Discontinuities of the ρ-invariant and an application to the L^2-ρ-invariant

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Summary

Let M be a closed connected oriented odd-dimensional manifold. The Atiyah–Patodi–Singer ρ-invariant assigns to a unitary representation α of the fundamental group a real number $\rho(M,\alpha)$. In other words, given M and a natural number k, we can view the ρ-invariant as a function defined on the variety $R_k(\pi_1(M))$ of k-dimensional unitary representations of the fundamental group of M. Levine determined subsets of $R_k(\pi_1(M))$ on which the ρ-invariant is continuous and showed that, when considered with values in $\mathbb{R}/\mathbb{Z}$, it is continuous on the entire variety. If the dimension of the manifold is $4n-1$, then Farber and Levine showed that the ρ-invariant with values in $\mathbb{R}/\mathbb{Z}$ is even locally constant.

Cheeger and Gromov defined an L^2-analogue of the ρ-invariant, called L^2-ρ-invariant, which assigns to a closed connected oriented $(4n-1)$-dimensional manifold and a group homomorphism $\phi\colon \pi_1(M) \to G$ a real number $\rho^{(2)}(M,\phi)$.

In this thesis, we consider a closed connected oriented 3-dimensional manifold M. We study the set of discontinuities of the ρ-invariant restricted to suitable tori in $R_k(\pi_1(M))$ and relate those discontinuities to the zeros of a multi-variable twisted Alexander polynomial. Given $\phi = (\phi_1,\dots,\phi_m)\colon \pi_1(M) \to \mathbb{Z}^m$ and $\alpha \in R_k(\pi_1(M))$, we consider the torus

$$\begin{aligned} S^1 \times \cdots \times S^1 &\to R_k(\pi_1(M)) \\ (z_1,\dots,z_m) &\mapsto \left(g \mapsto \alpha(g) \prod_{i=1}^{m} z_i^{\phi_i(g)} \right). \end{aligned}$$

In the one-dimensional case, i.e., a circle lying in the variety of unitary representations, we bound the heights of the jumps of the ρ-invariant by the degree of the Alexander polynomial associated to (M,ϕ,α). For a torus of arbitrary dimension we show that if the Alexander polynomial associated to (M,ϕ,α) is not zero, then the set of discontinuities of the ρ-invariant restricted to such a torus has measure zero. As a consequence, we deduce that the Riemann integral of the ρ-invariant over such a torus exists and, in case that α is trivial, equals the L^2-ρ-invariant $\rho^{(2)}(M,\phi)$.

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Chapter 1

Introduction

In 1975, Atiyah, Patodi and Singer introduced the η-invariant, which assigns to a closed connected oriented odd-dimensional Riemannian manifold M and a unitary representation α of its fundamental group a real number $\eta(M,\alpha)$. The η-invariant is defined as the value at zero of a meromorphic extension of a function involving the eigenvalues of a differential operator depending on the unitary representation of the fundamental group. Atiyah, Patodi and Singer studied the η-invariant in a series of articles [APS75a], [APS75b] and [APS76].

Let α be a k-dimensional unitary representation of the fundamental group and let τ_k be the trivial k-dimensional representation. Atiyah, Patodi and Singer [APS75b, Theorem 2.4] showed that the difference

$$\eta(M,\alpha) - \eta(M,\tau_k)$$

is independent of the Riemannian metric on M. It is usually referred to as the ρ-invariant $\rho(M,\alpha)$ associated to M and α.

In the case of a $(4n-1)$-dimensional manifold, the ρ-invariant can sometimes be calculated as a signature defect, i.e., a weighted difference of the ordinary and the twisted signature of a manifold having M as boundary. This is due to Atiyah, Patodi and Singer in the following form:

Theorem ([APS75b, Theorem 2.4]). *Let M be a closed connected oriented $(4n-1)$-dimensional manifold and let $\alpha\colon \pi_1(M) \to U(k)$ be a unitary representation. Assume there exists a compact oriented $4n$-dimensional manifold W with boundary consisting of r disjoint copies of M and α extends to a unitary representation $\beta\colon \pi_1(W) \to U(k)$. Then*

$$\rho(M,\alpha) = \frac{1}{r}\big(k\,\mathrm{sign}(W) - \mathrm{sign}_\beta(W)\big),$$

where $\mathrm{sign}(W)$ *and* $\mathrm{sign}_\beta(W)$ *denote the ordinary signature of W and the signature of W twisted with β, respectively.*

We denote by $R_k(\pi)$ the variety of k-dimensional unitary representations of π and consider the ρ-invariant as a map

$$\begin{aligned}\rho(M)\colon R_k(\pi_1(M)) &\to \mathbb{R}\\ \alpha &\mapsto \rho(M,\alpha).\end{aligned}$$

Levine determined sets on which the ρ-invariant is continuous. He proved the following:

Theorem ([Lev94, Theorem 2.1]). *Let M be a closed connected oriented odd-dimensional manifold. Let $r \in \mathbb{N} \cup \{0\}$ and let Σ_r be the subvariety of $R_k(\pi_1(M))$ given by*

$$\Sigma_r = \left\{ \alpha \in R_k(\pi_1(M)) \,\middle|\, \sum_{i=0}^{\infty} \dim_{\mathbb{C}} H_i^{\alpha}(M; \mathbb{C}^k) \geq r \right\}.$$

Then

$$\begin{aligned} \rho(M) \colon R_k(\pi_1(M)) &\to \mathbb{R} \\ \alpha &\mapsto \rho(M, \alpha) \end{aligned}$$

is continuous on $\Sigma_r \backslash \Sigma_{r+1}$. Furthermore, $\rho(M)$ is continuous when considered with values in $\mathbb{R}/\mathbb{Z}$.

If the dimension of the manifold M is of the form $4n-1$, then $\rho(M)$ is even locally constant in $\mathbb{R}/\mathbb{Z}$, as is shown in [FL96, Theorem 7.6].

By a result of Farber and Levine [FL96, Theorem 1.5], the integer jumps of the ρ-invariant can be calculated as a sum of signatures of suitable linking forms. In Chapter 6, we construct the homological linking forms by following the work of Farber and Levine and state their result.

In 1985, Cheeger and Gromov [CG85] defined the L^2-ρ-invariant, which assigns to a closed connected oriented $(4n-1)$-dimensional manifold and a group homomorphism $\phi \colon \pi_1(M) \to G$ a real number $\rho^{(2)}(M, \phi)$. As opposed to the situation for the classical ρ-invariant, the L^2-ρ-invariant can always be computed as an L^2-signature defect, see [Cha16b, Section 2.1]. Hence, the following can be used as a definition: Let W be a compact oriented $4n$-dimensional manifold with boundary consisting of r disjoint copies of M and let Γ be a group such that

- there exists a group monomorphism $i \colon G \to \Gamma$ and
- the group homomorphism $i \circ \phi \colon \pi_1(M) \to G \to \Gamma$ can be extended to a homomorphism $\Phi \colon \pi_1(W) \to \Gamma$.

Then the L^2-ρ-invariant is given as the difference of the ordinary and the L^2-signature $\operatorname{sign}^{(2)}_{\mathcal{N}\Gamma}(W)$ of W:

$$\rho^{(2)}(M, \phi) = \frac{1}{r}\left(\operatorname{sign}(W) - \operatorname{sign}^{(2)}_{\mathcal{N}\Gamma}(W)\right)$$

In fact, for a fixed M, there is always a choice of W such that for any group homomorphism ϕ a Γ satisfying the conditions above can be found. Cha used this fact to give a topological proof of the existence of upper bounds for the L^2-ρ-invariant, which first appeared in [CG85, (4.10)]. More precisely, Cha proved:

Theorem ([Cha16b, Theorem 1.3]). *Let M be a closed connected oriented $(4n-1)$-dimensional manifold. Then there is a constant C_M such that*

$$\left|\rho^{(2)}(M, \phi)\right| \leq C_M$$

holds for all groups G and all homomorphisms $\phi \colon \pi_1(M) \to G$.

Cha studied bounds on the L^2-ρ-invariant in terms of the complexity of M in [Cha16b,Cha16a]. Here the complexity of a manifold is defined to be the minimal number of top-dimensional simplices in a triangulation of the manifold. More concretely, Cha obtained an upper bound on the L^2-ρ-invariants of a 3-dimensional manifold which depends only linearly on the complexity of the manifold:

Theorem ([Cha16b, Theorem 1.5])**.** *Let M be a closed connected oriented 3-dimensional manifold with complexity n. Then*

$$\left|\rho^{(2)}(M,\phi)\right| \leq 363090n$$

holds for all groups G and all homomorphisms $\phi\colon \pi_1(M) \to G$.

Hence, the L^2-ρ-invariant gives lower bounds on the complexity of a 3-dimensional manifold and it turned out that these lower bounds are in many cases better than the previously known lower bounds (see [Cha16b] and [Cha16a]).

Although the classical ρ-invariant is in general not computable as a signature defect, at least in some cases it is related to the L^2-ρ-invariant. Let G be a finite group and $\alpha\colon \pi_1(M) \to G$ be a group homomorphism. Let $\phi_G\colon G \to U(|G|)$ be the regular representation. We define $\rho(M,\alpha) := \rho(M, \phi_G \circ \alpha)$. We often denote the group instead of the homomorphism if it is clear from the context which homomorphism we consider, i.e., $\rho(M,G) := \rho(M,\alpha)$. In case that G is a finite group it follows basically from the definitions that the classical and the L^2-ρ-invariant just differ by the order of G, namely

$$\rho^{(2)}(M,G) = \frac{1}{|G|}\rho(M,G).$$

More generally, the ρ- and L^2-ρ-invariant are also related in the case of a residually finite group G. If $\{G_i\}_{i\in\mathbb{N}}$ is a residual chain for G, we have

$$\rho^{(2)}(M,G) = \lim_{i\to\infty} \frac{1}{|G/G_i|}\rho\big(M, G/G_i\big).$$

This leads to the following question:

Question. *Let M be a closed connected oriented 3-dimensional manifold. Is the map*

$$\begin{aligned}\rho(M)\colon \mathcal{R}_k(\pi_1(M)) &\to \mathbb{R}\\ \alpha &\mapsto \rho(M,\alpha)\end{aligned}$$

bounded? If yes, can one determine upper bounds in terms of the complexity of M?

This question, which remains open, nonetheless served as the starting point for this thesis.

Given $\phi = (\phi_1,\ldots,\phi_m)\colon \pi_1(M) \to \mathbb{Z}^m$ and $\alpha \in R_k(\pi_1(M))$, we consider the torus

$$\begin{aligned}S^1 \times \cdots \times S^1 &\to R_k(\pi_1(M))\\ (z_1,\ldots,z_m) &\mapsto \left(g \mapsto \alpha(g)\prod_{i=1}^m z_i^{\phi_i(g)}\right)\end{aligned}$$

lying in the variety $R_k(\pi_1(M))$. In Chapter 7 we consider the one-dimensional case, which corresponds to a circle lying in $R_k(\pi_1(M))$, and bound the maximum height of the jumps of the ρ-invariant restricted to this circle by the degree of the twisted Alexander polynomial $\Delta^{\alpha\otimes\phi}$. It is defined as the order of the $\mathbb{C}[\mathbb{Z}]$-module $H_1^{\alpha\otimes\phi}\big(M;\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}]\big)$, and the reduced Alexander polynomial $\widetilde{\Delta}^{\alpha\otimes\phi}$ is the order of $\mathrm{Torsion}_{\mathbb{C}[\mathbb{Z}]}\left(H_1^{\alpha\otimes\phi}(M;\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}])\right)$. More precisely, we show:

Theorem 7.2. *Let M be a closed connected oriented 3-dimensional manifold. Furthermore, let $\phi\colon\pi_1(M)\to\mathbb{Z}$ and $\alpha\colon\pi_1(M)\to U(k)$. Given $t\in\mathbb{R}$ we define the group homomorphism*

$$\begin{aligned}\alpha_t^\phi\colon\pi_1(M)&\to U(k)\\ g&\mapsto\alpha(g)e^{it\phi(g)}.\end{aligned}$$

Let ρ_t be the ρ-invariant corresponding to α_t^ϕ. Then $\lim\limits_{t\searrow s}\rho_t$ exists for all $s\in\mathbb{R}$. If

$$\rho_s\neq\lim_{t\searrow s}\rho_t,$$

then the reduced Alexander polynomial $\widetilde{\Delta}^{\alpha\otimes\phi}$ of M has a zero at e^{is}. If $N(e^{is})$ denotes the multiplicity of this zero, then

$$\left|\rho_s-\lim_{t\searrow s}\rho_t\right|\leq N(e^{is}).$$

Furthermore, we have

$$\max\left\{|\rho_s-\rho_t|\mid s,t\in[0,2\pi]\right\}\leq\deg\left(\widetilde{\Delta}^{\alpha\otimes\phi}\right).$$

In Chapter 8 we more generally consider an m-dimensional torus lying in the variety of unitary representations. By using the result of Levine that the ρ-invariant is continuous on sets of the form

$$\left\{\alpha\in R_k\;\middle|\;\sum_{i=0}^{\infty}\dim_{\mathbb{C}}H_i^\alpha(M;\mathbb{C}^k)=r\right\},$$

we relate the set of discontinuities of the ρ-invariant to the zero set of the multivariable Alexander polynomial $\Delta^{\alpha\otimes\phi}$. We obtain the following result, which combines Corollary 8.7 and Corollary 8.12:

Corollary. *Let M be a closed connected oriented 3-dimensional manifold. Furthermore let $\alpha\colon\pi_1(M)\to U(k)$ and let $\phi=(\phi_1,\dots,\phi_m)\colon\pi_1(M)\to\mathbb{Z}^m$ be homomorphisms such that either*

1. *α is irreducible and α restricted to $\ker(\phi)$ is non-trivial or*
2. *α is trivial and ϕ is an epimorphism.*

Let $T^m:=\left(S^1\right)^m$ and given $z=(z_1,\dots,z_m)\in T^m$ consider the deformed representation

$$\begin{aligned}\alpha_z^\phi\colon\pi_1(M)&\to U(k)\\ g&\mapsto\alpha(g)\prod_{i=1}^m z_i^{\phi_i(g)}.\end{aligned}$$

Then

$$\begin{aligned}\rho(M)\colon T^m &\to \mathbb{R}\\ z &\mapsto \rho(M, \alpha_z^\phi)\end{aligned}$$

is constant on the connected components of $T^m \backslash \left(\left\{ z \in T^m \middle| \Delta^{\alpha\otimes\phi}(z) = 0 \right\} \cup \{(1, \ldots, 1)\} \right)$.

A classical example of a situation in which the second condition of the previous corollary is satisfied is that of a 3-dimensional manifold obtained by 0-framed surgery on a knot. Let $K \subset S^3$ be a knot and let M_K be obtained by 0-framed surgery on K. Let $\phi\colon \pi_1(M_K) \to \mathbb{Z}$ be the abelianization and for a given $z \in S^1$ denote by $\psi_z\colon \mathbb{Z} \to U(1)$ the homomorphism with $\psi_z(1) = z$. Then

$$\rho^{(2)}(M_K, \phi) = \int_{z\in S^1} \rho(M_K, \psi_z \circ \phi)\, \mathrm{d}z.$$

holds (see [COT03, Lemma 5.4]).

In Chapter 9 we generalize this result using the previous corollary by showing that the L^2-ρ-invariant of a closed connected oriented 3-dimensional manifold together with a homomorphism $\phi\colon \pi_1(M) \to \mathbb{Z}^m$ can be expressed as an integral over suitable ρ-invariants. More precisely, we obtain the following result:

Theorem 9.13. *Let M be a closed connected oriented 3-dimensional manifold. Furthermore, let $\phi\colon \pi_1(M) \to \mathbb{Z}^l$ be an epimorphism such that the Alexander polynomial Δ^ϕ is not zero. For any $z = (z_1, \ldots, z_l) \in T^l$ we consider the one-dimensional representation*

$$\begin{aligned}\psi_z\colon \mathbb{Z}^l &\to U(1)\\ (n_1, \ldots, n_l) &\mapsto z_1^{n_1} \cdots z_l^{n_l}.\end{aligned}$$

Then

$$\int_{T^l} \rho(M, \psi_z \circ \phi)\, \mathrm{d}z$$

exists and

$$\rho^{(2)}(M, \phi) = \int_{T^l} \rho(M, \psi_z \circ \phi)\, \mathrm{d}z.$$

Outline

In Chapter 2 we collect basic notions and results which we will need later on. In particular, flat connections on vector bundles over smooth manifolds are introduced. Their correspondence to unitary representations of the fundamental group will be crucial for the rest of the thesis.

In Chapter 3 we show that an analytic deformation of a connection of a vector bundle gives rise to an analytic deformation of a unitary representation of the fundamental group of a manifold. Going in the other direction is more subtle, hence we contend ourselves with proving that the analytic deformation obtained by following a circle in the variety of unitary representations of the fundamental group indeed gives rise to an analytic deformation of connections.

In Chapter 4 we give the definition of the Atiyah–Patodi–Singer η-invariant and ρ-invariant and collect their most important properties. In particular, we discuss situations in which the ρ-invariant can be expressed as a signature defect and show that this is the case for homomorphisms that factor through a finite group. We will also recall the known computation of the ρ-invariants of a manifold obtained by 0-framed surgery on a knot in terms of Levine–Tristam signatures.

In Chapter 5 we first give a short introduction to the group von Neumann algebra $\mathcal{N}G$ and the dimension function on $\mathcal{N}G$-modules which we will use to define the L^2-signature. Then we will give the analytic definition of the L^2-η- and the L^2-ρ-invariant as well as the possible definition of the L^2-ρ-invariant as a signature defect. We then use an approximation result of Lück and Schick to relate the L^2-ρ-invariant for a homomorphism to a residually finite groups to the classical ρ-invariant associated to quotients by finite-index subgroups. As an application, we derive the known result that the L^2-ρ-invariant of a manifold obtained by 0-framed surgery on a knot can be expressed as an integral over ρ-invariants.

In Chapter 6 we state the result of Farber and Levine that the heights of the jumps of the η-invariant can be calculated as sums of signatures of suitable linking forms. We construct the relevant linking forms and prove some of their properties by following the original article [FL96].

In Chapter 7 we consider the behavior of the ρ-invariant under the special type of analytic deformation of a unitary representation of the fundamental group of a manifold introduced in Chapter 3. For 3-dimensional manifolds, we will bound the heights of the jumps of the ρ-invariant that occur while deforming the unitary representation along a circle to the multiplicities of the zeros of a twisted Alexander polynomial of the manifold.

In Chapter 8 we consider a generalization of the previous setting in the form of an m-dimensional torus lying in the variety of k-dimensional representations. We relate the location of the jumps of the ρ-invariant appearing as the representation varies on the torus to the zero set of the corresponding Alexander polynomial.

In Chapter 9 we deduce that the Riemann integral of the ρ-invariants of a 3-dimensional manifold computed over a suitably chosen torus exists and equals the L^2-ρ-invariant of an epimorphism from the fundamental group of the manifold to a free abelian group. This generalizes the known result for knots stated in Chapter 5.

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Chapter 2

Preliminaries

In this chapter we collect basic notions and results which we will need later on.

2.1 Conventions

All manifolds, vector bundles over manifolds and sections of such vector bundles considered in this thesis are implicitly assumed to be smooth.

In this thesis, all rings are assumed to be unital and associative, but not necessarily commutative.

2.2 Twisted homology and cohomology

Let R be a ring with unit. An *involution* $i\colon R \to R$ is a map which satisfies

- $i(1) = 1$,
- $i(a+b) = i(a) + i(b)$ for all $a, b \in R$,
- $i(ab) = i(b)i(a)$ for all $a, b \in R$,
- $i(i(a)) = a$ for all $a \in R$.

If it is clear which involution we consider, then we often write $\overline{p} := i(p)$.

Let V be a right R-module. Then V becomes a left R-module, denoted by $\overline{V}$, by using the involution, namely

$$r \cdot v := v\overline{r} \qquad \text{for } r \in R \text{ and } v \in V.$$

If $f\colon V \to V'$ is an R-linear map of right R-modules, then the same map of underlying abelian groups defines an R-linear map $\overline{f}\colon \overline{V} \to \overline{V'}$ of left R-modules. In the same way we can turn a left R-module W into a right R-module $\overline{W}$, and similarly we define $\overline{f}$ for R-linear maps f between such modules.

Let X be a connected topological space with fundamental group π and universal covering $\widetilde{X}$. We always consider the singular chain complex $C_*\widetilde{X}$ as consisting of right $\mathbb{Z}\pi$-modules, where π acts on $C_*\widetilde{X}$ by deck transformations. We always view $\mathbb{Z}\pi$ as a ring with involution

$$\begin{aligned} i\colon \mathbb{Z}\pi &\to \mathbb{Z}\pi \\ \sum_{g\in\pi} a_g g &\mapsto \sum_{g\in\pi} a_g g^{-1}. \end{aligned}$$

Let A be a left $\mathbb{Z}\pi$-module. Then we can form the tensor product

$$C_*\widetilde{X} \otimes_{\mathbb{Z}\pi} A$$

and the *homology with coefficients in* A is defined by

$$H_i(X;A) \coloneqq H_i(C_*\widetilde{X} \otimes_{\mathbb{Z}\pi} A).$$

Since $C_*\widetilde{X}$ is a right $\mathbb{Z}\pi$-chain complex, we can consider the mapping chain complex $\operatorname{Hom}_{\mathbb{Z}\pi}(\overline{C_*\widetilde{X}}, A)$. Then the *cohomology of* X *with coefficients in* A is defined by

$$H^i(X;A) \coloneqq H_i\big(\operatorname{Hom}_{\mathbb{Z}\pi}\big(\overline{C_*\widetilde{X}}, A\big)\big).$$

2.3 Cup product with twisted coefficients

Let X be a connected topological space with fundamental group π and universal covering $\widetilde{X}$. Further let A and B be left $\mathbb{Z}\pi$-modules. Then $A \otimes_{\mathbb{Z}} B$ is a left $\mathbb{Z}\pi$-module, where the action is given by

$$\begin{aligned} \pi \times A \otimes_{\mathbb{Z}} B &\to A \otimes_{\mathbb{Z}} B \\ (g, a\otimes b) &\mapsto (g\cdot a) \otimes (g \cdot b). \end{aligned}$$

Let $\phi \in C^k(X;A) = \operatorname{Hom}_{\mathbb{Z}\pi}\big(\overline{C_k\widetilde{X}}, A\big)$ and $\psi \in C^l(X;B) = \operatorname{Hom}_{\mathbb{Z}\pi}\big(\overline{C_l\widetilde{X}}, B\big)$. We consider

$$\begin{aligned} \phi \cup \psi\colon C_{k+l}(\widetilde{X}) &\to A \otimes_{\mathbb{Z}} B \\ (\sigma\colon \Delta^{k+l} \to X) &\mapsto \big(\phi \circ \sigma([v_0, \cdots, v_k])\big) \otimes_{\mathbb{Z}} \big(\psi \circ \sigma([v_k, \cdots, v_{k+l}])\big), \end{aligned}$$

where Δ^{k+l} is the standard $(k+l)$-simplex on $k+l+1$ vertices $v_0, \ldots, v_{k+l}$.

One can show that the cup product descends to a map on cohomology as in [Fri19, Lemma 57.3].

Lemma 2.1. *Let X be a connected topological space with fundamental group π and universal covering $\widetilde{X}$. Let A and B be left $\mathbb{Z}\pi$-modules. Let $\phi \in C^k(X;A)$ and $\psi \in C^l(X;B)$. Then*

$$\cup\colon H^k(X;A) \times H^l(X;B) \to H^{k+l}(X; A \otimes_{\mathbb{Z}} B)$$

is a well-defined $\mathbb{Z}$-bilinear map. The action of π on $A \otimes_{\mathbb{Z}} B$ is given by the diagonal action

$$\begin{aligned} \pi \times A \otimes_{\mathbb{Z}} B &\to A \otimes_{\mathbb{Z}} B \\ (g, a\otimes b) &\mapsto (g\cdot a) \otimes (g \cdot b). \end{aligned}$$

We need the following well-known properties of the cup product later on.

Lemma 2.2 ([Fri19, Lemma 57.2]). *Let X be a connected topological space with fundamental group π and universal covering $\widetilde{X}$. Let A and B be left $\mathbb{Z}\pi$-modules. Let $\phi \in C^k(X;A)$ and $\psi \in C^l(X;B)$. Then*

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi \in C^{k+l+1}(X; A \otimes_{\mathbb{Z}} B).$$

Proposition 2.3 ([Fri19, Proposition 57.8]). *Let X be a connected topological space with fundamental group π and universal covering $\widetilde{X}$. Let A and B be left $\mathbb{Z}\pi$-modules. Let ν be the map*

$$\begin{aligned} \tau\colon A \otimes_{\mathbb{Z}} B &\to B \otimes_{\mathbb{Z}} A \\ a \otimes b &\mapsto b \otimes a. \end{aligned}$$

Let $\phi \in C^k(X;A)$ and $\psi \in C^l(X;B)$. Then

$$\phi \cup \psi = (-1)^{kl} \tau_*(\psi \cup \phi).$$

2.4 Cap product with twisted coefficients

Let X be a connected topological space with fundamental group π and universal covering $\widetilde{X}$. Let A and B be left $\mathbb{Z}\pi$-modules. Furthermore let $\phi \in C^k(X;A) = \operatorname{Hom}_{\mathbb{Z}\pi}(\overline{C_*\widetilde{X}}, A)$ and $\sigma_b = \sigma \otimes b \in C_l(X;B)$. For $l \geq k$ we define

$$\phi \cap \sigma_b := \underbrace{\sigma \circ [v_k, \cdots, v_l]}_{\Delta_{l-k} \to \widetilde{X}} \otimes_{\mathbb{Z}\pi} \Big(\underbrace{\phi \circ \sigma([v_0, \cdots, v_k])}_{\in A} \otimes_{\mathbb{Z}} b \Big) \in C_{l-k}(X; A \otimes_{\mathbb{Z}} B).$$

The cap product then defines a map on cohomology (see for example [Fri19, Lemma 59.2]).

Lemma 2.4. *Let X be a connected topological space with fundamental group π and universal covering $\widetilde{X}$. Let A and B be left $\mathbb{Z}\pi$-modules. Let $\phi \in C^k(X;A) = \operatorname{Hom}_{\mathbb{Z}\pi}(\overline{C_*\widetilde{X}}, A)$ and $\sigma \in C_l(X;B)$. If $l \geq k$, we get a well-defined map*

$$\cap\colon H^k(X;A) \times H_l(X;B) \to H_{l-k}(X; A \otimes_{\mathbb{Z}} B).$$

The following lemma relates the cap product to the cup product (see for example [Fri19, Lemma 59.7]).

Lemma 2.5. *Let X be a connected topological space with universal covering $\widetilde{X}$ and fundamental group π. Let A and B be left $\mathbb{Z}\pi$-modules. Let $l, m, n \in \mathbb{N} \cup \{0\}$ such that $n \geq m + l$. Let $x \in H^l(X;A)$, $y \in H^m(X;B)$ and $z \in H_n(X;\mathbb{Z})$. Then*

$$x \cap (y \cap z) = (y \cup x) \cap z \in H_{n-m-l}(X; A \otimes_{\mathbb{Z}} B).$$

2.5 Poincaré duality with twisted coefficients

We need the following version of Poincaré duality which is proven in [CLM18, Theorem 4.44 and Theorem 4.51].

Theorem 2.6. *Let M be a closed connected oriented n-dimensional manifold with fundamental group π. Let A be a left $\mathbb{Z}\pi$-module and $\alpha\colon \pi_1(M) \to \mathrm{Aut}_{\mathbb{Z}\pi}(A)$ a homomorphism. Let $[M] \in H_n(M;\mathbb{Z})$ be the fundamental class. Then the map*

$$\begin{aligned}\mathrm{PD}^{-1}\colon H^k(M;A) &\to H_{n-k}(M;A)\\ x &\mapsto x \cap [M]\end{aligned}$$

is an isomorphism for all $k \geq 0$.

For manifolds with boundary a similar statements holds (see [Wal04, Page 3]):

Theorem 2.7. *Let M be a connected oriented n-dimensional manifold with boundary ∂M and fundamental group π. Let A be a left $\mathbb{Z}\pi$-module and $\alpha\colon \pi_1(M) \to \mathrm{Aut}_{\mathbb{Z}\pi}(A)$ a homomorphism. Let $[M] \in H_n(M, \partial M;\mathbb{Z})$ be the fundamental class. Then the maps*

$$\begin{aligned}\mathrm{PD}^{-1}\colon H^k(M,\partial M;A) &\to H_{n-k}(M;A)\\ x &\mapsto x \cap [M]\end{aligned}$$

and

$$\begin{aligned}\mathrm{PD}^{-1}\colon H^k(M;A) &\to H_{n-k}(M,\partial M;A)\\ x &\mapsto x \cap [M]\end{aligned}$$

are isomorphisms for all $k \geq 0$.

2.6 Signature of a manifold

Let V be a finite dimensional $\mathbb{C}$-vector space and $h\colon V \times V \to \mathbb{C}$ a Hermitian form. Then h can be represented by a Hermitian matrix A_h. The *signature* of h, denoted by $\mathrm{sign}(h)$, is defined to be the number of positive minus the number of negative eigenvalues of A_h and does not depend on the choice of A_h. Let $k\colon V \times V \to \mathbb{C}$ be a skew-Hermitian form. Then $i \cdot k(-,-)$ is a Hermitian form and the signature of k is defined by $\mathrm{sign}(k) := \mathrm{sign}(i \cdot k)$.

Let M be a compact connected oriented manifold of dimension $2n$ with boundary ∂M, possibly empty, and fundamental group π. Let $[M] \in H_{2n}(M, \partial M;\mathbb{Z})$ be the fundamental class.

The untwisted *intersection form* on M is defined by

$$\begin{aligned}I_M\colon H_n(M;\mathbb{C}) \times H_n(M;\mathbb{C}) &\to \mathbb{C}\\ (a,b) &\mapsto \langle \mathrm{PD}(a) \cup \mathrm{PD}(b), [M]\rangle,\end{aligned}$$

where $\mathrm{PD}(a) \cup \mathrm{PD}(b) \in H^{2n}(M, \partial M; \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{C})$ is considered as an element in $H^{2n}(M, \partial M; \mathbb{C})$ by using the map

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{C} &\to \mathbb{C} \\ (a,b) &\mapsto a\overline{b}. \end{aligned}$$

The intersection form is well-known to be Hermitian if n is even and skew-Hermitian if n is odd. If M is closed, then I_M is non-degenerate. The *signature of* M is defined by

$$\mathrm{sign}(M) = \mathrm{sign}(I_M).$$

Let $\alpha \colon \pi_1(M) \to \mathrm{U}(k)$. Then we can consider homology with coefficients twisted by α, and we denote by $H_n^\alpha(M; \mathbb{C}^k)$ the homology $H_n(C_*(\widetilde{M}) \otimes_{\mathbb{Z}\pi} \mathbb{C}^k)$. The *intersection form on* M *twisted with* α is defined by

$$\begin{aligned} I_M^\alpha \colon H_n^\alpha(M; \mathbb{C}^k) \times H_n^\alpha(M; \mathbb{C}^k) &\to \mathbb{C} \\ (a,b) &\mapsto \left\langle \mathrm{PD}(a) \cup \mathrm{PD}(b), [M] \right\rangle, \end{aligned}$$

where, again, $\mathrm{PD}(a) \cup \mathrm{PD}(b) \in H^{2n}_{\alpha\otimes\alpha}(M, \partial M; \mathbb{C}^k \otimes_{\mathbb{Z}} \mathbb{C}^k)$ is considered as an element in $H^{2n}(M, \partial M; \mathbb{C})$ by using the left π-invariant map

$$\begin{aligned} \mathbb{C}^k \otimes_{\mathbb{Z}} \mathbb{C}^k &\to \mathbb{C} \\ (a,b) &\mapsto a^t\overline{b}. \end{aligned}$$

If M is closed, then I_M^α is again non-degenerate. It is Hermitian in case that n is even and skew-Hermitian in case that n is odd. The *signature of* M *twisted with* α, denoted by $\mathrm{sign}_\alpha(M)$, is defined to be the signature of I_M^α.

Note that if $\alpha \colon \pi_1(M) \to \mathbb{C}^k$ is trivial then $k\,\mathrm{sign}(M) = \mathrm{sign}_\alpha(M)$.

2.7 Order of a module

We want to define the order of a module over a commutative Noetherian unique factorization domain R. For example, if H is a free abelian group and if $\mathbb{F}$ is a field then the group ring $\mathbb{F}[H]$ is a commutative Noetherian unique factorization domain. In the following, we will mostly consider the ring $\mathbb{C}[\mathbb{Z}^l]$.

Let A be a finitely generated R-module. Since R is Noetherian, there is a resolution of A by free finite rank R-modules, called a *presentation* of A, of the form

$$R^r \xrightarrow{P\cdot} R^s \longrightarrow A \longrightarrow 0,$$

where P is a matrix with entries in R. Moreover, we can assume that $s \leq r$: If s was bigger than r, we could replace r by s and add $(s-r)$ zero columns to P.

Let $E(A)$ be the ideal in R generated by all $(s \times s)$-minors of P. It is known that $E(A)$ does not depend on the choice of a presentation of A (see for example [Tur01, Lemma 4.4]).

Since R is a unique factorization domain, there exists a unique smallest principal ideal of R that contains $E(A)$. A generator of this principal ideal is called the *order* of A. The order of A is well-defined up to multiplication by a unit in R. More details are given in [Tur01, Chapter 4.1].

The following well-known statement is for example shown in [Tur01, Remark 4.5].

Lemma 2.8. *Let A be a finitely generated module over a Noetherian unique factorization domain. Then* $\operatorname{ord}(A) \neq 0$ *if and only if A is an R-torsion module.*

2.8 Alexander polynomial

Let M be a compact connected manifold and $\phi\colon \pi_1(M) \to \mathbb{Z}^l$ a homomorphism. Let $\alpha\colon \pi_1(M) \to U(k)$ be a unitary representation. We consider the left action of π on the $\mathbb{C}[\mathbb{Z}^l]$-module $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]$ given by

$$\begin{aligned}\pi \times \Big(\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]\Big) &\to \mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]\\ (g,(v\otimes x)) &\mapsto (\alpha(g)v \otimes \phi(g)x).\end{aligned}$$

This situation will be common throughout the rest of this thesis, and we will denote a dependence of an invariant on such a twist by an exponent $\alpha \otimes \phi$.

Since M is compact, the cellular $\mathbb{C}[\mathbb{Z}^l]$-chain complex $C_*^{\mathrm{CW}}\big(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]\big)$ is finitely generated. As $\mathbb{C}\big[\mathbb{Z}^l\big]$ is Noetherian, it follows that $H_i^{\alpha\otimes\phi}\big(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]\big)$ is a finitely generated $\mathbb{C}[\mathbb{Z}^l]$-module for each i.

Definition 2.9. The *i-th twisted Alexander polynomial* $\Delta_i^{\alpha\otimes\phi}$ of (M,ϕ,α) is defined to be the order of $H_i^{\alpha\otimes\phi}\big(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]\big) \in \mathbb{C}\big[\mathbb{Z}^l\big]$.

Note that it is a consequence of Lemma 2.8 that the Alexander polynomial is zero unless $H_i^{\alpha\otimes\phi}\big(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]\big)$ is $\mathbb{C}\big[\mathbb{Z}^l\big]$-torsion. Therefore, it will prove useful to also study the Alexander polynomial of $\operatorname{Torsion}_{\mathbb{C}[\mathbb{Z}^l]}\big(H_i^{\alpha\otimes\phi}\big(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]\big)\big)$.

Definition 2.10. The *reduced i-th twisted Alexander polynomial* $\widetilde{\Delta}_i^{\alpha\otimes\phi}$ of (M,ϕ,α) is defined to be the order of $\operatorname{Torsion}_{\mathbb{C}[\mathbb{Z}^l]}\big(H_i^{\alpha\otimes\phi}\big(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]\big)\big)$. If $i=1$, we sometimes drop i from the notation.

Clearly, twisted Alexander polynomials are only well-defined up to multiplication by a unit in $\mathbb{C}\big[\mathbb{Z}^l\big]$, which are of the form ax with $0 \neq a \in \mathbb{C}$ and $x \in \mathbb{Z}^l$.

The twisted Alexander polynomial was for example studied by Friedl and Vidussi in [FV11] and by Friedl and Kim in [FK06].

Friedl and Vidussi showed in [FV11, Proposition 2] that under certain assumptions the zeroth Alexander polynomial is one.

Proposition 2.11. *Let M be a compact connected oriented manifold, whose boundary is empty or consists of tori. Let $\alpha\colon \pi_1(M) \to U(k)$ and $\phi\colon \pi_1(M) \to \mathbb{Z}$ be non-trivial. If α is irreducible and α restricted to* $\ker(\phi)$ *is non-trivial, then $\Delta_0^{\alpha\otimes\phi} = 1$ up to multiplication by a unit.*

Let $n, m \in \mathbb{Z}$. Let $p = \sum_{i=n}^{m} a_i t^i \in \mathbb{C}[\mathbb{Z}]$ with $a_m \neq 0$ and $a_n \neq 0$. Then the *degree* of p is defined to be $m - n$. Note that since Alexander polynomials are well-defined up to multiplication by a monomial, it follows that the degree of an Alexander polynomial in one variable is well-defined.

For an n-dimensional manifold M, we denote by $b_i(M)$ the i-th Betti number of M. In particular, $b_n(M) = 1$ if M is connected and closed and $b_n(M) = 0$ if M is connected and has non-empty boundary.

Let R be a commutative ring with involution i. For such a ring, the free modules R^n are equipped with R-sesquilinear scalar products given by

$$\langle v, w \rangle = v^t \cdot i(w).$$

Let $\beta\colon \pi_1(M) \to \mathrm{GL}(k, R)$. Then we denote by $\beta^\dagger\colon \pi_1(M) \to \mathrm{GL}(k, R)$ the unique representation which is determined by

$$\left\langle \beta(g^{-1})v, w \right\rangle = \left\langle v, \beta^\dagger(g)w \right\rangle.$$

The following proposition is proven in [FK06, Proposition 2.5].

Proposition 2.12. *Let M be a compact connected oriented 3-dimensional manifold whose boundary is empty or consists of tori and let $\phi\colon \pi_1(M) \to \mathbb{Z}$ be non-trivial. Furthermore let $\alpha\colon \pi_1(M) \to \mathrm{GL}(k, \mathbb{C})$ be a representation such that $\Delta_1^{\alpha\otimes\phi} \neq 0$.*

1. *If M is closed, then*

$$\Delta_2^{\alpha\otimes\phi}(t) = \Delta_0^{(\alpha\otimes\phi)^\dagger}(t^{-1})$$

up to multiplication by a unit.

2. *If M has non-empty boundary, then $\Delta_2^{\alpha\otimes\phi} = 1$ up to multiplication by a unit.*

In particular it is $\deg\left(\Delta_2^{\alpha\otimes\phi}\right) = b_3(M)\deg\left(\Delta_0^{(\alpha\otimes\phi)^\dagger}\right)$. If α is a unitary representation, then $\alpha = \alpha^\dagger$ and $(\alpha\otimes\phi)^\dagger = \alpha^\dagger \otimes (-\phi) = \alpha \otimes (-\phi)$ and hence,

$$\deg\left(\Delta_2^{\alpha\otimes\phi}\right) = b_3(M)\deg\left(\Delta_0^{\alpha\otimes\phi}\right).$$

2.9 Thurston norm

In 1986, Thurston defined a seminorm on $H^1(M;\mathbb{Z})$ of a compact orientable 3-dimensional manifold M by assigning to a cohomology class a truncated Euler characteristic of a representing embedded surface.

Let S be a surface with connected components S_i for $i = 1, \ldots, k$. We define

$$\chi_-(S) = \sum_{i=1}^{k} \max\{-\chi(S_i), 0\}.$$

Let M be a compact connected orientable 3-dimensional manifold. Let $\phi \in H^1(M;\mathbb{Z})$. The *Thurston norm* $\|\phi\|_T$ of ϕ is defined as

$$\|\phi\|_T = \min\{\chi_-(S) \mid S \subset M \text{ properly embedded surface representing } \mathrm{PD}^{-1}(\phi)\}.$$

If the manifold M is irreducible, it was shown by Thurston in [Thu86] that the Thurston norm defines a seminorm on $H^1(M;\mathbb{Z})$ which can be uniquely extended to a seminorm on $H^1(M;\mathbb{R})$.

Friedl and Kim showed in [FK06, Theorem 1.1] that the Thurston norm gives an upper bound on the degree of a one-variable Alexander polynomial:

Theorem 2.13. *Let M be a compact connected oriented 3-dimensional manifold whose boundary is empty or consists of tori. Furthermore let $\phi\colon \pi_1(M) \to \mathbb{Z}$ be non-trivial and $\alpha\colon \pi_1(M) \to \mathrm{GL}(k,\mathbb{C})$ a representation such that $\Delta_1^{\alpha\otimes\phi} \neq 0$. Then*

$$\|\phi\|_T \geq \frac{1}{k}\left(\deg\left(\Delta_1^{\alpha\otimes\phi}\right) - \deg\left(\Delta_0^{\alpha\otimes\phi}\right) - \deg\left(\Delta_2^{\alpha\otimes\phi}\right)\right).$$

2.10 Ring of germs of holomorphic functions

We denote by $\mathcal{O}$ the ring of germs of holomorphic functions at zero. An element in $\mathcal{O}$ can be represented by a holomorphic function $f\colon U \to \mathbb{C}$, where U is an open neighborhood of $0 \in \mathbb{C}$. In the following we will not distinguish between the function and its germ.

It is well-known that holomorphic functions can be written as power series

$$f(z) = \sum_{i=0}^{\infty} a_i z^i$$

with radius of convergence bigger than zero and the other way round that such power series are holomorphic inside their domain of convergence. Furthermore, a holomorphic function f defined in a neighborhood of 0 in $\mathbb{C}$ can be recovered from its values on the real line, and hence we will often restrict its domain of definition to a real interval $(-\epsilon,\epsilon)$ around 0. Note that the notion of the radius of convergence of a power series is independent of whether we consider its variable to take real or complex values.

We consider $\mathcal{O}$ with the involution which is induced by complex conjugation

$$\begin{aligned} -^*\colon \mathcal{O} &\to \mathcal{O} \\ h &\mapsto \left(z \mapsto \overline{h(\overline{z})}\right). \end{aligned}$$

Note that if $h(z) = \sum_{i=0}^{\infty} a_i z^i$, then $h^*(z) = \sum_{i=0}^{\infty} \overline{a_i} z^i$, which is a power series with the same radius of convergence. The involution readily extends to modules of the form $\mathcal{O}^k$ by coordinate-wise application.

We get an $\mathcal{O}$-valued form

$$\begin{aligned} \mathcal{O}^k \times \mathcal{O}^k &\to \mathcal{O} \\ (v,w) &\mapsto v^t w^*, \end{aligned}$$

which is $\mathcal{O}$-linear in the first variable and $\mathcal{O}$-anti-linear in the second variable.

Later on we will often consider homology with coefficients in $\mathcal{O}$, for which the following lemma will prove useful.

Lemma 2.14. *The ring $\mathcal{O}$ is a principal ideal domain.*

Proof. Let $I \subset \mathcal{O}$, $I \neq 0$ be an ideal. Let $a = \max\{n \in \mathbb{N} \cup \{0\} \mid p(z) \in z^n\mathcal{O} \text{ for all } p \in I\}$. Let J be the ideal which is generated by z^a. We want to show that $I = J$.

We first show $I \subset J$. Let $p \in I$. Since $p(z) \in z^a\mathcal{O}$, there exists $\tilde{p}(z) \in \mathcal{O}$ with $p(z) = \tilde{p}(z)z^a$. But that means $p(z) \in J$.

We now show $I \supset J$. We have to show that $z^a \in I$. It follows from the definition of a that there exists a $p \in I$ such that $p(z) = z^a\tilde{p}(z)$ and $\tilde{p}(z) \in \mathcal{O}$ with $\tilde{p}(0) \neq 0$. Then $1/\tilde{p}(z) \in \mathcal{O}$ and hence $z^a = p(z)/\tilde{p}(z) \in I$. □

2.11 Field of germs of meromorphic functions

We denote by $\mathcal{M}$ the field of germs of meromorphic functions at the origin, which arises as the quotient field of the integral domain $\mathcal{O}$. In the following we will not distinguish between a meromorphic function and its germ.

If f is a meromorphic function, there exists a neighborhood U of 0 in $\mathbb{C}$ and an $N \in \mathbb{N} \cup \{0\}$ such that f can be written on U as a bounded-below Laurent series

$$f(z) = \sum_{i=-N}^{\infty} a_i z^i,$$

where the non-negative part $\sum_{i=0}^{\infty} a_i z^i$ defines a holomorphic function on U. Vice versa, each power series of the form $\sum_{i=-N}^{\infty} a_i z^i$ such that $\sum_{i=0}^{\infty} a_i z^i$ has positive radius of convergence defines a meromorphic function on a neighborhood U of $0 \in \mathbb{C}$. Just as for holomorphic functions, a germ of a meromorphic function around 0 is determined by its values on the real line.

Similar to the ring $\mathcal{O}$ we consider $\mathcal{M}$ as a field with involution

$$\begin{aligned} -^*\colon \mathcal{M} &\to \mathcal{M} \\ h &\mapsto \big(z \mapsto \overline{h(\overline{z})}\big). \end{aligned}$$

Note that if $h \in \mathcal{M}$ and $h(z) = \sum_{i=-N}^{\infty} a_i z^i$, then $h^*(z) = \sum_{i=-N}^{\infty} \overline{a_i} z^i$, and hence the involution is again well-defined. We extend it to vectors of germs of meromorphic functions in the obvious way.

We get an $\mathcal{M}$-valued form

$$\begin{aligned} \mathcal{M}^k \times \mathcal{M}^k &\to \mathcal{M} \\ (v, w) &\mapsto v^t w^* \end{aligned}$$

which is $\mathcal{M}$-linear in the first variable and $\mathcal{M}$-anti-linear in the second variable.

2.12 Flat vector bundles and connections

This section is a short introduction to connections and flat vector bundles. A good reference on the properties of connections is [Kob14].

2.12.1 Connections

Let M be a manifold and $\mathcal{E}$ a complex vector bundle over M. We denote by $\Gamma(\mathcal{E})$ the $\mathbb{C}$-vector space of sections. A *differential form of degree* k is a section of the exterior power bundle $\Lambda^k T^*M$. To shorten the notation we write

$$\Omega^k(M) := \Gamma(\Lambda^k T^*M).$$

An $\mathcal{E}$*-valued differential form of degree* p is an element in

$$\Omega^p(M,\mathcal{E}) := \Gamma(\mathcal{E} \otimes \Lambda^p T^*M) = \Gamma(\mathcal{E}) \otimes_{\Omega^0(M)} \Gamma(\Lambda^p T^*M) = \Gamma(\mathcal{E}) \otimes_{\Omega^0(M)} \Omega^p(M).$$

Definition 2.15. Let M be a manifold and $\mathcal{E}$ a complex vector bundle over M. A *connection* on $\mathcal{E}$ is a $\mathbb{C}$-bilinear map

$$\begin{aligned} \nabla\colon \Gamma(TM) \times \Gamma(\mathcal{E}) &\to \Gamma(\mathcal{E}) \\ (X,s) &\mapsto \nabla_X s \end{aligned}$$

which satisfies for all tangent fields $X \in \Gamma(TM)$, all sections $s \in \Gamma(\mathcal{E})$ and smooth functions $f \in C^\infty(M)$ the following conditions

1. $\nabla_{fX} s = f \nabla_X s$,
2. $\nabla_X(fs) = f\nabla_X s + (Xf)s$,

where Xf denotes the directional derivative of f along the vector field X.

Equivalently, we can view a connection also as a first order differential operator

$$\nabla\colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E} \otimes T^*M)$$

satisfying for all $f\colon M \to \mathbb{C}$ smooth and all $s \in \Gamma(\mathcal{E})$

$$\nabla(fs) = s\,\mathrm{d}f + f\nabla s.$$

Then we can extend a connection ∇ to a map

$$\nabla\colon \bigoplus_{p\geq 0} \Gamma(\mathcal{E} \otimes \Lambda^p T^*M) \to \bigoplus_{p\geq 0} \Gamma(\mathcal{E} \otimes \Lambda^{p+1} T^*M)$$

by setting for $s \in \Gamma(\mathcal{E})$ and $\phi \in \Gamma(\Lambda^p T^*M)$

$$\nabla(s \otimes \phi) = \nabla s \wedge \phi + s\,\mathrm{d}\phi.$$

Lemma 2.16. *Let $\mathcal{E} \to M$ be a complex vector bundle and ∇_1, ∇_2 be two connections on $\mathcal{E}$. Then*

$$\nabla_1 - \nabla_2 \in \Omega^1(M; \mathrm{End}(\mathcal{E})).$$

Let ∇ be a connection on $\mathcal{E}$ and $A \in \Omega^1(M;\mathrm{End}(\mathcal{E}))$, then $\nabla + A$ is a connection on $\mathcal{E}$.

Proof. Let $s \in \Gamma(E)$ and $f \in C^\infty(M)$. Let ∇_1, ∇_2 be two connections. It follows easily from the properties of a connection that

$$(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 s - \nabla_2 s),$$

i.e., $(\nabla_1 - \nabla_2)$ is $C^\infty(M)$-linear and hence $\nabla_1 - \nabla_2 \in \Omega^1(M; \mathrm{End}(\mathcal{E}))$. If ∇ is a connection and $A \in \Omega^1(M; \mathrm{End}(\mathcal{E}))$, then the map

$$\begin{aligned} \Gamma(TM) \times \Gamma(\mathcal{E}) &\to \Gamma(\mathcal{E}) \\ (X, s) &\mapsto \nabla_X s + A(X)(s) \end{aligned}$$

satisfies

- $(\nabla + A)_{fX}(s) = f(\nabla_X(s) + A(X)(s)) = f(\nabla + A)_X(s)$,
- $(\nabla + A)_X(fs) = f\nabla_X s + (Xf)s + fA(X)(s) = f(\nabla + A)_X(s) + (Xf)s$. □

Put differently, the space of connections on a complex vector bundle $\mathcal{E} \to M$ is the affine space $\nabla + \Omega^1(M; \mathrm{End}(\mathcal{E}))$ for any fixed connection ∇.

Example 2.17. Let M be a manifold. We consider the trivial complex vector bundle over M of rank n. Let $f\colon M \to \mathbb{C}^n$ be smooth. Then

$$\begin{aligned} s\colon M &\to M \times \mathbb{C}^n \\ m &\mapsto (m, f(m)) \end{aligned}$$

is a section of $M \times \mathbb{C}^n$ and a connection on $M \times \mathbb{C}^n$ is defined by $\nabla_X s = Xf$ for any vector field X.

Lemma 2.18. *Every complex vector bundle admits a connection.*

Proof. Let $\mathcal{E} \to M$ be a complex vector bundle. Let $\{U_i\}_{i\in I}$ be a covering of M such that $\mathcal{E}|_{U_i}$ is trivial. Choose a connection ∇_i of $\mathcal{E}|_{U_i}$, for example the connection constructed in Example 2.17. Let $\{\rho_i\}_{i\in I}$ be a partition of unity such that the support of ρ_i is contained in U_i. Then it is easy to see that

$$\nabla s = \sum_{i\in I} \rho_i \nabla_i (s_{|U_i})$$

defines a connection on $\mathcal{E}$. □

Later on we will need the following constructions of canonical connections on tensor products and pull-backs of bundles with connections.

Theorem 2.19 ([AH11, Theorem 2.56])**.** *Let M and N be manifolds and let $f\colon M \to N$ be smooth. Let $\mathcal{E} \to N$ be a complex vector bundle over N with connection ∇. Then a connection $f^*\nabla$on $f^*\mathcal{E}$ is defined by*

$$(f^*\nabla_X)(f^* s) = f^*\big(\nabla_{\mathrm{d}f(X)} s\big) \qquad \text{for all } X \in TM \text{ and } s \in \Gamma(\mathcal{E}).$$

Lemma 2.20 ([RS17, Remark 1.5.9(3)])**.** *Let M be a manifold. Let $\mathcal{E}, \mathcal{F}$ be complex vector bundles over M with connections $\nabla_{\mathcal{E}}$ and $\nabla_{\mathcal{F}}$, respectively. Then a connection on the tensor product bundle $\mathcal{E} \otimes \mathcal{F}$ is given by*

$$(\nabla_{\mathcal{E}} \otimes \nabla_{\mathcal{F}})(s_{\mathcal{E}} \otimes s_{\mathcal{F}}) = \nabla_{\mathcal{E}} s_{\mathcal{E}} \otimes s_{\mathcal{F}} + s_{\mathcal{E}} \otimes \nabla_{\mathcal{F}} s_{\mathcal{F}}.$$

2.12.2 Parallel transport

Let $\mathcal{E} \to M$ be a complex vector bundle with connection ∇. Let $\gamma\colon [0,1] \to M$ be a path in M. We call a map $s\colon [0,1] \to \mathcal{E}$ a *flat section along* γ if $s(t) \in \mathcal{E}_{\gamma(t)}$ and

$$\nabla_{\gamma'(t)} s(t) = 0 \qquad \text{for all } t \in [0,1].$$

Note that finding for a given $v \in \mathcal{E}_{\gamma(0)}$ a flat section along γ corresponds to solving a system of ordinary differential equations with initial value $s(0) = v$. It follows that for $\gamma\colon [0,1] \to M$ and $v \in \mathcal{E}_{\gamma(0)}$ there exists a flat section along γ with $s(0) = v$ and it is uniquely determined. More details can be found in [Kob14, Chapter 1.1].

We call $t \mapsto s(t)$ the *parallel transport* of $s(0)$ along γ. For a fixed curve γ parallel transport along γ induces a linear map

$$\Gamma(\gamma)\colon \mathcal{E}_{\gamma(0)} \to \mathcal{E}_{\gamma(1)}.$$

Note that the map $\Gamma(\gamma)$ in general depends on the chosen path γ and not only on $\gamma(0)$ and $\gamma(1)$.

2.12.3 Flat connections

Definition 2.21. Let $\mathcal{E} \to M$ be a complex vector bundle. A (local) section s defined on a neighborhood $U \subseteq M$ is called *flat* if for all vector fields $X \in \Gamma(TM|_U)$

$$\nabla_X s = 0.$$

Definition 2.22. A connection on a vector bundle $\mathcal{E} \to M$ is called *flat* if for every $x \in M$ and $e \in \mathcal{E}_x$ there exists a flat local section s defined in a neighborhood around x with $s(x) = e$.

Definition 2.23. A vector bundle equipped with a flat connection is called a *flat vector bundle.*

Example 2.24. We consider the vector bundle $M \times \mathbb{C}^n \to M$. Let $f\colon M \to \mathbb{C}^n$ be smooth. Then

$$\begin{aligned} s\colon M &\to M \times \mathbb{C}^n \\ m &\mapsto (m, f(m)) \end{aligned}$$

is a section of $M \times \mathbb{C}^n$ and a connection on $M \times \mathbb{C}^n$ is defined by $\nabla_X s = Xf$. Then for every $v \in \mathbb{C}^n$ a flat global section is defined by

$$\begin{aligned} M &\to M \times \mathbb{C}^n \\ m &\mapsto (m, v). \end{aligned}$$

In case that $\mathcal{E} \to M$ is a flat vector bundle it is well-known that parallel transport only depends on the homotopy class of a path with fixed endpoints (see [Kob14, Chapter 1.2] for details).

Let $\mathcal{E} \to M$ be a complex vector bundle of rank k with flat connection ∇. Let γ be a path and denote by

$$\Gamma(\gamma): \mathcal{E}_{\gamma(0)} \to \mathcal{E}_{\gamma(1)}$$

the map which sends $v \in \mathcal{E}_{\gamma(0)}$ to an element $w \in \mathcal{E}_{\gamma(1)}$, where w is obtained by parallel transport of v along γ. If $\gamma(0) = \gamma(1)$, then $\Gamma(\gamma)$ can be represented by an element in $\mathrm{Aut}_{\mathbb{C}}(\mathcal{E}_{\gamma(0)})$. Since $\mathcal{E}$ is equipped with a flat connection, the map

$$\begin{aligned} \mathrm{hol}: \pi_1(M, m_0) &\to \mathrm{Aut}_{\mathbb{C}}(\mathcal{E}_{m_0}) \cong \mathrm{GL}(k, \mathbb{C}) \\ [\gamma] &\mapsto \Gamma(\gamma) \end{aligned}$$

does not depend on the element representing $[\gamma] \in \pi_1(M, m_0)$. The map hol is called the *holonomy of* $\mathcal{E}$ (see [Kob14, Chapter 1.2]).

Definition 2.25. Let $\mathcal{E} \to M$ be a complex vector bundle. A *flat structure* on $\mathcal{E}$ is a choice of local trivializations $\{U_i, \phi_i\}$ such that the transition maps

$$\phi_j^{-1} \circ \phi_i : (U_i \cap U_j) \times \mathbb{C}^k \to (U_i \cap U_j) \times \mathbb{C}^k$$

are of the form $(\phi_j^{-1} \circ \phi_i)(x, v) = (x, A^{i,j} v)$, where $A^{i,j} \in \mathrm{GL}(k, \mathbb{C})$ does not depend on the choice of $x \in (U_i \cap U_j)$.

We now give an important example of a vector bundle which can be equipped with a flat structure.

Example 2.26. Let M be a connected manifold and let α be a k-dimensional representation of the fundamental group. Denote by $\widetilde{M}$ the universal covering of M. We consider the vector bundle

$$\mathcal{E}_\alpha = \widetilde{M} \times \mathbb{C}^k / \sim,$$

where the equivalence relation is obtained from the right-action of π on $\widetilde{M} \times \mathbb{C}^k$ which is given by

$$\begin{aligned} \widetilde{M} \times \mathbb{C}^k \times \pi &\to \widetilde{M} \times \mathbb{C}^k \\ ((m, v), g) &\mapsto (m \cdot g, \alpha(g^{-1}) v). \end{aligned}$$

Then one can equip $\mathcal{E}_\alpha$ with a flat structure coming from the covering space.

In the next example we show that there is a canonical way to define a flat connection on a vector bundle equipped with a flat structure.

Example 2.27. Let $\mathcal{E} \to M$ be a complex vector bundle which is equipped with a flat structure $\{U_i, \phi_i\}$. Then we can define a flat connection ∇ on $\mathcal{E}$ in the following way: Let $i \in I$ and $e_1, \dots, e_k$ the standard basis of $\mathbb{C}^k$. Let

$$\begin{aligned} s_{ij}: U_i &\to U_i \times \mathbb{C}^k \xrightarrow{\phi_i} \mathcal{E} \\ m &\mapsto (m, e_j) \mapsto \phi_i(m, e_j). \end{aligned}$$

Then for all $X \in \Gamma(TM)$, $i \in I$ and $e_j \in \mathbb{C}^k$ we define

$$\nabla_X s_{ij} = 0.$$

Since the transition maps

$$\phi_j^{-1} \circ \phi_i : (U_i \cap U_j) \times \mathbb{C}^k \to (U_i \cap U_j) \times \mathbb{C}^k$$

are of the form $(\phi_j^{-1} \circ \phi_i)(x, v) = (x, A^{ij} v)$, where $A^{ij} \in \mathrm{GL}(k, \mathbb{C})$ does not depend on x, this indeed defines a global connection on $\mathcal{E}$.

A proof of the following lemma is given in [Kob14, Chapter 1.2, Page 5].

Lemma 2.28. *Let $\mathcal{E} \to M$ be a complex vector bundle of rank k with connection ∇. Then the following statements are equivalent:*

1. *∇ is flat.*
2. *$\mathcal{E}$ can be equipped with a flat structure $\{U_i, \phi_i\}_{i \in I}$ such that for all $i \in I$ and any standard basis vector $e_j \in \mathbb{C}^k$ the section*

$$\begin{aligned} s_{ij} : U_i &\to U_i \times \mathbb{C}^k \xrightarrow{\phi_i} \mathcal{E} \\ m &\mapsto (m, e_j) \mapsto \phi_i(m, e_j). \end{aligned}$$

is flat with respect to ∇.

Let $(\mathcal{E}, \nabla_{\mathcal{E}})$ and $(\mathcal{F}, \nabla_{\mathcal{F}})$ be two flat vector bundles. Then $(\mathcal{E}, \nabla_{\mathcal{E}})$ and $(\mathcal{F}, \nabla_{\mathcal{F}})$ are said to be *isomorphic*, if there exists an isomorphism $f : \mathcal{E} \to \mathcal{F}$ such that $\nabla_{\mathcal{E}} = f^*(\nabla_{\mathcal{F}})$.

The following theorem states that every flat vector bundle can be obtained as in Example 2.26. A proof of the theorem is given in [BP92, Theorem F.3.6]. In its formulation vector bundles equipped with a flat structure are used instead of flat connections, but this is equivalent by Lemma 2.28.

Theorem 2.29. *Let M be a connected manifold. To each conjugacy class of a homomorphism $\alpha : \pi_1(M) \to \mathrm{GL}(k, \mathbb{C})$ there corresponds a flat complex vector bundle of rank k with holonomy α. The complex vector bundle together with its flat structure is unique up to isomorphism.*

It follows from the above theorem and Example 2.26 that if $(\mathcal{F} \to M, \nabla)$ is a flat vector bundle over M with holonomy α, then it is isomorphic to the vector bundle

$$\mathcal{E}_\alpha = \widetilde{M} \times \mathbb{C}^k / \sim$$

equipped with its canonical flat connection as described in Examples 2.26 and 2.27.

2.12.4 Connections compatible with a Hermitian structure

Let $\mathcal{E}$ be a complex vector bundle. A *Hermitian structure* on $\mathcal{E}$ is a $\mathbb{C}$-valued metric h on the fibers which satisfies

- $h(\cdot, \cdot)$ is linear in the first argument,
- $h(\xi, \eta) = \overline{h(\eta, \xi)}$ for all $\xi, \eta \in \mathcal{E}_x$,
- $|\xi|^2 := h(\xi, \xi) > 0$ for all $\xi \neq 0$,
- if s_1 and s_2 are C^∞-sections then $h(s_1, s_2)$ is a C^∞-function.

Definition 2.30. Let $\mathcal{E}$ be a complex vector bundle with Hermitian structure h. A connection ∇ is *compatible with the Hermitian structure* if it preserves h, i.e., we have for all sections $\xi, \eta \in \Gamma(\mathcal{E})$ and all vector fields X

$$X(h(\xi, \eta)) = h(\nabla_X \xi, \eta) + h(\xi, \nabla_X \eta).$$

Lemma 2.31. *Let $\mathcal{E} \to M$ be a complex vector bundle and ∇ a connection which is compatible with the Hermitian structure. Let $\gamma\colon [0,1] \to M$ be a curve and let $v \in \mathcal{E}_{\gamma(0)}$. Let $t \mapsto s(\gamma(t)) \in \mathcal{E}$ be the parallel transport of v along γ. Then we have for all t that $|s(\gamma(t))| = |v|$.*

Proof. Since s is flat along γ and ∇ is compatible with the Hermitian structure, we obtain

$$\frac{\partial}{\partial t}\langle s(\gamma(t)), s(\gamma(t))\rangle = \langle \underbrace{\nabla_{\frac{\partial}{\partial t}} s(\gamma(t))}_{=0}, s(\gamma(t))\rangle + \langle s(\gamma(t)), \underbrace{\nabla_{\frac{\partial}{\partial t}} s(\gamma(t))}_{=0} \rangle = 0$$

and hence $|s(\gamma(t))| = |v|$. □

We obtain the following proposition from Theorem 2.29 and Lemma 2.31.

Proposition 2.32. *Let M be a connected manifold. To each conjugacy class of homomorphisms $\alpha\colon \pi_1(M) \to \mathrm{U}(k)$ there corresponds a flat Hermitian vector bundle of rank k with holonomy α and the connection on $\mathcal{E}$ is compatible with the Hermitian structure. The vector bundle together with its flat and Hermitian structure is unique up to isomorphism.*

2.13 Fréchet spaces

Definition 2.33. Let V be a topological vector space whose topology is induced by a family of seminorms $\{d_n(-)\}_{n\in\mathbb{N}}$, i.e., $U \subset V$ is open if for all $u \in U$ there exists $K > 0$ and $\epsilon > 0$ such that

$$\{x \in V \mid d_k(x-u) < \epsilon \text{ for all } k \leq K\} \subset U.$$

If V is Hausdorff and complete with respect to the metric defined by

$$d(x,y) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{d_n(x-y)}{1 + d_n(x-y)}$$

then V is called a *Fréchet space*.

Example 2.34. Let M be a compact manifold and $\mathcal{E} \to M$ a Hermitian vector bundle with connection ∇. The space of sections $\Gamma(\mathcal{E})$ is a $\mathbb{C}$-vector space, which we will now equip with a family of seminorms indexed over $\mathbb{N}$. For this, let $n \in \mathbb{N}$ and consider the map

$$\begin{aligned} \nabla^n\colon \big(\Gamma(TM)\big)^n \times \Gamma(\mathcal{E}) &\to \Gamma(\mathcal{E}) \\ ((X_1, \dots X_n), s) &\mapsto \nabla_{X_1} \circ \cdots \circ \nabla_{X_n} s. \end{aligned}$$

The operator norm of this map for a fixed section is given by

$$\|\nabla^n s\| := \sup\left\{ \left|(\nabla_{X_1} \circ \cdots \circ \nabla_{X_n} s)(x)\right| \;\middle|\; x \in M, X_i \in \Gamma(TM), \|X_i\| = 1 \right\}.$$

Then for $n \in \mathbb{N}$ and $s \in \Gamma(\mathcal{E})$

$$\|s\|_n := \sum_{i=0}^{n} \left\| \nabla^i s \right\|$$

defines a seminorm on $\Gamma(\mathcal{E})$. The vector space $\Gamma(\mathcal{E})$ with the family of seminorms $\{\|-\|_n\}_{n\in\mathbb{N}}$ is a Fréchet space (see [Sha93, Chapter 1]).

In the following we will always consider the space of sections of a Hermitian vector bundle as a Fréchet space with the topology induced by the family of seminorms defined in the example.

Definition 2.35. Let V be a complex topological vector space. Denote by V^* the $\mathbb{C}$-vector space of continuous linear functionals $f\colon V \to \mathbb{C}$.

1. Let $\Omega \subset \mathbb{C}$ be open. A map $g\colon \Omega \to V$ is called *weakly holomorphic* if for all $h \in V^*$ the map $h \circ g\colon \Omega \to \mathbb{C}$ is holomorphic.
2. Let $\Omega \subset \mathbb{C}$ be open. A map $g\colon \Omega \to V$ is called *analytic* or *strongly holomorphic* if for all $a \in \Omega$ there exists an open neighborhood $U \subset \Omega$ of a and $v_k \in V$ such that for all $z \in U$ we have
$$g(z) = \sum_{k=0}^{\infty} v_k (z-a)^k.$$
3. Let $(a,b) \subset \mathbb{R}$. A map $g\colon (a,b) \to V$ is called *analytic* if there exists an open neighborhood $\Omega \subset \mathbb{C}$ of (a,b) and an analytic function $f\colon \Omega \to V$ such that $f|_{(a,b)} = g$.

Clearly analytic functions are weakly holomorphic. If the vector space which we consider is a complex Fréchet space, then it is shown in [BD09, Theorem 4] that both definitions coincide.

Chapter 3

Analytic deformations of a connection

In this chapter we will introduce the notion of an analytic deformation of a flat connection on a Hermitian vector bundle. Later on we will study the ρ-invariant, which assigns a real number to a Hermitian vector bundle equipped with a flat connection. We will study how the value of this invariant changes under analytic deformations of the connection.

3.1 From analytic deformations of a connection to analytic deformations of the holonomy

Recall that for a Hermitian vector bundle $\mathcal{E} \to M$ over a manifold M the space $\Omega^1(M; \mathrm{End}(\mathcal{E}))$ of 1-forms taking values in the endomorphism bundle of $\mathcal{E}$ coincides with the space of sections $\Gamma(\mathrm{End}(\mathcal{E}) \otimes T^*M)$. We equip $\Omega^1(M; \mathrm{End}(\mathcal{E}))$ with the structure of a Fréchet space as described in Example 2.34, and in particular obtain a notion of convergence.

Definition 3.1. Let $\mathcal{E} \to M$ be a Hermitian vector bundle over a manifold M. Let $\{\nabla^t\}_{t \in (-\epsilon, \epsilon)}$ be a family of connections on $\mathcal{E}$. Then $\{\nabla^t\}_{t \in (-\epsilon, \epsilon)}$ is called an *analytic deformation* of ∇^0 if there exists $\Omega_i \in \Omega^1(M; \mathrm{End}(\mathcal{E}))$ such that for all $t \in (-\epsilon, \epsilon)$

$$\nabla^t = \nabla^0 + \sum_{i=1}^{\infty} t^i \Omega_i.$$

As we have seen in Proposition 2.32 a flat connection on a Hermitian vector bundle which is compatible with the Hermitian structure corresponds to a conjugacy class of unitary representations of the fundamental group. Hence a deformation of the connection corresponds to a deformation of a unitary representation of the fundamental group. In this section we want to study what sort of deformations of the unitary representation can arise by deforming the corresponding connection.

Definition 3.2. Let M be a connected manifold. Let $\{\alpha_t : \pi_1(M) \to U(k)\}_{t \in (-\epsilon, \epsilon)}$ be a family of unitary representations of the fundamental group. We call $\{\alpha_t\}_{t \in (-\epsilon, \epsilon)}$ an *analytic* *deformation of* α_0 if for fixed $g \in \pi_1(M)$ the entries of the unitary matrices $\alpha_t(g)$, viewed as functions of t, are holomorphic, i.e., $(t \mapsto \alpha_t(g)) \in U(k, \mathcal{O})$.

The first lemma considers the parallel transport corresponding to an analytic family of connections in a trivial vector bundle. The statement is that of [FL96, Lemma 5.2].

Lemma 3.3. *Let $\mathcal{E}$ be a Hermitian vector bundle over a closed n-dimensional ball M lying in the Euclidean space $\mathbb{R}^n$. Let $\{\nabla^t\}_{t\in(-\epsilon,\epsilon)}$ be a family of flat connections such that the deformation is analytic. Let $p \in M$ and $e \in \mathcal{E}_p$. For every $t \in (-\epsilon, \epsilon)$ let s_t be a section with $s_t(p) = e$ and $\nabla^t s_t = 0$. Then the curve of sections*

$$\begin{aligned}(-\epsilon, \epsilon) &\to \Gamma(\mathcal{E})\\ t &\mapsto s_t\end{aligned}$$

is an analytic map.

We now prove a version of the previous lemma for a general manifold M.

Lemma 3.4. *Let M be a connected manifold and let $\mathcal{E} \to M$ be a Hermitian vector bundle of rank k. Let $\{\nabla^t\}_{t\in(-\epsilon,\epsilon)}$ be an analytic deformation of ∇^0 such that for all $t \in (-\epsilon, \epsilon)$ the connection ∇^t is compatible with the Hermitian structure. Fix $p \in M$ and $[\gamma] \in \pi_1(M, p)$. For any $v \in \mathcal{E}_p$, let $\Gamma_t(v) \in \mathcal{E}_p$ be the parallel transport corresponding to ∇^t of v along γ. Under the canonical identification of the fiber $\mathcal{E}_p$ with $\mathbb{C}^k$ equipped with its standard Hermitian structure, the map $t \mapsto \Gamma_t(v) \in \mathbb{C}^k$ is analytic and the linear map*

$$\begin{aligned}\Gamma\colon \mathbb{C}^k &\to \mathcal{O}^k\\ v &\mapsto (t \mapsto \Gamma_t(v))\end{aligned}$$

can be represented by a unitary matrix with entries in $\mathcal{O}$.

Proof. Let $\{U_i\}_{i\in I}$ be an open covering of M such that $\mathcal{E}|_{U_i}$ is a trivial vector bundle for all i. Let $p \in M$ and $[\gamma] \in \pi_1(M, p)$ represented by $\gamma\colon [0, 1] \to M$. Choose $0 = k_0 < k_1 < \ldots < k_n = 1$ such that for every $i = 0, \ldots n-1$ there exists $j \in I$ such that $\gamma([k_i, k_{i+1}])$ is contained in U_j.

Recall that for each connection ∇^t and each $i = 0, \ldots n-1$ we have a linear map

$$\Gamma_t^{\gamma(k_i),\gamma(k_{i+1})}\colon \mathcal{E}_{\gamma(k_i)} \to \mathcal{E}_{\gamma(k_{i+1})}$$

which is given by parallel transport. Let $v \in \mathcal{E}_{\gamma(k_i)}$. For each $t \in (-\epsilon, \epsilon)$ there is a unique section s_t defined on $[k_i, k_{i+1}]$ that is flat with respect to ∇^t and satisfies $s_t(k_i) = v$.

It follows from Lemma 3.3 that the curve $t \mapsto s_t$ is analytic. We have $s_t(k_{i+1}) = \Gamma_t^{\gamma(k_i),\gamma(k_{i+1})}(v)$. Hence, the map

$$\begin{aligned}(-\epsilon, \epsilon) &\to \mathcal{E}_{\gamma(k_{i+1})}\\ t &\mapsto \Gamma_t^{\gamma(k_i),\gamma(k_{i+1})}(v)\end{aligned}$$

is analytic for each $v \in \mathcal{E}_{\gamma(k_i)}$. The map

$$\Gamma_t^{\gamma(k_0),\gamma(k_n)}\colon \mathcal{E}_{\gamma(k_0)} \to \mathcal{E}_{\gamma(k_n)}$$

is simply given by the composition

$$\Gamma_t^{\gamma(k_{n-1}),\gamma(k_n)} \circ \cdots \circ \Gamma_t^{\gamma(k_0),\gamma(k_1)}.$$

We obtain from Proposition 2.32 that for fixed $t \in (-\epsilon, \epsilon)$ the map $\Gamma_t^{\gamma(k_0),\gamma(k_n)}$ can be represented by a unitary matrix. Since the composition of analytic maps is analytic, it follows that

$$t \mapsto \Gamma_t^{\gamma(k_0),\gamma(k_n)}$$

is analytic and hence can be represented by a unitary matrix with entries in $\mathcal{O}$. □

3.2 Deforming the holonomy along a circle

Later on we will work with deformations of the unitary representation of the fundamental group instead of directly deforming the connection. We have already seen that an analytic deformation of the connection gives rise to an analytic deformation of the unitary representation of the fundamental group. We were not able to answer the question whether the other direction is true: that an analytic deformation of the unitary representation of the fundamental group always gives rise to an analytic deformation of a connection.

In this section we will deal with a special case. More precisely, we want to show the following: Let M be a connected manifold and let $\alpha\colon \pi_1(M) \to U(k)$ be a unitary representation. Let $\phi\colon \pi_1(M) \to \mathbb{Z}$. Let

$$\begin{aligned} \alpha_t\colon \pi_1(M) &\to U(k) \\ g &\mapsto \alpha(g) e^{it\phi(g)}. \end{aligned}$$

Let ∇^t be the connection corresponding to α_t (see Sections 2.12.3 and 2.12.4). We will show that $t \mapsto \nabla^t$ is analytic in t.

In the following we consider S^1 with the parametrization

$$\begin{aligned} \gamma\colon [-\pi, \pi) &\to S^1 \\ t &\mapsto e^{it} \end{aligned}$$

and we identify $T_pS^1 \cong \mathbb{R}\left\langle \frac{\partial}{\partial t} \right\rangle$ and $T_p^*S^1 \cong \mathbb{R}\langle \mathrm{d}t \rangle$.

Lemma 3.5. *We consider $S^1 \times \mathbb{C}$ as a vector bundle over S^1. For $s \in (-\epsilon, \epsilon)$, define $p_s\colon S^1 \to \mathbb{C}$ by $p_s(z) = z^s$ and denote by ∇^s the connection on $S^1 \times \mathbb{C}$ such that*

$$\begin{aligned} S^1 &\to S^1 \times \mathbb{C} \\ z &\mapsto \big(z, p_s(z)\big) \end{aligned}$$

is a parallel section with respect to ∇^s. Let $\tilde{r}$ be a (possibly local) section and let $X \in \Gamma(TS^1)$ be a vector field. Then

$$\nabla_X^s \tilde{r} = \nabla_X^0 \tilde{r} - is\, \mathrm{d}t(X) \tilde{r}.$$

In other words, $\nabla^s - \nabla^0 = -is\,\mathrm{d}t \in \Omega^1(S^1; \mathrm{End}(\mathbb{C}))$, where we identified $\mathrm{End}(\mathbb{C})$ with $\mathbb{C}$.

Proof. We denote by $r\colon S^1 \to \mathbb{C}$ the smooth function $\mathrm{pr}_2 \circ \tilde{r}$ and by $\tilde{p_s}\colon S^1 \to S^1 \times \mathbb{C}$ the section given by $z \mapsto (z, p_s(z))$. Since $\tilde{r}p_s = r\tilde{p}_s$, we obtain

$$\begin{aligned}
\nabla^s_{\frac{\partial}{\partial t}} \tilde{r} &= \nabla^s_{\frac{\partial}{\partial t}} \left(\frac{1}{p_s} \tilde{r} p_s \right) \\
&= \nabla^s_{\frac{\partial}{\partial t}} \left(\frac{1}{p_s} r \tilde{p}_s \right) \\
&= \frac{1}{p_s} r \underbrace{\nabla^s \tilde{p}_s}_{=0} + \frac{\partial}{\partial t} \left(r \frac{1}{p_s} \right) \tilde{p}_s \\
&= \left(\frac{\partial}{\partial t} r \right) \frac{1}{p_s} \tilde{p}_s + r \left(\frac{\partial}{\partial t} \frac{1}{p_s} \right) \tilde{p}_s
\end{aligned}$$

Thus, in the coordinates provided by the parametrization γ and abbreviating $a \coloneqq \gamma(t)$, we get

$$\begin{aligned}
(\mathrm{pr}_2 \circ \nabla^s_{\frac{\partial}{\partial t}} \tilde{r})(a) &= r'(a) \cdot p_0(a) + r(a) \cdot (-is) p_0(a) \\
&= r'(a) - isr(a) \\
&= (\mathrm{pr}_2 \circ \nabla^0_{\frac{\partial}{\partial t}} \tilde{r})(a) - isr(a),
\end{aligned}$$

and hence

$$\nabla^s_{\frac{\partial}{\partial t}} \tilde{r} = \nabla^0_{\frac{\partial}{\partial t}} \tilde{r} - is\tilde{r}.$$

Since ∇^s_X is $\mathcal{C}^\infty$-linear in X, it follows that

$$\nabla^s_X \tilde{r} = \nabla^0_X \tilde{r} - is\, \mathrm{d}t(X) \tilde{r}.$$

□

Lemma 3.6. *Let M be a connected manifold and $f\colon M \to S^1$ a smooth map. We consider the vector bundle $S^1 \times \mathbb{C}$ with connection ∇^s for some $s \in (-\epsilon, \epsilon)$ as defined previously in Lemma 3.5. Then the pullback bundle $(f^*(S^1 \times \mathbb{C}), f^*(\nabla^s))$ is a flat vector bundle with holonomy*

$$\begin{aligned}
\alpha\colon \pi_1(M) &\to U(1) \\
g &\mapsto e^{isf_*(g)}
\end{aligned}$$

and

$$f^*\left(\nabla^s_X\right) r = f^*\left(\nabla^0_X\right) r - is\, \mathrm{d}t(\mathrm{d}f(X)) r$$

for a (possibly local) section $r \in \Gamma(f^*(S^1 \times \mathbb{C}))$.

Proof. By definition of the pullback bundle we have the following commutative diagram:

$$\begin{array}{ccc}
f^*(S^1 \times \mathbb{C}) & \longrightarrow & S^1 \times \mathbb{C} \\
\downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1} \\
M & \xrightarrow{f} & S^1
\end{array}$$

Note that $f^*(S^1 \times \mathbb{C})$ is topologically just $M \times \mathbb{C}$, which is equipped with the connection $f^*(\nabla^s)$.

Let r be a section of $S^1 \times \mathbb{C}$. Recall from Lemma 2.19 that the pullback of the connection satisfies

$$\left(f^* \nabla^s_X\right)\left(f^* r\right) = f^*\left(\nabla^s_{\mathrm{d}f(X)} r\right) \qquad \text{for all } X \in \Gamma(TM).$$

and hence the pullback of a parallel section is parallel. Using the previous lemma, we obtain

$$\begin{aligned}
\left(f^* \nabla^s_X\right)\left(f^* r\right) &= f^*\left(\nabla^s_{\mathrm{d}f(X)} r\right) \\
&= f^*\left(\nabla^0_{\mathrm{d}f(X)} r - is\,\mathrm{d}t(\mathrm{d}f(X))r\right) \\
&= f^*\left(\nabla^0_{\mathrm{d}f(X)} r\right) - is\,\mathrm{d}t(\mathrm{d}f(X))\left(f^* r\right) \\
&= \left(f^* \nabla^0_X\right)\left(f^* r\right) - is\,\mathrm{d}t(\mathrm{d}f(X))\left(f^* r\right).
\end{aligned}$$

Since the difference of two connections on M is $C^\infty(M)$-linear in the section and any section of $M \times \mathbb{C}$ is a $C^\infty(M)$-linear combination of sections pulled back from S^1, we obtain for all $r \in \Gamma(M \times \mathbb{C})$

$$f^*\left(\nabla^s_X\right) r = f^*\left(\nabla^0_X\right) r - is\,\mathrm{d}t(\mathrm{d}f(X))r.$$

Let $[\gamma] \in \pi_1(M, m_0)$ for any fixed basepoint $m_0 \in M$. Parallel transport with respect to ∇^s of $(f(m_0), v) \in S^1 \times \mathbb{C}$ along $f \circ \gamma$ yields $\left(f(m_0), e^{isf_*([\gamma])}v\right)$ for any $v \in \mathbb{C}$. Hence, parallel transport with respect to $f^*(\nabla^s)$ of $(m_0, v) \in M \times \mathbb{C}$ along γ yields $\left(m_0, e^{isf_*([\gamma])}v\right)$. It follows that the holonomy is given by

$$\begin{aligned}
\alpha\colon \pi_1(M) &\to U(1) \\
[\gamma] &\mapsto e^{isf_*([\gamma])}
\end{aligned}$$

□

Lemma 3.7. *Let M be a connected manifold and $\alpha\colon \pi_1(M) \to U(k)$ a unitary representation of the fundamental group. Let $(\mathcal{E}, \nabla^\alpha)$ be a complex vector bundle with holonomy α. Let $f\colon M \to S^1$. For any $s \in (-\epsilon, \epsilon)$ let $(\mathbb{C}_s, \nabla^s)$ be the bundle $M \times \mathbb{C}$ with holonomy*

$$\begin{aligned}
\pi_1(M) &\to U(1) \\
g &\mapsto e^{isf_*(g)}.
\end{aligned}$$

Then $\mathcal{E}_\alpha \otimes \mathbb{C}_s$ is a bundle with holonomy

$$\begin{aligned}
\pi_1(M) &\to U(k) \\
g &\mapsto \alpha(g)e^{is\phi(g)}
\end{aligned}$$

and the connection satisfies

$$\left(\nabla^\alpha_Y \otimes \nabla^s_X\right)(\eta \otimes \nu) = \left(\nabla^\alpha_Y \otimes \nabla^0_X\right)(\eta \otimes \nu) - is\,\mathrm{d}t(\mathrm{d}f(X))(\eta \otimes \nu).$$

Proof. Recall from Lemma 2.20 that the connection induced on a tensor product $\mathcal{E} \otimes \mathcal{F}$ of bundles is given by

$$(\nabla_{\mathcal{E}} \otimes \nabla_{\mathcal{F}})(s_{\mathcal{E}} \otimes s_{\mathcal{F}}) = \nabla_{\mathcal{E}} s_{\mathcal{E}} \otimes s_{\mathcal{F}} + s_{\mathcal{E}} \otimes \nabla_{\mathcal{F}} s_{\mathcal{F}}$$

where $\nabla_{\mathcal{E}}$ is a connection on $\mathcal{E}$ and $\nabla_{\mathcal{F}}$ is a connection on $\mathcal{F}$.

Recall from Lemma 3.6 that

$$\nabla^s_X r = \nabla^0_X r - is\,\mathrm{d}t(\mathrm{d}f(X))r.$$

We now obtain for the connection on the tensor product bundle

$$\begin{aligned}\left(\nabla^\alpha_Y \otimes \nabla^s_X\right)(\eta \otimes \nu) &= \nabla^\alpha_Y \eta \otimes \nu + \eta \otimes \nabla^s_X(\nu)\\ &= \nabla^\alpha_Y \eta \otimes \nu + \eta \otimes \left(\nabla^0_X - is\,\mathrm{d}t\big(\mathrm{d}f(X)\big)\right)(\nu)\\ &= \left(\nabla^\alpha_Y \otimes \nabla^0_X\right)(\eta \otimes \nu) - is\,\mathrm{d}t\big(\mathrm{d}f(X)\big)(\eta \otimes \nu).\end{aligned}$$

It follows from the definition of the connection on a tensor product bundle that if η is a flat section with respect to ∇^α and ν is a section with respect to ∇^s then $\eta \otimes \nu$ is a flat section with respect to $\nabla^\alpha \otimes \nabla^s$. Hence, the holonomy of the product bundle is just the product of the holonomy of the single bundles. Therefore we obtain that the holonomy on $\mathcal{E}_\alpha \otimes \mathbb{C}_s$ is given by

$$\begin{aligned}\pi_1(M) &\to U(k)\\ g &\mapsto \alpha(g)e^{isf_*(g)}.\end{aligned}$$

□

Corollary 3.8. *Let M be a connected manifold and let $\alpha\colon \pi_1(M) \to U(k)$ be a unitary representation of the fundamental group. Let $f\colon M \to S^1$ and $f_*\colon \pi_1(M) \to \mathbb{Z}$. For $s \in (-\epsilon, \epsilon)$ we consider the deformation of α given by*

$$\begin{aligned}\alpha_s\colon \pi_1(M) &\to U(k)\\ g &\mapsto \alpha(g)e^{isf_*(g)}.\end{aligned}$$

Let $(\mathcal{E}_\alpha, \nabla^0)$ be a vector bundle with holonomy α. Let

$$\nabla^s_X r = \nabla^0_X r - is\,\mathrm{d}t(\mathrm{d}f(X))r.$$

Then $(\mathcal{E}_\alpha, \nabla^s)$ is a bundle with holonomy α_s.

Since the previous corollary expresses the difference $\nabla^s_X r - \nabla^0_X r$ as a power series — in fact, a linear polynomial — in s with coefficients in $\Omega_1(M; \mathrm{End}(\mathcal{E}))$, the family $\{\nabla^s\}_{s\in(-\epsilon,\epsilon)}$ defines an analytic deformation of ∇^0.

Chapter 4

η-invariant and ρ-invariant

In this chapter we first give the definition of the η-invariant, which depends on a closed connected oriented odd-dimensional Riemannian manifold and a flat connection on a Hermitian vector bundle over that manifold. Afterwards we will introduce the ρ-invariant, which is defined as a difference of η-invariants and has the advantage of being independent of the Riemannian structure.

4.1 Definition of the η-invariant

Atiyah, Patodi and Singer defined and studied the η-invariant in the series of articles [APS75a], [APS75b] and [APS76].

Let M be a closed connected oriented Riemannian manifold of odd dimension $2l-1$ and $\mathcal{E}$ be a flat Hermitian vector bundle of rank k over M, i.e., we are given

- a Hermitian structure h on each fiber $\mathcal{E}_x$ which varies smoothly with $x \in M$,
- a flat connection ∇ acting on the space of $\mathcal{C}^\infty$ forms on M with values in $\mathcal{E}$ such that the connection is compatible with the Hermitian structure on $\mathcal{E}$, i.e,
$$X(h(\xi,\eta)) = h(\nabla_X \xi, \eta) + h(\xi, \nabla_X \eta).$$
for all $\eta, \xi \in \Gamma(\mathcal{E})$.

Recall that we write
$$\Omega^p(M;\mathcal{E}) := \Gamma(\mathcal{E} \otimes \Lambda^p T^* M).$$
We denote by $\Omega^{\text{ev}}(M,\mathcal{E}) := \bigoplus_{n\geq 0} \Omega^{2n}(M;\mathcal{E})$ the $\mathcal{E}$-valued differential forms of even degree. We denote by $*{:}\,\Omega^k(M;\mathcal{E}) \to \Omega^{2l-1-k}(M;\mathcal{E})$ the Hodge duality operator. We consider the operator $B{:}\,\Omega^{\text{ev}}(M,\mathcal{E}) \to \Omega^{\text{ev}}(M,\mathcal{E})$, which maps $\phi \in \Omega^{2\text{p}}(M,\mathcal{E})$ to
$$B(\phi) = i^l(-1)^{p+1}(*\nabla - \nabla *)\phi.$$
The operator B is elliptic and self-adjoint (see [APS75b, Page 3]) and hence its spectrum is real and discrete. We consider the function
$$\widetilde{\eta}_B(s) = \sum_{\lambda \neq 0} \frac{\text{sign}(\lambda)}{|\lambda|^s},$$

where λ runs over all eigenvalues of B. It was shown by Atiyah, Patodi and Singer in [APS75a, Theorem 3.10] combined with [APS76, Page 74] that $\widetilde{\eta}_B(s)$ is holomorphic in the half-plane $\mathrm{Re}(s) > \dim(M)$. Furthermore it has a meromorphic continuation η_B to the whole complex plane, which is given by

$$\eta_B(2s) = \frac{1}{\Gamma(s+\frac{1}{2})} \int_0^\infty t^{s-\frac{1}{2}} \operatorname{trace}\left(Be^{-tB^2}\right) \mathrm{d}t,$$

where the Γ-function is defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, \mathrm{d}t$$

and the trace of the operator Be^{-tB^2} will be defined further below. The function $s \mapsto \eta_B(2s)$ is holomorphic if $\mathrm{Re}(s) > -\frac{1}{2}$ (see [APS75a, Theorem 4.14(iii)]). In particular $\eta_B(0)$ is finite. The *η-invariant* of B is then defined to be the value of the η-function at zero, i.e.,

$$\eta(M, \nabla) \coloneqq \eta_B(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{trace}\left(Be^{-tB^2}\right) \mathrm{d}t.$$

We will now describe the construction of the trace of Be^{-tB^2}. Denote by $\mathrm{End}(\mathcal{E}, \mathcal{E}) \to M \times M$ the endomorphism bundle over $M \times M$, where over a point $(u, v) \in M \times M$ the fiber is just $\mathrm{End}_{\mathbb{C}}(\mathcal{E}_u, \mathcal{E}_v)$. Pulling back this bundle along the projection $[0, \infty) \times M \times M \to M \times M$, we obtain a bundle $[0, \infty) \times \mathrm{End}(\mathcal{E}, \mathcal{E}) \to [0, \infty) \times M \times M$. It is shown by Cheeger and Gromov in [CG85, Chapter 4, Page 23f.] that there exists a smooth section k of this bundle such that

$$\left(Be^{-itB^2}\right)(\omega)(x) = \int_M k_t(x, y)\omega(y) \, \mathrm{d}y$$

for $x \in M$, i.e., k_t is a smooth kernel for the operator Be^{-itB^2} for any $t \geq 0$. The trace of Be^{-itB^2} is then defined to be

$$\operatorname{trace}(Be^{-itB^2}) \coloneqq \int_M \operatorname{trace}_{\mathbb{C}} k_t(x, x) \, \mathrm{d}x.$$

Let M be a closed connected oriented odd-dimensional manifold and $(\mathcal{E}, \nabla_{\mathcal{E}})$, $(\mathcal{F}, \nabla_{\mathcal{F}})$ flat vector bundles over M. Then one can consider the vector bundle $\mathcal{E} \oplus \mathcal{F}$ with connection $\nabla_{\mathcal{E}} \oplus \nabla_{\mathcal{F}}$. It follows easily from the definitions that

$$\eta(M, \nabla_{\mathcal{E}} \oplus \nabla_{\mathcal{F}}) = \eta(M, \nabla_{\mathcal{E}}) + \eta(M, \nabla_{\mathcal{F}}).$$

Let M be a closed connected oriented manifold. Recall from Section 2.12.4 that there is a one-to-one correspondence between

- flat Hermitian vector bundles of rank k over M such that the connection is compatible with the Hermitian structure up to isomorphism and
- conjugacy classes of k-dimensional unitary representations of the fundamental group of M.

Later on we will mostly work with unitary representations of the fundamental group instead of considering flat connections. Let $\alpha\colon \pi_1(M) \to U(k)$ and ∇_α be the corresponding connection, then we write

$$\eta(M,\alpha) := \eta(M,\nabla_\alpha).$$

Let $\alpha\colon \pi_1(M) \to U(k)$ and $\beta\colon \pi_1(M) \to U(l)$ be unitary representations. Let ∇_α and ∇_β be flat connections corresponding to α and β, respectively. Then $\nabla_\alpha \oplus \nabla_\beta$ corresponds to the unitary representation $\alpha \oplus \beta$ and hence,

$$\eta(M,\alpha\oplus\beta) = \eta(M,\alpha) + \eta(M,\beta).$$

4.2 Definition of the ρ-invariant and signature defect

Let M be a closed connected oriented odd-dimensional Riemannian manifold and let α be a k-dimensional unitary representation of the fundamental group. Then the *ρ-invariant* is defined as

$$\rho(M,\alpha) = \eta(M,\alpha) - \eta(M,\tau_k),$$

where $\eta(M,\tau_k)$ is the η-invariant corresponding to the trivial k-dimensional representation. Atiyah, Patodi and Singer showed that the ρ-invariant is independent of the Riemannian metric on M (see [APS75b, Theorem 2.4]).

Let M be an n-dimensional manifold and let $\alpha\colon \pi_1(M,m_0) \to G$ be a group homomorphism. Assume there exists an $(n+1)$-dimensional manifold with $\partial W = \bigsqcup_{i=1}^r M_i$, where the M_i are disjoint copies of M. We say $\alpha\colon \pi_1(M,m_0) \to G$ *extends* to a map $\beta\colon \pi_1(W,w_0) \to G$ if for each boundary component M_i of ∂W there exist paths $\gamma_{1,i},\gamma_{2,i}$ between $m_{0,i}$ and w_0 such that the following diagram commutes

$$\begin{tikzcd}
\pi_1(M_i,m_{0,i}) \arrow[r, "g\mapsto\gamma_{1,i}^{-1}*g*\gamma_{2,i}"] \arrow[d, "\cong"] & \pi_1(W,w_0) \arrow[d, "\beta"] \\
\pi_1(M,m_0) \arrow[r, "\alpha"] & G.
\end{tikzcd}$$

In certain special cases, the ρ-invariant can be calculated as a signature defect of the bounding manifold. Namely, the following theorem appears as [APS75b, Theorem 2.4].

Theorem 4.1. *Let M be a closed connected oriented $(4n-1)$-dimensional manifold and let $\alpha\colon \pi_1(M,m_0) \to U(k)$ be a group homomorphism. Assume there exists a compact $4n$-dimensional manifold W with ∂W consisting of r disjoint copies of M such that α extends to a group homomorphism $\beta\colon \pi_1(W) \to U(k)$. Then*

$$\rho(M,\alpha) = \frac{1}{r}\big(k\operatorname{sign}(W) - \operatorname{sign}_\beta(W)\big).$$

4.3 The ρ-invariant of manifolds obtained by 0-framed surgery along knots

Let K be a knot and let M_K be obtained by 0-framed surgery along K. Let A be a Seifert matrix of K. We consider

$$\begin{aligned}\sigma(K)\colon S^1 &\to \mathbb{Z}\\ z &\mapsto \operatorname{sign}\left((1-z)A + (1-\overline{z})A^t\right).\end{aligned}$$

The signature function $\sigma(K)$ is called the *Levine–Tristam signature function* of K and the value $\sigma_{-1}(K) := \sigma(K)(-1)$ is called *the signature* of K.

Note that

$$(1-z)A + (1-\overline{z})A^t = (\overline{z}-1)(zA - A^t).$$

Hence, if we denote the Alexander polynomial $\det(zA - A^t)$ of K by $\Delta_K(z)$, the function $z \mapsto \sigma_z(K)$ is constant on the connected components of $S^1 \backslash \left(\{z \in S^1 | \Delta_K(z) = 0\} \cup \{1\}\right)$. It is shown in [Lev69, Page 242] that $\sigma(K)$ is continuous at 1.

Example 4.2. Let K be the trefoil knot. The Alexander polynomial of K is $\Delta(t) = t - 1 + t^{-1}$, which has degree 2 and roots at $e^{\pm i\frac{\pi}{3}}$. The Seifert matrix of the left-handed trefoil is given by

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The Levine–Tristam signatures of the left-handed trefoil knot are then given by

$$\sigma_{e^{it}} = \begin{cases} 0 & |t| < \frac{\pi}{3} \\ 1 & |t| = \frac{\pi}{3} \\ 2 & \text{else.} \end{cases}$$

Denote by μ a meridian of K. Let $\alpha\colon \pi_1(M_K) \to U(1)$ be a unitary representation. Since α factorizes through $H_1(M_K, \mathbb{Z})$ and $H_1(M_K, \mathbb{Z}) \cong \mathbb{Z}$ is generated by a meridian, the map α is determined by the value $\alpha(\mu)$.

The following proposition was proved by Litherland (see [Lit84, Proof of Proposition 1]).

Proposition 4.3. *Let K be a knot and let M_K be obtained by 0-framed surgery along K. Let μ be a meridian of K and $\alpha\colon \pi_1(M_K) \to U(1)$ a unitary representation. Then*

$$\rho(M_K, \alpha) = \sigma_{\alpha(\mu)}(K).$$

4.4 The ρ-invariant for finite groups

Let G be a finite group and $g \in G$. The map

$$\begin{aligned}\Phi_g\colon \mathbb{C}[G] &\to \mathbb{C}[G]\\ z &\mapsto gz,\end{aligned}$$

is unitary. We denote by ϕ_G the group homomorphism

$$\begin{aligned}\phi_G\colon G &\to U(|G|)\\ g &\mapsto \Phi_g\end{aligned}$$

and call $\phi_G\colon G \to U(|G|)$ the *regular representation* of G.

Definition 4.4. Let M be a closed connected oriented odd-dimensional manifold. Let G be a finite group and $\alpha\colon \pi_1(M) \to G$ a group homomorphism. Then we define

$$\rho(M,\alpha) := \rho\big(M, \phi_G \circ \alpha\big).$$

We need the following elementary lemma.

Lemma 4.5. *Let $\phi_{\mathbb{Z}_l}\colon \mathbb{Z}_l \to U(l)$ be the regular representation and let*

$$\begin{aligned}\phi_{j,l}\colon \mathbb{Z}_l &\to U(1)\\ [n] &\mapsto e^{2\pi i j n/l}.\end{aligned}$$

The representation $\phi_{\mathbb{Z}_l}$ is reducible and

$$\phi_{\mathbb{Z}_l} = \bigoplus_{j=0}^{l-1} \phi_{j,l}.$$

Proof. Note that $\phi_{\mathbb{Z}_l}(1)$ is a permutation matrix with

$$a_{ij} = \begin{cases} 1 & i = j+1 \text{ or } i = 1, j = l\\ 0 & else. \end{cases}$$

Let $j \in \{0, \dots l-1\}$. The subspace V_j of $\mathbb{C}[\mathbb{Z}_l]$ generated by

$$v_j = \left(1, e^{2\pi i \frac{j}{l}}, e^{2\pi i \frac{2j}{l}}, \dots, e^{2\pi i \frac{(l-1)j}{l}}\right)^t$$

is invariant under the action of $\phi_{\mathbb{Z}_l}$. Moreover, the action of $\phi_{\mathbb{Z}_l}(1)$ on v_j is given by multiplication with $e^{2\pi i \frac{j}{l}}$. Hence, we obtain $\mathbb{C}[\mathbb{Z}_l] = \oplus_{j=0}^{l-1} V_j$ and

$$\phi_{\mathbb{Z}_l} = \bigoplus_{j=0}^{l-1} \phi_{j,l}.$$

□

Let M_K be obtained by 0-framed surgery along a knot K and let $\alpha\colon \pi_1(M_K) \to \mathbb{Z}_l$. We obtain the following useful corollary which says that in this case the ρ-invariant is given by a sum of Levine–Tristam signatures.

Corollary 4.6. *Let K be a knot and M_K be obtained by 0-framed surgery along K. Let $\alpha\colon \pi_1(M_K) \to \mathbb{Z}_l$ be the map which sends a meridian in M_K to $1 \in \mathbb{Z}_l$. Then*

$$\rho\big(M_K, \alpha\big) = \sum_{j=0}^{l-1} \sigma_{e^{2\pi i j/l}}(K).$$

Proof. Note that by definition $\rho\big(M_K,\alpha\big) := \rho\big(M_K,\phi_{\mathbb{Z}_l}\circ\alpha\big)$ and let

$$\begin{aligned}\phi_{j,l}\colon \mathbb{Z}_l &\to U(1)\\ [n] &\mapsto e^{2\pi i j n/l}.\end{aligned}$$

It follows from Lemma 4.5 that we have

$$\phi_{\mathbb{Z}_l} = \bigoplus_{j=0}^{l-1} \phi_{j,l}.$$

It follows from Proposition 4.3 that $\rho(M_K,\phi_{j,l}\circ\alpha) = \sigma_{e^{2\pi i j/l}}(K)$. Hence, we obtain

$$\begin{aligned}\rho\big(M_K,\alpha\big) := \rho\big(M_K,\phi_{\mathbb{Z}_l}\circ\alpha\big) &= \rho\big(M_K,\big(\oplus_{j=0}^{l-1}\phi_{j,l}\big)\circ\alpha\big)\\ &= \sum_{j=0}^{l-1}\rho\big(M_K,\phi_{j,l}\circ\alpha\big) = \sum_{j=0}^{l-1}\sigma_{e^{2\pi i j/l}}(K).\end{aligned}$$

□

4.5 Bordisms over a finite group

Let X be a CW-complex. For $i = 1,2$ let M_i be a closed oriented n-dimensional manifold and $\phi_i\colon M_i \to X$ a continuous map. Then M_1 and M_2 are *oriented bordant* over X if there exists a compact oriented $(n+1)$-dimensional manifold W with $\partial W = M_1 \sqcup \overline{M_2}$ and a continuous map $\Phi\colon W \to X$ such that $\Phi|_{M_i} = \phi_i$. We denote by $\Omega_n^{SO}(X)$ the set of oriented bordism classes of n-dimensional manifolds over X.

The following theorem is for example proved in [DK01, Chapter 9.3].

Theorem 4.7. *Let X be a CW-complex. Then we have an isomorphism*

$$\begin{aligned}\Omega_3^{\mathrm{SO}}(X) &\xrightarrow{\cong} H_3(X;\mathbb{Z})\\ (f\colon M\to X) &\mapsto f_*\big([M]\big).\end{aligned}$$

The following proposition is shown in [Bro94, Chapter 3, Corollary 10.2] for cohomology. The proof is verbatim the same for homology groups.

Proposition 4.8. *Let G be a finite group. Let M be a $\mathbb{Z}G$-module. Then for all $i > 0$ the groups $H_i(G;M)$ are annihilated by $|G|$.*

We obtain the following corollary.

Corollary 4.9. *Let M be a closed connected oriented 3-dimensional manifold and let $\alpha\colon\pi_1(M)\to G$ be a group homomorphism. If G is finite, then there exists a compact 4-dimensional manifold W with $\partial W = |G|M$ and α extends to a group homomorphism $\pi_1(W)\to G$.*

Proof. Let G be a finite group and let $\alpha\colon\pi_1(M)\to G$. Since

$$\begin{aligned}[M,BG] &\to \mathrm{Hom}(\pi_1(M),G)\\ f &\mapsto f_*\end{aligned}$$

is a bijection, there exists $\phi\colon M \to BG$ with $\phi_* = \alpha$. Then the pair $[(M,\phi)]$ defines a bordism class in $\Omega_3^{\mathrm{SO}}(BG)$.

It follows from Theorem 4.7 that $\Omega_3^{\mathrm{SO}}(BG) \cong H_3(BG;\mathbb{Z})$. Since G is finite, it follows from Proposition 4.8 that $H_3(BG;\mathbb{Z})$ and in particular the class $[(M,\phi)]$ is $|G|$-torsion. Hence, there exists a 4-dimensional manifold W equipped with a map $\psi\colon W \to BG$ such that $\partial W = |G|M$ and ψ extends ϕ and hence $\psi_*\colon \pi_1(W) \to G$ extends $\phi_* = \alpha$. □

It now follows from Theorem 4.1 and Corollary 4.9 that in case of of a closed connected oriented 3-dimensional manifold M and a homomorphism $\alpha\colon \pi_1(M) \to G$ to a finite group G the ρ-invariant $\rho(M,\alpha)$ can indeed be calculated as a signature defect.

Chapter 5

L^2-η-invariant and L^2-ρ-invariant

In this chapter we will give the definition of the L^2-η-invariant and the L^2-ρ-invariant, which are L^2-analogues of the classical versions. The L^2-η-invariant was first studied by Cheeger and Gromov in [CG85]. In order to give an equivalent topological definition of the L^2-ρ-invariant, we will need to introduce the L^2-signature, which in turn requires a brief introduction to the general L^2-theory.

5.1 Group von Neumann algebra and L^2-dimension

We first give a short introduction to the group von Neumann algebra $\mathcal{N}G$ and the dimension function on $\mathcal{N}G$-modules, which we will use to define an L^2-signature. A good reference is [Lüc02].

Let G be a countable discrete group. Then $\mathbb{C}G$ with the scalar product

$$\left\langle \sum_{g\in G} a_g g, \sum_{g\in G} b_g g \right\rangle = \sum_{g\in G} a_g \overline{b_g}$$

is a pre-Hilbert space. We denote by $\ell^2 G$ the completion of $\mathbb{C}G$. Note that

$$\ell^2 G = \left\{ \sum_{g\in G} a_g g \;\middle|\; \sum_{g\in G} |a_g|^2 < \infty \right\},$$

and the involution i on $\mathbb{C}G$ given by $i(\sum_{g\in G} a_g g) = \sum_{g\in G} \overline{a_g} g^{-1}$ extends to an involution on $\ell^2 G$.

The $\mathbb{C}G$-bimodule structure on the ring $\mathbb{C}G$ extends to a $\mathbb{C}G$-bimodule structure on $\ell^2 G$.

Definition 5.1. The *group von Neumann algebra* $\mathcal{N}G$ of a group G is defined to be the algebra of G-equivariant bounded operators $T\colon \ell^2 G \to \ell^2 G$, i.e.,

$$\mathcal{N}G = \left\{ T\colon \ell^2 G \to \ell^2 G \;\middle|\; T \text{ linear, bounded and } T(xg) = T(x)g \text{ for all } x \in \ell^2 G, g \in G \right\}.$$

The group von Neumann algebra $\mathcal{N}G$ becomes a left-$\mathbb{C}G$ module by post-composing an operator with the element of $\mathcal{N}G$ given by left multiplication with the element of $\mathbb{C}G$.

In general, a formal product $x \cdot y$ of $x, y \in \ell^2 G$ need not again define an element of $\ell^2 G$. But if $x \in \ell^2 G$ is such that $x \cdot y \in \ell^2 G$ for all $y \in \ell^2 G$, then multiplication by x defines an element of $\mathcal{N}(G)$.

We consider $\mathcal{N}G$ with the involution which sends $T \in \mathcal{N}G$ to its adjoint or, put differently, the involution is given by

$$\begin{aligned} \mathcal{N}\mathcal{G} &\to \mathcal{N}G \\ T &\mapsto \big(x \mapsto i(T(e)) \cdot x\big), \end{aligned}$$

where $e \in G \subset \ell^2 G$ is the unit element.

Example 5.2. If G is a finite group, then $\mathbb{C}G = \ell^2 G = \mathcal{N}G$.

Definition 5.3. The *von Neumann trace* on $\mathcal{N}G$ is defined by

$$\begin{aligned} \operatorname{trace}_{\mathcal{N}G} \colon \mathcal{N}G &\to \mathbb{C} \\ f &\mapsto \big\langle f(e), e \big\rangle, \end{aligned}$$

where $e \in G \subset \ell^2 G$ is the unit element.

Example 5.4. Let G be a finite group and let $f \in \mathcal{N}G$. Then $\mathcal{N}G$ is a $|G|$-dimensional $\mathbb{C}$-vector space with basis G and the classical trace of f is given by

$$\operatorname{trace}(f) = \sum_{g \in G} \big\langle f(g), g \big\rangle = \sum_{g \in G} \big\langle f(e)g, g \big\rangle = |G| \operatorname{trace}_{\mathcal{N}G}(f).$$

Definition 5.5. Let A be an $(n \times n)$-matrix with entries in $\mathcal{N}G$. We define the *von Neumann trace* of A to be

$$\operatorname{trace}_{\mathcal{N}G}(A) := \sum_{i=1}^{n} \big\langle a_{ii}(e), e \big\rangle,$$

where $e \in G \subset \ell^2(G)$ is the unit element.

Definition 5.6. Let P be a finitely generated projective $\mathcal{N}G$-module. Then there exists a square matrix A_P with entries in $\mathcal{N}G$ such that $A_P = A_P^2$ and the image of the map

$$\begin{aligned} l_{A_P} \colon \mathcal{N}G^n &\to \mathcal{N}G^n \\ x &\mapsto A_P x \end{aligned}$$

is isomorphic to P. It is shown in [Lüc02, Page 238] that $\operatorname{trace}_{\mathcal{N}G}(A_P)$ just depends on P and not on the chosen matrix A_P. The *dimension of* P is defined to be

$$\dim_{\mathcal{N}G}(P) = \operatorname{tr}(A_P) \in [0, \infty).$$

The following theorem is proved in [Lüc02, Theorem 6.5 and Theorem 6.7].

Theorem 5.7. *The dimension function* $\dim_{\mathcal{N}G}$*, which is a priori defined on finitely generated projective* $\mathcal{N}G$*-modules, can be extended to a function defined for all* $\mathcal{N}G$*-modules by*

$$\begin{aligned}\widetilde{\dim}_{\mathcal{N}G}\colon \{\mathcal{N}G\text{-modules}\} &\to [0,\infty] \\ N &\mapsto \sup\{\dim_{\mathcal{N}G}(P) \mid P \subset N \text{ finitely generated projective}\}.\end{aligned}$$

It is shown in [Lüc02, Theorem 6.7(4a)] that if P is a finitely generated projective $\mathcal{N}G$-module then

$$\widetilde{\dim}_{\mathcal{N}G}(P) = \dim_{\mathcal{N}G}(P).$$

From now on we write $\dim_{\mathcal{N}G}$ for the dimension function defined in Theorem 5.7 .

Lück showed that the dimension function is additive on short exact sequences:

Theorem 5.8 ([Lüc02, Theorem 6.7 (4b)])**.** *Let*

$$0 \to N \to M \to K \to 0$$

be an exact sequence of $\mathcal{N}G$*-modules. Then* $\dim_{\mathcal{N}G}(M) = \dim_{\mathcal{N}G}(N) + \dim_{\mathcal{N}G}(K)$.

The dimension of a non-trivial finitely generated (not necessarily projective) $\mathcal{N}G$-module is always a positive real number (see [Lüc02, Theorem 6.7(3) and Theorem 6.7(4e)]).

5.2 L^2-signature of manifolds

We now consider homology with local coefficients in $\mathcal{N}G$. Let M be a compact connected manifold with fundamental group π and let $j\colon \pi \to G$ be a group homomorphism, which induces a ring homomorphism $\mathbb{Z}\pi \to \mathbb{C}G$ again called j. Via j, the left $\mathbb{C}G$-module $\mathcal{N}G$ becomes a left $\mathbb{Z}\pi$-module. Hence, we can consider homology with coefficients in $\mathcal{N}G$.

In the following we want to show that $H_*(M;\mathcal{N}G)$ is a finitely generated $\mathcal{N}G$-module if M is compact. Before we can do so we need some theorems.

We need the following property of $\mathcal{N}G$ which was proved in [Lüc02, Theorem 6.5 and Theorem 6.7(1)].

Theorem 5.9. *The ring* $\mathcal{N}G$ *is semihereditary, i.e., any finitely generated submodule of a projective module is projective.*

We need the following two facts about finitely presented modules.

Lemma 5.10. *Let* R *be a ring and let* M *be a finitely generated projective* R*-module. Then* M *is finitely presented.*

Proof. Since M is finitely generated, there exists a surjective map $r\colon R^n \to M$. Since M is projective, the map r has a section and $R^n \cong M \oplus \ker(r)$. Let $p\colon R^n \to \ker(r)$ be the corresponding projection and $i\colon \ker(r) \to R^n$ the inclusion. Then

$$R^n \xrightarrow{i\circ p} R^n \xrightarrow{r} M \longrightarrow 0$$

is an exact sequence and hence M is a finitely presented R-module. □

The following theorem is shown in [Lam99, Proposition 4.26 (b)].

Theorem 5.11. *Let R be a ring and let M be a finitely presented R-module. Let $p\colon R^n \to M$ be a surjection. Then $\ker(p)$ is finitely generated.*

We now can show that $H_*(M;\mathcal{N}G)$ is indeed a finitely generated $\mathcal{N}G$-module.

Lemma 5.12. *Let M be a compact connected manifold and $\pi_1(M) \to G$ a homomorphism. Then $H_*(M;\mathcal{N}G)$ is a finitely generated $\mathcal{N}G$-module.*

Proof. Since M is compact, $C_*^{CW}\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \mathcal{N}G$ is a finitely generated $\mathcal{N}G$-module. Let

$$d_k\colon C_k^{CW}\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \mathcal{N}G \to C_{k-1}^{CW}\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \mathcal{N}G$$

be the boundary map. The image of d_k is a finitely generated $\mathcal{N}G$-module. Since $\mathcal{N}G$ is semihereditary and $C_{k-1}^{CW}\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \mathcal{N}G$ is projective, it follows from Theorem 5.9 that $\mathrm{Im}(d_k)$ is projective. We obtain from Theorem 5.10 that $\mathrm{Im}(d_k)$ is a finitely presented $\mathcal{N}G$-module. It follows from Theorem 5.11 together with the surjectivity of the map

$$C_k^{CW}\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \mathcal{N}G \xrightarrow{d_k} \mathrm{Im}(d_k)$$

that $\ker(d_k)$ is finitely generated $\mathcal{N}G$-module. Hence, we obtain that

$$H_k(M;\mathcal{N}G) = \frac{\ker(d_k)}{\mathrm{Im}(d_{k+1})}$$

is a finitely generated $\mathcal{N}G$-module. □

Let M be a finitely generated $\mathcal{N}G$-module and

$$h\colon M \times M \to \mathcal{N}G$$

a Hermitian form. It follows from [COT03, Chapter 5, Page 471] that for every Hermitian form over $\mathcal{N}(G)$ there is a spectral decomposition $M = V_+ \oplus V_- \oplus V_0$ such that h is positive definite on V_+, negative definite on V_- and zero on V_0. We define the L^2-signature of h to be

$$\mathrm{sign}^{(2)}_{\mathcal{N}G}(h) = \dim_{\mathcal{N}G}(V_+) - \dim_{\mathcal{N}G}(V_-).$$

Let W be an oriented n-dimensional manifold. Since $\mathcal{N}G$ is a $\mathbb{Z}\pi$-module, it follows from Theorem 2.6 that we have an isomorphism

$$\mathrm{PD}\colon H_m(W;\mathcal{N}G) \to H^{n-m}(W,\partial W;\mathcal{N}G).$$

If W be an oriented $4n$-dimensional manifold, we can thus define an intersection form

$$I_{\mathcal{N}G}\colon H_{2n}(W;\mathcal{N}G) \times H_{2n}(W;\mathcal{N}G) \to \mathcal{N}G$$

in the usual way and it is known to be Hermitian (see [COT03, Page 472]). We denote by $\mathrm{sign}^{(2)}_{\mathcal{N}G}(W)$ the *L^2-signature* of $I_{\mathcal{N}G}$.

5.3 Definition of the L^2-η-invariant and the L^2-ρ-invariant

Cheeger and Gromov defined in [CG85] the L^2-η-invariant depending on a closed connected oriented $(4n-1)$-dimensional Riemannian manifold M together with a group homomorphism $\alpha\colon \pi_1(M) \to G$. A good reference for the L^2-η-invariant is also [CT07]. Before we can give the definition of the L^2-η-invariant, we need to construct the signature operator of M.

Let M be a closed connected oriented Riemannian manifold of dimension $2l-1$. Denote by $*\colon \Omega^*(M) \to \Omega^{2l-1-*}(M)$ the Hodge duality operator and denote by $d\colon \Omega^*(M) \to \Omega^{*+1}(M)$ the exterior derivative. Then the *signature operator* D acts on even-dimensional forms and maps $\omega \in \Omega^{2p}(M)$ to

$$D(\omega) = i^l(-1)^{p+1}(*d - d*)\omega \in \Omega^{2(l-p-1)}(M) \oplus \Omega^{2(l-p)}(M).$$

It is of course possible to define the signature operator of an even-dimensional manifold, but one has to use a scalar factor different from $i^l(-1)^{p+1}$. We will not give the details as we will only deal with the case $2l-1 = 4n-1$.

Let $\alpha\colon \pi_1(M) \to G$ be a group homomorphism and let D_α be the signature operator on the α-cover of M, which is the cover of M corresponding to the subgroup $\ker(\alpha) \subseteq \pi_1(M)$. Analogous to the definition of the classical η-invariant in Section 4.1, we will consider for $t \geq 0$ the operator $D_\alpha e^{-tD_\alpha^2}$, which again admits a smooth kernel k_t with

$$\left(D_\alpha e^{-tD_\alpha^2}\right)(\omega)(x) = \int_M k_t(x,y)\omega(y)\,\mathrm{d}y.$$

The *G-trace* of $D_\alpha e^{-tD_\alpha^2}$ is then defined by

$$\mathrm{trace}_G(D_\alpha e^{-tD_\alpha^2}) = \int_{\mathcal{F}} \mathrm{trace}_{\mathbb{C}}\, k_t(x,x)\,\mathrm{d}x,$$

where $\mathcal{F}$ is a fundamental domain of the α-cover of M under deck transformations.

Definition 5.13. Let M be a closed connected oriented $(4n-1)$-dimensional Riemannian manifold and let $\alpha\colon \pi_1(M) \to G$ be a group homomorphism. Let D_α be the signature operator on the α-cover of M. Then the *L^2-η-invariant* is defined as

$$\eta^{(2)}(M,\alpha) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}}\,\mathrm{trace}_G\left(D_\alpha e^{-tD_\alpha^2}\right)\mathrm{d}t.$$

In the case of a finite group G, the L^2-η-invariant reduces to the classical η-invariant.

Lemma 5.14. *Let M be a closed connected oriented $(4n-1)$-dimensional manifold. Let G be a finite group and let $\alpha\colon \pi_1(M) \to G$ be a homomorphism. Let $\phi_G\colon G \to U(|G|)$ be the regular representation as introduced in Section 4.4. Then*

$$\eta^{(2)}(M,\alpha) = \frac{1}{|G|}\eta(M, \phi_G \circ \alpha).$$

Proof. Let $\mathcal{E}_\alpha$ be the bundle $\mathcal{E}_\alpha = \widetilde{M} \times \mathbb{C}^{|G|}/\sim$ where the equivalence relation is obtained from the right-action of π on $\widetilde{M} \times \mathbb{C}^{|G|}$ which is given by

$$\begin{aligned} \widetilde{M} \times \mathbb{C}^{|G|} \times \pi &\to \widetilde{M} \times \mathbb{C}^{|G|} \\ ((m,v),g) &\mapsto (m \cdot g, (\phi_G \circ \alpha)(g^{-1})v). \end{aligned}$$

Denote by M_G the α-cover of M. Since $\phi_G \circ \alpha$ factorizes through G, we have $\mathcal{E}_\alpha = M_G \times \mathbb{C}^{|G|}/\phi_G$. Let ∇^α be the canonical flat connection on $\mathcal{E}_\alpha$. Let $B_\alpha = \pm i^{2n}(*\nabla^\alpha - \nabla^\alpha *)\phi$ as defined in Section 4.1 and let D_α be the signature operator on M_G. Note that B_α can be locally written as a diagonal matrix where each non-zero entry is given by the signature operator on M_G. Let $k_t^{B_\alpha}$ be a smooth kernel for $B_\alpha e^{-tB_\alpha^2}$ and $k_t^{D_\alpha}$ be a smooth kernel for $D_\alpha e^{-tD_\alpha^2}$. Let $\tilde{x} \in M_G$ and let $x \in M$ be the image of $\tilde{x}$ under the covering map $M_G \to M$. Then

$$\operatorname{trace}_{\mathbb{C}} k_t^{B_\alpha}(x,x) = |G| \operatorname{trace}_{\mathbb{C}} k_t^{D_\alpha}(\tilde{x},\tilde{x})$$

and hence

$$\int_M \operatorname{trace}_{\mathbb{C}} k_t^{B_\alpha}(x,x)\,\mathrm{d}x = |G| \int_{\mathcal{F}} \operatorname{trace}_{\mathbb{C}} k_t^{D_\alpha}(x,x)\,\mathrm{d}x,$$

where $\mathcal{F}$ is a fundamental domain for M_G under deck transformations. It follows that

$$\eta^{(2)}(M,\alpha) = \frac{1}{|G|}\eta(M,\phi_G \circ \alpha). \qquad \square$$

Similar to the classical ρ-invariant one can define an L^2-ρ-invariant which is independent of the chosen Riemannian metric on M [CG85, Chapter 4, Page 23].

Definition 5.15. Let M be a closed connected oriented $(4n-1)$-dimensional Riemannian manifold and let $\alpha\colon \pi_1(M) \to G$ be a group homomorphism. Then the L^2-*ρ-invariant* is defined by

$$\rho^{(2)}(M,\alpha) = \eta^{(2)}(M,\alpha) - \eta(M,\tau_1),$$

where $\tau_1\colon \pi_1(M) \to U(1)$ denotes the trivial one-dimensional representation.

We obtain the following corollary from Lemma 5.14.

Corollary 5.16. *Let M be a closed connected oriented $(4n-1)$-dimensional manifold. Let G be a finite group and $\alpha\colon \pi_1(M) \to G$ a homomorphism. Then*

$$\rho^{(2)}(M,\alpha) = \frac{1}{|G|}\rho(M,\phi_G \circ \alpha).$$

Proof. We obtain the equalities

$$\begin{aligned} \rho^{(2)}(M,\alpha) = \eta^{(2)}(M,\alpha) - \eta(M,\tau_1) &= \frac{1}{|G|}\big(\eta(M,\phi_G \circ \alpha) - |G|\eta(M,\tau_1)\big) \\ &= \frac{1}{|G|}\big(\eta(M,\phi_G \circ \alpha) - \eta(M,\tau_{|G|})\big) = \frac{1}{|G|}\rho(M,\phi_G \circ \alpha), \end{aligned}$$

where the first equality holds by definition of the L^2-ρ-invariant and the second equality follows from Lemma 5.14. The third equality follows from the fact that $|G|\eta(M,\tau_1) = \eta(M,\tau_{|G|})$ holds for the trivial $|G|$-dimensional representation $\tau_{|G|}$ and the last equality is the definition of the ρ-invariant. $\square$

The following theorem is implicit in [Cha16b, Section 2.1].

Theorem 5.17. *Let M be a closed connected oriented $(4n-1)$-dimensional manifold and $\alpha\colon\pi_1(M)\to G$ be a group homomorphism. Let W be a compact oriented $4n$-dimensional manifold with $\partial W = rM$, where rM denotes r disjoint copies of M. Let Γ be a group such that there exists a monomorphism $m\colon G\hookrightarrow\Gamma$. Assume there exists a homomorphism $\beta\colon\pi_1(W)\to\Gamma$ such that the following diagram commutes:*

$$\begin{tikzcd}
\pi_1(M) \arrow[r, "\alpha"] \arrow[d, "i_*"] & G \arrow[d, "m"] \\
\pi_1(W) \arrow[r, "\beta"] & \Gamma.
\end{tikzcd}$$

Then

$$\rho^{(2)}(M,\alpha) = \frac{1}{r}\big(\operatorname{sign}(W) - \operatorname{sign}^{(2)}_{\mathcal{N}\Gamma}(W)\big),$$

where the $\mathcal{N}\Gamma$-L^2-signature on W is defined using the ring homomorphism $\mathbb{Z}\pi_1(W)\to\mathcal{N}\Gamma$ induced by β.

Moreover for a given closed connected oriented $(4n-1)$-manifold M and a group homomorphism $\alpha\colon\pi_1(M)\to G$ it is shown in [Cha16b, Section 2.1] that there always exists a compact oriented manifold W with boundary consisting of disjoint copies of M such that α extends to a group homomorphism $\beta\colon\pi_1(W)\to\Gamma$, where G injects into Γ. Hence, as opposed to the situation of the classical ρ-invariant, the L^2-ρ-invariant can always be computed as a signature defect and the statement of Theorem 5.17 can be used as a definition of the L^2-ρ-invariant.

5.4 The L^2-ρ-invariant of residually finite groups

Let M be a closed connected oriented $(4n-1)$-dimensional manifold. We have already seen in Corollary 5.16 that in the case of a group homomorphism $\pi_1(M)\to G$ to a finite group G the classical ρ-invariant and the L^2-ρ-invariant differ only by a constant factor, the order of G. In this section we consider residually finite groups.

Definition 5.18.

1. Let G be a group. A *residual chain of G* is a sequence $\{G_i\}_{i\in\mathbb{N}}$ of finite index normal subgroups of G such that
   - $G_i \subset G_{i-1}$ for all i
   - $\cap_{i\in\mathbb{N}} G_i = \{e\}$.
2. A group G is called *residually finite* if there exists a residual chain of G.

As holds for most flavors of L^2-invariants, $\rho^{(2)}(M,\phi\colon\pi_1(M)\to G)$ with G a residually finite group can be approximated by the L^2-ρ-invariants belonging to group homomorphisms to quotients by finite index subgroups of G.

The following theorem was proved in [LS05, Remark 1.29].

Theorem 5.19. *Let M be a closed connected oriented $(4n-1)$-dimensional manifold. Let G be a residually finite group and let $\alpha\colon\pi_1(M)\to G$. Choose a residual chain $\{G_i\}_{i\in\mathbb{N}}$ of G and* *let $p_{G_i}: G \to G/G_i$ be the projection map. Then*

$$\rho^{(2)}(M,\alpha) = \lim_{i\to\infty} \rho^{(2)}(M, p_{G_i} \circ \alpha).$$

We can now show the following corollary.

Corollary 5.20. *Let K be a knot and let M_K be obtained by 0-framed surgery along $K \subset S^3$. Let $\alpha: \pi_1(M_K) \to \mathbb{Z}$ be the homomorphism, which sends a meridian to $1 \in \mathbb{Z}$. Let*

$$\begin{aligned} \phi_z: \mathbb{Z} &\to U(1) \\ n &\mapsto z^n. \end{aligned}$$

Then

$$\rho^{(2)}(M_K, \mathbb{Z}) = \int_{S^1} \sigma_z(K)\,\mathrm{d}z = \int_{S^1} \rho(M_K, \phi_z \circ \alpha)\,\mathrm{d}z.$$

Proof. Consider the residual chain

$$\ldots \subset (k+1)!\mathbb{Z} \subset k!\mathbb{Z} \subset \ldots \subset \mathbb{Z}$$

of $\mathbb{Z}$. Let

$$\alpha_l: \pi_1(M_K) \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{p_l} \mathbb{Z}_l,$$

where p_l denotes the canonical projection map. Let

$$\begin{aligned} \phi_{j,l}: \mathbb{Z}_l &\to U(1) \\ [n] &\mapsto e^{2\pi i j n/l}. \end{aligned}$$

We obtain

$$\begin{aligned} \rho^{(2)}(M_K, \mathbb{Z}) &= \lim_{k\to\infty} \frac{1}{k!}\rho(M_K, \alpha_{k!}) = \lim_{k\to\infty} \frac{1}{k!} \sum_{j=0}^{k!-1} \rho(M_K, \phi_{j,k!} \circ \alpha_{k!}) \\ &= \lim_{k\to\infty} \frac{1}{k!} \sum_{j=0}^{k!-1} \sigma_{e^{2\pi i j/k!}} = \int_{S^1} \sigma_z(K)\,\mathrm{d}z. \end{aligned}$$

The first equality follows from Theorem 5.19 and the fact that for a finite group G

$$\rho^2(M_K, \alpha) = \frac{1}{|G|}\rho(M_K, \alpha).$$

holds. The second equality follows from Lemma 4.5 and the third equality follows from Proposition 4.3. The last equality follows from the fact that the Levine–Tristam signature function is continuous outside of a finite set and hence

$$\int_{S^1} \sigma_z(K)\,\mathrm{d}z$$

exists and the previous term is a Riemann sum approximation for it. Since, again by Proposition 4.3, $\sigma_z(K) = \rho(M, \phi_z \circ \alpha)$ holds, we also obtain

$$\rho^{(2)}(M_K, \mathbb{Z}) = \int_{S^1} \rho(M_K, \phi_z \circ \alpha)\,\mathrm{d}z. \qquad \square$$

Chapter 6

Results of Farber and Levine: Jumps of the η-invariant

In their article "Jumps of the η-invariant" Farber and Levine studied the behavior of the η-invariant under analytic deformations of a connection. Their main result, which we formulate in this chapter, is that the heights of the jumps of the η-invariant are always integers and can be calculated as sums of signatures of suitable linking forms. We construct the linking forms in detail and prove the relevant properties stated in [FL96].

6.1 The variety of unitary representations

Let G be a finitely generated group and $\alpha\colon G \to U(k)$ a unitary representation of G of rank k. We can alternatively view the representation as a homomorphism $\alpha\colon G \to \mathrm{Mat}(k \times k, \mathbb{C})$ with the additional assumption that $\alpha(g)$ is a unitary matrix for any element in a fixed finite generating set S of G. We denote the set of relations of G, expressed in the elements of S, by R, which may be an infinite set.

Viewed in this way, a unitary representation α of G is in fact uniquely determined by a choice for the $|S|$ unitary matrices $\alpha(s_1), \ldots, \alpha(s_{|S|})$ of rank k. In the opposite direction, any such choice for which the unitary matrices satisfy all the relations in R gives rise to a unitary representation of rank k of α. Summarizing the discussion, we obtain a correspondence between unitary representations of G of rank k and $|S|$-tuples of $(k \times k)$-matrices with entries in $\mathbb{C}$ that are unitary and satisfy the corresponding relations in R.

Our aim is to formulate these conditions as polynomials in the entries of the matrices. Since the condition for a matrix to be unitary involves complex conjugation, and therefore cannot be expressed in complex polynomial expressions, we instead consider the matrices as pairs of matrices with real entries, where one matrix corresponds to the real and the other to the imaginary part. It is then easy to see that both unitarity and the relations in R can be formulated as (possibly infinitely many) polynomial conditions with real coefficients in the $2k^2$ real entries of each of the $|S|$ matrices. We conclude that the set of unitary representations of G has the structure of a real algebraic subvariety of $\mathbb{R}^{2k^2|S|}$. Note that even though R may be an infinite set of relations, the polynomial ring $\mathbb{R}[x_1, \ldots, x_{2k^2|S|}]$ is Noetherian and hence the variety is cut out by finitely many polynomial equations.

Definition 6.1. Let G be a finitely generated group. Then we denote by $R_k(G)$ the *real algebraic variety of rank k unitary representations of G.*

The following result was shown by Levine [Lev94, Theorem 2.1].

Theorem 6.2. *Let M be a closed connected oriented odd-dimensional manifold. Let $r \in \mathbb{N} \cup \{0\}$ and let Σ_r be the subvariety*

$$\Sigma_r = \left\{ \alpha \in R_k(\pi_1(M)) \,\middle|\, \sum_{i=0}^{\infty} \dim_{\mathbb{C}} H_i^{\alpha}(M; \mathbb{C}^k) \geq r \right\}.$$

Then

$$\begin{aligned} \rho(M) \colon R_k(\pi_1(M)) &\to \mathbb{R} \\ \alpha &\mapsto \rho(M, \alpha) \end{aligned}$$

is continuous on $\Sigma_r \backslash \Sigma_{r+1}$. Furthermore $\rho(M)$ is continuous when considered with values in $\mathbb{R}/\mathbb{Z}$.

6.2 Homology and products over $\mathcal{O}$ and $\mathcal{M}$

Let M be a closed connected oriented odd-dimensional manifold. We denote the fundamental group of M by π.

Let $\{\alpha_t \colon \pi \to U(k)\}_{t \in (-\epsilon, \epsilon)}$ be an analytic deformation of α_0 (see Section 3.1).

Recall that $\mathcal{O}$ denotes the ring of germs of holomorphic functions as introduced in Section 2.10 and $\mathcal{M}$ is the field of germs of meromorphic functions as introduced in Section 2.11. In the following, we will also view these germs as being restricted to the real line, which incurs no loss of information as discussed previously.

An element of $\mathcal{O}^k$ is by definition a k-tuple $f = (f_1, \ldots, f_k)$ of germs at zero of holomorphic functions $f_1, \ldots, f_k$. For our purposes, it will often be convenient to view f as the germ at zero of a vector-valued function which is holomorphic in each coordinate.

We equip $\mathcal{O}^k$ with the structure of a left $\mathbb{Z}\pi$-module depending on the deformation α_t, where the action of π on $\mathcal{O}^k$ is given by

$$\begin{aligned} \pi \times \mathcal{O}^k &\to \mathcal{O}^k \\ (g, f) &\mapsto \big(t \mapsto \alpha_t(g) f(t)\big). \end{aligned}$$

Since the deformation is analytic, the vector-valued function $t \mapsto \alpha_t(g) f(t)$ is indeed an element of $\mathcal{O}^k$. In the same way π acts on $\mathcal{M}^k$ and $(\mathcal{M}/\mathcal{O})^k$.

Example 6.3. Let $\alpha \colon \pi \to U(k)$ be a k-dimensional representation of the fundamental group. Let $\phi \colon \pi_1(M) \to \mathbb{Z}$. Then

$$\begin{aligned} \alpha_t \colon \pi &\to U(k) \\ g &\mapsto \alpha(g) e^{it\phi(g)} \end{aligned}$$

for $t \in (\epsilon, -\epsilon)$ defines an analytic deformation of $\alpha_0 = \alpha$. The corresponding action of π on $\mathcal{O}^k$ is given by

$$\begin{aligned} \pi \times \mathcal{O}^k &\to \mathcal{O}^k \\ (g, f) &\mapsto \left(t \mapsto \alpha(g) e^{it\phi(g)} f(t)\right). \end{aligned}$$

Denote by $\widetilde{M}$ the universal covering of M. Recall from Section 2.2 that we consider the chain groups $C_n\big(\widetilde{M}\big)$ as right $\mathbb{Z}\pi$-modules. We now consider the cohomology of M with local coefficient system $\mathcal{O}^k$, i.e.,

$$H^n\big(M; \mathcal{O}^k\big) = H_n\big(\operatorname{Hom}_{\mathbb{Z}\pi}\big(\overline{C_* \widetilde{M}}, \mathcal{O}^k\big)\big).$$

We denote by $C_*^{CW}(M)$ the cellular chain complex of M. Since M is a compact manifold, $C_n^{CW}\big(\widetilde{M}\big)$ is a finitely generated $\mathbb{Z}\pi$-module and hence $\operatorname{Hom}_{\mathbb{Z}\pi}\big(\overline{C_n^{CW}\big(\widetilde{M}\big)}, \mathcal{O}^k\big)$ is finitely generated as an $\mathcal{O}$-module. Since $\mathcal{O}$ is a principal ideal domain,

$$H^n\big(M; \mathcal{O}^k\big) = H_n\big(\operatorname{Hom}_{\mathbb{Z}\pi}\big(\overline{C_*^{CW} \widetilde{M}}, \mathcal{O}^k\big)\big)$$

is a finitely generated $\mathcal{O}$-module.

Let $A, B \in \{\mathcal{O}^k, \mathcal{M}^k, (\mathcal{M}/\mathcal{O})^k\}$. We consider the diagonal action of π on the tensor product $A \otimes_{\mathcal{O}} B$, namely

$$\begin{aligned} \pi \times (A \otimes_{\mathcal{O}} B) &\to A \otimes_{\mathcal{O}} B \\ (g, f \otimes h) \quad &\mapsto \big(t \mapsto \alpha_t(g) f(t)\big) \otimes \big(t \mapsto \alpha_t(g) h(t)\big). \end{aligned}$$

We consider $\mathcal{M}/\mathcal{O}$ as a trivial $\mathbb{Z}\pi$-module. Since for fixed $t \in (-\epsilon, \epsilon)$ and $g \in \pi$ we have that $\alpha_t(g) \in U(k)$, we obtain that the map

$$\begin{aligned} \Phi \colon A \otimes_{\mathcal{O}} B &\to \mathcal{M}/\mathcal{O} \\ v \otimes w \quad &\mapsto \overline{v}^t w^* \end{aligned}$$

is left π-invariant, i.e., $\Phi(ga \otimes gb) = \Phi(a \otimes b)$ for all $g \in \pi$, $a \in A$ and $b \in B$. In the same way we consider $\mathcal{O}$ and $\mathcal{M}$ as trivial $\mathbb{Z}\pi$-modules and obtain left π-invariant maps

$$\begin{aligned} \mathcal{O}^k \otimes \mathcal{O}^k &\to \mathcal{O} \\ v \otimes w &\mapsto \overline{v}^t w^* \end{aligned} \qquad \begin{aligned} \mathcal{O}^k \otimes \mathcal{M}^k &\to \mathcal{M} \\ v \otimes w &\mapsto \overline{v}^t w^* \end{aligned} \qquad \begin{aligned} \mathcal{M}^k \otimes \mathcal{M}^k &\to \mathcal{O} \\ v \otimes w &\mapsto \overline{v}^t w^* \end{aligned}$$

6.3 Homological linking form

Let M be a closed connected oriented manifold of dimension $2l-1$. Let $\{\alpha_t \colon \pi \to U(k)\}_{t \in (-\epsilon, \epsilon)}$ be an analytic deformation of α_0. We consider $\mathcal{O}^k$, $\mathcal{M}^k$ and $(\mathcal{M}/\mathcal{O})^k$ with the π-actions as defined in Section 6.2 and we consider the homologies $H_*\big(M; \mathcal{O}^k\big)$, $H_*\big(M; \mathcal{M}^k\big)$ and $H_*\big(M; (\mathcal{M}/\mathcal{O})^k\big)$ twisted by $\{\alpha_t\}_t$, but omit the fixed family $\{\alpha_t\}_t$ from the notation.

The next step is to construct a linking form

$$\{-,-\}\colon \mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right) \times \mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right) \to \mathcal{M}/\mathcal{O}.$$

Since

$$0 \longrightarrow \mathcal{O}^k \longrightarrow \mathcal{M}^k \longrightarrow (\mathcal{M}/\mathcal{O})^k \longrightarrow 0$$

is a short exact sequence of $(\mathbb{Z}\pi,\mathcal{O})$-bimodules, we have a short exact sequence of $\mathcal{O}$-modules

$$0 \to C^*\left(M;\mathcal{O}^k\right) \to C^*\left(M;\mathcal{M}^k\right) \to C^*\left(M;(\mathcal{M}/\mathcal{O})^k\right) \to 0.$$

Hence, we get a long exact sequence

$$\dots\ H^{n-1}\left(M;\mathcal{M}^k\right) \longrightarrow H^{n-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right) \xrightarrow{\beta} H^n\left(M;\mathcal{O}^k\right) \xrightarrow{i^*} H^n\left(M;\mathcal{M}^k\right)\ \dots$$

of $\mathcal{O}$-modules. The map β is called the *Bockstein homomorphism* .

Lemma 6.4. *The image of the Bockstein homomorphism*

$$\beta\colon H^{n-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right) \to H^n\left(M;\mathcal{O}^k\right)$$

is $\mathrm{Torsion}_{\mathcal{O}}\left(H^n(M;\mathcal{O}^k)\right)$.

Proof. Since

$$\dots\ H^{n-1}\left(M;\mathcal{M}^k\right) \longrightarrow H^{n-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right) \xrightarrow{\beta} H^n\left(M;\mathcal{O}^k\right) \xrightarrow{i^*} H^n\left(M;\mathcal{M}^k\right)\ \dots$$

is an exact sequence, we have $\mathrm{Image}(\beta) = \ker(i^*)$.

Since $\mathcal{M}$ is a flat $\mathcal{O}$-module, we have $\mathrm{Tor}_{\mathcal{O}}(-;\mathcal{M}) = 0$. It follows from the universal coefficient theorem and the fact that $\mathrm{Hom}_{\mathbb{Z}\pi}\left(\overline{C_n\widetilde{M}},\mathcal{M}^k\right) \cong \mathrm{Hom}_{\mathbb{Z}\pi}\left(\overline{C_n\widetilde{M}},\mathcal{O}^k\right) \otimes_{\mathcal{O}} \mathcal{M}$ that

$$H^n\left(M;\mathcal{O}^k\right) \otimes \mathcal{M} \cong H^n\left(M;\mathcal{M}^k\right).$$

Hence, the kernel of $i^*\colon H^n\left(M;\mathcal{O}^k\right) \to H^n\left(M;\mathcal{M}^k\right)$ is precisely the $\mathcal{O}$-torsion submodule of $H^n\left(M;\mathcal{O}^k\right)$. □

Let ν be the map

$$\begin{aligned} \nu\colon (\mathcal{M}/\mathcal{O})^k \otimes_{\mathcal{O}} \mathcal{O}^k &\to \mathcal{M}/\mathcal{O} \\ v \otimes w \quad &\mapsto v^t w^*. \end{aligned}$$

Recall from Section 2.3 that, by using the cup product, we obtain a map

$$\cup\colon H^{l-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right) \times H^l\left(M;\mathcal{O}^k\right) \to H^{2l-1}\left(M;(\mathcal{M}/\mathcal{O})^k \otimes_{\mathcal{O}} \mathcal{O}^k\right) \xrightarrow{\nu_*} H^{2l-1}\left(M;\mathcal{M}/\mathcal{O}\right).$$

Let $[M] \in H_{2l-1}(M;\mathbb{Z})$ be the fundamental class. Recall from Section 2.4 that by using the cap product we get a map

$$H^{2l-1}(M;\mathcal{M}/\mathcal{O}) \xrightarrow{-\cap[M]} H_0(M;\mathcal{M}/\mathcal{O}) \xrightarrow{\cong} \mathcal{M}/\mathcal{O}.$$

We can now define a linking form on $\mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right)$.

Lemma 6.5. *Let* $x, y \in \operatorname{Torsion}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big)\big)$*. Let* $[M] \in H_{2l-1}(M;\mathbb{Z})$ *be the fundamental class and let* $z \in \beta^{-1}(x)$*. Then*

$$\begin{aligned} \{-,-\}\colon \operatorname{Torsion}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big)\big) \times \operatorname{Torsion}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big)\big) &\to \mathcal{M}/\mathcal{O} \\ (x,y) &\mapsto \nu_*(z \cup y) \cap [M] \end{aligned}$$

does not depend on the choice of $z \in \beta^{-1}(x)$*.*

Proof. Let $z_1, z_2 \in \beta^{-1}(x)$. Their difference is an element in $\ker(\beta)$. Let $z \in \ker(\beta)$ and let $y \in \operatorname{Torsion}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big)\big)$. We show that

$$\nu_*(z \cup y) = 0 \in H^{2l-1}(M;\mathcal{M}/\mathcal{O}).$$

Since

$$\ldots\; H^{l-1}\big(M;\mathcal{M}^k\big) \xrightarrow{p_*} H^{l-1}\big(M;(\mathcal{M}/\mathcal{O})^k\big) \xrightarrow{\beta} H^l\big(M;\mathcal{O}^k\big) \xrightarrow{i^*} H^l\big(M;\mathcal{M}^k\big)\; \ldots$$

is an exact sequence, we have that $\ker(\beta) = \operatorname{Im}(p_*)$. We consider the following commutative diagram

$$\begin{tikzcd} H^{l-1}(M;\mathcal{M}^k) \times \operatorname{Torsion}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big)\big) \arrow[r] \arrow[dd, "p_*\times\mathrm{id}"] & H^{l-1}\big(M;\mathcal{M}^k\big) \otimes_{\mathcal{O}} \operatorname{Torsion}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big)\big) \arrow[d, "\cup"] \\ & H^{2l-1}(M;\mathcal{M}) \arrow[d, "p_*"] \\ H^{l-1}\big(M;(\mathcal{M}/\mathcal{O})^k\big) \times \operatorname{Torsion}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big)\big) \arrow[r, "\nu_*(-\cup-)"'] & H^{2l-1}(M;\mathcal{M}/\mathcal{O}). \end{tikzcd}$$

Since $\operatorname{Torsion}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big)\big)$ is an $\mathcal{O}$-torsion module

$$H^{l-1}\big(M;\mathcal{M}^k\big) \otimes_{\mathcal{O}} \operatorname{Torsion}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big)\big) = 0.$$

It follows that the map

$$p_*\big(H^{l-1}\big(M;\mathcal{M}^k\big)\big) \times \operatorname{Torsion}_{\mathcal{O}}\big(H^{l-1}\big(M;\mathcal{O}^k\big)\big) \xrightarrow{\nu_*(-\cup-)} H^{2l-1}(M;\mathcal{M}/\mathcal{O})$$

is the zero map, i.e., if $z \in \ker(\beta) = p_*\big(H^{l-1}\big(M;\mathcal{M}^k\big)\big)$ and $y \in \operatorname{Torsion}\big(H^l\big(M;\mathcal{O}^k\big)\big)$, then

$$\nu_*(z \cup y) = 0 \in H^{2l-1}(M;\mathcal{M}/\mathcal{O}). \qquad \square$$

The proof of the next lemma is a slight variation of a statement which was proved in [Fri19, Lemma 74.18].

Lemma 6.6. *For any* $x \in H^m\big(M;(\mathcal{M}/\mathcal{O})^k\big)$ *and any* $y \in H^n\big(M;(\mathcal{M}/\mathcal{O})^k\big)$ *we have*

$$\nu_*(x \cup \beta(y)) = (-1)^{(n+1)(m+1)}\nu_*(y \cup \beta(x))^* \in H^{m+n+1}(M;\mathcal{M}/\mathcal{O}).$$

Proof. We consider the short exact sequence

$$0 \longrightarrow C^*\big(M;\mathcal{O}^k\big) \xrightarrow{i_*} C^*\big(M;\mathcal{M}^k\big) \xrightarrow{p_*} C^*\big(M;(\mathcal{M}/\mathcal{O})^k\big) \longrightarrow 0.$$

Let $x \in H^m\big(M;(\mathcal{M}/\mathcal{O})^k\big)$ and $y \in H^n\big(M;(\mathcal{M}/\mathcal{O})^k\big)$. We choose $f \in C^m\big(M;\mathcal{M}^k\big)$ and $g \in C^n\big(M;\mathcal{M}^k\big)$ with $[p_*(f)] = x$ and $[p_*(g)] = y$. We denote both boundary maps $C^*\big(M;(\mathcal{M}/\mathcal{O})^k\big) \to C^{*+1}\big(M;(\mathcal{M}/\mathcal{O})^k\big)$ and $C^*\big(M;\mathcal{M}^k\big) \to C^{*+1}\big(M;\mathcal{M}^k\big)$ by δ. Since $p_*(\delta(f)) = \delta(p_*(f)) = 0$ and likewise for g, there exist $\tilde{f} \in C^{m+1}\big(M;\mathcal{O}^k\big)$ and $\tilde{g} \in C^{n+1}\big(M;\mathcal{O}^k\big)$ with

$$\begin{aligned} i_*(\tilde{f}) &= \delta(f) \in C^{m+1}\big(M;\mathcal{M}^k\big) \\ i_*(\tilde{g}) &= \delta(g) \in C^{n+1}\big(M;\mathcal{M}^k\big). \end{aligned}$$

Note that it follows from the construction of the long exact sequence

$$\ldots \longrightarrow H^{*-1}\big(M;\mathcal{M}^k\big) \longrightarrow H^{*-1}\big(M;(\mathcal{M}/\mathcal{O})^k\big) \xrightarrow{\beta} H^*\big(M;\mathcal{O}^k\big) \longrightarrow H^*\big(M;\mathcal{M}^k\big) \longrightarrow \ldots$$

that

$$\begin{aligned} [\tilde{f}] &= \beta(x), \\ [\tilde{g}] &= \beta(y). \end{aligned}$$

We obtain the following equality in $H^{m+n+1}\big(M;\mathcal{M}^k \otimes \mathcal{M}^k\big)$

$$0 = [\delta(f \cup g)] = [\delta(f) \cup g + (-1)^m f \cup \delta(g)] = [i_*(\tilde{f}) \cup g + (-1)^m f \cup i_*(\tilde{g})],$$

where the first equality follows from Lemma 2.2.

Let

$$\begin{aligned} \mu\colon \mathcal{M}^k \otimes \mathcal{M}^k &\to \mathcal{M} \\ v \otimes w &\mapsto v^t w^*, \end{aligned}$$

$$\begin{aligned} \nu'\colon \mathcal{O}^k \otimes (\mathcal{M}/\mathcal{O})^k &\to \mathcal{M}/\mathcal{O} \\ v \otimes w &\mapsto v^t w^*. \end{aligned}$$

The following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}^k \otimes \mathcal{M}^k \arrow[r, "i\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes p"'] & \mathcal{M}^k \otimes \mathcal{M}^k \arrow[d, "\mu"] & \mathcal{M}^k \otimes \mathcal{O}^k \arrow[l, "\mathrm{id}\otimes i"'] \arrow[d, "p\otimes \mathrm{id}"] \\
\mathcal{O}^k \otimes (\mathcal{M}/\mathcal{O})^k \arrow[dr, "\nu'"'] & \mathcal{M} \arrow[d, "p"] & (\mathcal{M}/\mathcal{O})^k \otimes \mathcal{O}^k \arrow[dl, "\nu"] \\
 & \mathcal{M}/\mathcal{O} &
\end{tikzcd}$$

Note that, by Proposition 2.3, for any $a \in H^{m+1}(M;\mathcal{O}^k)$ and $b \in H^n(M;(\mathcal{M}/\mathcal{O})^k)$ the identity

$$\nu'_*(a \cup b) = (-1)^{n(m+1)}\nu_*(b \cup a)^*$$

holds, since the analogous identity holds on the level of coefficients. We have already seen that $[i_*(\tilde{f}) \cup g + (-1)^m f \cup i_*(\tilde{g})] = 0$, hence,

$$\begin{aligned}
0 &= (p \circ \mu)\big([i_*(\tilde{f}) \cup g + (-1)^m f \cup i_*(\tilde{g})]\big)\\
&= \big[\nu'_*(\tilde{f} \cup p_*(g)) + (-1)^m \nu_*(p_*(f) \cup \tilde{g})\big]\\
&= \nu'_*(\beta(x) \cup y) + (-1)^m \nu_*(x \cup \beta(y))\\
&= (-1)^{n(m+1)}\nu_*\big(y \cup \beta(x)\big)^* + (-1)^m \nu_*(x \cup \beta(y)).
\end{aligned}$$

□

We now deduce properties of the linking form defined in Lemma 6.5.

Corollary 6.7. *The map*

$$\begin{aligned}
\{-,-\}\colon \mathrm{Torsion}_{\mathcal{O}}(H^l(M;\mathcal{O}^k)) \times \mathrm{Torsion}_{\mathcal{O}}(H^l(M;\mathcal{O}^k)) &\to \mathcal{M}/\mathcal{O}\\
(x,y) &\mapsto \nu_*\big(\beta^{-1}(x) \cup y\big) \cap [M]
\end{aligned}$$

defines a $(-1)^l$-Hermitian form, i.e.,

$$\begin{aligned}
\{x,y\} &= -\{y,x\}^* && \text{if } l \text{ is odd}\\
\{x,y\} &= \{y,x\}^* && \text{if } l \text{ is even}
\end{aligned}$$

which is $\mathcal{O}$-linear in the first variable and $\mathcal{O}$-anti-linear in the second variable.

Proof. Let $x,y \in H^l\big(M;\mathcal{O}^k\big)$. Choose $\tilde{x},\tilde{y} \in H^{l-1}\big(M;(\mathcal{M}/\mathcal{O})^k\big)$ with $\beta(\tilde{x}) = x$ and $\beta(\tilde{y}) = y$. We obtain

$$\{x,y\} = \nu_*\big(\tilde{x} \cup \beta(\tilde{y})\big) \cap [M] = (-1)^{l^2}\nu_*\big(\tilde{y} \cup \beta(\tilde{x})\big)^* \cap [M] = (-1)^l\{y,x\}^*,$$

where the first and last equality follow from the definition and the fact that $(-1)^{l^2} = (-1)^l$. For the second equality we used Lemma 6.6. Note that the map

$$\cup\colon H^{l-1}\big(M;(\mathcal{M}/\mathcal{O})^k\big) \times H^l\big(M;\mathcal{O}^k\big) \to H^{2l-1}\big(M;(\mathcal{M}/\mathcal{O})^k \otimes_{\mathcal{O}} \mathcal{O}^k\big)$$

is $\mathcal{O}$-linear in both variables and

$$H^{2l-1}(M;\mathcal{M}/\mathcal{O}) \xrightarrow{-\cap[M]} H_0(M;\mathcal{M}/\mathcal{O}) \xrightarrow{\cong} \mathcal{M}/\mathcal{O}$$

is $\mathcal{O}$-linear. Since

$$\begin{aligned}
\nu\colon (\mathcal{M}/\mathcal{O})^k \otimes_{\mathcal{O}} \mathcal{O}^k &\to \mathcal{M}/\mathcal{O}\\
v \otimes w &\mapsto v^t w^*
\end{aligned}$$

is $\mathcal{O}$-linear in the first variable and $\mathcal{O}$-anti-linear in the second variable, this also holds for the map

$$H^{2l-1}\big(M;(\mathcal{M}/\mathcal{O})^k \otimes_{\mathcal{O}} \mathcal{O}^k\big) \to H^{2l-1}(M;\mathcal{M}/\mathcal{O}).$$

It follows that the map

$$\begin{aligned}\{-,-\}\colon \mathrm{Torsion}_{\mathcal{O}}\left(H^l\big(M;\mathcal{O}^k\big)\right) \times \mathrm{Torsion}_{\mathcal{O}}\left(H^l\big(M;\mathcal{O}^k\big)\right) &\to \mathcal{M}/\mathcal{O}\\ (x,y) &\mapsto \nu_*\big(\beta^{-1}(x)\cup y\big)\cap[M]\end{aligned}$$

is $\mathcal{O}$-linear in the first variable and $\mathcal{O}$-anti-linear in the second variable. □

Before we will prove that the form defined in Corollary 6.7 is indeed non-singular, we prove an elementary lemma.

Lemma 6.8. *Let R be a principal ideal domain and let M and L be modules over R with $L \subset M$ and M finitely generated. If M/L is isomorphic to M, then L is trivial.*

Proof. Assume for the sake of contradiction that there exists a non-trivial submodule $L_0 \subset M$ such that $M \cong M/L_0$. But then there exists a non-trivial submodule L_1 of M/L_0 such that $(M/L_0)/L_1$ is isomorphic to M/L_0 and hence to M. Repeating this construction, we get an infinite chain of projections to quotient modules:

$$M \xrightarrow{p_0} M/L_0 \xrightarrow{p_1} \big(M/L_0\big)/L_1 \xrightarrow{p_2} \Big(\big(M/L_0\big)/L_1\Big)/L_2 \xrightarrow{p_3} \dots$$

Let $N_i = \ker(p_i \circ \cdots \circ p_0)$, then we get an infinite chain of submodules of M:

$$N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \dots$$

But this a contradiction since M is a finitely generated module over a principal ideal domain and hence Noetherian. □

Corollary 6.9. *Let R be a principal ideal domain and let A, B and M be modules over R with M is finitely generated and $A \subseteq B \subseteq M$. If M/B is isomorphic to M/A, then $A = B$.*

Proof. Assume that M/B is isomorphic to M/A. Note that $(M/A)/(B/A) \cong M/B$ and hence, by assumption, isomorphic to M/A. It follows from Lemma 6.8 that B/A is trivial and hence $A = B$. □

We now show that the linking form $\{-,-\}$ is non-singular. Note that if R is a principal ideal domain and V is a torsion R-module, then every non-degenerate Hermitian form $V \times V \to R$ is non-singular (see for example [Hil12, Theorem 3.24]).

Lemma 6.10. *Let M be a closed connected oriented manifold of dimension $2l-1$. Let $[M] \in H_{2l-1}(M;\mathbb{Z})$ be the fundamental class. Then*

$$\begin{aligned}\{-,-\}\colon \mathrm{Torsion}_{\mathcal{O}}\left(H^l\big(M;\mathcal{O}^k\big)\right) \times \mathrm{Torsion}_{\mathcal{O}}\left(H^l\big(M;\mathcal{O}^k\big)\right) &\to \mathcal{M}/\mathcal{O}\\ (x,y) &\mapsto \nu_*\big(\beta^{-1}(x)\cup y\big)\cap[M]\end{aligned}$$

defines a non-singular form.

Proof. Fix $x \in \mathrm{Torsion}_{\mathcal{O}}\left(H^l\big(M;\mathcal{O}^k\big)\right)$. We show that if

$$\begin{aligned}\mathrm{Torsion}_{\mathcal{O}}\left(H^l\big(M;\mathcal{O}^k\big)\right) &\to \mathcal{M}/\mathcal{O}\\ y &\mapsto \nu_*\big(\beta^{-1}(x)\cup y\big)\cap[M]\end{aligned}$$

is the zero map, then $x = 0$. Recall from Corollary 6.7 that $\{-,-\}$ is $\mathcal{O}$-linear in the first variable and $\mathcal{O}$-anti-linear in the second. Hence,

$$\begin{aligned} H^l(M;\mathcal{O}^k) &\to \mathcal{M}/\mathcal{O} \\ y &\mapsto \nu_*\big(\beta^{-1}(x)\cup y\big)\cap[M] \end{aligned} \tag{6.1}$$

is $\mathcal{O}$-anti-linear. Let

$$\begin{aligned} \nu' \colon \mathcal{O}^k \otimes (\mathcal{M}/\mathcal{O})^k &\to \mathcal{M}/\mathcal{O} \\ v\otimes w &\mapsto v^t w^*. \end{aligned}$$

For the sake of notational convenience, we consider instead of (6.1) the $\mathcal{O}$-linear map which is given by

$$\begin{aligned} H^l(M;\mathcal{O}^k) &\to \mathcal{M}/\mathcal{O} \\ y &\mapsto \nu'_*\big(y\cup\beta^{-1}(x)\big)\cap[M]. \end{aligned} \tag{6.2}$$

Note that if the map in (6.2) is the zero map then so is the map in (6.1), and vice versa. Since $\nu' \colon \mathcal{O}^k \otimes_{\mathcal{O}} (\mathcal{M}/\mathcal{O})^k \to \mathcal{M}/\mathcal{O}$ is non-degenerate, we get an induced $\mathbb{Z}\pi$-linear isomorphism $\tilde{\nu}' \colon (\mathcal{M}/\mathcal{O})^k \to \operatorname{Hom}_{\mathcal{O}}\big(\mathcal{O}^k, \mathcal{M}/\mathcal{O}\big)$. This gives the first isomorphism of the following chain of natural isomorphisms:

$$\begin{aligned} \operatorname{Hom}_{\mathbb{Z}\pi}\big(\overline{C_*\widetilde{M}}, (\mathcal{M}/\mathcal{O})^k\big) &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}\pi}\big(\overline{C_*\widetilde{M}}, \operatorname{Hom}_{\mathcal{O}}\big(\mathcal{O}^k, \mathcal{M}/\mathcal{O}\big)\big) \\ &\xrightarrow{\cong} \operatorname{Hom}_{\mathcal{O}}\big(C_*\widetilde{M}\otimes_{\mathbb{Z}\pi}\mathcal{O}^k, \mathcal{M}/\mathcal{O}\big) \end{aligned}$$

We obtain the following isomorphisms, where we use for the third isomorphism that $\mathcal{O}$ is a domain and $\mathcal{M}/\mathcal{O}$ is divisible, and hence $\operatorname{Hom}_{\mathcal{O}}(-,\mathcal{M}/\mathcal{O})$ is an exact functor by Baer's criterion:

$$\begin{aligned} H^{l-1}\big(M;(\mathcal{M}/\mathcal{O})^k\big) &\xrightarrow{\cong} H_{l-1}\big(\operatorname{Hom}_{\mathbb{Z}\pi}\big(\overline{C_*\widetilde{M}}, (\mathcal{M}/\mathcal{O})^k\big)\big) \\ &\xrightarrow{\cong} H_{l-1}\big(\operatorname{Hom}_{\mathcal{O}}\big(C_*\widetilde{M}\otimes_{\mathbb{Z}\pi}\mathcal{O}^k, \mathcal{M}/\mathcal{O}\big)\big) \\ &\xrightarrow{\cong} \operatorname{Hom}_{\mathcal{O}}\big(H_{l-1}(C_*\widetilde{M}\otimes_{\mathbb{Z}\pi}\mathcal{O}^k), \mathcal{M}/\mathcal{O}\big) \\ &\xrightarrow{\cong} \operatorname{Hom}_{\mathcal{O}}\big(H_{l-1}\big(M;\mathcal{O}^k\big), \mathcal{M}/\mathcal{O}\big) \\ &\xrightarrow{\cong} \operatorname{Hom}_{\mathcal{O}}\big(H^l\big(M;\mathcal{O}^k\big), \mathcal{M}/\mathcal{O}\big). \end{aligned}$$

Tracing through the chain of isomorphisms, we obtain that the composite is given by

$$\begin{aligned} H^{l-1}\big(M;(\mathcal{M}/\mathcal{O})^k\big) &\xrightarrow{\cong} \operatorname{Hom}_{\mathcal{O}}\big(H^l(M;\mathcal{O}^k), \mathcal{M}/\mathcal{O}\big) \\ z &\mapsto \Big(y \mapsto \tilde{\nu}'_*(z)\cap\big(y\cap[M]\big)\Big), \end{aligned}$$

where $\tilde{\nu}'_*$ is the map on $H^{l-1}(M;-)$ induced by $\tilde{\nu}'$ and the cap product is understood to act via $\operatorname{Hom}_{\mathcal{O}}\big(\mathcal{O}^k,\mathcal{M}/\mathcal{O}\big)\otimes_{\mathcal{O}}\mathcal{O}^k \to \mathcal{M}/\mathcal{O}$ on the coefficients. Note that it follows from Lemma 2.5 that for $z\in H^{l-1}\big(M;(\mathcal{M}/\mathcal{O})^k\big)$ and $y\in H^l\big(M;\mathcal{O}^k\big)$ we have

$$\tilde{\nu}'_*(z)\cap\big(y\cap[M]\big) = \nu'_*(y\cup z)\cap[M].$$

Since $\mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right)$ is a direct summand in $H^l\left(M;\mathcal{O}^k\right)$, we get a surjective map

$$\Phi\colon H^{l-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right) \to \mathrm{Hom}_{\mathcal{O}}\left(\mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right), \mathcal{M}/\mathcal{O}\right). \tag{6.3}$$

Recall that we denote by $\beta\colon H^{l-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right) \to H^l\left(M;\mathcal{O}^k\right)$ the Bockstein homomorphism. We need to show that $\ker(\Phi) \subseteq \ker(\beta)$, since then the map of (6.2) is identically zero only if $0 = \beta(z) = \beta(\beta^{-1}(x)) = x$. In fact, the other inclusion holds by Lemma 6.5.

We obtain from (6.3) that

$$H^{l-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right)/\ker(\Phi) \cong \mathrm{Hom}_{\mathcal{O}}\left(\mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right), \mathcal{M}/\mathcal{O}\right).$$

Since $\mathrm{Torsion}_{\mathcal{O}}(H^l(M;\mathcal{O}^k))$ is an $\mathcal{O}$-torsion module and every ideal of $\mathcal{O}$ is generated by z^i for $i \in \mathbb{N}$, we conclude that $\mathrm{Torsion}_{\mathcal{O}}(H^l(M;\mathcal{O}^k))$ is isomorphic to a finite direct sum of cyclic modules of the form $\mathcal{O}/z^i\mathcal{O}$. Now it is easy to check that

$$\mathrm{Hom}_{\mathcal{O}}\left(\mathcal{O}/z^i\mathcal{O}, \mathcal{M}/\mathcal{O}\right) \cong \mathcal{O}/z^i\mathcal{O},$$

and hence

$$\mathrm{Hom}_{\mathcal{O}}\left(\mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right), \mathcal{M}/\mathcal{O}\right) \cong \mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right).$$

But at the same time

$$H^{l-1}\left(M;\left(\mathcal{M}/\mathcal{O}\right)^k\right)/\ker(\beta) \cong \mathrm{im}(\beta) = \mathrm{Torsion}_{\mathcal{O}}(H^l(M;\mathcal{O}^k)),$$

which implies that

$$H^{l-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right)/\ker(\beta) \cong H^{l-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right)/\ker(\Phi).$$

Since $\ker(\beta) \subseteq \ker(\Phi)$ and $H^{l-1}\left(M;(\mathcal{M}/\mathcal{O})^k\right)$ is a finitely generated $\mathcal{O}$-module, it follows from Corollary 6.9 that $\ker(\beta) = \ker(\Phi)$. $\square$

We now summarize the previous results in the following corollary. Recall that any power series in a variable z that has a positive radius of convergence defines an element of $\mathcal{O}$, and thus acts on any $\mathcal{O}$-module via scalar multiplication.

Corollary 6.11. *Let*

$$\begin{aligned}[][-,-]\colon \mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right) \times \mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right) &\to \mathbb{C} \\ (x,y) &\mapsto \mathrm{Res}_0\{x,y\}\end{aligned}$$

where $\mathrm{Res}_0\{x,y\}$ denotes the residue of the meromorphic function $\{x,y\}$ at zero. Then the map $[-,-]$ has the following properties:

1. *$[-,-]$ is $(-1)^l$-Hermitian, i.e.,*

$$\begin{aligned}[][x,y] &= -\overline{[y,x]} \qquad \text{if } l \text{ is odd}, \\ [x,y] &= \overline{[y,x]} \qquad \text{if } l \text{ is even},\end{aligned}$$

2. *$[-,-]$ is non-degenerate,*

3. $[zx, y] = [x, zy]$*, where* $z \in \mathcal{O}$ *is the power series with coefficients* $a_1 = 1, a_n = 0$ *for* $n \neq 1$*.*

Proof. First note that the map $[-,-]$ is well-defined since Res_0 does not depend on the particular representative of an element of $\mathcal{M}/\mathcal{O}$.

Clearly, for all meromorphic functions we have

$$\mathrm{Res}_0(f^*) = \overline{\mathrm{Res}_0(f)}.$$

We showed in Corollary 6.7 that $\{-,-\}$ is $(-1)^l$-Hermitian, and so is $[-,-] = \mathrm{Res}\{-,-\}$.

We now show that $[-,-]$ is non-degenerate. Let $y \in \mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right)$, $y \neq 0$. We have seen in Lemma 6.10 that $\{-,-\}$ is non-degenerate. Hence, there is $x \in \mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right)$ with $\{x, y\} \neq 0$. Since $\{x, y\}$ is a non-trivial meromorphic function, there exists $m \in \mathbb{Z}$ such that

$$\mathrm{Res}_0\left(z^m\{x,y\}\right) = \mathrm{Res}_0\left(\{z^m x, y\}\right) \neq 0.$$

Hence, $[z^m x, y] = \mathrm{Res}_0\left(\{z^m x, y\}\right) \neq 0$.

It follows from the fact that $\{-,-\}$ is $\mathcal{O}$-linear in the first variable and $\mathcal{O}$-anti-linear with respect to the involution $f \mapsto f^*$ in the second variable that $\{zx, y\} = \{x, zy\}$ and hence $[zx, y] = [x, zy]$. □

Let

$$T_i = \left\{x \in \mathrm{Torsion}_{\mathcal{O}}\left(H^l\left(M;\mathcal{O}^k\right)\right) \mid z^i x = 0\right\}.$$

We now consider the $(-1)^l$-Hermitian form

$$\begin{aligned} h_i \colon T_i \times T_i &\to \mathbb{C} \\ (x,y) &\mapsto [z^{i-1}x, y]. \end{aligned}$$

We have for all i that $T_{i-1} \subset T_i$ and $zT_{i+1} \subset T_i$. Let $x = zv \in zT_{i+1}$ and $y \in T_i$, then

$$h_i(x,y) = \left[z^i v, y\right] = \left[v, z^i y\right] = 0.$$

Hence, h_i descends to a $(-1)^l$-Hermitian form l_i on $V_i := T_i/zT_{i+1}$ given by

$$\begin{aligned} l_i \colon V_i \times V_i &\to \mathbb{C} \\ (x,y) &\mapsto \left[z^{i-1}x, y\right]. \end{aligned} \tag{6.4}$$

For all i the canonical map $V_{i-1} \hookrightarrow V_i$ is injective. Note that $l_i(x,y) = 0$ if $x \in V_{i-1}$ or $y \in V_{i-1}$.

Since M is a compact manifold and $\mathcal{O}$ is a principal ideal domain, the $\mathcal{O}$-module $H^l(M;\mathcal{O})$ is finitely generated. Furthermore, all cyclic $\mathcal{O}$-modules are of the form $\mathcal{O}/z^i\mathcal{O}$ and hence finite-dimensional $\mathbb{C}$-vector spaces. It follows that $\mathrm{Torsion}_{\mathcal{O}}\left(H^l(M;\mathcal{O})\right)$ is also a finite-dimensional $\mathbb{C}$-vector space since it is a finite direct sum of cyclic modules. Hence, we obtain that for all i the submodule T_i is a finite-dimensional $\mathbb{C}$-vector space, and thus V_i is as well. All in all, we have concluded that the l_i define Hermitian or skew-Hermitian forms on finite-dimensional $\mathbb{C}$-vector spaces.

Denote by $\sigma_{M,\alpha_t}(l_i)$ the signature of l_i as defined in (6.4). Since $\mathrm{Torsion}_{\mathcal{O}}\left(H^l(M;\mathcal{O})\right)$ is finite-dimensional, it follows that there exists $K \in \mathbb{N}$ such that for all $k > K$ we have $T_K = T_k$, and hence $V_K = V_k$. Since $l_i(x,y) = 0$ for $x \in V_{i-1}$, it follows that $\sigma_{M,\alpha_t}(l_k) = 0$ for $k > K$.

The next theorem was proved by Farber and Levine in [FL96, Theorem 1.5] and we will use it in the following to relate the jumps of the ρ-invariant to the zeros of the Alexander polynomial. Note that Farber and Levine formulated the theorem by using the η-invariant instead of the ρ-invariant, but for fixed M their difference

$$\eta(M,\alpha\colon \pi_1(M) \to U(k)) - \rho(M,\alpha\colon \pi_1(M) \to U(k)) = k\eta(M,\tau_1)$$

is a real number just depending on the rank of the unitary representation, where τ_1 denotes the trivial representation of rank 1.

Theorem 6.12 ([FL96, Theorem 1.5])**.** *Let M be a closed connected oriented $(2l-1)$-dimensional manifold. Let $\{\alpha_t\}_{t\in(-\epsilon,\epsilon)}$ be an analytic deformation of $\alpha_0\colon \pi_1(M) \to U(k)$ coming from an analytic deformation of the corresponding connection (see Section 3.1). Let $\sigma_{(M,\alpha_t)}(l_i)$ be the signature of the linking form l_i defined in* (6.4) *and let*

$$\rho_+ = \lim_{t\searrow 0} \rho(M,\alpha_t)$$
$$\rho_- = \lim_{t\nearrow 0} \rho(M,\alpha_t).$$

Then

$$\rho_+ = \rho(M,\alpha_0) + \sum_{i=1}^{\infty} \sigma_{(M,\alpha_t)}(l_i)$$
$$\rho_- = \rho(M,\alpha_0) + \sum_{i=1}^{\infty} (-1)^i \sigma_{(M,\alpha_t)}(l_i).$$

As remarked above, the signatures $\sigma_{(M,\alpha_t)}(l_i)$ are zero for i large enough, hence the sums above are actually finite.

We will consider two examples which combined will give us a bound on the signature of l_i.

Example 6.13. We consider the $\mathcal{O}$-torsion module $T = \mathcal{O}/z^n\mathcal{O}$, i.e.,

$$T = \left\{ \left[\sum_{k=0}^{n-1} a_k z^k \right] \,\middle|\, a_k \in \mathbb{C} \right\}.$$

Then

$$T_i = \left\{ x \in T \mid z^i x = 0 \right\} = \begin{cases} \left\{ \left[\sum_{k=n-i}^{n-1} a_k z^k \right] \,\middle|\, a_k \in \mathbb{C} \right\} & i < n \\ T & i \geq n \end{cases}.$$

We get for $i < n$

$$zT_{i+1} = z\left\{ \left[\sum_{k=n-i-1}^{n-1} a_k z^k \right] \,\middle|\, a_k \in \mathbb{C} \right\} = \left\{ \left[\sum_{k=n-i}^{n-1} a_{k-1} z^k \right] \,\middle|\, a_k \in \mathbb{C} \right\} = T_i$$

and for $i \geq n$

$$zT_{i+1} = zT = \left\{ \left[\sum_{k=1}^{n-1} a_k z^k \right] \middle| \, a_k \in \mathbb{C} \right\}.$$

Hence, we get

$$V_i = T_i / zT_{i+1} \cong \begin{cases} 0 & i < n \\ \mathbb{C} & i \geq n \end{cases}.$$

Assume that T is the $\mathcal{O}$-torsion part of the homology group $H^l\big(M;\mathcal{O}^k\big)$ of a $(2l-1)$-dimensional manifold M. Let l_i be the $(-1)^l$-Hermitian form on V_i as defined in (6.4). Let σ_i be the signature of l_i.

If $i < n$, then $l_i = 0$ since $V_i = 0$. If $i > n$, then $V_i = V_{i-1}$ and by definition of l_i we have that $l_i(x,y) = 0$ if $x \in V_{i-1}$, and hence $l_i = 0$ on V_i.

The only signature which might be different from zero is σ_n. In this case $l_n\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and we obtain $\sigma_n \in \{1, 0, -1\}$.

Example 6.14. Let $T = \bigoplus_{i \in I} \mathcal{O}/z^{n_i}\mathcal{O}$, where $|I| < \infty$ and $n_i \in \mathbb{N}$. Assume that T is the $\mathcal{O}$-torsion part of the homology group $H^l\big(M;\mathcal{O}^k\big)$ of a $(2l-1)$-dimensional manifold M. Let l_i be the $(-1)^l$-Hermitian form on V_i as defined in (6.4). Then we obtain

$$V_j \cong \mathbb{C}^{\#\{n_i | n_i \geq j\}}.$$

It follows from Example 6.13 that

$$\left|\sigma(l_j)\right| \leq \# \left\{ n_i | i \in I, n_i = j \right\}.$$

Moreover we will use the following theorem due to Farber and Levine [FL96, Theorem 7.6].

Theorem 6.15. *Let M be a closed connected oriented $(4n-1)$-dimensional manifold. Then*

$$\begin{aligned} \rho(M)_{\mathbb{R}/\mathbb{Z}}\colon R_k(\pi_1(M)) &\to \mathbb{R}/\mathbb{Z} \\ \alpha &\mapsto [\rho(M,\alpha)] \end{aligned}$$

is locally constant.

Put differently, $\rho(M)_{\mathbb{R}/\mathbb{Z}}$ is constant on the connected components of $R_k(\pi_1(M))$.

We obtain the following corollary from Theorem 6.2 and Theorem 6.15.

Corollary 6.16. *Let M be a closed connected oriented $(4n-1)$-dimensional manifold. Then*

$$\begin{aligned} \rho(M)\colon R_k(\pi_1(M)) &\to \mathbb{R} \\ \alpha &\mapsto \rho(M,\alpha) \end{aligned}$$

is constant on the connected components of

$$\left\{ \alpha \in R_k(\pi_1(M)) \middle| \sum_{i=0}^{\infty} \dim_{\mathbb{C}} H_i^{\alpha}(M;\mathbb{C}^k) = r \right\}.$$

Proof. Let

$$\Sigma_r = \left\{ \alpha \in R_k(\pi_1(M)) \,\middle|\, \sum_{i=0}^{\infty} \dim_{\mathbb{C}} H_i^{\alpha}(M;\mathbb{C}^k) \geq r \right\}.$$

Theorem 6.2 says that

$$\begin{aligned} \rho(M)\colon R_k(\pi_1(M)) &\to \mathbb{R} \\ \alpha &\mapsto \rho(M,\alpha) \end{aligned}$$

is continuous on $\Sigma_r \backslash \Sigma_{r+1}$. Hence, by Theorem 6.15 it is even constant on the connected components of

$$\Sigma_r \backslash \Sigma_{r+1} = \left\{ \alpha \in R_k(\pi_1(M)) \,\middle|\, \sum_{i=0}^{\infty} \dim_{\mathbb{C}} H_i^{\alpha}(M;\mathbb{C}^k) = r \right\}. \qquad \square$$

Chapter 7

Height of the jumps of the ρ-invariant for a circular deformation

Let M be a closed connected oriented 3-dimensional manifold with fundamental group π. In this chapter, we consider a circle lying in the variety of all k-dimensional unitary representations of π. We relate the height of the jumps of the ρ-invariant which appear by going around the circle to the zeros of the Alexander polynomial of M.

If $\alpha\colon G \to U(k)$ and $\hat{\phi}\colon G \to U(1) = S^1 \subset \mathbb{C}^*$ are unitary representations, we can consider the unitary representation $\alpha \cdot \hat{\phi}$ given by $(\alpha \cdot \hat{\phi})(g) = \alpha(g)\hat{\phi}(g)$, which is well-defined since $U(1)$ viewed as a subgroup of $\mathbb{C}^*$ lies in (in fact, is) the center of $U(k)$. In this way, the variety $R_1(G)$ becomes a group and acts on $R_k(G)$ for any $k \in \mathbb{N}$.

Definition 7.1. Let G be a finitely generated group and $\phi = (\phi_1, \dots, \phi_l)\colon G \to \mathbb{Z}^l$ a homomorphism. Fix a representation $\alpha \in R_k(G)$. The *torus T_α^ϕ in the variety of unitary representations* $R_k(G)$ is the image of the map

$$\begin{aligned} S^1 \times \ldots \times S^1 &\to R_k(G) \\ (z_1, \ldots, z_l) &\mapsto \big(g \mapsto \alpha(g) z_1^{\phi_1(g)} \cdot \ldots \cdot z_l^{\phi_l(g)}\big). \end{aligned}$$

If $l = 1$ and ϕ_1 is non-trivial, the torus is called a *circle*.

Let $\widetilde{M}$ be the universal covering of M. Recall that we consider the chain groups $C_*(\widetilde{M})$ as right $\mathbb{Z}\pi$-modules. Let α be a k-dimensional unitary representation of the fundamental group. Let $\phi\colon \pi_1(M) \to \mathbb{Z}$, $\phi \neq 0$. We consider the analytic deformation of α which is given by

$$\begin{aligned} \alpha_t^\phi\colon \pi_1(M) &\to U(k) \\ g &\mapsto \alpha(g) e^{it\phi(g)}, \end{aligned}$$

where t varies in $(-\epsilon, \epsilon)$. We want to relate the jumps of the map $t \mapsto \rho(M, \alpha_t^\phi)$ to the zeros of the reduced Alexander polynomial of M.

Recall that π acts on $\mathcal{O}^k$ via

$$\begin{aligned} \pi \times \mathcal{O}^k &\to \mathcal{O}^k \\ (g, f) &\mapsto \left(t \mapsto \alpha_t^\phi(g) f(t)\right). \end{aligned}$$

We consider the left action of $\mathbb{Z}\pi$ on $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\left[y^{\pm 1}\right]$ which is induced by

$$\begin{aligned} \alpha \otimes \phi \colon \pi &\to \operatorname{Aut}(\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\left[y^{\pm 1}\right]) \\ g &\mapsto \left(v \otimes f \mapsto (\alpha(g) v \otimes y^{\phi(g)} f(y))\right). \end{aligned}$$

We consider $\mathbb{C}\left[y^{\pm 1}\right]$ as a subring of $\mathcal{O}$ by identifying a finite power series $\sum a_j y^j$ with the holomorphic function $\sum a_j (e^{it})^j$ of t. Thus $\mathcal{O}$ becomes a left $\mathbb{C}\left[y^{\pm 1}\right]$-module via left multiplication. Note that

$$C_*\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \mathcal{O}^k \cong \left(C_*\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \left(\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\left[y^{\pm 1}\right]\right)\right) \otimes_{\mathbb{C}[y^{\pm 1}]} \mathcal{O}$$

and

$$H_1^{\{\alpha_t^\phi\}_t}\big(M; \mathcal{O}^k\big) = H_1^{\{\alpha_t^\phi\}_t}\big(C_*\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \mathcal{O}^k\big) \cong H_1^{\{\alpha_t^\phi\}_t}\left(C_*\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \left(\mathbb{C}^k \otimes \mathbb{C}\left[y^{\pm 1}\right]\right) \otimes_{\mathbb{C}[y^{\pm 1}]} \mathcal{O}\right).$$

Since $\mathbb{C}\left[y^{\pm 1}\right]$ is a principal ideal domain, it follows from the universal coefficient theorem that we have a short exact sequence

$$\begin{aligned} 0 \longrightarrow H_1^{\alpha \otimes \phi}\left(M; \mathbb{C}^k \otimes \mathbb{C}\left[y^{\pm 1}\right]\right) \otimes_{\mathbb{C}[y^{\pm 1}]} \mathcal{O} \longrightarrow H_1^{\{\alpha_t^\phi\}_t}\left(C_*\big(\widetilde{M}\big) \otimes_{\mathbb{Z}\pi} \left(\mathbb{C}^k \otimes \mathbb{C}\left[y^{\pm 1}\right]\right) \otimes_{\mathbb{C}[y^{\pm 1}]} \mathcal{O}\right) \\ \longrightarrow \operatorname{Tor}_{\mathbb{C}[y^{\pm 1}]}\left(H_0^{\alpha \otimes \phi}\left(M; \mathbb{C}^k \otimes \mathbb{C}\left[y^{\pm 1}\right]\right), \mathcal{O}\right) \longrightarrow 0. \end{aligned}$$

Clearly, $\mathcal{O}$ is a torsion-free $\mathbb{C}\left[y^{\pm 1}\right]$-module. It follows that

$$\operatorname{Tor}_{\mathbb{C}[y^{\pm 1}]}\left(H_0^{\alpha \otimes \phi}\left(M; \left(\mathbb{C}^k \otimes \mathbb{C}\left[y^{\pm 1}\right]\right)\right), \mathcal{O}\right) = 0$$

and therefore

$$H_1^{\{\alpha_t^\phi\}_t}\big(M; \mathcal{O}^k\big) \cong H_1^{\alpha \otimes \phi}\big(M; \mathbb{C}^k \otimes \mathbb{C}\left[y^{\pm 1}\right]\big) \otimes_{\mathbb{C}[y^{\pm 1}]} \mathcal{O}.$$

Recall that we denote by $\widetilde{\Delta}^{\alpha \otimes \phi}$ the reduced Alexander polynomial associated to (M, α, ϕ), i.e., $\widetilde{\Delta}^{\alpha \otimes \phi}$ is the order of the module $\operatorname{Torsion}_{\mathbb{C}[y^{\pm 1}]}\left(H_1^{\alpha \otimes \phi}\big(M; \mathbb{C}^k \otimes \mathbb{C}[y^{\pm 1}]\big)\right)$.

Theorem 7.2. *Let M be a closed connected oriented 3-dimensional manifold. Furthermore, let $\phi \colon \pi_1(M) \to \mathbb{Z}$ and $\alpha \colon \pi_1(M) \to U(k)$. Given $t \in \mathbb{R}$ we define the group homomorphism*

$$\begin{aligned} \alpha_t^\phi \colon \pi_1(M) &\to U(k) \\ g &\mapsto \alpha(g) e^{it\phi(g)}. \end{aligned}$$

Let ρ_t be the ρ-invariant corresponding to α_t^ϕ. Then $\lim_{t\searrow s}\rho_t$ exists for all $s\in\mathbb{R}$. If

$$\rho_s \neq \lim_{t\searrow s}\rho_t,$$

then the reduced Alexander polynomial $\widetilde{\Delta}^{\alpha\otimes\phi}$ of M has a zero at e^{is}. If $N(e^{is})$ denotes the multiplicity of this zero, then

$$\left|\rho_s - \lim_{t\searrow s}\rho_t\right| \leq N(e^{is}).$$

Furthermore, we have

$$\max\left\{|\rho_s-\rho_t| \mid s,t\in[0,2\pi]\right\} \leq \deg\left(\widetilde{\Delta}^{\alpha\otimes\phi}\right).$$

Proof. Note that $t\mapsto\alpha_t^\phi$ is an analytic deformation of α_0. For our particular choice of $\{\alpha_t^\phi\}_t$, it follows from Corollary 3.8 that $t\mapsto\alpha_t^\phi$ gives rise to an analytic deformation of the corresponding connection. Hence, we can apply Theorem 6.12 to estimate the height of the jumps.

Further note that $H_1^{\alpha\otimes\phi}\big(M;\mathbb{C}^k\otimes\mathbb{C}\big[y^{\pm1}\big]\big)$ is a finitely generated module over $\mathbb{C}\big[y^{\pm1}\big]$. Since $\mathbb{C}\big[y^{\pm1}\big]$ is a principal ideal domain, there exist $p_j\in\mathbb{C}\big[y^{\pm1}\big]$ such that

$$H_1^{\alpha\otimes\phi}\big(M;\mathbb{C}^k\otimes\mathbb{C}\big[y^{\pm1}\big]\big)\cong\mathbb{C}\big[y^{\pm1}\big]^\mu\oplus\bigoplus_{j=1}^{\nu}\mathbb{C}\big[y^{\pm1}\big]/p_j(y)\mathbb{C}\big[y^{\pm1}\big]$$

and the p_j are powers of suitable non-zero irreducible elements in $\mathbb{C}\big[y^{\pm1}\big]$. Hence, each p_j can be taken to be of the form $p_j(y)=(y-a_j)^{m_j}$ for some $m_j\in\mathbb{N}$ and $a_j\in\mathbb{C}\backslash\{0\}$. The reduced Alexander polynomial of M is given by

$$\widetilde{\Delta}^{\alpha\otimes\phi}(y)=\prod_{j=1}^{\nu}p_j(y).$$

Clearly,

$$H_1^{\alpha\otimes\phi}\big(M;\mathbb{C}^k\otimes\mathbb{C}[y^{\pm1}]\big)\otimes_{\mathbb{C}[y^{\pm1}]}\mathcal{O}\cong\mathcal{O}^\mu\oplus\bigoplus_{j=1}^{\nu}\mathcal{O}/p_j(e^{it})\mathcal{O}.$$

If $p_j(1)\neq0$, then $z\mapsto\frac{1}{p_j(e^{iz})}$ is a holomorphic function around zero and hence p_j is a unit in $\mathcal{O}$, which implies that $\mathcal{O}/p_j(e^{it})\mathcal{O}=0$. If $p_j(1)=0$, then $a_j=1$ and 1 is a zero of multiplicity m_j, and hence the map $z\mapsto p_j(e^{iz})$ has a zero at $z=0$ of multiplicity m_j. Thus, $\mathcal{O}/p_j(e^{it})\mathcal{O}$ is isomorphic to $\mathcal{O}/t^{m_j}\mathcal{O}$. We therefore get

$$\mathrm{Torsion}_{\mathcal{O}}\left(H_1^{\{\alpha_t^\phi\}_t}\big(M;\mathcal{O}^k\big)\right)\cong\bigoplus_{\substack{j=1,\dots,\nu\\ a_j=1}}\mathcal{O}/p_j(e^{it})\mathcal{O}=\bigoplus_{\substack{j=1,\dots,\nu\\ a_j=1}}\mathcal{O}/t^{m_j}\mathcal{O}.$$

Let

$$T_i=\left\{x\in H_1^{\{\alpha_t^\phi\}_t}\big(M;\mathcal{O}^k\big)\;\middle|\;t^ix=0\right\}.$$

Let $V_i = T_i/tT_{i+1}$. Let $l_i\colon T_i/tT_{i+1} \times T_i/tT_{i+1} \to \mathbb{C}$ be the Hermitian form defined in (6.4) using the notation introduced previously in Section 6.3. Note that there we considered a general $(2l-1)$-dimensional manifold with the linking forms being defined on subquotients of the l-th cohomology. In our current situation, l takes the value 2 and by Poincaré duality $H_1(M;\mathcal{O}^k)$ is identified with $H^2(M;\mathcal{O}^k)$.

Let $\sigma(l_i)$ be the signature of l_i. It follows from Example 6.14 that

$$|\sigma(l_i)| \leq \left\{\#p_j \mid m_j = i, a_j = 1\right\}.$$

It follows from Theorem 6.12 that

$$\left|\rho_0 - \lim_{t\searrow s}\rho_t\right| \leq \sum_{i=1}^{\infty}\left|\sigma(l_i)\right| \leq \sum_{i=1}^{\infty}\#\{p_j|m_j = i, a_j = 1\} = N(1)$$

and

$$\left|\rho_0 - \lim_{t\nearrow s}\rho_t\right| \leq \sum_{i=1}^{\infty}\left|\sigma(l_i)\right| \leq N(1).$$

Considering instead of $t = 0$ the jumps of the ρ-invariant at $t = s$, we obtain by a similar calculation

$$\left|\rho_s - \lim_{t\searrow s}\rho_t\right| \leq N(e^{is})$$

and

$$\left|\rho_s - \lim_{t\nearrow s}\rho_t\right| \leq N(e^{is}).$$

Hence, we obtain for all $s \in [0, 2\pi]$

$$\left|\lim_{t\searrow s}\rho_t - \lim_{t\nearrow s}\rho_t\right| \leq 2N(e^{is}).$$

In particular, we obtain that

$$\begin{aligned}[0, 2\pi] &\to \mathbb{R}\\ t &\mapsto \rho_t\end{aligned}$$

is continuous on $[0, 2\pi]\backslash\{t \in [0, 2\pi] \mid \widetilde{\Delta}^{\alpha\otimes\phi}(e^{it}) = 0\}$. Theorem 6.15 says that in case of a 3-dimensional manifold the map

$$\begin{aligned}\rho(M)_{\mathbb{R}/\mathbb{Z}}\colon R_k(\pi_1(M)) &\to \mathbb{R}/\mathbb{Z}\\ \alpha &\mapsto [\rho(M, \alpha)]\end{aligned}$$

is locally constant. Hence, the map

$$\begin{aligned}[0, 2\pi] &\to \mathbb{R}\\ t &\mapsto \rho_t\end{aligned}$$

is constant on the connected components of $[0,2\pi]\backslash\{t \in [0,2\pi] \mid \widetilde{\Delta}^{\alpha\otimes\phi}(e^{it}) = 0\}$.

Let $e^{i\lambda_1},\dots,e^{i\lambda_r}$ be the zeros of $\widetilde{\Delta}^{\alpha\otimes\phi}$ on S^1 with multiplicity N_i. Let $t,s \in [0,2\pi)$ with $t < s$. To simplify the notation assume that e^{it} and e^{is} are both not zeros of the Alexander polynomial $\widetilde{\Delta}^{\alpha\otimes\phi}$. Then

$$\left|\rho_s - \rho_t\right| \leq \sum_{\substack{i=1,\dots,r\\ \lambda_i\in(t,s)}} 2N_i \tag{7.1}$$

as well as

$$\left|\rho_s - \rho_t\right| \leq \sum_{\substack{i=1,\dots,r\\ \lambda_i\in[0,t)\cup(s,2\pi)}} 2N_i. \tag{7.2}$$

Hence, we obtain

$$2\left|\rho_s - \rho_t\right| \leq \sum_{\substack{i=1,\dots,r\\ \lambda_i\in(t,s)}} 2N_i + \sum_{\substack{i=1,\dots,r\\ \lambda\in[0,t)\cup(s,2\pi)}} 2N_i \leq 2\deg\left(\widetilde{\Delta}^{\alpha\otimes\phi}\right). \tag{7.3}$$

The same inequality holds in the case that one or both of e^{is} and e^{it} are zeros of the Alexander polynomial. Assuming that e^{is} is a zero of the Alexander polynomial, the only difference is that the respective summand N_i appears in both sums on the right-hand side in (7.1) and (7.2), and thus still contributes a total of $2N_i$. The same holds true for e^{it}.

Since (7.3) holds for all $s,t \in [0,2\pi)$, we conclude

$$\max\left\{\left|\rho_s - \rho_t\right| \middle| s,t \in [0,2\pi)\right\} \leq \deg\left(\widetilde{\Delta}^{\alpha\otimes\phi}\right). \qquad \square$$

As already mentioned in Section 2.9, the Thurston norm gives an upper bound on the degree of a sum of suitable Alexander polynomials. We obtain the following corollary.

Corollary 7.3. *Let M be a closed connected oriented 3-dimensional manifold. Furthermore let $\alpha\colon \pi_1(M) \to U(k)$ and $\phi\colon \pi_1(M) \to \mathbb{Z}$ be group homomorphisms. We consider the deformation α_t of α given by*

$$\begin{aligned} \alpha_t^{\phi}\colon \pi_1(M) &\to U(k) \\ g &\mapsto \alpha(g)e^{it\phi(g)}. \end{aligned}$$

Let ρ_t be the ρ-invariant corresponding to α_t^{ϕ}. If $\Delta_1^{\alpha\otimes\phi} \neq 0$, then

$$\max\{\left|\rho_s - \rho_t\right| \big| s,t \in [0,2\pi]\} \leq k\left\|\phi\right\|_T + 2\deg(\Delta_0^{\alpha\otimes\phi}).$$

If furthermore α is irreducible and α restricted to $\ker(\phi)$ *is non-trivial, we obtain*

$$\max\{\left|\rho_s - \rho_t\right| \big| s,t \in [0,2\pi]\} \leq k\left\|\phi\right\|_T.$$

Proof. Since the Alexander polynomial $\Delta_1^{\alpha\otimes\phi}$ is not zero, the module $H_1^{\alpha\otimes\phi}(M;\mathbb{C}^k \otimes \mathbb{C}\left[y^{\pm1}\right])$ is $\mathbb{C}\left[y^{\pm1}\right]$-torsion and hence $\Delta_1^{\alpha\otimes\phi} = \widetilde{\Delta}_1^{\alpha\otimes\phi}$.

In the case that $\Delta_1^{\alpha\otimes\phi} \neq 0$ it was shown by Friedl and Kim in [FK06, Theorem 1.1] that

$$\|\phi\|_T \geq \frac{1}{k}\left(\deg\left(\Delta_1^{\alpha\otimes\phi}\right) - \deg\left(\Delta_0^{\alpha\otimes\phi}\right) - \deg\left(\Delta_2^{\alpha\otimes\phi}\right)\right).$$

Since M is a closed manifold and α is unitary, we obtain from Proposition 2.12 that

$$\deg\left(\Delta_0^{\alpha\otimes\phi}\right) = \deg\left(\Delta_2^{\alpha\otimes\phi}\right).$$

Hence, we obtain

$$\max\left\{|\rho_s - \rho_t| \big| s,t \in [0,2\pi]\right\} \leq \deg\left(\Delta_1^{\alpha\otimes\phi}\right) \leq k\,\|\phi\|_T + 2\deg\left(\Delta_0^{\alpha\otimes\phi}\right).$$

Furthermore, if α is irreducible and α restricted to $\ker(\phi)$ is non-trivial, we obtain from Proposition 2.11 that

$$\deg\left(\Delta_0^{\alpha\otimes\phi}\right) = 0. \qquad \square$$

We now consider the ρ-invariants of a manifold which is obtained by 0-framed surgery on a knot as already done in Example 4.2.

Example 7.4. Let K be the left-handed trefoil knot. Let M_K be obtained by 0-framed surgery on K. Let

$$\alpha_t \colon \pi_1(M_K) \to H_1(M_K) \to U(1)$$

be the unitary representation which sends a generator of $H_1(M_K)$ to e^{it}. The Alexander polynomial is given by $\Delta(t) = t - 1 + t^{-1}$ and it has roots at $e^{\pm i\frac{2\pi}{6}}$, both of multiplicity one. Then we obtain from Theorem 7.2 that

$$\begin{aligned} \rho(M_K,\alpha_t) &= 0 \ \text{ for } \ |t| < \frac{\pi}{3} \\ |\rho(M_K,\alpha_t)| &\leq 1 \ \text{ for } \ |t| = \frac{\pi}{3} \\ |\rho(M_K,\alpha_t)| &\leq 2 \ \text{ else.} \end{aligned}$$

In the case of manifolds obtained by 0-framed surgery on knots the ρ-invariant can be calculated by using Levine–Tristam signatures (see Section 4.3) and are given by:

$$\rho(M_K,\alpha_t) = \begin{cases} 0 & |t| < \frac{\pi}{3} \\ 1 & |t| = \frac{\pi}{3} \\ 2 & \text{else.} \end{cases}$$

Hence in this case the bounds provided by Theorem 7.2 are sharp.

Chapter 8

Deforming along higher-dimensional tori in the variety of representations

In this chapter we consider closed connected oriented 3-dimensional manifolds and a torus lying in the variety of k-dimensional unitary representations. We relate the discontinuities of the ρ-invariant appearing as the representation varies on the torus to the zero set of the corresponding Alexander polynomial.

8.1 Notation and computational tools

Let M be a closed connected oriented manifold with fundamental group π. Furthermore let $\alpha\colon \pi \to U(k)$ be a unitary representation and $\phi = (\phi_1, \ldots, \phi_l)\colon \pi \to \mathbb{Z}^l$ a homomorphism. We denote by $T^l = (S^1)^l$ the l-dimensional torus. Let $z = (z_1, \ldots, z_l) \in T^l$.

We denote by α_z^ϕ the unitary representation given by

$$\begin{aligned} \alpha_z^\phi\colon \pi &\to U(k) \\ g &\mapsto \alpha(g) \prod_{j=1}^{l} z_j^{\phi_j(g)}. \end{aligned}$$

As $z \in T^l$ varies, the representation α_z^ϕ runs through the l-torus $T_\alpha^\phi \subset R_k(\pi)$.

For fixed α, ϕ and z the fundamental group π acts on $\mathbb{C}^k$ via

$$\begin{aligned} \pi \times \mathbb{C}^k &\to \mathbb{C}^k \\ (g, v) &\mapsto \alpha_z^\phi(g) v. \end{aligned}$$

We denote by $\alpha \otimes \phi$ the map

$$\begin{aligned} \alpha \otimes \phi\colon \pi &\to \operatorname{Aut}\left(\mathbb{C}^k \otimes \mathbb{C}[\mathbb{Z}^l]\right) \\ g &\mapsto \left(v \otimes x \mapsto \alpha(g) v \otimes \phi(g) x\right). \end{aligned}$$

In this chapter we want to relate the location of the jumps of the ρ-invariant to the zeros of the Alexander polynomial in two cases:

1. α is irreducible and α restricted to $\ker(\phi)$ is non-trivial,
2. α is trivial and ϕ is an epimorphism.

In the following we will compute the $\mathbb{C}$-dimension of the homology groups $H_i^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right)$ in the two cases mentioned above to apply Corollary 6.16. Before we will do so we state some results which we will use to compute the homology groups.

A less detailed version of the proof of the following lemma is also given in [FK06, Lemma 2.3].

Lemma 8.1. *Let M be a closed connected orientable n-dimensional manifold and R a principal ideal domain with involution. Let $\beta\colon \pi_1(M) \to \mathrm{GL}(k,R)$ be a representation, and denote by $\beta^\dagger\colon \pi_1(M) \to \mathrm{GL}(k,R)$ the unique representation determined by*

$$\left\langle \beta(g^{-1})v, w\right\rangle = \left\langle v, \beta^\dagger(g)w\right\rangle.$$

Then

$$H_{n-i}^{\beta}\left(M;R^k\right) \cong \mathrm{Hom}_R\left(H_i^{\beta^\dagger}\left(M;R^k\right),R\right) \oplus \mathrm{Ext}\left(H_{i-1}^{\beta^\dagger}\left(M;R^k\right),R\right)$$

as R-modules.

Proof. We consider $C_*\left(\widetilde{M}\right)$ as a right $\mathbb{Z}\pi$-module. We write R_β^k when we consider R^k with the left-action given by

$$\begin{aligned} \pi \times R^k &\to R^k \\ (g,v) &\mapsto \beta(g)v. \end{aligned}$$

and $R_{\beta^\dagger}^k$, when we consider R^k with the left-action given by

$$\begin{aligned} \pi \times R^k &\to R^k \\ (g,v) &\mapsto \beta^\dagger(g)v. \end{aligned}$$

We consider the map

$$\begin{aligned} \Phi\colon \mathrm{Hom}_{\mathbb{Z}\pi}\left(\overline{C_*(\widetilde{M})}, R_\beta^k\right) &\to \mathrm{Hom}_R\left(C_*(\widetilde{M}) \otimes_{\mathbb{Z}\pi} R_{\beta^\dagger}^k, R\right) \\ f &\mapsto \left((c\otimes v) \mapsto \langle f(c), v\rangle\right). \end{aligned}$$

The map Φ is well-defined since for $f \in \mathrm{Hom}_{\mathbb{Z}\pi}\left(\overline{C_*(\widetilde{M})}, R_\beta^k\right)$ and $g \in \pi$ we have

$$f(\sigma\cdot g) = f(g^{-1}\cdot\sigma) = g^{-1}\cdot f(\sigma) = \beta(g^{-1})f(\sigma)$$

and hence

$$\begin{aligned} \Phi(f)((\sigma\cdot g)\otimes v) &= \langle f(\sigma\cdot g), v\rangle = \left\langle \beta(g^{-1})f(\sigma), v\right\rangle \\ &= \left\langle f(\sigma), \beta^\dagger(g)v\right\rangle = \langle f(\sigma), g\cdot v\rangle = \Phi(f)(\sigma\otimes(g\cdot v)). \end{aligned}$$

We skip the proof that the map really defines an isomorphism. We obtain

$$\begin{aligned}
H_{n-i}\big(M;R^k_\beta\big) &\cong H^i\big(M;R^k_\beta\big)\\
&= H_i\Big(\mathrm{Hom}_{\mathbb{Z}\pi}\big(\overline{C_*(\widetilde{M})},R^k_\beta\big)\Big)\\
&\cong H_i\big(\mathrm{Hom}_R\big(C_*(M;R^k_{\beta^\dagger}),R\big)\big)\\
&\cong \mathrm{Hom}_R\big(H_i\big(M;R^k_{\beta^\dagger}\big),R\big)\oplus \mathrm{Ext}\big(H_{i-1}\big(M;R^k_{\beta^\dagger}\big),R\big).
\end{aligned}$$

The first isomorphism is Poincaré duality, the second the definition, the third is induced by the isomorphism above and for the last we used the Universal Coefficient Theorem. □

A proof of the following theorem is given in [Rot09, Theorem 10.90].

Theorem 8.2 (Künneth Homology Spectral Sequence)**.** *Let R be a ring, let $\mathbf{D}$ be a chain complex of flat right R-modules, and let $\mathbf{A}$ be a chain complex of left R-modules. Assume that both $\mathbf{D}$ and $\mathbf{A}$ are concentrated in non-negative degrees. Then there is a first quadrant spectral sequence*

$$E^2_{p,q} = \bigoplus_{s+t=q} \mathrm{Tor}^R_p\big(H_s(\mathbf{D}),H_t(\mathbf{A})\big) \Rightarrow H_{p+q}(\mathbf{D}\otimes_R\mathbf{A}).$$

Furthermore, we need the following theorem (see [Rot09, Theorem 10.31]).

Theorem 8.3. *Let M be a first quadrant bicomplex and*

$$\mathrm{Tot}(M)_m = \oplus_{p+q=m}M_{p,q}.$$

Let (E^r,d^r) be a first quadrant spectral sequence such that $E^2_{p,q}\Rightarrow H_n(\mathrm{Tot}(M))$. Then we have an exact sequence

$$H_2(\mathrm{Tot}(M)) \to E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \to H_1(\mathrm{Tot}(M)) \to E^2_{1,0}\to 0.$$

We can now calculate the needed homology groups in both cases.

8.2 Case 1: α irreducible and non-trivial restricted to $\ker(\phi)$

We first consider the case that α irreducible and α restricted to $\ker(\phi)$ is non-trivial. The proof of the next lemma is nearly the same as the proof of Proposition A.3 in [FKK12], but corrects a small mistake made right after applying the Universal Coefficient Theorem.

Lemma 8.4. *Let X be a topological space with universal covering and $\alpha\colon\pi_1(X)\to U(k)$ be an irreducible representation and $\phi\colon\pi_1(X)\to\mathbb{Z}^l$ such that α restricted to $\ker(\phi)$ is non-trivial. Then*

$$H_0^{\alpha\otimes\phi}\big(X;\mathbb{C}^k\otimes\mathbb{C}[\mathbb{Z}^l]\big)=0.$$

Proof. Let $\alpha\colon \pi_1(X) \to U(k)$ be a representation and $\phi\colon \pi_1(X) \to \mathbb{Z}^l$ such that α restricted to $\ker(\phi)$ is non-trivial. We show that if

$$H_0^{\alpha\otimes\phi}\big(X;\mathbb{C}^k \otimes \mathbb{C}\big[\mathbb{Z}^l\big]\big) \neq 0,$$

then α is a reducible representation, which proves the claim. We denote $\ker(\phi)$ by Γ. It follows from [HS97, Chapter VI.3] that for any connected topological space Z with $\pi_1(Z) \subseteq \pi_1(X)$ we have

$$H_0^{\alpha\otimes\phi}\big(Z;\mathbb{C}^k \otimes \mathbb{C}\big[\mathbb{Z}^l\big]\big) \cong \big(\mathbb{C}^k \otimes \mathbb{C}\big[\mathbb{Z}^l\big]\big)/\big\{(\alpha\otimes\phi)(g)v - v \mid v \in \mathbb{C}^k \otimes \mathbb{C}\big[\mathbb{Z}^l\big], g \in \pi_1(Z)\big\}.$$

Hence, by comparing the right-hand sides for $Z = B\Gamma$ and $Z = X$, we can read off that $H_0^{\alpha\otimes\phi}\big(X;\mathbb{C}^k \otimes \mathbb{C}\big[\mathbb{Z}^l\big]\big) \neq 0$ implies $H_0^{\alpha\otimes\phi}\big(\Gamma;\mathbb{C}^k \otimes \mathbb{C}\big[\mathbb{Z}^l\big]\big) \neq 0$. Thus, we have

$$\begin{aligned} 0 \neq H_0^{\alpha\otimes\phi}\big(\Gamma;\mathbb{C}^k \otimes \mathbb{C}\big[\mathbb{Z}^l\big]\big) &\cong \big(\mathbb{C}^k \otimes \mathbb{C}\big[\mathbb{Z}^l\big]\big)/\big\{(\alpha\otimes\phi)(g)v - v \mid v \in \mathbb{C}^k \otimes \mathbb{C}\big[\mathbb{Z}^l\big], g \in \Gamma\big\} \\ &\cong \mathbb{C}^k/\big\{\alpha(g)v - v \mid v \in \mathbb{C}^k, g \in \Gamma\big\} \otimes \mathbb{C}\big[\mathbb{Z}^l\big] = H_0^{\alpha}\big(\Gamma;\mathbb{C}^k\big) \otimes \mathbb{C}\big[\mathbb{Z}^l\big] \end{aligned}$$

and therefore $H_0^{\alpha}(\Gamma;\mathbb{C}^k) \neq 0$.

Let $E\Gamma$ be the universal covering of $B\Gamma$. Since α is a unitary representation, we have

$$\big\langle \alpha(g)v, w\big\rangle = \big\langle v, \alpha(g^{-1})w\big\rangle$$

for all $g \in \Gamma$ and all $v, w \in \mathbb{C}^k$. It follows as in the proof of Lemma 8.1 that

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}[\Gamma]}\left(\overline{C_*(E\Gamma)};\mathbb{C}^k_\alpha\right) &\xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}\left(C_*(E\Gamma)\otimes_{\mathbb{Z}[\Gamma]} \mathbb{C}^k_\alpha, \mathbb{C}\right) \\ f &\mapsto \Big((c\otimes w) \mapsto \big\langle f(c), w\big\rangle\Big) \end{aligned}$$

is an isomorphism. We obtain

$$H^i_\alpha\big(\Gamma;\mathbb{C}^k\big) = H_i\big(\mathrm{Hom}_{\mathbb{Z}[\Gamma]}\left(\overline{C_*\big(E\Gamma\big)}, \mathbb{C}^k_\alpha\right)\big) \cong H_i\big(\mathrm{Hom}_{\mathbb{C}}(C_*(E\Gamma)\otimes_{\mathbb{Z}[\Gamma]} \mathbb{C}^k_\alpha, \mathbb{C})\big).$$

Since $\mathbb{C}$ is a field, the Universal Coefficient Theorem implies that

$$\begin{aligned} H_i\big(\mathrm{Hom}_{\mathbb{C}}(C_*(E\Gamma)\otimes_{\mathbb{Z}[\Gamma]} \mathbb{C}^k_\alpha, \mathbb{C})\big) &\cong \mathrm{Hom}_{\mathbb{C}}\left(H_i(C_*(E\Gamma)\otimes_{\mathbb{Z}[\Gamma]} \mathbb{C}^k_\alpha), \mathbb{C}\right) \\ &= \mathrm{Hom}_{\mathbb{C}}\left(H_i^\alpha\big(\Gamma;\mathbb{C}^k\big), \mathbb{C}\right). \end{aligned}$$

Since $H_0^\alpha\big(\Gamma;\mathbb{C}^k\big) \neq 0$ and hence has a non-trivial dual, we obtain $H^0_\alpha\big(\Gamma;\mathbb{C}^k\big) \neq 0$. It follows from [HS97, Chapter VI.3] that

$$H^0_\alpha\big(\Gamma;\mathbb{C}^k\big) \cong \big\{v \in \mathbb{C}^k \;\big|\; \alpha(g)v = v \text{ for all}\, g \in \Gamma\big\}.$$

We now consider the subspace of $\mathbb{C}^k$ consisting of the fixed points of $\alpha|_\Gamma$,

$$V = \big\{v \in \mathbb{C}^k \;\big|\; \alpha(g)v = v \text{ for all}\, g \in \Gamma\big\}.$$

Note that $V \cong H^0_\alpha\big(\Gamma;\mathbb{C}^k\big)$ and we have already seen that $V \neq 0$. Since α restricted to $\ker(\phi)$ is non-trivial, we also have $V \neq \mathbb{C}^k$. Let W be the orthogonal complement of V in $\mathbb{C}^k$. Since α is unitary, we can write α with respect to the decomposition $\mathbb{C}^k = V \oplus W$ as

$$\alpha(g) = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & A(g) \end{pmatrix}.$$

for any $g \in \Gamma$. Let $\mu \in \pi$ be fixed and let

$$\alpha(\mu) = \begin{pmatrix} B & E \\ C & D \end{pmatrix}.$$

with respect to the decomposition $V \oplus W = \mathbb{C}^k$. Let $g \in \Gamma$ be arbitrary. Since $g\mu = \mu\big(\mu^{-1}g\mu\big)$ and $\big(\mu^{-1}g\mu\big) \in \Gamma$, we obtain that

$$\begin{pmatrix} \mathrm{id} & 0 \\ 0 & A(g) \end{pmatrix}\begin{pmatrix} B & E \\ C & D \end{pmatrix} = \alpha(g)\alpha(\mu) = \alpha(\mu)\alpha\big(\mu^{-1}g\mu\big) = \begin{pmatrix} B & E \\ C & D \end{pmatrix}\begin{pmatrix} \mathrm{id} & 0 \\ 0 & A(\mu^{-1}g\mu) \end{pmatrix}$$

and matrix multiplication yields

$$\begin{pmatrix} B & E \\ A(g)C & A(g)D \end{pmatrix} = \begin{pmatrix} B & EA(\mu^{-1}g\mu) \\ C & DA(\mu^{-1}g\mu) \end{pmatrix}.$$

In particular, we obtain $A(g)C = C$. We now show that $C = 0$.

Let $v \in V$. We obtain for all $g \in \Gamma$ that

$$\alpha(g)\left(\begin{pmatrix} 0 \\ Cv \end{pmatrix}\right) = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & A(g) \end{pmatrix}\begin{pmatrix} 0 \\ Cv \end{pmatrix} = \begin{pmatrix} 0 \\ A(g)Cv \end{pmatrix} = \begin{pmatrix} 0 \\ Cv \end{pmatrix}.$$

It follows from the definition of V that $(0, Cv)^t \in V$ and hence $(0, Cv)^t \in V \cap W = \{0\}$. Since v was arbitrary, it follows that $C = 0$ and hence

$$\alpha(\mu) = \begin{pmatrix} B & E \\ 0 & D \end{pmatrix}.$$

Since $\mu \in \pi$ was arbitrary, it follows that α is of the form

$$\alpha(g) = \begin{pmatrix} B(g) & E(g) \\ 0 & D(g) \end{pmatrix}.$$

for all $g \in \pi$. Since α is unitary and the decomposition $\mathbb{C}^k = V \oplus W$ is orthogonal, we get for $\mu \in \pi, v \in V$ and $w \in W$:

$$\begin{aligned} \big\langle E(\mu)w, v\big\rangle &= \big\langle E(\mu)w + D(\mu)w, v\big\rangle = \big\langle \alpha(\mu)\big((0,w)^t\big), v\big\rangle \\ &= \big\langle w, \alpha(\mu^{-1})\big((v,0)^t\big)\big\rangle = 0 \end{aligned}$$

It follows that $E(\mu) = 0$ for all $\mu \in \pi$, and hence α is reducible. □

The next step is to compute for $i \geq 0$ the homology groups $H_i^{\alpha_z^\phi}\big(M; \mathbb{C}^k\big)$ in the case that $H_0^{\alpha\otimes\phi}\big(M; \mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}\big[\mathbb{Z}^l\big]\big) = 0$. We identify $\mathbb{Z}^l$ with the free abelian group generated by $x_1, \ldots, x_l$. Given $z = (z_1, \ldots, z_l) \in T^l$ we consider

$$\begin{aligned} \psi_z \colon \mathbb{C}\big[\mathbb{Z}^l\big] = \mathbb{C}[x_1^{\pm1}, \ldots x_l^{\pm1}] &\to \mathbb{C} \\ \sum_{(n_1,\ldots,n_l)\in\mathbb{Z}^l} a_{(n_1,\ldots,n_l)} \prod_{i=1}^{l} x_i^{n_i} &\mapsto \sum_{(n_1,\ldots,n_l)\in\mathbb{Z}^l} a_{(n_1,\ldots,n_l)} \prod_{i=1}^{l} z_i^{n_i}. \end{aligned}$$

Lemma 8.5. *Let M be a connected manifold with fundamental group π and universal covering $\widetilde{M}$. If $H_0^{\alpha\otimes\phi}(M;\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l])=0$, then*

$$
\begin{aligned}
&H_0^{\alpha_z^\phi}(M;\mathbb{C}^k)=0\\
&H_1^{\alpha_z^\phi}(M;\mathbb{C}^k)=H_1^{\alpha\otimes\phi}(M;\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l])\otimes_{\mathbb{C}[\mathbb{Z}^l]}\mathbb{C}_{\psi_z}.
\end{aligned}
$$

Proof. Let $\mathbf{A}$ be the non-negative chain complex

$$0\to\mathbb{C}_{\psi_z}\to 0$$

of left $\mathbb{C}[\mathbb{Z}^l]$-modules with a single non-trivial module in degree 0. Then

$$H_i(\mathbf{A})=\begin{cases}\mathbb{C}_{\psi_z} & i=0\\ 0 & i\neq 0.\end{cases}$$

Let $\mathbf{D}$ be the non-negative chain complex whose n-th chain module is given by the free and hence flat right $\mathbb{C}[\mathbb{Z}^l]$-module

$$C_n(\widetilde{M})\otimes_{\mathbb{Z}\pi}(\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l]).$$

It follows from Theorem 8.2 that

$$
\begin{aligned}
E_{p,q}^2&=\operatorname{Tor}_p^{\mathbb{C}[\mathbb{Z}^l]}\left(H_q\left(C_*(\widetilde{M})\otimes_{\mathbb{Z}\pi}(\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l])\right),\mathbb{C}_{\psi_z}\right)\\
&\Rightarrow H_{p+q}\left(\left(C_*(\widetilde{M})\otimes_{\mathbb{Z}\pi}(\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l])\right)\otimes_{\mathbb{C}[\mathbb{Z}^l]}\mathbb{C}_{\psi_z}\right)=H_{p+q}^{\alpha_z^\phi}(M;\mathbb{C}^k).
\end{aligned}
$$

Since $E_{0,0}^2=\operatorname{Tor}_0^{\mathbb{C}[\mathbb{Z}^l]}\left(H_0^{\alpha\otimes\phi}(M;\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l]),\mathbb{C}_{\psi_z}\right)=0$ by assumption, we obtain that $H_0^{\alpha_z^\phi}(M;\mathbb{C}^k)=0$. It follows from Theorem 8.3 that we have an exact sequence

$$
\begin{aligned}
&\underbrace{\operatorname{Tor}_2^{\mathbb{C}[\mathbb{Z}^l]}\left(H_0^{\alpha\otimes\phi}(M;\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l]),\mathbb{C}_{\psi_z}\right)}_{=0}\to\operatorname{Tor}_0^{\mathbb{C}[\mathbb{Z}^l]}\left(H_1^{\alpha\otimes\phi}(M;\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l]),\mathbb{C}_{\psi_z}\right)\\
&\qquad\to H_1^{\alpha_z^\phi}(M;\mathbb{C}^k)\to\underbrace{\operatorname{Tor}_1\left(H_0^{\alpha\otimes\phi}(M;\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l]),\mathbb{C}_{\psi_z}\right)}_{=0}\to 0.
\end{aligned}
$$

Hence, we obtain

$$H_1^{\alpha_z^\phi}(M;\mathbb{C}^k)=H_1^{\alpha\otimes\phi}(M;\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{C}[\mathbb{Z}^l])\otimes_{\mathbb{C}[\mathbb{Z}^l]}\mathbb{C}_{\psi_z}.\qquad\square$$

Now we can show the following lemma.

Lemma 8.6. *Let M be a closed connected orientable 3-dimensional manifold. Furthermore let $\alpha\colon\pi_1(M)\to U(k)$ be an irreducible representation and let $\phi\colon\pi_1(M)\to\mathbb{Z}^l$ be such that α restricted to* $\ker(\phi)$ *is non-trivial. Let $z\in T^l$. If $\Delta^{\alpha\otimes\phi}(z)\neq 0$, then for all $i\geq 0$*

$$H_i^{\alpha_z^\phi}(M;\mathbb{C}^k)=0.$$

Proof. Since α is an irreducible representation and α restricted to $\ker(\phi)$ is non-trivial, we can apply Lemma 8.4 and obtain that $H_0(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}^l]) = 0$. It follows from Lemma 8.5 that

$$\begin{aligned} H_0^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) &= 0 \\ H_1^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) &= H_1^{\alpha\otimes\phi}\left(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}^l]\right) \otimes_{\mathbb{C}[\mathbb{Z}^l]} \mathbb{C}_{\psi_z}. \end{aligned}$$

Since $\mathbb{C}[\mathbb{Z}^l]$ is Noetherian and M is compact, there exists a free, finite rank presentation of the $\mathbb{C}[\mathbb{Z}^l]$-module $H_1^{\alpha\otimes\phi}\left(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}^l]\right)$, which we can assume to have the form

$$\mathbb{C}[\mathbb{Z}^l]^m \xrightarrow{P} \mathbb{C}[\mathbb{Z}^l]^n \to H_1^{\alpha\otimes\phi}\left(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}^l]\right) \to 0$$

for $m \geq n$, otherwise $\Delta^{\alpha\otimes\phi}$ would be zero. Since tensoring is right exact, we obtain an exact sequence

$$\mathbb{C}^m \xrightarrow{\psi_z(P)} \mathbb{C}^n \to H_1^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) \to 0$$

and hence,

$$H_1^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) \cong \mathbb{C}^n/\psi_z(P)\mathbb{C}^m.$$

Recall that $\Delta^{\alpha\otimes\phi}$ is defined as the order of $H_1^{\alpha\otimes\phi}(M;\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}^l])$. By definition of the Alexander polynomial, $\Delta^{\alpha\otimes\phi}$ is the greatest common divisor of all $(n \times n)$-subdeterminants of P. In particular, if z is not a zero of $\Delta^{\alpha\otimes\phi}$, then there is an $(n \times n)$-submatrix of P whose determinant does not have a zero at z. Note that $\psi_z(\Delta^{\alpha\otimes\phi}) = \Delta^{\alpha\otimes\phi}(z)$. Hence, the corresponding submatrix of $\psi_z(P)$ has non-zero determinant, and thus $\psi_z(P)$ has rank n. It follows that

$$H_1^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) \cong \mathbb{C}^n/\psi_z(P)\mathbb{C}^m = 0.$$

In fact, if $\Delta^{\alpha\otimes\phi}(z)$ were 0, then all $(n \times n)$-subdeterminants of $\psi_z(P)$ vanish and the matrix has rank strictly less than n, which necessarily means that $H_1^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right)$ would have non-zero dimension.

Since $\alpha_z^\phi\colon \pi_1(M) \to U(k)$ is a unitary representation, α_z^ϕ satisfies

$$\left\langle \alpha_z^\phi(g^{-1})v, w \right\rangle = \left\langle v, \alpha_z^\phi(g)w \right\rangle.$$

We obtain from Lemma 8.1 that

$$H_2^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) \cong \operatorname{Hom}_{\mathbb{C}}\left(H_1^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right), \mathbb{C}\right) \oplus \operatorname{Ext}_{\mathbb{C}}\left(H_0^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right), \mathbb{C}\right).$$

In case that $\Delta^{\alpha\otimes\phi}(z) \neq 0$ we have already seen that $H_0^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) = H_1^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) = 0$ and hence

$$H_2^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) = 0.$$

It follows from Lemma 8.1 that

$$H_3^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right) = \operatorname{Hom}_{\mathbb{C}}\left(\underbrace{H_0^{\alpha_z^\phi}\left(M;\mathbb{C}^k\right)}_{=0}, \mathbb{C}\right) = 0. \qquad \square$$

By using the result of Levine that the ρ-invariant is constant on suitable subset of $R_k(\pi_1(M))$ we obtain the following corollary:

Corollary 8.7. *Let M be a closed connected oriented 3-dimensional manifold. Furthermore let $\alpha\colon \pi_1(M) \to U(k)$ be an irreducible representation. Let $\phi\colon \pi_1(M) \to \mathbb{Z}^l$ be such that α restricted to $\ker(\phi)$ is non-trivial. Then*

$$\begin{aligned} \rho(M)\colon T^l &\to \mathbb{R} \\ z &\mapsto \rho(M, \alpha_z^\phi) \end{aligned}$$

is constant on the connected components of $T^l \backslash \left\{ z \in T^l \middle| \Delta^{\alpha\otimes\phi}(z) = 0 \right\}$.

Proof. We obtain from Lemma 8.6 that if $\Delta^{\alpha\otimes\phi}(z) \neq 0$ then $H_i^{\alpha_z^\phi}\left(M; \mathbb{C}^k\right) = 0$ for all $i \in \mathbb{N} \cup \{0\}$. Hence, $z \mapsto \sum_{i=0}^3 \dim_{\mathbb{C}} H_i^{\alpha_z^\phi}\left(M; \mathbb{C}^k\right)$ is identically zero on $T^l \backslash \left\{ z \in T^l \middle| \Delta^{\alpha\otimes\phi}(z) = 0 \right\}$. It now follows from Corollary 6.16 that

$$\begin{aligned} T^l &\to \mathbb{R} \\ z &\mapsto \rho(M, \alpha_z^\phi) \end{aligned}$$

is constant on the connected components of $T^l \backslash \left\{ z \in T^l \middle| \Delta^{\alpha\otimes\phi}(z) = 0 \right\}$. □

8.3 Case 2: α trivial and ϕ an epimorphism

We now consider the case where $\alpha\colon \pi_1(M) \to U(1)$ is the trivial unitary representation of rank 1 and $\phi\colon \pi_1(M) \to \mathbb{Z}^l$ is an epimorphism.

For any $m \in \mathbb{N}$, we identify $\mathbb{Z}^m$ with the free abelian group generated by $x_1, \dots, x_m$ and for $z = (z_1, \dots, z_m) \in T^m$ we consider the group homomorphism

$$\begin{aligned} \psi_z\colon \mathbb{Z}^m &\to U(1) \\ \prod_{i=1}^m x_i^{n_i} &\mapsto \prod_{i=1}^m z_i^{n_i}. \end{aligned}$$

The map ψ_z induces an action of $\mathbb{C}[\mathbb{Z}^m]$ on $\mathbb{C}$. We denote $\mathbb{C}$ equipped with this action by $\mathbb{C}_{\psi_z}$, which we sometimes abbreviate as just $\mathbb{C}_z$.

We use the following formula to reduce the proof of the subsequent lemma to the case $l = 1$.

Lemma 8.8. *Let A_1, A_2 be two groups with fixed actions on $\mathbb{C}$. For $i = 1, 2$ let C_i be a $\mathbb{Z}[A_i]$-chain complex. Then*

$$(C_1 \otimes_{\mathbb{Z}} C_2) \otimes_{\mathbb{Z}[A_1 \times A_2]} \mathbb{C} \cong (C_1 \otimes_{\mathbb{Z}[A_1]} \mathbb{C}) \otimes_{\mathbb{C}} (C_2 \otimes_{\mathbb{Z}[A_2]} \mathbb{C}).$$

Proof. Since $\mathbb{Z}[A_1 \times A_2]$ is canonically isomorphic to $\mathbb{Z}[A_1] \otimes_{\mathbb{Z}} \mathbb{Z}[A_2]$, we obtain an isomorphism

$$(C_1 \otimes_{\mathbb{Z}} C_2) \otimes_{\mathbb{Z}[A_1 \times A_2]} \mathbb{C} \xrightarrow{\cong} (C_1 \otimes_{\mathbb{Z}} C_2) \otimes_{\mathbb{Z}[A_1] \otimes_{\mathbb{Z}} \mathbb{Z}[A_2]} (\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}).$$

It is now easy to check that the map

$$(C_1 \otimes_{\mathbb{Z}} C_2) \otimes_{\mathbb{Z}[A_1] \otimes_{\mathbb{Z}} \mathbb{Z}[A_2]} (\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}) \to (C_1 \otimes_{\mathbb{Z}[A_1]} \mathbb{C}) \otimes_{\mathbb{C}} (C_1 \otimes_{\mathbb{Z}[A_1]} \mathbb{C})$$
$$(c_1 \otimes c_2) \otimes (z_1 \otimes z_2) \mapsto (c_1 \otimes z_1) \otimes (c_2 \otimes z_2)$$

defines an isomorphism. □

We first calculate the homology groups of a torus with twisted coefficients in $\mathbb{C}_{\psi_z}$.

Lemma 8.9. *Let $z = (z_1, \dots, z_l) \in T^l, z \neq (1, \dots, 1)$. Then for all $n \geq 0$*

$$H_n\left(T^l; \mathbb{C}_{\psi_z}\right) = 0.$$

Proof. We consider $T^1 = S^1$ with the canonical CW-structure consisting of one 0-cell and one 1-cell, and equip T^m with the product CW-structure. We write $\widetilde{T^m}$ for the universal covering of T^m with its induced CW-structure. Note that

$$\begin{aligned} C_*(\widetilde{T^l}) \otimes_{\mathbb{Z}[\mathbb{Z}^l]} \mathbb{C}_z &\cong \left(C_*(\widetilde{T^1}) \otimes_{\mathbb{Z}} C_*(\widetilde{T^{l-1}})\right) \otimes_{\mathbb{Z}[\mathbb{Z}^l]} \mathbb{C}_z \\ &\cong \left(C_*(\widetilde{T^1}) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{C}_{(z_1)}\right) \otimes_{\mathbb{C}} \left(C_*(\widetilde{T^{l-1}}) \otimes_{\mathbb{Z}[\mathbb{Z}^{l-1}]} \mathbb{C}_{(z_2,\dots,z_l)}\right) \end{aligned}$$

where the second isomorphism is that of Lemma 8.8. Since $\mathbb{C}$ is a field, it now follows from the Künneth theorem that

$$\begin{aligned} H_n\left(T^l; \mathbb{C}_{\psi_z}\right) &= H_n\left(C_*(\widetilde{T^l}) \otimes_{\mathbb{Z}[\mathbb{Z}^l]} \mathbb{C}_z\right) \\ &= H_n\left(\left(C_*(\widetilde{T^1}) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{C}_{(z_1)}\right) \otimes_{\mathbb{C}} \left(C_*(\widetilde{T^{l-1}}) \otimes_{\mathbb{Z}[\mathbb{Z}^{l-1}]} \mathbb{C}_{(z_2,\dots,z_l)}\right)\right) \\ &= \oplus_{p+q=n} H_p\left(C_*(\widetilde{T^1}) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{C}_{(z_1)}\right) \otimes_{\mathbb{C}} H_q\left(C_*(\widetilde{T^{l-1}}) \otimes_{\mathbb{Z}[\mathbb{Z}^{l-1}]} \mathbb{C}_{(z_2,\dots,z_l)}\right) \\ &= \oplus_{p+q=n} H_p(S^1; \mathbb{C}_{(z_1)}) \otimes_{\mathbb{C}} H_q\left(T^{l-1}; \mathbb{C}_{(z_2,\dots,z_l)}\right). \end{aligned} \tag{8.1}$$

The cellular chain complex of $\widetilde{S^1}$ coming from the standard CW-structure on S^1 is given by

$$0 \to \mathbb{Z}[t^{\pm 1}] \xrightarrow{1-t} \mathbb{Z}[t^{\pm 1}] \to 0.$$

Let $z \in S^1$, then tensoring with $\mathbb{C}_z$ yields

$$0 \to \mathbb{C} \xrightarrow{1-z} \mathbb{C} \to 0.$$

and thus we obtain for $i \geq 0$

$$H_i(S^1; \mathbb{C}_z) = \begin{cases} 0 & z \neq 1 \\ \mathbb{C} & z = 1. \end{cases}$$

Hence, by (8.1), we have that $H_i(T^l, \mathbb{C}_z) = 0$ for all $i \geq 0$ if $z_1 \neq 1$. The same argument applied to the other entries of the tuple z proves the claim. □

By using the previous lemma we can show:

Lemma 8.10. *Let* $z = (z_1, \dots, z_l) \in T^l, z \neq (1, \dots, 1)$*. Then for all* $p \geq 0$

$$\mathrm{Tor}_p^{\mathbb{C}[\mathbb{Z}^l]}(\mathbb{C}, \mathbb{C}_z) = 0.$$

Proof. We first want to show that

$$\mathrm{Tor}_p^{\mathbb{C}[\mathbb{Z}^l]}(\mathbb{C}, \mathbb{C}_z) = \mathrm{Tor}_p^{\mathbb{Z}[\mathbb{Z}^l]}(\mathbb{Z}, \mathbb{C}_z).$$

Let

$$\dots \to P_2 \to P_1 \to \mathbb{Z} \to 0.$$

be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\big[\mathbb{Z}^l\big]$. Since $\mathbb{C}$ is flat over $\mathbb{Z}$

$$\dots \to P_2 \otimes_{\mathbb{Z}} \mathbb{C} \to P_1 \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C} \to 0$$

is a projective resolution over $\mathbb{C}\big[\mathbb{Z}^l\big]$. Then

$$\begin{aligned}
\mathrm{Tor}_p^{\mathbb{C}[\mathbb{Z}^l]}(\mathbb{C}, \mathbb{C}_z) &= H_p\big(\dots \to P_2 \otimes_{\mathbb{Z}} \mathbb{C} \otimes_{\mathbb{C}[\mathbb{Z}^l]} \mathbb{C}_z \to P_1 \otimes_{\mathbb{Z}} \mathbb{C} \otimes_{\mathbb{C}[\mathbb{Z}^l]} \mathbb{C}_z \to 0\big) \\
&= H_p\big(\dots \to P_2 \otimes_{\mathbb{Z}[\mathbb{Z}^l]} \mathbb{C}_z \to P_1 \otimes_{\mathbb{Z}[\mathbb{Z}^l]} \mathbb{C}_z \to 0\big) \\
&= \mathrm{Tor}_p^{\mathbb{Z}[\mathbb{Z}^l]}(\mathbb{Z}, \mathbb{C}_z).
\end{aligned}$$

By definition of group homology, we have

$$H_p\big(\mathbb{Z}^l; \mathbb{C}_z\big) = \mathrm{Tor}_p^{\mathbb{Z}[\mathbb{Z}^l]}(\mathbb{Z}; \mathbb{C}_z).$$

Since T^l is a classifying space of $\mathbb{Z}^l$, $H_p\big(T^l; \mathbb{C}_z\big) = H_p\big(\mathbb{Z}^l; \mathbb{C}_z\big)$, and hence we obtain for all $p \geq 0$

$$\mathrm{Tor}_p^{\mathbb{C}[\mathbb{Z}^l]}(\mathbb{C}, \mathbb{C}_z) = H_p\big(T^l; \mathbb{C}_z\big) = 0$$

by Lemma 8.9. □

We now show that the homology groups $H_i^{\psi_z \circ \phi}(M; \mathbb{C}^k) = H_i^{\alpha_z^\phi}(M; \mathbb{C})$ often vanish.

Lemma 8.11. *Let* M *be a closed connected oriented 3-dimensional manifold with fundamental group* π*. Let* $\phi \colon \pi_1(M) \to \mathbb{Z}^l$ *be an epimorphism. Let* $z = (z_1, \dots, z_l) \in T^l, z \neq (1, \dots, 1)$*. If* $\Delta^\phi(z) \neq 0$*, then we obtain for all* $i \geq 0$ *that*

$$H_i^{\psi_z \circ \phi}\big(M; \mathbb{C}\big) = 0.$$

Proof. Since ϕ is an epimorphism and $\psi_z \colon \mathbb{Z}^l \to U(1)$ is non-trivial, we have

$$H_0^{\psi_z \circ \phi}\big(M; \mathbb{C}\big) = \mathbb{C} \big/ \big\langle (\psi_z \circ \phi)(g)v - v \mid g \in \pi_1 M, v \in \mathbb{C}\big\rangle = 0.$$

We will compute the homology groups by using Theorem 8.2. As in the proof of Lemma 8.5 let $\mathbf{A}$ be the complex

$$0 \to \mathbb{C}_{\psi_z} \to 0$$

of left $\mathbb{C}\big[\mathbb{Z}^l\big]$-modules with a single non-trivial module in degree 0. Then

$$H_i(\mathbf{A}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Denote by $\widetilde{M}$ the universal covering of M and let $\mathbf{D}$ be the complex whose n-th chain module is given by the free and hence flat right $\mathbb{C}\big[\mathbb{Z}^l\big]$-module

$$C_n(\widetilde{M}) \otimes_{\mathbb{Z}\pi} \mathbb{C}\big[\mathbb{Z}^l\big].$$

Let $\widehat{M}$ be the cover corresponding to $\ker(\phi)$, then its cellular chain complex is given by $\mathbf{D}$. Since $\widehat{M}$ is connected, we obtain

$$H_0^\phi\big(M;\mathbb{C}\big[\mathbb{Z}^l\big]\big) = H_0^\phi\big(M;\mathbb{Z}\big[\mathbb{Z}^l\big]\big) \otimes_{\mathbb{Z}} \mathbb{C} = H_0\big(\widehat{M}\big) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}.$$

We now want to use Theorem 8.2. In the notation of Theorem 8.2 we have

$$E^2_{p,q} = \operatorname{Tor}_p^{\mathbb{C}[\mathbb{Z}^l]}\big(H_q^\phi(M;\mathbb{C}\big[\mathbb{Z}^l\big]), \mathbb{C}_{\psi_z}\big).$$

It follows from Lemma 8.10 combined with $H_0^\phi\big(M;\mathbb{C}\big[\mathbb{Z}^l\big]\big) = \mathbb{C}$ that $E^2_{p,0} = 0$.

Hence, we obtain from $E^2_{2,0} = E^2_{1,0} = 0$ and Theorem 8.3 that

$$H_1^{\psi_z\circ\phi}\big(M;\mathbb{C}\big) \cong E^2_{0,1} \cong H_1^\phi\big(M;\mathbb{C}\big[\mathbb{Z}^l\big]\big) \otimes_{\mathbb{C}[\mathbb{Z}^l]} \mathbb{C}_{\psi_z}.$$

We denote by Δ^ϕ the Alexander polynomial associated to (M,ϕ), i.e., Δ^ϕ is the order of $H_1^\phi(M;\mathbb{C}[\mathbb{Z}^l])$. Analogous to the proof of Lemma 8.6 one can show that if $\Delta^\phi(z) \neq 0$ then $H_1^{\psi_z\circ\phi}\big(M;\mathbb{C}\big) = 0$. Note that

$$\big\langle(\psi_z\circ\phi)(g^{-1})v, w\big\rangle = \big\langle v, (\psi_z\circ\phi)(g)w\big\rangle.$$

Hence, we deduce from Lemma 8.1 that

$$\begin{aligned} H_3^{\psi_z\circ\phi}(M;\mathbb{C}) &= \operatorname{Hom}_{\mathbb{C}}\big(H_0^{\psi_z\circ\phi}(M;\mathbb{C}),\mathbb{C}\big) = 0, \\ H_2^{\psi_z\circ\phi}(M;\mathbb{C}) &= \operatorname{Hom}_{\mathbb{C}}\big(H_1^{\psi_z\circ\phi}(M;\mathbb{C}),\mathbb{C}\big) \oplus \operatorname{Ext}\big(H_0^{\psi_z\circ\phi}(M;\mathbb{C}),\mathbb{C}\big) \\ &= \operatorname{Hom}_{\mathbb{C}}\big(H_1^{\psi_z\circ\phi}(M;\mathbb{C}),\mathbb{C}\big). \end{aligned}$$

If $\Delta^\phi(z) \neq 0$, then $H_1^{\psi_z\circ\phi}(M;\mathbb{C}) = 0$ and hence

$$H_2^{\psi_z\circ\phi}(M;\mathbb{C}) = \operatorname{Hom}_{\mathbb{C}}\big(H_1^{\psi_z\circ\phi}(M;\mathbb{C}),\mathbb{C}\big) = 0. \qquad \square$$

Similar to the previous case, we can now deduce that the ρ-invariant is constant where the Alexander polynomial does not vanish.

Corollary 8.12. *Let M be a closed connected oriented 3-dimensional manifold. Furthermore let $\phi\colon \pi_1(M) \to \mathbb{Z}^l$ be an epimorphism. Given $z = (z_1,\ldots,z_l) \in T^l$ we consider the group homomorphism*

$$\begin{aligned} \psi_z\colon \mathbb{Z}^m &\to U(1) \\ \prod_{i=1}^l x_i^{n_i} &\mapsto \prod_{i=1}^l z_i^{n_i}. \end{aligned}$$

Then

$$\begin{aligned} \rho(M)\colon T^l &\to \mathbb{R} \\ z &\mapsto \rho(M, \psi_z \circ \phi) \end{aligned}$$

is constant on the connected components of $T^l \setminus \left(\left\{ z \in T^l \middle| \Delta^\phi(z) = 0 \right\} \cup \{1\} \right)$*, where* 1 *denotes the point* $(1, \ldots, 1) \in T^l$.

Proof. We obtain from Lemma 8.11 that if $\Delta^\phi(z) \neq 0$ and $z \neq (1, \ldots, 1)$ then all homology groups $H_j^{\psi_z \circ \phi}(M; \mathbb{C})$ are zero. It follows from Corollary 6.16 that

$$\begin{aligned} \rho(M)\colon T^l &\to \mathbb{R} \\ z &\mapsto \rho(M, \psi_z \circ \phi) \end{aligned}$$

is constant on the connected components of $T^l \setminus \left(\left\{ z \in T^l \middle| \Delta^\phi(z) = 0 \right\} \cup \{1\} \right)$. □

Chapter 9

The L^2-ρ-invariant as an integral over ρ-invariants

Let K be a knot and let M_K be obtained by 0-framed surgery on K. Furthermore let $\phi\colon \pi_1(M_K) \to H_1(M_K) \cong \mathbb{Z}$ be given by abelianization. We have seen in Corollary 5.20 that the L^2-ρ-invariant $\rho^{(2)}(M_K, \phi)$ is given by an integral over the ρ-invariants of M_K corresponding to the representations $\psi_z \circ \phi$ for $z \in S^1$. In this chapter we will obtain a more general result for a closed connected oriented 3-dimensional manifold M and homomorphisms $\phi\colon \pi_1(M) \to \mathbb{Z}^l$.

9.1 Zero sets of real and complex analytic functions

In this section we will study zero sets of real and complex analytic functions. Later on we will consider the zero set of a complex polynomial and the restriction of the zero set to the standard torus $T^l = \{z = (z_1, \dots, z_l) \in \mathbb{C}^l \mid |z_1| = \dots = |z_l| = 1\} \subset \mathbb{C}^l$.

In order to formulate our results in a more general way, we will consider a real submanifold L of $\mathbb{C}^l = \mathbb{R}^{2l}$ of real dimension l with the following property:

- For each $x \in L$ there exists an open neighborhood $U \subset \mathbb{C}^l$ of x and a biholomorphic map $\Phi\colon B_\epsilon(0) \subset \mathbb{C}^l \to U$ such that $\Phi\big(B_\epsilon(0) \cap \mathbb{R}^l\big) = U \cap L$.

First we show that the standard torus T^l indeed has this property.

Lemma 9.1. *For each $w \in T^l$ there exists an open neighborhood $U \subset \mathbb{C}^l$ of x and a biholomorphic map $\Phi\colon B_\epsilon(0) \to U$ such that $\Phi\big(B_\epsilon(0) \cap \mathbb{R}^l\big) = U \cap T^l$.*

Proof. Let $w = (w_1, \dots, w_l) \in T^l$. Let $\epsilon > 0$. We consider the map

$$\begin{aligned} \Psi\colon B_\epsilon(0) \subset \mathbb{C}^l &\to \mathbb{C}^l \\ (z_1, \dots, z_l) &\mapsto \left(e^{2\pi i z_1} w_1, \dots, e^{2\pi i z_l} w_l\right) \end{aligned}$$

For ϵ small enough, Φ is biholomorphic onto its image and $\Phi\left(B_\epsilon(0) \cap \mathbb{R}^l\right) = \Phi\left(B_\epsilon(0)\right) \cap T^l$. $\square$

Let $f: D \subset \mathbb{C}^l \to \mathbb{C}$ be a holomorphic function. We denote the zero set of f by

$$Z(f) := \{x \in D \mid f(x) = 0\}.$$

For $U \subset D$, we denote by $Z(f;U) = Z(f) \cap U$ the zero set of f in U.

Definition 9.2. A subset $A \subset \mathbb{R}^n$ is said to have *n-measure zero* if for any $\epsilon > 0$ there exists a countable collection of balls $B_i \subset \mathbb{R}^n$ with radii r_i such that $\sum_i r_i^n < \epsilon$.

A standard result of measure theory is that the property of having n-measure zero is preserved under smooth maps. The notion of an n-measure zero set readily extends to compact real n-manifolds:

Definition 9.3. A subset $A \subset L$ of a compact real n-dimensional manifold L is said to have *n-measure zero* if its intersection with any choice of charts has n-measure zero.

We will use the following result on zero sets of real analytic functions which was shown in [Mit15, Proposition 0].

Proposition 9.4. *Let f be a real analytic function defined on a connected open set $U \subset \mathbb{R}^n$. If f is not identically zero on U, then the zero set $Z(f;U)$ of f in U has n-measure zero.*

The following proposition follows from [Sha92, Chapter 1.2, Theorem 5 and the subsequent remark].

Proposition 9.5. *Let $f: D \subset \mathbb{C}^l \to \mathbb{C}$ be a holomorphic function and D be connected. If there exists an open subset $U \subset (\mathbb{R}^l \cap D)$ such that f vanishes on U, then $f \equiv 0$ on D.*

Combining the two results, we can prove that the zero set of a holomorphic function on $\mathbb{C}^l$ has measure zero even when restricted to certain submanifolds of real codimension l.

Lemma 9.6. *Let $L \subset \mathbb{C}^l$ be a compact real l-dimensional manifold. Assume that for each $x \in L$ there exists an open neighborhood $U \subset \mathbb{C}^l$ of x and a biholomorphic map $\Phi: B_\epsilon(0) \to U$ such that $\Phi(B_\epsilon(0) \cap \mathbb{R}^l) = U \cap L$. Let $p: \mathbb{C}^l \to \mathbb{C}$ be a holomorphic function which is not identically zero. Then $Z(p; L) \subset L$ has l-measure zero.*

Proof. We have to show that every point $x \in L$ has an open neighborhood U such that $Z(p;U) \subset L$ has l-measure zero. Let $x \in L$ and $x \in U \subset \mathbb{C}^l$ an open neighborhood such that there exists a biholomorphic map $\Phi: B_\epsilon(0) \to U$ with $\Phi(B_\epsilon(0) \cap \mathbb{R}^l) = U \cap L$.

Since p is not identically zero and Φ is a biholomorphic map, it follows that $p \circ \Phi$ is not identically zero. Hence, by Proposition 9.5, there exists $x \in B_\epsilon(0) \cap \mathbb{R}^l$ with $(p \circ \Phi)(x) \neq 0$. Thus, the restriction of $p \circ \Phi$ to $B_\epsilon(0) \cap \mathbb{R}^l$ is a real analytic function that is not identically zero on the open set $(B_\epsilon(0) \cap \mathbb{R}^l) \subset \mathbb{R}^l$. It follows from Proposition 9.4 that $Z(p \circ \Phi; B_\epsilon(0) \cap \mathbb{R}^l)$ has l-measure zero. Since Φ is a biholomorphism and in particular smooth, we conclude that $Z(p;U)$ has l-measure zero. ☐

We summarize the result in the following corollary.

Corollary 9.7. *Let $p: \mathbb{C}^l \to \mathbb{C}$ be a holomorphic function which is not identically zero. Then the zero set of p in T^l has l-measure zero.*

9.2 The L^2-ρ-invariant for epimorphisms $\pi_1(M^3) \to \mathbb{Z}^l$

Let M be a closed connected oriented 3-dimensional manifold. Let $\phi\colon \pi_1(M) \to \mathbb{Z}^l$ be an epimorphism for which the Alexander polynomial Δ^ϕ is not zero. As before given $z = (z_1, \dots, z_l) \in T^l$ we define

$$\begin{aligned} \psi_z\colon \mathbb{Z}^l &\to U(1) \\ (n_1, \dots, n_l) &\mapsto z_1^{n_1} \cdots z_l^{n_l}. \end{aligned}$$

In this section we want to show that

$$\int_{T^l} \rho(M, \psi_z \circ \phi)\, \mathrm{d}z$$

exists and equals the L^2-ρ-invariant $\rho^{(2)}(M, \phi)$, which generalizes the previously known result for 0-framed surgery on knots.

We have already seen in Corollary 8.12 that the map

$$\begin{aligned} \rho(M)\colon T^l &\to \mathbb{R} \\ z &\mapsto \rho(M, \psi_z \circ \phi) \end{aligned}$$

is constant on the connected components of $T^l \setminus \big(\{z \in T^l | \Delta^\phi(z) = 0\} \cup \{1\}\big)$, where 1 denotes the point $(1, \dots, 1) \in T^l$. We now show that $\rho(M)$ is bounded on this set. It suffices to prove that $T^l \setminus \big(\{z \in T^l | \Delta^\phi(z) = 0\} \cup \{1\}\big)$ has only finitely many connected components, for which we will need the notion of a semi-algebraic set:

Definition 9.8. A subset V of $\mathbb{R}^l$ is called *semi-algebraic* if there exist $n, m \in \mathbb{N} \cup \{0\}$ and polynomials $p_1, \dots, p_n, f_1, \dots, f_m \in \mathbb{R}[X_1, \dots, X_l]$ such that

$$V = \Big\{ x \in \mathbb{R}^l \mid p_i(x) = 0 \text{ for } i = 1, \dots, n \text{ and } f_i(x) > 0 \text{ for } i = 1, \dots, m \Big\}.$$

Theorem 9.9 ([BCR98, Theorem 2.3.6])**.** *Every semi-algebraic subset of $\mathbb{R}^l$ is a union of a finite number of connected semi-algebraic sets.*

Using this theorem, we can show:

Lemma 9.10. *Let $p \in \mathbb{C}[x_1, y_1, \dots, x_l, y_l]$. Consider the torus T^l as the subset of $\mathbb{R}^{2l}$ given by*

$$T^l = \Big\{ (x_1, y_1, \dots, x_l, y_l) \in \mathbb{R}^{2l} \,\Big|\, x_j^2 + y_j^2 = 1 \textit{ for } j = 1, \dots, l \Big\}.$$

Then the set $T^l \setminus \big\{ z \in T^l \,\big|\, p(z) = 0 \big\}$ consists of only finitely many connected components.

Proof. Note that we can write the set $T^l \setminus \big\{ z \in T^l \,\big|\, p(z) = 0 \big\}$ as

$$\begin{aligned} \big\{ (x_1, y_1, \dots, x_l, y_l) \in \mathbb{R}^{2l} | \operatorname{Re}(p(x_1, y_1, \dots, x_l, y_l))^2 + \operatorname{Im}(p(x_1, y_1, \dots, x_l, y_l))^2 > 0 \\ \text{and } x_j^2 + y_j^2 - 1 = 0 \text{ for all } j = 1, \dots, l \big\}. \end{aligned}$$

Hence $T^l \setminus \big\{ z \in T^l \,\big|\, p(z) = 0 \big\}$ is a semi-algebraic subset of $\mathbb{R}^{2l}$. It follows from Theorem 9.9 that $T^l \setminus \big\{ z \in T^l \,\big|\, p(z) = 0 \big\}$ has only a finite number of connected components. □

Corollary 9.11. *The set $T^l \setminus \big(\{z \in T^l | \Delta^\phi(z) = 0\} \cup \{1\}\big)$ consists of finitely many connected components.*

Proof. Note that the set

$$\Big\{(x_1, y_1, \ldots, x_l, y_l) \in \mathbb{R}^{2l} \,\Big|\, \Delta^\phi(x_1 + iy_1, \ldots, x_l + iy_l) = 0 \text{ or } x_j + iy_j = 1 \text{ for all } j = 1, \ldots, l\Big\}.$$

is the zero set of the polynomial

$$p(x_1, y_1, \ldots, x_l, y_l) = \Delta^\phi(x_1 + iy_1, \ldots, x_l + iy_l) \cdot \left(\sum_{i=1}^{l}(x_i - 1 + iy_i)(x_i - 1 - iy_i)\right).$$

The statement then follows from Lemma 9.10. □

We need the following fact on Riemann-integrable functions (see [Apo74, Theorem 14.5]):

Theorem 9.12. *Let f be a bounded function defined on a compact interval $I \subset \mathbb{R}^l$. Then f is Riemann-integrable if the set of discontinuities of f in I has l-measure zero.*

We can prove the main result of this chapter.

Theorem 9.13. *Let M be a closed connected oriented 3-dimensional manifold. Furthermore, let $\phi\colon \pi_1(M) \to \mathbb{Z}^l$ be an epimorphism such that the Alexander polynomial Δ^ϕ is not zero. For any $z = (z_1, \ldots, z_l) \in T^l$ we consider the one-dimensional representation*

$$\begin{aligned} \psi_z\colon \mathbb{Z}^l &\to U(1) \\ (n_1, \ldots, n_l) &\mapsto z_1^{n_1} \cdots z_l^{n_l}. \end{aligned}$$

Then

$$\int_{T^l} \rho(M, \psi_z \circ \phi)\, \mathrm{d}z$$

exists and

$$\rho^{(2)}(M, \phi) = \int_{T^l} \rho(M, \psi_z \circ \phi)\, \mathrm{d}z.$$

Proof. For any $k \in \mathbb{N}$, we denote by $\mathrm{pr_k}\colon \mathbb{Z}^l \to (\mathbb{Z}_k)^l$ the canonical projection. We consider the residual chain of $\mathbb{Z}^l$ given by

$$\cdots \subset \big((k+1)!\mathbb{Z}\big)^l \subset (k!\mathbb{Z})^l \subset \cdots \subset \mathbb{Z}^l.$$

Let $j = (j_1, \ldots, j_l) \in \mathbb{Z}^l$ and let

$$\begin{aligned} \psi_k^j\colon (\mathbb{Z}_k)^l &\to U(1) \\ ([n_1], \ldots, [n_l]) &\mapsto e^{2\pi i n_1 j_1/k} \cdots e^{2\pi i n_l j_l/k}. \end{aligned}$$

We obtain

$$\begin{aligned} \rho^{(2)}(M, \phi) &= \lim_{k \to \infty} \rho^{(2)}\big(M, \pi_1(M) \xrightarrow{\phi} \mathbb{Z}^l \xrightarrow{\mathrm{pr}_{k!}} (\mathbb{Z}_{k!})^l\big) \\ &= \lim_{k \to \infty} \frac{1}{k!^l}\, \rho\big(M, \pi_1(M) \xrightarrow{\phi} \mathbb{Z}^l \xrightarrow{\mathrm{pr}_{k!}} (\mathbb{Z}_{k!})^l\big) \\ &= \lim_{k \to \infty} \frac{1}{k!^l} \sum_{j \in \{0, \ldots, k!-1\}^l} \rho\big(M, \psi_{k!}^j \circ \mathrm{pr}_{k!} \circ \phi\big), \end{aligned} \tag{9.1}$$

where the first equality follows from Theorem 5.19. The second equality holds because for a finite group G we have the identity

$$\rho^2(M, \alpha) = \frac{1}{|G|}\rho(M, \alpha).$$

The last equality follows from Lemma 4.5.

It follows from Corollary 8.12 that $\rho(M, \psi_z \circ \phi)$ is constant on the connected components of $T^l \backslash \big(\{z \in T^l \mid \Delta^\phi(z) = 0\} \cup \{1\}\big)$ and from Corollary 9.11 that there are only finitely many of those. By Corollary 9.7, the zero set of Δ^ϕ in T^l has l-measure zero. Hence, the map

$$\begin{aligned} T^l &\to \mathbb{R} \\ z &\mapsto \rho(M, \psi_z \circ \phi) \end{aligned}$$

is bounded except for a set of measure zero.

We conclude from Theorem 9.12 that the Riemann integral $\int_{T^l} \rho(M, \psi_z \circ \phi)\,\mathrm{d}z$ exists. Hence, the expression of $\rho^{(2)}(M, \phi)$ as a Riemann sum provided by (9.1) converges and we obtain

$$\rho^{(2)}(M; \phi) = \lim_{k\to\infty} \frac{1}{k!^l} \sum_{j \in \{0,\dots,k!-1\}^l} \rho\big(M, \psi^j_{k!} \circ \phi\big) = \int_{T^l} \rho\big(M, \psi_z \circ \phi\big)\,\mathrm{d}z. \qquad \square$$

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