\partial\Omega})h = -(n_{\Sigma_*} \cdot [Dn_{\partial\Omega} n_{\Sigma_*}])h,$$

cf. the proof of Lemma 3.7 in [14]. Note that if e.g. Ω is a convex sphere, $S_{\partial\Omega} < 0$. The linearized problem around a stationary solution Σ_* now reads as

$$\begin{aligned}
\partial_t h &= -[\![n_{\Sigma_*} \cdot \nabla\mu]\!], && \text{on } \Sigma_*,\\
\mu|_{\Sigma_*} &= \Delta_{\Sigma_*} h + \kappa_*^2 h, && \text{on } \Sigma_*,\\
\Delta\mu &= 0, && \text{in } \Omega\backslash\Sigma_*,\\
n_{\partial\Omega} \cdot \nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\partial\Sigma_*,\\
\nabla_{\Sigma_*} h \cdot n_{\partial\Sigma_*} &= -S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})h, && \text{on } \partial\Sigma_*,\\
h(0) &= h_0, && \text{on } \Sigma_*.
\end{aligned} \tag{3.11}$$

Regarding the stationary solution we note that κ_* is constant and either equal to zero or $-1/R$, because Σ_* is flat or part of a circle with radius $R > 0$, respectively.

To identify relevant quantities in the stability analysis, let us formally consider the corresponding eigenvalue problem

$$\begin{aligned}
\lambda h &= -[\![n_{\Sigma_*} \cdot \nabla\mu]\!], && \text{on } \Sigma_*,\\
\mu|_{\Sigma_*} &= \Delta_{\Sigma_*} h + \kappa_*^2 h, && \text{on } \Sigma_*,\\
\Delta\mu &= 0, && \text{in } \Omega\backslash\Sigma_*,\\
n_{\partial\Omega} \cdot \nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\partial\Sigma_*,\\
\nabla_{\Sigma_*} h \cdot n_{\partial\Sigma_*} &= -S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})h, && \text{on } \partial\Sigma_*,
\end{aligned} \tag{3.12}$$

for some $\lambda \in \mathbb{C}$. Multiplying $(3.12)_1$ with $\Delta_{\Sigma_*}\bar{h} + \kappa_*^2\bar{h}$ in $L_2(\Sigma_*)$ gives

$$\lambda \int_{\Sigma_*} h(\Delta_{\Sigma_*}\bar{h} + \kappa_*^2\bar{h})d\mathcal{H}^1 = \int_\Omega |\nabla\mu|^2 dx.$$

Here, $d\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure, $d \in \mathbb{N}_0$. An integration by parts invoking the boundary conditions formally entails

$$\begin{aligned}
\lambda \left[\int_{\Sigma_*} |\nabla_{\Sigma_*} h|^2 d\mathcal{H}^1 + \int_{\partial\Sigma_*} S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})|h|^2 d\mathcal{H}^0 - \kappa_*^2 \int_{\Sigma_*} |h|^2 d\mathcal{H}^1\right] + \\
+ \int_\Omega |\nabla\mu|^2 dx = 0.
\end{aligned} \tag{3.13}$$

We now note that the term in brackets may change its sign in dependence of the curvature κ_*, the values of the form $S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})$ on the two boundary points of

$\partial\Sigma_*$, and the length of the curve Σ_*. The last dependence is somewhat hidden and stems from the scaling properties of the first term involving the gradient of h.

Referring to [**26**], we want to introduce a bilinear functional by

$$I_*(h,h) := \int_{\Sigma_*} |\nabla_{\Sigma_*} h|^2 d\mathcal{H}^1 + \int_{\partial\Sigma_*} S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})|h|^2 d\mathcal{H}^0 - \kappa_*^2 \int_{\Sigma_*} |h|^2 d\mathcal{H}^1.$$

Note that for $\lambda \neq 0$, integrating (3.12)$_1$ over Σ_* yields that necessarily $\int_{\Sigma_*} h d\mathcal{H}^1 = 0$, for any eigenfunction h to the eigenvalue $\lambda \neq 0$.

Hence it stems from (3.13) that positivity of I_* on mean value free functions gives $\lambda \leq 0$ for any possible eigenvalue λ. Hence studying the sign of I_* for mean value free functions is the crucial point in our stability analysis.

As a trivial consequence we want to point out that if $\kappa_* = 0$ and S is identically zero on $\partial\Sigma_*$, we obtain that $\lambda \leq 0$. This corresponds to the geometrical situation of a flat solution Σ_* and flat, perpendicular walls, which was already investigated in Section 3.2.

3.3.2. Flat stationary solutions. Let us start with the simpler case when the stationary solution is flat, $\kappa_* = 0$. Then by rotation, we can assume that $\Sigma_* = (0, L)$ for some $L > 0$. Let us rewrite (3.11) as an abstract evolution equation in the setting of Chapter 2. Let $3/2 < q < 2$ and

$$X_0 := W_q^{1-1/q}(\Sigma_*), \quad X_1 := W_q^{4-1/q}(\Sigma_*).$$

Define a linear operator $A : D(A) \subset X_1 \to X_0$ as follows. Let $Bu := [\![n_{\Sigma_*} \cdot \nabla u]\!]$ and $T_0 v$ be the unique solution of the two-phase elliptic problem

$$\begin{aligned} [\![\mu]\!] = 0, \quad \mu|_{\Sigma_*} = v, & \qquad \text{on } \Sigma_*, \\ \Delta\mu = 0, & \qquad \text{in } \Omega\backslash\Sigma_*, \\ n_{\partial\Omega} \cdot \nabla\mu|_{\partial\Omega} = 0, & \qquad \text{on } \partial\Omega\backslash\partial\Sigma_*. \end{aligned} \tag{3.14}$$

Then we define A by $Ah := BT_0(\Delta_{\Sigma_*} h)$, with domain

$$D(A) := X_1 \cap \{h : \nabla_{\Sigma_*} h \cdot n_{\partial\Sigma_*} = -S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})h, \text{ on } \partial\Sigma_*\}.$$

We can then rewrite (3.11) as

$$\dot{h}(t) + Ah(t) = 0, \; t > 0, \quad h(0) = h_0. \tag{3.15}$$

The main benefit of this formulation is now the fact that A has maximal regularity. More precisely, let $p \in (6, \infty)$, $q \in (19/10, 2) \cap (2p/(p+1), 2)$, and $J = (0, T)$, $0 < T < \infty$. Then, by a perturbation argument, the operator A has maximal L_p-regularity on J with respect to the base space X_0, cf. Section 2.5. Define the trace space as $X_\gamma := B_{qp}^{4-1/q-3/p}(\Sigma_*)$. Then it holds that

$$X_\gamma = (X_0, X_1)_{1-1/p,p}.$$

We now want to apply the generalized principle of linearized stability to deduce stability or instability results for (3.15), cf. [**57**], [**58**].

Let us simplify notation first. Since $\partial\Sigma_* = \{0, L\}$, we may rewrite the boundary conditions as

$$\partial_x h(0) = -\omega_1 h(0), \quad \partial_x h(L) = \omega_2 h(L),$$

where

$$\omega_1 := -S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})(0), \quad \omega_2 := -S_{\partial\Omega}(n_{\Sigma_*}, n_{\Sigma_*})(L).$$

In particular again if Ω is a convex sphere, $\omega_1, \omega_2 > 0$ since $S_{\partial\Omega} < 0$. We now want to analyse different geometries and their respective stability properties. Let us

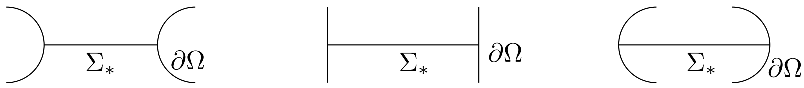

FIGURE 1. Different signs of ω_1, ω_2. Left: $\omega_1 = \omega_2 < 0$. Middle: $\omega_1 = \omega_2 = 0$. Right: $\omega_1 = \omega_2 > 0$.

discuss stability and instability results in the case of flat stationary solutions. We start with the left hand side case, where we can show stability of $h_* = 0$ for (3.15).

THEOREM 3.6. *Let $\omega_1, \omega_2 \leq 0$. Then $h_* = 0$ is normally stable, that is,*

(1) *The set of equilibria of (3.15) is the kernel of A, which is one-dimensional.*
(2) *The eigenvalue zero is semi-simple, $X_0 = N(A) \oplus R(A)$.*
(3) *The spectrum satisfies $\sigma(-A)\backslash\{0\} \subset \mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.*

In particular, $h_* = 0$ is stable in X_γ and there is some $\delta > 0$, such that if $|h_0|_{X_\gamma} \leq \delta$, the unique solution h of (3.15) exists globally in time, belongs to $W^1_p(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; D(A))$, and converges to some equilibrium solution in X_γ at an exponential rate.

PROOF. Consider some $\lambda \in \sigma(-A) \subset \mathbb{C}$ and the corresponding eigenvalue problem for the eigenfunction $h \in D(A)$,

$$\begin{cases} \lambda h = -[\![n_{\Sigma_*} \cdot \nabla\mu]\!], & \text{on } \Sigma_*, \\ [\![\mu]\!] = 0, \quad \mu|_{\Sigma_*} = \partial_x\partial_x h, & \text{on } \Sigma_*, \\ \Delta\mu = 0, & \text{in } \Omega\backslash\Sigma_*, \\ n_{\partial\Omega} \cdot \nabla\mu|_{\partial\Omega} = 0, & \text{on } \partial\Omega\backslash\partial\Sigma_*, \\ \partial_x h(0) = -\omega_1 h(0), \\ \partial_x h(L) = \omega_2 h(L). \end{cases} \tag{3.16}$$

Multiplying equation $(3.16)_1$ with $\partial_x\partial_x\bar{h}$, an integration by parts invoking the boundary conditions yields

$$\lambda\left[\int_0^L |\partial_x h|^2 dx - \omega_1 h(0)^2 - \omega_2 h(L)^2\right] + \int_\Omega |\nabla\mu|^2 = 0. \tag{3.17}$$

Let us characterize the kernel of A. Pick some $h \in N(A)$. Then (3.17) for $\lambda = 0$ entails that μ has to be constant, whence $\partial_x\partial_x h = c$ on $(0, L)$ for some $c \in \mathbb{R}$. In particular, by the fundamental theorem of calculus,

$$h(s) = h(0) + s\partial_x h(0) + \int_0^s \int_0^\tau \partial_x\partial_x h(\tau')d\tau' d\tau, \quad s \in [0, L],$$

whence invoking the boundary condition gives

$$h(s) = h(0) - \omega_1 h(0)s + cs^2/2, \quad s \in [0, L]. \tag{3.18}$$

Let us start now with the case where $\omega_1, \omega_2 < 0$. By differentiating (3.18) and invoking the boundary condition at $x = L$ we obtain that

$$\partial_x h(L) = -\omega_1 h(0) + cL = \omega_2 h(L).$$

Also from (3.18) we obtain that $h(L) = [1-\omega_1 L]h(0) + cL^2/2$. The linear system

$$\begin{bmatrix} h(0) \\ h(L) \end{bmatrix} = \begin{bmatrix} 0 & -\omega_2/\omega_1 \\ 1-\omega_1 L & 0 \end{bmatrix} \begin{bmatrix} h(0) \\ h(L) \end{bmatrix} + \begin{bmatrix} cL/\omega_1 \\ cL^2/2 \end{bmatrix} \tag{3.19}$$

now has a unique solution since $1+(1-\omega_1 L)\omega_2/\omega_1 \geq 1$ for any $\omega_1 < 0, \omega_2 < 0, L > 0$. Explicitly solving the linear system gives

$$h(0) = \frac{c(L - \omega_2 L^2/2)}{\omega_1 + \omega_2 - \omega_1\omega_2 L},$$

which gives in combination with (3.18) a unique solution h which depends linearly on c. Hence the kernel of A is truly one-dimensional.

With this at hand we may now prove that zero is a semi-simple eigenvalue. Since $D(A)$ compactly embeds into X_0, the resolvent of A_0 is compact and the spectrum only consists of at most countably many isolated eigenvalues. Furthermore, every spectral value in $\sigma(A)$ is a pole of finite algebraic multiplicity. Using Remark A.2.4 in [**43**] it suffices to show that $N(A) = N(A^2)$. Then the range of A is closed and $X_0 = N(A) \oplus R(A)$. So pick some $h \in N(A^2)$. Then $h_1 := Ah \in R(A) \cap N(A)$. Then h_1 is mean value free on $(0, L)$ and there is some $c_1 \in \mathbb{R}$ such that

$$h_1(x) = c_1 \left[\frac{(L - \omega_2 L^2/2)(1-\omega_1 x)}{\omega_1 + \omega_2 - \omega_1\omega_2 L} + \frac{x^2}{2} \right], \quad x \in [0, L].$$

A straightforward integration gives

$$\int_0^L h_1(x)dx = c_1 \frac{6L^2 + \omega_1\omega_2 L^4/2 - 2(\omega_1+\omega_2)L^3}{6(\omega_1+\omega_2-\omega_1\omega_2 L)}.$$

Now for any $L > 0, \omega_1, \omega_2 < 0$ the right hand side can only be zero if $c_1 = 0$. But then $h_1 = 0$ and $Ah = 0$. Hence h belongs to the kernel of A. Then $N(A) = N(A^2)$ and zero is semi-simple. Furthermore, equation (3.17) yields that necessarily λ is real and $\lambda \leq 0$. Hence the third assertion is proved. The rest of the statement is a consequence of the generalized principle of linearized stability of Prüss, Simonett, and Zacher [**58**].

For completeness we shall show that zero is also semi-simple in the simpler case where $\omega_1 = 0, \omega_2 < 0$. The case $\omega_1 < 0, \omega_2 = 0$, follows the same lines. In the case $\omega_1 = 0, \omega_2 < 0$, the kernel of A consists of functions h of form $h_c(s) = c(L/\omega_2 - L^2/2 + s^2/2)$ for $c \in \mathbb{R}$. Then the same arguments give that zero is semi-simple. Note that in the case $\omega_1 = \omega_2 = 0$, the kernel of A consists of the constant functions, cf. Lemma 3.1. □

Let us now be concerned with the case when $\omega_1, \omega_2 > 0$, cf. Figure 1. We will show the following result for (3.15). For simplicity we will assume that $\omega_1 = \omega_2 =: \omega_+ > 0$.

THEOREM 3.7. (1) *For fixed $L > 0$, there exists some $\delta = \delta(L) > 0$, such that if $\omega_+ \leq \delta$, the solution $h_* = 0$ is stable in X_γ. Furthermore, there exists some $\eta > 0$, such that if $|h_0|_\gamma \leq \eta$, the solution to the initial value h_0 exists on $\mathbb{R}_+$ and converges to the equilibrium point $h_\infty := \frac{1}{L}\int_0^L h_0 dx$ in X_γ at an exponential rate.*

(2) *For fixed $L > 0$, there exists some $K = K(L) > 0$, such that if $\omega_+ \geq K$, the solution $h_* = 0$ is normally hyperbolic and unstable in X_γ.*

(3) *For fixed $\omega_+ > 0$, there is some $\delta > 0$ such that if $L_0 \leq \delta$, the interface $\Sigma = (0, L_0)$ corresponding to $h_* = 0$ is stable in X_γ. Moreover, the second statement of (1) holds.*

(4) *For fixed $\omega_+ > 0$, there is some $K > 0$, such that if $L \geq K$, the interface $\Sigma_* = (0, L)$ corresponding to $h_* = 0$ is normally hyperbolic and unstable in X_γ.*

Proof. Let $L, \omega_+ > 0$ and $\Sigma_* = (0, L)$. Let A be the linear operator of (3.15). Let us be concerned with the kernel of A. Again if $Ah = 0$, the corresponding chemical potential $\mu = T_0 \partial_x \partial_x h$ is constant and therefore $\partial_x \partial_x h = c$ for some $c \in \mathbb{R}$. As before, h can be written as $h(s) = (1 - \omega_+ s)h(0) + cs^2/2$ for all $s \in [0, L]$. The corresponding linear system for $[h(0), h(L)] \in \mathbb{R}^2$ in (3.19) can be uniquely solved whenever $2 - L\omega_+ \neq 0$. Note that for either fixed $L > 0$ or $\omega_+ > 0$, this can be ensured by choosing $\delta > 0$ sufficiently small or $K > 0$ sufficiently large. In any case,

$$h(s) = h(0)[1 - \omega_+ s] + cs^2/2, \; s \in [0, L], \quad h(0) = c\frac{L - \omega_+ L^2/2}{\omega_+(2 - \omega_+ L)}. \tag{3.20}$$

Arguing as in the proof of Theorem 3.6 we can show that the kernel of A is truly one dimensional, given by functions of type (3.20) for $c \in \mathbb{R}$. Hence, $X_0 = N(A) \oplus R(A)$ and the eigenvalue zero is semi-simple.

(1) For some $0 \neq \lambda \in \sigma(-A) \subset \mathbb{C}$ and a corresponding eigenfunction $h \in D(A)$, the eigenvalue problem again reads as

$$\begin{aligned}
\lambda h &= -[\![n_{\Sigma_*} \cdot \nabla \mu]\!], && \text{on } \Sigma_*, \\
[\![\mu]\!] = 0, \quad \mu|_{\Sigma_*} &= \partial_x \partial_x h, && \text{on } \Sigma_*, \\
\Delta \mu &= 0, && \text{in } \Omega \backslash \Sigma_*, \\
n_{\partial\Omega} \cdot \nabla \mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega \backslash \partial\Sigma_*, \\
\partial_x h(0) &= -\omega_+ h(0), \\
\partial_x h(L) &= \omega_+ h(L).
\end{aligned} \tag{3.21}$$

Necessarily,

$$\lambda \left[\int_0^L |\partial_x h|^2 dx - \omega_+ h(0)^2 - \omega_+ h(L)^2 \right] + \int_\Omega |\nabla \mu|^2 = 0.$$

We aim to show that the term in brackets is still positive if $\omega_+ \leq \delta$ and $\delta > 0$ is small. Integrating $(3.21)_1$ over $(0, L)$ yields that h is mean value free. Hence we can use Poincaré-Wirtinger inequality to deduce

$$\begin{aligned}
&\int_0^L |\partial_x h|^2 dx - \omega_+ h(0)^2 - \omega_+ h(L)^2 \\
&\quad \geq c_0(L) |h|^2_{H_2^1(0,L)} - \omega_+ h(0)^2 - \omega_+ h(L)^2 \\
&\quad \geq \tilde{c}_0(L) |h|^2_{C^0([0,L])} - \omega_+ h(0)^2 - \omega_+ h(L)^2,
\end{aligned} \tag{3.22}$$

for some $c_0, \tilde{c}_0 > 0$, since $H_2^1(0, L) \hookrightarrow C^0([0, L])$. Hence the first claim follows if $\delta > 0$ is sufficiently small.

(2) Again we fix $L > 0$. We need to show that if $\omega_+ \geq K$ for $K > 0$ large, there is a positive eigenvalue $\lambda > 0$ of $-A$. For $\lambda > 0$ we can rewrite the eigenvalue

problem (3.21) as

$$\lambda h - D_{MS}\tilde{\Delta} h = 0, \tag{3.23}$$

where $\tilde{\Delta} : D(\tilde{\Delta}) \subset X_1 \to X_0$ is given by $\tilde{\Delta} h := \partial_x \partial_x h$ with domain $D(\tilde{\Delta}) := W_q^{4-1/q}(\Sigma_*) \cap \{\partial_x h(0) = -\omega_+ h(0),\ \partial_x h(L) = \omega_+ h(L)\}$. Furthermore, we define the Dirichlet-to-Neumann operator D_{MS} as follows. For given $g \in W_q^{2-1/q}(0, L)$, we solve the two-phase elliptic problem

$$\begin{aligned} \Delta\theta &= 0, && \text{in } \Omega\backslash\Sigma_*, \\ [\![\theta]\!] = 0, \quad \theta|_{\Sigma_*} &= g, && \text{on } \Sigma_*, \\ n_{\partial\Omega} \cdot \nabla\theta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\partial\Sigma_*, \end{aligned}$$

uniquely by $\theta \in W_q^2(\Omega\backslash\Sigma_*)$ and define $D_{MS} g := -[\![n_{\Sigma_*} \cdot \nabla\theta]\!]$. The inverse Neumann-to-Dirichlet operator

$$N_{MS} = [D_{MS}]^{-1} : W_{q,(0)}^{1-1/q}(\Sigma_*) \to W_{q,(0)}^{2-1/q}(\Sigma_*)$$

then admits a compact, selfadjoint extension to $L_{2,(0)}(\Sigma_*)$, cf. Lemma 5.5. It is also shown there that N_{MS} is injective on $L_{2,(0)}(\Sigma_*)$. Note however that $\partial_x \partial_x h$ is not mean value free on Σ_*, even though h is. We may however rewrite (3.23) as

$$\lambda h - D_{MS}(I - P_0)\tilde{\Delta} h = D_{MS} P_0 \tilde{\Delta} h,$$

where $P_0 v$ is the mean value of v. Next note that $D_{MS} P_0 \tilde{\Delta} h = 0$, since $P_0 \tilde{\Delta} h$ is constant. Applying N_{MS} then gives that (3.23) is equivalent to

$$\lambda N_{MS} h - (I - P_0)\tilde{\Delta} h = 0.$$

Hereby we understand $\tilde{\Delta}$ as the natural extension to $H_2^2(\Sigma_*)$. We may now follow the lines of [**64**], [**57**]. Define $B_\lambda := \lambda N_{MS} - (I - P_0)\tilde{\Delta}$ with natural domain $D(B_\lambda) := H_{2,(0)}^2(\Sigma_*) \cap \{\partial_x h(0) = -\omega_+ h(0),\ \partial_x h(L) = \omega_+ h(L)\}$. We will now show that

$$B_\lambda \text{ is } \begin{cases} \text{positive definite,} & \text{if } \lambda \geq \lambda_0 \text{ for some } \lambda_0 > 0, \\ \text{not positive definite,} & \text{if } \lambda > 0 \text{ is sufficiently small.} \end{cases} \tag{3.24}$$

Since N_{MS} is positive definite on $L_{2,(0)}(\Sigma)$, cf. the proof of Lemma 5.4, there is some $d_0 > 0$ such that

$$\begin{aligned} (B_\lambda h|h)_2 &= \lambda(N_{MS} h|h)_2 - (\partial_x \partial_x h|h)_2 + (P_0 \partial_x \partial_x h|h)_2 \\ &\geq \lambda d_0 |h|_2^2 + |\partial_x h|_2^2 - \omega_+[h(0)^2 + h(L)^2], \end{aligned}$$

since $(P_0 \partial_x \partial_x h|h)_2 = 0$. It remains to show that

$$(\lambda - 1) d_0 |h|_2^2 + |\partial_x h|_2^2 - \omega_+[h(0)^2 + h(L)^2] \geq 0, \tag{3.25}$$

if only $\lambda \geq \lambda_0$ for $\lambda_0 > 0$ sufficiently large. Then B_λ is positive definite for $\lambda \geq \lambda_0$. We now claim the following Young-type inequality. Note that the following lemma immediately implies (3.25).

LEMMA 3.8. *For every $\delta > 0$ there is a constant $C_\delta > 0$, such that*

$$h(j)^2 \leq \delta |\partial_x h|_2^2 + C_\delta |h|_2^2, \quad j = 0, L,$$

for any $h \in H_2^1(\Sigma_*)$.

PROOF. The proof follows the lines of [25]. Assume there is some $\delta > 0$ such that the statement is not true. Then there is a sequence $(h_n)_n \subset H_2^1(\Sigma_*)$, such that

$$1 = h_n(0)^2 > \delta|\partial_x h_n|_2^2 + n|h_n|_2^2, \quad \text{for all } n \in \mathbb{N}. \tag{3.26}$$

In particular, $|\partial_x h_n|_2^2 < 1/\delta$ and $|h_n|_2^2 < 1/n$ for each n. Hence $(h_n)_n$ is bounded in H_2^1 and there is a subsequence again denoted by $(h_n)_n$ converging weakly to some h in H_2^1. Furthermore, h_n converges strongly to zero in L_2. By uniqueness, h_n converges weakly to zero in H_2^1. By the compact embedding $H_2^1(\Sigma_*) \hookrightarrow\hookrightarrow C^0([0, L])$, h_n converges strongly in C^0-norm to zero as $n \to \infty$. This implies $h_n(0) \to 0$ as $n \to \infty$, which is a contradiction to (3.26). □

We now show the second part of (3.24). Note that

$$\lim_{\lambda\to 0,\lambda>0}(B_\lambda h|h)_2 = -(\partial_x\partial_x h|h)_2 = |\partial_x h|_2^2 - \omega_+[h(0)^2 + h(L)^2], \quad h \in D(B_\lambda),$$

since $\lim_{\lambda\to 0,\lambda>0} \lambda(N_{MS}h|h)_2 = 0$ for any fixed $h \in D(B_\lambda)$. It now remains to construct a function $\bar{h} \in D(B_\lambda)$ such that

$$|\partial_x \bar{h}|_2^2 - \omega_+[\bar{h}(0)^2 + \bar{h}(L)^2] < 0. \tag{3.27}$$

We start with the following construction. Let $\varepsilon > 0$ and define

$$\bar{g}(s) := \begin{cases} 1 - \omega_+ s, & s \in [0, \varepsilon], \\ 1 - \omega_+\varepsilon - (1 - \omega_+\varepsilon)(s - \varepsilon)/(L/2 - \varepsilon), & s \in [\varepsilon, L/2], \\ -\bar{g}(L - s), & s \in [L/2, L], \end{cases} \tag{3.28}$$

cf. Figure 2. Then $\bar{g}$ satisfies the boundary conditions $\partial_x \bar{g}(0) = -\omega_+ \bar{g}(0)$ and

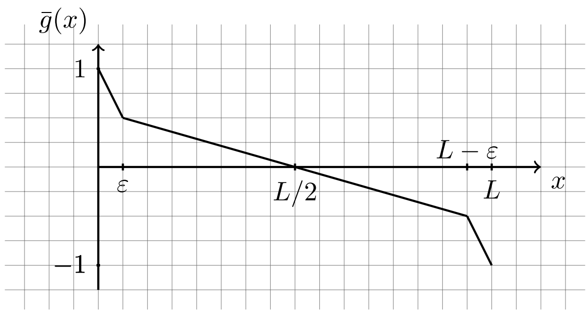

FIGURE 2. Construction of $\bar{g}$.

$\partial_x \bar{g}(L) = \omega_+ \bar{g}(L)$. Clearly, $g \in H^1_{2,(0)}(\Sigma_*)$. Furthermore a direct calculation shows

$$\begin{aligned} |\partial_x \bar{g}|^2_{L_2(0,L/2)} - \omega_+ \bar{g}(0)^2 &= \int_0^\varepsilon \omega_+^2\, dx + \int_\varepsilon^{L/2} \frac{(1-\omega_+\varepsilon)^2}{(L/2-\varepsilon)^2} dx - \omega_+ \\ &= \varepsilon\omega_+^2 + \frac{(1-\omega_+\varepsilon)^2}{L/2-\varepsilon} - \omega_+. \end{aligned}$$

In particular, the first two terms converge to $2/L$ as $\varepsilon \to 0$. If now $\omega_+ > 2/L$, the right hand side will be negative whenever $\varepsilon = \varepsilon(\omega_+) > 0$ is small enough. Note that the critical value for ω_+ is $2/L$, which is exactly the degeneracy of the linear system for $[h(0), h(L)] \in \mathbb{R}^2$ in (3.19): $2 - L\omega_+ = 0$. By approximating $\bar{g}$ with a smoother function in $D(B_\lambda)$ we have shown (3.27). Following [64] using (3.24) we obtain that there is indeed a positive eigenvalue $\lambda > 0$ as claimed.

(3) Fix now $\omega_+ > 0$. We now need to understand the dependence on L in estimate (3.22). Let us calculate the embedding constant of $H^1_{2,(0)}(0, L) \hookrightarrow C^0([0, L])$. Firstly,

$$h(t) = h(s) + \int_s^t \partial_x h(\tau) d\tau, \quad s, t \in [0, L].$$

Integrating in $s \in [0, L]$ and using that h is mean value free on $(0, L)$ gives

$$h(t) = \frac{1}{L} \int_0^L \int_s^t \partial_x h(\tau) d\tau ds, \quad s, t \in [0, L].$$

Hence

$$\sup_{t \in [0,L]} |h(t)| \leq \int_0^L |\partial_x h(\tau)| d\tau.$$

Using Hölders inequality gives

$$\sup_{t \in [0,L]} |h(t)|^2 \leq L \int_0^L |\partial_x h(\tau)|^2 d\tau.$$

In particular,

$$\begin{aligned}\int_0^L |\partial_x h|^2 dx - \omega_+ h(0)^2 - \omega_+ h(L)^2 \\ \geq \frac{1}{L} |h|^2_{C^0([0,L])} - \omega_+ [h(0)^2 + h(L)^2].\end{aligned}$$

Again we see that if $L = L(\omega_+) > 0$ is sufficiently small,

$$\int_0^L |\partial_x h|^2 dx - \omega_+ [h(0)^2 + h(L)^2] \geq (\frac{1}{L} - 2\omega_+) |h|^2_{C^0([0,L])} \geq 0.$$

Hence (3) follows.

(4) We fix $\omega_+ > 0$. Following the lines of the proof of (2), we only need to justify $(3.24)_2$, where B_λ is defined as before. In particular, we need to show that there is a function $\bar{h} \in D(B_\lambda)$ such that

$$|\partial_x \bar{h}|^2_{L_2(0,L)} - \omega_+ [\bar{h}(0)^2 + \bar{h}(L)^2] < 0,$$

where now $\omega_+ > 0$ is fixed, if we only choose $L > 0$ large enough. Let us consider the function $\bar{g}$ defined in (3.28). Again for $\varepsilon > 0$,

$$|\partial_x \bar{g}|^2_{L_2(0,L/2)} - \omega_+ \bar{g}(0) = \varepsilon \omega_+^2 + \frac{(1 - \omega_+ \varepsilon)^2}{L/2 - \varepsilon} - \omega_+.$$

Since $\omega_+ > 0$ is fixed, we may choose $\varepsilon > 0$ so small, such that $\varepsilon \omega_+^2 \leq \omega_+/2$. Then choosing $L = L(\omega_+) > 0$ sufficiently large we obtain that $|\partial_x \bar{g}|^2_{L_2(0,L/2)} - \omega_+ \bar{g}(0) < 0$. We can then follow the lines of the proof of (2) to conclude (4). □

REMARK 3.9. For monotonicity considerations of the spectral properties we refer to [25].

3.3.3. Curved stationary solutions. In this section we consider stationary solutions Σ_* with constant curvature $\kappa_* = -1/R$, for some $R > 0$. In particular, Σ_* is part of a circle. We can therefore introduce a parametrization by arc length,

$$\psi : (0, l) \to \Sigma_*, \ \sigma \mapsto \psi(\sigma),$$

where $l > 0$ is the length of the curve and σ the arc length parameter. Note that $l < 2\pi R = 2\pi/|\kappa_*|$. Note that this induces an extra restriction on κ_* and l,

$$|\kappa_*| l < 2\pi. \tag{3.29}$$

Corresponding to (3.11) we now want to make a linear stability analysis for

$$\begin{aligned} \partial_t \rho &= -[\![n_{\Sigma_*} \cdot \nabla \mu]\!], && \text{on } \Sigma_*, \\ \mu|_{\Sigma_*} &= \partial_\sigma \partial_\sigma \rho + \kappa_*^2 \rho, && \text{on } \Sigma_*, \\ \Delta \mu &= 0, && \text{in } \Omega \backslash \Sigma_*, \\ n_{\partial\Omega} \cdot \nabla \mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega \backslash \partial\Sigma_*, \\ \partial_\sigma \rho(0) &= -\omega_1 \rho(0), \\ \partial_\sigma \rho(l) &= \omega_2 \rho(l), \\ \rho(0) &= \rho_0, && \text{on } \Sigma_*. \end{aligned} \tag{3.30}$$

Note that by some abuse of notation we may identify $\sigma \in (0, l)$ and $\psi(\sigma) \in \Sigma_*$, since there is no danger of confusion.

Let us rewrite (3.30) again as an abstract evolution equation. Let $3/2 < q < 2$, $X_0 := W_q^{1-1/q}(0, l)$, and $X_1 := W_q^{4-1/q}(0, l)$. Define now a linear operator $A : D(A) \subset X_1 \to X_0$ by means of $A\rho := BT_0(\partial_\sigma \partial_\sigma \rho + \kappa_*^2 \rho)$, where $Bu := [\![n_{\Sigma_*} \cdot \nabla u]\!]$ and $T_0 v$ is the unique solution of the two-phase elliptic problem (3.14).

The domain of A is thereby given by

$$D(A) := X_1 \cap \{\rho : \partial_\sigma \rho(0) = -\omega_1 \rho(0), \ \partial_\sigma \rho(l) = \omega_2 \rho(l)\}. \tag{3.31}$$

We can then rewrite (3.30) as the abstract evolutionary problem

$$\dot\rho(t) + A\rho(t) = 0, \ t > 0, \quad \rho(0) = \rho_0. \tag{3.32}$$

Let again $p \in (6, \infty)$, $q \in (19/10, 2) \cap (2p/(p+1), 2)$, and $X_\gamma := B_{qp}^{4-1/q-3/p}(0, l)$.

We start with a positive result on stability for (3.32) of the trivial solution $\rho_* = 0$.

THEOREM 3.10. *Let $l > 0$ be fixed. Then there is some $\delta = \delta(l) > 0$, such than whenever $|\kappa_*| \in (0, \delta)$ and $\omega_1, \omega_2 \in (-\infty, \delta)$, the trivial equilibrium $\rho_* = 0$ is normally stable, that is,*

1. *A has maximal L_p-regularity.*
2. *The set of equilibria of (3.32) is the kernel of A, which has finite dimension $m \in \mathbb{N} \cup \{0\}, m < \infty$.*
3. *The eigenvalue zero is semi-simple, $X_0 = N(A) \oplus R(A)$.*
4. *The spectrum satisfies $\sigma(-A) \backslash \{0\} \subset \mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.*

In particular, $\rho_* = 0$ is stable in X_γ and there is some $\delta_1 > 0$, such that if $|\rho_0|_{X_\gamma} \le \delta_1$ the unique solution ρ to (3.32) with respect to the initial value ρ_0 exists globally in time,

$$\rho \in W_p^1(\mathbb{R}_+; X_0) \cap L_p(\mathbb{R}_+; D(A)),$$

and converges to some equilibrium solution in X_γ at an exponential rate.

Proof. For any κ_* constant we note that the term $\kappa_*\rho$ is a compact perturbation of $\partial_\sigma\partial_\sigma\rho$ in $W_q^{2-1/q}(0,l)$, whence A has maximal L_p-regularity by a perturbation argument, cf. Section 2.5. Let us now characterize the kernel of A. Since the domain $D(A)$ compactly embeds into X_0, the resolvent of A is compact. The spectrum then consists solely of isolated eigenvalues of finite multiplicity. In particular, the kernel, if it is nontrivial, has finite dimension $m < \infty$, cf. [**19**], [**43**], [**44**]. Pick some $\rho \in D(A)$ such that $A\rho = 0$. Then the solution of the corresponding elliptic problem is constant, hence $\partial_\sigma\partial_\sigma\rho + \kappa_*^2\rho$ is constant. Hence the kernel of A is given by the solutions ρ of

$$\begin{aligned} \partial_\sigma\partial_\sigma\rho + \kappa_*^2\rho &= c, && \text{on } (0,l),\\ \partial_\sigma\rho(0) &= -\omega_1\rho(0), \\ \partial_\sigma\rho(l) &= \omega_2\rho(l), \end{aligned}$$

where c is any constant $c \in \mathbb{R}$. The next thing we show is that zero is semi-simple. By Remark A.2.4 in [**43**] it suffices to prove that $N(A^2) = N(A)$. To this end pick some $\rho \in N(A^2)$. Let $\rho_1 := A\rho$. Then $A\rho_1 = 0$ and hence $\rho_1 \in N(A) \cap R(A)$. Note that then necessarily ρ_1 is mean value free, $\int_0^l \rho_1 = 0$. This stems from integrating $\rho_1 = A\rho$ over Σ_* and an integration by parts. Since ρ_1 also belongs to the kernel of A,

$$\begin{aligned} \partial_\sigma\partial_\sigma\rho_1 + \kappa_*^2\rho_1 &= c_1, && \text{on } (0,l),\\ \partial_\sigma\rho_1(0) &= -\omega_1\rho_1(0), \\ \partial_\sigma\rho_1(l) &= \omega_2\rho_1(l), \end{aligned} \tag{3.33}$$

for some constant c_1. Note that c_1 is determined by ρ_1. Since ρ_1 is mean value free, we can test $(3.33)_1$ with ρ_1 to the result

$$\int_0^l \partial_\sigma\partial_\sigma\rho_1\rho_1 + \kappa_*^2\int_0^l \rho_1\rho_1 = c_1\int_0^l \rho_1 = 0.$$

An integration by parts then gives

$$-\int_0^l |\partial_\sigma\rho_1|^2 + \kappa_*^2\int_0^l |\rho_1|^2 + \omega_1\rho_1(0)^2 + \omega_2\rho_1(l)^2 = 0.$$

Since ρ_1 is mean value free, we can use Poincaré-Wirtinger inequality to find some constant $c_0 = c_0(l) > 0$, such that

$$-c_0|\rho_1|_{H^1}^2 + \kappa_*^2|\rho_1|_{L^2}^2 + \omega_1\rho_1(0)^2 + \omega_2\rho_1(l)^2 \geq 0.$$

In particular, if κ_*^2 is sufficiently small and ω_1, ω_2 are negative or positive but small, the second, third and fourth term may be absorbed by the first one and we obtain

$$-\tilde{c}_0|\rho_1|_{H^1}^2 \geq 0,$$

for some $\tilde{c}_0 > 0$. Hence $\rho_1 = 0$, which implies $A\rho = \rho_1 = 0$ and $\rho \in N(A)$. This shows zero is a semi-simple eigenvalue.

Let us now consider the general eigenvalue problem $\lambda\rho = -A\rho$ for some $\rho \in D(A)$, which reads as

$$\begin{aligned}
\lambda\rho &= -[\![n_{\Sigma_*} \cdot \nabla\mu]\!], && \text{on } \Sigma_*,\\
\mu|_{\Sigma_*} &= \partial_\sigma\partial_\sigma\rho + \kappa_*^2\rho, && \text{on } \Sigma_*,\\
\Delta\mu &= 0, && \text{in } \Omega\backslash\Sigma_*,\\
n_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\partial\Sigma_*,\\
\partial_\sigma\rho(0) &= -\omega_1\rho(0),\\
\partial_\sigma\rho(l) &= \omega_2\rho(l).
\end{aligned} \tag{3.34}$$

Multiplying (3.34)$_1$ with $\partial_\sigma\partial_\sigma\bar\rho + \kappa_*^2\bar\rho$ and invoking boundary and transmission conditions gives

$$\lambda\left[|\partial_\sigma\rho|_{L^2}^2 - \kappa_*^2|\rho|_{L^2}^2 - \omega_1\rho(0)^2 - \omega_2\rho(l)^2\right] + |\nabla\mu|_{L^2}^2 = 0. \tag{3.35}$$

If $\lambda \neq 0$, any eigenfunction ρ associated to some eigenvalue λ is necessarily mean value free, whence again Poincaré-Wirtinger inequality gives that

$$\left[|\partial_\sigma\rho|_{L^2}^2 - \kappa_*^2|\rho|_{L^2}^2 - \omega_1\rho(0)^2 - \omega_2\rho(l)^2\right] \geq 0,$$

provided $|\kappa_*| \in [0,\delta)$ and $\omega_1, \omega_2 \in (-\infty, \delta)$ for $\delta > 0$ sufficiently small. Equation (3.35) then gives that λ is real and $\lambda \leq 0$. Hence (3) follows. The generalized principle of linearized stability of Prüss, Simonett, and Zacher [**58**] then gives the result. □

Let us show instability results for the evolution equation (3.32).

THEOREM 3.11. *Let A, X_0, X_γ, X_1 be as above in* (3.31)*.*

(1) *For fixed $l > 0$ and any small κ_*, there is some $K = K(l, \kappa_*) > 0$ such that if $\omega_1 = \omega_2 \geq K$, the trivial solution $\rho_* = 0$ is unstable in X_γ.*
(2) *For fixed $l > 0$ and any $\omega_1 = \omega_2$ small, there is some $K = K(l, \omega_1) > 0$ such that if $\kappa_*^2 \geq K$, the trivial solution $\rho_* = 0$ is unstable in X_γ.*
(3) *For any κ_* and $\omega_1 = \omega_2$ small, there is some $K = K(\kappa_*, \omega_1) > 0$ such that if $l \geq K$, the trivial solution $\rho_* = 0$ is unstable in X_γ.*
(4) *In (2) and (3) the constant K is not too large to violate the geometric condition between length and curvature of a circle* (3.29)*, that means there are (κ_*, l) fulfilling (2) or (3) which at the same time fulfil $|\kappa_*|l < 2\pi$.*

In particular, in any of these cases, $\sigma(-A) \cap [\zeta + i\mathbb{R}] = \emptyset$ and $\sigma(-A) \cap \{z \in \mathbb{C} : \operatorname{Re} z > \zeta\} \neq \emptyset$ for some $\zeta \in \mathbb{R}, \zeta \geq 0$.

PROOF. By the compact embedding $D(A) \hookrightarrow\hookrightarrow X_0$ we know that A has a compact resolvent. Hence the spectrum of A is isolated, consists only of eigenvalues and each eigenvalue has finite multiplicity. Furthermore, any eigenvalue λ is real and satisfies

$$\lambda\left[|\partial_\sigma\rho|_{L^2}^2 - \kappa_*^2|\rho|_{L^2}^2 - \omega_1\rho(0)^2 - \omega_2\rho(l)^2\right] + |\nabla\mu|_{L^2}^2 = 0,$$

cf. (3.35), where $\mu = T_0(\partial_\sigma\partial_\sigma\rho + \kappa_*^2\rho)$ and ρ is a corresponding eigenfunction to λ.

We now follow the lines of the proof of Theorem 3.7,(2). For $\lambda > 0$ we can rewrite the eigenvalue problem $\lambda\rho = A\rho$ as

$$\lambda\rho - D_{MS}(I - P_0)S\rho = 0, \tag{3.36}$$

where D_{MS} is as before the corresponding Dirichlet-to-Neumann operator with inverse $N_{MS} = [D_{MS}]^{-1}$, $P_0 f$ the mean value of f, and $S\rho := \partial_\sigma\partial_\sigma\rho + \kappa_*^2\rho$ with domain $D(S) := D(A)$. We can then extend the operators in a natural way and rewrite (3.36) as

$$\lambda N_{MS}\rho - (I - P_0)S\rho = 0.$$

Define $B_\lambda := \lambda N_{MS} - (I - P_0)S$ with natural domain $D(B_\lambda) := H_2^2(0,l) \cap \{\rho : \int_0^l \rho = 0,\ \partial_\sigma\rho(0) = -\omega_1\rho(0),\ \partial_\sigma\rho(l) = -\omega_1\rho(l)\}$.

Let us show that there is some $\lambda_0 > 0$ such that B_λ is positive definite on $L_{2,(0)}$ for all $\lambda \geq \lambda_0$. Since N_{MS} is positive definite on $L_{2,(0)}$, cf. Lemma 5.5, and ρ is mean value free,

$$\begin{aligned}
(B_\lambda\rho|\rho)_2 &= \lambda(N_{MS}\rho|\rho)_2 - ((I - P_0)(\partial_\sigma\partial_\sigma\rho + \kappa_*^2\rho)|\rho)_2 \\
&= \lambda(N_{MS}\rho|\rho)_2 - (\partial_\sigma\partial_\sigma\rho + \kappa_*^2\rho|\rho)_2 \\
&\geq \lambda d_0|\rho|_2^2 + |\partial_\sigma\rho|_2^2 - \omega_1\rho(0)^2 - \omega_2\rho(l)^2 - \kappa_*^2|\rho|_2^2,
\end{aligned}$$

for some $d_0 > 0$, for any $\rho \in D(B_\lambda)$. Lemma 3.8 then gives that for any ω_1, ω_2, and κ_*, the last three terms may be absorbed if $\lambda \geq \lambda_0$ for some $\lambda_0 = \lambda_0(\omega_1, \omega_2, \kappa_*) > 0$. Hence B_λ is positive definite on $L_{2,(0)}$ for all $\lambda \geq \lambda_0$.

It remains to construct a function $\bar\rho \in D(B_\lambda)$, such that $(B_\lambda\bar\rho|\bar\rho)_2 < 0$, if $\lambda > 0$ is sufficiently small. Since $\lambda(N_{MS}\rho|\rho)_2 \to 0$ as $\lambda \to 0$, it is enough to find some $\bar\rho \in D(B_\lambda)$ such that

$$|\partial_\sigma\bar\rho|_2^2 - \omega_1\bar\rho(0)^2 - \omega_2\bar\rho(l)^2 - \kappa_*^2|\bar\rho|_2^2 < 0. \tag{3.37}$$

We now want to find such $\bar\rho$ in all three cases stated in the theorem. To this end we start again with the prototype introduced in (3.28). Let $\varepsilon_1 > 0$ small. Define $\bar g : [0,l] \to \mathbb{R}, \sigma \mapsto \bar g(\sigma)$ by means of

$$\bar g(\sigma) := \begin{cases} 1 - \omega_1\sigma, & \sigma \in [0, \varepsilon_1], \\ 1 - \omega_1\varepsilon_1 - (1 - \omega_1\varepsilon_1)(\sigma - \varepsilon_1)/(l/2 - \varepsilon_1), & \sigma \in [\varepsilon_1, l/2], \\ -\bar g(l - \sigma), & \sigma \in [l/2, l]. \end{cases}$$

Note that $\bar g(0) = 1$, $\bar g(l) = -1$, $\bar g$ fulfils the boundary conditions, is mean value free, and piecewise smooth and continuous, hence in $H_2^1(0,l)$. We then explicitly calculate

$$\begin{aligned}
&|\partial_\sigma\bar g|^2_{L_2(0,l/2)} - \omega_1\bar g(0)^2 - \kappa_*^2|\bar g|^2_{L_2(0,l/2)} = \\
&= \varepsilon_1\omega_1^2 + \frac{(1 - \omega_1\varepsilon_1)^2}{l/2 - \varepsilon_1} - \omega_1 - \kappa_*^2\left[\frac{\varepsilon_1}{3}(3 - 3\omega_1\varepsilon_1 + \omega_1^2\varepsilon_1) + \frac{1}{6}(1 - \omega_1\varepsilon_1)^2(l - 2\varepsilon_1)\right].
\end{aligned}$$

Note that by symmetry it suffices to calculate the expressions on $(0, l/2)$. We now let formally $\varepsilon_1 \to 0$. The expression on the right hand side then converges to

$$\frac{2}{l} - \omega_1 - \kappa_*^2\frac{l}{6}. \tag{3.38}$$

We now distinguish the three cases in the theorem.

(1) Here we fix $l > 0$ and κ_*. It is clear that there is some $K > 0$ such that the expression in (3.38) gets strictly negative if $\omega_1 \geq K$. It even holds that $2/l - \omega_1 - \kappa_*^2 l/6 \to -\infty$ if $\omega_1 \to \infty$.

(2) Here we fix $l > 0$ and $|\omega_1|$ small. Note that in this case there is a geometric condition, $|\kappa_*|l < 2\pi$, so we can not choose $|\kappa_*|$ arbitrarily large. However, taking the limit as $|\kappa_*| \to 2\pi/l$ of expression (3.38), we obtain

$$\lim_{|\kappa_*| \to 2\pi/l} \left[\frac{2}{l} - \omega_1 - \kappa_*^2 \frac{l}{6} \right] = \frac{2}{l} \left[1 - \frac{\pi^2}{3} \right] - \omega_1 < 0,$$

provided $|\omega_1|$ is small enough.

(3) In this case we fix κ_* and $|\omega_1|$ small. Again we have to fulfil the relation $|\kappa_*|l < 2\pi$. Taking limits $l \to 2\pi/|\kappa_*|$,

$$\lim_{l \to 2\pi/|\kappa_*|} \left[\frac{2}{l} - \omega_1 - \kappa_*^2 \frac{l}{6} \right] = |\kappa_*| \left[\frac{1}{\pi} - \frac{\pi}{3} \right] - \omega_1 < 0,$$

provided again $|\omega_1|$ is small.

This way we now obtain the following result in all three cases: By choosing $\varepsilon_1 > 0$ very small, we can construct $\bar{g}$ as above such that the strict inequality (3.37) holds true. Since $\bar{g}$ is only H_2^1 and not H_2^2 we need to approximate $\bar{g}$ by a more regular function $\bar{\rho}$, which then belongs to the domain $D(B_\lambda)$ and also fulfils the strict inequality (3.37). This then shows that $(B_\lambda \bar{\rho}|\bar{\rho})_2 < 0$ for some $\bar{\rho} \in D(B_\lambda)$ if $\lambda > 0$ is sufficiently small, whence B_λ is not positive definite for this small $\lambda > 0$. Following the proof of [**64**], we then obtain the existence of a positive eigenvalue.

The fact that then $\rho_* = 0$ is unstable in X_γ follows from the generalized principle of linearized stability, cf. [**57**]. □

3.3.4. Summary on linearized stability and instability. In this section we shall summarize the results on linearized stability and provide representative figures.

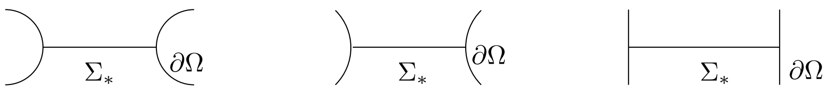

FIGURE 3. $\kappa_* = 0$. Stability for all $\omega_1, \omega_2 \leq 0$ regardless of $L > 0$.

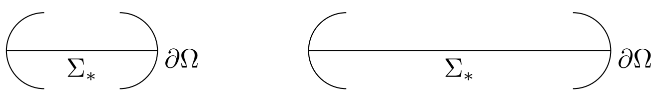

FIGURE 4. $\kappa_* = 0$ and fixed $\omega_1 = \omega_2 > 0$. Stability for small $L > 0$, instability for large $L > 0$.

FIGURE 5. $\kappa_* = 0$ and fixed $L > 0$. Stability for small $\omega_1 = \omega_2 > 0$, instability for large $\omega_1 = \omega_2$.

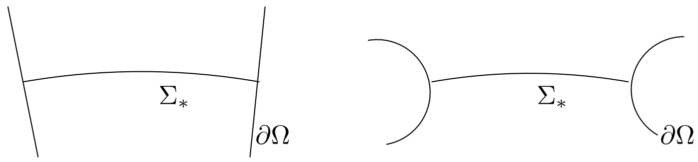

FIGURE 6. Fixed $l > 0$ and small $\kappa_* \neq 0$. Stability for $\omega_1, \omega_2 \leq 0$.

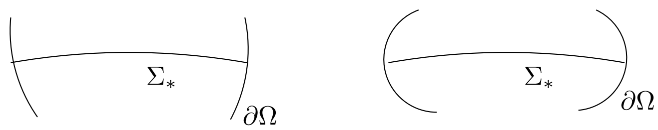

FIGURE 7. Fixed $l > 0$ and small $\kappa_* \neq 0$. Stability for $\omega_1, \omega_2 > 0$ small. Instability for $\omega_1, \omega_2 > 0$ large.

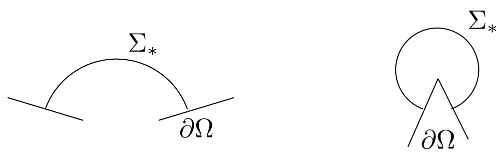

FIGURE 8. Fixed $l > 0$ and $\omega_1, \omega_2 = 0$. Stability for κ_*^2 small. Instability for κ_*^2 large.

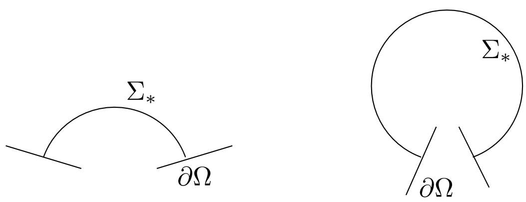

FIGURE 9. Fixed $\kappa_* \neq 0$ and $\omega_1, \omega_2 = 0$. Stability for $l > 0$ small. Instability for $l > 0$ large.

REMARK 3.12. Proving nonlinear stability for general geometries even in two space dimensions is an involved task. For a first result on nonlinear stability we refer to Section 7 in the article together with Harald Garcke [**27**].

CHAPTER 4

The two-phase Navier-Stokes/Mullins-Sekerka equations with ninety degree contact angle

4.1. Introduction

In this chapter we study the two-phase Navier-Stokes equations with surface tension coupled to the Mullins-Sekerka problem inside a bounded domain in two or three space dimensions. In our model, the interface separating the two fluids meets the boundary of the domain at a constant ninety degree angle. Let us introduce the precise model and its assumptions. We assume that the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, can be decomposed as $\Omega = \Omega^+(t) \dot\cup \mathring{\Gamma}(t) \dot\cup \Omega^-(t)$, where $\mathring{\Gamma}(t)$ denotes the interior of $\Gamma(t)$, an $(n-1)$-dimensional submanifold with boundary. We interpret $\Gamma(t)$ to be the interface separating the two phases, $\Omega^+(t)$ and $\Omega^-(t)$, which both will be assumed to be connected. The boundary of $\Gamma(t)$ will be denoted by $\partial\Gamma(t)$. Furthermore we assume $\Gamma(t)$ to be orientable, the unit normal vector field on $\Gamma(t)$ pointing from $\Omega^-(t)$ into $\Omega^+(t)$ will be denoted by $\nu_{\Gamma(t)}$.

Let us introduce some notation. Let $V_{\Gamma(t)}$ denote the normal velocity and $H_{\Gamma(t)}$ the mean curvature of the free interface $\Gamma(t)$. By $[\![\cdot]\!]$ we denote the jump of a quantity across $\Gamma(t)$ in direction of $\nu_{\Gamma(t)}$, that is,

$$[\![f]\!](x) := \lim_{\varepsilon \to 0+} [f(x + \varepsilon \nu_{\Gamma(t)}) - f(x - \varepsilon \nu_{\Gamma(t)})], \quad x \in \Gamma(t).$$

Furthermore, $a \otimes b$ is defined by $[a \otimes b]_{ij} := a_i b_j$ for vectors $a, b \in \mathbb{R}^n$ and $A^\top$ denotes the transposed matrix of A.

We assume that Ω is filled by two immiscible, incompressible fluids with respective constant densities $\rho^\pm > 0$ in the two phases. Their respective constant viscosities are denoted by $\mu^\pm > 0$ and $\sigma > 0$ is a given surface tension constant. To economize our notation, we let $\rho := \rho^+ \chi_{\Omega^+(t)} + \rho^- \chi_{\Omega^-(t)}$ and $\mu := \mu^+ \chi_{\Omega^+(t)} + \mu^- \chi_{\Omega^-(t)}$, where χ_M is the indicator function of a set M. In our model, u is the velocity of the fluids, p the pressure, η the chemical potential, and $\Gamma(t)$ the free interface at time $t \geq 0$.

Let us consider the case where the domain is a cylindrical container $\Omega = \Sigma \times (L_1, L_2)$, where $-\infty < L_1 < 0 < L_2 < \infty$, and $\Sigma \subset \mathbb{R}^2$ is bounded and has smooth boundary. We denote the walls of the cylinder Ω by $S_1 := \partial\Sigma \times (L_1, L_2)$, and bottom and top by $S_2 := \Sigma \times \{L_1, L_2\}$. As usual, $\nu_{\partial\Omega}$ denotes the unit normal vector field pointing outwards of Ω and $\nu_{S_1} = \nu_{\partial\Omega}$ on the walls S_1. The projection is defined as $P_{S_1} := I - \nu_{S_1} \otimes \nu_{S_1}$.

We naturally impose that $\mathring{\Gamma}(t) \subset \Omega$ and $\partial\Gamma(t) \subset S_1$ for all $t \geq 0$, that is, the interface stays inside the domain for positive times and the boundary of the interface is contained in the boundary of the domain as well.

In a cylindrical domain the full problem for two possibly different, constant densities $\rho^\pm > 0$ and viscosities $\mu^\pm > 0$ reads as

$$\begin{aligned}
\rho\partial_t u - \mu\Delta u + \operatorname{div}[(\rho u + [\![\rho]\!]\nabla\eta)\otimes u] + \nabla p &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
-[\![\mu(Du + Du^\top)]\!]\nu_{\Gamma(t)} + [\![p]\!]\nu_{\Gamma(t)} &= \sigma H_{\Gamma(t)}\nu_{\Gamma(t)}, && \text{on } \Gamma(t),\\
[\![u]\!] &= 0, && \text{on } \Gamma(t),\\
V_{\Gamma(t)} - u|_{\Gamma(t)}\cdot\nu_{\Gamma(t)} &= -[\![\nu_{\Gamma(t)}\cdot\nabla\eta]\!], && \text{on } \Gamma(t),\\
\nu_{\Gamma(t)}\cdot\nu_{S_1} &= 0, && \text{on } \partial\Gamma(t),\\
\Delta\eta &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\eta|_{\Gamma(t)} &= \sigma H_{\Gamma(t)}, && \text{on } \Gamma(t),\\
\nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Gamma(t),\\
P_{S_1}\left(\mu(Du + Du^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u &= 0, && \text{on } S_2,\\
u(0) &= u_0, && \text{on } \Omega\backslash\Gamma(0),\\
\Gamma(0) &= \Gamma_0.
\end{aligned} \tag{4.1}$$

Note that in (4.1), individual masses are conserved, i.e.,

$$\frac{d}{dt}|\Omega^\pm(t)| = 0, \quad t\in\mathbb{R}_+,$$

where we used $\Delta\eta = 0$ and $\operatorname{div} u = 0$ in the bulk phases $\Omega\backslash\Gamma(t)$ and the boundary conditions for η and u.

In this model (4.1) which is itself derived as a sharp interface limit by Abels, Garcke, and Grün in [**2**], the momentum balance $(4.1)_1$ contains an extra term involving the chemical potential η since the densities in the two phases are different. This term however is needed to get an energy structure for the system, cf. Section 5 in [**2**]. It is shown there that the energy

$$E(t) := \int_{\Gamma(t)} \sigma d\mathcal{H}^{n-1} + \frac{1}{2}\int_\Omega \rho(t)u(t)^2 dx$$

satisfies the energy-dissipation relation

$$\frac{d}{dt}E(t) = -D(t) := -\int_\Omega \mu|\mathbb{D}u(t)|^2 dx - \int_\Omega |\nabla\eta(t)|^2 dx.$$

Hereby, $\mathbb{D}u$ is the symmetric part of the gradient Du.

There is a remark in order regarding this extra term in $(4.1)_1$. Since $\operatorname{div} u = 0$ and $\Delta\eta = 0$ in the bulk phases $\Omega\backslash\Gamma(t)$, we obtain

$$\operatorname{div}[(\rho u + (\rho^+ - \rho^-)\nabla\eta)\otimes u] = \rho(u\cdot\nabla)u + (\rho^+ - \rho^-)(\nabla\eta\cdot\nabla)u, \quad \text{in } \Omega\backslash\Gamma(t).$$

In the case of equivalent densities, say for simplicity $\rho = 1$, the extra term $\mathrm{div}[(\rho^+ - \rho^-)\nabla\mu \otimes u]$ vanishes and the system reduces to

$$
\begin{aligned}
\partial_t u - \mu^{\pm}\Delta u + (u\cdot\nabla)u + \nabla p &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\Delta\eta = 0, \quad \mathrm{div}\, u &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
-[\![\mu^{\pm}(Du + Du^{\top})]\!]\nu_{\Gamma(t)} + [\![p]\!]\nu_{\Gamma(t)} &= \sigma H_{\Gamma(t)}\nu_{\Gamma(t)}, && \text{on } \Gamma(t),\\
V_{\Gamma(t)} - u|_{\Gamma(t)}\cdot\nu_{\Gamma(t)} &= -[\![\nu_{\Gamma(t)}\cdot\nabla\eta]\!], && \text{on } \Gamma(t),\\
\nu_{\Gamma(t)}\cdot\nu_{S_1} &= 0, && \text{on } \partial\Gamma(t),\\
[\![u]\!] = 0, \quad \eta|_{\Gamma(t)} &= \sigma H_{\Gamma(t)}, && \text{on } \Gamma(t),\\
\nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Gamma(t),\\
P_{S_1}\left(\mu^{\pm}(Du + Du^{\top})\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u &= 0, && \text{on } S_2,\\
u(0) = u_0, \quad \Gamma(0) &= \Gamma_0.
\end{aligned}
$$

Let us comment on the boundary conditions for u. To be able to use the reflection techniques of [**64**] we can not pose Dirichlet boundary conditions on the walls S_1. The next natural step is then to pose Navier boundary conditions. In a cylindrical container though, we need to resolve compatibility conditions on the ninety degree edges between the walls and the bottom/top part of the container, cf. [**64**].

However, if the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is bounded and has smooth boundary, we can also show local well-posedness for a system involving full Navier boundary conditions with positive friction coefficient. To prove this statement, we again start a localization procedure, this time even involving only the model problem with the contact line. Since there is no Dirichlet boundary anymore, we do not need the respective compatibility conditions. By performing a perturbation argument, we can include full Navier boundary conditions. However, since the reference surface might be curved, we need to introduce curvilinear coordinates in a neighbourhood of Σ and consider a more complicated transformed mean curvature operator, cf. Section 2.4, see also [**63**], [**3**], [**4**], [**22**].

This way, we can also deal with the problem

$$
\begin{aligned}
\rho\partial_t u - \mu\Delta u + \mathrm{div}[(\rho u + [\![\rho]\!]\nabla\eta)\otimes u] + \nabla p &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\Delta\eta = 0, \quad \mathrm{div}\, u &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
-[\![\mu(Du + Du^{\top})]\!]\nu_{\Gamma(t)} + [\![p]\!]\nu_{\Gamma(t)} &= \sigma H_{\Gamma(t)}\nu_{\Gamma(t)}, && \text{on } \Gamma(t),\\
V_{\Gamma(t)} - u|_{\Gamma(t)}\cdot\nu_{\Gamma(t)} &= -[\![\nu_{\Gamma(t)}\cdot\nabla\eta]\!], && \text{on } \Gamma(t),\\
\nu_{\Gamma(t)}\cdot\nu_{S_1} &= 0, && \text{on } \partial\Gamma(t),\\
[\![u]\!] = 0, \quad \eta|_{\Gamma(t)} &= \sigma H_{\Gamma(t)}, && \text{on } \Gamma(t),\\
\nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Gamma(t),\\
P_{S_1}(\mu(Du + Du^{\top})\nu_{S_1}) + \alpha u &= 0, && \text{on } \partial\Omega\backslash\partial\Gamma(t),\\
u\cdot\nu_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\partial\Gamma(t),\\
u(0) = u_0, \quad \Gamma(0) &= \Gamma_0,
\end{aligned}
$$

where $\alpha > 0$ is a constant friction coefficient. For a further discussion on boundary conditions, we also refer to the introduction of [**38**].

Let us give an overview of this chapter. We rewrite the free boundary problem of the moving interface as a nonlinear problem for the height function parametrizing the interface. We then deal with the analysis of the underlying linear problem, proving maximal regularity in an $L_p - L_q$ scale for the height function and an L_r scale for the velocity. A fixed point argument then renders that the nonlinear problem is also well-posed. The last part finally deals with qualitative behaviour, stability properties, and convergence to equilibrium solutions.

4.2. Reduction to a flat interface

In this section we transform the equations defined on the time-dependent domain $\Omega\backslash\Gamma(t)$ and the moving interface $\Gamma(t)$ to a fixed reference frame. We follow the ideas of [**64**]. Note that in this cylindrical geometry the transformation also takes a simpler form compared to a general domain, cf. Section 2.4. To simplify notation let $n = 3$, the modifications for $n = 2$ are obvious.

We now assume that the interface at time t is given as a graph over the fixed flat reference surface $\Sigma := \Omega \cap \{x_3 = 0\}$. More precisely, we assume that there is a height function $h : \Sigma \times [0, \infty) \to (L_1, L_2)$, such that

$$\Gamma(t) = \Gamma_h(t) := \{x \in \Sigma \times (L_1, L_2) : x_3 = h(x', t),\ x' = (x_1, x_2) \in \Sigma\}, \quad t \geq 0.$$

We will now construct a Hanzawa-type transformation, which is an isomorphism on Ω and maps the moving interface $\Gamma(t)$ to the reference surface Σ for every $t \geq 0$. To this end pick some smooth bump function $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ such that $\chi(s) = 1$ for $|s| \leq \delta/2$ and $\chi(s) = 0$ for $|s| \geq \delta$, where $\delta \leq \min\{-L_1, L_2\}/3$. It is easy to check that such a function χ does exist. Define now a mapping

$$\Theta_h : \Omega \times \mathbb{R}_+ \to \Omega, \quad \Theta_h(x, t) := x + \chi(x_3)h(x', t)e_3 =: x + \theta_h(x, t),$$

where $x = (x', x_3)$. Then

$$D\Theta_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_1 h\chi & \partial_2 h\chi & 1 + h\chi' \end{pmatrix}.$$

It clearly follows that $D\Theta_h$ is a regular matrix and hence Θ_h is invertible, if $h\chi'$ is sufficiently small. For instance, this is the case whenever

$$|h|_{L_\infty(L_\infty)} \leq \frac{1}{2|\chi'|_{L_\infty(\mathbb{R})}}.$$

Note that $|\chi'|_\infty$ can be bounded by a constant depending on δ only. Then, given invertibility, one easily computes the inverse to the result

$$(D\Theta_h)^{-1} = \frac{1}{1 + h\chi'} \begin{pmatrix} 1 + h\chi' & 0 & 0 \\ 0 & 1 + h\chi' & 0 \\ -\partial_1 h\chi & -\partial_2 h\chi & 1 \end{pmatrix}.$$

For the sequel we fix the bump function χ and choose $0 < d_0 < 1/(2|\chi'|_\infty)$ sufficiently small and assume that $|h|_{\infty,\infty} \leq d_0$. This way we ensure that the inverse $\Theta_h^{-1} : \Omega \to \Omega$ is well defined and maps the free interface $\Gamma(t)$ to the fixed reference surface Σ.

We will now calculate how the equations behave under this transformation. Define the transformed quantities

$$w(x,t) := u(\Theta_h(x,t),t), \quad q(x,t) := p(\Theta_h(x,t),t), \quad \vartheta(x,t) := \eta(\Theta_h(x,t),t), \tag{4.2}$$

for $x \in \Omega$, $t \in \mathbb{R}_+$. We now determine the equations which (w, q, ϑ) solve. Define $D\Theta_h^{-\top} := ((D\Theta_h)^{-1})^\top$, as well as the transformed quantities

$$\nabla_h := D\Theta_h^{-\top}\nabla, \quad \nabla_h u := (\nabla_h u_k^\top)_{k=1}^3, \quad \operatorname{div}_h := \operatorname{Tr}(\nabla_h), \quad \Delta_h := \operatorname{div}_h \nabla_h.$$

With this it is straightforward to check that

$$\begin{aligned} \nabla u(\Theta_h(x,t),t) = \nabla_h w(x,t), \quad [(u\cdot\nabla)u](\Theta_h(x,t),t) = [(w\cdot\nabla_h)w](x,t), \\ \Delta u(\Theta(x,t),t) = \Delta_h w(x,t), \quad \operatorname{div} u(\Theta(x,t),t) = \operatorname{div}_h w(x,t), \quad x\in\Omega, t\in\mathbb{R}_+. \end{aligned} \tag{4.3}$$

Furthermore,

$$\partial_t u(\Theta_h(x,t),t) = \partial_t w(x,t) + Dw(x,t)\partial_t\Theta_h^{-1}(\Theta_h(x,t),t), \quad x\in\Omega, t\in\mathbb{R}_+.$$

The upper unit normal at the free interface $\Gamma(t)$ and the normal velocity of which can both be expressed in terms of h by

$$\nu_{\Gamma(t)} = \frac{(-\nabla h, 1)^\top}{\sqrt{1+|\nabla h|^2}}, \quad V_{\Gamma(t)} = \frac{\partial_t h}{\sqrt{1+|\nabla h|^2}}, \quad x'\in\Sigma, t\in\mathbb{R}_+. \tag{4.4}$$

We are now able to transform the two-phase Navier-Stokes/Mullins-Sekerka system (4.1) to the fixed reference frame, the transformed system reads as

$$\begin{aligned}
\rho^\pm\partial_t w - \mu^\pm\Delta w + \nabla q &= a^\pm(h; D_x, D_x^2)(w,q) + \bar{a}(h,w,\vartheta), && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} w &= G_d(h,w), && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu^\pm(Dw+Dw^\top) - qI]\!]\nu_\Sigma &= \sigma\Delta_{x'}h\nu_\Sigma + G_S(h,w,q), && \text{on } \Sigma,\\
[\![w]\!] &= 0, && \text{on } \Sigma,\\
\partial_t h &= w\cdot\nu_\Sigma - [\![\partial_3\vartheta]\!] + G_\Sigma(h,w,\vartheta), && \text{on } \Sigma,\\
(-\nabla_{x'}h,1)^\top\cdot\nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\vartheta &= G_c(h,\vartheta), && \text{in } \Omega\backslash\Sigma,\\
\vartheta|_\Sigma - \sigma\Delta_{x'}h &= G_\kappa(h), && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\vartheta|_{\partial\Omega} &= G_N(h,\vartheta), && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu^\pm(Dw+Dw^\top)\nu_{S_1}\right) &= G_P^\pm(h,w), && \text{on } S_1\backslash\partial\Sigma,\\
w\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
w &= 0, && \text{on } S_2,\\
w(0) &= w_0, && \text{on } \Omega\backslash\Sigma,\\
h(0) &= h_0, && \text{on } \Sigma,
\end{aligned} \tag{4.5}$$

where $\nu_\Sigma = e_3$, and

$$
\begin{aligned}
a^\pm(h; D_x, D_x^2)(w,q) &:= \mu^\pm(\Delta_h - \Delta)w + (\nabla - \nabla_h)q,\\
\bar{a}(h,w,\vartheta) &:= Dw\cdot\partial_t\Theta_h^{-1} - (w\cdot\nabla_h)w - (\rho^+ - \rho^-)(\nabla_h\vartheta\cdot\nabla_h)w+\\
&\quad + \operatorname{div}((\llbracket\rho\rrbracket(\nabla_h - \nabla)\vartheta)\otimes u),\\
G_d(h,w) &:= (\operatorname{div} - \operatorname{div}_h)w,\\
G_S(h,w,q) &:= \llbracket\mu^\pm\left((D\Theta_h - I)Dw + Dw^\top(D\Theta_h - I)^\top\right)\rrbracket\nu_{\Gamma_h}+\\
&\quad + \llbracket\left(\mu^\pm(Dw + Dw^\top) - qI\right)(e_3 - \nu_{\Gamma_h})\rrbracket + \sigma(K(h)\nu_{\Gamma_h} - \Delta_{x'}he_3),\\
G_\Sigma(h,w,\vartheta) &:= w\cdot(-\nabla_{x'}h,0)^\top - \llbracket e_3\cdot(\nabla_h - \nabla)\vartheta\rrbracket - \llbracket(-\nabla_{x'}h,0)^\top\cdot\nabla_h\vartheta\rrbracket,\\
G_c(h,\vartheta) &:= (\Delta - \Delta_h)\vartheta,\\
G_\kappa(h) &:= \sigma(K(h) - \Delta_{x'}h),\\
G_N(h,\vartheta) &:= \nu_{\partial\Omega}\cdot(\nabla - \nabla_h)\vartheta,\\
G_P^\pm(h,w) &:= P_{S_1}\left(\mu^\pm\left((D\Theta_h - I)Dw + Dw^\top(D\Theta_h - I)^\top\right)\nu_{S_1}\right).
\end{aligned}
$$

Here, cf. [**22**],

$$
K(h) = H(\Gamma_h) = \operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right), \quad x\in\Sigma, t\in\mathbb{R}_+.
$$

Let us briefly explain how we transformed the equations. For instance, pick the evolution equation $(4.1)_5$ for the free interface, $V_{\Gamma(t)} = u|_{\Gamma(t)}\cdot\nu_{\Gamma(t)} - \llbracket\nu_{\Gamma(t)}\cdot\nabla\mu\rrbracket$ on $\Gamma(t)$. By employing (4.3) and (4.4), this reads as

$$
\frac{\partial_t h}{\sqrt{1+|\nabla_{x'}h|^2}} = w\cdot\frac{(-\nabla_{x'}h,1)^\top}{\sqrt{1+|\nabla_{x'}h|^2}} - \frac{\llbracket(-\nabla_{x'}h,1)^\top\cdot\nabla_h\eta\rrbracket}{\sqrt{1+|\nabla_{x'}h|^2}}, \quad x'\in\Sigma, t\in\mathbb{R}_+,
$$

where w is as in (4.2). Hence we obtain

$$
\partial_t h = w\cdot(-\nabla_{x'}h,1)^\top - \llbracket(-\nabla_{x'}h,1)^\top\cdot\nabla_h\eta\rrbracket, \quad x'\in\Sigma, t\in\mathbb{R}_+,
$$

or equivalently,

$$
\begin{aligned}
&\partial_t h - w\cdot e_3 + \llbracket\partial_3\eta\rrbracket =\\
&\quad = w\cdot(-\nabla_{x'}h,0)^\top - \llbracket e_3\cdot(\nabla_h - \nabla)\eta\rrbracket - \llbracket(-\nabla_{x'}h,0)^\top\cdot\nabla_h\eta\rrbracket, \quad x'\in\Sigma, t\in\mathbb{R}_+.
\end{aligned}
$$

Furthermore, we want to point out that we used the fact that in the deduction of (4.5) the normal ν_{S_1} is independent of x_3 and that the transformation Θ_h leaves the Dirichlet-boundary S_2 invariant.

Since $\nu_\Sigma = e_3$, one can also easily decompose the stress tensor condition $(4.5)_3$ into tangential and horizontal parts, cf. [**64**]. Then, $(4.5)_3$ reads as

$$
\begin{aligned}
-\llbracket\mu^\pm\partial_3(w_1,w_2)\rrbracket - \llbracket\mu^\pm\nabla_{x'}w_3\rrbracket &= (G_S(h,w,q))_{1,2}, && \text{on } \Sigma,\\
-2\llbracket\mu^\pm\partial_3 w_3\rrbracket + \llbracket q\rrbracket - \sigma\Delta_{x'}h &= (G_S(h,w,q))_3, && \text{on } \Sigma.
\end{aligned}
$$

To economize notation, let $G_S^{\|}(h,w,q) := (G_S(h,w,q))_{1,2}$ and $G_S^{\perp}(h,w,q) := (G_S(h,w,q))_3$.

4.3. Maximal regularity for the linear problem

The main goal of this section is to derive a maximal regularity result for the linearized problem.

4.3.1. Linearization, regularity and compatibility conditions. In this section we consider the linear part of the two-phase Navier-Stokes/Mullins-Sekerka system, which reads as

$$\begin{aligned}
\rho^{\pm}\partial_t u - \mu^{\pm}\Delta u + \nabla\pi &= g_1, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= g_2, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu^{\pm}\partial_3(u_1,u_2)]\!] - [\![\mu^{\pm}\nabla_{x'}u_3]\!] &= g_3, && \text{on } \Sigma,\\
-2[\![\mu^{\pm}\partial_3 u_3]\!] + [\![\pi]\!] - \sigma\Delta_{x'}h &= g_4, && \text{on } \Sigma,\\
[\![u]\!] &= g_5, && \text{on } \Sigma,\\
\partial_t h - (u_3^+ + u_3^-)/2 + [\![\partial_3\eta]\!] &= g_6, && \text{on } \Sigma,\\
(-\nabla_{x'}h, 0)^\top \cdot \nu_{S_1} &= g_7, && \text{on } \partial\Sigma,\\
\Delta\eta &= g_8, && \text{in } \Omega\backslash\Sigma,\\
\eta|_\Sigma - \sigma\Delta_{x'}h &= g_9, && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= g_{10}, && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu^{\pm}(Du + Du^\top)\nu_{S_1}\right) &= P_{S_1}g_{11}, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= g_{12}, && \text{on } S_1\backslash\partial\Sigma,\\
u &= g_{13}, && \text{on } S_2,\\
u(0) &= u_0, && \text{on } \Omega\backslash\Sigma,\\
h(0) &= h_0, && \text{on } \Sigma.
\end{aligned} \tag{4.6}$$

Here we take $(u_3^+ + u_3^-)/2$ instead of the trace of u in equation $(4.6)_6$, since u is allowed to have a jump across Σ. Hereby $u_3^\pm$ denote the directional traces of u_3 with respect to $\{x_3 \gtrless 0\}$.

4.3.2. Regularity of the solution. The question of function spaces is now a very delicate matter. The main idea already used by Abels and Wilke in the case of no boundary contact [4] is to treat the Navier-Stokes part of the evolution as lower order compared to the Mullins-Sekerka part. They consider some height function h as given and solve the two-phase Navier-Stokes equations in dependence of h by a function $u = u(h)$. Afterwards plugging in the solution $u(h)$ into the evolution equation for h they obtain a problem only dependent on h. If now u is sufficiently more regular as the other terms in the evolution equation for $\partial_t h$, the Navier-Stokes equations can be seen as a lower order perturbation of Mullins-Sekerka in the coupled problem. By choosing the time interval sufficiently small one gets well-posedness for the coupled system, stemming from the unique solvability of the pure Mullins-Sekerka evolution of h.

Let us begin by recalling the maximal regularity class for h of the pure Mullins-Sekerka system with boundary contact, cf. Section 2.5. For $p \in (6,\infty)$ and $q \in$

$(5/3, 2) \cap (2p/(p+1), 2)$, $T > 0$, we obtained a unique local in time strong solution

$$h \in W_p^1(0,T; W_q^{1-1/q}(\Sigma)) \cap L_p(0,T; W_q^{4-1/q}(\Sigma)),$$

and $\eta \in L_p(0,T; W_q^2(\Omega\backslash\Sigma))$, of the linearized Mullins-Sekerka with boundary contact, cf. Section 2.5.

We now need to make sure of two things: to be later able to treat the Navier-Stokes part as lower order perturbation, we have to have that $u|_\Sigma$ has better time regularity and at least as much space regularity as the other terms in $(4.6)_6$, namely

$$L_p(0,T; W_q^{1-1/q}(\Sigma)).$$

On the other hand, the linearized curvature term $\Delta_{x'} h$ has to be at least admissible data in the Stokes-part, that is, $\Delta_{x'} h$ has to be at least of the same regularity as $Du|_\Sigma$, cf. $(4.6)_4$. By choosing a setting where u is too regular, $\Delta_{x'} h$ fails to be admissible data, and by choosing u not regular enough, $u|_\Sigma$ may not be treated as a lower order perturbation. In the following lines we want to explain a setting of function spaces, in which the coupling is of lower order and u is still regular enough to control the nonlinear terms.

One possible choice would be an $L_p - L_p$-ansatz, where p as above is large. The vector field u would then be very regular, hence making the nonlinearities easy to handle since in particular $p > 5$. In this ansatz we search for

$$u \in W_p^1(0,T; L_p(\Omega)) \cap L_p(0,T; W_p^2(\Omega\backslash\Sigma)),$$

whence by classical theory, $u \in BUC([0,T]; W_p^{2-2/p}(\Omega\backslash\Sigma))$. Taking traces yields $u|_\Sigma \in BUC([0,T]; W_p^{2-3/p}(\Sigma))$. Clearly this is more regularity and hence it can be seen as a lower order perturbation in $L_p(0,T; W_q^{1-1/q}(\Sigma))$. However,

$$\Delta_{x'} h \in W_p^{2/3-1/(3q)}(0,T; L_q(\Sigma)) \cap L_p(0,T; W_q^{2-1/q}(\Sigma)),$$

on the other hand,

$$Du|_\Sigma \in W_p^{1/2-1/(2p)}(0,T; L_p(\Sigma)) \cap L_p(0;T; W_p^{1-1/p}(\Sigma)).$$

It is now a consequence of Sobolev-type embedding theorems to see that $W_q^{2-1/q}(\Sigma)$ does not embed into $W_p^{1-1/p}(\Sigma)$ in general, due to $5/3 < q < 2$ and $p > 6$. Hence this $L_p - L_p$ ansatz with large p does not work.

Alternatively, one can make an $L_q - L_q$ ansatz, searching for some

$$u \in W_q^1(0,T; L_q(\Omega)) \cap L_q(0,T; W_q^2(\Omega\backslash\Sigma)),$$

where $5/3 < q < 2$. Clearly, the function u possesses way less regularity in this ansatz. It is then easy to check that $\Delta_{x'} h$ is admissible data by comparing the regularity classes of $\Delta_{x'} h$ and $Du|_\Sigma$. Also,

$$u|_\Sigma \in L_{2q/(2-q)}(0,T; W_q^{1-1/q}(\Sigma)).$$

Note that as $q \to 2$, the time integral exponent $2q/(2-q)$ tends to $+\infty$. Hence the Stokes part may be treated as lower order whenever $q < 2$ is close to 2. However we want to point out that handling the nonlinearities may be more difficult since certain Sobolev embeddings fail since $q < 2$.

By choosing an $L_p - L_q$ approach one may get better regularity for u, however if one takes any trace of u on the boundary, for instance in the simplest case of the Dirichlet conditions on top and bottom of the container, one ends up with Triebel-Lizorkin spaces for the time regularity. It is well known that the optimal regularity for the trace of a function

$$u \in W_p^1(0,T;L_q(\Omega)) \cap L_p(0,T;W_q^2(\Omega\backslash\Sigma))$$

on the boundary, e.g. on S_2, is

$$u|_{S_2} \in F_{pq}^{1-1/(2q)}(0,T;L_q(S_2)) \cap L_p(0,T;W_q^{2-1/q}(S_2)).$$

It is particularly hard to treat this problem in a mixed $L_p - L_q$ setting, since even in the model problems it is not clear how to generalize for instance the results of Prüss and Simonett in [**56**] regarding the Dirichlet-to-Neumann operator. This operator is well understood in an $L_p - L_p$ setting, however the proof of Proposition 3.3 given in [**56**] is not easily generalizable to a mixed setting where $p \neq q$, since it heavily relies on real interpolation method and Triebel-Lizorkin spaces do not naturally appear as real interpolation spaces in general.

These approaches above motivate our introduction of a third integration scale. We will show that for given $q < 2$ sufficiently close to 2 and $6 < p < \infty$ finite but large, there is some exponent $3 < r = r(p,q) < 7/2 < \infty$ such that the following is true: $\Delta_{x'} h$ is admissible data in the Stokes part, and $u|_\Sigma$ is lower order in the evolution equation for h. This $L_r - L_r$ approach with $r > 3$ circumvents the problem of Triebel-Lizorkin data spaces in the Stokes part completely and hence makes the problem a lot easier to tackle. Also it allows to make use of known results of Prüss and Simonett in [**56**] and makes the nonlinearities easier to handle in the contraction estimates. We will give the precise choice of r below in Theorem 4.1 and prove the above assertions rigorously.

THEOREM 4.1. *Let $n = 3$, that is, $\dim \Sigma = n - 1 = 2$. Let $5/3 < q < 2$ and $6 < p < \infty$. Furthermore, let $0 < T \leq T_0$ for some fixed $T_0 < \infty$. Let*

$$2 \leq r < \frac{7}{6/q - 1}.$$

Then, for any $h \in W_p^1(0,T;W_q^{1-1/q}(\Sigma)) \cap L_p(0,T;W_q^{4-1/q}(\Sigma))$, we have that

$$\Delta_{x'} h \in W_r^{1/2-1/(2r)}(0,T;L_r(\Sigma)) \cap L_r(0;T;W_r^{1-1/r}(\Sigma)).$$

Furthermore, there is some $C = C(T) > 0$, such that

$$\begin{aligned} &|\Delta_{x'} h|_{W_r^{1/2-1/(2r)}(0,T;L_r(\Sigma))\cap L_r(0;T;W_r^{1-1/r}(\Sigma))} \\ &\qquad \leq C(T)|h|_{W_p^1(0,T;W_q^{1-1/q}(\Sigma))\cap L_p(0,T;W_q^{4-1/q}(\Sigma))}. \end{aligned} \tag{4.7}$$

Furthermore, if $9/5 < q < 2$, we can choose r to satisfy $3 < r < 7/2$. If $3 < r < 7/2$, we have

$$\begin{aligned} &W_r^1(0,T;L_r(\Omega)) \cap L_r(0,T;W_r^2(\Omega\backslash\Sigma)) \hookrightarrow \\ &\qquad \hookrightarrow L_\infty(0,T;L_\infty(\Omega)) \cap L_\infty(0,T;W_r^1(\Omega\backslash\Sigma)) \cap L_r(0,T;W_\infty^1(\Omega\backslash\Sigma)). \end{aligned} \tag{4.8}$$

Moreover,

$$\mathrm{tr}_\Sigma^\pm : W_r^1(0,T;L_r(\Omega^\pm)) \cap L_r(0,T;W_r^2(\Omega^\pm)) \to L_\infty(0,T;W_q^{1-1/q}(\Sigma)),$$

is bounded. By restricting to height functions h with initial time trace zero, $h(t=0)=0$, the embedding constant in (4.7) can be chosen to be independent of T and only depending on T_0. In particular, the embedding does not degenerate and the embedding constant stays bounded as $T \to 0$.

Restricting to vanishing traces at $t=0$ in (4.8), the embedding constant is also independent of $T \in (0, T_0]$.

Moreover, we can choose r to satisfy $3 < r < 7/2$ and additionally, for this r, every

$$u \in W^1_r(0,T; L_r(\Omega)) \cap L_r(0,T; W^2_r(\Omega\backslash\Sigma))$$

has a gradient $Du \in L_{\tilde{r}}(0,T; L_\infty(\Omega))$ for some $\tilde{r} > r$. The mapping $[u \mapsto Du]$ is bounded between these spaces. Restricted to functions u with vanishing trace at $t=0$, the operator norm is independent of $T \in (0, T_0]$.

Proof. Let $2 \le r \le p$ and $0 < T \le T_0$. Note that due to $p \ge r$, we have that $L_p(0,T;Z) \hookrightarrow L_r(0,T;Z)$ for a Banach space Z. The embedding constant here only depends on T_0, which stems from Hölder's inequality,

$$|f|_{L_r(0,T;Z)} \le T^{(p-r)/(pr)} |f|_{L_p(0,T;Z)} \le T_0^{(p-r)/(pr)} |f|_{L_p(0,T;Z)}, \quad f \in L_p(0,T;Z).$$

Due to Sobolev's embedding theorem, cf. [**1**], [**62**], $W^{2-1/q}_q(\Sigma) \hookrightarrow W^{1-1/r}_r(\Sigma)$, for $2 - 3/q > 1 - 3/r$. This gives an upper restriction on r,

$$r < \frac{3q}{3-q}. \tag{4.9}$$

Summing up, $L_p(0,T; W^{2-1/q}_q(\Sigma)) \hookrightarrow L_r(0,T; W^{1-1/r}_r(\Sigma))$, provided $r \le p$ and (4.9) holds.

Since we want to use the results of [**35**] on the half line, we now consider some

$$h \in W^1_p(\mathbb{R}_+; W^{1-1/q}_q(\Sigma)) \cap L_p(\mathbb{R}_+; W^{4-1/q}_q(\Sigma)).$$

Firstly, using Proposition 5.37 in [**35**] on the half line,

$$h \in H^\theta_p(\mathbb{R}_+; W^{1-1/q+3(1-\theta)}_q(\Sigma)),$$

whenever $\theta \in (0,1)$. It follows that

$$\Delta_{x'} h \in H^\theta_p(\mathbb{R}_+; W^{2-1/q-3\theta}_q(\Sigma)), \quad \theta \in (0,1).$$

Let $\epsilon > 0$ small. By choosing $\theta := 2/3 - 1/q + 2/(3r) - \epsilon \in (0,1)$, we obtain

$$\Delta_{x'} h \in H^{2/3-1/q+2/(3r)-\epsilon}_p(\mathbb{R}_+; W^{2/q-2/r+3\epsilon}_q(\Sigma)).$$

By Sobolev embeddings for Besov spaces,

$$\Delta_{x'} h \in H^{2/3-1/q+2/(3r)-\epsilon}_p(0,T; L_r(\Sigma)),$$

for any small $\epsilon > 0$.

Assume for a moment that

$$2/3 - 1/q + 2/(3r) > 1/2 - 1/(2r). \tag{4.10}$$

Then we may choose $\epsilon > 0$ so small, such that

$$2/3 - 1/q + 2/(3r) - \epsilon > 1/2 - 1/(2r).$$

Then $\Delta_{x'}h \in W_p^{1/2-1/(2r)}(0,T;L_r(\Sigma)) \hookrightarrow W_r^{1/2-1/(2r)}(0,T;L_r(\Sigma))$. Inequality (4.10) however is equivalent to $r < 7/(6/q-1)$ since $q < 2$. Estimate (4.7) is a direct consequence of these considerations. Furthermore, whenever h has vanishing trace at $t=0$, a standard extension argument allows to see that the estimate does not degenerate as $T \to 0$, that is, $C(T)$ stays bounded as $T \to 0$ since it only depends on T_0.

Note that $7/(6/q-1) \to 7/2$ as $q \to 2$. In particular, choosing $q > 9/5$ gives $7/(6/q-1) > 3$, hence we can choose $r > 3$. Now let $u \in W_r^1(0,T;L_r(\Omega)) \cap L_r(0,T;W_r^2(\Omega\backslash\Sigma))$ for $r > 3$. We may use the embedding

$$W_r^1(0,T;L_r(\Omega)) \cap L_r(0,T;W_r^2(\Omega\backslash\Sigma)) \hookrightarrow BUC([0,T];W_r^{2-2/r}(\Omega\backslash\Sigma)),$$

cf. [**6**], to see that $Du \in BUC([0,T];W_r^{1-2/r}(\Omega\backslash\Sigma))$, which then in turn yields $Du \in L_\infty(0,T;L_r(\Omega))$. It also follows that $Du \in L_r(0,T;L_\infty(\Omega))$. Regarding the trace operator, we note that

$$BUC([0,T];W_r^{2-2/r}(\Omega\backslash\Sigma)) \hookrightarrow L_\infty(0,T;W_q^1(\Omega\backslash\Sigma)),$$

whenever $r \geq 5q/(q+3)$, which is surely satisfied since $r > 3$ and $q < 2$. Let again $u \in W_r^1(0,T;L_r(\Omega)) \cap L_r(0,T;W_r^2(\Omega\backslash\Sigma))$ for $3 < r < 7/2$. Then by standard interpolation,

$$Du \in L_{\tilde{r}}(0,T;L_\infty(\Omega)), \quad \text{for all } 1 < \tilde{r} < \frac{1}{\frac{1}{r} - \left(\frac{1}{2} - \frac{3}{2r}\right)}. \tag{4.11}$$

If now $q > 99/50$, we obtain $7/(6/q-1) > 17/5$, hence we can in particular choose $r = 17/5$. By (4.11) we now see that $\tilde{r} = 4$ is possible. Hence we have proven that, provided $q > 99/50$, we can choose $r = 17/5$ and $Du \in L_4(0,T;L_\infty(\Omega))$. ☐

REMARK 4.2. (1) We can choose from now on $p \in (6,\infty)$, $q \in (99/50,2) \cap (2p/(p+1),2)$, and $r = 17/5$. In particular, the set of admissible indices is not empty.

(2) Let us explain the relations between p, q, and r a bit further. Note that for given $q \in (5/3,2)$, we basically obtained the upper restriction $r < \min(3q/(3-q); 7/(6/q-1))$. If we let now $q \to 2$, we see that formally this reduces to $r < \min(6; 7/2) = 7/2$. If we now take the limit as $r \to 7/2$ in the constraint of (4.11), we obtain again formally the bound $\tilde{r} < 14/3$. Again, the main idea of the proof of the last part is to choose $q < 2$ close enough to 2 such that these arguments are valid. We want to point out however that our proof gives exponents (p,q,r) with restrictions which do not depend on each other.

This motivates to choose the following setting for the solutions to the two-phase Navier-Stokes/Mullins-Sekerka system and its principal linearization (4.6).

Let $T \in (0,\infty)$ and $p \in (6,\infty)$, $q \in (99/50,2) \cap (2p/(p+1),2)$, and $r = 17/5$ as in Theorem 4.1. From now on, we will fix the integration scales p, q, and r. We are looking for solutions (u,π,h,μ) of (4.6) with

$$\begin{gathered} u\in W^1_r(0,T;L_r(\Omega))\cap L_r(0,T;W^2_r(\Omega\backslash\Sigma)),\quad \pi\in L_r(0,T;\dot H^1_r(\Omega)),\\ [\![\pi]\!]\in W_r^{1/2-1/(2r)}(0,T;L_r(\Sigma))\cap L_r(0,T;W_r^{1-1/r}(\Sigma)),\\ h\in W^1_p(0,T;W_q^{1-1/q}(\Sigma))\cap L_p(0,T;W_q^{4-1/q}(\Sigma)),\quad \mu\in L_p(0,T;W^2_q(\Omega\backslash\Sigma)). \end{gathered} \tag{4.12}$$

4.3.3. Regularity of the data. To be able to derive a maximal regularity result, we will now deduce optimal regularity classes for the data in problem (4.6). Given a solution $(u,\pi,[\![\pi]\!],h,\mu)$ in the classes of (4.12), we derive by standard trace theory the following necessary conditions for the data,

$$\begin{gathered} g_1\in L_r(0,T;L_r(\Omega)),\\ g_2\in L_r(0,T;W^1_r(\Omega\backslash\Sigma)),\\ g_3\in W_r^{1/2-1/(2r)}(0,T;L_r(\Sigma))\cap L_r(0;T;W_r^{1-1/r}(\Sigma)),\\ g_4\in W_r^{1/2-1/(2r)}(0,T;L_r(\Sigma))\cap L_r(0;T;W_r^{1-1/r}(\Sigma)),\\ g_5\in W_r^{1-1/(2r)}(0,T;L_r(\Sigma))\cap L_r(0,T;W_r^{2-1/r}(\Sigma)),\\ g_6\in L_p(0,T;W_q^{1-1/q}(\Sigma)),\\ g_7\in F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma))\cap L_p(0,T;W_q^{3-2/q}(\partial\Sigma)),\\ g_8\in L_p(0,T;L_q(\Omega)),\\ g_9\in L_p(0,T;W_q^{2-1/q}(\Sigma)),\\ g_{10}\in L_p(0,T;W_q^{1-1/q}(\partial\Omega)),\\ P_{S_1}g_{11}\in W_r^{1/2-1/(2r)}(0,T;L_r(S_1))\cap L_r(0;T;W_r^{1-1/r}(S_1)),\\ g_{12}\in W_r^{1-1/(2r)}(0,T;L_r(S_1))\cap L_r(0,T;W_r^{2-1/r}(S_1)),\\ g_{13}\in W_r^{1-1/(2r)}(0,T;L_r(S_2))\cap L_r(0,T;W_r^{2-1/r}(S_2)),\\ u_0\in W_r^{2-2/r}(\Omega\backslash\Sigma),\\ h_0\in B_{qp}^{4-3/p-1/q}(\Sigma). \end{gathered} \tag{4.13}$$

For the regularity of g_7 we refer to Section 2.2. At this point we note that in (4.13) the function g_2 does not have to have the time regularity of Du in $\Omega\backslash\Sigma$. This is due to the fact that there is some compatibility condition hidden in the system stemming from the divergence equation, which inherits a certain time regularity for (g_2,g_5,g_{12},g_{13}). This will be discussed in the next section regarding compatibility conditions. However we clearly want to point out that g_2 being $L_r(0,T;W^1_r(\Omega\backslash\Sigma))$ alone is a necessary but not a sufficient condition.

4.3.4. Compatibility conditions. We now shall discuss all the compatibility conditions for the data $((g_j)_{j=1}^{13},u_0,h_0)$ of system (4.6). In Lemma 4.3 below we rigorously show these conditions all occur and are well-defined. The following observations have already been made in Section 2.4 and [**64**].

At the starting point of the evolution at time $t = 0$ we have to have that

$$\begin{aligned} \operatorname{div} u_0 = g_2|_{t=0}, \quad -[\![\mu^\pm \partial_3 (u_0)_{1,2}]\!] - [\![\mu^\pm \nabla_{x'} (u_0)_3]\!] = g_3|_{t=0}, \\ [\![u_0]\!] = g_5|_{t=0}, \quad u_0 \cdot \nu_{S_1} = g_{12}|_{t=0}, \quad u_0|_{S_2} = g_{13}|_{t=0}, \\ (-\nabla_{x'} h_0, 1)^\top \cdot \nu_{S_1} = g_7|_{t=0}, \quad P_{S_1}(\mu^\pm (D u_0 + D u_0^\top) \nu_{S_1}) = P_{S_1} g_{11}|_{t=0}. \end{aligned} \tag{4.14}$$

These conditions follow by evaluating the respective equations at time zero. Here, $(u_0)_{1,2}$ denotes the vector in $\mathbb{R}^2$ with the first two entries of u_0, similarly $(u_0)_3$ denotes the last component of u_0.

Since $\partial\Sigma \subset S_1 \neq \emptyset$ and bottom, top and walls of the container have a common boundary, $\partial S_1 \cap \partial S_2 \neq \emptyset$, there are additional compatibility conditions. Simply by comparing equations we get

$$\begin{aligned} &[\![g_{12}]\!] = g_5 \cdot \nu_{S_1}, && \text{on } \partial\Sigma, \\ &[\![(g_{11} \cdot e_3)/\mu^\pm - \partial_3 g_{12}]\!] = \partial_{\nu_{S_1}} (g_5 \cdot e_3), && \text{on } \partial\Sigma, \\ &P_{\partial\Sigma}[(D_{x'} \Pi g_5 + (D_{x'} \Pi g_5)^\top) \nu_{\partial\Sigma}] = [\![P_{\partial\Sigma} \Pi g_{11}/\mu^\pm]\!], && \text{on } \partial\Sigma, \\ &g_3 \cdot (\nu_{S_1})_{1,2} = -[\![g_{11} \cdot e_3]\!], && \text{on } \partial\Sigma, \\ &g_{13} \cdot \nu_{S_1} = g_{12}, && \text{on } \partial S_2, \\ &P_{\partial\Sigma}[\mu^\pm (D_{x'} \Pi g_{13} + (D_{x'} \Pi g_{13})^\top) \nu_{\partial\Sigma}] = P_{\partial\Sigma} \Pi g_{11}, && \text{on } \partial S_2, \\ &\mu^\pm \partial_{\nu_{S_1}} (g_{13} \cdot e_3) + \mu^\pm \partial_3 g_{12} = g_{11} \cdot e_3, && \text{on } \partial S_2. \end{aligned} \tag{4.15}$$

Here, $\Pi v := (v_1, v_2) \in \mathbb{R}^2$ for $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $\nu_{\partial\Sigma} := \Pi \nu_{S_1}$. The projection then is given by $P_{\partial\Sigma} := I - \nu_{\partial\Sigma} \otimes \nu_{\partial\Sigma}$. For further discussion we refer to [**64**].

We want to point out that there is no additional compatibility condition for $\partial_t g_7$ on $\partial\Sigma$ as there is in [**64**], since g_7 does not have a well defined time derivative on $\partial\Sigma$ in our regularity class. This is due to the fact that we have a different maximal regularity class for h as in [**64**].

Finally we turn to the divergence equation and want to point out that there is another compatibility and regularity condition hidden in the system, which has already been investigated in [**64**]. For completeness we explain it here briefly.

Consider the divergence equation $\operatorname{div} u = g_2$ and multiply this equation with a testfunction $\varphi \in W^1_{r'}(\Omega)$, where $r' = r/(r-1)$ is the conjugate exponent. An integration by parts on the two Lipschitz domains $\Omega \cap \{x_3 \gtrless 0\}$ and using the equations entails that

$$\begin{aligned} \int_{\Omega\setminus\Sigma} g_2 \varphi dx - \int_{S_1} g_{12} \varphi|_{S_1} dS_1 - \int_{S_2} (g_{13} \cdot \nu_{S_2}) \varphi|_{S_2} dS_2 \\ + \int_\Sigma (g_5 \cdot \nu_\Sigma) \varphi|_\Sigma d\Sigma = - \int_{\Omega\setminus\Sigma} u \cdot \nabla \varphi dx, \end{aligned} \tag{4.16}$$

see also Proposition A.14 in [**64**]. Hence the functional $\varphi \mapsto \langle (g_2, g_5, g_{12}, g_{13}), \varphi \rangle$ defined by the left hand side of (4.16) is continuous on $W^1_{r'}(\Omega)$ with respect to the seminorm $|\nabla \cdot|_{L_{r'}(\Omega)}$. Since $C_0^\infty(\Omega) \subseteq W^1_{r'}(\Omega)$ is dense in the homogeneous space $\dot{H}^1_{r'}(\Omega)$ with respect to this seminorm, it follows that $\varphi \mapsto \langle (g_2, g_5, g_{12}, g_{13}), \varphi \rangle$ defines a functional on $\dot{H}^1_{r'}(\Omega)$. In other words, $(g_2, g_5, g_{12}, g_{13}) \in \hat{H}^{-1}_r(\Omega) := (\dot{H}^1_{r'}(\Omega))'$. The norm of $(g_2, g_5, g_{12}, g_{13})$ in $\hat{H}_r^{-1}(\Omega)$ is then given by

$$|(g_2, g_5, g_{12}, g_{13})|_{\hat{H}_r^{-1}(\Omega)} := \sup\{\langle(g_2, g_5, g_{12}, g_{13}), \varphi\rangle / |\nabla\varphi|_{L_{r'}(\Omega)} : \varphi \in W^1_{r'}(\Omega)\}.$$

We now turn again to the equations. Since $u \in W^1_r(0,T; L_r(\Omega))$, it follows from (4.16) that $\frac{d}{dt}(g_2, g_5, g_{12}, g_{13})$ is well defined and is in $L_r(0,T; \hat{H}_r^{-1}(\Omega))$. Consequently,

$$(g_2, g_5, g_{12}, g_{13}) \in W^1_r(0,T; \hat{H}_r^{-1}(\Omega)) \tag{4.17}$$

is another necessary compatibility and regularity condition.

We close this subsection by showing that the compatibility conditions we have deduced above are all well-defined conditions.

LEMMA 4.3. *Let $r > 3$. Then all appearing traces and hence the compatibility conditions are all well-defined.*

PROOF. For the compatibility condition stemming from the divergence equation we refer to the above discussion and (4.17). For g_j, $j = 3, 5, 7, 12, 13$, and $P_{S_1} g_{11}$ we note that all functions have a well-defined trace at $t = 0$ since $r > 3$. Indeed, the condition for g_7 is independent of r (and fulfilled by choice of p and q) and the rest easily follow by trace theory. Pick for instance g_3. Then g_3 surely has a trace at $t = 0$ whenever $1/2 - 1/(2r) - 1/r > 0$. This is however equivalent to $r > 3$. By taking traces in the spatial variables one easily sees that all the other traces are well-defined. □

4.3.5. Maximal regularity. Let us consider the linear problem

$$\begin{aligned}
\rho^\pm \partial_t u - \mu^\pm \Delta u + \nabla \pi &= g_1, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= g_2, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu^\pm \partial_3 (u_1, u_2)]\!] - [\![\mu^\pm \nabla_{x'} u_3]\!] &= g_3, && \text{on } \Sigma,\\
-2[\![\mu^\pm \partial_3 u_3]\!] + [\![\pi]\!] - \sigma \Delta_{x'} h &= g_4, && \text{on } \Sigma,\\
[\![u]\!] &= g_5, && \text{on } \Sigma,\\
\partial_t h - (u_3^+ + u_3^-)/2 + [\![\partial_3 \mu]\!] &= g_6, && \text{on } \Sigma,\\
(-\nabla_{x'} h, 1)^\top \cdot \nu_{S_1} &= g_7, && \text{on } \partial\Sigma,\\
\Delta \mu &= g_8, && \text{in } \Omega\backslash\Sigma,\\
\mu|_\Sigma - \sigma \Delta_{x'} h &= g_9, && \text{on } \Sigma,\\
\nu_{\partial\Omega} \cdot \nabla \mu|_{\partial\Omega} &= g_{10}, && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu^\pm (Du + Du^\top)\nu_{S_1}\right) &= P_{S_1} g_{11}, && \text{on } S_1\backslash\partial\Sigma,\\
u \cdot \nu_{S_1} &= g_{12}, && \text{on } S_1\backslash\partial\Sigma,\\
u &= g_{13}, && \text{on } S_2,\\
u(0) &= u_0, && \text{on } \Omega\backslash\Sigma,\\
h(0) &= h_0, && \text{on } \Sigma.
\end{aligned} \tag{4.18}$$

The main result on maximal regularity for (4.18) is the following.

THEOREM 4.4. *Let $\mu^\pm, \rho^\pm, \sigma > 0$ be constant, $-\infty < L_1 < 0 < L_2 < \infty$, (p,q,r) as in Theorem 4.1 and $\Sigma \subset \mathbb{R}^2$ be open, bounded and smooth. Let $\Omega := \Sigma \times (L_1, L_2)$,*

$S_1 := \partial\Sigma \times (L_1, L_2)$, and $S_2 := \Sigma \times \{L_1, L_2\}$. Let $0 < T < \infty$. The coupled linear system (4.18) then admits a unique solution $(u, \pi, [\![\pi]\!], h, \mu)$ with regularity (4.12), if and only if the data satisfy the regularity and compatibility conditions (4.13), (4.14), (4.15), and (4.17). Furthermore, the solution map $[((g_j)_{j=1,\dots,13}, u_0, h_0) \mapsto (u, \pi, [\![\pi]\!], h, \mu)]$ between the above spaces is continuous.

PROOF. First we reduce to trivial initial data by solving an auxiliary ninety degree angle linear Mullins-Sekerka problem of type

$$\begin{aligned}
\partial_t \bar{h} + [\![\partial_3 \bar{\mu}]\!] &= g_6, && \text{on } \Sigma,\\
(-\nabla_{x'} \bar{h}, 1)^\top \cdot \nu_{S_1} &= g_7, && \text{on } \partial\Sigma,\\
\Delta \bar{\mu} &= g_8, && \text{in } \Omega\backslash\Sigma,\\
\bar{\mu}|_\Sigma - \sigma \Delta_{x'} \bar{h} &= g_9, && \text{on } \Sigma,\\
\nu_{\partial\Omega} \cdot \nabla \bar{\mu}|_{\partial\Omega} &= g_{10}, && \text{on } \partial\Omega\backslash\Sigma,\\
\bar{h}(0) &= h_0, && \text{on } \Sigma,
\end{aligned}$$

by functions

$$\bar{h} \in W^1_p(0, T; W^{1-1/q}_q(\Sigma)) \cap L_p(0, T; W^{4-1/q}_q(\Sigma)), \quad \bar{\mu} \in L_p(0, T; W^2_q(\Omega\backslash\Sigma)).$$

Then we solve an auxiliary two-phase Stokes problem

$$\begin{aligned}
\rho^\pm \partial_t \bar{u} - \mu^\pm \Delta \bar{u} + \nabla \bar{\pi} &= g_1, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} \bar{u} &= g_2, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu^\pm \partial_3 (\bar{u}_1, \bar{u}_2)]\!] - [\![\mu^\pm \nabla_{x'} \bar{u}_3]\!] &= g_3, && \text{on } \Sigma,\\
-2[\![\mu^\pm \partial_3 \bar{u}_3]\!] + [\![\bar{\pi}]\!] &= g_4 - \sigma \Delta_{x'} \bar{h}, && \text{on } \Sigma,\\
[\![\bar{u}]\!] &= g_5, && \text{on } \Sigma,\\
P_{S_1} \left(\mu^\pm (D\bar{u} + D\bar{u}^\top) \nu_{S_1} \right) &= P_{S_1} g_{11}, && \text{on } S_1\backslash\partial\Sigma,\\
\bar{u} \cdot \nu_{S_1} &= g_{12}, && \text{on } S_1\backslash\partial\Sigma,\\
\bar{u} &= g_{13}, && \text{on } S_2,\\
\bar{u}(0) &= u_0, && \text{on } \Omega\backslash\Sigma,
\end{aligned} \tag{4.20}$$

using Theorem A.11 in [**64**] by functions

$$\bar{u} \in W^1_r(0, T; L_r(\Omega)) \cap L_r(0, T; W^2_r(\Omega\backslash\Sigma)), \quad \bar{\pi} \in L_r(0, T; \dot{H}^1_r(\Omega\backslash\Sigma)),$$

with

$$[\![\bar{\pi}]\!] \in W^{1/2-1/(2r)}_r(0, T; L_r(\Sigma)) \cap L_r(0, T; W^{1-1/r}_r(\Sigma)).$$

Here we want to point out two things: $\Delta_{x'} \bar{h}$ has sufficient regularity to be admissible data and that there is no compatibility condition stemming from $(4.20)_4$. Hence $g_4 - \sigma \Delta_{x'} \bar{h}$ is admissible data for the problem. Having now $(\bar{u}, \bar{\pi}, \bar{h}, \bar{\mu})$ at hand, we are left to solve

$$
\begin{aligned}
\rho^\pm\partial_t u - \mu^\pm\Delta u + \nabla\pi &= 0, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu^\pm\partial_3(u_1,u_2)]\!] - [\![\mu^\pm\nabla_{x'}u_3]\!] &= 0, && \text{on } \Sigma,\\
-2[\![\mu^\pm\partial_3 u_3]\!] + [\![\pi]\!] - \sigma\Delta_{x'}h &= 0, && \text{on } \Sigma,\\
[\![u]\!] &= 0, && \text{on } \Sigma,\\
\partial_t h - u_3|_\Sigma + [\![\partial_3\mu]\!] &= (\bar{u}_3^+ + \bar{u}_3^-)/2, && \text{on } \Sigma,\\
(-\nabla_{x'}h, 1)^\top\cdot\nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\mu &= 0, && \text{in } \Omega\backslash\Sigma,\\
\mu|_\Sigma - \sigma\Delta_{x'}h &= -\sigma\Delta_{x'}\bar{h}, && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu^\pm(Du + Du^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u &= 0, && \text{on } S_2,\\
u(0) &= 0, && \text{on } \Omega\backslash\Sigma,\\
h(0) &= 0, && \text{on } \Sigma.
\end{aligned}
$$

We do this as follows. Define the linear Mullins-Sekerka operator $L_{MS} : {}_0\mathbb{E}_{MS,T} \to {}_0\mathbb{F}_{MS,T}$ by

$$
L_{MS}(h,\mu) := \begin{pmatrix} \partial_t h - [\![\partial_3\mu]\!] \\ \Delta\mu \\ \mu|_\Sigma - \sigma\Delta_{x'}h \\ n_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega} \\ (-\nabla_{x'}h|_{\partial\Sigma}, 1)^\top\cdot\nu_{S_1} \end{pmatrix},
$$

where

$$
{}_0\mathbb{E}_{MS,T} := [{}_0W_p^1(0,T;W_q^{1-1/q}(\Sigma)) \cap L_p(0,T;W_q^{4-1/q}(\Sigma))] \times L_p(0,T;W_q^2(\Omega\backslash\Sigma)),
$$

and

$$
\begin{aligned}
{}_0\mathbb{F}_{MS,T} &:= L_p(0,T,W_q^{1-1/q}(\Sigma)) \times L_p(0,T;L_q(\Omega)) \times L_p(0,T;W_q^{1-1/q}(\Sigma))\\
&\times L_p(0,T;W_q^{1-1/q}(\partial\Omega)) \times [{}_0F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma)) \cap L_p(0,T;W_q^{3-2/q}(\partial\Sigma))].
\end{aligned}
$$

In Section 2.5 we have shown that $[L_{MS} : {}_0\mathbb{E}_{MS,T} \to {}_0\mathbb{F}_{MS,T}]$ is boundedly invertible. Define the linear Stokes operator

$$
L_S : {}_0\mathbb{E}_{MS,T} \to [{}_0W_r^1(0,T;L_r(\Omega)) \cap L_r(0,T;W_r^2(\Omega\backslash\Sigma))]
$$

by $L_S(h) := u$, where (u, π) is the unique solution of

$$\begin{aligned}
\rho^\pm \partial_t u - \mu^\pm \Delta u + \nabla \pi &= 0, && \text{in } \Omega \backslash \Sigma,\\
\operatorname{div} u &= 0, && \text{in } \Omega \backslash \Sigma,\\
-[\![\mu^\pm \partial_3 (u_1, u_2)]\!] - [\![\mu^\pm \nabla_{x'} u_3]\!] &= 0, && \text{on } \Sigma,\\
-2[\![\mu^\pm \partial_3 u_3]\!] + [\![\pi]\!] &= \sigma \Delta_{x'} h, && \text{on } \Sigma,\\
[\![u]\!] &= 0, && \text{on } \Sigma,\\
P_{S_1}\left(\mu^\pm (Du + Du^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1 \backslash \partial\Sigma,\\
u \cdot \nu_{S_1} &= 0, && \text{on } S_1 \backslash \partial\Sigma,\\
u &= 0, && \text{on } S_2,\\
u(0) &= 0, && \text{on } \Omega \backslash \Sigma,
\end{aligned}$$

cf. Theorem A.11 in [**64**]. It then stems from Theorem 4.1 that L_S is well defined, linear and bounded. Define $B : {}_0\mathbb{E}_{MS,T} \to {}_0\mathbb{F}_{MS,T}$ and $G(\bar{u}, \bar{h}) \in {}_0\mathbb{F}_{MS,T}$ by

$$B(h) := (-L_S(h)|_\Sigma, 0, 0, 0, 0)^\top, \quad G(\bar{u}, \bar{h}) := ((\bar{u}_3^+ + \bar{u}_3^-)/2, 0, -\sigma \Delta_{x'} \bar{h}, 0, 0)^\top.$$

We can hence rewrite the problem as

$$L_{MS}(h, \mu) = -B(h) + G(\bar{u}, \bar{h}), \quad \text{in } {}_0\mathbb{F}_{MS,T}.$$

We now solve this equation by a Neumann series argument. Clearly this equation is equivalent to

$$(I + L_{MS}^{-1} B)(h, \mu) = L_{MS}^{-1} G(\bar{u}, \bar{h}), \quad \text{in } {}_0\mathbb{F}_{MS,T},$$

hence it remains to show that $|L_{MS}^{-1} B|_{\mathcal{B}({}_0\mathbb{E}_{MS,T})} \leq \frac{1}{2}$, if $T > 0$ is small enough. Then by a Neumann series argument, $(I + L_{MS}^{-1} B)$ is invertible and the theorem is shown. Since we now have that L_{MS} is boundedly invertible and the norm of the inverse is independent of T since we only consider functions with vanishing time trace at $t = 0$, the claim follows from Theorem 4.1. Indeed,

$$\begin{aligned}
|B(h)|_{{}_0\mathbb{F}_{MS,\tau}} = |L_S(h)|_{L_p(0,\tau; W_q^{1-1/q}(\Sigma))} &\leq \tau^{1/p} |L_S(h)|_{L_\infty(0,\tau; W_q^{1-1/q}(\Sigma))}\\
&\leq \tau^{1/p} |h|_{{}_0\mathbb{E}_{MS,\tau}}, \quad \tau > 0.
\end{aligned}$$

Note that again since h has vanishing time trace, all embeddings in Theorem 4.1 are time-independent. In particular, by choosing $\tau > 0$ sufficiently small, we get a unique solution (h, μ) in the proper regularity class on $(0, \tau)$. Solving then the two-phase Stokes system for this particular h gives a proper (u, π) in the L_r-regularity scale, again on $(0, \tau)$.

Shifting back the equations via $\tilde{u}(t) := u(t - \tau)$, $\tilde{\pi}(t) := \pi(t - \tau)$, $\tilde{h}(t) := h(t - \tau)$, and $\tilde{\mu} := \mu(t - \tau)$ we can again apply this argument and solve again on the same length time interval $(0, \tau)$, which in turn gives us now a solution on $(0, 2\tau)$ in fact. Repeating the steps we can solve then the problem on $(0, T)$, cf. Section 2.3 in [**64**]. □

4.4. Nonlinear Well-Posedness

In this section we show local well-posedness for the full nonlinear problem (4.5). The main result is the following.

Theorem 4.5. *Let $\mu^\pm, \rho^\pm, \sigma > 0$ be constant, $-\infty < L_1 < 0 < L_2 < \infty$, (p,q,r) satisfy*

$$p \in (6,\infty), \quad q \in (99/50, 2) \cap (2p/(p+1), 2), \quad r = 17/5, \quad 1/r > 1/4 + 1/p. \tag{4.23}$$

Let $\Sigma \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Let $\Omega := \Sigma \times (L_1, L_2)$, $S_1 := \partial\Sigma \times (L_1, L_2)$ be the walls and $S_2 := \Sigma \times \{L_1, L_2\}$ bottom and top of the cylinder. Furthermore let

$$(u_0, h_0) \in W_r^{2-2/r}(\Omega\backslash\Sigma) \times B_{qp}^{4-1/q-3/p}(\Sigma)$$

satisfy the compatibility conditions

$$\begin{aligned} \operatorname{div} u_0 = G_d(h_0, u_0), \quad & \text{in } \Omega\backslash\Sigma, \\ -[\![\mu^\pm \partial_3 (u_0)_{1,2}]\!] - [\![\mu^\pm \nabla_{x'} (u_0)_3]\!] = G_S^{\|}(h_0, u_0),\ [\![u_0]\!] = 0, \quad & \text{on } \Sigma, \\ P_{S_1}(\mu^\pm (Du_0 + Du_0^\top)\nu_{S_1}) = 0,\ u_0 \cdot \nu_{S_1} = 0, \quad & \text{on } S_1, \\ u_0|_{S_2} = 0, \text{ on } S_2, \quad (-\nabla_{x'} h_0, 1)^\top \cdot \nu_{S_1} = 0, \ & \text{on } \partial\Sigma. \end{aligned} \tag{4.24}$$

Then the full nonlinear (transformed) problem (4.5) admits a unique local-in-time strong solution, that is, there is some $T_0 > 0$, such that for every $0 < T \le T_0$ there is some $\varepsilon = \varepsilon(T) > 0$, such that whenever the smallness condition

$$|u_0|_{W_r^{2-2/r}(\Omega\backslash\Sigma)} + |h_0|_{B_{qp}^{4-1/q-3/p}(\Sigma)} \le \varepsilon \tag{4.25}$$

is satisfied there is a unique strong solution $(u, \pi, [\![\pi]\!], h, \mu)$ of (4.5) on $(0,T)$ with regularity (4.12).

Proof. We first again reduce the problem to $(u_0, h_0) = 0$. This can be done by first solving an auxiliary Stokes problem as in Section 3.2 in [**64**] to reduce to $u_0 = 0$, and then a linearized Mullins-Sekerka problem, cf. Section 2.5, to reduce to $h_0 = 0$. Note that the respective compatibility conditions are satisfied by(5.5). The reference solution (u_*, π_*, h_*, μ_*) resolving the initial conditions then belongs to the proper regularity classes.

Let us now introduce notation. Let

$$\begin{aligned} {}_0\mathbb{E}_u(T) &:= {}_0W_r^1(0,T; L_r(\Omega)) \cap L_r(0,T; W_r^2(\Omega\backslash\Sigma)), \\ \mathbb{E}_\pi(T) &:= L_r(0,T; \dot{H}_r^1(\Omega\backslash\Sigma)), \\ {}_0\mathbb{E}_q(T) &:= {}_0W_r^{1/2-1/(2r)}(0,T; L_r(\Sigma)) \cap L_r(0,T; W_r^{1-1/r}(\Sigma)), \\ {}_0\mathbb{E}_h(T) &:= {}_0W_p^1(0,T; W_q^{1-1/q}(\Sigma)) \cap L_p(0,T; W_q^{4-1/q}(\Sigma)), \\ \mathbb{E}_\mu(T) &:= L_p(0,T; W_q^2(\Omega\backslash\Sigma)), \end{aligned}$$

and

$${}_0\mathbb{E}(T) := {}_0\mathbb{E}_u(T) \times \mathbb{E}_\pi(T) \times {}_0\mathbb{E}_q(T) \times {}_0\mathbb{E}_h(T) \times \mathbb{E}_\mu(T) \cap \{(u, \pi, q, h, \mu) : q = [\![\pi]\!]\}.$$

Moreover, let

$$
\begin{aligned}
\mathbb{F}_1(T) &:= L_r(0,T;L_r(\Omega)),\\
\mathbb{F}_2(T) &:= L_r(0,T;W_r^1(\Omega\backslash\Sigma)),\\
\mathbb{F}_3(T) &:= {}_0W_r^{1/2-1/(2r)}(0,T;L_r(\Sigma)) \cap L_r(0;T;W_r^{1-1/r}(\Sigma)),\\
\mathbb{F}_4(T) &:= {}_0W_r^{1/2-1/(2r)}(0,T;L_r(\Sigma)) \cap L_r(0;T;W_r^{1-1/r}(\Sigma)),\\
\mathbb{F}_5(T) &:= {}_0W_r^{1-1/(2r)}(0,T;L_r(\Sigma)) \cap L_r(0,T;W_r^{2-1/r}(\Sigma)),\\
\mathbb{F}_6(T) &:= L_p(0,T;W_q^{1-1/q}(\Sigma)),\\
\mathbb{F}_7(T) &:= {}_0F_{pq}^{1-2/(3q)}(0,T;L_q(\partial\Sigma)) \cap L_p(0,T;W_q^{3-2/q}(\partial\Sigma)),\\
\mathbb{F}_8(T) &:= L_p(0,T;L_q(\Omega)),\\
\mathbb{F}_9(T) &:= L_p(0,T;W_q^{2-1/q}(\Sigma)),\\
\mathbb{F}_{10}(T) &:= L_p(0,T;W_q^{1-1/q}(\partial\Omega)),\\
\mathbb{F}_{11}(T) &:= {}_0W_r^{1/2-1/(2r)}(0,T;L_r(S_1)) \cap L_r(0;T;W_r^{1-1/r}(S_1),\\
\mathbb{F}_{12}(T) &:= {}_0W_r^{1-1/(2r)}(0,T;L_r(S_1)) \cap L_r(0,T;W_r^{2-1/r}(S_1)),\\
\mathbb{F}_{13}(T) &:= {}_0W_r^{1-1/(2r)}(0,T;L_r(S_2)) \cap L_r(0,T;W_r^{2-1/r}(S_2)).
\end{aligned}
$$

Let

$$
{}_0\mathbb{F}(T) := \times_{j=1}^{13}\mathbb{F}_j(T) \cap \{(g_2, g_5, g_{12}, g_{13}) \in W_r^1(\mathbb{R}_+;\hat{H}_r^{-1}(\Omega))\}.
$$

Define a linear operator by the left hand side of (4.5), that is define $\mathsf{L} : {}_0\mathbb{E}(T) \to {}_0\mathbb{F}(T)$ via

$$
\mathsf{L}(u,\pi,q,h,\mu) := \begin{pmatrix}
\rho^\pm\partial_t u - \mu^\pm\Delta u + \nabla\pi\\
\operatorname{div} u\\
-[\![\mu^\pm\partial_3(u_1,u_2)]\!] - [\![\mu^\pm\nabla_{x'}u_3]\!]\\
-2[\![\mu^\pm\partial_3 u_3]\!] + q - \sigma\Delta_{x'}h\\
[\![u]\!]\\
\partial_t h - u_3|_\Sigma + [\![\partial_3\mu]\!]\\
(-\nabla_{x'}h, 1)^\top|_{\partial\Sigma}\cdot\nu_{S_1}\\
\Delta\mu\\
\mu|_\Sigma - \sigma\Delta_{x'}h\\
\nu_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega}\\
P_{S_1}\left(\mu^\pm(Du + Du^\top)\nu_{S_1}\right)\\
u|_{S_1}\cdot\nu_{S_1}\\
u|_{S_2}
\end{pmatrix}.
$$

We collect the right hand side in the operator $\mathsf{R} : \mathbb{E}(T) \to \mathbb{F}(T)$, defined by

$$\mathsf{R}(u,\pi,q,h,\mu) := \begin{pmatrix} a^{\pm}(h;D_x^2)(u,\pi) + \bar{a}(h,u) \\ G_d(u,h) \\ G_S(u,\pi,h)_{1,2} \\ G_S(u,\pi,h)_3 \\ 0 \\ G_\Sigma(u,h,\mu) \\ 0 \\ G_c(h,\mu) \\ G_\kappa(h) \\ G_N(h,\mu) \\ G_P^{\pm}(u,h) \\ 0 \\ 0 \end{pmatrix}.$$

Hereby $\mathbb{E}(T)$ and $\mathbb{F}(T)$ are defined similarly as above but without the trace properties at $t = 0$.

It is now clear that for $h \in {}_0\mathbb{E}(T)$ (which is a function having vanishing time trace) the compatibility condition $(-\nabla_{x'} h(t=0), 1)^\top|_{\partial\Sigma} \cdot \nu_{S_1} = 0$ is satisfied. Regarding the compatibility conditions for the Stokes system we refer to Section 3.1 in [**64**]. Therefore both operators are well defined.

Let $z := (u,\pi,q,h,\mu)$ and $z_* := (u_*, \pi_*, [\![\pi_*]\!], h_*, \mu_*)$ the reference solution as above. We can now rewrite the problem abstractly as

$$\mathsf{L}(z+z_*) = \mathsf{R}(z+z_*), \quad z \in {}_0\mathbb{E}(T).$$

Note that we already know that L is invertible from ${}_0\mathbb{E}(T)$ to ${}_0\mathbb{F}(T)$ with norms independent of T. This renders the fixed point equation

$$z = \mathsf{L}^{-1}(\mathsf{R}(z+z_*) - \mathsf{L}z_*), \quad \text{in } {}_0\mathbb{E}(T).$$

Define $\mathsf{K} : {}_0\mathbb{E}(T) \to {}_0\mathbb{E}(T)$ by means of $[z \mapsto \mathsf{L}^{-1}(\mathsf{R}(z+z_*) - \mathsf{L}z_*)]$. We now need to establish contraction estimates for R.

LEMMA 4.6. *There are $r_0, T_0 > 0$, such that*

$$|\mathsf{R}(z_1+z_*) - \mathsf{R}(z_2+z_*)|_{{}_0\mathbb{F}(T)} \le C(T^\alpha + |z_*|_{\mathbb{E}(T)} + |z_1|_{{}_0\mathbb{E}(T)} + |z_2|_{{}_0\mathbb{E}(T)})|z_1 - z_2|_{{}_0\mathbb{E}(T)},$$

for some $\alpha > 0$ and for all $z_1, z_2 \in \mathsf{B}(r,0) \subset {}_0\mathbb{E}(T)$, if $r \le r_0$ and $T \le T_0$.

Having these estimates at hand we proceed as in the proof of Theorem 2.28 to obtain a fixed point of K by Banach's contraction mapping principle by choosing $\varepsilon(T) > 0$ in (4.25) small enough. The uniqueness of h is understood in the sense of the proof of Theorem 2.28. This finishes the proof. $\square$

PROOF OF LEMMA 4.6. Let us first note that

$$[h \mapsto \Delta_h] \in C^1(U; \mathcal{B}(W_r^2(\Omega\backslash\Sigma); L_r(\Omega))), \tag{4.26}$$

$$[h \mapsto \nabla_h] \in C^1(U; \mathcal{B}(W_r^k(\Omega\backslash\Sigma); W_r^{k-1}(\Omega\backslash\Sigma))), \quad k = 1,2, \tag{4.27}$$

where $U \subset B_{qp}^{4-1/q-3/p}(\Sigma)$ is a sufficiently small neighbourhood of zero. This can be shown as in Lemma 2.17.

We estimate every nonlinearity separately. Let $h \in W$, where W is a sufficiently small neighbourhood of zero in $\mathbb{E}_h(T)$. We recall that $a^{\pm}(h; D_x)(u,\pi) = \mu^{\pm}(\Delta_h - \Delta)u - (\nabla - \nabla_h)\pi$. Clearly,

$$|(\Delta_h - \Delta)u|_{L_r(0,T;L_r(\Omega))} \leq |\Delta_h - \Delta|_{L_\infty(0,T;\mathcal{B}(W^2_r(\Omega\backslash\Sigma);L_r(\Omega)))}|u|_{L_r(0,T;W^2_r(\Omega\backslash\Sigma))}.$$

Using (4.26) this gives

$$|(\Delta_h - \Delta)u|_{L_r(0,T;L_r(\Omega))} \leq C|h|_{{}_0\mathbb{E}(T)}|u|_{L_r(0,T;W^2_r(\Omega\backslash\Sigma))}, \quad h \in W.$$

The same arguments give

$$|(\nabla_h - \nabla)\pi|_{L_r(0,T;L_r(\Omega))} \leq C|h|_{{}_0\mathbb{E}(T)}|\pi|_{L_r(0,T;\dot{H}^1_r(\Omega\backslash\Sigma))}, \quad h \in W,$$

since (4.27) is also true for the homogeneous counterparts $\dot{H}^k_r$ replacing W^k_r. Note that these estimates and the C^1-dependence of h and the linear structure in (u,π) of $a^{\pm}(h)(u,\pi)$ then automatically give rise to a contraction estimate of form

$$\begin{aligned}
&|a^{\pm}(h_1)(u_1,\pi_1) - a^{\pm}(h_2)(u_2,\pi_2)|_{L_r(0,T;L_r(\Omega))} \leq \\
&\quad \leq C|h_1 - h_2|_{{}_0\mathbb{E}(T)}\left(|u_1 - u_2|_{L_r(0,T;W^2_r(\Omega\backslash\Sigma))} + |\pi_1 - \pi_2|_{L_r(0,T;\dot{H}^1_r(\Omega\backslash\Sigma))}\right),
\end{aligned} \tag{4.28}$$

valid for all $h_1, h_2 \in W, u_1, u_2 \in L_r(0,T;W^2_r(\Omega\backslash\Sigma)), \pi_1, \pi_2 \in L_r(0,T;\dot{H}^1_r(\Omega\backslash\Sigma))$. Indeed, we have that $a^{\pm} \in C^2\left(W \times \mathbb{E}_u(T) \times \mathbb{E}_\pi(T); L_r(0,T;L_r(\Omega))\right)$, and $a^{\pm}(0) = 0$, $Da^{\pm}(0) = 0$. Alternatively, one can explicitly estimate the difference and end up with (4.28). Before we estimate $\bar{a}(u,h,\eta) := Du \cdot \partial_t\Theta_h^{-1} + (u\cdot\nabla_h)u + (\rho^+ - \rho^-)(\nabla_h\eta \cdot \nabla_h)u$, some remarks are in order. Firstly, $Du \cdot \partial_t\Theta_h^{-1} = -\chi\partial_t h(1 + h\chi')^{-1}\partial_3 u$, see [**57**].

Contracting the (transformed) convection term $(u\cdot\nabla_h)u$ is easy due to the fact that $\mathbb{E}_u(T) \hookrightarrow L_\infty(0,T;L_\infty(\Omega)) \cap L_\infty(0,T;W^1_r(\Omega\backslash\Sigma))$. More precisely,

$$\begin{aligned}
&|(u\cdot\nabla_h)u|_{L_r(0,T;L_r(\Omega))} \leq \\
&\qquad T^{1/r}|u|_{L_\infty(0,T;L_\infty(\Omega))}|\nabla_h|_{L_\infty(0,T;\mathcal{B}(W^1_r(\Omega\backslash\Sigma);L_r(\Omega)))}|u|_{L_\infty(0,T;W^1_r(\Omega\backslash\Sigma))}.
\end{aligned}$$

Regarding the other terms we recall that $Du \in L_4(0,T;L_\infty(\Omega))$ by Theorem 4.1. We then get by Hölder inequality that

$$\begin{aligned}
&|Du\cdot\partial_t\Theta_h^{-1}|_{L_r(0,T;L_r(\Omega))} \leq \\
&\quad |\chi(1+h\chi')^{-1}|_{L_\infty(0,T;L_\infty(\Omega))}|Du|_{L_{p_1}(0,T;L_\infty(\Omega))}|\partial_t h|_{L_p(0,T;L_r(\Sigma))}|1|_{L_{p_0}(0,T;L_\infty(\Omega))},
\end{aligned}$$

where $1 < p_0, p_1 < \infty$ are such that

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p} + \frac{1}{p_0}. \tag{4.29}$$

Choose $p_1 = 4$. By choice of $q < 2$ and $3 < r < 7/2$, we have $W^{1-1/q}_q(\Sigma) \hookrightarrow L_r(\Sigma)$ by Sobolev's embedding theorem. Since $1/r > 1/4 + 1/p$, $r = 17/5$, cf. (4.23), we find finite $1 < p_0 < \infty$ such that (4.29) is fulfilled. Note that these estimates may not optimal but sufficient in our case. We then obtain that there is some $\varepsilon = \varepsilon(p,q,r) > 0$ such that

$$|Du\cdot\partial_t\Theta_h^{-1}|_{L_r(0,T;L_r(\Omega))} \leq CT^\varepsilon|u|_{\mathbb{E}_u(T)}|h|_{\mathbb{E}_h(T)}, \quad h \in W.$$

Furthermore,

$$|(\nabla_h\eta\cdot\nabla_h)u|_{L_r(0,T;L_r(\Omega))} \leq |\nabla_h\eta|_{L_p(0,T;L_r(\Omega))}|\nabla_h u|_{L_{p_1}(0,T;L_\infty(\Omega))}|1|_{L_{p_0}(0,T;L_\infty(\Omega))},$$

where p_0, p_1 are as above. Again by Sobolev embedding, $W^1_q(\Omega\backslash\Sigma) \hookrightarrow L_r(\Omega)$, whence

$$|(\nabla_h\eta{\cdot}\nabla_h)u|_{L_r(0,T;L_r(\Omega))} \leq |\nabla_h\eta|_{L_p(0,T;W^1_q(\Omega\backslash\Sigma))}|\nabla_h u|_{L_{p_1}(0,T;L_\infty(\Omega))}|1|_{L_{p_0}(0,T;L_\infty(\Omega))}.$$

In view of (4.27), these estimates together with the smooth dependence of $\bar{a} \in C^\infty(\mathbb{E}_u(T) \times W; \mathbb{F}_1(T))$, as well as $\bar{a}(0,0,0) = 0$ and $D\bar{a}(0,0,0) = 0$ give rise to contraction estimates for $\bar{a}$.

For $G_d(u,h) := (\mathrm{div} - \mathrm{div}_h)u$, the estimate in $\mathbb{F}_2(T)$ is straightforward,

$$|G_d(u,h)|_{L_r(0,T;W^1_r(\Omega\backslash\Sigma))} \leq |\nabla - \nabla_h|_{L_\infty(0,T;\mathcal{B}(W^2_r(\Omega\backslash\Sigma);W^1_r(\Omega)))}|u|_{L_r(0,T;W^2_r(\Omega\backslash\Sigma))},$$

where we used that $G_d(u,h) = \mathrm{Tr}(\nabla - \nabla_h)u$.

The contraction estimates for $G_c(h,\eta) := (\Delta - \Delta_h)\eta = (\mathrm{div}\,\nabla - \mathrm{div}_h\,\nabla_h)\eta$ and $G_N(h,\eta) := \nu_{\partial\Omega}\cdot(\nabla - \nabla_h)\eta$ easily stem from (4.26)-(4.27) with q replacing r. Recall that in this graph situation, $K(h) = \mathrm{div}_{x'}(\nabla_{x'}h(1 + |\nabla_{x'}h|^2)^{-1/2})$, whence

$$G_\kappa(h) = \left(1 - \frac{1}{\sqrt{1 + |\nabla_{x'}h|^2}}\right)\Delta_{x'}h + \nabla_{x'}h\cdot\nabla_{x'}\left(\frac{1}{\sqrt{1+|\nabla_{x'}h|^2}}\right), \quad x' \in \Sigma. \tag{4.30}$$

Again using the product estimate

$$|\nabla h\cdot\nabla^2 h|_{\mathbb{F}_9(T)} \leq C|\nabla h|_{L_\infty(0,T;B^{3-1/q-3/p}_{qp}(\Sigma))}|\nabla^2 h|_{\mathbb{F}_9(T)} \leq C|h|^2_{{}_0\mathbb{E}_h(T)}$$

and the fact that $G_\kappa \in C^\infty(W;\mathbb{F}_9(T))$, $G_\kappa(0) = 0$, $DG_\kappa(0) = 0$, ensure the contraction property of G_κ.

Regarding $G^\pm_P(u,h)$ it is shown in Section 3.1 in [**64**], that

$$G^\pm_P(u,h) = P_{S_1}\left[\frac{1}{1+\chi' h}\left(\chi\partial_3 u((-\nabla_{x'}h,1)^\top\cdot\nu_{S_1}) + \begin{pmatrix}\chi\nabla_{x'}h\\ \chi' h\end{pmatrix}\partial_3(u\cdot\nu_{S_1})\right)\right].$$

Therefore, due to the fact that $u\cdot\nu_{S_1} = 0$ on $S_1\backslash\partial\Sigma$ and $(-\nabla_{x'}h,1)^\top\cdot\nu_{S_1} = 0$ on $\partial\Sigma\times(L_1,L_2)$, the nonlinearity $G^\pm_P(u,h)$ vanishes for the fixed point we obtain later. Hence we may replace $G^\pm_P(u,h)$ by zero in the definition of R.

Now, for $G_S(h,u,\pi)$ we split $G_S(h,u,\pi) = G^S_S(h,u,\pi) + G^\kappa_S(h)$, where

$$\begin{aligned} G^S_S(h,u,\pi) &:= [\![\mu^\pm\left((D\Theta_h - I)Du + Du^\top(D\Theta_h - I)^\top\right)]\!]\nu_{\Sigma_h} + \\ &\quad + [\![\left(\mu^\pm(Du + Du^\top) - \pi I\right)(e_3 - \nu_{\Sigma_h})]\!], \\ G^\kappa_S(h) &:= \sigma(K(h)\nu_{\Sigma_h} - \Delta_{x'}he_3). \end{aligned}$$

Regarding the estimates of $G^S_S(h,u,\pi)$ we refer to [**56**]. Note that due to Remark 1.2. (c) in [**56**] we may use these results since $r > 3$ and $\mathbb{E}_h(T) \hookrightarrow BUC([0,T];C^2(\Sigma))$.

Considering $G^\kappa_S(h)$ we may write $G^\kappa_S(h) = G_\kappa(h)e_3 + K(h)(\nu_{\Sigma_h} - e_3)$ and estimate each term separately. In particular, we have to control terms of the form $\nabla h\cdot\nabla^2 h$ in the norm of $\mathbb{F}_3(T) = {}_0W^{1/2-1/(2r)}_r(0,T;L_r(\Sigma)) \cap L_r(0,T;W^{1-1/r}_r(\Sigma))$. This stems from the observation in (4.30). Now, by Theorem 4.1 we already know that the space in which second derivatives of h live in embeds into $\mathbb{F}_3(T)$.

We may now use the product estimate of Proposition 5.7 in [**48**] to obtain

$$\begin{aligned} |\nabla h\cdot\nabla^2 h|_{W^{1/2-1/(2r)}_r(0,T;L_r(\Sigma))} &\lesssim |\nabla h|_{L_\infty(0,T;L_\infty(\Sigma))}|\nabla^2 h|_{W^{1/2-1/(2r)}_r(0,T;L_r(\Sigma))} + \\ &\quad + |\nabla h|_{W^{1/2-1/(2r)}_r(0,T;L_\infty(\Sigma))}|\nabla^2 h|_{L_\infty(0,T;L_r(\Sigma))}. \end{aligned}$$

Furthermore,

$$|\nabla h \cdot \nabla^2 h|_{L_r(0,T;W_r^{1-1/r}(\Sigma))} \lesssim |\nabla h|_{L_\infty(0,T;C^1(\Sigma))} |\nabla^2 h|_{L_r(0,T;W_r^{1-1/r}(\Sigma))}.$$

These estimates show that the product terms of form $\nabla h \cdot \nabla^2 h$ are well defined in $\mathbb{F}_3(T)$.

These observations allow us to conclude contraction estimates for G_S^κ since again $G_S^\kappa(0) = 0, DG_S^\kappa(0) = 0$.

Regarding $G_\Sigma(u, h, \mu) = u|_\Sigma \cdot (-\nabla_{x'} h, 0)^\top - [\![e_3 \cdot (\nabla - \nabla_h)\mu]\!] - [\![(-\nabla_{x'} h, 0)^\top \cdot \nabla_h \mu]\!]$, the last two terms can be controlled as before. Clearly the first term is smooth in (u, h) and quadratic and the bound

$$|u|_\Sigma \cdot (-\nabla_{x'} h, 0)^\top|_{L_p(0,T;W_q^{1-1/q}(\Sigma))} \le T^{1/p} |u|_{L_\infty(0,T;W_q^1(\Omega\backslash\Sigma))} |\nabla h|_{L_\infty(0,T;C^1(\Sigma))}$$

renders contraction estimates also for G_Σ. This concludes the proof of the contraction estimates. ☐

REMARK 4.7. We need the technical assumption (4.23) for the contraction estimates. Note that if $p \in (23, \infty)$, the last inequality in (4.23) is fulfilled automatically.

4.5. Qualitative behaviour

In this section we investigate the long-time behaviour of solutions starting close to equilibria. By a study of the spectrum of the linearization we will show that solutions starting close to certain equilibria converge to an equilibrium solution at an exponential rate.

Let us again consider the case of a cylindrical container $\Omega = \Sigma \times (L_1, L_2)$, where $-\infty < L_1 < 0 < L_2 < \infty$ and $\Sigma \subset \mathbb{R}^2$ is open, bounded and has smooth boundary. We want to study stability properties of

$$\begin{aligned}
\rho\partial_t u - \mu\Delta u + \operatorname{div}[(\rho u + [\![\rho]\!]\nabla\eta) \otimes u] + \nabla p &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
-[\![\mu(Du + Du^\top)]\!]\nu_{\Gamma(t)} + [\![p]\!]\nu_{\Gamma(t)} &= \sigma H_{\Gamma(t)}\nu_{\Gamma(t)}, && \text{on } \Gamma(t),\\
[\![u]\!] &= 0, && \text{on } \Gamma(t),\\
V_{\Gamma(t)} - u|_{\Gamma(t)} \cdot \nu_{\Gamma(t)} &= -[\![\nu_{\Gamma(t)} \cdot \nabla\eta]\!], && \text{on } \Gamma(t),\\
\nu_{\Gamma(t)} \cdot \nu_{S_1} &= 0, && \text{on } \partial\Gamma(t),\\
\Delta\eta &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\eta|_{\Gamma(t)} &= \sigma H_{\Gamma(t)}, && \text{on } \Gamma(t),\\
\nu_{\partial\Omega} \cdot \nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Gamma(t),\\
P_{S_1}\big(\mu(Du + Du^\top)\nu_{S_1}\big) &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u \cdot \nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u &= 0, && \text{on } S_2,\\
u(0) &= u_0, && \text{on } \Omega\backslash\Gamma(0),\\
\Gamma(0) &= \Gamma_0.
\end{aligned} \tag{4.31}$$

We recall that $\rho := \rho^+\chi_{\Omega^+(t)} + \rho^-\chi_{\Omega^-(t)}$ and $\mu := \mu^+\chi_{\Omega^+(t)} + \mu^-\chi_{\Omega^-(t)}$.

4.5.1. Equilibria and spectrum of the linearization. We note that the pressure p as well as the chemical potential μ may be reconstructed by the semiflow $(u(t), \Gamma(t))$ as follows. For given $\Gamma(t)$ we can solve the two-phase elliptic problem

$$\begin{cases} \Delta\eta = 0, & \text{in } \Omega\backslash\Gamma(t), \\ \eta|_{\Gamma(t)} = \sigma H_{\Gamma(t)}, & \text{on } \Gamma(t), \\ n_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} = 0, & \text{on } \partial\Omega, \end{cases}$$

and the weak transmission problem

$$\begin{cases} ([1/\rho]\nabla p|\nabla\varphi)_{L_2(\Omega)} \\ \qquad = ([\mu/\rho]\Delta u - [1/\rho]\operatorname{div}[(\rho u + [\![\rho]\!]\nabla\eta)\otimes u]|\nabla\varphi)_{L_2(\Omega)}, & \text{for all } \varphi \in W^1_{r'}(\Omega), \\ [\![p]\!] = [\![\mu(Du + Du^\top)\nu_{\Gamma(t)}\cdot\nu_{\Gamma(t)}]\!] + \sigma H_{\Gamma(t)}, & \text{on } \Gamma(t), \end{cases}$$

where $r' = r/(r-1)$, cf. Lemma A.7 in [**64**]. Therefore we may concentrate on the set of equilibria $\mathcal{E}$ for the semiflow $(u(t), \Gamma(t))$. Note that the set of equilibria for (4.31) is given by

$$\mathcal{E} = \{(u, \Gamma) : u = 0, H_\Gamma = \text{const.}\}.$$

In particular, also μ is constant, p is constant in the two phases of $\Omega\backslash\Gamma$ and also the jump $[\![p]\!]$ is constant on Γ.

REMARK 4.8. We want to point out that in the special case when Γ is a C^2-graph of a function h over Σ, we can even deduce that $H_\Gamma = 0$ and h is constant. A proof of this can be found in Section 3.2.

We now work again in the graph situation, that is, we assume the free interface $\Gamma(t)$ is the graph of a height function h over Σ.

The linearization of the transformed two-phase Navier-Stokes/Mullins-Sekerka problem (4.31) around the trivial equilibrium $(0, \Sigma) \in \mathcal{E}$ induces us to study the problem

$$\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla p &= f_u, && \text{in } \Omega\backslash\Sigma, \\ \operatorname{div} u &= 0, && \text{in } \Omega\backslash\Sigma, \\ -[\![\mu(Du + Du^\top)]\!]e_3 + [\![p]\!]e_3 + \sigma\Delta_{x'}he_3 &= 0, && \text{on } \Sigma, \\ [\![u]\!] &= 0, && \text{on } \Sigma, \\ \partial_t h - u_3 + [\![\partial_3\eta]\!] &= f_h, && \text{on } \Sigma, \\ (\nabla_{x'}h, -1)^\top\cdot\nu_{S_1} &= 0 && \text{on } \partial\Sigma, \\ \Delta\eta &= 0, && \text{in } \Omega\backslash\Sigma, \\ \eta|_\Sigma + \sigma\Delta_{x'}h &= 0, && \text{on } \Sigma, \\ \nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Sigma, \\ P_{S_1}\left(\mu(Du + Du^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma, \\ u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma, \\ u &= 0, && \text{on } S_2, \\ u(0) &= u_0, && \text{in } \Omega\backslash\Sigma, \\ h(0) &= h_0, && \text{on } \Sigma, \end{aligned} \tag{4.32}$$

where f_h is assumed to be mean value free. Let us note the following observations. Integrating equation $(4.32)_5$ over Σ yields $\int_\Sigma h(t)dx = \int_\Sigma h_0 dx$ for all $t \in \mathbb{R}_+$. In other words, whenever h_0 and f_h are mean value free, the solution h will stay mean value free for all times. Furthermore, applying $P_\Sigma = I - e_3 \otimes e_3$ to equation $(4.32)_3$ directly yields that $P_\Sigma([\![\mu^\pm(Du + Du^\top)]\!]e_3) = 0$ on Σ.

We want to write system (4.32) as an abstract evolution equation. To this end let

$$X_0 := L_{r,\sigma}(\Omega) \times W_q^{1-1/q}(\Sigma), \quad X_1 := (L_{r,\sigma}(\Omega) \cap W_r^2(\Omega\backslash\Sigma)) \times W_q^{4-1/q}(\Sigma),$$

and define a linear operator $A : D(A) \subset X_1 \to X_0$ by

$$A(u,h) := (-[\mu/\rho]\Delta u + \nabla p/\rho, -u_3 + [\![\partial_3\eta]\!])$$

with domain

$$\begin{aligned} D(A) := \{(u,h) \in X_1 : [\![u]\!] = 0 \text{ on } \Sigma,\ &P_{S_1}\left(\mu^\pm(Du + Du^\top)\nu_{S_1}\right) = 0 \text{ on } S_1\backslash\partial\Sigma, \\ &u\cdot\nu_{S_1} = 0 \text{ on } S_1\backslash\partial\Sigma,\ u = 0 \text{ on } S_2, \\ &P_\Sigma([\![\mu^\pm(Du + Du^\top)]\!]e_3) = 0 \text{ on } \Sigma,\ (\nabla_{x'}h, -1)^\top \cdot \nu_{S_1} = 0 \text{ on } S_1\}. \end{aligned}$$

Here, $p \in \dot{H}_r^1(\Omega\backslash\Sigma)$ solves the weak transmission problem

$$\begin{aligned} (\nabla p/\rho|\nabla\varphi)_{L_2(\Omega)} &= ([\mu/\rho]\Delta u|\nabla\varphi)_{L_2(\Omega)}, && \text{for all } \varphi \in W_{r'}^1(\Omega), \\ [\![p]\!] &= \sigma\Delta_{x'}h + ([\![\mu^\pm(Du + Du^\top)]\!]e_3|e_3)_{L_2(\Sigma)}, && \text{on } \Sigma, \end{aligned}$$

cf. Lemma A.7 in [**64**] and $\eta \in W_q^2(\Omega\backslash\Sigma)$ solves the elliptic problem

$$\begin{aligned} \Delta\eta &= 0, && \text{in } \Omega\backslash\Sigma, \\ \eta|_\Sigma + \sigma\Delta_{x'}h &= 0, && \text{on } \Sigma, \\ \partial_\nu\eta &= 0, && \text{on } \partial\Omega. \end{aligned}$$

As in [**4**] and [**64**], we will sometimes make use of the notation via solution operators T_1 and T_2 for the pressure, that is,

$$\nabla p/\rho = T_1[(\mu/\rho)\Delta u] + T_2[\sigma\Delta_{x'}h + ([\![\mu(Du + Du^\top)]\!]e_3|e_3)_{L_2(\Sigma)}], \tag{4.34}$$

cf. Lemma A.7 in [**64**]. Note that $\Delta_{x'}h \in W_q^{2-1/q}(\Sigma) \hookrightarrow W_r^{1-1/r}(\Sigma)$ for $h \in W_q^{4-1/q}(\Sigma)$ by Sobolev embedding, since $r < 3q/(3-q)$.

We can then rewrite problem (4.32) in a more compact form as

$$\dot{z}(t) + Az(t) = f(t), \quad t \in \mathbb{R}_+, \quad z(0) = z_0,$$

where $z := (u,h)$, $f := (f_u, f_h)$ and $z_0 := (u_0, h_0)$. We can now show a similar result as in [**4**] about properties of the operator A.

LEMMA 4.9. *Let $n = 2,3$, (p,q,r) as in Theorem 4.1, $\rho^\pm$, $\mu^\pm, \sigma > 0$ constant and X_0 and A as above. Then the following statements are true.*

(1) *The linear operator $-A$ generates an analytic C_0-semigroup e^{-At} in X_0.*
(2) *The spectrum $\sigma(-A)$ consists of countably many eigenvalues with finite algebraic multiplicity.*
(3) *$\lambda = 0$ is a semi-simple eigenvalue with multiplicity 1 and $X_0 = N(A) \oplus R(A)$.*
(4) *$\sigma(-A)\backslash\{0\} \subset \mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.*

(5) *The kernel* $N(A)$ *is isomorphic to the tangent space* $T_{(0,\Sigma)}\mathcal{E}$ *of* $\mathcal{E}$ *at the trivial equilibrium* $(0,\Sigma)\in\mathcal{E}$ *and is given by* $N(A)=\{(u,h): u=0, h=\mathit{const.}\}$.

(6) *The restriction of* e^{-At} *to* $R(A)$ *is exponentially stable.*

Proof. The first assertion follows from Theorem 4.4 and the proof of Proposition 1.2 in [**53**]. Since $D(A)$ compactly embeds into X_0, the resolvent of A is compact and therefore the spectrum of A consists only of countably many eigenvalues with finite multiplicity. By classical results, it does not depend on q and r, cf. [**8**], [**19**]. So let $\lambda\in\sigma(-A)$ be an eigenvalue with eigenfunctions $(u,h)\in D(A)$. The corresponding eigenvalue problem reads as

$$
\begin{aligned}
\lambda\rho u-\mu\Delta u+\nabla p&=0, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u&=0, && \text{in } \Omega\backslash\Sigma,\\
[\![u]\!]&=0, && \text{on } \Sigma,\\
-[\![\mu(Du+Du^\top)]\!]e_3+[\![p]\!]e_3-\sigma\Delta_{x'}he_3&=0, && \text{on } \Sigma,\\
\lambda h-u_3+[\![\partial_3\eta]\!]&=0, && \text{on } \Sigma,\\
(\nabla_{x'}h,-0)^\top\cdot\nu_{S_1}&=0 && \text{on } \partial\Sigma,\\
\Delta\eta&=0, && \text{in } \Omega\backslash\Sigma,\\
[\![\eta]\!]=0,\quad \eta|_\Sigma+\sigma\Delta_{x'}h&=0, && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega}&=0, && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu(Du+Du^\top)\nu_{S_1}\right)&=0, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1}&=0, && \text{on } S_1\backslash\partial\Sigma,\\
u&=0, && \text{on } S_2.
\end{aligned}
\tag{4.35}
$$

Testing equation $(4.35)_1$ with u in $L_2(\Omega)$ and invoking boundary and transmission conditions yields

$$
\lambda|\rho^{1/2}u|^2_{L_2(\Omega)}+|\mu^{1/2}(Du+Du^\top)|^2_{L_2(\Omega)}+\sigma\bar{\lambda}|\nabla_{x'}h|^2_{L_2(\Sigma)}+|\nabla\eta|^2_{L_2(\Omega)}=0. \tag{4.36}
$$

Let $\lambda=0$. Then $u=0$ by Korn's inequality and $\eta=\mathit{const.}$, whence $\Delta_{x'}h$ is constant on Σ. An integration over Σ together with the boundary condition $(4.35)_6$ yields that $\Delta_{x'}h=0$ on Σ. Hence h has to be constant. We obtain that the kernel $N(A)$ is one-dimensional and $N(A)=\{(u,h): u=0, h=\mathit{const.}\}$. Taking real parts in (4.36) yields $\operatorname{Re}\lambda\le 0$. We also easily obtain that $\sigma(-A)\cap i\mathbb{R}=\{0\}$, hence $\sigma(-A)\backslash\{0\}\subset\mathbb{C}_-$. Next we show that the eigenvalue $\lambda=0$ is semi-simple. Pick $z=(u,h)\in N(A^2)$. Then $z_1:=Az\in N(A)$, hence $z_1=(0,h_1)$ and h_1 is constant.

The problem for $z = (u,h)$ now reads as

$$\begin{aligned}
-\mu\Delta u + \nabla p &= 0, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu(Du + Du^\top)]\!]e_3 + [\![p]\!]e_3 - \sigma\Delta_{x'}he_3 &= 0, && \text{on } \Sigma,\\
[\![u]\!] &= 0, && \text{on } \Sigma,\\
-u_3 + [\![\partial_3\eta]\!] &= h_1, && \text{on } \Sigma,\\
(\nabla_{x'}h, -1)^\top \cdot \nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\eta &= 0, && \text{in } \Omega\backslash\Sigma,\\
\eta|_\Sigma + \sigma\Delta_{x'}h &= 0, && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\mu|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu(Du + Du^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma.\\
u &= 0, && \text{on } S_2.
\end{aligned} \tag{4.37}$$

Integrating $(4.37)_5$ over Σ and using the fact that h_1 is constant yields

$$-\int_\Sigma u_3 dx' + \int_\Sigma [\![\partial_3\eta]\!]dx' = h_1 \int_\Sigma 1dx'.$$

Note that due to the boundary conditions in (4.37), $\int_\Sigma u_3 dx' = \int_{\Omega^+} \operatorname{div} u = 0$ and $\int_\Sigma [\![\partial_3\eta]\!]dx' = \int_\Omega \Delta\eta dx = 0$. Hence $h_1 = 0$ and therefore $(u,h) \in N(A)$, whence $N(A^2) \subset N(A)$. Since A has compact resolvent, $R(A)$ is closed in X_0 and $\lambda = 0$ is a pole of $(\lambda - A)^{-1}$. Therefore $\lambda = 0$ is semi-simple, cf. [**43**], and $X_0 = N(A) \oplus R(A)$. Since also $\sigma(A|_{R(A)}) \subset \mathbb{C}_+$ we obtain that the restricted semigroup $e^{-At}|_{R(A)}$ is exponentially stable. ☐

Define now a linear operator $L : D(L) \subset X_1 \to P_0^\Sigma X_0$ by $L(u,h) := A(u,h)$, where

$$D(L) := D(A) \cap \{(u,h) \in X_1 : (h,1)_{L_2(\Sigma)} = 0\},$$

and $P_0^\Sigma X_0 := X_0 \cap \{(u,h) \in X_0 : P_0^\Sigma h = 0\}$. Hereby, $P_0^\Sigma h := (h|1)_{L_2(\Sigma)}/|\Sigma|$.

Then L is well-defined and $\sigma(-L) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq -\kappa < 0\}$ for some $\kappa > 0$, since the spectrum is discrete.

4.5.2. Parametrization of the nonlinear phase manifold. We recall, cf. (4.5), that the transformed equations around the trivial equilibrium $(0,\Sigma) \in \mathcal{E}$ read as

$$
\begin{aligned}
\rho\partial_t u - \mu\Delta u + \nabla p &= F_u(h,u,p), && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= G_d(h,u), && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu(Du+Du^\top) - pI]\!]e_3 &= \sigma\Delta_{x'}he_3 + G_S(h,u,p), && \text{on } \Sigma,\\
[\![u]\!] &= 0, && \text{on } \Sigma,\\
\partial_t h &= u_3 - [\![\partial_3\eta]\!] + G_\Sigma(h,u,\eta), && \text{on } \Sigma,\\
(-\nabla_{x'}h,0)^\top\cdot\nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\eta &= G_c(h,\eta), && \text{in } \Omega\backslash\Sigma,\\
\eta|_\Sigma - \sigma\Delta_{x'}h &= G_\kappa(h), && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= G_N(h,\eta), && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu(Du+Du^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u &= 0, && \text{on } S_2,\\
u(0) &= u_0, && \text{on } \Omega\backslash\Sigma,\\
h(0) &= h_0, && \text{on } \Sigma,
\end{aligned}
$$

where $F_u(h,u,p) := a^\pm(h;D_x)(u,p) + \bar{a}(h,u)$, cf. (4.5). The nonlinear phase manifold is given by

$$
\begin{aligned}
\mathsf{PM} := \{&(u,h)\in W_r^{2-2/r}(\Omega\backslash\Sigma)\cap B_{qp}^{4-1/q-3/p}(\Sigma) : \operatorname{div} u = G_d,\\
&P_\Sigma(\mu^\pm(Du+Du^\top)e_3) = ((G_S)_{1,2},0),\ [\![u]\!]=0,\ (\nabla_{x'}h|n_{\partial\Sigma})=0,\\
&(h|1)_{L_2(\Sigma)}=0,\ P_{S_1}\left(\mu^\pm(Du+Du^\top)\nu_{S_1}\right)=0,\ u\cdot\nu_{S_1}=0,\ u|_{S_2}=0\}.
\end{aligned}
$$

as a subset of $X_\gamma := W_r^{2-2/r}(\Omega\backslash\Sigma)\cap B_{qp}^{4-1/q-3/p}(\Sigma)$. The linear phase manifold is given by

$$
\begin{aligned}
\mathsf{PM}_0 := \{&(u,h)\in W_r^{2-2/r}(\Omega\backslash\Sigma)\cap B_{qp}^{4-1/q-3/p}(\Sigma) : \operatorname{div} u = 0,\\
&P_\Sigma(\mu^\pm(Du+Du^\top)e_3) = 0,\ [\![u]\!]=0,\ (\nabla_{x'}h|n_{\partial\Sigma})=0,\\
&(h|1)_{L_2(\Sigma)}=0,\ P_{S_1}\left(\mu^\pm(Du+Du^\top)\nu_{S_1}\right)=0,\ u\cdot\nu_{S_1}=0,\ u|_{S_2}=0\}.
\end{aligned}
$$

Following the lines of Section 4.2 in [**64**], we obtain that there is a local parametrization of PM over PM_0 around zero. More precisely, there is a small $r>0$, such that for every $(u_0,h_0)\in B(r,0)\subset\mathsf{PM}$ there is a C^2-function φ and a decomposition

$$(u_0,h_0) = (\tilde{u}_0,\tilde{h}_0) + (\varphi(\tilde{u}_0,\tilde{h}_0),0),\quad (\tilde{u}_0,\tilde{h}_0)\in\mathsf{PM}_0. \tag{4.39}$$

For details we refer to Proposition 4.3 and Section 4.2 in [**64**].

4.5.3. Convergence to equilibria. We now state and prove the main result.

THEOREM 4.10. *The trivial equilibrium $(0,\Sigma)\in\mathcal{E}$ is stable in the following sense. For each $\varepsilon>0$ there exists some $\delta=\delta(\varepsilon)>0$ such that for all initial values $(u_0,h_0)\in X_\gamma\cap\mathsf{PM}$ satisfying*

$$|u_0|_{W_r^{2-2/r}(\Omega\backslash\Sigma)} + |h_0|_{B_{qp}^{4-1/q-3/p}(\Sigma)} \le \delta(\varepsilon), \tag{4.40}$$

there exists some global in time solution

$$u \in W_r^1(\mathbb{R}_+; L_r(\Omega)) \cap L_r(\mathbb{R}_+; W_r^2(\Omega\backslash\Sigma)),$$
$$h \in W_p^1(\mathbb{R}_+; W_q^{1-1/q}(\Sigma)) \cap L_p(\mathbb{R}_+; W_q^{4-1/q}(\Sigma)),$$

such that

$$|u(t)|_{W_r^{2-2/r}(\Omega\backslash\Sigma)} + |h(t)|_{B_{qp}^{4-1/q-3/p}(\Sigma)} \le \varepsilon, \quad t \in \mathbb{R}_+.$$

Moreover,

$$\Big[|u(t)|_{W_r^{2-2/r}(\Omega\backslash\Sigma)} + |h(t) - P_0^\Sigma h_0|_{B_{qp}^{4-1/q-3/p}(\Sigma)}\Big] \to_{t\to\infty} 0,$$

where $P_0^\Sigma h_0 := \frac{1}{|\Sigma|}(h_0|1)_{L_2(\Sigma)}$ *is the mean value of* h_0. *The convergence is at an exponential rate.*

Proof. We follow the lines of [4] and [64]. Let $\varepsilon > 0$ be given and $(u_0, h_0) \in X_\gamma \cap \mathsf{PM}$ such that the smallness condition (4.40) holds for some $\delta > 0$ to be specified later. By (4.39), we can decompose the initial data

$$(u_0, h_0) = (0, P_0^\Sigma h_0) + (\tilde u_0, \tilde h_0) + (\varphi(\tilde u_0, \tilde h_0), 0),$$

where $(\tilde u_0, \tilde h_0) + (\varphi(\tilde u_0, \tilde h_0), 0) \in \mathsf{PM}$ and $(\tilde u_0, \tilde h_0) \in \mathsf{PM}_0$.

We now want to decompose the solution $(u(t), h(t))$ suitably and write

$$(u(t), h(t)) = (0, P_0^\Sigma h_0) + (\tilde u(t), \tilde h(t)) + (\bar u(t), \bar h(t)), \quad t \in \mathbb{R}_+,$$

where $(\tilde u(t), \tilde h(t)) \in \mathsf{PM}_0$ for $t \in \mathbb{R}_+$, and estimate each term separately. We consider the two coupled systems

$$\begin{aligned}
\omega\rho\bar u + \rho\partial_t\bar u - \mu^\pm\Delta\bar u + \nabla\bar\pi &= F_u(P_0^\Sigma h_0 + \tilde h + \bar h, \tilde u + \bar u, \tilde\pi + \bar\pi), && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div}\bar u &= G_d(P_0^\Sigma h_0 + \tilde h + \bar h, \tilde u + \bar u), && \text{in } \Omega\backslash\Sigma,\\
-P_\Sigma([\![\mu^\pm(D\bar u + D\bar u^\top)e_3]\!]) &= G_S^\parallel(P_0^\Sigma h_0 + \tilde h + \bar h, \tilde u + \bar u), && \text{on } \Sigma,\\
-2[\![\mu^\pm\partial_3\bar u_3]\!] + [\![\bar\pi]\!] - \sigma\Delta_{x'}\bar h &= G_S^\perp(P_0^\Sigma h_0 + \tilde h + \bar h, \tilde u + \bar u), && \text{on } \Sigma,\\
[\![\bar u]\!] &= 0, && \text{on } \Sigma,\\
\omega\bar h + \partial_t\bar h - \bar u_3 + [\![\partial_3\bar\eta]\!] &= G_\Sigma(P_0^\Sigma h_0 + \tilde h + \bar h, \tilde u + \bar u, \tilde\eta + \bar\eta), && \text{on } \Sigma,\\
(-\nabla_{x'}\bar h, 0)^\top \cdot \nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\bar\eta &= G_c(P_0^\Sigma h_0 + \tilde h + \bar h, \tilde\eta + \bar\eta), && \text{in } \Omega\backslash\Sigma,\\
\bar\eta|_\Sigma - \sigma\Delta_{x'}\bar h &= G_\kappa(P_0^\Sigma h_0 + \tilde h + \bar h), && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\bar\eta|_{\partial\Omega} &= G_N(P_0^\Sigma h_0 + \tilde h + \bar h, \tilde\eta + \bar\eta), && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu^\pm(D\bar u + D\bar u^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\bar u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\bar u &= 0, && \text{on } S_2,\\
\bar u(0) &= \varphi(\tilde u_0, \tilde h_0), && \text{on } \Omega\backslash\Sigma,\\
\bar h(0) &= 0, && \text{on } \Sigma,
\end{aligned} \tag{4.41}$$

where $\omega > 0$, and

$$\begin{aligned}
\rho\partial_t\tilde{u} - \mu^{\pm}\Delta\tilde{u} + \nabla\tilde{\pi} &= \omega\rho(I - T_1)\bar{u}, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div}\tilde{u} &= 0, && \text{in } \Omega\backslash\Sigma,\\
-P_\Sigma([\![\mu^{\pm}(D\tilde{u} + D\tilde{u}^\top)e_3]\!]) &= 0, && \text{on } \Sigma,\\
-2[\![\mu^{\pm}\partial_3 u_3]\!] + [\![\tilde{\pi}]\!] - \sigma\Delta_{x'}\tilde{h} &= 0, && \text{on } \Sigma,\\
[\![\tilde{u}]\!] &= 0, && \text{on } \Sigma,\\
\partial_t\tilde{h} - \tilde{u}_3 + [\![\partial_3\tilde{\eta}]\!] &= \omega(\bar{h} - P_0^\Sigma\bar{h}), && \text{on } \Sigma,\\
(-\nabla_{x'}\tilde{h}, 0)^\top \cdot \nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\tilde{\eta} &= 0, && \text{in } \Omega\backslash\Sigma,\\
\tilde{\eta}|_\Sigma - \sigma\Delta_{x'}\tilde{h} &= 0, && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\tilde{\eta}|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu^{\pm}(D\tilde{u} + D\tilde{u}^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\tilde{u}\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
\tilde{u} &= 0, && \text{on } S_2,\\
\tilde{u}(0) &= \tilde{u}_0, && \text{on } \Omega\backslash\Sigma,\\
\tilde{h}(0) &= \tilde{h}_0, && \text{on } \Sigma.
\end{aligned} \tag{4.42}$$

Let us note a few things here. T_1 in $(4.42)_1$ is the solution operator stemming from (4.34). The right hand side of $(4.42)_1$ can equivalently be written as $\omega\rho(I - T_1)\bar{u} = \omega\rho\bar{u} - \omega\rho\nabla\bar{q}$, where $\bar{q} \in \dot{H}^1_r(\Omega\backslash\Sigma)$ is the unique solution of the weak transmission problem

$$\begin{aligned}
(\nabla\bar{q}|\nabla\psi)_{L_2(\Omega)} &= (\bar{u}|\nabla\psi)_{L_2(\Omega)}, && \text{for all } \psi \in W^1_{r'}(\Omega),\\
[\![\bar{q}]\!] &= 0, && \text{on } \Sigma.
\end{aligned}$$

Furthermore, the initial value $\tilde{h}_0$ in (4.42) is mean value free. Note that the right hand side of $(4.42)_6$ is mean value free as well, hence an integration of $(4.42)_6$ over Σ yields that $\tilde{h}$ stays mean value free for all times $t > 0$. In particular, we can equivalently rewrite (4.42) in the projected base space $P_0^\Sigma X_0$ as

$$\frac{d}{dt}\tilde{z}(t) + L\tilde{z}(t) = R(\bar{z})(t), \quad t > 0, \quad z(0) = \tilde{z}_0 := (\tilde{u}_0, \tilde{h}_0). \tag{4.44}$$

Here, $\tilde{z} := (\tilde{u}, \tilde{h})$, $\bar{z} := (\bar{u}, \bar{h})$ and $R(\bar{z}) := (\omega(I - T_1)\bar{u}, (I - P_0^\Sigma)\bar{h})$.

Note that by Lemma 4.9, the spectral bound of $-L$ satisfies $s(-L) \leq -\kappa < 0$ and the restricted semigroup e^{-Lt} is exponentially stable on $P_0^\Sigma X_0$.

We now solve this evolution equation in exponentially time-weighted spaces to get suitable decay estimates, cf. [4] and [**64**].

Let us introduce notation. Let $\mathbb{E}_u(\mathbb{R}_+) := H^1_r(\mathbb{R}_+; L_r(\Omega)) \cap L_r(\mathbb{R}_+; H^2_r(\Omega\backslash\Sigma))$ and $\mathbb{E}_h(\mathbb{R}_+) := W^1_p(\mathbb{R}_+; W^{1-1/q}_q(\Sigma)) \cap L_p(\mathbb{R}_+; W^{4-1/q}_q(\Sigma))$. For $\beta \in [0, -s(-L))$

define

$$e^{-\beta t}\mathbb{E}_u(\mathbb{R}_+) := \{w \in L_r(\mathbb{R}_+; L_r(\Omega)) : e^{\beta t} w \in \mathbb{E}_u(\mathbb{R}_+)\},$$
$$e^{-\beta t}\mathbb{E}_h(\mathbb{R}_+) := \{w \in L_p(\mathbb{R}_+; L_q(\Omega)) : e^{\beta t} w \in \mathbb{E}_h(\mathbb{R}_+)\}.$$

In a similar way we define $e^{-\beta t} L_r(\mathbb{R}_+; L_r(\Omega))$. Since $0 \leq \beta < -s(-L)$, we obtain that for every

$$(f_u, f_h) \in e^{-\beta t}[L_r(\mathbb{R}_+; L_r(\Omega)) \times L_p(\mathbb{R}_+; W_q^{1-1/q}(\Sigma))],$$

and $(\hat{u}_0, \hat{h}_0) \in X_\gamma$ there is a unique solution $(u, h) \in e^{-\beta t}[\mathbb{E}_u(\mathbb{R}_+) \times \mathbb{E}_h(\mathbb{R}_+)]$ of the linear evolution problem

$$\partial_t(u, h) + L(u, h) = (f_u, f_h), \quad t \in \mathbb{R}_+, \quad (u, h)|_{t=0} = (\hat{u}_0, \hat{h}_0),$$

by maximal regularity in exponentially time-weighted spaces. Furthermore, there is some $M > 0$ such that

$$|(u, h)|_{e^{-\beta t}[\mathbb{E}_u(\mathbb{R}_+) \times \mathbb{E}_h(\mathbb{R}_+)]} \leq M |(f_u, f_h, \hat{u}_0, \hat{h}_0)|_{e^{-\beta t}[L_r(\mathbb{R}_+; L_r(\Omega)) \times L_p(\mathbb{R}_+; W_q^{1-1/q}(\Sigma))] \times X_\gamma}.$$

In particular, we may then easily solve (4.44) in dependence of $\bar{z} = (\bar{u}, \bar{h})$,

$$(\tilde{u}, \tilde{h}) = \left(\frac{d}{dt} + L, \operatorname{tr}|_{t=0}\right)^{-1} (\omega(I - T_1)\bar{u}, (I - P_0^\Sigma)\bar{h}, \tilde{u}_0, \tilde{h}_0). \tag{4.45}$$

Let us now discuss problem (4.41). For given $\omega > 0$, let L_ω be given by the left hand side of (4.41) and N the collection of nonlinearities on the right hand side. Then we can rewrite problem (4.41) in the shorter form

$$L_\omega \bar{w} = N(w_\infty + \tilde{w} + \bar{w}), \quad (\bar{u}, \bar{h})(0) = (\varphi(\tilde{u}_0, \tilde{h}_0), 0),$$

where $\bar{w} := (\bar{u}, \bar{h}, \bar{\pi}, \bar{\eta})$, $\tilde{w} := (\tilde{u}, \tilde{h}, \tilde{\pi}, \tilde{\eta})$ and $w_\infty := (0, P_0^\Sigma h_0, 0, 0)$. Note at this point that w_∞ is constant and N does not explicitly depend on w_∞. Furthermore, due to the first part of the proof, $\tilde{w}$ depends only on $(\tilde{u}_0, \tilde{h}_0, \bar{u}, \bar{h})$, cf. (4.45).

In order to solve problem (4.41) we need to resolve the initial data and the compatibility conditions at $t = 0$ properly. By solving certain auxiliary problems in exponentially weighted spaces, we may construct an extension operator

$$\operatorname{ext}_\beta : \bar{X}_\gamma \to e^{-\beta}[\mathbb{E}_u(\mathbb{R}_+) \times \mathbb{E}_h(\mathbb{R}_+)],$$

satisfying $\operatorname{ext}_\beta(v, g)|_{t=0} = (v, g)$ for all $(v, g) \in \bar{X}_\gamma$, where

$$\begin{aligned} \bar{X}_\gamma := \{(u, h) \in X_\gamma : & u|_{S_2} = 0,\ (u|\nu_{S_1}) = 0,\ P_{S_1}(\mu^\pm(Du + Du^\top)\nu_{S_1}) = 0, \\ & [\![u]\!] = 0,\ (\nabla_{x'} h|\nu_{\partial\Sigma}) = 0\}, \end{aligned}$$

cf. [**64**]. Now define

$$M(\tilde{u}_0, \tilde{h}_0, \bar{w}) := N(w_\infty + \tilde{w} + \bar{w} + \operatorname{ext}_\beta[(\varphi(\tilde{u}_0, \tilde{h}_0), 0) - (\bar{u}(0), \bar{h}(0))]).$$

By construction, $M(\tilde{u}_0, \tilde{h}_0, \bar{w})|_{t=0} = N(u_0, h_0, 0, 0)$. This allows us to solve the problem

$$L_\omega \bar{w} = M(\tilde{u}_0, \tilde{h}_0, \bar{w}), \quad (\bar{w}_1, \bar{w}_2)|_{t=0} = (\varphi(\tilde{u}_0, \tilde{h}_0), 0),$$

by the implicit function theorem, since all relevant compatibility conditions at $t = 0$ are satisfied. Following the lines of [**4**], we obtain that there is some small $\rho > 0$ and a ball $B(0, \rho) \subset X_\gamma \cap \mathsf{PM}_0$, such that there is a $\Phi \in C^1(B(0, \rho); e^{-\beta t}[\mathbb{E}_u(\mathbb{R}_+) \times$

$\mathbb{E}_h(\mathbb{R}_+)\times\mathbb{E}_\pi(\mathbb{R}_+)\times\mathbb{E}_\eta(\mathbb{R}_+)]$) satisfying $\bar{w}=\Phi(\tilde{u}_0,\tilde{h}_0)$. By construction, $\bar{w}$ is the solution of (4.41). Here, $\mathbb{E}_\pi(\mathbb{R}_+):=L_r(\mathbb{R}_+;\dot{H}^1_r(\Omega\backslash\Sigma))$, $\mathbb{E}_\eta(\mathbb{R}_+):=L_p(\mathbb{R}_+;W^2_q(\Omega\backslash\Sigma))$.

We then obtain that the convergence $(u(t),h(t))\to(0,P_0^\Sigma h_0)$ in X_γ is at an exponential rate. The proof is complete. $\square$

CHAPTER 5

Rayleigh-Taylor instability for the two-phase Navier-Stokes/Mullins-Sekerka equations with ninety degree contact angle

5.1. Introduction

In this chapter we study the two-phase Navier-Stokes equations with surface tension coupled to a Mullins-Sekerka problem for two immiscible, incompressible Newtonian fluids inside a bounded domain under the effects of gravitational acceleration. The gravitational force is hereby constant and acts only in one space direction. In our model the interface separating the two fluids meets the boundary of the domain at a constant ninety degree angle. This leads to a free boundary problem for the interface involving a contact angle problem at the boundary as well. We are especially interested in stability and instability properties of the problem, depending on the surface tension and the two different densities of the fluids.

Rayleigh-Taylor instability for the two-phase Navier-Stokes equations with surface tension was first studied in [**54**], where a full space problem in $\mathbb{R}^n$ was considered and the interface is a small perturbation of a flat $(n-1)$-dimensional plane. The case of a two-phase Navier-Stokes problem in a capillary domain with boundary contact was treated in [**64**]. The two-phase Navier-Stokes/Mullins-Sekerka equations were also studied in [**4**], for equal densities, no gravity, and without boundary contact. In this chapter we continue the investigation of the two-phase Navier-Stokes/Mullins-Sekerka equations in a capillary with boundary contact. For further discussion on two-phase Navier-Stokes and Mullins-Sekerka we also refer to [**22**], [**56**], and [**58**].

We assume that the domain $\Omega \subset \mathbb{R}^n$, $n = 2,3$, can be decomposed as $\Omega = \Omega^+(t)\dot{\cup}\mathring{\Gamma}(t)\dot{\cup}\Omega^-(t)$, where $\mathring{\Gamma}(t)$ denotes the interior of $\Gamma(t)$, a $(n-1)$-dimensional submanifold with boundary. We interpret $\Gamma(t)$ to be the interface separating the two phases, $\Omega^+(t)$ and $\Omega^-(t)$, which will be assumed to be connected. The boundary of $\Gamma(t)$ will be denoted by $\partial\Gamma(t)$. Furthermore we assume $\Gamma(t)$ to be orientable, the unit normal vector field on $\Gamma(t)$ pointing from $\Omega^-(t)$ into $\Omega^+(t)$ will be denoted by $\nu_{\Gamma(t)}$.

We denote by $V_{\Gamma(t)}$ the normal velocity and by $H_{\Gamma(t)}$ the mean curvature of the free interface $\Gamma(t)$. By $[\![\cdot]\!]$ we mean the jump of a quantity across $\Gamma(t)$ in direction of $\nu_{\Gamma(t)}$, that is,

$$[\![f]\!](x) := \lim_{\varepsilon\to 0+}[f(x+\varepsilon\nu_{\Gamma(t)}) - f(x-\varepsilon\nu_{\Gamma(t)})], \quad x \in \Gamma(t).$$

Let us discuss relevant quantities of our model. We assume $\rho^\pm > 0$ are the two positive constant densities of the fluids in the two phases, $\mu^\pm > 0$ their respective constant viscosities, and $\sigma > 0$ is a given surface tension constant. The gravitational acceleration constant will be denoted by $\mathsf{g} > 0$. To economize our notation, we let $\rho := \rho^+ \chi_{\Omega^+(t)} + \rho^- \chi_{\Omega^-(t)}$ and $\mu := \mu^+ \chi_{\Omega^+(t)} + \mu^- \chi_{\Omega^-(t)}$, where χ_M is the indicator function of a given set M.

We shall denote by u the velocity of the fluids, p the pressure, η the chemical potential, and $\Gamma(t)$ the free interface at time $t \geq 0$.

Let us consider the case where the domain is a cylindrical container $\Omega = \Sigma \times (L_1, L_2)$, where $-\infty < L_1 < 0 < L_2 < \infty$ and $\Sigma \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. We denote the walls of the cylinder by $S_1 := \partial\Sigma \times (L_1, L_2)$ and bottom and top by $S_2 := \Sigma \times \{L_1, L_2\}$. As usual, $\nu_{\partial\Omega}$ denotes the unit normal vector field pointing outwards of Ω and $\nu_{S_1} = \nu_{\partial\Omega}$ on the walls S_1. The projection is defined as $P_{S_1} := I - \nu_{S_1} \otimes \nu_{S_1}$.

In a cylindrical domain the full problem reads as

$$
\begin{aligned}
\rho\partial_t u - \mu\Delta u + \operatorname{div}[(\rho u + [\![\rho]\!]\nabla\eta) \otimes u] + \nabla p &= -\rho \mathsf{g} e_n, && \text{in } \Omega\backslash\Gamma(t),\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
-[\![\mu(Du + Du^\top)]\!]\nu_{\Gamma(t)} + [\![p]\!]\nu_{\Gamma(t)} &= \sigma H_{\Gamma(t)}\nu_{\Gamma(t)}, && \text{on } \Gamma(t),\\
[\![u]\!] &= 0, && \text{on } \Gamma(t),\\
V_{\Gamma(t)} - u|_{\Gamma(t)} \cdot \nu_{\Gamma(t)} &= -[\![\nu_{\Gamma(t)} \cdot \nabla\eta]\!], && \text{on } \Gamma(t),\\
\nu_{\Gamma(t)} \cdot \nu_{S_1} &= 0, && \text{on } \partial\Gamma(t),\\
\Delta\eta &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
[\![\eta]\!] = 0, \quad \eta|_{\Gamma(t)} &= \sigma H_{\Gamma(t)} + [\![\rho]\!]\mathsf{g} x_n, && \text{on } \Gamma(t),\\
\nu_{\partial\Omega} \cdot \nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Gamma(t),\\
P_{S_1}\big(\mu(Du + Du^\top)\nu_{S_1}\big) &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u \cdot \nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u|_{S_2} &= 0, && \text{on } S_2,\\
u(0) &= u_0, && \text{on } \Omega\backslash\Gamma(0),\\
\Gamma(0) &= \Gamma_0.
\end{aligned} \tag{5.1}
$$

Here we want to mention that we implicitly impose that

$$\mathring{\Gamma}(t) \subset \Omega, \quad \partial\Gamma(t) \subset S_1, \quad t \geq 0.$$

Note that the gravitational force in $(5.1)_1$ is given by $-\rho \mathsf{g} e_n$. Without gravitational effects, that is, $\mathsf{g} = 0$, this model was proposed by Abels, Garcke, and Grün in [**2**]. We want to note a few things here. The momentum balance $(5.1)_1$ contains an extra term involving the chemical potential η if the densities in the two phases are different. This term is already needed to get an energy structure for the system without gravitational effects, cf. Section 5 in [**2**]. Furthermore, we have to modify the equation for the chemical potential η, equation $(5.1)_8$.

We will below show that the energy

$$E(t) := \int_{\Gamma(t)} \sigma d\mathcal{H}^{n-1} + \frac{1}{2}\int_\Omega \rho(x,t)u(x,t)^2 dx + \int_\Omega \rho(x,t)\mathsf{g} x_n dx$$

satisfies the energy-dissipation relation

$$\frac{d}{dt}E(t) = -D(t) := -2\int_\Omega \mu|\mathbb{D}u(t)|^2dx - \int_\Omega |\nabla\eta(t)|^2dx \le 0, \tag{5.2}$$

cf. Lemma 5.1. Again, $\mathbb{D}u$ is the symmetric part of the gradient Du, $2\mathbb{D}u = Du + Du^\top$. There is a remark in order regarding this extra term in $(5.1)_1$. Since $\operatorname{div} u = 0$ and $\Delta\eta = 0$ in the bulk phases $\Omega\backslash\Gamma(t)$, we obtain

$$\operatorname{div}[(\rho u + (\rho^+ - \rho^-)\nabla\eta)\otimes u] = \rho(u\cdot\nabla)u + (\rho^+ - \rho^-)(\nabla\eta\cdot\nabla)u, \quad \text{in } \Omega\backslash\Gamma(t).$$

In the case of equal densities, say for simplicity $\rho = 1$, the extra term $\operatorname{div}[(\rho^+ - \rho^-)\nabla\mu\otimes u]$ vanishes. We also note that individual masses are conserved,

$$\frac{d}{dt}|\Omega^\pm(t)| = 0, \quad t\in\mathbb{R}_+,$$

due to the boundary conditions and $\Delta\eta = 0$ and $\operatorname{div} u = 0$ in the bulk phases $\Omega\backslash\Gamma(t)$. Note that it is convenient to introduce the modified pressure $\tilde p := p + \rho \mathrm{g} x_n$. This leads to the problem

$$\begin{aligned}
\rho\partial_t u - \mu\Delta u + \operatorname{div}[(\rho u + [\![\rho]\!]\nabla\eta)\otimes u] + \nabla\tilde p &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
-[\![\mu(Du + Du^\top)]\!]\nu_{\Gamma(t)} + [\![\tilde p]\!]\nu_{\Gamma(t)} &= \sigma H_{\Gamma(t)}\nu_{\Gamma(t)} + [\![\rho]\!]\mathrm{g}x_n\nu_{\Gamma(t)}, && \text{on } \Gamma(t),\\
[\![u]\!] &= 0, && \text{on } \Gamma(t),\\
V_{\Gamma(t)} - u|_{\Gamma(t)}\cdot\nu_{\Gamma(t)} &= -[\![\nu_{\Gamma(t)}\cdot\nabla\eta]\!], && \text{on } \Gamma(t),\\
\nu_{\Gamma(t)}\cdot\nu_{S_1} &= 0, && \text{on } \partial\Gamma(t),\\
\Delta\eta &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
[\![\eta]\!] = 0, \quad \eta|_{\Gamma(t)} &= \sigma H_{\Gamma(t)} + [\![\rho]\!]\mathrm{g}x_n, && \text{on } \Gamma(t),\\
\nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Gamma(t),\\
P_{S_1}\left(\mu(Du + Du^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Gamma(t),\\
u|_{S_2} &= 0, && \text{on } S_2,\\
u(0) &= u_0, && \text{on } \Omega\backslash\Gamma(0),\\
\Gamma(0) &= \Gamma_0.
\end{aligned} \tag{5.3}$$

It is also noteworthy that the difference of the densities enters the equation for the chemical potential, $(5.3)_8$, since we now let gravity act on the system, $\mathrm{g} > 0$. This is indeed needed and meaningful to obtain an energy structure for the system as we shall show below. For further discussion of the free energy we refer to [**2**]. We now close this introduction by showing that the system (5.3) is physically meaningful and thermodynamically consistent.

Lemma 5.1. *Suppose $(u, \tilde p, \Gamma, \eta)$ is a sufficiently smooth solution to (5.3). Then*

$$\frac{d}{dt}\left[\int_\Omega \rho\frac{|u|^2}{2}dx + \int_{\Gamma(t)}\sigma d\mathcal{H}^{n-1} + \int_\Omega \rho\mathrm{g}x_n dx\right] = -2\int_\Omega \mu|\mathbb{D}u|^2dx - \int_\Omega |\nabla\eta|^2dx.$$

Proof. By well known transport identities, cf. [**2**],

$$\frac{d}{dt}\int_\Omega \rho\frac{|u|^2}{2}dx = \int_\Omega \rho\partial_t u\cdot u dx - \int_{\Gamma(t)} [\![\rho]\!]\frac{|u|^2}{2}V_\Gamma d\mathcal{H}^{n-1}.$$

By equation (5.3)$_1$,

$$\frac{d}{dt}\int_\Omega \rho\frac{|u|^2}{2}dx =$$
$$= \int_\Omega u\,[2\mu\,\mathrm{div}\,\mathbb{D}u - \mathrm{div}[(\rho u + j)\otimes u] - \nabla\tilde{p}]\,dx - \int_{\Gamma(t)}[\![\rho]\!]\frac{|u|^2}{2}V_\Gamma d\mathcal{H}^{n-1},$$

where $j := [\![\rho]\!]\nabla\eta$. Note that since $\mathrm{div}\,u = 0$ and $\mathrm{div}\,j = 0$ in the bulk phases $\Omega\backslash\Gamma(t)$,

$$u\cdot\mathrm{div}[(\rho u + j)\otimes u] = u\cdot[(\rho u + j)\cdot\nabla]u = \frac{1}{2}\nabla|u|^2\cdot(\rho u + j),\quad \text{in } \Omega\backslash\Gamma(t).$$

Hence,

$$\frac{d}{dt}\int_\Omega \rho\frac{|u|^2}{2}dx = -2\int_\Omega \mu|\mathbb{D}u|^2dx + \int_{\Gamma(t)}\frac{1}{2}|u|^2\left([\![\rho]\!](u\cdot\nu_\Gamma - V_\Gamma) - [\![j]\!]\cdot\nu_\Gamma\right)d\mathcal{H}^{n-1}$$
$$+\int_{\Gamma(t)} u\cdot(-2[\![\mu\mathbb{D}u]\!]\nu_\Gamma + [\![\tilde{p}]\!]\nu_\Gamma)d\mathcal{H}^{n-1}.$$

We obtain by equations (5.3)$_3$ and (5.3)$_5$ that

$$\frac{d}{dt}\int_\Omega \rho\frac{|u|^2}{2}dx = -2\int_\Omega \mu|\mathbb{D}u|^2dx + \int_{\Gamma(t)}(u\cdot\nu_\Gamma)(\sigma H_\Gamma + [\![\rho]\!]\mathrm{g}x_n)d\mathcal{H}^{n-1}.$$

We may now use $\frac{d}{dt}\int_{\Gamma(t)}\sigma = -\int_{\Gamma(t)}\sigma H_\Gamma V_\Gamma$ to the result

$$\frac{d}{dt}\left[\int_\Omega \rho\frac{|u|^2}{2}dx + \int_{\Gamma(t)}\sigma d\mathcal{H}^{n-1}\right] = -2\int_\Omega \mu|\mathbb{D}u|^2dx +$$
$$+\int_{\Gamma(t)}(\sigma H_\Gamma + [\![\rho]\!]\mathrm{g}x_n)(u\cdot\nu_\Gamma - V_\Gamma)d\mathcal{H}^{n-1} + \int_{\Gamma(t)}[\![\rho]\!]\mathrm{g}x_n V_\Gamma d\mathcal{H}^{n-1}.$$

We have

$$\int_{\Gamma(t)}(\sigma H_\Gamma + [\![\rho]\!]\mathrm{g}x_n)(u\cdot\nu_\Gamma - V_\Gamma)d\mathcal{H}^{n-1} = \int_{\Gamma(t)}\eta|_\Gamma[\![\nabla\eta]\!]\cdot\nu_\Gamma d\mathcal{H}^{n-1} = -\int_\Omega|\nabla\eta|^2dx,$$

whence using

$$\int_{\Gamma(t)}[\![\rho]\!]\mathrm{g}x_n V_\Gamma d\mathcal{H}^{n-1} = -\frac{d}{dt}\int_\Omega \rho\mathrm{g}x_n dx$$

consequently gives the desired equality and the proof is complete. □

Let us give an overview of this chapter. We reduce the problem to a fixed reference geometry and a nonlinear problem for the height function by means of Hanzawa transform. Then we show local well-posedness and characterize spectral properties of the linearization around the trivial equilibrium. The main result is then proving presence of linearized Rayleigh-Taylor instability for the nonlinear problem. Moreover, we discuss a thermodynamically relevant Mullins-Sekerka problem in a cylidrical domain with gravitational force.

5.2. Reduction to a flat interface and well-posedness

In this section we transform the equations (5.3) defined on the time-dependent domain $\Omega\backslash\Gamma(t)$ with free boundary $\Gamma(t)$ to a fixed reference frame. We follow the same strategy as in Section 4.2. Again this is based on the ideas of [**57**], [**64**]. To simplify notation, let $n = 3$.

We again assume that the interface at time t is given as a graph over the fixed reference surface $\Sigma := \Omega \cap \{x_3 = 0\}$, that is, there is some height function $h : \Sigma \times [0, \infty) \to (L_1, L_2)$, such that

$$\Gamma(t) = \Gamma_h(t) := \{x \in \Sigma \times (L_1, L_2) : x_3 = h(x', t),\ x' = (x_1, x_2) \in \Sigma\}, \quad t \geq 0.$$

Following the lines of Section 4.2, we transform the two-phase Navier-Stokes/Mullins-Sekerka system with gravity (5.1) to the fixed reference frame, the transformed system reads as

$$\begin{aligned}
\rho^\pm \partial_t w - \mu^\pm \Delta w + \nabla q &= F_f(w, q, h, \vartheta), && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} w &= F_d(h, w), && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu^\pm(Dw + Dw^\top) - qI]\!]\nu_\Sigma &= \sigma\Delta_{x'} h \nu_\Sigma + [\![\rho]\!]\mathsf{g} h \nu_\Sigma + F_S(h, w, q), && \text{on } \Sigma,\\
[\![w]\!] &= 0, && \text{on } \Sigma,\\
\partial_t h &= w \cdot \nu_\Sigma - [\![\partial_3 \vartheta]\!] + F_\Sigma(h, w, \vartheta), && \text{on } \Sigma,\\
(-\nabla_{x'} h, 0)^\top \cdot \nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\vartheta &= F_c(h, \vartheta), && \text{in } \Omega\backslash\Sigma,\\
\vartheta|_\Sigma - \sigma\Delta_{x'} h - [\![\rho]\!]\mathsf{g} h &= F_\kappa(h), \quad [\![\vartheta]\!] = 0, && \text{on } \Sigma,\\
\nu_{\partial\Omega} \cdot \nabla\vartheta|_{\partial\Omega} &= F_N(h, \vartheta), && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu^\pm(Dw + Dw^\top)\nu_{S_1}\right) &= F_P(h, w), && \text{on } S_1\backslash\partial\Sigma,\\
w \cdot \nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
w &= 0, && \text{on } S_2,\\
w(0) &= w_0, && \text{on } \Omega\backslash\Sigma,\\
h(0) &= h_0, && \text{on } \Sigma,
\end{aligned} \tag{5.4}$$

where $\nu_\Sigma = e_3$, and

$$\begin{aligned}
F_f(w, q, h, \vartheta) &:= \mu(\Delta_h - \Delta)w + (\nabla - \nabla_h)q +\\
&\quad + \rho Dw \cdot \partial_t \Theta_h^{-1} - \rho(w \cdot \nabla_h)w - (\rho^+ - \rho^-)(\nabla_h \vartheta \cdot \nabla_h)w,\\
F_d(h, w) &:= (\operatorname{div} - \operatorname{div}_h)w,\\
F_S(h, w, q) &:= [\![\mu^\pm\left((D\Theta_h - I)Dw + Dw^\top(D\Theta_h - I)^\top\right)]\!]\nu_{\Gamma_h} +\\
&\quad + [\![\left(\mu^\pm(Dw + Dw^\top) - qI\right)(e_3 - \nu_{\Gamma_h})]\!] +\\
&\quad + \sigma(K(h)\nu_{\Gamma_h} - \Delta_{x'} h e_3) + [\![\rho]\!]\mathsf{g} h(e_3 - \nu_{\Gamma_h}),\\
F_\Sigma(h, w, \vartheta) &:= w \cdot (-\nabla_{x'} h, 0)^\top - [\![e_3 \cdot (\nabla_h - \nabla)\vartheta]\!] - [\![(-\nabla_{x'} h, 0)^\top \cdot \nabla_h \vartheta]\!],\\
F_c(h, \vartheta) &:= (\Delta - \Delta_h)\vartheta,\\
F_\kappa(h) &:= \sigma(K(h) - \Delta_{x'} h),
\end{aligned}$$

as well as

$$F_N(h,\vartheta) := \nu_{\partial\Omega}\cdot(\nabla-\nabla_h)\vartheta,$$
$$F_P(h,w) := P_{S_1}\left(\mu^\pm\left((D\Theta_h - I)Dw + Dw^\top(D\Theta_h-I)^\top\right)\right)\nu_{S_1}\right).$$

Recall that in this graph situation,

$$K(h) = H(\Gamma_h) = \operatorname{div}_{x'}\left(\frac{\nabla_{x'}h}{\sqrt{1+|\nabla_{x'}h|^2}}\right).$$

Furthermore, we want to point out that we used the fact that the normal ν_{S_1} is independent of x_3 and that the transformation Θ_h leaves the Dirichlet-boundary S_2 invariant.

Let us discuss local well-posedness of this transformed system. We again work with the same function spaces as in the case without gravitational effects, which were discussed and introduced in Section 4.3.2. Since the terms stemming from gravity only induce a lower order perturbation, we obtain that the principal linearization of (5.4) has maximal regularity using Theorem 4.4. Indeed, $[\![\rho]\!]\mathsf{g}h$ is of lower order with respect to $\Delta_{x'}h$ in $(5.4)_{3,8}$. We then may follow the lines of the proof of Theorem 4.5 to obtain local well-posedness for (5.4). For completeness, let us state the precise result.

Theorem 5.2. *Let $\mu^\pm,\rho^\pm,\sigma,\mathsf{g}>0$ be constant, $-\infty<L_1<0<L_2<\infty$, and (p,q,r) satisfy*

$$p\in(6,\infty),\quad q\in(99/50,2)\cap(2p/(p+1),2),\quad r=17/5,\quad 1/r>1/4+1/p.$$

Moreover let $\Sigma\subset\mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Sigma$. Let $\Omega:=\Sigma\times(L_1,L_2)$, $S_1:=\partial\Sigma\times(L_1,L_2)$, and $S_2:=\Sigma\times\{L_1,L_2\}$. Furthermore let

$$(w_0,h_0)\in W_r^{2-2/r}(\Omega\backslash\Sigma)\times B_{qp}^{4-1/q-3/p}(\Sigma),$$

satisfy the compatibility conditions

$$\begin{aligned}\operatorname{div} w_0 = F_d(h_0,w_0),\ &\text{in } \Omega\backslash\Sigma,\\ -[\![\mu^\pm\partial_3(w_0)_{1,2}]\!]-[\![\mu^\pm\nabla_{x'}(w_0)_3]\!] = F_S^{\parallel}(w_0,h_0),\ \text{on } \Sigma,\quad &[\![w_0]\!]=0,\ \text{on } \Sigma,\\ P_{S_1}(\mu^\pm(Dw_0+Dw_0^\top)\nu_{S_1})=0,\ \text{on } S_1,\quad &w_0\cdot\nu_{S_1}=0,\ \text{on } S_1,\\ w_0|_{S_2}=0,\ \text{on } S_2,\quad &(-\nabla_{x'}h_0,1)^\top\cdot\nu_{S_1}=0,\ \text{on } \partial\Sigma.\end{aligned}\tag{5.5}$$

Then (5.4) admits a unique local-in-time strong solution, that is, there is some $T_0>0$, such that for every $0<T\le T_0$ there is some $\varepsilon=\varepsilon(T)>0$, such that whenever the smallness condition

$$|(w_0,h_0)|_{W_r^{2-2/r}(\Omega\backslash\Sigma)\times B_{qp}^{4-1/q-3/p}(\Sigma)}\le\varepsilon$$

is satisfied there is a unique strong solution on $(0,T)$,

$$\begin{aligned}w\in W_r^1(0,T;L_r(\Omega))\cap L_r(0,T;W_r^2(\Omega\backslash\Sigma)),\quad q\in L_r(0,T;\dot H_r^1(\Omega\backslash\Sigma)),\\ [\![q]\!]\in W_r^{1/2-1/(2r)}(0,T;L_r(\Sigma))\cap L_r(0,T;W_r^{1-1/r}(\Sigma)),\\ h\in W_p^1(0,T;W_q^{1-1/q}(\Sigma))\cap L_p(0,T;W_q^{4-1/q}(\Sigma)),\quad \vartheta\in L_p(0,T;W_q^2(\Omega\backslash\Sigma)).\end{aligned}$$

5.3. Rayleigh-Taylor instability

5.3.1. Equilibria and spectrum of the linearization. In this section we characterize the equilibria of (5.3) and analyze the spectrum of the linearization around the trivial equilibrium.

As in the model without gravity we may note that the pressure π as well as the chemical potential η may be reconstructed by the semiflow $(u(t), \Gamma(t))$ by solving the two-phase elliptic problem

$$\begin{aligned} \Delta\eta &= 0, && \text{in } \Omega\backslash\Gamma(t), \\ [\![\eta]\!] = 0, \quad \eta|_{\Gamma(t)} &= \sigma H_{\Gamma(t)} + [\![\rho]\!]\mathrm{g}x_n, && \text{on } \Gamma(t), \\ n_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega, \end{aligned}$$

cf. Section A.2, and the weak transmission problem

$$\begin{aligned} (\nabla\pi|\nabla\varphi)_{L_2(\Omega)} &= (\mu\Delta u - \operatorname{div}[(\rho u + [\![\rho]\!]\nabla\eta)\otimes u]|\nabla\varphi)_{L_2(\Omega)}, \quad \text{for all } \varphi\in W^1_{r'}(\Omega), \\ [\![\pi]\!] &= [\![\mu(Du + Du^\top)\nu_{\Gamma(t)}\cdot\nu_{\Gamma(t)}]\!] + \sigma H_{\Gamma(t)} + [\![\rho]\!]\mathrm{g}x_n, \quad \text{on } \Gamma(t), \end{aligned}$$

where $r' = r/(r-1)$, cf. Lemma A.7 in [**64**].

Assume that we have a time-independent solution (u, π, η, Γ) of (5.3). By the energy-dissipation equality (5.2) we directly obtain that

$$|\sqrt{\mu}(Du + Du^\top)|^2_{L_2(\Omega)} + |\nabla\mu|^2_{L_2(\Omega)} = 0.$$

From this we deduce by Korn's inequality that u has to be constant. Since u vanishes on S_2 we obtain that $u = 0$. We also obtain that μ is constant.

Hence $\nabla\pi = 0$ in $\Omega\backslash\Gamma$ and hence also π is constant in $\Omega\backslash\Gamma$, with possibly different values in the two phases. Furthermore, since also the trace of μ on Γ is constant, we obtain

$$\sigma H_\Gamma + [\![\rho]\!]\mathrm{g}x_n = const.$$

In particular if $H_\Gamma = 0$, then x_n is constant on the interface. Hence flat interfaces belong to the set of equilibria. Assuming that Γ is the graph of a height function h over Σ, we obtain that h solves the elliptic quasilinear problem

$$\begin{aligned} \sigma\,\operatorname{div}_{x'}\left(\frac{\nabla_{x'}h}{\sqrt{1+|\nabla_{x'}h|^2}}\right) + [\![\rho]\!]\mathrm{g}h &= c, && x'\in\Sigma, \\ (n_{\partial\Sigma}|\nabla_{x'}h) &= 0, && x'\in\partial\Sigma, \end{aligned} \tag{5.8}$$

where $c := [\![\rho]\!]\mathrm{g}\frac{1}{|\Sigma|}\int_\Sigma h\,dx'$. Note that c is determined by integrating (5.8) over Σ and invoking the boundary condition. All admissible height functions solving this quasilinear problem belong to the set of equilibria.

We are interested in stability properties of the trivial equilibrium $(0, \Sigma)$. We consider now the linear problem

$$
\begin{aligned}
\rho\partial_t u - \mu\Delta u + \nabla\pi &= f_u, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u &= 0, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu(Du + Du^\top) - \pi I]\!]e_3 &= \sigma\Delta_{x'} h e_3 + [\![\rho]\!]\mathrm{g} h e_3, && \text{on } \Sigma,\\
[\![u]\!] &= 0, && \text{on } \Sigma,\\
\partial_t h &= u_3 - [\![\partial_3\eta]\!] + f_h, && \text{on } \Sigma,\\
(-\nabla_{x'} h, 1)^\top \cdot \nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\eta &= 0, && \text{in } \Omega\backslash\Sigma,\\
[\![\eta]\!] = 0, \quad \eta|_\Sigma - \sigma\Delta_{x'} h - [\![\rho]\!]\mathrm{g} h &= 0, && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Sigma,\\
P_{S_1}\left(\mu(Du + Du^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u\cdot\nu_{S_1} &= 0, && \text{on } S_1\backslash\partial\Sigma,\\
u &= 0, && \text{on } S_2,\\
u(0) &= u_0, && \text{on } \Omega\backslash\Sigma,\\
h(0) &= h_0, && \text{on } \Sigma,
\end{aligned}
\tag{5.9}
$$

where h_0 and f_h are assumed to be mean value free. Note that the system conserves this property during the evolution. This motivates to study problem (5.9) in the following setting, where h is mean value free. Let

$$X_0 := L_{r,\sigma}(\Omega) \times W_q^{1-1/q}(\Sigma) \cap \{(u, h) : (h|1)_{L_2(\Sigma)} = 0, (n_{\partial\Sigma}|\nabla_{x'} h) = 0 \text{ on } \partial\Sigma\},$$

and

$$X_1 := W_r^2(\Omega\backslash\Sigma) \times W_q^{4-1/q}(\Sigma).$$

Here, $L_{r,\sigma}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ functions with vanishing divergence with respect to the $L_r(\Omega)$-norm, as usual. Let $L : D(L) \subset X_1 \to X_0$ be given as

$$L(u, h) := \begin{pmatrix} (\mu^\pm/\rho^\pm)\Delta u - (1/\rho^\pm)\nabla\pi \\ u_3 - [\![\partial_3\eta]\!] \end{pmatrix}, \tag{5.10}$$

where

$$
\begin{aligned}
D(L) := W_r^2(\Omega\backslash\Sigma) \times W_q^{4-1/q}(\Sigma) \cap \{(u, h) &: \operatorname{div} u = 0, [\![u]\!] = 0,\\
&P_\Sigma([\![\mu^\pm(Du + Du^\top)]\!]e_3) = 0, P_{S_1}(\mu^\pm(Du + Du^\top)\nu_{S_1}) = 0,\\
&(u|\nu_{S_1}) = 0, (h|1)_{L_2(\Sigma)} = 0, (n_{\partial\Sigma}|\nabla_{x'} h) = 0\}.
\end{aligned}
$$

Here, $\pi \in \dot{H}_r^1(\Omega\backslash\Sigma)$ and $\eta \in W_q^2(\Omega\backslash\Sigma)$ in (5.10) are determined as the unique solutions of the corresponding weak transmission problem

$$
\begin{aligned}
([1/\rho^\pm]\nabla\pi|\nabla\psi)_{L_2(\Omega)} &= ([\mu^\pm/\rho^\pm]\Delta u|\nabla\psi)_{L_2(\Omega)}, \quad \text{for all } \psi \in W_{r'}^1(\Omega),\\
[\![\pi]\!] &= ([\![\mu^\pm(Du + Du^\top)]\!]e_3|e_3) + \sigma\Delta_{x'} h + [\![\rho]\!]\mathrm{g} h, \quad \text{on } \Sigma,
\end{aligned}
$$

and the solution of the elliptic problem

$$\begin{aligned} \Delta\eta &= 0, && \text{in } \Omega\backslash\Sigma, \\ [\![\eta]\!] = 0, \quad \eta|_\Sigma &= \sigma\Delta_{x'}h + [\![\rho]\!]\mathrm{g}h, && \text{on } \Sigma, \\ (\nu_{\partial\Omega}|\nabla\eta) &= 0, && \text{on } \partial\Omega\backslash\Sigma. \end{aligned}$$

From Lemma A.7 in [**64**] stems also the notation via solution operators,

$$(1/\rho^\pm)\nabla\pi = T_1[(\mu^\pm/\rho^\pm)\Delta u] + T_2[([\![\mu^\pm(Du + Du^\top)]\!]e_3|e_3) + \sigma\Delta_{x'}h + [\![\rho]\!]\mathrm{g}h].$$

We will now analyze the spectrum of the operator L. Note that the domain $D(L)$ compactly embeds into X_0, whence L has a compact resolvent and the spectrum of L only consists of eigenvalues with finite multiplicity.

Again note that for any $(u, h) \in D(L)$, the height function h is mean value free. Let $\lambda \in \mathbb{C}$ and consider the corresponding eigenvalue problem $\lambda(u, h) = L(u, h)$ for $(u, h) \in D(L)$, that is

$$\begin{aligned} \lambda\rho u - \mu\Delta u + \nabla\pi &= 0, && \text{in } \Omega\backslash\Sigma, \\ \operatorname{div} u &= 0, && \text{in } \Omega\backslash\Sigma, \\ -[\![\mu(Du + Du^\top) - \pi I]\!]e_3 &= \sigma\Delta_{x'}he_3 + [\![\rho]\!]\mathrm{g}he_3, && \text{on } \Sigma, \\ [\![u]\!] &= 0, && \text{on } \Sigma, \\ \lambda h - u_3 + [\![\partial_3\eta]\!] &= 0, && \text{on } \Sigma, \\ (-\nabla_{x'}h|\nu_{\partial\Sigma}) &= 0, && \text{on } \partial\Sigma, \\ \Delta\eta &= 0, && \text{in } \Omega\backslash\Sigma, \\ [\![\eta]\!] = 0, \quad \eta|_\Sigma - \sigma\Delta_{x'}h - [\![\rho]\!]\mathrm{g}h &= 0, && \text{on } \Sigma, \\ (\nu_{\partial\Omega}|\nabla\eta) &= 0, && \text{on } \partial\Omega\backslash\Sigma, \\ P_{S_1}\left(\mu(Du + Du^\top)\nu_{S_1}\right) &= 0, && \text{on } S_1\backslash\partial\Sigma, \\ (u|\nu_{S_1}) &= 0, && \text{on } S_1\backslash\partial\Sigma, \\ u &= 0, && \text{on } S_2. \end{aligned} \tag{5.11}$$

We test $(5.11)_1$ with u in $L_2(\Omega)$ to obtain

$$\lambda|\rho^{1/2}u|^2_{L_2(\Omega)} + |\mu^{1/2}(Du + Du^\top)|^2_{L_2(\Omega)} - (\sigma\Delta_{x'}he_3 + [\![\rho]\!]\mathrm{g}he_3|u)_{L_2(\Sigma)} = 0.$$

Equation $(5.11)_5$ entails

$$\lambda|\rho^{1/2}u|^2_{L_2(\Omega)} + |\mu^{1/2}(Du + Du^\top)|^2_{L_2(\Omega)} - (\sigma\Delta_{x'}h + [\![\rho]\!]\mathrm{g}h|\lambda h + [\![\partial_3\eta]\!])_{L_2(\Sigma)} = 0.$$

An integration by parts gives

$$\begin{aligned} &\lambda|\rho^{1/2}u|^2_{L_2(\Omega)} + |\mu^{1/2}(Du + Du^\top)|^2_{L_2(\Omega)} + \\ &+ \bar\lambda\left[\sigma|\nabla_{x'}h|^2_{L_2(\Sigma)} - [\![\rho]\!]\mathrm{g}|h|^2_{L_2(\Sigma)}\right] + |\nabla\eta|^2_{L_2(\Omega)} = 0. \end{aligned} \tag{5.12}$$

Using equality (5.12) we obtain important properties of the spectrum of L. These are stated in the next lemma. It is the crucial step in showing Rayleigh-Taylor instability of the coupled two-phase Navier-Stokes/Mullins-Sekerka system.

Theorem 5.3. *The operator $L : D(L) \subset X_1 \to X_0$ from (5.10) has the following spectral properties:*

(1) $\sigma(L) \cap i\mathbb{R} \subset \{0\}$, *and* $0 \in \sigma(L)$ *if and only if* $[\![\rho]\!]\mathrm{g}/\sigma \in \sigma(-\Delta_N)$.
(2) *If* $[\![\rho]\!] \leq 0$, *then* $\sigma(L) \subset \mathbb{C}_-$.
(3) *If* $[\![\rho]\!] > 0$ *and* $[\![\rho]\!]\mathrm{g}/\sigma < \lambda_1$, *then* $\sigma(L) \subset \mathbb{C}_-$.
(4) *If* $[\![\rho]\!] > 0$ *and* $[\![\rho]\!]\mathrm{g}/\sigma > \lambda_1$, *then* $\sigma(L) \cap \mathbb{C}_+ \neq \emptyset$.

Here, $-\Delta_N$ *denotes the negative Neumann-Laplacian in the space*

$$X := W_q^{2-1/q}(\Sigma) \cap \{h : (h|1)_{L_2(\Sigma)} = 0\}$$

with domain

$$D(-\Delta_N) := W_q^{4-1/q}(\Sigma) \cap X \cap \{h : \ n_{\partial\Sigma} \cdot \nabla_{x'} h = 0 \text{ on } \partial\Sigma\}.$$

We denote by $\sigma(-\Delta_N) \subset \mathbb{R}_+$ *its spectrum, and* $\lambda_1 > 0$ *its first eigenvalue.*

Proof. Let us prove the first statement. Let $\lambda = 0$. Then (5.12) implies that the symmetric gradient is zero and η is constant. By Korn's inequality, $u = 0$ since u vanishes on S_2. Also π is constant in the phases with possibly different values. Since then also $[\![\pi]\!]$ is constant, this entails that h solves the linear elliptic problem

$$\begin{aligned} \Delta_{x'} h + \frac{[\![\rho]\!]\mathrm{g}}{\sigma} h &= 0, && \text{on } \Sigma, \\ (\nabla_{x'} h | n_{\partial\Sigma}) &= 0, && \text{on } \partial\Sigma. \end{aligned}$$

This stems from integrating equation $(5.11)_3$ over Σ. Hence this problem has a nontrivial solution if and only if $[\![\rho]\!]\mathrm{g}/\sigma$ belongs to the spectrum of $-\Delta_N$. This shows that $0 \in \sigma(L)$ if and only if $[\![\rho]\!]\mathrm{g}/\sigma \in \sigma(-\Delta_N)$. Let now $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda = 0$. Then by the same arguments we conclude that u is zero and η is constant. In this case, equation $(5.11)_5$ reads $\lambda h = 0$, whence since h may not be trivial, $\lambda = 0$. This shows the first assersion.

Whenever $[\![\rho]\!] \leq 0$, the expression in brackets in (5.12) satisfies

$$\Big[\sigma|\nabla_{x'} h|^2_{L_2(\Sigma)} - [\![\rho]\!]\mathrm{g}|h|^2_{L_2(\Sigma)}\Big] \geq 0. \tag{5.14}$$

Hence taking real parts in (5.12) yields that $\operatorname{Re}\lambda \leq 0$ and the above arguments show that then $\operatorname{Re}\lambda < 0$, since $\mathrm{g}, \sigma > 0$.

Now let $[\![\rho]\!] > 0$ and $[\![\rho]\!]\mathrm{g}/\sigma < \lambda_1$. Following the lines of [64], the Poincare inequality then yields that

$$|\nabla_{x'} h|^2_{L_2(\Sigma)} - \frac{[\![\rho]\!]\mathrm{g}}{\sigma}|h|^2_{L_2(\Sigma)} \geq 0,$$

since h is mean value free. Note that this inequality renders equation (5.14) to be true, whence also in this case $\sigma(L) \subset \mathbb{C}_-$.

Let us finally consider the most involved case where $[\![\rho]\!] > 0$ and $[\![\rho]\!]\mathrm{g}/\sigma > \lambda_1$. We now aim to show that $\sigma(L) \cap \mathbb{C}_+$ is not empty. Let $3 < r < 7/2$. We will now, for given $\lambda \geq 0$ and given $g \in W_r^{1-1/r}(\Sigma)$, use Theorem A.13 in [64] to solve the linear two-phase Stokes problem

$$
\begin{aligned}
\lambda\rho^{\pm}u-\mu^{\pm}\Delta u+\nabla\pi&=0, && \text{in } \Omega\backslash\Sigma,\\
\operatorname{div} u&=0, && \text{in } \Omega\backslash\Sigma,\\
-[\![\mu^{\pm}(Du+Du^{\top})]\!]e_3+[\![\pi]\!]e_3&=ge_3, && \text{on } \Sigma,\\
[\![u]\!]&=0, && \text{on } \Sigma,\\
P_{S_1}(\mu^{\pm}(Du+Du^{\top})\nu_{S_1})&=0, && \text{on } S_1\backslash\partial\Sigma,\\
(u|\nu_{S_1})&=0, && \text{on } S_1\backslash\partial\Sigma,\\
u&=0 && \text{on } S_2,
\end{aligned}
\tag{5.15}
$$

uniquely by some $u\in W_r^2(\Omega\backslash\Sigma)\cap W_r^1(\Omega)$. Define the corresponding Neumann-to-Dirichlet operator $N_\lambda^S: W_r^{1-1/r}(\Sigma)\to W_r^{2-1/r}(\Sigma)$ of the Stokes problem (5.15) by $N_\lambda^S g:=(u|e_3)$. Regarding N_λ^S, Proposition 4.1 in [**64**] gives the following properties.

LEMMA 5.4. *Let $3<r<7/2$. The Neumann-to-Dirichlet operator N_λ^S of the Stokes problem (5.15) admits a compact, self-adjoint extension to $L_2(\Sigma)$ which has the following properties.*

(1) *If u denotes the solution of (5.15), then*

$$(N_\lambda^S g|g)_{L_2(\Sigma)}=\lambda|(\rho^{\pm})^{1/2}u|^2_{L_2(\Omega)}+\frac{1}{2}|(\mu^{\pm})^{1/2}(Du+Du^{\top})|^2_{L_2(\Omega)}$$

for all $g\in W_r^{1-1/r}(\Sigma)$ and $\lambda\geq 0$.

(2) *For each $\alpha\in(0,1/2)$, there exists $C_\alpha>0$, such that*

$$(N_\lambda^S g|g)_{L_2(\Sigma)}\geq\frac{(1+\lambda)^\alpha}{C_\alpha}|N_\lambda^S g|^2_{L_2(\Sigma)},$$

for all $g\in L_2(\Sigma)$ and $\lambda\geq 0$. In particular,

$$|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}\leq\frac{C_\alpha}{(1+\lambda)^\alpha}$$

for all $\lambda\geq 0$.

(3) *$N_\lambda^S g$ has mean value zero for all $g\in L_2(\Sigma)$, $\lambda\geq 0$.*

Let us be concerned with the elliptic problem of the Mullins-Sekerka equations. For given $g\in W_q^{2-1/q}(\Sigma)$, $5/3<q<2$, we may solve the two-phase elliptic problem

$$
\begin{aligned}
\Delta\eta&=0, && \text{in } \Omega\backslash\Sigma,\\
[\![\eta]\!]=0,\quad \eta|_\Sigma&=g, && \text{on } \Sigma,\\
(n_{\partial\Omega}|\nabla\eta)&=0, && \text{on } \partial\Omega,
\end{aligned}
$$

uniquely by a function $\eta\in W_q^2(\Omega\backslash\Sigma)$, cf. Appendix A, and define the corresponding Dirichlet-to-Neumann operator

$$D_{MS}: W_q^{2-1/q}(\Sigma)\to W_q^{1-1/q}(\Sigma)$$

by $D_{MS}g:=[\![\partial_3\eta]\!]$. The eigenvalue problem (5.11) can then, for $\lambda\geq 0$, equivalently be written as

$$\lambda h+N_\lambda^S(A_*h)+D_{MS}(A_*h)=0, \tag{5.16}$$

where $A_*h := -\sigma\Delta_N h - [\![\rho]\!]\mathrm{g}h$, equipped with domain

$$D(A_*) := W_q^{4-1/q}(\Sigma) \cap \{h : (h|1)_{L_2(\Sigma)} = 0, (n_{\partial\Sigma}|\nabla_{x'}h) = 0\}. \tag{5.17}$$

Note at this point that also A_*h is indeed mean value free for every $h \in D(A_*)$. We now want to invert $N_\lambda^S + D_{MS}$ and apply the inverse operator to equation (5.16). To this end, we need properties of the corresponding Neumann-to-Dirichlet operator $N_{MS} = [D_{MS}]^{-1}$, which is given as follows.

For given g we solve

$$\begin{aligned} \Delta\vartheta &= 0, && \text{in } \Omega\backslash\Sigma, \\ [\![\vartheta]\!] = 0, \quad [\![\partial_3\vartheta]\!] &= g, && \text{on } \Sigma, \\ (n_{\partial\Omega}|\nabla\vartheta) &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{5.18}$$

and define $N_{MS}g := \vartheta|_\Sigma$. Proposition 10.5.1 in [**57**] gives that whenever $g \in W_q^{1-1/q}(\Sigma)$ is mean value free, problem (5.18) is solvable and the solution is unique up to a constant. Hence if we restrict the problem to mean-value free functions, the operator $N_{MS} : W_{q,(0)}^{1-1/q}(\Sigma) \to W_{q,(0)}^{2-1/q}(\Sigma)$ is well defined. Here, $W_{q,(0)}^s(\Sigma) := W_q^s(\Sigma) \cap \{u : (u|1)_{L_2(\Sigma)} = 0\}$, $s > 0$, and $L_{2,(0)}(\Sigma) := L_2(\Sigma) \cap \{u : (u|1)_{L_2(\Sigma)} = 0\}$. Regarding N_{MS}, we now have the following result.

LEMMA 5.5. *The Neumann-to-Dirichlet operator*

$$N_{MS} : W_{q,(0)}^{1-1/q}(\Sigma) \to W_{q,(0)}^{2-1/q}(\Sigma)$$

of problem (5.18) admits a compact, selfadjoint extension to $L_{2,(0)}(\Sigma)$, which has the following properties.

(1) *For $g \in W_{q,(0)}^{1-1/q}(\Sigma)$ and $\vartheta \in W_{q,(0)}^2(\Omega\backslash\Sigma)$ the corresponding unique mean-value free solution of (5.18), we have*

$$(N_{MS}g|g)_{L_2(\Sigma)} = |\nabla\vartheta|_{L_2(\Omega)}^2. \tag{5.19}$$

(2) *The extension to $L_{2,(0)}(\Sigma)$ satisfies (5.19) as well, where now $g \in L_{2,(0)}(\Sigma)$ and $\vartheta \in W_{q,(0)}^1(\Omega)$ is the unique weak solution of (5.18).*

(3) *There exists a constant $c_0 > 0$, such that*

$$(N_{MS}g|g)_{L_2(\Sigma)} \geq c_0|N_{MS}g|_{L_2(\Sigma)}^2$$

for all $g \in W_{q,(0)}^{1-1/q}(\Sigma)$ and the extension satisfies

$$|N_{MS}|_{\mathcal{B}(L_{2,(0)}(\Sigma);L_2(\Sigma))} \leq C.$$

(4) *The extension of N_{MS} to $L_{2,(0)}(\Sigma)$ is injective.*

Let us postpone the proof of Lemma 5.5 to a later point. Having these results at hand, we can proceed as in [**64**] and [**57**]. We know that both N_λ^S and N_{MS} are selfadjoint and positive semi-definite on $L_{2,(0)}(\Sigma)$, hence $T_\lambda := [N_\lambda^S + D_{MS}]^{-1}$ is selfadjoint and positive semi-definite as well. Furthermore, $N_\lambda^S + D_{MS}$ is injective in the space of mean-value free functions. Indeed, whenever $(N_\lambda^S + D_{MS})g = 0$, by the above results the corresponding solutions are $u = 0$ and $\eta = const.$, whence g

has to be constant as well, hence zero. In particular, regarding equation (5.16), we may rewrite the eigenvalue problem as

$$\lambda T_\lambda h + A_* h = 0. \tag{5.20}$$

We now want to show there is a nontrivial solution to (5.20). To this end let $B_\lambda := \lambda T_\lambda + A_*$ with domain

$$D(B_\lambda) := W_2^2(\Sigma) \cap \{h : (h|1)_{L_2(\Sigma)} = 0, (n_{\partial\Sigma}|\nabla_{x'} h) = 0\}.$$

We now want to show that for $\lambda > 0$ sufficiently small, there is some $h_* \in D(B_\lambda)$ such that $(B_\lambda h_*|h_*)_{L_2(\Sigma)} < 0$.

To this end let e be an eigenfunction to the first nontrivial eigenvalue of $-\Delta_N$ in X, that is, $-\Delta_N \mathsf{e} = \lambda_1 \mathsf{e}$. Define $v_\lambda := T_\lambda \mathsf{e}$. Then, since e is mean value free,

$$\mathsf{e} = T_\lambda^{-1} v_\lambda = N_\lambda^S v_\lambda + D_{MS} v_\lambda = N_\lambda^S v_\lambda + [N_{MS}]^{-1} v_\lambda. \tag{5.21}$$

Let $P_0 : L_2(\Sigma) \to L_{2,(0)}(\Sigma)$ be the orthogonal projection onto the mean value free functions, $P_0 f := f - (f|1)_{L_2(\Sigma)}/|\Sigma|$. Applying P_0 to (5.21) yields

$$\mathsf{e} = P_0 \mathsf{e} = P_0 N_\lambda^S v_\lambda + P_0 [N_{MS}]^{-1} v_\lambda = P_0 N_\lambda^S v_\lambda + [N_{MS}]^{-1} v_\lambda,$$

since $[N_{MS}]^{-1} v_\lambda$ is mean value free. Hence

$$N_{MS} \mathsf{e} = N_{MS} P_0 N_\lambda^S v_\lambda + v_\lambda.$$

Testing with $N_\lambda^S v_\lambda$ yields

$$(N_{MS} \mathsf{e}|N_\lambda^S v_\lambda) = (N_{MS} P_0 N_\lambda^S v_\lambda | P_0 N_\lambda^S v_\lambda) + (v_\lambda | N_\lambda^S v_\lambda),$$

since $N_{MS} P_0 N_\lambda^S v_\lambda$ is mean value free. Invoking Lemma 5.4 and Lemma 5.5,

$$c_0 |N_{MS} P_0 N_\lambda^S v_\lambda|^2_{L_2(\Sigma)} + \frac{(1+\lambda)^{1/4}}{C} |N_\lambda^S v_\lambda|^2_{L_2(\Sigma)} \le (v_\lambda + N_{MS} P_0 N_\lambda^S v_\lambda | N_\lambda^S v_\lambda)_{L_2(\Sigma)},$$

for some $c_0 > 0$. In particular, there is a constant $C > 0$ independent of λ, such that

$$|N_\lambda^S v_\lambda|^2_{L_2(\Sigma)} \le C (v_\lambda + N_{MS} P_0 N_\lambda^S v_\lambda | N_\lambda^S v_\lambda)_{L_2(\Sigma)}, \quad \lambda > 0.$$

Let us estimate the right hand side. We have

$$(v_\lambda + N_{MS} P_0 N_\lambda^S v_\lambda | N_\lambda^S v_\lambda)_{L_2(\Sigma)} = (N_{MS} \mathsf{e} | N_\lambda^S v_\lambda)_{L_2(\Sigma)} = (N_{MS} \mathsf{e} | P_0 N_\lambda^S v_\lambda)_{L_2(\Sigma)},$$

since $N_{MS}\mathsf{e}$ is mean value free. Since N_{MS} is selfadjoint, the right hand side is equal to $(\mathsf{e}|N_{MS} P_0 N_\lambda^S v_\lambda)_{L_2(\Sigma)}$. Hence we obtain

$$(v_\lambda + N_{MS} P_0 N_\lambda^S v_\lambda | N_\lambda^S v_\lambda)_{L_2(\Sigma)} \le |\mathsf{e}|_{L_2(\Sigma)} |N_\lambda^S v_\lambda|_{L_2(\Sigma)}.$$

This shows that $|N_\lambda^S v_\lambda|^2_{L_2(\Sigma)}$ is bounded as $\lambda \to 0$. This implies

$$\lim_{\lambda \to 0} \lambda T_\lambda \mathsf{e} = \lim_{\lambda \to 0} \lambda v_\lambda = \lim_{\lambda \to 0} \lambda [N_{MS} \mathsf{e} - N_{MS} P_0 N_\lambda^S v_\lambda] = 0.$$

Therefore, because of $B_\lambda = \lambda T_\lambda + A_*$,

$$\lim_{\lambda \to 0} (B_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Sigma)} = (A_* \mathsf{e}|\mathsf{e})_{L_2(\Sigma)}.$$

By choice of e, $-\Delta_N \mathsf{e} = \lambda_1 \mathsf{e}$, whence

$$(A_* \mathsf{e}|\mathsf{e})_{L_2(\Sigma)} = \sigma \left(\lambda_1 - \frac{[\![\rho]\!] \mathsf{g}}{\sigma} \right) |\mathsf{e}|_{L_2(\Sigma)} < 0.$$

This shows that

$$\lim_{\lambda\to 0}(B_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Sigma)} < 0.$$

We are now interested in the behaviour of $(B_\lambda h|h)_{L_2(\Sigma)}$ as $\lambda \to \infty$. Recall that $T_\lambda = [N_S^\lambda + D_{MS}]^{-1}$. We can write

$$T_\lambda = (I + N_{MS}N_\lambda^S)^{-1}N_{MS} = N_{MS} - N_{MS}N_\lambda^S(I + N_{MS}N_\lambda^S)^{-1}N_{MS},$$

since $(I + N_{MS}N_\lambda^S)^{-1} = I - N_{MS}N_\lambda^S(I + N_{MS}N_\lambda^S)^{-1}$. Hence, since N_{MS} is self-adjoint,

$$\begin{aligned}(T_\lambda g|g)_{L_2(\Sigma)} &= (N_{MS}g|g)_{L_2(\Sigma)} - (N_{MS}N_\lambda^S(I + N_{MS}N_\lambda^S)^{-1}N_{MS}g|g)_{L_2(\Sigma)}\\ &= (N_{MS}g|g)_{L_2(\Sigma)} - (N_\lambda^S(I + N_{MS}N_\lambda^S)^{-1}N_{MS}g|N_{MS}g)_{L_2(\Sigma)}\end{aligned}$$

Furthermore, $(N_\lambda^S(I + N_{MS}N_\lambda^S)^{-1}N_{MS}g|N_{MS}g)_{L_2(\Sigma)}$ can be bounded by

$$|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}|(I + N_{MS}N_\lambda^S)^{-1}|_{\mathcal{B}(L_{2,(0)}(\Sigma);L_2(\Sigma))}|N_{MS}g|^2_{L_2(\Sigma)},$$

which itself is bounded by

$$\frac{|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}}{1 - |N_{MS}|_{\mathcal{B}(L_{2,(0)}(\Sigma);L_2(\Sigma))}|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}}|N_{MS}g|^2_{L_2(\Sigma)},$$

provided $|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}$ is small enough. Altogether,

$$\begin{aligned}&(T_\lambda g|g)_{L_2(\Sigma)} \geq\\ &\geq (N_{MS}g|g)_{L_2(\Sigma)} - \frac{|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}}{1 - |N_{MS}|_{\mathcal{B}(L_{2,(0)}(\Sigma);L_2(\Sigma))}|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}}|N_{MS}g|^2_{L_2(\Sigma)}.\end{aligned}$$

Lemma 5.5 renders there is some constant $c_0 > 0$, such that

$$\begin{aligned}&(T_\lambda g|g)_{L_2(\Sigma)} \geq\\ &\geq (N_{MS}g|g)_{L_2(\Sigma)} - \frac{1}{c_0}\frac{|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}}{1 - |N_{MS}|_{\mathcal{B}(L_{2,(0)}(\Sigma);L_2(\Sigma))}|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}}(N_{MS}g|g)_{L_2(\Sigma)},\end{aligned}$$

and Lemma 5.4 gives that $|N_\lambda^S|_{\mathcal{B}(L_2(\Sigma);L_2(\Sigma))}$ tends to zero as $\lambda \to \infty$. In particular, there exists some $\lambda_+ > 0$, such that

$$(T_\lambda g|g)_{L_2(\Sigma)} \geq \frac{1}{2}(N_{MS}g|g)_{L_2(\Sigma)}, \quad \lambda \geq \lambda_+.$$

Since N_{MS} is self-adjoint, positive semidefinite and invertible, N_{MS} is positive definite, cf. Proposition 1.3.6 in [**45**]. Hence there is a positive $d_0 > 0$ such that $(N_{MS}g|g)_{L_2(\Sigma)} \geq d_0|g|^2_{L_2(\Sigma)}$ for all $g \in L_{2,(0)}(\Sigma)$. Clearly,

$$\begin{aligned}(A_* g|g)_{L_2(\Sigma)} &= (-\sigma\Delta_N g|g)_{L_2(\Sigma)} - [\![\rho]\!]\mathsf{g}|g|^2_{L_2(\Sigma)}\\ &= \sigma|\nabla_{x'}g|^2_{L_2(\Sigma)} - [\![\rho]\!]\mathsf{g}|g|^2_{L_2(\Sigma)}\\ &\geq -[\![\rho]\!]\mathsf{g}|g|^2_{L_2(\Sigma)},\end{aligned}$$

for all $g \in W^2_{2,(0)}(\Sigma)$, whence

$$\begin{aligned}(B_\lambda g|g)_{L_2(\Sigma)} &\geq \frac{\lambda}{2}(N_{MS}g|g)_{L_2(\Sigma)} + (A_* g|g)_{L_2(\Sigma)}, && \lambda \geq \lambda_+,\\ &\geq \left[\frac{\lambda}{2}d_0 - [\![\rho]\!]\mathrm{g}\right]|g|^2_{L_2(\Sigma)}, && \lambda \geq \lambda_+. \end{aligned} \tag{5.22}$$

In particular, if $\lambda > 0$ is large enough, B_λ is positive definite. Define

$$\lambda_* := \sup\{\lambda > 0 : B_\mu \text{ is not positive semi-definite for each } \mu \in (0,\lambda]\}. \tag{5.23}$$

By what we have shown above, $0 < \lambda_* < \infty$ and B_λ has a negative eigenvalue for each $\lambda < \lambda_*$, again since the resolvent of B_λ is compact. It follows that since the eigenvalue has to cross the imaginary axis, $0 \in \sigma(B_{\lambda_*})$, cf. [**64**]. Hence there exists a nontrivial solution $h \in D(B_{\lambda_*})$ to $B_{\lambda_*}h = 0$ in $L_{2,(0)}(\Sigma)$. In other words,

$$\lambda_* T_{\lambda_*} h + A_* h = 0.$$

Hence $A_* h \in W^{2-1/q}_q(\Sigma)$. By regularity theory, $h \in W^{4-1/q}_q(\Sigma)$ and satisfies the other conditions in (5.17). We have shown that $\sigma(L) \cap \mathbb{C}_+ \neq \emptyset$ and the proof is complete. □

We close this section with the proof of Lemma 5.5.

Proof of Lemma 5.5. Let us for given $g \in W^{1-1/q}_{q,(0)}(\Sigma)$ solve problem (5.18) by $\vartheta \in W^2_{q,(0)}(\Omega\backslash\Sigma)$. We then obtain

$$(N_{MS}g|g)_{L_2(\Sigma)} = \int_\Sigma \vartheta|_\Sigma [\![\partial_3\vartheta]\!]dx' = |\nabla\vartheta|^2_{L_2(\Omega)}.$$

Since $\int_\Omega \vartheta dx = 0$, we obtain by Poincaré-Wirtinger inequality and a standard argument that

$$|\operatorname{tr}_\Sigma \vartheta|^2_{L_2(\Sigma)} \leq C|\nabla\vartheta|^2_{L_2(\Omega)} \tag{5.24}$$

for some $C > 0$. Hereby we used Sobolev embeddings and $[\![\vartheta]\!] = 0$ to show that $\vartheta \in H^1_2(\Omega)$. In particular, there is some $C > 0$ such that

$$|\vartheta|_{H^1_2(\Omega)} \leq C|\vartheta|_{W^2_q(\Omega\backslash\Sigma)}.$$

Interpolating estimates then gives (5.24). Consequently,

$$(N_{MS}g|g)_{L_2(\Sigma)} \geq \frac{1}{C}|N_{MS}g|^2_{L_2(\Sigma)} \tag{5.25}$$

for all $g \in W^{1-1/q}_{q,(0)}(\Sigma)$ for some $C > 0$ independent of g. Since $W^{1-1/q}_{q,(0)}(\Sigma)$ is dense in $L_{2,(0)}(\Sigma)$ we may extend N_{MS} to all of $L_{2,(0)}(\Sigma)$. Note that (5.25) implies

$$|N_{MS}g|_{L_2(\Sigma)} \leq C|g|_{L_2(\Sigma)}, \quad g \in L_{2,(0)}(\Sigma).$$

It remains to characterize the weak limit. To this end let $g \in L_{2,(0)}(\Sigma)$ and $(g_n)_n \subset W^{1-1/q}_{q,(0)}(\Sigma)$ such that $g_n \to g$ in $L_{2,(0)}(\Sigma)$. Let ϑ_n be the corresponding solution of (5.18) with right hand side g_n. An integration by parts in the two phases then yields that

$$\int_\Sigma g_n \varphi|_\Sigma = \int_{\Omega\backslash\Sigma} \nabla\vartheta_n \cdot \nabla\varphi, \quad \varphi \in C_0^\infty(\Omega), n \in \mathbb{N}. \tag{5.26}$$

By definition of the extension, $N_{MS}g = \lim_{n\to\infty} N_{MS}g_n$. This way,

$$|\nabla\vartheta_n|^2_{L_2(\Omega)} = (N_{MS}g_n|g_n)_{L_2(\Sigma)} \to_{n\to\infty} (N_{MS}g|g)_{L_2(\Sigma)}.$$

Therefore, $(\nabla\vartheta_n)_n$ is bounded in $L_2(\Omega)$. By Poincare-Wirtinger inequality, $(\vartheta_n)_n$ is bounded in $H^1_2(\Omega)$. Hence, for a subsequence, $\vartheta_n \rightharpoonup \tilde{\vartheta}$ in $H^1_2(\Omega)$ for some $\tilde{\vartheta}$. Passing to the limit in (5.26) yields that

$$\int_\Sigma g\varphi|_\Sigma = \int_{\Omega\backslash\Sigma} \nabla\tilde{\vartheta}\cdot\nabla\varphi, \quad \varphi \in C_0^\infty(\Omega),$$

whence $\tilde{\vartheta}$ is the unique weak solution of (5.18) with given right hand side g. The fact that $N_{MS}: W^{1-1/q}_{q,(0)}(\Sigma) \to W^{2-1/q}_{q,(0)}(\Sigma)$ is injective is obvious: If for some g we have $N_{MS}g = 0$, by (5.19) we know that ϑ is constant, where ϑ is the unique solution of (5.18). Hence $g = 0$ by (5.18). $\square$

5.4. Thermodynamically consistent Mullins-Sekerka equations with gravity

We will now formulate a Mullins-Sekerka problem with gravity which is itself thermodynamically consistent. For a related model of Cahn-Hilliard type with gravity see Section 3.6 in [18]. Again let $\Omega = \Sigma \times (L_1, L_2) \subset \mathbb{R}^n$, $n = 2, 3$, be a cylinder, where $-\infty < L_1 < 0 < L_2 < \infty$ and $\Sigma \subset \mathbb{R}^{n-1}$ is bounded with smooth boundary. Again we assume that $\rho^\pm > 0, \mathsf{g} > 0$ are constant and $\rho := \rho^+\chi_{\Omega^+(t)} + \rho^-\chi_{\Omega^-(t)}$, where χ is the indicator function. Let $S_1 := \partial\Sigma \times (L_1, L_2)$. We consider now a pure two-phase Mullins-Sekerka problem with ninety degree angle condition

$$\begin{aligned}
V_\Gamma &= -[\![\nabla\eta]\!]\cdot\nu_\Gamma, && \text{on } \Gamma(t),\\
\nu_\Gamma\cdot\nu_{S_1} &= 0, && \text{on } \partial\Gamma(t),\\
\Delta\eta &= 0, && \text{in } \Omega\backslash\Gamma(t),\\
\eta|_\Gamma &= \sigma H_\Gamma + [\![\rho]\!]\mathsf{g}x_n, && \text{in } \Omega\backslash\Gamma(t),\\
\nu_\Omega\cdot\nabla\eta &= 0, && \text{on } \partial\Omega\backslash\Gamma(t),\\
\Gamma(0) &= \Gamma_0.
\end{aligned} \tag{5.27}$$

Again we are interested in physical relevance, well-posedness, and stability properties.

5.4.1. Energy-dissipation equality. A similar calculation as in the proof of Lemma 5.1 using transport identities shows that if (Γ, η) is a sufficiently smooth solution to (5.27),

$$\frac{d}{dt}\left[\int_{\Gamma(t)} \sigma\, d\mathcal{H}^{n-1} + \int_\Omega \rho\mathsf{g}x_n\, dx\right] = -\int_\Omega |\nabla\eta|^2 dx \le 0. \tag{5.28}$$

Note that if $\mathsf{g} = 0$, this reduces to the classical estimate. Note that since $\Delta\eta = 0$ in the bulk phases $\Omega\backslash\Gamma(t)$, phases are conserved also for $\mathsf{g} > 0$,

$$\frac{d}{dt}|\Omega^\pm(t)| = 0, \quad t \in \mathbb{R}_+.$$

5.4.2. Well-Posedness. Assuming that the free interface $\Gamma(t)$ is a graph of a function h over Σ, we can transform the system to $\Omega\backslash\Sigma$ by means of Hanzawa transform,

$$\begin{aligned}
\partial_t h + [\![\partial_3\eta]\!] &= G_\Sigma(h,\eta), && \text{on } \Sigma,\\
(-\nabla_{x'}h,0)^\top\cdot\nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\eta &= G_c(h,\eta), && \text{in } \Omega\backslash\Sigma,\\
\eta|_\Sigma - \sigma\Delta_{x'}h - [\![\rho]\!]\mathsf{g}h &= G_\kappa(h), && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= G_N(h,\eta), && \text{on } \partial\Omega\backslash\Sigma,\\
h(0) &= h_0, && \text{on } \Sigma,
\end{aligned}\tag{5.29}$$

where G_Σ, G_c, G_κ and G_N are nonlinearities, similarly as before, cf. also Section 2.6. Note that well-posedness of (5.29) in the case $\mathsf{g}=0$ with a ninety degree angle condition was already proven in Section 2.6. Since again the extra term only induces a compact perturbation, we obtain well-posedness of (5.29) by a perturbation argument.

5.4.3. Stability properties. Let us discuss the equilibria of (5.27). Again we note the potential η may be reconstructed by solving a two-phase elliptic problem. By the energy-dissipation equailty (5.28) we easily see that any stationary solution (Γ,η) necessarily satisfies $\eta = const$. This implies that for (Γ,η),

$$\sigma H_\Gamma + [\![\rho]\!]\mathsf{g}x_n = const.$$

We readily see that flat interfaces belong to the set of equilibria.

Again we want to investigate properties of the trivial equilibrium $(0,\Sigma)$. Consider the linear problem

$$\begin{aligned}
\partial_t h &= -[\![\partial_3\eta]\!] + f_h, && \text{on } \Sigma,\\
(-\nabla_{x'}h,1)^\top\cdot\nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\
\Delta\eta &= 0, && \text{in } \Omega\backslash\Sigma,\\
\eta|_\Sigma &= \sigma\Delta_{x'}h + [\![\rho]\!]\mathsf{g}h, && \text{on } \Sigma,\\
\nu_{\partial\Omega}\cdot\nabla\eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Sigma,\\
h(0) &= h_0, && \text{on } \Sigma.
\end{aligned}\tag{5.30}$$

Suppose f_h and h_0 are mean value free. Then $\int_\Sigma h(t)dx = 0$ for all times $t\in\mathbb{R}_+$. Define

$$X_0 := W_q^{1-1/q}(\Sigma)\cap\{h : (h|1)_{L_2(\Sigma)} = 0\},\quad X_1 := W_q^{4-1/q}(\Sigma),$$

and $L : D(L)\subset X_1 \to X_0$ by $Lh := -[\![\partial_3\eta]\!]$ with natural domain

$$D(L) := X_1\cap\{h : (h|1)_{L_2(\Sigma)} = 0,\ (-\nabla_{x'}h,1)^\top\cdot\nu_{S_1} = 0 \text{ on } \partial\Sigma\}.\tag{5.31}$$

Clearly, for given $h\in D(L)$, $\eta\in W_q^2(\Omega\backslash\Sigma)$ is given as the unique solution of the corresponding linear two-phase elliptic problem $(5.30)_3 - (5.30)_5$.

Again since $D(L)$ embeds compactly in X_0, the resolvent of L is compact and hence the spectrum of L only consists of isolated eigenvalues with finite multiplicity. We now want to analyze the spectrum of L. To this end we consider the eigenvalue problem $\lambda h = Lh$ for some $\lambda \in \mathbb{C}$, $h \in D(L)$, that is, h is mean value free and satisfies

$$\begin{aligned} \lambda h &= -[\![\partial_3 \eta]\!], && \text{on } \Sigma,\\ (-\nabla_{x'} h, 0)^\top \cdot \nu_{S_1} &= 0, && \text{on } \partial\Sigma,\\ \Delta \eta &= 0, && \text{in } \Omega\backslash\Sigma,\\ \eta|_\Sigma &= \sigma \Delta_{x'} h + [\![\rho]\!] \mathrm{g} h, && \text{on } \Sigma,\\ \nu_{\partial\Omega} \cdot \nabla \eta|_{\partial\Omega} &= 0, && \text{on } \partial\Omega\backslash\Sigma. \end{aligned} \tag{5.32}$$

Testing the first equation $(5.32)_1$ with $\eta|_\Sigma$ in $L_2(\Sigma)$ yields

$$\lambda \left[\sigma |\nabla_{x'} h|^2_{L_2(\Sigma)} - [\![\rho]\!] \mathrm{g} |h|^2_{L_2(\Sigma)}\right] + |\nabla \eta|^2_{L_2(\Omega)} = 0. \tag{5.33}$$

Assume that $\lambda = 0$ for a moment. Then η is constant, whence h has to solve the elliptic problem

$$\Delta_{x'} h + \frac{[\![\rho]\!]\mathrm{g}}{\sigma} h = 0, \text{ on } \Sigma, \quad (n_{\partial\Sigma} | \nabla_{x'} h) = 0, \text{ on } \partial\Sigma. \tag{5.34}$$

Let again $-\Delta_N$ denote the negative Neumann-Laplacian in

$$X := W^{2-1/q}_q(\Sigma) \cap \{h : (h|1)_{L_2(\Sigma)} = 0\}$$

with domain

$$D(-\Delta_N) := W^{4-1/q}_q(\Sigma) \cap X \cap \{h : n_{\partial\Sigma} \cdot \nabla_{x'} h = 0 \text{ on } \partial\Sigma\},$$

as in Lemma 5.3. Then clearly h is a nontrivial solution to (5.34) if and only if $[\![\rho]\!]\mathrm{g}/\sigma$ belongs to the spectrum of $-\Delta_N$.

By (5.33), λ is necessary real. Hence $\sigma(L) \cap i\mathbb{R} \subset \{0\}$. Following the lines of the proof of Lemma 5.3, we readily obtain that $\sigma(L) \subset \mathbb{C}_-$ whenever $[\![\rho]\!]\mathrm{g}/\sigma < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta_N$ in X. Note that this includes the case where $[\![\rho]\!] \leq 0$. In particular in these cases, since 0 is not an eigenvalue and L has compact resolvent, we have a spectral gap, $\sigma(L) \subset \{\mathrm{Re}\, z \leq -\kappa < 0\}$ for some $\kappa > 0$. We now aim to show that $\sigma(L) \cap \mathbb{C}_+$ is not empty if $[\![\rho]\!]\mathrm{g}/\sigma > \lambda_1$.

Consider the eigenvalue problem $\lambda h = Lh$ in X_0 for $h \in D(L), \lambda \geq 0$. Define $A_* h := -\sigma \Delta_N h - [\![\rho]\!]\mathrm{g} h$, where $D(A_*) := D(L)$. Let T_0 be the solution operator to the elliptic two-phase problem and $D_{MS} g := [\![\partial_3 T_0 g]\!]$ the corresponding Dirichlet-to-Neumann operator. Then the eigenvalue problem can be written as

$$\lambda h + D_{MS} A_* h = 0. \tag{5.35}$$

By applying the Neumann-to-Dirichlet operator N_{MS}, cf. (5.18), the equation (5.35) can be rewritten as

$$\lambda N_{MS} h + A_* h = 0. \tag{5.36}$$

Define $B_\lambda := \lambda N_{MS} + A_*$ with natural domain $D(B_\lambda) := W^2_2(\Sigma) \cap \{h : (h|1) = 0,\ (n_{\partial\Sigma}|\nabla h) = 0\}$. Note that A_* by perturbation of $-\Delta_N$ and N_{MS} by Lemma 5.5 both admit suitable extensions to $L_{2,(0)}(\Sigma)$. Then, for $\lambda > 0$ sufficiently large, B_λ is positive definite. Indeed,

$$(B_\lambda h|h)_\Sigma = \lambda(N_{MS} h|h) + (A_* h|h) \geq \lambda d_0 |h|^2_{L_2(\Sigma)} + \sigma |\nabla_{x'} h|^2_{L_2(\Sigma)} - [\![\rho]\!]\mathrm{g}|h|^2_{L_2(\Sigma)},$$

for all $h \in \{h \in H^2(\Sigma) : (h|1)_2 = 0,\ n_{\partial\Sigma} \cdot \nabla_{x'} h = 0 \text{ on } \partial\Sigma\}$, where $d_0 > 0$ is as in (5.22). On the other hand, let $0 \neq h_* \in D(A_*)$ be a nontrivial eigenfunction of $-\Delta_N$ to the eigenvalue $\lambda_1 > 0$. Then

$$(B_\lambda h_*|h_*)_\Sigma = \lambda(N_{MS} h_*|h_*)_\Sigma + (A_* h_*|h_*)_\Sigma.$$

Taking limits as $\lambda \to 0$, $\lim_{\lambda\to 0}(B_\lambda h_*|h_*)_\Sigma = (A_* h_*|h_*)_{L_2(\Sigma)}$. By choice of h_*, we readily get

$$(A_* h_*|h_*)_{L_2(\Sigma)} = \sigma \left[\lambda_1 - \frac{[\![\rho]\!]\mathsf{g}}{\sigma}\right] |h_*|^2_{L_2(\Sigma)},$$

whence $(B_\lambda h_*|h_*)_\Sigma < 0$, provided $\lambda > 0$ is sufficiently small. Defining $\lambda_* > 0$ as in (5.23), the same arguments yield $0 \in \sigma(B_{\lambda_*})$, hence there is a nontrivial $h \in D(B_{\lambda_*})$ solving (5.36) in $L_{2,(0)}(\Sigma)$ and the same bootstrap argument as in the proof of Lemma 5.3 finally yields $h \in D(L)$.

We have shown the following result.

THEOREM 5.6. *Let $-\Delta_N$ and $\lambda_1 > 0$ be as in Lemma 5.3. The operator $L : D(L) \subset X_1 \to X_0$ from (5.31) has the following spectral properties.*

(1) $\sigma(L) \cap i\mathbb{R} \subset \{0\}$, *and* $0 \in \sigma(L)$ *if and only if* $[\![\rho]\!]\mathsf{g}/\sigma \in \sigma(-\Delta_N)$.
(2) *If* $[\![\rho]\!] \leq 0$, *or* $[\![\rho]\!] > 0$ *and* $[\![\rho]\!]\mathsf{g}/\sigma < \lambda_1$, *then* $\sigma(L) \subset \mathbb{C}_-$.
(3) *If* $[\![\rho]\!] > 0$ *and* $[\![\rho]\!]\mathsf{g}/\sigma > \lambda_1$, *then* $\sigma(L) \cap \mathbb{C}_+ \neq \emptyset$.

APPENDIX A

Auxiliary problems of elliptic type

A.1. Smooth domains

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with smooth boundary $\partial\Omega$. Furthermore let Σ be a smooth submanifold of $\mathbb{R}^n$ with boundary such that the interior $\mathring{\Sigma}$ is inside Ω and meets $\partial\Omega$ at a constant ninety degree angle.

In this chapter we are concerned with problems of elliptic type, namely,

$$\begin{aligned} (\eta - \Delta)u &= f, && \text{in } \Omega\backslash\Sigma, \\ u|_\Sigma &= g_1, && \text{on } \Sigma, \\ n_{\partial\Omega} \cdot \nabla u|_{\partial\Omega} &= g_2, && \text{on } \partial\Omega, \end{aligned} \tag{A.1}$$

where $\eta > 0$ is a fixed shift parameter, as well as the non-shifted version,

$$\begin{aligned} -\Delta u &= f, && \text{in } \Omega\backslash\Sigma, \\ u|_\Sigma &= g_1, && \text{on } \Sigma, \\ n_{\partial\Omega} \cdot \nabla u|_{\partial\Omega} &= g_2, && \text{on } \partial\Omega, \end{aligned} \tag{A.2}$$

where in both cases f, g_1 and g_2 are given data. We will show optimal solvability of this problem via a localization method. To this end we consider first the model problem of (A.3) on $\mathbb{R}^n_+$ with flat interface $\{x_n > 0, x_1 = 0\}$.

A.1.1. Flat interfaces.

THEOREM A.1. *Let $q \in (3/2, 2)$ and $\Sigma := \{x_n > 0, x_1 = 0\}$ be a flat interface. Then there is some $\eta_0 > 0$, such that for every $\eta \geq \eta_0$, $f \in L_q(\mathbb{R}^n_+)$, $g_1 \in W^{2-1/q}_q(\Sigma)$, and $g_2 \in W^{1-1/q}_q(\partial\mathbb{R}^n_+)$, there exists a unique solution $u \in W^2_q(\mathbb{R}^n_+\backslash\Sigma)$ of*

$$\begin{aligned} (\eta - \Delta)u &= f, && \text{in } \mathbb{R}^n_+\backslash\Sigma, \\ u|_\Sigma &= g_1, && \text{on } \Sigma, \\ \partial_n u|_{\partial\mathbb{R}^n_+} &= g_2, && \text{on } \partial\mathbb{R}^n_+. \end{aligned} \tag{A.3}$$

Furthermore, there is some $C(\eta) > 0$ and some $K > 0$ independent of $\eta \geq \eta_0$, such that

$$\begin{aligned} &|u|_{L_q(\mathbb{R}^n_+)} + \eta^{-1/2}|Du|_{L_q(\mathbb{R}^n_+\backslash\Sigma)} + \eta^{-1}|D^2u|_{L_q(\mathbb{R}^n_+\backslash\Sigma)} \\ &\quad \leq K\eta^{-1}|f|_{L_q(\mathbb{R}^n_+)} + C(\eta)|g_1|_{W^{2-1/q}_q(\Sigma)} + K\eta^{-1/(2q)-1/2}|g_2|_{W^{1-1/q}_q(\partial\mathbb{R}^n_+)} \end{aligned}$$

for all $\eta \geq \eta_0$.

Proof. We first solve the auxiliary upper half space problem

$$\begin{aligned} (\eta-\Delta)w &= f, && \text{in } \mathbb{R}^n_+, \\ \partial_n w|_{\partial\mathbb{R}^n_+} &= g_2, && \text{on } \partial\mathbb{R}^n_+, \end{aligned}$$

by a function $w \in W^2_q(\mathbb{R}^n_+)$. Then it suffices to solve

$$\begin{aligned} (\eta-\Delta)u &= 0, && \text{in } \mathbb{R}^n_+\backslash\Sigma, \\ u|_\Sigma &= \tilde{g}_1, && \text{on } \Sigma, \\ \partial_n u|_{\partial\mathbb{R}^n_+} &= 0, && \text{on } \partial\mathbb{R}^n_+, \end{aligned} \tag{A.4}$$

where $\tilde{g}_1 := g_1 - w|_\Sigma$, since then $u = w + v$ solves the initial problem. Since $\partial_n u = 0$ on the boundary, we may reflect problem (A.4) via an even reflection to obtain an elliptic problem on $\dot{\mathbb{R}} \times \mathbb{R}^{n-1}$. By Theorem 2.11 using $q < 2$ we obtain that $Rg_1 \in W^{2-1/q}_q(\tilde{\Sigma})$, where $\tilde{\Sigma} := \{x_1 = 0\}$. Here, R denotes the aforementioned even reflection in x_n-direction. The problem we are left to solve is now

$$\begin{aligned} (\eta-\Delta)v &= 0, && \text{in } \mathbb{R}^n\backslash\tilde{\Sigma}, \\ v|_{\tilde{\Sigma}} &= Rg_1, && \text{on } \tilde{\Sigma}. \end{aligned} \tag{A.5}$$

Let $x' := (x_2, ..., x_n)$. It is now well known that the operator $(\eta - \Delta_{x'})^{1/2}$ with domain $W^1_q(\mathbb{R}^{n-1})$ has maximal regularity on the half line $\mathbb{R}_+$ with respect to the base space $L_q(\mathbb{R}^{n-1})$ and the induced semigroup is analytic:

By Example 4.5.16 in [57], the fractional power of the shifted Laplacian $(\eta - \Delta_{x'})^{1/2}$ admits a bounded $\mathcal{H}^\infty$-calculus with base space $L^q(\mathbb{R}^{n-1})$ and domain $W^1_q(\mathbb{R}^{n-1})$, and the $\mathcal{H}^\infty$-angle, say φ_∞, is zero.

By the embedding (3.62) in [57], the operator $(\eta - \Delta_{x'})^{1/2}$ then also admits bounded imaginary powers with power angle $\varphi_P \leq \varphi_\infty$, from which it follows by Theorem 4.4.5 in [57] that $(\eta - \Delta_{x'})^{1/2}$ is $\mathcal{R}$-sectorial with spectral angle $\varphi_R \leq \varphi_P$. Since $\varphi_\infty = 0$, we have $\varphi_R \leq 0$ and therefore Theorem 4.4.4 in [57] yields that $(\eta - \Delta_{x'})^{1/2}$ has maximal regularity on finite time intervals with respect to the base space $L^q(\mathbb{R}^{n-1})$. Note that due to the shift parameter the spectral bound of the fractional power $(\eta - \Delta_{x'})^{1/2}$ is strictly less than zero, see Remark A.3. Therefore by well known results we have maximal regularity on the whole half-line $\mathbb{R}_+$. Note that the induced semigroup of this operator is also analytic, cf. [57].

From Theorem 6.1.8 in [57], the natural domain of $\eta - \Delta_{x'}$ with respect to the base space $L_q(\mathbb{R}^{n-1})$ equals the Sobolev space $W^2_q(\mathbb{R}^{n-1})$. Corollary 6.1.9 in [57] then renders the domain of the fractional operator to be $D((\eta - \Delta_{x'})^{1/2}) = W^1_q(\mathbb{R}^{n-1})$. To obtain a solution via maximal L_q-regularity with non-trivial initial value, the initial value has to be in the corresponding interpolation space, which is in this case explicitly given by

$$\Big(L_q(\mathbb{R}^{n-1}), D((\eta - \Delta_{x'})^{1/2})\Big)_{1-1/q,q} = W^{1-1/q}_q(\mathbb{R}^{n-1}).$$

More precisely, we have a unique solution

$$v \in W^1_q(\mathbb{R}_+; L^q(\mathbb{R}^{n-1})) \cap L^q(\mathbb{R}_+; W^1_q(\mathbb{R}^{n-1}))$$

to the evolution equation

$$\begin{aligned} \partial_{x_1} v - (\eta - \Delta_{x'})^{1/2} v &= 0, \qquad x_1 \in (0,\infty), \\ v(x_1 = 0) &= v_0, \end{aligned}$$

whenever $v_0 \in W_q^{1-1/q}(\mathbb{R}^{n-1})$. The solution to this problem is then given by application of the semigroup,

$$v_+(x_1, ..., x_n) = e^{-(\eta-\Delta_{x'})^{1/2} x_1} v_0(x_2, ..., x_n).$$

By choosing $v_0 = R\tilde{g}_1$, an easy calculation shows that $v = e^{-(\eta-\Delta_{x'})^{1/2} x_1} R\tilde{g}_1$ is a solution to (A.5), at least for $x_1 \geq 0$. Note that by a simple coordinate transform,

$$v_- = e^{(\eta-\Delta_{x'})^{1/2} x_1} R\tilde{g}_1$$

solves (A.5) for $x_1 \leq 0$, so altogether a solution of (A.5) is given by

$$v(x_1, x') = e^{-(\eta-\Delta_{x'})^{1/2} |x_1|} R g_1(x'), \quad x_1 \in \mathbb{R}, x' \in \mathbb{R}^{n-1}.$$

By maximal regularity,

$$\|v\|_{W_q^1(\mathbb{R}; L_q(\mathbb{R}^{n-1})) \cap L_q(\mathbb{R}; W_q^1(\mathbb{R}^{n-1}))} \leq C \|R\tilde{g}_1\|_{W_q^{1-1/q}(\mathbb{R}^{n-1})},$$

hence

$$\|v|_{\mathbb{R}^n_+}\|_{W_q^1(\mathbb{R}^n_+)} \leq C \|g_1\|_{W_q^{1-1/q}(\Sigma)}. \tag{A.6}$$

Note that by differentiating the equations we can also control second derivatives,

$$\|v|_{\mathbb{R}^n_+}\|_{W_q^2(\mathbb{R}^n_+)} \leq C \|g_1\|_{W_q^{2-1/q}(\Sigma)}. \tag{A.7}$$

Indeed, note that the semigroup is analytic, that $(\eta - \Delta)^{1/2}$ is invariant under translations and commutes with partial derivatives with respect to x_j, $j = 2, ..., n$, and that we can differentiate the equations, see Proposition 2.2.1 (i) and (iv) in [**43**]. We obtain that $\partial_{x_j} v, j = 2, ..., n$, solves

$$\begin{aligned} \partial_{x_1} \partial_{x_j} v - (\eta - \Delta)^{1/2} \partial_{x_j} v &= 0, \qquad x_1 \in (0,\infty), \\ \partial_{x_j} v(x_1 = 0) &= \partial_{x_j} v_0, \end{aligned}$$

for each $j = 2, ..., n$. Therefore maximal regularity just as above yields in this case

$$\|\partial_{x_j} v\|_{W_q^1(\mathbb{R}; L^q(\mathbb{R}^{n-1})) \cap L^q(\mathbb{R}; W_q^1(\mathbb{R}^{n-1}))} \leq C \|\partial_{x_j} v_0\|_{W_q^{1-1/q}(\mathbb{R}^{n-1})},$$

which then entails that

$$\|\partial_{x_1} \partial_{x_j} v\|_{L^q(\mathbb{R}^n_+)} + \|\partial_{x_j} \partial_{x_k} v\|_{L^q(\mathbb{R}^n_+)} \leq C \|v_0\|_{W_q^{2-1/q}(\mathbb{R}^{n-1})}$$

for all $j, k = 2, ..., n$. Note that since $\partial_{x_1} \partial_{x_1} v = (\eta - \Delta_{x'}) v$, we can also control the full W_q^2-norm, so altogether (A.7) holds true.

To obtain the explicit dependence of the shift parameter $\eta > 0$ we proceed as follows. By uniqueness of the solution, we know that $u(x) = u_\eta(\sqrt{\eta} x)$, where u_η is the unique solution of

$$\begin{aligned} (I - \Delta) u_\eta &= f_\eta, && \text{in } \mathbb{R}^n_+ \backslash \Sigma, \\ u_\eta|_\Sigma &= g^1_\eta, && \text{on } \Sigma, \\ \partial_n u_\eta &= g^2_\eta, && \text{on } \partial\mathbb{R}^n_+, \end{aligned}$$

where

$$f_\eta(x) := \frac{1}{\eta} f(\frac{1}{\sqrt{\eta}}x), \quad g^1_\eta(x) := g_1(\frac{1}{\sqrt{\eta}}x), \quad g^2_\eta(x) := \frac{1}{\sqrt{\eta}} g_2(\frac{1}{\sqrt{\eta}}x).$$

By the above arguments there now exists a constant $K > 0$ independent of η, such that

$$\|u_\eta\|_{W^2_q(\mathbb{R}^n_+\setminus\Sigma)} \le K \left(\|f_\eta\|_{L^q} + \|g^1_\eta\|_{W^{2-1/q}_q} + \|g^2_\eta\|_{W^{1-1/q}_q} \right).$$

A direct calculation shows that

$$\begin{aligned}
\|u_\eta\|_{W^2_q} &= \sqrt{\eta}^{n/q}\|u\|_{L^q} + \sqrt{\eta}^{n/q-1}\|Du\|_{L^q} + \sqrt{\eta}^{n/q-2}\|D^2u\|_{L^q},\\
\|f_\eta\|_{L^q} &= \sqrt{\eta}^{n/q-2}\|f\|_{L^q},\\
\|g^1_\eta\|_{W^{2-1/q}_q} &= \sqrt{\eta}^{(n-1)/q}\|g_1\|_{L^q} + \sqrt{\eta}^{(n-1)/q-1}\|Dg_1\|_{L^q} + \sqrt{\eta}^{n/q-2}[Dg_1]_{W^{1-1/q}_q},\\
\|g^2_\eta\|_{W^{1-1/q}_q} &= \sqrt{\eta}^{(n-1)/q-1}\|g_2\|_{L^q} + \sqrt{\eta}^{(n-1)/q-(1-1/q)}[g_2]_{W^{1-1/q}_q}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sqrt{\eta}^{n/q}\|u\|_{L^q} + \sqrt{\eta}^{n/q-1}\|Du\|_{L^q} + \sqrt{\eta}^{n/q-2}\|D^2u\|_{L^q}\\
&\qquad \le K\sqrt{\eta}^{n/q-2}\|f\|_{L^q} + K\sqrt{\eta}^{(n-1)/q}\|g_1\|_{W^{2-1/q}_q} + K\sqrt{\eta}^{(n-1)/q-1}\|g_2\|_{W^{1-1/q}_q},
\end{aligned}$$

where $K > 0$ is independent of $\eta \ge \eta_0$. Multiplying with $\sqrt{\eta}^{-n/q}$ gives the desired estimate. □

REMARK A.2. We can introduce weighted norms for the solution,

$$|u|_{2,q,\eta} := \|u\|_{L^q} + \sqrt{\eta}^{-1}\|Du\|_{L^q} + \sqrt{\eta}^{-2}\|D^2u\|_{L^q},$$

as well as the data,

$$|(f, g_1, g_2)|_{q,\eta} := \sqrt{\eta}^{-2}\|f\|_{L^q} + \sqrt{\eta}^{-1/q}\|g_1\|_{W^{2-1/q}_q} + \sqrt{\eta}^{-1/q-1}\|g_2\|_{W^{1-1/q}_q}.$$

This gives $|u|_{2,q,\eta} \le K|(f, g_1, g_2)|_{q,\eta}$, where $K > 0$ is independent of $\eta \ge \eta_0$. Note that for fixed $\eta \ge \eta_0$, these norms are equivalent norms in solution and data space.

REMARK A.3. Let $A_\eta := -(\eta - \Delta)^{1/2}, \eta > 0$, with base space $L^q(\mathbb{R}^{n-1})$ and domain $W^1_q(\mathbb{R}^{n-1})$. We now show that for every $\eta > 0$ there is $c_0(\eta) > 0$, such that the spectral bound satisfies $s(-(\eta - \Delta)^{1/2}) \le -c_0(\eta) < 0$. From Example 4.5.16 in [**57**] we get that A_η is invertible, has maximal regularity, and the semigroup is analytic. For fixed λ, we define the Fourier multiplication operator $M_\lambda := \mathcal{F}^{-1}m_\lambda\mathcal{F}$, where the symbol is given by

$$m_\lambda(\xi) := \frac{-(\eta + |\xi|^2)^{1/2}}{\lambda + (\eta + |\xi|^2)^{1/2}}, \quad \operatorname{Re}\lambda > -\eta^{1/2}.$$

Clearly, $(\eta + |\xi|^2)^{1/2} \ge \eta^{1/2}$ uniformly in $\xi \in \mathbb{R}^{n-1}$. If $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $|\lambda + (\eta + |\xi|^2)^{1/2}| \ge C(\lambda, \eta) > 0$. If $\lambda \in \mathbb{R} \cap \{\lambda \ge -\eta^{1/2}/2\}$, $|\lambda + (\eta + |\xi|^2)^{1/2}| \ge \eta^{1/2}/2$. Furthermore, using that $(\eta + |\xi|^2)^{1/2} \le 1 + (\eta + |\xi|^2)^{1/2} \le C(\eta)(\eta + |\xi|^2)^{1/2}$ for all $\xi \in \mathbb{R}^{n-1}$, we can conclude that m is holomorphic and bounded on

$$(\lambda, \xi) \in \big(\Sigma_{2\pi/3} - c_0(\eta)\big) \times (-\Sigma_{\pi/4} \cap \Sigma_{\pi/4})^{n-1}.$$

Hence m_λ satisfies the scalar Mikhlin condition for every $\lambda \in \Sigma_{2\pi/3} - c_0(\eta)$, see Remark A.4, whence M_λ can be extended to a bounded operator on $L^p(\mathbb{R}^{n-1})$.

So, for given $f \in L^p(\mathbb{R}^{n-1}), \lambda \in \Sigma_{2\pi/3} - c_0(\eta)$ we find a unique solution of $\lambda u - (\eta - \Delta)^{1/2} u = f$, namely by solving $A_\eta u = M_\lambda f$. This however shows that $\Sigma_{2\pi/3} - c_0(\eta) \subset \rho((\eta-\Delta)^{1/2})$, which immediately implies that the spectral bound is strictly negative, $s(-(\eta-\Delta)^{1/2}) \leq -c_0(\eta) < 0$.

Remark A.4. Suppose $\theta > 0$ and $f : (-\Sigma_\theta \cup \Sigma_\theta)^n \to \mathbb{C}$ is holomorphic and bounded, where $\Sigma_\theta := \{z \in \mathbb{C} : |\arg z| < \theta\}$. Then there is some $C(\theta, k) > 0$, such that

$$|D^k f(z)| \leq C(\theta, k)|f|_{L^\infty(-\Sigma_\theta \cup \Sigma_\theta, \mathbb{C})}|z|^{-k}, \quad z \in \dot{\mathbb{R}}^n, k \in \mathbb{N}_0.$$

In particular, f satisfies the scalar Mikhlin condition. There is also a more general statement for Banach space valued multipliers implying $\mathcal{R}$-boundedness of the operator family, see Proposition 4.3.10 in [**57**]. In our case the proof is much more simple.

Proof. The proof is a direct consequence of Cauchy's formula. For simplicity let $n = 1$. For $y \in \dot{\mathbb{R}}$, since the double sector is conical, there exists $\kappa = \kappa(\theta)$ independent of y such that

$$\overline{B_{\kappa(\theta)|y|}(y)} \subseteq -\Sigma_\theta \cup \Sigma_\theta.$$

In particular, $\kappa = \sin\theta/2$ is a possible choice. Cauchy's formula then gives

$$f^{(k)}(y) = \frac{k!}{2\pi i} \int_{\partial B_{\kappa(\theta)|y|}(y)} \frac{f(\zeta)}{(\zeta - y)^{k+1}} d\zeta, \quad k \in \mathbb{N}_0.$$

Hence,

$$|f^{(k)}(y)| \leq \frac{k!\kappa(\theta)^{-k-1}}{2\pi} |f|_{L^\infty(-\Sigma_\theta \cup \Sigma_\theta, \mathbb{C})}|y|^{-k-1} \mathrm{meas}(\partial B_{\kappa(\theta)|y|}(y)), \quad k \in \mathbb{N}_0,$$

which finishes the proof since $\mathrm{meas}(\partial B_{\kappa(\theta)|y|}(y))$ equals $2\pi\kappa(\theta)|y|$. □

A.1.2. Bent interfaces. We consider now the problem on a perturbed upper half space $\mathbb{R}^n_\gamma := \mathbb{R}^n \cap \{x_n > \gamma(x_1, ...x_{n-1})\}$ with perturbed interface $\Sigma_{\tilde\gamma} := \mathbb{R}^n_\gamma \cap \{x_1 = \tilde\gamma(x_2, ..., x_n)\}$,

$$\begin{aligned} (\eta - \Delta)u &= f, && \text{in } \mathbb{R}^n_\gamma \backslash \Sigma_{\tilde\gamma}, \\ u &= g_1, && \text{on } \Sigma_{\tilde\gamma}, \\ n_{\partial\mathbb{R}^n_\gamma} \cdot \nabla u &= g_2, && \text{on } \partial\mathbb{R}^n_\gamma. \end{aligned} \tag{A.9}$$

We will show that this system is also solvable if $\eta \geq \eta_0$ for η_0 sufficiently large and $\|\gamma\|_{C^1} + \|\tilde\gamma\|_{C^1} \leq \varepsilon_0$ for $\varepsilon_0 > 0$ sufficiently small.

Theorem A.5. *There is some $\eta_0 > 0$ and some $\varepsilon_0 = \varepsilon_0(\eta_0) > 0$, such that the following is true. For all $\eta \geq \eta_0$ and $\|\gamma\|_{C^1} + \|\tilde\gamma\|_{C^1} \leq \varepsilon_0$, there is a unique solution $u \in W^2_q(\mathbb{R}^n_\gamma \backslash \Sigma_{\tilde\gamma})$ of the η-shifted problem (A.9), if and only if $f \in L^q(\mathbb{R}^n_\gamma), g_1 \in W^{2-1/q}_q(\Sigma_{\tilde\gamma})$, and $g_2 \in W^{1-1/q}_q(\partial\mathbb{R}^n_\gamma)$.*

Proof. We show this by a perturbation argument. Transforming via F, cf. Lemma 2.19, the perturbed problem is equivalent to the regular upper half space problem

$$\begin{aligned} (\eta-\Delta)\bar{u} &= \bar{f}+\mathcal{A}_1(\bar{u}), && \text{in } \mathbb{R}^n_+\backslash\Sigma,\\ \bar{u} &= \bar{g}_1, && \text{on } \Sigma,\\ \partial_n\bar{u} &= \bar{g}_2+\mathcal{A}_2(\bar{u}), && \text{on } \partial\mathbb{R}^n_+, \end{aligned} \tag{A.10}$$

where $\bar{f}=f\circ F, \bar{g}_j=g_j\circ F, j=1,2$, and

$$\mathcal{A}_1(\bar{u})=\sum_{l,j}R_{jl}\partial_l\partial_j\bar{u}+\sum_l\tilde{R}_l\partial_l\bar{u},$$

cf. (2.49), as well as $\mathcal{A}_2(\bar{u})=(e_n-n_\gamma)\cdot\nabla\bar{u}+n_\gamma\cdot((DF^T-I)\nabla\bar{u})$. Here n_γ denotes the upper normal of $\partial\mathbb{R}^n_\gamma$. We now perform a Neumann Series argument. We equip $W^2_q(\mathbb{R}^n_+\backslash\Sigma)$ and the data space $[L_q(\mathbb{R}^n_+)\times W_q^{2-1/q}(\Sigma_{\tilde{\gamma}})\times W_q^{1-1/q}(\partial\mathbb{R}^n_\gamma)]$ with the weighted norms of Remark A.2. Then the operator L, defined by the left hand side of (A.10), is invertible. Its inverse is bounded by a constant K independent of η with respect to these η-weighted norms, cf. Remark A.2. We now show that

$$\|(\mathcal{A}_1,0,\mathcal{A}_2)\|_{\mathcal{B}(W^2_q(\mathbb{R}^n_+\backslash\Sigma);L_q(\mathbb{R}^n_+)\times W_q^{2-1/q}(\Sigma_{\tilde{\gamma}})\times W_q^{1-1/q}(\partial\mathbb{R}^n_\gamma))}\to 0,$$

with respect to the η-scaled norms as $\eta\to\infty$, and $|\beta|_{C^1}+|\gamma|_{C^1}\to 0$. By a standard Neumann series argument, $L+(\mathcal{A}_1,0,\mathcal{A}_2)$ is invertible if $T>0$ is small enough.

Now,

$$\begin{aligned} |\mathcal{A}_1(\bar{u})|_\eta &= \eta^{-1}|\mathcal{A}_1(\bar{u})|_{L_q(\mathbb{R}^n_+)}\\ &\lesssim \eta^{-1}\sum_{j,l}|\partial_l\partial_j\bar{u}|_{L_q(\mathbb{R}^n_+)}|R_{jl}|_{L_\infty(\mathbb{R}^n_+)}+\eta^{-1}\sum_j|\partial_j\bar{u}|_{L_q(\mathbb{R}^n_+)}|\tilde{R}_j|_{L_\infty(\mathbb{R}^n_+)}\\ &\lesssim (|\beta|_{C^1}+|\gamma|_{C^1})|\bar{u}|_\eta+C(\beta,\gamma)\eta^{-1/2}|\bar{u}|_\eta, \end{aligned}$$

where R_{jl} and $\tilde{R}_l$ are as in (2.50). Now note that

$$|\mathcal{A}_2(\bar{u})|_\eta=\sqrt{\eta}^{-1-1/q}|\mathcal{A}_2(\bar{u})|_{W_q^{1-1/q}(\partial\mathbb{R}^n_+)}.$$

Extend n_γ to a function on $\mathbb{R}^n_+$ by a bounded extension operator, cf. [**62**], and denote the extension by $\tilde{n}_\gamma$. Then

$$\begin{aligned} |\mathcal{A}_2(\bar{u})|_\eta \lesssim{}& \sqrt{\eta}^{-1-1/q}|(e_n-\tilde{n}_\gamma)\cdot\nabla\bar{u}|_{L_q(\mathbb{R}^n_+)}\\ &+\sqrt{\eta}^{-1-1/q}|\nabla(e_n-\tilde{n}_\gamma)\cdot\nabla\bar{u}|_{L_q(\mathbb{R}^n_+)}\\ &+\sqrt{\eta}^{-1-1/q}|(e_n-\tilde{n}_\gamma)\cdot\nabla^2\bar{u}|_{L_q(\mathbb{R}^n_+)}\\ &+\sqrt{\eta}^{-1-1/q}|n_\gamma|_{C^1}|(DF^T-I)\nabla\bar{u}|_{L_q(\mathbb{R}^n_+)}\\ &+\sqrt{\eta}^{-1-1/q}|n_\gamma|_{C^1}|\nabla(DF^T-I)\nabla\bar{u}|_{L_q(\mathbb{R}^n_+)}\\ &+\sqrt{\eta}^{-1-1/q}|n_\gamma|_{C^1}|(DF^T-I)\nabla^2\bar{u}|_{L_q(\mathbb{R}^n_+)}. \end{aligned}$$

In weighted norms,

$$
\begin{aligned}
|\mathcal{A}_2(\bar{u})|_\eta &\lesssim C(n_\gamma, DF)\sqrt{\eta}^{-1/q}\sqrt{\eta}^{-1}\|\nabla \bar{u}\|_{L_q(\mathbb{R}^n_+)}+\\
&\quad + \sqrt{\eta}^{1-1/q}(|n_\gamma - e_n|_{L_\infty} + \|DF - I\|_{L_\infty})\sqrt{\eta}^{-2}|D^2\bar{u}|_{L_q(\mathbb{R}^n_+)}\\
&\lesssim \sqrt{\eta}^{-1/q}C(n_\gamma, DF)|\bar{u}|_\eta + \sqrt{\eta}^{1-1/q}(|\beta|_{C^1} + |\gamma|_{C^1})|\bar{u}|_{2,\eta}.
\end{aligned}
$$

Fixing first $\eta_0 > 0$ large enough and then $\varepsilon_0 = \varepsilon_0(\eta_0) > 0$ small enough finishes the proof. □

A.1.3. Localization. By a localization argument we can now show that the shifted problem is solvable in the case of a bounded, smooth domain.

THEOREM A.6. *Let $q \in (3/2, 2)$, $\Omega \subset \mathbb{R}^n$ a bounded, smooth domain and Σ a smooth surface inside Ω intersecting the boundary $\partial\Omega$ at a nintey degree angle. Then there is some $\eta_0 \geq 0$, such that if $\eta \geq \eta_0$, for every $(f, g_1, g_2) \in L_q(\Omega) \times W_q^{2-1/q}(\Sigma) \times W_q^{1-1/q}(\partial\Omega)$ there is unique $u \in W_q^2(\Omega\backslash\Sigma)$ solving (A.3). Furthermore, the solution map $[(f, g_1, g_2) \mapsto u]$ is continuous between the above spaces.*

PROOF. We show solvability and uniqueness via a localization method, cf. [**57**]. To this end let $(\varphi_j)_{j=0,\dots,N}$ be a partition of unity as in Section 2.5.4. Suppose u is a solution. We then obtain localized equations for $u^j := \varphi_j u$, namely

$$
\begin{aligned}
(\eta - \Delta)u^j &= \varphi_j f + \mathcal{C}_1^j(u), && \text{in } \Omega\backslash\Sigma,\\
u^j|_\Sigma &= \varphi_j g_1 + \mathcal{C}_2^j(u), && \text{on } \Sigma,\\
n_{\partial\Omega}\cdot\nabla u^j &= \varphi_j g_2 + \mathcal{C}_3^j(u), && \text{on } \partial\Omega,
\end{aligned}
$$

where the perturbation operators are given by

$$
\mathcal{C}_1^j(u) = \varphi_j\Delta u - \Delta(\varphi_j u), \quad \mathcal{C}_2^j(u) = 0, \quad \mathcal{C}_3^j(u) = n_{\partial\Omega}\cdot(\nabla(\varphi_j u) - \varphi_j\nabla u).
$$

In other words, these operators are of commutator type and therefore of lower order,

$$
\mathcal{C}_1^j(u) = u\Delta\varphi_j + 2\nabla\varphi_j\cdot\nabla u, \quad \mathcal{C}_3^j(u) = n_{\partial\Omega}\cdot u\nabla\varphi_j.
$$

Let $\mathcal{C}^j := (\mathcal{C}_1^j, 0, \mathcal{C}_3^j)$. We now show that

$$
|\mathcal{C}^j u|_\eta \leq C(\eta)|u|_{2,\eta}, \tag{A.12}
$$

where the constant $C(\eta)$ satisfies $C(\eta) \to 0$ as $\eta \to \infty$. Then the proof follows the lines of Section 2.5.4. We have

$$
\begin{aligned}
|\mathcal{C}_1^j u|_\eta &= \frac{1}{\eta}\|u\Delta\varphi_j\|_{L^q} + \frac{2}{\eta}\|\nabla u\cdot\nabla\varphi_j\|_{L^q}\\
&\leq \frac{1}{\eta}\|u\|_{L^q}\|\Delta\varphi_j\|_{L^\infty} + \frac{2}{\sqrt{\eta}}\frac{1}{\sqrt{\eta}}\|\nabla u\|_{L^q}\|\Delta\varphi_j\|_{L^\infty}\\
&\leq \frac{1}{\eta}\|\varphi_j\|_{C^2}|u|_{2,\eta} + \frac{2}{\sqrt{\eta}}\|\varphi_j\|_{C^2}|u|_{2,\eta},
\end{aligned}
$$

as well as

$$\begin{aligned}|\mathcal{C}_3^j u|_\eta =& \sqrt{\eta}^{-1-1/q} \|n_{\partial\Omega} \cdot u \nabla \varphi_j\|_{W_q^{1-1/q}} \\ &\leq \sqrt{\eta}^{-1/q} \sqrt{\eta}^{-1} \|u\|_{W_q^1} \|n_{\partial\Omega} \cdot \nabla \varphi_j\|_{C^1} \\ &\leq \sqrt{\eta}^{-1/q} |u|_{2,\eta} \|n_{\partial\Omega} \cdot \nabla \varphi_j\|_{C^1}.\end{aligned}$$

Clearly, these two estimates give (A.12). □

A.1.4. The non-shifted problem. We will now show solvability of the non-shifted problem (A.2).

THEOREM A.7. *Let $q \in (3/2, 2)$. For every $(f, g_1, g_2) \in L_q(\Omega) \times W_q^{2-1/q}(\Sigma) \times W_q^{1-1/q}(\partial\Omega)$ there is unique $u \in W_q^2(\Omega\backslash\Sigma)$ solving (A.2). Furthermore, there is some constant $C > 0$, such that*

$$|u|_{W_q^2(\Omega\backslash\Sigma)} \leq C \left(|f|_{L_q(\Omega)} + |g_1|_{W_q^{2-1/q}(\Sigma)} + |g_2|_{W_q^{1-1/q}(\partial\Omega)} \right).$$

PROOF. First we choose $\eta > 0$ large enough and solve (A.3) by a function $v \in W_q^2(\Omega\backslash\Sigma)$. It therefore remains to solve

$$\begin{aligned} -\Delta w &= -\eta v, && \text{in } \Omega\backslash\Sigma, \\ w|_\Sigma &= 0, && \text{on } \Sigma, \\ n_{\partial\Omega} \cdot \nabla w|_{\partial\Omega} &= 0, && \text{on } \partial\Omega, \end{aligned}$$

since then $u := v + w$ solves (A.2). To this end define A to be the negative Laplacian $-\Delta$ in $L_q(\Omega)$ with domain

$$D(A) := \{w \in W_q^2(\Omega\backslash\Sigma) : w|_\Sigma = 0,\ n_{\partial\Omega} \cdot \nabla w|_{\partial\Omega} = 0\}.$$

Since $D(A)$ compactly embeds into $L_q(\Omega)$ by Sobolev embeddings, A has compact resolvent and the spectrum $\sigma(A)$ only consists of isolated eigenvalues of A with finite multiplicity. We will show that zero is not a possible eigenvalue, hence A is invertible.

Suppose $u \neq 0$ is a nontrivial eigenfunction to the eigenvalue λ. The corresponding eigenvalue problem reads as

$$\begin{aligned} -\lambda u &= \Delta u, && \text{in } \Omega\backslash\Sigma, \\ u|_\Sigma &= 0, && \text{on } \Sigma, \\ n_{\partial\Omega} \cdot \nabla u|_{\partial\Omega} &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Assume for a moment that $q = 2$. Testing the resolvent equation with u in $L_2(\Omega)$ and invoking the boundary condition, an integration by parts entails

$$-\lambda |u|_{L_2(\Omega)}^2 = \int_\Omega u \Delta u dx = -|\nabla u|_{L_2(\Omega)}^2.$$

Whence if $\lambda = 0$, then $u \in D(A)$ has to be a constant function, hence zero since u vanishes on Σ. This is a contradiction, hence $\lambda = 0$ is not a possible eigenvalue if $q = 2$. Note that the same arguments are still valid if $q > 2$, since then by Sobolev embeddings $W_q^2(\Omega) \hookrightarrow W_2^2(\Omega)$.

Consider now the case where $3/2 < q < 2$. Again consider the eigenvalue problem $\lambda u = Au$ in $L_q(\Omega)$. By Sobolev embedding, $W_q^2(\Omega) \hookrightarrow L_{\tilde{q}}(\Omega)$ for some

$\tilde{q} > 2$. We obtain that now $\lambda u = Au$ holds true in $L_{\tilde{q}}(\Omega)$. Therefore $Au \in L_{\tilde{q}}(\Omega)$. Since we already know that $A_{\tilde{q}} : D(A_{\tilde{q}}) \to L_{\tilde{q}}(\Omega)$ is invertible we obtain $u \in W^2_{\tilde{q}}(\Omega)$. Hereby, $A_{\tilde{q}}u := Au$ with domain

$$D(A_{\tilde{q}}) := \{w \in W^2_{\tilde{q}}(\Omega\backslash\Sigma) : w|_\Sigma = 0,\ n_{\partial\Omega} \cdot \nabla w|_{\partial\Omega} = 0\}.$$

The same arguments now yield that $\lambda = 0$ is not a possible eigenvalue.

Therefore we may uniquely solve (A.13) and the proof is complete since the proof of the estimates follows in a straightforward way. □

A.1.5. Relations between shifted and non-shifted problems. We conclude this section by the following observation, cf. [**57**]. Consider the special case where $(f, g_1, g_2) = (0, g, 0)$. Define solution operators as follows. Let $T_0 g$ be the solution of the non-shifted problem (A.2) for $(f, g_1, g_2) = (0, g, 0)$, and, for $\eta \geq \eta_0$, $T_\eta g$ the solution of (A.3) with $(f, g_1, g_2) = (0, g, 0)$. Then, $T_0 g - T_\eta g = \eta(\eta - \Delta_N)^{-1} T_0 g$. Hereby, $z := (\eta - \Delta_N)^{-1} f$ solves the two-phase problem

$$\begin{aligned} (\eta - \Delta)z &= f, && \text{in } \Omega\backslash\Sigma,\\ z|_\Sigma &= 0, && \text{on } \Sigma,\\ (n_{\partial\Omega}|\nabla z) &= 0, && \text{on } \partial\Omega. \end{aligned}$$

For details we refer to Section 6.6 in [**57**].

A.2. Cylindrical domains.

Let us now consider the elliptic problem in a cylindrical domain. For notation, let $n = 3$. In this subsection $\Omega \subset \mathbb{R}^3$ is a bounded cylinder. We now need a result for the elliptic model problem in the case where the top of the container meets the walls at a ninety degree angle. With this at hand we may solve the elliptic problem in a cylindrical domain with a localization argument as before. So let us consider the quarter space $G := \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ and the one-phase quarter space problem

$$\begin{aligned} (\eta - \Delta)u &= f, && \text{in } G,\\ \partial_1 u &= g_1, && \text{on } S_1 := \{x_1 = 0, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}_+\},\\ \partial_3 u &= g_2, && \text{on } S_2 := \{x_1 \in \mathbb{R}_+, x_2 \in \mathbb{R}, x_3 = 0\}. \end{aligned} \tag{A.14}$$

The key observation is now that the two Neumann boundary conditions on S_1 and S_2 are compatible whenever $q < 2$. Suppose that we want to find a solution $u \in W^2_q(G)$ of the problem. Then by trace theory,

$$\nabla u|_{S_j} \in W^{1-1/q}_q(S_j), \quad j = 1, 2.$$

This yields necessary conditions for the data. We see that on the set $\partial S_1 \cap \partial S_2 = \{x_1 = x_3 = 0\}$ where the two boundary conditions meet, there is no compatibility condition for the data g_1 and g_2 in the system. This is due to the fact that since $q < 2$ the functions $\nabla u|_{S_j}$ do not have a trace on ∂S_j.

So let the given data in (A.14) satisfy

$$f \in L_q(G), \quad g_j \in W^{1-1/q}_q(S_j),\ j = 1, 2.$$

By a simple reflection argument we can reduce the problem to a upper half-space problem with one Neumann condition and obtain full $W^2_q(G)$-regularity for the solution. Let us state this observation in the following theorem.

THEOREM A.8. *For all* $(f, g_1, g_2) \in [L_q(G) \times W_q^{1-1/q}(S_1) \times W_q^{1-1/q}(S_2)]$ *there exists a unique solution* $u \in W_q^2(G)$ *to problem* (A.14). *Furthermore, the solution map* $[(f, g_1, g_2) \mapsto u]$ *is continuous with respect to these spaces.*

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