Lisse 1-Motives

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Abstract. We study and compare the two different notions of rational lisse 1-motives due to Deligne and more recently due to Pepin Lehalleur. We establish a Néron-Ogg-Shafarevich criterion over normal base schemes of arbitrary dimension. As an application we obtain new “independence of ℓ”-results for ℓ-adic cohomology of curves and commutative group schemes.

Contents

1. Introduction

To a scheme X of finite type over a field K and ℓ a prime different from the characteristic of K, one can associate well-behaved ℓ-adic cohomology groups

$$\mathrm{H}^i_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell),$$

which carry an action of the absolute Galois group $\mathrm{Gal}(K^s/K)$ of K, hence are naturally étale sheaves on $\mathrm{Spec}(K)$. Suppose now that K is the function field of a noetherian, finite dimensional, integral normal scheme S on which ℓ is still invertible. Then one can ask under what circumstances the sheaves $\mathrm{H}^i_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell)$ are unramified over S, i.e. lie in the essential image of the fully faithful restriction functor

$$\mathrm{res}\colon \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S, \mathbb{Q}_\ell) \to \mathrm{Sh}_{\text{ét}}(K, \mathbb{Q}_\ell)$$

from constructible lisse étale sheaves over S to constructible étale sheaves over $\mathrm{Spec}(K)$.

This happens for instance if X is the generic fiber of a proper smooth S-scheme. In general, the geometric meaning of unramified cohomology groups seems to be somewhat mysterious. A seemingly more basic question is the following:

Question 1.1. *In the situation above, is unramifiedness of* $\mathrm{H}^i_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell)$ *independent of* ℓ*?*

Outside of characteristic 0, it is not clear how to answer such a question in general. In this work, we will use recent advances in the theory of motives to show that the answer is positive if X is a curve. Let us first explain what is known for smooth and proper curves, focusing on the case of $i = 1$. Let $(\mathrm{SmPrCurves})/K$ be the category of smooth and projective curves over K. Then we can form a commutative diagram

$$\begin{tikzcd}
(\mathrm{SmPrCurves})/K \arrow[rr, "\mathrm{H}^1_{\text{ét}}(-_{K^s}, \mathbb{Q}_\ell)"] \arrow[dr, "\mathrm{H}^1_{\mathrm{M}}(-, \mathbb{Q})"'] & & \mathrm{Sh}_{\text{ét}}(K, \mathbb{Q}_\ell) \\
& \mathrm{AbVar}(K, \mathbb{Q}) \arrow[ur, "R_\ell"'] & \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S, \mathbb{Q}_\ell) \arrow[u, "\mathrm{res}"'] \\
& \mathrm{AbSch}(S, \mathbb{Q}) \arrow[u, "\mathrm{res}"] \arrow[ur, "R_\ell"'] &
\end{tikzcd} \tag{1.1}$$

where $\mathrm{AbVar}(K, \mathbb{Q})$ (resp. $\mathrm{AbSch}(S, \mathbb{Q})$) is the isogeny category of abelian varieties over K (resp. abelian schemes over S), and where $\mathrm{H}^1_{\mathrm{M}}(X, \mathbb{Q})$ and $R_\ell(A)$ are slight variants of the Jacobian of a curve X and the ℓ-adic Tate module of an abelian scheme A. The key result in this situation is commonly known as the *Criterion of Néron-Ogg-Shafarevich*:

Theorem 1.2 ([ST68, Theorem 1], [Gro66b]). *Assume that S is either Dedekind or that all residue characteristics of S are* 0. *Then the square*

$$\begin{tikzcd} \mathrm{AbVar}(K,\mathbb{Q}) \arrow[r, "R_\ell"] & \mathrm{Sh}_{\text{ét}}(K,\mathbb{Q}_\ell) \\ \mathrm{AbSch}(S,\mathbb{Q}) \arrow[u, hook, "\mathrm{res}_{\mathrm{Ab}}"] \arrow[r, "R_\ell"'] & \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S,\mathbb{Q}_\ell) \arrow[u, hook, "\mathrm{res}_{\mathrm{Sh}}"'] \end{tikzcd}$$

satisfies the following condition:

(NOS) The vertical arrows are fully faithful and the essential image of $\mathrm{res}_{\mathrm{Ab}}$ *consists precisely of those objects which are mapped to the essential image of* $\mathrm{res}_{\mathrm{Sh}}$ *via R_ℓ.*

In other words, an abelian variety A over K is the generic fiber of some abelian scheme over S if and only if $R_\ell(A)$ is the restriction of a lisse étale sheaf over S.

From this, it is easy to conclude the following

Corollary 1.3 ([ST68, Corollary 1]). *If* $\dim(S) = 1$ *or all residue characteristics of S are* 0 *and if X is a smooth, proper curve over K, then the answer to Question 1.1 is positive.*

Indeed, with $i = 1$ being the only difficult case, we see that $\mathrm{H}^1_{\text{ét}}(X_{K^s},\mathbb{Q}_\ell)$ is unramified over S if and only if $\mathrm{H}^1_{\mathrm{M}}(X,\mathbb{Q})$ extends to an abelian scheme over S. The latter condition does not depend on ℓ.

There are two directions in which we will generalize the above result: Allowing general curves X and allowing general normal base schemes S.

Allowing general curves is not a difficult generalization: Deligne defined categories of lisse 1-motives $\mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q})$ [Del74, Définition 10.1.2, Variante 10.1.10], which generalize abelian schemes and for which a diagram analogous to (1.1) exists [Del74, Définition 10.3.4], without assuming the curves to be smooth or proper. We will recall his theory below in Section 2.1. The analogue of Theorem 1.2 for Deligne 1-motives was shown for Dedekind schemes in the thesis of Matev [Mat14] and is shown for base schemes in characteristic 0 below, see Corollary F. We do not whether this result can be extended to more general base schemes.

Passing to general normal schemes S is by far harder, because we do not know whether an analogue of Theorem 1.2 holds in this context. More precisely, note that Theorem 1.2 actually holds before passing to the isogeny categories. Such an integral statement is known to be false in positive and even in mixed characteristic (see [Gro66b, Remarque 4.6] and [dJO97, §6], respectively). Even rationally, Grothendieck expressed strong doubts about a positive answer[1] and the author is not aware of any substantial progress on this question since it was first proposed. As the category of abelian schemes is a full subcategory of Deligne's category of 1-motives, the question is also open for the latter.

We thus have use different categories of 1-motives, relying on recent advances in the theory of mixed motives. In full generality of Question 1.1, one would expect the following *motivic criterion of Néron-Ogg-Shafarevich*:

[1] "[...] une réponse affirmative à la question generale qu'on vient de soulever me semble cependant assez peu plausible." [Gro66b, Rermarque 4.8]

Conjecture 1.4. There are abelian categories $\mathrm{M}(T,\mathbb{Q})$ (and $\mathrm{M}^{\mathrm{ls}}(T,\mathbb{Q})$) of (lisse) mixed motives, naturally associated to any scheme T. Moreover, there are contravariant functors

$$\mathrm{H}^i_{\mathrm{M}}(-,\mathbb{Q})\colon (\mathrm{Sch})/T \to \mathrm{M}(T,\mathbb{Q})$$

as well as covariant realization functors

$$R_\ell\colon\ \mathrm{M}^{(\mathrm{ls})}(T,\mathbb{Q}) \to \mathrm{Sh}^{(\mathrm{ls})}_{\text{ét}}(T,\mathbb{Q}_\ell)$$

for any prime ℓ invertible on T. Finally, there is be a commutative Diagram

$$\begin{tikzcd}
(\mathrm{Sch})/K \arrow[rr, "\mathrm{H}^i_{\text{ét}}(-_{K^s},\mathbb{Q}_\ell)"] \arrow[dr, "\mathrm{H}^i_{\mathrm{M}}(-,\mathbb{Q})"] & & \mathrm{Sh}_{\text{ét}}(K,\mathbb{Q}_\ell) \\
& \mathrm{M}(K,\mathbb{Q}) \arrow[ur, "R_\ell"] \arrow[r, phantom, "(\dagger)"] & \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S,\mathbb{Q}_\ell) \arrow[u, hook, "\mathrm{res}"'] \\
& \mathrm{M}^{\mathrm{ls}}(S,\mathbb{Q}) \arrow[u, hook, "\mathrm{res}"] \arrow[ur, "R_\ell"] &
\end{tikzcd}$$

for any $i \geqslant 0$ and ℓ invertible on S, such that the square $(\dagger)$ satisfies the condition (NOS).

Such a result would immediately imply a positive answer to Question 1.1: Indeed, then $\mathrm{H}^i_{\text{ét}}(X_{K^s},\mathbb{Q}_\ell)$ would be unramified over S if and only if $\mathrm{H}^i_{\mathrm{M}}(-,\mathbb{Q})$ is the restriction of an object of $\mathrm{M}^{\mathrm{ls}}(S,\mathbb{Q})$. The latter condition clearly does not depend on ℓ.

Conjecture 1.4 can be deduced from standard conjectures on rational motivic sheaves $\mathrm{DM}(-,\mathbb{Q})$: It would follow from conservativity of ℓ-adic realizations and the existence of a motivic t-structure, see Proposition 2.8. Unfortunately, both of those assumptions seem to be far out of reach.

The outlook improves considerably if one restricts the attention from general schemes over K to curves. In this setting, Pepin Lehalleur has recently established a satisfying theory of mixed relative 1-motives. In particular, he defines categories $\mathbf{M}^1(-,\mathbb{Q})$ and $\mathbf{M}^{1,\mathrm{ls}}(-,\mathbb{Q})$ of (lisse) mixed 1-motives, which have outstanding formal properties recalled in Section 2.4. As one would expect, they are the heart of a t-structure on a suitable subcategory of $\mathrm{DM}(-,\mathbb{Q})$. Moreover, the ℓ-adic realization functors are indeed conservative when restricted to these subcategories. Finally, for a finite type separated morphism $f\colon Y \to X$, we have an adjunction

$$f^*\colon \mathbf{M}^1(Y) \leftrightarrows \mathbf{M}^1(X)\colon \tau^{1,\leqslant 0}\omega^1 f_*$$

defining a 1-motivic pushforward functor.

Unfortunately, deducing the analogue of Conjecture 1.4 in this setting is not quite formal, because of two reasons: First, the 1-motivic pushforward functor $\tau^{1,\leqslant 0}\omega^1 f_*$ does not coincide with the usual pushforward f_*, and its interaction with the ℓ-adic realization is unclear, see Question 1.5. An secondly, the categories $\mathbf{M}^1(-)$ are not symmetric monoidal, because the product of two curves is no longer a curve. These two problems notwithstanding, we prove the full analogue of Conjecture 1.4 for curves, under the mild assumption $(\star)$ on our schemes S.

($\star$) S is noetherian, finite dimensional and all finite type S-schemes satisfy resolution of singularities by alterations.

This assumption is in particular satisfied for schemes of finite type over an excellent noetherian surface.

Theorem A (Theorem 6.1). *Let S normal integral scheme satisfying ($\star$), let ℓ be a prime invertible on S, and let $j\colon U \to S$ be an open immersion. Then the following is true:*

- *For $N \in \mathbf{M}^{1,\mathrm{ls}}(S,\mathbb{Q})$, the canonical map*
$$N \to \tau^{1,\leqslant 0}\omega^1 j_* j^* N$$
is an isomorphism.
- *Let $M \in \mathbf{M}^{1,\mathrm{ls}}(U,\mathbb{Q})$ be a lisse 1-motive such that $R_\ell(M)$ is unramified over S. Then the pushed forward motive*
$$\tau^{1,\leqslant 0}\omega^1 j_* M$$
lies in $\mathbf{M}^{1,\mathrm{ls}}(S,\mathbb{Q})$.

In particular, the square

$$\begin{tikzcd} \mathbf{M}^{1,\mathrm{ls}}(U,\mathbb{Q}) \arrow[r, "R_\ell"] & \mathrm{Sh}_{\text{ét}}(K,\mathbb{Q}_\ell) \\ \mathbf{M}^{1,\mathrm{ls}}(S,\mathbb{Q}) \arrow[u, hook, "\mathrm{res}"] \arrow[r, "R_\ell"'] & \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S,\mathbb{Q}_\ell) \arrow[u, hook', "\mathrm{res}"'] \end{tikzcd}$$

satisfies (NOS).

As a consequence, we obtain the following positive answer to Question 1.1.

Corollary B (Corollary 6.15). *Let S satisfy ($\star$) and be normal, integral with function field K. Let X be a curve or a semi-abelian scheme over K. If ℓ, ℓ' are two primes invertible on S, then $\mathrm{H}^i_{\text{ét}}(X_{K^s},\mathbb{Q}_\ell)$ is unramified over S if and only if $\mathrm{H}^i_{\text{ét}}(X_{K^s},\mathbb{Q}_{\ell'})$ is.*

We also deduce the Zariski-Nagata purity statement

Corollary C (Corollary 6.10). *Let S be a regular scheme satisfying ($\star$) and $j\colon U \to S$ be an open dense immersion whose complement has codimension at least 2. Then*
$$j^*\colon \mathbf{M}^{1,\mathrm{ls}}(S,\mathbb{Q}) \to \mathbf{M}^{1,\mathrm{ls}}(U,\mathbb{Q})$$
is an equivalence of categories with inverse $\omega^1 j_*$; no truncation is needed.

Finally, we get the following general fact

Corollary D (Theorem 6.11). *Let S be a (not necessarily normal) scheme satisfying ($\star$). An object $M \in \mathbf{M}^1(S)$ is lisse if and only if $R_\ell(M)$ is a lisse sheaf.*

Besides the excellent formal properties of $\mathrm{DM}(-,\mathbb{Q})$, our main tool in proving Theorem A is Deligne's category of 1-motives $\mathbf{M}_1^{\mathrm{Del}}(-,\mathbb{Q})$. Pepin Lehalleur constructed a comparison functor
$$\Phi_S\colon \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \to \mathbf{M}^{1,\mathrm{ls}}(S,\mathbb{Q})$$
and showed that it is fully faithful if S is regular. We improve this result and obtain the following comparison

Theorem E (Corollaries 6.13 and 6.14)**.** *Let S be a normal integral scheme satisfying $(\star)$. Then the comparison functor*

$$\Phi_S\colon \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \to \mathbf{M}^{1,\mathrm{ls}}(S,\mathbb{Q})$$

is fully faithful. If $\dim(S) = 1$ *or S is of characteristic* 0*, it is even an equivalence of categories.*

In particular, Theorems A and E yield

Corollary F (Corollaries 4.14 and 4.15)**.** *Let S be a normal connected noetherian scheme with field of functions K and assume that either* $\dim(S) = 1$ *or that S is a $\mathbb{Q}$-scheme. Then a Deligne 1-motive over K whose ℓ-adic realization is unramified over S has good reduction over S.*

In fact, we will prove a slightly more precise version of this result first, see Theorem 4.10, and use it as a stepping stone in the proof of Theorem A. We refer the reader to Section 3 for an overview of how to prove Theorem A.

Open questions and future work. Let $j\colon U \to S$ be an open immersion between normal schemes. Our main theorem can be seen as understanding for lisse objects the behavior of the pushforward functor

$$\tau^{1,\leqslant 0}\omega^1 j_*\colon \mathbf{M}^1(U) \to \mathbf{M}^1(S)$$

on 1-motives. Understanding this functor both for more general motives as well as for more general morphisms seems very desirable. A positive answer to the following question would be helpful, as it would allow to control the functor using realizations:

Question 1.5. *Let $f\colon X \to Y$ be a finite type morphism of schemes and $M \in \mathbf{M}^1(X)$. Is the canonical transformation*

$$H^0(f_* R_\ell(M)) \to R_\ell(\tau^{1,\leqslant 0}\omega^1 f_* M)$$

an equivalence?

Note that Theorem A implies a positive answer if M is lisse, f is an open immersion and Y is normal.

In another direction it would be very interesting to generalize the results of [PL19, PL17] in some part from rational to integral motives (with all residue characteristics still inverted, say). Indeed, consider the following question.

Question 1.6. *Let $j\colon U \to S$ be an open immersion between Dedekind schemes, and let A be an abelian scheme over U with Néron model N over S. Is the canonical morphism*

$$\Phi_S(N) \to \omega^1 j_* \Phi_U(A)$$

an equivalence?

Discussions with Pepin Lehalleur have made it very plausible to the author that the answer is yes, and that indeed the component group of the Néron model appears as first cohomology of $\omega^1 j_* \Phi_U(A)$. With the component group being rationally trivial, this becomes interesting only if one works with some integral version of 1-motives. Finally, seeing that the pushforward for 1-dimensional bases seems to recover Néron models, it is a tantalizing question what happens for $\dim(S) > 1$, where Néron models might no longer exist: Theorem A implies that in the situation where A has unramified Tate module but no semi-abelian continuation over S (see [Gro66b, Remarque 4.6] for an example), the 1-motivic pushforward $\tau^{1,\leqslant 0}\omega^1 j_*\Phi_U(A)$ behaves like a Néron model would [BLR90, Corollary 8.3.6], even though no Néron model exists.

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2. Background and conventions

2.1. **Schemes and Deligne 1-motives.** All schemes in this paper are supposed to be noetherian, separated and finite dimensional. Except in Section 4, we will further assume that S is quasi-excellent and satisfies resolution of singularities by Galois-alterations, which we recall to be the case for schemes of finite type over an excellent noetherian scheme of dimension less or equal to 2 [dJ97, Theorem 5.13]. For S a scheme, ℓ a prime invertible on S and $\Lambda \in \{\mathbb{Z}_\ell, \mathbb{Q}_\ell\}$, we write $\mathrm{Sh}_{\text{ét}}(S, \Lambda)$ for the categories of *constructible* étale sheaves with Λ-coefficients defined in [SGA77, Exposé VI, Définition 1.1.1] and $\mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S, \Lambda)$ for constructible lisse étale sheaves with Λ-coefficients. We further write $\mathrm{D}_{\text{ét},c}(S, \mathbb{Z}_\ell)$ for Ekedahl's category $D^b_c(S - \mathbb{Z}_\ell)$ [Eke90, Theorem 6.3], a $\otimes$-triangulated category equipped with a $t-$structure having $\mathrm{Sh}_{\text{ét}}(S, \mathbb{Z}_\ell)$ as heart. We set $\mathrm{D}_{\text{ét},c}(S, \mathbb{Q}_\ell) = \mathrm{D}_{\text{ét},c}(S, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}} \mathbb{Q}$ and direct the reader to [CD16, 7.2.20, 7.2.21] for an explanation why we don't need to worry about the finiteness hypotheses of [Eke90, Chapter 6].

Definition 2.1 ([Del74, Variante 10.1.10])**.** Let S be a scheme. An *integral Deligne 1-motive* is a two-term complex

$$[L \xrightarrow{u} G]$$

of commutative group schemes over S where L is an étale lattice and G is a semi-abelian scheme sitting in an extension

$$1 \to T \to G \to A \to 1 \tag{2.1}$$

with T a torus and A an abelian scheme. Morphisms are morphisms of complexes of group schemes. We denote the resulting category by $\mathbf{M}^{\mathrm{Del}}_1(S, \mathbb{Z})$ and we write $\mathbf{M}^{\mathrm{Del}}_1(S, \mathbb{Q})$ for $\mathbf{M}^{\mathrm{Del}}_1(S, \mathbb{Z}) \otimes \mathbb{Q}$. Pulling back group schemes equips those categories with contravariant functoriality.

Note that, contrary to some other conventions in the literature, we do not Karoubi-complete $\mathbf{M}^{\mathrm{Del}}_1(S, \mathbb{Q})$. In all relevant cases, the category is going to be abelian anyways, so that this distinction does not matter. We direct the reader to [Mum65, Chapter 6] and [FC90, Chapter 2] for key facts about abelian and semi-abelian schemes; see also [BVK16, Org04] for an extensive treatise of Deligne 1-motives over a field and their relation to triangulated categories of mixed motives. We recall that an étale lattice is a group scheme which is étale locally isomorphic to a constant finitely generated free abelian group scheme, and that a torus is a group scheme étale locally isomorphic to $\mathbb{G}_m^r$. We call a lattice (resp. a torus) *split*, if it is isomorphic to $\mathbb{Z}^r$ (resp. $\mathbb{G}^r_{m,S}$). We also remind the reader that $\mathrm{Hom}(-, \mathbb{G}_{m,S})$ defines an anti-equivalence between lattices and tori, which is a special case of general Cartier-Duality of 1-motives [Del74, 10.2.10], which we recall as Proposition 4.5. Finally, recall that in (2.1), the semi-abelian scheme G determines the abelian part A and the toric part T uniquely. More generally, any Deligne 1-motive $[L \xrightarrow{u} G]$ comes with a so-called *weight filtration* given by

$$[0 \to T] \to [0 \to G] \to [L \to G]$$

whose graded pieces $[0 \to T], [0 \to A], [L \to 0]$ are called *pure* 1-motives. Note that this also gives a filtration on the ℓ-adic realization of $[L \xrightarrow{u} G]$, which we introduce now.

Proposition 2.2. *Let S be a scheme, and Λ be $\mathbb{Z}$ or $\mathbb{Q}$.*

- *Denote by $(\mathrm{Sm}/S)_{\text{ét}}$ the site given by the category of schemes smooth over S equipped with the étale topology. Putting the lattice part L of a Deligne 1-motive $[L \to G]$ in degree 0 and the semi-abelian part G in degree 1, we may see $\mathbf{M}_1^{\mathrm{Del}}(S, \Lambda)$ as a full subcategory of* $\mathrm{Cpl}(\mathrm{Sh}((\mathrm{Sm/S})_{\mathrm{et}}, \Lambda))$, *the category of complexes of Λ-linear étale sheaves on $(\mathrm{Sm}/S)_{\text{ét}}$. More is true: The induced functor*
$$\hat{\Phi}_S \colon \mathbf{M}_1^{\mathrm{Del}}(S, \Lambda) \to D(\mathrm{Cpl}(\mathrm{Sh}((\mathrm{Sm/S})_{\mathrm{et}}, \Lambda)))$$
into the derived category is fully faithful, i.e. any quasi-isomorphism of complexes already is an isomorphism of Deligne 1-motives [PL19, Lemma A.6].
- *If k is a field, then $\mathbf{M}_1^{\mathrm{Del}}(\mathrm{Spec}(k), \mathbb{Q})$ is abelian* [Org04, Proposition 3.2.2]. *If k'/k is a purely inseparable field extension, then the pullback functor*
$$\mathbf{M}_1^{\mathrm{Del}}(\mathrm{Spec}(k), \mathbb{Q}) \to \mathbf{M}_1^{\mathrm{Del}}(\mathrm{Spec}(k'), \mathbb{Q})$$
is an equivalence of categories [PL17, Proposition 3.6].

Definition 2.3. Let S be a scheme and let ℓ be a prime invertible on S. We define the ℓ-adic Tate module functor
$$T_\ell \colon \mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z}) \to \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S, \mathbb{Z}_\ell)$$
by sending a Deligne 1-motive $[L \xrightarrow{u} G]$ to the inverse system given by
$$\{[L \xrightarrow{u} G] \otimes \mathbb{Z}/\ell^r\}_{r \in \mathbb{N}}$$
where the $\otimes$ is the (left derived) tensor product in $D(\mathrm{Cpl}(\mathrm{Sh}((\mathrm{Sm/S})_{\mathrm{et}}, \mathbb{Z})))$. It is an easy check that the above is well-defined, i.e. gives a complex concentrated in degree 0. Rationalizing, we obtain a realization functor
$$R_\ell \colon \mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Q}) \to \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S, \mathbb{Q}_\ell).$$

Let us make explicit how the sheaves in the inverse system look like. If $[L \xrightarrow{u} G]$ is a Deligne 1-motive and $T \to S$ is étale, then the T-values of the étale sheaf $[L \xrightarrow{u} G] \otimes \mathbb{Z}/\ell^r$ are given by
$$\{(k, g) \in L(T) \times G(T) \mid u(k) = \ell^r \cdot g\}/\langle\{(\ell^r k, u(k))\}\rangle.$$

2.2. **Symmetric monoidal ∞-categories.** As customary, we write "∞-category" for "$(\infty, 1)$-category" and work with quasi-categories as developed by Joyal and Lurie. Except in Section 5 we do not distinguish between a 1-category and its nerve. Derived categories of abelian categories are understood as stable ∞-categories; we remind the reader that the notion of a t-structure is defined purely in terms of the homotopy category. Some of the key arguments rely on symmetric monoidal structures on the categories considered, see [Lur17, Definition 2.0.0.7] for the notion of a symmetric monoidal ∞-category and [Lur17, Remark 2.4.2.6] for the definition of the ∞-category of such gadgets. We write $\mathcal{P}r^{st,\otimes}$ for the ∞-category of stable presentably symmetric monoidal ∞-categories, i.e. symmetric monoidal ∞-categories whose underlying category is stable and presentable and such that the tensor product preserves small colimits in each variable separately, see [Lur17, Propositions 4.8.1.15, 4.8.2.18] for the definition of this category. We recall that $\mathcal{P}r^{st,\otimes}$ comes with a forgetful functor

$$\mathcal{P}r^{st,\otimes} \to \widehat{\mathrm{Cat}_\infty}$$

to the category of large ∞-categories. This is a limit preserving functor between complete categories by [Lur17, Proposition 4.8.1.15, Corollary 3.2.2.5] and [Lur09, Proposition 5.5.3.13].

Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. Following [Lur17, Definition 4.6.1.7] we call an object c of $\mathcal{C}$ *dualizable*, if there is an object $c^\vee$ such that $-\otimes c^\vee$ is left and right adjoint to $-\otimes c$ in the homotopy category $h(\mathcal{C})$. This notion can also be found under the names *strongly dualizable* and *rigid* in the literature. For later use, we record two well-known auxiliary statements:

Lemma 2.4 ([HPS97, Theorem A.2.5 (a)])**.** *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. The full subcategory $\mathcal{C}^{\mathrm{ls}}$ of dualizable objects is a thick subcategory.*

Lemma 2.5. *Let S be a scheme and ℓ invertible on S. An object of $\mathrm{Sh}_{\text{ét}}(S, \mathbb{Q}_\ell)$ is lisse if and only if it is dualizable in $\mathrm{D}_{\text{ét},c}(S, \mathbb{Q}_\ell)$.*

Proof. Both directions of the general case reduce to the strictly local case: On the one hand, a sheaf is lisse if and only if its specialization morphisms are isomorphisms by [Fu11, Proposition 5.8.9]. On the other hand, dualizability is a statement about certain morphisms being isomorphisms, which can be checked on stalks. Hence assume that S is strictly local. A lisse sheaf is then constant and thus dualizable.

For the other direction, it is enough to consider specializations along discrete valuation rings by our assumptions on the base scheme. Hence consider S to be the spectrum of a strictly hensenlian dvr with $i\colon Z \to S$ the inclusion of the closed point and $j\colon U \to S$ the inclusion of the open point. Take a constructible $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ on S which is dualizable in $\mathrm{D}_{\text{ét},c}(S, \mathbb{Q}_\ell)$. Denote by $\mathcal{G}$ the constant sheaf on S taking as values the global sections of $\mathcal{F}$. We thus get a canonical morphism

$$\phi : \mathcal{G} \to \mathcal{F}$$

which we claim to be an isomorphism. First observe that $i^!\mathcal{F}$ is concentrated in degree 2 by absolute purity and that we hence have an equality $\mathcal{F} \cong \tau^{\leqslant 0} j_* j^* \mathcal{F}$. This means that on the open point, ϕ is identified with the inclusion of the invariants into a module with a Galois action, and is in particular injective. Hence we see that ϕ is an injective morphism of sheaves. Denoting by $\mathcal{H}$ its cokernel, we hence get a short exact sequence

$$\mathcal{G} \to \mathcal{F} \to \mathcal{H}$$

of constructible étale sheaves. By Lemma 2.4 we see that $\mathcal{H}$ is still dualizable, but now satisfies $i^*\mathcal{H} = 0$. We will prove in what follows that this is enough to conclude $\mathcal{H} = 0$.

Indeed, let $\mathcal{H}^\vee$ be the dual of $\mathcal{H}$ in $\mathrm{D}_{\text{ét},c}(S, \mathbb{Q}_\ell)$ and consider the unit

$$\eta\colon \mathbb{Q}_\ell \to \mathcal{H} \otimes \mathcal{H}^\vee$$

of the adjunction witnessing dualizability of $\mathcal{H}$. As $\mathcal{H} \otimes \mathcal{H}^\vee$ is concentrated in τ-nonnegative degrees, it factors over

$$\bar{\eta} \colon \mathbb{Q}_\ell \to \tau^{\leqslant 0}(\mathcal{H} \otimes \mathcal{H}^\vee)$$

which is again just a morphism of usual étale sheaves. But the right hand side has

$$i^* \tau^{\leqslant 0}(\mathcal{H} \otimes \mathcal{H}^\vee) \cong \tau^{\leqslant 0}(i^*\mathcal{H} \otimes i^*\mathcal{H}^\vee) \cong 0$$

by $i^*\mathcal{H} \cong 0$, hence has no global sections over S and does thus not admit a nonzero map from the constant sheaf $\mathbb{Q}_\ell$. Thus, we see that $\bar{\eta}$ and hence η is the zero morphism. Now by the triangle identity for the dualizing adjunction, we see that the identity on $\mathcal{H}$ factors as $(\epsilon \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \eta)$ showing that it is the zero morphism, and that $\mathcal{H}$ is hence already zero. This shows the claim. $\square$

Remark 2.6. The author does not know whether the analogous statement for general objects of $\mathrm{Sh}_{\text{ét}}(S, \mathbb{Q}_\ell)$ is, or should expected to be, true. He once thought it to be obvious and thanks Denis-Charles Cisinski for setting him straight.

2.3. **Rational mixed motives.** While the abelian categories of mixed motives from Conjecture 1.4 have not been constructed yet, we do have reasonable candidates for its derived category. If the base is a field, such categories have been independently constructed by Hanamura, Levine and Voevodsky. This construction has been extended to more general bases by Ayoub, and later again by Cisinski and Déglise. Indeed, Ayoub constructs in [Ayo14, §3] a closed symmetric monoidal stable combinatorial model category whose homotopy category he denotes by $\mathrm{DA}^{\text{ét}}(S, \mathbb{Q})$. Recall that with rational coefficients, this and other candidate categories for the derived category of mixed motives are known to agree, see [CD19] and [CD16] for comparison results. We will write simply $\mathrm{DM}(S)$ for the underlying closed presentably symmetric monoidal compactly generated stable ∞-category, an object of $\mathcal{P}r^{st}$, see [Lur17, Proposition 4.1.7.10]. As usual, we denote by $\mathrm{DM}_c(S)$ the subcategory of compact objects. This formalism has been worked out in ∞-categorical language in [Hoy17] and [Kha16]; for a very nice synopsis, we direct the reader to [RS19, §2].

We write $\mathrm{DM}^{\mathrm{dl}}(S)$ for the full subcategory of $\mathrm{DM}(S)$ spanned by dualizable objects. We recall that the tensor unit is compact [Ayo14, Proposition 3.19], so that $\mathrm{DM}^{\mathrm{dl}}(S)$ is contained in $\mathrm{DM}_c(S)$.

Let ℓ be a prime invertible on S. We write R_ℓ for the ℓ-adic realization functor

$$R_\ell \colon \operatorname{ho}(\mathrm{DM}_c(S)) \to \mathrm{D}_{\text{ét},c}(S, \mathbb{Q}_\ell)$$

defined by Ayoub [Ayo14, Definition 9.6] and extended by Cisinski-Déglise [CD16, Theorem 7.2.24]. Beware that we define R_ℓ only on the level of closed symmetric monoidal triangulated categories, because we have not put any kind of ∞-categorical enhancement on our categories of ℓ-adic sheaves. We will abuse notation by passing implicitly to the homotopy categories whenever realization functors appear, as we never need additional structure. The reader who dislikes leaving the realm of stable ∞-categories may think about the ∞-category presented by $\mathrm{DM}_{h,c}(S, \mathbb{Q}_\ell)$ from [CD16], which effectively is an enhancement of $\mathrm{D}_{\text{ét},c}(S, \mathbb{Q}_\ell)$.

Finally, while not logically necessary for the rest of the paper, let us recall the dream of a motivic t-structure (see [And04, Chapitre 21, Conjecture 22.1.4.1]) and how it would in particular imply Conjecture 1.4:

Conjecture 2.7. Let S be a scheme and ℓ a prime invertible on S. Then there is a t-structure on $\mathrm{DM}(S)$ restricting to $\mathrm{DM}_c(S)$ such that the ℓ-adic realization $\mathrm{DM}_c(S) \to \mathrm{D}_{\text{ét},c}(S, \mathbb{Q}_\ell)$ is t-exact and conservative.

We note that by conservativity of the realization, the motivic t-structure is uniquely determined if it exists (although it might a priori depend on ℓ).

Proposition 2.8. *Conjecture 2.7 implies Conjecture 1.4.*

Proof. Define $\mathbf{M}(S, \mathbb{Q})$ to be the heart of the motivic t-structure τ^{M} on $\mathrm{DM}(S)$. For $f\colon X \to S$, set $M^i(X) = \mathrm{H}^i_{\mathrm{M}}(M(X)) = \mathrm{H}^i_{\mathrm{M}}(f_*\mathbb{Q}_X)$ and define further $\mathbf{M}^{\mathrm{ls}}(S, \mathbb{Q})$ to be the full subcategory of objects of $\mathbf{M}(S, \mathbb{Q})$ lying in $\mathrm{DM}^{\mathrm{dl}}(S)$. The formula $R_\ell(M^i(X)) = R^i f_* \mathbb{Q}_\ell$ holds because the ℓ-adic realization is compatible with pushforwards and is t-exact. Let $r\colon \operatorname{Spec}(K) \to S$ be the canonical map. The motivic restriction

$$\mathrm{res} = r^*\colon \mathbf{M}^{\mathrm{ls}}(S, \mathbb{Q}) \to \mathbf{M}(K, \mathbb{Q})$$

is fully faithful because for any $M \in \mathbf{M}^{\mathrm{ls}}(S, \mathbb{Q})$, the adjunction morphism

$$M \to \tau^{\mathrm{M}, \leqslant 0} r_* r^*$$

is an isomorphism. This follows from conservativity of R_ℓ together with the analogous statement for lisse étale sheaves. Let finally $N \in \mathbf{M}(K, \mathbb{Q})$ be an object such that $R_\ell(N)$ is the restriction of a lisse sheaf over S. Then

$$R_\ell(\tau^{\mathrm{M}, \leqslant 0} r_* N) = \mathrm{H}^0\, r_* R_\ell(N)$$

is lisse, hence dualizable in $\mathrm{D}_{\text{ét},c}(S, \mathbb{Q}_\ell)$ by Lemma 2.5. But as R_ℓ is a symmetric monoidal closed conservative functor, it reflects dualizability, so we can conclude that

$$\tau^{\mathrm{M}, \leqslant 0} r_* N \in \mathbf{M}^{\mathrm{ls}}(S, \mathbb{Q})$$

and hence the Conjecture follows. □

Remark 2.9. Given the conjectural definition of lisse motives above, one might be tempted to write $\mathrm{DM}^{\mathrm{ls}}(S)$ instead of $\mathrm{DM}^{\mathrm{dl}}(S)$. We would like to reserve the former notation for the full subcategory of $\mathrm{DM}(S)$ spanned by compact objects whose cohomology groups under the motivic t-structure lie in $\mathbf{M}^{\mathrm{ls}}(S, \mathbb{Q})$ (or, to give an unconditional definition, whose realizations all have lisse cohomology sheaves). Whether $\mathrm{DM}^{\mathrm{ls}}(S)$ and $\mathrm{DM}^{\mathrm{dl}}(S)$ should be expected to agree is unclear to the author and linked to remark 2.6.

2.4. **Pepin Lehalleur's 1-motives.** As noted above, Deligne's category of 1-motives $\mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Q})$ is expected to be the right one if the base S is the spectrum of a perfect field. As Pepin Lehalleur's category $\mathbf{M}^1(S, \mathbb{Q})$ proceeds via $\mathrm{DM}(S)$, it is worthwhile to quickly recall the relation between $\mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Q})$ and $\mathrm{DM}(S, \mathbb{Q})$. By Proposition 2.2 we have an embedding of $\mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Q})$ into $\mathrm{Cpl}(\mathrm{Sh}((\mathrm{Sm/S})_{\mathrm{et}}, \mathbb{Q}))$. Postcomposing with first the $\mathbb{A}^1$-localization and then the T-stabilization, we obtain a functor

$$\tilde{\Phi}_S(1)\colon \mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Q}) \to \mathrm{DM}(S)$$

and then define

$$\tilde{\Phi}_S := \big(\tilde{\Phi}_S(1)\big)(-1).$$

The final Tate twist is there for technical reasons: Using terminology introduced just below, it is because the image of $\tilde{\Phi}_S(1)$ consists of homological motives and we want to work with cohomological ones.

Definition 2.10 ([PL19, Definitions 1.1, 3.1])**.** Let S be a scheme.

- The category $\mathrm{DM}^{\mathrm{coh}}(S)$ is the smallest localizing subcategory of $\mathrm{DM}(S)$ containing
$$\{f_*\mathbb{Q}_X \mid f\colon X \to S \text{ proper}\}.$$
- The category $\mathrm{DM}^1(S)$ is the smallest localizing subcategory of $\mathrm{DM}(S)$ containing
$$\{f_*\mathbb{Q}_X \mid f\colon X \to S \text{ proper and of relative dimension at most } 1\}.$$
- The inclusion $\mathrm{DM}^1(S) \to \mathrm{DM}^{\mathrm{coh}}(S)$ preserves small sums, hence has a right adjoint
$$\omega^1\colon \mathrm{DM}^{\mathrm{coh}}(S) \to \mathrm{DM}^1(S).$$
by adjoint functor theorems.

Replacing the pushforward f_* along proper morphisms f by $f_\sharp$ for smooth f yields analogous homological categories. It is however unclear whether an analogue to ω^1 exists in this setup. It is easy to see that some restriction on f besides being of relative dimension less or equal to 1 is necessary for the definition of $\mathrm{DM}^1(S)$ to make sense. Indeed, open immersion are of relative dimension zero, but their pushforward picks up contributions from the closed complement by localization sequences, hence should not be covered by a theory of 1-motives. The categories introduced above satisfy the expected stability properties and a version of localization.

Proposition 2.11 ([PL19, Propositions 1.17, 1.12, Corollary 1.19])**.**

- *The subcategory* $\mathrm{DM}^1(-) \subset \mathrm{DM}(-)$ *is preserved by* f^* *for arbitrary morphisms* f *and by* $f_!$ *for quasi-finite morphisms* f*.*
- *The subcategory* $\mathrm{DM}^{\mathrm{coh}}(-) \subset \mathrm{DM}(-)$ *is preserved by pullback along arbitrary morphisms as wells as by* $f^!, f_*, f_!$ *as long as* f *is separated and of finite type.*
- *Let* $M \in \mathrm{DM}(S)$ *and let* $i\colon Z \to S$ *and* $j\colon U \to S$ *be complementary closed and open immersions. Then* M *lies in* $\mathrm{DM}^{\mathrm{coh}}(S)$ *(resp. in* $\mathrm{DM}^1(S)$*) if* i^*M *and* j^*M *lie in* $\mathrm{DM}^{\mathrm{coh}}(Z)$ *and* $\mathrm{DM}^{\mathrm{coh}}(U)$ *(resp.* $\mathrm{DM}^1(Z)$ *and* $\mathrm{DM}^1(U)$*).*

If one restricts to compact objects, the last point gives rise to a punctual reconnaissance criterion by continuity.

The fact that $\mathrm{DM}^{\mathrm{coh}}$ is preserved by pushforward along separated, finite type morphisms $f\colon S \to T$ distinguishes it from DM^1 and is the reason to consider it. Indeed, using the reflection ω^1, we get an adjunction
$$f^*\colon \mathrm{DM}^1(T) \leftrightarrows \mathrm{DM}^1(S) : \omega^1 f_*.$$
which, as we will see below, even restricts to compact objects.

The case where the base $S = \mathrm{Spec}(k)$ is the spectrum of a perfect field k has been extensively studied rather early. Voevodsky announced and Orgogozo proved the following comparison:

Theorem 2.12 ([Org04][PL19, Lemma 3.12]). *Let S be the spectrum of a perfect field. Then $\tilde{\Phi}_S$ has essential image in $\mathrm{DM}^1(S)$ and in fact induces an equivalence of categories*

$$D^b(\mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q})) \to \mathrm{DM}_c^1(S).$$

In particular, we can use this equivalence to transport the standard t-structure from the left hand side to a nice t-structure on the right hand side, having as $\mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q})$ as heart. Also, as ℓ-adic realization of Deligne 1-motives is conservative, we get a corresponding conservativity statement for $\mathrm{DM}_c^1(S)$.

In [BVK16], a "motivic Picard functor"

$$\omega^1 \colon \mathrm{DM}_c^{\mathrm{coh}}(S) \to \mathrm{DM}_c^1(S)$$

is constructed, again in the perfect field case. The t-structure and the motivic Picard functor are investigated for not necessarily compact motives in [ABV09] and [Ayo11].

Let us describe how to generalize these results to an arbitrary base scheme S. First, we have the following result on the comparison functor $\tilde{\Phi}_S$.

Theorem 2.13 ([PL19, Corollary 2.19]). *Let S be a scheme. The comparison functor $\tilde{\Phi}_S$ has essential image in $\mathrm{DM}_c^1(S)$, giving rise to*

$$\tilde{\Phi}_S \colon \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \to \mathrm{DM}_c^1(S).$$

If S is moreover geometrically unibranch, the essential image of $\tilde{\Phi}_S$ lies also in $\mathrm{DM}^{ls}(S)$.

This is a rather hard result. The difficult part is deriving the assertion for the abelian scheme part A, which builds on the relation between $\tilde{\Phi}_S(A)$ and the motive of A explored in [AEWH15] and [AHPL16]. The restriction to geometrically unibranch base schemes on the other hand is forced by the toric and lattice part. Presumably one could avoid this assumption by working systematically with the pro-étale topology instead of the étale topology. This technical point is discussed in detail in [PL19, Appendix A]. An easier result is the observation that conservativity of the ℓ-adic realization functor still holds over general bases, see [PL19, Proposition 4.1]. This follows quite directly from the localization property of $\mathrm{DM}(-)$ and the punctual conservativity statement.

The two main results which we now discussed in the case of spectra of perfect fields and we still have to generalize are ω^1 restricting to compact objects and the existence of a nice t-structure. Dropping the perfectness assumption is not a problem, see [PL17, Propositions 3.6,3.7; Lemma 3.8]. In order to pass to general bases, there are two different approaches available. Working exclusively with compact objects, Vaish has found a way to glue together the punctual motivic Picard functor as well as the punctual motivic t-structures to get global results, see [Vai17]. Pepin Lehalleur's approach [PL19, PL17] on the other hand develops everything for not necessarily compact objects first and deals with the problem of restricting to compact objects later. The first key result is the existence of a geometric motivic Picard functor:

Theorem 2.14 ([PL19, Theorem 3.21] resp. [Vai17, Theorem 5.2.2]). *The functor $\omega^1 \colon \mathrm{DM}^{\mathrm{coh}}(S) \to \mathrm{DM}^1(S)$ preserves compact objects.*

The proofs of this is result are rather nontrivial: Pepin Lehalleur relies on explicitly relating $\omega^1 f_* \mathbb{Q}$ to the relative Picard complex of f for a special class of functions f, while Vaish's method builds upon the results from [BVK16] on the motivic Albanese and Picard functors. Finally, Pepin Lehalleur defined a t-structure on $\mathrm{DM}^1(S)$:

Definition 2.15 ([PL19, Definition 4.10])**.** The 1-motivic t-structure τ^1 on $\mathrm{DM}^1(S)$ is the cohomological t-structure generated by the family

$$\{e_\# \tilde{\Phi}_S(\mathbb{M}) \mid e\colon U \to S \text{ étale},\ \mathbb{M} \in \mathbf{M}_1^{\mathrm{Del}}(U, \mathbb{Q})\}$$

of compact objects. We denote by $\mathbf{M}^1(S)$ its heart and by $\mathbf{M}^{1,\mathrm{ls}}(S)$ the full subcategory of $\mathbf{M}^1(S)$ spanned by objects which lie in $\mathrm{DM}^{\mathrm{dl}}(S)$.

We refer the reader to [Ayo07, Definition 2.1.68] for the notion of generated t-structures. In the above setup, it means that the full subcategory $\mathrm{DM}^{1,\leqslant 0}(S)$ of $\mathrm{DM}^1(S)$ is spanned by those objects which are left orthogonal to all $e_\# \tilde{\Phi}_S(\mathbb{M})$. Note that contrary to the homological conventions of [Ayo07, PL19, PL17] we use cohomological notation. We indicate this by using superscripts for the truncation functors and hope that it does not lead to confusion. This 1-motivic t-structure is non-degenerate by [PL19, Proposition 4.30] and agrees with the already constructed t-structure if the base is the spectrum of a field by [PL19, Proposition 4.21]. A key result is the fact that this t-structure restricts to compact objects:

Theorem 2.16 ([PL17, Theorem 4.1], resp. [Vai17, Theorem 5.2.4])**.** *The t-structure τ^1 restricts to compact objects.*

Let us also summarize the known exactness properties of the four functors with respect to τ^1.

Proposition 2.17 ([PL19, Proposition 4.14])**.** *Let f be a morphism of schemes. Then f^* is t-left exact and $\omega^1 f_*$ is t-right exact with respect to τ^1. If f is quasi-finite and separated, then $f_!$ is t-left exact, and hence $\omega^1 f^!$ is t-right exact. Finally, if f is finite, then f_* is t-exact.*

Recall that f_* preserves $\mathrm{DM}^1(-) \subset \mathrm{DM}(-)$ for finite morphisms, hence we do not need ω^1 in this case. If we restrict to compact objects, we obtain the following useful

Proposition 2.18 ([PL17, Theorem 4.1])**.** *For any morphism $f\colon T \to S$, the induced functor*

$$f^*\colon \mathrm{DM}^1_c(S) \to \mathrm{DM}^1_c(T)$$

is t-exact. Moreover, if ℓ is a prime invertible on S, the restriction of the rational ℓ-adic realization functor

$$R_\ell\colon \mathrm{DM}^1_c(S) \to \mathrm{D}_{\text{ét},c}(S, \mathbb{Q}_\ell)$$

is t-exact with respect to τ^1 on the left hand side and the standard t-structure on the right hand side.

This is highly non-formal and relies on an explicit geometric analysis of degenerating curves, which is the main content of [PL17].

Having discussed the t-structure on $\mathrm{DM}^1(S)$, we can now talk about comparing $\mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q})$ to $\mathbf{M}^1(S)$. As a first step, we see that the comparison functor $\tilde{\Phi}_S$ has image in the heart.

Theorem 2.19 ([PL19, Corollary 2.19, Theorem 4.22])**.** *The comparison functor $\tilde{\Phi}_S$ restricts to*

$$\Phi_S\colon \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \to \mathbf{M}^1(S).$$

If S is geometrically unibranch, then, as noted above, its essential image even lies in $\mathbf{M}^{1,\mathrm{ls}}(S)$.

Pepin Lehalleur also proved the following full faithfullness assertion:

Theorem 2.20 ([PL19, Theorem 4.31])**.** *Let S be a regular scheme. Then*

$$\Phi_S\colon \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \to \mathbf{M}^{1,\mathrm{ls}}(S)$$

is fully faithful.

As noted in the introduction, we will generalize this result to drop the strong regularity assumption and also investigate essential surjectivity, see Theorem 6.12 as well as Corollaries 6.13 and 6.14.

We also need compatibility of the two notions of realization along Φ_S:

Lemma 2.21. *Let S be a scheme, ℓ a prime invertible on S, and let $\mathbb{M} \in \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q})$ be a Deligne 1-motive. Then there is a canonical isomorphism*

$$R_\ell(\mathbb{M})(-1) \cong R_\ell(\Phi_S(\mathbb{M}))$$

in $\mathrm{Sh}^{\mathrm{ls}}_{\mathrm{\acute{e}t}}(S,\mathbb{Q}_\ell)$.

Proof. This is just unraveling the definitions. The additional Tate twist on the left hand side is forced by our choice of baking such a twist into the definition of Φ_S. By construction of both rational realizations, we choose the obvious integral model and reduce to understanding the mod ℓ^r-cases instead. There, rigidity tells us that composing

$$D(\mathrm{Cpl}(\mathrm{Sh}((\mathrm{Sm/S})_{\mathrm{\acute{e}t}},\mathbb{Z}/\ell))) \to \mathrm{D}_{\mathbb{A}^1,\mathrm{\acute{e}t}}(S,\mathbb{Z}/\ell) \cong \mathrm{D}_{\mathrm{\acute{e}t},c}(S,\mathbb{Z}/\ell)$$

is just the obvious restriction from Sm/S to Et/S. As the first functor on the integral level is monoidal, we are finally reduced to comparing the restriction to the small etale site of

$$\mathbb{M} \otimes \mathbb{Z}/\ell^r,$$

the (derived) tensor product taking place inside $D(\mathrm{Cpl}(\mathrm{Sh}((\mathrm{Sm/S})_{\mathrm{et}},\mathbb{Z})))$, with the ℓ-adic realization of Deligne 1-motives and we notice that they coincide by construction. $\square$

3. Strategy of proof

In this section, we sketch the proof of the main theorem.

Theorem A (Theorem 6.1). *Let S normal integral scheme satisfying $(\star)$, let ℓ be a prime invertible on S, and let $j\colon U \to S$ be an open immersion. Then the following is true:*

- *For $N \in \mathbf{M}^{1,\mathrm{ls}}(S,\mathbb{Q})$, the canonical map*
$$N \to \tau^{1,\leqslant 0}\omega^1 j_* j^* N$$
is an isomorphism.
- *Let $M \in \mathbf{M}^{1,\mathrm{ls}}(U,\mathbb{Q})$ be a lisse 1-motive such that $R_\ell(M)$ is unramified over S. Then the pushed forward motive*
$$\tau^{1,\leqslant 0}\omega^1 f_* M$$
lies in $\mathbf{M}^{1,\mathrm{ls}}(S,\mathbb{Q})$.

In particular, the square
$$\begin{tikzcd}
\mathbf{M}^{1,\mathrm{ls}}(U,\mathbb{Q}) \arrow[r, "R_\ell"] & \mathrm{Sh}_{\text{ét}}(K,\mathbb{Q}_\ell) \\
\mathbf{M}^{1,\mathrm{ls}}(S,\mathbb{Q}) \arrow[r, "R_\ell"'] \arrow[u, hook, "\mathrm{res}"] & \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S,\mathbb{Q}_\ell) \arrow[u, hook, "\mathrm{res}"']
\end{tikzcd}$$
satisfies (NOS).

We divide the statement in a uniqueness-type and a existence-type statement, which are proven quite differently.

Lemma U (Lemma 6.6). *Let $j\colon U \to S$ be an open immersion of normal schemes, and let $N \in \mathbf{M}^{1,\mathrm{ls}}(S)$ be a lisse 1-motive over S. Then the canoncial map*
$$N \to \tau^{1,\leqslant 0}\omega^1 j_* j^* N$$
is an isomorphism.

Note that this statement is strictly stronger than the assertion that the pullback
$$j^*\colon \mathbf{M}^{1,\mathrm{ls}}(S) \to \mathbf{M}^{1,\mathrm{ls}}(U)$$
is fully faithfull.

Lemma E (Lemma 6.7). *Let $j : U \to S$ be an open immersion between normal schemes, and let $M \in \mathbf{M}^{1,\mathrm{ls}}(U)$ be a lisse 1-motive whose ℓ-adic realization has good reduction over S. Then after passing to a finite étale cover of S, we find a lisse 1-motive $N \in \mathbf{M}^{1,\mathrm{ls}}(S)$ with $j^*N \cong M$.*

As the functor $\tau^{1,\leqslant 0}\omega^1 j_*$ is compatible with finite étale base change, the two statements together clearly imply the main theorem.

Proof of Lemma U. The proof the uniqueness statement is fairly straightforward. If S is regular, this was communicated to the author by Pepin Lehalleur; in that case, one can use absolute purity and dualizability of N to understand the exceptional inverse image of N under the inclusion $S\backslash U \to S$. As one knows how ω^1 interacts with Tate twists, this allows one to conclude. For the general case, assume for simplicity that there is a resolution of singularities $p\colon \tilde{S} \to S$ with S regular, which we can assume to be an isomorphism over U by shrinking U. Then the already shown regular case allows to reduce to showing in fact that

$$N \to \tau^{1,\leqslant 0}\omega^1 p_* p^* N$$

is an isomorphism. This we can do locally in the h-topology. Using then that p acquires a section after passing to the h-cover given by p itself, the statement follows from faithfullness of ℓ-adic realizations together with a similar statment for lisse ℓ-adic sheaves.

Proof of Lemma E, Case of dimension less or equal 2. The existence part of the statement is more involved. The essential idea is to construct the 1-motive N locally in the rh-topology and then glue the local pieces. For a Deligne 1-motive M, consider the following assumption

($\star$) The lattice part and the character lattice of the toric part are constant group schemes, and the ℓ^2-quotient $T_\ell(A)/\ell^2$ of the ℓ-adic Tate module of the abelian part A of M is also constant.

The assumption on the lattice and toric part makes those 1-motives easier to work with, but is not essential; the assumption on the abelian part could be also called "existence of a level ℓ^2-structure and is needed because the moduli functor paramtrizing polarized abelian schemes with level $\geqslant 4$-structure is representable by a scheme and not merely a DM-stack. At any rate, every Deligne 1-motive satisfies ($\star$) after pulling back along a finite étale cover, which we allowed ourselves in the statement of the theorem. The following result reduces extending Deligne 1-motives to extending abelian schemes:

Theorem 3.1 (Theorem 4.10)**.** *Let S be a normal connected noetherian scheme of finite dimension and ℓ be a prime number invertible on S. Let $M = [L \xrightarrow{u} G]$ be a Deligne 1-motive over the generic point η of S and suppose that its abelian part A and its ℓ-adic Tate module F both have good reduction. Then M has good reduction.*

This together with the classical criterion of Néron-Ogg-Shafarevich handles the case $\dim(S) = 1$ directly. Assume now $\dim(S) = 2$. By continuity and the uniqueness result, we can reduce to S being strictly local with closed point $s \in S$. By the 1-dimensional case, we can assume given a Deligne 1-motive $M \in \mathbf{M}_1^{\mathrm{Del}}(U, \mathbb{Z})$ such that its ℓ-adic Tate module extends to a lisse ℓ-adic sheaf $\mathcal{F}$ over S. In particular, M automatically satisfies condition ($\star$). We outline the construction of a lisse 1-motive $N \in \mathbf{M}^{1,\mathrm{ls}}(S)$ with $j^*N \cong \Phi_U(M)$.

Step 1: Extending M after modifying the base.

We find a proper morphism $p\colon \hat{S} \to S$ from a normal, connected $\hat{S}$ that is an isomorphism over U together with a Deligne 1-motive $\hat{M}/\hat{S}$ extending M, see Proposition 4.28. For the abelian part, this follows from the existence of a moduli scheme of polarized abelian varieties with level structure together with

our assumptions on existence of $\mathcal{F}$. Theorem 4.10 allows us to then deduce the statement for the whole of M.
Step 2: The 1-motive over the exceptional fiber is isoconstant.
In fact, Section 4.3 is devoted to proving the following

Theorem 3.2 (Theorem 4.18). *Let k be a field of exponential characteristic p and let $\varepsilon : S \to \operatorname{Spec}(k)$ be a geometrically connected finite type reduced k-scheme. Then the functor*

$$\varepsilon^* \colon \mathbf{M}_1^{\mathrm{Del}}(\operatorname{Spec}(k), \mathbb{Z}[1/p]) \to \mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z}[1/p])$$

is fully faithful with essential image those 1-motives $M \in \mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z}[1/p])$ whose ℓ-adic realization $R_\ell(M)$ is constant over $S \otimes \bar{k}$ for some (hence every) prime $\ell \neq p$.

In our situation, this produces a 1-motive $M_s \in \mathbf{M}_1^{\mathrm{Del}}(s, \mathbb{Z})$ together with an isogeny

$$\phi_s \colon M_s \times \hat{S}_s \to \hat{M}$$

of Deligne 1-motives over $\hat{S}_s$, where we denote by $\hat{S}_s := (\hat{S} \times_S s)_{red}$ the reduced special fiber.
Step 3: Using h-descent for motives.
As the square

$$\begin{tikzcd}
\hat{S} \arrow[d, "p"] & \hat{S}_s \arrow[l, "i'"'] \arrow[d, "p'"] \\
S & s \arrow[l, "i"']
\end{tikzcd}$$

is an abstract blow-up square (up to passing to reduced subschemes, which is inconsequential), we get an equivalence of symmetric monoidal stable ∞-categories

$$\mathrm{DM}(S) \to \mathrm{DM}(\hat{S}) \times_{\mathrm{DM}(\hat{S}_s)} \mathrm{DM}(s).$$

see Lemma 5.7.
Step 4: Input from category theory.
Objects of pullbacks of ∞-categories are easy to describe. In particular, the triple

$$\left(\Phi_{\hat{S}}(\hat{M}), \Phi_s(M_s), \Phi_{\hat{S}_s}(\phi_s) \colon (p')^*\Phi_s(M_s) \cong (i')^*\Phi_{\hat{S}}(\hat{M})\right)$$

defines an object N in the pullback above, which is strongly dualizable by a result of Lurie [Lur17, Proposition 4.6.1.11].
Step 5: End of the proof.
Observe that $j^*N \cong \Phi_U(M)$ by construction, so it remains to check that N is in $\mathrm{DM}^1(S)$ and actually in the heart of the t-structure. This can be checked on points of the basis S [PL19, Proposition 1.25] and [PL17, Theorem 4.1 (ii)], where it is obvious by construction: Over U, we have $j^*N \cong \Phi_U(M) \in \mathbf{M}^{1,\mathrm{ls}}(U)$ and over s we have $N_s \cong \Phi_s(M_s) \in \mathbf{M}^{1,\mathrm{ls}}(s)$.

Proof of Lemma E, Higher dimensional case. The strategy of proof for the surface case has to be adapted somewhat to work in the higher dimensional case. To fix ideas, consider the case $\dim(S) = 3$ and assume as before that S is strictly local, that $j\colon U \to S$ has complement of codimension 2, and that $M \in \mathbf{M}_1^{\mathrm{Del}}(U, \mathbb{Z})$ has ℓ-adic Tate module with good reduction over S, hence that M satisfies condition $(\star)$. We use the same numbering for steps as in the 2-dimensional case, and we will use the following notational convention: Starting with $S^0 := S$ and $U^0 := U$, schemes with upper indices will denote subschemes of S. Modifications thereof will be denoted with a hat and upper indices in curly brackets, e.g. $\hat{S}^{\{0\}}$ will denote a modification of S^0. A collection of upper indices refers to the reduced subscheme which underlies the fibered product of all schemes over S, e.g

$$S^{\{0,1\},2} := \left(\hat{S}^{\{0\}} \times_S \hat{S}^{\{1\}} \times_S S^2\right)_{red}.$$

Finally, canonical morphisms between such gadgets will be denoted with the letter p, carrying as superscript the index of the source and as subscript the index of the target. For example,

$$p_0^{\{0\}} : \hat{S}^{\{0\}} \to S^0$$

will be a modification.

Step 1: Extending M over a modification.

As before, we construct a modification

$$p_0^{\{0\}} \colon \hat{S}^{\{0\}} \to S^0$$

of S^0 with $\hat{S}^{\{0\}}$, such that $p^{\{0\}}$ is an isomorphism over U^0 and such that we have an extension $\hat{M}^0$ of M over $\hat{S}^{\{0\}}$. As announced above, we set $S^1 := (S^0 \backslash U^0)_{red}$ and

$$\hat{S}^{\{0\},1} := (\hat{S}^{\{0\}} \times_S S^1)_{red}$$

to be the reduced exceptional locus of the modification, giving a commuting diagram

$$\begin{tikzcd}
 & \hat{S}^{\{0\}} \arrow[dd, "p_0^{\{0\}}"] & \hat{S}^{\{0\},1} \arrow[l] \arrow[dd, "p_0^{\{0\},1}"] \\
U^0 \arrow[ur, "\hat{\jmath}^0"] \arrow[dr, "j^0"'] & & \\
 & S^0 & S^1. \arrow[l]
\end{tikzcd}$$

Step 2: The 1-motive over the exceptional locus comes generically from a 1-motive downstairs.

As $S \backslash U$ is no longer 0-dimensional, we can not apply Theorem 4.18 directly; however, by continuity, we do get a result generically on S^1. In fact, we find over a normal open dense $j^1\colon U^1 \to S^1$ a Deligne 1-motive M^1 together with an isogeny

$$\phi^{1,0}\colon \left(q_1^{\{0\},1}\right)^* M^1 \cong \hat{M}^0_{|(\hat{S}^{\{0\},1} \times_{S^1} U^1)}$$

where $q_1^{\{0\},1}$ is the base change of $p_1^{\{0\},1}$ along j^1. Note that S^1 has no reason to be normal, so we cannot expect to extend M^1 over the whole of S^1. This forces

us to repeat the first step with S^1 in place of S^0:
Step 1 again: Extending M^1 over a modification.
As above, we construct a diagram

$$\begin{tikzcd} & \hat{S}^{\{1\}} \arrow[d, "p_1^{\{1\}}"] & \hat{S}^{\{1\},2} \arrow[l] \arrow[d, "p_1^{\{1\},2}"] \\ U^1 \arrow[ur, "\hat{j}^1"] \arrow[dr, "j^1"'] & & \\ & S^1 & S^2 \arrow[l] \end{tikzcd}$$

and an extension $\hat{M}^1$ of M^1 over $\hat{S}^{\{1\}}$. More importantly, we can also extend the constructed isogeny $\phi^{1,0}$ in the following sense: Setting

$$\hat{S}^{\{0,1\}} := \left(\hat{S}^{\{1\}} \times_{S^0} \hat{S}^{\{0\}}\right)_{red}$$

and we obtain an isogeny

$$\hat{\phi}^{1,0}\colon \left(p_1^{\{0,1\}}\right)^* M^1 \to \left(p_0^{\{0,1\}}\right)^* M^0$$

by extending $\phi^{1,0}$ using descent results for morphism of 1-motives, see Lemma 4.9.

Step 2 again: M^1 is isoconstant over the special fiber of $p_0^{\{0\}}$.
By our assumptions on S, we know that S^2 is a single point and hence we find a Deligne 1-motive M^2 together with an isogeny

$$\phi^{2,0}\colon \left(p_0^{\{0\},2}\right)^* \hat{M}_0 \to \left(p_2^{\{0\},2}\right)^* M_2$$

on $S^{\{0\},2}$. As an iportant technical point regarding choice of $\{0\},2$ over $\{1\},2$, we note that in general $\hat{S}^{\{1\},2}$ will not be connected. However, $p_0^{\{0,2\}}$ still has geometrically connected fibers. Additionally, we need for later reference also a compatibility condition between $\hat{M}^1$ and M^2 on the fibers over S^2:
Step 2.5: A compatibility condition.
We need to construct additional isogenies

$$\hat{\phi}^{1,2}\colon \left(p_1^{\{1\},2}\right)^* \hat{M}^1 \to \left(p_2^{\{1\},2}\right)^* M^2$$

which come out of the already constructed ones and our descent results. They are constructed in a way to satsify an evident compatibility condition on $\hat{S}^{\{0,1\},2}$. Constructing all those data is the content of Section 4.4, with Corollary 4.33 being the main existence result. Notice that at the end of the construction, we only work on the modifications (except for S^2, because $\hat{S}^{\{2\}} = S^2$). In particular, objects with mixed upper indices such as $\hat{S}^{\{0\},1}$ do not play a role in the end. This is why we adopt a slightly different notational convention in Section 4.4, dropping the curly brackets again.
Step 3: Using h-descent for motives.

By an inductive h-descent argument, we see that we get a cartesian cube

$$\begin{tikzcd}
\mathrm{DM}\left(S^0\right) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{DM}\left(\hat{S}^{\{1\}}\right) \arrow[dd] \arrow[dr] & \\
& \mathrm{DM}\left(S^2\right) \arrow[rr] \arrow[dd] & & \mathrm{DM}\left(\hat{S}^{\{1\},2}\right) \arrow[dd] \\
\mathrm{DM}\left(\hat{S}^{\{0\}}\right) \arrow[rr] \arrow[dr] & & \mathrm{DM}\left(\hat{S}^{\{0,1\}}\right) \arrow[dr] & \\
& \mathrm{DM}\left(\hat{S}^{\{0\},2}\right) \arrow[rr] & & \mathrm{DM}\left(\hat{S}^{\{0,1\},2}\right)
\end{tikzcd}$$

of symmetric monoidal presentable stable ∞-categories.

Step 4: Describing objects in the limit of the more complicated diagram becomes more annoying in general. Calculating an explicit categorically fibrant replacement of an equivalent diagram is the subject of Section 5. The upshot is still the same, however: The constructed motives $\hat{M}^0, \hat{M}^1, M^2$ and isogenies

$$\hat{\phi}^{0,1}, \phi^{0,2}, \phi^{0,1}, \left(p^{\{0,1\},2}_{\{0,1\}}\right)^* \hat{\phi}^{0,1}, \left(p^{\{0,1\},2}_{\{0\},2}\right)^* \phi^{0,2}, \left(p^{\{0,1\},2}_{\{1\},2}\right)^* \phi^{1,2}$$

together define (after applying the comparsion functors Φ) a dualizable object N in the limit of this diagram.

Step 5: Up to some notational clutter, the rest of the proof proceeds exactly as above, see Lemma 6.7.

4. On Deligne 1-motives

This section contains various results on Deligne 1-motives which represent the algebraic geometry backbone of this work.

4.1. **Preliminaries on Deligne 1-motives.** For any scheme V, denote by $\mathcal{C}(V)$ one of the following categories:

- The category of abelian schemes over V;
- The category of étale lattices over V;
- The category of étale tori over V;
- The category of semi-abelian schemes over V which are globally extensions of abelian schemes by tori;
- The category of Deligne 1-motives over V;
- The category of smooth ℓ-adic sheaves over V, where ℓ is a prime number invertible over V.

In all those cases, the following result is well known:

Proposition 4.1. *Let S be normal connected with generic point η and let $\mathcal{C}$ be one of the above. Then the restriction functor*

$$\mathrm{Res}_{\mathcal{C}} \colon \mathcal{C}(S) \to \mathcal{C}(\eta)$$

is fully faithful.

Proof. The case of lattices, tori and ℓ-adic sheaves follows from their description as representations of the étale fundamental group and [SGA71, Exp. V Proposition 8.1]. The case of (semi-)abelian schemes is [FC90, Proposition 2.7], and those cases are put together to obtain the result for Deligne 1-motives in [PL19, Proposition A.11]. □

Definition 4.2. Let $S, \mathcal{C}$ be as above. An object in $\mathcal{C}(\eta)$ is said to *have good reduction* or to *extend over* S if it is in the essential image of $\mathrm{Res}_{\mathcal{C}}$.

In this context, let us recall the following continuity result, which always allows to extend objects over η to objects over some open U of S.

Proposition 4.3 (see [PL19, Proposition A.10])**.** *Let $S = \lim_{i\in I} S_i$ be a projective limit of qcqs schemes with affine transition maps. Assume that*

(1) $\mathcal{C}$ is the category of abelian schemes or
(2) $\mathcal{C}$ is any of the categories listed above with exception of the last one and all S_i are connected normal.

Then

$$\operatorname*{colim}_{i\in I} \mathcal{C}(S_i) \to \mathcal{C}(S)$$

is an equivalence of categories.

Proof. The case of abelian schemes follows from [Gro66a, Théorèmes 8.8.2, 8.10.5] and [Gro67, Proposition 17.7.8], which imply that the category of smooth proper schemes of finite presentation over S is the colimit of those over the S_i; then use that taking group objects of a category is compatible with passage to the colimit [Gro66a, Scholie 8.8.3] to deduce full faithfullness. In oder to see essential surjectivity, start with an abelian scheme A over S and find, by the observation above, a proper smooth group scheme A_i over some S_i which pulls back to S. Replacing A_i by the relative connected component of the identity we obtain an abelian scheme over S_i which still pulls back to A.

For the second case we first observe that the category of finite étale connected schemes over S is the colimit of the categories over the S_i. As any étale lattice on a connected normal scheme is split by a finite étale connected cover, we can reduce full faithfullness and essential surjectivity to the case of constant lattices and split tori everywhere. By a duality argument, the only thing left to do is to spread out morphisms from split lattices to semi-abelian varieties. As those morphisms are determined by their behavior on a basis which is a finitely presented S-scheme, we are done by the references above. $\square$

The duality between tori and lattices as well as Cartier duality of abelian schemes (which works without assumption on the base scheme) extend to give a Cartier duality theory on 1-motives. The following material is certainly well-known to experts, but the author could find no adequate reference in the literature. We thus give a quick sketch of the theory. Fix a base scheme S and write as shorthand

$$\mathcal{D} := D(\mathrm{Cpl}(\mathrm{Sh}((\mathrm{Sm/S})_{\mathrm{et}}, \mathbb{Z}))),$$

a symmetric monoidal closed stable ∞-category whose internal Hom we denote by $[,]$, trusting that this does not cause confusion with the shift functor. We see $\mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z})$ as a full subcategory of $\mathcal{D}$ and suppress the embedding functor from the notation. For a 1-motive M in $\mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z})$ we set

$$M^\vee := [M, \mathbb{G}_m[-1]] \in \mathcal{D}$$

where $\mathbb{G}_m = \mathbb{G}_{m,S}$ and we recall that by our grading convention $\mathbb{G}_m[-1]$ is a Deligne 1-motive.

Lemma 4.4. *Using the notations and conventions above, $M^\vee$ is again a Deligne 1-motive. Its lattice, toric and abelian part are the Cartier duals of the toric, lattice and abelian parts of M.*

Proof. First, note that the result is well known for pure 1-motives; the second sentence of the lemma being a tautology. Consider now $M = [L \to G]$ where G sits in the extension $1 \to T \to G \to A \to 1$, with T a torus and A an abelian scheme. Write for simplicity B, S, X for the Cartier duals of A, T, L. Assume first $L = 0$ to obtain from the extension above a fiber sequence

$$A^\vee[-1] \to M^\vee \to T^\vee$$

showing that $M^\vee = \mathrm{fib}(T^\vee \to A^\vee)$ is indeed just the Deligne 1-motive $[T^\vee \to A^\vee]$. Suppose on the other hand that $T = 0$, and note that we have a fiber sequence

$$L \to A \to M[1]$$

inducing

$$A^\vee \to L^\vee \to M^\vee$$

and we claim that $M^\vee = H[-1]$, where H is a semi-abelian variety, extension of B by S. Indeed, this follows from the above fiber sequence together with the vanishing of $\mathrm{Hom}(B, S)$. We thus have seen that the duality maps the category of 1-motives without lattice part to the category of 1-motives without toric part and vice versa. This is the only instance of Cartier duality which is going to be used in this work, but for completeness, we further explain the case of general 1-motives M. From the fiber sequence

$$M \to L \xrightarrow{u} G$$

we get a fiber sequence $G^\vee \xrightarrow{u^\vee} S[-1] \to M^\vee$. On the other hand, we have fiber sequences

$$(G)^\vee[1] \to X \to B$$

and

$$S \to H \to B$$

which yield a commutative diagram

$$\begin{tikzcd}
G^\vee \arrow[r] \arrow[d, "u^\vee"'] & X[-1] \arrow[r] \arrow[d, dashed, "\phi"'] & B[-1] \arrow[d, equal] \\
S[-1] \arrow[r] \arrow[d] & H[-1] \arrow[r] \arrow[d] & B[-1] \\
M^\vee \arrow[r, equal] & M^\vee &
\end{tikzcd}$$

of fiber sequences, showing that $M^\vee$ is indeed the 1-motive $[X \xrightarrow{\phi} H]$. Note that the vanishing of $\mathrm{Hom}(B, S)$ guarantees that ϕ is unique, as it should be. $\square$

Using this calculation, we obtain the full Cartier duality.

Proposition 4.5. *The functor* $(-)^\vee$ *defines an autoequivalence on* $\mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z})$. *There is a canonical natural isomorphism* $\mathrm{id} \to (-)^{\vee\vee}$. *Finally, if* ℓ *is a prime invertible on* S*, then we get a natural perfect pairing*

$$R_\ell(M) \otimes R_\ell(M^\vee) \to \mathbb{Z}_\ell(1).$$

Proof. With the category of Deligne 1-motives sitting fully faithfully inside $\mathcal{D}$, it is clear from the above lemma that the duality induces an endofunctor of $\mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z})$. From the adjunction between internal Hom and tensor product in $\mathcal{D}$, we get from the identity $[M, \mathbb{G}_m[-1]] \to [M, \mathbb{G}_m[-1]]$ a canonical morphism

$$[M, \mathbb{G}_m[-1]] \otimes M \to \mathbb{G}_m[-1]$$

which is classically called *Poincaré Biextension* of M, and which again by adjunction yields the natural morphism

$$M \to [[M, \mathbb{G}_m[-1]], \mathbb{G}_m[-1]] = M^{\vee\vee}.$$

By classical duality results, this is an isomorphism on the pure pieces of M, by naturality it is hence an isomorphism for all 1-motives M. Similarly, applying R_ℓ to the Poincaré Biextension yields the mentioned pairing on ℓ-adic realizations, coinciding with the Weil Pairing if M is an abelian scheme. Hence perfectness of the pairing is known for pure 1-motives and thus for all 1-motives. $\square$

We axiomatize a strategy of proving statements about 1-motives which relies on this duality.

Lemma 4.6 (Duality trick)**.** *Let* $(\mathcal{P})$ *be a statement about Deligne 1-motives such that*

(1) $(\mathcal{P})$ *is true for abelian schemes over* S

(2) $(\mathcal{P})$ is true for a 1-motive of the form $[0 \to G]$ if and only if it is true for its dual $[X \to A^\vee]$.
(3) If $(\mathcal{P})$ is true for a 1-motive $[0 \to G]$ in $\mathcal{C}_{sab}(S)$, it is also true for all 1-motives of the form $[L \to G]$ in $\mathbf{M}_1^{\mathrm{Del}}(S)$.

Then $(\mathcal{P})$ is true for all objects of $\mathbf{M}_1^{\mathrm{Del}}(S)$.

Proof. Let $[L \to G]$ be a 1-motive in $\mathbf{M}_1^{\mathrm{Del}}(S)$ with toric part T and abelian part A. As $(\mathcal{P})$ is true for $[0 \to A^\vee]$ by 1), it also holds for $[X \to A^\vee]$ by 3) and hence for $[0 \to G]$ by 2). Now we may apply 3) a second time to conclude. $\square$

In particular, in order to generalize a statement from abelian schemes to generalize 1-motives one can usually assume that the statement already holds for globally semi-abelian schemes.

Lemma 4.7. *Let S be a locally noetherian scheme, ℓ a prime invertible on S and $M_1, M_2 \in \mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z})$. Then the canonical map*

$$\mathrm{Hom}_{\mathbf{M}_1^{\mathrm{Del}}(S)}(M_1, M_2) \to \mathrm{Hom}_{\mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S,\mathbb{Z}_\ell)}(T_\ell(M_1), T_\ell(M_2))$$

is an injection.

Proof. The statement is well known for abelian schemes by schematic density of the ℓ^∞-torsion and is easy for lattices and tori. Denote the graded pieces of the weight filtration of M_i by $\mathrm{Gr_j(M_i)}$ and denote by

$$\mathrm{Hom}^{fil}(T_\ell(M_1), T_\ell(M_2)) \subset \mathrm{Hom}(T_\ell(M_1), T_\ell(M_2))$$

the subset of homormorphism that are compatible with the filtration coming from the weight filtration of 1-motives. The commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(M_1, M_2) \arrow[r, hook] \arrow[d] & \prod_{j=-2,-1,0} \mathrm{Hom}(Gr_j(M_1), Gr_j(M_2)) \arrow[d, hook] \\
\mathrm{Hom}^{fil}(T_\ell(M_1), T_\ell(M_2)) \arrow[r, hook] \arrow[d, hook] & \prod_{j=-2,-1,0} \mathrm{Hom}(T_\ell(Gr_j(M_1)), T_\ell(Gr_j(M_2))) \\
\mathrm{Hom}(T_\ell(M_1), T_\ell(M_2)) &
\end{tikzcd}$$

then finishes the proof. $\square$

Lemma 4.8. *Let S be locally noetherian reduced, M_1, M_2 be two Deligne 1-motives over S whose toric and lattice parts are split. Then the functor which associates to an S scheme S' the set*

$$\mathrm{Hom}_{\mathbf{M}_1^{\mathrm{Del}}(S',Z)}((M_1)_{S'}, (M_2)_{S'})$$

is representable by an S scheme H which is unramified and essentially proper (satisfies the valuative criterion for properness, but is not necessarily finitely presented).

Proof. If M_1, M_2 are abelian schemes, the statement follows from a result of Murre together with Proposition 4.1, as explained in [Gro66b, Proof of Proposition 1.2]. Applying the duality trick, we may assume that the statement is true for semi-abelian schemes. Let L_i (resp. G_i) be the lattice (resp. semi-abelian) part of M_i. The cartesian square of fppf-sheaves on S

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{M}_1^{\mathrm{Del}}(S')}(M_1, M_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{S'-grp}(G_1, G_2) \arrow[d] \\
\mathrm{Hom}_{S'-grp}(L_1, L_2) \arrow[r] & \mathrm{Hom}_{S'-grp}(L_1, G_2)
\end{tikzcd}$$

reduces us to showing that the bottom morphism is an unramified and essentially proper morphism of schemes. This is clear as L_1 is assumed to be split. $\square$

The following is an extension of [Gro66b, Proposition 1.2] to the context of 1-motives.

Lemma 4.9. *Let S be noetherian reduced with maximal points $\{\eta_i\}_{i\in I}$, ℓ be a prime invertible on S and let M_1, M_2 be two Deligne 1-motives over S whose toric and lattice part are split. Suppose we are given a morphism*

$$u\colon R_\ell(M_1) \to R_\ell(M_2)$$

as well as morphism

$$f_i\colon (M_1)_{\eta_i} \to (M_2)_{\eta_i}$$

with $R_\ell(f_i) = u_{\eta_i}$. Then there is a unique morphism $f : M_1 \to M_2$ with $f_i = f_{\eta_i}$ and $R_\ell(f) = u$.

Proof. By the continuity result Proposition 4.3, the morphisms f_i extend to a morphism f_U over some open dense normal $U \subset S$. Denote by H the scheme representing the functor from Lemma 4.8. The morphism f_U corresponds to a section $\phi_U\colon U \to H$ of $H \to S$ over U, which we need to extend over S. Applying [Gro66b, Lemme 1.2.1] we need to check that for any finite morphism $p\colon S' \to S$ with S' reduced such that $U' := S' \times_S U$ is dense in S' and any morphism $u' : S' \to H$ such that $u'|_{U'} = \phi_U \circ p$, the two composites in

$$S' \times_S S' \rightrightarrows S' \xrightarrow{u'} H$$

agree. This follows by applying Lemma 4.7 to this situation and using that u is defined over S. $\square$

4.2. **Good reduction of Deligne 1-motives.**

The main result of this section is the following.

Theorem 4.10. *Let S be a normal connected noetherian scheme of finite dimension and ℓ be a prime number invertible on S. Let $M = [L \xrightarrow{u} G]$ be a Deligne 1-motive over the generic point η of S and suppose that its abelian part A and its ℓ-adic Tate module F both have good reduction. Then M has good reduction.*

We will deduce Theorem 4.10 from the more specific Lemma 4.12 below.

Definition 4.11. Let S, ℓ be as above, let $\mathcal{G}$ be a semi-abelian variety that is an extension of an abelian variety by a torus over S, and let $v\colon S \to \mathcal{G}$ be a section. For $r \in \mathbb{N}$, let $V_{\ell,r}(v)$ be the locally constant étale sheaf of sets on S which on $T \to S$ evaluates to

$$V_{\ell,r}(v)(T) := \{y \in \mathcal{G}(T) \mid \ell^r \cdot y = v\}.$$

Lemma 4.12. *Let $S, \ell, \mathcal{G}$ be as above, and assume in addition that the toric part of $\mathcal{G}$ is a split torus. Let $j\colon U \to S$ be an open dense inclusion and $v\colon U \to \mathcal{G}$ be a section of $\mathcal{G}$ over U. Suppose that for all $r \in \mathbb{N}$, we are given locally constant étale sheaves of sets W_r on S together with isomorphisms $j^*W_r \cong V_{\ell,r}(v)$. Then v extends to a (necessarily unique) section $\tilde{v}\colon S \to \mathcal{G}$.*

Let us show that this lemma is enough to prove the main theorem.

Proof of Theorem 4.10. Denote the toric part of M by T, so G sits in an extension

$$1 \to T \to G \to A \to 1.$$

We first claim that extension of F implies in particular that T and L extend. Since F extends, the Tate modules of L and T certainly extend because they are subquotients of F and triviality of a group representation passes to subquotients.

To see that L extends we have to check that the associated homomorphism

$$\pi_1^{\text{ét}}(\eta, \bar{\eta}) \to \mathrm{GL}(L_{\bar{\eta}})$$

factors over $\pi_1^{\text{ét}}(S, \bar{\eta})$. However, the composition

$$\pi_1^{\text{ét}}(\eta, \bar{\eta}) \to \mathrm{GL}(L_{\bar{\eta}}) \hookrightarrow \mathrm{GL}(L_{\bar{\eta}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell),$$

which describes the Tate module of L, does. For T, we can apply the same argument to the cocharacter lattice and hence see that both L and T extend.

Hence assume that L, A, T and F have extensions $\mathscr{L}, \mathcal{A}, \mathcal{T}$ and $\mathcal{F}$ over S.

Using the duality trick, it is enough to prove the theorem under the assumption that G extends. Indeed, 1-motives of the form $[0 \to G]$ correspond via Cartier duality of 1-motives to 1-motives of the form $[X \to A^\vee]$ where $A^\vee$ is the dual abelian variety to A and $X = \mathrm{Hom}(T, \mathbb{G}_m)$, see 4.5, whose ℓ-adic Tate module is dual to that of G. Note that A extends if and only if $A^\vee$ extends and X extends if and only if T extends. Hence under our assumptions, both the lattice part and the semi-abelian part of $[X \to A^\vee]$ extend. Hence, as explained before, if we produce a proof under the assumption that the semi-abelian part extends, we can apply it first to $[X \to A^\vee]$ and then dualize to obtain an extension of G over S, and then apply it again to obtain an extension of $[L \to G]$.

Thus we assume that G extends to $\mathcal{G}$ in what follows, and we only need to worry about extending u in the general case. As extensions of u are necessarily unique, it is by descent sufficient to produce such an extension after passing to an étale cover of S, so we can assume $\mathcal{T}$ to be a split torus and $\mathscr{L}$ to be a constant lattice, and we directly reduce to the case $\mathscr{L} \cong \mathbb{Z}$. By continuity, see Proposition 4.3, we can assume that the 1-motive extends over a dense open subscheme $j\colon U \to S$, and we are reduced to showing that the section

$$v := u(1)\colon U \to \mathcal{G}$$

extends to a section $\tilde{v}\colon S \to \mathcal{G}$. Recall that, from the weight filtration of the 1-motive, we get a short exact sequence

$$0 \to T_\ell(G) \to T_\ell(M) \xrightarrow{\gamma} \mathbb{Z}_\ell \to 0$$

of étale sheaves over η, which by full faithfullness of the restriction functors extends to a short exact sequence over S. Consider for $r \in \mathbb{N}$ the resulting sequence

$$0 \to T_\ell(\mathcal{G})/\ell^r \to F/\ell^r \xrightarrow{\gamma_r} \mathbb{Z}/\ell^r \to 0$$

of étale sheaves of $\mathbb{Z}/\ell^r$-modules over S and set $W_r := \gamma_r^{-1}(\{1\})$. Inspecting the definition of $T_\ell(M)$, one directly sees $j^*W_r \cong V_{\ell,r}(v)$ and we can apply Lemma 4.12 to extend v, which concludes the proof of the theorem. $\square$

Proof of Lemma 4.12. First assume $\dim(S) = 1$. By continuity and étale descent, we may assume that S is the spectrum of a strictly henselian discrete valuation ring $\mathcal{O}_K$ with field of fractions K. By assumption on unramifiedness of roots, we see that v has ℓ^r-th roots in $\mathcal{G}(K)$ for all r, so we get a map

$$\mathbb{Z}[1/\ell] \to \mathcal{G}(K),\ 1 \mapsto v$$

by choosing compatible roots of v. Considering the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{T}(S) \arrow[r] \arrow[d, hook] & \mathcal{G}(S) \arrow[r] \arrow[d, hook] & \mathcal{A}(S) \arrow[r] \arrow[d, "\wr"'] & 0 \\
1 \arrow[r] & \mathcal{T}(K) \arrow[r] & \mathcal{G}(K) \arrow[r] & \mathcal{A}(K) \arrow[r] & 0
\end{tikzcd}$$

where the isomorphism on the right follows from the Néron mapping property [BLR90, Proposition 1.2.8], we obtain

$$\mathcal{G}(K)/\mathcal{G}(S) \cong \mathcal{T}(K)/\mathcal{T}(S) \cong (K^\times/\mathcal{O}_K^\times)^r \cong \mathbb{Z}^r,$$

which shows that the composite

$$\mathbb{Z}[1/\ell] \to \mathcal{G}(K) \to \mathcal{G}(K)/\mathcal{G}(S)$$

has to be the zero map (as they are no nonzero ℓ-divisible elements in $\mathbb{Z}^r$), so $v \in \mathcal{G}(S)$.

If $\dim(S) > 1$, we can extend the morphism v over any point of codimension 1 by the previous case. By continuity, see Proposition 4.3, we can hence assume that the complement $Z := S\backslash U$ has codimension at least 2 in S. As the torus $\mathcal{T}$ is assumed to be a split torus $\mathbb{G}_m^r$, the first terms of the fppf-cohomology sequence corresponding to $1 \to \mathcal{T} \to \mathcal{G} \to \mathcal{A} \to 1$ give, using [Sta19, Tag 03P8],

$$\begin{tikzcd}
1 \arrow[r] & (\mathcal{O}_S(S)^\times)^r \arrow[r] \arrow[d] & \mathcal{G}(S) \arrow[r] \arrow[d] & \mathcal{A}(S) \arrow[r] \arrow[d] & \mathrm{Pic}(S)^r \arrow[d] \\
1 \arrow[r] & (\mathcal{O}_S(U)^\times)^r \arrow[r] & \mathcal{G}(U) \arrow[r] & \mathcal{A}(U) \arrow[r] & \mathrm{Pic}(U)^r.
\end{tikzcd}$$

Here the leftmost vertical arrow is an isomorphism, and the rightmost is injective because S is normal and Z has codimension bigger or equal to 2 (combine [Gro67, Corollaire 21.6.11] with [Ful98, Proposition 1.8]). After a short diagram chase, we are reduced to finding a preimage under $\mathcal{A}(S) \to \mathcal{A}(U)$ of the image of v in $\mathcal{A}(U)$, so we may assume that the toric part of $\mathcal{G}$ is trivial.

Now let $\nu\colon Y \hookrightarrow \mathcal{A}$ be the reduced closure of $v\colon U \to \mathcal{A}$. Denoting by $\pi\colon Y \to S$ the projection, we see that π is proper and an isomorphism over U. We want to show that π is quasi-finite, because then it will be an isomorphism by [Sta19, Tag 0AB1] and [Sta19, Tag 02OG] and $\nu \circ \pi^{-1}$ will be the extension of v.

Let $\tilde{Y}$ be the connected component of the normalization of Y in which U is dense and $n\colon \tilde{Y} \to Y$ be the induced map. Set $\tilde{\pi} = \pi \circ n$ and $\tilde{\nu} = \nu \circ n$. As U

and $\tilde{Y}$ are normal and U is dense in $\tilde{Y}$, we get isomorphisms $V_{\ell,r}(\tilde{\nu}) \cong \tilde{\pi}^* W_r$ for all $r \in \mathbb{N}$. Let $i\colon \bar{s} \to S$ be a geometric point and consider the cartesian square

$$\begin{tikzcd} \tilde{Y}_{\bar{s}} \arrow[r, "i'"] \arrow[d, "\tilde{\pi}'"] & \tilde{Y} \arrow[d, "\tilde{\pi}"] \\ \bar{s} \arrow[r, "i"] & S. \end{tikzcd}$$

The induced section

$$\tilde{\nu}_{\bar{s}}\colon \tilde{Y}_{\bar{s}} \to \mathcal{A} \times_S \bar{s}$$

has $V_{\ell,r}(\tilde{\nu}_{\bar{s}}) \cong i'^* \pi^* W_r \cong \pi'^* i^* W_r$, which is a constant sheaf. Hence $\tilde{\nu}_{\bar{s}}$ has ℓ^r-th roots for all $r \in \mathbb{N}$ and hence factors on reduced connected components over $\bar{s}$ by Lemma 4.13 below. As $\tilde{Y}_{\bar{s}}$ has only finitely many connected components, this then shows that π is quasi-finite and thus finishes the proof. □

Lemma 4.13. *Let k be an algebraically closed field in which ℓ is invertible, A/k an abelian variety, Y/k of finite type, reduced and connected and $\alpha \in A(Y)$ a section that has ℓ^rth roots in $A(Y)$ for all $r \in \mathbb{N}$. Then α factors over $Y \to \mathrm{Spec}(k)$.*

Proof. Suppose the image of α in A is not a single closed point. Then we find some point $y \in Y$ which is not mapped to a closed point of A, i.e. α gives rise to a nonzero element $\beta \in A(y)/A(k)$. By assumption on the existence of roots, we find a nontrivial map

$$\mathbb{Z}[1/\ell] \to A(y)/A(k).$$

This is impossible because the group on the right hand side has no torsion (as torsion points are k-rational) and is finitely generated by the Lang-Néron-theorem [Con06, Theorem 2.1]. □

From Theorem 4.10 we can derive three variants of criteria for good reduction for Deligne 1-motives. First, the classical criterion of Néron, Ogg and Shafarevic [ST68] gives the following.

Corollary 4.14. *Let S be a connected Dedekind scheme with generic point η and ℓ be a prime invertible on S. Then a Deligne 1-motive $[L \to G]$ over η extends over S if and only if its Tate module F does.*

Grothendieck's extension of this criterium for schemes in characteristic 0 [Gro66b, Corollaire 4.2] lets us generalize this.

Corollary 4.15. *Let S be a connected normal scheme over $\mathbb{Q}$ with generic point η. Then a Deligne 1-motive $[L \to G]$ over η extends over S if and only if its Tate module F does.*

By Proposition 4.3, we get the following general result.

Corollary 4.16. *Let S be connected normal with generic point η, and let $[L \to G]$ be a Deligne-1-motive over η whose Tate module extends. Then there is an open subset $V \subset S$ whose complement $S \backslash V$ has codimension at least 2 in S such that $[L \to G]$ extends over V.*

One final application:

Corollary 4.17. *Let S be either a Dedekind scheme or a normal scheme over $\mathbb{Q}$. Then for U in S open, the categories $\mathbf{M}_1^{\mathrm{Del}}(U,\mathbb{Q})$ of Deligne 1-isomotives are abelian and satisfy Zariski-descent: For an open cover $S = U \cup V$, we obtain a 2-cartesian square*

$$\begin{array}{ccc} \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) & \longrightarrow & \mathbf{M}_1^{\mathrm{Del}}(U,\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathbf{M}_1^{\mathrm{Del}}(V,\mathbb{Q}) & \longrightarrow & \mathbf{M}_1^{\mathrm{Del}}(U\cap V,\mathbb{Q}) \end{array}$$

which, as all functors are fully faithful, is just an existence statement for objects.

Proof. If there is a prime ℓ invertible on S, then Corollary 4.14 or Corollary 4.15 show an equivalence

$$\mathbf{M}_1^{\mathrm{Del}}(U,\mathbb{Q}) \cong \mathbf{M}_1^{\mathrm{Del}}(\eta,\mathbb{Q}) \times_{\mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(\eta,\mathbb{Q}_\ell)} \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(U,\mathbb{Q}_\ell)$$

from which the claim follows. Otherwise, let ℓ_1, ℓ_2 be two prime numbers and $S_i := S[1/\ell_i]$ the corresponding open cover of S. Given $\mathbb{M}_1 \in \mathbf{M}_1^{\mathrm{Del}}(S_1), \mathbb{M}_2 \in \mathbf{M}_1^{\mathrm{Del}}(S_2)$ and an equivalence $(\mathbb{M}_1)_\eta \cong (\mathbb{M}_2)_\eta$, we can first assume that this equivalence is given by an honest morphism of Deligne 1-motives: Indeed, we may replace $\mathbb{M}_2$ by the extension of $(\mathbb{M}_1)_\eta$ over S_2. We can now use the fact that the integral categories satisfy descent to show the equivalence

$$\mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \to \mathbf{M}_1^{\mathrm{Del}}(S_1,\mathbb{Q}) \times_{\mathbf{M}_1^{\mathrm{Del}}(S_1\cap S_2,\mathbb{Q})} \mathbf{M}_1^{\mathrm{Del}}(S_2,\mathbb{Q})$$

which proves the claim. □

The author does not know whether the assumptions in the above corollary are optimal.

4.3. **1-motives with geometrically constant Tate module.**

The aim of this section is to prove the following theorem.

Theorem 4.18. *Let k be a field of exponential characteristic p and let $\varepsilon : S \to \mathrm{Spec}(k)$ be a geometrically connected finite type reduced k-scheme. Then the functor*

$$\varepsilon^* \colon \mathbf{M}_1^{\mathrm{Del}}(\mathrm{Spec}(k),\mathbb{Z}[1/p]) \to \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Z}[1/p])$$

is fully faithful with essential image those 1-motives $M \in \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Z}[1/p])$ whose ℓ-adic realization $R_\ell(M)$ is constant over $S \otimes \bar{k}$ for some (hence every) prime $\ell \neq p$.

We note that full faithfullness holds integrally if there are no inseparable field extensions floating around.

Lemma 4.19. *Let k be a field of exponential characteristic p and let $\varepsilon : S \to \mathrm{Spec}(k)$ be a geometrically connected finite type reduced k-scheme which is also geometrically reduced. Then*

$$\varepsilon^* \colon \mathbf{M}_1^{\mathrm{Del}}(\mathrm{Spec}(k),\mathbb{Z}) \to \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Z})$$

is fully faithful.

Proof. Fix a prime $\ell \neq p$. As S is geometrically connected, ε induces a surjection on étale fundamental groups and hence

$$\varepsilon^* \colon \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(\mathrm{Spec}(k), \mathbb{Z}_\ell) \to \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S, \mathbb{Z}_\ell)$$

is fully faithful. For $M, N \in \mathbf{M}_1^{\mathrm{Del}}(\mathrm{Spec}(k), \mathbb{Z})$ we obtain a commutative square of abelian groups

$$\begin{tikzcd}
\mathrm{Hom}_k(M, N) \arrow[r, hook] \arrow[d] & \mathrm{Hom}_k(R_\ell(M), R_\ell(N)) \arrow[d, equal] \\
\mathrm{Hom}_S(\varepsilon^* M, \varepsilon^* N) \arrow[r, hook] & \mathrm{Hom}_S(R_\ell(\varepsilon^* M), R_\ell(\varepsilon^* N))
\end{tikzcd}$$

which shows that pulling back 1-motives is faithful. As S is moreover geometrically reduced, we find a finite Galois extension $\gamma : \mathrm{Spec}(l) \to \mathrm{Spec}(k)$ such that S has an l-rational point, pulling back along which gives a diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(\varepsilon^* M, \varepsilon^* N) \arrow[r] \arrow[d, hook] & \mathrm{Hom}_l(\gamma^* M, \gamma^* N) \arrow[d, hook] \\
\mathrm{Hom}_k(R_\ell(M), R_\ell(N)) \arrow[r] & \mathrm{Hom}_l(\gamma^* R_\ell(M), \gamma^* R_\ell(N))
\end{tikzcd}$$

and we see that elements in the image of

$$\mathrm{Hom}_S(\varepsilon^* M, \varepsilon^* N) \to \mathrm{Hom}_l(\gamma^* M, \gamma^* N)$$

come with a canonical Galois-descent-datum from which we obtain an element of $\mathrm{Hom}_k(M, N)$ showing the claimed fullness. $\square$

We fix a set of prime numbers P such that all primes not in P are invertible on all schemes we consider in the following. As always, ℓ is a prime number not contained in P. In practice, P might consist of one prime number or of all prime numbers except ℓ. The reason we do so is that we hope to eventually generalize the results of this paper from $\mathbb{Q}$ to $\mathbb{Z}[1/P]$-coefficients. The following terminology allows us to restrict to isogenies of degree only divisible by primes in P.

Definition 4.20. We call the morphisms (resp. isomorphisms) in $\mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z}[1/P])$ *admissible quasi-morphisms* (resp. *admissible quasi-isogenies*). An admissible quasi-isogeny is called *admissible isogeny* if it is induced by a morphism in $\mathbf{M}_1^{\mathrm{Del}}(S, \mathbb{Z})$.

We recall the following consequence of the Tate Conjecture for abelian varieties proved by Zarhin and Faltings:

Theorem 4.21. *Let S be a reduced connected scheme, locally of finite type over a field or the integers, $A, B \in \mathrm{AbSch}(S)$ abelian schemes over S and let $u \colon T_\ell(A) \to T_\ell(B)$ be a morphism of ℓ-adic sheaves. Suppose that there is a point $s \in S$ such that $u_s \colon T_\ell(A_s) \to T_\ell(B_s)$ is induced by a honest morphism $v \colon A_s \to B_s$ of abelian schemes. Then there is an admissible quasi-morphism $v \colon A \to B$ with $T_\ell(v) = u$.*

That this result follows from the now known Tate conjecture for abelian varieties is already mentioned in the introduction of [Gro66b]. However, the author is not aware of any proof in the literature except an answer by Keerthi

Madapusi Pera on mathoverflow [MP13]. For the convenience of the reader, we repeat his argument here.

Proof. Applying [Gro66b, Proposition 1.2], it is enough to construct the morphism over the normalization of S. Using that S is connected and applying an inductive argument, we reduce to S connected normal. We want to reduce to the case that the function field F of S is finitely generated, which is automatic if S is of finite type over the integers. Let S be of finite type over some field k; we can assume S to be geometrically connected. We are going to apply essentially the argument of [Gro66b, 2.2], with a slight modification because we do not assume the characteristic of k to be 0. Namely we write k as inductive limit over its finitely generated subfields and obtain, via [Gro66a, Théorèmes 8.8.2, 8.10.5] and Proposition 4.3, a finitely generated subfield k' of k, a finite type normal integral scheme S' over k', two abelian schemes A', B' over S', a point s' of S' and a morphism $v' \colon A'_{s'} \to B'_{s'}$ such that S', A', B', s', v' base change to S, A, B, s, v along $\mathrm{Spec}(k) \to \mathrm{Spec}(k')$. Factor $k' \to k$ into $k' \to k'' \to k$ with $k \to k''$ separable and $k'' \to k$ purely inseparable, and denote with $(-)''$ the base changes along $\mathrm{Spec}(k'') \to \mathrm{Spec}(k')$. Then $S \to S''$ is radicial, so u gives rise to a morphism u'' by topological invariance of the étale site. We claim that u'' gives rise to a unique $u' \colon T_\ell(A') \to T_\ell(B')$ with $(u')_s = T_\ell(v')$. Indeed, setting $S''' := S'' \times_{S'} S''$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{S'}(T_\ell(A'), T_\ell(B')) \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{s'}(T_\ell((A')_{s'}), T_\ell((B')_{s'})) \arrow[d] \\
\mathrm{Hom}_{S''}(T_\ell(A''), T_\ell(B'')) \arrow[r, hook] \arrow[d, shift left] \arrow[d, shift right] & \mathrm{Hom}_{s''}(T_\ell((A'')_{s''}), T_\ell((B'')_{s''})) \arrow[d, shift left] \arrow[d, shift right] \\
\mathrm{Hom}_{S'''}(T_\ell(A'''), T_\ell(B''')) \arrow[r, hook] & \mathrm{Hom}_{s'''}(T_\ell((A''')_{s'''}), T_\ell((B''')_{s'''}))
\end{tikzcd}$$

of homomorphism groups of locally constant sheaves, where the columns are equalizers because $S'' \to S'$ is pro-étale [BS14, Proposition 2.3.3, Lemma 5.1.2]. Now using that $T_\ell(v'')$ descends to $T_\ell(v')$, one directly gets that u'' gives rise to a unique u'. Dropping the $(-)'$, we may thus assume that k and hence F and $k(s)$ are finitely generated. We now consider

$$\begin{tikzcd}
\mathrm{Hom}_S(A, B) \arrow[r] \arrow[d, "\phi"] & \mathrm{Hom}_S(T_\ell(A), T_\ell(B)) \arrow[d, "\phi_\ell"] \\
\mathrm{Hom}_s(A_s, B_s) \arrow[r] & \mathrm{Hom}_s(T_\ell(A_s), T_\ell(B_s)
\end{tikzcd}$$

where Hom denotes homomorphisms of commutative groups schemes and of locally constant sheaves, respectively. In this diagram, all arrows are injective and the horizontal arrows become isomorphisms after applying $- \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ to the left hand side by the Tate conjecture for abelian varieties [Fal83, Zar75]. In particular, the canonical map

$$\mathrm{coker}(\phi) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{coker}(\phi_\ell)$$

is an isomorphism. As the $\mathrm{coker}(\phi_\ell)$ does not have ℓ-torsion, we deduce that $\mathrm{coker}(\phi)$ does not have ℓ-torsion, which is moreover true for any prime not contained in P. Hence we see that

$$\operatorname{coker}(\phi) \otimes_{\mathbb{Z}} \mathbb{Z}[P^{-1}] \hookrightarrow \operatorname{coker}(\phi_\ell)$$

is an injection, from which one easily deduces the theorem. $\square$

We next show that $\mathbf{M}_1^{\mathrm{Del}}(\operatorname{Spec}(k), \mathbb{Z}[1/P])$ satisfies Galois-descent for objects.

Lemma 4.22. *Let k be a field of exponentiation characteristic in P, l/k a Galois extension and M/l a Deligne 1-motive. Suppose we have for $\sigma \in \operatorname{Gal}(l/k)$ admissible quasi-isogenies $b_\sigma \colon \sigma^* M \to M$ such that*

$$\sigma^* b_\tau \circ b_\sigma = b_{\tau\circ\sigma},$$

then there is a 1-motive N/k and an admissible quasi-isogeny $\phi\colon N_l \cong M$ such that $b_\sigma \circ \phi = \phi \circ (id \times \sigma)$.

Proof. The case where M is an abelian variety is a result of Ribbet [Rib04, Theorem 8.2] (note that his statement is slightly weaker in that it only covers $P = \{\text{all primes}\}$, but the proof is adequate also for the above formulation).

In order to prove the general case, we apply the duality trick and can assume that we are able to descend the semi-abelian part G of M. Write $M = [L \xrightarrow{u} G]$ and b_σ^{lat} (resp. b_σ^{sab}) for the restriction of b_σ to the lattice and semi-abelian part of M. Descending the lattice part up to isogeny is easy, so we are put in the following situation: We are given a lattice K and a semi-abelian variety H over k together with admissible isogenies $\phi^{sab}\colon H_l \to G$ and $\phi^{lat}\colon K_l \to L$ such that for any $\sigma \in \operatorname{Gal}(l/k)$, the diagram

$$\begin{tikzcd}
\sigma^* K_l \arrow[r, "\sigma^*\phi^{lat}"] \arrow[d, "\mathrm{id}_K \otimes \sigma"'] & \sigma^* L \arrow[r, "\sigma^* u"] \arrow[d, "b_\sigma^{lat}"'] & \sigma^* G \arrow[r, "\sigma^*(\phi^{sab})^{-1}"] \arrow[d, "b_\sigma^{sab}"] & \sigma^* H_l \arrow[d, "\mathrm{id}_H \otimes \sigma"] \\
K_l \arrow[r, "\phi^{lat}"'] & L \arrow[r, "u"'] & G \arrow[r, "(\phi^{sab})^{-1}"'] & H_l
\end{tikzcd}$$

commutes. Let m be a unit in $\mathbb{Z}[1/P]$ such that, setting $[m] \circ (\phi^{sab})^{-1} =: \psi$, the admissible quasi-isogeny $\psi \circ u \circ \phi^{lat}$ is an admissible isogeny of semi-abelian varieties. The above diagrams then induce new diagrams

$$\begin{tikzcd}
\sigma^* K_l \arrow[r, "\sigma^*\phi^{lat}"] \arrow[d, "\mathrm{id}_K \otimes \sigma"'] & \sigma^* L \arrow[r, "\sigma^* u"] \arrow[d, "b_\sigma^{lat}"'] & \sigma^* G \arrow[r, "\sigma^*\psi"] \arrow[d, "b_\sigma^{sab}"] & \sigma^* H_l \arrow[d, "\mathrm{id}_H \otimes \sigma"] \\
K_l \arrow[r, "\phi^{lat}"'] & L \arrow[r, "u"'] & G \arrow[r, "\psi"'] & H_l
\end{tikzcd}$$

where the outer squares are a Galois-descent datum yielding a morphism $v\colon K \to H$ over k such that $v_l = \psi \circ u \circ \phi^{lat}$. Setting $N := [K \xrightarrow{v} H]$ and defining

$$\phi := \left(\frac{1}{m}\phi^{lat}, \phi^{sab}\right) : N_l \to M$$

gives the required objects. $\square$

Lemma 4.23. *Let k be an algebraically closed field of exponential characteristic in P, $\varepsilon\colon S \to \operatorname{Spec}(k)$ a reduced connected finite-type k-scheme and $\mathcal{A}$ an abelian* *scheme over S whose ℓ-adic Tate module is constant. Then there is an abelian variety B over* $\mathrm{Spec}(k)$ *together with an admissible quasi-isogeny*

$$\phi\colon B \times S \to \mathcal{A}.$$

Proof. Choose a k-rational point $s\colon \mathrm{Spec}(k) \to S$ and set $B := s^*\mathcal{A}$. By Theorem 4.21 we obtain an admissible quasi-isogeny

$$B \times S \to \mathcal{A}$$

of abelian schemes over S which over s restricts to the canonical identification. □

Lemma 4.24. *Let k be an algebraically closed field of exponential characteristic in P, $\varepsilon : S \to \mathrm{Spec}(k)$ of finite type with S connected reduced. Let $M = [L \xrightarrow{u} G]$ be a Deligne 1-motive over S such that $T_\ell(M)$ is constant. Then there is a Deligne 1-motive N over* $\mathrm{Spec}(k)$ *together with an isogeny*

$$\psi\colon M \to \varepsilon^* N$$

over S.

Proof. Let us first note that the toric and the lattice part of M are necessarily split as the Tate module is constant. The case where M is an abelian scheme is precisely Lemma 4.23. We want to apply the usual duality trick. Let us hence assume for a moment that we are already given a semi-abelian scheme G' over $\mathrm{Spec}(k)$ together with an isogeny

$$\phi\colon G \to \varepsilon^* G'.$$

Define a 1-motive $\hat{N} := [L \xrightarrow{\phi\circ u} \varepsilon^* G']$ on S together with the morphism $\psi\colon M \to \hat{N}$ which is the identity on the lattice part and given by ϕ on the semi-abelian parts. To see that ψ is an isogeny, take $\phi' : \varepsilon^* G' \to G$ with $\phi' \circ \phi = [m]$ and observe that the morphism $\psi'\colon \hat{N} \to M$ given by

$$\begin{array}{ccc} [L & \xrightarrow{\phi\circ u} & G] \\ \downarrow{\scriptstyle [m]} & & \downarrow{\scriptstyle \phi'} \\ [L & \xrightarrow{u} & G'] \end{array}$$

satisfies $\psi' \circ \psi = [m]$. Let us see that there is a 1-motive N over $\mathrm{Spec}(k)$ with $\hat{N} \cong \varepsilon^* N$. As L is a split lattice, we directly reduce to $L = \mathbb{Z}$, where the datum of u is equivalent to giving a morphism $\alpha\colon S \to G$ over $\mathrm{Spec}(k)$. We have to check that α factors over $\mathrm{Spec}(k)$, i.e. that it's schematic image is a single closed point. As in the proof of Lemma 4.12, our assumption on the Tate module translates to the fact that α is ℓ-divisible in $G(S)$. Denote the toric part of G by T and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(S) & \longrightarrow & G(S) & \longrightarrow & A(S) & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & T(k) & \longrightarrow & G(k) & \longrightarrow & A(k) & \longrightarrow & 0 \end{array}$$

where the rows are exact as k is algebraically closed. We know by Lemma 4.13 that the image of α under $G(S) \to A(S)$ lies in the image of $A(k) \to A(S)$.

As $G(k) \to A(k)$ is surjective, we find $\beta \in G(k) \subset G(S)$ such that $\alpha\beta^{-1}$ maps to 0 in $A(S)$, hence lies in $T(S)$. Note that $\alpha\beta^{-1}$ is still ℓ-divisible by virtue of k being algebraically closed. We need to see that $\alpha\beta^{-1}$ lies in $T(k)$. As T is a split torus, we readily reduce to showing that any ℓ-divisible element in $\mathbb{G}_m(S)$ lies in $\mathbb{G}_m(k)$. This follows quite directly: As S is reduced, connected and of finite type over $\mathrm{Spec}(k)$, it is enough to see that every Zariski-point of S gets mapped to a closed point of $\mathbb{G}_m$. Considering the residue field, we obtain a finitely generated field extension L/k and an ℓ-divisible element $t \in L$. As k is algebraically closed, we get an inclusion $k(t^{1/\ell^\infty}) \to L$, but the left hand side field is not finitely generated over k for $t \notin k$. Hence we get the existence of N over k with an isomorphism $\varepsilon^* N \cong \hat{N}$, and we can conclude, at least conditional on the existence of G' and ϕ.
To obtain G' and ϕ, consider the 1-motive $[X \xrightarrow{v} A^\vee]$ dual to $[0 \to G]$ and apply the above arguments to it. Taking the inverse of the dual of the resulting isogeny yields ϕ as above, and we are done. $\square$

Lemma 4.25. *Let k be a field of exponential characteristic in P, $\varepsilon\colon S \to \mathrm{Spec}(k)$ a geometrically connected finite-type k-scheme which is reduced, $\mathcal{F}$ a smooth ℓ-adic sheaf over $\mathrm{Spec}(k)$ and M a Deligne 1-motive over S coming with an isomorphism*

$$u\colon \varepsilon^*\mathcal{F} \to T_\ell(M).$$

Then there is a 1-motive N over $\mathrm{Spec}(k)$ together with an admissible quasi-isogeny

$$\phi\colon \varepsilon^* N \to M$$

and an isomorphism $v\colon \mathcal{F} \to T_\ell(N)$ such that

$$u = T_\ell(\phi) \circ \varepsilon^* v.$$

Proof. We note that for any purely inseparable field extension k'/k, the induced functor

$$\mathbf{M}_1^{\mathrm{Del}}(k, \mathbb{Z}[1/P]) \to \mathbf{M}_1^{\mathrm{Del}}(k', \mathbb{Z}[1/P])$$

is an equivalence of categories by [PL17, Proposition 3.5]. The author states the Proposition with rational coefficients, because he relies on [Bri17, Theorem 3.11], which is formulated with rational coefficients. However, as the key ingredient [Bri17, Lemma 3.11] only relies on inverting the exponential characteristic of k, one easily sees that [Bri17, Theorem 3.11] and hence [PL17, Proposition 3.5] hold with $\mathbb{Z}[1/P]$ coefficients as well.

In particular, we may first replace k by the field of definition of S to assume S geometrically reduced and then base change to the perfection of k to assume k perfect. By continuity and Lemma 4.24, we find a finite Galois extension l/k, an 1-motive $\hat{N}$ over l and an admissible quasi-isogeny $\phi\colon \varepsilon_l^* \hat{N} \to M_l$, where M_l is the pullback of M along the base change $S_l \to S$ of $\mathrm{Spec}(l) \to \mathrm{Spec}(k)$. Using ϕ and the obvious descent datum on M_l, we can apply Lemma 4.22 to descend $\hat{N}$ to a 1-motive N over k, so we now only need to care about the quasi-isogeny ϕ. After multiplying with a sufficiently large unit of $\mathbb{Z}[1/P]$, we can replace ϕ with an admissible isogeny. It is then an easy check that $\phi\colon (\varepsilon^* N)_l \to M_l$ is Galois-equivariant and hence descend to the sought-after isogeny. $\square$

Proof of Theorem 4.18. We begin with full faithfullness. Using the same reduction as already explained in the beginning of the proof of Lemma 4.25, we may pass to a finite purely inseparable extension of k and hence assume S to be geometrically reduced. There, full faithfullness directly follows from the stronger Lemma 4.19. The characterization of the essential image follows from Lemma 4.25. $\square$

4.4. **Extending 1-motives over iterated modifications.** We now turn towards formalizing the idea of extending a Deligne 1-motive from an open dense subscheme to a modification of the base. In the whole section, ℓ is a prime invertible on all schemes considered.

Definition 4.26. We call *extendable datum* $(S, U, j, A, M, \mathcal{F}, u_M)$ the datum of a reduced scheme S, an open dense immersion $j\colon U \to S$ such that U is normal, a Deligne 1-motive M with abelian part A admitting an ℓ^2-level structure and such that the toric and the lattice part of M are split, a smooth ℓ-adic sheaf $\mathcal{F} \in \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S, \mathbb{Z}_\ell)$ and an isomorphism $u_M\colon R_\ell(M) \to j^*\mathcal{F}$.

We recall that admitting ℓ^2-level structure here is equivalent to there being an isomorphism $T_\ell(A)/\ell^2 \cong \mathbb{Z}/\ell^2$, i.e. the ℓ^2-part of the Tate module of A is constant. This is a technical assumption chosen because the moduli stack of polarized abelian varieties with N-level structure is a scheme if $N \geqslant 4$.

Definition 4.27. Let $(S, U, j, A, M, \mathcal{F}, u_M)$ be an extendable datum. An *extending modification* for it consists of the following data:

1. A proper morphism $p\colon \hat{S} \to S$ that is an isomorphism over U such that $\hat{S}$ is normal and $U \cong p^{-1}(U)$ is dense in $\hat{S}$.
2. An Deligne 1-motive $\hat{M}$ over $\hat{S}$.
3. An isomorphism of Deligne 1-motives $\hat{M}_{|U} \cong M$.
4. An isomorphism of ℓ-adic sheaves $u_{\hat{M}}\colon R_\ell(M) \to \mathcal{F}$ extending u_M.

Note that by normality of $\hat{S}$ and Proposition 4.1, the Deligne 1-motive $\hat{M}$ and the isomorphism $u_{\hat{M}}$ are uniquely determined by 1) and 3) once they exist.

The following result is an extension of a result of Grothendieck [Gro66b, Proof of Thèoréme 4.1] on abelian schemes:

Proposition 4.28. *Extending modifications exist for all extendable data.*

Proof. By passing to the normalization, we may assume S normal, connected. Then A admits a polarization of some degree d^2 by [Ray70], hence corresponds uniquely to a morphism $f\colon U \to \mathcal{M}$ where $\mathcal{M}$ is a corresponding fine moduli scheme parameterizing abelian varieties with a polarization of degree d^2 and an ℓ^2-level structure [Hid04, Theorem 6.20]. Let $\tilde{S}$ be the schematic closure of

$$U \xrightarrow{\Gamma_f} U \times_{\mathbb{Z}} \mathcal{M} \to S \times_{\mathbb{Z}} \mathcal{M}.$$

It comes with a canonical projection $\tilde{p}\colon \tilde{S} \to S$ which is an isomorphism over U. As U is normal, this is still true if we precompose with the normalization $\tilde{S}^n \to \tilde{S}$. Denote by $\hat{S}$ the closure of $U \in \tilde{S}^n$, a clopen subscheme, and by $\hat{p}\colon \hat{S} \to S$ the induced map. We claim that $\hat{S}$ is proper. Indeed, by the valuative criterion for noetherian schemes [GW10, Theorem 15.9], we have to produce a unique lift in each diagram

$$\begin{tikzcd} \tau \arrow[r] \arrow[d] & \hat{S} \arrow[d, "\hat{p}"] \\ T \arrow[r] & S \end{tikzcd}$$

where T is the spectrum of a discrete valuation ring with generic point $\tau \in T$. For this, we observe that the abelian scheme on $\hat{S}$ corresponding to $\hat{S} \to \tilde{S} \to \mathcal{M}$ has as ℓ-adic Tate module precisely $\hat{p}^*\mathcal{F}$ because $\hat{S}$ is normal and the statement is true on the open dense subscheme $U \subset \hat{S}$, see Proposition 4.1. Hence by the classical Néron-Ogg-Shafarevich criterion, we see that the map $\tau \to \mathcal{M}$ extends uniquely over T, which yields a lift $T \to \tilde{S}$. As $\hat{S} \to \tilde{S}$ is proper, this allows us to conclude that $\hat{p}$ is proper. Hence $\hat{p} : \hat{S} \to S$ is proper and an isomorphism over U, and it is clear by construction that the abelian scheme $\hat{A}$ corresponding to $\hat{S} \to \tilde{S} \to \mathcal{M}$ restricts to A over U. As $\hat{S}$ is normal and A extends to $\hat{A}$ we can then apply Theorem 4.10 to obtain the Deligne 1-motive $\hat{M}$ over $\hat{S}$ with abelian part $\hat{A}$ and ℓ-adic Tate module $p^*\mathcal{F}$. □

Definition 4.29. For an extendable datum $(S, U, j, A, M, \mathcal{F}, u_M)$ we add an extending modification and call the result $(S, U, j, A, M, \mathcal{F}, u_M, p, \hat{S}, \hat{M}, u_{\hat{M}})$ an *extended datum.*

The remainder of the section sets up the formalism needed in the case $\dim(S) > 2$, where we need to stratify the base scheme.

Definition 4.30. Let $D_0 := (S, U, j, A, M, \mathcal{F}, u_M, p, \hat{S}, \hat{M}, u_{\hat{M}})$ and $D_1 := (Z, V, k, B, N, i^*\mathcal{F}, u_N, \pi, \hat{Z}, \hat{N}, u_{\hat{N}})$ be two extended data and Let $i\colon Z \to S$ be a closed immersion. We say that D_0 and D_1 are *linked along* i, if the following holds: Setting $q = i \circ \pi$ and considering the diagram

$$\begin{tikzcd} (\hat{Z} \times \hat{S})_{\mathrm{red}} \arrow[r, "\hat{p}"] \arrow[d, "\hat{q}"] & \hat{Z} \arrow[d, "q"] \\ \hat{S} \arrow[r, "p"] & S, \end{tikzcd}$$

there is an admissible quasi-isogeny

$$\phi\colon \hat{p}^*\hat{N} \to \hat{q}^*\hat{M}$$

on $(\hat{Z} \times_S \hat{S})_{\mathrm{red}}$ such that the diagram

$$\begin{tikzcd} R_\ell(\hat{p}^*\hat{N}) \arrow[r, "R_\ell(\phi)"] \arrow[d, "\hat{p}^*(u_{\hat{N}})"'] & R_\ell(\hat{q}^*\hat{M}) \arrow[d, "\hat{q}^*u_M"] \\ \hat{p}^*q^*\mathcal{F} \arrow[r, equal] & \hat{q}^*p^*\mathcal{F} \end{tikzcd} \tag{4.1}$$

commutes.

Lemma 4.31. *Let $(S, U, j, A, M, \mathcal{F}, u_M, p, \hat{S}, \hat{M}, u_{\hat{M}})$ be an extended datum such that S is normal and connected. Let $i\colon Z \to S$ be the inclusion of a closed subscheme. Then there is an extended datum*

$$(Z, V, k, B, N, i^*\mathcal{F}, u_N, \pi, \hat{Z}, \hat{N}, u_{\hat{N}})$$

that is linked to the original datum along i.

Proof. Let $\eta = \{\eta_1, \dots, \eta_r\}$ be the scheme of generic points of Z. For each generic point η_k, we set $\hat{S}_k := (\hat{S} \times_S \eta_k)_{\mathrm{red}}$, consider the diagram

$$\begin{array}{ccc} \hat{S}_k & \xrightarrow{i_k} & \hat{S} \\ \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p} \\ \eta_k & \longrightarrow & S \end{array}$$

and note that the pair $p_k \colon \hat{S}_k \to \eta_k$ and $i_k^* M$ satisfy the conditions of Theorem 4.18: The scheme $\hat{S}_k$ is reduced and geometrically connected by Zariski's Main Theorem, and $R_\ell(i_k^* M)$ is isomorphic to $p_k^* \mathcal{F}_{|\eta_k}$. Hence we find a Deligne 1-motive N_k over η_k together with an admissible quasi-isogeny $\phi_k \colon p_k^* N_k \to i_k^* M$ and an isomorphism $u_{N,k} \colon T_\ell(N_k) \to \mathcal{F}_{|\eta_k}$, compatible with ϕ_k on $\hat{S}_k$.

Varying k, we obtain M_η, u_η and so on. Using the continuity result Proposition 4.3, we can extend N_η to a small open U of Z which is dense as Z is reduced and which we choose to be normal. Then the Tate module of N is just the restriction of $\mathcal{F}$, so the abelian part B of N admits an ℓ^2-level structure and we may choose an extending modification $\pi \colon \hat{Z} \to Z$ with extension $\hat{N} \in \mathrm{AbSch}(\hat{Z})$ and corresponding isomorphism $u_{\hat{N}}$

It remains to construct the admissible quasi-isogeny ϕ by extending ϕ_η to the whole of $(\hat{Z} \times_S \hat{S})_{\mathrm{red}}$. By Lemma 4.9 we can extend ϕ_η to ϕ_C over the reduced closure C of

$$(\hat{Z} \times_S \hat{S})_{\mathrm{red}} \times_{\hat{Z}} \eta$$

inside $(\hat{Z} \times_S \hat{S})_{\mathrm{red}}$. Now let $\alpha \in (\hat{Z} \times_S \hat{S})_{\mathrm{red}}$ be a maximal point not lying over η, let $z \in \hat{Z}$ be its image, and denote by $\hat{Z}_z$ the strict henselization of $\hat{Z}$ at z (this has nothing to do with completions, we apologize for the confusing notation). Set

$$\Gamma := (\hat{Z} \times \hat{S})_{\mathrm{red}} \times_{\hat{Z}} \hat{Z}_z$$

and denote by $\Gamma(z)$ and $\Gamma(\eta)$ the fibers of Γ over z and η. Write α_z for $\alpha \times_{\hat{Z}} \hat{Z}_z$. Note that Γ is reduced and $\Gamma(z) \to \Gamma$ is an isomorphism locally at α_z by our maximality assumption, so the inclusion $\Gamma(z)_{\mathrm{red}} \to \Gamma(z)$ is also an isomorphism locally around α_z. Let us use this to extend ϕ_η over α_z.

In order to do so, choose a closed point $\gamma \in \Gamma(z) \cap C \times_{\hat{Z}} \hat{Z}_z$ - such a point exists by properness of the map

$$\Gamma \to \hat{Z}_z$$

and the fact that $\eta \times_{\hat{Z}} \hat{Z}_z$ is dense in $\hat{Z}_z$. Endow γ with the reduced subscheme structure. For a geometrically connected T-scheme z write

$$\mathbf{M}_1^{\mathrm{Del}}(T/z, \mathbb{Z}[1/P])$$

for the full subcategory of $\mathbf{M}_1^{\mathrm{Del}}(T, \mathbb{Z}[1/P])$ spanned by those 1-motives whose ℓ-adic Tate module is a pullback from a lisse ℓ-adic sheaf over z, i.e. constant. We can then repeatedly apply Theorem 4.18 to obtain a chain of equivalence

$$\mathbf{M}_1^{\mathrm{Del}}(\gamma/z, \mathbb{Z}[1/P]) \cong \mathbf{M}_1^{\mathrm{Del}}(z, \mathbb{Z}[1/P]) \cong \mathbf{M}_1^{\mathrm{Del}}(\Gamma(z)_{\mathrm{red}}/z, Z[1/P])$$

compatible with the ℓ-adic realizations. We can now use this equivalence to extend $(\phi_C)_\gamma$ over the whole of $\Gamma(z)_{\mathrm{red}}$ and then restrict the obtained quasi-isogeny to α_z. Etale descent for morphisms of 1-motives then gives us ϕ_α, and we can apply Lemma 4.9 again to extend ϕ over the whole of $(\hat{Z} \times \hat{S})$ such that the diagram (4.1) commutes. □

Lemma 4.32. *Suppose we are given for $k \in \{0,1,2\}$ an extended datum*

$$D^k = (S^k, U^k, j^k, A^k, M^k, \mathcal{F}^k, u_{M^k}, p^k, \hat{S}^k, \hat{M}^k, u_{\hat{M}^k})$$

and closed immersions $i_0^1\colon S^1 \to S^0, i_0^2\colon S^2 \to S^0, i_1^2\colon S^2 \to S^1$ with $i_0^2 = i_0^1 \circ i_1^2$. Assume $\mathcal{F}^1 = (i_0^1)^*\mathcal{F}^0$ and $\mathcal{F}^2 = (i_0^2)^*\mathcal{F}^0$ and that S^0 is normal connected. Assume further that D^0 is linked with D^2 along i_0^2 and D^0 is linked with D^1 along i_0^1. Then D^1 is linked with D^2 along i_1^2.

Further: Given $M \subset N \subset \{0,1,2\}$ we write

$$\hat{S}^M := \left(\bigtimes_{m \in M} \hat{S}^m \right)_{\mathrm{red}}$$

and

$$\hat{p}^N_M := \hat{S}^M \to \hat{S}^N$$

for the obvious projection, such that we have isogenies

$$\phi^l_m\colon (\hat{p}^{l,m}_l)^*\hat{A}^l \to (\hat{p}^{l,m}_m)^*\hat{A}^m$$

witnessing the links. Then we have the following cocycle condition

$$(\hat{p}^{0,1,2}_{2,0})^*(\phi^{2,0}) = (\hat{p}^{0,1,2}_{1,2})^*(\phi^{1,2}) \circ (\hat{p}^{0,1,2}_{0,1})^*(\phi^{0,1}) \tag{4.2}$$

for morphisms of abelian schemes over $\hat{S}^{0,1,2}$.

Proof. As in the proof of the lemma before, we want to apply Lemma 4.9 to realize the given isomorphism of Tate modules on the abelian scheme level. The proof is marginally different because S^1 is not assumed normal, so $\hat{S}^{1,2}$ might have connected components whose fiber product with V is empty - this is the reason why we need to assume the existence of the ambient normal scheme S^0. It allows us to work on $\hat{S}^{0,1,2}$ and consider there the admissible quasi-isogeny

$$\psi := (\hat{p}^{0,1,2}_{2,0})^*(\phi^{2,0}) \circ (\hat{p}^{0,1,2}_{0,1})^*(\phi^{0,1})^{-1}\colon (\hat{p}^{0,1,2}_{2,1})^*(\hat{p}^{1,2}_2)^*\hat{A}^2 \to (\hat{p}^{0,1,2}_{2,1})^*(\hat{p}^{1,2}_1)^*\hat{A}^1$$

which induces the canonical identification on Tate modules. Now the projection $\hat{p}^{0,1,2}_{1,2}$ is surjective and has geometrically connected fibers as reduction of a base change of p^0, so we can use Theorem 4.18 to obtain the sought-after isogeny on each generic point of $\hat{S}^{1,2}$. This is enough by Lemma 4.9. The cocycle condition from the second part of the lemma is clear. □

In particular, we obtain the following result which is the formulation we will need in the following chapters:

Corollary 4.33. *Let S be connected, noetherian, normal and suppose we are given an extendable datum of the form $(S, U, j, A, M, \mathcal{F}, u_M)$. Then there is a filtration $S = S^n \supset \ldots \supset S^0$ by reduced closed subschemes with $U^k := S^k \backslash S^{k+1}$, and extended data*

$$D^k = (S^k, U^k, j^k, A^k, M^k, \mathcal{F}^k, u_{M^k}, p^k, \hat{S}^k, \hat{M}^k, u_{\hat{M}^k})$$

such that

$$D^0 = (S, U, j, A, M, \mathcal{F}, u_M, p, \hat{S}, \hat{M}, u_{\hat{M}})$$

and such that for $m > k$, the data D^m and D^k are linked along the evident closed immersion $i^k_m \colon S^k \to S^m$. In particular, if we use notations analogous to the ones of the above lemma, this means that for all $\varnothing \neq M \subset N \subset \{0, \ldots, n\}$ we have projections

$$\hat{p}^N_M \colon \hat{S}^M \to \hat{S}^N$$

and we have isogenies

$$\phi^l_m \colon (\hat{p}^{l,m}_l)^* \hat{A}^l \to (\hat{p}^{l,m}_m)^* \hat{A}^m$$

witnessing the links. Furthermore, for all sets $M \subset \{0, \ldots, n\}$ and $l, m, k \in M$ we have the cocycle condition

$$(\hat{p}^M_{l,m})^*(\phi^{l,m}) \circ (\hat{p}^M_{m,k})^*(\phi^{m,k}) = (\hat{p}^M_{l,k})^*(\phi^{l,k}). \tag{4.3}$$

Proof. We construct the data by descending induction starting from S^n using Lemma 4.31 at each step and setting $S^{k-1} = (S^k \backslash U^k)_{\mathrm{red}}$. Then Lemma 4.32 shows that all data are linked along all inclusions we want, and the generalized cocycle condition (4.3) follows from the cocycle conditions (4.1) in Lemma 4.32 by pulling back. □

5. Auxiliary results on quasi-categories

In this section, we will distinguish between a 1-category C and its nerve $N(C)$.

5.1. **Certain limits of quasi-categories.** For a natural number $n \in \mathbb{N}_{\geqslant 1}$, we write $\langle n \rangle := \{1, \ldots, n\}$ and

$$\mathcal{P}^*\langle n \rangle := \mathcal{P}\langle n \rangle \backslash \{\varnothing\} = \{A \subset \langle n \rangle \mid A \neq \varnothing\}$$

seen as a poset via $\supseteq$. We want to prove the following

Proposition 5.1. *Let $\mathcal{C}$ be a quasi-category having finite limits, let $K' := N(\mathcal{P}^*\langle n \rangle^{op})$ and let $p\colon K' \to \mathcal{C}$ be a diagram in $\mathcal{C}$. Writing $P_A := p(A)$ for $A \in \mathcal{P}^*\langle n \rangle$, consider the iterated pullback*

$$P_* := \left(\left(P_{\{1\}} \underset{P_{\{1,2\}}}{\times} P_{\{2\}} \right) \underset{P_{\{1,3\}} \underset{P_{\{1,2,3\}}}{\times} P_{\{2,3\}}}{\times} P_3 \right) \underset{\cdots}{\times} \cdots \underset{\cdots}{\times} P_{\{n\}}$$

in $\mathcal{C}$. Then

(1) P_* is the limit of the diagram p.

(2) Suppose $\mathcal{C}$ is the category of small quasi-categories. Given for all $i \in \langle n \rangle$ an object o_i of P_i and for all $A = \{a_1 < \ldots < a_r\}$ in $\mathcal{P}^*\langle n \rangle$ with $r \geqslant 2$ a chain

$$p(\{a_1\} \to A)(o_{a_1}) \xrightarrow{\sim} \ldots \xrightarrow{\sim} p(\{a_r\} \to A)(o_{a_r})$$

of equivalences, we obtain an object o_* of P_* mapping to o_i under the projection $P_* \to P_i$.

In case $n = 2$, the first assertion is clear and the second one is a description of objects in a pullback of quasi-categories due to Joyal: There is a model of the pullback $A_{\{1\}} \times_{A_{\{1,2\}}} A_{\{2\}}$ in which an object is given by an object of $A_{\{1\}}$, an object of $A_{\{2\}}$, and an equivalence between their images in $A_{\{1,2\}}$. The case of $n > 2$ requires some preparations.

5.2. **Certain pushouts in quasi-categories.** In order to prove Proposition 5.1, we need to rewrite the limit into a more convenient form. For ease of citations, we will show the dual statements in this subsection and work with colimits instead. In this subsection, $\mathcal{C}$ is a quasi-category having finite colimits and $n \geqslant 2$.

Lemma 5.2. *Let $K = N(\mathcal{P}^*\langle n \rangle)$ and let $p\colon K \to \mathcal{C}$ be a diagram. Let $K' := N(\mathcal{P}^*\langle n-1 \rangle)$ and let $q, \tilde{q}\colon K' \to K$ be induced by the obvious inclusion $i\colon \mathcal{P}^*\langle n-1 \rangle \hookrightarrow \mathcal{P}^*\langle n \rangle$ and by*

$$\tilde{i}\colon \mathcal{P}^*\langle n-1 \rangle \hookrightarrow \mathcal{P}^*\langle n \rangle, A \mapsto A \cup \{n\}$$

respectively. Then there is a natural pushout square

$$\begin{tikzcd} \mathrm{colim}(p \circ \tilde{q}) \arrow[r] \arrow[d] & \mathrm{colim}(p \circ q) \arrow[d] \\ p(\{n\}) \arrow[r] & \mathrm{colim}(p) \end{tikzcd}$$

in $\mathcal{C}$.

Proof. We first construct a pushout square of simplicial sets

$$\begin{tikzcd} K' \arrow[r, "i_0"] \arrow[d, "j"] & K' \times \Delta^1 \arrow[d, "\delta"] \\ K'^{\rhd} \arrow[r, "\eta"] & K \end{tikzcd}$$

that is obtained by taking nerves of the following diagram of posets and order-preserving maps

$$\begin{tikzcd} \mathcal{P}^*\langle n-1\rangle \arrow[r, "\hat{i}_0"] \arrow[d, "\hat{j}"] & \mathcal{P}^*\langle n-1\rangle \times \{0,1\} \arrow[d, "\hat{\delta}"] \\ \mathcal{P}^*\langle n-1\rangle \cup \{\varnothing\} \arrow[r, "\hat{\eta}"] & \mathcal{P}^*\langle n\rangle \end{tikzcd}$$

where the maps are given as follows:

- $\hat{j}$ is the obvious inclusion.
- $\hat{i}_0$ is given by
$$\hat{i}_0 \colon \mathcal{P}^*\langle n-1\rangle \to \mathcal{P}^*\langle n-1\rangle \times \{0,1\},\ A \mapsto (A,0).$$
- $\hat{\delta}$ is
$$\hat{\delta} \colon \mathcal{P}^*\langle n-1\rangle \times \{0,1\} \to \mathcal{P}^*\langle n\rangle,\ (A,\varepsilon) \mapsto \begin{cases} A & \text{if } \varepsilon = 1, \\ A \cup \{n\} & \text{if } \varepsilon = 0. \end{cases}$$
- Finally,
$$\hat{\eta} \colon \mathcal{P}^*\langle n-1\rangle \cup \{\varnothing\} \to \mathcal{P}^*\langle n\rangle,\ A \mapsto A \cup \{n\}.$$

We observe that the maps $\hat{\delta}$ and $\hat{\eta}$ are injective and jointly surjective, with $\hat{\delta}$ missing the set $\{n\}$ and $\hat{\eta}$ missing sets not containing n. Moreover, any chain

$$A_0 \supseteq \ldots \supseteq A_r$$

in $\mathcal{P}^*\langle n\rangle$ has either $n \in A_i$ for all i or else cannot contain the set $\{n\}$, so the maps δ and η are jointly surjective on r-simplices. Finally, any chain lying completely in the image of $\hat{\delta}$ and $\hat{\eta}$ has $A_r \supsetneq \{n\}$ and hence is of the form

$$A_0' \cup \{n\} \supseteq \ldots \supseteq A_r' \cup \{n\}$$

for a chain $A_0' \supseteq \ldots A_r'$ in $\mathcal{P}^*\langle n-1\rangle$. Thus the diagram of simplicial sets is a pushout. As i_0 is a monomorphism, we can apply [Lur09, Proposition 4.4.2.2] to get a pushout

$$\begin{tikzcd} \mathrm{colim}(p \circ \tilde{q}) \arrow[r] \arrow[d] & \mathrm{colim}(p \circ \delta) \arrow[d] \\ \mathrm{colim}(p \circ \eta) \arrow[r] & \mathrm{colim}(p) \end{tikzcd}$$

in $\mathcal{C}$, so we only need to identify the upper right and lower left colimits. By [Lur09, Proposition 4.1.1.3] and [Lur09, Corollary 4.1.1.11.] this follows from the fact that the inclusion of a right cone point is right anodyne. $\square$

Lemma 5.3. *Let $K = N(\mathcal{P}^*\langle n\rangle)$ and let $p\colon K \to \mathcal{C}$ be a diagram. Then there is a natural pushout square*

$$\begin{tikzcd} \coprod\limits_{A\in\mathcal{P}^*\langle n\rangle} \coprod\limits_{a\in A} p(A) \arrow[r] \arrow[d] & \coprod\limits_{a\in\langle n\rangle} p(\{a\}) \arrow[d] \\ \coprod\limits_{A\in\mathcal{P}^*\langle n\rangle} p(A) \arrow[r] & \operatorname{colim}(p) \end{tikzcd}$$

in $\mathcal{C}$ where the maps are induced by taking coproducts over A and $a \in A$ of the maps $p(A) \to p(\{a\})$ resp. $p(A) = p(A)$.

Proof. Define for $A \in \mathcal{P}^*\langle n\rangle$ the set

$$Q_A := \{B \in \mathcal{P}^*\langle n\rangle \mid A \subset B\}$$

which we see as a partially ordered subset of $\mathcal{P}^*\langle n\rangle$. Define partially ordered sets with trivial orderings J_1, J_2, J_3 as follows:

- $J_1 := \{(Q_A, a) \in \mathcal{P}(\mathcal{P}^*\langle n\rangle) \times \langle n\rangle \mid A \in \mathcal{P}^*\langle n\rangle, \#A \geqslant 2, a \in A\}$
- $J_2 := \{Q_A \in \mathcal{P}(\mathcal{P}^*\langle n\rangle) \mid A \in \mathcal{P}^*\langle n\rangle, \#A \geqslant 2\}$
- $J_3 := \{Q_{\{k\}} \in \mathcal{P}(\mathcal{P}^*\langle n\rangle) \mid k \in \langle n\rangle\}$

and set $J := J_1 \sqcup J_2 \sqcup J_3$ as a set. Equip J with a poset structure by declaring that for all $A \in \mathcal{P}^*\langle n\rangle, \#A \geqslant 2$ and $a \in A$ we have $(Q_A, a) \leqslant Q_A$ as well as $(Q_A, a) \leqslant Q_{\{a\}}$ and no other nontrivial relations. We obtain an order-preserving map

$$\hat{F}\colon J \to \mathcal{P}(\mathcal{P}^*\langle n\rangle), (Q_A, a) \mapsto Q_A, Q_B \mapsto Q_B$$

where we equip the right hand side with the poset structure coming from $\subset$. In particular, we get an order preserving map $F := N \circ \hat{F}$ from J into the collection of simplicial subsets of K, and we want to verify the condition of [Lur09, Remark 4.2.3.9] to be then able to apply [Lur09, Corollary 4.2.3.10] to our situation. For this, let $\sigma := (B_0 \supseteq \ldots \supseteq B_r)$ be an r-simplex of K and consider

$$J_\sigma := \{I \in J \mid \{B_0, \ldots, B_r\} \subseteq F(I)\}$$

with the induced poset structure, cf. [Lur09, Notation 4.2.3.7]. We need to show that $N(J_\sigma)$ is contractible. First of all, as there are no nontrivial compositions of arrows in J, we see that $N(J_\sigma)$ is a disjoint union of trees. To check that it is connected, we note

$$J_\sigma = \{(Q_A, a) \mid a \in A \subset B_r, \#A \geqslant 2\} \sqcup \{Q_A \mid A \subset B_r, \#A \geqslant 2\} \sqcup \{Q_{\{k\}} \mid k \in B_r\}$$

and we get for all $a \in A \subset B_r$ with $\# \geqslant 2$ a chain

$$Q_{B_r} \geqslant (Q_{B_r}, a) \leqslant Q_{\{a\}} \geqslant (Q_A, a) \leqslant Q_A$$

which shows that any 0-simplex of $N(J_\sigma)$ can be connected to Q_{B_r}, so $N(J_\sigma)$ is connected. Now applying [Lur09, Proposition 4.2.3.4] and [Lur09, Corollary 4.3.2.10], we obtain a map

$$q\colon N(J) \to \mathcal{C}$$

with the property $\operatorname{colim}(p) \cong \operatorname{colim}(q)$ and which is given on a 0-simplex S by $\operatorname{colim}(p|_{F(S)})$. But all $F(S)$ have final objects, so by the same argument as in the end of the proof of lemma 5.2, we can identify the diagram q with the one claimed. $\square$

5.3. **Proof of Proposition 5.1.** The first part of the proposition follows inductively by applying the dual of Lemma 5.2. For the second part, we note the following

Lemma 5.4. *Let $\mathcal{C}$ be a quasi-category and $k \in \mathbb{N}_{\geqslant 2}$. Let $\mathcal{C}^{(\Delta^k)}$ be the full subcategory of $\mathcal{C}^{\Delta^k}$ spanned by maps $\Delta^k \to \mathcal{C}$ that factor over the largest sub-Kan complex of $\mathcal{C}$. Then the diagonal $\delta^k \colon \mathcal{C} \to \mathcal{C}^{\times k}$ factors as*

$$\mathcal{C} \xrightarrow{\delta^{(k)}} \mathcal{C}^{(\Delta^{k-1})} \xrightarrow{(p_0,\dots,p_k)} \mathcal{C}^{\times k}$$

where the first map is an equivalence and the second one a fibration, both in the Joyal model structure.

Proof. The proof of [Joy08, Proposition 5.16] works in this setup. $\square$

With this tool in hand, we can go to the

Proof of Proposition 5.1, second part. By the first part of the proposition as well as the dual of Lemma 5.3, we already have a convenient description of the limit at our disposal. Lemma 5.4 gives as the defining property of P_* that the square

$$\begin{tikzcd} P_* \arrow[r] \arrow[d] & \bigtimes_{A \in \mathcal{P}^*\langle n \rangle} p(A)^{(\Delta^{\#A-1})} \arrow[d] \\ P_1 \times \ldots \times P_n \arrow[r] & \bigtimes_{A \in \mathcal{P}^*\langle n \rangle} p(A)^{\times \#A} \end{tikzcd}$$

is a homotopy pullback square in the Joyal model structure. However, as all objects are fibrant and the right vertical map is a fibration, the simplicial set pullback models the homotopy pullback (see e.g. [Lur09, Remark A.2.4.5]), and we may take the above square to be an actual pullback square of simplicial sets. The statement then follows by considering 0-simplices. $\square$

5.4. **Further auxiliary results.**

Lemma 5.5. *Let*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "k"] & C \arrow[d, "f"] \\ B \arrow[r, "g"] & D \end{tikzcd}$$

be a cartesian diagram of stable quasi-categories and exact functors, and let $l = f \circ h$. For any objects a, a' of A, the diagram

$$\begin{tikzcd} \mathrm{Map}_A(a, a') \arrow[r] \arrow[d] & \mathrm{Map}_C(h(a), h(a')) \arrow[d] \\ \mathrm{Map}_B(k(a), k(a)') \arrow[r] & \mathrm{Map}_D(l(a), l(a')) \end{tikzcd}$$

is cartesian in the quasi-category of spaces.

Proof. As in the proof of [NS17, Proposition II.1.5], one uses Lemma 5.3 to obtain a cartesian square

$$\begin{tikzcd}
\mathrm{Map}_A(a, a') \arrow[r] \arrow[d] & \mathrm{Map}_D(l(a), l(a')) \arrow[d] \\
\mathrm{Map}_B(k(a), k(a)') \times \mathrm{Map}_C(h(a), h(a')) \arrow[r] & \mathrm{Map}_D(l(a), l(a')) \times \mathrm{Map}_D(l(a), l(a'))
\end{tikzcd}$$

of spaces, which translates into the square of the Lemma. $\square$

Lemma 5.6. *Let*

$$\begin{tikzcd}
A \arrow[r, "h^*"] \arrow[d, "k^*"] & C \arrow[d, "f^*"] \\
B \arrow[r, "g^*"] & D
\end{tikzcd}$$

be a diagram of stable quasi-categories and exact functors admitting right adjoints h_*, k_*, f_*, g_* and let $l^* = f^* \circ h^*$ with right adjoint $l_* = h_* \circ f_*$. Suppose that

(1) The canonical maps $k^*h_* \to g_*f^*$ and $h^*k_* \to f_*g^*$ are equivalences.

(2) The canonical diagram

$$\begin{tikzcd}
\mathrm{id}_A \arrow[r] \arrow[d] & h_*h^* \arrow[d] \\
k_*k^* \arrow[r] & l_*l^*
\end{tikzcd}$$

in $\mathrm{Fun}^{\mathrm{ex}}(A, A)$ is cartesian.

(3) The canonical diagram

$$\begin{tikzcd}
k^*k_* \arrow[r] \arrow[d] & \mathrm{id}_B \arrow[d] \\
k^*k_*g_*g^* \arrow[r] & g_*g^*
\end{tikzcd}$$

in $\mathrm{Fun}^{\mathrm{ex}}(B, B)$ is cartesian.

(4) The diagram

$$\begin{tikzcd}
h^*h_* \arrow[r] \arrow[d] & \mathrm{id}_C \arrow[d] \\
h^*h_*f_*f^* \arrow[r] & f_*f^*
\end{tikzcd}$$

in $\mathrm{Fun}^{\mathrm{ex}}(C, C)$ is cartesian.

Then the canonical functor $A \to B \times_D C$ is an equivalence.

Proof. That the functor is fully faithful follows readily from Lemma 5.5 and (2). To show essential surjectivity, consider an element of $B \times_D C$ given, as in Proposition 5.1, by an object c of C, an object b of B, and an equivalence $g^*b \to f^*c$. Define an object a of A to be the pullback

$$\begin{tikzcd}
a \arrow[rrr] \arrow[d] & & & k_*b \arrow[d] \\
h_*c \arrow[r] & h_*f_*f^*c \arrow[r, "\sim"] & h_*f_*g^*b \arrow[r, "\sim"] & k_*g_*g^*b
\end{tikzcd}$$

in A. Hence we obtain

$$\begin{tikzcd} k^*a \arrow[r] \arrow[d] & k^*k_*b \arrow[r] \arrow[d] & b \arrow[d] \\ k^*h_*c \arrow[r] & k^*k_*g_*g^*b \arrow[r] & g_*g^*b \end{tikzcd}$$

were the left square is cartesian by (3) as k^* is exact and the right square is cartesian by assumption (4). The bottom composition is equivalent to the identity by (1), which shows that the natural map $k^*a \to b$ is an equivalence. By symmetry, we also see that $h^*a \to c$ is an equivalence and we are done because $B \times_D C \to B \times C$ is conservative. □

5.5. ***h*-descent for motives.** The results of this section are certainly well known, but the author could not find them in this form in the literature. Recall that for noetherian schemes, a square

$$\begin{tikzcd} \tilde{X} \arrow[d, "k"] & \tilde{Z} \arrow[l, "g"'] \arrow[d, "f"] \\ X & Z \arrow[l, "h"] \end{tikzcd}$$

is called an *abstract blowup square* if it is cartesian, h is a closed immersion, and k is proper and an isomorphism over $X \backslash Z$.

Lemma 5.7. *Let*

$$\begin{tikzcd} \tilde{X} \arrow[d, "k"] & \tilde{Z} \arrow[l, "g"'] \arrow[d, "f"] \\ X & Z \arrow[l, "h"] \end{tikzcd}$$

be an abstract blowup-square. Then the square

$$\begin{tikzcd} \mathrm{DM}(X) \arrow[r, "h^*"] \arrow[d, "k^*"] & \mathrm{DM}(Z) \arrow[d, "f^*"] \\ \mathrm{DM}(\tilde{X}) \arrow[r, "g^*"] & \mathrm{DM}(\tilde{Z}) \end{tikzcd}$$

is a pullback square of symmetric monoidal stable ∞-categories.

Proof. As the forgetful functor from symmetric monoidal stable ∞-categories to stable ∞-categories detects limits [Lur17, Corollary 3.2.2.5], we may check the assertion underlyingly. We check the prerequisites of Lemma 5.6. (1) follows from proper base change, (2) is [CD19, Proposition 3.3.10], and (4) follows from $h^*h_* \cong \mathrm{id}$ as h is a closed immersion. To see (3), we note that by localization, the functor $\mathrm{DM}(\tilde{X}) \to \mathrm{DM}(\tilde{X}\backslash\tilde{Z}) \times \mathrm{DM}(\tilde{Z})$ is conservative and hence reflects pullbacks. As the square is an abstract pullback square, k is an isomorphism on $\tilde{X}\backslash\tilde{Z}$ and the square restricts to a pullback square over $\tilde{X}\backslash\tilde{Z}$. On the other hand, after applying g^* and proper base change again, we see that the right vertical morphism becomes $g^* \to g^*g_*g^*$ and the left vertical morphism is $f^*f_*g^* \to f^*f_*g^*g_*g^*$ and both are equivalences because g is a closed immersion. □

In order to use resolution of singularities by alterations, we need a stronger version of this statement. If X is a scheme on which a finite group G acts, we will also write $X\colon G \to (\mathrm{Sch})$ for the corresponding functor from the one-object category G to schemes and $\mathrm{DM}(X,G)$ for the symmetric monoidal stable ∞-category presented by the model category described in [CD19, Proposition 3.1.6] and [CD19, Proposition 3.1.24]. We then have

Lemma 5.8. *Let*

$$\begin{tikzcd} \tilde{X} \arrow[d, "k"] & \tilde{Z} \arrow[l, "g"'] \arrow[d, "f"] \\ X & Z \arrow[l, "h"'] \end{tikzcd}$$

be a cartesian diagram of schemes where h is a closed immersion, and $U = X \backslash Z$ is normal. Assume further that a finite group G acts on $\tilde{X}$ such that k is equivariant with respect to the action of the trivial group on X, that $U \times_X \tilde{X}/G$ is a scheme and that the induced morphism $U \times_X \tilde{X}/G \to U$ is radicial. Then the square

$$\begin{tikzcd} \mathrm{DM}(X) \arrow[r, "h^*"] \arrow[d, "\bar{k}^*"] & \mathrm{DM}(Z) \arrow[d, "\bar{f}^*"] \\ \mathrm{DM}(\tilde{X},G) \arrow[r, "\bar{g}^*"] & \mathrm{DM}(\tilde{Z},G) \end{tikzcd}$$

is a pullback square of symmetric monoidal stable ∞-categories. Here we denote by $\bar{k}^*$, $\bar{f}^*$ and $\bar{g}^*$ the functors induced by the morphisms of diagrams $\bar{k}\colon (\tilde{X},G) \to (X,*)$, $\bar{f}\colon (\tilde{Z},G) \to (Z,*)$ and $\bar{g}\colon (\tilde{X},G) \to (\tilde{Z},G)$.

Proof. We may forget the symmetric monoidal structure and then check the hypotheses of Lemma 5.6 on the level of stable model categories. First we note that all functors have explicitly described right adjoints as in [CD19, Propositions 3.1.11, 3.1.15] - for instance, the right adjoints to $\bar{k}^*$ and $\bar{f}^*$ are given by $(k_*-)^G$ and $(f_*-)^G$, where the G-invariants are, of course, derived. We first consider the diagram

$$\begin{tikzcd} \mathrm{DM}(\tilde{X}) \arrow[r, "g^*"] & \mathrm{DM}(\tilde{Z}) \\ \mathrm{DM}(\tilde{X},G) \arrow[r, "\bar{g}^*"] \arrow[u, "i^*"] & \mathrm{DM}(\tilde{Z},G) \arrow[u, "j^*"] \end{tikzcd}$$

where i^*, j^* come from forgetting the G-action. We then observe that the exchange morphism $i^*\bar{g}_* \to g_*j^*$ is an isomorphism because its transpose $\bar{g}^*i_\# \to j_\# g^*$ (see also [CD19, Proposition 3.1.11]) is readily seen to be an isomorphism from the direct description in [CD19, (3.1.2.3)]. We then see $\bar{h}^*\bar{k}_* \to \bar{f}_*\bar{g}^*$ by proper base change and the fact that h^* commutes with taking G-invariants, and we see $\bar{k}^*h_* \simeq \bar{g}_*\bar{f}^*$ after applying the conservative functor i^*, the above exchange isomorphism, and usual proper base change (see also the proof of [Ayo07, Théorème 2.4.22]). Hence we see that condition (1) of Lemma 5.6 is satisfied in this situation. Condition (2) in this setting is precisely [CD19, Theorem 4.4.1]. For condition (3), we first observe that the pullback functors $\mathrm{DM}(\tilde{X},G) \to \mathrm{DM}(\tilde{Z},G)$ and $\mathrm{DM}(\tilde{X},G) \to \mathrm{DM}(\tilde{U},G)$ are jointly conservative, which follows by conservativity of i^*. Pulling back to $\mathrm{DM}(\tilde{Z}, G)$, the diagram in (3) becomes cartesian because $\bar{g}^*\bar{g}_* \to \mathrm{id}$ is an equivalence, as can be seen after applying j^* and the above noted exchange isomorphism, see also [Ayo07, Corollaire 2.4.19]. In $\mathrm{DM}(\widetilde{U}, G)$ we have that $\mathrm{DM}(U) \to \mathrm{DM}(\widetilde{U}, G)$ is an equivalence: The unit of the adjunction is an equivalence by [CD19, Corollary 3.3.9]; for the counit we can check the statement after pulling back along $\widetilde{U} \to U$, where the G-action becomes split. We then observe that for split G-actions the claim is obvious from the direct descriptions of the model categories in [CD19, Propositions 3.1.6, 3.1.11]. This shows condition (3) of the lemma. Finally, condition (4) follows again from $h^*h_* \simeq \mathrm{id}$. □

Lemma 5.9. *Let X be a scheme on which a finite group G acts. Then the functor*

$$i^* \colon \mathrm{DM}(X, G) \to \mathrm{DM}(X)$$

that is induced by forgetting the G-action reflects dualizability.

Proof. As the functor is conservative and strong monoidal, it is enough to show that it is also closed. By a game of adjunction, this is equivalent to the following projection formula: For any objects M and N of $\mathrm{DM}(X)$ resp. $\mathrm{DM}(X, G)$, the canonical map

$$i_\#(M \otimes i^*N) \to (i_\# i^*M) \otimes N$$

is an equivalence. We may check this after applying i^* again. By the proof of [CD19, Lemma 3.1.14], we get a canonical isomorphism

$$i^* i_\# K \cong \bigoplus_G K$$

for all K in $\mathrm{DM}(X)$, which finishes the proof. □

6. Proof of Theorem A and consequences

6.1. **The Néron-Ogg-Shafarevich criterion.** Let $j : U \to S$ be an open immersion of schemes. Recall from [PL19] that the adjunction

$$j^* \colon \mathrm{DM}(S) \rightleftarrows \mathrm{DM}(U) : j_*$$

restricts to $\mathrm{DM}^{\mathrm{coh}}$ and refines to an adjunction

$$j^* \colon \mathrm{DM}^1(S) \rightleftarrows \mathrm{DM}^1(U) : \omega^1 j_*$$

which restricts to $\mathrm{DM}^{1,\geqslant 0}(S)$ and finally refines to the adjunction

$$j^* \colon \mathbf{M}^1(S) \rightleftarrows \mathbf{M}^1(U) : \tau^{1,\leqslant 0}\omega^1 j_*.$$

We now come to the core of this work, namely the following Néron-Ogg-Shafaravich criterion for lisse 1-motives.

Theorem 6.1. *Let S be a finite dimensional noetherian normal integral scheme which admits resolutions of singularities by alterations, let ℓ be a prime invertible on S, and let $j \colon U \to S$ be an open immersion. Then the following is true:*

- *For $N \in \mathbf{M}^{1,\mathrm{ls}}(S, \mathbb{Q})$, the canonical map*

$$N \to \tau^{1,\leqslant 0}\omega^1 j_* j^* N$$

  is an isomorphism.
- *Let $M \in \mathbf{M}^{1,\mathrm{ls}}(U, \mathbb{Q})$ be a lisse 1-motive such that $R_\ell(M)$ is unramified over S. Then the pushed forward motive*

$$\tau^{1,\leqslant 0}\omega^1 f_* M$$

  lies in $\mathbf{M}^{1,\mathrm{ls}}(S, \mathbb{Q})$.

We will deduce this from Lemmas 6.6 and 6.7. The author thanks Pepin Lehalleur for sharing the following weaker version of Lemma 6.6, which served as inspiration.

Lemma 6.2. *Let S be regular connected and let $j \colon U \to S$ be an open immersion. For all $M \in \mathbf{M}^{1,\mathrm{ls}}(S)$, the unit of the adjunction $M \to \tau^{1,\leqslant 0}\omega^1 j_* j^* M$ is an equivalence. In particular the restriction*

$$j^* \colon \mathbf{M}^{1,\mathrm{ls}}(S) \to \mathbf{M}^{1,\mathrm{ls}}(U)$$

is fully faithful. More is true: If further the complement of U in S has codimension at least 2, then even the map $M \to \omega^1 j_* j^* M$ is an equivalence.

Proof. Let $i \colon Z \to S$ be the inclusion of the reduced complement. Applying ω^1 to the colocalization sequence and recalling that ω^1 commutes with $i_! = i_*$ ([PL19, Proposition 3.3.(iii)c)]), we obtain the fiber sequence

$$i_!\omega^1 i^! M \to M \to \omega^1 j_* j^* M.$$

Pepin Lehalleur shows in [PL17, Lemma 4.7] that in our setup, $\omega^1 i^! M$ is concentrated in degrees $\geqslant 2$ and vanishes if Z is of codimension $\geqslant 2$ in S. Using that $i_!$ is exact one derives the claim. □

Lemma 6.3. *Let $j : U \to S$ be an open dense immersion between normal schemes and ℓ a prime invertible on S. Then the pullback functor*

$$j^* \colon \mathbf{M}^{1,\mathrm{ls}}(S) \to \mathbf{M}^{1,\mathrm{ls}}(U)$$

is faithful and conservative.

Proof. We first note that it is conservative, because it fits into a commutative diagram

$$\begin{tikzcd} \mathbf{M}^{1,\mathrm{ls}}(S) \arrow[r, "R_\ell"] \arrow[d, "j^*"'] & \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S, \mathbb{Q}_\ell) \arrow[d, "j^*"] \\ \mathbf{M}^{1,\mathrm{ls}}(U) \arrow[r, "R_\ell"'] & \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(U, \mathbb{Q}_\ell) \end{tikzcd}$$

where the three other functors are conservative. Faithfullness now follows because an exact functor between abelian categories which is conservative is also faithful. ☐

Lemma 6.4. *Let $f\colon Y \to X$ be a proper morphism of schemes, equivariant with respect to the action of a finite group G on Y and a trivial action on X, such that G acts transitively on the geometric fibers of f. Then for $N \in \mathbf{M}^1(X), M \in \mathbf{M}^{1,\mathrm{ls}}(X)$ we have*

$$\mathrm{Hom}_{\mathbf{M}^1(X)}(N, M) \cong \mathrm{Hom}_{\mathbf{M}^1(Y)}(f^*N, f^*M)^G$$

functorially in N and M.

Proof. We first prove this statement in the case that f has a section $\gamma\colon X \to Y$, which we turn into a G-equivariant morphism Γ making the diagram

$$\begin{tikzcd} \coprod_G X \arrow[r, "\Gamma"] \arrow[d, "\Sigma"'] & Y \arrow[dl, "f"] \\ X & \end{tikzcd}$$

commute where Σ denotes the fold map. We obtain a corresponding diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{M}^1(\coprod_G X)}(\Sigma^*M, \Sigma^*N)^G & \mathrm{Hom}_{\mathbf{M}^1(Y)}(f^*M, f^*N)^G \arrow[l, "\Gamma^*"'] \\ \mathrm{Hom}_{\mathbf{M}^1(X)}(M, N) \arrow[u, equal, "\Sigma^*"] \arrow[ur, "f^*"'] & \end{tikzcd}$$

showing that Γ^* is a surjection. For injectivity, faithfulness of ℓ-adic realizations reduces us to showing the statement after applying R_ℓ. Here we have to show that $\mathcal{F} \cong (R^0 f_* f^* \mathcal{F})^G$ for a lisse ℓ-adic sheaf $\mathcal{F}$ on X. Proper base change reduces this to the case of a strictly henselian local ring, and being lisse to its residue field, where it is clear by our assumption on the fibers. This concludes the case where f has a section. For the general case, we use h-descent for DM, as f has a section after passing to the h-cover given by f itself. Consider thus the Cech-Nerve $C(f)$ of f and the diagram

$$\begin{tikzcd} \mathrm{DM}(X) \arrow[r, "\sim"] \arrow[d] & \lim_\Delta \mathrm{DM}(C(f)) \arrow[d] \\ \mathrm{DM}(Y) \arrow[r, "\sim"] & \lim\limits_{\Delta} \mathrm{DM}(C(f) \times_X Y) \end{tikzcd}$$

where the horizontal arrows are isomorphisms. Hence we obtain the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{DM}(X)}(N,M) \arrow[r, "\sim"] \arrow[d] & \lim\limits_{i\in\Delta} \mathrm{Hom}(N^i, M^i) \arrow[d] \\ \mathrm{Hom}_{\mathrm{DM}(Y)}(f^*N, f^*M) \arrow[r, "\sim"] & \lim\limits_{i\in\Delta} \mathrm{Hom}(f^*N^i, f^*M^i) \end{tikzcd}$$

where the right vertical arrow is an isomorphism by what we did above, and thus the left vertical arrow is also, which we wanted to show. □

We need a version of Zariski's connectedness principle which works for Galois alterations instead of only for modifications.

Lemma 6.5. *Let $p\colon \tilde{S} \to S$ be a Galois-alteration w.r.t a finite group G and normal target S. For any geometric point $s \in S$ and any lift $\tilde{s} \in \tilde{S}$, the map*

$$G \cdot \tilde{s} \to \pi_0(\tilde{S}_s)$$

is a surjection. In other words, the fibers are geometrically connected G-sets.

Proof. Considering the Stein factorization $\tilde{S} \to T \xrightarrow{\pi} S$ of p and applying Zariski's Main Theorem, we can replace $\tilde{S}$ by T and talk about geometric fibers of the finite G-morphism $\pi\colon T \to S$ instead. By assumption, the field extension $\kappa T/\kappa(S)$ factors as $\kappa(S) \to \lambda \to \kappa(T)$ with $\kappa(T)/\lambda$ Galois with group G and $\lambda/\kappa(S)$ purely inseparable. Let $s \in S$ be a point of S and $\mathcal{O}_{S,s}$ the corresponding normal local ring. Then connected components of $\pi^{-1}(s)$ inject G-equivariantly into maximal ideals of the integral closure of $\mathcal{O}_{S,s}$ in $\kappa(S)$. We first show that the integral closure Γ of $\mathcal{O}_{S,s}$ in λ has only one maximal ideal lying over s. By factoring the field extension $\lambda/\kappa(S)$, we may assume that its degree is p. The absolute Frobenius of K then induces morphisms $\mathrm{Spec}(\mathcal{O}_{S,s}) \to \mathrm{Spec}(\Gamma) \to \mathrm{Spec}(\mathcal{O}_{S,s})$ where the composition is a universal homeomorphism and the first map is surjective by finiteness, hence also a homeomorphism. Hence we may replace S by its integral closure in λ and assume the $\kappa(S)/\kappa(T)$ separable. But then $\mathcal{O}_S = (\pi_*\mathcal{O}_T)^G$ and the claim follows. □

We can finally prove the first part of our main theorem.

Lemma 6.6. *Let $j : U \to S$ be an open immersion between normal schemes, and take $M \in \mathbf{M}^{1,\mathrm{ls}}(S)$. Then*

$$\eta_M\colon M \to \tau^{1,\leqslant 0}\omega^1 j_* j^* M$$

is an isomorphism.

Proof. Let us first note that for all $N \in \mathbf{M}^1(S)$ we have a commutative square

$$\begin{tikzcd} \mathrm{Hom}(N,M) \arrow[r] \arrow[d, hook] & \mathrm{Hom}(j^*N, j^*M) \arrow[d, hook] \\ \mathrm{Hom}(R_\ell(N), R_\ell(M)) \arrow[r, hook] & \mathrm{Hom}(j^*R_\ell(N), j^*R_\ell(M)) \end{tikzcd}$$

where the indicated arrows are injective because the realizations are faithful (again, as conservative exact functors between abelian categories) and because the $R_\ell(M)$ is lisse. As a consequence, we see that η_M is a monomorphism in

$\mathbf{M}^1(S)$, and we only have to see that it is also an epimorphism. In particular, we are free to shrink U. Doing so, we may assume the existence of a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\tilde{j}"] \arrow[d, "e"'] & \tilde{S} \arrow[d, "p"] & \tilde{Z} \arrow[l, "\tilde{i}"'] \arrow[d, "q"] \\ U \arrow[r, hook, "j"'] & S & Z \arrow[l, "i"] \end{tikzcd}$$

where both squares are cartesian, $\tilde{S}$ is regular, $U = S \backslash Z$, i is a closed immersion, and the right square satisfies the conditions of Lemma 5.8 with finite group G. In particular, V/G exists as a scheme and $V/G \to U$ is radicial. We obtain a diagram

$$\begin{tikzcd} \mathrm{DM}(S) \arrow[r, "i^*"] \arrow[d, "\bar{p}^*"] \arrow[dd, bend right=60, "p^*"'] & \mathrm{DM}(Z) \arrow[d, "\bar{q}^*"'] \arrow[dd, bend left=60, "q^*"] \\ \mathrm{DM}(\tilde{S}; G) \arrow[r, "\bar{\tilde{i}}^*"] \arrow[d, "\kappa^*"] & \mathrm{DM}(\tilde{Z}; G) \arrow[d, "\lambda^*"'] \\ \mathrm{DM}(\tilde{S}) \arrow[r, "\tilde{i}^*"'] & \mathrm{DM}(\tilde{Z}) \end{tikzcd}$$

where the upper square is cartesian by Lemma 5.8 and all functors have right adjoints. Note that for any $N \in \mathbf{M}^1(S)$ we have functorial isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{M}^1(S)}(N, \tau^{1,\leqslant 0}\omega^1 \bar{p}_*\bar{p}^* M) &= \mathrm{Hom}_{\mathrm{DM}(S)}(N, \bar{p}_*\bar{p}^* M) \\ &= \mathrm{Hom}_{\mathrm{DM}(\tilde{S})}(p^*N, p^*M)^G \\ &= \mathrm{Hom}_{\mathbf{M}^1(\tilde{S})}(p^*N, \tau^{1,\leqslant 0}\omega^1 \tilde{j}_*\tilde{j}^* p^* M)^G \\ &= \mathrm{Hom}_{\mathbf{M}^1(\tilde{S})}(p^*N, \tau^{1,\leqslant 0}\omega^1 \tilde{j}_* e^* j^* M)^G \\ &= \mathrm{Hom}_{\mathbf{M}^1(V)}(e^*j^*N, e^*j^*M)^G \\ &= \mathrm{Hom}_{\mathbf{M}^1(U)}(j^*N, j^*M) \\ &= \mathrm{Hom}_{\mathbf{M}^1(S)}(N, \tau^{1,\leqslant 0}\omega^1 j_*j^*M) \end{aligned}$$

which shows $\tau^{1,\leqslant 0}\omega^1 \bar{p}_*\bar{p}^* M \cong \tau^{1,\leqslant 0}\omega^1 j_*j^*M$. Hence the result follows from the two preceding lemmas. □

Lemma 6.7. *Let $j : U \to S$ be an open immersion between normal schemes, and let $\mathbb{M} \in \mathbf{M}_1^{\mathrm{Del}}(U)$ be a Deligne 1-motive whose ℓ-adic realization has good reduction over S. Then after passing to a finite étale cover of S, we find a lisse 1-motive $N \in \mathbf{M}^{1,\mathrm{ls}}(S)$ with $j^*N \cong \Phi_U(\mathbb{M})$.*

Proof. As the lattice and the toric part of $\mathbb{M}$ extend over S, we may choose a finite étale cover $e \colon S' \to S$ such that $e_U^*\mathbb{M}$ satisfies the following: ($\star$) The toric and lattice part are split and the abelian part A admits an ℓ^2-level structure. Dropping e and S' from the notation, we assume that $\mathbb{M}$ already satisfies those properties; denote the abelian part of $\mathbb{M}$ by A.

By the assumptions, $(S, U, j, A, \mathbb{M}, \mathcal{F}, u_{\mathbb{M}})$ is an extendable datum to which we apply Corollary 4.33 to obtain linked extended data D^i, maps $\hat{p}_M^N$ and isogenies ϕ_m^l as specified there. We arrange the D^i and $\hat{p}_M^N$ to a functor

$$\hat{S}^\bullet \colon \mathcal{P}^* \langle k \rangle \to (\mathrm{Schemes})/S$$

and apply DM(−) to obtain a functor

$$\mathrm{DM}(\hat{S}^\bullet)\colon (\mathcal{P}^*\langle k\rangle)^{op} \to \mathcal{P}r^{st,\otimes}.$$

We note that the canonical morphism

$$\mathrm{DM}(S) \to \lim_{(\mathcal{P}^*\langle k\rangle)^{op}} \mathrm{DM}(\hat{S}^\bullet)$$

is an equivalence of symmetric monoidal stable ∞-categories. This follows by induction from Lemma 5.7 and Lemma 5.2. For $i \in \langle k\rangle$ define

$$o_i := \Phi_{\hat{S}^i}(\hat{\mathbb{M}}^i) \in \mathrm{DM}(\hat{S}^i)$$

and note that Corollary 4.33 gives in particular for $A = \{a_1 < \ldots < a_r\} \in \mathcal{P}^*\langle k\rangle$ a chain

$$(p^A_{a_1})^* o_1 \xrightarrow{f_{1,2}} (p^A_{a_2})^* o_2 \xrightarrow{f_{2,3}} \ldots \xrightarrow{f_{r-1,r}} (p^A_{a_r})^* o_r$$

of equivalences in $\mathrm{DM}(\hat{S}^A)$, where we set

$$f_{k,l} = (\hat{p}^A_{\{a_k,a_l\}})^*\big(\Phi_{\hat{S}^{k,l}}(\phi^{a_k,a_l})\big).$$

Hence by Proposition 5.1, we obtain $N \in \mathrm{DM}(S)$ satisfying $(i_0^n \circ \hat{p}^n)^* N \cong \Phi_{\hat{S}^n}(\hat{\mathbb{M}}^n)$. By [Lur17, Proposition 4.6.1.11], N is strongly dualizable. Further, denoting by $j^0\colon U \to \hat{S}$ the unique open immersion over S, we have

$$j^*N \cong (j^0)^*(\hat{p}^0)^*N \cong (j^0)^*\Phi_{\hat{S}^0}\hat{\mathbb{M}}^0 \cong \Phi_U(\hat{\mathbb{M}}^0_{|U}) = \Phi_U(\mathbb{M}).$$

Finally, we need to check that N is actually in $\mathbf{M}^1(S)$. It is enough to verify this after pulling back along inclusions $i\colon s \to S$ of points of S by [PL19, Proposition 1.25] and [PL17, Theorem 4.1 (ii)]. Let $r \in \langle k\rangle$ be the unique index such that $s \in S^r \backslash S^{r+1} = U^r$. Denoting by $j^r\colon U^r \to \hat{S}^r$ the unique open immersion over S, we find that i factors as

$$i = i_0^r \circ \hat{p}^r \circ j^r \circ i'$$

with $i'\colon S \to U^r$ and hence we find that

$$i^*N \cong (i')^*(j^r)^*\Phi_{\hat{S}^r}(\hat{\mathbb{M}}^r) \cong \Phi_s(\hat{\mathbb{M}}^r \times_{\hat{S}^r} s)$$

is in $\mathbf{M}^1(s)$ as required. □

Proof of Theorem 6.1. By transitivity of the pushforward functors, we may shrink U and assume by continuity that $M = \Phi_U(\mathbb{M})$ for a Deligne 1-motive $\mathbb{M}$. As being lisse can be checked on an étale cover and as $\tau^{1,\leqslant 0}, \omega^1$ and j_* all are compatible with finite étale base change, we can pass to a finite étale cover of S and apply Lemma 6.7 to produce $N \in \mathbf{M}^{1,\mathrm{ls}}(S)$ with $j^*N \cong M$. Then by Lemma 6.6 we have

$$N \cong \tau^{1,\leqslant 0}\omega^1 j_* j^* M \cong \tau^{1,\leqslant 0}\omega^1 j_* M$$

which is therefore lisse. □

6.2. **Regular schemes and purity.** We still have to discuss the special case of regular schemes and the resulting purity statement.

Lemma 6.8. *Let $j\colon U \to S$ be a dense open immersion between regular schemes, ℓ be a prime invertible on S, and M an object of $\mathbf{M}^{1,\mathrm{ls}}(U)$ whose ℓ-adic realization extends to a lisse ℓ-adic sheaf on S. After shrinking U, there is a factorization of j into a composition $U \xrightarrow{j_1} V \xrightarrow{j_2} S$ of open immersions such that $S \backslash V$ is of codimension at least 2 in S, such that $M_V := \tau^{1,\leqslant 0}\omega^1 j_{1*}M$ is of the form $\Phi_V(\mathbb{M})$ for a Deligne 1-motive $\mathbb{M} \in \mathbf{M}_1^{\mathrm{Del}}(V)$*

Proof. By shrinking U, we may assume that $M = \Phi_U(\mathbb{N})$ comes from a Deligne 1-motive $\mathbb{N} \in \mathbf{M}_1^{\mathrm{Del}}(U, \mathbb{Q})$, to which we apply Corollary 4.16 to find j_1, j_2 and $\mathbb{M} \in \mathbf{M}_1^{\mathrm{Del}}(V)$ with $j_1^*\mathbb{M} \cong \mathbb{N}$. Then

$$\Phi_V(\mathbb{M}) \to \tau^{1,\leqslant 0}\omega^1 j_{1*}j_1^*\Phi_V(\mathbb{M}) \cong \tau^{1,\leqslant 0}\omega^1 j_{1*}\Phi_U(\mathbb{N}) \cong \tau^{1,\leqslant 0}\omega^1 j_{1*}M$$

is an isomorphism by Lemma 6.2. □

Corollary 6.9. *Let S be regular, and let $j\colon U \to S$ be an open immersion whose complement has codimension at least 2. Let further $\mathbb{M}$ be a Deligne 1-motives over U and $M = \Phi_U(\mathbb{M})$ the corresponding lisse 1-motive. Then $\omega^1 j_*M$ is lisse.*

Proof. As dualizability can be checked locally for the étale topology and as $\omega^1 j_*$ commutes with étale pullback by [PL19, Proposition 3.3 v)c)] and smooth base change, we may assume that there is a prime ℓ invertible on S and that A admits an ℓ^2-level structure. By Zariski-Nagata purity, there is a lisse ℓ-adic sheaf $\mathcal{F} \in \mathrm{Sh}^{\mathrm{ls}}_{\text{ét}}(S, \mathbb{Z}_\ell)$ together with an isomorphism $u_A\colon j^*\mathcal{F} \to T_\ell(A)$. By Lemma 6.7, there is an $N \in \mathbf{M}^{1,\mathrm{ls}}(S)$ with $j^*N \cong M$. By Proposition 6.2 we have $N \cong \omega^1 j_*M$, so the latter is lisse. □

Corollary 6.10. *Let S be a regular scheme and $j\colon U \to S$ be an open dense immersion whose complement has every codimension at least 2. Then*

$$j^*\colon \mathbf{M}^{1,\mathrm{ls}}(S, \mathbb{Q}) \to \mathbf{M}^{1,\mathrm{ls}}(U, \mathbb{Q})$$

is an equivalence of categories with inverse $\omega^1 j_*$; no truncation is needed.

6.3. **Realization reflects being lisse.** Finally, we show that being lisse can be checked on realizations for not necessarily regular base schemes:

Theorem 6.11. *Let S be a scheme, ℓ a prime invertible on S, and $M \in \mathbf{M}^1(S)$ be a 1-motive whose ℓ-adic realization $R_\ell(M)$ is a lisse sheaf. Then $M \in \mathbf{M}^{1,\mathrm{ls}}(S)$.*

Proof. We first show this in the case that S is normal. By continuity, we find a dense open immersion $j\colon U \to S$ such that $j^*M \in \mathbf{M}^{1,\mathrm{ls}}(U)$. By Theorem 6.1, $\tau^{1,\leqslant 0}\omega^1 j_*M$ is lisse and the canonical map $M \to \tau^{1,\leqslant 0}\omega^1 j_*M$ is an isomorphism.

For the general case, we use noetherian induction and the key fact that dualizability of an object in a limit of symmetric monoidal stable ∞-categories can be checked on the individual categories appearing [Lur17, Proposition 4.6.1.11]. As closed coverings can be described by abstract blow-up squares, this result together with Lemma 5.7 reduces us to S being irreducible. Choosing a regular Galois-alteration of S [dJ97, Corollary 5.15] and applying Lemma 5.8 together with the case of regular schemes proven above then reduces us to schemes of lower dimension. As the case of 0-dimensional schemes is obvious, this shows the claim. □

6.4. **Comparing Deligne's and Pepin Lehalleur's categories.** We now come to the comparison between Deligne 1-motives and $\mathbf{M}^1(-)$. We first deal with the case of Dedekind schemes and that of regular schemes over $\mathbb{Q}$, and consider the normal case over $\mathbb{Q}$ afterwards.

Lemma 6.12. *Let S be regular and either one-dimensional or defined over $\mathbb{Q}$. Then the comparison functor*

$$\Phi_S \colon \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \to \mathbf{M}^{1,\mathrm{ls}}(S)$$

is an equivalence of categories.

Proof. Assume S to be connected with generic point η. We have a square

$$\begin{tikzcd} \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \arrow[r, hook, "\Phi_S"] \arrow[d, hook] & \mathbf{M}^{1,\mathrm{ls}}(S) \arrow[d, hook] \\ \mathbf{M}_1^{\mathrm{Del}}(\eta,\mathbb{Q}) \arrow[r, "\sim", "\Phi_\eta"'] & \mathbf{M}^{1,\mathrm{ls}}(\eta) \end{tikzcd}$$

where Φ_η is an equivalence of categories by [PL19, Proposition 4.21] and the vertical functors are fully faithful by Propositions 6.2 and 4.1. Furthermore, Φ_S is fully faithful by [PL19, Theorem 4.31]. To show essential surjectivity, take an object M of $\mathbf{M}^{1,\mathrm{ls}}(S)$ and note that by the above diagram, we only need to produce a Deligne 1-motive $\mathbb{M} \in \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q})$ restricting to M_η. If there is a prime ℓ which is invertible on S, we are done because $R_\ell(M)$ is a lisse ℓ-adic sheaf by Lemma 2.5 and we may apply Corollary 4.14 or Corollary 4.15 to extend $\Phi_\eta^{-1}(M_\eta)$ to $\mathbb{M}$ over S. Otherwise, apply Corollary 4.17 to reduce to this situation. □

Corollary 6.13. *Let S be a normal scheme and ℓ a prime invertible on S. Then the comparison functor*

$$\Phi_S \colon \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \to \mathbf{M}^{1,\mathrm{ls}}(S)$$

is fully faithful.

Proof. Let $j : U \to S$ be an open dense immersion with U regular. Let $\mathbb{M}_1, \mathbb{M}_2 \in \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q})$ and $M_i := \Phi_S(\mathbb{M}_i)$. We know that in the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q})}(\mathbb{M}_1,\mathbb{M}_2) \arrow[r, "\Phi_S"] \arrow[d, "j^*"'] & \mathrm{Hom}_{\mathbf{M}^{1,\mathrm{ls}}(S)}(M_1,M_2) \arrow[d, "j^*"] \\ \mathrm{Hom}_{\mathbf{M}_1^{\mathrm{Del}}(U,\mathbb{Q})}(j^*\mathbb{M}_1,j^*\mathbb{M}_2) \arrow[r, "\Phi_U"'] & \mathrm{Hom}_{\mathbf{M}^{1,\mathrm{ls}}(U)}(M_1,M_2) \end{tikzcd}$$

the left vertical and the bottom horizontal arrows are bijections by Proposition 4.1 and Theorem 6.12. As the right vertical map is injective by Lemma 6.3, we see that all morphisms are bijections which shows the claim. □

Corollary 6.14. *Let S be a normal $\mathbb{Q}$-scheme. Then the comparison functor*

$$\Phi_S \colon \mathbf{M}_1^{\mathrm{Del}}(S,\mathbb{Q}) \to \mathbf{M}^{1,\mathrm{ls}}(S)$$

is an equivalence of categories.

Proof. This follows from Corollaries 4.15 and 6.13. □

6.5. **Application to the cohomology of curves.** As the final application, we come back to Question 1.1 and give the promised answer in the case of curves.

Corollary 6.15. *Let S be a normal connected scheme with generic point η, ℓ a prime invertible on S, and let $f: X \to \eta$ be a curve or a semi-abelian scheme. Then unramifiedness of $R^i f_* \mathbb{Q}_\ell$ does not depend on ℓ.*

Proof. Assume X connected. For $i = 0$, unramifiedness is equivalent to the field of constants of X being a field extension of η which is unramified over S, hence it does not depend on ℓ. If X is a curve, its second cohomology is equal to the second cohomology of the normalizations of the proper components. Hence we can reduce the $i = 2$ case to the $i = 0$ case by Poincaré duality. Finally, if X is a semi-abelian group scheme, the first cohomology of X is unramified if and only if any higher cohomology group of X is. Hence we may restrict our attention to $i = 1$. If X is a semi-abelian group scheme, set $\mathbb{M} = [0 \to X]$. If X is a curve, let $\mathbb{M}$ be the Deligne 1-motive associated to the semi-normalization of X. Then $\mathbb{M}$ has as ℓ-adic realization the first cohomology as X by [Del74, Construction 10.3.6]. Use continuity to find a dense open immersion $j: U \to S$ and an extension $\mathbb{N}$ of $\mathbb{M}$ to U. By Theorem 6.1, the realization $R_\ell(\mathbb{M})$ has good reduction over S if and only if

$$\tau^{1,\leq 0}\omega^1 j_* \mathbb{N} \in \mathbf{M}^{1,\mathrm{ls}}(S)$$

and we see that this condition does not depend on ℓ. □

References

[ABV09] Joseph Ayoub and Luca Barbieri-Viale, *1-motivic sheaves and the Albanese functor*, J. Pure Appl. Algebra **213** (2009), no. 5, 809–839.

[AEWH15] Giuseppe Ancona, Stephen Enright-Ward, and Annette Huber, *On the motive of a commutative algebraic group*, Doc. Math. **20** (2015), 807–858.

[AHPL16] Giuseppe Ancona, Annette Huber, and Simon Pepin Lehalleur, *On the relative motive of a commutative group scheme*, Algebr. Geom. **3** (2016), no. 2, 150–178.

[And04] Yves André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 17, Société Mathématique de France, Paris, 2004.

[Ayo07] Joseph Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II*, Astérisque (2007), no. 315, vi+364 pp. (2008).

[Ayo11] ———, *The n-motivic t-structures for $n = 0$, 1 and 2*, Adv. Math. **226** (2011), no. 1, 111–138.

[Ayo14] ———, *La réalisation étale et les opérations de Grothendieck*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 1, 1–145.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990.

[Bri17] Michel Brion, *Commutative algebraic groups up to isogeny*, Doc. Math. **22** (2017), 679–725.

[BS14] Bhargav Bhatt and Peter Scholze, *The pro-etale topology for schemes*, arXiv:1309.1198v2, 2014.

[BVK16] Luca Barbieri-Viale and Bruno Kahn, *On the derived category of 1-motives*, Astérisque (2016), no. 381, xi+254.

[CD16] Denis-Charles Cisinski and Frédéric Déglise, *Étale motives*, Compos. Math. **152** (2016), no. 3, 556–666.

[CD19] ———, *Triangulated categories of mixed motives*, Springer, 2019.

[Con06] Brian Conrad, *Chow's K/k-image and K/k-trace, and the Lang-Néron theorem*, Enseign. Math. (2) **52** (2006), no. 1-2, 37–108.

[Del74] Pierre Deligne, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77.

[dJ97] Aise J. de Jong, *Families of curves and alterations*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 2, 599–621.

[dJO97] Aise J. de Jong and Frans Oort, *On extending families of curves*, J. Algebraic Geom. **6** (1997), no. 3, 545–562.

[Eke90] Torsten Ekedahl, *On the adic formalism*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 197–218.

[Fal83] Gerd Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366.

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford.

[Fu11] Lei Fu, *Etale cohomology theory*, Nankai Tracts in Mathematics, vol. 13, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.

[Gro66a] Alexander Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 28, 255.

[Gro66b] ———, *Un théorème sur les homomorphismes de schémas abéliens*, Invent. Math. **2** (1966), 59–78.

[Gro67] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361.

[GW10] Ulrich Görtz and Torsten Wedhorn, *Algebraic geometry I*, Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010, Schemes with examples and exercises.

[Hid04] Haruzo Hida, *p-adic automorphic forms on Shimura varieties*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2004.

[Hoy17] Marc Hoyois, *The six operations in equivariant motivic homotopy theory*, Adv. Math. **305** (2017), 197–279.

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610, x+114.

[Joy08] André Joyal, *The theory of quasi-categories and its applications*, 2008.

[Kha16] Adeel Khan, *Motivic homotopy theory in derived algebraic geometry*, Ph.D. thesis, 2016.

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Lur17] ———, *Higher algebra*, `http://www.math.harvard.edu/~lurie/papers/HA.pdf`, 2017.

[Mat14] Tzanko Matev, *Good reduction of 1-motives*, Ph.D. thesis, 2014.

[MP13] Keerthi Madapusi Pera, *Status of grothendieck' s conjecture on homomorphisms of abelian schemes*, MathOverflow, `https://mathoverflow.net/q/123454`, 2013.

[Mum65] David Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34, Springer-Verlag, Berlin-New York, 1965.

[NS17] Thomas Nikolaus and Peter Scholze, *On topological cyclic homology*, arXiv preprint arXiv:1707.01799 (2017).

[Org04] Fabrice Orgogozo, *Isomotifs de dimension inférieure ou égale à un*, Manuscripta Math. **115** (2004), no. 3, 339–360.

[PL17] Simon Pepin Lehalleur, *Constructible 1-motives and exactness*, arXiv preprint arXiv:1712.01180 (2017).

[PL19] ———, *Triangulated categories of relative 1-motives*, Adv. Math. **347** (2019), 473–596.

[Ray70] Michel Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Lecture Notes in Mathematics, Vol. 119, Springer-Verlag, Berlin-New York, 1970.

[Rib04] Kenneth A. Ribet, *Abelian varieties over* $\mathbf{Q}$ *and modular forms*, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 241–261.

[RS19] Timo Richarz and Jakob Scholbach, *The intersection motive of the moduli stack of shtukas*, arXiv preprint arXiv:1901.04919 (2019).

[SGA71] *Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics, Vol. 224, Springer-Verlag, Berlin-New York, 1971, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud.

[SGA77] *Cohomologie l-adique et fonctions L*, Lecture Notes in Mathematics, Vol. 589, Springer-Verlag, Berlin-New York, 1977, Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Edité par Luc Illusie.

[ST68] Jean-Pierre Serre and John Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.

[Sta19] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2019.

[Vai17] Vaibhav Vaish, *Punctual gluing of t-structures and weight structures*, arXiv preprint arXiv:1705.00790 (2017).

[Zar75] Juri G. Zarhin, *Endomorphisms of Abelian varieties over fields of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 2, 272–277, 471.