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A DEFINABLE p-ADIC ANALOGUE OF KIRSZBRAUN'S THEOREM ON EXTENSIONS OF LIPSCHITZ MAPS

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Abstract A direct application of Zorn's lemma gives that every Lipschitz map $f : X \subset \mathbb{Q}_p^n \to \mathbb{Q}_p^\ell$ has an extension to a Lipschitz map $\widetilde{f} : \mathbb{Q}_p^n \to \mathbb{Q}_p^\ell$. This is analogous to, but easier than, Kirszbraun's theorem about the existence of Lipschitz extensions of Lipschitz maps $S \subset \mathbb{R}^n \to \mathbb{R}^\ell$. Recently, Fischer and Aschenbrenner obtained a definable version of Kirszbraun's theorem. In this paper, we prove in the p-adic context that $\widetilde{f}$ can be taken definable when f is definable, where definable means semi-algebraic or subanalytic (or some intermediary notion). We proceed by proving the existence of definable Lipschitz retractions of $\mathbb{Q}_p^n$ to the topological closure of X when X is definable.

Keywords: p-adic semi-algebraic functions; p-adic subanalytic functions; Lipschitz continuous functions; p-adic cell decomposition; definable retractions

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1. Introduction

A Lipschitz continuous map f with Lipschitz constant 1 (a Lipschitz map, for short) from any subset $X \subset \mathbb{Q}_p^n$ to $\mathbb{Q}_p^\ell$ can be extended to a Lipschitz map $\widetilde{f} : \mathbb{Q}_p^n \to \mathbb{Q}_p^\ell$, by Zorn's lemma. This is explained in the proof of [2, Theorem 1.2], the key point being that, if X is moreover closed and $a \in \mathbb{Q}_p^n$ is arbitrary, then f can be extended to a Lipschitz map $X \cup \{a\} \to \mathbb{Q}_p^\ell$ by defining the value of a as $f(x_a)$ for a chosen $x_a \in X$ which lies closest to a among the elements of X. By Zorn's lemma and an easy passing to the topological closure like in Lemma 4 below, f can thus be extended to a Lipschitz map $\widetilde{f} : \mathbb{Q}_p^n \to \mathbb{Q}_p^\ell$. The aim of this paper is to render this construction of $\widetilde{f}$ constructive, when more is known about f. Such a question was raised to us by Aschenbrenner, after work by him and Fischer [1] on making such results constructive (more precisely definable) in the real case.

Let us first briefly recall the real situation, where we refer to [1] for a more complete context and history. A Lipschitz map $g : S \subset \mathbb{R}^n \to \mathbb{R}^\ell$ can always be extended to a Lipschitz map $\widetilde{g} : \mathbb{R}^n \to \mathbb{R}^\ell$, but the argument is more subtle than just applying Zorn's lemma. In the case that $\ell = 1$, the result was observed by McShane [20] and independently by Whitney [23] in 1934, and it can be explained in terms of moduli of continuity of f (see [1, Proposition 5.4]). The case of general ℓ is more delicate, and the result was obtained by Kirszbraun [16] also in 1934, partially relying on Zorn's lemma. Recently, Fischer and Aschenbrenner [1] showed that $\widetilde{g}$ can be taken definable when g is definable (in a very general sense). This can be seen as a constructiveness result. Results related to Whitney's extension theorem continue to play a role in differential topology (see, e.g., [19]).

We prove the definability of $\widetilde{f}$ in the p-adic case when f is definable, where definable can mean semi-algebraic or subanalytic, or some intermediary notion, coming respectively from [6, 11, 18]. We approach our p-adic result via showing that, for any closed definable subset $X \subset \mathbb{Q}_p^n$, there exists a definable Lipschitz retraction

$$r : \mathbb{Q}_p^n \to X,$$

namely, a Lipschitz map $r : \mathbb{Q}_p^n \to X$ such that $r(x) = x$ whenever $x \in X$; see Theorem 2 below. Lipschitz retractions in the real case exist onto convex closed sets (see [1, Corollary 2.14]), but not for general closed sets. The general existence of Lipschitz retractions in our setting may be somewhat surprising, but, in fact, the absence of a convexity condition in whatever form in the p-adic case reminds one of a similar absence in the results on piecewise Lipschitz continuity of [5].

In the p-adic case, there is in fact no difference in difficulty between the $\ell = 1$ case and the case of general ℓ, by the usual definition of the ultra-metric norm as the sup-norm. Naturally, the case when $n = 1$, namely, when the domain of f is a subset of $\mathbb{Q}_p$, is more easy than the case of general n, and it has been treated recently in [17]. We prove our results by an induction on n, where we use a certain form of cell decomposition/preparation with Lipschitz continuous centres, similar to such a result of [5] but which treated no form of preparation; see Theorem 16 below. This decomposition/preparation result is used to geometrically simplify the set X by replacing it by what we call a centred cell. Once we have reduced to the case that X is a centred cell, we use an almost explicit construction of the Lipschitz retraction r, with as the only nonexplicit part some choices of definable Skolem functions. On the way, we obtain a result on the existence of definable isometries with properties adapted to the geometry of X; see Proposition 10.

Our results also hold in families of definable functions (see the variants given by Theorems 19 and 20 at the end of the paper), and for any fixed finite field extension K of $\mathbb{Q}_p$.

1.1. Main results

To state our main results, we first fix some notation. We consider a finite extension K of $\mathbb{Q}_p$. We denote by $\mathrm{ord}\colon K \to \mathbb{Z} \cup \{+\infty\}$ the associated valuation and by $|\cdot| : K \to \mathbb{R}_+$ the associated norm, defined by $|x| = q^{-\operatorname{ord}(x)}$, with q the number of elements of the residue field of K. We equip K^n with the product metric, namely $d(x, y) = \max_{i=1\ldots n} |x_i - y_i|$ for $x = (x_1 \ldots x_n)$ and $y = (y_1 \ldots y_n)$ in K^n, and with the metric topology.

Write $\mathcal{O}_K$ for the valuation ring, $\mathcal{M}_K$ for the maximal ideal of $\mathcal{O}_K$, and k_K for the residue field. Let us fix ϖ some uniformizer of $\mathcal{O}_K$. We denote by $\overline{ac}_m : K \to \mathcal{O}_K/(\mathcal{M}_K^m)$ the map sending nonzero $x \in K$ to $x\varpi^{-\operatorname{ord}(x)} \bmod \mathcal{M}_K^m$, and sending zero to zero. This map is called the mth angular component map. We point out that the map $\overline{ac}_m$ is definable in the field structure of K (see [10, Lemma 2.1 4]).

We denote by RV the union of $K^\times/(1 + \mathcal{M}_K)$ and $\{0\}$, and by $rv : K \to RV$ the quotient map sending zero to zero. More generally, if m is a positive integer, we set $RV_m = K^\times/(1 + \mathcal{M}_K^m) \cup \{0\}$ and $rv_m : K \to RV_m$ the quotient map.

For $m, n > 0 \in \mathbb{N}$, we set

$$Q_{m,n} = \{x \in K^\times \mid \operatorname{ord}(x) \in n\mathbb{Z} \text{ and } \overline{ac}_m(x) = 1\}.$$

When $X \subset Y \subset K^n$, a retraction from Y to a X is a map $r : Y \to X$ which is the identity on X. By definable we mean either semi-algebraic, or subanalytic, or an intermediary structure given by an analytic structure on $\mathbb{Q}_p$, as in [6]. The notions of semi-algebraic sets and of subanalytic sets are recalled in [5] and are based on quantifier elimination results from [11, 18], and we refer to [6] for background on more general analytic structures. A function

$$f : X \subset \mathbb{Q}_p^n \to \mathbb{Q}_p^\ell$$

is called Lipschitz (in full, Lipschitz continuous with Lipschitz constant 1) when

$$|f(x) - f(y)| \leqslant |x - y| \text{ for all } x, y \in X.$$

We call a function $\widetilde{f} : A \to B$ an extension of a function $f : A_0 \to B$ when $A_0 \subset A$ and $\widetilde{f}$ coincides with f on A_0.

The following results, and their family versions given below as Theorems 19 and 20, are the main results of the paper.

Theorem 1 (Ext_n). *Let $X \subset K^n$ be a definable set. Let $f : X \to K^\ell$ be a definable and Lipschitz function. Then there exists a definable Lipschitz function $\widetilde{f} : K^n \to K^\ell$ which is an extension of f. Moreover, we can ensure that the range of $\widetilde{f}$ is contained in the topological closure of the range of f.*

Theorem 2 (Ret_n). *Let $X \subset K^n$ be a definable set. There exists a definable Lipschitz retraction $r : K^n \to \overline{X}$. Here, $\overline{X}$ is the topological closure of X in K^n.*

Theorem 1 can be seen as a consequence of Theorem 2, as follows.

Proof of Theorem 1 knowing Theorem 2. Let $\overline{f} : \overline{X} \to K$ be the unique definable Lipschitz extension of f to the topological closure $\overline{X}$ of X, as given by Lemma 4 below. Let $r : K^n \to \overline{X}$ be a definable Lipschitz retraction as given by Theorem 2. Then $\widetilde{f} = \overline{f} \circ r$ extends f, and is Lipschitz. □

In fact, we will prove these two theorems together with Proposition 10 by a joint induction on n in § 3.

Remark 3 (Some remarks about Theorem 2).

(1) One really needs to consider $\overline{X}$ in Theorem 2. For instance, there is no continuous retraction from K to $K^\times$.

(2) The Archimedean analogue of Theorem 2 is false. For instance, there is no continuous retraction $r : \mathbb{R} \to \{-1, 1\}$. However, when $X \subset \mathbb{R}^n$ is a closed convex set, the projection $r : \mathbb{R}^n \to X$ to the closest point of X is a Lipschitz retraction; see [1, Corollary 2.14].

(3) It would be interesting to know if Theorems 1 and 2 hold in some form for other classes of valued fields K. Natural examples would be $\mathbb{R}((t))$, $\mathbb{C}((t))$ or algebraically closed valued fields (see below). Some difficulties in more general settings are the absence of definable Skolem functions in general (they are used in the proof of Theorem 2), and infiniteness of the residue field (we use the finiteness of the residue field in Corollary 13).

(4) In this form, the analogue of Theorem 2 does not hold for ACVF, the theory of algebraically closed valued fields. Indeed, let L be an algebraically closed valued field, and let $X = \{x \in L \mid |x| > 1\}$. Then X is a closed set, but one can check that there is no Lipschitz retraction $r : L \to X$. However, in this example X might not be considered as a closed set, because it is defined by means of $<$. One might hope that for a 'good' notion of definable closed set (such as a set defined with $\leqslant$, $=$, finite unions, and intersections), an analogue of Theorem 2 holds in ACVF. For instance, there exists a definable Lipschitz retraction from L onto $\{x \in L \mid |x| \geqslant 1\}$.

2. Preliminary results

2.1. Dimension of definable sets

To a nonempty definable set $X \subset K^n$ one can associate a dimension, denoted by $\dim(X) \in \mathbb{N}$. It is defined as the maximum of the numbers $k \geqslant 0$ such that there is a coordinate projection $p : K^n \to K^k$ such that $p(X)$ has nonempty interior in K^k. This dimension, studied in [15] in the slightly more general context of P-minimal structures, enjoys some nice properties that we will freely use. Let us mention that $\dim(\overline{X}) = \dim(X)$ and that if $f : X \to Y$ is a definable map then $\dim(f(X)) \leqslant \dim(X)$.

2.2. Presburger sets

A Presburger set is a subset of $\mathbb{Z}^n$ defined in the language $\mathcal{L}_{Pres}$ consisting of $+, -, 0, 1, <$ and, for each $n > 0$, the binary relation $\cdot \equiv_n \cdot$ for congruence modulo n. Since $(\mathbb{Z}, \mathcal{L}_{Pres})$ eliminates quantifiers by [22], and 1 generates $\mathbb{Z}/n\mathbb{Z}$ for any integer n, any Presburger set can be seen to be a finite union of sets of the following form:

$$\{(u, v) \in \mathbb{Z}^n \mid u \in U,\ v \in a\mathbb{Z} + b \text{ and } \alpha(u) \square_1 v \square_2 \beta(u)\},$$

where $U \subset \mathbb{Z}^{n-1}$ is a Presburger set, a, b are positive integers, $\alpha, \beta : U \to \mathbb{Z}$ are definable functions, and each $\square_i$ is $<$ or no condition; see [3] for related results and some background.

2.3. Retractions

We begin with three basic lemmas.

Lemma 4. *Let $X \subset K^n$ and $Y \subset K^\ell$ be definable sets. Let $f : X \to Y$ be a definable Lipschitz function. Then $\overline{X}$ and $\overline{Y}$ are also definable, and there exists a unique Lipschitz extension $\overline{f} : \overline{X} \to \overline{Y}$ which is also definable.*

Proof. The proof is a simple topological and definability argument. □

Lemma 5. *Let $X \subset Y \subset K^n$. Let $r : Y \to X$ be a Lipschitz retraction. Then, for all $y \in Y$, $|r(y) - y| = d(y, X)$, where $d(y, X)$ is the infimum of $d(y, x)$ for $x \in X$.*

Proof. Let us assume that $|r(y) - y| > d(y, X)$, and let $x \in X$ such that $|r(y) - y| > |x - y|$. Then

$$|r(y) - r(x)| = |r(y) - x| = |(r(y) - y) + (y - x)| = |r(y) - y| > |y - x|.$$

This contradicts the fact that r is Lipschitz. □

The following result is inspired by [17, Lemma 11].

Lemma 6 (Gluing lemma). *Let $X \subset K^n$ be a definable set. Let $X_i \subset X$ for $i = 1, \dots, m$ be a finite collection of definable sets, and let $r_i : X \to X_i$ be definable Lipschitz retractions. Then there exists a definable Lipschitz retraction*

$$r : X \to \bigcup_{i=1}^{m} X_i.$$

Proof. With an easy induction on m, we can assume that $m = 2$. So, we have two definable sets X_1, X_2. Let us define r by

$$\begin{aligned} r : X &\to X_1 \cup X_2 \\ x &\mapsto \begin{cases} r_1(x) & \text{if } d(x, X_1) \leqslant d(x, X_2) \\ r_2(x) & \text{otherwise.} \end{cases} \end{aligned}$$

Let $x, y \in X$, and let us prove that $|r(x) - r(y)| \leqslant |x - y|$. This is obvious, except when for example $d(x, X_1) \leqslant d(x, X_2)$ and $d(y, X_1) > d(y, X_2)$. So let us assume that these two inequalities hold.

This implies that $r(x) = r_1(x) \in X_1$ and $r(y) = r_2(y) \in X_2$. We obtain

$$|x - r(y)| \geqslant d(x, X_2) \geqslant d(x, X_1) = |r(x) - x|. \tag{1}$$

The last equality follows from Lemma 5. Then

$$|x - r(y)| \geqslant \max(|r(x) - x|, |x - r(y)|) \geqslant |r(x) - r(y)|, \tag{2}$$

where the first inequality follows from (1). Moreover,

$$|y - r(x)| \geqslant d(y, X_1) > d(y, X_2) = |y - r(y)|. \tag{3}$$

The last equality follows from Lemma 5 again. Then

$$|r(y) - r(x)| = |(r(y) - y) + (y - r(x))| \overset{(3)}{=} |y - r(x)|. \tag{4}$$

With one more step, we get

$$|x - r(y)| \overset{(2)}{\geqslant} |r(x) - r(y)| \overset{(4)}{=} |y - r(x)| \overset{(3)}{>} |y - r(y)|. \tag{5}$$

Finally,

$$|x - y| = |(x - r(y)) + (r(y) - y)| \overset{(5)}{=} |x - r(y)| \overset{(2)}{\geqslant} |r(x) - r(y)|,$$

which is what we have to show. □

2.4. Centred cells

Definition 7 (Centred cells). Let $n' \leqslant n$ be integers. We say that a definable set $C \subset (K^\times)^n$ is an open centred cell if there exists an integer m such that C is of the form

$$C = rv_m^{-1}(G) \cap (K^\times)^n,$$

for some set $G \subset (RV_m)^n$; that is,

$$C = \{x \in (K^\times)^n \mid (rv_m(x_1), \ldots, rv_m(x_n)) \in G\}.$$

Furthermore, if $C' \subset K^{n'}$ is an open centred cell, we say that

$$C := C' \times \{\underbrace{(0, \ldots, 0)}_{n-n' \text{ times}}\} \subset K^n$$

is a centred cell.

Note that, in the above definition, C and C' have the same dimension.

A centred cell is thus just a coordinatewise pullback under rv_m of some definable subset G of RV_m^n for some m, where we call a subset A of $K^n \times \mathbb{Z}^\ell \times \prod_{i=1}^N RV_{m_i}$ a definable set whenever its natural pullback in $K^{n+\ell+N}$ (coordinatewise under ord and rv_{m_i}) is a definable set.

Definition 8 (Monomial function). Let $C' \subset K^{n'}$ be an open centred cell, and let $C = C' \times \{(0, \ldots, 0)\} \subset K^n$ be the associated centred cell. We say that a definable map $f : C \to \mathbb{Z}$ is a monomial function if there exist an integer m and a definable map $f_0 : (RV_m)^{n'} \to \mathbb{Z}$ such that $f = f_0 \circ \pi$, where $\pi : C \to (RV_m)^{n'}$ is defined by

$$(x_1, \ldots, x_n) \mapsto (rv_m(x_1), \ldots, rv_m(x_{n'})).$$

A monomial function is thus just a definable function induced by a function purely on the RV_m side for some m.

The following lemma illustrates how monomial functions are useful to build new centred cells.

Lemma 9. *Let $C = rv_m^{-1}(G) \subset K^n$ be an open centred cell. Let $C' \subset K^{n+1}$ be a set of the form*

$$C' = \{(y,t) \in C \times K \mid |\alpha(y)| \square_1 |t| \square_2 |\beta(y)| \text{ and } t \in \lambda Q_{m',n'}\},$$

with the $\square_i$ either $<$ or no condition for $i = 1, 2$, with m', n' some integers, and with some nonzero $\lambda \in K$. In addition, let us assume that $\operatorname{ord}\alpha$ and $\operatorname{ord}\beta$ are monomial functions on C. Then C' is an open centred cell.

Proof. It is well known that a set like C' is definable, since the condition $Q_{m',n'}$ is a definable set. The centred cell comes with an m, as an RV_m-pullback, and so do the monomial functions $\operatorname{ord}\alpha$ and $\operatorname{ord}\beta$ come with integers m_1, m_2, witnessing the definition of monomial function. Increasing some of these m, m_1, m_2, and m' if necessary, there is no harm in assuming that they are all equal. Now the lemma follows easily. □

The following result forms part of the induction scheme for the proofs of Theorems 1 and 2, and it will be proved together with these theorems in § 3.2.

Proposition 10 (Mon_n). *Let $X \subset K^n$ be a definable set. For each $i = 1, \ldots, m$, let $f_i : X \to \mathbb{Z}$ be a definable map. Then there exists a decomposition $X = \coprod_{j=1}^{\ell} A_j$ in disjoint definable sets such that for each index $j \in \{1, \ldots, \ell\}$ there is a definable isometry $\varphi_j : K^n \xrightarrow{\sim} K^n$ such that $\varphi_j^{-1}(A_j)$ is a centred cell, and, for each $i = 1 \ldots m$, $f_i \circ \varphi_j$ is a monomial function on $\varphi_j^{-1}(A_j)$.*

Remark 11. In [13, 14], it is conjectured that the trees of balls $T(X)$ of definable sets $X \subset \mathbb{Z}_p^n$ are characterized by a simple combinatorial condition. The trees satisfying this combinatorial condition are called trees of level d. One can easily check that, if $X \subset \mathbb{Z}_p^n$ is a centred cell, its associated tree $T(X)$ is a tree of level d, where $d = \dim(X)$. Hence Proposition 10 implies that, up to cutting a definable set in finitely many pieces, [13, Conjecture 1.1] holds.

2.5. Retraction of centred cells

Lemma 12. *Let $C \subset (K^\times)^n$ be an open centred cell. Put $G := \operatorname{ord}(C) \subset \mathbb{Z}^n$ and $C' := \operatorname{ord}^{-1}(G) \subset (K^\times)^n$. Suppose that the function*

$$x \mapsto (\overline{ac}_m(x_1), \ldots, \overline{ac}_m(x_n))$$

is constant on C. Then there is a definable Lipschitz retraction r from C' to C.

Proof. The set $(\mathcal{O}_K^\times/1 + \mathcal{M}_K^m)$ is finite, of size $N = (q-1)q^{m-1}$. Let

$$\xi_1, \xi_2, \ldots, \xi_N \in \mathcal{O}_K^\times$$

be a set of representatives of $(\mathcal{O}_K^\times/1 + \mathcal{M}_K^m)$ with the extra condition that

$$\xi_1 = 1. \tag{6}$$

For $(u_1, \ldots, u_n) \in (RV_m)^n$, let us set

$$(\gamma_1, \ldots, \gamma_n) := \operatorname{ord}(u_1, \ldots, u_n) \in \mathbb{Z}^n$$

$$A := rv_m^{-1}(u_1 \dots, u_n) \subset (K^\times)^n$$
$$B := \mathrm{ord}^{-1}(\gamma_1, \dots, \gamma_n) \subset (K^\times)^n.$$

One has the following decomposition:

$$B = \coprod_{(i_1 \dots i_n) \in \{1 \dots N\}^n} (\xi_{i_1}, \dots, \xi_{i_n}) \cdot A. \tag{7}$$

By definition of C and C', if $x \in C'$, there exists a unique n-tuple $(i_1 \dots i_n) \in \{1, \dots, N\}^n$ such that $(\xi_{i_1} x_1, \dots, \xi_{i_n} x_n)$ lies in C. We define r as follows.

$$\begin{aligned} r : C' &\to C \\ x &\mapsto (\xi_{i_1}, \dots, \xi_{i_n}) \cdot x, \end{aligned}$$

where $(i_1 \dots i_n)$ is the unique n-tuple of $\{1, \dots, N\}^n$ such that $(\xi_{i_1}, \dots, \xi_{i_n}) \cdot x \in C$.

Let us check that r is Lipschitz by a case analysis. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be in C'. Let $(\xi_{i_1}, \dots, \xi_{i_n})$ (respectively, $(\xi'_{i_1}, \dots, \xi'_{i_n})$) be the n-tuple that appears in the definition of r for x (respectively, for y). Let us fix some index $j \in \{1, \dots, n\}$, and let us check that $|r(x)_j - r(y)_j| \leqslant |x_j - y_j|$.

Case 1. $|x_j| = |y_j|$.

Case 1.1. $\overline{ac}_m(x_j) = \overline{ac}_m(y_j)$.

In this case, the constancy condition of the lemma implies that $\xi_{i_j} = \xi'_{i_j}$. So

$$|r(x)_j - r(y)_j| = |\xi_{i_j}(x_j - y_j)| = |x_j - y_j|.$$

Case 1.2. $\overline{ac}_m(x_j) \neq \overline{ac}_m(y_j)$.

The case condition implies that

$$|\varpi^m| \cdot |x_j| \leqslant |x_j - y_j|. \tag{8}$$

By the constancy hypotheses, and according to the definition of r,

$$\overline{ac}_m(r(x)_j) = \overline{ac}_m(r(y)_j)$$

and

$$|r(x)_j| = |x_j| = |y_j| = |r(y)_j|.$$

It follows that

$$rv_m(r(x)_j) = rv_m(r(y)_j). \tag{9}$$

So (8) and (9) imply that

$$|r(x)_j - r(y)_j| \overset{(9)}{\leqslant} |\varpi^m| \cdot |x_j| \overset{(8)}{\leqslant} |x_j - y_j|.$$

Case 2. $|x_j| < |y_j|$.

In this case, we have

$$|r(x_j)| = |x_j| < |y_j| = |r(y)_j|.$$

So,

$$|r(x)_j - r(y)_j| = |r(y)_j| = |y_j| = |x_j - y_j|,$$

and we are done. □

Corollary 13. *Let $C \subset (K^\times)^n$ be an open centred cell. Put $G := \mathrm{ord}(C) \subset \mathbb{Z}^n$ and $X := \mathrm{ord}^{-1}(G) \subset (K^\times)^n$. Then there exists a definable Lipschitz retraction $r : X \to C$.*

Proof. Since $(\mathcal{O}_K^\times/1+\mathcal{M}_K^m)$ is finite, there exists a finite partition $C = \coprod_{j\in J} C_j$ such that, for each $j \in J$, C_j is an open centred cell, and such that the map $(\overline{ac}_m)^n : C_j \to (\mathcal{O}_K^\times/1+\mathcal{M}_K^m)$ is constant. Thanks to Lemma 12, for each $j \in J$ there exists a definable Lipschitz retraction $r_j : X \to C_j$. Finally, by Lemma 6, there exists a definable Lipschitz retraction $r : X \to C$. □

Lemma 14. *Let $U \subset \mathbb{Z}^{n-1}$ be a Presburger set. Let a, b be positive integers. Let $\alpha, \beta : U \to b + a\mathbb{Z}$ be definable functions. Let us set*

$$V' = \{(u, v) \in U \times \mathbb{Z} \mid \alpha(u) \,\square_1\, v \,\square_2\, \beta(u)\},$$
$$V = V' \cap (\mathbb{Z}^{n-1} \times (b + a\mathbb{Z})),$$

where $\square_i$ is $<$ or no condition for $i = 1, 2$. Then there exists a definable Lipschitz retraction $r : \mathrm{ord}^{-1}(V') \to \mathrm{ord}^{-1}(V)$.

Proof. Define r by

$$\begin{aligned} r : \mathrm{ord}^{-1}(V') &\to \mathrm{ord}^{-1}(V) \\ (x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_{n-1}, \varpi^i x_n), \end{aligned}$$

where i is the unique index $i \in \{0, \ldots, a-1\}$ such that $(x_1, \ldots, x_{n-1}, \varpi^i x_n) \in \mathrm{ord}^{-1}(V)$. Thanks to the definitions of V and V', r is well defined. Let us prove that r is Lipschitz.

So, let $x = (x_1, \ldots, x_n)$ and $x' = (x'_1, \ldots, x'_n) \in \mathrm{ord}^{-1}(V')$, and let us prove that $|r(x) - r(x')| \leqslant |x - x'|$. Since r does not change the first $n-1$ coordinates, it suffices to check that $|r(x)_n - r(x')_n| \leqslant |x_n - x'_n|$. Let i (respectively, i') be the integer such that $r(x)_n = \varpi^i x_n$ (respectively, $r(x')_n = \varpi^{i'} x'_n$).

Case 1. $|x_n| = |x'_n|$.

In this case, $i = i'$ because i (respectively, i') is the smallest integer $\geqslant 0$ such that $\mathrm{ord}(x_n) + i \in b + a\mathbb{Z}$ (respectively, $\mathrm{ord}(x'_n) + i' \in a\mathbb{Z} + b$). So

$$|r(x)_n - r(x')_n| = |\varpi^i (x_n - x'_n)| \leqslant |x_n - x'_n|.$$

Case 2. $|x_n| > |x'_n|$.

We have

$$|r(x)_n - r(x')_n)| \leqslant \max(|\varpi^i x_n|, |\varpi^{i'} x'_n|) \leqslant |x_n| = |x_n - x'_n|,$$

which finishes the proof. □

2.6. Cell decomposition and preparation with Lipschitz centres

In this section, we improve [5, Proposition 4.6] (and [4, Proposition 2.4]) by adding a kind of preparation to a cell decomposition statement. Instead of reproving Proposition 4.6 completely and observing that the preparation can be ensured as required, we give a blueprint on how to derive preparation from cell decomposition, in our context. This blueprint does not yet seem to work in more general P-minimal structures as in [15]. Recall that the idea of cell decomposition/preparation in the p-adic context goes back to Cohen [7] and Denef [9]. Let us first recall the notion of cells in our context, slightly adapted from the notion of [4, Definition 3.1].

Definition 15 (p-adic cells). Let Y be a definable subset of $\mathbb{Z}^M \times K^N$. A cell $A \subset K \times Y$ over Y is a (nonempty) set of the form

$$A = \{(t, y) \in K \times Y \mid y \in Y',\ \alpha(y) \,\square_1\, \mathrm{ord}(t - c(y)) \,\square_2\, \beta(y),\ t - c(y) \in \lambda Q_{m,n}\}, \quad (10)$$

with $Y' \subset Y$ a definable set, integers $n > 0$, $m > 0$, λ in K, $\alpha, \beta\colon Y' \to \mathbb{Z}$ and $c\colon Y' \to K$ all definable functions, and $\square_i$ either $<$ or no condition, and such that A projects surjectively onto Y'. A cell is often considered with extra (usually nonunique) data which comes from the cell description, with terminology as follows. We call c the centre of the cell A, $\lambda Q_{m,n}$ the coset of A, α and β the boundaries of A, the $\square_i$ the boundary conditions of A, and Y' the base of A. If $\lambda = 0$ we call A a 0-cell, and if $\lambda \neq 0$ we call A a 1-cell.

Theorem 16 (p-adic cell decomposition/preparation with Lipschitz centres). *Let Y be a definable subset of $\mathbb{Z}^M \times K^N$, let $X \subset Y \times K^n$ be a definable set, and let $f_i : X \to \mathbb{Z}$ be definable functions, for $i = 1 \ldots m$. Then there exists a partition $X = \coprod_{j=1}^{\ell} X_j$ such that, for each j, for a well-chosen coordinate projection $\pi_j : Y \times K^n \to Y \times K^{n-1}$, one has the following.*

(a) *X_j is a cell over $\pi_j(Y \times K^n)$ with centre $c_j : \pi_j(X_j) \to K$ which is Lipschitz with respect to the variables of K^{n-1} (i.e., for all $y \in Y$, the map $c_j(y, \cdot)$ is Lipschitz on $\{\hat{x} \mid (y, \hat{x}) \in \pi_j(X_j)\}$) and with coset $\lambda_j Q_{m_j,n_j}$ for some $\lambda_j \in K$ and positive integers m_j, n_j.*

(b) *For all $i = 1 \ldots m$ and $j = 1 \ldots \ell$, there exist a rational number $a_{i,j} \in \frac{1}{n_j}\mathbb{Z}$ and a definable map $h_{i,j} : \pi_j(X_j) \to \mathbb{Z}$ such that, for all $(y, x) \in X_j$,*

$$f_i(y, x) = h_{i,j}(y, \hat{x}) + a_{i,j}\, \mathrm{ord}\left(\frac{x_n - c_j(y, \hat{x})}{\lambda_j}\right), \quad (11)$$

where we write $(y, x) = (y, \hat{x}, x_n) = (\pi_j(y, x), x_n)$, and with the convention that $0/0 = 1$ in (11).

The argument that we will give derives preparation from cell decomposition and some additional properties which may be compared to results from [8] for more general P-minimal structures than our structures. We now give two of these additional properties.

A definable function $f : \mathbb{Z}^m \times K^n \to \mathbb{Z}^\ell$ is piecewise (with definable pieces) equal to the restriction to the piece of a function

$$\gamma(z) + h(x)$$

for definable functions $\gamma : \mathbb{Z}^m \to \mathbb{Z}^\ell$ and $h : K^n \to \mathbb{Z}^\ell$, essentially by quantifier elimination. This can be seen as follows. By the quantifier elimination result of [21] (respectively, of [6]) and by the finiteness of the residue field of K, one has quantifier elimination in the two-sorted structure with sorts K and $\mathbb{Z}$, the Macintyre language on K (respectively, enriched with an analytic structure if we are in a situation as in [11] or [6]), the language $\mathcal{L}_{Pres}$ on $\mathbb{Z}$, and the symbol ord for the valuation (extended, say, by zero on zero). Now the claim about piecewise existence of γ and h follows from a syntactical analysis of quantifier-free formulae in such a two-sorted language, and basic properties about the additive ordered group $\mathbb{Z}$.

Secondly, and similarly, a definable function $g : \mathbb{Z}^m \times K^n \to K^\ell$ is piecewise (with definable pieces) induced by definable functions $h : K^n \to K^\ell$, again by two-sorted quantifier elimination following from [21] (respectively in the analytic case following from [6]), a syntactical analysis of quantifier-free formulae, and definable Skolem functions in the one-sorted structure K.

These properties will be used in the following proof of Theorem 16. We first give an extra definition (by induction on n) and a lemma.

Definition 17. Let Y and $X \subset Y \times K^n$ be definable sets. We call X a full cell over Y with full centres $(c_1, \ldots, c_n)$ if it is a cell over $Y \times K^{n-1}$ with centre c_n, and if the base of X is itself a full cell over Y with full centres $(c_1, \ldots, c_{n-1})$. For such a full cell X over Y, the image in $Y \times (\mathbb{Z} \cup \{+\infty\})^n$ of X under the map

$$(y, z) \in X \mapsto (y, \mathrm{ord}(z_1 - c_1(y)), \ldots, \mathrm{ord}(z_n - c_n(y, z_1, \ldots, z_{n-1}))$$

is called the skeleton of the full cell X over Y.

Lemma 18. *Let $X \subset \mathbb{Z}^m \times K^n$ be a full cell over $\mathbb{Z}^m$ with skeleton $A \subset \mathbb{Z}^{m+n}$ over $\mathbb{Z}^m$. Suppose that, for each natural number N, there are infinitely many tuples $z \in \mathbb{Z}^m$ such that $A_z := \{w \in \mathbb{Z}^n \mid (z, w) \in A\}$ is of cardinality at least N. Then there are z and z' in $\mathbb{Z}^m$ with $z \neq z'$ such that $X_z := \{x \in K^n \mid (z, x) \in X\}$ and $X_{z'}$ have nonempty intersection.*

Proof. The lemma follows from the cell decomposition theorem for Presburger sets, namely [3, Theorem 1], and the property about g mentioned just before Definition 17, applied to the occurring centres of the full cell X. □

Proof of Theorem 16. First note that the result including (a) but without (b) is [5, Proposition 4.6], for a general definable subset Y of $\mathbb{Z}^M \times K^N$ for $M, N \geqslant 0$.

Clearly we are allowed to work piecewise, and hence, by induction on m, we may assume that the f_i do not vanish on X.

The simple case that the function $x \in X_y \mapsto f_i(y, x)$ is constant for each $y \in Y$ and each i is immediate, where $X_y = \{x \in K^n \mid (y, x) \in X\}$.

Let us now consider the graph X' of the function

$$F : (y, x) \in X \mapsto (f_1(y, x), \ldots, f_m(y, x)) \in \mathbb{Z}^m,$$

and let us put $Y' := Y \times \mathbb{Z}^m$ so that X' can be naturally seen as a subset of $Y' \times K^n$. Let f_i' be the induced function on X' coming from f_i and the natural bijection between X and X'. By the simple case treated above, the theorem holds for X' and the functions f_i', yielding cells X_j', functions $h_{i,j}'$, and so on. We now derive from this the result for X and the f_i. By the induction hypothesis (with induction on n) applied recursively to the base of the cell

X', and up to changing notation and focusing on one part which we denote by X' again, we may suppose that X' is a full cell over Y'. Let A' be its skeleton. By the graph construction, the fibres $X'_{z,y} := \{x \in K^n \mid (z, y, x) \in X'\}$ are all disjoint when z runs over $\mathbb{Z}^m$ and when y is any fixed value in Y. By Lemma 18, this implies that we can partition X' further without changing the centres, and reduce to the case that $A'_{z,y} := \{s \in \mathbb{Z}^n \mid (z, y, s) \in A'\}$ is either empty or a singleton for each $z \in \mathbb{Z}^m$ and each $y \in Y$. Now the theorem for X and the f_i follows from the first property mentioned below Theorem 16 and the piecewise linearity of Presburger definable functions of the cell decomposition theorem for Presburger sets, namely [3, Theorem 1]. Indeed, the dependence of z on s for any fixed y under the condition $(z, y, s) \in A'$ is a piecewise linear function, uniformly so in y. $\square$

3. Proofs of the main result by a joint induction

We prove Theorems 1 and 2 and Proposition 10 by a joint induction; that is, we prove the properties (Mon_n), (Ret_n), and (Ext_n) by induction on n.

3.1. Proofs for the case $n = 1$

When $n = 1$, Proposition 10 follows from the p-adic cell decomposition (Theorem 16). For this, one does not need the Lipschitz assertion on the centres of the cells.

To prove Theorem 2 for $n = 1$, thanks to Lemma 6, Theorem 16, and up to translating with the centre, one is reduced to finding a definable Lipschitz retraction $r : K \to \overline{C}$, where $C \subset K^\times$ is an open centred cell. By Corollary 13, one can assume that $C = \mathrm{ord}^{-1}(G)$ for a Presburger set $G \subset \mathbb{Z}$. Such a set is a finite union of sets of the form

$$\{g \in \mathbb{Z} \mid g \in a\mathbb{Z} + b, \text{ and } \alpha \,\square_1\, g \,\square_2\, \beta\},$$

where $a, b, \alpha, \beta \in \mathbb{Z}$ and each $\square_i$ is $<$ or no condition. Thanks to Lemma 14, we can drop the congruence relation $g \in a\mathbb{Z} + b$ and assume that

$$G = \{g \in \mathbb{Z} \mid \alpha \,\square_1\, g \,\square_2\, \beta\},$$

and we are reduced to constructing a definable Lipschitz retraction $r : K \to \overline{C}$. Depending on the values of the $\square_i$ (namely $<$ or no condition), this leaves four cases.

(1) $C = K^\times$. Then, $\overline{C} = K$, and we take $r = id$.

(2) $C = \{x \in K \mid 0 < |x| \leqslant s\}$, so $\overline{C} = \{x \in K \mid |x| \leqslant s\}$. Then we take

$$\begin{aligned} r : K &\to \overline{C} \\ x &\mapsto \begin{cases} x & \text{if } |x| \leqslant s \\ 0 & \text{if } |x| > s. \end{cases} \end{aligned}$$

(3) $C = \{x \in K \mid s \leqslant |x| \leqslant s'\} = \overline{C}$. Let us pick $x_0 \in K$ such that $|x_0| = s$. Then we consider

$$\begin{aligned} r : K &\to \overline{C} \\ x &\mapsto \begin{cases} x & \text{if } |x| \in C \\ x_0 & \text{otherwise.} \end{cases} \end{aligned}$$

(4) $C = \{x \in K \mid s \leqslant |x|\} = \overline{C}$. Let us pick $x_0 \in K$ such that $|x_0| = s$. Then we consider

$$\begin{aligned} r : K &\to \overline{C} \\ x &\mapsto \begin{cases} x & \text{if } |x| \in C \\ x_0 & \text{otherwise.} \end{cases} \end{aligned}$$

One easily checks the required conditions in each of these cases. This proves Theorem 2 when $n = 1$, and implies Theorem 1 for $n = 1$ as explained on page 41.

3.2. Proofs for $n > 1$

We now prove by induction the properties (Mon_n), (Ret_n) and (Ext_n). The basis of the induction has been obtained in § 3.1.

So let us fix an integer $n > 1$, and let us assume that (Ret_{n-1}), (Mon_{n-1}), and (Ext_{n-1}) hold. We will prove the statements in the following order: (Mon_n), (Ret_n), and (Ext_n).

Proof of Proposition 10 (Mon_n)**.** Let us apply Theorem 16 to the functions f_i. So we can assume that $X \subset K^n$ is a cell

$$X = \{(y, t) \in Y \times K \mid |\alpha(y)| \,\square_1\, |t - c(y)| \,\square_2\, |\beta(y)| \text{ and } t - c(y) \in \lambda Q_{m',n'}\}$$

over K^{n-1} with a base Y and a Lipschitz centre $c : Y \to K$, and that for each $i \in \{1, \ldots, m\}$ there is some $a_i \in \mathbb{Q}$ and $h_i : Y \to K$ some definable function such that, for each $y \in Y, t \in K$ with $(y, t) \in X$, one has that $f_i(y, t)$ is of a prepared form, coming from (11). By induction hypothesis (Ext_{n-1}), we can extend $c : Y \to K$ to a definable Lipschitz function $\overline{c} : K^{n-1} \to K$. Let us consider the definable isometry $\varphi : K^n \to K^n$ defined by $(y, t) \in K^{n-1} \times K \mapsto (y, t + \overline{c}(y))$. Then, considering $\varphi^{-1}(X)$, we can assume that $c \equiv 0$. Hence, for any $(y, t) \in X$ one has

$$f_i(y, t) = h_i(y) + a_i \operatorname{ord}(t/\lambda),$$

for some rational numbers a_i and some definable functions h_i. By induction hypothesis (Mon_{n-1}) applied to the functions $h_i, \operatorname{ord}\alpha, \operatorname{ord}\beta$, and up to cutting X in finitely many pieces, we can assume that the following hold.

– $Y \subset K^{n-1}$ is a centred cell.

– X is the cell over Y defined by

$$X = \{(y, t) \in Y \times K \mid |\alpha(y)| \,\square_1\, |t| \,\square_2\, |\beta(y)| \text{ and } t \in \lambda Q_{m',n'}\}.$$

– For each index i and each $(y, t) \in X$, one has

$$f_i(y, t) = h_i(y) + a_i \operatorname{ord}(t/\lambda).$$

– The functions h_i, $\operatorname{ord}\alpha$, and $\operatorname{ord}\beta$ are monomial functions on Y.

Then, according to Lemma 9, X is a centred cell, and each f_i is a monomial function. □

Proof of Theorem 2 (Ret_n). *Step 1.* Let us show first that, when $\dim(X) < n$, there exists a definable Lipschitz retraction $r : K^n \to \overline{X}$.

By Lemma 6, we can cut X in definable pieces. Thanks to Proposition 10, we can assume that there exists a definable set $X' \subset K^{n-1}$ such that $X = X' \times \{0\}$. So we can apply our induction hypothesis (Ret_{n-1}) to X' and conclude.

Step 2. According to (Mon_n), we can assume that $X = \overline{C}$, where C is a centred cell, let us say of the form $C = C' \times \{(0, \ldots, 0)\}$ for some open centred cell $C' \subset K^{n'}$.

If $n' < n$, we are reduced to Step 1.

If $n = n'$, then C is an open centred cell. So, according to Corollary 13, we can assume that

$$X = \mathrm{ord}^{-1}(G)$$

for some Presburger set $G \subset \mathbb{Z}^n$.

Step 3. For each $i \in \{1, \ldots, n\}$, let

$$G_i := \{(g_1, \ldots, g_n) \in G \mid g_i \geqslant g_j \text{ for all } j = 1, \ldots, n\}.$$

Then

$$G = \bigcup_{i=1}^{n} G_i$$

and

$$X = \bigcup_{i=1}^{n} \mathrm{ord}^{-1}(G_i).$$

By Lemma 6, we are reduced to proving the result for each $X = \mathrm{ord}^{-1}(G_i)$. So, replacing G by the G_i, and up to a permutation of the coordinates, we can assume that G satisfies the following condition:

$$g_n \geqslant g_j \text{ for all } j = 1, \ldots, n \text{ and all } (g_1, \ldots, g_n) \in G. \tag{12}$$

Using the results mentioned in § 2.2, one can assume that G still satisfies condition (12) and is moreover of the form

$$G = \{(u, v) \in \mathbb{Z}^n \mid u \in U,\ v \in a\mathbb{Z} + b \text{ and } \alpha(u) \square_1 v \square_2 \beta(u)\},$$

where $U \subset \mathbb{Z}^{n-1}$ is a Presburger set, $\alpha, \beta : U \to \mathbb{Z}$ are definable functions, and each $\square_i$ is $<$ or no condition. Then, using Lemma 14, we can remove the congruence condition $v \in a\mathbb{Z} + b$.

Step 4. We are reduced to the following assertion. We take as given the following data.

- A Presburger set $U \subset \mathbb{Z}^{n-1}$.
- Definable functions $\alpha, \beta : U \to \mathbb{Z}$ such that for any $u \in U$ there exists some $v \in \mathbb{Z}$ with $\alpha(u) < v < \beta(u)$ and such that for any $(u_1, \ldots, u_{n-1}) \in U$, we have, by (12),

$$\alpha(u) \geqslant \max_{i=1,\ldots,n-1} u_i.$$

– A Presburger set

$$G = \{(u, v) \in \mathbb{Z}^n \mid u \in U,\ v \in \mathbb{Z} \text{ and } \alpha(u) \,\square_1\, v \,\square_2\, \beta(u)\}.$$

We then have to show that there exists a definable Lipschitz retraction

$$r : K^n \to \overline{\mathrm{ord}^{-1}(G)}.$$

Step 4.1. We first construct a definable Lipschitz map

$$r : \mathrm{ord}^{-1}(U \times \mathbb{Z}) \to \overline{\mathrm{ord}^{-1}(G)} \tag{13}$$

such that $r(x) = x$ for all $x \in \mathrm{ord}^{-1}(G)$.

If both $\square_1$ and $\square_2$ are no condition, then $G = U \times \mathbb{Z}$, and the map

$$r : x \in \mathrm{ord}^{-1}(U \times \mathbb{Z}) \mapsto x$$

works. So we can assume that one of $\square_1$ or $\square_2$ is $<$.

Recall that the theory of our one-sorted structure K has definable Skolem functions (see [12, Theorem 3.2] in the semi-algebraic case and [11] in the subanalytic case), but this also follows directly from the above cell decomposition result in any of our settings. If $\square_2$ is $<$, we let

$$f_- : \mathrm{ord}^{-1}(U) \to K^\times$$

be a definable Skolem function satisfying

$$\mathrm{ord}(f_-(u)) = \beta(u) - 1 \text{ for all } u \in \mathrm{ord}^{-1}(U).$$

Similarly, if $\square_1$ is $<$, we let

$$f_+ : \mathrm{ord}^{-1}(U) \to K^\times$$

be a definable Skolem function which satisfies

$$\mathrm{ord}(f_+(u)) = \alpha(u) + 1 \text{ for all } u \in \mathrm{ord}^{-1}(U).$$

Let us write H for the union of the graph of f_- (when f_- is defined) and the graph of f_+ (when f_+ is defined), namely,

$$H = \{(u, f_+(u)) \mid u \in \mathrm{ord}^{-1}(U)\} \cup \{(u, f_-(u)) \mid u \in \mathrm{ord}^{-1}(U)\} \quad \text{when } \square_1 \text{ and } \square_2 \text{ are } <$$
$$H = \{(u, f_+(u)) \mid u \in \mathrm{ord}^{-1}(U)\} \quad \text{when } \square_1 \text{ is } < \text{ and } \square_2 \text{ is no condition}$$
$$H = \{(u, f_-(u)) \mid u \in \mathrm{ord}^{-1}(U)\} \quad \text{when } \square_1 \text{ is no condition and } \square_2 \text{ is } < .$$

By construction, $H \subset \mathrm{ord}^{-1}(G)$, so $\overline{H} \subset \overline{\mathrm{ord}^{-1}(G)}$ and $\dim(H) = \dim(\overline{H}) = n - 1$. According to step 1, we can find a definable Lipschitz retraction

$$s : K^n \to \overline{H}.$$

We now define our Lipschitz map as desired for (13) like this:

$$\begin{aligned} r : \mathrm{ord}^{-1}(U \times \mathbb{Z}) &\to \overline{\mathrm{ord}^{-1}(G)} \\ z &\mapsto \begin{cases} z & \text{if } z \in \mathrm{ord}^{-1}(G) \\ s(z) & \text{if } z \notin \mathrm{ord}^{-1}(G). \end{cases} \end{aligned}$$

The key remaining work is to prove that r is Lipschitz. Let us consider $z=(y,x)$ and $z'=(y',x')\in \mathrm{ord}^{-1}(U\times\mathbb{Z})$, and let us prove that $|r(z)-r(z')|\leqslant |z-z'|$.

If $z=(y,x)$ and $z'=(y',x')$ belong simultaneously to $\mathrm{ord}^{-1}(G)$, or to its complement $\mathrm{ord}^{-1}(G)^c$, then $|r(z)-r(z')|\leqslant |z-z'|$, because the identity map and the function s are Lipschitz. So we will assume that $\mathrm{ord}(z)\in G$ and $\mathrm{ord}(z')\notin G$.

Since $|(y,x)-(y',x')|=\max(|y-y'|,|x-x'|)$, it follows that $|(y,x)-(y',x)|$ and $|(y',x)-(y',x')|$ are less than or equal to $|(y,x)-(y',x')|$. So we can assume that $x=x'$ or $y=y'$.

Case 1. $x=x'$. So $z=(y,x)\in \mathrm{ord}^{-1}(G)$ and $z'=(y',x)\notin \mathrm{ord}^{-1}(G)$.

For simplicity of notation, let us assume that $|y_1|\neq |y_1'|$. In particular, $|z-z'|\geqslant |y_1|$. If $\square_2$ is $<$, let $z''=(y,f_-(y))\in \mathrm{ord}^{-1}(G)$ (otherwise $\square_2$ is no condition, $\square_1$ is $<$, and then we set $z''=(y,f_+(y))\in \mathrm{ord}^{-1}(G)$ and the undermentioned reasoning can also been applied). Since $z,z''\in \mathrm{ord}^{-1}(G)$, according to condition (12), $|f_-(y)|\leqslant |y_1|$ and $|x|\leqslant |y_1|$. So

$$|z-z''|=|x-f_-(y)|\leqslant |y_1|\leqslant |z-z'|.$$

Likewise,

$$|z''-z'|=|(y,f_-(y))-(y',x)|=\max(|y-y'|,|x-f_-(y)|)\leqslant \max(|y-y'|,|y_1|)\leqslant |z-z'|.$$

So it suffices to show that $|r(z)-r(z'')|\leqslant |z-z''|$ and that $|r(z'')-r(z')|\leqslant |z''-z'|$. Since $z,z''\in \mathrm{ord}^{-1}(G)$, $r(z)=z$ and $r(z'')=z''$, and this implies the first inequality. Finally, $z''\in H$ and $z'\notin \mathrm{ord}^{-1}(G)$, so, by definition of r, $r(z'')=s(z'')$ and $r(z')=s(z')$. Since s is Lipschitz, $|r(z'')-r(z')|=|s(z'')-s(z')|\leqslant |z''-z'|$.

Case 2. $y=y'$. So $z=(y,x)\in \mathrm{ord}^{-1}(G)$ and $z'=(y,x')\notin \mathrm{ord}^{-1}(G)$.

Case 2.1. $\square_2$ is the condition $<$, and $\beta(\mathrm{ord}\, y)\leqslant \mathrm{ord}\, x'$, where $\mathrm{ord}\, y$ stands for $(\mathrm{ord}\, y_1,\dots,\mathrm{ord}\, y_{n-1})$.

So $|z-z'|=|x-x'|=|x|$. Let us consider $z''=(y,f_-(y))$. Then

$$|x'|<|f_-(y)|\leqslant |x|.$$

So

$$|z'-z''|=|f_-(y)|\leqslant |x|\leqslant |z-z'|.$$

Likewise,

$$|z''-z|=|f_-(y)-x|\leqslant |x|\leqslant |z-z'|.$$

So it suffices to show that $|r(z'')-r(z')|\leqslant |z''-z'|$ and that $|r(z)-r(z'')|\leqslant |z-z''|$.

Let us show the first inequality. Since $z''\in H$ and $H\subset \mathrm{ord}^{-1}(G)$ by construction, we get that $r(z'')=z''$. Since s is a retraction on $\overline{H}$, we also obtain that $s(z'')=z''$. Hence, $r(z'')=s(z'')$. Since $\mathrm{ord}(z')\notin G$, by definition of r we get that $r(z')=s(z')$. The inequality follows, since s is Lipschitz.

To prove the second inequality, we remark that $z,z''\in \mathrm{ord}^{-1}(G)$. So, by definition of r, we get that $r(z)=z$ and $r(z'')=z''$, which implies the second inequality.

Case 2.2. $\square_1$ is the condition $<$, and $\alpha(\mathrm{ord}\, y)\geqslant \mathrm{ord}\, x'$, where $\mathrm{ord}\, y$ stands for $(\mathrm{ord}\, y_1,\dots,\mathrm{ord}\, y_{n-1})$.

So $|z - z'| = |x - x'| = |x'|$. Let us consider $z'' = (y, f_+(y))$. Then

$$|f_+(y)| < |x'|.$$

So

$$|z' - z''| = |x' - f_+(y)| = |x'| = |z - z'|.$$

Likewise,

$$|z'' - z| = |f_+(y) - x| \leqslant |f_+(y)| < |x'| = |z - z'|.$$

So it suffices to show that $|r(z'') - r(z')| \leqslant |z'' - z'|$, and that $|r(z) - r(z'')| \leqslant |z - z''|$. This is shown as at the end of Case 2.1.

Step 4.2. By induction hypothesis (Ret_{n-1}), there is a definable Lipschitz retraction

$$\sigma : K^{n-1} \to \overline{\mathrm{ord}^{-1}(U)}.$$

It induces a definable Lipschitz retraction

$$\begin{aligned} \tau : K^n &\to \overline{\mathrm{ord}^{-1}(U \times \mathbb{Z})} = \overline{\mathrm{ord}^{-1}(U)} \times K \\ (x_1, \ldots, x_n) &\mapsto (\sigma(x_1, \ldots, x_{n-1}), x_n). \end{aligned}$$

Using Lemma 4, we can extend r of (13) as constructed in Step 4.1 in a Lipschitz way to

$$\overline{r} : \overline{\mathrm{ord}^{-1}(U \times \mathbb{Z})} \to \overline{\mathrm{ord}^{-1}(G)},$$

which is a definable Lipschitz retraction. Then $\overline{r} \circ \tau : K^n \to \overline{\mathrm{ord}^{-1}(V)}$ is a definable Lipschitz retraction, as desired for property (Ret_n). □

Proof of Theorem 1 (Ext_n)**.** Property (Ext_n) follows from Property (Ret_n) similarly to how Theorem 1 is obtained from Theorem 2 just below Theorem 2. □

To end the paper, we give family versions of Theorems 1 and 2.

Theorem 19. *Let $Y \subset K^m$ and $X \subset Y \times K^n$ be definable sets. Let $f : X \to K^\ell$ be a definable function, and suppose that, for each $y \in Y$, the function f_y sending z to $f(y, z)$ is a Lipschitz function on $X_y := \{z \mid (y, z) \in X\}$. There exists a definable function $\widetilde{f} : Y \times K^n \to K^\ell$ such that, for each $y \in Y$,*

$$\widetilde{f}_y : z \mapsto \widetilde{f}(y, z)$$

is a Lipschitz extension of f_y.

Theorem 20. *Let $Y \subset K^m$ and $X \subset Y \times K^n$ be definable sets. There exists a definable function $r : Y \times K^n \to Y \times K^n$ such that, for each $y \in Y$, r_y is a definable Lipschitz retraction from K^n to $\overline{X_y}$, with $X_y = \{z \mid (y, z) \in X\}$ and $\overline{X_y}$ its topological closure.*

These two theorems follow by noting that the proofs of Theorems 1 and 2 and of Proposition 10 work completely uniformly in $y \in Y$, and that definable Skolem functions can be used to pick the elements like x_0 in § 3.1 for the case $n = 1$. In these last theorems, Y is not allowed to be a definable subset of $\mathbb{Z}^m \times K^{m'}$, since that would render impossible the above usage of definable Skolem functions, which indeed only exist in the one-sorted structure K.

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