# Real-valued differential forms on non-archimedean abelian varieties with totally degenerate reduction

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# Chapter 1

# Introduction

Let $A$ be an abelian variety over $\mathbb{C}$ of dimension $n$. The associated complex analytic space $A(\mathbb{C})$ is a connected compact complex Lie group, hence a complex torus. Denote by $V$ the tangent space of $A(\mathbb{C})$ at the identity. The exponential map $\exp\colon V \to A(\mathbb{C})$ is a surjective morphism of Lie groups whose kernel $\Lambda$ is a complete lattice in $V$. Via this morphism we can compute the Dolbeault cohomology groups $\mathrm{H}^{p,q}_{\overline{\partial}}(A(\mathbb{C}))$ of $A(\mathbb{C})$. If $T := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{T} := \mathrm{Hom}_{\mathbb{C}-\text{antilinear}}(V, \mathbb{C})$, then we have a canonical isomorphism of complex vector spaces

$$\mathrm{H}^{p,q}_{\overline{\partial}}(A(\mathbb{C})) \cong \Lambda^p T \otimes_{\mathbb{C}} \Lambda^q \overline{T}. \tag{1.1}$$

In particular we have $\dim_{\mathbb{C}} \mathrm{H}^{p,q}_{\overline{\partial}}(A(\mathbb{C})) = \binom{n}{p}\binom{n}{q}$.

Let $\Omega$ be an open subset of a finite dimensional real vector space $N_{\mathbb{R}}$. For $k \in \mathbb{N}$ let $A^k(\Omega, \mathbb{R})$ be the space of real valued smooth differential $k$-forms on $\Omega$. We denote by

$$A^{p,q}(\Omega) := A^p(\Omega, \mathbb{R}) \otimes_{C^\infty(\Omega)} A^q(\Omega, \mathbb{R})$$

the space of superforms of bidegree $(p, q)$ on $\Omega$. There are natural differential operators

$$d'\colon A^{p,q}(\Omega) \to A^{p+1,q}(\Omega) \quad \text{and} \quad d''\colon A^{p,q}(\Omega) \to A^{p,q+1}(\Omega),$$

which make $(A^{\bullet,\bullet}(\Omega), d', d'')$ into a bicomplex of real vector spaces. Let $\Lambda$ be a full lattice in $N_{\mathbb{R}}$ and denote by $A^{p,q}(N_{\mathbb{R}})^{\Lambda}$ the subspace $A^{p,q}(N_{\mathbb{R}})$ given by $\Lambda$-invariant superforms of bidegree $(p, q)$ on $N_{\mathbb{R}}$.

**Proposition 1.1.** *There is a canonical isomorphism of real vector spaces*

$$\mathrm{H}^q(A^{p,\bullet}(N_{\mathbb{R}})^{\Lambda}, d'') \cong \Lambda^p N^*_{\mathbb{R}} \otimes_{\mathbb{R}} \Lambda^q N^*_{\mathbb{R}}.$$

Now we consider the non-archimedean case. Let $K$ be a non-trivially valued complete, algebraically closed, non-archimedean field, for example $\mathbb{C}_p$ for a prime number $p$ or the completion of the field of complex Puiseux series $\mathbb{C}\{\{t\}\}$. In [CLD12] the authors define a bicomplex $(A_X^{\bullet,\bullet}, d', d'')$ of real valued smooth differential forms on a $K$-analytic space $X$. Let $X$ be an algebraic variety over $K$ and $X^{\mathrm{an}}$ its analytification in the sense of Berkovich [Ber90]. In [Gub16] the author also defines a bicomplex $(A_{X^{\mathrm{an}}}^{\bullet,\bullet}, d', d'')$ of sheaves of real valued smooth $(p, q)$-differential forms on $X^{\mathrm{an}}$ and shows that the definition of $A_{X^{\mathrm{an}}}^{\bullet,\bullet}$ agrees with the one from [CLD12]. We briefly recall the definition of $A_{X^{\mathrm{an}}}^{p,q}$ in the formalism of [Gub16]. A tropical chart of $X^{\mathrm{an}}$ is a pair $(V, \varphi_U)$ where $V \subseteq X^{\mathrm{an}}$ is an open subset and $\varphi_U$ is a universal closed embedding of an open subset $U \subseteq X$ into an algebraic torus $T_U$ over $K$ and $\varphi_{U,\mathrm{trop}}(V)$ is an open subset of a particular subset of a finite dimensional real vector space $N_{U,\mathbb{R}}$. The definition of $\varphi_U$ requires a choice, however the definition of differential forms on $V$ will be independent of this choice. Hence the idea to define differential forms on an open subset of $X^{\mathrm{an}}$ is to locally pull back differential forms on a specific subset of a real vector space along the tropical chart $\varphi_{U,\mathrm{trop}}$. More precisely, an element $\alpha \in A_{X^{\mathrm{an}}}^{p,q}(V)$ for an open subset $V \subseteq X^{\mathrm{an}}$ is given by an equivalence class $[(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}]$ where $(V_i)_{i\in I}$ is an open cover of $X^{\mathrm{an}}$, the pairs $(V_i, \varphi_{U_i})_{i\in I}$ are tropical charts and $\alpha_i$ is a superform on the open subset $\varphi_{U,\mathrm{trop}}(V)$ of the polyhedral complex $\varphi_{U,\mathrm{trop}}(U^{\mathrm{an}})$ in $N_{U,\mathbb{R}}$ satisfying a compatibility condition.

Let $A$ be an abelian variety over $K$ of dimension $n$. The analytification $A^{\mathrm{an}}$ is a connected smooth compact $K$-analytic group in the sense of Berkovich [Ber90]. In analogy to the complex case in this thesis we study the tropical Dolbeault cohomology groups

$$\mathrm{H}_{d''}^{p,q}(A^{\mathrm{an}}) := \frac{\ker(d''\colon A_{A^{\mathrm{an}}}^{p,q}(A^{\mathrm{an}}) \to A_{A^{\mathrm{an}}}^{p,q+1}(A^{\mathrm{an}}))}{\mathrm{im}(d''\colon A_{A^{\mathrm{an}}}^{p,q-1}(A^{\mathrm{an}}) \to A_{A^{\mathrm{an}}}^{p,q}(A^{\mathrm{an}}))}, \tag{1.2}$$

of $A^{\mathrm{an}}$ when $A$ has totally degenerate reduction. To understand the latter term we recall some points about the non-archimedean uniformization of abelian varieties. This is a very important technique that has been introduced by Mumford, Faltings and Chai in the formal setup and by Raynaud, Bosch and Lütkebohmert in the rigid setup. The theory of the latter has been reformulated for Berkovich analytic spaces in [Ber90], [Gub10], [BR15], et al. and the new feature is that contrary to rigid spaces, on Berkovich spaces there is an actual topology and not only a Grothendieck topology. To an abelian variety $A$ over $K$ we can up to isomorphism canonically associate an exact sequence

$$1 \to T \to E \to B \to 0$$

of algebraic groups over $K$, where $T$ is an algebraic torus over $K$ and $B$ is an abelian variety over $K$ such that $B^{\mathrm{an}}$ has good reduction, and a morphism of $K$-analytic groups

$$p\colon E^{\mathrm{an}} \to A^{\mathrm{an}}$$

which is a topological universal covering, whose kernel $M := \ker(p)$ is a discrete subgroup of $E(K)$ such that $A^{\mathrm{an}} \to E^{\mathrm{an}}/M$ is an isomorphism of $K$-analytic groups.

More precisely, the morphism $p\colon E^{\mathrm{an}} \to A^{\mathrm{an}}$ is a topological universal covering in the sense that $E^{\mathrm{an}}$ is a simply connected topological space and $p$ is a covering map. After fixing a point of the fiber over the identity of $A^{\mathrm{an}}$ we can endow $E^{\mathrm{an}}$ with a unique structure of a $K$-analytic group such that $p$ becomes a morphism of $K$-analytic groups and a local isomorphism. The action of $M$ on $E^{\mathrm{an}}$ by translation is free and properly discountinous. Then $A$ is uniformizable in the sense that we have an isomorphism of $K$-analytic groups $A^{\mathrm{an}} \cong E^{\mathrm{an}}/M$ induced by $p$.

If $T = 0$, $M = 0$ and $E = B$ we say that $A$ has *good reduction*, if $B = 0$ and $E = T$ we say that $A$ has *totally degenerate reduction*.

Let $N$ denote the cocharacter group of $T$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ the base change to $\mathbb{R}$. The tropicalization map $\mathrm{trop}_T\colon T^{\mathrm{an}} \to N_{\mathbb{R}}$ maps $M$ bijectively onto a lattice $\Lambda$ of $N_{\mathbb{R}}$.

In the non-archimedean setup we have the following result.

**Theorem 1.1.** *Let $A$ be an abelian variety over $K$ of dimension $n$ with totally degenerate reduction. There exists a canonical morphism of real vector spaces*

$$\Phi_A^{p,q}\colon \Lambda^p N_{\mathbb{R}}^* \otimes_{\mathbb{R}} \Lambda^q N_{\mathbb{R}}^* \to \mathrm{H}_{d''}^{p,q}(A^{\mathrm{an}}) \tag{1.3}$$

*which is injective for all $p, q \in \mathbb{Z}$.*

We sketch the construction of the morphism $\Phi_A^{p,q}$ from (1.3). From now on let $A$ be an abelian variety over $K$ of dimension $n$ with totally degenerate reduction. By the previous paragraph this means that we have morphisms

$$p\colon T^{\mathrm{an}} \to A^{\mathrm{an}} \quad \text{and} \quad \mathrm{trop}_T\colon T^{\mathrm{an}} \to N_{\mathbb{R}}.$$

To study the tropical Doulbeault cohomology groups of $A^{\mathrm{an}}$ we will introduce a new class of tropical charts of $A^{\mathrm{an}}$ called *refined tropical charts*. These will encode information about the uniformization and induce tropical charts of $A^{\mathrm{an}}$ in the usual sense. They are defined as follows. Let $\widetilde{\Omega}$ be $\Lambda$-small open subset of $N_{\mathbb{R}}$, i.e. $\Omega$ does not intersect any non-trivial $\Lambda$-translates of itself. The open subset

$\mathrm{trop}_T^{-1}(\widetilde{\Omega}) =: \widetilde{V}$ of $T^{\mathrm{an}}$ is $M$-small and $p$ maps $\widetilde{V}$ isomorphically onto an open subset $V'$ of $A^{\mathrm{an}}$. Then we define an analytic moment map $f_{V'}\colon V' \to T^{\mathrm{an}}$ which can be locally approximated by algebraic moment maps. From the latter we get a tropical chart $(V, \varphi_U)$ and an integral $\mathbb{R}$-affine morphism $F$ which we use to pull back a differential form $\alpha \in A^{p,q}(N_{\mathbb{R}})$ to the the real vector space $N_{U,\mathbb{R}}$. In order to get a well-defined differential form on $A^{\mathrm{an}}$ we furthermore have to assume that the superform $\alpha$ that we want to pull back is $\Lambda$-invariant. Hence we will consider the subspace $A^{p,q}(N_{\mathbb{R}})^{\Lambda}$ of $A^{p,q}(N_{\mathbb{R}})$ of $\Lambda$-invariant $(p,q)$-superforms. In this way we get a canonical morphism of real vector spaces

$$\phi_A^{p,q}\colon A^{p,q}(N_{\mathbb{R}})^{\Lambda} \to A^{p,q}(A^{\mathrm{an}}) \tag{1.4}$$

which is compatible with the differential $d''$ and therefore induces a morphism

$$\Phi_A^{p,q}\colon A_{\mathrm{cl}}^{p,q}(N_{\mathbb{R}})^{\Lambda} \to \mathrm{H}_{d''}^{p,q}(A^{\mathrm{an}}).$$

To construct $\phi_A^{p,q}$ it is essential that the toric part of $A^{\mathrm{an}}$ is non-trivial. In fact, the general idea behind the proof of the main result is to relate properties of a differential form $\alpha \in A^{p,q}(N_{\mathbb{R}})^{\Lambda}$ with properties of its image $\phi_A^{p,q}(\alpha)$ on $A^{\mathrm{an}}$ and this can be done by studying the relations between the tropicalization $\mathrm{trop}_T$ and the universal covering $p$.

We prove Theorem 1.1 first for top-dimensional forms and then for arbitrary degrees via integration of differential forms. For a top-dimensional superform $\alpha \in \Lambda^n N_{\mathbb{R}}^* \otimes_{\mathbb{R}} \Lambda^n N_{\mathbb{R}}^*$ with constant coefficients we use a partition of unity argument to show that if the integral of $\Phi_A^{p,q}(\alpha)$ vanishes over $A^{\mathrm{an}}$ then $\alpha = 0$. For the general case we reduce ourselves to the top-dimensional case by pairing $\alpha \in \Lambda^p N_{\mathbb{R}}^* \otimes_{\mathbb{R}} \Lambda^q N_{\mathbb{R}}^*$ with a complementary form $\beta \in \Lambda^{n-p} N_{\mathbb{R}}^* \otimes_{\mathbb{R}} \Lambda^{n-q} N_{\mathbb{R}}^*$.

A result of Jell and Wanner ([JW18, Theorem 5.1.]) implies that for $n = 1$ this morphism is in fact an isomorphism. Indeed an abelian variety with totally degenerate reduction of dimension 1 is a Mumford curve.

Note that although the setup is very similar to the complex one, many results and tools from complex geometry are not available in the context of tropical Dolbeault cohomology. For instance in the complex case to compute the cohomology groups (1.1) one can use the Künneth formula, but in the non-archimedean case the Künneth formula does not hold in general (see [Liu17b, Example 1.7.] for a counterexample). The surjectivity of $\Phi_A^{p,q}$ remains an open question. The main problem is the fact that for an arbitrary differential forms $\alpha$ on $A^{\mathrm{an}}$ given by $(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}$ we do not have any control over the tropicalizations $\varphi_{U_i,\mathrm{trop}}$ in the sense that contrarily to what we do in the construction of the morphism $\Phi_A^{p,q}$, we do not know whether all the forms $\alpha_i$ come from a differential form on (an open subset) of $N_{\mathbb{R}}$.

**Tropical cycle class map**

Let $X$ be a smooth algebraic variety over $K$. Denote by $\mathrm{CH}^q(X)$ the ring of algebraic cycles of codimension $q$ on $X$ modulo rational equivalence and put $\mathrm{CH}^q(X)_{\mathbb{Q}} := \mathrm{CH}^q(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. In [Liu17a] Liu defines and studies the tropical cycle class map

$$\mathrm{cl}_{\mathrm{trop}} \colon \mathrm{CH}^q(X)_{\mathbb{Q}} \longrightarrow \mathrm{H}^{q,q}_{d''}(X^{\mathrm{an}}),$$

which relates algebraic cycles to tropical Dolbeault cohomology classes. Let $\overline{L} := (L, ||\cdot||)$ be a smoothly metrized line bundle. In [CLD12] the authors define the first Chern form $c_1(\overline{L}) \in A^{1,1}(X^{\mathrm{an}})$ associated to $\overline{L}$. Let $[c_1(\overline{L})]$ denote the corresponding cohomology class in $\mathrm{H}^{1,1}_{d''}(X^{\mathrm{an}})$. In the last part of this thesis we compare the image of the first Chern class under the troipcal cycle classe map with the cohomology class of the first Chern form associated to $\overline{L}$.

**Proposition 1.2.** *We have*

$$\mathrm{cl}_{\mathrm{trop}}(c_1(L)) = [c_1(\overline{L})]$$

*in* $\mathrm{H}^{1,1}_{d''}(X^{\mathrm{an}})$.

## Structure of the thesis

In the Chapter 2 we recall some definitions and results from non-archimedean analytic geometry in order to give a short but self-contained reminder on the uniformization of abelian varieties over non-archimedean fields (Section 2.2). In Chapter 3 we recall the definition of differential forms on the analytification of an algebraic variety over $K$ in the formalism of [Gub16]. In Section 3.4 we show that the natural inclusion of differential complexes

$$(\Lambda^p N^*_{\mathbb{R}} \otimes_{\mathbb{R}} \Lambda^\bullet N^*_{\mathbb{R}}, d'') \to (A^{p,\bullet}(N_{\mathbb{R}})^\Lambda, d'')$$

is a quasi-isomorphism (see Proposition 3.4.26). In Section 3.4 we will present a result about integration on the product of analytifications of algebraic varieties over $K$ (see Proposition 3.4.21).

Let $A$ be an abelian variety over $K$ with totally degenerate reduction. In Chapter 4 we study the Dolbeault cohomology of $A^{\mathrm{an}}$. In Section 4.1 we will fix the setup in the totally degenerate case and show a basic relation between the tropicalization map and the universal covering $p$. In Section 4.2 we study this even more by introducing refined tropical charts and prove some basic properties. These will allow us to define the morphism $\Phi^{p,q}_A \colon \Lambda^p N^*_{\mathbb{R}} \otimes_{\mathbb{R}} \Lambda^q N^*_{\mathbb{R}} \to \mathrm{H}^{p,q}_{d''}(A^{\mathrm{an}})$ in Section 4.3. Finally we prove Theorem 1.1 (see Theorem 4.3.7) by using integration.

Let $X$ be a smooth algebraic variety over $K$. In Chapter 5 we consider the tropical cycle class map $\mathrm{cl}_{\mathrm{trop}}\colon \mathrm{CH}^q(X)_{\mathbb{Q}} \to \mathrm{H}^{q,q}_{d''}(X^{\mathrm{an}})$. First we will explain the simplifications that can be made we look at the case $q = 1$ and in the last part we prove Proposition 1.2 (see Proposition 5.2.3) by using Čech cohomology.


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## Notation and conventions

Throughout the whole thesis $K$ will denote a non-trivially valued complete, algebraically closed, non-archimedean field. Denote by $K^\circ$ its valuation ring, by $K^{\circ\circ}$ the maximal ideal of $K^\circ$ and by $k := K^\circ/K^{\circ\circ}$ its residue field. Note that $k$ is also algebraically closed.

A variety over $K$ is a separated integral scheme of finite type over $K$. An abelian variety $A$ over $K$ is a proper algebraic group over $K$.

If not stated otherwise, all $K$-analytic spaces are $K$-analytic spaces in the sense of [Ber90]. We will also consider many results from [CLD12] where the authors consider more general $K$-analytic spaces and where the spaces that we will consider are called *good* $K$-analytic spaces (see also Remark 2.1.6).

# Chapter 2

# Preliminaries

## 2.1 Non-archimedean analytic geometry

In this section we recall basic definitions and results from non-archimedean analytic geometry. The main focus will be put on the analytification of algebraic varieties and on group objects in the categories of $K$-analytic spaces and of formal $K$-analytic spaces.

In what follows all $K$-affinoid algebras are assumed to be strict [Ber90, § 2.1.].

### Affinoid domains and reduction map

What follows can be found in [Ber90, § 2.2. and § 2.4.]. Let $\mathcal{A}$ be a $K$-affinoid algebra and $X = \mathcal{M}(\mathcal{A})$.

**- 2.1.1.** The topological space $X$ can be provided with a sheaf of rings $\mathcal{O}_X$ (see [Ber90, § 2.3.]) which turns $X$ into a locally ringed space called *$K$-affinoid space*. A *morphism of $K$-affinoid spaces* $X = \mathcal{M}(\mathcal{A}) \to Y = \mathcal{M}(\mathcal{B})$ is a morphism of locally ringed spaces which comes from a bounded homomorphism $\mathcal{B} \to \mathcal{A}$ of $K$-affinoid algebras. We denote by ($K$-Aff) the category fo $K$-affinoid spaces.

A closed subset $V \subseteq X$ is called an *affinoid domain* in $X$ if there exists a bounded homomorphism of $K$-affinoid algebras $\varphi\colon \mathcal{A} \to \mathcal{A}_V$ satisfying the following universal property. For every bounded homomorphism $\psi\colon \mathcal{A} \to \mathcal{B}$ of $K$-affinoid algebras such that $\mathcal{M}(\psi)(\mathcal{M}(\mathcal{B}))$ lies in $V$, there exists a unique bounded homomorphism $\delta\colon \mathcal{A}_V \to \mathcal{B}$ with $\psi = \delta \circ \varphi$. By [Ber90, Proposition 2.2.4.(i)] if $V$ is an affinoid domain in $X$ then there is an isomorphism $\mathcal{M}(\mathcal{A}_V) \cong V$ for a $K$-affinoid algebra $\mathcal{A}_V$.

**- 2.1.2.** For $f \in \mathcal{A}$ we denote by $\rho(f) = \lim_{n\to\infty} \sqrt[n]{||f^n||}$ the *spectral radius* of $f$, where $||\cdot||$ denotes the norm on $\mathcal{A}$. By [Ber90, Corollary 1.3.3.] the function

$\rho\colon \mathcal{A} \to \mathbb{R}_{\geq 0}$ is a bounded seminorm called *spectral norm.* We say that $\mathcal{A}$ is *distinguished* if the spectral norm on $\mathcal{A}$ is equal to the residue norm for some epimorphism $K\{T_1, \dots T_n\} \to \mathcal{A}$ (see [Ber90, § 4.3., p. 81]). We say that the $K$-affinoid space $X = \mathcal{M}(\mathcal{A})$ is *distinguished* if $\mathcal{A}$ is a distinguished $K$-affinoid algebra.

For a point $x \in X$ let $\mathcal{H}(x)$ denote the *completed residue field* of $x$. The set $\mathcal{A}^\circ := \{f \in \mathcal{A} \mid \rho(f) \leq 1\}$ is a subring and $\mathcal{A}^{\circ\circ} := \{f \in A \mid \rho(f) < 1\}$ is an ideal of $\mathcal{A}^\circ$. Denote the residue ring $\mathcal{A}^\circ/\mathcal{A}^{\circ\circ}$ by $\widetilde{\mathcal{A}}$. Every point $x \in \mathcal{M}(\mathcal{A})$ corresponds to an equivalence class of characters $\chi_x\colon \mathcal{A} \to \mathcal{H}(x)$ on $\mathcal{A}$. Denote by $\widetilde{\chi}_x\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{H}}(x)$ the induced homomorphism. Then $\ker(\widetilde{\chi}_x)$ is a prime ideal.

**Definition 2.1.3.** The *reduction map* is defined as

$$\pi\colon \mathcal{M}(\mathcal{A}) \to \operatorname{Spec} \widetilde{\mathcal{A}}, \quad x \mapsto \ker(\widetilde{\chi}_x).$$

Set $\widetilde{X} := \operatorname{Spec}(\widetilde{\mathcal{A}})$.

**Remark 2.1.4.** By [Ber90, § 2.4.] the reduction map is surjective and the preimage of a closed (respectively open) subset of $\widetilde{X}$ under the reduction map is an open (respectively closed) subset of $X$.

## $K$-Analytic spaces and groups

We recall some basic notions about $K$-analytic spaces and the analytification functor. Main references are [Ber90, § 3.1-3.4] and [Ber93, § 1.3., § 2.6.].

**- 2.1.5.** In the following we will not need the precise definition of $K$-analytic space, therefore we only recall the main points. A *$K$-analytic space* is given by a topological space together with an atlas whose charts $\varphi_V$ are pairs $(V, \mathcal{A}_V)$ given by a $K$-affinoid domain $V$, a $K$-affinoid algebra $\mathcal{A}_V$ and a homeomeorphism $V \cong \mathcal{M}(\mathcal{A}_V)$ subject to some compatibility conditions (for more details see [Ber93, Definition 1.2.3.]). A *morphism of $K$-analytic spaces* $f\colon X \to Y$ is given by a continous map such that for all charts $\varphi_V$ on $X$ there is a chart $\varphi_W$ of $Y$ with $f(V) \subseteq W$, and for every such chart $\varphi_V$ there is a morphism of $K$-affinoid spaces $(V, \mathcal{A}_V) \to (W, \mathcal{A}_W)$ subject to some compatibility conditions (see [Ber93, Definition 1.2.7.]). A $K$-analytic space is called *good* if every point has a $K$-affinoid neighborhood ([Ber93, p. 22 at the bottom]).

**Remark 2.1.6.** Our main reference for $K$-analytic spaces will be [Ber90] where $K$-analytic spaces are good $K$-analytic spaces in the sense of 2.1.5. Hence from now on all $K$-analytic spaces are assumed to be good.

**Definition 2.1.7.** A morphism $\varphi\colon Y \to X$ of $K$-analytic spaces is called an *analytic domain* if $\varphi$ induces a homeomorphism of $Y$ with its image in $X$ and for any morphism $\psi\colon Z \to X$ of $K$-analytic spaces with $\psi(Z) \subseteq \varphi(Y)$, there exists a unique morphism $\sigma\colon Z \to Y$ of $K$-analytic spaces such that $\psi = \varphi \circ \sigma$.

## The analytification functor

Let ($K$-Sch) denote the category of schemes of locally finite type over $K$ and ($K$-An) the category of $K$-analytic spaces. By [Ber90, § 3.1., top of p. 48] the category ($K$-An) admits fibered products. For $X, Y$ in ($K$-Sch) respectively in ($K$-An) we will denote by $X \times_K Y$ the fibered product of $X$ with $Y$ over Spec($K$) respectively $\mathcal{M}(K)$.

**- 2.1.8.** Let $X \in$ ($K$-Sch) . We consider the functor

$$F\colon (K\text{-An}) \to (\text{Sets}), Z \to \text{Hom}_{\text{LRS}}(Z, X), \tag{2.1}$$

where $\text{Hom}_{\text{LRS}}(Z, X)$ denotes the set of morphisms of locally ringed spaces from $Z$ to $X$. Then by [Ber90, Theorem 3.4.1.], the functor $F$ is representable by a pair $(X^{\text{an}}, \pi)$, where $X^{\text{an}}$ is a $K$-analytic space and $\pi\colon X^{\text{an}} \to X$ is a faithfully flat morphism of locally ringed spaces. The correspondence

$$(K\text{-Sch}) \to (K\text{-An}), \quad X \mapsto X^{\text{an}},$$

is a functor which commutes with fibered products over $K$. This follows by a formal argument which can be found almost word for word in [GR02, XII, 1.] for complex analytic spaces. Let $X, Y$ be schemes of locally finite type over $K$ and denote by $(X^{\text{an}}, \pi_X)$ and $(Y^{\text{an}}, \pi_Y)$ the pairs representing the functor $F$ for $X$ and $Y$ as in (2.1). Denote by $p_X\colon X^{\text{an}} \times_K Y^{\text{an}} \to X^{\text{an}}$ and $p_Y\colon X^{\text{an}} \times_K Y^{\text{an}} \to Y^{\text{an}}$ the canonical projections onto the first and second factor. By using [GW10, Proposition 3.4.] one can show that $X \times_K Y$ is the fibered product of $X$ and $Y$ in the category of locally ringed spaces. Put $\pi := p_X \circ \pi_X \times_K p_Y \circ \pi_Y\colon X^{\text{an}} \times_K Y^{\text{an}} \to X \times_K Y$. Then the pair $(X^{\text{an}} \times_K Y^{\text{an}}, \pi)$ represents the functor $F$ for $X \times_K Y$ and so does the pair $((X \times_K Y)^{\text{an}}, \pi_{X \times_K Y}\colon (X \times_K Y)^{\text{an}} \to X \times_K Y)$. By uniqueness we get an isomorphism of $K$-analytic spaces $(X \times_K Y)^{\text{an}} \cong X^{\text{an}} \times_K Y^{\text{an}}$.

**Definition 2.1.9.** We call the functor $X \mapsto X^{\text{an}}$ the *analytification functor* and $X^{\text{an}} = (X^{\text{an}}, \pi)$ the *analytification* of $X$. Let $f\colon X \to Y$ be a morphism of schemes of locally finite type. Then we will denote by $f^{\text{an}}\colon X^{\text{an}} \to Y^{\text{an}}$ the canonical morphism on analytifications. Let $\mathcal{F}$ be an $\mathcal{O}_X$ module, then we denote $\mathcal{F}^{\text{an}}$ the $\mathcal{O}_{X^{\text{an}}}$-module $\pi^*\mathcal{F}$.

**- 2.1.10.** We introduce $K$-analytic groups as done in [Ber90, Chapter 5]. Hence we will assume that all $K$-analytic spaces are assumed to be separated, this means that the diagonal morphism is a closed immersion ([Ber90, § 3.1., p. 50]). A *$K$-analytic group $G$* (respectively a *$K$-affinoid group*) is a group object in the category ($K$-An) of $K$-analytic spaces (respectively in the category ($K$-Aff) from 2.1.1). Recall that this means that $G$ is a $K$-analytic space (respectively a $K$-affinoid space) and there are three morphisms

$$\mu\colon G \times_K G \to G, \quad \iota\colon G \to G \quad \epsilon\colon \mathcal{M}(K) \to G \tag{2.2}$$

that satisfy the usual groups axioms. A *morphism of $K$-analytic groups* (respectively *of $K$-affinoid spaces*) is a morphism of $K$-analytic spaces (respectively of $K$-affinoid spaces) that respects the group structure.

**Example 2.1.11.** Recall that by 2.1.8 the analytification functor commutes with fibered products. An important class of examples of $K$-analytic groups is given by the analytification of group varieties over $K$, i.e. group objects in the category ($K$-Var) of varieties over $K$. Indeed let $G = (G, m, i, e)$ be a group variety over $K$. Denote by $\psi$ the isomorphism of $K$-analytic spaces $G^{\mathrm{an}} \times_{\mathrm{Spec}(K)^{\mathrm{an}}} G^{\mathrm{an}} \cong (G \times_K G)^{\mathrm{an}}$. The tuple $(G^{\mathrm{an}}, \mu, \iota, \epsilon)$ where

$$\begin{aligned} \mu :=& m^{\mathrm{an}} \circ \psi\colon G^{\mathrm{an}} \times_{\mathrm{Spec}(K)^{\mathrm{an}}} G^{\mathrm{an}} \to G^{\mathrm{an}}, \\ \iota :=& i^{\mathrm{an}}\colon G^{\mathrm{an}} \to G^{\mathrm{an}}, \quad \epsilon := e^{\mathrm{an}}\colon \mathrm{Spec}(K)^{\mathrm{an}} \to G^{\mathrm{an}}. \end{aligned}$$

is a $K$-analytic group. We show associativity of $\mu$. By definition we have $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$, then

$$\begin{aligned} \mu \circ (\mu \times \mathrm{id}_{G^{\mathrm{an}}}) =& m^{\mathrm{an}} \circ \psi \circ (m^{\mathrm{an}} \circ \psi \times \mathrm{id}_{G^{\mathrm{an}}}) \\ =& m^{\mathrm{an}} \circ \psi \circ (m^{\mathrm{an}} \times \mathrm{id}_{G^{\mathrm{an}}}) \circ (\psi \times \mathrm{id}_{G^{\mathrm{an}}}) \\ =& m^{\mathrm{an}} \circ (m \times \mathrm{id}_G)^{\mathrm{an}} = (m \circ (m \times \mathrm{id}_G))^{\mathrm{an}} \\ =& (m \circ (\mathrm{id}_G \times m))^{\mathrm{an}} = m^{\mathrm{an}} \circ \psi \circ (\mathrm{id}_{G^{\mathrm{an}}} \times m^{\mathrm{an}} \circ \psi) \\ =& \mu \circ (\mathrm{id}_{G^{\mathrm{an}}} \times \mu). \end{aligned}$$

The other properties of a group object can be verified in a similar way.

**Example 2.1.12.** Let $\mathbb{G}_{m,K} = \mathrm{Spec}(K[Z^{\pm 1}])$ be the multiplicative group over $K$, then by Example 2.1.11 its analytification $\mathbb{G}_{m,K}^{\mathrm{an}}$ is a $K$-analytic group. The set $\mathbb{G}_{m,K,1}^{\mathrm{an}} := \{|\cdot|_t \in \mathbb{G}_{m,K}^{\mathrm{an}} \mid |Z|_t = 1\}$ is an affinoid subgroup $\mathbb{G}_{m,K}^{\mathrm{an}}$ (see [Ber90, Example 5.1.4., § 6.3.]).

**Definition 2.1.13.** (i) A *$K$-analytic torus* is a $K$-affinoid group which is isomorphic to $(\mathbb{G}_{m,K}^{\mathrm{an}})^r$ for some natural number $r \geq 0$.

(ii) A *$K$-affinoid torus* $T_1^{\mathrm{an}}$ is a $K$-affinoid group which is isomorphic to $(\mathbb{G}_{m,K,1}^{\mathrm{an}})^r$ for some natural number $r \geq 0$.

**Remark 2.1.14.** By [Bos71, § 3., p. 13] there exists a one-to-one correspondence between algebraic tori $T$ over $K$ and $K$-affinoid tori $T_1^{\mathrm{an}}$.

## Formal $K$-analytic spaces and groups

Throughout this section all $K$-analytic spaces are assumed to be strict and separated (see [Ber90, § 2.1., §3.1.]), so that we can use the correspondence between rigid analytic and (Berkovich) analytic spaces explained in [Ber90, Proposition 3.3.1]. We follow the exposition and terminology from [Ber90, § 4.3. and §6.4] and [Gub10, § 2.5.]. Recall from Definition 2.1.3 that for a $K$-affinoid space $X = \mathcal{M}(\mathcal{A})$ there exists a reduction map $\pi\colon \mathcal{M}(\mathcal{A}) \to \operatorname{Spec} \widetilde{\mathcal{A}}$ where $\operatorname{Spec} \widetilde{\mathcal{A}}$ is a scheme over the residue field $k$ of $K$.

**- 2.1.15.** Let $\mathcal{A}$ be a $K$-affinoid algebra and set $X := \mathcal{M}(\mathcal{A})$ and $\widetilde{X} := \operatorname{Spec}(\mathcal{A})$. An affinoid domain $V$ in a $K$-affinoid space $X$ is called *formal* if the induced morphism $\widetilde{V} \to \widetilde{X}$ is an open immersion. Formal affinoid domains generate a topology on $X$ called the *formal topology.* A *formal affinoid variety* Spf($\mathcal{A}$) over $K$ is a ringed space $(X_{\text{f-An}}, \mathcal{O}_{X_{\text{f-An}}})$ where $X_{\text{f-An}} = X$ and $\mathcal{O}_{X_{\text{f-An}}}$ is the sheaf $\mathcal{O}_X$ seen as a sheaf with respect to the formal topology defined in (i). A *morphism* of affinoid varieties over $K$ is a morphism of locally ringed spaces which is induced by a reverse morphism of the corresponding $K$-affinoid algebras.

**- 2.1.16.** A *formal* affinoid covering $\mathcal{U} = \{U_i\}_{I \in i}$ of a $K$-analytic space $X$ is an affinoid covering of $X$ which is locally finite and such that the intersection $U_i \cap U_j$ is a formal subdomain of $U_i$ for all $i, j \in I$. Two formal coverings $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ are called *equivalent* if $U_i \cap V_j$ is a formal subdomain of both $U_i$ and $V_j$ for all $i \in I$ and $j \in J$. A formal affinoid covering $\mathcal{U} = \{U_i\}_{I \in i}$ of $X$ is called *distinguished* if $U_i$ is a distinguished $K$-affinoid space for every $i \in I$ (see 2.1.2).

**Definition 2.1.17.** A *formal $K$-analytic space* is a pair $(X,\mathcal{U}) = X$ where $X$ is a $K$-analytic space and $\mathcal{U}$ is a fixed equivalence class of formal coverings of $X$ by affinoid subdomains isomorphic to formal affinoid varieties Spf($\mathcal{A}$) over $K$. A *morphism of formal $K$-analytic spaces* $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a morphism of $K$-analytic spaces $X \to Y$ such that $f(U) \subseteq V$ for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

**- 2.1.18.** Denote by (f-$K$-An) the category of formal $K$-analytic spaces. By [Ber90, § 6.4.] the category (f-$K$-An) admits fibered products. A *formal analytic $K$-group* $(G,\mathcal{U})$ is a group object in the category of formal analytic $K$-spaces. We say that

$(G,\mathcal{U})$ is *distinguished* if $\mathcal{U}$ is a distinguished formal affinoid covering. By *loc. cit.* the functor

$$(\text{f-}K\text{-An}) \to (K\text{-An})$$

is fully faithful. Hence a $K$-analytic group can have at most one formal $K$-analytic group structure.

**- 2.1.19.** A formal covering $\mathcal{U} = \{U_i\}_{i\in I}$ of a $K$-analytic space $X$ gives rise to a reduced scheme $\widetilde{X}$ of locally finite type over $k$ and to a map $\pi\colon X \to \widetilde{X}$ that extends all the reduction maps $\pi_i\colon U_i \to \widetilde{U}_i$ (see Definition 2.1.3). Equivalent formal coverings of $X$ give rise to the same reduction ([Ber90, § 4.3.]). Note that if $G$ is a formal $K$-analytic group then the reduction $\widetilde{G}$ is a group scheme over $k$. We recall the main points of the construction of $\pi\colon X \to \widetilde{X}$ by following [Bos77, pp. 8-9]. By definition of formal affinoid covering, we can glue the affine $k$-schemes $\widetilde{U}_i$ along the open subsets $\widetilde{U_i \cap U_j}$ to get a reduced $k$-scheme $\widetilde{X}$ locally of finite type, where we define

$$\widetilde{U_i \cap U_j} := \pi_i(U_i \cap U_j) = \pi_j(U_i \cap U_j).$$

For $x, y \in X$ we consider the equivalence relation on $X$ given by $x \sim y$ if and only if for every formal open subset $U$ of $X$ with $x \in U$ we also have $y \in U$. We consider the projection $\pi\colon X \to X/\sim$ and endow the set $X/\sim$ with the quotient topology. One can show that $\widetilde{X}$ is homeomorphic to $X/\sim$. Finally we consider the subsheaves $\mathcal{O}_X^\circ$ and $\mathcal{O}_X^{\circ\circ}$ of $\mathcal{O}_X$ given by

$$V \mapsto \mathcal{O}_X^\circ(V) := \{f \in \mathcal{O}_X(V) \mid \rho(f) \leq 1\}, \text{ and}$$
$$V \mapsto \mathcal{O}_X^{\circ\circ}(V) := \{f \in \mathcal{O}_X(V) \mid \rho(f) < 1\}.$$

Denote by $\widetilde{\mathcal{O}}_X$ as the sheaf associated to the presheaf $\mathcal{O}_X^\circ/\mathcal{O}_X^{\circ\circ}$, then one can show that $\pi_*\widetilde{\mathcal{O}}_X \cong \mathcal{O}_{\widetilde{X}}$ and $\pi : (X, \mathcal{O}_X) \to (X/\sim, \pi_*\widetilde{\mathcal{O}}_X)$ is the desired reduction map. By [Ber90, § 6.4., p.121] a morphism $\varphi\colon X \to Y$ of formal $K$-analytic spaces induces a morphism $\widetilde{\varphi}\colon \widetilde{X} \to \widetilde{Y}$ of $k$-schemes which commutes with the reduction maps.

**Definition 2.1.20.** A $K$-analytic group $G$ is said to have *good reduction* if $G$ is a distinguished formal $K$-analytic group.

**Remark 2.1.21.** In [Ber90, § 6.4., pp. 121-122] a $K$-analytic group has good reduction if it is a distinguished formal $K$-analytic group whose reduction $\widetilde{G}$ is geometrically reduced. In our setup the latter condition is redundant since $K$ is algebraically closed and by Remark 2.1.19 the reduction is a reduced scheme. In Definition 2.1.20 we require the $K$-analytic group $G$ to be distinguished in order to induce a $k$-group scheme structure on $\widetilde{G}$. Indeed if $A, B$ are distinguished $K$-affinoid algebras such that $\widetilde{A}\widehat{\otimes}_k\widetilde{B}$ is reduced, then by [Bos69, Satz 6.5.] we have

$$\widetilde{A\widehat{\otimes}_K B} = \widetilde{A}\widehat{\otimes}_k\widetilde{B}.$$

For more details see [Bos76, § 1] and [Bos69, § 3 and § 6].

## 2.2 Non-archimedean uniformization

We use the terminology introduced in Section 2.1 to recall the main facts about the uniformization of abelian varieties over $K$. We will mainly follow the exposition from [Ber90, § 6.5.], but also add some more informations from [BR15, § 4], [FRSS18, § 3] and [Gub10, § 4].

**- 2.2.1.** All group objects in the category ($K$-Var) of varieties over $K$, (f-$K$-An) of formal $K$-analytic spaces and ($K$-An) of $K$-analytic spaces defined in Section 2.1 are commutative. A sequence

$$0 \to T \to G \to B \to 0$$

of group objects in ($K$-Var) is called *exact* if the associated sequence of fppf sheaves in abelian groups is exact. There is also a notion of exactness in (f-$K$-An) which we will not need in the following. For details one may refer to [Sch13, Chapter 3] or to [Ser59, Chapter VII].

**- 2.2.2.** In this first part we follow the exposition from [Ber90, § 6.5.]. Let $A$ be an abelian variety over $K$. There exists a unique compact $K$-analytic subgroup $A_1$ of $A^{\mathrm{an}}$ which is an analytic domain (Definition 2.1.7), has the structure of a formal $K$-analytic group (Definition 2.1.17) and has semi-abelian reduction, i.e. the reduction is an extension of an abelian variety by a torus.

The $K$-analytic group $A_1$ contains a unique closed $K$-analytic subgroup that is a $K$-affinoid torus $T'$ that is isomorphic to $T_1^{\mathrm{an}}$ (see Example 2.1.12) for some (split) algebraic torus $T = \mathbb{G}_{m,K}^r$ over $K$ (see also Remark 2.1.14). We call $r$ the *torus rank* of $A$.

The closed immersion $T_1^{\mathrm{an}} \to A_1$ uniquely extends to a homomorphism of $K$-analytic groups $T^{\mathrm{an}} \to A^{\mathrm{an}}$. Then we define $E^{\mathrm{an}}$ as the push-out of $A_1$ and $T^{\mathrm{an}}$ over $T_1^{\mathrm{an}}$ in the category of formal $K$-analytic groups.

By [Ber90, Corollary 6.5.2.] the morphism $p\colon E^{\mathrm{an}} \to A^{\mathrm{an}}$ is a universal covering in the topological sense, i.e. $p$ is a covering map and $E^{\mathrm{an}}$ is simply connected. Moreover $E^{\mathrm{an}}$ has the structure of $K$-analytic group and $p$ is a morphism of $K$-analytic groups which uniquely extends the morphism $A_1 \to A^{\mathrm{an}}$ and $T^{\mathrm{an}} \to$

$A^{\mathrm{an}}$. Its kernel $M := \ker(p)$ is a discrete subgroup of $E^{\mathrm{an}}(K)$ isomorphic to the fundamental group $\pi_1(A^{\mathrm{an}})$ of $A^{\mathrm{an}}$ at the identity which by *loc. cit.* is a free abelian group whose rank is equal to the torus rank of $A$. Note that the action of $M$ on $E^{\mathrm{an}}$ by translation is free and properly discountinous, then $A$ is uniformizable in the sense that the morphism of $K$-analytic groups $E^{\mathrm{an}}/M \to A^{\mathrm{an}}$ induced by $p$ is an isomorphism (see [Gub10, § 4.1.]). Note that by *loc.cit.* the $K$-analytic group $E^{\mathrm{an}}$ is the analytification of an algebraic variety $E$ over $K$.

**- 2.2.3.** We reformulate the uniformization result a bit differently following [Gub10, § 4.1.-4.3.] and [FRSS18, § 3.2.]. We keep the same notation as in 2.2.2. To an abelian variety $A$ over $K$ one can uniquely (up to isomorphism) associate the datum of an exact sequence

$$1 \to T \to E \to B \to 0 \tag{2.3}$$

of algebraic groups over $K$, where $T$ is an algebraic torus over $K$ and $B$ is an abelian variety over $K$ such that $B^{\mathrm{an}}$ has good reduction (Definition 2.1.20) and a morphism of $K$-analytic groups

$$p\colon E^{\mathrm{an}} \to A^{\mathrm{an}}$$

which is a topological universal covering, whose kernel $M := \ker(p)$ is a discrete subgroup of $E(K)$ such that there is an isomorphism of $K$-analytic groups $E^{\mathrm{an}}/M \cong A^{\mathrm{an}}$ induced by $p$.

**Definition 2.2.4.** We say that $A$ has *totally degenerate reduction* if in (2.3) we have $B = 0$ and $E = T$. On the other hand we say that $A$ has *good reduction* if the toric part $T$ and $M$ are trivial and we have $A = B$.

# Chapter 3

# Differential forms on the analytification of an algebraic variety

## 3.1 Superforms

In this section define $(p, q)$-superforms on a real vector space, introduce some basic terminology from convex geometry and then consider superforms on polyhedral complexes. We begin with the definition of superforms on real vector spaces introduced by [Lag12]. We follow the exposition from [Gub16, § 2-3] and [Gub13, Appendix A] for the definitions about convex geometry.

Let $N$ be a free abelian group of finite rank $r$. Denote by $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual and by $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R})$ respectively $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ the base change to $\mathbb{R}$. Let $e_1, \ldots, e_r$ be a basis of $N$ and $x_1, \ldots, x_r$ the induced basis on $N_{\mathbb{R}}$. Write $\langle \cdot, \cdot \rangle \colon M \times N \to \mathbb{Z}, (m, n) \mapsto \langle m, n \rangle := m(n)$ for the pairing between $M$ and $N$.

Let $N'$ be another free abelian group of rank $r'$. A morphism $F \colon N'_{\mathbb{R}} \to N_{\mathbb{R}}$ is called *affine* if it is of the form $F := L_{\mathbb{R}} + a$ for a linear map $L_{\mathbb{R}} \colon N'_{\mathbb{R}} \to N_{\mathbb{R}}$ and $a \in N_{\mathbb{R}}$. It is called *integral* $\mathbb{R}$*-affine* if it is affine and the linear map $L_{\mathbb{R}} \colon N'_{\mathbb{R}} \to N_{\mathbb{R}}$ is induced by a homomorphism on the underlying abelian groups $L \colon N' \to N$.

By a *complete lattice* $\Lambda$ in a finite dimensional real vector space $V$ we mean a discrete additive subgroup of $V$ with $\mathrm{rank}_{\mathbb{Z}} \Lambda = \dim_{\mathbb{R}} V$.

An *integral* $\mathbb{R}$*-affine space* is a pair $(\mathbb{A}, N)$ consisting of a real affine space and a complete lattice $N$ of the underlying real vector space of $\mathbb{A}$.

**- 3.1.1.** Let $\Omega \subseteq N_{\mathbb{R}}$ be an open subset.

**Definition 3.1.2.** A *(Lagerberg) superform of bidegree* $(p, q)$, or $(p, q)$*-superform* on $\Omega$ is an element of

$$A^{p,q}(\Omega) := A^p(\Omega, \mathbb{R}) \otimes_{C^\infty(\Omega)} A^q(\Omega, \mathbb{R}) = C^\infty(\Omega) \otimes_{\mathbb{Z}} \Lambda^p M \otimes_{\mathbb{Z}} \Lambda^q M.$$

In the coordinates $x_1, \ldots, x_n$, an element $\alpha \in A^{p,q}(\Omega)$ is given by

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} d' x_I \wedge d'' x_J,$$

where $I$ respectively $J$ are given by $i_1 < \ldots < i_p$ and $j_1 < \ldots < j_q$, $\alpha_{IJ} \in C^\infty(\Omega)$ and we use the notation

$$d' x_I \wedge d'' x_J := (dx_{i_1} \wedge \ldots \wedge dx_{i_p} \otimes dx_{j_1} \wedge \ldots \wedge dx_{j_q}).$$

**Remark 3.1.3.** (i) For two superforms

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} d' x_I \wedge d'' x_J \in A^{p,q}(\Omega),$$
$$\beta = \sum_{|K|=r, |L|=s} \beta_{IJ} d' x_K \wedge d'' x_L \in A^{r,s}(\Omega)$$

we can define a wedge product

$$\wedge \colon A^{p,q}(\Omega) \times A^{r,s}(\Omega) \to A^{p+r,q+s}(\Omega), (\alpha, \beta) \mapsto \alpha \wedge \beta,$$

which in coordinates is given by

$$\alpha \wedge \beta = \sum_{|I|=p, |J|=q, |K|=r, |L|=s} (-1)^{sq} \alpha_{IJ} \beta_{KL} d' x_I \wedge d' x_K \wedge d'' x_J \wedge d'' x_L.$$

(ii) We can also define differential operators $d'$ and $d''$ on the space $A(\Omega) := \bigoplus_{p,q \leq r} A^{p,q}(\Omega)$ which turn $A(\Omega)$ into a bigraded differential $C^\infty(\Omega)$-algebra. More precisely, the differential operators $d'$ and $d''$ are given by $d' := D \otimes \mathrm{id}$ on $A^{p,q}(\Omega) := A^p(\Omega, \mathbb{R}) \otimes \Lambda^q M$ respectively $d'' := \mathrm{id} \otimes D$ on $\Lambda^p M \otimes A^q(\Omega, \mathbb{R})$. Here $D$ denotes the usual differential operator on the space $A^k(\Omega, \mathbb{R})$ for $k \in \mathbb{Z}_{\geq 0}$. In coordinates $d''\alpha$ is given by

$$\begin{aligned} d''\alpha &= \sum_{k=1}^{q} \sum_{|I|=p, |J|=q} \frac{\partial \alpha_{IJ}}{\partial x_k} d'' x_k \wedge d' x_I \wedge d'' x_J \\ &= (-1)^p \sum_{k=1}^{q} \sum_{|I|=p, |J|=q} \frac{\partial \alpha_{IJ}}{\partial x_k} d' x_I \wedge d'' x_k \wedge d'' x_J \in A^{p,q+1}(\Omega), \end{aligned}$$

and similarly for $d'\alpha$. We will denote by $A^{p,q}_{\mathrm{cl}}(\Omega)$ the space of *closed* superforms of bidegree $(p, q)$ on $\Omega$.

(iii) The assignment $\Omega \mapsto A^{p,q}(\Omega)$ defines a sheaf on $N_{\mathbb{R}}$. The *support* of a $(p, q)$ superform $\alpha \in A^{p,q}(\Omega)$, is defined as

$$\operatorname{supp}(\alpha) := \{x \in \Omega \mid \alpha_x \neq 0\},$$

where $\alpha_x$ denotes the germ of $\alpha$ at $x$. We denote by $A^{p,q}_c(\Omega)$ the space of $(p, q)$ superforms with compact support in $\Omega$.

(iv) Let $N'$ be a free abelian group of rank $r'$ and $F := L_{\mathbb{R}} + a\colon N'_{\mathbb{R}} \to N_{\mathbb{R}}$ an affine morphism with $F(\Omega') \subseteq \Omega$ for an open subset $\Omega'$ of $N'_{\mathbb{R}}$, then we can define the pullback

$$F^*\colon A^{p,q}(\Omega) \to A^{p,q}(\Omega')$$

as follows. Let $\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \in A^{p,q}(\Omega)$, then

$$F^*\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} \circ F d'L^*_{\mathbb{R}}x_I \wedge d''L^*_{\mathbb{R}}x_J$$

Affine pullback commutes with the wedge product and with the differential operators $d'$ and $d''$.

(v) There exists a $C^\infty(\Omega)$-linear isomorphism $J^{p,q}\colon A^{p,q}(\Omega) \to A^{q,p}(\Omega)$ given by

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \mapsto J^{p,q}(\alpha) = \sum_{|I|=q,|J|=p} \alpha_{IJ} d'x_J \wedge d''x_I.$$

**- 3.1.4.** Let $\Lambda \subseteq N_{\mathbb{R}}$ be a complete lattice. For $\lambda \in \Lambda$ denote by $\tau_\lambda\colon N_{\mathbb{R}} \to N_{\mathbb{R}}$, $v \mapsto v + \lambda$ the translation by $\lambda$ on $N_{\mathbb{R}}$.

**Definition 3.1.5.** Let $\Omega \subseteq N_{\mathbb{R}}$ be an open subset, $\alpha \in A^{p,q}(\Omega)$ and $\lambda \in \Lambda$. We say that $\alpha$ is $\Lambda$-*invariant* if

$$\tau^*_\lambda \alpha = \alpha, \quad \text{for all } \lambda \in \Lambda.$$

For $\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J$ this means that

$$\tau^*_\lambda \alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} \circ \tau_\lambda d'x_I \wedge d''x_J = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J = \alpha,$$

i.e. the $\alpha_{IJ}$ are $\Lambda$-invariant. We will denote by $A^{p,q}(\Omega)^\Lambda$ the space of $\Lambda$-invariant superforms of bidegree $(p, q)$ on $\Omega$ and by $A^{p,q}_{\mathrm{cl}}(\Omega)^\Lambda$ the subspace of closed $\Lambda$-invariant superforms of bidegree $(p, q)$ on $\Omega$.

**Example 3.1.6.** Let $\Lambda^{p,q}N_{\mathbb{R}}^* = \Lambda^{p,q}M_{\mathbb{R}}$ denote the space of superforms of bidegree $(p,q)$ on $N_{\mathbb{R}}$ with constant coefficients. For $\alpha \in \Lambda^{p,q}N_{\mathbb{R}}^*$ in coordinates we have

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J,$$

where $\alpha_{IJ} \in \mathbb{R}$. Then $\alpha$ is $\Lambda$-invariant and since every superform with constant coefficients is also closed we have an inclusion

$$\Lambda^{p,q}N_{\mathbb{R}}^* \subseteq A_{\mathrm{cl}}^{p,q}(N_{\mathbb{R}})^{\Lambda}.$$

**Definition 3.1.7.** We say that an open subset $\Omega \subseteq N_{\mathbb{R}}$ is $\Lambda$-*small* if for every non-zero element $\lambda \in \Lambda$ we have

$$\Omega \cap \tau_{\lambda}^{-1}(\Omega) = \emptyset.$$

**- 3.1.8.** In order to define superforms on polyhedral complexes we have to recall some notions from convex geometry from [Gub13, Appendix A] and [Gub16, § 3.1.].

**Definition 3.1.9.** (i) A polyhedron $\sigma$ is a finite intersection of finitely many closed half-spaces

$$\sigma := \bigcap_{i=1}^{m} \{n \in N_{\mathbb{R}} \mid \langle m_i, n \rangle \leq c_i\}, \quad m_i \in M_{\mathbb{R}}, c_i \in \mathbb{R}, i \in \{1, \ldots, m\}$$

in $N_{\mathbb{R}}$. We denote by $\mathbb{A}_\sigma$ the real affine space generated by $\sigma$ and by $\mathbb{L}_\sigma$ its underlying real vector space. The *dimension of* $\sigma$ is defined as $\dim(\sigma) := \dim_{\mathbb{R}} \mathbb{L}_\sigma$. We call $\sigma$ an *integral* $\mathbb{R}$-*affine* polyhedron if we can choose $m_i \in M$ for all $i \in \{1, \ldots, m\}$.

(ii) Let $\sigma$ be a polyhedron. A *closed face* of $\sigma$ is either $\sigma$ or given by $\sigma \cap \partial H$, where $\partial H$ is the boundary of a closed half-space containing $\sigma$. If $\tau$ is a closed face (respectively a proper closed face) of $\sigma$, then we write $\tau \preceq \sigma$ (respectively $\tau \prec \sigma$). The *relative interior* of $\sigma$ is defined as

$$\mathrm{relint}(\sigma) := \sigma \setminus \bigcup_{\tau \prec \sigma} \tau.$$

**Remark 3.1.10.** Let $\sigma$ be an integral $\mathbb{R}$-affine polyhedron of dimension $n$ in $N_{\mathbb{R}}$ and $\mathbb{A}_\sigma$, respectively $\mathbb{L}_\sigma$ as in Definition 3.1.9.

(i) There is a canonical integral $\mathbb{R}$-affine structure $(\mathbb{A}_\sigma, N_\sigma)$ on $\mathbb{A}_\sigma$, where $N_\sigma := N \cap \mathbb{L}_\sigma$. Note that $N_\sigma$ is a complete lattice in $\mathbb{L}_\sigma$.

(ii) Let $\widetilde{\Omega} \subseteq N_{\mathbb{R}}$ and $\Omega := \widetilde{\Omega} \cap \sigma$. Define $A_{\sigma}^{p,q}(\Omega)$ as the space of $(p,q)$-superforms on the open subset $\widetilde{\Omega} \cap \operatorname{relint}(\sigma)$ of $\mathbb{A}_{\sigma}$ given by restriction of elements in $A^{p,q}(\widetilde{\Omega})$. A partition of unity argument shows that the definition is independent of the choice of $\widetilde{\Omega}$ (see [GK17, § 2.2.]).

**Remark 3.1.11.** (i) Let $m \in M$ and $H := \{n \in N_{\mathbb{R}} \mid \langle n, m \rangle \leq 0\}$ be a half-space. Its boundary $\partial H$ is a subspace of $N_{\mathbb{R}}$. There exists an isomorphism of groups $N/(N \cap \partial H) \cong \mathbb{Z}$. Let $[\omega_{\partial H,H}]$ be the outward pointing generator of $N/(N \cap \partial H)$. This means that there is $u_{\partial H,H} \in M$ such that

$$\langle h, u_{\partial H,H} \rangle \leq 0, \text{ for all } h \in H, \text{ and } \langle \omega_{\partial H,H}, u_{\partial H,H} \rangle = 1.$$

Then we choose a representative $\omega_{\partial H,H} \in N$. Note that after a translation, we can perform the same construction for any integral $\mathbb{R}$-affine half-space.

(ii) Let $\Omega$ be an open subset of $N_{\mathbb{R}}$ and $\sigma$ an $r$-dimensional (top dimension) integral $\mathbb{R}$-affine polyhedron contained in $\Omega$. For any closed face $\rho$ of codimension 1, we put $\omega_{\sigma,\rho} := \omega_{\partial H,H}$ as in (i), where $\partial H$ is the affine hyperplane generated by $\rho$ and $H$ the corrsponding halfspace containing $\sigma$. The element $\omega_{\sigma,\rho} \in N$ is unique up to addition with elements in $N_{\rho} = N \cap \mathbb{L}_{\rho}$.

**Definition 3.1.12.** (i) A *polyhedral complex* $\mathcal{C}$ is a finite set of polyhedra with the following properties

(i) every polyhedron in $\mathcal{C}$ has all its closed faces in $\mathcal{C}$;

(ii) if $\sigma$ and $\tau$ are elements in $\mathcal{C}$, then $\sigma \cap \tau$ is either empty or a closed face of both $\sigma$ and $\tau$.

We say that a polyhedral complex $\mathcal{C}$ is *integral* $\mathbb{R}$*-affine* if every polyhedron $\sigma \in \mathcal{C}$ is integral $\mathbb{R}$-affine. The *support* of a polyhedral complex $\mathcal{C}$ is given by

$$|\mathcal{C}| := \bigcup_{\sigma \in \mathcal{C}} \sigma.$$

A *maximal polyhedron* in $\mathcal{C}$ is a polyhedron which is maximal with respect to inclusion. We say that $\mathcal{C}$ is *pure* $n$*-dimensional* if every maximal polyhedron in $\mathcal{C}$ has dimension $n$. For $k \in \mathbb{N}$ we write $\mathcal{C}_k := \{\sigma \in \mathcal{C} \mid \dim \sigma = k\}$. A polyhedral complex $\mathcal{D}$ *subdivides* the polyhedral complex $\mathcal{C}$ if $|\mathcal{D}| = |\mathcal{C}|$ and if every polyhedron of $\mathcal{D}$ is contained in a polyhedron of $\mathcal{C}$.

(iii) Let $\mathcal{C}$ be a polyhedral complex in $N_{\mathbb{R}}$ and $\Omega \subseteq \mathcal{C}$ open. A *superform* $\alpha$ *of bidegree* $(p,q)$ on $\Omega$ is an equivalence class $[(\widetilde{\alpha}, \widetilde{\Omega})]$, such that

(i) the set $\widetilde{\Omega}$ is an open subset of $N_{\mathbb{R}}$ and $\widetilde{\alpha} \in A^{p,q}(\widetilde{\Omega})$, and

(ii) two pairs $(\widetilde{\alpha}, \widetilde{\Omega})$ and $(\widetilde{\beta}, \widetilde{\Omega})$ are *equivalent* if their restriction to $A^{p,q}_{\sigma}(\widetilde{\Omega} \cap \sigma)$ agrees, for every maximal polyhedron $\sigma \in \mathcal{C}$.

As in Remark 3.1.10 (ii), a partition of unity argument shows that this definition does not depend on the choice of the open subset $\widetilde{\Omega}$. Denote by $A^{p,q}_{\mathcal{C}}(\Omega)$ the space of superforms of bidegree $(p, q)$ on $\mathcal{C}$. If the polyhedral complex $\mathcal{C}$ is clear from the context we will only write $A^{p,q}(\Omega)$.

(iv) The assignment $\Omega \mapsto A^{p,q}_{\mathcal{C}}(\Omega)$ defines a sheaf on $\mathcal{C}$. The *support* of a superform $\alpha \in A^{p,q}_{\mathcal{C}}(\Omega)$ is defined as the support in the sense of sheaves as in Remark 3.1.3(iv). Denote by $A^{p,q}_{\mathcal{C},c}(\Omega)$ the space of $(p, q)$-superforms with compact support in $\Omega$.

**Remark 3.1.13.** Let $\mathcal{C}$ be a polyhedral complex in $N_{\mathbb{R}}$, $\Omega \subseteq \mathcal{C}$ open and $\widetilde{\Omega} \subseteq N_{\mathbb{R}}$ open such that $\Omega = \widetilde{\Omega} \cap |\mathcal{C}|$.

(i) By restricting the wedge product and differential operators $d'$ and $d''$ from $A^{p,q}(\widetilde{\Omega})$ we get a wedge product and differential operators on $A^{p,q}_{\mathcal{C}}(\Omega)$. This gives $A_{\mathcal{C}}(\Omega) := \bigoplus_{p,q} A^{p,q}_{\mathcal{C}}(\Omega)$ the structure of an alternating bigraded differential algebra. For example, let $\alpha \in A^{p,q}_{\mathcal{C}}(\Omega)$ be given by $\widetilde{\alpha} \in A^{p,q}(\widetilde{\Omega})$ for an open subset $\widetilde{\Omega} \subseteq N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathcal{C}|$ then $d''\alpha$ is given by $d''\widetilde{\alpha} \in A^{p,q+1}(\widetilde{\Omega})$. Since affine pull-back commutes with differentials, it follows that the differential $d''$ of equivalent superforms in $A^{p,q}_{\mathcal{C}}(\Omega)$ will be equivalent in $A^{p,q+1}_{\mathcal{C}}(\Omega)$.

(ii) Let $N'$ be a free abelian group of rank $r'$ and let $F\colon N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an affine map. Suppose that $\mathcal{C}'$ is a polyhedral complex of $N'_{\mathbb{R}}$ with $F(|\mathcal{C}'|) \subseteq |\mathcal{C}|$ and put $\Omega' := \Omega \cap |\mathcal{C}'|$. Then there exists a well-defined pull-back

$$F^*\colon A^{p,q}_{\mathcal{C}}(\Omega) \to A^{p,q}_{\mathcal{C}'}(\Omega')$$

defined as follows. Let $\widetilde{\Omega} \subseteq N_{\mathbb{R}}$ be an open subset such that $\Omega = \widetilde{\Omega} \cap |\mathcal{C}|$, then $F^{-1}(\widetilde{\Omega})$ is open in $N'_{\mathbb{R}}$ and we define $\Omega' := F^{-1}(\widetilde{\Omega}) \cap |\mathcal{C}'|$. Let $\alpha \in A^{p,q}_{\mathcal{C}}(\Omega)$ be given by $\widetilde{\alpha} \in A^{p,q}(\widetilde{\Omega})$ then $F^*\widetilde{\alpha} \in A^{p,q}(F^{-1}(\widetilde{\Omega}))$ is given restricting $F^*\widetilde{\alpha}$ to $\Omega'$. The definition is independent of the choice of the representative $(\widetilde{\alpha}, \widetilde{\Omega})$.

(iii) Suppose that $\mathcal{C}$ has dimension $n$. Then by [CLD12, Lemme (1.4.5.)] for $n < \max\{p, q\}$ we have $A^{p,q}_{\mathcal{C}} = 0$.

**Definition 3.1.14.** (i) Let $\mathcal{C}$ be a pure $n$-dimensional polyhedral complex. A *weight* on $\mathcal{C}$ is an integer valued function $m : \mathcal{C}_n \to \mathbb{Z}, \sigma \mapsto m(\sigma) =: m_{\sigma}$. We call the pair $(\mathcal{C}, m)$ a *weighted polyhedral complex.*

(ii) Let $C := (\mathcal{C}, m)$ be a weighted integral $\mathbb{R}$-affine pure $n$-dimensional polyhedral complex. For a codimension one face $\tau$ of a polyhedron $\sigma \in \mathcal{C}_n$ let $\omega_{\sigma,\tau} \in N_\sigma$ be a representative as in Remark 3.1.11. We say that the weighted polyhedral complex $C$ satisfies the *balancing condition* if for all $\tau \in \mathcal{C}_{n-1}$ we have

$$\sum_{\sigma \in \mathcal{C}_n, \sigma \succ \tau} m_\sigma \omega_{\sigma,\tau} \in N_\tau.$$

(iii) A *tropical cycle* $C = (\mathcal{C}, m)$ *of dimension* $n$ in $N_\mathbb{R}$ is a weighted integral $\mathbb{R}$-affine pure $n$-dimensional polyhedral complex which satisfies the balancing condition.

## 3.2 Differential forms on the analytification of an algebraic variety

In this section we first define tropical charts on the analytification $X^{\text{an}}$ of an algebraic variety $X$ over $K$. These are given by closed immersions of the analytification of very affine open subsets into an algebraic torus and will be seen as local coordinate charts on $X^{\text{an}}$. This will allow us to define differential forms on open subsets of $X^{\text{an}}$. In the last part of the section we will look at some basic results about the support of the sheaf of real valued smooth differential forms on $X^{\text{an}}$.

Let $X$ be an algebraic variety over $K$.

**- 3.2.1.** We recall the main definitions and properties of algebraic moment maps and tropical charts as done in [Gub13, § 3], [Gub16, § 4] and of analytic moment maps following [CLD12, § 2].

**Definition 3.2.2.** Let $T := \operatorname{Spec}(K[M])$ be an algebraic torus over $K$ with character group $M$ and cocharacter group $N$. Denote by $N_\mathbb{R}$ the base extension $N \otimes_\mathbb{Z} \mathbb{R}$ and for $m \in M$ denote by $\chi^m := 1 \cdot m$ its associated cocharacter in $K[M]$.

(i) The *tropicalization map* is given by

$$\operatorname{trop}_T \colon T^{\text{an}} \longrightarrow N_\mathbb{R}, |\cdot|_t \mapsto \operatorname{trop}_T(t) := [m \mapsto -\log |\chi^m|_t].$$

If we fix coordinates $Z_1, \ldots, Z_r$ on $T$, then $M \cong \mathbb{Z}^r$ and $N_\mathbb{R} \cong \mathbb{R}^r$ and the tropicalization map is given by

$$\operatorname{trop}_T \colon T^{\text{an}} \longrightarrow \mathbb{R}^r, |\cdot|_t \mapsto (-\log |Z_1|_t, \ldots, -\log |Z_r|_t).$$

(ii) Let $U$ be an open subset of $X$. An *algebraic moment map* is a morphism $\varphi\colon U \to T$ of algebraic varieties over $K$. The *tropicalization* of $\varphi$ will be denoted by

$$\varphi_{\mathrm{trop}}\colon U^{\mathrm{an}} \xrightarrow{\varphi^{\mathrm{an}}} T^{\mathrm{an}} \xrightarrow{\mathrm{trop}} N_{\mathbb{R}}.$$

(iii) A moment map $\varphi'\colon U' \to T'$ of the open subset $U'$ of $X$ *refines* the moment map $\varphi\colon U \to T$ if $U' \subseteq U$ and if there is an affine homomorphism $\psi\colon T' \to T$ such that $\varphi|_{U'} = \psi \circ \varphi'$. An affine homomorphism is a group homomorphism $T \to T'$ composed with a multiplicative translation on $T$.

(iv) An *analytic moment map* is a morphism of analytic spaces $f\colon Y \to T$ from an analytic space $Y$ to an analytic torus $T$ (Definition 2.1.13(i)). Denote the composition

$$Y \xrightarrow{f} T \xrightarrow{\mathrm{trop}_T} N_{\mathbb{R}},$$

by $f_{\mathrm{trop}}$.

**Remark 3.2.3.** Note that the affine homomorphism $\psi\colon T' \to T$ from Definition 3.2.2(iii) induces a homomorphism $M \to M'$ on character lattices by precomposition with $\psi$. Its dual is the linear part of an integral $\mathbb{R}$-affine map $\mathrm{Trop}(\psi)\colon N'_{\mathbb{R}} \to N_{\mathbb{R}}$ such that $(\varphi|_{U'})_{\mathrm{trop}} = \mathrm{Trop}(\psi) \circ \varphi'_{\mathrm{trop}}$.

**- 3.2.4.** By [Gub10, § 4.12.] for every affine open subset $U$ of $X$ we can define a moment map $\varphi_U\colon U \to T_U$ where $T_U = \mathrm{Spec}(K[M_U])$ and $M_U := \mathcal{O}_X(U)^\times/K^\times$. The moment map $\varphi_U$ is canonical up to multiplicative translation by an element of $T_U(K)$ and is called *canonical moment map.* Let $f\colon X' \to X$ be a morphism of algebraic varieties over $K$ and let $U'$ be an affine open subset of $X'$ such that $f(U') \subseteq U$. Then by *loc. cit.* there exists a canonical affine homomorphism $\psi_{U,U'}\colon T_{U'} \to T_U$ of canonical tori such that $\psi_{U,U'} \circ \varphi_{U'} = \varphi_U \circ f$ on $U'$.

**Remark 3.2.5.** Let $U$ be an affine open subset of $X$, then $\varphi_U$ refines every other moment map on $U$. Indeed let $\phi\colon U \to T$ be another moment map on $U$ and denote by $M := \mathrm{Hom}(T, \mathbb{G}_m)$ the character group of the torus $T$. We have a morphism of $\mathbb{Z}$-modules $M \to M_U$ given by composition with $\phi$. We get an induced map of $K$-algebras $K[M] \to K[M_U]$ and consequently an affine homomorphism in the opposite direction on the tori $\psi_U\colon T_U \to T$. If we denote by $N_U$ respectively $N$ the dual space of $M_U$ and $M$, then we get an induced integral $\mathbb{R}$-affine morphism

$$\mathrm{Trop}(\psi_{U,T}) := \mathrm{trop} \circ \psi_U^{\mathrm{an}}\colon N_{U,\mathbb{R}} \to N_{\mathbb{R}}$$

such that $\mathrm{Trop}(\psi_{U,T}) \circ \varphi_{U,\mathrm{trop}} = \phi_{\mathrm{trop}}$ on $U$.

**Remark 3.2.6.** Let $Y$ be closed subvariety of dimension $n$ of an algebraic torus $T$ over $K$ and $\varphi\colon Y \to T$ the corresponding closed immersion. As explained in [Gub16, § 4.6.], by the Bieri-Groves theorem and further results from tropical geometry, the image $\varphi_{\text{trop}}(Y^{\text{an}})$ has the structure of an integral $\mathbb{R}$-affine polyhedral complex of dimension $n$ which is unique up to subdivision. Moreover, it is possible to endow $\varphi_{\text{trop}}(Y^{\text{an}})$ with a positive canonical weight function $m$ which satisfies the balancing condition (see [Gub16, § 4.7.]). Hence $(\varphi_{\text{trop}}(Y^{\text{an}}), m)$ is a tropical cycle.

**- 3.2.7.** An open subset $U$ of $X$ is called *very affine* if $U$ has a closed embedding into a multiplicative torus. Let $U \subseteq X$ be an affine open subset. By [Gub16, § 4.13.] the subset $U$ is very affine open if and only if the canonical moment map $\varphi_U$ is a closed immersion. Moreover the intersection of two very affine open subsets of $X$ is very affine open subset of $X$. By *loc. cit.* the very affine open subsets form a basis for the Zariski topology on $X$.

**Lemma 3.2.8.** *Let $U$ and $U'$ be very affine open subsets of $X$. Then with the same notation as in* (3.2.4) *there exists a canonical isomorphism of abelian groups*

$$M_U \oplus M_{U'} \cong M_{U \times_K U'}.$$

*Proof.* This is known as Rosenlicht's theorem and can be found in [Ros61, Theorem 2]. Alternatively there is proof using a more modern language on the website of Brian Conrad (see [Con, Theorem 4.1.]). □

**Definition 3.2.9.** (i) A tropical chart on $X^{\text{an}}$ is a pair $(V, \varphi_U)$ consisting of an open subset $V$ of $X^{\text{an}}$ contained in $U^{\text{an}}$ for a very affine open subset $U$ of $X$ with $V = \varphi_{U,\text{trop}}^{-1}(\Omega)$ for some open subset $\Omega$ of $\varphi_{U,\text{trop}}(U^{\text{an}})$.

(ii) A tropical chart $(V', \varphi_{U'})$ is a *tropical subchart* of $(V, \varphi_U)$ if $V' \subseteq V$ and $U' \subseteq U$.

**Proposition 3.2.10.** *The tropical charts on $X^{\text{an}}$ have the following properties.*

*(i) They form a basis on $X^{\text{an}}$, i.e. for every open subset $W$ of $X^{\text{an}}$ and for every $x \in W$ there is a tropical chart $(V, \varphi_U)$ with $x \in V \subseteq W$.*

*(ii) The intersection $(V \cap V', \varphi_{U \cap U'})$ of tropical charts $(V, \varphi_U)$ and $(V', \varphi_{U'})$ is a tropical subchart of both.*

*(iii) If $(V, \varphi_U)$ is a tropical chart and if $U''$ is a very affine open subset of $U$ with $V \subseteq (U'')^{\text{an}}$, then $(V, \varphi_{U''})$ is a tropical subchart of $(V, \varphi_U)$.*

*Proof.* See [Gub16, Proposition 4.16.]. □

**Remark 3.2.11.** Let $(V, \varphi_U)$ and $(V', \varphi_{U'})$ be two tropical charts. Then by Proposition 3.2.10(ii) also $(V \cap V', \varphi_{U \cap U'})$ is a tropical chart and we get a canonical affine homomorphism $\psi_{U,U\cap U'} \colon T_{U\cap U'} \to T_U$ of the underlying tori such that $\varphi_U = \psi_{U,U\cap U'} \circ \varphi_{U\cap U'}$. The associated affine map $\mathrm{Trop}(\psi_{U,U\cap U'}) \colon N_{U\cap U',\mathbb{R}} \to N_{U,\mathbb{R}}$ maps the tropical variety $\varphi_{U\cap U',\mathrm{trop}}((U\cap U')^{\mathrm{an}})$ onto $\varphi_{U,\mathrm{trop}}(U^{\mathrm{an}})$ by [Gub16, § 5.1.].

**Definition 3.2.12.** With the notation from Remark 3.2.11 we define the *restriction* $\alpha|_{V\cap V'}$ of $\alpha \in A^{p,q}(\varphi_{U,\mathrm{trop}}(V))$ to $V \cap V'$ by

$$\alpha|_{V\cap V'} := \mathrm{Trop}(\psi_{U\cap U',U})^* \alpha \in A^{p,q}(\varphi_{U\cap U',\mathrm{trop}}(V \cap V'))$$

**Proposition 3.2.13.** *Let $W \subseteq X^{\mathrm{an}}$ be an open subset and $\varphi \colon W \to T^{\mathrm{an}}$ be an analytic moment map defined on an open subset $W$ of $X^{\mathrm{an}}$. For every $x \in W$, there is a very affine open subset $U$ of $X$ with an algebraic moment map $\varphi' \colon U \to T$ and an open neighborhood $V$ of $x$ in $W \cap U^{\mathrm{an}}$ such that $\varphi_{\mathrm{trop}} = \varphi'_{\mathrm{trop}}$ on $V$.*

*Proof.* See [Gub16, Proposition 7.2.]. □

**Remark 3.2.14.** It is clear that every algebraic moment map induces an analytic one. Proposition 3.2.13 gives a local converse, in the sense that analytic moment maps can locally be approximated by algebraic ones. This will allow us to translate constructions for $K$-analytic spaces appearing in [CLD12] into the formalism of [Gub16].

**Lemma 3.2.15.** *Let $W \subseteq X^{\mathrm{an}}$ be open and $f \colon W \to T^{\mathrm{an}}$ an analytic moment map. For every point $x \in W$ there exists a triple $(F, V, \varphi_U)$, where $V$ is an open neighborhood of $x$ in $W$, the pair $(V, \varphi_U)$ is a tropical chart on $X^{\mathrm{an}}$ and $F \colon N_{U,\mathbb{R}} \to N_{\mathbb{R}}$ is an integral $\mathbb{R}$-affine morphism with $F \circ \varphi_{U,\mathrm{trop}} = f_{\mathrm{trop}}$ on $V$.*

*Proof.* Let $N$ denote the cocharacter group of $T$ and put $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. By Proposition 3.2.13 there exists a very affine open subset $U'$ of $X$ and an algebraic moment map $\phi_{U'} \colon U' \to T$ such that $f_{\mathrm{trop}}|_{V'} = \phi_{U',\mathrm{trop}}|_{V'}$ for a neighborhood $V'$ of $x$ in $V \cap U'^{\mathrm{an}}$. By Proposition 3.2.10(i) there exists a tropical chart $(V, \varphi_U)$ with $x \in V \subseteq V'$. Moreover by Proposition 3.2.10(iii) after possibly replacing $U$ with $U' \cap U$ we have that $U$ is a very affine open subset of $U'$. By Remark 3.2.3 and 3.2.5 there exists an $\mathbb{R}$-affine morphism $F' \colon N_{U',\mathbb{R}} \to N_{\mathbb{R}}$ such that

$$F' \circ \varphi_{U',\mathrm{trop}} = \phi_{U',\mathrm{trop}}$$

on $(U')^{\mathrm{an}}$. As $U \subseteq U'$ by (3.2.4) there exists a canonical integral $\mathbb{R}$-affine morphism $G \colon N_{U,\mathbb{R}} \to N_{U',\mathbb{R}}$ such that

$$G \circ \varphi_{U,\mathrm{trop}} = \varphi_{U',\mathrm{trop}}$$

on $U^{\mathrm{an}}$. Put $F := F' \circ G \colon N_{U,\mathbb{R}} \to N_{\mathbb{R}}$, then on $V$ we have

$$\begin{aligned} f_{\mathrm{trop}} = \phi_{U',\mathrm{trop}} &= F' \circ \varphi_{U',\mathrm{trop}} \\ &= F' \circ G \circ \varphi_{U,\mathrm{trop}} = F \circ \varphi_{U,\mathrm{trop}}. \end{aligned}$$

□

**Definition 3.2.16.** A *differential form* $\alpha$ of bidegree $(p,q)$ on an open subset $V$ of $X^{\mathrm{an}}$ is given by the datum $[(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}]$, where:

(i) $(V_i)_{i\in I}$ is an open covering of $V$;

(ii) $(V_i, \varphi_{U_i})_{i\in I}$ are tropical charts, and

(iii) for every $i \in I$ we have $\alpha_i \in A^{p,q}(\varphi_{U_i,\mathrm{trop}}(V))$ and for all $i, j \in I$ we have $\alpha_i|_{V_i\cap V_j} = \alpha_j|_{V_i\cap V_j}$ in the sense of Remark 3.2.11.

Two differential forms $\alpha = (V_i, \varphi_{U_i}, \alpha_i)_{i\in I}, \alpha' = (V_j', \varphi_{U_j'}, \alpha_j')_{j\in J}$ are *equivalent* if for all $i \in I, j \in J$ we have $\alpha_i|_{V_i\cap V_j'} = \alpha_j'|_{V_i\cap V_j'}$. We will denote by $A^{p,q}_{X^{\mathrm{an}}}(V)$ the space of differential $(p,q)$-forms on $V$. If the variety $X$ over $K$ is clear from the context, we will just write $A^{p,q}(V)$.

**Remark 3.2.17.** In [CLD12] the authors define differential $(p,q)$-forms on arbitrary $K$-analytic spaces using analytic moment maps. In [Gub16, Remark 4.17.] the author shows with the help of Proposition 3.2.13 that on the analytification of an algebraic variety over $K$ the two definitions agree. Hence we can use the results from [CLD12].

**Remark 3.2.18.** (i) To define the wedge product of differential forms on an open subset of $X^{\mathrm{an}}$ as well as differential operators $d'$ and $d''$ we follow [Jel16, Definition 3.2.15.]. Let $\alpha \in A^{p,q}(V)$ and $\beta \in A^{r,s}(V)$ be differential forms on $V$ given by $(V_i, \varphi_{U_i}, \alpha_i)_{i\in I}$ respectively $(W_j, \varphi_{Z_j}, \beta_j)_{j\in J}$. After choosing a common refinement we may assume that $\alpha$ and $\beta$ are given by $(W_i, \varphi_{Z_i}, \alpha_i)_{i\in I}$ and $(W_i, \varphi_{Z_i}, \beta_i)_{i\in I}$. Then we define $\alpha \wedge \beta$ and $D\alpha$ to be given by

$$(W_i, \varphi_{Z_i}, \alpha_i \wedge \beta_i)_{i\in I}, \text{ and } (V_i, \varphi_{U_i}, D\alpha_i)_{i\in I},$$

where $D \in \{d', d''\}$. By *loc.cit.* these definitions are independent of the chosen representations for $\alpha$ and $\beta$.

(ii) Let $f \colon X \to Y$ be a morphism of algebraic varieties over $K$ and $f^{\mathrm{an}} \colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ the induced morphism on the analytification. Let $W \subseteq Y^{\mathrm{an}}$ be an open subset. There exists a canonical pull-back of differential forms

$$f^*\colon A^{p,q}_{Y^{\mathrm{an}}}(W) \to A^{p,q}_{X^{\mathrm{an}}}(f^{\mathrm{an},-1}(W)),$$

defined as follows. Let $\alpha \in A^{p,q}_{Y^{\mathrm{an}}}(W)$ be given by $(W_i, \varphi_{Z_i}, \alpha_i)_{i\in I}$. Put

$$f^{\mathrm{an},-1}(W_i) =: V_i, \quad \text{and} \quad f^{-1}(Z_i) =: U_i$$

and note that $U_i$ is a Zariski open subset which may not be very affine. By 3.2.7 there exists an open cover $(U_{ij})_{j\in J}$ of $U_i$ by very affine open subsets. Let $\varphi_{U_{ij}}\colon U_{ij} \to T_{U_{ij}}$ be the canonical moment map and put $V_{ij} := V_i \cap U_{ij}^{\mathrm{an}}$. Since $f(U_{ij}) \subseteq Z_i$, by 3.2.4 we get a canonical affine homomorphism of tori $\psi_{Z_i,U_{ij}}\colon T_{U_{ij}} \to T_{Z_i}$ such that $\psi_{Z_i,U_{ij}} \circ \varphi_{U_{ij}} = \varphi_{Z_i} \circ f$ on $U_{ij}$. Then $(V_{ij}, \varphi_{U_{ij}})$ is a tropical chart and we define $f^*\alpha \in A^{p,q}_{X^{\mathrm{an}}}(f^{-1,\mathrm{an}}(W))$ to be given by

$$(V_{ij}, \varphi_{U_{ij}}, \mathrm{Trop}(\psi_{Z_i,U_{ij}})^*\alpha_i)_{i\in I, j\in J}.$$

This construction is independent of the chosen representative for $\alpha$ ([Jel16, Remark 3.2.18.]).

(iii) By [Gub16, § 5.3.] the differential $(p,q)$-forms on $X^{\mathrm{an}}$ form a sheaf which we will denote by $A^{p,q}_{X^{\mathrm{an}}}$. Let $\alpha$ be a differential form on an open subset $V$ of $X^{\mathrm{an}}$. As in Remark 3.1.3 (iv), we define, the *support* of $\alpha$ of a differential $(p,q)$-form on an open subset $V$ of $X^{\mathrm{an}}$ as the support in the sense of sheaves. For an open subset $V$ of $X^{\mathrm{an}}$ we will denote by $A^{p,q}_c(V)$ the space of differential $(p,q)$-forms on $V$ with compact support.

(iv) Let $n := \dim X$. Then by [CLD12, (3.1.2.)] for $n < \max\{p,q\}$ we have $A^{p,q}_{X^{\mathrm{an}}} = 0$.

**- 3.2.19.** We explain how to pull-back differential forms on the analytifications of algebraic varieties along (purely) analytic morphisms in the formalism of [Gub16] by using the local approximation of analytic moment maps by algebraic ones as in Proposition 3.2.13.

Let $X, Y$ be algebraic varieties and $W \subseteq Y^{\mathrm{an}}$ an open subset. Let $f\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be a morphism of $K$-analytic spaces. Suppose that $\alpha' \in A^{p,q}(W)$ is given by $(W, \varphi_Z, \alpha)$. The composition

$$\varphi := \varphi_Z^{\mathrm{an}} \circ f|_{f^{-1}(W)}\colon f^{-1}(W) \to T_Z^{\mathrm{an}}$$

is an analytic moment map. By Lemma 3.2.15 there exists a family of triples $(V_i, \varphi_{U_i}, F_i)_{i\in I}$ where $(V_i)_{i\in I}$ is an open cover of $f^{-1}(V)$, the pairs $(V_i, \varphi_{U_i})_{i\in I}$ are tropical charts on $f^{-1}(W)$ and $F_i : N_{U_i,\mathbb{R}} \to N_{Z,\mathbb{R}}$ is an integral $\mathbb{R}$-affine map such that $F_i \circ \varphi_{U_i} = \varphi_{\mathrm{trop}}$ on $V_i$ for every $i \in I$. Then we define $f^*\alpha'$ by

$$(V_i, \varphi_{U_i}, F_i^*\alpha)_{i\in I}. \tag{3.1}$$

This construction is well-defined, indeed let $(V'_j, \varphi_{U'_j}, F'_j)_{j\in J}$ be another family of triples as in Lemma 3.2.15. For all $i \in I$ and $j \in J$ let $\mathrm{Trop}(\psi_{U_i,U_i\cap U'_j})$ and $\mathrm{Trop}(\psi_{U'_j,U_i\cap U'_j})$ be the canonical integral $\mathbb{R}$-affine morphisms from 3.2.4. On $V_i \cap V'_j$ we have

$$\begin{aligned} F_i \circ \mathrm{Trop}(\psi_{U_i,U_i\cap U'_j}) \circ \varphi_{U_i\cap U'_j,\mathrm{trop}} &= F_i \circ \varphi_{U_i,\mathrm{trop}} = \varphi_{\mathrm{trop}} \\ &= F'_j \circ \varphi_{U'_j,\mathrm{trop}} = F'_j \circ \mathrm{Trop}(\psi_{U'_j,U_i\cap U'_j}) \circ \varphi_{U_i\cap U'_j,\mathrm{trop}}. \end{aligned}$$

Then

$$\begin{aligned} F_i^*\alpha|_{V_i\cap V'_j} &= \mathrm{Trop}(\psi_{U_i,U_i\cap U'_j})^* F_i^*\alpha \\ &= \mathrm{Trop}(\psi_{U'_j,U_i\cap U'_j})^* F'^*_j\alpha = F'^*_j\alpha|_{V_i\cap V'_j}, \end{aligned}$$

hence $[(V_i, \varphi_{U_i}, F_i^*\alpha)_{i\in I}] = [(V'_j, \varphi_{U'_j}, F'^*_j\alpha)_{j\in J}]$ in $A^{p,q}(W)$. The next proposition shows that we can apply the same idea for an arbitrary differential form $\alpha \in A^{p,q}(W)$.

**Proposition 3.2.20.** *There exists a canonical pull-back of differential forms*

$$f^*\colon A^{p,q}_{Y^{\mathrm{an}}}(W) \to A^{p,q}_{X^{\mathrm{an}}}(f^{-1}(W)), \tag{3.2}$$

*with the following properties.*

*(i) The pull-back $f^*$ coincides with the pull-back from [CLD12, (3.1.7.)].*

*(ii) Suppose that $g\colon X \to Y$ is a morphism of algebraic varieties over $K$. Then the pull-back $g^{\mathrm{an},*}$ from* (3.2) *coincides with the pullback $g^*$ from Remark 3.1.3(ii).*

*(iii) The pull-back defined in* (3.2) *is functorial. This means that if $g : Y^{\mathrm{an}} \to Z^{\mathrm{an}}$ is another morphism of $K$-analytic spaces, we have*

$$(g \circ f)^* = f^* \circ g^*.$$

*Proof.* Let $\alpha \in A^{p,q}(W)$ be given by $(W_i, \varphi_{Z_i}, \alpha_i)_{i\in I}$. Then we define $f^*\alpha$ as

$$(V_{ij}, \varphi_{U_{ij}}, F_{ij}^*\alpha_i)_{I\in I, j\in J}, \tag{3.3}$$

where for every $i \in I$ the family of triples $(V_{ij}, \varphi_{U_{ij}}, F_{ij}^*\alpha_i)_{j\in J}$ is given as in (3.1). We show the compatibility condition from Definition 3.2.16. By assumption we have

$$\mathrm{Trop}(\psi_{Z_i,Z_i\cap Z_j})^*\alpha_i = \mathrm{Trop}(\psi_{Z_j,Z_i\cap Z_j})^*\alpha_j \tag{3.4}$$

for all $i \in I$ and $j \in J$. Fix $i \in I$ and $j \in J$. We want to show that

$$F_{ik}^*\alpha_i|_{V_{ik}\cap V_{jl}} = F_{jl}^*\alpha_j|_{V_{ik}\cap V_{jl}}, \tag{3.5}$$

for all $k \in K$ and $l \in L$. We claim that for every $p \in V_{ik} \cap V_{jl}$ there exists an open neighborhood $V_p = V_p(k,l)$ such that

$$F_{ik}^*\alpha_i|_{V_p} = F_{jl}^*\alpha_j|_{V_p}.$$

First note that since $V_{ik} \subseteq f^{-1}(Z_i)$ and $V_{jl} \subseteq f^{-1}(Z_j)$, the composition

$$\varphi\colon V_{ik} \cap V_{jl} \to f^{-1}(Z_i \cap Z_j) \xrightarrow{f|_{f^{-1}(Z_i\cap Z_j)}} T_{Z_i\cap Z_j}$$

is an analytic moment map and hence for every $p \in V_{ik} \cap V_{jl}$ there exists a triple $(V_p, \varphi_{U_p}, F_p)$ as in Lemma 3.2.15. After possibly replacing $U_p$ by $U_p \cap U_{ik} \cap U_{jl}$ we may assume that $U_p \subseteq U_{ik}$ and $U_p \subseteq U_{jl}$. Then on $\varphi_{U_p,\mathrm{trop}}(V_p)$ we have

$$\begin{aligned} F_{ik} \circ \mathrm{Trop}(\psi_{U_{ik},U_p}) &= \mathrm{Trop}(\psi_{Z_i,Z_i\cap Z_j}) \circ F_p \quad \text{and} \\ F_{jl} \circ \mathrm{Trop}(\psi_{U_{jl},U_p}) &= \mathrm{Trop}(\psi_{Z_j,Z_i\cap Z_j}) \circ F_p, \end{aligned}$$

where $\mathrm{Trop}(\psi_{U_{ik},U_p}), \mathrm{Trop}(\psi_{Z_i,Z_i\cap Z_j})$, etc. denote the canonical integral $\mathbb{R}$-affine morphisms from (3.2.4) induced by the inclusions $U_p \subseteq U_{ik}$, $Z_i \cap Z_j \subseteq Z_i$. Finally we get

$$\begin{aligned} F_{ik}^*\alpha_i|_{V_p} &= \mathrm{Trop}(\psi_{U_{ik},U_p})^* F_{ik}^*\alpha_i = F_p^* \,\mathrm{Trop}(\psi_{Z_i,Z_i\cap Z_j})^*\alpha_i \\ &= F_p^* \,\mathrm{Trop}(\psi_{Z_j,Z_i\cap Z_j})^*\alpha_j = \mathrm{Trop}(\psi_{U_{jl},U_p})^* F_{jl}^*\alpha_j \\ &= F_{jl}^*\alpha_j|_{V_p}, \end{aligned}$$

where in the third equality we used (3.4). Let $(V_p(k,l))_{p\in V_{ik}\cap V_{jl}}$ be an open cover of $V_{ik} \cap V_{jl}$ then (3.5) follows since $A^{p,q}_{X^{\mathrm{an}}}$ is a sheaf. In the same way one can show that this construction does not depend on the chosen representation for $\alpha$. Part (i) follows by construction and Remark 3.2.17. For part (ii) let $\alpha \in A^{p,q}(W)$ be given by $(W_i, \varphi_{Z_i}, \alpha_i)_{i\in I}$. After possibly passing to a common refinement we may assume that $g^*\alpha$ and $g^{\mathrm{an},*}\alpha$ are given on the same cover. Then they agree by construction. Part (iii) immediately follows from part (i) since the pull-back defined in [CLD12, (3.1.7.)] is functorial. □

**- 3.2.21.** We review some results about the support of differential forms on the analytification that we will need for the definition of the integral in Section 3.4.

**Definition 3.2.22.** Let $V \subseteq X^{\mathrm{an}}$ be an open subset. We say that $\alpha \in A_c^{p,q}(X^{\mathrm{an}})$ is given by *a single form* if $\alpha = [(V, \varphi_U, \alpha_U)]$, for $\alpha_U \in A^{p,q}_{\varphi_{U,\mathrm{trop}}(U^{\mathrm{an}})}(\varphi_{U,\mathrm{trop}}(V))$.

**Proposition 3.2.23.** *Let $(V, \varphi_U)$ be a tropical chart of $X^{\mathrm{an}}$ and let $\alpha = [(V, \varphi_U, \alpha_U)] \in A^{p,q}(V)$ be given by a single form, with $\alpha_U \in A^{p,q}(\varphi_{U,\mathrm{trop}}(V))$. Then $\alpha = 0$ in $A^{p,q}(V)$ if and only if $\alpha_U = 0$ in $A^{p,q}(\varphi_{U,\mathrm{trop}}(V))$.*

*Proof.* See [Gub16, Proposition 5.6.] □

**Corollary 3.2.24.** *Let $(V, \varphi_U)$ be a tropical chart of $X^{\mathrm{an}}$ and $\alpha = [(V, \varphi_U, \alpha_U)] \in A^{p,q}(V)$ be given by a single form, with $\alpha_U \in A^{p,q}(\varphi_{U,\mathrm{trop}}(V))$. Then $\mathrm{supp}(\alpha_U) = \varphi_{U,\mathrm{trop}}(\mathrm{supp}(\alpha))$.*

*Proof.* See [Gub16, Remark 5.4.] and [CLD12, Corollaire 3.2.3.]. □

**Lemma 3.2.25.** *Let $W \subseteq X^{\mathrm{an}}$ be an open subset, $\alpha \in A^{p,q}(W)$ and $\beta \in A^{r,s}(W)$. Let $Y$ be an algebraic variety over $K$ and $f : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ a morphism of $K$-analytic spaces. Then we have the following identities*

*(i)* $\mathrm{supp}(\alpha \wedge \beta) \subseteq \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$;

*(ii)* $\mathrm{supp}(f^*\alpha) \subseteq f^{-1}(\mathrm{supp}(\alpha))$.

*Proof.* Both identities follow from the definitions. We show part (i). Let $x \in X$ be such that $\alpha_x = 0$ or $\beta_x = 0$. Then there exists an open neighborhood $V$ of $x$ where $\alpha = 0$ or $\beta = 0$. We may assume that $\alpha$ and $\beta$ are given by $(V, \varphi_U, \alpha_U)$ and $(V, \varphi_U, \beta_U)$. Then $\alpha \wedge \beta$ is given by $(V, \varphi_U, \alpha_U \wedge \beta_U)$ on $V$ and by Proposition 3.2.23 we get $(\alpha \wedge \beta)_x = 0$ which proves the claim. □

**Lemma 3.2.26.** *Let $W$ be an open subset of $X^{\mathrm{an}}$ and let $U$ be a Zariski open subset of $X$. If $\alpha \in A^{p,q}(W)$ with $\dim(X \setminus U) < \max\{p, q\}$, then $\mathrm{supp}(\alpha) \subseteq W \cap U^{\mathrm{an}}$.*

*Proof.* [Gub16, Lemma 5.11.]. □

## 3.3 Tropical Dolbeault cohomology

In the next section we define tropical Dolbeault cohomology groups by closely following [Jel16, § 3.4].

Let $X$ be an algebraic variety over $K$ of dimension $n$.

**Definition 3.3.1.** Let $W \subseteq X^{\mathrm{an}}$ be open. The *tropical Dolbeault cohomology* and the *tropical Dolbeault cohomology with compact support* on $W$ are defined as

$$\mathrm{H}^{p,q}_{d''}(W) := \mathrm{H}^q(A^{p,\bullet}_{X^{\mathrm{an}}}(W), d'') \quad \text{and} \quad \mathrm{H}^{p,q}_{d'',c}(W) := \mathrm{H}^q(A^{p,\bullet}_{X^{\mathrm{an}},c}(W), d'')$$

**Definition 3.3.2.** Let $p \in \mathbb{N}$. Define a sheaf of $A^{0,0}_{X^{\mathrm{an}}}$-modules on $X^{\mathrm{an}}$ by

$$\mathcal{L}^p_X := \ker(d'' : A^{p,0}_{X^{\mathrm{an}}} \to A^{p,1}_{X^{\mathrm{an}}}).$$

**Proposition 3.3.3.** *There exists a canonical isomorphism*

$$\underline{\mathbb{R}} \cong \mathcal{L}_X^0,$$

*where* $\underline{\mathbb{R}}$ *denotes the constant sheaf with stalks* $\mathbb{R}$*.*

*Proof.* See [Jel16, Lemma 3.4.5.]. □

**Theorem 3.3.4.** *The complex*

$$0 \to \mathcal{L}_X^p \to A_{X^{\mathrm{an}}}^{p,0} \xrightarrow{d''} A_{X^{\mathrm{an}}}^{p,1} \xrightarrow{d''} \dots \xrightarrow{d''} A_{X^{\mathrm{an}}}^{p,n} \to 0$$

*of sheaves on* $X^{\mathrm{an}}$ *is exact. In fact it is an acyclic resolution, hence*

$$\mathrm{H}^q(X^{\mathrm{an}}, \mathcal{L}_X^p) \cong \mathrm{H}_{d''}^{p,q}(X^{\mathrm{an}}) \quad \textit{and} \quad \mathrm{H}_c^p(X^{\mathrm{an}}, \mathcal{L}_X^p) \cong \mathrm{H}_{d'',c}^{p,q}(X^{\mathrm{an}}).$$

*Moreover we have isomorphisms*

$$\mathrm{H}_{\mathrm{sing}}^q(X^{\mathrm{an}}) \cong \mathrm{H}^q(X^{\mathrm{an}}, \underline{\mathbb{R}}) \cong \mathrm{H}_{d''}^{0,q}(X^{\mathrm{an}})$$

*and*

$$\mathrm{H}_{\mathrm{sing},c}^q(X^{\mathrm{an}}) \cong \mathrm{H}_c^q(X^{\mathrm{an}}, \underline{\mathbb{R}}) \cong \mathrm{H}_{d'',c}^{0,q}(X^{\mathrm{an}}).$$

*Proof.* See [Jel16, Corollary 3.4.6.]. □

**- 3.3.5.** We collect known results about the tropical Dolbeault cohomology groups from [Jel16], [JW18].

**Proposition 3.3.6.** *Let* $X$ *be a variety. Then the real vector space* $\mathrm{H}^{0,q}(X^{\mathrm{an}})$ *is finite dimensional for all* $q \in \mathbb{N}$*.*

*Proof.* See [Jel16, Theorem 3.4.9.]. □

**Proposition 3.3.7.** *Let* $X$ *be a variety of dimension* $n$*. Then there exists for all* $q \in \mathbb{N}$ *a homomorphism*

$$\mathrm{H}_{d''}^{0,q}(X^{\mathrm{an}}) \to \mathrm{H}_{d''}^{q,0}(X^{\mathrm{an}}).$$

*If* $X$ *is proper then this map is injective for* $q \in \{0, 1, n\}$*.*

*Proof.* See [Jel16, Proposition 3.4.11.]. □

**Definition 3.3.8.** A smooth projective curve $X$ over $K$ of genus $g \geq 1$ is called a *Mumford curve* if there is a semistable model of $X$ such that all the irreducible components of the special fiber are rational (see [JW18, Definition 2.27.] and [Ber90, Theorem 4.4.1.] for other characterizations).

**Proposition 3.3.9.** *Let* $X$ *be either* $\mathbb{P}_K^1$ *or a Mumford curve of over* $K$*. Let* $g$ *denote the genus of* $X$*. Then*

$$\dim_{\mathbb{R}} \mathrm{H}^{0,1}(X^{\mathrm{an}}) = \dim_{\mathbb{R}} \mathrm{H}^{1,0}(X^{\mathrm{an}}) = g,$$
$$\dim_{\mathbb{R}} \mathrm{H}^{0,0}(X^{\mathrm{an}}) = \dim_{\mathbb{R}} \mathrm{H}^{1,1}(X^{\mathrm{an}}) = 1, \quad \dim_{\mathbb{R}} \mathrm{H}^{p,q}(X^{\mathrm{an}}) = 0, \textit{else}.$$

*Proof.* See [JW18, Theorem 5.1.]. □

## 3.4 Integration

In this section we recall how to integrate top-dimensional differential forms on open subsets of $X^{\mathrm{an}}$. In Proposition 3.4.21 we will prove a formula about the integration on the product of the analytifications of algebraic varieties. We mainly follow [Gub16, § 2-5].

We use the same notation and conventions as in Section 3.1 and 3.2.

**- 3.4.1.** Let $\Omega$ be an open subset of $N_{\mathbb{R}}$ and let $\mathrm{vol}(x_1, \ldots, x_r)$ denote the Lebesgue measure on $N_{\mathbb{R}}$ normalized such that $\mathrm{vol}(N_{\mathbb{R}}/N) = 1$. Let $\alpha \in A_c^{r,r}(\Omega)$ be given by $\alpha = f\ d\mathrm{vol}(x_1, \ldots, x_r)$ for $f \in C_c^\infty(\Omega)$. We define

$$\int_\Omega \alpha := \int_\Omega f$$

as in [Jel16, Definition 2.1.7.].

**- 3.4.2.** We can also define integration in terms of contractions as done in [Gub16, § 2.6.]. Let $\Omega \subseteq N_{\mathbb{R}}$ be open. We view a superform $\alpha \in A_c^{p,q}(\Omega)$ as a multilinear map

$$N_{\mathbb{R}}^{p,q} \to C^\infty(\Omega), \quad (v_1, \ldots, v_{p+q}) \mapsto \alpha(v_1, \ldots, v_{p+q}) \tag{3.6}$$

which is alternating in $(v_1, \ldots, v_p)$ and in $(v_{p+1}, \ldots, v_{p+q})$.

**Definition 3.4.3.** Let $\alpha \in A^{p,q}(\Omega)$, and $I \subseteq \{1, \ldots, p+q\}$ a subset of cardinality $s$ with $s'$ elements contained in $\{1, \ldots, p\}$ and $s''$ elements contained in $\{p+1, \ldots, p+q\}$. Given vectors $v_1, \ldots, v_s \in N_{\mathbb{R}}$, then the *contraction* of $\langle \alpha; v_1, \ldots, v_s\rangle_I \in A^{p-s',q-s''}(\Omega)$ is given by inserting $v_1, \ldots, v_s$ for the variables $(n_i)_{i\in I}$ of (3.6). Let $\alpha \in A_c^{r,r}(\Omega)$, then we view $\langle \alpha; e_1, \ldots, e_r\rangle_{\{1,\ldots,r\}} \in A_c^{0,r}(\Omega)$ is a $r$-form on $\Omega$ and the integral is given by

$$\int_\Omega \alpha := (-1)^{\frac{r(r-1)}{2}} \int_\Omega \langle \alpha; e_1, \ldots, e_r\rangle_{\{1,\ldots,r\}}.$$

and on the right-hand side we use usual integration of a top-dimensional form.

**- 3.4.4.** Let $\sigma$ be an integral $\mathbb{R}$-affine polyhedron of dimension $n$ and consider the integral $\mathbb{R}$-affine structure $(\mathbb{A}_\sigma, N_\sigma)$ associated to $\sigma$. Put $\Omega := \widetilde{\Omega} \cap \sigma$ for an open subset $\widetilde{\Omega} \subseteq N_{\mathbb{R}}$. Let $\alpha \in A_\sigma^{n,n}(\Omega)$ and consider the affine morphism $F \colon \mathbb{L}_\sigma = N_{\sigma,\mathbb{R}} \to \mathbb{A}_\sigma$, then we define

$$\int_\sigma \alpha := \int_{F^{-1}(\widetilde{\Omega}\cap \mathrm{relint}(\sigma))} F^*\alpha$$

where $F^*\alpha = F^*(\widetilde{\alpha})|_{\widetilde{\Omega}\cap \mathrm{relint}(\sigma)}$ for some representative $\widetilde{\alpha} \in A_c^{n,n}(\widetilde{\Omega})$ and on the right-hand side we integrate as in 3.4.1. With the same arguments as in Remark 3.1.13 one can show that this definition does not depend on the choice of the representative $\widetilde{\alpha}$.

**- 3.4.5.** Let $(\mathcal{C}, m)$ be a weighted integral $\mathbb{R}$-affine polyhedral complex of pure dimension $n$ as in Definition 3.1.14. Let $\Omega \subseteq |\mathcal{C}|$ be an open subset and $\alpha \in A^{n,n}_{\mathcal{C},c}(\Omega)$, we set

$$\int_{(\mathcal{C},m)} \alpha := \sum_{\sigma\in\mathcal{C}_n} m_\sigma \int_\sigma \alpha.$$

**Remark 3.4.6.** By [Gub16, § 3.9.] it is possible to push-forward tropical cycles by integral $\mathbb{R}$-affine maps $F\colon N'_{\mathbb{R}} \to N_{\mathbb{R}}$, i.e. if $(\mathcal{C}', m')$ is a weighted integral $\mathbb{R}$-affine polyhedral complex of pure dimension $n$ in $N'_{\mathbb{R}}$, then we can define a tropical cycle $(F_*\mathcal{C}', m)$ on $N_{\mathbb{R}}$ called the *push-forward of* $(\mathcal{C}', m')$ *by* $F$.

**Proposition 3.4.7.** *With the same notation and assumptions as in Remark 3.4.6 let* $\Omega'$ *be an open subset of* $N'_{\mathbb{R}}$ *and* $\alpha \in A^{n,n}_{F_*(\mathcal{C}'),c}(\Omega')$. *Then we have the following projection formula*

$$\int_{F_*(\mathcal{C}',m)} \alpha = \int_{(\mathcal{C}',m)} F^*(\alpha).$$

*Proof.* See [Gub16, Proposition 3.10.]. □

**Proposition 3.4.8.** *Let* $\varphi'\colon U' \to T'$ *be a moment map for a non-empty open* $U' \subseteq T'$ *which refines the moment map* $\varphi\colon U \to T$, *i.e. there is an affine homomorphism* $\psi\colon T' \to T$ *such that* $\varphi = \psi \circ \varphi'$ *on* $U'$. *Then we have an equality of tropical cycles*

$$\mathrm{Trop}(\psi)_*(\mathrm{Trop}(\varphi'_*(U'))) = \mathrm{Trop}(\varphi_*(U)).$$

*Proof.* See [Gub16, Proposition 4.11.]. □

**- 3.4.9.** Let $X$ be an algebraic variety of dimension $n$.

**Proposition 3.4.10.** *Let* $\alpha \in A^{p,q}_c(X^{\mathrm{an}})$ *be a differential form with* $\max\{p, q\} = n$. *Then there is a very affine open subset* $U$ *of* $X$ *with associated tropical chart* $(V, \varphi_U)$ *such that* $\mathrm{supp}(\alpha) \subseteq U^{\mathrm{an}}$ *and such that* $\alpha$ *on* $U^{\mathrm{an}}$ *is given by a superform* $\alpha_U \in A^{p,q}_c(\varphi_{U,\mathrm{trop}}(V))$. *When* $p, q = n$ *we call the pair* $(U, \alpha_U)$ *a* very affine chart of integration for $\alpha$.

*Proof.* See [Gub16, Proposition 5.16.]. □

**Definition 3.4.11.** Let $\alpha \in A_c^{n,n}(W)$ for an open subset $W$ of $X^{\mathrm{an}}$. Let $(U, \alpha_U)$ be a very affine chart of integration for $\alpha$. The *integral of $\alpha$ over $W$* is given by

$$\int_W \alpha := \int_{\varphi_{U,\mathrm{trop}}(U^{\mathrm{an}})} \alpha_U$$

and we integrate as in 3.4.5.

**Lemma 3.4.12.** *For $\alpha \in A_c^{n,n}(W)$, the following properties hold*

*(i) If $(U, \alpha_U)$ is a very affine chart of integration for $\alpha$, then every non-empty very affine open subset $U'$ of $U$ is a very affine chart of integration $(U', \alpha_{U'})$ for $\alpha$.*

*(ii) The definition of $\int_W \alpha$ is independent of the choice of very affine chart of integration for $\alpha$.*

*Proof.* See [Gub16, Lemma 5.15.]. □

**Proposition 3.4.13.** *Let $\alpha \in A_c^{2n-1}(W)$, then we have*

$$\int_W d'\alpha = 0 \quad \textit{and} \quad \int_W d''\alpha = 0.$$

*Proof.* See [Gub16, Theorem 5.17.]. □

**Remark 3.4.14.** The proof of 3.4.13 relies on a Stokes' formula for integrals over tropical cycles. Since we will not need it later we will not explain it here. For more details see [Gub16, Proposition 3.5.].

## Integration on the product

We prove an integration formula on the product of the analyitifcations of algebraic varieties over $K$. This result was originally meant to be applied to the general case of the study of the tropical Dolbeault cohomology of an abelian variety over $K$. This problem presented some difficulties that unfortunately could not be solved. Nevertheless the following result can be interesting for other purposes, so I decided to leave it in the thesis.

**- 3.4.15.** For $i = 1, 2$ let $M_i$ be free abelian groups of rank $r_i$, let $N_i := \mathrm{Hom}(M_i, \mathbb{Z})$ be their dual and $N_{i,\mathbb{R}} := N_i \otimes_{\mathbb{Z}} \mathbb{R}$ their extension to $\mathbb{R}$. Let $e_1, \ldots, e_{r_1}$ respectively $b_1 \ldots, b_{r_2}$ be a basis of $N_1$ respectively $N_2$ and $x_1, \ldots, x_{r_1}$ respectively $y_1, \ldots, y_{r_2}$ be the induced basis on $N_{1,\mathbb{R}}$ respectively on $N_{2,\mathbb{R}}$. Denote by $j_1 : N_{1,\mathbb{R}} \to N_{1,\mathbb{R}} \times N_{2,\mathbb{R}}, x \mapsto (x, 0)$ respectively $j_2 : N_{2,\mathbb{R}} \to N_{1,\mathbb{R}} \times N_{2,\mathbb{R}}, x \mapsto (0, x)$ the canonical inclusions. For polyhedra $\sigma_i \subseteq N_{i,\mathbb{R}}$ of dimension $\dim(\sigma_i) =: r_i$ we define the *product of* $\sigma_1$ *with* $\sigma_2$ by

$$\sigma_1 \times \sigma_2 := j_1(\sigma_1) \times j_2(\sigma_2) \subseteq N_{1,\mathbb{R}} \times N_{2,\mathbb{R}}.$$

Note that $\sigma_1 \times \sigma_2$ is a polyhedron of dimension $r_1 + r_2$. The faces of $\sigma_1 \times \sigma_2$ are given as products $\tau_1 \times \tau_2$ of faces $\tau_1$ of $\sigma_1$ and faces $\tau_2$ of $\sigma_2$. Finally note that we have $\mathrm{relint}(\sigma_1 \times \sigma_2) = \mathrm{relint}(\sigma_1) \times \mathrm{relint}(\sigma_2)$.

**Lemma 3.4.16.** *For* $i = 1, 2$ *let* $\Omega_i \subseteq N_{i,\mathbb{R}}$ *be open subsets and* $\alpha_i \in A_c^{r_i,r_i}(\Omega_i)$. *Then we have*

$$\int_{\Omega_1 \times \Omega_2} p_{\Omega_1}^* \alpha_1 \wedge p_{\Omega_2}^* \alpha_2 = \int_{\Omega_1} \alpha_1 \cdot \int_{\Omega_2} \alpha_2.$$

*Here* $p_{\Omega_i}$ *denotes the canonical projection* $\Omega_1 \times \Omega_2 \to \Omega_i$.

*Proof.* Using the notation from paragraph 3.4.1 let $\alpha_1 = f\ d\mathrm{vol}(x_1, \dots, x_{r_1})$ and $\alpha_2 = g\ d\mathrm{vol}(y_1, \dots, y_{r_2})$ for $f, g \in C_c^\infty(\Omega_i)$. Then we get

$$\int_{\Omega_1 \times \Omega_2} p_{\Omega_1}^* \alpha_1 \wedge p_{\Omega_2}^* \alpha_2 = \int_{\Omega_1 \times \Omega_2} f \cdot g = \int_{\Omega_1} f \cdot \int_{\Omega_2} g = \int_{\Omega_1} \alpha_1 \cdot \int_{\Omega_2} \alpha_2,$$

where in the second equality we used Fubini's theorem. □

**Lemma 3.4.17.** *Let* $\sigma_i \subseteq N_{i,\mathbb{R}}$ *be a* $r_i$*-dimensional polyhedron and set* $\Omega_i := \widetilde{\Omega}_i \cap \sigma_i$ *for* $\widetilde{\Omega}_i \subseteq N_{i,\mathbb{R}}$. *Let* $\alpha_i \in A_{\sigma_i,c}^{r_i,r_i}(\Omega_i)$, *then*

$$\int_{\sigma_1 \times \sigma_2} p_{\sigma_1}^* \alpha_1 \wedge p_{\sigma_2}^* \alpha_2 = \int_{\sigma_1} \alpha_1 \cdot \int_{\sigma_2} \alpha_2.$$

*Here* $p_{\sigma_i}$ *denotes the canonical projection* $\sigma_1 \times \sigma_2 \to \sigma_i$.

*Proof.* Consider the affine map $F_i \colon N_{\sigma_i,\mathbb{R}} \to \mathbb{A}_{\sigma_i}$ and let $\widetilde{\alpha}_i \in A_c^{r_i,r_i}(\widetilde{\Omega}_i)$ be superforms with $\widetilde{\alpha}_i|_{\widetilde{\Omega}_i \cap \mathrm{relint}(\sigma_i)} = \alpha_i$. Put $\Omega_i' := F_i^{-1}(\widetilde{\Omega}_i \cap \mathrm{relint}(\sigma_i))$. By definition we have

$$\begin{aligned}\int_{\sigma_1} \alpha_1 \cdot \int_{\sigma_2} \alpha_2 &= \int_{\Omega_1'} F_1^* \alpha_1 \cdot \int_{\Omega_2'} F_2^* \alpha_2 \\ &= \int_{\Omega_1' \times \Omega_2'} p_{\Omega_1'}^* F_1^* \alpha_1 \wedge p_{\Omega_2'}^* F_2^* \alpha_2,\end{aligned}$$

where in the last equality we used Lemma 3.4.16. Note that since $F_i \circ p_i = p_{\mathbb{A}_{\sigma_i}} \circ (F_1 \times F_2)$ we have

$$p_{\Omega_1'}^* F_1^* \alpha_1 \wedge p_{\Omega_2'}^* F_2^* \alpha_2 = (F_1 \times F_2)^* (p_{\sigma_1}^* \alpha_1 \wedge p_{\sigma_2}^* \alpha_2). \tag{3.7}$$

By the same identities as in Lemma 3.2.25 for differential forms on polyhedral complexes, we have $\operatorname{supp}(p_{\sigma_1}^*\alpha_1 \wedge p_{\sigma_2}^*\alpha_2) \subseteq p_{\sigma_1}^{-1}(\operatorname{supp}(\alpha_1)) \cap p_{\sigma_2}^{-1}(\operatorname{supp}(\alpha_2))$ and hence

$$\operatorname{supp}(p_{\sigma_1}^*\alpha_1 \wedge p_{\sigma_2}^*\alpha_2) \subseteq \widetilde{\Omega}_1 \times \widetilde{\Omega}_2 \cap \operatorname{relint}(\sigma_1) \times \operatorname{relint}(\sigma_2) \tag{3.8}$$

From the discussion in 3.4.15 we have

$$\widetilde{\Omega}_1 \times \widetilde{\Omega}_2 \cap \operatorname{relint}(\sigma_1) \times \operatorname{relint}(\sigma_2) = \widetilde{\Omega}_1 \times \widetilde{\Omega}_2 \cap \operatorname{relint}(\sigma_1 \times \sigma_2). \tag{3.9}$$

Finally by applying first (3.8), (3.9) and then (3.7) we conclude that

$$\begin{aligned}\int_{\Omega_1' \times \Omega_2'} p_{\Omega_1'}^* F_1^* \alpha_1 \wedge p_{\Omega_2'} F_2^* \alpha_2 &= \int_{\widetilde{\Omega}_1 \times \widetilde{\Omega}_2 \cap \operatorname{relint}(\sigma_1 \times \sigma_2)} (F_1 \times F_2)^* (p_{\sigma_1}^* \alpha_1 \wedge p_{\sigma_2}^* \alpha_2) \\ &= \int_{\sigma_1 \times \sigma_2} p_{\sigma_1}^* \alpha_1 \wedge p_{\sigma_2}^* \alpha_2.\end{aligned}$$

□

**- 3.4.18.** For $i = 1, 2$ let $\mathcal{C}_i \subseteq N_{i,\mathbb{R}}$ be a pure $n_i$-dimensional polyhedral complexes. Then we define their product by

$$\mathcal{C}_1 \times \mathcal{C}_2 := \{\sigma_1 \times \sigma_2 \mid \sigma_1 \in \mathcal{C}_1,\ \sigma_2 \in \mathcal{C}_2\}.$$

This is a polyhedral complex in $N_{1,\mathbb{R}} \times N_{2,\mathbb{R}}$ of dimension $n_1 + n_2$. Note that we have $(\mathcal{C}_1 \times \mathcal{C}_2)_{n_1+n_2} = \mathcal{C}_{1,n_1} \times \mathcal{C}_{2,n_2}$, hence if $(\mathcal{C}_1, m_1)$ and $(\mathcal{C}_2, m_2)$ are weighted polyhedral complexes, we define the weighted polyhedral complex

$$(\mathcal{C}_1 \times \mathcal{C}_2, m_1 \circ p_1 \cdot m_2 \circ p_2),$$

where $p_i \colon N_{1,\mathbb{R}} \times N_{2,\mathbb{R}} \to N_{i,\mathbb{R}}$ denote the canonical projections.

**Lemma 3.4.19.** *For $i = 1, 2$ let $(\mathcal{C}_i, m_i)$ be weighted integral $\mathbb{R}$-affine polyhedral complexes of dimension $n_i$ and $\Omega_i := \widetilde{\Omega}_i \cap \mathcal{C}_i$ for open subsets $\widetilde{\Omega}_i \subseteq N_{i,\mathbb{R}}$. For $\alpha_i \in A_{\mathcal{C}_i,c}^{n_i,n_i}(\Omega_i)$ we have*

$$\int_{(\mathcal{C}_1 \times \mathcal{C}_2, m_1 \circ p_1 \cdot m_2 \circ p_2)} p_{\mathcal{C}_1}^* \alpha_1 \wedge p_{\mathcal{C}_2}^* \alpha_2 = \int_{(\mathcal{C}_1, m_1)} \alpha_1 \cdot \int_{(\mathcal{C}_2, m_2)} \alpha_2.$$

*Proof.* This follows from linearity of the integral and Lemma 3.4.17. Indeed we have

$$\begin{aligned}
&\int_{(\mathcal{C}_1\times\mathcal{C}_2, m_1\circ p_1\cdot m_2\circ p_2)} p^*_{\mathcal{C}_1}\alpha_1\wedge p^*_{\mathcal{C}_2}\alpha_2\\
&=\sum_{\sigma_1\times\sigma_2} m_1\circ p_1(\sigma_1\times\sigma_2)\cdot m_2\circ p_2(\sigma_1\times\sigma_2)\int_{\sigma_1\times\sigma_2} p^*_{\sigma_1}\alpha_1\wedge p^*_{\sigma_2}\alpha_2\\
&=\sum_{\sigma_1} m_1(\sigma_1)\sum_{\sigma_2} m_2(\sigma_2)\int_{\sigma_1}\alpha_1\cdot\int_{\sigma_2}\alpha_2\\
&=\int_{(\mathcal{C}_1,m_1)}\alpha_1\cdot\int_{(\mathcal{C}_2,m_2)}\alpha_2,
\end{aligned}$$

where in the second to last equality we used 3.4.17. □

**- 3.4.20.** For $i=1,2$ let $X_i$ be an algebraic variety over $K$ of dimension $n_i$. Recall from 2.1.8 that the analytification functor commutes with fibered products. Let $U_i\subseteq X_i$ be a very affine open subset. Denote by $M_i$ the free abelian group $M_{U_i}:=\mathcal{O}(U_i)/K^\times$ (see 3.2.4) of rank $r_i$ and by $N_i:=\mathrm{Hom}_{\mathbb{Z}}(M_i,\mathbb{Z})$ their dual. Note that $N_1\times N_2\cong\mathrm{Hom}_{\mathbb{Z}}(M_1\times M_2,\mathbb{Z})$. Let $e_1,\dots,e_{r_1}$ respectively $b_1\dots,b_{r_2}$ be a basis of $N_1$ respectively $N_2$ and $x_1,\dots,x_{r_1}$ respectively $y_1,\dots,y_{r_2}$ be the induced basis on $N_{1,\mathbb{R}}$ respectively on $N_{2,\mathbb{R}}$. Denote by $p_i : X_1\times X_2\to X_i$ the canonical projection onto the $i$-th factor. We will also denote by $p_i$ the induced map $p_i^{\mathrm{an}}$ on the analytifications.

**Proposition 3.4.21.** *For $i=1,2$ let $\alpha_i\in A_c^{n_i,n_i}(X_i^{\mathrm{an}})$. Then*

$$\int_{(X_1\times X_2)^{\mathrm{an}}} p_1^*\alpha_1\wedge p_2^*\alpha_2=\int_{X_1^{\mathrm{an}}}\alpha_1\cdot\int_{X_2^{\mathrm{an}}}\alpha_2.$$

*Proof.* For $i=1,2$ let $(U_i,\alpha_{U_i})$ be a very affine chart of integration for $\alpha_i$. This means that we have $\mathrm{supp}(\alpha_i)\subseteq U_i^{\mathrm{an}}$ and $\alpha_i$ is given by $\alpha_{U_i}$ on $\varphi_{U_i,\mathrm{trop}}(U_i^{\mathrm{an}})$. Denote by $\varphi_i : U_i\to T_{U_i}$ the canonical moment maps. We will write $\alpha_i|_{U^{\mathrm{an}}}=\varphi^*_{U_i,\mathrm{trop}}\alpha_{U_i}$. The projections $\pi_i : T_{U_1}\times T_{U_2}\to T_{U_i}$ are morphisms of tori, hence we get commutative diagrams:

$$\begin{array}{ccccc}
(U_1\times U_2)^{\mathrm{an}}\cong U_1^{\mathrm{an}}\times U_2^{\mathrm{an}} & \xrightarrow{\varphi_{U_1}^{\mathrm{an}}\times\varphi_{U_2}^{\mathrm{an}}} & T_{U_1}^{\mathrm{an}}\times T_{U_2}^{\mathrm{an}}\cong T_{U_1\times U_2}^{\mathrm{an}} & \xrightarrow{\mathrm{trop}_{T_{U_1\times U_2}}} & N_{U_1\times U_2,\mathbb{R}}\\
\downarrow p_{U_i} & & \downarrow \pi & & \downarrow \mathrm{Trop}(\pi_i)\\
U_i^{\mathrm{an}} & \xrightarrow[\varphi_{U_i}^{\mathrm{an}}]{} & T_{U_i}^{\mathrm{an}} & \xrightarrow[\mathrm{trop}_{T_{U_i}}]{} & N_{U_i,\mathbb{R}}
\end{array}$$

where the isomorphism $T_{U_1}\times T_{U_2}\cong T_{U_1\times U_2}$ follows from Lemma 3.2.8. We claim that $(U_1\times U_2,\mathrm{Trop}(\pi_1)^*\alpha_1\wedge\mathrm{Trop}(\pi_2)^*\alpha_2)$ is a chart of integration for $p_1^*\alpha_1\wedge p_2^*\alpha_2$.

By Lemma 3.2.25 we have

$$\begin{aligned}\operatorname{supp}(p_1^*\alpha_1 \wedge p_2^*\alpha_2) &\subseteq \operatorname{supp}(p_1^*\alpha_1) \cap \operatorname{supp}(p_2^*\alpha_2) \subseteq p_1^{-1}(\operatorname{supp}\alpha_1) \cap p_2^{-1}(\operatorname{supp}\alpha_2)\\ &\subseteq \operatorname{supp}(\alpha_1) \times \operatorname{supp}(\alpha_2) \subseteq U_1^{\mathrm{an}} \times U_2^{\mathrm{an}}.\end{aligned}$$

Finally we show that $p_1^*\alpha_1 \wedge p_2^*\alpha_2$ is given by $\operatorname{Trop}(\pi_1)^*\alpha_1 \wedge \operatorname{Trop}(\pi_2)^*\alpha_2$ on $\varphi_{U_1\times U_2,\mathrm{trop}}((U_1\times U_2)^{\mathrm{an}}) = \varphi_{U_1,\mathrm{trop}}(U_1^{\mathrm{an}}) \times \varphi_{U_2,\mathrm{trop}}(U_2^{\mathrm{an}})$. Indeed the diagram immediately gives

$$\varphi^*_{U_1\times U_2,\mathrm{trop}} \operatorname{Trop}(\pi_i)^*\alpha_{U_i} = p^*_{U_i}\varphi^*_{U_i,\mathrm{trop}}\alpha_{U_i} = p^*_{U_i}(\alpha_i|_{U_i^{\mathrm{an}}}) = (p_i^*\alpha_i)|_{U_1^{\mathrm{an}}\times U_2^{\mathrm{an}}}.$$

In conclusion we have

$$\begin{aligned}\int_{X_1^{\mathrm{an}}\times X_2^{\mathrm{an}}} p_1^*\alpha_1 \wedge p_2^*\alpha_2 &= \int_{\varphi_{U_1\times U_2,\mathrm{trop}}((U_1\times U_2)^{\mathrm{an}})} \operatorname{Trop}(\pi_1)^*\alpha_1 \wedge \operatorname{Trop}(\pi_2)^*\alpha_2\\ &= \int_{\varphi_{U_1,\mathrm{trop}}(U_1^{\mathrm{an}})\times\varphi_{U_2,\mathrm{trop}}(U_2^{\mathrm{an}})} \operatorname{Trop}(\pi_1)^*\alpha_1 \wedge \operatorname{Trop}(\pi_2)^*\alpha_2\\ &= \int_{\varphi_{U_1,\mathrm{trop}}(U_1^{\mathrm{an}})} \alpha_1 \cdot \int_{\varphi_{U_2,\mathrm{trop}}(U_2^{\mathrm{an}})} \alpha_2 = \int_{X_1}\alpha_1 \cdot \int_{X_2}\alpha_2,\end{aligned}$$

where in the third equality we used Lemma 3.4.19. □

## Invariant superforms

**- 3.4.22.** Let $\Lambda \subseteq N_{\mathbb{R}}$ be a complete lattice. In Example 3.1.6 we considered the subspace $A^{p,q}_{\mathrm{cl}}(\Omega)^\Lambda$ of $A^{p,q}(N_{\mathbb{R}})$ of closed $\Lambda$-invariant $(p,q)$-superforms on $N_{\mathbb{R}}$ and the subspace $\Lambda^{p,q}N^*_{\mathbb{R}}$ of $(p,q)$-superforms with constant coefficients. In Proposition 3.4.26 we will consider the cohomology groups $\mathrm{H}^q(A^{p,\bullet}(N_{\mathbb{R}})^\Lambda, d'')$ and in analogy to the complex case we will show that the canonical inclusion

$$(\Lambda^{p,\bullet}N^*_{\mathbb{R}}, d'') \to (A^{p,\bullet}(N_{\mathbb{R}})^\Lambda, d'')$$

of differential complexes is a quasi-isomorphism.

Let $N_{\mathbb{R}}$ be a finite dimensional real vector space and $\Lambda$ a complete lattice in $N_{\mathbb{R}}$. Let $e_1,\ldots,e_{r_1}$ be a basis $\Lambda$ and $x_1,\ldots,x_{r_1}$ the induced basis on $N_{\mathbb{R}}$. For $v \in N_{\mathbb{R}}$ denote by $\tau_v \colon N_{\mathbb{R}} \to N_{\mathbb{R}}$ the morphism given by translation by $v$. Let $\operatorname{vol}(x_1\ldots,x_r)$ denote the Lebesgue measure on $N_{\mathbb{R}}$ normalized such that $\operatorname{vol}(N_{\mathbb{R}}/\Lambda) = 1$.

**Lemma 3.4.23.** *Let $v \in N_{\mathbb{R}}$ and let $\alpha \in A^{p,q}(N_{\mathbb{R}})^\Lambda$ be $d''$-closed. Then $\tau_v^*\alpha - \alpha$ is $d''$-exact.*

*Proof.* We can closely follow the proof given in [Deb05, § 3.4.]. We will use the definition of the integral via contractions as explained in Definition 3.4.3. Let $w \in N_{\mathbb{R}}$. Let $\eta_v$ be the $(p, q-1)$-form on $N_{\mathbb{R}}$ given by

$$\langle \eta_v(w); x_1, \dots, x_p, x_1, \dots x_{q-1} \rangle_{\{1,\dots,p+q-1\}} = \int_0^1 \langle \alpha(w+tv); x_1, \dots, x_p, v, x_1, \dots x_{q-1} \rangle_{\{1,\dots,p+q\}}\, dt.$$

Note that $\eta_v$ is $\Lambda$-invariant. We claim that $d''\eta_v(w) = \alpha(w+tv) - \alpha(w)$. We will prove it in the case $p, q = 1$. For the general case we refer to the footnote of [Deb05, p. 25]. Let $\alpha = \sum_{1 \leq i,j \leq r} \alpha_{ij} d'x_i \wedge d''x_j \in A^{1,1}(N_{\mathbb{R}})$, then

$$\langle \alpha(w); x_i, v \rangle_{\{1,2\}} = \sum_{1 \leq i,j \leq r} \alpha_{ij} \langle d'x_i \wedge d''x_j; x_i, v \rangle_{\{1,2\}} = \sum_{1 \leq i,j \leq r} \alpha_{ij}(w) \cdot v_j.$$

and we set $\eta_v = \sum_{i=1}^r \eta_{v,i} d'x_i \in A^{1,0}(N_{\mathbb{R}})^{\Lambda}$. Since $\alpha$ is $d''$-closed we also have $\frac{\partial \alpha_{ij}}{\partial x_k} = \frac{\partial \alpha_{ik}}{\partial x_j}$. Fix $i, k \in \{1, \dots r\}$, then

$$\begin{aligned}\frac{\partial \eta_{v,i}}{\partial x_k}(w) \langle d'x_i; x_i \rangle_{\{1\}} &= \int_0^1 \sum_{1 \leq j \leq r} \frac{\partial \alpha_{ij}(w+tv)}{\partial x_k} \cdot v_j \, dt = \int_0^1 \sum_{1 \leq j \leq r} \frac{\partial \alpha_{ik}(w+tv)}{\partial x_j} \cdot v_j \, dt \\ &= \int_0^1 \frac{d\alpha_{ik}(x+tv)}{dt}\, dt = \alpha_{ik}(w+tv)|_0^1 = \alpha_{ik}(w+v) - \alpha_{ik}(w).\end{aligned}$$

By summing over $i, k \in \{1, \dots, r\}$ we finally get

$$\begin{aligned} d''\eta_v(w) &= \sum_{1 \leq i,k \leq r} \frac{\partial \eta_{v,i}}{\partial x_k}(w) d'x_i \wedge d''x_k \\ &= \sum_{1 \leq i,k \leq r} (\alpha_{ik}(w+v) - \alpha_{ik}(w)) d'x_i \wedge d''x_k = \alpha(w+v) - \alpha(w). \end{aligned}$$

□

**Remark 3.4.24.** (i) Let $F$ be a fundamental domain in $N_{\mathbb{R}}$ given by a $\mathbb{Z}$-basis of $\Lambda$. Let $g\colon N_{\mathbb{R}} \to \mathbb{R}$ be a function. Then we define the *average* $\widetilde{g}\colon N_{\mathbb{R}} \to \mathbb{R}$ of $g$ by

$$\widetilde{g}(w) := \frac{\int_F g(w+x)}{\int_F 1},$$

(ii) If $g$ is $\Lambda$-invariant, then the average $\widetilde{g}$ is constant.

(iii) For $\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \in A^{p,q}(N_\mathbb{R})$ we define

$$\widetilde{\alpha} := \sum_{|I|=p,|J|=q} \widetilde{\alpha}_{IJ} d'x_I \wedge d''x_J,$$

where $\widetilde{\alpha}_{IJ}$ denotes the average of $\alpha_{IJ}$.

*Proof of part (ii).* It suffices to show that for $w, z \in F$ we have $\widetilde{g}(w) = \widetilde{g}(z)$, since for every $x \in N_\mathbb{R}$ there exists an element $\lambda \in \Lambda$ such that $x - \lambda \in F$. We show the claim in the case $r = 1$. Let $b$ be a basis of $N$, after a change of basis we may assume that $b = e_1$ is the standard basis vector (in the definition of $\widetilde{g}$ we are taking a quotient, so every renormalization vanishes). Denote by $x$ the corresponding coordinate function on $N_\mathbb{R}$ and assume that $w > z$. Then

$$\begin{aligned}\widetilde{g}(w) = \frac{\int_0^1 g(w+x)\ dx}{\int_0^1 dx} &= \int_{w-z}^{1+(w-z)} g(z+x)\ dx = \int_{w-z}^{1+(w-z)} g(z+x)\ dx \\ &= \int_0^1 g(z+x)\ dx - \int_0^{w-z} g(z+x)\ dx + \int_1^{1+(w-z)} g(z+x)\ dx = \widetilde{g}(z),\end{aligned}$$

where in the last equality we used that $g$ is $\Lambda$-invariant.

□

**Lemma 3.4.25.** *We keep the same notation as in Remark 3.4.24. Let $\alpha \in A^{p,q}_{\mathrm{cl}}(N_\mathbb{R})^\Lambda$. Then $\widetilde{\alpha} \in \Lambda^{p,q} N^*_\mathbb{R}$ and $\alpha$ and $\widetilde{\alpha}$ are cohomologous.*

*Proof.* The first part of the statement immediately follows from the discussion from Remark 3.4.24. for the second part of the statement let $w \in N_\mathbb{R}$ we have

$$\begin{aligned}&\langle \widetilde{\alpha}(w) - \alpha(w); x_1, \ldots, x_p, x_1, \ldots, x_q \rangle_{\{1,\ldots,p+q\}} \\ &\quad = \frac{\int_F \langle \alpha(w+x) - \alpha(w); x_1, \ldots, x_p, x_1, \ldots, x_q \rangle_{\{1,\ldots,p+q\}}\ dx}{\int_F dx} \\ &\quad = \frac{\int_F \langle d''\eta_x(w); x_1, \ldots, x_p, x_1, \ldots, x_q \rangle_{\{1,\ldots,p+q\}}\ dx}{\int_F dx} \\ &\quad = \frac{d''(\int_F \langle \eta_x(w); x_1, \ldots, x_p, x_1, \ldots, x_{q-1} \rangle_{\{1,\ldots,p+q-1\}}\ dx)}{\int_F dx},\end{aligned}$$

where in the second equality we used Lemma 3.4.23 and in the last equality differentiaton under the integral sign. Note that since the left-hand side is $\Lambda$-invariant and by Remark 3.1.3(v) the differential $d''$ commutes with affine pull-back, also the right-hand side is $\Lambda$-invariant.

□

**Proposition 3.4.26.** *There is a canonical isomorphism of real vector spaces*

$$\Psi\colon \Lambda^{p,q} N_{\mathbb{R}}^* \xrightarrow{\cong} \mathrm{H}^q(A^{p,\bullet}(N_{\mathbb{R}})^\Lambda, d'').$$

*Proof.* Injectivity of $\Psi$ follows from Example 3.1.6. For the surjectiviy let $\alpha \in A^{p,q}_{\mathrm{cl}}(N_{\mathbb{R}})^\Lambda$ be a $d''$-closed superform. Then by Lemma 3.4.25 there exists a superform $\beta \in A^{p,q-1}(N_{\mathbb{R}})^\Lambda$ with $d''\beta = \alpha - \widetilde{\alpha}$ for $\widetilde{\alpha} \in \Lambda^{p,q} N_{\mathbb{R}}^*$. Hence $\Psi(\widetilde{\alpha}) = \alpha$ in $\mathrm{H}^q(A^{p,\bullet}(N_{\mathbb{R}})^\Lambda, d'')$. □

# Chapter 4

# Abelian varieties with totally degenerate reduction

## 4.1 Setup

Let $K$ be a non-trivially valued complete, algebraically closed, non-archimedean field. Let $A$ be an abelian variety over $K$ with totally degenerate reduction of dimension $n$. Recall from 2.2.2 and Definition 2.2.4 that this means that there is an isomorphism of $K$-analytic groups $A^{\mathrm{an}} \cong T^{\mathrm{an}}/M$, where $T = (\mathbb{G}_{m,K}^{\mathrm{an}})^r$ is a (split) algebraic torus over $K$ of rank $r$ and $M$ is a discrete subgroup of $T(K)$. Note that then $n = r$. In contrast to the notation introduced in Chapter 3, here $M$ does not denote the character group of $T$. However we have $\mathrm{rank}_{\mathbb{Z}}(M) = r$. Denote by $N$ the cocharacter group of $T$ and by $\mathrm{trop}_T \colon T^{\mathrm{an}} \to N_{\mathbb{R}}$ the tropicalization map. By [Gub10, §4.2] the tropicalization map $\mathrm{trop}_T$ maps $M$ bijectively onto a complete lattice $\Lambda$ of $N_{\mathbb{R}}$. In this section we prove a first relation between the tropicalization $\mathrm{trop}_T$ and the morphism $p \colon T^{\mathrm{an}} \to A^{\mathrm{an}}$.

**- 4.1.1.** For an element $x \in T(K)$ and $v \in N_{\mathbb{R}}$ we will denote by $\tau_x \colon T^{\mathrm{an}} \to T^{\mathrm{an}}$ and $\tau_v \colon N_{\mathbb{R}} \to N_{\mathbb{R}}$ the morphisms corresponding to translation by $x$, respectively by $v$.

**Definition 4.1.2.** Let $V \subseteq T^{\mathrm{an}}$ be an open subset. We say that $V$ is *$M$-small* if for every non-zero element $m \in M$ we have

$$V \cap \tau_m^{-1}(V) = \emptyset.$$

**Lemma 4.1.3.** *(i) Let $x \in T(K)$ and set $\lambda := \mathrm{trop}_T(x)$. Then $\tau_\lambda \circ \mathrm{trop}_T = \mathrm{trop}_T \circ \tau_x$.*

*(ii) Let $\Omega \subseteq N_{\mathbb{R}}$ be a $\Lambda$-small open subset. Then $\mathrm{trop}_T^{-1}(\Omega)$ is a $M$-small open subset of $T^{\mathrm{an}}$.*

*Proof.* For part (i) let $T = \mathrm{Spec}(K[Z_1^{\pm 1}, \dots, Z_r^{\pm 1}])$ and $x = (x_1, \dots, x_r) \in T(K) = (K^\times)^r$. Consider the morphism of $K$-algebras

$$f_x \colon K[Z_1^{\pm 1}, \dots, Z_r^{\pm 1}] \to K[Z_1^{\pm 1}, \dots, Z_r^{\pm 1}], \quad Z_i \mapsto Z_i \cdot x_i$$

which induces the translation morphism $\tau_x^{\mathrm{alg}}$ on $T$. Note that $\tau_x \colon T^{\mathrm{an}} \to T^{\mathrm{an}}$ is then given by

$$| \cdot |_y \to \tau_x^{\mathrm{an}}(| \cdot |_x) = | \cdot |_y \circ f_x.$$

Let $| \cdot |_y \in T^{\mathrm{an}}$. By [CLD12, (2.2.2)] invertible analytic functions on a torus are given by monomials. Hence it suffices to check the claim for $Z_i$ for every $i = 1, \dots, r$

$$\tau_x(| \cdot |_y)(Z_i) = | \cdot |_y \circ f_x(Z_i) = |x_i \cdot Z_i|_y = |x_i| \cdot |Z_i|_y = |Z_i|_x |Z_i|_y.$$

Hence

$$\begin{aligned} \mathrm{trop}_T(\tau_x(| \cdot |_y)) &= (-\log(\tau_x(|Z_1|_y)), \dots, -\log(\tau_x(|Z_r|_y))) \\ &= (-\log(|Z_1|_x|Z_1|_y), \dots, -\log(|Z_r|_x|Z_r|_y) \\ &= (-\log(|Z_1|_y), \dots, -\log(|Z_n|_y)) + \lambda = \tau_\lambda(\mathrm{trop}_T(| \cdot |_y)), \end{aligned}$$

where $\lambda := \mathrm{trop}_T(x)$.

For part (ii) let $m \in M$ be non-trivial and put $\lambda := \mathrm{trop}_T(m)$ which is non-trivial since $M$ maps bijectively onto $\Lambda$. By part (i) we get

$$\tau_m(\mathrm{trop}_T^{-1}(\Omega)) \cap \mathrm{trop}_T^{-1}(\Omega) = \mathrm{trop}_T^{-1}(\tau_\lambda(\Omega)) \cap \mathrm{trop}_T^{-1}(\Omega) = \emptyset$$

since by assumption we have $\tau_\lambda(\Omega) \cap \Omega = \emptyset$. □

**Remark 4.1.4.** Note that for all $m \in M$ we have $p \circ \tau_m = p$ and if $V$ is a $M$-small open subset of $T^{\mathrm{an}}$, then so is $\tau_m^{-1}(V)$. Moreover, since $M$ acts freely and properly discontinuously on $T^{\mathrm{an}}$, for every point there is an $M$-small neighborhood.

## 4.2 Refined tropical charts

In this section we introduce the family of refined tropical charts on $A^{\mathrm{an}}$. These are tropical charts that also encode information about the torus $T^{\mathrm{an}}$ and induce tropical charts in the sense of Definition 3.2.9.

**Definition 4.2.1.** Let $a \in A^{\mathrm{an}}$. A *refined tropical chart* around $a$ in $A^{\mathrm{an}}$ is a tuple $(\Omega, F, V, \varphi_U)$ where

- the subset $\Omega$ is a $\Lambda$-small open neighborhood of $\mathrm{trop}_T(t)$ in $N_{\mathbb{R}}$, for an element $t \in p^{-1}(\{a\}) \subseteq T^{\mathrm{an}}$;

- the pair $(V, \varphi_U)$ is a tropical chart around $a$ on $A^{\mathrm{an}}$;
- the subset $V$ is open subset of $A^{\mathrm{an}}$ with $V \subseteq p(\mathrm{trop}_T^{-1}(\Omega))$;
- the map $F\colon N_{U,\mathbb{R}} \to N_{\mathbb{R}}$ is an integral $\mathbb{R}$-affine map such that the following diagram

$$\begin{array}{ccccc} \widetilde{V} & \xrightarrow{i} & T^{\mathrm{an}} & \xrightarrow{\mathrm{trop}_T} & N_{\mathbb{R}} \\ {\scriptstyle p}\downarrow & & & & \uparrow{\scriptstyle F} \\ V & \xrightarrow[\varphi_U^{\mathrm{an}}|_V]{} & T_U^{\mathrm{an}} & \xrightarrow[\mathrm{trop}_{T_U}]{} & N_{U,\mathbb{R}} \end{array} \tag{4.1}$$

commutes, where $\widetilde{V} := p|^{-1}_{\mathrm{trop}_T^{-1}(\Omega)}(V)$ and $i\colon \widetilde{V} \hookrightarrow T^{\mathrm{an}}$ denotes the natural inclusion. In short, we have $F \circ \varphi_{U,\mathrm{trop}} \circ p = \mathrm{trop}_T \circ i$ on $\widetilde{V}$.

**Lemma 4.2.2.** *Let $(\Omega, F, V, \varphi_U)$ be a refined tropical chart around $a \in A^{\mathrm{an}}$. Then $(\tau_\lambda(\Omega), \tau_\lambda^{-1} \circ F, V, \varphi_U)$ is also a refined tropical chart around $a$ for every $\lambda \in \Lambda$.*

*Proof.* Let $t \in p^{-1}(\{a\})$ be such that $\Omega$ is a $\Lambda$-small open neighborhood of $\mathrm{trop}_T(t)$. Then $\tau_\lambda(\Omega)$ is a $\Lambda$-small open neighborhood of $\tau_\lambda(t)$. Let $m \in M$ be such that $\mathrm{trop}_T(m) = \lambda$ then by Remark 4.1.4 we have

$$p(\mathrm{trop}_T^{-1}(\tau_\lambda(\Omega))) = p(\tau_m(\mathrm{trop}_T^{-1}(\Omega))) = p(\mathrm{trop}_T^{-1}(\Omega)),$$

hence $V$ is an open neighborhood of $a$ with $V \subseteq p(\mathrm{trop}_T^{-1}(\tau_\lambda(\Omega)))$. Put

$$\widetilde{V} := p|^{-1}_{\mathrm{trop}_T^{-1}(\Omega)}(V), \text{ and } \widetilde{V}_\lambda := p|^{-1}_{\mathrm{trop}_T^{-1}(\tau_\lambda(\Omega))}(V)$$

and let $i\colon \widetilde{V} \hookrightarrow T^{\mathrm{an}}$ and $i_m\colon \widetilde{V}_\lambda \hookrightarrow T^{\mathrm{an}}$ denote the canonical inclusions. Then we have $i_m = \tau_m^{-1} \circ i \circ \tau_m^{-1}$ on $\widetilde{V}_\lambda$ and

$$\begin{aligned} \mathrm{trop}_T \circ i_m &= \mathrm{trop}_T \circ \tau_m^{-1} \circ i \circ \tau_m^{-1} = \tau_\lambda^{-1} \circ \mathrm{trop}_T \circ i \circ \tau_m^{-1} \\ &= \tau_\lambda^{-1} \circ F \circ \varphi_{U,\mathrm{trop}} \circ p \circ \tau_m^{-1} = \tau_\lambda^{-1} \circ F \circ \varphi_{U,\mathrm{trop}} \circ p, \end{aligned}$$

where in the second equality we used the relation (4.1), Lemma 4.1.3 and that $\tau_m^{-1}(\widetilde{V}_\lambda) \subseteq \widetilde{V}$. Hence $(\tau_\lambda(\Omega), \tau_\lambda^{-1} \circ F, V, \varphi_U)$ is a refined tropical chart around $a$. □

**Definition 4.2.3.** Let $(\Omega, F, V, \varphi_U)$ be a refined tropical chart around $a$ in $A^{\mathrm{an}}$. We say that $(\Omega', F', V', \varphi_{U'})$ is a *refined tropical subchart* of $(\Omega, F, V, \varphi_U)$ if $\Omega' \subseteq \Omega$ and if $(V', \varphi_{U'})$ is a tropical subchart of $(V, \varphi_U)$.

**Remark 4.2.4.** Let $(\Omega', F', V', \varphi_{U'})$ be a refined tropical subchart of a refined tropical chart $(\Omega, F, V, \varphi_U)$. Since $(V', \varphi_{U'})$ is a tropical subchart of $(V, \varphi_U)$, by 3.2.4 there exists a canonical integral $\mathbb{R}$-affine morphism $G\colon N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ such that $G \circ \varphi_{U',\mathrm{trop}} = \varphi_{U,\mathrm{trop}}$ on $(U')^{\mathrm{an}}$.

**- 4.2.5.** In what follows we want to show that for every point $a$ of $A^{\mathrm{an}}$ there exists a refined tropical chart.

**Lemma 4.2.6.** *For every $t \in T^{\mathrm{an}}$ there exists an $M$-small open neighborhood of $t$ of the form $\mathrm{trop}_T^{-1}(\Omega)$ for a $\Lambda$-small open subset $\Omega$ in $N_{\mathbb{R}}$.*

*Proof.* Since $\Lambda$ is a discrete subgroup of $N_{\mathbb{R}}$ it acts freely and properly discontinuously on $N_{\mathbb{R}}$. Hence there exists a $\Lambda$-small open neighborhood $\Omega$ around $\mathrm{trop}_T(t)$. By Lemma 4.1.3(ii) we conclude that $\mathrm{trop}_T^{-1}(\Omega)$ is $M$-small. □

**Proposition 4.2.7.** *For every point $a \in A^{\mathrm{an}}$ there exists a refined tropical chart $(\Omega, F, V, \varphi_U)$ on $A^{\mathrm{an}}$ with $a \in V$.*

*Proof.* Fix a point $t \in p^{-1}(\{a\}) \subseteq T^{\mathrm{an}}$. Let $\Omega$ be a $\Lambda$-small open subset of $N_{\mathbb{R}}$ as in Lemma 4.2.6. Then by Lemma 4.1.3(ii), the subset $\widetilde{V}' := \mathrm{trop}_T^{-1}(\Omega)$ is $M$-small. Put $V' := p(\widetilde{V}')$ and note that $p|_{\widetilde{V}'}\colon \widetilde{V} \to V'$ is an isomorphism. Define the analytic moment map $f_{V'}$ as the composition

$$f_{V'}\colon V' \xrightarrow{p|_{\widetilde{V}'}^{-1}} \widetilde{V}' \xrightarrow{i_{\widetilde{V}'}} T^{\mathrm{an}}$$

and on $\widetilde{V}'$ we have $f_{V'} \circ p = i_{\widetilde{V}'}$, where $i_{\widetilde{V}'}\colon \widetilde{V}' \hookrightarrow T^{\mathrm{an}}$ denotes the canonical inclusion. By Lemma 3.2.15 there exists a triple $(F, V, \varphi_U)$ where $(V, \varphi_U)$ is a tropical chart with $a \in V \subseteq V'$ and $F\colon N_{U,\mathbb{R}} \to N_{\mathbb{R}}$ is an integral $\mathbb{R}$-affine morphism with $F \circ \varphi_{U,\mathrm{trop}} = f_{V',\mathrm{trop}}$ on $V$. The tuple

$$(\Omega, F, V, \varphi_U)$$

is a refined tropical chart around $a$. We show that $F$ fits into the commutative diagram (4.1). Put $\widetilde{V} := p|_{\widetilde{V}}^{-1}(V)$ and denote by

$$i\colon \widetilde{V} \to \widetilde{V}' \xrightarrow{i_{\widetilde{V}'}} T^{\mathrm{an}}$$

the canonical inclusion. Since on $V'$ we have $\mathrm{trop}_T \circ i_{\widetilde{V}'} \circ p|_{\mathrm{trop}_T^{-1}(\Omega)}^{-1} = F \circ \varphi_{U,\mathrm{trop}}$ and since $V \subseteq V'$ we conclude that on $\widetilde{V}$ we have $F \circ \varphi_{U,\mathrm{trop}} \circ p = \mathrm{trop}_T \circ i$. □

**Lemma 4.2.8.** *Let $(\Omega, F, V, \varphi_U)$ and $(\Omega', F', V', \varphi_{U'})$ be refined tropical charts around $a$ in $A^{\mathrm{an}}$. Then for every point $y \in V \cap V'$ there exists a refined tropical chart $(\Omega'', F'', V'', \varphi_{U''})$ with $y \in V''$ which is a refined tropical subchart of both $(\Omega, F, V, \varphi_U)$ and $(\tau_\lambda(\Omega'), \tau_\lambda^{-1} \circ F', V', \varphi_{U'})$ for some $\lambda \in \Lambda$.*

*Proof.* First we show that there exists a $\Lambda$-small open subset $\Omega''$ of $N_{\mathbb{R}}$ which lies in $\mathrm{trop}_T^{-1}(\Omega) \cap \mathrm{trop}_T^{-1}(\tau_\lambda(\Omega'))$ for some $\lambda \in \Lambda$. Let $t, t' \in p^{-1}(\{y\}) \subseteq T^{\mathrm{an}}$ be such that $\Omega$ respectively $\Omega'$ are $\Lambda$-small open neighborhoods of $\mathrm{trop}_T(t)$ respectively of $\mathrm{trop}_T(t')$. By Remark 4.1.4 it follows that there exists $\lambda \in \Lambda$ such that $\mathrm{trop}_T(t) = \tau_\lambda(\mathrm{trop}_T(t'))$ and $\Omega \cap \tau_\lambda(\Omega') \neq \emptyset$. Hence the intersection $\Omega \cap \tau_\lambda(\Omega')$ is an open neighborhood of $\mathrm{trop}_T(t)$ in $N_{\mathbb{R}}$. By Lemma 4.2.6 there exists a $M$-small open neighborhood $\mathrm{trop}_T^{-1}(\widetilde{\Theta})$ of $t$, where $\widetilde{\Theta}$ is a $\Lambda$-small open subset of $N_{\mathbb{R}}$. Put

$$\Omega'' := \widetilde{\Theta} \cap \Omega \cap \tau_\lambda(\Omega').$$

Next we show that there exists a tropical chart $(V'', \varphi_{U''})$ which is a tropical subchart of $(V \cap V', \varphi_{U \cap U'})$. The composition $f_{V \cap V'}$

$$V \cap V' \to p^{-1}(\mathrm{trop}_T^{-1}(\Omega) \cap \mathrm{trop}_T^{-1}(\tau_\lambda(\Omega'))) \to \mathrm{trop}_T^{-1}(\Omega) \cap \mathrm{trop}_T^{-1}(\tau_\lambda(\Omega')) \to T^{\mathrm{an}}$$

is an analytic moment map. By Lemma 3.2.15 there exists a triple $(F'', V'', \varphi_{U''})$, where $(V'', \varphi_{U''})$ is a tropical chart with $a \in V'' \subseteq V \cap V'$ and an integral $\mathbb{R}$-affine morphism $F'' : N_{U'',\mathbb{R}} \to N_{\mathbb{R}}$ such that $F'' \circ \varphi_{U'',\mathrm{trop}} = f_{V \cap V',\mathrm{trop}}$ on $V''$. Since $V'' \subseteq (U'')^{\mathrm{an}}$, $V'' \subseteq (U)^{\mathrm{an}}$ and $V'' \subseteq (U')^{\mathrm{an}}$, we have $V'' \subseteq (U'')^{\mathrm{an}}$. By Proposition 3.2.10 after replacing $U''$ by $U \cap U' \cap U''$ conclude that $(V'', \varphi_{U''})$ is a tropical subchart of $(V \cap V', \varphi_{U \cap U'})$. Hence the refined tropical chart $(\Omega'', F'', V'', \varphi_{U''})$ is a refined tropical subchart of both $(\Omega, F, V, \varphi_U)$ and $(\tau_\lambda(\Omega'), \tau_\lambda^{-1} \circ F', V', \varphi_{U'})$. □

**Lemma 4.2.9.** *With the same notation as in Lemma 4.2.8 we have*

$$F \circ G = F'' = \tau_\lambda^{-1} \circ F' \circ G'$$

*on* $\varphi_{U'',\mathrm{trop}}(V'')$, *where* $G \colon N_{U'',\mathbb{R}} \to N_{U',\mathbb{R}}$ *and* $G' \colon N_{U'',\mathbb{R}} \to N_{U',\mathbb{R}}$ *are the canonical integral* $\mathbb{R}$*-affine morphisms from Remark 4.2.4.*

*Proof.* For convenience we introduce the following notation. Let $(\Omega, F, V, \varphi_U)$ be a refined tropical chart in $A^{\mathrm{an}}$ with the same notation as in (4.1) we will denote by $i_{\widetilde{V}}$ the canonical inclusion $i_{\widetilde{V}} \colon \widetilde{V} \hookrightarrow T^{\mathrm{an}}$. Put $\widetilde{V}_\lambda := p|^{-1}_{\mathrm{trop}_T^{-1}(\tau_\lambda(\Omega))}(V)$ and let $v \in \varphi_{U'',\mathrm{trop}}(V'')$ and $x \in V''$ with $\varphi_{U'',\mathrm{trop}}(x) = v$. Then

$$\begin{aligned} F''(v) &= F''(\varphi_{U'',\mathrm{trop}}(x)) = \mathrm{trop}_T \circ i_{\widetilde{V}''} \circ p^{-1}(x) = \mathrm{trop}_T \circ i_{\widetilde{V}} \circ p^{-1}(x) \\ &= F \circ \varphi_{U,\mathrm{trop}} \circ p \circ p^{-1}(x) = F \circ G \circ \varphi_{U'',\mathrm{trop}}(x) \\ &= F \circ G(v). \end{aligned}$$

and

$$\begin{aligned} F''(v) &= F''(\varphi_{U'',\mathrm{trop}}(x)) = \mathrm{trop}_T \circ i_{\widetilde{V}''} \circ p^{-1}(x) = \mathrm{trop}_T \circ i_{\widetilde{V}_\lambda} \circ p^{-1}(x) \\ &= \tau_\lambda^{-1} \circ F' \circ \varphi_{U',\mathrm{trop}} \circ p \circ p^{-1}(x) = \tau_\lambda^{-1} \circ F' \circ G' \circ \varphi_{U'',\mathrm{trop}}(x) \\ &= \tau_\lambda^{-1} \circ F' \circ G'(v). \end{aligned}$$

□

## 4.3 Cohomological results

In this section we will prove the main result of this chapter. First we will define a canonical morphism
$$\Phi_A^{p,q}\colon A_{\mathrm{cl}}^{p,q}(N_{\mathbb{R}})^{\Lambda} \to \mathrm{H}_{d''}^{p,q}(A^{\mathrm{an}}).$$
Then by using integration, in Theorem 4.3.7 we will show that $\Phi_A^{p,q}$ is injective.

Recall from Example 3.1.6 that for $p, q \in \mathbb{Z}$ we denote by $A^{p,q}(N_{\mathbb{R}})^{\Lambda}$ (respectively $A_{\mathrm{cl}}^{p,q,}(N_{\mathbb{R}})^{\Lambda}$) the space of ($d''$-closed) $\Lambda$-invariant superforms on $N_{\mathbb{R}}$.

**Proposition 4.3.1.** *There exists a canonical morphism*

$$\phi_A^{p,q}\colon A^{p,q}(N_{\mathbb{R}})^{\Lambda} \to A^{p,q}(A^{\mathrm{an}}),$$

*compatible with the differentials $d', d''$ which induces a morphism of real vector spaces*
$$\Phi_A^{p,q}\colon A_{\mathrm{cl}}^{p,q}(N_{\mathbb{R}})^{\Lambda} \to \mathrm{H}_{d''}^{p,q}(A^{\mathrm{an}}).$$

*Proof.* Let $\alpha \in A^{p,q}(N_{\mathbb{R}})^{\Lambda}$. By Lemma 4.2.7 there exist refined tropical charts $(\Omega_i, F_i, V_i, \varphi_{U_i})_{i\in I}$, where $(V_i)_{i\in I}$ is an open cover of $A^{\mathrm{an}}$, and for every $i \in I$ the morphism $F_i$ denotes an integral $\mathbb{R}$-affine morphism as in Definition 4.2.1 and $(V_i, \varphi_{U_i})$ is a tropical chart on $A^{\mathrm{an}}$. Then we define $\phi_A^{p,q}(\alpha)$ to be given by

$$(V_i, \varphi_{U_i}, F_i^*\alpha)_{i\in I} \tag{4.2}$$

We show that for all $i, j \in I$ we have $\alpha_i|_{V_i\cap V_j} = \alpha_j|_{V_i\cap V_j}$. For every point $y \in V_i \cap V_j$ let $(\Omega'', F'', V'', \varphi_{U''})$ be a refined tropical chart around $y$ as in Lemma 4.2.8. By Lemma 4.2.9 there exists $\lambda \in \Lambda$ such that

$$F_i \circ G_i = F'' = \tau_\lambda^{-1} \circ F_j \circ G_j$$

on $\varphi_{U'',\mathrm{trop}}(V'')$, where $G_i\colon N_{U'',\mathbb{R}} \to N_{U_i,\mathbb{R}}$ and $G_j\colon N_{U'',\mathbb{R}} \to N_{U_j,\mathbb{R}}$ are canonical integral $\mathbb{R}$-affine morphisms. Then

$$\begin{aligned} F_i^*\alpha|_{V''} &= G_i^* F_i^*\alpha = (F'')^*\alpha \\ &= G_j^* F_j^* \tau_\lambda^{-1,*}\alpha = G_j^* F_j^*\alpha = F_j^*\alpha|_{V''}, \end{aligned}$$

where in the second to last equality we used the $\Lambda$-invariance of $\alpha$. We can cover $V_i \cap V_j$ by open subsets given as in Lemma 4.2.8 and since $A^{p,q}$ is a sheaf, we get compatibility on overlaps. The same argument also shows that the definition of $\phi_A^{p,q}(\alpha)$ does not depend on the chosen covering. Compatibility with the differentials $d'$ and $d''$ follows from Remarks 3.1.3(v) and 3.2.18(i). □

**- 4.3.2.** Let $\alpha \in A^{p,q}(N_\mathbb{R})$ and $(V_i, \varphi_{U_i})_{i\in I}$ be an open cover of $T^{\mathrm{an}}$. For every $i \in I$, the subset $U_i$ is a very affine subset of $T$, therefore we have an algebraic moment map $\varphi_i \colon U_i \to T$ and by Remark 3.2.5 it follows that for every $i \in I$ there exists an integral $\mathbb{R}$-affine morphism $F_i \colon N_{U_i,\mathbb{R}} \to N_\mathbb{R}$ such that $F_i \circ \varphi_{U_i,\mathrm{trop}} = \varphi_{i,\mathrm{trop}}$ on $U_i^{\mathrm{an}}$. Then the family of triples $(V_i, \varphi_{U_i}, F_i^*\alpha)_{i\in I}$ defines a differential form of bidegree $(p, q)$ on $T^{\mathrm{an}}$. After possibly passing to a common refinement one sees that this construction does not depend on the chosen cover, hence we get a canonical morphism of real vector spaces

$$\phi_T \colon A^{p,q}(N_\mathbb{R}) \to A^{p,q}(T^{\mathrm{an}}), \alpha \mapsto [(V_i, \varphi_{U_i}, F_i^*\alpha)_{i\in I}]. \tag{4.3}$$

**Lemma 4.3.3.** *Let* $\alpha \in A^{p,q}(N_\mathbb{R})^\Lambda$*, then* $p^*\phi_A^{p,q}(\alpha) = \phi_T^{p,q}(\alpha)$ *in* $A^{p,q}(T^{\mathrm{an}})$.

*Proof.* This follows from the definitions. Let $\phi_A^{p,q}(\alpha)$ be given by $(W_i, \varphi_{Z_i}, F_i'^*\alpha)_{i\in I}$ as in (4.2) in particular we have

$$F_i \circ \varphi_{Z_i,\mathrm{trop}} \circ p = \mathrm{trop}_T \circ i$$

on $\widetilde{W}_i$ (see (4.1)). By the description of the pull-back in (3.2) the form $p^*\phi_A^{p,q}(\alpha)$ is given by

$$(V_{ij}, \varphi_{U_{ij}}, (F'_{ij})^*(F'_i)^*\alpha)_{i\in I, j\in J},$$

where $F'_{ij} \colon N_{U_{ij},\mathbb{R}} \to N_\mathbb{R}$ is an integral $\mathbb{R}$-affine morphism with

$$F'_{ij} \circ \varphi_{U_{ij},\mathrm{trop}} = \mathrm{trop}_T \circ i'$$

on $V_{ij}$, where $i' \colon p^{-1}(W_i) \to T_{Z_i}^{\mathrm{an}}$ is an analytic moment map. After passing to a common refinement, we may assume that $\phi_T^{p,q}(\alpha)$ is given by $(V_{ij}, \varphi_{U_{ij}}, F_{ij}^*\alpha)_{i\in I, j\in J}$. Since $V_{ij} \subseteq \widetilde{W}_i$ it follows that on $\varphi_{U_{ij},\mathrm{trop}}(V_{ij})$ we have $F'_i \circ F'_{ij} = F_{ij}$. Then for all $i \in I, j \in J$ we have

$$F_{ij}^*\alpha = (F'_{ij})^*(F'_i)^*\alpha,$$

hence $p^*\phi_A^{p,q}(\alpha) = \phi_T^{p,q}(\alpha)$ in $A^{p,q}(T^{\mathrm{an}})$. □

**- 4.3.4.** Let $\pi \colon N_\mathbb{R} \to N_\mathbb{R}/\Lambda$ be the universal covering. By [Deb05, § 4] there is a bijection between superforms in $A^{p,q}(N_\mathbb{R})^\Lambda$ and superforms in $A^{p,q}(N_\mathbb{R}/\Lambda)$. For $\alpha \in A^{p,q}(N_\mathbb{R})^\Lambda$ denote by $\alpha'$ the corresponding form in $A^{p,q}(N_\mathbb{R}/\Lambda)$.

**Lemma 4.3.5.** *Let* $(\Omega, F, V, \varphi_U)$ *be a refined tropical chart in* $A^{\mathrm{an}}$*. With the same notation as in* (4.3.4) *let* $\alpha \in A^{r,r}(N_\mathbb{R})^\Lambda$ *be a superform with* $\mathrm{supp}(\alpha') \subseteq \pi(\Omega)$ *and suppose that* $\phi_A^{r,r}(\alpha) \in A^{r,r}(A^{\mathrm{an}})$ *is given by a single form* $(V, \varphi_U, \alpha_U)$ *in the sense of Definition 3.2.22. Then*

$$\int_{A^{\mathrm{an}}} \phi_A^{r,r}(\alpha) = \int_{N_\mathbb{R}/\Lambda} \alpha'.$$

*Proof.* First note that by Lemma 4.3.3 we have $\phi_T^{r,r}(\alpha) = p^*\phi_A^{r,r}(\alpha)$. Since $\phi_A^{r,r}(\alpha)$ is given by a single chart by Corollary 3.2.24 we have $\mathrm{supp}(\phi_A^{r,r}(\alpha)) \subseteq V$. By definition of refined tropical chart $V \subseteq p^{-1}(\mathrm{trop}_T^{-1}(\Omega))$ and the restriction

$$p\colon \mathrm{trop}_T^{-1}(\Omega) \to p^{-1}(\mathrm{trop}_T^{-1}(\Omega))$$

is an isomorphism. Therefore by functoriality of the pull-back along analytic morphisms and Lemma 3.2.25(ii) we conclude that

$$\mathrm{supp}(\phi_T^{r,r}(\alpha)) = \mathrm{supp}(p^*\phi_A^{r,r}(\alpha)) \subseteq p^{-1}(V)$$

and the support is compact since $p$ is an isomorphism. Then $(T^{\mathrm{an}}, \alpha)$ is a very affine chart of integration for $\phi_T^{r,r}(\alpha)$. Hence

$$\int_{N_{\mathbb{R}}/\Lambda} \alpha' = \int_{\Omega} \alpha = \int_{N_{\mathbb{R}}} \alpha = \int_{T^{\mathrm{an}}} \phi_T^{r,r}(\alpha), \tag{4.4}$$

where in the second equality we used the $\Lambda$-invariance of $\alpha$ and in the third we use Definition 3.4.11. By Lemma 4.3.3 we get

$$\int_{T^{\mathrm{an}}} \phi_T^{r,r}(\alpha) = \int_{T^{\mathrm{an}}} p^*\phi_A^{r,r}(\alpha).$$

The proposition then follows from the following equality

$$\int_{V} \phi_A^{r,r}(\alpha) = \int_{p^{-1}(V)} p^*\phi_A^{r,r}(\alpha). \tag{4.5}$$

Let $\phi_A^{r,r}(\alpha)$ be given by

$$(V, \varphi_U, \alpha_U),$$

where $\alpha_U = F^*\alpha$. By Lemma 3.2.26, we have $\mathrm{supp}(\phi_A^{r,r}(\alpha)) \subseteq U^{\mathrm{an}}$. Hence $(U, \alpha_U)$ is a very affine chart of integration for $\phi_A^{r,r}(\alpha)$. By (3.2) the pull-back $p^*\phi_A^{r,r}(\alpha)$ is given by $(V_i, \varphi_{U_i}, \mathrm{Trop}(\psi_{U,U_i})^*F^*\alpha)_{i\in I}$. By Lemma 3.2.26 we have that $(U_i, \mathrm{Trop}(\psi_{U,U_i})^*F^*\alpha)$ is a very affine chart of integration for some $i \in I$. The projection formula in 3.4.7 implies

$$\begin{aligned}\int_{p^{-1}(V)} p^*\phi_A^{r,r}(\alpha) &= \int_{\varphi_{U_i,\mathrm{trop}}(U_i^{\mathrm{an}})} \mathrm{Trop}(\psi_{U,U_i})^*F^*\alpha \\ &= \int_{\mathrm{Trop}(\psi_{U,U_i})_*\varphi_{U_i,\mathrm{trop}}(U_i^{\mathrm{an}})} F^*\alpha.\end{aligned}$$

Finally, by the Sturmfels-Tevelev multiplicity formula ([Gub16, Proposition 4.11.]) we have

$$\int_{\mathrm{Trop}(\psi_{U,U_i})_*\varphi_{U_i,\mathrm{trop}}(U_i^{\mathrm{an}})} \alpha_U = \int_{\varphi_{U,\mathrm{trop}}(U^{\mathrm{an}})} F^*\alpha = \int_V \phi_A^{r,r}(\alpha).$$

By Lemma 3.4.12 the integral does not depend on the chosen very affine chart of integration, hence (4.5) holds.

□

**Proposition 4.3.6.** *Let $\alpha \in A^{r,r}(N_{\mathbb{R}})^{\Lambda}$. Then*

$$\int_{A^{\mathrm{an}}} \phi_A^{r,r}(\alpha) = \int_{N_{\mathbb{R}}/\Lambda} \alpha.$$

*Proof.* Let $\phi_A^{r,r}(\alpha)$ be given by $(V_i, \varphi_{U_i}, F_i^* \alpha)_{i \in I}$ as in (4.2). Since $\pi \colon N_{\mathbb{R}} \to N_{\mathbb{R}}/\Lambda$ is a covering map and $\Omega_i$ is $\Lambda$-small for every $i \in I$ the subset $\pi(\Omega_i) =: \Theta_i$ is an open subset of $N_{\mathbb{R}}/\Lambda$. Since $(\Omega_i)_{i \in I}$ is an open cover of $N_{\mathbb{R}}$, then $(\Theta_i)_{i \in I}$ is an open cover of $N_{\mathbb{R}}/\Lambda$. Let $(\chi_k)_{k \in K}$ be a partition of unity with compact supports on $N_{\mathbb{R}}/\Lambda$ subordinate to the cover $(\Theta_i)_{i \in I}$. This means that there exists a map $s \colon K \to I$ such that for $k \in s^{-1}(i)$ we have $\operatorname{supp}(\chi_k) \subseteq \Omega_i$. By (4.3.4) we first observe that $\pi^* \chi_k \in C^{\infty}(N_{\mathbb{R}})^{\Lambda}$ for every $k \in K$. Fix $k \in s^{-1}(i)$, put $\beta_k := \pi^* \chi_k \alpha$ and note that in the notation of (4.3.4) we have $\beta_k' = \chi_k \alpha'$. Then

$$\operatorname{supp}(\beta_k') \subseteq \Theta_i.$$

After passing to a common refinement we may assume that

$$\phi_A^{0,0}(\pi^* \chi_k) = [(V_i, \varphi_{U_i}, F_i^* \pi^* \chi_k)_{i \in I}].$$

Then as in Remark 3.2.18(i)

$$\phi_A^{0,0}(\chi_k) \cdot \phi_A^{r,r}(\alpha) = [(V_i, \varphi_{U_i}, F_i^* \pi^* \chi_k \cdot F_i^* \alpha)_{i \in I}] = [(V_i, \varphi_{U_i}, F_i^* (\pi^* \chi_k \cdot \alpha))_{i \in I}] = \phi_A^{r,r}(\beta_k).$$

In particular $\phi_A^{r,r}(\beta_k)$ is given by a single form $(V_i, \varphi_{U_i}, F_i^* \beta_k)$. This means that we are in the situation of Lemma 4.3.5, and we get

$$\int_{A^{\mathrm{an}}} \phi_A^{r,r}(\beta_k) = \int_{N_{\mathbb{R}}/\Lambda} \beta_k'.$$

Note that we have $\sum_{k \in K} \phi_A^{0,0}(\pi^* \chi_k) = 1$, indeed

$$\sum_{k \in K} \phi_A^{0,0}(\pi^* \chi_k) = [(V_i, \varphi_{U_i}, \sum_{k \in K} F_i^* \pi^* \chi_k)_{i \in I}] = [(V_i, \varphi_{U_i}, F_i^* \pi^* (\sum_{k \in K} \chi_k)_{i \in I}] = 1.$$

Hence

$$\int_{N_{\mathbb{R}}/\Lambda} \alpha' = \sum_{i \in I} \sum_{k \in s^{-1}(i)} \int_{\Omega_i} \beta_k = \sum_{i \in I} \sum_{k \in s^{-1}(i)} \int_{V_i} \phi_A^{0,0}(\pi^* \chi_k) \cdot \phi_A^{r,r}(\alpha) = \int_{A^{\mathrm{an}}} \phi_A^{r,r}(\alpha).$$

□

**Theorem 4.3.7.** *Let $p, q \in \mathbb{Z}$. The morphism of real vector spaces*

$$\Phi_A^{p,q} \colon \Lambda^{p,q} N_{\mathbb{R}}^* \longrightarrow \mathrm{H}_{d''}^{p,q}(A^{\mathrm{an}}), \tag{4.6}$$

*is injective.*

*Proof.* First assume that $\max\{p, q\} > r$ or $p < 0$ or $q < 0$, then both sides are zero and the claim trivially holds. Hence suppose that $0 \leq p, q \leq r$. Next we fix the notation. Let $I \subseteq \{1, \ldots, r\}$ be a subset. We will denote by $I^c := \{1, \ldots, r\} \setminus I$ its complementary set. Let

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} d'x_I \wedge d''x_J$$

be a superform on $N_{\mathbb{R}}$ with constant coefficients. Let $I, J \subseteq \{1, \ldots, r\}$ be such that $|I| = p$ and $|J| = q$ and consider the form $\beta^{IJ} = d'x_{I^c} \wedge d''x_{J^c}$ which is $d''$-closed in $A^{r-p,r-q}(N_{\mathbb{R}})^{\Lambda}$. In particular $\phi_A^{r,r}(\alpha \wedge \beta^{IJ})$ is $d''$-closed in $A^{r,r}(A^{\mathrm{an}})$. If $\Phi_A^{r,r}(\alpha)$ is $d''$-exact in $A^{p,q}(A^{\mathrm{an}})$, then by the Leibniz rule, so is $\phi_A^{r,r}(\alpha \wedge \beta^{IJ})$. Let vol denote the Haar measure on $N_{\mathbb{R}}$ normalized such that $\mathrm{vol}(N_{\mathbb{R}}/\Lambda) = 1$. With the notation from 3.4.1 we write $\alpha \wedge \beta^{IJ} = f d\mathrm{vol}(x_1, \ldots, x_r)$ where $f \in \mathbb{R}$. Since $\alpha \wedge \beta^{IJ}$ has constant coefficients, Proposition 4.3.6 implies

$$0 = \int_{A^{\mathrm{an}}} \phi_A^{r,r}(\alpha \wedge \beta^{IJ}) = f \cdot \mathrm{vol}(N_{\mathbb{R}}/\Lambda) = f,$$

where the first equality follows from Proposition 3.4.13 as $\Phi_A^{r,r}(\alpha \wedge \beta^{IJ})$ is $d''$-exact. Since this holds for all subsets $I, J$ with $|I| = p$ and $|J| = q$, we conclude that $\alpha = 0$. □

**Corollary 4.3.8.** *Let $A$ be an abelian variety with totally degenerate reduction of dimension $n$. Then for $0 \leq p, q \leq n$, the tropical Dolbeault cohomology is non-trivial. In fact we have*

$$\dim_{\mathbb{R}} \mathrm{H}_{d''}^{p,q}(A^{\mathrm{an}}) \geq \binom{n}{p} \cdot \binom{n}{q}. \tag{4.7}$$

*Moreover for $p = 0$ and $q \in \{0, \ldots, n\}$ the cohomology groups are finite-dimensional and for $p = 0$ and $q \in \{0, 1\}$ the inequality in (4.7) is an equality.*

*Proof.* The inequality (4.7) immediately follows from Theorem 4.3.7. The first part of the last statement follows from Proposition 3.3.6. Finally by Theorem 3.3.4 we have a canonical isomorphism

$$\mathrm{H}_{d''}^{0,1}(A^{\mathrm{an}}) \cong \mathrm{H}_{\mathrm{sing}}^{1}(A^{\mathrm{an}}, \mathbb{R})$$

By the universal coefficient theorem we also have a canonical isomorphism

$$\mathrm{H}^1_{\mathrm{sing}}(A^{\mathrm{an}}, \mathbb{R}) \cong \mathrm{Hom}(\mathrm{H}_1(A^{\mathrm{an}}, \mathbb{R}), \mathbb{R})$$

and the claim now follows from the observations at the end of 2.2.2 since $\pi_1(A^{\mathrm{an}})$ has rank $n$. □

**Remark 4.3.9.** An abelian variety $A$ over $K$ of dimension 1 with totally degenerate reduction is a Mumford curve. By [JW18, Theorem 5.1.] the morphism in (4.6) is an isomorphism.

### The Jacobian of a curve

For this section we follow [BR15, § 4].

**- 4.3.10.** Let $X$ be a smooth projective curve over $K$ with $X(K) \neq \emptyset$ and denote by $J$ its Jacobian. Fix a point $x \in X(K)$ and let $\alpha \colon X \to J$ denote the Abel-Jacobi map. Over the complex numbers, Abel's theorem says that we have an isomorphism $\mathrm{H}_1(X(\mathbb{C}), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}_1(J(\mathbb{C}), \mathbb{Z})$. The same holds in the analytic setup.

**Proposition 4.3.11.** *The homomorphism on singular homology groups*

$$\alpha_* \colon \mathrm{H}_1(X^{\mathrm{an}}, \mathbb{Z}) \to \mathrm{H}_1(J^{\mathrm{an}}, \mathbb{Z})$$

*is an isomorphism.*

*Proof.* See [BR15, Proposition 4.7.]. □

**Proposition 4.3.12.** *The Abel-Jacobi map induces an isomorphism on tropical Dolbeault cohomology groups*

$$\alpha^* \colon \mathrm{H}^{0,1}_{d''}(J^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{H}^{0,1}_{d''}(X^{\mathrm{an}}).$$

*Proof.* Consider the following diagram

$$\begin{array}{ccc}
\mathrm{H}^{0,1}_{d''}(J^{\mathrm{an}}) & \longrightarrow & \mathrm{H}^{0,1}_{d''}(X^{\mathrm{an}}) \\
\downarrow & & \downarrow \\
\mathrm{H}^1_{\mathrm{sing}}(J^{\mathrm{an}}, \mathbb{R}) & \longrightarrow & \mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{R}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbb{R}}(\mathrm{H}_1(J^{\mathrm{an}}, \mathbb{R}), \mathbb{R}) & \xrightarrow[\mathrm{Hom}(\alpha_*)]{\cong} & \mathrm{Hom}_{\mathbb{R}}(\mathrm{H}_1(X^{\mathrm{an}}, \mathbb{R}), \mathbb{R})
\end{array}$$

First note that by Theorem 3.3.4 we have a canonical isomorphism

$$\mathrm{H}^{1,0}_{d''}(X^{\mathrm{an}}) \cong \mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{R}).$$

By the universal coefficient theorem we have a canonical isomorphism

$$\mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathrm{H}_1(X^{\mathrm{an}}, \mathbb{Z}), \mathbb{R}).$$

Consider the isomorphism $\alpha_* \colon \mathrm{H}_1(X^{\mathrm{an}}, \mathbb{Z}) \to \mathrm{H}_1(J^{\mathrm{an}}, \mathbb{Z})$ from Proposition 4.3.11. Then $\alpha_*$ extends to an isomorphism

$$\alpha_* \otimes_{\mathbb{Z}} \mathbb{R} \colon \mathrm{H}_1(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{H}_1(J^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

and by the universal coefficient theorem we have

$$\mathrm{H}_1(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{H}_1(X^{\mathrm{an}}, \mathbb{R}) \quad \text{and} \quad \mathrm{H}_1(J^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{H}_1(J^{\mathrm{an}}, \mathbb{R}).$$

By exactness we finally get that also $\mathrm{Hom}(\alpha_*)$ is an isomorphism. By naturality of the sequence in the universal coefficient theorem and construction of the isomorphism from Theorem 3.3.4, both diagrams commute. Since all the vertical arrows and the bottom horizontal arrow are isomorphisms we get that $\alpha^*$ is an isomorphism as well.

□

# Chapter 5

# The first tropical Chern form

## Introduction

Let $X$ be a smooth algebraic variety over $K$. Denote by $\mathrm{CH}^q(X)$ the ring of algebraic cycles of codimension $q$ on $X$ modulo rational equivalence and put $\mathrm{CH}^q(X)_{\mathbb{Q}} := \mathrm{CH}^q(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. For $q \geq 0$ consider the *tropical cycle class map*

$$\mathrm{cl}_{\mathrm{trop}} \colon \mathrm{CH}^q(X)_{\mathbb{Q}} \to \mathrm{H}^{q,q}_{d''}(X^{\mathrm{an}})$$

introduced by Liu in [Liu17a]. In what follows we want to study the tropical cycle class map in the case $q = 1$. We will compare the image of the first Chern class of a line bundle $L$ and the tropical cycle class map with the cohomology class of the first Chern form associated to $L$ endowed with a smooth metric. In fact in Proposition 5.2.3 we will show that they agree.

## 5.1 Tropical cycle class map

We recall the construction of the tropical cycle class map as done in [Liu17a]. Let $q \geq 0$ and recall from Definition 3.3.2 that $\mathcal{L}^q_{X^{\mathrm{an}}}$ denotes the sheaf $\ker(d'' \colon A^{q,0}_{X^{\mathrm{an}}} \to A^{q,1}_{X^{\mathrm{an}}})$.

**Definition 5.1.1.** Let $(X, \mathcal{O}_X)$ be a locally ringed space. The *$q$-th sheaf of Milnor $K$-theory* $\mathcal{K}^q_X$ for $(X, \mathcal{O}_X)$ is the sheaf associated to the presheaf that assigns to every open subset $U$ of $X$ the $\mathbb{Q}$-vector space $K^M_q(\mathcal{O}_X(U)) \otimes_{\mathbb{Z}} \mathbb{Q}$, where for $p \geq 1$ we define $K^M_q(\mathcal{O}_X(U))$ to be the abelian group generated by the symbols $\{f_1, \dots, f_q\}$ where $f_1, \dots, f_q, f'_i \in \mathcal{O}_X(U)^\times$ modulo the relations

(1) $\{f_1, \dots, f_i f'_i, \dots, f_q\} = \{f_1, \dots, f_i, \dots, f_q\} + \{f_1, \dots, f'_i, \dots, f_q\}$;

(2) $\{f_1, \dots, f, \dots, 1 - f, \dots, f_q\} = 0$.

We put $K_0^M(\mathcal{O}_X(U)) = \mathbb{Z}$ and $K_q^M(\mathcal{O}_X(U)) = 0$ for $q < 0$. If the locally ringed space $(X, \mathcal{O}_X)$ is clear from the context, we will write $\mathcal{K}^q$ instead of $\mathcal{K}_X^q$.

**Lemma 5.1.2.** *Let $X$ be a smooth scheme of finite type over $K$. For every $p \geq 0$ there exists a canonical isomorphism*

$$\mathrm{H}^q(X, \mathcal{K}_X^q) \cong \mathrm{CH}^q(X)_{\mathbb{Q}}.$$

*Proof.* See [Liu17a, Lemma 2.2(1)]. □

**- 5.1.3.** In [Liu17a] the author considers arbitrary $K$-analytic spaces. Since we are only interested in differential forms on the analytification of an algebraic variety, we will only consider this case and use the formalism of [Gub16] (see also Remark 3.2.17). Let $X$ be a variety over $K$. Define the morphism of sheaves

$$\tau_{X^{\mathrm{an}}}^q : \mathcal{K}_{X^{\mathrm{an}}}^q \to \mathcal{L}_{X^{\mathrm{an}}}^q$$

as follows. Let $V$ be an open subset of $X^{\mathrm{an}}$ and $\{f_1, \dots, f_q\} \in \mathcal{K}_{X^{\mathrm{an}}}^q(V)$, where $f_1, \dots, f_q \in \mathcal{O}_{X^{\mathrm{an}}}(V)^\times$. Then consider the analytic moment map $f := (f_1, \dots, f_q) : V \to (\mathbb{G}_{m,K}^{\mathrm{an}})^q$. Let $\alpha := d'x_1 \wedge \ldots \wedge d'x_q$ where we view $x_i := f_{i,\mathrm{trop}}$ as coordinates on $\mathbb{R}^q$. Using Lemma 3.2.15 we define

$$\tau_{X^{\mathrm{an}}}^q(\{f_1, \dots, f_q\}) := [(V_i, \varphi_{U_i}, F_i^* \alpha)_{i \in I}] \in A_{X^{\mathrm{an}}}^{q,0}(V)$$

where $(V_i, \varphi_{U_i})_{i \in I}$ is an open cover of $V$ by tropical charts and $F_i$ is an integral $\mathbb{R}$-affine morphism which satisfies $F_i \circ \varphi_{U_i,\mathrm{trop}} = f_{\mathrm{trop}}$ on $V_i$ for ever $i \in I$. Since $\alpha$ has constant coefficients, we clearly have

$$d'' \tau_{X^{\mathrm{an}}}^q(\{f_1, \dots, f_q\}) = 0$$

and hence $\tau_{X^{\mathrm{an}}}^q(\{f_1, \dots, f_q\}) \in \mathcal{L}_{X^{\mathrm{an}}}^q(V)$. Put $\mathcal{T}_{X^{\mathrm{an}}}^q := \mathcal{K}_{X^{\mathrm{an}}}^q / \ker \tau_{X^{\mathrm{an}}}^q$.

**- 5.1.4.** Let $p, q \in \mathbb{Z}$. By [Liu17a, Proposition 3.4.] and [Liu17a, Corollary 3.5.] there exists a canonical isomorphism of sheaves

$$\mathcal{T}_{X^{\mathrm{an}}}^q \otimes_{\underline{\mathbb{Q}}} \underline{\mathbb{R}} \cong \mathcal{L}_{X^{\mathrm{an}}}^q.$$

which induces canonical isomorphism of real vector spaces

$$\mathrm{H}^q(X^{\mathrm{an}}, \mathcal{T}_{X^{\mathrm{an}}}^p) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{H}_{d''}^{p,q}(X^{\mathrm{an}}).$$

In particular, the real vector space $\mathrm{H}_{d''}^{p,q}(X^{\mathrm{an}})$ has a canonical rational structure.

**Definition 5.1.5.** Let $X$ be a smooth variety over $K$. Define the *tropical cycle class map* $\mathrm{cl}_{\mathrm{trop}}$ as the composition

$$\begin{aligned}\mathrm{cl}_{\mathrm{trop}} : \mathrm{CH}^q(X)_{\mathbb{Q}} \cong \mathrm{H}^q(X, \mathcal{K}_X^q) \to \mathrm{H}^q(X^{\mathrm{an}}, \mathcal{K}_{X^{\mathrm{an}}}^q) \to \\ \to \mathrm{H}^q(X^{\mathrm{an}}, \mathcal{T}_{X^{\mathrm{an}}}^q) \hookrightarrow \mathrm{H}^q(X^{\mathrm{an}}, \mathcal{T}_{X^{\mathrm{an}}}^q) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{H}_{d''}^{q,q}(X).\end{aligned}$$

**- 5.1.6.** Finally we recall the definition of the first Chern form associated to a line bundle by following [GK17, § 8.] and [CLD12, § 6.2.]. Let $X$ be an algebraic variety over $K$.

**Definition 5.1.7.** A *continuous* (respectively *smooth*) *metric* $||\cdot||$ on $L$ is an assignment to each open subset $U \subseteq X^{\mathrm{an}}$ and local section $s \in L^{\mathrm{an}}(U)$ associates a continuous (respectively smooth) function

$$||s||\colon U \to \mathbb{R}_{\geq 0},$$

such that

(i) it is compatible with restriction to smaller open subsets;

(ii) for all $p \in U$ we have $||s(p)|| = 0$ if, and only if $s(p) = 0$;

(iii) for any $\lambda \in \mathcal{O}_{X^{\mathrm{an}}}(U)$ we have $||(\lambda s)(p)|| = |\lambda(p)|||s(p)||$.

The pair $\overline{L} = (L, ||\cdot||)$ is called a *metrized line bundle.*

**Proposition 5.1.8.** *There exists a smooth metric $||\cdot||$ on $L$.*

*Proof.* See [CLD12, Prop. 6.2.6.]. □

Let $\overline{L} = (L, ||\cdot||)$ a metrized line bundle on $X$ endowed with a smooth metric $||\cdot||$. Then by [CLD12, (6.4.1.)] there exists a canonical smooth form $c_1(L, ||\cdot||) = c_1(\overline{L}) \in A^{1,1}(X^{\mathrm{an}})$ called the *first Chern form* of $\overline{L}$ which for a regular section $s$ in $L^{\mathrm{an}}$ over an open subset $U \subseteq X^{\mathrm{an}}$ is given by

$$c_1(\overline{L})|_U = -d'd'' \log ||s||.$$

## 5.2 Comparison of Chern classes

Next we want to study the cycle class map from Definition 5.1.5 for $q = 1$. In this case we can make some significant simplifications. We adapt the results from Section 5.1 to our setup. Let $X$ be a smooth variety over $K$. By Definition 5.1.1 there is an isomorphism of sheaves

$$\mathcal{K}_X^1 \cong \mathcal{O}_X^\times \otimes_{\underline{\mathbb{Z}}} \underline{\mathbb{Q}}.$$

It is a classical result in algebraic geometry that there exist isomorphisms

$$\mathrm{CH}^1(X) \cong \mathrm{Pic}^1(X) \cong \mathrm{H}^1(X, \mathcal{O}_X^\times),$$

(for the first one see [Liu02, Ch. 7, Proposition 2.16.] and for the second one [Liu02, Ch. 7, Corollary 1.19.]) which extend to isomorphisms

$$\mathrm{CH}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{Pic}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{H}^1(X, \mathcal{O}_X^\times) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{5.1}$$

**Definition 5.2.1.** Let $E$ be a locally free sheaf of finite rank over $X$ and for every integer $k \geq 1$ let $c_k^{\mathrm{CH}}(E) \in \mathrm{CH}^k(X)$ denote its $k$-th Chern class. We define the *$k$-th tropical Chern class* as

$$c_{k,\mathrm{trop}}(E) := \mathrm{cl}_{\mathrm{trop}}(c_k^{\mathrm{CH}}(E)) \in \mathrm{H}_{d''}^{k,k}(X^{\mathrm{an}}).$$

**- 5.2.2.** We recall some basic notation about Čech cohomology as done in [Har77, III, § 4]. Let $X$ be a topological space, $\mathcal{F}$ a sheaf of abelian groups and $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of $X$. For every integer $p \geq 0$ the *abelian group of $p$-cochains* is defined as

$$\check{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{(i_0,\ldots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0\ldots i_p}),$$

where $U_{i_0\ldots i_p} := U_{i_0} \cap \ldots \cap U_{i_p}$. There is a differential

$$d\colon \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, \mathcal{F}),\ f \mapsto df \tag{5.2}$$

where

$$(df)_{i_0,\ldots,i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k f_{i_0\ldots\widehat{i}_k\ldots i_p+1}|_{U_{i_0\ldots i_{p+1}}}$$

By [Har77, p. 218] we have $d^2 = 0$ hence $(\check{C}^\bullet(\mathcal{U}, \mathcal{F}), d)$ defines a complex of abelian groups and we will denote by

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) := \mathrm{H}^p(\check{C}^\bullet(\mathcal{U}, \mathcal{F}), d)$$

the $p$-th cohomology group of this complex. Suppose that $\mathcal{V}$ is another open cover of $X$ that is a refinement of $\mathcal{U}$, then by [Har77, III, Exercise 4.4.] for every $p \geq 0$ there exists a natural morphism

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^p(\mathcal{V}, \mathcal{F}). \tag{5.3}$$

By *loc. cit.* the property of being a refinement defines an inductive order on the set of coverings of $X$, hence we can consider the inductive limit

$$\varinjlim_{\mathcal{U}} \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) =: \check{\mathrm{H}}^p(X, \mathcal{F}).$$

Recall that for every open cover $\mathcal{U}$ of $X$ we have a canonical morphism to the direct limit

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^p(X, \mathcal{F}). \tag{5.4}$$

For every $p \geq 0$ there exists a natural morphism

$$\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) \to \mathrm{H}^p(X, \mathcal{F})$$

to the sheaf cohomology groups of $X$ which are compatible with the refinement maps (5.3). By the universal property of direct limits for every $p \geq 0$ we get a canonical morphism

$$\check{\mathrm{H}}^p(X, \mathcal{F}) \to \mathrm{H}^p(X, \mathcal{F}).$$

By *loc. cit.* for $p = 1$ the canonical morphism

$$\check{\mathrm{H}}^1(X, \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{F}) \tag{5.5}$$

is an isomorphism.

**Proposition 5.2.3.** *Let $X$ be a smooth variety over $K$ and $\overline{L} := (L, ||\cdot||)$ be a line bundle over $X$ endowed with a smooth metric $||\cdot||$. Denote by $[c_1(\overline{L})]$ the cohomology class of the first Chern form $c_1(\overline{L}) \in A^{1,1}(X^{\mathrm{an}})$ defined in 5.1.6. Then*

$$c_{1,\mathrm{trop}}(L) = [c_1(\overline{L})] \in \mathrm{H}^{1,1}_{d''}(X^{\mathrm{an}}). \tag{5.6}$$

*Proof.* Let $L$ be given by $(U_i, \varphi_i)_{i\in I}$ where $\mathcal{U} := (U_i)_{i\in I}$ is an open cover of $X$ and $\varphi_i \colon L|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$ is an isomorphism of $\mathcal{O}_X|_{U_i}$-modules for every $i \in I$. For every $i, j \in I$, denote by $U_{ij} := U_i \cap U_j$ the intersection of two open subsets. For $i \in I$ let $s_i \in L(U_i)$ be the section with $\varphi(s_i) = 1$ in $\mathcal{O}_X(U_i)$. On $U_{ij}$ we have $s_j = f_{ij} \cdot s_i$ where $f_{ij} \in \mathcal{O}_X(U_{ij})^\times$. We will use the same notation for the analytification of these open subsets and of these morphisms. Consider the resolution of sheaves on $X^{\mathrm{an}}$ defined in Theorem 3.3.4

$$0 \to \mathcal{L}^0_X \to A^{1,0}_{X^{\mathrm{an}}} \xrightarrow{d''} A^{1,1}_{X^{\mathrm{an}}} \xrightarrow{d''} A^{1,2}_{X^{\mathrm{an}}} \xrightarrow{d''} \dots,$$

and the truncated part in lower degree

$$0 \to \mathcal{L}^0_X \xrightarrow{i} A^{1,0}_{X^{\mathrm{an}}} \xrightarrow{d''} \ker^1_{X^{\mathrm{an}}} \to 0,$$

where $i$ denotes the inclusion and $d''$ the differential and

$$\ker^1_{X^{\mathrm{an}}} := \ker(d'' \colon A^{1,1}_{X^{\mathrm{an}}} \xrightarrow{d''} A^{1,2}_{X^{\mathrm{an}}}).$$

We analyze the left-hand side and the right-hand side of (5.6) separately. We start with the right-hand side. By definition we have $c_1(\overline{L}) \in \ker^1_{X^{\mathrm{an}}}(X) = \check{\mathrm{H}}^0(\mathcal{U}, \ker^1_{X^{\mathrm{an}}})$. Then $\alpha = (\alpha_i)_{i\in I} := (-d' \log ||s_i||)_{i\in I}$ is an element in $\check{C}^0(\mathcal{U}, A^{1,0}_{X^{\mathrm{an}}})$

with $d''\alpha_i = c_1(\overline{L})|_{U_i} \in \ker^1_{X^{\mathrm{an}}}(U_i)$ for every $i \in I$ hence $d''\alpha = c_1(\overline{L})$ in $\check{C}^0(\mathcal{U}, \ker^1_{X^{\mathrm{an}}})$. Let $d$ denote the Čech coboundary operator from (5.2). Then we have

$$\begin{aligned}(d\alpha)_{ij} &= (\alpha_i - \alpha_j)|_{U_{ij}} = (-d' \log ||s_i|| - d' \log ||s_j||)|_{U_{ij}} \\ &= (-d' \log ||s_i|| - d' \log |f_{ij}| - d' \log ||s_i||)|_{U_{ij}} \\ &= (-d' \log |f_{ij}|)|_{U_{ij}} \in A^{1,0}(U_{ij}).\end{aligned}$$

Hence $d\alpha \in \check{C}^1(\mathcal{U}, A^{1,0}_{X^{\mathrm{an}}})$. On the left-hand side we start with

$$f \in \check{\mathrm{H}}^1(X, \mathcal{O}_X^\times)$$

which is the image of $c_1(L)$ under the isomorphisms (5.1) and (5.5). Let $\mathcal{V}$ be an open cover of $X$ and $f_{\mathcal{V}} \in \check{\mathrm{H}}^1(\mathcal{V}, \mathcal{O}_X^\times)$ whose image under the morphism (5.4) in $\check{\mathrm{H}}^1(X, \mathcal{O}_X^\times)$ is $f$. After chosing a common refinement of $\mathcal{U}$ and $\mathcal{V}$ we may assume that $\mathcal{V} = \mathcal{U}$ and

$$f_{\mathcal{U}} \in \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^\times).$$

Let $f'_{\mathcal{U}} = (f_{ij})_{i,j \in I}$ be a representative of $f_{\mathcal{U}}$ in $\check{C}^1(\mathcal{U}, \mathcal{O}_X^\times)$. For every $i, j \in I$ we have $f_{ij} \in \mathrm{Hom}(U_{ij}, \mathbb{G}^{\mathrm{an}}_{m,K})$. Via $\tau^1_{X^{\mathrm{an}}}$ we get a Čech cocycle

$$(\tau^1_{X^{\mathrm{an}}}(f'_{\mathcal{U}}))_{ij} := (d'(-\log |f_{ij}|))_{ij} \in \check{C}^1(\mathcal{U}, \mathcal{L}^0_X)$$

and

$$i(\tau^1_{X^{\mathrm{an}}}(f'_{\mathcal{U}})) = d\alpha \in \check{C}^1(\mathcal{U}, A^{1,0}_{X^{\mathrm{an}}}). \tag{5.7}$$

Note that on $\check{C}^0(\mathcal{U}, \mathcal{O}_X^\times)$ we have $d \circ \tau^1_{X^{\mathrm{an}}} = \tau^1_{X^{\mathrm{an}}} \circ d$ hence (5.7) is independent of the choice of the representative $f'_{\mathcal{U}}$. Then we have the following equality

$$c_{1,\mathrm{trop}}(L) = [i(\tau^1_{X^{\mathrm{an}}}(f'_{\mathcal{U}}))] = [d\alpha] = [c_1(\overline{L})] \in \check{\mathrm{H}}^1(\mathcal{U}, \ker^1_{X^{\mathrm{an}}}).$$

Via the morphism (5.4) and the isomorphism (5.5) we conclude that the the equality $c_{1,\mathrm{trop}}(L) = [c_1(\overline{L})]$ also holds in $\mathrm{H}^{1,1}_{d''}(X^{\mathrm{an}})$. □

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