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Weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities and nonlocal free energies

Helmut Abels[1] | **Yutaka Terasawa**[2]

[1]Fakultät für Mathematik, Universität Regensburg, Regensburg, Germany
[2]Graduate School of Mathematics, Nagoya University, Nagoya, Japan

Correspondence
Helmut Abels, Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany.
Email: helmut.abels@mathematik.uni-regensburg.de

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We consider a diffuse interface model for the flow of two viscous incompressible Newtonian fluids with different densities in a bounded domain in two and three space dimensions and prove existence of weak solutions for it. In contrast to earlier contributions, we study a model with a singular nonlocal free energy, which controls the $H^{\alpha/2}$-norm of the volume fraction. We show existence of weak solutions for large times with the aid of an implicit time discretization.

KEYWORDS
Cahn-Hilliard equation, diffuse interface model, mixtures of viscous fluids, Navier-Stokes equation, nonlocal operators, two-phase flow

MSC CLASSIFICATION
35Q30; 35Q35; 76D03; 76D05; 76D27; 76D45

1 | INTRODUCTION

In this contribution, we consider a two-phase flow for incompressible fluids of different densities and different viscosities. The two fluids are assumed to be macroscopically immiscible and to be miscible in a thin interface region; ie, we consider a diffuse interface model (also called phase field model) for the two-phase flow. In contrast to sharp interface models, where the interface between the two fluids is a sufficiently smooth hypersurface, diffuse interface model can describe topological changes due to pinch off and droplet collision.

There are several diffuse interface models for such two-phase flows. Firstly, in the case of matched densities, ie, the densities of both fluids are assumed to be identical, there is a well-known model H, cf Hohenberg and Halperin or Gurtin et al.[1,2] In the case that the fluid densities do not coincide, there are different models. On one hand, Lowengrub and Truskinovsky[3] derived a quasi-incompressible model, where the mean velocity field of the mixture is in general not divergence free. On the other hand, Ding et al[4] proposed a model with a divergence free mean fluid velocities. But this model is not known to be thermodynamically consistent. In Abels et al,[5] a thermodynamically consistent diffuse interface model for two-phase flow with different densities and a divergence free mean velocity field was derived, which we call AGG model for short. The existence of weak solutions of the AGG model was shown in Abels et al.[6] For analytic result in the case of matched densities, ie, the model H, we refer to Abels[7] and Giorgini et al[8] and the reference given there. Existence of weak and strong solutions for a slight modification of the model by Lowengrub and Truskinovsky was proven in Abels.[9,10]

Concerning the Cahn-Hilliard equation, Giacomin and Lebowitz[11,12] observed that a physically more rigorous derivation leads to a nonlocal equation, which we call a nonlocal Cahn-Hilliard equation. There are two types of nonlocal

Cahn-Hilliard equations. One is the equation where the second order differential operator in the equation for the chemical potential is replaced by a convolution operator with a sufficiently smooth even function. We call it a nonlocal Cahn-Hilliard equation with a regular kernel in the following. The other is one where the second order differential operator is replaced by a regional fractional Laplacian. We call it a nonlocal Cahn-Hilliard equation with a singular kernel, since the regional fractional Laplacian is defined by using singular kernel. The nonlocal Cahn-Hilliard equation with a regular kernel was analyzed in previous works.[12-16] On the other hand, the nonlocal Cahn-Hilliard equation with a singular kernel was first analyzed in Abels et al,[17] where they proved the existence and uniqueness of a weak solution of the nonlocal Cahn-Hilliard equation, its regularity properties, and the existence of a (connected) global attractor.

Concerning the nonlocal model H with a regular kernel, where the convective Cahn-Hilliard equation is replaced by the convective nonlocal Cahn-Hilliard equation with a regular kernel, first studies were done in references[18-20]; see also Frigeri[21] and the references there for more recent results. More recently, the nonlocal AGG model with a regular kernel, where the convective Cahn-Hilliard equation is replaced by the convective nonlocal Cahn-Hilliard equation with a regular kernel, was studied by Frigeri,[22] and he showed the existence of a weak solution for that model. The method of the proof in Frigeri[22] is based on the Faedo-Galerkin method of a suitably mollified system and the method of passing to the limit with two parameters tending to zero. The method is different from Abels,[6] which is based on implicit time discretization and a Leray-Schauder fixed point argument.

In this contribution, we consider a nonlocal AGG model with a singular kernel, where a convective Cahn-Hilliard equation in the AGG model is replaced by a convective nonlocal Cahn-Hilliard equation with a singular kernel. Our aim is to prove the existence of a weak solution of such a system.

In this contribution, we consider existence of weak solutions of the following system, which couples a nonhomogeneous Navier-Stokes equation system with a nonlocal Cahn-Hilliard equation:

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes(\rho\mathbf{v}+\widetilde{\mathbf{J}})) - \operatorname{div}(2\eta(\varphi)D\mathbf{v}) + \nabla p = \mu\nabla\varphi \qquad \text{in } Q, \tag{1}$$

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } Q, \tag{2}$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}(m(\varphi)\nabla\mu) \qquad \text{in } Q, \tag{3}$$

$$\mu = \Psi'(\varphi) + \mathcal{L}\varphi \qquad \text{in } Q, \tag{4}$$

where $\rho = \rho(\varphi) := \frac{\tilde{\rho}_1+\tilde{\rho}_2}{2} + \frac{\tilde{\rho}_2-\tilde{\rho}_1}{2}\varphi$, $\widetilde{\mathbf{J}} = -\frac{\tilde{\rho}_2-\tilde{\rho}_1}{2}m(\varphi)\nabla\mu$, $Q = \Omega\times(0,\infty)$. We assume that $\Omega\subset\mathbb{R}^d$, $d = 2,3$, is a bounded domain with C^2-boundary. Here and in the following $\mathbf{v}$, p, and ρ are the (mean) velocity, the pressure, and the density of the mixture of the two fluids, respectively. Furthermore, $\tilde{\rho}_j, j = 1,2$, are the specific densities of the unmixed fluids, φ is the difference of the volume fractions of the two fluids, and μ is the chemical potential related to φ. Moreover, $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T)$, $\eta(\varphi) > 0$ is the viscosity of the fluid mixture, and $m(\varphi) > 0$ is a mobility coefficient. The term $\widetilde{\mathbf{J}}$ describes the mass flux; ie, we have

$$\partial_t\rho = -\operatorname{div}\widetilde{\mathbf{J}}.$$

It is important to have the term with $\widetilde{\mathbf{J}}$ in (1) in order to obtain a thermodynamically consistent model, cf Abels et al[5] for the case with a local free energy.

Finally, $\mathcal{L}$ is defined as

$$\begin{aligned}\mathcal{L}u(x) &= \text{p.v.}\int_\Omega (u(x)-u(y))k(x,y,x-y)dy \\ &= \lim_{\varepsilon\to 0}\int_{\Omega\setminus B_\varepsilon(x)} (u(x)-u(y))k(x,y,x-y)dy \qquad \text{for } x\in\Omega\end{aligned} \tag{5}$$

for suitable $u : \Omega\to\mathbb{R}$. Here, the kernel $k : \mathbb{R}^d\times\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})\to\mathbb{R}$ is assumed to be $(d+2)$-times continuously differentiable and to satisfy the conditions

$$k(x,y,z) = k(y,x,-z), \tag{6}$$

$$|\partial_x^\beta\partial_y^\gamma\partial_z^\delta k(x,y,z)| \leqslant C_{\beta,\gamma,\delta}|z|^{-d-\alpha-|\delta|}\ , \tag{7}$$

$$c_0|z|^{-d-\alpha} \leqslant k(x,y,z) \leqslant C_0|z|^{-d-\alpha}\ . \tag{8}$$

for all $x,y,z\in\mathbb{R}^d$, $z\neq 0$ and $\beta,\gamma,\delta\in\mathbb{N}_0^d$ with $|\beta|+|\gamma|+|\delta| \leqslant d+2$ and some constants $C_{\beta,\gamma,\delta}, c_0, C_0 > 0$. Here, α is the order of the operator, cf Abels and Kassmann.[23] We restrict ourselves to the case $\alpha\in(1,2)$. If $\omega\in C_b^{d+2}(\mathbb{R}^d)$, then $k(x,y,z) = \omega(x,y)|z|^{-d-\alpha}$ is an example of a kernel satisfying the previous assumptions.

We add to our system the boundary and initial conditions

$$\mathbf{v}|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{9}$$

$$\partial_{\mathbf{n}}\mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{10}$$

$$(\mathbf{v},\varphi)|_{t=0} = (\mathbf{v}_0,\varphi_0) \qquad \text{in } \Omega. \tag{11}$$

Here, $\partial_{\mathbf{n}} = \mathbf{n}\cdot\nabla$ and $\mathbf{n}$ denotes the exterior normal at $\partial\Omega$. We note that (9) is the usual no-slip boundary condition for the velocity field and $\partial_{\mathbf{n}}\mu|_{\partial\Omega} = 0$ describes that there is no mass flux of the fluid components through the boundary. Furthermore, we complete the system above by an additional boundary condition for φ, which will be part of the weak formulation, cf Definition 1. If φ is smooth enough (eg, $\varphi(t) \in C^{1,\beta}(\overline{\Omega})$ for every $t \geq 0$) and k fulfills suitable assumptions, then

$$\mathbf{n}_{x_0}\cdot\nabla\varphi(x_0) = 0 \qquad \text{for all } x_0 \in \partial\Omega, \tag{12}$$

where $\mathbf{n}_{x_0}$ depends on the interaction kernel k, cf Abels et al,[17, theorem 6.1] and $x_0 \in \partial\Omega$.

The total energy of the system at time $t \geq 0$ is given by

$$E_{\text{tot}}(\varphi,\mathbf{v}) = E_{\text{kin}}(\varphi,\mathbf{v}) + E_{\text{free}}(\varphi), \tag{13}$$

where

$$E_{\text{kin}}(\varphi,\mathbf{v}) = \int_\Omega \rho\frac{|\mathbf{v}|^2}{2}\,dx, \qquad E_{\text{free}}(\varphi) = \int_\Omega \Psi(\varphi)\,dx + \frac{1}{2}\mathcal{E}(\varphi,\varphi)$$

are the kinetic energy and the free energy of the mixture, respectively, and

$$\mathcal{E}(u,v) = \int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))k(x,y,x-y)\,dx\,dy \tag{14}$$

for all $u,v \in H^{\frac{\alpha}{2}}(\Omega)$ is the natural bilinear form associated to $\mathcal{L}$, which will also be used to formulate the natural boundary condition for φ weakly. Every sufficiently smooth solution of the system above satisfies the energy identity

$$\frac{d}{dt}E_{\text{tot}}(\varphi,\mathbf{v}) = -\int_\Omega 2\eta(\varphi)|D\mathbf{v}|^2\,dx - \int_\Omega m(\varphi)|\nabla\mu|^2\,dx$$

for all $t \geq 0$. This can be shown by testing (1) with $\mathbf{v}$, (3) with μ, and (4) with $\partial_t\varphi$, where the product of $\mathcal{L}\varphi$ and $\partial_t\varphi$ coincides with

$$\mathcal{E}(\varphi(t),\partial_t\varphi(t))$$

under the same natural boundary condition for $\varphi(t)$ as before, cf (12).

We consider a class of singular free energies, which will be specified below and which includes the homogeneous free energy of the so-called regular solution models used by Cahn and Hilliard[24]:

$$\Psi(\varphi) = \frac{\vartheta}{2}\left((1+\varphi)\ln(1+\varphi) + (1-\varphi)\ln(1-\varphi)\right) - \frac{\vartheta_c}{2}\varphi^2, \quad \varphi \in [-1,1], \tag{15}$$

where $0 < \vartheta < \vartheta_c$. This choice of the free energies ensures that $\varphi(x,t) \in [-1,1]$ almost everywhere. In order to deal with these terms, we apply techniques, which were developed in Abels and Wilke[25] and extended to the present nonlocal Cahn-Hilliard equation in Abels et al.[17]

Our proof of existence of a weak solution of (1) to (4) together with a suitable initial and boundary condition follows closely the proof of the main result of Abels et al.[6] The following are the main differences and difficulties of our paper compared with Abels et al.[6] Since we do not expect H^1-regularity in space for the volume fraction φ of a weak solution of our system, we should eliminate $\nabla\varphi$ from our weak formulation taking into account the incompressibility of $\mathbf{v}$. Implicit time discretization has to be constructed carefully, using a suitable mollification of φ and an addition of a small Laplacian term to the chemical potential equation taking into account of the lack of H^1-regularity in space of φ. While the arguments for the weak convergence of temporal interpolants of weak solutions of the time-discrete problem are similar to Abels et al,[6] the function space used for the order parameter has less regularity in space since the nonlocal operator of order less than 2 is involved in the equation for the chemical potential. For the convergence of the singular term $\Psi'(\varphi)$, we employ the argument in Abels et al.[17] The only difference is that we work in space-time domains directly. For the validity of the energy inequality, additional arguments using the equation of chemical potential and the fact that weak convergence together with norm convergence in uniformly convex Banach spaces imply strong convergence are needed.

The structure of the contribution is as follows: In Section 2, we present some preliminaries, fix notations, and collect the needed results on nonlocal operator. In Section 3, we define weak solutions of our system and state our main result concerning the existence of weak solutions. In Section 4, we define an implicit time discretization of our system and show the existence of weak solutions of an associated time-discrete problem using the Leray-Schauder theorem. In Section 5, we obtain compactness in time of temporal interpolants of the weak solutions of time-discrete problem and obtain weak solutions of our system as weak limits of a suitable subsequence.

2 | PRELIMINARIES

As usual, $a \otimes b = (a_i b_j)_{i,j=1}^d$ for $a, b \in \mathbb{R}^d$ and $A_{sym} = \frac{1}{2}(A + A^T)$ for $A \in \mathbb{R}^{d\times d}$. Moreover,

$$\langle f, g\rangle \equiv \langle f, g\rangle_{X',X} = f(g), \qquad f \in X', g \in X$$

denotes the duality product, where X is a Banach space and X' is its dual. We write $X \hookrightarrow\hookrightarrow Y$ if X is compactly embedded into Y. For a Hilbert space H, its inner product is denoted by $(\cdot,\cdot)_H$.

Let $M \subseteq \mathbb{R}^d$ be measurable. As usual $L^q(M)$, $1 \le q \le \infty$, denotes the Lebesgue space, $\|.\|_q$ its norm and $(.\,,.)_M = (.\,,.)_{L^2(M)}$ its inner product if $q = 2$. Furthermore, $L^q(M;X)$ denotes the set of all $f : M \to X$ that are strongly measurable and q-integrable functions/essentially bounded functions. Here, X is a Banach space. If $M = (a, b)$, we denote these spaces for simplicity by $L^q(a, b; X)$ and $L^q(a, b)$. Recall that $f : [0,\infty) \to X$ belongs $L^q_{\text{loc}}([0,\infty); X)$ if and only if $f \in L^q(0, T; X)$ for every $T > 0$. Furthermore, $L^q_{\text{uloc}}([0,\infty); X)$ is the *uniformly local* variant of $L^q(0,\infty; X)$ consisting of all strongly measurable $f : [0,\infty) \to X$ such that

$$\|f\|_{L^q_{\text{uloc}}([0,\infty);X)} = \sup_{t\ge 0} \|f\|_{L^q(t,t+1;X)} < \infty.$$

If $T < \infty$, we define $L^q_{\text{uloc}}([0, T); X) := L^q(0, T; X)$.

For a domain $\Omega \subset \mathbb{R}^d$, $m \in \mathbb{N}_0$, $1 \le q \le \infty$, the standard Sobolev space is denoted by $W^m_q(\Omega)$. $W^m_{q,0}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^m_q(\Omega)$, $W^{-m}_q(\Omega) = (W^m_{q',0}(\Omega))'$, and $W^{-m}_{q,0}(\Omega) = (W^m_{q'}(\Omega))'$. $H^s(\Omega)$ denotes the L^2-Bessel potential of order $s \ge 0$.

Let $f_\Omega = \frac{1}{|\Omega|}\int_\Omega f(x)\,dx$ denote the mean value of $f \in L^1(\Omega)$. For $m \in \mathbb{R}$, we define

$$L^q_{(m)}(\Omega) := \{f \in L^q(\Omega) : f_\Omega = m\}, \qquad 1 \le q \le \infty.$$

Then the orthogonal projection onto $L^2_{(0)}(\Omega)$ is given by

$$P_0 f := f - f_\Omega = f - \frac{1}{|\Omega|}\int_\Omega f(x)\,dx \qquad \text{for all } f \in L^2(\Omega).$$

For the following, we denote

$$H^1_{(0)} \equiv H^1_{(0)}(\Omega) = H^1(\Omega) \cap L^2_{(0)}(\Omega), \qquad (c, d)_{H^1_{(0)}(\Omega)} := (\nabla c, \nabla d)_{L^2(\Omega)}.$$

Because of Poincaré's inequality, $H^1_{(0)}(\Omega)$ is a Hilbert space. More generally, we define for $s \ge 0$

$$\begin{aligned} H^s_{(0)} \equiv H^s_{(0)}(\Omega) = H^s(\Omega) \cap L^2_{(0)}(\Omega), \qquad & H^s_{(0)}(\Omega) = (H^s_{(0)}(\Omega))', \\ H^s_0(\Omega) = (H^s(\Omega))', \qquad & H^s(\Omega) = (H^s_0(\Omega))'. \end{aligned}$$

Finally, $f \in H^s_{\text{loc}}(\Omega)$ if and only if $f|_{\Omega'} \in H^s(\Omega')$ for every open and bounded subset Ω' with $\overline{\Omega'} \subset \Omega$.

We denote by $L^2_\sigma(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^2(\Omega)^d$, where $C^\infty_{0,\sigma}(\Omega)$ is the set of all divergence free vector fields in $C_0^\infty(\Omega)^d$. The corresponding Helmholtz projection, ie, the L^2-orthogonal projection onto $L^2_\sigma(\Omega)$, is denoted by P_σ, cf, eg, Sohr.[26]

Let $I = [0, T]$ with $0 < T < \infty$ or $I = [0, \infty)$ if $T = \infty$ and let X is a Banach space. The Banach space of all bounded and continuous $f : I \to X$ is denoted by $BC(I; X)$. It is equipped with the supremum norm. Moreover, $BUC(I; X)$ is defined as the subspace of all bounded and uniformly continuous functions. Furthermore, $BC_w(I; X)$ is the set of all bounded and weakly continuous $f : I \to X$. $C_0^\infty(0, T; X)$ denotes the vector space of all smooth functions $f : (0, T) \to X$ with $\operatorname{supp} f \subset\subset (0, T)$. By definition $f \in W^1_p(0, T; X)$, $1 \leq p < \infty$, if and only if $f, \frac{df}{dt} \in L^p(0, T; X)$. Furthermore, $W^1_{p,\mathrm{uloc}}([0, \infty); X)$ is defined by replacing $L^p(0, T; X)$ by $L^p_{\mathrm{uloc}}([0, \infty); X)$, and we set $H^1(0, T; X) = W^1_2(0, T; X)$ and $H^1_{\mathrm{uloc}}([0, \infty); X) := W^1_{2,\mathrm{uloc}}([0, \infty); X)$. Finally, we note the following:

Lemma 1. *Let X, Y be two Banach spaces such that $Y \hookrightarrow X$ and $X' \hookrightarrow Y'$ densely. Then $L^\infty(I; Y) \cap BUC(I; X) \hookrightarrow BC_w(I; Y)$.*

For a proof, see, eg, Abels.[9]

2.1 | Properties of the nonlocal elliptic operator $\mathcal{L}$

In the following, let $\mathcal{E}$ be defined as in (14). Assumptions (6) to (8) yield that there are positive constants c and C such that

$$c\,\|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)} \leqslant |u_\Omega|^2 + \mathcal{E}(u, u) \leqslant C\,\|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)} \qquad \text{for all } u \in H^{[\frac{\alpha}{2}]}(\Omega).$$

This implies that the following norm equivalences hold:

$$\mathcal{E}(u, u) \sim \|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)} \qquad \text{for all } u \in H^{\frac{\alpha}{2}}_{(0)}(\Omega), \tag{16}$$

$$\mathcal{E}(u, u) + |u_\Omega|^2 \sim \|u\|^2_{H^{\frac{\alpha}{2}}(\Omega)} \qquad \text{for all } u \in H^{\frac{\alpha}{2}}(\Omega), \tag{17}$$

cf Abels et al.[17, lemma 2.4 and corollary 2.5]

In the following, we will use a variational extension of the nonlocal linear operator $\mathcal{L}$ (see (1)) by defining $\mathcal{L} : H^{\frac{\alpha}{2}}(\Omega) \to H_0^{\frac{\alpha}{2}}(\Omega)$ as

$$\langle \mathcal{L}u, \varphi\rangle_{H_0^{-\frac{\alpha}{2}}, H^{\frac{\alpha}{2}}} = \mathcal{E}(u, \varphi) \quad \text{for all } \ \varphi \in H^{\frac{\alpha}{2}}(\Omega).$$

This implies

$$\langle \mathcal{L}u, 1\rangle = \mathcal{E}(u, 1) = 0.$$

We note that $\mathcal{L}$ agrees with (1) as soon as $u \in H^\alpha_{\mathrm{loc}}(\Omega) \cap H^{\frac{\alpha}{2}}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, cf Abels and Kassmann.[23, lemma 4.2] But this weak formulation also includes a natural boundary condition for u, cf Abels et al,[17, theorem 6.1] for a discussion.

We will also need the following regularity result, which essentially states that the operator $\mathcal{L}$ is of lower order with respect to the usual Laplace operator. This result is from Abels et al.[17, lemma 2.6]

Lemma 2. *Let $g \in L^2_{(0)}(\Omega)$ and $\theta > 0$. Then the unique solution $u \in H^1_{(0)}(\Omega)$ for the problem*

$$-\theta\int_\Omega \nabla u \cdot \nabla\varphi + \mathcal{E}(u, \varphi) = (g)_{L^2}\varphi \qquad \text{for all } \ \varphi \in H^1_{(0)}(\Omega) \tag{18}$$

belongs to $H^2_{loc}(\Omega)$ and satisfies the estimate

$$\theta\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{H^{\alpha/2}(\Omega)} \leqslant C\|g\|^2_{L^2(\Omega)},$$

where C is independent of $\theta > 0$ and g.

For the following, let $\phi : [a, b] \to \mathbb{R}$ be continuous and define $\phi(x) = +\infty$ for $x \notin [a, b]$. As in Abels set al,[17, section 3] we fix $\theta \geqslant 0$ and consider the functional

$$F_\theta(c) = \frac{\theta}{2}\int_\Omega |\nabla c|^2\, dx + \frac{1}{2}\mathcal{E}(c, c) + \int_\Omega \phi(c(x))\, dx \tag{19}$$

where

$$\begin{aligned} \operatorname{dom} F_0 &= \left\{ c \in H^{\alpha/2}(\Omega) \cap L^2_{(m)}(\Omega) : \phi(c) \in L^1(\Omega) \right\}, \\ \operatorname{dom} F_\theta &= H^1(\Omega) \cap \operatorname{dom} F_0 \qquad \text{if} \quad \theta > 0 \end{aligned}$$

for a given $m \in (a, b)$. Moreover, we define

$$\mathcal{E}_\theta(u, v) = \theta \int_\Omega \nabla u \cdot \nabla v \, dx + \mathcal{E}(u, v)$$

for all $u, v \in H^1(\Omega)$ if $\theta > 0$ and $u, v \in H^{\alpha/2}(\Omega)$ if $\theta = 0$.

In the following, $\partial F_\theta(c) : L^2_{(m)}(\Omega) \to \mathcal{P}(L^2_{(0)}(\Omega))$ denotes the subgradient of F_θ at $c \in \operatorname{dom} F$, ie, $w \in \partial F_\theta(c)$ if and only if

$$(w, c' - c)_{L^2} \leqslant F_\theta(c') - F_\theta(c) \qquad \text{for all} \;\; c' \in L^2_{(m)}(\Omega).$$

The following characterization of $\partial F_\theta(c)$ is an important tool for the existence proof.

Theorem 1. *Let $\phi : [a, b] \to \mathbb{R}$ be a convex function that is twice continuously differentiable in (a, b) and satisfies $\lim_{x \to a} \phi'(x) = -\infty$, $\lim_{x \to b} \phi'(x) = +\infty$. Moreover, we set $\phi(x) = +\infty$ for $x \notin (a, b)$ and let F_θ be defined as in (19). Then $\partial F_\theta : \mathcal{D}(\partial F_\theta) \subseteq L^2_{(m)}(\Omega) \to L^2_{(0)}(\Omega)$ is a single valued, maximal monotone operator with*

$$\begin{aligned} \mathcal{D}(\partial F_0) = \{ c \in H^\alpha_{loc}(\Omega) \cap H^{\alpha/2}(\Omega) \cap L^2_{(m)}(\Omega) : \phi'(c) \in L^2(\Omega), \exists f \in L^2(\Omega) : \\ \mathcal{E}(c, \varphi) + \int_\Omega \phi'(c) \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \, \varphi \in H^{\alpha/2}(\Omega) \} \end{aligned}$$

if $\theta = 0$ and

$$\begin{aligned} \mathcal{D}(\partial F_\theta) = \{ c \in H^2_{loc}(\Omega) \cap H^1(\Omega) \cap L^2_{(m)}(\Omega) : \phi'(c) \in L^2(\Omega), \exists f \in L^2(\Omega) : \\ \mathcal{E}_\theta(c, \varphi) + \int_\Omega \phi'(c) \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \, \varphi \in H^1(\Omega) \} \end{aligned}$$

if $\theta > 0$ as well as

$$\partial F_\theta(c) = -\theta \Delta c + \mathcal{L} c + P_0 \phi'(c) \quad \text{in} \quad \mathcal{D}'(\Omega) \qquad \text{for} \;\; \theta \geqslant 0.$$

Moreover, the following estimates hold

$$\begin{aligned} \theta \|c\|^2_{H^1} + \|c\|^2_{H^{\alpha/2}} + \|\phi'(c)\|^2_2 &\leqslant C \left(\|\partial F_\theta(c)\|^2_2 + \|c\|^2_2 + 1 \right) \\ \int_\Omega \int_\Omega (\phi'(c(x)) - \phi'(c(y)))(c(x) - c(y)) k(x, y, x - y) \, dx \, dy & \\ &\leqslant C \left(\|\partial F_\theta(c)\|^2_2 + \|c\|^2_2 + 1 \right) \\ \theta \int_\Omega \phi''(c) |\nabla c|^2 \, dx &\leqslant C \left(\|\partial F_\theta(c)\|^2_2 + \|c\|^2_2 + 1 \right) \end{aligned} \tag{20}$$

for some constant $C > 0$ independent of $c \in \mathcal{D}(\partial F_\theta)$ and $\theta \geqslant 0$.

The result follows from Abels et al.[17, corollary 3.2 and theorem 3.3]

3 | WEAK SOLUTIONS AND MAIN RESULT

In this section, we define weak solutions for the system (1)-(4) and (9)-(11) together with a natural boundary condition for φ given by the bilinear form $\mathcal{E}$, summarize the assumptions, and state the main result.

Assumption 1. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with C^2-boundary. The following conditions hold true:

1. $\rho(\varphi) = \frac{1}{2}(\tilde{\rho}_1 + \tilde{\rho}_2) + \frac{1}{2}(\tilde{\rho}_2 - \tilde{\rho}_1)\varphi$ for all $\varphi \in [-1, 1]$.
2. $m \in C^1(\mathbb{R})$, $\eta \in C^0(\mathbb{R})$ and there are constants $m_0, K > 0$ such that $0 < m_0 \leq m(s), \eta(s) \leq K$ for all $s \in \mathbb{R}$.
3. $\Psi \in C([-1, 1]) \cap C^2((-1, 1))$ and

$$\lim_{s \to \pm 1} \Psi'(s) = \pm\infty , \quad \Psi''(s) \geq -\kappa \text{ for some } \kappa \in \mathbb{R} . \tag{21}$$

A standard example for a homogeneous free energy density Ψ satisfying the previous conditions is given by (15). Since for solutions we will have $\varphi(x, t) \in [-1, 1]$ almost everywhere, we only need the functions m, η on this interval. But for simplicity, we assume m, η to be defined on $\mathbb{R}$.

Definition 1. Let $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ and $\varphi_0 \in H^{\alpha/2}(\Omega)$ with $|\varphi_0| \leq 1$ almost everywhere in Ω and let Assumption 1 be satisfied. Then $(\mathbf{v}, \varphi, \mu)$ such that

$$\begin{aligned}
&\mathbf{v} \in BC_w([0, \infty); L^2_\sigma(\Omega)) \cap L^2(0, \infty; H^1_0(\Omega)^d) , \\
&\varphi \in BC_w([0, \infty); H^{\alpha/2}(\Omega)) \cap L^2_{\mathrm{uloc}}([0, \infty); H^\alpha_{\mathrm{loc}}(\Omega)) , \qquad \Psi'(\varphi) \in L^2_{\mathrm{uloc}}([0, \infty); L^2(\Omega)) , \\
&\mu \in L^2_{\mathrm{uloc}}([0, \infty); H^1(\Omega)) \text{ with } \nabla\mu \in L^2(0, \infty; L^2(\Omega))
\end{aligned}$$

is called a weak solution of (1)-(4) and (4)-(9) if the following conditions hold true:

$$\begin{aligned}
-(\rho\mathbf{v}, \partial_t\psi)_Q + (\operatorname{div}(\rho\mathbf{v} \otimes \mathbf{v}), \psi)_Q + (2\eta(\varphi)D\mathbf{v}, D\psi)_Q - \left((\mathbf{v} \otimes \widetilde{\mathbf{J}}), \nabla\psi\right)_Q& \\
= -(\varphi\nabla\mu, \psi)_Q&
\end{aligned} \tag{22}$$

for all $\boldsymbol{\psi} \in C^\infty_0(\Omega \times (0, \infty))^d$ with $div\boldsymbol{\psi} = 0$,

$$-(\varphi, \partial_t\psi)_Q + (\mathbf{v} \cdot \nabla\varphi, \psi)_Q = -(m(\varphi)\nabla\mu, \nabla\psi)_Q \tag{23}$$

$$\int_0^\infty \int_\Omega \mu\psi \, dx \, dt = \int_0^\infty \int_\Omega \Psi'(\varphi)\psi \, dx \, dt + \int_0^\infty \mathcal{E}(\varphi(t), \psi(t)) \, dt \tag{24}$$

for all $\psi \in C^\infty_0((0, \infty); C^1(\overline{\Omega}))$ and

$$(\mathbf{v}, \varphi)|_{t=0} = (\mathbf{v}_0, \varphi_0) . \tag{25}$$

Recall $\widetilde{\mathbf{J}} = -\frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} m(\varphi)\nabla\mu$. Finally, the energy inequality

$$\begin{aligned}
E_{\mathrm{tot}}(\varphi(t), \mathbf{v}(t)) + \int_s^t \int_\Omega 2\eta(\varphi) \, |D\mathbf{v}|^2 \, dx \, d\tau + \int_s^t \int_\Omega m(\varphi)|\nabla\mu|^2 \, dx \, d\tau& \\
\leq E_{\mathrm{tot}}(\varphi(s), \mathbf{v}(s))&
\end{aligned} \tag{26}$$

holds true for all $t \in [s, \infty)$ and almost all $s \in [0, \infty)$ (including $s = 0$). Here E_{tot} is as in (13).

The main result of this contribution is as follows:

Theorem 2 (Existence of weak solutions). *Let Assumption 1 hold and $\alpha \in (1, 2)$. Then for every $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ and $\varphi_0 \in H^{\alpha/2}(\Omega)$ such that $|\varphi_0| \leq 1$ almost everywhere and $(\varphi_0)_\Omega \in (-1, 1)$, there exists a weak solution $(\boldsymbol{v}, \varphi, \mu)$ of (1)-(4) and (9)-(11).*

Remark 1. Using, eg, $\varphi\nabla\mu \in L^2(0, \infty; L^2(\Omega))$, one can consider this term in (1) as a given right-hand side and obtain the existence of a pressure such that (1) holds in the sense of distributions in the same way as for the single Navier-Stokes equations, cf, eg, Sohr.[26]

4 | APPROXIMATION BY AN IMPLICIT TIME DISCRETIZATION

Let Ψ be as in Assumption 1. We define $\Psi_0 : [-1,1] \to \mathbb{R}$ by $\Psi_0(s) = \Psi(s) + \kappa\frac{s^2}{2}$ for all $s \in [a,b]$. Then $\Psi_0 : [-1,1] \to \mathbb{R}$ is convex and $\lim_{s\to\pm1}\Psi_0'(s) = \pm\infty$. A basic idea for the following is to use this decomposition to split the free energy E_{free} into a singular convex part E and a quadratic perturbation. In the equations, this yields a decomposition into a singular monotone operator and a linear remainder. To this end, we define an energy $E : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ with domain

$$\operatorname{dom} E = \{\varphi \in H^{\alpha/2}(\Omega) \mid -1 \le \varphi \le 1 \text{ a.e.}\}$$

given by

$$E(\varphi) = \begin{cases} \frac{1}{2}\mathcal{E}(\varphi,\varphi) + \int_\Omega \Psi_0(\varphi)\,dx & \text{for } \varphi \in \operatorname{dom} E\,, \\ +\infty & \text{else}\,. \end{cases} \tag{27}$$

This yields the decomposition

$$E_{\text{free}}(\varphi) = E(\varphi) - \frac{\kappa}{2}\|\varphi\|_{L^2}^2 \qquad \text{for all } \ \varphi \in \operatorname{dom} E.$$

Moreover, E is convex and $E = F_0$ if one chooses $\phi = \Psi_0$ and F_0 is as in Subsection 2.1. This is a key relation for the following analysis in order to make use of Theorem 1, which in particular implies that $\partial E = \partial F_0$ is a maximal monotone operator.

To prove our main result, we discretize our system semi-implicitly in time in a suitable manner. To this end, let $h = \frac{1}{N}$ for $N \in \mathbb{N}$ and $\mathbf{v}_k \in L^2_\sigma(\Omega)$, $\varphi_k \in H^1(\Omega)$ with $\varphi_k(x) \in [-1,1]$ almost everywhere and $\rho_k = \frac{1}{2}(\tilde\rho_1 + \tilde\rho_2) + \frac{1}{2}(\tilde\rho_2 - \tilde\rho_1)\varphi_k$ be given. Then $\Psi(\varphi_k) \in L^1(\Omega)$. We also define a smoothing operator P_h on $L^2(\Omega)$ as follows. We choose u as the solution of the following heat equation:

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T)\,, \\ u|_{t=0} = \varphi' & \text{on } \Omega\,, \\ \partial_\nu u|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $\varphi' \in L^2(\Omega)$, and set $P_h\varphi' := u|_{t=h}$. Then $P_h\varphi' \in H^2(\Omega)$ and $P_h\varphi' \to \varphi'$ in $L^2(\Omega)$ as $h \to 0$ for all $\varphi' \in L^2(\Omega)$. Moreover, we have $|P_h\varphi'| \le 1$ in Ω if $|\varphi'(x)| \le 1$ almost everywhere and $P_h\varphi' \to_{h\to0} \varphi'$ in $H^{\frac{\alpha}{2}}(\Omega)$ as $h \to 0$ for all $\varphi' \in H^{\frac{\alpha}{2}}(\Omega)$.

Now, we determine $(\mathbf{v},\varphi,\mu) = (\mathbf{v}_{k+1},\varphi_{k+1},\mu_{k+1})$, $k \in \mathbb{N}$, successively as solution of the following problem: Find $\mathbf{v} \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, $\varphi \in \mathcal{D}(\partial E)$ and

$$\mu \in H^2_n(\Omega) = \{u \in H^2(\Omega) \mid \partial_{\mathbf{n}} u|_{\partial\Omega} = 0 \ \text{ on } \ \partial\Omega\},$$

such that

$$\begin{aligned} \left(\frac{\rho\mathbf{v} - \rho_k\mathbf{v}_k}{h}, \psi\right)_\Omega &+ (\operatorname{div}(\rho(P_h\varphi_k)\mathbf{v}\otimes\mathbf{v}),\psi)_\Omega + (2\eta(\varphi_k)D\mathbf{v}, D\psi)_\Omega + \left(\operatorname{div}(\mathbf{v}\otimes\widetilde{\mathbf{J}}),\psi\right)_\Omega \\ &= -((P_h\varphi_k)\nabla\mu,\psi)_\Omega \end{aligned} \tag{28}$$

for all $\psi \in C^\infty_{0,\sigma}(\Omega)$,

$$\frac{\varphi - \varphi_k}{h} + \mathbf{v}\cdot\nabla P_h\varphi_k = \operatorname{div}(m(P_h\varphi_k)\nabla\mu) \ \text{ almost everywhere in } \ \Omega, \tag{29}$$

and

$$\int_\Omega \left(\mu + \kappa\,\frac{\varphi+\varphi_k}{2}\right)\psi\,dx = \mathcal{E}(\varphi,\psi) + \int_\Omega \Psi_0'(\varphi)\psi\,dx + h\int_\Omega \nabla\varphi\cdot\nabla\psi\,dx \tag{30}$$

for all $\psi \in H^{\alpha/2}(\Omega)$, where

$$\widetilde{\mathbf{J}} \equiv \widetilde{\mathbf{J}}_{k+1} := -\frac{\tilde\rho_2 - \tilde\rho_1}{2}m(P_h\varphi_k)\nabla\mu_{k+1} = -\frac{\tilde\rho_2 - \tilde\rho_1}{2}m(P_h\varphi_k)\nabla\mu\,.$$

For the following, let

$$E_{\text{tot},h}(\varphi,\mathbf{v}) = \int_\Omega \rho\frac{|\mathbf{v}|^2}{2}\,dx + \int_\Omega \Psi(\varphi)\,dx + \frac{1}{2}\mathcal{E}(\varphi,\varphi) + \frac{h}{2}\int_\Omega |\nabla\varphi|^2\,dx \tag{31}$$

denote the total energy of the system (28)-(30).

Remark 2.

1. As in 6. Abels et al,[6] we obtain the important relation

$$-\frac{\rho-\rho_k}{h} - \mathbf{v}\cdot\nabla\rho(P_h\varphi_k) = \operatorname{div}\widetilde{\mathbf{J}}\,,$$

   by multiplication of (29) with $-\frac{\tilde\rho_2-\tilde\rho_1}{2} = \frac{\partial\rho(\varphi)}{\partial\varphi}$. Because of $\operatorname{div}(\mathbf{v}\otimes\widetilde{\mathbf{J}}) = (\operatorname{div}\widetilde{\mathbf{J}})\mathbf{v} + \left(\widetilde{\mathbf{J}}\cdot\nabla\right)\mathbf{v}$, this yields that

$$\begin{aligned}&\left(\frac{\rho\mathbf{v}-\rho_k\mathbf{v}_k}{h},\psi\right)_\Omega + (\operatorname{div}(\rho(P_h\varphi_k)\mathbf{v}\otimes\mathbf{v}),\psi)_\Omega + (2\eta(\varphi_k)D\mathbf{v},D\psi)_\Omega\\ &+\left(\left(\operatorname{div}\widetilde{\mathbf{J}} - \frac{\rho-\rho_k}{h} - \mathbf{v}\cdot\nabla\rho(P_h\varphi_k)\right)\frac{\mathbf{v}}{2},\psi\right)_\Omega + \left(\left(\widetilde{\mathbf{J}}\cdot\nabla\right)\mathbf{v},\psi\right)_\Omega = -((P_h\varphi_k)\nabla\mu,\psi)_\Omega \qquad (32)\end{aligned}$$

   for all $\psi\in C^\infty_{0,\sigma}(\Omega)$ to (28), which will be used to derive suitable a priori estimates.
2. Integrating (29) in space, one obtains $\int_\Omega\varphi\,dx = \int_\Omega\varphi_k\,dx$ because of $div\,\mathbf{v}=0$ and the boundary conditions.

The following lemma is important to control the derivative of the singular free energy density $\Psi'(\varphi)$.

Lemma 3. *Let $\varphi\in\mathcal{D}(\partial F_h)$ and $\mu\in H^1(\Omega)$ be a solution of (30) for given $\varphi_k\in H^1(\Omega)$ with $|\varphi_k(x)|\le 1$ almost everywhere in Ω such that*

$$\varphi_\Omega = \frac{1}{|\Omega|}\int_\Omega\varphi\,dx = \frac{1}{|\Omega|}\int_\Omega\varphi_k\,dx \in (-1,1)\,.$$

Then there is a constant $C = C(\int_\Omega\varphi_k,\Omega)>0$, independent of φ,μ,φ_k, such that

$$\begin{aligned}\|\Psi_0'(\varphi)\|_{L^2(\Omega)} + \left|\int_\Omega\mu\,dx\right| &\le C(\|\nabla\mu\|_{L^2} + \|\nabla\varphi\|^2_{L^2} + 1)\ \textit{and}\\ \|\partial F_h(\varphi)\|_{L^2(\Omega)} &\le C\left(\|\mu\|_{L^2}+1\right)\,.\end{aligned}$$

Proof. The proof is an adaptation of the corresponding result in Abels et al.[6] For the convenience of the reader, we give the details. First, we choose $\psi = \varphi-\varphi_\Omega$ in (30) and get

$$\begin{aligned}&\int_\Omega\mu(\varphi-\varphi_\Omega)\,dx + \int_\Omega\kappa\frac{\varphi+\varphi_k}{2}(\varphi-\varphi_\Omega)\,dx\\ =&\mathcal{E}(\varphi,\varphi) + \int_\Omega\Psi_0'(\varphi)(\varphi-\varphi_\Omega)\,dx + h\int_\Omega\nabla\varphi\cdot\nabla\varphi\,dx\,. \qquad (33)\end{aligned}$$

Let $\mu_0 = \mu-\mu_\Omega$. Then $\int_\Omega\mu(\varphi-\varphi_\Omega)\,dx = \int_\Omega\mu_0\varphi\,dx$.

In order to estimate the second term in (32), we use that $\overline{\varphi}\in(-1+\varepsilon,1-\varepsilon)$ for sufficiently small $\varepsilon>0$ and that $\lim_{\varphi\to\pm1}\Psi_0'(\varphi) = \pm\infty$. Hence, for sufficiently small ε, one obtains the inequality $\Psi_0'(\varphi)(\varphi-\varphi_\Omega)\ge C_\varepsilon|\Psi_0'(\varphi)| - \tilde C_\varepsilon$, which implies

$$\int_\Omega\Psi_0'(\varphi)(\varphi-\varphi_\Omega)\,dx \ge C\int_\Omega|\Psi_0'(\varphi)|\,dx - C_1\,.$$

Together with (32), we obtain

$$\begin{aligned}\int_\Omega|\Psi_0'(\varphi)|\,dx \le&\ C\|\mu_0\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + C\int_\Omega\frac{\kappa}{2}|\varphi+\varphi_k||\varphi-\varphi_\Omega|\,dx + C_1\\ \le&\ C(\|\mu_0\|_{L^2(\Omega)} + \|\varphi\|^2_{L^2(\Omega)} + 1)\\ \le&\ C(\|\nabla\mu\|_{L^2(\Omega)}+1)\,,\end{aligned}$$

because of $|\varphi|,|\varphi_k|\le 1$. Next, we choose $\psi\equiv 1$ in (30). This yields

$$\int_\Omega\mu\,dx = \int_\Omega\Psi_0'(\varphi)\,dx - \int_\Omega\frac{\kappa}{2}\left(\varphi+\varphi_k\right)\,dx\,.$$

Altogether, this leads to

$$\left|\int_\Omega \mu \, dx\right| \le C(\|\nabla\mu\|_{L^2(\Omega)} + 1) \,.$$

Finally, the estimates of $\partial F_h(\varphi)$ and $\Psi_0'(\varphi)$ in $L^2(\Omega)$ follow directly from (30) and (20). □

Now, we will prove existence of solution to the time-discrete system. We basically follow the line of the corresponding arguments in Abels et al[6] here. As before, we denote

$$H^2_n(\Omega) := \{u \in H^2(\Omega) : \mathbf{n}\cdot\nabla u|_{\partial\Omega} = 0\}.$$

Lemma 4. *For every $\mathbf{v}_k \in L^2_\sigma(\Omega)$, $\varphi_k \in H^1(\Omega)$ with $|\varphi_k(x)| \le 1$ almost everywhere, and $\rho_k = \frac{1}{2}(\tilde\rho_1+\tilde\rho_2)+\frac{1}{2}(\tilde\rho_2-\tilde\rho_1)\varphi_k$, there is some solution $(\mathbf{v},\varphi,\mu) \in \left(H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)\right)\times\mathcal{D}(\partial F_h)\times H^2_n(\Omega)$ of the system (29)-(30) and (32). Moreover, the solution satisfies the discrete energy estimate*

$$\begin{aligned} &E_{\mathrm{tot,h}}(\varphi,\mathbf{v}) + \int_\Omega \rho_k\frac{|\mathbf{v}-\mathbf{v}_k|^2}{2}\,dx + \int_\Omega \frac{|\nabla\varphi-\nabla\varphi_k|^2}{2}\,dx + \frac{1}{2}\mathcal{E}(\varphi-\varphi_k,\varphi-\varphi_k) \\ &\quad + h\int_\Omega 2\eta(\varphi_k)|D\mathbf{v}|^2\,dx + h\int_\Omega m(\varphi_k)|\nabla\mu|^2\,dx \le E_{\mathrm{tot,h}}(\varphi_k,\mathbf{v}_k)\,. \end{aligned} \tag{34}$$

Proof. As first step, we prove the energy estimate (34) for any solution $(\mathbf{v},\varphi,\mu) \in \left(H^1_0(\Omega)^d\cap L^2_\sigma(\Omega)\right)\times\mathcal{D}(\partial F_h)\times H^2_n(\Omega)$ of (29)-(30) and (22).

We choose $\boldsymbol{\psi} = \mathbf{v}$ in (32) and use that

$$\int_\Omega \left((\operatorname{div}\widetilde{\mathbf{J}})\frac{\mathbf{v}}{2} + \left(\widetilde{\mathbf{J}}\cdot\nabla\right)\mathbf{v}\right)\cdot\mathbf{v}\,dx = \int_\Omega \operatorname{div}\left(\widetilde{\mathbf{J}}\frac{|\mathbf{v}|^2}{2}\right)\,dx = 0.$$

Then we derive as in Abels et al[6], proof of lemma 4.3

$$\int_\Omega \left(\operatorname{div}(\rho(P_h\varphi_k)\mathbf{v}\otimes\mathbf{v}) - (\nabla\rho(P_h\varphi_k)\cdot\mathbf{v})\frac{\mathbf{v}}{2}\right)\cdot\mathbf{v}\,dx = \int_\Omega \operatorname{div}\left(\rho(P_h\varphi_k)\mathbf{v}\frac{|\mathbf{v}|^2}{2}\right)\,dx = 0\,,$$

due to $div\mathbf{v} = 0$. Next, one easily gets

$$\frac{1}{h}(\rho\mathbf{v}-\rho_k\mathbf{v}_k)\cdot\mathbf{v} = \frac{1}{h}\left(\rho\frac{|\mathbf{v}|^2}{2} - \rho_k\frac{|\mathbf{v}_k|^2}{2}\right) + \frac{1}{h}(\rho-\rho_k)\frac{|\mathbf{v}|^2}{2} + \frac{1}{h}\rho_k\frac{|\mathbf{v}-\mathbf{v}_k|^2}{2}\,.$$

Therefore, (32) with $\boldsymbol{\psi} = \mathbf{v}$ yields

$$0 = \int_\Omega \frac{\rho|\mathbf{v}|^2 - \rho_k|\mathbf{v}_k|^2}{2h}\,dx + \int_\Omega \rho_k\frac{|\mathbf{v}-\mathbf{v}_k|^2}{2h}\,dx + \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}|^2\,dx + \int_\Omega P_h\varphi_k\nabla\mu\cdot\mathbf{v}\,dx\,. \tag{35}$$

Moreover, multiplying (29) with μ and using the boundary condition for μ, one concludes

$$0 = \int_\Omega \frac{\varphi-\varphi_k}{h}\,\mu\,dx + \int_\Omega (\mathbf{v}\cdot\nabla P_h\varphi_k)\,\mu\,dx + \int_\Omega m(P_h\varphi_k)|\nabla\mu|^2\,dx\,. \tag{36}$$

Furthermore, choosing $\psi = \frac{1}{h}(\varphi-\varphi_k)$ in (30), we obtain

$$\begin{aligned} 0 &= \int_\Omega \nabla\varphi\cdot\nabla(\varphi-\varphi_k)\,dx + \int_\Omega \Psi_0'(\varphi)\frac{\varphi-\varphi_k}{h}\,dx + \frac{1}{h}\mathcal{E}(\varphi,\varphi-\varphi_k) \\ &\quad - \int_\Omega \mu\,\frac{\varphi-\varphi_k}{h}\,dx - \int_\Omega \kappa\frac{\varphi^2-\varphi_k^2}{2h}\,dx\,. \end{aligned} \tag{37}$$

Summation of (35) to (4) yields

$$
\begin{aligned}
0 =& \int_\Omega \frac{\rho|\mathbf{v}|^2 - \rho_k|\mathbf{v}_k|^2}{2h}\, dx + \int_\Omega \rho_k \frac{|\mathbf{v}-\mathbf{v}_k|^2}{2h}\, dx + \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}|^2\, dx + \int_\Omega m(P_h\varphi_k)|\nabla\mu|^2\, dx \\
&+ \int_\Omega \Psi_0'(\varphi)\frac{\varphi-\varphi_k}{h}\, dx - \int_\Omega \kappa \frac{\varphi^2-\varphi_k^2}{2h}\, dx \\
&+ \int_\Omega \nabla\varphi\cdot\nabla(\varphi-\varphi_k)\, dx + \frac{1}{h}\mathcal{E}(\varphi,\varphi-\varphi_k) \\
\geq& \int_\Omega \frac{\rho|\mathbf{v}|^2 - \rho_k|\mathbf{v}_k|^2}{2h}\, dx + \int_\Omega \rho_k \frac{|\mathbf{v}-\mathbf{v}_k|^2}{2h}\, dx + \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}|^2\, dx + \int_\Omega m(P_h\varphi_k)|\nabla\mu|^2\, dx \\
&+ \frac{1}{h}\int_\Omega (\Psi_0(\varphi)-\Psi_0(\varphi_k))\, dx - \int_\Omega \frac{\kappa}{2}\,\frac{\varphi^2-\varphi_k^2}{h}\, dx \\
&+ \int_\Omega \frac{|\nabla\varphi-\nabla\varphi_k|^2}{2}\, dx + \int_\Omega \left(\frac{|\nabla\varphi|^2}{2} - \frac{|\nabla\varphi_k|^2}{2}\right) dx \\
&+ \frac{1}{h}\frac{\mathcal{E}(\varphi,\varphi)}{2} - \frac{1}{h}\frac{\mathcal{E}(\varphi_k,\varphi_k)}{2} + \frac{1}{h}\frac{\mathcal{E}(\varphi-\varphi_k,\varphi-\varphi_k)}{2}\ ,
\end{aligned}
$$

because of $\int_\Omega P_h\varphi_k\nabla\mu\cdot\mathbf{v}\, dx = -\int_\Omega(\mathbf{v}\cdot\nabla P_h\varphi_k)\mu\, dx$,

$$
\begin{aligned}
\Psi_0'(\varphi)\,(\varphi-\varphi_k) &\geq \Psi_0(\varphi)-\Psi_0(\varphi_k)\ , \\
\nabla\varphi\cdot\nabla(\varphi-\varphi_k) &= \frac{|\nabla\varphi|^2}{2} - \frac{|\nabla\varphi_k|^2}{2} + \frac{|\nabla\varphi-\nabla\varphi_k|^2}{2}\ , \quad \text{and} \\
\mathcal{E}(\varphi,\varphi-\varphi_k) &= \frac{\mathcal{E}(\varphi,\varphi)}{2} - \frac{\mathcal{E}(\varphi_k,\varphi_k)}{2} + \frac{\mathcal{E}(\varphi-\varphi_k,\varphi-\varphi_k)}{2}\ .
\end{aligned}
$$

This shows (34).

We will prove existence of weak solutions with the aid of the Leray-Schauder principle. In order to obtain a suitable reformulation of our time-discrete system, we define suitable $\mathcal{L}_k, \mathcal{F}_k : X \to Y$, where

$$
\begin{aligned}
X &= \left(H_0^1(\Omega)^d \cap L^2_\sigma(\Omega)\right) \times \mathcal{D}(\partial F_h) \times H^2_n(\Omega)\ , \\
Y &= \left(H_0^1(\Omega)^d \cap L^2_\sigma(\Omega)\right)' \times L^2(\Omega) \times L^2(\Omega)
\end{aligned}
$$

and

$$
\mathcal{L}_k(\mathbf{w}) = \begin{pmatrix} L_k(\mathbf{v}) \\ -\mathrm{div}(m(P_h\varphi_k)\nabla\mu) + \int_\Omega \mu\, dx \\ \varphi + \partial F_h(\varphi) \end{pmatrix}
$$

for every $\mathbf{w} = (\mathbf{v}, \varphi, \mu) \in X$ and

$$
\langle L_k(\mathbf{v}), \psi\rangle = \int_\Omega 2\eta(\varphi_k) D\mathbf{v} : D\psi\, dx \quad \text{for all } \psi \in H_0^1(\Omega)^d \cap L^2_\sigma(\Omega).
$$

Moreover, we define

$$
\mathcal{F}_k(\mathbf{w}) = \begin{pmatrix} -\frac{\rho\mathbf{v}-\rho_k\mathbf{v}_k}{h} - \mathrm{div}(\rho(P_h\varphi_k)\mathbf{v}\otimes\mathbf{v}) - \nabla\mu P_h\varphi_k - \left(\mathrm{div}\widetilde{\mathbf{J}} - \frac{\rho-\rho_k}{h} - \mathbf{v}\cdot\nabla\rho(P_h\varphi_k)\right)\frac{\mathbf{v}}{2} - \left(\widetilde{\mathbf{J}}\cdot\nabla\right)\mathbf{v} \\ -\frac{\varphi-\varphi_k}{h} - \mathbf{v}\cdot\nabla P_h\varphi_k + \int_\Omega \mu\, dx \\ \varphi + \mu + \tilde{\kappa}\frac{\varphi+\varphi_k}{2} \end{pmatrix}
$$

for $\mathbf{w} = (\mathbf{v}, \varphi, \mu) \in X$. By construction, $\mathbf{w} = (\mathbf{v}, \varphi, \mu) \in X$ is a solution of (28) to (30) if and only if

$$
\mathcal{L}_k(\mathbf{w}) - \mathcal{F}_k(\mathbf{w}) = 0\ .
$$

In Abels et al, [section 4.2][6] it is shown that

$$L_k : H^1_0(\Omega)^d \cap L^2_\sigma(\Omega) \to \left(H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)\right)'$$

is invertible and that for every $f \in L^2(\Omega)$,

$$-\mathrm{div}(m(P_h\varphi_k)\nabla\mu) + \int_\Omega \mu \, dx = f \text{ in } \Omega , \quad \partial_{\mathbf{n}}\mu|_{\partial\Omega} = 0 \tag{38}$$

has a unique solution $\mu \in H^2_n(\Omega)$. This follows from the Lax-Milgram Theorem and elliptic regularity theory. Moreover, in Abels et al,[6, section 4.2] the estimate

$$\|\mu\|_{H^2(\Omega)} \le C_k \left(\|\mu\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\right) \tag{39}$$

is shown.

Because of Theorem 1, ∂F_h is maximal monotone, and therefore,

$$I + \partial F_h : \mathcal{D}(\partial F_h) \to L^2(\Omega)$$

is invertible. Moreover, $(I+\partial F_h)^{-1} : L^2(\Omega) \to H^1(\Omega)$ is continuous, which can be shown as in the proof of proposition 7.5.5 in Abels.[27] Since now, a nonlocal operator is involved, we provide the details for the convenience of the reader. Let $f_l \to_{l\to\infty} f$ in $L^2(\Omega)$ such that $f_l = u_l + \partial F(u_l)$ and $f = u + \partial F(u)$ be given. Then $u_l \to u$ in $H^1(\Omega)$ since

$$\begin{aligned}
\|u_l - u\|^2_{L^2} + h\|\nabla u_l - \nabla u\|^2_{L^2} + \mathcal{E}(u_l - u, u_l - u) &\le \|u_l - u\|^2_{L^2} + (\partial F_h(u_l) - \partial F_h(u), u_l - u)_{L^2} \\
&\le \|u_l + \partial F_h(u_l) - (u + \partial F_h(u))\|_{L^2} \, \|u_l - u\|_{L^2} \\
&\le \frac{1}{2}\|f_l - f\|^2_{L^2} + \frac{1}{2}\|u_l - u\|^2_{L^2} .
\end{aligned}$$

Altogether, $\mathcal{L}_k : X \to Y$ is invertible with continuous inverse $\mathcal{L}_k^{-1} : Y \to X$.

We introduce the following auxiliary Banach spaces

$$\begin{aligned}
\tilde{X} &:= \left(H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)\right) \times H^1(\Omega) \times H^2_n(\Omega), \\
\tilde{Y} &:= L^{\frac{3}{2}}(\Omega)^d \times W^1_{\frac{3}{2}}(\Omega) \times H^1(\Omega)
\end{aligned}$$

in order to obtain a completely continuous mapping in the following. Because of the considerations above, $\mathcal{L}_k^{-1} : Y \to \tilde{X}$ is continuous. Because of the compact embedding $\tilde{Y} \hookrightarrow\hookrightarrow Y$, $\mathcal{L}_k^{-1} : \tilde{Y} \to \tilde{X}$ is compact.

Next, we show that $\mathcal{F}_k : \tilde{X} \to \tilde{Y}$ is continuous and bounded. To this end, one uses the estimates:

$$\begin{aligned}
\|\rho\mathbf{v}\|_{L^{\frac{3}{2}}(\Omega)} &\le C\|\mathbf{v}\|_{H^1(\Omega)}(\|\varphi\|_{L^2(\Omega)} + 1) , & \|\mathrm{div}(\rho(P_h\varphi_k)\mathbf{v}\otimes\mathbf{v})\|_{L^{\frac{3}{2}}(\Omega)} &\le C_k\|\mathbf{v}\|^2_{H^1(\Omega)} , \\
\|\nabla\mu P_h\varphi_k\|_{L^{\frac{3}{2}}(\Omega)} &\le C_k\|\nabla\mu\|_{L^2(\Omega)} , & \|(\mathrm{div}\widetilde{\mathbf{J}})\mathbf{v}\|_{L^{\frac{3}{2}}(\Omega)} &\le C_k\|\mathbf{v}\|_{H^1(\Omega)}\|\mu\|_{H^2(\Omega)} , \\
\|(\widetilde{\mathbf{J}}\cdot\nabla)\mathbf{v}\|_{L^{\frac{3}{2}}(\Omega)} &\le C\|\mathbf{v}\|_{H^1(\Omega)}\|\mu\|_{H^2(\Omega)} , & \|\mathbf{v}\cdot\nabla\varphi_k\|_{W^1_{\frac{3}{2}}(\Omega)} &\le C_k\|\mathbf{v}\|_{H^1(\Omega)} .
\end{aligned}$$

Note that $P_h\varphi_k$ and therefore $\rho(P_h\varphi_k)$ belong to $H^2(\Omega)$). More precisely,

1. For the estimate of $div(\rho(P_h\varphi_k)\mathbf{v}\otimes\mathbf{v})$ in $L^{\frac{3}{2}}(\Omega)$, one has to estimate a term of the form $\rho(P_h\varphi_k)\partial_l\mathbf{v}_i\mathbf{v}_j$ in $L^{\frac{3}{2}}(\Omega)$, which are a product of functions in $L^\infty(\Omega)$, $L^2(\Omega)$ and $L^6(\Omega)$. Therefore, the term is bounded in $L^{\frac{3}{2}}(\Omega)$. Moreover, there are terms of the form $\partial_l\rho(P_h\varphi_k)\mathbf{v}_i\mathbf{v}_j$ in $L^{\frac{3}{2}}(\Omega)$, where each factor belongs to $L^6(\Omega)$.
2. To estimate $(\mathrm{div}\widetilde{\mathbf{J}})\mathbf{v}$ in $L^{\frac{3}{2}}(\Omega)$, one has terms of the form $m'(P_h\varphi_k)\partial_i P_h p_k\partial_j\mu\mathbf{v}_l$ and of the form $m(P_h\varphi_k)\partial_i\partial_j\mu\mathbf{v}_l$. For the first type of terms, the first factor is in $L^\infty(\Omega)$, and the other three are in $L^6(\Omega)$, which yields the bound in $L^{\frac{3}{2}}(\Omega)$. The second type are products of functions in $L^\infty(\Omega)$, $L^2(\Omega)$, and $L^6(\Omega)$.

3. The bound of $(\mathbf{J}\cdot\widetilde{\nabla})\mathbf{v}$ in $L^{\frac{3}{2}}(\Omega)$ follows easily since the factors in $m(P_h\varphi_k)\partial_i\mu\partial_j\mathbf{v}_l$ are bounded in $L^\infty(\Omega)$, $L^6(\Omega)$ and $L^2(\Omega)$, respectively.

The estimates of the other terms are more easy and left to the reader. These estimates show the boundedness of $\mathcal{F}_k$. Using analogous estimates for differences of the terms, one can show the continuity of $\mathcal{F}_k : \tilde{X} \to \tilde{Y}$.

We will now apply the Leray-Schauder principle on $\tilde{Y}$. To this end, we use that $\mathcal{L}_k(\mathbf{w}) - \mathcal{F}_k(\mathbf{w}) = 0$ for $\mathbf{w} \in X$ is equivalent to

$$\mathbf{f} - \mathcal{F}_k \circ \mathcal{L}_k^{-1}(\mathbf{f}) = 0 \quad \text{for } \mathbf{f} = \mathcal{L}_k(\mathbf{w}) \,. \tag{40}$$

Therefore, we define $\mathcal{K}_k := \mathcal{F}_k \circ \mathcal{L}_k^{-1} : \tilde{Y} \to \tilde{Y}$. We remark that $\mathcal{K}_k$ is a compact operator since $\mathcal{L}_k^{-1} : \tilde{Y} \to \tilde{X}$ is compact and $\mathcal{F}_k : \tilde{X} \to \tilde{Y}$ is continuous. Hence, (40) is equivalent to the fixed-point equation

$$\mathbf{f} = \mathcal{K}_k(\mathbf{f}) \qquad \text{for } \mathbf{f} \in \tilde{Y}.$$

Now, we have to show that there is some $R > 0$ such that

$$\text{If } \mathbf{f} \in \tilde{Y} \text{ and } 0 \le \lambda \le 1 \text{ fulfill } \mathbf{f} = \lambda\mathcal{K}_k\mathbf{f}\,, \text{ then } \|\mathbf{f}\|_{\tilde{Y}} \le R\,. \tag{41}$$

To this end, we assume that $\mathbf{f} \in \tilde{Y}$ and $0 \le \lambda \le 1$ are such that $\mathbf{f} = \lambda\mathcal{K}_k\mathbf{f}$. Let $\mathbf{w} = \mathcal{L}_k^{-1}(\mathbf{f})$. Then

$$\mathbf{f} = \lambda\mathcal{K}_k(\mathbf{f}) \quad \Leftrightarrow \quad \mathcal{L}_k(\mathbf{w}) - \lambda\mathcal{F}_k(\mathbf{w}) = 0\,.$$

The latter equation is equivalent to

$$\begin{aligned}
&\int_\Omega 2\eta(\varphi_k)D\mathbf{v} : D\psi\, dx + \lambda\int_\Omega \frac{\rho\mathbf{v} - \rho_k\mathbf{v}_k}{h}\cdot\psi\, dx + \lambda\int_\Omega \operatorname{div}(\rho(P_h\varphi_k)\mathbf{v}\otimes\mathbf{v})\cdot\psi\, dx\\
&+ \lambda\int_\Omega\left(\operatorname{div}\widetilde{\mathbf{J}} - \frac{\rho-\rho_k}{h} - \mathbf{v}\cdot\nabla\rho(P_h\varphi_k)\right)\frac{\mathbf{v}}{2}\cdot\psi\, dx + \lambda\int_\Omega\left(\widetilde{\mathbf{J}}\cdot\nabla\right)\mathbf{v}\cdot\psi\, dx\\
&= -\lambda\int_\Omega \nabla\mu P_h\varphi_k\cdot\psi\, dx
\end{aligned} \tag{42}$$

for all $\psi \in H_0^1(\Omega)^d \cap L_\sigma^2(\Omega)$ and

$$\lambda\frac{\varphi-\varphi_k}{h} + \lambda\mathbf{v}\cdot\nabla P_h\varphi_k - \lambda\int_\Omega \mu\, dx = \operatorname{div}(m(P_h\varphi_k)\nabla\mu) - \int_\Omega \mu\, dx\,, \tag{43}$$

$$\varphi + \partial F_h(\varphi) = \lambda\varphi + \lambda\mu + \lambda\tilde{\kappa}\frac{\varphi+\varphi_k}{2}\,. \tag{44}$$

As in the proof of (34), we choose $\boldsymbol{\psi} = \mathbf{v}$ in (32), test (43) with μ, and multiply (44) with $\frac{1}{h}(\varphi-\varphi_k)$. In the same way as before, one obtains

$$\begin{aligned}
0 =&\lambda\frac{1}{h}\int_\Omega\left(\frac{\rho|\mathbf{v}|^2}{2} - \frac{\rho_k|\mathbf{v}_k|^2}{2}\right) + \lambda\frac{1}{h}\int_\Omega\rho_k\frac{|\mathbf{v}-\mathbf{v}_k|^2}{2} + \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}|^2 + (1-\lambda)\left(\int_\Omega\mu\right)^2\\
&+ \int_\Omega m(\varphi_k)|\nabla\mu|^2 + (1-\lambda)\frac{1}{h}\int_\Omega\varphi(\varphi-\varphi_k) + \int_\Omega\nabla\varphi\cdot(\nabla\varphi-\nabla\varphi_k)\\
&+ \frac{1}{h}\mathcal{E}(\varphi,\varphi-\varphi_k) + \frac{1}{h}\int_\Omega\Psi_0'(\varphi)(\varphi-\varphi_k) - \lambda\frac{1}{h}\int_\Omega\kappa\frac{\varphi^2-\varphi_k^2}{2}
\end{aligned}$$

$$\geq \lambda\frac{1}{h}\int_\Omega\left(\frac{\rho|\mathbf{v}|^2}{2}-\frac{\rho_k|\mathbf{v}_k|^2}{2}\right)+\lambda\frac{1}{h}\int_\Omega\rho_k\frac{|\mathbf{v}-\mathbf{v}_k|^2}{2}+\int_\Omega 2\eta(\varphi_k)|D\mathbf{v}|^2+(1-\lambda)\left(\int_\Omega\mu\right)^2$$
$$+\int_\Omega m(\varphi_k)|\nabla\mu|^2+(1-\lambda)\frac{1}{h}\int_\Omega\left(\frac{\varphi^2}{2}-\frac{\varphi_k^2}{2}\right)+\int_\Omega\left(\frac{|\nabla\varphi|^2}{2}-\frac{|\nabla\varphi_k|^2}{2}\right)$$
$$+\frac{1}{h}\frac{\mathcal{E}(\varphi,\varphi)}{2}-\frac{1}{h}\frac{\mathcal{E}(\varphi_k,\varphi_k)}{2}+\frac{1}{h}\frac{\mathcal{E}(\varphi-\varphi_k,\varphi-\varphi_k)}{2}$$
$$+\frac{1}{h}\int_\Omega(\Psi_0(\varphi)-\Psi_0(\varphi_k))-\lambda\frac{1}{h}\int_\Omega\kappa\frac{\varphi^2-\varphi_k^2}{2}\ .$$

For brevity, we omitted the integration element dx. Thus, we obtain

$$h\int_\Omega 2\eta(\varphi_k)|D\mathbf{v}|^2+h\int_\Omega m(\varphi_k)|\nabla\mu|^2+\frac{h}{2}\int_\Omega|\nabla\varphi|^2$$
$$+\int_\Omega\Psi(\varphi)+(1-\lambda)\left(\int_\Omega\mu\,dx\right)^2+\frac{\mathcal{E}(\varphi,\varphi)}{2}$$
$$\leq\int_\Omega\frac{\rho_k|\mathbf{v}_k|^2}{2}+\frac{1}{2}\int_\Omega\varphi_k^2+\frac{h}{2}\int_\Omega|\nabla\varphi_k|^2+\int_\Omega\Psi_0(\varphi_k)+\int_\Omega|\kappa|\frac{\varphi_k^2}{2}+\frac{\mathcal{E}(\varphi_k,\varphi_k)}{2}\ .$$

Here, we used $-\lambda\int_\Omega\tilde{\kappa}\frac{\varphi_k^2}{2}\,dx\leq\lambda\int_\Omega|\tilde{\kappa}|\frac{\varphi_k^2}{2}\,dx$ and in addition estimated every λ resp. $(1-\lambda)$ on the right side by 1. Because of $\mathbf{w}=(\mathbf{v},\varphi,\mu)=\mathcal{L}_k^{-1}(\mathbf{f})\in X$, $\varphi\in\mathcal{D}(\partial F_h)$ and therefore $\varphi\in[-1,1]$ almost everywhere. In particular, we have $\rho\geq 0$. Moreover, $\int_\Omega\Psi(\varphi)\,dx$ is bounded.

Altogether, we conclude

$$(1-\lambda)\left(\int_\Omega\mu\,dx\right)^2+h\int_\Omega 2\eta(\varphi_k)|D\mathbf{v}|^2\,dx+h\int_\Omega m(\varphi_k)|\nabla\mu|^2\,dx$$
$$+\frac{h}{2}\int_\Omega|\nabla\varphi|^2\,dx+\frac{\mathcal{E}(\varphi,\varphi)}{2}\leq C_k\ ,\tag{45}$$

for some C_k independent of $(\mathbf{v},\varphi,\mu)$. Using $\|\varphi\|_{L^\infty}\leq 1$, Korn's inequality, (17), and the fact that η, m, and a are bounded from below by a positive constant, we obtain

$$\sqrt{1-\lambda}\left|\int_\Omega\mu\,dx\right|+\|\mathbf{v}\|_{H^1(\Omega)}+\|\nabla\mu\|_{L^2(\Omega)}+\|\varphi\|_{H^1(\Omega)}\leq C_k\ .\tag{46}$$

In order to estimate $\|\mu\|_{L^2}$, we distinguish the cases $\lambda\in[\frac{1}{2},1]$ and $\lambda\in[0,\frac{1}{2})$. In the case $\lambda\in[\frac{1}{2},1]$, we simply use $\frac{1}{2}|\int_\Omega\mu\,dx|\leq\lambda|\int_\Omega\mu\,dx|$ and conclude as in the proof of Lemma 3 together with (46) from (44) that

$$\left|\int_\Omega\mu\,dx\right|\leq C_k\ .$$

In the case $\lambda\in[0,\frac{1}{2})$, we conclude directly from (46) that $\left|\int_\Omega\mu\,dx\right|\leq C_k$. Thus, (46) can be improved to

$$\|\mathbf{v}\|_{H^1(\Omega)}+\|\mu\|_{H^1(\Omega)}+\|\varphi\|_{H^1(\Omega)}\leq C_k\ .\tag{47}$$

With the help of (39), we can estimate $\|\mu\|_{H^2(\Omega)}$ and derive

$$\|\mathbf{v}\|_{H^1(\Omega)}+\|\mu\|_{H^2(\Omega)}+\|\varphi\|_{H^1(\Omega)}\leq C_k\ .\tag{48}$$

Using (44), we also have $\|\partial F_h(\varphi)\|_{L^2(\Omega)}\leq C_k$. Altogether, we conclude

$$\|\mathbf{w}\|_{\tilde{X}}+\|\partial F_h(\varphi)\|_{L^2(\Omega)}=\|(\mathbf{v},\varphi,\mu)\|_{\tilde{X}}+\|\partial F_h(\varphi)\|_{L^2(\Omega)}\leq C_k\ .$$

Finally, we can estimate $\mathbf{f} = \mathcal{L}_k(\mathbf{w})$ in $\tilde{Y}$ by using that $\mathbf{f} - \lambda\mathcal{F}_k\mathcal{L}_k^{-1}(\mathbf{f}) = 0$ implies $\mathbf{f} = \lambda\mathcal{F}_k(\mathbf{w})$ together with the boundedness of $\mathcal{F}_k : \tilde{X} \to \tilde{Y}$. Thus, we obtain

$$\|\mathbf{f}\|_{\tilde{Y}} = \|\lambda\mathcal{F}_k(\mathbf{w})\|_{\tilde{Y}} \leq C'_k .$$

Thus, the condition of the Leray-Schauder principle is satisfied, which proves the existence of a solution. □

5 | PROOF OF THEOREM 2

5.1 | Compactness in time

In order to prove our main result Theorem 2, we send $h \to 0$ resp. $N \to \infty$ for the approximate solution, which are obtained by suitable interpolations of our time-discrete solutions. To this end, let $N \in \mathbb{N}$ be given and let $(\mathbf{v}_{k+1}, \varphi_{k+1}, \mu_{k+1})$, $k \in \mathbb{N}$, be chosen successively as a solution of (28) to (30) with $h = \frac{1}{N}$ and $(\mathbf{v}_0, \varphi_0^N)$ where $\varphi_0^N = P_h\varphi_0$ as initial value.

As in Abels et al,[6] we define $f^N(t)$ for $t \in [-h, \infty)$ by the relation $f^N(t) = f_k$ for $t \in [(k-1)h, kh)$, where $k \in \mathbb{N}_0$ and $f \in \{\mathbf{v}, \varphi, \mu\}$. Moreover, let $\rho^N = \frac{1}{2}(\tilde{\rho}_1 + \tilde{\rho}_2) + \frac{1}{2}(\tilde{\rho}_2 - \tilde{\rho}_1)\varphi^N$. Furthermore, we introduce the notation

$$\left(\Delta_h^+ f\right)(t) := f(t+h) - f(t) , \qquad \left(\Delta_h^- f\right)(t) := f(t) - f(t-h) ,$$
$$\partial_{t,h}^{\pm} f(t) := \frac{1}{h}\left(\Delta_h^{\pm} f\right)(t) , \qquad f_h := \left(\tau_h^* f\right)(t) = f(t-h) .$$

In order to derive the weak formulation in the limit, let $\psi \in \left(C_0^\infty(\Omega \times (0,\infty))\right)^d$ with $div\boldsymbol{\psi} = 0$ be arbitrary and choose $\tilde{\psi} := \int_{kh}^{(k+1)h} \psi\, dt$ as test function in (28). By summation with respect to $k \in \mathbb{N}_0$, this yields

$$\begin{aligned}\int_0^\infty \int_\Omega \partial_{t,h}^-(\rho^N\mathbf{v}^N)\cdot\psi\, dx\, dt + \int_0^\infty\int_\Omega \operatorname{div}\left(\rho_h^N\mathbf{v}^N \otimes \mathbf{v}^N\right)\cdot\psi\, dx\, dt + \int_0^\infty\int_\Omega 2\eta(\varphi_h^N)D\mathbf{v}^N : D\psi\, dx\, dt \\ - \int_0^\infty\int_\Omega \left(\mathbf{v}^N \otimes \widetilde{\mathbf{J}}^N\right) : D\psi\, dx\, dt = -\int_0^\infty\int_\Omega \nabla\mu^N\varphi_h^N\cdot\psi\, dx\, dt \end{aligned} \tag{49}$$

for all $\psi \in \left(C_0^\infty(\Omega\times(0,\infty))\right)^d$ with $div\boldsymbol{\psi} = 0$. Here, $\rho_h^N = (\rho^N)_h$ and $\varphi_h^N = (\varphi^N)_h$. Using a simple change of variable, one sees

$$\int_0^\infty\int_\Omega \partial_{t,h}^-(\rho^N\mathbf{v}^N)\cdot\psi\, dx\, dt = -\int_0^\infty\int_\Omega (\rho^N\mathbf{v}^N)\cdot\partial_{t,h}^+\psi\, dx\, dt$$

for sufficiently small $h > 0$. In the same way, one derives

$$\int_0^\infty\int_\Omega \partial_{t,h}^-\varphi^N\,\zeta\, dx\, dt + \int_0^\infty\int_\Omega \mathbf{v}^N\varphi_h^N\cdot\nabla\zeta\, dx\, dt = \int_0^\infty\int_\Omega m(\varphi_h^N)\nabla\mu^N\cdot\nabla\zeta\, dx\, dt \tag{50}$$

for all $\zeta \in C_0^\infty((0,\infty); C^1(\overline{\Omega}))$ as well as

$$\begin{aligned}\int_0^\infty\int_\Omega (\mu^N + \kappa\,\frac{\varphi^N + \varphi_h^N}{2})\psi\, dx\, dt = \int_0^\infty \mathcal{E}(\varphi^N,\psi)\, dt + \int_0^\infty\int_\Omega \Psi_0'(\varphi^N)\psi\, dx\, dt \\ + h\int_0^\infty\int_\Omega \nabla\varphi^N\cdot\nabla\psi\, dx\, dt\end{aligned} \tag{51}$$

for all $\psi \in C_0^\infty((0,\infty); C^1(\overline{\Omega}))$.

Let $E^N(t)$ be defined as

$$E^N(t) = \frac{(k+1)h - t}{h}E_{\text{tot}}(\varphi_k, \mathbf{v}_k) + \frac{t - kh}{h}E_{\text{tot}}(\varphi_{k+1}, \mathbf{v}_{k+1}) \text{ for } t \in [kh, (k+1)h)$$

and set

$$D^N(t) := \int_\Omega 2\eta(\varphi_k)|D\mathbf{v}_{k+1}|^2\,dx + \int_\Omega m(\varphi_k)|\nabla\mu_{k+1}|^2\,dx$$

for all $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}_0$. Then (4) yields

$$-\frac{d}{dt}E^N(t) = \frac{E_{\mathrm{tot}}(\varphi_k, \mathbf{v}_k) - E_{\mathrm{tot}}(\varphi_{k+1}, \mathbf{v}_{k+1})}{h} \geq D^N(t) \tag{52}$$

for all $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}_0$. Integration implies

$$\begin{aligned} E_{\mathrm{tot}}(\varphi^N(t), \mathbf{v}^N(t)) + \int_s^t \int_\Omega \left(2\eta(\varphi_h^N)|D\mathbf{v}^N|^2 + m(\varphi_h^N)|\nabla\mu^N|^2\right) dx\,d\tau \\ \leq E_{\mathrm{tot}}(\varphi^N(s), \mathbf{v}^N(s)) \end{aligned} \tag{53}$$

for all $0 \leq s \leq t < \infty$ with $s, t \in h\mathbb{N}_0$.

Because of Lemma 3 and since $E_{\mathrm{tot}}(\varphi_0^N, \mathbf{v}_0)$ is bounded, we conclude that

$$\begin{aligned} &(\mathbf{v}^N)_{N\in\mathbb{N}} \subseteq L^2(0,\infty; H^1(\Omega)^d) \cap L^\infty(0,\infty; L^2(\Omega)^d)\ , \\ &(\nabla\mu^N)_{N\in\mathbb{N}} \subseteq L^2(0,\infty; L^2(\Omega)^d)\ , \\ &(\varphi^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,\infty; H^{\frac{\alpha}{2}}(\Omega))\ , \quad \text{and} \\ &(h^{\frac{1}{2}}\nabla\varphi^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,\infty; L^2(\Omega)) \end{aligned} \tag{54}$$

are bounded. Moreover, there is a nondecreasing $C : (0,\infty) \to (0,\infty)$ such that

$$\int_0^T \left|\int_\Omega \mu^N\,dx\right| dt \leq C(T) \text{ for all } 0 < T < \infty\ .$$

Therefore, there are subsequences (denoted again by the index $N \in \mathbb{N}$, $h > 0$, respectively) such that

$$\begin{aligned} \mathbf{v}^N &\rightharpoonup \mathbf{v} \text{ in } L^2(0,\infty; H^1(\Omega)^d)\ , \\ \mathbf{v}^N &\rightharpoonup^* \mathbf{v} \text{ in } L^\infty(0,\infty; L^2(\Omega)^d)\ , \\ \varphi^N &\rightharpoonup^* \varphi \text{ in } L^\infty(0,\infty; H^{\frac{\alpha}{2}}(\Omega))\ , \\ \mu^N &\rightharpoonup \mu \text{ in } L^2(0,T; H^1(\Omega)) \text{ for all } 0 < T < \infty\ , \\ \nabla\mu^N &\rightharpoonup \nabla\mu \text{ in } L^2(0,\infty; L^2(\Omega)^d)\ , \end{aligned}$$

where $\mu \in L^2_{uloc}([0,\infty); H^1(\Omega))$.

In the following, $\tilde{\varphi}^N$ denotes the piecewise linear interpolant of $\varphi^N(t_k)$ in time, where $t_k = kh$, $k \in \mathbb{N}_0$. Then $\partial_t\tilde{\varphi}^N = \partial_{t,h}^-\varphi^N$, and therefore,

$$\|\tilde{\varphi}^N - \varphi^N\|_{H^{-1}(\Omega)} \leq h\|\partial_t\tilde{\varphi}^N\|_{H^{-1}(\Omega)}\ . \tag{55}$$

Using that $\mathbf{v}^N\varphi^N$ and $\nabla\mu^N$ are bounded in $L^2(0,\infty; L^2(\Omega)^d)$ and (50), we conclude that $\partial_t\tilde{\varphi}^N \in L^2(0,\infty; H^{-1}_{(0)}(\Omega))$ is bounded. Since $(\varphi^N)_{N\in\mathbb{N}}$ and therefore $(\tilde{\varphi}^N)_{N\in\mathbb{N}}$ are bounded in $L^\infty(0,\infty; H^{\frac{\alpha}{2}}(\Omega))$, the lemma of Aubin-Lions yields

$$\tilde{\varphi}^N \to \tilde{\varphi} \text{ in } L^2(0,T; L^2(\Omega)) \tag{56}$$

for all $0 < T < \infty$ for some $\tilde{\varphi} \in L^\infty(0,\infty; L^2(\Omega))$ (and a suitable subsequence). In particular, $\tilde{\varphi}^N(x,t) \to \tilde{\varphi}(x,t)$ almost every $(x,t) \in \Omega \times (0,\infty)$. Because of (55),

$$\|\tilde{\varphi}^N - \varphi^N\|_{L^2(-h,\infty; H^{-1}(\Omega))} \to 0, \tag{57}$$

and thus, $\tilde{\varphi} = \varphi$. Since $\tilde{\varphi}^N \in H^1_{\mathrm{uloc}}([0,\infty); H^{-1}(\Omega)) \cap L^\infty([0,\infty); H^{\frac{\alpha}{2}}(\Omega)) \hookrightarrow BUC([0,\infty); L^2(\Omega))$ and $\tilde{\varphi}^N \in L^\infty(0,\infty; H^{\frac{\alpha}{2}}(\Omega))$ are bounded, Lemma 1 implies $\varphi \in BC_w([0,\infty); H^{\frac{\alpha}{2}}(\Omega))$. Moreover, $(\tilde{\varphi}^N - \varphi^N)_{N\in\mathbb{N}} \subseteq L^\infty(-h,\infty; H^{\frac{\alpha}{2}}(\Omega))$ is bounded since

$(\varphi^N)_{N\in\mathbb{N}}, (\tilde{\varphi}^N)_{N\in\mathbb{N}} \subseteq L^\infty(-h,\infty;H^{\frac{\alpha}{2}}(\Omega))$ are bounded. By interpolation with (57), we conclude

$$\tilde{\varphi}^N - \varphi^N \to 0 \text{ in } L^2(-h,T;L^2(\Omega)) \tag{58}$$

and therefore

$$\varphi^N \to \varphi \text{ in } L^2(0,T;L^2(\Omega)) \tag{59}$$

for all $0 < T < \infty$. Moreover, we have

$$\begin{aligned}\|\varphi_h^N - \varphi\|_{L^2(0,T;L^2(\Omega))} &\le \|\varphi_h^N - \varphi_h\|_{L^2(0,T;L^2(\Omega))} + \|\varphi_h - \varphi\|_{L^2(0,T;L^2(\Omega))}\\ &\le h^{\frac{1}{2}}\|\varphi_0^N\|_{L^2(\Omega)} + \|\varphi^N - \varphi\|_{L^2(0,T-h;L^2(\Omega))} + \|\varphi_h - \varphi\|_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{60}$$

Because of $\|\varphi_h - \varphi\|_{L^2(0,T;L^2(\Omega))} \to_{h\to 0} 0$, we obtain $\|\varphi_h^N - \varphi\|_{L^2(0,T;L^2(\Omega))} \to_{h\to 0} 0$.

Finally, using the bounds of $\tilde{\varphi}^N$ in $H^1(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$ for all $0 < T < \infty$ as well as $\tilde{\varphi}^N \to \varphi$ in $L^2(0,T;L^2(\Omega))$, we conclude $\tilde{\varphi}^N(0) \to \varphi(0)$ in $L^2(\Omega)$. Since $\tilde{\varphi}^N(0) = \varphi_0^N \to_{N\to\infty} \varphi_0$ in $L^2(\Omega)$, we derive $\varphi(0) = \varphi_0$.

Since ρ^N depends affine linearly on φ^N, the conclusions hold true for ρ^N.

To pass to the limit in (5.1), we closely follow the corresponding argument in Abels et al.[17] The only difference is that we work on the space-time domains directly, while they work on the spacial domains fixing a time variable in Abels et al.[17] We include the argument here for completeness. We first observe that $\Psi_0'(\varphi^N)$ are bounded in $L^2_{uloc}([0,\infty);L^2(\Omega))$ using Lemma 3 and the boundedness of $\nabla\mu_N$ in $L^2(0,\infty;L^2(\Omega))$. Using this bound, we can pass to a subsequence such that $\Psi_0'(\varphi^N)$ converges weakly in $L^2(0,T;L^2(\Omega))$ to χ for all $0 < T < \infty$ as N tends to infinity. Let $\psi \in C_0^\infty((0,\infty);C^1(\overline{\Omega}))$. Thanks to the convergences listed above, we can pass to the limit $N \to \infty$ in (51) to find

$$\int_0^\infty \int_\Omega (\mu + \kappa\varphi)\psi \, dx\, dt = \int_0^\infty \mathcal{E}(\varphi,\psi)\, dt + (\chi,\psi)_{L^2((0,\infty)\times\Omega)}.$$

To show (24), we only have to identify the weak limit $\chi = \lim_{N\to\infty}\Psi_0'(\varphi^N)$. Let $T > 0$. Since (59) holds, passing to a subsequence, we have $\varphi^N \to \varphi$ almost everywhere in $\Omega \times (0,T)$. On the other hand, thanks to Egorov's theorem, there exists a set $Q_m \subset \Omega \times (0,T)$ such that $|Q_m| \ge |\Omega \times (0,T)| - \frac{1}{2m}$ and on which $\varphi^N \to \varphi$ uniformly. We now use (uniform with respect to N) estimate on $\Psi_0'(\varphi^N)$ in $L^2(\Omega\times(0,T))$. By definition, the quantity

$$M_{\delta,N} = \left|\left\{(x,t) \in \Omega \times (0,T) \middle| |\varphi^N(x,t)| > 1-\delta\right\}\right|$$

is decreasing in δ for all $n \in \mathbb{N}$. Since $\Psi_0'(y)$ is unbounded for $y \to \pm 1$, we set

$$c_\delta := \inf_{|c|\ge 1-\delta} |\Psi_0'(c)| \to_{\delta\to 0} \infty,$$

we have by the Tschebychev inequality

$$\int_{\Omega\times(0,T)} |\Psi_0'(\varphi^N)|^2\, dx\, dt \ge c_\delta^2 |M_{\delta,N}|.$$

From the uniform (with respect to N) estimate of the norm of $\Psi_0'(\varphi^N)$ in $L^2(\Omega\times(0,T))$, we obtain $M_{\delta,n} \to 0$ for $\delta \to 0$ uniformly in $n \in \mathbb{N}$. Therefore, we deduce

$$\lim_{\delta\to 0}\left|\left\{(x,t) \in \Omega \times (0,T) \middle| |\varphi^N(x,t)| > 1-\delta\right\}\right| = 0$$

uniformly in $N \in \mathbb{N}$. Thus, there exists $\delta = \delta(m)$ independent of N, such that

$$\left|\left\{(x,t) \in \Omega \times (0,T) \middle| |\varphi^N(x,t)| > 1-\delta\right\}\right| \le \frac{1}{2m}, \qquad \forall N \in \mathbb{N}$$

Consider now $N \in \mathbb{N}$ so large that by uniform convergence, we have $|\varphi^{N'}(x,t) - \varphi^N(x,t)| < \frac{\delta}{2}$ for all $N' \geq N$ and all $(x,t) \in Q_m$. Moreover, let $Q'_{mN} \subset Q_m$ be defined by

$$Q'_{mN} = Q_m \cap \left\{(x,t) \in \Omega \times (0,T) | |\varphi^N(x,t)| \leq 1 - \delta\right\} .$$

By the above construction, we immediately deduce that $|Q'_{mN}| \geq |\Omega \times (0,T)| - \frac{1}{m}$ and that $\left|\varphi^{N'}(x,t)\right| < 1 - \frac{\delta}{2}$ for all $N' \geq N$ and for all $(x,t) \in Q_{m,N}$. Therefore, by the regularity assumptions on the potential Ψ'_0, we deduce that $\Psi'_0(\varphi^N) \to \Psi'_0(\varphi)$ uniformly on Q'_{mN}. Since m is arbitrary, we have $\Psi'_0(\varphi^N) \to \Psi'_0(\varphi)$ almost everywhere in $\Omega \times (0,T)$. By a diagonal argument, passing to a subsequence, we have $\Psi'_0(\varphi^N) \to \Psi'_0(\varphi)$ almost everywhere in $\Omega \times (0,\infty)$ and $\Psi'_0(\varphi^N) \to \Psi'_0(\varphi)$ as $h \to 0$ in $L^q(Q_T)$ for every $1 \leq q < 2$ and $0 < T < \infty$. Finally, the uniqueness of weak and strong limits gives $\chi = \Psi'_0(\varphi)$ as claimed.

Next, we show $\mathbf{v}^N \to \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$ for all $0 < T < \infty$ and almost everywhere. We note that $\partial_t \left(\widetilde{\rho\mathbf{v}}^N\right) = \partial^-_{t,h}\left(\rho^N\mathbf{v}^N\right)$ since $\widetilde{\rho\mathbf{v}}^N$ is the piecewise linear interpolant of $\left(\rho^N\mathbf{v}^N\right)(t_k)$. Using that

$$\begin{aligned}
\rho_h^N\mathbf{v}^N \otimes \mathbf{v}^N &\text{ is bounded in } L^2(0,T;L^{\frac{3}{2}}(\Omega)) ,\\
D\mathbf{v}^N &\text{ is bounded in } L^2(0,T;L^2(\Omega)) ,\\
\mathbf{v}^N \otimes \nabla\mu^N &\text{ is bounded in } L^{\frac{8}{7}}(0,T;L^{\frac{4}{3}}(\Omega)) ,\\
\nabla\mu^N\varphi_h^N &\text{ is bounded in } L^2(0,T;L^2(\Omega)) ,
\end{aligned}$$

together with (49), we obtain that $\partial_t\left(\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)\right)$ is bounded in $L^{\frac{8}{7}}(0,T;(W^1_6(\Omega))')$ for all $0 < T < \infty$. Here, we remark that the boundedness of $\nabla\mu^N \in L^2(0,T;L^2(\Omega))$ and $\varphi_h^N \in L^\infty(0,T;L^\infty(\Omega))$ imply that $\nabla\mu^N\varphi_h^N \in L^2(0,T;L^2(\Omega))$ is bounded.

Since ρ^N is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega)^d)$ and $\mathbf{v}^N$ is bounded in $L^2(0,T;H^1(\Omega)^d)$, using a product rule for Besov spaces, cf Runst and Sickel,[28] suitable Sobolev embeddings and the boundedness of $\mathbb{P}_\sigma$ in Sobolev spaces, we have the boundedness of $\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N)$ in $L^2(0,T;H^\epsilon(\Omega)^d)$ for some $\epsilon > 0$.

Hence, the lemma of Aubin-Lions implies

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{v}}^N) \to \mathbf{w} \text{ in } L^2(0,T;L^2(\Omega)^d)$$

for all $0 < T < \infty$ for some $\mathbf{w} \in L^\infty(0,\infty;L^2(\Omega)^d)$. Since the projection $\mathbb{P}_\sigma : L^2(0,T;L^2(\Omega)^d) \to L^2(0,T;L^2_\sigma(\Omega))$ is weakly continuous, we conclude from the weak convergence $\widetilde{\rho\mathbf{v}}^N \rightharpoonup \rho\mathbf{v}$ in $L^2(0,T;L^2(\Omega))$ that $\mathbf{w} = \mathbb{P}_\sigma(\rho\mathbf{v})$. This yields

$$\int_0^T\int_\Omega \rho^N|\mathbf{v}^N|^2 = \int_0^T\int_\Omega \mathbb{P}_\sigma(\rho^N\mathbf{v}^N)\cdot\mathbf{v}^N \to \int_0^T\int_\Omega \mathbb{P}_\sigma(\rho\mathbf{v})\cdot\mathbf{v} = \int_0^T\int_\Omega \rho\,|\mathbf{v}|^2$$

because of $\mathbb{P}_\sigma(\rho^N\mathbf{v}^N) \to_{N\to\infty} \mathbb{P}_\sigma(\rho\mathbf{v})$ in $L^2(0,T;L^2(\Omega)^d)$. Since weak convergence and convergence of the norms imply strong convergence in a Hilbert space, we conclude $(\rho^N)^{\frac{1}{2}}\mathbf{v}^N \to (\rho)^{\frac{1}{2}}\mathbf{v}$ in $L^2(0,T;L^2(\Omega)^d)$. Because of

$$\rho^N \to \rho \text{ almost everywhere in } (0,\infty)\times\Omega \text{ and } |\rho^N| \geq c > 0 ,$$

we derive

$$\mathbf{v}^N = (\rho^N)^{-\frac{1}{2}}\left((\rho^N)^{\frac{1}{2}}\mathbf{v}^N\right) \to_{N\to\infty} \mathbf{v} \text{ in } L^2(0,T;L^2(\Omega)^d) .$$

This yields $\mathbf{v}^N \to_{N\to\infty} \mathbf{v}$ almost everywhere in $(0,\infty)\times\Omega$ (for a subsequence).

Now, we can pass to the limit in (51) and (50) to get (22) and (23) with the aid of the previous results using that for all divergence free $\boldsymbol{\psi}$

$$\int_0^T\int_\Omega \nabla\mu^N P_N\varphi_h^N\cdot\psi\,dx\,dt \to_{N\to\infty} \int_0^T\int_\Omega \nabla\mu\varphi\cdot\psi\,dx\,dt .$$

The initial condition $\mathbf{v}(0) = \mathbf{v}_0$ in $L^2(\Omega)^d$ is shown in the same way as in Abels et al.[6] Therefore, we omit the proof.

Finally, using (4), $\Psi'(\varphi) \in L^2_{uloc}([0,\infty);L^2(\Omega))$, and the local regularity result due to Abels and Kassmann,[23, lemma 4.3] we obtain $\varphi \in L^2_{uloc}([0,\infty);H^\alpha(\Omega'))$ for every open Ω' with $\overline{\Omega'} \subseteq \Omega$, i.e., $\varphi \in L^2_{uloc}([0,\infty);H^\alpha_{loc}(\Omega))$.

5.2 Proof of the energy inequality

It remains to show the energy inequality (26). If we show that $\varphi^N(t) \to \varphi(t)$ in $H^{\frac{\alpha}{2}}_{(m)}$ for almost every $t \in (0,\infty)$ and $\sqrt{h}\nabla\varphi^N \to 0$ in $(L^2(\Omega))^d$ for almost every $t \in (0,\infty)$, the rest of the proof is almost the same as in Abels et al,[6] and we omit it. To this end, it suffices to show $(\varphi^N, \sqrt{h}\nabla\varphi^N)$ converges strongly to $(\varphi, 0)$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)}(\Omega)\times(L^2(\Omega))^d)$ for every $T>0$. If we take $\psi = \varphi^N$ in (51) (after a standard approximation), we have

$$\int_0^\infty\int_\Omega \left(\mu^N + \kappa\,\frac{\varphi^N+\varphi^N_h}{2}\right)\varphi^N\,dx\,dt = \int_0^\infty \mathcal{E}(\varphi^N,\varphi^N)\,dt + \int_0^\infty\int_\Omega \Psi_0'(\varphi^N)\varphi^N\,dx\,dt + h\int_0^\infty\int_\Omega \nabla\varphi^N\cdot\nabla\varphi^N\,dx\,dt\,. \tag{61}$$

Since $\varphi^N \to \varphi$ in $L^2(Q_T)$, $\mu^N \rightharpoonup \mu$ in $L^2(Q_T)$ and $\Psi_0'(\varphi^N) \rightharpoonup \Psi_0'(\varphi)$ in $L^2(Q_T)$ as $N\to\infty$, we have

$$\lim_{N\to\infty}\left\{\int_0^\infty \mathcal{E}(\varphi^N(t),\varphi^N(t))\,dt + h\int_0^\infty\int_\Omega \nabla\varphi^N\cdot\nabla\varphi^N\,dx\,dt\right\} = \int_0^\infty\int_\Omega(\mu\varphi+\kappa\varphi^2)\,dx\,dt - \int_0^\infty\int_\Omega\Psi_0'(\varphi)\varphi\,dx\,dt = \int_0^\infty \mathcal{E}(\varphi(t),\varphi(t))\,dt \tag{62}$$

because of (24).

Next, we show $\varphi^N \rightharpoonup \varphi$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$ and $\sqrt{h}\nabla\varphi^N \rightharpoonup 0$ in $L^2(0,T;L^2)$ as $N\to\infty$ for any $T>0$. Let $T>0$ be arbitrarily fixed. $(\varphi^N)_{N\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}_{(m)})$, hence, also in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$. Then there exists some $\varphi' \in L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$ such that $\varphi^N \rightharpoonup \varphi'$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$. Since $\varphi^N \to \varphi$ in $L^2(Q_T)$, $\varphi = \varphi'$. Hence, $\varphi^N \rightharpoonup \varphi$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)})$.

For any fixed $\boldsymbol{\psi} \in C_0^\infty(Q_T)^d$,

$$\int_{Q_T}\sqrt{h}\;\nabla\varphi_N\cdot\psi\;d(x,t) = -\int_{Q_T}\sqrt{h}\;\varphi^N\;\operatorname{div}\;\psi\;d(x,t)$$

tends to zero as $N\to\infty$ since $\varphi^N \to \varphi$ in $L^2(Q_T)$. Since $\sup_{N\in\mathbb{N}}\|\sqrt{h}\nabla\varphi^N\|_{L^2(Q_T)^d} < \infty$ and $\overline{C_0^\infty(Q_T)^d}^{\|\cdot\|_{L^2(Q_T)^d}} = L^2(Q_T)^d$, we have $\sqrt{h}\nabla\varphi^N \rightharpoonup 0$ in $L^2(Q_T)^d$. Hence, we have $(\varphi^N, \sqrt{h}\nabla\varphi^N) \rightharpoonup (\varphi,0)$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)}\times(L^2)^d)$.

Because of (62), we also have the convergence of the norms of $(\varphi^N,\sqrt{h}\nabla\varphi^N)$ to that of $(\varphi,0)$ in $L^2(0,T;H^{\frac{\alpha}{2}}_{(m)}\times(L^2)^d)$. Hence, we have shown the claim.

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ORCID

Helmut Abels https://orcid.org/0000-0002-6606-4352

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